On large gaps between zeros of $L$-functions from branches

André LeClair

Cornell University, Physics Department, Ithaca, NY 14850

Abstract

It is commonly believed that the normalized gaps between consecutive ordinates $t_n$ of the zeros of the Riemann zeta function on the critical line can be arbitrarily large. In particular, drawing on analogies with random matrix theory, it has been conjectured that

$$\lambda' = \limsup_{n \to \infty} (t_{n+1} - t_n) \frac{\log(t_n/2\pi e)}{2\pi}$$

equals \infty. In this article we provide arguments, although not a rigorous proof, that $\lambda'$ is finite. Conditional on the Riemann Hypothesis, we show that if there are no changes of branch between consecutive zeros then $\lambda' \leq 3$, otherwise $\lambda'$ is allowed to be greater than 3. Additional arguments lead us to propose $\lambda' \leq 5$. This proposal is consistent with numerous calculations that place lower bounds on $\lambda'$. We present the generalization of this result to all Dirichlet $L$-functions and those based on cusp forms.

\textsuperscript{a} andre.leclair@gmail.com
Let $\rho = \frac{1}{2} + it_n$ denote the $n$-th zero of the Riemann zeta function on the upper critical line, with $n = 1, 2, \ldots$. An interesting question which has received a great deal of attention concerns just how large the gaps between consecutive $t_n$ can be. Since on average the gaps are known to be $2\pi / \log t_n$, one is led to study the appropriately normalized gaps

$$g_n = (t_{n+1} - t_n) \frac{\log t_n}{2\pi}$$

and

$$\lambda = \limsup_{n \to \infty} g_n$$

Montgomery conjectured that $\lambda = \infty$ based on analogies with random matrix theory [1, 2].

Obtaining lower bounds on $\lambda$ is a very difficult problem. Extensive analysis by many authors indicates that if $\lambda$ is indeed $\infty$, then it must approach it very slowly, if at all. The first unconditional result is due to Hall [3]: $\lambda > 2.34$. Assuming the Riemann Hypothesis (RH), Montgomery and Odlyzko [4] obtained $\lambda > 1.9799$, and Conrey, Ghosh, and Gonek [5] improved the result to $\lambda > 2.337$. If one assumes the generalized Riemann Hypothesis (GRH), one can do slightly better. The previously mentioned authors obtained $\lambda > 2.68$ [6]. Ng [7] obtained $\lambda > 3$ and Bui [8] showed $\lambda > 3.033$. The current best unconditional result is due to Bui and Milinovich, $\lambda > 3.18$, based on the method of Hall [9]. All of these results were obtained by similar methods involving studying higher moments of the zeta function. Despite these difficult analyses, the results have improved only incrementally, and are still very far from the expected $\lambda = \infty$. It should be mentioned that Hughes [10] has proposed some ideas on how to perhaps get to $\lambda = \infty$.

The above results indicate that either new methods are needed, or the conjecture $\lambda = \infty$ is false. In this short note, we will argue the latter from a simple argument based on the branches of the zeta function. Thus the main goal of this article is to propose an upper bound on $\lambda$.

Let us modify slightly the definition of $g_n$ and $\lambda$:

$$g'_n = (t_{n+1} - t_n) \frac{\log(t_n/2\pi e)}{2\pi}, \quad \lambda' = \limsup_{n \to \infty} g'_n$$

The extra $2\pi e$ is inconsequential in the limit $n \to \infty$ where $\lambda' = \lambda$, but as we will explain, it is more instructive for performing numerical checks of our ideas at large but finite $n$. Also, integer values of $\lambda'$ will have a special significance. As we will show, assuming the Riemann Hypothesis, a simple argument leads to $\lambda' \leq 3$ if there are no changes of branch between
consecutive zeros. The value 3 is already very close to the best lower bound \( \lambda' > 3.18 \) [4]. Additional arguments lead to the much more specific proposal

\[
\lambda' \leq 5 \tag{4}
\]

To our knowledge, this is the first proposal for an upper bound on the normalized gaps. Although we are unable to provide a rigorous proof, it is worthwhile to elucidate the arguments leading to this proposal, since they are new and lead to a definite but unexpected result which is in the opposite direction of previous results. Furthermore, it appears to be closer to the reality suggested by numerical results. We have checked numerically that \( g'_n < 3 \) for all \( n < 10^9 \). For clarity, we itemize the hypotheses behind our proposal which we could not rigorously prove.

