Abstract

We study the effect of boundary conditions on vacuum polarization for charged scalar fields in two space-time dimensions. We find that both Dirichlet and Neumann boundary conditions lead to screening. In the Dirichlet case, the vacuum polarization charge density vanishes at the boundary, whereas it attains its maximum there for Neumann boundary conditions.

1 Introduction

Historically, vacuum polarization was one of the first effects of quantum electrodynamics that were theoretically studied [1]. On the practical side, it provides corrections to atomic energy levels, cf. the review [2]. In recent years, there has been revived interest in the topic, in particular in its dynamical version, the Schwinger effect.

However, very little seems to be known about the influence of boundaries on vacuum polarization. To the best of our knowledge, the only work in which the effect of different boundary conditions in external electric fields was analyzed is [3]. They considered the (massless) charged scalar field in two space-time dimensions, confined to an interval and subject to a constant electric field. It was found that, at the linearized level, Dirichlet boundary condition exhibit screening (with the maximal charge density near the boundaries) and Neumann boundary conditions anti-screening. These findings are rather surprising and counter-intuitive. One would expect vacuum polarization to be always screening. Furthermore, due to the repulsive (attractive) nature of Dirichlet (Neumann) boundary conditions, one would expect the screening to be stronger for Neumann boundary conditions.

The purpose of this note is to show that these surprising results are due to two major flaws in the calculation: First, a mode sum formula for the charge density is used, which, as pointed out in [4], is not gauge invariant and leads to incorrect results. Second, the perturbative expansion used in [3] breaks down for the zero mode in the massless case with Neumann boundary conditions. Hence, this mode was neglected in [3]. However, the zero mode is the mode that is most sensitive to external fields and thus contributes most to vacuum polarization. Here, we use a corrected version of the mode sum formula and consider massive fields to avoid the problem with the zero mode in the Neumann case. We obtain the expected behavior of vacuum polarization: It is screening and more pronounced for Neumann than for Dirichlet boundary conditions. More precisely, we find that the charge density vanishes near the boundary in the
Dirichlet case and attains its maximum at the boundary in the Neumann case. This also com-
plies nicely with the finding that the current density vanishes near Dirichlet boundaries in an
Aharonov-Bohm type setting with toroidally compactified dimensions [5].

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2 Setup

We follow the conventions of [3], i.e., we use signature (+, −), and define the covariant derivative
as

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi.$$  \hspace{1cm} (1)

We denote our coordinates by \(x = (t, z)\). The field \(\phi\) satisfies the Klein-Gordon equation

$$(D_\mu D^\mu + m^2)\phi = 0$$  \hspace{1cm} (2)

and the corresponding charge density, i.e., the 0 component of the current, is given by

$$\rho = ie(\phi^* D_0 \phi - \phi (D_0 \phi)^*).$$  \hspace{1cm} (3)

We will be interested in the vacuum polarization on a finite interval, whose length we
normalize to 1 so that \(z \in [0, 1]\), in the presence of a constant electric field \(E\). The latter is
implemented by

$$A_0 = -E(z - \frac{1}{2}), \hspace{1cm} A_1 = 0.$$  \hspace{1cm} (4)

With the separation ansatz

$$\phi = \phi_n(z)e^{-i\omega_n t},$$  \hspace{1cm} (5)

the equation of motion [2] can be explicitly solved in terms of parabolic cylinder functions as

$$\phi_n(z) = a_n D_{\frac{m^2}{2\pi^2} - \frac{1}{2}}(i\frac{1}{\sqrt{\lambda}}(\omega_n + \lambda(z - \frac{1}{2}))) + b_n D_{\frac{m^2}{2\pi^2} - \frac{1}{2}}(i\frac{1}{\sqrt{\lambda}}(\omega_n + \lambda(z - \frac{1}{2}))),$$  \hspace{1cm} (6)

where

$$\lambda = eE.$$  \hspace{1cm} (7)

