SOME REMARKS ON DESCENT DATA WITH APPLICATIONS TO GALOIS DESCENT

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ABSTRACT. We present a proof of the equivalence of the standard definition of descent data on schemes with another one mentioned in the literature that involves certain cartesian diagrams. Using this equivalence, we discuss the Galois descent of both schemes and morphisms of schemes.

0. Introduction

In this expository paper we present a detailed proof of the equivalence of the standard definition of descent data on schemes with another one mentioned in the literature that involves certain cartesian diagrams. See Section 2. To our knowledge, no detailed proof of this equivalence has appeared in print. As an application, we provide in Section 3 the missing details of the discussion of Galois descent contained in [BLR, §6.2, Example B, pp. 139-140]. The subject of Galois descent is discussed amply in the literature, but mostly over a field. Even when a more general base scheme is allowed, some important details are omitted (for another example of such omissions, see [GW, comments after (14.20.1), p. 457]). In Section 4 as an application of the detailed discussion of Section 3 we generalize the standard result on Galois descent for morphisms of schemes over a field [J, Proposition 2.8] to an arbitrary base scheme. More precisely, we show that, if $S' \to S$ is a finite Galois covering of schemes with Galois group $\Gamma$ and $\delta: X_1 \to X_2$ is an $S'$-morphism of $\Gamma$-schemes that descend to $S$, then $\delta$ descends to $S$ if, and only if, $\delta$ is invariant under the action of $\Gamma$ on morphisms defined by (4.1).

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1. Preliminaries

The identity morphism of an object $A$ of a category will be denoted by $1_A$.
If $S$ is a scheme, $(\text{Sch}/S)$ will denote the category of $S$-schemes. If $X$ is an $S$-scheme, $\text{Aut}(X/S)$ will denote the group of $S$-automorphisms of $X$.

Given morphisms of schemes $X \rightarrow f S \leftarrow g T$, we will write $X \times_{f,S,g} T$ for the fiber product of $f$ and $g$. When $f$ and $g$ are not relevant, we will write $X \times_S T$ for $X \times_{f,S,g} T$. If $u: X \rightarrow Y$ is an $S$-morphism of schemes, $u \times_S T$ will denote the $T$-morphism of schemes $u \times_S 1_T: X \times_S T \rightarrow Y \times_S T$.

Recall that a commutative diagram in a category $\mathcal{C}$

\[ Z \xrightarrow{u} U \xrightarrow{\pi} V \xrightarrow{w} W \]

is cartesian if for every commutative diagram of solid arrows in $\mathcal{C}$

\[ T \xrightarrow{h} Z \xrightarrow{\psi} Y \xrightarrow{\psi} V \xrightarrow{w} W \]

there exists a unique arrow $h: T \rightarrow Z$ in $\mathcal{C}$ such that the full diagram (1.2) commutes. It is easy to check that (1.1) is cartesian if both horizontal arrows $u$ and $w$ are isomorphisms. Further, if (1.1) is cartesian and $\psi: Y \rightarrow Z$ is an isomorphism in $\mathcal{C}$, then

\[ Y \xrightarrow{u \circ \psi} U \xrightarrow{\pi} V \xrightarrow{w} W \]

is cartesian as well. We will also need the following fact.

**Lemma 1.1.** If

\[ Z \xrightarrow{g_i} U \xrightarrow{e_i} V \xrightarrow{k_i} W \]
is a cartesian square in the category of schemes for every $i$ in some index set, then the diagram

$$
\begin{array}{cccc}
\coprod Z_i & \overset{\coprod g_i}{\longrightarrow} & U \\
\downarrow \coset & & \downarrow \\
\coprod V_i & \overset{\coprod k_i}{\longrightarrow} & W.
\end{array}
$$

is cartesian as well.

Proof. (Sketch) This may be verified by starting with a commutative diagram

and considering, for every $j$, the commutative diagram

For more information on fiber products and coproducts of schemes and cartesian diagrams, see \cite{EGA I new} Chapter 0, §1.2, and Chapter I, §3.1).

\section{Descent data on schemes}

In this Section we reformulate the standard definitions of covering data and descent data on schemes (these standard definitions can be found, for example, in \cite{BLR} Chapter 6]). We focus on schemes, but similar considerations apply to quasi-coherent modules.

Let $f: S' \to S$ be a morphism of schemes, set $S'' = S' \times_S S'$ and let $p_i: S'' \to S'$ ($i = 1, 2$) be the canonical projection onto the $i$-th factor. Then the following
The diagram is cartesian
\begin{equation}
\begin{array}{c}
S'' \\
p_1 \\
p_2 \\
\xrightarrow{f} \xrightarrow{p} S' \\
S \\
\end{array}
\end{equation}

Further, set \( S''' = S' \times S' \times S' \) and let \( p_{jk} : S''' \to S'' \) be given (set-theoretically) by \((s_1, s_2, s_3) \mapsto (s_j, s_k)\), where \((j, k) = (1, 2), (1, 3)\) or \((2, 3)\). Note that
\begin{align}
p_1 \circ p_{12} &= p_1 \circ p_{13} \\
p_1 \circ p_{23} &= p_2 \circ p_{12} \\
p_2 \circ p_{23} &= p_2 \circ p_{13},
\end{align}
where the first (respectively, second, third) common composition \( S''' \to S' \) is the projection onto the first (respectively, second, third) factor.

