Oscillating Effects in the Nambu – Jona-Lasinio Model

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1. INTRODUCTION

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The concept of dynamical chiral symmetry breaking (DCSB) plays an essential role in elementary particle physics and quantum field theory (QFT). In QFT this phenomenon is well observed in Nambu – Jona-Lasinio (NJL) type models – four-dimensional models with four-fermionic interactions. The simplest one is presented in the single fundamental multiplet (one flavor case) of color SU(N) group:

\[ L_\psi = \bar{\psi}_k i \partial_k \psi_k + \frac{G}{2N} (|\bar{\psi}_k \psi_k|^2 + |\bar{\psi}_k i \gamma_5 \psi_k|^2), \tag{1} \]

(here summation over color index \( k = 1, \ldots, N \) is implied). Moreover, \( L_\psi \) is invariant under continuous chiral transformations

\[ \psi_k \rightarrow e^{i \theta \gamma_5} \psi_k; \quad (k = 1, \ldots, N). \tag{2} \]

Since there are no closed physical systems in nature, the influence of different external factors on the DCSB mechanism is of great interest. In these realms, special attentions have been given to analysis of the vacuum structure of the NJL type models at nonzero temperature and chemical potential, in the presence of external (chromo-)magnetic fields, with allowance for curvature and nontrivial space-time topology. Combined action of external electromagnetic and gravitational fields on DCSB effect in four-fermion field theories were investigated in [4,5].

In the present paper we consider the phase structure and related oscillating effects of the four dimensional Nambu – Jona-Lasinio model in two cases: 1) in non-simply connected space-time of the form \( R^3 \times S^1 \) (space coordinate is compactified and the length of the circle \( S^1 \) is \( L \)) with nonzero chemical potential \( \mu \) and 2) in Minkowski space-time at nonzero values of \( \mu, H \), where \( H \) is the external magnetic field.

**NJL model at \( \mu \neq 0 \).** First of all let us prepare the basis for investigations in following sections and consider phase structure of the model (1) at \( \mu \neq 0 \) in Minkowski space-time.

Recall some well-known vacuum properties of the theory (1) at \( \mu = 0 \). The introduction of an auxiliary Lagrangian

\[ \bar{L} = \bar{\psi} i \partial_\psi - \bar{\psi} (\sigma_1 + i \sigma_2 \gamma_5) \psi - \frac{N}{2G} (\sigma_1^2 + \sigma_2^2) \tag{3} \]

greatly facilitates the problem under consideration. (In (3) and other formulae below we have omitted the fermionic color index \( k \) for simplicity.) On the equations of motion for auxiliary bosonic fields \( \sigma_{1,2} \) the theory (3) is equivalent to the (1) one.

From (3) one can find in the leading order of \( 1/N \) - expansion the effective potential (which is the same as in 1-loop approximation) of the model at \( \mu = 0 \):

\[ \frac{1}{N} V_0(\Sigma) = \frac{\Sigma^2}{2G} - \frac{1}{16\pi^2} \left\{ \Lambda^4 \ln \left( 1 + \frac{\Sigma^2}{\Lambda^2} \right) + \Lambda^2 \Sigma^2 - \Sigma^4 \ln \left( 1 + \frac{\Lambda^2}{\Sigma^2} \right) \right\}, \tag{4} \]

where \( \Lambda \) is the UV cutoff parameter, \( \Sigma = \sqrt{\sigma_1^2 + \sigma_2^2} \).

Now one can easily see that at \( G < G_c = 4\pi^2/\Lambda^2 \) the global minimum point (GMP) of the potential (4) equals to zero. Hence, in this case fermions are massless, and chiral invariance (2) is not broken. If \( G > G_c \), then the GMP of (4) is \( \Sigma_0(G,\Lambda) \neq 0 \). This mean that spontaneous breaking of the symmetry (2) takes place. Moreover, fermions acquire mass \( M \equiv \Sigma_0(G,\Lambda) \).