With these mode functions, the evaluation of the vacuum polarization has to proceed numeri-
cally. It is thus more instructive to proceed perturbatively, i.e., to consider the electric field as a
perturbation and compute the vacuum polarization at first order in \(\lambda\). For this, the corrections
of first order in \(\lambda\) of the frequencies and the mode functions have to be determined. For this
purpose, it is useful to write, as in [3], the mode frequencies \(\omega_n\) and solutions \(\phi_n\) as solutions
to a time-dependent Schrödinger equation, albeit in a space of indefinite metric. Introducing

$$\Psi = \begin{pmatrix} \phi \\ \pi^* \end{pmatrix},$$  \hspace{1cm} (8)

with \(\pi^* = D_0 \phi\) the momentum conjugate to \(\phi^*\), we can write the equation of motion in the
form

$$i\partial_t \Psi = H \Psi$$  \hspace{1cm} (9)

with

$$H = i \begin{pmatrix} 0 & 1 \\ D_t^2 - m^2 & 0 \end{pmatrix} + \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix} = H_0 + H_1.$$  \hspace{1cm} (10)
This operator is hermitean w.r.t. the inner product
\[
\langle \Psi_1 | \Psi_2 \rangle = i \int dz \left( \phi_1^* \pi_2 - \pi_1 \phi_2 \right). \tag{11}
\]

In the case of Dirichlet boundary conditions, \( \Psi(0) = \Psi(1) = 0 \), a basis of eigenvectors \( \phi_n^D \) of \( H_0 \) with eigenvalues \( \omega_n^D \) is given by
\[
\omega^D_n = \text{sgn}(n) \sqrt{m^2 + \pi^2 n^2}, \quad \phi^D_n = |\omega^D_n|^{-\frac{1}{2}} \sin \pi n z, \tag{12}
\]
for \( n \in \mathbb{Z} \setminus \{0\} \) and with \( \pi_n = -i \omega_n \phi_n \). For Neumann boundary conditions \( \Psi'(0) = \Psi'(1) = 0 \), one has
\[
\omega^N_{n \neq 0} = \text{sgn}(n) \sqrt{m^2 + \pi^2 n^2}, \quad \phi^N_{n \neq 0} = |\omega^N_n|^{-\frac{1}{2}} \cos \pi n z, \tag{13}
\]
\[
\omega^N_{\pm 0} = \pm m, \quad \phi^N_{\pm 0} = (2m)^{-\frac{1}{2}}. \tag{14}
\]
Note that the modes are normalized to \( \text{sgn}(n) \), with \( \text{sgn}(\pm 0) = \pm 1 \), and that in the massless limit, the two zero modes \( \phi^N_{\pm 0} \) of the Neumann case are not normalizable. Physically, this implies the absence of a vacuum in that case.

