MIXED INEQUALITIES OF FEFFERMAN-STEIN TYPE FOR SINGULAR INTEGRAL OPERATORS

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Abstract
We give Fefferman-Stein type inequalities related to mixed estimates for Calderón-Zygmund operators. More precisely, given \( \delta > 0 \), \( q > 1 \), \( \varphi(z) = z(1 + \log^+ z)^\delta \), a nonnegative and locally integrable function \( u \) and \( v \in RH_\infty \cap A_q \), we prove that the inequality

\[
\int \left\{ x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t \right\} \leq C \frac{\int_{\mathbb{R}^n} |f| (M_{p,\varphi^{1-q'}} u)(\Psi(v))}{t}
\]

holds with \( \Psi(z) = z^{p' + 1 - q'} \chi_{[0,1]}(z) + z^{p'} \chi_{[1,\infty)}(z) \), for every \( t > 0 \) and every \( p > \max\{ q, 1 + 1/\delta \} \). This inequality provides a more general version of mixed estimates for Calderón-Zygmund operators proved in [6]. It also generalizes the Fefferman-Stein estimates given in [17] for the same operators. We further get similar estimates for operators of convolution type with kernels satisfying an \( L^p \)–Hörmander condition, generalizing some previously known results which involve mixed estimates and Fefferman-Stein inequalities for these operators.

Keywords Calderón-Zygmund operators · Young functions · Muckenhoupt weights

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Introduction and main results

In 1985, E. Sawyer proved an endpoint estimate on the real line for the Hardy-Littlewood maximal operator \( M \) which involved two different weights (see [20]). More precisely, if \( u, v \in A_1 \) then the inequality

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\[ uv \left( \left\{ x \in \mathbb{R} : \frac{M(f)(x)}{v(x)} > t \right\} \right) \leq C \int \frac{|f(x)|u(x)v(x)}{t} dx \]  

(1.1) holds for every positive \( t \). This estimate, which can be seen as the weak \((1, 1)\) type inequality of \( Sf = M(fv)/v \) with respect to the measure \( d\mu(x) = u(x)v(x) dx \), allowed to give an alternative proof of the boundedness of \( M \) in \( L^p(w) \) when \( w \in A_p \), a result due to Muckenhoupt in [15]. Different extensions of (1.1) were obtained, see for example [6, 16] and [11] for \( M \) and Calderón-Zygmund operators (CZO), [3] for commutators of CZO, [4] for fractional operators, [1] and [2] for generalized maximal operators associated to Young functions.

On the other hand, in [9], it was shown that if \( w \) is a nonnegative and locally integrable function and \( 1 < p < \infty \) then

\[
\int_{\mathbb{R}^n} (Mf(x))^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p Mw(x) dx,
\]

where \( C \) depends only on \( p \). We shall refer to this type of estimate as Fefferman-Stein inequalities. Regarding CZO, the first result due to Córdoba and Fefferman [5] established that if \( w \) is a nonnegative and locally integrable function then

\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C_{p,r} \int_{\mathbb{R}^n} |f(x)|^p M(M,w)(x) dx,
\]

for \( 1 < p, r < \infty \). Later on, Wilson improved the estimate above in [21] for rough singular integrals, obtaining the operator \( M^2 \) on the right-hand side, which is pointwise lesser than \( M(M) \). Other estimates for CZO were proved by Pérez in [17], where the maximal operators involved are related to Young functions satisfying certain properties (see Theorem 4).

Concerning the Fefferman-Stein estimates for mixed inequalities, in [3], we prove a result involving a radial power function \( v \) that fails to be locally integrable in \( \mathbb{R}^n \) and a nonnegative function \( u \) given by

\[
uw \left( \left\{ x \in \mathbb{R}^n : \frac{M\phi(f)(x)}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f| v}{t} \right) Mu,
\]

where \( \Phi \) is a Young function of \( L \log L \) type and \( w \) depends on \( v \) and \( \Phi \). This result generalizes a previous estimate proved in [16], where the authors set a counterexample for the Hardy-Littlewood maximal operator \( M \), showing that the estimate above fails to be true for pairs \((u, M^2 u)\) and \( \nu \) in \( RH^\infty \).

In this paper, we study Fefferman-Stein inequalities for mixed estimates involving CZO. We shall be dealing with a linear operator \( T \), bounded on \( L^2 = L^2(\mathbb{R}^n) \) and such that for \( f \in L^2 \) with compact support we have the representation

\[
Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy, \ x \notin \text{supp} f,
\]

(1.2)

where \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) is a measurable function defined away from the origin. We say that \( T \) is a CZO if \( K \) is a standard kernel, which means that it satisfies a size condition given by

\[
|K(x)| \leq \frac{1}{|x|^n},
\]

and the following smoothness condition also holds

\[
|K(x-y) - K(x-z)| \leq \frac{|x-z|}{|x-y|^{n+1}}, \text{ if } |x-y| > 2|x-z|.
\]

(1.3)

The notation \( A \lesssim B \) means, as usual, that there exists a positive constant \( c \) such that \( A \leq cB \). When \( A \lesssim B \) and \( B \lesssim A \), we shall write \( A \approx B \).

We are now in a position to state our main results.

**Theorem 1** Let \( 0 \leq u \in L^1_{\text{log}^1} q > 1 \), and \( v \in RH^\infty \cap A_q \). Let \( T \) be a CZO, \( \delta > 0 \), and \( \varphi(z) = z(1 + \log^+ z)^\delta \). Then for every \( p > \max\{q, 1 + 1/\delta\} \), the inequality
\[
uv \left( \left\{ x \in \mathbb{R}^n : \frac{|T(f)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} |f(x)| M_{\varphi,1-q} u(x) M(\Psi)(v)(x) \, dx
\]

holds for every positive \( t \) and every bounded function \( f \) with compact support, where \( \Psi(z) = z^{p' + 1 - q'} X_{[0,1]}(z) + z^{q'} X_{[1,\infty]}(z) \).

When \( v = 1 \), the theorem above gives the result proved in [17] for CZO. It also corresponds to the case \( m = 0 \) of the commutator operator given in [19]. This type of estimate is not only an extension of the well-known weak endpoint inequality for the operator \( T \) but also provides an estimate of the type \((u, \tilde{M}u)\) for mixed inequalities, where \( \tilde{M} \) is an adequate maximal function.

