On Hausdorff dimension in inhomogeneous Diophantine approximation over global function fields

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Abstract

In this paper, we study inhomogeneous Diophantine approximation over the completion $K_v$ of a global function field $K$ (over a finite field) for a discrete valuation $v$, with affine algebra $R_v$. We obtain an effective upper bound for the Hausdorff dimension of the set

$$\text{Bad}_A(\epsilon) = \left\{ \theta \in K_v^m : \liminf_{(p,q) \in R_v^m \times R_v^n, |q| \to \infty} \|q^n Aq - p^m \| \geq \epsilon \right\},$$

of $\epsilon$-badly approximable targets $\theta \in K_v^m$ for a fixed matrix $A \in \mathcal{M}_{m,n}(K_v)$, using an effective version of entropy rigidity in homogeneous dynamics for an appropriate diagonal action on the space of $R_v$-grids. We further characterize matrices $A$ for which $\text{Bad}_A(\epsilon)$ has full Hausdorff dimension for some $\epsilon > 0$ by a Diophantine condition of singularity on average. Our methods also work for the approximation using weighted ultrametric distances.

1 Introduction

In the theory of inhomogeneous Diophantine approximation of real numbers by rational ones (in several variables), one studies the distribution of the vectors $Ax \in \mathbb{R}^m$ modulo $\mathbb{Z}^m$, as $x$ varies over $\mathbb{Z}^n$, near a vector $b \in \mathbb{R}^m$ for a $m \times n$ real matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$. For instance, if $m, n \geq 1$ and $\langle \xi \rangle = \inf_{x \in \mathbb{Z}^m} \| \xi - x \|$ denotes the distance from $\xi \in \mathbb{R}^m$ to a nearest integral vector with respect to the Euclidean norm $\| \cdot \|$ on $\mathbb{R}^m$, using the inhomogeneous Khintchine-Groshev theorem of [Sch1 Theorem 1], we have

$$\liminf_{x \in \mathbb{Z}^n, \|x\| \to \infty} \|x\|^n \langle Ax - b \rangle^m = 0$$

for almost every $(A, b) \in \mathcal{M}_{m,n}(\mathbb{R}) \times \mathbb{R}^m$.

Let us consider the exceptional set of solutions $(A, b)$ of the above equation. We call $A$ badly approximable for $b$ if

$$\liminf_{x \in \mathbb{Z}^n, \|x\| \to \infty} \|x\|^n \langle Ax - b \rangle^m > 0.$$ 

If the left hand side is at least $\epsilon$, we say that $A$ is $\epsilon$-bad for $b$. It is known that given any $b \in \mathbb{R}^m$, the set of badly approximable matrices $A \in \mathcal{M}_{m,n}(\mathbb{R})$ has zero Lebesgue measure but full Hausdorff dimension, see [Sch2 ET].

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In this paper, replacing $\mathbb{R}$ by a completion of a function field and using weighted norms, given an $m \times n$ matrix $A$, under a necessary assumption of non-singularity on average, we prove that the set of $m$-dimensional vectors with respect to which $A$ is $\epsilon$-bad does not have full Hausdorff dimension, and we obtain an explicit upper bound: there exists $c(A) > 0$, depending only on $m$, $n$ and $A$, such that for every $\epsilon > 0$, the Hausdorff dimension of the set of $m$-dimensional vectors that are $\epsilon$-bad for $A$ is bounded from above by $m - c(A) \ln(1/\epsilon)$.

See [KKL] for analogous results in the case of the real field. Let us state our main results, referring to Section 2.1 for more precise definitions.

Let $K$ be any global function field over a finite field $\mathbb{F}_q$ of $q$ elements for a prime power $q$, that is, the function field of a geometrically connected smooth projective curve $C$ over $\mathbb{F}_q$. The most studied example in Diophantine approximation in positive characteristic is the case of the field $K = \mathbb{F}_q(Z)$ of rational fractions in one variable $Z$ over $\mathbb{F}_q$, where $C = \mathbb{P}^1$ is the projective line, but we emphasize the fact that our work applies in the general situation above.

We fix a (normalized) discrete valuation $v$ on $K$. Let $K_v$ and $\mathcal{O}_v$ be the completion of $K$ with respect to $v$ and its valuation ring, respectively. We fix a uniformizer $\pi_v \in K$, which satisfies $v(\pi_v) = 1$. Let $k_v = \mathcal{O}_v/\pi_v \mathcal{O}_v$ be the residual field and let $q_v$ be its cardinality. The (normalized) absolute value $| \cdot |$ associated with $v$ is defined by $| x | = q_v^{-v(x)}$. For every $\sigma \in \mathbb{Z}_{\geq 1}$, let $\| \cdot \| : K_v^\sigma \to [0, +\infty[$ be the norm $(\xi_1, \ldots, \xi_\sigma) \mapsto \max_{1 \leq i \leq \sigma} | \xi_i |$. We denote by $\dim_{\text{Haus}}$ the Hausdorff dimension of the subsets of $K_v^m$ for this standard norm.

The discrete object analogous to the set of integers $\mathbb{Z}$ in $\mathbb{R}$ is the affine algebra $R_v$ of the curve $C$ minus the point $v$. If $K = \mathbb{F}_q(Z)$ and $v = [1 : 0]$ is the standard point at infinity of $C = \mathbb{P}^1$, then $R_v = \mathbb{F}_q[Z]$ is the ring of polynomials in $Z$ over $\mathbb{F}_q$.

Let $m, n \in \mathbb{Z}_{\geq 1}$. Let us fix, throughout the paper, two weights consisting of a $m$-tuple $r = (r_1, \ldots, r_m)$ and a $n$-tuple $s = (s_1, \ldots, s_n)$ of positive integers such that we have $|r| = \sum_{1 \leq i \leq m} r_i = \sum_{1 \leq j \leq n} s_j$. The $r$-quasinorm of $\xi \in K_v^m$ and $s$-quasinorm of $\theta \in K_v^n$ are given by

$\| \xi \|_r = \max_{1 \leq i \leq m} | \xi_i |^{1/r_i}$ \quad and \quad $\| \theta \|_s = \max_{1 \leq j \leq n} | \theta_j |^{1/s_j}$.

We denote by $\langle \xi \rangle_r = \inf_{x \in R_v^m} \| \xi - x \|_r$ the (weighted) distance from $\xi$ to the set $R_v^m$ of integral vectors in $K_v^m$.

Let $\epsilon > 0$. A matrix $A \in \mathcal{M}_{m,n}(K_v)$ is said to be $\epsilon$-bad for a vector $\theta \in K_v^m$ if

$$\liminf_{x \in R_v^m, \| x \|_s \to \infty} \| x \|_s \langle A x - \theta \rangle_r \geq \epsilon .$$

(1)

Denote by $\text{Bad}_A(\epsilon)$ the set of vectors $\theta \in K_v^m$ such that $A$ is $\epsilon$-bad for $\theta$. Given two subsets $U$ and $V$ of a given set, we denote $U - V = \{ x \in U : x \notin V \}$. We say that a matrix $A \in \mathcal{M}_{m,n}(K_v)$ is $(r, s)$-singular on average if for every $\epsilon > 0$, we have

$$\lim_{N \to \infty} \frac{1}{N} \text{Card}\{ \ell \in \{1, \ldots, N\} : \exists y \in R_v^m - \{0\}, \langle A y \rangle_r \leq \epsilon q_v^{-\ell}, \| y \|_s \leq q_v^\ell \} = 1 .$$

(2)

For the basic example of function field, when $K = \mathbb{F}_q(Z)$ and $v = [1 : 0]$, Bugeaud and Zhang [BZ] found a sufficient condition (and an equivalent one when $n = m = 1$) on $A$ for the Hausdorff dimension of $\text{Bad}_A(\epsilon)$ to be full. We first strengthen and extend their result to general function fields.
Theorem 1.1 Let $A \in \mathcal{M}_{m,n}(K_v)$ be a matrix. The following assertions are equivalent:
(1) there exists $\varepsilon > 0$ such that the set $\text{Bad}_A(\varepsilon)$ has full Hausdorff dimension,
(2) the matrix $A$ is $(r,s)$-singular on average.

We also provide an effective upper bound on the Hausdorff dimension in terms of $\varepsilon$, which is a new result even in the basic case $K = \mathbb{F}_q[Z]$ and $v = [1 : 0]$.

Theorem 1.2 For every $A \in \mathcal{M}_{m,n}(K_v)$ which is not $(r,s)$-singular on average, there exists a constant $c(A) > 0$ depending only on $A$, $r$, $s$, such that for every $\varepsilon > 0$, we have
$$\dim_{\text{Haus}} \text{Bad}_A(\varepsilon) \leq m - c(A) \frac{\varepsilon^r}{m(1/\varepsilon)}.$$

The proofs of the above main theorems of this paper are largely divided into two parts. Firstly, assuming the singular on average property in order to prove the full Hausdorff dimension property, we give a lower bound on the Hausdorff dimension of appropriately chosen subsets of $K_v$, using new function fields versions of classical tools in Diophantine approximation such as geometry of numbers, transference principle and best approximation such as geometry of numbers, transference principle and best approximation principle. Secondly, in order to prove the upper bound in Theorem 1.2, we use technics of homogeneous dynamics of diagonal actions and in particular the entropy method (see for instance [Kle, EL, LSS, ELW]). Let us explain briefly the latter part.

Let $d = m + n$. The dynamical space relevant to inhomogeneous Diophantine approximation is the space $\mathcal{Y}$ of unimodular grids $A + b$ in $K_v$, that is of (Haar-covolume 1) $R_v$-lattices $\Lambda$ of $K_v$ translated by vectors $b \in K_v$, endowed with the affine action of the diagonal subgroup of $\text{SL}_d(K_v)$. This is in higher dimension more convenient than the study of the commuting actions of $\text{SL}_d(R_v)$ and of the diagonal group on the Bruhat-Tits building associated with $\text{SL}_d(K_v)$ (see [BPP] Part III) when $d = 2$. Given the above weights $r$ and $s$, we consider the affine action on $\mathcal{Y}$ of the 1-parameter diagonal subgroup $(a^k)_{k \in \mathbb{Z}}$ where
$$a = \text{diag}(\pi_v^{-r_1}, \ldots, \pi_v^{-r_m}, \pi_v^{s_1}, \ldots, \pi_v^{s_n}).$$

The space $\mathcal{Y}$ of unimodular grids $a$-equivariantly projects onto the space of unimodular $R_v$-lattices $\mathcal{X} = \text{SL}_d(K_v)/\text{SL}_d(R_v)$ by the map sending $A + b$ to $\Lambda$. We say that a unimodular $R_v$-lattice $\Lambda$ diverges on average under the action of $a$ if for every compact subset $Q$ of $\mathcal{X}$, we have
$$\lim_{N \to \infty} \frac{1}{N} \text{Card} \{ \ell \in \{1, \ldots, N\} : a^\ell \Lambda \notin Q \} = 1.$$

Following Dani's path, we prove in Section 5.2 that the lattice $\Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v$ diverges on average under $a$ if and only if $A$ is $(r,s)$-singular on average.

As developed in the last Section 6 the main idea of the entropy method in our situation is that if the point $\Lambda_A$ does not diverge on average, then the Hausdorff dimension of $\text{Bad}_A(\varepsilon)$ provides a lower bound on the conditional entropy of $a$ with respect to a measure $\mu$ constructed by well-separated sets on the fibers of the projection $\mathcal{Y} \to \mathcal{X}$. An effective control of the maximal conditional entropy by a control of the support of $\mu$ on the thin/thick parts of $\mathcal{Y}$ hence gives an effective upper bound. See [KKL] for the real case.

Our paper is organized as follows. In Section 2 we recall basic facts on the geometry of numbers, define the best approximation sequence and prove transference principle for the weighted case, which generalize previous results of Bugeaud-Zhang [BZ]. In Section 3...
we give a characterization of the singular on average property with weights in terms of the best approximation sequence. In Section 3, we establish the lower bound on the Hausdorff dimension by constructing a subsequence with controlled growth of the best approximation sequence for a matrix whose transpose is singular on average. In Section 5, we recall some background on homogeneous dynamics and conditional entropy, and prove an effective and positive characteristic version of the variational principle for conditional entropy of [EL] §7.55 (see [KKL] in the real case). Finally, in Section 6, we implement the entropy method.

We remark that in [KKL], the first two authors, with Wooyeon Kim, also show that given any vector \( b \in \mathbb{R}^m \), the set of matrices \( A \in \mathcal{M}_{m,n}(\mathbb{R}) \) that are \( \epsilon \)-bad for \( b \) does not have full Hausdorff dimension, and estimate an explicit upper bound. Thus it seems very interesting to obtain a similar result in the global function field case.

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2 Background material for the lower bound

2.1 On global function fields

We refer for instance to [Gos, Ros], as well as [BPP] §14.2, for the content of this section. Let \( \mathbb{F}_q \) be a finite field with \( q \) elements, where \( q \) is a positive power of a positive prime. Let \( K \) be the function field of a geometrically connected smooth projective curve \( C \) over \( \mathbb{F}_q \), or equivalently an extension of \( \mathbb{F}_q \) with transcendence degree 1, in which \( \mathbb{F}_q \) is algebraically closed. We denote by \( g \) the genus of \( C \). There is a bijection between the set of closed points of \( C \) and the set of normalized discrete valuations \( v \) of \( K \), the valuation of a given element \( f \in K \) being the order of the zero or the opposite of the order of the pole of \( f \) at the given closed point. We fix such an element \( v \) throughout this paper, and use the notation \( K_v, \mathcal{O}_v, \pi_v, k_v, q_v, | \cdot | \) defined in the introduction. We furthermore denote by \( \deg_v \) the degree of \( v \), so that

\[
q_v = q^{\deg_v}.
\]

We denote by \( \text{vol}_v \) the normalized Haar measure on the locally compact additive group \( K_v \) such that \( \text{vol}_v(\mathcal{O}_v) = 1 \). For any positive integer \( d \), let \( \text{vol}_v^d \) be the normalized Haar measure on \( K_v^d \) such that \( \text{vol}_v^d(\mathcal{O}_v^d) = 1 \). Note that for every \( g \in \text{GL}_d(K_v) \) we have

\[
d \text{vol}_v^d(gx) = | \det(g) | \, d \text{vol}_v^d(x),
\]

where \( \det \) is the determinant of a matrix. For every discrete additive subgroup \( \Lambda \) of \( K_v^d \), we again denote by \( \text{vol}_v^d \) (and simply \( \text{vol}_v \) when \( d = 1 \)) the measured induced on \( K_v^d/\Lambda \) by \( \text{vol}_v^d \).

Note that the completion \( \hat{K}_v \) of \( K \) for \( v \) is the field \( k_v((\pi_v)) \) of Laurent series \( x = \sum_{i \in \mathbb{Z}} x_i(\pi_v)^i \) in the variable \( \pi_v \) over \( k_v \), where \( x_i \in k_v \) is zero for \( i \in \mathbb{Z} \) small enough. We have

\[
| x | = q_v^{-\sup\{j \in \mathbb{Z} : \forall i < j, x_i = 0 \}},
\]

and \( \mathcal{O}_v = k_v[[\pi_v]] \) is the local ring of power series \( x = \sum_{i \in \mathbb{Z} \geq 0} x_i(\pi_v)^i \) in the variable \( \pi_v \) over \( k_v \).
Recall that the affine algebra \( R_v \) of the affine curve \( C - \{v\} \) consists of the elements of \( K \) whose only poles are at the closed point \( v \) of \( C \). Its field of fractions is equal to \( K \), hence we can write elements of \( K \) as \( x/y \) with \( x, y \in R_v \) and \( y \neq 0 \). By for instance \([\text{BPP}]\) Eq. (14.2)], we have
\[
R_v \cap \mathcal{O}_v = \mathbb{F}_q .
\]
For every \( \xi \in K_v \), we denote by
\[
|\langle \xi \rangle| = \inf_{x \in R_v} \| \xi - x \|
\]
the distance in \( K_v \) from \( \xi \) to the set \( R_v \) of integral points of \( K_v \).

For instance, if \( C \) is the projective line \( \mathbb{P}^1 \), if \( \infty = [1 : 0] \) is its usual point at infinity and if \( Z \) is a variable name, then \( g = 0 \), \( K = \mathbb{F}_q(Z) \), \( \pi_\infty = Z^{-1} \), \( K_\infty = \mathbb{F}_q((Z^{-1})) \), \( \mathcal{O}_\infty = \mathbb{F}_q[[Z^{-1}]] \), \( k_\infty = \mathbb{F}_q \), \( q_\infty = q \) and \( R_\infty = \mathbb{F}_q[Z] \). In this setting, there are numerous results on Diophantine approximation in the fields of formal power series, see for instance \([\text{Las}], [\text{Bug}, \text{Chap. 9}]\). On the other hand, little is known about Diophantine approximation over general global function fields, see for instance \([\text{KIST}]\) (for a single valuation in positive characteristic) for the ground work on the geometry of number for function fields.

### 2.2 On the geometry of numbers and Dirichlet’s theorem

Let \( d \) be a positive integer. An \( R_v \)-lattice \( \Lambda \) in \( K_v^d \) is a discrete \( R_v \)-submodule in \( K_v^d \) that generates \( K_v^d \) as a \( K_v \)-vector space. The \textit{covolume} of \( \Lambda \), denoted by \( \text{Covol}(\Lambda) \), is defined as the measure of the (compact) quotient space \( K_v^d/\Lambda \):
\[
\text{Covol}(\Lambda) = \text{vol}_v(K_v^d/\Lambda) .
\]

For example, \( R_v^d \) is an \( R_v \)-lattice in \( K_v^d \), and by for instance \([\text{BPP}, \text{Lem. 14.4}]\), we have
\[
\text{Covol}(R_v^d) = q^{(g-1)d} .
\]

Let \( \overline{B}(0, r) \) be the closed ball of radius \( r \) centered at zero in \( K_v^d \) with respect to the norm \( \| \cdot \| : (\xi_1, \ldots, \xi_d) \mapsto \max_{1 \leq i \leq d} |\xi_i| \). For every integer \( k \in \{1, \ldots, d\} \), the \( k \)-th \textit{minimum} of an \( R_v \)-lattice \( \Lambda \) is defined by
\[
\lambda_k(\Lambda) = \min \{ r > 0 : \dim_{K_v}(\text{span}_{K_v}(\overline{B}(0, r) \cap \Lambda)) \geq k \},
\]
where \( \text{span}_{K_v} \) denotes the \( K_v \)-linear span of a subset of a \( K_v \)-vector space and \( \dim_{K_v} \) is the dimension of a \( K_v \)-vector space. Note that \( \lambda_1(\Lambda), \ldots, \lambda_d(\Lambda) \in q_v^{-1} \). The next result follows from \([\text{KIST}], \text{Theo. 4.4}\) and Equation (14).

**Theorem 2.1 (Minkowski’s theorem)** For every \( R_v \)-lattice \( \Lambda \) in \( K_v^d \), we have
\[
q^{-(g-1)d} \text{ Covol}(\Lambda) \leq \lambda_1(\Lambda) \cdots \lambda_d(\Lambda) \leq q_v^d \text{ Covol}(\Lambda) . \quad \square
\]

Since \( \lambda_1(\Lambda) \leq \cdots \leq \lambda_d(\Lambda) \), the following result follows immediately from Minkowski’s theorem.2.1

**Corollary 2.2** For every \( R_v \)-lattice \( \Lambda \) in \( K_v^d \), we have
\[
\lambda_1(\Lambda) \leq q_v \text{ Covol}(\Lambda)^\frac{1}{d} . \quad \square
\]
The following result generalizes [GG] Theo. 2.1, which is proved only when $K = \mathbb{F}_q(Z)$ and $v = \infty$, to all function fields $K$ and valuations $v$. See also [KlW] Theo. 1.3 in the case of the field $\mathbb{Q}$.

**Theorem 2.3 (Dirichlet’s theorem)** For every matrix $A \in M_{m,n}(K_v)$ whose rows are denoted by $A_1, \ldots, A_m$, for all $(r_1', \ldots, r_m') \in \mathbb{Z}_{\geq 0}^m$ and $(s_1', \ldots, s_n') \in \mathbb{Z}_{\geq 0}^n$ with

$$r_i' > 1 + \frac{q-1}{\deg v} \quad \text{and} \quad \sum_{i=1}^m r_i' = \sum_{j=1}^n s_j',$$

there exists an element $y = (y_1, \ldots, y_n) \in R_v^n - \{0\}$ such that, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we have

$$|\langle A_i y \rangle| \leq q_v q^{q-1} q_v^{-r_i'} \quad \text{and} \quad |y_j| \leq q_v q^{q-1} q_v^{s_j'}.$$

**Proof.** With $A, r_1', \ldots, r_m'$ and $s_1', \ldots, s_n'$ as in the statement, we apply Corollary 2.2 with $d = m + n$ to the $R_v$-lattice

$$\Lambda = \begin{pmatrix} \pi_v^{-r_1'} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \pi_v^{-r_m'} & \cdots & \pi_v^{-s_n'} \\ 0 & \cdots & \pi_v^{-s_n'} \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d,$$

where $I_k$ is the $k \times k$ identity matrix. Since the above two matrices have determinant 1 by the assumption $\sum_{i=1}^m r_i' = \sum_{j=1}^n s_j'$, and by Equation (1), we have $\text{Covol}(\Lambda) = q^{(q-1)d}$.

Corollary 2.2 hence says that there exists $(x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_n)) \in R_v^n - \{0\}$ such that

$$\max \left\{ \max_{i=1, \ldots, m} |\pi_v^{-r_i'}(x_i + A_i y)|, \max_{j=1, \ldots, n} |\pi_v^{s_j'} y_j| \right\} \leq q_v \text{Covol}(\Lambda)^{\frac{1}{d}} = q_v q^{q-1}.$$

Assume for a contradiction that $y = 0$. Then for all $i = 1, \ldots, m$, since $|\pi_v| = q_v^{-1}$, we have the inequality $|x_i| \leq q_v q^{q-1} q_v^{-r_i'}$. Since $r_i' > 1 + \frac{q-1}{\deg v}$, this would imply that $|x_i| < 1$. By Equation (3), we have $\{z \in R_v : |z| < 1\} = \{0\}$. Since $x_i \in R_v$, we would have that $x = 0$, contradicting the fact that $(x, y) \neq 0$. Therefore $y \neq 0$ and the result follows. □

The following corollary is due to [Kr1] Theo. 1.1 (see also [BZ] Theo. 3.2 where the assumption that $cm$ is divisible by $n$ is implicit) in the special case when $K = \mathbb{F}_q(Z)$ and $v = \infty$ and without weights.

Let $\min r = \min_{1 \leq i \leq m} r_i$ and similarly for $\max s$, $\min r$ and $\max s$.

**Corollary 2.4** For all $A \in M_{m,n}(K_v)$ and $\alpha \in \mathbb{Z}_{\geq 0}$ with $\alpha > \frac{1}{\min r} + \frac{q-1}{(\min r)(\deg v)}$, there exists $y \in R_v^n - \{0\}$ such that

$$\langle Ay \rangle_r \leq q^{\deg v + q-1} q_v^\alpha \quad \text{and} \quad \|y\|_s \leq q^{\deg v + q-1} q_v^\alpha.$$

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Proof. We apply Theorem 2.3 with \( r_i' = \alpha r_i > 1 + \frac{q_{v-1}}{\deg v} \) for \( i = 1, \ldots, m \) and \( s'_j = \alpha s_j \) for \( j = 1, \ldots, n \), noting that \( \sum_{i=1}^m r'_i = \sum_{j=1}^n s'_j \) since \( \sum_{i=1}^m r_i = \sum_{j=1}^n s_j \).

Remark. When \( r = (n, n, \ldots, n) \) and \( s = (m, m, \ldots, m) \), the above result says that for every integer \( \alpha > \frac{1}{n} + \frac{q_{v-1}}{n \deg v} \), there exists \( y \in R^n_v - \{0\} \) such that

\[
\min_{x \in R^n_v} \| A y - x \| \leq q_v q^{g-1} q_v^{-\alpha n} \text{ and } \| y \| \leq q_v q^{g-1} q_v^{\alpha m},
\]

where \( \| \cdot \| \) is the sup norm.

2.3 Best approximation sequences with weights

In this subsection, we construct a version with weights, valid for all function fields, of the best approximation sequences associated with a completely irrational matrix by Bugeaud-Zhang [BZ].

A matrix \( A \in \mathcal{M}_{m,n}(K_v) \) is said to be completely irrational if \( \langle A y \rangle_r \neq 0 \) for every \( y \in R^n_v - \{0\} \). Note that this does not depend on the weight \( r \), and that the fact that \( A \) is completely irrational not necessarily imply that \( t_A \) is completely irrational.

Remark 2.5 Let \( A \in \mathcal{M}_{m,n}(K_v) \) be such that \( t_A \) is not completely irrational.

(1) The matrix \( t_A \) is \((s, r)\)-singular on average.

(2) For every \( \epsilon > 0 \) small enough, the set \( \text{Bad}_A(\epsilon) \) has full Hausdorff dimension.

Proof. By assumption, there exist \( x \in R^n_v \) and \( y = (y_1, \ldots, y_m) \in R^m_v - \{0\} \) such that \( t_A y - x = 0 \).

(1) For every \( \epsilon > 0 \), if \( \ell_0 = \lceil \log q_v \| y \|_r \rceil \) then for all integers \( N \geq \ell_0 \) and \( \ell \in \{\ell_0, \ldots, N\} \), we have \( \langle t_A y \rangle_s = 0 \leq \epsilon q_v^\ell \) and \( \| y \|_r \leq q_v^\ell \), hence \( t_A \) is \((s, r)\)-singular on average (see Equation (2)).

(2) For every \( \theta = (\theta_1, \ldots, \theta_m) \in K^m_v \), let

\[
y \cdot \theta = \sum_{j=1}^m y_j \theta_j \in K_v.
\]

For every \( \epsilon \in [0, \frac{1}{\|y\|_r}] \), let \( U_{y, \epsilon} = \{ \theta \in K^m_v : \langle y \cdot \theta \rangle \geq (\epsilon \| y \|_r)^{\min r} \} \). If \( \epsilon \) is small enough, then the set \( U_{y, \epsilon} \) contains a closed ball of positive radius: For instance, let \( j_0 \in \{1, \ldots, m\} \) be such that \( y_{j_0} \neq 0 \); define \( \theta_{0,j} = 0 \) if \( j \neq j_0 \), \( \theta_{0,j_0} = \frac{y_{j}}{y_{j_0}} \) and \( \theta_0 = (\theta_{0,1}, \ldots, \theta_{0,m}) \); then it is easy to check using the ultrametric inequality that the closed ball \( \overline{B}(\theta_0, \frac{1}{\|y\|_r}) \) is contained in \( U_{y, \epsilon} \) if \( \epsilon < \frac{1}{\min r} \| y \|^{-1}_r \).

Let us prove that \( \text{Bad}_A(\epsilon) \) contains \( U_{y, \epsilon} \), which implies that \( \dim_{\text{Haus}}(\text{Bad}_A(\epsilon)) = m \) if \( \epsilon \) is small enough. Let \( \theta \in U_{y, \epsilon} \) and \( (y', x') \in R^n_v \times (R^n_v - \{0\}) \).

If \( \| y \|_r \| Ax' + y' - \theta \|_r \geq 1 \), then since \( x' \in R^n_v - \{0\} \) so that \( \| x' \|_s \geq 1 \), we have

\[
\| x' \|_s \| Ax' + y' - \theta \|_r \geq \frac{1}{\| y \|_r} \| y \|_r \| Ax' + y' - \theta \|_r \geq \frac{1}{\| y \|_r} \geq \epsilon.
\]
If \( \| y \|_r \| A x' + y' - \theta \|_r \leq 1 \), then since \( y \cdot (A x' + y') = (A y) \cdot x' + y \cdot y' = x \cdot x' + y \cdot y' \in R_v \), and since \( \theta \in U_{y, r} \), we have

\[
\| x' \|_s \| A x' + y' - \theta \|_r \geq \frac{1}{\| y \|_r} \| y \|_r \| A x' + y' - \theta \|_r \\
\geq \frac{1}{\| y \|_r} \left( \max_{1 \leq j \leq m} \| y \|_{r_j} \| A x' + y' - \theta \|_{r_j} \right)^{\frac{1}{\min r}} \\
\geq \frac{1}{\| y \|_r} \| y \cdot (A x' + y' - \theta) \|_{\min r} \geq \frac{1}{\| y \|_r} \langle y \cdot \theta \rangle^{\frac{1}{\min r}} \geq \epsilon.
\]

Therefore \( \theta \in \mathrm{Bad}_A(\epsilon) \), as wanted. \( \square \)

For every matrix \( A \in \mathcal{M}_{m,n}(K_v) \), a best approximation sequence for \( A \) with weights \((r, s)\) is a sequence \((y_i)_{i \geq 1}\) in \( R_v^n \) such that, with \( Y_i = \| y_i \|_s \) and \( M_i = \langle A y_i \rangle_r \),

- the sequence \((Y_i)_{i \geq 1}\) is positive and strictly increasing,
- the sequence \((M_i)_{i \geq 1}\) is positive and strictly decreasing, and
- for every \( y \in R_v^n - \{0\} \) with \( \| y \|_s < Y_{i+1} \), we have \( \langle A y \rangle_r \geq M_i \).

We denote by \( \text{lcm} \) the least common multiple of \( r_1, \ldots, r_m \), and similarly for \( \text{lcm} \ s \).

**Lemma 2.6** Assume that \( A \in \mathcal{M}_{m,n}(K_v) \) is completely irrational.

1. There exists a best approximation sequence \((y_i)_{i \geq 1}\) for \( A \) with weights \((r, s)\).
2. If \((y_i)_{i \geq 1}\) is a best approximation sequence for \( A \) with weights \((r, s)\), then
   - i) we have \( M_i \leq q_i^\frac{1}{\text{lcm} r} \) and \( M_i \leq q_i^\frac{1}{\text{lcm} s} \) if \( i \) is large enough,
   - ii) we have \( Y_i \geq q_i^\frac{1}{\text{lcm} s} \) and \( Y_i \geq q_i^\frac{1}{\text{lcm} r} \) for every \( i \geq 1 \),
   - iii) the sequence \((M_i Y_{i+1})_{i \geq 1}\) is uniformly bounded.

