CONSTRUCTIONS OF MORSE MAPS FOR KNOTS AND LINKS,
AND UPPER BOUNDS ON THE MORSE–NOVIKOV NUMBER

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Abstract. The Morse–Novikov number $MN(L)$ of an oriented link $L \subset S^3$ is the minimum number of critical points of a Morse map $S^3 \setminus L \to S^1$ representing the class of a Seifert surface for $L$ in $H_1(S^3, L; \mathbb{Z})$ (e.g., $MN(L) = 0$ if and only if $L$ is fibered). We develop various constructions of Morse maps (Milnor maps, Stallings twists, splicing along a link which is a closed braid with respect to a Morse map, Murasugi sums, cutting a Morse map along an arc on a page) and use them to bound Morse–Novikov numbers from above in terms of other knot and link invariants (free genus, crossing number, braid index, wrapping genus and layered wrapping genus).

1. Introduction; statement of results

An oriented link $L \subset S^3$ determines a cohomology class $\xi_L \in H^1(S^3 \setminus L; \mathbb{Z}) \cong \pi_0(\text{Map}(S^3 \setminus L, K(\mathbb{Z}, 1)))$. The homotopy class of maps $S^3 \setminus L \to S^1 = K(\mathbb{Z}, 1)$ corresponding to $\xi_L$ contains smooth maps which are Morse (that is, have no degenerate critical points), and which restrict to a standard fibration in a neighborhood of $L$ (so they have only finitely many critical points). The minimum number of critical points of such a map is the Morse–Novikov number $MN(L)$. Tautologously, $MN(L) = 0$ if and only if $L$ is a fibered link. It is natural to ask how to calculate, or estimate, $MN(L)$ for general $L$. Moreover, for fibered links, there exist both nice characterizations in other terms (e.g., $MN(L) = 0$ if and only if the kernel of $\pi_1(S^3 \setminus L) \to \mathbb{Z} : [\gamma] \mapsto \int_\gamma \xi_L$ is finitely generated) and an array of interesting constructions (e.g., links of singularities and Milnor fibrations [24], Murasugi sums [26, 48, 6], Stallings twists [48], splicing [2]). Again, it is natural to ask what happens in general.

Some progress on these questions was made in [32]. There, the Morse–Novikov theory of maps from manifolds to the circle (introduced by Novikov [31], and previously applied to knots in $S^3$ by Lazarev [20]) was applied to give lower bounds for $MN(L)$: for example, it was shown (using analogues for Novikov homology of the Morse inequalities for ordinary homology) that for all $n \geq 0$, there exists a knot $K$ with genus $g(K) = n$ and $MN(K) \geq 2n$; it was also shown that there are knots with vanishing Novikov homology and non-zero Morse–Novikov number. Subadditivity

$$MN(L_0 \# L_1) \leq MN(L_0) + MN(L_1).$$

of Morse–Novikov number over connected sum was established by an explicit construction, and it was conjectured that a restatement in terms of Seifert surfaces extends to arbitrary Murasugi sums. (In fact, earlier work of Goda [12, 13] immediately implies [2]; Goda’s results are stated in terms of his “handle number”

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of a Seifert surface \( R \), not the Morse–Novikov number of the link \( \partial R \), and his exposition and proofs use Gabai’s language of sutured manifolds \([7]\) and \( C \)-product decompositions \([8]\), not that of Morse maps.)

The present paper continues the investigations of \([32]\). Section 2 assembles preliminary material and generalities on Morse maps. Section 3 constructs Milnor maps and provides two simple, but fundamental, examples of Milnor maps which are Morse maps but not fibrations: \( u \), a minimal Morse map for the (non-fibered) 2-component unlink \( U \), and \( o_1 \), a Morse map with two critical points for the (fibered) unknot \( O \). Section 4 describes Stallings twists of Morse maps. Section 5 shows how to a Morse map \( f \) : \( S^3 \setminus L \to S^1 \) (satisfying a condition much weaker than being a fibration) are associated certain maps of surfaces—monodromies and adiexodons (the latter being trivial when, and only when, \( f \) is a fibration)—from which \( f \) can be reconstructed. Section 6 introduces closed \( f \)-braids in \( S^3 \setminus L \), and constructs Morse maps by splicing along closed \( f \)-braids. Section 7 constructs Morse maps as Murasugi sums of simpler Morse maps, and in particular by cutting a Morse map along an arc on a page; Murasugi sums provide an alternative approach to Goda’s results, Cor. 9.1.

In Sections 8 and 9, Murasugi sums and our other constructions are used to relate the Morse–Novikov number to other knot invariants, as follows. Let \( K \) be a knot, \( \beta \in B_n \) an \( n \)-string braid. The free genus \( g_f(K) \) of \( K \) is the least genus of a Seifert surface \( S \) for \( K \) for which \( \pi_1(S^3 \setminus S) \) is a free group. The braid index \( \text{brin}(K) \) is the least \( m \) such that \( K \) can be represented as a closed \( m \)-string braid. The \( k \)-twisted, \( \pm \)-clased Whitehead double of \( K \) is the knot \( D(K, k; \pm) \) bounding the Seifert surface \( A(K, k) \ast A(O, \mp 1) \) plumbed along transverse arcs of an annulus \( A(K, k) \) having Seifert matrix \( [k] \) with \( K \subset \partial A(K, k) \) and a Hopf annulus \( A(O, \mp 1) \). The wrapping genus \( g_{\text{wr}}(K) \) (resp., layered wrapping genus \( g_{\text{wr}}(K) \)) of \( K \) is the least \( n \) such that \( K \) lies on a Heegaard surface of genus \( n \) (resp., \( K \) is isotopic to a closed 1-string \( o_n \)-braid, where \( o_n \) is the connected sum of \( n \) copies of \( o_1 \)). The crossing number \( c(K) \) of \( K \) is the least number of crossings in a knot diagram for \( K \).

**Free Genus Estimate.** \( \text{MN}(K) \leq 4g_f(K) \).

**Braid Index Estimate.** \( \text{MN}(D(K, m; \pm)) \leq 4\text{brin}(K) - 2 \).

**Wrapping Genera Estimate.** \( \text{MN}(D(K, m; \pm)) \leq 2(g_{\text{wr}}(K) + 1) \leq 2(g_{\text{wr}}(K) + 1) \).

**Crossing Number Estimate.** \( \text{MN}(D(K, m; \pm)) \leq 2(c(K) + 2) \).

In some cases, an upper bound deduced from one of these estimates coincides with the lower bound from \([32]\), so \( \text{MN}(L) \) is known precisely. More often, unfortunately, a large gap remains: the strongest inequality accessible for any knot \( K \) using the results of \([32]\) is \( \text{MN}(K) \geq 2g(K) \), and we are not aware of any technique which could be used to show that \( \text{MN}(K) > 2g(K) \) for some \( K \).

**Question.** Does there exist a knot \( K \) with \( \text{MN}(K) > 2g(K) \)?

This paper supercedes, and considerably extends, the second author’s preprint \([13]\) (in particular, the proof of the Free Genus Estimate in \([13]\) was inadequate, and \([12, 13]\) had been overlooked). Both authors thank Hiroshi Goda, Andrei Pajitnov, and Claude Weber for helpful conversations and communications, and Walter Neumann for comments on a draft of this paper. The Section de Mathématiques of the University of Geneva provided extensive hospitality during much of this research.

2. Preliminaries and generalities

The symbol \( \square \) signals either the end or the omission of a proof. The notations \( A := B \) and \( B : =: A \) both define \( A \) to mean \( B \). Terms being defined are set in
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Spaces, maps, etc., are smooth ($C^\infty$) unless otherwise stated. The set of critical points of a map $f: M \to N$ is denoted $\text{crit}(f)$; for $x \in \text{crit}(f)$, let $\text{ind}(f; x)$ denote the index of $f$ at $x$. Manifolds may have boundary, but corners only if so noted. A closed manifold is one which is compact and has empty boundary. Manifolds are (not only orientable, but) oriented unless otherwise noted; in particular, $\mathbb{R}$, $\mathbb{C}^n$, $D^{2n} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 \leq 1\}$, and $S^{2n-1} := \partial D^{2n}$ are equipped with standard orientations, as is $S^2$ when it is identified with the Riemann sphere $\mathbb{P}_1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$. The manifold $M$ with its orientation reversed is denoted $-M$. The interior (resp., boundary) of $M$ is denoted by $\text{Int} M$ (resp., $\partial M$).

For suitable $Q \subset M$, let $\text{Nb}(Q \hookrightarrow M)$ denote a closed regular neighborhood of $Q$ in $(M, \partial M)$, and let $\text{Ext}(Q \hookrightarrow M)$ denote $M \setminus \text{Int} \text{Nb}(Q \hookrightarrow M)$, the exterior of $Q$. A submanifold $Q \subset M$ is proper if $\partial Q = Q \cap \partial M$. If $Q$ is a codimension–2 submanifold of $M$ with trivial normal bundle, then a trivialization $\tau: Q \times D^2 \to \text{Nb}(Q \hookrightarrow M)$ is adapted to a map $f: M \setminus Q \to S^1$ if $\tau(\xi, 0) = \xi$ and $f(\tau(\xi, z)) = z/|z|$ for $z \neq 0$.

An arc is a manifold diffeomorphic to $[0, 1]$. A surface is a compact 2-manifold. A link $L$ is a non-empty closed 1-submanifold of $S^3$; a knot is a connected link. A spanning surface for a link $L$ is a surface $S \subset S^3$ with $\partial S = L$; a Seifert surface for $L$ is a spanning surface for $L$ without closed components (every $L$ has a Seifert surface).

If $K$ is a knot and $k \in \mathbb{N}$, then a $k$-twisted annulus of type $K$ is any annulus $A(K,k) \subset S^3$ such that $K \subset \partial A(K,k)$ and the linking number in $S^3$ of the 1-cycles $K$ and $\partial A(K,k) \setminus K$ is $-k$. If $S \subset S^3$ is a surface and $K \subset S$ is a knot, then the $S$-framing of $K$ is the integer $k$ such that $\text{Nb}(K \hookrightarrow S) = A(K,k)$.

A handlebody of genus $g$ is a boundary-connected sum $(S^1 \times D^2)_1 \sqcup \cdots \sqcup (S^1 \times D^2)_g \cong: \mathcal{H}^g$ of $g \geq 0$ solid tori. A Heegaard surface for $K$ is $\mathcal{H}^g \subset S^3$ is a Seifert surface for $K$ and $\partial \mathcal{H}^g \subset S$ is a Heegaard handlebody. A genus-$g$ Heegaard splitting of $S^3$ is a pair $(\mathcal{H}^g_1, \mathcal{H}^g_2)$ where $\mathcal{H}^g_1$ is Heegaard (so $\mathcal{H}^g_2$ and $\partial \mathcal{H}^g_1 = \partial \mathcal{H}^g_2$ are Heegaard as well). According to Waldhausen [20], up to isotopy there is only one genus-$g$ Heegaard handlebody or surface in, or splitting of, $S^3$.

Let $L \subset S^3$ be a link. The image of the fundamental class $[S] \in H_2(S,L;\mathbb{Z})$ in $H_2(S^3,L;\mathbb{Z}) \cong H^1(S^3 \setminus L;\mathbb{Z})$ is independent of the choice of spanning surface $S$ for $L$; let $\xi_L \in H^1(S^3 \setminus L;\mathbb{Z}) \cong \pi_0(\text{Map}(S^3 \setminus L, S^1))$ correspond to $[S]$. Call $f: S^3 \setminus L \to S^1$ simple if $\xi_L \cong [f] \in \pi_0(\text{Map}(S^3 \setminus L, S^1))$, and Morse if it is smooth and has no degenerate critical points. The Morse–Novikov number of $L$, written $\text{MN}(L)$, is the least $n$ such that some simple Morse map $f: S^3 \setminus L \to S^1$ has $n$ critical points.

**Definitions.** Let $f: S^3 \setminus L \to S^1$ be a simple Morse map. The binding of $f$ is the link $L$. A page of $f$ is any $S(f, \theta) := L \cup f^{-1}(\exp(i\theta))$ for $\exp(i\theta) \in S^1$; the page $S(f, \theta)$ is smooth if $\exp(i\theta) \in S^3 \setminus \text{crit}(f)$, singular if $\exp(i\theta) \in \text{crit}(f)$. Say that $f$ is:

(a) boundary-regular if $f$ has an adapted trivialization $\tau: L \times D^2 \to \text{Nb}(L \hookrightarrow S^3)$; (b) moderate if $\text{ind}(f; x) \in \{1, 2\}$ for all $x \in \text{crit}(f)$; (c) self-indexed if $|f| \circ \text{crit}(f)$ factors as $x \mapsto \text{ind}(f; x)$ followed by an injection; (d) minimal if $\text{card}(\text{crit}(f)) \leq \text{card}((\text{crit}(g))$ for all simple Morse $g: S^3 \setminus L \to S^1$; (e) boundary-connected if every page of $f$ has trivial second homology (equivalently, if no page of $f$ contains a non-empty closed surface); (f) connected if every page of $f$ is connected.

**Proposition 1.** Let $f: S^3 \setminus L \to S^1$ be a simple Morse map.

1. If $\text{card}(\text{crit}(f)) < \infty$, then up to proper isotopy $f$ is boundary-regular.

2. If $f$ is boundary-regular, then: (a) $\text{card}(\text{crit}(f))$ is finite and even; (b) every smooth page of $f$ is a spanning surface for $L$; and (c) $f$ is boundary-connected if and only if every smooth page of $f$ is a Seifert surface for $L$. 

(3) If either (a) \( f \) is minimal or (b) \( f \) is boundary-connected, then \( f \) is moderate.
(4) If \( f \) is moderate, then up to isotopy \( f \) is moderate and self-indexed; if also \( \text{card}(\text{crit}(f)) < \infty \), then the isotopy may be taken to be proper.

(5) If \( f \) is connected, then \( f \) is boundary-connected.
(6) If \( f \) is boundary-connected and every Seifert surface for \( L \) is connected (e.g., if \( L \) is a knot), then \( f \) is connected.

(7) Let \( f \) be boundary-regular, moderate, and self-indexed. (a) If \( \text{crit}(f) = \emptyset \), then: (i) \( f \) is connected; (ii) \( f \) is a fibration over \( S^3 \); and (iii) any two of the pages of \( f \) are isotopic (rel. \( L \)). (b) If \( \text{crit}(f) \neq \emptyset \), then: (i) half the critical points of \( f \) are of index 1 and half of index 2; (ii) \( \int f^{-1}(S^3 \setminus f(\text{crit}(f))) \) is a trivial fibration over each of the two components of \( S^3 \setminus f(\text{crit}(f)) \); (iii) the smooth pages of \( f \) fall into two isotopy classes (rel. \( L \)); and (iv) if \( S(f, \theta_1) \) and \( S(f, \theta_2) \) belong to these two isotopy classes, then \(|\chi(S(f, \theta_1)) - \chi(S(f, \theta_2))| = \text{card}(\text{crit}(f))\).