Below we will present the generalization of the above result to all \( L \)-functions based on Dirichlet characters and cusp (modular) forms. In order to avoid introducing too many definitions that are well-known for these cases, we will present the main arguments for the Riemann zeta function itself, and subsequently simply state its generalization to these other \( L \)-functions. As will be clear, our arguments assume the RH appropriate to the \( L \)-function in question is true, although indirectly. For instance, we do not need to assume the GRH to study \( \lambda' \) for Riemann zeta itself, but only the original RH.

Let \( \vartheta(t) \) denote the Riemann-Siegel \( \vartheta \) function:

\[
\vartheta(t) = 3 \log \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) - t \log \sqrt{\pi} \tag{5}
\]

and \( a(t) \) the argument of the zeta function defined in the following specific manner:

\[
a(t) = \lim_{\delta \rightarrow 0^+} \arg \zeta \left( \frac{1}{2} + \delta + it \right). \tag{6}
\]

In [11] [12] it was proposed that the ordinate \( t_n \) of the \( n \)-th zeta zero on the upper critical line, with \( n = 1, 2, 3, \ldots \), satisfies the exact transcendental equation

\[
\vartheta(t_n) + a(t_n) = \left( n - \frac{3}{2} \right) \pi \tag{7}
\]

Several remarks regarding this equation are in order. The equation was obtained by putting the zeros in one to one correspondence with the zeros of the cosine function. In obtaining this formula, we did not assume Backlund’s counting formula for \( N(T) \), i.e. the number of zeros up to height \( T \) in the entire critical strip, \( N(T) = \vartheta(T)/\pi + 1 + S(T) \), where
$S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$. The equation (7) contains more information that $N(T)$. It is important to note that $S(T)$ and $a(t)$ are not equivalent. For instance $S(T)$ is not defined at the ordinate of a zero, unlike $a(t)$. See [13] for a review of known properties of $S(T)$. It is also important that $a(t)$ is defined precisely as in (6), i.e. as a limit from the right of the critical line with $\delta$ not allowed to be strictly zero. The behavior of $a(t)$ would be completely different if it were defined as a limit from the left of the line or along the line. (See Remark 1.) If there is a unique solution to (7) for every positive integer $n$, then the RH is true since this implies that the number of zeros on the critical line saturate the formula $N(T)$ [11, 12]. If one ignores the $a(t_n)$ term there is in fact a unique solution for every $n$ that can be expressed in terms of the Lambert $W$ function to a very good approximation. However we were unable to rigorously prove there exists a unique solution including the $a(t_n)$ term; thus we take it as our first hypothesis:

**Hypothesis 1.** There is a unique solution to the equation (7) for every positive integer $n$. This effectively assumes the RH and that the zeros are simple [12].

Using the Stirling formula,

$$\vartheta(t) = \frac{t}{2} \log \left( \frac{t}{2\pi e} \right) - \frac{\pi}{8} + O(1/t). \quad (8)$$