Let us now imagine that \( \mu > 0 \). In this case the effective potential \( V_\mu(\Sigma) \) looks like [3]:

\[ V_\mu(\Sigma) = V_0(\Sigma) - \frac{N \theta(\mu - \Sigma)}{16\pi^2} \left\{ \frac{10}{3} \mu (\mu^2 - \Sigma^2)^{3/2} - 2\mu^3 \sqrt{\mu^2 - \Sigma^2} + \Sigma^4 \ln \left( \mu + \sqrt{\mu^2 - \Sigma^2} \right)^2/\Sigma^2 \right\}. \tag{5} \]
where \( \theta(x) \) is the step function. It follows from (5) that in the case \( G < G_c \) and at arbitrary values of chemical potential chiral symmetry (2) is not broken. However, at \( G > G_c \) the model has a rich phase structure. On the phase portrait one can see critical curves of the second- and first-order phase transitions, respectively. Furthermore, there are two tricritical points, two massive phases with spontaneously broken chiral invariance as well as the symmetric massless phase on the phase portrait of the NJL model (detailed calculations of the vacuum structure of the NJL model one can find in \([1]\)).

2. Oscillations in the \( R^3 \times S^1 \) Space-Time

It is well-known that unified theory of all forces (including gravitation) of the nature is not constructed up to now. Since in early Universe the gravity was sufficiently strong, so one should take it into account, many physicists study quantum field theories in space-times with nontrivial metric and topology. At this, NJL model is the object of special attention (see the review \([11]\)), because the idea of dynamical chiral symmetry breaking is the underlying concept in elementary physics. There is a copious literature on this subject \([1]\), \([12]\). In particular, the investigation of four-fermion theories in the space-time of the form \( R^d \times S^1 \times \cdots \times S^1 \) is of great interest \([12]\). The matter is that such space-time topology occurs in string theories, in description of Casimir type effects and so on.

In the present section the NJL model in the \( R^3 \times S^1 \) space-time and at \( \mu \neq 0 \) is considered, since great amount of physical phenomena take place at nonzero particle density, i.e. at nonzero chemical potential. Here space coordinate is compactified and the circumference \( S^1 \) has the length \( L \). For simplicity we study only the case with periodic boundary conditions: \( \psi(t, x + L, y, z) = \psi(t, x, y, z) \).

In the leading order over \( N \) the effective potential has the following form \([3]\):

\[
V_{\mu L}(\Sigma) = V_L(\Sigma) - \frac{N\lambda}{6\pi} \sum_{n=0}^{\infty} \alpha_n \theta(\mu - \sqrt{\Sigma^2 + (2\pi\lambda n)^2}) \cdot (\mu - \sqrt{\Sigma^2 + (2\pi\lambda n)^2})^2 (\mu + 2\sqrt{\Sigma^2 + (2\pi\lambda n)^2}),
\]

where

\[
V_L(\Sigma) = V_0(\Sigma) - \frac{2N}{\pi^2 L} \int_0^\infty dx x^2 \ln[1 - e^{(-L\sqrt{x^2 + \Sigma^2})}],
\]

\( \alpha_n = 2 - \delta_{n0} \). Investigating potential (6) one can find \([2]\) at \((0.917...)G_c < G < G_c \) on the phase portrait of the model in the plane \((\mu, \lambda)\), where \( \lambda = 1/L \), there are only two massive nonsymmetric phases. In contrast, there are infinitely many massive phases in the NJL theory if \( G_c < G < (1.225...)G_c = G_1 \). In both cases there are also infinitely many symmetric massless phases in the NJL model.

In the case \( G_c < G < G_1 \) the boundary \( \mu_c(\lambda) \) between symmetric and nonsymmetric phases is the critical curve of second order phase transitions. This curve oscillates at \( \lambda \to 0 \), since

\[
\mu_c(\lambda) \approx \frac{2\pi\lambda_0}{\sqrt{6}} \left\{ 1 + \frac{3\lambda^2}{\pi^2 \lambda_0^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi\lambda_0L/\sqrt{6})}{n^2} \right\},
\]

where \( \lambda_0 \) is defined by the relation \( \frac{\pi^2}{2\lambda^2} - \frac{\lambda^2}{8} = -\frac{\pi^2}{16} \lambda_0^2 \).

