ON UNIVALENCE OF EQUIVARIANT RIEMANN DOMAINS
OVER THE COMPLEXIFICATION OF A NON-COMPACT,
RIEMANNIAN SYMMETRIC SPACE

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Abstract. Let $G/K$ be a non-compact, rank-one, Riemannian symmetric space and let $G^\mathbb{C}$ be the universal complexification of $G$. We prove that a holomorphically separable, $G$-equivariant, Riemann domain over $G^\mathbb{C}/K^\mathbb{C}$ is necessarily univalent, provided that $G$ is not a covering of $SL(2,\mathbb{R})$. As a consequence of the above statement one obtains a univalence result for holomorphically separable, $G \times K$-equivariant Riemann domains over $G^\mathbb{C}$. Here $G \times K$ acts on $G^\mathbb{C}$ by left and right translations. The proof of such results involves a detailed study of the $G$-invariant complex geometry of the quotient $G^\mathbb{C}/K^\mathbb{C}$, including a complete classification of all its Stein $G$-invariant subdomains.

1. Introduction

Let $G$ be a connected Lie group and let $Y$ be a complex $G$-manifold, i.e. a complex manifold endowed with a real-analytic action of $G$ by holomorphic transformations. Consider the action of $G$ on its universal complexification $G^\mathbb{C}$ by left translations. A $G$-equivariant local biholomorphism $p : Y \longrightarrow G^\mathbb{C}$ is by definition a $G$-equivariant Riemann domain over $G^\mathbb{C}$. It is of interest to determine conditions under which $p$ is injective, i.e. under which the Riemann domain is univalent.

One motivation comes from the classical problem of describing the envelope of holomorphy of a domain in a Stein manifold. If $G = \mathbb{R}^n$, as a consequence of Bochner’s tube theorem, the envelope of holomorphy of a $G$-invariant domain in $G^\mathbb{C}$ is a univalent $G$-equivariant Riemann domain over $G^\mathbb{C}$. An analogous statement for $G$ a compact Lie group is due to O. S. Rothaus ([Rt]). The above results were later generalized to arbitrary holomorphically separable, $G$-equivariant Riemann domains over $G^\mathbb{C}$. They were also extended to a larger class of Lie groups, including for example the product of a compact and a simply connected nilpotent Lie group (see [CL], [Ia], [CIT]). Note that since $G^\mathbb{C}$ is Stein (see [He2]), holomorphic separability of $Y$ is a necessary condition for $p$ to be injective.

Another motivation comes from the problem of extending to a global action the local $G^\mathbb{C}$-action induced by a $G$-action on a reduced complex space. Indeed the univalence of $G$-equivariant Riemann domains over $G^\mathbb{C}$ turns out to be a necessary condition for the existence of such an extension (see [Pa], [HI], [CIT]).

When $G$ is a non-compact, real semisimple Lie group, univalence of holomorphically separable $G$-equivariant Riemann domains over $G^\mathbb{C}$ does not hold in general. For $G = SL(2,\mathbb{R})$, a Stein counter-example was pointed out to us by K. Oeljeklaus (see Sect. 8). The image of this Riemann domain in $G^\mathbb{C}$ is also invariant under right $K$-translations and its construction is based on the existence of non-trivial coverings of the $K$-orbits in $G^\mathbb{C}$. Here $K$ a maximal compact subgroup.
in $G$. Observe also that $SL(2, \mathbb{C})/SL(2, \mathbb{R})$ is simply connected. Thus this example gives a negative answer to the question whether the simple-connectivity of the quotient $G^C/G$ is a sufficient condition for univalence of $G$-equivariant Riemann domains over $G^C$ (cf. [CL]).

Let $G$ be a connected, non-compact, real simple Lie group and let $K$ be a maximal compact subgroup of $G$. The group $G$ is not necessarily embedded in $G^C$, but it is assumed to have finite center. Consider the action of the product group $G \times K$ on $G^C$ by left and right translations. One of the results of this paper is the following theorem (Thm. 8.1).

**Theorem.** Let $G/K$ be a non-compact, rank-one, Riemannian symmetric space. A holomorphically separable, $G \times K$-equivariant Riemann domain $p: Y \to G^C$ is univalent, provided that $G$ is not a covering of $SL_2(\mathbb{R})$.

Note that since $Y$ embeds equivariantly into its envelope of holomorphy (cf. [Ro] and Sect. 2), there is no loss of generality in assuming that $Y$ is Stein. Then a result of P. Heinzner ([He1]) implies that the categorical quotient $Y/K$ is also Stein. By performing categorical $K$-reduction on both $Y$ and $G^C$, one can associate to $p: Y \to G^C$ a Stein, $G$-equivariant Riemann domain $q: Y/K \to G^C/K^C$. A suitable characterization of the univalence of $q$ (see Prop. 8.1) implies that $p$ is univalent if $q$ is univalent. Then the above theorem is a consequence of the following one, which is the main result of the paper (Thm. 7.6 and Rem. 7.8).

**Theorem.** A holomorphically separable, $G$-equivariant Riemann domain $q: \Sigma \to G^C/K^C$ is univalent, provided that $G$ is not a covering of $SL_2(\mathbb{R})$.

The proof of this theorem is carried out as follows. First we show that, with few exceptions, the map $q$ is injective on every $G$-orbit. For principal $G$-orbits this is done by determining their topology. The result is then extended to the remaining $G$-orbits by a general argument (Sect. 5). As a consequence there exists a $G$-invariant domain in $\Sigma$ on which $q$ is injective.

Next we show that such domain can be enlarged to the whole $\Sigma$. This is done by successively lifting to $\Sigma$ local slices for principal $G$-orbits in $G^C/K^C$. Since such slices are one-dimensional and $q$ is injective on $G$-orbits, each lifting determines a $G$-invariant domain in $\Sigma$ on which $q$ is injective. The main difficulty is to ensure monodromy around singular $G$-orbits. For this we combine a detailed description of the $G$-orbit structure of $G^C/K^C$ with the complex-geometric properties of certain non-Stein, $G$-invariant domains in $G^C/K^C$.

By the above univalence result, all Stein, $G$-equivariant Riemann domains over $G^C/K^C$ can be regarded as Stein, invariant domains in $G^C/K^C$. We carry out their classification in Theorem 6.1.

For $G/K$ of arbitrary rank, recent investigations due to several authors have indicated an interplay between the complex geometry of distinguished Stein, $G$-invariant domains in $G^C/K^C$ (see [KS], [FHW] and references therein) and the harmonic analysis on $G$-symmetric spaces contained in $G^C/K^C$. Envelopes of holomorphy of $G$-invariant domains in $G^C/K^C$ might give new insights in this context. We hope the present paper to be a first step for further investigations on symmetric spaces of higher rank.

The paper is organized as follows. In Section 2 we recall some basic notions and results from geometric invariant theory. In Section 3, from a Stein $G \times K$-equivariant Riemann domain $p: Y \to G^C$ we obtain a Stein, $G$-equivariant, Riemann domain $q: Y/K \to G^C/K^C$. We also show that $p$ is univalent if $q$ is univalent.

In Section 4 we give a detailed description of the $G$-orbit structure of $G^C/K^C$ when $G/K$ is a non-compact, rank-one, Riemannian symmetric space. We also
describe an explicit model for the space \( G^\mathbb{C}/K^\mathbb{C} \) in the cases \( G = SO_0(n,1) \) and \( G = SU(n,1) \).

In Section 5 we show that, with few exceptions, a \( G \)-equivariant Riemann domain \( q: Y \to G^\mathbb{C}/K^\mathbb{C} \) is univalent on every \( G \)-orbit.

In Section 6 we carry out a complete classification of Stein, \( G \)-invariant domains in \( G^\mathbb{C}/K^\mathbb{C} \). When \( G = SU(n,1) \) some of these domains appear to be new.

In Section 7 we prove the univalence result for holomorphically separable, \( G \)-equivariant Riemann domains over \( G^\mathbb{C}/K^\mathbb{C} \).

In Section 8 we obtain the result for holomorphically separable, \( G \times K \)-equivariant Riemann domains over \( G^\mathbb{C} \). We also discuss some examples.

In the Appendix we compute the Levi form of all non-closed hypersurface \( G \)-orbits in \( G^\mathbb{C}/K^\mathbb{C} \). The results of this computation are used in Sections 6 and 7.

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2. Preliminaries

Let $G$ be a connected, real Lie group. A complex Lie group $G^C$ together with a Lie group homomorphism $\iota: G \to G^C$ is called a universal complexification of $G$ if for every Lie group homomorphism $\psi$ from $G$ to a complex Lie group $H$ there exists a holomorphic homomorphism $\psi^C: G^C \to H$ such that $\psi = \psi^C \circ \iota$. A universal complexification of $G$ always exists and is unique up to biholomorphisms (see [Ho]).

Assume that $G$ is a connected, real semisimple Lie group. Denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{g}^C := \mathfrak{g} \oplus i\mathfrak{g}$ its complexification. Then the universal complexification of $G$ is a complex semisimple Lie group $G^C$ with Lie algebra $\mathfrak{g}^C$. If $G$ is a real form of a simply connected complex semisimple Lie group $G^C$, then its universal complexification is $G^C$. Furthermore, if $\Gamma$ is a central subgroup of $G$, then the universal complexification of the quotient group $G/\Gamma$ is given by $G^C/\Gamma$. Note that every automorphism of $G$ uniquely extends to a holomorphic automorphism of its universal complexification $G^C$.

Let $K$ be a compact Lie group and $X$ a Stein $K$-space, i.e. a reduced Stein space with a real-analytic action of $K$ by holomorphic transformations. The categorical quotient $X//K$ of $X$ is defined by the following equivalence relation: $x \sim y$ if and only if $f(x) = f(y)$ for every $K$-invariant holomorphic function $f$ on $X$.

We recall some basic properties of the categorical quotient (see [He1]).

**Theorem 2.1.** Let $K$ be a compact Lie group and $X$ a Stein $K$-space. Then

(i) the categorical quotient $X//K$ equipped with the algebra $O(X)^K$ of holomorphic $K$-invariant functions on $X$ is a Stein space and the projection $\pi: X \to X//K$ is holomorphic,

(ii) for every $K$-invariant holomorphic map $\psi$ from $X$ to a complex space $Y$ there exists a unique holomorphic map $\hat{\psi}: X//K \to Y$ making the diagram

\[\begin{array}{ccc}
X & \xrightarrow{\pi} & X//K \\
| & \psi \downarrow & \\
Y & & \hat{\psi}
\end{array}\]

commute.

If the $K$-action on $X$ is the restriction of a $K^C$-action, then the algebras of $K$-invariant and of $K^C$-invariant holomorphic functions on $X$ coincide. In particular they induce the same equivalence relation on $X$ and $X//K \cong X//K^C$. In this case, if all $K^C$-orbits are closed, then $X//K^C$ coincides with the usual orbit space $X/K^C$ (cf. [Sn], Thm. 3.8). A $K$-action on a Stein space can always be extended to a $K^C$-action, as shown by the following theorem due to Heinzner ([He1]).

**Theorem 2.2.** Let $K$ be a compact Lie group and $X$ a Stein $K$-space. Then there exist a Stein $K^C$-space $X^*$ and a $K$-equivariant holomorphic embedding $\iota: X \hookrightarrow X^*$ with the following properties:

(i) the map $\iota$ is open and $K^C \cdot \iota(X) = X^*$,

(ii) for every $K$-equivariant holomorphic map $\varphi$ from $X$ into a complex $K^C$-space $Z$, there exists a unique $K^C$-equivariant holomorphic map $\varphi^*: X^* \to Z$ making the
diagram

\[
\begin{array}{ccc}
X & \xymatrix{\subset} & X^* \\
\varphi \downarrow & \nearrow \varphi^* \\
Z & & \\
\end{array}
\]

(iii) the inclusion \(X \hookrightarrow X^*\) induces an isomorphism between the categorical quotients \(X//K\) and \(X^*///K^C\).

Observe that, since \(K^C \cdot \iota(X) = X^*\), if \(X\) is non-singular, then \(X^*\) is also non singular. Let \(X\) be a complex manifold and let \(G\) be a Lie group. A Riemann domain over \(X\) is a complex manifold \(Y\) together with a locally biholomorphic map \(p : Y \to X\). If both \(X\) and \(Y\) are \(G\)-manifolds and the map \(p\) is \(G\)-equivariant, then we refer to \(p : Y \to X\) as a \(G\)-equivariant Riemann domain. If \(X\) is Stein and \(Y\) is holomorphically separable, then \(Y\) embeds as an open domain in its envelope of holomorphy \(\hat{Y}\) and the map \(p\) extends to a local biholomorphism \(\hat{p} : \hat{Y} \to X\) (see \[Ro\]). Moreover the \(G\)-action on \(Y\) extends to \(\hat{Y}\) and the map \(\hat{p}\) is \(G\)-equivariant, i.e. \(\hat{p} : \hat{Y} \to X\) is a Stein, \(G\)-equivariant Riemann domain. A Riemann domain \(p : Y \to X\) is called univalent if the map \(p\) is injective. Assume that \(X\) is Stein and \(Y\) is holomorphically separable. If \(\hat{p}\) is univalent, then \(p\) is also univalent. Aiming at univalence results for holomorphically separable Riemann domains over \(G^C\), it is therefore not restrictive to start with Riemann domains which are Stein.

## 3. From Riemann domains over \(G^C\) to Riemann domains over \(G^C/K^C\)

Let \(G\) be a connected, non-compact, real semisimple Lie group, \(K \subset G\) a maximal compact subgroup and \(G^C\) the universal complexification of \(G\). Let \(G \times K\) act on \(G^C\) by left and right translations, i.e.

\[
(g,k) \cdot z := gzk^{-1}, \quad \text{for} \quad (g,k) \in G \times K, \quad z \in G^C.
\]

In this section, to every Stein, \(G \times K\)-equivariant Riemann domain \(p : Y \to G^C\) we associate a Stein, \(G\)-equivariant Riemann domain \(q : \Sigma \to G^C/K^C\). We also show that the univalence of \(q\) implies that of \(p\).

Let \(X\) be a Stein \(K^C\)-manifold and let \(p : Y \to X\) be a Stein, \(K\)-equivariant Riemann domain. By Theorem 2.2 there exist a Stein \(K^C\)-manifold \(Y^*\), a \(K\)-equivariant holomorphic open embedding \(\iota : Y \hookrightarrow Y^*\) and a \(K^C\)-equivariant holomorphic map \(p^* : Y^* \to X\) such that the diagram

\[
\begin{array}{ccc}
Y & \xymatrix{\subset} & Y^* \\
p \downarrow & \nearrow \iota & \\
X & & \\
\end{array}
\]

commutes. Since \(p\) is locally biholomorphic, \(p^*\) is \(K^C\)-equivariant and \(Y^* = K^C\cdot Y\), one has that \(p^*\) is locally biholomorphic as well, i.e. it defines a Stein \(K^C\)-equivariant Riemann domain. By Theorem 2.2 the spaces \(Y^*//K^C\) and \(Y//=K\) are biholomorphic. Therefore Theorem 2.1 implies there exists a holomorphic map \(q : Y//=K \to X//=K^C\) making the diagram
commute. Here the horizontal arrows denote the categorical quotient maps.

Assume that all $K^C$-orbits in $X$ are closed. We claim that all $K^C$-orbits in $Y^*$ are closed as well. Suppose by contradiction that there exists a non closed orbit $K^C \cdot y$ in $Y^*$. Let $K^C \cdot z$ be a lower dimensional orbit in its closure (see [Sn], Prop. 2.3). Since $p^*$ is locally biholomorphic and $K^C$-equivariant, the orbit $K^C \cdot p^*(z)$ lies in the closure of $K^C \cdot p^*(y)$ and has lower dimension. In particular such orbits are distinct. It follows that the orbit $K^C \cdot p^*(y)$ is not closed, contradicting the assumption.

By the above claim, the categorical quotients $X/K^C$ and $Y^*/K^C$ coincide with the orbit spaces $X/K^C$ and $Y^*/K^C$, respectively. As a consequence, the map

$$q: Y//K \to X/K^C$$

is locally biholomorphic, i.e. it defines a Stein Riemann domain. We refer to it as the Riemann domain induced by $p: Y \to X$. Next we prove a general univalence result for Stein, $K$-equivariant Riemann domains.

**Proposition 3.1.** Let $X$ be a Stein $K^C$-manifold, all of whose $K^C$-orbits are closed and have connected isotropy subgroups. Let $p: Y \to X$ be a Stein, $K$-equivariant Riemann domain and $p^*: Y^* \to X$ its extension to the $K^C$-globalization $Y^*$ of $Y$. Then

(i) the induced Stein, Riemann domain $q: Y//K \to X/K^C$ is univalent if and only if $p^*: Y^* \to X$ is univalent,

(ii) if $q: Y//K \to X/K^C$ is univalent, then $p: Y \to X$ is univalent.

**Proof.** (i) If $p^*$ is injective, then it maps distinct $K^C$-orbits in $Y^*$ onto distinct $K^C$-orbits in $X$. As we already noticed, since all $K^C$-orbits in $X$ are closed, the categorical quotients $X//K^C$ and $Y^*/K^C$ coincide with the orbit spaces $X/K^C$ and $Y^*/K^C$, respectively. It follows that the induced map $Y^*/K^C \to X/K^C$ is injective. Moreover, by Theorem 2.2 the space $Y//K$ can be identified with $Y^*/K^C$. As a result the induced Riemann domain $q: Y//K \to X/K^C$ is univalent.

Conversely, assume that $q: Y//K \to X/K^C$ is univalent, i.e. that the map $Y^*/K^C \to X/K^C$ is injective. By assumption the $K^C$-isotropy subgroups in $X$ are connected, thus $p^*$ is injective on every $K^C$-orbit in $Y^*$. It follows that $p^*: Y^* \to X$ is globally injective. This concludes the proof of (i). Statement (ii) is a direct consequence of (i). \qed

**Remark 3.2.** In general, under the assumptions of the above proposition, the univalence of $p: Y \to X$ does not imply that of $q: Y//K \to X/K^C$. For instance, let $C^*$ act on $\mathbb{C} \times C^*$ and on $X := C^* \times C^*$ by multiplication on the second factor. Define $p^*: \mathbb{C} \times C^* \to X$ by $(z,w) \to (e^z,w)$ and consider

$$Y := \{ (z,w) \in \mathbb{C} \times C^* : \text{Im } z < |w| < 2\pi + \text{Im } z \}.$$ 

Then $Y$ is a Stein $S^1$-invariant subdomain of $Y^* = \mathbb{C} \times C^*$ and the map $p := p^*|_Y$ is injective. Nevertheless the induced map $q: Y//S^1 \cong \mathbb{C} \to X/C^* \cong C^*$, given by $z \to e^z$, is not injective. \qed
Consider now the case when \( X \) is the group \( G^C \) endowed with the \( G \times K \)-action by left and right translations. Let \( p: Y \to G^C \) be a Stein, \( G \times K \)-equivariant Riemann domain. Note that the actions of \( G \) and \( K \) commute on \( G^C \). Thus they also commute on \( Y \), due to the fact that \( p \) is equivariant and locally injective. Since the \( K \)-action on \( G^C \) is the restriction of a \( K^C \)-action all of whose orbits are closed, the spaces \( G^C // K \) and \( G^C / K^C \) are biholomorphic.

By the universality property of the categorical quotient (cf. Theorem 2.1), the \( G \)-actions on \( Y \) and on \( G^C \) induce \( G \)-actions on \( Y//K \) and on \( G^C / K^C \), respectively. Moreover the induced Stein, Riemann domain

\[ q: Y//K \to G^C / K^C \]

is \( G \)-equivariant. By applying Proposition 3.1 to this situation, one obtains the following result.

**Corollary 3.3.** Let \( p: Y \to G^C \) be a Stein, \( G \times K \)-equivariant Riemann domain over \( G^C \) and let \( q: Y//K \to G^C / K^C \) be the induced Stein, \( G \)-equivariant Riemann domain over \( G^C / K^C \). If \( q \) is univalent, then \( p \) is univalent.

4. **\( G \)-Orbit Structure of \( G^C / K^C \)**

Let \( G \) be a connected, non-compact, real simple Lie group, \( K \subset G \) a maximal compact subgroup and \( G^C \) the universal complexification of \( G \). Assume that \( G \) is embedded in \( G^C \). The quotient \( G/K \) is a Riemannian symmetric space of the non-compact type. In this section we obtain a complete description of the \( G \)-orbit structure of \( G^C / K^C \) in the case when \( G/K \) has rank one.

We recall some basic facts which hold for \( G/K \) of arbitrary rank. Let \( \sigma \) denote the anti-holomorphic involution of \( G^C \) relative to \( G \) and \( \tau: G^C \to G^C \) the holomorphic extension of the Cartan involution \( \theta \) of \( G \) with respect to \( K \). Note that the fixed point set of \( \tau \) in \( G^C \) contains the complexification \( K^C \) of \( K \). The commuting composition \( \sigma \circ \tau = \tau \circ \sigma \) is a Cartan involution of \( G^C \). Denote by \( U \) the corresponding compact real form. The \( U \)-orbit of the base point \( eK^C \) in \( G^C / K^C \) is diffeomorphic to the compact dual symmetric space \( U/K \), and is embedded in \( G^C / K^C \) transversally to \( G/K \).

**Remark 4.1.**

(i) For every triple \( (G, K, G^C) \) as above, the manifold \( G^C / K^C \) is simply connected. To see this, denote by \( \tilde{G}^C \) and \( \tilde{U} \subset \tilde{G}^C \) the universal coverings of \( G^C \) and \( U \) respectively. Let \( \tilde{G} \) be the real form of \( \tilde{G}^C \) relative to the lifting of \( \sigma \) to \( \tilde{G}^C \). The group \( \tilde{G} \) is connected (cf. [St]) and is a finite covering of \( \tilde{G}^C \). Hence \( \tilde{G} = \tilde{G} / \Gamma \), where \( \Gamma \) is a finite central subgroup of \( \tilde{G} \). Similarly \( \tilde{K} = \tilde{K} / \Gamma \), where \( \tilde{K} \) is a maximal compact subgroup of \( \tilde{G} \). One has \( \tilde{G}^C \cong \tilde{G}^C / \Gamma \) (cf. Sect. 2) and consequently \( \tilde{U} = \tilde{U} / \Gamma \). As a consequence there are isomorphisms

\[ U/K \cong \tilde{U} / \Gamma / \tilde{K} / \Gamma \cong \tilde{U} / \tilde{K} \]

Since \( \tilde{K} \) is connected, the quotient \( \tilde{U} / \tilde{K} \) is simply connected. Moreover \( U/K \) is a topological retract of \( G^C / K^C \). Hence the claim follows.

(ii) From different triples \( (G, K, G^C) \) as above associated with the same Riemannian symmetric space one obtains the same complexification \( G^C / K^C \). Indeed the map \( \tilde{G}^C / \tilde{K}^C \to G^C / K^C \), given by \( \gamma \tilde{K}^C \to \gamma \tilde{K}^C \), defines a biholomorphism. Moreover the center of \( G \) acts trivially on \( G^C / K^C \). As a consequence, different triples \( (G, K, G^C) \) yield the same \( G \)-orbit structure of \( G^C / K^C \) and \( G \)-equivariantly diffeomorphic orbits. \( \square \)
Closed $G$-orbits of maximal dimension form an open dense subset of $G^C/K^C$ and come in a finite number of orbit types. We refer to them as principal $G$-orbits. They have real codimension equal to the rank of $G/K$. Singular orbits are closed $G$-orbits which are not principal.

The $G$-orbit structure of $G^C/K^C$ is closely related to the $G \times K^C$-orbit structure of $G^C$. Then, slices for the closed $G$-orbits in $G^C/K^C$ can be obtained by applying Matsuki’s results on double coset decompositions of groups arising from two involutions ([Ma], Sect. 4).

Let $\mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $K$ and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Following Matsuki, we denote by

$$\mathcal{A} := \exp i \mathfrak{a} K^C$$

the image of the compact torus $\exp i \mathfrak{a}$ in $G^C/K^C$. The set $\mathcal{A}$ is a slice for all closed $G$-orbits intersecting the compact dual symmetric space $U/K$ in $G^C/K^C$. It is called the fundamental Cartan subset. The remaining slices for closed $G$-orbits in $G^C/K^C$ are of the form

$$\mathcal{C} := \exp i \mathfrak{c} \cdot z,$$

where $\mathfrak{c}$ is an abelian semisimple subalgebra of $\mathfrak{g}$ of the same dimension as $\mathfrak{a}$ and $z \in \mathcal{A}$ is a base point sitting on a singular closed $G$-orbit. Such sets $\mathcal{C}$ are called standard Cartan subsets.

By [Ge2], every standard Cartan subset $\mathcal{C}$ admits a base point $z$ with the following properties:

- there exists a subgroup $G' \subseteq G$ (possibly $G'$ is equal to $G$) such that the isotropy subgroup of $z$ in $G'$ coincides with the isotropy subgroup $G_z$ of $z$ in $G$.
- the quotient $G'/G_z$ is a pseudo-Riemannian symmetric space of the same rank as $G/K$.
- the slice representation of $G_z$ at $z$ is equivalent to the isotropy representation of $G'/G_z$.

More precisely, let $\mathfrak{g}' = \mathfrak{g}_z \oplus \mathfrak{q}'$ be the decomposition of the Lie algebra of $G'$ corresponding to the symmetric space $G'/G_z$ (when $G' = G$, one has $\mathfrak{g} = \mathfrak{g}_z \oplus \mathfrak{q}$). Denote by $T(G \cdot z)_z$ the tangent space of the orbit $G \cdot z$ at $z$ and by $N_z$ a complementary subspace of $T(G \cdot z)_z$ in $T(G^C/K^C)_z$. Then $N_z \cong \mathfrak{q}'$ and the slice representation at $z$ is equivalent to the Adjoint representation of $G_z$ on $\mathfrak{q}'$. Moreover, both $\mathfrak{a}$ and $\mathfrak{c}$ are maximal abelian subalgebras in $\mathfrak{q}'$.

