Exponential Randomized Response: Boosting Utility in Differentially Private Selection

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Abstract—A differentially private selection algorithm outputs from a finite set the item that approximately maximizes a data-dependent quality function. The most widely adopted mechanisms tackling this task are the pioneering exponential mechanism and permute-and-flip, which can offer utility improvements of up to a factor of two over the exponential mechanism. This work introduces a new differentially private mechanism for private selection and conducts theoretical and empirical comparisons with the above mechanisms. For reasonably common scenarios, our mechanism can provide utility improvements of factors significantly larger than two over the exponential and permute-and-flip mechanisms. Because the utility can deteriorate in niche scenarios, we recommend our mechanism to analysts who can tolerate lower utility for some datasets.

Index Terms—differential privacy, mechanism, permute-and-flip, disclosure risk, error analysis

I. INTRODUCTION

Differentially private selection enables analysts to retrieve the item in a dataset that approximately optimizes a data-dependent quality function. Thanks to the formal guarantee of privacy, the analyzed individuals enjoy plausible deniability. The exponential mechanism, firstly introduced in 2007 [1], facilitated this type of private analysis; it was the de facto algorithm until the permute-and-flip mechanism was formulated in 2020 [2], which offers a utility improvement of up to 2×. Both algorithms are easy to implement, run in linear time, conform to regularity, and are easily applicable to tasks such as the mode and median. Furthermore, practitioners can integrate them into other differentially private mechanisms for training machine learning models [3], [4], localization mechanisms [5], heavy hitter estimation [6], frequent itemsets mining [7], synthetic data generation [8], [9], and discovering causal relationships in graphs [10].

Herein, we introduce the exponential randomized response mechanism [11], which shares similar properties and provides improvements over the exponential and permute-and-flip mechanisms for differentially private selection, impacting many practical implementations such as the above. Exponential randomized response provides utility improvements over the exponential mechanism of up to \( \frac{1}{1-p} \), where \( 0 \leq p < 1 \) is the user-defined bias of a coin. Furthermore, it provides better utility than permute-and-flip up to at least \( \frac{1}{2(1-p)} \times \), where \( 0.5 \leq p < 1 \). These improvements are possible when the quality score of the optimal candidate is above the rest by a defined amount, the sensitivity of the quality function. Thus, there exist niche scenarios where the utility of exponential randomized response can deteriorate, not providing these utility improvements. Therefore, we recommend our mechanism to “risk-neutral” analysts, who may not resent lower utility for some datasets, while we suggest “risk-averse” analysts to rely on the permute-and-flip mechanism. Lastly, to help analysts, we provide an approach to choose \( \varepsilon \).

The remaining of the paper is structured as follows. We introduce background in § II. We present the exponential randomized response mechanism in § III, compare it to the other mechanisms in § IV and § V, and perform experiments with real-world data in § VI. Lastly, we present related work in § VII and discuss our conclusion in § VIII.

II. BACKGROUND

McKenna and Sheldon [2] introduced the permute-and-flip mechanism, but also a framework to compare mechanisms of the same class. Thus, for consistency and ease of comparison, we inspire in their publication to provide parts of the background and error comparison with the other mechanisms. Specifically, this section provides background on differential privacy, randomized response, the problem of private selection, and existing differentially private mechanisms.

A. Differential Privacy

A dataset \( D \) belongs to the universe of possible datasets \( \mathcal{D} \) and can be modeled as a (multi)set of records from individuals. Let \( D' \sim D \) denote neighboring datasets, i.e., \( D' \) has one more or less individual than \( D \) s.t. \( ||D|| - ||D'|| = 1 \) (Table I includes notation). Differential privacy, introduced in 2006 by Dwork [11], mathematically formalizes a definition of privacy that probabilistically guarantees the inability of an attacker to distinguish among neighboring datasets based on the output of an analysis. The level of indistinguishability is parameterized by \( \varepsilon \); the lower the \( \varepsilon \), the harder it is to distinguish datasets.

**Definition 1:** (Differential Privacy) A randomized mechanism \( \mathcal{M} : D \rightarrow \mathcal{R} \) is \( \varepsilon \)-differentially private iff for any
neighboring dataset \( D' \sim D \), and any set of possible outputs \( S \subseteq \text{Range}(M) \):
\[
\Pr[M(D) \in S] \leq e^\epsilon \times \Pr[M(D') \in S].
\]

Mechanisms guarantee differential privacy by adding carefully-chosen random noise, typically to the output of a deterministic function. Such a function extracts information from a dataset, e.g. the mode or the median. Beyond \( \epsilon \), the sensitivity of a function \( \Delta \) also affects the scale of the noise. Sensitivity determines how much can the outputs of a function differ among neighboring datasets.

**Definition 2**: (\( \ell_1 \)-Sensitivity). The \( \ell_1 \)-sensitivity of a real-valued function \( q : D \rightarrow \mathbb{R}^n \), executed over neighboring datasets \( D \sim D' \), is defined as:
\[
\Delta = \max_{D \sim D'} \| q(D) - q(D') \|_1.
\]

### B. Randomized Response

Randomized Response (RR) was first introduced in 1965 by Warner [12] to eliminate bias in surveys and provide plausible deniability to interviewees, which encouraged them to answer sensitive questions truthfully. Furthermore, RR is well-known to be differentially private [13]. In practice, the interviewees would:

1. Flip a coin with bias \( p \).
2. If \( \text{Bernoulli}(p) \), answer truthfully.
3. Else, flip the coin again and respond “Yes” if \( \text{Bernoulli}(p) \) and “No” otherwise.

### C. Differentially Private Selection

In the context of the differentially private selection problem, let an individual data in \( D \) (a record) take a fixed value \( r \) from the finite set of possible candidates \( \mathcal{R} \). A quality score (real-valued) function \( q(\cdot) \) assigns to each candidate \( r \) a score \( q_r \), which is bounded above by the sensitivity \( \Delta \) of \( q(\cdot) \). \( q_r \) captures how well \( r \) fulfills a statistical property of the dataset \( D \) (e.g., frequency or distance to the median). A selection mechanism \( M \) returns the candidate \( r \) that approximately maximizes \( q_r(D) \). Our goal is to design \( M \) s.t. it satisfies differential privacy. Given that \( \tilde{q} \) is derived from \( D \), for notational convenience, we use \( M(\tilde{q}) \) instead of \( M(D, \tilde{q}) \). We will compare our mechanism with the following most widely used \( \epsilon \)-differentially private selection mechanisms:

**Exponential Mechanism** \( (M_{EM}) \). Firstly introduced by McSherry and Talwar in 2007 [1], \( M_{EM} \) is a differentially private mechanism designed to answer selection queries without adding noise directly to the computed quantity [13].

**Definition 3**: (The Exponential Mechanism). The exponential mechanism \( M_{EM}(\tilde{q}) \) returns outputs \( r \in \mathcal{R} \) with probability \( p_r \propto \exp(\frac{\epsilon}{2\Delta} q_r) \).

Unconventionally, one can represent sampling \( M_{EM} \) with the rejection sampling algorithm in Algorithm [1] which repeatedly samples uniformly at random a candidate \( r \) from the set \( \mathcal{R} \) with replacement and returns \( r \) with probability \( p_r \).