If \( \pi : X \to S' \) is an \( S' \)-scheme and \( i = 1 \) or \( 2 \), we will write \( p_i^*X = X \times_{S'} p_i^*S'' \) and regard it as an \( S'' \)-scheme via \( p_i^*(\pi) \). Further, we will write \( p_{i, X} = (p_i)_X : p_i^*X \to X \). The \( S''' \)-schemes \( p_{jk}^*p_i^*X \) (with structural morphisms \( p_{jk}^*p_i^*(\pi) \)) and morphisms \( p_{jk, p_i^*X} : p_{jk}^*p_i^*X \to p_i^*X \) are defined similarly. The equalities \ref{eqs1}-\ref{eqs3} induce various identifications among these objects. For example, by \ref{eqs3},
\begin{equation}
p_{23, p_2^*X} = p_{13, p_3^*X}.
\end{equation}

Recall that a covering datum on \( X \) relative to \( f \) is an isomorphism of \( S'' \)-schemes \( p_1^*X \simeq p_2^*X \).

Set \( X'' = p_1^*X \) and let \( \varphi : X'' \xrightarrow{\sim} p_2^*X \) be a covering datum on \( X \). In particular, \( p_1^*(\pi) = p_2^*(\pi) \circ \varphi \). Define
\begin{align}
q_1 &= p_{1, X} \\
q_2 &= p_{2, X} \circ \varphi.
\end{align}

Then the following diagram, which is an instance of diagram \ref{diag}, is cartesian for \( i = 1 \) and \( 2 \):
\begin{equation}
\begin{array}{c}
X'' \\
\xrightarrow{q_i} X \\
\xrightarrow{p_i} \xrightarrow{\pi} S' \\
S'' \xrightarrow{p_i^*(\pi)} S'.
\end{array}
\end{equation}

Conversely, assume that there exist cartesian diagrams of the form \ref{diag} such that \ref{eq} holds. Then there exist unique morphisms \( \varphi : X'' \to p_2^*X \) and \( \psi : p_2^*X \to X'' \).
such that the following diagrams commute:

\[
\begin{array}{c}
\begin{array}{ccc}
X'' & \xrightarrow{q_2} & X \\
\downarrow{\varphi} & & \downarrow{\pi} \\
S'' & \xrightarrow{p_2} & S' \\
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
p_2^*X & \xrightarrow{p_2.X} & X \\
\downarrow{\psi} & & \downarrow{\pi} \\
S'' & \xrightarrow{p_2} & S' \\
\end{array}
\end{array}
\]

Since the diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
p_2^*X & \xrightarrow{p_2.X} & X \\
\downarrow{\varphi \circ \psi} & & \downarrow{\pi} \\
S'' & \xrightarrow{p_2} & S' \\
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
X'' & \xrightarrow{q_2} & X \\
\downarrow{\psi \circ \varphi} & & \downarrow{\pi} \\
S'' & \xrightarrow{p_2} & S' \\
\end{array}
\end{array}
\]

commute, \(\varphi \circ \psi\) (respectively, \(\psi \circ \varphi\)) is the identity morphism of \(p_2^*X\) (respectively, \(X''\)). Thus we obtain an \(S''\)-isomorphism \(\varphi: X'' \cong p_2^*X\) (i.e., a covering datum on \(X\) relative to \(f\)) such that (2.7) holds.

We conclude that to give a covering datum on \(X\) relative to \(f\) is equivalent to giving a pair of cartesian diagrams (2.8) such that (2.6) holds.

Now let \(\varphi: X'' \cong p_2^*X\) again be a covering datum on \(X\) relative to \(f\) and define \(q_1\) and \(q_2\) by (2.6) and (2.7), respectively. Further, write \(X''' = p_{12}^*p_1^*X = p_{13}^*p_1^*X\) (2.2) and note that \(p_{12}^*p_2^*X = p_{23}^*X''\) (2.3). Then we may discuss the \(S'''\)-isomorphism...
\( p_{12}^* \varphi : X'''' \xrightarrow{\sim} p_{23}^* X''\) in analogy to the foregoing discussion of the \( S''\)-isomorphism \( \varphi : X'''' \xrightarrow{\sim} p_2^* X \). Thus we define

\[
q_{12} = p_{12,X'''} \quad (2.9)
\]

\[
q_{13} = p_{13,X''} \quad (2.10)
\]

\[
q_{23} = p_{23,X''} \circ p_{12}^* \varphi. \quad (2.11)
\]