Consider the twisted bundle $G \times_{G_z} \mathfrak{q}'$ defined as the orbit space of $G \times \mathfrak{q}'$ under the action of $G_z$ given by $h \cdot (g, X) := (gh^{-1}, Ad_hX)$. The group $G$ acts on $G \times_{G_z} \mathfrak{q}'$ by $\tilde{g} \cdot [g, X] := [\tilde{g} g, X]$. By Luna’s slice Theorem ([Lu], Prop. 1.2), there exists an open $Ad_{G_z}$-invariant neighborhood $V$ of $0$ in $\mathfrak{q}'$ such that the map

$$G \times_{G_z} V \to G^C/K^C, \quad [g, X] \to g \exp i X \cdot z$$

is a $G$-equivariant diffeomorphism onto an open $G$-invariant neighborhood of $z$ in $G^C/K^C$. Non-closed $G$-orbits in $G \times_{G_z} V$ correspond to non-closed $Ad_{G_z}$-orbits in $V$. The standard Cartan subset $\mathcal{C}$ in $G^C/K^C$ is the image of the set $\{e\} \times \mathfrak{c}$ via the above map.

Let us now assume that $G/K$ has rank one. Then the $G$-orbit space of $G^C/K^C$ can be completely determined. Let $\Delta_\mathfrak{a}$ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and let

$$\mathfrak{g} = Z_\mathfrak{g}(\mathfrak{a}) \bigoplus_{\alpha \in \Delta_\mathfrak{a}} \mathfrak{g}^\alpha, \quad \text{with} \quad Z_\mathfrak{g}(\mathfrak{a}) = Z_{\mathfrak{e}}(\mathfrak{a}) \oplus \mathfrak{a},$$

be the corresponding root decomposition. Here $Z_\mathfrak{g}(\mathfrak{a})$ and $Z_{\mathfrak{e}}(\mathfrak{a})$ denote the centralizers of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{e}$, respectively. Let $\Gamma$ be the lattice in $\mathfrak{a}$ given by the kernel of
the map $a \to U/K$ defined by $X \to \exp(iX)K$. Since the symmetric space $U/K$ is simply connected (cf. Remark [I]), the lattice $\Gamma$ is given by
\[ \Gamma = \bigoplus_{\alpha \in \Delta} \mathbb{Z} \pi h_\alpha, \]
where $h_\alpha \in a$ is uniquely determined by $\alpha(h_\alpha) = 2$ (cf. [HI], Thm. 8.5, p. 322). Denote by $W_K(a)$ the Weyl group of $a$ and let the semidirect product $W_K(a) \ltimes \Gamma$ act on $a$ by
\[(k, \gamma) \cdot A := \text{Ad}_k A + \gamma.\]
Denote by $a_0$ a fundamental domain for this action and define $A_0 := \exp i a_0 K^C$. Then every closed $G$-orbit through the fundamental Cartan subset $A$ intersects $A_0$ in a single point (cf. [Ma], Thm. 3).

Let $z \in A_0$ be a base point for a standard Cartan subset $C$. By [Ge2] and by the local linearization (1), the $G$-orbit structure of $G^C/K^C$ in a neighbourhood of $z$ is modelled on the orbit structure of the tangent space of a rank-one, pseudo-Riemannian symmetric space under the isotropy representation. It can be described as follows.

**Remark 4.2.** Let $G/H$ a rank-one, pseudo-Riemann symmetric space. Assume that the group $H$ is connected. Let $g = h \oplus q$ be the corresponding Lie algebra decomposition and $q \cap t \oplus q \cap \mathfrak{p}$ the Cartan decomposition of $q$. The isotropy representation of $G/H$ is equivalent to the Adjoint representation of $H$ on $q$. Denote by $B$ both the Killing form of $g$ and its restriction to $q \setminus \{0\}$. The signature of $B$ on $q$ is given by $(s^+, s^-)$, with
\[ s^+ := \dim(q \cap \mathfrak{p}) \quad s^- := \dim(q \cap t). \]
For $r \in \mathbb{R}$, denote by $B_r$ the level hypersurface $\{B = r\}$ in $q \setminus \{0\}$. In diagonalized form one has
\[ B_r = \{x_1^2 + \cdots + x_s^2 - y_1^2 - \cdots - y_s^2 = r\}. \]
Since $G/K$ has rank one, every $\text{Ad}_H$-orbit in $q \setminus \{0\}$ is a hypersurface. Thus, by the connectedness of $H$ and that $\text{Ad}_H$-invariance of $B$, it coincides with a connected component of some $B_r$. We distinguish four cases.

(a) Assume $s^+ = s^- = 1$. For every $r \neq 0$ the level set $B_r$ consists of two connected components. They intersect either $a = q \cap \mathfrak{p}$ or $\mathfrak{c} = q \cap \mathfrak{t}$ in opposite points, depending on whether $r > 0$ or $r < 0$. The nilcone $B_0$ consists of four non-closed $\text{Ad}_H$-orbits.

(b) Assume $s^+ > 1$ and $s^- = 1$. For $r > 0$ the level set $B_r$ consists of a single component intersecting $q \cap \mathfrak{p}$ in a sphere. Thus for every non-zero vector $A \in q \cap \mathfrak{p}$, and every $t > 0$, the points $tA$ and $-tA$ belong to the same $\text{Ad}_H$-orbit. If $r < 0$ the level set $B_r$ consists of two connected components, which intersect $\mathfrak{c} = q \cap \mathfrak{t}$ in opposite points. The nilcone $B_0$ consists of two non-closed $\text{Ad}_H$-orbits.

(b) Assume $s^+ = 1$ and $s^- > 1$. If $r > 0$, the level set $B_r$ consists of two connected components, which intersect $a = q \cap \mathfrak{p}$ in opposite points. If $r < 0$, the level set $B_r$ intersects $q \cap \mathfrak{t}$ in a sphere. Thus for every non-zero vector $C \in q \cap \mathfrak{t}$ and every $s > 0$, the points $sC$ and $-sC$ belong to the same $\text{Ad}_H$-orbit. The nilcone $B_0$ consists of two non-closed $\text{Ad}_H$-orbits.

(d) Assume $s^+ > 1$ and $s^- > 1$. For every $r \neq 0$ the level set $B_r$ consists of a single connected component. It intersects either $q \cap \mathfrak{p}$ or $q \cap \mathfrak{t}$ in a sphere, depending on whether $r > 0$ or $r < 0$. Thus for every non-zero vector $A \in q \cap \mathfrak{p}$ and every $t > 0$, the points $tA$ and $-tA$ belong to the same $\text{Ad}_H$-orbit. A similar statement holds true for points $sC$ and $-sC$. 

with \( C \) a non-zero vector in \( q \cap k \) and \( s > 0 \). The nilcone \( B_0 \) consists of a unique non-closed \( Ad_H \)-orbit.

In order to give further details, we recall the classification of rank-one, Riemannian symmetric spaces of the non-compact type. For each space \( M \) we list its real dimension, its standard presentation \( G/K \), and the dimensions of the restricted roots spaces of \( g \) (cf. [Wo], p. 294 and [H], p. 532).

**Table 4.0**

| \( M \) | \( \dim M \) | \( G/K \) | \( \dim g^\alpha \) | \( \dim g^{2\alpha} \) |
|---|---|---|---|---|
| \( H^n(\mathbb{R}) \) | \( n \) | \( SO_0(n,1)/SO(n), \ n \geq 2 \) | \( n-1 \) | \( 0 \) |
| \( H^n(\mathbb{C}) \) | \( 2n \) | \( SU(n,1)/U(n), \ n \geq 2 \) | \( 2(n-1) \) | \( 1 \) |
| \( H^n(\mathbb{H}) \) | \( 4n \) | \( Sp(n,1)/Sp(n) \times Sp(1), \ n \geq 2 \) | \( 4(n-1) \) | \( 3 \) |
| \( H^2(\text{Cay}) \) | \( 16 \) | \( F_4^*/\text{Spin}(9) \) | \( 8 \) | \( 7 \) |

**Remark.** The two dimensional symmetric space \( S_0(2,1)/SO(2) \) can also be identified with \( SU(1,1)/U(1) \) or \( SL(2,\mathbb{R})/SO(2) \). The symmetric space \( S_0(3,1)/SO(3) \) can be identified with \( SL(2,\mathbb{C})/SU(2) \).

### 4.1 The reduced case

Assume that the restricted root system of \( g \) is reduced, i.e. it consists of two roots \( \{ \pm \alpha \} \). This is the case of the spaces \( H^n(\mathbb{R}) \) in Table 4.0. A fundamental domain for the action of \( W_K(a) \times \Gamma \) on \( a \) is given by \( a_0 = \{ A \in a : 0 \leq \alpha(A) \leq \pi \} \) and there are three singular orbits intersecting \( A_0 := \exp ia_0 K^C \). Their base points are given by \( z_j = g_j k^C \), for \( j = 1, 2, 3 \). Here \( g_j = \exp iA_j \) and the elements \( A_j \in a_0 \) satisfy the conditions

\[
\alpha(A_1) = 0, \quad \alpha(A_2) = \pi/2, \quad \alpha(A_3) = \pi,
\]

respectively. The \( G \)-orbits through \( z_1 \) and \( z_3 \) are diffeomorphic to the symmetric space \( G/K \) and are embedded in \( G^C/K^C \) as totally real submanifolds of maximal dimension. Moreover, the \( G \)-orbit through \( z_2 \) is a rank-one, pseudo-Riemannian symmetric space \( G/H \) with involution \( \tau_{z_2} = Ad_{g_2} \circ \tau \circ Ad_{g_2}^{-1} \). The space \( G/H \) is embedded in \( G^C/K^C \) as a closed, totally real submanifold of maximal dimension (see Lemma 2.11 and Rem. 2.13 in [Ge1]). A standard Cartan subset starting at \( z_2 \) is given by \( C = \exp i\cdot z_2 \), where \( c = \mathbb{R}(X + \theta(X)) \) and \( X \) is a non-zero vector in \( g^\alpha \). In the next lemma we determine the \( G \)-orbit structure of \( G^C/K^C \) in a neighbourhood of \( z_2 \). Fix a generator \( C \) of \( c \).

**Lemma 4.3.** Assume that the restricted root system of \( g \) is reduced. Let \( z_2 \in A_0 \) be the base point of the Cartan subset \( C \).

(i) If \( \dim G/K > 2 \), then the orbit \( G \cdot z_2 \) is simply connected. In particular, the isotropy subgroup \( H \) of \( z_2 \) in \( G \) is connected.

(ii) For every \( s > 0 \), the points \( \exp(isC) \cdot z_2 \) and \( \exp(-isC) \cdot z_2 \) lie on the same \( G \)-orbit in \( G^C/K^C \) if and only if \( \dim g^\alpha > 1 \).

(iii) If \( \dim g^\alpha > 1 \), there are two non-closed \( G \)-orbits in \( G^C/K^C \) containing \( G \cdot z_2 \) in their closure. If \( \dim g^\alpha = 1 \), such orbits are four.
Proof. (i) Using the hyperquadric model (cf. Example 4.4), one can verify that the orbit of $z_2$ is diffeomorphic to $SO_0(n, 1)/SO_0(n-1, 1)$. In particular, it is topologically equivalent to a sphere of dimension $n-1$ and is simply connected for $n > 2$. In that case, the isotropy subgroup $H$ is connected, since $G$ is connected by assumption. When $n = 2$, the orbit $G/H$ is not simply connected. The isotropy subgroup of $z_2$ is either connected (when $G = SO_0(2, 1)$) or its quotient by the ineffectivity subgroup is connected (when $G$ is a non trivial covering of $SO_0(2, 1)$).

As a consequence, (ii) and (iii) of the lemma follow from Remark 4.2 provided that $\dim(q \cap p) = 1$ and $\dim(q \cap \mathfrak{t}) = \dim g^\alpha$. In order to show this, define $g[\alpha] := g^\alpha \oplus g^{-\alpha}$. Then $g[\alpha]$ is a $\theta$-stable subspace of $g$ of dimension equal to $2\dim g^\alpha$. Let $g[\alpha] = g[\alpha]_I \oplus g[\alpha]_p$ be its Cartan decomposition. The components $g[\alpha]_I$ and $g[\alpha]_p$ are generated by vectors of the form

$$X + \theta(X) \quad \text{and} \quad X - \theta(X)$$

respectively, where $X$ ranges through the elements of a basis of $g^\alpha$. In particular $\dim g[\alpha]_I = \dim g[\alpha]_p = \dim g^\alpha$. Consider the decomposition $g = Z_{\mathfrak{t}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus g[\alpha]$ and note that $\tau_{z_2} = Ad_{\mathfrak{g}_2} \circ \tau \circ Ad_{\mathfrak{g}_2}^{-1} = Ad_{\mathfrak{g}_2} \circ \theta$. Since $Ad_{\exp iA_2} = e^{ad(iA_2)}$, one has

$$\tau_{z_2} = Id \quad \text{on} \quad Z_{\mathfrak{t}}(\mathfrak{a}), \quad \tau_{z_2} = -Id \quad \text{on} \quad \mathfrak{a}.$$ 

Since $\alpha(A_2) = \pi/2$, one has $\tau_{z_2} = -\theta$ on $g[\alpha]$. It follows that $q := Fix(-\tau_{z_2}, g) = \mathfrak{a} \oplus g[\alpha]_I$. In particular, $\dim(q \cap p) = \dim \mathfrak{a} = 1$ and $\dim(q \cap \mathfrak{t}) = \dim g[\alpha]_I = \dim g^\alpha$, as wished. \hfill \Box

From the above discussion and Table 4.0 it follows that in the reduced case the $G$-orbit space of $G^C/K^C$ can be described by the following diagrams.

1. $G/K = SO_0(2, 1)/SO(2)$.

$$\begin{array}{c}
\vskip.3cm
\begin{array}{c}
\ell_2(I_2)
\end{array}
\vskip.3cm
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\begin{array}{c}
\ell_4(I_4)
\end{array}
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\begin{array}{c}
\ell_1(I_1)
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\begin{array}{c}
\bullet\bullet\bullet
\end{array}
\end{array}$$

2. $G/K = SO_0(n, 1)/SO(n), \; n > 2$.

$$\begin{array}{c}
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\begin{array}{c}
\ell_2(I_2)
\end{array}
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\begin{array}{c}
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\begin{array}{c}
\ell_1(I_1)
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\end{array}$$

Set $I_1 = I_3 = (0, 1)$. For $j = 1, 3$, the maps $\ell_j : I_j \to G^C/K^C$, defined by

$$\ell_j(t) := \exp(-itA_2) \cdot z_j, \quad \ell_3(t) := \exp(itA_2) \cdot z_2,$$
parametrize the principal $G$-orbits through $\mathcal{A}_0$. One has
\[ \mathcal{A}_0 = z_1 \cup \ell_1(I_1) \cup z_2 \cup \ell_3(I_3) \cup z_3. \]
Set $I_2 = I_4 = (0, \infty)$. For $j = 2, 4$, the maps $\ell_j : I_j \to G^C/K^C$, defined by
\[ \ell_2(s) := \exp(isC) \cdot z_2, \quad \ell_4(s) := \exp(-isC) \cdot z_2, \]
parametrize the principal closed $G$-orbits through the standard Cartan subset $C$ and
\[ C = \ell_2(I_2) \cup z_2 \cup \ell_4(I_4). \]
The points $w_1, w_2, w_3, w_4$ represent the non-closed $G$-orbits containing the singular orbit $G \cdot z_2$ in their closure.

**Example 4.4.** The complex hyperquadric. Let $G = SO_0(n, 1)$, with $n \geq 2$, and let $G^C = SO(n, 1, \mathbb{C})$ be its universal complexification. By definition $G^C$ is the subgroup of $SL(n + 1, \mathbb{C})$ leaving invariant the quadratic form of signature $(n, 1)$. The space $G^C/K^C$ can be identified with the $G^C$-orbit through $(0, \ldots, 1)$ which coincides with the $n$-dimensional complex hyperquadric
\[ M^C = \{ (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{C}^{n+1} : \xi_1^2 + \cdots + \xi_n^2 - \xi_{n+1}^2 = -1 \}. \]
Fix the elements
\[ A_2 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\pi}{2} \\ 0 & \cdots & \frac{\pi}{2} & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -2 & 0 \\ 0 & \cdots & 2 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \]
in $\mathfrak{g}$ as generators of $\mathfrak{a}$ and $\mathfrak{c}$, respectively. Then points on the singular orbits in $M^C$ satisfying conditions [2] are given by
\[ z_1 = (0, \ldots, 0, 1), \quad z_2 = (0, \ldots, 0, i, 0), \quad z_3 = (0, \ldots, 0, -1). \]
The $G$-orbit of $z_2$ is diffeomorphic to the pseudo-Riemannian symmetric space $G/H \cong SO_0(n, 1)/SO_0(n - 1, 1)$. The slices $\ell_1$ and $\ell_3$ are given by
\begin{align*}
\ell_1(t) &= (0, \ldots, 0, i \sin \frac{\pi}{2}(1-t), \cos \frac{\pi}{2}(1-t)), \quad t \in (0, 1), \\
\ell_3(t) &= (0, \ldots, 0, i \sin \frac{\pi}{2}(1+t), \cos \frac{\pi}{2}(1+t)), \quad t \in (0, 1).
\end{align*}
The slices $\ell_2$ and $\ell_4$ are given by
\begin{align*}
\ell_2(s) &= (0, \ldots, 0, \sinh 2s, i \cosh 2s, 0), \quad s > 0, \\
\ell_4(s) &= (0, \ldots, 0, -\sinh 2s, i \cosh 2s, 0), \quad s > 0.
\end{align*}
The slice representation at $z_2$ is equivalent to the linear action of $SO_0(n - 1, 1)$ on $\mathbb{R}^n$. When $n = 2$, we can choose representatives of the four non-closed hypersurface $G$-orbits containing $G \cdot z_2$ in their closure to be
\[ w_1 = (-1, i, -1), \quad w_2 = (1, i, -1), \quad w_3 = (1, i, 1), \quad w_4 = (-1, i, 1). \]
When $n > 2$, the slice representation identifies $\ell_2$ and $\ell_4$ and representatives of the two non-closed hypersurface $G$-orbits containing $G \cdot z_2$ in their closure are for example
\[ w_1 = (-1, 0, \ldots, 0, i, -1) \quad \text{and} \quad w_2 = (1, 0, \ldots, 0, i, -1). \]
4.2 The non-reduced case.

Assume that the restricted root system of \( \mathfrak{g} \) is non-reduced, i.e. it consists of four roots \( \{ \pm \alpha, \pm 2\alpha \} \). This is the case of \( H^n(C), H^n(H) \) and \( H^2(Cay) \) in Table 4.0. A fundamental domain for the action of \( W_K(a) \rtimes \Gamma \) in \( a \) is given by \( a_0 = \{ A \in a : 0 \leq \alpha(A) \leq \pi/2 \} \), and there are three singular orbits intersecting \( \mathcal{A}_0 \). Their base points are given by \( z_j = g_jK^C \), for \( j = 1, 2, 3 \). Here \( g_j = \exp iA_j \) and the elements \( A_j \in a_0 \) satisfy the conditions
\[
\alpha(A_1) = 0, \quad \alpha(A_2) = \pi/4, \quad \alpha(A_3) = \pi/2, \tag{7}
\]
respectively. The \( G \)-orbits through \( z_1 \) is isomorphic to the symmetric space \( G/K \), the one through \( z_3 \) is isomorphic to a rank-one, pseudo-Riemannian symmetric space \( G/H \). Both orbits are embedded in \( G^C/K^C \) as totally real submanifolds of maximal dimension. (see Lemma 2.11 and Rem. 2.13 in [Ge1]). The orbit of \( z_2 \) is a homogeneous space \( G/H' \), with \( H' := G_{z_2} \), and \( \dim G/H' > \dim G/K \) (see Lemma 2.14 and Rem. 2.15 in [Ge1]). Set \( G' := Z_G(g_2^2) \), where \( Z_G(g_2^2) \) denotes the centralizer of \( g_2^2 \) in \( G \). Then \( H' \) is contained in \( G' \) and \( G'/H' \) is a rank-one, pseudo-Riemannian symmetric space with involution \( \tau_{z_2} = Ad_{g_2} \circ \tau \circ Ad_{g_2^{-1}} \).

Moreover, the slice representation at \( z_2 \) is equivalent to the isotropy representation of \( G'/H' \) (cf. [Ge2]). The standard Cartan subset starting at \( z_2 \) is given by \( C' = \exp i\epsilon \cdot z_2 \), where \( \epsilon = R(X + \theta(X)) \) and \( X \) is a non-zero vector in \( g^{2\alpha} \). If \( Z_\epsilon(a) \oplus a \oplus g^{\pm \alpha} \oplus g^{\pm 2\alpha} \) is the restricted root decomposition of \( \mathfrak{g} \), then the Lie algebra of \( G' \) is given by
\[
\mathfrak{g}' = Z_\epsilon(a) \oplus a \oplus g^{\pm 2\alpha}. \tag{8}
\]
Moreover, if \( \mathfrak{h}' \oplus \mathfrak{q}' \) is the \( \tau_{z_2} \)-decomposition of \( \mathfrak{g}' \), then \( \epsilon' \) is a maximal abelian subalgebra in \( \mathfrak{q}' \). Fix a generator \( C' \) of \( \epsilon' \).

Lemma 4.5. Assume that the restricted root system of \( \mathfrak{g} \) is non-reduced. Let \( z_2 \in \mathcal{A}_0 \) be the base point of the Cartan subset \( C' \).

(i) The isotropy subgroup \( H' \) of \( z_2 \) in \( G \) is connected.

(ii) For every \( t > 0 \), the points \( \exp(itC') \cdot z_2 \) and \( \exp(-itC') \cdot z_2 \) sit on the same \( G \)-orbit if and only if \( \dim g^{2\alpha} > 1 \).

(iii) If \( \dim g^{2\alpha} > 1 \), there are two non-closed \( G \)-orbits in \( G^C/K^C \) containing \( G \cdot z_2 \) in their closure. If \( \dim g^{2\alpha} = 1 \), such orbits are four.

Proof. (i) The group \( H' \) is connected if and only if \( H' \cap K \) is connected. Note that \( G' = Z_G(g_2^2) \) is \( \theta \)-stable, since so is \( G \) and \( \theta(g_2^2) = g_2^{-4} \). Therefore \( H' \cap K \) is the common fixed point subgroup of the two involutions \( \tau_{z_2} \) and \( \theta \) of \( G' \). As a result, \( H' \cap K = Z_K(g_2^2) \). Now regard \( z_2 \) as a point on the compact dual symmetric space \( U/K \) endowed with the \( K \)-action by left translations. Denote by \( K_{z_2} \) the isotropy subgroup of \( z_2 \) in \( K \). On the one hand, \( K_{z_2} = Z_K(g_2^2) \). On the other hand, since the isotropy subalgebra \( \mathfrak{t}_{z_2} \) is given by \( \mathfrak{t} \cap Ad_{z_2}(\mathfrak{t}) \), one sees that \( \mathfrak{t}_{z_2} \) has minimal dimension and coincides with \( Z_\epsilon(a) \) if and only if \( \alpha(A_2) \neq m\pi \), for \( m \in \mathbb{Z} \). By (7), it follows that \( K_{z_2} \) is principal and consequently is equal to \( Z_K(a) \). Finally \( Z_K(a) \) is connected for all rank-one, Riemannian symmetric spaces of dimension greater than two (see [Kn] or Lemma 5.1 for a direct proof). In conclusion
\[
H' \cap K = Z_K(g_2^2) = K_{z_2} = Z_K(a)
\]
implies (i).

Parts (ii) and (iii) of the lemma follow by applying Remark 4.2 to the symmetric space \( G'/H' \), provided that
\[
\dim \mathfrak{q}' \cap \mathfrak{p} = 1, \quad \dim \mathfrak{q}' \cap \mathfrak{t} = \dim \mathfrak{g}^{2\alpha}.
\]
In order to show this, define \( g[2\alpha] := g^{2\alpha} \oplus g^{-2\alpha} \). Then \( g[2\alpha] \) is \( \theta \)-stable subspace of \( g \) of dimension equal to \( 2 \dim g^{2\alpha} \). Let \( g[2\alpha] = g[2\alpha]_t \oplus g[2\alpha]_p \) be its Cartan decomposition. The components \( g[2\alpha]_t \) and \( g[2\alpha]_p \) are generated by vectors of the form

\[ X + \theta(X) \quad \text{and} \quad X - \theta(X) \]

respectively, where \( X \) ranges through the elements of a basis of \( g^{2\alpha} \). In particular \( \dim g[2\alpha]_t = \dim g[2\alpha]_p = \dim g^{2\alpha} \). One sees that

\[ \tau_{z_3} = \text{Id} \quad \text{on} \quad Z_t(a), \quad \tau_{z_2} = -\text{Id} \quad \text{on} \quad a, \quad \tau_{z_2} = -\theta \quad \text{on} \quad g[2\alpha]. \]

Consequently \( q' := \text{Fix}(-\tau_{z_2}, \theta') = a \oplus g[2\alpha]_t \) and \( \dim(q' \cap p) = \dim a = 1 \). Similarly, \( \dim(q' \cap t) = \dim g[2\alpha]_t = \dim g^{2\alpha} \), as wished.