**Algorithm 1**: \( M_{EM}(\tilde{q}) \) (cf. adapted [2])

1. \( q_s = \max q_r \)
2. repeat
3. \( r \sim \text{Uniform}[\mathcal{R}] \)
4. \( p_r = \exp \left( \frac{\epsilon}{2\Delta} (q_r - q_s) \right) \)
5. until \( \text{Bernoulli}(p_r) \)
6. return \( r \)

**Permute-and-Flip** \( (M_{PF}) \). Designed by Mckenna and Sheldon in 2020 [2], \( M_{PF} \) is a differentially private mechanism based on \( M_{EM} \), where the sampling is performed without replacement instead. Such modification discards for future consideration candidates that probably have a low quality score, making \( M_{PF} \) yield lower error than \( M_{EM} \). \( M_{PF} \) rejection sampling algorithm is equivalent to Algorithm [1] with the addition of “\( \mathcal{R} = \mathcal{R} \setminus r \)” after line 4 inside the loop.

### III. EXPONENTIAL RANDOMIZED RESPONSE

We propose a new mechanism for differentially private selection, \( M_{ERR} \): “exponential randomized response”. To better explain the mechanism, we first introduce the naive form of \( M_{ERR} \), followed by the derivation of its complete version. \( M_{ERR} \) is easy to implement and runs in linear time like \( M_{EM} \) and \( M_{PF} \).

**A. Naive \( M_{ERR} \)**

Naive \( M_{ERR} \) is depicted in Fig. 1 while Algorithm 2 details its rejection sampling algorithm. The setting is as follows, the owner of a dataset \( D \) sets the bias of a coin flip, and an analyst executes a query \( M_{ERR}(\tilde{q}) \) over \( D \), e.g., the mode. First, \( M_{ERR} \) flips the biased coin. If the coin comes up heads, \( M_{ERR} \) returns the mode of \( D \) (truthful answer); otherwise, the mechanism defaults to \( M_{EM} \) with the frequency as quality function \( q(\cdot) \). Intuitively, \( M_{ERR} \) performs better regarding utility than \( M_{EM} \) because the truthful output probability is increased wrt to the rest of possible outputs. Furthermore, note that \( M_{ERR} \) only needs one more input than \( M_{EM} \), the bias of the coin.

The naive version assumes datasets whose optimal quality vector score \( q_s \) is more than a sensitivity \( \Delta \) apart from any other score. For example, given the histogram \( \tilde{q} = (r_1, r_2, r_3) = (5, 3, 1) \), the mode of \( \tilde{q} \) and of any neighboring
### Algorithm 2: Naive \( M_{ERR}(\vec{q}) \)

**Input:** \( \mathcal{R} \); \( \vec{q} \); \( p \).

1. if Bernoulli(\( p \)):
2. return arg max \( q_r \)
3. \( q_* = \max q_r \)
4. repeat
5. \( r \sim \text{Uniform}[\mathcal{R}] \)
6. \( p_r = (1-p) \exp \left( \frac{\varepsilon}{2\Delta} (q_r - q_*) \right) \)
7. until Bernoulli(\( p_r \))
8. return \( r \)

\( \vec{q}' \) such as \((4, 3, 1)\) or \((5, 4, 1)\), would still yield \( r_1 \) as the optimal candidate. In this setting, naive \( M_{ERR} \) fulfills differential privacy. Moreover, we later prove in Theorem 3 that \( M_{ERR} \) is never worse than \( M_{EM} \) and, therefore, it improves the expected error by a factor of \( \frac{1}{\varepsilon} \), where \( 0 \leq p < 1 \).

However, without assuming a \( \vec{q} \) whose optimal \( q_* \) is more than \( \Delta \) apart from the rest of scores, naive \( M_{ERR} \) does not fulfill differential privacy. Imagine instead the histogram \( \vec{q}' = (5, 4, 1) \), whose optimal candidate is \( r_1 \). There exist neighboring quality vectors such as \((4, 4, 1)\) or \((5, 5, 1)\) where the candidates that maximize \( \vec{q}' \) are \( r_1 \) or \( r_2 \)—what is truthful in \( D \) might not be truthful or be the single truth in \( D' \). This discrepancy changes the output probabilities of the same candidate across neighboring datasets and ultimately breaks the property of differential privacy. Nonetheless, we formulate a complete version of \( M_{ERR} \) that solves this problem and is differentially private for any quality vector \( \vec{q}' \).

### B. Complete \( M_{ERR} \)

We first present complete \( M_{ERR} \) in Fig. 2 the difference being that the coin can have more than two sides. Algorithm 3 contains its rejection sampling algorithm. \( M_{ERR} \) works by first taking as input a decreasingly ordered quality vector \( \vec{q}' \). Next, \( M_{ERR} \) checks whether the optimal candidate quality score \( q_1 \) is at least a sensitivity \( \Delta \) apart from the second-highest candidate \( q_2 \). If it is not, then Algorithm 3 behaves equivalently to Algorithm 2. If it is, then \( q_2 \) is added to the set \( T \), and \( M_{ERR} \) proceeds to check whether \( q_2 \) and \( q_3 \) are at least \( \Delta \) apart. It continues to perform this check for the following \( q_r \) values, and it will stop once they are not at least \( \Delta \) apart. One can see this process as finding the bounds of an interval \( I = [q_* - c\Delta, q_* + \Delta] \), where \( q_* \) is the optimal quality score, and \( c \in \mathbb{N}^+ \).

The search process to build \( T \) is what sets \( M_{ERR} \) apart from its naive implementation, and what makes \( M_{ERR} \) fulfill differential privacy for any quality vector. Let us again take the example with \( \vec{q} = (r_1, r_2, r_3) = (5, 4, 1) \). Under complete \( M_{ERR} \), the optimal candidates are \( r_1 \) or \( r_2 \) because they are \( \Delta \) apart, i.e., their quality scores are within the bounding interval \( I = [4, 6] \). These optimal candidates are preserved across any neighboring dataset, e.g., for \((4, 4, 1)\), \((5, 5, 1)\) or \((6, 4, 1)\) the optimal candidates are also \( r_1 \) or \( r_2 \), whose quality scores are within \( I \) as well.

In the proof of \( M_{ERR} \): 
\[ \Pr[M_{ERR}(\vec{q}) = r] \leq \exp \varepsilon, \]
The terms in the numerator and denominator follow the same probability formulation across neighboring datasets thanks to considering the set \( T \). If these terms did not follow the same expression—as it can happen with naive \( M_{ERR} \)—then the mechanism would not fulfill differential privacy. However, this solution comes at a utility cost in niche scenarios, where \( q_* \) is not more than \( \Delta \) apart from at least another candidate (see § V). Due to this uncertainty, we recommend \( M_{ERR} \) for “risk-neutral” analysts, who may not resent obtaining less accurate outputs on some occasions, while we recommend \( M_{PF} \) for “risk-averse” analysts.

