Bianchi IX cosmologies and the golden ratio

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Abstract
Special solutions to the Einstein equations in the asymptotic limit for the Bianchi IX cosmologies in the vacuum are examined using Ellis–MacCallum–Wainwright (‘expansion-normalized’) variables. Using an iterative map (the B-map) obeyed by two of the dynamical variables (the normalized shear components) in the ‘asymptotic regime’ close to the cosmological singularity, two period 3 solutions are constructed. These are the simplest of an infinite number of periodic solutions and represent the transition from one vacuum Bianchi I Kasner solution to another. It is shown that the full 3-cycle solutions for the remaining variables (the logarithms of the normalized curvatures) generate a set of self-similar golden rectangles in a graphical time series representation of their dynamics as the normalized time parameter is run backwards towards the initial singularity.

Keywords: golden ratio, self-similarity, anisotropic cosmologies, Bianchi models

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear dynamical systems can display a number of interesting phenomena such as deterministic chaos, self-similarity (both continuous and discrete), pattern formation, critical behaviour, and self-organization. As a set of quasi-linear partial differential equations, the Einstein equations are nonlinear in the metric and its first derivatives and thus form a nonlinear dynamical system. Some of the better known nonlinear phenomena that appear in general relativity are the critical behaviour associated with gravitational collapse and black hole formation [1] and the deterministic chaos occurring in the geodesic motion of test particles and...
in the evolution of some anisotropic cosmologies [2]. One system in particular, the Bianchi IX spacetime, has been studied using a variety of methods for more than a half century. Some results, particularly those concerned with the dynamics close to the initial singularity, have remained controversial. A discussion of what is known and what is conjectured about the behaviour of the Bianchi IX evolution can be found in a review by Heinzle and Uggla [3].

The different approaches to understanding the dynamics of the Bianchi cosmologies can be classified into three main categories: (i) the metric formalism and piece-wise approach to the singularity, (ii) the Hamiltonian formulation and (iii) the dynamical systems analysis. Each of these approaches has its advantages and disadvantages in describing different aspects of the evolution of Bianchi IX cosmologies. Taken together they provide a broader description of these cosmologies than can be obtained by a single approach alone.

The metric approach taken by Belinskii, Khalatnikov and Lifshitz [4], known as the BKL method, uses the metric components and their time derivatives as the dynamical variables. In the Hamiltonian approach [5, 6] the vacuum field equations are reduced to a time-dependent Hamiltonian system for a particle, ‘the universe point’, in two dimensions. Finally, the approaches of Bogoyavlensky and Novikov [7] and Ellis, MacCallum and Wainwright [8] describe the evolution of the universe by orbits of a system of differential equations in \( \mathbb{R}^5 \). The different approaches have utilized both analytic and numerical techniques, but most of these deal with specific rather than general behaviours. More rigorous and more general results, by Ringström [9] and others [10–12], have begun to turn beliefs about generic behaviour into facts, particularly with respect to the oscillatory behaviour near the cosmological singularity.

One feature of the critical phenomena in gravitational collapse is the existence of a discrete self-similarity that occurs at the onset of criticality. While a continuous self-similarity exists in the dynamics of Friedmann–Lemaître–Robertson–Walker (FLRW) cosmologies, the existence of discrete self-similarities in cosmological solutions has not received a great deal of attention. In this work we analyze one interesting example of a discrete self-similarity that arises in the dynamics of the vacuum Bianchi IX cosmologies and show that it has properties that are related to the golden ratio, \( \phi = (1 + \sqrt{5})/2 \).

One advantage of studying spatially homogeneous spacetimes is that the Einstein equations reduce to a set of nonlinear ordinary differential equations (ODEs) and this simplifies the subsequent analysis of the dynamics. Near the singularity it has been shown that important dynamical properties of the ODEs can be captured by algebraic and geometric iterative maps and this leads to further simplifications. While iterative maps can provide information on the sequence of different events that take place, more detailed information such as the time intervals over which those sequences occur and the continuous changes in the magnitude of the dynamical variables require the construction of solutions to the full set of ODEs. In most cases these equations cannot be solved analytically and numerical methods have been employed to construct solutions from a variety of initial conditions. See [13] for a review.

The goal of this work is to provide some details that arise out of an analysis of period 3 solutions to an iterative map derived from the full set of Einstein equations for the Bianchi IX cosmologies. The existence of a period 3 solution implies, by the Sharkovskii theorem [14], that an infinite number of higher period solutions also exist. In addition, the dynamics of a system with period 3 solutions may exhibit chaotic behaviour [15]. For the Bianchi IX cosmologies, period 3 solutions of the iterative map are used to provide initial conditions for the full set of ODEs. This leads to a graphical time series representation of the full dynamics that forms a set of self-similar golden rectangles. An appropriate rescaling of both the time parameterization and the amplitudes of the dynamical variables leads to purely periodic solutions to the Einstein equations.

