2-connected equimatchable graphs on surfaces

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Abstract
A graph $G$ is equimatchable if any matching in $G$ is a subset of a maximum-size matching. It is known that any 2-connected equimatchable graph is either bipartite or factor-critical. We prove that for any vertex $v$ of a 2-connected factor-critical equimatchable graph $G$ and a minimal matching $M$ that isolates $v$ the graph $G \setminus (M \cup \{v\})$ is either $K_{2n}$ or $K_{n,n}$ for some $n$. We use this result to improve the upper bounds on the maximum size of 2-connected equimatchable factor-critical graphs embeddable in the orientable surface of genus $g$ to $4\sqrt{g}+17$ if $g \leq 2$ and to $12\sqrt{g}+5$ if $g \geq 3$. Moreover, for any nonnegative integer $g$, $h$, and $k$ we provide a construction of arbitrarily large 2-connected equimatchable bipartite graphs with orientable genus $g$, respectively nonorientable genus $h$, and a genus embedding with face-width $k$.

Keywords: graph, matching, equimatchable, factor-critical, embedding, genus, bipartite.

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1 Introduction
A graph $G$ is called equimatchable if any maximal matching of $G$ has maximum size. Equimatchable graphs constitute a classical topic of matching theory investigated for several decades since appearing in [4], [9], and [12]. In particular, Grünbaum [4] asked for a characterisation of all equimatchable graphs. The first step in this direction was a characterisation of all randomly-matchable graphs – equimatchable graphs with a perfect matching. By a result of Sumner [19], connected randomly-matchable graphs are exactly the complete graphs $K_{2n}$ and complete bipartite graphs $K_{n,n}$ for $n \geq 1$. The fundamental work [8] provides a structural characterisation of equimatchable graphs without a perfect matching using Gallai-Edmonds decomposition, yielding also a polynomial-time algorithm for recognizing equimatchable graphs. Equimatchable factor-critical graphs with a cut-vertex are investigated in [2] where it is proved that they contain exactly one cut-vertex $v$, every component of $G - v$ is either $K_{2n}$ or $K_{n,n}$, and $v$ is adjacent to at least two adjacent vertices of every component of $G - v$. A similar description of 2-connected equimatchable factor-critical graphs with respect to a 2-cut $\{u, v\}$ is given as Theorem 2.2 of [2]: $G \setminus \{u, v\}$ has exactly two components which differ from a complete or complete bipartite graph by at most one edge or by at most two vertices. Furthermore, it is proved in [17] that if $G$ is a 3-connected planar graph, $v$ a vertex of $G$, and $M$ a minimal matching isolating $v$, then $G \setminus (V(M) \cup \{v\})$ is randomly matchable and connected, where a matching $M$ is isolating a vertex $v$ if $\{v\}$ is a component of $G \setminus V(M_v)$. It can be easily seen that every component of $G \setminus V(M_v)$ except $\{v\}$ is randomly matchable for every factor-critical graph $G$ and a minimal matching $M_v$ isolating a vertex $v$ of $G$. Our main theorem below extends these results by showing that $G \setminus (V(M) \cup \{v\})$ has exactly one component.

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Theorem. Let $G$ be a 2-connected, factor-critical equimatchable graph. Let $v$ be a vertex of $G$ and $M_v$ a minimal matching isolating $v$. Then $G \setminus (V(M_v) \cup \{v\})$ is isomorphic with $K_{2n}$ or $K_{n,n}$ for some nonnegative integer $n$.

The relationship between embeddings of graphs and matching extensions was extensively studied, see for instance [1], [5], or [10]. The characterisation of equimatchable graphs in [5] implies that any 2-connected equimatchable graph is either bipartite or factor-critical. A bipartite graph cannot be factor-critical, since otherwise it would have an odd number of vertices and removing a vertex from the smaller partite set cannot yield a graph with a perfect matching. Therefore, these two classes are disjoint. All 3-connected planar equimatchable graphs are characterised in [7] – there are 23 such graphs and none of them is bipartite. Let $G$ be a 3-connected equimatchable graph with an embedding $\Pi$ in the surface of genus $g$. In [6] it is proved that if $G$ is either factor-critical, or bipartite and $\Pi$ has face-width at least 3, then the number of vertices of $G$ is bounded from above by $c \cdot g^{3/2}$ for some constant $c$. The proof uses the fact that there is no such bipartite graph at all and proceeds to restrict the size of equimatchable factor-critical graphs embeddable in a fixed surface. First it is shown that if a 3-connected graph has many vertices (a number linear in the genus of the graph), then it has a vertex $v$ isolated by a matching $M_v$ of size at most 4. The proof is finished by showing that $G \setminus (V(M_v) \cup \{v\})$ has at most $\binom{\delta}{2}(4g + 3)$ components.

To bound the maximum size of equimatchable factor-critical graphs embeddable in a fixed surface, we employ a slightly different strategy: while we allow larger isolating matchings, we use a more precise description of $G \setminus (V(M_v) \cup \{v\})$ given in our main result, Theorem 5, which implies that it has at most one component. As a complete or complete bipartite graph embeddable in the surface of genus $g$ has at most $O(\sqrt{g})$ vertices, it suffices to bound the size of isolating matchings. Note that any vertex of degree $d$ admits an isolating matching of size at most $d$. The last ingredient of our proof is Lemma 12 showing that either the total number of vertices of the graph, or the minimum degree, is sufficiently small.

Concerning the methods of the paper, while we repeatedly use the characterisation of randomly matchable graphs from [19], the Gallai-Edmonds decomposition is not used beyond the fact that every 2-connected equimatchable graph is either bipartite or factor-critical. The constants in the orientable and the nonorientable case are different, hence we state our results explicitly for both cases. However, most of the proofs are virtually identical and in such cases, we omit the proof of the nonorientable case.

The paper is organized as follows. In Section 2 we briefly collect the necessary terms, definitions, and notation regarding matchings and embeddings. In Section 3 we present a proof of our main result stating that the graph $G \setminus (V(M_v) \cup \{v\})$ is connected for any 2-connected factor-critical equimatchable graph $G$ and a minimal matching $M_v$ isolating a vertex $v$. Section 4 is devoted to lower and upper bounds on the maximum size of an equimatchable graph embeddable in a fixed surface.