Proof. Straightforward. (For (3a) and (7b), see [32]. A slightly more precise statement of (7(b)IV) appears in Cor. [6,1].)

In case (7a), \( L \) is (as usual) called a fibered link and \( S(f, \theta) \) is called a fiber surface of \( f \). (Also as usual, any Seifert surface for \( L \) isotopic to a page of \( f \) is called a fiber surface for \( L \).) In case (7b), any smooth page of smaller (resp., larger) Euler characteristic will be called a large (resp., small) page, spanning surface, or Seifert surface, as the case may be, of \( f \) (note: not “of \( L \”) . Any fiber surface of a fibration \( f \) may be called either large or small, as suits convenience.

Convention. Henceforth, all Morse maps are boundary-regular and simple.

Proposition 2. If every Seifert surface for \( L \) is connected, then every minimal Morse map \( f: S^3 \setminus L \to S^1 \) is boundary-connected (and so, by Prop. 1[6], connected).

Proof. Given any link \( L \), and a Morse map \( f: S^3 \setminus L \to S^1 \) which is not boundary-connected, there is a spanning surface \( S(f, \theta_0) \) such that the union \( S' \) of all closed components of \( S(f, \theta_0) \) is non-empty. If the Seifert surface \( S'' := S(f, \theta_0) \setminus S' \) is connected, then by Alexander duality \( H_2(S^3 \setminus S''; \mathbb{Z}) \cong \widetilde{H}_0(S''; \mathbb{Z}) = \{0\} \), so by a standard argument there is a compact 3–submanifold \( M \subset S^3 \setminus S'' \) such that \( \partial M \neq \emptyset \) is a component of \( S' \). Every 1–cycle in \( M \), being disjoint from \( S'' \), has linking number 0 with \( L \). It follows that the restriction \( f|\bar{M}: M \to S^1 \) has a continuous lift through \( \mathbb{R} \to S^1: \theta \mapsto \exp(i\theta) \) to \( f: M \to \mathbb{R} \); \( f \) is a Morse function rel. \( S' \), and has the same critical points, with the same indices, as \( f|\bar{M} \). Since \( M \) is compact, \( f \) has (global) extrema in \( \text{Int} M \), which are local extrema of \( f|\text{Int} M \) and thus of \( f \), so \( f \) is not moderate. By Prop. 1[6a], \( f \) is not minimal. 

For many links \( L \) (e.g., knots, fibered links) the hypothesis of Prop. 2 is satisfied. However, for many other links (e.g., split links) it fails; for at least some such links, the conclusion of Prop. 2 also fails (see Example 3).

Questions. (1) Does there exist a link \( L \) for which the conclusion of Prop. 2 holds although the hypothesis fails? (2) Does there exist a link \( L \) for which no minimal Morse function is boundary-connected?

3. Morse maps from Milnor maps

The first explicit Morse maps (in fact, fibrations) for an infinite class of links were given by Milnor’s celebrated Fibration Theorem [24], where they appear as (the instances for \( n = 2 \) of) what are now called the “Milnor maps” associated to singular
points of complex analytic functions $\mathbb{C}^n \to \mathbb{C}$. For present and future purposes, it is useful to extend somewhat the framework in which Milnor studied these maps. Given a non-constant meromorphic function $F: M \sim \mathbb{P}(1)$ on a complex manifold $M$, let $\mathbb{D}(F)$ be the (possibly singular) complex hypersurface which is the closure in $M$ of $F^{-1}(0) \cup F^{-1}(\infty)$.

**Definitions.** The argument of $F$ is $\arg(F) := F/|F|: M \setminus \mathbb{D}(F) \to S^1$. For $M = \mathbb{C}^n$, the Milnor map of $F$ is $\varphi_F := \arg(F)(S^{2n-1} \setminus \mathbb{D}(F))$; for $r > 0$, the Milnor map of $F$ at radius $r$, denoted by $\varphi_{F,r}$, is the Milnor map of $(z_1, \ldots, z_n) \mapsto F(z_1/r, \ldots, z_n/r)$.

**Lemma A.** Let $F: \mathbb{C}^n \sim \mathbb{P}(1)$ be meromorphic and not constant. A necessary and sufficient condition for $(z_1, \ldots, z_n) \in S^{2n-1} \setminus \mathbb{D}(F)$ to be a critical point of $\varphi_F$ is that the complex vectors

$$
(\overline{z}_1, \ldots, \overline{z}_n), \quad \frac{1}{iF(z_1, \ldots, z_n)} \left( \frac{\partial F}{\partial z_1}(z_1, \ldots, z_n), \ldots, \frac{\partial F}{\partial z_n}(z_1, \ldots, z_n) \right)
$$

be linearly dependent over $\mathbb{R}$.

**Proof.** For holomorphic $F$, this is [24] Lemma 4.1, and Milnor’s proof there applies equally well to meromorphic $F$. \qed

**Construction 1 (Milnor maps).** Let $F: \mathbb{C}^2 \sim \mathbb{P}(1)$ be meromorphic, not constant, and suppose each irreducible analytic component of $\mathbb{D}(F)$ has multiplicity 1 (i.e., $F$ has no repeated factors in the algebra of meromorphic functions $\mathbb{C}^2 \sim \mathbb{P}(1)$). If we let

$$
m(F) := \inf \{|z|^2 + |w|^2 : (z, w) \in \mathbb{D}(F)\} = \sup \{r : rS^3 \cap \mathbb{D}(F) = \emptyset\},
$$

then the reasoning in [24] shows that there is a set $X(F) \subset \mathbb{P}(1)$, $r$ finite in case $F$ is rational, and discrete in $\mathbb{P}(1)\setminus \mathbb{P}(1)$, such that: (a) if $r \in \mathbb{P}(1)$ intersects $rS^3$ transversally, so that $L(F, r) := (1/r)(\mathbb{D}(F) \cap rS^3)$ is a link in $S^3$; (b) if $r$ and $r'$ are in the same component of $\mathbb{P}(1)$ then $L(F, r)$ and $L(F, r')$ are isotopic; and (c) if $m(F) < r \notin X(F)$, then $\varphi_{F,r}$ is simple, and there is a trivialization of $\text{Nb}(L(F, r) \hookrightarrow S^3)$ which is adapted to $\varphi_{F,r}$—so that, if also $\varphi_{F,r}$ has no degenerate critical points, then $\varphi_{F,r}$ is a Morse map (in the sense of the convention on p. 4). In practice, Lemma A makes it easy to locate the critical points of a Milnor map and check them for non-degeneracy.

Several special cases of this construction have special names.

If $F$ is holomorphic and $(0,0) \in \mathbb{D}(F) = F^{-1}(0)$ (so $m(F) = 0$), then for every $r \in \mathbb{P}(1)$, $\inf X(F)$, the link $L(F, r)$ (known as the link of the singularity of $F$ at $(0,0)$) is fibered, and $\varphi_{F,r}$ is a fibration (known as the Milnor fibration of $F$). Up to isotopy, the link of the singularity and the Milnor fibration are independent of $r < \inf X(F)$.

If $F$ is meromorphic, $(0,0) \in \mathbb{D}(F)$ (so $m(F) = 0$), and neither $F$ nor $1/F$ is holomorphic at $(0,0)$ (that is, $(0,0)$ is a point of indeterminacy of $F$: it is in both the closure of $F^{-1}(0)$ and the closure of $F^{-1}(\infty)$), then we will call $L(F, r)$ the link of indeterminacy of $F$ at $(0,0)$. Up to isotopy, the link of indeterminacy is independent of $r < \inf X(F)$.

If $F$ is a polynomial (i.e., both holomorphic and rational), then up to isotopy the link $L(F, r)$ is independent of $r > \sup X(F)$; it is called the link at infinity of $F$ (cf. [34]) and denoted $L(F, \infty)$. (Warning: in [29], [28], and elsewhere, the phrase “link at infinity of $F$” denotes what we will call the generic link at infinity of $F$, that is, the intersection of a generic fiber of $F$ with any sufficiently large 3–sphere; that link can differ substantially from the link at infinity of $F$ defined here.)

If $F$ is rational and neither a polynomial nor the reciprocal of a polynomial, then up to isotopy $L(F, r)$ is independent of $r > \sup X(F)$; we will call it the link of indeterminacy at infinity of $F$ and denote it $L(F, \infty)$. 
Several of the following examples of Milnor maps will be used in later sections.

**Example 1.** Given \( p, q \geq 0 \) with \( p+q = 1 \) in case \( pq = 0 \), let \( F_{p,q} : \mathbb{C}^2 \to \mathbb{C} : (z, w) \mapsto p z^p + q w^q \); \( X(F_{p,q}) = \emptyset \) because \( F_{p,q} \) is weighted-homogeneous. The link of the singularity of \( F_{0,1}^{-1}(0) \) at \((0,0)\) (of course \( F_{0,1}^{-1}(0) \) is not in fact singular at \((0,0)\)), namely, \( \{(z, w) \in S^3 : w = 0\} \), is denoted \( O \), and the Milnor fibration of \( F_{0,1} = pr_2 \) will be denoted \( o \). More generally, the link of the singularity of \( F_{p,q}^{-1}(0) \) at \((0,0)\) is denoted \( O\{p, q\} \), and the Milnor fibration of \( F_{p,q} \) will be denoted \( o\{p, q\} \).

**Definitions.** A knot isotopic to \( O \) is called an unknot; in particular \( O \) itself is called the horizontal unknot, and \( O':=\{(z, w) \in S^3 : z = 0\} \) is called the vertical unknot. A link isotopic to \( O\{p, q\} \) (resp., to the mirror image of \( O\{p, q\} \)) is called a torus link of type \((p, q)\) (resp., type \((p, -q)\), or equivalently \((-p, q)\)). A torus link of type \((2, 2)\) (resp., \((2, -2)\)) is a positive (resp., negative) Hopf link, and its fiber surface \( A(O,-1) \) is a positive (resp., negative) Hopf annulus.

A link of indeterminacy is never a knot, and is always obtained from a (multicomponent) link of a singularity by reversing the orientation of some, but not all, components; a link of indeterminacy is never isotopic to the link of a singularity.

Recent work of Pichon [33] confirms an empirical observation (about certain real polynomial maps \( \mathbb{R}^4 \to \mathbb{R}^2 \)) recorded offhandedly at the end of [33, Example 4.7], and immediately implies that the link of indeterminacy of \( F \) is a fibered link if and only if \( \varphi_{F,r} \) is a fibration for all sufficiently small \( r > 0 \).

**Example 2.** Calculations using Lemma [A] show that the link of indeterminacy of \( \left(z^2 + w^3\right)/\left(z^3 + w^2\right) \) at \((0,0)\) is fibered by the Milnor map at small radius. The link of indeterminacy of \( z/w \) at \((0,0)\) (and at infinity) is a negative Hopf link and is fibered by its Milnor map. The Milnor map of \( \left(z^2 + w^3\right)/\left(z^2 - w^2\right) \) at any radius \( r > 0 \) is Morse but not a fibration; the link of indeterminacy at \((0,0)\) (and at infinity) is not fiberable, since it has a disconnected Seifert surface (two disjoint annuli), whereas for homological reasons every Seifert surface of a fibered link is connected.

Like links of indeterminacy, many links at infinity are fibered (for instance, if \( L(F, \infty) \) is a knot, then it is fibered [29]) but some are non-fiberable; in the fibered case, the Milnor map at sufficiently large radius is a fibration, and in every case the Milnor map is Morse provided that its critical points are non-degenerate. There is a very limited overlap between the classes of links at infinity and links of singularities (for instance, since \( X(F_{p,q}) = \emptyset \), the link at infinity of \( F_{p,q} \) is \( O\{p, q\} \), and—as it happens—any knot which is both a link at infinity and the link of a singularity is a torus knot of type \((p, q)\) for positive, relatively prime \( p \) and \( q \) [34, 28]. The polynomials with fibered links at infinity have been variously characterized by several authors (see Bodin [2] and references therein).

**Example 3.** Both \( F : (z, w) \mapsto (w^2 - z)^2 - z^5 - 4z^4w \) and \( G : (z, w) \mapsto z(zw + 1) \) have links at infinity which are fibered by their Milnor maps at sufficiently large radius. The knot \( L(F, \infty) \) is not the link of a singularity: \( L(F, \infty) \) is a non-trivial cable on a non-trivial torus knot (as is readily seen from the parametrization \( \zeta \mapsto (\zeta^4, \zeta^5 + \zeta^6) \) of \( F^{-1}(0) \)), and therefore not itself a torus knot. The generic link at infinity of \( G \) is the link at infinity of \( (z, w) \mapsto (z + t)((z + t)w + 1) - t = z^2w + 2tzw + z + t^2w \) for \( t \neq 0 \), and is not fiberable (see [29, corrigendum]).

Like a link of indeterminacy, a link of indeterminacy at infinity is never a knot, and is always obtained from a (multicomponent) link at infinity by reversing the orientation of some, but not all, components. A link of indeterminacy at infinity can be isotopic to a link at infinity. Some links of indeterminacy at infinity are fibered, others are non-fiberable.


**Figure 1.** Can this fibered knot be fibered by a Milnor map?

**Figure 2.** A small and a large Seifert surface of $u$.

**Question.** If the link of indeterminacy at infinity of $F$ is fibered, is $\varphi_{F,r}$ is a fibration for all sufficiently large $r$? (Can the methods of [33] be adapted to this context?)

**Example 4.** Let $G_0: \mathbb{C}^2 \sim \mathbb{P}_1(\mathbb{C}) : (z,w) \mapsto zw/(4z - 1)$. Easily, $X(G_0) = \{1/4\}$. Calculation with Lemma A shows that, for every $r > 0$, $\varphi_{G_0,r}$ has no critical points. Thus the link of indeterminacy at infinity $L(G_0, \infty) = L(G_0, 1)$ is fibered by $\varphi_{G_0}$. (Alternatively, though less explicitly, it is easy to recognize that $L(G_0, \infty)$ is isotopic to the connected sum of a positive and a negative Hopf link; and the connected sum of fibered links is well known to be fibered—a proof from the point of view of Morse maps is given in [32].) More generally, if $k \in \mathbb{Z}$ and $G_k: \mathbb{C}^2 \sim \mathbb{P}_1(\mathbb{C}) : (z,w) \mapsto zw/(4z - w^k)$, then $L(G_k, \infty) = L(G_k, 1)$ is fibered by $\varphi_{G_k}$. (Notice that $L(G_1, \infty)$ is isotopic to the link at infinity $L(G, \infty)$ in Example 3.)

