On maximum $k$-edge-colorable subgraphs of bipartite graphs

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Abstract

If $k \geq 0$, then a $k$-edge-coloring of a graph $G$ is an assignment of colors to edges of $G$ from the set of $k$ colors, so that adjacent edges receive different colors. A $k$-edge-colorable subgraph of $G$ is maximum if it is the largest among all $k$-edge-colorable subgraphs of $G$. For a graph $G$ and $k \geq 0$, let $\nu_k(G)$ be the number of edges of a maximum $k$-edge-colorable subgraph of $G$. In 2010 Mkrtchyan et al. proved that if $G$ is a cubic graph, then $\nu_2(G) \leq \frac{|V| + 2\delta(G)}{4}$. This result implies that if the cubic graph $G$ contains a perfect matching, in particular when it is bridgeless, then $\nu_2(G) \leq \frac{\nu_1(G) + \nu_3(G)}{2}$. One may wonder whether there are other interesting graph-classes, where a relation between $\nu_2(G)$ and $\frac{\nu_1(G) + \nu_3(G)}{2}$ can be proved. Related with this question, in this paper we show that $\nu_k(G) \geq \frac{\nu_{k-i}(G) + \nu_{k+i}(G)}{2}$ for any bipartite graph $G$, $k \geq 0$ and $i = 0, 1, ..., k$.

Keywords: Edge-coloring; bipartite graph; $k$-edge-colorable subgraph; maximum $k$-edge-colorable subgraph.

1. Introduction

In this paper graphs are assumed to be finite, undirected and without loops, though they may contain multiple edges. The set of vertices and edges of a graph $G$ is denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex $u$ of $G$ is denoted by $d_G(u)$. Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree of a vertex of $G$. A graph $G$ is regular, if $\delta(G) = \Delta(G)$. The girth of the graph is the length of the shortest cycle in its underlying simple graph.

A bipartite graph is a graph whose vertices can be divided into two disjoint sets $U$ and $W$, such that every edge connects a vertex in $U$ to one in $W$. A graph is nearly bipartite, if it contains a vertex, whose removal results into a bipartite graph.

A matching in a graph $G$ is a subset of edges such that no vertex of $G$ is incident to two edges from the subset. A maximum matching is a matching that contains the largest possible number of edges.

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If $k \geq 0$, then a graph $G$ is called $k$-edge colorable, if its edges can be assigned colors from a set of $k$ colors so that adjacent edges receive different colors. The smallest integer $k$, such that $G$ is $k$-edge colorable is called chromatic index of $G$ and is denoted by $\chi'(G)$. The classical theorem of Shannon states that for any graph $G \Delta(G) \leq \chi'(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$ [18, 21]. On the other hand, the classical theorem of Vizing states that for any graph $G \Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ [21, 22]. Here $\mu(G)$ is the maximum multiplicity of an edge of $G$. A graph is class I if $\chi'(G) = \Delta(G)$, otherwise it is class II.

If the edges of $G$ are colored, then for a color $\alpha$ let $E_\alpha$ be the set of edges of $G$ that are colored with $\alpha$. Observe that $E_\alpha$ is a matching. We say that a vertex $v$ is incident to the color $\alpha$, if $v$ is incident to an edge from $E_\alpha$. If $v$ is not incident to the color $\alpha$, then we say that $v$ misses the color $\alpha$. Now, if we have two different colors $\alpha$ and $\beta$, then consider the subgraph of $G$ induced by $E_\alpha \cup E_\beta$. Observe that the components of this subgraph are paths or even cycles. The components which are paths are usually called $\alpha-\beta$-alternating paths or Kempe chains [21]. If $P$ is an $\alpha-\beta$-alternating path connecting the vertices $u$ and $v$, then we can exchange the colors on $P$ and obtain a new edge-coloring of $G$. Observe that if $u$ is incident to the color $\alpha$ in the former edge-coloring, then in the new one it will miss the color $\alpha$.

