Why Polyhedra Matter in Non-Linear Equation Solving

J. Maurice Rojas

To my sister, Clarissa Amelia, on her 12th birthday.

Abstract. We give an elementary introduction to some recent polyhedral techniques for understanding and solving systems of multivariate polynomial equations. We provide numerous concrete examples and illustrations, and assume no background in algebraic geometry or convex geometry. Highlights include the following:

(1) A completely self-contained proof of an extension of Bernstein’s Theorem. Our extension relates volumes of polytopes with the number of connected components of the complex zero set of a polynomial system, and allows any number of polynomials and/or variables.

(2) A near optimal complexity bound for computing mixed area — a quantity intimately related to counting complex roots in the plane.

1. Introduction

In a perfect world, a scientist or engineer who wishes to solve a system of polynomial equations arising from some important application would simply pick up a book on algebraic geometry, look through the table of contents, and find a well-explained, provably fast algorithm which solves his or her problem. (Algebraic geometry began 2000 years ago as the study of polynomial equations, didn’t it?) He or she would then surf the web to download a good (free) implementation which would run quickly enough to be useful.

Once one stops laughing at how the real world compares, one realizes what is missing: the standard classical algebraic geometry texts (e.g., EGA1, EGA2, EGA3.1, EGA3.2, EGA4.1, EGA4.2, EGA4.3, SGA1, SGA2, SGA3.1, SGA3.2, SGA3.3, SGA4.1, SGA4.2, SGA4.3, SGA4.4, SGA4.5, SGA6, SGA7, Mum95, Mum99, Har77, Sha94, GH94) rarely contain algorithms and none contains a complexity analysis of any algorithm. Furthermore, one soon learns from experience that the specific structure underlying one’s equations is rarely if ever exploited by a general purpose computational algebra package.

This research was partially supported by a grant from the Texas A&M College of Science and NSF Grant DMS-0211458.

1In fairness, it should be noted that the major thrust of 20th century algebraic geometry was understanding the topological nature of zero sets of polynomials, rather than efficiently approximating the location of these zeros.
Considering the ubiquity of polynomial equations in applications such as geometric modelling [Man98, Gol03], control theory [Sus98, NM99], cryptography [Dod01], radar imaging [FH95], learning theory [Vid97, VR02], chemistry [GH99, Gat01], game theory [McL97, Roj97], and kinematics [Can93, EM99] (just to mention a few applications), it then becomes clear that we need an algorithmic theory of algebraic geometry that is practical as well as rigorous. One need only look at the active research in numerical linear algebra (e.g., eigenvalue problems for large sparse matrices) to see how far we are from a completely satisfactory theory for the numerical solution of general systems of multivariate polynomial equations.

More recently, the introduction of algorithmic and combinatorial ideas has invigorated computational algebraic geometry. Here we give an elementary introduction to one recent aspect of computational algebraic geometry: polyhedral methods for solving systems of multivariate polynomial equations. The buzz-word for the cognicenti is **toric varieties** [Ful93, Cox03, Sot03]. However, rather than deriving algorithms from toric variety theory as an afterthought, we will begin directly with concrete examples and see how convex geometry naturally arises from solving equations.

Simply put, polyhedral methods are a first step toward a new class of algorithms which adapt themselves to the intrinsic nature of the underlying system of equations. For the purposes of this paper, this means that our polynomial equations will be expressed as sums of monomial terms, and the techniques we describe will exploit the combinatorial structure of which monomial terms appear.

**Example 1.0.1.** Suppose one has the following 3 equations in 3 unknowns $x$, $y$, and $z$:

\begin{align*}
&c_1 + c_{12}x + c_{13}y + c_{14}z + c_{15}y^2 + c_{16}y^3 + c_{17}y^4 + c_{18}y^5 + c_{19}y^6 + c_{20}y^7 + c_{21}y^8 + c_{22}y^9 + c_{23}y^{10} + c_{24}y^{11} = 0 \\
&c_5 + c_{15}x + c_{16}y + c_{17}z + c_{18}y^2 + c_{19}y^3 + c_{20}y^4 + c_{21}y^5 + c_{22}y^6 + c_{23}y^7 + c_{24}y^8 + c_{25}y^9 + c_{26}y^{10} + c_{27}y^{11} = 0 \\
&c_7 + c_{17}x + c_{18}y + c_{19}z + c_{20}y^2 + c_{21}y^3 + c_{22}y^4 + c_{23}y^5 + c_{24}y^6 + c_{25}y^7 + c_{26}y^8 + c_{27}y^9 + c_{28}y^{10} + c_{29}y^{11} = 0,
\end{align*}

where the $c_{i,j}$ are any given complex numbers. One may reasonably guess that such a system of equations, being neither over-determined or under-determined, will have only finitely many roots $(x, y, z) \in \mathbb{C}^3$ with probability 1, for any continuous probability distribution on the coefficient space $\mathbb{C}^{36}$. In fact, with probability 1, the number of roots will always be the same number (cf. Theorems 3.2.1 and 3.2.2 of Section 3.2). What then is this “generic” number of roots?

Noting that the maximum of the sum of the exponents in any summand of the first, second, or third equation is 28 (i.e., our polynomials each have **total degree** 28), a classical theorem of Bézout [Sha94, Ex. 1, Pg. 198] gives us an upper bound of 21952 = 28^3. A slightly more refined variant which uses degrees with respect to different variables, the **multi-graded** version of Bézout’s Theorem [MS87], yields a sharper upper bound of 6000 = 6 \cdot 10^3.

However, the true generic number of roots is 321. This number was calculated by using the correct concept in our setting: the **convex hull** of the exponent vectors (also known as the **Newton polytopes**) of our polynomials. In this case, all our

---

\[2\] Recall that a set $B \subseteq \mathbb{R}^n$ is convex iff for all $x, y \in B$, the line segment connecting $x$ and $y$ is also contained in $B$. The **convex hull** of $B$, $\text{Conv}(B)$, is then simply the smallest convex set containing $B$, and the computational complexity of convex hulls of finite point sets is fairly well-understood [FS84].
Newton polytopes are identical, and the volume (suitably normalized) of any one serves as the correct generic number of complex roots.

The key idea to keep in mind is that the complexity of solving a system of polynomial equations, or even approximating a single root, depends strongly on the total number of complex roots. Since one does not usually know this number a priori, the algorithm one uses ultimately makes implicit use of some upper bound on this number, usually one of the three we just mentioned. So, much as how our preceding comparison of bounds turned out, algorithms which take monomial term structure into account are preferable over those that do not.

A natural question, especially relevant to geometric modelling, then arises: Is there an analogous theory for systems of equations expressed in other bases? In particular, the systems of equations arising from B-splines are expressed in the so-called Bernstein-Bezier basis which uses sums of terms like $\prod (1 - x_i)^{a_i}$.

The short answer is that an analogous theory for such bases does not yet exist. However, the philosophies of fewnomial theory [Kho91, LRW03, Roj03], not to mention polyhedral methods [Roj94, HS95, Li97, Roj99b, Ver00, EP02, McD02, MR03], are bringing us closer to a theory that can efficiently handle such questions.

We now outline the main results we explain in our paper. The second result below is new, while the first is older. (Related earlier results will be reviewed throughout this paper.) However, we emphasize that the statement and proof of the first result has been considerably simplified, we provide many more illustrations and examples than what is usually found in the earlier literature (e.g., [Ber75, Kus75, Kus76, Ful93, HS95]), and we have made an effort to keep all prerequisites explicit and contained in this paper.

**Notation 1.0.1.** Let $\mathbf{0}$ denote the origin in $\mathbb{R}^n$ and let $e_1, \ldots, e_n$ denote the standard basis vectors $[1, 0, \ldots, 0]$, $[0, 1, \ldots, 0]$ and $[0, \ldots, 0, 1]$ in $\mathbb{R}^n$. Also, for any $B \subseteq \mathbb{R}^n$, let $\text{Conv}(B)$ denote the smallest convex set containing $B$. Also, we let $\text{Vol}(\cdot)$ denote the usual $n$-dimensional volume in $\mathbb{R}^n$, renormalized so that $\text{Vol}(\text{Conv}(0, e_1, \ldots, e_n)) = 1$. Finally, we will abuse notation slightly by setting $\text{Vol}(A) := \text{Vol}(\text{Conv}(A))$ whenever $A$ is a finite subset of $\mathbb{R}^n$.

**Notation 1.0.2.** For any $c \in \mathbb{C}^*: = \mathbb{C} \setminus \{0\}$ and $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and call $cx^a$ a monomial term. Also, for any polynomial of the form $f(x) = \sum_{a \in A} c_a x^a$, we call $\text{Supp}(f) := \{a \mid c_a \neq 0\}$ the support of $f$, and define $\text{Newt}(f) := \text{Conv}(\text{Supp}(f))$ to be the Newton polytope of $f$. We will assume henceforth that $F := (f_1, \ldots, f_k)$ where, for all $i$, $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ and $\text{Supp}(f_i) \subseteq A_i$. We call $F$ a $k \times n$ polynomial system (over $\mathbb{C}$) with support $(A_1, \ldots, A_k)$. Finally, we let $Z_{\mathbb{C}}(F)$ denote the set of $z \in \mathbb{C}^n$ with $f_j(z) = \cdots = f_k(z) = 0$.

**Theorem 1.** (Special Case (full version in Sec. 6)) Following the notation above, the number of connected components of $Z_{\mathbb{C}}(F)$ is no more than $\text{Vol}(B)$, where $B := \{\mathbf{0}, e_1, \ldots, e_n\} \cup \bigcup_{i=1}^k A_i$. In particular, if the number of complex roots of $F$ is finite, then it is no more than $\text{Vol}(B)$.
As might be expected, a sharper estimate on the generic number of complex roots comes at a price: the resulting formula is more difficult to evaluate. However, one can get an explicit and optimal complexity estimate for the case of a pair of bivariate equations.

**Theorem 2.** Following the notation of Theorem 1, suppose \( k = n = 2 \). Then the generic number of complex roots of a polynomial system \( F = (f_1, f_2) \) with support \( (A_1, A_2) \) can be computed within \( O(b \bar{N} + \bar{N} \log \bar{N}) \) bit operations, and takes at least \( \Omega(b \bar{N}) \) bit operations in the worst case, where \( \bar{N} \) denotes the sum of the cardinalities of the \( A_i \), and \( b \) is the maximum bit-length of any coordinate of any \( A_i \).

Numerous examples of our two main theorems will appear as we review some of the background necessary for the applications and proofs of our theorems. Theorem 1 is proved three times: the simplified version above is proved in Section 3 and then two proofs of the full version appear respectively in Sections 4 and 5. The last proof uses the main combinatorial construction detailed in this paper: the **mixed subdivision** of \( n \) polytopes in \( n \) dimensions. Theorem 2 is proved in Section 5 as a simple consequence of mixed subdivisions in the plane.

**2. From Binomial Systems to Volumes of Pyramids**

Perhaps the best and simplest place to begin to understand the connection between polytopes and polynomials is the special case of binomial systems, i.e., polynomial systems where each polynomial has exactly 2 monomial terms. For such systems, there is an immediate connection to linear algebra over the integers.

**Example 2.0.2.** Suppose we want to find all the complex solutions of

\[
\begin{align*}
xy^7z^7w^4 &= c_1 \\
x^6y^4z^9w^6 &= c_2 \\
x^2y^3z^2w^6 &= c_3 \\
x^6y^4z^8w^5 &= c_4
\end{align*}
\]

where the \( c_{i,j} \) are given nonzero complex numbers. Note in particular that this implies that any root of our system must satisfy \( xyzw \neq 0 \). A particularly elegant trick we’ll generalize shortly is the following: Consider the \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
1 & 7 & 7 & 4 \\
6 & 4 & 9 & 6 \\
2 & 3 & 2 & 6 \\
6 & 4 & 8 & 5
\end{pmatrix}
\]

whose \( i^{th} \) row vector is the exponent vector of the \( i^{th} \) equation above. Then multiplying and dividing the equations above is easily seen to be equivalent to performing row operations on \( E \). For example, doing a pivot operation to zero out all but the top entry of the first column of \( E \) is just the computation of the matrix factorization

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-6 & 1 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
-6 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 7 & 7 & 4 \\
6 & 4 & 9 & 6 \\
2 & 3 & 2 & 6 \\
6 & 4 & 8 & 5
\end{pmatrix} = \begin{pmatrix}
1 & -38 & -33 & -18 \\
0 & -11 & -12 & -2 \\
0 & -38 & -34 & -19
\end{pmatrix},
\]

which is in turn equivalent to observing that

Equation 1 is...

\[
x \cdot y^7 \cdot z^7 \cdot w^4 = c_1
\]

(Equation 2)/(Equation 1): is...

\[
y \cdot -38 \cdot z^{33} \cdot w^{-18} = c_1 \cdot -6 \cdot c_2
\]

(Equation 3)/(Equation 1): is...

\[
y \cdot -11 \cdot z^{12} \cdot w^{-2} = c_1 \cdot -2 \cdot c_3
\]

(Equation 4)/(Equation 1): is...

\[
y \cdot -38 \cdot z^{34} \cdot w^{-19} = c_1 \cdot -6 \cdot c_4
\]

Note also that this new binomial system has exactly the same roots as our original system. (This follows easily from the fact that our left-most matrix above is invertible, and the entries of the inverse are all integers.) So we can solve the last 3
equations for \((y, z, w)\) and then substitute into the first equation to solve for \(x\) and be done. \(\diamond\)

Note, however, that if we wish to complete the solution of our example above, we must continue to use row operations on \(E\) that are invertible over the integers\(^3\). This can be done by performing a simple variant of Gauss-Jordan elimination where one uses no divisions. In essence, one uses elementary integer row operations to minimize the absolute value of the entries in a given column, instead of reducing them to zero.

