Ridge energy for thin nematic polymer networks

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Minimizing the elastic free energy of a thin sheet of nematic polymer network among smooth isometric immersions of the flat surface representing the configuration at the time of crosslinking, as purported by the mainstream theory, may clash with the scarcity of such immersions. In this paper, to broaden the class of admissible spontaneous deformations, we consider ridged isometric immersions, which can cause a sharp ridge in the immersed surfaces. We propose a model to compute the extra energy distributed along such ridges. This energy comes from bending and is shown to scale quadratically with the sheet’s thickness, falling just in between stretching and bending energies. We put our theory to the test by studying the spontaneous deformation of a disk on which a radial hedgehog was imprinted at the time of crosslinking. We predict the number of ridges that develop in terms of the degree of order induced in the material by external agents (such as heat and illumination).

I. INTRODUCTION

Nematic elastomers are rubber materials with a fluid-like component constituted by elongated, rod-like molecules appended to the crosslinked polymer strands that form the background matrix. The fluid component is ordered as nematic liquid crystals can be, which makes these solid materials very susceptible to external stimuli, such as heat, light, and environmental humidity. The prompt response to these stimuli, so characteristic of liquid crystals, once transferred to the solid matrix, makes it possible to do work and change the shape of bodies with no direct contact. The possible technological applications of these materials are boundless (see, for example, the papers [1–9], and above all the review [10]), but a number of theoretical challenges remain open [11]; this paper is concerned with one of them.

The order established in the material by the mutual interaction of nematic molecules is described by a scalar order parameter, representing the degree of molecular alignment, and a director, representing the average direction of alignment. Actually, there are two sets of these order parameters, namely, the pair \((s_0, m)\) for the reference configuration of the rubber matrix, which here will be taken to be the configuration where the crosslinking takes place, and the pair \((s, n)\) for the current (deformed) configuration, the one the rubber matrix takes on in response to an applied stimulus (more details are given in Sect. II below). The director \(n\) can be tied to the deformation of the body in several ways, the spectrum going from complete independence to complete enslaving. Following the terminology introduced in [10], we call nematic polymer networks the nematic elastomers in which the crosslinking is so tight that the nematic director remains enslaved to the deformation[1]; these are the specific nematic elastomers treated here. The reason for this choice will soon become clear.

The most striking manifestation of the ability of nematic polymer networks to perform changes in shape is perhaps achieved when they are thin sheets. We represent one such sheet as a slab \(S\) of thickness \(2h\) extending in the reference configuration on both sides of a flat surface \(S\). The director \(m\) is blueprinted on \(S\) (in its own plane), uniformly reproduced across the thickness, with

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1 This name has not yet met with universal acceptance. Some also say that these are liquid crystal glasses [12][13], while others prefer to say that they are simply nematic elastomers with a locked (or frozen) director [16].
a given scalar order parameter \( s_0 \). External stimuli may act on the degree of order, changing \( s_0 \) into \( s \), in a programmable way. The system is thus carried out of equilibrium and a deformation ensues, for the free energy to attain a minimum under the changed circumstances.

An elastic free-energy density, \( f_e \), is available for bulk materials in three space dimensions since the pioneering work \( 17 \); it is delivered by the “trace formula”, derived from assuming an anisotropic Gaussian distribution of the polymer chains that constitute the rubber matrix. This formula features both the deformation \( f \) of the three-dimensional body \( \mathcal{B} \) occupied by the material and measures of anisotropy in both reference and current configurations of \( \mathcal{B} \) (see Sect. [11]). For a sufficiently thin slab \( S \), however, one’s desire is to reduce \( f_e \) to a function of the mapping \( y \) that only changes the flat reference mid surface \( S \) into a curved surface \( \mathcal{I} \) in the current configuration.

In a nematic polymer network, for which \( f_e \) eventually depends only on \( f \), such a dimension reduction was performed in \( 21 \) by revisiting (and extending) a standard method of the theory of plates, known as the Kirchhoff-Love hypothesis \( 22 \). As expected, this method delivers a surface elastic energy with two components, a stretching energy \( f_s \) scaling like \( h \), and a bending energy \( f_b \) scaling like \( h^3 \): \( f_s \) depends only on the two-dimensional stretching (or metric) tensor \( C := (\nabla y)^\top (\nabla y) \), while \( f_b \) also depends on the invariant measures of curvature of \( \mathcal{I} \) and the relative orientation of \( n \) in the frame of principal directions of curvature. Not only do \( f_s \) and \( f_b \) scale differently with \( h \), they are also basically different things. By Gauss’ theorem egregium \( 23 \), p. 139, the Gaussian curvature \( K \) of \( \mathcal{I} \) is fully determined by the metric tensor \( C \), thus deserving the name of intrinsic curvature. As a consequence, \( f_b \) depends only on the intrinsic curvature, whereas \( f_s \) also depends on extrinsic measures of curvature, relating on how \( \mathcal{I} \) is embedded in three-dimensional space.

For moderately curved surfaces \( \mathcal{I} \) and sufficiently thin slabs \( S \), for which \( f_b \) can be neglected relative to \( f_s \), the energy minimizing shapes are isometric immersion \( \mathcal{I} \) of the metric tensor \( C_0 \) that minimizes \( f_s \). The search for such immersions corresponding to a variety of imprinted \( m \) fields has been the subject of a vast, elegant literature (see, among others, \( 15 \) [24–30]). This may seem to solve the direct morphic mechanics problem for nematic polymer networks, namely, how to identify the shapes produced by a certain imprinted director field \( m \). More difficult (and less visited), but affordable is the inverse problem of assigning \( m \) so as to produce a desired shape upon stimulation \( 34 \).

As reassuring as this picture may appear, things are unfortunately more complicated than they look like: there are at least two unresolved issues, which, as it were, answer the contrasting calls of extreme scarcity and overwhelming abundance. As for scarcity, a smooth isometric immersion with prescribed metric tensor \( C_0 \) may altogether fail to exist in the large. As for abundance, if we renounce the smoothness requirement for the immersion the number of admissible solutions becomes embarrassingly too large \( 3 \).

A remedy to scarcity was proposed by the theory of geometric elasticity \( 32 \) [33]. If the target metric corresponding to \( C_0 \) is geometrically incompatible with a smooth immersion, this theory proposes to replace it with the one that minimizes an appropriate \( L^2 \)-distance from it. It is a viable approximation, if you do not wish to renounce regularity.

A remedy to abundance would be provided by a selection criterion that single out one shape out of many, preferably on energetic grounds. Here the essential question is: what extra energy should be attached to a singular shape? This is the avenue taken here. We allow \( \mathcal{I} \) to have ridges, that is, lines along which the outer unit normal \( \nu \) suffers a discontinuity. As for the

\[ \text{Critiques have been moved to this formula. A noticeable improvement was achieved in } 19 \text{ through a successful extension of Edward’s tube model } 20 \text{ for entangled rubber elasticity. Here, however, we shall abstain from dwelling any further on possible extensions of the trace formula, as desirable as these may be.} \]

\[ \text{Here we may be guilty of some abuse of language, as the metric induced on } \mathcal{I} \text{ by } y \text{ differs from the Euclidean metric on } S \text{ whenever } C \neq 1. \text{ However, we may think of endowing } S \text{ with the metric described by any given symmetric, positive tensor } C \text{ and ask whether } S, \text{ so endowed, can be immersed in three-dimensional Euclidean space preserving the metric. In this sense, which will always be understood here, the word } \text{isometry} \text{ is justified.} \]

\[ \text{In Sect. [V] below, we shall provide plenty of examples for continuous isometric immersions with continuous } m, \text{ but discontinuous } \nu. \]
extra energy cost to be associated with a ridge, we extract it from the bending energy density $f_b$. We conceive a ridge as a limiting tight fold, for which we justify an expression for a ridge linear density $f_r$, which depends (in a symmetric way) on the traces $\nu_1$ and $\nu_2$, of the unit normal $\nu$ on both sides of the ridge. It turns out that $f_r$ scales like $h^2$, juts in between $f_s$ and $f_b$. This justifies an approximation other than geometric elasticity: finding piecewise isometric $C^2$-immersions that minimize the total ridge energy.