By Lemma 2.11 and Remark 2.13 in [Ge1], the \( G \)-orbit of \( z_3 \) is a rank-one, pseudo-Riemannian symmetric space \( G/H \) with involution \( \tau_{z_3} = \text{Ad}_{g_3} \circ \tau \circ \text{Ad}_{g_3}^{-1} \). The space \( G/H \) is embedded in \( G^C/K^C \) as a closed, totally real submanifold of maximal dimension. The standard Cartan subset starting at \( z_3 \) is given by \( C = \exp i\cdot z_3 \), where \( c = \mathbb{R}(X + \theta(X)) \) and \( X \) is a non-zero vector in \( g^{\alpha} \). If \( g = h \oplus q \) is the \( \tau_{z_2} \)-decomposition of \( g \), then \( c \) is a maximal abelian subalgebra in \( q \). Fix a generator \( C \) of \( c \).

**Lemma 4.6.** Assume that the restricted root system of \( g \) is non-reduced. Let \( z_3 \in A_0 \) be the base point of the Cartan subset \( C \).

(i) The orbit \( G \cdot z_3 \) is simply connected. In particular the isotropy subgroup \( H \) of \( z_3 \) in \( G \) is connected.

(ii) For every \( t > 0 \), the points \( \exp(itC) \cdot z_3 \) and \( \exp(-itC) \cdot z_3 \) sit on the same \( G \)-orbit in \( G^C/K^C \).

(iii) There is precisely one non-closed \( G \)-orbit in \( G^C/K^C \) containing \( G \cdot z_3 \) in its closure.

**Proof.** (i) Since by assumption \( G \) is connected, we prove that \( H \) is connected by showing that the orbit \( G \cdot z_3 \) is simply connected. In order to do this, by Remark 4.1, it is sufficient to choose \( G \) as in the standard presentation in Table 4.0. Let \( G = SU(n, 1) \). By direct computations (cf. Example 4.7) one finds that \( G \cdot z_3 \cong SU(n, 1)/U(n - 1, 1) \). This quotient is topologically equivalent to the complex projective space \( \mathbb{C}P^{2n-1} \). In particular, it is simply connected.

Consider then \( G = Sp(n, 1) \) or \( G = F_4^* \). In both cases the group \( G \) is simply connected. Since \( H \) is the fixed point subgroup of an involution of \( G \), it is connected (cf. [Si]). It follows that the quotient is simply connected.

Parts (ii) and (iii) follow from Remark 4.2 provided that \( \dim(q \cap p) = 1 + g^{2\alpha} \) and \( \dim(q \cap t) = \dim g^{\alpha} \). In order to show this, define \( g[\alpha] := g^{\alpha} \oplus g^{-\alpha} \) and \( g[2\alpha] := g^{2\alpha} \oplus g^{-2\alpha} \). Then both \( g[\alpha] \) and \( g[2\alpha] \) are \( \theta \)-stable subspaces of \( g \) of dimension equal to \( \dim g^{\alpha} \) and \( 2 \dim g^{2\alpha} \) respectively. Let \( g[\alpha]_t, g[\alpha]_p, g[2\alpha]_t \) and \( g[2\alpha]_p \) be the components of the respective Cartan decompositions. The same arguments as in the proof of Lemma 4.3 and Lemma 4.5 show that

\[ \dim g[\alpha]_t = \dim g[\alpha]_p = \dim g^{\alpha} \quad \dim g[2\alpha]_t = \dim g[2\alpha]_p = \dim g^{2\alpha}. \]

Moreover, one sees that

\[ \tau_{z_3} = \text{Id} \quad \text{on} \quad Z_t(a), \quad \tau_{z_3} = -\text{Id} \quad \text{on} \quad a, \quad \tau_{z_3} = -\theta \quad \text{on} \quad g[\alpha], \quad \tau_{z_3} = \theta \quad \text{on} \quad g[2\alpha]. \]

Since

\[ g = Z_t(a) \oplus a \oplus g[\alpha] \oplus g[2\alpha], \]
it follows that $q := Fix(-\tau z_3, g) = a \oplus [\alpha]_t \oplus [2\alpha]_p$. In particular, $\dim(q \cap p) = 1 + \dim g^{2\alpha}$ and $\dim(q \cap t) = \dim g^{\alpha}$, as claimed. \qed

As a consequence of the above lemmas and Table 4.0, in the non-reduced case the $G$-orbit space of $G^C/K^C$ can be represented by the following diagrams.

\[ G/K = SU(n, 1)/U(n), \ n \geq 2. \]

\[ G/K = Sp(n, 1)/Sp(n) \times Sp(1), \ n \geq 2 \]
\[ G/K = F^*_4/Spin(9) \]

Set $I_1 = I_3 = (0, 1)$. For $j = 1, 3$, define $\ell_j : I_j \rightarrow G^C/K^C$ by

\[ \ell_1(t) = \exp(-itA_2) \cdot z_2, \quad \ell_3(t) = \exp(itA_2) \cdot z_2. \]

The slices $\ell_1$ and $\ell_3$ parametrize the principal $G$-orbits through $A_0$ and $A_0 = z_1 \cup \ell_1(I_1) \cup z_2 \cup \ell_3(I_3) \cup z_3$.

Set $I_2 = I_4 = (0, \infty)$. For $j = 2, 4$, define $\ell_j : I_j \rightarrow G^C/K^C$ by

\[ \ell_2(s) = \exp(sC') \cdot z_2, \quad \ell_4(s) = \exp(-sC') \cdot z_2. \]

The slices $\ell_2$ and $\ell_4$ parametrize the principal $G$-orbits through the Cartan subset $C'$ with base point $z_2$ and $C' = \ell_2(I_2) \cup z_2 \cup \ell_4(I_4)$.

Finally, set $I_5 = (0, \infty)$ and define $\ell_5 : I_5 \rightarrow G^C/K^C$ by

\[ \ell_5(s) = \exp(sC) \cdot z_3. \]

The slice $\ell_5$ parametrizes the principal $G$-orbits through the standard Cartan subset $C$ with base point $z_3$. The points $w_1, \ldots, w_4$ represent the non-closed orbits containing $G \cdot z_2$ in their closure. The point $w_5$ represents the non-closed orbit containing $G \cdot z_3$ in its closure.
Example 4.7. A model in the non-reduced case. Let $G = SU(n,1)$, with $n \geq 2$, be the subgroup of $SL(n+1,\mathbb{C})$ leaving invariant the hermitian form $(z,w)_{n,1} = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n - z_{n+1} \bar{w}_{n+1}$ in $\mathbb{C}^{n+1}$. Denote by $\sigma$ the conjugation of $G^c = SL(n+1,\mathbb{C})$ relative to $G$, namely $\sigma(g) = I_{n+1} \bar{g}^{-1} I_{n+1}$. Denote by $\mathbb{P}^n$ the complex projective space endowed with the opposite complex structure, i.e. the one for which the map $\mathbb{P}^n \to \mathbb{P}^n$, $[z] \mapsto [\bar{z}]$ is holomorphic. The group $G^c$ acts holomorphically on $\mathbb{P}^n \times \mathbb{P}^n$ by

$$g \cdot ([z],[w]) := ([g \cdot z],[\sigma(g) \cdot w]).$$

Under this action $\mathbb{P}^n \times \mathbb{P}^n$ consists of two orbits: a closed one given by $\{(z,w) \in \mathbb{P}^n \times \mathbb{P}^n : (z,w)_{n,1} = 0\}$ and an open one given by its complement. The quotient $G^c/K^c$ can be identified with the open orbit

$$M^c := G^c \cdot \{([0 : \ldots : 0 : 1],[0 : \ldots : 0 : 1]) = \mathbb{P}^n \times \mathbb{P}^n \setminus \{(z,w)_{n,1} = 0\}. $$

Fix the elements

$$A_2 = \begin{pmatrix} 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \frac{4}{i} \\
0 & \ldots & \frac{4}{i} & 0 \\
\end{pmatrix}, \quad C' = \begin{pmatrix} 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & i & 0 \\
0 & \ldots & 0 & -i \\
\end{pmatrix}, \quad C = \begin{pmatrix} 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 0 \\
\end{pmatrix}$$

in $\mathfrak{g}$ as generators of $\mathfrak{a}$, $\mathfrak{c}'$ and $\mathfrak{c}$, respectively. Then points on the singular orbits in $M^c$ satisfying conditions (7), are given by

$$z_1 = ([0 : \ldots : 0 : 1],[0 : \ldots : 0 : 1]), \quad z_2 = ([0 : \ldots : 0 : i : 1],[0 : \ldots : 0 : -i : 1])$$
$$z_3 = ([0 : \ldots : 0 : 1 : 0],[0 : \ldots : 0 : 1 : 0]).$$

The $G$-orbit of $z_2$ is diffeomorphic to the homogeneous space $G/H'$, where $H' \cong U(n-1) \times SO(1,1)$. The group $G'$ is isomorphic to $U(n-1) \times SU(1,1)$ and the quotient $G'/H'$ is diffeomorphic to the two-dimensional rank-one, pseudo-Riemannian symmetric space $SU(1,1)/SO(1,1)$. The $G$-orbit of $z_3$ is diffeomorphic to the pseudo-Riemannian symmetric space $SU(n,1)/SU(n-1,1)$. The slices $\ell_1$ and $\ell_3$ are given by

$$\ell_1(t) = ([0 : \ldots : i \sin \frac{\pi}{4}(1-t) : \cos \frac{\pi}{4}(1-t)],[0 : \ldots : -i \sin \frac{\pi}{4}(1-t) : \cos \frac{\pi}{4}(1-t)),$$
$$\ell_3(t) = ([0 : \ldots : i \sin \frac{\pi}{4}(1+t) : \cos \frac{\pi}{4}(1+t)],[0 : \ldots : -i \sin \frac{\pi}{4}(1+t) : \cos \frac{\pi}{4}(1+t)),$$

where $t \in (0,1)$. The slices $\ell_2$ and $\ell_4$ are given by

$$\ell_2(s) = ([0 : \ldots : ie^{-s} : e^s],[0 : \ldots : -ie^s : e^{-s}]),$$
$$\ell_4(s) = ([0 : \ldots : ie^s : e^{-s}],[0 : \ldots : -ie^{-s} : e^s]),$$

with $s > 0$. Finally the slice $\ell_5$ is given by

$$\ell_5(s) = ([0 : \ldots : \sinh s : i \cosh s : 0],[0 : \ldots : \sinh s : -i \cosh s : 0]),$$

with $s > 0$. The slice representation at $z_2$ is equivalent to the standard action of $SO(1,1)$ on $\mathbb{R}^2$. So there are four non-closed $G$-orbits containing $G \cdot z_2$ in their closure. We can choose representatives of such orbits to be

$$w_1 = ([0 : \ldots : 0 : 1],[0 : \ldots : -i : 1]), \quad w_2 = ([0 : \ldots : i : 1],[0 : \ldots : 1 : 0]),$$
$$w_3 = ([0 : \ldots : 1 : 0],[0 : \ldots : -i : 1]), \quad w_4 = ([0 : \ldots : i : 1],[0 : \ldots : 0 : 1]).$$

A representative for the unique non-closed orbit containing $G \cdot z_3$ in its closure is given by

$$w_5 = ([0 : \ldots : 1 : -i : 1],[0 : \ldots : 1 : i : 1]).$$

□
Remark 4.8. When $G = SU(1,1)$, the restricted root system of $\mathfrak{g}$ is reduced. The quotient $G^C/K^C$ can be identified with $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{(z,w), 1 = 0\}$ and the $G$-orbit space can be described as above, except for the fact that the slice $\ell_5$ and the point $w_5$ must to be omitted. Moreover the $G$-orbit through $z_5$ is diffeomorphic to the symmetric space $G/K$. Note that $SU(1,1)^C/U(1)^C$ is biholomorphic to $SO_0(2,1)^C/SO(2)^C$. Thus it can also be identified with the two-dimensional hyperquadric described in Example 4.4. □

5. UNIVALENCE ON $G$-ORBITS IN $G^C/K^C$

Let $G$ be a connected, non-compact, real simple Lie group, $K \subset G$ a maximal compact subgroup and $G^C$ the universal complexification of $G$. Assume that $G$ is embedded in $G^C$. Consider a $G$-equivariant Riemann domain $q: \Sigma \to G^C/K^C$.

The main goal of this section is to prove that $q$ is injective on $G$-orbits, if $G/K$ is a rank-one, Riemannian symmetric space of dimension greater than three. We first prove the result for principal $G$-orbits and later we extend it to all $G$-orbits by a general argument. In most cases, the injectivity of $q$ on principal $G$-orbits follows from their simple connectedness. The cases $\dim(G/K) = 2, 3$ are discussed separately.

Recall that by (ii) of Remark 4.1 different triples $(G, K, G^C)$ associated with the same Riemannian symmetric space $G/K$ yield $G$-equivariantly diffeomorphic orbits in $G^C/K^C$. Let $A_0$, $C'$ and $C$ be the standard Cartan subsets in $G^C/K^C$. Let $H$ be the isotropy subgroup of the base point of $C$ and $H'$ the isotropy subgroup of the base point of $C'$ (see Lemmas 4.3, 4.5 and 4.6). By Prop. 3.4 and Prop. 3.15 in [Ge1], the principal orbits intersecting $A_0$, $C$ and $C'$ have isotropy type $Z_K(a)$, $Z_H(c)$ and $Z_{H'}(c')$, respectively.

Lemma 5.1. Principal $G$-orbits of isotropy type $Z_K(a)$ are simply connected if and only if $\dim(G/K) > 2$.

Proof. An orbit $G/Z_K(a)$ is topologically equivalent to $K/Z_K(a)$. Consider the isotropy representation of $K$ on $\mathfrak{p}$. The non-zero $K$-orbits in $\mathfrak{p}$ are diffeomorphic to $K/Z_K(a)$. Since $G/K$ has rank one, they are also diffeomorphic to spheres of dimension $\dim(G/K) - 1$. Hence the statement follows. □

Remark 5.2. When $G = SO_0(2,1)$, the isotropy subgroup $Z_K(a)$ is trivial. Therefore principal orbits of type $G/Z_K(a)$ are diffeomorphic to $SO_0(2,1)$ and topologically equivalent to $SO(2)$. In particular, they are not simply connected. □

Lemma 5.3. Principal $G$-orbits of isotropy type $Z_H(c)$ are simply connected, except when $G$ is one of the groups $SO_0(2,1)$, $SO_0(3,1)$ or $SU(2,1)$.
Proof. An orbit \( G/Z_H(\epsilon) \) is topologically equivalent to \( K/Z_{K\cap H}(\epsilon) \). We prove the lemma by discussing each case separately. Let \( G = SO_0(n,1) \). Using the hyperquadric model given in Example 4.4 one checks that
\[
H \cong SO_0(n-1,1), \quad Z_H(\epsilon) \cong SO_0(n-2,1), \quad K/Z_{H\cap K}(\epsilon) \cong SO(n)/SO(n-2).
\]
In particular, \( K/Z_{K\cap H}(\epsilon) \) is diffeomorphic to a Stiefel manifold, which is simply connected for \( n > 3 \).

Consider next the case \( G = SU(n,1) \), with \( n \geq 3 \). Direct computations on the model in Example 4.4 show that the isotropy subgroup \( Z \) is simply connected.

Finally, consider \( G = Sp(n,1) \) or \( G = F_4^* \). Note that in both cases \( K \) is simply connected. Therefore \( K/Z_{K\cap H}(\epsilon) \) is simply connected, provided that \( Z_{K\cap H}(\epsilon) \) is connected. In order to show this, consider the compact, rank-one, symmetric space \( K/H \) and the corresponding isotropy representation of \( K/H \) on \( \mathfrak{f} \cap \mathfrak{q} \). The non-zero \( K \cap H \)-orbits in \( \mathfrak{f} \cap \mathfrak{q} \) are of type \( K \cap H/Z_{K\cap H}(\epsilon) \) and are diffeomorphic to spheres of dimension \( \dim(\mathfrak{f} \cap \mathfrak{q}) - 1 \). Since \( \dim(\mathfrak{f} \cap \mathfrak{q}) = \dim \mathfrak{g}^a > 2 \) (cf. Table 4.0), they are simply connected. By Lemma 4.3 or Lemma 4.6, the group \( K \) and likewise its maximal compact subgroup \( K \cap H \) are connected. Then the exact homotopy sequence of the quotient \( K \cap H/Z_{K\cap H}(\epsilon) \), implies that the group \( Z_{K\cap H}(\epsilon) \) is connected, as wished. This completes the proof of the lemma.

\( \square \)

Remark 5.4. When \( G = SO_0(2,1) \), direct computations using the model described in Example 4.7 show that the isotropy subgroup \( Z_H(\epsilon) \) is trivial. Therefore principal orbits of type \( G/Z_H(\epsilon) \) are diffeomorphic to \( SO_0(2,1) \) and topologically equivalent to \( SO(2) \). In particular, they are not simply connected.

Similarly, when \( G = SO_0(3,1) \) the isotropy subgroup \( Z_H(\epsilon) \) is isomorphic to \( SO_0(1,1) \), which is connected. Therefore principal orbits of type \( G/Z_H(\epsilon) \) are topologically equivalent to \( SO(3) \) and are not simply connected.

When \( G = SU(2,1) \), direct computations using the model described in Example 4.7 show that the isotropy subgroup \( Z_{K\cap H}(\epsilon) \) is isomorphic to \( S(U(1) \times U(1)) \), which is connected. Principal orbits of type \( G/Z_H(\epsilon) \) are topologically equivalent to \( K/Z_{K\cap H}(\epsilon) \cong U(2)/U(1) \cong SO(3) \). Hence they are not simply connected.

Note that in all the above cases, despite the fact that the orbits are not simply connected, the corresponding isotropy subgroups are connected.

\( \square \)

Lemma 5.5. All principal \( G \)-orbits of type \( Z_{H}(\epsilon') \) are simply connected.

Proof. An orbit of type \( G/Z_{H}(\epsilon') \) is topologically equivalent to \( K/Z_{H\cap K}(\epsilon') \). We prove that the latter quotient is simply connected by discussing each case separately.

Consider first \( G = SU(n,1) \). Direct computations using the model constructed in Example 4.4 show that \( Z_{H\cap K}(\epsilon') \cong U(n-1) \). Hence the quotient \( K/Z_{H\cap K}(\epsilon') \cong U(n)/U(n-1) \) is diffeomorphic to the sphere \( S^{2n-1} \). In particular, it is simply connected for all \( n \geq 2 \).

Next let \( G = Sp(n,1) \) or \( G = F_4^* \). Both \( G \) and \( K \) are simply connected. So the quotient \( K/Z_{H\cap K}(\epsilon') \) is simply connected provided that \( Z_{H\cap K}(\epsilon') \) is connected.

\( \square \)
In order to show this, denote by \( K' \) the maximal compact subgroup of \( G' \) (see Sect. 4.2). Since \( H' \) is contained in \( G' \), the groups \( H' \cap K \) and \( H' \cap K' \) coincide and are both connected by Lemma 4.3. Consider the compact, rank-one, symmetric space \( K'/(K' \cap H') \subset G'/H' \). The non-zero orbits of the isotropy representation of \( K' \cap H' \) on \( \mathfrak{g} \) are of type \( K' \cap H'/Z_{K' \cap H'}(\mathfrak{c}') \) and are diffeomorphic to spheres of dimension equal to \( \dim g^{2n} - 1 \). Since \( \dim g^{2n} > 2 \) (cf. Table 4.0), they are simply connected. As \( H' \cap K' \) is connected, by the exact homotopy sequence of the quotient \( K' \cap H'/Z_{K' \cap H'}(\mathfrak{c}') \), the groups \( Z_{K' \cap H'}(\mathfrak{c}') \) are also connected. It follows that the quotients \( K/Z_{H' \cap K}(\mathfrak{c}') \) and \( G/Z_{H' \cap K}(\mathfrak{c}') \) are simply connected, as desired.

**Lemma 5.6.** Let \( q : \Sigma \to Z \) be a \( G \)-equivariant Riemann domain. Assume that every \( z \) in \( Z \) admits an arbitrary small neighbourhood \( V \) and a sequence \( \{ z_n \} \) converging to \( z \) with the property that both the isotropy subgroups \( G_{z_n} \) and the intersections \( G \cdot z_n \cap V \) are connected. Then \( q \) is injective on every \( G \)-orbit of \( \Sigma \).

**Proof.** Assume by contradiction that the map \( q \) is not injective on the \( G \)-orbit through some \( \zeta \) in \( \Sigma \). Then there exists \( h \in G \) with \( h \cdot \zeta \neq \zeta \) such that \( q(h \cdot \zeta) = q(\zeta) \). Since \( q \) is locally injective, one can choose an open neighborhood \( V \) of \( z := q(\zeta) \) in \( Z \) as in the assumption, and open neighbourhoods \( W_\zeta \) and \( W_{h \cdot \zeta} \) of \( \zeta \) and \( h \cdot \zeta \) in \( \Sigma \), such that \( W_\zeta \cap W_{h \cdot \zeta} = \emptyset \) and the restrictions \( q|_{W_\zeta} : W_\zeta \to V \) and \( q|_{W_{h \cdot \zeta}} : W_{h \cdot \zeta} \to V \) are bijective. Then there exists a sequence \( \{ z_n \} \) in \( Z \), converging to \( z \), with the property that both the isotropy subgroups \( G_{z_n} \) and the intersections \( G \cdot z_n \cap V \) are connected.

Consider the sequence \( \{ \zeta_n := (q|_{W_\zeta})^{-1}(z_n) \} \) in \( W_\zeta \). Since \( \{ \zeta_n \} \) converges to \( \zeta \), for \( n \) large enough, the points \( h \cdot \zeta_n \) lie in \( W_{h \cdot \zeta} \). Consequently their images \( q(h \cdot \zeta_n) = h \cdot q(\zeta_n) = h \cdot z_n \) lie in \( V \). Since both \( G_{z_n} \) and \( G \cdot z_n \cap V \) are connected, the set \( \Omega_n := \{ g \in G : g \cdot z_n \in V \} \) is connected. Note that both \( e \) and \( h \) belong to \( \Omega_n \). Hence there exists a continuous path \( \gamma : [0, 1] \to \Omega_n \) with \( \gamma(0) = e \) and \( \gamma(1) = h \). By the \( G \)-equivariance of \( q \) both paths

\[
t \mapsto (q|_{W_\zeta})^{-1}(\gamma(t) \cdot z_n) \quad \text{and} \quad t \mapsto \gamma(t) \cdot \zeta_n
\]

in \( \Sigma \) are liftings of \( t \mapsto \gamma(t) \cdot z_n \), with initial point \( \zeta_n \). On the other hand \( (q|_{W_\zeta})^{-1}(\gamma(1) \cdot z_n) \in W_\zeta \) while \( \gamma(1) \cdot \zeta_n \in W_{h \cdot \zeta} \), giving a contradiction.

As a consequence of the previous lemmas one obtains the main result of this section.

**Proposition 5.7.** Let \( G \) be a connected, non-compact, real simple Lie group such that the Riemannian symmetric space \( G/K \) has rank one. Assume that \( G \) is embedded in its universal complexification \( G^C \) and is different from the groups \( SL(2, \mathbb{R}) \) and \( Spin(3, 1) \). Let \( q : \Sigma \to G^C/K^C \) be a \( G \)-equivariant Riemann domain. Then \( q \) is injective on every \( G \)-orbit.

**Proof.** We begin by proving the following claim.

**Claim.** The isotropy subgroups of all principal \( G \)-orbits are connected.

**Proof of the claim.** Since \( G \) is connected, the isotropy subgroups of simply connected orbits are necessarily connected. Hence by Lemmas 5.1 and 5.3 we only need to discuss the isotropy types \( Z_K(a) \) when \( G \) has Lie algebra \( so_0(2, 1) \) and the isotropy types \( Z_H(c) \) when \( G \) has Lie algebra \( so_0(2, 1), so_0(3, 1) \) and \( su(2, 1) \).

Let \( g = so(2, 1) \). When \( G = SO_0(2, 1) \) the isotropy subgroups of all principal \( G \)-orbits are connected, by Remarks 5.2 and 5.3. Observe that \( SO_0(2, 1) \) is centerless and that \( SL(2, \mathbb{R}) \) is a double covering of \( SO(2, 1) \). Since the universal
complexification of $SL(2, \mathbb{R})$ is $SL(2, \mathbb{C})$, which is simply connected, no covering of $SO_0(2, 1)$ other than $SL(2, \mathbb{R})$ admits an embedding into its universal complexification. Hence the claims follows for every group $G \neq SL(2, \mathbb{R})$ which has Lie algebra $so(2, 1)$ and embeds in its universal complexification.

Let $g = so(3, 1)$. When $G = SO_0(3, 1)$ the isotropy subgroup $Z_H(e)$ is connected, by Remark 5.4. Note that $SO_0(3, 1)$ is centerless and $Spin(3, 1)$ is the only non-trivial covering of $SO_0(3, 1)$ which embeds in its universal complexification. Hence the claims follows for every group $G \neq Spin(3, 1)$, which has Lie algebra $so(3, 1)$ and embeds in its universal complexification.

Finally, let $g = su(2, 1)$. When $G = SU(2, 1)$, the isotropy subgroup $Z_H(e)$ is connected, by Remark 5.4. Thus the same holds true for every connected real Lie group covered by $SU(2, 1)$. Since no covering group of $SU(2, 1)$ admits an embedding in its universal complexification, the claim holds true for every $G$ which has Lie algebra $su(2, 1)$ and embeds in its universal complexification. This concludes the proof of the claim.

In order to complete the proof of the proposition, recall that the union of principal $G$-orbits forms an open dense subset of $G^C/K^C$. Hence, by the above claim every point in $G^C/K^C$ can be approximated by points with connected isotropy subgroups. Due to this fact and the description of the slice representation at closed $G$-orbits (cf. Lemma 4.2 and diagrams in Sect. 4), all assumptions of Lemma 5.6 are met and the statement follows. □

**Remark 5.8.** When $G = SL(2, \mathbb{R})$, the isotropy subgroups of all principal $G$-orbits in $G^C/K^C$ consist of the central elements $\{ \pm I_2 \}$. As we shall see in Example 7.7 in this case there exist Stein, $G$-equivariant Riemann domains which are not injective on $G$-orbits. Similarly, one can construct $G$-equivariant Riemann domains which are not injective on $G$-orbits in the case $G = Spin(3, 1)$. However, by Theorem 7.5 such Riemann domains cannot be Stein.

