SPOCC: Scalable POssibilistic Classifier Combination - toward robust aggregation of classifiers

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Abstract

We investigate a problem in which each member of a group of learners is trained separately to solve the same classification task. Each learner has access to a training dataset (possibly with overlap across learners) but each trained classifier can be evaluated on a validation dataset.

We propose a new approach to aggregate the learner predictions in the possibility theory framework. For each classifier prediction, we build a possibility distribution assessing how likely the classifier prediction is correct using frequentist probabilities estimated on the validation set. The possibility distributions are aggregated using an adaptive t-norm that can accommodate dependency and poor accuracy of the classifier predictions. We prove that the proposed approach possesses a number of desirable classifier combination robustness properties.

keywords: robust classifier combination, aggregation, fusion, possibility theory

1 Introduction

Classification is a supervised machine learning task consisting of assigning objects (inputs) to discrete categories (classes). When several predictors have been trained to solve the same classification task, a second level of algorithmic procedure is necessary to reconcile the classifier predictions and deliver a single one. Such a procedure is known as classifier combination, fusion or aggregation. When each individual classifier is trained using the same training algorithm (but under different circumstances) the aggregation procedure is referred to as an ensemble method. When each classifier may be generated by different training algorithms, the aggregation procedure is referred to as a multiple classifier system. In both cases, the set of individual classifiers is called a classifier ensemble.

Classifier combination comes either from a choice of the programmer or is imposed by context. In the first case, combination is meant to increase classification performances by either increasing the learning capacity or mitigating
overfitting. For instance, boosting [21] and bagging [4] can be regarded as such approaches. In the second case, it is not possible to learn a single classifier. A typical situation of this kind occurs when the dataset is dispatched on several machines in a network and sequential learning (such as mini-batch gradient descent) is not possible either to preserve network load or for some privacy or intellectual property reasons. In this decentralized learning setting, a set of classifiers are trained locally and the ensemble is later aggregated by a meta-learner.

In this article, we address classifier aggregation in a perspective that is in line with the decentralized setting assuming that the meta-learner has access to a validation set of data which is not used when training the individual classifiers. We introduce a number of desirable robustness properties for the aggregation procedure in this context. We investigate fault tolerance (ability to discard classifiers whose predictions are noise), robustness to adversarial classifiers (ability to thwart classifier with abnormal error rates) and robustness to redundant information (when classifier prediction are highly dependent).

We introduce an aggregation procedure in the framework of possibility theory. We prove that these robustness properties are verified asymptotically (when the size of the validation set is large) for this new approach. The mechanism governing the aggregation essentially relies on estimates of probabilities of class labels, classifier predictions or classifier correct predictions. There are many related works [15, 16, 17] dealing with classifier combination using similar information. We believe we are the first to do so in the framework of possibility theory but more importantly these above referenced work are not proved to possess theoretical robustness guarantees. An asymptotic optimality property is verified by an approach from Balakrishnan and Mojirsheibani [1]. This property is stronger than most of the properties that we state except for robustness to classifier dependency. Also, two technical conditions are necessary for the property to hold while our results have no such conditions to check. Similar remarks hold w.r.t. [3] which shares some ideas with [1]. Another piece of work with strong properties (oracle inequalities) is exponential weight aggregation [19] but the properties are non-exact and hold in expectation or with high probability while our properties rely on almost sure convergence. Also, exponential weight aggregation is a linear combination model while our method is non-linear.

In the next section, we recall the classifier aggregation problem and formally define the robustness properties that we seek. In section 3, we introduce a new aggregation technique in the framework of possibility theory and we show that the desired properties hold asymptotically for this technique. Section 4 contains numerical experiments illustrating our results.

2 Problem statement

2.1 Classification

Let $\Omega$ denote a set of $\ell$ class labels $\Omega = \{1, \ldots, \ell\}$. Let $x$ denote an input example with $d$ entries. Most of the time, $x$ is a vector and lives in $\mathbb{R}^d$ but sometimes some of its entries are categorical data and $x$ lives in an abstract

\footnote{Error rate is proved to convergence to the vicinity of the optimal one not exactly to the optimal one.}
space which does not necessarily have a vector space structure. Without loss of
generality, we suppose that \( x \) is a vector in the rest of this article.

A classification task consists in determining a prediction function \( c \) that
maps any input \( x \) with its actual class \( y \in \Omega \). This function is obtained from a
training set \( D_{\text{train}} \) which contains pairs \( (x^{(i)}, y^{(i)}) \) where \( y^{(i)} \) is the class
label of example \( x^{(i)} \). Given \( K \) classifiers, the label \( y \) assigned by the \( k \)th classifier
to the input \( x \) is denoted by \( c_k(x) \).

From a statistical point of view, training examples are instances of a random
vector \( X \) whose distribution is unknown. Likewise, class labels are instances of
a random variable \( Y \) whose distribution is also unknown. The training set is
often alleged to contain i.i.d. samples of the joint distribution of \((X, Y)\).

2.2 Classifier performance estimates

The ultimate goal of machine learning is to obtain predictor that generalize well
(w.r.t. unseen data at training time). Mathematically speaking, this means
achieving the lowest possible expected loss between predictions and true values.
When misclassification errors do not have different costs, the 0-1 loss function
\( L \) is the standard choice:

\[
L(y, c_k(x)) = \begin{cases} 
0 & \text{if } y = c_k(x) \\
1 & \text{otherwise}
\end{cases}
\]

In this case, the expected loss is the misclassification error rate of \( c_k \). It is well
known, that the error rate minimizer is Bayes classifier \( c_\star \):

\[
c_\star(x) = \arg \max_{y \in \Omega} p(Y = y|X = x) .
\]

Obviously, since the conditional distributions of \( Y \) given \( X = x \) are unknown,
we must try to find proxys of the Bayes classifier. The error rate of classifier \( c_k \)
is denoted by \( r[c_k] \).

Although our goal is to achieve the lowest possible error rate, the performances
of a classifier are not, in general, constant across true class labels and
predicted ones. This finer grained information will be instrumental to elicit
our possibilistic ensemble of classifiers. This information is contained in the
confusion matrix \( M(k) \). Each entry of this matrix reads

\[
M_{i,j}^{(k)} = \sum_{(x,y) \in D_{\text{val}}} I\{y = i\} I\{c_k(x) = j\} , \tag{1}
\]

where \( I \) denotes the indicator function. It is important to compute the confusion
matrices using a validation set \( D_{\text{val}} \) disjoint from \( D_{\text{train}} \) otherwise the estimates
drawn from the matrix are biased. Actually, if \( n_{\text{val}} \) is the size of the validation
set, then \( \frac{M_{i,j}^{(k)}}{n_{\text{val}}} \) is the maximum likelihood estimate of the joint probability
\( p(Y = i, c_k(X) = j) \). Also, the sum of the non-diagonal entries of \( M(k) \) over
\( n_{\text{val}} \) is an unbiased estimate of the error rate of \( c_k \). Many other performance
criterion estimates can be derived from a confusion matrix.