The paper is organized as follows. In Sect. II, we recall both stretching and bending energies for nematic polymer networks, as they emerged from the dimension reduction of the bulk energy density delivered by the trace formula. In Sect. III we construct our ridge energy as a limit of the bending energy entrapped in an appropriately folded sheet. Section IV is concerned with the general equations that govern piecewise $C^2$-immersions with discontinuities of the unit normal field $\nu$ concentrated along smooth curves; such ridged isometric immersions are the shape competing for a minimum in our theory. In Sect. V, we show how many ridged isometric immersions can be generated when $S$ is a disk in a plane and the imprinted director field $m$ is the radial hedgehog (in the same plane); we compute the ridge energy that acts as an obstruction to the proliferation of shapes and we determine the minimizers provided by our (approximate) theory. Section VI is where we draw our conclusions and comment on other unresolved issues. The paper is closed by an appendix, where we illustrate a geometric construction apt to produce the analytic solution proposed for the ridged isometric immersion of a hedgehog in Sect. V.

II. STRETCHING AND BENDING ENERGIES

In this section, we recall the outcomes of the dimension reduction method applied in [21] to the trace formula of the neo-classical theory for nematic elastomers (for which we refer the reader to Chap. 6 of [18]). Two director fields feature in this theory; these are $m$, defined in the reference configuration $B$ of the body, and $n$, defined in the current configuration $f(B)$ obtained from $B$ through the deformation $f$. $B$ is a region in three-dimensional Euclidean space $E$ and $f: B \to E$ is a diffeomorphism of $B$.

The directors $m$ and $n$ represent the average alignment of the elongated molecules appended to the rubber polymeric matrix in the reference and current configurations. They are properly defined through the tensorial measures of anisotropy that characterize the end-to-end Gaussian distribution of polymer strands. These are the polymer step tensors $L_m$ and $L_n$, in the reference and current configurations, respectively, which, following [34] and [35], we write as

$$L_m := a_0(I + s_0 m \otimes m)$$

and

$$L_n := a(I + s n \otimes n).$$

Here $I$ is the identity (in three-dimension space), $a_0$ and $a$ are fixed positive parameters (representing the persistence lengths perpendicular to $m$ and $n$, respectively), $s_0$ and $s$ are nematic scalar order parameters, which can be expressed as $s_0 = r_0 - 1$ and $s = r - 1$ in terms of the ratios $r_0$ and $r$ of the parallel (along $m$ and $n$) and perpendicular (across $m$ and $n$) step chain lengths in the reference and current configurations, respectively.

The neo-classical theory of nematic elastomers expresses the elastic free-energy density $f_e$ (per unit volume in the reference configuration) as

$$f_e := \frac{1}{2} \mu \text{tr}(P^T L_n^{-1} F L_m),$$
where \( F := \nabla f \) is the deformation gradient and \( \mu > 0 \) is an elastic modulus (which scales linearly with both absolute temperature and number density of polymer chains). This is usually called the *trace formula*.

In nematic elastomers, \( n \) and \( F \) are fully independent. In contrast, in nematic polymer networks, \( n \) is enslaved to \( F \). In these materials, with which we are concerned in this paper, the director field \( m \) is blueprinted in the elastic matrix [36] and conveyed by the deformation into \( n \), which is thus delivered by

\[
 n = \frac{Fm}{|Fm|} \tag{3}
\]

In general, elastomers are *incompressible*, and so \( F \) must satisfy

\[
 \det F = 1 \tag{4}
\]

Both (3) and (4) will be enforced as constraints on all admissible deformations \( f \) of \( \mathcal{B} \).

With \( m \) (and \( s_0 \)) imprinted in the reference configuration at the time of crosslinking and \( n \) enslaved to the deformation, the only residual freedom lies with \( s \), which can be changed by either thermal or optical stimuli. For example, by heating the sample above the crosslinking temperature, we reduce the nematic order of the chains, so that \( s < s_0 \); this in turns induces a spontaneous deformation so as to minimize the total elastic free energy. Thus, \( s \) can be regarded as the *activation parameter* of our theory, driven by external stimuli. For definiteness, we shall assume that both \( s_0 \) and \( s \) range in the interval \((-1, 1)\).

It was shown in [21] that by use of (1) and (3) \( f_e \) can be given the following form

\[
 f_e = \frac{1}{2} \mu a_0 \frac{a}{a} F(C_f), \tag{5}
\]

where \( C_f := F^T F \) is the right Cauchy-Green tensor associated with the deformation \( f \) and

\[
 F(C_f) = \text{tr} C_f + \frac{s_0}{s + 1} m \cdot C_f m - \frac{s}{s + 1} \frac{m \cdot C^2_f m}{m \cdot C_f m}. \tag{6}
\]

The properties of this function will illuminate the role of \( s \) as activation parameter.

As a consequence of [1], \( C_f \) is also subject to the constraint

\[
 \det C_f = 1. \tag{7}
\]

The tensors \( C_f \) that make \( F(C_f) \) stationary subject to (7) are solutions to the equation

\[
 \frac{\partial F}{\partial C_f} = \lambda \frac{\partial}{\partial C_f} (\det C_f), \tag{8}
\]

where \( \lambda \) is a Lagrange multiplier. It is not difficult to see that this equation reduces to

\[
 I + \frac{1}{s + 1} \left( s_0 + s \frac{m \cdot C^2_f m}{(m \cdot C_f m)^2} \right) m \otimes m - \frac{s}{s + 1} \frac{1}{m \cdot C_f m} (C_f m \otimes m + m \otimes C_f m) = \lambda C_f^{-1}. \tag{9}
\]

It follows from (9) that changing \( m \) into \( Q m \), for any orthogonal tensor \( Q \), transforms a solution \( C_f \) into \( QC_f Q^T \), which makes any solution \( C_f \) of (9) an isotropic tensor-symmetric-valued function of \( m \). By the representation theorem of such functions [37] and (7), we know that \( C_f \) must have the form

\[
 C_f = \lambda_f^2 m \otimes m + \frac{1}{\lambda_f} (I - m \otimes m). \tag{10}
\]
Making use of (10) in (9), we readily conclude that
\[ \lambda = \frac{1}{\lambda_f} \quad \text{and} \quad \lambda_f = \sqrt[3]{\frac{s + 1}{s_0 + 1}}. \]  
(11)

By expressing \( F(C_f) \) in terms of \( \lambda_f \) with the aid of (10), it is easy to show that for \( \lambda_f \) as in (11) this function attains its unique minimum. Thus, when \( s < s_0 \), the spontaneous deformation induced in the material would be a contraction along \( m \), accompanied by a dilation in the plane orthogonal to \( m \), to preserve the volume. Of course, it remains to be seen whether, for an assigned \( m \), a deformation with a metric that minimizes \( F \) locally is indeed geometrically compatible in the large; differently put, whether there is an isometric immersion in three space dimensions of the desired target metric \( C_f \) as in (10).

