Abstract. The aim of this paper is to develop the $L_p$ John ellipsoid for the geometry of log-concave functions. Using the results of the $L_p$ Minkowski theory for log-concave function established in [26], we characterize the $L_p$ John ellipsoid for log-concave function, and establish some inequalities of the $L_p$ John ellipsoid for log-concave function. Finally, the analog of Ball’s volume ratio inequality for the $L_p$ John ellipsoid of log-concave function is established.

1. INTRODUCTION

Let $K$ be a convex body in $\mathbb{R}^n$, among all ellipsoids contained in $K$, there exists a unique ellipsoid $JK$ with the maximum volume, this ellipsoid is called the John’s ellipsoid of $K$. It plays an important role in convex geometry and Banach space geometry (see, e.g., [12–14, 32, 34, 38, 39, 48]). One of the most important results concerning the John ellipsoid is the Ball’s volume-ratio inequality, which states that: if $K$ is an origin symmetric convex body in $\mathbb{R}^n$, then

$$\frac{V(K)}{V(JK)} \leq \frac{2^n}{\omega_n},$$

with equality if and only if $K$ is a parallelotope. Here $V(K)$ denotes the $n$-dimensional volume and $\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}$ denotes the volume of a unit ball in $\mathbb{R}^n$.

In 1990’s, the $L_p$ Brunn-Minkowski theory was firstly initiated by Lutwak (see [43, 44]), during the last two decades, it has achieved great development and expanded rapidly (see, e.g., [21, 28, 36, 37, 42, 45, 47, 54–58]). The $L_p$ extension of the John ellipsoid is given by Lutwak, Yang and Zhang [46].

Given a smooth convex body $K \in \mathbb{R}^n$ that contains the origin in its interior. Let $f_p(K, \cdot)$ be the $L_p$ curvature function of $K$, $p > 0$, find

$$\min_{\phi \in SL(n)} \int_{S^{n-1}} f_p(\phi K, \cdot) dS(u),$$

where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. The minimum is actually attained at some $\phi_p \in SL(n)$, and defines an ellipsoid $E_pK$, which $\phi_p$ maps it into the unit ball $B$, that is, $\phi_p E_p K = B$. The ellipsoid is unique and is called the volume-normalized $L_p$ John ellipsoid of $K$. The equivalent ways to state the

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above problem is given by the following two Optimization Problems [46]: Given a convex body \( K \) in \( \mathbb{R}^n \) that contains the origin in its interior, find an ellipsoid \( E \), amongst all origin-centered ellipsoids, which solves the following constrained maximization problem:

\[
\max \left( \frac{V(E)}{\omega_n} \right) \quad \text{subject to} \quad V_p(K, E) \leq 1.
\]  

(1.2)

A maximal ellipsoid will be called an \( S_p \) solution for \( K \). The dual problem is to find \( E \) such that

\[
\min V_p(K, E) \quad \text{subject to} \quad \left( \frac{V(E)}{\omega_n} \right)^\frac{1}{p} \geq 1.
\]  

(1.3)

A minimal ellipsoid will be called an \( \overline{S}_p \) solution for \( K \). Where

\[
\overline{V}_p(K, E) = \left[ \int_{S^{n-1}} \left( \frac{h_E}{h_K} \right)^p dV_K \right]^{\frac{1}{p}}, \quad p > 0,
\]

is the normalized \( L_p \) mixed volume of \( K \) and \( E \). More details about the solution of the two problems \( S_p, \overline{S}_p \) and related inequalities see [46]. The Orlicz extension of the John ellipsoid is done by Zou and Xiong [59]. Recently, the study of the geometry of log-concave functions in the field of convex geometry has emerged, with a quite natural idea is to replace the volume of a convex body by the integral of a log-concave function. To establish functional version of the problems from the convex geometric analysis of convex body has attracted a lot of authors interest (see, e.g., [1–4, 7, 16–20, 22, 24, 35, 41, 52]). Also an extension of the John ellipsoids to the case of log-concave functions has attracted a lot of authors interest, for example, in [5], the authors extend the notion of John’s ellipsoid to the setting of integrable log-concave functions and obtain integral ratio of a log-concave function and establish the reverse functional affine isoperimetric inequality. The extension of the LYZ ellipsoid to the log-concave functions is done by Fang and Zhou [27].

The Löwner ellipsoid function for log-concave function is invested by Li, Schütt and Werner [40]. Extensive research has been devoted to extend the concepts and inequalities from convex bodies to the setting of log-concave functions (see, e.g., [25, 29]). In fact, it was observed that the Prékopa-Leindler inequality is the functional analog of the Brunn-Minkowski inequality (see e.g., [15, 30, 49]) for convex bodies. Much progress has been made see [6, 8, 10, 23].

Let \( f \) be a log-concave functions of \( \mathbb{R}^n \) such that

\[
f : \mathbb{R}^n \to \mathbb{R}, \quad f = e^{-u},
\]

where \( u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a convex function. We always consider in this paper that a log-concave function \( f \) is integrable and such that \( f \) is nondegenerate, i.e., the interior of the support of \( f \) is non-empty, \( int(supp f) \neq \emptyset \). This implies that \( 0 < \int_{\mathbb{R}^n} f dx < \infty \).

Let \( f = e^{-u}, \ g = e^{-v} \) be log-concave functions, for any real \( \alpha, \beta > 0 \), the Asplund sum and scalar multiplication of two log-concave functions are defined as,

\[
\alpha \cdot f \oplus \beta \cdot g := e^{-w}, \quad \text{where} \quad w^* = \alpha u^* + \beta v^*.
\]  

(1.4)
Here $\omega^*$ denotes as usual the Fenchel conjugate of the convex function $\omega$. Correspond to the volume $V(K)$ of a convex body $K$ in $\mathbb{R}^n$, the total mass $J(f)$ of a log-concave function $f$ in $\mathbb{R}^n$ is firstly considered in [24]. The functional counterpart of Minkowski’s first inequality and related isoperimetric inequalities are established. The $L_p$ extension of the Asplund sum and scalar multiplication of two log-concave functions are discussed in [26], the functional $L_p$ Minkowski’s first inequality and the functional $L_p$ Minkowski problem also been discussed.

Our main goals in this paper are to discuss the functional $L_p$ John ellipsoid, based on the $L_p$ Asplund sum and $L_p$ scalar multiplication of two log-concave functions. Owing to the functional $L_p$ Minkowski’s first variation of $f$ and $g$, we focus on the following:

Problem $S_p$. Given a log-concave function $f \in A_0$, find a Gaussian function $\gamma_\phi$ which solves the following constrained maximization problem:

$$
\max \left( \frac{J(\gamma_\phi)}{c_n} \right) \text{ subject to } \delta J_p(f, \gamma_\phi) \leq 1.
$$

(1.5)

Where $c_n = (2\pi)^{\frac{n}{2}}$ and $\phi \in GL(n)$, $\gamma_\phi = e^{-\frac{\|\phi x\|^2}{2}}$ is the Gaussian function. $\delta J_p(f, \gamma_\phi)$ is the normalized first variation of the total mass $J(f)$ with respect to the $L_p$ Asplund sum.

In section 3, we prove that there exists a unique Gaussian function which solves the Problem $S_p$. The unique Gaussian function which solves the problem $S_p$ is called the $L_p$ John ellipsoid for the log-concave function $f$, and denoted by $E_p f$. Moreover, we characterize a Gaussian function which is the solution of the problem $S_p$.

In section 4, we focus on the continuity of the $L_p$ John ellipsoid, we prove that the $L_p$ John ellipsoid $E_p f$ is continuous with respect to $f$ and $p$. By the $L_p$ Minkowski’s first inequality for log-concave function, we prove that the total mass of the $L_p$ John ellipsoid $E_p f$ is no more than the total mass of $f$. In the end of this section, we show the similar Ball’s volume rate inequality is also holds for log-concave function.

2. Preliminaries

2.1. Convex bodies. In this paper, we work in $n$-dimensional Euclidean space, $\mathbb{R}^n$, endowed with the usual scalar product $\langle x, y \rangle$ and norm $\|x\|$. Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denote the standard unit ball and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}_o^n$ be the subclass of convex bodies $K$ whose relative interior $int(K)$ is nonempty. For $i \leq n$, let $\mathcal{H}^i$ be the $i$-dimensional Hausdorff measure, we indicate by $V(K) = \mathcal{H}^n(K)$ the $n$-dimensional volume.

