t–Deformations of quantum Schubert polynomials

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For my daughter’s 14-th birthday

Abstract

We construct a certain solution $F(t)$ to the Witten–Dijkgraaf–Verlinde–Verlinde equation related to the small quantum cohomology ring of flag variety, and study the $t$–deformation of quantum Schubert polynomials corresponding to this solution.

1 Introduction

The cohomology ring of the flag variety $Fl_n = SL_n/B$ is isomorphic to the quotient of the polynomial ring $P_n := \mathbb{Z}[x_1, \ldots, x_n]$ by the ideal $I_n$ generated by symmetric polynomials without constant term:

$$H^*(Fl_n, \mathbb{Z}) \cong P_n/I_n.$$ 

The Schubert cycles $X_w := \overline{BwB}/B$, $w \in W$, form a linear basis of the homology group $H_*(Fl_n, \mathbb{Z})$ and via the Poincare duality they are represented by the (geometric) Schubert polynomials $X_w(x)$ in the cohomology ring $H^*(Fl_n, \mathbb{Z})$. It was discovered by A. Lascoux and M.-P. Schützenberger, [LS1], [LS2], that there exists the set of distinguish representatives $S_w(x) \in P_n$ of the geometric Schubert polynomials $X_w(x) \in P_n/I_n$ with nice algebraic, combinatorial and geometric properties. Follow A. Lascoux and M.-P. Schützenberger [LS1] these distinguish representatives $S_w(x)$ are called Schubert polynomials. These polynomials form a stable, homogeneous, orthonormal basis in the ring of polynomials $P_n = \mathbb{Z}[x_1, \ldots, x_n]$ indexed by permutations $w \in S^{(n)} = \{w \in S_{\infty}, l(c(w)) \leq n\}$. We refer the reader to [M] and [F] for detailed account on Schubert polynomials. One of the most deep and fundamental properties of the Schubert
polynomials $S_w(x)$ is that their structural constants $c_{uv}^w$, defined from the decomposition

$$S_u S_v = \sum_w c_{uv}^w S_w \mod I_n,$$

appear to be nonnegative integers. The only known proof of this fact appeals to algebraic geometry and is based on an interpretation of the structural constant $c_{uv}^w$ as the intersection number of the Schubert cycles $X_u, X_v$ and $X_{w_0 w}$. One of the most fundamental problems of the Schubert Calculus is to find an algebraic proof of nonnegativity of the structural constants $c_{uv}^w$ and give their combinatorial interpretation (Littlewood–Richardson’s problem for Schubert polynomials).

Since the quantum cohomology of compact complex Kahler manifolds, was introduced by C. Vafa [V], the computation of the (big) quantum cohomology rings of flag varieties became a priority problem of Quantum Schubert Calculus. We refer the reader to [FP] and [MS] where the definition and basic properties of quantum cohomology may be found.

The essential new ingredient of the quantum cohomology theory in comparison with the classical one is the existence of the Gromov–Witten potential $F(t)$ which contains all information about the multiplication rules in the quantum cohomology ring. The Gromov–Witten potential $F(t)$ satisfies the celebrated Witten–Dijkgraaf–Verlinde–Verlinde equation (WDVV–equation):

$$\sum_{\nu,\mu} \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_\nu} g^{\nu\mu} \frac{\partial^3 F}{\partial t_\mu \partial t_\nu \partial t_i} = \sum_{\nu,\mu} \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_\nu} g^{\nu\mu} \frac{\partial^3 F}{\partial t_\mu \partial t_j \partial t_i},$$  

(1.1)

for all $i, j, k, l$, where $g^{\nu\mu}$ stands for the inverse matrix of $g_{\mu\nu}$, where $g_{\mu\nu}$ is the intersection matrix:

$$g_{\mu\nu} = \frac{\partial^3 F}{\partial t_0 \partial t_\nu \partial t_\mu}.$$  

In particular case of the flag variety $Fl_n$, the WDVV–equation takes the following form

$$\sum_{w \in S_n} \frac{\partial^3 F}{\partial t_{w_1} \partial t_{w_2} \partial t_v} \cdot \frac{\partial^3 F}{\partial t_{w_0 v} \partial t_{w_3} \partial t_4} = \sum_{w \in S_n} \frac{\partial^3 F}{\partial t_{w_1} \partial t_{w_2} \partial t_v} \cdot \frac{\partial^3 F}{\partial t_{w_0 v} \partial t_{w_1} \partial t_{w_4}},$$  

(1.2)

for all $w_1, w_2, w_3, w_4 \in S_n$, where $w_0$ is the longest element of $S_n$, and $t = (t_w, w \in S_n)$ stands for the set of independent variables $t_w$ parameterized by permutations $w \in S_n$.

