Well-Posedness of the
Kadomtsev–Petviashvili Hierarchy, Mulase
Factorization, and Frölicher Lie Groups

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Abstract. We recall the notions of Frölicher and diffeological spaces, and we build regular Frölicher Lie groups and Lie algebras of formal pseudo-differential operators in one independent variable. Combining these constructions with a smooth version of Mulase’s deep algebraic factorization of infinite-dimensional groups based on formal pseudo-differential operators, we present two proofs of the well-posedness of the Cauchy problem for the Kadomtsev–Petviashvili (KP) hierarchy in a smooth category. We also generalize these results to a KP hierarchy modelled on formal pseudo-differential operators with coefficients which are series in formal parameters, we describe a rigorous derivation of the Hamiltonian interpretation of the KP hierarchy, and we discuss how solutions depending on formal parameters can lead to sequences of functions converging to a class of solutions of the standard KP-II equation.

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1. Introduction

In the 1980s, M. Mulase published two fundamental papers on the algebraic structure and formal integrability properties of the Kadomtsev–Petviashvili (KP) hierarchy, see [37,38]. A common theme in these papers was the use of a powerful theorem on the factorization of a group of formal pseudo-differential operators of infinite order which integrated the algebra of formal pseudo-differential operators: this factorization—a delicate algebraic generalization of the Birkhoff decomposition of loop groups appearing for example in [23,45]—allowed him to solve the Cauchy problem for the KP hierarchy in an algebraic context.
In this paper, we adapt Mulase’s results and constructions to a rigorous setting suitable for analysis, and we prove well-posedness of the Cauchy problem for the KP hierarchy in a smooth category. In fact, we provide two solutions for the Cauchy problem. The first one is modelled after the theory of $r$-matrices [46,50], while the second one uses an infinite-dimensional version of the Ambrose–Singer theorem (see [32,33], and [53] for the classical finite-dimensional case). As our smooth category, we choose the setting of regular Frölicher Lie groups and algebras, see [25], which is not too different from the so-called convenient setting described in the same reference. This context allows us to construct genuine Lie groups and Lie algebras structures out of spaces of formal pseudo-differential operators.

It is well known that endowing spaces of formal pseudo-differential operators with rigorous manifold structures derived from topological structures on infinite-dimensional vector spaces, is a non-trivial issue. For example, we may recall that there is no natural Lie group attached to the algebra of differential operators on $S^1$ (see [27, Sect. 4], [28, II.4.4], or [23, Chp. 10]) and that the manifolds modelled on “direct limit of ILH algebras” appearing in [2, Section 5] possess underlying topological structures which need to be treated with extreme care, since there appear, e.g. locally convex topological vector spaces which are not complete. We need groups such as the one mentioned above in order to establish a smooth version of Mulase’s factorization, our main tool for proving well-posedness. Our constructions in a Frölicher setting are indeed flexible and user-friendly enough so that they allow us to distinguish clearly which properties depend on smoothness or on topology. In our approach via Frölicher Lie groups, properties where topology appears, such as integration, are clearly delimited. On the other hand, properties which we obtain via differentiation, which are numerous, are also explicitly described. Moreover, see Remark 2.6, our chosen approach is compatible with more standard settings, in the sense that smoothness in the Frölicher category coincides with smoothness in the category of the $(c^\infty)$-convenient setting [25] and with Gateaux smoothness. If we restrict to Fréchet manifolds. These facts allow us to carry out, without using topological arguments which could lead us to restricting the field of applications, explicit computations leading, for example, to the announced smooth version of Mulase’s factorization, to solving Cauchy problems and to a rigorous hamiltonian formulation for the KP hierarchy. They also allow us to examine explicit examples: motivated by the theory of pseudo-differential operators with rough coefficients, see [35], we propose a deformed KP hierarchy and we remark that its solutions determine solutions to the KP-II equation as it appears for instance in [9,51].

Our paper is structured as follows. Section 2 is a summary of the necessary notions in the theory of diffeological and of Frölicher spaces which we use herein\(^1\) and it is also a contribution to the theory of regular Frölicher Lie groups and Lie algebras, as we now explain. We introduce and study the main

\(^1\)As explained in the previous paragraphs, our favoured setting is the category of Frölicher spaces. This category is a full subcategory of the category of diffeological spaces, and therefore it is natural to begin by considering the latter spaces, as we do in Sect. 2.
properties of diffeological and Frölicher spaces in the first five subsections of Sect. 2. Then, we introduce diffeological groups and the more refined notion of Frölicher Lie groups. Interestingly, it is not obvious how to define tangent spaces—let alone Lie algebras—of diffeological groups. We do it in detail and we prove (see Proposition 2.20) that in fact tangent spaces to diffeological groups are diffeological vector spaces. This proposition is a precise version of a result announced in [30]. We then consider (regular) Frölicher Lie groups and (regular) Frölicher Lie algebras. In the context of Lie groups modelled on e.g. Fréchet spaces, this notion has led to many investigations, see [2,3,14,20,22,34,42,43], and also [40] for an overview of presently open problems. We discuss regularity in Sects. 2.6 and 2.7, and we answer a question raised by Kriegl and Michor in [25] (after Omori [42]) on the existence of a non-regular (in a precise sense appearing in 2.7 below) Lie group. We finish Sect. 2 with a summary of the theory of principal fibre bundles and the Ambrose–Singer theorem in the context of diffeological spaces after [32]. We need these tools for our second proof of the well-posedness of the Cauchy problem of the KP hierarchy.

Section 3 is on the algebra of formal pseudo-differential operators in one independent variable and its “integration” to a regular Frölicher Lie group. We begin with a review of some aspects of Mulase’s work including his factorization theorem, see [37,38] and the more recent review [17], and then we adapt his constructions to our smooth setting, so as to obtain a regular Frölicher Lie group $G(\Psi(A_t))$ of infinite-order formal pseudo-differential operators with regular Frölicher Lie algebra. This Lie algebra is given by formal pseudo-differential operators with appropriate coefficients which are reminiscent of Mulase’s choice of “time-dependent” coefficients, see [37,38]. These constructions allow us to prove a smooth version of Mulase’s factorization theorem, our main tool in the solution of the Cauchy problem for the KP hierarchy.

The actual analysis of the Cauchy problem for the KP hierarchy in our smooth category is carried out in Sect. 4. As announced, we present two proofs of well-posedness: one uses an infinite-dimensional version of the Reyman–Semenov-Tian-Shansky integration theorem, see [46], and the other uses infinite-dimensional principal bundles and the version of the Ambrose–Singer theorem presented in Sect. 2. Our proofs are much enriched versions of previous work by the authors, see [17,32]: in [17] is shown that the Reyman–Semenov-Tian-Shansky integration theorem yields formally solutions to the equations of the KP hierarchy via Mulase’s factorization theorem, but no comments are made in that paper with respect to the existence of smooth solutions. On the other hand, the second proof builds upon tools developed for the study of a $q$-deformed version of the KP hierarchy in [32,33], and it completes these two works.

The KP hierarchy with formal series coefficients, with emphasis on solutions which are functions with low regularity, is considered on its own right in Sect. 5. We argue that the deformed initial data is not only an appropriate tool for the analysis of standard KP, but also that it is a natural way to study some dynamical systems with rough initial data. In fact, we expect that our work can be used to investigate actual existence of conservation laws or blow-up
phenomena. We also consider in this section the hamiltonian structure of the KP and deformed KP hierarchies, extending to a smooth setting the approach of [17]. Hamiltonian structures have been studied before for non-deformed KP hierarchy posed on formal pseudo-differential operators with coefficients for instance in $C^\infty(S^1, \mathbb{R})$ (see e.g. [13, 28, 55, 56]), but we propose a general non-formal version valid in our smooth category of Frölicher spaces. As a final remark, in Sect. 5.3 we point out the need to compare the solutions to the KP-II equation which are deduced from our solutions to the deformed KP hierarchy, with the ones considered in [9, 51]. We believe this comparison is an important open problem in the theory of non-formal analysis of nonlinear integrable systems.

We close this introduction on a personal note. The second named author was fortunate to be Prof. Dickey’s colleague at the University of Oklahoma during two years. Leonid Aleksandrovich was a kind and generous human being, in addition to being a brilliant mathematician. It is an honour to dedicate this work to his memory.

2. Preliminaries on Diffeological Lie Groups

2.1. Souriau’s Diffeological Spaces and Frölicher Spaces

Definition 2.1 [52], see e.g. [24]. Let $X$ be a set.

- A $p$-parametrization of dimension $p$ (or $p$-plot) on $X$ is a map from an open subset $O$ of $\mathbb{R}^p$ to $X$.

- A diffeology on $X$ is a set $\mathcal{P}$ of parametrizations on $X$, called plots of the diffeology, such that, for all $p \in \mathbb{N}$,
  - any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$;
  - Let $I$ be an arbitrary set of indexes; let $\{f_i : O_i \to X\}_{i \in I}$ be a family of compatible maps that extend to a map $f : \bigcup_{i \in I} O_i \to X$. If $\{f_i : O_i \to X\}_{i \in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.
  - Let $f \in \mathcal{P}$, defined on $O \subset \mathbb{R}^p$. Let $q \in \mathbb{N}$, $O'$ an open subset of $\mathbb{R}^q$ and $g$ a smooth map (in the usual sense) from $O'$ to $O$. Then, $f \circ g \in \mathcal{P}$.

- If $\mathcal{P}$ is a diffeology on $X$, then $(X, \mathcal{P})$ is called a diffeological space. Let $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ be two diffeological spaces; a map $f : X \to X'$ is differentiable (=smooth) if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

Remark 2.2. Any diffeological space $(X, \mathcal{P})$ can be endowed with the weakest topology such that all the maps that belong to $\mathcal{P}$ are continuous. We do not dwell deeper on this fact in this work because it is not closely related to the main themes of this paper.

We now introduce Frölicher spaces, see [18], using terminology from [25].

Definition 2.3. A Frölicher space is a triple $(X, \mathcal{F}, \mathcal{C})$ such that

- $\mathcal{C}$ is a set of paths $\mathbb{R} \to X$,
- $\mathcal{F}$ is the set of functions from $X$ to $\mathbb{R}$, such that a function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
A path $c : \mathbb{R} \to X$ is in $\mathcal{C}$ (i.e. is a contour) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces; a map $f : X \to X'$ is differentiable (=$\text{smooth}$) if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})$.

Any family of maps $\mathcal{F}_g$ from $X$ to $\mathbb{R}$ generates a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$ by setting, after [25]:

- $\mathcal{C} = \{ c : \mathbb{R} \to X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R}) \}$
- $\mathcal{F} = \{ f : X \to \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R}) \}$

In this case, we call $\mathcal{F}_g$ a generating set of functions for the Frölicher structure $(X, \mathcal{F}, \mathcal{C})$. We can easily see that $\mathcal{F}_g \subset \mathcal{F}$. This notion is useful for this paper since it allows us to describe a Frölicher structure in a simple way, see for instance Proposition 2.16. A Frölicher space $(X, \mathcal{F}, \mathcal{C})$ carries a natural topology, which is the pull-back topology of $\mathbb{R}$ via $\mathcal{F}$. We note that in the case of a finite-dimensional differentiable manifold $X$, we choose $\mathcal{F}$ to be the set of all smooth maps from $X$ to $\mathbb{R}$, and $\mathcal{C}$ to be the set of all smooth paths from $\mathbb{R}$ to $X$. In this case, the underlying topology of the Frölicher structure is the same as the manifold topology [25]. In the infinite-dimensional case, there is to our knowledge no complete study of the relation between the Frölicher topology and the manifold topology; our intuition is that these two topologies can differ.

We also remark that if $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we can define a natural diffeology on $X$ by using the following family of maps $f$ defined on open domains $D(f)$ of Euclidean spaces (see [31]):

$$P_\infty(\mathcal{F}) = \coprod_{p \in \mathbb{N}} \{ f : D(f) \to X; \mathcal{F} \circ f \in C^\infty(D(f), \mathbb{R}) \text{ (in the usual sense)} \}. $$

(2.1)

If $X$ is a differentiable manifold, this diffeology has been called the nébulese diffeology by Souriau, see [52]; it is also discussed by P. Iglesias-Zemmour, see [24]. We can easily show the following:

**Proposition 2.4** [31]. Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces. A map $f : X \to X'$ is smooth in the Frölicher sense if and only if it is smooth for the underlying diffeologies $P_\infty(\mathcal{F})$ and $P_\infty(\mathcal{F}')$.

Thus, we can also state:

Smooth manifold $\Rightarrow$ Frölicher space $\Rightarrow$ Diffeological space .

A deeper analysis of these implications has been given in [57]. The next remark is inspired on this work and on [31]; it is based on [25, p.26, Boman’s theorem].

**Remark 2.5.** The set of contours $\mathcal{C}$ of a Frölicher space $(X, \mathcal{F}, \mathcal{C})$ does not give us a diffeology, because a diffeology needs to be stable under restriction of domains. In the case of paths in $\mathcal{C}$, the domain is always $\mathbb{R}$. However, $\mathcal{C}$ defines a “minimal diffeology” $P_1(\mathcal{F})$ whose plots are smooth parametrizations which are locally of the type $c \circ g$, where $g \in P_\infty(\mathbb{R})$ and $c \in \mathcal{C}$. Within this setting, we can replace $P_\infty$ by $P_1$ in Proposition 2.4.
We also remark that given an algebraic structure, we can define a corresponding compatible diffeological structure. For example, a $\mathbb{R}$-vector space equipped with a diffeology is called a diffeological vector space if addition and scalar multiplication are smooth (with respect to the canonical diffeology on $\mathbb{R}$). An analogous definition holds for Frölicher vector spaces. We will consider diffeological and Frölicher groups in Sect. 2.6 below.

**Remark 2.6.** Frölicher, $c^\infty$ (the “smooth convenient setting” of [25]) and Gâteaux smoothness are the same notion if we restrict to a Fréchet context, see [25, Theorem 4.11]. Indeed, for a smooth map $f : (F, \mathcal{P}_1(F)) \to \mathbb{R}$ defined on a Fréchet space with its 1-dimensional diffeology, we have that $\forall (x, h) \in F^2$, the map $t \mapsto f(x + th)$ is smooth as a classical map in $C^\infty(\mathbb{R}, \mathbb{R})$. Hence, it is Gâteaux smooth. The converse is obvious.

**Definition 2.7** (Smooth Homotopy). Let $X$ and $Y$ be diffeological spaces and let $f_i : X \to Y$, $i \in \{0, 1\}$ be smooth maps. The maps $f_0$ and $f_1$ are (smoothly) homotopic if there is a smooth map $H : X \times \mathbb{R} \to Y$ such that $H(\cdot, 0) = f_0$ and $H(\cdot, 1) = f_1$. We call the map $H$ a smooth homotopy.

This definition enables us to define straightforwardly smooth homotopy equivalences, fundamental groups and related objects, see [24] for details.

### 2.2. Tangent Space

Let $X$ be a diffeological space. There exist two main ways to define the tangent space at a point $x \in X$. The internal tangent space at $x \in X$ described in [10], and the external tangent space $^eTX$, defined simply as the set of derivations on $C^\infty(X, \mathbb{R})$, see [24,25]. It is known that these two constructions coincide for finite-dimensional manifolds and in other important cases, see [25, Section 28] and [10]. For us, it is actually enough to define tangent cones after [16].

For each $x \in X$, we consider the set of paths $C_x = \{ c \in C^\infty(\mathbb{R}, X) | c(0) = x \}$ and take the equivalence relation $\mathcal{R}$ given by $c \mathcal{R} c' \iff \forall f \in C^\infty(X, \mathbb{R}), \quad \partial_t(f \circ c)|_{t=0} = \partial_t(f \circ c')|_{t=0}.$

The tangent cone at $x$ is the quotient $^iT_xX = C_x/\mathcal{R}$.

Equivalence classes of $\mathcal{R}$ are denoted by $V = \partial_t c(t)|_{t=0} = \partial_t c(0) \in ^iT_xX$. We also use the notation $Df(V) = \partial_t(f \circ c)|_{t=0}$.

It is shown in [16, Section 2] that there exist examples of diffeological spaces for which the tangent cone at a point $x$ is not a vector space, hence the need for more sophisticated definitions as in [10,24,25]. However, in Sect. 2.6 we show that this difficulty is absent in the case of diffeological groups.
2.3. Differential Forms

**Definition 2.8** [52]. Let \((X, \mathcal{P})\) be a diffeological space and let \(V\) be a vector space equipped with a differentiable structure. A \(V\)-valued \(n\)-differential form \(\alpha\) on \(X\) (noted \(\alpha \in \Omega^n(X, V)\)) is a map

\[
\alpha : \{p : O_p \to X\} \in \mathcal{P} \mapsto \alpha_p \in \Omega^n(O_p; V)
\]

such that

- Let \(x \in X\). \(\forall (p, p') \in \mathcal{P}^2\) such that \(x \in Im(p) \cap Im(p')\), the forms \(\alpha_p\) and \(\alpha_{p'}\) are of the same order \(n\).
- Let \(y \in O_p\) and \(y' \in O_{p'}\), and assume that \((X_1, ..., X_n)\) are \(n\) germs of paths in \(Im(p) \cap Im(p')\). If there exists two systems of \(n\)-vectors \((Y_1, ..., Y_n) \in (T_yO_p)^n\) and \((Y'_1, ..., Y'_n) \in (T_{y'}O_{p'})^n\) such that \(p_*(Y_1, ..., Y_n) = p'_*(Y'_1, ..., Y'_n) = (X_1, ..., X_n)\), then

\[
\alpha_p(Y_1, ..., Y_n) = \alpha_{p'}(Y'_1, ..., Y'_n).
\]

We note by

\[
\Omega(X; V) = \bigoplus_{n \in \mathbb{N}} \Omega^n(X, V)
\]

the set of \(V\)-valued differential forms.

We make two remarks for the reader:

- If there do not exist \(n\) linearly independent vectors \((Y_1, ..., Y_n)\) in the last point of the definition, then \(\alpha_p = 0\) at \(y\).
- Let \((\alpha, p, p') \in \Omega(X, V) \times \mathcal{P}^2\). If there exists \(g \in C^\infty(D(p); D(p'))\) (in the usual sense) such that \(p' \circ g = p\), then \(\alpha_p = g^*\alpha_{p'}\).

