On a generalisation of Krein’s example

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We generalise a classical example given by Krein in 1953. We compute the difference of the resolvents and the difference of the spectral projections explicitly. We further give a full description of the unitary invariants, i.e., of the spectrum and the multiplicity. Moreover, we observe a link between the difference of the spectral projections and Hankel operators.

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1 Introduction and main results

1.1 Introduction

Krein presented in [13] a rigorous definition of the spectral shift function \( \xi = \xi(\bullet, A_1, A_0) \in L_1(\mathbb{R}) \) defined via

\[
\text{tr}(\chi(A_1) - \chi(A_0)) = \int_{\mathbb{R}} \chi'(\vartheta)\xi(\vartheta) \, d\vartheta,
\]

whenever \( \chi \) belongs to a suitable class of functions and \( A_1 - A_0 \) is of trace class. In a naive definition, one would choose the indicator function \( \chi = 1_{(-\infty, \vartheta)} \), as the above formula then becomes formally

\[
\text{tr}(1_{(-\infty, \vartheta)}(A_1) - 1_{(-\infty, \vartheta)}(A_0)) = \xi(\vartheta).
\]

Unfortunately, formula (1.1) is not true: even if \( A_1 - A_0 \) is a rank 1 perturbation (and hence of trace class), the difference of the spectral projections \( 1_{(-\infty, \vartheta)}(A_1) - 1_{(-\infty, \vartheta)}(A_0) \) need not to be of trace class, i.e., the left hand side of (1.1) is not defined. Krein presented such an example in his paper [13], where \( A_1 = (H+1)^{-1} \) and \( A_0 = (H_D+1)^{-1} \) are the resolvents at the spectral point \(-1\) of the Neumann and Dirichlet Laplacian \( H = (-\frac{d^2}{dx^2})^N \) and \( H_D = (-\frac{d^2}{dx^2})^D \) on the half-line \( \mathbb{R}_+ = (0, \infty) \), respectively. Krein showed that the difference of the spectral projections is an integral operator given by

\[
\left(1_{(-\infty, \vartheta)}(A_0) - 1_{(-\infty, \vartheta)}(A_1)\right) \psi(t) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\sin((\frac{t}{2} - 1)^{1/2}(t + \tau))}{t + \tau} \psi(\tau) \, d\tau
\]

\[1\] Maybe this is a fortune as it gave rise to new research ...
for $t \in \mathbb{R}_+$, $0 < \vartheta < 1$ and $\psi \in C_c(\mathbb{R}_+)$, and hence not Hilbert Schmidt. Kostrykin and Makarov diagonalised the integral operator of (1.2) and proved that it has a simple purely absolutely continuous spectrum filling in the interval $[-1, 1]$; in particular, the integral operator of (1.2) is not compact, see [12]. Note that the kernel function of the integral operator of (1.2) depends only on the sum of the variables; such operators on $L_2(\mathbb{R}_+)$ are called Hankel (integral) operators. We refer to Peller’s monograph [19] for an overview on Hankel operators.

Relations between differences of spectral projections and Hankel operators are also discussed in the work of Pushnitski [22, 23, 24] and together with Yafaev [25, 26] in the framework of scattering theory, related to an idea of Peller [18]. We also refer to [28] for an approach based on a result of Megretski˘ı, Peller, and Treil [16].

In this paper we generalise Krein’s example by considering operators of the type

$$H = \left( -\frac{d^2}{dt^2} \right)^N \otimes \text{id} + \text{id} \otimes L \quad \text{and} \quad H^D = \left( -\frac{d^2}{dt^2} \right)^D \otimes \text{id} + \text{id} \otimes L \text{ in } L_2(\mathbb{R}_+) \otimes \mathcal{H},$$

(1.3)

where $\mathcal{H} \neq \{0\}$ is a separable complex Hilbert space and $L$ is a self-adjoint nonnegative operator on $\mathcal{H}$ (precise definitions are given in Section 3). We call $H$ resp. $H^D$ the (abstract) Neumann resp. Dirichlet operator. In particular, this framework includes

(a) Krein’s example of the half-line $\mathbb{R}_+$ with $L = 0$ and $\mathcal{H} = \mathbb{C}$;

(b) the example of the classical half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ with $L = -\Delta_{\mathbb{R}^{n-1}}$ and $n \geq 2$;

(c) the case when $L$ is (minus) the Laplacian on a generally noncompact manifold $Y$, e.g. on the cylinder $\mathbb{R}_+ \times Y$ with Neumann resp. Dirichlet boundary conditions on $\{0\} \times Y$.

We consider the resolvents

$$A_0 = (H^D + 1)^{-1} \quad \text{and} \quad A_1 = (H + 1)^{-1}$$

(1.4)

of the operators $H^D$ and $H$ defined in (1.3) at the spectral point $-1$. The difference $A_1 - A_0$ of the resolvents will be computed with the help of a Krein-type resolvent formula from the theory of boundary pairs [21].

Next we would like to compute the difference $\mathbf{1}_{(-\infty, \vartheta)}(A_0) - \mathbf{1}_{(-\infty, \vartheta)}(A_1)$ of the spectral projections for all $0 < \vartheta < 1$. It is generally hard to compute differences of spectral projections explicitly. In our example, however, the computation can be performed, using the transformation formula for spectral measures and the above mentioned convolution-type formula from [29]. This idea is borrowed from Krein’s example.

We give a full description of the unitary invariants of the resolvent difference and of the difference of the spectral projections. Moreover, the spectral properties establish a link between the difference of the spectral projections and Hankel operators.

Operators of the type (1.3) have been studied before; criteria for self-adjointness (see, e.g., Schmüdgen’s monograph [27]), the spectrum (see, e.g., [27] or Weidmann’s monograph [29]), and a convolution-type formula for the spectral projection (see [29]) are known and will be very useful in this paper. There are classical works on spectral theory of self-adjoint boundary value problems with operator-valued potential as in (1.3), see, e.g., Gorbachuk and Kutovoi [7, 8, 10, 11, 14] and the monograph [9]. Gorbachuk and Kutovoi showed in [10] that $A_1 - A_0$ is trace class if and only if (in the present notation) $(L + 1)^{-1}$ is trace class. Sufficient
1.2 Main results

Criteria for $A_1 - A_0$ to belong to Schatten classes can be found in [11]. The proofs rely on the resolvent identities and the ideal properties of Schatten classes; the resolvent difference is not computed explicitly in [10, 11].

Abstract boundary value problems have often been treated using operator theory. We refer to the review article [4] for an overview on boundary triplets and also to [21] for the concept of boundary pairs, see also the references therein. Such concepts allow for example to calculate differences of resolvents of operators with different boundary conditions. There are related works by Boitsev, Neidhardt, and Popov [3] on tensor products of boundary triplets (with bounded operator $L$), Malamud and Neidhardt [15] for unitary equivalence and regularity properties of different self-adjoint realisations, Gesztesy, Weikard, and Zinchenko [5, 6] for a general spectral theory of Schrödinger operators with bounded operator potentials, and Mogilevskii [17], see also the references therein. Moreover, when finishing this paper, the authors of the present paper have learned about the recent paper [2], where Boitsev, Brasche, Malamud, Neidhardt and Popov construct a boundary triplet for the adjoint of the symmetric operator $T \otimes \text{id} + \text{id} \otimes L$ with $T$ being symmetric and $L$ being self-adjoint. This generalises the situation of (1.3), where $T = -d^2/dt^2$ on $L^2(\mathbb{R}^+)$. The focus in [2] is on self-adjoint extensions which do not respect the tensor structure (1.3) as models for quantum systems coupled to a reservoir. Note that in [15, 2] one has to “regularise” the boundary triplet (i.e., one has to modify the boundary map and spectrally decompose $L$ into bounded operators) in order to treat also unbounded operators $L$. In our approach, we can directly treat unbounded operators $L$ without changing the boundary map or decomposing $L$. The special case of operators $L$ with purely discrete spectrum has been treated e.g. in [21, Sec. 6.4] or in a slightly different setting in [20, Sec. 3.5.1].

The results of this paper will be part of the PhD thesis of the second author at Johannes Gutenberg University Mainz.

1.2 Main results

Let $A_0$ and $A_1$ be the resolvents defined in (1.4) of the (abstract) Dirichlet and Neumann operators given in (1.3) above.

1.1 Theorem. (a) The resolvent difference $A_1 - A_0$ acts on elementary tensors as follows:

$$
([A_1 - A_0](\psi \otimes \varphi))(t) = \int_{\mathbb{R}^+} \psi(\tau) \exp(-((L + 1)^{1/2}(t + \tau))(L + 1)^{-1/2}\varphi \, d\tau
$$

for all $\psi \in C_c(\mathbb{R}^+)$ and all $\varphi \in \mathcal{G}$.

(b) Let $0 < \vartheta < 1$ and let $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$. Then the difference of the spectral projections of $A_0$ and $A_1$ associated with the open interval $(-\infty, \vartheta)$ acts on elementary tensors as follows:

$$
\left(\left[1_{(-\infty, \vartheta)}(A_0) - 1_{(-\infty, \vartheta)}(A_1)\right](\psi \otimes \varphi)\right)(t) = \frac{2}{\pi} \int_{\mathbb{R}^+} \psi(\tau) 1_{[0, \alpha(\vartheta)]}(L) \frac{\sin((\alpha(\vartheta) - L)^{1/2}(t + \tau))}{t + \tau} \varphi \, d\tau
$$

for all $\psi \in C_c(\mathbb{R}^+)$ and all $\varphi \in \mathcal{G}$.
If we represent $L$ as multiplication operator by the independent variable on a von Neumann direct integral (see below), then a scaling transformation yields the following beautiful representation with separated variables for the resolvent difference $A_1 - A_0$:

1.2 Theorem. The resolvent difference $A_1 - A_0$ is unitarily equivalent to

$$
\left(\left[\left(-\frac{d^2}{dt^2}\right)^N + 1\right]^{-1} - \left[\left(-\frac{d^2}{dt^2}\right)^D + 1\right]^{-1}\right) \otimes (L + 1)^{-1} \text{ on } L_2(\mathbb{R}_+) \otimes \mathcal{G}.
$$

For brevity let us write $\sigma = \sigma(L)$ for the spectrum of $L$. It is well known that $L$ is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral $\int_\sigma \mathcal{G}(\lambda) \, d\nu(\lambda)$, see Theorem 2.3 below. Moreover, from Krein’s example \cite{13} we know that the first factor (the difference of the Neumann and Dirichlet resolvent) in the previous theorem is a rank 1 operator with eigenvalue 0 of infinite multiplicity and simple eigenvalue 1/2. Hence we conclude:

1.3 Corollary. One has

$$\sigma(A_1 - A_0) = \{0\} \cup \left\{\frac{1}{2(\lambda + 1)} : \lambda \in \sigma\right\},$$

and the spectral decomposition of $A_1 - A_0$ is as follows:

(a) $0$ is an eigenvalue of infinite multiplicity.