We shall also consider operators as in (1.2) associated to kernels with less regularity properties, which appeared in the study of Coifman type estimates for these operators. It was proved in [14] that the classical Hörmander condition on the kernel fails to achieve the desired estimate (see also [13]). We now introduce the notation related to this topic. Given a Young function \( \varphi \), we denote

\[
\|f\|_{\varphi,|\cdot|^{-s}} = \left\| f \chi_{|y|^{-s}} \right\|_{\varphi,B(0,2s)}
\]

where \( |x| \sim s \) means that \( s \leq |x| \leq 2s \) and \( \| \cdot \|_{\varphi,B(0,2s)} \) denotes the Luxemburg average over the ball \( B(0, 2s) \) (see the “Preliminaries and basic definitions” section for further details).

We say that \( K \) satisfies the \( L^p \)-Hörmander condition, and we denote it by \( K \in H_{\varphi} \), if there exist constants \( c \geq 1 \) and \( C_{\varphi} > 0 \) such that the inequality

\[
\sum_{k=1}^{\infty} (2^k R)^p \| K(\cdot - y) - K(\cdot) \|_{\varphi,|\cdot|^{-2^k R}} \leq C_{\varphi}
\]

holds for every \( y \in \mathbb{R}^n \) and \( R > c |y| \). When \( \varphi(t) = t' \), \( r' \geq 1 \), we write \( H_{\varphi} = H_r \).

In [12], the authors prove certain Fefferman-Stein inequalities for these types of operators. Concretely, if \( \Phi \) is a Young function and there exists \( 1 < p < \infty \) and Young functions \( \eta, \varphi \) such that \( \eta \in B_{p'} \) and \( \eta^{-1}(z) \varphi^{-1}(z) \leq \Phi^{-1}(z) \) for \( z \geq z_0 \geq 0 \), then the inequality

\[
w(\{ x \in \mathbb{R}^n : |T(f)(x)| > r \}) \leq C \int_{\mathbb{R}^n} |f(x)| M_{\varphi,\eta} w(x) \, dx
\]

holds with \( \varphi, \eta(z) = \varphi(z^{1/p}) \), and where \( \Phi \) is the complementary Young function of \( \Phi \) (see the “Preliminaries and basic definitions” section).

Given \( 0 < p < \infty \), we say that a Young function \( \varphi \) has an upper type \( p \) if there exists a positive constant \( C \) such that \( \varphi(st) \leq C s^p \varphi(t) \), for every \( s \geq 1 \) and \( t \geq 0 \). If \( \varphi \) has an upper type \( p \) then it has an upper type \( q \), for every \( q \geq p \). We also say that \( \varphi \) has a lower type \( p \) if there exists \( C > 0 \) such that the inequality \( \varphi(st) \leq C s^p \varphi(t) \) holds for every \( 0 \leq s \leq 1 \) and \( t \geq 0 \). When \( \varphi \) has a lower type \( p \), it also has a lower type \( q \) for every \( q \leq p \).

For operators associated to kernels satisfying a regularity of Hörmander type, we have the following result.

**Theorem 2** Let \( \Phi \) be a Young function such that \( \Phi \) has an upper type \( r \) and a lower type \( s \), for some \( 1 < s < r \). Let \( T \) be an operator as in (1.2), with kernel \( K \in H_{\varphi} \). Assume that there exist \( 1 < p < p' \) and Young functions \( \eta, \varphi \) such that \( \eta \in B_{p'} \) and \( \eta^{-1}(z) \varphi^{-1}(z) \leq \Phi^{-1}(z) \) for every \( z \geq z_0 \). If \( 0 \leq u \in L^1_{\text{loc}} \) and \( v \in \mathbb{R}_{\text{loc}} \cap A_q \) with \( q = 1 + (p - 1)/r \) then the inequality

\[
uv \left( \left\{ x \in \mathbb{R}^n : \frac{|T(f)(x)|}{v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} |f(x)| M_{\varphi,1-q} u(x) M(\Psi)(v)(x) \, dx
\]

holds for every \( t > 0 \), where \( \varphi, \eta(z) = \varphi(z^{1/p}) \) and \( \Psi(z) = z^{p' + 1 - q'} X_{[0,1]}(z) + z^{q'} X_{[1,\infty]}(z) \).

We now give an example in order to show that the class of functions satisfying the hypotheses on Theorem 2 is non-empty. Let \( r > 1, 1 < p < p', \delta \geq 0, 0 < \varepsilon < \min\{r - 1, p' - r\} \), and \( \Phi(t) = t^{-\varepsilon + \delta}(1 + \log t)^\delta \). Observe that \( \Phi \approx \tilde{\Phi} \), so \( \Phi \) is
a Young function since it is the complementary of a Young function. We also take \( \eta(t) = t^{p'-\tau} \), with \( 0 < \tau < p' - r - \epsilon \). Then, we have that \( \Phi \) has an upper type \( r \) and a lower type \( s \) for every \( 1 < s < r \), and \( \eta \in B_{p'} \). Furthermore,

\[
\eta^{-1}(t) \approx t^{1/(p'-\tau)} \quad \text{and} \quad \Phi^{-1}(t) \approx t^{1/(p'-s)(\log t)^{-1/(r-\epsilon)}} \quad \text{for } t \geq e.
\]

Therefore, if we take \( \varphi(t) = t^q(1 + \log^+ t)^{q/(r-\epsilon)} \) where \( 1/q = 1/(r-\epsilon) - 1/(p' - \tau) \), we have the relation \( \eta^{-1}(t)\varphi^{-1}(t) \approx \Phi^{-1}(t) \), for \( t \geq e \).

Theorem 2 can be seen as a generalization of (1.5), corresponding to \( \nu = 1 \).