Since \( p_{12}^* p_1^*(\pi) = p_{13}^* p_1^*(\pi) = p_{12}^* p_2^*(\pi) \circ p_{12}^* \varphi = p_{23}^* p_1^*(\pi) \circ p_{12}^* \varphi \), the following diagram is cartesian for \((j,k) = (1,2), (1,3)\) and \((2,3)\) as an instance of diagram (1.3):

\[
\begin{array}{ccc}
X'''' & \xrightarrow{q_{jk}} & X'' \\
p_{12}^* p_1^*(\pi) & \downarrow & p_{1}^*(\pi) \\
S''' & \xrightarrow{\sim} & S''.
\end{array}
\]

Now, by the commutativity of

\[
p_{jk}^* X'' \xrightarrow{p_{jk}^* X'''} \xrightarrow{p_{jk}^* \varphi} p_{2}^* X
\]

and the equalities (2.6), (2.7), (2.9), (2.10) and (2.11), we have

\[
q_1 \circ q_{12} = q_1 \circ q_{13} \quad (2.13)
\]

\[
q_1 \circ q_{23} = q_2 \circ q_{12} \quad (2.14)
\]

Assume now that \( \varphi \) is, in fact, a descent datum on \( X \) relative to \( f \), i.e., the following diagram of isomorphisms of \( S'''\)-schemes, where the equalities are induced by (2.2), (2.3) and (2.4), commutes:

\[
\begin{array}{ccc}
p_{12}^* p_1^* X = p_{13}^* p_1^* X & \xrightarrow{p_{13}^* \varphi} & p_{23}^* p_2^* X = p_{13}^* p_2^* X \\
p_{12}^* \varphi & \sim & p_{23}^* \varphi \\
p_{12}^* p_2^* X & \xrightarrow{\sim} & p_{23}^* p_2^* X
\end{array}
\]

Then

\[
q_2 \circ q_{23} = q_2 \circ q_{13}. \quad (2.16)
\]

Indeed, by (2.5), (2.7), (2.10), (2.11), (2.12) and (2.15), we have

\[
q_2 \circ q_{23} = p_{2,X} \circ (\varphi \circ p_{23,X''}) \circ p_{12}^* \varphi = p_{2,X} \circ (p_{23,X} \circ p_{23}^* \varphi) \circ p_{12}^* \varphi = p_{2,X} \circ \varphi \circ p_{13,X''} = q_2 \circ q_{13}.
\]
Thus we obtain six commutative diagrams

\[(2.17)\]

\[
\begin{array}{ccc}
X^{m} & \xrightarrow{q_{jk}'} & X' \\
p_{12}^{*}p_{1}^{*} & \downarrow & p_{12}^{*} \\
S^{m} & \xrightarrow{p_{jk}} & S' \\
\end{array}
\]

where \(i = 1 \) or \(2, \ (j,k) = (1,2), (1,3) \) or \((2,3)\), the squares are cartesian, equations \((2.6), (2.7), (2.9), (2.10)\) and \((2.11)\) hold (where \(\varphi\) is the covering datum on \(X\) determined by the right-hand square in \((2.17)\) for \(i = 2\)) and the various top horizontal compositions satisfy the relations \((2.13), (2.14)\) and \((2.16)\).

Conversely, assume that there exist commutative diagrams of the form \((2.17)\) with cartesian squares such that \((2.6), (2.7)\) (where \(\varphi\) is the covering datum on \(X\) determined by the right-hand square in \((2.17)\) for \(i = 2\)), \((2.9), (2.10), (2.14)\) and \((2.16)\) hold. Then \((2.13)\) also holds since it follows from \((2.2), (2.6), (2.9)\) and \((2.10)\). We will show that \((2.11)\) holds as well and that diagram \((2.15)\) commutes, i.e., \(\varphi\) is a descent datum on \(X\) relative to \(f\).

By \((2.7), (2.9)\) and the commutativity of \((2.12)\), the following diagram commutes

\[
\begin{array}{ccc}
X^{m} & \xrightarrow{q_{12} \circ q_{12}} & X' \\
p_{12}^{*} & \downarrow & p_{12}^{*} \\
S^{m} & \xrightarrow{p_{12}} & S' \\
\end{array}
\]

Further, \(p_{12}^{*} \varphi\) is the unique morphism such that \(p_{2,\chi} \circ p_{12,\chi}^{*} = q_{2} \circ q_{12}\).