**Proposition 2.9.** The set \(\mathcal{P}(\Omega^n(X, V))\) of all maps \(q : x \mapsto \alpha(x)\) from an open subset \(O_q\) of \(V\) to \(\Omega^n(X, V)\) such that for each \(p \in \mathcal{P}\),

\[
\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, V)),
\]

is a diffeology on \(\Omega^n(X, V)\).

Working on plots of the diffeology, we can define the wedge product and the exterior derivative of differential forms. They have the same properties as the usual wedge product and exterior derivative of differential forms on finite-dimensional manifolds, see for instance [24].

**Definition 2.10.** Let \((X, \mathcal{P})\) be a diffeological space.

- \((X, \mathcal{P})\) is **finite-dimensional** if and only if

\[
\exists n_0 \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad n \geq n_0 \Rightarrow dim(\Omega^n(X, \mathbb{R})) = 0.
\]

Then, we set

\[
dim(X, \mathcal{P}) = \max\{n \in \mathbb{N}|dim(\Omega^n(X, \mathbb{R})) > 0\}.
\]

- If not, \((X, \mathcal{P})\) is called **infinite-dimensional**.

Let us make a few remarks on this definition. If \(X\) is a manifold with \(dim(X) = n\), its nêbuleuse diffeology is such that

\[
dim(X, \mathcal{P}_\infty) = n.
\]
On the other hand, if \((X, \mathcal{F}, C)\) is the natural Frölicher structure on the finite-dimensional manifold \(X\), and we take \(\mathcal{P}_1\) defined as before, then it is an easy exercise to show that

\[
\dim(X, \mathcal{P}_1) = 1.
\]

Therefore, dimension depends on the diffeology considered. Now, \(\mathcal{F}\) is not only the set of smooth maps \((X, \mathcal{P}_1) \to (\mathbb{R}, \mathcal{P}_1(\mathbb{R}))\), but also the set of smooth maps \((X, \mathcal{P}_1) \to (\mathbb{R}, \mathcal{P}_\infty(\mathbb{R}))\), and also the set of smooth maps \((X, \mathcal{P}_\infty) \to (\mathbb{R}, \mathcal{P}_\infty(\mathbb{R}))\). These observations follow from adapting arguments of [31] based on Boman’s theorem as it appears in [25], see e.g. [57].

We are led to the following definition, since \(\mathcal{P}_\infty(\mathcal{F})\) is clearly the diffeology with the largest dimension associated to \((X, \mathcal{F}, C)\):

**Definition 2.11.** The **dimension** of a Frölicher space \((X, \mathcal{F}, C)\) is the dimension of the diffeological space \((X, \mathcal{P}_\infty(\mathcal{F}))\).

### 2.4. Push-Forward, Quotients and Subsets

We state only the results that will be used hereafter.

**Proposition 2.12** [24,52]. Let \((X, \mathcal{P})\) be a diffeological space, and let \(X'\) be a set. Let \(f : X \to X'\) be a map. We define the **push-forward diffeology** as the coarsest (i.e. the smallest for inclusion) among the diffeologies on \(X'\), which contains \(f \circ \mathcal{P}\).

We have now the tools needed to describe the diffeology on a quotient:

**Proposition 2.13.** Let \((X, \mathcal{P})\) be a diffeological space and let \(\mathcal{R}\) be an equivalence relation on \(X\). Then, there is a natural diffeology on \(X/\mathcal{R}\), noted by \(\mathcal{P}/\mathcal{R}\), defined as the push-forward diffeology on \(X/\mathcal{R}\) induced by the quotient projection \(X \to X/\mathcal{R}\).

Finally, we consider a subset \(X_0 \subset X\), where \(X\) is a Frölicher space or a diffeological space. We can define a subset structure on \(X_0\), induced by \(X\) as follow:

- If \(X\) is equipped with a diffeology \(\mathcal{P}\), we can define a diffeology \(\mathcal{P}_0\) on \(X_0\), called the **subset diffeology** [24,52] by setting
  \[
  \mathcal{P}_0 = \{ p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0 \}.
  \]
- If \((X, \mathcal{F}, C)\) is a Frölicher space, we take as a generating set of maps \(\mathcal{F}_g\) on \(X_0\) the restrictions of the maps \(f \in \mathcal{F}\). In that case, the contours (resp. the induced diffeology) on \(X_0\) are the contours (resp. the plots) on \(X\) whose images are subsets of \(X_0\).

### 2.5. Cartesian Products and Projective Limits

We review the diffeological structures of products and projective limits since they are crucial for our discussion of Frölicher Lie groups of formal pseudo-differential operators. We begin with the following important proposition (see [52]):
Proposition 2.14. Let \((X, \mathcal{P})\) and \((X', \mathcal{P}')\) be two diffeological spaces. There exists a diffeology \(\mathcal{P} \times \mathcal{P}'\) on \(X \times X'\) made of all plots \(g : O \to X \times X'\) which decompose as \(g = f \times f'\), where \(f : O \to X \in \mathcal{P}\) and \(f' : O \to X' \in \mathcal{P}'\). We call it the product diffeology.

We apply this result to the case of Frölicher spaces, and we derive very easily (compared with e.g. [25]) the following:

Proposition 2.15. Let \((X, \mathcal{F}, \mathcal{C})\) and \((X', \mathcal{F}', \mathcal{C}')\) be two Frölicher spaces equipped with their natural diffeologies \(\mathcal{P}\) and \(\mathcal{P}'\). There is a natural structure of Frölicher space on \(X \times X'\) whose contours \(\mathcal{C} \times \mathcal{C}'\) are the 1-plots of \(\mathcal{P} \times \mathcal{P}'\).

We can even state the above results for infinite products: we simply take Cartesian products of the plots or of the contours. Let us now consider projective limits of Frölicher and diffeological spaces.

Proposition 2.16. Let \(\Lambda\) be an infinite set of indexes.

- Let \(\{ (X_\alpha, \mathcal{P}_\alpha) \}_{\alpha \in \Lambda}\) be a family of diffeological spaces indexed by \(\Lambda\) and totally ordered by inclusion. We write \(\{ i_{\beta, \alpha} : X_\alpha \to X_\beta \}_{(\alpha, \beta) \in \Lambda^2}\) for the corresponding family of inclusion maps, and we assume that they are smooth. If \(X = \bigcap_{\alpha \in \Lambda} X_\alpha\), then \(X\) carries the projective diffeology \(\mathcal{P}\), which is the pull-back of the diffeologies \(\mathcal{P}_\alpha\) of each \(X_\alpha\) via the family of inclusion maps \(\{ f_\alpha : X \to X_\alpha \}_{\alpha \in \Lambda}\). The diffeology \(\mathcal{P}\) is made of all plots \(g : O \to X\) such that for each \(\alpha \in \Lambda\), 
\[ f_\alpha \circ g \in \mathcal{P}_\alpha. \]

This is the largest diffeology for which the maps \(f_\alpha\) are smooth.

- Let \(\{ (X_\alpha, \mathcal{F}_\alpha, \mathcal{C}_\alpha) \}_{\alpha \in \Lambda}\) be a family of Frölicher spaces indexed by \(\Lambda\) and totally ordered by inclusion. There is a natural structure of Frölicher space on \(X = \bigcap_{\alpha \in \Lambda} X_\alpha\).

A generating set of functions for this Frölicher structure is the set of maps of the type:
\[ \bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha \circ f_\alpha. \]

2.6. Regular Frölicher Lie Groups
This subsection is mostly inspired in [32], see also [33].

Definition 2.17. Let \(G\) be a group equipped with a diffeology \(\mathcal{P}\). We call \(G\) a diffeological group if multiplication and inversion are smooth maps.

An analogous definition holds for Frölicher groups.

After Iglesias-Zemmour, see [24], we note that an arbitrary diffeological group does not necessarily possess a Lie algebra. Now we give conditions for the existence of a tangent space at the identity element \(e\) (of course, \(e\) is precisely the identity mapping if the group \(G\) is a group of transformations).

Definition 2.18. The diffeological group \(G\) is a diffeological Lie group if and only if \(\mathcal{T}_e G\) is a diffeological vector space and the derivative of the adjoint action of \(G\) on \(\mathcal{T}_e G\) defines a Lie bracket. In this case, we call \(\mathcal{T}_e G\) the Lie algebra of \(G\), and we denote it generically by \(\mathfrak{g}\).
Remark 2.19. Let $G$ be a diffeological Lie group with Lie algebra $\mathfrak{g} = \{ \partial tc(0) : c \in C_e \text{ and } c(0) = e \}$.

We note that this definition coincides with the classical definition of the Lie algebra of a finite-dimensional Lie group via germs of paths at $e$. We have:

- Let $(X,Y) \in \mathfrak{g}^2$. Then, $X+Y = \partial_t(c.d)(0)$ where $c,d \in C_e^2$, $c(0) = d(0) = e$, $X = \partial tc(0)$ and $Y = \partial td(0)$.
- Let $(X,g) \in \mathfrak{g} \times G$. Then, $Ad_g(X) = \partial_t(gcg^{-1})(0)$ where $c \in C_e$, $c(0) = e$, and $X = \partial tc(0)$.
- Let $(X,Y) \in \mathfrak{g}^2$. Then, $[X,Y] = \partial_t(Ad_{c(t)}Y)$ where $c \in C_e$, $c(0) = e$, and $X = \partial tc(0)$.

All these operations are smooth (and thus well defined) in the context of Frölicher Lie groups and Frölicher Lie algebras as well.

We now present a complete proof of [30, Proposition 1.6] which states that if we start with a diffeological group $G$, we only need to check that $^iT_eG$ admits a Lie bracket in order for $G$ to be a diffeological Lie group with Lie algebra $^iT_eG$.

Proposition 2.20. Let $G$ be a diffeological group. Then the tangent cone at the identity element, $^iT_eG$, is a diffeological vector space.

Proof. Let $(X,Y) \in ^iT_eG$, $X = \partial_t x(t)|_{t=0}$ and $Y = \partial_t y(t)|_{t=0}$ From [16,30], see e.g. [24], we know that $^iT_eG$ is a cone, and we only have to complete the proof of [30] to show that the smooth operation

$$X + Y = \partial_t(x(t),y(t))|_{t=0}$$

transforms $^iT_eG$ into a vector space (smoothness of the operation + follows from [30, Lemma 1.2]).

First, we remark that

$$\forall \lambda \in \mathbb{R}, \quad \lambda X + \lambda Y = \partial_t(x(\lambda t),y(\lambda t))|_{t=0} = \lambda \partial_t(x(t),y(t))|_{t=0} = \lambda (X + Y).$$

Thus, we only have to prove that $X + Y = Y + X$, that is

$$\partial_t(x(t),y(t))|_{t=0} = \partial_t(y(t),x(t))|_{t=0}.$$

For this, we assume that $G$ is equipped with its nèbuleuse diffeology $\mathcal{P}_\infty$. We recall that this diffeology consists of all smooth maps $p : D(p) \subset \mathbb{R}^d \rightarrow G$ where $D(p)$ is equipped with the diffeology $\mathcal{P}_1(D(p))$ generated by

$$\{ \gamma \in C^\infty([a,b], D(p)) \text{ (in the usual sense) } | (a,b) \in \mathbb{R}^2 \land a < b \} \cup \{ \{0\} \rightarrow G \}. $$

For the diffeology $\mathcal{P}_\infty$, the group multiplication and inversion are also smooth, since they are based on smooth paths. This fact comes from a difficult theorem, [25, Boman’s theorem], which implies that multiplication and inversion are smooth with respect to $\mathcal{P}_\infty$ if an only if they are smooth with respect to the $\mathcal{P}_1$ diffeology of $G$ which is defined in Remark 2.5. This change of diffeology is necessary at this stage in order to avoid longer arguments in this already very long proof; we will relax this assumption momentarily.
**First step**: \( \partial_t x^{-1}(t)|_{t=0} = (-1) \cdot \partial_t x(t)|_{t=0}. \)

If \( y(t) = (x(t))^{-1} \), then \( x(t), y(t) = e, X + Y = 0 \), and we get in the same way that \( Y + X = 0 \). Let \( (-1) \cdot X \) be denoted by \(-X\). For any \( f \in C^\infty(G, \mathbb{R}) \) we have

\[
Df((-X) + X + Y) = \partial_t f(x(-t).x(t).y(t))|_{t=0}.
\]

Let \( a, b, c \) be three smooth paths such that \( a(0) = b(0) = c(0) = e \). Then the map

\[
\Phi_3 : (t_1, t_2, t_3) \mapsto f(a(t_1).b(t_2).c(t_3)),
\]

defined on an open neighbourhood of \((0, 0, 0) \subset \mathbb{R}^3\), is a (classical) smooth map, and

\[
D_{(0,0,0)} \Phi_3(v_1, v_2, v_3) = \partial_t f \circ a(t)|_{t=0}.v_1 + \partial_t f \circ b(t)|_{t=0}.v_2 + \partial_t f \circ c(t)|_{t=0}.v_3.
\]

Let us apply this to \( a(t) = x(-t), b(t) = x(t) \) and \( c(t) = y(t) = (x(t))^{-1} \), with \( v_1 = v_2 = v_3 = 1 \). We obtain

\[
Df(-X) = Df(-X) + Df(X + Y)
= Df(-X) + Df(\partial_t (x(t).y(t)))|_{t=0}
= Df(-X) + \partial_t f(x(t).y(t))|_{t=0}
= \partial_t f \circ x(-t)|_{t=0} + \partial_t f \circ x(t)|_{t=0} + \partial_t f \circ y(t)|_{t=0}
= Df(-X + X) + Df(Y)
= Df(Y).
\]

Thus \( Y = -X \) when \( G \) is equipped with \( \mathcal{P}_\infty \).

**second step**: \( X + Y = Y + X. \)

Here, \( y \) is no longer equal to \( x^{-1} \). Let \( f \in C^\infty(G, \mathbb{R}) \) and let us define

\[
\Phi_4(t_1, t_2, t_3, t_4) = f(a(t_1)b(t_2)c(t_3)d(t_4))
\]
on an open neighbourhood of \((0, 0, 0, 0) \subset \mathbb{R}^4\), where \( a, b, c, d \) are smooth paths such that \( a(0) = b(0) = c(0) = d(0) = e \). Then, mimicking the arguments of the first step, \( \Phi_4 \) is smooth, and

\[
D_{(0,0,0,0)} \Phi_4(v_1, v_2, v_3, v_4) = \partial_t f \circ a(t)|_{t=0}.v_1 + \partial_t f \circ b(t)|_{t=0}.v_2 + \partial_t f \circ c(t)|_{t=0}.v_3 + \partial_t f \circ d(t)|_{t=0}.v_4.
\]

Setting \( v_1 = v_2 = v_3 = v_4 = 1 \), with \( a = x, b = y, c = x^{-1}, d = y^{-1} \), we get

\[
\partial_t f(x(t).y(t).x^{-1}(t).y^{-1}(t)) = Df(X) + Df(Y) + Df(-X) + Df(-Y)
= Df(X + Y) + (-Df(X + Y))
= 0
\]
on the one hand. And on the other hand,

\[
\partial_t f(x(t).y(t).x^{-1}(t).y^{-1}(t)) = Df(X) + Df(Y) + Df(-X) + Df(-Y)
= \partial_t f(x(t)y(t)) + \partial_t f(x^{-1}(t)y^{-1}(t))
= \partial_t f(x(t)y(t)) + \partial_t f((y(t)x(t))^{-1})
= Df((X + Y) + (-(Y + X))).
\]
Thus in $^iT_eG$,

$$X + Y = Y + X,$$

and hence $^iT_eG$ is a vector space if $G$ is equipped with the $\mathcal{P}_\infty$ diffeology.

**Third step: relaxing the nèbuleuse diffeology** Let $\mathcal{P}_1(G)$ be the diffeology generated by 0- and 1-dimensional plots of $\mathcal{P}$, and let $\mathcal{P}_\infty(G)$ be defined as before. The first two steps of the foregoing proof are based on the fact that the maps

$$(t_1, t_2, t_3) \mapsto a(t_1)b(t_2)c(t_3)$$

and

$$(t_1, t_2, t_3, t_4) \mapsto a(t_1)b(t_2)c(t_3)d(t_4)$$

are plots of the nèbuleuse diffeology. But they are also smooth maps for the $\mathcal{P}_1(G)$ diffeology, and hence if $f \in C^\infty((G, \mathcal{P}_1(G)), \mathbb{R})$, the maps $\Phi_3$ and $\Phi_4$ remain smooth, and hence the first and second steps remain valid. Let us now consider $\mathcal{P}$ a diffeology on $G$ such that $(G, \mathcal{P})$ is a diffeological group. Then

$$\mathcal{P}_1(G) \subset \mathcal{P} \subset \mathcal{P}_\infty(G)$$

and

$$C^\infty((G, \mathcal{P}_1(G)), \mathbb{R}) \supset C^\infty((G, \mathcal{P}), \mathbb{R}) \supset C^\infty((G, \mathcal{P}_\infty(G)), \mathbb{R}).$$

We have, $\forall f \in C^\infty((G, \mathcal{P}_1(G)), \mathbb{R})$, and for all smooth paths $x(t)$ and $y(t)$ in $\mathcal{P}$ and hence in $\mathcal{P}_1(G)$ as above,

$$\partial_t x^{-1}(t)|_{t=0} = (-1).\partial_t x(t)|_{t=0}$$

and

$$Df(X + Y) = Df(Y + X).$$

This is then true for each $f \in C^\infty((G, \mathcal{P}), \mathbb{R})$, and this fact completes the proof. \(\square\)

Now we go back to the problem of equipping $^iT_eG$ with a Lie algebra structure. We remark that the condition on the existence of a Lie bracket obtained from differentiation of the adjoint action of the group $G$ on $^iT_eG$ cannot be relaxed: it has to be assumed in order to give a structure of Lie algebra to $^iT_eG$. However, if both $G$ and $^iT_eG$ are smoothly embedded into a diffeological algebra $A$, and if group multiplication and $^iT_eG$-addition are the pull-back of the operations on $A$, then the Lie bracket exists and it is defined by the standard relation

$$[u, v] = uv - vu$$ (2.2)

as long as $^iT_eG$ is stable under (2.2). We can ask the following natural question, somewhat reminiscent of Hilbert’s classical problem on the relation between topological groups and Lie groups:

**Does there exist a diffeological (or Frölicher) group which is not a diffeological (or Frölicher) Lie group?**
Let us now concentrate on diffeological and Frölicher Lie groups. We write $g = i^T e G$. The basic properties of adjoint and coadjoint actions, and of Lie brackets, remain globally the same as in the case of finite-dimensional Lie groups, and the proofs are similar: we only need to replace charts by plots of the underlying diffeologies (see e.g. [30] for further details, and [5] for the case of Frölicher Lie groups), as soon as we have checked that the Lie algebra $g$ is a diffeological Lie algebra, i.e. a diffeological vector space equipped with a smooth Lie bracket.