Let $h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be the support function of $K$; i.e., for $x \in \mathbb{R}^n$,

$$
h_K(x) = \max \{ \langle x, y \rangle : y \in K \}.
$$

Let $n_K(x)$ be the unit outer normal at $x \in \partial K$, then $h_K(n_K(x)) = \langle n_K(x), x \rangle$. It is shown that the sublinear support function characterizes a convex body and, conversely, every sublinear function on $\mathbb{R}^n$ is the support function of a nonempty
compact convex set. By the definition of the support function, if \( \phi \in GL(n) \), then the support function of the image \( \phi K := \{ \phi y : y \in K \} \) is given by

\[
h_{\phi K}(x) = h_K(\phi^t x),
\]

where \( \phi^t \) denotes the transpose of \( \phi \). Let \( K \in \mathcal{K}_o^n \) be a convex body that contains the origin in its interior, the polar body \( K^\circ \) is defined by

\[
K^\circ = \{ y \in \mathbb{R}^n : \langle y, x \rangle \leq 1, \text{ for all } x \in K \}.
\]

Obviously, for \( \phi \in GL(n) \), then \( (\phi K)^\circ = \phi^{-t}K^\circ \). The gauge function \( \| \cdot \|_K \) is defined by

\[
\|x\|_K = \min \{ a \geq 0 : x \in aK \} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x).
\]

It is clear that

\[
\|x\|_K = 1 \quad \text{whenever} \quad x \in \partial K.
\]

Recall that the \( L_p \) (\( p \geq 1 \)) Minkowski combination of convex bodies \( K \) and \( L \) is defined as

\[
h_{K + \epsilon L}(x)^p = h_K(x)^p + \epsilon h_L(x)^p.
\]

One of the most important inequality related to the \( L_p \) Brunn-Minkowski combination of convex bodies \( K \) and \( L \) is

\[
V(K + \epsilon L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}},
\]

with equality if and only if \( K \) and \( L \) are dilation of each other. The \( L_p \) surface area measure of \( K \) is defined by

\[
dS_p(K, \cdot) = h_{K}^{1-p}dS(K, \cdot),
\]

where \( dS(K, \cdot) \) is the classical surface area measure, which is given by

\[
\lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = \int_{S^{n-1}} h_Q(u)dS(K, u).
\]

It is easy to say that for \( \lambda > 0 \), \( S_p(\lambda K, \cdot) = \lambda^{n-p}S_p(K, \cdot) \). If \( K \in \mathcal{K}_o^n \), then \( K \) has a curvature function, then \( f_p(K, \cdot) : S^{n-1} \to \mathbb{R} \), the \( L_p \)-curvature function of \( K \), is defined by

\[
f_p(K, \cdot) = h_{K}^{1-p}f(K, \cdot),
\]

where \( f(K, \cdot) \) is the curvature function, \( f(K, \cdot) : S^{n-1} \to \mathbb{R}^n \) defined as the Radon-Nikodym derivative

\[
f(K, \cdot) = \frac{dS(K, \cdot)}{dS},
\]

and \( dS \) is the standard Lebesgue measure on \( S^{n-1} \).

For quick reference about the definition and notations in convex geometry, good references are Gardner [31], Gruber [33], Schneider [53].
2.2. **Functional setting.** In the following, we discuss in the functional setting in $\mathbb{R}^n$. Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, that is $u((1-t)x + ty) \leq (1-t)u(x) + tu(y)$ for $t \in (0,1)$. We set $\text{dom}(u) = \{ x \in \mathbb{R}^n : u(x) \in \mathbb{R} \}$, by the convexity of $u$, $\text{dom}(u)$ is a convex set in $\mathbb{R}^n$. We say that $u$ is proper if $\text{dom}(u) \neq \emptyset$, and $u$ is of class $C^2_+$ if it is twice differentiable on $\text{int}(\text{dom}(u))$, with a positive definite Hessian matrix. Recall that the Fenchel conjugate of $u$ is the convex function defined by

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - u(x) \}. \quad (2.3)$$

It is obvious that $u(x) + u^*(y) \geq \langle x, y \rangle$ for $x, y \in \mathbb{R}^n$, there is an equality if and only if $x \in \text{dom}(u)$ and $y$ is in the subdifferential of $u$ at $x$, that means

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x) \rangle. \quad (2.4)$$

The convex function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, if the subset $\{ x \in \mathbb{R}^n : u(x) > t \}$ is an open set for any $t \in (-\infty, +\infty]$. Moreover, if $u$ is a lower semi-continuous convex function, then also $u^*$ is a lower semi-continuous convex function, and $u^{**} = u$.

The infimal convolution of functions $u$ and $v$ from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ is defined by

$$u \square v(x) = \inf_{y \in \mathbb{R}^n} \{ u(x - y) + v(y) \}. \quad (2.5)$$

The right scalar multiplication by a nonnegative real number $\alpha$:

$$(u\alpha)(x) := \begin{cases} \alpha u(\frac{x}{\alpha}), & \text{if } \alpha > 0; \\ I_{(0)}, & \text{if } \alpha = 0. \end{cases} \quad (2.6)$$

The following results below gather some elementary properties of $u$, the Fenchel conjugate and the infimal convolution, which can be found in [24,50].

**Lemma 2.1.** Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, then there exist constants $a$ and $b$, with $a > 0$, such that, for $\forall x \in \mathbb{R}^n$

$$u(x) \geq a\|x\| + b. \quad (2.7)$$

Moreover $u^*$ is proper, and satisfies $u^*(y) > -\infty$, $\forall y \in \mathbb{R}^n$.

**Proposition 2.2.** Let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then:

1. $(u \square v)^* = u^* + v^*$;
2. $(u\alpha)^*(x) = \alpha u^*(\frac{x}{\alpha})$, $\alpha > 0$;
3. $\text{dom}(u \square v) = \text{dom}(u) + \text{dom}(v)$;
4. it holds $u^*(0) = -\inf(u)$, in particular if $u$ is proper, then $u^*(y) > -\infty$; $\inf(u) > -\infty$ implies $u^*$ is proper.

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called log-concave if for $x, y \in \mathbb{R}^n$ and $0 < t < 1$, we have

$$f((1-t)x + ty) \geq f^{1-t}(x)f^t(y).$$
If \( f \) is a strictly positive log-concave function on \( \mathbb{R}^n \), then there exist a convex function \( u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) such that \( f = e^{-u} \). Following the notations in paper [24], let
\[
\mathcal{L} = \{ u : \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty\} | \text{ proper, convex, } \lim_{|x| \to +\infty} u(x) = +\infty \}.
\]
\[
\mathcal{A} = \{ f : \mathbb{R}^n \to \mathbb{R} | f = e^{-u}, u \in \mathcal{L} \}.
\]

Let \( f \in \mathcal{A} \) be a log-concave, according to a series of papers by Artstein-Avidan and Milman [9], Rotem [51], the support function of \( f = e^{-u} \) is defined as,
\[
h_f(x) = (-\log f(x))^* = u^*(x).
\] (2.8)

Here the \( u^* \) is the Fenchel conjugate of \( u \). The definition of \( h_f \) is a proper generalization of the support function \( h_K \), in fact, one can easily checks \( h_{\chi_K} = h_K \). Obviously, the support function \( h_f \) shares the most of the important properties of \( h_K \).

The polar function of \( f = e^{-u} \) is defined by \( f^o = e^{-u^*} \). Specifically,
\[
f^o(y) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{e^{-\langle x, y \rangle}}{f(x)} \right\},
\]
it follows that, \( f^o \) is also a log-concave function.

Let \( \phi \in GL(n) \), we always write \( f \circ \phi(x) = \phi f(x) = f(\phi x) \). The following proposition shows that \( h_f \) is \( GL(n) \) covariant which is proved in [27].

**Proposition 2.3.** Let \( f \in \mathcal{A} \). For \( \phi \in GL(n) \) and \( x \in \mathbb{R}^n \), then
\[
h_{\phi f}(x) = h_f(\phi^{-t}x).
\]
Moreover, for the polar function of \( f \),
\[
(\phi f)^o = \phi^{-t} f^o.
\]

The class of log-concave functions \( \mathcal{A} \) can be endowed with an algebraic structure which extends in a natural way as the usual of the Minkowski’s structure on \( \mathcal{K}^n \). For example, the Asplund sum of two log-concave functions is corresponded to the classical Minkowski sum of two convex bodies. See [24] for more about the Asplund sum and the related inequalities of the total mass of the log-concave function in \( \mathcal{A} \) which correspond to the convex bodies in \( \mathcal{K}^n \). In very recently, the \( L_p \) Asplund sum of log-concave functions are studied by author Fang, Xing and Ye [26]. Let
\[
\mathcal{A}_0 = \{ e^{-u} : u \in \mathcal{L}_0 \} \subset \mathcal{A}
\]
with
\[
\mathcal{L}_0 = \{ u \in \mathcal{L} : u \geq 0, (u^*)^* = u \text{ and } u(o) = 0 \}.
\]
Clearly, if \( u \in \mathcal{L}_0 \), then \( u \) has its minimum attained at the origin \( o \).