Note that a best approximation sequence might be not unique (and the terminology "best", though traditional, is not very appropriate). When \( m = n = r_1 = s_1 = 1 \), \( K = \mathbb{F}_q(Z) \) and \( v = \infty \), then \( A \in K_v \) is completely irrational if and only if \( A \in K_v - K \), and with \((\frac{P}{Q_i})_{i \geq 1}\) the sequence of convergents of \( A \) (see for instance [Las]), we may take \( y_i = Q_{i-1} \) for all \( i \geq 1 \).

If \( A \in \mathcal{M}_{m,n}(K_v) \) is not completely irrational, a best approximation sequence for \( A \) with weights \((r, s)\) is a finite sequence \((y_i)_{1 \leq i \leq i_0}\) in \( R_v^n \), such that, with \( Y_i = \| y_i \|_s \) and \( M_i = \langle A y_i \rangle_r \),

- \( 1 = Y_1 < \cdots < Y_{i_0} \),
- \( M_1 > \cdots > M_{i_0} = 0 \),
- for all \( i \in \{1, \ldots, i_0 - 1\} \) and \( y \in R_v^n - \{0\} \) with \( \| y \|_s < Y_{i+1} \), we have \( \langle A y \rangle_r \geq M_i \),
- which stops at the first \( i_0 \) such that there exists \( z \in R_v^n \) with \( 0 < \| z \|_s \leq Y_{i_0} \) and \( \langle A z \rangle_r = 0 \).

The proof of Lemma 2.6 is similar to the one given after [BZ] Def. 3.3] in the particular case when \( K = \mathbb{F}_q(Z) \), \( v = \infty \) and without weights.

**Proof.** (1) Let us prove by induction on \( i \geq 1 \) that there exist \( y_1, \ldots, y_i \) in \( R_v^n \) such that, with \( Y_j = \| y_j \|_s \) and \( M_j = \langle A y_j \rangle_r \) for every \( 1 \leq j \leq i \), we have \( 1 = Y_1 < \cdots < Y_i \), \( M_1 > \cdots > M_i > 0 \), and (using \( M_0 = +\infty \) by convention)

(a) we have \( \langle A y \rangle_r \geq M_{i-1} \) for every \( y \in R_v^n - \{0\} \) with \( \| y \|_s < Y_i \),
\((b_1)\) we have \(\langle Ay \rangle_r \geq M_i\) for every \(y \in R^n_v - \{0\}\) with \(\|y\|_s \leq Y_i\).

Note that \(\{x \in R_v : |x| \leq 1\} = R_v \cap \mathcal{O}_v = \mathbb{F}_q\) by Equation (3). Hence the elements with smallest \(s\)-quasinorm in \(R^n_v - \{0\}\) are the elements in the finite set \(\mathbb{F}_q^n - \{0\}\), which is the set of elements in \(R^n_v\) with \(s\)-quasinorm 1. Furthermore, the set \(\{\|y\|_s : y \in R^n_v - \{0\}\}\) is contained in \(\bigcup_{n=1}^{\infty} \mathbb{F}_q^n = \mathbb{F}_q^\infty\). Similarly, for every \(x \in K^n_m - \{0\}\), we have \(\langle x \rangle_r \in q^{\min S Z}\).

Therefore there exists an element \(y_1 \in R^n_v\) with \(\|y_1\|_s = 1\) such that

\[\langle Ay_1 \rangle_r = \min\{\langle Ay \rangle_r : y \in R^n_v, \|y\|_s = 1\}\]

We thus have \(Y_1 = \|y_1\|_s = 1\) and \(M_1 = \langle Ay_1 \rangle_r > 0\) since \(A\) is completely irrational. There is no \(y \in R^n_v - \{0\}\) with \(\|y\|_s < Y_1\), and if \(\|y\|_s = Y_1\), then \(\langle Ay \rangle_r \geq M_1\), hence the claims \((a_1)\) and \((b_1)\) are satisfied.

Assume by induction that \(y_1, \ldots, y_i\) as above are constructed. Let

\[S = \{y \in R^n_v : \|y\|_s > Y_i, \langle Ay \rangle_r < M_i\}\]

Note that the set \(\{z \in R^n_v, 0 < \|z\|_s \leq Y_i\}\) is finite by the discreteness of \(R^n_v\), and \(\epsilon_i = \min\{\langle Az \rangle_r : z \in R^n_v, 0 < \|z\|_s \leq Y_i\}\) is positive, since \(A\) is completely irrational. Corollary \([2,3]\) of Dirichlet’s theorem implies in particular, by taking in its statement \(\epsilon\) large enough, that for every \(\epsilon > 0\), there exists \(y \in R^n_v - \{0\}\) such that \(\langle Ay \rangle_r < \epsilon\). Applying this with \(\epsilon = \min\{M_i, \epsilon_i\} > 0\) proves that the set \(S\) is nonempty. Hence the set \(S_{\min}\) of elements of \(S\) with minimal \(s\)-quasinorm, which is finite again by the discreteness of \(R^n_v\), is nonempty. Therefore there exists \(y_{i+1} \in S_{\min}\) such that

\[\langle Ay_{i+1} \rangle_r = \min\{\langle Az \rangle_r : z \in S_{\min}\}\]

Then \(Y_{i+1} = \|y_{i+1}\|_s = \min\{\|y\|_s \geq Y_i\}\). The definition of the set \(S\). We also have that \(M_{i+1} = \langle Ay_{i+1} \rangle_r < M_i\) since \(y_{i+1} \in S_{\min} \subseteq S_{\min}\), and again by the definition of \(S\).

Let us now prove that \(y_{i+1}\) satisfies the properties \((a_{i+1})\) and \((b_{i+1})\).

- Let \(y \in R^n_v - \{0\}\) be such that \(\|y\|_s < Y_{i+1}\). If \(\|y\|_s < Y_i\), then by the induction hypothesis \((b_i)\), we have \(\langle Ay \rangle_r \geq M_i\), as wanted for Property \((a_{i+1})\). If \(\|y\|_s > Y_i\), then by the definition of \(S\), we have \(\langle Ay \rangle_r \geq M_i\) as wanted for Property \((a_{i+1})\), otherwise \(y\) would be an element of \(S\) with \(s\)-quasinorm strictly less than the minimum \(s\)-quasinorm of the elements of \(S\), a contradiction.

- Let \(y \in R^n_v - \{0\}\) be such that \(\|y\|_s \leq Y_{i+1}\). Either \(\|y\|_s < Y_{i+1}\), in which case, as just seen, \(\langle Ay \rangle_r \geq M_i\), \(M_{i+1}\), as wanted for Property \((b_{i+1})\). Or \(\|y\|_s = Y_{i+1} > Y_i\), in which case either \(\langle Ay \rangle_r \geq M_i\), \(M_{i+1}\), as wanted for Property \((b_{i+1})\), or \(\langle Ay \rangle_r < M_i\), so that \(y\) belongs to \(S_{\min}\), hence \(\langle Ay \rangle_r \geq \min\{\langle Az \rangle_r : z \in S_{\min}\} = M_{i+1}\).

By induction, this proves Assertion (1) of Lemma \([2,6]\).

\[(2)\ i)\] This follows from the facts that \(M_i < \frac{1}{\min S Z}\) and that \(M_{i+1} < M_i\).

ii) Since \(Y_1 = 1\), this follows by induction from the facts that \(Y_i < \frac{1}{\min S Z}\) and that \(Y_{i+1} > Y_i\).

iii) Let \(\alpha = \left\lfloor \log_q (\frac{\deg v + g - 1}{\min S, v_{i+1}}) \right\rfloor - 1\), which satisfies \(\alpha > \frac{1}{\min F(\deg v)} + \frac{g - 1}{\prod_{q^k} (\deg v)}\) if \(i\) is large enough, by Assertion \((2)\ ii)\). By Corollary \([2,3]\) there exists \(y \in R^n_v - \{0\}\) such that

\[\|y\|_s \leq \frac{\deg v + g - 1}{\min S, v_{i+1}} q^\alpha < \frac{\deg v + g - 1}{\min S, v_{i+1}} q^{\alpha} \log_q (\frac{\deg v + g - 1}{\min S, v_{i+1}} Y_{i+1}) = Y_{i+1}\]
and

\[
\langle Ay \rangle_r \leq q^{\frac{\deg v + g - 1}{\min r}} q_v^{-\alpha} \\
\leq q^{\frac{\deg v + g - 1}{\min r}} q_v^{-\left(\log q_v \left( q^{\frac{\deg v + g - 1}{\min s}} Y_{i+1}^{-1} - 2 \right) \right) = q^{(\deg v + g - 1)\left(\frac{1}{\min r} + \frac{1}{\min s}\right) + 2 \deg v (Y_{i+1})^{-1}}.
\]

Since \( M_i \leq \min \{ \langle Ay \rangle_r : y \in R_v^n, 0 < \|y\|_s < Y_{i+1} \} \) by the definition of a best approximation sequence, the result follows. \( \square \)

### 2.4 Transference theorems with weights

In this section, we will show that a matrix \( A \in A_{m,n}(K_v) \) is singular on average if and only if its transpose \(^tA\) is singular on average. To do this, following [Cas, Chap. V], we prove a transference principle between two problems of homogeneous approximations with weights. See also [GE, Ger] in the disjoint case of the field \( \mathbb{Q} \).

Let \( d \in \mathbb{Z}_{\geq 2} \) be a positive integer at least 2. For all \( \xi = (\xi_1, \ldots, \xi_d) \) and \( \theta = (\theta_1, \ldots, \theta_d) \) in \( K_v^d \), we denote

\[
\xi \cdot \theta = \sum_{k=1}^d \xi_k \theta_k.
\]

Let \( \alpha_1, \ldots, \alpha_d \in \mathbb{Z} \) be integers and let \( \alpha = \sum_{k=1}^d \alpha_k \). We consider the parallelepiped

\[
\mathcal{P} = \{ \xi = (\xi_1, \ldots, \xi_d) \in K_v^d : \forall \ k = 1, \ldots, d, \ |\xi_k| \leq q_v^{\alpha_k} \}.
\]

Following Schmidt's terminology [Sch3 page 109] in the case of the field \( \mathbb{Q} \) (building on Mahler's compound one), we call the parallelepiped

\[
\mathcal{P}^* = \{ \xi = (\xi_1, \ldots, \xi_d) \in K_v^d : \forall \ k = 1, \ldots, d, \ |\xi_k| \leq \frac{1}{q_v^{\alpha_k}} \prod_{i=1}^d q_v^{\alpha_i} = q_v^{\alpha - \alpha_k} \}
\]

the pseudocompound of \( \mathcal{P} \). Note that \( \mathcal{P} \) and \( \mathcal{P}^* \) are preserved by the multiplication of the components of their elements by elements of \( \mathcal{O}_v \).

**Theorem 2.7** With \( \mathcal{P} \) and \( \mathcal{P}^* \) as above, for every \( F \in \text{SL}_d(K_v) \),

if \( \mathcal{P}^* \cap {}^tF^{-1}(R_v^d) \neq \{0\} \), then \( \pi_v^{-\beta_d} \mathcal{P} \cap F(R_v^d) \neq \{0\} \),

where

\[
\beta_d = \left\lfloor \frac{1}{d-1} \left( d + 1 + \frac{(g-1)d}{\deg v} \right) \right\rfloor.
\]

**Remark.** The \( R_v \)-lattice \( {}^tF^{-1}(R_v^d) \) is called the dual lattice of the \( R_v \)-lattice \( F(R_v^d) \) since we have \( z \cdot w \in R_v \) for all \( z \in {}^tF^{-1}(R_v^d) \) and \( w \in F(R_v^d) \). They have the same covolume as \( R_v^d \), since \( \det(F) = 1 \).

**Proof.** Let \( z = (z_1, \ldots, z_d) \in \mathcal{P}^* \cap {}^tF^{-1}(R_v^d) - \{0\} \) and \( \kappa_0 = \max \{ k \in \mathbb{Z}_{\geq 0} : z \in \pi_v^k \mathcal{P}^* \} \).

Up to permuting the coordinates, we may assume that, for all \( k = 2, \ldots, d \), we have

\[
|z_1| = q_v^{\alpha_1 - \kappa_0} \quad \text{and} \quad |z_k| \leq q_v^{\alpha_k - \kappa_0}.
\]
With $F_k$ the $k$-th row of $F$, let us consider the $R_v$-lattice $\Lambda = M(R_v^d)$ where

$$M = \begin{pmatrix} \pi_v^{-1} \sum_{k=1}^{d} z_k F_k \\ \pi_v^{\beta_d+\alpha_2} F_2 \\ \vdots \\ \pi_v^{\beta_d+\alpha_d} F_d \end{pmatrix}.$$  

By subtracting to the first row a linear combination of the other rows, and since $\det F = 1$, the determinant of the above matrix $M$ is equal to $\pi_v^{(d-1)\beta_d+\alpha-\alpha_1-1} z_1$. By Equations (5) and (4), we thus have

$$\text{Covol}(\Lambda) = \det(M) \text{Covol}(R_v^d) = q_v^{1-\kappa_0-(d-1)\beta_d} q_v^{(g-1)d}.$$  

Since $d \geq 2$ and $\beta_d \geq \frac{1}{d-1}(d + 1 + \frac{(g-1)d}{\deg v})$, Corollary 2.2 applied to the $R_v$-lattice $\Lambda$ gives that

$$\lambda_1(\Lambda) \leq q_v \text{Covol}(\Lambda)^{\frac{d}{d-1}} \leq 1.$$  

Hence, by the definition of the first minimum $\lambda_1(\Lambda)$, there exists $w \in R_v^d - \{0\}$ such that for every $k = 2, \ldots, d$, we have

$$|z \cdot F(w)| \leq q_v^{-1} < 1 \quad \text{and} \quad |F_k(w)| \leq q_v^{\beta_d+\alpha_k}.$$  

Since $z \in \mathcal{I}^{-1}(R_v^d)$ and $w \in R_v^d$, we have $z \cdot F(w) \in R_v^d$ by the above Remark. The first inequality of Equation (6) hence implies that $z \cdot F(w) = 0$, which means that

$$z_i F_i(w) = - \sum_{k=2}^{d} z_k F_k(w).$$  

By the ultrametric property of $|\cdot |$, by Equations (5) and (6), we have

$$q_v^{\alpha-\alpha_1-\kappa_0}|F_1(w)| = |z_1 F_1(w)| \leq \max_{2 \leq k \leq d} |z_k F_k(w)| \leq \max_{2 \leq k \leq d} q_v^{\alpha_0-\kappa_0} q_v^{\beta_d+\alpha_k} = q_v^{\alpha+\beta_d-\kappa_0}.$$  

Therefore $|F_1(w)| \leq q_v^{\beta_d+\alpha_1}$ and with the second inequality of Equation (6), we conclude that $F(w) \in \pi_v^{\beta_d} \mathcal{G}$.

\begin{flushright} $\Box$ \end{flushright}

**Corollary 2.8** There exist $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$ with $\kappa_2 > 0$, depending only on $m$, $n$, $g$, $\deg v$, $r$, and $s$, such that for all $A \in \mathcal{M}_{m,n}(K_v)$ and $\varepsilon \in q_v^{\kappa_3-1}$, for every large enough $Y \in q_v^{Z_{\geq 1}}$, if there exists $y \in R_v^n - \{0\}$ such that

$$\langle Ay \rangle_r \leq \varepsilon Y^{-1} \quad \text{and} \quad \|y\|_s \leq Y,$$  

then there exists $x \in R_v^m - \{0\}$ such that

$$\langle Ax \rangle_s \leq q_v^{\kappa_3} \varepsilon^{\kappa_2} X^{-1} \quad \text{and} \quad \|x\|_r \leq X,$$  

where $X = q_v^{\kappa_3} \varepsilon^{-\kappa_4} Y$.  

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Proof. Let \(|s| = \sum_{j=1}^{n} s_j\). Denoting \(\alpha_\epsilon = -\log q_v \epsilon \in \mathbb{Z}_{\geq 1}\) and \(\alpha_y = \log q_v Y \in \mathbb{Z}_{\geq 1}\), we define \(\delta = q_v^{-\alpha_\delta}\) and \(Z = q_v^{\alpha_Z} Y\) where

\[
\alpha_\delta = \left\lceil \frac{\alpha_\epsilon - 1}{\|s\|(1/r + 1/m) - 1} \right\rceil \quad \text{and} \quad \alpha_Z = \left\lceil \left(\frac{|s|}{\min s} - 1\right)\alpha_\delta \right\rceil.
\] (9)

Note that \(\alpha_\delta\) is well defined since \(\frac{|s|}{\min s} \geq 1\), and that \(\alpha_\delta\) and \(\alpha_Z\) are nonnegative. We have

\[
\left(\frac{|s|}{\min s} - 1\right)\alpha_\delta = \alpha_\epsilon - 1,
\]

hence \(\left(\frac{|s|}{\min s} - 1\right)\alpha_\delta + 1 \leq \alpha_\epsilon - \frac{|s|}{\min s} \alpha_\delta\),

therefore \(\left(\frac{|s|}{\min s} - 1\right)\alpha_\delta \leq \alpha_Z \leq \alpha_\epsilon - \frac{|s|}{\min s} \alpha_\delta\). (10)

Let \(d = m + n \geq 2\). Let us consider the following parallelepipeds

\[
\mathcal{Q} = \left\{ \xi = (\xi_1, \ldots, \xi_d) \in K^d_v : \forall i = 1, \ldots, m, \ |\xi_i| \leq e^{v_i} Y^{-r_i} \right\},
\]

\[
\mathcal{P} = \left\{ \xi = (\xi_1, \ldots, \xi_d) \in K^d_v : \forall j = 1, \ldots, n, \ |\xi_{m+j}| \leq Y^{s_j} \right\}.
\]

Since \(\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j\), the pseudocompound \(\mathcal{P}^*\) of \(\mathcal{P}\) is equal to

\[
\mathcal{P}^* = \left\{ \xi = (\xi_1, \ldots, \xi_d) \in K^d_v : \forall i = 1, \ldots, m, \ |\xi_i| \leq \delta|s| Z^{-r_i} \right\}.
\]

By the right inequality of Equation (10), for every \(i = 1, \ldots, m\), we have

\[
\delta|s| Z^{-r_i} = q_v^{-\alpha_\delta} \delta Z^{-r_i} Y^{-r_i} \geq q_v^{-r_i(\alpha_Z + \frac{|s|}{\min s} \alpha_\delta)} Y^{-r_i} \geq \epsilon^{v_i} Y^{-r_i}.
\]

By the left inequality of Equation (10), for every \(j = 1, \ldots, n\), we have

\[
\delta|s|^{-s_j} Z^{s_j} = q_v^{-\alpha_\delta} \delta Z^{-s_j} Y^{s_j} \geq q_v^{-s_j(\alpha_Z - \frac{|s|}{\min s} \alpha_\delta)} Y^{s_j} \geq Y^{s_j}.
\]

Therefore \(\mathcal{Q}\) is contained in \(\mathcal{P}^*\).

Now, by the assumption of Corollary (2.8) let \(y \in R^n_v - \{0\}\) be such that the inequalities (7) are satisfied. Then there exists \((x', y) \in R^n_v \times (R^n_v - \{0\})\) such that

\[
\|Ay - x'\|_F \leq \epsilon Y^{-1} \quad \text{and} \quad \|y\|_s \leq Y.
\]

Therefore

\[
\mathcal{Q} \cap \left( I_m \begin{array}{c} A \end{array} \\ 0 \end{array} \right) R^d_v \neq \{0\}.
\]

Since \(\mathcal{Q} \subset \mathcal{P}^*\), this implies that

\[
\mathcal{P}^* \cap \left( I_m \begin{array}{c} A \end{array} \\ 0 \end{array} \right) R^d_v \neq \{0\}.
\]
By Theorem 2.7, we have
\[ \pi_v^{-\beta_d} \mathcal{P} \cap \left( \begin{array}{cc} I_m & 0 \\ -tA & I_n \end{array} \right) R_v^d \neq \{0\}. \]
Then there exists \((x, y') \in (R_v^m \times R_v^n) - \{0\}\) such that
\[ \|\pi_v^{\beta_d} x\|_r \leq Z \quad \text{and} \quad \|\pi_v^{\beta_d} (-tA x - y')\|_s \leq \delta Z^{-1}. \]
Equation (11) gives
\[ \|x\|_r \leq q_v^{\beta_d} Y = q_v^{\beta_d} + \varepsilon \frac{\|x\|_r}{\|x\|_r - \|x\|_s} \]
\[ \leq q_v^{\beta_d} + 1 + \frac{1}{\varepsilon(\frac{|\alpha_{\min} - 1}{|\alpha_{\min}| + |\alpha_{\max}|} - 1) Y}. \]
If \(k_3 = \frac{\beta_d}{\alpha_{\min}} + 1 > 0\) and \(k_4 = \frac{|\alpha_{\min} - 1}{|\alpha_{\min}| + |\alpha_{\max}|} - 1 \geq 0\), this proves the right inequality in Equation (8) with \(X = q_v^{k_3} \epsilon^{-k_4} Y\).

The right inequality in Equation (11), since \(\beta_d \geq 0\) and by using the left inequality in Equation (10) and the definition (9) of \(\alpha_{\delta}\), gives
\[ \langle tA\rangle_x \leq q_v^{\beta_d} \delta Z^{-1} = q_v^{\beta_d} \alpha_{\delta} - \alpha Z Y^{-1} \leq q_v^{\beta_d} \frac{|\alpha_{\min} - 1}{|\alpha_{\min}| + |\alpha_{\max}|} \epsilon^{-k_3} X^{-1} \]
\[ \leq q_v^{\beta_d} \frac{|\alpha_{\min} - 1}{|\alpha_{\min}| + |\alpha_{\max}|} \epsilon^{-k_3} X^{-1} \]
This proves the left inequality in Equation (8) for appropriate positive constants \(k_1\) and \(k_2\).

If \(x = 0\), then we have \(y' \neq 0\) and \(\|y'\|_s \leq q_v^{k_1 - k_3} \epsilon^{k_4 + k_4} Y^{-1}\), which contradicts the fact that \(y' \in R_v^n\) if \(Y\) is large enough. This concludes the proof of Corollary 2.8.

**Corollary 2.9** Let \(m, n\) be positive integers and \(A \in \mathcal{M}_{m,n}(K_v)\). Then \(A\) is \((r, s)\)-singular on average if and only if \(tA\) is \((s, r)\)-singular on average.

**Proof.** This follows from Corollary 2.8.

It follows from this corollary and from Remark 2.3 that if \(A \in \mathcal{M}_{m,n}(K_v)\) is such that \(tA\) is not completely irrational, then \(A\) is \((r, s)\)-singular on average.

### 3 Characterisation of singular on average property

In this section, we give a characterisation of the singular on average property with weights in terms of an asymptotic property in average of the best approximation sequence with weights. In the real case, the relation between the singular property and the best approximation sequence has been studied in [Chev, Chev, CC, LSST]. Also in the real case, and with weights, the relation (similar to the one below) between the singular on average property and the best approximation sequence has been studied in [KSL, Prop. 6.7].

For the sake of later applications, we work with transposes of matrices.
Theorem 3.1 Let $A \in \mathcal{M}_{m,n}(K)$ and let $(y_i)_{i \geq 1}$ be a best approximation sequence in $K^m$ for $A$ with weights $(s, r)$. The following statements are equivalent.

1. For all $a > 1$ and $\epsilon > 0$, we have
   \[
   \lim_{N \to \infty} \frac{1}{N} \text{Card} \{ \ell \in \{1, \ldots, N\} : \exists y \in R^m \setminus \{0\}, \langle tA y \rangle_s \leq \epsilon a^{-\ell}, \|y\|_r \leq a^\ell \} = 1.
   \]

2. The matrix $tA$ is $(s, r)$-singular on average.

3. There exists $a > 1$ such that for every $\epsilon > 0$, we have
   \[
   \lim_{N \to \infty} \frac{1}{N} \text{Card} \{ \ell \in \{1, \ldots, N\} : \exists y \in R^m \setminus \{0\}, \langle tA y \rangle_s \leq \epsilon a^{-\ell}, \|y\|_r \leq a^\ell \} = 1.
   \]

4. For every $\epsilon' > 0$, we have
   \[
   \lim_{k \to \infty} \frac{1}{\log_{q_v} Y_k} \text{Card} \{ i \leq k : M_i Y_{i+1} > \epsilon' \} = 0.
   \]

Proof. Since Assertion (2) is Assertion (1) for $a = q_v > 1$, it is immediate that (1) implies (2) implies (3).

Let us first prove that Assertion (3) implies Assertion (1). Let $a > 1$ be as in Assertion (3) and let $\epsilon' \in [0, 1[^+$. Let $\epsilon = \frac{\epsilon'}{a} > 0$.

We may assume that the set $I = \{ i \in \mathbb{Z}_{\geq 1} : M_i Y_{i+1} > \epsilon' \}$ is infinite, otherwise Assertion (4) is clear since $\lim_{k \to \infty} Y_k = +\infty$. We consider the increasing sequence $(i_j)_{j \in \mathbb{Z}_{\geq 1}}$ of positive integers such that $I = \{ i_j : j \geq 1 \}$. For every $j \geq 1$, by taking the logarithm in base $a$, we thus have $\log_a \epsilon' - \log_a M_{i_j} < \log_a Y_{i_j+1}$, hence
   \[
   \log_a \epsilon - \log_a M_{i_j} < \log_a Y_{i_j+1} - 1. \tag{12}
   \]

Note that for every $i \geq 1$ and $X \in [Y_i, Y_{i+1}[,$ the system of inequalities
   \[
   \langle tA y \rangle_s \leq \epsilon X^{-1} \quad \text{and} \quad 0 < \|y\|_r \leq X \tag{13}
   \]
has a solution $y \in R^m$ if and only if $M_i \leq \epsilon X^{-1}$. Indeed, if the latter inequality is satisfied, then $y_i$ is a solution of the system (13) since $M_i = \langle tA y_i \rangle_s$ and $X \geq Y_i = \|y_i\|_r$. Conversely, if this system has a solution, then since
   \[
   M_i \leq \min \{ \langle tA y \rangle_s : y \in R^m, 0 < \|y\|_r < Y_{i+1} \}
   \]
by the definition of a best approximation sequence, the inequality $M_i \leq \epsilon X^{-1}$ holds since $X < Y_{i+1}$. Hence, for every integer $\ell \in [\log_a Y_i, \log_a Y_{i+1}[,$ the system of inequalities (13) has no integral solutions for $X = a^\ell$ if and only if
   \[
   \log_a \epsilon - \log_a M_i < \ell < \log_a Y_{i+1}. \tag{14}
   \]

There exists an integer $j_0 \geq 1$ such that for every integer $j \geq j_0$, we have $\log_a Y_{i_{j+1}} \geq 2$ by Lemma (2.6) (2) ii). If $\ell$ is the integer in the interval $[\log_a Y_{i_{j+1}} - 1, \log_a Y_{i_{j+1}}[,$ (which is half-open and has length 1, hence does contain one and only one integer), then $\ell \geq 1$ and by Equations (12) and (14), the system (13) has no integral solutions for $X = a^\ell$.

Let $u = \lceil (\log r)(\log_a a) \rceil$, which belongs to $\mathbb{Z}_{\geq 1}$. By Lemma (2.6) (2) ii), for every $k \in \mathbb{Z}_{\geq 1}$, since the sequence $(i_{j})_{j \in \mathbb{Z}_{\geq 1}}$ is increasing, we have
   \[
   Y_{i_{k+u+1}} \geq q_v^{-m} Y_{i_k+1} \geq a Y_{i_k+1}.
   \]
The intervals $[\log a Y_{i_{j+1}} - 1, \log a Y_{i_{j+1}}]$ and $[\log a Y_{i_{j(j+1)}} - 1, \log a Y_{i_{j(j+1)+1}}]$ are hence disjoint for every $j \in \mathbb{Z}_{\geq 1}$. Thus, if $j$ is large enough, with $N_j = [\log a Y_{i_{j+1}}]$, the number $n(N_j)$ of integers $\ell \in \{1, \ldots, N_j\}$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$ is at least $j - j_0$. Therefore $\frac{j - j_0}{\log a Y_{i_{j+1}}}$ tends to 0 as $j \to +\infty$, by Assertion (3). This implies that $\frac{j - j_0}{\log a Y_{i_{j+1}}}$ tends to 0 as $j \to +\infty$.

For every integer $k \geq 1$, let $i_j(k)$ be the unique positive integer such that we have $i_j(k) \leq k < i_j(k) + 1$, so that $j(k) = \text{Card}\{i \leq k : M_i Y_{i+1} > \epsilon^j\}$. Hence, since $(Y_i)_{i \geq 1}$ is increasing, we have

$$
\lim_{k \to \infty} \frac{1}{\log a Y_k} \text{Card}\{i \leq k : M_i Y_{i+1} > \epsilon^j\} \leq \frac{\ln q_v}{\ln a} \lim_{k \to \infty} \frac{j(k)}{\log a Y_{i_j(k)}} = 0,
$$

which proves Assertion (1).

Let us now prove that Assertion (1) implies Assertion (4). Let $a > 1$ and $\epsilon \in [0,1[$. By Lemma 2.6 (2) iii), let $c \geq 1$ be such that for every $i \geq 1$, we have $M_i Y_{i+1} \leq a^c$. By Equation (14), since the number of integer points in an open interval is at most equal to its length, for every $i \geq 1$, the number of integers $\ell \in [\log a Y_i, \log a Y_{i+1}]$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$ is at most

$$
\log a Y_{i+1} - (\log a \epsilon - \log a M_i) = (\log a M_i Y_{i+1} - \log a \epsilon).
$$

For every $N \geq 1$ large enough, let $k_N \geq 1$ be such that $N \in [\log a Y_{k_N}, \log a Y_{k_N+1}]$ and let $n'(N)$ be the number of integers $\ell \in \{1, \ldots, N\}$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$. Then

$$
\frac{n'(N)}{N} \leq \frac{1}{N} \sum_{i=1}^{k_N} \max \{0, \log a M_i Y_{i+1} - \log a \epsilon\}
\leq (c - \log a \epsilon) \frac{1}{\log a Y_{k_N}} \text{Card}\{i \leq k_N : M_i Y_{i+1} > \epsilon\}.
$$

This last term tends to 0 as $N \to +\infty$ by Assertion (1) applied with $\epsilon' = \epsilon$. Therefore $\lim_{N \to +\infty} \frac{n'(N)}{N} = 0$, thus proving Assertion (4).

