Algebraic Bethe ansatz for a quantum integrable derivative nonlinear Schrödinger model

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Abstract

We find that the quantum monodromy matrix associated with a derivative nonlinear Schrödinger (DNLS) model exhibits $U(2)$ or $U(1, 1)$ symmetry depending on the sign of the related coupling constant. By using a variant of quantum inverse scattering method which is directly applicable to field theoretical models, we derive all possible commutation relations among the operator valued elements of such monodromy matrix. Thus, we obtain the commutation relation between creation and annihilation operators of quasi-particles associated with DNLS model and find out the $S$-matrix for two-body scattering. We also observe that, for some special values of the coupling constant, there exists an upper bound on the number of quasi-particles which can form a soliton state for the quantum DNLS model.

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1 Introduction

Quantum integrable field models and spin chains in low dimensions have recently attracted much attention due to their close connection with diverse areas of physics as well as mathematics. By using algebraic Bethe ansatz, which occurs naturally in the framework of quantum inverse scattering method (QISM), one can find out the spectrum and various correlation functions of quantum integrable models with short-range interactions [1-4]. The nonlinear Schrödinger (NLS) model is a well known example of such quantum integrable field model in 1 + 1-dimension [1-8]. For the case of derivative nonlinear Schrödinger (DNLS) model, however, the situation is a little bit complicated due to the following reason. There exists one type of classical DNLS model with equation of motion like [9]

\[ iq_t + q_{xx} - 4i\xi(q^*q)_x = 0, \]  

where the subscripts denote the derivatives with respect to corresponding variables. By using an equal time ‘nonultralocal’ Poisson bracket (PB) structure given by \( \{ q(x), q(y) \} = \{ q^*(x), q^*(y) \} = 0, \{ q(x), q^*(y) \} = \delta_x(x - y) \), one can show that the infinite number of conserved quantities (including the Hamiltonian) associated with this DNLS equation of motion yield vanishing PB relations among themselves [10]. This fact establishes the classical integrability of DNLS model \((1.1)\) in the Liouville sense. However, due to the appearance of nonultralocal commutation relations among the basic field operators, quantum version of such DNLS model can not be handled by QISM and therefore quantum integrability can not be established for this case.

On the other hand, there exists another type of DNLS model with equation of motion like [11]

\[ i\psi_t + \psi_{xx} - 4i\xi(\psi^*\psi)\psi_x = 0. \]  

The Hamiltonian

\[ H = \int_{-\infty}^{+\infty} (-\psi^*\psi_{xx} + 2i\xi(\psi^*\psi)\psi^*\psi_x)dx, \]  

and an equal time ‘ultralocal’ PB structure

\[ \{ \psi(x), \psi(y) \} = \{ \psi^*(x), \psi^*(y) \} = 0, \{ \psi(x), \psi^*(y) \} = -i\delta(x - y), \]  

yield eqn.\((1.2)\) through a canonical evolution: \( \psi_t = \{ \psi, H \} \). By using the ultralocal PB structure \((1.4)\) and Lax operator given by

\[ \tilde{U}(x, \lambda) = i \left( \begin{array}{cc} \xi\psi^*(x)\psi(x) - \lambda^2/4 & \sqrt{\xi}\lambda\psi^*(x) \\ \sqrt{\xi}\lambda\psi(x) & -\xi\psi^*(x)\psi(x) + \lambda^2/4 \end{array} \right), \]  

where \( \lambda \) is a spectral parameter, one can prove the classical integrability for the DNLS model \((1.2)\) in the Liouville sense. Furthermore, by taking advantage of ultralocal commutation relations among the basic field operators, integrability of the corresponding quantum DNLS model can also be established through QISM [12].
Similar to the case of other integrable systems, the monodromy matrix plays a key role in deriving the conserved quantities and studying exact solvability of the DNLS model \((1.2)\) as well as its quantum analogue. Though the PB relations (commutation relations) among various elements of classical (quantum) monodromy matrix associated with such DNLS model have been computed earlier \([12]\), this problem should be reinvestigated due to the following reasons. First of all, it was assumed earlier that the monodromy matrix of classical (quantum) DNLS model exhibits \(SU(2)\) (\(U(2)\)) symmetry for any value of the corresponding coupling constant \(\xi\). However, in comparison to the case of NLS model where the symmetry of monodromy matrix depends on the sign of the corresponding coupling constant, this seems to be a rather questionable assumption. Therefore, it is necessary to properly study the symmetry properties of the classical (quantum) monodromy matrix of DNLS model and use those symmetries as an input for deriving the PB relations (commutation relations) among the elements of the monodromy matrix.

Secondly, a rather cumbersome method was adopted earlier for finding out the commutation relations among the elements of quantum monodromy matrix at the infinite interval limit. As it will be clear from the discussions in Sec.4 of the present article, the above mentioned method yields an incorrect commutation relation between creation and annihilation operators associated with the quasi-particles of quantum DNLS model. Since such commutation relation plays a crucial role in finding out the norm of Bethe eigenstates, the \(S\)-matrix for two-body scattering and various correlation functions, it is necessary to calculate very carefully the commutation relations among the elements of quantum monodromy matrix at infinite interval limit. Finally, it may be noted that two different approaches were taken earlier for treating the classical and quantum DNLS model \([12]\). The integrability of quantum DNLS model was established by first discretising the system on a lattice, evaluating the commutation relations among the elements of the monodromy matrix defined on that lattice and finally taking the continuum limit. For the case of classical DNLS model, however, no such lattice regularisation was taken and PB relations among the elements of monodromy matrix were evaluated directly for the continuum model.

The aim of the present article is to shed some light on the above mentioned issues and especially study the quantum DNLS model by using a variant of QISM \([2]\) which can be applied to continuum field models without performing any lattice regularisation. In Sec.2 of this article, we find out the symmetry properties of monodromy matrix associated with the classical DNLS model \((1.2)\) and subsequently use those symmetries for evaluating PB relations among various elements of this monodromy matrix. In Sec.3, we construct the quantum monodromy matrix of DNLS model on a finite interval and derive all possible commutation relations among the elements of such monodromy matrix through QISM. In Sec.4, we take the infinite interval limit of these commutation relations and construct exact eigenstates for the diagonal elements of quantum monodromy matrix through algebraic Bethe ansatz. Furthermore, we obtain the commutation relation between creation and annihilation operators of quasi-particles associated with DNLS model and find out the \(S\)-
matrix of two-body scattering among such quasi-particles. Sec.5 is the concluding section.

2 Integrability of the classical DNLS model

To investigate the symmetry properties of monodromy matrix associated with the classical DNLS model (1.2), we start with a Lax operator of the form

\[
U(x, \lambda) = i \begin{pmatrix} \xi \psi^*(x) \psi(x) - \lambda^2/4 & \xi \lambda \psi^*(x) \\ \lambda \psi^*(x) \psi(x) + \lambda^2/4 & -\xi \psi^*(x) \psi(x) \end{pmatrix}.
\]

(2.1)

Note that the off-diagonal elements of this Lax operator differ from the corresponding elements of previously given Lax operator (1.5) through some scale factors. However, as will be shown later, the traces of monodromy matrices associated with the Lax operators (1.5) and (2.1) coincide with each other and consequently both of these Lax operators correspond to the same DNLS model (1.2).