**Definition 2.1** ([26]). Let \( f = e^{-u}, g = e^{-v} \in \mathcal{A}_0 \), and \( \alpha, \beta \geq 0 \). The \( L_p \) (\( p \geq 1 \)) Asplund sum and multiplication of \( f \) and \( g \) is defined as
\[
\alpha \cdot_p f \oplus_p \beta \cdot_p g = e^{-[(u^*)^p]_{[1]}(v^*)^p_{[1]}},
\] (2.9)
where
\[
(u \cdot_p \alpha) \Box_p (v \cdot_p \beta) = \left[ (\alpha(u^*)^p + \beta(v^*)^p)^{\frac{1}{p}} \right]^*.
\]
The $L_p$ Asplund sum is an extension of the Asplund sum on $\mathcal{A}_0$. Specially, when $p = 1$, it reduces to the Asplund sum of two functions on $\mathcal{A}_0$, that is
\[
(\alpha \cdot f \oplus \beta \cdot g)(x) = \sup_{y \in \mathbb{R}^n} f \left( \frac{x - y}{\alpha} \right) \alpha \left( \frac{y}{\beta} \right)^\beta.
\] (2.10)
Moreover, when $\alpha = 0$ and $\beta > 0$, we have $(\alpha \cdot f \oplus \beta \cdot g)(x) = g \left( \frac{x}{\beta} \right)^\beta$; when $\alpha > 0$ and $\beta = 0$, then $(\alpha \cdot f \oplus \beta \cdot g)(x) = f \left( \frac{x}{\alpha} \right)^\alpha$; finally, when $\alpha = \beta = 0$, we set $f \oplus g = I_{[0]}$. We say that the $L_p$ Asplund sum for log-concave functions is closely related to the $L_p$ Minkowski sum for convex bodies in $\mathbb{R}^n$. For examples, $K, L \in \mathcal{K}^n$, let
\[
\chi_K(x) = e^{-I_K(x)} = \begin{cases} 1, & \text{if } x \in K; \\ 0, & \text{if } x \notin K, \end{cases}
\] (2.11)
where $I_K$ is the indicator function of $K$, and it is a lower semi-continuous convex function,
\[
I_K(x) = \begin{cases} 0, & \text{if } x \in K; \\ \infty, & \text{if } x \notin K. \end{cases}
\] (2.12)
The characteristic function $\chi_K$ is log-concave functions with $u = I_K$ belongs to $\mathcal{L}$, $w^* = h_K$ belongs to $\mathcal{L}$ if $0 \in \text{int}(K)$, for $p \in [1, +\infty)$, the function
\[
\left( (I_K \cdot \alpha) \Box_p (I_L \cdot \beta) \right) = \left[ (\alpha(I_K^*)^p + \beta(I_L^*)^p) \right]^{\frac{1}{p}}
\] = \left[ (\alpha h_K^p + \beta h_L^p) \right]^{\frac{1}{p}} = I_{\alpha \cdot p K + \beta \cdot p L}.
\] Then $\alpha \cdot \chi_K \oplus \beta \cdot \chi_L = e^{-[I_K \oplus \alpha \Box_p I_L \cdot \beta]} = \chi_{\alpha \cdot p K + \beta \cdot p L}$.

The following Proposition assert that the $L_p$ Asplund sum of log-concave functions is closed in $\mathcal{A}_0$.

**Proposition 2.4** ([26]). Let $f$ and $g$ belong both to the same class $\mathcal{A}_0$, and $\alpha, \beta \geq 0$. Then $f \cdot \alpha \oplus \beta \cdot g$ belongs to $\mathcal{A}_0$.

The total mass function of $f$ is defined as
\[
J(f) = \int_{\mathbb{R}^n} f(x) dx.
\] (2.13)
Clearly, when $f = \chi_K$, $J(f) = V(K)$. Similar to the integral expression of mixed volume $V(K, L)$, for $f = e^{-u}$ and $g = e^{-v}$ in $\mathcal{A}_0$, the quantity $\delta J(f, g)$, which is called as the first variation of $J$ at $f$ along $g$ is defined by (see [24])
\[
\delta J(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}.
\]
It has been shown that $\delta J(f, g)$ has the following integral expression,
\[
\delta J(f, g) = \int_{\mathbb{R}^n} h_g d\mu(f, x),
\] (2.14)
where $\mu(f, x)$ is the surface area measure of $f$ on $\mathbb{R}^n$ and is given by
\[
 u(f, x) = (\nabla u(x))_x f(\mathcal{H}^n),
\] (2.15)
here $\nabla u$ is the gradient of $u$ in $\mathbb{R}^n$, that means, for any Borel function $g \in \mathcal{A}$,
\[
 \int_{\mathbb{R}^n} g(x) d\mu(f, x) = \int_{\mathbb{R}^n} g(\nabla u(x)) e^{-u(x)} dx.
\] (2.16)
Specially, if take $f = g$ in (2.14), then
\[
 \delta J(f, f) = n J(f) + \int_{\mathbb{R}^n} f \log f dx = J(n f + f \log f).
\] (2.17)
In the following sections we write $J(n f + f \log f)$ in terms of $J(f^o)$ for simplicity.

The $L_p$ surface area measure of $f$, denoted by $\mu_p(f, \cdot)$ is given in [26].

**Definition 2.2 ([26]).** Let $f = e^{-u} \in \mathcal{A}_0$ be a log-concave function, the $L_p$ surface area measure of $f$, denoted as $\mu_p(f, \cdot)$, is the Borel measure on $\Omega$ such that
\[
 \int_{\Omega} g(y) d\mu_p(f, y) = \int_{\{x \in dom(u); \nabla u(x) \in \Omega\}} g(\nabla u(x)) (h_f(\nabla u(x)))^{1-p} f(x) dx,
\] (2.18)
holds for every Borel function $g$ such that $g \in L^1(\mu_p(f, \cdot))$.

Similarly, the first variation of the total mass at $f$ along $g$ with respect to the $L_p$ Asplund sum is defined as,

**Definition 2.3 ([26]).** Let $f, g \in \mathcal{A}_0$. For $p \geq 1$, the first variation of the total mass of $f$ along $g$ with respect to the $L_p$ Asplund sum is defined by
\[
 \delta J_p(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus_p t \cdot g) - J(f)}{t},
\] (2.19)
whenever the limit exists.

The following integral expression of $\delta J_p(f, g)$ is $L_p$ extension of (2.14) which is established in [26]. Specially, if take $f = e^{-I_K(x)}$ and $g = e^{-I_L(x)}$, where $I_K$ and $I_L$ are the indicator function of $K$ and $L$. So $J(f) = V(K)$ and $J(g) = V(L)$, then we have $\delta J(f, g) = V_p(K, L)$.

**Theorem 2.5 ([26]).** Let $f = e^{-u} \in \mathcal{A}_0$ and $g = e^{-u} \in \mathcal{A}_0$. For $p \geq 1$, assume that $g$ is an admissible $p$-perturbation for $f$. In addition, suppose that there exists a constant $k > 0$ such that
\[
 det(\nabla^2 h_f) \leq k(h_f)^{n(p-1)} det(\nabla^2 h_f),
\]
holds for all $x \in \mathbb{R}^n \setminus \{o\}$. Then
\[
 \delta J_p(f, g) = \frac{1}{p} \int_{\mathbb{R}^n} (h_g)^p d\mu_p(f, x).
\] (2.20)

Note that the support function of the log-concave function is nondecreasing, it’s easy to get that if $g_1 \leq g_2$, then
\[
 \delta J_p(f, g_1) \leq \delta J_p(f, g_2).
\]
In the following, we normalize the $\delta J_p(f, g)$. For $f = e^{-u}$, $g = e^{-v} \in A_0$, and $1 \leq p < \infty$, we define

$$\overline{\delta} J_p(f, g) = \left( \frac{p \cdot \delta J_p(f, g)}{J(f^\circ)} \right)^\frac{1}{p} = \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} \left( \frac{h_g}{h_f} \right)^p h_f d\mu(f, x) \right]^\frac{1}{p}, \quad (2.21)$$

Note that $\frac{h_g(x)}{h_f(x)}$ is a probability measure on $\mathbb{R}^n$. For $p = \infty$ define

$$\overline{\delta} J_\infty(f, g) = \max \left\{ \frac{h_g(x)}{h_f(x)} : x \in \mathbb{R}^n \right\}. \quad (2.22)$$

Unless $\frac{h_g(x)}{h_f(x)}$ is a constant on $\mathbb{R}^n$, by the Jensen’s inequality, it follows that $\overline{\delta} J_p(f, g) < \overline{\delta} J_q(f, g)$, for $1 \leq p < q < \infty$. For $p = \infty$, we have $\lim_{p \to \infty} \overline{\delta} J_p(f, g) = \overline{\delta} J_\infty(f, g)$. Moreover, we have the following Lemma.

