THE STOKES PHENOMENON FOR CERTAIN PDES IN A CASE
WHEN INITIAL DATA HAVE A FINITE SET OF SINGULAR
POINTS

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Abstract. We study the Stokes phenomenon via hyperfunctions for the solutions
of the 1-dimensional complex heat equation under the condition that the Cauchy
data are holomorphic on \( \mathbb{C} \) but a finitely many singular or branching points with
the appropriate growth condition at the infinity. The main tool are the theory of
summability and the theory of hyperfunctions, which allows us to describe jumps
across Stokes lines.

1. Introduction

This paper deals with the 1-dimensional complex heat equation
\( \partial_t u(t,z) = \partial_z^2 u(t,z), u(0,z) = \varphi(z) \). The aim of this work is to describe jumps across the Stokes lines in
terms of hyperfunctions in the case when the initial data \( \varphi(z) \) have a finite set of singular
points. First, we consider the function \( \varphi(z) \) which has a single-valued singular point and
we derive the jump in a form of convergent series (see Theorem 1). Then we discuss the
case when the function \( \varphi(z) \) has a multi-valued singular point and we give the integral
representation of the jump (see Theorem 2). Thus we obtain a full characterization of
the Stokes phenomenon for the considered equation. At the end, we extend our results
to the generalization of the heat equation.

The important point to note here is that D.A. Lutz, M. Miyake and R. Schäfke in [7]
considered the similar problem for the heat equation when the Cauchy data is a function
\( \varphi(z) = 1/z \) with singularity at 0. They proved that the heat kernel was given by a
function as a jump of Borel sum (see [7, Theorem 5.1]).

It is worth pointing out that this work is a continuation of the paper [9] in which we
study the heat equation with the Cauchy data given by a meromorphic function with a
simple pole or finitely many poles.

2. Notation. Gevrey’s asymptotics and \( k \)-summability

In the paper we use the following notation.
A set of the form
\( S = S_d(\alpha, R) = \{ z \in \tilde{\mathbb{C}} : z = re^{i\phi}, r \in (0, R), \phi \in (d - \alpha/2, d + \alpha/2) \} \)
defines a sector \( S \) in a direction \( d \in \mathbb{R} \) with an opening \( \alpha > 0 \) and a radius
\( R \in \mathbb{R}_+ \) in the universal covering space \( \tilde{\mathbb{C}} \) of \( \mathbb{C} \setminus \{0\} \),
\( D_r = \{ z \in \mathbb{C} : |z| < r \} \)
defines a complex disc \( D_r \) in \( \mathbb{C} \) with a radius \( r > 0 \).

In the case that
1. \( R = +\infty \), then this sector is called unbounded and one can write \( S = S_d(\alpha) \) for short,
2. the opening \( \alpha \) is not essential, then the sector \( S_d(\alpha) \) is denoted briefly by \( S_d \).

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3. the radius $r$ is not essential, the set $D_r$ will be designate by $D$.
To simplify the notation, we abbreviate a set $S_\alpha(\alpha) \cup D$ (resp. $S_\alpha \cup D$) to $\hat{S}_\alpha(\alpha)$ (resp. $\hat{S}_\alpha$).

If $f$ is a holomorphic function on a domain $G \subset C^n$, then it will be written as $f \in \mathcal{O}(G)$.

The set of all formal power series (i.e. a power series $\sum_{n=0}^{\infty} a_n t^n$ created for a sequence of complex numbers $(a_n)_{n=0}^{\infty}$) will be represented by the symbol $C[[t]]$. Similarly, $\mathcal{O}(D_r)[[t]]$ stands for the set of all formal power series $\sum_{n=0}^{\infty} a_n(z) t^n$ with $a_n(z) \in \mathcal{O}(D_r)$ for all $n \in \mathbb{N}_0$.

**Definition 1.** Assume that $k > 0$ and $f \in \mathcal{O}(S)$. The function $f$ is called of exponential growth of order at most $k$, if for every proper subsector $S^* \prec S$ (i.e. $S^* \setminus \{0\} \subseteq S$) there exist constants $C_1, C_2 > 0$ such that $|f(x)| \leq C_1 e^{C_2 |x|^k}$ for every $x \in S^*$.

If the function $f$ is of exponential growth of order at most $k$, then one can write $f \in \mathcal{O}^k(S)$.

**Definition 2.** A power series $\sum_{n=0}^{\infty} a_n t^n \in C[[t]]$ is called a formal power series of Gevrey order $s$ ($s \in \mathbb{R}$), if there exist positive constants $A, B > 0$ such that $|a_n| \leq AB^n (n!)^s$ for every $n \in \mathbb{N}_0$. The set of all such formal power series is denoted by $C[[t]]_s$ (resp. $\mathcal{O}(D_r)[[t]]_s$).

**Remark 1.** (see [1]) If $k < 0$ then $u \in C[[t]]_k \iff u$ is convergent and $u \in \mathcal{O}^{-\frac{1}{k}}(C)$.

**Definition 3.** Assume that $s \in \mathbb{R}$, $S$ is a given sector in $\hat{C}$ and $f \in \mathcal{O}(S)$. A power series $f(t) = \sum_{n=0}^{\infty} a_n t^n \in C[[t]]_s$ is called Gevrey’s asymptotic expansion of order $s$ of the function $f$ in $S$ (in symbols $f(t) \sim_s \hat{f}(t)$ in $S$) if for every $S^* \prec S$ there exist positive constants $A, B > 0$ such that for every $N \in \mathbb{N}_0$ and every $t \in S^*$

$$|f(t) - \sum_{n=0}^{N} a_n t^n| \leq AB^N (N!)^s |t|^{N+1}.$$ 

To introduce the notion of summability, by Balser’s theory of general moment summability ([1] Section 6.5), in particular ([1] Theorem 38), we may take Ecalle’s acceleration and deceleration operators instead of the standard Laplace and Borel transform.