2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. A matching is a set of independent edges, that is, a set of edges with no endvertices in common. A matching $M$ of a graph $G$ is called perfect if every vertex of $G$ is incident with an edge of $M$. A graph $G$ is factor-critical if $G \setminus \{v\}$ has a perfect matching for any vertex $v$ of $G$ and equimatchable if any its maximal matching is maximum. A graph is called randomly matchable if it is equimatchable and has a perfect matching. By [19], the connected randomly-matchable graphs are exactly the even complete graphs $K_{2n}$ and complete regular bipartite graphs $K_{n,n}$ for all $n \geq 1$. For a matching $M$, by $|M|$ we denote the size of $M$, that is, the number of edges of $M$, and by $V(M)$ we denote the set of vertices incident with the edges of $M$. A vertex is called covered by a matching $M$ if it is incident with an edge of $M$, or equivalently, if it lies in $V(M)$. For a vertex $v$, a matching $M$ is called a matching isolating $v$ if $\{v\}$ is a component of $G \setminus V(M)$. A matching $M$ isolating a vertex $v$ is called minimal if no subset of $M$ isolates $v$. For a graph $G$ and its vertex $v$, the set difference $G \setminus \{v\}$ is for brevity denoted by $G - v$. By $G \cup H$ we denote the disjoint union of graphs $G$ and $H$. An edge with endvertices $u$ and $v$ is denoted by $uv$. By $\delta(G)$ we denote the minimal degree of $G$ and by $N(v)$ the set of neighbours of a vertex $v$. For a deeper account of matching theory the reader is referred to [11].

A surface is a connected 2-dimensional manifold without boundary. The sphere with $g$ handles (respectively $h$ crosscaps) attached forms a model for orientable surfaces of genus $g$ (nonorientable surfaces of genus $h$) and is denoted by $S_g$ ($N_h$). Indeed, the classification theorem for orientable (nonorientable) surfaces states that for any orientable (nonorientable) surface there is exactly one $g \geq 0$ such that $S$ is
homeomorphic with $S_g$ (exactly one $h \geq 1$ such that $S$ is homeomorphic with $N_h$), see $[3]$. The number $g$ (respectively $h$) is called the orientable (nonorientable) genus of the surface. For instance, $S_0$ is the sphere, $S_1$ is the torus, and $N_1$ is the projective plane. The characteristic of a surface $S$, denoted by $\chi(S)$, equals $2 - 2g$ if $S$ is homeomorphic with $S_g$, or $2 - h$ if $S$ is homeomorphic with $N_h$. An embedding of a graph $G$ in a surface $S$ is a representation of $G$ on $S$ with the following properties. The vertices of $G$ are represented by distinct points of $S$, and the edges of $G$ are represented by disjoint images of the open unit interval, and any open neighbourhood of the image of a vertex intersects images of all edges incident with that vertex, see $[3]$ or $[20]$ for more details. An embedding of a graph in a surface is called cellular (or 2-cell) if every face of the embedding is homeomorphic with an open disc; we consider only cellular embeddings. The Euler-Poincaré formula (see $[3]$ or $[20]$) states that if a graph $G$ with $p$ vertices and $q$ edges is cellularly embedded in a surface $S$ with $r$ faces, then $p - q + r = \chi(S)$. The orientable (nonorientable) genus of a graph $G$ is the minimum orientable (nonorientable) genus of a surface into which $G$ can be cellularly embedded and is denoted by $\gamma(G)$, respectively $\tilde{\gamma}(G)$. Face-width, sometimes called also representativity or planar-width, of an embedding $\Pi$ in a surface $S$ is the minimum number of faces of $\Pi$ whose union contains a noncontractible cycle in the surface $S$. Several equivalent definitions and further details about face-width can be found in $[18]$.

We now present a well-known upper bound on the number of faces of an embedding of a simple graph.

**Proposition 1.** Let $G$ be a simple graph with $q$ edges embedded with $r$ faces. Then $2q \geq 3r$.

**Proof.** As $G$ is simple, any face of the embedding has length at least 3. The result follows from the fact that the union of face boundaries contains every edge precisely twice.

We repeatedly use the following result due to Ringel and Youngs and Ringel.

**Theorem 2** ($[17, 14, 15, 16]$). The orientable and nonorientable genera of complete and complete bipartite graphs are given by the following formulae:

$$\gamma(K_n) = \left\lceil \frac{(n - 3)(n - 4)}{12} \right\rceil, n \geq 3; \quad \tilde{\gamma}(K_n) = \left\lceil \frac{(n - 3)(n - 4)}{6} \right\rceil, n \geq 3 \text{ and } n \neq 7, \quad \tilde{\gamma}(K_7) = 3;$$

$$\gamma(K_{m,n}) = \left\lceil \frac{(m - 2)(n - 2)}{4} \right\rceil, m, n \geq 2; \quad \tilde{\gamma}(K_{n,m}) = \left\lceil \frac{(m - 2)(n - 2)}{2} \right\rceil, m, n \geq 2.$$

For a more detailed treatment of topological graph theory the reader is referred to $[3]$ or $[20]$.

### 3 Isolating matchings 2-connected equimatchable graphs

This section is devoted to the proof of our main result stated as Theorem $[5]$. We start with two lemmas concerning isolating matchings.

**Lemma 3.** Let $G$ be a factor-critical graph. For every vertex $v$ of $G$ there is a matching $M_v \subseteq E(G)$ isolating $v$ such that $|M_v| \leq \deg(v)$.

**Proof.** Since $G$ is factor critical, the graph $G' = G - v$ has a perfect matching $M'$. Clearly, every neighbour of $v$ is incident with exactly one edge of the matching $M'$. Consider a set $M \subseteq M'$ such that $M$ contains precisely those edges from $M'$ that are incident with at least one neighbour of $v$. Then $M$ is the desired matching $M_v$ containing at most $\deg(v)$ edges and isolating $v$.