**Lemma 2.6.** Suppose $f = e^{-u}$, $g = e^{-v} \in A_0$, $1 \leq p < q < \infty$. Then

$$\overline{\delta} J_1(f, g) \leq \overline{\delta} J_p(f, g) \leq \overline{\delta} J_q(f, g) \leq \overline{\delta} J_\infty(f, g). \quad (2.23)$$

In order to establish the continuity of the $L_p$ John ellipsoid for log-concave function in section 4, we give the following Lemma of the $\overline{\delta} J_p(f, g)$.

**Lemma 2.7.** Let $f = e^{-u}$, $g = e^{-v}$, $g_0 = e^{-v_0} \in A_0$, then

$$|\overline{\delta} J_p(f, g) - \overline{\delta} J_p(f, g_0)| \leq \frac{\|h_g - h_{g_0}\|_\infty}{\min\{|h_f| : x \in \mathbb{R}^n\}}, \quad (2.24)$$

for all $p \in [1, \infty]$, where $\| \cdot \|_\infty$ denotes the $\infty$ norms.

**Proof.** First suppose that $p < \infty$, by (2.21) and the triangle inequality for $L_p$ norms, we have

$$|\overline{\delta} J_p(f, g) - \overline{\delta} J_p(f, g_0)| \leq \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} \frac{h_g}{h_f} - \frac{h_{g_0}}{h_f} \right]^p \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} \frac{h_f^{p-1}}{h_f} h_f d\mu(f, x) \right]^\frac{1}{p}$$

$$\leq \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} 1 \right] \frac{1}{|h_f|} \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} h_f d\mu(f, x) \right]^\frac{1}{p} \|h_g - h_{g_0}\|_\infty$$

$$\leq \frac{\|h_g - h_{g_0}\|_\infty}{\min\{|h_f| : x \in \mathbb{R}^n\}}.$$ 

The third inequality we use the fact that $\frac{h_f d\mu(f, x)}{J(f^\circ)}$ is a probability measure on $\mathbb{R}^n$. For $p \to \infty$, the continuous of the $L_p$ norm with $p$ shows that (2.24) holds for $p = \infty$ as well. \hfill \square

The following Lemma shows some Properties of $\delta J_p(f, g)$ and its normalizer.

**Lemma 2.8.** Suppose that $f = e^{-u}$, $g = e^{-v} \in A_0$, then

1. $\delta J_p(f, f) = \frac{1}{p} J(f^\circ)$.
2. $\overline{\delta} J_p(f, f) = 1$.
3. $\delta J_p(f, \lambda \cdot g) = \lambda \delta J_p(f, g)$, for $\lambda > 0$.
4. $\overline{\delta} J_p(f, \lambda \cdot g) = \lambda \overline{\delta} J_p(f, g)$, for $\lambda > 0$.
5. $\delta J_p(\phi f, g) = |\det \phi|^{-1} \delta J_p(f, \phi^{-1} g)$, for all $\phi \in GL(n)$.
6. $\overline{\delta} J_p(\phi f, g) = \overline{\delta} J_p(f, \phi^{-1} g)$, for all $\phi \in GL(n)$. 


Proof. By formula (2.20) and Definition 2.3, it immediately gives (1) and (2).

In order to prove (3), by Definition 2.1, we have \( h_{\lambda, f} = \lambda h_f \). So we have \( \delta J_p(f, \lambda \cdot g) = \lambda \delta J_p(f, g) \). By (3), it yields (4) directly.

By the integral formula of the first variation (2.20), and note that \( \nabla x(\phi u) = \phi \nabla x u \), we have

\[
\delta J_p(\phi f, g) = \frac{1}{p} \int_{\mathbb{R}^n} h_g^{1-p}(\nabla x(\phi u)) e^{-\phi u} dx
\]

\[
= \frac{1}{p} \int_{\mathbb{R}^n} h_{\phi^{-1} g}^{1-p}(\nabla x u) h_f^{1-p}(\nabla x u) e^{-u(\phi x)} dx
\]

\[
= |\det \phi|^{-1} \frac{1}{p} \int_{\mathbb{R}^n} h_{\phi^{-1} g}^{1-p}(\nabla u) h_f^{1-p}(\nabla u) e^{-u} dx
\]

\[
= |\det \phi|^{-1} \delta J_p(f, \phi^{-1} g).
\]

On other hand, note that \( J((\phi f)\phi^0) = |\det \phi|^{-1} J(f\phi^0) \) and together with (5), it follows that \( \delta J_p(\phi f, g) = \delta J_p(f, \phi^{-1} g) \). So we complete the proof. \( \square \)

The following result will be used in the next section (see \([52]\)).

**Proposition 2.9** (\([52]\)). Let \( D \) be a relatively open convex sets, and \( f_1, f_2, \cdots \), be a sequence of finite convex functions on \( D \). Suppose that the real number \( f_1(x), f_2(x), \cdots \), is bounded for each \( x \in D \). It is then possible to selected a subsequence of \( f_1, f_2, \cdots \), which converges uniformly on closed bounded subset of \( D \) to some finite convex function \( f \).

### 3. \( L_p \) John ellipsoid for log-concave functions

Let \( \gamma = e^{-\frac{\|x\|^2}{2}} \) be the standard Gaussian function. In the following, we set

\[
\gamma_{\phi}(x) = e^{-\frac{\|\phi x\|^2}{2}},
\]

where \( \phi \in \text{GL}(n) \). It is worth noting that the Gaussian function \( \gamma_{\phi} \) plays an important role in the study of the extremal problem of log-concave functions as the ellipsoids do for the study of the extremal problems of convex bodies. In fact, it is the unique function of \( \mathcal{A} \) which is self-dual, that is

\[
f = e^{-\frac{\|x\|^2}{2}} \iff f^0 = f.
\]

Now, Let us consider the following optimization problem.

The \( L_p \) optimization problem \( S_p \) \((p \geq 1)\) for log-concave functions \( f \): Given a log-concave function \( f \in \mathcal{A}_0 \), find a Gaussian function \( \gamma_{\phi} \) which solves the following constrained maximization problem:

\[
\max \left( \frac{J(\gamma_{\phi})}{c_n} \right) \text{ subject to } \delta J_p(f; \gamma_{\phi}) \leq 1. \tag{3.1}
\]
The dual $L_p$ optimization problem $\overline{S}_p$ ($p \geq 1$) for log-concave functions $f$: Given a log-concave function $f \in A_0$, find a Gaussian function $\gamma_\phi$ which solves the following constrained minimization problem:

$$\min \delta J_p(f, \gamma_\phi) \quad \text{subject to} \quad \frac{J(\gamma_\phi)}{c_n} \geq 1. \quad (3.2)$$

**Lemma 3.1.** These above optimization problems for log-concave functional are equivalent to:

1. The problem $S_p$ is equivalent to:
   $$\max \left( \frac{J(\gamma_\phi)}{c_n} \right) \quad \text{subject to} \quad \overline{J}_p(f, \gamma_\phi) = 1.$$ 

2. The dual problem $\overline{S}_p$ is equivalent to:
   $$\min \delta J_p(f, \gamma_\phi) \quad \text{subject to} \quad \frac{J(\gamma_\phi)}{c_n} = 1.$$ 

**Proof.** (1) Assume that $\gamma_\phi = e^{-\frac{\|\phi(x)\|^2}{2}}$ is the solution of the problem $S_p$, if $\frac{J(\gamma_\phi)}{c_n}$ obtains the maximum, but $\delta J_p(f, \gamma_\phi) \neq 1$, assume that $\delta J_p(f, \gamma_\phi) < 1$, then let $\overline{\gamma}_\phi = 1 - \frac{\delta J_p(f, \gamma_\phi)}{\delta J_p(f, \gamma_\phi)^p} \cdot \gamma_\phi$.