Similarly, there exists an \(S^{m}\)-isomorphism \(g: X^{m} \xrightarrow{\sim} p_{23}^{*}X^{m} = p_{12}^{*}p_{2}^{*}X\) such that the following diagram commutes

\[
\begin{array}{ccc}
X^{m} & \xrightarrow{q_{23}} & X' \\
p_{23}^{*} & \downarrow & p_{23}^{*} \\
S^{m} & \xrightarrow{p_{23}} & S' \\
\end{array}
\]

where we have used \((2.6)\). Moreover, \(g\) is the unique morphism that satisfies the identity \(p_{1,\chi} \circ p_{23,\chi} \circ g = q_{1} \circ q_{23}\). Now \((2.3), (2.14)\) and the preceding uniqueness statements imply that \(g = p_{12}^{*} \varphi\), whence \(q_{23} = p_{23,\chi} \circ p_{12}^{*} \varphi\), i.e., \((2.11)\) holds.

Finally, the diagram with cartesian square (where the equalities come from \((2.4)\)
and (2.16)

\[ X' = \frac{p_2^* p_2^* X}{(p_2 \circ p_{23})_X = (p_2 \circ p_{13})_X} \]

commutes for \( h = p_{23}^* \varphi \circ p_{12}^* \varphi \) and \( h = p_{13}^* \varphi \). Indeed, since (2.12) commutes and (2.7), (2.10) and (2.11) hold, we have

\[ (p_2 \circ p_{23})_X \circ p_{23}^* \varphi \circ p_{12}^* \varphi = p_{2, x} \circ \varphi \circ p_{23} \circ X'' \circ p_{12}^* \varphi = q_2 \circ q_{23} \]

and

\[ (p_2 \circ p_{13})_X \circ p_{13}^* \varphi = p_{2, x} \circ \varphi \circ p_{13} \circ X'' = q_2 \circ q_{13}. \]

Thus \( p_{23}^* \varphi \circ p_{12}^* \varphi = p_{13}^* \varphi \), i.e., the cocycle condition (2.15) is satisfied.

We conclude that to give a descent datum on \( X \) relative to \( f \) is equivalent to giving six commutative diagrams of the form (2.17) consisting of cartesian squares such that (2.6), (2.7) (where \( \varphi \) is the covering datum on \( X \) determined by the right-hand square in (2.17) for \( i = 2 \)), (2.9), (2.10), (2.14) and (2.16) hold.

To conclude this Section, we observe that, if \( Y \) is an \( S \)-scheme, then the \( S' \)-scheme \( Y' = f^* Y = Y \times_S S' \) is endowed with a canonical descent datum \( c_Y : p_Y^* Y' \simeq p_Y^* Y' \), namely the composite \( S''\)-isomorphism

\[ p_Y^* Y' = p_Y^* f^* Y \simeq (f \circ p_1)^*(Y) = (f \circ p_2)^*(Y) \simeq p_Y^* Y', \]

where the second equality holds by the commutativity of (2.1). Set-theoretically, \( c_Y \) can be described by the formula

\[ c_Y(y, s', s', t') = (y, t', s', t'), \]

where \((y, -) \in Y' \) and \((s', t') \in S''\).

3. Galois descent of schemes

In this Section we use the developments of the previous Section to discuss Galois descent of schemes. Compare with [BLR, §6.2, Example B, pp. 139-140].

Recall that a morphism of schemes \( f : S' \to S \) is said to be finite and locally free if \( f \) is affine and \( f_* O_{S'} \) is a finite and locally free \( O_S \)-module. Equivalently, \( f \) is finite, flat and locally of finite presentation.