(b) For $\bullet \in \{p, ac, sc\}$ one has $\sigma_\bullet(A_1 - A_0) \setminus \{0\} = \left\{\frac{1}{2(\lambda + 1)} : \lambda \in \sigma_\bullet\right\}$, and the multiplicity of $\frac{1}{2(\lambda + 1)}$ (with respect to $A_1 - A_0$) coincides with the multiplicity of $\lambda$ (with respect to $L$) for $d\mu_\bullet$-almost all $\lambda$.

In particular, $A_1 - A_0$ is compact if and only if $L$ has a purely discrete spectrum.\footnote{This is equivalent with $(L + 1)^{-1}$ being compact, cf. \cite{10}, \cite{11}.}

The spectral decomposition of the difference of the spectral projections looks as follows:

1.4 Theorem. Let $0 < \vartheta < 1$ and let $\alpha(\vartheta) = \frac{1}{\vartheta} - 1 > 0$. Then one has:

(a) $\sigma_p(\mathbb{I}_{(-\infty, \vartheta)}(A_0) - \mathbb{I}_{(-\infty, \vartheta)}(A_1)) = \left\{-1, 1\right\} \text{ if } \mu(\sigma \cap [0, \alpha(\vartheta))] > 0 \text{ and } \mu(\sigma \cap [0, \alpha(\vartheta)]) = 0.

(b) $\sigma_p(\mathbb{I}_{(-\infty, \vartheta)}(A_0) - \mathbb{I}_{(-\infty, \vartheta)}(A_1)) = \left\{0\right\} \text{ if } \mu(\sigma \cap [\alpha(\vartheta), \infty]) = 0 \text{ and } \mu(\sigma \cap [\alpha(\vartheta), \infty]) > 0.

If $\mu(\sigma \cap [\alpha(\vartheta), \infty]) > 0$ then the multiplicity of the eigenvalue 0 is infinite.

(c) $\sigma_{ac}(\mathbb{I}_{(-\infty, \vartheta)}(A_0) - \mathbb{I}_{(-\infty, \vartheta)}(A_1)) = \left\{-1, 1\right\} \text{ if } \mu(\sigma \cap [0, \alpha(\vartheta))] > 0 \text{ and } \mu(\sigma \cap [0, \alpha(\vartheta)]) = 0.

If $\mu(\sigma \cap [0, \alpha(\vartheta))] > 0$ then the (uniform) multiplicity of the absolutely continuous spectrum equals the dimension of $\int_{\sigma(0, \alpha(\vartheta))} \mathcal{G}(\lambda) \, d\nu(\lambda)$.\footnote{This is equivalent with $(L + 1)^{-1}$ being compact, cf. \cite{10}, \cite{11}.}
1.3 Structure of the article

(d) The singular continuous spectrum is empty.

Let us close this subsection with a remark and an example.

1.5 Remark (Link to Hankel operators). Observe that $1_{(-\infty, \vartheta)}(A_0) - 1_{(-\infty, \vartheta)}(A_1)$ is unitarily equivalent to its negative, that its kernel is either trivial or infinite dimensional, and that zero belongs to its spectrum, for all $0 < \vartheta < 1$. Consequently, the characterisation theorem of bounded self-adjoint Hankel operators [16, Theorem 1] implies that $1_{(-\infty, \vartheta)}(A_0) - 1_{(-\infty, \vartheta)}(A_1)$ is always unitarily equivalent to a Hankel integral operator on $L_2(\mathbb{R}_+)$.

1.6 Example (Classical half-space). If $L$ is the free Laplacian on $\mathbb{R}^{n-1}$ for some $n \geq 2$ then the difference of the spectral projections associated with $(-\infty, \vartheta)$ has infinite dimensional kernel, and its (absolutely continuous) spectrum equals $[-1, 1]$ and is of infinite multiplicity, for all $0 < \vartheta < 1$.

1.3 Structure of the article

In Section 2 we briefly present the main tool of our analysis, namely the concept of boundary pairs, some facts on the tensor product of operators, and the von Neumann direct integral decomposition of a self-adjoint operator. In Section 3 we apply the theory of boundary pairs to our example and calculate the related objects explicitly. In particular, we establish Theorem 1.1 (a). Section 4 contains the proof of Theorem 1.2. In Section 5 we establish Theorem 1.1 (b) and Theorem 1.4.

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2 Tools

2.1 Boundary pairs

Let us briefly explain the concept of boundary pairs which is basically an abstract version of boundary value problems for elliptic operators defined via their quadratic forms. Details can be found in [21].

Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{h}$ a closed and densely defined quadratic form with domain $\mathcal{H}^1 = \text{dom}(\mathfrak{h})$ (i.e., $\mathcal{H}^1$ with its intrinsic norm defined by $\|u\|_\mathfrak{h}^2 = \mathfrak{h}(u) + \|u\|_\mathcal{H}^2$ is complete).

A boundary pair $(\Gamma, \mathcal{G})$ associated with $\mathfrak{h}$ is a pair given by another Hilbert space $\mathcal{G}$ and a bounded map $\Gamma: \mathcal{H}^1 \rightarrow \mathcal{G}$ such that the kernel (null space) $\ker(\Gamma)$ is dense in $\mathcal{H}$ and such that the range $\mathcal{G}^{1/2} = \text{ran}(\Gamma)$ is dense in $\mathcal{G}$.

Given a boundary pair $(\Gamma, \mathcal{G})$ associated with $\mathfrak{h}$, we can define the following objects:

- the (abstract) Neumann operator $H$ as the operator associated with the closed form $\mathfrak{h}$;
• the (abstract) Dirichlet operator $H^D$ as the operator associated with the closed form $\mathbf{h}|_{\ker \Gamma}$;
• the space of weak solutions $\mathcal{N}^1(z) = \{ h \in \mathcal{H}^1 \mid \langle \mathbf{h} (h, f) = z \langle h, f \rangle \forall f \in \ker \Gamma = \mathcal{H}^{1, D} \}$;
• for $z \notin \sigma(H^D)$, $\mathcal{H}^1 = \mathcal{H}^{1, D} + \mathcal{N}^1(z)$ (direct sum with closed subspaces); in particular, the (Dirichlet) solution operator $S(z) = \left( \Gamma^l \right)^{-1} : \ker \Gamma = \mathcal{H}^{1/2} \rightarrow \mathcal{N}^1(z) \subset \mathcal{H}^1$ is defined;
• for $z \notin \sigma(H^D)$, the Dirichlet-to-Neumann (sesquilinear) form $I_z$ is defined via $I_z(\varphi, \psi) = \langle \mathbf{h} - z1 \rangle(S(z)\varphi, S(-1)\psi)$, $\varphi, \psi \in \mathcal{H}^{1/2}$;
• we endow $\mathcal{H}^{1/2}$ with the norm given by $\| \varphi \|_{\mathcal{H}^{1/2}}^2 = I_{-1}(\varphi) = \| S\varphi \|^2$.

We say that a boundary pair $(\Gamma, \mathcal{G})$ associated with $\mathbf{h}$ is elliptically regular if the associated Dirichlet solution operator $S = S(-1) : \mathcal{H}^{1/2} \rightarrow \mathcal{H}$ extends to a bounded operator $\bar{S} : \mathcal{G} \rightarrow \mathcal{H}$, or equivalently, if there exists a constant $c > 0$ such that $\| S\varphi \|_{\mathcal{H}} \leq c\| \varphi \|_{\mathcal{G}}$ for all $\varphi \in \mathcal{G}$. We call $\bar{S}$ the extended solution operator. For an elliptic boundary pair, the Dirichlet-to-Neumann form $I_z$ is sectorial, and the associated operator, the Dirichlet-to-Neumann operator $\Lambda(z)$ has domain independent of $z$.

The main example is the following: let $X$ be an open subset of $\mathbb{R}^n$ with smooth boundary $Y = \partial X$. Let $\mathcal{H} = L_2(X), f(u) = \int_X |\nabla u(x)|^2 \, dx$, $\text{dom}(\mathbf{h}) = H^1(X)$. Moreover, let $\Gamma u = u|_Y$, i.e., $\Gamma$ is the (Sobolev) trace map. Under suitable conditions (e.g., $Y$ is compact or some curvature assumptions of $Y$), $\Gamma : H^1(X) \rightarrow L_2(Y)$ is bounded, where we consider $Y$ as a Riemannian manifold with its natural $(n-1)$-dimensional measure. In our example above we have $X = \mathbb{R}^n_+$ and $Y = \{0\} \times \mathbb{R}^{n-1}$. Then $H$ resp. $H^D$ is the Neumann resp. Dirichlet Laplacian; $\mathcal{N}^1(z)$ the space of weak solutions of $(-\Delta - z)h = 0$ with $h \in H^1(X)$; $S(z)$ is the solution operator, associating to $\varphi \in \ker(\Gamma)$ the weak solution $h$ with $\Gamma h = \varphi$. Moreover, $\Lambda(z)$ is the classical Dirichlet-to-Neumann operator, associating to a boundary function $\varphi : Y \rightarrow \mathbb{C}$ the normal derivative of the function $h \in \mathcal{N}^1(z)$ with $\Gamma h = \varphi$.

For elliptic boundary pairs, we have the following Krein-type formula
\[ R(z) - R^D(z) = \bar{S}(z)\Lambda(z)^{-1}\bar{S}(\bar{z})^*, \]
(see [21 Thm. 4.2 (ii)]), where $R(z) = (H - z)^{-1}$ and $R^D(z) = (H^D - z)^{-1}$ are the resolvents of the Neumann resp. Dirichlet operator.

### 2.2 Tensor product of operators

In this subsection we fix some notation and briefly discuss how a result from [27] about cores for certain self-adjoint product type operators carries over to the forms associated with these operators; furthermore, we present three facts on operators of this product type.

Let $T_k \geq 0$ be a self-adjoint operator on a complex Hilbert space $\mathcal{H}_k$ with domain $\text{dom}(T_k)$, where $k = 1, 2$. We write $\mathcal{H}_1 \otimes \mathcal{H}_2$ for the usual Hilbert space tensor product and $\mathcal{H}_1 \odot \mathcal{H}_2$ for the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Let $T \in \{T_1, T_2\}$. Recall (see [27, p. 145]) that a vector $f \in \bigcap_{m=1}^{\infty} \text{dom}(T^m)$ is called bounded for $T$ if there exists a constant $B_f > 0$ such that $\| T^m f \| \leq B_f^m$ for every $m \in \mathbb{N}$. In this case we write $f \in \mathcal{D}^b(T)$.