Favaron $[2]$, Theorem 1.1] proved that any connected factor-critical equimatchable graph $G$ with a cut-vertex contains precisely one cut-vertex $v$ and every component of $G - v$ is either $K_{2n}$ or $K_{n,n}$. For equimatchable factor-critical graphs with a 2-cut $\{u, v\}$, it is still possible to give a description of the structure of $G' = G \setminus \{u, v\}$, albeit it is more complicated; $G'$ has exactly two components and these components are almost complete or complete bipartite, see $[2]$ Theorem 2.2] for the precise statement and details. Removing isolating matchings instead of vertex-cuts allows us to obtain a similar description for graphs with arbitrary connectivity in the lemma below. The underlying idea of its proof is well known, in particular, it was applied in $[2]$ and $[6]$ to prove more specific variants of the result.

**Lemma 4.** Let $G$ be a connected factor-critical equimatchable graph and $M$ a minimal matching isolating $v$. Then every component of $G \setminus V(M)$ except $\{v\}$ is isomorphic with either $K_{2n}$, or $K_{n,n}$ for some integer $n$. 


Proof. Let \( G' = G \setminus (V(M) \cup \{v\}) \) and denote by \( M' \) any maximal matching of \( G' \). Clearly, \( M = M' \cup M_v \) is a maximal matching of \( G \). The graph \( G \) is factor-critical and equimatchable, hence \( M \) leaves only the vertex \( v \) uncovered and \( M' \) must be a perfect matching of \( G' \). Since arbitrary maximal matching \( M' \) of \( G' \) is a perfect matching of \( G' \), \( G' \) is randomly matchable and by \[9\] all of its components are either complete with even number of vertices or complete regular bipartite.

Note that since \( G \) is factor-critical, there always exists a matching isolating any fixed vertex \( v \) of \( G \).

We say that a subgraph \( H_1 \) (such as a vertex, edge, or component) of a graph \( G \) is linked with other subgraph \( H_2 \) of same graph \( G \) if there are vertices \( k_1 \) of \( H_1 \) and \( k_2 \) of \( H_2 \) such that \( k_1 k_2 \in E(G) \). We are now ready to prove our main result, which sharpens Lemma 4 by showing that \( G \) has only one component and generalizes \[4\] Lemma 1.6], which proves that \( G' \) has only one component if \( G \) is 3-connected and planar.

Theorem 5. Let \( G \) be a 2-connected, factor-critical equimatchable graph. Let \( v \) be a vertex of \( G \) and \( M_v \) a minimal matching isolating \( v \). Then \( G \setminus (V(M_v) \cup \{v\}) \) is isomorphic with \( K_{2n} \) or \( K_{n,n} \) for some nonnegative integer \( n \).

Proof. We prove the theorem by a series of claims. Let \( G' = G \setminus (V(M_v) \cup \{v\}) \).

Claim 1. If \( xy \) is an arbitrary edge of matching \( M_v \), then \( x \) and \( y \) cannot be linked to different components of \( G' \).

Proof of Claim 1. We prove the claim by contradiction. Let \( C \) and \( D \) be different components of \( G' \) and suppose that \( x \) is adjacent to a vertex \( x' \) of \( C \) and \( y \) is adjacent to a vertex \( y' \) of \( D \). Let \( M \) be defined by \( M = (M_v \setminus \{xy\}) \cup \{xx',yy\} \). It is easy to see that \( M \) is a matching of \( G \). Furthermore, \( C - x' \) and \( D - y' \) are components of \( G' \setminus M \). From Lemma \[4\] follows that \( C \) and \( D \) have even number of vertices and hence both \( C - x' \) and \( D - y' \) have odd number of vertices. It follows that any maximal matching \( M' \) such that \( M \subseteq M' \) leaves uncovered at least one vertex of both \( C - x' \) and \( D - y' \). This is a contradiction with the fact that \( G \) is equimatchable and factor-critical.

Claim 2. Let \( C \) be a component of \( G' \) and \( xy \) an edge of matching \( M_v \) such that \( x \) is linked to some vertex \( x' \) from \( C \). Then \( y \) is linked either to \( v \) or to some vertex \( y' \) of \( C \) such that \( y' \neq x' \).

Proof of Claim 2. Suppose that \( y \) is linked neither with \( C \) nor with \( v \). Let \( M \) be defined by \( M = (M_v \setminus \{xy\}) \cup \{xx',yy\} \). It is easy to see that \( M \) is a matching of \( G \). As all neighbours of \( v \) are covered by \( M \), any maximal matching \( M' \) of \( G \) such that \( M \subseteq M' \) leaves \( v \) uncovered. Since \( x \) is linked with \( C \), by Claim \[4\] \( y \) cannot be linked to any other component of \( G' \). According to our assumption, \( y \) is not linked with \( v \) or \( C \). Therefore, \( M' \) leaves uncovered both \( v \) and \( y \). This is a contradiction with the fact that \( G \) is equimatchable and factor-critical, which completes the proof of the claim.

Claim 3. For any edge \( e \) of \( M_v \) linked with a component \( C \) of \( G' \), there are two independent edges joining the endvertices of \( e \) with \( v \) and \( C \), respectively.

Proof of Claim 3. Let \( e = xy \) and suppose that \( x \) is linked with a vertex \( x' \) of \( C \). By Claim \[4\] \( y \) is linked either with \( v \) or with some vertex \( y' \) of \( C \). If \( y \) is linked with \( v \), then \( xx' \) and \( yy' \) are the two desired edges and we are done. If \( y \) is not linked with \( v \), then by the minimality of \( M_v \), \( v \) is linked with \( x \). In this case \( xv \) and \( yv' \) are the desired edges, which completes the proof.

Claim 4. Let \( C \) be an arbitrary component of \( G' \) and \( xy \) an edge of matching \( M_v \) linked with \( C \). If \( G' \) has at least two components, then there are two independent edges joining \( x \) and \( y \) with \( C \).