Let \( f : S' \to S \) be a finite, surjective and locally free morphism (in particular, \( f \) is faithfully flat and quasi-compact) and let \( \Gamma \) be a subgroup of \( \text{Aut}(S'/S) \). If \( X \) is an \( S \)-scheme, an action of \( \Gamma \) on \( X \) over \( S \) (via automorphisms) is a group homomorphism \( \rho : \Gamma \to \text{Aut}(X/S) \).
For every scheme $X$, set $\Gamma \times X = \coprod_{\sigma \in \Gamma} X$. Then $\rho$ induces an action of the $S$-group scheme $\Gamma \times S$ on $X$ over $S$, i.e., an $S$-morphism $(\Gamma \times S) \times_{S} X \rightarrow X$ subject to well-known conditions. We will henceforth identify $(\Gamma \times S) \times_{S} X$ and $\Gamma \times X$ so that the preceding morphism will be written as $\Gamma \times X \rightarrow X$. Now set
\[
b = \coprod_{\sigma \in \Gamma} 1_{S'}: \Gamma \times S' \rightarrow S', (\sigma, s') \mapsto s'.
\]
We will regard $\Gamma \times S'$ as an $S$-scheme via $f \circ b$ (whence $b$ is an $S$-morphism). The canonical action of $\Gamma$ on $S'$ over $S$, i.e., the $S$-morphism $\prod_{\sigma \in \Gamma} \sigma: \prod_{\sigma \in \Gamma} S' \rightarrow S'$ will be written as
\[
a: \Gamma \times S' \rightarrow S', (\sigma, s') \mapsto \sigma s'.
\]
We now assume that $f$ is a Galois covering with Galois group $\Gamma$, i.e., the morphism of $S$-schemes
\[
\vartheta = (b, a): \Gamma \times S' \rightarrow S'', (\sigma, s') \mapsto (s', \sigma s'),
\]
is an isomorphism.
For example, if $K/k$ is a finite Galois extension of fields with Galois group $\Gamma$, then the canonical morphism $f: \text{Spec } K \rightarrow \text{Spec } k$ is a Galois covering. In effect, in this case $\vartheta$ is the isomorphism of $k$-schemes induced by the isomorphism of $k$-algebras $K \otimes_k K \xrightarrow{\sim} \prod_{\sigma \in \Gamma} K, x \otimes y \mapsto \prod_{\sigma \in \Gamma} x\sigma(y)$.
Clearly, the following diagrams commute
\[
\begin{array}{ccc}
\Gamma \times S' & \xrightarrow{\vartheta} & S'' \\
b \downarrow & & \downarrow p_1 \\
S' & \xrightarrow{p_1} & S''
\end{array}
\]
and
\[
\begin{array}{ccc}
\Gamma \times S' & \xrightarrow{\vartheta} & S'' \\
a \downarrow & & \downarrow p_2 \\
S' & \xrightarrow{p_2} & S''
\end{array}
\]
Further, since $\vartheta$ is an isomorphism, the morphism of $S$-schemes
\[
\varphi: \Gamma \times \Gamma \times S' \rightarrow S'', (\sigma, \tau, s') \mapsto (s', \tau s', (\sigma \tau)s'),
\]
is an isomorphism as well.
We now define $S$-morphisms $\tilde{\varphi}_{jk}: \Gamma \times \Gamma \times S' \rightarrow \Gamma \times S'$ by the formulas
\[
\tilde{\varphi}_{12}(\sigma, \tau, s') = (\tau, s'),
\]
\[
\tilde{\varphi}_{13}(\sigma, \tau, s') = (\sigma \tau, s'),
\]
\[
\tilde{\varphi}_{23}(\sigma, \tau, s') = (\sigma, \tau s').
\]
Then the following diagram commutes for \((j,k) = (1,2), (1,3)\) and \((2,3)\):

\[
\begin{array}{ccc}
\Gamma \times \Gamma \times S' & \xrightarrow{\theta} & S''
\\
\sim & \quad \sim
\\
\tilde{p}_{jk} & & p_{jk}
\end{array}
\]

Let \(\pi : X \to S'\) be an \(S'\)-scheme and recall the schemes \(X'' = p_1^*X = X \times_{\pi,S,p_1} S''\) and \(X''' = p_{12}^*p_1^*X = p_{13}^*p_1^*X\). We will make the identifications

\[
\Gamma \times X = X \times_{S',b} (\Gamma \times S') = b^*X
\]

\[
1_\Gamma \times \pi = b^*(\pi) = \varphi^*(p_1^*(\pi)) \quad \text{(see 3.3)}
\]

\[
\Gamma \times \Gamma \times X = (b \circ \tilde{p}_{12})^*X = (b \circ \tilde{p}_{13})^*X
\]

\[
1_\Gamma \times 1_\Gamma \times \pi = (b \circ \tilde{p}_{12})^*(\pi) = \tilde{p}_{12}^*(\varphi^*p_1^*(\pi)) = \varphi^*(p_{12}^*p_1^*(\pi)) \quad \text{(see 3.9)}.
\]

Via the above identifications, \(\vartheta\) and \(\varrho\) induce isomorphisms

\[
\vartheta_{X''} : \Gamma \times X \xrightarrow{\sim} X'', (\sigma, x) \mapsto (x, \pi(x), \sigma \pi(x)),
\]

and

\[
\varrho_{X'''} : \Gamma \times \Gamma \times X \xrightarrow{\sim} X''', (\sigma, \tau, x) \mapsto (x, \pi(x), \tau \pi(x), (\sigma \tau) \pi(x)),
\]

where we have used the commutativity of (3.3) and (3.9) to obtain the indicated set-theoretic formulas.

Now let \(\rho : \Gamma \to \text{Aut}(X/S)\) be an action of \(\Gamma\) on \(X\) over \(S\) which is compatible with the canonical action of \(\Gamma\) on \(S'\) over \(S\), i.e., if \(\Gamma \times X \to X\) is the \(S\)-morphism induced by \(\rho\), then the following diagram of \(S\)-morphisms commutes

\[
\begin{array}{ccc}
\Gamma \times X & \xrightarrow{1_\Gamma \times \pi} & X
\\
\downarrow \pi & & \downarrow \pi
\\
\Gamma \times S' & \xrightarrow{a} & S'
\end{array}
\]

where \(a\) is given by (3.1). We will show that \(\rho\) defines a descent datum on \(X\) relative to \(f\) by constructing a diagram of the form (2.17) with cartesian squares such that (2.6), (2.7) (where \(\varphi\) is the covering datum on \(X\) determined by the right-hand square in (2.17) for \(i = 2\)), (2.9), (2.10), (2.14) and (2.16) hold (see the previous Section).
We begin by noting that (3.12) may be written as

\[
\prod_{\sigma \in \Gamma} X \xrightarrow{\rho(\sigma)} X \\
\Pi \pi \\
\prod_{\sigma \in \Gamma} S' \xrightarrow{\sigma} S'.
\]