**Definition 2.21.** A Frölicher Lie group $G$ with Lie algebra $g$ is called **regular** if and only if there is a smooth map

$$
Exp : C^\infty([0, 1], g) \to C^\infty([0, 1], G)
$$

such that $g(t) = Exp(v(t))$ is the unique solution of the differential equation

$$
\begin{cases}
g(0) = e \\
\frac{dg(t)}{dt} g(t)^{-1} = v(t), \quad v(t) \in C^\infty([0, 1], g).
\end{cases}
$$

We define the exponential function as follows:

$$
exp : g \to G \\
v \mapsto exp(v) = g(1),
$$

where $g$ is the image by $Exp$ of the constant path $v$.

We can also define the Riemann integral of smooth $g$-valued functions, see [33].

**Definition 2.22.** Let $(V, \mathcal{F}, \mathcal{C})$ be a Frölicher vector space, i.e. a vector space $V$ equipped with a Frölicher structure compatible with vector space addition and scalar multiplication. The space $(V, \mathcal{F}, \mathcal{C})$ is **regular** if there is a smooth map

$$
\int_0^\cdot : C^\infty([0; 1]; V) \to C^\infty([0; 1], V)
$$

such that $\int_0^\cdot v = u$ if and only if $u$ is the unique solution of the differential equation

$$
\begin{cases}
u(0) = 0 \\
u'(t) = v(t).
\end{cases}
$$

This definition applies, for instance, if $V$ is a complete locally convex topological vector space equipped with its natural Frölicher structure given by the Frölicher completion of its nébuleuse diffeology, see [24,31,32].

**Definition 2.23.** Let $G$ be a Frölicher Lie group with Lie algebra $g$. Then, $G$ is **fully regular** if both $G$ and $g$ are regular in the sense of definitions 2.21 and 2.22, respectively.
For completeness, let us mention that—following terminology used in the early investigations on infinite-dimensional Lie theory ([42]; see also [40])—a regular Lie algebra \( \mathfrak{g} \) is said to be **enlargeable** if there exists a (not necessarily regular) Frölicher Lie group \( G \) with Lie algebra \( \mathfrak{g} \).

**Theorem 2.24** [32]. Let \( G \) be a regular Frölicher Lie group with Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{g}_1 \) be a Lie subalgebra of \( \mathfrak{g} \), and set \( G_1 = \text{Exp}(\mathcal{C}^\infty([0;1]; \mathfrak{g}_1))(1) \). If \( \text{Ad}_{G_1 \cup G_1^{-1}}(\mathfrak{g}_1) = \mathfrak{g}_1 \), i.e. \( \forall g \in \text{Exp}(\mathcal{C}^\infty([0;1]; \mathfrak{g}_1))(1) \), \( \forall v \in \mathfrak{g}_1 \), \( \text{Ad}_g v \in \mathfrak{g}_1 \) and \( \text{Ad}_{g^{-1}} v \in \mathfrak{g}_1 \), then \( G_1 \) is a Frölicher subgroup of \( G \).

We finish this subsection stating two further key results from [32,33] which we need in Sect. 3.

**Theorem 2.25.** Let \( (A_n)_{n \in \mathbb{N}^*} \) be a sequence of (Frölicher) vector spaces which are regular, equipped with a graded smooth multiplication operation on \( \bigoplus_{n \in \mathbb{N}^*} A_n \), i.e. a multiplication such that for each \( n, m \in \mathbb{N}^* \), \( A_n A_m \subset A_{n+m} \) is smooth with respect to the corresponding Frölicher structures. Let us define the (non unital) algebra of formal series \( \mathcal{A} = \left\{ \sum_{n \in \mathbb{N}^*} a_n \right\} \), equipped with the Frölicher structure of an infinite product. Then, the set

\[
1 + \mathcal{A} = \left\{ 1 + \sum_{n \in \mathbb{N}^*} a_n \mid \forall n \in \mathbb{N}^*, a_n \in A_n \right\}
\]

is a Frölicher Lie group with regular Frölicher Lie algebra \( \mathcal{A} \). Moreover, the exponential map defines a smooth bijection \( \mathcal{A} \to 1 + \mathcal{A} \).

**Notation:** for each \( u \in \mathcal{A} \), we denote by \([u]_n\) the \( A_n\)-component of \( u \).

**Theorem 2.26.** Let

\[
1 \longrightarrow K \overset{i}{\longrightarrow} G \overset{p}{\longrightarrow} H \longrightarrow 1
\]

be an exact sequence of Frölicher Lie groups, such that there is a smooth section \( s : H \to G \), and such that the subset diffeology from \( G \) on \( i(K) \) coincides with the push-forward diffeology from \( K \) to \( i(K) \). We let

\[
0 \longrightarrow \mathfrak{k} \overset{i'}{\longrightarrow} \mathfrak{g} \overset{p}{\longrightarrow} \mathfrak{h} \longrightarrow 0
\]

be the corresponding sequence of Lie algebras. Then,

- The Lie algebras \( \mathfrak{k} \) and \( \mathfrak{h} \) are regular if and only if the Lie algebra \( \mathfrak{g} \) is regular;
- The Frölicher Lie groups \( K \) and \( H \) are regular if and only if the Frölicher Lie group \( G \) is regular.
2.7. Existence of Non-regular Lie Groups

It is shown in [34] that the Frölicher Lie group $\text{Diff}^+([0;1[)$ of all increasing (Frölicher) diffeomorphisms of the open interval is non-regular in the sense of Definition 2.21. The proof is based on the observation that the unit vector field $1_{[0;1[}$ is in the Lie algebra of $\text{Diff}^+([0;1[)$ and that it generates a local flow acting by translations. This local flow cannot be global.

After this example, we consider a question raised in [25] on the existence of non-regular Lie groups. There are two main definitions of regular Lie groups. The first one was given by Omori [42,43] in the context of “generalized Lie groups”, which are topological groups without atlas but with additional structures, see [42]. Omori’s definition assumes that Eq. (2.3) can be solved by a convergent Euler method (details are given below, see Definition 2.27). The second one was given by Kriegl and Michor in [25]; it does not assume convergence of the Euler numerical scheme; instead, it assumes the existence of a solution to (2.3), and the groups considered are “convenient” $c^\infty$-Lie groups. Definition 2.21 used for Frölicher Lie groups is an adaptation of the one given in [25].

Are there non-regular Lie groups in the sense of Omori?

Let us comment more on this question. There are actually many groups for which we cannot prove existence of the exponential map, even in the extended sense of [25] or Definition 2.21. For example, the Frölicher Lie group $\text{Diff}^+([0,1[)$, which is not a Lie group because there is no known atlas on it, is not regular. On the other hand, there are interesting groups which are regular, for example the group of units of a (“nice”) algebra, modelled on a complete, Mackey complete, locally convex topological vector space, see [22]: the “minimal reasonable setting” for infinite-dimensional Lie groups is indeed difficult to determine [40].

Let us now prove that there exists a Lie group modelled on a locally convex topological vector space, which is not Omori-regular. We consider Omori’s definition of a regular Lie group in a specialized setting.

Definition 2.27 [42]. Let $\mathcal{A}$ be a topological algebra, for which $\mathcal{A}^*$ is open in $\mathcal{A}$ and it is a Lie group modelled on $\mathcal{A}$. Then $\mathcal{A}^*$ is Omori-regular if and only if, for any $v \in C^\infty([0,1],\mathcal{A})$, there is a unique smooth path $g \in C^\infty([0,1],\mathcal{A}^*)$, such that

1. $g^{-1}dg = v$ (equation for the right logarithmic derivative)
2. $g(0) = 1$ (initial condition)
3. $g(t) = \lim_{n \to +\infty} \prod_{i=0}^{n-1} \left(1 + \frac{t}{n} v \left(\frac{i}{n}\right)\right)$ (convergence of the Euler method).

Let us consider

$$\mathbb{R}((X)) = \left\{ \sum_{n \in \mathbb{Z}} a_n X^n \mid \exists N \in \mathbb{Z}, \quad \forall n < N, a_n = 0 \right\}.$$

The space $\mathbb{R}((X))$ is a vector subspace of $\mathbb{R}^\mathbb{Z}$, where the latter is equipped with its (product) Fréchet topology. $\mathbb{R}((X))$ is a well-known (commutative) field, and hence $\mathbb{R}((X))^* = \mathbb{R}((X)) - \{0\}$.
Proposition 2.28. The set of invertible elements $\mathbb{R}((X))^*$ is an open subset of $\mathbb{R}((X))$, and multiplication and inversion are smooth functions. As a consequence, $\mathbb{R}((X))^*$ is a Lie group modelled on a locally convex topological vector space.

Proof. Since $\mathbb{R}((X))^* = \mathbb{R}((X)) - \{0\}$, it is open in $\mathbb{R}((X))$, and hence it is a smooth submanifold of $\mathbb{R}((X))$. Let us now consider multiplication. Let

$$\left( \sum_{n \in \mathbb{Z}} a_n X^n; \sum_{n \in \mathbb{Z}} b_n X^n \right) \in \mathbb{R}((X))^2,$$

and let $\sum_{n \in \mathbb{Z}} c_n X^n$ be the product Laurent series. We have

$$\forall n \in \mathbb{Z}, \quad c_n = \sum_{i+j=n} a_i b_j,$$

and this sum is well defined since it has only a finite number of nonzero terms. Moreover, for fixed indexes $i$ and $j$, the map $(a_i, b_j) \mapsto a_i b_j$ is smooth. So that, for the product Frölicher structure, multiplication is smooth.

Let us now consider inversion. Let $\sum_{n \geq N} a_n X^n \in \mathbb{R}((X))$, where $N = \min\{n \in \mathbb{Z} | a_n \neq 0\}$. Let $\sum_{n \geq -N} b_n X^n$ the inverse series. The coefficients of the inverse series can be calculated by induction with the formula:

$$\sum_{i+j=n} a_i b_j = \delta_{n,0} \quad \text{(Kronecker symbol)}.$$

First, we get $b_{-N} = a_N^{-1}$ and by induction on $p \in \mathbb{N}^*$,

$$b_{-N+p} = -a_N^{-1} \sum_{k=0}^{p-1} a_{N+p-k} b_{-N+k},$$

which shows that inversion is smooth by the same arguments as above. □

Lemma 2.29. The sequence $\left( 1 + \frac{X^{-1}}{n} \right)^n$ has no limit in $\mathbb{R}((X))$.

Proof. The Frölicher structure of $\mathbb{R}((X))$ makes all the indexwise projections $[.]_m$ on the $m$-th coefficient of the Laurent series smooth. We remark that for any $m > 0$,

$$\left[ \left( 1 + \frac{X^{-1}}{n} \right)^n \right]_m = 0$$

and that, for $m \in \mathbb{N}$,

$$\lim_{n \to +\infty} \left[ \left( 1 + \frac{X^{-1}}{n} \right)^n \right]_{-m} = \lim_{n \to +\infty} \frac{1}{n^m} \binom{n}{m} = \frac{1}{m!} \neq 0.$$

Thus this sequence converges in $\mathbb{R}^\mathbb{Z}$, but not in $\mathbb{R}((X))$. □

These results imply the following theorem:

Theorem 2.30. $\mathbb{R}((X))^*$ is not regular in the sense of Omori.
Proof. We have that $X^{-1} \in \mathbb{R}((X))$ so that the constant path $t \in [0, 1] \mapsto v(t) = X^{-1} \in C^\infty([0, 1], \mathbb{R}((X)))$.

The Euler method gives:

$$\prod_{i=0}^{n-1} \left(1 + \frac{t}{n} v \left( \frac{i}{n} \right) \right) = \left(1 + \frac{tX^{-1}}{n} \right)^n.$$

By Lemma 2.29, this sequence does not converge for $t = 1$, and hence $\mathbb{R}((X))$ is not regular in the sense of Omori. □

2.8. Principal Bundles, Connections and the Ambrose–Singer Theorem

Let $P$ be a diffeological space and let $G$ be a Frölicher Lie group. Let us assume that there exists a smooth right action $P \times G \to P$, such that $\forall (p, p', g) \in P \times P \times G$, we have $p.g = p'.g \Rightarrow p = p'$. Let $M = P/G$, equipped with the quotient diffeology. $P$ is called a principal $G$-bundle with base $M$.

In [24, Article 8.32] Iglesias-Zemmour gives a definition of a connection on a principal $G$-bundle in terms of paths on the total space $P$, generalizing the classical notion of path lifting for principal bundles with finite-dimensional Lie groups as structure groups.

Definition 2.31. Let $G$ be a diffeological group, and let $\pi: P \to X$ be a principal $G$-bundle. Denote by $\text{Paths}_{\text{loc}}(P)$ the diffeological space of local paths (see [24, Article 1.63]), and by $\text{tpath}(P)$ the tautological bundle of local paths

$$\text{tpath}(P) := \{ (\gamma, t) \in \text{Paths}_{\text{loc}}(P) \times \mathbb{R} \mid t \in D(\gamma) \}.$$

A diffeological connection is a smooth map $H: \text{tpath}(P) \to \text{Paths}_{\text{loc}}(P)$ satisfying the following properties for any $(\gamma, t_0) \in \text{tpath}(P)$:

1. the domain of $\gamma$ equals the domain of $H(\gamma, t_0)$,
2. $\pi \circ \gamma = \pi \circ H(\gamma, t_0)$,
3. $H(\gamma, t_0)(t_0) = \gamma(t_0)$,
4. $H(\gamma \cdot g, t_0) = H(\gamma, t_0) \cdot g$ for all $g \in G$,
5. $H(\gamma \circ f, s) = H(\gamma, f(s)) \circ f$ for any smooth map $f$ from an open subset of $\mathbb{R}$ into $D(\gamma)$,
6. $H(H(\gamma, t_0), t_0) = H(\gamma, t_0)$.

Another formulation of this definition can be found in [32] under the terminology of path lifting.

Remark 2.32. Diffeological connections satisfy many of the usual properties which classical connections on a principal $G$-bundle (where $G$ is a finite-dimensional Lie group) enjoy; in particular, they admit unique horizontal lifts of paths in $\text{Paths}_{\text{loc}}(M)$ [24, Article 8.32], and they pull back by smooth maps [24, Article 8.33].

Proposition 2.33. Let $V$ be a vector space. Then, $G$ acts smoothly from the right on the space $\Omega(P, V)$ of $V$-valued differential forms on $P$ by setting

$$\forall (g, \alpha) \in \Omega^n(P, V) \times G, \forall p \in \mathcal{P}(P), \quad (g \ast \alpha)_p = \alpha_p \circ (dg^{-1})^n.$$
Proof. $G$ acts smoothly on $P$ so that, if $p \in \mathcal{P}(P)$, $g.p \in \mathcal{P}(P)$. The right action is now well defined, and smoothness is trivial.

**Definition 2.34.** Let $\alpha \in \Omega(P; g)$. The differential form $\alpha$ is **right-invariant** if and only if, for each $p \in \mathcal{P}(P)$, and for each $g \in G$,

$$\alpha_{g.p} = \text{Ad}_{g^{-1}} \circ g_* \alpha_p.$$ 

Now, we turn to connections and holonomy. Let $p \in P$ and let $\gamma$ be a smooth path in $P$ starting at $p$.

**Definition 2.35.** A **connection** on $P$ is a $g$-valued right-invariant 1-form $\theta$ such that, for each $v \in g$, for any path $c : \mathbb{R} \to G$ satisfying

\[\begin{align*}
c(0) &= e_G \\
\partial_t c(t)|_{t=0} &= v,
\end{align*}\]

and for each $p \in P$, we have:

$$\theta(\partial_t(p.c(t))_{t=0}) = v.$$ 

Now, let $p \in P$ and let $\gamma$ be a smooth path in $P$ starting at $p$, defined on $[0,1]$. Let $H_{\theta \gamma}(t) = \gamma(t)g(t)$, where $g(t) \in C^\infty([0,1]; g)$ is a path satisfying the differential equation:

\[\begin{align*}
\theta(\partial_t H_{\theta \gamma}(t)) &= 0 \\
H_{\theta \gamma}(0) &= \gamma(0).
\end{align*}\]

The first line of this equation is equivalent to the differential equation

$$g^{-1}(t)\partial_t g(t) = -\theta(\partial_t \gamma(t))$$

which is integrable, and the second line is equivalent to the initial condition $g(0) = e_G$. This shows that horizontal lifts are well defined, as in the standard case of finite-dimensional manifolds. Moreover, the map $H_{\theta}(.)$ defines trivially a diffeological connection. This enables us to consider the holonomy group of a connection. Notice that a straightforward adaptation of the arguments of [31] shows that the holonomy group does not depend (up to conjugation and up to the choice of connected component of $M$) on the choice of the base point $p$.

Now we assume that $\dim\([0,1]\] \geq 2$ and we fix a connection $\theta$ on $P$.

**Definition 2.36.** Let $\alpha \in \Omega(P; g)$ be a $G$-invariant 1-form. Let $\nabla \alpha = d\alpha - \frac{1}{2}[\theta, \alpha]$ be the horizontal derivative of $\alpha$. The curvature 2-form induced by $\theta$ is

$$\Omega = \nabla \theta.$$ 

This definition allows us to consider reductions of the structure group.

**Theorem 2.37** [32]. We assume that $G_1$ and $G$ are regular Frölicher groups with regular Lie algebras $g_1$ and $g$. Let $\rho : G_1 \hookrightarrow G$ be an injective morphism of Lie groups. If there exists a connection $\theta$ on $P$, with curvature $\Omega$, such that for any smooth 1-parameter family $H_{\theta \gamma t}$ of horizontal paths starting at $p$, and for any smooth vector fields $X, Y$ in $M$, the map

$$s, t \in [0,1]^2 \rightarrow \Omega_{H_{\theta \gamma t}}(X, Y)$$ (2.4)
is a smooth $g_1$-valued map (for the $g_1$-diffeology), and if $M$ is simply connected, then the structure group $G$ of $P$ reduces to $G_1$, and the connection $\theta$ also reduces.

We now state the announced Ambrose–Singer theorem, using the terminology of [48] for the classification of groups via properties of the exponential map:

**Theorem 2.38** [32]. Let $P$ be a principal bundle whose structure group is a fully regular Frölicher Lie group $G$. Let $\theta$ be a connection on $P$ and $H_\theta$ the associated diffeological connection.