Proof of Claim 4. Without loss of generality assume that \( x \) is adjacent to a vertex \( x' \) of \( C \) and suppose to the contrary that \( y \) is not adjacent to a vertex of \( C \) different from \( x' \). Let \( D \) be a component of \( G' \) different from \( C \). Since \( G \) is 2-connected, \( D \) is linked with at least two vertices of \( G' \setminus V(D) \). Furthermore, the fact that \( v \) is not linked with \( D \) implies that these two vertices must be vertices of \( M_v \). Because \( x \) is linked with \( C \), from Claim \[4\] we get that \( y \) cannot be linked with \( D \) and thus at least one of the vertices of \( M_v \) linked with \( D \) is different from both \( x \) and \( y \). Let \( x_1 y_1 \) be an edge of \( M_v \) linked with \( D \) such that \( x_1 y_1 \neq xy \). According to Claim \[4\] we can assume that \( x_1 \) is adjacent to a vertex \( x'_1 \) of \( D \) and \( y_1 \) is adjacent to \( v \). It is clear that the set \( M \) defined by \( M = (M_v \setminus \{xy, x_1 y_1\}) \cup \{xx',x_1 x'_1,y_1 v\} \) is a matching of \( G \). Claim \[4\] implies that \( y \) is not linked with any component of \( G' \) different from \( C \) and in particular, it is not linked with \( D \). According to our assumption, \( y \) is not adjacent to any vertex of \( C - x' \). It follows
that any maximal matching $M'$ such that $M \subseteq M'$ leaves uncovered $y$ and one vertex of both $C$ and $D$. This contradicts equimatchability and factor-criticality of $G$ and completes the proof of the claim.

Claim 5. Let $e$ and $f$ be two edges of $M_v$ linked with two different components of $G'$. Then $e$ and $f$ are not linked.

Proof of Claim 5. Let $e = x_1y_1$ and $f = x_2y_2$. Assume that $e$ is linked with a component $C$ of $G'$ and $f$ is linked with a component $D$ of $G'$. Claim 4 implies that both $x_1$ and $y_1$ are linked with $C$ and both $x_2$ and $y_2$ are linked with $D$. Suppose to the contrary that that $e$ and $f$ are linked; we can assume that they are linked by edge $x_1x_2$. Let $y_1'$ be a vertex of $C$ adjacent to $y_1$ and $y_2'$ a vertex of $D$ adjacent to $y_2$. Clearly, the set $M$ defined by $M = (M_v \setminus \{x_1y_1, x_2y_2\}) \cup \{x_1x_2, y_1'y_1, y_2y_2'\}$ is a matching of $G$ and any maximal matching $M'$ such that $M \subseteq M'$ leaves unmatched $v$ and at least one vertex of both $C$ and $D$, again contradicting the equimatchability and factor-criticality of $G$.

Claim 6. Let $e$, $f_1$, and $f_2$ be edges of $M_v$ and $C$ and $D$ two different components of $G'$ such that $C$ is linked with $f_1$ and $D$ is linked with $f_2$. If $e$ is not linked with $C$, then it is not linked with $f_1$.

Proof of Claim 6. Let $e = uw$, $f_1 = x_1y_1$, and $f_2 = x_2y_2$, and for the contrary suppose that $e$ is linked with $f_1$. Since $e$ and $f_1$ are linked, by Claim 5 $e$ is not linked to any component of $G'$ different from $C$. Moreover, by our assumption $e$ is not linked with $C$. By Claim 4 there are two independent edges joining $f_1$ and $C$ and two independent edges joining $f_2$ and $D$. Therefore, we can assume that $u$ is linked with $x_1$. As $M_v$ is minimal, $f_2$ is linked with $v$; let $x_2$ be adjacent to $v$. Let $y_1'$ be a vertex of $C$ adjacent to $y_1$ and $y_2'$ a vertex of $D$ adjacent to $y_2$. It is clear that the set $M$ defined by $M = (M_v \setminus \{e, f_2\}) \cup \{ux_1, y_1'y_1, x_2y_2'\}$ is a matching of $G$. Since $e$ is not linked with any component of $G'$, any maximal matching $M'$ of $G$ such that $M \subseteq M'$ leaves unmatched the vertex $w$ and one vertex of both $C$ and $D$, which contradicts the fact that $G$ is equimatchable and factor-critical.

Claim 7. If $G'$ has at least two components, then $v$ is a cutvertex.

Proof of Claim 7. Our aim is to show that in $G - v$ there is no path between arbitrary two components of $G'$. We proceed by contradiction: suppose there is such a path and among all such paths, choose a path that minimizes the number $k$ of edges of $M_v$ incident with it. Denote one of the paths with $k$ minimal by $P$ and by $C$ and $D$ the components of $G'$ joined by $P$. From the fact that $C$ and $D$ are components of $G'$ follows that they cannot be linked directly, and consequently $k > 0$. Let $e$ and $f$ be the first, respectively the last, edge of $M_v$ incident with $P$. As no other component of $G'$ is linked with either $C$ or $D$, we get that $e$ is linked with $C$ and $f$ is linked with $D$. From Claim 4 it follows that both endvertices of $e$ are linked with $C$ and then Claim 4 implies that $e$ is not linked with $D$. Therefore, $e$ and $f$ are distinct and $k > 1$. Notice that $k = 2$ is equivalent with $e$ and $f$ being linked, which is not possible due to Claim 5. Suppose that $k \geq 3$. By the minimality of $k$, there is an edge $a$ of $M_v$ such that $a$ is linked with $e$, but not with $C$. However, this contradicts Claim 4 and hence $k \geq 3$ is not possible. We conclude that any path between $C$ and $D$ contains $v$. Since $G$ is connected, there is at least one such path. Consequently, $v$ is a cutvertex of $G$, which completes the proof of the claim.

From the fact that $G$ is 2-connected and from Claim 7 it follows that $G'$ has only one component. Lemma 4 implies that this component is either $K_{2n}$ or $K_{n,n}$, which completes the proof.

The characterisation of equimatchable factor-critical graphs with a cut-vertex in 2 implies that in such graphs $G \setminus V(M_v)$ can have arbitrarily-many components and therefore, Theorem 5 cannot be extended to graph that are not 2-connected.

4 Size of 2-connected equimatchable graphs on surfaces

The aim of this section is to obtain good lower and upper bounds on the maximum size of equimatchable factor-critical graphs embeddable in the surface of arbitrary fixed genus using Theorem 5. We start by showing that there are arbitrarily large equimatchable factor-critical graphs with a cutvertex and any given genus.

