Convex Regression in Multidimensions: Suboptimality of Least Squares Estimators

Gil Kur, Fuchang Gao, Adityanand Guntuboyina, and Bodhisattva Sen

Abstract: The least squares estimator (LSE) is shown to be suboptimal in squared error loss in the usual nonparametric regression model with Gaussian errors for \( d \geq 5 \) for each of the following families of functions: (i) convex functions supported on a polytope (in fixed design), (ii) bounded convex functions supported on a polytope (in random design), and (iii) convex Lipschitz functions supported on any convex domain (in random design). For each of these families, the risk of the LSE is proved to be of the order \( n^{-2/d} \) (up to logarithmic factors) while the minimax risk is \( n^{-4/(d+4)} \), for \( d \geq 5 \). In addition, the first rate of convergence results (worst case and adaptive) for the full convex LSE are established for polytopal domains for all \( d \geq 1 \). Some new metric entropy results for convex functions are also proved which are of independent interest.

MSC 2010 subject classifications: Primary 62G08.

Keywords and phrases: Adaptive risk bounds, bounded convex regression, Dudley’s entropy bound, Lipschitz convex regression, lower bounds on the risk of least squares estimators, metric entropy, nonparametric maximum likelihood estimation, Sudakov minoration.
1. Introduction

We consider the following nonparametric regression problem:

\[ Y_i = f_0(X_i) + \xi_i, \quad \text{for } i = 1, \ldots, n, \]

where \( f_0 : \Omega \to \mathbb{R} \) is an unknown convex function on a full-dimensional compact convex domain \( \Omega \subseteq \mathbb{R}^d \) (\( d \geq 1 \)), \( X_1, \ldots, X_n \) are design points that may be fixed in \( \Omega \) or i.i.d. from the uniform distribution \( \mathbb{P} \) on \( \Omega \), and \( \xi_1, \ldots, \xi_n \) are i.i.d. unobserved errors having the \( N(0, \sigma^2) \) distribution with \( \sigma^2 > 0 \) being unknown. Given data \( \{(Y_i, X_i)\}_{i=1}^n \), the goal is to recover \( f_0 \). This is the classical problem of convex regression which has a long history in statistics and related fields. Standard references include Hildreth [32], Hanson and Pledger [31], Groeneboom et al. [23], Groeneboom and Jongbloed [22], Seijo and Sen [42], Kuosmanen [34], Lim and Glynn [38] and Balázs [3]. Applications of convex regression can be found in Varian [45], Varian [46], Allon et al. [2], Matzkin [40], Aït-Sahalia and Duarte [1], Keshavarz et al. [33] and Toriello et al. [43].

A natural way to estimate \( f_0 \) in (1) is to use the method of least squares, i.e., minimize the sum of squared errors subject to the convexity constraint. Formally, the convex least squares estimator (LSE) is defined as

\[ \hat{f}_n \in \text{argmin}_{f \in \mathcal{C}(\Omega)} \sum_{i=1}^n (Y_i - f(X_i))^2 \]

where the minimization is over \( \mathcal{C}(\Omega) \) which is defined as the class of all real-valued convex functions defined on \( \Omega \). \( \hat{f}_n \) coincides with the maximum likelihood estimator as we have assumed that the errors \( \xi_1, \ldots, \xi_n \) are normally distributed. \( \hat{f}_n \) is uniquely defined at the design points \( X_1, \ldots, X_n \) and can be extended to other points in \( \Omega \) in a piecewise affine fashion. The convex LSE does not involve any tuning parameters. Seijo and Sen [42] performed a detailed study of the characterization and computation of \( \hat{f}_n \) (see also Kuosmanen [34] and Lim and Glynn [38]) and Mazumder et al. [41] (see also Chen and Mazumder [13]) demonstrated that it can be efficiently computed for fairly large values of the dimension \( d \) and the sample size \( n \).

The theoretical properties of \( \hat{f}_n \) are fairly well-understood in the univariate case \( d = 1 \). Hanson and Pledger [31] proved (uniform) consistency on compact subintervals contained in the interior of \( \Omega \) and Dümbgen et al. [17] strengthened these results by proving rates of convergence. Groeneboom et al. [23] proved pointwise rates of convergence and asymptotic distributions under smoothness assumptions and these results were extended by Chen and Wellner [14] and Ghosal and Sen [21]. Guntuboyina and Sen [27], Chatterjee et al. [12], Bellec [7] and Chatterjee [11] proved risk bounds for the convex LSE under the equally-spaced fixed design setting. These results imply that the univariate convex LSE achieves the minimax rate \( n^{-4/5} \) for estimating general convex functions while also achieving faster parametric rates (up to logarithmic multiplicative factors) for estimating piecewise affine convex functions.
In the multivariate case $d \geq 2$, consistency of the convex LSE was proved by Seijo and Sen [42] (see also Lim and Glynn [38]). However, rates of convergence have not been proved previously and one of the main goals of the present paper is to fill this gap. Rates of convergence are available, however, for certain alternative estimators such as the Lipschitz convex LSE and the bounded convex LSE. The Lipschitz convex LSE is defined as the LSE over $C_L(\Omega)$:

$$
\hat{f}_n(C_L(\Omega)) \in \arg\min_{f \in C_L(\Omega)} \sum_{i=1}^{n} (Y_i - f(X_i))^2
$$

where $C_L(\Omega)$ is the class of all convex functions on $\Omega$ that are $L$-Lipschitz. The bounded convex LSE is defined as the LSE over $C_B(\Omega)$:

$$
\hat{f}_n(C_B(\Omega)) \in \arg\min_{f \in C_B(\Omega)} \sum_{i=1}^{n} (Y_i - f(X_i))^2
$$

where $C_B(\Omega)$ is the class of all convex functions on $\Omega$ that are uniformly bounded by $B$. Rates of convergence for the Lipschitz convex LSE can be found in Balázs et al. [4], Lim [37] and Mazumder et al. [41] while rates for the bounded convex LSE are in Han and Wellner [30]. It should be noted that these alternative estimators $\hat{f}_n(C_L(\Omega))$ and $\hat{f}_n(C_B(\Omega))$ crucially depend on tuning parameters (specifically $L$ and $B$) while the convex LSE is tuning parameter free.

In Section 2, we provide the first rate of convergence results for the convex LSE for $d \geq 2$. Let us describe these results at a high-level here (we ignore logarithmic multiplicative factors in this Introduction; see the actual theorems for the full bounds). We assume that $\Omega$ is a polytope and that the design points $X_1, \ldots, X_n$ form a fixed regular rectangular grid intersected with $\Omega$. As is common in fixed design analysis, we work with the loss function

$$
\ell^2_{\mathbb{P}_n}(f, g) := \int (f(g))^2 \, d\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - g(X_i))^2
$$

where $\mathbb{P}_n$ is the empirical distribution of the design points $X_1, \ldots, X_n$ (note that $\mathbb{P}_n$ is non-random here as we are working in fixed-design). The risk of $\hat{f}_n$ is defined as $\mathbb{E}_{f_0} \ell^2_{\mathbb{P}_n}(\hat{f}_n, f_0)$ and we prove bounds for $\mathbb{E}_{f_0} \ell^2_{\mathbb{P}_n}(\hat{f}_n, f_0)$ that hold for finite samples. Our first main result is Theorem 2.1 which proves that the risk of the convex LSE is bounded from above by

$$
r_{n,d} := \begin{cases} 
    n^{-4/(d+4)} : d \leq 4 \\
    n^{-2/d} : d \geq 5 
\end{cases}
$$

We also prove adaptive risk bounds for the convex LSE which imply that the convex LSE converges at rates faster than $r_{n,d}$ when $f_0$ is a piecewise affine convex function. Specifically we prove in Theorem 2.3 that when $f_0$ is a piecewise affine convex function with $k$ affine pieces, the risk of $\hat{f}_n$ is bounded from above by

$$
a_{k,n,d} := \begin{cases} 
    k n^4/d : d \leq 4 \\
    \left(\frac{k}{n}\right)^{4/d} : d \geq 5 
\end{cases}
$$
up to a logarithmic term which depends on the number of facets of each affine piece. It is interesting that the rate $a_{k,n,d}$ switches from a parametric rate for $d \leq 4$ to a slower nonparametric rate for $d \geq 5$.

We also prove lower bounds for the performance of the convex LSE which imply that the rates $r_{n,d}$ and $a_{k,n,d}$ are not merely loose upper bounds but accurately describe the behavior of $f_n$. These lower bound results are only interesting for $d \geq 5$ because for $d \leq 4$, $r_{n,d}$ is already the minimax rate (see (14)) and $a_{k,n,d}$ is the parametric rate. In Theorem 2.2, we prove the existence of a bounded, Lipschitz convex function $f_0$ on $\Omega$ where the risk of the convex LSE is bounded from below by $n^{-2/d}$ (note that logarithmic factors are being ignored in this Introduction). This function where the LSE is shown to achieve the $n^{-2/d}$ rate will be a piecewise affine convex function whose number of affine pieces is of the order $\sqrt{n}$ (see Lemma 2.4). This proves that $r_{n,d}$ correctly describes the worst case risk behavior of the convex LSE. Moreover, in Theorem 2.5, we show, for every $1 \leq k \leq \sqrt{n}$, the existence of a convex function that is piecewise affine with $\sim k$ affine pieces where the risk of the convex LSE is bounded from below by $(k/n)^{4/d}$. This shows that $a_{k,n,d}$ correctly describes the adaptive behavior of the convex LSE for $d \geq 5$ and $1 \leq k \leq \sqrt{n}$. Note that $a_{k,n,d}$ cannot be expected to be a tight bound for $k \gg \sqrt{n}$ as then $(k/n)^{4/d}$ will dominate the worst case risk bound of $n^{-2/d}$.

Our results imply the minimax suboptimality of the convex LSE for $d \geq 5$. Indeed, the minimax risk for the class of all bounded, Lipschitz convex functions is of the order $n^{-4/(d+4)}$. From a comparison of $r_{n,d}$ with the minimax risk $n^{-4/(d+4)}$, we can immediately conclude that the convex LSE is minimax suboptimal for $d \geq 5$.

We also proved that this minimax suboptimality extends to the bounded convex LSE and Lipschitz convex LSE. Here we worked in the random design framework to facilitate comparison with existing risk results on these estimators in [4, 30, 37, 41]. We thus assume that $X_1, \ldots, X_n$ are independently distributed according to the uniform distribution $\mathbb{P}$ on $\Omega$ and consider the loss function

$$\ell_2^2(f, g) := \int_{\Omega} (f - g)^2 d\mathbb{P}. \quad (8)$$

For the bounded convex LSE, it was proved in Han and Wellner [30] that its risk is bounded from above by $r_{n,d}$ (defined in (6)) when $\Omega$ is a polytope. In Theorem 3.1, we prove that there exists a bounded, Lipschitz convex function on $\Omega$ where the risk of the bounded convex LSE is bounded from below by $n^{-2/d}$. This implies that the bounded convex LSE is minimax suboptimal in random design when the domain $\Omega$ is a polytope. This contrasts intriguingly with the recent result of Kur et al. [35] who proved that the bounded convex LSE is minimax optimal when $\Omega$ is a smooth convex body (such as the unit ball). Some insight into this is given in Section 6.

For the Lipschitz convex LSE, it was proved, in Balázs et al. [4], Lim [37], Mazumder et al. [41], that its risk is bounded again by $r_{n,d}$ for all convex bodies $\Omega$ (regardless of whether $\Omega$ is polytopal or smooth). In Theorem 3.2, we prove the existence of a bounded, Lipschitz convex function on $\Omega$ where the risk of the Lipschitz convex LSE is bounded from below by $n^{-2/d}$. This implies that the Lipschitz convex LSE is minimax
suboptimal when $d \geq 5$ in random design for every convex domain $\Omega$.

The main highlight among our results is the minimax suboptimality of the convex LSE and the bounded convex LSE over polytopal domains as well as the Lipschitz convex LSE over general convex domains for $d \geq 5$. In a recent paper Kur et al. [36] involving two of the authors of the present paper, it was shown that the LSE over the class of support functions of all convex bodies was suboptimal for $d \geq 6$ for the problem of estimating an unknown convex body based on support function measurements. The fact that LSEs and related Empirical Risk Minimization procedures can be suboptimal has been observed before. For example, it was observed in Birgé and Massart [8, Section 4] that it is possible to design pathological function classes where the LSE is provably minimax suboptimal. However, the class of all convex functions, bounded convex functions, Lipschitz convex functions and support functions of convex bodies are quite far from pathological and it is quite surprising that the LSE over these natural classes is minimax suboptimal unless $d$ is small.

Our risk results required proving novel metric entropy bounds for convex functions. These results are given in Section 4. For our fixed design risk bounds, we prove, in Theorem 4.1, a metric entropy upper bound for the class

$$\{ f \in \mathcal{C}(\Omega) : \ell_{\mathbb{P}_n}(f, f_0) \leq t \}$$

for affine functions $f_0$ under the $\ell_{\mathbb{P}_n}$ pseudometric. This result is different from existing metric entropy results in [9, 15, 20, 25, 26] for convex functions in that it deals with the discrete $\ell_{\mathbb{P}_n}$ pseudometric while all existing results deal with continuous $L_p$ metrics. Also the constraint $\ell_{\mathbb{P}_n}(f, f_0) \leq t$ on the convex functions in the above class is comparatively weak. For our random design results, we prove, in Theorem 4.4, bracketing $L_2(\mathbb{P})$ metric entropy bounds for

$$\left\{ f \in \mathcal{C}(\Omega) : \ell_{\mathbb{P}}(f, f_0) \leq t, \sup_{x \in \Omega} |f(x)| \leq B \right\}$$

for polytopal $\Omega$, piecewise affine $f_0 \in \mathcal{C}(\Omega)$ and $t > 0$. This result also improves existing results in certain aspects; see Section 4 for details.

Let us now quickly summarize the contents of the rest of the paper. The results for the convex LSE in fixed design are given in Section 2. Results for the bounded convex LSE and Lipschitz convex LSE in random design are in Section 3. Metric entropy results are in Section 4. The main technical ideas behind the proofs are briefly described in Section 5. Section 6 contains a discussion of issues related to our main results. Section 7 contains the proofs of the main results from Section 2. Section 8 contains the proofs of the results from Section 3. Section 9 contains the proofs of the metric entropy results from Section 4. Some additional proofs are relegated to Section 10.

2. Risk bounds for the convex LSE

In this section, we prove rates of convergence for the convex LSE $\hat{f}_n$ (defined as in (2)). These are the first rate of convergence results for the convex LSE for $d \geq 2$. Throughout
the paper, \( c_d, C_d, \kappa_d \) etc. denote constants that depend on the dimension \( d \) alone and their exact value might change from appearance to appearance. We sometimes refer to these constants as dimensional constants (“dimensional” here refers to the dependence on \( d \)).

Let us first describe our assumptions on \( \Omega \) that we use throughout this section. We assume that the domain \( \Omega \) is a polytope. \( \Omega \) can be written in the form:

\[
\Omega = \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i \text{ for } i = 1, \ldots, F \}
\]  

(9)

for some positive integer \( F \), unit vectors \( v_1, \ldots, v_F \) and real numbers \( a_1, \ldots, a_F, b_1, \ldots, b_F \). We also assume that \( \Omega \) is contained in the unit ball (this can be achieved by scaling). Our bounds will be nonasymptotic and hold even when \( \Omega \) changes with \( n \). We assume however that \( F \) is bounded from above by a constant depending on \( d \) alone. We also assume that the volume of \( \Omega \) is bounded from below by a constant depending on \( d \) alone.

We do not address the problem of finding rates of convergence of the convex LSE in the non-polytopal case where \( \Omega \) is a smooth convex body. Rates of convergence in this case will be quite different from the rates derived in this section. See Section 6 for more details.

In this section, we work with the fixed design setting where \( X_1, \ldots, X_n \) form a fixed regular rectangular grid in \( \Omega \) and \( Y_1, \ldots, Y_n \) are generated according to (1). Specifically, for \( \delta > 0 \), let

\[
S := \{ (k_1\delta, \ldots, k_d\delta) : k_i \in \mathbb{Z}, 1 \leq i \leq d \}
\]  

(10)

denote the regular \( d \)-dimensional \( \delta \)-grid in \( \mathbb{R}^d \). We assume that \( X_1, \ldots, X_n \) are an enumeration of the points in \( S \cap \Omega \) with \( n \) denoting the cardinality of \( S \cap \Omega \). By the usual volumetric argument and the assumption that the volume of \( \Omega \) is bounded from above and below by constants depending on \( d \) alone, there exists a small enough constant \( \kappa_d \) such that whenever \( \delta \leq \kappa_d \), we have

\[
2 \leq c_d \delta^{-d} \leq n \leq C_d \delta^{-d}
\]  

(11)

for dimensional constants \( c_d \) and \( C_d \). We shall assume throughout this section that \( \delta \leq \kappa_d \) so that the above inequality holds.

We study the performance of the LSE \( \hat{f}_n \) under the loss function (5). The next couple of results prove upper bounds for the risk \( \mathbb{E}_{f_0} \ell_{\mathcal{P}_n}^2 (\hat{f}_n, f_0) \) of the convex LSE. Let \( \mathcal{C}(\Omega) \) denote the class of all real-valued convex functions on \( \Omega \) and let \( \mathcal{A}(\Omega) \) denote the class of all affine functions on \( \Omega \). For each \( f \in \mathcal{C}(\Omega) \), let

\[
\mathcal{L}(f) := \inf_{g \in \mathcal{A}(\Omega)} \ell_{\mathcal{P}_n}(f, g)
\]

where, it may be recalled, that \( \ell_{\mathcal{P}_n}(f, g) \) is our loss function defined via (5). The following result proves an upper bound involving \( \mathcal{L}(f_0) \) for \( \mathbb{E}_{f_0} \ell_{\mathcal{P}_n}^2 (\hat{f}_n, f_0) \) for arbitrary \( f_0 \in \mathcal{C}(\Omega) \). Note that the number \( F \) appearing in the bound (12) below is the number of parallel halfspaces or slabs defining \( \Omega \) (see (9)) and this number is assumed to be bounded by a constant depending on \( d \) alone.
Theorem 2.1. Fix \( f_0 \in \mathcal{C}(\Omega) \) with \( \mathcal{L} := \mathcal{L}(f_0) \). There exists positive constants \( C_d \) and \( \gamma_d \) depending only on \( d \) such that

\[
\mathbb{E}_{f_0 \in \mathcal{L}(f_0) \leq \mathcal{L}} \ell_n^2(f_n, f_0) \leq \begin{cases} 
C_d \max \left\{ \frac{2^{2d/(4+d)}}{n^2 (\log n)^F}, \frac{2^2 (\log n)^F}{n^2} \right\} & \text{for } d \leq 3 \\
C_4 \max \left\{ \frac{\sigma_0^2}{\sqrt{n} (\log n)^{1+F/2}}, \frac{\sigma_0^2 (\log n)^{2+F}}{n^2} \right\} & \text{for } d = 4 \\
C_d \max \left\{ \sigma \mathcal{L} \left( \frac{(\log n)^F}{n} \right)^{2/d}, \sigma^2 \left( \frac{\log n}{n} \right)^{4/d} \right\} & \text{for } d \geq 5.
\end{cases}
\]

(12)

When \( \mathcal{L} = \mathcal{L}(f_0) \) and \( \sigma \) are fixed positive constants (not changing with \( n \)) and \( n \) is sufficiently large, the leading terms in the right hand side of (12) are the first terms inside the maxima. More precisely, from (12), it easily follows that for every \( \mathcal{L} > 0 \) and \( \sigma > 0 \), there exist constants \( C_d \) (depending on \( d \) alone) and \( N_{d,\sigma/\mathcal{L}} \) (depending only on \( d \) and \( \sigma/\mathcal{L} \)) such that

\[
\sup_{f_0 \in \mathcal{C}(\Omega) : \mathcal{L}(f_0) \leq \mathcal{L}} \mathbb{E}_{f_0 \in \mathcal{L}(f_0) \leq \mathcal{L}} \ell_n^2(f_n, f_0) \leq \begin{cases} 
C_d 2^{2d/(4+d)} (\frac{\sigma^2}{n} (\log n)^F)^{4/(d+4)} & \text{for } d = 1, 2, 3 \\
C_4 \frac{\sigma_0^2}{\sqrt{n} (\log n)^{1+F/2}} & \text{for } d = 4 \\
C_d \sigma \mathcal{L} \left( \frac{(\log n)^F}{n} \right)^{2/d} & \text{for } d \geq 5.
\end{cases}
\]

(13)

for \( n \geq N_{d,\sigma/\mathcal{L}} \). The risk upper bound obtained above can be compared with the following minimax risk characterization (which can be proved by routine arguments; see e.g., [24, Proof of Theorem 3.2]). Let \( \mathcal{C}^L(\Omega) \) denote the class of all convex functions on \( \Omega \) that are \( L \)-Lipschitz and uniformly bounded by \( L \). Then there exist constants \( c_d, C_d \) and \( N_{d,\sigma/\mathcal{L}} \) such that

\[
c_d L^{2d/(4+d)} \left( \frac{\sigma^2}{n} \right)^{4/(d+4)} \leq \inf_{f_0 \in \mathcal{C}^L(\Omega)} \sup_{f_n \in \mathcal{C}^L(\Omega)} \mathbb{E}_{f_0 \in \mathcal{C}^L(\Omega)} \ell_n^2(f_n, f_0) \leq C_d L^{2d/(4+d)} \left( \frac{\sigma^2}{n} \right)^{4/(d+4)}
\]

(14)

for \( n \geq N_{d,\sigma/\mathcal{L}} \). Letting \( \mathcal{C}^L(\Omega) \) be the class of all convex functions on \( \Omega \) that are uniformly bounded by \( L \), it is easy to see that

\[\mathcal{C}^L(\Omega) \subseteq \mathcal{C}^L(\Omega) \subseteq \left\{ f_0 \in \mathcal{C}(\Omega) : \mathcal{L}(f_0) \leq L \right\}\]

which implies that the minimax lower bound in (14) also holds for the larger classes \( \mathcal{C}^L(\Omega) \) and \( \left\{ f_0 \in \mathcal{C}(\Omega) : \mathcal{L}(f_0) \leq L \right\} \).

A comparison of (13) and (14) implies that the convex LSE \( \hat{f}_n \) is nearly minimax optimal (up to logarithmic multiplicative factors) over the class \( \left\{ f_0 \in \mathcal{C}(\Omega) : \mathcal{L}(f_0) \leq L \right\} \) (or over the smaller classes \( \mathcal{C}^L(\Omega) \) or \( \mathcal{C}^L(\Omega) \)) for \( d \leq 4 \). However, the rate given by (13) is strictly suboptimal compared to the minimax rate for \( d \geq 5 \).

The next result shows that, for each \( d \geq 5 \), there exists a bounded Lipschitz convex function \( f_0 \) for which \( \mathbb{E}_{f_0 \in \mathcal{L}(f_0)} \ell_n^2(f_n, f_0) \) is bounded from below by \( n^{-2/d} \) up to a logarithmic multiplicative factor. Comparing this to (14) (and noting that \( n^{-2/d} \gg n^{-4/(d+4)} \) for
$d \geq 5$), we immediately conclude that, when $\Omega$ is a polytope and $d \geq 5$, the convex LSE is minimax suboptimal over $C^d_L(\Omega)$ (or over the larger classes $C^L(\Omega)$ or $\{f \in C(\Omega) : \mathcal{L}(f) \leq L\}$).

**Theorem 2.2.** Fix $d \geq 5$, $L > 0$ and $\sigma > 0$. There exist constants $c_d$ and $C_{d,\sigma/L}$ such that

$$
\sup_{f_0 \in C^d_L(\Omega)} \mathbb{E}_{f_0} f_n^2 (\hat{f}_n, f_0) \geq c_d \sigma^2 \log n - \frac{d}{4} \left( \log n \right)^d - \frac{d}{4} \sigma^2 \log n \quad \text{for } n \geq C_{d,\sigma/L}.
$$

We shall now present risk bounds when $f_0$ is a piecewise affine convex function. To motivate these results, let us first examine inequality (12) when $f_0$ is an affine function. Here $\mathcal{L} = \mathcal{L}(f_0) = 0$ so we have

$$
\sup_{f_0 \in \mathcal{A}(\Omega)} \mathbb{E}_{f_0} f_n^2 (\hat{f}_n, f_0) \leq \begin{cases} 
C_d \sigma^2 \left( \log n \right)^F & \text{for } d = 1, 2, 3 \\
C_d \sigma^2 \left( \log n \right)^2 + F & \text{for } d = 4 \\
C_d \sigma^2 \left( \frac{\log n}{n} \right)^{d/4} & \text{for } d \geq 5.
\end{cases}
$$

The bounds given in (16) are of a smaller order than those given by (13) which means that the LSE $\hat{f}_n$ adapts to affine functions by converging to them at faster rates compared to other convex functions in $\{f \in C(\Omega) : \mathcal{L}(f) \leq \mathcal{L} \}$. In the next result, we prove that a similar adaptation holds for a larger class of piecewise affine convex functions. For $k \geq 1$ and $h \geq 1$, let $\mathcal{C}_{k,h}(\Omega)$ denote all functions $f_0 \in C(\Omega)$ for which there exist $k$ convex subsets $\Omega_1, \ldots, \Omega_k$ satisfying the following properties:

1. $f_0$ is affine on each $\Omega_i$,
2. each $\Omega_i$ can be written as an intersection of at most $h$ slabs (i.e., as in (9) with $F = h$), and
3. $\Omega_1 \cap \mathcal{S}, \ldots, \Omega_k \cap \mathcal{S}$ are disjoint with $\cup_{i=1}^k (\Omega_i \cap \mathcal{S}) = \Omega \cap \mathcal{S}$.