Here we are interested in thin sheets and in the appropriate dimension reduction of \( F(C_f) \) to be attributed to the mid surface \( S \) of the slab \( S \) of thickness \( 2h \). Formally, \( S \) is a flat region in the \( (x_1, x_2) \) plane of a fixed Cartesian frame \( (e_1, e_2, e_3) \) and \( S \) is the set in three-space defined as \( S := \{(x, x_3) \in S \times [-h, h]\} \). The mapping \( f : S \to \mathcal{E} \) describes the deformation of \( S \) into the surface \( \mathcal{S} = y(S) \) in the deformed slab \( f(S) \); we shall assume that \( y \) is of class \( C^2 \) and that \( m \) is a two-dimensional field imprinted on \( S \), so that \( m \cdot e_3 \equiv 0 \) (see Fig. 1).  

\[ F(C_f) \]

Figure 1: The flat surface \( S \) in the \( (x_1, x_2) \) plane of a fixed Cartesian frame \( (e_1, e_2, e_3) \) is deformed by the mapping \( y \) into a smooth surface \( \mathcal{S} \) embedded in three-dimensional Euclidean space \( \mathcal{E} \). The blueprint orientation is denoted by \( m \) in the reference configuration and by \( n \) in the current one; \( e_3 \) is the outer unit normal to \( S \), while \( \nu \) is the outer unit normal to \( \mathcal{S} \); correspondingly, \( m_\perp := e_3 \times m \) and \( n_\perp := \nu \times n \).

The (two-dimensional) deformation gradient has the following general representation,
\[ \nabla y = a \otimes m + b \otimes m_\perp, \]  
(12)
where \( m_\perp := e_3 \times m \). In (12), \( a \) and \( b \) are vector fields defined on \( S \); they live in \( \mathcal{E} \), the translation space of \( \mathcal{S} \), and are everywhere tangent to \( \mathcal{S} \). It follows from (12) that the two-dimensional stretching tensor \( C \) is represented as
\[ C = (\nabla y^T)(\nabla y) = a^2 m \otimes m + a \cdot b(m \otimes m_\perp + m_\perp \otimes m) + b^2 m_\perp \otimes m_\perp, \]  
(13)

5 Clearly, still according to (11), for \( s > s_0 \), which is achieved upon cooling the sample below the crosslinking temperature, the material would expand along \( m \) and contract transversely.

6 In \( S \), \( m \) is extended uniformly away from \( S \), so as to be independent of the \( x_3 \) coordinate.
where \(a^2 := a \cdot a\) and \(b^2 := b \cdot b\). We shall require that \(S\) is inextensible, which amounts to the constraint \(|a \times b| = 1\). Thus, since \(\det C = a^2b^2 - (a \cdot b)^2 = |a \times b|^2\), we shall require that

\[
\det C = 1.
\]

(14)

Under this constraint, the outer unit normal \(\nu\) to \(S\) will be delivered by

\[
\nu = a \times b.
\]

(15)

Applying (3) to the present setting, we obtain that

\[
n = \frac{(\nabla y)m}{|\nabla y|m},
\]

(16)

and so we may write \(a = an\) and define \(n_\perp := \nu \times n\), so that the frame \((n, n_\perp, \nu)\) is equally oriented as \((m, m_\perp, e_3)\) (see Fig. 1).

In [21], we extended the classical Kirchhoff-Love hypothesis [22] to obtain a dimension reduction of \(F(C_f)\) in (6), that is, a method that converts \(f_e\) in (5) into a surface energy-density (to be integrated over \(S\)). As standard in the theory of plates, such a surface energy is delivered by a polynomial in odd powers of \(h\), conventionally truncated so as to retain the first two relevant ones, the first and the third power. The former is the stretching energy \(f_s\), accounting for the work done to alter distances and angles in \(S\), while the latter is the bending energy \(f_b\), accounting for the work to fold \(S\). Thus, dropping the scaling constant \(\frac{1}{2} \mu a_0\), which has the dimensions of an energy per unit volume, we can write

\[
f_e = f_s + f_b + O(h^5),
\]

(17)

where (to within an inessential additive constant)

\[
f_s = \frac{2h}{s + 1} \left( \text{tr} C + s_0 m \cdot C m + \frac{s}{m \cdot C m} \right),
\]

(18a)

\[
f_b = \frac{2h^3}{3} \left\{ 2(8H^2 - K) + \frac{1}{s + 1} \left[ \left( \frac{3s}{a^2} - a^2 s_0 - \text{tr} C \right) K - \frac{4s}{a^2} (2H - \kappa_n) \kappa_n \right] \right\},
\]

(18b)

where \(H\) and \(K\) are the mean and Gaussian curvatures of \(\mathcal{S}\), defined as

\[
H := \frac{1}{2} \text{tr}(\nabla \nu) \quad \text{and} \quad K := \det(\nabla \nu)
\]

(19)

in terms of the (two-dimensional) curvature tensor \(\nabla \nu\), and

\[
\kappa_n := n \cdot (\nabla \nu)n.
\]

(20)

The total elastic free energy then reduces to the functional

\[
\mathcal{F}[y] := \int_S (f_s + f_b) dA,
\]

(21)

where \(A\) is the area measure.

A perturbation approach to the minimization of \(\mathcal{F}\) is justified when the length scale associated with the average radius of curvature of \(\mathcal{S}\) is large compared to \(h\), which is the smallest length in the system: then \(f_s\) and \(f_b\) are well scale-separated and the latter can be viewed as a higher-order correction to the former. In this approach, it is justified to ask what stretching tensor \(C_0\) would minimize \(f_s\), the leading term in \(\mathcal{F}\). The answer is easily obtained [21],

\[
C_0 = \lambda^2_1 m \otimes m + \lambda^2_2 m_\perp \otimes m_\perp,
\]

(22)
where

\[ \lambda_1 := \sqrt[2]{\frac{s+1}{s_0+1}} \quad \text{and} \quad \lambda_2 = \frac{1}{\lambda_1}. \]  

(23)

A deformation \( y \) for which (22) is valid is an isometric immersion as it minimizes the (leading) stretching energy. The problem is then whether such immersions do exist and how many they are. This is when the bending energy comes to play. If there are no isometric immersions, it means that \( f_s \) must be blended with \( f_b \) and more elaborate minimizing shapes \( \mathcal{S} \) must be sought for, presumably exhibiting regions where the average radius of curvature is not much larger than \( h \). On the other hand, if there are many isometric immersions, we may hope to use the bending energy as a selection criterion, choosing the isometric immersion with the least bending energy.

Both scenarios, however, are overoptimistic. The first, because minimizing the blended energy is not an easy task, also numerically, as the functional \( F \) depends on the second as well as the first gradient of \( y \). The second, because the two-step minimization, which unleashes \( f_b \) over the minimizers of \( f_s \), may actually turn out to be rather disappointing; for example, only spheres are allowed among surfaces \( \mathcal{S} \) with positive \( K \), if one insists in minimizing \( f_b \) uniformly \[ 21 \].

So far we thought of isometric immersions as smooth mappings. The regularity issue now becomes relevant and opens up new perspectives. As shown in Sect. [IV] one can easily jump from no isometric immersion to too many by relaxing the requirement that \( y \) be \( C^2 \). We shall thus consider mappings \( y \) that are piecewise \( C^1 \), with \( \nabla y \) allowed to jump across one or several ridges, which will be assumed to be smooth curves of class \( C^1 \). One such mapping is a piecewise isometric immersion if \( \langle \nabla y \rangle^T (\nabla y) \equiv C_0 \) on the whole domain \( S \), despite the discontinuities of \( \nabla y \) across ridges.

