Interior-point methods for second-order cone optimization based on the generalized trigonometric barrier function

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Abstract. In this article, primal-dual interior-point methods (IPMs) for second-order cone optimization (SOCO) based on the generalized trigonometric barrier function are studied. Furthermore, we derive the iteration bounds of large- and small-update IPMs for SOCO.

1. Introduction
Consider the SOCO problem

\[(P) \text{ minimize } \{c^T x : Ax = b, x \in K\},\]

and its dual problem

\[(D) \text{ maximize } \{b^T y: A^T y + s = c, s \in K\}.\]

Here \(K = K^1 \times \cdots \times K^K \subseteq \mathbb{R}^n\), with \(K^j = A^{N_j} := \{(x_1, \ldots, x_N) \in \mathbb{R}^n : x_1^2 + \cdots + x_N^2, x_i \geq 0\}\), and \(n = n_1 + \cdots + n_N\). Let \(x = (x^1; \cdots; x^N)\), \(s = (s^1; \cdots; s^N)\), \(c = (c^1; \cdots; c^N)\) and \(A = (A^1, \cdots, A^K)\) with \(x^j, s^j \in K^j\), \(c^j \in \mathbb{R}^{n_j}\), \(A^j \in \mathbb{R}^{m \times n_j}\), and \(b \in \mathbb{R}^m\). Without loss of generality, we assume that rank \((A) = m\).

Recently, kernel function based IPMs for linear optimization (LO) [1-3], \(P(\kappa)\) -linear complementarity problems (LCPs) [4-7], SOCO [8-11] and semidefinite optimization (SDO) [12,13] are proposed. Particularly, the so-called kernel function with the trigonometric barrier term was first introduced in [1]. Furthermore, they present the primal-dual IPMs for LO and derived the iteration bounds for large-update IPMs.

In this paper, we develop the primal-dual IPMs for SOCO based on the following parametric kernel function

\[\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{p\pi}(\tan^p(h(t)) - 1), p \geq 2,\]

where

\[h(t) = \frac{\pi}{2t + 2}.\]
This parametric kernel function was first considered for LO in [2], was later extended to $P_\ell(\kappa)$-LCPs in [4], SDO in [8] and convex quadratic SDO in [12].

2. Second-order cones

Let $x \circ s := \left( x^{(1)} \circ s^{(1)}, \ldots, x^{(N)} \circ s^{(N)} \right)^T, x, s \in K$. The multiplication and quadratic representation operators are given by $L(x) := \text{diag}(L(x^{(1)}), \ldots, L(x^{(N)}))$ and $P(x) := \text{diag}(P(x^{(1)}), \ldots, P(x^{(N)}))$, where

$$L(x^{(j)}) := \begin{pmatrix} x_1^{(j)} & \left( x_2^{(j)} \right)^T \\ x_2^{(j)} & x_1^{(j)} E_{n-1} \end{pmatrix},$$

and

$$P(x^{(j)}) := \begin{bmatrix} \| x^{(j)} \|^2 & 2x_1^{(j)} (x_2^{(j)})^T \\ 2x_1^{(j)} x_2^{(j)} & \det(x^{(j)}) E_{n-1} + 2x_1^{(j)} (x_2^{(j)})^T \end{bmatrix}.$$ 

The spectral decomposition of $x^{(j)} \in K^{(j)}$ is given by

$$x^{(j)} = \lambda_{\max}(x^{(j)}) z_{1}^{(j)} + \lambda_{\min}(x^{(j)}) z_{2}^{(j)}$$

where $z_{1}^{(j)} := \frac{1}{2} \left( 1; \frac{x_2^{(j)}}{\| x_2^{(j)} \|} \right)$ and $z_{2}^{(j)} := \frac{1}{2} \left( 1; -\frac{x_2^{(j)}}{\| x_2^{(j)} \|} \right)$. Let $\| x_2^{(j)} \| = 0$ if $x_2^{(j)} = 0$.

The trace and the determinant of $x \in K$ are given by

$$\text{tr}(x) = 2 \sum_{j=1}^{N} x_1^{(j)},$$

and

$$\det(x) = \prod_{j=1}^{N} (\| x^{(j)} \|^2 - \| x_2^{(j)} \|^2).$$

Then

$$\| x \| = \sqrt{\sum_{j=1}^{N} (\lambda_{\max}(x^{(j)})^2 + \lambda_{\min}(x^{(j)})^2)}.$$ 

It follows that $\text{tr}(e) = 2N$ and $| \lambda_{\max}(x) | \leq \| x \|$, $| \lambda_{\min}(x) | \leq \| x \|$, where

$$\lambda_{\max}(x) = \max\{ \lambda_{\max}(x^{(j)}): j \in N \}, \quad \lambda_{\min}(x) = \min\{ \lambda_{\min}(x^{(j)}): j \in N \}.$$ 