**Theorem 2.3.** For every $k \geq 1$ and $h \geq 1$, we have

$$
\sup_{f_0 \in \mathcal{C}_{k,h}(\Omega)} \mathbb{E}_{f_0} f_n^2 (\hat{f}_n, f_0) \leq \begin{cases} 
C_d \sigma^2 \left( \frac{k}{n} \right) \left( \log n \right)^h & \text{for } d = 1, 2, 3 \\
C_d \sigma^2 \left( \frac{k}{n} \right) \left( \log n \right)^{h+2} & \text{for } d = 4 \\
C_d \sigma^2 \left( \frac{k}{n} \right)^{d/4} & \text{for } d \geq 5
\end{cases}
$$

for a constant $C_d$ depending on $d$ alone.

**Remark 2.1.** Note that Theorem 2.3 generalizes the bound (16) because $\mathcal{A}(\Omega) = \mathcal{C}_{1,F}(\Omega)$.

The logarithmic factors in (17) have powers involving $h$ which means that they cannot be ignored when $h$ grows with $n$. Thus Theorem 2.3 only gives something useful when $h$ is either constant or grows very slowly with $n$.

If we ignore the logarithmic factors in (17), we can see that the risk bound in (17) switches from a parametric rate for $d \leq 4$ to a slower nonparametric rate for $d \geq 5$. Focussing on the $d \geq 5$ case, it is also easy to see that the rate given by (17) becomes
larger than that given by (13) when \( k > \sqrt{n\sigma^{-d/4}} \) so Theorem 2.3 is only interesting (for \( d \geq 5 \)) for \( k \) in the range \( 1 \leq k \leq \sqrt{n\sigma^{-d/4}} \). In the next result, we show that for every \( k \) satisfying \( 1 \leq k \leq \sqrt{n\sigma^{-d/4}} \), there exists a piecewise affine convex function on \( \Omega \) with no larger than \( C_d k \) affine pieces where the risk of the LSE is bounded from below by \( (k/n)^{4/d} \) (up to logarithmic multiplicative factors). The function where the rate \( (k/n)^{4/d} \) is achieved can be taken to be a piecewise affine approximation to a smooth convex function such as the quadratic function. Its existence is guaranteed by the following lemma.

**Lemma 2.4.** Let \( f_0(x) := \|x\|^2 \). There exists a positive constant \( C_d \) (depending on the dimension alone) such that the following is true. For every \( k \geq 1 \), there exist \( m \leq C_d k \) -simplices \( \Delta_1, \ldots, \Delta_m \) and a convex function \( \tilde{f}_k \) such that

1. \( \Omega = \bigcup_{i=1}^{m} \Delta_i \),
2. \( \Delta_i \cap \Delta_j \) is contained in a facet of \( \Delta_i \) and a facet of \( \Delta_j \) for each \( i \neq j \),
3. \( \tilde{f}_k \) is affine on each \( \Delta_i, i = 1, \ldots, m \),
4. \( \sup_{x \in \Omega} |f_0(x) - \tilde{f}_k(x)| \leq C_d k^{-2/d} \),
5. \( \tilde{f}_k \in C_{C_d}(\Omega) \).

The next result shows that the LSE achieves the rate \( (k/n)^{4/d} \) for the functions \( \tilde{f}_k \) given by the above lemma.

**Theorem 2.5.** Fix \( d \geq 5 \). There exist positive constants \( c_d \) and \( N_d \) such that for \( n \geq N_d \) and

\[
1 \leq k \leq \min \left( \sqrt{n\sigma^{-d/4}}, c_d n \right),
\]

we have

\[
\mathbb{E}_{f_k} \ell_P^2(\hat{f}_n, \tilde{f}_k) \geq c_d \sigma^2 \left( \frac{k}{n} \right)^{4/d} (\log n)^{-4(d+1)/d} \tag{19}
\]

where \( \tilde{f}_k \) is the function from Lemma 2.4.

The above result immediately implies that the adaptive risk bounds in (17) cannot be improved for all \( k \) satisfying (18) (note that \( \min (\sqrt{n\sigma^{-d/4}}, c_d n) \) will equal \( \sqrt{n\sigma^{-d/4}} \) unless \( \sigma \) is of smaller order than \( n^{-2/d} \)). This implies, in particular, that the LSE cannot adapt at near parametric rates for affine functions for \( d \geq 5 \).

The lower bound given by (19) for \( k = \sqrt{n\sigma^{-d/4}} \) is of the same order as that given by Theorem 2.2. In other words, the adaptive lower bound in Theorem 2.5 implies minimax suboptimality of the convex LSE.

### 3. Suboptimality of constrained convex LSEs in two settings

The highlight of the results of Section 2 is the minimax suboptimality of the convex LSE in the fixed gridded design setting when \( d \geq 5 \). In this section, we show that this suboptimality also extends to the bounded convex LSE (when \( \Omega \) is a polytope) and the Lipschitz convex LSE (for general \( \Omega \)). We consider here the random design setting where the observed data are \((X_1, Y_1), \ldots, (X_n, Y_n)\) with \( X_1, \ldots, X_n \) being independent
having the uniform distribution $\mathbb{P}$ on $\Omega$ and $Y_1, \ldots, Y_n$ being generated according to (1) for independent $N(0, \sigma^2)$ errors $\xi_1, \ldots, \xi_n$. We also assume that $\xi_1, \ldots, \xi_n, X_1, \ldots, X_n$ are independent random variables and work with the $\ell^2_\mathbb{P}$ loss function (8).

Let us first state our result for the bounded convex LSE (defined as in (4)). This result assumes that the domain $\Omega$ is a polytope. The risk $\mathbb{E}_{f_0} \ell^2_\mathbb{P}(\hat{f}_n(C^B(\Omega)), f_0)$ of the bounded convex LSE was studied in Han and Wellner [30] who proved matching upper and lower bounds (up to logarithmic factors in $n$) for $d \leq 4$ and just upper bounds for $d \geq 5$. When $\Omega$ is a polytope and $d \geq 5$, it was proved in Han and Wellner [30, Theorem 3.6] that

$$\sup_{f_0 \in C^B(\Omega)} \mathbb{E}_{f_0} \ell^2_\mathbb{P}(\hat{f}_n(C^B(\Omega)), f_0)$$

is bounded from above by $n^{-2/d}$ (ignoring multiplicative factors that are logarithmic in $n$ and that depend on $B$ and $\sigma$). On the other hand, the minimax rate over $C^B(\Omega)$ under the $\ell^2_\mathbb{P}$ loss function equals $n^{-4/(d+4)}$ for all $d \geq 1$ (see e.g., Han and Wellner [30, Theorem 2.3 and Theorem 2.4]). There is therefore a gap between the upper bound of $n^{-2/d}$ and the minimax risk of $n^{-4/(d+4)}$ for $d \geq 5$. The next result shows that, for $d \geq 5$, there exist functions $f_0$ in $C^B(\Omega)$ where the risk of $\hat{f}_n(C^B(\Omega))$ is bounded from below by $n^{-2/d}$ (up to logarithmic multiplicative factors) thereby proving that the bounded convex LSE is minimax suboptimal. The fact that $\Omega$ is polytopal is crucial here for the LSE becomes optimal when $\Omega$ is the unit ball as recently shown in Kur et al. [35]. We provide an explanation of this in Section 6.

**Theorem 3.1.** Let $\Omega$ be a polytope whose number of facets is bounded by a constant depending on $d$ alone. Assume also that $\Omega$ is contained in the unit ball and contains a ball of constant (depending on $d$ alone) radius. Fix $d \geq 5$. There exist constants $c_d$ and $N_{d,\sigma/B}$ such that for every $B > 0$ and $\sigma > 0$, we have

$$\sup_{f_0 \in C^B(\Omega)} \mathbb{E}_{f_0} \ell^2_\mathbb{P}(\hat{f}_n(C^B(\Omega)), f_0) \geq c_d \sigma B n^{-2/d} (\log n)^{-4(d+1)/d} \quad \text{whenever } n \geq N_{d,\sigma/B}.$$ 

(20)

The next result is for the Lipschitz convex LSE (defined as in (3)). The following result shows that the same lower bound $n^{-2/d}$ (up to logarithmic factors) holds for the Lipschitz convex LSE for essentially every convex domain $\Omega$ (regardless of whether $\Omega$ is polytopal or smooth).

**Theorem 3.2.** Suppose $\Omega$ is a convex body that is contained in the unit ball and contains a ball centered at zero of constant (depending on $d$ alone) radius. Fix $d \geq 5$. There exist positive constants $c_d$ and $N_{d,\sigma/L}$ such that for every $L > 0$ and $\sigma > 0$, we have

$$\sup_{f_0 \in C_L(\Omega)} \mathbb{E}_{f_0} \ell^2_\mathbb{P}(\hat{f}_n(C_L(\Omega)), f_0) \geq c_d \sigma L n^{-2/d} (\log n)^{-4(d+1)/d} \quad \text{whenever } n \geq N_{d,\sigma/L}.$$ 

(21)

4. Metric entropy results

Our risk results from the previous two sections are based on new metric entropy results for convex functions. Specifically, the risk bounds for the convex LSE in Section 2
are proved via a metric entropy bound for convex functions satisfying a discrete $L_2$ constraint and the risk lower bounds in Section 3 are proved via a bracketing entropy bound for bounded convex functions with an additional $L_2$ constraint. The goal of this section is to describe these entropy results. We would like to start however with a brief description of existing entropy results for convex functions.

Bronštejn [9] proved that the metric entropy of bounded Lipschitz convex functions defined on a fixed convex body $\Omega$ in $\mathbb{R}^d$ is of the order $\epsilon^{-d/2}$ under the supremum $(L_\infty)$ metric. The Lipschitz constraint can be removed if one is only interested in $L_p$ metrics for $p < \infty$. Indeed, it was shown (by Gao [18] and Dryanov [16] for $d = 1$ and by Guntuboyina and Sen [26] for $d \geq 2$) that the metric entropy of bounded convex functions on $\Omega = [0, 1]^d$ is of the order $\epsilon^{-d/2}$ under the $L_p$ metric for every $1 \leq p < \infty$.

The boundedness constraint can further be relaxed to a $L_q$-norm constraint ($1 \leq q \leq \infty$) in which case the aforementioned result will hold in the $L_p$ metric for $1 \leq p < q$ (see Guntuboyina [25]). The case of more general convex bodies $\Omega$ was considered by Gao and Wellner [20] who proved that the same bounds hold when $\Omega$ is an arbitrary polytope.

Gao and Wellner [20] also studied the case where $\Omega$ is not a polytope. For example, when $\Omega$ is the unit ball, they showed that the metric entropy of bounded convex functions on $\Omega$ is of the order $\epsilon^{-(d-1)}$ (which is larger than $\epsilon^{-d/2}$) in the $L_2$ metric when $d \geq 3$.

We shall now state our results and explain how they are related to the existing ones. The $\epsilon$-covering number of a set $S$ under a pseudometric $d$ will be denoted by $N(\epsilon, S, d)$. Also, the $\epsilon$-bracketing number of a set $S$ of functions under a pseudometric $d$ will be denoted by $N([\epsilon, S, d])$.

Our first main metric entropy result is the following. We use notation that is similar to that in Section 2. Recall that $\mathcal{S}$ is the regular $d$-dimensional $\delta$-grid defined in (10). The resolution of the grid $\delta$ will appear in the bounds below (note that, by (11), $\delta$ will be of order $n^{-1/d}$). Let $\Omega$ be a convex body such that $\Omega \cap \mathcal{S} \neq \emptyset$. For $1 \leq p < \infty$ and a function $f$ on $\Omega$, we define the quasi-norm

$$
\ell_S(f, \Omega, p) = \left( \frac{1}{\#(\Omega \cap \mathcal{S})} \sum_{s \in \Omega \cap \mathcal{S}} |f(s)|^p \right)^{1/p},
$$

where $\#(\Omega \cap \mathcal{S})$ denotes the cardinality of $\Omega \cap \mathcal{S}$. Furthermore, for any fixed function $f_0$ on $\Omega$, and any $t > 0$, denote

$$
B^p_S(f_0; t; \Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is convex on } \Omega, \ell_S(f - f_0, \Omega, p) \leq t \}.
$$

We are interested in the metric entropy of $B^p_S(f_0; t; \Omega)$ under the $\ell_S(\cdot, \Omega, p)$ quasi-metric. The following result deals with the case when $f_0 \in \mathcal{A}(\Omega)$ (recall that $\mathcal{A}(\Omega)$ denotes the class of all affine functions on $\Omega$). We state and prove this for every $1 \leq p < \infty$ (as the result could be of independent interest) even though we only require the case $p = 2$ for proving the risk bounds of Section 2.

**Theorem 4.1.** Let $\Omega$ be a $d$-dimensional convex polytope that equals the intersection of at most $F$ pairs of halfspaces with distance no larger than 1 (i.e., as in (9) with
max_{1 \leq i \leq r}(b_i - a_i) \leq 1). There exists a constant \(c_{d,p}\) depending only on \(d\) and \(p\) such that for every \(f_0 \in \mathcal{A}(\Omega)\), \(\varepsilon > 0\) and \(t > 0\), we have

\[
\log N(\varepsilon, B^p_S(f_0; t; \Omega), \ell_S(\cdot, \Omega, p)) \leq \left[ c_{d,p} \log(1/\delta) \right]^F (t/\varepsilon)^{d/2}.
\] (22)

Theorem 4.1 differs from existing entropy results for convex functions in the following ways. First, it deals with the discrete \(\ell_S(\cdot, \Omega, p)\) metric while all results previously have studied the continuous \(L_p\) metrics. Second, the constraint on functions in \(B^p_S(f_0; t; \Omega)\) is

\[
\frac{1}{\#(\Omega \cap S)} \sum_{s \in \Omega \cap S} |f(s) - f_0(s)|^p \leq t^p
\]

which is much weaker than imposing uniform boundedness on the class. When \(\delta \downarrow 0\), one might view the above constraint as a constraint on the continuous \(L_p\) norm of \(f\), but it must be noted that the \(L_p\) metric entropy under such an \(L_p\) constraint equals \(\infty\) (more generally, \(L_p\) metric entropy under an \(L_q\) norm constraint is finite if and only if \(p < q\); see [20, 25]). The bound (22) also approaches infinity as \(\delta \downarrow 0\) but only logarithmically in \(1/\delta\) and this only leads to additional logarithmic terms in our risk bounds.

Theorem 4.1 implies, via the triangle inequality, bounds for \(\log N(\varepsilon, B^p_S(f_0; t; \Omega), \ell_S(\cdot, \Omega, p))\) for arbitrary not necessarily affine \(f_0\). Indeed, the triangle inequality gives

\[
B^p_S(f_0; t; \Omega) \subseteq B^p_S(f; t + \ell_S(f - f_0, \Omega, p); \Omega)
\]

for every \(f, f_0\). Applying this for affine functions \(f\), we obtain from Theorem 4.1 that

\[
\log N(\varepsilon, B^p_S(f_0; t; \Omega), \ell_S(\cdot, \Omega, p)) \leq \left[ c_{d,p} \log(1/\delta) \right]^F \left( \frac{t + \inf_{f \in A(\Omega)} \ell_S(f - f_0, \Omega, p)}{\varepsilon} \right)^{d/2}.
\] (23)

While the above inequality is useful (we use it in the proof of Theorem 2.1), it is loose in the case when \(f_0\) is piecewise affine and \(t\) is small. For piecewise affine \(f_0\), we use instead the following two corollaries of Theorem 4.1. Corollary 4.2 will be used to prove Theorem 2.3 while Corollary 4.3 will be used in the proof of Theorem 2.5.

**Corollary 4.2.** Suppose \(f_0\) is a piecewise affine convex function on \(\Omega\). Suppose \(\Omega_1, \ldots, \Omega_k\) are convex subsets of \(\Omega\) such that

1. \(f_0\) is affine on each \(\Omega_i\).
2. Each \(\Omega_i\) equals an intersection of at most \(s\) pairs of parallel halfspaces.
3. \(\Omega_1 \cap S, \ldots, \Omega_k \cap S\) are disjoint with \(\bigcup_{i=1}^k (\Omega_i \cap S) = \Omega \cap S\).

Then

\[
\log N(\varepsilon, B^p_S(f_0; t; \Omega), \ell_S(\cdot, \Omega, p)) \leq k \left( \frac{t}{\varepsilon} \right)^{d/2} \left( c_{d,p} \log \frac{1}{\delta} \right)^s
\]

for a constant \(c_{d,p}\) that depends on \(d\) and \(p\) alone.

The next result can be seen as a consequence of the above Corollary when each polytope \(\Omega_i\) is a \(d\)-simplex.
Corollary 4.3. Suppose \( f_0 \) is a piecewise affine convex function on \( \Omega \). Suppose that \( \Omega \) can be written as the union of \( k \) \( d \)-simplices \( \Delta_1, \ldots, \Delta_k \) such that \( f_0 \) is affine on each \( \Delta_i \) and such that \( \Delta_i \cap \Delta_j \) is contained in a facet of \( \Delta_i \) and a facet of \( \Delta_j \) for each \( i \neq j \). Then

\[
\log N(\epsilon, B^p_{\ell_S}(f_0; t, \Omega), \ell_S(\cdot, \Omega, p)) \leq C_{d,p} k \left( \frac{t}{\epsilon} \right)^{d/2} \left( \log \frac{1}{\delta} \right)^{d+1}
\]

for a constant \( C_{d,p} \) that depends only on \( d \) and \( p \).

We next state our main bracketing entropy result. This is crucial for our risk lower bounds in Section 3. Recall that \( \mathcal{C}^\Gamma(\Omega) \) denotes the class of all convex functions on \( \Omega \) that are uniformly bounded by \( \Gamma \). We state the next result for every \( 1 \leq p < \infty \) for completeness although we only use it for \( p = 2 \).

Theorem 4.4. Let \( \Omega \) be a convex body in \( \mathbb{R}^d \) with volume bounded by 1. Let \( f_0 \) be a convex function on \( \Omega \) that is bounded by \( \Gamma \). For a fixed \( 1 \leq p < \infty \) and \( t > 0 \), let

\[
B^p_\Gamma(f_0; t; \Omega) = \left\{ f \in \mathcal{C}^\Gamma(\Omega) : \int_\Omega |f(x) - f_0(x)|^p \, dx \leq t^p \right\}.
\]

Suppose \( \Delta_1, \ldots, \Delta_k \subseteq \Omega \) are \( d \)-simplices with disjoint interiors such that \( f_0 \) is affine on each \( \Delta_i \). Then for every \( 0 < \epsilon < \Gamma \) and \( t > 0 \), we have

\[
\log N_{[\| \cdot \|_{p, \cup_{i=1}^k \Delta_i}]}(\epsilon, B^p_\Gamma(f_0; t; \Omega), \| \cdot \|_{p, \cup_{i=1}^k \Delta_i}) \leq C_{d,p} k \left( \frac{\log \frac{\Gamma}{\epsilon}}{\epsilon} \right)^{d+1} \left( \frac{t}{\epsilon} \right)^{d/2}
\]

for a constant \( C_{d,p} \) that depends on \( p \) and \( d \) alone. The left hand side above denotes bracketing entropy with respect to the \( L_p \) metric on the set \( \Delta_1 \cup \cdots \cup \Delta_k \).

To see how Theorem 4.4 compares to existing bracketing entropy results, consider the special case when \( \Omega \) is a \( d \)-simplex and when \( f_0 \equiv 0 \). In that case, the conclusion of Theorem 4.4 (for \( k = 1 \) and \( \Delta_1 = \Omega \)) becomes:

\[
\log N_{[\| \cdot \|_{p, \Delta_1}]}(\epsilon, B^p_\Gamma(f_0; t; \Omega), \| \cdot \|_{p, \Delta_1}) \leq C_{d,p} \left( \frac{\log \frac{\Gamma}{\epsilon}}{\epsilon} \right)^{d+1} \left( \frac{t}{\epsilon} \right)^{d/2}
\]

The class of convex functions above has both an \( L_\infty \) constraint (uniform boundedness) as well as an \( L_p \) constraint and (26) does not hold if either of the two constraints are dropped. Indeed, the entropy becomes infinite if the \( L_\infty \) constraint is dropped. On the other hand, if the \( L_p \) constraint is dropped, then the bracketing entropy is of the order \( (\Gamma/\epsilon)^{d/2} \) as proved by Gao and Wellner [20] (see also Doss [15]). In contrast to \( (\Gamma/\epsilon)^{d/2} \), (26) only has a logarithmic dependence on \( \Gamma \) and is much smaller when \( t \) is small. Han and Wellner [30, Lemma 3.3] proved a weaker bound for the left hand side of (25) having additional multiplicative factors involving \( k \) (these factors cannot be neglected since we care about the regime \( k \sim \sqrt{n} \)).
5. Proof ideas

We briefly describe here the key ideas underlying the proofs of the main results of the paper. The risk upper bounds for the convex LSE (Theorem 2.1 and Theorem 2.3) in Section 2 are based on standard techniques [10] for analyzing LSEs combined with our metric entropy results of Section 4. The main novelty here is in the metric entropy results. The worst case risk lower bound for the convex LSE in Theorem 2.2 follows from the adaptive lower bound in Theorem 2.5 by taking $k \sim \sqrt{n}$. The main ideas behind the proof of Theorem 2.5 are as follows. Chatterjee [10] proved that the risk of the LSE at $\tilde{f}_k$ (this is the function given by Lemma 2.4) behaves as $t^2_{\tilde{f}_k}$ where $t_{\tilde{f}_k}$ is the maximizer of the function

$$
t \mapsto H_{\tilde{f}_k}(t) := \mathbb{E} \sup_{f \in C(\Omega); \ell_{h_n}(f_k, f) \leq t} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left( f(X_i) - \tilde{f}_k(X_i) \right) - \frac{t^2}{2} \tag{27}$$

over $t \in [0, \infty)$. The task then boils down to proving that $t_{\tilde{f}_k}$ is bounded from below by $(k/n)^{2/d}$ up to logarithmic factors. This requires proving upper bounds and lower bounds for the function $H_{\tilde{f}_k}(t)$. We prove upper bounds using Dudley’s entropy bound and our metric entropy result of Corollary 4.3. The lower bounds are proved by Sudakov minoration as well as a metric entropy lower bound for local balls around $f_0(x) := \|x\|^2$ (see Lemma 7.3).

The proof of Theorem 3.1 uses the same basic strategy as that of Theorem 2.5 but is technically more involved because of the random design setting. We use conditional versions of many arguments used in the proof of Theorem 2.5 including the conditional version of the result of Chatterjee [10]. The bracketing entropy upper bound from Theorem 4.4 as well as the metric entropy lower bound from Lemma 8.2 are crucial for this proof.

The proof of Theorem 3.2 involves taking a large polytopal region $S$ inside the general domain $\Omega$, using ideas from the proof of Theorem 3.1 on the subset $S$, and using the Lipschitz constraint to deal with the relatively small set $\Omega \setminus S$. The Lipschitz constraint is crucial here as it allows the use of covering numbers in the supremum ($L_\infty$) metric due to Bronstein [9].

Our ideas behind the proofs for the lower bounds on the performance of the LSEs have been used in a simpler setting in Kur et al. [36]. Specifically, the fixed design lower bound in Kur et al. [36] only works in the regime $k \sim \sqrt{n}$ and so it does not yield the adaptive lower bounds in Theorem 2.5. The random design lower bound in Kur et al. [36] uses an assumption on the Koltchinskii-Pollard entropy (or $\infty$-covering) which is not available in the present setting.

The main proof ideas for the metric entropy results are as follows. Let us start with Theorem 4.4 because its proof is technically simpler. The key is to consider a polytopal domain of the form

$$\Omega := \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i, i = 1, \ldots, d + 1 \}$$
for some unit vectors $v_1, \ldots, v_{d+1}$ and prove the bound (26). Results of Gao and Wellner [20] can be used to show this if we consider the $L_p$ norm on the smaller set
\[ \Omega_0 := \{ x \in \mathbb{R}^d : a_i + \eta(b_i - a_i) \leq v_i^T x \leq b_i - \eta(b_i - a_i), i = 1, \ldots, d + 1 \} \]
for a fixed $\eta > 0$ (see Corollary 9.4). The challenge then is to extend this from $\Omega_0$ to all of $\Omega$. We do this via induction by sequentially extending to each domain
\[ T_r(\Omega) := \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i, 1 \leq i \leq r \text{ and } a_i + \eta(b_i - a_i) \leq v_i^T x \leq b_i - \eta(b_i - a_i), r < i \leq d + 1 \} \]
for $r = 0, \ldots, d + 1$ (note that $T_0(\Omega) = \Omega_0$ and $T_{d+1}(\Omega) = \Omega$). Details can be found in the statement and proof of Lemma 9.1.