From (7) it follows that \( \mu_c(\lambda) \) has an oscillating part, which oscillates at \( L \to \infty \) with frequency \( \lambda_0/(2\sqrt{6}) \).

Some other physical parameters of the NJL system such as particle density, fermionic condensate and pressure also oscillate at \( L \to \infty \). Let us consider in detail oscillations of pressure and particle density in the ground state at \( \mu > \mu_c(0) \).

This constraint means that for sufficiently large values of \( L \) the global minimum point of the effective potential equals to zero. In this case the thermodynamic potential (TDP) \( \Omega(\mu, L) \) of the model equals to \( V_{\mu L}(0) \). Hence, using (6) we have

\[
\Omega(\mu, L) = V_L(0) - \frac{N\lambda}{6\pi} \sum_{n=0}^{\infty} \alpha_n \theta(\mu - 2\pi\lambda n)(\mu - 2\pi\lambda n)^2(\mu + 4\pi\lambda n) \cdot dx. \]

In order to extract from (8) the oscillating part one should use the following Poisson’s summation formula \([13]\):

\[
\sum_{n=0}^{\infty} \alpha_n \Phi(n) = 2 \sum_{k=0}^{\infty} \alpha_k \int_0^\infty \Phi(x) \cos(2\pi kx) dx. \]

Then, at \( L \to \infty \) the TDP \( \Omega(\mu, \lambda) \) oscillates with frequency \( \mu/(2\pi) \), because it looks like \([3]\):

\[
\Omega(\mu, L) = V_L(0) - \frac{N\mu^4}{12\pi^2} - N \sum_{k=0}^{\infty} \left[ \frac{4\lambda^4}{\pi^2 k^4} - \frac{2\mu^3}{\pi^2 k^4} \sin(\mu k L) - \frac{4\lambda^4}{\pi^2 k^4} \cos(\mu k L) \right].
\]

The pressure \( p \) and the particle density \( n \) in the ground state of the system is defined as \( p = -\partial(\Omega L)/\partial L \) and \( n = -\partial\Omega/\partial \mu \), respectively. Using the above expression for \( \Omega(\mu, L) \), we see that these quantities also oscillates with frequency \( \mu/(2\pi) \).

One can interpret the case under consideration as the ground state of the NJL system, located between two parallel plates with periodic boundary conditions. The force which acts on each of plates is known as generalized Casimir force. Evidently, this force is proportional to the pressure in the ground state of the system. Hence, at nonzero chemical potential the Casimir force of the constrained fermionic vacuum is oscillates at \( L \to \infty \).
Finally, we should like to underline that one can observe oscillations of some physical parameters due to a presence in a phase structure of the NJL model of cascades of massless as well as massive phases.

3. MAGNETIC OSCILLATIONS AT $\mu \neq 0$

In the present section we shall study the magnetic properties of the NJL vacuum. At $\mu = 0$ this problem was considered in $[3]$. It was shown in $[3]$ that at $G > G_c$ the chiral symmetry is spontaneously broken for arbitrary values of external magnetic field $H$, and even for $H = 0$. At $G < G_c$ the NJL system has a symmetric vacuum at $H = 0$. However, if the external (arbitrary small) magnetic field is switched on, then for all $G \in (0, G_c)$ one has a spontaneous breaking of initial symmetry $[8]$. This is a so called effect of dynamical chiral symmetry breaking (DCSB) catalysis by external magnetic field (for the first time this effect was observed in the framework of (2+1)-dimensional Gross - Neveu model in $[3]$ and then was explained in $[4]$).