### C. Derivation

To define \( M_{ERR} \), we start by describing its probability mass function in Lemma 1 (see Fig. 3). We continue by proving that \( M_{ERR} \) is regular in Lemma 2 (see Appendix A). Regularity ensures that there is no built-in bias towards specific outcomes. Most importantly, in Theorem 1 (see Fig. 4), we prove that \( M_{ERR} \) is \( \varepsilon \)-differentially private. Note that for monotonic quality functions such as the mode, the accuracy of \( M_{ERR} \) can be improved. That is, for any \( D \sim D' \), we have either \( \forall r \in \mathcal{R}, q_r \geq q'_r \) or \( \forall r \in \mathcal{R}, q_r \leq q'_r \). Thus, we can increase the accuracy of \( M_{ERR} \) by using \( \exp \left( \frac{\varepsilon}{2\Delta} q_r \right) \) instead of \( \exp \left( \frac{\varepsilon}{2\Delta} q'_r \right) \). Such approach also satisfies differential privacy. Observe that Eq. 1 is a product of two terms, whenever the first term is \( \geq 1 \) the second is \( \leq 1 \). Therefore, we can upper bound the first term by \( \exp(\varepsilon) \) to fulfill differential privacy, and vice-versa because of the symmetry of neighboring.
Lemma 1: The probability mass function (pmf) of \( \mathcal{M}_{\text{ERR}} \) is expressed as:

\[
\Pr[\mathcal{M}_{\text{ERR}}(\bar{q}) = r] = \begin{cases} 
\frac{p r T}{m} + (1 - p) \frac{\exp(\frac{p r q_r}{\Delta})}{\sum_{r \in \mathcal{T}} \exp(\frac{p r q_r}{\Delta})} & \text{if } r \in \mathcal{T} \\
(1 - p) \frac{\exp(\frac{(1 - p) q_r}{\Delta})}{\sum_{r \not\in \mathcal{T}} \exp(\frac{(1 - p) q_r}{\Delta})} & \text{if } r \not\in \mathcal{T}
\end{cases}
\]

where \( p \in [0, 1] \) is the bias of a coin, \( \mathcal{T} \) stands for “true” and \( F \) for “false”, \( \mathcal{R} = \{1, 2, \ldots, n\} \) is a finite set of candidates, \( q_r \) is the output of a quality score (real-valued) function \( q(\cdot) \) for candidate \( r \), \( \Delta \) is the sensitivity of \( q(\cdot) \), \( \varepsilon \geq 0 \) is the privacy parameter of differential privacy, and \( m = |\mathcal{T}| \).

\( \mathcal{T} = \{ r : q_r \in I \} \subseteq \mathcal{R} \) is the set of candidates whose quality scores are within the bounding interval \( I \), where \( I = [q_s - c\Delta, q_s + \Delta] \), \( q_s \) is the optimal quality score, and \( c \in \mathbb{N}^+ \). Together, \( c\Delta \) defines the distance from \( q_s \) s.t. if we had \( I' = [q_s - (c + 1)\Delta, q_s + \Delta] \) and \( \mathcal{T}' = \{ r : q_r \in I' \} \), then \( \mathcal{T}' = \mathcal{T} \).

Proof: The pmf follows from randomized response and the exponential mechanism, \( \mathcal{M}_{\text{EM}} \). With probability \( p/m \) the mechanism outputs one of the candidates in \( \mathcal{T} \), i.e., either the candidate that maximizes \( q(\cdot) \) or others \( q_r \in I \). We consider \( r \in \mathcal{T} \) as \( T \) candidates, and \( r \notin \mathcal{T} \) as \( F \). On the other hand, with probability \( 1 - p \), the mechanism defaults to \( \mathcal{M}_{\text{EM}} \), which returns any candidate \( r \in \mathcal{R} \) with probability \( p_r \propto \exp(\frac{p r q_r}{\Delta}) \).

Fig. 3. Probability mass function of \( \mathcal{M}_{\text{ERR}} \).

Algorithm 3: Complete \( \mathcal{M}_{\text{ERR}}(\bar{q}) \)

Input: \( \mathcal{R} \); \( p \); \( \bar{q} \), decreasingly ordered vector of quality scores s.t. \( q_r \geq q_{r+1} \forall r \in \mathcal{R} \).

1: \( \mathcal{T} = \{ q_1 \} \)
2: while \( (q_r - q_{r+1}) \leq \Delta \) and \( r < |\mathcal{R}| \):
3: \( \mathcal{T} = \mathcal{T} \cup r + 1 \)
4: \( r++ \)
5: \( t = |\mathcal{T}| \)
6: repeat
7: \( r \sim \text{Uniform}[\mathcal{T}] \)
8: \( p_r = \frac{t}{r} \)
9: \( \mathcal{T} = \mathcal{T} \setminus r \)
10: if Bernoulli(\( p_r \)):
11: \hspace{1em} return \( r \)
12: until \( \mathcal{T} == \emptyset \)
13: \( q_s = q_1 \)
14: repeat
15: \( r \sim \text{Uniform}[\mathcal{R}] \)
16: \( p_r = (1 - p) \exp\left(\frac{q_r - q_s}{\Delta}\right) \)
17: until Bernoulli(\( p_r \))
18: return \( r \)

IV. PRIVACY COMPARISON WITH \( \mathcal{M}_{\text{EM}} \)

A. Disclosure risk

We examine whether \( \mathcal{M}_{\text{ERR}} \) has at least the same risk of disclosing an individual in a dataset as \( \mathcal{M}_{\text{EM}} \). To accomplish this goal, first, we define a strong adversary model inspired by the work of Lee and Clifton [14].

Given a dataset \( D \), we assume an attacker has complete knowledge of the universe of possible datasets \( \mathcal{D} \), i.e., the attacker would have access to all the records of \( D \), and knows the bias of the coin flip \( p \). The only thing the attacker does not know is which individual is missing in the neighboring dataset \( D' \); however, the attacker knows that \( D' \) consists of records of a particular property and can make a query over \( D' \) (e.g., the mode). For example, this adversary model can correspond to a hospital worker that knows all the cancer patient records in \( D \) but does not know which patient has dropped chemotherapy. Assuming the adversary has infinite computing power, the adversary’s goal is to discover who is missing in \( D' \) based on the information of \( D \). Our task is to prevent the adversary from guessing the missing individual correctly with high probability.

The adversary’s attack model assembles a set of possible worlds \( \Omega \) (containing possible neighboring datasets \( \omega \) of \( D' \)) derived from the combinations of records in \( D \). Assuming a uniform prior, the attacker’s posterior belief \( \beta \) represents the change in belief towards each possible world after observing an analysis output; specifically, the belief that the queried \( D' \) is \( \omega \). In practice, the attacker would execute a query on \( D' \), obtaining a noisy output. With this noisy answer, the adversary would compute the posterior believes \( \beta \) of all possible worlds \( \omega_i \), selecting the world with the highest posterior belief as to the most confident answer. In the context of this study, we employ \( \bar{q}' \) instead of \( D' \) for convenience.

Definition 4: (Posterior belief). Given the mechanism \( \mathcal{M}(\bar{q}) \) and its response \( r \), for each possible world \( \omega \), the adversary’s posterior belief on \( \omega \) is defined as:

\[
\beta(\omega) = \Pr(\bar{q}' = \omega | r) = \frac{\Pr(\mathcal{M}(\omega) = r)}{\sum_{\omega' \in \Omega} \Pr(\mathcal{M}(\omega') = r)}.
\]

(2)

The higher the posterior, the higher the probability that the record is disclosed; this is our measure of disclosure risk. Definition 5 is adapted from [14]:

Definition 5: (Disclosure risk). Given the set of possible worlds \( \Omega \), the risk \( \rho \) of disclosing the presence or absence
Lemma 3: There is a constant $C > 0$ such that for any $\epsilon > 0$ and $m > 0$,

$$
\frac{1}{\Omega} \leq \beta(\epsilon) \leq \frac{1}{1 + (|\Omega| - 1) \exp(-\epsilon)}.
$$

Theorem 1: $M_{\text{ERR}}$ is $\epsilon$-differentially private.