The next section will briefly describe the Ellis–MacCallum–Wainwright (EMW) orthogonal tetrad approach for understanding the dynamics of Bianchi cosmologies. Specifically,
the presentation will focus on the ODEs describing Bianchi IX spacetimes. In addition the 
iterative map (B-map) in a two dimensional subspace of the full phase space is discussed, 
along with the heteroclinic orbits that connect the equilibria of the map. Section 4 analyzes 
the period 3 solutions to the B-map both analytically and numerically. A time series representa-
tion of the numerical solution produces self-similar rectangles that in section 4 are shown to 
be exactly golden rectangles. A proof that the magnitude of the shear has a specific lower 
bound is also provided. Finally, a concluding discussion introduces a rescaling of the golden 
rectangle dynamics to show the existence of periodic solutions to the ODEs.

2. The Bianchi IX cosmology as a dynamical system

The work described here will follow the dynamical systems approach as interpreted by Ellis 
and MacCallum and further extended by Wainwright and his collaborators. See [8, 16] for 
a detailed description.

Ellis and MacCallum [17] introduced non-zero commutation func-
tions associated with the frame vectors \( \{ e_i, e_j \} \). For diagonal class A Bianchi models these are 
chosen so that

\[
[e_0, e_i] = \vartheta_i e_i, \quad i = 1, 2, 3 \quad \text{(no summation over } i) \\
[e_j, e_k] = n_{jk} e_i \quad \text{(cycle over } i, j, k = 1, 2, 3)
\]

where the \( \vartheta_i(t) \) are the eigenvalues of the expansion tensor and the three structure constants \( n_i \) 
are also the spatial curvature components. It is convenient to replace the components \( \vartheta_i \) with 
the expansion scalar \( \Theta \) and the components of a traceless shear tensor where

\[
\sigma_+ = \frac{3}{2}(\vartheta_2 + \vartheta_3 - \frac{2}{3} \Theta) \\
\sigma_- = \frac{\sqrt{3}}{2}(\vartheta_2 - \vartheta_3).
\]

The matter content consists of a perfect fluid with a linear equation of state

\[
p = (\gamma - 1) \rho \quad (1)\]

where \( p \) and \( \rho \) are, respectively, the pressure and mass density of the fluid in the spacetime 
and \( \gamma \) is a constant. The curvature and shear variables are then normalized with respect to the 
expansion scalar yielding \( N_i = n_i / \Theta \) and \( \Sigma_{\pm} = \sigma_{\pm} / \Theta \) while the density parameter \( \Omega = 3 \rho / \Theta^2 \). 
Finally a time variable \( \tau \) is defined by

\[
\frac{d \tau}{dt} = \frac{\Theta}{3}.
\]

Following Wainwright and Hsu [16] the Einstein equations take the form:

\[
N_1' = (q - 4 \Sigma_+) N_1 \\
N_2' = (q + 2 \Sigma_+ + 2 \sqrt{3} \Sigma_-) N_2 \\
N_3' = (q + 2 \Sigma_+ - 2 \sqrt{3} \Sigma_-) N_3 \\
\Sigma_+ = -(2 - q) \Sigma_+ + 3 \Sigma_+ \\
\Sigma_- = -(2 - q) \Sigma_- + 3 \Sigma_-(2)
\]

where the prime (’) indicates a derivative with respect to \( \tau \), and

2 It should be noted that different numerical factors appear in the field equations given in [8, 16]. These result from 
the use of different normalizations. Wainwright and Hsu use the expansion scalar \( \Theta \) whereas Ellis and Wainwright 
use the Hubble parameter \( H = \Theta / 3 \).
In the vacuum, the density parameter $\Omega = 0$ leading to $q = 2\Sigma^2$. When constructing numerical solutions to the system of ODEs (2), the curvature variables $N_i$ present challenges as they approach the initial singularity. Two components always tend toward zero, while the other one grows. Numerical accuracy can be lost very quickly and the computer program either crashes or leads to incorrect solutions. This problem can be avoided by transforming from the $N_i$s to a new set of variables, $Z_i$s, defined by

$$N_i = \exp(-Z_i)$$

and making the change $\tau \rightarrow -\tau$ so that for the $Z_i$s, $\tau \rightarrow \infty$ as the singularity is approached. The new equations are

$$
\begin{align*}
Z_1' &= q - 4\Sigma_+ \\
Z_2' &= q + 2\Sigma_+ + 2\sqrt{3}\Sigma_- \\
Z_3' &= q + 2\Sigma_+ - 2\sqrt{3}\Sigma_- \\
\Sigma_+' &= (2 - q)\Sigma_+ + 3\Sigma_- \\
\Sigma_-' &= (2 - q)\Sigma_- + 3\Sigma_-
\end{align*}
$$