\section{Full Hausdorff dimension for singular on average matrices}

\subsection{Modified Bugeaud-Zhang sequences}

In this subsection, we construct a subsequence with controlled growth of the best approximation sequence with weights of a matrix, assuming that its transpose is singular on average for those weights. We use as inspiration [BZ, page 470] in the special case of $K = \mathbb{F}_q(Z)$ and $v = v_{\mathbb{Z}_2}$, and the first claim of the proof of [BuKLR, Theo. 2.2] in the case of the field $\mathbb{Q}$ (with characteristic zero).

**Proposition 4.1** Let $A \in \mathcal{M}_{m,n}(K_v)$ be such that $^tA$ is completely irrational and $(s, r)$-singular on average. Let $(y_i)_{i \in \mathbb{Z}_{\geq 1}}$ be a best approximation sequence in $K_v^m$ for $^tA$ with weights $(s, r)$, and let $c > 0$ be such that $M_i Y_{i+1} \leq q_v^c$ for every $i \in \mathbb{Z}_{\geq 1}$. For all $a > b > 0$, there exists an increasing map $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ such that
(1) for every $i \in \mathbb{Z}_{\geq 1}$, we have

$$Y_{\varphi(i+1)} \geq q_v^b Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq q_v^{b+c}.$$  

(2) we have

$$\limsup_{k \to \infty} \frac{k}{\log_q Y_{\varphi(k)}} \leq \frac{1}{a}. \quad (16)$$

**Proof.** Let $A, (Y_i)_{i \in \mathbb{Z}_{\geq 1}}$ and $a, b$ be as in the statement. We start by proving a particular case, that will be useful in two of the four cases below.

**Lemma 4.2** If furthermore we have \( \lim_{k \to \infty} Y_k^{1/k} = +\infty \), then there exists an increasing map $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ such that Equations (15) and (16) are satisfied.

**Proof.** The fact that \( \lim_{k \to \infty} Y_k^{1/k} = +\infty \) implies that the set

$$\mathcal{J}_0 = \{ j \in \mathbb{Z}_{\geq 1} : Y_{j+1} \geq q_v^b Y_j \}$$

is infinite. We construct the increasing sequence \((\varphi(i))_{i \in \mathbb{Z}_{\geq 1}}\) of positive integers by stacks \(\{\varphi(i+1), \ldots, \varphi(i+k)\}\) with \(i_k+1 > i_k\), by induction on \(k \in \mathbb{Z}_{\geq 0}\). For \(k = 0\), let \(i_0 = 0\), let \(i_1 = 1\) and let \(\varphi(1)\) be the smallest element of \(\mathcal{J}_0\).

For \(k \in \mathbb{Z}_{\geq 0}\), assume that \(i_k\) and \(\varphi(i_k)\) are constructed such that \(\varphi(i_k) \in \mathcal{J}_0\) and Equation (15) holds for every \(i \leq i_k - 1\). Let us construct \(i_{k+1}\) and \(\varphi(i_{k+1})\) such that \(\varphi(i_{k+1}) \in \mathcal{J}_0\) and Equation (15) holds for every \(i \leq i_{k+1} - 1\). Let \(j_0\) be the smallest element of \(\mathcal{J}_0\) greater than \(\varphi(i_k)\). Let \(r' = 0\) if the set \(\{ j > \varphi(i_k) : Y_j \geq q_v^b Y_j \}\) is empty. Otherwise, let \(r' \in \mathbb{Z}_{\geq 1}\) be the maximal integer such that by induction there exist \(j_1, j_2, \ldots, j_{r'} \in \mathbb{Z}_{\geq 1}\) such that for \(\ell = 1, \ldots, r'\), the set \(\{ j > \varphi(i_k) : Y_{j-1} \geq q_v^b Y_j \}\) is nonempty and for \(\ell = 1, \ldots, r'+1\) the integer \(j_{\ell}\) is its largest element. Since the sequence \((Y_i)_{i \in \mathbb{Z}_{\geq 1}}\) is increasing, this in particular implies that \(j_{\ell-1} > j_{\ell}\) for \(\ell = 1, \ldots, r'+1\), which itself ensures the finiteness of \(r'\). Now we define \(i_{k+1} = i_k + r' + 1\) and

$$\varphi(i_k + 1) = j_{r'}, \quad \varphi(i_k + 2) = j_{r'-1}, \ldots, \varphi(i_k + r') = j_1, \quad \varphi(i_k + 1) = j_0.$$

By construction, for \(\ell = 1, \ldots, r'\), we have

$$Y_{\varphi(i_k+\ell+1)} = Y_{j_{\ell}+\ell} \geq q_v^b Y_{j_{\ell}+\ell} = q_v^b Y_{\varphi(i_k+\ell)}.$$

As \(\varphi(i_k + 1) = j_{r'} > \varphi(i_k)\), we have \(Y_{\varphi(i_k+1)} \geq Y_{\varphi(i_k)+1} \geq q_v^b Y_{\varphi(i_k)}\) since \(\varphi(i_k) \in \mathcal{J}_0\). Note that \(\varphi(i_k+1) = j_0 \in \mathcal{J}_0\). This proves the claim on the left hand side of Equation (15) for \(i \leq i_k-1\).

By the maximality property of \(j_{\ell-1}\) in the above construction, for every \(\ell = 1, \ldots, r'\), we have \(Y_{\varphi(i_k+\ell+1)} = Y_{j_{\ell}+\ell} < q_v^b Y_{j_{\ell}+\ell+1} = q_v^b Y_{\varphi(i_k+\ell)+1}\). By the maximality of \(r'\) in the above construction, we have \(Y_{\varphi(i_k+1)} < q_v^b Y_{\varphi(i_k)+1}\). Hence, by the definition of \(c\), for every \(\ell = 0, \ldots, r'\), we have

$$M_{\varphi(i_k+\ell)} Y_{\varphi(i_k+\ell+1)} \leq M_{\varphi(i_k+\ell)} Y_{\varphi(i_k+\ell)+1} q_v^b \leq q_v^{b+c}.$$

This proves the claim on the right hand side of Equation (15) for \(i \leq i_k-1\).
Since $\lim_{k \to \infty} \frac{k}{\log_{q_v} Y_k} = 0$, Equation (16) is satisfied for $\varphi$, and this concludes the proof of Lemma 4.2.

Now in what follows, we will discuss four cases on the configuration in $Z_{\geq 1}$ of the set

$$\mathcal{J} = \{ j \in Z_{\geq 1} : M_j Y_{j+1} \leq q_v^{b+c-3a} \} .$$

By Theorem 3.1 (4) applied with $\epsilon' = q_v^{b+c-3a}$, we have

$$\lim_{k \to \infty} \frac{1}{\log_{q_v} Y_k} \text{Card}\{ i \leq k : i \in \mathcal{J} \} = 0 .$$

(17)

**Case 1.** Assume first that $\mathcal{J}$ is finite.

By Equation (17), we then have $\lim_{k \to \infty} \frac{k}{\log_{q_v} Y_k} = 0$, hence Proposition 4.1 follows from Lemma 4.2.

**Case 2.** Let us now assume that there exists $j_\ast \in Z_{\geq 1}$ such that $j \in \mathcal{J}$ for every $j \geq j_\ast$.

Let us consider the auxiliary increasing sequence $(\psi(i))_{i \in Z_{\geq 1}}$ of positive integers defined by induction by setting $\psi(1) = \min\{ j_\ast \in Z_{\geq 1} : \forall j \geq j_\ast, j \in \mathcal{J} \}$ and, for every $i \geq 1$,

$$\psi(i+1) = \min\{ j \in Z_{\geq 1} : q_u^a Y_{\psi(i)} \leq Y_j \} .$$

Since the sequence $(Y_i)_{i \in Z_{\geq 1}}$ is increasing and converges to $+\infty$, this is well defined, and $\psi$ is increasing, hence takes value in $\mathcal{J}$ by the assumption of Case 2. Let us now define the sequence $(\varphi(i))_{i \in Z_{\geq 1}}$ by, for every $i \in Z_{\geq 1}$,

$$\varphi(i) = \begin{cases} \psi(i) & \text{if } M_{\psi(i)} Y_{\psi(i+1)} \leq q_v^{b+c-a} , \\ \psi(i+1) - 1 & \text{otherwise.} \end{cases}$$

Note that the sequence $(\varphi(i))_{i \in Z_{\geq 1}}$ is increasing with $\varphi \geq \psi$.

Let $i \in Z_{\geq 1}$. Let us prove that

$$Y_{\varphi(i+1)} \geq q_u^a Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq q_v^{b+c} ,$$

(18)

by discussing on the values of $\varphi(i)$ and $\varphi(i+1)$. This implies that Equation (15) is satisfied since $a \geq b$, and that Equation (16) is satisfied since by induction $Y_{\varphi(k)} \geq q_u^a (k-1) Y_{\varphi(1)}$ for every $k \in Z_{\geq 1}$.

- Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+1)$. By the definition of $\psi(i+1)$, we have

$$Y_{\psi(i+1)} = Y_{\psi(i)} \geq q_v^a Y_{\psi(i)} = q_u^a Y_{\varphi(i)} .$$

If $\psi(i) \neq \psi(i+1) - 1$, then by the definition of $\varphi(i)$, we have

$$M_{\varphi(i)} Y_{\psi(i+1)} = M_{\psi(i)} Y_{\psi(i+1)} \leq q_v^{b+c-a} \leq q_v^{b+c} .$$

If $\psi(i) = \psi(i+1) - 1$, then $\varphi(i+1) = \varphi(i) + 1$ and by the definition of $c$, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\varphi(i)} Y_{\varphi(i)+1} \leq q_v^c \leq q_v^{b+c} .$$

This proves Equation (18).
• Assume that \( \varphi(i) = \psi(i) \) and \( \varphi(i + 1) = \psi(i + 2) - 1 \). Since the sequence \( (Y_i)_{i \in \mathbb{Z}_{\geq 1}} \) is increasing and by the definition of \( \psi(i + 1) \), we have

\[
Y_{\varphi(i+1)} = Y_{\psi(i+2) - 1} \geq Y_{\psi(i+1)} \geq q_v^a \psi(i) = q_v^a Y_{\varphi(i)}.
\]

We have \( q_v^a Y_{\varphi(i+1)} > Y_{\psi(i+2) - 1} \) by the minimality property of \( \psi(i + 2) \). If \( \psi(i + 1) > \psi(i) + 1 \), then \( M_{\psi(i)} Y_{\psi(i+1)} \leq q_v^{b+c-a} \) by the dichotomy in the definition of \( \varphi(i) \). Hence

\[
M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i)} Y_{\psi(i+2) - 1} \leq M_{\psi(i)} Y_{\psi(i+1)} q_v^a \leq q_v^{b+c-a} q_v^a = q_v^{b+c}.
\]

If \( \psi(i + 1) = \psi(i) + 1 \), then \( M_{\psi(i)} Y_{\psi(i) + 1} \leq q_v^{b+c-3a} \) since \( \psi(i) \in \mathcal{J} \). Hence

\[
M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i)} Y_{\psi(i)+1} \leq M_{\psi(i)} Y_{\psi(i) + 1} q_v^a \leq q_v^{b+c-3a} q_v^a \leq q_v^{b+c}.
\]

This proves Equation (18).

• Assume that \( \varphi(i) = \psi(i + 1) - 1 \) and \( \varphi(i + 1) = \psi(i + 1) \). Since \( \psi(i + 1) - 1 \in \mathcal{J} \), we have

\[
M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i+1) - 1} Y_{\psi(i+1)} \leq q_v^{b+c-3a} \leq q_v^{b+c}.
\]

If \( \psi(i + 1) - 1 = \psi(i) \), then by the definition of \( \psi(i + 1) \), we have

\[
\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)+1}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)}} \geq q_v^a.
\]

If \( \psi(i + 1) - 1 > \psi(i) \), then we have \( M_{\psi(i)} Y_{\psi(i+1)} > q_v^{b+c-a} \) by the dichotomy in the definition of \( \varphi(i) \), we have \( Y_{\psi(i+1) - 1} < q_v^a Y_{\psi(i)} \leq q_v^a Y_{\psi(i) + 1} \) by the minimality property of \( \psi(i + 1) \), and we have \( M_{\psi(i)} Y_{\psi(i) + 1} \leq q_v^{b+c-3a} \) since \( \psi(i) \in \mathcal{J} \). Therefore

\[
\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)+1}} = \frac{M_{\psi(i)} Y_{\psi(i)+1}}{M_{\psi(i)} Y_{\psi(i)+1}} \geq \frac{q_v^{b+c-a}}{q_v^a} q_v^{b+c-3a} q_v^a = q_v^a.
\]

This proves Equation (18).

• Assume that \( \varphi(i) = \psi(i + 1) - 1 \) and \( \varphi(i + 1) = \psi(i + 2) - 1 \). By the previous case computations, we have

\[
\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+2) - 1}}{Y_{\psi(i)+1}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)}+1} \geq q_v^a.
\]

We have \( q_v^a Y_{\psi(i+1)} > Y_{\psi(i+2) - 1} \) by the minimality property of \( \psi(i + 2) \). Hence since \( \psi(i + 1) - 1 \in \mathcal{J} \), we have

\[
M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i+1) - 1} Y_{\psi(i)+1} \leq M_{\psi(i+1) - 1} Y_{\psi(i)+1} \left( \frac{Y_{\psi(i)+1}}{Y_{\psi(i+1)}} \right)
\]

\[
\leq q_v^{b+c-3a} q_v^a \leq q_v^{b+c}.
\]

This proves Equation (18) and concludes the proof of Case 2.

**Case 3.** Let us now assume that \( \mathcal{J} \) and \( \mathcal{J}^c \) are both infinite, and that the number of sequences of consecutive elements of \( \mathcal{J} \) with length at least \( 3a \) is finite.
Let \( j_0 = \min \mathcal{J} \). Let us write the set \( \mathbb{Z}_{\geq j_0} = \bigcup_{i \in \mathbb{Z}_{\geq 1}} C_i \cup D_i \) as the disjoint union of nonempty finite sequences \( C_i \) of consecutive integers in \( \mathcal{J} \) and finite nonempty sequences \( D_i \) of consecutive integers in \( \mathcal{J} \) with \( \max C_i < \min D_i \leq \max D_i < \min C_{i+1} \) for all \( i \in \mathbb{Z}_{\geq 1} \). Under the assumption of Case 3, let \( i_0 \in \mathbb{Z}_{\geq 1} \) be such that \( \text{Card } C_i < 3a \) for every \( i \geq i_0 \). Let \( k_0 = \min C_{i_0} \).

Then there exists an element of \( \mathcal{J} \) in any finite sequence of \( 3[a] + 1 \) consecutive integers at least \( k_0 \), so that for every \( k \in \mathbb{Z}_{\geq 1} \) we have

\[
\frac{k}{\log q_k Y_k} \leq \frac{k_0 + (3[a] + 1) \text{Card}\{i \leq k : i \in \mathcal{J}\}}{\log q_k Y_k},
\]

which converges to 0 as \( k \to +\infty \) by Equation (17) and since \( \lim_{k \to \infty} Y_k = +\infty \). Therefore \( \lim_{k \to \infty} Y_k^{\frac{1}{k}} = +\infty \), and Lemma 4.2 implies Proposition 4.1.

**Case 4.** Let us finally assume that \( \mathcal{J} \) and \( \mathcal{J} \) are both infinite, and that there are infinitely many sequences of consecutive elements of \( \mathcal{J} \) with length at least \( 3a \).

With the notation \((C_i)_{i \in \mathbb{Z}_{\geq 1}}\) and \((D_i)_{i \in \mathbb{Z}_{\geq 1}}\) of the beginning of Case 3, let \((i_k)_{k \in \mathbb{Z}_{\geq 1}}\) be the increasing sequence of positive integers such that \( \{i \in \mathbb{Z}_{\geq 1} : \text{Card } C_i \geq 3a\} = \{i_k : k \in \mathbb{Z}_{\geq 1}\} \).

For every \( k \in \mathbb{Z}_{\geq 1} \), let us define an increasing finite sequence \((\psi_k(i))_{1 \leq i \leq m_k+1}\) of positive integers by setting \( \psi_k(1) = \min C_{i_k} \) and by induction

\[
\psi_k(i + 1) = \min \{ j \in C_{i_k} : q_k^a Y_{\psi_k(i)} \leq Y_j \},
\]

as long as this set is nonempty. Since \( C_{i_k} \) is a finite sequence of consecutive positive integers with length at least \( 3a \) and \( Y_{i + 1} \geq q_k \min Y_i \) for every \( i \in \mathbb{Z}_{\geq 1} \), there exists \( m_k \in \mathbb{Z}_{\geq 2} \) such that \( \psi_k(i) \) is defined for \( i = 1, \ldots, m_k + 1 \). Note that \( \psi_k(i) \) belongs to \( \mathcal{J} \) for \( i = 1, \ldots, m_k + 1 \) since \( C_{i_k} \subset \mathcal{J} \).

As in Case 2, let us define an increasing finite sequence \((\phi_k(i))_{1 \leq i \leq m_k}\) of positive integers by

\[
\phi_k(i) = \begin{cases} 
\psi_k(i) & \text{if } M_{\psi_k(i)} Y_{\psi_k(i)} \leq q_k^{b+c-a}, \\
\psi_k(i + 1) - 1 & \text{otherwise}.
\end{cases}
\]

As in the proof of Case 2, since for \( i = 1, \ldots, m_k \), the integers \( \psi_k(i) \), \( \psi_k(i + 1) \) as well as \( \psi_k(i + 1) - 1 \) belong to \( \mathcal{J} \), we have, for every \( i = 1, \ldots, m_k - 1 \),

\[
Y_{\psi_k(i + 1)} \geq q_k^a Y_{\psi_k(i)} \quad \text{and} \quad M_{\psi_k(i)} Y_{\psi_k(i + 1)} \leq q_k^{b+c}.
\]

Since \( \phi_k(m_k) \in C_{i_k} \) and \( \phi_{k+1}(1) \in C_{i_{k+1}} \), we have \( \phi_k(m_k) < \phi_{k+1}(1) \). Let us define an increasing finite sequence \((\phi_k'(i))_{1 \leq i \leq r_k+1}\) of positive integers that will allow us to interpolate between \( \phi_k(m_k) \) and \( \phi_{k+1}(1) \). Let \( j_0 = \phi_{k+1}(1) \). If \( \{j \in \mathbb{Z}_{\geq \phi_k(m_k)} : Y_{j_0} \geq q_k^a Y_j\} \) is empty, let \( r_k' = 0 \) and \( \phi_k'(1) = j_0 = \phi_{k+1}(1) \). Otherwise, by decreasing induction, let \( r_k' \in \mathbb{Z}_{\geq 1} \) be the maximal positive integer such that there exist \( j_1, \ldots, j_{r_k'}' \in \mathbb{Z}_{\geq 1} \) such that for \( \ell = 1, \ldots, r_k' \), the set \( \{j \in \mathbb{Z}_{\geq \phi_k(m_k)} : Y_{j_{\ell - 1}} \geq q_k^a Y_j\} \) is nonempty and for \( \ell = 1, \ldots, r_k' + 1 \), the integer \( j_{\ell} \) is its largest element. As in the part of the proof of Case 1 that does not need some belonging to \( \mathcal{J} \), the sequence \((\phi_k'(i))_{1 \leq i \leq r_k'+1}\) is well defined, it is contained in \([\phi_k(m_k), \phi_{k+1}(1)]\), and for \( i = 1, \ldots, r_k' \), we have

\[
Y_{\phi_k'(i + 1)} \geq q_k^a Y_{\phi_k'(i)} \quad \text{and} \quad M_{\phi_k'(i)} Y_{\phi_k'(i + 1)} \leq q_k^{b+c}.
\]
Putting alternatively together the sequences \((\varphi_k(i))_{1 \leq i \leq m_k - 1}\) and \((\varphi'_k(i))_{1 \leq i \leq r'_k}\) as \(k\) ranges over \(\mathbb{Z}_{\geq 1}\), we now define (with the standard convention that an empty sum is zero) \(N_k = \sum_{\ell=1}^{k-1}(m_{\ell} - 1 + r'_{\ell})\) and
\[
\varphi(i) = \begin{cases} 
\varphi_k(i - N_k) & \text{if } 1 + N_k \leq i \leq m_k - 1 + N_k \\
\varphi'_k(i + 1 - m_k - N_k) & \text{if } m_k + N_k \leq i \leq r'_k - 1 + m_k + N_k.
\end{cases}
\]
By Equation (19) for \(i = 1, \ldots, m_k - 2\), by Equation (20) for \(i = 1, \ldots, r'_k\), and since \(\varphi'_k(r'_k + 1) = \varphi_{k+1}(1)\), in order to prove that the map \(\varphi\) satisfies Equation (15), hence Assertion (1) of Proposition 4.1, we only have to prove the following lemma.

**Lemma 4.3** For every \(k \in \mathbb{Z}_{\geq 1}\), we have
\[
Y_{\varphi'_k(1)} \geq q_v^b Y_{\varphi_k(m_k - 1) - 1} \quad \text{and} \quad M_{\varphi_k(m_k - 1)} Y_{\varphi'_k(1)} \leq q_v^{b+c}.
\]

**Proof.** Since \(\varphi'_k(1) \geq \varphi_k(m_k)\), hence \(Y_{\varphi'_k(1)} \geq Y_{\varphi_k(m_k)}\), the left hand side of Equation (21) follows from the left hand side of Equation (19) with \(i = m_k - 1\). If \(\varphi'_k(1) = \varphi_k(m_k)\), then the right hand side of Equation (21) follows from the right hand side of Equation (19) with \(i = m_k - 1\).

Let us hence assume that \(\varphi'_k(1) > \varphi_k(m_k)\), so that
\[
Y_{\varphi'_k(1)} \leq q_v^a Y_{\varphi_k(m_k)}
\]
by the maximality of \(r'_k\). Let us prove that \(\varphi_k(m_k) = \psi_k(m_k)\). For a contradiction, assume otherwise that \(\varphi_k(m_k) = \psi_k(m_k + 1) > \psi_k(m_k)\). As in the third subcase of Case 2, we have \(M_{\psi_k(m_k)} Y_{\psi_k(m_k) - 1} > q_v^{b+c-a}\) by the dichotomy in the definition of \(\varphi_k(m_k)\), we have \(Y_{\psi_k(m_k) - 1} < q_v^a Y_{\psi_k(m_k)} \leq q_v^a Y_{\psi_k(m_k) + 1}\) by the minimality property of \(\psi_k(m_k + 1)\), and we have \(M_{\psi_k(m_k)} Y_{\psi_k(m_k) + 1} \leq q_v^{b+c-3a}\) since \(\psi_k(m_k) \in J\). Therefore, as in the third subcase of Case 2, we have
\[
\frac{Y_{\psi_k(m_k) - 1}}{Y_{\psi_k(m_k) - 1} - 1} = \frac{M_{\psi_k(m_k)} Y_{\psi_k(m_k) + 1}}{M_{\psi_k(m_k)} Y_{\psi_k(m_k) + 1} - 1} \geq q_v^a.
\]
Hence by the construction of \(\varphi'_k(1)\), we have \(\varphi'_k(1) = \varphi_k(m_k)\), a contradiction to our assumption that \(\varphi'_k(1) > \varphi_k(m_k)\). We now discuss on the two possible values of \(\varphi_k(m_k - 1)\).

First assume that \(\varphi_k(m_k - 1) = \psi_k(m_k - 1)\). If \(\psi_k(m_k - 1) \neq \psi_k(m_k - 1)\) then \(M_{\psi_k(m_k - 1)} Y_{\psi_k(m_k - 1)} \leq q_v^{b+c-a}\) by the dichotomy in the definition of \(\varphi_k(m_k - 1)\). If on the contrary \(\psi_k(m_k - 1) = \psi_k(m_k - 1)\) then \(M_{\psi_k(m_k - 1)} Y_{\psi_k(m_k - 1)} \leq q_v^{b+c-3a} \leq q_v^{b+c-a}\) since the integer \(\psi_k(m_k - 1)\) belong to \(J\) as \(m_k \geq 2\). Since \(\varphi_k(m_k) = \psi_k(m_k)\) by Equation (22), we have
\[
M_{\varphi_k(m_k - 1)} Y_{\varphi'_k(1)} = M_{\psi_k(m_k - 1)} Y_{\psi_k(m_k)} \left(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\right) \leq q_v^{b+c-a} q_v^a = q_v^{b+c}.
\]
This proves the right hand side of Equation (21).

Now assume that \(\varphi_k(m_k - 1) = \psi_k(m_k - 1)\). Again since \(\varphi_k(m_k) = \psi_k(m_k)\), since the integer \(\psi_k(m_k - 1)\) belongs to \(J\) as \(m_k \geq 2\), and by Equation (22), we have
\[
M_{\varphi_k(m_k - 1)} Y_{\varphi'_k(1)} = M_{\psi_k(m_k - 1)} Y_{\psi_k(m_k)} \left(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\right) \leq q_v^{b+c-3a} q_v^a \leq q_v^{b+c}.
\]
This proves the right hand side of Equation (21), and concludes the proof of Lemma 4.3.

Finally, let us prove Assertion (2) of Proposition 4.1. Since there exists an element of \( e^j \) in any finite sequence of \( 3[a] + 1 \) consecutive integers in the complement of \( \bigcup_{k \in \mathbb{Z}_{\geq 1}} C_{ik} \), there exists \( c_0 \geq 0 \) such that, for every \( k \in \mathbb{Z}_{\geq 1} \), we have

\[
\frac{\text{Card}\{ j \leq \varphi(k) : j \notin \bigcup_{k \in \mathbb{Z}_{\geq 1}} C_{ik} \}}{\text{log}_q Y_{\varphi(k)}} \leq c_0 + (3[a] + 1) \frac{\text{Card}\{ j \leq \varphi(k) : j \notin e \}}{\text{log}_q Y_{\varphi(k)}},
\]

which converges to 0 as \( k \to +\infty \) as seen at the end of the proof of Case 3. Let us define \( n(k) = \text{Card}\{ j \leq k : Y_{\varphi(j)} \geq a^j Y_{\varphi(j+1)} \} \). For every \( \ell \in \mathbb{Z}_{\geq 1} \), since \( Y_{j+1} \geq q_{\ell}^{m_{\ell}} Y_{j} \) for every \( j \in \mathbb{Z}_{\geq 1} \), and by the maximality of \( m_{\ell} \) in the construction of \( (\varphi_{\ell}(i))_{1 \leq i \leq m_{\ell}} \), we have

\[
\text{Card}\{ j \in C_{i_k} : j \geq \varphi(\ell)(m_{\ell}) \} \leq 2|a| \min r. \quad \text{If } \varphi(i) \text{ belongs to } C_{i_k} \text{ but } \varphi(i+1) \text{ does not, then } \varphi(i) \geq \varphi(\ell)(m_{\ell}).
\]

Since when \( \varphi(i) \) and \( \varphi(i+1) \) belong to \( C_{i_k} \) for some \( \ell \in \mathbb{Z}_{\geq 1} \), then \( \varphi \) and \( \varphi_{\ell} \) coincide on \( i \) and \( i + 1 \), and since Equation (19) holds, we hence have

\[
k - n(k) = \text{Card}\{ j \leq k : Y_{\varphi(j)} < a^j Y_{\varphi(j+1)} \} \leq 2|a| \min r \text{ Card}\{ j \leq \varphi(k) : j \notin \bigcup_{k \in \mathbb{Z}_{\geq 1}} C_{ik} \}. \]

Hence

\[
\limsup_{k \to +\infty} \frac{k}{\text{log}_q Y_{\varphi(k)}} = \limsup_{k \to +\infty} \frac{n(k) + k - n(k)}{\text{log}_q Y_{\varphi(k)}} = \limsup_{k \to +\infty} \frac{n(k)}{\text{log}_q Y_{\varphi(k)}} \leq \limsup_{k \to +\infty} \frac{n(k)}{\text{log}_q a^{1/\varphi(1)} Y_{\varphi(1)}} = \frac{1}{a}.
\]

This proves Equation (16) and concludes the proof of Proposition 4.1.

### 4.2 Lower bound on the Hausdorff dimension of \( \text{Bad}_A(\epsilon) \)

In this subsection, we use the scheme of proof in the real case of [CCGMS, Theo. 6.1], which is a weighted version of [BuKLR, Theo. 5.1], in order to estimate the lower bound on the Hausdorff dimension of the \( e \)-bad sets of \((\mathbf{r}, \mathbf{s})\)-singular in average matrices.

For a given sequence \((\mathbf{y}_i)_{i \geq 1}\) in \( R_{\mathbf{v}}^m - \{0\} \) and for every \( \delta > 0 \), let

\[
\text{Bad}_{(\mathbf{y}_i)_{i \geq 1}}^\delta = \{ \theta \in (\pi_{\mathbf{v}} \mathcal{O}_{\mathbf{v}})^m : \forall i \geq 1, \ |\langle \theta \cdot \mathbf{y}_i \rangle| \geq \delta \}.
\]

**Proposition 4.4** Let \( A \in \mathcal{M}_{m,n}(K_{\mathbf{v}}) \) be such that \( ^tA \) is completely irrational and let \((\mathbf{y}_i)_{i \geq 1}\) be a best approximation sequence in \( K_{\mathbf{v}}^m \) for \(^tA\) with weights \((s, \mathbf{r})\). Suppose that there exist \( b, c > 0 \) and an increasing function \( \varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \) such that

\[
\forall i \in \mathbb{Z}_{\geq 1}, \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq q_{\mathbf{v}}^{b+c}.
\]

Then for every \( \delta \in ]0, 1], \), if \( \epsilon = \delta \min_{\mathbf{v}} q_{\mathbf{v}}^{-b-c} \), then the set \( \text{Bad}_{(\mathbf{y}_i)_{i \geq 1}}^\delta \) is contained in the set \( \text{Bad}_A(\epsilon) \).

