On boundary controllability and stabilizability of the 1D wave equation in non-cylindrical domain

Mokhtari Yacine
March 26, 2020

Contents

1 Introduction and preliminaries

2 Main results
   2.1 Controllability result
   2.2 Stability result

3 Construction of the exact solution

4 Proof of main results
   4.1 Proof of the controllability theorem
   4.2 Proof of the stability theorem

5 Further remarks and open questions

Abstract

In this paper, we deal with boundary controllability and boundary stabilizability of the 1D wave equation in non-cylindrical domain of the form \((\alpha(t) < x < \beta(t))\). By using the characteristics method, we prove under a natural assumption on the boundary functions that the 1D wave equation is controllable and stabilizable from one side of the boundary. Furthermore, the control function and the decay rate of the solution are given explicitly.

1 Introduction and preliminaries

In this work, we are interested in the boundary controllability and stabilizability of the one dimensional wave equation in non-cylindrical domain. More precisely, define \(Q\) by

\[
Q = \{(t, x) \in \mathbb{R}^2, \ x \in (\alpha(t), \beta(t)), \ \alpha(t) < \beta(t), \ t \in (0, \infty)\},
\]

with \(\alpha(0) = 0\) and \(\beta(0) = 1\), and consider the following two systems

\[
\begin{cases}
y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\
y(t, \alpha(t)) = \frac{1}{4}u(t), \ y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\
y(0, x) = y_0(x), \ y_t(0, x) = y_1(x), & \text{in } (0, 1),
\end{cases}
\]

(1)
and
\[
\begin{aligned}
&\begin{cases}
  y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\
y_t(0, x) = y_0(x), & \text{in } (0, 1).
\end{cases}
\end{aligned}
\]
(2)

The functions \( u \in H^1_{loc}(0, \infty) \), and \( f \in C([0, \infty)) \) in (1) and (2) represent the control force and the feedback function respectively.

Controllability of system (1) has been extensively studied in the recent past years; most of the papers in this direction dealt with the case of one moving endpoint with boundary conditions of the form
\[
y(t, 0) = 0, \quad y(t, kt + 1) = u(t), \quad k \in (0, 1), \quad t \in (0, \infty).
\]

In [3], it has been shown that exact controllability holds for all times \( T > \frac{2k(k+1)}{2} \). The same authors came back in [4] and improved the latter result to \( T > \frac{2k(k+1)}{(1-k)r-1} \). Later, in [12], the controllability time has been improved to be \( T > \frac{2}{1-k} \). In these papers, only a sufficient condition is provided for the exact controllability.

Concerning the two moving endpoints case, the boundary functions considered in [10] are of the form
\[
\alpha(t) = -kt, \quad \beta(t) = rt + 1, \quad t \in (0, \infty), \quad k, r \in [0, 1) \text{ with } r + k > 0.
\]

It has been shown that exact controllability holds if, and only if \( T \geq \frac{2}{(1-k)(1-r)} \). More general boundary functions are considered in [6] with boundary conditions of the form
\[
y(t, 0) = 0, \quad y(t, s(t)) = u(t), \quad t \in (0, \infty),
\]
where \( s : [0, \infty) \to (0, \infty) \) is assumed to be \( C^1 \) function satisfying \( \|s'\|_{L^{\infty}(0, \infty)} < 1 \). Furthermore, it has been assumed that \( s \) must be in some admissible class of curves (see [6] for more details). Under these assumptions, the authors proved that exact controllability holds if, and only if \( T \geq s^+ \circ (s^-)^{-1}(0) \), where \( s^\pm(t) = t \pm s(t) \). Also, they provided a controllability result when the control is located on the non-moving part of the boundary. By considering the boundary conditions
\[
y(t, 0) = u(t), \quad y(t, s(t)) = 0, \quad t \in (0, \infty),
\]
they proved that exact controllability holds if, and only if \( T \geq (s^-)^{-1}(1) \). In all the cited works, the proofs rely on the multipliers technique or the non-harmonic Fourier analysis.

Recently, in [11], a new Carleman estimate has been established for the wave equation in time-dependent domain in more general settings. As a consequence, it has been shown for a general boundary conditions of the form
\[
y(t, \lambda_1(t)) = u(t), \quad y(t, \lambda_2(t)) = 0, \quad t \in (0, \infty),
\]

Figure 1: The curves \((t, \alpha(t))_{t \geq 0}\) in red and \((t, \beta(t))_{t \geq 0}\) in blue.
where $\lambda_i(t) < \lambda_2(t)$, $t \in (0, \infty)$, are smooth functions satisfying $\|\lambda_i\|_{L^\infty(0,\infty)} < 1$, $i = 1, 2$, that system (1) is exactly controllable at time $T$ if $T > T^* = (\lambda_1^+)^{-1} \circ \lambda_2^+ \circ (\lambda_2^-)^{-1}(0)$ and not exactly controllable if $T < T^*$ (the functions $\lambda_i^\pm$ are defined by $\lambda_i^\pm(t) = t \pm \lambda_i(t)$, $i = 1, 2$). However, the result doesn’t cover the critical case $T = T^*$.

As for the boundary stability of system (2), to the best of our knowledge, the only existing result in the literature is in [1] where the authors dealt with the same system but with only one moving endpoint, i.e.

$$y(t,0) = 0, \quad y_t(t,a(t)) + f(t)y_x(t,a(t)) = 0, \quad t \in (0,\infty),$$

with $a$ is strictly positive $1$-periodic function and $f$ is the feedback function. The authors proved exponential stability of system (2) for a particular class of feedbacks $f$. The proof relies on transforming problem (2) which is posed on non-cylindrical domain into a problem posed on cylindrical one, then making use of some known results of boundary stability of the $1D$ wave equation and composition of periodic functions. Stability problem of system (2) has not yet been studied even for the cylindrical domain case.

In this paper, we will improve all the previous results either for the boundary control or the boundary stability of the $1D$ wave equation by using the characteristics method. We shall build the unique exact solution to both systems (1) and (2) in an appropriate energy space. To do so, we proceed by transforming both of systems to a first order hyperbolic system by introducing the Riemann invariants

$$\begin{cases} p = y_t - y_x, \\ q = y_t + y_x. \end{cases} \tag{3}$$

An elementary computation shows that system (1) transforms into

$$\begin{cases} p_t + p_x = 0, & \text{in } Q, \\ q_t - q_x = 0, & \text{in } Q, \\ (p + q)(t,\alpha(t)) = u(t), & (p + q)(t,\beta(t)) = 0, & \text{in } (0,\infty), \\ p(0,x) = \tilde{p}(x), & q(0,x) = \tilde{q}(x). & \text{in } (0,1). \end{cases} \tag{4}$$

In the same way, system (2) becomes

$$\begin{cases} p_t + p_x = 0, & \text{in } Q, \\ q_t - q_x = 0, & \text{in } Q, \\ (p + F(t)q)(t,\alpha(t)) = 0, & (p + q)(t,\beta(t)) = 0, & \text{in } (0,\infty), \\ p(0,x) = \tilde{p}(x), & q(0,x) = \tilde{q}(x), & \text{in } (0,1). \end{cases} \tag{5}$$

where $F(t) = \frac{1-f(t)}{1+f(t)}$ with $1 + f(t) \neq 0, \forall t \geq 0$. Henceforth, we use the following notations:

- the spaces family $[L^2(\alpha(t),\beta(t))]_{t \geq 0}$ will be denoted by $L^2(\alpha(t),\beta(t))$.

- The spaces family $[H^1(\beta(t))(\alpha(t),\beta(t))]_{t \geq 0}$ will be denoted by $H^1(\beta(t))(\alpha(t),\beta(t))$ where

$$H^1(\beta(t))(\alpha(t),\beta(t)) = \{h \in H^1(\alpha(t),\beta(t)), \ h(\beta(t)) = 0, \ t \geq 0\}.$$

- For any function $z$, the functions $z^\pm$ will represent the quantities $z^\pm(t) = t \pm z(t)$.

- $C$ denotes a generic positive constant which might be different from line to line.

The Riemann coordinates introduced in (3) guarantee the equivalence of the transformed systems (4),(5), to the original systems (1),(2) up to an additive constant. All the results for the transformed systems will be proved in $[L^2(\alpha(t),\beta(t))]^2$, then the results for the original ones can be deduced by inverting the transformation.
Since our approach consists in constructing the unique exact solutions to systems (4) and (5), instead of studying each system separately, we consider the following system
\[
\begin{aligned}
\frac{p_t + p_x}{q_t - q_x} &= 0, & \text{in } & Q, \\
(p + F(t) q)(t, \alpha(t)) &= v(t), & \text{in } & Q, \\
(p + q)(t, \beta(t)) &= 0, & \text{in } & (0, \infty), \\
p(0, x) &= \tilde{p}(x), & \text{in } & (0, 1), \\
q(0, x) &= \tilde{q}(x).
\end{aligned}
\]
(6)
where \(v \in L^2_{\text{loc}}(0, \infty)\) stands for \(u'\). Note that if \(F \equiv 1\) then system (6) turns to be (4), and if \(v \equiv 0\), system (6) turns to be (5). Observe that the solutions to the first and the second equations of (6) satisfy
\[
\frac{d}{dt} p(t, c + t) = \frac{d}{dt} q(t, c - t) = 0, \quad t \geq 0, \quad c \in \mathbb{R},
\]
hence, \(p\) (resp. \(q\)) is constant along the characteristic lines \(x - t = c\) (resp. \(x + t = c\)). The idea is to use the boundary conditions
\[
(p + F(t) q)(t, \alpha(t)) = v(t), \quad (p + q)(t, \beta(t)) = 0, \quad t > 0,
\]
and the reflection of the characteristic lines \(x \pm t = c, \ c \in \mathbb{R}\), on the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) to find the unique solution to system (6). Along this work, we assume that the boundary curves satisfy
\[
\alpha, \beta \in C^1(0, \infty), \quad \|\alpha'\|_{L^\infty(\mathbb{R}_+)} \leq 1, \quad \alpha(t) < \beta(t), \quad \forall t > 0.
\]
(9)
The size assumption in (9) guarantees that the characteristic lines \(x = t + c\) (resp. \(x = c - t\)) meet the curve \((t, \alpha(t))_{t \geq 0}\) (resp. \((t, \beta(t))_{t \geq 0}\)) in finite time; also, they serve to ensure that the characteristic lines \(x \pm t = c\) are not gliding on the boundary curves or are not out of \(Q\). In fact, assumption (9) is necessary for the existence of solutions. One can construct a boundary function which does not satisfy (9) for which does not exist any solution. A straightforward consequence of assumption (9) is that the functions \(\alpha^\pm(t) := t \pm \alpha(t)\) and \(\beta^\pm(t) := t \pm \beta(t)\) are invertible from \([0, \infty) \to [0, \infty)\) and \([0, \infty) \to [-1, \infty)\) respectively. In the sequel, we use the standard notations to denote their inverses \((\alpha^\pm)^{-1}\) and \((\beta^\pm)^{-1}\).