**Remark 1** From the hypothesis, we have that \( \varphi_p(t) \geq \Phi(t) \geq t \) for \( t \geq t_0 \). The second inequality is immediate since \( \Phi \) is a Young function. For the first one, given \( t \geq t_0 \), we can see that \( \eta(t) \leq \varphi \) since \( \eta \in B_{p'} \). This implies that \( t^{1/p'} \varphi^{-1}(t) \leq t \) or equivalently, \( \varphi^{-1}(t) \leq t^{1/p} \). Then, again by hypothesis

\[
\Phi^{-1}(t) \geq t^{1/p'} \varphi^{-1}(t) \geq (\varphi^{-1}(t))^{p/p'} \varphi^{-1}(t) = (\varphi^{-1}(t))^p = \varphi_p^{-1}(t),
\]

which directly implies that \( \varphi_p(t) \geq \Phi(t) \). These relations will be useful in the proof of Theorem 2.

The article is organized as follows: in the “Preliminaries and basic definitions” section, we give the preliminaries and definitions. The “Auxiliary results” section contains some technical results that will be useful for the proof of the main theorems in the “Proof of the main results” section.

**Preliminaries and basic definitions**

By a weight \( w \), we understand a locally integrable function such that \( 0 < w(x) < \infty \) for almost every \( x \). Given \( 1 < p < \infty \), the Muckenhoupt \( A_p \) class is defined as the collection of weights \( w \) such that the inequality

\[
\left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} \leq C
\]

holds for some positive constant \( C \) and every cube \( Q \) in \( \mathbb{R}^n \) with sides parallel to the coordinate axes. When necessary, we shall denote by \( A_Q \) and \( \ell(Q) \) the center and the side-length of the cube \( Q \), respectively.

We say that \( w \) belongs to \( A_1 \) if there exists a positive constant \( C \) such that the inequality

\[
\frac{1}{|Q|} \int_Q w \leq C w(x)
\]

holds for every cube \( Q \) and almost every \( x \in Q \). Finally, for \( p = \infty \), the \( A_{\infty} \) class is understood as the collection of all \( A_p \) classes, that is, \( A_{\infty} = \bigcup_{p \geq 1} A_p \).

Given \( 1 \leq p < \infty \), the smallest constant for which the corresponding inequality above holds is denoted by \( [w]_{A_p} \). It is well-known that \( A_p \) classes are increasing in \( p \), that is, \( A_p \subset A_q \) for \( p < q \) and that every \( w \in A_p \) is doubling, that is, there exists a constant \( C > 1 \) such that \( w(2Q) \leq C w(Q) \), for every cube \( Q \).

For further properties and details about Muckenhoupt classes, see, for example, [8] or [10].

An important property of Muckenhoupt weights is that they satisfy a reverse Hölder condition. Given a real number \( s > 1 \), we say that \( w \in RH_s \) if the inequality

\[
\left( \frac{1}{|Q|} \int_Q w^s \right)^{1/s} \leq C \frac{1}{|Q|} \int_Q w,
\]

holds for some positive constant \( C \) and every cube \( Q \). The \( RH_{\infty} \) class is defined as the set of weights that verify

\[
\sup_Q w \leq C \frac{1}{|Q|} \int_Q w.
\]
for some $C > 0$ and every cube $Q$. Given $1 < s \leq \infty$, the smallest constant for which the corresponding inequality above holds is denoted by $[w]_{\text{RH}}$. It is well-known that reverse Hölder classes are decreasing on $s$, that is, $\text{RH}_\infty \subset \text{RH}_r \subset \text{RH}_s$ for every $1 < t < s$.

The next lemma establishes some useful properties of $\text{RH}_\infty$ weights that we shall use later. A proof can be found in [7].

**Lemma 3** Let $w$ be a weight.

(a) If $w \in \text{RH}_\infty \cap A_p$, then $w^{1-p'} \in A_1$;
(b) if $w \in \text{RH}_\infty$, then $w^r \in \text{RH}_\infty$ for every $r > 0$;
(c) if $w \in A_1$, then $w^{-1} \in \text{RH}_\infty$.

We say that $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is convex, strictly increasing and also satisfies $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The generalized inverse $\varphi^{-1}$ of $\varphi$ is defined by

$$
\varphi^{-1}(t) = \inf \{ s \geq 0 : \varphi(s) \geq t \},
$$

where we understand $\inf \emptyset = \infty$. When $\varphi$ is a Young function that verifies $0 < \varphi(t) < \infty$ for every $t > 0$, it can be seen that $\varphi$ is invertible and the generalized inverse of $\varphi$ is its actual inverse function. Throughout this paper, we shall deal with this type of Young functions.

The complementary function of the Young function $\varphi$ is denoted by $\tilde{\varphi}$ and defined for $t \geq 0$ by

$$
\tilde{\varphi}(t) = \sup \{ ts - \varphi(s) : s \geq 0 \}.
$$

It is well-known that $\tilde{\varphi}$ is also a Young function and further

$$
\varphi^{-1}(t) \tilde{\varphi}^{-1}(t) \approx t. \quad (2.1)
$$

Given a Young function $\varphi$ and a Muckenhoupt weight $w$, the generalized maximal operator $M_{\varphi, w}$ is defined, for $f$ such that $\varphi(f) \in L^1_{\text{loc}}$, by

$$
M_{\varphi, w}f(x) = \sup_{Q} \| f \|_{\varphi, Q, w},
$$

where $\| f \|_{\varphi, Q, w}$ is an average of Luxemburg type given by the expression

$$
\| f \|_{\varphi, Q, w} = \inf \left\{ \lambda > 0 : \frac{1}{w(Q)} \int_Q \varphi \left( \frac{|f(y)|}{\lambda} \right) w(y) \, dy \leq 1 \right\},
$$

and this infimum is actually a minimum, since it is easy to see that

$$
\frac{1}{w(Q)} \int_Q \varphi \left( \frac{|f(y)|}{\| f \|_{\varphi, Q, w}} \right) w(y) \, dy \leq 1.
$$

When $w = 1$, we simply write $\| f \|_{\varphi, Q}$ and $M_{\varphi, w} = M_\varphi$. When $\varphi(t) = t$, the operator $M_{\varphi, w}$ is just the classical Hardy-Littlewood maximal function with respect to the measure $d\mu(x) = w(x) \, dx$.

Notice that when $\varphi$ is a Young function which has both a lower type $r_1$ and an upper type $r_2$ with $1 < r_1 < r_2$, we have $M_{r_1} \lesssim M_\varphi \lesssim M_{r_2}$.