The ideas behind the proof of Theorem 4.1 are similar but more technically involved because of the discrete metric and the lack of any uniform boundedness. Even the first step of proving the result in the strict interior (such as $\Omega_0$ above) of the full domain $\Omega$ is challenging as there are no prior results (such as those in Gao and Wellner [20]) in this discrete unbounded setting. This result is proved in Proposition 9.5. The induction step (carried out in Proposition 9.11 and Lemma 9.12) is also more delicate.

6. Discussion

This section has some high-level remarks on the results of the paper. Our minimax suboptimality results (for $d \geq 5$) are based on constructions involving piecewise affine functions. Specifically, we prove that the suboptimal rate $n^{-2/d}$ is realized at the piecewise affine function $\tilde{f}_k$ for $k \sim \sqrt{n}$. One might wonder if the convex LSE achieves the same rate $n^{-2/d}$ or a faster rate (such as the minimax rate $n^{-4/(d+4)}$) at smooth convex functions such as $f_0(x) = \|x\|^2$. This appears to be challenging to resolve. The main difficulty stems from the fact that for such $f_0$, the function $t \mapsto H_{f_0}(t)$ (defined as in (27) with $\tilde{f}_k$ replaced by $f_0$) will take values of the same order ($n^{-2/d}$ up to logarithmic factors) for $n^{-2/d} < t \lesssim n^{-1/d}$. Indeed, the upper bound of order $n^{-2/d}$ (up to a logarithmic factor) can be proved by the arguments involved in the proof of Theorem 2.1 and the lower bound of $n^{-2/d}$ follows from Lemma 7.3 or Lemma 8.2 via Sudakov minoration (Lemma 7.4). The (square of the) maximizer of $H_{f_0}(\cdot)$ determines the rate of convergence of the LSE (by Chatterjee [10, Theorem 1.1]) and the fact that $H_{f_0}(\cdot)$ takes values of the same order in the large interval $n^{-2/d} < t \lesssim n^{-1/d}$ makes it difficult to accurately pin down the location of its maximizer.

As already mentioned previously, our minimax suboptimality result for the bounded convex LSE when the domain is a polytope (for $d \geq 5$) contrasts with the recent result of Kur et al. [35] who proved that the bounded convex LSE is minimax optimal when $\Omega = B_2$ ($B_2$ is the unit ball in $\mathbb{R}^d$) for all $d$. Why is the LSE suboptimal over $C^B([0, 1]^d)$ but optimal over $C^B(B_2)$? The class $C^B(B_2)$ is much larger than $C^B([0, 1]^d)$ in the sense of metric entropy under the $L_2$ norm with respect to the Lebesgue measure. Indeed the $\epsilon$-entropy of $C^B([0, 1]^d)$ is of the order $\epsilon^{-d/2}$ while that of $C^B(B_2)$ is of the order
This increased metric entropy of $C^B(\mathcal{B}_d)$ is driven by the curvature (of the boundary) of $\mathcal{B}_d$. Specifically, one can obtain disjoint spherical caps $S_1, \ldots, S_N$ (with $N \sim e^{-(d-1)}$) of height $r^2$ such that the indicators of $\cup_{i \in H} S_i$ for sufficiently separated subsets $H \subseteq \{1, \ldots, N\}$ form an $\epsilon$-packing subset of $C^B(\mathcal{B}_d)$ in the $L_2$ metric with respect to Lebesgue measure (strictly speaking, these indicator functions are not convex but they can be approximated by piecewise affine convex functions).

In other words, the complexity of $C^B(\mathcal{B}_d)$ is driven by the complexity of these well-separated subsets (unions of spherical caps) of $\mathcal{B}_d$. This aspect of $C^B(\mathcal{B}_d)$ is crucially used in Kur et al. [35] to prove the optimality of the LSE for $C^B(\mathcal{B}_d)$. In contrast, the complexity of $C^B([0,1]^d)$ (or more generally $C^B(\Omega)$ when $\Omega$ is a polytope) is not driven by indicator-like functions of subsets of the domain. Here $\epsilon$-packing sets can be constructed by local perturbations of a smooth convex function such as $f_0(x) := \|x\|^2$.

This seems to be the main difference between $C^B(\mathcal{B}_d)$ and $C^B([0,1]^d)$ which is causing the LSE to switch from minimax optimality to suboptimality.

An interesting observation is that, in both the polytopal and the smooth cases, the worst case risk of the LSE over $C^B(\Omega)$ equals, up to logarithmic factors, the global Gaussian complexity:

$$\mathbb{E} \sup_{f \in C^B(\Omega)} \frac{1}{n} \sum_{i=1}^n \xi_i f(X_i)$$

where $\xi_1, \ldots, \xi_n, X_1, \ldots, X_n$ are independent with $\xi_1, \ldots, \xi_n$ distributed as normal with mean zero and variance $\sigma^2$ and $X_1, \ldots, X_n$ distributed according to the uniform distribution $\mathbb{P}$ on $\Omega$. When $\Omega$ is a polytope, (28) is of the order $n^{-2/d}$. To see this, one can upper bound (28) by using standard empirical process bounds via $L_2(\mathbb{P})$ bracketing entropy bounds (see e.g., van de Geer [44, Theorem 5.11] restated as inequality (44)) and lower bound (28) by Sudakov minoration along with the metric entropy lower bound in Lemma 8.2. When $\Omega = \mathcal{B}_d$, this strategy of upper bounding (28) via $L_2(\mathbb{P})$ bracketing entropy gives a suboptimal upper bound as explained by Kur et al. [35]. The reason is that the $L_2(\mathbb{P}_n)$ bracketing $\epsilon$-entropy (here $\mathbb{P}_n$ is the empirical measure of $X_1, \ldots, X_n$) is different from the $L_2(\mathbb{P})$ bracketing $\epsilon$-entropy for $\epsilon$ smaller than $n^{-1/(d-1)}$. Thus, to prove the sharp bound for (28) in the ball case, Kur et al. [35] resort to a different technique via level sets and chaining using $L_1$ bracketing numbers.

Isotonic regression is another shape constrained regression problem where the LSE is known to be minimax optimal for all dimensions (see Han et al. [29]). The class of coordinatewise monotone functions on $[0,1]^d$ is similar to $C^B(\mathcal{B}_d)$ in that its metric entropy is driven by well-separated subsets of $[0,1]^d$ (see Gao and Wellner [19, Proof of Proposition 2.1]). Other examples of such classes where the LSE is optimal for all dimensions can be found in Han [28].

We proved our fixed-design risk bounds for the full convex LSE (in Section 2) in the case where the domain $\Omega$ is a polytope. A natural question is to extend these to the case where $\Omega$ is a smooth convex body such as the unit ball. Based on the results of Kur et al. [35], it is reasonable to conjecture that convex LSE will be minimax optimal in fixed design when the domain is the unit ball. However it appears nontrivial to prove
this as the level set reduction employed in Kur et al. [35] cannot be used in the absence of uniform boundedness. We hope to address this in future work.

7. Proofs of results from Section 2

This section has proofs for Theorems 2.1, 2.2, 2.3 and 2.5 (Lemma 2.4 is proved in Section 10). The metric entropy results: inequality (23), Corollary 4.2 and Corollary 4.3 stated in Section 4 are crucial for these proofs. Let us also recall here some general results that will be used in these proofs starting with the following result of Chatterjee [10]. We use the following notation. For a function \( f \) on \( \Omega \), a class of functions \( \mathcal{F} \) on \( \Omega \) and \( t > 0 \), let

\[
\mathcal{B}_{\mathcal{F}}^{\mathcal{P}_n}(f, t) := \{ g \in \mathcal{F} : \ell_{\mathcal{P}_n}(f, g) \leq t \}.
\]

(29)

where \( \ell_{\mathcal{P}_n} \) is given in (5).

**Theorem 7.1** (Chatterjee). Consider data generated according to the model:

\[
Y_i = f(X_i) + \xi_i \quad \text{for } i = 1, \ldots, n
\]

where \( X_1, \ldots, X_n \) are fixed deterministic design points in a convex body \( \mathcal{X} \subseteq \mathbb{R}^d \), \( f \) belongs to a convex class of functions \( \mathcal{F} \) and \( \xi_1, \ldots, \xi_n \) are independently distributed according to the normal distribution with mean 0 and variance \( \sigma^2 \). Consider the LSE

\[
\hat{f}_n(\mathcal{F}) \in \arg\min_{g \in \mathcal{F}} \sum_{i=1}^{n} (Y_i - g(X_i))^2
\]

and define \( t_f := \arg\max_{t \geq 0} H_f(t) \) where

\[
H_f(t) := \mathbb{E} \sup_{g \in \mathcal{B}_{\mathcal{F}}^{\mathcal{P}_n}(f, t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i (g(X_i) - f(X_i)) - \frac{t^2}{2}
\]

where \( \mathcal{B}_{\mathcal{F}}^{\mathcal{P}_n}(f, t) \) is defined in (29). Then \( H_f(\cdot) \) is a concave function on \([0, \infty)\), \( t_f \) is unique and the following pair of inequalities hold for positive constants \( c \) and \( C \):

\[
\mathbb{P} \left\{ 0.5t_f^2 \leq \ell_{\mathcal{P}_n}^2(\hat{f}_n(\mathcal{F}), f) \leq 2t_f^2 \right\} \geq 1 - 6 \exp \left( -\frac{cn{t_f^2}}{\sigma^2} \right)
\]

(30)

and

\[
0.5t_f^2 - \frac{C\sigma^2}{n} \leq \mathbb{E} \ell_{\mathcal{P}_n}^2(\hat{f}_n(\mathcal{F}), f) \leq 2t_f^2 + \frac{C\sigma^2}{n}.
\]

Upper bounds for \( t_f \) can be obtained via:

\[
t_f \leq \inf \{ r > 0 : H_f(r) \leq 0 \}
\]

(31)

and lower bounds for \( t_f \) can be obtained via:

\[
t_f \geq r_1 \quad \text{if } 0 \leq r_1 < r_2 \text{ are such that } H_f(r_1) \leq H_f(r_2).
\]

(32)
Let us also recall the Dudley metric entropy bound for the supremum of a Gaussian process.

**Theorem 7.2** (Dudley). Let \( \xi_1, \ldots, \xi_n \) be independently distributed according to the normal distribution with mean 0 and variance \( \sigma^2 \). Then for every deterministic \( X_1, \ldots, X_n \in X \), every class of functions \( F, f \in F \) and \( t \geq 0 \), we have

\[
E \sup_{g \in B_{\frac{F}{P_n}}(f, t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i (g(X_i) - f(X_i)) \leq \sigma \inf_{0 < \theta \leq t/2} \left( \frac{12}{\sqrt{n}} \int_{\theta}^{t/2} \sqrt{\log N(\epsilon, B_{\frac{F}{P_n}}(f, t), \ell_{P_n})} d\epsilon + 2\theta \right).
\]

\( C_d \) is a constant depending on \( d \) alone, we get

\[
\log N(\epsilon, B_{\frac{F}{P_n}}(f_0, t), \ell_{P_n}) \leq C_d (\log n)^{F/2} \left( \frac{t + \Omega}{\epsilon} \right)^{d/2}.
\]

for every \( t > 0 \) and \( \epsilon > 0 \) where \( \Omega = \Omega(f_0) = \inf_{f \in A(\Omega)} \ell_{P_n}(f_0, f) \). We now control

\[
G(t) := E \sup_{g \in C(\Omega)} \frac{1}{n} \sum_{i=1}^{n} \xi_i (g(X_i) - f_0(X_i))
\]

so we can bound \( E f_0 \ell_{P_n}^2(\hat{f}_n, f_0) \) by Theorem 7.1. Theorem 7.2 along with (33) gives

\[
\frac{G(t)}{\sigma} \leq C_d (\log n)^{F/2} \left( \frac{t + \Omega}{\epsilon} \right)^{d/4} \int_{\theta}^{t/2} \left( \frac{t}{\epsilon} \right)^{d/4} d\epsilon + 2\theta
\]

\leq \left\{ \int_{\theta}^{t/2} \left( \frac{t}{\epsilon} \right)^{d/4} d\epsilon + \int_{\theta}^{t/2} \left( \frac{\Omega}{\epsilon} \right)^{d/4} d\epsilon \right\} + 2\theta
\]

for every \( 0 < \theta \leq t/2 \). Below, we replace \( C_d 2^{d/4} / 4 \) by just \( C_d \).

Before proceeding further, it is convenient to split into the three cases \( d \leq 3, d = 4 \) and \( d \geq 5 \). When \( d \leq 3 \), we take \( \theta = 0 \) to get

\[
G(t) \leq \frac{C_d \sigma}{\sqrt{n}} (\log n)^{F/2} \left( t + \Omega^{d/4} t^{1-d/4} \right).
\]

Because

\[
\frac{C_d \sigma}{\sqrt{n}} (\log n)^{F/2} t \leq \frac{t^2}{4} \quad \text{if and only if} \quad t \geq \frac{4C_d \sigma}{\sqrt{n}} (\log n)^{F/2}
\]

and

\[
\frac{C_d \sigma}{\sqrt{n}} (\log n)^{F/2} \Omega^{d/4} t^{1-d/4} \leq \frac{t^2}{4} \quad \text{iff} \quad t \geq (4C_d)^{d/4} \left( \frac{\sigma (\log n)^{F/2}}{\sqrt{n}} \right)^{d/4} \Omega^{d/4},
\]
we deduce that
\[ H(t) := G(t) - \frac{t^2}{2} \leq 0 \quad \text{for } t \geq C_d \max \left( \left( \frac{\sigma (\log n)^{F/2}}{\sqrt{n}} \right)^{\frac{1}{1+\epsilon}} \frac{\sigma}{\sqrt{n}}, \frac{\sigma (\log n)^{F/2}}{\sqrt{n}} \right). \]

It follows from (31) that
\[ t_{f_0} \leq C_d \max \left( \left( \frac{\sigma (\log n)^{F/2}}{\sqrt{n}} \right)^{\frac{1}{1+\epsilon}} \frac{\sigma}{\sqrt{n}}, \frac{\sigma (\log n)^{F/2}}{\sqrt{n}} \right) \]
so Theorem 2.1 for \( d \leq 3 \) follows directly from Theorem 7.1.

For \( d = 4 \), (35) leads to
\[ G(t) \leq \frac{C_d \sigma}{\sqrt{n}} (t + \mathcal{L}) (\log n)^{1+(F/2)} \]

Choosing \( \theta = t/(2\sqrt{n}) \), we obtain
\[ G(t) \leq \frac{C_d \sigma}{\sqrt{n}} (t + \mathcal{L}) (\log n)^{1+(F/2)} \]
from which we can deduce as before that
\[ H(t) = G(t) - \frac{t^2}{2} \leq 0 \quad \text{for } t \geq C_d \max \left( \frac{\sigma \mathcal{L}(\log n)^{(F/4)+(1/2)}}{n^{1/4}}, \frac{\sigma (\log n)^{1+(F/2)}}{\sqrt{n}} \right) \]
which proves Theorem 2.1 for \( d = 4 \).

Finally, for \( d \geq 5 \), (35) leads to the bound
\[ G(t) \leq \frac{C_d \sigma (\log n)^{F/2}}{\sqrt{n}} \left\{ \int_{\theta}^{\infty} \left( \frac{t}{\epsilon} \right)^{d/4} d\epsilon + \int_{\theta}^{\infty} \left( \frac{\mathcal{L}}{\epsilon} \right)^{d/4} d\epsilon \right\} + 2\sigma \theta \]
\[ \leq \frac{C_d \sigma (\log n)^{F/2}}{\sqrt{n}} (t + \mathcal{L})^{d/4} \theta^{1-(d/4)} + 2\sigma \theta \]
for every \( \theta > 0 \). The choice
\[ \theta = \left( \frac{C_d (\log n)^{F/2}}{\sqrt{n}} \right)^{d/4} (t + \mathcal{L}) \]
gives
\[ G(t) \leq 2\sigma \left( \frac{C_d (\log n)^{F/2}}{\sqrt{n}} \right)^{4/d} (t + \mathcal{L}) \]
from which it follows that
\[ H(t) = G(t) - \frac{t^2}{2} \leq 0 \quad \text{for } t \geq \max \left( \sqrt{\sigma \mathcal{L}} \left( \frac{(\log n)^{F/2}}{\sqrt{n}} \right)^{2/d}, \sigma \left( \frac{(\log n)^{F/2}}{\sqrt{n}} \right)^{4/d} \right) \]
which concludes the proof of Theorem 2.1.
7.2. Proof of Theorem 2.2

This basically follows from Theorem 2.5. Let $c_d$ and $N_d$ be as given by Theorem 2.5. Letting $k = \sqrt{n\sigma^{-d/4}}$ and assuming that $n \geq \max(N_d, (c_d^2)\sigma^{-d/2})$, we obtain from Theorem 2.5 that

$$\sup_{f_0 \in C_{C_d}(\Omega)} \mathbb{E}_{f_0} \ell_n^2(\hat{f}_n, f_0) \geq c_d \sigma n^{-2/d} (\log n)^{-4(d+1)/d}.$$  

where $C_d$ is such that $\hat{f}_k \in C_{C_d}(\Omega)$ (existence of such a $C_d$ is guaranteed by Lemma 2.4). The required lower bound (15) on the class $C_L(\Omega)$ for an arbitrary $L > 0$ can now be obtained by an elementary scaling argument.

7.3. Proof of Theorem 2.3

Theorem 2.3 follows from a straightforward application of the metric entropy bound in Corollary 4.2 and the general results Theorem 7.1 and Theorem 7.2. Indeed, combining Corollary 4.2 and Theorem 7.2, we get

$$G(t) \leq C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2} \int_{\theta}^{t/2} \left( \frac{t}{\epsilon} \right)^{d/4} d\epsilon + 2 \sigma \theta$$  

for every $0 < \theta \leq t/2$ where $G(t)$ is as in (34). We now split into the three cases $d \leq 3$, $d = 4$ and $d \geq 5$. When $d \leq 3$, we take $\theta = 0$ to obtain

$$G(t) \leq C_dt \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2}$$

so that $G(t) \leq t^2/2$ for $t \geq 2C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2}$. This proves Theorem 2.3 for $d \leq 3$.

When $d = 4$, we get

$$G(t) \leq C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2} t \log \frac{t}{2\theta} + 2 \sigma \theta.$$

Choosing $\theta := t/(2\sqrt{n})$, we get

$$G(t) \leq C_d \sigma t \sqrt{\frac{k}{n}} (\log n)^{1+h/2}.$$

This gives $G(t) \leq t^2/2$ for $t \geq 2C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{1+h/2}$ which proves Theorem 2.3 for $d = 4$.

For $d \geq 5$, we get

$$G(t) \leq C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2} \int_{\epsilon}^{\infty} \left( \frac{t}{\epsilon} \right)^{d/4} d\epsilon + 2 \sigma \theta$$

$$\leq C_d \sigma \sqrt{\frac{k}{n}} (\log n)^{h/2} t^{d/4} \theta^{1-(d/4)} + 2 \sigma \theta.$$
Take $\theta = t \left( C_d (\log n)^{h/2} \sqrt{\frac{k}{n}} \right)^{4/d}$ to get

$$G(t) \leq C_d \sigma t \left( \log n \right)^{h/2} \left( \sqrt{\frac{k}{n}} \right)^{4/d}. \quad (37)$$

This clearly implies that $G(t) \leq t^2/2$ for

$$t \geq 2 C_d \sigma \left( \log n \right)^{h/2} \left( \sqrt{\frac{k}{n}} \right)^{4/d}$$

which completes the proof of Theorem 2.3.

### 7.4. Proof of Theorem 2.5

In addition to Theorem 7.1, Theorem 7.2, Lemma 2.4 and Corollary 4.3, this proof will need the following two results. The proof of the first result below is given in Section 10 while the second result is standard. Recall the notation (29).

**Lemma 7.3.** Let $\Omega$ be a convex body contained in the unit ball whose volume is bounded from below by a constant depending on $d$ alone. Let $f_0(x) := \|x\|^2$. There exist two positive constants $c_1$ and $c_2$ depending on $d$ alone such that

$$\log N(c_1 n^{-2/d}, \mathcal{C}_n(f_0, t), \ell_{P_n}) \geq \frac{n}{8} \quad \text{for } t \geq c_2 n^{-2/d}. \quad (38)$$

**Lemma 7.4** (Sudakov minoration). Let $\xi_1, \ldots, \xi_n$ be independently distributed according to the normal distribution with mean 0 and variance $\sigma^2$. Then for every deterministic $X_1, \ldots, X_n \in X$, every class of functions $F$ and $t \geq 0$, we have

$$\mathbb{E} \sup_{g \in \mathcal{C}_n(f, t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i (g(X_i) - f(X_i)) \geq \frac{\beta \sigma}{\sqrt{n}} \sup_{\epsilon > 0} \left\{ \epsilon \sqrt{\log N(\epsilon, \mathcal{B}_{P_n}(f, t), \ell_{P_n})} \right\}. \quad (39)$$

**Proof of Theorem 2.5.** By Lemma 2.4, $\tilde{f}_k$ satisfies

$$\ell_{P_n}(f_0, \tilde{f}_k) \leq \sup_{x \in \Omega} \left| f_0(x) - \tilde{f}_k(x) \right| \leq C_d k^{-2/d}. \quad (38)$$

where $f_0(x) := \|x\|^2$. Theorem 7.1 says that $\mathbb{E} \sup_{\epsilon > 0} \left\{ \epsilon \sqrt{\log N(\epsilon, \mathcal{B}_{P_n}(f_0, t), \ell_{P_n})} \right\}$. Note that we are working in the fixed design setting so $X_1, \ldots, X_n$ are non-random and the expectation above is being taken with respect to the randomness in $\xi_1, \ldots, \xi_n$. 