6. **$G$-invariant Stein domains in $G^C/K^C$**

Let $G/K$ be a non-compact, rank-one, Riemannian symmetric space. In this section we exhibit a complete classification of Stein $G$-invariant domains in $G^C/K^C$. The main ingredient is the computation of the Levi form of hypersurface $G$-orbits in $G^C/K^C$, which is carried out in [Ge1] and in Appendix 9. Most of the Stein domains in our list are known. However, for $G = SU(n, 1)$ we present some examples which appear to be new. By working out an explicit model of $G^C/K^C$, we show that they are all biholomorphic to $\mathbb{B}^n \times \mathbb{C}^n$.

The classification result is stated for the the standard presentations of $G/K$ given in Table 4.0. This is no loss of generality, since by Remark 4.4 the $G$-orbit structure of $G^C/K^C$ as well as the $CR$-structure and topology of $G$-orbits do not depend on the presentation of the symmetric space $G/K$.

Retain the notation used in diagrams (3), (4), (9), (10) of Section 4. Consider the $G$-invariant domains in $G^C/K^C$ defined by

$$D_1(a) = G \cdot (z_1 \cup \ell_1((a, 1))), \quad D_2(a) = G \cdot (z_3 \cup \ell_3((a, 1))), \quad 0 \leq a < 1,$$

$$S_1(b) = G \cdot \ell_2((b, \infty)), \quad S_2(b) = G \cdot \ell_4((b, \infty)), \quad 0 \leq b < \infty.$$ (14)
Indeed, by Prop. 5.6 and Prop. 5.21 in [Ge1], all principal orbits have definite Levi domains in $G/K$. From the description of the domains in $G$ respectively. One such biholomorphism is given for example by the map $w·a$, for $a$ in $A_5$, where $g_3 = \exp ia_3$, with $\alpha(a_3) = \frac{T}{2}$ (cf. (2)). Note that $g_3 \in SO(n, 1) \setminus \{SO_0(n, 1)\}$, therefore $g_3G = Gg_3$. As a consequence, the above map exchanges the singular orbits $G·z_1$ and $G·z_3$ and maps $G·\ell_1(a)$ onto $G·\ell_3(a)$, for $0 < a < 1$.

When $G = SO_0(n, 1)$, with $n \geq 2$, the domain $D_2(0)$ and its subdomains $D_2(a)$, for $0 < a < 1$, are Stein since they are biholomorphic to $D_1(0)$ and $D_1(a)$, respectively. One such biholomorphism is given for example by the map

$$G^C/K^C \to G^C/K^C, \quad gK^C \to g_3gK^C,$$

where $g_3 = \exp ia_3$, with $\alpha(a_3) = \frac{T}{2}$ (cf. (2)). Note that $g_3 \in SO(n, 1) \setminus \{SO_0(n, 1)\}$, therefore $g_3G = Gg_3$. As a consequence, the above map exchanges the singular orbits $G·z_1$ and $G·z_3$ and maps $G·\ell_1(a)$ onto $G·\ell_3(a)$, for $0 < a < 1$.

When $G = SO_0(2, 1)$, the domains $S_1(0)$ and $S_2(0)$ and their subdomains $S_1(b)$ and $S_2(b)$, for $0 < b < \infty$, were shown to be Stein in [Ge].

The last four domains in the list contain in their interior one of the non-closed orbits $G·w_i$, for some $i = 1, \ldots, 4$. Their boundary consists of two non closed $G$-orbits and the singular orbit in their closure. All of them are Stein if $G = SO_0(2, 1) \cong SU(1, 1)/\{\pm Id\}$ and $G = SU(n, 1)$, with $n > 1$. These facts are proved in Example 6.3 by constructing explicit models of such domains.

In order to complete the classification, it remains to show that no $G$-invariant domains in $G^C/K^C$ are Stein other than the ones listed in Table 6.0. When $G = SO_0(2, 1) \cong SU(1, 1)/\{\pm Id\}$ and $G = SU(n, 1)$, with $n \geq 2$, this is proved in Example 6.3.

In all other cases, namely $SO_0(n, 1)$, with $n > 1$, $Sp(n, 1)$ and $F_4^*$, this follows from the description of the $G$-orbit space of $G^C/K^C$ given in diagrams 11, 9, 10, and from the computation of the Levi form of the hypersurface $G$-orbits in $G^C/K^C$. Indeed, by Prop. 5.6 and Prop. 5.21 in [Ge], all principal orbits have indefinite Levi
form, except for the ones intersecting the slice \( \ell_1 \) (the domain \( D_1(a) \) is Stein) and, only when the restricted root system of \( \mathfrak{g} \) is reduced, the slice \( \ell_3 \) (the domain \( D_2(a) \) is Stein for \( G = SO_0(n, 1) \)). Moreover, by Remarks 9.10 and 9.18 the Levi form of the non-closed hypersurface orbits \( G \cdot w_2 \) and \( G \cdot w_3 \) is indefinite. Since the boundary of a Stein domain cannot have indefinite Levi form, the theorem follows.

\( \square \)

Let us illustrate the result of Theorem 6.1 on the model of \( G^C/K^C \) described in Example 4.2. The Stein, \( G \)-invariant domains are studied by means of an appropriate \( G \)-invariant function on \( G^C/K^C \).

**Example 6.2.** Let \( G = SO_0(n, 1) \). By Example 4.2 the quotient \( G^C/K^C \) can be identified with \( M^C := \{ \xi \in \mathbb{C}^{n+1} : \xi_1^2 + \ldots + \xi_n^2 - \xi_{n+1}^2 = -1 \} \). Assume \( n > 2 \). Consider the \( G \)-invariant function \( f : M^C \rightarrow \mathbb{R} \) defined by

\[
   f(\xi_1, \ldots, \xi_{n+1}) := |\xi_1|^2 + \ldots + |\xi_n|^2 - |\xi_{n+1}|^2 - 1.
\]

For every \( 0 < a < 1 \), the \( G \)-invariant domains \( D_1(a) \) and \( D_2(a) \) coincide with the two connected components of the set \( \{ \xi \in M^C : f(\xi) < r \} \), for some \(-2 < r < 0\). Every such domain is bounded by a single \( G \)-orbit on which the Levi form of \( f \) is positive definite. Hence it is Stein.

The \( G \)-invariant domains \( D_1(0) \) and \( D_2(0) \) coincide with the two connected components of the set \( \{ \xi \in M^C : f(\xi) = 0 \} \). They are bounded by the non-smooth hypersurfaces \( \partial D_1(0) = G \cdot (z_2 \cup w_1) \) and \( \partial D_2(0) = G \cdot (z_2 \cup w_2) \), respectively. At all smooth points of \( \partial D_1(0) \) and \( \partial D_2(0) \) the Levi form of \( f \) has \( n - 2 \) positive eigenvalues and one zero eigenvalue. This is consistent with the fact that \( D_1(0) \) and \( D_2(0) \) are Stein. The Levi form of \( f \) is indefinite on all remaining hypersurface \( G \)-orbits. Thus there are no other Stein \( G \)-invariant domains in \( M^C \).

Next we determine all Stein, \( G \)-invariant domains in \( G^C/K^C \) in the case \( G = SU(n, 1) \) by using the model of \( G^C/K^C \) described in Example 4.2 and Remark 4.3. This settles the missing cases in the proof of Theorem 6.1.

**Example 6.3.** Let \( G = SU(n, 1) \), with \( n \geq 1 \). By Example 4.2 the quotient \( G^C/K^C \) can be identified with \( M^C := \mathbb{P}^n \times \mathbb{P}^n \setminus \{ (z, w)_{n,1} = 0 \} \). Consider the \( G \)-invariant function \( f : M^C \rightarrow \mathbb{R} \) defined by

\[
   f([z], [w]) = -\langle z, z \rangle_{n,1} \langle w, w \rangle_{n,1} / |\langle z, w \rangle_{n,1}|^2.
\]

- **\( G = SU(1,1) \)**
  
  By computing the Levi form of \( f \) on the \( G \)-orbits in the level set \( \{ f = r \} \), with \( r < 0 \), one shows that the domains \( D_1(a) \) and \( D_2(a) \) are Stein for all \( 0 < a < 1 \). Similarly one shows that \( S_1(b) \) and \( S_2(b) \) are Stein for every \( b > 0 \). One can also verify that the Levi form of \( f \) on all non-closed hypersurface orbits \( G \cdot w_1, \ldots, G \cdot w_4 \) is identically zero. This is consistent with the fact that the domains \( D_1(0), D_2(0), S_1(0) \) and \( S_2(0) \) are Stein. We claim that the domains

\[
   W_{1,1} := D_1(0) \cup G \cdot w_1 \cup S_1(0), \quad W_{1,2} := D_1(0) \cup G \cdot w_4 \cup S_2(0),
   W_{2,1} := D_2(0) \cup G \cdot w_2 \cup S_1(0), \quad W_{2,2} := D_2(0) \cup G \cdot w_3 \cup S_2(0),
\]

are Stein as well. By evaluating the hermitian forms \( \langle z, z \rangle_{n,1} \) and \( \langle w, w \rangle_{n,1} \) on the slices described in Example 4.2 and Remark 4.3, one sees that such domains can be characterized as follows

\[
   W_{1,1} = \{ \langle z, w \rangle_{1,1} \neq 0 \text{ and } \langle z, z \rangle_{1,1} < 0 \},
\]
\[ W_{1,2} = \{ \langle z, w \rangle_{1,1} \neq 0 \text{ and } \langle w, w \rangle_{1,1} < 0 \}, \]
\[ W_{2,1} = \{ \langle z, w \rangle_{1,1} \neq 0 \text{ and } \langle w, w \rangle_{1,1} > 0 \}, \]
\[ W_{2,2} = \{ \langle z, w \rangle_{1,1} \neq 0 \text{ and } \langle z, z \rangle_{1,1} > 0 \}. \]

As a consequence, the maps defined by
\[ \Delta \times \mathbb{C} \to W_{1,1}, \quad (u, v) \to ([u : 1], [\bar{v} : 1 + \bar{u}v]), \]
\[ \mathbb{C} \times \Delta \to W_{1,2}, \quad (u, v) \to ([u : 1 + uv], [\bar{v} : 1]), \]
\[ \Delta \times \mathbb{C} \to W_{2,1}, \quad (u, v) \to ([1 + uv : u], [\bar{v} : 1]) \]
\[ \mathbb{C} \times \Delta \to W_{2,2}, \quad (u, v) \to ([1 : u], [1 + \bar{u}v : \bar{v}]), \]
are biholomorphisms. Here \( \Delta \) denotes the unit disk in \( \mathbb{C} \). In particular the domains \( W_{1,1}, W_{2,2} \) are Stein, as claimed.

Other \( G \)-domains in \( M^C \) which are possibly Stein can only be obtained as arbitrary unions of domains \( W_{k,l} \), for \( k, l = 1, 2 \). We claim that such unions are not Stein. For instance, let us show that \( W_{1,1} \cup W_{2,1} \) is not Stein. Consider the Stein local chart
\[ \phi : \mathbb{C}^2 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad (u, v) \to ([u : 1], [1 : \bar{v}]). \]

Since the preimage
\[ \phi^{-1}(W_{1,1} \cup W_{2,1}) = \{ (u, v) \in \mathbb{C}^2 : u \neq v \text{ and either } |u| < 1 \text{ or } |v| < 1 \} \]
is not Stein, the domain \( W_{1,1} \cup W_{2,1} \) is not Stein either. An analogous argument applies to the remaining cases.

- \( G = SU(n, 1) \), with \( n \geq 1 \).

Using the \( G \)-invariant function \( f \), one can prove that the domains \( D_1(a) \), for \( a > 0 \), are Stein. One can also verify that \( D_1(0) \) coincides with a connected component of the set \( \{ z \in M^C : f(z) < 0 \} \) and that on the smooth part of its boundary \( \partial D_1(0) = G \cdot (w_1 \cup z_2 \cup w_4) \) the Levi form of \( f \) has non-negative eigenvalues. This is consistent with the fact that \( D_1(0) \) is Stein.

Moreover, the Levi form of \( f \) on the principal \( G \)-orbits through the slices \( \ell_2, \ell_3, \ell_4, \ell_5 \) and on the non-closed hypersurface orbit \( G \cdot w_5 \) is indefinite. On the other hand, the Levi form of \( f \) is definite on the non-closed hypersurface orbits \( G \cdot w_2 \) and \( G \cdot w_3 \). As a result, other \( G \)-invariant domains in \( M^C \) which are possibly Stein are only
\[ W_{1,1} := D_1(0) \cup G \cdot w_1 \cup S_1(0), \quad W_{1,2} := D_1(0) \cup G \cdot w_3 \cup S_2(0), \quad W_{1,1} \cup W_{1,2}. \]

First we show that \( W_{1,1} \) and \( W_{1,2} \) are indeed Stein. By evaluating \( < z, z >_{n,1} \) and \( < w, w >_{n,1} \) on the slices described in Example 4.7 one sees that such domains can be characterized as follows
\[ W_{1,1} = \{ ([z], [w]) \in \mathbb{P}^n \times \mathbb{P}^n : < z, w >_{n,1} \neq 0 \text{ and } < z, z >_{n,1} < 0 \} \]
\[ W_{1,2} = \{ ([z], [w]) \in \mathbb{P}^n \times \mathbb{P}^n : < z, w >_{n,1} \neq 0 \text{ and } < w, w >_{n,1} < 0 \}. \]

As a consequence the maps
\[ \mathbb{B}^n \times \mathbb{C}^n \to W_{1,1}, \quad (u, v) \to ([u : 1], [v : 1 + \bar{v} \bar{u} + \ldots + \bar{v}_n \bar{u}_n]) \]
\[ \mathbb{C}^n \times \mathbb{B}^n \to W_{1,2}, \quad (u, v) \to ([u : 1 + u_1 v_1 + \ldots + u_n v_n], [\bar{v} : 1]), \]
are biholomorphisms. Here \( \mathbb{B}^n \) denotes the unit ball in \( \mathbb{C}^n \). In particular \( W_{1,1} \) and \( W_{1,2} \) are Stein, as claimed.

Next we show that the domain \( \Omega := W_{1,1} \cup W_{1,2} \), with boundary \( \partial \Omega = G \cdot (w_2 \cup z_2 \cup w_3) \), is not Stein. Assume by contradiction that \( \Omega \) is Stein. Let \( c' \) be the abelian subalgebra generating the Cartan subset \( C' \) (cf. Example 1.7). Let \( T = \exp c' \) be the corresponding compact torus in \( G \). Consider the \( T \)-action on
\(\Omega\) and the induced local holomorphic \(T^C\)-action. By the globalization theorem in [He1], Section 6.6, the domain \(\Omega\) embeds in its \(T^C\)-globalization \(\Omega^\ast\) as a \(T\)-invariant, orbit-convex subset. By definition, this means that the intersection of \(\Omega\) with an \(\exp i\cdot \text{orbit}\) in \(\Omega^\ast\) is connected.

Every \(T^C\)-orbit through the slice \(\ell_1\) is contained in \(\Omega\). Indeed in \(M^C\) one can verify that \(\exp(isC^\prime) \cdot \ell_1(t) = ([0 : \ldots : e^{s}i \sin \frac{\pi}{4}(1-t) : e^{-s} \cos \frac{\pi}{4}(1-t)], [0 : \ldots : -e^{-s}i \sin \frac{\pi}{4}(1-t) : e^{s} \cos \frac{\pi}{4}(1-t)]).\)

Thus for fixed \(0 < t < 1\), the function \(\mathbb{R} \rightarrow \mathbb{R}\), defined by \(s \mapsto f(\exp(isC^\prime) \cdot \ell_1(t))\) is given by
\[
(e^{2s} \sin^2 \frac{\pi}{4}(1-t) - e^{-2s} \cos^2 \frac{\pi}{4}(1-t))(e^{-2s} \sin^2 \frac{\pi}{4}(1-t) - e^{2s} \cos^2 \frac{\pi}{4}(1-t))
\]
and vanishes exactly twice, namely on \(G \cdot w_1\) and on \(G \cdot w_4\). It follows that \(\exp(i\ell) \cdot \ell_1(t)\) never crosses the boundary of \(\Omega\) and consequently the complex orbit \(T^C \cdot \ell_1(t)\) is entirely contained in \(\Omega\), as claimed. Moreover, for every fixed \(s > 0\), one has
\[
\lim_{n \rightarrow \infty} \exp(isC^\prime) \cdot \ell_1(1/n) = \ell_2(s) \in \Omega,
\]
\[
\lim_{n \rightarrow \infty} \exp(-isC^\prime) \cdot \ell_1(1/n) = \ell_4(s) \in \Omega.
\]

Then the orbit-convexity of \(\Omega\) in \(\Omega^\ast\) implies that the sequence \(\{\ell_1(1/n)\}_n\) has a limit point in \(\Omega\). On the other hand, in \(G^C/K^C\) one has \(\lim_n \ell_1(1/n) = z_2\), which is not in \(\Omega\). This yields a contradiction and proves that \(\Omega\) is not Stein. The classification of all Stein \(G\)-invariant domains in \(M^C\) is now complete. \(\Box\)

We conclude this section with a remark which is a consequence of Theorem 6.1 and is often used in the sequel.

**Remark 6.4.** Let \(D\) be a domain in \(G^C/K^C\) with smooth boundary \(\partial D\). It is well known that if \(D\) is not pseudoconvex at \(z \in \partial D\), then no holomorphic function on \(D\) diverges in the vicinity of \(z\). Let \(\ell : I \rightarrow G^C/K^C\) be a slice for principal \(G\)-orbits in \(G^C/K^C\). By the classification of Stein, \(G\)-invariant domains in \(G^C/K^C\) given in Theorem 6.1 the following facts hold.

(i) Assume that the Levi form of the orbits parametrized by \(\ell\) is definite. Let \((c, d) \subset I\) be an interval with \(0 \leq c < d\) and \(d \in I\). Then no holomorphic function on the invariant domain \(G \cdot \ell((c, d))\) diverges in the vicinity of the boundary orbit \(G \cdot \ell(d)\) (for instance, if \(I = (0, 1)\), the domain \(D_1(d)\) is strictly pseudoconvex at every point of the boundary orbit \(G \cdot \ell_1(d)\). Thus the domain \(G \cdot \ell_1((c, d))\) is not pseudoconvex at every point of \(G \cdot \ell_1(d)\).

(ii) Assume that the Levi form of the orbits parametrized by \(\ell\) is indefinite. Let \((c, d) \subset I\) be an interval, with \(c \in I\). Then no holomorphic function on the invariant domain \(G \cdot \ell((c, d))\) diverges in the vicinity of the boundary orbit \(G \cdot \ell(c)\). Similarly, if \(d \in I\), then no holomorphic function diverges in the vicinity of \(G \cdot \ell(d)\). \(\Box\)
7. Univalence over $G^C/K^C$

Let $G$ be a connected, non-compact, real simple Lie group, $K \subset G$ a maximal compact subgroup and $G^C$ the universal complexification of $G$. Assume that the center $\Gamma$ of $G$ is finite and $G$ is not a covering of $SL(2, \mathbb{R})$. In this section we show that a holomorphically separable, $G$-equivariant Riemann domain $q : \Sigma \to G^C/K^C$ is necessarily univalent, if the rank of $G/K$ is equal to one (cf. Thm. 7.6 and Rem. 7.8).

In most cases the map $q$ is injective on every $G$-orbit (cf. Sect. 5). So we are reduced to prove the injectivity of $q$ over the global slices for the $G$-action defined by diagrams (3), (4), (9), (10) in Section 4. Recall that the slices parametrizing principal $G$-orbits are diffeomorphic to open intervals of $\mathbb{R}$ and that a local diffeomorphism of a one-dimensional smooth manifold into the real line $\mathbb{R}$ is necessarily injective. As a consequence, $q$ is injective on every connected component of $\Sigma$ over a domain in $G^C/K^C$ consisting of principal orbits.

However, in order to ensure monodromy around the singular orbit $G \cdot z_2$ (cf. diagrams in Sect. 4), it is necessary to combine the uniqueness property of path liftings for Riemann domains with the complex geometry of the $G$-invariant domains in $G^C/K^C$. Before proving the main result of this section some preliminary lemmas are needed.

Let $\ell : I \to G^C/K^C$ be one of the slices for principal $G$-orbits defined in (5), (9), (11), (12) and (13) of Section 4. Define

$$\tilde{I} := (0,1], \quad \text{when} \quad I = (0,1), \quad \text{and} \quad \tilde{I} := I, \quad \text{when} \quad I = \mathbb{R}^+.$$

Recall that $I = (0,1)$ only when $\ell = \ell_1$ or $\ell = \ell_3$. In those cases extend $\ell$ to $\tilde{I} = (0,1]$ by defining

$$\ell_1(1) := eK^C, \quad \ell_3(1) := \exp(iA_3)K^C.$$

We refer to $\ell : \tilde{I} \to G^C/K^C$ as an extended slice. Note that the images of the extended slices $\ell_1$ and $\ell_3$ include the points $z_1$ and $z_3$, respectively.

Let $q : \Sigma \to G^C/K^C$ be a $G$-equivariant Riemann domain and let $\ell : \tilde{I} \to G^C/K^C$ be an extended slice. A local lifting of $\ell$ is a smooth path $\tilde{\ell} : J \to \Sigma$, defined on a non-empty interval $J$ open in $\tilde{I}$, and satisfying the condition $q \circ \tilde{\ell} = \ell$ on $J$. A local lifting $\tilde{\ell} : J \to \Sigma$ is maximal if it cannot be extended to a larger interval $J'$, with $J \subset J' \subset \tilde{I}$.

**Lemma 7.1.** Assume that $G$ is embedded in its universal complexification $G^C$ and is different from $SL(2, \mathbb{R})$ and $Spin(3,1)$. Let $q : \Sigma \to G^C/K^C$ be a Stein, $G$-equivariant Riemann domain and let $\ell : \tilde{I} \to G^C/K^C$ be a maximal local lifting of an extended slice $\ell : \tilde{I} \to G^C/K^C$.

(i) if the Levi form of the principal orbits parametrized by $\ell$ is definite, then the invariant domain $G \cdot \ell(J)$ in $G^C/K^C$ is Stein (see Theorem 6.7).

(ii) If the Levi form of the principal orbits parametrized by $\ell$ is indefinite, then $J$ coincides with $\tilde{I}$.

**Proof.** (i) Consider first the case $\tilde{I} = \mathbb{R}^+$ (see diagram 3, Ex. 6.3 and Rem. 4.8). By Theorem 6.1 we need to show that $J = (b, +\infty)$, for some $b \geq 0$. Assume by contradiction that $J = (b, d)$, with $0 \leq b < d < \infty$. Since the lifting $\tilde{\ell}(J)$ is a one-dimensional real-analytic submanifold of $\Sigma$, the local diffeomorphism $q|_{\tilde{\ell}(J)}$ is
By the Steinness of $\Sigma$, there exists a subsequence of $\{Gq\}_{n}$ leaves every given compact subset in $\Sigma$. Since $\Sigma$ is Stein, there exists a holomorphic function $f \in \mathcal{O}(\Sigma)$ such that $\lim_{n \to \infty} |f(\ell(d-1/n))| = \infty$.

On the other hand, the push-forward of $f$ by $q|_{G\ell(J)}$ defines a holomorphic function in $\mathcal{O}(G \cdot \ell(J))$ which diverges in the vicinity of the boundary orbit $G \cdot \ell(d)$. This contradicts (ii) of Remark 6.4 implying that $J$ is of the form $(b, \infty)$, as claimed.

Consider now the case $I = (0, 1)$. This only occurs for $\ell = \ell_1$ or, when the restricted root system of $\mathfrak{g}$ is reduced, for $\ell = \ell_3$ (cf. diagrams in Sect. 4 and [Ge1], Prop. 5.6). By Theorem 6.1 we need to show that $J = (a, 1]$, for some $a \geq 0$. Assume by contradiction that $J = (a, d)$ with $0 \leq a < d \leq 1$. The argument used in the previous case shows that $J = (a, 1)$ and there exists a holomorphic function $f \in \mathcal{O}(\Sigma)$ such that $\lim_{n \to \infty} |f(\ell(1-1/n))| = \infty$. Moreover, the push-forward of $f$ by $q|_{G\ell(J)}$ defines a holomorphic function $\tilde{f} \in \mathcal{O}(G \cdot \ell(J))$ which diverges in the vicinity of the boundary orbit $G \cdot \ell(1)$. On the other hand, such orbit is a totally real submanifold of $G \cdot \ell((a, 1))$. Thus $\tilde{f}$ extends to a holomorphic function on $G \cdot \ell((a, 1))$. This yields a contradiction, implying that $J = (a, 1]$, as desired.

(ii) Assume first that $\tilde{I} = \mathbb{R}^+$. Then (ii) of Remark 6.4 and an analogous argument as in the proof of (i) show that $J = \tilde{I}$. Consider then the case $\tilde{I} = (0, 1]$. This only occurs when the restricted root system of $\mathfrak{g}$ is non-reduced and $\ell = \ell_3$ (cf. diagrams in Sect. 4 and [Ge1], Prop. 5.6). A similar argument as in the proof of (i) shows that if a lifting $\tilde{\ell}_3: J \to \Sigma$ is maximal, then either $J = (0, 1]$ or $J = (0, 1)$.