Then $h_{\overline{\gamma}_\phi} = \frac{1}{\delta J_p(f, \gamma_\phi)} h_{\gamma_\phi}$. Moreover, we have

$$\overline{\delta J}_p(f, \overline{\gamma}_\phi) = \left[ \frac{1}{J(f^\phi)} \int_{\mathbb{R}^n} h_{\gamma_\phi}^p \mu_p(f, x) \right]^{\frac{1}{p}} = \frac{1}{\delta J_p(f, \gamma_\phi)} \left[ \frac{1}{J(f^\phi)} \int_{\mathbb{R}^n} h_{\gamma_\phi}^p \mu_p(f, x) \right]^{\frac{1}{p}} = 1.$$ 

On the other hand, since $\gamma_\phi = e^{-\frac{\|\phi(x)\|^2}{2}}$, by the Definition 2.1 of the $L_p$ multiplication, a simple computation shows that, for $\lambda \in \mathbb{R}$,

$$J(\lambda^p \cdot \gamma_\phi) = \int_{\mathbb{R}^n} e^{-\lambda^2 \frac{\|\phi(x)\|^2}{2}} dx = \lambda^2 J(\gamma_\phi). \quad (3.3)$$ 

Note that $\delta J_p(f, \gamma_\phi)^{-\frac{q}{2}} > 1$, then we have

$$J(\frac{1}{\delta J_p(f, \gamma_\phi)^p} \cdot \gamma_\phi) = \overline{\delta J}_p(f, \gamma_\phi)^{-\frac{q}{2}} J(\gamma_\phi) \geq J(\gamma_\phi).$$

This means that $\gamma_\phi$ is not a solution to the problem $S_p$, which is contract with our assumption, so we complete the proof of the first one. The similar with the proof of Problem $\overline{S}_p$, so we complete the proof. \qed

**Lemma 3.2.** Suppose $f \in A_0$. If $\gamma_\phi$ is a Gaussian function that is an solution for the problem $S_p$ of $f$, then

$$\left( \frac{c_n}{J(\gamma_\phi)} \right)^{\frac{2m}{n}} \cdot \gamma_\phi$$
is an solution for problem \( \overline{S}_p \) of \( f \). Conversely, if \( \gamma_\varphi \) is a Gaussian function that is an solution for the \( \overline{S}_p \) problem of \( f \), then

\[
\frac{1}{\delta J_p(f, \gamma_\varphi)} \cdot \gamma_\varphi
\]

is an solution for problem \( S_p \) of \( f \).

**Proof.** Let \( \gamma_T = e^{-\frac{\|T\|_2^2}{2}} \), where \( T \in \text{GL}(n) \), and satisfies \( J(\gamma_T) \geq c_n \). By Lemma 2.8, it obviously has \( \overline{\delta J}_p \left( f, \frac{1}{\delta J_p(f, \gamma_T)} \cdot \gamma_T \right) = 1 \). Since \( \gamma_\phi \) is an \( S_p \) solution for \( f \), then

\[
J(\gamma_\phi) \geq J \left( \frac{1}{\delta J_p(f, \gamma_T)} \cdot \gamma_T \right).
\]

By (3.3) it shows

\[
J \left( \frac{1}{\delta J_p(f, \gamma_T)} \cdot \gamma_T \right) = \overline{\delta J}_p(f, \gamma_T)^{-\frac{2}{p}} J(\gamma_T).
\] (3.4)

According to Lemma 3.1, we have \( \overline{\delta J}_p(f, \gamma_\phi) = 1 \). Then

\[
\overline{\delta J}_p(f, \gamma_T) \geq \left( \frac{J(\gamma_T)}{J(\gamma_\phi)} \right)^{\frac{2}{n}} \geq \left( \frac{c_n}{J(\gamma_\phi)} \right)^{\frac{2}{n}} = \overline{\delta J}_p \left( f, \left( \frac{c_n}{J(\gamma_\phi)} \right)^{\frac{2}{n}} \cdot \gamma_\phi \right).
\]

On the other hand, since \( J \left( \frac{c_n}{J(\gamma_\phi)} \right)^{\frac{2}{n}} \cdot \gamma_\phi \) is a solution to Problem \( \overline{S}_p \). We complete the first assertion’s proof.

Let \( \gamma_\mathcal{T} = e^{-\frac{\|\mathcal{T}\|_2^2}{2}} \), \( \mathcal{T} \in \text{GL}(n) \), such that \( \overline{\delta J}_p(f, \gamma_\mathcal{T}) \leq 1 \). Then we have

\[
J \left( \frac{c_n}{J(\gamma_\mathcal{T})} \right)^{\frac{2}{n}} \cdot \gamma_\mathcal{T} \geq c_n.
\]

Since \( \gamma_\varphi \) is solution of \( \overline{S}_p \), we have

\[
\left( \frac{c_n}{J(\gamma_\varphi)} \right)^{\frac{2}{n}} \overline{\delta J}_p(f, \gamma_\varphi) = \overline{\delta J}_p \left( f, \left( \frac{c_n}{J(\gamma_\varphi)} \right)^{\frac{2}{n}} \cdot \gamma_\varphi \right) \geq \overline{\delta J}_p(f, \gamma_\varphi).
\]

By Lemma 3.1 we have \( J(\gamma_\varphi) = c_n \). Hence the above inequality can be rewritten as

\[
\frac{J(\gamma_\varphi)}{\overline{\delta J}_p(f, \gamma_\varphi)^{\frac{2}{n}}} \geq \left( \frac{c_n}{J(\gamma_\mathcal{T})} \right)^{\frac{2}{n}} \cdot \gamma_\mathcal{T} \geq \overline{\delta J}_p(\overline{\gamma}_\varphi).
\]

By formula (3.4) again, it means

\[
J \left( \frac{1}{\overline{\delta J}_p(f, \gamma_\varphi)} \cdot \gamma_\varphi \right) \geq J \left( \frac{1}{\overline{\delta J}_p(f, \gamma_\mathcal{T})} \cdot \gamma_\mathcal{T} \right) \geq J(\gamma_\mathcal{T}).
\]

On the other hand, since

\[
\overline{\delta J}_p \left( f, \frac{1}{\overline{\delta J}_p(f, \gamma_\varphi)} \cdot \gamma_\varphi \right) = 1.
\]

This completes the proof. \( \square \)
Theorem 3.3. Suppose that $f \in A_0$. Then there exist a solution, $\gamma_{p}$, satisfies the Optimization Problem $\overline{S}_p$.

Proof. By the definition of the optimization problem $\overline{S}_p$, let $\phi \in GL(n)$, if $\gamma_{p}$ is Gaussian function subject to $J(\gamma_{p}) = c_n$, then we have $\phi \in SL(n)$. The existence of the solution of the Problem $\overline{S}_p$ is equivalent to find the $\phi_0 \in SL(n)$ which solves the following. Let $\phi \in SL(n)$, choose $\epsilon_0 > 0$ sufficiently small so that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, $I + \epsilon \phi$ is invertible. For $\epsilon \in (-\epsilon_0, \epsilon_0)$ define $\phi_{\epsilon} \in SL(n)$ by

$$\phi_{\epsilon} = \frac{I + \epsilon \phi}{\det(I + \epsilon \phi)^{\frac{1}{n}}}.$$ 

here $I$ is the identity matrix and $|\det \phi_{\epsilon}| = 1$. Then find $\phi_0$ such that

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \delta J_p(f, \gamma_{\phi_{\epsilon}}) = 0.$$ 

(3.5)

It is equivalent to

$$\int_{\mathbb{R}^n} \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left( \frac{\|\phi_{\epsilon}^{-1} x\|_2^2}{2} \right)^p d\mu_p(f, x) = 0.$$ 

Since the norm of the vector and $t^p$ are continuous functions, it grants there exists a solution of the above equation. So there exists a $\phi \in SL(n)$ such that $\overline{S}_p(f, \gamma_{\phi})$. $\square$

We say that the $\gamma_{\phi}$ is the solution of the $L_p$ functional optimization problem $S_p$, and we rewrite it as the $\gamma_f$. The following Corollary are obviously.

Corollary 3.4. Suppose $f \in A_0$, $\gamma_f$ be the solution of the optimization problem $S_p$, then

$$\overline{\delta} J_p(f, \gamma_f) = 1.$$ 

(3.6)

By Lemma 3.2 and Theorem 3.3, it guarantees that there exists a unique solution for the optimization problem $\overline{S}_p$.

Theorem 3.5. Let $f \in A_0$. Then problem $\overline{S}_p$ has a unique solution. Moreover a Gaussian function $\gamma_{\phi}$ solves $\overline{S}_p$ if and only if it satisfies

$$\delta J_p(f, \gamma_{\phi}) h_{\gamma_{\phi}}(y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h_{\gamma_{\phi}}^{p-1}(x) d\mu_p(f, x),$$ 

(3.7)

for any $y \in \mathbb{R}^n$.

In order to prove Theorem 3.5, we need the following Lemma.