It was conjectured in [KM] that the WDVV–equation (1.2) defines uniquely the Gromov–Witten potential $F(t)$ for the flag variety $Fl_n$ if the following conditions are satisfied:

1) Normalization:

$$\frac{\partial^3 F}{\partial t_1 \partial t_v \partial t_w} = \delta_{v,w_0 w};$$  

2) Initial conditions:

$$\frac{\partial^3 F}{\partial t_{s_k} \partial t_{s_k} \partial t_{w_0}} = q_k, \quad 1 \leq k \leq n - 1,$$
where for each $k \in [1, n-1]$, $q_k$ is a certain constant, and $s_k = (k, k+1)$ denotes the transposition that interchanges $k$ and $k+1$, and fixes all other elements of $[1, n]$;

3) Degree conditions:

$$\frac{\partial^3 \mathcal{F}}{\partial t_u \partial t_v \partial t_w} = 0,$$

if either $l(u) + l(v) + l(w) < l(w_0)$ or difference $l(u) + l(v) - l(w_0)$ is an odd positive integer.

In particular case when $t_w = 0$ for all $w \in S_n$ such that $l(w) \geq 2$, the quantum cohomology ring $\text{QH}^*(F\ell_n)$ (the so-called small quantum cohomology ring) was computed by A. Givental and B. Kim [GK], see also [C1]. The study of (small) quantum Schubert polynomials corresponding to the small quantum cohomology ring $\text{QH}^*(F\ell_n)$ of the flag variety $F\ell_n$ was initiated in [FGP] and independently in [KM], see also [C2]. The Grassmannian case was considered earlier in [B], [C1] and [W].

The main goal of the present paper is to construct a toy model for big quantum cohomology ring and big quantum Schubert polynomials for the flag variety $F\ell_n$. More precisely, to construct the (toy) Gromov–Witten potential $\mathcal{F}(t)$ which appears to be a solution to the WDVV–equation (1.2). Based on this solution, we define the $t$–deformation $\tilde{S}_w'$ of the quantum Schubert polynomials $\tilde{S}_w$ and investigate some of their properties. The (toy) Gromov–Witten potential $\mathcal{F}(t)$ is defined to be the Grothendieck residue of the function $U(x) = \exp\left(\sum_{w \in S_n} t_w \tilde{S}_w(x)\right)$ with respect to the ideal $\tilde{I}_n$:

$$\mathcal{F}(t) = \langle \exp\left(\sum_{w \in S_n} t_w \tilde{S}_w(x)\right) \rangle_{\tilde{I}_n}.$$

Here the function $\mathcal{F}(t)$ is a natural generalization of the quantum generating volume function $V(z; q)$ introduced by A. Givental and B. Kim [GK].

The content of the paper is arranged as follows. In Section 2 we review the definition and some basic properties of the quantum Schubert polynomials to be used in Sections 3 and 4. In Section 3 we introduce and study the (toy) Gromov–Witten potential $\mathcal{F}(t)$ and the $t$–deformation of quantum Schubert polynomials related to the potential $\mathcal{F}(t)$. In Section 4 we construct a certain Lax pair related to yet another deformation $X'_w$ of the quantum Schubert polynomials.

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2 Quantum Schubert polynomials

In this section we review the definition and basic properties of the quantum Schubert polynomials. The study of (small) quantum Schubert polynomials for flag variety $Fl_n$ was initiated in [FGP] and, independently, in [KM]. The case of Grassmannian varieties was considered earlier in [B], [C1] and [W]. In our exposition we follow [KM].

Let $X_n = (x_1, \ldots, x_n), Y_n = (y_1, \ldots, y_n)$ be two sets of independent variables, put

$$\tilde{S}_{w_0}(x, y) := \tilde{S}_{w_0}^{(q)}(X_n, Y_n) = \prod_{i=1}^{n-1} \Delta_i(y_{n-i} | X_i),$$

where $\Delta_k(t|X_k) := \sum_{j=0}^{k} t^{k-j} e_j(X_k|q_1, \ldots, q_{k-1})$ is the generating functions for the quantum elementary symmetric polynomials in the variables $X_k = (x_1, \ldots, x_k)$, i.e.

$$\Delta_k(t|X_k) := \sum_{i=0}^{k} e_i(X_k|q)t^{k-i}$$

The determinant (2.1) was introduced in connection to the (small) quantum cohomology ring of flag varieties by A. Givental and B. Kim [GK].