We shall lower bound $t_{\tilde{f}_k}$ by proving suitable upper and lower bounds for the function $\tilde{G}(\cdot)$. Note first that by the properties of $\tilde{f}_k$ given in Lemma 2.4, we can apply the metric entropy bound in Corollary 4.3 along with Theorem 7.2 to get (similar to the calculation underlying (37)) a constant $\Upsilon_d$ such that

$$\tilde{G}(t) \leq \Upsilon_d \sigma t (\log n)^{2(d+1)/d} \left( \frac{k}{n} \right)^{2/d} \quad \text{for every } t > 0 \tag{39}$$

We now prove the following lower bound for $\tilde{G}(t)$: there exist positive constants $\gamma_d$ and $\Gamma_d$ depending on $d$ alone such that

$$\tilde{G}(t) \geq \gamma_d \sigma t \left( \frac{k}{n} \right)^{2/d} \quad \text{for all } t \leq \gamma_d k^{-2/d} \tag{40}$$

provided $k \leq \Gamma_d n$. To prove this, suppose first that $t = 2C_d k^{-2/d}$ where $C_d$ is the constant from (38). For this choice of $t$, it follows from the triangle inequality (and (38)) that

$$\mathcal{B}_{P_n}^{C(\Omega)}(\tilde{f}_k, t) \supseteq \mathcal{B}_{P_n}^{C(\Omega)}(f_0, C_d k^{-2/d})$$

where, it may be recalled, $\mathcal{B}_{P_n}^{C(\Omega)}(f, s) := \{ g \in C(\Omega) : \ell_{P_n}(f, g) \leq s \}$. This immediately implies

$$\tilde{G}(t) \geq G(C_d k^{-2/d}).$$

where $G(t)$ is defined as in (34). Lemma 7.4 now gives

$$\tilde{G}(t) \geq G(C_d k^{-2/d}) \geq \frac{\beta \sigma}{\sqrt{n}} \sup_{\epsilon > 0} \left\{ \epsilon \sqrt{\log N(\epsilon, \mathcal{B}_{P_n}(f_0, C_d k^{-2/d}), \ell_{P_n})} \right\}.$$

Using the lower bound on the metric entropy given by Lemma 7.3 for $\epsilon = c_1 n^{-2/d}$ gives

$$\tilde{G}(t) \geq \frac{\beta \sigma}{\sqrt{n}} (c_1 n^{-2/d}) \sqrt{\frac{n}{8}} = \frac{\beta c_1}{\sqrt{8}} \sigma n^{-2/d}$$

provided

$$k \leq \left( \frac{C_d}{c_2} \right)^{d/2} n. \tag{41}$$

The condition above is necessary for the inequality $c_2 n^{-2/d} \leq C_d k^{-2/d}$ which is required for the application of Lemma 7.3. This gives

$$\tilde{G}(t) \geq \frac{\beta c_1}{\sqrt{8}} \sigma n^{-2/d} \quad \text{for } t = 2C_d k^{-2/d}.$$

Now for $t \leq 2C_d k^{-2/d}$, we use the fact that $x \mapsto \tilde{G}(x)$ is concave on $[0, \infty)$ (and that $\tilde{G}(0) = 0$) to deduce that

$$\frac{\tilde{G}(t)}{t} \geq \tilde{G}(2C_d k^{-2/d}) \geq \sigma \left( \frac{k}{n} \right)^{2/d} \frac{\beta c_1}{2\sqrt{8}C_d} \quad \text{for all } t \leq 2C_d k^{-2/d}.$$
This proves (40) for
\[ \gamma_d = \min \left( \frac{\beta c_1}{2\sqrt{8C_d}}, 2C_d \right) \quad \text{and} \quad \Gamma_d = \left( \frac{c_3 C_d}{c_2} \right)^{d/2}. \]

We shall now bound the quantity \( t_{f_k}^2 \) (defined in Theorem 7.1) from below using (39) and (40). By the lower bound in (40), we get
\[ \sup_{t > 0} \left( \tilde{G}(t) - \frac{t^2}{2} \right) \geq \sup_{t \leq \gamma_d k^{-2/d}} \left( \gamma_d \sigma t \left( \frac{k}{n} \right)^{2/d} - \frac{t^2}{2} \right). \]

Taking \( t = \gamma_d \sigma (k/n)^{2/d} \) and noting that
\[ t = \gamma_d \sigma \left( \frac{k}{n} \right)^{2/d} \leq \gamma_d k^{-2/d} \quad \text{if and only if} \quad k \leq \sqrt{n} \sigma^{-d/4}, \]
we get that
\[ \sup_{t > 0} \left( \tilde{G}(t) - \frac{t^2}{2} \right) \geq \frac{\gamma_d^2 \sigma^2}{2} \left( \frac{k}{n} \right)^{4/d}. \]

The above inequality, combined with (39) and the fact that \( t_{f_k} \) maximizes \( \tilde{G}(t) - t^2/2 \) over all \( t > 0 \), yield
\[ \frac{\gamma_d^2 \sigma^2}{2} \left( \frac{k}{n} \right)^{4/d} \leq \sup_{t > 0} \left( \tilde{G}(t) - \frac{t^2}{2} \right) = \tilde{G}(t_{f_k}) - \frac{t_{f_k}^2}{2} \leq \tilde{G}(t_{f_k}) \leq \Upsilon_d \sigma t_{f_k} (\log n)^{2(d+1)/d} \left( \frac{k}{n} \right)^{2/d}. \]

This implies
\[ t_{f_k} \geq \frac{\gamma_d^2 \sigma}{2 \Upsilon_d} \left( \frac{k}{n} \right)^{2/d} \left( \log n \right)^{-2(d+1)/d}. \]

Theorem 7.1 then gives
\[ \mathbb{E}_{f_k} \ell_{\tilde{v}_n}^2 \left( \tilde{f}_n, \tilde{f}_k \right) \geq \frac{\gamma_d^4}{8 \Upsilon_d^2} \sigma^2 \left( \frac{k}{n} \right)^{4/d} \left( \log n \right)^{-4(d+1)/d} - \frac{C \sigma^2}{n}. \]

It is now clear that the first term on the right hand side above dominates the second term when \( n \) is larger than a constant depending on \( d \) alone. This completes the proof of Theorem 2.5.

8. Proofs of results from Section 3

We provide here the proofs for Theorem 3.1 and Theorem 3.2. These proofs are similar in spirit to that of Theorem 2.5 with some differences that are necessary to deal with the random design setting. Let us first state some general results that will be used in these proofs.
The proof of Theorem 2.5 needed the ingredients: Theorem 7.1, Theorem 7.2, Lemma 7.3, Lemma 2.4 and Lemma 7.4. Modified forms of these ingredients to cover the random design setting (as described below) are used for the proof of Theorem 3.1 and Theorem 3.2.

As in the proof of Theorem 2.5, a key role will be played by Theorem 7.1 of Chatterjee [10]. Theorem 7.1 holds for the fixed design setting with no restriction on the design points which means that it also applies to the random design setting provided we condition on the design points $X_1, \ldots, X_n$. In particular, for our random design setting with $\ell_{\pi_n}$ defined as in (5), inequality (30) becomes:

$$
P \left\{ 0.5t_f^2 \leq \ell_{\pi_n}^2(\hat{f}_n, f) \leq 2t_f^2 \bigg| X_1, \ldots, X_n \right\} \geq 1 - 6 \exp \left( -\frac{cnt_f^2}{\sigma^2} \right) \quad (42)$$

where

$$t_f = t_f(X_1, \ldots, X_n) := \arg\max_{t \geq 0} H_f(t) \quad (43)$$

with

$$H_f(t) := \mathbb{E} \left[ \sup_{g \in \mathcal{B}_{P_n}(f, t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i(g(X_i) - f(X_i)) \bigg| X_1, \ldots, X_n \right] - \frac{t^2}{2}.$$ 

Here $t_f = t_f(X_1, \ldots, X_n)$ is random as it depends on the random design points $X_1, \ldots, X_n$.

Instead of Dudley’s theorem (Theorem 7.2), we shall use the following theorem on the suprema of empirical processes. The first conclusion of the theorem below is taken from van de Geer [44, Theorem 5.11] while the second conclusion essentially follows from van de Geer [44, Proof of Lemma 5.16].

**Theorem 8.1.** Suppose $X_1, \ldots, X_n$ are independently distributed according to a distribution $\mathbb{P}$ on $\Omega$. Suppose $\mathcal{F}$ is a class of real-valued functions on $\Omega$ that are uniformly bounded by $\Gamma > 0$. Then the following two statements are true:

1. There exists a positive constant $C$ such that

$$\mathbb{E} \sup_{f \in \mathcal{F}} |\mathbb{P}_n f - \mathbb{P} f| \leq C \inf \left\{ a \geq \frac{\Gamma}{\sqrt{n}} : a \geq C \sqrt{n} \int a \sqrt{\log N_\mathbb{P}(u, \mathcal{F}, L_2(\mathbb{P}))} du \right\}. \quad (44)$$

2. There exists a positive constant $C$ such that

$$\mathbb{P} \left\{ \sup_{f,g \in \mathcal{F}} (\ell_{\mathbb{P}}(f,g) - 2\ell_{\mathbb{P}_n}(f,g)) > Ca \right\} \leq C \exp \left( -\frac{na^2}{C\Gamma^2} \right) \quad (45)$$

and

$$\mathbb{P} \left\{ \sup_{f,g \in \mathcal{F}} (\ell_{\mathbb{P}_n}(f,g) - 2\ell_{\mathbb{P}}(f,g)) > Ca \right\} \leq C \exp \left( -\frac{na^2}{C\Gamma^2} \right) \quad (46)$$

provided

$$na^2 \geq C \log N_\mathbb{P}(a, \mathcal{F}, L_2(\mathbb{P})). \quad (47)$$
Instead of Lemma 7.3, we shall use the following result which proves the same lower bound as in Lemma 7.3 in the random design setting with high probability. Recall that $C_L^F(\Omega)$ denotes the class of all convex functions on $\Omega$ that are $L$-Lipschitz and uniformly bounded by $L$.

**Lemma 8.2.** Let $\Omega$ be a convex body that contains a ball of constant (depending on $d$ alone) radius. Let $f_0(x) := \|x\|^2$. Then there exist positive constants $c_1, c_2, c_3, c_4$ and $C$ depending on $d$ alone such that

$$
\mathbb{P} \left\{ \log N(\epsilon, \mathfrak{B}_{\ell_p}^F(\Omega)) \geq c_1 \epsilon^{-d/2} \right\} \geq 1 - \exp(-c_2 n) 
$$

for each fixed $\epsilon, t, L$ satisfying $L \geq C$ and $c_3 n^{-2/d} \leq \epsilon \leq \min(c_4, t/4)$.

Lemma 2.4 will also be crucially used in the proof of Theorem 3.1. The following analogue of Lemma 2.4 for the case when $\Omega$ is not necessarily a polytope will be used in the proof of Theorem 3.2.

**Lemma 8.3.** Suppose $\Omega$ is a convex body that is contained in the unit ball and contains a ball of constant (depending on $d$ alone) radius centered at zero. Let $f_0(x) := \|x\|^2$. There exists a positive constant $C_d$ (depending on dimension alone) such that the following is true. For every $k \geq 1$, there exist $m \leq C_d k$ $d$-simplices $\Delta_1, \ldots, \Delta_m \subseteq \Omega$ having disjoint interiors and a convex function $\tilde{f}_k$ such that

1. $(1 - C_d k^{-1/d})\Omega \subseteq \bigcup_{i=1}^m \Delta_i \subseteq \Omega$,
2. $\tilde{f}_k$ is affine on each $\Delta_i$, $i = 1, \ldots, m$,
3. $\sup_{x \in \Omega} |f_0(x) - \tilde{f}_k(x)| \leq C_d k^{-2/d}$,
4. $\tilde{f}_k \in C_{C_d}(\Omega)$.

Lemma 7.4 will be used in the proofs of Theorem 3.1 and Theorem 3.2 in the following conditional form:

$$
\mathbb{E} \left[ \sup_{g \in \mathfrak{B}_{\ell_p}^F(f, t)} \frac{1}{n} \sum_{i=1}^n \xi_i (g(X_i) - f(X_i)) \bigg| X_1, \ldots, X_n \right] \geq \frac{\beta \sigma}{\sqrt{n}} \sup_{\epsilon > 0} \{ \epsilon \sqrt{\log N(\epsilon, \mathfrak{B}_{\ell_p}^F(f, t), \ell_p)} \}.
$$

We are now ready for the proofs of Theorem 3.1 and Theorem 3.2.

**8.1. Proof of Theorem 3.1**

*Proof of Theorem 3.1.* It is enough to prove (20) when $B$ is a fixed dimensional constant. From here, the inequality for arbitrary $B > 0$ can be deduced by an elementary scaling argument.

Let $f_0(x) := \|x\|^2$ and $\tilde{f}_k$ be as given by Lemma 2.4. Below we shall assume that $B$ is a large enough dimensional constant so that $\tilde{f}_k \in C^B(\Omega)$. The main task in this proof will be to bound the quantity $t_{\tilde{f}_k}$ (defined via (43)) from below where $t_{\tilde{f}_k}$ maximizes

$$
H_{\tilde{f}_k}(t) := G_{\tilde{f}_k}(t) - \frac{t^2}{2}
$$
over all \( t \geq 0 \) where

\[
G_{f_k}(t) := \mathbb{E} \left[ \sup_{g \in \mathbb{E}_{n}^{B(\cdot)(\tilde{f}_k,t)}} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left( g(X_i) - \tilde{f}_k(X_i) \right) \right| X_1, \ldots, X_n].
\] (50)

We shall prove a lower bound for \( t_{f_k} \) that holds with high probability over the randomness in \( X_1, \ldots, X_n \). Specifically, we shall prove the existence of three dimensional constants \( \gamma_d, c_d \) and \( C_d \) and a constant \( N_{d,\sigma} \) which depends on \( d \) and \( \sigma \) such that

\[
\mathbb{P} \{ t_{\tilde{f}_k} \geq c_d n^{-1/d} \sqrt{\sigma} (\log n)^{-2(d+1)/d} \} \geq 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right)
\] (51)

for \( k = \gamma_d \sqrt{n} \sigma^{-d/4} \) and \( n \geq N_{d,\sigma} \).

Before proceeding with the proof of (51), let us first show how (51) completes the proof of Theorem 3.1. Note first that

\[
\sup_{f \in \mathbb{C}^B(\Omega)} \mathbb{E}_f \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), f) \geq \mathbb{E}_{\tilde{f}_k} \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k)
\]

so it is enough to prove that the right hand side of (20) is a lower bound for \( \mathbb{E}_{\tilde{f}_k} \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \). We shall assume therefore that the data have been generated from the true function \( \tilde{f}_k \). Let \( \rho_n \) be the lower bound on \( t_{\tilde{f}_k} \) given by (51) i.e.,

\[
\rho_n := c_d n^{-1/d} \sqrt{\sigma} (\log n)^{-2(d+1)/d}.
\] (52)

Inequality (42) clearly implies

\[
\mathbb{P} \left\{ \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} t^2_{\tilde{f}_k} \left| X_1, \ldots, X_n \right. \right\} \geq 1 - 6 \exp \left( -\frac{c n t^2_{\tilde{f}_k}}{\sigma^2} \right).
\]

As a result,

\[
\mathbb{P} \left\{ \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \rho^2_n \right\} \geq \mathbb{P} \left\{ \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} t^2_{\tilde{f}_k}, t_{\tilde{f}_k} \geq \rho_n \right\}
\]

\[
= \mathbb{E} \left[ I \{ t_{\tilde{f}_k} \geq \rho_n \} \mathbb{P} \left\{ \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} t^2_{\tilde{f}_k} \left| X_1, \ldots, X_n \right. \right\} \right]
\]

\[
= \mathbb{E} \left[ I \{ t_{\tilde{f}_k} \geq \rho_n \} \left( 1 - 6 \exp \left( -\frac{c n t^2_{\tilde{f}_k}}{\sigma^2} \right) \right) \right]
\]

\[
\geq \left( 1 - 6 \exp \left( -\frac{c n \rho^2_n}{\sigma^2} \right) \right) \mathbb{P} \{ t_{\tilde{f}_k} \geq \rho_n \}.
\]

We can now use (51) to obtain

\[
\mathbb{P} \left\{ \ell^2_{\mathbb{P}}(\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \rho^2_n \right\} \geq \left( 1 - 6 \exp \left( -\frac{c n \rho^2_n}{\sigma^2} \right) \right) \left( 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right) \right)
\]

\[
\geq 1 - 6 \exp \left( -\frac{c n \rho^2_n}{\sigma^2} \right) - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right).
\]
Clearly if \( N_{d, \sigma} \) is chosen appropriately, then, for \( n \geq N_{d, \sigma} \),
\[
\frac{n \rho_n^2}{\sigma^2} = \frac{c_d^2}{\sigma} n^{(d-2)/d} (\log n)^{-4(d+1)/d}
\]
will be larger than any constant multiple of \( n^{(d-4)/d} \) which gives
\[
\mathbb{P} \left\{ \ell_{\infty}^2 (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \rho_n^2 \right\} \geq 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right). \tag{53}
\]

We shall now argue that a similar inequality also holds for \( \ell_{\infty}^2 (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \). For every \( a > 0 \), we have
\[
\mathbb{P} \left\{ \ell_{\infty}^2 (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right) \right\} \geq \mathbb{P} \left\{ \ell_{\infty}^2 (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{\rho_n^2}{2}, \sup_{f, g \in C^B(\Omega)} (\ell_{\infty} (f, g) - 2\ell_{\infty} (f, g)) \leq a \right\} \geq \mathbb{P} \left\{ \ell_{\infty}^2 (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{\rho_n^2}{2} \right\} + \mathbb{P} \left\{ \sup_{f, g \in C^B(\Omega)} (\ell_{\infty} (f, g) - 2\ell_{\infty} (f, g)) \leq a \right\} - 1 \tag{54}
\]
\[
\geq \mathbb{P} \left\{ \sup_{f, g \in C^B(\Omega)} (\ell_{\infty} (f, g) - 2\ell_{\infty} (f, g)) \leq a \right\} - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right).
\]

To bound the probability above, we use (46). Gao and Wellner \cite[Theorem 1.5]{GaoWellner} gives
\[
\log N[\epsilon, C^B(\Omega), \ell_{\infty}] \leq C_d \left( \frac{B}{\epsilon} \right)^{d/2} \tag{55}
\]
The requirement (47) is therefore satisfied when \( a = n^{-2/(d+4)} L^{d/(d+4)} \) multiplied by a large enough dimensional constant. Inequality (46) then gives
\[
\mathbb{P} \left\{ \sup_{f, g \in C^B(\Omega)} (\ell_{\infty} (f, g) - 2\ell_{\infty} (f, g)) \leq C_d n^{-2/(d+4)} B^{d/(d+4)} \right\} \geq 1 - C_d \exp \left( -\frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}} \right) \tag{56}
\]
which then implies that
\[
\mathbb{P} \left\{ \ell_{\infty} (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \left( c_d n^{-1/d} \sqrt{\sigma (\log n)^{-2(d+1)/d}} - C_d n^{-2/(d+4)} B^{d/(d+4)} \right) \right\} \geq 1 - C_d \exp \left( -\frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}} \right) - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right).
\]

Because \( n^{-2/(d+4)} \) is of a smaller order than \( n^{-1/d} \) and \( n^{d/(d+4)} \) is of a larger order than \( n^{(d-4)/d} \) (and \( B \) is a dimensional constant), we obtain
\[
\mathbb{P} \left\{ \ell_{\infty} (\hat{f}_n(C^B(\Omega)), \tilde{f}_k) \geq c_d n^{-1/d} \sqrt{\sigma (\log n)^{-2(d+1)/d}} \right\} \geq 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right).
\]
provided \( n \geq N_{d,\sigma} \) where \( N_{d,\sigma} \) is a constant depending on \( d \) and \( \sigma \) alone. Finally, note that \( N_{d,\sigma} \) can be chosen so that

\[
P \left\{ \ell_P (\hat{f}_n (C^B (\Omega)), \tilde{f}_k) \geq c_d n^{-1/d} \sqrt{\sigma} (\log n)^{-2(d+1)/d} \right\} \geq \frac{1}{2}
\]

which immediately gives that

\[
\mathbb{E}_{\hat{f}_k} \ell_P^2 (\hat{f}_n (C^B (\Omega)), \tilde{f}_k) \geq \frac{c_d^2}{4} \sigma n^{-2/d} (\log n)^{-4(d+1)/d}.
\]

(57)

This completes the proof of Theorem 3.1 assuming that (51) is true.

Let us now start the proof of (51). For this purpose, we shall require both upper and lower bounds for \( G_{\tilde{f}_k} (t) \) (defined in (50)) for appropriate values of \( t \). We start to prove upper bounds. Note first that \( B_{C^B (\Omega)} P (\tilde{f}_k, t) \subseteq B_{C^B (\Omega)} P (\tilde{f}_k, 2t + C_d n^{-2/(d+4)} B^{d/(d+4)}) \)

with probability at least

\[
1 - C \exp \left( - \frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}} \right).
\]

(59)

Here we are using the notation

\[
\mathcal{B}^F_P (f, t) := \{ g \in \mathcal{F} : \ell_P (f, g) \leq t \}.
\]

(60)

where \( \ell_P \) is given in (8). (58) is a consequence of

\[
P \left\{ \sup_{f, g \in C^B (\Omega)} (\ell_P (f, g) - 2\ell_P (f, g)) \leq C_d n^{-2/(d+4)} B^{d/(d+4)} \right\} \geq 1 - C \exp \left( - \frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}} \right)
\]

(61)

whose proof follows from the same argument as the proof of (56). Thus for

\[
t \geq C_d n^{-2/(d+4)} B^{d/(d+4)},
\]

(62)

we get

\[
\mathcal{B}^C_P (\tilde{f}_k, t) \subseteq \mathcal{B}^C_P (\tilde{f}_k, 3t)
\]

with probability at least (59). We deduce consequently that, for a fixed \( t \) satisfying (62), the event:

\[
G_{\tilde{f}_k} (t) \leq \mathcal{G}_{\tilde{f}_k} (3t) := \mathbb{E} \left[ \sup_{g \in \mathcal{B}^C_P (\tilde{f}_k, 3t)} \frac{1}{n} \sum_{i=1}^n \xi_i (g(X_i) - \tilde{f}_k (X_i)) \right] X_1, \ldots, X_n
\]
holds with probability at least \((59)\). It is easy to see that the function
\[
T(x_1, \ldots, x_n) := \mathbb{E} \left[ \sup_{g \in \mathcal{B}_p^{C_B}(\tilde{f}_k, 3t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left( g(X_i) - \tilde{f}_k(X_i) \right) \bigg| X_1 = x_1, \ldots, X_n = x_n \right]
\]
satisfies the bounded differences condition:
\[
|T(x_1, \ldots, x_n) - T(x_1', \ldots, x_n')| \leq \frac{2B\sigma}{n} \sum_{i=1}^{n} I\{x_i \neq x_i'\}
\]
and the bounded differences concentration inequality consequently gives
\[
\mathbb{P} \left\{ \mathcal{G}_{\tilde{f}_k}(3t) \leq \mathbb{E} \mathcal{G}_{\tilde{f}_k}(3t) + x \right\} \geq 1 - \exp \left( -\frac{nx^2}{2B^2\sigma^2} \right) \text{ (63)}
\]
for every \(x > 0\). We next control
\[
\mathbb{E} \mathcal{G}_{\tilde{f}_k}(3t) = \mathbb{E} \left[ \sup_{g \in \mathcal{B}_p^{C_B}(\tilde{f}_k, 3t)} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left( g(X_i) - \tilde{f}_k(X_i) \right) \right]
\]
where the expectation on the left hand side is with respect to \(X_1, \ldots, X_n\) while the expectation on the right hand side is with respect to \(\xi_1, \ldots, \xi_n, X_1, \ldots, X_n\). Clearly
\[
\mathbb{E} \mathcal{G}_{\tilde{f}_k}(3t) = \mathbb{E} \sup_{h \in \mathcal{H}} (Q_n h - Q h) \text{ (64)}
\]
where \(\mathcal{H}\) consists of all functions of the form \((\xi, x) \mapsto \xi \left( g(x) - \tilde{f}_k(x) \right)\) as \(g\) varies over \(\mathcal{B}_p^{C_B}(\tilde{f}_k, 3t)\), \(Q_n\) is the empirical measure corresponding to \((\xi_i, X_i), i = 1, \ldots, n\), and \(Q\) is the distribution of \((\xi, X)\) where \(\xi\) and \(X\) are independent with \(\xi \sim N(0, \sigma^2)\) and \(X \sim \mathbb{P}\).

We now use the bound \((44)\) requires us to control \(N_{\|}(\epsilon, \mathcal{H}, L_2(\mathbb{Q}))\). This is done by Theorem 4.4 which states that
\[
\log N_{\|}(\epsilon, \mathcal{B}_{\mathbb{P}}^{C_B}(\tilde{f}_k, t), L_2(\mathbb{P})) \leq C_d k \left( \log \frac{C_d B}{\epsilon} \right)^{d+1} \left( \frac{t}{\epsilon} \right)^{d/2}. \text{ (65)}
\]
Theorem 4.4 is stated under the unnormalized integral constraint \(\int_{\Omega} (f - \tilde{f}_k)^2 \leq t^2\) and for bracketing numbers under the unnormalized Lebesgue measure but this implies \((65)\) as the volume of \(\Omega\) is assumed to be bounded on both sides by dimensional constants. We now claim that
\[
N_{\|}(\epsilon, \mathcal{H}, L_2(\mathbb{Q})) \leq N_{\|}(\epsilon\sigma^{-1}, \mathcal{B}_{\mathbb{P}}^{C_B}(\tilde{f}_k, 3t), L_2(\mathbb{P})). \text{ (66)}
\]
Inequality (66) is true because of the following. Let \( \{[g_L, g_U], g \in G\} \) be a set of covering brackets for the set \( \mathcal{B}_F^{\mathcal{H}}(f_k, 3t) \). For each bracket \([g_L, g_U]\), we associate a corresponding bracket \([h_L, h_U]\) for \( \mathcal{H} \) as follows:

\[
h_L(\xi, x) := \xi \left( g_L(x) - \tilde{f}_k(x) \right) I\{\xi \geq 0\} + \xi \left( g_U(x) - \tilde{f}_k(x) \right) I\{\xi < 0\}
\]

and

\[
h_U(\xi, x) := \xi \left( g_U(x) - \tilde{f}_k(x) \right) I\{\xi \geq 0\} + \xi \left( g_L(x) - \tilde{f}_k(x) \right) I\{\xi < 0\}.
\]

It is now easy to check that whenever \( g_L \leq g \leq g_U \), we have \( h_L \leq h_g \leq h_U \) where \( h_g(\xi, x) = \xi \left( g(x) - \tilde{f}_k(x) \right) \). Further, \( h_U - h_L = |\xi| (g_U - g_L) \) and thus \( Q(h_U - h_L)^2 = \sigma^2 P \left( g_U - g_L \right)^2 \) which proves (66). Inequality (65) then gives that for every \( a \geq B/\sqrt{n} \), we have

\[
\int_a^B \sqrt{\log N[|\{u, \mathcal{H}, L_2(Q)\}|]} du \leq C_d \sqrt{k} \int_a^B \left( \log \frac{C_d B \sigma}{u} \right)^{(d+1)/2} \left( \frac{t \sigma}{u} \right)^{d/4} du
\]

\[
\leq C_d \sqrt{k} (t \sigma)^{d/4} \left( \log \frac{C_d B \sigma}{a} \right)^{(d+1)/2} \int_a^B u^{-d/4} du
\]

\[
\leq C_d \sqrt{k} (t \sigma)^{d/4} \left( \log \frac{C_d B \sigma}{a} \right)^{(d+1)/2} a^{1-(d/4)}
\]

\[
\leq C_d \sqrt{k} (t \sigma)^{d/4} \left( \log(C_d \sigma \sqrt{n}) \right)^{(d+1)/2} a^{1-(d/4)}
\]

where, in the last inequality, we used \( a \geq B/\sqrt{n} \). The inequality \( a \geq C n^{-1/2} \int_a^B \sqrt{\log N[|\{u, \mathcal{H}, L_2(Q)\}|]} du \) will therefore be satisfied for

\[
a \geq C_d t \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d}
\]

for an appropriate constant \( C_d \). The bound (44) then gives

\[
\mathbb{E} \mathcal{G}_{f_k}(3t) \leq C_d t \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d} + \frac{B}{\sqrt{n}}.
\]

Assuming now that

\[
t \geq \frac{B k^{-2/d}}{\sigma} n^{(4-d)/(2d)},
\]

we deduce

\[
\mathbb{E} \mathcal{G}_{f_k}(3t) \leq (1 + C_d) t \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d}
\]

Putting the above steps together, we obtain that for every \( x > 0 \), the inequality

\[
G_{f_k}(t) \leq \mathcal{G}_{f_k}(3t) \leq \mathbb{E} \mathcal{G}_{f_k}(t) + x \leq C_d t \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d} + x
\]
holds with probability at least
\[
1 - C \exp \left(- \frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}}\right) - \exp \left(\frac{-nx_0^2(t)}{2B^2\sigma^2}\right)
\]
for every fixed \( t \) satisfying (62) and (67). We take
\[
x = x_0(t) := C_d t \sigma \left(\frac{k}{n} \right)^{2/d} \left(\log(C_d \sigma \sqrt{n})\right)^{2(d+1)/d}
\]
to deduce that
\[
H_{\tilde{f}_k}(t) \leq G_{\tilde{f}_k}(t) \leq C_d t \sigma \left(\frac{k}{n} \right)^{2/d} \left(\log(C_d \sigma \sqrt{n})\right)^{2(d+1)/d}
\]
with probability at least
\[
1 - C \exp \left(- \frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}}\right) - \exp \left(\frac{-nx_0^2(t)}{2B^2\sigma^2}\right)
\]
provided \( t \) satisfies (62) and (67).