Proposition 6. For any nonnegative integers $g$, $h$, and $k$ there exist connected factor-critical equimatchable graphs $G$ and $	ilde{G}$ with at least $k$ vertices such that $G$ has orientable genus $g$ and $	ilde{G}$ has nonorientable genus $h$. 

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Proof. Let \( n \) be an integer such that \( K_{2n+1} \) has orientable genus \( g \) and \( v \) an arbitrary vertex of \( K_{2n+1} \). Take \( k \) copies of the triangle \( K_3 \) and designate one vertex in each copy. It is easy to verify that the graph obtained by vertex amalgamation of \( K_{2n+1} \) at \( v \) and \( k \) triangles at the designated vertices is a connected factor-critical equimatchable graph with genus \( g \) and at least \( k \) vertices. The proof of the nonorientable case is analogous. \( \square \)

It is easy to see and well-known that any complete bipartite graph \( K_{m,n} \) is equimatchable.

**Proposition 7.** For any integers \( m \) and \( n \) such that \( m \geq n \) the complete bipartite graph \( K_{m,n} \) is equimatchable and its maximum matching has size \( n \). \( \square \)

The next three results yield a construction of large 2-connected equimatchable factor-critical graphs embeddable in any fixed surface.

**Lemma 8.** Let \( u \) and \( v \) be adjacent vertices of \( K_{n,n} \) and \( x \) and \( y \) different vertices from the larger partite set of \( K_{m+1,m} \) for some \( m \) and \( n \). Then the graph \( G \) defined by \( G = K_{n,n} \cup K_{m+1,m} \cup \{ux, vy\} \) is factor-critical and equimatchable.

Proof. Denote by \( H_1 \) the copy of \( K_{n,n} \) and by \( H_2 \) the copy of \( K_{m+1,m} \) in \( G \), thus \( G = H_1 \cup H_2 \cup \{ux, vy\} \). First, we show that \( G \) is factor-critical, that is, the graph \( G - w \) has a perfect matching for any vertex \( w \) of \( G \). Denote by \( A \) and \( B \) the larger, respectively the smaller partite set of \( H_2 \). We distinguish three cases.

**Case 1:** \( w \) is a vertex of \( H_1 \). We can assume that \( w \) is in same partite set as \( v \). Clearly, there is a perfect matching \( M_1 \) of \( H_1 \) \( \setminus \{u, w\} \) and a perfect matching \( M_2 \) of \( H_2 - x \). It follows that matching \( M \) defined by \( M = M_1 \cup M_2 \cup \{ux\} \) is a perfect matching of \( G - w \).

**Case 2:** \( w \) is a vertex of \( A \). Take any perfect matching \( M_1 \) of \( H_1 \) and any perfect matching \( M_2 \) of \( H_2 - w \). The matching \( M \) defined by \( M = M_1 \cup M_2 \) is clearly a perfect matching of \( G - w \).

**Case 3:** \( w \) is a vertex of \( B \). Take any perfect matching \( M_1 \) of \( H_1 \) \( \setminus \{u, v\} \) and any perfect matching \( M_2 \) of \( H_2 \) \( \setminus \{w, x, y\} \). It is easy to see that the matching \( M \) defined by \( M = M_1 \cup M_2 \cup \{ux, vy\} \) is a perfect matching of \( G - w \).

Now we show that \( G \) is equimatchable by proving that any matching \( M \) of \( G \) is a subset of a maximum matching. As \( G \) is factor-critical, any maximum matching of the graph \( G \) leaves precisely one vertex uncovered. We distinguish three cases according to which of the edges \( ux \) and \( vy \) lie in \( M \).

**Case 1:** neither \( ux \) nor \( vy \) is an edge of \( M \). Clearly, \( M \) is a disjoint union of matchings \( M_1 \) of \( H_1 \) and \( M_2 \) of \( H_2 \). Since both \( H_1 \) and \( H_2 \) are equimatchable by Proposition 7, the matchings \( M_1 \) and \( M_2 \) can be extended to maximum matchings \( M'_1 \) of \( H_1 \) and \( M'_2 \) of \( H_2 \), respectively. Clearly, the matching \( M'_1 \cup M'_2 \) covers all vertices of \( H_1 \) and \( M'_2 \) covers all but one vertices of \( H_2 \). Therefore, the matching \( M' \) defined by \( M' = M'_1 \cup M'_2 \cup \{ux, vy\} \) is a maximum matching of \( G \).

**Case 2:** either \( ux \) or \( vy \) is an edge of \( M \), but not both. We can assume that \( ux \) is an edge of \( M \) and \( vy \) is not an edge of \( M \). Let \( H'_1 = H_1 - u \) and \( H'_2 = H_2 - x \). Observe that \( H'_1 \) is isomorphic with \( K_{n-1,n-1} \) and \( H'_2 \) is isomorphic with \( K_{m-1,m} \). Consequently, by Proposition 7, \( H'_1 \) is equimatchable and any its maximum matching misses exactly one vertex and \( H'_2 \) is equimatchable and has a perfect matching. The matching \( M \) is a disjoint union of matching \( M_1 \) of \( H'_1 \), matching \( M_2 \) of \( H'_2 \), and the edge \( ux \). By equimatchability of \( H'_1 \) and \( H'_2 \), the matching \( M_1 \) extends to a matching \( M'_1 \) of \( H'_1 \) missing exactly one vertex and \( M_2 \) extends to a perfect matching of \( H'_2 \). The matching \( M' \) defined by \( M' = M'_1 \cup M'_2 \cup \{ux\} \) is the desired matching of \( G \) missing exactly one vertex.

**Case 3:** both \( ux \) and \( vy \) are edges of \( M \). Let \( H'_1 = H_1 \setminus \{u, v\} \) and \( H'_2 = H_2 \setminus \{x, y\} \). Observe that \( H'_1 \) is isomorphic with \( K_{n-1,n-1} \) and \( H'_2 \) is isomorphic with \( K_{m-1,m} \) and thus, by Proposition 7 both are equimatchable, \( H'_1 \) admitting a perfect matching and \( H'_2 \) a matching missing exactly one vertex. Clearly, \( M \) is a disjoint union of matchings \( M_1 \) of \( H'_1 \), \( M_2 \) of \( H'_2 \), and edges \( ux \) and \( vy \). Again, \( M_1 \) extends to a perfect matching \( M'_1 \) of \( H'_1 \) and \( M_2 \) extends to a matching \( M'_2 \) of \( H'_2 \) missing exactly one vertex. Therefore, the matching \( M' \) defined by \( M' = M'_1 \cup M'_2 \cup \{ux, vy\} \) is a matching of \( G \) missing exactly one vertex. \( \square \)

Although we need the following lemma only for \( K_{n,n} \) and \( K_{m+1,m} \), we state it in a general form since the proof is identical.