It follows from [27, Theorem 7.23] and [27, Exercise 17.a] that the operator $T_1 \otimes \text{id} + \text{id} \otimes T_2$ is self-adjoint and that the subspace
\[ \mathcal{D}_b = \text{span}\{f_1 \otimes f_2 : f_1 \in \mathcal{D}^b(T_1), f_2 \in \mathcal{D}^b(T_2)\} \quad (2.1) \]
2.1 Proposition. The subspace $D_b$ of $H_1 \otimes H_2$ defined in (2.1) is a core for the form associated with the self-adjoint operator $T_1 \otimes \text{id} + \text{id} \otimes T_2$.

Proof. For brevity let us write $H = T_1 \otimes \text{id} + \text{id} \otimes T_2$ and $H = H_1 \otimes H_2$. It suffices to show that $D_b$ is a core for the self-adjoint operator $H^{1/2}$, see [27, Proposition 10.5].

It is well known (see, e.g., [27, Corollary 4.14]) that the domain of $T_1 \otimes \text{id} + \text{id} \otimes T_2$ is a Hilbert space. The induced norm is denoted by $\| \cdot \|$. Since $D_b$ is a core for $H^{1/2}$, we can choose a sequence $(x_m)$ in $D_b$ such that $x_m \to x$ in $H$ and $Hx_m \to Hx$ in $H$ as $m \to \infty$. It follows directly from the functional calculus for self-adjoint operators and the obvious inequality $\lambda \leq 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$ that $H^{1/2}x_m \to H^{1/2}x$ in $H$ as $m \to \infty$. Consequently $D_b$ is a core for $H^{1/2}$, as claimed.

Here are three more facts on operators of the type $T_1 \otimes \text{id} + \text{id} \otimes T_2$.

2.2 Proposition. Let, as above, $T_1$ and $T_2$ be nonnegative self-adjoint operators.

(a) $\sigma(T_1 \otimes \text{id} + \text{id} \otimes T_2) = \{t_1 + t_2 : t_k \in \sigma(T_k), \ k = 1, 2\}$.

(b) For all $\alpha \in \mathbb{R}$, all $f_1, g_1 \in H_1$, and all $f_2, g_2 \in H_2$ one has

$$\langle 1_{(-\infty, \alpha)}(T_1 \otimes \text{id} + \text{id} \otimes T_2)(f_1 \otimes f_2), g_1 \otimes g_2 \rangle_{H_1 \otimes H_2} = \int_{-\infty}^{\alpha} \langle 1_{(-\infty, \alpha - \lambda)}(T_1)f_1, g_1 \rangle_{H_1} \, d\langle 1_{\lambda}(T_2)f_2, g_2 \rangle_{H_2}.$$

(c) The operator $T_1 \otimes \text{id} + \text{id} \otimes T_2$ has a purely absolutely continuous spectrum if $T_1$ has a purely absolutely continuous spectrum.

Proof. Part (a) follows from [27, Corollary 7.25] and [27, Exc. 18.a]; for Part (b), see [29, Thm 8.34] and for Part (c) see [15, Prp A.2 (iv)].

2.3 The von Neumann direct integral

The theory of von Neumann direct integrals is one of the main tools in this paper; for a theoretical background we refer to [1, Chapter 7]. In this subsection we fix some notation and discuss how the theory of von Neumann direct integrals can be applied in our example.

Given a positive finite Borel measure $\mu$ on $\mathbb{R}$ we denote the von Neumann direct integral of separable complex Hilbert spaces $\mathcal{H}(\lambda)$ by $\mathcal{H} = \int_{\mathbb{R}} \mathcal{H}(\lambda) \, d\mu(\lambda)$. Any element $\varphi \in \mathcal{H}$ takes values $\varphi(\lambda) \in \mathcal{H}(\lambda)$ for $d\mu$-almost all $\lambda \in \sigma$. We will use the notation $\varphi = \int_{\mathbb{R}} \varphi(\lambda) \, d\mu(\lambda)$. The von Neumann direct integral $\mathcal{H}$ together with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \langle \varphi_1(\lambda), \varphi_2(\lambda) \rangle_{\mathcal{H}(\lambda)} \, d\mu(\lambda), \quad \varphi_1, \varphi_2 \in \mathcal{H},$$

is a Hilbert space. The induced norm is denoted by $\| \cdot \|_{\mathcal{H}}$. We assume without loss of generality that $\mathcal{H}(\lambda) \neq \{0\}$ for $d\mu$-almost every $\lambda$. Further we identify the Hilbert spaces $\int_{\mathbb{R}} \mathcal{H}(\lambda) \, d\mu(\lambda)$ and $\int_{\text{supp}(\mu)} \mathcal{H}(\lambda) \, d\mu(\lambda)$, where $\text{supp}(\mu)$ denotes the support of the measure $\mu$. We will make use of the following well-known fact:
2.3 **Theorem ([11, Theorem 1, p.177]).** Every self-adjoint operator on a separable complex Hilbert space is unitarily equivalent to the multiplication operator by the independent variable on a von Neumann direct integral.

Except for Subsection 3.7 we will suppose in Sections 3–5:

2.4 **Assumption.** The operator \( L \) in (1.3) acts by multiplication by the independent variable on a von Neumann direct integral \( G = \int_R^\oplus \mathcal{G}(\lambda) \, d\mu(\lambda) \neq \{0\} \).

2.5 **Remark.** With this assumption we do not forfeit generality. This is clear in view of Theorem 1, Corollary 1.3, and Theorem 1.4. In view of Theorem 1.1 we will show in Subsections 3.7 and 5.1 below that the corresponding results from Proposition 3.18 and Lemma 5.2 naturally carry over to the situation when \( L \) is not necessarily a multiplication operator.

3 **The boundary pair of the generalised half-space problem**

Let \( \mathcal{G} \) be a (non-trivial) separable Hilbert space and \( \mathcal{H} = L_2(\mathbb{R}_+, \mathcal{G}) \). As \( \mathcal{H} \) and \( L_2(\mathbb{R}_+) \otimes \mathcal{G} \) are naturally isometrically isomorphic, we will very often identify \( \psi(\bullet) \varphi \) with \( \psi \otimes \varphi \) for all \( \psi \in L_2(\mathbb{R}_+) \) and \( \varphi \in \mathcal{G} \).

3.1 **The form and its associated operator**

Let us consider the nonnegative form \( h \) on \( \mathcal{H}^1 = H^1(\mathbb{R}_+, \mathcal{G}) \cap L_2(\mathbb{R}_+, \text{dom}(L^{1/2})) \) defined by

\[
h(u) = \int_{\mathbb{R}_+} \left( \|u'(t)\|^2_{\mathcal{G}} + \|L^{1/2}(u(t))\|^2_{\mathcal{G}} \right) dt,
\]

where \( \text{dom}(L^{1/2}) \) is equipped with the graph norm of \( L^{1/2} \). It is easy to see that \( h \) is closed.

Let \( H \) be the self-adjoint operator

\[
H = \left( -\frac{d^2}{dt^2} \right)^N \otimes \text{id} + \text{id} \otimes L \quad \text{on} \quad L_2(\mathbb{R}_+) \otimes \mathcal{G}.
\]

Using the above-mentioned identification of \( \mathcal{H} = L_2(\mathbb{R}_+, \mathcal{G}) \) with \( L_2(\mathbb{R}_+) \otimes \mathcal{G} \), one can show that

\[
\text{dom}(H) = \{ u \in H^2(\mathbb{R}_+, \mathcal{G}) \cap L_2(\mathbb{R}_+, \text{dom}(L)) : u'(0) = 0 \},
\]

see [15, Proposition 5.2].

3.1 **Lemma.** The operator \( H \) is associated with the form \( h \).

**Proof.** For all \( u \in \text{dom}(H) \) and all \( v \in \mathcal{H}^1 \) we have

\[
h(u, v) = \int_{\mathbb{R}_+} \{ \langle u'(t), v'(t) \rangle_{\mathcal{G}} + \langle L^{1/2}(u(t)), L^{1/2}(v(t)) \rangle_{\mathcal{G}} \} \, dt

= \int_{\mathbb{R}_+} \{ \langle -u''(t), v(t) \rangle_{\mathcal{G}} + \langle L(u(t)), v(t) \rangle_{\mathcal{G}} \} \, dt = \langle Hu, v \rangle_{\mathcal{H}},
\]

where we used integration by parts and the self-adjointness of \( L^{1/2} \). Since \( H \) is self-adjoint the claim follows. \( \square \)
Recall that
\[
\mathcal{D}_b = \text{span}\{\psi \otimes \varphi : \psi \in \mathcal{D}(\mathbb{R}; (-d^2/dt)^N), \varphi \in \mathcal{D}(L)\} \subset L^2(\mathbb{R}_+) \otimes \mathcal{G}
\]
is a core for \(H\) as well as for \(\mathfrak{h}\) by Subsection 2.2.

Functions of the type
\[
h: \mathbb{R}_+ \to \mathcal{G}, \quad t \mapsto h(t) = \int_{\sigma}^{\oplus} \exp(i\sqrt{z} - \lambda t)\varphi(\lambda) \, d\mu(\lambda) \quad (3.1)
\]
will play an important role in this paper. Here, \(\sqrt{z}\) is the square root cut along the positive half-axis. First of all we have to check that \(h\) is in \(\mathcal{H}\) for all \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\) and all \(\varphi \in \mathcal{G}\).

**3.2 Lemma.** Let \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\) and let \(\varphi \in \mathcal{G}\). Then the function \(h: \mathbb{R}_+ \to \mathcal{G}\) defined in (3.1) is continuous and \(h \in \mathcal{H}\).

**Proof.** For every \(t \in \mathbb{R}_+\) one has \(\|h(t)\|_\mathcal{G} \leq \|\varphi\|_\mathcal{G} < \infty\) so \(h\) is \(\mathcal{G}\)-valued. By the dominated convergence theorem we see that \(\mathbb{R}_+ \ni t \mapsto h(t) \in \mathcal{G}\) is continuous. Consequently, \(h\) is measurable and we compute
\[
\|h\|_\mathcal{H}^2 \leq \int_{\mathbb{R}_+} dt \int_{\sigma}^{\oplus} d\mu(\lambda) \exp(-2^{1/2} (|z| - \Re(z))^{1/2} t) \|\varphi(\lambda)\|_{\mathcal{G}(\lambda)} \quad (3.2)
\]
\[
= \frac{1}{2^{1/2} (|z| - \Re(z))^{1/2}} \|\varphi\|_\mathcal{G}^2 < \infty.
\]

Next we show:

**3.3 Lemma.** Let \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\) and let \(\varphi \in \text{dom}(L^{1/4})\). Then the function \(h: \mathbb{R}_+ \to \mathcal{G}\) defined in (3.1) is also in \(\mathcal{H}^1\).