The classifier combination that we introduce in section 3 essentially relies
on the information contained in those matrices. Computing those matrices can
thus be regarded as the training phase of the combination method.
2.3 Combining classifiers
Let \( C \) denote the random vector spanned by plugging \( X \) into the ensemble of classifiers:

\[
C = \begin{pmatrix}
    c_1(X) \\
    \vdots \\
    c_K(X)
\end{pmatrix}.
\]

A realization of this random vector is denoted by \( c \) or \( c(x) \) whenever the dependence on inputs must be made explicit. We place ourselves in the context where vectors \( c \) can be pictured as new (learned) representation of inputs. In this context, the best aggregate classifier \([1]\) is thus

\[
c_*(x) = \arg \max_{y \in \Omega} p(Y = y | C = c(x)).
\] (2)

Again, the distributions involved in the above definition are unknown. Since \( c \) lives in the discrete space \( \Omega^K \), it is possible to try to learn these distributions \([1, 16]\) but such statistical learning approaches do not scale well w.r.t. either \( \ell \) or \( K \). Generally speaking, classifier combination consists in finding a function \( f \) capturing the relation between vectors \( c \) and class labels \( y \) that achieves the closest possible performances as compared to \( c_* \).

2.4 Desirable properties for classifier combination
In terms of purely error rate related performances, the most desirable property for some aggregation function \( f \) is

\[
r[f(c)] \to r[c_*].
\] (3)

The aggregation technique studied in \([1]\) achieves a result of this kind (under two technical assumptions). Indeed this technique, which elaborates on \([15]\), amounts to compute maximum likelihood estimates of the probabilities involved in \([2]\). But classifier aggregation can also bring other types of guarantees which we refer to as robustness. Robustness is understood here as a form of fault tolerance, i.e. the ability to maintain a good level of predictions in several circumstances involving malfunctioning individual classifiers. There may be different causes behind malfunctioning classifiers, e.g. hardware failure or malicious hacks.

Among other possibilities, we have identified the following desirable properties in this scope:

(a) robustness to random guess: if the error rate of \( c_k \) is \( \frac{\ell - 1}{\ell} \) then \( f(c) = f(c_{-k}) \) where \( c_{-k} \in \Omega^{K-1} \) is the same vector as \( c \) but with its \( k \)th entry deleted.

Property (a) means that if the predictions of \( c_k \) are in average no better than random guess then \( c_k \) has no influence on the aggregated classifier.

(b) robustness to adversarial classifiers: if \( c_k \) has an error rate larger than random guess, i.e. \( r[f(c)] > \frac{\ell - 1}{\ell} \), then there is a classifier \( c_k^{\text{rec}} \) with an error rate lower than random guess such that \( f(c) = f(\tilde{c}) \) where \( \tilde{c}_s = c_s \) for any \( s \neq k \) and \( \tilde{c}_k = c_k^{\text{rec}} \).
Property (b) means that we can somewhat rectify the incorrect predictions of classifier \( c_k \) so that the aggregated classifier is identical to the one obtained from a non-adversarial situation.

(c) robustness to redundant information: if there are two individual classifiers such that \( c_k(x) = c_k'(x) \) for any \( x \), then \( f(c) = f(c_{-k}) \).

Property (c) means that copies of classifiers have no influence on the aggregated classifier.

In the above, we assume that the aggregation function \( f \) is produced by a given algorithmic procedure and that this procedure applies for any \( K > 1 \). So \( f(c_{-k}) \) is not the restriction of \( f(c) \) but another function learned from the same algorithm by omitting classifier \( c_k \).

Obviously, one can think of other properties or reshape them in different ways. For instance, a soft version of property (c) would be better in the sense that an ensemble contains rarely identical copies of a predictor but it contains very often highly dependent ones. This a first attempt to formalize desirable robustness properties for classifier combination and we hope that more advanced declinations of these will be proposed in the future.

For the time being, our goal in this paper is to introduce an aggregation procedure that is compliant with properties (a) to (c). We will prove that these properties hold for the possibilistic approach that we introduce in the next section at least asymptotically for some of them. Observe that (3) asymptotically implies properties (a) to (c) so the added value of our approach as compared to [11] relies on memory complexity and scalability w.r.t. \( \ell \) as numerical experiments will illustrate in section 4.

3 Robust combination in the possibilistic framework

In this section, we introduce a new classifier combination approach in the possibilistic theory framework. Possibility theory [26, 10] is an uncertainty representation framework. It has strong connections with belief functions [9, 7], random sets [14, 18, 20], imprecise probabilities [11, 6] or propositional logic [2]. For a concise but thorough overview of possibility theory, the reader is referred to [12]. In this paper, we adopt a knowledge based system view of this theory. In this regard classifier predictions are expert knowledge to which a degree of belief is attached in the form of possibility distributions. Following a normative approach, experts are reconciled by designing a conjunctive rule that must obey the desirable properties presented in the previous section.

3.1 Possibility theory basics

A possibility distribution \( \pi \) maps each element of \( \Omega \) to the unit interval \([0; 1]\) whose extreme values correspond respectively to impossible and totally possible epistemic states. If \( \pi(y) = 1 \) then this class label is totally possible (meaning that we have no evidence against \( y \)). If \( \pi(y) = 0 \) then \( y \) is ruled out as a possible class label.
Given a subset $A$ of $\Omega$, a possibility measure $\Pi$ is given by:

$$\Pi(A) = \max_{y \in A} \pi(y)$$

(4)

which means that the possibility of a subset $A$ is equal to the maximum possibility degree in this subset. A possibility measure is thus maxitive: $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ as opposed to probability measures which are summative. Observe that this property accounts for the fact that the possibility distribution is enough information to compute the possibility measure of any subset.

### 3.2 From classifier confusion matrices to possibility distributions

If one normalizes the $j$th column of the confusion matrix $M^{(k)}$, then we obtain an estimate of the probability distribution $p(Y = i | c_k = j)$. So, if the $k$th classifier predicts $j$ for some input $x$, we can adopt these frequentist probabilities as our beliefs on the class label of $x$. But unless, unrealistic conditional independence assumptions\textsuperscript{2} are made, probabilistic calculus rules will not easily allow to combine beliefs arising from several classifiers.

As an alternative to this approach, we propose to build a possibility distribution from $p(Y = i | c_k = j)$ as information fusion in the possibilistic framework can mitigate dependency issues and does not lead to intractable computations. To cast the problem in the possibilistic framework, we use Dubois and Prade transform (DPT) [9]. For some arbitrary probability distribution $p$ on $\Omega$, let $\text{per}$ denote a permutation on $\{1; \ldots ; \ell\}$ such that probability masses of $p$ are sorted in descending order and $p' = p \circ \text{per}$, i.e. $p'(1) \geq p'(2) \geq \ldots \geq p'(\ell)$. The (unique) possibility distribution $\pi$ arising from $p$ through DPT is given by

$$\pi(\text{per}(i)) = \begin{cases} 1 & \text{if } i = 1 \\ \pi(\text{per}(i) - 1) & \text{if } i > 1 \text{ and } p'(i) = p'(i - 1) \\ \sum_{q=i}^{\ell} p'(q) & \text{otherwise} \end{cases}$$

(5)

If $p_{Y|c_k=j}$ denotes the distribution of class labels when the $k$th classifier predicts $j$, the corresponding possibility distribution is denoted by $\pi_{k|c_k=j} = \text{DPT} \{p_{Y|c_k=j}\}$. For each input $x$, the $K$ classifier predictions are turned into $K$ expert opinions in the form of possibility distributions $(\pi_{k|c_k(x)})_{k=1}^K$.

### 3.3 Aggregation of possibility distributions

Formally speaking, any $K$-ary operator on the set of possibility distributions is an admissible combination operator. Triangular norms, or t-norms, are instrumental to yield well defined aggregation operators for possibility distributions. A t-norm $T : [0; 1]^2 \rightarrow [0, 1]$ is a commutative and associative mapping therefore it is easy to build a $K$-ary version of it using successive pairwise operations:

$$T(a_1, \ldots a_K) = T(a_1, T(\ldots, T(a_{K-1}, a_K)))$$

\textsuperscript{2}These assumptions and the corresponding probabilistic approach are described in section [9].
for any \((a_k)_{k=1}^K \subseteq [0; 1]_K\).