Figure 2: An example of a boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) that don’t satisfy assumption (9). The values of the solution are not defined on the green part of the characteristic lines lying under or above these curves.
2 Main results

We start by giving the well-posedness result for system (6).

**Theorem 1.** Let \((p, q, v, F) \in \left[L^2(0, 1)\right]^2 \times L^1_{\text{loc}}(0, \infty) \times C([0, \infty))\). Assume that (9) holds. Then, there exists a unique solution to system (6) satisfying

\[
(p, q) \in C \left(0, t; \left[L^2(\alpha(t), \beta(t))\right]^2\right), \quad t \geq 0.
\]  

(10)

The proof of this theorem is a straightforward consequence of the explicit construction of the unique solution that will be done in section 3.

**Remark 2.** By inverting the transformation given in (3), we obtain

\[
y_t = \frac{p + q}{2}, \quad y_x = \frac{q - p}{2},
\]

hence, for \((y_0, y_1, u, f) \in H^1_1(0, 1) \times L^2(0, 1) \times H^{1}_{\text{loc}}(0, \infty) \times C([0, \infty))\) the solutions to systems (1) and (2) satisfy the regularity

\[
y \in C \left(0, t; H^1_{\beta(\iota)}(t, \beta(t))\right) \cap C^1 \left(0, t; L^2(\alpha(t), \beta(t))\right), \quad t \geq 0.
\]

2.1 Controllability result

**Definition 3.** System (1) is said to be exactly controllable at time \(T > 0\) if for any initial state \((y_0, y_1) \in H^1_1(0, 1) \times L^2(0, 1)\) and for any target state \((h, k) \in H^1_{\beta(T)}(\alpha(T), \beta(T)) \times L^2(\alpha(T), \beta(T))\), there exists \(u \in H^1_{\text{loc}}(0, \infty)\) such that \((y(T), y_1(T)) = (h, k)\).

We have the following result:

**Theorem 4.** Let \((y_0, y_1) \in H^1_1(0, 1) \times L^2(0, 1)\). Assume that the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) satisfy (9). System (1) is exactly controllable at time \(T > 0\) if, and only if \(T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\). Further, if \(T = T^*\), there exists a unique control \(u \in H^1(0, T^*)\) steering the solution \((y, y_1)\) to the equilibrium point \((0, 0)\) given by

\[
u(t) = \begin{cases} 
\int_0^t y_1 (\alpha^+(s)) \, ds + y_0 (\alpha^+(t)), & \text{if } t \in \left(0, (\alpha^+)^{-1}(1)\right), \\
y_0 \left(\beta^- \circ (\beta^-)^{-1} \circ \alpha^+(t)\right) + \int_0^{(\alpha^+)^{-1}(1)} y_1 (\alpha^+(s)) \, ds & \text{if } t \in \left((\alpha^+)^{-1}(1), T^*\right), \\
-\int_{(\alpha^+)^{-1}(1)}^t y_1 (\beta^- \circ (\beta^-)^{-1} \circ \alpha^+(s)) \, ds, & \text{if } t \in \left(T^*, +\infty\right). 
\end{cases}
\]

(11)

**Remark 5.** The controllability result still makes sense even if the boundary curves \((t, \alpha(t))_{t \geq 0}\) and \((t, \beta(t))_{t \geq 0}\) are allowed to intersect in time larger than \(T^*\).

**Remark 6.** Let us consider the particular case \(\alpha(t) = kt, \beta(t) = rt + 1, k, r \in (-1, 1),\) with \(\frac{2(k-r)}{(1-r)\eta(1+k)} < \frac{1}{2}\). (The last assumption guarantees that the boundary curves don’t intersect before \(T^*\)). In this case, it can be checked that \(T^*\) is given by \(T^* = \frac{2}{(1-r)\eta(1+k)}\), which is the same time found in [10]. In particular, if \(\alpha \equiv 0\) and \(\beta \equiv 1\), we obtain the classical result \(T^* = 2\).

**Remark 7.** The minimal time \(T^*\) is precisely the time where a characteristic line with speed 1 emerging from the point \((0, 0)\) and meeting the curve \((t, \beta(t))_{t \geq 0}\) and reflected to meet the curve \((t, \alpha(t))_{t \geq 0}\) at the point \((T^*, \alpha(T^*))\). If the control \(u\) is located on the curve \((t, \beta(t))_{t \geq 0}\) instead of \((t, \alpha(t))_{t \geq 0}\), then \(T^*\) is the time where a characteristic line emerging from the point \((0, 1)\) and meeting the curve \((t, \alpha(t))_{t \geq 0}\) and reflected again to meet the curve \((t, \beta(t))_{t \geq 0}\) at the point \((T^*, \beta(T^*))\). In this case \(T^* = (\beta^-)^{-1} \circ \alpha^+ \circ (\beta^+)^{-1}(1)\).
Figure 3: $T^*$ is the minimal time if the control $u$ is located on the curve $(t, \alpha(t))_{t \geq 0}$ and $T^{**}$ is the minimal time of controllability if the control is located on the curve $(t, \beta(t))_{t \geq 0}$. If these two curves are symmetric with respect to $x = \frac{1}{2}$ we get $T^* = T^{**}$.

**Remark 8.** Consider the particular case $\alpha = 0$ and $\beta(t) = t + 1$. It is easy to see that controllability cannot occur since the emerging characteristic from the point $(0, x)$ for any $x \in (0, 1)$ will never come back, and hence, the control located on the curve $(t, \alpha(t))_{t \geq 0}$ will never be transformed to the component $q$.

### 2.2 Stability result

For the sake of lighting notations, we introduce the function $\phi := \phi(\alpha, \beta)$ defined by

$$\phi := \alpha^- \circ (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}. \quad (12)$$

By assumption (9), the function $\phi : [-1, \infty) \to [\alpha^- \circ (\alpha^+)^{-1}(1), \infty)$ is well defined and increasing function as composition of increasing functions, and hence invertible with inverse

$$\phi^{-1} := \beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha^-)^{-1}.$$ 

Let $(\psi_n)_{n \geq 0}$ be a sequence of functions such that

$$\psi_n : [0, \phi(0)) \to [0, \infty)\quad (13)$$

$$\tau \mapsto \psi_n(\tau) = \prod_{i=0}^{n} \left| F \left( (\alpha^-)^{-1} \circ \phi^{[i]}(\tau) \right) \right|.$$ 

The notation $\phi^{[n]}$ refers to the $n^{th}$ composed of $\phi^{[n]}$ i.e.

$$\phi^{[n]} = \underbrace{\phi \circ \phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}},$$

with the convention $\phi^{[0]} = I$. Then, we have:

**Theorem 9.** Let $(y_0, y_1) \in H^1_1(0, 1) \times L^2(0, 1)$. Assume that the boundary curves $(t, \alpha(t))_{t \geq 0}$ and $(t, \beta(t))_{t \geq 0}$ satisfy (9). In addition, assume that

$$\phi(\tau) < \cdots < \phi^{[n]}(\tau) < \phi^{[n+1]}(\tau) \to \infty, \forall \tau \in [0, \phi(0)), \quad (14)$$

}
then, the solution to (2) \((y(t), y_i(t))\) decays to zero in \(H^1(\beta(t)) \times L^2(\alpha(t), \beta(t))\), if, and only if
\[
\lim_{n \to \infty} \psi_n(\tau) = 0, \quad \forall \tau \in [0, \phi(0)).
\] (15)

If there exists \(g \in C(\mathbb{R}, (0, \infty))\) such that
\[
\psi_n(\tau) \sim \frac{\ln(\psi_n(\tau))}{\phi_n(\tau)}, \quad \forall \tau \in [0, \phi(0)),
\] (16)
then, the solution to (2) decays like \(g(t)\), i.e. there exists a positive constant \(C\) such that
\[
\|(y(t), y_i(t))\|_{H^1(\beta(t)) \times L^2(\alpha(t), \beta(t))} \leq C g(t) \|(y_0, y_1)\|_{H^1(0,1) \times L^2(0,1)}.
\] (17)

In particular, the solution to (2) \((y(t), y_i(t))\) decays exponentially to zero with growth \(\omega > 0\), i.e. there exists \(M \geq 1\) such that
\[
\|(y(t), y_i(t))\|_{H^1(\beta(t)) \times L^2(\alpha(t), \beta(t))} \leq Me^{-\omega t} \|(y_0, y_1)\|_{H^1(0,1) \times L^2(0,1)}, \forall t \geq 0,
\] if, and only if
\[
\sup_{\tau \in [0, \phi(0))] \lim_{n \to \infty} \frac{\ln(\psi_n(\tau))}{\phi_n(\tau)} = -\omega.
\] (18)