**Example 2.0.3.** Let us consider the lower right 3 × 3 block of our last matrix from our last example. The variant of Gauss-Jordan elimination we propose is easily depicted below.

\[
\begin{pmatrix}
-38 & -33 & -19 \\
-11 & -12 & -2 \\
-38 & -34 & -19
\end{pmatrix} \rightarrow \begin{pmatrix}
-11 & -12 & -18 \\
-38 & -33 & -18 \\
-38 & -34 & -19
\end{pmatrix} \rightarrow \begin{pmatrix}
-11 & -12 & -18 \\
-5 & 3 & -12 \\
-5 & 2 & -13
\end{pmatrix} \rightarrow \text{etc.} \rightarrow \begin{pmatrix}
-1 & -18 & 22 \\
0 & 93 & -122 \\
0 & -1 & -1
\end{pmatrix}
\]

Proceeding with the same strategy on the lower 2 × 2 block of the last matrix, and then continuing recursively, one finally obtains that our system of equations from Example 2.0.3 is equivalent to the following simpler system:

\[
\begin{align*}
x &= u^{62}c^{-3}c_1^{23}c_2^{11}c_3^{-26}c_4^{-1} \\
y &= u^{175}c_1^{8}c_2^{66}c_3^{31}c_4^{75} \\
z &= c_1^{-10}c_2^{38}c_3^{-93}c_4
\end{align*}
\]

Note in particular that the exponent vectors above correspond exactly to the third and first matrices in the identity below:

\[
\begin{pmatrix}
-3 & 23 & 11 & -26 \\
-8 & 66 & 31 & -75 \\
0 & 1 & 0 & -1 \\
-10 & 82 & 38 & -93
\end{pmatrix} = \begin{pmatrix}
1 & 7 & 7 & 4 \\
6 & 4 & 9 & 6 \\
2 & 3 & 2 & 6 \\
6 & 4 & 8 & 5
\end{pmatrix} - \begin{pmatrix}
1 & 0 & 0 & 62 \\
0 & 1 & 0 & 175 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 215
\end{pmatrix}
\]

and that the left-most matrix \(U\) has determinant \(-1\) (and thus the entries of \(U^{-1}\) are all integral). One should of course note that our system is now very easy to solve: since the roots of \(w^d = c\) are \(\{c|^{1/d}, c|^{1/d}e^{2\pi i/d}, \ldots, c|^{1/d}e^{2\pi i(d-1)/d}\}\), we can simply substitute the 215 resulting values for \(w\) into the first, second, and third equations to solve for \(x\), \(y\), and \(z\), thus finding all 215 complex roots of our system. \(\diamond\)

This motivates the following definition from 19th century algebra.

**Definition 2.0.1.** Let \(\mathbb{Z}^{n \times n}\) denote the set of all \(n \times n\) matrices with all entries integral, and let \(\mathbb{GL}_n(\mathbb{Z})\) denote the set of all matrices in \(\mathbb{Z}^{n \times n}\) with determinant \(\pm 1\) (the set of unimodular matrices). Recall that any \(m \times n\) matrix \([u_{ij}]\) with \(u_{ij} = 0\) for all \(i > j\) is called upper triangular. Then, given any \(M \in \mathbb{Z}^{n \times n}\), we call any identity of the form \(UM = H\), with \(H = [h_{ij}] \in \mathbb{Z}^{n \times n}\) upper triangular and \(U \in \mathbb{GL}_n(\mathbb{Z})\), a Hermite factorization of \(M\). Also if, in addition, we have:

1. \(h_{ij} \geq 0\) for all \(i, j\).
2. for all \(i\), if \(j\) is the smallest \(j'\) such that \(h_{ij} \neq 0\) then \(h_{ij} > h_{i'j}\) for all \(i' \leq i\).

then we call \(H\) the Hermite normal form of \(M\). \(\diamond\)

---

\(^3\)While one could simply use rational operations on \(E\), and thus radicals on our equations, this quickly introduces some unpleasant ambiguities regarding choices of \(d\)-th roots. Hence the need for our integrality restriction.

\(^4\)Recall that Gaussian elimination is what one does to reduce a matrix to upper triangular form (see Definition 2.0.1 below), while Gauss-Jordan elimination is what one does to reduce a matrix to diagonal form \(\text{Str98}\).
Lemma 2.0.1. For any $M \in \mathbb{Z}^{n \times n}$, a Hermite factorization can be computed within $O((n + h_M)^{0.375})$ bit operations, where $h_M := \log(2n + \max_{i,j} |m_{ij}|)$ and $M = [m_{ij}]$. Furthermore, the Hermite normal form exists uniquely for $M$, and can also be computed within the preceding bit complexity bound. ■

By extending the tricks from our last examples, we can easily obtain the following lemma.

Lemma 2.0.2. Suppose $a_1, \ldots, a_n \in \mathbb{Z}^n$ and $c_1, \ldots, c_n \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Let $E$ denote the $n \times n$ matrix whose $i$th row is the vector $a_i$. Then the complex roots of the binomial system $F := (x^{a_1} - c_1, \ldots, x^{a_n} - c_n)$ are exactly the complex solutions of the binomial system

$$
\begin{align*}
x_1^{u_{11}} & \cdots & x_n^{u_{1n}} \\
\vdots & & \vdots \\
x_1^{u_{n1}} & \cdots & x_n^{u_{nn}}
\end{align*}
$$

where $|u_{ij}|E = [h_{ij}]$ is any Hermite factorization of $E$. In particular, the complex roots of $F$ can be expressed explicitly as monomials in $\sqrt[n]{c_1}, \ldots, \sqrt[n]{c_n}$, where $h := \prod_{i=1}^n h_{ii}$. ■

Letting $(\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$, we then easily obtain the following corollary.

Definition 2.0.2. Given any $k \times n$ polynomial system $F$, its Jacobian matrix is $\text{Jac}(F) := \left[ \begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{array} \right]$. We then say that a root $\zeta \in \mathbb{C}^n$ of $F$ is degenerate iff $\text{rank} \text{Jac}(F)|_{x=\zeta} < k$, and smooth otherwise. ◦

Corollary 2.0.1. Suppose $F = (f_1, \ldots, f_n)$ is any $n \times n$ binomial system and, for all $i$, $v_i$ is either vector defined by the difference of the exponent vectors of $f_i$. Then $F$ has only finitely many roots in $(\mathbb{C}^*)^n \implies F$ has exactly $|\det M|$ many, where $M$ is the $n \times n$ matrix whose $i$th row is $v_i$. In particular, the last quantity is exactly $\text{Vol}(\{O, v_1, \ldots, v_n\})$.

Also, every root of $F$ in $(\mathbb{C}^*)^n$ is non-degenerate iff $F$ has exactly $\text{Vol}(\{O, v_1, \ldots, v_n\})$ roots in $(\mathbb{C}^*)^n$. Finally, fixing the support of $F$, there is an algebraic hypersurface $\Delta$ in the coefficient space $\mathbb{C}^{2n}$ such that for all coefficient specializations specializations outside of $\Delta$, $F$ has exactly $\text{Vol}(\{O, v_1, \ldots, v_n\})$ roots in $(\mathbb{C}^*)^n$. ■

We illustrate the last portion of our corollary with the following example.

Example 2.0.4. Let us find all $(c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}) \in \mathbb{C}^6$ such that

$$
\begin{align*}
c_{1,1}x^2y^7z^5 &= c_{1,2} \\
c_{2,1}x^4y^{14}z^{12} &= c_{2,2} \\
c_{3,1}x^8y^{10}z^{14} &= c_{3,2}
\end{align*}
$$
has infinitely many solutions in \((\mathbb{C}^*)^3\). In particular, by Lemma 2.0.2, the roots of our system are exactly those of

\[
x^2 y^7 z^5 = \frac{c_{1,2}}{c_{1,1}}
\]

\[
y^{18} z^6 = \left(\frac{c_{1,2}}{c_{1,1}}\right)^4 \left(\frac{c_{3,2}}{c_{3,1}}\right)^{-1}
\]

\[1 = \left(\frac{c_{1,2}}{c_{1,1}}\right)^{-2} \frac{c_{2,2}}{c_{2,1}}
\]

Clearly, then, our system has infinitely many roots in \((\mathbb{C}^*)^3\) iff \(c_{1,2}^2 c_{2,1} = c_{2,2} c_{1,1}^2\) (and no roots whatsoever if \(c_{1,2} c_{2,1} \neq c_{2,2} c_{1,1}^2\)). So in this example, we can take

\[
\Delta = \{(c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}) \in \mathbb{C}^6 \mid c_{1,2}^2 c_{1,2} = c_{2,2} c_{1,1}^2\}.
\]

We conclude this section with a similar result for a slightly more complicated class of polynomial systems.

**Definition 2.0.3.** Let \(F\) be any \(k \times n\) polynomial system with support \(\{A_1, \ldots, A_k\}\). Then we say that \(F\) is of type \((m_1, \ldots, m_k)\) iff \(\# A_i = m_i\) for all \(i\). Also, we say that \(F\) is unmixed iff \(A_1 = \cdots = A_k\). Finally, writing \(f_i(x) = \sum_{a \in A_i} c_{i,a}\) for all \(i\), we say a property \(P\) regarding \(F\) holds generically iff there is an algebraic hypersurface \(H \subset \mathbb{C}^\Sigma_i \# A_i\) such that \((c_{i,a} \mid i \in \{1, \ldots, n\}, a \in A_i) \in \mathbb{C}^\Sigma_i \# A_i \setminus H \implies P\) holds.

**Proposition 2.0.1.** Following the notation above, if \(P_1, \ldots, P_t\) are properties of \(F\) that hold generically, then their conjunction \(P_1 \wedge \cdots \wedge P_t\) holds generically as well.

**Corollary 2.0.2.** Given any unmixed \(n \times n\) polynomial system \(F = (f_1, \ldots, f_n)\) of type \((n + 1, \ldots, n + 1)\), let \(A\) be the support of any \(f_i\). Then \(F\) either has exactly \(\text{Vol}(A)\) roots in \((\mathbb{C}^*)^n\), no roots in \((\mathbb{C}^*)^n\), or infinitely many roots in \((\mathbb{C}^*)^n\). Furthermore, the first possibility holds generically and implies that all the roots of \(F\) in \((\mathbb{C}^*)^n\) are non-degenerate. Finally, however many roots \(F\) has in \((\mathbb{C}^*)^n\), they can always be expressed explicitly as monomials in \(\text{Vol}(A)^\Sigma\)-roots of linear combinations of the coefficients of \(F\) (and possibly some additional free parameters).

**Proof:** By Gauss-Jordan elimination, \(F\) is equivalent to a binomial system (i.e., one considers the monomials of \(F\) as new variables, thus obtaining a linear system that we can place into reduced row echelon form). So by Corollary 2.0.1, and some additional care with the Hermite normal when \(F\) has infinitely many roots, we are done.

Corollary 2.0.2 will be the cornerstone of our proof of the special case of Theorem 2.0.1 where \(k = n\) and \(F\) is unmixed (also known as Kushnirenko’s Theorem). Note in particular that any Newton polytope from a polynomial system as in Corollary 2.0.2, when \(\text{Vol}(A) > 0\), is an \(n\)-simplex in \(\mathbb{R}^n\), i.e., the \(n\)-dimensional analogue of a 3-dimensional pyramid with a triangular base.

### 3. Subdividing Polyhedra and Kushnirenko’s Theorem

Here we prove the following central result which gives a strong connection between polytope volumes and the number of complex roots of polynomial systems.
Theorem 3.0.1 (Kushnirenko’s Theorem). Following the notation of Definition 2.0.3 of Section 3, suppose $F = (f_1, \ldots, f_n)$ is an $n \times n$ polynomial system with $\text{Supp}(f_i) \subseteq A$ for all $i$. Then $F$ has only finitely many roots in $(\mathbb{C}^*)^n \implies F$ has \leq \text{Vol}(A)$ roots in $(\mathbb{C}^*)^n$. Furthermore, for fixed $A$, $F$ generically has exactly $\text{Vol}(A)$ roots in $(\mathbb{C}^*)^n$.

This result is originally due to Anatoly Georievich Kushnirenko [Kus75, Kus76]. So while we certainly claim no originality in our proof, we have strived to simplify the known proofs from the literature and include as much of the necessary background as reasonably possible.

Before laying the technical foundations for our proof, let us first see a concrete illustration of the main ideas. In essence, one proves Kushnirenko’s Theorem by deforming $F$ (preserving the number of roots along the way) into a collection of simpler systems. Making this rigourous and efficient then provides a natural motivation for a new space (containing an embedded copy of $(\mathbb{C}^*)^n$) in which our roots will live.

Example 3.0.5. Consider the special case $n = 2$ with

\begin{align*}
f_1(x, y) &:= -2 + x^2 - 3y + 5x^7y^5 + 4x^6y^7 \\
f_2(x, y) &:= 3 + 2x^2 + y + 4x^7y^5 + 2x^5y^7.
\end{align*}

The Newton polygon boundary and support appear below:

According to Theorem 3.0.1, $F$ either has \leq 35 roots in $(\mathbb{C}^*)^2$ or infinitely many. (The standard and multi-graded Bézout bounds respectively reduce to $169 = 13^2$ and $98 = 2 \cdot 7 \cdot 7$.) The true number of roots for our example turns out to be exactly 35, and these roots are all non-degenerate.