Moreover, a t-norm has 1 as neutral element, 0 as absorbing element and possesses the monotonicity property which reads: for any \(a, b, c, d \in [0; 1]\) such that \(a \leq c\) and \(b \leq d\), then \(T(a, b) \leq T(c, d)\). Finally, a t-norm is upper bounded by the minimum of its operands.

To combine possibility distributions using a t-norm, we can simply apply a t-norm elementwise. For instance, if \(\pi_{kk'}\) is the aggregated possibility distribution obtained by applying a t-norm to distributions \(\pi_k\) and \(\pi_{k'}\), then

\[
\pi_{kk'}(y) = T(\pi_k(y), \pi_{k'}(y)), \forall y \in \Omega.
\]

We will use the same t-norm symbol to stand for the overall combination or possibility distributions and we will write \(\pi_{kk'} = T(\pi_k, \pi_{k'})\). Examples of t-norms are elementwise multiplication \(T \times\) and elementwise minimum \(T \land\).

Decision making based on maximum expected utility is also justified using non-additive measures (capacities) such as possibility measures. Consequently, the possibilistic aggregated classifier, denoted \(c_{\text{ens}}\), is given by

\[
c_{\text{ens}}(x) = \arg\max_{y \in \Omega} \pi_{\text{ens}}(y),
\]

with \(\pi_{\text{ens}} = T(\pi_{1|c_1(x)}, \ldots, \pi_{K|c_K(x)})\).

Algorithm 1 explains what computations should be anticipated as part of a training phase and algorithm 2 summarizes how an input \(x\) class label is predicted at test time. The procedure corresponding to these algorithms is referred to as Scalable POssibilistic Classifier Combination (SPOCC). Note that there may be several class labels maximizing \(\pi_{\text{ens}}\) therefore the aggregated classifier prediction \(c_{\text{ens}}(x)\) may be set-valued. Working with set-valued predictions is out of the scope of this paper and will be considered in future works. In the advent of a class label tie, and for any probabilistic, possibilistic or deterministic aggregation approach, one of these labels is chosen at random.

**Data:** validation set \(D_{\text{val}}\), classifiers \((c_k)_{k=1}^K\).

for \(k \in \{1, \ldots, K\}\) do

Compute confusion matrix \(M^{(k)}\) as in (1).

for \(j \in \Omega\) do

Compute conditional probability estimates

\[
\hat{p}_{\text{mle}}(Y = i|c_k = j) \leftarrow \frac{M_{ij}^{(k)}}{\sum_i M_{ij}^{(k)}}, \forall i \in \Omega.
\]

Compute possibility distribution using (5)

\[
\pi_{k|j} \leftarrow \text{DPT}\{\hat{p}_{\text{mle}}(Y = i|c_k = j)\}.
\]

end

end

Return possibility distributions \((\pi_{k|j})_{1 \leq k \leq K, 1 \leq j \leq \ell}\).

**Algorithm 1:** SPOCC - training phase
**Data:** input $\mathbf{x}$, classifiers $(c_k)_{k=1}^K$, possibility distributions $(\pi_{kj})_{1 \leq k \leq K, 1 \leq j \leq \ell}$ and t-norm $\mathcal{T}$.

**for** $k \in \{1, \ldots, K\}$ **do**

- Compute individual classifier prediction $j_k \leftarrow c_k(x)$.

**end**

Compute $\pi_{\text{ens}} \leftarrow \mathcal{T}(\pi_{1j_1}, \ldots, \pi_{Kj_K})$.

Return $c_{\text{ens}}(\mathbf{x}) \leftarrow \arg \max_y \pi_{\text{ens}}(y)$.

**Algorithm 2:** SPOCC - test phase

### 3.4 Adaptive aggregation w.r.t. dependency

The predictions of an ensemble of individual classifiers are usually significantly dependent because they are trained to capture the same bound between inputs and class labels. So if classifiers are at least weak classifiers, they will often produce identical predictions. More importantly, from an information fusion standpoint, if a majority of the classifiers are highly dependent and have a larger error rate than the remaining ones, they are likely to guide the ensemble toward their level of performances making classifier fusion counter-productive.

In the approach introduced in this paper, it is possible to mitigate dependency negative impact by choosing an idempotent t-norm such as elementwise minimum $\mathcal{T}_\lambda$. Indeed, in the worst dependency case, classifier $c_k$ is a copy of classifier $c_k'$ therefore they have an unjustified weight in the ensemble predictions. But if two individual classifiers are identical they will also yield identical possibility distributions and if these latter are combined using $\mathcal{T}_\lambda$, then these two classifiers will be counted as one. This is exactly the spirit of property (c).

Two difficulties arise from this quest for robustness w.r.t. classifier redundancy:

(i) It is not recommended to systematically use an idempotent combination mechanism because it is also possible that two poorly dependent classifiers yield identical possibility distributions in which case it appears justified that their common prediction impacts on the ensemble aggregated decision.

(ii) There are different levels of dependency among subsets of individual classifiers therefore, using a single t-norm to jointly aggregate them is not the best option.

To address the first issue, we propose to use the following parametric family $(\mathcal{T}_\lambda)_{\lambda \in [1, +\infty)}$ of t-norms:

$$
\mathcal{T}_\lambda(a_1, a_2) = e^{-\left(\frac{1}{n_{\text{val}}} \log a_1^{\lambda} + \frac{1}{n_{\text{val}}} \log a_2^{\lambda}\right)^\frac{1}{\lambda}}, \forall a_1, a_2 \in [0; 1].
$$

This family is known as Aczel-Alsina t-norms and is such that $\mathcal{T}_1 = \mathcal{T}_\infty$ and $\mathcal{T}_\infty = \mathcal{T}_\lambda$. We can thus tune $\lambda$ all the higher as the level of dependence between classifiers is high.

To assess the dependence level $\lambda$ among two classifiers $c_k$ and $c_k'$, we use the following definition

$$
\kappa(c_k, c_k') = 1 - \exp\left(-\frac{1}{n_{\text{val}}} \log \left(\frac{L_0}{L_1}\right)\right),
$$

(8)
where \( \frac{L_0}{L_1} \) is the likelihood ratio of the independence model over the joint model. These likelihoods are given by

\[
L_0 = \prod_{i=1}^{n_{val}} \hat{p}_{\text{mle}}(c_k(x^{(i)})) \hat{p}_{\text{mle}}(c_{k'}(x^{(i)})),
\]
and

\[
L_1 = \prod_{i=1}^{n_{val}} \hat{p}_{\text{mle}}(c_k(x^{(i)}), c_{k'}(x^{(i)})).
\]

These likelihoods are computed using all training examples contained in the validation set \( D_{\text{val}} \). The probabilities involved in the computation of \( L_0 \) are the maximum likelihood estimates of the parameters of the multinomial marginal distributions \( p(c_k(X)) \) and \( p(c_{k'}(X)) \) respectively. The probabilities involved in the computation of \( L_1 \) are the maximum likelihood estimates of the parameters of the multinomial joint distribution \( p(c_k(X), c_{k'}(X)) \).