Now we turn to a more general situation when $H, \mu \neq 0$. All the information about the vacuum structure of the NJL model is contained in the effective potential which in 1-loop approximation and at $H, \mu \neq 0$ has the following form:

$$V_{H\mu}(\Sigma) = V_H(\Sigma) - \frac{NeH}{4\pi^2} \sum_{k=0}^{\infty} \alpha_k \theta(\mu - s_k) \left\{ \mu \sqrt{\mu^2 - s_k^2} - s_k^2 \ln \left[ \left( \mu + \sqrt{\mu^2 - s_k^2} \right)/s_k \right] \right\},$$

(10)

where $s_k = \sqrt{\Sigma^2 + 2eHk}$, $\epsilon$ is the N-th part of the proton electric charge (due to this fact, the expression (10) is only the 1-loop, but is not the leading order over $N$ one for effective potential),

$$V_H(\Sigma) = V_0(\Sigma) - \frac{NeH^2}{2\pi^2} \left\{ \xi(-1, x) - \frac{1}{2} \right\} \ln x + \frac{x^2}{4},$$

$x = \Sigma^2/(2eH)$, $\xi(\nu, x)$ is the generalized Riemann zeta-function,

$[13]$, $\xi(-1, x) = d\xi(\nu, x)/d\nu|_{\nu=-1}$.

At $\mu = 0$ the effective potential (10) reduces to $V_H(\Sigma)$.

The last potential has nonzero global minimum point $\Sigma_0(H)$, such that at $G < G_c$ (only this case we shall study in the present report)

$$\Sigma_0(H) \approx \frac{eH}{\pi} \sqrt{G/12} \at\ H \to \infty,$$

$$\Sigma_0^2(H) \approx \frac{eH}{\pi} \exp\left\{ -\frac{1}{eH} \left( 4\pi^2 - \Lambda^2 \right) \right\} \at\ H \to 0.$$

Therein, $\Sigma_0(H)$ is a monotonically increasing function versus $H$.

Hence, at $G < G_c$ and $H = 0$ the NJL vacuum is chirally symmetric one, but arbitrary small value of external magnetic field $H$ induces the DCSB, and fermions acquire nonzero mass $\Sigma_0(H)$ (the effect of magnetic catalysis of DCSB).

In order to get a phase portrait of the model at $\mu \neq 0$ one should find a one-to-one correspondence between points of $(\mu, H)$-plane and global minimum points of the function (10). It is possible to show that above critical curve

$$\mu_c(H) = \frac{2\pi}{N\sqrt{eH}} \left[ V_H(0) - V_H(\Sigma_0(H)) \right]^{1/2}$$

(11)
one has $(\mu, H)$-points, corresponding to the zero global minimum of the potential (10). If $\mu < \mu_c(H)$ we have nontrivial global minimum, which equals to $\Sigma_0(H)$.

Hence, at $\mu > \mu_c(H)$ ($G < G_c$) there is a region, where the vacuum of the NJL model is symmetric one. The external magnetic field ceases to induce the DCSB at $\mu > \mu_c(H)$ (or at sufficiently small values of magnetic field $H < H_c(\mu)$, where $H_c(\mu)$ is the inverse function to $\mu_c(H)$). But, under the critical curve (11) (or at $H > H_c(\mu)$) due to a presence of external magnetic field the chiral symmetry is spontaneously broken. Here magnetic field induces dynamical fermion mass $\Sigma_0(H)$, which is not $\mu$-dependent value. Indeed, the line (11) is the critical curve of first order phase transitions.

At first sight, properties of the symmetric vacuum are slightly varied, when parameters $\mu$ and $H$ are changed. However, it is not so and in the region $\mu > \mu_c(H)$ we have infinitely many massless symmetric phases of the theory as well as a variety of critical curves of the second order phase transitions. On the experiment this cascade of phases is identified with oscillations of such physical quantities as magnetization and particle density. Let us prove it.

It is well-known that a state of the thermodynamic equilibrium ($\equiv$ the ground state) of arbitrary quantum system is described by the thermodynamic potential (TDP) $\Omega$, which is a value of the effective potential in its global minimum point. In the case under consideration the TDP $\Omega(\mu, H)$ at $\mu > \mu_c(H)$ has the form

$$\Omega(\mu, H) \equiv V_{H\mu}(0) = V_H(0) - \frac{NeH}{4\pi^2} \sum_{k=0}^{\infty} \alpha_k \theta(\mu - \epsilon_k) \cdot$$

$$\cdot \left\{ \mu \sqrt{\mu^2 - \epsilon_k^2} - \epsilon_k \ln \left[ \left( \mu + \sqrt{\mu^2 - \epsilon_k^2} \right)/\epsilon_k \right] \right\},$$

(12)

where $\epsilon_k = \sqrt{2\Sigma^2 Hk}$. We shall use the following criterion of the phase transitions: if at least one first (second) partial derivative of $\Omega(\mu, H)$ is a discontinuous function at some point, then it is a point of the first (second) order phase transition.