In order to prove that $J = (0, 1]$, suppose by contradiction that $J = (0, 1)$. Consider a sequence $\{z_n\}$ in $G \cdot \ell_3(J)$, converging to a point on the boundary orbit $G \cdot w_5$, say $w_5$. Since the Levi form of $G \cdot w_5$ is indefinite (see Remark 9.11), no holomorphic function on $G \cdot \ell_3(J)$ diverges on $\{z_n\}$. Note that the restriction $q|_{G\ell(J)}: G \cdot \ell_3(J) \to G \cdot \ell_3(J)$ is a biholomorphism. Hence no holomorphic function of $G \cdot \ell_3(J)$ diverges on the sequence $\{z_n\}$ in $\Sigma$, defined by $\zeta_n := (q|_{G\ell(J)})^{-1}(z_n)$. By the Steinness of $\Sigma$, there exists a subsequence of $\{\zeta_n\}$ converging to a point $\zeta_5$ in $\Sigma$. Since $q$ is continuous, one has $q(\zeta_5) = w_5$.

By the $G$-equivariance of $q$, the description of the slice representation at $z_3$ given in Remark 11.2 and Proposition 5.7 there exists a $G$-invariant neighbourhood $V$ of $\eta_5$ in $\Sigma$ on which $q$ is injective. Its image $q(V)$ intersects the slice $\ell_5$ in $\ell_5((0, \epsilon))$, for some $\epsilon > 0$. By statement (i) of this lemma, the local lifting $s \mapsto (q|_{V})^{-1}(\ell_5(s))$, with $s \in (0, \epsilon)$, extends to a lifting $\tilde{\ell}_5: \tilde{I} \to \Sigma$ of $\ell_5$. Note that $q$ maps the $G$-invariant domain $W := G \cdot (\tilde{\ell}_3(J) \cup \eta_5 \cup \tilde{\ell}_5(\tilde{I}_5))$ in $\Sigma$ biholomorphically onto the domain $q(W) = G \cdot (\ell_3(J) \cup \eta_5 \cup \ell_5(\tilde{I}_5))$ in $G^2/K^\Sigma$. Since $G \cdot \ell_3(1)$ is a totally real submanifold of $q(W) \cup G \cdot \ell_3(1)$ (see Lemma 2.11 and Rem. 2.13 in [Ge1]), every holomorphic function on $q(W)$ extends to a holomorphic function on $q(W) \cup G \cdot \ell_3(1)$. As a consequence, no holomorphic function on $W$ can diverge on the sequence $\{\ell_3(1 - \frac{1}{n})\}_{n}$ in $\Sigma$.

On the other hand, by the maximality of $\tilde{\ell}_3$, when $n$ grows the sequence $\{\tilde{\ell}_3(1 - \frac{1}{n})\}_{n}$ leaves every given compact subset in $\Sigma$. Since $\Sigma$ is Stein, there exists a holomorphic function $f \in \mathcal{O}(\Sigma)$ such that $\lim_{n \to \infty} |f(\tilde{\ell}_3(1 - 1/n))| = \infty$. This yields a contradiction, implying that $J$ necessarily coincides with $(0, 1]$, as claimed.

\hfill $\Box$

Let $\ell_1$ and $\ell_3$ be the slices parametrizing the principal orbits through the fundamental Cartan subset $\mathcal{A}$. Denote by $\mathcal{C} = \exp \iota \cdot z_2$ the standard Cartan subspace with base point $z_2$ and define $\mathcal{C}^* := \mathcal{C} \setminus \{z_2\}$. Recall that in the reduced case,
\( \mathfrak{c} = \mathbb{R}(X + \theta(X)) \), for some non-zero vector \( X \in \mathfrak{g}^n \), and \( z_2 = \exp(iA_2)K^C \) with \( \alpha(A_2) = \pi/2 \). In the non-reduced case, \( \mathfrak{c} = \mathbb{R}(X + \theta(X)) \), for some non-zero vector \( X \in \mathfrak{g}^{2n} \), and \( z_2 = \exp(iA_2)K^C \) with \( \alpha(A_2) = \pi/4 \). In both cases, \( \exp \mathfrak{c} \) is a compact, one-dimensional, real torus in \( G \) which we denote by \( T \). Both \( T \) and its universal complexification \( T^C \cong \mathbb{C}^* \) act on \( G^C/K^C \) by left translations.

In the next proposition, we single out two distinguished \( G \)-invariant domains \( \Omega \) and \( \Omega' \) in \( G^C/K^C \) containing all \( T^C \)-orbits through the slices \( \ell_1(I_1) \) and \( \ell_3(I_3) \), respectively.

**Lemma 7.2.** Let \( G/K \) be a non-compact, rank-one, Riemannian symmetric space. Consider the domain in \( G^C/K^C \) defined by
\[
\Omega := G \cdot (\ell_1(I_1) \cup w_1 \cup w_4 \cup \mathbb{C}^*). \]
Then for every point \( z \in \ell_1(I_1) \), the complex orbit \( T^C \cdot z \) is contained in \( \Omega \).

Similarly, define
\[
\Omega' := G \cdot (\ell_3(I_3) \cup w_2 \cup w_3 \cup \mathbb{C}^*). \]
Then for every point \( z \in \ell_3(I_3) \), the complex orbit \( T^C \cdot z \) is contained in \( \Omega' \).

**Proof.** We first assume that \( G = SO_0(2,1) \) and prove the statement by using the model \( M^C \) of \( G^C/K^C \) constructed in Example 4.3. Let \( C \) be the generator of \( \mathfrak{c} \) chosen there. Then, for \( s \in \mathbb{R} \) and \( t \in (0,1) \), one has
\[
\exp(isC) \cdot \ell_1(t) = (\sinh(2s) \sin \pi t, i \cosh(2s) \sin \pi t, \cos \pi t) .
\]

Since \( z_2 = (0, i, 0) \) and the entries of the matrix group \( G \) are real, from the above expression one easily verifies that \( \exp is \cdot \ell_1(I_1) \cap G \cdot z_2 = \emptyset \). Consider then the \( G \)-invariant function \( f(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - 1 \) defined on \( M^C \). The function \( f \) vanishes on the real hypersurface \( G \cdot \{z_2 \cup _j w_j\} \), is negative on the sets \( G \cdot \ell_1(I_j) \), for \( j = 1, 3 \) and positive on the sets \( G \cdot \ell_2(I_j) \), for \( j = 2, 4 \). Moreover, for every fixed \( t_0 \in (0,1) \), one sees that
\[
f(\exp(isC) \cdot \ell_1(t_0)) = (\sinh^2 2s + \cosh^2 2s) \sin^2 \pi t_0 - \cos^2 \pi t_0 - 1
\]
is strictly increasing for \( |s| \to \infty \). Thus it vanishes exactly twice. As a consequence, the path \( \exp(isC) \cdot \ell_1(t_0) \) crosses the hypersurface \( f^{-1}(0) \setminus \{G \cdot z_2\} \) exactly twice, namely on the orbits \( G \cdot w_1 \) and \( G \cdot w_2 \). It follows that \( \exp(isC) \cdot \ell_1(t_0) \in \Omega \), for every \( s \in \mathbb{R} \). Thus the \( T^C \)-orbit through \( \ell_1(t_0) \) is entirely contained in \( \Omega \), as stated. An analogous argument proves the statement for the higher dimensional hyperquadrics. By (ii) of Remark 4.1, this settles the case when \( \mathfrak{g} \) has a reduced restricted root system.

Consider now the case when the restricted root system of \( \mathfrak{g} \) is non-reduced. We prove the statement by reducing to the two-dimensional case. Set \( \mathfrak{g} := so(2,1) \) and fix a basis in \( \mathfrak{g} \) of the form \( \{\hat{X}, \theta(\hat{X}), \hat{A} = [\theta(\hat{X}), \hat{X}]\} \), where \( \hat{X} \) is a root vector in \( \mathfrak{g}^+ \) and \( \alpha(\hat{A}) = \frac{\pi}{2} \). Define \( \hat{C} = \hat{X} + \theta(\hat{X}) \). Choose a root vector \( X \in \mathfrak{g}^{2n} \) and normalize the triple \( \{X, \theta(X), A = [\theta(X), X]\} \) in \( \mathfrak{g} \) so that \( \alpha(A) = \frac{\pi}{2} \). Such a triple generates a three-dimensional \( \theta \)-stable subalgebra of \( \mathfrak{g} \) isomorphic to \( \mathfrak{g} \). In particular, there exists an injective Lie algebra homomorphism
\[
\varphi_* : \mathfrak{g} \to \mathfrak{g}
\]
mapping \( \hat{X}, \hat{A} \) and \( \theta(\hat{X}) \) into \( X, A \) and \( \theta(X) \) respectively. Clearly \( \varphi_* \) maps \( \hat{C} = \hat{X} + \theta(\hat{X}) \) into \( C = X + \theta(X) \) as well. Let \( \tilde{K} = SO(2) \) be the maximal compact subgroup of \( \tilde{G} := SO_0(2,1) \) and \( \tilde{\mathfrak{k}} \) its Lie algebra. Note that \( \tilde{\mathfrak{f}} \) and \( \mathfrak{f} \) are generated by \( \tilde{C} \) and \( C \), respectively. One can check that the \( \mathbb{C} \)-linear extension
\( \hat{\mathfrak{g}}^c \to \mathfrak{g}^c \) of \( \varphi_* \) induces a Lie group morphism \( \varphi : \hat{G}^C \to G^C \) mapping \( \hat{K}^C \) to \( K^C \). As a consequence one obtains a holomorphic map (denoted by the same symbol)
\[
\varphi : \hat{G}^C / \hat{K}^C \to G^C / K^C.
\]
Let \( \hat{\Omega} \) be the domain
\[
\hat{\Omega} = \hat{G} \cdot (\hat{\ell}_1(I_1) \cup \hat{\ell}_2(I_2) \cup \hat{\ell}_4(I_4))
\]
in \( \hat{G}^C / \hat{K}^C \) considered in (i). We claim that \( \varphi(\hat{\Omega}) \subset \Omega \). The map \( \varphi \) is “equivariant” with respect to the action of \( \hat{G} \), that is \( \varphi(g \cdot x) = \varphi(g) \cdot \varphi(x) \), for every \( g \in \hat{G} \) and \( x \in \hat{G}^C / \hat{K}^C \). By the definition of \( \varphi_* \) one has \( \varphi(\exp(itA)) = \exp(itA) \) and \( \varphi(\exp(it\hat{A})) = \exp(it\hat{C}) \). It follows that
\[
\varphi(\hat{\ell}_1(I_1)) = \ell_1(I_1), \quad \varphi(\hat{\ell}_2(I_2)) = \ell_2, \quad \varphi(\hat{\ell}_4(I_4)) = C.
\]
We conclude the proof of the claim by showing that \( \varphi(\hat{\omega}_1) \in G \cdot \omega_1 \) and \( \varphi(\hat{\omega}_4) \in G \cdot \omega_4 \) (possibly the orbit \( G \cdot \omega_4 \) and \( G \cdot \omega_1 \) coincide). By (1) there is a commutative diagram
\[
\begin{array}{ccc}
\hat{G} \times \hat{G} \hat{\omega}_2 \hat{V}_2 & \xrightarrow{\phi} & G \times G \omega_2 V_2 \\
\downarrow & & \downarrow \\
\hat{G}^C / \hat{K}^C & \xrightarrow{\phi} & G^C / K^C.
\end{array}
\]
The vertical arrows correspond to the equivariant embeddings given in (1) and the map \( \phi \) is defined by \( [\hat{g}, \hat{X}] \to [\varphi(\hat{g}), \varphi_* (\hat{X})] \). Since \( \varphi_* \) is an injective homomorphism, \( \varphi(\hat{\omega}_1) \) does not lie on the singular orbit \( G \cdot \omega_2 \). Indeed in the twisted bundle \( G \times G \omega_2 V_2 \) such orbit corresponds to the set \( \{ [g, 0] : g \in G \} \). On the other hand \( \varphi(\hat{\omega}_1) \in \hat{G} \cdot \hat{\ell}_1(I_1) \cap G \cdot \hat{\ell}_2(I_2) \). Therefore the image \( \varphi(\hat{\omega}_1) \) necessarily lies on the orbit \( G \cdot \omega_1 \). Similarly one proves that \( \varphi(\hat{\omega}_4) \in G \cdot \omega_1 \). In conclusion \( \hat{\Omega} \) is mapped by \( \varphi \) into \( \Omega \), as claimed.

Observe that \( \exp \hat{G} \cdot \hat{\ell}_1(I_1) = \varphi(\exp \hat{G} \cdot \hat{\ell}_1(I_1)) \) and recall that in the 2-dimensional case we already showed that \( \exp \hat{G} \cdot \hat{\ell}_1(I_1) \subset \hat{\Omega} \). Then, by the above claim, for every \( z \in \ell_1(I_1) \) one has
\[
T^C \cdot \ell_1(z) \subset \Omega,
\]
as required. The statement regarding the domain \( \Omega' \) follows from similar arguments. \hfill \Box

**Lemma 7.3.** Assume that \( G \) is embedded in its universal complexification \( G^C \) and is different from the groups \( SL(2, \mathbb{R}) \) and \( Spin(3, 1) \). Let \( q : \Sigma \to G^C / K^C \) be a Stein, \( G \)-equivariant Riemann domain.

(i) Let \( \hat{\ell}_1 : I_1 \to \Sigma \) be a lifting of the slice \( \ell_1 \). Assume that the closure of \( G \cdot \hat{\ell}_1(I_1) \) in \( \Sigma \) contains points \( \eta_1 \) and \( \eta_4 \) mapped by \( q \) into the non-closed orbits \( G \cdot \omega_1 \) and \( G \cdot \omega_4 \), respectively (possibly the orbits \( G \cdot \omega_1 \) and \( G \cdot \omega_4 \) coincide). Then the singular orbit \( G \cdot \omega_2 \) is contained in \( q(\Sigma) \).

(ii) Let \( \hat{\ell}_3 : I_3 \to \Sigma \) be a lifting of the slice \( \ell_3 \). Assume that the closure of \( G \cdot \hat{\ell}_3(I_3) \) in \( \Sigma \) contains points \( \eta_2 \) and \( \eta_3 \) mapped by \( q \) into the non-closed orbits \( G \cdot \omega_2 \) and \( G \cdot \omega_3 \), respectively (possibly the orbits \( G \cdot \omega_2 \) and \( G \cdot \omega_3 \) coincide). Then the singular orbit \( G \cdot \omega_2 \) is contained in \( q(\Sigma) \).
Proof. (i) We begin by showing that $\Sigma$ contains an open $G$-invariant set which is biholomorphic to the domain $\Omega = G \cdot (\ell_1(I_1) \cup w_1 \cup w_4 \cup C^*)$ of Lemma 7.2. By the $G$-equivariance of $q$, the description of the slice representation at $z_2$ given in Remark 4.2 and Proposition 5.7, there exists a $G$-invariant neighbourhood $V$ of $\eta_1$ in $\Sigma$ on which $q$ is injective. Its image $q(V)$ intersects the slice $\ell_2$ in $\ell_2((0, \epsilon))$, for some $\epsilon > 0$. By (i) of Lemma 7.1 the map $s \mapsto (q|_V)^{-1}(\ell_2(s))$, with $s \in (0, \epsilon)$, extends to a lifting $\tilde{\ell}_2: I_2 \to \Sigma$ of $\ell_2$. A similar argument yields a lifting $\tilde{\ell}_4: I_4 \to \Sigma$ of $\ell_4$. Since $q$ is injective on the set $\tilde{\ell}_2(I_1) \cup \eta_1 \cup \tilde{\ell}_2(I_2) \cup \eta_4 \cup \tilde{\ell}_4(I_4)$, as well as on every $G$-orbit (cf. Proposition 5.7), it is injective on the $G$-invariant subdomain of $\Sigma$ given by

$$W := G \cdot \tilde{\ell}_1(I_1) \cup \eta_1 \cup \tilde{\ell}_2(I_2) \cup \eta_4 \cup \tilde{\ell}_4(I_4) .$$

Note that $q(W) = \Omega$. In particular $W$ is biholomorphic to $\Omega$, as claimed.

Let $C = \exp \mathfrak{c} \cdot z_2$ be the standard Cartan subset in $G^C/K_C$ starting at $z_2$. Recall that $T := \exp \mathfrak{c}$ is a compact torus in $G$. By Heinzner’s globalization theorem (He11, Sect. 6.6), the space $\Sigma$ can be embedded in its $T_C$-globalization $\Sigma^*$, as a $T$-invariant, orbit-convex domain. By definition, this means that the intersection of $\Sigma$ with an exp $i \cdot t$-orbit in $\Sigma^*$ is necessarily connected.

Consider now the induced local $T_C$-orbit of a point $\zeta \in \tilde{\ell}_1(I_1)$ in $\Sigma$. Since $q|_W$ is biholomorphic and $G$-equivariant, by Lemma 7.2, such orbit is in fact global. Let $C$ be a generator of the abelian subalgebra $\mathfrak{c}$. For every fixed $s > 0$, one has

$$\lim_{n \to \infty} \exp(isC) \cdot \tilde{\ell}_1(1/n) = \tilde{\ell}_2(s) \in W,$$

and

$$\lim_{n \to \infty} \exp(-isC) \cdot \tilde{\ell}_1(1/n) = \tilde{\ell}_4(s) \in W .$$

By the orbit-convexity of $\Sigma$ in its $T_C$-globalization, the sequence $\{ \tilde{\ell}_1(1/n) \}_n$ converges to a point $\zeta_2 \in \Sigma$. By the continuity of $q$, one has $q(\zeta_2) = z_2$. Therefore $z_2 \in q(\Sigma)$, as required.

Part (ii) is proved by showing that $\Sigma$ contains an open subset biholomorphic to the domain $\Omega'$ of Lemma 7.2 and arguing as in the previous case. \hfill \Box

Let $G$ be a connected Lie group and let $\tilde{G} \to G = \tilde{G}/\Gamma$ be a covering of $G$. If $X$ is a $G$-manifold, it can be regarded as a $\tilde{G}$-manifold by letting $\Gamma$ act trivially on it.

Lemma 7.4. Let $G$ be a connected, real Lie group and let $\tilde{G} \to G = \tilde{G}/\Gamma$ be a finite covering of $G$. Let $X$ be a complex $G$-manifold with the property that every Stein, $G$-equivariant Riemann domain over $X$ is univalent. Let $q: \Sigma \to X$ be a Stein, $G$-equivariant Riemann domain. Then

(i) the image $q(\Sigma)$ is biholomorphic to the quotient $\Sigma/\Gamma$ and $q: \Sigma \to q(\Sigma)$ can be identified with the quotient map,

(ii) $q$ is a $\tilde{G}$-equivariant covering.

In particular $q(\Sigma)$ is Stein.

Proof. (i) Since $\Gamma$ is a finite subgroup of $\tilde{G}$, the quotient $\Sigma/\Gamma$ can be regarded as the categorical quotient of $\Sigma$ with respect to $\Gamma$. Then $\Sigma/\Gamma$ is a Stein space and the quotient map $\pi: \Sigma \to \Sigma/\Gamma$ is holomorphic (cf. Thm. 211). Moreover, since $q$ is
\(\Gamma\)-invariant, there exists a \(G\)-equivariant holomorphic map \(\tilde{q} : \Sigma/\Gamma \to X\) making the diagram

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\pi} & \Sigma/\Gamma \\
q & \downarrow & \sqrt{\tilde{q}} \\
X
\end{array}
\]

commute. Since \(q = \tilde{q} \circ \pi\) is locally biholomorphic, then \(\pi\) is also locally biholomorphic. In particular \(\Sigma/\Gamma\) is a manifold and \(\tilde{q} : \Sigma/\Gamma \to X\) is a Stein, \(G\)-equivariant Riemann domain. By the assumption on \(X\), the map \(\tilde{q}\) is injective, implying (i).

(ii) Without loss of generality one may assume that \(\Gamma\) acts effectively on \(\Sigma\). Then the statement follows by showing that \(\Gamma\) acts freely on \(\Sigma\). Assume by contradiction that this is not the case. Then there exists \(\gamma \in \Gamma\) whose fixed point set \(\text{Fix}(\gamma) := \{\zeta \in \Sigma : \gamma \cdot \zeta = \zeta\}\) is not empty. Since \(\text{Fix}(\gamma)\) is a proper analytic subset of \(\Sigma\), it has no interior point. In particular there exist \(\zeta \in \text{Fix}(\gamma)\) and a sequence \(\{\zeta_n\}_n\) in the complement of \(\text{Fix}(\gamma)\) in \(\Sigma\) such that \(\zeta_n \to \zeta\). Note that by the continuity of \(\gamma\), one has \(\gamma \cdot \zeta_n \to \gamma \cdot \zeta = \zeta\).

Let \(U\) be an open neighborhood of \(\zeta\) on which \(\pi\) is injective. Then, for \(n\) large enough, both \(\zeta_n\) and \(\gamma \cdot \zeta_n\) lie in \(U\). Since \(\Gamma\) acts trivially on \(\Sigma/\Gamma\), it follows that \(\pi(\zeta_n) = \gamma \cdot \pi(\zeta_n) = \pi(\gamma \cdot \zeta_n)\). On the other hand since \(\zeta_n \notin \text{Fix}(\gamma)\), one has \(\gamma \cdot \zeta_n \neq \zeta_n\). This gives a contradiction and concludes the proof of the lemma. \(\square\)

Recall the following consequence of the uniqueness of path-liftings on Riemann domains, which will be often used in the proof of the main theorem of this section.

**Lemma 7.5.** Let \(q : \Sigma \to Z\) be a Riemann domain and let \(W\) be a domain of \(\Sigma\) such that the restriction \(q|_W : W \to Z\) is bijective. Then \(W = \Sigma\).

Next comes the main result of this section.

**Theorem 7.6.** Let \(G/K\) be a non-compact, rank-one, Riemannian symmetric space. Assume that \(G\) is a connected, simple, real Lie group which is embedded in its universal complexification \(G^C\) and is different from \(SL(2, \mathbb{R})\). Then a holomorphically separable, \(G\)-equivariant Riemann domain \(q : \Sigma \to G^C/K^C\) is univalent.

**Proof.** Recall that \(\Sigma\) admits a \(G\)-equivariant holomorphic embedding into its envelope of holomorphy. Thus we may assume that \(\Sigma\) is Stein (cf. Sect. 2). We prove the theorem in the case when the \(G\)-orbit diagram of \(G^C/K^C\) is of type \([\mathbb{F}]\), namely for \(g = \mathfrak{su}(n, 1)\). In all remaining cases, but \(G = \text{Spin}(3, 1)\) which is discussed separately, the statement follows from the same arguments with fewer steps.

So we first assume that \(G\) is different from \(\text{Spin}(3, 1)\) and divide the proof in three subcases, depending on the image of \(\Sigma\) in \(G^C/K^C\). Finally we discuss the case \(G = \text{Spin}(3, 1)\).

(i) The image \(q(\Sigma)\) contains the singular orbit \(G \cdot z_2\). We begin by proving that there exists a \(G\)-invariant domain \(V \subset \Sigma\) with the properties that \(q\) is injective on \(V\) and

\[
q(V) = G \cdot \left(\ell_1(1) \bigcup_{j=1}^4 (\ell_j(I_j) \cup W_j) \cup z_2\right).
\]

The extended slices \(\ell_j : I_j \to G^C/K^C\) are defined in \([\mathbb{E}]\). Let \(\zeta_2\) be an element in \(q^{-1}(z_2)\) and let \(U\) be an open neighborhood of \(\zeta_2\) in \(\Sigma\) on which the restriction
Moreover, since \( q \) intersects \( G \) non-closed \( G \)-orbits containing \( G \cdot z_2 \) in their closures. By Lemma 7.1 each extended slice \( \ell_j \) admits a lifting \( \tilde{\ell}_j : \tilde{I}_j \to \Sigma \) such that

\[
\tilde{\ell}_j(t) = (q|U)^{-1}\ell_j(t), \quad t \in (0, \epsilon).
\]

For \( j = 1, \ldots, 4 \), choose points \( \eta_j \in (q|U)^{-1}(G \cdot w_j) \). Consider then the open \( G \)-invariant set in \( \Sigma \)

\[
V := G \cdot \left( \tilde{\ell}_1(1) \bigcup_{j=1}^{4} (\tilde{\ell}_j(I_j) \cup \eta_j) \cup \zeta_2 \right).
\]

Since \( q \) is injective on each lifted slice \( \tilde{\ell}_j \) and on all \( G \)-orbits (cf. Proposition 5.7), it is injective on \( V \) as well. Hence \( V \) is the open \( G \)-invariant domain in \( \Sigma \) with the required properties.

Consider a sequence \( \{z_n\} \) in \( G \cdot \ell_3(J) \), converging to a point on the boundary orbit \( G \cdot z_2 \). Recall that the Levi form of \( G \cdot w_5 \) is indefinite (see Rem. 0.10). Then, by arguing as in the proof of (ii) in Lemma 7.1, the domain \( V \) can be enlarged to an invariant domain \( W \) in \( \Sigma \) with the properties that the restriction \( q|W \) is injective and \( q(W) = G^C/K^C \). By Lemma 7.3 one has \( W = \Sigma \) and the theorem follows.