**Definition 2.1** ([KM]) For each permutation $w \in S_n$, the quantum double Schubert polynomial $\tilde{S}_w(x, y)$ is defined to be

$$\tilde{S}_w(x, y) = \partial_{w,w_0}^{(y)} \tilde{S}_{w_0}(x, y),$$

where divided difference operator $\partial_{w,w_0}^{(y)}$ acts on the $y$ variables.

**Definition 2.2** (cf. [KM]) For each permutation $w \in S_n$, the (small) quantum Schubert polynomial $\tilde{S}_w(x)$ is defined to be the $y = 0$ specialization of the quantum double Schubert polynomials $\tilde{S}_w(x, y)$:

$$\tilde{S}_w(x, y) = \partial_{w,w_0}^{(y)} \tilde{S}_{w_0}(x, y)|_{y=0}.$$
Now we are going to review some basic properties of (small) quantum Schubert polynomial which will be used in the next sections. We start with reminding the result of A. Givental and B. Kim [GK], and I. Ciocan-Fontanine [C1], on the structure of the small quantum cohomology ring $\text{QH}^*(F_{l_n})$ of the flag variety $F_{l_n}$:

$$\text{QH}^*(F_{l_n}) \cong \mathbb{Z}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/\bar{I}_n,$$

where the ideal $\bar{I}_n$ is generated by the quantum elementary symmetric polynomials $\bar{e}_i(x) := e_i(X_n, q_1, \ldots, q_{n-1})$, $1 \leq i \leq n$, with generating function $\Delta_n(t|X_n)$, see (2.1).

There exists a natural pairing $\langle f, g \rangle_Q$ on the ring of polynomials $\mathbb{Z}[X_n, q_1, \ldots, q_{n-1}]$, and the small quantum cohomology ring $\text{QH}^*(F_{l_n}) \cong \mathbb{Z}[X_n, q_1, \ldots, q_{n-1}]/\bar{I}_n$ which is induced by the Grothendieck residue with respect to the ideal $\bar{I}_n$, see, e.g., [GK], [KM]:

$$\langle f, g \rangle_Q = \text{Res}_{\bar{I}_n}(f, g), \quad f, g \in \mathbb{Z}[X_n, q_1, \ldots, q_{n-1}].$$

The pairing $\langle f, g \rangle_Q$ satisfies the following properties

1) $\langle f, g \rangle_Q = 0$ if $f \in \bar{I}_n$;

2) $\langle f, g \rangle_Q$ defines a nondegenerate pairing in $\mathbb{Z}[x, q]/\bar{I}_n \cong \text{QH}^*(F_{l_n})$.

Let us denote by $\Lambda_n[q] = \Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$, $\mathcal{H}_n[q] = \mathcal{H}_n \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$, where $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]\mathcal{S}_n$ stands for the ring of symmetric functions, and $\mathcal{H}_n$ stands for the $\mathbb{Z}$-span generated by monomials $x^I = x_1^{i_1} \cdots x_n^{i_n}$, where $I \subset \delta_n = (n-1, n-2, \ldots, 1, 0)$.

**Proposition 2.3** ([FGP], [KM])

- The quantum Schubert polynomials $\bar{S}_w$, $w \in S_n$, form a $\Lambda_n[q]$-basis of $P_n \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$.

- The quantum Schubert polynomials $\bar{S}_w$, $w \in S_n$, form a $\mathcal{H}_n[q]$-linear basis of $\mathcal{H}_n[q]$.

- (Orthogonality) Let $u, v \in S_n$. Then

$$\langle \bar{S}_u, \bar{S}_v \rangle_Q = \begin{cases} 1, & \text{if } u = w_0v, \\ 0, & \text{otherwise.} \end{cases}$$

- (Gromov–Witten’s invariants of order 3) Let $u, v, w \in S_n$, then there exists a polynomial $\bar{c}^w_{uv}(q) \in \mathbb{Z}[q_1, \ldots, q_{n-1}]$ with nonnegative integer coefficients such that

$$\bar{S}_u \bar{S}_v \equiv \sum_w \bar{c}^w_{uv}(q) \bar{S}_w \mod \bar{I}_n.$$  

Let us remind that the coefficient $\bar{c}^w_{uvd}$ of the polynomial

$$\bar{c}^w_{uv}(q) = \sum_{d \geq 0} \bar{c}^w_{uvd} q^d$$

is called the Gromov–Witten invariant of order 3 and degree $d$ corresponding to the Schubert cycles $X_u$, $X_v$, and $X_w$.  