**Proof.** First consider the case when \(\varphi \in \text{dom}(L)\). By Lemma 3.2 we know that \(h \in \mathcal{H}\), and it is straightforward to show that \(h \in \mathcal{H}^1\); note that \(h^t\) exists in the strong sense.

Now consider the case when \(\varphi \in \text{dom}(L^{1/4})\). Again, Lemma 3.2 shows that \(h\) is in \(\mathcal{H}\). Since \(\text{dom}(L)\) is a core for \(L^{1/4}\) we can approximate \(\varphi\) by a sequence \((\varphi_m) \subset \text{dom}(L)\) with respect to the graph norm of \(L^{1/4}\). Straightforward computations show that \(\|h - h_m\|_\mathcal{H} \xrightarrow{m \to \infty} 0\) and \(\mathfrak{h}(h_k - h_m) \xrightarrow{k,m \to \infty} 0\), where \(h_m = \int_{\sigma}^{\oplus} \exp(i\sqrt{z} - \lambda t)\varphi_m(\lambda) \, d\mu(\lambda)\) for all \(m \in \mathbb{N}\). Consequently, the closedness of \(\mathfrak{h}\) yields:
\[
h \in \mathcal{H}^1\quad \text{and} \quad \|h - h_m\|_\mathfrak{h} \xrightarrow{m \to \infty} 0. \quad (3.3)
\]

This completes the proof of the lemma.

**3.2 The boundary operator**

As boundary operator we will choose the restriction to \(\mathcal{H}^1\) of the usual boundary operator on the Sobolev space \(H^1(\mathbb{R}_+, \mathcal{G})\) that evaluates a given function at zero, i.e., we define the boundary operator \(\Gamma: \mathcal{H}^1 \to \mathcal{G}\) by \(\Gamma u = u(0)\).

**3.4 Lemma.** One has \(\|\Gamma\| \leq 2\).
3.2 The boundary operator

**Proof.** Let \( u \in \mathcal{H}^1 \). Define the Lipschitz continuous function \( \chi : [0, \infty) \to [0, 1] \) by

\[
\chi(t) = 1 - t \text{ if } 0 \leq t < 1 \quad \text{and} \quad \chi(t) = 0 \text{ if } t \geq 1.
\]

Then one has

\[
u(0) = -((\chi \cdot u)(1) - (\chi \cdot u)(0)) = -\int_0^1 (\chi \cdot u)'(t) \, dt = -\int_0^1 (\chi'(t) \cdot u(t) + \chi(t) \cdot u'(t)) \, dt.
\]

The result now follows from

\[
\|\Gamma u\|_{\mathcal{G}}^2 \leq 2 \int_0^1 \|\chi'(t) \cdot u(t) + \chi(t) \cdot u'(t)\|_{\mathcal{G}}^2 \, dt
\leq 4 \int_0^1 (\|u(t)\|_{\mathcal{G}}^2 + \|u'(t)\|_{\mathcal{G}}^2) \, dt
\leq 4\|u\|_{\mathcal{H}}^2.
\]

The proof of the following lemma is straightforward:

**3.5 Lemma.** The kernel of \( \Gamma \) is dense in \( \mathcal{H} \) with respect to the norm \( \|\cdot\|_{\mathcal{H}} \), and the range of \( \Gamma \) is dense in \( \mathcal{G} \).

Next we define the form \( h^D = h|_{\mathcal{H}^{1,0}} \) on the closed subspace \( \mathcal{H}^{1,0} = \ker(\Gamma) \) of \( \mathcal{H}^1 \). Then \( h^D \) is a densely defined nonnegative closed form. We call \( H^D \), the self-adjoint operator associated with \( h^D \), the Dirichlet operator. We shall show that the Dirichlet operator coincides with the self-adjoint operator

\[
\left(-\frac{d^2}{dt^2}\right)^D \otimes \text{id} + \text{id} \otimes L \quad \text{on } L_2(\mathbb{R}_+) \otimes \mathcal{G}.
\]

We know (see Subsection 2.2 above) that

\[
\mathcal{D}_b^D = \text{span}\{\psi \otimes \varphi : \psi \in \mathcal{D}((-\frac{d^2}{dt^2})^D), \varphi \in \mathcal{D}^b(L)\} \subset L_2(\mathbb{R}_+) \otimes \mathcal{G}
\]

is an invariant core for \((-\frac{d^2}{dt^2})^D \otimes \text{id} + \text{id} \otimes L\). Note that \( \mathcal{D}_b^D \subset \ker(\Gamma) \).

**3.6 Lemma.** The Dirichlet operator is given by \( H^D = \left(-\frac{d^2}{dt^2}\right)^D \otimes \text{id} + \text{id} \otimes L \).

**Proof.** For brevity we shall write \( \tilde{H}^D = \left(-\frac{d^2}{dt^2}\right)^D \otimes \text{id} + \text{id} \otimes L \). We will show that \( \tilde{H}^D \) is associated with \( h^D \). This is proven in three steps:

**Step 1.** Integration by parts yields \( h^D(u, f) = \langle \tilde{H}^D u, f \rangle \) for all \( u, f \in \mathcal{D}_b^D \).

**Step 2.** Let \( u \in \mathcal{D}_b^D \) and let \( \tilde{f} \in \ker(\Gamma) \). Choose \( (f_k) \subset \mathcal{D}_b \) with \( \|\tilde{f} - f_k\|_b \underset{k \to \infty}{\longrightarrow} 0 \). Integration by parts yields \( h(u, f_k) = \langle \tilde{H}^D u, f_k \rangle - \langle \Gamma(u'), \Gamma f_k \rangle \) for \( u \in \text{dom}(L) \subset \mathcal{G} \).

As \( k \) tends to infinity we obtain that \( h(u, f_k) \to h(u, \tilde{f}) = h^D(u, \tilde{f}) \) and, on the other hand,

\[
\langle \tilde{H}^D u, f_k \rangle - \langle \Gamma(u'), \Gamma f_k \rangle \to \langle \tilde{H}^D u, f \rangle - \langle \Gamma(u'), \Gamma f \rangle = \langle \tilde{H}^D u, f \rangle.
\]

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Step 3. Let \( \tilde{u} \in \text{dom}(\tilde{H}^D) \) and let \( \tilde{f} \in \ker(\Gamma) \). Choose \( (u_m) \subset \mathcal{D}_b^D \) with \( \|\tilde{u} - u_m\|_{\tilde{H}^D} \xrightarrow{m \to \infty} 0 \).

Then, by Step 1 and the positivity of \( \tilde{H}^D \), one has

\[
\mathcal{H}^D(u_k - u_m) = \langle \tilde{H}^D(u_k - u_m), u_k - u_m \rangle_{\mathcal{H}} \leq \|u_k - u_m\|_{\mathcal{H}}^2 \text{ for all } k, m \in \mathbb{N}
\]

so \( (u_m)_m \) is Cauchy with respect to \( \|\cdot\|_{\mathcal{H}} \). Since \( \mathcal{H}^D \) is closed it follows that \( \tilde{u} \in \ker(\Gamma) \) and \( \|\tilde{u} - u_m\|_{\mathcal{H}} \xrightarrow{m \to \infty} 0 \). As \( m \) tends to infinity we obtain that \( \mathcal{H}^D(u_m, \tilde{f}) \to \mathcal{H}^D(\tilde{u}, \tilde{f}) \) and, on the other hand, \( \langle \tilde{H}^D u_m, \tilde{f} \rangle_{\mathcal{H}} \to \langle \tilde{H}^D \tilde{u}, \tilde{f} \rangle_{\mathcal{H}} \). Consequently,

\[
\mathcal{H}^D(\tilde{u}, \tilde{f}) = \langle \tilde{H}^D \tilde{u}, \tilde{f} \rangle_{\mathcal{H}}
\]

and thus \( \tilde{H}^D \subset H^D \). Since \( \tilde{H}^D \) and \( H^D \) are both self-adjoint we conclude that \( \tilde{H}^D = H^D \). \( \square \)

3.7 Lemma.  
(a) The operators \( H \) and \( H^D \) are unitarily equivalent.

(b) The spectrum of \( H \) is purely absolutely continuous filling in the interval \( [\min \sigma, \infty) \); the same is true for \( H^D \).

Proof. (a) Since the Neumann and Dirichlet Laplacians on \( L_2(\mathbb{R}^+) \) are unitarily equivalent it follows that \( H \) and \( H^D \) are also unitarily equivalent. Part (b) follows from Proposition 2.2 \( \square \)

3.8 Remark. One can actually show that the domain of \( H^D \) is given by

\[
\text{dom}(H^D) = \{ u \in H^2(\mathbb{R}^+, \mathcal{G}) \cap L_2(\mathbb{R}^+, \text{dom}(L)) : u(0) = 0 \},
\]

see \([15] \) Proposition 5.2.

3.3 The solution operator and the range of the boundary operator

Let \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \). Define

\[
\mathcal{N}^1(z) = \{ h \in \mathcal{H}^1 : \mathcal{H}(h, f) = z \langle h, f \rangle_{\mathcal{H}} \text{ for all } f \in \ker(\Gamma) \}.
\]

The so-called solution operator, given formally by \( S(z) = (\Gamma|_{\mathcal{N}^1(z)})^{-1} \), associates to a boundary value \( \varphi \in \text{ran}(\Gamma) \) the unique element \( h \in \mathcal{N}^1(z) \) such that \( \Gamma h = \varphi \) (see \([21] \) Prp 2.9).