The vector-valued function is defined by

$$\psi(v) = (\psi^{(1)}(v); \cdots; \psi^{(N)}(v)),$$

where

$$\psi(x^{(j)}) := \psi^\prime(\lambda_{\max}(x^{(j)}) z_{1}^{(j)} + \psi^\prime(\lambda_{\min}(x^{(j)}) z_{2}^{(j)}), j = 1, \ldots, N.$$ 

Then we define the barrier function as follows

$$\Psi(x) := \sum_{j=1}^{N} (\psi^\prime(\lambda_{\max}(x^{(j)})) + \psi^\prime(\lambda_{\min}(x^{(j)})))$$

according to the kernel function $\psi (t)$ given by (1). One can conclude that

$$\Psi(x) = 0 \iff \psi(x) = 0 \iff \psi'(x) = 0 \iff x = e.$$ 

3. Kernel function and the corresponding barrier function

The exponential convexity plays an important role in the analysis of primal-dual IPMs based on the so-called kernel functions [1-3], see Lemma 3.2 in [2].

Lemma 3.1 Let $t > 0$. Then

$$\Psi(x) = 0 \iff \psi(x) = 0 \iff \psi'(x) = 0 \iff x = e.$$
From Lemma 3.1, we can obtain the following important result, see Lemma 3.2 in [9].

**Theorem 3.2** Let \( x, s \) and \( v \in \text{int} K \). If \( \det v \circ v = \det(x) \det(s) \) and \( \text{tr}(v \circ v) = \text{tr}(x \circ s) \), we have

\[
\Psi(v) \leq \frac{\Psi(x) + \Psi(s)}{2}.
\]

Lemma 3.3 in [2] yields an upper bound of the kernel function \( \psi(t) \) given by (1).

**Lemma 3.3** Let \( t > 1 \). Then

\[
\psi(t) \leq \frac{p\pi + 8}{8} (t - 1)^2.
\]

Let

\[
\delta(v) := \frac{1}{\sqrt{2}} \| \psi'(v) \|.
\]

Then \( \delta(v) \geq 0 \), and \( \delta(v) = 0 \Leftrightarrow \Psi(v) = 0 \). The following theorem gives a lower bound on \( \delta(v) \) in terms of the barrier function, see Theorem 3.11 in [9].

**Theorem 3.4** Let \( \rho(s) : [0, \infty) \to [1, \infty] \) be the inverse function of \( \phi(t) \) for \( t \geq 1 \) and \( v \in \text{int} K \). Then

\[
\delta(v) \geq \frac{\psi'(\rho(\Psi(v)))}{2}.
\]

**Corollary 3.5** Let \( v \in \text{int} K \) and \( \Psi(v) \geq 1 \). Then

\[
\delta(v) \geq \frac{\sqrt{\Psi(v)}}{6}.
\]

Let \( \rho(s) : [0, \infty) \to (0,1] \) is the inverse function of \( -\psi'(t) / 2 \) for \( t \in (0,1] \). The following theorem gives an upper bound for the effect of a \( \mu \)-update on the barrier function.

**Theorem 3.6** Let \( v \in \text{int} K \) and \( \beta \geq 1 \). Then

\[
\Psi(\rho v) \leq 2N \psi(\beta \rho(\Psi(v)/2N)).
\]

**Corollary 3.7** Let \( 0 \leq \theta < 1 \) and \( v_\theta = \frac{v}{\sqrt{1 - \theta}} \). If \( \Psi(v) \leq \tau \), then

\[
\Psi(v_\theta) \leq 2N \psi(\rho(\tau/2N)/\sqrt{1 - \theta}).
\]

## 4. IPMs for SOCO

### 4.1. Central path

We assume that there exists \((x^0, y^0, s^0)\) such that \(Ax^0 = b, x^0 \in \text{int} K, A^Ty^0 + s^0 = c, s^0 \in \text{int} K\).

Under the above condition, the optimality condition for (P) and (D) is given by

\[
Ax = b, x \in K, A^Ty + s = c, s \in K, x \circ s = 0.
\]

Then we replace \( x \circ s = 0 \) by \( x \circ s = \mu e \) with \( \mu > 0 \). This yields

\[
Ax = b, x \in K, A^Ty + s = c, s \in K, x \circ s = \mu e.
\]

It is well-known that the parameterized system (4) has a unique solution \((x(\mu), y(\mu), s(\mu))\) for any \( \mu > 0 \). The set of \((x(\mu), y(\mu), s(\mu))\) gives a homotopy path, i.e., the central path. If \( \mu \to 0 \) then the limit of the central path exists and satisfies \( x \circ s = 0 \). This gives an optimal solution of SOCO.
4.2. NT search directions
Similar to the SDO case, the obtained Newton-type system from the system (4) doesn't always have a unique solution. The main reason is that $x$ and $s$ do not operator commute in general. This difficulty can be solved by applying a scaling scheme as described below, see Lemma 28 in [14].