**Lemma 9.** Let \( a, b, c, d \) be positive integers such that \( c > d \). Let \( u \) and \( v \) be two adjacent vertices of \( K_{a,b} \). Then there are two distinct vertices \( x \) and \( y \) from the larger partite set of \( K_{c,d} \) such that the graph
Let $G$ defined by $G = K_{a,b} \cup K_{c,d} \cup \{ux, vy\}$ has the genus equal to $\gamma(K_{a,b}) + \gamma(K_{c,d})$. Similarly, there are two distinct vertices $\tilde{x}$ and $\tilde{y}$ from the larger partite set of $K_{c,d}$ such that the graph $\tilde{G} = K_{a,b} \cup K_{c,d} \cup \{\tilde{u}x, \tilde{v}y\}$ has the genus equal to $\tilde{\gamma}(K_{a,b}) + \tilde{\gamma}(K_{c,d})$.

**Proof.** We start by constructing the desired graph $G$ and its embedding of genus $\gamma(K_{a,b}) + \gamma(K_{c,d})$. Denote by $H_1$ a copy of $K_{a,b}$ and by $H_2$ a copy of $K_{c,d}$. Let $H_i$ be a minimum-genus embedding of $H_i$ for $i \in \{1, 2\}$. Since the vertices $u$ and $v$ are adjacent, there is a face $F_1$ of $H_1$ such that both $u$ and $v$ lie on the boundary of $F_1$. Because $H_2$ is bipartite, any face of $H_2$ has length at least four and thus contains at least two vertices from the larger partite set of $H_2$. Let $x$ and $y$ be arbitrary two vertices of the larger partite set of $H_2$ that lie together on the boundary of a face $F_2$ of $H_2$ and let $G = H_1 \cup H_2 \cup \{ux, vy\}$. Adding one end of the edge $ux$ into the interior of $F_1$ and the other end of $ux$ into the interior of $F_2$ merges these faces into one face $F$, producing an embedding $\Pi$ of connected graph $H_1 \cup H_2 \cup \{ux, vy\}$ in the surface of genus $\gamma(H_1) + \gamma(H_2)$.

Consequently, both $v$ and $y$ lie on the boundary of $F$ and the edge $vy$ can be added into $\Pi$ without raising the genus, yielding the desired embedding of $G$ in the surface of genus $\gamma(H_1) + \gamma(H_2)$. Since $H_1$ and $H_2$ are disjoint subgraphs of $G$, we get that $\gamma(G) \geq \gamma(H_1) + \gamma(H_2)$, which completes the proof of the orientable case. The proof of the nonorientable case is completely analogous.

**Theorem 10.** For any nonnegative integers $g$ and $h$ there exist 2-connected factor-critical equimatchable graphs $G$ and $\tilde{G}$ such that $G$ has orientable genus $g$ and at least $4\lfloor \sqrt{2g} \rfloor + 5$ vertices and $\tilde{G}$ has nonorientable genus $h$ and at least $4\lfloor \sqrt{h} \rfloor + 5$ vertices.

**Proof.** Let $n$ and $m$ be maximum integers such that $K_{n,n}$ is embeddable in the orientable surface of genus $[g/2]$ and $K_{m+1,m}$ is embeddable in the orientable surface of genus $[g/2]$. Let $u$ and $v$ be two adjacent vertices of $K_{n,n}$. By Lemma 11 there are two vertices $x$ and $y$ of $K_{m+1,m}$ such that the graph $G$ defined by $G = K_{n,n} \cup K_{m+1,m} \cup \{ux, vy\}$ is 2-connected with orientable genus $\gamma(K_{n,n}) + \gamma(K_{m+1,m}) = \lfloor g/2 \rfloor + \lfloor g/2 \rfloor = g$. By Lemma 10 the graph $G$ is equimatchable and factor-critical.

To complete the proof it suffices to bound the number of vertices of $G$ from below by calculating the value of $n$ and $m$. First suppose that $g$ is even. It is not difficult to verify that $n = \lfloor \sqrt{2g} \rfloor + 2$ and that $m = \lfloor (3 + \sqrt{8g + 1})/2 \rfloor$. Since $2\alpha \geq \lfloor \alpha \rfloor \geq \lfloor 2\alpha \rfloor - 1$ holds for any positive real number $\alpha$, we get that $K_{m+1,m}$ has $2m + 1 \geq 3 + \lfloor \sqrt{8g + 1} \rfloor \geq 3 + 2\lfloor \sqrt{2g} \rfloor$ vertices. Consequently, $G$ has at least $4\lfloor \sqrt{2g} \rfloor + 7$ vertices. If $g$ is odd, then $n = \lfloor \sqrt{2g - 2} \rfloor + 2$ and $m = \lfloor (3 + \sqrt{8g + 9})/2 \rfloor$. Since $\lfloor \sqrt{2g - 2} \rfloor \geq \lfloor \sqrt{2g} \rfloor - 1$ for any positive integer $g$, $K_{n,n}$ has $2(2 + \lfloor \sqrt{2g - 2} \rfloor) \geq 2 + 2\lfloor \sqrt{2g} \rfloor$ vertices. Similarily as in the case of even $g$ we get that $K_{m+1,m}$ has at least $3 + 2\lfloor \sqrt{2g} \rfloor$ vertices. Therefore, $G$ has at least $4\lfloor \sqrt{2g} \rfloor + 5$ vertices, which completes the proof of the orientable case. The nonorientable case is analogous.

The following four lemmas enable us to obtain upper bounds on the size of 2-connected equimatchable factor-critical graphs embeddable in a fixed surface.