The definition of the dependence level \( \kappa \) can be extended to more than two classifiers but this will turn out to be unnecessary because we will use hierarchical agglomerative clustering [23] (HAC) to address issue (ii). HAC will produce a dendrogram \( \mathcal{G} \), i.e. a t-norm computation binary tree. Each leaf in this tree is in bijective correspondence with one of the possibility distributions \( \pi_k \) induced by a classifier. There are thus \( K \) leaves in \( \mathcal{G} \). Furthermore, each non-leaf node in the tree stands for a t-norm operation involving two operands only. Consequently, each non-leaf node has exactly two children and there \( K - 1 \) such nodes, one of them being the root node. Figure 1 gives an illustrative example of a dependence dendrogram allowing to compute the aggregated possibility distribution.

\[
\begin{align*}
\pi_{1245} &= T_{\lambda_4}(\pi_{12}, \pi_{345}) \\
\pi_{345} &= T_{\lambda_3}(\pi_{34}, \pi_5) \\
\pi_{34} &= T_{\lambda_2}(\pi_3, \pi_4) \\
\pi_{12} &= T_{\lambda_1}(\pi_1, \pi_2) \\
\pi_1, \pi_2, \pi_3, \pi_4, \pi_5 &
\end{align*}
\]

Figure 1: Example of a dendrogram for \( K = 5 \). Leaf nodes are at the bottom. For each of the four non-leaf nodes, a specific dependence level \( \lambda_a \) \((a \in \{1;2;3;4\})\) must be determined to compute the aggregated possibility distribution.

HAC relies on a classifier dissimilarity matrix \( D \). In our case, entries of this \( K \times K \) matrix are simply given by \( D_{kk'} = 1 - \kappa(c_k, c_{k'}) \).

The t-norm based possibility distribution aggregation method described in the above paragraphs is meant to replace the penultimate step of Algorithm 2 but most of the computations pertaining to this dependency adaptive aggregation can be done at training time. Indeed, the computation of the pairwise
dependence levels and the dendrogram do not depend on the unseen example \( x \) that we will try to classify at test time. For a minimal test phase computation time, we need to assign to each non-leaf node \( V_a \) of \( G \) the appropriate dependence level \( \lambda_a \) (as illustrated in Figure 1) during the training phase. The corresponding array is denoted by \( \lambda \mapsto \lambda_\alpha \). The function \( J_{G,\lambda} \) maps the set of possibility distributions \( (\pi_k)_{1 \leq k \leq K} \) to the aggregated distribution \( \pi_{\text{ens}} \) by executing the computation graph \( G \) and using the dependence levels contained in \( \lambda \). There are \( K - 1 \) hyperparameters in array \( \lambda \) that need to be tuned. They will be automatically set to appropriate values by heuristic search, see A for a presentation of this heuristic.

### 3.5 Adaptive aggregation w.r.t. informational content

When the predictions delivered by classifier \( c_k \) are poorer than those of another classifier \( c_{k'} \), it is instrumental to reduce the impact of \( c_k \) on the decisions issued by the ensemble. Regardless of the formal definition behind what are called "poor predictions", we propose to use the following mechanism to gradually fade classifier \( c_k \) out of the ensemble: for a given scalar \( \alpha_k \in [0; 1] \), we update all conditional possibility distributions related to \( c_k \) as follows:

\[
\pi_{kj} \leftarrow (1 - \alpha_k) \pi_{kj} + \alpha_k, \forall j \in \Omega.
\]  

(12)

This mechanism is equivalent to an operation known as discounting [22]. When \( \alpha = 0 \), then classifier \( c_k \) influence on the ensemble is not reduced. When \( \alpha_k = 1 \), classifier \( c_k \) is discarded from the ensemble since we obtain constant one possibility distributions which are the neutral element of t-norms and the t-norm based aggregation method introduced in the previous subsection.

Obviously, we need to find a value of the discounting coefficient tailored for each classifier and in line with what poor predictions are meant to be. Again, it is tempting to set this \( K \) hyperparameters using grid search but the corresponding complexity calls for a more subtle strategy. Similarly as for dependency hyperparameters, we will resort to a heuristic search.

Among other possibilities, our solution consists in binding the discounting rates together using the following formula:

\[
\alpha_k = 1 - \left( \frac{1 - \hat{r} [c_k]}{1 - \min_{k'} \hat{r} [c_{k'}]} \right)^\rho,
\]  

(13)

where \( \hat{r} \) is the estimated error rate on the validation set and \( \rho \in [0; +\infty] \) is a hyperparameter to tune by grid search. Using the above equation, the best base classifier is not discounted and we have \( \hat{r} [c_k] \leq \hat{r} [c_{k'}] \Rightarrow \alpha_k \leq \alpha_{k'} \).

### 3.6 Fully adaptive aggregation

The fully adaptive version (w.r.t. both dependence and informative content) of SPOCC is referred to as adaSPOCC. The corresponding training and test phases are described in Algorithm 3 and 4 respectively.
**Data:** validation set $\mathcal{D}_{val}$ and classifiers $(c_k)_{k=1}^K$.

Execute SPOCC - training phase (algorithm 1)

for $k \in \{1, \ldots, K\}$ do
  for $k' \in \{k, \ldots, K\}$ do
    Compute the dissimilarity $D_{kk'} \leftarrow 1 - \kappa(c_k, c_{k'})$ using (9).
    Assign $D_{k'k} \leftarrow D_{kk'}$.
  end
end

Obtain dendrogram $\mathcal{G}$ by applying HAC to dissimilarity matrix $D$.

Apply heuristic to set parameters $\lambda_a \in \lambda$ (see $\Lambda$).

Compute parameters $(\alpha_k)_{k=1}^K$ as in (13).

Update all conditional possibility distributions as in (12).

Return possibility distributions $(\pi_k|j)_{1 \leq k \leq K, 1 \leq j \leq \ell}$, dendrogram $\mathcal{G}$, array $\lambda$.

**Algorithm 3:** adaSPOCC - training phase

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**Data:** input $x$, classifiers $(c_k)_{k=1}^K$, possibility distributions $(\pi_{k|j})_{1 \leq k \leq K, 1 \leq j \leq \ell}$, dendrogram $\mathcal{G}$, array $\lambda$.

for $k \in \{1, \ldots, K\}$ do
  Compute individual classifier prediction $j_k \leftarrow c_k(x)$.
end

$\pi_{ens} \leftarrow J_{\mathcal{G} : \lambda} (\pi_{1|j_1}, \ldots, \pi_{K|j_K})$ (computation graph execution).

Return $c_{ens}(x) \leftarrow \arg \max_{y \in \Omega} \pi_{ens}(y)$.

**Algorithm 4:** adaSPOCC - test phase

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3.7 Properties of the possibilistic ensemble

In this paper, we adopt a normative approach for the selection of a classifier decision aggregation mechanism. In this subsection, we give sketches of proofs showing that robustness properties (a) to (c) hold for adaSPOCC asymptotically:

- Property (a): if $c_k$ is a random classifier then when $n_{val} \to \infty$, each conditional distribution $p_{Y|c_k=j}$ converges to a uniform distribution so DPT turns it into a constant one possibility distribution, which is the neutral element of $\mathcal{T}_\lambda$.