To see why this is so, let us define a toric deformation $\hat{F} := (\hat{f}_1, \hat{f}_2)$ as follows:

\begin{align*}
\hat{f}_1(x, y, t) &:= -2t + x^2 - 3y + 5x^7y^5 + 4x^6y^7t \\
\hat{f}_2(x, y, t) &:= 3t + 2x^2 + y + 4x^7y^5 + 2x^5y^7t,
\end{align*}

and $\hat{A} := \text{Supp}(\hat{f}_1) = \text{Supp}(\hat{f}_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. (We will see later in Lemma 3.1.4 that most choices of powers of $t$ for each monomial would have worked just as well for proving Kushnirenko’s Theorem for our example.) Note that $F(1, x, y) = F(x, y)$. Intuitively, one would expect 2 equations in 3 unknowns to generically define a curve (cf. Theorem 2.2.1 and the Implicit Function Theorem from calculus), and this turns out to be the case for our example. More to the point, the number of roots of $\hat{F}$ in $(\mathbb{C}^*)^2$ is constant for all $t \in \mathbb{C} \setminus \Sigma$, where $\Sigma$ is a finite set not containing 1. So to show that $F$ has 35 roots in $(\mathbb{C}^*)^2$, it suffices to show

\footnote{This crucial fact is elaborated a bit later in this section — specifically, Lemma 3.2.1.}
that the number of roots of $\hat{F}$ in $(\mathbb{C}^*)^2$ is 35 for some suitable fixed $t$. At least initially, this seems no easier than counting the roots of $F$.

The key trick then is to count something else which, for fixed $t$ sufficiently close to 0, is easily provable to be the same as the number of roots of $\hat{F}$ in $(\mathbb{C}^*)^2$. This is where polyhedral subdivisions come into play almost magically. First, note that our new system is still unmixed but now has a 3-dimensional Newton polytope:

Next, note that any root $(x, y, t) \in (\mathbb{C}^*)^3$ of $\hat{F}$ lies on a parametric curve of the form $C_{(x_0, y_0, w)} := \{(s^{w_1} a_1, s^{w_2} a_2, s^{w_3} a_3) \mid s \in \mathbb{C}^*\}$. However, we will see momentarily that the set of $w \in \mathbb{Z}^n$ for which the roots of $F_t$ in $(\mathbb{C}^*)^3$ approach a $C_{(x_0, y_0, w)}$ as $t \to 0$ is dictated by the face structure of $\text{Conv}(\hat{A})$, and all the roots of $F$ in $(\mathbb{C}^*)^3$ approach some $C_{(x_0, y_0, w)}$ as $t \to 0$. ∙

Let us pause now to review some basic convex geometric definitions.

**Definition 3.0.4.** A (closed) half-space (with (inner) normal $\mathbf{a} = (a_1, \ldots, a_n)$), $H_0 \subset \mathbb{R}^n$, is any set of the form $\{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid a_1 y_1 + \cdots + a_n y_n \geq c\}$ for some real number $c$. A polyhedron is any finite intersection of half-spaces. Also, a d-flat in $\mathbb{R}^n$ is any translate of a d-dimensional subspace of $\mathbb{R}^n$. Finally, a (convex) polytope in $\mathbb{R}^n$ is the convex hull of any finite point set in $\mathbb{R}^n$, and an n-simplex is the convex hull of any $n+1$ points which do not lie in an $(n-1)$-flat. ∙

**Definition 3.0.5.** For any $w := (w_1, \ldots, w_n) \in \mathbb{R}^n$ and any compact set $B \subset \mathbb{R}^n$, we let $B^w$ — the face of $B$ with (inner) normal $w$ — be the set of all $y := (y_1, \ldots, y_n) \in B$ minimizing the inner product $w \cdot y := w_1 y_1 + \cdots + w_n y_n$. We call a face $Q$ of $B$ lower (resp. upper) iff the last coordinate of any inner normal of $Q$ is positive (resp. lower). Also, the dimension of a face $Q$ of $B$, $\dim Q$, is the dimension of the smallest flat containing $Q$. Finally, for any d-dimensional polytope, its faces of dimension 0, 1, $d-2$, $d-1$, or $d$ are respectively called vertices, edges, ridges, facets, or improper. ∙

The most important thing we’ll do with polytopes, after taking their faces, is to subdivide them.

**Definition 3.0.6.** A subdivision of a polytope $P$ is a collection of polytopes $\{Q_i\}_{i=1}^N$ called cells satisfying the following conditions.

1. $\bigcup_{i=1}^n Q_i = P$
2. For any $w \in \mathbb{R}^n$ and $j \in \{1, \ldots, N\}$, we have $Q^w_j \in \{Q_i\}_{i=1}^N$.

In particular, if all the $Q_i$ are simplices then we say that $\{Q_i\}_{i=1}^N$ is a triangulation. Finally, we call a cell $Q_j$ of $\{Q_i\}_{i=1}^N$ full-dimensional iff $\dim Q_j = \dim P$. ∙
In particular, it is clear that one way to compute the volume of a polytope is to take any triangulation and add the volumes of all its full-dimensional cells. From basic linear algebra, we know that this reduces to a finite sum of absolute values of determinants of matrices of edge vectors.

**Example 3.0.5 (Continued).** Let us now examine the lower hull of $\hat{A}$, projected onto the $(x, y)$-plane, and its inner lower facet normals.

In particular, the projections of the faces of the lower hull of $\hat{A}$ onto $A$ induce a triangulation $\{Q_i\}$ of $\text{Conv}(A)$.

Picking $w = (1, 2, 2)$ to examine the curves $C(x_0, y_0, w)$, we see that $\hat{F}(s^{w_1}x_0, s^{w_2}y_0, s^{w_3})$ is exactly

$$s^2(-2 + x_0^2 - 3y_0) + \text{Higher Order Terms in } s$$

$$s^2(3 + 2x_0^2 + y_0) + \text{Higher Order Terms in } s.$$  

In particular, the $(x_0, y_0) \in (\mathbb{C}^*)^2$ which tend to a well-defined limit as $s \to 0$ while satisfying $\hat{F}(s^{w_1}x_0, s^{w_2}y_0, s^{w_3}) = 0$ must also satisfy

$$-2 + x_0^2 - 3y_0$$

$$3 + 2x_0^2 + y_0.$$  

(This follows easily from the Implicit Function Theorem upon observing that the roots of $(-2 + x_0^2 - 3y_0, 3 + 2x_0^2 + y_0)$ are all non-degenerate.) So by Corollary 2.0.4 of the last section, there are exactly $\text{Vol}(\{(0, 0), (2, 0), (0, 1)\}) = 2$ such points.

Put another way, the number of $(x_0, y_0) \in (\mathbb{C}^*)^2$ for which $\hat{F}$ has roots in $(\mathbb{C}^*)^3$ approaching $C(x_0, y_0, (1, 2, 2))$ as $t \to 0$ is exactly 2.

Let us call the last system an **initial term** system and observe that its Newton polytope is exactly the cell of $\{Q_i\}$ corresponding to $w = (1, 2, 2)$. Proceeding similarly with the other inner lower facet normals of $\hat{A}$, there are exactly $\text{Vol}(\{(0, 1), (7, 5), (6, 7)\}) = 18$ curves of the form $C(x_0, y_0, (4, -7, 18))$, and exactly $\text{Vol}(\{(2, 0), (0, 1), (7, 5)\}) = 15$ curves of the form $C(x_0, y_0, (0, 0, 1))$, approached by roots.
of $\hat{F}$ in $(\mathbb{C}^*)^3$ as $t \to 0$. Also, the last two initial term systems have Newton polytopes respectively equal to the cell of $\{Q_i\}$ with inner lower facet normal $(4, -7, 18)$ or $(0, 0, 1)$.

To conclude, note that $w$ not a multiple of $(1, 2, 2), (4, -7, 18)$, or $(0, 0, 1) \implies$ the resulting initial term system has a Newton polytope of dimension $\leq 1$. Since $C(x_0, y_0, w) = C(x_0, y_0, \omega w)$ for any $\omega \in \mathbb{Z}$ and $w \in \mathbb{Z}^3$, another application of Corollary 2.0.2 then tells us that we have found all $C(x_0, y_0, w)$ (with $(x_0, y_0) \in (\mathbb{C}^*)^2$ and $w \in \mathbb{Z}^3$) that are approached by roots of $\hat{F}$ in $(\mathbb{C}^*)^3$ as $t \to 0$. Since there are $35 = \text{Vol}(A)$ such curves, and since they don’t intersect at any fixed $t$, this implies that $\hat{F}_t$ has exactly $35$ roots in $(\mathbb{C}^*)^2$ for any $t$ with $|t|$ sufficiently small. So, assuming every root of $\hat{F}$ in $(\mathbb{C}^*)^3$ converges to some $C(x_0, y_0, w)$ as $t \to 0$, $\hat{F}$ has exactly $35$ roots and we are done. $\diamond$

The preceding argument can be made completely general (not to mention rigorous) with just a little more work. In particular, we can prove our last assumption by constructing a space in which the roots of $\hat{F}$ all converge to well-defined limits as $t \to 0$. This is one of the main motivations behind toric varieties, which provide a useful and elegant way to compactify $(\mathbb{C}^*)^n$.

**Remark 3.0.1.** The idea of examining the behavior of the zero set of $F$ along a monomial curve is really not so far-fetched. In many practical computations (see, e.g., [WV91, LI97]), some roots of polynomial systems depending on parameters tend to have unbounded coordinates, even if the parameters are all finite. So monomial curves are a natural approach to formalize what it means for a root to “approach infinity.” $\diamond$

### 3.1. Enter Toric Varieties Corresponding to Point Sets.

Before going further, let us first give a more succinct definition of initial term systems and formalize our constructions of $A$ and $\tilde{F}$.

**Definition 3.1.1.** For any $f \in \mathbb{C}[x_1, \ldots, x_n]$ of the form $\sum_{a \in A} c_a x^a$, let its initial term polynomial with respect to the weight $w$ be $\text{Init}_w(f) := \sum_{a \in A^w} c_a x^a$. $\diamond$

**Definition 3.1.2.** Given any $A \subset \mathbb{Z}^n$, a lifting function for $A$ is any function $\omega : A \to \mathbb{R}^n$ and we let $A := \{(a, \omega(a)) \mid a \in A\}$. Also, letting $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the natural projection which forgets the last coordinate, we call $A_\omega := \{\text{Conv}(\pi(\tilde{A}^w)) \mid w \in \mathbb{R}^n \setminus \{0\}\}$ the subdivision of $\text{Conv}(A)$ induced by $\omega$. Finally, we say that $\omega$ is a generic lifting iff $A_\omega$ is a triangulation of $\text{Conv}(A)$. $\diamond$

**Definition 3.1.3.** Following the notation above, if we have in addition that $\omega(A) \subset \mathbb{Z}^n$, then for any polynomial $f(x) = \sum_{a \in A} c_a x^a$, it’s lift with respect to $\omega$ is the polynomial $\hat{f}(x, t) := \sum_{a \in A} c_a x^a t^{\omega(a)}$. Finally, the lift with respect to $(\omega_1, \ldots, \omega_n)$ of a $k \times n$ polynomial system $F := (f_1, \ldots, f_k)$ is simply $\hat{F} := (\hat{f}_1, \ldots, \hat{f}_k)$, where $\hat{f}_i$ is the lift of $f_i$ with respect to $\omega_i$ for all $i$. $\diamond$

**Lemma 3.1.1.** Following the notation of Definition 3.1.3, we have that for any fixed $A$, generic lifting functions occur generically. More precisely, there is a finite union, $H_A$, of proper flats in $\mathbb{R}^A$ such that $\omega(A) \in \mathbb{R}^A \setminus H \implies \omega$ is a generic lifting for $A$. $\blacksquare$
We will now refine the approach of Example 3.0.5 as follows: After building \( \hat{A} \) and \( \hat{F} \) via a generic lifting function, we will build a new point set \( \tilde{A} \) and a space \( Y_\tilde{A} \) with the following properties:

1. \( Y_\tilde{A} \) is compact
2. There is an \( h \)-to-1 map from \((\mathbb{C}^*)^n\) to a dense open subset of \( Y_\tilde{A} \), for some positive integer \( h \).
3. \( \hat{F} \) has a well-defined complex zero set \( \tilde{Z} \in Y_\tilde{A} \).
4. There is a natural map \( \pi: Y_\tilde{A} \to \mathbb{P}^1_\mathbb{C} \), where \( \mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{\infty\} \) is the usual complex projective line, such that for all \( t_0 \in \mathbb{C}^* \), \( h\#(\pi^{-1}(t_0) \cap \tilde{Z}) \) is exactly the number of roots of \( \hat{F} \) in \((\mathbb{C}^*)^n\) with \( t \)-coordinate \( t_0 \).

Our proof of Kushnirenko’s Theorem will then focus instead on (a) showing that \( \#(\pi^{-1}(1) \cap \tilde{Z}) = \#(\pi^{-1}(0) \cap \tilde{Z}) \) generically, and (b) computing \( \#\pi^{-1}(t) \) at \( t = 0 \) to avoid the use of limits. We’ve actually already seen an example of (b), from an elementary point of view, in Example 3.0.5 of the last section. So let us now elaborate the framework needed for (a).