The problem with such ridged immersions is that they effectively encapsulate a bending energy in the ridges across which the outer unit normal to \( S \) jumps abruptly. In the following section, by regarding each of these ridges as tight folds with continuous principal curvatures, one of order \( 1/h \), we shall extract out of \( f_b \) an elastic ridge energy-density (per unit length) \( f_r \). This energy, which scales like \( h^2 \), will serve better than \( f_b \) the purpose of mitigating the multiplicity of ridged isometric immersions. It will form the basis of our (simplified) model for nematic polymer networks.

III. RIDGE ENERGY

In this section, we describe how we envisage a ridge on \( \mathcal{S} \) and derive from \( f_b \) in (18b) the energy that can be associated with it. Let \( C \) be a smooth (plane) curve on \( S \) (say, of class \( C^1 \)) splitting \( S \) in two sides, \( S_1 \) and \( S_2 \) and let \( e \) be a unit tangent vector to \( C \) (see Fig. 2a). A deformation \( y: S \rightarrow \mathcal{E} \), continuous across \( C \) but with discontinuous gradient \( \nabla y \), must obey the following kinematic compatibility condition,

\[ [\nabla y]e = 0, \]  

(24)

where the jump \( [\cdot] := (\cdot)_2 - (\cdot)_1 \) is taken on the two sides \( S_2 \) and \( S_1 \) of \( C \). The deformation \( y \) transforms \( C \) into a smooth curve \( \mathcal{C} \) on \( \mathcal{S} \) along which the outer unit normal \( \mathbf{v} \) is discontinuous; we call \( \mathcal{C} \) a ridge of \( \mathcal{S} \). We shall denote by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) the traces of \( \mathbf{v} \) taken on the sides \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), corresponding to the sides \( S_1 \) and \( S_2 \) of \( C \), respectively.\[ 7 \] 

\[ 8 \] Here we do not discuss boundary conditions, which may have a disquieting role. We think only of shapes in space, which are then defined to within arbitrary translations and rotations. Of course, one could easily envision more complicated splittings of \( S \) (and, correspondingly, more complicated ridges on \( \mathcal{S} \)). Here, we prefer to keep things simple at first, and then generalize in an assumption the result suggested by our simple construction.
(a) The surface $S$ is split by a smooth curve in two sides, $S_1$ and $S_2$, which a deformation $y$, continuous through $C$, but with discontinuous gradient, maps into the sides $S_1$ and $S_2$ of the ridge $C$. The unit tangent vector $t$ to $C$ is related by (25) to the unit tangent vector $e$ to $C$.

(b) Cross-section of the tube surface described by (29). The outer unit normal $\nu$ to the tube is expressed through (30) in terms of the principal normal $n^*$ and the binormal $b$ of $C$. The normals $\nu_1$ and $\nu_2$ delimit the tube connecting $S_1$ and $S_2$. The unit tangent $t$ to $C$ is entering the page.

Figure 2: A single ridge and the tube construction around it, perceivable only at the shortest length scale admissible in our model, that is, $h$.

We designate by $\gamma$ the parameterization of $C$ in the arc-length $\ell$ and correspondingly we call $t(\ell) = \gamma'(\ell)$ its unit tangent vector; $t$ is related to $e$ through the equation

$$t = \frac{(\nabla y)_i e}{|\nabla y|_e},$$

(25)

where by $24$ $i$ can take either value $i = 1, 2$, irrespective of the side upon which $C$ is approached. The unit vectors $n^*$ and $b$, designating the principal normal and the binormal of $C$, complete its Frenet-Serret frame $(t, n^*, b)$, under the assumption that the curvature $\kappa$ of $C$ does not vanish.

The mapping $y$ is a ridged isometric immersion if

$$[(\nabla y)T(\nabla y)] = 0,$$

(26)

meaning that the stretching tensor $C$ is continuous across $C$. The field $m$ is taken to be continuous across $C$, but by (16) $n$ generally fails to be continuous across $C$. However, because of the identities

$$[(\nabla y)m \cdot (\nabla y)m] = 0, \quad [(\nabla y)e \cdot (\nabla y)e] = 0, \quad [(\nabla y)e \cdot (\nabla y)m] = 0,$$

(27)

which follow from (26) and the continuity of $m$, we arrive to

$$[n \cdot t] = 0,$$

(28)

so that the projection of $n$ along the ridge must be the same on both its sides.

Consider now a tube surface (see, for example, pp. 649–650 of [38]) generated by the motion of a sphere of radius $R$ whose center travels along $C$. Such a surface can be parameterized as follows,

$$p(\ell, \theta) = \gamma(\ell) + R(-\cos \theta n^*(\ell) + \sin \theta b(\ell)) \quad \text{for} \quad 0 \leq \ell \leq L \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

(29)

\[9\] Possibly chosen as in [22], though this is not necessary for the validity of our conclusions here.
In these parameters, the outer unit normal $\nu$ reads as
\begin{equation}
\nu = -\cos \theta n^*(\ell) + \sin \theta b(\ell),
\end{equation}
see Fig. 2b, and it is not difficult to see that the curvature tensor is given by \[ p. 650 \]
\begin{equation}
\nabla_s \nu = \frac{\kappa \cos \theta}{1 + R \kappa \cos \theta} t \otimes t + \frac{1}{R} t_\perp \otimes t_\perp,
\end{equation}
where $t_\perp : \nu \times t$, so that
\begin{equation}
H = \frac{1}{2} \left( \frac{1}{R} + \frac{\kappa \cos \theta}{1 + R \kappa \cos \theta} \right) \quad \text{and} \quad K = \frac{1}{R} \frac{\kappa \cos \theta}{1 + R \kappa \cos \theta}.
\end{equation}
The area element is correspondingly delivered by
\begin{equation}
dA = R(1 + R \kappa \cos \theta).
\end{equation}

Now we imagine that at the shortest length scale meaningful in our model, the ridge $C$ is rounded off by a tube surface that connects smoothly (in a $C^1$-fashion) $S_1$ and $S_2$. This amounts to take $R = h$ in the preceding formulæ and to let two angles $\theta_1(\ell)$ and $\theta_2(\ell)$ delimit the connecting tube. By inserting these latter functions in (30), we obtain the normals $\nu_1(\ell)$ and $\nu_2(\ell)$ to $S_1$ and $S_2$, respectively, already introduced in our coarser description of the ridge $C$.

The aim of this construction is to extract from $f_b$ the bending energy concentrated in a jump of $\nu$ and assign it to $C$ as an energy distributed over its length. To this end, we assume that $\kappa = O(h^0)$ and estimate $f_b$ in (18b) at the leading order in $h$. It readily follows from (31) and (32) that
\begin{equation}
H = \frac{1}{2h} + O(1), \quad K = \frac{1}{h} + O(1), \quad \text{and} \quad \kappa_n = \frac{1}{h} (n \cdot t_\perp)^2 + O(1).
\end{equation}
Making use of (34) in (18b), we arrive at
\begin{equation}
f_b = \frac{8}{3} h \left( 1 - \frac{s}{s+1} \frac{1}{a^2} (n \cdot t)^2 (n \cdot t_\perp)^2 \right) + O(h^2),
\end{equation}
where we recall that $a^2 = m \cdot C m$. Integrating this in the tube delimited by $\theta_1$ and $\theta_2$, since by (28) and the constraint $n \cdot n = 1$ both $(n \cdot t)^2$ and $(n \cdot t_\perp)^2$ are continuous across $C$ and can be taken as independent of $\theta$ over the tube, by (33) we estimate a single ridge energy $\mathcal{F}_r$ as
\begin{equation}
\mathcal{F}_r := \frac{8}{3} h^2 \int_0^L d\ell \int_{\theta_1}^{\theta_2} d\theta \left( 1 - \frac{s}{s+1} \frac{1}{a^2} (n \cdot t)^2 (n \cdot t_\perp)^2 \right) + O(h^3),
\end{equation}
so that $\mathcal{F}_r$ is reduced to a line integral along $C$ with density (per unit length)
\begin{equation}
f_r := \frac{8}{3} h^2 \arccos(\nu_1 \cdot \nu_2) \left( 1 - \frac{s}{s+1} \frac{1}{a^2} (n \cdot t)^2 (n \cdot t_\perp)^2 \right).
\end{equation}

This is the main outcome of our tube construction. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are isometrically immersed (with stretching energy scaling like $h$) and meet at the ridge $C$, this latter is endowed with the extra energy $\mathcal{F}_r$ in (36). In case of multiple ridges $C_j$, we assume that
\begin{equation}
\mathcal{F}_r := \sum_{j=1}^N \int_{C_j} f_r d\ell,
\end{equation}
where $N$ is the total number of ridges present on $S$. This is the total ridge energy that we shall assign here to a ridged isometric immersion.