First order perturbation theory in an indefinite inner product space proceeds analogously to that on Hilbert space. Given a basis \( \Psi_n^{(0)} \) of eigenvectors of \( H_0 \) with non-degenerate eigenvalues \( \omega_n^{(0)} \), the first order corrections are given by
\[
\omega_n^{(1)} = \frac{\langle \Psi_n^{(0)} | H_1 | \Psi_n^{(0)} \rangle}{\langle \Psi_n^{(0)} | \Psi_n^{(0)} \rangle}, \quad \Phi_n^{(1)} = \sum_{k \neq n} \frac{1}{\langle \Psi_k^{(0)} | \Psi_k^{(0)} \rangle} \frac{\langle \Psi_k^{(0)} | H_1 | \Psi_n^{(0)} \rangle}{\omega_n^{(0)} - \omega_k^{(0)}} \Psi_k^{(0)}. \tag{15}
\]
Note however that this breaks down in the presence of an eigenvector \( \Psi_k^{(0)} \) of vanishing norm, \( \langle \Psi_k^{(0)} | \Psi_k^{(0)} \rangle = 0 \). In particular, this implies that perturbation theory can not be applied to the massless case with Neumann boundary conditions. For the mode solutions for Dirichlet and Neumann boundary conditions, one finds that there are no corrections to the frequencies \( \omega_n \), while for the solutions one obtains, after a lengthy but straightforward calculation,
\[
\phi_n^D = (m^2 + \pi^2 n^2)^{-\frac{1}{4}} \left[ \sin \pi n z + \lambda \sqrt{m^2 + \pi^2 n^2} \left( \frac{1}{\pi n} (\frac{1}{2} - z) \sin \pi n z - z (1 - z) \cos \pi n z \right) \right], \tag{16}
\]
\[
\phi_n^N_{n \neq 0} = (m^2 + \pi^2 n^2)^{-\frac{1}{4}} \left[ \cos \pi n z + \lambda \sqrt{m^2 + \pi^2 n^2} \left( \frac{1}{\pi n} (\frac{1}{2} - z) \cos \pi n z + (z (1 - z) + (\pi n)^{-2}) \sin \pi n z \right) \right], \tag{17}
\]
\[
\phi^N_{\pm 0} = (2m)^{-\frac{1}{2}} \mp \lambda \sqrt{2m} \left( \frac{1}{24} - \frac{1}{4} z^2 + \frac{1}{6} z^3 \right), \tag{18}
\]
up to corrections of \( O(\lambda^2) \). In the massless limit, the result for Dirichlet boundary conditions coincides with the expression found in [3], cf. eq. (3.5) there. Regarding the result for the Neumann case, it seems that the \( \pm 0 \) modes were neglected in [3], and that the massless case was considered in spite of the presence of non-normalizable modes. It seems that disregarding the \( \pm 0 \) modes is the origin of the anti-screening effect found in [3]. Properly including these modes, which are the most affected by the external field, leads to the expected screening behaviour, as discussed below.
3 Quantization and point-split renormalization

Quantization based on the normalized mode solutions discussed in the previous section proceeds by writing the quantum field as

$$\phi(t, z) = \sum_{n>0} a_n \phi_n(z)e^{-i\omega_n t} + \sum_{n<0} b_n^\dagger \phi_n(z)e^{-i\omega_n t},$$

(19)

with the operators $a_n$, $b_n$, fulfilling the commutation relations

$$[a_n, a_m^\dagger] = \delta_{nm},$$
$$[b_n, b_m^\dagger] = \delta_{nm},$$

(20)

with all other commutators vanishing. In the case of Neumann boundary conditions, the $+0$ mode is included in the first sum in (19), while the $-0$ mode is included in the second. The vacuum state $|0\rangle$ is defined by the property that it is annihilated by $a_n$ and $b_n$.

The point-wise products appearing in the charge density are ill-defined and have to be renormalized. A well-controlled way to do this is by point-split renormalization w.r.t. a Hadamard parametrix. Physically reasonable states, vacuum states in particular [6], have two-point functions

$$w_{\Omega^*}(x, x') = \langle \Omega | \phi(x)\phi^*(x')|\Omega \rangle$$

(21)

of Hadamard form, i.e., their singular behavior as $x' \to x$ is of a universal form, which is determined entirely by the background fields in a neighborhood of $x$, but is independent of the state $\Omega$, cf. [4], Section 2, for a review. For a charged scalar field in 1+1 dimension, it is of the form

$$w_{\Omega^*}(x, x') = -\frac{1}{4\pi} U(x, x') \log(-(x-x')^2 + i\varepsilon(x-x')^0) + R_{\Omega^*}(x, x'),$$

(22)

with $U(x, x')$ and $R_{\Omega^*}(x, x')$ smooth functions. While $R_{\Omega^*}(x, x')$ is state dependent, $U(x, x')$ is fixed and given by the parallel transport w.r.t. the covariant derivative $\nabla$ along the straight line from $x'$ to $x$, up to corrections that are irrelevant for the determination of the vacuum polarization,

$$U(x, x') = \exp \left[-ie \int_0^1 A_\mu(x' + s(x-x'))(x-x')^\mu ds \right] + O((x-x')^2).$$

(23)