If \(f \equiv 1\), the solution to system (2) vanishes in finite time \(T\) if, and only if \(T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\), i.e.
\[
y(T) \equiv y_i(T) \equiv 0, \quad \forall T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0).
\]

Let us illustrate the previous theorem by some examples:

**Example 10** (Cylindrical domain). If \(Q\) is cylindrical domain, i.e. \(\alpha \equiv 0\) and \(\beta \equiv 1\), the function \(\phi\) defined in (12) is given by \(\phi(\tau) = \tau + 2\), therefore, the functions sequence \((\psi_n)_{n \geq 0}\) defined in (13) takes the form
\[
\psi_n : [0, 2) \to [0, \infty),
\]
\[
\tau \mapsto \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)|.
\] (19)

In this case, assumptions of theorem (9) can be checked easily. Note that since system (2) is non-autonomous (\(f\) is time dependent), the decay rate is not necessarily exponential. Below, we illustrate this fact by several examples:

- **Exponential decay:**
  Let \(f(t) = \frac{2 - \sin(\pi \tau)}{2 + \sin(\pi \tau)}\), therefore, \(F(t) = \frac{\sin(\pi \tau)}{2}\), thus,
  \[
  \psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \left(\frac{\sin(\pi \tau)}{2}\right)^{n+1}.
  \]

  By (18), we have
  \[
  \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \to \infty} \frac{\ln(\psi_n(\tau))}{\phi_n(\tau)} = \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \to \infty} \frac{(n+1) \ln \left(\frac{\sin(\pi \tau)}{2}\right)}{\tau + 2n} = \sup_{\tau \in (0,1) \cup (1,2)} \frac{1}{2} \ln \left\{ \frac{\sin(\pi \tau)}{2} \right\} = -\ln 2.
  \]
  therefore, exponential decay occurs with growth bound \(\omega = \frac{\ln 2}{2}\).
• **Polynomial decay:**

Let \( f(t) = \frac{t^{-(t+1)-(t+3)^{-s}}}{(t+1)^{-(t+3)^{-s}}} \), \( s > 0 \), then \( F(t) = \left( \frac{t+1}{t+3} \right)^{-s} \), consequently, the functions sequence \( (\psi_n)_{n \geq 0} \) defined in (19) takes the form

\[
\psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \prod_{i=0}^{n} \left| \frac{\tau + 2i + 3}{\tau + 2i + 1} \right|^{-s} = \left( \frac{\tau + 2n + 3}{\tau + 1} \right)^{-s}.
\]

Set \( g(t) = (t+1)^{-s}, s > 0 \). A simple computation shows that

\[
\lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\phi[n])} = \lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\tau + 2n)} = \frac{1}{\tau + 1}, \quad \tau \in [0, 2),
\]

thus, by (17), the solution to system (2) decays like \((t + 1)^{-s}, s > 0\).

• **Logarithmic decay:**

Let \( f(t) = \frac{\log^-(t+1) - \log^-(t+3)}{\log^-(t+1)+\log^-(t+3)}, s > 0 \), then \( F(t) = \left( \frac{\log(t+3)}{\log(t+1)} \right)^{-s} \), consequently, we obtain

\[
\psi_n(\tau) = \prod_{i=0}^{n} |F(\tau + 2i)| = \prod_{i=0}^{n} \left| \frac{\log(\tau + 2i + 3)}{\log(\tau + 2i + 1)} \right|^{-s} = \left( \frac{\log(\tau + 2n + 3)}{\log(\tau + 1) \tau + 1} \right)^{-s}.
\]

By letting \( g(t) = \log^{-s}(t + 1), s > 0 \), we get

\[
\lim_{n \to \infty} \frac{\psi_n(\tau)}{g(\phi[n])} = \log^s(\tau + 1) \lim_{n \to \infty} \left| \frac{\log(\tau + 2n + 3)}{\log(\tau + 2n + 1)} \right|^{-s} = \log^s(\tau + 1), \quad \tau \in [0, 2),
\]

hence, (17) is satisfied with \( g(t) = \log^{-s}(t + 1), s > 0 \).

• **Stability in finite time:**

Let \( f(t) = \frac{t}{t+1} \), therefore, \( F(t) = \frac{1}{t+1} \), consequently, we obtain

\[
\psi_n(\tau) = \prod_{i=0}^{n} \frac{1}{\tau + 2i + 1} = \frac{1}{2^{n+1}} \prod_{i=0}^{n} \left( \frac{\tau + 1}{2} + i \right) = \frac{1}{2^{n+1}} \left[ \prod_{i=0}^{n} \left( \frac{\tau + 1}{2} + i \right) \right]^{-1} = \frac{1}{2^{n+1}} \left[ \left( \frac{\tau + 1}{2} \right)_{n+1} \right]^{-1},
\]

where \((\frac{\tau + 1}{2})_{n+1}\) is the Pochhammer symbol which has as asymptotics

\[
\left( \frac{\tau + 1}{2} \right)_{n+1} = \frac{2\pi}{\Gamma(\frac{\tau + 1}{2})} e^{-n-1}(n+1)^{n+1+\frac{\pi}{2}} \left( 1 + O \left( \frac{1}{n+1} \right) \right).
\]

So, we get

\[
\psi_n(\tau) \sim_{n \to \infty} C n^{n+1} (n+1)^{-n-1-\frac{\pi}{2}},
\]

which by (15) implies that the solution to system (2) decays to zero. To prove stability in finite time it suffices to show that the growth bound defined in (18) is infinite. By using (20) we obtain

\[
\lim_{n \to \infty} \frac{\ln \psi_n(\tau)}{\phi[n](\tau)} = \lim_{n \to \infty} \frac{n + 1 - (n + 1 + \frac{\pi}{2}) \ln n}{2n + \tau} = -\infty = -\omega.
\]

This phenomena is due to the fact that \( f(t) \to 1 \) which by theorem (9) leads to stability in finite time.
Example 11 (Non cylindrical domain). Things are more delicate in the non-cylindrical case. Consider a boundary functions of the form \( \alpha(t) = rt, \beta(t) = kt + 1, r, k \in (-1,1). \) To guarantee that \( \alpha(t) \neq \beta(t), \forall t \geq 0, \) we assume that \( k \geq r. \) The function \( \phi \) defined in (12) will be given by

\[
\phi(\tau) = \frac{(1+k)(1-r)}{(1-k)(1+r)} \tau + \frac{2(1-r)}{(1-k)(1+r)} = a\tau + b,
\]

therefore, we obtain

\[
\phi^{[n]}(\tau) = \begin{cases} 
  a^n \left( \tau - \frac{b}{1-a} \right) + \frac{b}{1-a}, & \text{if } r < k, \\
  \tau + \frac{2n}{1+r}, & \text{if } r = k.
\end{cases}
\]

Consequently,

\[
(\alpha^-)^{-1} \circ \phi^{[n]}(\tau) = \begin{cases} 
  a^n \left( \tau - \frac{b}{1-a(1-r)} \right) + \frac{b}{1-a(1-r)}, & \text{if } r < k, \\
  \tau + \frac{2n}{1+r(1-r)}, & \text{if } r = k.
\end{cases}
\] (21)

For simplicity, let us take \( f \) as in the previous example, \( f(t) = \frac{1}{2+t} \) which implies that \( F(t) = \frac{1}{t+1}. \) So, we have:

- If \( r < k : \)

  From (21), we can check that (14) is satisfied if, and only if \( a > 1. \) To verify (15), it is enough to estimate its asymptotics. So, we have

\[
\psi_n(\tau) = \prod_{i=0}^{n} \left| a^i \left( \tau - \frac{b}{1-a(1-r)} \right) + \frac{b}{1-a(1-r)} \right| = \prod_{i=0}^{n} \frac{1}{a^i s(\tau) + z}.
\]

Since \( a > 1, \) the series \( \sum_{i=0}^{\infty} \ln \left( 1 + \frac{z}{a^i s(\tau)} \right) \) converges, so,

\[
\psi_n(\tau) \sim C a^{-\frac{n+1}{2}} s^{-n-1}(\tau), \quad \forall \tau \in [0,b). \quad (22)
\]

In view of (22), assumption (15) is satisfied if \( a > 1, \) namely,

\[
\frac{(1+k)(1-r)}{(1-k)(1+r)} > 1, \quad (23)
\]

i.e. the solution to system (2) decays to zero if (23) is satisfied. On the contrary of the cylindrical domain case, even with this choice of \( f, \) (18) is not satisfied, and hence, exponential stability cannot occur. Indeed, we have

\[
\lim_{n \to \infty} \ln \psi_n(\tau) = - \lim_{n \to \infty} \frac{n^2}{2a^n} \ln |a| = 0, \quad \forall \tau \in [0,b).
\]

We still be able to get an idea about the decay rate. From (22), we observe that the term that really matters is \( a^{-\frac{n+1}{2}}, \) so, for \( g(t) = e^{-\frac{1}{2} \ln^2(t)} \), we obtain

\[
a^{-\frac{n}{2}} \sim C g(a^n s(\tau) + z), \quad \forall \tau \in [0,b).
\]

Note that we didn’t lose too much since \( g \) decays to zero faster than any polynomial function. This loss can be justified by the fact that the characteristic lines will need a larger time to reflect on the two endpoints when \( t \) becomes larger.
If \( k = r \):

In this case, the lines \( x = rt \) and \( x = kt + 1 \) are parallel, therefore, the characteristic speeds are the same for all time, so we might expect stability in finite time with this choice of \( f \). Let us first check that the solution to system (2) decays exponentially. By (20), the functions sequence \( \left( \psi_n \right)_{n \geq 1} \) behaves like

\[
\psi_n(\tau) = \prod_{i=0}^{n} \frac{1}{\tau - r + 1 + \frac{2i}{(1+r)(1-r)}}
= \frac{(1+r)^{n+1}(1-r)^{n+1}}{2^{n+1} n!} \prod_{i=0}^{n} \left[ \frac{(\tau + 1 - r)(1 + r)}{2} + i \right]^{-1}
= \frac{(1+r)^{n+1}(1-r)^{n+1}}{2^{n+1} n!} \left[ \frac{(\tau + 1 - r)(1 + r)}{2} \right]^{-n+1}
\sim C \left( 1 + \frac{(\tau + 1 - r)(1 + r)}{2} \right)^{-n+1}
\times e^{n+1} (n+1)^{\frac{(\tau + 1 - r)(1 + r)}{2}-n-\frac{1}{2}}.
\]

By using (18), we get

\[
\lim_{n \to \infty} \ln \psi_n(\tau) = \lim_{n \to \infty} \ln \frac{(n+1) \ln \left( \frac{\tau - r}{f} + 1 \right)}{2(n+1) \ln \left( \frac{\tau - r}{f} + 1 \right) + n + \frac{1}{2}} \ln (n + 1)
= -\infty = -\omega,
\]

therefore, the solution will vanish in finite time. This is due to the fact that \( f(t) \to 1 \).