5
3 t–Deformation

In this Section we introduce and study the (toy) Gromov–Witten potential \( F(t) \) and define the \( t \)–deformation of the quantum Schubert polynomials corresponding to this potential.

To start, let us consider the function

\[
\exp \left( \sum_{w \in S_n} t_w \tilde{S}_w(x) \right) \in \mathbb{Q}[[X, q_1, \ldots, q_{n-1}, \{ t_w \}_{w \in S_n}]],
\]

and denote by \( K(x, t) = \left[ \exp \left( \sum_{w \in S_n} t_w \tilde{S}_w(x) \right) \right] \) the projection of the function (3.1) on the subspace \( \overline{\mathcal{H}}_n(t, q) := \mathcal{H}_n \otimes_\mathbb{Z} \mathbb{Z}[[q_1, \ldots, q_{n-1}, \{ t_w \}_{w \in S_n}]] \) generated by the quantum Schubert polynomials \( \tilde{S}_w(x) \), \( w \in W \), which called kernel. Thus, we have

\[
K(x, t) = \sum_{w \in S_n} \varphi_w(t) \tilde{S}_{w0}(x),
\]

where coefficients \( \varphi_w(t) \) belong to the formal power series ring \( \mathbb{Z}[[q_1, \ldots, q_{n-1}, \{ t_w \}_{w \in S_n}]] \).

**Definition 3.1** Define the (toy) Gromov–Witten potential \( F(t) \) to be the Grothendieck residue of the kernel \( K(x, t) \) with respect to the ideal \( \overline{I}_n: \quad \mathcal{F}(t) = \langle K(x, t) \rangle_{\overline{I}_n}. \)

In other words, \( \mathcal{F}(t) = \varphi_{id}(t) \), i.e.

\[
K(x, t) = \mathcal{F}(t) \tilde{S}_{w0}(x) + \sum_{w>1} \varphi_w(t) \tilde{S}_{w0}(x).
\]

**Lemma 3.2** For each \( w \in S_n \),

\[
\frac{\partial}{\partial t_w} \mathcal{F}(t) = \varphi_w(t). \tag{3.3}
\]

**Proof.** By definition,

\[
\frac{\partial}{\partial t_w} \mathcal{F}(t) = \langle \frac{\partial}{\partial t_w} \exp \left( \sum_{w \in S_n} t_w \tilde{S}_w(x) \right) \rangle_{\overline{I}} =
\]

\[
= \langle \tilde{S}_w(x) \cdot K(x, t) \rangle_{\overline{I}} = \langle K(x, t), \tilde{S}_w(x) \rangle_Q = \varphi_w(t).
\]

It follows from Lemma 3.2 that

\[
K(x, t) = \left( \sum_{w \in S_n} \frac{\partial}{\partial t_w} \tilde{S}_{w0}(x) \right) \cdot \mathcal{F}(t).
\]

From now we will use the notation \([f], f \in \mathbb{Z}[X_n][[q_1, \ldots, q_{n-1}, \{ t_w \}_{w \in S_n}]]\) to denote the projection of \( f \) on the subspace \( \mathcal{H}_n(t, q) := \mathcal{H}_n \otimes_\mathbb{Z} \mathbb{Z}[[q_1, \ldots, q_{n-1}, \{ t_w \}_{w \in S_n}]] \) generated by the small quantum Schubert polynomials \( \tilde{S}_w, w \in S_n \).
Definition 3.3 Define the $t$–deformation $\tilde{\mathcal{S}}_w^t(x)$ of the quantum Schubert polynomial $\tilde{\mathcal{S}}_w(x)$ by the following rule

$$\tilde{\mathcal{S}}_w^t(x) = [K(x, t)\tilde{\mathcal{S}}_w(x)].$$

It is clear from the very definition that

$$\langle \tilde{\mathcal{S}}_w^t(x) \rangle_I = \varphi_w(t).$$

Assume that $\tilde{\mathcal{S}}_w^t(x) = \sum u \alpha_{u, w}(t) \tilde{\mathcal{S}}_w^0(x)$.