**Example 5.** It can be shown (for instance, by using the techniques of [37]) that, for all sufficiently small $\varepsilon > 0$, if $F(z,w) = 1 - z^2 + 3z^6 + (\varepsilon w)^3 - 3\varepsilon w$, then $L(F, 1)$ is the 8-crossing knot pictured in Fig. 1. This knot is not the link of a singularity nor a link at infinity. Although it is fibered (being a closed positive braid, [38]), calculations using Lemma A show that $\varphi_F$ has at least 2 critical points. We do not know if there exists $G$ such that some link $L(G, r)$ is isotopic to $L(F, 1)$ and $\varphi_{G,r}$ is a fibration.

**Example 6.** If $F: \mathbb{C}^2 \to \mathbb{C} : (z,w) \mapsto z(2z - 1)$ and $G: \mathbb{C}^2 \sim \mathbb{P}_1(\mathbb{C}) : (z,w) \mapsto z^{-1}(2z - 1) = 2 - z^{-1}$, then $L(G, \infty)$ is obtained from $L(F, \infty)$ by reversing the orientation of one component; since both links are evidently split links of two unknotted components, they are isotopic. For $r \geq 1/4$, and in particular for $r > \sup X(F) = 1/2$, $\varphi_{F,r}$ has a 1–sphere of degenerate critical points, so is not Morse. By contrast, $\varphi_{G,r}$ is Morse for $r \geq 1/2 = \sup X(G)$, with one critical point of index 1 and one of index 2. Let $U := L(G, 1) = L(G, \infty)$, $u := \varphi_G$. (The situation is pictured in Fig. 2.)

**Example 7.** A Milnor map which is a Morse map need not be minimal. Let $F(z,w) = 4w + 3(w^2 + z^2)$. Calculations using Lemma A show that $X(F) = \{4/3\}$; for $0 < r < 4/3$, $L(F,r)$ is isotopic to the link of the singularity of $F$ at $(0,0)$, an unknot, while for $4/3 < r < \infty$, $L(F,r)$ is isotopic to $O\{2,2\}$, the link at infinity of $F$. For
onto which $\varphi$ is a fibration; for $r = 2/3$, $\text{crit}(\varphi_{F,r}) = \{(z, w) : \text{Re}(z) = 0, \text{Re}(w) = -1/3, |z|^2 + |w|^2 = 4/9\}$ is a circle of degenerate critical points; for $2/3 < r < 4/3$, $\varphi_{F,r}$ is Morse, and $\text{crit}(\varphi_{F,r}) = \{(0, w) : |w| = r, |w + 1| = 1/3\}$ consists of two points, one of index 1 and one of index 2. In particular, $\varphi_F$ is a Morse map for the unknot $L(F, 1)$. Let $o_1 : S^3 \setminus O \to S^1$ be the Morse map for $O$ onto which $\varphi_F$ is carried by an isotopy carrying $L(F, 1)$ onto $O$. A small and a large Seifert surface of $o_1$ are pictured in Fig. 3.

The non-fibrations $u$ and $o_1$, trivial though they be, are ingredients of fundamental importance throughout the following sections.

**Proposition 3.** $\text{MN}(U) = 2$ and $u$ is minimal.

**Proof.** Since $U$ has a disconnected Seifert surface, it is not fibered. \qed

**Example 8.** If $\text{MN}(L) = 0$, then any two minimal Morse maps for $L$ are isotopic: this simply rephrases the well-known fact that, up to isotopy, a fibered link has a unique fibration. If $\text{MN}(L) > 0$, then there may be more than one isotopy class of minimal Morse map. We illustrate this by constructing a minimal Morse map for $U$ which is not isotopic to $u$.

First, construct two Morse maps—non-moderate, and barely non-minimal—for $O$, as follows. Let $B^3 \subset S^3 \setminus O$ be a ball with $1 \in o(\text{Int} B^3)$ and $-1 \neq o(B^3)$, and such that (a continuous branch of) $-i \log(o|\partial B^3)$ is a Morse function on $\partial M$ with exactly two critical points. By a standard argument (see [23]), for ext $\in \{\text{min}, \text{max}\}$ there is a smooth homotopy (supported off a neighborhood of $\partial B^3$) from $-i \log(o|B^3)$ to a function $g_{\text{ext}} : B^3 \to [-\pi/2, \pi/2]$ such that: (a) $g_{\text{ext}}$ has exactly two critical points in $\text{Int} B^3$, both non-degenerate, having indices 0 and 1 in the case of $g_{\text{min}}$ and indices 2 and 3 in the case of $g_{\text{max}}$; (b) 0 is a regular value of $g_{\text{ext}}$; (c) the critical values of $g_{\text{ext}}$ are of opposite sign. Then there is a smooth homotopy (supported in $\text{Int} B^3$) from $o$ to a non-moderate Morse map $o_{\text{ext}} : S^3 \setminus O \to S^1$ with $o_{\text{ext}}(B^3) = \exp(i g_{\text{ext}})$ and $o_{\text{ext}}(B^3) = o(B^3)$. Clearly $S(o_{\text{ext}}, -1)$ is a 2–disk and $S(o_{\text{ext}}, 1)$ is the disjoint union of a 2–disk and a 2–sphere $S^2_{\text{ext}}$. Let $B^3_{\text{ext}} \subset S^3$ be the 3–ball with $\partial B^3_{\text{ext}} = S^2_{\text{ext}}$ and $S(o_{\text{ext}}, -1) \subset \text{Int} B^3_{\text{ext}}$. The identification space $\Sigma := (B^3_{\text{min}} \cup B^3_{\text{max}})/\equiv$, where $\equiv$ identifies $S^2_{\text{min}}$ to $S^2_{\text{max}}$ by a diffeomorphism, is a piecewise-smooth 3–sphere which is easily given a smooth structure such that

$$v := (o_{\text{min}}(B^3_{\text{min}} \setminus O) \cup o_{\text{max}}(B^3_{\text{max}} \setminus O)) /\equiv$$

has exactly two critical points, both nondegenerate, of indices 1 and 2. (The same construction one dimension lower is pictured in Fig. 4.) If $\delta : S^3 \to \Sigma$ is a diffeomorphism with $\delta(U) = \partial S(o_{\text{min}}, -1) \cup \partial S(o_{\text{max}}, -1)$, then $\omega := v \circ \delta$ is a non-boundary-connected minimal Morse map for $U$; a large Seifert surface of $\omega$ is the union of two disjoint 2–disks, and a small spanning surface is the disjoint union of a 2–sphere and two 2–disks which are separated by the 2–sphere.
Figure 4. Top left: a Morse modification, with a local minimum and a saddlepoint, of $-i$ times the logarithm of the Milnor fibration of the identity map of $\mathbb{P}_1(\mathbb{C}) \setminus \{0, \infty\}$. Bottom left: a similar map with a saddlepoint and a local maximum. Left: $-i$ times the logarithm of a Morse map (with two saddlepoints) $S^2 \setminus \{x_1, x_2, x_3, x_4\} \cong (\mathbb{P}_1(\mathbb{C}) \setminus \{0, \infty\}) \# (\mathbb{P}_1(\mathbb{C}) \setminus \{0, \infty\}) \to S^1$.

Links of singularities, links at infinity, links of indeterminacy, and links of indeterminacy at infinity are all graph links in $S^3$ as defined by Eisenbud & Neumann [5]. Graph links are highly atypical of links $L$ which have Milnor maps $S^3 \setminus L \to S^1$. A large class of such links (namely, precisely those such links whose Milnor maps come from holomorphic—not merely meromorphic—maps $F: \mathbb{C}^2 \to \mathbb{C}$) consists of the transverse $\mathbb{C}$–links in the sense of [10]; by Boileau & Orevkov [3], transverse $\mathbb{C}$–links are exactly the same, up to isotopy, as quasipositive links (defined and studied in [36] and its sequels). A typical quasipositive link—for example, the knot in Example 5—is very far from being a graph link.

We conclude this section with a series of questions about links which have, up to isotopy, a Milnor map which is a minimal Morse map—briefly, links which are M–M–good. Prop. 3 asserts that $U$ is M–M–good; Milnor’s Fibration Theorem [24] proves that the link of a singularity is M–M–good; Pichon’s results [33] show that a fibered link of indeterminacy is M–M–good. Although the knot in Example 5 has a Milnor map which is Morse, we do not know if it is M–M–good.

Questions. (1) Is every transverse $\mathbb{C}$–link M–M–good? (2) If (1) cannot be answered in the affirmative (or until it is), is at least every strongly quasipositive link M–M–good? (By definition, $L$ is strongly quasipositive if and only if $L$ has a quasipositive Seifert surface, as defined in [35]; by [39], $L$ is strongly quasipositive if and only if $L$ bounds a subsurface of a fiber surface of a torus link of type of $\{p, q\}$ for some $p, q \geq 1$. A strongly quasipositive link is quasipositive; many quasipositive links are not strongly quasipositive.) (3) If (2) cannot be answered in the affirmative (or until it is), is at least every strongly quasipositive fibered link M–M–good—that is, does every strongly quasipositive fibered link have a fibration which, up to isotopy, is the Milnor map of a holomorphic function? (It follows from [29] [38] [39], and deep results of Giroux [10] [11] that a fibered link $L$ is strongly quasipositive if and only if $L$ is a stable plumbing of positive Hopf links; see p. [24]) (4) Is at least the knot in Fig. 1 which is fibered and strongly quasipositive, M–M–good?

4. Stallings twists of Morse maps

Define $\Xi: S^3 \to S^3$ by $\Xi(z, w) = (z \arg(w), w)$ for $w \neq 0, \Xi(z, 0) = (z, 0)$. Although the bijection $\Xi$ is discontinuous at each point of the standard unknot $O$, its restriction
\( \Xi(S^3 \setminus O) : S^3 \setminus O \to S^3 \setminus O \) is a diffeomorphism (as is \( \Xi = \text{id}_O : O \to O \)). If \( X \subset S^3 \setminus O \) is a manifold, then the manifold \( \varsigma_n(X, O) := \Xi^n(X) \subset S^3 \) is diffeomorphic to \( X \), but may or may not be isotopic to \( X \) for \( n \neq 0 \). (Fig. 5 illustrates the relationship among \( X \), \( O \), and \( \varsigma_1(X, O) \) when \( X = L \) is a link.) More generally, given any unknot \( \gamma \subset S^3 \), and a diffeomorphism \( \delta : S^3 \to S^3 \) with \( \gamma = \delta(O) \), let \( \Xi_\gamma := \delta^{-1} \circ \Xi \circ \delta \), and for \( X \subset S^3 \setminus \gamma \), let \( \varsigma_n(X, \gamma) := \Xi^n_\gamma(X) \). Up to isotopy, \( \varsigma_n(X, \gamma) \) does not depend on the choice of \( \delta \).

Let \( M \subset S^3 \) and \( Q \) be manifolds. Say a smooth map \( f : M \to Q \) is flat near \( O \) when \( O \) has a regular neighborhood \( \text{Nb}(O \to S^3) \) such that \( f|(M \cap \text{Nb}(O \to S^3)) \) factors through \( (z, w) \mapsto ((1 - |w|^2)^1/2, w) \). More generally, given an unknot \( \gamma \) and a diffeomorphism \( \delta : S^3 \to S^3 \) with \( \gamma = \delta(O) \), say that \( f \) is flat near \( \gamma \) if \( f \circ \delta^{-1} \) is flat near \( O \). (This does depend on the choice of \( \delta \).) If \( f \) is flat near \( \gamma \), then \( f \) cannot see that \( \Xi_\gamma \) is discontinuous—in fact, \( f \circ \Xi_\gamma : M \to Q \) is smooth, \( \text{crit}(f \circ \Xi_\gamma) = \Xi^{-1}_\gamma(\text{crit}(f)) \), and if \( Q = \mathbb{R} \) or \( Q = S^1 \), then \( \Xi_\gamma|_{\text{crit}(f \circ \Xi_\gamma)} \) is index-preserving.

**Construction 2 (Stallings twists).** Let \( L \subset S^3 \) be a link, \( f : S^3 \setminus L \to S^1 \) a Morse map. If \( (a) \) \( \gamma \) is an unknot, \( (b) \) for some \( \theta, \gamma \subset \text{Int}(S(f, \theta) \setminus \text{crit}(f)), \) and \( (c) \) the \( S(f, \theta) \)-framing of \( \gamma \) is 0, then it is easy to check that \( f \) is flat near \( \gamma \) for an appropriate choice of \( \delta \). Having made that choice, call \( \varsigma_n(f, \gamma) := f \circ \delta \circ \Xi^n : S^3 \setminus \varsigma_n(L, \gamma) \to S^1 \) the \( n \)-fold Stallings twist of \( f \) along \( \gamma \).

**Proposition 4.** The \( n \)-fold Stallings twist of \( f \) along \( \gamma \) depends (up to isotopy) only on \( L \) and \( f \) (up to isotopy), \( \gamma \) (up to isotopy respecting property [11], allowing \( \theta \) to vary), and \( n \in \mathbb{Z} \), and \( \varsigma_n(f, \gamma) \) is a Morse map for \( \varsigma_n(L, \gamma) \) with the same number of critical points of each index as the Morse map \( f \).

**Corollary 4.1.** If \( L \subset S^3 \) is a link, \( f : S^3 \setminus L \to S^1 \) is a Morse map, and \( \gamma \subset \text{Int}(S(f, \theta)) \setminus \text{crit}(f) \) is an unknot with \( S(f, \theta) \)-framing 0, then, for every \( n \in \mathbb{Z} \), \( \text{MN}(\varsigma_n(L, \gamma)) = \text{card}(\text{crit}(f)) \leq \text{MN}(L) \).

**Example 9.** In Example 4, the \( S(G_0, \theta) \)-framing of \( O \subset L(G_0, 1) \) is 0 for all \( \theta \). If \( \gamma := A(O, 0) \setminus O \), then \( \varsigma_n(L(G_0, 1), \gamma) \) is (isotopic to) \( L(G_n, 1) \) and \( \varsigma_n(f, \gamma) \) is (isotopic to) \( \varphi_{G_n} \), for all \( n \in \mathbb{Z} \).