**Proof.** Fix \( \delta \in ]0, 1] \) and \( \theta \in \text{Bad}_{(\mathbf{y}_i)_{i \geq 1}}^\delta \). Let \( \epsilon_1 = \delta \min_{\mathbf{v}} q_{\mathbf{v}}^{-b-c} \). For every \((\mathbf{y}', \mathbf{x}')\) in \( R_{\mathbf{v}}^m \times R_{\mathbf{v}}^n \) such that \( \| \mathbf{x}' \|_s \geq \epsilon_1 Y_{\varphi(1)} \), let \( k \) be the unique element of \( \mathbb{Z}_{\geq 1} \) for which

\[
Y_{\varphi(k)} \leq \epsilon_1^{-1} \| \mathbf{x}' \|_s < Y_{\varphi(k+1)}.
\]

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which exists since \( \|x'\|_s \geq \epsilon_1 Y_{\varphi(1)} \) and since the sequence \((Y_{\varphi(i)})_{i \geq 1}\) is increasing, converging to \(+\infty\). Let \( x_{\varphi(k)} \in R^N_u \) be such that \( M_{\varphi(k)} = \|A y_{\varphi(k)} - x_{\varphi(k)}\|_s \). Then by the ultrametric inequality, the assumption of the proposition, the fact that \( \epsilon_1 \|q_v^{b+c} - \delta_{\min s} \|_s \leq 1 \) and the definition of \( \text{Bad}_{(\varphi(i))_{i \geq 1}} \), we have

\[
\left\| (A y_{\varphi(k)} - x_{\varphi(k)}) \cdot x' \right\| \leq \max_{1 \leq i \leq n} M_{\varphi(i)} \|x'\|_s \leq \max_{1 \leq i \leq n} (\epsilon_1 M_{\varphi(k)} Y_{\varphi(k+1)}) \|x'\|_s \leq (\epsilon_1 q_v^{b+c}) \min s = \delta \leq \min_{\ell \in R_u} \| y_{\varphi(k)} \cdot \theta - \theta' \|.
\]

(23)

Observe that

\[
y_{\varphi(k)} \cdot \theta = y_{\varphi(k)} \cdot (A x') + y_{\varphi(k)} \cdot y' - y_{\varphi(k)} \cdot (A x' + y' - \theta) = (A y_{\varphi(k)} - x_{\varphi(k)}) \cdot x' - y_{\varphi(k)} \cdot (A x' + y' - \theta),
\]

where \( \ell = x_{\varphi(k)} \cdot x' + y_{\varphi(k)} \cdot y' \in R_u \). Thus we have, using the equality case of the ultrametric inequality for the second equality below with the strict inequality in Equation (23), and again the definition of \( \text{Bad}_{(\varphi(i))_{i \geq 1}} \) for the last inequality below,

\[
|y_{\varphi(k)} \cdot (Ax' + y' - \theta)| = \left| \left( (A y_{\varphi(k)} - x_{\varphi(k)}) \cdot x' - y_{\varphi(k)} \cdot \theta + \ell \right) \right| = \max \left\{ |(A y_{\varphi(k)} - x_{\varphi(k)}) \cdot x'|, |y_{\varphi(k)} \cdot \theta - \ell| \right\} = |y_{\varphi(k)} \cdot \theta - \ell| \geq |(y_{\varphi(k)} \cdot \theta)| \geq \delta.
\]

Hence, we have

\[
\delta \leq |y_{\varphi(k)} \cdot (Ax' + y' - \theta)| \leq \max_{1 \leq j \leq m} Y_{\varphi(k)}^{(r_j)} \| Ax' + y' - \theta \|_r^{(r_j)},
\]

which implies, since \( \delta \leq 1 \), that

\[
Y_{\varphi(k)} \| Ax' + y' - \theta \|_r \geq \min_{1 \leq j \leq m} Y_{\varphi(k)}^{(r_j)} \| Ax' + y' - \theta \|_r \geq \delta_{\min r}^{\frac{1}{r_j}} = \delta_{\min r}^{\frac{1}{r_j}}.
\]

Finally, for every \((y', x') \in R^m_u \times R^n_u\) such that \( \| x' \|_s \geq \epsilon_1 Y_{\varphi(1)} \), we have

\[
\| x' \|_s \| Ax' + y' - \theta \|_r \geq \epsilon_1 Y_{\varphi(k)} \| Ax' + y' - \theta \|_r \geq \delta_{\min r}^{\frac{1}{r_j}} \| q_v^{b+c} \|_{\min s} Y_{\varphi(k)} \| y \|_r.
\]

By Equation (11), this implies that \( \theta \in \text{Bad}_{A}(\epsilon) \) for \( \epsilon = \delta_{\min r}^{\frac{1}{r_j}} \| q_v^{b+c} \|_{\min s} \).

Proposition 4.5 For every \( \delta \in [0, \frac{1}{q_v}] \), there exist \( b = b(\delta) > 0 \) and \( C = C(\delta) > 0 \) such that for every sequence \((y_i)_{i \in \mathbb{Z}_{\geq 1}}\) in \( R^m_u - \{0\} \) satisfying \( \| y_{i+1} \|_r \geq q_v^b \| y_i \|_r \) for all \( i \in \mathbb{Z}_{\geq 1} \), we have

\[
\dim_{\text{Haus}} \text{Bad}_{(y_i)_{i \geq 1}} \geq m - C \limsup_{k \to \infty} k \frac{1}{\log_{q_v} \| y_k \|_r}.
\]

Proof. Fix \( \delta \in [0, \frac{1}{q_v}] \). Let

\[
b = b(\delta) = \frac{- \log_{q_v} \delta}{\min r},
\]

(24)
which is positive since $\delta < 1$. By the mass distribution principle (see for instance [Fal page 60]), it is enough to prove that there exist a (Borel, positive) measure $\mu$, supported on $\text{Bad}^{\phi}_{(y_i)_{i \geq 1}}$, and constants $C, C_0, r_0 > 0$, with $C$ depending only on $\delta$, such that, for every closed ball $B$ of radius $r < r_0$, we have

$$
\mu(B) \leq C_0 r^{-m} \limsup_{k \to \infty} \frac{k}{\log q_v} \frac{1}{y_k}.
$$

We adapt by modifying it quite a lot the measure construction in the proof of [CGGMS Theo. 6.1].

By convention, let $Y_0 = 1$ and $n_{0,j} = 0$ for $j = 1, \ldots, m$. For every $k \in \mathbb{Z}_{\geq 1}$, define $Y_k = \|y_k\|_r$, which is at least 1 since $y_k \in R_v^m - \{0\}$, and for every $j = 1, \ldots, m$, let $n_{k,j} \in \mathbb{Z}_{\geq 0}$ be such that

$$
q_v^{-n_{k,j}} \leq Y_k^{-r_j} < q_v^{-n_{k,j}+1}.
$$

Note that the sequence $(n_{k,j})_{k \in \mathbb{Z}_{\geq 0}}$ is nondecreasing, for all $j = 1, \ldots, m$.

For every $k \in \mathbb{Z}_{\geq 0}$, let us consider the polydisc

$$
\Pi(Y_k) = \overline{B}(0, \frac{1}{q_v} Y_k^{-r_1}) \times \cdots \times \overline{B}(0, \frac{1}{q_v} Y_k^{-r_m}) = \overline{B}(0, q_v^{-n_{k,1}-1}) \times \cdots \times \overline{B}(0, q_v^{-n_{k,m}-1}),
$$

where $\overline{B}(0, r')$ is the closed ball of radius $r' > 0$ and center 0 in $K_v$. Note that $\Pi(Y_0) = (\pi_v \mathcal{O}_v)^m$ is the open unit ball of $K_v^m$ and that $\Pi(Y_k)$ is an additive subgroup of $K_v^m$. Since the residual field $k_v = \mathcal{O}_v/\pi_v \mathcal{O}_v$ lifts as a subfield of order $q_v$ of $K_v$, for every $\ell \in \mathbb{Z}_{\geq 0}$, we have a disjoint union

$$
\overline{B}(0, q_v^{-\ell}) = \bigsqcup_{a \in k_v} \left( a \pi_v^\ell + \overline{B}(0, q_v^{-\ell-1}) \right).
$$

Hence by induction, the polydisc $\Pi(Y_k)$ is the disjoint union of

$$
\Delta_{k+1} = \prod_{1 \leq j \leq m} q_v^{n_{k+1,j} - n_{k,j}}
$$

translates of the polydisc $\Pi(Y_{k+1})$. Note that

$$
\Delta_{k+1} \geq \prod_{1 \leq j \leq m} Y_{k+1}^{-r_j} Y_k^{-r_j} q_v^{-1} = q_v^{-m} (Y_{k+1} Y_k^{-1})^{|\mathbf{r}|}. \tag{26}
$$

For every $k \in \mathbb{Z}_{\geq 0}$, let us fix some elements $\theta_{1,k+1}, \ldots, \theta_{\Delta_{k+1},k+1}$ in $(\pi_v \mathcal{O}_v)^m$ (which are not unique in the ultrametric space $K_v^m$) such that

$$
\Pi(Y_k) = \bigsqcup_{i=1}^{\Delta_{k+1}} \left( \theta_{i,k+1} + \Pi(Y_{k+1}) \right).
$$

By convention, let us define $Z_{0,\delta} = \emptyset$ and $I_0 = \{\Pi(Y_0)\}$. For every $k \in \mathbb{Z}_{\geq 1}$, let us define

$$
Z_{k,\delta} = \{ \theta \in (\pi_v \mathcal{O}_v)^m : |\langle y_k \cdot \theta \rangle| < \delta \}
$$

and

$$
I_k = \{ \theta_{i_1,1} + \cdots + \theta_{i_k,k} + \Pi(Y_k) : \forall j \in \{1, \ldots, k\}, \ 1 \leq i_j \leq \Delta_j \}. \tag{27}
$$
Lemma 4.6 For every \( k \in \mathbb{Z}_{\geq 1} \), we have

(1) for every \( I' \in I_{k+1} \), if \( I' \cap Z_{k,\delta} \neq \emptyset \) then \( I' \subset Z_{k,\delta} \).

(2) for every \( I \in I_k \), we have \( \text{vol}^n(I \cap Z_{k,\delta}) \leq \delta Y_{-|r|}^k \).

Proof. (1) If \( I' \in I_{k+1} \) and \( I' \cap Z_{k,\delta} \neq \emptyset \), let \( \theta \in I' \cap Z_{k,\delta} \). Then for every \( \theta' \in I' \), if \( x, x' \in R_v \) are such that \( \langle y_k \cdot \theta \rangle = \langle y_k \cdot \theta' \rangle - (x + x') \leq \max \{ \langle (y_k \cdot \theta') - x \rangle, \langle (y_k \cdot (\theta' - \theta)) - x' \rangle \} \), then by the ultrametric inequality, since \( \theta \in Z_{k,\delta} \) and \( \theta' - \theta \in F(Y_{k+1}) \), by the assumption of Proposition 4.5 and by the definition of \( b \), we have

\[
\langle y_k \cdot \theta' \rangle \leq \delta \langle y_k \cdot \theta \rangle \leq \delta \langle y_k \cdot (\theta' - \theta) \rangle \leq \delta \langle y_k \cdot (\theta' - \theta - \delta) \rangle \leq \delta \langle y_k \cdot (\theta' - \theta) \rangle \leq \delta \langle y_k \cdot (\theta' - \theta) \rangle \leq \delta.
\]

This inequality \( \langle y_k \cdot \theta' \rangle \leq \delta \) is actually strict, since \( \langle y_k \cdot \theta \rangle < \delta \) and by Equation (24), we have \( q_v^{-b \min \{ \theta' \}} = q_v^{-b \delta} < \delta \). Since \( I' \) is contained in \( F(Y_0) = (\pi_v \mathcal{O}_v)^m \), we thus have that \( \theta' \in Z_{k,\delta} \) and this proves Assertion (1).

(2) Let \( j_0 \in \{1, \ldots, m\} \) be such that \( Y_k = \langle y_{k,j_0} \rangle \) where \( y_k = (y_{k,1}, \ldots, y_{k,m}) \). In particular, \( y_{k,j_0} \) is nonzero. For every \( z \in R_v \), let

\[
L_k(z) = \{ \theta \in K_v^m : y_k \cdot \theta = z \},
\]

which is an affine hyperplane of \( K_v^m \) transverse to the \( j_0 \)-axis, and let

\[
\mathcal{N}(k, z) = \{ \theta' \in (\pi_v \mathcal{O}_v)^m : \exists u' \in L_k(z), | \theta'_{j_0} - u'_{j_0} | \leq \delta Y_{-r_{j_0}}^k \text{ and } \forall j \neq j_0, \theta'_{j} = u'_{j} \},
\]

which is the intersection with the open unit ball in \( K_v^m \) of the \( (\delta Y_{-r_{j_0}}^k) \)-thickening along the \( j_0 \)-axis of the affine hyperplane \( L_k(z) \).

Fix \( I \in I_k \). Since \( \text{vol}_v(B(\theta', r')) = q_v^{\log q_v r'} \leq r' \) for all \( r' > 0 \), and by Fubini’s theorem, we have

\[
\text{vol}^n(I \cap \mathcal{N}(k, z)) \leq \delta Y_{-r_{j_0}}^k \prod_{j \neq j_0} Y_{-r_j}^k = \delta Y_{-|r|}^k.
\]

Claim 1. Let us prove that the set \( Z_{k,\delta} \) is contained in the union of the sets \( \mathcal{N}(k, z) \) for \( z \in R_v \).

Proof. Let \( \theta = (\theta_1, \ldots, \theta_m) \in Z_{k,\delta} \) and let \( z \in R_v \) be such that \( \langle y_k \cdot \theta \rangle = \langle y_k \cdot \theta - z \rangle \). Let us define \( u_j = \theta_j \) if \( j \neq j_0 \),

\[
u_{j_0} = \frac{z - \sum_{j \neq j_0} y_{k,j} \theta_j}{y_{k,j_0}}
\]

and \( u = (u_1, \ldots, u_m) \), which is the projection of \( \theta \) on the affine hyperplane \( L_k(z) \) along the \( j_0 \)-axis. Then, since \( \theta \in Z_{k,\delta} \), we have

\[
| \theta_{j_0} - u_{j_0} | = | y_{k,j} \cdot \theta - z | = \frac{\langle y_k \cdot \theta \rangle}{\langle y_k, \theta \rangle} \leq \delta Y_{-r_{j_0}}^k.
\]

Since \( Z_{k,\delta} \) is contained in \((\pi_v \mathcal{O}_v)^m\), this proves Claim 1. \( \square \)
Claim 2. Let us prove that there exists a unique \( z \in R_v \) such that \( I \cap Z_{k,\delta} \) is contained in \( I \cap \mathcal{N}(k,z) \).

Proof. By Claim 1, the set \( I \cap Z_{k,\delta} \) is contained in \( \bigcup_{z \in R_v} I \cap \mathcal{N}(k,z) \). Assume for a contradiction that there exist two distinct elements \( z, z' \) in \( R_v \) such that there exist \( \theta \in I \cap \mathcal{N}(k,z) \) and \( \theta' \in I \cap \mathcal{N}(k,z') \). Let \( u \in L_k(z) \) and \( u' \in L_k(z') \) be the projections of \( \theta \) and \( \theta' \) along the \( j_0 \)-axis on \( L_k(z) \) and \( L_k(z') \) respectively.

Let \( j \in \{1, \ldots, m\} \). Note that \( \theta - \theta' \in \Pi(Y_k) \) since \( I \in I_k \). If \( j \neq j_0 \), then

\[
|u_j - u'_j| = |\theta_j - \theta'_j| \leq \frac{1}{q_v} Y_k^{-r_{j_0}}.
\]

Furthermore, by the ultrametric inequality, since \( \theta \) (respectively \( \theta' \)) is contained in the \( (\delta Y_k^{-r_{j_0}}) \)-thickening along the \( j_0 \)-axis of \( L_k(z) \) (respectively \( L_k(z') \)), and since \( \delta \leq \frac{1}{q_v} \), we have

\[
|u_{j_0} - u'_{j_0}| = |(u_{j_0} - \theta_{j_0}) + (\theta_{j_0} - \theta'_{j_0}) + (\theta'_{j_0} - u'_{j_0})| \\
\leq \max\{|u_{j_0} - \theta_{j_0}|, |\theta_{j_0} - \theta'_{j_0}|, |\theta'_0 - u'_{j_0}|\} \\
\leq \max\{\delta Y_k^{-r_{j_0}}, \frac{1}{q_v} Y_k^{-r_{j_0}}\} = \frac{1}{q_v} Y_k^{-r_{j_0}}.
\]

This implies since \( u \in L_k(z) \) and \( u' \in L_k(z) \) that

\[
1 \leq |z - z'| = |y_k \cdot u - y_k \cdot u'| \leq \max_{1 \leq j \leq m} |y_{k,j}| |u_j - u'_j| \leq \max_{1 \leq j \leq m} Y_k^{r_{j_0}} \frac{1}{q_v} Y_k^{-r_{j_0}} = \frac{1}{q_v},
\]

which is a contradiction since \( q_v > 1 \). This proves Claim 2. \( \square \)

By Equation (27), Claim 2 concludes the proof of Assertion (2) of Lemma 4.6. \( \square \)

Since every element \( I' \) of \( I_{k+1} \) is a translate of \( \Pi(Y_{k+1}) \), and by Equation (25), we have

\[
\operatorname{vol}_v^m(I') = \operatorname{vol}_v^m(\Pi(Y_{k+1})) = \prod_{j=1}^m q_v^{-n_{k+1,j}} \geq q_v^{-2m} Y_{k+1}^{m-|r|}.
\]

For every \( I \in I_k \), there are \( \Delta_{k+1} \) elements \( I' \in I_{k+1} \) contained in \( I \), they are pairwise disjoint and they have the same volume \( \operatorname{vol}_v^m(I) \operatorname{vol}_v^m(\Pi(Y_{k+1})) \). Among them, those who meet \( Z_{k,\delta} \) are actually contained in \( I \cap Z_{k,\delta} \) by Lemma 4.6 (1), thus their number is at most

\[
\frac{\operatorname{vol}_v^m(I \cap Z_{k,\delta})}{\operatorname{vol}_v^m(\Pi(Y_{k+1}))}.
\]

Therefore, by Equation (26) and Lemma 4.6 (2), we have

\[
\operatorname{Card}\{I' \in I_{k+1} : I' \subset I, \ I' \cap Z_{k,\delta} = \emptyset\} \geq \Delta_{k+1} - \frac{\operatorname{vol}_v^m(I \cap Z_{k,\delta})}{\operatorname{vol}_v^m(\Pi(Y_{k+1}))} \\
\geq q_v^{-m} (Y_{k+1} Y_k^{m-|r|}) \frac{\delta Y_k^{m-|r|}}{q_v^{-2m} Y_{k+1}^{m-|r|}} \\
= c_1 (Y_{k+1} Y_k^{-1})^{|r|}, \tag{28}
\]

where \( c_1 = q_v^{-m} - q_v^{-2m} \delta \) belongs to \([0,1] \) by the assumption on \( \delta \).

Now, let us define by induction \( J_0 = I_0 \) and for every \( k \in \mathbb{Z}_{\geq 0} \),

\[
J_{k+1} = \bigcup_{J \in J_k} \{I \in I_{k+1} : I \subset J, \ I \cap Z_{k,\delta} = \emptyset\}.
\]

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By Equation (28) and by induction, we have

\[
\text{Card } J_{k+1} \geq \prod_{j=1}^{k} c_1 \left( Y_{j+1} Y_{j}^{-1} \right)^{|j|} = c_k^k \left( Y_{k+1} Y_1^{-1} \right)^{|1|} .
\]  

(29)

By Lemma 4.6 (1) and by induction, we have

\[J_{k+1} = \{ J \in I_{k+1} : \forall j \in \{1, \ldots, k\}, J \cap Z_j \delta = \emptyset \} = \{ J \in I_{k+1} : J \subset \bigcap_{j=1}^{k} \overline{cZ}_{j, \delta} \},\]

where \(\overline{c}\) denotes the complement in \((\pi_v O_v)^m\). Hence \((\bigcup J_k)_{k \geq 1}\) is a decreasing sequence of compact subsets of \((\pi_v O_v)^m\), whose intersection is contained in \(\bigcap_{k \geq 1} \overline{cZ}_{k, \delta} = \text{Bad}^\delta_{(Y_i)_{i \geq 1}}\).

For every \(k \in \mathbb{Z}_{\geq 0}\), let us define a measure

\[\mu_k = \left( \text{vol}^m (\Pi(Y_k)) \text{ Card } J_k \right)^{-1} \sum_{J \in J_k} \text{vol}^m |J|,\]

which is a probability measure with support \(\bigcup J_k\). By the compactness of \((\pi_v O_v)^m\), any weak-star accumulation point \(\mu\) of the sequence \((\mu_k)_{k \geq 1}\) is a probability measure with support in \(\text{Bad}^\delta_{(Y_i)_{i \geq 1}}\).

For every closed ball \(B\) in \((\pi_v O_v)^m\) with radius \(r' \in [0, r_0 = Y_{1 - \min r}]\), let \(k \in \mathbb{Z}_{\geq 0}\) be such that

\[Y_{k+1} - \min r < r' \leq Y_{k} - \min r .\]

(30)

Note that \(|t| \leq t + 1 \leq q_v t\) if \(t \geq 1\), and that \(r' q_v^{n_{k+1,j}+1} \geq Y_{k+1} - \min r Y_{k+1} r_j q_v \geq 1\) for every \(j = 1, \ldots, m\), by Equation (25). Then \(B\) can be covered by a subset of \(I_{k+1}\) with cardinality at most

\[\prod_{j=1}^{m} \left( r' q_v^{n_{k+1,j}+1} \right) \leq (r')^m q_v^{3m} Y_{k+1} |r| .\]

Let \(C = \frac{-\log q_v - c_1}{\min r} > 0\), which depends (besides on \(m, q_v\) and \(r\)) only on \(\delta\). Defining \(C_0 = q_v^{3m} Y_{1} |r|\), by Equations (29) and (30), we thus have

\[\mu_{k+1}(B) \leq q_v^{3m} (r')^m Y_{k+1} |r| (\text{Card } J_{k+1})^{-1} \leq q_v^{3m} (r')^m c_1^{k} Y_{1} |r| \leq C_0 (r')^{m-C \frac{k}{\log q_v} Y_{1} |r|} .\]

Therefore, since the ball \(B\) is closed and open and since \(r' \leq r_0 \leq 1\), we have

\[\mu(B) \leq \limsup_{k \to \infty} C_0 (r')^{m-C \frac{k}{\log q_v} Y_{1} |r|} = C_0 (r')^{m-C \limsup_{k \to \infty} \frac{k}{\log q_v} Y_{1} |r|} ,\]

which concludes the proof of Proposition 4.5. 

\[\square\]
4.3 Proof that Assertion (2) implies Assertion (1) in Theorem 1.1

Suppose that $A$ is $(r,s)$-singular on average. Then by Corollary 2.9 the matrix $^tA$ is also $(s,r)$-singular on average. By Remark 2.5 (2), in order to prove that there exists $\epsilon > 0$ such that Bad$_A(\epsilon)$ has full Hausdorff dimension, we may assume that the matrix $^tA$ is completely irrational.

By Lemma 2.6 let $(y_k)_{k \in \mathbb{Z}_{\geq 1}}$ be a best approximation sequence in $K_v^m$ for the matrix $^tA$ with weights $(s,r)$, and let $c > 0$ be such that $M_i y_{i+1} \leq q_i^c$ for every $i \in \mathbb{Z}_{\geq 1}$. Fix some $\delta \in \left]0, \frac{1}{q_{2m}} \right]$ and let $b = b(\delta) > 0$ and $C = C(\delta) > 0$ as in Proposition 4.5. By Proposition 4.4 for every $a > b$, we have a subsequence $(y_{\varphi(k)})_{k \geq 1}$ such that the properties (15) and (16) are satisfied. Proposition 4.4, whose assumption is satisfied by the first inequality in Equation (15), and using Equation (16) for the last inequality, we have

$$\dim_{\text{Haus}} \text{Bad}_A(\epsilon) \geq \dim_{\text{Haus}} \text{Bad}_{\delta}(y_{\varphi(i)})_{i \geq 1} \geq m - C \limsup_{k \to \infty} \frac{k}{\log q_v y_{\varphi(k)}} \geq m - \frac{C}{a}.$$ 

Letting $a$ tend to $+\infty$, this concludes the proof that Assertion (2) implies Assertion (1) in Theorem 1.1.

5 Background material for the upper bound

5.1 Homogeneous dynamics

Let $K_v, \mathcal{O}_v, \pi_v, R_v, q_v$ be as in Subsection 2.1. Let $m, n \in \mathbb{N} - \{0\}$ and $d = m + n$. We fix some weights $r = (r_1, \ldots, r_m)$ and $s = (s_1, \ldots, s_n)$ as in the introduction. In this subsection, we introduce the space of unimodular grids $\mathcal{Y}$ in $K_v^d$ and the diagonal flow $(a^t)_{t \in \mathbb{Z}}$ acting on this space. Let

$$G_0 = \text{SL}_d(K_v) \quad \text{and} \quad G = \text{ASL}_d(K_v) = \text{SL}_d(K_v) \ltimes K_v^d,$$

and let

$$\Gamma_0 = \text{SL}_d(R_v) \quad \text{and} \quad \Gamma = \text{ASL}_d(R_v) = \text{SL}_d(R_v) \ltimes R_v^d.$$

The product in $G$ is given by

$$(g, u) \cdot (g', u') = (gg', u + gu')$$

for all $g, g' \in G_0$ and $u, u' \in K_v^d$. We also view $G$ as a subgroup of $\text{SL}_{d+1}(K_v)$ by

$$G = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in \text{SL}_d(K_v), \ u \in K_v^d \right\}.$$

We shall identify $G_0$ with the corresponding subgroup of $G$. We consider the one-parameter diagonal subgroup $(a^t)_{t \in \mathbb{Z}}$ of $G_0$, where $a = \text{diag}(a_-, a_+)$ and

$$a_- = \text{diag}(\pi_v^{-r_1}, \ldots, \pi_v^{-r_m}) \in \text{GL}_m(K_v) \quad \text{and} \quad a_+ = \text{diag}(\pi_v^{s_1}, \ldots, \pi_v^{s_n}) \in \text{GL}_n(K_v).$$
Note that for all $\theta \in K_v^m$, $\xi \in K_v^n$ and $\ell \in \mathbb{Z}$, we have

$$\| a^\ell \theta \|_r = q_v^\ell \| \theta \|_r \quad \text{and} \quad \| a^\ell \xi \|_s = q_v^{-\ell} \| \xi \|_s . \quad (32)$$

We denote by $G^+$ the unstable horospherical subgroup for $a$ in $G$ and by $U$ the unipotent radical of $G$, that is,

$$G^+ = \{ g \in G : \lim_{\ell \to \infty} a^\ell g a^{-\ell} = I_{d+1} \} \quad \text{and} \quad U = \left\{ \begin{pmatrix} I_d & u \\ 0 & 1 \end{pmatrix} : u \in K_v^d \right\} .$$

Let $U^+ = G^+ \cap U = \left\{ \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} : w \in K_v^m \right\}$, which is a closed subgroup in $G^+$ normalized by $a$.

Let us define

$$\mathcal{X} = G_0 / \Gamma_0 \quad \text{and} \quad \mathcal{Y} = G / \Gamma .$$

Even though we have Covol$(R_v^d) = q_v^{-(g-1)d}$ by Equation $(33)$, we say that an $R_v$-lattice $\Lambda$ in $K_v^d$ is unimodular if Covol$(\Lambda) = \text{Covol}(R_v^d)$. A translate in the affine space $K_v^d$ of an unimodular lattice is called an unimodular grid. We identify the homogeneous space $\mathcal{X} = \text{SL}_d(K_v)/\text{SL}_d(R_v)$ with the space of unimodular lattices in $K_v^d$ by the equivariant homeomorphism

$$x = g \Gamma_0 \mapsto \Lambda_x = g R_v^d ,$$

and the homogeneous space $\mathcal{Y} = \text{ASL}_d(K_v)/\text{ASL}_d(R_v)$ with the space of unimodular grids by the equivariant homeomorphism

$$y = \begin{pmatrix} g \\ u \\ 1 \end{pmatrix} \Gamma \mapsto \tilde{\Lambda}_y = g R_v^d + u . \quad (33)$$

We denote by $\pi : \mathcal{Y} \to \mathcal{X}$ the natural projection map (forgetting the translation factor), which is a proper map. Note that the fibers of $\pi$ are exactly the orbits of $U$ in $\mathcal{Y}$, and in particular each orbit under $U^+$ in $\mathcal{Y}$ is contained in some fiber of $\pi$ (see Lemma 5.3 for a precise understanding of the $U^+$-orbits).