1. For each $p \in P$, the holonomy group $H_p^\theta$ is a diffeological subgroup of $G$, which does not depend on the choice of $p$ up to conjugation.
2. There exists a second holonomy group $H^{\text{red}} \subset H_\theta$, which is the smallest structure group for which there is a subbundle $P'$ to which $\theta$ reduces. Its Lie algebra is spanned by the curvature elements, i.e. it is the smallest integrable Lie algebra which contains the curvature elements.
3. If $G$ is a Lie group (in the classical sense) of type I or II, there is a (minimal) closed Lie subgroup $\overline{H}^{\text{red}}$ (in the classical sense) such that $H^{\text{red}} \subset \overline{H}^{\text{red}}$, whose Lie algebra is the closure in $g$ of the Lie algebra of $H^{\text{red}}$. $\overline{H}^{\text{red}}$ is the smallest closed Lie subgroup of $G$ among the structure groups of closed sub-bundles $\overline{P}'$ of $P$ to which $\theta$ reduces.

From [32] again, we have the following result:

**Proposition 2.39.** If the connection $\theta$ is flat and $M$ is connected and simply connected, then for any path $\gamma$ starting at $p \in P$, the map

$$\gamma \mapsto H_\theta \gamma (1)$$

depends only on $\pi(\gamma(1)) \in M$, and it defines a global smooth section $M \to P$. Therefore, $P = M \times G$.

Let us precise a little bit more this result (see [25, section 40.2] for an analogous statement in the $C^\infty$-setting):

**Theorem 2.40.** Let $(G, g)$ be a regular Lie group with regular Lie algebra and let $X$ be a simply connected Frölicher space. Let $\alpha \in \Omega^1(M, g)$ such that

$$d\alpha + [\alpha, \alpha] = 0. \quad (2.5)$$

Then there exists a smooth map

$$f : X \to G$$

such that

$$df \cdot f^{-1} = \alpha.$$

Moreover, we move from one solution $f$ to another by applying the adjoint action of $G$, pointwise in $x \in X$.

We remark that the theorem also holds if we consider the equation

$$d\alpha - [\alpha, \alpha] = 0$$
instead of (2.5); we only need to change left logarithmic derivatives for right logarithmic derivatives, and adjoint action for coadjoint action. The correspondence between solutions is given by the inverse map $f \mapsto f^{-1}$ on the group $C^\infty(X,G)$.

3. On Groups and Algebras of Formal Series and Pseudo-Differential Operators

3.1. The Algebra of Formal Pseudo-Differential Operators

We let $A$ be a commutative $K$-algebra with unit 1, in which $K$ is any topologically complete field of characteristic zero. Let $A^*$ be the group of units of $A$, i.e. the group of invertible elements of $A$. We assume that $A$ is equipped with a derivation, that is, with a $K$-linear map $\partial : A \to A$ satisfying the Leibnitz rule $\partial(f \cdot g) = (\partial f) \cdot g + f \cdot (\partial g)$ for all $f, g \in A$.

Let $\xi$ be a formal variable not in $A$. The algebra of symbols over $A$ is the vector space $\Psi_\xi(A) = \left\{ P_\xi = \sum_{\nu \in \mathbb{Z}} a_\nu \xi^\nu : a_\nu \in A, a_\nu = 0 \text{ for } \nu \gg 0 \right\}$ equipped with the associative multiplication $\circ$ given by

$$P_\xi \circ Q_\xi = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k P_\xi}{\partial \xi^k} \partial^k Q_\xi, \quad (3.1)$$

with the prescription that multiplication on the right-hand side of (3.1) is standard multiplication of Laurent series in $\xi$ with coefficients in $A$, see [13,41]. The algebra $A$ is included in $\Psi_\xi(A)$. Operation (3.1) mirrors the extension of the Leibnitz rule to negative powers of the derivative $\partial$ as explained, for example, in [41].

The algebra of formal pseudo-differential operators over $A$ is the vector space

$$\Psi(A) = \left\{ P = \sum_{\nu \in \mathbb{Z}} a_\nu \partial^\nu : a_\nu \in A, a_\nu = 0 \text{ for } \nu \gg 0 \right\}$$

equipped with the unique multiplication which makes the map $\sum_{\nu \in \mathbb{Z}} a_\nu \xi^\nu \mapsto \sum_{\nu \in \mathbb{Z}} a_\nu \partial^\nu$ an algebra homomorphism. The algebra $\Psi(A)$ is associative but not commutative. It becomes a Lie algebra over $K$ if we define, as usual,

$$[P, Q] = PQ - QP, \quad (3.2)$$

and it is also an example of an infinite-dimensional non-commutative Poisson algebra in the sense of Kubo, see [29].

The order of $P \neq 0 \in \Psi(A)$, $P = \sum_{\nu \in \mathbb{Z}} a_\nu \partial^\nu$, is $N$ if $a_N \neq 0$ and $a_\nu = 0$ for all $\nu > N$. If $P$ is of order $N$, the coefficient $a_N$ is called the leading term or principal symbol of $P$. We denote by $\Psi^N(A)$ the vector space...
of pseudo-differential operators $P$ as above satisfying $k > N \Rightarrow a_k = 0$, that is,
\[ \Psi^N(A) = \{ P \in \Psi(A) \mid \text{order}(P) \leq N \}. \]
The vector space $\Psi^0(A)$ is of particular interest, since direct computations show that it is an algebra. We denote by $\Psi^{0,*}(A)$ its group of units, i.e. the group of invertible elements of $\Psi^0(A)$. In the following lemma we collect some classical properties of $\Psi(A)$.

**Lemma 3.1.** (1) If $P$ and $Q$ are formal pseudo-differential operators over $A$, and the leading terms of $P$ and $Q$ are not divisors of zero in $A$, then
\[ \text{order}(PQ) = \text{order}(P) + \text{order}(Q), \]
and
\[ \text{order}([P, Q]) \leq \text{order}(P) + \text{order}(Q) - 1. \]
(2) Every nonzero formal pseudo-differential operator $P$ for which its leading term is invertible has an inverse in $\Psi(A)$.

This lemma is proven for instance in [41]. After [33], we assume now, and till the end of Sect. 3.1, that the algebra $A$ is a Frölicher algebra. Thus, addition, scalar multiplication and multiplication are smooth, and inversion is a smooth operation on $A^*$, in which $A^*$ is equipped with the subset Frölicher structure. We also assume that $\partial$ is smooth. Then, identifying a formal pseudo-differential operator $P \in \Psi(A)$ with its sequence of partial symbols, we obtain that $\Psi(A)$, as a linear subspace of $A^Z$, carries a natural Frölicher structure. We obtain the following special case of [33, Proposition 3.25]:

**Proposition 3.2.** $\Psi(A)$ is a Frölicher algebra.

**Remark 3.3.** In more classical settings, $\Psi(A)$ can have a quite complicated structure. For example, if $A$ is a locally convex algebra, so is $\Psi(A)$, but this space is not complete for the product topology even if $A$ is complete, see [2, Section 5].

We further assume, until the end of Sect. 3.1, that $A^*$ is a Frölicher Lie group with Lie algebra $\mathfrak{g}_A$.

We notice that if $A$ is a complete locally convex topological algebra equipped with its natural Frölicher structure, and $A^*$ is open in $A$, then $\mathfrak{g}_A = A$. For the needs of integration, we also have to assume (till the end of Sect. 3.1) that $A$ is regular as a Frölicher vector space. We now extend a result proved in [22] for the case when $A$ is a suitable topological algebra and the group of units is open in $A$. In that paper, Glöckner was able to adapt the standard proof of the Lie group structure of $GL(H)$ when $H$ is a Hilbert space and to prove that for any algebra in his class, the group of units is an analytic Lie group (this notion appears in [22] and references therein); since in our setting $A$ can be quite general, our proof needs to be different to Glöckner’s. It is based on Theorems 2.25 and 2.26.
Lemma 3.4. The group $1 + \Psi^{-1}(A)$ is a fully regular Frölicher Lie group with regular Lie algebra $\Psi^{-1}(A)$.

Proof. By integration componentwise, we already know that $\Psi^{-1}(A)$ is regular. We now remark that $\Psi^{-1}(A)$ is graded by the order. Then, setting $A_n = \{a_{-n}\partial^{-n}|a_{-n} \in A\}$, we obtain the hypotheses needed in order to apply Theorem 2.26.

Theorem 3.5. There exists a short exact sequence of groups

$$1 \longrightarrow 1 + \Psi^{-1}(A) \longrightarrow \Psi^0(A) \longrightarrow A^* \longrightarrow 1$$

such that:

1. The injection $1 + \Psi^{-1}(A) \rightarrow \Psi^0(A)$ is smooth
2. The principal symbol map $\sigma_0 : \Psi^0(A) \rightarrow A^*$ is smooth and it has a global section which is the restriction to $A^*$ of the canonical inclusion $A \rightarrow \Psi^0(A)$.

As a consequence, $A^*$ is a fully regular Frölicher Lie group if and only if $\Psi^0(A)$ is a fully regular Frölicher Lie group with regular Lie algebra $g_A \oplus \Psi^{-1}(A)$.

Proof. Let $a, b \in (\Psi^0(A))^2$. Then, $\sigma_0(ab) = \sigma_0(a)\sigma_0(b)$. If $a$ is invertible, setting $b = a^{-1}$, we get $\sigma_0(a)\sigma_0(a^{-1}) = 1_A$. Thus, $\sigma_0(a) \in A^*$. Thus, the map $a \mapsto \sigma_0(a)$ is a morphism of diffeological groups. Moreover, the section map $A^* \rightarrow \Psi^0(A)$ is a smooth morphism of Frölicher Lie groups, and the inclusion $1 + \Psi^{-1}(A) \rightarrow \Psi^0(A)$ is obviously a morphism of Frölicher Lie groups. Lemma 3.4 implies that we can apply Theorem 2.26, thereby completing the proof.

As explained in Remark 2.6, the results stated above remain valid for classical Gâteaux differentiability if $A$ is a Fréchet space.

Let us now prove the following:

Theorem 3.6. There is no Frölicher subalgebra $A$ of $\Psi(A)$ whose group of units is a Frölicher Lie subgroup of $\Psi(A)^*$ with Lie algebra $g$ and $\partial \in g$, which is Omori-regular. In particular $\Psi(A)^*$ is not Omori-regular.

Proof. Substituting $X = \partial^{-1}$ in Lemma 2.29 we have that the sequence

$$u_n = \left(1 + \frac{\partial}{n}\right)^n$$

converges in $\mathbb{R}^Z$ but not in $\mathbb{R}((\partial^{-1})) = \mathbb{R}^Z \cap \Psi(A)$.

The fact that the Euler method fails to define an exponential map motivates us, after [37,38], to regularize $\Psi(A)$ into some “deformed” algebra, see Definitions 3.8, 3.17 and 4.3 below.
3.2. Mulase Formal Lie Group and Frölicher Structures

Following Bourbaki [8, Algebra, Chapters 1–3, p. 454–457], we let \( T \) be the additive monoid of all sequences of natural numbers \( t = (n_i)_{i \in \mathbb{N}} \) such that \( n_i = 0 \) except for a finite number of indices, and we let \( R \) be a commutative ring equipped with a derivation \( \partial \). A formal power series is a function \( u \) from \( T \) to \( R \), \( u = (u_t)_{t \in T} \). Consider an infinite number of formal variables \( \tau_i \), \( i \in I \), and set \( \tau = (\tau_1, \tau_2, \cdots) \). Then, the formal power series \( u \) can be written as \( u = \sum_{t \in T} u_t t^t \) in which \( t^t = \tau_1^{n_1} \tau_2^{n_2} \cdots \). We say that \( u_t \in R \) is a coefficient and that \( u_t \tau^t \) is a term.

Operations on the set \( R[[\tau]] := R[[\tau_1, \tau_2, \cdots]] \) are defined in an usual manner: If \( u = (u_t)_{t \in T} \), \( v = (v_t)_{t \in T} \), then

\[
    u + v = (u_t + v_t)_{t \in T} \quad \text{and} \quad uv = w,
\]

in which \( w = (w_t)_{t \in T} \) and

\[
    w_t = \sum_{r,s \in T} u_r v_s, \quad \text{where} \quad r + s = t.
\]

It is shown in [8, p. 455] that this multiplication is well defined and that \( A_t = R[[\tau]] \) equipped with these two operations is a commutative algebra with unit. The derivation \( \partial \) on \( R \) extends to a derivation on \( A_t \) via

\[
    \partial u = \sum_{t \in T} (\partial u_t) t^t.
\]

We define the valuation of a power series following [37], p. 59. Let \( u \in A_t \), \( u \neq 0 \). We write \( u = \sum_{t \in T} u_t t^t \), and if \( t = (n_i)_{i \in \mathbb{N}} \), we set \( |t| = \sum in_i \), which is equivalent to Mulase’s assumption \( \text{ord}(\tau_i) = i \). The terms \( u_t t^t \) such that \( |t| = p \) are called monomials of valuation \( p \). The formal power series \( u_p \), whose terms of valuation \( p \) are those of \( u \), and whose other terms are zero, is called the homogeneous part of \( u \) of valuation \( p \). The series \( u_0 \), the homogeneous part of \( u \) of valuation \( 0 \), is identified with an element of \( R \) called the constant term of \( u \). For a formal series \( u \neq 0 \), the least integer \( p \geq 0 \) such that \( u_p \neq 0 \) is called the valuation of \( u \), and it is denoted by \( \text{val}_t(u) \). We extend this definition to the case \( u = 0 \) by setting \( \text{val}_t(0) = +\infty \) (see [8, p. 457]). The following properties hold: if \( u, v \) are formal power series then

\[
    \text{val}_t(u + v) \geq \inf \{\text{val}_t(u), \text{val}_t(v)\}, \quad \text{if} \quad u + v \neq 0, \quad (3.3)
\]

\[
    \text{val}_t(u + v) = \inf \{\text{val}_t(u), \text{val}_t(v)\}, \quad \text{if} \quad \text{val}_t(u) \neq \text{val}_t(v), \quad (3.4)
\]

\[
    \text{val}_t(uv) \geq \text{val}_t(u) + \text{val}_t(v), \quad \text{if} \quad uv \neq 0. \quad (3.5)
\]

Remark 3.7. Other definitions of \( \text{val}_t \) are possible, see [11,12] and [17].

Definition 3.8 (Mulase [38]). We define spaces of formal pseudo-differential and differential operators of infinite order, \( \mathcal{P}(A_t) \) and \( \mathcal{D}_A_t \), respectively, as

\[
    \mathcal{P}(A_t) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : a_\alpha \in A_t \text{ and } \exists (C,N) \in \mathbb{R} \times \mathbb{N} \text{ such that } \text{val}_t(a_\alpha) > C\alpha - N \forall \alpha \gg 0 \right\}
\]

(3.6)
and
\[ \hat{\mathcal{D}}_{A_t} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : P \in \hat{\Psi}(A_t) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}. \quad (3.7) \]

In addition, we define
\[ \mathcal{I}_{A_t} = \Psi^{-1}(A_t) = \left\{ P \in \widehat{\Psi}(A_t) : \forall \alpha > 0, a_\alpha = 0 \right\} \]
and
\[ G_{A_t} = 1 + \mathcal{I}(A_t). \]

Notice that \( A_t \subset \hat{\Psi}(A_t) \) as the 0-order term, that
\[ G_{A_t} \subset \Psi(A_t) \subset \hat{\Psi}(A_t), \]
and that \( \hat{\mathcal{D}}_{A_t} \not\subset \Psi(A_t) \). The operations on \( \hat{\Psi}(A_t) \) are extensions of the operations on \( \Psi(A_t) \). More precisely we have (see [38], see also an explicit proof in [17]):

**Lemma 3.9** The space \( \hat{\Psi}(A_t) \) has an algebra structure, and \( \hat{\mathcal{D}}_{A_t} \) is a subalgebra of \( \hat{\Psi}(A_t) \).

**Definition 3.10** Let \( K \) be the ideal of \( A_t \) generated by \( \tau_1, \tau_2, \cdots \). If \( P \in \hat{\Psi}(A_t) \), we denote by \( P|_{\tau=0} \) the equivalence class \( P \mod K \) (i.e. the projection on the constant coefficient of the \( \tau \)-series), and we identify it with an element of \( \Psi(A) \). We also define the spaces
\[ G(\hat{\Psi}(A_t)) = \left\{ P \in \hat{\Psi}(A_t) : P|_{\tau=0} \in G_{A_t} \right\} \quad (3.8) \]
and
\[ \hat{\mathcal{D}}_{A_t}^\times = \left\{ P \in \hat{\mathcal{D}}_{A_t} : P|_{\tau=0} = 1 \right\}. \quad (3.9) \]

**Remark 3.11** We note that if \( R \) is a Fréchet ring, then the spaces defined in (3.6) and (3.7) are locally convex but not necessarily complete, and the same remark holds for the spaces defined in Lemma 3.12 below. On the other hand, we will see (Theorem 3.13) that these spaces have Frölicher structures and that therefore they can be studied in a rather complete way without the need to consider delicate convergence issues.

Hereafter we assume that \( R \) is a Frölicher \( \mathbb{K} \)-algebra and that \( \partial : R \to R \) is a smooth derivation. These assumptions imply the following (we use the notations introduced at the beginning of this subsection):

**Lemma 3.12** \( \Psi(R)[[\tau]] \) is a Frölicher algebra and
\[ G\Psi_t(R) = \left\{ u \in \Psi(R)[[\tau]] : \text{the monomial of valuation 0 equals 1} \right\} \]
is a Frölicher Lie group with Lie algebra
\[ g\Psi_t(R) = \left\{ u \in \Psi(R)[[\tau]] : \text{the monomial of valuation 0 equals 0} \right\}. \]
Proof First of all we note that $Ψ(R)$ is a Frölicher algebra because of Proposition 3.2. Now, since the space of power series $Ψ(R)(τ)$ can be identified with the vector space of functions $Ψ(R)^T ⊂ Ψ(R) × T$, we can conclude that $Ψ(R)(τ)$ admits a natural Frölicher structure, thanks to the results reviewed in Sects. 2.4 and 2.5. We need to show that the algebra operations are smooth with respect to this structure. Addition and scalar multiplication are those of the vector space $Ψ(R)^T$, so that they are smooth. Now, if 

$$u = \sum_{t ∈ T} a_t τ^t \quad \text{and} \quad v = \sum_{t ∈ T} b_t τ^t,$$

then $u.v = w$, in which $w$ is given by the formulae recalled above, when we introduced operations in the algebra of series $A_t$: fixing $t ∈ T$, we see that the coefficient of $τ^t$ in $Ψ(R)$ is a (finite) linear combination of multiplications $a_{t'}b_{t''}$ with $t' + t'' = t$. Thus multiplication is smooth, and so $Ψ(R)(τ)$ is a Frölicher algebra.