Lemma 3.6. Let $f = e^{-u} \in A_0$ and $\phi \in GL(n)$. If the Gaussian function $\gamma_{\phi}$ solves the optimization functional problem $\overline{S}_p$, then

$$\delta J_p(f, \gamma_{\phi}) h_{\gamma_{\phi}}(y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h_{\gamma_{\phi}}^{p-1}(x) d\mu_p(f, x).$$
Proof. By the $SL(n)$ invariance of the $\delta J_p(f, g)$, we may assume that $\gamma^\sigma_\phi = \gamma$ is the solution of problem $\delta J_p$. Let $\phi \in SL(n)$, choose $\epsilon_0 > 0$ sufficiently small so that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, $I + \epsilon \phi$ is invertible. For $\epsilon \in (-\epsilon_0, \epsilon_0)$ define $\phi_\epsilon \in SL(n)$ by

$$
\phi_\epsilon = \frac{I + \epsilon \phi}{\det(I + \epsilon \phi)^{\frac{1}{n}}},
$$

here $I$ is the identity matrix and $|\det \phi_\epsilon| = 1$. Then

$$
\delta J_p(f, \gamma_{\phi_\epsilon}) \leq \delta J_p(f, \gamma_{\phi_\epsilon}),
$$

for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. That means

$$
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \delta J_p(f, \gamma_{\phi_\epsilon}) = 0.
$$

(3.8)

On the other hand, by Proposition 2.3, we have $\gamma^\sigma_\phi = \phi^{-1}_\epsilon \gamma$. Then $h_{\gamma_{\phi_\epsilon}}(x) = h_{\phi^{-1}_\epsilon \gamma}(x) = h_\gamma(\phi x)$. By the definition of $\delta J_p(f, g)$, we have

$$
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_{\mathbb{R}^n} h^p_{\gamma_{\phi_\epsilon}}(x) d\mu_p(f, x) = 0.
$$

It is equivalent to the following

$$
\int_{\mathbb{R}^n} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left( \frac{\|\phi_\epsilon x\|^2}{2} \right)^p d\mu_p(f, x) = 0.
$$

Or equivalently,

$$
\int_{\mathbb{R}^n} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left( \det(I + \epsilon \phi)^{-\frac{2p}{n}} \left( \langle x \cdot x \rangle + 2\epsilon \langle x \cdot \phi x \rangle + \epsilon^2 \langle \phi x \cdot \phi x \rangle \right) \right) d\mu_p(f, x) = 0.
$$

Since $\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \det(I + \epsilon \phi) = \text{trace}(\phi)$, by a simple computation, we have,

$$
\text{trace}(\phi) \delta J_p(f, \gamma) = \frac{n}{2p} \int_{\mathbb{R}^n} \langle x, \phi x \rangle \left( \frac{\|x\|^2}{2} \right)^{p-1} d\mu_p(f, x).
$$

Choosing an appropriate $\phi$ for each $i, j \in \{1, 2, \cdots, n\}$ gives

$$
\delta_{i,j} \delta J_p(f, \gamma) = \frac{n}{2p} \int_{\mathbb{R}^n} \langle x, e_i \rangle \langle x, e_j \rangle h^{p-1}_\gamma(x) d\mu_p(f, x),
$$

where $e_1, \cdots, e_n$ is an orthonormal basis of $\mathbb{R}^n$, and $\delta_{i,j}$ is the Koronecker symbols. Which in turn gives

$$
\|y\|^2 \delta J_p(f, \gamma) = \frac{n}{2p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h^{p-1}_\gamma(x) d\mu_p(f, x).
$$

(3.9)

That means

$$
\delta J_p(f, \gamma) h_\gamma(y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h^{p-1}_\gamma(x) d\mu_p(f, x).
$$

(3.10)

So we have

$$
\delta J_p(f, \gamma^\sigma_\phi) h_{\gamma_{\phi_\epsilon}}(y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h^{p-1}_{\gamma_{\phi_\epsilon}}(x) d\mu_p(f, x).
$$

We complete the proof.
Now we prove Theorem 3.5.

Proof of Theorem 3.5. Lemma 3.6 grants that if $\gamma_{\phi}$ is a $S_p$ solution of $f$, then the above formula holds.

Conversely, without loss of generality, we may prove that (3.7) holds when $\gamma_{\phi} = \gamma$, that is $\phi = I$. Then for any $\phi_1 \in GL(n)$, we shall prove that if $J(\gamma_{\phi_1}) = c_n$ for some $\phi_1 \in GL(n)$,

$$\delta J_p(f, \gamma_{\phi_1}) \geq \delta J_p(f, \gamma),$$

with equality if and only if $\gamma_{\phi_1} = \gamma$. Equivalently, we shall prove that if $\phi_1$ is a positive definite symmetric matrix with $|\phi_1| = 1$, and note that $\gamma_{\phi_1} = e^{-\frac{1}{2} |x|} \phi_1$ then

$$\frac{1}{p\delta J_p(f, \gamma)} \int_{\mathbb{R}^n} h^p_{\gamma_{\phi_1}}(x) d\mu_p(f, x) \geq 1,$$

or equivalently,

$$\frac{1}{p\delta J_p(f, \gamma)} \int_{\mathbb{R}^n} \left( \frac{\phi_1^{-t}x}{\|x\|} \right)^p \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x) \geq 1,$$

with equality if and only if $\frac{\phi_1^{-t}x}{\|x\|} = 1$, for $x \in \mathbb{R}^n$.

Write $\phi_1^{-t} = OTP^t$, where $T = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $O$ is an orthogonal matrix. To establish our inequality we need to show that if $\phi_1$ is a positive definite symmetric matrix with $\det \phi_1 = 1$, then

$$\int_{\mathbb{R}^n} \log \left( \frac{\|\phi_1^{-t}x\|}{\|x\|} \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x) \geq 0.$$

On the other hand, since $\nabla(\phi u) = \phi^t \nabla_{\phi} u$ for each $\phi \in GL(n)$, and $(\phi u)^* = \phi^{-t} u^*$, we have

$$\int_{\mathbb{R}^n} \log \left( \frac{\|T x\|^2}{\|x\|^2} \right)^p \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(\phi f, x)$$

$$= 2p \int_{\mathbb{R}^n} \log \frac{\|T O^t \nabla_{O} u\|}{\|O^t \nabla_{O} u\|} \left( \frac{\|O^t \nabla_{O} u\|^2}{2} \right)^p (u^* (O^{-1} O^t \nabla_{O} u))^{-p} e^{O^t u} dx$$

$$= 2p \int_{\mathbb{R}^n} \log \frac{\|T O^t u\|}{\|\nabla u\|} \left( \frac{\|\nabla u\|^2}{2} \right)^p (u^* (\nabla u))^{-p} e^{-u} dx \quad (3.11)$$

$$= 2p \int_{\mathbb{R}^n} \log \frac{\|T O^t x\|}{\|x\|} \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x).$$
Then we have
\[
\int_{\mathbb{R}^n} \log \left( \frac{\|\phi_1^{-t}x\|^2}{\|x\|^2} \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x)
= 2p \int_{\mathbb{R}^n} \log \left( \frac{\|TO^t x\|}{\|x\|} \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x)
= 2p \int_{\mathbb{R}^n} \log \left( \frac{\|Tx\|}{\|x\|} \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(0, x)
= p \int_{\mathbb{R}^n} \log \left( \sum_{i=1}^n x_i^2 \lambda_i \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(0, x)
\geq p \int_{\mathbb{R}^n} \left[ \sum_{i=1}^n x_i^2 \log \lambda_i \right] \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(0, x)
= 2p \delta_J_p(0, \gamma) \sum_{i=1}^n \log \lambda_i = 0.
\]

Where \( x_i = \langle \frac{x}{\|x\|}, e_i \rangle \). Then we have
\[
\left[ \frac{1}{p \delta_J_p(f, \gamma)} \int_{\mathbb{R}^n} h_{\phi_1}^p(x) d\mu_p(f, x) \right]^{\frac{1}{p}}
\geq \exp \left[ \frac{2}{\delta_J_p(f, \gamma)} \int_{\mathbb{R}^n} \log \left( \frac{\|\phi_1^{-t}x\|}{\|x\|} \right) \left( \frac{\|x\|^2}{2} \right)^p d\mu_p(f, x) \right]^{\frac{1}{p}}
\geq 1.
\]

The first inequality in (3.13) is a consequence of the Jensen’s inequality, with equality holds if and only if there exist a constant \( c > 0 \) such that \( \|\phi_1^{-t}x\| = c \) for all \( x \in \mathbb{R}^n \).