If $\Phi, \Psi$ and $\varphi$ are Young functions satisfying

$$
\Phi^{-1}(t)\Psi^{-1}(t) \lesssim \varphi^{-1}(t)
$$

for $t \geq t_0 \geq 0$, then

$$
\varphi(st) \lesssim \Phi(s) + \Psi(t), \quad (2.2)
$$
for every \( s, t \geq 0 \). As a consequence of this estimate, we obtain the generalized Hölder inequality
\[
\|fg\|_{\varphi_E} \lesssim \|f\|_{\varphi_E} \|g\|_{\varphi_E},
\]
for every doubling weight \( w \) and every measurable set \( E \) such that \( |E| < \infty \). Particularly, in views of (2.1), we get that
\[
\int_{\mathbb{R}^n} |f|^p \varphi(x) \, dx \lesssim \|f\|_{\varphi} \|\varphi\|_{\psi} \int_{\mathbb{R}^n} |g|^q \varphi(x) \, dx.
\]
We say that a Young function \( \varphi \) belongs to \( B_p \), \( p > 1 \), if there exists a positive constant \( c \) such that
\[
\int_{c}^{\infty} \varphi(t) \, dt \leq \infty.
\]
These classes were introduced in [18] and played a fundamental role in the Fefferman-Stein estimates for CZO.

**Auxiliary results**

In this section, we state and prove some estimates that will be useful in the proof of our main results. The theorem below establishes a strong type Fefferman-Stein estimate for CZO.

**Theorem 4** ([17]) Let \( T \) be a CZO, \( 1 < p < \infty \), and \( \varphi \) a Young function that verifies \( \varphi \in B_p \). Then, there exists a positive constant \( C \) such that for every weight \( w \) we have that
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_{\varphi} w(x) \, dx.
\]

The following result states a Coifman type estimate for operators associated to kernels with less regularity.

**Theorem 5** ([13]) Let \( \Phi \) be a Young function and \( T \) as in (1.2), with kernel \( K \in H_{\Phi} \). Then for every \( 0 < p < \infty \) and \( w \in A_{\infty} \), the inequality
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} (M_{\Phi} f(x))^p w(x) \, dx
\]
holds for every \( f \) such that the left hand-side is finite.

The following lemma will be an important tool in the sequel.

**Lemma 6** Let \( \varphi \) be a Young function, \( w \) a doubling weight, \( f \) such that \( M_{\varphi,w} f(x) < \infty \) almost everywhere and \( Q \) be a fixed cube. Then,
\[
M_{\varphi,w}(f\chi_{R\mathbb{R}^n \setminus RQ})(x) \approx M_{\varphi,w}(f\chi_{R\mathbb{R}^n \setminus RQ})(y)
\]
for every \( x, y \in Q \), where \( R = 4\sqrt{n} \).

**Proof** Fix \( x \) and \( y \) in \( Q \) and let \( Q' \) be a cube containing \( x \). We can assume that \( Q' \cap \mathbb{R}^n \setminus RQ \neq \emptyset \), since \( \|f\|_{\varphi,Q,w} = 0 \) otherwise. We shall prove that
\[
\mathcal{L}(Q') \geq \frac{3}{4} \mathcal{L}(Q).
\]
Indeed, let \( B_{Q'} = B(x_Q', \mathcal{L}(Q')/2) \) and \( B_Q = B(x_Q, \mathcal{L}(Q)/\sqrt{n}/2) \). Observe that \( Q' \cap \mathbb{R}^n \setminus RQ \neq \emptyset \) implies that \( B_{Q'} \cap \mathbb{R}^n \setminus RQ \neq \emptyset \). If (3.1) does not hold, for \( z \in B_{Q'} \), we would have
\begin{equation*}
|z - x_Q| \leq |z - x| + |x - x_Q| \\
\leq \sqrt{n} \ell(Q') + \frac{\sqrt{n}}{2} \ell(Q) \\
< \left( \frac{3\sqrt{n}}{4} + \frac{\sqrt{n}}{2} \right) \ell(Q) \\
< \frac{R}{2} \ell(Q).
\end{equation*}

which yields \( B_Q \subseteq B(x_Q, R\ell(Q)/2) \subseteq RQ \), a contradiction. Therefore, (3.1) holds. Then, we have that \( Q \subseteq RQ' \). Indeed, if \( z \in Q \), we get

\begin{equation*}
|z - x_Q| \leq |z - x| + |x - x_Q| \\
\leq \sqrt{n} \ell(Q) + \frac{\sqrt{n}}{2} \ell(Q') \\
\leq \left( \frac{4\sqrt{n}}{3} + \frac{\sqrt{n}}{2} \right) \ell(Q') \\
< 2\sqrt{n} \ell(Q') = \frac{R}{2} \ell(Q'),
\end{equation*}

which implies that \( Q \subseteq B(x_Q, \ell(Q')/2) \subseteq RQ' \). Thus,

\begin{equation*}
\frac{1}{w(Q')} \int_{Q'} \varphi \left( \frac{|f|}{\|f\|_{\varphi,RQ'},w} \right) w \leq \frac{w(RQ')}{w(Q')} \frac{1}{w(Q')} \int_{RQ'} \varphi \left( \frac{|f|}{\|f\|_{\varphi,RQ'},w} \right) w \leq C,
\end{equation*}

since \( w \) is doubling. This yields \( \|f\|_{\varphi,Q',w} \leq C \|f\|_{\varphi,RQ',w} \leq C M_{\varphi,w} f(y) \), for every \( Q' \) containing \( x \), which finally implies that \( M_{\varphi,w} f(x) \leq C M_{\varphi,w} f(y) \). The other inequality can be achieved analogously by interchanging the roles of \( x \) and \( y \). \qed

The following result gives a relation between \( M_{\varphi,w} \) and the unweighted version \( M_{\varphi} \), when \( w \) belongs to the \( A_1 \) class.

**Lemma 7** Let \( w \in A_1 \) and \( \varphi \) be a Young function.

(a) There exists a positive constant \( C \) such that

\( M_{\varphi,w} f(x) \leq C M_{\varphi,w} f(x) \),

for every \( f \) such that \( M_{\varphi,w} f(x) < \infty \) a.e.;

(b) If \( w^r \in A_1 \) for some \( r > 1 \), then

\( M_{\varphi,w} f(x) \leq C M_{\varphi,w} f(x) \),

for every \( f \) such that \( M_{\varphi,w} f(x) < \infty \) a.e.