3.9 Lemma. One has \( \text{dom}(L^{1/4}) \subset \text{ran}(\Gamma) \) and, for every \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \),

\[
S(z)|_{\text{dom}(L^{1/4})} \varphi = \int_{\sigma} \exp(i\sqrt{z - \lambda} \cdot \bullet) \varphi(\lambda) \, d\mu(\lambda).
\]  

Proof. The lemma is proven in two steps. First we show that \( \text{dom}(L) \subset \text{ran}(\Gamma) \) and \( \text{(3.4)} \) holds on \( \text{dom}(L) \). Then, by approximation, we obtain that \( \text{dom}(L^{1/4}) \subset \text{ran}(\Gamma) \) and \( \text{(3.4)} \) holds on \( \text{dom}(L^{1/4}) \).
Step 1. Let \( \varphi \in \text{dom}(L) \) and let \( h = \int_{\sigma}^\oplus \exp \left( i \sqrt{z - \lambda} \cdot \right) \varphi(\lambda) \, d\mu(\lambda) \). By Lemma 3.3 we know that \( h \in \mathcal{H}^1 \) and hence \( \Gamma h = \varphi \). It remains to show that \( h \in \mathcal{N}^1(z) \). This is proven as follows:

Let \( \Phi \in \mathcal{D}_b \). A straightforward computation shows that

\[
\mathfrak{h}(h, \Phi) = \langle h', \Phi' \rangle_{\mathcal{H}} + \int_{\mathbb{R}^+} \langle L(h(t)), \Phi(t) \rangle_{\mathcal{Y}} \, dt = z \langle h, \Phi \rangle_{\mathcal{H}} - i \int_{\sigma} \langle \sqrt{z - \lambda} \varphi(\lambda), (\Gamma \Phi)(\lambda) \rangle_{\mathcal{Y}} \, d\mu(\lambda).
\]

Now let \( f \in \ker(\Gamma) \). Choose a sequence \( (\Phi_m) \subset \mathcal{D}_b \) with \( \| f - \Phi_m \|_b \xrightarrow{m \to \infty} 0 \). Clearly \( \mathfrak{h}(h, \Phi_m) \xrightarrow{m \to \infty} \mathfrak{h}(h, f) \) and \( z \langle h, \Phi_m \rangle_{\mathcal{H}} \xrightarrow{m \to \infty} z \langle h, f \rangle_{\mathcal{H}} \), and an easy computation shows that \( -i \int_{\sigma} \langle \sqrt{z - \lambda} \varphi(\lambda), (\Gamma \Phi_m)(\lambda) \rangle_{\mathcal{Y}} \, d\mu(\lambda) \xrightarrow{m \to \infty} 0 \). It follows that \( h \in \mathcal{N}^1(z) \).

Step 2. Let \( \varphi \in \text{dom}(L^{1/4}) \) and let \( h = \int_{\sigma}^\oplus \exp \left( i \sqrt{z - \lambda} \cdot \right) \varphi(\lambda) \, d\mu(\lambda) \). Again, we know by Lemma 3.3 that \( h \in \mathcal{H}^1 \) and hence \( \Gamma h = \varphi \).

Now choose a sequence \( (\varphi_m) \subset \text{dom}(L) \) with \( \| \varphi - \varphi_m \|_{L^{1/4}} \xrightarrow{m \to \infty} 0 \). By Step 1 we know that \( h_m = \int_{\sigma}^\oplus \exp \left( i \sqrt{z - \lambda} \cdot \right) \varphi_m(\lambda) \, d\mu(\lambda) \in \mathcal{N}^1(z) \) for all \( m \in \mathbb{N} \), and (3.3) implies that \( \| h - h_m \|_b \xrightarrow{m \to \infty} 0 \). Consequently, \( h \in \mathcal{N}^1(z) \). This completes the proof of the lemma.

The following proposition shows that \( \text{ran}(\Gamma) \subset \text{dom}(L^{1/4}) \) so, in fact, \( S(z) \mid_{\text{dom}(L^{1/4})} = S(z) \).

3.10 Proposition. One has \( \text{ran}(\Gamma) \subset \text{dom}(L^{1/4}) \).

Proof. We decompose \( \mathcal{H}^1 \) into the orthogonal sum of \( \mathcal{N}^1 = \mathcal{N}^1(-1) \) and \( \ker(\Gamma) \). Since \( \Gamma \) is linear it suffices to show that \( \Gamma h \in \text{dom}(L^{1/4}) \) for all \( h \in \mathcal{N}^1 \). This is proven in four steps:

Step 1. Let \( h \in \mathcal{N}^1 \). Choose a sequence \( (\tilde{h}_m) \subset \mathcal{D}_b \) with \( \| h - \tilde{h}_m \|_b \xrightarrow{m \to \infty} 0 \). Put

\[
\tilde{h}_m = P_{\mathcal{N}^1} \tilde{h}_m, \quad m \in \mathbb{N},
\]

where \( P_{\mathcal{N}^1} \) denotes the orthogonal projection of \( \mathcal{H}^1 \) onto \( \mathcal{N}^1 \).

Step 2. Let \( m \in \mathbb{N} \) and set \( \varphi_m = \Gamma \tilde{h}_m \). Then one has:

\[
\varphi_m = \Gamma P_{\mathcal{N}^1} \tilde{h}_m = \Gamma P_{\mathcal{N}^1} \tilde{h}_m + \Gamma P_{\ker(\Gamma)} \tilde{h}_m = \Gamma \tilde{h}_m \in \text{dom}(L),
\]

where \( P_{\ker(\Gamma)} \) denotes the orthogonal projection of \( \mathcal{H}^1 \) onto \( \ker(\Gamma) \). By Lemma 3.3 we know that

\[
\int_{\sigma}^\oplus \exp \left( - (1 + \lambda)^{1/2} \cdot \right) \varphi_m(\lambda) \, d\mu(\lambda) \in \mathcal{N}^1 \quad \text{and} \quad \Gamma \left( \int_{\sigma}^\oplus \exp \left( - (1 + \lambda)^{1/2} \cdot \right) \varphi_m(\lambda) \, d\mu(\lambda) \right) = \varphi_m.
\]

Since \( \Gamma\mid_{\mathcal{N}^1} \) is injective we thus obtain:

\[
h_m = \int_{\sigma}^\oplus \exp \left( - (1 + \lambda)^{1/2} \cdot \right) \varphi_m(\lambda) \, d\mu(\lambda).
\]
Step 3. Clearly \( \|h - h_m\|_h = \|P_{\mathcal{H}}(h - \tilde{h}_m)\|_h \leq \|h - \tilde{h}_m\|_h \xrightarrow{m \to \infty} 0 \). It follows that
\[
\|\Gamma h - \varphi\|_\mathcal{G} = \|\Gamma h - \Gamma h_m\|_\mathcal{G} \xrightarrow{m \to \infty} 0.
\]

Step 4. We already know that \((h_m)_m\) is a Cauchy sequence with respect to \(\|\cdot\|_h\). A straightforward computation shows that
\[
\|h_k - h_m\|_h^2 \geq \int_{\sigma} \lambda^{1/2} \|\varphi_k(\lambda) - \varphi_m(\lambda)\|_{\mathcal{G}(\lambda)}^2 \exp\left(-2(1 + \lambda)^{1/2} t\right) dt d\mu(\lambda)
\]
\[
= \frac{1}{2} \int_{\sigma} \left(\frac{\lambda}{1 + \lambda}\right)^{1/2} \lambda^{1/2} \|\varphi_k(\lambda) - \varphi_m(\lambda)\|_{\mathcal{G}(\lambda)}^2 d\mu(\lambda)
\]
for all \(k, m \in \mathbb{N}\). Choose \(\lambda_0 > 0\) large enough such that \((\lambda/(1 + \lambda))^{1/2} \geq 1/2\) for all \(\lambda \geq \lambda_0\). Then we obtain:
\[
\|\varphi_k - \varphi_m\|_{L^{1/4}}^2 \leq \left(1 + \lambda_0^{1/2}\right) \|\varphi_k - \varphi_m\|_{\mathcal{G}}^2 + 4\|h_k - h_m\|_h^2 \xrightarrow{k, m \to \infty} 0.
\]

Since \(\text{dom}(L^{1/4})\) is complete with respect to \(\|\cdot\|_{L^{1/4}}\) there exists \(\varphi \in \text{dom}(L^{1/4})\) such that \(\|\varphi - \varphi_m\|_{L^{1/4}} \xrightarrow{m \to \infty} 0\). Consequently, one has \(\Gamma h = \varphi \in \text{dom}(L^{1/4})\), as claimed.

3.11 Remark. \(\Gamma\) is surjective if and only if \(L\) is bounded.

We have thus computed the solution operator \(S(z)\) at every point \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\). In particular, if \(z = -1\) and \(\varphi \in \text{dom}(L^{1/4})\) then \((3.2)\) tells us that
\[
\|S(-1)\varphi\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|\varphi\|_{\mathcal{G}}^2.
\]
This inequality proves:

3.12 Lemma. The boundary pair \((\Gamma, \mathcal{G})\) is elliptically regular.

3.4 The extended solution operator and its adjoint

Let \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\). According to \((3.2)\) we know that
\[
\mathcal{G} \ni \varphi \mapsto \int_{\sigma} \exp(i\sqrt{z - \lambda} \bullet) \varphi(\lambda) d\mu(\lambda) \in \mathcal{H}
\]
defines a bounded operator. In the preceding subsection we have shown that the solution operator \(S(z): \text{ran}(\Gamma) \to \mathcal{H}^1 \subset \mathcal{H}\) is given by \(S(z)\varphi = \int_{\sigma} \exp(i\sqrt{z - \lambda} \bullet) \varphi(\lambda) d\mu(\lambda)\). As, by Lemma \(3.10\) \(\text{ran}(\Gamma)\) is dense in \(\mathcal{G}\), we can extend this formula to all of \(\mathcal{G}\):

3.13 Lemma. If \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\) then the unique bounded extension of \(S(z)\) to \(\mathcal{G}\) is given by
\[
\tilde{S}(z): \mathcal{G} \to \mathcal{H}, \quad \tilde{S}(z) \varphi = \int_{\sigma} \exp(i\sqrt{z - \lambda} \bullet) \varphi(\lambda) d\mu(\lambda).
\]

Next we compute the adjoint of the extended solution operator.
3.14 Lemma. If \( z \in \mathbb{C} \setminus \left[ \min \sigma, \infty \right) \) then the bounded operator \((\bar{S}(\bar{z}))^* : H \to G\) acts on elementary tensors as follows:

\[
(\bar{S}(\bar{z}))^* (\psi \otimes \eta) = \int_{\sigma} \left( \int_{\mathbb{R}^+} \psi(t) \exp(i\sqrt{z - \lambda} t) \, dt \right) \eta(\lambda) \, d\mu(\lambda)
\]

for all \( \psi \in C_c(\mathbb{R}^+) \) and all \( \eta \in G \). Consequently, \((\bar{S}(\bar{z}))^*\) can be evaluated explicitly on the dense subspace \( C_c(\mathbb{R}^+) \odot G \) of \( H \).