**Definition 4** (see [1] Section 11.1). Let $d \in \mathbb{R}$, $\tilde{k} > k > 0$ and $k := (1/k - 1/\tilde{k})^{-1}$.

The acceleration operator in a direction $d$ with indices $k$ and $\tilde{k}$, denoted by $A_{\tilde{k}, k, d}$, is defined for every $g(t) \in \mathcal{O}^{\tilde{k}}(\hat{S}_\tilde{k})$ by

$$(A_{\tilde{k}, k, d} g)(t) := t^{-k} \int_{e^{id} \mathbb{R}^+} g(s)C_{\tilde{k}/k}(s/t)^{\tilde{k}} \, ds,$$

where the Ecalle kernel $C_{\tilde{k}/k}$ is defined by

$$C_{\tilde{k}/k}(\tau) := \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n! \Gamma(1 + \frac{n}{\tilde{k}})} \quad \text{for} \quad \alpha > 1 \tag{1}$$

and the integration is taken over the ray $e^{id} \mathbb{R}^+ := \{re^{id} : r \geq 0\}$.

The formal deceleration operator with indices $k$ and $\tilde{k}$, denoted by $\hat{D}_{\tilde{k}, k}$, is defined for every $\hat{f}(t) = \sum_{n=0}^{\infty} a_n t^n \in C[[t]]$ by

$$(\hat{D}_{\tilde{k}, k} \hat{f})(t) := \sum_{n=0}^{\infty} a_n t^n \frac{\Gamma(1 + n/\tilde{k})}{\Gamma(1 + n/k)}.$$ 

**Definition 5.** Let $k > 0$ and $d \in \mathbb{R}$. A formal power series $\hat{f}(t) = \sum_{n=0}^{\infty} a_n t^n \in C[[t]]$ is called $k$-summable in a direction $d$ if

$$g(t) = (\hat{D}_{k, d} \hat{f})(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(1 + n)}{\Gamma(1 + n + 1/k)} t^n \in \mathcal{O}^k(\hat{S}_d(\varepsilon)) \quad \text{for some} \quad \varepsilon > 0.$$
Moreover, the \( k \)-sum of \( \hat{f}(t) \) in the direction \( d \) is given by
\[
(2) \quad f^d(t) = S_{k,d} \hat{f}(t) := (A_1 \Leftrightarrow d_1, \Leftrightarrow d_2, \Leftrightarrow d_3) \hat{f}(t) \quad \text{with} \quad \theta \in (d - \varepsilon/2, d + \varepsilon/2).
\]

**Definition 6.** If \( \hat{f} \) is \( k \)-summable in all directions \( d \) but (after identification modulo \( 2\pi \)) finitely many directions \( d_1, \ldots, d_n \) then \( \hat{f} \) is called \( k \)-summable and \( d_1, \ldots, d_n \) are called singular directions of \( \hat{f} \).

### 3. The Stokes phenomenon and hyperfunctions

#### 3.1. The Stokes phenomenon for \( k \)-summable formal power series

Now let us recall the concept of the Stokes phenomenon [2 Definition 7].

**Definition 7.** Assume that \( \hat{f} \in \mathbb{C}[[t]]_{1/k} \) (resp. \( \hat{u} \in \mathcal{O}(D)[[t]]_{1/k} \)) is \( k \)-summable with finitely many singular directions \( d_1, d_2, \ldots, d_n \). Then for every \( l = 1, \ldots, n \) a set \( L_{d_l} = \{ t \in \hat{C} : \arg t = d_l \} \) is called a Stokes line for \( \hat{f} \) (resp. \( \hat{u} \)). Of course every such Stokes line \( L_{d_l} \) for \( \hat{f} \) (resp. \( \hat{u} \)) determines so called anti-Stokes lines \( L_{d_l \pm 2\pi/k} \) for \( \hat{f} \) (resp. \( \hat{u} \)).

Moreover, if \( d_l^+ \) (resp. \( d_l^- \)) denotes a direction close to \( d_l \) and greater (resp. less) than \( d_l \), and let \( f^{d_l^+} = S_{k,d_l^+} \hat{f} \) (resp. \( f^{d_l^-} = S_{k,d_l^-} \hat{f} \)) then the difference \( f^{d_l^+} - f^{d_l^-} \) is called a jump for \( \hat{f} \) across the Stokes line \( L_{d_l} \). Analogously we define the jump for \( \hat{u} \).

**Remark 2.** Let \( r(t) := f^{d_l^+}(t) - f^{d_l^-}(t) \) for all \( t \in S_{d_l}(\frac{\pi}{k}) \). Then \( r(t) \sim_{1/k} 0 \) on \( S \).

#### 3.2. Laplace type hyperfunctions

We will describe jumps across the Stokes lines in terms of hyperfunctions. The similar approach to the Stokes phenomenon one can find in [3, 8, 10]. For more information about the theory of hyperfunctions we refer the reader to [3].