Now since

\[
\prod \rho(\sigma) \\
\sim \\
\prod \pi \\
\prod \sigma \in \Gamma S' \sim S'
\]

is cartesian for every \(\sigma \in \Gamma\), Lemma (1.1) shows that the equivalent diagrams (3.12) and (3.13) are cartesian as well. We now observe that, if

\[
q_2 = (\prod \rho(\sigma)) \circ \vartheta_{X''}^{-1},
\]

then the cartesian square (3.12) decomposes as

\[
\begin{array}{ccc}
\Gamma \times X & \xrightarrow{\vartheta_{X''}} & X'' \\
\sim & \downarrow & \sim \\
\Gamma \times S' & \xrightarrow{\vartheta} & S'' \\
\end{array}
\]

where the lower part of the diagram commutes by the commutativity of (3.4). We conclude that the right-hand square in (3.16) is cartesian. Thus, setting \(q_1 = p_{1,X}\) (whence (2.6) holds), there exist cartesian diagrams for \(i = 1\) and 2

\[
\begin{array}{ccc}
X'' & \xrightarrow{q_i} & X \\
p_i(\pi) & \downarrow & \pi \\
S'' & \sim & S'
\end{array}
\]

that define a covering datum \(\varphi: X'' \sim p_2^*X\) on \(X\) relative to \(f\) such that (2.7) holds.
Now let $\tilde{q}_{jk}: \Gamma \times \Gamma \times X \to \Gamma \times X$ be given by the formulas

\begin{align}
\tilde{q}_{12}(\sigma, \tau, x) &= (\tau, x), \\
\tilde{q}_{13}(\sigma, \tau, x) &= (\sigma \tau, x), \\
\tilde{q}_{23}(\sigma, \tau, x) &= (\sigma, \rho(\tau)x).
\end{align}

Then (2.6), (3.10), (3.17) and (3.19) yield

\begin{align}
q_1 \circ \vartheta_{X''} \circ \tilde{q}_{23} &= (\prod \rho(\sigma)) \circ \tilde{q}_{12}.
\end{align}

Further, since $\rho(\sigma)(\rho(\tau)x) = \rho(\sigma\tau)x$ for all $(\sigma, \tau, x) \in \Gamma \times \Gamma \times X$, we have

\begin{align}
(\prod \rho(\sigma)) \circ \tilde{q}_{23} &= (\prod \rho(\sigma)) \circ \tilde{q}_{13}.
\end{align}

Define

$q_{jk} = \vartheta_{X''} \circ \tilde{q}_{jk} \circ \tilde{q}_{X''}^{-1}$.

Then (2.14) and (2.16) follow at once from (3.15), (3.20) and (3.21). Further, since $\tilde{q}_{jk} = \tilde{p}_{jk, \Gamma \times X}$ for $(j, k) = (1, 2)$ and (1, 3), the commutativity of (3.9) shows that $q_{jk} = p_{jk, X''}$ for such $(j, k)$, i.e., (2.9) and (2.10) hold. Next, the diagram

\begin{align}
\Gamma \times \Gamma \times X \xrightarrow{\tilde{q}_{jk}} \Gamma \times X \xrightarrow{1_{\Gamma} \times 1_{\Gamma} \times \pi} \Gamma \times S' \xleftarrow{\tilde{p}_{jk}} S'
\end{align}

is cartesian for $(j, k) = (1, 2), (1, 3)$ and (2, 3). This is clear if $(j, k) = (1, 2)$ or (1, 3). If $(j, k) = (2, 3)$, then (3.22) is cartesian because (3.12) is cartesian. Now (3.22) decomposes as

\begin{align}
\Gamma \times \Gamma \times X \xrightarrow{\vartheta_{X''}} X' \xrightarrow{q_{jk}} X'' \xleftarrow{\vartheta_{X''}^{-1}} \Gamma \times X
\end{align}

where the bottom part of the diagram commutes by the commutativity of (3.9). Consequently, the central square above is cartesian. Thus we obtain the desired
commutative diagrams with cartesian squares

\[ \begin{array}{ccc}
X'' & \xrightarrow{q_{jk}} & X'' \\
\downarrow{p_1 p_1(p_1)} & & \downarrow{p_1(p_1)} \\
S'' & \xrightarrow{p_{jk}} & S''
\end{array} \]

such that (2.6), (2.7), (2.9), (2.10), (2.14) and (2.16) hold.