Proof: For any two neighboring dataset $D \sim D'$ and their corresponding quality score vectors $\vec{q} \sim \vec{q}'$ and any $r \in R$,

$$
\exp \left( \frac{\epsilon}{2\Delta} q_r \right) \leq \exp(\epsilon/2) \exp \left( \frac{\epsilon}{2\Delta} q'_r \right).
$$

Because of the symmetry of neighboring, we also have $\forall r$,

$$
\exp \left( \frac{\epsilon}{2\Delta} q'_r \right) \leq \exp(\epsilon/2) \exp \left( \frac{\epsilon}{2\Delta} q_r \right).
$$

Case 1: For any candidate $r \in T$, to show

$$
\frac{\Pr[M_{\text{ERR}}(\vec{q}) = r]}{\Pr[M_{\text{ERR}}(\vec{q}') = r]} = \frac{p_r + (1 - p_r) \sum_{r' \in R} \exp(\frac{\epsilon}{2\Delta} q_{r'})}{p_r + (1 - p_r) \sum_{r' \in R} \exp(\frac{\epsilon}{2\Delta} q'_{r'})} \leq \exp(\epsilon).
$$

Case 2: For any candidate $r \notin T$, it is easy to show that $M_{\text{ERR}}$ is differentially private, as $M_{\text{EM}}$ is well known to fulfill differential privacy.

![Fig. 4. $M_{\text{ERR}}$ is differentially private.](image)

of any individual in a dataset is defined by the adversary’s maximum posterior belief, $\varrho = \arg \max_{\omega} \beta(\omega)$.

Luckily, the probability of a successful attack can be bounded. Our goal is to find these bounds so that a practitioner can set $\epsilon$ s.t. the disclosure risk $\varrho$ is within a tolerable range. We employ the posterior belief (Eq. 2) to calculate the bounds of $\varrho$ in Theorem 2 (Appendix B), obtaining the following expression:

$$
\frac{1}{|\Omega|} \leq \varrho \leq \frac{1}{1 + (|\Omega| - 1) \exp(-\epsilon)},
$$

where $\Omega$ is the set of possible neighboring datasets. We prove in Lemma 3 that these bounds can be used for $M_{\text{EM}}$; thus showing that the disclosure risk for $M_{\text{ERR}}$ is never worse than with $M_{\text{EM}}$. Note that the disclosure risk bounds are independent of the bias of the coin $p$, and that as long as $p \neq 1$, individuals still enjoy plausible deniability.

B. Choosing $\epsilon$

By re-arranging the disclosure risk bounds of Eq. 4, we derive a closed form expression for the bounds of $\epsilon$,

$$
0 \leq \epsilon \leq \ln \left( \frac{|\Omega| - 1}{1 - \varrho} \right).
$$

Fig. 5 shows the bounds of the disclosure risk as the number of candidates in $R$ and $\epsilon$ vary. Without loss of generality, we

\[^2\text{Note that the lower bound is not represented in Fig. 5}\]
(ii) Fig. 6(b). Quality vectors where where \( v_1 = n \) and any other \( v_i \) is picked uniformly at random from the discrete interval \([0, v_1 - 2\Delta]\). Plateauing is observed because at large enough \( n \) values, the quality vectors start approaching the quality vectors of the form depicted in (i). In these experiments, \( M_{ERR} \) outperforms \( M_{EM} \) in all the domain for \( p \geq 0.25 \); however, the improvements are not as large as in (i). \( M_{PF} \) performs similar to \( M_{ERR} \) for \( p = 0.25 \), but \( M_{ERR} \) provides more utility for larger \( p \) values.

(iii) Fig. 7. Quality vectors of the form \( \vec{q} = (v, v, \ldots, 0) \in \mathbb{R}^n \) for \( v < -\Delta \). In summary, this comparison indicates that \( M_{ERR} \) also performs better than \( M_{EM} \) and \( M_{PF} \) for any number of candidates \( n \) with \( p \geq 0.25 \) and \( p \geq 0.5 \), respectively. Fig. 7 is covered in detail in § V.C.

Fig. 5. Upper bound on the risk of disclosure of \( M_{ERR} \) as a function of \( \varepsilon \) for a varying number of candidates.

We have set \( |\Omega| = 2|\mathcal{R}| + 1 \), because adding or removing records from individuals belonging to the same candidate will affect its corresponding quality score only by 1 (\( q_r + 1 \) or \( q_r - 1 \)); yielding two possible \( D' \) per candidate in addition to the original \( D \). A privacy officer dealing with a sensitive dataset of, e.g., 100 candidates, may only allow a disclosure risk of 20%, granting analysts to make queries with a maximum privacy budget of \( \varepsilon \approx 4 \).

V. UTILITY COMPARISON WITH \( M_{EM} \)

We compare utility by benchmarking mechanisms based on the error random variable, a measure of accuracy:

\[
\mathcal{E}(M, \vec{q}) = q_* - q_{M(\vec{q})},
\]

where \( q_* = \max_r q_r \) is the optimal quality score and \( q_{M(\vec{q})} \) is a r.v. that takes the value of the quality score of the output candidate from \( M(\vec{q}) \).

A. Optimal setting for \( M_{ERR} \)

In Appendix C, we prove in Proposition 1 that the optimal scenario for \( M_{ERR} \) occurs when there is only one candidate in \( \mathcal{T} \). In this setting, we prove in Theorem 3 that \( M_{ERR} \) is never worse than \( M_{EM} \), and in Lemma 5 we prove that there is always a range of \( p \) values (coin bias) for which \( M_{ERR} \) is never worse than \( M_{PF} \). To show these improvements, we carried out several empirical experiments with synthetic data. Without loss of generality, the quality vectors fulfilling \( |\mathcal{T}| = 1 \) are decreasingly ordered vectors of the form \( \vec{q} = \{v_1, v_2, \ldots, v_n\} \) such that \( v_1 - v_2 \geq \Delta \). Different vector arrangements fulfill this condition; we have experimented with three arrangements with the mode as the task (\( \Delta = 1 \)):

(i) Fig. 6(a). Quality vectors where \( v_1 = n \), \( v_1 - v_2 = 2\Delta \) and every consecutive candidate \( v_i - v_{i+1} = \Delta \) for \( i = (2, \ldots, n) \). The experiments show how the expected errors are approximately parallel lines as the number of candidates \( n \) increases and plateau at \( n \approx 100 \). Plateauing indicates that the probability assigned to new candidates vanishes due to their relatively small quality scores. The expected error of \( M_{EM} \) is always lower for at least \( p \geq 0.25 \). Moreover, the improvements that \( M_{PF} \) brings over \( M_{EM} \) are low for \( \varepsilon = 0.1 \).
B. Worst-Case setting for $M_{ERR}$

In Appendix C, Proposition 11 also proves that the worst-case setting for $M_{ERR}$ occurs when all candidates are in $\mathcal{T}$. In such a setting, Lemma 6 shows that there are quality vectors where $M_{EM}$ can outperform $M_{ERR}$. The vector arrangement we picked to showcase this scenario is decreasingly ordered and of the form $\vec{q} = (v_1, v_2, ..., v_n)$ such that every consecutive $v_i$ is $\Delta$ apart from the immediate previous candidate. Fig. 6(c) shows that as the number of candidates increases, the error for $M_{ERR}$ keeps increasing above the errors of $M_{EM}$ and $M_{PF}$. This is because, in this setting, $M_{ERR}$ with probability $p$ defaults to picking uniformly at random any candidate in $\mathcal{R}$, reducing useful output information. Because of this niche setting, we recommend $M_{ERR}$ to “risk-neutral” analysts.