By using the Lax operator (2.1) along with its asymptotic form at \( |x| \to \infty \), we define the monodromy matrices of DNLS model on finite and infinite intervals as

\[
T_{x_2}^{x_1}(\lambda) = \mathcal{P} \exp \int_{x_1}^{x_2} U(x, \lambda) dx
\]

and

\[
T(\lambda) = \lim_{x_2 \to +\infty \atop x_1 \to -\infty} e(-x_2, \lambda) \left\{ e(x_2, \lambda) \right\} e(x_1, \lambda)
\]

respectively, where \( \mathcal{P} \) denotes the path ordering and \( e(x, \lambda) = e^{-\frac{i}{2} \lambda^2 \sigma_3 x} \). To find out the symmetries of such monodromy matrices, we note that the Lax operator (2.1) satisfies the relations

\[
U(x, \lambda)^* = K U(x, \lambda^*) K^{-1}, \quad U(x, -\lambda) = L U(x, \lambda) L^{-1},
\]

(2.4a, b)

where \( K = \begin{pmatrix} 0 & \sqrt{-\xi} \\ 1/\sqrt{-\xi} & 0 \end{pmatrix} \) and \( L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). By using these relations we find that the symmetries of monodromy matrix (2.3) are given by

\[
T(\lambda)^* = K T(\lambda^*) K^{-1}, \quad T(-\lambda) = L T(\lambda) L^{-1}.
\]

(2.5a, b)

By exploiting the symmetry relation (2.5a) and restricting to the case when \( \lambda \) is a real parameter, one may now express \( T(\lambda) \) in the form

\[
T(\lambda) = \begin{pmatrix} a(\lambda) & -\xi b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix}.
\]

(2.6)

Since the Lax operator (2.1) is a traceless matrix, we also get \( \text{det} T(\lambda) = 1 \) or, equivalently, \( |a(\lambda)|^2 + \xi |b(\lambda)|^2 = 1 \). Moreover, by using the symmetry relation (2.5b), it is easy to see
that $a(-\lambda) = a(\lambda)$ and $b(-\lambda) = -b(\lambda)$. So, for the case of real $\lambda$, $T(\lambda)$ within the range $\lambda \geq 0$ contains all informations about the scattering data. Therefore, in the following we shall consider the PB relations among the elements of monodromy matrix (2.6) only within the range $\lambda \geq 0$.

In analogy with the monodromy matrix $T(\lambda)$ (2.3) which is defined through Lax operator (2.1), one can also define the monodromy matrix $\tilde{T}(\lambda)$ corresponding to Lax operator (1.5) as

$$\tilde{T}(\lambda) = \lim_{x_1 \to -\infty} \lim_{x_2 \to +\infty} e(-x_2, \lambda) \left\{ \mathcal{P} \exp \int_{x_1}^{x_2} \tilde{U}(x, \lambda) dx \right\} e(x_1, \lambda). \quad (2.7)$$

It was assumed earlier that this $\tilde{T}(\lambda)$ can be written in a $SU(2)$ symmetric form for any value of the corresponding coupling constant $\xi$ [12]. For the purpose of properly investigating the symmetry properties of $\tilde{T}(\lambda)$, we observe that the Lax operators (1.5) and (2.1) are related through a symmetry transformation given by

$$MU(x, \lambda)M^{-1} = \tilde{U}(x, \lambda), \quad (2.8)$$

where $M = \begin{pmatrix} \xi^{-\frac{1}{2}} & 0 \\ 0 & \xi^{\frac{1}{2}} \end{pmatrix}$. Consequently, the corresponding monodromy matrices would also be related as $\tilde{T}(\lambda) = MT(\lambda)M^{-1}$. By using this relation along with eqn.(2.9), we can express $\tilde{T}(\lambda)$ for real $\lambda$ as

$$\tilde{T}(\lambda) = \begin{pmatrix} \tilde{a}(\lambda) & -\rho \tilde{b}^*(\lambda) \\ \tilde{b}(\lambda) & \tilde{a}^*(\lambda) \end{pmatrix}, \quad (2.9)$$

where $\rho = \text{sign } \xi$ and

$$\tilde{a}(\lambda) = a(\lambda), \quad \tilde{a}^*(\lambda) = a^*(\lambda), \quad \tilde{b}(\lambda) = \sqrt{\xi} b(\lambda), \quad \tilde{b}^*(\lambda) = \rho \sqrt{\xi} b^*(\lambda). \quad (2.10)$$

Now from eqn.(2.9) it is evident that the monodromy matrix $\tilde{T}(\lambda)$ takes the form of a $SU(2)$ group valued object when $\xi > 0$ and $SU(1,1)$ group valued object when $\xi < 0$. Thus we find that, similar to the case of NLS model, symmetry of the monodromy matrix associated with DNLS model is also determined through the sign of the corresponding coupling constant. As a result PB relations among the scattering data of DNLS model, which were derived earlier by assuming $\tilde{T}(\lambda)$ to be a $SU(2)$ group valued object, actually correspond to the case $\xi > 0$.

It may be noted that, the diagonal elements of the monodromy matrices $T(\lambda)$ (2.6) and $\tilde{T}(\lambda)$ (2.3) coincide with each other. So, by using the results of Ref.12 where $\ln \tilde{a}(\lambda)$ was expanded in powers of $\frac{1}{\lambda}$, we can write

$$\ln a(\lambda) = \ln \tilde{a}(\lambda) = \sum_{n=0}^{\infty} \frac{i C_n}{\lambda^{2n}}, \quad (2.11)$$

5
and find out the first few $C_n$s as

$$
C_0 = -\xi \int_{-\infty}^{+\infty} \psi^* \psi \, dx, \quad C_1 = 4i\xi \int_{-\infty}^{+\infty} \psi^* \psi_x \, dx, \\
C_2 = 8\xi \int_{-\infty}^{+\infty} (\psi^* \psi_{xx} - 2i\xi(\psi^* \psi)\psi_x) \, dx.
$$

(2.12a, b)

(2.12c)

Note that the Hamiltonian (1.3) of DNLS model is related to the third expansion coefficient (2.12c) as $H = -\frac{1}{8\xi} C_2$. As a result, the monodromy matrices $\tilde{T}(\lambda)$ (2.9) and $T(\lambda)$ (2.6) correspond to the same DNLS model (1.2).