Two comments are in order. First, by direct inspection of (37), it is evident that for $s > 0$ the ridge energy-density would promote an alignment of $n$ at $\pi/2$ with the ridge (on both adjoining sides), whereas for $s < 0$ it would equally promote an alignment either parallel or orthogonal to the ridge. Second, and more importantly, since $\mathcal{F}_e$ scales like $h^2$, it dominates over the bending energy distributed over the smooth components $\mathcal{F}_b$ of $\mathcal{F}$ isometrically immersed in three-space. Thus, $\mathcal{F}_e$ should suffice to decide which ridged isometric immersion is energetically favorable; minimizing $\mathcal{F}_e$ in (38) should provide the selection criterion that we seek.

In the following section, we shall write the equations that describe a ridged immersion in a special representation. An example of $\mathcal{F}_e$ will be computed explicitly in Sect. IV.

IV. REPRESENTING RIDGED ISOMETRIC IMMERSIONS

Away from possible point defects (the only ones allowed here), the director field $m$ imprinted on $S$ and its orthogonal companion $m_\perp$ are assumed to have continuous gradients, which can be represented as

$$\nabla m = m_\perp \otimes c \quad \text{and} \quad \nabla m_\perp = -m \otimes c$$

in terms of the planar connector field $c$ [21]. We shall use the frame $(m, m_\perp, e_3)$ to represent a deformation $y$ of $S$,

$$y = y_1 m + y_2 m_\perp + y_3 e_3,$$

where $y_i$ are smooth scalar fields on $S$, so that by (39)

$$\nabla y = y_1 m_\perp \otimes c + m \otimes \nabla y_1 - y_2 m \otimes c + m_\perp \otimes \nabla y_2 + e_3 \otimes \nabla y_3.$$  

Letting, similarly, $c = c_1 m + c_2 m_\perp$, we easily see that the vectors $a$ and $b$ in (12) can be given the representation

$$a = (\nabla y)m = (y_{1,1} - c_1 y_2) m + (y_{2,1} + c_1 y_1) m_\perp + y_{3,1} e_3,$$

$$b = (\nabla y)m_\perp = (y_{1,2} - c_2 y_1) m + (y_{2,2} + c_2 y_1) m_\perp + y_{3,2} e_3,$$

where we have used the expressions

$$\nabla y_1 = y_{1,1} m + y_{1,2} m_\perp + y_{1,3} e_3,$$

$$\nabla y_2 = y_{2,1} m + y_{2,2} m_\perp + y_{2,3} e_3,$$

$$\nabla y_3 = y_{3,1} m + y_{3,2} m_\perp + y_{3,3} e_3.$$  

Now, also in view of (13), we see that requiring $y$ in (40) to be an isometric immersion satisfying (22) reduces to enforcing the following conditions

$$a^2 = \lambda_1^2, \quad b^2 = \lambda_2^2, \quad a \cdot b = 0.$$  

These, with the aid of (12), read explicitly as

$$(y_{1,1} - c_1 y_2)^2 + (y_{2,1} + c_1 y_1)^2 + y_{3,1}^2 = \lambda_1^2, \quad (45a)$$

$$(y_{1,2} - c_2 y_1)^2 + (y_{2,2} + c_2 y_1)^2 + y_{3,2}^2 = \lambda_2^2, \quad (45b)$$

$$(y_{1,1} - c_1 y_2)(y_{1,2} - c_2 y_1) + (y_{2,1} + c_1 y_1)(y_{2,2} + c_2 y_1) + y_{3,1} y_{3,2} = 0, \quad (45c)$$
which constitute a non-linear system of PDEs for the unknown functions \( y_1, y_2, \) and \( y_3 \). As a consequence of Gauss’ theorem egregium, an isometric immersion characterized by \( K \) has Gaussian curvature dictated by \( m \) through the equation \[ K = (\lambda_1^2 - \lambda_2^2) (c_2^2 - c_1^2 + c_{12}), \] (46)

where we have set \( c_{12} = m \cdot (\nabla e) m_\perp \).

Moreover, for a ridged isometry, equations (45) must be supplemented by the form appropriate to this setting of the jump condition in (24). Since both \( m \) and \( m_\perp \) are continuous across any plane curve \( C \) (with unit tangent \( e \)), by (12), (24) becomes

\[ (m \cdot e)[a] + (m_\perp \cdot e)[b] = 0. \] (47)

Letting \( e = \cos \chi m + \sin \chi m_\perp \), since both \( y \) and \( e \) are continuous across \( C \), (47) reduces to the three scalar equations

\[ [y_{1,1}] \cos \chi + [y_{1,2}] \sin \chi = 0, \quad [y_{2,1}] \cos \chi + [y_{2,2}] \sin \chi = 0, \quad [y_{3,1}] \cos \chi + [y_{3,2}] \sin \chi = 0. \] (48)

While equations (45) hold on the whole of \( S \), despite the jumps that the gradients \( \nabla y_1, \nabla y_2, \) and \( \nabla y_3 \) may suffer across the curves \( C_j \) that \( y \) transforms into the ridges \( C_j \), equations (48) are valid only along such curves.

In the following section, we shall find solutions to (45) and (48) in a special case. It will be expedient to compute on a ridge the inner product \( \nu_1 \cdot \nu_2 \), which features in the expression for \( f_\epsilon \) in (37). To this end, we first recall (15) and remark that

\[ \nu_2 = (a_1 + [a]) \times (b_1 + [b]) = \nu_1 + a_1 \times [b] + [a] \times b_1, \] (49)

where use has also been made of (47). Since \( a_1 = \lambda_1 n_1 \) and \( b_1 = \lambda_2 \nu_1 \times n_1 \), by (44) it follows from (49) that

\[ \nu_1 \cdot \nu_2 = \frac{1}{\lambda_1^2} a_1 \cdot a_2 + \lambda_2^2 b_1 \cdot b_2 - 1, \] (50)

where we also employed (23). With \( a_i \) and \( b_i \) given by (12), we easily revert (50) into an expression featuring the traces of the gradient components \( y_{i,j} \) on the two sides of the ridge under consideration.

V. RIDGED CONES

It is time now to put our theory to the test. In this section, we shall consider a classical example, already treated within the traditional theory \[ 24 \], that of a disk \( S \) of radius \( R \) upon which the planar radial hedgehog \( m \) has been imprinted. In polar coordinates \((\varrho, \vartheta)\), with associated orthonormal frame \((e_\varrho, e_\vartheta)\), \( m = e_\varphi \) and \( m_\perp = e_\vartheta \). It is an easy exercise to check with the aid of (39) that then \( c_1 = 0, c_2 = \frac{1}{2}, \) and \( c_{12} = -\frac{1}{2} \), so that by (16) \( K = 0 \), independently of the prescribed principal stretches \( \lambda_1 \) and \( \lambda_2 \).