C. Worst-Case Setting for $M_{EM}$

We proceed to analyse how $\mathcal{E}(M_{ERR}, \vec{q})$ compares in the worst-case scenario for $M_{EM}$ (and $M_{PF}$). Such worst-case setting occurs when the quality vectors are of the form $\vec{q} = (v, v, ..., v, 0)$ for some $v \leq 0$ [2]. The expected error expressions are presented in Eq. 15 (Appendix C). For $v < -\Delta$, $\mathbb{E}[\mathcal{E}(M_{ERR}, \vec{q})]$ takes the form of Eq. 15 the optimal setting for $M_{ERR}$. While when $-\Delta \leq v \leq 0$, $\mathbb{E}[\mathcal{E}(M_{ERR}, \vec{q})]$ becomes Eq. 16 which corresponds to the worst-case setting for $M_{ERR}$.

Nonetheless, $M_{ERR}$ decreases in accuracy when $-\Delta \leq v \leq 0$ and therefore $T = \mathcal{R}$. Such decrease in accuracy is appreciated in the abrupt step in the bottom plots at $v = -1$, where $\mathbb{E}[\mathcal{E}(M_{ERR}, \vec{q})]$ switches from Eq. 15 to Eq. 16. However, the ratios hover around 1.

Lastly, we computed the worst-case error as a function of the number of candidates $n$ by numerically maximizing the expected error over $v$. Fig. 8 shows the upper and lower bounds of the error for $M_{EM}$ and $M_{PF}$ (dashed lines in [2]), and how $M_{ERR}$ is able to outperform the lower bound for some $p$ values. Moreover, note how the utility improvement of $M_{PF}$ over $M_{EM}$ decreases as $n$ increases, while the improvement of $M_{ERR}$ does not decay.

VI. REAL-WORLD EXPERIMENTS

To bring our comparisons to real-world problems, we empirically compared the accuracy of $M_{ERR}$ to $M_{EM}$ and $M_{PF}$ for different $\varepsilon$ values with real datasets. To prevent bias towards datasets favoring $M_{ERR}$, we chose the same five datasets employed to compare $M_{PF}$ to $M_{EM}$ in [2] for the mode and median problems ($\Delta = 1$). They were originally used by the benchmark framework DPBenchmark: HEPTH, ADULTFRANK, MEDCOST, SEARCHLOGS, and PATENT [15]. In these datasets, the candidates map to the 1024 bins of a discretized domain. We extracted the quality vector from each dataset for the mode and median and calculated the expected error for each mechanism analytically. For the mode, the quality function is the number of individuals in a bin. For the median, we used the negated number of individuals that need to be removed from or added to a bin to become the median.

Across all results in Figures 9, 10, and 11 the expected errors behave as approximately parallel lines and $M_{ERR}$ provides asymptotically constant multiplicative improvements in accuracy over $M_{EM}$ of up to $\frac{1}{p} \times$, where $0 \leq p < 1$, which brings additive savings of $\varepsilon$. $M_{ERR}$ is also more accurate than $M_{PF}$ by a factor of at least $\frac{1}{2(1-p)} \times$, where $0.5 \leq p < 1$. The latter is clearly shown in Fig. 11 for both the (a) mode and (b) median, where we fixed the expected error of $M_{EM}$ to 100 and employed the same $\varepsilon$ for the rest.
of mechanisms. Thus, for the five selected real-world datasets, $M_{ERR}$ dominates $M_{EM}$ and can outperform $M_{PF}$.

VII. RELATED WORK

Randomized Response. Since the inception of randomized response (RR) [12], other RR models have been introduced: the unrelated question model [16], two-stage RR models [17], [18], or the forced response model [19]. In recent years, RR has been used for local differential privacy [20]–[23], and improved private weighted histogram aggregation in crowdsourcing by leveraging multivariate RR [24]. Moreover, researchers devised an optimal differentially private mechanism for a binary range of candidate outputs [25]. Although our new mechanism $M_{ERR}$ leverages RR, it does not enable local differential privacy; instead, $M_{ERR}$ can execute differentially private selection queries over datasets with an arbitrary range of candidate outputs.

Differential privacy selection. There are numerous studies advancing differentially private selection [1], [2], [13], [26]–[29] and leveraging their algorithms in practical applications [3]–[10].

Most notably, the pioneering exponential mechanism [1], and permute-and-flip [2], which Pareto dominates the ex-
Exponential mechanism. Furthermore, another commonly used mechanism for private selection is report noisy max [13], which adds independently generated Laplace noise to each quality score, returning the corresponding candidate with the most significant noisy score. While there might be quality vectors where report noisy max outperforms the exponential and permute-and-flip mechanisms, [2] shows similar performance metrics between the exponential and the report noisy max mechanisms. Other proposed mechanisms for private selection are the large margin mechanism that guarantees approximate differential privacy [26], or the more scalable subsampled exponential mechanism [27]. Other studies propose considering the quality functions themselves differentially private for cases where one cannot use the exponential mechanism [28], or a generalization of the exponential mechanism to handle quality functions of varying sensitivity [29]. In contrast, while our new mechanism also tackles differentially private selection, it is unique in the use of RR to increase the accuracy of the output.

These advances and new mechanisms can be plugged in different applications, such as using the exponential mechanism to privately select weights of machine learning models [3], train random forest algorithms [4], extracting location information [5], estimating heavy hitters [6], mining frequent itemsets [7], generating synthetic data [8], [9], and discovering causal relationships in graphs [10].

VIII. CONCLUSION

With this study, we introduce a new mechanism for differentially private selection: exponential randomized response, $M_{ERR}$. We have shown that $M_{ERR}$ provides utility improvements over the exponential ($M_{EM}$) and permute-and-flip ($M_{PF}$) mechanisms. We first demonstrated that $M_{ERR}$ is never worse than $M_{EM}$ regarding disclosure risk, and provided an approach to choose $\varepsilon$. Moreover, we have outlined an optimal and non-optimal setting for $M_{ERR}$. In the optimal setting, based on our theoretical utility comparisons, we have proven that $M_{ERR}$ is never worse than $M_{EM}$, and that there is always a range of values $p$ within $[0, 1]$ where $M_{ERR}$ is never worse than $M_{PF}$. Lastly, to showcase the benefits of $M_{ERR}$, we have carried out synthetic experiments to systematically compare the mechanisms and real-world experiments to validate the results in a realistic setting. The conclusions are the following.

Experiments with synthetic datasets. We have benchmarked $M_{ERR}$ under three different scenarios for the mode query. (§ V-A) In all the experiments under the optimal setting, $M_{ERR}$ outperforms $M_{EM}$ for any $p$, and also $M_{PF}$ for at least $p \geq 0.5$. (§ V-C) Moreover, in the worst-case scenario for $M_{EM}$ and $M_{PF}$, $M_{ERR}$ maintains such utility improvements for most quality vectors. (§ V-B) However, in the non-optimal scenario for $M_{ERR}$, the other mechanisms $M_{EM}$ and $M_{PF}$ can show better utility.

Experiments with real-world datasets. $M_{ERR}$ provides across all datasets the same significant utility improvements as in § V-A for both the mode and median.