- Property (b): let $c_k$ denote an adversarial classifier, i.e. $r[c_k] > \ell - \frac{1}{\ell}$. (ada)SPOCC uses the following rectified classifier $c_k^{(rec)} = h \circ c_k$ defined as

$$c_k^{(rec)}(x) = \arg \max_{y \in \Omega} p(Y = y|c_k(x)). \quad (14)$$

We have

$$1 - r\left[c_k^{(rec)}\right] = \sum_{y' \in \Omega} p\left(Y = y'|c_k^{(rec)} = y'\right) p\left(c_k^{(rec)} = y'\right). \quad (15)$$

Moreover, we can write

$$p\left(Y = y'|c_k^{(rec)} = y'\right) = \sum_{y'' \in \Omega} p\left(Y = y'|c_k^{(rec)} = y', c_k = y''\right) p\left(c_k = y''|c_k^{(rec)} = y'\right). \quad (16)$$

Given the definition of $c_k^{(rec)}$ we know that $p\left(c_k = y''|c_k^{(rec)} = y'\right) = 0$ if $y'' \not\in h^{-1}(y')$. The definition also gives

$$p\left(Y = y'|c_k^{(rec)} = y', c_k = y''\right) = \max_{y \in \Omega} p(Y = y|c_k = y''). \quad (17)$$

The maximal probability value of a discrete variable is always greater or equal than $\frac{1}{\ell}$ therefore

$$p\left(Y = y'|c_k^{(rec)} = y'\right) \geq \frac{1}{\ell} \sum_{y'' \in h^{-1}(y')} p\left(c_k = y''|c_k^{(rec)} = y'\right) \quad (18)$$

$$\geq \frac{1}{\ell} p\left(c_k \in h^{-1}(y') \mid c_k^{(rec)} = y'\right). \quad (19)$$

Again, given the definition of $c_k^{(rec)}$ we know that $p\left(c_k \in h^{-1}(y') \mid c_k^{(rec)} = y'\right) = 1$. Since $c_k$ is not the random classifier, at least one of the conditional distributions $p_{Y|c_k}$ is not uniform in which case the inequality is strict. We thus obtain $1 - r\left[c_k^{(rec)}\right] > \frac{1}{\ell}$.

Finally, when $n_{val} \to \infty$, if $c_k(x) = y$ and $c_k^{(rec)}(x) = y'$, the $y$th column of $M^{(k)}$ will be identical to the $y'$th column of the confusion matrix of $c_k^{(rec)}$ so they will be mapped to identical possibility distributions.
• Property (c): when $n_{\text{val}} \to \infty$, the likelihood ratio appearing in (9) writes

$$\frac{L_0}{L_1} = \prod_{i=1}^{n_{\text{val}}} \frac{p\left(c_k\left(x^{(i)}\right)\right)}{p\left(c_k\left(x^{(i)}\right), c_{k'}\left(x^{(i)}\right)\right)}.$$  \hspace{1cm} (20)

If $c_{k'}$ is a copy of $c_k$ then $p\left(c_k\left(x^{(i)}\right), c_{k'}\left(x^{(i)}\right)\right) = p\left(c_k\left(x^{(i)}\right)\right) = p\left(c_{k'}\left(x^{(i)}\right)\right)$ and

$$\frac{L_0}{L_1} = \prod_{i=1}^{n_{\text{val}}} p\left(c_k\left(x^{(i)}\right)\right).$$  \hspace{1cm} (21)

If $c_k$ is not a constant function, then probabilities are smaller than one and $\frac{L_0}{L_1} \to 0$. The pair of classifiers $(c_k, c_{k'})$ will thus be detected as maximally dependent by HAC and they will be aggregated using $T_1 = T_h$, hence property (c) holds in this case.

When $c_k$ is a constant function, then both $c_k$ and $c_{k'}$ will yield identical possibility distributions that are a Dirac function. In this case, the output of $T_h$ will also be this Dirac function, meaning that idempotence is always true in these circumstances. Note that, however, the procedure described in [3.3] will fail to detect the dependency between $c_k$ and $c_{k'}$. There are plenty of ways to thwart this issue as constant classifiers are not difficult to detect. In practice, we will use add-one Laplace smoothing to estimate probabilities $p\left(c_k\right)$ so we will never obtain a Dirac function as possibility distribution.

Properties (a) to (c) rely on asymptotic estimates of multinomial distribution parameters which, from the strong law of numbers, converge almost surely to their exact values. Consequently, the properties do not hold only in expectation or with high probability but systematically (when $n_{\text{val}}$ is large).

Although properties (a) to (c) are not as strong as (3), adaSPOCC is a scalable aggregation technique as the number of parameters it requires to learn from the validation set is in $O(\ell K)$ while the number of parameters to learn from $D_{\text{val}}$ in [1] is in $O(\ell^K)$ and is therefore doomed to overfit when $K$ is large.

4 Experimental results

In this section, we present a number of experimental results allowing to prove the robustness of SPOCC and adaSPOCC as compared to other aggregation techniques. The section starts with results obtained when the base classifiers are trained on a synthetic dataset and are meant to highlight performances discrepancies in simple situations where robustness is required. Another set of experiments on real datasets are also presented to prove that the method is not only meaningful on toy examples.

4.1 General setup

Designing numerical experiments allowing to compare aggregation methods is not a trivial task. A crucial aspect consists in training a set of base classifiers that achieve a form of diversity [25] so that the fusion of their predictions has a
significant impact on performances. Among other possibilities [5], we chose to
induce diversity by feeding the base classifiers with different disjoint subsets of
data points at training time. The subsets are not chosen at random but instead
in a deterministic way allowing each base classifier to focus on some regions of
the input space and thus learn significantly different decision frontiers.

Each aggregation technique is fed only with the predictions of the base clas-
sifiers on the validation set in order to tune hyperparameters or learn the combi-
nation itself. Consequently, SPOCC and adaSPOCC are only compared to well
established methods that use the same level of information. The benchmarked
aggregation techniques are:

- classifier selection\footnote{Selection can be regarded as a special type of fusion.} based on estimated accuracies of the base classifiers,
- weighted vote aggregation based on estimated accuracies of the base clas-
sifiers,
- exponentially weighted vote aggregation based on estimated accuracies of
  the base classifiers,
- naive Bayes aggregation,
- Bayes aggregation,
- stacking.

In the exponentially weighted vote aggregation, accuracies are not directly
used as vote weight (as in standard weighted vote aggregation) but are mapped
to weights using a softmax function. This function has a positive temperature
hyperparameter that regulates the assignment of weights. When this parameter
is zero, then we retrieve unweighted vote aggregation whereas when it is very
large, then we retrieve classifier selection.

Bayes aggregation relies on (2). The conditional distributions involved in
this equation are learned from the validation set. Naive Bayes aggregation uses
conditional independence assumptions that allow to factorize probabilities as

\[ p(y | c(x)) \propto p(y) \prod_{k=1}^{K} p(c_k(x) | y) . \] (22)

The conditional independence assumptions are not realistic but yield a model
with far less parameters to learn as compared to Bayes aggregation.

For each of the estimated probabilities involved in the mechanism behind
SPOCC, adaSPOCC, Bayes or Naive Bayes aggregation, we perform add-one-
Laplace smoothing to avoid computational issues related to zero probabilities.

Finally, we also train a logistic regression to map classifier predictions to the
true class labels. This approach belong to a methodology known as stacking
\footnote{Selection can be regarded as a special type of fusion.}. An \( L_2 \) regularization term is added to the cross-entropy loss. A positive
hyperparameter regulates the relative importance of the regularization term.

All hyperparameters are tuned automatically using a cross-validated grid
search on the validation set. For each hyperparameter, the grid contains 100
Figure 2: Synthetic dataset obtained from four Gaussian distributions. Examples belonging to class $\omega_0$ are in blue while those belonging to class $\omega_1$ are in cyan. Optimal decision frontiers are in magenta.

Figure 3: Subsets of the data seen respectively by $c_1$ to $c_4$ points. When the hyperparameter is unbounded, a logarithmic scale is used to design the grid.