2.1. The Bianchi I vacuum subset

Of the two possible Bianchi IX state spaces, the one consistent with (4) is characterized by the conditions $N_i > 0$. Setting one or more of the $N_i$s to zero leads to important boundary subsets. The Bianchi I vacuum subset of the system (2) is the Kasner ring (also called the Kasner circle)

$$K = \{\Sigma_+^2 + \Sigma_-^2 = 1, \quad N_i = N_2 = N_3 = 0\},$$

where each fixed point on $K$ represents a Kasner solution. This solution is the simplest spatially homogeneous, anisotropic spacetime and its vacuum metric can be expressed in terms of three explicitly time-dependent scale factors as [18],

$$ds^2 = dt^2 - p_1^2 dx^2 - p_2^2 dy^2 - p_3^2 dz^2$$

where $p_1, p_2$ and $p_3$ are any three numbers that satisfy the conditions

$$p_1 + p_2 + p_3 = 1 \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 = 1.$$  

Given the restrictions (7) between the $p_i$s only a single parameter, $u$, is required to distinguish between different Kasner solutions
where the \( p_i \)s are arranged as \( p_1 < p_2 < p_3 \). All possible sets of \( p_i \) values are generated if \( u \) is allowed to run through all values in the range \( u \geq 1 \). Based on the sign distribution (two positive and one negative) implied by (8), the Kasner vacuum universe expands in two spatial dimensions and contracts in one.

Bogoyavlensky [7] demonstrated that the vacuum equations lead to an iterative map from the Kasner ring onto itself. This will be referred to as the B-map and can be described as follows: given a Kasner solution, represented by an angle \( \theta_n \) (measured in radians from the \( \Sigma^+ \)-axis), the subsequent solution, \( \theta_{n+1} \), can be found by using

\[
p_1(u) = \frac{-u}{1 + u + u^2}, \quad p_2(u) = \frac{1 + u}{1 + u + u^2}, \quad p_3(u) = \frac{u(1 + u)}{1 + u + u^2},
\]

when \( \theta_n \) is in the given range. For angles in other ranges, the formula is reflected about the \( \Sigma^+ \)-axis, and/or rotated by \( \frac{2\pi}{3} \) once or twice. Figure 1(a) provides an example of the action of the iterative map (9) determining transitions from one angular position to another on the Kasner ring.

This map has an alternate construction described by Ma and Wainwright [19]. First circumscribe an equilateral triangle with sides tangent to the Kasner ring at \( T_1, T_2 \) and \( T_3 \). The vertices of this triangle are \( P_1 = (\Sigma^+, \Sigma^-) = (2, 0), P_2 = (-1, \sqrt{3}) \) and \( P_3 = (-1, -\sqrt{3}) \) as shown in figure 1(b). Consider the point labelled 0 in figure 1(b) as the initial point of the B-map dynamics. Except for the \( T_i \)s, each point on the Kasner ring will have a unique triangle vertex that is closest to it. Take the equilateral triangle vertex which is closest to point 0 (i.e. \( P_1 \)) and construct a line that passes through it and the point on the Kasner ring. That line can be extended to intersect another point on the Kasner ring (in this case point 1). This point represents the new Kasner solution that follows after the Bianchi IX transition. The procedure continues indefinitely or until the solution hits one of the tangent points \( T_i \). Each of these points is mapped onto itself since they are equidistant from the two nearest vertices of the equilateral triangle \( \Delta P_1P_2P_3 \).

Figure 1. (a) Action of the B-map given by (9). (b) A geometric construction of the B-map.
2.2. The Bianchi II vacuum subset

A second subset, defined by the vanishing of two curvature components, is the Bianchi II subset. This consists of a set of heteroclinic orbits connecting two Kasner solutions. This subset referred to as the heteroclinic cap (other names include Kasner cap or Taub cap) is defined by:

\[ \Sigma^+_i + \Sigma^-_i + \frac{3}{4}N_i^2 = 1, \quad N_{i-1} = N_{i+1} = 0, \quad i \mod 3 \].

With \( N_i > 0 \) this describes an ellipsoidal half-surface in the \( (\Sigma^+_i, \Sigma^-_i, N_i) \) subspace. Figure 2(a) shows some of the heteroclinic orbits with \( N_i \neq 0 \) that connect two Kasner states on \( \mathcal{K} \). The projection of the orbits onto the \( \Sigma^+_i - \Sigma^-_i \) plane define a set of rays that emanate from the vertex \( P_1 \) and connect two points on the Kasner ring. These rays form the basis for the procedure described above (see [19]). Similarly, when \( N_2 \) (or \( N_3 \)) is non-zero the B-map transitions are described by rays originating from \( P_2 \) (or \( P_3 \)). See figure 2(b).