The two point function

$$w_{\Omega^*}(x, x') = \langle \Omega | \phi^*(x)\phi(x')|\Omega \rangle$$

(24)

has the same form, with $U(x, x')$ replaced by $U(x, x')^* = U(x', x)$. The idea of Hadamard point-split renormalization, which goes back to Dirac [7] and was re-discovered in the context of quantum field theory on curved space-times, cf. [8] for a recent review, is to define the expectation value of a local expression quadratic in fields by

$$\langle \Omega | D_\alpha \phi(x)(D_\beta \phi)^*(x)|\Omega \rangle = \lim_{x' \to x} \left[ D_\alpha D_\beta^* \left( w_{\Omega^*}(x, x') - H_{\phi^*}(x, x') \right) \right].$$

(25)

Here $\alpha$ and $\beta$ are symmetrized multi-indices, $D_\mu^*$ stands for the application of $D_\mu^* = \partial_\mu - ieA_\mu$ on the primed variable, and $H_{\phi^*}(x, x')$ is the first term on the r.h.s. of (22).

For the evaluation of the vacuum polarization, we have two such expressions to evaluate. We perform the point-splitting in the time direction, so that $x' = (t + \tau, z)$, and we obtain

$$\langle 0 | \phi^*(x)D_0\phi(x')|0 \rangle = -i \sum_{n<0} |\phi_n(z)|^2(\omega_n - eA_0)e^{-i\omega_n(t+ie)},$$

(26)

$$\langle 0 | \phi(x)D_0^*\phi^*(x')|0 \rangle = i \sum_{n>0} |\phi_n(z)|^2(\omega_n - eA_0)e^{i\omega_n(t+ie)},$$

(27)
where we added an $i\varepsilon$ prescription to ensure convergence. On the other hand, we compute

$$D_0^* H^{\phi\phi}(x, x') = -\frac{1}{2\pi} \frac{1}{\tau + i\varepsilon} U(x', x) = -\frac{1}{2\pi} \left( \frac{1}{\tau + i\varepsilon} + ieA_0 \right) + O(\tau),$$

(28)

$$D_0^* H^{\phi\phi}(x, x') = -\frac{1}{2\pi} \frac{1}{\tau + i\varepsilon} U(x', x) = -\frac{1}{2\pi} \left( \frac{1}{\tau + i\varepsilon} - ieA_0 \right) + O(\tau).$$

(29)

For the vacuum polarization, we thus obtain

$$\rho(z) = ie\langle 0 | \left( \phi^* D_0 \phi - \phi D_0^* \phi^* \right) | 0 \rangle$$

$$= e \lim_{\tau \to 0} \left( \sum_{n<0} |\phi_n(z)|^2 (\omega_n - eA_0) e^{-i\omega_n(\tau + i\varepsilon)} + \sum_{n>0} |\phi_n(z)|^2 (\omega_n - eA_0) e^{i\omega_n(\tau + i\varepsilon)} \right) + \frac{e^2}{\pi} A_0(z).$$

We see that the singular parts of (28) and (29) cancel. In the case of interest to us, we have $\omega_n = -\omega_n$, so that we may combine the two terms into a single sum over $n > 0$. Formally taking the limit $\tau \to 0$ (we will proceed more carefully in the following), one then obtains

$$\rho(z) = e \sum_{n>0} \left( |\phi_n(z)|^2 (\omega_n - eA_0) - |\phi_{-n}(z)|^2 (\omega_n + eA_0) \right) + \frac{1}{\pi} e^2 A_0(z) \quad \text{(FORMALLY)}.$$ 

(30)

(31)

Up to the last term, which came from the Hadamard point-split procedure, this coincides with the expression used in [3]. Alternatively, this last term can also be seen as a consequence of including a parallel transport in the point-split prescription, as required in [9] in the context of Dirac fields, cf. also the discussion in Section 2.2 of [4]. In particular, it is analogous to the correction term $\frac{e^2}{\pi \tau} A_0$, which is necessary to render the mode sum expression of Wichmann and Kroll [10] for the third order contribution to the vacuum polarization in 3+1 dimensions gauge invariant, cf. [11] for example.