We shall use the representation (40) for \( y \) with

\[ y_1 = f \sin \psi \cos \varphi, \quad y_2 = f \sin \psi \sin \varphi, \quad y_3 = f \cos \psi, \] (51)

where \( f, \psi, \) and \( \varphi \) are assumed to be piecewise \( C^2 \)-functions of \((\varrho, \vartheta)\). Thus \( \psi \) and \( \varphi \) represents the polar and azimuthal angles of \( y \) in the movable frame \((e_\varrho, e_\vartheta, e_\psi)\), while \( f \) measures radial
dilation (or contraction). Some labour is required to see that with this choice the isometry conditions (45) become

\begin{align}
&f^2 + f^2(\psi_\varrho^2 + \varphi_\varrho^2 \sin^2 \psi) = \lambda_1^2, \\
&\frac{1}{\varrho^2} \left[ f^2 + f^2(\psi_\varphi^2 + (1 + \varphi, \varrho)^2 \sin \psi^2) \right] = \lambda_2^2, \\
&\frac{1}{\varrho} \left[ f, \varrho f, \varphi + f^2 [\psi, \varphi, \varrho + \varphi, \varphi(1 + \varphi, \varrho) \sin^2 \psi] \right] = 0,
\end{align}

where commas denote partial derivatives in the variables $(\varrho, \varphi)$.

We shall look for solutions of (52) under the simplifying assumption that $f$ is a positive function of $\varrho$ only and both $\psi$ and $\varphi$ are functions of $\vartheta$ only. The deformed surface $S$ is thus conical, in the same class employed to represent crumpled sheets of paper in [39, 40]. Further requiring that the centre of the disk $S$ is held fixed in a spontaneous deformation, we see that (52a) has the unique (positive) solution

\[ f(\varrho) = \lambda_1 \varrho, \]

and that (52c) is identically satisfied, while (52b) reduces to

\[ \psi'^2 + (1 + \varphi')^2 \sin^2 \psi = \mu^2, \]

where use has also been made of (53a) and we have set

\[ \mu := \frac{\lambda_2}{\lambda_1} = \frac{1}{\lambda_1}. \]

Hereafter in this section, a prime $'$ will denote differentiation with respect to $\vartheta$.

Our efforts will now focus on solving (53b) subject to the periodic boundary conditions

\[ \psi(0) = \psi(2\pi), \quad \varphi(0) = \varphi(2\pi). \]

For a given function $\psi$ satisfying the first equality in (54) and such that $\sin \psi \neq 0$ and $\psi'^2 \leq \mu^2$, (53b) is solved by integrating

\[ \varphi' = -1 + \frac{\sqrt{\mu^2 - \psi'^2}}{\sin \psi}, \]

provided that the following condition is met,

\[ I := \int_0^{2\pi} \frac{\sqrt{\mu^2 - \psi'^2}}{\sin \psi} d\vartheta = 2\pi, \]

for the second equality in (53b) to be valid too. Clearly, for $\mu \leq 1$, (55) is solved by $\psi \equiv \arcsin \mu$, which corresponds to a circular cone when $\mu < 1$. But, for $\mu > 1$, (55) is quite a demanding request, which cannot be satisfied by a $2\pi$-periodic function $\psi$ of class $C^2(\mathbb{R})$.

\[ \text{---} \]

10 This assumption is strongly motivated by the requirement that $K = 0$, which demands that $S$ be developable. Strictly speaking, all these surfaces are singular at the tip, where all radii meet. This singularity could formally be removed by expunging the centre of the disk $S$. We rather prefer to keep it in place and tolerate the singularity it bears.

11 It is perhaps worth noting that, under the assumption that $f$ depends only on $\varrho$ and both $\psi$ and $\varphi$ depend only on $\vartheta$, all three equations (55) also follow from simply requiring that $\det C = 1$.

12 Differently said, in addition to (54), we also require that both first and second derivatives of $\psi$ agree at 0 and $2\pi$. This, also in view of (55), is indeed necessary for the surface $S$ represented by (51) to be $C^2$. 

---
To see this, we study the minimum of the functional \( I \) defined in (56) and show that for \( \mu > 1 \) it exceeds \( 2\pi \). The Euler-Lagrange equation associated with \( I \) is

\[
\psi'' = (\mu^2 - \psi'^2) \cot \psi, \tag{57}
\]

which possesses \( \psi \equiv \frac{\pi}{2} \) as a special solution (for all \( \mu \)) with \( I = 2\mu \pi \). All other solutions can be obtained by quadrature, which gives the conservation law

\[
\psi'^2 = \mu^2 \left( 1 - \frac{B^2}{\sin^2 \psi} \right), \tag{58}
\]

where \( 0 < B < 1 \) is an arbitrary integration constant. This equation represents two branches of solutions for \( \psi \), which can be connected in a \( C^2 \)-fashion where \( \psi' = 0 \), describing a function that oscillates between the barriers \( \psi_1 = \arcsin B \) and \( \psi_2 = \pi - \arcsin B \). Formally, the inverse of \( \psi \) on each branch satisfies

\[
\pm \tan (\mu(\vartheta - C)) = \frac{\cos \psi \sqrt{\sin^2 \psi - B^2}}{\sin \psi}, \tag{59}
\]

where \( C \) is another constant of integration, which just shifts a single solution branch. Alternating decreasing and increasing branches, from (59), we can obtain a periodic function \( \psi \), bounded in the interval \([\psi_1, \psi_2]\), with semiperiod \( \Delta \vartheta = \frac{\pi}{\mu} \). Thus, for \( \psi \) to be \( 2\pi \)-periodic, there must be an integer \( n \), such that \( 2n\Delta \vartheta = 2\pi \), which amounts to \( \mu = n \).

This shows that all smooth stationary points of \( I \) besides \( \psi \equiv \frac{\pi}{2} \) are attained when \( \mu \) is an integer. To compute the corresponding value of \( I \), we first remark that this is independent of \( C \), as \( C \) can be freely chosen by translating the origin of \( \vartheta \). Setting \( C = 0 \), amounts to set \( \psi(0) = \frac{\pi}{2} \).

Moreover, by use of (58), for a periodic solution \( \psi \) as in (59), we may give \( I \) the form

\[
I = 4nB \int_{\frac{\pi}{2}}^{\pi - \arcsin B} \frac{d\psi}{\sin \psi \sqrt{\sin^2 \psi - B^2}} = 2n\pi > 2\pi, \tag{60}
\]

which is only valid for \( \mu = n \). Thus, when \( \mu \) is an integer, \( I \) attains the same minimum on two periodic functions (one being constant), when \( \mu \) is not an integer, the only minimizer is \( \psi \equiv \frac{\pi}{2} \). However, for \( \mu > 1 \), all these minima are greater than \( 2\pi \), and so the condition (56) cannot be met in the class of periodic \( C^2 \)-functions. Consequently, a smooth isometric immersion fails to exist in the class of conical shapes (51).

We now turn to look for ridged isometric immersions representable in the class (51). First, we see how to write the jump conditions (48) in the present context. Here \( \chi = 0 \), as jumps may only occur along radii of the disk \( S \). Moreover, by (53a), equations (51) imply that \( [y_{1,1}] = [y_{2,1}] = [y_{3,1}] = 0 \), independently of \( \psi \), so that (48) is identically satisfied. We shall thus look for solutions \( \psi \) of (53d) that are piecewise of class \( C^1 \). Since (53d) must be valid on the whole of \( S \), a jump in \( \psi' \) is admissible only if

\[
[\psi'^2] = 0, \tag{61}
\]

while, by (55), no jump in \( \varphi' \) is allowed.