If $k < \chi'(G)$, we cannot color all edges of $G$ with $k$ colors. Thus it is reasonable to investigate the maximum number of edges that one can color with $k$ colors. A subgraph $H$ of a graph $G$ is called maximum $k$-edge-colorable, if $H$ is $k$-edge-colorable and contains maximum number of edges among all $k$-edge-colorable subgraphs. For $k \geq 0$ and a graph $G$ let $$\nu_k(G) = \max \{|E(H)| : H \text{ is a } k\text{-edge-colorable subgraph of } G\}.$$ Clearly, a $k$-edge-colorable subgraph is maximum if it contains exactly $\nu_k(G)$ edges.

There are several papers where the ratio $\frac{|E(H)|}{|V(G)|}$ has been investigated. Here $H_k$ is a maximum $k$-edge-colorable subgraph of $G$. [5, 10, 13, 16, 23] prove lower bounds for the ratio when the graph is regular and $k = 1$. For regular graphs of high girth the bounds are improved in [7]. Albertson and Haas have investigated the problem in [1, 2] when $G$ is a cubic graph. See also [13], where the authors proved that for every cubic graph $G \nu_2(G) \geq \frac{5}{6}|V(G)|$ and $\nu_3(G) \geq \frac{7}{6}|V(G)|$. Moreover, [3] shows that for any cubic graph $G \nu_2(G) + \nu_3(G) \geq 2|V(G)|$.

Bridgeless cubic graphs that are not 3-edge-colorable are usually called snarks [6], and the problem for snarks is investigated by Steffen in [10, 20]. This lower bound has also been investigated in the case when the graphs need not be cubic in [5, 11, 17]. Kosovski and Rizzi have investigated the problem from the algorithmic perspective [12, 17]. Since the problem of constructing a $k$-edge-colorable graph in an input graph is NP-complete for each fixed $k \geq 2$, it is natural to investigate the (polynomial) approximability of the problem. In [12], for each $k \geq 2$ an algorithm for the problem is presented. There for each fixed value of $k \geq 2$, algorithms are proved to have certain approximation ratios and they are tending to 1 as $k$ tends to infinity.

Some structural properties of maximum $k$-edge-colorable subgraphs of graphs are proved
in [3, 14]. In particular, there it is shown that every set of disjoint cycles of a graph with 
$\Delta = \Delta(G) \geq 3$ can be extended to a maximum $\Delta$-edge colorable subgraph. Also there it is shown that a maximum $\Delta$-edge colorable subgraph of a simple graph is always class I. Finally, if $G$ is a graph with girth $g \in \{2k, 2k + 1\}$ ($k \geq 1$) and $H$ is a maximum $\Delta$-edge colorable subgraph of $G$, then 
$\frac{|E(H)|}{|E(G)|} \geq \frac{2k}{2k+1}$ and the bound is best possible is a sense that there is an example attaining it.

In [13] Mkrtchyan et al. proved that for any cubic graph $\nu_2(G) \leq \left\lfloor \frac{V(G)+2e(G)}{4} \right\rfloor$. For bridgeless cubic graphs, which by Petersen theorem have a perfect matching, this inequality becomes, $\nu_2(G) \leq \frac{\nu_1(G)+\nu_3(G)}{2}$. One may wonder whether there are other interesting graph-classes, where a relation between $\nu_2(G)$ and $\frac{\nu_1(G)+\nu_3(G)}{2}$ can be proved. In [9], the following conjecture is stated:

**Conjecture 1.** (9) For each $k \geq 1$ and a nearly bipartite graph $G$

$$
\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.
$$

In the same paper the bipartite analogue of this conjecture is stated, which says that for bipartite graphs the statement of the Conjecture 1 holds without the sign of floor. Note that [9] verifies Conjecture 1 and its bipartite analogue when $G$ contains at most one cycle.

The present paper is organized as follows: In Section 2, some auxiliary results are stated. Section 3 proves the main result of the paper, which states that for any bipartite graph $G$, $k \geq 0 \ \nu_k(G) \geq \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}$, where $i = 0, 1, ..., k$. Section 4 discusses the future work.

Terms and concepts that we do not define, can be found in [24].

2. Auxiliary results

In this section, we present some auxiliary results that will be useful later. The first two of them are simple consequences of a classical theorem due to König [21, 24], which states for any bipartite graph $G$, we have $\chi'(G) = \Delta(G)$.