The statistical significance of the reported results are given in terms of 95% confidence interval estimated from bootstrap sampling. When the accuracies of two aggregation methods have overlapping confidence intervals, the performance discrepancy is not significant.

4.2 Synthetic Data

In this subsection, we use a very simple generating process to obtain example/label pairs. Data points are sampled from four isotropic Gaussian distributions. The centers of these Gaussian distributions are located at each corner of centered square of a 2D input space ($d = 2$). The standard deviations of each of the distribution is 1. There are $\ell = 2$ possible class labels: $\Omega = \{\omega_0; \omega_1\}$. Points such that $x_1$ and $x_2$ are both positive belong to $\omega_0$. Points such that $x_1$ and $x_2$ are both negative also belong to $\omega_0$. All the other points belong to $\omega_1$. Figure 2 shows one such dataset obtained from this generating process.

In this series of experiments, the dataset is divided in four disjoint subsets as depicted in Figure 3. Then 80% of one such subset are used to train one of the base classifier. The remaining 20% are used for the validation set. Each base classifier $c_k$ sends the corresponding set of prediction/label pairs $\{(c_k(x), y)\}_{(x,y) \in D_{val}}$ to the aggregation method.
Given the shape of optimal decision frontiers, the base classifiers trained in this subsection are decision trees with a maximal depth of two.

4.2.1 Robustness w.r.t. adversaries

Among other possibilities, adversarial predictions are simulated by sampling from a Bernoulli distribution $Z \sim \text{Ber} (\theta)$. Given $Z = 1$, the prediction of a base classifier is replaced with another (arbitrarily selected) class label that will not coincide with the classifier prediction. When $Z = 0$, the classifier prediction is unchanged. Consequently, an adversarial classifier built in this way from a base classifier with an error rate lower than random guess will achieve an error rate greater than random guess as $\theta \to 1$.

The evolution of the classification accuracy of the benchmarked aggregation methods as the number of adversaries grows can be witnessed on Figure 4. For simplicity, all adversaries are built from the same base classifier ($c_1$) with $\theta = 0.5$. Two methods cannot maintain the same level of performances when the number of adversaries increases: weighted vote ensemble and Bayes aggregation. For the weighted vote ensemble, this is explained by the fact that the number of misleading classifiers outnumber legitimate classifiers and start to obtain a majority of votes. For Bayes aggregation, the performances are degrading simply because of overfitting.

4.2.2 Robustness w.r.t. faults

Erroneous predictions are simulated by sampling from a Bernoulli distribution $Z \sim \text{Ber} (\theta)$. Given $Z = 1$, the prediction of a base classifier is replaced with an (arbitrarily selected) class label that will coincide with the classifier prediction with probability $\frac{1}{\ell}$. When $Z = 0$, the classifier prediction is unchanged. Consequently, a noisy classifier built in this way from a base classifier will achieve an error rate equal to $\ell^{-1}$ (random guess) as $\theta \to 1$.

The evolution of the classification accuracy of the benchmarked aggregation methods as the number of noisy classifiers grows can be witnessed on Figure 5. For simplicity, all noisy classifiers are built from the same base classifier ($c_1$) with $\theta = 0.9$. Similarly as for adversarial classifiers, weighted vote ensemble and Bayes aggregation cannot maintain the same level of performances when the number of perturbed classifiers increases. The same reasons are also behind these performance decays (majority of incorrect classifiers for the weighted vote ensemble and overfitting for Bayes aggregation).

4.2.3 Robustness w.r.t. informational redundancy

Redundancy in classifier predictions is simulated by adding several copies of one of the base classifiers (classifier $c_1$ in our experiments). As shown in Figure 6, this very simple setting allows to observe severe performance decays for the weighted vote ensemble, the exponentially weighted vote ensemble and the naïve Bayes aggregation. Vote based ensembles are very sensitive to changes of majority. Naïve Bayes aggregation is also sensitive to this phenomenon and suffers from its inability to capture dependency relations between the base classifiers.

Unlike the previous experiment, it can be noted that Bayes aggregation maintains the same level of performances as the number of clones of $c_1$ increases.
Figure 4: Evolution of accuracy distributions (violin plots) for several aggregation methods w.r.t. the number of adversaries. SPOCC and adaSPOCC are in orange while other methods are in blue.
Figure 5: Evolution of accuracy distributions (violin plots) for several aggregation methods w.r.t. the number of noisy classifiers. SPOCC and adaSPOCC are in orange while other methods are in blue.
Because clones will always produce identical predictions as \( c_1 \), training the Bayes aggregation in these conditions is equivalent to learn from \( K = 4 \) base classifiers regardless how many copies of \( c_1 \) are added. However, should these copies be slightly perturbed, then we would observe the same overfitting issues as in the previous experiments.

4.2.4 Summarizing synthetic data experiment results

In the previous paragraphs, we have seen which methods are tolerant to adversaries, faults and redundancy and scale well w.r.t. \( K \). Only SPOCC, adaSPOCC and stacking seem to be robust w.r.t. each of these forms of difficulties. Beside robustness, their absolute performances also matter. Averages\(^4\) performances are reported in Table 1 for each experiment as well as the global average on all experiments. We also provide as reference the optimal (Bayes) classifier accuracy as well as the performances of the best base classifier \( c_k \), i.e. optimal selection.

In terms of accuracies, SPOOC or adaSPOOC are always the top 1 or top 2 aggregation approach. While the naive Bayes aggregation is slightly better than SPOOC or adaSPOOC in the two first series of experiments, it performs very significantly worse in the last one and it is outperformed on global average.

\(^4\)Averages w.r.t the either the number of adversaries, noisy classifiers or clones respectively.
Table 1: Average performances of aggregation methods on the synthetic data. The first figure is the average accuracy followed by the semi-width of the 95% confidence interval width of this latter and the average standard deviation. Best accuracies (or those not statistically significantly different) are in bold characters.

| Method          | Adversaries | Faults | Resilience | Global Average |
|-----------------|-------------|--------|------------|----------------|
| Clf. Selection  | 72.26%      | 82.56% | 72.25%     | 72.25%         |
|                 | ±0.52       | ±0.59  | ±0.51      | ±0.28          |
|                 | std. 8.56%  | std. 8.56% | std. 8.55% | std. 8.56%    |
| Weighted Vote   | 81.47%      | 81.89% | 76.46%     | 79.04%         |
|                 | ±0.25       | ±0.23  | ±0.52      | ±0.22          |
|                 | std. 4.23%  | std. 3.83% | std. 8.86% | std. 6.57%    |
| Exp. Weighted Vote | 84.34%     | 84.35% | 82.70%     | 84.81%         |
|                 | ±0.23       | ±0.23  | ±0.39      | ±0.17          |
|                 | std. 3.86%  | std. 3.77% | std. 6.46% | std. 4.92%    |
| Stacking        | 81.44%      | 83.90% | 83.38%     | 81.44%         |
|                 | ±0.28       | ±0.29  | ±0.36      | ±0.17          |
|                 | std. 4.75%  | std. 4.82% | std. 5.89% | std. 5.18%    |
| Naive Bayes Agg.| 85.23%      | 85.22% | 80.27%     | 83.07%         |
|                 | ±0.13       | ±0.14  | ±0.46      | ±0.19          |
|                 | std. 2.25%  | std. 2.25% | std. 8.12% | std. 5.55%    |
| Bayes Agg.      | 66.46%      | 66.96% | 81.16%     | 71.24%         |
|                 | ±0.77       | ±0.79  | ±0.51      | ±0.46          |
|                 | std. 13.45% | std. 13.43% | std. 8.46% | std. 13.90%   |
| SPOCC           | 84.42%      | 84.42% | 83.92%     | 84.10%         |
|                 | ±0.23       | ±0.23  | ±0.34      | ±0.16          |
|                 | std. 3.82%  | std. 3.82% | std. 5.76% | std. 4.61%    |
| adaSPOCC        | 84.01%      | 84.01% | 84.54%     | 84.16%         |
|                 | ±0.23       | ±0.23  | ±0.72      | ±0.13          |
|                 | std. 3.89%  | std. 3.89% | std. 4.12% | std. 4.00%    |
| Best base Clf.  | 82.66%      | 82.66% | 82.66%     | 82.66%         |
|                 | ±0.41       | ±0.41  | ±0.41      | ±0.41          |
|                 | std. 6.87%  | std. 6.87% | std. 6.87% | std. 6.87%    |
| Optimal Clf.    | 87.92%      | 87.92% | 87.92%     | 87.92%         |
|                 | ±≈ 0        | ±≈ 0   | ±≈ 0       | ±≈ 0           |
|                 | std. ≈ 0%   | std. ≈ 0% | std. ≈ 0% | std. ≈ 0%     |