**Definition 3.1.4.** Let \( N := \#A \) and let

\[
\mathbb{P}^N_\mathbb{C} := \{[x_0 : \cdots : x_N] \mid x_0, \ldots, x_N \in \mathbb{C} \text{ not all } 0\}
\]

denote complex \( N \)-dimensional projective space. Then, given any finite subset \( A \subset \mathbb{Z}^n \), we let \( \varphi_A: (\mathbb{C}^*)^n \to \mathbb{P}^{N-1}_\mathbb{C} \) — the **generalized Veronese map** — be the map defined by \( x \mapsto [x^{a_1} : \cdots : x^{a_n}] \), where \( \{a_1, \ldots, a_N\} = A \). We then let \( Y_A \) — the **toric variety corresponding to the point set \( A \)** — denote the closure of \( \varphi_A((\mathbb{C}^*)^n) \) in \( \mathbb{P}^{N-1}_\mathbb{C} \). \( \blacksquare \)

Being a closed subset of a compact space, we thus see that \( Y_A \) is compact as a topological space and this will be important later for guaranteeing that certain limits of curves exist. However, one may wonder if \( Y_A \) actually compactifies \((\mathbb{C}^*)^n\) in any reasonable way and what the closure above really means. Here’s one way to make this precise.

**Lemma 3.1.2.** Following the notation of Definition 3.1.4, let \( [a_1, \ldots, a_n] = a_i \) for all \( i \). Also let \( E \) (resp. \( \hat{E} \)) be the \( N \times n \) (resp. \( N \times (n+1) \)) matrix whose \( i \)-th row is \((a_{i1}, \ldots, a_{in})\) (resp. \((a_{i1}, \ldots, a_{in}, 1)\)). Finally, let \( H \) be the Hermite normal form of \( E \), let \( \hat{U}E = H \) be any Hermite factorization of \( E \), and let \( \bar{u}_i \) (resp. \( h \)) denote the \( i \)-th row of \( \hat{U} \) (resp. the product of the diagonal elements of \( H \)). Then \( Y_A = \{[p_1 : \cdots : p_N] \in \mathbb{P}^{N-1}_\mathbb{C} \mid p^{\bar{u}_r+1} = p^{\bar{u}_{r+1}}, \ldots, p^{\bar{u}_N} = p^{\bar{u}_{N+r}} \} \), where \( r \) is the rank of \( H \) and, for all \( i \), \( \bar{u}_i^+ - \bar{u}_i^- = \bar{u}_i \) and \( \bar{u}_i^+ \in \mathbb{R}^+_0 \). Furthermore, \( \varphi_A \) is an \( h \)-to-1 map, i.e., \( \#\varphi_A^{-1}(p) = h \) for all \( p \in \varphi_A((\mathbb{C}^*)^n) \). \( \blacksquare \)

The \( p_i \) above are sometimes called **toric coordinates**. The proof of Lemma 3.1.2 is a routine application of the Hermite normal form we introduced in the last section. Let us see an example of \( Y_A \) now.

**Example 3.1.1.** Taking \( A \) as in our last example, we obtain

\[
\varphi_A(x, y) = [1 : x^2 : y : x^7y^5 : x^6y^7], \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \\ 7 & 5 & 1 \\ 6 & 7 & 1 \end{bmatrix}, \quad \text{and} \quad \hat{E} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \\ 7 & 5 & 1 \end{bmatrix}.
\]
Using the \texttt{ihermite} command in Maple, we then easily obtain that \( H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \) is the Hermite normal form for \( E \) and \( \bar{E} = \begin{bmatrix} 7 & -3 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 15 & -7 & -10 & 20 \end{bmatrix} \) is a Hermite factorization for \( \bar{E} \). So Lemma 3.1.4 tells us that our \( Y_A \) here can also be defined as the zero set in \( \mathbb{P}^4_\mathbb{C} \) of the following collection of binomials:
\[
(p_1^{15} p_4^2 - p_2^7 p_3^{10}, p_1 p_5^9 - p_2^3 p_3^{7})
\]
(Note, for instance, that \( 1^{15}(x^7 y^5)^2 - (x^2)^7 y^{10} = x^{14} y^{10} - x^{14} y^{10} = 0 \).) Furthermore, since \( h = 1 \cdot 1 \cdot 1 = 1 \), our map \( \varphi_A \) here is thus a bijection. \( \diamond \)

The most relevant combinatorial/geometric properties of \( Y_A \) can be summarized as follows (see the companion tutorials [Cox03, Sot03] in this volume for other aspects and points of view).

**Definition 3.1.5.** Following the notation above, let \( N = \# A \) and \( \{a_1, \ldots, a_N\} = A \). Then, for any face \( Q \) of \( \text{Conv}(A) \), the orbit \( O_Q \) is the subset
\[
\{[p_1 : \cdots : p_N] \in Y_A \mid p_i = 0 \implies a_i \notin Q\}.
\]
Also, for any \( p \in O_Q \) with \( Q \) a proper face, we say that \( p \) lies at toric infinity.

Finally, given any \( f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n] \) of the form \( f_i(x) = \sum_{a \in A} c_{i,a} x^a \) for all \( i \), the zero set of \( F = (f_1, \ldots, f_k) \) in \( Y_A \) is simply the set of all \([p_1 : \cdots : p_N] \in Y_A \) with \( \sum_{j=1}^N c_{i,a} p_j = 0 \) for all \( i \). \( \diamond \)

Just as a polytope can be expressed as a disjoint union of the relative interiors of its faces, \( Y_A \) can always be expressed as disjoint union of the \( O_Q \). The lemma below follows routinely from Lemma 3.1.2 and Definition 3.1.3. Recall that a (complex) algebraic set is simply a subset of \( \mathbb{C}^N \) or \( \mathbb{P}^N_\mathbb{C} \) defined by the zero set of a system of polynomial equations.

**Lemma 3.1.3.** Following the notation of Definition 3.1.3, \( O_Q \) is a dense open subset of a \( d \)-dimensional algebraic subset of \( \mathbb{P}^{N-1}_\mathbb{C} \), where \( d = \dim Q \). Also, \( Y_A \) is the disjoint union
\[
\bigsqcup_{Q \text{ a face of Conv}(A)} O_Q
\]
and
\[
Y_A \setminus \varphi_A ((\mathbb{C}^*)^N) = \bigsqcup_{Q \text{ a proper face of Conv}(A)} O_Q.
\]
Finally, \( F \) has a root in \( O_{\text{Conv}(A)} = \text{Init}_w(F) \) has a root in \( (\mathbb{C}^*)^N \). \( \blacksquare \)

Since all faces of \( \text{Conv}(A) \) have a well-defined inner normal, Lemma 3.1.3 thus gives a complete characterization of when a root of \( F \) lies at toric infinity, as well as which piece of toric infinity. This is what will allow us to replace the cumbersome curves \( C(x_0, y_0, w) \) mentioned earlier in Example 3.0.3 with a single algebraic curve in \( Y_A \).

### 3.2. The Smooth Case of Kushnirenko’s Theorem.

Let us now review the last tool we’ll need to start our proof of Kushnirenko’s Theorem: Simplified characterizations of the \textbf{A-discriminant} and \textbf{Cayley trick}, and some basic facts on algebraic curves.
Definition 3.2.1. Given any \( k \times n \) polynomial system \( F \), the toric Jacobian matrix of \( F \) is \( \text{ToricJac}(F) = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \). Assuming \( F \) is unmixed, we then say that \( F \) has a degenerate root at \( p = [p_1 : \cdots : p_n] \in Y_A \) if \( p \) is a root of \( F \) and rank ToricJac(\( F \))\( |_{p} < k \). We then let the discriminant variety, \( \Delta(A_1, \ldots, A_k) \), denote the set of all \((c_1, a_1, \ldots, c_{1,a_N}) \times \cdots \times (c_k, a_1, \ldots, c_{k,a_N}) \in (\mathbb{C}^N)^k \) such that \( F \) has a degenerate root in \( Y_A \). Finally, given any \( k \)-tuple of point sets from \( \mathbb{R}^n \), \((A_1, \ldots, A_k)\), its Cayley configuration is the point set \( \text{Cay}(A_1, \ldots, A_k) := (A_1 \times \{0, \ldots, 0\}) \cup (A_2 \times e_{n+1}) \cup \cdots \cup (A_k \times e_{n+k-1}) \subset \mathbb{R}^{n+k-1} \).

Example 3.2.1. Returning to Example 3.0.5 one last time, consider the root \( q = [1 : \sqrt{-1} : -1 : 0 : 0] \in Y_A \) of

\[
\begin{align*}
 f_1(x,y) &= -2t + x^2 - 3y + 5x^7y^5 + 4x^6y^7t \\
 f_2(x,y) &= 3t + 2x^2 + y + 4x^7y^5 + 2x^6y^7t,
\end{align*}
\]

where \( \hat{A} := \text{Supp}(\hat{f}_1) = \text{Supp}(\hat{f}_2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & 7 \end{bmatrix} \). Note in particular that \( q \in O_{1,2,3} \) and thus lies at the portion of toric infinity corresponding to the smallest triangular cell of \( A_\omega \). The toric Jacobian matrix, in toric coordinates, is then

\[
\begin{bmatrix}
 2p_2 + 35p_1 + 24p_5 \\
 4p_2 + 28p_4 + 12p_5 \\
 p_3 + 20p_4 + 14p_5 \\
 3p_1 + 2p_5
\end{bmatrix}
\]

Evaluating at \( q \), our matrix then becomes

\[
\begin{bmatrix}
 2\sqrt{-1} & 3 & -2 \\
 4\sqrt{-1} & -1 & 3
\end{bmatrix}
\]

which clearly has rank 2, so \( q \) is a non-degenerate root.

Theorem 3.2.1. Following the notation of Definition 3.2.1, there is a homogeneous polynomial \( D_A \in \mathbb{C}[c_{a_1}, \ldots, c_{a_N}] \) such that

\[
 D_A(c_{a_1}, \ldots, c_{a_N}) \neq 0 \iff (c_{a_1}, \ldots, c_{a_N}) \notin \Delta_A,
\]

i.e., \( \Delta_A \) is always contained in an algebraic hypersurface in \( \mathbb{C}^N \). In particular,

\[
 D_{\text{Cay}}(A_1, \ldots, A_k) \neq 0 \iff (c_{1,a_1}, \ldots, c_{1,a_N}) \times \cdots \times (c_{k,a_1}, \ldots, c_{k,a_N}) \notin \Delta_{A_k}.
\]

Corollary 3.2.1. Suppose \( F \) is an \( n \times n \) polynomial system with support \((A_1, \ldots, A_n)\) and that there is an \((n-1)\)-flat containing translates of \( A_1, \ldots, A_n \). Then for fixed \((A_1, \ldots, A_n)\), \( F \) generically has no roots in \((\mathbb{C}^*)^n \). In particular, in the unmixed case, \( F \) generically has no roots in \( Y_A \).

Obtaining the discriminant of a system of equations via the discriminant of a single larger equation via the Cayley configuration is sometimes called the Cayley trick [GKZ94]. Theorem 3.2.1 can actually be derived directly from our framework here via the toric resultant (see, e.g., [Mou02] in this volume or [EP02]). However, for the sake of brevity we omit the proof. The final additional fact we’ll need follows easily from the Implicit Function Theorem.
DEFINITION 3.2.2. If \( X \subseteq \mathbb{P}^N_{\mathbb{C}} \) and \( Y \subseteq \mathbb{P}^N_{\mathbb{C}} \) are algebraic curves, then a morphism \( \psi : X \rightarrow Y \) is simply a well-defined map of the form \( [p_1 : \cdots : p_N] \rightarrow [\phi_1(p_1, \ldots, p_N) : \cdots : \phi_N(p_1, \ldots, p_N)] \), where \( \phi_1, \ldots, \phi_N \) are homogeneous polynomials of the same degree. \( \square \)

LEMMA 3.2.1. Suppose \( C \subseteq \mathbb{P}^N_{\mathbb{C}} \) is a smooth algebraic curve (not necessarily connected) and \( \psi : C \rightarrow \mathbb{P}^1_{\mathbb{C}} \) is any morphism. Then either \( \#\psi(X) < \infty \) or \( \psi(X) = \mathbb{P}^1_{\mathbb{C}} \). In the latter case, there is a positive integer \( m \) and a finite set \( \text{Crit}_\psi \subseteq \mathbb{P}^1_{\mathbb{C}} \), the critical values of \( \psi \), such that \( \#\psi^{-1}(t) = m \iff t \in \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma \).

Finally, in the special case where \( C \) is the zero set in \( Y_{\tilde{A}} \) of an \( n \times (n + 1) \) polynomial system \( \tilde{F}(x_1, \ldots, x_n, t) \) with \( \text{Supp}(\tilde{f}) \subseteq \tilde{A} \) for all \( i, \tilde{A} \subseteq \tilde{A} \), and \( \psi(\varphi_\tilde{A}(x_1, \ldots, x_n, t)) = [1 : t] \) for all \( t \in \mathbb{C}^* \), we have that \( t_0 \in \mathbb{C} \) lies in \( \text{Crit}_\psi \iff (\tilde{F}, t - t_0) \) has a degenerate root in \( Y_{\tilde{A}} \).

THEOREM 3.2.2. Following the notation of Theorem 2.0.4, fix \( A \). Then \( F \) generically has exactly \( \text{Vol}(A) \) roots in \( (\mathbb{C}^*)^n \), all of which are non-degenerate.