The descent datum \( \varphi: X'' \xrightarrow{\sim} p_2^*X \) on \( X \) relative to \( f \) thus associated to \( \rho \) may be described (set-theoretically) as follows. By (2.7) and (3.15), we have

\[ \prod \rho(\sigma) = p_{2,X} \circ \varphi \circ \vartheta_{X''}. \]

It then follows that \( \varphi \) is given by the formula

\[ \varphi(x, \pi(x), s') = (\rho(\sigma)x, \pi(x), s'), \]

where \( \sigma \) is the unique element of \( \Gamma \) such that \( s' = \sigma \pi(x) \).

Example 3.1. Let \( Y \) be an \( S \)-scheme. Then \( Y' = Y \times_S S' \) is canonically endowed with an action of \( \Gamma \) over \( S \) that is compatible with \( a \), namely \( \prod(1_Y \times_S \sigma): \Gamma \times Y' \rightarrow Y' \). The associated descent datum on \( Y' \) (relative to \( f \)) is the isomorphism of \( S'' \)-schemes \( c_{Y'}: p_1^*Y' \xrightarrow{\sim} p_2^*Y' \).

4. Galois Descent of Morphisms

We keep the notation and hypotheses of the previous Section. In this Section we generalize the standard result [4 Proposition 2.8] on the Galois descent of morphisms of \( k \)-schemes, where \( k \) is a field, to an arbitrary base scheme \( S \).

For \( i = 1 \) or \( 2 \), let \( \pi_i: X_i \rightarrow S' \) be an \( S' \)-scheme equipped with an action \( \rho_i: \Gamma \rightarrow \text{Aut}(X_i/S) \) that is compatible with the canonical action of \( \Gamma \) on \( S' \) over \( S \). If \( \delta: X_1 \rightarrow X_2 \) is an \( S' \)-morphism, i.e., \( \pi_2 \circ \delta = \pi_1 \), then the commutativity of (3.14) (for both \( \rho_1 \) and \( \rho_2 \)) shows that \( \rho_2(\sigma) \circ \delta \circ \rho_1(\sigma)^{-1}: X_1 \rightarrow X_2 \) is a morphism of \( S' \)-schemes for every \( \sigma \in \Gamma \). Thus we may define a left action of \( \Gamma \) on the set \( \text{Hom}_{S'}(X_1, X_2) \) by

\[ \Gamma \times \text{Hom}_{S'}(X_1, X_2) \rightarrow \text{Hom}_{S'}(X_1, X_2), (\sigma, \delta) \mapsto \rho_2(\sigma) \circ \delta \circ \rho_1(\sigma)^{-1}. \]

Now, for \( i = 1 \) and \( 2 \), let \( \varphi_i: p_1^*X_i \xrightarrow{\sim} p_2^*X_i \) be the descent datum on \( X_i \) associated to \( \rho_i \) in the previous Section. Note that \( p_2^*(\pi_i) \circ \varphi_i = p_1^*(\pi_i) \) for \( i = 1 \) and \( 2 \).

Proposition 4.1. Let \( \delta \in \text{Hom}_{S'}(X_1, X_2) \). Then \( \delta \) is invariant under the action of \( \Gamma \) if, and only if, the diagram

\[ \begin{array}{ccc}
p_1^*X_1 & \xrightarrow{\varphi_1} & p_2^*X_1 \\
\downarrow{p_1(\delta)} & & \downarrow{p_2(\delta)} \\
p_1^*X_2 & \xrightarrow{\varphi_2} & p_2^*X_2
\end{array} \]
commutes.

**Proof.** By the definition (4.1), we need to show that (4.2) commutes if, and only if,

\[
\begin{array}{c}
\Gamma \times X_1 \quad \Downarrow_{\rho_1(\sigma)} \\
\delta \\
\Gamma \times X_2 \quad \Downarrow_{\rho_2(\sigma)}
\end{array}
\]

commutes. By (3.24) applied to both $\rho_1$ and $\rho_2$, the preceding diagram decomposes as

\[
\begin{array}{c}
\Gamma \times X_1 \quad \overset{\delta}{\longrightarrow} \quad p_1^*X_1 \quad \overset{\varphi_1}{\longrightarrow} \quad p_2^*X_1 \quad \overset{p_{2,1}}{\longrightarrow} \quad X_1 \\
\delta \\
\Gamma \times X_2 \quad \overset{\delta}{\longrightarrow} \quad p_1^*X_2 \quad \overset{\varphi_2}{\longrightarrow} \quad p_2^*X_2 \quad \overset{p_{2,2}}{\longrightarrow} \quad X_2
\end{array}
\]

where the left-hand and right-hand squares commute. Thus, if (4.2) commutes, then (4.3) commutes as well. Conversely, assume that (4.3), i.e., the outer diagram in (4.4), commutes. To show that (4.2) commutes, it suffices to check that the diagram with cartesian square