For every $N \in \mathbb{N} - \{0\}$, we denote by $d_{\text{SL}_N(K_v)}$ the right-invariant distance on $\text{SL}_N(K_v)$ defined by

$$\forall g, h \in \text{SL}_N(K_v), \quad d_{\text{SL}_N(K_v)}(g, h) = \max\{ \ln(1 + \| gh^{-1} - \text{id} \|), \ln(1 + \| hg^{-1} - \text{id} \|) \} ,$$

where $\| \| \| \|$ is the operator norm on $\mathcal{M}_N(K_v)$ defined by the sup norm $\| \|$ on $K_v^N$. We endow every closed subgroup $H$ of $G$ with the right-invariant distance $d_H$ on $H$, which is the restriction to $H$ of the distance $d_{\text{SL}_d+1(K_v)}$. For instance, identifying the additive group $K_v^m$ with $U^+$ by the map $w \mapsto \hat{w} = \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have

$$\forall w, w' \in K_v^m, \quad d_{U^+}(\hat{w}, \hat{w'}) = \ln(1 + \| w - w' \|) . \quad (34)$$

We also consider the distance $d_{U^+, \| \|}$ on $U^+$ such that the map $w \mapsto \hat{w}$ is an isometry between $K_v^m$ endowed with the distance induced by the norm $\| \|$ and $(U^+, d_{U^+, \| \|}) : \forall w, w' \in K_v^m, \quad d_{U^+, \| \|}(\hat{w}, \hat{w'}) = \| w - w' \| . \quad (35)$
Observe that the distances $d_{U^+,||}$ and $d_{U^+}$ on $U^+$ are locally equivalent. More precisely, for all $w, w' \in K^m$, \[ \text{if } d_{U^+}(\hat{w}, \hat{w}') \leq \ln 2 \text{ then } \frac{1}{2} \|w - w'\| \leq d_{U^+}(\hat{w}, \hat{w}') \leq \|w - w'\|. \] (36)

We endow $\mathcal{Y} = G/\Gamma$ with the quotient distance $d_{\mathcal{Y}}$ of the distance $d_G$ on $G$, defined by

$$d_{\mathcal{Y}}(y, y') = \min_{\gamma \in \Gamma} d_G(\tilde{y}\gamma, \tilde{y}'\gamma)$$

for any representatives $\tilde{y}$ and $\tilde{y}'$ of the classes $y$ and $y'$ in $G/\Gamma$ respectively. This is a well defined distance since the canonical projection $G \rightarrow \mathcal{Y}$ is a covering map and the distance $d_G$ on $G$ is right-invariant. Given any closed subgroup $H$ of $G$, we denote by $B_H(x, r)$ (respectively $B_{\mathcal{Y}}(x, r)$) the open ball of center $x$ and radius $r > 0$ for the distance $d_H$ (respectively $d_{\mathcal{Y}}$), and by $B_{\mathcal{Y}}^H$ the open ball $B_H(\text{id}, r)$. Note that for all $y \in \mathcal{Y}$ and $r > 0$, we have (for the left action of subsets of $G$ on $\mathcal{Y}$)

$$B_{\mathcal{Y}}(y, r) = B^G_r y.$$ We denote by $B_{\mathcal{Y}}^{U^+,||}$ the open ball of center $\text{id}$ and radius $r > 0$ for the distance $d_{U^+,||}$ on $U^+$. Remark that $B_{\mathcal{Y}}^{U^+||} = B^U_{\ln(1+r)}$ for every $r > 0$, by Equations (34) and (35).

**Lemma 5.1** For all $\epsilon > 0$ and $k \in \mathbb{Z}_{\geq 0}$, we have

$$a^{-k}B_{\epsilon}^{U^+,||}a^k \subset B_{\epsilon q_v^{-k}\text{min} r}^{U^+} \quad \text{and} \quad a^{-k}B_{\epsilon}^{U^+}a^k \subset B_{\ln(1+(\epsilon^{-1})q_v^{-k}\text{min} r)}^{U^+}.$$

Similarly, we have

$$a^kB_{\epsilon}^{U^+,||}a^{-k} \subset B_{\epsilon q_v^k\text{max} r}^{U^+} \quad \text{and} \quad a^kB_{\epsilon}^{U^+}a^{-k} \subset B_{\ln(1+(\epsilon^{-1})q_v^k\text{max} r)}^{U^+}.$$

**Proof.** The proof of the second assertion being similar, we only prove the first one. For every $w = (w_1, \ldots, w_m) \in K^m$, we have $a^{-k}\hat{w}a^k = \tilde{a}^{-k}w$ and

$$\|a^{-k}w\| = \max_{1 \leq i \leq m} |a^{\epsilon^{-1}}w_i| \leq q_v^{-k}\text{min} r \|w\|.$$

The part on the left of the first assertion hence follows from Equation (35). The part on the right of the first assertion follows from the remark just before the statement of Lemma 5.1. \[ \square \]

Given a point $x$ in $\mathcal{Y}$ (and similarly for $x$ in $\mathcal{X}$), we define the injectivity radius of $\mathcal{Y}$ at $x$ to be

$$\text{inj}(x) = \sup \{ r > 0 : \forall \gamma \in \Gamma - \{\text{id}\}, \ B_G(\tilde{x}, r) \cap B_G(\tilde{x}\gamma, r) = \emptyset \},$$

which does not depend on the choice of $\tilde{x} \in G$ such that $x = \tilde{x}\Gamma$, and is positive and finite since the canonical projection $G \rightarrow \mathcal{Y}$ is a nontrivial covering map. For every $r > 0$, we denote the $r$-thick part of $\mathcal{Y}$ by

$$\mathcal{Y}(r) = \{ x \in \mathcal{Y} : \text{inj}(x) \geq r \}.$$ It follows from the finiteness of a Haar measure of the homogeneous space $\mathcal{Y}$ that $\mathcal{Y}(r)$ is a compact subset of $\mathcal{Y}$ for every $r > 0$, and that the Haar measure of the $r$-thin part $\mathcal{Y} - \mathcal{Y}(r)$ tends to 0 as $r$ goes to 0. For every compact subset $K$ of $\mathcal{Y}$, there exists $r > 0$ such that $K \subset \mathcal{Y}(r)$.
5.2 Dani correspondence

In this subsection, we give an interpretation of the property for a matrix $A \in \mathcal{M}_{m,n}(K_v)$ to be $(r,s)$-singular on average in terms of dynamical properties of the action of the one-parameter diagonal subgroup $(d^t)_{t \in \mathbb{Z}}$ on the space of unimodular lattices, as originally developed by Dani (see for instance [Kle] §4). For every $A \in \mathcal{M}_{m,n}(K_v)$, let $u_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_0$.

**Proposition 5.2** A matrix $A \in \mathcal{M}_{m,n}(K_v)$ is $(r,s)$-singular on average if and only if the forward orbit $\{a^t u_A R^d_v : t \in \mathbb{Z}_{\geq 0}\}$ in $X$ of the lattice $u_A R^d_v$ under $a$ diverges on average in $X$, that is, if and only if for any compact subset $Q$ of $X$, we have

$$\lim_{N \to \infty} \frac{1}{N} \text{Card}\{t \in \{1, \ldots, N\} : a^t u_A \in Q\} = 0.$$

**Proof.** Let $Q$ be a compact subset of $X$. By Mahler’s compactness criterion (see for instance [KIST, Theo. 1.1]), there exists $\varepsilon \in ]0,1[$ such that $Q$ is contained in

$$X_{\varepsilon} = \{g R^d_v \in X : \forall (\theta, \xi) \in g R^d_v - \{0\} \subset K_v^m \times K_v^n, \ max\{\|\theta\|_r, \|\xi\|_s\} > \varepsilon\},$$

which is the subset of $X$ consisting of the unimodular lattices with systole (for an appropriate quasinorm) larger than $\varepsilon$. Observe that by Equation (32), for all sufficiently large $t \in \mathbb{Z}_{\geq 1}$, there exists an element $y \in R^d_v - \{0\}$ such that $\langle Ay \rangle_r \leq \varepsilon q_v^{-\ell}$ and $\|y\|_s \leq \varepsilon q_v^\ell$ if and only if we have $a^t u_A R^d_v = \begin{pmatrix} a^t & 0 \\ 0 & a^t \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R^d_v \in X - X_{\varepsilon}$.

With $\ell = \lceil -\log q_v \rceil$, it follows that

$$0 \leq \text{Card}\{t \in \{1, \ldots, N\} : a^t u_A R^d_v \in Q\} \leq \text{Card}\{t \in \{1, \ldots, N\} : a^t u_A R^d_v \in X_{\varepsilon}\} = \text{Card}\{t \in \{1, \ldots, N\} : \#y \in R^n_v - \{0\}, \langle Ay \rangle_r \leq \varepsilon q_v^{-\ell}, \|y\|_s \leq \varepsilon q_v^\ell\} \leq \text{Card}\{t \in \{1, \ldots, N\} : \#y \in R^n_v - \{0\}, \langle Ay \rangle_r \leq \frac{\varepsilon^2}{q_v^{\ell - \ell}}, \|y\|_s \leq \varepsilon q_v^\ell\} \leq \ell \varepsilon + \text{Card}\{t \in \{1, \ldots, N - \ell\} : \#y \in R^n_v - \{0\}, \langle Ay \rangle_r \leq \frac{\varepsilon^2}{q_v^{\ell}}, \|y\|_s \leq \varepsilon q_v^\ell\}.$$ 

After dividing by $N$ (or equivalently by $N - \ell$) this last expression, its limit as $N$ tends to $0$ exists and is equal to 0 if $A$ is $(r,s)$-singular on average (see Equation (2)). Hence we have $\lim_{N \to \infty} \frac{1}{N} \text{Card}\{t \in \{1, \ldots, N\} : a^t u_A \in Q\} = 0$ by the above string of inequalities.

The converse implication follows similarly by taking for the compact set $Q$ the subset $X_{\varepsilon}$.

We denote by $\| \cdot \|_{s,r}$ the quasi-norm on $K^d_v = K_v^m \times K_v^n$, defined by

$$\| (\theta, \xi) \|_{s,r} = \max \{\|\theta\|_r^d, \|\xi\|_s^d\}.$$ 

Let $\varepsilon > 0$. We define the $\varepsilon$-compact part of $\mathcal{Y}$ to be

$$\mathcal{L}_\varepsilon = \{y \in \mathcal{Y} : \forall u \in \tilde{\Lambda}_y, \|u\|_{r,s} \geq \varepsilon\}.\tag{38}$$

By Mahler’s compactness criterion (see for instance [KIST, Theo. 1.1]) and since the natural projection $\pi : \mathcal{Y} \to X$ is proper, the subset $\mathcal{L}_\varepsilon$ is compact.

For every $\theta \in K_v^m$, we denote by $y_{A,\theta}$ the unimodular grid $u_A R^d_v - (\theta \ 0)$.
Lemma 5.3 For every $A \in \mathcal{M}_{m,n}(K_v)$, the map $K_v^m \to \mathcal{Y}$ defined by $\theta \mapsto y_{A,\theta}$ induces a bilipschitz homeomorphism $\phi_A$ from $\mathbb{T}^m = K_v^m/R_v^m$ endowed with the quotient distance $d_{\mathbb{T}^m}$ of the distance on $K_v^m$ defined by the standard norm $\|\cdot\|$, and the $U^+$-orbit $U^+y_{A,0}$ endowed with the restriction of the distance $d_{\mathcal{Y}}$ of $\mathcal{Y}$. Furthermore, the map $\phi_A$ is an isometry between ($\mathbb{T}^m, d_{\mathbb{T}^m}$) and the $U^+$-orbit $U^+y_{A,0}$ endowed with the quotient distance $d_{U^+y_{A,0},||\cdot||}$ of the distance $d_{U^+,||\cdot||}$ of $U^+$ defined in Equation (35).

Proof. The map $K_v^m \to \mathcal{Y}$ defined by $\theta \mapsto y_{A,\theta}$ is clearly invariant under translations by $R_v^m$, and induces a bijection

$$\phi_A : \theta \mod R_v^m \mapsto y_{A,\theta}$$

from $\mathbb{T}^m = K_v^m/R_v^m$ to the orbit $U^+y_{A,0}$. This orbit is contained in the fiber $\pi^{-1}(x_A)$ of $x_A = u_A R_v^m$ for the natural projection $\pi : \mathcal{Y} \to \mathcal{X}$, as already seen.

For all $A \in \mathcal{M}_{m,n}(K_v)$ and $\theta \in K_v^m$, let $u_{A,\theta} = \begin{pmatrix} 1_m & A & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$, so that we have $y_{A,\theta} = u_{A,\theta} \Gamma$. For all $\theta, \theta' \in \mathbb{T}^m$, denoting lifts of them to $K_v^m$ by $\tilde{\theta}, \tilde{\theta}'$ respectively, identifying $K_v^d$ with $K_v^m \times K_v^n$, and using Equations (39) and (31), we have

$$d_{\mathcal{Y}}(\phi_A(\theta), \phi_A(\theta')) = \inf_{\gamma \in \Gamma} d_G(u_{A,\theta} \gamma, u_{A,\theta'})$$

$$= \inf_{x \in R_v^m, y \in R_v^n} d_G((u_{A}g, (x + Ay - \tilde{\theta}, y)), (u_{A}, (-\tilde{\theta}', 0)))$$

$$= \inf_{x \in R_v^m} d_{U^+}((\text{id}, (x - \tilde{\theta}, 0)), (\text{id}, (-\tilde{\theta}', 0)))$$

$$= \inf_{x \in R_v^m} \ln(1 + \|\tilde{\theta} - \tilde{\theta}' - x\|) = \ln(1 + d_{\mathbb{T}^m}(\theta, \theta')).$$

The last claim of Lemma 5.3 hence follows from Equation (34). By the properties of the logarithm map and by the compactness of $\mathbb{T}^m$, which implies that the diameter $\delta$ of $(\mathbb{T}^m, d_{\mathbb{T}^m})$ is finite, we have

$$\frac{\ln(1 + \delta)}{\delta} d_{\mathbb{T}^m}(\theta, \theta') \leq d_{\mathcal{Y}}(\phi_A(\theta), \phi_A(\theta')) \leq d_{\mathbb{T}^m}(\theta, \theta').$$

This proves the first claim of Lemma 5.3.

Proposition 5.4 Let $\varepsilon > 0$. For every $(A, \theta) \in \mathcal{M}_{m,n}(K_v) \times K_v^m$ such that $\theta \in \text{Bad}_A(\varepsilon)$, one of the following statements holds.

1. There exists $y \in R_v^n$ such that $\langle Ay - \theta \rangle_r = 0$. Note that given $A$, there are only countably many $\theta$ satisfying this statement.

2. The forward $a$-orbit of the point $y_{A,\theta}$ is eventually in the $\varepsilon$-compact part $\mathcal{L}_\varepsilon$, that is, there exists $T \geq 0$ such that for every $\ell \geq T$, we have $a^\ell y_{A,\theta} \in \mathcal{L}_\varepsilon$.

Proof. Assume for a contradiction that both statements do not hold. Then there exist infinitely many $\ell \in \mathbb{Z}_{\geq 1}$ such that $a^\ell y_{A,\theta} \not\in \mathcal{L}_\varepsilon$, hence such that there exists $y_\ell \in R_v^n$ with $\langle Ay_\ell - \theta \rangle_r < q_\ell \varepsilon^\frac{1}{m}$ and $\|y_\ell\|_s < q_\ell \varepsilon^\frac{1}{m}$. Since the statement (1) does not hold, the inequality

$$\|y\|_s \langle Ay - \theta \rangle_r < \varepsilon$$

has infinitely many solutions $y \in R_v^n$, which contradicts the assumption $\theta \in \text{Bad}_A(\varepsilon)$. \qed
5.3 Entropy, partition construction, and effective variational principle

In this subsection, after recalling the basic definitions and properties about entropy (using [ELW] as a general reference, and in particular its Chapter 2), we give the preliminary constructions of σ-algebras and results on entropy that will be needed in Section 6. In particular, we give an effective and positive characteristic version of the variational principle for conditional entropy on [EL] §7.55. See [KKL] for the real case.

Let \((X, \mathcal{B}, \mu)\) be a standard Borel probability space. For every set \(E\) of subsets of \(X\), we denote by \(\sigma(E)\) the σ-algebra of subsets of \(X\) generated by \(E\). Let \(\mathcal{P}\) be a (finite or) countable \(\mathcal{B}\)-measurable partition of \(X\). Let \(\mathcal{A}, \mathcal{C}\) and \(\mathcal{C}'\) be sub-σ-algebras of \(\mathcal{B}\). Suppose that \(\mathcal{C}\) and \(\mathcal{C}'\) are countably generated.

For every \(x \in X\), we denote by \([x]_{\mathcal{P}}\) the atom of \(x\) for \(\mathcal{P}\), which is the element of the partition \(\mathcal{P}\) containing \(x\). We denote by \([x]_{\mathcal{C}}\) the atom of \(x\) for \(\mathcal{C}\), which is the intersection of all elements of \(\mathcal{C}\) containing \(x\). Note that \([x]_{\sigma(\mathcal{P})} = [x]_{\mathcal{P}}\). We denote by \((\mu^{\mathcal{P}}_x)_{x \in X}\) an \(\mathcal{A}\)-measurable family of (Borel probability) conditional measures of \(\mu\) with respect to \(\mathcal{A}\) on \(X\), given for instance by [EL] Theo. 5.9.

Using the standard convention \(0 \log 0 = 0\) and using \(\log_q v\) instead of \(\log v\) for computational purposes in the field \(K_v\), the entropy of the partition \(\mathcal{P}\) with respect to \(\mu\) is defined by

\[
H_{\mu}(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log_q \mu(P) \in [0, \infty].
\]

Recall the (logarithmic) cardinality majoration

\[
H_{\mu}(\mathcal{P}) \leq \log_q \text{(Card } \mathcal{P}) .
\] (40)

The information function of \(\mathcal{C}\) given \(\mathcal{A}\) with respect to \(\mu\) is the measurable map \(I_{\mu}(\mathcal{C}|\mathcal{A}) : X \rightarrow [0, \infty]\) defined by

\[
\forall x \in X, \quad I_{\mu}(\mathcal{C}|\mathcal{A})(x) = -\log_q \mu_{\mathcal{C}}^{\mathcal{A}}([x]_{\mathcal{C}}).
\]

The conditional entropy of \(\mathcal{C}\) given \(\mathcal{A}\) with respect to \(\mu\) is defined by

\[
H_{\mu}(\mathcal{C}|\mathcal{A}) = \int_X I_{\mu}(\mathcal{C}|\mathcal{A}) \, d\mu.
\] (41)

Recall the additivity property \(H_{\mu}(\mathcal{C} \lor \mathcal{C}'|\mathcal{A}) = H_{\mu}(\mathcal{C}|\mathcal{A}) + H_{\mu}(\mathcal{C}'|\mathcal{A})\) (see for instance [ELW] Prop. 2.13) so that if \(\mathcal{A} \subset \mathcal{C}' \subset \mathcal{C}\), we have

\[
H_{\mu}(\mathcal{C}|\mathcal{A}) = H_{\mu}(\mathcal{C}|\mathcal{C}') + H_{\mu}(\mathcal{C}'|\mathcal{A}).
\] (42)

Let \(T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)\) be a measure-preserving transformation. Assume that the σ-algebra \(\mathcal{A}\) is strictly \(T\)-invariant, i.e., that \(T^{-1}\mathcal{A} = \mathcal{A}\). If the partition \(\mathcal{P}\) has finite entropy with respect to \(\mu\), let

\[
h_{\mu}(T, \mathcal{P}|\mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}|\mathcal{A}\right) = \inf_{n \geq 1} \frac{1}{n} H_{\mu}\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}|\mathcal{A}\right).
\]

The conditional (dynamical) entropy of \(T\) given \(\mathcal{A}\) is

\[
h_{\mu}(T|\mathcal{A}) = \sup_{\mathcal{P}} h_{\mu}(T, \mathcal{P}|\mathcal{A}),
\]

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where the upper bound is taken on all countable \( \mathcal{B} \)-measurable partitions \( \mathcal{P} \) of \( X \) with finite entropy with respect to \( \mu \).

Let \( \mathcal{E} = \{ B \in \mathcal{B} : \mu(T^{-1}B \Delta B) = 0 \} \) be the sub-\( \sigma \)-algebra of \( T \)-invariant elements of \( \mathcal{B} \) modulo \( \mu \). We denote by \((\mu^T_x)_{x \in X}\) the associated family of conditional measures. The following result is proven for instance in [ELW, Theo. 2.34].

**Proposition 5.5 (Entropy and ergodic decomposition)** If \( \mathcal{A} \) is strictly \( T \)-invariant, then we have

\[
h_\mu(T|\mathcal{A}) = \int_X h_{\mu^T_x}(T|\mathcal{A}) \, d\mu(x) .
\]

We now work in the standard Borel space \( \mathcal{Y} \) of unimodular grids, endowed with the distance \( d_\mathcal{Y} \) (see Subsection 5.1). Let \( \delta > 0 \). For every subset \( B \) of \( \mathcal{Y} \), we define the \( \delta \)-boundary \( \partial_\delta B \) of \( B \) by

\[
\partial_\delta B = \{ y \in \mathcal{Y} : \inf_{y' \in B} d_\mathcal{Y}(y, y') + \inf_{y'' \in \mathcal{Y} \setminus B} d_\mathcal{Y}(y, y'') < \delta \}
\]

if \( B \) and \( \mathcal{Y} \setminus B \) are nonempty, and \( \partial_\delta B = \emptyset \) otherwise. Note that for all subsets \( B \) and \( B' \) of \( \mathcal{Y} \), we have

\[
\partial_\delta (B \cup B') \subseteq \partial_\delta B \cup \partial_\delta B' \quad \text{and} \quad \partial_\delta (B \cap B') \subseteq \partial_\delta B \cup \partial_\delta B' .
\]

We also have \( \partial_\delta B \subseteq \partial_{\delta'} B \) if \( \delta \leq \delta' \). Given any set \( \mathcal{P} \) of subsets of \( \mathcal{Y} \), we define the \( \delta \)-boundary \( \partial_\delta \mathcal{P} \) of \( \mathcal{P} \) by

\[
\partial_\delta \mathcal{P} = \bigcup_{B \in \mathcal{P}} \partial_\delta B .
\]

**Lemma 5.6** For every \( r > 0 \), there exist \( \delta_r \in \mathbb{I}_{0,r} \) and a finite measurable partition \( \mathcal{P} = \{P_1, \ldots, P_N, P_\infty\} \) by closed and open subsets of \( \mathcal{Y} \) such that

1. the subset \( P_\infty \) is contained in the \( r \)-thin part \( \mathcal{Y} \setminus \mathcal{Y}(r) \),
2. for every \( i \in \{1, \ldots, N\} \), there exists \( y_i \in \mathcal{Y}(r) \) such that \( B^G_{y_i} \in P_i \subseteq B^G_{y_i} \),
3. the set \( \partial_\delta \mathcal{P} \) is empty.

**Proof.** Choose a finite maximal \( r \)-separated subset \( \{y_1, \ldots, y_N\} \) of \( \mathcal{Y}(r) \) for the distance \( d_\mathcal{Y} \), which exists by the compactness of \( \mathcal{Y}(r) \). By induction on \( i = 1, \ldots, N \), we define a Borel subset \( P_i \) of \( \mathcal{Y} \) by

\[
P_i = B^G_{y_i} - \left( \bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N} B^G_{y_j} \right) .
\]

Define \( P_\infty = \mathcal{Y} \setminus \bigcup_{j=1}^{N} P_j \), which is also a Borel subset of \( \mathcal{Y} \).

By construction, we have \( P_i \subseteq B^G_{y_i} \). Since the set \( \{y_1, \ldots, y_N\} \) is \( r \)-separated, the intersection of open balls \( B^G_{y_i} \cap B^G_{y_j} = B_\mathcal{Y}(y_i, \frac{r}{2}) \cap B_\mathcal{Y}(y_j, \frac{r}{2}) \) is empty if \( j > i \). By construction, the intersection \( B^G_{y_i} \cap P_j \) is empty if \( j < i \). Therefore \( P_i \) contains \( B^G_{y_i} \), and Assertion (ii) follows.

By construction, we have \( \bigcup_{j=1}^{N} P_j \subseteq \bigcup_{j=1}^{N} B^G_{y_j} = \bigcup_{j=1}^{N} B_\mathcal{Y}(y_j, r) \), and the later union contains \( \mathcal{Y}(r) \), since the \( r \)-separated set \( \{y_1, \ldots, y_N\} \) is maximal. Assertion (i) follows.
For every \( s > 0 \), let \( n_s = \left[ \frac{\ln(e^s - 1)}{\ln \delta_s} \right] \in \mathbb{Z} \) and \( \delta'_s = \ln \left( \frac{1 + q^{n_s}}{1 + q^{n_s - 1}} \right) > 0 \). For all \( \delta > 0 \) and \( y \in \mathcal{Y} \), assume that there exists a point \( z \in \partial \mathcal{B}(y, s) \). Let \( z' \in B_{\mathcal{Y}}(y, s) \) and \( z'' \notin B_{\mathcal{Y}}(y, s) \) be such that \( d_{\mathcal{Y}}(z, z') + d_{\mathcal{Y}}(z, z'') < \delta \). Since the operator norm on \( \mathcal{M}_{d+1}(K) \) has values in \( \{0\} \cup q^n \), the set \( \{d_{\mathcal{Y}}(y, y') : y, y' \in \mathcal{Y}\} \) of values of the distance function \( d_{\mathcal{Y}} \) on \( \mathcal{Y} \) is contained in \( \{0\} \cup \{\ln(1 + q^n) : n \in \mathbb{Z}\} \). Since \( s \in [\ln(1 + q^{n_s - 1}), \ln(1 + q^{n_s})] \), we have \( d_{\mathcal{Y}}(y, y') \leq \ln(1 + q^{n_s - 1}) \) since \( z' \in B_{\mathcal{Y}}(y, s) \) and \( d_{\mathcal{Y}}(y, z'') \geq \ln(1 + q^{n_s}) \) since \( z'' \notin B_{\mathcal{Y}}(y, s) \). Therefore by the triangle inequality and the inverse triangle inequality, we have

\[
\delta > d_{\mathcal{Y}}(z, z') + d_{\mathcal{Y}}(z, z'') \geq d_{\mathcal{Y}}(z', z'') \geq d_{\mathcal{Y}}(y, z'') - d_{\mathcal{Y}}(y, z') \\
\geq \ln(1 + q^{n_s}) - \ln(1 + q^{n_s - 1}) = \delta'_s.
\]

Hence \( \partial \mathcal{B}(y, s) \) is empty for every \( \delta \in [0, \delta'_s] \).

By Equation (43), for every \( \delta > 0 \), we have

\[
\partial \mathcal{Y} \subset \bigcup_{j=1}^{N} \partial \mathcal{B}_{y_j}^{G} \cup \bigcup_{j=1}^{N} \partial \mathcal{B}_{y_j}^{Z}.
\]

Hence Assertion (iii) follows with \( \delta_r = \min \{\delta'_r, r\} \).

Note that since the distance \( d_G \) has values in \( \{0\} \cup \{\ln(1 + q^n) : n \in \mathbb{Z}\} \), the open balls in \( G \) are open and compact, and since the canonical projection \( G \to \mathcal{Y} \) is open and continuous, the subsets \( P_i \) of \( \mathcal{Y} \) are by construction open and compact, and \( P_\infty \) is closed and open.

Let \( \mathcal{C} \) be a countably generated \( \sigma \)-algebra of subsets of \( \mathcal{Y} \). Note that for every \( j \in \mathbb{Z} \), the \( \sigma \)-algebra \( \mathcal{A}^j \mathcal{C} \) is also countably generated and

\[
[y]_{\mathcal{A}^j \mathcal{C}} = \mathcal{A}^j [a^{-j} y]_{\mathcal{C}}.
\]

We say that \( \mathcal{C} \) is \( a^{-1} \)-descending if \( \mathcal{A} \mathcal{C} \) is contained in \( \mathcal{C} \). If \( \mathcal{C} \) is \( a^{-1} \)-descending, for all \( y \in \mathcal{Y} \) and \( j \in \mathbb{Z}_{\geq 0} \), we have

\[
[y]_{\mathcal{C}} \subset [y]_{\mathcal{A}^j \mathcal{C}}.
\]

Given a Borel probability measure \( \mu \) on \( \mathcal{Y} \) and a closed subgroup \( H \) of \( G \), we say that \( \mathcal{C} \) is \( H \)-subordinate modulo \( \mu \) if for \( \mu \)-almost every \( y \in \mathcal{Y} \), there exists \( r = r_y \in [0, 1] \) such that we have

\[
B_r y \subset [y]_{\mathcal{C}} \subset B_1 y.
\]

If \( \mathcal{C} \) is \( U^+ \)-subordinate modulo \( \mu \) and \( \mu \) is \( \mathcal{A} \)-invariant, since \( \mathcal{A} \) normalises \( U^+ \) and by Lemma [5.1] for every \( j \in \mathbb{Z} \), the \( \sigma \)-algebra \( \mathcal{A}^j \mathcal{C} \) is also \( U^+ \)-subordinate modulo \( \mu \).

For every \( \sigma \)-algebra \( \mathcal{A} \) of subsets of \( \mathcal{Y} \), for all \( a, b \in \mathbb{Z} \cup \{\pm \infty\} \) with \( a < b \), we define a \( \sigma \)-algebra \( \mathcal{A}_{a}^{b} \) of subsets of \( \mathcal{Y} \) by

\[
\mathcal{A}_{a}^{b} = \bigcup_{i=a}^{b} \mathcal{A}^i \mathcal{A} = \sigma \left( \bigcup_{a \leq i \leq b} \mathcal{A}^i \mathcal{A} \right).
\]

Note that if \( \mathcal{A} \) is countably generated, then so is \( \mathcal{A}_{a}^{b} \).
Proposition 5.7 For every \( r \in ]0,1[ \), there exists a countably generated sub-\( \sigma \)-algebra \( \mathcal{A}^{U^+} \) of the Borel \( \sigma \)-algebra of \( \mathcal{Y} \) such that

1. the countably generated \( \sigma \)-algebra \( \mathcal{A}^{U^+} \) is \( a^{-1} \)-descending,
2. for every \( y \in \mathcal{Y} (r) \), we have \([y]_{\mathcal{A}^{U^+}} \subset B_r^U y\),
3. for every \( y \in \mathcal{Y} \), we have \( B_{\delta_r}^U y \subset [y]_{\mathcal{A}^{U^+}} \), where \( \delta_r \in ]0, r] \) is as in Lemma 5.6.

Let \( \mu \) be a Borel \( a \)-invariant ergodic probability measure on \( \mathcal{Y} \) with \( \mu (\mathcal{Y} (r)) > 0 \). Then \( \mathcal{A}^{U^+} \) is \( U^+ \)-subordinate modulo \( \mu \).