Henceforth it is implicit that we are considering the limit of large $n$ since we use the above asymptotic behavior, i.e. we will not always display the lim sup. Assuming Hypothesis 1 we can consider the difference of the equation for two consecutive $n$’s. Clearly one has

$$(t_{n+1} - t_n) \log t_n < t_{n+1} \log t_{n+1} - t_n \log t_n, \quad \text{thus}$$

$$g'_n < 1 - \frac{1}{\pi} (a(t_{n+1}) - a(t_n)) \quad (9)$$

First of all, the equation (9) shows that if $a(t)$ is finite, then so is $\lambda'$. See Remark 1 for comments concerning this finiteness. In order to constrain $\lambda'$, we need one additional hypothesis. Let us write

$$a(t_n) = A_n + 2\pi b_n, \quad b_n \in \mathbb{Z} \quad (10)$$

where $A_n$ denotes the principal branch: $A_n = \lim_{\delta \to 0^+} \text{Arg} \zeta(\frac{1}{2} + \delta + it_n)$ with $-\pi < A_n < \pi$.

We will refer to $b_n$ as the branch number, where $b_n = 0$ corresponds to the principal branch. Since $|A_{n+1} - A_n| \leq 2\pi$, equation (9) implies

$$g'_n \leq 3 + \Delta b_{\text{max}} \quad (11)$$
where

\[ \Delta b_{\text{max}} = \max (b_n - b_{n+1}) \] (12)

From this equation we deduce that if there are no changes of branch between consecutive zeros, then \( g'_n \leq 3 \) and \( \lambda' \leq 3 \). Changes of branch allow \( \lambda' \geq 3 \), but do not necessarily imply it. Based on the result \( \lambda > 3.18 \) in [9] we should conclude that branch changes do occur; however all the way up to \( n = 10^9 \) we found \( g'_n < 3 \) (see below). The main shortcoming of our approach is that it leads to bounds on \( \lambda' \) that are only integers; it would be impossible to constrain it further to specific values like \( \lambda > 3.18 \) with this reasoning. However we think the approach is useful since it endows values like \( \lambda' = 3 \) some special significance.

An upper bound on \( \lambda' \) would follow from a bound on how much the branch number \( b_n \) can decrease between consecutive zeros. Let us assume that \( a(t)/\pi \) behaves like \( S(t) \) as far a branch changes are concerned. Between zeros, \( a(t) \) always decreases. Assuming Hypothesis [1] since the zeros are simple, the only place where the branch number of \( a(t)/\pi \) can increase is at the jumps by 1 at each zero. Therefore there is an upper bound to increases of branch number:

\[ b_{n+1} - b_n \leq 1 \] (13)

Now since the average of \( g'_n \) is known to be 1, the average of \( a(t_{n+1}) - a(t_n) \) is zero. This suggests that the average of \( b_{n+1} - b_n \) equals zero. A reasonable assumption then is that there is no preference between increases verses decreases of branch number. This leads to our second hypothesis, which is likely to be the more difficult one to establish:

**Hypothesis 2.** There is no dichotomy between increases and decreases of branch number.

Then [13] implies

\[ |b_{n+1} - b_n| \leq 1 \] (14)

This hypothesis implies \( \Delta b_{\text{max}} \leq 1 \). Combined with (11) this leads to \( \lambda' \leq 5 \). Note that, in contrast, \( \lambda' = \infty \) would require an infinite number of branch changes between two consecutive zeros.

**Remark 1.** Based on the validity of the Euler product formula to the right of the critical line (for \( L \) functions based on principal characters, the product must be truncated in a
well-prescribed fashion), it was argued in [14, 15] that \( a(t) \) is finite, which is consistent with an upper bound on \( \lambda' \): if \( a(t) \) is finite, then so is \( \lambda' \). Let us summarize the arguments that led to this conjecture. Consider first \( L \)-functions based on a non-principal Dirichlet character \( \chi \). We conjectured that \( C_N = \sum_{n=1}^{N} \chi(p_n) \), where \( p_n \) is the \( n \)-th prime, behaves like a random walk and is thus \( O(\sqrt{N}) \) (up to logs). Further support for this conjecture was obtained using a variant of Cramér’s random model for the primes [16]. Assuming this random walk behavior, we proved that the Euler product converges to the right of the critical line. For the present work, since we are assuming the GRH for non-principal characters, we can actually use the result \( C(x) = \sum_{p \leq x} \chi(p) = O(\sqrt{x} \log^2 x) \) ([17] page 125). The \( \log^2 x \) does not spoil the convergence arguments in [14], so the GRH implies the validity of the Euler product formula to the right of the critical line for non-principal Dirichlet characters.