(ii) The image \( q(\Sigma) \) does not contain the orbit \( G \cdot z_2 \), but contains a non-closed \( G \)-orbit. Assume for example that \( w_1 \in q(\Sigma) \) and let \( \eta_1 \in q^{-1}(w_1) \). By the \( G \)-equivariance of \( q \), the description of the slice representation at \( z_2 \) given in Remark 5.2 and Proposition 5.7 there exists a \( G \)-invariant neighbourhood \( V \) of \( \eta_1 \) in \( \Sigma \) on which \( q \) is injective. Its image \( q(V) \) intersects the slices \( \ell_1 \) and \( \ell_2 \) in the sets \( \ell_1((0, \epsilon)) \) and \( \ell_2((0, \epsilon)) \), for some \( \epsilon > 0 \). Arguing as in previous case, one can construct a \( G \)-invariant domain \( V \subset \Sigma \) with the properties that \( q \) is injective on \( V \) and

\[
q(V) = G \cdot (\ell_1(\tilde{I}_1) \cup w_1 \cup \ell_2(\tilde{I}_2)).
\]

If \( V = \Sigma \) (this is possible by Theorem 6.1), then the map \( q \) is injective, as desired. If \( V \neq \Sigma \), then there exists a point \( \eta \) in the closure of \( V \) in \( \Sigma \) which is mapped by \( q \) into one of the non-closed orbits \( G \cdot w_2 \) or \( G \cdot w_4 \). Assume that \( q(\eta) \) lies in \( G \cdot w_4 \). Then by (i) of Lemma 7.3 the image \( q(\Sigma) \) necessarily contains \( G \cdot z_2 \), contradicting the current assumption.

If \( q(\eta) \in G \cdot w_2 \), iterating the procedure of lifting slices and orbits, we can enlarge \( V \) to an invariant domain \( W \) in \( \Sigma \) on which \( q \) is injective and such that

\[
q(W) = G \cdot (\ell_1(\tilde{I}_1) \cup w_1 \cup \ell_2(\tilde{I}_2) \cup w_2 \cup \ell_3(\tilde{I}_3)).
\]

In particular \( W \) is biholomorphic to \( q(W) \), which is not Stein by Theorem 6.1. Hence \( W \) is a proper subset of \( \Sigma \) and there exists a point \( \eta \) in the closure of \( W \) in \( \Sigma \) whose image \( q(\eta) \) lies either in \( G \cdot w_3 \) or in \( G \cdot w_4 \). In both cases Lemma 7.3 implies that \( q(\Sigma) \) contains \( G \cdot z_2 \), contradicting the current assumption. In conclusion, if \( q(\Sigma) \) does not contain the singular orbit \( G \cdot z_2 \) but contains the non-closed orbit \( G \cdot w_1 \), then \( q \) is injective. For the other non-closed \( G \)-orbits, the theorem can be proved by arguing in a similar way.

(iii) The image \( q(\Sigma) \) contains no non-closed \( G \)-orbits. This assumption implies that the image \( q(\Sigma) \) contains none of the singular orbits lying in the closure of a non-closed \( G \)-orbit. More precisely \( q(\Sigma) \) contains neither \( G \cdot z_2 \) nor \( G \cdot z_3 \). Note that the hypersurfaces \( G \cdot \left( z_2 \bigcup_{j=1}^{4} w_j \right) \) and \( G \cdot (z_3 \cup w_5) \) disconnect \( G^C/K^C \). Therefore there exists a slice \( \ell = \ell_j \), for some \( j = 1, \ldots, 5 \), such that \( q(\Sigma) = G \cdot \ell(J) \), for some interval \( J \subset \hat{I} \) which is open in \( \hat{I} \). Define \( M := q^{-1}(\ell(J)) \). One has that \( \Sigma = G \cdot M \). Moreover, since \( q \) is injective on \( G \)-orbits (see Prop. 5.7) and every orbit in \( q(\Sigma) \) intersects \( \ell(J) \) in a single point, every \( G \)-orbit in \( \Sigma \) intersects \( M \) in a single point.
as well. As a consequence, the surjective map \( \Pi : \Sigma \to M \), given by \( \zeta \mapsto G \cdot \zeta \cap M \), is well defined.

**Claim.** The map \( \Pi \) is continuous.

**Proof of the Claim.** Let \( N \) be an open set in \( M \). We prove the claim by showing that for every \( m \in N \) and \( \zeta \in \Pi^{-1}(m) \), there exists an open neighbourhood of \( \zeta \) in \( \Sigma \) which is contained in \( \Pi^{-1}(N) \). By construction \( \zeta = g \cdot m \), for some \( g \in G \). Let \( V \) be an open neighborhood of \( m \) in \( \Sigma \) on which \( q \) is injective. Choose an open interval \( J' \subset J \) such that \( q(m) \in \ell(J') \subset \ell(V) \). Note that \( q(m) \) either sits on a principal \( G \)-orbit, or on the singular orbit \( G \cdot z_1 \cong \mathbb{G}/K \). Let \( \ell(J') \) be the lifting of \( \ell(J') \) via the restriction \( q|_V \). By shrinking \( \ell' \) if necessary, one can find an open neighborhood \( U \) of the identity in \( G \) such that \( U \cdot \ell(J') \) is open and contained in \( q(V) \). This fact is clear if \( q(m) \) lies on a principal \( G \)-orbit (see diagram (9)). If \( q(m) \) lies on the singular orbit \( G \cdot z_1 \), it follows from the equivariant embedding \( (\Pi) \) at \( z_1 \) and the compactness of the isotropy subgroup \( G_{z_1} \cong K \).

As a result, \( U \cdot \ell(J') = (q|_V)^{-1}(U \cdot \ell(J')) \) is an open neighbourhood of \( m \) in \( \Sigma \) and \( gU \cdot \ell(J') \) is an open neighbourhood of \( \zeta \) contained in \( \Pi^{-1}(N) \). Hence \( \Pi^{-1}(N) \) is open in \( \Sigma \), as wished (one can show that \( M \cong \Sigma/G \) and that \( \Pi \) can be identified with the quotient map).

By the above claim, \( M \) is connected and is a one-dimensional real-analytic submanifold of \( \Sigma \). It follows that \( q \) is injective on \( M \). Moreover \( M \) and \( q(M) \) are slices for the \( G \)-action in \( \Sigma \) and \( q(\Sigma) \), respectively. Since \( q \) is injective on \( G \)-orbits, it is injective on \( \Sigma \) implying the theorem.

(iv) **The group \( G \) is Spin\((3,1)\).** Assume by contradiction that \( q : \Sigma \to G^C/K^C \) is not univalent. Recall that the center of \( G \) acts trivially on \( G^C/K^C \) and that by (i), (ii) and (iii) the statement holds true for the group \( SO_0(3,1) \). Then Lemma 7.4 applies to show that the restriction of \( q \) to every \( G \)-orbit is a double covering and the image \( q(\Sigma) \) is Stein. On the other hand, by Theorem [6.7] all Stein \( G \)-invariant domains in \( G^C/K^C \) contain a singular orbit diffeomorphic to \( G/K \). Since \( G/K \) is simply connected, this gives a contradiction. This concludes the proof of the theorem. \( \square \)

When \( G = SL(2,\mathbb{R}) \), non-injective, Stein \( G \)-equivariant Riemann domains over \( G^C/K^C \) do exist. Next we construct one such Riemann domain explicitly. It turns out that such an example is essentially the only possible one. Indeed by Lemma 7.4 if \( q : \Sigma \to G^C/K^C \) is a Stein, \( G \)-equivariant Riemann domain which is not univalent, then the center \( \Gamma = \{ \pm I_2 \} \) acts freely on \( \Sigma \). Moreover, \( q \) is a \( G \)-equivariant covering onto its image \( q(\Sigma) \) which turns out to be Stein. It follows that the restriction of \( q \) to every \( G \)-orbit is a double covering. Thus the singular \( G \cdot z_1 \) and \( G \cdot z_3 \), which are simply connected, cannot lie in \( q(\Sigma) \). Then, by Theorem [6.7] the image \( q(\Sigma) \) coincides with a domain \( S_i(b) \), for some \( i = 1, 2 \) and \( b \geq 0 \). For every \( S_i(b) \) there is exactly one \( G \)-equivariant double covering. In the example below we carry out its construction for \( q(\Sigma) = S_1(0) \).

**Example 7.7.** Let \( G = SL(2,\mathbb{R}) \). Consider the Stein domain \( S_1(0) \) in \( G^C/K^C \) defined in [14]. Let

\[
\ell_2 : \mathbb{R}^{>0} \to G^C/K^C, \quad \ell_2(s) := \exp(isC)z_2
\]

be the slice map defined in (9) of Section 4. The isotropy subgroup in \( G \) of every point \( \ell_2(s) \) coincides with \( \{ \pm I_2 \} \) (cf. Remarks [5.4] and [4.1]). It follows that \( S_1(0) := G \cdot \ell_2(\mathbb{R}^{>0}) \) is topologically equivalent to \( SO_0(2,1) \times \mathbb{R}^{>0} \). Define \( \Sigma := G \times \mathbb{R}^{>0} \). Since \( G \) is a double covering of \( SO_0(2,1) \), the map

\[
q : \Sigma \to S_1(0), \quad (g, s) \to g\ell_2(s)
\]
defines a double covering of $S_1(0)$. As a consequence, with the complex structure pulled back from $S_1(0)$ the manifold $\Sigma$ is Stein (\textbf{St}) and the map $q$ is a holomorphic covering. In other words, $q: \Sigma \rightarrow S_1(0)$ defines a non-univalent Stein, $G$-equivariant Riemann domain over $G^C/K^C$. \hfill \Box

\textbf{Remark 7.8.} By the results of Lemma \textbf{7.4} one can show that Theorem \textbf{7.6} also holds for $G$ not embedded in $G^C$, provided that the center $\Gamma$ of $G$ is finite and $G$ is not a covering of $SL(2, \mathbb{R})$ (cf. (iv) in the proof of Theorem \textbf{7.6}). If $G$ is a covering of $SL(2, \mathbb{R})$, a construction similar to the one in Example \textbf{7.7} yields a non-univalent, Stein $G$-equivariant Riemann domain over $G^C/K^C$. \hfill \Box

8. Univalence over $G^C$

Let $G$ be a connected, non-compact, real simple Lie group, $K \subset G$ a maximal compact subgroup and $G^C$ its universal complexification. In this section we prove a univalence result for $G \times K$-equivariant Riemann domains over $G^C$, when the symmetric space $G/K$ has rank one. We also discuss some examples.

\textbf{Theorem 8.1.} Let $G/K$ be a non-compact, rank-one, Riemannian symmetric space. Assume that $G$ is a connected, simple, real Lie group which has finite center and is not a covering of $SL(2, \mathbb{R})$. Then a holomorphically separable, $G \times K$-equivariant Riemann domain $p: Y \rightarrow G^C$ is univalent.

\textbf{Proof.} Recall that $Y$ admits a $G \times K$-equivariant holomorphic embedding into its envelope of holomorphy. Thus we may assume that $Y$ is Stein (cf. Sect. 2). Consider the induced Stein, $G$-equivariant Riemann domain $q: Y//K \rightarrow G^C/K^C$ constructed in Section 3. By Remark \textbf{7.8} the map $q$ is injective. Then, by Corollary \textbf{3.2} the Riemann domain $p: Y \rightarrow G^C$ is univalent, as wished. \hfill \Box

When $G$ is either $SL(2, \mathbb{R})$ or a non-trivial covering of $SL(2, \mathbb{R})$, a construction similar to the one in Example \textbf{7.7} yields examples of non-univalent, Stein, $G \times K$-equivariant Riemann domains over $G^C$.

\textbf{Example 8.2.} Let $G = SL(2, \mathbb{R})$ and let $S_1(0)$ be the Stein, $G$-invariant domain in $G^C/K^C$ defined in (\textbf{14}). As we observed in Example \textbf{7.7} the domain $S_1(0)$ is diffeomorphic to $SO_0(2,1) \times \mathbb{R}^>$. Define $\Omega := \pi^{-1}(S_1(0))$, where $\pi: G^C \rightarrow G^C/K^C$ is the canonical projection. Since $\pi$ is holomorphic and both $S_1(0)$ and $G^C$ are Stein, the domain $\Omega$ is Stein as well. Consider the slice $\ell_2: \mathbb{R}^> \rightarrow G^C/K^C$ (cf. (6) of Sect. 4) and its lifting to $G^C$ defined by $\tilde{\ell}_2(s) := \exp(i \theta \bar{C}) \exp(iA_2)$. Note that the map $SO_0(2,1) \times \mathbb{R}^> \times K^C$, given by $(g, s, k) \mapsto g\tilde{\ell}_2(s)k^{-1}$, is a diffeomorphism. Define $Y := G \times \mathbb{R}^> \times K^C$. By construction the map

$$p: Y \rightarrow \Omega, \quad (g, s, k) \mapsto g\tilde{\ell}_2(s)k^{-1}$$

is a double covering. With the complex structure pulled back from $\Omega$, the manifold $Y$ is Stein (\textbf{St}) and the map $p$ is holomorphic. Let $G \times K$ act on $Y$ by $(l, h) \cdot (g, s, k) := (lg, s, hk)$ and on $\Omega$ by left and right translations. Then $p$ defines a non-univalent, Stein, $G \times K$-equivariant Riemann domain over $G^C$. \hfill \Box
Let $G = K \times N$ be the product of a compact Lie group and a simply connected nilpotent Lie group. Then a holomorphically separable, $G$-equivariant Riemann domain over $G^C$ is necessarily univalent (see [CI], [Ia], [CIT]). The above example shows that an analogous statement does not hold for a semisimple Lie group $G$. Next we exhibit a different counter-example for $G = SO_0(2,1)$, a group which meets the assumptions of Theorem 8.1. Such example was pointed out to us by K. Oeljeklaus. We are not aware of similar constructions in higher dimension. That is, if dimension of $G/K$ is greater than two, univalence of holomorphically separable, $G$-equivariant Riemann domains over $G^C$ seems to be an open question.

**Example 8.3.** Let $G = SO_0(2,1)$. Then $G^C = SO(2,1,\mathbb{C})$ and $K^C = SO(2,\mathbb{C})$. Let $S_1(0)$ be the $G$-invariant Stein domain in $G^C/K^C$ defined in (14) and let $\Omega = \pi^{-1}(S_1(0))$, where $\pi: G^C \rightarrow G^C/K^C$ is the canonical projection. As we already observed in Example 8.2, the domain $\Omega$ is a Stein, $G$-invariant domain in $G^C$ which is diffeomorphic to $G \times \mathbb{R}^{>0} \times K^C$. Denote by $K^C$ the universal covering of $K^C$ and by \( \psi: K^C \rightarrow K^C \) the covering homomorphism. Define $Y := G \times \mathbb{R}^{>0} \times \tilde{K}^C$ and let $G$ act on $Y$ by left translations. Consider the slice $\tilde{\ell}_2: \mathbb{R}^{>0} \rightarrow G^C/K^C$ (cf. [10] of Sect. 4) and its lifting to $G^C$ given by $\tilde{\ell}_2(s) := \exp(isC)\exp(iA_2)$. Define a $G$-equivariant covering of $\Omega$ by $p: Y \rightarrow \Omega$, \[(g, s, k) \mapsto g\tilde{\ell}_2(s)\psi(k^{-1}).\]

With the complex structure pulled back from $\Omega$, the manifold $Y$ is Stein ([St]) and the map $p$ is holomorphic. In particular $p: Y \rightarrow \Omega$ defines a non-univalent, Stein, $G$-equivariant Riemann domain over $G^C$. 

**Remark.** One can show that $\Omega$ is a holomorphically trivial $\mathbb{C}^*$-bundle over $S_1(0)$. Thus it is biholomorphic to $S_1(0) \times \mathbb{C}^*$ and consequently $Y$ is biholomorphic to $S_1(0) \times \mathbb{C}$. After identifying $S_1(0)$ with $SO_0(2,1) \times \mathbb{R}^{>0}$, one sees that the map $SO_0(2,1) \times \mathbb{R}^{>0} \rightarrow G^C$, given by $(g, s) \mapsto g\tilde{\ell}_2(s)$, defines a global $C^\infty$-section of the holomorphic $\mathbb{C}^*$-bundle $\pi|\Omega: \Omega \rightarrow S_1(0)$. Hence such bundle is differentiably trivial and, by Oka principle, is also holomorphically trivial (cf. [Gi]), as claimed.

For the sake of completeness, we explicitly construct a trivialization on the model of $G^C/K^C$ discussed in Example 1.7 and Remark 1.8.

Let $G = SU(1,1)$ and identify $G^C/K^C$ with $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{(z,w)_{1,1} = 0\}$. Note that $S_1(0)$ corresponds to the subset $\{(1 : u, [\bar{v} : 1]) : u, v \in \Delta, u \neq v\}$ (cf. Example 8.2). Denote by $D$ the diagonal in $\Delta \times \Delta$. Then the injective holomorphic map $\Delta \times \Delta \setminus D \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{(z,w)_{1,1} = 0\}$ defined by 

\[(u,v) \mapsto ([1 : u], [\bar{v} : 1])\]

identifies $\Delta \times \Delta \setminus D$ with $S_1(0)$. The map $\Delta \times \Delta \setminus D \rightarrow G^C$ given by 

\[(u,v) \mapsto \left( \begin{array}{c} \frac{1}{v} \\ \frac{u-v}{w-v} \end{array} \right)\]

defines a global holomorphic section of the $\mathbb{C}^*$-bundle $\pi|\Omega: \Omega \rightarrow S_1(0)$, since one has 

\[
\left( \begin{array}{c} \frac{1}{v} \\ \frac{u-v}{w-v} \end{array} \right) \cdot ([0 : 1], [0 : 1]) = ([1 : u], [\bar{v} : 1]).
\]

As a consequence the map $(\Delta \times \Delta) \setminus D \times \mathbb{C}^* \rightarrow \Omega$, given by 

\[(u,v,\lambda) \mapsto \left( \begin{array}{c} \frac{1}{\lambda} \\ \frac{1}{\lambda v-u} \end{array} \right) \cdot \left( \begin{array}{cc} \lambda^{-1} & 0 \\ 0 & \lambda \end{array} \right)\]

defines a biholomorphism from $S_1(0) \times \mathbb{C}^*$ onto $\Omega$. \qed
9. Appendix. The Levi form of non-closed hypersurface orbits.

In this section we outline the computation of the Levi form of non-closed hypersurface $G$-orbits in $G^C/K^C$. The results of this computation were used in Section 6, where we completed the classification of Stein $G$-invariant domains in $G^C/K^C$. Recall that every real hypersurface $S$ in a complex manifold inherits a $CR$-structure of hypersurface type. Let $J$ denote the complex structure of the ambient manifold.

For every $x \in S$, the tangent space to $S$ at $x$ decomposes as

$$TS_x = T_C S_x \oplus NS_x,$$

where $T_C S_x = TS_x \cap J(TS_x)$ is a complex subspace of $TS_x$, called the complex tangent space, and $NS_x$ is a one-dimensional real subspace. Denote by $TS = T_C S \oplus NS$ the tangent bundle of $S$. The subbundle $(T_C S)^C \subset TS^C$ of the complexified tangent bundle $TS^C$ decomposes as $HS \oplus AS$, where $HS$ and $AS$ denote its $(1,0)$ and $(0,1)$ components, respectively. Let $Z$ be a tangent vector in $T_C S_x$ and $\hat{Z}$ an arbitrary extension of $Z$ to a local section of $T_C S$. Then the vector fields $\frac{i}{4}(\hat{Z} - iJ\hat{Z})$ and $\frac{i}{4}(\hat{Z} + iJ\hat{Z})$ define local sections of the bundles $HS$ and $AS$, respectively. The Levi form of $S$ at $z$ is the hermitian form $L_x : T_C S_x \times T_C S_x \to (TS_x)^C/(T_C S_x)^C$ defined by

$$L_x(Z, W) := \frac{i}{4}[(\hat{Z} - iJ\hat{W})_x \mod (T_C S)^C].$$

In the hypersurface case, $(TS_x)^C/(T_C S_x)^C$ is a one-dimensional complex vector space. When $Z$ varies in $T_C S_x$, the image of the quadratic form $L_x(Z, Z)$ is contained in its real part, which can be identified with $NS_x \cong \mathbb{R}$. We say that the Levi form of $S$ is definite if $\{L_x(Z, Z)\}$ is a halfline in $NS_x$; that it is indefinite if $\{L_x(Z, Z)\}$ coincides with $NS_x$; that it is identically zero if $\{L_x(Z, Z)\} = \{0\}$ (for more details, see [16]).

9.1. Non-closed orbits with a totally real singular orbit in their closure.

We first consider non-closed $G$-orbits which contain in their closure the orbit of a point $z = \exp iAK^C \in A_0$, satisfying the condition $\alpha(A) = \pi/2$, with $\alpha$ a simple restricted root (see 2 and 7 in Sect. 4). The singular orbit $G \cdot z$ is diffeomorphic to a rank-one, pseudo-Riemannian symmetric space $G/H$, embedded in $G^C/K^C$ as a totally real submanifold of maximal dimension. Let $(g = h \oplus q, \tau)$ be the corresponding symmetric algebra. Non-closed $G$-orbits in $G^C/K^C$ containing $G \cdot z$ in their closure are in one-to one correspondence with the nilpotent $Ad_H$-orbits in $q$ (cf. (1) and Remark 4.2).

Let $X$ be an element in $q$ and let $x = \exp iX \cdot z$ be the corresponding point in $G^C/K^C$. Denote by $S$ the $G$-orbit of $x$. Denote by $\pi : G^C \to G^C/K^C$ the canonical projection and by $\pi_*$ its differential. Then the tangent space to $S$ at $x$ is generated by the vector fields induced on $G^C/K^C$ by the one-parameter subgroups in $G$, via the map

$$(17) \quad \pi^* : g \to T(G^C/K^C)_x, \quad X \mapsto X^* := (\pi_*)_x \left( \frac{d}{dt} \big|_{t=0} \exp tX \right).$$

Observe that $T(G^C/K^C)_z \cong q^C$ and $T(G^C/K^C)_x \cong Ad_x q^C$. Hence the vector $X^*$ is the $Ad_x q^C$-component of $X$ in the decomposition $g^C = Ad_x h^C \oplus Ad_x q^C$.

To explicitly determine base points for such non-closed orbits and their tangent spaces, we decompose $g$ by an appropriate restricted root system. Fix a maximal abelian subalgebra $b \subset h \cap p$. Since $g$ has real rank one, $\dim b = 1$ and $Z_g(b) = b \oplus Z_r(b)$. Let $\Delta_b$ be the restricted root system of $g$ with respect to $b$ and let

$$g = g^0 \oplus g^{\pm \lambda} \oplus g^{\pm 2\lambda}, \quad g^0 = Z_g(b)$$

be the decomposition of $g$.
be the corresponding restricted root decomposition. Every root space $g^\lambda$ is $\tau_\varphi$-stable. For every $\mu \in \Delta_b \cup \{0\}$, we indicate by $g^\mu_b$ and $g^\mu_q$ the intersections of $g^\mu$ with $h$ and $q$, respectively. In particular, we have a combined decomposition

$$g = h \oplus q,$$

where $h = g^0_{q \cap h} \oplus g^{+\lambda} + g^{-2\lambda} \oplus b$, and $q = g^0_{q \cap q} \oplus g^{+\lambda} \oplus g^{+2\lambda}$. Here $g^0_{q \cap h}$ and $g^0_{q \cap q}$ denote the intersections of $Z_T(b)$ with $h$ and $q$ respectively. Note that, by the real rank one condition, $g^0_{q \cap r}$ coincides with $g^0_q$. If the restricted root system $\Delta_b$ is reduced, then $g^{\pm 2\lambda} = \{0\}$.

**Lemma 9.1.** Let $g$ be a simple real Lie algebra of real rank one, with reduced restricted root system (i.e. $g = so(n,1)$). Then the following facts hold:

(i) $\dim g^{\pm\lambda}_q = 1$;

(ii) $[g^{\lambda}_q, g^{-\lambda}_h] = g^0_q$, $[g^{-\lambda}_q, g^\lambda_h] = g^0_q$.