**Proposition 5.** If \( L \subset S^3 \) is a link, \( f : S^3 \setminus L \to S^1 \) is a Morse map, and \( \gamma \subset \text{Int}(S(f, \theta)) \setminus \text{crit}(f) \), then for every \( n \) there is a Morse map \( \varsigma_n(f, \gamma) \to S^1 \) which has exactly two more critical points than \( f \), one of index 1 and one of index 2.

**Proof.** If necessary, change coordinates on \( S^1 \) (by rotation, say) to arrange that \( \exp(i\theta) \) is such that \( S(o_1, \theta) \) is a large page of the Morse map \( o_1 : S^3 \setminus O \to S^1 \) described in Example 4. Let \( f \neq o_1 \) be a connected sum of \( f \) and \( o_1 \) (as defined, say, in [22] or see Construction 11) along any component of \( L \) which belongs to the boundary of the component of \( S(f, \theta) \) that contains \( \gamma \); \( f \neq o_1 \) is Morse, with exactly two more critical points than \( f \) (one of index 1 and one of index 2), and the page \( S(f \neq o_1, \theta) \)}
may be identified with an appropriate boundary-connected sum \( S(f, \theta) \sqcup S(\theta_1, \theta) \). A glance at Fig. 5 shows that, for every \( k \), \( S(\theta_1, \theta) \) contains an unknot \( \gamma_k \) with \( S(\theta_1, \theta) \)-framing \( k \). If the \( S(f, \theta) \)-framing of \( \gamma \) is \( s \), the unknot \( \gamma \not\subset S(f \neq \theta) \) has \( S(f \neq \theta) \)-framing \( 0 \). Evidently the pairs \( (L, \gamma) \) and \( (L \neq \theta, \gamma \not\subset \gamma) \) are isotopic, so the links \( \varsigma_n(L, \gamma) \) and \( \varsigma_n(L \neq \theta, \gamma \not\subset \gamma) \) are isotopic for any \( n \).

**Corollary 5.1.** For any link \( L \subset S^3 \), any unknot \( \gamma \) interior to a (smooth) page of a minimal Morse map \( S^3 \setminus L \to S^1 \), and all \( n \), \( \text{MN}(\varsigma_n(L, \gamma)) \leq \text{MN}(L) + 2 \).

**Example 10.** Let \( L \) be \( O\{2, 2\} \), the fibered positive Hopf link, with fiber surface \( A(O, -1) \), the positive Hopf annulus. Let \( \gamma \subset A(O, -1) \) be a core circle. Evidently \( \varsigma_n(L, \gamma) = \partial A(O, n - 1) \); by Cor. 5.1 \( \text{MN}(\partial A(O, k)) \leq 2 \) for all \( k \).

**Corollary 5.2.** \( \text{MN}(\partial A(O, k)) = 2 \) for \( k \neq \pm 1 \), and \( \text{MN}(\partial A(O, \pm 1)) = 0 \).

**Historical note.** Stallings [13] observed that, if \( L \) is a fibered link and \( \gamma \) is an unknot on a fiber surface \( F \) of \( L \) with \( F \)-framing \( 0 \), then \( \varsigma_n(L, \gamma) \) is a fibered link for any \( n \); he called the operation in question simply “twisting”. In [14], Harer observed that, if \( L \) is a fibered link and \( \gamma \) is an unknot on a fiber surface \( F \) of \( L \) with \( F \)-framing \( \pm 2 \), then \( \varsigma_{\mp 1}(L, \gamma) \) is a fibered link; he called this operation and the one introduced in [13] both “Stallings twists”. It is clear that, like Stallings’s original twisting, Harer twists can be extended from fibered links and fibrations to arbitrary links and Morse functions, and that Prop. 4 and Cor. 4.1 extend in obvious ways.

5. **Monodromies and adiexodons for Morse maps**

Let \( L \subset S^3 \) be a link, \( f: S^3 \setminus L \to S^1 \) a moderate, self-indexed Morse map. Assume, changing coordinates on \( S^1 \) if need be, that \( S(f, 0) \) (resp., \( S(f, \pi) \)) is a small (resp., large) spanning surface of \( f \) (as defined on p. 3) and that, if \( x \in \text{crit}(f) \), then \( f(x) = \exp((11+\text{ind}(f; x)i\pi/4) \). Let \( \mathcal{H}_s := L \cup f^{-1}\{z \in S^1 : \text{Re}(z) \leq 0\} \), \( \mathcal{H}_s := L \cup f^{-1}\{z \in S^1 : \text{Re}(z) \geq 0\} \).

**Lemma B.** \( \mathcal{H}_s \) and \( \mathcal{H}_\ell \) are smooth 3–manifolds.

**Proof.** Since \( \text{crit}(f) \subset \text{Int} \mathcal{H}_s \), the (topological) boundary of \( \mathcal{H}_s \) (and of \( \mathcal{H}_\ell \)) is the union \( S(f, \pi/2) \cup S(f, -\pi/2) \) of two spanning surfaces of \( f \) along their common boundary \( L \), and so is a closed topological 2–manifold. Although the manifold structure of \( \partial \mathcal{H}_s \) is not \( a \) priori smooth along \( L \) (where there might be a corner), in fact the convention that \( f \) is boundary-regular ensures that \( \partial \mathcal{H}_s \) is a closed smooth surface, so \( \mathcal{H}_s \) and \( \mathcal{H}_\ell \) are smooth. (Note that as geometric 2–cycles, \( \partial \mathcal{H}_s = \partial \mathcal{H}' - \partial \mathcal{H}' \) and \( \partial \mathcal{H}_\ell = -\partial \mathcal{H}_s = \partial \mathcal{H}_s + \partial \mathcal{H}_s \).)

Let \( \text{Nb}(L \hookrightarrow S^3) \) be a regular neighborhood with a trivialization \( \tau: L \times D^2 \to \text{Nb}(L \hookrightarrow S^3) \) adapted to \( f \), so that \( \mathcal{H}_s := S(f, \pi/2) \cap \text{Ext}(L \hookrightarrow S^3) \) (resp., \( S_s := S(f, 0) \cap \text{Ext}(L \hookrightarrow S^3) \)) is isotopic in \( S^3 \setminus L \to S(f, \pi/2) \) (resp., \( S(f, 0) \)). Let \( g := \text{rank } H_1(\mathcal{H}_s; \mathbb{Z}) \), \( g_s := \text{rank } H_1(S_s; \mathbb{Z}) \), so that, by Prop. 4.2, \( (g - g_s)/2 =: \nu \geq 0 \) is an integer.

**Proposition 6.** There exists a 3–manifold with corners, equipped with a handle decomposition

\[
S_s \times [-1,1] \cup \bigcup_{k=1}^{2\nu} h_k^{(i)}
\]

for which the 1–handle \( h_k^{(i)} \) is attached to \( S_s \times [-1,1] \) along \( S_s \times \{1\} \) for \( k = 1, \ldots, \nu \) and along \( S_s \times \{1\} \) for \( k = \nu + 1, \ldots, 2\nu \), and diffeomorphisms of 3–manifolds with
Proof. By construction, \( \mathcal{H}_\ell \cap \text{Ext}(L \hookrightarrow S^3) \) and \(-\mathcal{H}_s \cap \text{Ext}(L \hookrightarrow S^3)\) are both relative cobordisms from \(S(f, \pi/2) \cap \text{Ext}(L \hookrightarrow S^3)\) to \(S(f, -\pi/2) \cap \text{Ext}(L \hookrightarrow S^3)\). By the assumptions on \( f(\text{crit}(f)), S(f, \pi/2) \cap \text{Ext}(L \hookrightarrow S^3)\) and \(S(f, -\pi/2) \cap \text{Ext}(L \hookrightarrow S^3)\) are both diffeomorphic to \(S_\ell\). The claims follow immediately upon application of the usual dictionary between Morse functions and cobordisms (see [23]) to continuous branches of \(-i \log(f|\mathcal{H}_\ell \cap \text{Ext}(L \hookrightarrow S^3))\) and \(-i \log(f|\mathcal{H}_s \cap \text{Ext}(L \hookrightarrow S^3))\).

**Corollary 6.1.** \(S_\ell\) is diffeomorphic to \(S_s\) with \(\nu\) hollow handles attached. □

**Corollary 6.2.** The following are equivalent. (1) \(S_s\) is connected. (2) \(f\) is connected. (3) \(S_\ell\) is diffeomorphic to a connected sum of \(S_s\) and a closed surface of genus \(\nu\). (4) The interior of \(\text{Ext}(L \hookrightarrow S^3)\) contains arcs \(\alpha_+, \alpha_-\) and handlebodies \(\mathcal{H}_\ell^\nu, \mathcal{H}_s^\nu\) with the following properties. (a) \(f(\mathcal{H}_\ell^\nu) \subset \{ \exp(i\theta) : |\theta \mp \pi/4| \leq 1/10 \}\). (b) The arc \(\alpha_\pm\) has one endpoint in \(\text{Int} S_s\) and the other in \(\partial \mathcal{H}_\ell^\nu\), and is mapped diffeomorphically onto \([0, \pi/4 - 1/10]\) by the restriction of \(\mp i \log(f)\). (c) The ambient connected sum of \(\partial \mathcal{H}_s^\nu\) and \(S_s\) along \(\alpha_\pm\) is isotopic to \(S_\ell\). (With an appropriate choice of the “connecting tube” which is guided by \(\alpha_\pm\), the isotopy can be achieved by following the gradient flow of \(\mp i \log(f)\).) □

Let \(S\) be a surface. Say that a relative 1–handle decomposition of a 3–manifold

\[ M = (S \times [-1, 1]) \cup \bigcup_{k=1}^{n} h_k^{(1)}, \quad (\dagger) \]

in which each 1–handle \(h_k^{(1)}\) is attached either to \(S \times \{-1\}\) or to \(S \times \{1\}\), is involutized by a map \(\iota : M \to M\) provided that \(\iota\) is an orientation-reversing involution such that \(\iota(S \times [-1, 1]) = S \times [-1, 1]\) and, for some \(\varepsilon > 0\), \(\iota(S \times [-\varepsilon, \varepsilon]) : (x, t) \mapsto (x, -t)\).

**Lemma C.** (1) If the decomposition (\dagger) can be involutized, then \(n\) is even. (2) If \(n = 0\) or if \(S\) is connected, then the decomposition (\dagger) can be involutized. □

When the relative 1–handle decomposition (\h) of the domain of \(\eta_s\), constructed (implicitly) in the course of the proof of Prop. (12) by applying Morse theory to \(-i \log(f)\), can be involutized, it may also be said that \(f\) can be involutized. For example, if \(f\) is connected (e.g., if \(f\) is a fibration), then \(f\) can be involutized; on the other hand, \(u\) can be involutized although the small Seifert surface of \(u\) is not connected. The Morse map \(\omega\) described in Example (12) cannot be involutized.

**Corollary 6.3.** If \(f\) can be involutized, then \(\mathcal{H}_\ell\) and \(\mathcal{H}_s\) are handlebodies of genus \(g\), that is, \((\mathcal{H}_\ell, \mathcal{H}_s)\) is a genus-\(g\) Heegaard splitting of \(S^3\). □

**Definitions.** Suppose \(f\) can be involutized. Choose diffeomorphisms \(\eta_s\) and \(\eta_\ell\), a relative handle decomposition (\h) of the domain of \(\eta_s\), and an involution \(\iota_s\) which involutizes \(f\). The monodromy of \(f\) (given these choices) is the diffeomorphism \(h := \text{pr}_1 \circ \eta_s^{-1} \circ \eta_s \circ \iota_s \circ \eta_s^{-1} \circ \eta_s \circ (\text{id}_{S_\ell}, 1) : S_\ell \to S_\ell\) (pictured schematically in Fig. (\dagger)). Suppose further that \(f\) is connected. Choose diffeomorphisms \(\delta_\pm : S_\ell \to S_\ell \# \mathcal{H}_s^\nu\) at the ends of isotopies as in Cor. (12). Choose a smooth map \(c_\pm : S_\ell \# \mathcal{H}_s^\nu \to S_s\).
which (in an obvious way) “collapses the second connected-summand to a point”. The ±-adiexodon of $f$ (given these choices) is the smooth map $A_{\pm} := c_{\pm} \circ \delta_{\pm}: S_{\ell} \to S_{s}$.

A monodromy for $u$ (the identity map of an annulus) is pictured in Fig. 7 A monodromy for $o_1$ (in effect, the map $S^1 \times S^1 : (z, w) \mapsto (w, z)$ restricted to $\text{Ext}(\{(1, 1)\} \hookrightarrow S^1 \times S^1)$) is pictured in Fig. 8 and a corresponding Heegaard splitting in Fig. 9.

**Remarks.** (1) In case $f$ is a fibration, the monodromy of $f$ is a geometric monodromy (or “holonomy”, [48]) of $f$ in the usual sense, somewhat justifying so naming it in general (but see note (1) on page 13); both adiexodons of a fibration may be taken to be the identity. (2) It is well known that, when $f$ is a fibration, both the embedding of $L$ into $S^3$ and the map $f$ can be reconstructed (up to isotopy) from $h$ alone, via the obvious construction of $\text{Ext}(L \hookrightarrow S^3)$ as the mapping torus of $h$ (that is, the identification space $(S_{\ell} \times [-1, 1])/\equiv_h$, where the non-trivial equivalence classes of $\equiv_h$ are the pairs $\{(x, -1), (h(x), 1)\}$ for $x \in S_{\ell}$). In general, both the embedding of $L$ into $S^3$ and the map $f$ can be reconstructed (up to isotopy) similarly, using the extra data provided
Figure 8. A monodromy for $o_1$; the middle arrow represents the composition $\eta_s \circ \iota_s \circ \eta_s^{-1} : \mathcal{H}_s \cap \text{Ext}(O \leftrightarrow S^3) \to \mathcal{H}_s \cap \text{Ext}(O \leftrightarrow S^3)$.