\[
\begin{array}{c}
p_1^*X_1 \\
\delta \\
p_2^*X_2 \\
\varphi \circ \delta
\end{array}
\]

commutes for $h = \varphi_2 \circ p_1^*(\delta)$ and $h = p_2^*(\delta) \circ \varphi_1$. The above diagram clearly commutes if $h = \varphi_2 \circ p_1^*(\delta)$. Now, since $\delta_{p_1^*X_1}$ is an isomorphism and the outer diagram and left-hand square in (4.4) commute, we have $p_{2,1} \circ p_2^*(\delta) \circ \varphi_1 = p_{2,2} \circ \varphi_2 \circ p_1^*(\delta)$, i.e., the top triangle of diagram (4.5) commutes when $h = p_2^*(\delta) \circ \varphi_1$. The commutativity of the lower triangle in (4.5) when $h = p_2^*(\delta) \circ \varphi_1$ can be checked using the identities $\pi_2 \circ \delta = \pi_1$ and $p_2^*(\pi_i) \circ \varphi_i = p_1^*(\pi_i)$ ($i = 1$ and 2). \qed

Recall now that the descent datum $\varphi_i : p_1^*X_i \sim p_2^*X_i$ is said to be **effective** if there exist $S$-schemes $Y_i$ and $S'$-isomorphisms $\theta_i : X_i \sim Y_i'$ such that the diagram

\[
\begin{array}{c}
p_1^*X_i \quad \overset{\varphi_i}{\sim} \quad p_2^*X_i \\
p_1^*(\theta_i) \sim p_2^*(\theta_i) \\
p_1^*Y_i' \quad \overset{e_Y}{\sim} \quad p_2^*Y_i'
\end{array}
\]
commutes. If this is the case, then we say that $X_i$ descends to $Y_i$ (or to $S$). By [SGA1 VIII, Corollary 7.6], $X_i$ descends to $S$ if $\pi_i: X_i \to S'$ is quasi-projective.

**Corollary 4.2.** Assume that, for $i = 1$ and $2$, $X_i$ descends to $Y_i$ and let $\theta_i: X_i \sim Y_i'$ be the corresponding isomorphism of $S'$-schemes. Let $\delta: X_1 \to X_2$ be an $S'$-morphism and define $\varepsilon: Y_1' \to Y_2'$ by the commutativity of the diagram

\[
\begin{array}{ccc}
X_1 & \delta & X_2 \\
\theta_1 \sim & \sim & \theta_2 \\
Y_1 & \varepsilon & Y_2'
\end{array}
\]

Then $\varepsilon = \psi \times_S S'$ for some $S$-morphism $\psi: Y_1 \to Y_2$, if, and only if, $\varepsilon$ is invariant under $\Gamma$ (4.1), i.e., for every $\sigma \in \Gamma$, the diagram

\[
\begin{array}{ccc}
Y'_1 & 1_{Y_1} \times \sigma & Y'_1 \\
\varepsilon & & \varepsilon \\
Y'_2 & 1_{Y_2} \times \sigma & Y'_2
\end{array}
\]

commutes.

**Proof.** By [SGA1 Theorem 5.2 and comment after the statement], $\varepsilon = \psi \times_S S'$ for some $S$-morphism $\psi: Y_1 \to Y_2$, if, and only if, the diagram (which is an instance of (4.2))

\[
p_1 Y'_1 \sim p_1 Y'_1 \\
p_1(\varepsilon) \sim p_1(\varepsilon) \\
p_2 Y'_2 \sim p_2 Y'_2
\]

commutes (see the next remark). By the proposition, the latter is the case if, and only if, $\varepsilon$ is invariant under the action of $\Gamma$. □

**Remark 4.3.** In [SGA1 Theorem 5.2 and comment after the statement], the schemes $p_1^* Y'_1$ and $p_2^* Y'_2$ have been identified via $c_{Y_1}$. Thus the condition in [loc.cit.] that $p_1^*(\varepsilon)$ and $p_2^*(\varepsilon)$ be equal is indeed equivalent to the commutativity of diagram (4.6).

**References**

[BLR] Bosch, S., Lütkebohmert, W. and Raynaud, M.: Néron models. Erg. der Math. Grenz. 21, Springer-Verlag, Berlin, 1990.

[GW] Görtz, U. and Wedhorn, T.: Algebraic geometry I. Schemes with examples and exercises. Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010.

[SGA1] Grothendieck, A.: Revêtements étalés et groupe fondamental (SGA 1). Séminaire de géométrie algébrique du Bois Marie 1960–61. Lecture Notes in Math. 224, Springer-Verlag 1971.
[EGA I](new) Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique I. Le langage des schémas. Grundlehren Math. Wiss. **166**, Springer-Verlag, Berlin, 1971.

[J] Jahnel, J.: The Brauer-Severi variety associated with a central simple algebra (unpublished). Available at https://www.math.uni-bielefeld.de/lag/man/052.pdf

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