**Proof.** Observe that $\theta g^{\lambda}_q = g^{-\lambda}_q$. Hence $g_q[\lambda] := g^{\lambda}_q \oplus g^{-\lambda}_q$ is a $\theta$-stable subspace of $q$ and $\dim g_q[\lambda] \cap p = \dim g^\lambda_q$. Since $g^0_q \subset \mathfrak{t}$ and $\dim \mathfrak{p} \cap \mathfrak{q} = 1$ (see the proof of Lemma 4.3 (ii)), statement (i) holds. Statement (ii) can be verified directly. □

**Lemma 9.2.** Let $g$ be a real simple Lie algebra of real rank one, with non-reduced restricted root system (i.e. $g = su(n,1), sp(n,1)$, or $f_4$). Then the following facts hold:

(i) The root spaces $g^{\pm 2\lambda}_q$ are contained in $h$. Therefore $g^{\pm 2\lambda}_q = \{0\}$;

(ii) $\dim g^{\pm\lambda}_q > 1$;

(iii) Fix $X^0_\lambda \in g^\lambda_q$ and denote by $(g^\lambda_q)_0$ a complement of $\mathbb{R}X^0_\lambda$ in $g^\lambda_q$ (resp. by $(g^{-\lambda}_q)_0$ a complement of $\mathbb{R}\theta X^{-\lambda}_\lambda$ in $g^{-\lambda}_q$). Then

$$[X^0_\lambda, g^{0}_{h \cap \mathfrak{t}}] = (g^\lambda_q)_0, \quad [X^0_\lambda, g^{-0}_{h \cap \mathfrak{t}}] = g^0_q, \quad [X^0_{-\lambda}, g^{0}_{h \cap \mathfrak{t}}] = (g^{-\lambda}_q)_0.$$

**Proof.** Real rank one Lie algebras with a non-reduced restricted root system are equal-rank. Hence the root system $\Delta$ of $g^\mathbb{C}$, with respect to a maximally split Cartan subalgebra of $g$ extending $b$, has a real root. Since $\dim g^{2\lambda}_q$ is odd, the restriction of such a root to $b$ coincides with the restricted root $2\lambda$ (cf. [HI], p. 584). Further, by Remark 2.13 in [Ge1], the subalgebra $h$ is a non-compact real form of $Ad_\mathbb{R}^\mathbb{C} \cong \mathfrak{t}^\mathbb{C}$, with respect to the conjugation $\sigma_{\tau_\varphi}|Ad_\mathbb{R}^\mathbb{C}$. Precisely, if $g = su(n,1), sp(n,1)$, or $f_4$, then $h$ is given by $u(n-1,1) \oplus u(1), sp(n-1,1) \oplus sp(1)$ and $so(8,1)$, respectively. Since $h$ is equal-rank, the root spaces $g^{\pm 2\lambda}_q$ have non-trivial intersection with $h$. Statements (i) and (ii) follow then by looking at the dimensions of the restricted root spaces of $h$ and $g$ (cf. [HI], p. 532). Statement (iii) can be verified directly. □

**Reference points for non-closed $G$-orbits.** Let $C = \exp i\mathbf{z}$ be the standard Cartan subset in $G^\mathbb{C}/K^\mathbb{C}$ with base point $z$. Recall that $\mathbf{e} = \mathbb{R}(X + \theta(X))$, where $X$ is a non-zero vector in $g^\alpha$ (here $g^\alpha$ is a restricted root space with respect to the adjoint action of $a \subset p$, as in Sect. 4). Normalize the triple $\{X, \theta(X), A := [\theta(X), X]\}$ so that $\alpha(A) = 2$. Define $B := X - \theta(X)$ and $b := \mathbb{R}(X - \theta(X))$. One easily verifies that $b$ is a maximal abelian subalgebra in $h \cap p$. If the restricted root system $\Delta_b$ is reduced, then

$$X^0_\lambda = \frac{1}{2}(A - (X + \theta X)), \quad X^0_{-\lambda} = \frac{1}{2}(A + (X + \theta X)).$$
are generators of the one-dimensional spaces $g_\lambda^1$ and $g_{\lambda}^{-1}$, respectively. They satisfy the relations
\[ [B, X_0^0] = 2X_\lambda^0, \quad [B, X_{-\lambda}^0] = -2X_{-\lambda}^0, \quad [X_\lambda^0, X_{0\lambda}^0] = B, \quad \theta X_\lambda^0 = -X_{-\lambda}^0. \]
The vectors
\[ X_\lambda^0, \quad X_{-\lambda}^0, \quad -X_0^0, \quad -X_{-\lambda}^0 \]
are a complete set of representatives of the nilpotent $Ad_H$-orbits in $\mathfrak{q}$. The corresponding points in $G^C/K^C$
\[ x_0 = \exp iX_\lambda^0 \cdot z, \quad x_1 = \exp iX_{-\lambda}^0 \cdot z, \quad y_0 = \exp(-iX_\lambda^0) \cdot z, \quad y_1 = \exp(-iX_{-\lambda}^0) \cdot z \]
lie on non-closed $G$-orbits containing the singular orbit $G \cdot z$ in their closures. In the orbit diagram (2), the $G$-orbits of $x_0, x_1, y_0, y_1$ are represented by $w_3, w_2, w_1, w_4$, respectively. If $\dim G/K > 2$ the points $x_0$ and $x_1$ lie on the same $G$-orbit and likewise the points $y_0$ and $y_1$ (cf. diagram (2)). When the restricted root system $\Delta_b$ is non-reduced, all points $x = \exp iX_\lambda \cdot z$, with $X_\lambda \in g_\lambda^1$, and $y = \exp iX_{-\lambda} \cdot z$, with $X_{-\lambda} \in g_{-\lambda}^1$, lie on the same $G$-orbit. They are represented by $w_3$ in the orbit diagrams (9) and (10).

**Remark 9.3.** When the restricted root system $\Delta_b$ is reduced, the points $x_0$ and $x_1$ lie on the boundary of the Stein domain $D_2(0)$ (cf. Theorem 6.1). The points $y_0$ and $y_1$ lie on the boundary of the Stein domain $D_1(0)$ (cf. Theorem 6.1).

### The tangent space to the $G$-orbit of $x_0$
Denote by $S$ the $G$-orbit of the point $x_0 = \exp iX_\lambda^0 \cdot z$, with $X_\lambda^0 \in g_\lambda^1$. In the next Lemma we determine the generators of the tangent space to $S$ at $x_0$, namely the vectors $X^* \in TS_{x_0}$, for $X \in \mathfrak{g}$.

**Lemma 9.4.**
(i) Let $Y_{2\lambda} \in g_{2\lambda}^b$ and $Y_{\lambda} \in g_\lambda^1$. Then $Y_{2\lambda}^* = Y_{\lambda}^* = 0$.
(ii) Let $X_\lambda \in g_\lambda^1$. Then $X_\lambda^* = Ad_{x_0} X_\lambda$.
(iii) Let $B \in \mathfrak{b}$. Then $B^* = i\lambda(B) Ad_{x_0} X_\lambda^0$.
(iv) Let $W_0 \in g_{0\lambda}^b$. Then $W_0^* = -iAd_{x_0} [X_\lambda^0, W_0]$.
(v) Let $Z_0 \in g_\lambda^0$. Then $Z_0^* = Ad_{x_0} Z_0$.
(vi) Let $Y_{-\lambda} \in g_{-\lambda}$.
(vii) Let $X_{-\lambda} \in g_{-\lambda}$. Then $X_{-\lambda}^* = -iAd_{x_0} [X_{-\lambda}^0, \mathfrak{y}_{\lambda}]$.
(viii) Let $Y_{-2\lambda} \in g_{-2\lambda}$.

\[ x_0 = \exp iX_\lambda^0 \cdot z, \quad x_1 = \exp iX_{-\lambda}^0 \cdot z, \quad y_0 = \exp(-iX_\lambda^0) \cdot z, \quad y_1 = \exp(-iX_{-\lambda}^0) \cdot z \]

\[ X_\lambda^0, \quad X_{-\lambda}^0, \quad -X_0^0, \quad -X_{-\lambda}^0 \]

are generators of the one-dimensional spaces $g_\lambda^1$ and $g_{\lambda}^{-1}$, respectively. They satisfy the relations
\[ [B, X_0^0] = 2X_\lambda^0, \quad [B, X_{-\lambda}^0] = -2X_{-\lambda}^0, \quad [X_\lambda^0, X_{0\lambda}^0] = B, \quad \theta X_\lambda^0 = -X_{-\lambda}^0. \]

**Proof.** All statements are obtained by combining the formula $Ad_{\exp i\lambda Y} = \exp ad_{i\lambda Y}$ with the bracket relations among root vectors. We omit the computations, which are long but straightforward.

Fix $\theta X_\lambda^0 \in g_{-\lambda}$ and denote by $(g_{-\lambda})_0$ a complementary subspace to $\mathbb{R}\theta X_\lambda^0$ in $g_{-\lambda}$.

By (iii) of Lemma 9.2 and Lemma 9.4 the tangent space to $S$ at $x_0$ is given by
\[ TS_{x_0} = T_C S_{x_0} \oplus NS_{x_0}, \]
where
\[ T_C S_{x_0} = Ad_{x_0} (g_{-\lambda}^C) \oplus Ad_{x_0} (g_{\lambda}^1)^C \oplus Ad_{x_0} (g_{-\lambda}^1)^C, \quad NS_{x_0} = \mathbb{R} Ad_{x_0} \theta X_\lambda^0. \]

Note that if $\Delta_b$ is reduced, by (i) of Lemma 9.1 one has $(g_{-\lambda})_0 = \{0\}$. 

Remark 9.5. There exists a basis of $\mathfrak{g}$ so that the above decomposition of $TS_{x_0}$ is orthogonal with respect to the Killing form $B$ of $\mathfrak{g}^C$. If the restricted root system $\Delta_0$ is non-reduced, one can construct it starting from a basis of $\mathfrak{g}^C/\mathfrak{s}^C$ consisting of root vectors with respect to a maximally split Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ extending $\mathfrak{b}$. In the reduced case, this is immediate by (i) of Lemma 9.1. □

The Levi form of the $G$-orbit of $x_0$. The same arguments used in Section 4 of [Ge1] yield the following formulas for the Levi form of $S$ at $x_0$. Let $Z, W$ be vectors in $T_{x_0}S_{x_0}$. Then

$$(20) \quad L_{x_0}(Z, W) = \frac{1}{2}[(*)^{-1}JW, Z] - i\frac{1}{2}[(*)^{-1}W, Z] \mod (T_{x_0}S_{x_0})^C,$$

where $(*)^{-1}JW$ and $(*)^{-1}W$ are arbitrary elements in the preimages of $JW$ and $W$ by the map defined in (17). In the next lemma we compute the Levi form of $S$ at $x_0$. Fix $F^0_\lambda := Ad_{x_0}0_{\lambda}$ as a generator of $NS_{x_0}$.

Lemma 9.6.

(i) Let $X_\lambda \in (\mathfrak{g}_q^{-\lambda})_0$. Set $F^-_\lambda := Ad_{x_0}X_\lambda$. Then

$$L_{x_0}(F^-_\lambda, F^-_\lambda) = -\frac{1}{6}Ad_{x_0}[X_\lambda, 0]_\mathfrak{g} = pF^0_\lambda \mod (T_{x_0}S_{x_0})^C, \quad p \geq 0.$$

(ii) Let $Z_0 \in \mathfrak{g}_q^{-\lambda}$. Write $Z_0 = [X_\lambda, Y_\lambda]$, for some $Y_\lambda \in \mathfrak{g}_q^{-\lambda}$ (cf. Lemma 9.2) and set $F_0 := Ad_{x_0}Z_0$. Then

$$L_{x_0}(F_0, F_0) = -\frac{1}{2}Ad_{x_0}[Y_\lambda, Z_0] = nF^0_\lambda \mod (T_{x_0}S_{x_0})^C, \quad n \leq 0.$$

(iii) Let $X_\lambda \in \mathfrak{g}_q^\lambda$ and set $F_\lambda := Ad_{x_0}X_\lambda$. Then

$$L_{x_0}(F_\lambda, F_\lambda) = 0.$$

(iv) Let $X_\lambda \in (\mathfrak{g}_q^{-\lambda})_0$ and $Z_0 \in \mathfrak{g}_q^0$. Set $F^-_\lambda := Ad_{x_0}X_\lambda$ and $F_0 := Ad_{x_0}Z_0$. Then

$$L_{x_0}(F^-_\lambda, F_0) = 0.$$

(v) Let $X_\lambda \in (\mathfrak{g}_q^{-\lambda})_0$ and $X_\lambda \in \mathfrak{g}_q^\lambda$. Set $F^-_\lambda := Ad_{x_0}X_\lambda$ and $F_\lambda := Ad_{x_0}X_\lambda$. Then

$$L_{x_0}(F^-_\lambda, F_\lambda) = aF^0_\lambda, \quad a \in \mathbb{C}.$$

(vi) Let $X_\lambda \in \mathfrak{g}_q^\lambda$ and $Z_0 \in \mathfrak{g}_q^0$. Set $F_\lambda := Ad_{x_0}X_\lambda$ and $F_0 := Ad_{x_0}Z_0$. Then

$$L_{x_0}(F_0, F_\lambda) = 0.$$

Proof. By way of example, we prove the first two statements of the Lemma. The remaining ones follow in a similar way, and the details are omitted.

(i) Let $F^-_\lambda := Ad_{x_0}X_\lambda$. In order to compute the brackets (20), we invert the relations in Lemma 9.4, and decompose the results in $\mathfrak{g}^C = Ad_{x_0}\mathfrak{h}^C + Ad_{x_0}\mathfrak{q}^C$. Write $X_\lambda = [X_\lambda^0, Y_\lambda]$, for some $Y_{\lambda 0} \in \mathfrak{g}_q^\lambda$ (cf. Lemma 9.2). Then

$$(*)^{-1}JF^-_\lambda = -Y_{\lambda 0} + \frac{1}{6}ad^2_{X_\lambda^0}(Y_{\lambda 0}) =$$

$$= -Ad_{x_0}Y_{\lambda 0} + iAd_{x_0}ad_{X_\lambda^0}(Y_{\lambda 0}) + \frac{1}{2}Ad_{x_0}ad^2_{X_\lambda^0}(Y_{\lambda 0}) -$$

$$- \frac{i}{6}Ad_{x_0}ad^3_{X_\lambda^0}(Y_{\lambda 0}) + \frac{3}{8}Ad_{x_0}ad^4_{X_\lambda^0}(Y_{\lambda 0}).$$

and

$$(*)^{-1}F^-_\lambda = ad_{X_\lambda^0}(Y_{\lambda 0}) + \frac{1}{2}ad^2_{X_\lambda^0}(Y_{\lambda 0}) =$$
By computations similar to the above ones, we have
\[(21)\]
\[= \operatorname{Ad}_{x_0} \operatorname{ad}_{X^0}(Y_{-2\lambda}) - i \operatorname{Ad}_{x_0} \operatorname{ad}_{X^0}(Y_{-2\lambda}) + \frac{i}{6} \operatorname{Ad}_{x_0} \operatorname{ad}_{X^0}(Y_{-2\lambda}).\]

By formulas (20), we obtain
\[L_{x_0}(F_{-\lambda}, F_{-\lambda}) = -\frac{1}{6} \operatorname{Ad}_{x_0}[\operatorname{ad}_{X^0}(Y_{-2\lambda}), \operatorname{ad}_{X^0}(Y_{-2\lambda})] = -\frac{1}{6} \operatorname{Ad}_{x_0}[[X^0_{\lambda}, X_{-\lambda}], X_{-\lambda}] \mod (T_{C}S_{x_0})^C.\]

To complete the proof of the statement, set \(F^0_{\lambda} := \operatorname{Ad}_{x_0} X^0_{\lambda}\) and note that by Remark 9.5, the component \(p F^0_{-\lambda}\) of the above brackets in \(NS_{x_0}\) is given by
\[B(L_{x_0}(F_{-\lambda}, F_{-\lambda}), F^0_{\lambda}) = pB(F^0_{-\lambda}, F^0_{\lambda}).\]

Since \(B(F^0_{-\lambda}, F^0_{\lambda}) = B(X^0_{-\lambda}, t X^0_{-\lambda})\) is negative, the real number \(p\) has the same sign as
\[(21)\]
\[B([[X^0_{\lambda}, X_{-\lambda}], X_{-\lambda}], X^0_{\lambda}) = -B([X^0_{\lambda}, X_{-\lambda}], [X^0_{\lambda}, X_{-\lambda}]).\]

By Lemmas 9.1 and 9.2, the brackets \([X^0_{\lambda}, X_{-\lambda}]\) lie in \(\mathfrak{t}\), so \(B([X^0_{\lambda}, X_{-\lambda}], [X^0_{\lambda}, X_{-\lambda}])\) is non-positive. It follows that \(p \geq 0\), as claimed.

(ii) Write \(Z_0 = [X^0_{\lambda}, Y_{-\lambda}]\), for some \(Y_{-\lambda} \in g_{-\lambda}^0\) (cf. Lemma 9.1 and Lemma 9.2). By computations similar to the above ones, we have
\[(*)_1 J F_0 = -Y_{-\lambda} = -\operatorname{Ad}_{x_0} Y_{-\lambda} + i \operatorname{Ad}_{x_0} \operatorname{ad}_{X^0}(Y_{-\lambda}) + \frac{1}{2} \operatorname{Ad}_{x_0} \operatorname{ad}_{X^0}(Y_{-\lambda}),\]
and
\[L_{x_0}(F_0, F_0) = -\frac{1}{2} \operatorname{Ad}_{x_0} [Y_{-\lambda}, [X^0_{\lambda}, Y_{-\lambda}]] = -\frac{1}{2} \operatorname{Ad}_{x_0} [Y_{-\lambda}, Z_0] \mod (T_{C}S_{x_0})^C.\]

To complete the proof of the statement, observe that \(n = \frac{B(L(F_0, F_0), F^0)}{B(F^0_{-\lambda}, F^0_{\lambda})}\) has the same sign as
\[B([Y_{-\lambda}, [X^0_{\lambda}, Y_{-\lambda}]], X^0_{\lambda}) = B([X^0_{\lambda}, Y_{-\lambda}], [X^0_{\lambda}, Y_{-\lambda}]).\]

Since \([X^0_{\lambda}, Y_{-\lambda}]\) lies in \(\mathfrak{t}\), the above expression is non-positive and \(n \leq 0\), as claimed.

\[\square\]

**Proposition 9.7.** Let \(S\) be the \(G\)-orbit of the point \(x_0 = \exp i X^0_{\lambda} \cdot z\).

If the restricted root system \(\Delta_b\) is reduced, then the Levi form of the orbit \(S\) is definite provided that \(\dim G/K > 2\). It is identically zero if \(\dim G/K = 2\).

If the restricted root system \(\Delta_b\) is non-reduced, then the Levi form of the orbit \(S\) is indefinite.

**Proof.** If the restricted root system \(\Delta_b\) is reduced, then only (ii),(iii) and (vi) of Lemma 9.6 apply. By (ii) of Lemma 9.6, for every \(F_0 \in \operatorname{Ad}_{x_0}(g^0_{-\lambda})^C\), the real numbers \(B(L(F_0, F_0), F^0_{\lambda})\) all have the same sign. In other words, the restriction of the Levi form to \(\operatorname{Ad}_{x_0}(g^0_{-\lambda})^C \subset T_{C}S_{x_0}\) is either definite or identically zero. It is identically zero when \(\operatorname{ad}_{X^0_{\lambda}} : g_{-\lambda}^0 \to g_{-\lambda}^0\) is the zero-map. This happens if and only if \(g = \mathfrak{sl}(2, \mathbb{R})\) and \(\dim G/K = 2\).

If the restricted root system \(\Delta_b\) is non-reduced, then \(\dim G/K > 2\). In this case, the restriction of the Levi form to \(\operatorname{Ad}_{x_0}(g^0_{-\lambda})^C \subset T_{C}S_{x_0}\) is definite. Moreover, by (i) of Lemma 9.6 and (iii) of Lemma 9.2 the restriction of the Levi form to \(\operatorname{Ad}_{x_0}(g^0_{-\lambda})^C \subset T_{C}S_{x_0}\) is definite with opposite sign. As a result, the Levi form of \(S\) is indefinite, as stated.

\[\square\]
The Levi form of the G-orbit $y_0$. By the same methods, one can compute the tangent space and the Levi form of the G-orbit $S$ of the point $y_0 = \exp(-iX^0_0) \cdot z$. As we already remarked, the orbits $G \cdot x_0$ and $G \cdot y_0$ are distinct only when the restricted root system of $g$ is reduced. So we focus on this case. For the tangent space to $S$ at $y_0$ one has $TS_{y_0} = T_{\mathbb{C}}S_{y_0} \oplus NS_{y_0}$, where
\[
T_{\mathbb{C}}S_{y_0} = \text{Ad}_y(\mathfrak{g}^0_\mathbb{C})^C + \text{Ad}_y(\mathfrak{g}^\lambda_\mathbb{C})^C, \quad NS_{y_0} = \mathbb{R}\text{Ad}_y(\theta X^0_\lambda).
\]
Fix $F^0_\lambda := \text{Ad}_y(\theta X^0_\lambda)$ as a generator of $NS_{y_0}$. For the Levi form, one has the following results.

Lemma 9.8. Let $Z_0 \in \mathfrak{g}^\lambda_\mathbb{R}$, write $Z_0 = [X^0_\lambda, Y_{-\lambda}]$, for some $Y_{-\lambda} \in \mathfrak{g}^{-\lambda}_\mathbb{R}$ (cf. Lemma 9.2) and set $F_0 := \text{Ad}_y Z_0$. Then
\[
L_{y_0}(F_0, F_0) = \frac{1}{2}\text{Ad}_y[Y_{-\lambda}, Z_0] = sF^0_\lambda \mod (T_{\mathbb{C}}S_{y_0})^C, \quad s \geq 0.
\]

(i) Let $X_\lambda \in \mathfrak{g}^\lambda_\mathbb{R}$ and set $F_\lambda := \text{Ad}_y X_\lambda$. Then
\[
L_{y_0}(F_\lambda, F_\lambda) = 0.
\]

(ii) Let $Z_0 \in \mathfrak{g}^0_\mathbb{R}$ and $X_\lambda \in \mathfrak{g}^\lambda_\mathbb{R}$. Set $F_0 := \text{Ad}_y Z_0$ and $F_\lambda := \text{Ad}_y X_\lambda$. Then
\[
L_{y_0}(F_0, F_\lambda) = 0.
\]

Proposition 9.9. Let $S$ be the G-orbit of the point $y_0$. If the restricted root system $\Delta_b$ is reduced, then the Levi form of the orbit $S$ is definite provided that $\dim G/K > 2$. It is identically zero if $\dim G/K = 2$.

Remark 9.10. By Proposition 9.7 and Proposition 9.9, if the restricted root system $\Delta_b$ is reduced, then the Levi form of the orbits represented by $w_1$ and $w_2$ in diagram (1) is definite. This is consistent with the fact that these orbits lie in the boundary of the Stein domains $D_1(0)$ and $D_2(0)$, respectively (cf. Theorem 6.1). If $\dim G/K = 2$, all orbits represented by $w_3, \ldots, w_4$ in diagram (4) are Levi flat. We refer to Example 6.3 for a classification of G-invariant Stein domains bounded by such orbits. If the restricted root system $\Delta_b$ is reduced, then the Levi form of the orbit represented by $w_5$ in diagrams (9) and (10) is indefinite. As a consequence this orbit cannot lie in the boundary of a Stein G-invariant domain in $G^C/K^C$.

9.2. Non-Closed orbits with a CR singular orbit in their closure. We consider now non-closed G-orbits containing in their closure the orbit of a point $z = gK^C = \exp(iA)^{\mathbb{C}} \in \mathcal{A}_0$, satisfying the condition $\alpha(A) = \pi/4$, with $\alpha$ a simple restricted root (see (7) in Sect. 4.2). In this case the singular orbit $G \cdot z$ has dimension greater than $\dim G/K$. Recall from Section 4.2 that the isotropy subgroup $H'$ of $z$ in $G$ is contained in $G' := \mathbb{Z}_G(g^0)$ and that $G'/H'$ is diffeomorphic to a rank-one, pseudo-Riemannian symmetric space, totally real in $G^C/K^C$. Let $(g' = h' \oplus q', \tau_z)$ be the associated symmetric algebra. Non-closed G-orbits in $G^C/K^C$ containing $G \cdot z$ in their closure are in one-to-one correspondence with the nilpotent $Ad_{H'}$-orbits in $g'$ (cf. (11) and Remark 1.2).

To explicitly determine reference points for such non-closed orbits and their tangent spaces, we decompose $g$ and $g'$ by an appropriate restricted root system. Let $C' = \exp(\mathfrak{c}' \cdot z)$ be the standard Cartan subsystem with base point $z$. Recall that $C' = \mathbb{R}(X + \theta(X))$, where $X$ is a non-zero vector in $\mathfrak{g}^{2\alpha}$. In particular, $\mathfrak{c}'$ is contained in $\mathfrak{g}'$ (see (53)). Define $b' = \mathbb{R}(X - \theta(X))$. Then $b'$ is a maximal abelian subalgebra in $\mathfrak{h}' \cap p$ and the restricted root decompositions of $g$ with respect to $b'$ is given by
\[
g = Z_4(b') \oplus b' \oplus g^{\pm \lambda} \oplus g^{\pm \lambda}.
\]
In order to determine how the above root decomposition restricts to the subalgebra \( g' \), observe that in general \( g' \) is not simple, but is the direct sum of a copy of \( so(m, 1) \), with \( m = \dim g^{2\lambda} + 1 \) (even), and a compact subalgebra \( l \) entirely contained in \( b' \)

\[
g' = l \oplus so(m, 1), \quad b' = l \oplus so(m - 1, 1).
\]

Observe also that all real rank one Lie algebras with a non-reduced restricted root system are equal-rank. Hence the root system \( \Delta \) of \( g^C \) with respect to a maximally split Cartan subalgebra \( g \) extending \( b' \) contains a real root. Since \( g^{2\lambda} \) is odd-dimensional (cf. Table 4.0), the restriction of this real root to \( b' \) coincides with the restricted root \( 2\lambda \) (see [11], p. 584). Since \( g' \) has a reduced restricted root system (cf. (8)) and \( so(m, 1) \), with \( m \) even, is equal-rank, then \( g' \cap g^{2\lambda} \neq \{0\} \). It follows that the restricted root decomposition of \( g' \) with respect to \( b' \) is given by

\[
g' = Z_l(b') \oplus b' \oplus g^{\pm 2\lambda}.
\]

Let

\[
g' = b' \oplus q', \quad \text{with} \quad b' = g^{0}_{0} \oplus g^{\pm 2\lambda}_{0} \oplus b' \quad \text{and} \quad q' = g_{q}^{0} \oplus g_{q}^{\pm 2\lambda},
\]

be the combined decomposition of \( g' \). Note that \( g' \) has real rank one as well. Therefore \( g^{0}_{q} \subset \mathfrak{t} \) and an analogue of Lemma 9.1 holds for \( g' \). Set \( g[\lambda] := g^\lambda \oplus g^{-\lambda} \) and \( g[\alpha] := g^\alpha \oplus g^{-\alpha} \) (here \( \alpha \) is a restricted root in \( \Delta_a \), as in Sect. 4).

**Lemma 9.11.** The following facts hold:

(i) \( \dim g^{\pm 2\lambda}_{q} = 1 \);

(ii) \( [g^{\pm 2\lambda}_{q}, g^{\pm 2\lambda}_{q}] = g^{0}_{q} \); \( [g^{\pm 2\lambda}_{q}, g^{0}_{q}] = g^{0}_{q} \);

(iii) the decomposition \( g = g' \oplus g[\alpha] \) is \( ad_{b'} \)-stable. In particular \( g[\alpha] = g[\lambda] \).

**Proof.** Statement (i) follows from the fact that \( \dim q' \cap \mathfrak{p} = 1 \) (see the proof of (ii) in Lemma 9.1), while (ii) can be checked directly.