Next, we want to derive the PB relations among the elements of $T(\lambda)$ (2.6) for both positive and negative values of the coupling constant $\xi$. To this end, we apply (1.4) to evaluate the PB relations among the elements of Lax operator (2.1) and find that

$$
\{U(x, \lambda) \otimes U(y, \mu)\} = [r(\lambda, \mu), U(x, \lambda) \otimes 1 + 1 \otimes U(y, \mu)] \delta(x - y),
$$

(2.13)

where

$$
r(\lambda, \mu) = -\xi \left( t^c \sigma_3 \otimes \sigma_3 + s^c(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \right)
$$

(2.14)

with $t^c = \frac{\lambda^2 + \mu^2}{2(\lambda^2 - \mu^2)}$, $s^c = \frac{2\lambda \mu}{\lambda^2 - \mu^2}$. Next, by employing a standard technique [1] for deriving PB relations among the elements of monodromy matrix (2.3) with the help of eqn.(2.13), we obtain

$$
\{T(\lambda) \otimes T(\mu)\} = r_+ (\lambda, \mu) T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu) r_(\lambda, \mu),
$$

(2.15)

where $r_+(\lambda, \mu)$ matrices are given by

$$
r_+ (\lambda, \mu) = E^{\pm 1}(-L, \lambda) \otimes E^{\pm 1}(-L, \mu) r(\lambda, \mu) E^{\mp 1}(-L, \lambda) \otimes E^{\mp 1}(-L, \mu)
$$

$$
= -\xi \left( t^c \sigma_3 \otimes \sigma_3 + s^c_+ \sigma_+ \otimes \sigma_- + s^c_- \sigma_- \otimes \sigma_+ \right),
$$

with $s^c_+ = \pm 2i\pi \lambda^2 \delta(\lambda^2 - \mu^2)$. By substituting the symmetric form of $T(\lambda)$ (2.6) to eqn.(2.15) and expressing it in elementwise form, we finally obtain

$$
\{a(\lambda), a(\mu)\} = 0, \quad \{a(\lambda), a^*(\mu)\} = 0,
$$

(2.16a, b)

$$
\{a(\lambda), b(\mu)\} = \xi \left( \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} \right) a(\lambda)b(\mu) - 2i\pi \xi \lambda^2 \delta(\lambda^2 - \mu^2) b(\lambda)a(\mu),
$$

(2.16c)

$$
\{a(\lambda), b^*(\mu)\} = -\xi \left( \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} \right) a(\lambda)b^*(\mu) + 2i\pi \xi \lambda^2 \delta(\lambda^2 - \mu^2) b^*(\lambda)a(\mu),
$$

(2.16d)

$$
\{b(\lambda), b^*(\mu)\} = -4i\pi \lambda^2 \delta(\lambda^2 - \mu^2) |a(\lambda)|^2.
$$

(2.16e)

The above PB relations among the scattering data of the DNLS model are evidently valid for all values of the coupling constant $\xi$. Since from eqns.(2.16a) and (2.11) it follows that $\{C_m, C_n\} = 0$ for all $m, n$, the DNLS model (1.2) represents a classical integrable system in the Liouville sense. With the help of transformation (2.10), one can also recast eqns.(2.16a-e) as the PB relations among the elements of monodromy matrix $\tilde{T}(\lambda)$ (2.9) and compare such PB relations with their counterparts in Ref.12. It is easy to see that the
forms of eqns.(2.16a-d) remain unaltered through the transformation (2.10) and coincide with their counterparts as given earlier (after taking care of slight changes in notations and in the definition of fundamental PB relation). However, by using eqns.(2.16e) and (2.10) we find that

\[
\{\tilde{b}(\lambda), \tilde{b}^*(\mu)\} = -4i\pi|\xi|\lambda^2\delta(\lambda^2 - \mu^2)|\tilde{a}(\lambda)|^2.
\]

(2.17)

It is interesting to note that, for the case \(\xi < 0\), the above equation differs from its counterpart [12] through a sign factor. Thus, we are able to derive here the correct PB relation between \(\tilde{b}(\lambda)\) and \(\tilde{b}^*(\mu)\) for the case \(\xi < 0\). It is well known that the commutation relation between the quantum analogues of \(\tilde{b}(\lambda)\) and \(\tilde{b}^*(\mu)\) plays a crucial role in finding out the norm of the Bethe eigenstates, the \(S\)-matrix for two-body scattering and various correlation functions. As a first step for evaluating this commutation relation and other commutation relations which would represent the quantum counterparts of the classical PB relations (2.16), in the following we shall quantise the monodromy matrix (2.2) of DNLS model on a finite interval.

3 Commutation relations for the quantum monodromy matrix on a finite interval

In the quantised version of the DNLS model (1.2), the fundamental PB relations (1.4) are replaced by equal time commutation relations among the basic field operators:

\[
[\psi(x), \psi(y)] = \left[\psi^\dagger(x), \psi^\dagger(y)\right] = 0, \quad \left[\psi(x), \psi^\dagger(y)\right] = \delta(x - y),
\]

(3.1)

and vacuum state is defined as \(\psi(x)|0\rangle = 0\). By using the ultralocal commutation relations (3.1) and a version of QISM which can be applied to field models without performing any lattice regularisation [2], at present we shall construct the quantum monodromy matrix of DNLS model on a finite interval and derive all commutation relations among the elements of such quantum monodromy matrix. To this end, we assume that the quantised form of the classical Lax operator (2.1) is given by

\[
U_q(x, \lambda) = i \begin{pmatrix}
 f\rho(x) - \lambda^2/4 & \xi\lambda\psi^\dagger(x) \\
 \lambda\psi(x) & -g\rho(x) + \lambda^2/4
\end{pmatrix},
\]

(3.2)

where \(\rho(x) = \psi^\dagger(x)\psi(x)\), and \(f, g\) are two yet undetermined parameters. It will be shown later that \(f \to \xi, \ g \to \xi\) at the classical limit and, as a result, \(U_q(x, \lambda)\) (3.2) correctly reproduces \(U(x, \lambda)\) (2.1) in this limit. By using \(U_q(x, \lambda)\), we define the quantum monodromy matrix of DNLS model on a finite interval as

\[
T_{x_1}^{x_2}(\lambda) = \mathcal{P}\exp\int_{x_1}^{x_2} U_q(x, \lambda)dx,
\]

(3.3)
where the symbol :: denotes the normal ordering of operators. It is evident that this quantum monodromy matrix (3.3) satisfies a differential equation given by

\[
\frac{\partial}{\partial x_2} T_{x_1}^{x_2}(\lambda) = : U_q(x_2, \lambda) T_{x_1}^{x_2}(\lambda) :
\]

\[
= -\frac{i}{4} \lambda^2 \sigma_3 T_{x_1}^{x_2}(\lambda) + i \xi \lambda \psi \dagger(x_2) \sigma_+ T_{x_1}^{x_2}(\lambda) + i \lambda \sigma_- T_{x_1}^{x_2}(\lambda) \psi(x_2) + i f \psi \dagger(x_2) e_{11} T_{x_1}^{x_2}(\lambda) \psi(x_2) - i g \psi \dagger(x_2) e_{22} T_{x_1}^{x_2}(\lambda) \psi(x_2),
\]

(3.4)

where \( e_{11} = \frac{1}{2}(1 + \sigma_3) \) and \( e_{22} = \frac{1}{2}(1 - \sigma_3) \). For the purpose of applying QISM, however, it is needed to find out the differential equation satisfied by the product \( T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) \).