Figure 9. Abstract handlebodies diffeomorphic to the Heegaard handlebodies $\mathcal{H}_s, \mathcal{H}_\ell \subset S^3$ defined by $o_1 : S^3 \setminus O \to S^1$. On the surfaces bounding the handlebodies, corresponding loops are drawn in the same style; in particular, the heavy, dark separating curves represent $O \subset S^3$.

by the adiexodons. In particular, all isotopy invariants of $L \subset S^3$ or $f$ are determined by the monodromy and adiexodons of $f$—though the calculation of any particular invariant may be more or less opaque (a familiar fact even when $f$ is a fibration). The Alexander invariant (in particular, the Alexander polynomial) and Novikov homology, among others, do have reasonably transparent calculations in these terms. Likewise, a suggestive realization of the kernel of $\pi_1(S^3 \setminus L) \to \mathbb{Z} : [\gamma] \mapsto \int_\gamma \xi_L$ as an infinite free product with amalgamation, in the style of [30] and [4], falls out naturally from any monodromy and adiexodons of $f : S^3 \setminus L \to S^1$. However, we will not undertake such calculations in this paper. (3) The triples $(\mathcal{H}_s, S(f, \pi/2), -S(f, -\pi/2))$ and $(\mathcal{H}_\ell, -S(f, \pi/2), S(f, -\pi/2))$ are “sutured manifolds” as defined in [12] (equivalent to the original definition, [7]). The language of sutured manifolds could have
been used throughout this section, although the language of Morse maps to \( S^1 \) seems more natural for the study of monodromy and closed \( f \)-braids.

**Historical notes.** (1) Of course the combination of Greek roots in “monodromy” means, more or less, “single road” (a denotation of \( \text{MONOΔΡΟΜΟΣ} \) in modern Greek is “one-way street”). As Nicholas Katz has pointed out to us \([17]\), “In 19th century mathematics, monodromic meant ‘single-valued’. The ‘monodromy group’ of a differential equation measures the *failure* of its solutions to be monodromic, and by extension from that usage, ‘monodromy’ has come to mean what should really be called polydromy.” Since the word has, however unfortunately, become standard to describe the “first return map” from which a fibration over \( S^1 \) can be reconstructed, we feel somewhat justified in extending its use to the analogous part of the geometric data from which a more general Morse map from a link complement to \( S^1 \) can be reconstructed. (2) As noted after Definitions \([5]\) to reconstruct a Morse map which is not a fibration more is needed than a monodromy. In a sense, the extra information which is needed is carried by paths that, rather than “going once around” (whether to close up or keep going), go to—or come from—nowhere. We are indebted to the classicist William Wyatt \([52]\) for suggesting the word “adiexodon”, coined by Aristotle in a discussion of the infinite \([1]\), and parseable as (more or less) “that from which there is no way out by going through”.

6. **Splicing and closed \( f \)-braids**

The theory of splicing (multi)links in homology 3–spheres was introduced (for empty links) by Siebenmann \([46]\), and extensively developed by Eisenbud & Neumann \([5]\). We use only a special case, of which the following example is the paradigm.

**Example 11.** Let \( L_0 := L(G_0, \infty) \subset S^3 \) be the fibered link of indeterminacy at infinity of the rational function \( G_0 : (z, w) \mapsto zw/(4z - 1) \) introduced in Example \([4]\) The three components of \( L_0 \) are unknots: the horizontal and vertical unknots \( O \) and \( O' \) (as on p. \([4]\)) and \( O_0 := \{(z, w) \in S^3 : z = 1/4\} \). With the orientation imposed on \( L_0 \) by \( \varphi_{G_0} \), \( O' - O_0 \) is the oriented boundary of an annulus \( A(O', 0) \subset S^3 \setminus O \). This fact (or direct calculation) shows that, if \( \tau_0 : L_0 \times D^2 \to \text{Nb}(L_0 \leftrightarrow S^3) \) is a trivialization which is adapted to \( \varphi_{G_0} \), then \( \tau_0 | O \times D^2 \) is isotopic (by an isotopy that can be made arbitrarily small by shrinking the regular neighborhoods) to a trivialization \( \tau : O \times D^2 \to \text{Nb}(O \leftrightarrow S^3) \) which is adapted to \( o \). Of course the exterior \( \text{Ext}(O \leftrightarrow S^3) \) of \( O \) is also a regular neighborhood \( \text{Nb}(O' \leftrightarrow S^3) \) of \( O' \). From that point of view, up to an arbitrarily small isotopy \( L_0 \) and \( \varphi_{G_0} \) can be thought of as being constructed from \( O \) and \( o \) by replacing the triple

\[
(\text{Nb}(O' \leftrightarrow S^3), O', o | (\text{Nb}(O' \leftrightarrow S^3) \setminus O'))
\]

with the triple

\[
(\text{Ext}(O \leftrightarrow S^3), O' \cup O_0, \varphi_{G_0} | (\text{Ext}(O \leftrightarrow S^3) \setminus O' \cup O_0)).
\]

At the level of a single fiber surface, this can be done in two steps: first, replace a 2-disk \( S(o, \theta) \) by \( S(o, \theta) \cup A(O', 0) \), a Seifert ribbon (see \([35]\)) bounded by \( L_0 \), immersed with doublepoints along the arc \( S(o, \theta) \cap A(O', 0) \); then disingularize \( S(o, \theta) \cup A(O', 0) \) by “opening up” its arc of doublepoints into an unknot. (The situation is illustrated in Fig. \([10]\).) Actually, as the construction shows, this can be done for all fiber surfaces simultaneously—a striking, though elementary, fact.

Entirely similarly, for all \( k \) the fibered link of indeterminacy at infinity \( L_k := L(G_k, \infty) \subset S^3 \) of the rational function \( G_k : (z, w) \mapsto zw/(4z - w^k) \) (also introduced
in Example 14, and its fibration $\phi_{G_k}$, can be constructed from $O$ and $o$ by replacing

$$(\text{Nb}(O' \hookrightarrow S^3), O', o \big| (\text{Nb}(O' \hookrightarrow S^3) \setminus O'))$$

with

$$(\text{Ext}(O \hookrightarrow S^3), O' \cup O_k, \phi_{G_k} \big| (\text{Ext}(O \hookrightarrow S^3) \setminus O' \cup O_k)),$$

where now $O_k := \{(z, w) \in S^3 : z = w^k/4\} = L_k \setminus (O \cup O')$, so that $O' - O_k$ is the oriented boundary of an annulus $A(O', k) \subset S^3 \setminus O$; again, each fiber surface is constructed from a Seifert ribbon by opening up an arc of doublepoints into an unknot.

What makes this example work is that $O'$ is a closed braid around the axis $O$. More precisely, $O'$ is a closed 1-string $o$-braid in the sense of the following definition.

**Definition.** Let $f : S^3 \setminus L \to S^1$ be a Morse map. If $B \subset S^3 \setminus L$ is a link such that

1. $B \cap \text{crit}(f) = \varnothing$,
2. $B$ intersects every page of $f$ transversely, and
3. every intersection of $B$ with a page of $f$ is positive,

then $f(B) : B \to S^1$ is a local diffeomorphism and an orientation-preserving covering projection of some degree $n \geq 1$. Call such a link $B$ a **closed $n$-string $f$-braid**. Call $B$ pure if it has the same number of components as it has strings, that is, if each component of $B$ is a closed 1-string $f$-braid.

**Construction 3 (splicing along a closed $f$-braid).** Let $L \subset S^3$ be a link, $f : S^3 \setminus L \to S^1$ a Morse map. A **framed closed $n$-string $f$-braid** is a pair $(B, k)$ where $B$ is a closed $n$-string $f$-braid and $k : B \to \mathbb{Z}$ is a framing of $B$ (i.e., $k$ is continuous). As a closed $f$-braid, $B$ has a regular neighborhood with a trivialization $\tau : B \times D^2 \to \text{Nb}(B \hookrightarrow S^3 \setminus L)$ such that

1. $\tau(x, 0) = x$ and $f(\tau(x, z)) = f(x)$ for all $(x, z) \in B \times D^2$, and
2. for each component $C \subset B$, $\tau(C \times [0, 1/2])$ is an annulus $A(C, k(C))$. Given such a trivialization $\tau$, orient $\tau(C \times [0, 1/2])$ so that its boundary contains $C$ (rather than $-C$), and let the **spliced link of $(B, k)$ along $f$** be

$$\Psi(f, B, k) := L \cup \bigcup_{C \subset B} \partial \tau(C \times [0, 1/2]).$$

Clearly, $\Psi(f, B, k)$ depends (up to isotopy) only on the embeddings of $L$ in $S^3$ and $B$ in $S^3 \setminus L$ (up to isotopy) and the framing $k$, and not on $f$ (except insofar as $B$ is restricted to be a closed $f$-braid) nor on the choices of $\text{Nb}(B \hookrightarrow S^3 \setminus L)$ and $\tau$.

Equally clearly, a Seifert surface for $\Psi(f, B, k)$ can be constructed from any smooth page $S(f, \theta)$, by taking $S(f, \theta) \cup \tau(C \times [0, 1/2])$ (a Seifert ribbon for $\Psi(f, B, k)$ with $n$ arcs of doublepoints, the components of $S(f, \theta) \cap \tau(C \times [0, 1/2])$) and open up each arc of doublepoints into an unknot as in Example 14. Up to isotopy, this Seifert surface depends only on $S(f, \theta)$, $B$, and $k$.

The **spliced map of $(B, k)$ along $f$, $\Phi(f, B, k) : S^3 \setminus \Psi(f, B, k) \to S^1$**, depends (up to isotopy) on $f$ as well as on $L$, $B$, and $k$. In case $B$ is pure, we mimic Example 14.
exactly for each component $C$ of $B$, replacing

$$(\text{Nb}(C \hookrightarrow S^3), C, f|\text{(Nb}(C \hookrightarrow S^3) \setminus C))$$

with

$$(\text{Ext}(O \hookrightarrow S^3), O' \cup O_k, \varphi_{G_0}|\text{(Ext}(O \hookrightarrow S^3) \setminus O' \cup O_k)).$$

For general $B$, if the component $C \subset B$ is a closed $\nu$-string $f$-braid, then

$$\text{Nb}(C \hookrightarrow S^3), C, f|\text{(Nb}(C \hookrightarrow S^3) \setminus C)$$

is replaced with

$$(\text{Ext}(O \hookrightarrow S^3), O' \cup O_{k(C)}, \varphi_{G_0}|\text{(Ext}(O \hookrightarrow S^3) \setminus O' \cup O_{k(C)})).$$

For more details (from a rather different, much more general, viewpoint), see [5].

**Example 12.** For all $f$ and $B$, and any framings $k, k'$ of $B$, $\Phi(f, B, k')$ is a composition of Stallings twists of $\Phi(f, B, k)$ along the unknots produced by opening up the arcs of doublepoints on $S(f, \theta) \cup \tau(C \times [0, 1/2])$. (In light of Example 11, this is a generalization of Example 10.)

**Proposition 7.** For all $f$, $B$, and $k$, $\text{crit}(\Phi(f, B, k)) = \text{crit}(f)$, and $\text{ind}(\Phi(f, B, k); x) = \text{ind}(f; x)$ for all $x \in \text{crit}(f)$. In particular, if $f$ is a fibration (resp., moderate; self-indexed), then so is $\Phi(f, B, k)$. □

**Corollary 7.1.** For all $L$, for every Morse map $f$: $S^3 \setminus L \to S^1$ and every framed $f$-braid $(B, k)$, $\text{MN}(\Phi(f, B, k)) \leq \text{MN}(L)$. □

To make good use of Construction 3 we need an extensive supply of closed $f$-braids. For the rest of this section, assume that $f$ is moderate, self-indexed, and connected. In this case, closed $n$-string $f$-braids are especially easy to describe—and, somewhat surprisingly, plentiful and diverse even for $n = 1$ unless $f$ is isotopic to $o$.

Let $f$: $S^3 \setminus L \to S^1$ be normalized as at the beginning of Sect. 3. Fix $n \geq 1$ and $X \subset \text{Int Nb}(\partial S_n \hookrightarrow S_n)$ with $\text{card}(X) = n$.

**Proposition 8.** If $B$ is a closed $n$-string $f$-braid, then, up to isotopy through closed $n$-string $f$-braids:

1. $B \subset \text{Ext}(L \hookrightarrow S^3);
2. B \cap \mathcal{H}_k = \eta_k(B_k)$, where $\text{pr}_2|B_k: B_k \to [-1, 1]$ is an orientation-preserving covering projection (i.e., $B_k \subset S_k \times [-1, 1]$ is a geometric $n$-string $S^1$-braid);
3. (a) $B \cap \mathcal{H}_s \subset \eta_s(S_s \times [-1, 1])$, and in fact (b) $B \cap \mathcal{H}_s = \eta_s(X \times [-1, 1])$.

Conversely, if $B$ has properties (1), (2), and (3), then $B$ is a closed $n$-string $f$-braid.

**Proof.** Suppose $B$ is a closed $n$-string $f$-braid. Since $B \subset S^3 \setminus L$ is compact, certainly $B \cap N = \emptyset$ for some regular neighborhood $N$ of $L$ in $S^3$ which is concentric to $\text{Nb}(L \hookrightarrow S^3)$ with respect to some trivialization of $\text{Nb}(L \hookrightarrow S^3)$ that is adapted to $f$. An appropriate isotopy of $S^3$ (rel. $L$) supported near $\text{Nb}(L \hookrightarrow S^3)$, preserving level sets of $f$ and thus the class of closed $n$-string $f$-braids, carries $N$ onto $\text{Nb}(L \hookrightarrow S^3)$ and $B$ onto a closed $n$-string $f$-braid with property (1). Now property (2) is immediate from Prop. 4 and 5.

Suppose $B$ has property (1). From the construction of $\eta_s$ and the handle decomposition (3) of its domain, for $k = 1, \ldots, \nu$ and $r = 0, 1$ there exists a transverse 2–disk $\Delta_{k+r\nu} \subset \eta_s(h_{k+r\nu}^{(3)}) \subset \mathcal{H}_s \cap \text{Ext}(L \hookrightarrow S^3)$ of the handlebody $\mathcal{H}_s$ such that $f|\Delta_{k+r\nu}$ is Morse with exactly one critical point $x_{k+r\nu}$ of index $2r$, and $\eta_s(h_{k+r\nu}^{(1)}) \cap \text{crit}(f) = \Delta_{k+r\nu} \cap f^{-1}(\exp((-1)^{r+1}i\pi/4) = \{\xi_{k+r\nu}\}$. By (3) and (4), the finite set $B \cap \bigcup_{k,r} \Delta_{k+r\nu}$ is disjoint from $\text{crit}(f|\bigcup_{k,r} \Delta_{k+r\nu})$. Now the process adequately sketched in Fig. 11 (swapping a collar of their common boundary from one handlebody to the other to
shrink the transverse disks and make $B \cap \bigcup_{k,r} \Delta_{k+r} = \emptyset$; shortening the 1–handles around their shrunken transverse disks to make $B \cap \bigcup_{k,r} \eta_s(h^{(1)}_{k+r}) = \emptyset$; isotoping the handlebody decomposition with shrunken, shortened 1–handles back to the original one, and carrying $B$ along by the isotopy) lets $B$ acquire property (3a) while preserving properties (1) and (2). Since $S_s$ is connected, any two embeddings of the same finite set into $\text{Int} \ S_s$ are isotopic, so it is easy to endow $B$ with the further property (3b) while preserving properties (1), (2), and (3a).