Lemma 3.4

$$\alpha_{u, w}(t) = \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} F(t). \tag{3.4}$$

Proof. By definition,

$$\frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} F(t) = \langle \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} K(x, t) \rangle_I = \langle \tilde{\mathcal{S}}_u^t(x) \tilde{\mathcal{S}}_w(x) K(x, t) \rangle_I$$

$$= \langle \tilde{\mathcal{S}}_u^t(x) \tilde{\mathcal{S}}_u^t(x) \rangle_I = \langle \tilde{\mathcal{S}}_u^t(x), \tilde{\mathcal{S}}_u^t(x) \rangle Q = \alpha_{u, w}(t).$$

It follows from (3.3) that $\alpha_{u, w}(t) = \frac{\partial}{\partial t_u} \varphi_w(t)$.

Lemma 3.5 For each $w \in S_n$,

$$\frac{\partial}{\partial t_w} K(x, t) = \tilde{\mathcal{S}}_w^t(x). \tag{3.5}$$

Proof. We have,

$$\frac{\partial}{\partial t_w} K(x, t) = [\tilde{\mathcal{S}}_w(x) K(x, t)] = \tilde{\mathcal{S}}_w^t(x).$$

Corollary 3.6 For each $u, w \in S_n$, let us consider the operator

$$\Delta_{u, w} = \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} - \sum \tilde{c}_{w, w_0 \tau}^{u, \tau} \frac{\partial}{\partial t_\tau},$$

where $\tilde{c}_{w, w_0}^{u, \tau} = \tilde{c}_{w, w_0}^{u, \tau}(q)$ stands for the structural constants for quantum Schubert polynomials, see Section 2. Then

$$\Delta_{u, w} F = 0. \tag{3.6}$$
Proof. It follows from (3.5), that
\[
\frac{\partial}{\partial t} K(x, t) = \left[ \tilde{S}_w(x) K(x, t) \right] = \sum_{\tau} \left[ \tilde{S}_w(x) \tilde{S}_{w_{\tau}}(x) \varphi_\tau(t) \right]
\]
\[
= \sum_{u} \left[ \sum_{\tau} \varphi_\tau(t) \tilde{c}_{w_{u \tau}}^{w_{u \tau}} \right] \tilde{S}_{w_{u \tau}}.
\]
Hence,
\[
\alpha_{u, w}(t) = \frac{\partial}{\partial t} \varphi_\tau(t) = \sum_{\tau} \varphi_\tau(t) \tilde{c}_{w_{u \tau}}^{w_{u \tau}}.
\]
Equation (3.6) follows from the following relations:
\[
\alpha_{u, w}(t) = \frac{\partial}{\partial t} \varphi_\tau(t) \frac{\partial}{\partial t} F, \quad \text{see Lemma 3.4}, \quad \text{and}
\]
\[
\varphi_\tau(t) = \frac{\partial}{\partial t} F, \quad \text{see Lemma 3.2}.
\]

Now we are going to study the structural constants for multiplication of the $t$–deformed quantum Schubert polynomials $\tilde{S}_w^t(x)$. Let us define functions $\Lambda_{u w \tau}(t)$ from the decomposition
\[
\left[ \tilde{S}_u^t \tilde{S}_w^t \right] = \sum_{\tau} \Lambda_{u w \tau}(t) \tilde{S}_{w_{\tau}}^t := \tilde{S}_u^t \circ \tilde{S}_w^t.
\]

**Lemma 3.7**
\[
\Lambda_{u w \tau}(t) = \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} \frac{\partial}{\partial t_\tau} F(t).
\]