4 Evaluation of the vacuum polarization

We finally want to evaluate the expression (30). In the perturbative approach, at first order in $\lambda$, this is possible analytically in the massless case with Dirichlet boundary conditions. We rewrite (30) as

$$\rho(z) = e \lim_{\tau \to 0} \sum_{n=1}^{\infty} \left( |\phi_n(z)|^2 (\pi n - eA_0) - |\phi_{-n}(z)|^2 (\pi n + eA_0) \right) e^{i\pi n(\tau + i\varepsilon)} + \frac{1}{\pi} e^2 A_0(z).$$

(32)

Up to first order in $\lambda$, we have

$$|\phi_n(z)|^2 = \frac{1}{\pi |n|} \left( \sin^2 \pi nz - \lambda \sin \pi nz \left[ \frac{1}{\pi n} \left( z - \frac{1}{2} \right) \sin \pi nz + z(1 - z) \cos \pi nz \right] \right)$$

(33)

and thus obtain, to first order in $\lambda$,

$$\rho(z) = -2e\lambda \lim_{\tau \to 0} \sum_{n=1}^{\infty} z(z - 1) \sin \pi nz \cos \pi nz e^{i\pi n(\tau + i\varepsilon)} - \frac{1}{\pi} e\lambda \left( z - \frac{1}{2} \right)$$

$$= -e\lambda z(z - 1) \lim_{\tau \to 0} \frac{1}{2i} \left( e^{i\pi (2z + \tau + i\varepsilon)} - e^{i\pi (-2z + \tau + i\varepsilon)} \right) - \frac{1}{\pi} e\lambda \left( z - \frac{1}{2} \right)$$

$$= -e\lambda \frac{z(z - 1)}{2} \cot \pi z - \frac{1}{\pi} e\lambda \left( z - \frac{1}{2} \right).$$

(34)
Figure 1: In blue the vacuum polarization for Dirichlet boundary conditions in the massless case, according to [34]. In orange the result of [3].

Figure 2: Vacuum polarization for Dirichlet boundary conditions to first order in $\lambda$ for $m = 0$ (blue), $m = 1$ (orange), and $m = 5$ (green).

Up to the last term, this coincides with the result of [3]. However, this term makes a qualitative difference for the resulting vacuum polarization, as seen in Figure 1. The vacuum polarization charge density is not only much smaller in magnitude, but it vanishes exactly at the boundaries, as one would naively expect for Dirichlet, i.e., repulsive, boundary conditions.

In the massive case, a completely analytic treatment is not possible. However, in the perturbative treatment, the numerical evaluation of the expression (30), using the perturbative expansion of the mode functions, is straightforward. For the Dirichlet case, results are shown in Figure 2. We see that vacuum polarization is suppressed for higher masses.

Results for the case of Neumann boundary conditions are shown in Figure 3. Again, we see that vacuum polarization is suppressed for increasing mass (note that $m\rho$ is plotted). In contrast to the Dirichlet case, the vacuum polarization is not vanishing at the boundary, it is in fact maximal there, as one would expect for attractive boundary conditions. In any case, it is screening, as opposed to the anti-screening behaviour that was found in [3]. As discussed above, it is the proper inclusion of the $\pm 0$ modes, which also necessitated to work with a finite
mass, which explains the difference to the results of [3] (apart from the inclusion of the last term on the r.h.s. of (30)). Finally, we remark that for Neumann boundary conditions and high enough masses, the vacuum polarization changes sign within the interval $\left(0, \frac{1}{2}\right)$, and analogously in $\left(\frac{1}{2}, 1\right)$. This means that, in the present approximation, as one approaches the boundary at $z = 0$ from $z = \frac{1}{2}$, the perceived charge at $z = 0$ first decreases, and then increases. Such a somewhat counterintuitive behaviour is also present in the vacuum polarization in the Coulomb potential [11], although there it occurs upon including effects of higher order in the external field.

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