To this end, we borrow a notation of Ref.2 where the sign for normal arrangement of operator factors is taken as ::. The sign ::, applied to the product of several operator factors (including \( \psi \) and \( \psi \dagger \)), ensures the arrangement of all \( \psi \dagger \) on the left, and all \( \psi \) on the right, without altering the order of the remaining factors. For example,

\[
: X \psi \psi \dagger Y : = \psi \dagger X Y \psi,
\]

where \( X \) and \( Y \) may be taken as some elements of the quantum monodromy matrix (3.3). Now by using the basic commutation relations (3.1) and the method of ‘extension’ [2], we find that the product of two monodromy matrices satisfies the following differential equation (details of this calculation are given in the Appendix):

\[
\frac{\partial}{\partial x_2} (T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu)) = : \mathcal{L}(x_2, \lambda, \mu) T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) ::,
\]

(3.5)

where

\[
\mathcal{L}(x; \lambda, \mu) = U_q(x, \lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U_q(x, \mu) + \mathcal{L}_\Delta(x; \lambda, \mu),
\]

(3.6)

with

\[
\mathcal{L}_\Delta(x; \lambda, \mu) = \begin{pmatrix}
-f^2 \rho(x) & -\mu \xi \psi \dagger(x) & 0 & 0 \\
0 & gf \rho(x) & 0 & 0 \\
-\lambda f \psi(x) & -\lambda \mu \xi & gf \rho(x) & \mu \xi \psi \dagger(x) \\
0 & \lambda g \psi(x) & 0 & -g^2 \rho(x)
\end{pmatrix}.
\]

Next, let us consider a \((4 \times 4)\) \( R(\lambda, \mu) \) matrix of the form

\[
R(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s(\lambda, \mu) & t(\lambda, \mu) & 0 \\
0 & t(\lambda, \mu) & s(\lambda, \mu) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(3.7)

with \( t(\lambda, \mu) = \frac{\lambda^2 - \mu^2}{\lambda^2 - \mu^2 q^{-2}} \), \( s(\lambda, \mu) = \frac{(q - q^{-1}) \lambda \mu}{\lambda^2 - \mu^2 q^{-2}} \), and \( q = e^{-i \alpha} \), \( \alpha \) being an yet undetermined real parameter. It is easy to check that the \( R(\lambda, \mu) \) matrix (3.7) and \( \mathcal{L}(x; \lambda, \mu) \) (3.6) satisfy an equation given by

\[
R(\lambda, \mu) \mathcal{L}(x; \lambda, \mu) = \mathcal{L}(x; \mu, \lambda) R(\lambda, \mu),
\]

(3.8)
provided the parameters $f$, $g$, $\alpha$ and the coupling constant $\xi$ are related as

$$\xi = -\sin \alpha, \quad f = \frac{\xi e^{-i\alpha/2}}{\cos \alpha/2}, \quad g = \frac{\xi e^{i\alpha/2}}{\cos \alpha/2}. \quad (3.9a, b, c)$$

By using eqns.\(3.5\) and \(3.8\), we find that the monodromy matrix \(3.3\) of DNLS model satisfies the quantum Yang-Baxter equation (QYBE):

$$R(\lambda, \mu)T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) = T_{x_1}^{x_2}(\mu) \otimes T_{x_1}^{x_2}(\lambda)R(\lambda, \mu). \quad (3.10)$$

Expressing this QYBE in elementwise form, one may explicitly obtain all possible commutation relations among the elements of quantum monodromy matrix \(3.3\).

Note that the relations \((3.9a, b, c)\) are obtained as a necessary condition for satisfying QYBE \((3.10)\). By solving eqns.\((3.9a, b, c)\), the parameters $f$, $g$ and $\alpha$ can be determined as some functions of the known coupling constant $\xi$. Thus QYBE fixes all undetermined parameters in the quantum Lax operator \((3.2)\) and corresponding $R(\lambda, \mu)$ matrix \((3.7)\). Due to eqn.\((3.9a)\) it is evident that the coupling constant of quantum DNLS model must be restricted within a range given by $|\xi| \leq 1$ and the parameter $\alpha$ has a one-to-one correspondence with such coupling constant for $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. It may be noticed that, by putting $x_2 = x_1 + \Delta$ in eqn.\((3.3)\) (here $\Delta$ is a small positive parameter) and retaining terms only up to the order $\Delta$, one obtains

$$T_{x_1}^{x_1+\Delta}(\lambda) \sim I + i\Delta \begin{pmatrix} f\rho_n - \lambda^2/4 & \xi\lambda\psi_n^\dagger \\ \lambda\psi_n & -g\rho_n + \lambda^2/4 \end{pmatrix}, \quad (3.11)$$

where $\psi_n^\dagger = \frac{1}{\Delta} \int_{x_1}^{x_1+\Delta} \psi(x) dx$, $\psi_n = \frac{1}{\Delta} \int_{x_1}^{x_1+\Delta} \psi(x) dx$, $\rho_n = \frac{1}{\Delta} \int_{x_1}^{x_1+\Delta} \rho(x) dx$. It is interesting to observe that this $T_{x_1}^{x_1+\Delta}(\lambda) \quad (3.11)$ satisfies QYBE \((3.10)\) up to order $\Delta$ and reproduces (up to a gauge transformation) the lattice Lax operator for quantum DNLS model [12] when the parameter $\Delta$ is identified as a lattice constant. In this article, however, we shall not use any lattice discretisation and directly work with the quantum monodromy matrix \(3.3\) which satisfies QYBE exactly.

Next, for the purpose of investigating the classical limit of the quantum Lax operator \((3.2)\), we replace the last commutation relation in eqn.\((3.1)\) by $[\psi(x), \psi^\dagger(y)] = \hbar \delta(x - y)$, where $\hbar$ is the Planck’s constant, and then repeat all calculations of this section. It turns out that eqns.\((3.9b, c)\) remain unchanged, but eqn.\((3.9a)\) is modified as: $\hbar \xi = -\sin \alpha$. Consequently, for any fixed value of $\xi$, $\alpha \to 0$ limit is essentially equivalent to $\hbar \to 0$ limit. Since from eqns.\((3.9b, c)\) it follows that $f \to \xi$ and $g \to \xi$ when $\alpha \to 0$, the quantum Lax operator \((3.2)\) indeed reproduces the classical Lax operator \((2.1)\) at $\hbar \to 0$ limit.
4 Algebraic Bethe ansatz for the quantum monodromy matrix on an infinite interval

For the purpose of taking infinite interval limit of QYBE (3.10), we define the quantum analogue of classical monodromy matrix (2.3) as

\[ T(\lambda) = \lim_{x_2 \to +\infty, x_1 \to -\infty} e(-x_2, \lambda) T_{x_1}^{x_2}(\lambda) e(x_1, \lambda), \]  