**Example 12** (Constant feedback). Consider the case when \( f \) is a constant such that \( f \neq 1 \) with keeping \( \alpha \) and \( \beta \) as in the previous example. A simple computation yields

\[
\psi_n = F^{n+1} = \left| \frac{f - 1}{f + 1} \right|^{n+1},
\]

therefore, by using the formula in (18), we arrive at:

- If \( r < k \):

  We can check that the decay is not exponential. Indeed,

  \[
  \lim_{n \to \infty} \ln \psi_n(\tau) = \lim_{n \to \infty} \ln \left( \frac{n+1}{{\alpha}^n} \right) = 0, \quad \forall \tau \in [0,b).
  \]

  Nonetheless, by (17), we can determine the decay rate for a particular values of \( f \). Let \( g(t) = a^{-st} \).

  It is easy to check that if \( a^s = \frac{f-1}{f+1} \) for some \( s > 0 \) then

  \[
  \lim_{n \to \infty} \psi_n(\tau) = a^s \left| \frac{f - 1}{f + 1} \right|^{n+1} = 1, \quad \forall \tau \in [0,b),
  \]

  and hence, the solution decays like \( t^{-s}, s > 0 \).

- If \( r = k \):

  In this case, we have

  \[
  \lim_{n \to \infty} \ln \psi_n(\tau) = \lim_{n \to \infty} \ln \left( \frac{n+1}{{\alpha}^n} \right) = (\frac{1+r}{2}) \ln \left( \frac{f - 1}{f + 1} \right) = -\omega,
  \]
hence, exponential decay occurs with growth bound \( \omega \). In particular, if \( Q \) is a cylindrical domain \( (r = 0) \), system (2) is exponentially stable if, and only if

\[
\frac{1}{2} \ln \left| \frac{f - 1}{f + 1} \right| = -\omega < 0,
\]

which is a known result from [19].

**Remark 13.** By setting

\[
F(t) = \frac{g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t))}, \quad \forall t \geq 0,
\]

with \( g(t) \neq 0 \), for all \( t \geq 0 \), we obtain

\[
\psi_{n}(\tau) = \prod_{i=0}^{n} \left| F \left( (\alpha^{-})^{-1} \circ \phi[i](\tau) \right) \right| = \prod_{i=0}^{n} \left| \frac{g(\phi[i+1](\tau))}{g(\phi[i](\tau))} \right| = \left| \frac{g(\phi[n+1](\tau))}{g(\phi[0](\tau))} \right|.
\]

In this case, (16) is automatically satisfied, and since \( F = \frac{1-f}{1+f} \), we obtain

\[
\frac{g(\alpha^{-}(t)) - g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t)) + g(\phi \circ \alpha^{-}(t))} = f_{g}(t), \quad \forall t \geq 0.
\]

The last expression provides an explicit relation between the decay rate and the feedback function \( f \). This means that \( f \) can be determined based on the desired decay rate. Formula (24) has been used to construct \( f \) in the second and the third points in example (10).

**Remark 14.** Examples (11) and (12) illustrate the big influence of the boundary curves nature on the decay rate of the solution to system (2).

**Remark 15.** Observe that the time of extinction of the solution to system (2) for \( f \equiv 1 \) is the time of exact controllability in theorem (4). This can be explained by the fact that exponential stability implies exact controllability for time reversible systems (see for instance [7, Remark 1.5] or [9]). Even though our system is not time reversible (because of the boundary functions), we have seen that this implication remains true.

### 3 Construction of the exact solution

The aim now is to find the solution \((p, q)\) to system (6) in all \( Q \). To this end, let us start by splitting \( Q \) into infinite parts. Namely

\[
Q = \bigcup_{n \geq 0} \Sigma_{0}^{p} = \bigcup_{n \geq 0} \Sigma_{n}^{p}, \quad \Sigma_{i}^{p} \cap \Sigma_{j}^{p}, \Sigma_{i}^{q} \cap \Sigma_{j}^{q} = \emptyset, \ i \neq j,
\]

where \( \Sigma_{0}^{p}, \Sigma_{n}^{p} \) are given for \( n = 0, 1 \), by

\[
\Sigma_{0}^{p} = \{ (t, x) \in Q, \ t \in [0, x) \}, \quad (25)
\]

\[
\Sigma_{1}^{p} = \{ (t, x) \in Q, \ t - x \in [0, \alpha^{-} \circ (\alpha^{+})^{-1} (1)) \}, \quad (26)
\]

\[
\Sigma_{0}^{q} = \{ (t, x) \in Q, \ t \in [0, 1 - x) \}, \quad (27)
\]

\[
\Sigma_{1}^{q} = \{ (t, x) \in Q, \ t + x \in [1, \beta^{+} \circ (\beta^{-})^{-1} (0)) \}, \quad (28)
\]

and for all \( n \geq 1 \)

\[
\Sigma_{2n}^{p} = \{ (t, x) \in Q, \ t - x \in [\phi[n-1] \circ \alpha^{-} \circ (\alpha^{+})^{-1} (1), \phi[n](0)) \}, \quad (29)
\]

\[
\Sigma_{2n+1}^{p} = \{ (t, x) \in Q, \ t - x \in [\phi[n](0), \phi[n] \circ \alpha^{-} \circ (\alpha^{+})^{-1} (1)) \}, \quad (30)
\]
\[ \Sigma^q_{2n} = \left\{ (t,x) \in Q, \ t + x \in \left[ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1} (0), \xi^{[n]}(1) \right] \right\}, \] (31)

\[ \Sigma^q_{2n+1} = \left\{ (t,x) \in Q, \ t + x \in \left[ \xi^{[n]}(1), \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1} (0) \right] \right\}, \] (32)

where \( \xi \) is defined by
\[ \xi := \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}. \] (33)

The construction of these regions relies on the reflection of the principal characteristic lines of positive and negative slopes emerging from the points \((0,0)\) and \((0,1)\) and reflected along the boundary curves. More precisely, the lines \(x = t\) and \(x = -t + 1\) starting respectively from \((0,0)\) and \((0,1)\) meet the curves \((t,\beta(t))_{t \geq 0}\) and \((t,\alpha(t))_{t \geq 0}\) in the points \((\beta^-(0))^{-1},\beta((\beta^-)^{-1}(0))\) and \((\alpha^+)^{-1}(1),\alpha((\alpha^+)^{-1}(1))\) respectively. The regions \( \Sigma^p_0 \) and \( \Sigma^q_0 \) are those located between \( t = 0 \) and these lines. We can do similarly to construct the regions \( \Sigma^p_n, \Sigma^q_n, n \geq 1 \), given above. In the sequel, we denote by \( p_n \) and \( q_n \) the restriction of \( p \) and \( q \) solutions of (6) on \( \Sigma^p_n \) and \( \Sigma^q_n, n \geq 0 \).

**Figure 4:** The regions \( \Sigma^p_i \) are those between the red line and \( \Sigma^q_i \) are those between the blue lines.

**Remark 16.** In particular, if \( \alpha \equiv 0 \) and \( \beta \equiv 1 \), the regions \( \Sigma^p_n, \Sigma^q_n, n \geq 0 \), are simply given by
\[ \Sigma^p_n = \left\{ (t,x) \in \mathbb{R}_+ \times [0,1], \ t - x \in [n-1,n) \right\}, \]
\[ \Sigma^q_n = \left\{ (t,x) \in \mathbb{R}_+ \times [0,1], \ x + t \in [n,n+1) \right\}. \]

During the construction below, we use the standard density argument by assuming first that the initial states are sufficiently regular then passing to the limit. So, the constructed solutions must be understood in the weak sense. Let us start by finding \( p_0 \) and \( q_0 \):

**Lemma 17.** Let \( (\bar{p}, \bar{q}) \in [L^2(0,1)]^2 \). The solution \( (p_0, q_0) \) to system (6) is given by
\[ p_0(t,x) = \bar{p}(x-t), \quad q_0(t,x) = \bar{q}(x+t). \] (34)

**Proof.** The proof readily follows from (7). \( \blacksquare \)

Now, let us find the solution in the regions \( \Sigma^p_1, \Sigma^q_1 \):
Lemma 20. Let \((\tilde{p}, \tilde{q}) \in [L^2(0,1)]^2\). The solution \((p_1, q_1)\) to system (6) is given by
\[
p_1(t, x) = v \left( (\alpha^-)^{-1} (t - x) \right) - F \left( (\alpha^-)^{-1} (t - x) \right) \tilde{q} \left( \alpha^+ \circ (\alpha^-)^{-1} (t - x) \right),
\]
\[
q_1(t, x) = -\tilde{p} \left( -\beta^- \circ (\beta^+)^{-1} (x + t) \right).
\]

Proof. By using (34), we have at the boundary curves
\[
p_0(\tau, \beta(\tau)) = \tilde{p} \left( -\beta^- (\tau) \right), \quad \tau \in \left[ 0, (\beta^-)^{-1} (0) \right),
\]
\[
q_0(\chi, \alpha(\chi)) = \tilde{q} \left( \alpha^+ (\chi) \right), \quad \chi \in \left[ 0, (\alpha^+)^{-1} (1) \right).
\]