We shall next prove a lower bound for \( H_{\tilde{f}_k}(t) \). The key ingredients here are Lemma 8.2 and the conditional form (49) of Sudakov’s minoration. Let us first prove lower bounds for \( G_{\tilde{f}_k}(t) \). Note that
\[
B \subseteq B \subseteq \mathbb{P}_n(f_0, t/2)
\]
whenever \( \ell_{\mathbb{P}_n}(f_0, \tilde{f}_k) \leq t/2 \).
Because \( \ell_{\mathbb{P}_n}(f_0, \tilde{f}_k) \leq \sup_{x \in \Omega} |f_0(x) - \tilde{f}_k(x)| \leq L_d k^{-2/d} \), the condition \( \ell_{\mathbb{P}_n}(f_0, \tilde{f}_k) \leq t/2 \) will be satisfied for
\[
t \geq 2L_d k^{-2/d}.
\]
Thus for \( t \) satisfying the above,
\[
G_{\tilde{f}_k}(t) \geq \mathbb{E} \left[ \sup_{g \in \mathcal{B}_{\mathbb{P}_n}(f_0, t/2)} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left(g(X_i) - \tilde{f}_k(X_i)\right) \left| X_1, \ldots, X_n\right\} X_1, \ldots, X_n \right]
\]
\[
= \mathbb{E} \left[ \sup_{g \in \mathcal{B}_{\mathbb{P}_n}(f_0, t/2)} \frac{1}{n} \sum_{i=1}^{n} \xi_i \left(g(X_i) - f_0(X_i)\right) \left| X_1, \ldots, X_n\right\} X_1, \ldots, X_n \right].
\]
Inequality (49) then gives
\[
G_{\tilde{f}_k}(t) \geq \frac{\beta \sigma}{\sqrt{n}} \sup_{\epsilon > 0} \left\{ \epsilon \sqrt{\log N(\epsilon, \mathcal{B}_{\mathbb{P}_n}(f_0, t/2), \mathbb{P}_n)} \right\}
\]
We now use Lemma 8.2 with \( \epsilon = c_3 n^{-2/d} \) to claim that for \( B \geq C \) and
\[
t \geq 16c_3 n^{-2/d},
\]
we have
\[ G_{\tilde{f}_k}(t) \geq \beta \sqrt{c_1 \sigma c_3^{1-(d/4)}} n^{-2/d} \]
with probability at least \( 1 - \exp(-c_2 n) \). This gives the following lower bound on \( H_{\tilde{f}_k}(t) \):
\[ H_{\tilde{f}_k}(t) \geq \beta \sqrt{c_1 \sigma c_3^{1-(d/4)}} n^{-2/d} - \frac{t^2}{2}. \]
Taking \( t = t_0 \) where
\[ t_0^2 = \beta \sqrt{c_1 \sigma c_3^{1-(d/4)}} n^{-2/d} \]
gives us that
\[ H_{\tilde{f}_k}(t_0) \geq \frac{t_0^2}{2} = \frac{\beta}{2} \sqrt{c_1 \sigma c_3^{1-(d/4)}} n^{-2/d} \]
with probability at least \( 1 - \exp(-c_2 n) \) provided \( t = t_0 \) satisfies (71) and (72). The condition (71) is equivalent to
\[ k \geq \left( \frac{4L_2^2}{\beta \sqrt{c_1 \sigma c_3^{1-(d/4)}}} \right)^{d/4} \sqrt{n} \sigma^{-d/4} \]
and (72) is equivalent to
\[ n \geq \left( \frac{256 c_3^2}{\beta \sqrt{c_1 \sigma c_3^{1-(d/4)}}} \right)^{d/2} \sigma^{-d/2}. \]

We shall now combine (69) and (74). Suppose \( t_1 > 0 \) is such that
\[ C_d k \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d} = \frac{\beta}{2} \sqrt{c_1 \sigma c_3^{1-(d/4)}} n^{-2/d} \]
where \( C_d \) is as in (69) and the other constants \( (\beta, c_1 \) and \( c_3 \)) are from (74). The above equality is the same as
\[ t_1 = \frac{\beta \sqrt{c_1 c_3^{1-(d/4)}}}{2C_d} k^{-2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{-2(d+1)/d}. \]
In that case, (69) and (74) together imply that
\[ H_{\tilde{f}_k}(t_1) \leq H_{\tilde{f}_k}(t_0) \]
with probability at least
\[ 1 - C \exp \left( -\frac{n^{d/(d+4)}}{C_d B^{8/(d+4)}} \right) - \exp \left( \frac{-n x_0^2(t_1)}{2 B^2 \sigma^2} \right) - \exp(-c_2 n). \]
If we now assume that
\[ k \geq (2C_d)^{-d/2} \left( \beta \sqrt{c_1} c_3^{1-(d/4)} \right)^{d/4} \sqrt{n} \sigma^{-d/4}, \] (78)
then \( t_1 < t_0 \). Inequality (32) then gives that
\[ t_{\hat{f}} \geq t_1 = \frac{\beta \sqrt{c_1} c_3^{1-(d/4)}}{2C_d} k^{-2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{-2(d+1)/d}. \]
with probability at least (77).

We shall now take \( k = \gamma_d \sqrt{n} \sigma^{-d/4} \) where \( \gamma_d \) is the larger of the two dimensional constants on the right hand sides of (75) and (78) and this will obviously ensure that both (75) and (78) are satisfied. The quantity \( t_1 \) then equals
\[ t_1 = c_d n^{-1/d} \sqrt{\sigma} \left( \log(C_d n \sigma^{1-(d/4)}) \right)^{-2(d+1)/d} \] (79)
for a specific \( c_d \) and
\[ x_0(t_1) = C_d c_d^{2/d} \gamma_d \sigma n^{-2/d}. \]
The probability in (77) then equals
\[ 1 - C \exp \left( -\frac{n^{d/(d+4)}}{C_d B^{s/(d+4)}} \right) - \exp \left( -\frac{-C_d^2 c_d^2 (\gamma_d)^{4/d} n^{(d-4)/d}}{2B^2} \right) - \exp(-c_2 n). \]
Now if we assume that \( B \geq 1 \), then
\[ \frac{n^{d/(d+4)}}{C_d B^{s/(d+4)}} \geq \frac{n^{d/(d+4)}}{C_d B^2} \geq \frac{n^{(d-4)/d}}{B^2} \]
and also \( n \geq n^{(d-4)/d}/B^2 \). We thus deduce that the probability in (77) is bounded from below by
\[ 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2 B^2} \right). \]
which can be further simplified to
\[ 1 - C_d \exp \left( -\frac{n^{(d-4)/d}}{C_d^2} \right). \]
as \( B \) is a constant that only depends on \( d \). If we now take \( n \) to be larger than a constant \( N_{d,\sigma} \) depending on \( d \) and \( \sigma \) alone, then the conditions (62) and (71) will be satisfied for \( t = t_1 \) and (76) will also be satisfied. Finally the logarithmic term in (79) can be further simplified by the bound \( \log(C_d n \sigma^{1-(d/4)}) \leq c_d \log n \). This completes the proof of (51) and consequently Theorem 3.1.
8.2. Proof of Theorem 3.2

Proof of Theorem 3.2. It is enough to prove (21) when \( L \) is a fixed dimensional constant. From here, the inequality for arbitrary \( L > 0 \) follows by a scaling argument.

Let \( f_0(x) := \|x\|^2 \) and \( \hat{f} \) be as given by Lemma 8.3. Let \( L \) be a dimensional constant large enough so that \( \hat{f} \in \mathcal{C}_L^4(\Omega) \). As in the proof of Theorem 3.1, the key is to prove (51) where \( t_{\hat{f}} \) is defined as the maximizer of

\[
H_{\hat{f}}(t) := G_{\hat{f}}(t) - \frac{t^2}{2}
\]

over all \( t \geq 0 \) and

\[
G_{\hat{f}}(t) := \mathbb{E} \left[ \sup_{g \in \mathcal{C}_L^4(\Omega) \setminus \{\hat{f}\}} \frac{1}{n} \sum_{i=1}^n \xi_i \left( g(X_i) - \hat{f}(X_i) \right) \right] |X_1, \ldots, X_n|.
\]

Before proceeding with the proof of (51), let us first show how (51) completes the proof of Theorem 3.2. Because \( \hat{f} \in \mathcal{C}_L^4(\Omega) \), it is enough to prove that the right hand side of (21) is a lower bound for \( \mathbb{E}_{\hat{f}} \ell_n^2(\hat{f}_n(C_L(\Omega)), \hat{f}) \). We shall assume therefore that the data have been generated from the true function \( \hat{f} \). Note first that, as shown in the proof of inequality (53) in Theorem 3.1, inequality (51) leads to

\[
\mathbb{P} \left\{ \ell_n^2(\hat{f}_n(C_L(\Omega)), \hat{f}) \geq \frac{1}{2} \rho_n^2 \right\} \geq 1 - C_d \exp \left( \frac{-n(d-4)/d}{C_d^2} \right)
\]

where \( \rho_n \) is given by (52) and \( n \geq N_{d,\sigma} \) for a large enough constant \( N_{d,\sigma} \) depending only on \( d \) and \( \sigma \). A similar inequality will be shown below for \( \ell_n^2(\hat{f}_n(C_L(\Omega)), \hat{f}) \). For every \( a > 0 \), we write

\[
\mathbb{P} \left\{ \ell_n(\hat{f}_n(C_L(\Omega)), \hat{f}) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right) ; \hat{f}_n(C_L(\Omega)) \in \mathcal{C}_L^{4L}(\Omega) \right\}
\]

\[
= \mathbb{P} \left\{ \ell_n(\hat{f}_n(C_L^{4L}(\Omega)), \hat{f}) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right) ; \hat{f}_n(C_L(\Omega)) \in \mathcal{C}_L^{4L}(\Omega) \right\}
\]

\[
\geq \mathbb{P} \left\{ \ell_n^2(\hat{f}_n(C_L^{4L}(\Omega)), \hat{f}) \geq \frac{\rho_n^2}{2} , \sup_{f,g \in \mathcal{C}_L^{4L}(\Omega)} (\ell_n(f,g) - 2\ell_n(f,g)) \leq a , \hat{f}_n(C_L(\Omega)) \in \mathcal{C}_L^{4L}(\Omega) \right\}
\]

\[
\geq \mathbb{P} \left\{ \ell_n^2(\hat{f}_n(C_L^{4L}(\Omega)), \hat{f}) \geq \frac{\rho_n^2}{2} , \hat{f}_n(C_L(\Omega)) \in \mathcal{C}_L^{4L}(\Omega) \right\} - 1
\]

\[
+ \mathbb{P} \left\{ \sup_{f,g \in \mathcal{C}_L^{4L}(\Omega)} (\ell_n(f,g) - 2\ell_n(f,g)) \leq a \right\}.
\]

We now bound the probability:

\[
\mathbb{P} \left\{ \ell_n(\hat{f}_n(C_L(\Omega)), \hat{f}) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right) ; \hat{f}_n(C_L(\Omega)) \not\in \mathcal{C}_L^{4L}(\Omega) \right\}
\]
For this, we first make the following claim:

\[ f \in C_L(\Omega), f \not\in C_{LL}^4(\Omega) \implies \min \left( \ell_{\mathbb{P}_n}(f, \tilde{f}_k), \ell_{\mathbb{P}}(f, \tilde{f}_k) \right) > L. \]  

(82)

To see (82), note that assumptions \( f \in C_L(\Omega) \) and \( f \not\in C_{LL}^4(\Omega) \) together imply that \( f(x) > 4L \) for some \( x \in \Omega \). By the Lipschitz property of \( f \), the fact that \( \Omega \) has diameter \( \leq 2 \) and the fact that \( \tilde{f}_k \) is bounded by \( L \), we have

\[ f(y) - \tilde{f}_k(y) \geq f(x) - L\|y - x\| - L > L \quad \text{for all } y \in \Omega \]

which clearly implies that both \( \ell_{\mathbb{P}_n}(f, \tilde{f}_k) \) and \( \ell_{\mathbb{P}}(f, \tilde{f}_k) \) are larger than \( L \). This proves (82).

Assume now that \( N_{d,\sigma} \) is large enough so that \( \rho_n \) is at most \( L \) for \( n \geq N_{d,\sigma} \). The fact (82) clearly implies that

\[
\mathbb{P} \left\{ \ell_{\mathbb{P}}(\hat{f}_n(C_L(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right), \hat{f}_n(C_L(\Omega)) \not\in C_{LL}^4(\Omega) \right\}
\]

\[
= \mathbb{P} \{ \hat{f}_n(C_L(\Omega)) \not\in C_{LL}^4(\Omega) \} = \mathbb{P} \{ \ell_{\mathbb{P}_n}(\hat{f}_n(C_L(\Omega)), \tilde{f}_k) \geq \frac{\rho_n^2}{2}, \hat{f}_n(C_L(\Omega)) \not\in C_{LL}^4(\Omega) \}
\]

Combining the above with (81), we get

\[
\mathbb{P} \left\{ \ell_{\mathbb{P}}(\hat{f}_n(C_L(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \left( \frac{\rho_n}{\sqrt{2}} - a \right) \right\}
\]

\[
\geq \mathbb{P} \left\{ \ell_{\mathbb{P}_n}^2(\hat{f}_n(C_L(\Omega)), \tilde{f}_k) \geq \frac{1}{2} \rho_n^2 \right\} + \mathbb{P} \left\{ \sup_{f,g \in C_{LL}^4(\Omega)} (\ell_{\mathbb{P}_n}(f, g) - 2\ell_{\mathbb{P}}(f, g)) \leq a \right\} - 1.
\]

This inequality is analogous to inequality (54) in the proof of Theorem 3.1. From here, one can deduce

\[
\mathbb{E}_{\hat{f}_k} \| \hat{f}_n(C_L(\Omega)), \tilde{f}_k \| \geq \frac{\sigma^2}{4} \rho_n^{-2/d}(\log n)^{-4(d+1)/d}.
\]

(83)

in the same way that (57) was derived from (54). The only difference is that, instead of (55), we now use the following result due to Bronštein [9]:

\[
\log N(||\epsilon, C_{LL}^4(\Omega), \ell_{\mathbb{P}}) \leq \log N(\epsilon, C_{LL}^4(\Omega), \ell_{\infty}) \leq Cd \left( \frac{L}{\epsilon} \right)^{d/2}
\]

(84)

where, of course, \( \ell_{\infty} \) refers to the metric \((f, g) \mapsto \sup_{x \in \Omega} |f(x) - g(x)| \) (recall that \( \ell_{\infty} \) covering numbers dominate bracketing numbers with respect to \( L^2(P) \) for every probability measure \( P \)). (83) clearly completes the proof of Theorem 3.2.

Let us now provide the proof of (51). This was proved in Theorem 3.1 on the basis of the inequalities (69) and (74). Below we shall establish (69) and (74) in the present setting with slight modification. From these, (51) will follow via the same argument.
used in Theorem 3.1. Note that the difference between the current proof and the proof of Theorem 3.1 is that \(G_{\tilde{f}}(t)\) is now defined as in (80) involving a supremum of \(g \in B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t)\) while, in the proof of Theorem 3.1, \(G_{\tilde{f}}(t)\) was defined as in (50) involving a supremum of \(g \in B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t)\).

Let us start with the proof of (69). For this, note first that (82) immediately implies
\[
B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t) = B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t) \quad \text{for all } t \leq L.
\]
Let us assume therefore that \(t \leq L\) so we can work with the class of bounded Lipschitz convex functions \(C_{L}^{c_{L}}(\Omega)\). We write
\[
G_{\tilde{f}}(t) \leq G_{\tilde{f}}^{I}(t) + G_{\tilde{f}}^{II}(t)
\]
where
\[
G_{\tilde{f}}^{I}(t) := \mathbb{E} \left[ \sup_{g \in B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t)} \frac{1}{n} \sum_{i=1}^{n} I\{X_i \in \bigcup_{i=1}^{m} \Delta_i\} \xi_i \left( g(X_i) - \tilde{f}(X_i) \right) \mid X_1, \ldots, X_n \right]
\]
and
\[
G_{\tilde{f}}^{II}(t) := \mathbb{E} \left[ \sup_{g \in B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t)} \frac{1}{n} \sum_{i:X_i \notin \bigcup_{i=1}^{m} \Delta_i} \xi_i \left( g(X_i) - \tilde{f}(X_i) \right) \mid X_1, \ldots, X_n \right].
\]
Here \(\Delta_1, \ldots, \Delta_m\) are the \(d\)-simplices given by Lemma 8.3. We shall provide upper bounds for both \(G_{\tilde{f}}^{I}(t)\) and \(G_{\tilde{f}}^{II}(t)\). The bound for \(G_{\tilde{f}}^{I}(t)\) is very similar to the bound (69) obtained for \(G_{\tilde{f}}(t)\) in the proof of Theorem 3.1 with the following two differences. Instead of the metric entropy bound (55), we use the result (84) due to Bronštein [9]. Inequality (84) allows us to replace \(B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, t)\) by \(B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, 3t)\) with high probability. Also, instead of (65), we shall use (which also follows from Theorem 4.4)
\[
N_{[\varepsilon]}(\{x \mapsto g(x)I\{x \in \bigcup_{i=1}^{m} \Delta_i\} : g \in B_{\mathbb{F}_n}^{c_{L}(\Omega)}(\tilde{f}_k, 3t)\}, L_2(\mathbb{P})) \leq C_d k \left( \log \frac{C_d L}{\varepsilon} \right)^{d+1} \left( \frac{t}{\varepsilon} \right)^{d/2}.
\]
With these changes, following the proof of inequality (69) in Theorem 3.1 with \(B\) replaced by \(4L\) allows us to deduce that
\[
G_{\tilde{f}}^{I}(t) \leq C_d \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{2(d+1)/d}
\]
with probability at least
\[
1 - C \exp \left( - \frac{n^{d/(d+4)}}{C_d L^8/(d+4)} \right) - \exp \left( - \frac{-nt^2}{C_d L^2} \left( \frac{k}{n} \right)^{4/d} \left( \log(C_d \sigma \sqrt{n}) \right)^{4(d+1)/d} \right).
\]
for every $0 < t \leq L$ satisfying
\[
t \geq C_d n^{-2/(d+4)} L^{d/(d+4)} \quad \text{and} \quad t \geq \frac{4L k^{-2/d}}{\sigma} n^{(4-d)/(2d)}.
\] (86)

We shall now bound $G_{\bar{f}_k}^{II}(t)$. For this, let $
 \bar{n} := \sum_{i=1}^{m} I\{X_i \notin \cup_{i=1}^{m} \Delta_i\}$ and use Dudley’s bound (Theorem 7.2) and (84) to write

\[
G_{\bar{f}_k}^{II}(t) = \frac{\bar{n}}{n} \mathbb{E} \left[ \sup_{g \in \mathcal{A}^{dL}(\bar{f}_k,t)} \frac{1}{\bar{n}} \sum_{i: X_i \notin \cup_{i=1}^{m} \Delta_i} \xi_i \left( g(X_i) - \bar{f}_k(X_i) \right) \bigg| X_1, \ldots, X_n \right]
\]

\[
\leq \frac{\sigma \bar{n}}{n} \inf_{\delta > 0} \frac{12}{\sqrt{n}} \int_{\delta}^{\infty} \sqrt{\log N(\epsilon, C_{4L}^4(\Omega), \ell_{\infty})} d\epsilon + 2\delta
\]

\[
\leq C_d \frac{\sigma \bar{n}}{n} \inf_{\delta > 0} \left( \frac{1}{\sqrt{n}} \int_{\delta}^{\infty} \left( \frac{L}{\epsilon} \right)^{d/4} d\epsilon + \delta \right).
\]

The choice $\delta = L(\bar{n})^{-2/d}$ then gives

\[
G_{\bar{f}_k}^{II}(t) \leq C_d L \frac{\sigma}{n} (\bar{n})^{1-(2/d)}.
\] (87)

$\bar{n}$ is binomially distributed with parameters $n$ and \( \tilde{p} := \text{Vol}(\Omega \setminus (\cup_{i=1}^{m} \Delta_i))/\text{Vol}(\Omega) \). Because $(1 - C_d k^{-2/d}) \Omega \subseteq \cup_{i=1}^{m} \Delta_i \subseteq \Omega$ (see Lemma 8.3), we have

\[
\tilde{p} \leq 1 - (1 - C_d k^{-1/d})^d \leq d C_d k^{-1/d}
\]

because $(1 - u)^d \geq 1 - du$. Hoeffding’s inequality:

\[
\mathbb{P} \{ \text{Bin}(n, \tilde{p}) \leq np + u \} \geq 1 - \exp \left( \frac{-u^2}{2n} \right) \quad \text{for every } u \geq 0
\]

gives (below $C_d$ is such that $\tilde{p} \leq C_d k^{-1/d}$)

\[
\mathbb{P} \{ \bar{n} \leq 2 C_d n k^{-1/d} \} \geq \mathbb{P} \{ \bar{n} - n \tilde{p} \leq C_d n k^{-1/d} \} \geq 1 - \exp \left( -\frac{C_d^2}{2} n k^{-2/d} \right).
\]

Combining the above with (87), we get that

\[
G_{\bar{f}_k}^{II}(t) \leq C_d L \sigma n^{-2/d} k^{-1/d} k^{2/d^2}
\]

with probability at least $1 - \exp(-C_d n k^{-2/d})$. Combining this bound with the bound (85) obtained for $G_{\bar{f}_k}^{I}(t)$, we get (below $H_{\bar{f}_k}(t) := G_{\bar{f}_k}(t) - t^2/2$)

\[
H_{\bar{f}_k}(t) \leq G_{\bar{f}_k}(t) \leq C_d t \sigma \left( \frac{k}{n} \right)^{2/d} \left( \log(C_d \sqrt{n}) \right)^{(2(d+1))/d} + C_d L \sigma n^{-2/d} k^{-1/d} k^{2/d^2}
\] (88)
with probability at least
\[
1 - C \exp \left( - \frac{n^{d+1}}{C_d L^{d+2}} \right) - \exp \left( - \frac{nL^2}{C_d L^2} \left( \frac{k}{n} \right)^{d/2} \left( \log(C_d \sqrt{n}) \right)^{4(d+1)/d} \right) - \exp(-C_dn k^{-1/2})
\]
for every \(0 \leq t \leq L\) satisfying (86). Under the condition
\[
t \left( \log(C_d \sqrt{n}) \right)^{4(d+1)/d} \geq Lk^{2/d} k^{-3/d},
\]
the second term on the right hand side of (88) is dominated by the first term leading to inequality (69).