**Proposition 1.** Let $G$ be a bipartite graph and let $k \geq 0$. Then a subgraph $F$ of $G$ is $k$-edge-colorable, if and only if $\Delta(F) \leq k$.

**Proposition 2.** Let $k \geq 0$ and let $G$ be a $k$-regular bipartite graph. Then for $i = 0, 1, ..., k$ we have $\nu_i(G) = i \cdot \frac{|V(G)|}{2}$.

Our next auxiliary result follows from an observation that a vertex can be incident to at most $k$ edges in a $k$-edge-colorable graph.

**Proposition 3.** If $G$ is a graph, $v$ is a vertex of $G$ and $k \geq 0$. Then

$$
\nu_k(G) \leq \nu_k(G - v) + k.
$$

The next result states that if one is removing an edge from a graph, then $\nu_k(G)$ can decrease by at most one.
**Proposition 4.** If $G$ is a graph, $e$ is an edge of $G$ and $k \geq 0$. Then

$$\nu_k(G-e) \leq \nu_k(G) \leq \nu_k(G-e) + 1.$$ 

In order to prove our next auxiliary result, we will use alternating paths.

**Lemma 1.** Let $G$ be a bipartite graph, $e = uv$ be an edge of $G$, and $j \geq 0$. Then for any maximum $j$-edge-colorable subgraph $H_j$ with $e \notin E(H_j)$, we have $d_{H_j}(u) = j$ or $d_{H_j}(v) = j$.

**Proof.** Assume that there is a maximum $j$-edge-colorable subgraph $H_j$ that does not contain $e$ and with $d_{H_j}(u) \leq j - 1$ and $d_{H_j}(v) \leq j - 1$. Then there are colors $\alpha$ and $\beta$ of $H_j$ such that $\alpha$ misses at $u$ and $\beta$ misses at $v$. Clearly, $\alpha$ must be present at $v$ and $\beta$ must be present at $u$, since $H_j$ is maximum $j$-edge-colorable. Consider the $\alpha - \beta$ alternating paths starting at $u$ and $v$. If they are the same, then we get an odd cycle contradicting the fact that $G$ is bipartite. Hence they are different. Exchange the colors $\alpha$ and $\beta$ on one of them and color $e$. Observe that we have got a $j$-edge-colorable subgraph of $G$ with $|E(H_j)| + 1$ edges contradicting the maximality of $H_j$. Thus the statement of the lemma should be true. \(\square\)

If $M$ is a matching in a graph $G$, then a simple odd path $P$ is said to be $M$-augmenting, if the odd edges of $P$ lie outside $M$, the even edges of $P$ belong to $M$, and the end-points of $P$ are not covered by $M$. It is easy to see that if $G$ contains an $M$-augmenting path, then $M$ is not a maximum matching in $G$. The classical theorem of Berge [4], states that if $M$ is not a maximum matching in $G$, then $G$ must contain an $M$-augmenting path. In the end of this section, we prove the analogue of this result for $k$-edge-colorable subgraphs of bipartite graphs. It is quite plausible that our result can be derived using the general result about maximality of so-called $c$-matchings (Theorem 2 of Section 8, page 152 of [4]), however, here we will give a direct proof that works only for bipartite graphs.

We will require some definitions. For a positive integer $k \geq 1$, bipartite graph $G$ and a $k$-edge-colorable subgraph $A_k$ of $G$ define an $A_k$-augmenting path as follows.

**Definition 1.** A simple $u-v$-path $P$ is $A_k$-augmenting, if it is of odd length, the even edges of $P$ belong to $A_k$, the odd edges lie outside $A_k$ and $d_{A_k}(u) \leq k - 1$, $d_{A_k}(v) \leq k - 1$.

Observe that if $G$ contains an $A_k$-augmenting path $P$, then $|E(A_k)| < \nu_k(G)$. In order to see this, consider a subgraph $B_k$ of $G$ obtained from $A_k$ by removing the even edges of $P$ from $A_k$ and adding the odd edges. Observe that any vertex $w$ of $G$ has degree at most $k$ in $B_k$, hence $B_k$ is $k$-edge-colorable by Proposition [1]. Moreover, $|E(B_k)| = |E(A_k)| + 1$.