We will consider the space
\[
\mathcal{H}^k(L_d) := \mathcal{O}^k(D \cup (S_d \setminus L_d))/\mathcal{O}^k(\hat{S}_d)
\]
of Laplace type hyperfunctions supported by \( L_d \) with exponential growth of order \( k \). It means that every hyperfunction \( G \in \mathcal{H}^k(L_d) \) may be written as
\[
G(s) = [g(s)]_d = \{ g(s) + h(s) : h(s) \in \mathcal{O}^k(\hat{S}_d) \}
\]
for some defining function \( g(s) \in \mathcal{O}^k(D \cup (S_d \setminus L_d)) \).

By the Köthe type theorem [3] one can treat the hyperfunction \( G = [g(s)]_d \) as the analytic functional defined by
\[
G(s)[\varphi(s)] := \int_{\gamma_d} g(s)\varphi(s)\,ds \quad \text{for sufficiently small} \quad \varphi \in \mathcal{O}^{-k}(\hat{S}_d)
\]
with \( \gamma_d \) being a path consisting of the half-lines from \( e^{id^-} \infty \) to \( 0 \) and from \( 0 \) to \( e^{id^+} \infty \), i.e. \( \gamma_d = -\gamma_{d^-} + \gamma_{d^+} \) with \( \gamma_{d^+} = L_{d^+} \).

#### 3.3. The description of jumps across the Stokes lines in terms of hyperfunctions

Assume that \( \hat{f} \) is \( k \)-summable and \( d \) is a singular direction. By (2) the jump for \( \hat{f} \) across the Stokes line \( L_d \) is given by
\[
f^{d^+}(t) - f^{d^-}(t) = (A_1 \Leftrightarrow d^+ - A_1 \Leftrightarrow d^-)\hat{D}_1 \Leftrightarrow \hat{f}(t).
\]

We will describe this jump in terms of hyperfunctions. To this end, observe that we can treat \( g(t) := \hat{D}_1 \Leftrightarrow \hat{f}(t) \in \mathcal{O}^k(D \cup (S_d \setminus L_d)) \) as a defining function of the hyperfunction \( G(s) := [g(s)]_d \in \mathcal{H}^k(L_d) \).
So, for sufficiently small \( r > 0 \) and \( t \in S_{\delta}(\pi, r) \) this jump is given as the Ecalle acceleration operator \( A_{1, \frac{r}{1+ kr}} \) acting on the hyperfunction \( G(s) \). Precisely, we have

\[
\begin{align*}
f^{d+}(t) - f^{d-}(t) &= (A_{1, \frac{r}{1+ kr}} G)(t) := G(s) \left[ \frac{1}{1+k} C_{\frac{r}{1+ kr}} \left((s/t)^{1+kr}\right) \right] \\
&= G(s^{1+k}) \left[ (t^{-1+k})(s/t)^{1+kr} \right],
\end{align*}
\]

where \( G(s)[\varphi(s)] \) is defined by \((\ref{eq:Ecalle})\), and the last equality holds by the change of variables, because if \( G(s) = [g(s)]_d \) then \( G(s^p) = [g(s^p)]_{dp} \) for every \( p > 0 \).

### 4. Characterization of the Stokes phenomenon in a case when the initial data have a finite set of singular points

In this section we specify a form of the jumps across the Stokes lines based on the solution of the heat equation in a case when the initial data have a finite set of singular points. Due to the linearity of the equation, it is enough to consider the case that the singularity occurs only at one point – singular or branching point.

Recall the following proposition

**Proposition 1** \([9, \text{Theorem 4}]\). Suppose that \( \hat{u} \) is a unique formal solution of the Cauchy problem of the heat equation

\[
\begin{align*}
\frac{\partial_t u}{\partial_t} &= \partial_{zz} u \\
u(0, z) &= \varphi(z)
\end{align*}
\]

with

\[
\varphi \in \mathcal{O}^2 \left( D \cup S_{\frac{\pi}{2}} \right) \text{ for some } \varepsilon > 0.
\]

Then \( \hat{u} \) is 1-summable in the direction \( d \) and for every \( \theta \in \left( d - \frac{\pi}{2}, d + \frac{\pi}{2} \right) \) and for every \( \varepsilon \in (0, \varepsilon) \) there exists \( r > 0 \) such that its 1-sum \( \hat{u} \) is represented by

\[
\begin{align*}
u(t, z) = \hat{u}^0(t, z) &= \frac{1}{\sqrt{4\pi t}} \int_0^{\frac{1}{4\pi t}} \left( \varphi(z + s) + \varphi(z - s) \right) e^{-s^2} ds
\end{align*}
\]

for \( t \in S_\delta(\pi, r) \) and \( z \in D_r \).

Now consider the heat equation \((\ref{eq:heat_eq})\) with \( \varphi(z) \in \mathcal{O}^2(\mathbb{C} \setminus \{z_0\}) \) for some \( z_0 \in \mathbb{C} \setminus \{0\} \). First, observe that in this case \( L_\delta \) with \( \delta := 2\theta := 2 \arg z_0 \) is a Stokes line for \( \hat{u} \).