Based on all the experiment results under the optimal scenario, $M_{ERR}$ provides asymptotically constant multiplicative improvements in accuracy over $M_{EM}$ of up to $\frac{1}{1-p} \times$, where $0 \leq p < 1$. Whereas $M_{PF}$ provides improvements over $M_{EM}$ up to a factor of $2 \times$. Likewise, $M_{ERR}$ provides more accuracy than $M_{PF}$ up to a factor of at least $\frac{1}{2(1-p)} \times$, where $0.5 \leq p < 1$. Given the significant utility improvements that $M_{ERR}$ can provide to numerous real-world applications, for some analysts, the benefits of $M_{ERR}$ may outweigh its drawbacks in a non-optimal scenario. Overall, we recommend exponential randomized response to “risk-neutral” analysts, who can tolerate lower utility in a few analyses, while we suggest “risk-averse” analysts the use of permute-and-flip.

Future work. It would be useful to explore how to ameliorate the utility loss of $M_{ERR}$ in its worst-case setting, e.g., by devising methods that employ minimal privacy budget to choose $M_{ERR}$ or $M_{PF}$ based on the dataset. Furthermore, performing more empirical experiments with other real-world datasets and tasks to compare $M_{ERR}$ with $M_{PF}$ would further outline the benefits of one mechanism over the other. Moreover, $M_{ERR}$ could potentially use $M_{PF}$ instead of...
\[ \mathcal{M}_{\text{EM}} \text{, if combining } \mathcal{M}_{PF} \text{ with randomized response proves to be differentially private. In an optimal scenario, such proof would make the new } \mathcal{M}_{\text{ERR}} \text{ outperform } \mathcal{M}_{PF} \text{ also by a factor of } \frac{1}{\epsilon + \Delta}\]. Lastly, it would be interesting to explore whether \( \mathcal{M}_{\text{ERR}} \) is a competitive algorithm for private dataset publishing. For every record in an attribute, randomized response outputs the record with probability \( p \); otherwise, the algorithm triggers \( \mathcal{M}_{\text{EM}} \) with the mode as quality function.

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**APPENDIX A**

**REGULARITY AND DIFFERENTIAL PRIVACY**

**Lemma 2:** \( \mathcal{M}_{\text{ERR}} \) is regular.

**Proof:** Regularity is achieved if the a mechanism \( \mathcal{M} : \mathbb{R}^n \to \mathbb{R} \) fulfills three conditions: symmetry, shift-invariance, and monotonicity.

We will define and prove each condition consecutively. The definitions are adopted from and the regularity proofs inspired by (1).

**Definition 6:** (Symmetry). For any permutation \( \pi : \mathcal{R} \to \mathcal{R} \) and associated permutation matrix \( \Pi \in \mathbb{R}^{n \times n} \),

\[
\Pr[\mathcal{M}(\vec{q}) = r] = \Pr[\mathcal{M}(\Pi \vec{q}) = \pi(r)].
\]
Considering \( p_{r,T} \) and \( p_{r,F} \) in the definition of \( M_{\text{ERR}} \) (Lemma 1), let all the probabilities for each \( r \) form a vector \( \vec{p} \). Let \( \vec{p} = \pi \vec{t} \) denote the same vector on the permuted quality scores. The permutations are equally likely in both \( \vec{p} \) and \( \vec{p}' \), and the differences are \( p_{r,T} = p'_{\pi(r),T} \) and \( p_{r,F} = p'_{\pi(r),F} \). Hence we satisfy Equation 6, and therefore, \( M_{\text{ERR}} \) is symmetric.

**Definition 7:** (Shift-invariance). For all constants \( c \in \mathbb{R} \),

\[
\Pr[M(\vec{q} + c \mathbf{1}) = r] = \Pr[M(\vec{q}) = r].
\]

\( M_{\text{ERR}} \) only depends on \( \vec{q} \) through \( q_r \). Adding a constant to \( \vec{q} \) does not change the hierarchy of \( q_r \) values. Therefore, \( M_{\text{ERR}} \) is shift-invariant.

**Definition 8:** (Monotonicity). If \( q_r \leq q'_r \) and \( q_s \geq q'_s \forall s \neq r \), then

\[
\Pr[M(\vec{q} + c \mathbf{1}) = r] \leq \Pr[M(\vec{q}') = r].
\]

Given \( p_{r,T} \) and \( p_{r,F} \), clearly, the probability mass function is monotonically increasing in \( p_{r,T} \) and \( p_{r,F} \) (and hence \( q_r \)), and monotonically decreasing in \( p_{r,T} \) and \( p_{r,F} \) (and hence in \( q_r \)). Consequently, \( M_{\text{ERR}} \) is monotonic.

Because \( M_{\text{ERR}} \) is symmetric, shift-invariant and monotonic for a given \( p \), \( M_{\text{ERR}} \) is regular.

### APPENDIX B

**PRIVACY: DISCLOSURE RISK**

**Theorem 2:** The lower and upper bounds of the disclosure risk \( \varrho \) of \( M_{\text{ERR}} \) are expressed as:

\[
\frac{1}{|\Omega|} \leq \varrho \leq \frac{1}{1 + (|\Omega| - 1) \exp(-2\Delta)},
\]

where \( \varrho \) is the disclosure risk, and \( \Omega \) is the set of possible worlds.

**Proof:** We prove each bound as follows:

**Lower bound.** The disclosure risk \( \varrho \) cannot be lower than picking at random one of the possible worlds in \( \Omega \). Thus, we trivially prove \( \frac{1}{|\Omega|} \leq \varrho \).

**Upper bound.** We must assume the worst-case scenario for the calculation of the upper bound, i.e., the setting where the correct answer seems to be most likely for the adversary (\( M_{\text{ERR}}(\omega_i) = r \)). Thus, we must use the expression \( p_{r,T} \) from the pmf of \( M_{\text{ERR}} \) for the \( r \) that corresponds to the true answer. Furthermore, we find the upper bounds using the differential privacy property of \( M_{\text{ERR}} \) from Theorem 1. For notational convenience, we denote \( q(\omega_i, r) \) as \( q_r \), i.e., the quality score of candidate \( r \) in world \( \omega_i \). Note that all possible worlds are neighboring, which we depict by using \( q_r \sim q'_r \). Substituting in Eq. 7

\[
\beta(\omega_i) = \frac{\frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)}{\sum_{r \in \mathbb{R}} \frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)}
\]

\[
\varrho = \frac{1}{1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)} = \frac{1}{1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)}
\]

\[
1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)
\]

\[
1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)
\]

\[
\sum_{r \in \mathbb{R}} \frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)
\]

\[
\sum_{r \in \mathbb{R}} \frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)
\]

\[
= \frac{1}{1 + (|\Omega| - 1) \exp(-2\Delta)}.
\]

**Lemma 3:** \( M_{\text{ERR}} \) is never worse than \( M_{\text{EM}} \) wrt disclosure risk.

**Proof:** The lower bound is equal for both mechanisms, as it solely depends on the number of neighboring datasets. Regarding the upper bound, because \( M_{\text{EM}} \) is differentially private, then Eq. 5 and Eq. 6 in Theorem 2 can be re-written as:

\[
\frac{1}{1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)} \leq \frac{1}{1 + \sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)}
\]

\[
\sum_{r \in \mathbb{R}} \frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)
\]

\[
\sum_{r \in \mathbb{R}} \frac{p}{m} + (1 - p) \sum_{r \in \mathbb{R}} \exp(\frac{\epsilon}{2} q'_r)
\]

\[
\sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)
\]

\[
\sum_{j=1}^{[\Omega]} \exp(\frac{\epsilon}{2} q'_j)
\]

\[
= \frac{1}{1 + (|\Omega| - 1) \exp(-2\Delta)}.
\]

Showing that the disclosure risk lower and upper bound for \( M_{\text{EM}} \) and for \( M_{\text{ERR}} \) are equal concludes the proof.