Now let $u ∈ GΨ_t(R)$. The coefficients of $u^{-1}$ are obtained from the equation

$$uu^{-1} = 1$$

which gives, by induction on the valuation in $t$ of each monomial:

$$[u^{-1}]_0 = ([u]_0)^{-1} = 1$$

and

$$[u^{-1}]_t = - \sum_{t' + t'' = t, t' ≠ 0} [u]_{t'}[u^{-1}]_{t''}$$

which shows that inversion is smooth, since $Ψ(R)$ is a Frölicher algebra.

Since $Ψ(R)(τ)$ is a Frölicher algebra, the adjoint map

$$GΨ_t(R) × gΨ_t(R) → gΨ_t(R)$$

$$(g, v) ↦ gvg^{-1}$$

differentiates to the classical Lie bracket

$$gΨ_t(R) × gΨ_t(R) → Ψ_t(R)$$

$$(v_1, v_2) ↦ [v_1, v_2] = v_1 v_2 - v_2 v_1,$$

see the discussion leading to (2.1). Now

$$val_t (v_1 v_2 - v_2 v_1) ≥ \inf (val_t(v_1), val_t(v_2)) ≥ 1,$$

and hence $[v_1, v_2] ∈ gΨ_t(R)$, which ends the proof. □

Theorem 3.13 The following algebras are Frölicher algebras:

(1) $A_t$;  (2) $Ψ(A_t)$;  (3) $\tilde{Ψ}(A_t)$;  (4) $\tilde{D}_{A_t}$.

Proof (1) We argue as in the previous lemma. Let us look at monomials of the series $u ∈ A_t$. Addition and scalar multiplication are those induced by the vector space $R^T$, so that they are smooth. If $u.v = w$, then for any fixed $t ∈ T$, the $R$-value of the $t$-monomial is a (finite) linear combination of multiplications the $R$-values of the $t'$- and $t''$-monomials of $u$ and $v$, respectively, where $t' + t'' = t$. Thus multiplication is smooth.
(2) It follows from Proposition 3.2 and the previous item.

(3) We recall that the notation $\tau^t$ makes sense because of the conventions at the beginning of this subsection. We follow the proof of Lemma 3.12. Let $P = \sum_{n \in \mathbb{Z}} a_n \partial^n \in \hat{Ψ}(A_t)$. Then we have

$$\sum_{n \in \mathbb{Z}} a_n \partial^n = \sum_{n \in \mathbb{Z}} \left( \sum_{t \in T} u_{nt} \tau^t \right) \partial^n = \sum_{t \in T} \left( \sum_{n \in \mathbb{Z}} u_{nt} \partial^n \right) \tau^t$$

(3.10)

for $u_{nt} \in R$, and so we can identify $\hat{Ψ}(A)$ with a subalgebra of $Ψ(R)[[\tau]]^2$. By Proposition 3.2, $Ψ(R)$ is a Frölicher algebra, and by Lemma 3.12, $Ψ(R)[[\tau]]$ is a Frölicher algebra. Thus, by Proposition 2.15, $\hat{Ψ}(A)$ is a Frölicher algebra.

(4) $\hat{D}_{A_t}$ is a subalgebra of $\hat{Ψ}(A_t)$ and therefore it is a Frölicher algebra.

We obtain the following result at the group level:

**Theorem 3.14** The following assertions hold:

1. $\hat{D}_{A_t}^\times$ is a Frölicher group.
2. $G_R = 1 + \Psi^{-1}(R)$ and $G_{A_t} = 1 + \Psi^{-1}(A_t)$ are Frölicher Lie groups.
3. $G(\hat{Ψ}(A_t))$ is a Frölicher group.

**Proof** (1) We proceed as in Theorem 3.13, see Eq. (3.10). The group $\hat{D}_{A_t}^\times$ can be identified with a subgroup of $GΨ_t(R)$, on which multiplication and inversion are smooth.

(2) These facts follow from Lemma 3.4 and Theorem 3.13.

(3) Recall that we denote by $P|_{\tau=0}$ the projection of the coefficients of $P$ on their constant coefficient, see Definition 3.10. Of course, this coincides with “evaluating at $\tau_1 = \tau_2 = ... = 0$”, which explains the notation.

Since

$$P|_{\tau=0}(G(\hat{Ψ}(A_t))) = G_R = 1 + \Psi^{-1}(R)$$

we have the following exact sequence:

$$0 \to Ker (P|_{\tau=0}) \to G(\hat{Ψ}(A_t)) \to G_R \to 0$$

and there is a global section $G_R \to G(\hat{Ψ}(A_t))$, given by the canonical inclusion componentwise, which is smooth. Then, adapting the classical (algebraic) construction of the semi-direct product

$$G(\hat{Ψ}(A_t)) = Ker (P|_{\tau=0}) \rtimes G_R$$

we remark that all the necessary operations to build the multiplication and the inversion group structure of $G(\hat{Ψ}(A_t))$ from $Ker (P|_{\tau=0})$ and $G_R$ are smooth, for the subset Frölicher structures inherited from $\hat{Ψ}(A_t)$.

We are ready to state and prove an enriched version of Mulase’s algebraic factorization theorem, see [37,38], for the Frölicher group $G(\hat{Ψ}(A_t))$:
Theorem 3.15  (1) For any $U \in G(\hat{\Psi}(A_t))$, there exist unique $S \in G_{A_t}$ and $Y \in \hat{D}_A^\times$ such that

$$U = S^{-1}Y.$$ 

In other words, there exists a unique global factorization of the Frölicher Lie group $G(\hat{\Psi}(A_t))$ as a group defined by matched pairs,

$$G(\hat{\Psi}(A_t)) = G_{A_t} \hat{D}_A^\times.$$  

(2) The factorization $U \mapsto (S, Y)$ is (Frölicher) smooth. Hence, it is smooth in the sense of Gateaux if $R$ is a Fréchet ring.

In order to prove this theorem, we need to introduce some notation: we let $u \mapsto u_-$ be the linear projection on $\Psi^{-1}(A_t)$ with respect to the $\partial$-components of $u \in \hat{\Psi}(A_t)$, and we set $u_+ = u - u_-$. 

Proof Part (1) is already proved in [37,38] (see also the later exposition [17]), but we have to examine carefully the original proof in order to check that the map $U \mapsto S$ is smooth. Once we have this first result, we finish the proof of the theorem by remarking that $Y = SU$.

Let us follow the proof of [37, Lemma 3]. We solve the equation

$$(SU)_- = 0$$  \hspace{1cm} (3.11)$$

for a given

$$U = \sum_{\nu \in \mathbb{Z}} u_\nu \partial^\nu$$

and unknown

$$S = \sum_{\mu \in \mathbb{Z} - N} s_\mu \partial^\mu.$$ 

Equation (3.11) turns into the infinite system

$$\sum_{\nu \in \mathbb{Z} - N} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{\mu+i-\nu} \right) s_\nu = -u_\mu \quad \text{for } \mu \in \mathbb{Z} - N,$$

with the convention

$$\binom{\nu}{i} = (-1)^{|\nu|} \binom{|\nu|}{i} \quad \text{if } \nu < 0.$$ 

We solve this infinite system by induction on $\mu$. For $\mu = -1$, let us study the $t$-monomials

$$\left[ \sum_{\nu \in \mathbb{Z} - N} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-1+i-\nu} \right) s_\nu \right]_0 = \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) s_{-1} \right]_0 = -[s_{-1}]_0$$
thus \([s_{-1}]_0 = [u_{-1}]_0\). Then, at the monomial of valuation 1, which is just the \(t_1\)-monomial, we have

\[
\left[ \sum_{\nu \in \mathbb{Z} - N} \left( \sum_{i \in N} \begin{pmatrix} \nu \\ i \end{pmatrix} \partial^i u_{-1+i-\nu} \right) \right]_{s_{-1}} = \left[ \left( \sum_{i \in N} \begin{pmatrix} -1 \\ i \end{pmatrix} \partial^i u_i \right) \right]_{t_1} \left[ [s_{-1}]_0 \right]_{t_1} = \left[ \left( \sum_{i \in N} \begin{pmatrix} -1 \\ i \end{pmatrix} \partial^i u_i \right) \right]_{0} [s_{-1}]_{t_1}
\]

and therefore we get

\[
[s_{-1}]_1 = [u_{-1}]_1 + \left[ \left( \sum_{i \in N} \begin{pmatrix} -1 \\ i \end{pmatrix} \partial^i u_i \right) \right]_{t_1} [u_{-1}]_0 - [s_{-1}]_{t_1}
\]

which shows that the maps \(u \mapsto [s_{-1}]_0\) and \(u \mapsto [s_{-1}]_{t_1}\) are smooth. Similar calculations can be carried out for any \(t\)-monomial appearing in \(s_{-1}\), and they show that \(u \mapsto s_{-1}\) is smooth. Let us now assume that \(s_{-1}, \ldots, s_{-k}\) are determined and that they are smooth with respect to \(U\), and let us determine \(s_{-k-1}\). We write, for \(\mu = -k - 1\),

\[
\sum_{\nu \leq -k-1} \left( \sum_{i \in N} \begin{pmatrix} \nu \\ i \end{pmatrix} \partial^i u_{-k-1+i-\nu} \right) s_{\nu} = -u_{-k-1} - \sum_{\nu = -k}^{1} \left( \sum_{i \in N} \begin{pmatrix} \nu \\ i \end{pmatrix} \partial^i u_{-k-1+i-\nu} \right) s_{\nu}
\]

noticing that the right hand side is fully determined. We carry out induction with respect to the valuation on \(t\). Let us us sketch the first two steps as we did before:

- for the \(t = 0\) term:
  \[
  [s_{-k-1}]_0 = (-1)^{-k-1} \left[ u_{-k-1} + \sum_{\nu = -k}^{1} \left( \sum_{i \in N} \begin{pmatrix} \nu \\ i \end{pmatrix} \partial^i u_{-k-1+i-\nu} \right) s_{\nu} \right]_0,
  \]

- for the \(t_1\)-monomial:
  \[
  [s_{-k-1}]_{t_1} = (-1)^{-k-1} \left( \left[ u_{-k-1} + \sum_{\nu = -k}^{1} \left( \sum_{i \in N} \begin{pmatrix} \nu \\ i \end{pmatrix} \partial^i u_{-k-1+i-\nu} \right) s_{\nu} \right]_1 + \left[ \left( \sum_{i \in N} \begin{pmatrix} -1 \\ i \end{pmatrix} \partial^i u_i \right) \right]_{t_1} [s_{-k-1}]_0 \right).
  \]

Thus, we have a construction of \(S\) consisting only of smooth operations, so that the map

\[
U \mapsto S
\]
is smooth. Hence the map

\[ U \mapsto Y = SU \]

is also smooth. This completes the proof. \( \square \)

3.3. Differentiable Structures and Regular Subgroups of \( G(\Psi(A_t)) \)

Let us stress that we have established smoothness of the operations on the groups considered by Mulase, but that we have not proved the existence of an exponential map. This is principally because it is difficult to show the existence of the exponential map on \( \hat{\Psi}(A_t) \); also, because it is difficult to determine the tangent space \( T_1 G(\hat{\Psi}(A_t)) \), and even more, to differentiate an hypothetical adjoint action of \( G(\hat{\Psi}(A_t)) \) on \( T_1 G(\hat{\Psi}(A_t)) \). Most of the difficulties come from the very general definition of \( \hat{\Psi}(A_t) \). This is why we construct a Frölicher subalgebra \( \Psi(A_t) \subset \hat{\Psi}(A_t) \), motivated by the next lemma. We denote by \( \exp_t \) the exponential defined on formal series of elements of \( \Psi(A) \).

Lemma 3.16 Let \( (P_n)_{n \in \mathbb{N}^*} \in \Psi(A)^{\mathbb{N}^*} \) such that order\( (P_n) \leq n \). Then

\[ \exp_t \left( \sum_{i \in \mathbb{N}^*} t_i P_i \right) \in \hat{\Psi}(A), \]

and the map \( (P_n)_{n \in \mathbb{N}^*} \mapsto \exp_t(\sum_{i \in \mathbb{N}^*} t_i P_i) \) is smooth with respect to the Frölicher structures of \( \Psi(A)^{\mathbb{N}^*} \) and \( \hat{\Psi}(A) \). Moreover, if we write

\[ \exp_t \left( \sum_{i \in \mathbb{N}^*} t_i P_i \right) = \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha, \]

then we have

\[ \alpha \leq \text{val}_t a_{\alpha} \]

for all \( \alpha \in \mathbb{Z} \).

Proof Each monomial of \( \exp_t(\sum_{i \in \mathbb{N}^*} t_i P_i) \) is, up to a scalar multiplication, a finite sum of finite products of terms of the sequence \( (P_n) \). Since multiplication is smooth on \( \Psi(A) \), we get that the map \( (P_n)_{n \in \mathbb{N}^*} \mapsto \exp_t(\sum_{i \in \mathbb{N}^*} t_i P_i) \) is smooth. Thus, we only need to check that \( \forall \alpha \in \mathbb{Z}, \alpha \leq \text{val}_t a_{\alpha} \), which will ensure us that \( \exp_t(\sum_{i \in \mathbb{N}^*} t_i P_i) \in \hat{\Psi}(A) \). Setting \( P_f(\mathbb{N}) \) the set of finite subsets of \( \mathbb{N} \), we get that for any \( I \in P_f(\mathbb{N}) \) and any multi-index \( (\alpha_i)_{i \in I} \), the \( (\prod_{i \in I} t_i^{\alpha_i}) \)-monomial \( p_{(t_i), (\alpha_i)} \) is of \( (\theta\)-maximal) order

\[ \alpha = \sum_{i \in I} \alpha_i, \]

by property \((3.5)\). Then, from the classical formulas that compute \( a_{\alpha} \), we obtain

\[ \text{val}_t a_{\alpha} \geq \sum_{I \in P_f(\mathbb{N}), \sum_i \alpha_i = \alpha} \alpha_i = \alpha, \]

using \((3.3)\) and \((3.4)\). \( \square \)
**Definition 3.17** The regular spaces of formal pseudo-differential and differential operators of infinite order are, respectively, \( \Psi(A_t) \) and \( \mathcal{D}_{A_t} \), in which

\[
\Psi(A_t) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \in \hat{\Psi}(A_t) : \text{val}_t(a_\alpha) \geq \alpha \right\}
\]

and

\[
\mathcal{D}_{A_t} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : P \in \Psi(A_t) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}.
\]

In addition, we define

\[
G(\Psi(A_t)) = \{ P \in \Psi(A_t) : P|_{\tau=0} \in G_{A_t} \}
\]

and

\[
\mathcal{D}_{A_t}^\times = \{ P \in \mathcal{D}_{A_t} : P|_{\tau=0} = 1 \}.
\]

Complementing Remark 3.11, we note that if \( R \) is a Fréchet ring, then \( \Psi(A_t) \) and \( \mathcal{D}_{A_t} \) are Fréchet algebras. In our general case, we have:

**Lemma 3.18** \( \Psi(A_t) \) is a Frölicher algebra, and \( G(\Psi(A_t)) \) is a Frölicher group.

**Proof** From Theorems 3.13 and 3.14, we only have to prove that \( \Psi(A_t) \) and \( G(\Psi(A_t)) \) are stable under algebraic operations. In order to do this, we only need to check some estimates:

- Let \( (\sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha, \sum_{\alpha \in \mathbb{Z}} b_\alpha \partial^\alpha) \in \Psi(A_t)^2 \). Then

\[
\sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha + \sum_{\alpha \in \mathbb{Z}} b_\alpha \partial^\alpha = \sum_{\alpha \in \mathbb{Z}} (a_\alpha + b_\alpha) \partial^\alpha.
\]

Using property (3.3) we obtain

\[
\text{val}_t(a_\alpha + b_\alpha) \geq \inf(\text{val}_t(a_\alpha), \text{val}_t(b_\alpha)) \geq \alpha.
\]

- Proceeding in the same way and with the same notations, we see that the equation

\[
\left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right) \left( \sum_{\alpha \in \mathbb{Z}} b_\alpha \partial^\alpha \right) = \sum_{(\alpha, \beta, k) \in \mathbb{Z}^2 \times \mathbb{N}} \frac{1}{k!} (\partial^k a_\alpha) b_\beta \partial^{\alpha + \beta - k}
\]

gives:

\[
\text{val}_t \left( (\partial^k a_\alpha) b_\beta \right) \geq \text{val}_t (a_\alpha b_\beta)
\]

and by property (3.5),

\[
\text{val}_t \left( (\partial^k a_\alpha) b_\beta \right) \geq \alpha + \beta \geq \alpha + \beta - k.
\]
• Mimicking the proof of Lemma 3.12, if $\sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \in G(\Psi(A_t))$, then

$$\left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_t = - \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_0^{-1} \sum_{t'+t''=t, t' \neq 0} \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_{t'} \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_{t''}.$$

Since

$$\left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_0 \in G_R,$$

we have

$$\text{val}_t \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_0 = 0,$$

and hence, considering each sum of monomials of constant valuation $k$,

$$\text{ord} \left( \sum_{|t|=k} \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_t \right) \leq \sup_{|t|=k} \text{ord} \left( \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_0^{-1} \sum_{t'+t''=t, t' \neq 0} \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_{t'} \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_{t''} \right) \leq 0 + \sup_{|t|=k} \text{ord} \left( \sum_{t'+t''=t, t' \neq 0} \left[ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right]_{t'} \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_{t''} \right) \leq \sup_{|t|=k} \sup_{t'+t''=t, t' \neq 0} \text{ord} \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)_{t'} + \text{ord} \left( \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_{t''} \right) \leq \sup_{|t|=k} \sup_{t'+t''=t, t' \neq 0} \text{val}_t \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)_{t'} + \text{val}_t \left( \left[ \left( \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \right)^{-1} \right]_{t''} \right) \leq k.$$
for the reader. It can be proved either by direct computations or by considering the intersection of the “hatted” sets appearing in Theorems 3.13 and 3.14 with $\Psi(A_t)$.