By using the boundary conditions given in (8), we get
\[
p_1(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_0(\tau, \alpha(\tau)) \quad (37)
\]
\[
= v(\tau) - F(\tau)\tilde{q}(\alpha^+ (\tau)), \quad \tau \in \left[ 0, (\alpha^+)^{-1} (1) \right),
\]
\[
qu_1(\chi, \beta(\chi)) = -p_0(\chi, \beta(\chi)) = -\tilde{p}(-\beta^- (\chi)), \quad \chi \in \left[ 0, (\beta^-)^{-1} (0) \right). (38)
\]

Consider the latter values as initial states on both regions \(\Sigma^p, \Sigma^q\) and use (7), we write
\[
p_1(t, c - t) = p_1(\tau, \tau - s), \quad q_1(t, c + \chi) = q_1(\chi, c + \chi). (39)
\]

By using the fact that \(p\) and \(q\) are constant along the characteristic lines \(x = t - \alpha^- (\tau)\) and \(x = -t + \beta^+ (\chi)\) respectively, we obtain
\[
p_1(t, t - \alpha^- (\tau)) = p_1(\tau, \alpha(\tau)) = v(\tau) - F(\tau)\tilde{q}(\alpha^+ (\tau)), (40)
\]
and
\[
qu_1(t, -t + \beta^+ (\chi)) = q_1(\chi, \beta(\chi)) = -\tilde{p}(-\beta^- (\chi)). (41)
\]

Now, letting \((\alpha^-)^{-1} (t - x) = \tau\) in (40) and \(\chi = (\beta^+)^{-1} (x + t)\) in (41) yields the desired result. \(\square\)

Remark 19. Note that \(\alpha^+ \circ (\alpha^-)^{-1} (t - x), (t, x) \in \Sigma^p\) and \(-\beta^- \circ (\beta^+)^{-1} (x + t), (t, x) \in \Sigma^q\) belong to \((0, 1)\) and the above expressions make perfectly sense. To clarify more things, let \((t, x) \in \Sigma^p\) and let \(\tilde{x}(s) = s - t + x\) the line passing through the point \((t, x)\). By moving backwards, this line meets the curve \((s, \alpha(s))_{s \geq 0}\) at the point \((\alpha^-)^{-1} (t - x), \alpha(\alpha^-)^{-1} (t - x))\) where \((\alpha^-)^{-1} (t - x) \in \left[ 0, (\alpha^-)^{-1} (1) \right)\). We use again the reflection of the characteristic line of negative slope passing through the latter point. i.e. \(\tilde{x}(s) = -s + \alpha^+ \circ (\alpha^-)^{-1} (t - x)\) lying in \(\Sigma^p\), for \(s = 0\), we obtain \(\tilde{x}(0) = \alpha^+ \circ (\alpha^-)^{-1} (t - x) \in (0, 1)\). We can do similarly for \(-\beta^- \circ (\beta^+)^{-1} (x + t), (t, x) \in \Sigma^q\).

Lemma 20. Let \((\tilde{p}, \tilde{q}) \in [L^2(0,1)]^2\). The solution \((p_2, q_2)\) to system (6) is given by
\[
p_2(t, x) = v \left( (\alpha^-)^{-1} (t - x) \right) + F \left( (\alpha^-)^{-1} (t - x) \right) \tilde{p} \left( -\phi^{-1}(t - x) \right), (42)
\]
\[
q_2(t, x) = -v \left( (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1} (x + t) \right) + F \left( (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1} (x + t) \right) \tilde{q} \left( \xi^{-1}(x + t) \right), (43)
\]

where \(\phi\) and \(\xi\) are defined in (12) and (33).
Proof. From (35) and (36), we have at the boundary curves
\[
p_1(\tau, \beta(\tau)) = v \left( (\alpha)^{-1} \circ \beta^-(\tau) \right) - F((\alpha)^{-1} \circ \beta^-)(\tau) \tilde{q} \left( \alpha^+ \circ (\alpha)^{-1} \circ \beta^-(\tau) \right),
\]
\[
\tau \in \left[ (\beta)^{-1}(0), (\beta)^{-1} \circ \alpha^- \circ (\alpha)^{-1}(1) \right]
\]
and
\[
q_1(\chi, \alpha(\chi)) = -\tilde{p} \left( -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\chi) \right),
\]
\[
\chi \in \left[ (\alpha)^{-1}(1), (\alpha)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0) \right].
\]
In order to find \(p_2\) and \(q_2\), we use the boundary conditions (8) and the values of \(p_1\) and \(q_1\) at the boundary curves given in (44) and (45) as initial states. Namely, for any \(\tau \in \left[ (\beta^-)^{-1}(0), (\beta^-)^{-1} \circ \alpha^- \circ (\alpha)^{-1}(1) \right]\) and \(\chi \in \left[ (\alpha)^{-1}(1), (\alpha)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0) \right]\), we have along the lines \(x = t - \alpha^- (\tau)\) and \(x = -t + \beta^+ (\chi)\) respectively
\[
p_2(t, t - \alpha^- (\tau)) = p_2(\tau, \alpha(\tau)) = v(\tau) - F(\tau) q_1(\tau, \alpha(\tau)),
\]
\[
q_2(t, \beta^+(\chi) - t) = q_2(\chi, \beta(\chi)) = -p_1(\chi, \beta(\chi)).
\]
Plugging (44) and (45) in (46) and (47), we get
\[
p_2(t, t - \alpha^- (\tau)) = v(\tau) + F(\tau) \tilde{p} \left( -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\chi) \right),
\]
and
\[
q_2(t, \beta^+(\chi) - t) = -v \left( (\alpha)^{-1} \circ \beta^- (\chi) \right) + F \left( (\alpha)^{-1} \circ \beta^- (\chi) \right) \tilde{q} \left( \alpha^+ \circ (\alpha)^{-1} \circ \beta^- (\chi) \right).
\]
The proof follows immediately for \(\tau = (\alpha)^{-1}(t - x)\) and \((\beta^+)^{-1}(x + t) = \chi\).

\[\square\]

Remark 21. In the same spirit of remark (19), the expressions in (42) and (43) make perfectly sense. We can use the same reasoning to show that
\[
-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha)^{-1}(t - x) \in (0, 1), \forall (t, x) \in \Sigma^p_2,
\]
\[
\alpha^+ \circ (\alpha)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + t) \in (0, 1), \forall (t, x) \in \Sigma^q_2.
\]
More generally, we have:

Lemma 22. Let \((\tilde{p}, \tilde{q}) \in [L^2(0, 1)]^2\). The solutions \(p_{2n+1}, p_{2n+2}, q_{2n+1}, q_{2n+2}, n \geq 1\), to system (6) are given by
\[
p_{2n+1}(t, x)
= \sum_{k=0}^{n} v \left( (\alpha)^{-1} \circ (\phi^{-1})^k(t - x) \right) \prod_{i=0}^{k-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^i(t - x) \right) - \tilde{q} \left( (\xi^{-1})^n \circ \alpha^+ \circ (\alpha)^{-1}(t - x) \right) \prod_{k=0}^{n} F \left( (\alpha)^{-1} \circ (\phi^{-1})^k(t - x) \right),
\]
Proof. The above expressions can be proved by induction. Let us start by proving (with the convention \( \beta(0) = 0 \)), \( q_{2n+1}(t, x) \)

\[
q_{2n+1}(t, x) = -\sum_{k=0}^{n-1} v \left( (\alpha)^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} (x + t) \right) \times \\
\quad \sum_{i=0}^{k-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} (x + t) \right) \\
- p \left( - (\phi^{-1})^n \circ \beta^{-1} (x + t) \right) \times \\
\quad \sum_{k=0}^{n-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^{k} \circ \beta^{-1} (x + t) \right),
\]

\[
q_{2n+2}(t, x) = -\sum_{k=0}^{n} v \left( (\alpha)^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} (x + t) \right) \times \\
\quad \sum_{i=0}^{k-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} (x + t) \right) \\
+ \tilde{q} \left( (\xi^{-1})^n \circ \alpha^+ \circ (\alpha)^{-1} \circ \beta^{-1} (x + t) \right) \times \\
\quad \sum_{k=0}^{n} F \left( (\alpha)^{-1} \circ (\phi^{-1})^{k} \circ \beta^{-1} (x + t) \right).
\]

with the convention \( \sum_{k=0}^{-1} = 1 \). The functions \( \phi \) and \( \xi \) are defined in (12) and (33).

\[
p_{2n+2}(t, x)
\]

\[
= \sum_{k=0}^{n} v \left( (\alpha)^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} (t - x) \right) \prod_{i=0}^{k-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} (t - x) \right)
\]

\[
+ \tilde{p} \left( - (\phi^{-1})^n \circ \beta^{-1} (t - x) \right) \prod_{k=0}^{n} F \left( (\alpha)^{-1} \circ (\phi^{-1})^{k} \circ \beta^{-1} (t - x) \right),
\]

\[
p_{2n+1}(t, \beta(t))
\]

\[
= \sum_{k=0}^{n} v \left( (\alpha)^{-1} \circ (\phi^{-1})^k \circ \beta^{-1} (t) \right) \prod_{i=0}^{k-1} F \left( (\alpha)^{-1} \circ (\phi^{-1})^i \circ \beta^{-1} (t) \right)
\]

\[
- \tilde{q} \left( (\xi^{-1})^n \circ \alpha^+ \circ (\alpha)^{-1} \circ \beta^{-1} (t) \right) \prod_{k=0}^{n} F \left( (\alpha)^{-1} \circ (\phi^{-1})^{k} \circ \beta^{-1} (t) \right).
\]

Now, we use the boundary condition given in (8), i.e.