Proof of Theorem 3.2.2: Let \( N := \#A \) and \( \{a_1, \ldots, a_N\} := A \) as before and pick any generic lifting function \( \omega \) with integral range. Following the notation of Definition 3.1.2, let us then define \( \tilde{A} := A \cup \{a, \omega(a) + 1 \mid a \in A \} \) and order the coordinates \( p = [p_1 : \cdots : p_{2N}] \) of \( \mathbb{P}^N_{\mathbb{C}} \) so that

\[
\varphi_{\tilde{A}}(x, t) = [x^{a_1} \omega(a_1) : \cdots : x^{a_N} \omega(a_N) : x^1 \omega(a_1) + 1 : \cdots : x^{a_N} \omega(a_N) + 1].
\]

Note in particular that \( \tilde{F} \) has a well-defined zero set in \( Y_{\tilde{A}} \), as well as \( Y_{\tilde{A}} \), since the coordinates \( p_1, \ldots, p_N \) of \( Y_{\tilde{A}} \) can be identified with an obvious subset of the coordinates of \( Y_{\tilde{A}} \).

We can now at last define our promised map \( \pi : Y_{\tilde{A}} \rightarrow \mathbb{P}^1_{\mathbb{C}} \) by \( p \mapsto [p_1 : p_{N+1}] \). Defining \( \tilde{Z} \) (resp. \( Z \)) to be the zero set of \( \tilde{F} \) in \( Y_{\tilde{A}} \) (resp. \( F \) in \( Y_A \)), note that there is an isomorphism (an algebraic bijection) between \( \pi^{-1}(1) \cap \tilde{Z} \) and \( Z \) defined by \( [p_1 : \cdots : p_{2N}] \leftrightarrow [p_1 : \cdots : p_N] \). (This is easily checked since Lemma 3.1.2 tells us that the binomials that define \( Y_{\tilde{A}} \) are exactly those defining \( Y_A \) union \( \langle p_{N+1} p_2 - p_{N+2} p_1, \cdots, p_{N+1} p_{N+1} - p_1 p_{N+2} \rangle \).

Now note that \( \pi \) also induces a natural morphism from \( \tilde{Z} \) to \( \mathbb{P}^1_{\mathbb{C}} \). Let \( H \) be the Hermite normal form of \( A \) and \( h \) the product of the diagonal elements of \( H \). Since the first \( n \) columns of the Hermite normal forms of \( A \) and \( \tilde{A} \) are the same, Lemma 3.1.2 then tells us that the number of roots of \( F \) is exactly \( h\#(\pi^{-1}(1) \cap \tilde{Z}) \). By applying Theorem 3.2.3 to \( (A, \ldots, A) \) (and Proposition 2.0.1) it thus suffices to show that \( h\#(\pi^{-1}(1) \cap \tilde{Z}) = \text{Vol}(A) \) generically.

Next, note that by construction, all the initial term systems of \( F \) will be unmixed and have Newton polytopes of volume 0. In particular, by Corollary 3.2.1 any particular initial term system will generically have no roots. Similarly, by Corollary 2.0.2 the initial term systems of \( \tilde{F} \) will have each have smooth zero set generically. So by Lemma 3.1.3 and Proposition 2.0.1, it will be generically true that \( F \) will have no roots at toric infinity in \( Y_A \), and all the roots of \( \tilde{F} \) at toric infinity in \( Y_{\tilde{A}} \) will be non-degenerate. Furthermore, by applying Theorem 3.2.1 to \( (A, \ldots, A) \), we know that \( \tilde{Z} \) is generically smooth. So by Proposition 2.0.3 again, it thus suffices to show that \( \tilde{Z}, Z, \) and \( \tilde{Z} \cap (\text{Toric Infinity in } Y_{\tilde{A}}) \) smooth and \( Z \subseteq \varphi_{\tilde{A}}((\mathbb{C}^*)^n) \) \( \implies h\#(\pi^{-1}(1) \cap \tilde{Z}) = \text{Vol}(A) \).
So let us now assume the hypothesis of the last implication. By Lemma 3.2.1, \( Z \) (resp. \( \tilde{Z} \cap \pi^{-1}(0) \)) smooth \( \implies \) 1 (resp. 0) is not a critical value of \( \pi \). Also, by the Implicit Function Theorem, the smoothness of \( \tilde{Z} \) implies that \( \pi(Z) \) contains a small open ball about 1. So by the first part of Lemma 3.2.1, \( \pi(\tilde{Z}) = \mathbb{P}^1_k \).

Clearly, \( \mathbb{P}^1_k \) remains path-connected even after a finite set of points is removed, so let \( L \) be any continuous path connecting 0 and 1 in \( \mathbb{P}^1_k \setminus \text{Crit}(\pi|_Z) \). By the Implicit Function Theorem once more, and the fact that \( L \) is compact (by virtue of the compactness of \( \mathbb{P}^1_k \)), we must have that \( \#(\pi^{-1}(L) \cap \tilde{Z}) \) is constant on \( L \). So we now need only show that \( h\#(\pi^{-1}(0) \cap \tilde{Z}) = \text{Vol}(A) \).

To conclude, note that \( \hat{A} \) and \( \hat{A} \) have the same lower hull, so Lemmata 3.1.3 and 3.1.4 then imply that \( \pi^{-1}(0) \cap \tilde{Z} \) is nothing more than
\[
\left\{ [p_1 : \cdots : p_{2N}] \in Y_{\hat{A}} \mid \sum_{a_j \in \Omega} c_{i,a_j} p_j = 0 \text{ for all } i \in \{1, \ldots, n\} \text{ for some cell } Q \text{ of } A_\omega \right\}.
\]
In particular, by our smoothness assumption on \( \pi^{-1}(0) \cap \tilde{Z} \), Corollary 2.0.2 tells us that \( \pi^{-1}(0) \cap \tilde{Z} \) is actually
\[
\left\{ [p_1 : \cdots : p_{2N}] \in Y_{\hat{A}} \mid \sum_{a_j \in \Omega} c_{i,a_j} p_j = 0 \text{ for all } i \in \{1, \ldots, n\} \text{ for some full-dimensional cell } Q \text{ of } A_\omega \right\}.
\]
Since \( A_\omega \) is a triangulation, Corollary 2.0.2 and Lemma 3.1.2 (along with another application Hermite factorization) tells us that
\[
\#\pi^{-1}(0) \cap \tilde{Z} = \sum_{Q \text{ a full-dimensional cell of } A_\omega} \frac{\text{Vol}(Q)}{h} = \frac{\text{Vol}(A)}{h},
\]
so we are done. \( \blacksquare \)

**Remark 3.2.1.** Our proof generalizes quite easily to arbitrary algebraically closed fields and positive characteristic, e.g., the algebraic closure of a finite field. One need only use a little algebra to extend Lemma 3.2.4 to algebraically closed fields and then one can use the same proof above almost verbatim. \( \diamond \)

### 3.3. Path Following, Compactness, and the Degenerate Case of Kushnirenko.
Let us now allow degeneracies for the zero set of \( F \) and arrive at a strengthening of Kushnirenko’s Theorem.

**Theorem 3.3.1.** Following the notation of Theorem 3.0.4, let \( Z_A \) be the zero set of \( F \) in \( Y_A \), and let \( \{Z_i\} \) be the collection of path-connected components of \( Z_A \). Then there is a natural, well-defined positive intersection multiplicity \( \mu : \{Z_i\} \to \mathbb{N} \) such that \( \sum_i \mu(Z_i) = \text{Vol}(A) \) and \( \mu(Z_i) = 1 \) if \( Z_i \) is a non-degenerate root.

We actually have all the technical preliminaries we’ll need, except for one last simple proposition on path-connectedness.

**Proposition 3.3.1.** If \( \mathcal{H} \) is any algebraic hypersurface in \( \mathbb{C}^N \) then \( B \setminus \mathcal{H} \) is path-connected for any open ball \( B \) in \( \mathbb{C}^N \). \( \blacksquare \)

**Proof of Theorem 3.3.1:** Let \( N := \#A \) as usual and let \( P(A) \) denote the space of all polynomials in \( \mathbb{C}[x_1, \ldots, x_n] \) with support contained in \( A \). Clearly, we can naturally identify \( P(A) \) with \( \mathbb{C}^N \). Since zero sets of polynomials are unchanged
under scaling of the coefficients, we will then let $P(A, \ldots, A) := (P^n_{\mathbb{C}})^n$ be the space we’ll use to consider our possible $F$.

Note now that if $F \in (P^n_{\mathbb{C}})^n \setminus \Delta(A, \ldots, A)^n$ then we are done by Theorem 3.2.2 (simply setting $\mu(Z_i) = 1$ for every root $Z_i$). Indeed, since $(P^n_{\mathbb{C}})^n \setminus \Delta(A, \ldots, A)^n$ is path-connected by Proposition 3.3.1, the Implicit Function Theorem tells us that $F$ had better have the same number of roots in $Y_A$ as an $F$ which has smooth zero set and no roots at toric infinity.

Essentially the same idea can be used for $F \in \Delta(A, \ldots, A)^n$. In particular, for such $F$, let $F^{(i)}$ be any sequence such that $F^{(i)} \to F$. Then, letting $Z^{(i)}$ be the zero set of $F^{(i)}$ in $Y_A$, define $\zeta$ to be any limit point of $\{Z^{(i)}\}$. We must then have $F(\zeta) = 0$ and thus $Z_A$ must be non-empty.

Now let $\{U_i\}$ be disjoint open sets with $Z_i \subset U_i$ for all $i$. (Such a set of open sets must exist since $Y_A$ is compact and the $Z_i$ must be of positive distance from each other, using the usual distance in $P^n_{\mathbb{C}}$.) Note then that $Y_A \cup \bigcup_i U_i$ must be compact. By the continuity of $F$ as a function of its variables and coefficients, there must then be a ball $B$ about $F$ in $(P^n_{\mathbb{C}})^n$ such that the roots of any $G \in B \setminus \Delta(A, \ldots, A)^n$ are contained in $\bigcup_i U_i$.

We may now define $\mu(Z_i)$ as follows: Take any $G \in B \setminus \Delta(A, \ldots, A)^n$ and define $\mu(Z_i)$ to be the number of roots of $G$ in $U_i$. Since $B \setminus \Delta(A, \ldots, A)^n$ is path-connected by Proposition 3.3.1, the Implicit Function Theorem tells us that the number of roots is independent of whatever $G \in B \setminus \Delta(A, \ldots, A)^n$ we picked.

**Remark 3.3.1.** Essentially the same argument was used Mike Shub in [Shu93] to prove an extended version of Bézout’s Theorem (see also [BCSS98], Pg. 199). Note that neither theorem generalizes the other, but the theorems overlap in the special case where $A$ is the set of integral points in a scaled standard simplex. On the other hand, our next theorem will generalize both Kushnirenko’s Theorem and Bézout’s Theorem. The elegance of Shub’s approach is that it gives a rigorous and simple approach to intersection theory for a broad class of polynomial systems. ♦

**Remark 3.3.2.** Combining Theorems 3.3.1 and 3.2.2 we immediately obtain our earlier simpler statement of Kushnirenko’s Theorem (Theorem 7.0.7). ♦

4. Multilinearity and Reducing Bernstein’s Theorem to Kushnirenko’s Theorem

The big question now is how to count the roots of a mixed polynomial system, since being unmixed is such a strong restriction. Toward this end, let us consider another consequence of the basic properties of discriminant varieties.

**Lemma 4.0.1.** Let $F$ and $G$ be any $n \times n$ polynomial systems with support contained in $(A_1, \ldots, A_n)$ component-wise. Then, generically, $F$ and $G$ share no roots in $(\mathbb{C}^*)^n$. Furthermore, the number of roots of $F$ is generically a fixed constant. ■
As an immediate consequence, we obtain the following preliminary answer to our big question.

**Definition 4.0.1.** Let \( S_1, \ldots, S_k \) be any subsets of \( \mathbb{R}^n \). Then their **Minkowski sum** is simply \( S_1 + \cdots + S_k := \{ y_1 + \cdots + y_k \mid y_i \in S_i \text{ for all } i \} \).

It is easily proved that \( \text{Newt}(fg) = \text{Newt}(f) + \text{Newt}(g) \) (once one observes that the vertices of \( \text{Newt}(fg) \) are themselves Minkowski sums of vertices of \( \text{Newt}(f) \) and \( \text{Newt}(g) \)). So it should come as no surprise that Minkowski sums will figure importantly in our discussion relating polyhedra and polynomials.

**Lemma 4.0.2.** Let \( \mathcal{N}(A_1, \ldots, A_n) \) denote the generic number of roots in \( (\mathbb{C}^+)^n \) of an \( n \times n \) polynomial system \( F \) with support \( (A_1, \ldots, A_n) \). Then \( \mathcal{N}(A_1, \ldots, A_n) \) is a non-negative symmetric function of \( \text{Conv}(A_1), \ldots, \text{Conv}(A_n) \) which is multilinear with respect to Minkowski sum.

**Proof:** That \( \mathcal{N}(A_1, \ldots, A_n) \) is a well-defined non-negative symmetric function of \( A_1, \ldots, A_n \) is clear (thanks in part to the last part of **Lemma 1.0.1**). The formula for \( \mathcal{N}(A_1, \ldots, A_n) \) in the unmixed case then follows immediately from **Theorem 3.2.2**.

Translation invariance follows easily since the roots of \( F \) in \( (\mathbb{C}^+)^n \) are the same as the roots of \((x^{u_1} f_1, \ldots, x^{u_n} f_n)\) in \((\mathbb{C}^+)^n\). Defining \( x^u := (x_1^{u_1} \cdots x_n^{u_n}) \) for any \( n \times n \) matrix \( [u_{ij}] \), it is then easily checked that \( \text{Supp}(F(x^u)) = (U A_1, \ldots, U A_n) \) and \( U \) **unimodular** (cf. **Definition 2.0.1**). Let \( \mathcal{N}(U A_1, \ldots, U A_n) = \mathcal{N}(A_1, \ldots, A_n) \).