**Proposition 3.19** We have the following properties:

1. $I_{A_t} \subset \Psi(A_t)$.
2. $G_{A_t} \subset G(\Psi(A_t))$.
3. $\overline{D}_{A_t}^\times$ is a Frölicher subgroup of $G(\Psi(A_t))$.
4. For any $U \in G(\Psi(A_t))$ there exist unique $S \in G_{A_t}$ and $Y \in \overline{D}_{A_t}^\times$ such that

$$U = S^{-1}Y.$$ 

In other words, there exists a unique global factorization of the Frölicher Lie group $G(\Psi(A_t))$ as a group defined by matched pairs,

$$G(\Psi(A_t)) = G_{A_t} \overline{D}_{A_t}^\times.$$ 

5. The factorization $U \mapsto (S, Y)$ is smooth.

We now turn to regular Lie groups. For this, and till the end of the paper, we assume that $A_t$ is a regular algebra, and that $A_t^*$ is a regular Lie group. This is equivalent to saying that $R$ is a regular vector space and $R^*$ is a regular Lie group.

**Lemma 3.20** $\overline{\Psi}(A_t)$ is a regular vector space.

**Proof** Let $s \mapsto \sum_{n \in \mathbb{Z}} a_\alpha(s) \partial^\alpha \in C^\infty([0; 1], \overline{\Psi}(A_t))$.

- If $val_t(a_\alpha(s)) < \alpha$, then $a_\alpha(s) = 0$ and hence $\int_0^s a_\alpha = 0$.
- If $val_t(a_\alpha(s)) \geq \alpha$, then $s \mapsto \int_0^s a_\alpha$ exists since $R$ is regular.

This integral does not change the inequalities on $val_t$ satisfied by $a_\alpha(s)$, and so

$$s \mapsto \int_0^s \sum_{n \in \mathbb{Z}} a_\alpha(s) \partial^\alpha \in C^\infty([0; 1], \overline{\Psi}(A_t)).$$

\[ \square \]

**Theorem 3.21** The group $\overline{D}_{A_t}^\times$ is a regular Frölicher Lie group, with regular Lie algebra

$$\overline{D}_{A_t}^\times = \{ P \in \hat{D}_{A_t} : P|_{\tau=0} = 0 \}.$$ 

**Proof** The vector space $\overline{D}_{A_t}^\times$ is by construction a Frölicher vector space, which is regular as it can be seen by adapting the proof of Lemma 3.20. From Proposition 3.19, $\overline{D}_{A_t}^\times$ is a Frölicher group. Let

$$c(s) = a_0(s) + \sum_{\alpha \geq 0} a_\alpha(s) \partial^\alpha$$

be a path in $\overline{D}_{A_t}^\times$ starting at 1. From Eq. (3.15) we have

$$\partial a_0(0) = 0.$$
Moreover, \( \text{val}_t(a_\alpha) \geq \alpha \), and hence
\[
\text{val}_t(d_s a_\alpha(0)) \geq \alpha.
\]
Thus,
\[
\partial_s c(0) \in \mathfrak{D}_{A_t}^\times.
\]
Conversely, let \( w \in \mathfrak{D}_{A_t}^\times \). Since \( \text{val}_t(w) \geq 1 \), we have that the path
\[
c(s) = 1 + sw \in \mathfrak{D}_{A_t}^\times.
\]
Since \( \mathfrak{D}_{A_t}^\times \) is a Frölicher algebra and \( \mathfrak{D}_{A_t}^\times \) is a Frölicher group, we conclude that the adjoint map
\[
\mathfrak{D}_{A_t}^\times \times \mathfrak{D}_{A_t}^\times \rightarrow \mathfrak{D}_{A_t}^\times,
\]
\[
(D, d) \mapsto Ddd^{-1}
\]
is well defined, with values in \( \mathfrak{D}_{A_t}^\times \), and that it differentiates to the standard Lie bracket
\[
\langle d_1, d_2 \rangle = d_1 d_2 - d_2 d_1,
\]
which shows that \( \mathfrak{D}_{A_t}^\times \) is a Frölicher Lie group with regular Lie algebra \( \mathfrak{D}_{A_t}^\times \). We complete the proof by applying Theorem 2.25, with \( \mathcal{A}_n \) defined as the vector space of all \((t, \partial)\)-monomials of valuation equal to \( n \), and of order \( \leq n \).

We now give the corresponding theorem for \( G(\Psi(A_t)) \).

**Theorem 3.22** The group \( G(\Psi(A_t)) \) is a regular Frölicher Lie group, with regular Frölicher Lie algebra
\[
\mathfrak{g}(\Psi(A_t)) = \{ \mathfrak{P} \in \Psi(A_t) : \mathfrak{P}|_\tau = 0 \in \Psi^{-1}(R) \}.
\]

**Proof** We have the following exact sequence:
\[
0 \rightarrow \text{Ker}(\langle \cdot \rangle|_\tau = 0) \rightarrow G(\Psi(A_t)) \rightarrow \mathcal{G}_{\tau = 0} \rightarrow 0
\]
with smooth inclusion \( \mathcal{G}_{\tau = 0} \rightarrow G(\Psi(A_t)) \) on the \( \tau = 0 \) component. Thus, by Theorem 2.26 and Lemma 3.4, we only need to prove that \( \text{Ker}(\langle \cdot \rangle|_\tau = 0) \) is a regular Lie group with Lie algebra
\[
\mathcal{I} = \{ \mathfrak{P} \in \Psi(A_t) : \mathfrak{P}|_\tau = 0 = 0 \}.
\]
We notice that
\[
\text{Ker}(\langle \cdot \rangle|_\tau = 0) = \mathcal{G}\Psi_t(R) \cap \Psi(A_t)
\]
and hence \( \text{Ker}(\langle \cdot \rangle|_\tau = 0) \) is a Frölicher Lie group. Let us check the following:

- \( \mathcal{I} \) is regular as a vector space. It is proved in Lemma 3.20 that any path in \( C^\infty([0, 1], 0) \) integrates to a path on \( \Psi(A_t) \), and it is easy to see that the \( \tau = 0 \) component, which is equal to 0, integrates to 0.
- The exponential map \( \exp : C^\infty([0, 1], 1) \rightarrow C^\infty([0, 1], \text{Ker}(\mathfrak{P}|_\tau = 0)) \)

We apply Theorem 2.25 setting \( \mathcal{A}_m \) as the vector space of \( \tau \)-monomials of valuation equal to \( m \) and whose order is not greater than \( m \).
4. Integration of the Kadomtsev–Petviashvili Hierarchy

The Kadomtsev–Petviashvili (KP) hierarchy reads
\[ \frac{dL}{dt_k} = \left[(L^k)_+, L\right], \quad k \geq 1, \] (4.1)
with initial condition \( L(0) = L_0 \in \partial + \Psi^{-1}(R) \). The dependent variable \( L \) is chosen to be of the form
\[ L = \partial + \sum_{\alpha \leq -1} u_\alpha \partial^\alpha \in \Psi^1(A_t). \]

A standard reference on (4.1) is L.A. Dickey’s treatise [13], see also [27,37,38]. The following result gives a solution to the Cauchy problem for the KP hierarchy (4.1).

**Theorem 4.1** (1) Consider the KP hierarchy
\[ \frac{dL}{d\tau_k} = \left[(L^k)_+, L\right], \quad k \geq 1, \] (4.2)
with initial condition \( L(0) = L_0 \). Then,
(a) There exists a pair \((S, Y) \in G_{A_t} \times \overset{\circ}{D}(A_t)^\times\) such that the unique solution to Eq. (4.2) with \( L|_{\tau=0} = L_0 \) is
\[ L(\tau_1, \tau_2, \cdots) = YL_0Y^{-1} = SL_0S^{-1}. \]
(b) The pair \((S, Y)\) is uniquely determined by the smooth decomposition problem
\[ \exp\left(\sum_{k \in \mathbb{N}} \tau_k L_0^k\right) = S^{-1}Y \]
and the solution \( L \) depends smoothly on the initial condition \( L_0 \).

(2) Consider the KP hierarchy (4.1) with initial condition \( L(0) = L_0 \). Set \( \mathbb{K}^\infty = \bigcup_{n \in \mathbb{N}} \mathbb{K}^n \) and equip this set with the structure of a locally convex topological space given by the inductive limit. The algebra \( A_t \) is to be considered as an algebra of power series in infinitely many variables, and each monomial of a given series in \( A_t \) takes values in \( \mathbb{K}^\infty \).
(a) There exists a pair \((S, Y) \in G_{A_t} \times \overset{\circ}{D}(A_t)^\times\) such that the unique solution to Eq. (4.1) with \( L(0) = L_0 \) is
\[ L(t_1, t_2, \cdots) = YL_0Y^{-1} = SL_0S^{-1}. \]
(b) The pair \((S, Y)\) is uniquely determined by the smooth decomposition problem
\[ \exp\left(\sum_{k \in \mathbb{N}} t_k L_0^k\right) = S^{-1}Y \]
posed in the regular Frölicher Lie group \( G(\overline{\Psi}(A_t)) \).
(c) The solution operator \( L \) depends smoothly on the variable \( t \) and on the initial value \( L_0 \). This means that the map

\[
(L_0, s) \in (\partial + \Psi^{-1}(R)) \times \mathbb{K}^\infty \mapsto \sum_{n \in \mathbb{N}} \left( \sum_{|t|=n} [L(s)]_t \right) 
\in (\partial + \Psi^{-1}(A_t))^N
\]

is smooth.

The existence of an algebraic decomposition as in Parts 1(a), 1(b) of Theorem 4.1 appears already in Mulase’s seminal paper [37], and a formal solution to (4.1) as in Part (1) is in [17]. The richness of Theorem 4.1 stems from the fact that we pose the KP hierarchy in the Frölicher algebra \( \Psi(A_t) \) (Theorem 3.13) and that then we solve the corresponding Cauchy problem using analytically rigorous factorizations in Frölicher Lie groups (e.g. Theorem 3.14, Proposition 3.19, Theorems 3.21 and 3.22 for the infinite-dimensional regular Frölicher Lie group \( G(\Psi(A_t)) \)). We propose two proofs: one inspired by the formal arguments of [17] and the other based on the work of [32] on a \( q \)-deformed KP hierarchy. As is plain from the discussions below, the new arguments used herein are intrinsically interwoven with the ones used in [17,32].

4.1. First Proof

We prove Part (2). Motivated by [17], we model our proof after Reyman and Semenov-Tian-Shansky’s [46], see also the later exposition [44]. We set

\[
U = \exp \left( \sum_{k \in \mathbb{N}} t_k L_0^k \right).
\]

Then, \( U \in G(\Psi(A_t)) \) by Lemma 3.16 applied to \( P_n = L_0^n \) and Theorem 3.22. We consider the smooth decomposition

\[
U \mapsto (S, Y) \in G_{A_t} \times \overline{D}(A_t)^\times
\]

following Proposition 3.19, and we set \( L = Y L_0 Y^{-1} \). We make two obvious observations:

1. \( L^k = Y L_0^k Y^{-1} \).
2. \( U L_0^k U^{-1} = L_0^k \), since \( L_0 \) commutes with \( U = \exp(\sum_k t_k L_0^k) \).

It follows that \( L^k = Y L_0^k Y^{-1} = SS^{-1}Y L_0^k Y^{-1} SS^{-1} = SL_0^k S^{-1} \).

We take \( t_k \)-derivative of \( U \) for each \( k \geq 1 \). We obtain the equation

\[
L_0^k U = -S^{-1} S t_k S^{-1} Y + S^{-1} Y t_k
\]

and so, using \( U = S^{-1} Y \), we obtain the decomposition

\[
SL_0^k S^{-1} = -S t_k S^{-1} + Y t_k Y^{-1}.
\]

Since \( S t_k S^{-1} \in \mathcal{I}_A \) and \( Y t_k Y^{-1} \in \mathcal{D}_A \), we conclude that

\[
(L^k)_+ = Y t_k Y^{-1} \quad \text{and} \quad (L^k)_- = -S t_k S^{-1}.
\]
Now we take $t_k$-derivative of $L$:

$$\frac{dL}{dt_k} = Y_{t_k}L_0Y^{-1} - YL_0Y^{-1}Y_{t_k}Y^{-1}$$

$$= Y_{t_k}Y^{-1}YL_0Y^{-1} - YL_0Y^{-1}Y_{t_k}Y^{-1}$$

$$= (L^k)_+ L - L(L^k)_+$$

$$= [(L^k)_+, L].$$

We check the initial condition: We have $L(0) = Y(0)L_0Y(0)^{-1}$, but $Y(0) = 1$ since $Y \in \mathcal{D}(A_t)^\times$.

Smoothness with respect to $t$ is already proved by construction, and we have established smoothness of the map $L_0 \mapsto Y$ at the beginning of the proof. Thus, the map $L_0 \mapsto L(t) = Y(t)L_0Y^{-1}(t)$ is smooth in $\overline{\Psi}(A_t)$ and 2(c) follows.

In order to finish the proof, we need to prove that the solution $L$ belongs to $\partial + \Psi^{-1}(A_t)$. For this, we consider the above-mentioned relation $L^k = SL^k_0S^{-1}$, which for $k = 1$, implies that

$$L = L_0 + S[L_0, S^{-1}].$$

Now, $[L_0, S^{-1}] \in \Psi^{-1}(A_t)$, $S \in \Psi_0(A_t)$ and by hypothesis $L_0 \in \partial + \Psi^{-1}(R)$, so that

$$L \in \partial + \Psi^{-1}(A_t).$$

**Remark 4.2** The pseudo-differential operator $L$ we have constructed gives a full solution to the KP hierarchy, in contradistinction to the result proven in [17] in which (4.1) is considered as an infinite system of equations for the coefficients of $L$ in two independent variables, the “time variable” $t_k$ and the “space variable” modelled by the derivation $\partial$. We can give another construction of the operators $S, Y$ and $L$ appearing above using induction and taking advantage of the fact that we are considering our infinitely many time variables to determine points in $\mathcal{K}^\infty$.

First, we fix $t_2 = t_3 = \cdots = 0$, and we obtain a first operator $U_1(t_1) = \exp(t_1L_0) = S_1^{-1}Y_1$ such that $L_1 = Y_1L_0Y_1^{-1} = S_1L_0S_1^{-1}$ is the unique solution of the $t_1$-equation of the KP hierarchy

$$\frac{dL_1}{dt_1} = [(L_1)_+, L_1].$$

Then, applying this first result to $R = R[[t_1]]$, and noticing that $R[[t_1]] = R[[t_1]] [[t_2]]$, we set

$$U_2(t_1, t_2) = \exp(t_2L_1) = S_2^{-1}Y_2 \in \overline{G\Psi}(R[[t_1, t_2]]),$$

thereby obtaining a solution

$$L_2(t_1, t_2) = Y_2L_1Y_2^{-1} = Y_2Y_1L_0Y_1^{-1}Y_2^{-1}$$
or equivalently
\[ L_2(t_1, t_2) = S_2 L_1 S_2^{-1} = S_2 S_1 L_0 S_1^{-1} S_2^{-1}, \]
and so on. The total solution \( L \) is built by an inductive limit process which gives the operators \( S \) and \( Y \), and we check smoothness in \( t \) of this full solution by remarking that \( \sum_{|t| = n} [L(s)]_t \) is a finite sum of \( t \)-monomials.

### 4.2. Second Proof

In this second approach, we model our proof after [32, 38, 39]. As in [32], we apply a scaling to the KP hierarchy in order to work with fully regular Frölicher Lie groups.

We let \( L \in \partial + \Psi^{-1}(A_t) \) and, as before, we consider the space \( \mathcal{K}^\infty = \bigcup_{n \in \mathbb{N}} \mathcal{K}^n \) equipped with the structure of locally convex topological vector space given by the inductive limit. Since we have \( [L^n, L] = 0 \), we obtain from (4.1)

\[ \frac{dL}{dt_n} = [L^n, L] = -[L^n, L_n], \quad n \geq 1. \]  
(4.3)

Then, setting \( dt = \sum_{k \in \mathbb{N}^*} dt_k \), we can define

\[ Zdt = \sum_{n \in \mathbb{N}^*} L^n dt_n \quad \text{and} \quad Z^c dt = -\sum_{n \in \mathbb{N}^*} L^n dt_n. \]

The form \( Z^c dt \) is a bona fide one-form,

\[ Z^c dt \in \Omega^1(\mathcal{K}^\infty, \Psi^{-1}(A_t)), \]

where \( A_t \) is understood as set of smooth maps \( \mathcal{K}^\infty \to R \) and \( \Psi(A_t) \) as a set of smooth maps \( \mathcal{K}^\infty \to \Psi(R) \). Thus, in this context derivations with respect to \( t_n, n \geq 1 \), appearing in (4.3) correspond to derivations on a locally convex topological vector space.

It follows from Eq. (4.3) that the equation

\[ dZ^c - [Z^c, Z^c] = 0 \]

holds, and our constructions imply that this equation is a rigorous zero-curvature equation.

In order to build a solution to (4.3), we proceed to deform the KP hierarchy and we work with deformed versions of \( Zdt \) and \( Z^c dt \). After [32], we use the scaling \( t_n \mapsto q^n t_n \) to build the following operator in \( \Psi(A_t)[[q]] \):

\[ \tilde{L}(t_1, t_2, ...) = qL(q t_1, q^2 t_2, ...). \]  
(4.4)

We note that if we define \( \delta : \mathcal{K}^\infty \to \mathcal{K}^\infty \) by \( \delta(t_1, t_2, ...) = (qt_1, q^2 t_2, ...) \) then

\[ \frac{d\tilde{L}}{dt_n} = q \frac{d(L \circ \delta)}{dt_n} \bigg|_{(t_1, t_2, ...)} = q^{n+1} \frac{dL}{dt_n} \bigg|_{\delta(t_1, t_2, ...)}, \]

an easy observation which will be of use presently. We also consider the 1-forms

\[ \tilde{Z}_{L, +}(t_1, ...) = \sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_+ dt_n \]  
(4.5)
and

\[ \tilde{Z}_{L,-}(t_1, \ldots) = -\sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_{-} dt_n. \]  

(4.6)

By construction, \( \tilde{Z}_+ \) and \( \tilde{Z}_- \) are smooth 1-forms in \( \Omega^1(T, \Psi(A_t)[[q]]) \). Now, instead of Eq. (4.3), we write down an equation for \( \tilde{L} \). Using (4.3) we obtain, after cancellation of the \( q_{n+1} \)-factors, the “deformed KP hierarchy”

\[ \frac{d\tilde{L}}{dt_n} = [\tilde{L}^n_+, \tilde{L}] = -[\tilde{L}_n^-, \tilde{L}], \quad n \geq 1. \]  

(4.7)

These equations are equivalent to

\[ d(\tilde{L}) = [\tilde{Z}_{L,+}, \tilde{L}] = -[\tilde{Z}_{L,-}, \tilde{L}], \]  

(4.8)

and we conclude that the corresponding zero-curvature equations

\[ d\tilde{Z}_{L,+} + [\tilde{Z}_{L,+}, \tilde{Z}_{L,+}] = 0 \]  

(4.9)

and

\[ d\tilde{Z}_{L,-} - [\tilde{Z}_{L,-}, \tilde{Z}_{L,-}] = 0 \]  

(4.10)

also hold.