The converse is trivial. $\square$

**Corollary 8.1.** If a knot $K$ is contained in a page of $f$, then $K$ is isotopic in $S^3$ to a closed 1-string $f$-braid $B$.

**Proof.** Let a link $K$ be contained in the page $S(f, \theta)$ of $f$. Let $x \in K$ be a basepoint. If $S(f, \theta)$ is singular and $r$ is the index of $f$ at any (and therefore every) point of $\text{crit}(f) \cap S(f, \theta)$, then (because $r \in \{1, 2\}$) it is easy to construct a vectorfield $V$ on a small neighborhood of $K$ such that $V$ is non-zero off $L$, $V$ equals $(-1)^{r+1}$ times the gradient of $-i \log(f)$ outside a smaller neighborhood of $\text{crit}(f) \cap S(f, \theta)$, and the flow of $V$ carries $K$ onto a smooth page of $f$.

Suppose $S(f, \theta)$ is a smooth page of $f$. There is an arbitrarily small isotopy of $S^3$ which carries $S(f, \theta)$ (and therefore $K$) into $\text{Int} \ S(f, \theta)$. By another (possibly lengthy) isotopy, $K$ can be moved into $S_\ell$ and $x$ into $\text{Int} \ \text{Nb}(\partial S_\ell \hookrightarrow S_\ell)$.

Now suppose that $K$ is a knot. Let $x_s \in \text{Int} \ \text{Nb}(\partial S_\ell \hookrightarrow S_\ell)$ be the point such that $\eta_s(x_s, \pm 1) = \eta_s(x, \mp 1)$. Let $\gamma: ([−1, 1], 1) \to (K, x)$ be a $C^\infty$ map onto $K$ such that $][1 − 1, 1]$ is a bijective immersion and $\gamma$ is infinitely flat at $\pm 1$. If $B_\ell \subset S_\ell \times [−1, 1]$ is the geometric 1-string $S_\ell$ braid parametrized by $t \mapsto (\gamma(t), t)$, then $B := \eta_s(B_\ell) \cup \eta_s(\{x_s\} \times [−1, 1])$ is a closed 1-string $f$-braid isotopic to $K$ in $S^3$. (The situation is illustrated in Fig. 12) $\square$

Prop. (2) shows immediately that (up to isotopy) any closed 1-string $f$-braid may be constructed in a similar manner, starting from an immersed (not necessarily embedded) proper arc $\gamma$ on a smooth page $S(f, \theta)$ and treating $\gamma$ as an “$f$–ascending knot diagram” on $S(f, \theta)$. This construction is illustrated in Fig. 13.

The *wrapping genus* $g_{wr}(K)$ of a knot $K$ has been defined as the least $n$ such that $K$ embeds on a Heegaard surface of genus $n$ (so, for instance, $g_{wr}(K) = 1$ if and only if $K$ is a non-trivial torus knot). Since a large Seifert surface of the $g$–fold connected sum $o_g \equiv \#_1^g o_1$ is the exterior of a point on a Heegaard surface of genus $g$, it is equivalent to define $g_{wr}(K)$ as the least $g$ such that, up to isotopy, $K$ is embedded on a large Seifert surface of $o_g$.  Let $g_{wr}(K)$, the *layered wrapping genus* of $K$, be the least $g$ such that $K$ is isotopic to a closed 1-string $o_g$–braid. (The name is appropriate in view of the description of closed 1-string $f$-braids in the previous paragraph.)
Figure 12. Top: $K$, a torus knot of type $(2,3)$ lying on the large Seifert surface of $o_1$ (left, $S(o_1,-\pi/2)$) as pictured in Fig. 8; right, $S(o_1,-\pi/2)$ with its unknotted boundary in standard position, together with the disk $S(o_1,\pi/2)$). Bottom: a closed 1-string $o_1$-braid $B$ isotopic to $K$ in $S^3$ (left, the part of $B$ inside the torus $S(o_1,\pi/2) \cup S(o_1,\pi)$ is represented by dashed lines, and the rest of $B$ by solid and dotted lines; right, all of the torus but $O$ has been removed).

Figure 13. Left: an $o_1$-ascending knot diagram with eight crossings, on a large Seifert surface of $o_1$. Right: the union of the 1-string $o_1$-braid with the given diagram, and the binding $O$ of $o_1$; this link is isotopic to the union of a figure-8 knot and one of its meridional circles.

**Corollary 8.2.** For all $K$, $gw\ell(K) \leq gw_{tr}(K)$.

Since the figure-8 knot is not a torus knot, Fig. 13 shows that the inequality in Cor. 8.2 can be sharp.

**Corollary 8.3.** If $K$ is a knot and $D \subset S^3$ is a 2-disk such that $D \cap A(K,k)$ is a transverse arc of an annulus $A(K,k)$, then $MN(\partial D \cup \partial A(K,k)) \leq 2gw_{tr}(K)$ for all $k \in \mathbb{Z}$. 
Proof. We may assume that $K$ is a closed 1-string $g_{wt}(K)$-braid. By Cor. 3 and Prop. 4 card(crit($\Phi(o_{g_{wt}(K)}, K, k)) = \text{card}(\text{crit}(g_{wt}(K))) = 2g_{wt}(K)$. But $\Psi(g_{wt}(K), K, k)$ and $\partial D \cup \partial A(K, k)$ are isotopic. □

Remarks. (1) We thank Claude Weber for pointing out to us that Construction 3 is an instance of splicing. (2) The first author (re)discovered Construction 3, for (1) A closed $f$–patches) on a smooth page of Hopf plumbing and the geometry of the enhanced Milnor number, [10]; cf. [38, 47]. (3) Eisenbud & Neumann [5, Theorem 4.2] prove that a (multi)link is fibered if and only if it is non-split (“irreducible”) and each of its (multi)link “splice components” is fibered. It is clear that the methods of [5] could be used to derive quite general (sub)additivity formulæ for Morse–Novikov number over appropriate splicings (including cablings).

Historical note. (1) A closed $o$–braid is simply a closed braid with axis $O$ in the usual sense. There is some published work [35, 47, 50] on closed $f$–braids for a fibration $f: S^3 \setminus L \to S^1$ (and, more generally, for a fibration $f: M \to S^1$ where the 3–manifold $M$ is not necessarily a link complement in $S^3$). (2) Stillwell [49] calls $g_{wt}(K)$ the “handle number” of $K$; Goda’s “handle number” [13] is something else entirely (see the note on p. 25). (3) We are not aware of previous work on closed $f$–braids for Morse maps $f$ which are not fibrations, nor on layered wrapping genus.

7. Murasugi sums of Morse maps

The underlying idea of the construction of a $2n$–gonal Murasugi sum $f_0 *_{(N_0, N_1, \zeta)} f_1$ of two Morse maps $f_\zeta: S^3_s \setminus L_s \to S^1$ is simple: given appropriate $2n$–gonal 2–cells (so-called $n$–patches) on a smooth page of $f_0$ and a smooth page of $f_1$, with appropriate regular neighborhoods $N_s \subset S^3_s$ (each a 3–disk) such that $f_s| N_s$ is “standard” in a suitable sense, and in particular has no critical points in $\text{Int} N_s$, the exteriors $E_s = S^3 \setminus \text{Int}(N_s)$ (which are also 3–disks) may be glued together along their boundaries to produce a 3–sphere on which the restrictions $f_s| E_s$ glue together to give a Morse map (on the complement of an appropriate link, denoted $f_0 *_{(N_0, N_1, \zeta)} f_1$) whose critical points are the disjoint union of the critical points of $f_0$ and $f_1$. The details, unfortunately, are somewhat technical.

For $n \in \mathbb{N}$, write $G_n := \{z \in \mathbb{C} : z^n = 1\}$. Let $p_n: \mathbb{P}_1(\mathbb{C}) \setminus G_{2n} \to S^1$ be the argument of the rational function $\mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C}) : z \mapsto (1 + z^n)/(1 - z^n)$, viz.,

$$p_n(z) = \frac{(1 + z^n)/(1 + z^n)}{(1 - z^n)/(1 - z^n)} \text{ for } z \notin G_{2n} \cup \{\infty\}, \ p_n(\infty) = -1. $$

Define $P_n: \mathbb{P}_1(\mathbb{C}) \setminus G_{2n} \times [0, \pi] \to S^1$ by $P_n(z, \theta) = \exp(i\theta)p_n(z)$. Note that $p_n(\zeta z) = p_n(z)$ and $P_n(z, \theta) = P_n(\zeta z, \theta)$ for any $\zeta \in G_n$.

Lemma D. (1) $p_1$ is a fibration with fiber $]-1, 1[ = p_1^{-1}(1)$. For $n > 1$, $p_n$ has exactly two critical points, $0 \in p_n^{-1}(1)$ and $\infty \in p_n^{-1}(-1)$, at each of which the germ of $p_n$ is smoothly conjugate to the germ of $z \mapsto \text{Im}(z^n)$ at 0. (2) $P_1$ is a fibration with fiber $]-1, 1[ \times [0, \pi] = P_1^{-1}(1)$. For all $n$, $P_n$ has no critical points, and there is a trivialization $\tau_{P_n}: (G_{2n} \times [0, \pi]) \times D^2 \to \text{Nb}(G_{2n} \times [0, \pi]) \hookrightarrow \mathbb{P}_1(\mathbb{C}) \times [0, \pi]$ which is adapted to $P_n$. □

Let $Q_n(\theta)$ denote the closure of $P_n^{-1}(\exp(i\theta))$ in $\mathbb{P}_1(\mathbb{C}) \times [0, \pi]$. By inspection (and Lemma [10]), for all $\exp(i\theta) \in S^1$, $Q_1(\theta)$ is a 4–gonal 2–disk with smooth interior bounded by the union of $G_{2} \times [0, \pi]$ and 2 semicircles in $\mathbb{P}_1(\mathbb{C}) \times [0, \pi]$; for $n > 1$ and all $\exp(i\theta) \in S^1 \setminus \{1, -1\}$, $Q_n(\theta)$ is a 4$n$–gonal 2–disk with smooth interior bounded.
by the union of $\mathbb{G}_{2n} \times [0, \pi]$ and $2n$ circular arcs in $\mathbb{P}_1(\mathbb{C}) \times \{0, \pi\}$. (The cases $n = 1, 2$, \(\theta = \pi/2\), are pictured in Fig. 14.)

**Lemma E.** For all $\exp(i\theta) \in S^1$, the restriction $\text{pr}_2\big|Q_1(\theta)\colon Q_1(\theta) \to [0, \pi]$ has no critical points. For $n > 1$ and $\exp(i\theta) \in S^1 \setminus \{1, -1\}$, there is exactly one critical point of $\text{pr}_2\big|Q_n(\theta)\colon Q_n(\theta) \to [0, \pi]$ (to wit, $(0, \theta)$ for $0 < \theta < \pi$ and $(\infty, \theta - \pi)$ for $\pi < \theta < 2\pi$), at which the germ of $\text{pr}_2\big|Q_n(\theta)$ is smoothly conjugate to the germ of $z \mapsto \text{Im}(z^n)$ at $0$. \(\Box\)

By further inspection, for $0 < \theta < 2\pi$, $\theta \neq \pi$, $Q_n(\theta)$ is isotopic (by a piecewise-smooth isotopy fixing $\partial Q_n(\theta) \cup (Q_n \cap \mathbb{P}_1(\mathbb{C}) \times \{\theta - \lfloor \theta/\pi \rfloor \pi\}$ pointwise) to a piecewise-smooth $4n$–gonal $2$–disk $Q'_n(\theta)$ so situated that $Q'_n(\theta) \cap (\mathbb{P}_1(\mathbb{C}) \times [0, \theta - \lfloor \theta/\pi \rfloor \pi])$ and $Q'_n(\theta) \cap (\mathbb{P}_1(\mathbb{C}) \times [\theta - \lfloor \theta/\pi \rfloor \pi, \pi])$ are both piecewise-smooth $4n$–gonal $2$–disks, while $Q'_n(\theta) \cap (\mathbb{P}_1(\mathbb{C}) \times \{\theta - \lfloor \theta/\pi \rfloor \pi\})$ is a (smooth) $2$–disk in $\mathbb{P}_1(\mathbb{C}) \times \{\theta - \lfloor \theta/\pi \rfloor \pi\}$ naturally endowed with the structure of a $2n$–gon. (The cases $n = 1, 2$, $\theta = \pi/2$, are pictured in Fig. 15. Versions of $Q'_2(\pi/2)$ and $Q'_2(3\pi/2)$ in $\text{Nb}(S^2 \hookrightarrow S^3) \cong \mathbb{P}_1(\mathbb{C}) \times [0, \pi]$ are pictured in Fig. 16.)

Let $L \subset S^3$ be a link, $f\colon S^3 \setminus L \to S^1$ a Morse map, and $\psi$ an $n$–star, in the sense of [11], on a smooth page $S(f, \theta)$: that is, $\psi$ is the union of $n$ arcs $\alpha_s$, pairwise disjoint except for a common endpoint $* \psi \in \text{Int} S(f, \theta)$, and $\partial S(f, \theta) \cap \alpha_s = \partial \alpha_s \setminus * \psi$ for each $s).$ A regular neighborhood $\text{Nb}(\psi \hookrightarrow S^3)$ is $f$–good provided that (a) $\text{Nb}(\psi \hookrightarrow S^3) \cap S(f, \theta)$ is a regular neighborhood $\text{Nb}(\psi \hookrightarrow S(f, \theta))$ (and thus an $n$–patch on $S(f, \theta)$ in the sense of [11]: that is, a $2$–disk naturally endowed with the structure of a $2n$–gon whose edges are alternately boundary arcs and proper arcs in $S(f, \theta)$), and (b) there exists a diffeomorphism $h\colon (\mathbb{P}_1(\mathbb{C}), \mathbb{G}_{2n}) \to (\partial \text{Nb}(\psi \hookrightarrow S^3), L \cap \partial \text{Nb}(\psi \hookrightarrow S^3))$ with $(f \circ h)(\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{G}_{2n}) = p_n$.