Moreover, note that from the strict concave of the log-concave function that equality in (3.12) is possible only if \( x_1 \cdots x_n \neq 0 \) which implies \( \lambda_1 = \cdots = \lambda_n \), for \( x \in \mathbb{R}^n \). Thus \( \|Tx\| = \lambda_i \) when \( x_i \neq 0 \). Now equality in (3.13) would force \( \|\phi_1^{-t}x\| = c \) or equivalently \( \|T^{-t}x\| = c \) for \( x \in \mathbb{R}^n \), so that \( \lambda_i = c \) for all \( i \). This together with the fact \( \lambda_1 \cdots \lambda_n = 1 \) shows that equality in (3.13) would imply \( T = I \) and hence \( \phi_1 = I \).

**Theorem 3.7.** Let \( f \in A_0 \). Then problem \( S_p \) has a unique solution. Moreover a Gaussian function \( \gamma_\phi \) solves \( S_p \) if and only if it satisfies
\[
\left( \frac{c_n}{J(\gamma_\phi)} \right)^{\frac{2}{p}} \delta_J_p(f, \gamma_\phi) h_\gamma(\sqrt{\alpha} \phi^{-t}x)(y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h_\gamma(\sqrt{\alpha} \phi^{-t}y)^{p-1} d\mu_p(f, x).
\]

for any \( y \in \mathbb{R}^n \).

**Proof.** By computation shows \( \alpha^p \cdot_p e^{-v} = e^{-\alpha v(\frac{1}{\alpha})} \), then \( \alpha^p \cdot_p \gamma_\phi = \gamma_{\phi \sqrt{\alpha}} \). So we obtain \( h(\alpha^p \cdot_p \gamma_\phi) = h_\gamma(\sqrt{\alpha} \phi^{-t}x) \).
Together with Lemma 3.2 and Theorem 3.5 we have
\[
\left(\frac{c_n}{J(\gamma_{\phi})}\right)^{\frac{2}{p}} \delta J_p(f, \gamma_{\phi}) h_{\gamma}(\sqrt{\alpha_{\phi}}^{-1} y) = \frac{n}{4p} \int_{\mathbb{R}^n} |\langle x, y \rangle|^2 h_{\gamma}(\sqrt{\alpha_{\phi}}^{-1} y)^{p-1} d\mu_p(f, x).
\]
□

Now we define \(L_p\) John ellipsoid for log-concave functions.

**Definition 3.1.** Let \(f \in A_0\) be log-concave function, the unique Gaussian function that solves the constrained maximization problem
\[
\max J(\phi \gamma) \quad \text{subject to} \quad \delta J_p(f, \phi \gamma) \leq 1,
\]
(3.15)
is called the \(L_p\) John ellipsoid of \(f\) and denoted by \(E_p f\). The unique Gaussian function that solves the constrained minimization problem
\[
\min \delta J_p(f, \phi \gamma) \quad \text{subject to} \quad \frac{J(\phi \gamma)}{c_n} = 1,
\]
(3.16)
is called the normalized \(L_p\) John ellipsoid of \(f\) and denoted by \(\overline{E}_p f\).

Specially, if we take \(f = e^{-\|x\|^2/2}\) for \(K \in \mathcal{K}_0^n\), since the Gaussian function \(\gamma_{\phi}\) can be viewed as \(\gamma_{\phi} = e^{-\|\phi x\|^2/2}\), where \(E\) is the origin-centered ellipsoid. Then Definition 3.1 deduce the definition of \(L_p\) John ellipsoid defined in [46].

Since \(\overline{\delta} J_p(\phi f, g) = \overline{\delta} J_p(f, \phi^{-1} g)\) for \(\phi \in GL(n)\), then we have the following result.

**Proposition 3.8.** Let \(f \in A_0\). Then
\[
E_p(T f) = T(E_p f).
\]
(3.17)

**Proof.** By the definition of problem \(S_p\), set \(E_p f = \gamma_{\phi}\), since \(\overline{\delta} J_p(f, \gamma_{\phi}) = 1\). By the Lemma 2.8, we have
\[
\overline{\delta} J_p(T f, T \gamma_{\phi}) = \overline{\delta} J_p(f, T^{-1} T \gamma_{\phi}) = \overline{\delta} J_p(f, \gamma_{\phi}) = 1.
\]
(3.18)

By the uniqueness of the problem \(S_p\), we have \(E_p(T f) = T(E_p f)\). So we complete the proof. □

If we take \(f = \gamma\), then we have \(E_p \gamma = \gamma\). Moreover by Proposition 3.8, we have the following.

**Corollary 3.9.** Let \(\gamma_{\phi} = e^{-\frac{\|\phi x\|^2}{2}}\), for \(\phi \in GL(n)\), then
\[
E_p(\gamma_{\phi}) = \gamma_{\phi}.
\]

4. Continuity

In this section, we will show that the family of \(L_p\) John ellipsoids for log-concave functions is continuous in \(p \in [1, \infty)\). First let \(f = e^{-u} \in A_0\), note that if \(\overline{E}_p f\) is a solution of problem \(\overline{S}_p\), then there exist a \(\overline{\phi} \in SL(n)\) such that \(\overline{E}_p f = \gamma_{\overline{\phi}} = e^{-\frac{\|\overline{\phi} x\|^2}{2}}\).
Theorem 4.1. Let $f$ and $\{f_i\}$ in $A_0$, such that $\lim_{i \to \infty} f_i = f$ on $\mathbb{R}^n$. Then

$$\lim_{i \to \infty} E_p f_i = E_p f. \quad (4.1)$$

To prove Theorem 4.1, we need the following Theorem.

First we prove that $d\mu_p(f_i, x) \to d\mu_p(f, x)$

Lemma 4.2. Let $f = e^{-u}$, $f_i = e^{-u_i} \in A_0$, if $f_i \to f$, then $d\mu_p(f_i, x) \to d\mu_p(f, x)$.

Proof. We only need to prove that, for any function $g \in L^1(\mu_p(f, ))$,

$$\lim_{i \to \infty} \int_{\mathbb{R}^n} g(x) d\mu_p(f_i, x) = \int_{\mathbb{R}^n} g(x) d\mu_p(f, x). \quad (4.2)$$

Set $a = \max\{|g(x)| : x \in \mathbb{R}^n\}$, $b = \max\{|f(x)| : x \in \mathbb{R}^n\}$, $b_i = \max\{|f_i(x)| : x \in \mathbb{R}^n\}$, $c = \max\{|h_f(x)| : x \in \mathbb{R}^n\}$. Note that $f$ and $f_i \in A_0$ are integrable, and $h_{f_i} \to h_f$ whenever $f_i \to f$, then there exist an $N_1 \in \mathbb{N}$ such that $|f_i - f| < \frac{a}{2b} \epsilon$ for $i \geq N_1$, and $N_2 \in \mathbb{N}$ such that $|h_{f_i} - h_f| < \frac{a}{2b} \epsilon$ for $i \geq N_2$. Since $d\mu_p(f, x) = h_f^{-p} f dx$, so we can choose $i \geq \max\{N_1, N_2\}$, then

$$\left| \int_{\mathbb{R}^n} g d\mu_p(f_i, x) - \int_{\mathbb{R}^n} g d\mu_p(f, x) \right| \leq \int_{\mathbb{R}^n} |g h_f^{p_i} f_i - g h_f^p f| dx,$$

$$\leq \int_{\mathbb{R}^n} |g h_f^{p_i} f_i - g h_f^p f| dx + \int_{\mathbb{R}^n} |g f| h_f^{p_i} f_i - h_f^p f| dx,$$

$$\leq \epsilon.$$

So we complete the proof. \hfill \Box

Theorem 4.3. Suppose $f = e^{-u}$, $f_i = e^{-u_i}$, $g = r^{-v}$, $g_j = e^{-v_j} \in A_0$, where $i, j \in \mathbb{N}$. If $f_i \to f$, $g_i \to g$, then

$$\lim_{i,j \to \infty} \delta J_p(f_i, g_j) = \delta J_p(f, g). \quad (4.3)$$

Proof. Since $f_i \to f$, $g_i \to g$, by the Definition of $\delta J_p(f, g)$, we have

$$\left| \int_{\mathbb{R}^n} h_g^{p_i} d\mu_p(f_i, x) - \int_{\mathbb{R}^n} h_g^p d\mu_p(f, x) \right| \leq \int_{\mathbb{R}^n} |h_g^{p_i} f_i - h_g^p f| d\mu_p(f, x). \quad (4.4)$$

Note that $h_g$ and $h_g_i$ are bounded, set $c_i = \max\{|g_i(x)| : x \in \mathbb{R}^n\}$ and $c_0 = \max\{|g(x)| : x \in \mathbb{R}^n\}$, then $c_i$, $c_0$ are bounded. Let $c = \sum_{j=1}^{p-1} c_i^j c_0^{p-1-j}$ and $|h_g_i - h_g| \leq \frac{\epsilon}{2}$, then

$$\int_{\mathbb{R}^n} |h_g^{p_i} - h_g^p| d\mu_p(f, x) \leq \frac{\epsilon}{2}. \quad (4.5)$$