**Proof** For (a), fix \( x \) and a cube \( Q \ni x \). Since \( w \in A_1 \), we have that

\begin{align*}
\frac{1}{|Q|} \int_Q \varphi \left( \frac{|f|}{\lambda} \right) w^r & = \frac{w(Q)}{|Q|} \frac{1}{w(Q)} \int_Q \varphi \left( \frac{|f|}{\lambda} \right) w^{-1} \\
& \leq \left( \sup_Q w^{-1} \right) [w]_{A_1} \left( \inf_Q w \right) \frac{1}{w(Q)} \int_Q \varphi \left( \frac{|f|}{\lambda} \right) w \\
& \leq [w]_{A_1},
\end{align*}
Lemma 8 Let $\delta \geq 0$ and $\varphi(t) = t(1 + \log^+ t)^\delta$. For every $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon, \delta)$ such that
\[
\varphi(t) \leq Ct^{1+\varepsilon}, \quad \text{for } t \geq 1.
\]
Moreover, the constant $C$ can be taken as $C = \max \left\{ 1, (\delta/\varepsilon)^\delta \right\}$.

Proof of the main results

Proof (Proof of Theorem 1) Let us first assume that $u$ is bounded. We fix $t > 0$ and perform the Calderón-Zygmund decomposition of $f$ at level $t$ with respect to the measure $d\mu(x) = v(x)\, dx$. Let us observe that $v \in RH_{\infty}$, so that $v \in A_{\infty}$ and therefore $\mu$ is a doubling measure. We obtain a collection of disjoint dyadic cubes $\{Q_j\}_{j=1}^\infty$ satisfying $t < f_Q^* \leq Ct$, where $f_Q^*$ is given by
\[
\frac{1}{v(Q_j)} \int_{Q_j} f(y)v(y)\, dy.
\]
If we write $\Omega = \bigcup_{j=1}^\infty Q_j$, then we have that $f(x) \leq t$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. We also decompose $f$ as $f = g + h$, where
\[
g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega; \\ f_Q^*, & \text{if } x \in Q_j,
\end{cases}
\]
and $h(x) = \sum_{j=0}^\infty h_j(x)$, with
\[
h_j(x) = \left( f(x) - f_Q^* \right) 1_{Q_j}(x).
\]
It follows that $g(x) \leq Ct$ almost everywhere, every $h_j$ is supported on $Q_j$ and
\[
\int_{Q_j} h_j(y)v(y)\, dy = 0. \quad (4.1)
\]
Let $Q_j^* = R Q_j$, where $R = 4\sqrt{n}$ as in Lemma 6 and $\Omega^* = \bigcup_j Q_j^*$. We proceed as follows.
\[
\mathcal{M}(v_{\Omega}^{1-p'}) \leq C \mathcal{M}_{\phi}(u_{\Omega}^{1-p'}),
\]

Let us estimate \( M_{\phi}(u_{\Omega}^{1-p'}) \). Recall that we have \( v \in \text{RH}_{\infty} \cap A_p \) since \( p > q \), so by item (a) of Lemma 3 we get \( v_{\Omega}^{1-p'} \in A_1 \).

We shall prove that there exists a positive constant \( C \) verifying
\[
M_{\phi}(u_{\Omega}^{1-p'}) = C \mathcal{M}_{\phi, v^{1-p'}}(u_{\Omega}^{1-p'}(x) v^{-q'}(x) \Psi(v(x))) \quad \text{for a.e. } x. \tag{4.2}
\]

Fix \( x \) and \( Q \) a cube containing \( x \). By taking \( \lambda = \|u_{\Omega}\|_{\phi, v^{1-p'}} \), we have that
\[
\frac{1}{|Q|} \int_{Q} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\lambda} \right) = \frac{1}{|Q|} \int_{Q \cap \{v^{1-p'} \leq \epsilon\}} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\lambda} \right) + \frac{1}{|Q|} \int_{Q \cap \{v^{1-p'} > \epsilon\}} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\lambda} \right)
\]
\[= I_1 + I_2. \]

By using that \( \varphi \) is submultiplicative and Lemma 7, for \( I_1 \), we have that
\[
I_1 \leq \frac{C}{|Q|} \int_{Q} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\|u_{\Omega}\|_{\phi, v^{1-p'}}} \right) \leq \frac{C}{|Q|} \int_{Q} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\|u_{\Omega}\|_{\phi, Q}} \right) \leq C.
\]

In order to estimate \( I_2 \), let \( \epsilon = (q' - 1)/(p' - 1) - 1 \). Observe that \( \epsilon > 0 \) since \( p' < q' \). By applying Lemma 8, we get that
\[
I_2 \leq \frac{C}{|Q|} \int_{Q} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\|u_{\Omega}\|_{\phi, Q}} \right) \leq \frac{C}{|Q|} \int_{Q} \varphi \left( \frac{u_{\Omega}^{1-p'}}{\|u_{\Omega}\|_{\phi, Q}} \right) \leq C[\|v^{1-q'}\|_{A_1}] \max \{1, v^{1-q'}(x)\},
\]

since \( v^{1-q'} \in A_1 \). Therefore, we can conclude that
\[ ||u^\ast v^{1-p'}||_{\varphi,Q} \leq C\lambda \max \left\{ 1, v^{1-q'}(x) \right\} \]

\[ = C \max \left\{ 1, v^{1-q'}(x) \right\} ||u^\ast||_{\varphi,Q,\varphi^{1-q'}} \]

\[ \leq CH(x)M_{\varphi,\varphi^{1-q'}}u^\ast(x), \]

for every cube \( Q \) that contains \( x \) and where \( H(x) = \max \left\{ 1, v^{1-q'}(x) \right\} \). This finally yields

\[ M_{\varphi}(u^\ast v^{1-p'}) (x) \leq CH(x)M_{\varphi,\varphi^{1-q'}}(x). \]

To obtain (4.2), it only remains to show that \( H(x)v^{q'}(x) \leq \Psi(v(x)). \) This can be achieved by noting that when \( v \leq 1 \) we have \( v^{q'}H = v^{q'+1-q'} \), and we get \( v^{q'}H = v^{q'} \) otherwise.