**Proof.** Standard arguments show that

\[
\int_{\sigma} \left( \int_{\mathbb{R}^+} \psi(t) \exp(i\sqrt{z - \lambda} t) \, dt \right) \eta(\lambda) \, d\mu(\lambda) \in G.
\]

(3.5)

Let \( \varphi \in G \). By Fubini’s theorem,

\[
\left\langle (\bar{S}(\bar{z}))^* (\psi \otimes \eta), \varphi \right\rangle_G = \left\langle \psi \otimes \eta, \bar{S}(\bar{z}) \varphi \right\rangle_H
= \int_{\sigma} \int_{\mathbb{R}^+} \left\langle \psi(t) \exp(i\sqrt{z - \lambda} t) \eta(\lambda), \varphi(\lambda) \right\rangle_G \, dt \, d\mu(\lambda).
\]

It is easily seen that \( \exp(i\sqrt{z - \lambda} t) = \exp(i\sqrt{z - \lambda} t) \). Therefore, (3.5) implies

\[
\left\langle (\bar{S}(\bar{z}))^* (\psi \otimes \eta), \varphi \right\rangle_G = \left\langle \int_{\sigma} \left( \int_{\mathbb{R}^+} \psi(t) \exp(i\sqrt{z - \lambda} t) \, dt \right) \eta(\lambda) \, d\mu(\lambda), \varphi \right\rangle_G.
\]

Since \( \varphi \in G \) was arbitrary this proves the lemma. \( \square \)

3.5 The Dirichlet-to-Neumann operator

We can think of the Dirichlet-to-Neumann operator \( \Lambda(z) \) as follows (see [21, top of p. 1053]): it maps certain boundary values \( \varphi \in \text{dom}(\Lambda(z)) \subset \text{dom}(L^{1/4}) \) to the “normal” derivative \( \partial_n h \) of the corresponding Dirichlet solution \( h \). In our situation this means:

\[
\Lambda(z) \varphi = -\frac{\partial}{\partial t} \left. \left( S(z) \varphi \right) \right|_{t=0}
= -\int_{\sigma} i\sqrt{z - \lambda} \exp(i\sqrt{z - \lambda} t) \varphi(\lambda) \, d\mu(\lambda) \bigg|_{t=0}
= -\int_{\sigma} i\sqrt{z - \lambda} \varphi(\lambda) \, d\mu(\lambda).
\]

As we will show in Lemma 3.17 below, this formal computation indeed gives us the correct result.

Let \( z \in \mathbb{C} \setminus \left[ \min \sigma, \infty \right) \). Define \( I_z : \text{dom}(L^{1/4}) \times \text{dom}(L^{1/4}) \to \mathbb{C} \) by

\[
I_z(\varphi, \eta) = h(S(z) \varphi, S(-1) \eta) - z(S(z) \varphi, S(-1) \eta),_G.
\]

Then, by [21] Theorem 2.12, \( I_z \) is a bounded form. We call \( I_z \) the **Dirichlet-to-Neumann form**. One has:
3.5 The Dirichlet-to-Neumann operator

3.15 Lemma. If \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \) then \( I_z \) is given by

\[
I_z(\varphi, \eta) = \int_{\sigma} \left( -i \sqrt{z - \lambda} \right) \langle \varphi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}(\lambda)} \, d\mu(\lambda)
\]  

(3.6)

for all \( \varphi, \eta \in \text{dom}(L^{1/4}) \).

Proof. The lemma is proven in two steps. First we show (3.6) for \( \varphi, \eta \in \text{dom}(L) \), and then we complete the proof by approximation.

Step 1. Let \( \varphi, \eta \in \text{dom}(L) \). Using Lemma 3.3 and Fubini’s theorem we compute:

\[
I_z(\varphi, \eta) = \int_{\sigma} \langle \varphi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}(\lambda)} \cdot \int_{\mathbb{R}^+} \exp \left( \left( i \sqrt{z - \lambda} - (1 + \lambda)^{1/2} \right) t \right) dt \left( i \sqrt{z - \lambda} \left( - (1 + \lambda)^{1/2} + \lambda - z \right) \right) d\mu(\lambda)
\]

\[
= \int_{\sigma} \left( -i \sqrt{z - \lambda} \right) \langle \varphi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}(\lambda)} d\mu(\lambda).
\]

Step 2. Let \( \varphi, \eta \in \text{dom}(L^{1/4}) \). Choose two sequences \( \{ \varphi_m \} \subset \text{dom}(L) \) and \( \{ \eta_m \} \subset \text{dom}(L) \) such that \( \| \varphi - \varphi_m \|_{L^{1/4}}^{m \to \infty} \to 0 \) and \( \| \eta - \eta_m \|_{L^{1/4}}^{m \to \infty} \to 0 \). By (3.3) we know that

\[
\| S(z) \varphi - S(z) \varphi_m \|_b^{m \to \infty} \to 0 \quad \text{and} \quad \| S(z) \eta - S(z) \eta_m \|_b^{m \to \infty} \to 0.
\]

Consequently,

\[
h \left( S(z) \varphi_m, S(-1) \eta_m \right) - z \left( S(z) \varphi_m, S(-1) \eta_m \right)_{\mathcal{H}} \xrightarrow{m \to \infty} h \left( S(z) \varphi, S(-1) \eta \right) - z \left( S(z) \varphi, S(-1) \eta \right)_{\mathcal{H}}.
\]

Furthermore a straightforward computation shows that

\[
\int_{\sigma} \left( -i \sqrt{z - \lambda} \right) \langle \varphi_m(\lambda), \eta_m(\lambda) \rangle_{\mathcal{G}(\lambda)} d\mu(\lambda) \xrightarrow{m \to \infty} \int_{\sigma} \left( -i \sqrt{z - \lambda} \right) \langle \varphi(\lambda), \eta(\lambda) \rangle_{\mathcal{G}(\lambda)} d\mu(\lambda).
\]

Thus (3.6) holds and the lemma is proven. \( \square \)

As the boundary pair \( (\Gamma, \mathcal{G}) \) is elliptically regular, it follows from [21, Theorem 3.8] that the Dirichlet-to-Neumann form is closed and sectorial for all \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \). Let \( \Lambda(z) \) be the closed operator associated with \( I_z \), i.e.,

\[
\text{dom}(\Lambda(z)) = \{ \varphi \in \text{dom}(L^{1/4}) : \exists \zeta \in \mathcal{G} \forall \eta \in \text{dom}(L^{1/4}), I_z(\varphi, \eta) = \langle \zeta, \eta \rangle_{\mathcal{G}} \}
\]

(3.7)

and \( \Lambda(z) \varphi = \zeta \). We call \( \Lambda(z) \) the Dirichlet-to-Neumann operator. One has:

3.16 Lemma. If \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \) then \( \text{dom}(\Lambda(z)) \supset \text{dom}(L^{1/2}) \) and

\[
\Lambda(z) \left|_{\text{dom}(L^{1/2})} \right. \varphi = \int_{\sigma} \left( -i \sqrt{z - \lambda} \right) \varphi(\lambda) \, d\mu(\lambda).
\]
3.5 The Dirichlet-to-Neumann operator

Proof. Let \( \varphi \in \text{dom}(L^{1/2}) \). Then

\[
\zeta = \int_\sigma (-i \sqrt{z - \lambda}) \varphi(\lambda) \, d\mu(\lambda) \quad \text{is in } \mathcal{G}.
\]

Therefore, Lemma \[3.15\] implies that \( I_z(\varphi, \eta) = (\zeta, \eta)_G \) for all \( \eta \in \text{dom}(L^{1/4}) \). This proves the lemma.

Furthermore it follows from [21, Theorem 3.8] that \( \text{dom}(\Lambda(z)) = \text{dom}(\Lambda(-1)) \) is independent of \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \). The next lemma shows that \( \text{dom}(\Lambda(-1)) \subset \text{dom}(L^{1/2}) \) so, in fact, \( \Lambda(z)|_{\text{dom}(L^{1/2})} = \Lambda(z) \).

3.17 Lemma. One has \( \text{dom}(\Lambda(-1)) \subset \text{dom}(L^{1/2}) \).

Proof. First we observe that for all \( \eta \in \text{dom}(L^{1/4}) \) we have

\[
\int_\sigma (1 + \lambda)^{-1/4} \eta(\lambda) \, d\mu(\lambda) \in \text{dom}(L^{1/4}).
\]

Now let \( \varphi \in \text{dom}(\Lambda(-1)) \). Choose \( \zeta \in \mathcal{G} \) according to (3.7). Then clearly

\[
\int_\sigma (1 + \lambda)^{-1/4} \zeta(\lambda) \, d\mu(\lambda) \in \mathcal{G}
\]

and, since \( \text{dom}(\Lambda(-1)) \subset \text{dom}(L^{1/4}) \),

\[
\int_\sigma (1 + \lambda)^{1/4} \varphi(\lambda) \, d\mu(\lambda) \in \mathcal{G}.
\]

Consequently, for all \( \eta \in \text{dom}(L^{1/4}) \), Lemma \[3.15\] implies:

\[
0 = I_{-1}(\varphi, \int_\sigma (1 + \lambda)^{-1/4} \eta(\lambda) \, d\mu(\lambda)) - \left( \zeta, \int_\sigma (1 + \lambda)^{-1/4} \eta(\lambda) \, d\mu(\lambda) \right)_G
\]

\[
= \left( \int_\sigma (1 + \lambda)^{1/4} \varphi(\lambda) \, d\mu(\lambda) - \int_\sigma (1 + \lambda)^{-1/4} \zeta(\lambda) \, d\mu(\lambda), \eta \right)_G.
\]

As \( \text{dom}(L^{1/4}) \) is dense in \( \mathcal{G} \) we obtain that, for \( d\mu \)-almost all \( \lambda \) in \( \sigma \),

\[
(1 + \lambda)^{1/4} \varphi(\lambda) = (1 + \lambda)^{-1/4} \zeta(\lambda).
\]

Therefore, \( \int_\sigma (1 + \lambda)^{1/2} \varphi(\lambda) \, d\mu(\lambda) = \zeta \in \mathcal{G} \) and thus \( \varphi \in \text{dom}(L^{1/2}) \), as claimed.

In particular, for all \( z \in \mathbb{C} \setminus [\min \sigma, \infty) \), the Neumann-to-Dirichlet operator

\[
\Lambda(z)^{-1} : \mathcal{G} \to \mathcal{G}, \quad \Lambda(z)^{-1} \varphi = \int_\sigma \frac{i}{\sqrt{z - \lambda}} \varphi(\lambda) \, d\mu(\lambda),
\]

is bounded.
3.6 A Krein-type resolvent formula

We have now computed the extended solution operator as well as its adjoint and the Neumann-
to-Dirichlet operator. Putting these results together we obtain, since the boundary pair \((\Gamma, \mathcal{G})\)
is elliptically regular, the following Krein-type resolvent formula for \((H - z)^{-1} - (H^D - z)^{-1}\).