**Lemma 11.** If $G$ is a randomly matchable graph embeddable in the orientable surface of genus $g$ (nonorientable genus $h$), then $|V(G)| \leq 4 + 4\sqrt{g}$, respectively $|V(G)| \leq 4 + 2\sqrt{h}$.

**Proof.** If $G$ is a complete graph embeddable in the orientable surface of genus $g$, then $|V(G)| \leq (7 + \sqrt{1 + 48g})/2$ by Theorem 2. If $G$ is a complete regular bipartite embeddable in the orientable surface of genus $g$, then $|V(G)| \leq 4 + \sqrt{g}$ by Theorem 2. The inequality $(7 + \sqrt{1 + 48g})/2 \leq 4 + 4\sqrt{g}$, which holds for any $g \geq 0$, implies the result in the orientable case. The proof of the nonorientable case is analogous.

**Lemma 12.** If $G$ has a cellular embedding in a surface $S$ and more than

$$\frac{6\chi(S)}{5 - d}$$

vertices for some $d \geq 6$, then $\delta(G) \leq d$.

**Proof.** We prove the lemma by contradiction. Suppose that $\delta(G) \geq d + 1$ and consider an embedding of $G$ in the surface $S$. Denote by $p, q$, and $r$ the number of vertices and edges of $G$ and the number of faces of the embedding, respectively. As $\delta(G) \geq d + 1$ we have $2q \geq (d + 1)p$. Since $G$ is a simple graph,
2q ≥ 3r holds by Proposition 3. Substituting the expressions for p and r from the last two inequalities into Euler-Poincaré formula yields
\[ \chi(S) = p - q + r ≤ \frac{2q}{d+1} - q + \frac{2q}{3} = \frac{q(5-d)}{3(d+1)}. \]

Using \( d ≥ 6 \) and \( 2q ≥ (d+1)p \) we have
\[ \frac{q(5-d)}{3(d+1)} ≤ \frac{p(d+1)}{2}, \quad \frac{5-d}{3(d+1)} \]
and therefore
\[ \chi(S) ≤ \frac{p(5-d)}{6}, \]
which contradicts the assumption of the lemma.

**Lemma 13.** Let \( G \) be a 2-connected, factor-critical equimatchable graph embeddable in the surface with orientable genus \( g \), respectively nonorientable genus \( h \). If \( G \) has a vertex of degree at most \( d \), then \( |V(G)| ≤ 5 + 2d + 4\sqrt{g} \), respectively \( |V(G)| ≤ 5 + 2d + 2\sqrt{2h} \).

**Proof.** Let \( v \) be a vertex of \( G \) with degree \( d \) in \( G \) and \( M_v \), a minimal matching that isolates \( v \). By Lemma 8 \( M_v \) covers at most \( 2d \) vertices. Let \( G' = G \setminus (V(M_v) \cup \{v\}) \). By Theorem 5 \( G' \) has at most one component, this component is randomly matchable, and Lemma 13 yields that \( |V(G')| ≤ 4 + 4\sqrt{g} \), respectively \( |V(G')| ≤ 4 + 2\sqrt{2h} \). Hence \( G \) is a union of vertex \( v \), matching \( M_v \), and \( G' \), and in the orientable case we have
\[ |V(G)| = |\{v\}| + |V(M_v)| + |V(G')| ≤ 1 + 2d + |V(G')| ≤ 1 + 2d + 4 + 4\sqrt{g} ≤ 5 + 2d + 4\sqrt{g}. \]
In the nonorientable case \( |V(G)| ≤ 1 + 2d + |V(G')| ≤ 5 + 2d + 2\sqrt{2h} \), which completes the proof.

**Lemma 14.** For any \( d ≥ 6 \) such that
\[ \frac{6(2-2g)}{5-d} ≤ 5 + 2d + 4\sqrt{g}, \quad \text{respectively} \quad \frac{6(2-h)}{5-d_0} ≤ 5 + 2d_0 + 2\sqrt{2h}, \]
the maximum size of a 2-connected factor-critical equimatchable graph embeddable in the surface with orientable genus \( G \) (nonorientable genus \( h \)) is at most \( 5 + 2d + 4\sqrt{g} \), respectively \( 5 + 2d + 2\sqrt{2h} \) vertices.

**Proof.** We prove the lemma by contradiction. Let \( d \) be an integer such that \( d ≥ 6 \) and let \( G \) be a 2-connected factor-critical equimatchable graph embeddable in the orientable surface of genus \( g \) with \( |V(G)| > 5 + 2d + 4\sqrt{g} \). By our assumption
\[ |V(G)| > \frac{6(2-2g)}{5-d} \]
and thus by Lemma 12 \( G \) has a vertex with degree \( d' \) such that \( d' ≤ d \). Consequently, by Lemma 13 \( G \) has at most \( 5 + 2d' + 4\sqrt{g} ≤ 5 + 2d + 4\sqrt{g} \) vertices, which is a contradiction. The nonorientable case is analogous.

**Theorem 15.** Let \( m(g) \), respectively \( \tilde{m}(h) \), denote the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the orientable surface of genus \( g \), respectively nonorientable surface of genus \( h \). Then the following inequalities hold.

i) If \( g ≤ 2 \) and \( h ≤ 2 \), then
\[ 4\sqrt{2g} + 1 ≤ m(g) ≤ 4\sqrt{g} + 17 \quad \text{and} \quad 4\sqrt{h} + 1 ≤ \tilde{m}(h) ≤ 2\sqrt{2h} + 17. \]

ii) If \( g ≥ 3 \) and \( h ≥ 3 \), then
\[ 4\sqrt{2g} + 1 ≤ m(g) ≤ c_g\sqrt{g} + 5 \quad \text{and} \quad 4\sqrt{h} + 1 ≤ \tilde{m}(h) ≤ \tilde{c}_h\sqrt{h} + 5, \]
where \( c_g ≤ 12 \) and \( \tilde{c}_h ≤ 10 \) are positive real constants such that the sequences \( (c_g)_{g=3}^∞ \) and \( (\tilde{c}_h)_{h=3}^∞ \) are decreasing, \( \lim_{g→∞} c_g = 2\sqrt{7} + 2 < 7.3 \), and \( \lim_{h→∞} \tilde{c}_h = \sqrt{2}(\sqrt{7} + 1) < 5.2. \)
Proof. The lower bounds follow from Theorem 11 and the inequality $|\alpha| > \alpha - 1$ which holds for any real number $\alpha$. To prove the upper bounds, we distinguish two cases.