We thus need only show that \( \mathcal{N} \) is a multilinear function of the convex hulls. To see the multilinearity, note that the zero set of \((f_1, f_2, \ldots, f_n)\) in \((\mathbb{C}^+)^n\) is exactly the union of the zero sets of \((f_1, f_2, \ldots, f_n)\) \((f_1, f_2, \ldots, f_n)\). So by the first part of **Lemma 4.0.1** and the symmetry of \( \mathcal{N} \), multilinearity follows.

Recall now the **polarization identity**: 

\[
m(x_1, \ldots, x_n) = \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{n-\# I} \binom{n}{\# I} m \left( \sum_{i \in I} x_i, \ldots, \sum_{i \in I} x_i \right),
\]

valid for any symmetric multilinear function. (The identity is not hard to prove via inclusion-exclusion [GKP94]. See also [Gol03] in this volume for another point of view.) Therefore, we must have 

\[
\mathcal{N}(A_1, \ldots, A_n) = \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{n-\# I} \binom{n}{\# I} \mathcal{N} \left( \sum_{i \in I} A_i, \ldots, \sum_{i \in I} A_i \right),
\]

and thus \( \mathcal{N}(A_1, \ldots, A_n) \) depends only the convex hulls of \( A_1, \ldots, A_n \). \( \blacksquare \)

So we have answered our big question, assuming we know a function \( \mathcal{M}(P_1, \ldots, P_n) \), defined on \( n \)-tuples \((P_1, \ldots, P_n)\) of polytopes in \( \mathbb{R}^n \), that satisfies the obvious analogues of the properties of \( \mathcal{N}(A_1, \ldots, A_n) \) specified in **Lemma 4.0.2**. However, such a function indeed exists: it is called the **mixed volume** and we denote it by \( \mathcal{M}(\cdot) \). Abusing notation slightly by setting \( \mathcal{M}(A_1, \ldots, A_n) := \mathcal{M}(\text{Conv}(A_1), \ldots, \text{Conv}(A_n)) \), we immediately obtain the following result.

**Theorem 4.0.2** (Bernstein’s Theorem). Suppose \( F \) is any \( n \times n \) polynomial system with fixed support \( A_1, \ldots, A_n \). Then \( F \) generically has exactly \( \mathcal{M}(A_1, \ldots, A_n) \) roots in \((\mathbb{C}^+)^n\). 

Of course, we now appear to have an even bigger question: what is mixed volume? This we now answer.

**Remark 4.0.1**

David N. Bernstein was the first to prove Theorem 4.0.2 in a slightly stronger form, along with an algebraic condition for when \( F \) would have exactly mixed volume many roots \[\text{Ber75}\]. Interestingly, Felix Minding appears to have been the first to prove the special case \( n = 2 \) in 1841, and mixed volume wasn’t even defined until near the end of the 19th century by Hermann Minkowski. ☑

We also point out that Minkowski was born on 22 June, 1864, in a town named Alexotas. This town used to belong to what was the Russian empire at the time but is now the Lithuanian city of Kaunas.

### 5. Mixed Subdivisions and Mixed Volumes from Scratch

Let us begin with an illustration of one of the simplest non-trivial examples of a Minkowski sum:

\[
\begin{align*}
\text{red} + \text{green} & = \text{yellow} \\
\end{align*}
\]

There are many different definitions of mixed volume but the two most important use Minkowski sums in an essential way. More to the point, if one can subdivide \( P_1 + \cdots + P_n \) in a special way, then one is well on the way to computing mixed volume. This is where **mixed subdivisions** enter.

**Definition 5.0.2.** [HS95] Given polytopes \( P_1, \ldots, P_k \subset \mathbb{R}^n \), a **subdivision** of \( (P_1, \ldots, P_k) \) is a finite collection of \( k \)-tuples \( \{(C_i^\alpha, \ldots, C_i^\beta)\}_{\alpha \in S} \) satisfying the following axioms:

1. \( \bigcup_{\alpha \in S} C_i^\alpha = P_i \) for all \( i \)
2. \( C_i^\alpha \cap C_i^\beta \) is a face of both \( C_i^\alpha \) and \( C_i^\beta \) for all \( \alpha, \beta, i \).
3. \( C_i^\beta \) a face of \( C_i^\beta \) for all \( i \Rightarrow C_i^\beta, \ldots, C_i^\alpha \) have a common inner normal.

Furthermore, if we have in addition that \( \sum_i \dim C_i^\alpha = \dim \sum_i C_i^\alpha \) for all \( \alpha \), then we call \( \{(C_i^\alpha, \ldots, C_i^\alpha)\}_{\alpha \in S} \) a **mixed subdivision**. ☑
Example 5.0.1.

Here we see a very special kind of subdivision \(\{Q_i\}\) of the Minkowski sum of two polygons \(P_1\) and \(P_2\), each with many vertices. In particular, the subdivision of \(P_1 + P_2\) above is built in such a way as to encode a mixed subdivision \(\{(C_1', C_2')\}\) of \((P_1, P_2)\). In particular, we see that each \(P_i\) has a distinguished vertex \(v_i\), and that we can read off a mixed subdivision of \((P_1, P_2)\) as follows: there are two cells \((P_1, v_2)\) and \((v_1, P_2)\), corresponding to the two cells \(P_1 + v_2\) and \(v_1 + P_2\) of \(\{Q_i\}\). The remaining cells of \(\{(C_1', C_2')\}\) are parallelograms of the form \((E_1, E_2)\) where \(E_i\) is an edge of \(P_i\) for all \(i\).

It is easily verified that any subdivision of \((P_1, \ldots, P_k)\) immediately induces a subdivision of \((\lambda P_1, \ldots, \lambda P_k)\), for any \(\lambda_1, \ldots, \lambda_k \geq 0\).

Example 5.0.2.

Note in particular that the areas of the cells of our induced subdivision of \(\lambda P_1 + \mu P_2\) scale according to their type. In particular, it is clear that for our above
example, \( \text{Area}(\lambda P_1 + \mu P_2) = \text{Area}(P_1)\lambda^2 + M\lambda\mu + \text{Area}(P_2)\mu^2 \), where \( M \) is the sum of the areas of all the mixed cells (the parallelograms).  

**Definition 5.0.3.** Following the notation above, the type of a cell \((C_1^\alpha, \ldots, C_n^\alpha)\) of a subdivision of \((P_1, \ldots, P_n)\) is simply the vector \((\dim C_1^\alpha, \ldots, \dim C_n^\alpha)\). In particular, the cells of type \((1, \ldots, 1)\) are called mixed cells.  

The following two lemmata are then immediate. The first follows from a slight modification of the proof of Lemma 3.1.1, while the second follows almost immediately from the first.

**Lemma 5.0.3.** Following the notation of Definition 3.1.2, recall that

\[
\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n
\]

is the natural projection which forgets the last coordinate. Then, given finite point sets \(A_1, \ldots, A_n \subseteq \mathbb{Z}^n\) and lifting functions \(\omega_i\) for \(A_i\) for all \(i\), the collection \((A_1, \ldots, A_n)\omega := \{w \in \mathbb{R}^n \setminus \{O\} | w \in \mathbb{R}^n \cap \{O\}\} \) always forms a subdivision of \((\dim C_1^\alpha, \ldots, \dim C_n^\alpha)\) — the subdivision of \((\dim C_1^\alpha, \ldots, \dim C_n^\alpha)\) induced by \((\omega_1, \ldots, \omega_n)\). In particular, for fixed \((A_1, \ldots, A_n)\), \((A_1, \ldots, A_n)_\omega\) will generically be a mixed subdivision.

**Lemma 5.0.4.** For \(\lambda_1, \ldots, \lambda_n \geq 0\), and any polytopes \(P_1, \ldots, P_n \subseteq \mathbb{R}^n\), the quantity \(Q(\lambda_1, \ldots, \lambda_n) := \text{Vol}(\sum_{i=1}^n \lambda_i P_i)\) is a homogeneous polynomial of degree \(n\) with nonnegative coefficients.

We then at last arrive at the following definition of the mixed volume.

**Definition 5.0.4.** Given any polytopes \(P_1, \ldots, P_n \subseteq \mathbb{R}^n\), their mixed volume is the coefficient of \(\lambda_1\lambda_2 \cdots \lambda_n\) in the above polynomial \(Q(\lambda_1, \ldots, \lambda_n)\).

**Example 5.0.3 (The Unmixed Case).** It is easily checked that \(M(P, \ldots, P) = \text{Vol}(P)\). Note also that the multilinearity of \(M(\cdot)\) with respect to Minkowski sum also follows immediately from the preceding definition.

**Example 5.0.4 (Line Segments).** It is also easily checked that \(M(\{0, a_1\}, \ldots, \{0, a_n\}) = |\det[a_1, \ldots, a_n]|\), where \(a_1, \ldots, a_n\) are any points in \(\mathbb{R}^n\) and \([a_1, \ldots, a_n]\) is the matrix whose columns are \(a_1, \ldots, a_n\).

The next two characterizations follow easily from the last two lemmata, and inclusion-exclusion [GKP94].

**Lemma 5.0.5.** For any mixed subdivision \(\{(C_1^\alpha, \ldots, C_n^\alpha)\}\) of \((P_1, \ldots, P_n)\),

\[
M(P_1, \ldots, P_n) := \sum_{(C_1^\alpha, \ldots, C_n^\alpha) \text{ a cell of type } (1, \ldots, 1)} \text{Vol} \left( \sum_{i} C_i \right).
\]

Furthermore, we have \(M(P_1, \ldots, P_n) := \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{n-|I|} \text{Vol} \left( \sum_{i \in I} P_i \right)\).

**Example 5.0.5 (Cornered Spikes).** A less trivial puzzle is the following formula:

\[
M(\{0, a_{11} e_1, a_{1n} e_n\}, \ldots, \{0, a_{11} e_1, a_{1n} e_n\}) = \max_{\sigma} \left\{ \prod_{i=1}^n a_{\sigma(i)} \right\}, \text{ where } a_{ij} \text{ are any non-negative real numbers and } \sigma \text{ ranges over all permutations of } \{1, \ldots, n\}.
\]

This gives some indication that mixed volume includes many simple functions as a special
case. However, our next example shows that mixed volume includes rather non-trivial functions as well.

**Example 5.0.6 (Bricks, a.k.a. the fine multigraded case).** Via multilinearity, it easily follows that \( M([0, d_{11}] \times \cdots \times [0, d_{n1}], \ldots, [0, d_{nn}] \times \cdots \times [0, d_{nn}]) = \text{Perm}[d_{ij}], \)
where Perm denotes the permanent. \( \diamond \)

In particular, this immediately shows that computing mixed volume is \( \# \text{P}-hard \) [Pap95, DGH98]. \( \diamond \)

Let us now finally prove Theorem 2.

**Proof of Theorem 2:** Note that by Bernstein’s Theorem, it suffices to find an algorithm for computing \( M(A_1, A_2) \) with bit complexity \( O(bN + N \log N) \). The main idea of the proof can then already be visualized in the first mixed subdivision we illustrated: one computes the mixed area of \((A_1, A_2)\) by first efficiently computing the convex hulls of \(A_1\) and \(A_2\), and then expressing the sum of the areas of the mixed cells compactly without building the entire mixed subdivision. This is not a contradiction, provided one views the mixed cells in the right way.

More precisely, first recall that the convex hulls of \(A_1\) and \(A_2\) can be computed within \( O(N \log N) \) bit operations, via the usual well-known 2-dimensional convex hull algorithms [PS85]. In particular, with this much work, we can already assume we know the inner edge normals of \( P_1 := \text{Conv}(A_1) \) and \( P_2 := \text{Conv}(A_2) \), and the vertices of \( P_1 \) and \( P_2 \) in counter-clockwise order.

Let us then pick a vertices \( v_1 \in P_1 \) and \( v_2 \in P_2 \) such that their angle cones are disjoint. Then there is a mixed subdivision (which we will never calculate explicitly!) with exactly 2 non-mixed cells — \((P_1, v_2)\) and \((v_1, P_2)\) — and several other mixed cells. (This is easily seen by picking a lifting function \( \omega_1 \) for \( P_1 \) that is identically zero, and a linear lifting function \( \omega_2 \) for \( P_2 \) that is minimized at \( v_2 \) and is constant on a line that intersects the angle cones of \( v_1 \) and \( v_2 \) only at the origin.)

Note then that the union of the mixed cells can be partitioned into a union of strips. In particular, by construction, there are disjoint contiguous sequences of edges \((E_{1}^{(i)}, \ldots, E_{a_{i}}^{(i)})\) and \((E_{1}^{(j)}, \ldots, E_{a_{j}}^{(j)})\), with \( E_{1}^{(i)} \) and \( E_{1}^{(j)} \) incident to \( v_i \), for all \( i \). Furthermore, every mixed cell of \( (A_1, A_2)_\omega \) is of the form \((E_{1}^{(i)}, E_{j}^{(2)})\) or \((E_{j}^{(1)}, E_{j}^{(2)})\), and every \( E_{j}^{(i)} \) and \( E_{j}^{(i)} \) is incident to some mixed cell.