The two-parameter vacuum Taub solution [20] that describes the heteroclinic orbits can be written in so-called ‘Kasner form’ (see the discussion in [8]):

\[ ds^2 = A^2 dt^2 - t^{2b} A^2 (dx + 4p_i b dz dy)^2 - t^{2b} A^2 dy^2 - t^{2b} A^2 dz^2 \]  

with

\[ A^2 = 1 + b^2 t^{4b}, \quad p_1 + p_2 + p_3 = 1 \quad p_1^2 + p_2^2 + p_3^2 = 1. \]

When the parameter \( b = 0 \) the solution reduces to the Kasner case. When \( b \neq 0 \) one and only one of the \( N_i \) is non-zero. A sequence of Kasner transitions connected by heteroclinic orbits has been called a heteroclinic chain. The so-called Mixmaster attractor is the union of the Kasner ring and the heteroclinic cap. The quantity

\[ N^2 = (N_1 N_2)^2 + (N_2 N_3)^2 + (N_3 N_1)^2 \]

provides a measure of how rapidly the evolution of a Bianchi IX spacetime approaches the Mixmaster attractor, since it will always vanish on that boundary subset. Recent efforts to provide rigorous proofs that the asymptotic state of the Bianchi IX spacetime (for almost all orbits) is described by

\[ \lim_{t \to -\infty} \Omega = 0 \quad \text{and} \quad \lim_{t \to -\infty} N^2 = 0 \]

has been the subject of [9–12].

For arbitrary initial values, the Kasner condition, \( \Sigma^2 = \Sigma^+_2 + \Sigma^-_2 = 1 \), will not hold, nor will it hold during a transient stage. However as the spacetime approaches the singularity it
will hold asymptotically since $\Sigma^2 \to 1$ and $N^2 \to 0$ as $\tau \to -\infty$. The numerical evidence that this is the case for most initial conditions is illustrated by an example shown in figure 3. A fourth-order Runge–Kutta–Fellberg method (with adaptive time step control on the fifth-order error) was implemented to solve the system of equations (5). The first few transitions do not occur on the Kasner ring but the solution quickly converges to a set of transitions that do. Time series representations of the same simulation are shown in figure 4. Clearly two of the $N$s approach zero at late times and during much of the evolution the Kasner condition holds (except during transitions taking place along the heteroclinic orbits).

**Figure 3.** A $(\Sigma_+, \Sigma_-)$ plot of a simulation solving Einstein’s equations, showing the transient phase and then B-map convergence. The initial data for this simulation are $Z_1 = -\ln(0.2)$, $Z_2 = -\ln(3.1)$, $Z_3 = -\ln(2.9)$, $\Sigma_+ = .90$, $\Sigma_- = .25$, $\gamma = 4/3$.

**Figure 4.** A time series representation of the simulation used to produce figure 3. (a) The $\ln(N_i)$ versus $-\tau$ graph. (b) The $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2$ versus $-\tau$ graph.
3. The B-map 3-cycle

Special solutions to the B-map have been shown to exist [3, 10, 21] and a particular heteroclinic chain that consists of three repeating Kasner transitions has received some special attention. The period 3 B-map solutions are shown in figure 5. There are only two possibilities in this case, depending on whether the transitions follow a clockwise or counter-clockwise pattern around the Kasner ring.

3.1. The 3-cycle: identifying the Kasner states

The 3-fold symmetry of the periodic B-map shown in figure 5 implies that the Kasner states are separated one from the next by an angular displacement \( \Delta \theta = \frac{2\pi}{3} \). The points on the counter-clockwise 3-cycle are labelled as \( \theta_0 \), \( \theta_1 \), and \( \theta_2 \). The B-map equation (9) must hold, and maps the angle, \( \theta_0 \), to \( \theta_0 + \frac{2\pi}{3} \). Therefore

\[
\cos(\theta_0 + \frac{2\pi}{3}) = \frac{4 - 5 \cos \theta_0}{5 - 4 \cos \theta_0}.
\]

Using the trigonometric identity for the cosine of the sum of two angles leads to:

\[
\cos \theta_0 \cos \frac{2\pi}{3} - \sin \theta_0 \sin \frac{2\pi}{3} = \frac{4 - 5 \cos \theta_0}{5 - 4 \cos \theta_0}.
\]

Replacing the trigonometric functions of \( 2\pi/3 \) with their numerical values, substituting \( \sqrt{1 - \cos^2 \theta_0} \) for \( \sin \theta_0 \), squaring the result and collecting common powers of \( \cos \theta_0 \) leads to a quartic equation for \( \cos \theta_0 \):

\[
64 \cos^4 \theta_0 - 80 \cos^3 \theta_0 - 12 \cos^2 \theta_0 + 40 \cos \theta_0 - 11 = 0.
\]

The solutions to this quartic equation are all real, namely,

\[
\cos \theta_0 = \frac{1 \pm 3\sqrt{5}}{8} \quad \text{and} \quad \cos \theta_0 = \frac{1}{2},
\]

the last value being a double root. The correct cosine value for the first point of the B-map counter-clockwise 3-cycle is \( \cos \theta_0 = \frac{1 + 3\sqrt{5}}{8} \). The other solutions are extraneous, since they
do not satisfy the original equation before the squaring operation was performed. The corresponding \( \theta_0 \) value is about 15.52\(^\circ\).