The following lemma states that the converse is also true.

**Lemma 2.** Let $G$ be a bipartite graph, $k \geq 1$ and let $A_k$ be a $k$-edge-colorable subgraph with $|E(A_k)| < \nu_k(G)$. Then $G$ contains an $A_k$-augmenting path.

**Proof.** For the $k$-edge-colorable subgraph $A_k$ consider all maximum $k$-edge-colorable subgraphs $H_k$ and choose one maximizing $|E(A_k) \cap E(H_k)|$. By an alternating component, we will mean a path or an even cycle of $G$ whose edges belong to $E(A_k)\setminus E(H_k)$ and $E(H_k)\setminus E(A_k)$, alternatively. Observe that any alternating component is either an even
cycle or an even path or an odd path. Moreover, since $|E(A_k)| < \nu_k(G)$, there is at least one edge in $E(H_k) \setminus E(A_k)$, hence $G$ contains at least one alternating component.

We claim that $G$ contains no alternating component $C$ that is an even cycle. On the opposite assumption, consider a subgraph $H'_k$ of $G$ obtained from $H_k$ by exchanging the edges on $C$. Observe that the degree of any vertex of $G$ is the same as it was in $H_k$. Hence $H'_k$ is $k$-edge-colorable by Proposition 1. Moreover, $|E(H'_k)| = |E(H_k)| = \nu_k(G)$, hence $H'_k$ is maximum $k$-edge-colorable. However $|E(A_k) \cap E(H'_k)| > |E(A_k) \cap E(H_k)|$, which contradicts our choice of $H_k$.

Now, consider all alternating components $C$ of $G$ and among them choose one maximizing $|E(C)|$. From the previous paragraph we have that $C$ is a path. Let us show that $C$ is an odd path. Assume that $C$ is an even path connecting vertices $u$ and $v$. Assume that $u$ is incident to an edge of $E(H_k) \setminus E(A_k)$ and $v$ is incident to $E(A_k) \setminus E(H_k)$ on $C$. Let us show that $d_{H_k}(v) \leq k - 1$. If $d_{H_k}(v) = k$, then $v$ is incident to an edge $e = vw \in E(H_k) \setminus E(A_k)$. Observe that $w \notin V(C)$. If $w \in V(C)$, then either we have an alternating component that is a cycle, or we have an odd cycle. Both of the cases are contradictory. Thus $w \notin V(C)$. Now observe that $C \cup \{e\}$ forms an alternating component with more edges than $C$. This contradicts our choice of $C$.

Thus $d_{H_k}(v) \leq k - 1$. Consider a subgraph $H'_k$ of $G$ by exchanging the edges on $C$. Observe that the degree of any vertex of $G$ is the same as it was in $H_k$ except $v$ which has degree at most $k$ and $u$ whose degree has decreased by one. Hence $H'_k$ is $k$-edge-colorable by Proposition 1. Moreover, $|E(H'_k)| = |E(H_k)| = \nu_k(G)$, hence $H'_k$ is maximum $k$-edge-colorable. However $|E(A_k) \cap E(H'_k)| > |E(A_k) \cap E(H_k)|$, which contradicts our choice of $H_k$.

Thus $C$ is an odd path. Again let the end-points of $C$ be $u$ and $v$. If $u$ and $v$ are incident to edges $E(A_k) \setminus E(H_k)$ on $C$, then similarly to previous paragraph, one can show that $d_{H_k}(u) \leq k - 1$ and $d_{H_k}(v) \leq k - 1$. If we exchange the edges of $H_k$ on $C$ we would find a larger $k$-edge-colorable subgraph, contradicting the maximality of $H_k$.

Thus, $u$ and $v$ are incident to edges $E(H_k) \setminus E(A_k)$ on $C$. Similarly to previous paragraph, one can show that $d_{A_k}(u) \leq k - 1$ and $d_{A_k}(v) \leq k - 1$. Now, it is not hard to see that $C$ is an $A_k$-augmenting path. The proof of the lemma is complete. 