### APPENDIX C

**UTILITY: ERROR ANALYSIS**

**A. Optimal and worst-case scenarios for \( M_{\text{ERR}} \)**

**Proposition 1:** Given a quality vector \( \vec{q} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n \) and \( M_{\text{ERR}} \), the optimal expected error occurs in scenarios where only one \( v_i \) is in \( T \), while the worst-case expected error occurs in scenarios where all \( v_i \) are in \( T \).

**Proof:** The expected error \( E[\mathcal{E}(M_{\text{ERR}}, \vec{q})] \) can be expressed as:

\[
E[\mathcal{E}(M_{\text{ERR}}, \vec{q})] = q_* - \sum_{r \in \mathbb{R}} q_r \Pr[M_{\text{ERR}}(\vec{q}) = r]
\]

\[
= q_* - \sum_{r \in \mathbb{R}} q_r (1 - p) \frac{q_r}{\sum_{r' \in R} p_{r'}} - \sum_{r \in T} \frac{m}{m} q_r.
\]

Eq. 8 clearly shows that the expected error of the set of quality vectors \( \vec{q} \) s.t. \( m = 1 \) is smaller than for another set where
\[ \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q})] - \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q})] = 0 + \int_{(c\Delta, \infty)} (\Pr [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q}) \geq t] - \Pr [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q}) \geq t]) \, dt \leq 0. \] (7)

Fig. 12. The expected of \( \mathcal{M}_{\text{ERR}} \) is less than the expected error of \( \mathcal{M}_{\text{EM}} \) when \( |T| = 1 \).

\( m > 1 \). Furthermore, Eq. [8] also shows that the worst-case scenario occurs when \( m = |R| \), as the last term becomes as small as possible.

**B. Utility comparison under the optimal scenario for \( \mathcal{M}_{\text{ERR}} \)**

We prove that \( \mathcal{M}_{\text{ERR}} \) performs always better than \( \mathcal{M}_{\text{EM}} \) under the optimal setting for \( \mathcal{M}_{\text{ERR}} \). It is helpful to first introduce Lemma [4] which gives a fact about partial sums of a non-decreasing sequence. Included verbatim from [2]:

**Lemma 4:** Let \( \tilde{f} \in \mathbb{R}^n \) be an arbitrary vector satisfying:

1. \( f_1 \leq f_2 \leq \ldots \leq f_n \)
2. \( \sum_{r=1}^{n} f_r = 0 \)

Then, the following holds

\[ \sum_{r=1}^{s} f_r \leq 0. \]

**Proof:** Let \( m \) be any index satisfying \( f_m \leq 0 \) and \( f_{m+1} \geq 0 \). If \( t \leq m \), the claim is clearly true, as it is a sum of non-positive terms. If \( t > m \), we have \( \sum_{r=1}^{s} f_r \leq \sum_{r=1}^{n} f_r = 0 \). In either case the partial sum is non-negative, and the claimed bound holds.

We inspire from [2] to prove Theorem [3] and we employ the pmf of \( \mathcal{M}_{\text{ERR}} \).

**Theorem 3:** The expected error of \( \mathcal{M}_{\text{ERR}} \) is never worse than \( \mathcal{M}_{\text{EM}} \) for datasets where \( |T| = 1 \). That is, for quality score vectors s.t. \( q_s \leq q_o - t \) where \( s \) is at least the second largest index and \( t > c\Delta \),

\[ \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q})] \leq \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q})], \]

\[ \Pr [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q}) \geq t] \leq \Pr [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q}) \geq t]. \]

**Proof:** We divide the proof into two parts. The first looks at the candidates \( r \notin T \) and the second at \( r \in T \).

**Part 1:** \( \forall r \notin T \).

Assume without loss of generality (by symmetry) that \( q_1 \leq q_2 \leq \ldots \leq q_n \). Let \( \forall r \notin T \),

\[ f_r (\tilde{q}) = \Pr [\mathcal{M}_{\text{ERR}} (\tilde{q}) = r] - \Pr [\mathcal{M}_{\text{EM}} (\tilde{q}) = r]. \] (9)

We, therefore, have

\[ \Pr [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q}) \geq t] - \Pr [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q}) \geq t] = \sum_{r=1}^{s} f_r (\tilde{q}). \]

The goal is to show for all \( s \in \{1, \ldots, n - 1\} \),

\[ \sum_{r=1}^{s} f_r (\tilde{q}) \leq 0. \]

Note that because \( s \) is at least the second largest index and \( q_s \leq q_o - t \) where \( t > c\Delta \), we must use \( \Pr [\mathcal{M}_{\text{ERR}} (\tilde{q}) = s] = p_s, F \) from the pmf of \( \mathcal{M}_{\text{ERR}} \). Next, we argue that \( f_r \) monotonically increases with \( q_r \), i.e., \( f_1 \leq f_2 \leq \ldots \leq f_n \).

Substituting in Eq. [9] we show that \( f_r (\tilde{q}) \leq 0 \) \( \forall r \notin T \):

\[ f_r (\tilde{q}) = p_{r,F} - \sum_{r' \in R} p_{r'} \]

\[ = (1 - p_r) \frac{p_r}{\sum_{r' \in R} p_{r'}} - \frac{p_r}{\sum_{r' \in R} p_{r'}} \leq 0, \]

(10)

where \( p_r = \exp (\frac{c\Delta}{n} q_r) \).

Given that \( p \geq 0 \) and \( p_r \geq 0 \), Eq. [10] can only take negative values. This fact together with Lemma [4] shows \( \sum_{r=1}^{n} f_r (\tilde{q}) \leq 0 \), as desired.

The ordering of expected errors follows directly. As we have already shown that \( \Pr [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q}) \geq t] \leq \Pr [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q}) \geq t] \), it follows:

\[ \int_{(c\Delta, \infty)} (\Pr [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q}) \geq t] - \Pr [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q}) \geq t]) \, dt \leq 0. \]

This expression proves that \( \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q})] \leq \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q})] \) for only a part of the domain: \((c\Delta, \infty)\). In part 2, we prove the statement in the range \([0, c\Delta]\).

**Part 2:** For \( r \in T \).

For the given quality vector described in the formulation of the theorem, there is only one candidate in the range \([0, c\Delta]\), i.e., the one that optimizes \( \tilde{q} \), which corresponds to \( n \). Thus, the error and its expected value in both \( \mathcal{M}_{\text{ERR}} \) and \( \mathcal{M}_{\text{EM}} \) is 0 in the range \([0, c\Delta]\).

With parts 1 and 2, we conclude in Fig. 12 that \( \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{ERR}}, \tilde{q})] \leq \mathbb{E} [\mathcal{E} (\mathcal{M}_{\text{EM}}, \tilde{q})] \) as desired.

Next, in the same setting and procedure as Theorem 3 we compare \( \mathcal{M}_{\text{ERR}} \) to \( \mathcal{M}_{PF} \) in Lemma 5 for a given \( p \).

**Lemma 5:** There is always a range of \( p \) values within \([0, 1]\) where the expected error of \( \mathcal{M}_{\text{ERR}} \) is never worse than \( \mathcal{M}_{PF} \) for datasets where \( |T| = 1 \).