The Cartesian coordinates of the initial Kasner solution \((\Sigma_+, \Sigma_-) = (\cos \theta_0, \sin \theta_0)\) can then be computed easily since

\[
\sin \theta_0 = \sqrt{1 - \left(\frac{1 + 3\sqrt{5}}{8}\right)^2} = \frac{\sqrt{3}}{8}(\sqrt{5} - 1).
\]

By using \( \cos \theta = \cos(\theta_0 + \frac{2\pi}{3}) \) and similar relations, the coordinates of all the points of the 3-cycle can be found. The results are:

\[
\begin{align*}
(\Sigma_{i0}, \Sigma_{i-0}) &= (\cos \theta_0, \sin \theta_0) = \left(\frac{1 + 3\sqrt{5}}{8}, \frac{\sqrt{3}(\sqrt{5} - 1)}{8}\right) \\
(\Sigma_{i1}, \Sigma_{i-1}) &= (\cos \theta_1, \sin \theta_1) = \left(\frac{1 - 3\sqrt{5}}{8}, \frac{\sqrt{3}(\sqrt{5} + 1)}{8}\right) \\
(\Sigma_{i2}, \Sigma_{i-2}) &= (\cos \theta_2, \sin \theta_2) = \left(-\frac{1}{4}, -\frac{\sqrt{3}\sqrt{5}}{4}\right).
\end{align*}
\]

The clockwise 3-cycle angles, \( \theta_i' \), can be computed similarly or found simply by recognizing from figure 5 that these angles are obtained as the mirror image of the counter-clockwise angles, reflected across the \( \Sigma_+ \)-axis.

3.2. The 3-cycle: numerical simulations

While iterative maps can provide some information about transitions between different states of some phase space variables, they do not provide detailed information regarding the passage of time that occurs between the transitions or some of the continuous changes that occur in the dynamic variables. In order to recover this information one must return to the original differential equations.

The values of \( \Sigma_+ \) and \( \Sigma_- \) are special, as are the values of the \( N_i \)s. Therefore initial conditions for numerical solutions to the full dynamics that converge to the B-map 3-cycles must be carefully chosen. If they are close to the Kasner solution then the values of \( \Sigma_+ \) and \( \Sigma_- \) are given by one of the pairs in (18). A discussion of the choice for initial spatial curvature values is provided in the appendix. However when two \( N_i \)s are close to zero and another has a value between zero and one, a reasonably close approximation to the 3-cycle dynamics of all five variables can be obtained for long-term simulations.

It can be expected that the simulations will show some sign of repetition, as shown in figures 6(a) and (b), where a repetitive pattern of self-similar quadrilaterals occurs in the \( \ln(N_i) \) time series. Another characteristic of this graph is that the oscillations take place only in the triangular upper-half of the plot. In addition, the \( \Sigma^2 \) time series indicates that 0.25 acts as a lower bound on \( \Sigma_+^2 \). It will be shown that not only are these observations exactly true, but the quadrilaterals shown in figure 6(a) are self-similar golden rectangles. It is well known that the golden ratio appears in continued fraction representations of solutions to iterative maps such as the Gauss map, Farey-map, and B-map [3, 22, 23] (and in particular see [24]) describing certain discrete, Bianchi IX, dynamical variable transitions. The appearance of golden rectangles in the asymptotic solutions to the full ODEs indicates a much deeper connection between the dynamical system and the golden ratio.
3.3. The 3-cycle: the time derivatives of the $Z_i$s

In order to prove that the self-similar rectangles appearing in the numerical simulations are golden rectangles, it is necessary to know the asymptotic values of the time derivatives of the $Z_i$s. These can be computed directly from the vacuum ODEs where $\Omega = 0$ and $\Sigma = 12$ imply $q = 2$. Then using the 3-cycle $\Sigma_{\pm}$ coordinates given in (18) one obtains for $\theta_0$,

$$Z_1' = (2) - 4\left(\frac{1 + 3\sqrt{5}}{8}\right) = 3\left(\frac{1 - \sqrt{5}}{2}\right) = 3(-\frac{1}{\phi})$$

$$Z_2' = (2) + 2\left(\frac{1 + 3\sqrt{5}}{8}\right) + 2\sqrt{3}\left(\frac{\sqrt{5} - 1}{8}\right) = 3\left(\frac{1 + \sqrt{5}}{2}\right) = 3(\phi)$$

$$Z_3' = (2) + 2\left(\frac{1 + 3\sqrt{5}}{8}\right) - 2\sqrt{3}\left(\frac{\sqrt{5} - 1}{8}\right) = 3(1)$$

with similar expressions for the two other 3-cycle Kasner ring points. These results appear in table 1, along with those for the clockwise cycles obtained by changing the signs of the $\Sigma$ values. The slopes for the clockwise cycle are the same as those in the counter-clockwise cycle except that the indices 2 and 3 are exchanged.