One can therefore calculate \( a(t) \) from the Euler product:

\[
a(t) = -\lim_{\delta \to 0^+} \Im \sum_p \log \left( 1 - \frac{\chi(p)}{p^{1/2 + \delta + it}} \right) \tag{15}
\]

If the product converges, then \( a(t) \) and \( \lambda' \) are finite. For principal characters the situation is more subtle and one has to truncate the Euler product [15, 18, 19] at \( p = p_{N_c} \) where \( N_c \sim t^2 \), however this does not spoil the conclusion that \( a(t) \) is finite. In fact, if one is interested in \( \lim \sup \), the truncation is not necessary since \( N_c \to \infty \) as \( t \to \infty \). Furthermore, approximating the expression in (15) using the prime number theorem led us to propose that \( a(t) \) is nearly always on the principal branch [15] (based on equation (38) in [15]). The latter approximation also supports Hypothesis 2 since there is no dichotomy between the oscillations above versus below zero. Consequently in regions where \( a(t_n) \) is on the principal branch, \( g'_n \) < 3; we will provide numerical evidence for this below.

We can easily extend the above arguments to two additional classes of \( L \)-functions. First consider \( L \)-functions based on any primitive Dirichlet character mod \( q \). Denote the zeros on the critical line as \( \rho = \frac{k}{2} + it_n \). Repeating the above arguments using the transcendental equations in [12] one finds that the proper normalization depends on \( q \):

\[
\lambda' = \lim_{n \to \infty} \sup (t_{n+1} - t_n) \frac{\log(q t_n/2\pi e)}{2\pi}, \tag{16}
\]

Next consider \( L \) functions based on cusp forms mod \( k \). Here the zeros on the line are \( \rho = \frac{k}{2} + it_n \). In this case the proper normalization does not depend on \( k \) however there is a
FIG. 1. The normalized gaps $g'_n = (t_{n+1} - t_n) \log(t_n/2\pi e)/2\pi$ of the zeta function for $10^7 - 10,000 < n < 10^7$.

difference by an overall factor of 2:

$$\lambda' = \limsup_{n \to \infty} (t_{n+1} - t_n) \frac{\log(t_n/2\pi e)}{\pi},$$

(17)

In both cases the previous arguments again imply $\lambda' \leq 5$.

Let us now provide numerical support for the above proposals. We limit ourselves to the zeta function. All the way up to $n = 10^9$ we found that $g'_n < 3$, with some cases coming close to this upper bound. In Figure 1 we display a region around $n = 10^7$. The different definitions of $\lambda$ and $\lambda'$ are significant here in that for some of the extreme gaps where $g'_n$ is just under 3, one can check that $g_n > 3$. As displayed in Figure 2, the rare points where $g_n > 3.18$ in this range still have $g'_n < 3$. The explanation of these results is simple based on the above ideas: in this whole range, $a(t_n)$ is always on the principal branch with $b_n = 0$ for all $n$, and by (11), $g'_n < 3$. (See Remark 1) Had we used the definition $g_n$ instead of $g'_n$, the rare points where $g_n > 3$ would not so easily be explained. We also wish to point out that if $a(t_n)$ is never very far from the principal branch, then $\lambda' \approx 3$, which suggests that Bui and Milinovich’s result $\lambda > 3.18$ may actually be close to the true upper bound.
FIG. 2. The normalized gaps $g_n = (t_{n+1} - t_n) \log(t_n)/2\pi$ of the zeta function for $10^7 - 10,000 < n < 10^7$, to be compared with Figure 1. The horizontal line is 3.18, based on the prediction in [9] that $\lambda' > 3.18$.

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