**Lemma F.** Every neighborhood of $\psi$ in $S^3$ contains an $f$–good regular neighborhood.
Proof. First suppose that $L = O$ and $f = o$. Stereographic projection

$$\sigma: S^3 \setminus \{(0, -i)\} \to \mathbb{C} \times \mathbb{R}: (z, w) \mapsto (z, \text{Re}(w))/(1 + \text{Im}(w))$$

maps $O$ to $S^1 \times \{0\}$ and $S(o, \pi/2)$ to $D^2 \times \{0\}$; $\psi_1 := \sigma^{-1}([0, 1] \times \{0\})$ is a 1-star in $S(o, \pi/2)$, and the preimage $\sigma^{-1}(B)$ of an appropriate ellipsoidal 3-disk $B \subset \mathbb{C} \times \mathbb{R}$ (say, with one focus at $(0, 0)$, center at $(1 - \varepsilon, 0)$ for sufficiently small $\varepsilon > 0$, and minor axes much shorter than the major axis), is an $o$-good regular neighborhood $\text{Nb}(\psi_1 \hookrightarrow S^3)$. The smooth map

$$(\mathbb{C} \times \mathbb{R}) \cup \{\infty\} \to (\mathbb{C} \times \mathbb{R}) \cup \{\infty\}: (\zeta, t) \mapsto (\zeta^n, t), \infty \mapsto \infty,$$

can be modified in a neighborhood of $\infty$ so as to induce via $\sigma^{-1}$ a cyclic branched covering $c_n^*: S^3 \to S^3$ of degree $n$, branched along $A := \{0\} \times S^1 \subset S^3$, with $c_n^{-1}(S(o, \pi/2)) = S(o, \pi/2)$; then $\psi_n := c_n^{-1}(\psi_1)$ is an $n$-star in $S(o, \pi/2)$, and (if minimal care has been taken) $c_n^{-1}(\text{Nb}(\psi_1 \hookrightarrow S^3))$ is an $o$-good regular neighborhood $\text{Nb}(\psi_n \hookrightarrow S^3)$. (The cases $n = 1, 2, 3$ are pictured in Fig. 18.) Clearly any neighborhood of $\psi_n$ in $S^3$ contains an $o$-good regular neighborhood of the type just constructed.

The general case follows immediately, upon observing that, for any link $L$, Morse map $f: S^3 \setminus L \to S^1$, and spanning surface $S(f, \theta)$, if $\psi \subset S(f, \theta)$ is an $n$-star on $S(f, \theta)$, then there is a diffeomorphism $h: M \to h(M)$ from a (not necessarily $f$-good) neighborhood $M$ of $\psi$ in $S^3$ to a neighborhood $h(M)$ of $\psi_n$ in $S^3$ such that $h(M \cap S(f, \theta)) = h(M) \cap S(o, \pi/2)$, and a diffeomorphism $k: (S^1, \exp(i\theta)) \to (S^1, i)$ such that $k \circ f|(M \setminus L) = o \circ h|(M \setminus L)$. (A “side view” of part of one such $M$ is pictured in Fig. 18.)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig16.png}
\caption{$Q'_2(\pi/2)$ and $Q'_2(3\pi/2)$ (the latter with viewports), as they appear in in $\text{Nb}(S^2 \hookrightarrow S^3) \cong \mathbb{P}_1(\mathbb{C}) \times [0, \pi]$.
\label{fig16}}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig17.png}
\caption{Some level sets of $o|(\partial \text{Nb}(\psi_n \hookrightarrow S^3) \setminus O)$ for $n = 1, 2, 3$.
\label{fig17}}
\end{figure}

\textbf{Construction 4 (Murasugi sum of Morse maps).} Let $L_s \subset S^3$ be a link, $f_s: S^2 \setminus L_s \to S^1$ a Morse map, $\exp(i\Theta) \in S^1 \setminus (f_0(\text{crit}(f_0)) \cup f_1(\text{crit}(f_1)) \cup \{0, \pi\})$, $\psi_{n,s} \subset S(f_s, \Theta)$ an $n$-star for $s = 0, 1$ and $n > 0$. Let $N_s := \text{Nb}(\psi_{n,s} \hookrightarrow S^3_s)$ be $f_s$-good, $E_s := \text{Ext}(\psi_{n,s} \hookrightarrow S^3_s)$, $d_s: E_s \to D^3$ a diffeomorphism. Let $h_s: (\mathbb{P}_1(\mathbb{C}), \mathbb{G}_{2n}) \to (\partial N_s, L \cap \partial N_s)$ be a diffeomorphism such that $(f_s \circ h_s)|(\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{G}_{2n}) = p_n$. For a fixed $\zeta \in \mathbb{G}_n$, define $\equiv_{\zeta}$ by $(z, s) \equiv_{\zeta} h_s(\zeta^s z, s) (z \in \mathbb{P}_1(\mathbb{C}), s = 0, 1)$. The identification
space \[ \Sigma(N_0, N_1, \zeta) := (E_0 \cup (\mathbb{P}_1(\mathbb{C}) \times [0, \pi]) \cup E_1)/\equiv_\zeta \]

has a natural piecewise-smooth structure, imposed on it by the identification map \( \Pi \), with respect to which the 1–submanifold and map

\[ L(N_0, N_1, \zeta) := (L_0 \cap E_0 \cup \mathbb{G}_{2n} \times [0, \pi] \cup L_1 \cap E_1)/\equiv_\zeta, \]

\[ f(N_0, N_1, \zeta) := ((f_0 \cup P_n \cup f_1)/\equiv_\zeta)|{\Sigma(N_0, N_1, \zeta) \setminus L(N_0, N_1, \zeta)} \]

are smooth where this is meaningful (i.e., in the complement of the identification locus \( \Pi(\mathbb{P}_1(\mathbb{C}) \times \{0, \pi\}) \), along which \( \Sigma(N_0, N_1, \zeta) \) is itself a priori only piecewise-smooth). It is not difficult to give \( \Sigma(N_0, N_1, \zeta) \) a smooth structure everywhere, in which \( L(N_0, N_1, \zeta) \) and \( f(N_0, N_1, \zeta) \) are everywhere smooth. The smoothing can be done very naturally off \( \Pi((\mathbb{G}_{2n} \cup \{0, \infty\}) \times \{0, \pi\}) \), using the fact that by construction (see Lemmas 11 and 12) the function \( (||d_0||^2 - 1) \cup pr_2 \cup (\pi + 1 - ||d_1||^2)/\equiv_\zeta \) and the map \( f(N_0, N_1, \zeta) \) are piecewise-smoothly transverse (in an evident sense) on \( \Pi((\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{G}_{2n} \cup \{0, \infty\}) \times \{0, \pi\}) \). A somewhat less natural, but not difficult, construction smooths \( \Sigma(N_0, N_1, \zeta) \) on \( \Pi(\{0, \infty\} \times \{0, \pi\}) \) in such a way as to make \( f(N_0, N_1, \zeta) \) smooth there also (in the process, \( (||d_0||^2 - 1) \cup pr_2 \cup (\pi + 1 - ||d_1||^2)/\equiv_\zeta \) may be forced to be not smooth at those points). Nor is there any difficulty in smoothing \( \Sigma(N_0, N_1, \zeta) \) on \( \Pi(\mathbb{G}_{2n} \times \{0, \pi\}) \). Further details will be suppressed.

When \( \Sigma(N_0, N_1, \zeta) \), with the smooth structure just constructed, is identified with \( S^3 \), \( L(N_0, N_1, \zeta) \) (resp., \( f(N_0, N_1, \zeta) \)) will be called a 2n–gonal Murasugi sum of \( L_0 \) and \( L_1 \) (resp., of \( f_0 \) and \( f_1 \)) and denoted by \( L_0 *_{(N_0, N_1, \zeta)} L_1 \) (resp., \( f_0 *_{(N_0, N_1, \zeta)} f_1 \)) or simply by \( L_0 * L_1 \) (resp., \( f_0 * f_1 \)). A 2–gonal Murasugi sum of links is simply a connected sum, and a 2–gonal Murasugi sum of Morse maps \( f_0 \) and \( f_1 \) may be denoted \( f_0 \neq f_1 \). A more detailed description of the construction in the case of connected sums, which goes more smoothly than the general case, is given in [32].)

**Theorem 9.** Let \( L_s \subset S^3_s \) be a link and \( f_s : S^3_s \setminus L_s \to S^1 \) a Morse map for \( s = 0, 1 \).

1. Any Murasugi sum \( L_0 *_{(N_0, N_1, \zeta)} L_1 \) is a link. 2. Any Murasugi sum \( f_0 *_{(N_0, N_1, \zeta)} f_1 \) is a Morse map, and its critical points \( \text{crit}(f_0 *_{(N_0, N_1, \zeta)} f_1) = \text{crit}(f_0) \cup \text{crit}(f_1) \) have indices that are inherited unchanged from those of \( f_0 \) and \( f_1 \); in particular, if \( f_0 \) and \( f_1 \) are moderate, then \( f_0 *_{(N_0, N_1, \zeta)} f_1 \) is moderate.

With \( \Theta \) as in Construction 4 the Seifert surface \( S_{(f_0, \Theta)} \cup Q_n(\Theta) \cup (S(f_1, \Theta) \cap E_1) \) (up to smoothing), and thus piecewise-smoothly isotopic to

\[ \Pi(S(f_0, \Theta) \cap E_0 \cup Q_n(\Theta) \cup S(f_1, \Theta) \cap E_1) = \]

\[ \Pi(S(f_0, \Theta) \cap E_0 \cup Q'_n(\Theta) \cap \mathbb{P}_1(\mathbb{C}) \times [0, \Theta - [\Theta/\pi] \pi]) \cup \]

\[ \Pi(Q'_n(\Theta) \cap \mathbb{P}_1(\mathbb{C}) \times [\Theta - [\Theta/\pi] \pi, \pi] \cup S(f_1, \Theta) \cap E_1) := S'_0 \cup S'_1 \]
where $S'_s$ is piecewise-smoothly isotopic to $S(f_s, \Theta)$ by an isotopy carrying the 2–disk $S'_s \cap S'_1$ (with its natural structure of $2n$–gon, noted after Lemma 13) to the $n$–patch $N_s \cap S(f_s, \Theta)$. Thus, for a suitable diffeomorphism $H: N_0 \cap S(f_0, \Theta) \rightarrow N_1 \cap S(f_1, \Theta)$ (determined by $\zeta$, up to isotopy), $S(f_0 *_{(N_0, N_1, \zeta)} f_1, \Theta)$ is a $2n$–gonal Murasugi sum $S(f_0, \Theta) *_H S(f_1, \Theta)$ as described in [41] (see also the primary sources [26, 48, 6]).

On the level of Seifert surfaces, a 2–gonal Murasugi sum $S_0 *_H S_1$ is the same as a boundary-connected sum $S_0 \natural S_1$. After boundary-connected sum, the most commonly encountered case of Murasugi sum is 4–gonal, and the most familiar and probably most useful 4–gonal Murasugi sums are annulus plumbings, as described in [42] (see also the primary sources [45, 9], as well as [44]).

Specifically, on any annulus $A(K, n)$, let $\gamma(K)$ be a proper arc joining the two components of $\partial A(K, n)$, so that $\text{Nb}(\gamma(K) \rightarrow A(K, n)) =: C_1$ is a 2–patch (though $\gamma(K)$, as given, is not a 2–star). Let $S$ be a Seifert surface $S$, $\alpha \subset S$ a proper arc, $C_0 \subset S$ a 2–patch with $\alpha \subset \partial C_0$ (respecting orientation). Let $H: (C_0, \alpha) \rightarrow (C_1, C_1 \cap K)$ be a diffeomorphism. Each of $\gamma(K)$, $C_0$, and $H$ is unique up to isotopy, so the 4–gonal Murasugi sum $S *_H A(K, n)$ depends (up to isotopy) only on $\alpha$, and there is no abuse in denoting it by $S *_{\alpha} A(K, n)$. When $S = A(K', n')$ is also an annulus, it is slightly abusive—but handy—to write simply $A(K', n') * A(K, n)$, with the understanding that $\alpha = \gamma(K')$.

**Example 13.** $S(f_0 \# f_1, \theta)$ is a boundary-connected sum $S(f_0, \theta) \natural S(f_1, \theta)$, for any $f_0$ and $f_1$.

**Example 14.** As pictured in Fig. 19 (redrawn piecewise-smoothly in Fig. 20) the large Seifert surface of $u$ is $A(O, 0)$. Of course the small Seifert surface of $u$ is two 2–disks. By the construction and Thm. 9 there is a 4–gonal Murasugi sum $u * u$ with 4 critical points and 4 critical values. Each non-singular page of $u * u$ is either a once-punctured torus bounded by an unknot or a 2–disk bounded by an unknot (see Fig. 20); note that, on the level of Murasugi sums of surfaces, a 2–disk is not a Murasugi sum of two pairs of two 2–disks. The Morse map $u * u$ may be modified in the standard way to yield a self-indexed Morse map $w$, with small surface a 2–disk and large surface a once-punctured surface of genus 2; $w$ is not a Murasugi sum of Morse maps.
Example 15. Since $o(2, 2)$ is a fibration, the 4-gonal Murasugi sum $u \ast o(2, 2)$ is moderate and self-indexed, with binding isotopic to $O$; in fact, $u \ast o(2, 2)$ is isotopic to $o_1$. Note that the large and small Seifert surfaces of $u \ast o(2, 2)$ are isotopic to the appropriate smooth pages of $u \ast u$.

Example 16. As in [11], a basket is a Seifert surface of the form $D^2_{*a_1}A(O, k_1) \cdots_{a_n}A(O, k_n)$, where $a_1, \ldots, a_n \subset D^2$ are proper arcs with pairwise disjoint endpoints. By Cor. 5.2 and Construction 4, if $S = D^2_{*a_i}A(O, k_1) \cdots_{a_n}A(O, k_n)$ is a basket, then $\mathcal{M}N\partial S \leq 2m$, where $m := \text{card}(\{j : 1 \leq j \leq n \text{ and } |k_n| \neq 2\})$. In particular, by [12], if $L$ is a special arborescent link in the sense of Sakuma [44] (that is, if $L$ bounds an arborescent plumbing of annuli $A(O, k)$), then $L$ bounds a basket (which is isotopic to such an arborescent plumbing) and $\mathcal{M}N\partial S \leq 2m$, where again $m$ is the number of annular plumbings which are not Hopf annuli.