The derivatives of the $Z_i$s are the eigenvalues, $\lambda_i$, of the 3-cycle vacuum curvature equations (2). Using the coordinates of $\theta_1$ in the counter-clockwise B-map 3-cycle leads to eigenvalues given by

$$\lambda_1 = 2 - 4\Sigma_+ = 3\phi$$

$$\lambda_2 = 2 + 2\Sigma_+ + 2\sqrt{3}\Sigma_- = 3$$

$$\lambda_3 = 2 + 2\Sigma_+ - 2\sqrt{3}\Sigma_- = -3\phi$$

which means that the two unstable eigenvalues are larger in magnitude than the stable eigenvalue. Liebscher et al [10] also obtained this result but did not mention the interesting relationship of the eigenvalues to the golden ratio.

The simulation shown in figure 6(a) approaches vacuum conditions from non-vacuum initial data. This would indicate that the appearance of the golden ratio is a feature of the vacuum
ODEs themselves. Therefore it is not surprising that iterative maps with golden ratio solutions arise in approximations to the ODEs in some limit. The plots in figure 6(a) are for the \( N \ln i \)s and since \( Z_i = -\ln N_i \) the signs of the slopes given in table 1 are opposite those appearing in the time series plots. A feature of figure 6(a), (that will prove useful later) is that the transitions between Kasner states occur when the two decreasing line segments intersect.

4. The self-similar golden rectangle structure

In section 3.3, it was discovered that the graphical time series representations for the \( Z_i \)s have slopes related to the golden ratio. In this section, the patterns observed in figure 6 are explored more thoroughly to prove that the quadrilaterals appearing in figure 6(a) form a sequence of self-similar golden rectangles. For simplicity, the discussion to follow will refer to figure 7, which is obtained from figure 6(a) by making the following modifications: (1) the time \( \tau \) is re-scaled to remove the factor of 3 appearing in table 1, (This is equivalent to using the normalization introduced in [8].) and (2) the plots shown in figure 7 are time series of the \( Z_i \)s (rather than the \( \ln N_i \)s) so that the sign of the slopes is consistent with those given in table 1. In what follows, attention will be focused on the two largest quadrilaterals appearing in figure 7.

The following propositions summarize some observations regarding the geometry of figure 7. Proofs of these propositions require only simple geometric arguments and properties of the B-map and original ODEs.

**Proposition 1.** The \( Z_i(\tau) \) variables follow straight-line segments that form a sequence of quadrilaterals. The highest and lowest vertices of each quadrilateral are aligned (i.e. they have the same \( \tau \)-coordinate). The slopes of the line segments forming the quadrilaterals repeatedly cycle through the sequence: 1, \( -\frac{1}{\phi} \), \( \phi \) (\( \tau \) rescaling has removed the factor of 3). The quadrilaterals are rectangles.

**Proposition 2.** The highest vertex of each rectangle lies on the line \( Z_i = \tau \).

**Proposition 3.** The lowest vertex of each rectangle lies on the line \( Z_i = 0 \), i.e. for each \( i \), \( \min(Z_i) = 0 \). In addition, the minimum value of \( \Sigma Z^2 \) during a 3-cycle transition is exactly \( \frac{1}{4} \), verifying the observation made in section 3.2.

**Proposition 4.** Each rectangle of the sequence is golden (i.e. its length-to-width ratio is \( \phi \)).

**Proposition 5.** The rectangles of the sequence are self-similar with a linear scaling factor of \( \phi^2 \).

| \( \theta_0 \) | \( \theta_1 \) | \( \theta_2 \) | \( \theta'_0 \) | \( \theta'_1 \) | \( \theta'_2 \) |
|---|---|---|---|---|---|
| \( Z_1 \) | \( 3(-\frac{1}{\phi}) \) | \( 3(\phi) \) | \( 3(1) \) | \( 3(-\frac{1}{\phi}) \) | \( 3(\phi) \) | \( 3(1) \) |
| \( Z_2 \) | \( 3(\phi) \) | \( 3(1) \) | \( 3(-\frac{1}{\phi}) \) | \( 3(1) \) | \( 3(-\frac{1}{\phi}) \) | \( 3(\phi) \) |
| \( Z_3 \) | \( 3(1) \) | \( 3(-\frac{1}{\phi}) \) | \( 3(\phi) \) | \( 3(\phi) \) | \( 3(1) \) | \( 3(-\frac{1}{\phi}) \) |
4.1. Proof of proposition 1

The B-map 3-cycle coordinates of $\Sigma_+$ and $\Sigma_-$ were used in section 3.3 to find the time derivatives of the $Z_i$s while they are in Kasner states. Each Kasner state has a constant $dZ_i/d\tau$, which means that the $Z_i(\tau)$ curves consist of straight line segments. The Kasner transitions take place simultaneously so the points $A$ and $B$ occur at the same value of $\tau$, as do the points $C$ and $D$. Therefore line segments $AB$ and $CD$ are both perpendicular to the $\tau$-axis.

The time derivatives $dZ_i/d\tau$ cycle through the values $1, -\phi, \phi$. (See table 1.) From figure 7, all vertices $A, B, C$ and $D$ as well as the intersection point $E$ are points where two line segments meet: one with positive slope $\phi$ and one with negative slope $-1/\phi$. Therefore at each of these points the product of the two slopes is $\phi \times (-\phi^{-1}) = -1$, i.e. the line segments meet at right angles, making all the quadrilaterals rectangles.