Proof. Fix \( r \in ]0,1[ \). Let \( \mathcal{P} = \{P_1, \ldots, P_N, P_\infty \} \) be a partition given by Lemma 5.6 for this \( r \). We prove a preliminary result on the countably generated sub-\( \sigma \)-algebra \( \sigma (\mathcal{P})_0^\infty \).

Lemma 5.8 For every \( y \in \mathcal{Y} \), we have \( B_{\delta_r}^U y \subset [y]_{\sigma (\mathcal{P})_0^\infty} \).

Proof. Let \( h \in B_{\delta_r}^U \). Assume for a contradiction that \( hy \notin [y]_{\sigma (\mathcal{P})_0^\infty} \). Then there exists \( k \in Z_{\geq 0} \) such that \( a^{-k}hy \) and \( a^{-k}y \) belong to different atoms of the partition \( \mathcal{P} \). Let \( \alpha = \min r > 0 \). By Lemma 5.1, we have

\[
d_\mathcal{Y} (a^{-k}hy, a^{-k}y) \leq d_G (a^{-k}ha^k, id) = d_{U^+} (a^{-k}ha^k, id) < \ln (1 + (e^{\delta_r} - 1) q^{-k\alpha}) \leq \delta_r \leq r.
\]

(45)

It follows that both \( a^{-k}hy \) and \( a^{-k}y \) belong to the \( \delta_r \)-boundary \( \partial_{\delta_r} \mathcal{P} \) of \( \mathcal{P} \). But the set \( \partial_{\delta_r} \mathcal{P} \) is empty by Lemma 5.6 (3), which gives a contradiction. \( \square \)

By Lemma 5.6 for every \( i \in \{1, \ldots, N\} \), there exist \( y_i \in \mathcal{Y} (r) \) and a Borel subset \( V_i \) of \( G \) contained in \( B_i^G \) such that \( P_i = V_i y_i \). Let \( \mathcal{U}^+ \) be the sub-\( \sigma \)-algebra of the Borel \( \sigma \)-algebra of \( \mathcal{Y} \) generated by the subsets \( P_\infty \cap \pi^{-1} (W) \), where \( W \) is a Borel subset of \( \mathcal{X} \), and the subsets \((U^+ B) \cap V_i)y_i\), where \( i \in \{1, \ldots, N\} \) and \( B \) is a Borel subset of \( G \). Then \( \mathcal{U}^+ \) is countably generated, since the Borel \( \sigma \)-algebra of \( \mathcal{X} \) is countably generated and \( U^+ \) is a closed subgroup of \( G \). For every \( y \in \mathcal{Y} \), the atom of \( y \) for \( \mathcal{U}^+ \) is equal to

\[
[y]_{\mathcal{U}^+} = \begin{cases} Uy & \text{if } y \in P_\infty \\ P_i \cap (B_r^U y) & \text{if } \exists i \in \{1, \ldots, N\}, y \in P_i. \end{cases}
\]

(46)

Let us now define

\[
\mathcal{A}^{U^+} = (\mathcal{U}^+)_{\sigma (\mathcal{P})_0^\infty},
\]

(47)

which is a countably generated sub-\( \sigma \)-algebra of the Borel \( \sigma \)-algebra of \( \mathcal{Y} \), since so is \( \mathcal{U}^+ \). Note that \( \mathcal{A}^{U^+} = (\mathcal{U}^+)_{\sigma (\mathcal{P})_0^\infty} \subset \mathcal{A}^{U^+} \), which proves Assertion (1).

For every \( y \in \mathcal{Y} (r) \), since \( P_\infty \subset \mathcal{Y} - \mathcal{Y} (r) \) by Lemma 5.6 (1), and by Equation (46), we have \([y]_{\mathcal{A}^{U^+}} \subset [y]_{\mathcal{U}^+} \subset B_{\delta_r}^U y\), which proves Assertion (2).

In order to prove the last Assertion (3), let us take \( y \in \mathcal{Y} \) and \( h \in B_{\delta_r}^U \) and let us prove that \( hy \in [y]_{\mathcal{A}^{U^+}} \). Since we have \( hy \in [y]_{\sigma (\mathcal{P})_0^\infty} \) by Lemma 5.8 for every \( k \geq 0 \), there exists \( i \in \{1, \ldots, N, \infty\} \) such that the points \( a^{-k}y \) and \( a^{-k}hy = a^{-k}ha^k (a^{-k}y) \) both belong to \( P_i \). If \( i = \infty \), then by Equation (46), the points \( a^{-k}y \) and \( a^{-k}hy \) lie in the same atom \([a^{-k}y]_{\mathcal{A}^{U^+}} = Ua^{-k}y \) since \( a^{-k}ha^k \in U^+ \). Assume that \( 1 \leq i \leq N \). Since \( h \in B_{\delta_r}^U \), it follows from Equation (45) that \( a^{-k}ha^k \in B_{\delta_r}^U \). Hence by Equation (46), the points \( a^{-k}y \) and \( a^{-k}hy \) lie in the same atom \([a^{-k}y]_{\mathcal{U}^+} = P_i \cap (B_{\delta_r}^U a^{-k}y) \) of \( \mathcal{U}^+ \). This proves Assertion (3).
Now let $\mu$ be an $\alpha$-invariant ergodic probability measure on $\mathcal{Y}$ with $\mu(\mathcal{Y}(r)) > 0$. By ergodicity, for $\mu$-almost every $y \in \mathcal{Y}$, there exists $k \in \mathbb{Z}_{\geq 1}$ such that $a^{-k}y \in \mathcal{Y}(r)$. Since $a^k\mathcal{U}^+ \subset \mathcal{U}^+$, by Assertion (1) and by Lemma [5.1] we have

$$[y]_{\mathcal{U}^+} = \{a^k[a^{-k}y] \in \mathcal{U}^+ \subset a^kB^U_{e^r} \mid a^{-k}y \in B^U_{e^r} \}.$$  

With Assertion (3), this proves that $\mathcal{U}^+$ is $U^+$-subordinate modulo $\mu$. \hfill $\square$

Let us introduce some material before stating and proving our next Lemma [5.9] The map $d_{K_v^m,r} : K_v^m \times K_v^m \to [0, +\infty]$ defined by

$$\forall \xi, \xi' \in K_v^m, \quad d_{K_v^m,r}(\xi, \xi') = \|\xi - \xi'\|_r$$

is an ultrametric distance on $K_v^m$, since the $r$-pseudonorm $\|\cdot\|_r$ satisfies the ultrametric inequality: for all $\xi, \xi' \in K_v^m$, we have

$$\|\xi + \xi'\|_r \leq \max\{\|\xi\|_r, \|\xi'\|_r\},$$

with equality if $\|\xi\|_r \neq \|\xi'\|_r$. Note that the map similar to $d_{K_v^m,r}$ in the real case of $[K\ell]$ is not a distance if $m \geq 2$ for general $r$. For every $\epsilon > 0$, we denote by $B_{\epsilon K_v^m,r}$ the open ball of center 0 and radius $\epsilon$ in $K_v^m$ for $d_{K_v^m,r}$. Note that the distance $d_{K_v^m,r}$ is bi-hölder equivalent to the standard one: For all $\xi, \xi' \in K_v^m$ such that $\|\xi - \xi'\|_r \leq 1$, we have

$$\|\xi - \xi'\|_{\max r} \leq d_{K_v^m,r}(\xi, \xi') \leq \|\xi - \xi'\|_{\max r}.$$  

We also endow the quotient space $\mathbb{T}^m = K_v^m/R_v^m$ with the quotient distance $d_{\mathbb{T}^m,r}$ of the distance $d_{K_v^m,r}$ on $K_v^m$ defined by Equation (40). For every $A \in \mathbb{M}_{m,n}(K_v)$, we denote by $d_{U^+y_A,0,r}$ the distance on the orbit $U^+y_A,0$ defined by requiring the homeomorphism $\phi_A$ given in Lemma [5.3] to be an isometry for the distances $d_{\mathbb{T}^m,r}$ and $d_{U^+y_A,0,r}$, that is,

$$\forall \theta, \theta' \in \mathbb{T}^m, \quad d_{U^+y_A,0,r}(\phi_A(\theta), \phi_A(\theta')) = d_{\mathbb{T}^m,r}(\theta, \theta').$$

Using the identification $w \mapsto \hat{w}$ between $K_v^m$ and $U^+$ (see Subsection 5.1), for every $\epsilon > 0$, we denote by $B_{\epsilon U^+}^+$ the open ball of radius $\epsilon$ in $U^+$ centered at the identity element for the distance $d_{U^+}^+$ on $U^+$ corresponding to the distance $d_{K_v^m,r}$ on $K_v^m$. The map $u \mapsto uy_A,0$ from $U^+$ onto $U^+y_A,0$ is 1-Lipschitz and locally isometric for the distances $d_{U^+}^+$ and $d_{U^+y_A,0,r}$. Improving Lemma [5.1] for all $\epsilon > 0$ and $k \in \mathbb{Z}$, we have

$$a^{-k}B_{\epsilon}^U \subset a^kB_{\epsilon q_v^k}.$$  

Again using the (locally compact) topological group identification $w \mapsto \hat{w}$ between $(K_v^m,+)$ and $U^+$, we endow $U^+$ with the Haar measure $m_{U^+}$ which corresponds to the normalized Haar measure $\text{vol}_v^m$ of $K_v^m$ (see Section 2.1). For every $j \in \mathbb{Z}$, the Jacobian $\text{Jac}_j$ with respect to the measure $m_{U^+}$ of the homeomorphism $\varphi_j : u \mapsto a^j u a^{-j}$ from $U^+$ to $U^+$ (which is constant since $\varphi_j$ is a group automorphism and $m_{U^+}$ is bi-invariant) is easy to compute: we have

$$\text{Jac}_j = q_v^j |r|.$$  

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Using Equation (57), we consider the following tail $\sigma$-algebra:

$$\mathcal{A}_\infty^+ = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \mathcal{A}_i^U = \bigcap_{k=1}^{\infty} (\mathcal{A}_k^U)^c = \bigcap_{k=1}^{\infty} (\mathcal{P}_k^U)^c. \tag{53}$$

This $\sigma$-algebra may not be countably generated, but it is strictly $\alpha$-invariant, i.e., we have $\alpha \mathcal{A}_\infty^+ = \mathcal{A} \mathcal{A}_\infty^+ = \alpha^{-1} \mathcal{A}_\infty^+$. We will use this $\sigma$-algebra in order to observe the relative entropy of $\alpha^{-1}$.

**Lemma 5.9** For every $r \in [0, 1[$, let $\mathcal{A}_\infty^+$ be as in Proposition 5.7 and $\mathcal{A}_\infty^+$ be as in Equation (53). Let $\mu$ be an $\alpha$-invariant ergodic probability measure on $\mathcal{Y}$. Then

$$h_\mu(\alpha^{-1} \mid \mathcal{A}_\infty^+) \leq |r|.$$

Furthermore, if $\mu(\mathcal{Y}(r)) > 0$, then

$$h_\mu(\alpha^{-1} \mid \mathcal{A}_\infty^+) = H_\mu(\mathcal{A}_\infty^+ \mid \alpha \mathcal{A}_\infty^+) .$$

**Proof.** Let us prove the first assertion. By [EL Prop. 7.44], there exists a countable Borel-measurable partition $\mathcal{G}$ with finite entropy which is a generator for the invertible transformation $\alpha$ modulo $\mu$, such that $\sigma(\mathcal{G})_0^\infty$ is $\alpha^{-1}$-descending and $G^+$-subordinate modulo $\mu$. It follows from [ELW Kolmogorov-Sinai Theo. 2.20 and Prop. 2.19 (8)] that

$$h_\mu(\alpha^{-1} \mid \mathcal{A}_\infty^+) = h_\mu(\alpha^{-1} \mid \mathcal{A}_\infty^+) = H_\mu(\sigma(\mathcal{G})_0^\infty \cap \mathcal{A}_\infty^+) .$$

Using the continuity and monotonicity of entropy [ELW Prop. 2.12 and Prop. 2.13 (2)], we have

$$H_\mu(\sigma(\mathcal{G})_0^\infty \cap \mathcal{A}_\infty^+) = \lim_{\ell \to \infty} H_\mu(\sigma(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty) \leq \lim_{\ell \to \infty} H_\mu(\sigma(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty) \leq H_\mu(\mathcal{A}_\infty^+ \mid \alpha(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty) .$$

Note that for every integer $\ell \geq 1$ the $\sigma$-algebra $\sigma(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty$ is countably generated, $\alpha^{-1}$-descending, and $U^+$-subordinate since $[y]_{(\mathcal{A}_\infty^+)_0^\infty} \subset U \cdot \mathcal{Y}$ for all $y \in \mathcal{Y}$ and since $\sigma(\mathcal{G})_0^\infty$ is $G^+$-subordinate. Thus by [EL Prop. 7.34] (recalling that we are using logarithms with base $q_v$), we have

$$H_\mu(\sigma(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty) \leq \lim_{\ell \to \infty} H_\mu(\sigma(\mathcal{G})_0^\infty \cap (\mathcal{A}_\infty^+)_0^\infty) = \log_q \mu_{\alpha^{-1}}(\mathcal{A}_\infty^+ \mid \mathcal{A} \mathcal{A}_\infty^+) \leq |r| .$$

where $\mu_{\alpha^{-1}}$ is the leafwise measure of $\mu$ at $x \in \mathcal{Y}$ with respect to $U^+$ as defined in [EL Theo. 6.3]. By [EL Theo. 6.30] (which applies since $U^+$ is abelian, hence unimodular) and by Equation (52) (see also [EL §7.42]), we have

$$\limsup_{k \to \infty} \frac{\mu_{\alpha^{-1}}(\mathcal{A}_\infty^+ \mid \mathcal{A} \mathcal{A}_\infty^+)}{k^2} = 0 .$$

Hence we have

$$\lim_{k \to \infty} \frac{\log_q \mu_{\alpha^{-1}}(\mathcal{A}_\infty^+ \mid \mathcal{A} \mathcal{A}_\infty^+)}{k} \leq |r| .$$
This proves the first assertion of the lemma.

In order to prove the second assertion, let us take an increasing sequence of finite partitions \((\mathcal{U}^+_k)_{k \geq 1}\) of \(\mathcal{Y}\) such that the union \(\bigcup_{k \geq 1} \mathcal{U}^+_k\) generates \(\mathcal{U}^+_1\), which is possible since the \(\sigma\)-algebra \(\mathcal{U}^+_1\) is countably generated. Since \(\mu\) is ergodic and \(\mu(\mathcal{Y}(r)) > 0\), for \(\mu\)-almost every \(y \in \mathcal{Y}\), there exists an increasing sequence of positive integers \((k_i)_{i \geq 1}\) such that \(a^{k_i}y \in \mathcal{Y}(r)\). By Proposition 5.7 [2], we have \(a^{k_i}y_{\mathcal{U}^+_1} \subset B_r^{U^+} a^{k_i}y\) for all \(i \geq 1\). Hence it follows from Equation (44) and Lemma 5.1 that

\[
[y]_{(\mathcal{U}^+_1)_{k_i}} = a^{-k_i} [a^{k_i}y]_{(\mathcal{U}^+_1)_{k_i}} \subset a^{-k_i} B_r^{U^+} a^{k_i}y \subset B_{\ln(1+(r^{-1}) q_{v^1}^{-k_i} \min r)}^{\mathcal{U}^+_1} y.
\]

Taking \(i \to \infty\), we have \([y]_{(\mathcal{U}^+_1)_{k_i}} = \{y\}\) for \(\mu\)-almost every \(y \in \mathcal{Y}\). This means that \((\mathcal{U}^+_1)_{k_i} = \mathcal{B}_{\mathcal{Y}}\) modulo \(\mu\), where \(\mathcal{B}_{\mathcal{Y}}\) is the Borel \(\sigma\)-algebra of \(\mathcal{Y}\). It follows that \(\bigvee_{k=1}^{\infty} \sigma((\mathcal{U}^+_k)_{k_i}) = (\mathcal{U}^+_1)_{k_i} = \mathcal{B}_{\mathcal{Y}}\) modulo \(\mu\), and \(\sigma((\mathcal{U}^+_k)_{k_i}) \subset \sigma((\mathcal{U}^+_k)_{k_i})_{k_i} = \mathcal{U}^+_1\) for each \(k \geq 1\). Using respectively the Kolmogorov-Sinai theorem for sequences [ELW Theo. 2.20], the Future formula [ELW Prop. 2.19 (8)] and the continuity of entropy [ELW Prop. 2.12], we have

\[
h_{\mu}(a^{-1} | \mathcal{U}^+_1) = \lim_{k \to \infty} h_{\mu}(a^{-1}, \mathcal{U}^+_k | \mathcal{U}^+_1) = h_{\mu}(\mathcal{U}^+_1 | \sigma((\mathcal{U}^+_1)_{k_i})_{k_i} = h_{\mu}(\mathcal{U}^+_1 | \sigma((\mathcal{U}^+_1)_{k_i} = h_{\mu}(\mathcal{U}^+_1 | a \mathcal{U}^+_1).
\]

This proves the second assertion of the lemma.\[\square\]

Let us introduce some more material before stating and proving our final Proposition 5.10 of Subsection 5.3. Let \(\mathcal{A}\) be a countably generated sub-\(\sigma\)-algebra of the Borel \(\sigma\)-algebra of \(\mathcal{Y}\). For all \(j \in \mathbb{Z}_{\geq 0}\) and \(y \in \mathcal{Y}\), let

\[
r_{y,j} = \inf \{ r > 0 : [y]_{a^j \mathcal{A}} \subset B_r^{U^+} y \} \in [0, +\infty],
\]

with the usual convention that \(\inf \emptyset = +\infty\). Note that \(r_{y,j}\) depends measurably on \(y\) and that \(r_{a^{-j}y,0} = r_{y,j} q_{v^{-j}}\) by Equation (51). We now define

\[
V_{y}^{a^j \mathcal{A}} = \{ u \in B_{r_{y,j}}^{U^+} : u y \in [y]_{a^j \mathcal{A}} \},
\]

which is a Borel subset of \(U^+\), called the \(U^+\)-shape of the atom \([y]_{a^j \mathcal{A}}\). Note that for every \(j \in \mathbb{Z}_{\geq 0}\), we have

\[
V_{y}^{a^j \mathcal{A}} = a^j V_{a^{-j}y}^{a^{-j} \mathcal{A}} a^{-j}.
\]

Let us define a Borel-measurable family \(\{ \tau_{y}^{a^j \mathcal{A}} \}_{y \in \mathcal{Y}}\) of Borel measures on \(\mathcal{Y}\), that we call the \(U^+\)-subordinate Haar measure of \(a^j \mathcal{A}\), as follows:

- if \(m_{U^+}(V_{y}^{a^j \mathcal{A}})\) is equal to 0 or \(\infty\), we set \(\tau_{y}^{a^j \mathcal{A}} = 0\),
- otherwise, \(\tau_{y}^{a^j \mathcal{A}}\) is the push-forward of the normalized measure \(\frac{1}{m_{U^+}(V_{y}^{a^j \mathcal{A}})} m_{U^+} | V_{y}^{a^j \mathcal{A}}\) by the map \(u \mapsto u y\).

Now let \(\mu\) be a Borel \(a\)-invariant probability measure on \(\mathcal{Y}\), such that \(\mathcal{A}\) is \(U^+\)-subordinate modulo \(\mu\). In particular, for \(\mu\)-almost every \(y \in \mathcal{Y}\), the atom \(V_{y}^{a^j \mathcal{A}}\) has positive and finite \(m_{U^+}\)-measure, hence the measure \(\tau_{y}^{a^j \mathcal{A}}\) is a probability measure with
support in $[y]_{\sigma^1\mathcal{A}}$. Furthermore, if $z \in [y]_{\sigma^1\mathcal{A}}$ then there exists $u \in U^+$ such that $z = uy$, $V_{\sigma^1\mathcal{A}} = V_{\sigma^1\mathcal{A}} u^{-1}$, and $\tau_{\sigma^1\mathcal{A}} = \tau_{\sigma^1\mathcal{A}}$, by the right-invariance of $m_{U^+}$.

The following proposition is a function field analog of the effective real case version [KKL, Prop. 2.10, §2.4] of [EL, §7.55].

**Proposition 5.10** Let $\mu$ be a Borel $\mathfrak{a}$-invariant ergodic probability measure on $\mathcal{Y}$ and let $\mathcal{A}$ be a countably generated sub-$\sigma$-algebra of the Borel $\sigma$-algebra of $\mathcal{Y}$ which is $\mathfrak{a}^{-1}$-descending and $U^+$-subordinate modulo $\mu$. Fix $j \in \mathbb{Z}_{\geq 1}$ and a $U^+$-saturated Borel subset $K'$ of $\mathcal{Y}$. Suppose that there exists $\epsilon > 0$ such that $[z]_{\mathcal{A}} \subset B^{U^+}_\epsilon y$ for every $z \in K'$. Then we have

$$H_\mu(\mathcal{A}|\mathcal{A}^1\mathcal{A}) \leq j |r| + \int_{\mathcal{Y}} \log \tau_{\sigma^1\mathcal{A}} ((\mathcal{Y} - K') \cup B^{U^+}_\epsilon \text{Supp } \mu) \, d\mu(y).$$

**Proof.** We fix $\mu$, $\mathcal{A}$, $j$, $K'$ and $\epsilon$ as in the statement. By for instance [EL] Theo. 5.9], let $(\mu_{y}^{\mathcal{A}})_{y \in \mathcal{Y}}$ be a measurable family of conditional measures of $\mu$ with respect to $\mathcal{A}^1\mathcal{A}$, so that for $\mu$-almost every $y \in \mathcal{Y}$, the measure $\mu_y^{\mathcal{A}}$ is a probability measure on $\mathcal{Y}$ giving full measure to the atom $[y]_{\mathcal{A}^1\mathcal{A}}$, with $\mu_{z}^{\mathcal{A}} = \mu_{y}^{\mathcal{A}}$ if $z \in [y]_{\mathcal{A}^1\mathcal{A}}$, and such that the following disintegration formula holds true:

$$\mu = \int_{y \in \mathcal{Y}} \mu_{y}^{\mathcal{A}} \, d\mu(y). \quad (55)$$

Let $p_y : y \mapsto \mu_{y}^{\mathcal{A}}([y]_{\mathcal{A}})$ and $p_{\tau} : y \mapsto \tau_{\sigma^1\mathcal{A}}([y]_{\mathcal{A}})$, which are nonnegative and measurable functions on $\mathcal{Y}$. Since $\mathcal{A}$ is $\mathfrak{a}^{-1}$-descending and $U^+$-subordinate modulo $\mu$, the atom $[y]_{\mathcal{A}}$ contains an open neighborhood of $y$ in the atom $[y]_{\mathcal{A}^1\mathcal{A}}$ for $\mu$-almost every $y \in \mathcal{Y}$. In particular, the function $p_{\tau}$ is $\mu$-almost everywhere positive.

Since $\mathcal{A}$ is countably generated and $\mathfrak{a}^{-1}$-descending, for every $y \in \mathcal{Y}$, the atom of $\mathfrak{a}$ for $\mathcal{A}^1\mathcal{A}$ is countably partitioned into atoms for $\mathcal{A}$ up to measure 0, that is, there exist a finite or countable subset $I_y$ of $[y]_{\mathcal{A}^1\mathcal{A}}$ and a $\mu_{y}^{\mathcal{A}^1\mathcal{A}}$-measure zero subset $N_y$ of $[y]_{\mathcal{A}^1\mathcal{A}}$ such that

$$[y]_{\mathcal{A}^1\mathcal{A}} = N_y \sqcup \bigcup_{x \in I_y} [x]_{\mathcal{A}}. \quad (56)$$

Let $I'_y = \{ x \in I_y : [x]_{\mathcal{A}} \cap \text{Supp } \mu \neq \emptyset \}$.

**Lemma 5.11** Let $x \in I_y$

1. If $x \notin I'_y$, then $\mu_{y}^{\mathcal{A}}([x]_{\mathcal{A}}) = 0$.
2. If $x \in I'_y$, then $[x]_{\mathcal{A}}$ is contained in $(\mathcal{Y} - K') \cup B^{U^+}_\epsilon \text{Supp } \mu$.

**Proof.** (1) This follows since $\text{Supp } \mu_{y}^{\mathcal{A}}$ is contained in $\text{Supp } \mu$.

(2) If $x \in I'_y$, there exists $z \in [x]_{\mathcal{A}} \cap \text{Supp } \mu$. For every $z' \in [x]_{\mathcal{A}}$, we have either $z' \in \mathcal{Y} - K'$ or $z' \in K'$. In the second case, since $\mathcal{A}$ is $U^+$-subordinate and $K'$ is $U^+$-saturated, we have $z \in [x]_{\mathcal{A}} = [z]_{\mathcal{A}} \subset U^+ z' \subset K'$. Hence by the assumption of Proposition 5.10 we have $z' \in [x]_{\mathcal{A}} = [z]_{\mathcal{A}} \subset B^{U^+}_\epsilon \subset B^{U^+}_\epsilon \text{Supp } \mu$, which proves the result. \qed
By the definition of the $U^+$-subordinate Haar measure of $\mathcal{A}^\mathcal{A}$, for $\mu$-almost every $y \in \mathcal{Y}$, we have

$$p_r(y) = \frac{m_{U^+}(V_y^\mathcal{A})}{m_{U^+}(V_y^\mathcal{A})} = \frac{m_{U^+}(V_y^\mathcal{A})}{\mu(U^+ \cap \mathcal{A}^{-j} Y)} = \frac{m_{U^+}(V_y^\mathcal{A})}{\mu(U^+ \cap \mathcal{A}^{-j} Y)}. $$

Hence, by the $\mathcal{A}$-invariance of $\mu$ and by Equation (52), we have

$$\int_{x \in \mathcal{X}} \log_{q_v} p_r(z) \, d\mu(z) = -\log_{q_v} \text{Jac}_{\mu} = -j |r|. $$

Respectively
- by the definition of the conditional entropy in Equation (41),
- by the disintegration formula (55),
- since $\mu_y^{\mathcal{A}^\mathcal{A}}$ gives full measure to $[y]^{\mathcal{A}^\mathcal{A}}$ which is partitioned as in Equation (56), and by Lemma 5.11 (1),
- since when $z$ varies in $[x]^{\mathcal{A}^\mathcal{A}} \subset [y]^{\mathcal{A}^\mathcal{A}}$, the values $p_\mu(z) = \mu_z^{\mathcal{A}^\mathcal{A}}([z]^{\mathcal{A}^\mathcal{A}}) = \mu_y^{\mathcal{A}^\mathcal{A}}([x]^{\mathcal{A}^\mathcal{A}})$ and $p_r(z) = \tau_y^{\mathcal{A}^\mathcal{A}}([z]^{\mathcal{A}^\mathcal{A}}) = \tau_y^{\mathcal{A}^\mathcal{A}}([x]^{\mathcal{A}^\mathcal{A}})$ are constant,
- by the concavity property of the logarithm,
- by Lemma 5.11 (2),

we hence have

$$H_\mu(\mathcal{A}^\mathcal{A} | \mathcal{A}^{\mathcal{A}}) - j |r|$$

$$= -\int_{x \in \mathcal{X}} \log_{q_v} p_\mu(z) - \log_{q_v} p_r(z) \, d\mu(z)$$

$$= \int_{y \in \mathcal{Y}} \int_{x \in \mathcal{X}} \int_{x \in \mathcal{X}} \log_{q_v} p_\mu(z) - \log_{q_v} p_r(z) \, d\mu_y^{\mathcal{A}^\mathcal{A}}(z) \, d\mu(y)$$

$$= \int_{y \in \mathcal{Y}} \sum_{x \in I_y^\mathcal{X}} \log_{q_v} p_\mu(z) - \log_{q_v} p_r(z) \, d\mu_y^{\mathcal{A}^\mathcal{A}}(z) \, d\mu(y)$$

$$= \int_{y \in \mathcal{Y}} \sum_{x \in I_y^\mathcal{X}} \log_{q_v} \frac{\tau_y^{\mathcal{A}^\mathcal{A}}([x]^{\mathcal{A}^\mathcal{A}})}{\mu_y^{\mathcal{A}^\mathcal{A}}([x]^{\mathcal{A}^\mathcal{A}})} \, d\mu(y)$$

$$\leq \int_{y \in \mathcal{Y}} \log_{q_v} \left( \sum_{x \in I_y^\mathcal{X}} \tau_y^{\mathcal{A}^\mathcal{A}}([x]^{\mathcal{A}^\mathcal{A}}) \right) \, d\mu(y)$$

$$\leq \int_{y \in \mathcal{Y}} \log_{q_v} \left( \tau_y^{\mathcal{A}^\mathcal{A}}(\mathcal{Y} - K^r ) \cup B_{\epsilon}^{U^+} \right) \, d\mu(y) \, d\mu(y).$$

This proves the result. \hfill \Box

6 Upper bound on the Hausdorff dimension of Bad$_A(\epsilon)$

6.1 Constructing measures with large entropy

In this subsection, we construct, as in [KKL Prop. 4.1] in the real case, an $\mathcal{A}$-invariant probability measure on $\mathcal{Y}$ giving an appropriate lower bound on the conditional entropy of $\mathcal{A}$ relative to the $\sigma$-algebra $\mathcal{A}^{\mathcal{A}}$ defined in Equation (53) using the $\sigma$-algebra $\mathcal{A}^{\mathcal{A}}$ constructed in Proposition 5.7 (see Equation (41)).
For any point \( x \) in a measurable space, we denote by \( \Delta_x \) the weak-star convergence of Borel measures on any locally compact space.