**Proof.** By definition,
\[
\tilde{S}_u^t \tilde{S}_w^t = \left[ \tilde{S}_u \tilde{S}_w^t K(x, t) \right] = \sum_{\alpha} \left[ \varphi_{w_{\alpha \tau}} \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha K(x, t) \right]
\]
\[
= \sum_{\tau} \left[ \sum_{\alpha} \varphi_{w_{\alpha \tau}} \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha \tilde{S}_\tau \rangle \tilde{S}_{w_{\tau}} K(x, t) \right] = \sum_{\tau} \left( \sum_{\alpha} \varphi_{w_{\alpha \tau}} \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha \tilde{S}_\tau \rangle \right) \tilde{S}_{w_{\tau}}.
\]
Hence,
\[
\Lambda_{u w \tau}(t) = \sum_{\alpha} \varphi_{w_{\alpha \tau}} \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha \tilde{S}_\tau \rangle, \quad \text{where}
\]
\[
\langle \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha \tilde{S}_\tau \rangle = \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\alpha \tilde{S}_\tau \rangle \tilde{i}.
\]
But by the very construction,
\[
\frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} \frac{\partial}{\partial t_\tau} F = \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\tau K(x, t) \rangle \tilde{i}
\]
\[
= \sum_{\alpha} \varphi_{w_{\alpha \tau}} \langle \tilde{S}_u \tilde{S}_w \tilde{S}_\tau \tilde{S}_\alpha \rangle \tilde{i} = \Lambda_{u w \tau}(t).
\]

\[\text{□}\]
Corollary 3.8 The Gromov–Witten potential $F(t) := \langle K(x, y) \rangle_{\text{eq}}$ satisfies the WDVV–equation (1.2).

Proof. Follows from associativity of multiplication of $t$–deformed Schubert polynomials.

Corollary 3.9

$$\langle \tilde{S}_u \tilde{S}_w \tilde{S}_\tau \rangle = \frac{\partial^3}{\partial t_u \partial t_w \partial t_\tau} F(t)|_{t=0}.$$ 

Proof. If $t_w = 0$ for all permutations $w \in S_n$, then $K(x, 0) = 1$, $\tilde{S}_w(x)|_{t=0} = \tilde{S}_w(x)$ and $\Lambda_{uw\tau}(0) = \langle \tilde{S}_u \tilde{S}_v \tilde{S}_\tau \rangle$.

Lemma 3.10

$$\frac{\partial}{\partial t_u} \tilde{S}_w^t(x) = \sum_\tau \tilde{c}_{uw}^\tau \tilde{S}_\tau^t(x).$$

Corollary 3.11 The kernel $K(x, t)$ satisfies the following system of differential equations

$$\left( \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_w} - \sum_\tau \tilde{c}_{uw}^\tau \frac{\partial}{\partial t_\tau} \right) K(x, t) = 0.$$

The $t$–deformed quantum Schubert polynomials $\tilde{S}_w^t$ do not orthogonal with respect to the quantum pairing $\langle , \rangle_Q$ any more, but with respect to a new scalar product $\langle , \rangle_t$ on the quantum cohomology ring:

$$\langle f, g \rangle_t = \langle fgK^{-2}(x, t) \rangle_{\tilde{I}}$$

they do orthogonal:

Lemma 3.12 The $t$–deformed quantum Schubert polynomials $\tilde{S}_w^t$ are orthogonal with respect to the pairing $\langle , \rangle_t$.

Proof. It is clear that

$$\langle \tilde{S}_u^t, \tilde{S}_w^t \rangle_t = \langle \tilde{S}_u^t, \tilde{S}_w^t \rangle_Q = \left\{ \begin{array}{ll} 1, & \text{if } v = w_0 w, \\ 0, & \text{otherwise}. \end{array} \right.$$ 

In the last part of this Section we are going to study the action of operators $\frac{\partial}{\partial t_u}$ on the $t$–deformed quantum Schubert polynomials $\tilde{S}_w^t$ (toy analog of Dubrovin’s connection).
Lemma 3.13
\[
\frac{\partial}{\partial t} \tilde{\mathcal{S}}_u^t = \tilde{\mathcal{S}}_u \circ \tilde{\mathcal{S}}_u^t.
\]

Proof. It is clear that
\[
\tilde{\mathcal{S}}_u \circ \tilde{\mathcal{S}}_u^t = \left[ \tilde{\mathcal{S}}_u \tilde{\mathcal{S}}_u K(x, t) \right] = \sum_{\tau} \Lambda_{uw\tau}(t) \tilde{\mathcal{S}}_{w\tau}.
\]

Similarly, \( \frac{\partial}{\partial t} \tilde{\mathcal{S}}_w^t = \left[ \tilde{\mathcal{S}}_u \tilde{\mathcal{S}}_w K(x, t) \right]. \)