When $G$ is not bipartite, $G$ may possess an augmenting path with respect to a maximum $k$-edge-colorable subgraph. Consider the graph from Figure 1 and let $A_2$ be the subgraph colored with $\alpha$ and $\beta$. It is easy to see that $A_2$ is maximum 2-edge-colorable in $G$, however $G$ contains an $A_2$-augmenting path.

![Figure 1: The statement of Lemma 2 is not true when $G$ is not bipartite.](image)

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3. The main results

In this section, we obtain the main result of the paper. Our first theorem proves a lower bound for \( \nu_k(G) \) in terms of the average of \( \nu_{k-1}(G) \) and \( \nu_{k+1}(G) \).

**Theorem 1.** For any bipartite graph \( G \) and \( k \geq 1 \)

\[
\nu_k(G) \geq \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.
\]

**Proof.** Assume that the statement of the theorem is wrong. Let \( G \) be a counter-example minimizing \( |V(G)| + |E(G)| \). We prove a series of claims that establish various properties of \( G \).

**Claim 1.** \( G \) is connected and \( |V(G)| \geq 2 \).

**Proof.** If \( G \) is the graph with one vertex, then clearly it is bipartite and \( \nu_i(G) = 0 \) for any \( i \geq 0 \), hence it is not a counter-example to our theorem. Thus, \( |V(G)| \geq 2 \). Let us show that \( G \) is connected. Assume that \( G \) contains \( t \geq 2 \) components, which are \( G^{(1)}, \ldots, G^{(t)} \). We have that for \( i \geq 0 \)

\[
\nu_i(G) = \nu_i(G^{(1)}) + \ldots + \nu_i(G^{(t)}),
\]

hence

\[
\nu_k(G) = \nu_k(G^{(1)}) + \ldots + \nu_k(G^{(t)}) \geq \frac{\nu_{k-1}(G^{(1)}) + \nu_{k+1}(G^{(1)})}{2} + \ldots + \frac{\nu_{k-1}(G^{(t)}) + \nu_{k+1}(G^{(t)})}{2}
\]

\[
= \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.
\]

Thus, \( G \) is not a counter-example to our statement contradicting our assumption. Here we used the fact that \( G^{(1)}, \ldots, G^{(t)} \) are smaller than \( G \), hence they are not counter-examples to our theorem. The proof of the claim is complete.

**Claim 2.** For any maximum \((k-1)\)-edge-colorable subgraph \( H_{k-1} \) and any maximum \((k+1)\)-edge-colorable subgraph \( H_{k+1} \), we have

\[
E(H_{k-1}) \cup E(H_{k+1}) = E(G).
\]

**Proof.** If \( E(H_{k-1}) \cup E(H_{k+1}) \neq E(G) \) for some \( H_{k-1} \) and \( H_{k+1} \), then there exist an edge \( e \) such that \( e \) lies outside \( H_{k-1} \) and \( H_{k+1} \). Hence

\[
\nu_{k-1}(G - e) = \nu_{k-1}(G)
\]

and

\[
\nu_{k+1}(G - e) = \nu_{k+1}(G),
\]

Thus, \( G \) is not a counter-example to our statement contradicting our assumption. Here we used the fact that \( G^{(1)}, \ldots, G^{(t)} \) are smaller than \( G \), hence they are not counter-examples to our theorem. The proof of the claim is complete.

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therefore we get:

\[ \nu_k(G) \geq \nu_k(G-e) \geq \frac{\nu_{k-1}(G-e) + \nu_{k+1}(G-e)}{2} = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}. \]

Here we used the fact that the bipartite graph \( G-e \) is not a counter-example. \( \Box \)

Our next claim states that removing an edge from \( G \) does not decrease the size of \( \nu_k(G) \).