We now return to the estimate of \( I \). We have that

\[ I \leq \frac{C}{v^{q'}} \int_{\mathbb{R}^n} |g|v^{q'}(M_{\varphi,\varphi^{1-q'}}u^\ast)\Psi(v) \]

\[ \leq \frac{C}{t} \int_{\mathbb{R}^n} |f|(M_{\varphi,\varphi^{1-q'}}u^\ast)\Psi(v) \]

\[ \leq \frac{C}{t} \int_{\mathbb{R}^n} |f|(M_{\varphi,\varphi^{1-q'}}u^\ast)M(\Psi(v)) + \frac{C}{t} \int_{\Omega} |f|_{Q}(M_{\varphi,\varphi^{1-q'}}u^\ast)\Psi(v). \]

Let \( u^\ast_j = u_{\chi_{R^n \setminus \mathbb{K}_{Q_j}}} \). For the integral over \( \Omega \), we shall use Lemma 6 to obtain

\[ \frac{C}{t} \int_{\Omega} |f|_{Q_j}(M_{\varphi,\varphi^{1-q'}}u^\ast)\Psi(v) \leq \frac{C}{t} \sum_j \int_{Q_j} |f|_{Q_j}(M_{\varphi,\varphi^{1-q'}}u^\ast)\Psi(v) \]

\[ \leq \frac{C}{t} \sum_j \inf_{Q_j} \left( M_{\varphi,\varphi^{1-q'}}u^\ast \right) (\Psi(v))(Q_j) v(Q_j) \int_{Q_j} |f| \]

\[ \leq \frac{C}{t} [v]_{RH_w} \left( \inf_{Q_j} \left( M_{\varphi,\varphi^{1-q'}}u^\ast \right) \right) (\Psi(v))(Q_j) \int_{Q_j} |f| \]

\[ \leq \frac{C}{t} [v]_{RH_w} \sum_j \int_{Q_j} |f|(M_{\varphi,\varphi^{1-q'}}u^\ast)M(\Psi(v)) \]

\[ \leq \frac{C}{t} [v]_{RH_w} \int_{\Omega} |f|(M_{\varphi,\varphi^{1-q'}}u^\ast)M(\Psi(v)), \]

so we achieved the desired estimate for \( I \).

For \( II \), by virtue of Lemma 3 and the fact that \( v \) is doubling, we have

\[ uv(\Omega_j) \leq v^{1-q'}(\Omega_j)||u||_{\varphi,\Omega_j,\varphi^{1-q'}} \left[ \frac{1}{v^{1-q'}(\Omega_j)} \int_{\Omega_j} \varphi \left( \frac{u}{||u||_{\varphi,\Omega_j,\varphi^{1-q'}}} \right) v^{1-q'} \right] (\sup_{\Omega_j}v^{q'}) \]

\[ \leq [v^{q'}]_{RH_w} \frac{v^{1-q'}(\Omega_j)}{|\Omega_j|} v^{q'}(\Omega_j)||u||_{\varphi,\Omega_j,\varphi^{1-q'}} \]

\[ \leq C [v^{q'}]_{RH_w} [u^{1-q'}]_{A_1} v(\Omega_j)||u||_{\varphi,\Omega_j,\varphi^{1-q'}} \]

\[ \leq \frac{C}{t} \int_{Q_j} |f|v(M_{\varphi,\varphi^{1-q'}}u) \]

\[ \leq \frac{C}{t} \int_{Q_j} |f|(M_{\varphi,\varphi^{1-q'}}u)M(\Psi(v)), \]

where in the last inequality we have used that \( \Psi(s) \geq s \). Therefore,
For every fixed $J$, we denote $A_{j,k} = \{ x : 2^{j-1}r_j < |x - x_j| \leq 2^jr_j \}$, where $r_j = R'(Q_j)/2$ and use the integral representation of $T$ given by (1.2). By combining (4.1) with the smoothness condition (1.3) on $K$, we get

\[
\text{III} \leq u^J \left( \sum_{j} \frac{|T(h_jv)|}{v} > t \right)
\]

\[
\leq C \sum_{j} \int_{Q_j^c} |T(h_jv)(x)|u(x) \, dx
\]

\[
\leq C \sum_{j} \int_{Q_j^c} |h_j(y)v(y)(K(x) - K(x - x_j))|u(x) \, dx
\]

\[
\leq C \sum_{j} \int_{Q_j^c} |h_j(y)v(y)| \int_{Q_j^c} |K(x) - K(x - x_j)|u(x) \, dx\, dy
\]

\[
\leq C \sum_{j} \int_{Q_j} |h_j(y)v(y)| \sum_{k=1}^{\infty} \int_{A_{j,k}} |K(x) - K(x - x_j)|u(x) \, dx\, dy
\]

For every fixed $y \in Q_j$, we have that

\[
\sum_{k=1}^{\infty} \int_{A_{j,k}} \frac{2^{-k}}{|x - x_j|^{n+1}} u^J(x) \, dx \leq C \sum_{k=1}^{\infty} \frac{\sqrt{\lambda}(Q_j) 2^{-1}}{2^{2jr_j}n} \int_{B(x_j, 2r_j)} u^J(x) \, dx
\]

\[
\leq CMu^J(y) \sum_{k=1}^{\infty} 2^{-k}
\]

Therefore, by Lemma 6, we obtain

\[
\text{III} \leq C \sum_{j} \int_{Q_j} |f|^v \left( \inf_{Q_j} Mu^J \right) + C \sum_{j} \int_{Q_j} |f|^v \left( \inf_{Q_j} Mu^J \right)
\]

\[
= A + B.
\]

Applying Lemma 7, we have that $Mu \leq M_{\phi, \nu} u \leq CM_{\phi, \nu, \nu - \nu} u$ and this yields

\[
A \leq C \int_{\Omega} |f|(M_{\phi, \nu, \nu - \nu} u)M(\Psi(v)).
\]

On the other hand,
\[
B \leq \frac{C}{t} \sum_j \int_{Q_j} |f| \left( \inf_{Q_j} M_{q_j}^* u \right) \\
\leq \frac{C}{t} \sum_j \int_{Q_j} |f| M_{p_j} u \\
\leq \frac{C}{t} \int_{\mathbb{R}^n} |f| \left( M_{\varphi, p_j}^* u \right) M(\Psi(v)).
\]