It is a simple matter to show that by for a conical surface described by (51) the vectors \( a \) and \( b \) in (42) can be given, also by (53a), the following expressions,

\[
a = \lambda_1 (\sin \psi \cos \varphi e_\varphi + \sin \psi \sin \varphi e_\theta + \cos \psi e_3), \tag{62a}
\]

\[
b = \lambda_1 \left( [\psi' \cos \varphi \sin \varphi - (1 + \varphi') \sin \psi \sin \varphi] e_\varphi + [\psi' \cos \varphi \sin \varphi + (1 + \varphi') \sin \psi \cos \varphi] e_\theta - \psi' \sin \psi e_3 \right). \tag{62b}
\]
It follows from these equations and (50) that on a ridge
\[ \nu_1 \cdot \nu_2 = 1 - 2 \frac{\psi'^2}{\mu^2}, \]  
(63)

where use has also been made of both (53b) and (61). Moreover, since \( t_1 \cdot n = 0 \), by (37) the ridged energy density (scaled to \( \frac{4 h^2}{3} \)) simply reduces to
\[ f_r = \arccos(\nu_1 \cdot \nu_2). \]

We must still enforce (54) and (56). We do so by means of a geometric construction, which is fully substantiated in Appendix A. For simplicity, but with no loss of generality, we assume that \( S \) is the unit disk, stretched circumferentially by \( \mu \) into a ridged immersion through the following steps (see Fig. 3). (1) For a given \( \mu > 1 \), choose an integer \( n \geq \mu \) and take a circular sector of \( S \) of amplitude \( \alpha := \frac{\pi}{2n} \) (the reference sector). (2) Stretch it so that its amplitude becomes \( \mu \alpha \). (3) Rotate the stretched sector around one of its edges by an appropriately chosen angle \( \beta \). (4) Reflect the rotated sector across the vertical plane containing the lifted edge. (5) Rotate by \( \pi \) the reflected sector around its edge lying on \( S \). (6) Repeat \( n - 1 \) times the preceding steps.

(a) Step 1: Choose an integer \( n \geq \mu \) and select a reference sector with amplitude \( \alpha = \frac{\pi}{2n} \). Here \( \mu = 1.7 \) and \( n = 3 \).
(b) Step 2: Stretch uniformly the reference sector into one with amplitude \( \mu \alpha \).
(c) Step 3: Rotate around \( e_1 \) by the angle \( \beta \).
(d) Step 4: Reflect across the vertical plane containing the lifted edge.
(e) Step 5: Rotate by \( \pi \) around the edge lying on \( S \).
(f) Step 6: Repeat \( n - 1 \) times.

Figure 3: Geometric construction of a ridged immersion of the unit disk \( S \).

This construction works, making sure that the generated surface closes on itself, if \( \beta \) is such that the reflection plane in step (4) cuts \( S \) where lied the edge of the reference sector (see Fig. 3d), that is, if (see Appendix A).

\[ \beta := \begin{cases} \arccos \frac{\tan \alpha}{\tan(\mu \alpha)} & \text{if } \mu < n, \\ \frac{\pi}{2} & \text{if } \mu = n. \end{cases} \]  
(64)
A continuous, piecewise differentiable 4α-periodic function \( \psi \) is obtained in (A5) by extending over \( \mathbb{R} \) the function \( \psi^* \) defined on \([0, \alpha]\) by
\[
\psi^*(\vartheta) := \arccos(\sin(\beta \sin(\mu \vartheta))).
\] (65)
Correspondingly, the function \( \varphi^* \) associated with \( \psi^* \) is given in \([0, \alpha]\) by (see again Appendix A)
\[
\varphi^*(\vartheta) := -\vartheta + \arccos \frac{\cos(\mu \vartheta)}{\sin \psi^*(\vartheta)}.
\] (66)
Its \( C^1 \)-extension \( \varphi \) over \( \mathbb{R} \) compatible with (55) is recorded in (A6). It is a boring, but simple exercise to check that functions \( \psi^* \) and \( \varphi^* \) satisfy (53b) identically, and so do their extensions \( \psi \) and \( \varphi \).

Figure 4 shows examples of the functions \( \psi \) and \( \varphi \) so generated alongside with the conical surface produced by the corresponding ridged immersions of the unit disk \( S \). It should be noted that whenever \( \mu = n \) this construction becomes singular, though it is still applicable. In such a case, all ridges are degenerate and lie on the vertical axis \( e_3 \) of the disk \( S \); all faces of \( S \) are vertical as well and \( \varphi \) becomes discontinuous (as shown in Fig. 5 for \( \mu = 3 \)).

The above construction produces \( N = 2n \) ridges, at each of which \( \psi'^2 = \psi'^2(\alpha) \), so that by (63) the total (dimensionless) ridge energy \( F_r \) is (see (38) and (A7))
\[
F_r(n, \mu) = 2n \arccos \left( 1 - \frac{2}{\sin^2 \left( \frac{\mu \pi}{2n} \right)} \left[ \cos^2 \left( \frac{\pi}{2n} \right) - \cos^2 \left( \frac{\mu \pi}{2n} \right) \right] \right).
\] (67)
Plots of \( F_r \) against \( \mu \) for several values of \( n \) are depicted in Fig. 6. They show that, for a given \( \mu \), \( F_r \) is minimized for \( n = \lceil \mu \rceil \), when the number of ridges is the least possible (as was perhaps to be expected).

We have also constructed other ridged immersions of the unit disk by letting \( \varphi \equiv 0 \) in (53b) and solving for \( \psi \) subject to the first of (54). Here we omit this construction because it delivered a total ridge energy larger than \( F_r \) for all \( \mu > 1 \). We cannot say whether the construction illustrated in this section delivers the least ridge energy among all conical lifting of the radial hedgehog, though this seems likely to us.

VI. CONCLUSIONS

Common wisdom has it that in sufficiently thin sheets of nematic polymer networks, as in all elastic material for that matter (see, for example, [41, p. 396] or [42, p. 404]), the bending energy (which scales as the cube of the thickness) may be neglected relative to the stretching energy (which scales linearly in the thickness). When activated, a nematic polymer network suffers a spontaneous deformation that attempts to transfer on the current shape the metric tensor that minimizes the stretching energy, which (with a slight abuse of language) we called an isometric immersion, for short. Such an immersion would generally depend on the nematic director imprinted on the sheet at the time of crosslinking, and, as is well known, it may fail to exist.

We started from relaxing the requirement of smoothness for an isometric immersion, thus removing a possible obstacle to its existence. We allowed for ridges in the immersed surfaces \( \mathcal{S} \) representing deformed sheets; these are lines where the normal to \( \mathcal{S} \) suffers a jump. Clearly,

\[\text{By } \lceil \mu \rceil, \text{ we mean the smallest integer greater than or equal to } \mu.\]
(a) $\mu = 2.7$ and $n = 3$.

(b) $\mu = 2.7$ and $n = 4$.

(c) $\mu = 2.7$ and $n = 5$.

Figure 4: On the left: Functions $\psi$ (red) and $\varphi$ (blue) for $\mu = 2.7$ and different values of $n$. On the right: Corresponding ridged immersions of the unit disk $S$. 
Figure 5: When the geometric construction of a ridged immersion of $S$ becomes singular. On the left: Functions $\psi$ (red) and $\varphi$ (blue) for $\mu = n = 3$. On the right: Corresponding ridged immersion of $S$ with all ridges collapsed on the vertical axis of $S$.