To prove statement (iii), note that \( ad_{b'}g' \subset g' \). Moreover, \( ad_{b'}(g^{\alpha} \oplus g^{-\alpha}) \subset (g^{\alpha} \oplus g^{-\alpha}) \). By (8) and (23) it follows that the decomposition \( g = g' \oplus g[\alpha] \) is \( ad_{b'} \)-stable and \( g[\alpha] = g[\lambda] \). \( \square \)

**Reference points for non-closed \( G \)-orbits.** Reference points for non-closed orbits containing \( G \cdot z \) in their closures can be obtained by applying the methods of the previous section to the symmetric space \( G'/H' \) (cf. [13]). In this case take \( X \in g^{2\alpha}, \theta X \) and \( A := [\theta X, X] \), normalized so that \( 2\alpha(A) = 2 \). Then

\[
X_{2\lambda}^{0} = \frac{1}{2}(A - (X + \theta X)), \quad X_{-2\lambda}^{0} = \frac{1}{2}(A + (X + \theta X))
\]

are generators of \( g^{2\lambda}_{q} \) and \( g^{-2\lambda}_{q} \) respectively, and the points

\[
x_{0} = \exp iX_{2\lambda}^{0} \cdot z, \quad x_{1} = \exp iX_{-2\lambda}^{0} \cdot z, \quad y_{0} = \exp(-iX_{2\lambda}^{0}) \cdot z, \quad y_{1} = \exp(-iX_{-2\lambda}^{0}) \cdot z
\]

lie on non-closed \( G \)-orbits in \( G^C/K^C \) containing the singular orbit \( G \cdot z \) in their closures. If the orbit diagram is of type (9) there are four such orbits, represented by \( w_3, w_2, w_1 \) and \( w_4 \), respectively. If the orbit diagram is of type (10) the points \( x_{0} \) and \( x_{1} \) lie on the same \( G \)-orbit, represented by \( w_2 \). Similarly, the points \( y_{0} \) and \( y_{1} \) lie on the same \( G \)-orbit represented by \( w_1 \). The \( G \)-orbits of \( y_{0} \) and \( y_{1} \) lie on the boundary of the Stein domain \( D_{1}(0) \) (cf. Theorem 6.1).

**The tangent space to the \( G \)-orbit of \( x_{0} \).** Denote by \( S \) the \( G \)-orbit of the point \( x_{0} = \exp iX_{2\lambda}^{0} \cdot z \). In order to compute the tangent space \( TS_{x_{0}} \), observe that at the point \( z \)

\[
T(G \cdot z)_z = q' \oplus V_z, \quad \text{and} \quad T(G^C/K^C)_z = Ad_{z}p^C = (q')^C \oplus V_z,
\]
where \( q' = T(G' \cdot z) \) and \( V_z = Ad_z \mathfrak{g}[\alpha]^C \) is a complex subspace of \( \mathfrak{g}[\alpha]^C \) (see [Ge1], Prop. 3.2). It follows that
\[
TS_{x_0} \subset Ad_{x_0}(q')^C \oplus Ad_{x_0}V_z.
\]
In order to determine generators for \( TS_{x_0} \), fix a maximally split Cartan subalgebra \( \mathfrak{s} \) of \( \mathfrak{g} \) extending \( \mathfrak{h}' \) and entirely contained in \( \mathfrak{h}' \) (one can check that in all cases under consideration \( \mathfrak{h}' \) has the same rank as \( \mathfrak{g} \) and such a Cartan subalgebra indeed exists). Let
\[
\mathfrak{g}^C = \mathfrak{s}^C \bigoplus_{\beta \in \Delta} \mathfrak{g}^\beta
\]
be the corresponding root decomposition of \( \mathfrak{g}^C \) and let \( \{ Z_\beta \}_{\beta \in \Delta} \) be a complex basis of \( \mathfrak{g}^C/\mathfrak{s}^C \) consisting of root vectors \( Z_\beta \in \mathfrak{g}^\beta \). Choose compatible orderings of \( \Delta_{\mathfrak{h}'} \) and \( \Delta \) (i.e. a root \( \beta \in \Delta \) is positive if its restriction to \( \mathfrak{h}' \) is). Fix \( \lambda \in \Delta_{\mathfrak{h}'} \), (either a positive or a negative short restricted root) and denote by \( \Delta_\lambda \) the set of roots in \( \Delta \) which restricted to \( \mathfrak{h}' \) are equal to \( \lambda \). The set \( \Delta_\lambda \) consists of pairs of complex roots
\[
\beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m, \quad m = \frac{1}{2} \dim \mathfrak{g}^\lambda,
\]
all with the same real part, equal to \( \lambda \). For \( \beta_i, \bar{\beta}_i \in \Delta_\lambda \), choose root vectors \( Z_{\beta_i} \in \mathfrak{g}^{\beta_i} \) and \( \sigma Z_{\beta_i} \in \mathfrak{g}^{\bar{\beta}_i} \). Then the vectors defined as
\[
X_\lambda^i = Z_{\beta_i} + \sigma Z_{\bar{\beta}_i}, \quad \text{and} \quad Y_\lambda^i = -i(Z_{\beta_i} - \sigma Z_{\bar{\beta}_i}), \quad i = 1, \ldots, m,
\]
belong to \( \mathfrak{g} \) and form a basis of the restricted root space \( \mathfrak{g}^\lambda \).

**Lemma 9.12.** The following facts hold:
(i) for all \( i = 1, \ldots, m \), one has \( \tau_z Z_{\beta_i} = -Z_{\beta_i} \) and \( i\tau_z X_\lambda^i = Y_\lambda^i \);
(ii) for every \( i = 1, \ldots, m \), the brackets \( [X_\lambda^i, i\tau_z X_\lambda^i] \) lie in \( \mathfrak{g}_{2\lambda}^\lambda \). For at least one index \( i \), such brackets are non-zero;
(iii) for all \( i, j = 1, \ldots, m \), with \( i \neq j \), the brackets \( [X_\lambda^i, i\tau_z X_\lambda^j] \) have no components in \( \mathfrak{g}_{2\lambda}^\lambda \).

**Proof.** (i) Since the Cartan subalgebra \( \mathfrak{s} \) lies in \( \mathfrak{h}' \), it is pointwise fixed by \( \tau_z \). As a consequence, all root spaces \( \mathfrak{g}^\beta \), with \( \beta \in \Delta \), are \( \tau_z \)-stable. The inclusion \( V_z \subset Ad_z \mathfrak{p}^C \) (see (25)) implies that \( \tau_z Z_{\beta_i} = -Z_{\beta_i} \), for \( i = 1, \ldots, m \). Since \( \sigma \tau_z = -\tau_z \sigma \) on \( V_z \subset \mathfrak{g}[\alpha]^C \), one has \( i\tau_z X_\lambda^i = Y_\lambda^i \), as desired.
(ii) By the definitions of \( X_\lambda^i \) and \( Y_\lambda^i \), one has
\[
[X_\lambda^i, i\tau_z X_\lambda^i] = [X_\lambda^i, Y_\lambda^i] = 2i[Z_{\beta_i}, \sigma Z_{\bar{\beta}_i}] \in \mathfrak{g}_{2\lambda}^\lambda.
\]
By (i) and the fact that \( \tau_z \sigma = -\sigma \tau_z \) on \( \mathfrak{g}[\lambda]^C = \mathfrak{g}[\alpha]^C \), one also has
\[
\tau_z(2i[Z_{\beta_i}, \sigma Z_{\bar{\beta}_i}]) = -2i[Z_{\beta_i}, \sigma Z_{\bar{\beta}_i}].
\]
This implies that \( [X_\lambda^i, i\tau_z X_\lambda^i] \) lies in \( \mathfrak{g}_{2\lambda}^\lambda \), as claimed. To prove the second part of the statement, consider the set \( \Delta_{2\lambda} \) consisting of the roots in \( \Delta \) which restricted to \( \mathfrak{h}' \) coincide with \( 2\lambda \). Since \( \Delta_{2\lambda} \) contains a real root in \( \Delta \) and such root is not simple (cf. Satake diagrams in [Hi], p.532), then there exist \( \beta, \bar{\beta} \in \Delta_\lambda \) such that \( \beta + \bar{\beta} = 2\lambda \). This shows that at least one of the brackets \( [X_\lambda^i, i\tau_z X_\lambda^i] \) has a non-zero component in \( \mathfrak{g}_{2\lambda}^\lambda \).
(iii) Let \( \beta_i, \bar{\beta}_j \) be roots in \( \Delta_\lambda \), with \( \beta_j \neq \beta_i, \bar{\beta}_i \). If either \( \beta_i + \beta_j \) or \( \beta_i + \bar{\beta}_j \) is a root in \( \Delta_\lambda \), then it is a root in \( \Delta_{2\lambda} \), with non-zero imaginary part. Since the root spaces relative to the real root in \( \Delta_{2\lambda} \) are contained in \( (\mathfrak{g}_{2\lambda}^\lambda)^C \) and \( \dim(\mathfrak{g}_{2\lambda}^\lambda)^C = 1 \) (cf. Lemma 9.2), then the root spaces relative to the remaining roots in \( \Delta_{2\lambda} \) are necessarily contained in \( (\mathfrak{g}_{2\lambda}^\lambda)^C \). Hence the statement follows.

For \( \lambda \in \Delta_\lambda^+ \), fix bases of \( \mathfrak{g}^\lambda \) and \( \mathfrak{g}^{-\lambda} \) of the form
\[
X_\lambda^1, i\tau_z X_\lambda^1, \ldots, X_\lambda^m, i\tau_z X_\lambda^m, \quad X_{-\lambda}^1, i\tau_z X_{-\lambda}^1, \ldots, X_{-\lambda}^m, i\tau_z X_{-\lambda}^m,
\]
respectively. For $i, j = 1, \ldots, m$, define

\[ w_i := \frac{1}{2} \text{Ad} \tau_0 (X_i^\lambda - \tau z X_i^\lambda) \quad \text{and} \quad v_j := \frac{1}{2} \text{Ad} \tau_0 (X_j^\lambda - \tau z X_j^\lambda). \]

In the next lemma we compute the images of the vectors in (28) under the map \( \cdot^*: g \to TS_{x_0} \), defined in (17). We omit the proof which consists of straightforward computations.

**Lemma 9.13.** The images of the vectors in (28) under the map (17) are given as follows.

(i) \( (X_i^\lambda)^* = w_i \),

(ii) \( (i \tau z X_i^\lambda)^* = -iw_i \),

(iii) \( (X_j^\lambda)^* = v_j - i w' \), where \( w' = \text{Ad} \tau_0 [X_2^0, X_j^{\lambda}] \),

(iv) \( (i \tau z X_j^\lambda)^* = -iv_j - iw'' \), where \( w'' = \text{Ad} \tau_0 [X_2^0, i \tau z X_j^\lambda] \).

Denote by \( W^+ \) the complex subspace of \( W_{x_0} \) spanned by the vectors \( \{w_1, \ldots, w_m\} \) and by \( W^- \) the one spanned by \( \{v_1, \ldots, v_m\} \). By (26), the results of Section 9.1 applied to the symmetric space \( G'/H' \) and Lemma 9.13, the tangent space to \( S \) at \( x_0 \) is given by \( TS_{x_0} = TS_{x_0} \oplus NS_{x_0} \), where

\[ (29) \quad T_S S_{x_0} = T_S (G' \cdot x_0)_{x_0} \oplus W^+_{x_0} \oplus W^-_{x_0} \quad NS_{x_0} = \mathbb{R} \text{Ad} \tau_0 X_2^0. \]

Fix \( F^0_{-2\lambda} := \text{Ad} \tau_0 X_2^0 \) as a generator of \( NS_{x_0} \).

**Lemma 9.14.** The following facts hold.

(i) The decomposition of \( T_S S_{x_0} \) given in (29) is orthogonal with respect to the Levi form.

(ii) Let \( W \in W^+_{x_0} \). Then \( L_{x_0} (W, W) = 0 \).

(iii) Let \( W \in W^-_{x_0} \). Then \( L_{x_0} (W, W) = b F^0_{-2\lambda} \) with \( b \geq 0 \).

(iv) Let \( Z \in T_S (G' \cdot x_0)_{x_0} \). Then \( L_{x_0} (Z, Z) = n F^0_{-2\lambda} \), with \( n \leq 0 \).

**Proof.** (i) Let \( Z \in T_S (G' \cdot x_0)_{x_0} \) and \( W \in W_{x_0} \). In order to show that \( L(Z, W) = 0 \), observe that both \( (\cdot)^{-1} JZ \), and \( (\cdot)^{-1} Z \) belong to \( g' = h' \oplus q' \), and can be written as

\[ (\cdot)^{-1} JZ = \text{Ad}_{x_0} X_0 + \text{Ad}_{x_0} X_{2\lambda} + \text{Ad}_{x_0} X_{-2\lambda}, \quad (\cdot)^{-1} Z = \text{Ad}_{x_0} Y_0 + \text{Ad}_{x_0} Y_{2\lambda} + \text{Ad}_{x_0} Y_{-2\lambda}, \]

according to the \( ad_{g} \)-root decomposition of \( g' \) given in (23). Similarly by (25), the vector \( W \in W^+_{x_0} \oplus W^-_{x_0} = \text{Ad}_{x_0} \mathfrak{g} [\mathbf{1}]_0^C \) can be written as

\[ W = \text{Ad}_{x_0} \text{Ad}_{z} P_{\lambda} + i \text{Ad}_{x_0} \text{Ad}_{z} Q_{\lambda}, \]

where

\[ \text{Ad}_{z} P_{\lambda} = U_{\lambda} + i V_{-\lambda} - \theta U_{\lambda} + i \theta V_{-\lambda}, \quad \text{and} \quad \text{Ad}_{z} Q_{\lambda} = U'_{\lambda} + i V'_{-\lambda} - \theta U'_{\lambda} + i \theta V'_{-\lambda}, \]

with \( U_{\lambda}, U'_{\lambda} \in g_{\lambda} \) and \( V_{-\lambda}, V'_{-\lambda} \in g^{-\lambda} \). One can verify that none of the brackets in (20) has a component in \( \text{Ad}_{x_0} \mathfrak{g} [-2\lambda] \) and \( L_{x_0} (Z, W) = 0 \), as required.

Let \( w_i \in W^+_{x_0} \) and \( v_j \in W^-_{x_0} \). Then, modulo \( (T_S S_{x_0})^C \), the Levi form is given by

\[ 2 L_{x_0} (w_i, v_j) = - \frac{1}{2} \text{Ad}_{x_0} [i \tau z X_i^\lambda, X_j^{\lambda}] - \frac{1}{2} \text{Ad}_{x_0} [X_i^{\lambda}, (X_j^{\lambda} - \tau z X_j^{\lambda})]. \]

In particular, \( L_{x_0} (w_i, v_j) = 0 \), for all \( i, j = 1, \ldots, m \). This concludes the proof of (i).

In the same way one shows that \( L(w_i, w_j) = 0 \), for all \( w_i, w_j \in W^+_{x_0} \), proving (ii).

(iii) Similar calculations and (iii) of Lemma 9.12 imply that \( L_{x_0} (v_i, v_j) = 0 \), for all \( v_i, v_j \in W^-_{x_0} \), with \( i \neq j \). When \( i = j \), one has

\[ L_{x_0} (v_i, v_i) = \text{Ad}_{x_0} [i \tau z X_i^{\lambda}, i \tau z X_i^{\lambda}] = \text{Ad}_{x_0} [Z_{-\beta_i}, \sigma Z_{-\beta_i}] = b_i F^0_{-2\lambda}, \quad b_i \in \mathbb{R}. \]
In order to prove that \( b_i \geq 0 \) observe that, by (iii) of Lemma 9.11 one can write
\[ X_i^+ - \chi = X_i^+ + X_i^\chi \]
for appropriate \( X_i^+ \in g^\alpha \) and \( X_i^\chi \in g^{-\alpha} \). Since \( z = \exp iAK \), with \( A \in a \) and \( \alpha(A) = \pi/4 \), one also has \( i\tau \cdot X_i^\chi = \theta X_i^+ - \theta X_i^\chi \) and
\[ [X_i^+, i\tau \cdot X_i^\chi] = ([X_i^+, \theta X_i^\chi] - [X_i^-, \theta X_i^\chi]) - ([X_i^-, \theta X_i^\chi] + [X_i^+, \theta X_i^\chi]) \in a \oplus Z_i(a). \]

By Lemma 5.1(i) in [Ge1], the first two terms of the above sum can be written as
\[ [X_i^+, \theta X_i^\chi] = B(X_i^+, \theta X_i^\chi)A_\alpha \]
and \( [\theta X_i^-, \theta X_i^\chi] = B(X_i^-, \theta X_i^\chi)A_\alpha \), where \( A_\alpha \) is an element in \( a \) satisfying the condition \( \alpha(A_\alpha) > 0 \). By the normalization of the reference points chosen in (24), one has \( \theta X_0 = -X_0^\chi \). Hence \( L_x(\nu, \nu) = b_i Ad_{x_0} \theta X_0^\chi \), for some real number \( b_i \geq 0 \), as claimed.
This concludes the proof of (iii).

(iv) Recall that the symmetric space \( G'/H' \) has a reduced restricted root system and that the Lie algebra \( g' \) is given by (29). Then the Levi form on \( T\gamma(G'/x_0)_{x_0} \) can be computed by the methods of Section 9.1. By (19), one has
\[ T\gamma(G'/x_0)_{x_0} = Ad_{x_0}(q^0)_{\gamma} \oplus Ad_{x_0}(q^0)_{\gamma}^G \]
and \( N(G'/x_0)_{x_0} = \Re Ad_{x_0} \theta X_0^\chi \).

Let \( F_0 \in Ad_{x_0}(q^0)_{\gamma} \) and \( F_2 \in Ad_{x_0}(q^0)_{\gamma}^G \). Then by Lemma 9.10 one has
\[ L_{x_0}(F_2, F_2) = L_{x_0}(F_0, F_2) = 0, \quad L_{x_0}(F_0, F_0) = nF_0^{\chi}_{X_0^\chi}, \quad n \leq 0. \]

The next proposition is a direct consequence of Lemma 9.12 and of Lemma 9.14.

**Proposition 9.15.** Let \( S \) be the \( G \)-orbit of \( x_0 \). The Levi form of \( S \) at \( x_0 \) is indefinite if \( g = sp(n, 1) \) or \( g = sp(1) \). It is definite if \( g = su(n, 1) \).

**Proof.** By Lemma 9.12 and (iii) of Lemma 9.14 the Levi form \( L_{x_0} \) is definite on \( W_{x_0} \). If \( g = su(n, 1) \), then \( \dim G'/H' = 1 \) and the Levi form is identically zero on \( T\gamma(G'/x_0)_{x_0} \). As a result, in this case the Levi form \( L_{x_0} \) is definite.

If \( g = sp(n, 1) \) or \( g = sp(1) \), then \( \dim G'/H' > 2 \) and the Levi form \( L_{x_0} \) on \( T\gamma(G'/x_0)_{x_0} \) is definite of the opposite sign as on \( W_{x_0} \) (cf. Proposition 9.7 and Lemma 9.14). As a result, \( L_{x_0} \) is indefinite, as claimed.

**The Levi form of the \( G \)-orbit of \( y_0 \).** By the same methods, one can compute the tangent space and the Levi form of the \( G \)-orbit \( S \) of the point \( y_0 = \exp i(-X_0^\chi) \cdot z \).
The tangent space to \( S \) at \( y_0 \) is given by \( TS_{y_0} = T\gamma S_{y_0} \oplus NS_{y_0} \), where
\[ T\gamma S_{y_0} = T\gamma(G'/y_0)_{y_0} \oplus W_{x_0}^+ \oplus W_{x_0}^- \]
and \( NS_{y_0} = \Re Ad_{y_0} \theta X_0^\chi \).

Fix \( F_{\chi 2} := Ad_{y_0} \theta X_0^\chi \) as a generator of \( NS_{y_0} \).

**Lemma 9.16.** The following facts hold.
(i) The decomposition of \( T\gamma S_{y_0} \) given in (30) is orthogonal with respect to the Levi form.
(ii) Let \( W \in W_{y_0}^+ \). Then \( L_{y_0}(W, W) = 0 \).
(iii) Let \( W \in W_{y_0}^- \). Then \( L_{y_0}(W, W) = bF_{\chi 2} \), with \( b \geq 0 \).
(iv) Let \( Z \in T\gamma(G'/y_0)_{y_0} \). Then \( L_{y_0}(Z, Z) = pF_{\chi 2} \), with \( p \geq 0 \).

**Proof.** The proof follows the same pattern as the proof of Lemma 9.14. One can check that the Levi form is not identically zero on \( W_{y_0} \) and has the same signature as on \( W_{x_0} \). Part (iv) follows from Lemma 9.8.

**Proposition 9.17.** Let \( S \) be the \( G \)-orbit of \( y_0 \). The Levi form of \( S \) at \( y_0 \) is definite.
Proof. The proposition follows from Lemma \ref{9.16} and the fact that the Levi form $L_{y_0}$ on $W_{y_0}$ is not identically zero.

\hfill $\blacksquare$

Remark 9.18. If the restricted root system $\Delta_b$ is non-reduced, then Proposition \ref{9.17} says that the Levi form of the orbits represented by $w_1$ and $w_4$ in diagram (9) and in diagram (10) is definite. This is consistent with the fact that these orbits lie in the boundary of the Stein domain $D_1(0)$ (cf. Theorem \ref{6.1}). When $g = \mathfrak{su}(n, 1)$, by Proposition \ref{9.15} the same is true for the Levi form of the orbits represented by $w_2$ and $w_3$ in diagram (9). We refer to Example 6.3 for a classification of the $G$-invariant Stein domains in $G^C/K^C$ bounded by these orbits. Proposition \ref{9.15} also says that the Levi form of the orbit represented by $w_2$ in diagram (10) is indefinite. Hence this orbit cannot lie in the boundary of a Stein $G$-invariant domain in $G^C/K^C$.

References

[AG] Akhiezer D. N., Gindikin S. G. On Stein extensions of real symmetric spaces. Math. Ann. 286 (1990) 1–12
[AGG] Burns D., Halverscheid S., Hind R. The Geometry of Grauert Tubes and Complexification of Symmetric Spaces. Duke Math. J. 118 (2003) 465–491
[Bo] Boggess A. CR–manifolds and tangential Cauchy-Riemann complex. Studies in Advanced Math., C.R.C. Press, 1991.
[CTT] Casadio-Tarabusi E., Iannuzzi A., Trapani S. Globalizations, fiber bundles and envelopes of holomorphy. Math. Zeit. 233 (2000) 535–551
[CL] Cœuré G., Loeb J. J. Univalence de certaines enveloppes d’holomorphie. C. R. Acad. Sci. Paris Sér. I Math. 302 (1986) 59–61
[FHW] Fels G., Huckleberry A. T., Wolf, J. A. Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint. Progress in Mathematics 245, Birkhuser, Boston 2005
[KS] Krötz B., Stanton R. Holomorphic extensions of representations II: geometry and harmonic analysis. GAFA A, Geom. Funct. Anal. 15 (2005) 190-245.
[Ge1] Geatti L. Invariant domains in the complexification of a noncompact Riemannian symmetric space. J. of Alg. 251 (2002) 619–685
[Ge2] Geatti L. A remark on the orbit structure of complexified symmetric spaces. Preprint (2006)
[Gr] Grauert H. Analytische Faserungen über holomorph-vollstän digen Räumen. Math. Ann. 135 (1958) 263–273
[GK] Gindikin S., Krötz B. Invariant Stein Domains in Stein Symmetric Spaces and a Nonlinear Complex Convexity Theorem. Inter. Math. Res. Not. 18 (2002) 959-971
[He1] Heinzner P. Geometric invariant theory on Stein spaces. Math. Ann. 289 (1991) 631–662
[He2] Heinzner P. Equivariant holomorphic extensions of real analytic manifolds. Bull. Soc. Math. France 121 (1993) 445–463
[HI] Helgason S. Differential geometry, Lie groups and symmetric spaces. GSM 34, AMS, Providence, 2001
[Hi] Heinzner P., Iannuzzi A. Integration of local actions on holomorphic fiber spaces. Nagoya Math. J. 146 (1997) 31–53
[Ho] Hochshild G. The structure of Lie groups Holden-day, San Francisco, 1965
[In] Iannuzzi A. Characterizations of $G$-tube domains. Man. Math. 98 (1999) 425–445
[Kn] Knapp A. W. Lie groups beyond an introduction. Birkhäuser, Boston, 2004
[Lu] Luna D. Sur certaines opérations différentiables des groupes de Lie. Amer. J. Math. 97 (1975) 172-181
[Ma] Matsuki T. Double coset decomposition of reductive Lie groups arising from two involutions. J. of Alg. 197 (1997) 49–91
[Ne]  NEEB K.H. On the complex geometry of invariant domains in complexified symmetric spaces, Ann. Inst. Fourier, Grenoble, 49 (1999) 177–225

[Pa]  PALAIS R. A global formulation of the Lie theory of transformation groups, Mem. Amer. Math. Soc., 22, 1955

[Ro]  ROSSI H. On envelopes of holomorphy, Comm. pure and applied Mathematics 16 (1963) 9–17

[Rt]  ROTHAUSS O. S. Envelopes of holomorphy of domains in complex Lie groups, Problem in analysis, pp. 309–317, Princeton Univ. Press, Princeton, 1970

[Sn]  SNOW D. M. Reductive Group Actions on Stein Spaces, Math. Ann. 259 (1982) 79–97

[St]  STEIN K. Überlagerungen holomorph-vollständiger komplexer Räume. Arch. Math. 7 (1956) 354–361

[Sb]  STEINBERG R. Endomorphisms of real algebraic groups, Mem. Amer. Math. Soc., 80, 1968

[Wo]  WOLF J.A. Spaces of constant curvature, Publish or Perish, Wilmington, Delaware, 1984

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