\[
q_{2n+2}(\chi, \beta(\chi)) = -p_{2n+1}(\chi, \beta(\chi)), \quad \chi \in \left\{ (\beta^{-1})^{-1} \circ \phi^{[n]}(0), \ (\beta^{-1})^{-1} \circ \phi^{[n]} \circ \alpha^{-1} \circ (\alpha^+)^{-1}(1) \right\},
\]

\[
15
\]
we find
\begin{equation}
q_{2n+2}(\chi, \beta(\chi)) = - \sum_{k=0}^{n} v \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^{-1}(\chi) \right) \prod_{i=0}^{k-1} F \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^{-1}(t) \right) \\
+ \overline{q} \left( (\xi^{-1})^{[n]} \circ \alpha^+ \circ (\alpha^{-1})^{-1} \circ \beta^{-1}(\chi) \right) \prod_{k=0}^{n} F \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^{-1}(\chi) \right).
\end{equation}

Since \( q \) is constant along the characteristic lines of the form \( x = c - t \), in particular, on the line \( x = \beta^+(\chi) - t \), we have
\[
q_{2n+2}(t, \beta^+(\chi) - t) = q_{2n+2}(\chi, \beta(\chi)).
\]

Finally, by letting \( \chi = (\beta^+)^{-1}(x + t) \) in (52), we obtain the formula in (51). Let us do similarly for \( p_{2n+2} \). By taking (50) for \( x = \alpha(t) \), we obtain
\begin{equation}
p_{2n+1}(t, \alpha(t)) = - \sum_{k=0}^{n-1} v \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \times \\
\prod_{i=0}^{k-1} F \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \\
\prod_{k=0}^{n-1} F \left( (\alpha^{-1})^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^{-1} \circ (\beta^+)^{-1} \circ \alpha^+(t) \right).
\end{equation}

Using the boundary condition
\[
p_{2n+2}(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)), \quad \tau \in \left[ (\alpha^+)^{-1} \circ \xi^{[n]}(1), (\alpha^+)^{-1} \circ \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) \right],
\]
and the fact that \( q \) is constant along the characteristic lines \( x = c - t \), in particular, on the line \( x = t - \alpha^-(\tau) \), we obtain
\begin{equation}
p_{2n+2}(\tau, t - \alpha^-(\tau)) = v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)).
\end{equation}

By letting \( \tau = (\alpha^-)^{-1}(t - x) \) in (53) and plugging the result in (54) then using the definition of \( \phi \) given in (12), we get
\[
p_{2n+2}(t, x) = v((\alpha^-)^{-1}(t - x)) + \sum_{k=0}^{n-1} v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k+1]}(t - x) \right) \times \\
\prod_{i=0}^{k-1} F((\alpha^-)^{-1}(t - x))F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i+1]}(t - x) \right) \\
+ F((\alpha^-)^{-1}(t - x))\overline{p} \left( (\phi^{-1})^{[n+1]}(t - x) \right) \times \\
\prod_{k=0}^{n-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k+1]}(t - x) \right).
\]

After some manipulation we obtain the formula in (49).
Remark 23. From what precede, we can see that the solution \((p,q)\) satisfies the regularity given in (10).

Remark 24. More generally, if \((\tilde{p},\tilde{q},v,F)\in \left[L^6(0,1)\right]^2 \times L^6_{loc}(0,\infty) \times L^6(0,\infty)\), we can see from (48)-(51) that the solution \((p,q)\) to problem (6) satisfies the regularity

\[(p,q) \in C(0,t;[L^r(\alpha(t),\beta(t))]^2),\ t \geq 0,\]

with \(\frac{1}{\theta} + \frac{1}{\eta} = \frac{1}{r}\), \(\theta, \eta \in [1, \infty)\).

4 Proof of main results

4.1 Proof of the controllability theorem

Let \(F \equiv 1\) in (35),(36),(42) and (43). The solution \(p_1\) sees the control immediately for \(t \geq 0\), on the contrary, the component \(q_1\) has to wait one more reflection on the curve \((t,\alpha(t))_{t \geq 0}\) to see it as soon as \(t \geq (\beta^-)^{-1}(0)\). Let us start by proving the necessary part:

Proposition 25. If \(T < T^* = (\alpha^+)^{-1} \circ (\beta^+)^{-1} \circ (\beta^-)^{-1}(0)\), then system (4) is not exactly controllable at time \(T\).

Proof. To prove this lemma, we make use of the expressions of the exact solution given in (36) and (43). Let \(T^*_\varepsilon = T^* - \varepsilon\) for sufficiently small \(\varepsilon > 0\); the solution \(q\) at this time is given by

\[q(T^*_\varepsilon, x) = \begin{cases} q_1^+(T^*_\varepsilon, x) & \text{if } x \in \left[\alpha(T^*_\varepsilon), T^*_\varepsilon - \beta^+ \circ (\beta^-)^{-1}(0)\right], \\ q_2^+(T^*_\varepsilon, x) & \text{if } x \in \left[T^*_\varepsilon - \beta^+ \circ (\beta^-)^{-1}(0), \beta(T^*_\varepsilon)\right). \end{cases} \]

Thus, system (1) will be never exactly controllable since we have for any initial state \(\tilde{p}\) and any target state \(k\)

\[q(T^*_\varepsilon, x) = -\tilde{p}\left(-\beta^- \circ (\beta^+)^{-1}(x + T^*_\varepsilon)\right) = k(x), \ x \in \left[\alpha(T^*_\varepsilon), T^*_\varepsilon - \beta^+ \circ (\beta^-)^{-1}(0)\right], \]

which is clearly a violating of the initial states. \(\Box\)

Now, we prove the sufficient part:

Proposition 26. If \(T \geq T^* = (\alpha^+)^{-1} \circ (\beta^+)^{-1} \circ (\beta^-)^{-1}(0)\), then system (6) is exactly controllable at time \(T\).
Proof. It suffices to prove it for $T = T^*$. Let $(h,k) \in L^2(\alpha(T^*),\beta(T^*))$ be a target state and let $T^{**} = (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}$ (1). We have three possible configurations:

**Case 1: $T^{**} = T^*$**

In this case, we have $p(T^*) = p_2(T^*)$ and $q(T^*) = q_2(T^*)$, then by making use of (42) and (43) we obtain

$$h(x) = p_2(T^*,x) = v \left( (\alpha^-)^{-1} (T^* - x) \right) + \tilde{p} \left( -\phi^{-1}(T^* - x) \right), \quad x \in (\alpha(T^*),\beta(T^*))$$

$$k(x) = -v \left( (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1} (x + T^*) \right) + \tilde{q} \left( \xi^{-1} (x + T^*) \right), \quad x \in (\alpha(T^*),\beta(T^*))$$

Therefore, the control $v(t)$ is given by

$$v(t) = \begin{cases} 
  h(T^* - \alpha^-(t)) - \tilde{p} \left( -\phi^{-1} \circ \alpha^-(t) \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), (\alpha^+)^{-1} \right) \\
  \tilde{q} \left( \alpha^+(t) \right) - k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^* \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^-(T^*) \right) 
\end{cases}$$

**Case 2: $T^{**} < T^*$**

In this case, $p(T^*)$ and $q(T^*)$ are defined by

$$p(T^*,x) = \begin{cases} 
  p_1(T^*,x), & \text{if } x \in \left( T^* - \alpha^- \circ (\alpha^+)^{-1} (1), \beta(T^*) \right) \\
  p_2(T^*,x), & \text{if } x \in \left( \alpha(T^*), T^* - \alpha^- \circ (\alpha^+)^{-1} (1) \right) 
\end{cases}$$

and $q(T^*) = q_2(T^*)$. Thus, by making use of (35), (42) and (43), then making some variabl substitutions, we arrive at

$$v(t) = \begin{cases} 
  h_1(T^* - \alpha^-(t)) + \tilde{q} \left( \alpha^+(t) \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), (\alpha^+)^{-1} \right) \\
  h_2(T^* - \alpha^-(t)) - \tilde{p} \left( -\phi^{-1} \circ \alpha^-(t) \right), & \text{if } t \in \left( (\alpha^+)^{-1} (1), T^* \right) \\
  \tilde{q} \left( \alpha^+(t) \right) - k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^* \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^-(T^*) \right) 
\end{cases}$$

where $h_1$ and $h_2$ are the restrictions of the target state $h$ on the regions $\Sigma_1^p$ and $\Sigma_2^p$ respectively.

**Case 3: $T^{**} > T^*$**

In this case, we have $p(T^*) = p_2(T^*)$, and $q(T^*)$ is defined by

$$q(T^*,x) = \begin{cases} 
  q_2(T^*,x), & \text{if } x \in (\alpha(T^*), \xi(1) - T^*) \\
  q_3(T^*,x), & \text{if } x \in (\xi(1) - T^*, \beta(T^*)) \end{cases}$$

By using (42), (43) and (50) for $n = 1$ and $t = T^*$, then making some variabl substitutions, we obtain

$$v(t) = \begin{cases} 
  h(T^* - \alpha^-(t)) - \tilde{p} \left( -\phi^{-1} \circ \alpha^-(t) \right), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), T^* \right) \\
  \tilde{q} \left( \alpha^+(t) \right) - k_2(\beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^*), & \text{if } t \in \left( 0, (\alpha^+)^{-1} (1) \right) \\
  -\tilde{p} \left( -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) - k_3(\beta^+ \circ (\beta^-)^{-1} \circ \alpha^- (t) - T^*), & \text{if } t \in \left( (\alpha^+)^{-1} (1), (\alpha^-)^{-1} \circ \beta^-(T^*) \right) 
\end{cases}$$

where $k_2$ and $k_3$ are the restrictions of the target state $k$ on the regions $\Sigma_2^q$ and $\Sigma_3^q$ respectively. The above expressions are well defined and the control $v$ is uniquely determined on $[0,T^*)$. In particular, from (42) and
(43), we can see that the control
\[
v(t) = \begin{cases} 
\tilde{q}(\alpha^+(t)), & \text{if } t \in \left[0, (\alpha^+)^{-1}(1)\right), \\
-\tilde{p}\left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t)\right), & \text{if } t \in \left((\alpha^+)^{-1}(1), T^*\right), \\
0, & \text{if } t \geq T^*,
\end{cases}
\]

makes $p_2$ and $q_2$ vanish, then by the boundary conditions given in (8) all the solutions $p_n, q_n, n \geq 2,$ will be zero. To get an explicit formula for $u$, it suffices to inverse the transformation defined in (3), then using the compatibility condition $\varphi_0(0) = u(0)$ to obtain (11).