(4.1)

where \( T_{x_1}^{x_2}(\lambda) \) is given by eqn.(3.3). It may be observed that, exactly like the case of classical Lax operator (2.1), the quantum Lax operator (3.2) also satisfies the following relations:

\[ U_q(x, \lambda)^* = K U_q(x, \lambda^*) K^{-1}, \quad U_q(x, -\lambda) = L U_q(x, \lambda) L^{-1}, \]  

(4.2a, b)

where \( K \) and \( L \) are same matrices which have appeared in (2.4a,b). By using eqn.(4.2a) and assuming \( \lambda \) to be a real parameter, it is easy to show that the quantum monodromy matrix (4.1) can be expressed in a symmetric form given by

\[ T(\lambda) = \begin{pmatrix} A(\lambda) & -\xi B^\dagger(\lambda) \\ B(\lambda) & A^\dagger(\lambda) \end{pmatrix}. \]  

(4.3)

Since from eqn.(4.2b) it follows that \( A(-\lambda) = A(\lambda) \) and \( B(-\lambda) = -B(\lambda) \), it is necessary to consider the quantum monodromy matrix (4.3) only within the range \( \lambda \geq 0 \). In analogy with \( T(\lambda) \) (4.1), which is defined by quantising the classical monodromy matrix \( T(\lambda) \) (2.3), one may also define the quantum analogue of the classical monodromy matrix \( \tilde{T}(\lambda) \) (2.7). By using arguments similar to the classical case, it is easy to show that such quantum monodromy matrix can be written in a symmetric form given by

\[ \tilde{T}(\lambda) = \begin{pmatrix} \tilde{A}(\lambda) & -\rho \tilde{B}^\dagger(\lambda) \\ \tilde{B}(\lambda) & \tilde{A}^\dagger(\lambda) \end{pmatrix}, \]  

(4.4)

where \( \rho = \text{sign} \xi \) and \( \tilde{A}(\lambda) = A(\lambda), \tilde{B}(\lambda) = \sqrt{\xi} B(\lambda) \). Thus we find that, the quantum monodromy matrix \( \tilde{T}(\lambda) \) can be expressed like a \( U(2) \) group valued object when \( \xi > 0 \) and \( U(1,1) \) group valued object when \( \xi < 0 \). As a result, the commutation relations which were derived earlier by assuming \( \tilde{T}(\lambda) \) to be a \( U(2) \) group valued object [12] should correspond to the case \( \xi > 0 \).

For finding out the commutation relations among the operator elements of monodromy matrix (4.1), it is required to get rid of the oscillatory terms which arise from the product \( T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) \) at the asymptotic limits \( x_1, x_2 \to \pm \infty \). To this end, we split the \( \mathcal{L}(x; \lambda, \mu) \) matrix (3.6) into two parts:

\[ \mathcal{L}(x; \lambda, \mu) = \mathcal{L}_0(x; \lambda, \mu) + \mathcal{L}_1(x; \lambda, \mu), \]
where $\mathcal{L}_0(\lambda, \mu)$ is given by
\[
\mathcal{L}_0(\lambda, \mu) = \lim_{|x| \to \infty} \mathcal{L}(x; \lambda, \mu) = \begin{pmatrix}
-\frac{i}{4}(\lambda^2 + \mu^2) & 0 & 0 & 0 \\
0 & -\frac{i}{4}(\lambda^2 - \mu^2) & 0 & 0 \\
0 & 0 & -\xi \lambda \mu & \frac{i}{4}(\lambda^2 - \mu^2) \\
0 & 0 & 0 & \frac{i}{4}(\lambda^2 + \mu^2)
\end{pmatrix},
\]
and $\mathcal{L}_1(x; \lambda, \mu)$ is the field dependent part of $\mathcal{L}(x; \lambda, \mu)$, which vanishes at $x \to \pm \infty$. Due to eqn. (3.8) it follows that
\[
R(\lambda, \mu) \varepsilon(x; \lambda, \mu) = \varepsilon(x; \mu, \lambda) R(\lambda, \mu),
\]
where $\varepsilon(x; \lambda, \mu) = e^{\mathcal{L}_0(\lambda, \mu)x}$. By using the above mentioned splitting of $\mathcal{L}(x; \lambda, \mu)$, one can derive the integral form of differential equation (3.5) as
\[
\mathcal{T}_{x_1}^{x_2}(\lambda) \otimes \mathcal{T}_{x_1}^{x_2}(\mu) = \varepsilon(x_2 - x_1; \lambda, \mu) + \int_{x_1}^{x_2} dx \varepsilon(x_2 - x; \lambda, \mu) : \mathcal{L}_1(x; \lambda, \mu) \mathcal{T}_{x_1}^{x_2}(\lambda) \otimes \mathcal{T}_{x_1}^{x_2}(\mu) :.
\]
Due to the presence of field dependent matrix $\mathcal{L}_1(x; \lambda, \mu)$, the second term in the r.h.s. of above integral equation vanishes at the limit $x_1, x_2 \to \pm \infty$. Consequently, at this limit, one gets
\[
\mathcal{T}_{x_1}^{x_2}(\lambda) \otimes \mathcal{T}_{x_1}^{x_2}(\mu) \rightarrow \varepsilon(x_2 - x_1; \lambda, \mu),
\]
which is an oscillatory term. To get rid of this unwanted term, we define an operator like
\[
W(\lambda, \mu) = \lim_{x_1 \to -\infty} \lim_{x_2 \to +\infty} \varepsilon(-x_2; \lambda, \mu) T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) \varepsilon(x_1; \lambda, \mu),
\]
which is clearly well behaved at the infinite interval limit. By using (3.10) and (4.5), it is easy to verify that this $W(\lambda, \mu)$ (4.6) satisfies an equation given by
\[
R(\lambda, \mu) W(\lambda, \mu) = W(\mu, \lambda) R(\lambda, \mu),
\]
which represents QYBE for the infinite interval limit.