**Claim 3.** For any edge \( e \) of \( G \), we have \( \nu_k(G) = \nu_k(G-e) \).

*Proof.* If \( \nu_k(G) = 1 + \nu_k(G-e) \) (Proposition 4), then

\[ \nu_k(G) = 1 + \nu_k(G-e) \geq 1 + \frac{\nu_{k-1}(G-e) + \nu_{k+1}(G-e)}{2} = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}. \]

Here we used the fact that \( G-e \) is not a counter-example and Proposition 4 twice. \( \Box \)

Our final claim establishes some relations for maximum and minimum degrees of \( G \). Its proof makes use of the fan-argument by Vizing [21, 22].

**Claim 4.** \( \Delta(G) \leq 2k \) and \( \delta(G) \leq k \).

*Proof.* Let \( H_{k-1} \) and \( H_{k+1} \) be a maximum \((k-1)\)-edge-colorable and a maximum \((k+1)\)-edge-colorable subgraphs of \( G \), respectively. By Claim 2 \( G \) is a union of \( H_{k-1} \) and \( H_{k+1} \), hence it is a union of \( 2k \) matchings. Thus \( \Delta(G) \leq 2k \).

Let us show that \( \delta(G) \leq k \). Assume that \( \delta(G) \geq k + 1 \). If \( \Delta(G) \leq k + 1 \), then \( G \) is \((k+1)\)-regular, hence from Proposition 2 we have \( \nu_i(G) = i \cdot \frac{|V|}{2} \) for \( i = k-1, k, k+1 \). Therefore

\[ \nu_k(G) = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}. \]

Thus, \( G \) is not a counter-example. Hence, we can assume that \( \Delta(G) \geq k + 2 \), and therefore \( E(H_{k-1}) \setminus E(H_{k+1}) \neq \emptyset \). Let \( e = uw \) be an edge from this set. Then \( u \) or \( v \) must be incident to all \((k+1)\) colors of \( H_{k+1} \) (apply Lemma 1 with \( j = k+1 \)). Assume that this vertex is \( v \). Let us show that \( u \) is incident to all \((k+1)\) colors of \( H_{k+1} \) as well.

On the opposite assumption, assume that \( u \) misses a color \( \beta \) of \( H_{k+1} \). Then \( v \) must be incident to an edge \( e_w = vw \) of color \( \beta \) in \( H_{k+1} \), as \( d_{H_{k+1}}(v) = k+1 \). Since \( d_{H_{k+1}}(v) \leq k-1 \) and \( d_{H_{k+1}}(v) = k+1 \), there is an edge \( e_z = vz \) incident to \( v \) such that \( e_z \in E(H_{k+1}) \setminus E(H_{k-1}) \). Let the color of \( e_z \) in \( H_{k+1} \) be \( \alpha \).

If \( \alpha \) is missing at \( u \), then consider a subgraph \( H_{k+1}' \) of \( G \) obtained from \( H_{k+1} \) by removing the edge \( e_z \), adding \( e \) to \( H_{k+1}' \) and coloring \( e \) with \( \alpha \). Observe that \( H_{k+1}' \) is \((k+1)\)-edge-colorable, \( |E(H_{k+1})| = |E(H_{k+1})| = \nu_k(G) \). Hence \( H_{k+1}' \) is maximum \((k+1)\)-edge-colorable. However, \( e_z \notin E(H_{k-1}) \cup E(H_{k+1}) \) violating Claim 2.

Thus, we can assume that \( \alpha \) is present at \( u \), hence it is different from \( \beta \). Consider the \( \alpha - \beta \) alternating path \( P_u \) of \( H_{k+1} \) starting from \( u \). We claim that \( P_u \) passes through \( v \). If
not, we could have exchanged the colors on $P_u$, remove $e_z$ from $H_{k+1}$, add $e$ to $H_{k+1}$, color it with $\alpha$ and get a new maximum $(k+1)$-edge-colorable subgraph violating Claim 2. Thus, $P_u$ passes through $v$. We claim that it passes first via $z$, then via $v$ and $w$. If $P_u$ first passes via $w$, then together with $e$ we get an odd cycle contradicting our assumption.