4.2. Proof of proposition 2

The line segments (e.g. $BD$) that are not a part of the rectangle sequence also take part in the Kasner transitions and therefore must meet the rectangle vertices. The slopes of these line segments must all be unity (see table 1) and must connect to each other to form a single line $Z_i = \tau + k$ for some $k$. Since the $\tau = 0$ point is arbitrarily chosen, a time translation can always be performed to re-set the initial time to ensure that $k = 0$. This is true only if the initial values are consistent with the asymptotic vacuum conditions near the singularity. It cannot be expected that this structure will exist during the transient stage.

4.3. Proof of proposition 3

Proposition 3 asserts that the lowest vertex of each rectangle lies on the line $Z_i = 0$. This requires showing that $\min(Z_i) = 0$ for each $i$. Since $Z_i = -\ln(N_i)$ and $N_i \geq 0$ it must be shown that $\max(N_i) = 1$. Using (10), $N_i = \left[\frac{1}{k}(1 - \Sigma^2)^{1/2}\right]$, so $\max(N_i) = 1$ implies $\min(\Sigma^2) = 1/4$.

Proof of this last condition is accomplished using the geometry of the 3-cycle B-map. Figure 8 reproduces the counter-clockwise 3-cycle shown in figure 5(a). The angular coordinates $\theta_0, \theta_1$, and $\theta_2$ are replaced with their Cartesian counterparts $G_0, G_1$, and $G_2$ given by:
Consider the transition from $G_0$ to $G_1$ where $P_1$ is the origin of the line segment connecting the two Kasner states. The heteroclinic orbit connecting $G_0$ to $G_1$ is described by $N_1 = N_2 = 0$ with

$$N_1 = \sqrt{\frac{4}{3}(1 - \Sigma^2)}.$$  

Since $\Sigma^2$ is the square of the magnitude of the position vector for any point in the $\Sigma_+\Sigma_-$ plane, it must be shown that the minimum distance between the line through $G_0$ and $G_1$ and the origin equals $\frac{1}{2}$.

Referring to figure 8, by the chord-radius theorem in circle geometry, $\overline{OH} \perp \overline{G_0G_1}$ implies $\overline{OH}$ bisects $\overline{G_0G_1}$. In other words, $H$ can be calculated as the midpoint of $\overline{G_0G_1}$ and its magnitude taken, yielding the minimum distance.

$$H = \frac{1}{2} \overline{G_0G_1}$$

$$= \frac{1}{2} \left( \Sigma_0 + \Sigma_1, \frac{1}{2}(\Sigma_0 + \Sigma_1) \right)$$

$$= \left( \frac{1}{8}, \frac{\sqrt{15}}{8} \right)$$
Computing the magnitude of $H$ leads to

$$|H| = \sqrt{(H_x)^2 + (H_y)^2} = \frac{1}{2}.$$  

This proves the result for $Z_1$ and shows that $\min(\Sigma^2) = H^2 = \frac{1}{4}$ exactly. This also verifies the observation regarding the time series plot of $\Sigma^2$ shown in figure 6(b). The minimum values of $Z_2$ and $Z_3$ are also zero when $N_2 \neq 0$ and $N_3 \neq 0$, respectively, since a rotation of the B-map diagram by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ leaves the geometry unaltered. The clockwise 3-cycle leads to the same conclusion.

4.4. Proof of proposition 4

Referring to figure 7, two arbitrary consecutive rectangles of the sequence are shown, with labelled points. Using a similar triangle argument,

$$\triangle \triangle$$

$$\angle BEA = \angle APE = 90^\circ$$

$$\Rightarrow \triangle BEA \sim \triangle APE$$

\[\text{(20)}\]

\[\therefore \frac{AE}{BE} = \frac{EP}{AP} = \phi \quad \text{(since } AE \text{ has slope } \phi),\]  

\[\text{(21)}\]

proving that the smaller rectangle (which was arbitrarily chosen), has a length-to-width ratio of $\phi$, and is therefore golden. Thus, all rectangles in the sequence are golden.

4.5. Proof of proposition 5

Once again, using a similar triangle argument and referring to figure 7,

$$\triangle \triangle$$

$$\angle BEA = \angle AEC = 90^\circ$$

$$\Rightarrow \triangle BEA \sim \triangle AEC$$

\[\text{(22)}\]

\[\therefore \frac{EC}{BE} = \frac{EC}{AE} = \phi \cdot \phi = \phi^2 \quad \text{(by } (21) \text{ and } (22))\]  

\[\text{(23)}\]

proving that rectangles in the sequence scale (in their linear dimensions), one to the next, by a factor of $\phi^2$.

5. Discussion

Using modified Ellis–MacCallum–Wainwright variables, a vacuum Bianchi IX cosmology near its initial singularity was considered. The B-map, an iterative map with interesting geometric structure, was used to find the shear variable values for the 3-cycle solutions. These values when used as partial initial data for the Einstein equations (the full set of ODEs) produced a time series representation that exhibited a discrete self-similar golden rectangle structure.