Next, we want to express QYBE (4.7) through the direct product of two monodromy matrices of the form (4.4). To this end, we note that $W(\lambda, \mu)$ (4.6) may be rewritten as
\[
W(\lambda, \mu) = C_+(\lambda, \mu) \mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) C_-(\lambda, \mu),
\]
where
\[
C_+(\lambda, \mu) = \lim_{x \to \infty} \varepsilon(-x; \lambda, \mu) E(x; \lambda, \mu), \quad C_-(\lambda, \mu) = \lim_{x \to -\infty} E(-x; \lambda, \mu) \varepsilon(x; \lambda, \mu),
\]
with $E(x; \lambda, \mu) = e(x, \lambda) \otimes e(x, \mu)$. Substituting the explicit forms of $E(x; \lambda, \mu)$ and $\varepsilon(x; \lambda, \mu)$ to (4.9a,b), and extracting the limits in the principal value sense: \( \lim_{x \to \pm \infty} P \left( e^{\frac{ikx}{\hbar}} \right) = \pm i\pi \delta(k) \)
we obtain
\[
C_+(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \rho_+(\lambda, \mu) & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad C_-(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \rho_-(\lambda, \mu) & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[ \rho_\pm(\lambda, \mu) = \mp \frac{2i\xi\lambda\mu}{\lambda^2 - \mu^2} + 2\pi\lambda\mu\delta(\lambda^2 - \mu^2) = \mp \frac{2i\xi\lambda\mu}{\lambda^2 - \mu^2 \mp ie}. \]

By using the expression (4.8), we rewrite QYBE (4.7) for the infinite interval limit as

\[ R(\lambda, \mu)C_+(\lambda, \mu)T(\lambda) \otimes T(\mu)C_-(\lambda, \mu) = C_+(\mu, \lambda)T(\mu) \otimes T(\lambda)C_-(\mu, \lambda)R(\lambda, \mu). \quad (4.11) \]

By inserting the explicit forms of \( R(\lambda, \mu) \) (3.7), \( C_+(\lambda, \mu) \) (4.10), and \( T(\lambda) \) (4.3) to QYBE (4.11) and comparing its matrix elements from both sides, we finally obtain

\[ [A(\lambda), A(\mu)] = 0, \quad [A(\lambda), A^\dagger(\mu)] = 0, \quad (4.12a, b) \]

\[ A(\lambda)B^\dagger(\mu) = \frac{\mu^2q - \lambda^2q^{-1}}{\mu^2 - \lambda^2 - ie}B^\dagger(\mu)A(\lambda) \]

\[ = \frac{\mu^2q - \lambda^2q^{-1}}{\mu^2 - \lambda^2}B^\dagger(\mu)A(\lambda) - 2\pi\xi\lambda\mu\delta(\lambda^2 - \mu^2)B^\dagger(\lambda)A(\mu), \quad (4.12c) \]

\[ B(\mu)A(\lambda) = \frac{\mu^2q - \lambda^2q^{-1}}{\mu^2 - \lambda^2 - ie}A(\lambda)B(\mu) \]

\[ = \frac{\mu^2q - \lambda^2q^{-1}}{\mu^2 - \lambda^2}A(\lambda)B(\mu) - 2\pi\xi\lambda\mu\delta(\lambda^2 - \mu^2)A(\mu)B(\lambda), \quad (4.12d) \]

\[ B(\mu)B^\dagger(\lambda) = \tau(\lambda, \mu)B^\dagger(\lambda)B(\mu) + 4\pi\lambda\mu\delta(\lambda^2 - \mu^2)A^\dagger(\lambda)A(\lambda), \quad (4.12e) \]

where \( q = e^{-i\alpha} \) and \( \tau(\lambda, \mu) = \left[ 1 + \frac{8\xi^2\lambda^2\mu^2}{(\lambda^2 - \mu^2)^2} - \frac{4\xi^2\lambda^2\mu^2}{(\lambda^2 - \mu^2 - ie)(\lambda^2 - \mu^2 + ie)} \right] \). The above commutation relations among the elements of quantum monodromy matrix (4.1) are evidently valid for both positive and negative values of the coupling constant \( \xi \). With the help of transformations like \( \tilde{A}(\lambda) = A(\lambda), \tilde{B}(\lambda) = \sqrt{\xi}B(\lambda) \), one can also recast eqns.(4.12a-e) as the commutation relations among the elements of the monodromy matrix \( \tilde{T}(\lambda) \) (4.4) and compare such commutation relations with their counterparts in Ref.12. It is easy to see that the forms of commutation relations (4.12a-d) remain unaltered through the above mentioned transformation and match with their counterparts as given earlier. However, by using the transformation \( \tilde{A}(\lambda) = A(\lambda), \tilde{B}(\lambda) = \sqrt{\xi}B(\lambda) \), equation (4.12e) may be expressed as

\[ \tilde{B}(\mu)\tilde{B}^\dagger(\lambda) = \tau(\lambda, \mu)\tilde{B}^\dagger(\lambda)\tilde{B}(\mu) + 4\pi\lambda\mu|\xi|\delta(\lambda^2 - \mu^2)\tilde{A}^\dagger(\lambda)\tilde{A}(\lambda), \]

which does not match at all with its counterpart [12] for either positive or negative value of \( \xi \). It can be shown that, due to a problem which arises while taking the infinite interval limit of QYBE, an incorrect commutation relation was obtained earlier between the operators \( \tilde{B}(\lambda) \) and \( \tilde{B}^\dagger(\mu) \) for both positive and negative values of \( \xi \). It is interesting to observe that, for the case \( \lambda \neq \mu \), eqn.(4.12e) gives \( [\tilde{B}(\lambda), \tilde{B}^\dagger(\mu)] \neq 0 \). On the other hand, from eqn.(2.16e) it follows that \( \{b(\lambda), b^*(\mu)\} = 0 \) for \( \lambda \neq \mu \). Thus we find that, similar to the case of NLS model [6], the correspondence between Poisson brackets and commutators among some elements of monodromy matrix may turn out to be a quite nontrivial one.
Due to eqn.(4.12a) it follows that all operator valued coefficients occurring in the expansion of \( \ln A(\lambda) \) in powers of \( \lambda \) would commute among themselves and, as a consequence, the monodromy matrix \([4.1]\) of DNLS model leads to a quantum integrable system. With the help of eqn.(4.1), it is easy to find that \( A(\lambda)|0\rangle = |0\rangle \). By using this relation and eqn.(4.12c), it can be shown that

\[
A(\lambda) |\mu_1, \mu_2, \cdots, \mu_N\rangle = \prod_{r=1}^{N} \left( \frac{\mu_r^2 q - \lambda^2 q^{-1}}{\mu_r^2 - \lambda^2 - i\epsilon} \right) |\mu_1, \mu_2, \cdots, \mu_N\rangle ,
\]

(4.13)

where \( \mu_j \)s are all distinct real numbers and \( |\mu_1, \mu_2, \cdots, \mu_N\rangle \equiv B^\dagger(\mu_1)B^\dagger(\mu_2)\cdots B^\dagger(\mu_N)|0\rangle \). Thus the states \( |\mu_1, \mu_2, \cdots, \mu_N\rangle \) diagonalise the generator of conserved quantities for the quantum DNLS model. However, by using eqn.(4.13), one finds that the eigenvalues corresponding to different expansion coefficients of \( \ln A(\lambda) \) would be complex quantities in general. To extract real eigenvalues, we define another operator \( \hat{A}(\lambda) \) through the relation: \( \hat{A}(\lambda) \equiv A(\lambda e^{-\frac{i\epsilon}{2}}) \) and expand \( \ln \hat{A}(\lambda) \) as

\[
\ln \hat{A}(\lambda) = \sum_{n=0}^{\infty} \frac{i C_n}{\lambda^{2n}} .
\]