Let us denote by \( \overline{\mathcal{X}} = \mathcal{X} \cup \{\infty, \mathcal{Y}\} \) and \( \overline{\mathcal{Y}} = \mathcal{Y} \cup \{\infty, \mathcal{Y}\} \) the one-point compactifications of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. We denote by \( \pi : \overline{\mathcal{Y}} \to \overline{\mathcal{X}} \) the unique continuous extension of the natural projection \( \pi : \mathcal{Y} \to \mathcal{X} \), mapping \( \infty, \mathcal{Y} \) to \( \infty, \mathcal{X} \). The left actions of \( \alpha \) on \( \mathcal{X} \) and \( \mathcal{Y} \) continuously extend to actions on \( \overline{\mathcal{X}} \) and \( \overline{\mathcal{Y}} \) fixing the points at infinity \( \infty, \mathcal{X} \) and \( \infty, \mathcal{Y} \). For every countably generated \( \sigma \)-algebra \( \mathcal{A} \) of subsets of \( \mathcal{X} \) or \( \mathcal{Y} \), we denote by \( \overline{\mathcal{A}} \) the countably generated \( \sigma \)-algebra of subsets of \( \overline{\mathcal{X}} \) or \( \overline{\mathcal{Y}} \) generated by \( \mathcal{A} \) and its point at infinity. For a finite partition \( \mathcal{P} = \{Q_1, \ldots, Q_N, \mathcal{Q}_x\} \) of \( \mathcal{Y} \) with only one unbounded atom \( Q_x \), we denote by \( \mathcal{F} \) the fine partition \( \{Q_1, \ldots, Q_N, \mathcal{Q}_x\} \) of \( \overline{\mathcal{Y}} \). Note that \( \bigvee_{i=a}^{b} a^{-i} \mathcal{P} = \bigvee_{i=a}^{b} a^{-i} \mathcal{F} \) for all \( a, b \in \mathbb{Z} \) with \( a < b \).

For every \( \eta \in [0, 1] \), we say that an element \( x \in \mathcal{X} \) has \( \eta \)-escape of mass on average under the action of \( \alpha \) if for every compact subset \( Q \) of \( \mathcal{X} \),

\[
\liminf_{N \to \infty} \frac{1}{N} \text{Card} \{ \ell \in \{1, \cdots, N\} : \alpha^{\ell} x \notin Q \} \geq \eta.
\]

When \( \eta = 1 \), as defined in the Introduction and in Proposition 5.2, we say that \( x \) diverges on average in \( \mathcal{X} \) under the action of \( \alpha \). For every \( A \in \mathcal{M}_{m,n}(K_v) \), we denote by \( x_A = u_A R_v^m \in \mathcal{X}^* \) its associated unimodular lattice (see Section 5.2), and by \( \eta_A \in [0, 1] \) the upper bound of the elements \( \eta \in [0, 1] \) such that \( x_A \) has \( \eta \)-escape of mass on average. Note that this upper bound is actually a maximum.

**Proposition 6.1** For every \( A \in \mathcal{M}_{m,n}(K_v) \), there exists a Borel probability measure \( \mu_A \) on \( \overline{\mathcal{X}} \) with \( \mu_A(\mathcal{X}) = 1 - \eta_A \) such that for every \( \epsilon > 0 \), there exists an \( \alpha \)-invariant Borel probability measure \( \overline{\mu} \) on \( \overline{\mathcal{Y}} \) satisfying the following properties.

1. The support of \( \overline{\mu} \) is contained in \( \mathcal{L}_\epsilon \cup \{\infty, \mathcal{Y}\} \), where the \( \epsilon \)-compact part \( \mathcal{L}_\epsilon \) is defined in Equation (53).
2. We have \( \overline{\pi} \overline{\mu} = \mu_A \). In particular, there exists an \( \alpha \)-invariant Borel probability measure \( \mu \) on \( \mathcal{Y} \) such that

\[
\overline{\mu} = (1 - \eta_A) \mu + \eta_A \Delta_{\infty, \mathcal{Y}}.
\]
3. For every \( r \in \{0, 1\} \), let \( \mathcal{A}^{U^+} \) be the \( \sigma \)-algebra of subsets of \( \mathcal{Y} \) constructed in Proposition 5.7 and let \( \mathcal{A}_x^{U^+} \) be as in Equation (58). Then

\[
\h_{\overline{\pi}}(a^{-1} | \mathcal{A}_x^{U^+}) = \h_{\pi}(a \mid \mathcal{A}_x^{U^+}) \geq |r|(1 - \eta_A) - \max(m - \dim_{\text{Haus}} \text{Bad}_A(e)).
\]

**Proof.** Since \( x_A \) has \( \eta_A \)-escape of mass on average but does not have \( (\eta_A + \delta) \)-escape of mass on average for any \( \delta > 0 \), there exists an increasing sequence of positive integers \( (k_i)_{i \in \mathbb{Z}_+} \) such that, for the weak-star convergence of Borel probability measures on the compact space \( \overline{\mathcal{X}} \), as \( i \to +\infty \), we have

\[
\frac{1}{k_i} \sum_{k=0}^{k_i-1} \Delta_{\alpha^{k} x_A} \overset{*}{\rightarrow} \mu_A,
\]

and \( \mu_A \) is a Borel probability measure on \( \overline{\mathcal{X}} \) with \( \mu_A(\mathcal{X}) = 1 - \eta_A \). This is equivalent to \( \mu_A(\{\infty, \mathcal{X}\}) = \eta_A \).
Let $\epsilon > 0$. For every $T \in \mathbb{Z}_{\geq 0}$, with the notation of Subsection 5.2 (see in particular Equations (58) and (59)), let

$$R_T = \{ \theta \in \mathbb{T}^m : \forall k \geq T, a^k \phi_A(\theta) \in \mathcal{L}_\epsilon \} \cap \text{Bad}_A(\epsilon) .$$

By Proposition 5.4 since a countable subset of $K_v^m$ has Hausdorff dimension 0, we have

$$\dim_{\text{Haus}} \left( \bigcup_{T=1}^{\infty} R_T \right) = \dim_{\text{Haus}} \text{Bad}_A(\epsilon) .$$

Thus, for every $j \in \mathbb{Z}_{\geq 1}$, there exists $T_j \in \mathbb{Z}_{\geq 0}$ satisfying

$$\dim_{\text{Haus}} R_{T_j} \geq \dim_{\text{Haus}} \text{Bad}_A(\epsilon) - \frac{1}{j} .$$

For all $i, j \in \mathbb{Z}_{\geq 1}$ such that $k_i \geq T_j$, let $S_{i,j}$ be a maximal $q_v^{-k_i}$-separated subset of $R_{T_j}$ for the distance $d_{\mathbb{T}^m}$, defined after Equation (50). Then $R_{T_j}$ can be covered by open balls of radius $q_v^{-k_i}$ for $d_{\mathbb{T}^m}$, and by $\prod_{j=1}^{m} q_v^{-k_j} / q_v^{-k_i} \max r = q_v^{k_i(m \max r - |r|)}$ open balls of radius $q_v^{-k_i} \max r$ with respect to the standard distance $d_{\mathbb{T}^m}$ (defining the Hausdorff dimension of subsets of $\mathbb{T}^m$). Since the lower Minkowski dimension is at least equal to the Hausdorff dimension, we have

$$\liminf_{i \to \infty} \frac{\log q_v \left( q_v^{k_i(m \max r - |r|)} \text{Card } S_{i,j} \right)}{- \log q_v \left( q_v^{-k_i} \max r \right)} \geq \dim_{\text{Haus}} R_{T_j} \geq \dim_{\text{Haus}} \text{Bad}_A(\epsilon) - \frac{1}{j} ,$$

which implies that

$$\liminf_{i \to \infty} \frac{\log q_v \text{ Card } S_{i,j}}{k_i} \geq |r| - \max r \left( m + \frac{1}{j} - \dim_{\text{Haus}} \text{Bad}_A(\epsilon) \right) . \quad (58)$$

Let us define the Borel probability measures

$$\nu_{i,j} = \frac{1}{\text{Card } S_{i,j}} \sum_{\theta \in S_{i,j}} \Delta_{\phi_A(\theta)} ,$$

which is the normalized counting measure on the finite subset $\phi_A(S_{i,j})$ of the $U^+$-orbit $\phi_A(\mathbb{T}^m) = U^+ y_{A,0} \subset \pi^{-1}(x_A)$, and

$$\tilde{\nu}_{i,j} = \frac{1}{k_i} \sum_{0 \leq k \leq k_i - 1} a^k \nu_{i,j} ,$$

which is the average of the previous one on the first $k_i$ points of the $a$-orbit. Since $\mathcal{Y}$ is compact, extracting diagonally a subsequence if necessary, we may assume that $\tilde{\nu}_{i,j}$ weak-star converges as $i \to +\infty$ towards an $a$-invariant Borel probability measure $\tilde{\mu}_j$, and that $\tilde{\mu}_j$ weak-star converges as $j \to +\infty$ towards an $a$-invariant Borel probability measure $\tilde{\mu}$. Let us prove that $\tilde{\mu}$ satisfies the three assertions of Proposition 6.1.

For all $k \geq T_j$ and $\theta \in S_{i,j} \subset R_{T_j}$, we have $a^k \phi_A(\theta) \in \mathcal{L}_\epsilon$ by the definition of $R_{T_j}$. Since $a^k \nu_{i,j}$ is a probability measure, we hence have

$$\tilde{\nu}_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{k_i-1} a^k \nu_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{T_j} a^k \nu_{i,j}(\mathcal{Y} - \mathcal{L}_\epsilon) \leq \frac{T_j}{k_i} .$$

Since $\mathcal{L}_\epsilon \cup \{ x_{\mathcal{Y}} \}$ is closed in $\mathcal{Y}$ and by taking limits first as $i \to +\infty$ then as $j \to +\infty$, we therefore have $\tilde{\mu}(\mathcal{Y} - \mathcal{L}_\epsilon) = 0$. This proves Assertion 1.

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(2) Since $\phi_A(S_{i,j})$ is contained in the fiber above $x_A$ of $\pi$ and since $\nu_{i,j}$ is a probability measure, we have $\pi_*\nu_{i,j} = \Delta x_A$. By the linearity and equivariance of $\pi_*$, we hence have

$$\pi_*\nu_{i,j} = \frac{1}{k_i} \sum_{0 \leq k \leq k_i-1} \Delta_{\alpha^k x_A} \cdot$$

By the weak-star continuity of $\pi_*$ and Equation (57), we thus have

$$\pi_*\mu = \lim_{j \to +\infty} \lim_{i \to +\infty} \pi_*\nu_{i,j} = \lim_{j \to +\infty} \mu_A = \mu_A .$$

Note that the point at infinity $\infty_{\mathcal{F}}$ is an isolated point in the support of $\mu$ by Assertion (1), since $\mathcal{L}_t$ is compact. We hence have

$$\mu(\{\infty_{\mathcal{F}}\}) = \mu(\pi^{-1}(\{\infty_{\mathcal{F}}\})) = \mu_A(\{\infty_{\mathcal{F}}\}) = \eta_A . \tag{59}$$

This proves Assertion (2).

(3) Suppose that $\mathcal{Q}$ is any finite Borel-measurable partition of $\mathcal{F}$ satisfying

(i) the partition $\mathcal{Q}$ contains an atom $Q_\infty$ of the form $\pi^{-1}(Q^\ast)$, where $Q^\ast - Q_\infty$ has compact closure,

(ii) there exists $t_0 \in \mathbb{Z}_{\geq 1}$ such that for every atom $Q \in \mathcal{Q}$ different from $Q_\infty$ and for every $y' \in Q$, we have $\dim(U+y' \cap Q) < q_v^{-t_0 \max \{r\} \text{ defined in Lemma 5.3}}$.

(iii) for all $Q \in \mathcal{Q}$ and $j \in \mathbb{Z}_{\geq 1}$, we have $\tilde{\mu}_j(\partial Q) = 0$ and $\mu(\partial Q) = 0$.

We first prove the following entropy bound: For every $M \in \mathbb{Z}_{\geq 1}$,

$$\frac{1}{M} H_\pi \left( \sigma(\mathcal{Q}^{(M)}) \mid \mathcal{Q}^{(M)} \right) \geq |r|(1 - \mu(\mathcal{Q}^{\infty})) - \max \{m - \dim \text{Haus Bad}_A(\epsilon)\} , \tag{60}$$

where $\mathcal{Q}^{(M)} = \sqrt{\prod_{k=0}^{M-1} \alpha^{-k} \mathcal{Q}}$. Since Equation (60) is clear if $\mu(\mathcal{Q}^{\infty}) = 1$, we may assume that $\mu(\mathcal{Q}^{\infty}) < 1$, hence that $\tilde{\mu}_j(\mathcal{Q}^{\infty}) < 1$ for all large enough $j \geq 1$. Now, we fix such a $j \geq 1$.

Take $\rho > 0$ small enough so that $\tilde{\mu}_j(\mathcal{Q}^{\infty}) + \rho < 1$ and let

$$\beta = \tilde{\mu}_j(\mathcal{Q}^{\infty}) + \rho . \tag{61}$$

Then for all large enough $i \in \mathbb{Z}_{\geq 1}$, since $\phi_A(S_{i,j}) \subset \pi^{-1}(x_A)$ and $Q_\infty = \pi^{-1}(Q^\ast)$ by Property (i) of $\mathcal{Q}$, we have

$$\beta = \tilde{\mu}_j(\mathcal{Q}^{\infty}) + \rho > \tilde{\nu}_{i,j}(Q_\infty) = \frac{1}{k_i} \frac{\Delta_{\alpha^k \phi_A(\theta)}(Q_\infty)}{\text{Card } S_{i,j}} \geq \frac{1}{k_i} \sum_{k=0}^{k_i-1} \sum_{\theta \in S_{i,j}} \Delta_{\alpha^k x_A}(Q_\infty)$$

$$= \frac{1}{k_i} \sum_{k=0}^{k_i-1} \Delta_{\alpha^k x_A}(Q^\ast) .$$

Thus, for every $\theta \in \mathbb{R}^m$, since $\alpha^k \phi_A(\theta) \in Q_\infty$ implies that $\alpha^k x_A \in Q^\ast$ by Property (i) of $\mathcal{Q}$, we have

$$\text{Card}\{k \in \{0, \ldots, k_i - 1\} : \alpha^k \phi_A(\theta) \in Q_\infty\} < \beta k_i . \tag{62}$$

Let us prove the following counting lemma inspired by [ELMW, Lem. 4.5] and [LSS, Lem. 2.4], where $\ell_0$ is given by Property (ii) of $\mathcal{Q}$.
Lemma 6.2 There exists a constant $C > 0$ depending only on $r$ and $\ell_0$ such that for all $A \in \mathcal{M}_{m,n}(K_v)$, $\theta \in \mathbb{T}^m$ and $T \in \mathbb{Z}_{\geq 0}$, defining $y = \phi_A(\theta)$, $I = \{k \in \mathbb{Z}_{\geq 0} : a^k y \in Q_\infty\}$, and

$$E_{y,T} = \{z \in U^y : \forall k \in \{0, \ldots, T\} - I, \ d_{U^y + a^k y, \| \cdot \|_1}(a^k y, a^k z) < q_v^{-\ell_0} \} ,$$

the set $E_{y,T}$ can be covered by $C q_v^{\lceil \text{Card}(I \cap \{0, \ldots, T\}) \rceil}$ closed balls of radius $q_v^{-(\ell_0 + T)}$ for the distance $d_{U^y + y, r}$.

Proof. As in the proof of [LSS] Lemma 2.4, we proceed by induction on $T$.

By the compactness of $\mathbb{T}^m$, there exists a constant $C \in \mathbb{Z}_{\geq 1}$ depending only on $r$ and $\ell_0$ such that the metric space $(\mathbb{T}^m, d_{\mathbb{T}^m})$ can be covered by $C$ closed balls of radius $q_v^{-\ell_0}$. Since $\phi_A : \mathbb{T}^m \to U^y$ is an isometry for the distances $d_{\mathbb{T}^m}$ and $d_{U^y + y, r}$, the orbit $U^y$ can be covered by $C$ closed balls for $d_{U^y + y, r}$ of radius $q_v^{-\ell_0}$. Thus the lemma holds for $T = 0$.

Let $N_T = C q_v^{\lceil \text{Card}(I \cap \{0, \ldots, T\}) \rceil}$.

Assume by induction that $E_{y,T-1}$ can be covered by $N_{T-1}$ balls for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T-1)}$. Note that for every $k \in \mathbb{Z}$, since $\pi_v^k \mathcal{O}_v/(\pi_v^{k+1} \mathcal{O}_v)$ has order $q_v$, every closed ball in $K_v$ of radius $q_v^{-k}$ is the disjoint union of $q_v$ closed balls of radius $q_v^{-k-1}$. Hence every closed ball for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T-1)}$ in $U^y$ can be covered by $q_v^{\lceil r \rceil}$ closed balls for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T)}$. Therefore, if $T \in I$, then $E_{y,T} = E_{y,T-1}$ can be covered by $N_T = q_v^{\lceil r \rceil} N_{T-1}$ closed balls for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T)}$.

Suppose conversely that $T \notin I$, so that in particular $N_T = N_{T-1}$. Denote the above covering of $E_{y,T-1}$ by $\{ B_i : i = 1, \ldots, N_{T-1} \}$. Since we have $E_{y,T} \subset E_{y,T-1}$, the set $\{E_{y,T} \cap B_i : i = 1, \ldots, N_{T-1} \}$ is a covering of $E_{y,T}$.

Claim. For all $i = 1, \ldots, N_{T-1}$ and $z_1, z_2 \in E_{y,T} \cap B_i$, we have $d_{U^y + y, r}(z_1, z_2) \leq q_v^{-(\ell_0 + T)}$.

Proof. Since $T \notin I$, we have $d_{U^y + a^T y, \| \cdot \|_1}(a^T y, a^T z_j) < q_v^{-\ell_0 r}$ for each $j = 1, 2$. Thus we have $d_{U^y + a^T y, \| \cdot \|_1}(a^T z_1, a^T z_2) < q_v^{-\ell_0 r}$ by the ultrametric inequality property of $\| \cdot \|$. Note that since $z_1, z_2 \in U^y = U^y A \theta_2$, there exist $\theta_1 = (\theta_1, \ldots, \theta_1, m)$ and $\theta_2 = (\theta_2, \ldots, \theta_2, m)$ in $\mathbb{T}^m$ such that (denoting in the same way lifts of $\theta_1$ and $\theta_2$ to $K_v^m$) we have $z_1 = y_A \theta_1$ and $z_2 = y_A \theta_2$. With $\langle \cdot, \cdot \rangle$ the map defined after Equation (3) and by Lemma 5.3, it follows that we have

$$\max_{1 \leq i \leq m} q_v^{T} \langle \theta_1, \theta_2 \rangle = d_{\mathbb{T}^m}(a^T \theta_1, a^T \theta_2) = d_{U^y + a^T y, \| \cdot \|_1}(a^T y_A \theta_1, a^T y_A \theta_2)$$

$$= d_{U^y + a^T y, \| \cdot \|_1}(a^T z_1, a^T z_2) < q_v^{-\ell_0 r} .$$

Hence, we have

$$d_{U^y + y, r}(z_1, z_2) = d_{\mathbb{T}^m, r}(\theta_1, \theta_2) = \max_{1 \leq i \leq m} |\langle \theta_1, \theta_2 \rangle |^{\frac{1}{\ell_0}} < q_v^{-(\ell_0 + T)} ,$$

which concludes the claim. \qed

By the above claim, the intersection $E_{y,T} \cap B_i$ is contained in a single ball for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T)}$ for each $i = 1, \ldots, N_{T-1}$. Thus $E_{y,T}$ can be covered by $N_T = N_{T-1}$ balls for $d_{U^y + y, r}$ of radius $q_v^{-(\ell_0 + T)}$. \qed

Recall that as constructed in the proof of Proposition 5.7, there exist a Borel-measurable partition $\mathcal{P} = \{ P_1, \ldots, P_N, P_\infty \}$ of $\mathcal{Y}$ with $N + 1$ elements, and a countably generated
Borel-measurable \( \mathcal{P} U^+ \) of subsets of \( \mathcal{Y} \), with \( [y]_{\mathcal{P} U^+} = [y]_{\mathcal{P}} \cap B^+_{\mathbb{R}} y \) for every \( y \in \mathcal{Y}(r) \) by Equation \( (13) \), such that we have \( \mathcal{P} U^+ = (\mathcal{P} U^+)_{\ell}^\mathcal{Y} \) as defined in Equation \( (47) \). We now consider the sequence of \( \sigma \)-algebras \( ((\mathcal{P} U^+)_{\ell}^\mathcal{Y})_{\ell\geq 1} \), which is a decreasing sequence of \( \sigma \)-algebras whose intersection is \( \mathcal{F} U^+ \) by Equation \( (53) \). Note that for every \( \ell \in \mathbb{Z}_{\geq 1} \), the \( \sigma \)-algebra \( (\mathcal{P} U^+)_{\ell}^\mathcal{Y} \) is countably generated.

Let \( Q \) be an atom of the finite partition \( \mathcal{Q}^{(k_i)} = \bigvee_{k_i=0}^{k_i-1} a^{-k} \mathcal{Q} \) of \( \mathcal{Y} \). First assume that \( \phi_A(S_{i,j}) \cap Q \) is nonempty. Fix any \( y \in \phi_A(S_{i,j}) \cap Q \) and for every \( k = 0, \ldots, k_i - 1 \), let \( Q_k \in \mathcal{Q} \) be such that \( Q = \bigcap_{k=0}^{k_i-1} a^{-k}Q_k \). For all \( k \in \{0, \ldots, k_i - 1\} - \{k \in \mathbb{Z}_{\geq 0} : a^k y \in Q_{\infty}\} \) and \( z \in \phi_A(S_{i,j}) \cap Q \), we have \( z \in U^+ \) and \( a^k y, a^k z \in a^k(U^+ y \cap a^{-k}Q_k) = U^+ a^k y \cap Q_k \) since \( a \) normalizes \( U^+ \). By Property (ii) of \( \mathcal{Q} \), we hence have

\[
d_{U^+ a^k y, |} (a^k y, a^k z) \leq \text{diam}_{U^+ a^k y, |} Q_k < q_{U}\ell_0 \max r .
\]

Therefore the intersection \( \phi_A(S_{i,j}) \cap Q \) is contained in the set \( E_{y,k_i - 1} \) defined in Lemma \( (62) \). It follows from Lemma \( (62) \) and Equation \( (62) \) that \( \phi_A(S_{i,j}) \cap Q \) can be covered by \( C q_{U}^{\deg \beta_{k_i}} \) closed balls for \( d_{U^+ y, r} \) of radius \( q_{U}^{\ell_0 k_i - 1} = q_{U}^{-\ell_0 + 1} q_{U}^{-k_i} \), where \( C \) depends only on \( r \) and \( \ell_0 \). Since \( S_{i,j} \) is \( q_{U}^{-k_i} \)-separated (hence \( q_{U}^{-\ell_0 + 1} q_{U}^{-k_i} \)-separated since \( \ell_0 \geq 1 \)) with respect to \( d_{T^m, r} \), and since \( \phi_A : (T^n, d_{T^m, r}) \to (U^+ y, d_{U^+ y, r}) \) is an isometry, we have

\[
\text{Card}(\phi_A(S_{i,j}) \cap Q) \leq C q_{U}^{\deg \beta_{k_i}} .
\]

This formula is also satisfied if \( \phi_A(S_{i,j}) \cap Q = \emptyset \).

**Claim.** For every large enough \( \ell \in \mathbb{Z}_{\geq 1} \), we have \( H_{\nu_{ij}}(\sigma(\mathcal{Q}^{(k_i)})) \leq H_{\nu_{ij}}(\mathcal{Q}^{(k_i)}) \).

**Proof.** Observe that using Proposition \( (5.7) \) (3), for every \( \ell \in \mathbb{Z}_{\geq 1} \), we have

\[
[y, A_0]_{(\mathcal{P} U^+)_{\ell}^\mathcal{Y}} = a^\ell [a^{-\ell} y A_0]_{(\mathcal{P} U^+)_{\ell}^\mathcal{Y}} = a^\ell [a^{-\ell} y A_0] \cup a^\ell B_{S_{ij}} a^{-\ell} y A_0 .
\]

As in the proof of Lemma \( (5.11) \) we have

\[
a^\ell B_{S_{ij}} a^{-\ell} \supset B_{U^+}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0} \leq B_{U^+}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0} \subseteq B_{U^+}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0} \subseteq B_{U^+}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0} \subseteq B_{\infty}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0}
\]

which implies that

\[
[y, A_0]_{(\mathcal{P} U^+)_{\ell}^\mathcal{Y}} \supset B_{U^+}^{\nu_{ij}}(a^{-\ell} y A_0)_{\ell_0} .
\]

It follows that \( [y, A_0]_{(\mathcal{P} U^+)_{\ell}^\mathcal{Y}} = U^+ y A_0 \) for all large enough \( \ell \in \mathbb{Z}_{\geq 1} \) since the orbit \( U^+ y A_0 \) is compact. Hence we have \( H_{\nu_{ij}}(\sigma(\mathcal{Q}^{(k_i)})) \leq H_{\nu_{ij}}(\mathcal{Q}^{(k_i)}) \) for all large enough \( \ell \in \mathbb{Z}_{\geq 1} \) which supports on a single atom of \( (\mathcal{P} U^+)_{\ell}^\mathcal{Y} \). This proves the claim. \( \square \)

Since the map \( \Psi = -\log q_{U} \) is nonincreasing, for every large enough \( \ell \in \mathbb{Z}_{\geq 1} \), it hence follows that

\[
\begin{aligned}
H_{\nu_{ij}}(\sigma(\mathcal{Q}^{(k_i)})) &\leq H_{\nu_{ij}}(\mathcal{Q}^{(k_i)}) = \sum_{Q \in \mathcal{Q}^{(k_i)}} \nu_{ij}(Q) \Psi(\nu_{ij}(Q)) \\
&= \sum_{Q \in \mathcal{Q}^{(k_i)}} \nu_{ij}(Q) \frac{\text{Card}(\phi_A(S_{i,j}) \cap Q)}{\text{Card} S_{i,j}} \\
&\geq \Psi \left( \frac{C q_{U}^{\deg \beta_{k_i}}}{\text{Card} S_{i,j}} \right) \sum_{Q \in \mathcal{Q}^{(k_i)}} \nu_{ij}(Q) = \log q_{U} \left( \text{Card} S_{i,j} \right) - |r| \beta k_i - \log q_{U} C .
\end{aligned}
\]
By taking $\ell \to +\infty$, it follows from the continuity of entropy that
\[
H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_{i})})|\mathcal{A}_{X}^{U^{+}}) \geq \log_{q_e}(\text{Card } S_{i,j}) - |r|\beta k_{i} - \log_{q_e} C . \tag{63}
\]

Since $\mathcal{A}_{X}^{U^{+}}$ is strictly $\alpha$-invariant, by the subadditivity and concavity properties of the entropy as in the proof of [LSS, Eq. (2.9)], for every $M \in \mathbb{Z}_{\geq 1}$, we have
\[
\frac{1}{M}H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_{X}^{U^{+}}) \geq \frac{1}{k_{i}}H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(k_{i})})|\mathcal{A}_{X}^{U^{+}}) - \frac{2M \log_{q_e}(\text{Card } \mathcal{Q})}{k_{i}} . \tag{64}
\]
Therefore, since $\nu_{i,j}(\mathcal{X}_{\mathcal{Y}}) = 0$, it follows from Equations (64) and (63) that
\[
\frac{1}{M}H_{\nu_{i,j}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_{X}^{U^{+}}) \geq \frac{1}{k_{i}}(\log_{q_e}(\text{Card } S_{i,j}) - |r|\beta k_{i} - \log_{q_e} C - 2M \log_{q_e}(\text{Card } \mathcal{Q})) .
\]
Now we can take $i \to \infty$ since the atoms $Q$ of the partition $\mathcal{Q}$ and hence of the partition $\mathcal{Q}^{(M)}$, satisfy $\bar{\mu}_{j}(\partial Q) = 0$ by the property (iii) of $\mathcal{Q}$. Also, the constants $C$ and Card $\mathcal{Q}$ are independent of $k_{i}$. Thus it follows from Equation (58) that
\[
\frac{1}{M}H_{\bar{\nu}_{j}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_{X}^{U^{+}}) \geq |r|(1 - \beta) - \max r (m + \frac{1}{j} - \dim_{\text{Haus}} \text{Bad}_{A}(\epsilon)) .
\]
By taking $\rho \to 0$ in Equation (61), we have
\[
\frac{1}{M}H_{\bar{\nu}_{j}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_{X}^{U^{+}}) \geq |r|(1 - \bar{\mu}_{j}(\mathcal{Q}_{\mathcal{X}})) - \max r (m + \frac{1}{j} - \dim_{\text{Haus}} \text{Bad}_{A}(\epsilon)) .
\]
Hence, it follows by taking $j \to \infty$ and by using the property (iii) of $\mathcal{Q}$ that
\[
\frac{1}{M}H_{\bar{\nu}}(\sigma(\mathcal{Q}^{(M)})|\mathcal{A}_{X}^{U^{+}}) \geq |r|(1 - \bar{\nu}(\mathcal{Q}_{\mathcal{X}})) - \max r (m - \dim_{\text{Haus}} \text{Bad}_{A}(\epsilon)) ,
\]
which proves Equation (60).