The partition into strips then arises as follows: the mixed cells of \( (A_1, A_2)_\omega \) can be partitioned into lists of one of the following two forms:

\[
(E_{j}^{(1)}, E_{m_{1}}^{(2)}) \ldots (E_{1}^{(1)}, E_{n_{1}}^{(2)})
\]

\[
(E_{j}^{(1)}, E_{m_{2}}^{(2)}) \ldots (E_{1}^{(1)}, E_{n_{2}}^{(2)})
\]

where \( j \in \{1, \ldots, a_{1}\} \) (resp. \( j \in \{1, \ldots, a_{2}\} \)), \( m_{1} \leq n_{j} \), and \( n_{j} \leq a_{2} \) (resp. \( n_{j} \leq a_{2} \)). In particular, the union of the mixed cells in any such list is simply the Minkowski sum of a continuous portion of the boundary of \( P_2 \) and an edge of \( P_1 \), and its area can thus be expressed as the absolute value of a determinant of differences of vertices of the \( P_t \). Furthermore, each formula can easily be found by a binary search on the sorted edge normals using \( O(N \log N) \) comparisons.

Since there are no more than \( N \) such strips, the total work we do is bounded above by the specified complexity bound, so our upper bound is proved.

\( \diamond \)Recall that this function can be defined as the variant of the determinant where all alternating signs in the full determinant expansion are replaced by \( +1 \)’s.
To obtain our lower bound, note that the mixed area of \((A_1, A_2)\) is zero iff \([P_1\text{ or } P_2\text{ is a point}]\) or \([P_1\text{ and } P_2\text{ are parallel line segments}]\). So just knowing whether the mixed area is positive or not amounts to a rank computation on a matrix of size \(O(\bar{N})\) and thus can take no less than \(\Omega(bN)\) bit operations in the worst case [BCS97]. ■

6. A Stronger Bernstein Theorem Via Mixed Subdivisions

Here we prove the following generalization of Theorem 3.3.1. It is at this point that we will use a slightly more high-brow type of toric variety: the toric variety \(X_P\) corresponding to a polytope \(P\). In essence, the key properties that we needed from \(Y_A\) (that it compactify \((\mathbb{C}^*)^n\) and have a partition into orbits corresponding to the faces of a polytope) continue to hold for \(X_P\). We make this change to avoid technicalities in defining the zero set of \(F\) in \(Y_A\). Since \(X_P\) is discussed elsewhere in this volume at greater length [Cox03, Sot03], we proceed with the statement of our theorem.

**Theorem 6.0.3.** Following the notation of Theorem 4.0.2, let \(P := \text{Conv}(A_1) + \cdots + \text{Conv}(A_n)\), let \(Z_P\) be the zero set of \(F\) in \(X_P\), and let \(\{Z_i\}\) be the collection of path-connected components of \(Z_P\). Then there is a natural, well-defined positive intersection multiplicity \(\mu : \{Z_i\} \to \mathbb{N}\) such that \(\sum_i \mu(Z_i) = \text{Vol}(A)\) and \(\mu(Z_i) = 1\) if \(Z_i\) is a non-degenerate root.

The proof will be almost exactly the same as that of our extended version of Kushnirenko’s Theorem, so let us first see an illustration of a toric deformation for a mixed system.

**Example 6.0.7.** Take \(n = 2\) and

\[
f_1(x, y) := c_{1,0} + c_{1,0} x^\alpha y^\beta + c_{1,0} x^\alpha y^\beta
\]

\[
f_2(x, y) := c_{2,0} + c_{2,0} x^\gamma + c_{2,0} x^\gamma y^\delta + c_{2,0} x^\gamma y^\delta.
\]

By Bernstein’s Theorem, the number of roots should be \(\alpha \delta + \beta \gamma\), so let us try to prove this.

Let us take the following lifting of \(F\):

\[
\hat{f}_1(x, y, t) := c_{1,0} + c_{1,0} x^\alpha t + c_{1,0} x^\alpha y^\beta t + c_{1,0} x^\alpha y^\beta
\]

\[
\hat{f}_2(x, y) := c_{2,0} + c_{2,0} x^\gamma + c_{2,0} x^\gamma y^\delta + c_{2,0} x^\gamma y^\delta t
\]

In particular, we see that there will be exactly one mixed cell for \((A_1, A_2)\) and its corresponding initial term system will be

\[
\text{Init}_{(0,0,1)}(\hat{F})(x, y, t) = (c_{1,0} + c_{1,0} x^\alpha y^\beta, c_{2,0} + c_{2,0} x^\gamma + c_{2,0} x^\gamma y^\delta)
\]
The idea of our proof of Bernstein’s Theorem then mimics our earlier proof of Kushnirenko’s Theorem: our lifting induces a lifted version \( \hat{P} = \text{Conv}(\hat{A}_1) + \text{Conv}(\hat{A}_2) \) of \( P \) and we’ll then try to build a map from our lifted zero set to the projective line. To do so, we’ll define \( \tilde{P} := \hat{P} \times [0,1] \) and this is illustrated below.

In particular, the only portion of the lower hull of \( \hat{P} \) (i.e., the “lower portion” of toric infinity on \( X_{\hat{P}} \)) which is touched by zero set of \( \hat{F} \) in \( X_{\tilde{P}} \) is the parallelogram facet, and the projection of this facet has area exactly \( \alpha \delta + \beta \gamma \).

**Proof of Theorem 6.0.3** We will first prove the generic case, and then derive the degenerate case, just as we did for the unmixed case.

At this point, we could just use Theorem 4.0.2 to get the generic case and proceed with our proof of the degenerate case. However, let us observe that we could
instead use mixed subdivisions to directly obtain Theorem 4.0.2 without reducing to the unmixed case. The proof proceeds exactly like the proof of Theorem 3.0.1 except for the following differences:

1. We work with $X_{\bar{P}}$ instead of $Y_{\bar{A}}$, where $\bar{P} = (\bar{P}_1 + \cdots + \bar{P}_n) \times [0, 1]$.
2. The map $\pi$ is essentially the same but is instead defined via the Cox coordinate ring [Cox03].
3. The only portions of toric infinity in $X_{\bar{P}}$ that intersect $\pi^{-1}(0) \cap \bar{Z}$ are those corresponding to facets on the lower hull of $\bar{P}$ that project to mixed cells of $(P_1, \ldots, P_n)_\omega$.
4. The final count of roots becomes a sum of roots of a collection of binomial systems.

To prove the degenerate case, we proceed exactly as in the proof of Theorem 3.3.1, except with the following minor modifications:

1. We use the notational changes above.
2. The space of $F$ we work with is instead $\mathbb{P}^{N_i-1}_\mathbb{C} \times \cdots \times \mathbb{P}^{N_n-1}_\mathbb{C}$, where $N_i = \#A_i$ for all $i$. ■

We are now finally ready to state and prove the full version of Theorem 1:

**Theorem 1** (Full Version) Suppose $F$ is any $k \times n$ polynomial system with $\text{Supp}(f_i) \subseteq A_i \subset (\mathbb{N} \cup \{0\})^n$ for all $i$ and let $Z_\mathbb{C}(F)$ denote the zero set of $F$ in $\mathbb{C}^n$. Then the number of connected components of $Z_\mathbb{C}(F)$ is no more than $\text{Vol} \left( \{ (O,e_1,\ldots,e_n) \cup \bigcup_{i=1}^k A_i \} \right) + \mathcal{M}(\{O,e_1\} \cup A_1, \ldots, \{O,e_k\} \cup A_k)$, according as $k < n$ or $k \geq n$.

**Proof:** Let $B := \{O,e_1,\ldots,e_n\} \cup \bigcup_{i=1}^k A_i$. If $k < n$ then we can simply set $f_{k+1} = \cdots = f_n = f_1$ and then apply Theorem 3.3.1, noting that the $\text{Supp}(f_i) \subseteq B$ for all $i$. In particular, it is easily checked that $Y_B$ actually contains an embedded copy of $\mathbb{C}^n$.

To prove the case $k > n$, note that we can reconsider such an $F$ as a $k \times k$ polynomial system with $\text{Supp}(f_i) \subseteq \{0,e_i\} \cup A_i$ for all $i$. Once again, by virtue of the fact that $\bar{P} := \text{Conv}(A_1) + \cdots + \text{Conv}(A_k)$ contains the non-negative orthant as one of the cones in its normal fan [Cox03], we have that our underlying toric variety (this time, $X_{\bar{P}}$) has an embedded copy of $\mathbb{C}^n$. So by Theorem 6.0.3 we’re done. ■

**References**

[Ber75] Bernstein, D. N., “The Number of Roots of a System of Equations,” Functional Analysis and its Applications (translated from Russian), Vol. 9, No. 2, (1975), pp. 183–185.

[BCSS98] Blum, Lenore; Cucker, Felipe; Shub, Mike; and Smale, Steve, *Complexity and Real Computation*, Springer-Verlag, 1998.

[Can88] Canny, John F., “Some Algebraic and Geometric Computations in PSPACE,” Proc. 20th ACM Symp. Theory of Computing, Chicago (1988), ACM Press.

[BCS97] Bürgisser, Peter; Clausen, Michael; and Shokrollahi, M. Amin, *Algebraic complexity theory*, with the collaboration of Thomas Lickteig, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 315, Springer-Verlag, Berlin, 1997.

[Can93] Canny, John F., “Computing roadmaps of general semi-algebraic sets,” Comput. J. 36 (1993), no. 5, pp. 504–514.

[Cox03] Cox, David A., “What is a Toric Variety,” Tutorial for a conference on Algebraic Geometry and Geometric Modelling (Vilnius, Lithuania, July 29 – August 2, 2002), submitted for publication, downloadable from http://www.cs.amherst.edu/~dac/lectures/tutorial.ps.
[CLO92] Cox, David A., Little, John, and O’Shea, Donal, *Ideals, Varieties, and Algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, 1992.

[CLO98] Cox, David A., Little, John, and O’Shea, Donal, *Using Algebraic Geometry*, Graduate Texts in Mathematics 185, Springer-Verlag (1998).

[Csa76] Csanky, L., “Fast Parallel Matrix Inversion Algorithms,” SIAM J. Comput. 5 (1976), no. 4, pp. 618–623.

[EGA1] Dieudonné, Jean and Grothendieck, Alexander, *Éléments de géométrie algébrique I: Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. No. 4, 1960.

[EGA2] *Éléments de géométrie algébrique II: Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. No. 8, 1961.

[EGA3.1] *Éléments de géométrie algébrique III: Étude cohomologique des faisceaux cohérents I*, Inst. Hautes Études Sci. Publ. Math. No. 11, 1961.

[EGA3.2] *Éléments de géométrie algébrique III: Étude cohomologique des faisceaux cohérents II*, Inst. Hautes Études Sci. Publ. Math. No. 17, 1963.

[EGA4.1] *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas I*, Inst. Hautes Études Sci. Publ. Math. No. 20, 1964.

[EGA4.2] *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas II*, Inst. Hautes Études Sci. Publ. Math. No. 24, 1965.

[EGA4.3] *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas III*, Inst. Hautes Études Sci. Publ. Math. No. 28, 1966.

[Dod01] Dodis, Yevgeny, Personal Communication, e-mailed from Courant Institute, New York.

[DGH98] Dyer, Martin; Gritzmann, Peter; and Hufnagel, Alexander, “On the Complexity of Computing Mixed Volumes,” SIAM J. Comput. 27 (1998), no. 2, pp. 356–400.

[DMcK72] Dym, H. and McKean, H. P., *Fourier Series and Integrals*, Probability and Mathematical Statistics, vol. 14, Academic Press, 1972.

[EM99] Emiris, Ioannis Z. and Mourrain, Bernard, “Computer algebra methods for studying and computing molecular conformations,” Algorithmica 25 (1999), no. 2–3, pp. 372–402.

[EP02] Emiris, Ioannis Z. and Pan, Victor Y., “Symbolic and Numeric Methods for Exploiting Structure in Constructing Resultant Matrices,” Journal of Symbolic Computation, Vol. 33, No. 4, April 1, 2002.

[FG05] Forsythe, Keith and Hatke, Gary, “A Polynomial Rooting Algorithm for Direction Finding,” preprint, MIT Lincoln Laboratories, 1995.

[Ful93] Fulton, William, *Introduction to Toric Varieties*, Annals of Mathematics Studies, no. 131, Princeton University Press, Princeton, New Jersey, 1993.

[GK94] Gel’fand, I. M., Kapranov, M. M., and Zelevinsky, A. V., *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[Gol03] Goldman, Ron, “Polar Forms in Geometric Modelling and Algebraic Geometry,” presentation at a conference on Algebraic Geometry and Geometric Modelling (Vilnius, Lithuania, July 29 – August 2, 2002), submitted for publication.

[GKP94] Graham, R. L., Knuth, D. E., and Patashnik, O., *Concrete Mathematics: A Foundation for Computer Science, 2nd edition*, Addison-Wesley, 1994.

[GH94] Griffiths, Phillip and Harris, Joseph, *Principles of Algebraic Geometry*, Reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.
WHY POLYHEDRA MATTER IN NON-LINEAR EQUATION SOLVING

[SGA1] Grothendieck, Alexander, et. al., *Revêtements étals et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois-Marie, 1960–1961, directed by Alexandre Grothendieck, with essays by M. Raynaud, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971.

[SGA2] *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, with an essay by Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2, North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968.

[SGA3.1] *Schémas en groupes, vol. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois-Marie, 1962/1964, directed by M. Demazure and A. Grothendieck, Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin-New York 1962/1964.

[SGA3.2] *Schémas en groupes, vol. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois-Marie, 1962/1964, directed by M. Demazure and A. Grothendieck, Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York 1962/1964.