This completes the proof when \( u \) is bounded, with a constant \( C \) that does not depend on \( u \). For the general case, given \( u \) we set \( u_N(x) = \min\{u(x), N\} \) for every \( N \in \mathbb{N} \). Then, we have that

\[
u N \left( \left\{ x \in \mathbb{R}^n : \left| \frac{T(fv)(x)}{v(x)} \right| > t \right\} \right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| \left( M_{\varphi, p_j}^* u_N \right) M(\Psi(v))
\]

for every \( N \in \mathbb{N} \) and with a positive constant \( C \) that does not depend on \( N \). Since \( u_N \not\sim u \), the monotone convergence theorem allows us to show that the estimate for \( u \) also holds. \( \square \)

**Proof** (Proof of Theorem 2) First, we shall consider the case \( u \) bounded. Fix \( t > 0 \) and, as in the proof of Theorem 1, perform the Calderón-Zygmund decomposition of \( f \) at level \( t \) with respect to the measure \( d\mu(x) = v(x) \, dx \). Therefore, we obtain a collection of disjoint dyadic cubes \( (Q_j)_j \in \mathbb{N} \), \( \Omega \), and \( h \) as in that proof. We take \( Q_j^* = cRQ_j^* \), where \( R \) is the dimensional constant given by Lemma 6 and \( c \geq 1 \) is the constant appearing on the \( L^\Phi \)–Hörmander condition for \( K \). By using the same notation as in Theorem 1, we get

\[
u v \left( \left\{ x \in \mathbb{R}^n : \left| \frac{T(fv)(x)}{v(x)} \right| > t \right\} \right) \leq \nu v \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^*: \left| \frac{T(gv)}{v} \right| > \frac{t}{2} \right\} \right) + \nu v(\Omega^*)
\]

\[
+ \nu v \left( \left\{ x \in \mathbb{R}^n \setminus \Omega^*: \left| \frac{T(hv)}{v} \right| > \frac{t}{2} \right\} \right) = I + II + III.
\]

Since \( \Phi \) has a lower type \( s \), we have \( M_s \leq M_\Phi \). Recall that \( p' > r \) since \( p < r' \). By using the fact that \( M_s g \) is an \( A_1 \) weight for every measurable and nonnegative function \( g \) such that \( M_s g \) is finite almost everywhere, we apply Tchebychev inequality with \( p' \) and Theorem 5 in order to get

\[
I \leq \frac{C}{p'} \int_{\mathbb{R}^n} |T(gv)|^{p'} u^{1/p'} X_{\mathbb{R}^n \setminus \Omega}.
\]

\[
\leq \frac{C}{p'} \int_{\mathbb{R}^n} |T(gv)|^{p'} M_s(u^{1/p'})
\]

\[
\leq \frac{C}{p'} \int_{\mathbb{R}^n} |M_s(gv)|^{p'} M_s(u^{1/p'})
\]

\[
\leq \frac{C}{p'} \int_{\mathbb{R}^n} |g|^p v^{p'} M_s(u^{1/p'})
\]

\[
\leq \frac{C}{p'} \int_{\mathbb{R}^n} |g|^p v^{p'} M_\Phi(u^{1/p'})
\]

Notice that we could apply Theorem 5 because \( \|T(gv)\|_{L^{p'}(w)} < \infty \), where \( w = M_s(u^{1/p'}) \in A_1 \). Indeed, if we first assume \( w \in A_1 \cap L^\infty \), we get

\[
\int_{\mathbb{R}^n} |T(gv)|^{p'} w \leq \|w\|_{L^\infty} \int_{\mathbb{R}^n} |T(gv)|^{p'} \leq C \|w\|_{L^\infty} \int_{\mathbb{R}^n} |g|^p \, v^{p'} < \infty.
\]
since \( f \) is bounded with compact support and \( T \) is bounded in \( L^{p'} \) because \( K \in H_{\Phi} \subset H_1 \) (see, for example, [8]). For the general case, we can take \( w_N = \min\{w, N\} \) for every \( N \in \mathbb{N} \). Then, every \( w_N \) belongs to \( A_1 \) and \( \|w_N\|_{A_1} \leq \|w\|_{A_1} \). This allows us to deduce the inequality in Theorem 5 for \( w_N \) and \( C \) independent of \( N \). By letting \( N \to \infty \), we are done.

We proceed now to estimate \( M_{\Phi}(u^*/v^{1-p'}) (x) \). We shall prove that

\[
M_{\Phi}(u^*/v^{1-p'}) (x) \leq C \left( M_{\phi_p,v^{1/q'}}(u^*) (x) v^{1-p'} (x) \right) \varphi(v(x)) \quad \text{for a.e. } x. \tag{4.3}
\]

Fix \( x \) and \( Q \) a cube containing \( x \). By hypothesis and Lemma 3, we have that \( v^{1-q'} \) is an \( A_1 \) weight. By taking \( \lambda = \|u^*\|_{\phi_p,v^{1-q'}} \), we have that

\[
\frac{1}{|Q|} \int_Q \Phi \left( \frac{u^*/\lambda}{v^{1-p'}} \right) = \frac{1}{|Q|} \int_{Q \cap \{|v^{1-p'}| \leq 1\}} + \frac{1}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} = A + B.
\]

It is easy to see that \( A \) is bounded by a constant \( C \), since \( \Phi(z) \leq \phi_p(z) \) for large \( z \) and \( \|u^*\|_{\phi_p,Q} \leq \lambda \). In order to estimate \( B \), we shall apply the upper type of \( \Phi \) combined with (2.2), the fact that \( \eta(t) \leq C t^{p'} \) and \( t \leq \phi_p(t) \) (see Remark 1) to get

\[
B \leq \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} \Phi \left( \frac{u^*/\lambda}{v^{1-p'}} \right) = \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| \leq 1\}} + \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} = A + B.
\]