By Lemma 4.2, we can show that

$$\left| \int_{\mathbb{R}^n} h_g^{p_i} d\mu_p(f_i, x) - \int_{\mathbb{R}^n} h_g^{p_i} d\mu_p(f, x) \right| \leq \frac{\epsilon}{2}. \quad (4.6)$$
So together with formulas (4.4), (4.5) and (4.6) we can choose unified $N$ such that

$$\left| \int_{\mathbb{R}^n} h_y^p d\mu_p(f_i, x) - \int_{\mathbb{R}^n} h_y^p d\mu_p(f, x) \right| \leq \epsilon.$$ 

We complete the proof. □

Now we give a proof of Theorem 4.1

**Proof of Theorem 4.1.** By Theorem 4.3, and $J(f_i^\circ) \to J(f^\circ)$ when $i \to \infty$, then we have

$$\lim_{i \to \infty} \delta J_p(f_i, E_p f_i) = \lim_{i \to \infty} \min_{J(\gamma_\phi) = \epsilon_n} \delta J_p(f_i, \gamma_\phi) = \min_{J(\gamma_\phi) = \epsilon_n} \delta J_p(f, \gamma_\phi) = \delta J_p(f, \overline{E_p f}).$$

We complete the proof. □

Note that $E_p f$ is bounded for $p \in [0, \infty]$. Thus in order to establish the continuity of $E_p f$ in $p \in [0, \infty]$, (2.24) shows that the $\delta J_p(K, \cdot)$ is continuity of $p \in [1, \infty]$.

**Lemma 4.4.** If $p_0 \in [1, \infty)$, then

$$\lim_{p \to p_0} \delta J_p(f, \gamma_\phi) = \delta J_{p_0}(f, \gamma_\phi). \tag{4.7}$$

for some $\phi \in SL(n)$.

**Theorem 4.5.** Let $f = e^{-u} \in A_0$, and $1 \leq p \leq q < \infty$, $E_p f$ be the solution of constrained maximization problem, then

$$J(E_{\infty} f) \leq J(E_q f) \leq J(E_p f) \leq J(E_1 f). \tag{4.8}$$

**Proof.** The definition of $\delta J_p(f, g)$, together with Jensen’s inequality, for $1 \leq p \leq q < \infty$, we have

$$\delta J_p(f, \gamma) = \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} \left( \frac{h_y}{h_f} \right)^p h_f d\mu_p(f, x) \right]^{\frac{1}{p}} \leq \left[ \frac{1}{J(f^\circ)} \int_{\mathbb{R}^n} \left( \frac{h_y}{h_f} \right)^q h_f d\mu_p(f, x) \right]^{\frac{1}{q}} = \delta J_q(f, \gamma).$$

By Definition 3.1, we have

$$E_q f = \max \left\{ E^f : \delta J_q(f, Ef) \leq 1 \right\} \leq \max \left\{ E^f : \delta J_q(f, Ef) \leq 1 \right\} = E_p f.$$ 

This implies $J(E_q f) \leq J(E_p f)$. For $p \to \infty$, by the definition of (2.22), and the continuity of $p \in [1, \infty]$, we have $E_{\infty} f = \lim_{p \to \infty} E_p f$, we complete the proof. □
Theorem 4.6. Let $f = e^{-u} \in \mathcal{A}_0$, such that $J(f) > 0$. Let $1 \leq p < \infty$, $E_p f$ be the solution of constrained maximization problem, then

$$J(E_p f) \leq J(f).$$

(4.9)

Proof. By the definition 3.1, we have

$$1 = \delta J_p(f, E_p f)^p = \frac{p \cdot \delta J_p(f, E_p f)}{J(f^o)}.$$

The $L_p$ Minkowski inequality for log-concave functions (see [26]) says that

$$\frac{J(f^o)}{p} \geq \delta J_p(f, f) + J(f) \log \frac{J(E_p f)}{J(f)}.$$

This means that

$$J(f) \log \frac{J(E_p f)}{J(f)} \leq 0.$$

Since $J(f) > 0$, then $\log \frac{J(E_p f)}{J(f)} \leq 0$. That means

$$J(E_p f) \leq J(f).$$

□

The functional Blaschke-Santaló inequality, proved for even functions in [11], and given in full generality in [6] says that for log-concave function $f \in \mathcal{A}_0$, the following inequality holds

$$P(f) \leq P(\gamma),$$

that is

$$J(f) J(f^o) \leq J(\gamma)^2 = (2\pi)^n.$$

So combine with the Theorem 4.6, we have the following Theorem.

Theorem 4.7. Let $f = e^{-u} \in \mathcal{A}_0$, such that $J(f) > 0$. Let $1 \leq p < \infty$, $E_p f$ be the solution of constrained maximization problem, then

$$J(E_p f) J(E_p f^o) \leq c_n^2.$$

(4.10)

where $c_n = (2\pi)^{\frac{n}{2}}$.

In the following, denoting by $\triangle_n$ and $B_{\infty}^n$ the regular simplex centered at the origin and the unit cube in $\mathbb{R}^n$. To establish the $L_p$ Ball’s ratio inequality for log-concave function, we need the following results, more details see [5].

Theorem 4.8 ([5]). Let $f \in \mathcal{A}_0$, $Ef$ be the functional John ellipsoid of $f$, then

$$\frac{J(f)}{J(Ef)} \leq \frac{J(g_c)}{J(Eg_c)},$$

(4.11)

where $g_c(x) = e^{-\|x\|_{\triangle_n} - c}$ for any $c \in \triangle_n$. Furthermore, there is equality if and only if $\frac{f}{\|f\|_{\infty}} = Tg_c$ for some affine map $T$ and some $c \in \triangle_n$. If we assume $f$ to be even, then

$$\frac{J(f)}{J(Ef)} \leq \frac{J(g)}{J(Eg)},$$

(4.12)
where \( g(x) = e^{-\|x\|_{\infty}^n} \), with equality if and only if \( \|f\|_{\infty} = Tg \) for some linear map \( T \in GL(n) \).

Moreover, by compute the right hand of the above formulas, it gives

**Lemma 4.9** ([5]). Let \( f \in \mathcal{A}_0 \), \( Ef \) be the functional John ellipsoid of \( f \), then

\[
\frac{J(f)}{J(Ef)} \leq \frac{e(n!)^{\frac{1}{n}}}{|E\Delta_n|} |\Delta_n|,
\]

(4.13)

If we assume \( f \) to be even, then

\[
I.rat(f) \leq \frac{e(n!)^{\frac{1}{n}}}{|E\Delta_n|} |\Delta_n|.
\]

(4.14)

Now we give the Ball’s volume ration inequality for log-concave function.

**Theorem 4.10.** Let \( f = e^{-u} \in \mathcal{A}_0 \), such that \( J(f) > 0 \). Let \( 1 \leq p < \infty \), \( E_p f \) be the solution of constrained maximization problem, then

\[
\frac{J(f)}{J(E_p f)} \leq \frac{n^{\frac{a-d}{n}}(n+1)^{\frac{a+b}{n}}e}{(n!)^{\frac{a}{n}}\omega_n}.
\]

(4.15)

If \( f \) is even then

\[
\frac{J(f)}{J(E_p f)} \leq \frac{e(n!)^{\frac{1}{n}}2^n}{\omega_n}.
\]

(4.16)

where \( \omega_n \) is the volume of ball \( B^n \).

**Proof.** By Theorem 4.5, and the fact \( E_{\infty} f = Ef \), then we have

\[
\frac{J(f)}{J(E_p f)} \leq \frac{e(n!)^{\frac{1}{n}}}{|E\Delta_n|} |\Delta_n|.
\]

On the other hand, note that the volume of \( \Delta_n \) is given by \( |\Delta_n| = \frac{\sqrt{2n+1}}{2\pi(n+1)^n} \), and the inradius of \( \Delta_n \) is given by \( r_{\Delta_n} = \frac{1}{\sqrt{2n(n+1)}} \). Then by a simple computation gives

\[
\frac{J(f)}{J(E_p f)} \leq \frac{e(n!)^{\frac{1}{n}}}{|E\Delta_n|} |\Delta_n| = \frac{n^{\frac{a-d}{n}}(n+1)^{\frac{a+b}{n}}e}{(n!)^{\frac{a}{n}}\omega_n}.
\]

If \( f \) is even, then

\[
\frac{J(f)}{J(E_p f)} \leq \frac{e(n!)^{\frac{1}{n}}}{|J\Delta_n|} |\Delta_n| = \frac{e(n!)^{\frac{1}{n}}2^n}{\omega_n}.
\]

So we complete the proof.

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