i) From Lemma 12 follows that if $G$ has more than $12(g - 1)$ vertices, then it has a vertex of degree at most 6, and hence by Lemma 13 at most $17 + 4\sqrt{g}$ vertices. The proof is concluded by observing that $17 + 4\sqrt{g} > 12(g - 1)$ holds for any $g \leq 2$. The nonorientable case is analogous.

ii) We start by determining the smallest $d$ such that $d \geq 6$ and

$$\frac{6(2 - 2g)}{5 - d} \leq 5 + 2d + 4\sqrt{g}$$

for a fixed integer $g \geq 3$. Solving (1) for $d$ we get that

$$d_g = \frac{5 - 4\sqrt{g} + \sqrt{112g + 120\sqrt{g} + 129}}{4}$$

is minimal such $d$ and it is easy to verify that for $g \geq 3$ is indeed $d_g \geq 6$. Therefore, by Lemma 14 $m(g) \leq 5 + 2d_g + 4\sqrt{g}$. Clearly, for the sequence $(c_g)_{g=3}^{\infty}$ defined by

$$c_g = \frac{5 + 4\sqrt{g} + \sqrt{112g + 120\sqrt{g} + 129}}{2\sqrt{g}}$$

$m(g) \leq c_g\sqrt{g} + 5$ for every $g \geq 3$. It can be verified by standard methods that the sequence is decreasing and has the claimed limit, which completes the proof of the orientable case. The nonorientable case is analogous.

In the investigation of 3-connected equimatchable graphs embeddable in a fixed surface Kawarabayashi and Plummer [6] proved that there is no such bipartite graph embeddable with face-width at least 3 at all. It is easy to see that there are arbitrarily large planar bipartite 2-connected equimatchable graphs.

Proposition 16. For any positive integer $k$ there is a planar 2-connected bipartite equimatchable graph with at least $k$ vertices.

Proof. Clearly, for any integer $k \geq 2$ the complete bipartite graph $K_{k,2}$ has the desired properties.

The following theorem shows that there are infinitely-many 2-connected bipartite equimatchable graphs with any given genus and face-width.

Theorem 17. For any positive integers $n$, $g$, and $k$ there exists a 2-connected bipartite equimatchable graph $G$ with at least $n$ vertices, orientable genus $g$, and an embedding in $S_g$ with face-width $k$. Similarly, for any positive integers $n$, $h$, and $k$ there exists a 2-connected bipartite equimatchable graph $G$ with at least $n$ vertices, nonorientable genus $h$, and an embedding in $N_h$ with face-width $k$.

Proof. We prove only the orientable case, since the nonorientable case is analogous. Take a 2-connected graph $G'$ with at least $n$ vertices, genus $g$, and with a genus embedding $\Pi'$ with face-width $k$, for example any sufficiently large 2-connected triangulation with a given genus and face-width; it is well known that such triangulations exist. We construct the desired graph $G$ starting from $G'$ by replacing every edge $e$ of $G'$ by $l$ parallel edges $e_1, \ldots, e_l$ for some fixed $l \geq 2$ and subdividing every edge $e_i$ by a new vertex $y_{e_i}$. Denote by $B$ the set of all vertices $y_{e_i}$ of $G$, that is, $B = \{y_{e_i} : e \in E(G'), 1 \leq i \leq l\}$. Let $A = V(G) \setminus B$.

Clearly, $G$ is bipartite and the vertices of $A$ form the smaller partite set of $G$. By [8, Theorem 3] a connected bipartite graph is equimatchable if and only if for any vertex $v$ from the smaller partite set there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| \leq |X|$. We prove that $G$ is equimatchable by exhibiting such set $X_v$ for every vertex $v$ from $A$. If a vertex $v$ is in $G'$ incident with an edge $e = uv$, then let $X_v = \{y_{e_1}, \ldots, y_{e_l}\}$. In $G$ we have $X_v \subseteq N(v)$ and $N(X_v) = \{u, v\}$, with possibly $u = v$ if $uv$ is a loop. Since $l \geq 2$, we have $|N(X_v)| \leq |X_v|$. Therefore, for every vertex $v$ from $A$ there exists a non-empty set $X_v$ such that $X_v \subseteq N(v)$ and $|N(X_v)| \leq |X_v|$, and hence by [8] $H$ is equimatchable. It is easy to see that multiplying and subdividing edges does not change the genus of the graph, and thus $\gamma(G) = \gamma(H)$. To construct the desired genus embedding $\Pi$ of $G$ with face-width $k$, start with $\Pi'$. For any edge $e = uv$ of $G'$, choose the preferred direction of $e$. If the preferred direction of $e$ is from $u$ to $v$, then in the rotation at $u$ replace the occurrence of $e$ by $e_1 \ldots e_k$ and replace the occurrence of $e^{-1}$ in the rotation at $v$ by $e_k \ldots e_1$. Finally, subdivide every edge $e_i$ by the new vertex $y_{e_i}$. Clearly, the
subdivided edges $e_1, \ldots, e_k$ bound $l - 1$ faces of length 4. Moreover, the occurrence of $e = uv$ in its face boundary is replaced by a sequence of two edges $(uye_1)(ye_1v)$ and the occurrence of $e^{-1}$ in its faces boundary is replaced by $(yge_k)(ye_ku)$. It is not difficult to see that union of any $m$ faces of $\Pi$ is union of at most $m$ faces of $\Pi'$ and hence the face-width of $\Pi$ is at least $k$. Since in $\Pi'$ there is a noncontractible curve of minimum length that intersects only vertices of $G'$ (see [13]), there is a homotopically equivalent noncontractible curve whose intersection with $G$ consists from precisely $k$ vertices of $G$. Thus face-width of $\Pi$ is at most $k$, which completes the proof.

Theorem 17 and the results of [6] suggest the following open problem.

**Problem.** Are there infinitely-many 3-connected bipartite equimatchable graphs embeddable in a given surface with face-width at most 2?

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