Hence, by taking $M \to \infty$, we have
\[
h_{\bar{\nu}}(a^{-1}|\mathcal{A}_{X}^{U^{+}}) = h_{\bar{\nu}}(a|\mathcal{A}_{X}^{U^{+}}) \geq |r|(1 - \bar{\nu}(\mathcal{Q}_{\mathcal{X}})) - \max r (m - \dim_{\text{Haus}} \text{Bad}_{A}(\epsilon)) ,
\]
provided that we have a partition $\mathcal{Q}$ satisfying the above requirements (i), (ii) and (iii). After taking a sufficiently small neighborhood of infinity $Q^{\mathcal{Y}}_{\mathcal{X}}$ in $\mathcal{X}$, so that if $Q_{\mathcal{X}} = \pi^{-1}(Q^{\mathcal{Y}}_{\mathcal{X}})$, then $\bar{\nu}(\mathcal{X}_{\mathcal{Y}})$ is sufficiently close to $\bar{\nu}(\mathcal{X}_{\mathcal{Y}}) = \eta_{A}$, we can indeed construct a finite Borel-measurable partition $\mathcal{Q}$ of $\mathcal{Y}$ satisfying Properties (i), (ii) and (iii), by following the procedure in [LSS, Proof of Theorem 4.2, Claim 2]. This proves Assertion (3). \hfill \Box

### 6.2 Effective upper bound on $\dim_{\text{Haus}} \text{Bad}_{A}(\epsilon)$

For every $\ell \in \mathbb{Z}_{\leq 1}$, with $\lambda_{1}$ the shortest length function of a nonzero vector of an $R_{e}$-lattice (see Subsection 2.2), we define
\[
\mathcal{X}^{\geq q_{e}^{\ell}} = \{x \in \mathcal{X}: \lambda_{1}(x) \geq q_{e}^{\ell}\} \quad \text{and} \quad \mathcal{Y}^{\geq q_{e}^{\ell}} = \pi^{-1}(\mathcal{X}^{\geq q_{e}^{\ell}}) .
\]
Note that by Corollary 2.2, we have $\lambda_{1}(x) \leq q_{e}$ for all $x \in \mathcal{X}$, thus $\mathcal{X} = \bigcup_{\ell = -\infty}^{1} \mathcal{X}^{\geq q_{e}^{\ell}}$. By Mahler’s compactness criterion (see for instance [KIST, Theo. 1.1]), the subsets $\mathcal{X}^{\geq q_{e}^{\ell}}$ and $\mathcal{Y}^{\geq q_{e}^{\ell}}$ are compact.
Lemma 6.3 Let $\mu'$ be an $a$-invariant Borel probability measure on $\mathcal{Y}$ and let $\mathcal{A}$ be a countably generated sub-$\sigma$-algebra of the Borel $\sigma$-algebra of $\mathcal{Y}$ which is $a^{-1}$-descending and $U^+$-subordinate modulo $\mu'$. For all $r' \geq \delta' > 0$, $\epsilon \in ]0,1]$ and $\ell \in \mathbb{Z}_{\leq 0}$, let $j_1, j_2$ be integers satisfying

$$j_1 > \frac{d - (d - 1)\ell}{\min r} - \log q_v \delta'$$

and

$$j_2 > \frac{d - (d - 1)\ell}{\min s} - \frac{n}{d} \log q_v \epsilon.$$ 

If $y \in \mathcal{Y}_{q_v}^+$ satisfies $B_{y^U}^{U^+} a^{-j_1} y \subset [a^{-j_1} y]_{\mathcal{A}} \subset B_{y}^{U^+} a^{-j_1} y$, then we have

$$\tau_{y}^{a^1_{\mathcal{A}}} (a^{-j_2} x) \leq 1 - \frac{1}{q_v^j} \left( q_v^{-(j_1 + j_2)} (r')^{-1} \epsilon_{x_{\mathcal{A}}} \right)^{|r|}.$$

Proof. Let $x = \pi(y)$, which belongs to $\mathcal{Y}_{q_v}^+$. Since $x$ is a unimodular $R^d$, by Minkowski’s theorem 2.1, we hence have

$$q_v^{d-1} \lambda_d(x) \leq (\lambda_1(x))^{d-1} \lambda_d(x) \leq \lambda_1(x) \lambda_2(x) \cdots \lambda_d(x) \leq q_v^d,$$

therefore $\lambda_d(x) \leq q_v^{d-(d-1)\ell}$. There are linearly independent vectors $v_1, \ldots, v_d$ in the $R^d$ lattice $x$ such that $\|v_i\| \leq q_v^{d-(d-1)\ell}$. Let $\Delta$ be the parallelepiped in $K_v$ generated by $v_1, \ldots, v_d$, that is,

$$\Delta = \{ t_1 v_1 + \cdots + t_d v_d \in K_v^d : \forall i = 1, \ldots, d, \ |t_i| \leq 1 \}.$$

We identify $K_v^d$ with $K_{v}^m \times K_{v}^n$. Then for every $b = (b^-, b^+) \in \Delta$ with $b^- \in K_{v}^m$ and $b^+ \in K_{v}^n$, we have $\|b\| \leq q_v^{d-(d-1)\ell} \epsilon$, hence $\|b^-\|_r \leq q_v^{-\min r}$ and $\|b^+\|_s \leq q_v^{-\min s}$ since $\ell \leq 0$. Note that the fiber $\pi^{-1}(x)$ can be parametrized as follows: Fixing $g \in G_0$ with $x = g \Gamma$, since $\Delta$ is a fundamental domain for the action of $R^d_v$ on $K_v^d$, we have

$$\pi^{-1}(x) = \{ w(b) g \Gamma : b \in \Delta \}, \text{ where } w(b) = \begin{pmatrix} I_d & b \\ 0 & 1 \end{pmatrix}.$$

In particular, there exists $b_0 = (b_0^-, b_0^+) \in \Delta$ such that $y = w(b_0) g \Gamma$.

With a slightly simplified notation, let $V_y$ be the $U^+$-shape of the atom $[y]_{a^1_{\mathcal{A}}}$ (see Equation (54)), so that we have $V_y y = [y]_{a^1_{\mathcal{A}}}$. Let $\Xi = \{ \theta \in K_v^m : w(\theta, 0) \in V_y \}$ be the Borel set corresponding to $V_y$ by the canonical bijection $\theta \mapsto w(\theta, 0)$ (see above Equation (35)) between $K_v^m$ and $U^+$. Note that $0 \in \Xi$ as $I_{d+1} \in V_y$. Since $a^1_{\mathcal{A}}$ expands the $r$-quasinorm on $K_v^m$ with ratio exactly $q_v^j$ (see Equation (52)), and by the assumption on $y$ in the statement of Lemma 6.3, we have $B_{y^U}^{U^+} y \subset [y]_{a^1_{\mathcal{A}}} \subset B_{y}^{U^+} y$. Hence by the definition of the $U^+$-shapes of atoms in Equation (54), we have

$$B_{q_v^{j+1} r}^{K_v^m} \subset \Xi \subset B_{q_v^{j+1} r'}^{K_v^m}.$$ 

(65)

The atom $[y]_{a^1_{\mathcal{A}}}$ can be parametrized by

$$[y]_{a^1_{\mathcal{A}}} = \{ w(b) g \Gamma : \exists b^- \in b_0^- + \Xi, \ b = (b^-, b_0^+) \},$$

and $\tau_{y}^{a^1_{\mathcal{A}}}$ is the pushforward measure of the normalized Haar measure on the Borel set (with positive measure) $b_0^- + \Xi$ of $K_v^m$.  

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Let us consider the sets
\[ \Theta^- = \{ b^- \in K^m_v : \| b^- \|_r < q_v^{-j_2} \epsilon \| \} \quad \text{and} \quad \Theta^+ = \{ b^+ \in K^m_v : \| b^+ \|_s < q_v^{j_2} \epsilon \} . \]

If \( b = (b^-, b^+) \in \Theta^- \times \Theta^+ \), then \( \| a^{j_2} b^- \|_r < \epsilon \) and \( \| a^{j_2} b^+ \|_s < \epsilon \) by Equation (32). By the definition of \( \mathcal{L}_\epsilon \) in Equation (38), and since the grid \( a^{j_2} g \mathcal{R}^m_v + (a^{j_2} b^-, a^{j_2} b^+) \) contains the vector \( (a^{j_2} b^-, a^{j_2} b^+) \), we have
\[ a^{j_2} w(b) g \Gamma = w(a^{j_2} b^-, a^{j_2} b^+) a^{j_2} g \Gamma \notin \mathcal{L}_\epsilon . \]

Hence we have \( w(b) g \Gamma \notin a^{-j_2} \mathcal{L}_\epsilon \), so that
\[ [y]_{a^{j_2} \mathcal{L}_\epsilon} - a^{-j_2} \mathcal{L}_\epsilon \supset w( ((b^- + \Xi) \times \{ b^+_0 \} ) \cap (\Theta^- \times \Theta^+) ) g \Gamma . \tag{66} \]

**Claim.** We have the inclusion \( \Theta^- \times \{ b^+_0 \} \subset ( (b^-_0 + \Xi) \times \{ b^+_0 \} ) \cap (\Theta^- \times \Theta^+) \).

**Proof.** We only have to prove that \( b^+_0 \in \Theta^+ \) and that \( \Theta^- \subset \Theta^-_0 + \Xi \). Since \( (b^-_0, b^+_0) \in \Delta \), we have \( \| b^+_0 \|_s \leq q_v^{m \min s} \), hence the former assertion follows from the assumption that \( j_2 > \frac{d - (d - 1) \ell}{\min \{ \| \cdot \|_r \}} - \frac{n}{d} \log q_v \epsilon \).

In order to prove the latter assertion, let us fix \( b^- \in \Theta^- \). Recall that the \( r \)-quasinorm \( \| \cdot \|_r \) satisfies the ultrametric inequality property, see Equation (10). Hence, it follows from the assumptions \( j_2 > \frac{d - (d - 1) \ell}{\min \{ \| \cdot \|_r \}} - \frac{n}{d} \log q_v \epsilon \) and \( j_1 > \frac{d - (d - 1) \ell}{\min \{ \| \cdot \|_r \}} - \log q_v \delta' \), since \( \epsilon \leq 1 \), that
\[ \| b^- - b^-_0 \|_r \leq \max \{ \| b^- \|_r, \| b^+_0 \|_r \} \leq \max \{ q_v^{-j_2} \frac{m}{\epsilon}, q_v^{d - (d - 1) \ell} \} \]
\[ \leq \max \{ q_v^{d - (d - 1) \ell} \frac{\min \{ \| \cdot \|_r \}}{\| b^+_0 \|_s}, q_v^{d - (d - 1) \ell} \} = q_v^{d - (d - 1) \ell} \frac{\min \{ \| \cdot \|_r \}}{\| b^+_0 \|_s} < q_v^{j_2} \delta' . \]

Hence by the left inclusion in Equation (69), we have \( b^- \in b^-_0 + B^{K^m_v, r}_{q_v^{j_2} \delta'} \subset b^-_0 + \Xi \), which concludes the latter assertion.

Now by Equation (66) and by the above claim, since the canonical isomorphism \( w \mapsto \hat{w} \) (see the line above Equation (33)) from \( K^m_v \) to \( U^+ \) maps the Haar measure \( \text{vol}^m_v \) of \( K^m_v \) to the Haar measure \( m_{U^+} \) of \( U^+ \), we have
\[ 1 - \tau^a_{\hat{y}} (a^{-j_2} \mathcal{L}_\epsilon) = \tau^a_{\hat{y}} (\| y \|_{a^{-j_2} \mathcal{L}_\epsilon} - a^{-j_2} \mathcal{L}_\epsilon) > \frac{\text{vol}^m_v (\Theta^-)}{\text{vol}^m_v (b^-_0 + \Xi)} . \tag{67} \]

On one hand, an open ball in \( K_v \) of radius \( \eta > 0 \) having Haar measure \( q_v^{\lfloor \log q_v \eta \rfloor} \), we have
\[ \text{vol}^m_v (\Theta^-) = \text{vol}^m_v (B^{K^m_v, r}_{q_v^{-j_2} \epsilon}) = \prod_{i=1}^m \text{vol}_v (B^{K^m_v}_{q_v^{-j_2} \epsilon}) \]
\[ > \prod_{i=1}^m \frac{1}{q_v^{m}} (q_v^{-j_2} \epsilon)^{m} = \frac{1}{q_v^{m}} (q_v^{-j_2} \epsilon)^{m} \| r \| . \tag{68} \]

On the other hand, by the right inclusion in Equation (65), we have
\[ \text{vol}^m_v (b^-_0 + \Xi) = \text{vol}^m_v (\Xi) \leq \text{vol}^m_v (B^{K^m_v, r}_{q_v^{j_2} \epsilon}) \]
\[ = \prod_{i=1}^m \text{vol}_v (B^{K^m_v}_{q_v^{j_2} \epsilon}) \leq \prod_{i=1}^m (q_v^{j_2} \epsilon)^{m} = (q_v^{j_2} \epsilon)^{m} \| r \| . \tag{69} \]
Lemma 6.3 follows by combining the inequalities (67), (68) and (69).

**Proof of Theorem 1.2.** We fix a matrix $A \in \mathcal{M}_{m,n}(K_a)$ which is not $(r,s)$-singular on average, or equivalently by Proposition 5.2 and the definition of $\eta_A$ just before Lemma 6.1, we assume that $\eta_A < 1$. We also fix $\epsilon \in [0,1]$.

By Proposition 6.1 there exist an $a$-invariant Borel probability measure $\overline{\mu}$ on $\mathcal{Y}$ (depending on $\epsilon$) and an $a$-invariant Borel probability measure $\mu$ on $\mathcal{Y}$ (unique since $\eta_A < 1$) such that

$$\text{Supp} \overline{\mu} \subset \mathcal{L} \cup \{x_{\mathcal{Y}}\}, \quad \pi_* \overline{\mu} = \mu_A, \quad \text{and} \quad \overline{\mu} = (1-\eta_A)\mu + \eta_A\Delta_{x_{\mathcal{Y}}}.$$ 

Take a compact subset $K_0$ of $\mathcal{X}$ such that $\mu_A(K_0) > 0.99\mu_A(\mathcal{X}) = 0.99(1-\eta_A)$. Write $K = \pi^{-1}(K_0)$ and choose $r \in ]0, \ln 2[$ such that $K \subset \mathcal{Y}(r)$. Then $\mu(\mathcal{Y}(r)) = 0.99$ since $\eta_A < 1$. Note that the choices of $K$ and $r$ are independent of $\epsilon$ since the measure $\mu_A$ depends only on $A$ (see Proposition 6.1 and Equation (57)).

For such an $r > 0$, let $\mathcal{A}^{U^+}$ be the $\sigma$-algebra of subsets of $\mathcal{Y}$ constructed in Proposition 5.7. Proposition 6.1 (3) gives the inequality

$$h_\overline{\mu}(a^{-1} \mid \mathcal{A}^{U^+}) \geq |r|(1 - \eta_A) - \max r (m - \dim_{\text{Haus}} \text{Bad}_A(\epsilon)).$$

By the linearity of entropy (and since the entropy of $a^{-1}$ vanishes on the fixed set $\{x_{\mathcal{Y}}\}$), we have

$$h_\mu(a^{-1} \mid \mathcal{A}^{U^+}) = \frac{1}{1 - \eta_A} h_\overline{\mu}(a^{-1} \mid \mathcal{A}^{U^+}) \geq |r| - \frac{\max r}{1 - \eta_A} (m - \dim_{\text{Haus}} \text{Bad}_A(\epsilon)). \quad (70)$$

In order to use Lemma 5.9 and Proposition 5.10 we need an ergodicity assumption on the measures that appear in these statements. We will choose an appropriate ergodic component of $\mu$. Using the notation of Proposition 5.5 with $X = \mathcal{Y}$ and $T = a^{-1}$, let us denote the ergodic decomposition of $\mu$ (see for instance [ILW, Theo. 2.7]) by

$$\mu = \int_{y \in \mathcal{Y}} \mu_y^\epsilon \, d\mu(y).$$

Let $E = \{y \in \mathcal{Y} : \mu_y^\epsilon(K) > 0.9\}$. It follows from the inequality $\mu(K) > 0.99$ that

$$0.99 < \int_{\mathcal{Y}} \mu_y^\epsilon(K) \, d\mu(y) \leq \mu(E) + 0.9 \mu(\mathcal{Y} - E) = 0.9 + 0.1 \mu(E),$$

hence $\mu(E) > 0.9$. By Proposition 5.5 with $T = a^{-1}$ and $\mathcal{A} = \mathcal{A}^{U^+}_X$, and by Equation (70), we have

$$\int_{\mathcal{Y}} h_{\mu_y^\epsilon}(a^{-1} \mid \mathcal{A}^{U^+}_X) \, d\mu(y) = h_\mu(a^{-1} \mid \mathcal{A}^{U^+}_X) \geq |r| - \frac{\max r}{1 - \eta_A} (m - \dim_{\text{Haus}} \text{Bad}_A(\epsilon)).$$

Since $h_{\mu_y^\epsilon}(a^{-1} \mid \mathcal{A}^{U^+}_X) \leq |r|$ for every $y \in \mathcal{Y}$ by Lemma 5.9 we have

$$\int_{\mathcal{Y} - E} h_{\mu_y^\epsilon}(a^{-1} \mid \mathcal{A}^{U^+}_X) \, d\mu(y) \leq |r| \mu(\mathcal{Y} - E).$$

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Hence
\[
\int_E h_{\mu^q}(a^{-1}|\mathcal{A}_\infty^+ \mid y) \, d\mu(y) \geq |r| \mu(E) - \frac{\max r}{1 - \eta_A} (m - \dim_{\mathrm{Haus}} \mathcal{B}_A(\epsilon)) \geq \mu(E) \left( |r| - \frac{\max r}{0.9 (1 - \eta_A)} (m - \dim_{\mathrm{Haus}} \mathcal{B}_A(\epsilon)) \right).
\]

Therefore, there exists \( z \in \mathcal{U} \) such that \( \mu^q(\mathcal{U}) > 0.9 \) and
\[
h_{\mu^q}(a^{-1}|\mathcal{A}_\infty^+ \mid y) \geq |r| - \frac{\max r}{0.9 (1 - \eta_A)} (m - \dim_{\mathrm{Haus}} \mathcal{B}_A(\epsilon)) = : \lambda^q(\mathcal{U}^+ \mid y).
\]
We denote \( \lambda = \mu^q \) for such a \( z \in \mathcal{U} \). Then \( \lambda \) is an \( a \)-invariant ergodic Borel probability measure on \( \mathcal{U} \) and \( \operatorname{Supp} \lambda \subseteq \operatorname{Supp} \mu \subseteq \mathcal{L}_\epsilon \). By Lemma 5.9, we have
\[
H_\lambda(\mathcal{A}^+ \mid a^k \mathcal{U}^+) \geq |r| - \frac{\max r}{0.9 (1 - \eta_A)} (m - \dim_{\mathrm{Haus}} \mathcal{B}_A(\epsilon)).
\]

We will apply Lemma 5.3 with \( \mu' = \lambda \) and \( \mathcal{A} = a^{-k} \mathcal{A}_\infty^+ \) for some \( k \geq 1 \). Take an integer \( \ell \leq 0 \) such that \( K \subset \mathcal{U}(r) \), which depends only on \( A \). Set
\[
j_1 = \left\lfloor \frac{d - (d - 1)\ell}{\min r} - \log q_y \delta' \right\rfloor + 1 \quad \text{and} \quad j_2 = \left\lfloor \frac{d - (d - 1)\ell}{\min s} - \frac{n}{d} \log q_y \epsilon \right\rfloor + 1,
\]
where \( \delta' \) will be determined later on.

Let \( k = \left\lfloor \log q_y \left( \frac{r_{\max r}}{\max r} - \frac{m}{d} \right) \right\rfloor + j_2 + 1 \) and \( \mathcal{A} = a^{-k} \mathcal{A}_\infty^+ \). By the properties of \( \mathcal{A}_\infty^+ \) given in Proposition 5.7 and since \( K \subset \mathcal{U}(r) \), for every \( y \in K \), we have
\[
B_{\delta_r} U^+ y \subseteq [y]_{\mathcal{A}^+} \subset B_{\delta_r}^+ y.
\]
It follows from Equations 56 and 50 that since \( \delta_r \leq r \leq \ln 2 \), for every \( y \in K \), we have
\[
B_{\delta_r/2}^+ y \subseteq \bigcup_{\delta_r} B_{\delta_r/2}^+ y = [y]_{\mathcal{A}^+} \subseteq B_{\delta_r}^+ y \subset B_{\delta_r}^+ y \subseteq B_{\delta_r}^+ y.
\]
Hence, by Equation 51, we have
\[
B_{\delta_r}^+ y \subseteq \bigcup_{\delta_r} [a^{-k} y]_{\mathcal{A}^+} = [a^{-k} y]_{\mathcal{A}^+} \subseteq B_{\delta_r}^+ y \subseteq B_{\delta_r}^+ y.
\]
Thus for every \( y \in a^k K \), we have
\[
B_{\delta_r}^+ y \subseteq [y]_{\mathcal{A}^+} \subset B_{\delta_r}^+ y,
\]
where, by the definition of \( k \), we take
\[
r' = q_y^{-j_2 - 1} \frac{m}{d} \quad \text{and} \quad \delta' = q_y^{-1} \frac{1}{\max r} \left( \delta_r/2 \right) \frac{1}{\min r} r'.
\]
Equation 73 implies that for every \( y \in a^{j_1 + k} K \), we have
\[
B_{\delta_r}^+ y \subseteq [a^{-j_1} y]_{\mathcal{A}^+} \subset B_{\delta_r}^+ y,
\]
where
\[
\delta_r = \min r \quad \text{and} \quad r' = q_y^{-j_2 - 1} \frac{m}{d}.
\]
Now, we will use Proposition 5.10 with \( j = j_1, K' = a^kK \) (which is \( U^+ \)-saturated since so is \( K \) and as \( a \) normalizes \( U^+ \)), and \( \epsilon = \epsilon' \) (which satisfies the assumption of Proposition 5.10 by Equation (73)). We claim that
\[
B^{U^+,r}_{q^{-1}e^{-n}} \mathcal{L}_\epsilon \subset \mathcal{L}_\epsilon. \tag{75}
\]

Indeed, for all \( y \in \mathcal{L}_\epsilon \) and \( \theta \in K_0^\epsilon \) such that \( \| \theta \|_r \leq q^{-1}e^{-n} \), for every vector \( u = (u^-, u^+) \) in the grid \( w(\theta, 0)y \), we can write \( u = v + (\theta, 0) \) for some \( v = (v^-, v^+) \) in the grid \( \tilde{\Lambda}_y \) associated with \( y \) (see Equation (83)). Since \( y \in \mathcal{L}_\epsilon \), we have (see Equation (83))
\[
\| v \|_{r, s} = \max \{ \| v^- \|_{r, s}, \| v^+ \|_{r, s} \} \geq \epsilon. \quad \text{Since } u^+ = v^+, \text{ if } \| v^+ \|_{s, \alpha} \geq \epsilon, \text{ then } w(\theta, 0)y \in \mathcal{L}_\epsilon.
\]

Otherwise \( \| v^- \|_{r, s} \geq \epsilon \). We then have \( \| \theta \|_r \leq q^{-1}e^{-n} < \epsilon < \epsilon \leq \| v^- \|_r \). It follows from the equality case of the ultrametric inequality property of \( \| \|_r \) that
\[
\| u^- \|_r = \| \theta + v^- \|_r = \max \{ \| \theta \|_r, \| v^- \|_r \} = \| v^- \|_r \geq \epsilon. \tag{85}
\]

Hence \( w(\theta, 0)y \in \mathcal{L}_\epsilon \), which proves Equation (75).

By Proposition 5.10, the \( \sigma \)-algebra \( \mathcal{A}^{U^+} \) is \( a^{-1} \)-descending and \( U^+ \)-subordinate modulo \( \lambda \), and so is \( \mathcal{A} = a^{-k} \mathcal{A}^{U^+} \) since \( a \) normalizes \( U^+ \). Note that \( \text{Supp} \lambda \subset a^{-j_2} \mathcal{L}_\epsilon \) since \( \lambda \) is \( a \)-invariant. By Equations (51) and (75), we have
\[
B^{U^+,r}_{q^{-1}e^{-n}} a^{-j_2} \mathcal{L}_\epsilon = a^{-j_2} B^{U^+,r}_{q^{-1}e^{-n}} \mathcal{L}_\epsilon = a^{-j_2} B^{U^+,r}_{q^{-1}e^{-n}} \mathcal{L}_\epsilon \subset a^{-j_2} \mathcal{L}_\epsilon.
\]

Note that we have
\[
\tau^a_{\mathcal{A}'} ((\mathcal{A} - a^k K) = 0
\]

for \( \lambda \)-almost every \( y \in a^k K \), since then (see just above Proposition 5.10) the support \( \text{Supp} \tau^a_{\mathcal{A}'} \) is contained in \( [y]_{\mathcal{A}'} \), which is contained in \( U^+ y \), hence in \( a^k K \) since \( a \) normalizes \( U^+ \) and \( K = \pi^{-1}(K_0) \) is \( U^+ \)-saturated. Therefore, it follows from Proposition 6.10 for the first line, from the fact that the integrated function is nonpositive (hence its integral on a smaller domain is larger) for the third line, that
\[
H_\lambda(\mathcal{A} | a^{j_1} \mathcal{A}) \leq j_1 |r| + \int_{\mathcal{A}'} \log_q \tau_{\mathcal{A}'} ((\mathcal{A} - a^k K) \cup B^{U^+,r}_{q^{-1}e^{-n}} \text{Supp} \lambda) \, d\lambda(y)
\]
\[
\leq j_1 |r| + \int_{\mathcal{A}'} \log_q \tau_{\mathcal{A}'} ((\mathcal{A} - a^k K) \cup a^{-j_2} \mathcal{L}_\epsilon) \, d\lambda(y)
\]
\[
\leq j_1 |r| + \int_{a^k K \cap a^{j_1-k} K \cap \mathcal{A} \geq q^\epsilon} \log_q \tau_{\mathcal{A}'} ((\mathcal{A} - a^k K) \cup a^{-j_2} \mathcal{L}_\epsilon) \, d\lambda(y)
\]
\[
= j_1 |r| + \int_{a^k K \cap a^{j_1-k} K \cap \mathcal{A} \geq q^\epsilon} \log_q \tau_{\mathcal{A}'} (a^{-j_2} \mathcal{L}_\epsilon) \, d\lambda(y).
\]

We now apply Lemma 6.3 with as said above \( \mu' = \lambda \) and \( \mathcal{A} = a^{-k} \mathcal{A}^{U^+} \), with \( j_1 \) and \( j_2 \) defined in Equation (72), and with \( y \in a^{j_1+k} K \cap \mathcal{A} \geq q^\epsilon \) which satisfies the assumption of Lemma 6.3 by Equation (74). Thus
\[
\tau_{\mathcal{A}'} (a^{-j_2} \mathcal{L}_\epsilon) \leq 1 - \frac{1}{q^{-\epsilon}} \left( q^{-\epsilon(j_1+j_2)} r^{-1} \epsilon \right)^{|r|} = 1 - q^{-\epsilon(j_1-1)|r|} \epsilon. \tag{83}
\]
Hence

\[- \log_{q_v} \tau_y^{\mathcal{A}_i \mathcal{A}}(a^{-j_2} \mathcal{L}) \geq - \log_{q_v} \left( 1 - q_v^{-\left( j_1 - 1 \right) |r| - m} \right) \geq \frac{q_v^{-(j_1 - 1)|r| - m}}{\ln q_v} .\]

Note that \( \lambda(a^k K \cap a^{j_1 + k} K \cap \mathcal{Y} \equiv q_v^d) \geq 1/2 \) since \( \lambda \) is \( a \)-invariant, \( K \subset \mathcal{Y} \equiv q_v^d \) and \( \lambda(K) > 0.9 \), so that the three sets \( a^k K \), \( a^{j_1 + k} K \) and \( \mathcal{Y} \equiv q_v^d \) have \( \lambda \)-measure > 0.9, hence their pairwise intersections have \( \lambda \)-measure > 2 \( \times \) 0.9 − 1 = 0.8 , and their triple intersection has \( \lambda \)-measure > 2 \( \times \) 0.8 − 1 = 0.6 . It follows from Equation (42) and the invariance under \( a \) of \( \lambda \), hence of the conditional entropy, that

\[
|r| - H_\lambda(\mathcal{A}^U \mid a \mathcal{A}^U) = |r| - \frac{1}{j_1} H_\lambda(\mathcal{A}^U \mid a^{j_1} \mathcal{A}^U) = |r| - \frac{1}{j_1} H_\lambda(\mathcal{A} \mid a^{j_1} \mathcal{A}^U) \\
\geq -\frac{1}{j_1} \int_{a^k K \cap a^{j_1 + k} K \cap \mathcal{Y} \equiv q_v^d} \log_{q_v} \tau_y^{\mathcal{A}_i \mathcal{A}}(a^{-j_2} \mathcal{L}) \, d\lambda(y) \\
\geq \frac{q_v^{\left| r \right| - m} q_v^{- j_1 |r|}}{2 \ln q_v \, j_1} .
\]

Therefore, by Equation (71), we have

\[
\frac{\max r}{0.9 \left( 1 - \eta_A \right)} \left( m - \dim_{\text{Haus}} \text{Bad}_A(\epsilon) \right) \geq \left| r \right| - H_\lambda(\mathcal{A}^U \mid a \mathcal{A}^U) \geq \frac{q_v^{\left| r \right| - m} q_v^{- j_1 |r|}}{2 \ln q_v \, j_1} .
\]

Observe that

\[
j_1 = \left[ \frac{d - (d - 1) \ell}{\min r} - \log_{q_v} \delta' \right] + 1 = \left[ \frac{d - (d - 1) \ell}{\min r} - \log_{q_v} \left( \frac{\left( \delta_r / 2 \right)_{\min \ell}}{q_v^2 \left( \max r \right)} q_v^{- j_2 \epsilon \mathcal{Y}} \right) \right] + 1
\]

\[
= \left[ \frac{d - (d - 1) \ell}{\min r} + \left[ \frac{d - (d - 1) \ell}{\min s} - \frac{n}{d} \log_{q_v} \epsilon \right] \right] + 1 - \frac{m}{d} \log_{q_v} \epsilon - \log_{q_v} \left( \frac{\delta_r / 2}{q_v^2 \left( \max r \right)} \right) + 1 \\
\leq \left( d - (d - 1) \ell \right) \left( \frac{1}{\min r} + \frac{1}{\max s} \right) - \log_{q_v} \left( \frac{\delta_r / 2}{q_v^2 \left( \max r \right)} \right) + 4 - \log_{q_v} \epsilon .
\]

The constants \( \eta_A \), \( \ell \), \( \delta_r \), and \( r \) depend only on the fixed matrix \( A \in \mathcal{M}_{m,n}(K_v) \). Hence there exists a constant \( c(A) > 0 \) depending only on \( d \), \( r \), \( s \) and \( A \) such that

\[
m - \dim_{\text{Haus}} \text{Bad}_A(\epsilon) \geq c(A) \frac{\epsilon |r|}{\log_{q_v} \left( 1/\epsilon \right)} .
\]

This proves Theorem 1.2 \( \square \)

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