Stacking and all other methods obtain worse (or sometimes comparable) results as compared to (ada)SPOCC. Moreover, observe that adaSPOCC achieves the smallest variance, meaning that its performances are more stable across dataset draws.

4.3 Real Data

To upraise the ability of the benchmarked methods to be deployed in more realistic situations (such as decentralized learning), we also need to test them on sets of real data. Since this is essentially useful in a big data context, we chose eight from moderate to large public datasets. The specifications of these datasets are reported in Table 2.

Example entries from the 20newsgroup data set are word counts obtained using the term frequency - inverse document frequency statistics. We reduced the dimensionality of inputs using a latent semantic analysis which is a standard practice for text data. We kept 100 dimensions. Also, as recommended, we stripped out each text from headers, footers and quotes which lead to overfitting. Besides, for the Wine and Avila datasets, the number of class labels is originally 10 and 12 respectively. We binarized these classification tasks because some classes have very small cardinalities which is problematic for our experimental design in which datasets are divided into several distinct subsets. Indeed, some subsets may possess no example at all of some classes which leads to imprecise labeling which is beyond the scope of this paper. To circumvent
Table 2: Real dataset specifications

| Name         | Size | Dim. | Nbr. of classes | Data type | Source               |
|--------------|------|------|-----------------|-----------|----------------------|
| 20newsgroup  | 18846| 100  | 20              | text      | sklearn              |
| MNIST        | 70000| 784  | 10              | image     | sklearn              |
| Satellite    | 6435 | 36   | 6               | image     | UCI repo. (Statlog)  |
| Wine         | 6497 | 11   | 2               | chemical  | UCI repo. (Wine Quality) |
| Spam         | 4601 | 57   | 2               | text      | UCI repo. (Spam)     |
| Avila        | 10430| 10   | 2               | layout    | UCI repo. (Avila)    |
| Drive        | 58569| 48   | 11              | current   | UCI repo. (Sensorless Drive Diagnosis) |
| Particle     | 130064| 50  | 2               | signal    | UCI repo. (MiniBooNE particle identification) |

these acute class imbalance issues, classes were merged as follows:

- In the Wine data set, class labels are wine quality scores. Two classes are obtained by comparing scores to a threshold of 5.
- In the Avila dataset, class labels are middle age bible copyist identities. The five first copyists are grouped in one class and the remaining ones in the other class.

Unlike synthetic data sets, we need to separate the original dataset into a train set and a test set. To avoid a dependency of the reported performances w.r.t train/test splits, we perform 2-fold cross validation (CV). Also, we shuffled at random examples and repeated the training and test phases 100 times.

To induce diversity in the base classifiers, we separated the training data into 5 distinct pieces using the following procedure: for each data set, for each class,

1. apply principal component analysis to the corresponding data,
2. project this data on the dimension with highest eigenvalue,
3. sort the projected values and split them into 5 subsets of cardinality \( n_i/5 \) where \( n_i \) is the proportion of examples belonging to class \( \omega_i \).

We argue that this way of splitting data leads to challenging fusion tasks because some base classifiers may see data that are a lot easier to separate than it should and will consequently not generalize very well. Actually, the training data to which classifier \( c_k \) has access is a non-iid sample of the distribution of \( (X, Y) \).

We used logistic regression with an \( L_2 \) regularization term to train the base classifiers. The regularization hyperparameter is set to default (i.e. 1.0).

To make sure that robustness observations from the previous subsection are confirmed on real data, we also add one copy of base classifier \( c_1 \), one adversary and one noisy classifier to the ensemble. Both the adversarial and noisy classifiers are built from \( c_1 \) with \( \theta = 0.9 \).

Average accuracies over random shuffles and CV-folds are given in Table 3 for \( K = 8 \) base classifiers. Train/validation split ratio are identical to the
Table 3: Classification accuracies (with bootstrap confidence intervals and standard deviations) for several real data sets when 5 base classifiers were trained separately on disjoint subsets of the datasets. Datasets were split in a non-iid way using a PCA based protocol. A clone of classifier $c_1$, an adversarial and noisy version of $c_1$ were added so there are $K = 8$ base classifiers to aggregate.

| Method       | MNIST | Satelite | Wac | Span | Avila | Drive | Particle |
|--------------|-------|----------|-----|------|-------|-------|----------|
| Cll. Selection | 70.86% ± 0.12% | 78.23% ± 0.12% | 64.92% ± 0.12% | 61.31% ± 0.12% | 60.66% ± 0.12% | 51.55% ± 0.12% | 82.11% ± 0.12% |
| Weighted Vote | 70.48% ± 0.15% | 75.34% ± 0.15% | 65.60% ± 0.15% | 65.22% ± 0.15% | 60.77% ± 0.15% | 56.72% ± 0.15% | 79.35% ± 0.15% |
| Exp. Weighted Vote | 70.10% ± 0.15% | 71.80% ± 0.15% | 65.63% ± 0.15% | 65.65% ± 0.15% | 62.35% ± 0.15% | 57.01% ± 0.15% | 82.64% ± 0.15% |
| Stacking | 65.52% ± 0.15% | 61.72% ± 0.15% | 66.65% ± 0.15% | 69.67% ± 0.15% | 65.58% ± 0.15% | 55.10% ± 0.15% | 84.70% ± 0.15% |
| Naive Bayes Agg. | 78.00% ± 0.10% | 75.55% ± 0.10% | 63.60% ± 0.10% | 66.74% ± 0.10% | 61.91% ± 0.10% | 66.86% ± 0.10% | 73.51% ± 0.10% |
| Bayes Agg. | 78.06% ± 0.10% | 75.55% ± 0.10% | 63.60% ± 0.10% | 66.74% ± 0.10% | 61.91% ± 0.10% | 66.86% ± 0.10% | 73.51% ± 0.10% |
| SPOCC | 79.13% ± 0.15% | 78.59% ± 0.15% | 64.92% ± 0.15% | 69.26% ± 0.15% | 63.10% ± 0.15% | 67.75% ± 0.15% | 82.13% ± 0.15% |
| adaSPOCC | 79.13% ± 0.15% | 78.59% ± 0.15% | 64.92% ± 0.15% | 69.26% ± 0.15% | 63.10% ± 0.15% | 67.75% ± 0.15% | 82.13% ± 0.15% |
| Best base Cll. | 78.80% ± 0.15% | 78.22% ± 0.15% | 60.66% ± 0.15% | 67.73% ± 0.15% | 62.00% ± 0.15% | 52.96% ± 0.15% | 82.33% ± 0.15% |
| Centralized Cll. | 77.52% ± 0.04% | 82.43% ± 0.04% | 72.87% ± 0.04% | 92.18% ± 0.04% | 76.23% ± 0.04% | 74.92% ± 0.04% | 82.56% ± 0.04% |
| AdaBoost | 41.31% ± 0.36% | 41.31% ± 0.36% | 41.31% ± 0.36% | 41.31% ± 0.36% | 41.31% ± 0.36% | 41.31% ± 0.36% | 41.31% ± 0.36% |