The next step is to prove a lower bound for \(H_{\tilde{f}}(t)\). Here the same argument as in the proof of Theorem 3.1 applies and we can deduce that inequality (74) holds with probability at least \(1 - \exp(-c_2n)\) provided the conditions (75) and (76) are satisfied (note that \(t_0\) in (74) is given by (73)).

We have thus proved inequalities (69) and (74). From these, we can follow the same argument as in that proof to deduce (51). The following additional constraint (which was not present in the proof of Theorem 3.1) needs to be checked here:
\[
Lk^{2/d} k^{-3/d} \left( \log(ek\sqrt{n}) \right)^{-4(d+1)/d} \leq t_1 \leq L
\]
where \(t_1\) is defined in (79) and \(k = \gamma_d \sqrt{n} \sigma^{-d/4}\) (for a large enough \(\gamma_d\)). This holds as long as \(n\) is larger than a constant \(N_{d,\sigma}\) depending on \(d\) and \(\sigma\) alone (note that \(L\) is a constant depending on \(d\) alone). Note also that the probability (89) has an additional term compared to (70) but this additional term \(\exp(-C_dn k^{-2/d})\) is easily seen to be bounded by \(\exp(-n^{(d-4)/d}/C_d^2)\) for \(k = \gamma_d \sqrt{n} \sigma^{-d/4}\) provided \(n \geq N_{d,\sigma}\) for a large enough constant \(N_{d,\sigma}\). The proof of (51) is thus complete. \(\square\)

9. Proofs of Metric Entropy Results

9.1. Proof of Theorem 4.4

First, we state the key Lemma, and prove it later (in Subsection 9.2).

Lemma 9.1. If \(\Omega\) is a \(d\)-dimensional convex body defined by \(\Omega = \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i, 1 \leq i \leq d+1 \} \), where \(v_i\) are fixed unit vectors. Then for any \(0 < \varepsilon < 1\), there exists a set \(G\) consisting of no more than \(\exp(C_3|\Omega|^{d/2p}[\log(1/\varepsilon)]^{(d+1)(t/\varepsilon)^{d/2}})\) brackets such that for every
\[
f \in B_p^\Gamma(\Omega, t) := \{ f \text{ is convex in } \Omega, \|f\|_p \leq t, \|f\|_\infty \leq \Gamma \},
\]
there exists a bracket \([g, h] \in G\) such that \(g(x) \leq f(x) \leq h(x)\) for all \(x \in \Omega\), and
\[
\int_\Omega |h(x) - g(x)|^p dx < \varepsilon^p.
\]
Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. Assume $\Omega = \bigcup_{i=1}^{m} \Delta_i$, where $\Delta_i$, $1 \leq i \leq m$ are $d$-simplices. For each $f \in B^p_g(f_0, \Omega, t)$, we define $t_i(f)$ as the smallest positive integer $t_i$ such that

$$\int_{\Delta_i} |f(x) - f_0(x)|^p dx \leq t_i^p \Delta_i.$$

Because $|f - f_0| \leq 2\Gamma$, we have $t_i \leq 2\Gamma/t$. Thus, there are no more than $(2\Gamma/t)^m$ choices of the sequence $t_1, t_2, \ldots, t_m$. For every such sequence $T = \{t_1, t_2, \ldots, t_m\}$, we define

$$\mathcal{F}_T = \left\{ f \in B^p_g(f_0, \Omega, t) : (t_i - 1)^p \Delta_i \leq \int_{\Delta_i} |f(x) - f_0(x)|^p dx \leq t_i^p \Delta_i, 1 \leq i \leq m \right\}.$$

Thus, for every $f \in \mathcal{F}_T$, we have

$$\sum_{i=1}^{m} (t_i - 1)^p \Delta_i \leq \sum_{i=1}^{m} \int_{\Delta_i} |f(x) - f_0(x)|^p dx \leq t^p,$$

i.e. $\sum_{i=1}^{m} (t_i - 1)^p \Delta_i \leq 1$. Hence,

$$\sum_{i=1}^{m} t_i^p \Delta_i \leq 2^{p-1} \sum_{i=1}^{m} [(t_i - 1)^p + 1] \Delta_i \leq 2^p.$$

Furthermore, for each $f \in \mathcal{F}_T$ and $1 \leq i \leq m$, the restriction of $f - f_0$ to $\Delta_i$ belongs to $B^p_{2\Gamma} (\Delta_i, t_i \Delta_i^{1/p})$ (since $f_0$ is linear on each $\Delta_i$). Since each simplex can be written as an intersection of $d + 1$ slabs, by Lemma 9.1, there exists a set $\mathcal{G}_i$ consisting of no more than $\exp(C(d, p) [\log(2\Gamma/\varepsilon_i)]^{d+1} t_i^{d/2} [\Delta_i]^{d/2} \varepsilon_i^{-d/2})$ brackets, such that for each $f \in \mathcal{F}_T$, there exists a bracket $[g_i, h_i] \in \mathcal{G}_i$ such that $g_i(x) + f_0(x) \leq f(x) \leq h_i(x) + f_0(x)$ for all $x \in \Delta_i$, and $\int_{\Delta_i} |h_i(x) - g_i(x)|^p dx \leq \varepsilon_i^p$. If we define $g(x) = g_i(x)$ and $h(x) = h_i(x)$ for $x \in \Delta_i$, $1 \leq i \leq m$. Then, we clearly have $g(x) \leq f(x) \leq h(x)$ for all $x \in \Omega$, and

$$\int_{[0,1]^d} |h(x) - g(x)|^p dx = \sum_{i=1}^{m} \int_{\Delta_i} |h_i(x) - g_i(x)|^p dx \leq \sum_{i=1}^{m} \varepsilon_i^p.$$

We choose

$$\varepsilon_i = \max(2^{-1-2/p} t_i |\Delta_i|^{1/p}, (4m)^{-1/p}) \cdot \varepsilon,$$

where we used the fact that if $|f - f_0| \leq 2\Gamma$ then $\int_{\Omega_{\infty}} |f(x) - f_0(x)|^p \leq \varepsilon^p/2$.

Then

$$\sum_{i=1}^{m} \varepsilon_i^p \leq \left( \frac{\varepsilon^p}{2^{p+2}} \sum_{i=1}^{m} t_i^p |\Delta_i| + \frac{1}{4} \varepsilon^p \right) \leq \frac{\varepsilon^p}{4}.$$

Thus, $[g, h]$ is an $\varepsilon$-bracket.
Note that for each fixed $T$, the total number of brackets $[g, h]$ is at most

$$N := \prod_{i=1}^{m} \exp(C(d, p)[\log(\Gamma/\varepsilon_i)]^{d+1}t^{d/2}|\Delta_i|^{d/2p}t^{d/2} \varepsilon_i^{-d/2})$$

$$\leq \prod_{i=1}^{m} \exp(C(d, p)[\frac{1}{p}\log(4m) + \log \Gamma + \log(1/\varepsilon)]^{d+1}(2^{1+2/p}t/\varepsilon)^{d/2})$$

$$\leq \exp \left(C'(d, p)m[\log m + \log \Gamma + \log(1/\varepsilon)]^{d+1}(t/\varepsilon)^{d/2}\right)$$

These with all the possible choices of $T$, the number of realizations of the brackets $[g, h]$ is at most

$$(2\Gamma/t)^m \cdot N \leq \exp \left(C''(d, p)m[\log m + \log \Gamma + \log(1/\varepsilon)]^{d+1}(t/\varepsilon)^{d/2}\right),$$

and the claim follows.

\[ \square \]

9.2. Proof of Lemma 9.1

Our starting point is the following results proved in Lemma 5 and Theorem 1(ii) of [20] respectively:

Proposition 9.2. If $\Omega$ is a convex body in $[0,1]^d$ with volume $|\Omega| \geq 1/d!$, then for any $0 < \delta < 1$, there exists a constant $\Lambda$ depending only on $d$, $p$ and $\delta$, such that $C_p(\Omega) \subset C_\infty(\Omega_\delta, \Lambda)$, where $\Omega_\eta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \eta \}$ and

$$C_p(\Omega) := \{ f \text{ is convex in } \Omega, \|f\|_p \leq 1 \}.$$

Proposition 9.3. If $\Omega$ is a convex body that can be triangulated into $m$ simplices of dimension $d$, then there exists a constant $C$ depending only on $d$ and $p$ such that for all $0 < \varepsilon < 1$, we have

$$\log N[|\varepsilon, C_\infty(\Omega), \| \cdot \|_p] \leq Cm|\Omega|^{d/2p} \varepsilon^{-d/2},$$

where $\|f\|_p = (\int_\Omega |f(x)|^p \, dx)^{1/p}$.

Using the last two propositions:

Corollary 9.4. Let $\Omega \subset [0,1]^d$ is a $d$-dimensional convex body which is defined by

$$\Omega = \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i, 1 \leq i \leq m \},$$

where $v_i$ are fixed unit vectors, $m \geq d + 1$. Then for any $0 < \eta < 1/5, t > 0$ and any $0 < \varepsilon < t$, the following holds:

$$\log N[|\varepsilon, C_p(\Omega, t), \| \cdot \|_{L_p(\Omega_0)}] \leq C_1m(t/\varepsilon)^{-d/2},$$

where $C_p(\Omega, t) := \{ f \text{ is convex in } \Omega, \|f\|_p \leq t \}$, $C_1$ is a constant depending only on $d$, $p$ and $\eta$, and

$$\Omega_0 := \{ x \in \mathbb{R}^d : a_i + \eta(b_i - a_i) \leq v_i^T x \leq b_i - \eta(b_i - a_i), 1 \leq i \leq m \}.$$
Proof. Observe that on $\Omega_0 \|f\|_\infty \leq C(\eta, d)\frac{t}{\log(\frac{\Gamma}{\varepsilon})}$. To see this, assume in contradiction that it is not the case. Using similar arguments to the proof of Proposition 9.2, there is a set of volume of $c(\eta, d)|\Omega|$ (a "cap/corner") such that $f \geq \frac{t}{\log(\frac{\Gamma}{\varepsilon})}$ which will contradict the definition of $C_p(\Omega, t)$. Then, using the Proposition 9.3 scaled by $\frac{t}{\log(\frac{\Gamma}{\varepsilon})}$, gives the corollary.

We will prove that if we replace $C_p(\Omega, t)$ by

$$B_p^r(\Omega, t) = \{ f \in C_p(\Omega, t) : \|f\|_\infty \leq \Gamma \},$$

then we can replace $\Omega_0$ by $\Omega$ at a cost of logarithmic factors in the rate of bracketing entropy.

For any domain $\mathcal{D}$ of an intersection of $d + 1$ "slabs" ($\mathcal{D} := \{ x \in \mathbb{R}^d : a_i \leq v_i^T x \leq b_i, 1 \leq i \leq d + 1 \}$, where $v_i$ are fixed unit vectors) and for every $0 \leq r \leq d + 1$ we define the operator $T_r$.

$$T_r(\mathcal{D}) = \{ x \in \mathcal{D} : a_i \leq v_i^T x \leq b_i \text{ for } i \leq r, a_j + \eta(b_j - a_j) \leq v_j^T x \leq b_j - \eta(b_j - a_j) \text{ for } r < j \leq d+1 \}.$$

Now, we are ready to prove the lemma.

Proof of Lemma 9.1. Because the desired inequality is invariant under affine transformation, we can assume $\Omega$ is contained in $[0, 1]^d$ and $|\Omega| \geq \frac{1}{d!}$. Fix $0 < \eta < 1/5$. We prove the following: There exist two constants $C_1(d, p)$ and $C_2(d, p)$ such that for all $r = 1, \ldots, d + 1$ the following holds:

$$\log N_{\|\cdot\|_{L^p(T_r(\Omega))}}(\varepsilon) \leq C_1[C_2(\log(\frac{\Gamma}{\varepsilon}))^r|\Omega|^{d/2p}d^d \varepsilon^{-d/2}].$$

We prove the statement by induction on $r$. Clearly,

$$T_0(\Omega) = \Omega_0 := \{ x \in \mathbb{R}^d : a_j + \eta(b_j - a_j) \leq v_j^T x \leq b_j - \eta(b_j - a_j) \text{ for } 1 \leq j \leq d + 1 \}.$$

By Corollary 9.4 and the assumptions on $\Omega$ the statement is true when $r = 0$. Suppose the statement is true for $r = k - 1$. We define

$$K_0 = T_{k-1}(\Omega) = \{ x \in T_k(\Omega) : a_k + \eta(b_k - a_k) \leq v_k^T x \leq b_k - \eta(b_k - a_k) \}. $$

For $s = 1, 2, \ldots, m$ define

$$K_{2s+1} = \{ x \in T_k(\Omega) : a_k + 2^{-s-1}\eta(b_k - a_k) \leq v_k^T x < a_k + 2^{-s}\eta(b_k - a_k) \},$$

$$K_{2s+2} = \{ x \in T_k(\Omega) : b_k - 2^{-s}\eta(b_k - a_k) < v_k^T x \leq b_k - 2^{-s-1}\eta(b_k - a_k) \},$$

Furthermore, we define

$$K_L = \{ x \in T_k(\Omega) : a_k \leq v_k^T x < a_k + 2^{-m-1}\eta(b_k - a_k) \},$$

$$K_R = \{ x \in T_k(\Omega) : b_k - 2^{-m-1}\eta(b_k - a_k) < v_k^T x \leq b_k \}.$$
Then, $K_0$, $K_L$, $K_R$, $K_{2s+1}$ and $K_{2s+2}$, $1 \leq s \leq m$ form a partition of $T_k(\Omega)$. Now, we aim to use the induction step. For this purpose we denote the inflated sets

\[ \hat{K}_{2s+1} = \{ x \in \Omega : a_k + 2^{-s-2} \eta(b_k - a_k) \leq v_k^T x < a_k + 3 \cdot 2^{-s-1} \eta(b_k - a_k) \}, \]

\[ \hat{K}_{2s+2} = \{ x \in \Omega : b_k - 3 \cdot 2^{-s-1} \eta(b_k - a_k) < v_k^T x \leq b_k - 2^{-s-2} \eta(b_k - a_k) \}. \]

Now, we will apply the operator $T_{k-1}$ on these sets:

\[ T_{k-1}(\hat{K}_{2s+1}) = \{ x \in \hat{K}_{2s+1} : a_i \leq v_i^T x \leq b_i \text{ for } i \leq k-1, \]

\[ a_j + \eta(b_j - a_j) \leq v_j^T x \leq b_j - \eta(b_j - a_j) \text{ for } k+1 \leq j \leq d+1; \]

\[ a_k + 2^{-s-2} \eta(b_k - a_k) + \eta^2(5 \cdot 2^{-s-2}(b_k - a_k)) \leq v_k^T x < a_k + 3 \cdot 2^{-s-1} \eta(b_k - a_k) - \eta^2(5 \cdot 2^{-s-2}(b_k - a_k)) \}

\[ \supset K_{2s+1}, \]

provided that $5\eta < 1$. Similarly, $T_{k-1}(\hat{K}_{2s+2}) \supset K_{2s+2}$.

We choose $m$ so that $\Gamma^p|K_L| \leq \frac{1}{4}2^{-p}\varepsilon^p$, and $\Gamma^p|K_R| \leq \frac{1}{4}2^{-p}\varepsilon^p$ in a way that $(\int_{K_L} f(x)^p dx)^{1/p}$ and $(\int_{K_R} f(x)^p dx)^{1/p}$ are negligible. This can be done by choosing $m = C_1(d, \Gamma) \log(\Gamma/\varepsilon)$.

To see this, observe that $K_R$, $K_L$ are slabs with width $5\eta \cdot 2^{-(m-1)}$ intersected with the unit cube. Thus, their volume can be bounded by $C(d) \cdot 2^{-(m-1)} \eta$, which implies that

\[ \left( \int_{K_L} f(x)^p dx \right)^{1/p} \leq C(d) 2^{-(m-1)} \eta \Gamma \leq \varepsilon/2. \]

For every $f \in B^p_p(\Omega, t)$, we define $t_i$ as the smallest integer that satisfies the following

\[ \int_{\hat{K}_i} |f(x)|^p dx \leq |\hat{K}_i| t_i^p. \]

Because any point in $\Omega$ is contained in $\hat{K}_i$ for at most three different $i$, we have

\[ \sum_{i=0}^{2m+2} (t_i - 1)^p t_i^p |\hat{K}_i| \leq 3 t^p. \]  

(90)

This implies that

\[ \sum_{i=0}^{2m+2} t_i^p |\hat{K}_i| \leq 2^{p-1} \sum_{i=0}^{2m+2} [(t_i - 1)^p + 1] |\hat{K}_i| \leq 3 \cdot 2^p, \]

where we used the fact that $\sum_{i=0}^{2m+2} |\hat{K}_i| \leq 3|\Omega| \leq 3$. Since $\|f\|_\infty \leq \Gamma$, we have $t_i \leq \Gamma/t$. Thus, the total number of choices of the sequence $t_0, t_1, t_2, \ldots, t_{2m+2}$ is at most $(\Gamma/t)^{2m+3}$. For each ordered sequence $T = \{t_0, t_1, t_2, \ldots, t_{2m+2}\}$ satisfying (90), we define

\[ \mathcal{F}_T = \left\{ f \in B_p(\Omega, t) : \forall 0 \leq i \leq 2m+2, (t_i - 1)^p t_i^p |\hat{K}_i| < \int_{\hat{K}_i} |f(x)|^p dx \leq t_i^p t_i^p |\hat{K}_i| \right\}. \]
Then, \( f \in B_p(\hat{K}_i, t_i |\hat{K}_i|^{1/p}) \). Clearly, for all \( 0 \leq i \leq 2m + 2 \), \( K_i \) satisfies the induction assumption. Therefore, for any \( 0 \leq i \leq 2m + 2 \), there exists two sets \( G_i \) and \( \hat{G}_i \), each consisting of \( \exp(C[C_2 \log(\Gamma/\varepsilon)]^{k-1}|K_i|^{d/2} t_i^{d/2} \varepsilon_i^{-\alpha}/2}) \) elements, such that for every \( f \in B_p(\hat{K}_i, t_i |\hat{K}_i|^{1/p}) \), there exists \( g_i \in G_i \) and \( \hat{g}_i \in \hat{G}_i \) such that \( g_i(x) \leq f(x) \leq \hat{g}_i(x) \) for all \( x \in K_i \), and
\[
\int_{K_i} |\hat{g}_i(x) - g_i(x)|^p \leq \varepsilon_i^p.
\]
We define \( \hat{f}(x) = g_i(x) \) if \( x \in K_i \), and \( \hat{f}(x) = -\Gamma \) for \( x \in K_L \cup K_R \). Similarly, we define \( \hat{g}(x) = \hat{g}_i(x) \) if \( x \in K_i \), and \( \hat{g}(x) = \Gamma \) for \( x \in K_L \cup K_R \). Then, we have \( \hat{f}(x) \leq f(x) \leq \hat{g}(x) \) for all \( x \in T_k(\Omega) \) and
\[
\int_{T_k(\Omega)} |\hat{g}(x) - \hat{f}(x)|^p dx \leq \sum_{i=0}^{2m+2} \int_{K_i} |\hat{g}_i(x) - g_i(x)|^p dx + \int_{K_L \cup K_R} |\hat{g}(x) - \hat{f}(x)|^p dx
\]
\[
\leq \sum_{i=0}^{2m+2} \varepsilon_i^p + \frac{1}{2} \varepsilon^p.
\]
If we choose
\[
\varepsilon_i = \frac{1}{2 \cdot 6^{1/p} t_i |\hat{K}_i|^{1/p} \varepsilon},
\]
then,
\[
\sum_{i=0}^{2m+2} \varepsilon_i^p = \frac{1}{6^{1/p}} \sum_{i=0}^{2m+2} t_i^p |\hat{K}_i| \varepsilon^p \leq \frac{1}{2} \varepsilon^p.
\]
This implies that
\[
\int_{T_k(\Omega)} |\hat{g}(x) - \hat{f}(x)|^p dx \leq \varepsilon^p.
\]
Now, let us count the number of possible realizations of the brackets \([\hat{f}, \hat{g}]\). For each fixed \( T \), the number of choices of the brackets \([\hat{f}, \hat{g}]\) is at most
\[
N := \prod_{i=0}^{2m+2} \exp(C_1[C_2 \log(\Gamma/\varepsilon)]^{k-1}|\hat{K}_i|^{d/2} t_i^{d/2} \varepsilon_i^{-\alpha}/2})
\]
\[
= \exp\left(C_1[C_2 \log(\Gamma/\varepsilon)]^{k-1}(2m + 3)(t/\varepsilon)^{d/2}\right)
\]
\[
\leq \exp\left(C_3[\log(\Gamma/\varepsilon)]^{k}(t/\varepsilon)^{d/2}\right).
\]
The total number of \( \varepsilon \)-brackets under the \( L_p(T_k(\Omega)) \) distance needed to cover \( B_p^\Gamma(\Omega, t) \) is then bounded by
\[
(\Gamma/t)^{2m+3} \cdot N \leq \exp\left(C_3[\log(\Gamma/\varepsilon)]^{k}(t/\varepsilon)^{d/2}\right)
\]
provided the constant \( C_3 \) is large enough, and Lemma 9.1 follows. \( \square \)
9.3. Proof of Theorem 4.1

Let $\Omega$ be a convex body, and let $c$ be the center of its John ellipsoid (i.e., the unique ellipsoid of maximum volume contained in $\Omega$). For any $\lambda > 0$, define $\Omega_\lambda = c + \lambda(\Omega - c)$. It is clear that $|\Omega_\lambda| = \lambda^d|\Omega|$, where $|\Omega|$ denotes the volume of $\Omega$.

9.3.1. Away from the boundary

The goal of this sub-section is to prove the following proposition, which is the equivalent "discrete" version of Corollary 9.4.

**Proposition 9.5.** Let $S$ be the regular $d$-dimensional $\delta$-grid, and let $\Omega$ be a convex body in $\mathbb{R}^d$. For any $t > 0$ and any $0 < \varepsilon < 1$, there exists a set $N$ consisting of $\exp(\gamma_d \cdot (t/\varepsilon)^{d/2})$ functions such that for every $f \in B_q^\delta_s(0; t; \Omega)$, there exists $g \in N$ satisfying $|f(x) - g(x)| < \varepsilon$ for all $x \in \Omega_{0.9}$, where $\gamma_d$ is a constant depending only on $d$.

To prove Proposition 9.5, we need some preparations.

**Lemma 9.6.** Let $S$ be a regular $\delta$-grid on $\mathbb{R}^d$, and let $\Omega$ be a convex body containing a ball of radius $r_0 \geq 10d^3/\delta$. We have

$$
\frac{9}{10}|\Omega|\delta^{-d} \leq \#(\Omega \cap S) \leq \frac{11}{10}|\Omega|\delta^{-d}.
$$

*Proof.* Let $s_1, \ldots, s_n$ be the grid points contained in $\Omega$. We have

$$
\bigcup_{i=1}^n (s_i + [-\delta/2, \delta/2]^d) \subset \Omega + [-\delta/2, \delta/2]^d \subset \Omega + \frac{\sqrt{d\delta}}{2}B_d.
$$

Note that when $\Omega$ contains a ball of radius $r_0$,

$$
|\Omega + \frac{\sqrt{d\delta}}{2}B_d| \leq \left(1 + \frac{\sqrt{d\delta}}{2r_0}\right)^d |\Omega| \leq \left(1 + \frac{1}{20d}\right)^{1/d} |\Omega| \leq \frac{11}{10}|\Omega|.
$$

Volume comparison gives us

$$
n \leq \frac{11}{10}|\Omega|\delta^{-d}.
$$

On the other hand, let $U$ be the union of the cubes $s_i + [-\delta/2, \delta/2]^d$. The volume of $U$ is $n\delta^d$. Since the union of $s_i + [-\delta, \delta]^d$ covers $\Omega$, we have $U + [-\delta/2, \delta/2]^d \supset \Omega$. In particular, $U$ contains the set

$$
\{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \sqrt{d\delta}/2\}.
$$

Since $\Omega$ contains a ball of radius $r_0$. If we let $c$ be the center of this ball, and define

$$
\hat{\Omega} = c + \left(1 - \frac{\sqrt{d\delta}}{2r_0}\right)(\Omega - c),
$$

...
then the distance between any \( x \in \hat{\Omega} \) and \( \partial \Omega \) is at least \( \sqrt{d}\delta/2 \). Hence \( U \supset \hat{\Omega} \). Consequently

\[
n = |U|\delta^{-d} \geq |\hat{\Omega}|\delta^{-d} = \left(1 - \frac{\sqrt{d}\delta}{2r_0}\right)^d |\Omega|\delta^{-d} \geq \frac{9}{10} |\Omega|\delta^{-d}.
\]

This finishes the proof of the lemma. \( \square \)

**Lemma 9.7.** Let \( S \) be a regular \( d \)-dimensional \( \delta \)-grid. Let \( \Omega \subset [0, 1]^d \) be a convex body that contains a ball of radius at least \( 10d^{3/2}\delta \). Then for every \( f \in B^d_2(0, t, \Omega) \), \( f \geq -20dt \).