Figure 6: The function $F_r$ in \[67\] is plotted against $\mu$ in the intervals $[1, n]$, for several values of $n$ (blue graphs). The lower envelope (red graph) is the plot of $F_r(\lceil \mu \rceil, \mu)$ against $\mu$.

ridges do not come for free (nothing does). If they did, we would be overwhelmed with an superabundance of shapes, for which we would lack a selecting energy criterion (as all would have the same stretching energy$^{14}$).

We thought of ridges as concentrations of bending energy; we put forward a model to compute the energy they bear distributed along their length. To accomplish this task, we employed a formula for the bending energy recently derived from the “trace formula” valid in three space

\[14\] And the same bending energy too, in the example worked out in Sect.\[V\]
dimensions [21].

We showed that the ridge energy density (per unit length) $f_r$ scales quadratically with the sheet’s thickness, and so it represents a contribution intermediate between stretching and bending energies. The formula we obtained for $f_r$ not only depends (symmetrically) on the normals to the adjoining sides (as was perhaps to be expected), but also on the orientation of the nematic director relative to the tangent to the ridge.

We applied our theory to the case where a planar hedgehog is imprinted on a flat disk at the time of crosslinking. We studied the total elastic energy, including the new ridge energy, in a class of conical deformations not new in the literature. We proved that in the regime in which the radii of the reference disk shrink and the circumferences expand, no smooth isometric immersion exists; the ridge energy then acts as the desired selection criterion.

The ridged cones that we found as energy minimizers are neither the developable cones of [39, 43–47] nor the excess cones of [40], as they do not share the degree of smoothness that both the latter and the former have in common.

A question, however, should be asked: How real these ridged cones are? Truth be told, we cannot expect that a rubber like material will spontaneously take on sharp ridges when activated by a change in its internal material organization. Our model has more the flavour of an asymptotic extrapolation, valid in the limit as the sheet’s thickness vanishes. It is, nevertheless, predictive.

For example, we have learned that a planar hedgehog will exhibit a number $N$ of ridges, given by the following formula in terms of the activation parameter $s$ of the theory,

$$N = 2 \left\lceil \sqrt{\frac{s_0 + 1}{s + 1}} \right\rceil \geq 4 \quad \text{for} \quad s < s_0.$$  \hspace{1cm} (68)

We expect that there is a critical value of the sheet’s thickness, below which (68) would reproduce the number of folds predicted by an elastic theory based on the full blown energy, where stretching and bending components are blended together and compete on different length scales. Only such a complete theory could estimate the critical thickness. We trust that the theory presented in this paper can be regarded as a first, viable approximation to such a complete theory.

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Appendix A: Geometric Construction

In this appendix, we provide further details on the geometric construction employed in Sect. V to produce a ridged isometric immersion of the unit disk $S$. In particular, our objective will be to justify both the formula for $\beta$ in (64) and that for $\psi^*$ in (65).

Take a circular sector of $S$ with amplitude $\alpha$ (the reference sector) and stretch it uniformly to obtain a sector with amplitude $\mu \alpha$. Then rotate the stretched sector around $e_1$ by $\beta$ and reflect the rotated sector across the vertical plane passing through the unit vector $e(\alpha)$ :=
Finally, rotate by $\pi$ the reflected sector around $e(2\alpha)$. Thus, we have ridged-immersed a sector with amplitude $4\alpha$. By replicating $n-1$ times this immersion, we finally obtain the entire surface $S$. Figure 7 shows the geometric details of the first immersed sector (with amplitude $\alpha$).

More precisely, for all $\vartheta \in [0, \alpha]$ we define the mapping $e(\vartheta) \mapsto E(\vartheta) := R_{e_1, \beta} e(\mu \vartheta)$, \hfill (A1)

where $R_{e_1, \beta}$ is the rotation around $e_1$ by the angle $\beta$. Explicitly, $E(\vartheta)$ is given by

$$E(\vartheta) = (I + \sin \beta W(e_1) - (1 - \cos \beta) P(e_1)) e(\vartheta) = \cos(\mu \vartheta) e_1 + \sin(\mu \vartheta) \cos \beta e_2 + \sin(\mu \vartheta) \sin \beta e_3,$$  \hfill (A2)

where $P(e_1)$ denotes the projection onto the plane orthogonal to $e_1$ and $W(e_1)$ is the skew-symmetric tensor associated with $e_1$. It follows from (51) and (A2) that for all $\vartheta \in [0, \alpha]$ the polar angle of the first immersed sector is given by (65) in the main text.

The appropriate value of $\beta$ is determined by requiring that the plane spanned by $e(\alpha)$ and $E(\alpha)$ is the vertical plane used to reflect the first immersed sector in step 4 of our construction (see Fig. 3d). Formally, this geometric requirement demands that $P(e_3) E(\alpha) = \delta e_3$, for some $\delta \in [0, 1]$, where $P(e_3)$ is the projection onto the plane orthogonal to $e_3$, that is,

$$\cos(\mu \alpha) e_1 + \sin(\mu \alpha) \cos \beta e_2 = \begin{cases} 
\delta (\cos \alpha e_1 + \sin \alpha e_2) & \text{if } \mu \alpha < \frac{\pi}{2}, \\
0 & \text{if } \mu \alpha = \frac{\pi}{2}.
\end{cases}$$  \hfill (A3)

Solving (A3), we easily find that $\beta$ is given by (64), while

$$\delta = \frac{\cos \mu \alpha}{\cos \alpha}.$$  \hfill (A4)

Since, by (51), $E(\vartheta) \cdot e_1 = \sin \psi \cos(\varphi + \vartheta)$ (see Fig. 7c), from (A2) we also retrieve (66).

Finally, we note that the function $\psi^*$ can be extended continuously to the entire $\mathbb{R}$ by making it periodic with period $4\alpha$ and oscillating symmetrically about $\frac{\pi}{2}$. The extended function $\psi$ is
represented as follows in one period,

$$\psi(\vartheta) = \begin{cases} 
\psi^*(\vartheta) & \text{if } \vartheta \in [0, \alpha], \\
\psi^*(2\alpha - \vartheta) & \text{if } \vartheta \in [\alpha, 2\alpha], \\
\pi - \psi^*(\vartheta - 2\alpha) & \text{if } \vartheta \in [2\alpha, 3\alpha], \\
\pi - \psi^*(4\alpha - \vartheta) & \text{if } \vartheta \in [3\alpha, 4\alpha]. 
\end{cases} \quad (A5)$$

Similarly, the periodic $C^1$-extension of $\varphi^*$ compatible with \((55)\) is given by

$$\varphi(\vartheta) = \begin{cases} 
\varphi^*(\vartheta) & \text{if } \vartheta \in [0, \alpha], \\
-\varphi^*(2\alpha - \vartheta), & \text{if } \vartheta \in [\alpha, 2\alpha], \\
\varphi^*(\vartheta - 2\alpha) & \text{if } \vartheta \in [2\alpha, 3\alpha], \\
-\varphi^*(4\alpha - \vartheta), & \text{if } \vartheta \in [3\alpha, 4\alpha]. 
\end{cases} \quad (A6)$$

Since both $\sin\psi$ and $\psi^2$ are $2\alpha$-periodic functions, by \((55)\) so is also $\varphi$. Moreover, it is easy to check that

$$\psi^2((2k + 1)\alpha) = \frac{\mu^2}{\sin^2\alpha} (\cos^2\alpha - \cos^2(\mu\alpha)) \quad \text{and} \quad \varphi(k\alpha) = 0 \quad \text{for all } k \in \mathbb{N}, \quad (A7)$$

as illustrated in Fig. 4.

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