In addition, it was found directly from the ODEs that the time derivatives of the $Z_i$ variables always equalled $\phi$, $1$ and $-\phi^{-1}$ using an appropriately re-scaled time variable. The scaling of the lengths (and widths) of the self-similar golden rectangles in the time series of $Z_i$s goes as $\phi^2$. Such a scaling provides a new set of curvature variables $\Phi_i = Z_i/\tau$ and time variable $T = \ln(\tau)$ such that a time series in the new variables clearly converges asymptotically to a
simple periodicity (see figure 9) [21]. The non-vacuum behaviour in the early evolution is magnified in these plots, and only at late times does the period 3 behaviour appear. It must be remarked that deviations from this behaviour will eventually occur at even later times in any numerical simulations due to the instability of the dynamical system. It is conjectured that the use of these new variables will lead to a compact state space which is one of the requirements for understanding chaotic behaviour in a system of ODEs.

While the golden ratio appears in the analysis of many natural and engineered systems, often it does so in an approximate form [25, 26]. For the 3-cycle Bianchi IX vacuum solution the appearance of the golden ratio is exact and depends on special initial conditions. That the Einstein equations lead to the golden ratio exactly is somewhat of a mystery, but the ubiquitousness of the number $\phi$ and its appearance in a large number of totally unrelated systems remains one of the great mysteries of mathematics.

From a mathematical point of view, cosmological expansion and gravitational collapse are related phenomena [27]. Discrete self-similarity in cosmological evolution and its relationship (if any) to self-similar collapse in black hole formation needs to be better understood. For example, is there a parameter that appears in critical black hole collapse that is exactly the golden ratio? The BKL conjecture regarding the generic form of spacetime singularities in inhomogeneous collapse remains an open question. Hopefully the study of self-similar Bianchi IX solutions can provide some insight into singularity formation in more general forms of gravitational collapse [13].

Interleaving the pattern shown in figure 7 with similar ones leads to a tiling of the Euclidean plane that yields geometric proofs of identities involving the golden ratio, the Fibonacci numbers and the number $\pi$. The details are provided in another manuscript by the authors [28].

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Appendix. Initial conditions for numerical simulations

In section 3.3, it was proven that the B-map 3-cycle $\Sigma_+$ and $\Sigma_-$ values (found in section 3.1) produce the golden-ratio-related time derivatives for the $Z_i$s. If one wishes to re-construct the full dynamics of the 3-cycle, then the $\Sigma_\pm$ values along with appropriate initial conditions for the $Z_i$s are required in order to obtain the correct 3-cycle evolution for a large number of transitions. Numerical errors arising from an inaccurate choice of curvature variables eventually produce deviations away from self-similarity, typically after five to eight transitions. At the same time it was found that the minimum value of $\Sigma^2$ would dip below the value of $\frac{1}{4}$. In order to obtain results for a larger number of transitions, one needs to begin with a set of initial values that are consistent with the B-map 3-cycles. A bit of hindsight based upon what is known about the self-similar structure helps in choosing the appropriate initial data.

Figure A1 shows a section of the self-similar golden rectangle sequence with some points labelled and the slope values shown. We set $\tau = \tau_1$ as the point in time at which two of the $Z_i$ values are equal. If $k$ is the time since the last transition, then the vertical line segments at time $\tau$ are as shown. First, $\Delta BGF$ is a 45-45-90 triangle, therefore $BG = FG = k$. The right triangles $\Delta AEP$ and $\Delta EBG$ are both similar to $\Delta BAE$ which has two perpendicular sides obeying $AE/BE = \phi$. This makes $EP = k\phi$ and $GE = k/\phi$. Then $FE = FG + GE = k + \frac{k}{\phi} = k(1 + \frac{1}{\phi}) = k\phi = EP$. Thus the value of the third $Z_i$ (at $F$) is twice the common value of the other two.

Since a time translation does not affect the overall dynamics, any small value, $\tau_1$, can be taken as the initial time, $\tau_0$. One of the initial $Z_i$ values should equal $\tau_0$, and the other two should equal $\tau_0/2$. For a given 3-cycle $(\Sigma_+, \Sigma_-)$ point there is a one-in-three chance of picking the correct $Z_i$ as the largest one.

For the numerical simulation responsible for the time series shown in figure 6, the initial data was chosen to be, $\tau_0 = 0.1$; $(\Sigma_+, \Sigma_-) = (\Sigma_{+2}, \Sigma_{-2}) = (\frac{1}{3}, \frac{\sqrt{5}}{3})$, $Z_1 = \tau_0/2$, $Z_2 = \tau_0$ and $Z_3 = \tau_0/2$. The matter content was given by $\gamma = 4/3$. Even after ten transitions between $\tau = 0.1$ and $\tau = 22,000$, there was no significant deviation away from the discrete self-similar pattern and the minimum value for $\Sigma^2$ never fell below 0.25.
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