[SGA3.3] *Schémas en groupes, vol. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois-Marie, 1962/1964, directed by M. Demazure and A. Grothendieck, Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin-New York 1962/1964.

[SGA4.1] *Théorie des topos et cohomologie étale des schémas, vol. 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois-Marie, 1963–1964, directed by M. Artin, A. Grothendieck, and J. L. Verdier, with the collaboration of N. Bourbaki, P. Deligne and B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972.

[SGA4.2] *Théorie des topos et cohomologie étale des schémas, vol. 2*, Séminaire de Géométrie Algébrique du Bois-Marie, 1963–1964, directed by M. Artin, A. Grothendieck, and J. L. Verdier, with the collaboration of N. Bourbaki, P. Deligne and B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin-New York, 1972.

[SGA4.3] *Théorie des topos et cohomologie étale des schémas, vol. 3*, Séminaire de Géométrie Algébrique du Bois-Marie, 1963–1964, directed by M. Artin, A. Grothendieck, and J. L. Verdier, with the collaboration of P. Deligne and B. Saint-Donat, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York, 1972.

[SGA4.12] Deligne, Pierre, *Cohomologie étale*, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4, 12, with the collaboration of J. F. Boutot, A. Grothendieck, L. Illusie and J. L. Verdier, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977.

[SGA5] Grothendieck, Alexander, et. al., *Cohomologie l-adique et fonctions L*, unpublished.

[SGA6] Grothendieck, Alexander, et. al., *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois-Marie, 1966–1967, directed by P. Berthelot, A. Grothendieck, and L. Illusie, with the collaboration of D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud and J. P. Serre, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin-New York, 1971.

[SGA7] *Groupes de monodromie en géométrie algébrique I*, Séminaire de Géométrie Algébrique du Bois-Marie, 1967–1969, directed by A. Grothendieck, with the collaboration of M. Raynaud and D. S. Rim, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, Berlin-New York, 1972.

[HMP] Häggele, Klemens; Morais, Juan Enrique; Pardo, Luis Miguel; Sombra, Martin, “On the Intrinsic Complexity of the Arithmetic Nullstellensatz,” Journal of Pure and Applied Algebra 146 (2000), no. 2, pp. 103–183.

[Har77] Hartshorne, Robin, *Algebraic Geometry,* Graduate Texts in Mathematics, No. 52, Springer-Verlag.

[Hir94] Hirsch, Morris, *Differential Topology,* corrected reprint of the 1976 original, Graduate Texts in Mathematics, 33, Springer-Verlag, New York, 1994.

[HS95] Huber, Birk and Sturmfels, Bernd, “A Polyhedral Method for Solving Sparse Polynomial Systems,” Math. Comp. 64 (1995), no. 212, pp. 1541–1555.
Maurice Rojas

[Ili89] Iliopoulos, Costas S., “Worst Case Complexity Bounds on Algorithms for Computing the Canonical Structure of Finite Abelian Groups and the Hermite and Smith Normal Forms of an Integer Matrix,” SIAM Journal on Computing, 18 (1989), no. 4, pp. 658–669.

[JKSS03] Jeronimo, Gabriela; Krick, Teresa; Sabia, Juan; and Sombra, Martín, “The Computational Complexity of the Chow Form,” Math ArXiv preprint [math.AG/0210009].

[KLS97] Kannan, Ravi; Lovász, László; and Simonovits, Miki, “Random Walks and an $O^*(n^5)$ Volume Algorithm for Convex Bodies,” Random Structures Algorithms, 11 (1997), no. 1, pp. 1–50.

[KM97] Karpinski, Marek and Macintyre, Angus J., “Polynomial bounds for VC dimension of sigmoidal and general Pfaffian neural networks,” J. Comp. Sys. Sci., 54, pp. 169–176, 1997.

[Khe02] Khetan, Amit, “Determinental Formula for the Chow Form of a Toric Surface,” Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC) 2002, ACM Press, to appear.

[Kho91] Fewnomials, AMS Press, Providence, Rhode Island, 1991.

[Koi96] Koiran, Pascal, “Hilbert’s Nullstellensatz is in the Polynomial Hierarchy,” DIMACS Technical Report 96-27, July 1996. (Note: This preprint considerably improves the published version which appeared in Journal of Complexity in 1996.)

[Koi97] “Randomized and Deterministic Algorithms for the Dimension of Algebraic Varieties,” Proceedings of the 38th Annual IEEE Computer Society Conference on Foundations of Computer Science (FOCS), Oct. 20–22, 1997, ACM Press.

[KP01] Krick, Teresa; Pardo, Luis Miguel; and Sombra, Martin, “Sharp Arithmetic Nullstellensatz,” Duke Mathematical Journal 109 (2001), no. 3, pp. 521–598.

[Kus75] Kushnirenko, Anatoly Georgievich, “A Newton Polytope and the Number of Solutions of a System of $k$ Equations in $k$ Unknowns,” Usp. Matem. Nauk., 30, no. 2, pp. 266–267 (1975).

[Kus76] “Newton Polytopes and the Bézout Theorem,” Functional Analysis and its Applications (translated from Russian), vol. 10, no. 3, July–September (1976), pp. 82–83.

[Lec00] Lecerf, Grégoire, “Computing an Equidimensional Decomposition of an Algebraic Variety by Means of Geometric Resolutions,” Proceedings of the International Symposium on Symbolic Algebra and Computation (ISSAC) 2000.

[Li97] Li, Tien Yien, “Numerical solution of multivariate polynomial systems by homotopy continuation methods,” Acta numerica, 1997, pp. 399–436, Acta Numer., 6, Cambridge Univ. Press, Cambridge, 1997.

[LRW03] Li, Tien-Yien; Rojas, J. Maurice; and Wang, Xiaoshen, “Counting Real Connected Components of Trinomial Curves Intersections and $m$-nomial Hypersurfaces,” Discrete and Computational Geometry, to appear.

[LiW91] Li, Tien Yien and Wang, Xiaoshen, “Solving deficient polynomial systems with homotopies which keep the subschemes at infinity invariant,” Math. Comp. 56 (1991), no. 194, pp. 693–710.

[MR03] Malajovich, Gregorio and Rojas, J. Maurice, “High Probability Analysis of the Condition Number of Sparse Polynomial Systems,” submitted for publication, also available as math ArXiv preprint [math.NA/0212179].

[Man98] Manocha, Dinesh, “Numerical Methods for Solving Polynomial Equations,” Applications of Computational Algebraic Geometry (San Diego, CA, 1997), pp. 41–66, Proc. Sympos. Appl. Math., 53, Amer. Math. Soc., Providence, RI, 1998.

[McD02] McDonald, John, “Fractional power series solutions for systems of equations,” Discrete Comput. Geom. 27 (2002), no. 4, pp. 501–529.

[McL97] McLennan, Andrew, “The maximal number of regular totally mixed Nash equilibria,” J. Econom. Theory 72 (1997), no. 2, pp. 411–425.

[MS87] Morgan, Alexander and Sommese, Andrew, “A homotopy for solving general polynomial systems that respects $m$-homogeneous structures,” Appl. Math. Comput. 24 (1987), no. 2, pp. 101–113.

[Mou02] Mourrain, Bernard, “Results,” presentation at a conference on Algebraic Geometry and Geometric Modelling (Vilnius, Lithuania, July 29 – August 2, 2002), submitted for publication.

[MP98] Mourrain, Bernard and Pan, Victor, “Asymptotic Acceleration of Solving Multivariate Polynomial Systems of Equations,” Proc. STOC ’98, pp. 488–496, ACM Press, 1998.
WHY POLYHEDRA MATTER IN NON-LINEAR EQUATION SOLVING  

[174x690]WHY POLYHEDRA MATTER IN NON-LINEAR EQUATION SOLVING  

[127x666]Mumford, David, \textit{Algebraic Geometry I: Complex Projective Varieties}, reprint of the 1976 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.  

[Mum95] Mumford, David, \textit{The red book of varieties and schemes}, second, expanded edition, includes the Michigan lectures (1974) on curves and their Jacobians, with contributions by Enrico Arbarello, Lecture Notes in Mathematics, 1358, Springer-Verlag, Berlin, 1999.  

[142x656][NR96] Neff, C. Andrew and Reif, John, “An Efficient Algorithm for the Complex Roots Problem,” Journal of Complexity 12 (1996), no. 2, pp. 81–115.  

[142x626][NM99] Nešić, D. and Mareels, Ivan M. Y., “Controllability of structured polynomial systems,” IEEE Trans. Automat. Control 44 (1999), no. 4, pp. 761–764.  

[Pap95] Papadimitriou, Christos H., \textit{Computational Complexity}, Addison-Wesley, 1995.  

[Pla84] Plaisted, David A., “New NP-Hard and NP-Complete Polynomial and Integer Divisibility Problems,” Theoret. Comput. Sci. 31 (1984), no. 1–2, 125–138.  

[PS85] Preparata, Franco P. and Shamos, Michael Ian, \textit{Computational Geometry: An Introduction}, Texts and Monographs in Computer Science, Springer-Verlag, New York-Berlin, 1985.  

[Roj94] Rojas, J. Maurice, “A Convex Geometric Approach to Counting the Roots of a Polynomial System,” Theoretical Computer Science (1994), vol. 133 (1), pp. 125–138. (Additional notes and corrections available on-line at http://www.math.tamu.edu/~rojas/list2.html.)  

[Roj97] Rojas, J. Maurice, “A New Approach to Counting Nash Equilibria,” Proceedings of the IEEE/IAFE Conference on Computational Intelligence for Financial Engineering, Manhattan, New York, March 23–25, 1997, pp. 130–136.  

[Roj99a] “Toric Intersection Theory for Affine Root Counting,” Journal of Pure and Applied Algebra, vol. 136, no. 1, March, 1999, pp. 67–100.  

[Roj99b] “Solving Degenerate Sparse Polynomial Systems Faster,” Journal of Symbolic Computation, vol. 28 (special issue on elimination theory), no. 1/2, July and August 1999, pp. 155–186.  

[Roj00a] “Algebraic Geometry Over Four Rings and the Frontier to Tractability,” Contemporary Mathematics, vol. 270, Proceedings of a Conference on Hilbert’s Tenth Problem and Related Subjects (University of Gent, November 1–5, 1999), edited by Jan Denef, Leonard Lipschitz, Thanasas Pheidas, and Jan Van Geel, pp. 275–321, AMS Press (2000).  

[Roj00b] “Some Speed-Ups and Speed Limits for Real Algebraic Geometry,” Journal of Complexity, FoCM 1999 special issue, vol. 16, no. 3 (sept. 2000), pp. 552–571.  

[Roj01] “Computational Arithmetic Geometry I. Sentences Nearly in the Polynomial Hierarchy,” J. Comput. System Sci., STOC ’99 special issue, vol. 62, no. 2, march 2001, pp. 216–235.  

[Roj02] “Additive Complexity and the Roots of Polynomials Over Number Fields and \textit{p}-adic Fields,” Proceedings of the 5th Annual Algorithmic Number Theory Symposium (ANTS V), Lecture Notes in Computer Science #2369, pp. 506–515, Springer-Verlag (2002).  

[Roj03] “Arithmetic Multivariate Descartes’ Rule,” Math ArXiV preprint \texttt{math.NT/0110322}, submitted for publication.  

[RY02] Rojas, J. M. and Ye, Yinyu, “On Solving Fewnomials Over an Interval in Fewnomial Time,” submitted for publication, also available as Math ArXiV preprint \texttt{math.NA/0106225}.  

[Sha94] Shafarevich, Igor R., \textit{Basic Algebraic Geometry I}, second edition, Springer-Verlag (1994).  

[Shu93] Shub, Mike, “Some Remarks on Bézout’s Theorem and Complexity Theory,” From Topology to Computation: Proceedings of the Smalefest (Berkeley, 1990), pp. 443–455, Springer-Verlag, 1993.  

[Sma00] Smale, Steve, “Mathematical Problems for the Next Century,” Mathematics: Frontiers and Perspectives, pp. 271–294, Amer. Math. Soc., Providence, RI, 2000.  

[Smi61] Smith, H. J. S., “On Systems of Integer Equations and Congruences,” Philos. Trans. 151, pp. 293–326 (1861).  

[Sot03] Sottile, Frank, “Toric Ideals, Real Toric Varieties, and the Moment Map,” presentation at a conference on Algebraic Geometry and Geometric Modelling (Vilnius, Lithuania, July 29 – August 2, 2002), submitted for publication, also available as Math ArXiV preprint \texttt{math.AG/0212044}.  

[Str98] Strang, Gilbert, \textit{Introduction to Linear Algebra}, Wellesley-Cambridge Press, 1998.  

[Sus98] Sussmann, Héctor J., “Some Optimal Control Applications of Real-Analytic Stratifications and Desingularization,” Singularities Symposium — Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996), 211–232, Banach Center Publ., 44,  


[Ver00] Verschelde, Jan, “Toric Newton method for polynomial homotopies,” Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998), J. Symbolic Comput. 29 (2000), no. 4–5, pp. 777–793.

[Vid97] Vidyasagar, M., A theory of learning and generalization, With applications to neural networks and control systems. Communications and Control Engineering Series, Springer-Verlag London, Ltd., London, 1997.

[VR02] Vidyasagar, M. and Rojas, J. Maurice, “An Improved Bound on the VC-Dimension of Neural Networks with Polynomial Activation Functions,” submitted for publication.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, TAMU 3368, COLLEGE STATION, TEXAS 77843-3368, USA.

E-mail address: rojas@math.tamu.edu http://www.math.tamu.edu/~rojas