3.18 Proposition. Let \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\). Then \((H - z)^{-1} - (H^D - z)^{-1} : \mathcal{H} \rightarrow \mathcal{H}\) satisfies
\[
(H - z)^{-1} - (H^D - z)^{-1} = \bar{S}(z)\Lambda(z)^{-1}(\bar{S}(z))^*.
\] (3.9)

This operator acts on elementary tensors as follows:
\[
(\bar{S}(z)\Lambda(z)^{-1}(\bar{S}(z))^*(\psi \otimes \varphi))(t) = \int_{\sigma}^{\infty} \frac{i}{\sqrt{z - \lambda}} \varphi(\lambda) \int_{\mathbb{R}^+} \psi(\tau) \exp(i\sqrt{z - \lambda}(t + \tau)) d\tau \, d\mu(\lambda)
\]
for all \(\psi \in C_c(\mathbb{R}^+\big)\) and all \(\varphi \in \mathcal{G}\). Consequently, the difference of the resolvents from (3.9)
can be evaluated explicitly on the dense subspace \(C_c(\mathbb{R}^+) \otimes \mathcal{G}\) of \(\mathcal{H}\).

Proof. By Lemma 3.12 we know that the boundary pair \((\Gamma, \mathcal{G})\) is elliptically regular. Therefore, [21, Theorem 1.2] implies (3.9). The explicit representation of (3.9) on \(C_c(\mathbb{R}^+) \otimes \mathcal{G}\) follows directly from Lemma 3.13, (3.8), and Lemma 3.14.

3.7 Explicit formulas for the boundary pair of the generalised half-space problem

Let us summarise the explicit formulas we have found for the boundary pair of the generalised half-space problem, written in a more handy version without referring to the direct integral representation of \(L\):

3.19 Proposition. Let \(z \in \mathbb{C} \setminus [\min \sigma, \infty)\). One has:

(a) The solution operator \(S(z) : \text{dom}(L^{1/4}) \rightarrow \mathcal{H}^1\) is given by
\[
(S(z)\varphi)(t) = \exp(i\sqrt{z - L} t) \varphi.
\]
In particular, \(\|S(-1)\varphi\|^2_{\mathcal{H}} \leq \frac{1}{2}\|\varphi\|^2_{\mathcal{G}}\) for every \(\varphi \in \text{dom}(L^{1/4})\) so \((\Gamma, \mathcal{G})\) is an elliptically regular boundary pair.

(b) The Dirichlet-to-Neumann operator \(\Lambda(z) : \text{dom}(L^{1/2}) \rightarrow \mathcal{G}\) is given by
\[
\Lambda(z)\varphi = i\sqrt{z - L} \varphi.
\]

(c) The difference of the resolvents of \(H\) and \(H^D\) acts on elementary tensors as follows:
\[
\left(\left[(H - z)^{-1} - (H^D - z)^{-1}\right] (\psi \otimes \varphi)\right)(t) = i \int_{\mathbb{R}^+} \psi(\tau) \exp(i\sqrt{z - L}(t + \tau))(\sqrt{z - L})^{-1}\varphi \, d\tau
\]
for all \(\psi \in C_c(\mathbb{R}^+)\) and all \(\varphi \in \mathcal{G}\).

\(^3\)In the case when \(L\) is bounded, cf. [15, equation (4.3)].

\(^4\)In the case when \(L\) is bounded, cf. [15, Lemma 4.2].

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4 A formula with separated variables for the difference of the resolvents

Proof. The results from Lemma 3.9, Proposition 3.10, Lemma 3.12, Lemma 3.16, Lemma 3.17, and Proposition 3.18 carry over to the situation when $L$ is not necessarily a multiplication operator, using Theorem 2.3 and the functional calculus.

Proof of Theorem 1.1. Set $z = -1$ in Proposition 3.19 (c).

4 A formula with separated variables for the difference of the resolvents

In this section we will establish Theorem 1.2. The outline of the proof is as follows:

Step 1. We change the order of evaluation with respect to the variables $t \in \mathbb{R}_+$ and $\lambda \in \sigma$ in the representation formula from Proposition 3.18. Then, for $d\mu$-almost all $\lambda$ in $\sigma$, we will obtain a vector-valued Hankel-type integral operator.

Step 2. The application of a scaling transformation will lead to a unitarily equivalent representation of (3.9) with separated variables, as claimed.

Step 1 will be performed in Subsection 4.1 and Step 2 will be performed in Subsection 4.2. Finally, in Subsection 4.3 we will deduce Corollary 1.3 from Theorem 1.2.

4.1 Proof of Theorem 1.2. Step 1

First, we observe that

$$W : \mathcal{C}_c(\mathbb{R}_+) \otimes \mathcal{G} \subset \mathcal{H} \to \int_{\sigma}^\oplus L_2(\mathbb{R}_+) \otimes \mathcal{G}(\lambda) \, d\mu(\lambda), \quad W(\psi \otimes \varphi) = \int_{\sigma}^\oplus \psi \otimes \varphi(\lambda) \, d\mu(\lambda),$$

defines an isometric operator with dense range. We denote the unique bounded extension of $W$ to $\mathcal{H}$ by the same symbol $W$. Obviously, $W$ is a unitary operator from $\mathcal{H}$ onto $\int_{\sigma}^\oplus L_2(\mathbb{R}_+) \otimes \mathcal{G}(\lambda) \, d\mu(\lambda)$. The similarity transformation with respect to the natural unitary operator $W$ leads to the expected result:

4.1 Lemma. If $z \in \mathbb{C} \setminus [\min \sigma, \infty)$ then, for all $\psi \in \mathcal{C}_c(\mathbb{R}_+)$ and all $\varphi \in \mathcal{G}$, one has

$$\left( W \tilde{S}(z) \Lambda(z)^{-1} (\tilde{S}(z))^* (\psi \otimes \varphi) \right)(\lambda) = \frac{i}{\sqrt{z} - \lambda} \int_{\mathbb{R}_+} \psi(\tau) \exp(i\sqrt{z} - \lambda (\bullet + \tau)) \, d\tau \otimes \varphi(\lambda)$$

for $d\mu$-almost all $\lambda$ in $\sigma$.

Proof. This is a consequence of Proposition 3.18 and Fubini’s theorem.

In particular, Lemma 4.1 shows that

$$W^{-1}[(H - z)^{-1} - (H^D - z)^{-1}]W = \int_{\sigma}^\oplus T_\lambda \, d\mu(\lambda),$$

where

$$T_\lambda(\psi \otimes \varphi_\lambda) = \frac{i}{\sqrt{z} - \lambda} \int_{\mathbb{R}_+} \psi(\tau) \exp(i\sqrt{z} - \lambda (\bullet + \tau)) \, d\tau \otimes \varphi_\lambda$$

(4.1)

for all $\psi \in \mathcal{C}_c(\mathbb{R}_+)$, all $\varphi_\lambda \in \mathcal{G}(\lambda)$ and $d\mu$-almost all $\lambda \in \sigma$. We write $T = \int_{\sigma}^\oplus T_\lambda \, d\mu(\lambda)$. 

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4.2 Proof of Theorem 1.2. Step 2

For the rest of this subsection we assume that

$$z \in (-\infty, \min \sigma).$$

It is then clear that $\lambda - z > 0$ and hence $i\sqrt{z - \lambda} = -(\lambda - z)^{1/2}$ for all $\lambda \in \sigma$. Therefore,

$$U_\lambda : L_2(\mathbb{R}_+, \mathcal{G}(\lambda)) \rightarrow L_2(\mathbb{R}_+, \mathcal{G}(\lambda)), \quad (U_\lambda f)(t) = (\lambda - z)^{1/4} f((\lambda - z)^{1/2} t),$$

is a unitary operator for every fixed $\lambda$ outside a set of $d\mu$-measure 0, and the operator-valued function $U = \int_\sigma U_\lambda \, d\mu(\lambda)$ defines a unitary operator on $\int_\sigma L_2(\mathbb{R}_+, \mathcal{G}(\lambda)) \, d\mu(\lambda)$. Note that $U$ depends on $z$, but we will suppress the dependency of $z$ in the notation (as we already did for $T$ in the previous subsection).

Let us now perform the scaling transformation of $T$ with respect to $U$. As both operators are fibred with respect to the direct integral over $\lambda$, we have $U^{-1}TU = \int_\sigma U^{-1}_\lambda T U_\lambda \, d\mu(\lambda)$. Moreover, for $\psi \in C_c(\mathbb{R}_+)$ and $\varphi_\lambda \in \mathcal{G}(\lambda)$ we calculate

$$(U^{-1}_\lambda T U_\lambda)(\psi \otimes \varphi_\lambda) = \int_{\mathbb{R}_+} \exp\left(-\left(t + \tau\right)\right) \psi(\tau) \, d\tau \otimes \frac{\varphi_\lambda}{\lambda - z}$$

for $d\mu$-almost all $\lambda \in \sigma$.

Let $\Psi_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be the function defined by $\Psi_0(t) = \exp(-t)$. It is well known that the difference of the resolvents (at $-1$) of the Neumann and Dirichlet Laplacians on the semi-axis is given by

$$\left( -\frac{d^2}{dt^2} \right) + 1 \left( -\frac{d^2}{dt^2} \right)^D + 1 = \langle \bullet, \Psi_0 \rangle_{L_2(\mathbb{R}_+)} \Psi_0. \quad (4.2)$$

Since $L$ is the multiplication operator by the independent variable on $\mathcal{G}$ one has

$$(L - z)^{-1} \varphi = \int_\sigma \frac{\varphi(\lambda)}{\lambda - z} \, d\mu(\lambda).$$

We have thus shown Theorem 1.2. \hfill $\square$

4.3 The spectral properties of the difference of the resolvents

Theorem 1.2 allows us to determine the spectral properties of the difference of the resolvents as stated in Corollary 1.3.

Proof of Corollary 1.3. Denote by $B_{\Psi_0}$ the self-adjoint rank one operator on $L_2(\mathbb{R}_+)$ from equation (1.2), where $\Psi_0(t) = \exp(-t)$. By Theorem 1.2 we know that $(H - z)^{-1} - (H^D - z)^{-1}$ on $\mathcal{H}$ is unitarily equivalent to

$$B_{\Psi_0} \otimes (L - z)^{-1} \quad \text{on} \quad L_2(\mathbb{R}_+) \otimes \mathcal{G}. \quad (4.3)$$
Denote by \( \{ \Psi_0 \}^\perp \) the orthogonal complement of \( C \Psi_0 \) in \( L_2(\mathbb{R}_+) \). Then the operator from (4.3) is unitarily equivalent to the block diagonal operator

\[
0 \oplus \left[ \frac{1}{2} (L - z)^{-1} \right] \quad \text{on} \quad \{ \Psi_0 \}^\perp \otimes G \oplus G,
\]

because the range of \( B \Psi_0 \) is spanned by \( \Psi_0 \) and \( \langle \Psi_0, \Psi_0 \rangle_{L_2(\mathbb{R}_+)} = \frac{1}{2} \). Now, standard arguments from spectral theory (see, e.g., [1, Chapter 7]) complete the proof.