For every sufficiently small \( \varepsilon > 0 \) there exists \( r > 0 \) such that for every fixed \( z \in D_r \) the jump is given by

\[
u^{d+}(t, z) - \nu^{d-}(t, z) = F_z(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-s^2} \right],
\]

where \( t \in S_\delta(\pi - \varepsilon, r) \) and

\[
F_z(s) = \left[ \varphi(s + z) + \varphi(z - s) \right]_{\theta_z} \quad \text{and} \quad \theta_z = \arg(z_0 - z).
\]

**Remark 3.** In the remainder of this section we assume that \( t \in S_\delta(\pi - \varepsilon, r) \) and fixed \( z \in D \) \((\varepsilon, r > 0)\).

Now we consider the case when \( z_0 \) is a single-valued singular point of the function \( \varphi(z) \in \mathcal{O}^2(\mathbb{C} \setminus \{z_0\}) \).
**Theorem 1.** Suppose that \( \varphi(z) = \sum_{n=1}^{\infty} \frac{a_n}{(z-z_0)^n} + \phi(z) \), where \( a_1, a_2, \ldots \in \mathbb{C} \) and \( \lim_{n \to \infty} \sqrt[4]{|a_n|} < 1 \), \( z_0 \in \mathbb{C} \setminus \{0\} \), \( \phi(z) \in O^2(\mathbb{C}) \). Then

\[
F_\delta(s) = \left[ \sum_{n=1}^{\infty} \frac{a_n}{(z+s-z_0)^n} \right]_{\theta_s} = -2\pi i \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(z+s-z_0),
\]

where \( \delta \) is the Dirac function and \( \delta^{(n-1)} \) denotes its \( (n-1) \)-th derivative.

Moreover, the jump is given by the convergent series

\[
u^{\delta^+}(t,z) - \nu^{\delta^-}(t,z) = -i \left. \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} e^{\frac{s^2}{4t^2}} \right|_{s=z_0-z}.
\]

**Proof.** Observe that since \( \delta(x) = \left[ -\frac{1}{2\pi i} \right] \) (see [3]), then \( \delta(x-a) = \left[ -\frac{1}{2\pi i(s-a)} \right] \) (where \( a \in \mathbb{R} \)) and differentiating it \( n \)-times one can easily obtain

\[
\delta^{(n)}(x-a) = \left[ -\frac{(-1)^n n!}{2\pi i(s-a)^{n+1}} \right] \Rightarrow -2\pi i \delta^{(n-1)}(x-a) = \left[ \frac{1}{(s-a)^n} \right].
\]

Notice that the same holds for \( a = z_0 - z \in \mathbb{C} \).

Hence we derive

\[
F_\delta(s) = \left[ \sum_{n=1}^{\infty} \frac{a_n}{(z+s-z_0)^n} \right]_{\theta_s} = -2\pi i \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(s+z-z_0).
\]

Thus

\[
u^{\delta^+}(t,z) - \nu^{\delta^-}(t,z) = F_\delta(s) \left[ \frac{1}{\sqrt{\pi t}} e^{-\frac{z^2}{4t^2}} \right]
= -2\pi i \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(s+z-z_0) \left[ \frac{1}{\sqrt{\pi t}} e^{-\frac{z^2}{4t^2}} \right]
= -i \left. \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} e^{\frac{s^2}{4t^2}} \right|_{s=z_0-z}.
\]

It remains to prove the convergence of the series above.

Notice that by Remark [1] since \( s \mapsto e^{-\frac{s^2}{4t^2}} \in O^2(\mathbb{C}) \), then there exist \( A, B > 0 \) such that for every \( t \in S(\theta, \pi - \tilde{\epsilon}, r) \) and \( z \in D_r \) (for every sufficiently small \( \epsilon > 0 \) and \( \tilde{\epsilon} \in (0, \epsilon) \)) we have that

\[
\left| \delta^{(n-1)}(z+s-z_0) e^{-\frac{z^2}{4t^2}} \right| = \left| \frac{d^{n-1}}{ds^{n-1}} e^{-\frac{s^2}{4t^2}} \right|_{s=z_0-z} \leq AB^{n-1}((n-1)!)^{\frac{1}{2}},
\]

so

\[
\left| \nu^{\delta^+}(t,z) - \nu^{\delta^-}(t,z) \right| = \left| -i \left. \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{a_n(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(z+s-z_0) e^{-\frac{z^2}{4t^2}} \right|_{s=z_0-z}
\leq \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{|a_n|}{(n-1)!} \left| \delta^{(n-1)}(z+s-z_0) e^{-\frac{z^2}{4t^2}} \right|
\leq \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{|a_n|}{(n-1)!} A\bar{B}^{n-1}((n-1)!)^{\frac{1}{2}} = \sqrt{\frac{\pi}{t}} \bar{A} \sum_{n=1}^{\infty} \frac{|a_n|}{(n-1)!} \left( \frac{AB^{n-1}((n-1)!)^{\frac{1}{2}}}{\sqrt{t}} \right) < \infty,
\]

because \( \lim_{n \to \infty} \sqrt[4]{|a_n|} < 1 \). Thus this implies the convergence of \( \nu^{\delta^+}(t,z) - \nu^{\delta^-}(t,z) \). \( \square \)