We define the following algebras and groups, in which \( \text{val}_q \) is valuation of \( q \)-series, after [32,33]:

**Definition 4.3** We set

\[
\Psi_q(R) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha \in \Psi(R)[[q]] : \text{val}_q(a_{\alpha}) \geq \alpha \right\},
\]

\[
G\Psi_q(R) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha \in \Psi_q(R) : a_0 = 1 + b_0, \quad \text{val}_q(b_0) \geq 1 \right\},
\]

\[
G_{R,q} = \left\{ A \in G\Psi_q(R) : A = 1 + B, \quad B \in \Psi^{-1}(R)[[q]] \right\},
\]

\[
D_q(R) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha \in \Psi_q(R) : a_\alpha = 0 \text{ if } \alpha < 0 \text{ and } a_0 = 1 + b_0, \quad \text{val}_q(b_0) \geq 1 \right\}.
\]

The following lemma is proved in [32,33], but now we can give an alternative, shorter proof, by using some of the results already stated in this paper.

**Lemma 4.4** \( \Psi_q(R) \) is a Frölicher algebra and a regular Lie algebra, and the groups \( G\Psi_q(R) \), \( G_{R,q} \) and \( D_q(R) \) are fully regular Frölicher Lie groups with Lie algebras given, respectively, by:

\[
\mathfrak{g}\Psi_q(R) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha \in \Psi_q(R) : \text{val}_q(a_0) \geq 1 \right\},
\]

\[
\mathfrak{g}_{R,q} = \Psi^{-1}(R)[[q]],
\]

and

\[
\mathfrak{d}_q(R) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_{\alpha} \partial^\alpha \in D_q(R) : a_0 = 0 \text{ if } \alpha < 0 \quad \text{val}_q(a_0) \geq 1 \right\}.
\]
Proof We simply remark that by substituting $q^n$ for $t_n$, we obtain maps

$$\overline{\Psi}(A_t) \to \Psi_q(R),$$

$$G(\overline{\Psi}(A_t)) \to G\Psi_q(R),$$

$$G_{A_t} \to G_{R,q},$$

and

$$\overline{D}(A_t) \to D_q(R).$$

By straightforward computation on the coefficients of the series, we check that the first map $\overline{\Psi}(A_t) \to \Psi_q(R)$ is a (smooth) morphism of Frölicher algebras which commutes with the valuations $\text{val}_t$ and $\text{val}_q$, and with integration along paths. Thus, identifying as formal variables e.g. $t_1$ with $q$, we have slice maps

$$\overline{\Psi}(A_t) \leftarrow \Psi_q(R),$$

$$G(\overline{\Psi}(A_t)) \leftarrow G\Psi_q(R),$$

$$G_{A_t} \leftarrow G_{R,q}.$$

Pulling back the Frölicher structure of $\overline{\Psi}(A_t)$, we see that $\Psi_q(R)$ is a regular Frölicher algebra. The rest of the lemma follows easily using similar arguments.

Now we adapt some of the algebraic arguments of [38, 39] to our Frölicher context. We remark that the operator $\tilde{L}$ we have considered so far is not, at this step of the proof, a solution to (4.7), but simply a $q$-deformation of an operator $L \in \partial + \Psi^{-1}(A_t)$. Our construction of a true solution to the KP hierarchy satisfying a given initial condition $L_0$ is based on the observation that the operator $q\partial$ is a (stationary) solution to our deformed KP hierarchy (4.7).

We consider the one-forms $Z_{q\partial,+}$ and $Z_{q\partial,-}$ defined as in (4.6) and (4.6) with $q\partial$ instead of $\tilde{L}$. Then the equation

$$d\tilde{Z}_{q\partial,+} + [\tilde{Z}_{q\partial,+}, \tilde{Z}_{q\partial,+}] = 0$$

(4.11)

obviously holds, while

$$d\tilde{Z}_{q\partial,-} - [\tilde{Z}_{q\partial,-}, \tilde{Z}_{q\partial,-}] = 0$$

(4.12)

is a trivial identity. Equation (4.12) allows us to apply the Ambrose–Singer Theorem 2.40 on the trivial principal bundle $\mathbb{K}^\infty \times G\Psi_q(R)$. More precisely, we choose an initial condition $L_0 \in \partial + \Psi^{-1}(R)$ and we choose $S_0 \in G_{R,q}$ such that $qL_0 = S_0(q\partial)S_0^{-1}$. Theorem 2.40 implies that there exists a unique section

$$\tilde{U}(t) : \mathbb{K}^\infty \to \mathbb{K}^\infty \times G\Psi_q(R)$$

such that

$$d\tilde{U} \cdot \tilde{U}^{-1} = \tilde{Z}_{q\partial,+}$$

(4.13)

with initial condition $\tilde{U}(0) = S_0^{-1} \cdot Y_0$, in which $Y_0 = 1$. This smooth section $\tilde{U}$ is a “$q$-deformation” of the dressing operator present in our first proof, and it replaces Mulase’s formal operator $U$ appearing in [38, Theorem 1.4].
Now we use the slice maps built in the proof of Lemma 4.4 and we obtain 
\(\tilde{U}(t) \in G(\tilde{V}(A_t))\) satisfying \(\tilde{U}(0) = \tilde{S}_0^{-1}\), in which \(\tilde{S}_0\) is the lift of \(S_0\). We use our factorization result (Proposition 3.19) to obtain unique \(\tilde{S}(t) \in G_{A_t}\) and 
\(\tilde{Y}(t) \in \tilde{D}^\times_{A_t}\) satisfying the equation

\[\tilde{U}(t) = \tilde{S}(t)^{-1} \cdot \tilde{Y}(t),\]

and the initial conditions \(\tilde{S}(0) = \tilde{S}_0\), and \(\tilde{Y}(0) = 1\). We return to \(G\Psi_q(R)\) (using again the maps appearing in the proof of Lemma 4.4), and we obtain the unique factorization

\[\check{U}(t) = \check{S}(t)^{-1} \cdot \check{Y}(t)\]  
(4.14)

with \(\check{S}(t) \in G_{R,q}, \check{Y}(t) \in \check{D}_q(R), \check{S}(0) = S_0, \) and \(\check{Y}(0) = 1\). In order to finish this second proof, we set

\[\check{L} = \check{S}(q\partial)\check{S}^{-1} \in C^\infty(T, \Psi_q(R))\]  
(4.15)

(so that \(\check{L}\) satisfies the \(q\)-deformed initial condition \(\check{L}(0) = qL_0\)) and we also define the smooth one-forms

\[\check{Z}_+ = \check{S}\check{Z}_{q\partial, +}\check{S}^{-1} + d\check{S} \cdot \check{S}^{-1},\]

\[\check{Z}_- = d\check{S} \cdot \check{S}^{-1}.\]  
(4.16)

The one-form \(\check{Z}_{q\partial, +}\) satisfies (4.12) and \(\check{Z}_+\) is obtained from \(\check{Z}_{q\partial, +}\) via a gauge transformation, so that \(\check{Z}_+\) satisfies the zero-curvature Eq. (4.10). Also, \(\check{Z}_-\) satisfies (4.10), since it is “pure gauge”.

By construction \(\check{Z}_-\) takes values in the Lie algebra of \(G_{R,q}\), and we claim that \(\check{Z}_+\) takes values in the Lie algebra of \(\check{D}_q(R)\). Indeed, we compute using Eq. (4.14):

\[d\check{U} = d(\check{S}^{-1} \cdot \check{Y}) = -\check{S}^{-1} d\check{S} \cdot \check{S}^{-1} \check{Y} + \check{S}^{-1} d\check{Y};\]

multiplying by \(\check{U}^{-1}\) on the right and using (4.13), we obtain

\[\check{Z}_{q\partial, +} = -\check{S}^{-1} d\check{S} + \check{S}^{-1} d\check{Y} \cdot \check{U}^{-1}.\]

Then

\[\check{S}\check{Z}_{q\partial, +}\check{S}^{-1} = -d\check{S}\check{S}^{-1} + d\check{Y} \check{U}^{-1}\check{S}^{-1}.\]

Adding \(d\check{S}\check{S}^{-1}\) to both sides of this equality and using (4.17), we obtain that \(\check{Z}_+\) can be written in terms of the slice \(\check{Y} \in C^\infty(\mathbb{I}K^\infty, \check{D}_q(R))\) as

\[\check{Z}_+ = d\check{Y} \cdot \check{Y}^{-1},\]

which proves our claim.

We are ready to check that \(\check{L}\), as defined in (4.15), satisfies the \(q\)-deformed KP hierarchy. Indeed:

\[\check{Z}_+ - \check{Z}_- = \check{S}\check{Z}_{q\partial, +}\check{S}^{-1} = \sum_{n=1}^\infty q^n (\check{S}q\partial\check{S}^{-1})^n dt_n = \sum_{n=1}^\infty q^n \check{L}^n dt_n\]

\[= \sum_{n=1}^\infty q^n \left[ (\check{L}^n)_+ - (\check{L}^n)_- \right] dt_n,\]
so that
\[ \tilde{Z}_+ = \sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_+ \, dt_n \quad \text{and} \quad \tilde{Z}_- = - \sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_- \, dt_n. \]

Now, the operator \( q\partial \) is covariantly constant with respect to \( \tilde{Z}_{q\partial,+} \), this is,
\[ d(q\partial) = [\tilde{Z}_{q\partial,+}, q\partial], \]
and therefore applying a gauge transformation with gauge \( \tilde{S} \) we obtain
\[ d(\tilde{S}(q\partial)\tilde{S}^{-1}) = [\tilde{S}\tilde{Z}_{q\partial,+}\tilde{S}^{-1} + d\tilde{S} \cdot \tilde{S}^{-1}, \tilde{S}(q\partial)\tilde{S}^{-1}], \]
this is,
\[ d(\tilde{L}) = [\tilde{Z}_{+,\tilde{L}} - \sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_+ \, dt_n, \tilde{L}] = [\tilde{Z}_{L,+}, \tilde{L}]. \]

This equation says that the operator \( \tilde{L} \) defined in (4.15) satisfies the first equation appearing in (4.8), which we know is equivalent to the deformed KP hierarchy (4.7). In an analogous fashion, using that \( q\partial \) is covariantly constant with respect to \( \tilde{Z}_{q\partial,-} = 0 \), we can check that
\[ d(\tilde{L}) = [\tilde{Z}_{+,\tilde{L}} - \sum_{n=1}^{+\infty} q^n (\tilde{L}^n)_- \, dt_n, \tilde{L}] = [\tilde{Z}_{L,-}, \tilde{L}], \]
this is, \( \tilde{L} \) given by (4.15) also satisfies the second equation appearing in (4.8).

It remains to check that our solution is unique. The only arbitrary choice made in this proof is the choice of \( S_0 \in G_{R,q} \) such that \( qL_0 = S_0(q\partial)S_0^{-1} \). The fact that \( \tilde{L} \) is independent on \( S_0 \) follows from the following argument motivated by [39]:

We write
\[ \tilde{L} = \tilde{S}(q\partial)\tilde{S}^{-1} = \tilde{Y}\tilde{U}^{-1}(q\partial)\tilde{U}\tilde{Y}^{-1} \]
and claim that \( \tilde{U}^{-1}(q\partial)\tilde{U} \) is constant. Indeed, the equation
\[ \tilde{Z}_{q\partial,+} = \sum_{n=1}^{+\infty} q^n (q\partial)^n \, dt_n = d\tilde{U} \cdot \tilde{U}^{-1} = \sum_{n=1}^{+\infty} \frac{\partial \tilde{U}}{\partial t_n} \tilde{U}^{-1} \, dt_n \]
implies
\[ \frac{\partial \tilde{U}}{\partial t_n} \tilde{U}^{-1} = q^n \partial^n \]
and so
\[ \frac{\partial}{\partial t_n} [\tilde{U}^{-1}(q\partial)\tilde{U}] = -\tilde{U}^{-1} \frac{\partial \tilde{U}}{\partial t_n} \tilde{U}^{-1}(q\partial)\tilde{U} + \tilde{U}^{-1} q\partial \frac{\partial \tilde{U}}{\partial t_n} = 0. \]
It follows that
\[ \tilde{L} = \tilde{Y}\tilde{U}^{-1}(0) \, (q\partial) \, \tilde{U}(0)\tilde{Y}^{-1} = \tilde{Y}S_0 \, (q\partial) \, S_0^{-1}\tilde{Y}^{-1}, \]
that is, our solution \( \tilde{L} \) given by (4.15) can be also written as
\[ \tilde{L} = \tilde{Y}(qL_0)\tilde{Y}^{-1}, \]
and this expression shows that $\tilde{L}$ does not depend on $S_0$.

Finally, we consider the infinite jets
\[ S_q = j^\infty(\tilde{S}), \]
\[ Y_q = j^\infty(\tilde{Y}), \]
and
\[ L_q = j^\infty(\tilde{L}). \]

By classical properties of infinite jets$^2$, all relations above involving relations on $\tilde{S}$, $\tilde{Y}$ and $\tilde{L}$ remain valid for $S_q$, $Y_q$ and $L_q$, and the $t$-monomials of the infinite jet series are polynomials in $q$. Then, they are $q$-smooth, where $q \in \mathbb{K}$ is no longer a formal variable. Setting
\[ (S, Y, L) = \lim_{q \to 1} (S_q, Y_q, L_q), \]
we recover Theorem 4.1.

5. Perturbed Solutions and Hamiltonian Approach

5.1. Hamiltonian Formulation of KP

We consider a regular Frölicher Lie algebra of formal pseudo-differential operators $\Psi(A)$ in which $A$ is an arbitrary Frölicher algebra equipped with a smooth derivation $\partial$. First of all, we recall that if $P = \sum_{-\infty < \nu \leq N} a_\nu \partial^\nu \in \Psi(A)$, the residue of $P$ (see [4]) is
\[ \text{res}(P) = a_{-1}. \]

The most important property of $\text{res}$ (see Adler, [4]; proofs also appear in Dickey’s [13] and in [41, Chapter 5]) is the following:

**Lemma 5.1** If $P, Q \in \Psi(A)$, then $\text{res}([P, Q]) = \partial(f)$ for some $f \in A$.

Let us suppose that there exists a $\mathbb{K}$-linear function $I : A \to B$, in which $B$ is an unitary Frölicher algebra, such that
\[ I(\partial u) = 0 \text{ for all } u \in A. \quad (5.1) \]

We define the trace form on $\Psi(A)$ by
\[ \text{Trace}(P) = I(\text{res}(P)) \text{ for all } P \in \Psi(A). \quad (5.2) \]

Hereafter we assume, generalizing the standard case in which $A = C^\infty(S^1, \mathbb{K})$ and the map $I$ is definite integration, that the trace form (5.2) satisfies the non-degeneracy condition

$^2$Jets are considered in [54], in the classical paper [7] and in the more recent review [47]; we note that in [54] the author introduces jets within the category $DS$ of differential spaces, but it is known that Frölicher spaces form a subcategory of $DS$, see [32]. Moreover, as explained in Sect. 2.5 (see also [19,32]) the category of Frölicher spaces is Cartesian closed, complete and cocomplete, and so infinite jets in Frölicher spaces can be defined by adapting the constructions of [7,47].
The pairing $\langle \cdot, \cdot \rangle : \Psi(A) \times \Psi(A) \rightarrow B$ given by $\langle P, Q \rangle = \text{Trace}(PQ)$ is non-degenerate, that is, $\langle P, Q \rangle = 0$ for all $Q \in \Psi(A)$ implies $P = 0$. (5.3)

Condition (5.1) implies that we can “integrate by parts”, that is, the identity $I(\partial(u) \cdot v) = -I(u \cdot \partial(v))$ holds for all $u, v \in A$. This fact is crucial for the proof of the following standard lemma:

**Lemma 5.2** The $B$-valued pairing $\langle \cdot, \cdot \rangle$ defined in (5.3) is $K$-bilinear, symmetric, non-degenerate, and it satisfies

$$\langle [P, Q], S \rangle = \langle [S, P], Q \rangle$$ (5.4)

for all $P, Q, S \in \Psi(A)$.

Now we consider the Frölicher Lie algebra $\Psi(A)$ and we define the regular dual space

$$\Psi(A)' = \{ \mu \in L(\Psi(A), B) : \mu = \langle P, \cdot \rangle \text{ for some } P \in \Psi(A) \}.$$ These definitions allow us to adapt standard results of Hamiltonian mechanics (see for instance [36] or the recent summary [17] and references therein) to our infinite-dimensional context as follows:

Inspired by [21,26], we let $f : \Psi(A)' \rightarrow B$ be a polynomial function of the type

$$f(\mu) = \sum_{k=0}^{n} a_k \text{Trace}(P^k)$$ (5.5)

with $\mu = \langle P, \cdot \rangle$. The functional derivative of $f$ at $\mu \in \Psi(A)'$ is the unique element $\delta f/\delta \mu$ of $\Psi(A)$ determined by

$$\left\langle \nu \bigg| \frac{\delta f}{\delta \mu} \right\rangle = df_\mu(Q)$$ (5.6)

for all $\nu = \langle Q, \cdot \rangle \in \Psi(A)'$. If $f = \text{Trace}\left( \sum_{k=0}^{n} a_k P^k \right)$, direct computations show that $\frac{\delta f}{\delta \mu}$ is given by (algebraic) derivation of the polynomials (5.5) in agreement with [13].