Let $P_w$ be the final part of $P_u$ that starts from $w$. Consider a $(k+1)$-edge-colorable subgraph $H_{k+1}'$ of $G$ obtained from $H_{k+1}$ as follows: exchange the colors on $P_w$, color $e$ with $\beta$, color $e_w$ with $\alpha$ and remove $e_z$ from $H_{k+1}$. Observe that $H_{k+1}'$ is $(k+1)$-edge-colorable, $|E(H_{k+1}')| = |E(H_{k+1})| = \nu_{k+1}(G)$. Hence $H_{k+1}'$ is maximum $(k+1)$-edge-colorable. However, $e_z \notin E(H_{k+1}) \cup E(H_{k+1}')$ violating Claim 2.

Thus $u$ and $v$ must be incident to any $(k+1)$ colors of $H_{k+1}$, in particular, $d(u) \geq k+2$ and $d(v) \geq k+2$. Observe that by Claim 2 any vertex of degree at least $k+2$ must be incident to an edge from $E(H_{k-1}) \setminus E(H_{k+1})$. Consider the bipartite graph $J = G - (E(H_{k-1}) \setminus E(H_{k+1}))$. Observe that $J$ is a $(k+1)$-regular bipartite graph with $V(J) = V(G)$. Hence from Proposition 2 we have $\nu_i(G) = i \cdot \frac{|V|}{2}$ for $i = k - 1, k, k+1$, and therefore

$$\nu_k(G) = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.$$

This means that $G$ is not a counter-example to our statement contradicting our assumption. Hence $\delta(G) \leq k$. The proof of the claim is complete. \qed

We are ready to prove the theorem. By Claim 4 $\delta(G) \leq k$, hence there is a vertex $u$ with $d_G(u) \leq k$. On the other hand, by Claim 1 $G$ is connected and $|V| \geq 2$, hence $d_G(u) \geq 1$. Thus, there is an edge $e = uv$ incident to $u$. By Claim 3 there is a maximum $k$-edge-colorable subgraph $H_k$ that does not contain $e$. By Lemma 1 $d_{H_k}(u) = k$ or $d_{H_k}(v) = k$ for any such $H_k$. Since $d_{G-e}(u) \leq k - 1$, we have $d_{H_k}(v) = k$ for any maximum $k$-edge-colorable subgraph $H_k$ that does not contain $e$.

By Proposition 3 we have $\nu_k(G) \leq \nu_k(G - v) + k$. Let us show that $\nu_k(G) = \nu_k(G - v) + k$. Assume that $\nu_k(G) \leq \nu_k(G - v) + k - 1$. Since $\nu_k(G) = \nu_k(G - e)$ (Claim 3) and $G - e - v = G - v$, we have $\nu_k(G - e) \leq \nu_k(G - e - v) + k - 1$.

Choose a maximum $k$-edge-colorable subgraph $H^{(0)}$ of $G - e - v$. If $H^{(0)}$ is maximum in $G - e$, then since $e$ does not lie in $H^{(0)}$, we have a contradiction with $d_{H^{(0)}}(v) = k$ as $d_{H^{(0)}}(v) = 0$. Thus $H^{(0)}$ is not maximum in $G - e$. By Lemma 2 there is a $k$-edge-colorable subgraph $H^{(1)}$ which is obtained from $H^{(0)}$ by shifting the edges on an $H^{(0)}$-augmenting path in $G - e$. Observe that $d_{H^{(1)}}(v) \leq 1$. If $H^{(1)}$ is maximum in $G - e$, then we have a contradiction with $d_{H^{(1)}}(v) = k$ as $d_{H^{(1)}}(v) = 1$. Thus $H^{(1)}$ is not maximum in $G - e$. By repeating the argument and applying Lemma 2 at most $(k - 1)$ times, we will obtain a maximum $k$-edge-colorable subgraph $H^{(i)}$ of $G - e$ with $d_{H^{(i)}}(v) \leq k - 1$ contradicting the fact that $d_{H_k}(v) = k$ for any maximum $k$-edge-colorable subgraph $H_k$ of $G$ that does not contain $e$. 