In particular, from the above theorem we obtain the following examples.
Example 1. Assume now \( \varphi(z) = \sum_{n=0}^{N} \frac{a_n}{(z - z_0)^n} + \phi(z) \), for some \( z_0 \in \mathbb{C} \setminus \{0\} \), where \( N \in \mathbb{N} \setminus \{0\} \), \( a_1, a_2, \ldots a_N \in \mathbb{C} \) and \( \phi(z) \in \mathcal{O}^2(\mathbb{C}) \). Then

\[
F_z(s) = \left[ \sum_{n=1}^{N} \frac{a_n}{(z + s - z_0)^n} \right]_{\theta_z} = -2\pi i \sum_{n=1}^{N} \frac{a_n(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(z + s - z_0),
\]

and the jump is given by

\[
u_s^+ (t, z) - \nu_s^- (t, z) = -i \sqrt{\frac{\pi}{t}} \sum_{n=1}^{N} \frac{a_n(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} e^{-z^2} \bigg|_{s=z_0 - z}.
\]

Example 2. Let \( \varphi(z) = e^{-|z|} \) for some \( z_0 \in \mathbb{C} \setminus \{0\} \). Then

\[
F_z(s) = \left[ e^{-|z|} \right]_{\theta_z} = -2\pi i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + 1)!} \delta^{(k)}(z + s - z_0),
\]

and the jump is given by

\[
u_s^+ (t, z) - \nu_s^- (t, z) = -i \sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + 1)!} \frac{d^k}{ds^k} e^{-\pi s^2} \bigg|_{s=z_0 - z}.
\]

Let us now consider the general case. For this purpose fix \( z \in D \). For \( s \in \mathcal{L}_{\theta_z} \), define (similarly as in \([3\) and \([10]\)) a function on \( \mathcal{L}_{\theta_z} \) by

\[
\text{var} F_z(s) = \begin{cases} 0 & \text{if } |s| < |z_0 - z| \\ \varphi(z_0 + (s + z - z_0) e^{2\pi i}) - \varphi(s + z) & \text{if } |s| > |z_0 - z|, \end{cases}
\]

and a Heaviside function in a direction \( \theta_z \) by

\[
H_{\theta_z}(xe^{i\theta_z}) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0, \end{cases}
\]

thus \( F_z(s) = \left[ \varphi(s + z) \right]_{\theta_z} = -\text{var} F_z(s) = -\text{var} F_z(s) H_{\theta_z}(s + z - z_0). \)

So \( \nu_s^+ (t, z) - \nu_s^- (t, z) = -\text{var} F_z(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \right] \) where, in general, \( -\text{var} F_z(s) \) is an analytic functional on \( \mathcal{L}_{\theta_z} \).

Notation. The set of all measurable functions \( f : \mathcal{L}_{\theta_z} \to \mathbb{C} \) such that \( \int_K |f| ds < \infty \) for all compact sets \( K \subset \mathcal{L}_{\theta_z} \) will be denoted by \( L^1_{\text{loc}}(\mathcal{L}_{\theta_z}) \).

Theorem 2. Under the above assumptions we have several cases to discuss

1. \( \text{var} F_z(s) \in L^1_{\text{loc}}(\mathcal{L}_{\theta_z}) \) and is an analytic function of exponential growth of order at most 2 for \( |s| > |z_0 - z| \).

Then for every sufficiently small \( \varepsilon > 0 \) there exists \( r > 0 \) such that the jump is given by

\[
u_s^+ (t, z) - \nu_s^- (t, z) = -\int_{z_0 - z}^{e^{i\theta_z}} \text{var} F_z(s) \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} ds,
\]

for \( (t, z) \in S_\varepsilon(\pi - \varepsilon, r) \times D \).

2. \( \text{var} F_z(s) \) is a distribution on \( \mathcal{L}_{\theta_z} \) and is an analytic function of exponential growth of order at most 2 for \( |s| > |z_0 - z| \).

Then there exist \( m \in \mathbb{N} \) and \( \text{var} F_z(s) \) satisfying the assumptions of the case (1) such that

\[
\frac{d^m}{ds^m} \text{var} F_z(s) = \text{var} F_z(s).
\]
Moreover, for every sufficiently small \( \varepsilon > 0 \) there exists \( r > 0 \) such that the jump is given by

\[
\begin{align*}
\Delta u^+(t, z) - \Delta u^-(t, z) &= -\var F_z(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \right] \\
&= -\frac{d^m}{ds^m} \var F(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \right] = -\var F(s) \left[ (-1)^m \frac{d^m}{ds^m} \left( \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \right) \right] \\
&= - \int_{z_0-z}^{\infty} \var F_z(s) (-1)^m \frac{d^m}{ds^m} \left( \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} \right) ds,
\end{align*}
\]

for \((t, z) \in S_\delta(\pi - \varepsilon, r) \times D\).

\[\text{(3) } \var F_z(s) \text{ is an analytic functional on } \mathcal{L}_\theta.\]

Then \( \var F_z(s) = \sum_{n=0}^\infty \var F_{z,n}(s) \), where \( \var F_{z,n}(s) \) satisfy the assumptions of the case (2). So for every \( n \in \mathbb{N} \) there exists \( k_n \in \mathbb{N} \) and \( \var F_{z,n}(s) \) satisfying the assumptions of the case (1) such that

\[
\var F_{z,n}(s) = \frac{d^{k_n}}{ds^{k_n}} \var F_z(s).
\]

Moreover, for every sufficiently small \( \varepsilon > 0 \) there exists \( r > 0 \) such that the jump is given by

\[
\begin{align*}
\Delta u^+(t, z) - \Delta u^-(t, z) &= -\var F_z(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \right] = -\sum_{n=0}^\infty \var F_{z,n}(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} \right] \\
&= - \sum_{n=0}^\infty \int_{z_0-z}^{\infty} \var F_{z,n}(s) (-1)^m \frac{d^m}{ds^m} \left( \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} \right) ds,
\end{align*}
\]

for \((t, z) \in S_\delta(\pi - \varepsilon, r) \times D\).