**Definition 5.3** The $B$-valued Lie-Poisson bracket on the regular dual space $\Psi(A)'$ is defined as follows: for all polynomial functions $F, G : \Psi(A)' \rightarrow B$ of the type (5.5) and $\mu \in \Psi(A)'$,

$$\{F, G\}(\mu) = \left\langle \mu \bigg| \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle.$$ (5.7)

Now we fix a polynomial function $H : \Psi(A)' \rightarrow B$ with $H$ as in (5.5). Then, Eq. (5.7) determines a derivation $X_H$ on polynomial functions $F : \Psi(A)' \rightarrow B$ via

$$X_H(\mu) \cdot F = \{F, H\}(\mu) = \left\langle \mu \bigg| \left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle$$ (5.8)

for all $\mu \in \Psi(A)'$. 
Lemma 5.4 Let $H : \Psi(A)' \to B$ be a smooth function on $\Psi(A)'$. Then, with the previous notations, Hamilton’s equations

$$\frac{d}{dt}(F \circ \mu) = X_H(\mu) \cdot F$$

(5.9)

for $\mu(t) = \langle P(t), \cdot \rangle \in \Psi(A)'$ can be written as equations on $\Psi(A)$ as follows:

$$\frac{dP}{dt} = \left[ \frac{\delta H}{\delta \mu}, P \right].$$

(5.10)

Proof Let $P \in C^\infty(\mathbb{R}, \Psi(A))$ and let $\mu = \langle P, \cdot \rangle \in C^\infty(\mathbb{R}, \Psi(A)')$. We set up the differential equation

$$\frac{d}{dt}(F \circ \mu) = X_H(\mu) \cdot F$$

for $\mu(t) = \langle P(t), \cdot \rangle \in \Psi(A)'$ and $F$ as above. If we set $F(\mu) = \text{Trace} \left( \sum_{k=0}^n a_k P^k \right)$, we can compute the left and right sides of (5.9) separately. Using (5.8) and (5.4), we obtain

$$\frac{d}{dt}(F \circ \mu) = \frac{d}{dt} \text{Trace} \left( \sum_{k=0}^n a_k P^k \right)$$

$$= \sum_{k=0}^n a_k \text{Trace} \left( \frac{dP^k}{dt} \right)$$

$$= \sum_{k=0}^n a_k \text{Trace} \left( kP^{k-1} \frac{dP}{dt} \right)$$

$$= \left\langle \frac{dP}{dt}, \frac{\delta F}{\delta \mu} \right\rangle$$

and

$$X_H(\mu) \cdot F = \left\langle P, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle$$

$$= \left\langle \left[ P, \frac{\delta H}{\delta \mu} \right], \frac{\delta F}{\delta \mu} \right\rangle,$$

and the result follows. \[\square\]

This proof suggests that we can identify $X_H(\mu)$ with the germ $\frac{dP}{dt}$. The equations for the integral curves of $X_H$ are Hamilton’s equations on $\Psi(A)'$ corresponding to the Hamiltonian function $H$.

Now we work in the context of $r$-matrices, see [50]. The decomposition $\Psi(A) = \Psi^{-1}(A) \oplus \mathcal{D}_A$, in which $\mathcal{D}_A = \Psi(A) - \Psi^{-1}(A)$, allows us to consider a new Lie bracket on the regular dual space $\Psi(A)'$ given by

$$[P, Q]_0 = [P_+, Q_+] - [P_-, Q_-],$$

(5.11)

in which $P_\pm = \pi_\pm(P)$, $Q_\pm = \pi_\pm(Q)$, and $\pi_\pm$ are the projection maps from $\Psi(A)$ onto $\mathcal{D}_A$ and $\mathcal{I}_A$, respectively. This bracket determines a new Poisson structure $\{ , \}_0$ on $\Psi(A)'$, simply by replacing the original Lie product for $[,]_0$...
in (5.7). Using again the non-degenerate pairing (5.3), we obtain the following version of Lemma 5.4:

**Lemma 5.5** Let $H : \Psi(A)' \to B$ be a smooth function on $\Psi(A)'$ such that

$$
\left\langle \mu \left| \frac{\delta H}{\delta \mu}, \cdot \right. \right\rangle = 0 \quad \text{for all } \mu \in \Psi(A)'.
$$

(5.12)

Then, as equations on $\Psi(A)$, the Hamiltonian equations of motion with respect to the $\{, \}_0$ Poisson structure of $\Psi(A)'$ are

$$
\frac{dP}{dt} = \left[ \frac{\delta H}{\delta \mu} + \cdot, P \right].
$$

(5.13)

We now use some specific functions $H$. Let us recall the following results (see for example [13] or the more recent [17]):

**Proposition 5.6** We define the functions $H_k(L) = \text{Trace}((L^k))$, $k = 1, 2, 3, \cdots$, for $L \in \Psi(A)$. Then, $\frac{\delta H_k}{\delta L} = kL^{k-1}$. In particular, the functions $H_k$ satisfy (5.12).

It follows that we can apply Lemma 5.5. It yields:

**Proposition 5.7** Let us equip the Lie algebra $\Psi(A)$ with the non-degenerate pairing (5.3). Write $\Psi(A) = \Psi^{-1}(A) \oplus D_A$ and consider the Hamiltonian functions

$$
\mathcal{H}_k(\mu) = \frac{1}{k} \text{Trace}((L^{k+1}))
$$

(5.14)

for $\mu = \langle L, \cdot \rangle$. The corresponding Hamiltonian equations of motion with respect to the $\{, \}_0$ Poisson structure of $\Psi(A)'$ are

$$
\frac{dL}{dt_k} = [(L^k)_{+}, L].
$$

(5.15)

Equation (5.15) is the $k$th Kadomtsev–Petviashvili equation of the KP hierarchy posed on the space $\Psi(A)$.

**5.2. Perturbed Solutions**

From now onwards, we set $R = C^\infty(S^1, \mathbb{K})$ where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and we consider the algebra

$$
R_z = \left\{ \sum_{n \in \mathbb{N}} z^n a_n : (a_n)_{\mathbb{N}} \in R^\mathbb{N} \right\}
$$

in the whole previous picture. The algebra $R_z$ is a Fréchet algebra with set of units given by

$$
R_z^* = \left\{ \sum_{n \in \mathbb{N}} z^n a_n \in R_z : a_0 \in R^* \right\},
$$

as it can be seen using standard techniques on formal series. According to Remark 2.6, smoothness in the Frölicher sense is equivalent to smoothness
in the sense of Gateaux derivatives in the category of Fréchet spaces. As a consequence, the operator $\partial$ on $R$ extends (smoothly) componentwise, to $R_z$.

We wish to develop KP in this setting, but before proceeding, let us give our motivation.

It is well known that there exists a theory of pseudo-differential operators with symbols of limited smoothness, see e.g. [6,33,35], but in this case the algebra of formal symbols cannot be constructed as in the standard case considered e.g. in [42] and references therein, principally because of the lack of continuity of the multiplication of functions in Sobolev classes for low Sobolev orders. However, for functions $f \in H^s_0$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subseteq (C^\infty_c)^\mathbb{N}$ such that $\lim f_n = f$ in $H^s$. Setting $u_0 = f_0$ and $u_n = f_n - f_{n-1}$ for $n > 0$, we obtain

$$f = \lim_{z \to 1} \sum_{n \in \mathbb{N}^*} z^n u_n,$$

so that we can naturally investigate the KP hierarchy on spaces of formal pseudo-differential operators with coefficients of low regularity by using $R_z$, as already sketched in [33] with other notations.

In order to ensure that we can apply our previous results in this setting, we have to check:

1. $R_z$ is a commutative $\mathbb{K}$-algebra with unit 1;
2. $\partial$ is a derivation on $R_z$;
3. $R_z$ is a Frölicher algebra;
4. $\partial$ is smooth;
5. $R_z$ is a Frölicher Lie group with Lie algebra $g_{R_z}$;
6. $R_z$ is regular as a Frölicher vector space.

The proofs of (1)–(4) and (6) are straightforward. The proof of (5) is an application of Theorems 2.25 and 2.26 to the short exact sequence

$$1 \to \left\{ \sum_{n \in \mathbb{N}} z^n a_n \in R_q : a_0 = 1 \right\} \to R_z^* \to R^* \to 1,$$

as we have done for other groups of series.

Let us consider the Hamiltonian formulation of KP in this context. We define the function $I : R_z \to \mathbb{R} = \mathbb{R}[\![z]\!]$ as

$$I \left( \sum_{n \in \mathbb{N}} z^n f_n \right) = \sum_{n \in \mathbb{N}} z^n \int_{S^1} f_n.$$

The function $I$ clearly satisfies (5.1), and the corresponding paring given by $\langle P, Q \rangle = I(\text{res}(PQ))$ for all $P, Q \in \Psi(R_z)$, is non-degenerate. We can then apply Proposition 5.7 and obtain the following result:

**Corollary 5.8.** Let us equip the Lie algebra $\Psi(R_z)$ with the non-degenerate pairing $\langle P, Q \rangle = I(\text{res}(PQ))$ for all $P, Q \in \Psi(R_z)$. Write $\Psi(R_z) = \Psi^{-1}(R_z) \oplus D_{R_z}$ and consider the Hamiltonian functions

$$\mathcal{H}_k(L_z) = \frac{1}{k} I \left( \text{res}(L_z^{k+1}) \right), \quad L_z \in \Psi(R_z)' = \Psi(R_z),$$

(5.16)
on $\Psi(R_z)'$. The corresponding Hamiltonian equations of motion with respect to the $\{,\}_0$ Poisson structure of $\Psi(R_z)'$ are

$$\frac{dL_z}{dt_k} = [(L_z^k)_+, L_z].$$

(5.17)

As in the classical theory, see [13] or [17], the Hamiltonian functions $H_k$ furnish an infinite family of $\langle [z]\rangle$-conservation laws for the flow of (5.17).

### 5.3. The KP Equation and Perturbed Solutions

In this final subsection, we consider briefly the perturbed solutions $\tilde{L}_z \in C^\infty(T, \Psi^1((R_z)_q))$ to (4.7) obtained by using the second proof of Theorem 4.1, and we discuss how to connect them with solutions to the ($q$-deformed) KP-II equation. We note that in writing $\tilde{L}_z$ we are following the notations of Sect. 4.2 and Corollary 5.8. In order to avoid complicated symbols, hereafter we write $L$ instead of $\tilde{L}_z$, and we also set $R_z,q = (R_z)_q$. We remark once again (see Sect. 4.2) that derivatives with respect to $t$ are derivatives of smooth functions and also that within this approach the deformation parameter $q$ is mandatory, because we have to use fully regular Lie groups as in Lemma 4.4. We first observe:

**Theorem 5.9** Let $L$ be a solution to (4.7). The pseudo-differential operator $L$ satisfies the system of partial differential equations

$$\frac{\partial (L^i_+)}{\partial t_j} - \frac{\partial (L^j_+)}{\partial t_i} + [L^i_+, L^j_+] = 0.$$  

(5.18)

Proof The $q = 1$ case is, for instance, in Dickey’s book [13]. The general case is proven in a similar way: we first observe that the equation

$$\frac{dL^m}{dt_n} = [L^n_+, L^m]$$

holds. Then, we use this equation and (4.7) to check that the expression

$$\left\{ \frac{d}{dt_n}(L^m_+) - \frac{d}{dt_m}(L^n_+) - [L^m_+, L^n_+] \right\} - \frac{d}{dt_n}(-L^m_+) - \frac{d}{dt_m}(-L^n) - [-L^n_-, -L^m]$$

equals zero. Since the round bracketed expression is a differential operator, while the curly bracketed expression is an integral operator, both must be zero.

We consider $j = 2$ and $i = 3$, and we fix $t_1$ and $t_n$ for $n \geq 4$. Our definition (4.4) yields

$$L_2^2 = q^2 (\partial^2 + 2u_1(x, qt_1, q^2 t_2, q^3 t_3, \cdots))$$

$$L_3^2 = q^3 (\partial^3 + 3 u_1(x, qt_1, q^2 t_2, q^3 t_3, \cdots) \partial + 3 u_1, x(x, qt_1, q^2 t_2, q^3 t_3, \cdots) + 3 u_2(x, qt_1, q^2 t_2, q^3 t_3, \cdots)).$$

We replace these expressions into Eq. (5.18) and we set $t_2 = y, t_3 = t, 2u_{-1} = u$, so that the functions $u$ and $u_2$ depend on $x, q^2 y$ and $q^3 t$, and on extra
parameters $qt_1, q^4 t_4, q^5 t_5, \cdots$. Note that the variable $x$ has not been scaled. In other words, we do not identify the space variable $x$ with $t_1$ as it is sometimes done, see for instance [13]. Equation (5.18) yields, after cancelling a common $q^5$ factor,

$$u_{xx} + 4u_{2,x} = u_y$$
$$\frac{1}{3}u_{xxx} + 2u_{2,xx} - u_x u = u_{xy} + 2u_{2,y} - \frac{2}{3}u_t.$$

We eliminate $u_2$ from these equations, and we obtain

$$\frac{1}{2}u_{yy} = \partial_x \left( -\frac{1}{6}u_{xxx} - uu_x + \frac{2}{3}u_t \right). \quad (5.19)$$

This is the standard KP-II equation (for the function $u(x, q^2 y, q^3 t)$) as it appears in [9,13,37].

We would like to compare briefly our viewpoint to other approaches to KP-II. Let us consider, for definitiveness, the particular initial condition

$$L_0 = \partial + u_0 \partial^{-1}$$

where $u_0 \in R_z$. Proceeding as in Sect. 4.2, we obtain the solution

$$L = \partial + u_{-1} \partial^{-1} + \text{(lower order terms)} \in C^\infty(\mathbb{R}^\infty, \Psi_q(R_z))$$

to (4.7), and therefore Theorem 5.9 and our foregoing discussion imply that $u = 2u_{-1} \in R_{z,q}$ solves the $q$-deformed KP-II Eq. (5.19). We stress that $u$ is a “perturbed solution” depending on the extra parameters $z$ and $q$. The $z$-parameter is connected with the fact that we are solving KP with coefficients with low regularity, as we explain in Sect. 5.2; the $q$-parameter is a “time-scaling” parameter motivated by geometric considerations, as pointed out before. Intuitively, in order to interpret our derivations and series, we may assume that we are in the following situation:

The points $(x, y) \in [-\pi; \pi]^2 \sim S^1 \times [-\pi; \pi] \sim \mathbb{T}^2$, the operators $\partial_x, \partial_y$ are classical derivatives with respect to coordinates, and $\partial_x^{-1}$ is a well defined primitive map. By expansion on (e.g. Fourier) series of a map $\phi \in C^\infty([-\pi; \pi], \mathbb{R})$, we obtain an element $u_0 \in R_z$ which converges to $\phi$ when $z \to 1$, uniformly on any compact subset of $[-\pi; \pi]$.

Let us consider the operator $S_0$ connecting the initial condition $L_0 = \partial + u_0 \partial^{-1}$ to $\partial = \partial_x$. We solve the equation $qL_0 = S_0 \cdot (q\partial) \cdot S_0^{-1}$ in which

$$S_0 = 1 + \sum_{k \in \mathbb{N}^*} s_{-n} \partial^{-n}$$

for the “deformed initial data” $qL_0$. Expanding both sides of the equation $L_0 S_0 = S_0 \partial$, we obtain

$$s_{-1,x} = -u_0,$$  \quad (5.20)

and proceeding recursively we find $s_{-n}$ in terms of $u_0$ and its derivatives. Equation (5.20) implies

$$\int_{S^1} u_0 = 0. \quad (5.21)$$
This necessary condition on the function $u_0$ appears also in [9, Equation (1.8)], as condition on the initial data for the KP-II equation.

Summarizing, our solution $u = 2u_{-1}$ to the $q$-deformed KP-II Eq. (5.19) is a $q$-series with coefficients in $C^\infty(\mathbb{IK}^\infty, R_z)$—in which we recall that $R_z = C^\infty(S^1, \mathbb{IK})[[z]]$—and initial data $u_0 \in R_z$ satisfying (5.21). In other words, we begin with a “rough function of $x$” as our given data (our $z$-series $u_0$), and we obtain a unique solution $u(x, q^2y, q^3t)$ for which the full $y$-dependence, including at $t = 0$, is obtained by integration of the hierarchy. This certainly is different to what Bourgain proves in [9] for standard KP-II: in this reference he shows that the Cauchy problem for KP-II is globally well-posed for arbitrary initial data $u(x, y, 0) \in L^2(S^1 \times S^1)$. Thus, it would seem that we can reach only a restricted class of solutions to KP-II$^3$; we wonder if these solutions can be compared with the “meromorphic” solutions appearing in [51], in which one (complex) variable is required for the solutions described therein. These matters are currently under investigation.

The foregoing observations are not restricted to KP-II. In fact, we could conclude from the interesting paper [15] and our previous remarks that the KP hierarchy furnishes only some of the solutions to the equation

$$w_{xxxy} + 3w_{xy}w_x + 3w_yw_{xx} + 2w_{yt} - 3w_{xx} = 0,$$

precisely the next equation after KP-II which can be deduced using Theorem 5.9. This discussion leads us to hypothesize on the existence of a possible equation extending the KP hierarchy, which could furnish a full description of all the solutions of the KP-II equation. There is already a candidate for this hypothetical equation, as it is a known fact that the KP hierarchy may be seen as a special case of a self-dual Yang-Mills hierarchy, see [1,49]. This situation seems to be a non-trivial instance of the well-known problem of differential inclusion, and this again is reason for us to suspect the existence of another equation, perhaps as the ones considered in [1,49], for which the KP hierarchy would be a specialization.

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$^3$The following example was essentially suggested by the referee: let us fix a smooth function $u_0 : S^1 \to \mathbb{IK}$ and think of it as a (trivial) element of $R_z = C^\infty(S^1, \mathbb{IK})[[z]]$. Then, we obtain a unique solution $u = \sum \sum f_n(x, qt_1, q^2y, q^3t, qt_4, \cdots)z^nq^k$ to the $q$-deformed KP-II Eq. (5.19), with $y$-dependence of $u$ fixed by integration. On the other hand, let us choose a smooth function $f$ on $S^1$ with $f(0) = 0$ and such that $\tilde{u}_0(x, q^2y) = qu_0(x) + qf(q^2y)$ is different from $u(x, 0, q^2y, 0, \cdots)$. Then, the solution to KP-II with initial data $(1/q)\tilde{u}_0$ arising via Bourgain’s theorem [9] induces, as explained in Sect. 4.2, a solution $\tilde{u}$ to our $q$-deformed KP-II equation. The function $\tilde{u}$ cannot coincide with our solution $u$, even though they coincide at $t = y = 0$. 
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