As Example 15 shows, if no restriction is placed on the Seifert surfaces along which a Murasugi sum $L_0 \ast L_1$ is formed, then it can easily happen that $\mathcal{M}N(L_0) + \mathcal{M}N(L_1) < \mathcal{M}N(L_0 \ast L_1)$. Much worse is true: as observed in [15], any knot $K$ bounds a Seifert surface $S$ such that $S = S_0 \ast S_1$ is a Murasugi sum and $K_0 := \partial S_0$, $K_1 := \partial S_1$ are unknots; so $\mathcal{M}N(K_0 \ast K_1) - (\mathcal{M}N(K_0) + \mathcal{M}N(K_1))$ can be arbitrarily large, since [32] for every $m$ there is a knot $K$ with $\mathcal{M}N(K) > m$. However, Thm. 9 does lead immediately to the following generalization of the inequality [3] stated in Section 1.

**Corollary 9.1.** If $f_s : S^3_s \setminus L_s \to S^1$ are minimal Morse maps ($s = 0, 1$), then

$$\mathcal{M}N(L_0 \ast (N_0, N_1, \zeta) L_1) \leq \mathcal{M}N(L_0) + \mathcal{M}N(L_1)$$

for any Murasugi sum $f_0 \ast (N_0, N_1, \zeta) f_1$. \hfill $\square$

One special, very simple case of Murasugi sum is so useful for applications that we draw explicit attention to it.

**Construction 5 (cutting).** Let $f : S^3 \setminus L \to S^1$ be a Morse map. Let $\alpha \subset S(f, \theta)$ be a proper arc on a smooth page of $f$. Among the smooth pages of $f \ast_{\alpha} u$ are surfaces isotopic to $S(f, \theta) \ast_{\alpha} A(O, 0)$ (that is, $S(f, \theta)$ with an untwisted, unknotted 1-handle attached “running above $\alpha$”) and surfaces isotopic to $S(f, \theta) \setminus \alpha := \text{Ext}(\alpha \hookrightarrow S(f, \theta))$ (that is, $S(f, \theta)$ “cut along $\alpha$”). Let the Morse map $f \setminus \alpha := f \ast_{\alpha} u$ be called $f$ cut along $\alpha$; denote the binding of $f \setminus \alpha$ by $L \setminus \alpha$ and call it $L$ cut along $\alpha$. Of course $L \setminus \alpha$, $\partial(S(f, \theta) \ast_{\alpha} A(O, 0))$, and $\partial(S(f, \theta) \setminus \alpha)$ are mutually isotopic.

**Lemma G.** If $f : S^3 \setminus L \to S^1$ is a Morse map, and $\alpha \subset S(f, \theta)$ is a proper arc on any smooth page of $f$, then $\mathcal{M}N(L \setminus \alpha) \leq \mathcal{M}N(L) + 2$. \hfill $\Box$

Given a Seifert surface $S$ and a proper arc $\alpha \subset S$, let $\alpha^* \subset S$ be a proper arc such that $\alpha^* \subset \text{Nb}(\alpha \hookrightarrow S)$ and $\alpha^*$ has exactly one point of intersection with $\alpha$, at which the arcs are transverse. Let $\varepsilon$ be 1 if the arc of $\partial S \cap \text{Nb}(\alpha \hookrightarrow S)$ is oriented from $\alpha$ to $\alpha^*$, $-1$ in the contrary case.

**Lemma H.** The links $\partial S$ and $\partial(S \ast_{\alpha} A(O, 0) \ast_{\alpha^*} A(O, -\varepsilon))$ are isotopic.

**Proof.** See Fig. 21. \hfill $\square$

**Historical note.** After the second author had prepared [13], the first author brought to his attention work of Goda [12, 13] which immediately implies [14], although Goda’s results are stated in the language of “handle number of a Seifert surface” and “Murasugi sum of Seifert surfaces” rather than that of “Morse–Novikov number of a link” and “Murasugi sum of Morse maps”, and Goda’s proofs correspondingly use the techniques of sutured manifolds [7] and C-product decompositions [8] rather than those of Morse maps. We have given the proof using Morse maps for the sake of variety.
8. Morse maps and free Seifert surfaces

A Seifert surface $S \subset S^3$ is called free if and only if $S$ is connected and the group $\pi_1(S^3 \setminus S)$ is free; alternatively, $S$ is free if and only if $S^3 \setminus \text{Int} \text{Nb}(S \hookrightarrow S^3)$ is a handlebody $\mathcal{H}^g$ (necessarily of genus $g = b_1(S)$, where for any $X$, we write $b_1(X) := \text{rank} H_1(X; \mathbb{Z})$). A Morse map $f$ is free provided that $f$ is connected and every Seifert surface $S(f, \theta)$ is free. It is well known (and obvious) that, if $f$ is a fibration, then $f$ is free. The following proposition is Lemma 4.2 of [32].

**Proposition 10.** If $f$ is a moderate self-indexed Morse map, then a large Seifert surface of $f$ is free.

**Corollary 10.1.** If $f$ is a moderate self-indexed Morse map (in particular, if $f$ is a self-indexed minimal Morse map), then $f$ is free if and only if a small Seifert surface of $f$ is free.

**Theorem 11.** If $S$ is a free Seifert surface, then there is a free moderate self-indexed Morse map $f: S^3 \setminus \partial S \to S^1$, with exactly $2b_1(S)$ critical points, such that $S$ is the small Seifert surface of $f$.

Proof. Given a connected Seifert surface $S$ with $b_1(S) = n$, enlarge it to a Seifert surface $S^+$ as follows. Let $\alpha_1(S), \ldots, \alpha_n(S) \subset S$ be pairwise disjoint proper arcs such that $S \cup \alpha_1(S) \cup \alpha_2(S) \cup \cdots \cup \alpha_n(S)$ is a 2–disk. There is a handlebody $\mathcal{H}^n \subset S^3$ such that $S \subset \partial \mathcal{H}^n$ and $\mathcal{H}^n$ is the trace of an isotopy (rel. $\partial S$) from $S$ to $\partial \mathcal{H}^n \setminus \text{Int} S$. The trace of $\alpha_\ast(S)$ by a suitable such isotopy is a meridional 2–disk of $\mathcal{H}^n$ with unknotted boundary $\gamma_\ast(S)$, and $\text{Nb}(\gamma_\ast(S) \hookrightarrow \partial \mathcal{H}^n)$ is an annulus $A(\gamma_\ast(S), 0)$. Evidently

$$S^+ := S \cup \bigcup S \gamma_\ast(S), 0) = S *_{\alpha_1(S)} A(\gamma_1(S), 0) *_{\alpha_2(S)} A(\gamma_2(S), 0) *_{\alpha_n(S)} A(\gamma_n(S), 0)$$

is a subsurface of $\partial \mathcal{H}^n$ bounded by an unknotted. If also $S$ is free, so that $\mathcal{H}^n$ is Heegaard, then by Waldhausen’s uniqueness theorem [31] up to isotopy the pair $(\partial \mathcal{H}^n, S^+)$ depends only on $n$ and is otherwise independent of $S$. In particular, if $T$ is a fiber surface with $b_1(T) = n$ (which always exists: e.g., a boundary-connected sum of $n$ Hopf annuli), then $T^+$ is isotopic to $S^+$. Now, $T$ is a large Seifert surface of a fibration $f_T$; by Construction 4 Thm. 9 and Construction 5 $T^+$ is a large Seifert surface of a self-indexed moderate Morse map $f_T \cup \alpha_1(T) \cup \alpha_2(T) \cup \cdots \cup \alpha_n(T)$ with $2n$ critical points. Let $\beta_1, \ldots, \beta_n \subset T^+$ be the arcs (which without loss of generality can be chosen to be proper) which are the images of suitably chosen $\alpha_1(S)^\times, \ldots, \alpha_n(S)^\times \subset S \subset S^+$ by some isotopy carrying $(\partial \mathcal{H}^n, S^+)$ onto $(\partial \mathcal{H}^n, T^+)$. (The situation is illustrated in Fig. 22) Again by Construction 4 and Thm. 9

$$(f_T \cup \alpha_1(T) \cup \alpha_2(T) \cup \cdots \cup \alpha_n(T)) *_{\beta_1} o\{2, -2\varepsilon_1\} *_{\beta_n} o\{2, -2\varepsilon_n\}$$
Figure 22. A pair of isotopic triples \((\partial \mathcal{H}^2, S^+, (\alpha_1(S)^\ast, \alpha_2(S)^\ast))\) and \((\partial \mathcal{H}^2, T^+, (\beta_1, \beta_2))\) constructed from a free Seifert surface \(S\) and a fiber surface \(T\).
is a self-indexed moderate Morse map with 2n critical points, and by Lemma \[1\] its binding \(\partial(T^+ \ast \beta_1, O\{2, -2\varepsilon_1\} \ast \cdots \ast \beta_n, O\{2, -2\varepsilon_n\})\) is isotopic to \(\partial S\), while evidently its small Seifert surface is isotopic to \(S\).

The free genus of a knot \(K\) is \(g_f(K) := \min \{g(S) : K = \partial S, S\) is a free Seifert surface\}. More generally, for any link \(L\) let \(b_f(L) := \min \{n : L = \partial S, S\) is a free Seifert surface\}; so, for instance, \(b_f(K) = 2g_f(K)\) if \(L = K\) is a knot, and \(b_f(L) = 1\) if and only if \(L = \partial A(O, n)\) for some \(n \in \mathbb{Z}, n \neq 0\).

**Free Genus Estimate.** For any link \(L\), \(\mathcal{MN}(L) \leq 2b_f(L)\). In particular, for any knot \(K\), \(\mathcal{MN}(K) \leq 4g_f(K)\).

**Remarks.** (1) Let \(g(K)\), as usual, denote the genus of a knot \(K\). Of course \(g_f(K) \geq g(K)\) for every knot \(K\). For many knots \(K\) (satisfying a suitable condition on the Alexander polynomial \(\Delta_K(t)\)), it follows from [32] that \(\mathcal{MN}(K) \geq 2g(K)\). We know no example of a knot \(K\) for which it can be shown that \(\mathcal{MN}(K) > 2g(K)\). There are knots \(K\) with \(g(K) = 1\) and \(g_f(K)\) arbitrarily large [25, 21]. (2) Clearly the **Free Genus Estimate** can be formally strengthened to aver that \(\mathcal{MN}(L) \leq 2b_f(L)\) for any link \(L\) and \(\mathcal{MN}(K) \leq 4g_f(K)\) for any knot \(K\), where the free Morse–Novikov number \(\mathcal{MN}(L)\) of \(L\) is the minimum possible number of critical points of a free Morse map with binding \(L\). There exist knots and links with minimal Morse maps which are not free. It would be interesting to know whether there exists \(L\) with \(\mathcal{MN}(L) < \mathcal{MN}(L)\).

**Historical note.** Neuwirth [30] was apparently the first to consider, in effect, the notion of “free Seifert surface”: in a footnote, he called a Seifert surface \(S\) “algebraically knotted” if \(\pi_1(S^3 \setminus S)\) is not free. Neuwirth’s language was adapted by Murasugi [27], Lyon [22], and others. By 1976, Problem 1.20 (attributed to Giffen and Siebenmann) in Kirby’s problem list [18] uses the phrase “free Seifert surface”, points out that \(S\) is free if and only if \(S^3 \setminus S\) is an open handlebody, apparently introduces the terminology “free genus”, and includes the imperative “Relate the free genus to other invariants of knots.” A number of later authors (e.g., Moriah [25], Livingston [21], M. Kobayashi & T. Kobayashi [19]) have studied free Seifert surfaces.

9. **Morse maps of doubled knots**

In this section, we estimate the Morse–Novikov number of a Whitehead double \(D(K, m, \pm) := \partial A(K, k) \ast A(O, \mp 1)\) of a knot \(K\) in terms of various invariants of \(K\).

**Braid Index Estimate.** For any knot \(K\) and all \(m \in \mathbb{Z}\), \(\mathcal{MN}(D(K, m, \pm)) \leq 4\text{brin}(K) - 2\).

**Proof.** Let \(n := \text{brin}(K)\). We may assume that \(K\) is a closed \(n\)-string o-braid. By Cor. [4.1] \(\Phi(o, K, k)\) is a fibration for any \(k \in \mathbb{Z}\). As illustrated in Fig. [23], it is easy to take a standard fiber surface \(S\) for \(\Phi(o, K, k)\) (as in Construction [3]) and find \(2n - 1\) proper arcs \(\alpha_i \subset S\) such that

\[
S \setminus \alpha_1 \setminus \alpha_2 \setminus \cdots \setminus \alpha_{2n-1} =: A(K, m)
\]

is an annulus; by adjusting \(k\), any value of \(m\) can be achieved. By Lemma \[8\] \(\mathcal{MN}(A(K, m)) \leq 4\text{brin}(K) - 2\). Thus \(\mathcal{MN}(D(K, m, \pm)) = \mathcal{MN}(A(K, m) \ast A(O, \mp 1)) \leq 4\text{brin}(K) - 2\).

**Wrapping Genera Estimate.** For any knot \(K\) and all \(m \in \mathbb{Z}\), \(\mathcal{MN}(D(K, m, \pm)) \leq 2(g_{wr}(K) + 1) \leq 2(g_{wr}(K) + 1)\).
Proof. Let $n := g_{\text{wfl}}(K)$. We may assume that $K$ is a closed 1-string $o_n$-braid. By Cor. 23.1 $\text{MN}(\Psi(O, K, m - 1) \leq 2n$. Entirely similarly to the situation pictured in Fig. 23, it is easy to take the standard Seifert surface $S$ for $\Phi(O, K, m - 1)$, and find on it a proper arc $\alpha$ such that $S \wr \alpha$ is an annulus $A(K, m)$. By Lemma G, $\text{MN}(A(K, m)) \leq 2n + 2$. Thus $\text{MN}(D(K, m, \pm)) = \text{MN}(A(K, m) \ast A(O, \mp 1)) \leq 2(g_{\text{wfl}}(K) + 1)$. As already noted in Cor. 8.2, $g_{\text{wfl}}(K) \leq g_{\text{wr}}(K)$. \hfill $\square$

Remark. The first author has observed that, for every knot $K$, there exists (at least one) $k$ such that $\text{MN}(D(K, m, \pm)) \leq 2g_{\text{wr}}(K)$ for $m = k - 1$ and $m = k + 1$. The proof will be deferred to another paper; it uses the technique of sutured manifolds.

Crossing Number Estimate. For any knot $K$ and all $m \in \mathbb{Z}$, $\text{MN}(D(K, m, \pm)) \leq 2(c(K) + 2)$.

Proof. This follows immediately from the estimate $\text{MN}(D(K, m, \pm)) \leq 2(g_{\text{wfl}}(K) + 1)$, given the trivial observation that $g_{\text{wfl}}(K) \leq c(K) + 1$. \hfill $\square$

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