**Proof:** Adapting the formulation of Theorem 3 that lead to Eq. [10] and using the probability mass function of \( \mathcal{M}_{PF} \) from [2],

\[ f_r (\tilde{q}) = \left(1 - p_r \right) \frac{p_r}{\sum_{r' \in R} p_{r'}} - \frac{p_r}{\sum_{r' \in R} p_{r'}} \leq 0, \]
where \( p_r = \exp\left(\frac{\epsilon}{\Delta} q_r\right) \).

Publication [2] shows that \( M_{PF} \) is never worse than \( M_{EM} \). Thus, for \( p = 0 \), \( M_{PF} \) performs better because \( M_{ERR} \) defaults to \( M_{EM} \)—Eq. [11] can only be positive. However, for \( p = 1 \), Eq. [11] can only be negative. Based on this fact, and because \( h_r \) is constant-valued as \( r \) ranges from 1 to \( n \), \( g_r \) is non-decreasing, and \( p_r \) is non-negative and non-decreasing, the errors of \( M_{EM} \) and \( M_{PF} \) must intersect at a point as \( p \) increases from 0 to 1, after which the expected error of \( M_{ERR} \) is never worse than \( M_{PF} \).

**C. Utility comparison under the non-optimal scenario for \( M_{EM} \)**

**Lemma 6:** The expected error of \( M_{ERR} \) can be worse than \( M_{EM} \) for datasets where \( |T| > 1 \).

**Proof:** Adapting the formulation of Theorem [3] let \( s \) denote the largest index for quality vectors satisfying \( q_s \leq q^* - t \), where \( t \geq 0 \). Because \( t \geq 0 \), there can be more than one candidate whose \( \Pr[M_{ERR}(\vec{q}) = s] = p_{s,T} \). Therefore, there can be another candidate in \( T \) that does not maximize \( \vec{q} \), whose corresponding error with \( M_{ERR} \) might be larger than with \( M_{EM} \). Nonetheless, for the candidates whose quality scores satisfy \( q^* - q_s > c\Delta \), \( M_{ERR} \) still provides better error than \( M_{EM} \). Following the steps that led to Eq. [10] we have for this setting instead:

\[
f_r(\vec{q}) = p_{r,T} - \frac{p_r}{\sum_{r' \in R} p_{r'}} = \frac{p}{m} + (1 - p) \frac{p_r}{\sum_{r' \in R} p_{r'}} - \frac{p_r}{\sum_{r' \in R} p_{r'}} \leq 0.
\]

Rearranging:

\[
\frac{1}{m} \leq \frac{p_r}{\sum_{r' \in R} p_{r'}},
\]

where \( m = |T|, p \geq 0 \), and \( p_r = \exp\left(\frac{\epsilon}{\Delta} q_r\right) \). The expression indicates \( M_{ERR} \) performs better than \( M_{EM} \) when the probability of randomly picking a non-optimal candidate in \( T \) is smaller than the probability of picking it with \( M_{EM} \). Overall, there might be vectors with which \( M_{ERR} \) yields a larger error than \( M_{EM} \) when \( |T| > 1 \).

**D. Worst-case scenario for \( M_{EM} \)**

We have shown that \( M_{ERR} \) performs at least as good as \( M_{EM} \) when \( |T| = 1 \), but \( M_{ERR} \) could perform worse when \( |T| > 1 \). In this section, we look into a scenario where \( M_{EM} \) (and \( M_{PF} \)) performs the worst. In such a setting, we calculate in Proposition [2] see Fig. [13] the analytical formulation for the expected error of \( M_{ERR} \), so that we can analyse and compare the errors across mechanisms in the worst-case scenario for \( M_{EM} \). Furthermore, we show in Equations [15] and [16] the two expressions the expected error of \( M_{ERR} \) can take in the worst-case scenario for \( M_{EM} \).
The worst-case expected errors for both $\mathcal{M}_{EM}$ and $\mathcal{M}_{PF}$ occur when $\vec{q} = (v, v, ..., v) \in \mathbb{R}^n$ for some $v \leq 0$ [2].

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{EM}, \vec{q})] = \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[1 - \frac{1}{1 + (n-1)p'}\right],
\]

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{PF}, \vec{q})] = \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[1 - \frac{1}{n p'}\right],
\]

where $p' = \exp\left(\frac{r}{2\Delta} v\right)$, and $n = |\mathcal{R}|$.

**Proposition 2:** When $\vec{q} = (v, v, ..., v) \in \mathbb{R}^n$ for some $v \leq 0$, the expected error of $\mathcal{M}_{ERR}$ is:

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{ERR}, \vec{q})] = (1 - p) \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[1 - \frac{1}{1 + (n-1)p'}\right] + \frac{2\Delta}{\varepsilon} \frac{q}{m} + (m - 1) \log\left(\frac{1}{p'}\right),
\]

where $p' = \exp\left(\frac{r}{2\Delta} v\right)$, $n = |\mathcal{R}|$, and $m = |\mathcal{T}|$.

**Proof:** We follow the parametrization of [2], and use Eq. [8] to prove the Proposition. Without loss of generality, assuming $q_\ast = 0$, and $\log(p_r) = \frac{r}{2\Delta} q_r$.

\[
-\mathbb{E}[\mathcal{E}(\mathcal{M}_{ERR}, \vec{q})] = -q_\ast + \sum_{r \in \mathcal{R}} q_r (1 - p) \frac{p_{r'}}{p_r} \sum_{r' \in \mathcal{R}} p_{r'} \log(p_{r'}) + \frac{2\Delta}{\varepsilon} \sum_{r \in \mathcal{T}} \frac{p_r}{m} \log(p_r).
\]

Plugging in $p_n = 1$ and $p_r = p' = \exp\left(\frac{r}{2\Delta} v\right)$ for $r < n$, we obtain:

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{ERR}, \vec{q})] = -(1 - p) \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[1 - \frac{1}{1 + (n-1)p'}\right] + \frac{2\Delta}{\varepsilon} \frac{q}{m} + (m - 1) \log\left(\frac{1}{p'}\right).
\]

The expected error of $\mathcal{M}_{ERR}$ can take two forms in the context of the worst-case scenario for $\mathcal{M}_{EM}$.

**Case 1.** For $v < -c \Delta$, then $m = |\mathcal{T}| = 1$:

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{ERR}, \vec{q})] = (1 - p) \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[1 - \frac{1}{1 + (n-1)p'}\right].
\]

**Case 2.** For $-c \Delta \leq v \leq 0$, that is $\mathcal{T} = \mathcal{R}$, then $m = n$:

\[
\mathbb{E}[\mathcal{E}(\mathcal{M}_{ERR}, \vec{q})] = \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left[(1 - p) \frac{1}{1 + (n-1)p'} + \frac{(n-1)p}{n}\right]
\]

\[
= \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left(\frac{(n - 1)p(n - 1)p' + (n - 1)p^2p'}{n + (n - 1)p'}\right)
\]

\[
= \frac{2\Delta}{\varepsilon} \log\left(\frac{1}{p'}\right) \left(\frac{n - 1}{np' - p' + 1}\right).
\]

Fig. 13. The expected errors of $\mathcal{M}_{EM}$, $\mathcal{M}_{PF}$, and the two possible expected error expressions of $\mathcal{M}_{ERR}$ under the conditions of the worst-case scenario of $\mathcal{M}_{EM}$. 