**Proof.** Let \( x_0 \) be the minimizer of \( f \) on \( \Omega \). If \( f(x_0) \geq 0 \), then there is nothing to prove; otherwise, the set \( K := \{ x \in \Omega \mid f(x) \leq 0 \} \) is a closed convex set containing \( x_0 \). Denote \( K_t = x_0 + t(K - x_0) \), and let \( \hat{K} = K_{1+\sigma} \setminus K_{1-\sigma} \), where \( \sigma = (10d)^{-1} \). We show that for all \( x \in \Omega \setminus \hat{K} \), \( |f(x)| \geq \sigma |f(x_0)| \). Indeed, we define a function \( g \) on \( \Omega \) so that \( g(x_0) = f(x_0) \), \( g(\gamma) = f(\gamma) \) for all \( \gamma \in \partial K \), and \( g \) is linear on \( L_\gamma := \{ x = x_0 + t(\gamma - x_0) \in \Omega \mid t \geq 0 \} \). Then, by the convexity of \( f \) on each \( L_\gamma \), we have \( |f(x)| \geq |g(x)| \) on \( \Omega \). Thus, for all \( x \in \Omega \setminus \hat{K} \),

\[
|f(x)| \geq |g(x)| = |g(\gamma)| + \frac{|x - \gamma|}{\|x_0 - \gamma\|}|f(x_0)| \geq \sigma |f(x_0)|.
\]

Next, we show that most of the grid points in \( \Omega \) are outside \( \hat{K} \). Indeed, If \( s \) is a grid point in \( \hat{K} \), then \( s + [-\delta/2, \delta/2]^d \subset K_{1+\sigma} \cap \Omega + [-\delta/2, \delta/2]^d \) and at least one half of the cube \( s + [-\delta/2, \delta/2]^d \) lies outside \( K_{1-\sigma} \). Thus, the number of grid points in \( \hat{K} \) is bounded by

\[
2|\left(K_{1+\sigma} \cap \Omega + [-\delta/2, \delta/2]^d\right) \setminus K_{1-\sigma}|\delta^{-d}.
\]

Since \( |(A + B) \setminus A| \) can be expressed as a sum of products of mixed volumes of \( A \) and \( B \), and smaller sets have smaller mixed volumes, we have

\[
|(A + B) \setminus B| \leq |(C + D) \setminus D|
\]

for all convex sets \( C \supset A \) and \( D \supset B \). Applying this inequality for \( A = C = [-\delta/2, \delta/2]^d \), \( B = (K_{1+\sigma} \cap \Omega \) and \( D = [-1, 1]^d \), we have

\[
|(K_{1+\sigma} \cap \Omega + [-\delta/2, \delta/2]^d) \setminus K_{1+\sigma} \cap \Omega| \leq |([-1, 1]^d + [-\delta/2, \delta/2]^d) \setminus [-1, 1]^d| = (2+\delta)^d - 2d,
\]

while \( |K_{1+\sigma} \cap \Omega \setminus K_{1-\sigma}| \leq \left[1 - \left(\frac{\sigma}{1+\sigma}\right)^d\right] |K_{1+\sigma} \cap \Omega| \), we have

\[
|(K_{1+\sigma} \cap \Omega + [-\delta/2, \delta/2]^d) \setminus K_{1-\sigma}| \leq \left[1 - \left(\frac{\sigma}{1+\sigma}\right)^d\right] |K_{1+\sigma} \cap \Omega| + (2+\delta)^d - 2d \leq 3d\sigma |\Omega|.
\]

Thus, the number of grid points in \( \hat{K} \) is bounded by

\[
6d\sigma |\Omega|\delta^{-d} \leq 6d\sigma \cdot \frac{10}{9} \cdot \#(S \cap \Omega) \leq 7d\sigma \cdot \#(S \cap \Omega).
\]
Hence,

\[ \#(S \cap \Omega) \cdot t^q \geq \sum_{s \in S \cap (\Omega, K)} |f(s)|^q \geq (1 - 7d\sigma) \cdot \#(S \cap \Omega) \cdot (\sigma|f(x_0)|)^q, \]

which implies that \( f(x_0) \geq -2^{1/3}\sigma^{-1}t \geq -20dt \) by using \( \sigma = (10d)^{-1}. \]

Lemma 9.8. Suppose that a convex body \( \Omega \) contains \( n \) grid points of a regular \( \delta \)-grid in \( \mathbb{R}^d \), and a ball of radius at least \( 400d^{3/2}\delta \). Then, at any point \( P \) on the boundary of \( \Omega_{0.95} \), any hyperplane passing through \( P \) cuts \( \Omega \) into two parts. The part that does not contain the center of John ellipsoid of \( \Omega \) as its interior point contains at least \( (20d)^{-d-1} \cdot n \) grid points.

Proof. For any point \( P \) on the boundary of \( \Omega_{0.95} \). Any hyperplane passing through \( P \) cuts \( \Omega \) into two parts. Suppose \( \Omega \) is a part that does not contain the center of John ellipsoid of \( \Omega \) as its interior. We prove that \( \text{convexity of } \Omega \) implies that \( \text{the John ellipsoid of } \Omega \) contains a ball of radius at least \( 400d^{3/2}\delta \). Because the distance from \( (T\Omega)_{0.95} \) to the boundary of \( T\Omega \) is at least \( \frac{1}{20} \), \( TL \) contains half of the side of the hyperplane \( T \) with center at \( TP \) and radius \( \frac{1}{20} \). Thus, \( TL \) has volume at least \( \frac{1}{2}20^{-d}|B_d| \). Since \( T\Omega \) is contained in the ball of radius \( d \), we have \( |T\Omega| \leq d^d|B_d| \). This implies that \( |TL| \geq \frac{1}{2d}(20d)^{-d}|T\Omega| \). Hence \( |L| \geq \frac{1}{20}(20d)^{-d}|\Omega| \).

Because the John ellipsoid of \( \Omega \) contains a ball of radius at least \( 400d^{3/2}\delta \), the distance from \( \Omega_{0.95} \) to the boundary of \( \Omega \) is at least \( 20d^{3/2}\delta \). Thus, \( \Omega \) contains a ball of radius at least \( 10d^{3/2}\delta \). By Lemma 9.6, the number of grid points in it is at least \( \frac{9}{20d}(20d)^{-d} |\Omega|^{-d} \). The statement of Lemma 9.8 then follows by using Lemma 9.6 one more time.

Lemma 9.9. Let \( S \) be the regular \( d \)-dimensional \( \delta \)-grid. Let \( \Omega \) be a convex body in \( R^d \) containing a ball of radius \( 400d^{3/2}\delta \). For every \( f \in B_S^d(0, t, \Omega) \), \( f(x) \leq (20d)^{\frac{d+1}{d}} t \) for all \( x \in \Omega_{0.95} \).

Proof. Let \( z \) be the maximizer of \( f \) on \( \Omega_{0.95} \). By the convexity of \( f \), \( z \) must be on the boundary of \( \Omega_{0.95} \). If \( f(z) \leq 0 \), there is nothing to prove. So we assume \( f(z) > 0 \). The convexity of \( f \) implies that \( z \) lies on the boundary of the convex set \( K : \{ x \in \Omega : f(x) \leq f(z) \} \supseteq \Omega_{0.95} \). There exists a hyperplane \( z \) so that the convex set \( \{ x \mid f(x) \leq f(z) \} \) lies entirely on one side of the hyperplane. Let \( L \) be the portion of \( \Omega \) that lies on the other side of the hyperplane that supports \( K \) at \( z \). This hyperplane cuts \( \Omega \) into two parts. Let \( L \) be the part that does not contain \( K \). Then, \( f(x) \geq f(z) \) for all \( x \in L \). By Lemma 9.8, we have

\[ \#(L \cap \Omega) \geq (20d)^{-d-1} \cdot (\Omega \cap \Omega). \]

Since \( f(x) \geq f(z) > 0 \) for all \( x \in L \), we have

\[ \#(L \cap \Omega) \cdot f(z)^q \leq \sum_{s \in L \cap \Omega} |f(s)|^q \leq \sum_{s \in \Omega \cap \Omega} |f(s)|^q \leq \#(\Omega \cap \Omega) \cdot t^q, \]
This implies that \( f(z) \leq (20d)^{\frac{d+1}{d}} t. \)

**Lemma 9.10** (Bronshtein). There exists a constant \( \beta \) depending only on \( d \) such that
for any \( \varepsilon > 0 \), any \( M > 0 \), and any convex set \( \Omega \subset B_d \), there exists a set \( \mathcal{G} \) consisting of \( \exp(\beta(M/\varepsilon)^{d/2}) \) functions, such that for any convex function \( f \) on \( \Omega \) that is bounded by \( M \) and has a Lipschitz constant bounded by \( M \), there exists \( g \in \mathcal{G} \) such that \( |f(x) - g(x)| < \varepsilon \) for all \( x \in \Omega \).

**Proof of Proposition 9.5** If \( \Omega \) contains \( N \leq 400d^4 \) grid points. Denote these grid points by \( s_1, s_2, \ldots, s_N \). Then
\[
\{(f(s_1), f(s_2), \ldots, f(s_N)) : f \in B^2_d(0, t, \Omega)\}
\]
is a subset of
\[
\{(x_1, x_2, \ldots, x_N) : |x_1|^q + |x_2|^q + \cdots + |x_N|^q \leq Nt^q\},
\]
i.e., the \( \ell^N_q \)-ball of radius \( N^{1/q}t \). By volume comparison, it can be covered by no more than \( (1 + \frac{2d}{q})^N \) \( \ell^N_q \)-balls of radius \( N^{1/q} \varepsilon \). Thus, by choosing \( \gamma_d \geq 400d^4 \), the statement of Proposition 9.5 is true when \( \Omega \) contains no more than \( 400d^4 \) grid points.

For the remaining case, we prove by induction.

If \( d = 1 \), and \( \Omega \) contains more than 400 grid points, then by Lemma 9.7 and Lemma 9.9, we have \(-20t \leq f(x) \leq 400t\) for all \( x \in \Omega_{0.95} \). Let \( T \) be a linear transform that maps the interval \( \Omega_{0.95} \) to the interval \([-1, 1]\). Then, \( f \circ T^{-1} \) are convex functions on \([-1, 1]\) satisfying \(-20t \leq f(T^{-1}x) \leq 400t\) for all \( x \in [-1, 1] \). By convexity of \( f \circ T^{-1} \), we have
\[
|(f \circ T^{-1}(x))'| \leq \max \left\{ \frac{|f \circ T^{-1}(1) - f \circ T^{-1}(0.95)|}{|1 - 0.95|}, \frac{|f \circ T^{-1}(-1) - f \circ T^{-1}(-0.95)|}{|(-1) - (-0.95)|} \right\} \leq 8400t.
\]
By Lemma 9.10, there exists a set \( \mathcal{G} \) consisting of no more than \( \exp(\beta\sqrt{8400}t\varepsilon^{-1/2}) \) functions such that for each \( f \in B^2_d(0; t; \Omega) \), there exists \( g \in \mathcal{G} \) satisfying \( |f \circ T^{-1}(x) - g(x)| < \varepsilon \) for all \( x \in [-0.95, 0.95] \). Because \( T\Omega_{0.9} \subset (T\Omega_{0.95})_{0.95} = [-0.95, 0.95] \), we have \( |f(z) - g \circ T(z)| < \varepsilon \) for all \( z \in \Omega_{0.9} \). Thus, the statement of the lemma is true with \( \mathcal{N} = \{g \circ T : g \in \mathcal{G}\} \) if we choose \( \gamma_1 = \max(400, \sqrt{8400} \beta) \).

Suppose the statement is true for \( d < k \). Consider the case \( d = k \). If the minimum number of parallel hyperplanes needed to cover all the grid points in \( \Omega \) is more than \( 400d^4 \). Then the lattice width \( w(\Omega, S) \) is at least \( 400d^4 \). Let \( \mu(\Omega, S) \) be the covering radius, i.e., the smallest number \( \mu \) such that \( \mu \Omega + S \supset \mathbb{R}^d \). By Khinchine’s flatness theorem, (c.f. [5], [6]) we have \( w(\Omega, S) \cdot \mu(\Omega, S) \leq C d^{d/2} \), which implies that the covering radius of \( \Omega \) is at most \( C(800d^{5/2})^{-1} \). Thus, \( \Omega \) contains a cube of side length \( C^{-1}1800d^{5/2} \delta \), and hence a ball of radius \( C^{-1}400d^{5/2} \delta \). Thus, all the previous lemmas are applicable to \( \Omega \). By Lemma 9.7 and Lemma 9.9, we have \(-20dt \leq f(x) \leq (20d)^{d+1}t\) for all \( x \in \Omega_{0.95} \). If \( T \) is an affine transformation so that the John ellipsoid of \( T\Omega_{0.95} \) is the unit ball \( B_d(1) \). Because \( \Omega_{0.9} \subset (\Omega_{0.95})_{0.95} \), by the proof of Lemma 9.8, the distance between the
boundary of $T(\Omega_{0.95})$ and the boundary of $T(\Omega_{0.9})$ is at least $\frac{1}{20}$. If we define convex function $\tilde{f}$ on $T(\Omega)$ by $\tilde{f}(y) = f(T^{-1}(y))$. Then, $-20dt \leq \tilde{f}(y) \leq (20d)^{d+1}t$ for all $y \in T(\Omega_{0.95})$. For any $u, v \in T(\Omega_{0.9})$, without loss of generality we assume $\tilde{f}(u) \leq \tilde{f}(v)$. Consider the half-line starting from $u$ and passing through $v$. Suppose the half-line intersects the boundary of $T(\Omega_{0.9})$ at $p$ and the boundary of $T(\Omega_{0.95})$ at $q$. By the convexity of $\tilde{f}$ on this half-line, we have

$$0 \leq \frac{\tilde{f}(v) - \tilde{f}(u)}{\|v - u\|} \leq \frac{|\tilde{f}(q) - \tilde{f}(p)|}{\|q - p\|} \leq 20[(20d)^{d+1}t + 20dt] := M.$$ 

This implies that $\tilde{f}$ is a convex function on $T(\Omega_{0.9})$ that has a Lipschitz constant $M$. Of course $f$ is also bounded by $M$ on $T(\Omega_{0.9})$. Thus, by Lemma 9.10, there exists a set of function $G$ consisting of at most $\exp(\beta \cdot (M/\epsilon)^{d/2})$ functions such that for every $f \in B^d_S(0, t, K)$, there exists a function $g \in G$, such that $|\tilde{f}(y) - g(y)| < \epsilon$ for all $y \in T(\Omega_{0.9})$. This implies $|f(x) - g(Tx)| < \epsilon$ for all $x \in \Omega_{0.9}$. Thus, by setting $N = \{g \circ T \mid g \in G\}$ the lemma follows with $\gamma_d \geq \beta M^{d/2}$.

If the minimum number of parallel hyperplanes needed to cover all the grid points in $\Omega$ is less than $400d^4$. Then, by applying the lemma for $d = k - 1$ for the grid points on each hyperplane, the lemma follows as long as we have $\gamma_d \geq 400d^2 \gamma_{d-1}$. \hfill \Box

### 9.3.2. Reaching the Boundary

Now, we try to reach closer to the boundary of $\Omega$. More precisely, we will extend Proposition 9.5 from $\Omega_{0.9}$ to the set $\Omega_0$ defined below.

Let $\Omega$ be a convex polytope with the center of John ellipsoid at $c$. Then, we can describe $\Omega$ as $\{x \in \mathbb{R}^d : -a_i \leq v_i^T(x - c) \leq b_i, 1 \leq i \leq F\}$, where $a_i > 0$, $b_i > 0$ and $v_i$ are unit vectors in $\mathbb{R}^d$. Let $m_i$ and $n_i$ be the smallest integer such that $2^{-m_i}a_i \leq \delta$ and $2^{-n_i}b_i \leq \delta$. Let

$$\Omega_0 = \{x \in \mathbb{R}^d : -(1 - 2^{-m_i})a_i \leq v_i^T(x - c) \leq (1 - 2^{-n_i})b_i, 1 \leq i \leq F\}.$$ 

Then the Hausdorff distance between $\Omega$ and $\Omega_0$ is no larger than $\delta$. Thus, $\Omega_0$ is indeed close to $\Omega$.

The following proposition suggests that to achieve our goal, we only need to properly decompose $\Omega_0$.

**Proposition 9.11.** If $D_i$, $1 \leq i \leq m$ is a sequence of convex subsets of $\Omega$ such that no point in $\Omega$ is contained in more than $M$ subsets in the sequence. Then, for $\Omega_0 \subset \bigcup_{i=1}^m (D_i)_{0.9}$, we have

$$\log N(\epsilon, B^d_S(0; t; \Omega), (\mathcal{L}^d_S(\cdot; \Omega_0)) \leq cM \frac{d}{2}(\frac{\delta}{\epsilon})^{d/2}.$$ 

**Proof.** Let $G_i$ be the set of grid points in $D_i$, and $S_i$ be the grid points in $(D_i)_{0.9} \setminus \bigcup_{j<i}(D_j)_{0.9}$. For every $f \in B^d_S(0; t; \Omega)$, define $t_i = t_i(f)$ to be the smallest positive
integer, such that
\[ \sum_{x \in G_i} |f(x)|^p \leq |G_i| t_i^p. \]

We have
\[ \sum_{i=1}^m |G_i| (t_i - 1)^p \leq \sum_{i=1}^m \sum_{x \in G_i} |f(x)|^p \leq Mnt^p. \]

Thus, there are no more than \( \binom{Mn+m}{m} \) possible values of the \( m \)-tuple \( (t_1, t_2, \ldots, t_m) \). Let
\[ \mathcal{K} = \{ (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m |G_i|(k_i - 1) \leq Mn \}. \]

For any \( K = (k_1, k_2, \ldots, k_m) \in \mathcal{K} \), define \( \mathcal{F}_K = \{ f \in B_S^p(0; t; \Omega) : t_i(f) = k_i, 1 \leq i \leq m \} \). By Proposition 9.5, there exists a set \( \mathcal{G}_i \) consisting of \( \exp(\gamma [k_i^{1/p}] d/2 \varepsilon_i^{-d/2}) \) functions, such that for every \( f \in \mathcal{F}_K \), there exists \( g_i \in \mathcal{G}_i \) satisfying
\[ \sum_{x \in S_i} |f(x) - g_i(x)|^p \leq |G_i| \varepsilon_i^p. \]

If we define \( g(x) = g_i(x) \) for \( x \in S_i \), then we have
\[ \sum_{x \in S \cap \Omega_0} |f(x) - g(x)|^p \leq \sum_{i=1}^m |G_i| \varepsilon_i^p = n \varepsilon^p, \]

where the last inequality holds if we let
\[ \varepsilon_i = \frac{t_i^{p(d+2p)}}{\left( \sum_{i=1}^m |G_i| k_i \right)^{d/(d+2p)}} \cdot n^{1/p \varepsilon}. \]

The total number of realizations of \( g \) is
\[ \exp \left( \gamma \sum_{i=1}^m [k_i^{1/p}]^{d/2} \varepsilon_i^{-d/2} \right) = \exp \left\{ \left( \sum_{i=1}^m (|G_i| k_i)^{d/(d+2p)} \right)^{d/(d+2p)} \cdot n^{-\frac{d}{2p} (t/\varepsilon)^{d/2}} \right\}. \]

Using the inequalities
\[ \sum_{i=1}^m (|G_i| k_i)^{d/(d+2p)} \leq \left( \sum_{i=1}^m |G_i| k_i \right)^{d/(d+2p)} m^{2p/(d+2p)} \]

and
\[ \sum_{i=1}^m |G_i| k_i = \sum_{i=1}^m |G_i| (k_i - 1) + \sum_{i=1}^m |G_i| \leq Mn + Mn = 2Mn, \]
we can bound the total number of realizations of $g$ by

$$\exp \left( \gamma (2M)^{d^2} m(t/\varepsilon)^{d/2} \right).$$

Consequently, we have

$$\log N(\varepsilon, B^{n}(0; t; \Omega), \ell^{n}_{s}(\cdot, \Omega_{0})) \leq \log \left( \frac{Mn + m}{m} \right) + \gamma (2M)^{d^2} m(t/\varepsilon)^{d/2} \leq cmM^{d^2} (t/\varepsilon)^{d/2}. \qed$$

Now, let us decompose $\Omega_{0}$.

**Lemma 9.12.** There exists convex sets $\hat{D}_{i}, 1 \leq i \leq N := \prod_{i=1}^{F} (m_{i} + n_{i})$ contained in $\Omega$, such that no point in $\Omega$ is contained in more than $4^{F}$ of these sets, and

$$\Omega_{0} \subset \bigcup_{i=1}^{N} (\hat{D}_{i})_{0.8}.$$

**Proof.** Let

$$K = \{(k_{1}, k_{2}, \ldots, k_{F}) : -m_{i} \leq k_{i} \leq n_{i} - 1, 1 \leq i \leq F\}.$$ 

There are $\prod_{i=1}^{F} (m_{i} + n_{i})$ elements in $K$. For each $K = (k_{1}, k_{2}, \ldots, k_{F}) \in K$, define

$$D_{K} = \{x \in \mathbb{R}^{d} : \alpha_{i}(k_{i}) \leq v_{i}^{T}(x - c) \leq \alpha_{i}(k_{i} + 1)\},$$

where

$$\alpha_{i}(t) = \begin{cases} 
-(1 - 2^{t})a_{i}, & t \leq 0 \\
(1 - 2^{-t})b_{i}, & t > 0
\end{cases}.$$

$D_{K}$ is a convex set. The union of all $D_{K}, K \in K$ is the set

$$\{x \in \mathbb{R}^{d} : -(1 - 2^{-m_{i}})a_{i} \leq v_{i}^{T}(x - c) \leq (1 - 2^{-n_{i}})b_{i}\}.$$

Similarly, we define

$$\hat{D}_{K} = \{x \in \mathbb{R}^{d} : \beta_{i}(k_{i}) \leq v_{i}^{T}(x - c) \leq \gamma_{i}(k_{i})\},$$

where

$$\beta_{i}(k_{i}) = \alpha_{i}(k_{i}) - \frac{1}{4}[\alpha_{i}(k_{i} + 1) - \alpha_{i}(k_{i})], \quad \gamma_{i}(k_{i}) = \alpha_{i}(k_{i} + 1) + \frac{1}{4}[\alpha_{i}(k_{i} + 1) - \alpha_{i}(k_{i})].$$

Let $c_{K}$ be the center of John ellipsoid of $\hat{D}_{K}$. We have

$$\hat{D}_{K} = \{x \in \mathbb{R}^{d} : \beta_{i}(k_{i}) - v_{i}^{T}(c_{K} - c) \leq v_{i}^{T}(x - c_{K}) \leq \gamma_{i}(k_{i}) - v_{i}^{T}(c_{K} - c)\}.$$ 

Thus,

$$(\hat{D}_{K})_{0.8} = \{x \in \mathbb{R}^{d} : 0.8[\beta_{i}(k_{i}) - v_{i}^{T}(c_{K} - c)] \leq v_{i}^{T}(x - c_{K}) \leq 0.8[\gamma_{i}(k_{i}) - v_{i}^{T}(c_{K} - c)]\}$$

$$= \{x \in \mathbb{R}^{d} : 0.8\beta_{i}(k_{i}) + 0.2v_{i}^{T}(c_{K} - c) \leq v_{i}^{T}(x - c) \leq 0.8\gamma_{i}(k_{i}) + 0.2v_{i}^{T}(c_{K} - c)\}$$

$$\supset \{x \in \mathbb{R}^{d} : 0.8\beta_{i}(k_{i}) + 0.2\alpha_{i}(k_{i} + 1) \leq v_{i}^{T}(x - c_{K}) \leq 0.8\gamma_{i}(k_{i}) + 0.2\alpha_{i}(k_{i})\}$$

$$= \{x \in \mathbb{R}^{d} : \alpha_{i}(k_{i}) \leq v_{i}^{T}(x - c_{K}) \leq \alpha_{i}(k_{i} + 1)\} = D_{K},$$
where in the second to the last equality we used the fact that $0.8\beta_i(k_i) + 0.2\alpha_i(k_i + 1) = \alpha_i(k_i)$ and $0.8\gamma_i(k_i) + 0.2\alpha_i(k_i) = \alpha_i(k_i + 1)$.

It is not difficult to check that if integer $k_i \neq 0, -1$, the interval $(\beta_i(k_i), \gamma_i(k_i))$ and $(\beta_i(j_i), \gamma_i(j_i))$ intersect only when $|k_i - j_i| \leq 1$, or one of the two cases: $t_i = 0$ or $t_i = -1$. Hence, where are at most four possibilities. Hence, no point can be covered by more than $4F$ different sets $\hat{D}_K$. The lemma follows by renaming these sets as $\hat{D}_i$, $1 \leq i \leq N$.