\[\text{Proof. } \text{Ad.}(1) \text{ First observe that for every } z \in D \text{ the function } s \mapsto \var F_z(s) \text{ is analytic on } \mathcal{L}_\theta \setminus \{z_0 - z\}, \text{ locally integrable and has an exponential growth of order at most 2 as } s \to \infty, s \in \mathcal{L}_\theta. \text{ Hence for every sufficiently small } \varepsilon > 0 \text{ there exists } r > 0 \text{ such that the integral } \Delta u^+(t, z) - \Delta u^-(t, z) \text{ is well defined for } (t, z) \in S_\delta(\pi - \varepsilon, r) \times D.\]

For \( z_0 - z = x_0 e^{i\theta_z} \) and \( s = x e^{i\theta_s} \), where \( x_0, x > 0 \), we obtain

\[
\begin{align*}
\Delta u^+(t, z) - \Delta u^-(t, z) &= F_z(s) \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{t^2}{4t}} \right] \\
&= \frac{1}{\sqrt{4\pi t}} \lim_{\varepsilon \to 0^+} \int_0^\infty \var \left( (x + i\varepsilon)e^{i\theta_z} + z_0 - x_0 e^{i\theta_z} \right) e^{-\frac{x^2}{4t}} \left( (x + i\varepsilon)e^{i\theta_z} \right)^2 e^{i\theta_s} dx \\
&- \int_0^\infty \var \left( (x - i\varepsilon)e^{i\theta_z} + z_0 - x_0 e^{i\theta_z} \right) e^{-\frac{x^2}{4t}} \left( (x - i\varepsilon)e^{i\theta_z} \right)^2 e^{i\theta_s} dx \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{x^2}{4t}} \left( x e^{i\theta_z} \right)^2 e^{i\theta_s} \lim_{\varepsilon \to 0^+} \left\{ \var \left( (x + i\varepsilon - x_0)e^{i\theta_z} + z_0 \right) \\
&- \var \left( (x - i\varepsilon - x_0)e^{i\theta_z} + z_0 \right) \right\} dx = (\ast)
\end{align*}
\]

Observe that

- for \( x - x_0 > 0 \), we have

\[
\lim_{\varepsilon \to 0^+} \left\{ \var \left( (x + i\varepsilon - x_0)e^{i\theta_z} + z_0 \right) - \var \left( (x - i\varepsilon - x_0)e^{i\theta_z} + z_0 \right) \right\} = \var \left( (x - x_0)e^{i\theta_z} + z_0 \right) - \var \left( (x - x_0)e^{i\theta_z} e^{2\pi i} + z_0 \right)
\]
Assume that \( n \) and the jump is given by \( \varphi(z) = \ln(z - z_0) \) for some \( z_0 \in \mathbb{C} \setminus \{0\} \). Then \( \varphi(z) = \ln(z - z_0) \) satisfies the assumptions of the case (2) and based on results in \([4]\) we can write \( \varphi(z) = \sum_{n=0}^{\infty} \varphi(z_n) \), where \( \varphi(z_n) \) satisfy the assumptions of the case (2). Then for every \( n \in \mathbb{N} \) there exists \( k_n \in \mathbb{N} \) and \( \varphi(z_n) \) satisfying the assumptions of the case (1) such that \( \varphi(z_n) = \frac{d^{kn}}{dz^{kn}} \varphi(z) \). The rest of the proof is analogous to the proof of the case (1).

Ad.(2) Observe that since \( \varphi(z) = \ln(z - z_0) \) is continuous on \( L_{\theta_z} \setminus \{z_0 - z\} \), by the locally structure theorem for distributions (see Proposition 7.1 \([2]\) ), there exist \( m \in \mathbb{N} \) and \( \varphi(z) = \ln(z - z_0) \) such that

\[
\frac{d^m}{dz^m} \varphi(z) = \varphi(z).
\]

Furthermore, \( \varphi(z) = \ln(z - z_0) \) has exponential growth of order at most 2 as \( s \to \infty, s \in L_{\theta_z} \), then also \( \varphi(z) = \ln(z - z_0) \) has an exponential growth of order at most 2 as \( s \to \infty, s \in L_{\theta_z} \). The rest of the proof is analogous to the proof of the case (1).

Ad.(3) Notice that since \( \varphi(z) = \ln(z - z_0) \) obey the assumptions of the case (2) and based on results in \([4]\) we can write \( \varphi(z) = \sum_{n=0}^{\infty} \varphi(z_n) \), where \( \varphi(z_n) \) satisfy the assumptions of the case (2). Then for every \( n \in \mathbb{N} \) there exists \( k_n \in \mathbb{N} \) and \( \varphi(z_n) \) satisfying the assumptions of the case (1) such that \( \varphi(z_n) = \frac{d^{kn}}{dz^{kn}} \varphi(z) \). The rest of the proof is also similar to the proof of the case (1).

Now we give two examples of the function \( \varphi(z) \) satisfying the case (1) of Theorem 2.