(4.14)

With the help of eqns.(4.13) and (4.14) it is easy to see that \( C_n \)s satisfy eigenvalue equations like \( C_n |\mu_1, \mu_2, \cdots, \mu_N\rangle = \kappa_n |\mu_1, \mu_2, \cdots, \mu_N\rangle \), where the first few \( \kappa_n \)s are explicitly given by

\[
\kappa_0 = \alpha N , \quad \kappa_1 = 2 \sin \alpha \sum_{j=1}^{N} \mu_j^2 , \quad \kappa_2 = \sin 2\alpha \sum_{j=1}^{N} \mu_j^4 .
\]

(4.15)

In analogy with the classical case, one may now define the Hamiltonian for quantum DNLS model as \( \mathcal{H} = -\frac{1}{8\xi} C_2 \). Eigenvalue equations corresponding to this Hamiltonian are evidently given by

\[
\mathcal{H}|\mu_1, \mu_2, \cdots, \mu_N\rangle = \frac{1}{4} \sqrt{1 - \xi^2} \left( \sum_{j=1}^{N} \mu_j^4 \right) |\mu_1, \mu_2, \cdots, \mu_N\rangle .
\]

(4.16)

Till now we have assumed that \( \mu_j \)s are some real parameters, for which \( |\mu_1, \mu_2, \cdots, \mu_N\rangle \) represents a scattering state. One can also construct the quantum \( N \)-soliton state for DNLS model [12] by choosing complex values of \( \mu_j \) given by

\[
\mu_j = \mu \exp \left[ -i\alpha \left( \frac{N+1}{2} - j \right) \right] ,
\]

(4.17)

where \( \mu \) is a real parameter and \( j \in [1, 2, \cdots N] \). Thus \( \mu_j \)s are uniformly distributed on a circle of radius \( \mu \). For this choice of \( \mu_j \), eqn.(4.13) reduces to a simple form like

\[
A(\lambda) |\mu_1, \mu_2, \cdots, \mu_N\rangle = q^{-N} \left( \frac{\lambda^2 - \mu^2 q^{N+1}}{\lambda^2 - \mu^2 q^{-N+1}} \right) |\mu_1, \mu_2, \cdots, \mu_N\rangle ,
\]

(4.18)
where the eigenvalue of \( A(\lambda) \) has only one zero and one pole on the complex \( \lambda^2 \)-plane. By using (4.18), we obtain the energy eigenvalue corresponding to the quantum \( N \)-soliton state as

\[
H|\mu_1, \mu_2, \cdots, \mu_N\rangle = \frac{\mu^4 \sin(2\alpha N)}{8 \sin \alpha}|\mu_1, \mu_2, \cdots, \mu_N\rangle.
\] (4.19)

In general, we may choose any positive integer value of \( N \) (greater than one) for constructing a quantum soliton state. However, we now consider the D NLS model with some particular values of coupling constant given by \( \xi = -\sin \alpha = -\sin \left(\frac{2\pi m}{n}\right) \), where \( m \) and \( n \) are nonzero integers which do not have any common factor. By using eqn.(4.17), one obtains \( \mu_j = \mu_{j+n} \) for this case. Since all \( \mu_j \) must take distinct values, we get \( N \leq n \) as a restriction on the number of quasi-particles which form a bound state for the quantum D NLS model corresponding to coupling constant \( \xi = -\sin \left(\frac{2\pi m}{n}\right) \).

Thus, by applying the method of algebraic Bethe ansatz, we are able to construct the exact eigenstates for quantum D NLS model. The commutation relation (4.12e) also plays an important role in the framework of algebraic Bethe ansatz, since by using this commutation relation one should be able to calculate the norm of eigenstates \( |\mu_1, \mu_2, \cdots, \mu_N\rangle \) and various correlation functions of the D NLS system. However, it may be noted that the commutation relation (4.12e) contains product of generalised functions \( (\lambda^2 - \mu^2 - i\epsilon)^{-1}(\lambda^2 - \mu^2 + i\epsilon)^{-1} \), which does not make sense at the limit \( \lambda \to \mu \). As a result, the action of operators \( B^\dagger(\lambda), B(\mu) \) are not well defined on the Hilbert space [8] and eigenstates like \( |\mu_1, \mu_2, \cdots, \mu_N\rangle \) are not normalised on the \( \delta \)-function. However, it is well known that, one can avoid this type of problem in the case of NLS equation by considering the quantum analogue of classical reflection operators [1,6,7]. So, in analogy with the case of NLS equation, at present we consider a reflection operator given by

\[
R^\dagger(\lambda) = A^{-1}(\lambda)B(\lambda),
\] (4.20)

and its adjoint \( R(\lambda) \). By using eqns.(4.12a-e), we find that such reflection operators satisfy well defined commutation relations like

\[
R(\lambda)R(\mu) = S^{-1}(\lambda, \mu) R(\mu)R(\lambda),
\]

\[
R^\dagger(\lambda)R^\dagger(\mu) = S^{-1}(\lambda, \mu) R^\dagger(\mu)R^\dagger(\lambda),
\]

\[
R(\lambda)R^\dagger(\mu) = S(\lambda, \mu) R^\dagger(\mu)R(\lambda) + 4\pi \lambda^2 \delta(\lambda^2 - \mu^2),
\] (4.21)

where

\[
S(\lambda, \mu) = \frac{\lambda^2 q - \mu^2 q^{-1}}{\lambda^2 q^{-1} - \mu^2 q}.
\] (4.22)

It is evident that these commutation relations among reflection operators of D NLS model are nicely encoded in a form of Zamolodchikov-Faddeev algebra [1,13] and \( S(\lambda, \mu) \) represents the nontrivial \( S \)-matrix element of two-body scattering between the related quasi-particles. It is easy to check that this \( S(\lambda, \mu) \) satisfies the conditions given by

\[
S^{-1}(\lambda, \mu) = S(\mu, \lambda) = S^*(\lambda, \mu),
\] (4.23)
and remains nonsingular at the limit $\lambda \to \mu$. As a result, the action of operators $R^\dagger(\lambda), R(\mu)$ would be well defined on the Hilbert space and eigenstates like $R(\mu_1)R(\mu_2) \cdots R(\mu_N)|0\rangle$ can be normalised on the $\delta$-function.

5 Concluding Remarks

We find that the classical monodromy matrix for DNLS model can be written as a $SU(2)$ ($SU(1, 1)$) group valued object for positive (negative) value of the corresponding coupling constant. By using such symmetric form of classical monodromy matrix, we derive Poisson bracket relations among the scattering data of the DNLS model for all values of the coupling constant. We also quantise the monodromy matrix of DNLS model on a finite interval. A variant of quantum inverse scattering method, which can be applied to field models without performing any lattice regularisation, fixes all parameters in the quantum monodromy matrix of DNLS model in a nontrivial way. Similar to the classical case, this quantum monodromy matrix exhibits $U(2)$ ($U(1, 1)$) symmetry for positive (negative) value of the coupling constant. By applying quantum inverse scattering method, we derive all possible commutation relations among the elements of such monodromy matrix. Infinite interval limits of these commutation relations enable us to construct the exact eigenstates for quantum DNLS model through algebraic Bethe ansatz. In this context, we consider the DNLS model with some special values of coupling constant given by $\xi = -\sin \alpha = -\sin \left(\frac{2\pi m}{n}\right)$, where $m$ and $n$ are nonzero integers which do not have any common factor. It turns out that the number of quasi-particles, which form a bound state for such quantum DNLS model, can not exceed the value $n$.