One can define a new multiplication \(*\) on the ring \( Q^*_t(\text{Fl}_n) = QH^*(\text{Fl}_n, \mathbb{Z}) \otimes \mathbb{Q}[t_w] : \)

\[
\tilde{\mathcal{S}}_u * \tilde{\mathcal{S}}_w = \sum_{\tau} \Lambda_{uw\tau}(t) \tilde{\mathcal{S}}_{w\tau}.
\]

Lemma 3.14 For each \( u \in S_n \) let us define an operator \( \nabla_u : Q^*_t(\text{Fl}_n) \to Q^*_t(\text{Fl}_n) \) by the following rule \( \nabla_u \tilde{\mathcal{S}}_w(x) = \frac{\partial}{\partial t} \tilde{\mathcal{S}}_u^t(x). \) Then

\[
\nabla_u \tilde{\mathcal{S}}_w = \tilde{\mathcal{S}}_u * \tilde{\mathcal{S}}_w.
\]

Proof. \( \nabla_u \tilde{\mathcal{S}}_w = \frac{\partial}{\partial t} \tilde{\mathcal{S}}_u^t = \sum_{\tau} \Lambda_{uw\tau}(t) \tilde{\mathcal{S}}_{w\tau} = \tilde{\mathcal{S}}_u * \tilde{\mathcal{S}}_w. \)

4 Lax pair

In this Section we construct the Lax pair related to yet another deformation \( X^t_w \) of the quantum Schubert polynomials.

Define a new scalar product

\[
\langle f, g \rangle_t = \langle fg K(x, t) \rangle_{\tilde{\mathcal{I}}_n}.
\]

Let \( X^t_w(x) \) be Gram–Schmidt’s orthogonalization of the lexicographically ordered monomials \( x^I, I \subset \delta_n, \) with respect to the pairing \( \langle , \rangle_t, \) then

Lemma 4.1

\[
\frac{\partial}{\partial t_w} \langle f, g \rangle_t = \langle \frac{\partial}{\partial t_w} f, g \rangle_t + \langle f, \frac{\partial}{\partial t_w} g \rangle_t + \langle \tilde{\mathcal{S}}_w f, g \rangle_t.
\]
Let us define

\[ \tilde{s}_w X^t_v = \sum_u \varphi_{uw}^v X^t_u \]  
\[ \frac{\partial}{\partial t_w} X^t_v = \sum_u \psi_{uw}^v X^t_u, \]  
(4.1) (4.2)

and let us introduce the following matrices

- \( L_w = ((L_w)_{uv}) \), where \((L_w)_{uv} = \varphi_{uw}^v(t)\),
- \( M_w = ((M_w)_{uv}) \), where \((M_w)_{uv} = \psi_{uw}^v(t)\).

Then we can rewrite (4.1) and (4.2) as follows

\[ \tilde{s}_w \cdot X = L_w \cdot X, \]
\[ \frac{\partial}{\partial t_w} \cdot X = M_w \cdot X, \]

where \( X = (X^t_w, w \in W)^t \) is a vector of length \( n! \).

**Lemma 4.2** Let \( u, v \in S_n \), then

\[ \frac{\partial}{\partial t_u} L_w = [M_u, L_w] = M_u L_w - L_w M_u. \]

**Proof.** Let us compute the following expression \( \frac{\partial}{\partial t_u} \tilde{s}_w \cdot X^t_v \) in two ways, using the fact that operators \( \frac{\partial}{\partial t_u} \) and \( \tilde{s}_w \) are commute.

Let us define \( \mathcal{F}_u(t) := \langle X^t_u, X^t_u \rangle \neq 0 \).

**Lemma 4.3**

\[ \varphi_{uw}^v \cdot \mathcal{F}_v = \varphi_{uv}^w \cdot \mathcal{F}_u. \]

In other words, if we define \( \bar{g} = \text{diag}(\mathcal{F}_v, v \in W) \) then matrix \( \bar{L}_w := L_w \bar{g} \) is symmetric and the matrix \( L_{\bar{g}} \) is a symmetrizable.

Let us define \( M_w = M_{\bar{g}} \bar{g} \), then

**Lemma 4.4**

\[ \bar{M}_w + \bar{M}'_w + \bar{L}_w = \frac{\partial}{\partial t_w} \bar{g}. \]

**Proof.** Let us consider \( \frac{\partial}{\partial t_w} \langle X^t_u, X^t_u \rangle_t = \frac{\partial}{\partial t_w} \mathcal{F}_u \cdot \delta_{uv} \), and apply Lemma 4.3.
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