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Thus, $\nu_k(G) = \nu_k(G - v) + k$. We have

$$
\nu_k(G) = k + \nu_k(G - v) \geq k + \frac{\nu_{k-1}(G - v) + \nu_{k+1}(G - v)}{2}
$$

$$
= \frac{\nu_{k-1}(G - v) + k - 1 + \nu_{k+1}(G - v) + k + 1}{2} \geq \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.
$$

This contradicts the fact that $G$ is a counter-example to our statement. Here we used the fact that $G - v$ is not a counter-example and Proposition 3 twice. The proof of the theorem is complete.

The proved theorem is equivalent to the following

**Remark 1.** If $G$ is a bipartite graph, then

$$
\nu_1(G) - \nu_0(G) \geq \nu_2(G) - \nu_1(G) \geq \nu_3(G) - \nu_2(G) \geq \ldots.
$$

Below we derive the main result of the paper as a corollary to the theorem proved above:

**Corollary 1.** Let $G$ be a bipartite graph and let $k \geq 0$. Then for $i = 0, 1, \ldots, k$ we have

$$
\nu_k(G) \geq \frac{\nu_{k-i}(G) + \nu_{k+i}(G)}{2}.
$$

**Proof.** We prove the statement by induction on $i$. When $i = 0$, the statement is trivial. When $i = 1$, it follows from Theorem 1. We will assume that the statement is true for $i - 1$, and prove it for $i$.

By induction hypothesis we have

$$
\nu_k(G) \geq \frac{\nu_{k-i+1}(G) + \nu_{k+i-1}(G)}{2}.
$$

By applying Theorem 1 on $\nu_{k-i+1}(G)$ and $\nu_{k+i-1}(G)$ we have

$$
\nu_k(G) \geq \frac{\nu_{k-i+2}(G) + \nu_{k-i}(G) + \nu_{k+i-2}(G) + \nu_{k+i}(G)}{4} = \frac{\nu_{k-i}(G) + \nu_{k+i}(G)}{4} + \frac{\nu_{k-i+2}(G) + \nu_{k+i-2}(G)}{4}.
$$

So, in order to complete the proof of the corollary, we need to show

$$
\nu_{k-i+2}(G) + \nu_{k+i-2}(G) \geq \nu_{k-i}(G) + \nu_{k+i}(G).
$$

Using Remark 1, we have

$$
\nu_{k-i+1}(G) - \nu_{k-i}(G) \geq \nu_{k-i+2}(G) - \nu_{k-i+1}(G) \geq \cdots \geq \nu_{k+i-1}(G) - \nu_{k+i-2}(G) \geq \nu_{k+i}(G) - \nu_{k+i-1}(G).
$$

The last inequality implies

$$
[\nu_{k-i+1}(G) - \nu_{k-i}(G)] + [\nu_{k-i+2}(G) - \nu_{k-i+1}(G)] \geq [\nu_{k+i-1}(G) - \nu_{k+i-2}(G)] + [\nu_{k+i}(G) - \nu_{k+i-1}(G)],
$$
or
\[
\nu_{k-i+2}(G) - \nu_{k-i}(G) \geq \nu_{k+i}(G) - \nu_{k+i-2}(G),
\]
which is equivalent to [1]. The proof of the corollary is complete. \qed

4. Future Work

For a (not necessarily bipartite) graph \( G \), let \( b(G) \) be the smallest number of vertices of \( G \) whose removal results into a bipartite graph. One can easily see that \( b(G) \) coincides with the minimum number of vertices of \( G \), such that any odd cycle of \( G \) contains a vertex from these vertices. \( b(G) \) is a well studied parameter frequently appearing in various papers on Graph theory and Algorithms. It can be easily seen that a graph \( G \) is bipartite if and only if \( b(G) = 0 \), and is nearly bipartite if and only if \( b(G) \leq 1 \).

We suspect that:

Conjecture 2. Let \( G \) be a graph and let \( k \geq 0 \). Then for \( i = 0, 1, ..., k \) we have
\[
\nu_k(G) \geq \frac{\nu_{k-i}(G) + \nu_{k+i}(G) - b(G)}{2}.
\]

Observe that when \( G \) is bipartite, we get the statement of Corollary [1]. On the other hand, when \( G \) is nearly bipartite and \( i = 1 \), we get the statement of the Conjecture [1].

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