Using this criterion one can show that lines $l_k = \{(\mu, H) : \mu = \sqrt{2\xi_k^2 Hk}\}$ ($k = 1, 2, \ldots$), are critical lines of second order phase transitions. Indeed, from (12) it follows that the first derivative $\partial \Omega/\partial \mu$ is a continuous function at all lines $l_k$. However, the second derivative $\partial^2 \Omega/(\partial \mu)^2$ has an infinite jump at each line $l_k$:
\[
\frac{\partial^2 \Omega}{(\partial \mu)^2} \bigg|_{(\mu, H) \to \mu_k^+} - \frac{\partial^2 \Omega}{(\partial \mu)^2} \bigg|_{(\mu, H) \to \mu_k^-} = -\frac{\text{Ne}H\mu}{2\pi^2 \sqrt{\mu^2 - \epsilon_k}} \bigg|_{\mu \to \epsilon_k} \to -\infty.
\]

So, these lines are critical curves of second order phase transitions and we have in the NJL model an infinite set of massless symmetric phases.

As was shown in [3], an infinite cascade of massless phases is the basis for different oscillating phenomena in the NJL model. First of all, using (9) one can present oscillating part for \(\Omega(\mu, H)\) in a manifest analytic form:

\[
\Omega(\mu, H) = \Omega_{\text{mon}}(\mu, H) + \Omega_{\text{osc}}(\mu, H),
\]

where \((\nu = \mu^2/(eH))\). In this formula

\[
\Omega_{\text{mon}} = V_H(0) - \frac{N\mu^4}{12\pi^2} - \frac{N(eH)^2}{4\pi^3} \int_0^\nu \int_{k_1} \cdots \int_{k_n} \frac{1}{k} P(\pi ky),
\]

\[
\Omega_{\text{osc}} = \frac{N\mu}{4\pi^3} \sum_{k=1}^\infty \left( \frac{eH}{\pi k} \right)^{3/2} \left[ Q(\pi ky) \cos(\pi ky + \pi/4) + P(\pi ky) \cos(\pi ky - \pi/4) \right].
\]

Functions \(P(x)\) and \(Q(x)\) entering in these formulæ are connected with Fresnel’s integrals \(C(x)\) - \(S(x)\) [10]:

\[
C(x) = \frac{1}{2} + \sqrt{\frac{x}{2\pi}} P(x) \sin x + Q(x) \cos x,
\]

\[
S(x) = \frac{1}{2} - \sqrt{\frac{x}{2\pi}} P(x) \cos x - Q(x) \sin x.
\]

They have at \(x \to \infty\) following asymptotics [11]:

\[
P(x) = x^{-1} - \frac{3}{4} x^{-3} + \ldots, \quad Q(x) = -\frac{1}{2} x^{-2} + \frac{15}{8} x^{-4} + \ldots
\]

The formula (14) presents exact oscillating part of the TDP (12) for the NJL model at \(G < G_c\). Since in the present case the TDP is proportional to the pressure of the system, one can conclude that pressure in the NJL model oscillates, when \(H \to 0\). It follows from (14) that frequency of oscillations at large values of a parameter \((eH)^{-1}\) equals to \(\mu^2/2\). Then, starting from (14) one can easily find manifest expression for oscillating parts of the particle density \(n = -\partial \Omega/\partial \mu\) and magnetization \(m = -\partial \Omega/\partial H\). These quantities oscillate at \(H \to 0\) with the same frequency \(\mu^2/2\) and have a rather involved form, so we do not present it here.

Magnetic oscillations in the NJL model with two light quarks as well as dependence of dynamical quark mass on the chemical potential at \(G > G_c\) were found in [17].

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