**Remark 27.** Since we have an explicit formula of the solution for all $t \geq 0$, we can prove that exact controllability holds at any time $T > T^*$ with loss of the uniqueness of the control.

### 4.2 Proof of the stability theorem

In this subsection, we let $v \equiv 0$. We start by proving the sufficient part.

At time $t \geq 0$, the components $p(t)$ and $q(t)$ might involve at most three values of the restrictive solutions $p_n(t)$ and $q_n(t)$ respectively on the contrary of the cylindrical case where $p(t)$ and $q(t)$ might involve at most two values (see Figure 3). More precisely, we have for the component $p$:

**Case 1:** $t \in \left[\left(\alpha^-\right)^{-1} \circ \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \left(\alpha^-\right)^{-1} \circ \phi^{[n]}(0)\right]$. 

In this case, $p(t)$ might expressed in function of $p_{2n-1}(t), p_{2n}(t), p_{2n+1}(t)$

\[
p(t, x) = \begin{cases} 
p_{2n-1}(t, x), & \text{if } x \in I_1(t) := \left[t - \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \beta(t)\right], \\
p_{2n}(t, x), & \text{if } x \in I_2(t) := \left[t - \phi^{[n]}(0), t - \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)\right], \\
p_{2n+1}(t, x), & \text{if } x \in I_3(t) := \left[\alpha(t), t - \phi^{[n]}(0)\right].
\end{cases}
\]

By definition of the regions $\Sigma^p_n, n \geq 0$, given in (25)-(32), we have for $k = 1, 2, 3$

\[
\{(t, x) \in \left[\left(\alpha^-\right)^{-1} \circ \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), (\alpha^-)^{-1} \circ \phi^{[n]}(0)\right] \times I_k(t)\} \subset \Sigma^p_{2n+k-2}.
\]

Consequently,

\[
\|p(t)\|^2_{L^2(\alpha(t), \beta(t))} = \sum_{k=1}^3 \|p_{2n+k-2}(t, x)\|^2_{L^2(I_k(t))} \leq \sum_{k=1}^3 \int_{(t, x) \in \Sigma^p_{2n+k-2}} |p_{2n+k-2}(t, x)|^2 \, dx,
\]

which leads us to estimate the right hand side of (56). By using the exact solution formulas given in (48) and (49), we obtain for $k = 1, 2, 3$

\[
\begin{align*}
\sum_{k=1}^3 \int_{(t, x) \in \Sigma^p_{2n+k-2}} |p_{2n+k-2}(t, x)|^2 \, dx & \leq \|\tilde{\beta}, \tilde{q}\|^2_{L^2(0,1)} \sum_{k=1}^3 \sup_{x, (t, x) \in \Sigma^p_{2n+k-2}} \prod_{i=0}^{n-1+\left[\frac{k-1}{2}\right]} \left|F\left((\alpha^-)^{-1} \circ \phi^{-1}^{[i]}(t - x)\right)\right|.
\end{align*}
\]

By definition of the regions $\Sigma^p_n, n \geq 0$, given in (25)-(32), we have

\[
(t, x) \in \Sigma^p_{2n} \iff t - x \in \left[\phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \phi^{[n]}(0)\right],
\]

\[
(t, x) \in \Sigma^p_{2n+1} \iff t - x \in \left[\phi^{[n]}(0), \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)\right],
\]

19
therefore, there exist $\tau_1 := \tau^n(t, x) \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)$ and $\tau_2 := \tau^n(t, x) \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)$ such that

\[
(t, x) \in \Sigma^n_{2n} \iff t - x = \phi^n(\tau_1),
\]

\[
(t, x) \in \Sigma^n_{2n+1} \iff t - x = \phi^n(\tau_2),
\]

Thus, \(n\),

\[
\sum_{k=1}^{3} \sup_{t, x, k} \prod_{i=0}^{n+k-2} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^i (t - x) \right) \right| \leq \sup_{\tau_2 \in [0, \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^i \circ \phi^{n-1}(\tau_2) \right) \right|
\]

\[
+ \sup_{\tau_1 \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^i \circ \phi^{n-1}(\tau_1) \right) \right|
\]

\[
= \sup_{\tau \in [0, \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(s) + \sup_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau)
\]

So,

\[
\|p(t)\|_{L^2(\gamma(t), \beta(t))}^2 \leq \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [0, \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau)
\]

\[
+ \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau)
\]

\[
+ \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [0, \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau).
\]

**Case 2:** \(t \in \left[ (\alpha^-)^{-1} \circ \phi^{n}(0), (\alpha^-)^{-1} \circ \phi^{n} \circ (\alpha^- \circ (\alpha^+)^{-1}(1) \right] \).

In this case, \(p(t)\) might be expressed in function of \(p_{2n}(t), p_{2n+1}(t), p_{2n+2}(t)\)

\[
p(t, x) = \begin{cases} 
p_{2n}(t, x), & \text{if } x \in I_4(t) := \left[ t - \phi^n(0), \beta(t) \right), \\
p_{2n+1}(t, x), & \text{if } x \in I_5(t) := \left[ t - \phi^n \circ \alpha^- \circ (\alpha^+)^{-1}(1), t - \phi^n(0) \right], \\
p_{2n+2}(t, x), & \text{if } x \in I_6(t) := \left[ \alpha(t), t - \phi^n \circ \alpha^- \circ (\alpha^+)^{-1}(1) \right].
\end{cases}
\]

In the same way, we obtain the estimate

\[
\|p(t)\|_{L^2(\gamma(t), \beta(t))}^2 \leq \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau)
\]

\[
+ \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau)
\]

\[
+ \|P(t)\|_{L^2(0,1)}^2 \sup_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0)]} \psi^n_1(\tau).
\]

Analogously, arguing for the component \(q\) as before.

**Case 1:** \(t \in \left[ (\beta^+)^{-1} \circ \xi^{n-1} \circ \beta^+ \circ (\beta^-)^{-1}(0), (\beta^+)^{-1} \circ \xi^{n}(1) \right] \).
The expression of \( q(t) \) might involve the expressions of \( q_{2n-1}(t), q_{2n}(t), q_{2n+1}(t) \)
\[
q(t, x) = \begin{cases} 
q_{2n-1}(t, x), & \text{if } x \in J_1(t) := \left[ \alpha(t), \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t \right], \\
q_{2n}(t, x), & \text{if } x \in J_2(t) := \left[ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t, \xi^{[n]}(1) - t \right], \\
q_{2n+1}(t, x), & \text{if } x \in J_3(t) := \left[ \xi^{[n]}(1) - t, \beta(t) \right].
\end{cases}
\] (62)

So, we have
\[
\|q(t)\|^2_{L^2(\alpha(t), \beta(t))} \leq \sum_{k=1}^{3} \int_{(t,x) \in \Sigma_{2n+k-2}} |q_{2n+k-2}(t,x)|^2 \, dx \leq \|\overline{p}, \overline{q}\|^2_{L^2(0,1)} \times \
\sum_{k=1}^{3} \sup_{x, (t,x) \in \Sigma_{2n+k-2}} \prod_{i=0}^{n-2+\left\lfloor \frac{k-1}{2} \right\rfloor} \left| F \left( (\alpha^-)^{(-1)} \circ (\phi^{-1})^{[i]} \circ \beta^- \circ (\beta^+)^{-1} (x + t) \right) \right|.
\] (63)

By definition of the regions \( \Sigma_{2n} \), \( n \geq 0 \) given in (25)-(32), we have
\[
(t, x) \in \Sigma_{2n} \Leftrightarrow t + x \in \left[ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1}(0), \xi^{[n]}(1) \right], \quad (64)
\]
\[
(t, x) \in \Sigma_{2n+1} \Leftrightarrow t + x \in \left[ \xi^{[n]}(1) \circ \beta^+ \circ (\beta^-)^{-1}(0) \right], \quad (65)
\]
and since \( \xi \) is defined as
\[
\xi = \beta^+ \circ (\beta^-)^{-1} \circ \phi \circ \beta^- \circ (\beta^+)^{-1},
\]
(64) and (65) turns to
\[
(t, x) \in \Sigma_{2n} \Leftrightarrow t + x \in \left[ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(0), \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]} \circ \beta^- \circ (\beta^+)^{-1}(1) \right], \quad (66)
\]
\[
(t, x) \in \Sigma_{2n+1} \Leftrightarrow t + x \in \left[ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]} \circ \beta^- \circ (\beta^+)^{-1}(1), \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(0) \right], \quad (67)
\]
therefore, there exist \( \chi_1 := \chi^n(t, x) \in \left[ 0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1) \right] \) and \( \chi_2 := \chi^n(t, x) \in \left[ \phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0) \right] \) such that
\[
(t, x) \in \Sigma_{2n} \Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(\chi_1),
\]
\[
(t, x) \in \Sigma_{2n+1} \Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(\chi_2).
\]
Thus, by combining (63),(66) and (67), we obtain
\[
\sum_{k=1}^{3} \sup_{x, (t,x) \in \mathbb{B}_{2n+k-2}} \prod_{i=0}^{n-2+\left[k-\frac{1}{2}\right]} \left| F\left(\left(\alpha^{-1} \circ (\phi^{-1})[k] \circ \beta^- \circ (\beta^+)^{-1}\right)(x+t)\right) \right|
\leq \sup_{\chi_2 \in \{\phi \circ \beta^- \circ (\beta^+)^{-1}\}} \prod_{i=0}^{n-2} \left| F\left(\left(\alpha^{-1} \circ (\phi^{-1})[i] \circ \beta^- \circ (\beta^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(\chi_2)\right)\right) \right|
+ \sup_{\chi_1 \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}]} \prod_{i=0}^{n-2} \left| F\left(\left(\alpha^{-1} \circ (\phi^{-1})[i] \circ \beta^- \circ (\beta^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(\chi_2)\right)\right) \right|
\leq \sup_{\chi_2 \in \{\phi \circ \beta^- \circ (\beta^+)^{-1}\}} \prod_{i=0}^{n-2} \left| F\left(\left(\alpha^{-1} \circ \phi^{[n-1]}(\chi_2)\right)\right) \right|
= \sup_{\chi_1 \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}]} \prod_{i=0}^{n-2} \left| F\left(\left(\alpha^{-1} \circ \phi^{[n]}(\chi_2)\right)\right) \right|.
\]