**Proof of Theorem 4.1**

By using the lemma above and Proposition 9.11, we have

$$\log N(\varepsilon, B^p_S(0; t; \Omega), \ell^p_S(\cdot; \Omega_0)) \leq c2^{dF} [\log(1/\delta)]^F (t/\varepsilon)^{d/2}.$$  

Because the distance between the boundary of $\Omega \setminus \Omega_0$ can be decomposed into at most $2F$ piece of width $\delta$. By Khinchine’s flatness theorem, the grid points in $\Omega \setminus \Omega_0$ contained in $cF$ hyperplanes for some constant $c$. The intersection of $\Omega$ and each of these hyperplanes is a $(d - 1)$ dimensional convex polytope. This enables us to obtain covering number estimates on $\Omega \setminus \Omega_0$ using induction on dimension. This concludes the proof of Theorem 4.1.

**9.4. Proofs of Corollary 4.2 and Corollary 4.3**

**Proof of Corollary 4.2.** Let $n$ denote the cardinality of $\Omega \cap S$ and let $n_i$ denote the cardinality of $\Omega_i \cap S$ for each $i = 1, \ldots, k$. We can assume that each $n_i > 0$ for otherwise we can simply drop that $\Omega_i$. For $f \in B^p_S(f_0; t; \Omega)$ and $1 \leq i \leq k$, let $\sigma_i(f)$ be the smallest positive integer for which

$$\sum_{s \in \Omega_i \cap S} |f(s) - f_0(s)|^p \leq n_i \sigma_i(f) t^p.$$  

It is clear then that $1 \leq \sigma_i(f) \leq n$ for each $i$. Also because $\sigma_i(f)$ is the smallest integer satisfying the above, we have

$$\sum_{s \in \Omega_i \cap S} |f(s) - f_0(s)|^p \geq n_i (\sigma_i(f) - 1) t^p,$$

which implies that

$$\sum_{i=1}^{k} n_i (\sigma_i(f) - 1) t^p \leq \sum_{i=1}^{k} \sum_{s \in \Omega_i \cap S} |f(s) - f_0(s)|^p = \sum_{s \in \Omega \cap S} |f(s) - f_0(s)|^p \leq t,$$

leading to

$$\sum_{i=1}^{k} n_i \sigma_i(f) \leq \sum_{i=1}^{k} n_i = n.$$  

Let

$$\Sigma := \{(\sigma_1(f), \ldots, \sigma_k(f)) : f \in B^p_S(f_0; t; \Omega)\}.$$
and note that the cardinality of \( \Sigma \) is at most \( n^k \) as \( 1 \leq \sigma_i(f) \leq n \) for each \( i \). For every \( (\sigma_1, \ldots, \sigma_k) \in \Sigma \), let
\[
\mathcal{F}_{\sigma_1, \ldots, \sigma_k} = \{ f \in B^p_{S}(f_0; t; \Omega) : \sigma_i(f) = \sigma_i, 1 \leq i \leq k \}.
\]
Observe now that if
\[
\ell^p_{S}(f - f_0, \Omega_i) \leq \epsilon \sigma_i^{1/p} \quad \text{for } i = 1, \ldots, k,
\]
then
\[
\ell^p_{S}(f - f_0, \Omega) = \left( \frac{1}{n} \sum_{i=1}^{k} \sum_{s \in \Omega_i \cap S} |f(s) - f_0(s)|^p \right)^{1/p} \leq \left( \epsilon \sum_{i=1}^{k} \frac{n_i \sigma_i}{n} \right)^{1/p} \leq \epsilon.
\]
This gives
\[
\log N(\epsilon, \mathcal{F}_{\sigma_1, \ldots, \sigma_k}, \ell^p_{S}(\cdot, \Omega)) \leq \sum_{i=1}^{k} \log N(\epsilon \sqrt{\sigma_i}, B^p_{S}(f_0; t \sqrt{\sigma_i}; \Omega_i), \ell^p_{S}(\cdot, \Omega_i)).
\]
Because \( f_0 \) is affine on each \( \Omega_i \), we can apply Theorem 4.1 to obtain
\[
\log N(\epsilon, \mathcal{F}_{\sigma_1, \ldots, \sigma_k}, \ell^p_{S}(\cdot, \Omega)) \leq k \left( \frac{t}{\epsilon} \right)^{d/2} \left( c_d \log \frac{1}{\delta} \right)^s.
\]
Because
\[
B^p_{S}(f_0; t; \Omega) = \bigcup_{(\sigma_1, \ldots, \sigma_k) \in \Sigma} \mathcal{F}_{\sigma_1, \ldots, \sigma_k}
\]
and the cardinality of \( \Sigma \) is at most \( n^k \), we deduce that
\[
\log N(\epsilon, B^p_{S}(f_0; t; \Omega), \ell^p_{S}(\cdot, \Omega)) \leq k \left( \frac{t}{\epsilon} \right)^{d/2} \left( c_d \log \frac{1}{\delta} \right)^s + k \log n
\]
Because \( \log n \leq C_d \log(1/\delta) \) (inequality (11)), the second term on the right hand side above is dominated by the first term. This completes the proof of Corollary 4.2.

**Proof of Corollary 4.3.** We use Corollary 4.2 with the following choice of \( \Omega_1, \ldots, \Omega_k \). Take \( \Omega_1 = \Delta_1 \) and define \( \Omega_i \) for \( 2 \leq i \leq k \) recursively as follows. Consider \( \Delta_i \) and its facets that have a non-empty intersection with \( \Delta_1, \ldots, \Delta_{i-1} \). If any of these facets contain points in \( S \), then we move the facets slightly inward so that they do not contain any grid points. This will ensure that \( \Omega_1, \ldots, \Omega_k \) are \( d \)-simplices satisfying the conditions of Corollary 4.2. The conclusion of Corollary 4.3 then directly follows from Corollary 4.2 (note that every \( d \)-simplex can be written as an intersection of \( d + 1 \) halfspaces so in particular it can be written as an intersection of at most \( d + 1 \) pairs of halfspaces). \( \square \)
10. Additional proofs

This section contains proofs of Lemma 2.4, Lemma 7.3, Lemma 8.2 and Lemma 8.3.

Proof of Lemma 2.4. For a fixed \( \eta > 0 \), let \( \mathcal{C}_\eta \) be the collection of all cubes of the form

\[
[k_1 \eta, (k_1 + 1) \eta] \times \cdots \times [k_d \eta, (k_d + 1) \eta]
\]

for \((k_1, \ldots, k_d) \in \mathbb{Z}^d\) which intersect \( \Omega \). Because \( \Omega \) is contained in the unit ball, there exists a dimensional constant \( c_d \) such that the cardinality of \( \mathcal{C}_\eta \) is at most \( C_d \eta^{-d} \) for \( \eta \leq c_d \).

For each \( B \in \mathcal{C}_\eta \), the set \( B \cap \Omega \) is a polytope whose number of facets is bounded from above by a constant depending on \( d \) alone. This polytope can therefore be triangulated into at most \( C_d \) number of \( d \)-simplices. Let \( \Delta_1, \ldots, \Delta_m \) be the collection obtained by the taking the all of the aforementioned simplices as \( B \) varies over \( \mathcal{C}_\eta \). These simplices clearly satisfy the first two requirements of Lemma 2.4. Moreover

\[
m \leq C_d \eta^{-d}
\]

and the diameter of each simplex \( \Delta_i \) is at most \( C_d \eta \). Now define \( \tilde{f}_\eta \) to be a piecewise affine convex function that agrees with \( f_0(x) = \|x\|^2 \) for each vertex of each simplex \( \Delta_i \) and is defined by linearity everywhere else on \( \Omega \). This function is clearly affine on each \( \Delta_i \), belongs to \( C^{C_d}(\Omega) \) for a sufficiently large \( C_d \) and it satisfies

\[
\sup_{x \in \Delta_i} |f_0(x) - \tilde{f}_\eta(x)| \leq C_d (\text{diameter}(\Delta_i))^2 \leq C_d \eta^2.
\]

Now given \( k \geq 1 \), let \( \eta = c_d k^{-1/d} \) for a sufficiently small dimensional constant \( c_d \) and let \( \tilde{f}_k \) to be the function \( f_\eta \) for this \( \eta \). The number of simplices is now \( m \leq C_d k \) and

\[
\sup_{x \in \Delta_i} |f_0(x) - \tilde{f}_\eta(x)| \leq C_d k^{-2/d}
\]

which completes the proof of Lemma 2.4. \( \square \)

Proof of Lemma 7.3. Let

\[
g(x_1, x_2, \ldots, x_d) = \left\{ \begin{array}{ll} \sum_{i=1}^d \cos^3(\pi x_i) & (x_1, x_2, \ldots, x_d) \in [-1/2, 1/2]^d \\ 0 & (x_1, x_2, \ldots, x_d) \notin [-1/2, 1/2]^d \end{array} \right.
\]

Note that \( g \) is smooth, \( \frac{\partial^2 g}{\partial x_i \partial x_j} = 0 \) for \( i \neq j \) and

\[
\left| \frac{\partial^2 g}{\partial x_i^2}(x_1, \ldots, x_d) \right| \leq \frac{4\sqrt{2}}{3} \pi^2
\]

which means that the Hessian of \( g \) is dominated by \((4\sqrt{2}\pi^2/3)\) times the identity matrix. It is also easy to check that the Hessian of \( g \) equals zero on the boundary of \([-0.5, 0.5]^d\).
Now for every grid point \( s := (k_1\delta, \ldots, k_d\delta) \) in \( S \cap \Omega \), consider the function
\[
g_s(x_1, \ldots, x_d) := \delta^2 g \left( \frac{x_1 - k_1\delta}{\delta}, \ldots, \frac{x_d - k_d\delta}{\delta} \right). \]

Clearly \( g_s \) is supported on the cube
\[
[(k_1 - 1/2)\delta, (k_1 + 1/2)\delta] \times [(k_2 - 1/2)\delta, (k_2 + 1/2)\delta] \times \cdots \times [(k_d - 1/2)\delta, (k_d + 1/2)\delta] \quad (91)
\]
and observe that these cubes for different grid points have disjoint interiors.

We now consider binary vectors in \( \{0,1\}^n \). We shall index each \( \xi \in \{0,1\}^n \) by \( \xi_s, s \in S \cap \Omega \). For every \( \xi = (\xi_s, s \in S \cap \Omega) \in \{0,1\}^n \), consider the function
\[
G_\xi(x) = f_0(x) + \frac{3}{4\sqrt{2\pi}} \sum_{s \in S \cap \Omega} \xi_s g_s(x).
\]

It can be verified that \( G_\xi \) is convex because \( f_0 \) has constant Hessian equal to 2 times the identity, the Hessian of \( g_s \) is bounded by \( (4\sqrt{2\pi}/3) \) and the supports of \( g_s, s \in S \cap \Omega \) have disjoint interiors. Note further that for \( \xi, \xi' \in \{0,1\}^n \) and \( s \in S \cap \Omega \),
\[
G_\xi(s) - G_{\xi'}(s) = \frac{3d\delta^2}{4\sqrt{2\pi}} (\xi_s - \xi'_s).
\]

This implies that
\[
\ell_{P_n}(G_\xi, G_{\xi'}) = \frac{3d\delta^2}{4\sqrt{2\pi}} \sqrt{\Upsilon(\xi, \xi')} \frac{\sqrt{n}}{n}
\]
where \( \Upsilon(\xi, \xi') := \sum_{s \in S \cap \Omega} I\{\xi_s \neq \xi'_s\} \) is the Hamming distance between \( \xi \) and \( \xi' \). The Varshamov-Gilbert lemma (see e.g., Massart [39, Lemma 4.7]) asserts the existence of a subset \( W \) of \( \{0,1\}^n \) with cardinality \( |W| \geq \exp(n/8) \) such that \( \Upsilon(\xi, \xi') \geq n/4 \) for all \( \xi, \xi' \in W \) with \( \xi \neq \xi' \). We then have
\[
\ell_{P_n}(G_\xi, G_{\xi'}) \geq \frac{3d\delta^2}{8\sqrt{2\pi}} \quad \text{for all } \xi, \xi' \in W \text{ with } \xi \neq \xi'.
\]

Inequality (11) then gives
\[
\ell_{P_n}(G_\xi, G_{\xi'}) \geq c_1 n^{-2/d} \quad \text{for all } \xi, \xi' \in W \text{ with } \xi \neq \xi'.
\]

for a constant \( c_1 \) depending on \( d \) alone. On the other hand, one can also check that
\[
\ell_{P_n}(G_\xi, f_0) \leq \frac{3d\delta^2}{4\sqrt{2\pi}} \leq c_2 n^{-2/d}
\]
for another constant \( c_2 \) depending on \( d \) alone. This completes the proof of Lemma 7.3. \( \square \)
Proof of Lemma 8.2. We first claim the existence of constants $c_1, c_2, C$ all depending on $d$ alone such that for every $\epsilon \leq c_1$ and $L \geq C$, there exist an integer $N$ with

$$c_2\epsilon^{-d/2} \leq \log N \leq 2c_2\epsilon^{-d/2}$$

and functions $f_1, \ldots, f_N \in C^L_L(\Omega)$ such that

$$\min_{1 \leq i \neq j \leq N} \ell_P(f_i, f_j) \geq \sqrt{2}\epsilon \quad \text{and} \quad \max_{1 \leq i \leq N} \sup_{x \in \Omega} |f_i(x) - f_0(x)| \leq 4\epsilon.$$

This basically follows from a similar construction as in the proof of Lemma 7.3 (with $\delta = \epsilon^2$). To facilitate calculations with $\ell_P$, it will be convenient to restrict the sum in (92) to all points $s = (k_1\delta, \ldots, k_d\delta)$ such that the cube (91) is fully contained in $\Omega$. The number of such grid points is also at least $c_d\delta^{-d}$ provided $\delta$ is smaller than a dimensional constant. Note also that each function (92) is bounded by $L$ and $L$-Lipschitz for a dimensional constant $L$.

Lemma 8.2 follows from the above claim and the Hoeffding inequality (note that $\sup_{x \in \Omega} (f_j(x) - f_k(x))^2 \leq 64\epsilon^2$). Indeed, for every $t > 0$, Hoeffding inequality followed by a union bound allows us to deduce that

$$\mathbb{P}\left\{ \ell_{\hat{P}_n}(f_j, f_k) - \ell_P(f_j, f_k) \geq -tn^{-1/2} \text{ for all } j, k \right\} \geq 1 - N^2 \exp\left(\frac{-t^2}{\Gamma}\right).$$

for a universal constant $\Gamma$. Taking $t = \epsilon^2\sqrt{n}$, we get

$$\mathbb{P}\left\{ \ell_{\hat{P}_n}(f_j, f_k) \geq \epsilon \text{ for all } j, k \right\} \geq 1 - N^2 \exp\left(\frac{-n}{\Gamma}\right) \geq 1 - \exp\left(4c_2\epsilon^{-d/2} - \frac{n}{\Gamma}\right).$$

Assuming now that

$$\epsilon \geq n^{-2/d}(8c_2\Gamma)^{2/d},$$

we get

$$\mathbb{P}\left\{ \ell_{\hat{P}_n}(f_j, f_k) \geq \epsilon \text{ for all } j, k \right\} \geq 1 - \exp\left(\frac{-n}{2\Gamma}\right).$$

Note finally that each $f_j$ belongs to $\mathcal{B}^{C^L_L(\Omega)}_{\hat{P}_n}(f_0, t)$ for $t \geq 4\epsilon$. This completes the proof of Lemma 8.2. \qed

Proof of Lemma 8.3. This proof is similar to that of Lemma 2.4. For a fixed $\eta > 0$, let $\mathcal{D}_\eta$ be the collection of all cubes of the form

$$[k_1\eta, (k_1 + 1)\eta] \times \cdots \times [k_d\eta, (k_d + 1)\eta]$$

for $(k_1, \ldots, k_d) \in \mathbb{Z}^d$ which are contained in the interior of $\Omega$. Because $\Omega$ is contained in the unit ball and contains a ball around zero of constant (depending on $d$ alone) radius and the diameter of $B$ is at most $\eta\sqrt{d}$, it follows that

$$(1 - C_d\eta) \Omega \subseteq \cup_{B \in \mathcal{D}_\eta} B \subseteq \Omega.$$
It is also easy to see that the cardinality of $D_\eta$ at most $C_d \eta^{-d}$ for $\eta \leq c_d$. We now triangulate each cube in $D_\eta$ into a constant number of $d$-simplices. Let $\Delta_1, \ldots, \Delta_m$ be the collection obtained by taking all of the aforementioned simplices as $B$ varies over $D_\eta$. These simplices clearly have disjoint interiors and the diameter of each simplex $\Delta_i$ is at most $C_d \eta$. Moreover

$$m \leq C_d \eta^{-d}.$$ 

Now define $\tilde{f}_\eta$ to be a piecewise affine convex function that agrees with $f_0(x) = \|x\|^2$ for each vertex of each simplex $\Delta_i$ and is defined by linearity everywhere else on $\Omega$. This function is clearly affine on each $\Delta_i$, belongs to $C_{C_d}(\Omega)$ for a sufficiently large $C_d$ and it satisfies

$$\sup_{x \in \Delta_i} |f_0(x) - \tilde{f}_\eta(x)| \leq C_d (\text{diameter}(\Delta_i))^2 \leq C_d \eta^2.$$ 

Now for each fixed $k \geq 1$, we take $\eta = c_d k^{-1/d}$ for a sufficiently small enough $c_d$ and let $\tilde{f}_k$ to be the function $\tilde{f}_\eta$ for this $\eta$. This completes the proof of Lemma 8.3.

References

[1] Aït-Sahalia, Y. and J. Duarte (2003). Nonparametric option pricing under shape restrictions. *J. Econometrics* 116(1-2), 9–47. Frontiers of financial econometrics and financial engineering.

[2] Allon, G., M. Beenstock, S. Hackman, U. Passy, and A. Shapiro (2007). Nonparametric estimation of concave production technologies by entropic methods. *J. Appl. Econometrics* 22(4), 795–816.

[3] Balázs, G. (2016). *Convex regression: theory, practice, and applications*. Ph. D. thesis, University of Alberta.

[4] Balázs, G., A. György, and C. Szepesvári (2015). Near-optimal max-affine estimators for convex regression. In *AISTATS*.

[5] Banaszczyk, W., A. E. Litvak, A. Pajor, and S. J. Szarek (1999). The flatness theorem for nonsymmetric convex bodies via the local theory of banach spaces. *Mathematics of operations research* 24(3), 728–750.

[6] Bárán, I. and D. G. Larman (1998). The convex hull of the integer points in a large ball. *Mathematische Annalen* 312(1), 167–181.

[7] Bellec, P. C. (2018). Sharp oracle inequalities for least squares estimators in shape restricted regression. *Ann. Statist.* 46(2), 745–780.

[8] Birgé, L. and P. Massart (1993). Rates of convergence for minimum contrast estimators. *Probab. Theory Related Fields* 97(1-2), 113–150.

[9] Bronšteĭn, E. M. (1976). $\varepsilon$-entropy of convex sets and functions. *Sibirsk. Mat. Ž.* 17(3), 508–514, 715.

[10] Chatterjee, S. (2014). A new perspective on least squares under convex constraint. *Ann. Statist.* 42(6), 2340–2381.

[11] Chatterjee, S. (2016). An improved global risk bound in concave regression. *Electron. J. Stat.* 10(1), 1608–1629.
[12] Chatterjee, S., A. Guntuboyina, and B. Sen (2015). On risk bounds in isotonic and other shape restricted regression problems. *Ann. Statist.* **43**(4), 1774–1800.

[13] Chen, W. and R. Mazumder (2020). Multivariate convex regression at scale. *arXiv preprint arXiv:2005.11588*.

[14] Chen, Y. and J. A. Wellner (2016). On convex least squares estimation when the truth is linear. *Electron. J. Stat.* **10**(1), 171–209.

[15] Doss, C. R. (2020). Bracketing numbers of convex and \( m \)-monotone functions on polytopes. *J. Approx. Theory* **256**, 105425.

[16] Dryanov, D. (2009). Kolmogorov entropy for classes of convex functions. *Constr. Approx.* **30**(1), 137–153.

[17] Dümbgen, L., S. Freitag, and G. Jongbloed (2004). Consistency of concave regression with an application to current-status data. *Math. Methods Statist.* **13**(1), 69–81.

[18] Gao, F. (2008). Entropy estimate for \( k \)-monotone functions via small ball probability of integrated Brownian motion. *Electron. Commun. Probab.* **13**, 121–130.

[19] Gao, F. and J. A. Wellner (2007). Entropy estimate for high-dimensional monotonic functions. *J. Multivariate Anal.* **98**(9), 1751–1764.

[20] Gao, F. and J. A. Wellner (2017). Entropy of convex functions on \( \mathbb{R}^d \). *Constr. Approx.* **46**(3), 565–592.

[21] Ghosal, P. and B. Sen (2017). On univariate convex regression. *Sankhya A* **79**(2), 215–253.

[22] Groeneboom, P. and G. Jongbloed (2014). *Nonparametric estimation under shape constraints*, Volume 38. Cambridge University Press.

[23] Groeneboom, P., G. Jongbloed, and J. A. Wellner (2001). Estimation of a convex function: characterizations and asymptotic theory. *Ann. Statist.* **29**(6), 1653–1698.

[24] Guntuboyina, A. (2012). Optimal rates of convergence for convex set estimation from support functions. *Ann. Statist.* **40**(1), 385–411.

[25] Guntuboyina, A. (2016). Covering numbers of \( \mathbb{L}_p \)-balls of convex functions and sets. *Constructive Approximation* **43**(1), 135–151.

[26] Guntuboyina, A. and B. Sen (2013). Covering numbers for convex functions. *IEEE Trans. Inf. Th.* **59**(4), 1957–1965.

[27] Guntuboyina, A. and B. Sen (2015). Global risk bounds and adaptation in univariate convex regression. *Probab. Theory Related Fields* **163**(1-2), 379–411.

[28] Han, Q. (2019). Global empirical risk minimizers with ”shape constraints” are rate optimal in general dimensions. *arXiv preprint arXiv:1905.12823*.

[29] Han, Q., T. Wang, S. Chatterjee, and R. J. Samworth (2019). Isotonic regression in general dimensions. *Ann. Statist.* **47**(5), 2440–2471.

[30] Han, Q. and J. A. Wellner (2016). Multivariate convex regression: global risk bounds and adaptation. *arXiv preprint arXiv:1601.06844*.

[31] Hanson, D. L. and G. Pledger (1976). Consistency in concave regression. *Ann. Statist.* **4**(6), 1038–1050.

[32] Hildreth, C. (1954). Point estimates of ordinates of concave functions. *J. Amer. Statist. Assoc.* **49**, 598–619.
[33] Keshavarz, A., Y. Wang, and S. Boyd (2011). Imputing a convex objective function. In Intelligent Control (ISIC), 2011 IEEE International Symposium on, pp. 613–619. IEEE.

[34] Kuosmanen, T. (2008). Representation theorem for convex nonparametric least squares. The Econometrics Journal 11(2), 308–325.

[35] Kur, G., Y. Dagan, and A. Rakhlin (2019). Optimality of maximum likelihood for log-concave density estimation and bounded convex regression. arXiv preprint arXiv:1903.05315.

[36] Kur, G., A. Guntuboyina, and A. Rakhlin (2020). On suboptimality of least squares with application to estimation of convex bodies. Accepted for presentation and publication in the 33rd Annual Conference on Learning Theory (COLT) July 9-12, 2020.

[37] Lim, E. (2014). On convergence rates of convex regression in multiple dimensions. INFORMS J. Comput. 26(3), 616–628.

[38] Lim, E. and P. W. Glynn (2012). Consistency of multidimensional convex regression. Oper. Res. 60(1), 196–208.

[39] Massart, P. (2007). Concentration inequalities and model selection. Lecture notes in Mathematics, Volume 1896. Berlin: Springer.

[40] Matzkin, R. L. (1991). Semiparametric estimation of monotone and concave utility functions for polychotomous choice models. Econometrica 59(5), 1315–1327.

[41] Mazumder, R., A. Choudhury, G. Iyengar, and B. Sen (2019). A computational framework for multivariate convex regression and its variants. J. Amer. Statist. Assoc. 114(525), 318–331.

[42] Seijo, E. and B. Sen (2011). Nonparametric least squares estimation of a multivariate convex regression function. Ann. Statist. 39(3), 1633–1657.

[43] Toriello, A., G. Nemhauser, and M. Savelsbergh (2010). Decomposing inventory routing problems with approximate value functions. Naval Res. Logist. 57(8), 718–727.

[44] van de Geer, S. A. (2000). Applications of empirical process theory, Volume 6 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.

[45] Varian, H. R. (1982). The nonparametric approach to demand analysis. Econometrica 50(4), 945–973.

[46] Varian, H. R. (1984). The nonparametric approach to production analysis. Econometrica 52(3), 579–597.