**Example 3.** Assume that \( \varphi(z) = \ln(z - z_0) \) for some \( z_0 \in \mathbb{C} \setminus \{0\} \). Then

\[
\varphi(z) = 2\pi i H_{\theta_z}(s + z - z_0),
\]

and the jump is given by

\[
u^+(t, z) - \nu^-(t, z) = -i \sqrt{\frac{\pi}{t}} \int_{z_0 - z}^{\infty} e^{-\frac{s^2}{4t}} ds.
\]

Indeed, for \( |s| > |z_0 - z| \) we derive

\[
\varphi(z) = \ln(z_0 + (s + z - z_0) e^{2\pi i}) - \ln(s + z) =
\]

\[
= \ln((s + z - z_0)e^{2\pi i} - z_0) - \ln(s + z - z_0) =
\]

\[
= \ln((s + z - z_0)e^{2\pi i}) - \ln(s + z - z_0) = 2\pi i.
\]

**Example 4.** Let \( \varphi(z) = (z - z_0)^{\lambda} \) for some \( z_0 \in \mathbb{C} \setminus \{0\}, \lambda \notin \mathbb{Z} \) and \( \lambda > -1 \). Then

\[
\varphi(z) = (z - z_0)^{\lambda},
\]

and the jump is given by

\[
\nu^+(t, z) - \nu^-(t, z) = -i \sqrt{\frac{\pi}{t}} \int_{z_0 - z}^{\infty} e^{-\frac{s^2}{4t}} (-s - z + z_0)^{\lambda} \sin(\lambda \pi) ds.
\]
More precisely, for $|s| > |z_0 - z|$

\[
\text{var} F_z(s) = \varphi(z_0 + (z + s - z_0)e^{2\pi i}) - \varphi(z + s) = \left((z_0 + (z + s - z_0)e^{2\pi i}) - z_0\right)^\lambda - \left((z + s) - z_0\right)^\lambda = (z + s - z_0)^\lambda (e^{2\pi i\lambda} - 1) = 2i(-1)^\lambda (z + s - z_0)^\lambda \sin(\pi \lambda),
\]

because

\[
\sin(\pi \lambda) = \frac{e^{i\pi \lambda} - e^{-i\pi \lambda}}{2i} = \frac{e^{2i\pi \lambda} - 1}{2i} \implies e^{2i\pi \lambda} - 1 = 2ie^{i\pi \lambda} \sin(\pi \lambda) = 2i(-1)^\lambda \sin(\pi \lambda).
\]

Now we present an example of the function $\varphi(z)$ satisfying the case (2) of Theorem 2

**Example 5.** Let again $\varphi(z) = (z - z_0)^\lambda$ for some $z_0 \in \mathbb{C} \setminus \{0\}$, $\lambda \notin \mathbb{Z}$ and $\lambda < -1$. Then for $m = [-\lambda]$ we can define $\text{var} \tilde{F}_z(s) \in L^1_{\text{loc}}(\mathcal{L}_z)$ by

\[
\text{var} \tilde{F}_z(s) = \frac{2i(-1)^{\lambda+m} \sin(\pi(\lambda + m))}{(\lambda + 1)(\lambda + 2)\ldots(\lambda + m)} (s + z - z_0)^{\lambda+m} H_{\theta_z}(s + z - z_0),
\]

thus

\[
\text{var} F_z(s) = \frac{d^m}{ds^m} \text{var} \tilde{F}_z(s) = \frac{d^m}{ds^m} \left\{ \frac{2i(-1)^{\lambda+m} \sin(\pi(\lambda + m))}{(\lambda + 1)(\lambda + 2)\ldots(\lambda + m)} (s + z - z_0)^{\lambda+m} H_{\theta_z}(s + z - z_0) \right\},
\]

and the jump is given by

\[
u^+(t, z) - \nu^-(t, z) = -i \sqrt{\pi t} \int_{z_0 - z}^{\infty} \frac{(-1)^\lambda (s + z - z_0)^{\lambda+m} \sin((\lambda + m)\pi)}{(\lambda + 1)(\lambda + 2)\ldots(\lambda + m)} \frac{d^m}{ds^m} \left( e^{-s^2} \right) ds.
\]

Finally, we give an example of the function $\varphi(z)$ that satisfies the case (3) of Theorem 2

**Example 6.** Assume now that $\varphi(z) = e^{\frac{|z-z_0|^2}{2z}}$ where $z_0 \in \mathbb{C} \setminus \{0\}$, $\lambda \notin \mathbb{Q}$ and $\lambda > 0$. Then for $k_n = [\lambda n]$ we can define functions $\text{var} \tilde{F}_{z,n}(s) \in L^1_{\text{loc}}(\mathcal{L}_z)$ by

\[
\text{var} \tilde{F}_{z,n}(s) = \frac{2i(-s - z + z_0)^{-\lambda n+k_n} \sin((-\lambda n + k_n)\pi)}{n!(-\lambda n + 1)(-\lambda n + 2)\ldots(-\lambda n + k_n)} H_{\theta_z}(s + z - z_0),
\]

and

\[
\text{var} F_z(s) = \sum_{n=0}^{\infty} \text{var} F_{z,n}(s) = \sum_{n=0}^{\infty} \frac{d^{k_n}}{ds^{k_n}} \text{var} \tilde{F}_{z,n}(s) = \left( \sum_{n=0}^{\infty} \frac{d^{k_n}}{ds^{k_n}} \frac{2i(-s - z + z_0)^{-\lambda n+k_n} \sin((-\lambda n + k_n)\pi)}{n!(-\lambda n + 1)(-\lambda n + 2)\ldots(-\lambda n + k_n)} \right) H_{\theta_z}(s + z - z_0).
\]
Then the jump is given by
\[
u^+ (t,z) - \nu^- (t,z) = -\int_{z_0 - z}^{e^{\lambda_n} \infty} \sum_{n=0}^{\infty} \frac{i(-s - z + z_0)^{-\lambda_n + k_n} \sin((-\lambda_n + k_n)\pi)}{\sqrt{\pi n!((-\lambda_n + 1)(-\lambda_n + 2)\ldots(-\lambda_n + k_n))}} \left((-1)^{k_n} \frac{d^{k_n}}{ds^{k_n}} \left(e^{-\frac{s^2}{2}} \right)\right) ds.
\]