We also obtain the commutation relation between creation and annihilation operators associated with quasi-particles of DNLS model and find out the $S$-matrix for two-body scattering. Such a commutation relation between creation and annihilation operators should play an important role in a future study, since by using it one might be able to calculate the norm of Bethe eigenstates and various correlation functions of the DNLS system. It may be noted that, there exist quantum integrable multicomponent generalisations of NLS model which can be diagonalised through algebraic Bethe ansatz [14-17]. A large class of multicomponent classical DNLS models, having infinite number of conserved quantities, are also studied in the literature [18,19]. However, the Hamiltonian structure of such multicomponent DNLS models have not yet received much attention. So it might be interesting to investigate whether there exist some multicomponent generalisations of classical DNLS model which are integrable in the Liouville sense and retain their integrability property even after quantisation.
Appendix

A direct attempt to calculate $\frac{\partial}{\partial x_2} \left( T_{x_1}^{x_2}(\lambda) \otimes T_{x_1}^{x_2}(\mu) \right)$ by using eqn.(3.4) evidently leads to indeterminate expressions of the form $\left[ T_{x_1}^{x_2}(\lambda), \psi^\dagger(x_2) \right]$. To bypass this problem, we follow the method of extension [2] which shifts the upper limit of one monodromy matrix (say $T_{x_1}^{x_2}(\lambda)$) by introducing a small parameter $\epsilon$ and takes $\epsilon \to 0$ limit only after differentiating the product $T_{x_1}^{x_2+\epsilon}(\lambda) \otimes T_{x_1}^{x_2}(\mu)$ with respect to $x_2$. Thus, by using eqn.(3.4), we obtain

$$
\frac{\partial}{\partial x_2} \left( T_{x_1}^{x_2+\epsilon}(\lambda) \otimes T_{x_1}^{x_2}(\mu) \right) = : \left( U_q(x_2 + \epsilon, \lambda) \otimes \mathbb{I} + \mathbb{I} \otimes U_q(x_2, \mu) \right) T_{x_1}^{x_2+\epsilon}(\lambda) \otimes T_{x_1}^{x_2}(\mu) : + K_+ + K_-, \tag{A1}
$$

where

$$
K_+ = i\xi \mu \left[ T_{x_1}^{x_2+\epsilon}(\lambda), \psi^\dagger(x_2) \right] \otimes \sigma_+ T_{x_1}^{x_2}(\mu) + i f \left[ T_{x_1}^{x_2+\epsilon}(\lambda), \psi(x_2) \right] \otimes e_{11} T_{x_1}^{x_2}(\mu) \psi(x_2),
$$

$$
K_- = i\lambda \sigma_- T_{x_1}^{x_2+\epsilon}(\lambda) \otimes \left[ \psi(x_2 + \epsilon), T_{x_1}^{x_2}(\mu) \right] + i f \psi^\dagger(x_2 + \epsilon) e_{11} T_{x_1}^{x_2+\epsilon}(\lambda) \otimes \left[ \psi(x_2 + \epsilon), T_{x_1}^{x_2}(\mu) \right].
$$

Now we consider the case $\epsilon > 0$. Since $\psi(x_2 + \epsilon)$ commutes with $\psi(x)$, $\psi^\dagger(x)$ for all $x$ lying within $x_1$ and $x_2$, we can write $\left[ \psi(x_2 + \epsilon), T_{x_1}^{x_2}(\mu) \right] = 0$ and $K_- = 0$ for this case. So, we have to calculate only the nontrivial commutator $\left[ T_{x_1}^{x_2+\epsilon}(\lambda), \psi^\dagger(x_2) \right]$ which appears in the expression of $K_+$. To this end, we consider a ‘transformation’ $\Omega$ which replaces the classical variables $\psi(x)$ and $\psi^*(x)$ by quantum operators $\psi(x)$ and $\psi^\dagger(x)$ respectively ($\Omega^{-1}$ denotes the reverse transformation). By applying a correspondence principle [2] to the present case, we obtain

$$
\left[ T_{x_1}^{x_2+\epsilon}(\lambda), \psi^\dagger(x_2) \right] = i : \Omega \left\{ T_{x_1}^{x_2+\epsilon}(q; \lambda), \psi^*(x_2) \right\} : , \tag{A2}
$$

where $T_{x_1}^{x_2+\epsilon}(q; \lambda)$ represents a classical monodromy matrix given by

$$
T_{x_1}^{x_2+\epsilon}(q; \lambda) = \mathcal{P} \exp \int_{x_1}^{x_2+\epsilon} U_q(x, \lambda) dx ,
$$

and $U_q(x, \lambda) = \Omega^{-1} U_q(x, \lambda)$. By using the fundamental PB relations (1.4), it is easy to find that

$$
\left\{ T_{x_1}^{x_2+\epsilon}(q; \lambda), \psi^*(x_2) \right\} = \int_{x_1}^{x_2+\epsilon} dx T_{x_2}^{x_2+\epsilon}(q; \lambda) \left\{ U_q(x, \lambda), \psi^*(x_2) \right\} T_{x_1}^{x_2}(q; \lambda)
$$

$$
= T_{x_2}^{x_2+\epsilon}(q; \lambda) (f \psi^*(x_2) e_{11} - g \psi^\dagger(x_2) e_{22} + \lambda \sigma_-) T_{x_1}^{x_2}(q; \lambda) .
$$

Taking $\epsilon \to 0$ limit of the above expression and inserting it to (A2), we obtain

$$
\lim_{\epsilon \to 0} \left[ T_{x_1}^{x_2+\epsilon}(\lambda), \psi^\dagger(x_2) \right] = i \left( f \psi^\dagger(x_2) e_{11} - g \psi^\dagger(x_2) e_{22} + \lambda \sigma_- \right) T_{x_1}^{x_2}(\lambda). \tag{A3}
$$

Taking $\epsilon \to 0$ limit also in eqn.(A1) and using (A3), we finally obtain the differential equation (3.3). Note that, instead of $\epsilon > 0$, we could have chosen $\epsilon < 0$ in eqn.(A1). Only the commutator $\left[ \psi(x_2 + \epsilon), T_{x_1}^{x_2}(\mu) \right]$ gives a nontrivial contribution for this case. However, repeating similar steps as outlined above and finally taking the $\epsilon \to 0$ limit, we get again the same differential equation (3.3).
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