5 The difference of the spectral projections

In this section we will establish Theorem 1.1 (b) and Theorem 1.4. In Subsection 5.1 we use Proposition 2.2 to compute the difference \( 1_{(-\infty, \alpha)}(H) - 1_{(-\infty, \alpha)}(H^D) \) of the spectral projections for every \( \alpha > 0 \). Then, we will establish Theorem 1.1 (b). In Subsection 5.2 we will change the order of evaluation with respect to the variables \( t \in \mathbb{R}_+ \) and \( \lambda \in \sigma \) in the formula for \( 1_{(-\infty, \alpha)}(H) - 1_{(-\infty, \alpha)}(H^D) \). We will obtain, for \( d\mu \)-almost all \( \lambda \) in \( \sigma \), a vector-valued Hankel-type integral operator. In Subsection 5.3 we will see that these vector-valued Hankel-type integral operators are closely related to the Hankel integral operator from Krein’s example [13] discussed above. After this observation we will be able to complete the proof of Theorem 1.4 using the above mentioned result from Kostrykin and Makarov [12].

5.1 Proof of Theorem 1.1 (b)

Since \( H \geq 0 \) and \( H^D \geq 0 \) both have a purely absolutely continuous spectrum we may, without loss of generality, assume that \( \alpha > 0 \). By Proposition 2.2 [11], formula (1.2), and Fubini’s theorem we obtain that

\[
\langle (1_{(-\infty, \alpha)}(H) - 1_{(-\infty, \alpha)}(H^D)) (\psi \otimes \varphi), \xi \otimes \eta \rangle_{\mathcal{H}}
= \frac{2}{\pi} \int_{\mathbb{R}_+} dt \int_{\sigma} d\mu(\lambda) \int_{\mathbb{R}_+} d\tau \langle \psi(\tau) \mathbb{1}_{[0, \alpha)}(\lambda) \varphi(\lambda), \xi(t) \eta(\lambda) \rangle_{\mathcal{H}} \sin((\alpha - \lambda)^{1/2}(t + \tau)) \quad \frac{t + \tau}{t + \tau}
\]

for all \( \psi, \xi \in C_c(\mathbb{R}_+) \) and all \( \varphi \in \mathcal{G} \), \( \eta \in \text{dom}(L) \).

5.1 Remark. Alternatively, this can also be computed using Proposition 3.18 and Stone’s formula for spectral projections.

Further one proves for all \( t \in \mathbb{R}_+ \) that

\[
h(t) = \frac{2}{\pi} \int_{\sigma} \int_{\mathbb{R}_+} \psi(\tau) \mathbb{1}_{[0, \alpha)}(\lambda) \sin((\alpha - \lambda)^{1/2}(t + \tau)) \quad \frac{t + \tau}{t + \tau} \quad \text{d}\tau \varphi(\lambda) \quad \text{d}\mu(\lambda) \in \mathcal{G}. \quad (5.1)
\]

By the dominated convergence theorem we obtain that \( \mathbb{R}_+ \ni t \mapsto h(t) \in \mathcal{G} \) is continuous. Consequently, \( h \) is measurable and we compute

\[
\| h \|_{\mathcal{H}} \leq \| \varphi \|_{\mathcal{G}} \quad \frac{1}{\pi} \max_{\tau \in \mathbb{R}_+} |\psi(\tau)| \int_{\text{supp}(\psi)} d\tau < \infty,
\]
where $\tau_0 = \min\left(\text{supp}(\psi)\right) > 0$. We have shown that
\[
\langle (1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D))(\psi \otimes \varphi), \xi \otimes \eta \rangle_{\mathcal{H}} = \langle h, \xi \otimes \eta \rangle_{\mathcal{H}}
\]
for all $\xi \in C_c(\mathbb{R}_+)$ and all $\eta \in \text{dom}(L)$. Since $C_c(\mathbb{R}_+) \otimes \text{dom}(L)$ is dense in $\mathcal{H}$ we have established the following result:

5.2 Lemma. If $\alpha > 0$ then $(1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D))(\psi \otimes \varphi) = h$ for all $\psi \in C_c(\mathbb{R}_+)$ and all $\varphi \in \mathcal{F}$, where $h \in \mathcal{H}$ is defined as in (5.1) above.

We can now prove Theorem 1.1 (b).

**Proof of Theorem 1.1 (b).** The result from Lemma 5.2 carries over to the situation when $L$ is not necessarily a multiplication operator, using Theorem 2.3 and the functional calculus:
\[
\left(1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D)\right)(\psi \otimes \varphi)(t) = \frac{2}{\pi} \int_{\mathbb{R}_+} \psi(\tau) \mathbf{1}_{[0,\alpha]}(\varphi) \frac{\sin((\alpha(\varphi) - L)^{1/2}(t + \tau))}{t + \tau} \varphi \, d\tau
\]
for all $\psi \in C_c(\mathbb{R}_+)$ and all $\varphi \in \mathcal{F}$, where $0 < \varphi < 1$ and $\alpha(\varphi) = \frac{1}{2} - 1$. Last, observe that
\[
1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D) = 1_{(-\infty,\alpha)}(A_0) - 1_{(-\infty,\alpha)}(A_1). \tag{5.2}
\]
Now the proof of Theorem 1.1 (b) is complete. \hfill \Box

5.2 Proof of Theorem 1.4. Step 1

Analogously to Lemma 4.1. one shows:

5.3 Lemma. Let $\alpha > 0$ and let $\psi \in C_c(\mathbb{R}_+)$, $\varphi \in \mathcal{F}$. Then one has
\[
\left(W(1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D))(\psi \otimes \varphi)(\lambda) = \frac{2}{\pi} \int_{\mathbb{R}_+} \mathbf{1}_{[0,\alpha]}(\lambda) \psi(\tau) \frac{\sin((\alpha - \lambda)^{1/2}(\bullet + \tau))}{\bullet + \tau} \, d\tau \otimes \varphi(\lambda)
\]
for $d\mu$-almost all $\lambda$ in $\sigma$, where $W : \mathcal{H} \rightarrow \int_\sigma^\oplus L_2(\mathbb{R}_+) \otimes \mathcal{F}(\lambda) \, d\mu(\lambda)$ is the unitary operator defined in Subsection 4.1 above.

5.3 Proof of Theorem 1.4. Step 2

Lemma 5.3 shows that if $\mu(\sigma \cap [0, \alpha]) = 0$ then $1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D) = 0$. Let us now consider the more interesting case when $\mu(\sigma \cap [0, \alpha]) > 0$. Lemma 5.3 implies in this case that $1_{(-\infty,\alpha)}(H) - 1_{(-\infty,\alpha)}(H^D)$ is unitarily equivalent to the block diagonal operator
\[
\left[\int_\sigma^\oplus T_\lambda \, d\mu(\lambda)\right] \oplus 0 \quad \text{on} \quad \left[\int_\sigma^\oplus L_2(\mathbb{R}_+, \mathcal{F}(\lambda)) \, d\mu(\lambda)\right] \oplus \left[\int_\sigma^\oplus L_2(\mathbb{R}_+, \mathcal{F}(\lambda)) \, d\mu(\lambda)\right],
\]
where for every fixed $\lambda \in \sigma \cap [0, \alpha)$ outside a set of $d\mu$-measure 0
\[ \overline{T}_\lambda(\psi \otimes \varphi_\lambda) = \frac{2}{\pi} \int_{\mathbb{R}_+} \psi(\tau) \frac{\sin((\alpha - \lambda)^{1/2} (\bullet + \tau))}{\bullet + \tau} d\tau \otimes \varphi_\lambda \]
for all $\psi \in C_c(\mathbb{R}_+)$ and all vectors $\varphi_\lambda \in \mathcal{G}(\lambda)$. We will write $\overline{T} = \int_{\sigma \cap [0, \alpha)} \overline{T}_\lambda d\mu(\lambda)$.

Next we define the unitary operator \[ \overline{U} = \int_{\mathcal{G}(\lambda)} \overline{U}_\lambda d\mu(\lambda) \quad \text{on} \quad \int_{\mathcal{G}(\lambda)} L_2(\mathbb{R}_+, \mathcal{G}(\lambda)) d\mu(\lambda), \]
where $\overline{U}_\lambda$ is the unitary scaling operator on $L_2(\mathbb{R}_+, \mathcal{G}(\lambda))$ given by
\[ (\overline{U}_\lambda f)(t) = (\alpha - \lambda)^{1/4} f((\alpha - \lambda)^{1/2} t) \]
for $d\mu$-almost all $\lambda \in \sigma \cap [0, \alpha)$. Note that $\overline{U}$ depends also on $\alpha$, but as before for $U$, we suppress this dependency. Again, both operators $\overline{U}$ and $\overline{T}$ are fibred with respect to the direct integral over $\lambda$, hence $\overline{U}^{-1} \overline{T} \overline{U} = \int_{\mathcal{G}(\lambda)} \overline{U}_\lambda^{-1} \overline{T}_\lambda \overline{U}_\lambda d\mu(\lambda)$. Moreover, for $\psi \in C_c(\mathbb{R}_+)$ and $\varphi_\lambda \in \mathcal{G}(\lambda)$ we compute
\[ (\overline{U}_\lambda^{-1} \overline{T}_\lambda \overline{U}_\lambda)(\psi \otimes \varphi_\lambda) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\sin(\bullet + \tau)}{\bullet + \tau} \psi(\tau) d\tau \otimes \varphi_\lambda = K \psi \otimes \varphi_\lambda \]
for $d\mu$-almost all $\lambda \in \sigma \cap [0, \alpha)$, where $K$ is given by
\[ (K \psi)(t) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\sin(t + \tau)}{t + \tau} \psi(\tau) d\tau, \quad \psi \in C_c(\mathbb{R}_+). \]

In [12], Kostrykin and Makarov have shown that $K$ has a simple and purely absolutely continuous spectrum filling in the interval $[-1, 1]$. Consequently, the operator
\[ \overline{U}^{-1} \overline{T} \overline{U} \quad \text{on} \quad \int_{\mathcal{G}(\lambda)} L_2(\mathbb{R}_+, \mathcal{G}(\lambda)) d\mu(\lambda) \]
is unitarily equivalent to the multiplication operator by the independent variable on
\[ L_2([-1, 1], \int_{\mathcal{G}(\lambda)} d\mu(\lambda)). \]
Now, an application of the transformation rule for spectral measures (see (5.2) above) completes to proof of Theorem [14].

**5.4 Remark.** Note that $K$ defined above is the Hankel integral operator on $L_2(\mathbb{R}_+)$ from Krein’s example in the case when $\vartheta = 1/2$, see [12] above.
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