Table 4: Maximal accuracy discrepancy w.r.t. the best approach. Max is taken over the 8 datasets.

| Method     | Cll. Selection | Weighted Vote | Exp. Weighted Vote | Stacking | Naive Bayes Agg. | Bayes Agg. | SPOCC | adaSPOCC |
|------------|----------------|---------------|-------------------|---------|------------------|-----------|-------|---------|
| max. disc. | 16.2%          | 10.74%        | 4.82%             | 12.52%  | 18.44%           | 2.25%     |       |         |
Memory occupation became problematic for Bayes aggregation whenever $\ell > 6$. It achieves unsurprisingly poor performances when $\ell = 6$ (Satellite dataset) confirming its inability to scale w.r.t. $\ell$.

- adaSPOCC is one of the most efficient aggregation approach. It achieves the highest average rank (over the 8 datasets). adaSPOCC has average rank of 2.1 followed by the exponentially weighted vote which is in average the top 3 approach.

- adaSPOCC is robust in the sense that it achieves the minimal maximal discrepancy w.r.t. the best concurrent approach. Absolute values of maximal discrepancies (over the 8 datasets) w.r.t. the best approach are reported in Table [4].

5 Conclusion

In this article, a new classifier aggregation technique is introduced. This technique relies on the framework of possibility theory. Conditional probabilities of class labels given a classifier prediction are estimated on a validation set and transformed in possibility distributions. For each input to be classified, the set of possibility distributions issued by the classifier predictions are regarded as a set of propositions that are conjunctively combined using a t-norm. The obtained method, called SPOCC, is scalable w.r.t. to both the number of class labels and the number of base classifiers.

An adaptive version of this method, called adaSPOCC is also introduced. It is proposed to perform hierarchical agglomerative clustering to identify subsets of classifiers which are not statistically independent. Each such cluster can thus be combined sequentially with different t-norms. T-norms are chosen from the Aczel-Alsina parametric family which allows to reduce the impact of redundant predictions when necessary. Moreover, the individual impact of a base classifier can also be regulated by setting discounting coefficients. When one such coefficient is set to one, the corresponding classifier is discarded from the fusion process. These coefficients as well as the t-norm parameters are automatically tuned using heuristic search monitoring the ensemble accuracy on the validation set.

The adaptive version of this non-probabilistic aggregation method possesses a number of nice statistical properties. These properties are well supported by several numerical experiments on both synthetic and real data sets and clearly show its ability to tolerate adversaries, faults or information redundancy.

There are several future research tracks that we plan to investigate to further develop this contribution. One of them consists in investigating to what extent the approach is modular w.r.t. imprecise classifiers, i.e. classifiers that can only discriminate between subsets of class labels. Since possibility theory is natively compatible with set theory, this modularity seems not too challenging to achieve as opposed to many other concurrent approaches.

Another interesting question is to adapt the proposed classifier aggregation method to regression tasks. This will require to use possibilities on the real line for instance which may be computationally more demanding.

Finally, a sequential version of adaSPOCC would also be desirable. This essentially amounts to find a sequential version of hierarchical agglomerative
clustering and of the hyperparameter tuning heuristics.

A Heuristic search for dependency parameters

In this appendix, we explain how to set hyperparameters \((\lambda_a)_{a=1}^{K-1}\) which are necessary to execute the computation graph \(G\) as part of adaSPOCC. Each hyperparameter \(\lambda_a\) regulates the levels of dependency between operands aggregated using the Aczel-Alsina t-norm \(T_{\lambda_a}\). Each \(\lambda_a\) lives in \([1; +\infty]\) and we can use a logarithmic grid and the validation set to assess the impact of a given value of \(\lambda_a\) in terms of classification accuracy of the ensemble. However, resorting to baseline grid search has exponential complexity in \(K\) and we will thus employ a heuristic search to keep computation time at bay.

To cleverly browse possible values for \((\lambda_a)_{a=1}^{K-1}\), we can remark that HAC agglomerates classifiers from most dependent ones to least dependent ones. This implies that if \(V_a\) is a child node of \(V_{a'}\) in \(G\) then \(\lambda_a \geq \lambda_{a'}\). Consequently, instead of systematically visit all configurations, we will start by performing a grid search on the full grid for the lowest nodes in the hierarchy and freeze the corresponding hyperparameters. Then, we will move to their parent nodes and perform grid search only for smaller values. This sequential grid search has a maximal complexity equal to \(K - 1\) times the cost of a 1D grid search.

Another trick allowing to improve the procedure consists in jointly setting a subset of hyperparameters. Clusters can be obtained from \(G\). This is actually the original intent behind HAC. For a given number of clusters \(N_c\), clusters are obtained by thresholding cophenetic correlation coefficients between pairs of classifiers. More precisely, if a pair of classifiers have a cophenetic correlation distance below the threshold, they are considered to belong to different clusters. Starting with a sufficiently high value of the threshold so that all classifiers are in one unique cluster, the threshold is lowered until the constraint on \(N_c\) is violated, i.e. further lowering it yields \(N_c + 1\) clusters. Note that the obtained clusters always correspond to non-overlapping branches of \(G\). This strategy is wrapped up by iterating on \(N_c\), starting from \(N_c = 2\) to \(K\).

Algorithm 5 summarizes the proposed heuristic search for dependency parameters. The 1D grid search for one \(\lambda_a\) is performed on a predefined grid and the retained value is the one achieving highest accuracy of the classifier ensemble.

References

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Initialize $\lambda_a \leftarrow 1, \forall V_a \in G$.

for $N_c$ from 2 to $K$ do

Obtain $N_c$ clusters denoted by $C_1 \ldots C_{N_c}$.

for $i$ from 1 to $N_c$ do

Find $V_a$ s.t. its descendants contain $C_i$ and no other leaf node.
Perform grid search jointly for $\lambda_a$ and all $\lambda_a'$ in correspondence with non-leaf descendants of $V_a$.
Append node $V_a$ and all its descendants to the set $\text{Treated}$.

end

Obtain list $\text{Remain}$ of those nodes in $G$ which do not belong to $\text{Treated}$.
Sort list $\text{Remain}$ so that $\text{Remain}[i]$ cannot be an antecedent of $\text{Remain}[j]$ if $i < j$.

for $i$ from 1 to length of $\text{Remain}$ do

Perform grid search for $\lambda_a$ where $V_a = \text{Remain}[i]$ and subject to $\lambda_a \leq \lambda_a'$ for any $V_a'$ that is a descendant of $V_a$.

end

Save in $r_i$ the empirical error rate achieved with this value of $\lambda$.

if $N_c \geq 3$ and $r_i \geq r_{i-1}$ then

Revert to the value of $\lambda$ obtained at the previous iteration.

Stop looping.

end

end

Return $\lambda_a$.

**Algorithm 5:** Heuristic search for dependency hyperparameters

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