Observe that by Remark 7 since \(s \mapsto e^{-\frac{s^2}{2}} \in \mathcal{O}(\mathbb{C})\), then there exist \(A,B > 0\) such that for every \(t \in S(\theta, \pi - \bar{\varepsilon}, r)\) and \(z \in D_r\) (for every sufficiently small \(\varepsilon > 0\) and \(\bar{\varepsilon} \in (0, \varepsilon)\)) we have that
\[
\left|(-1)^{k_n} \frac{d^{k_n}}{ds^{k_n}} \left(e^{-\frac{s^2}{2}} \right)\right| \leq AB^{\lambda_n} (n!)^{\frac{3}{2}} \text{ and}
\]
\[
\left|\frac{1}{n!(-\lambda_n + 1)(-\lambda_n + 2)\ldots(-\lambda_n + k_n)}\right| \leq \frac{1}{(n!)^{\frac{3}{2} + 1}} < \infty
\]
hence analogously to the proof of Theorem 7 we obtain the convergence of the above series.

At the end of this section, we similarly derive jumps for the following generalization of the heat equation
\[
\begin{align*}
\partial_t^\alpha u(t,z) &= \partial_t^\alpha u(t,z), \quad p,q \in \mathbb{N}, \quad 1 \leq p < q \\
u(0,z) &= \varphi(z) \in \mathcal{O}(D) \\
\partial_t^j u(0,z) &= 0 \quad \text{for} \quad j = 1,2,\ldots,p-1,
\end{align*}
\]
with \(\varphi(z) \in \mathcal{O}^{\frac{p}{q}}\left(D \cup \bigcup_{q=1}^{q-1} S_{\frac{p}{q}} \left(\frac{q/p}{p}\right)\right)\) for some \(\varepsilon > 0\).

Then a unique formal solution \(\tilde{u}(t,z)\) of this Cauchy problem is \(\frac{p}{q}\)-summable in the direction \(d\) and for every \(\psi \in (d - \frac{\varepsilon}{2},d + \frac{\varepsilon}{2})\) and for every \(\bar{\varepsilon} \in (0,\varepsilon)\) there exists \(\varepsilon > 0\) such that its \(\frac{p}{q}\)-sum \(u \in \mathcal{O}(S_{\delta}(\frac{q/p}{p} - \bar{\varepsilon}, r) \times D)\) is given by (see [9, Theorem 6])
\[
u(t,z) = \varphi(z) + \sum_{i=0}^{\infty} \varphi(z + e^{-\frac{2\pi i q}{p}(s)}) C^\frac{q}{p} \left(s/\sqrt{p}\right) ds.
\]
As in the case of the heat equation (3), we assume that \(\varphi(z) \in \mathcal{O}^{\frac{p}{q}}(\mathbb{C} \setminus \{z_0\})\).

Then \(L_\delta\) with \(\delta := q\theta/p := q \arg z_0/p\) is a separate Stokes line for \(\tilde{u}\), such that \(\delta_z = q \arg (z_0 - z)/p\) for every sufficiently small \(z\).

For every sufficiently small \(\varepsilon > 0\) there exists \(\varepsilon > 0\) such that for every fixed \(z \in D_r\) the jump is given by
\[
u^+ (t,z) - \nu^- (t,z) = F_z(s) \left[\frac{1}{q \sqrt{p}} C^\frac{q}{p} (s/\sqrt{p})\right]
\]
\[
= \left[\varphi(z + s) + \cdots + \varphi(z + e^{-\frac{2\pi i q}{p}s})\right] \left[\frac{1}{q \sqrt{p}} C^\frac{q}{p} (s/\sqrt{p})\right]
\]
\[
= \left[\varphi(z + s)\right] \left[\frac{1}{q \sqrt{p}} C^\frac{q}{p} (s/\sqrt{p})\right],
\]
(the last equality arising from the fact that in this case all singular points appear in the function \(\varphi(z + s)\)), where the hyperfunction \(F_z(s) = \left[\varphi(z + s) + \cdots + \varphi(z + e^{-\frac{2\pi i q}{p}s})\right] \theta_z\) belongs to the space \(\mathcal{O}^{\frac{p}{q}}\left(D \cup (S_{\theta}(\alpha) \setminus L_{\theta_z})\right) / \mathcal{O}^{\frac{p}{q}}(D \cup S_{\theta}(\alpha))\) with \(\theta_z = \arg (z_0 - z)\).
Thus, we obtain analogous results as for the heat equation \( \text{(4)} \), only that in Theorem \( \text{1} \) and Theorem \( \text{2} \) we replace 
\[
\frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}
\]
by 
\[
\frac{1}{q \sqrt{t}} C_{\frac{q}{p}} \left( \frac{s}{\sqrt{t}} \right).
\]

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