Finally, we get
\[
\|q(t)\|_{L^2(\alpha(t), \beta(t))}^2 \leq C \|\tilde{\rho}(\tilde{\eta})\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}] \cup [0, \phi \circ \beta^- \circ (\beta^+)^{-1}]} \psi_{n-1}(\chi),
\]
(68)
\[
+ C \|\tilde{\rho}(\tilde{\eta})\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}]} \psi_{n-1}(\chi).
\]

Case 2: \( t \in \left(\beta^+ \circ (\beta^-)^{-1} \circ \xi^{[n]}(1), (\beta^+)^{-1} \circ \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0)\right) \).

As previously, \( q(t) \) might involve the values of \( q_{2n}(t), q_{2n+1}(t), q_{2n+2}(t) \)
\[
q(t, x) = \begin{cases} 
q_{2n}(t, x), & \text{if } x \in J_4(t) := \left[\alpha(t), \xi^{[n]}(1) - t\right), \\
q_{2n+1}(t, x), & \text{if } x \in J_5(t) := \left[\xi^{[n]}(1) - t, \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t\right), \\
q_{2n+2}(t, x), & \text{if } x \in J_6(t) := \left[\xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t, \beta(t)\right].
\end{cases}
\]
(69)

In the same way, the following estimate holds
\[
\|q(t)\|_{L^2(\alpha(t), \beta(t))}^2 \leq C \|\tilde{\rho}(\tilde{\eta})\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}]} \psi_{n-1}(\chi),
\]
(70)
\[
+ C \|\tilde{\rho}(\tilde{\eta})\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}]} \psi_{n-1}(\chi) + C \|\tilde{\rho}(\tilde{\eta})\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}]} \psi_{n}(\chi).
\]

From (59),(61),(68) and (70), we deduce that
\[
\sup_{\tau \in [0, \phi(0)]} \psi_n(\tau) \to 0 \quad \text{as} \quad n \to \infty \implies \|q(t)\|_{L^2(\alpha(t), \beta(t))} \to 0,
\]
(71)
which finishes the proof of the first statement of theorem (2). The proof of the second and the third statements are just a consequence of (71). By definition of the regions $\Sigma^p_n, \Sigma^q_n, n \geq 0$, given in (25)-(32), we can see that letting $t \to \infty$ is the same as $\phi(\tau) \to \infty, \forall \tau \in [0, \phi(0)],$ so, if there exists a positive function $g$ such that
\[
C g \left( \phi^{[n]}(\tau) \right) \sim \psi_n(\tau), \forall \tau \in [0, \phi(0)),
\]
then obviously (17) holds. In particular, exponential stability follows immediately from
\[
\sup_{\tau \in [0, \phi(0))} \psi_n(\tau) = \sup_{\tau \in [0, \phi(0))} \exp \left( \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} \right).
\]

The proof of the necessary part is straightforward. From (48),(49) and (57), we have
\[
\int_{(t,x) \in \Sigma^p_{2n+1}} |p_{2n+1}(t,x)|^2 \, dx + \int_{(t,x) \in \Sigma^p_{2n+1}} |p_{2n+1}(t,x)|^2 \, dx
\]
\[
\geq C \left( \int_{\tau \in [0, \phi(0))} \inf_{\tau \in [\phi(0), \phi(1))} \left( \inf_{\tau \in [0, \phi(0))} \psi_n(\tau) \right) \right),
\]
therefore, if (15) is not satisfied then clearly stability cannot occur.

Let us prove the second claim of theorem (9). If $f \equiv 1$ then $F \equiv 0$. In this case, we infer from the exact formula of solutions given in (35),(36) and (43) that we have $p_1 \equiv 0$ while $q_1 \neq 0$, and since $q$ is constant along the characteristic lines, $q$ is identically zero from the time that $q_2$ will be zero, that is $t \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$ which is the same time for the boundary controllability of system (1).

5 Further remarks and open questions

Let us discuss briefly some possible variations and generalization of the obtained results in this work.

• It is our hope that the tools developed in this paper may help in dealing with the distributed control case
\[
\begin{align*}
    y(t,x) &= y_x(t,x) + \chi_{\omega_T} h(t,x), & \text{in } Q_T, \\
    y(t,\alpha(t)) &= y(t,\beta(t)) = 0, & \text{in } (0,T), \\
    y(0,x) &= y_0(x), & y_1(x) = y_1(x), & \text{in } (0,1),
\end{align*}
\]

where $\omega_T$ is a moving subset of $Q_T := (0,T) \times (0,1)$ defined by
\[
\omega_T = \{ (t,x) \in Q_T, \, x \in (a(t),b(t)) \}.
\]

and $(y_0, y_1, h) \in H^1_0(0,1) \times L^2(0,1) \times L^2(\omega_T)$. Actually, we can determine the minimal time for which the time-dependent geometric control condition introduced in [8, Definition 1.6] is satisfied. The latter condition states that every generalized bicharacteristic must meet the moving control region at some time $T$. It is easy to verify this condition in the one dimensional settings. Indeed, under assumption (9) with $a, b \in C^1(0,T)$ and $\|a\|_{L^\infty(0,T)} \cdot \|b\|_{L^\infty(0,T)} < 1$, we find that all the characteristics of positive slope or negative slope emerging from the point $(0,x)$, for any $x \in (0,1)$ meet $\omega_T$ if, and only if $T \geq T^*$ where $T^*$ is given by
\[
T^* = \max \{ T_1, T_2 \} = \max \left\{ b^+ \circ \beta^+ \circ (\beta^-)^{-1} \circ b(0), a^- \circ \alpha^- \circ (\alpha^+)^{-1} \circ a(0) \right\}.
\]

23
In particular, if $\alpha \equiv 0$ and $\beta \equiv 1$, the time $T^*$ is given by $T^* = 2 \max \{a, 1 - b\}$ which is exactly the time given in [14].

It worths to mention that problem (72) has been recently studied in [5] with very particular boundary curves and moving control support. The critical time of control seems to be far from being optimal. We can also find a result in [2] for moving control support in cylindrical domain. Exact controllability is proved for all time $T \geq 2$ which is clearly not optimal.

- Note that we have not used the $L^2$ settings in a crucial way. The same results can be proved in $L^p$, $p \in [1, \infty)$ or in the space of continuous functions.

Acknowledgment

I would like to thank my supervisor, Ammar Khodja Farid, for his continuous support and valuable remarks.

References

[1] K. Ammari, A. Bchatnia, and K. El Mufti, Stabilization of the wave equation with moving boundary. Eur. J. Control, 39 (2018), 35-38.

[2] C. Castro, A. Munch, and N. Cindea, Controllability of the linear one-dimensional wave equation with inner moving forces, SIAM J. Control Optim. 52 (2014), no. 6, 4027-4056.

[3] L. Cui, X. Liu, and H. Gao, Exact controllability for a one-dimensional wave equation in non-cylindrical domains. J. Math. Anal. Appl., 402 (2013), 612-625

[4] L. Cui, Y. Jiang, and Y. Wang, Exact controllability for a one-dimensional wave equation with the fixed endpoint control. Bound. Value Probl. 1 (2015), 1-10.

[5] L. Cui, Exact controllability of wave equations with locally distributed control in non-cylindrical domain. Journal of Mathematical Analysis and Applications, 482 (2020), no. 1, 123532.

[6] B. H. Haak, D. T. Hoang, Exact observability of a 1-dimensional wave equation on a noncylindrical domain, SIAM J. Control Optim. 57 (2019), no. 1, 570-589.

[7] V. Komornik, Rapid boundary stabilization of the wave equation, SIAM J. Control Optim., 29 (1991), 197-208.

[8] J. Le Rousseau, G. Lebeau, P. Terpolilli, E. Trélat, Geometric control condition for the wave equation with a time-dependent observation domain. Analysis & PDE, 10 (2017), 983-1015.
[9] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Rev, 20 (1978), 639-739.

[10] A. Sengouga, Exact boundary observability and controllability of the wave equation in an interval with two moving endpoints, Mathematical Control and Related Fields, 9(2020), 1-25.

[11] A. Shao, On Carleman and observability estimates for wave equations on time-dependent domains, Proc. Lond. Math. Soc. 119 (2019), no. 4, 998–1064.

[12] H. Sun, H. Li, and L. Lu. Exact controllability for a string equation in domains with moving boundary in one dimension. Electron. J. Diff. Equations, 98 (2015), 1-7.

[13] Rideau, P. Contrôle d’un assemblage de poutres flexibles par des capteurs actionneurs ponctuels: étude du spectre du système. Thèse, Ecole. Nat. Sup. des Mines de Paris, Sophia-Antipolis, France, 1985.

[14] E, Zuazua, Exact controllability for the semilinear wave equation in one space dimension, Ann. IHP, Analyse non Linéaire 10 (1996) 109-129.