We also use (see Remark 1) to get

\[
B \leq \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} \Phi \left( \frac{u^*/\lambda}{v^{1-p'}} \right) = \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| \leq 1\}} + \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} \varphi_p \left( \frac{u^*/\lambda}{v^{1-p'}} \right) v^{1-q'} + C \left( \frac{1}{|Q|} \int_{Q \cap \{|v^{1-p'}| \leq 1\}} + \frac{C}{|Q|} \int_{Q \cap \{|v^{1-p'}| > 1\}} \varphi_p \left( \frac{u^*/\lambda}{v^{1-p'}} \right) v^{1-q'} \right)
\]

By Lemma 7, we have that

\[
\|u^*/v^{1-p'}\|_{\phi_p,Q} \leq C \max \{1, v^{1-q'}(x) \} \lambda = C \max \{1, v^{1-q'}(x) \} \|u^*\|_{\phi_p,v^{1-q'}}
\]

and by proceeding similarly as in the proof of Theorem 1 we can obtain (4.3). This allows us to finish the estimate of \( I \) by following similar lines as on page 11. For \( II \), we use again that \( t \leq \phi_p(t) \) combined with the fact that \( v^{1-q'} \in A_1 \). We also notice that \( p' < q' \) since \( r > 1 \), so we get \( \Psi(v) \geq v \). By following the same argument as on page 12, we get the desired bound.

We finish with the estimate of \( III \). We denote \( A_{j,k} = \{ x : 2^{k-1}r_j < |x - x_0| \leq 2^kr_j \} \), where \( r_j = cR'(Q_j)/8 \) and use the integral representation of \( T \) given by (1.2). By combining (4.1) with the \( L^p \)-Hörmander condition on (1.4) \( K \), we get
Thus, by Lemma 6, we get
\[ \text{that for every } y \in Q_j. \]

We shall prove that there exists a positive constant $C$ such that
\[ F_j(y) \leq CM_\Phi u_j^*(y), \]
for every $y \in Q_j$. Indeed, by applying generalized Hölder inequality with the functions $\Phi$ and $\bar{\Phi}$, since $K \in H_\Phi$, we have that
\[ F_j(y) \leq C \sum_{k=1}^{\infty} (2^k r_j)^p \| (K(\cdot - (y - x_{Q_j})) - K(\cdot)) \|_{L^p} \| u_j^* \|_{L^p(\bar{\Phi}(x_{Q_j}^*, 2^k r_j))} \]
\[ \leq C_\Phi M_{\Phi} u_j^*(y). \]

Thus, by Lemma 6, we get
\[ III \leq \frac{C}{t} \sum_j \int_{Q_j} |T(y)| v \left( \inf_{Q_j} M_\Phi u_j^* \right) + C \sum_j \int_{Q_j} |f_j| v \left( \inf_{Q_j} M_\Phi u_j^* \right). \]

Recall that $M_\Phi \lesssim M_{\Phi, \nu, \psi}. \quad$ This allows us to conclude the estimate similarly as we did on page 13. The proof is complete when $u$ is bounded. For the general case, we can proceed as in the proof of Theorem 1.

\[ \qed \]

\textbf{REFERENCES}
\begin{enumerate}
\item F. Berra. Mixed weak estimates of Sawyer type for generalized maximal operators. \textit{Proc. Amer. Math. Soc.}, 147(10):4259–4273, 2019.
\item F. Berra, M. Carena, and G. Pradolini. Improvements on Sawyer type estimates for generalized maximal functions. \textit{Math. Nachr.}, 293(10):1911–1930, 2020.
\item F. Berra, M. Carena, and G. Pradolini. Mixed weak estimates of Sawyer type for commutators of generalized singular integrals and related operators. \textit{Michigan Math. J.}, 68(3):527–564, 2019.
\item F. Berra, M. Carena, and G. Pradolini. Mixed weak estimates of Sawyer type for fractional integrals and some related operators. \textit{J. Math. Anal. Appl.}, 479(2):1490–1505, 2019.
\item A. Cordoba and C. Fefferman. A weighted norm inequality for singular integrals. \textit{Studia Math.}, 57(1):97–101, 1976.
\end{enumerate}
6. D. Cruz-Uribe, J. M. Martell, and C. Pérez. Weighted weak-type inequalities and a conjecture of Sawyer. *Int. Math. Res. Not.*, (30):1849–1871, 2005.
7. D. Cruz-Uribe and C. J. Neugebauer. The structure of the reverse Hölder classes. *Trans. Amer. Math. Soc.*, 347(8):2941–2960, 1995.
8. J. Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.
9. C. Fefferman and E. M. Stein. Some maximal inequalities. *Amer. J. Math.*, 93:107–115, 1971.
10. L. Grafakos. *Classical and modern Fourier analysis*. Pearson Education, Inc., Upper Saddle River, NJ, 2004.
11. K. Li, S. Ombrosi, and C. Pérez. Proof of an extension of E. Sawyer’s conjecture about weighted mixed weak-type estimates. *Math. Ann.*, 374(1-2):907–929, 2019.
12. M. Lorente, J. M. Martell, C. Pérez, and M. S. Riveros. Generalized Hörmander conditions and weighted endpoint estimates. *Studia Math.*, 195(2):157–192, 2009.
13. M. Lorente, M. S. Riveros, and A. de la Torre. Weighted estimates for singular integral operators satisfying Hörmander’s conditions of Young type. *J. Fourier Anal. Appl.*, 11(5):497–509, 2005.
14. José María Martell, Carlos Pérez, and Rodrigo Trujillo-González. Lack of natural weighted estimates for some singular integral operators. *Trans. Amer. Math. Soc.*, 357(1):385–396, 2005.
15. B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.
16. S. Ombrosi and C. Pérez. Mixed weak type estimates: examples and counterexamples related to a problem of E. Sawyer. *Colloq. Math.*, 145(2):259–272, 2016.
17. C. Pérez. Weighted norm inequalities for singular integral operators. *J. London Math. Soc. (2)*, 49(2):296–308, 1994.
18. C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted $L^p$-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.
19. Carlos Pérez and Gladis Pradolini. Sharp weighted endpoint estimates for commutators of singular integrals. *Michigan Math. J.*, 49(1):23–37, 2001.
20. E. Sawyer. A weighted weak type inequality for the maximal function. *Proc. Amer. Math. Soc.*, 93(4):610–614, 1985.
21. J. Michael Wilson. Weighted norm inequalities for the continuous square function. *Trans. Amer. Math. Soc.*, 314(2):661–692, 1989.

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