**Lemma 4.1** Let $u \in \text{int} \ K$. Then $x \circ s = \mu e$ if and only if $P(u)x \circ P(u)^{-1}s = \mu e$.

From Lemma 4.1, we replace the $x \circ s = \mu e$ by $P(u)x \circ P(u)^{-1}s = \mu e$. Applying Newton's method again, we have

$$A\Delta x = 0, A^T \Delta y + \Delta s = 0, P(u)^{-1}(s) \circ P(u)\Delta x + P(u)(x) \circ P(u)^{-1}\Delta s = \mu e - P(u)(x) \circ P(u)^{-1}(s). \quad (5)$$

Now, we consider the Nesterov and Todd scaling scheme, see Lemma 3.2 in [15].

**Lemma 4.2** Let $x, s \in \text{int} K$. Then there exists a unique $w \in \text{int} K$ such that

$$x = P(w)s$$

where

$$w = P(x^{1/2})(P(x^{1/2})s)^{-1/2}. \quad (6)$$

Let $u = w^{-1/2}, v = P(w)^{1/2}x/\sqrt{\mu}, \quad \bar{A} = P(w)^{1/2}A P(w)^{1/2}x, \quad dx := P(w)^{-1/2}\Delta x / \sqrt{\mu}, \quad ds := P(w)^{-1/2}\Delta s / \sqrt{\mu}.$

Then

$$\bar{A}dx = 0, \bar{A}^T\Delta y + ds = 0, \quad dx + ds = v^{-1} - v.$$

We can conclude that the system (6) has a unique solution.

Now, by replacing $v^{-1} - v$ in the system (6) by $-\psi'(v)$, we get

$$\bar{A}dx = 0, \bar{A}^T\Delta y + ds = 0, \quad dx + ds = -\psi'(v). \quad (7)$$

The new search direction $(dx, ds)$ is obtained by solving the system (7). Then we have $(\Delta x, \Delta s)$. If $(x, s) \neq (x(\mu), s(\mu))$, then $(\Delta x, \Delta s)$ is nonzero. By choosing the step size $\alpha$, we have

$$x_+ := x + \alpha \Delta x, \quad y_+ := y + \alpha \Delta y, \quad s_+ := s + \alpha \Delta s.$$

Note that

$$x \circ s = \mu e \iff v = e \iff \psi'(v) = 0 \iff \psi(v) = 0 \iff \Psi(v) = 0. \quad (8)$$

This means that the barrier function can be considered as a measure for the distance between the given iterate $(x, y, s)$ and $(x(\mu), y(\mu), s(\mu))$.

4.3. Primal-dual IPMs
Primal-dual IPMs for SOCO is now described below.

**Algorithm 1**

**Step 0** Input a threshold parameter $0 < \tau < 1$, an accuracy parameter $\varepsilon > 0$, a fixed barrier update parameter $0 < \theta < 1$, a strictly feasible $(x^0, y^0, s^0)$ and $\mu^0 = 1$ such that $\Psi(x^0, s^0; \mu^0) < \tau$. Set $x := x^0; y := y^0; s := s^0; \mu := \mu^0$.

**Step 1** If $\Psi(v) \leq \varepsilon$, stop, $(x, y, s)$ is an optimal solution; otherwise, update $\mu := (1 - \theta)\mu$, go to Step 2.

**Step 2** If $\Psi(v) \leq \tau$, go back to **Step 1**; otherwise, go to **Step 3**.

**Step 3** Update $x := x + \alpha \Delta x; y := y + \alpha \Delta y; s := s + \alpha \Delta s$, go back to **Step 2**.

5. Analysis and complexity of the algorithms
After a feasible step, we have, by Lemma 3.2,

$$\Psi(v_+) \leq \frac{\Psi(v + \alpha dx) + \Psi(v + \alpha ds)}{2}.$$

Let
\[ f(\alpha) := \Psi(v_{+}) - \Psi(v). \]

Then
\[ f(\alpha) \leq f_{1}(\alpha) \text{ and } f(0) = f_{1}(0) = 0. \]

where
\[ f_{1}(\alpha) := \frac{1}{2}(\Psi(v + \alpha dx) + \Psi(v + \alpha dx) - \Psi(v)). \]

This means that \( f_{1}(\alpha) \) gives an upper bound for the decrease of the barrier function \( \Psi(v) \). It follows that
\[ f_{1}'(\alpha) = \frac{1}{2}(\text{tr}(\psi'(v + \alpha dx) \circ dx) + \text{tr}(\psi'(v + \alpha dx) \circ ds)), \]
and
\[ f_{1}''(\alpha) = \frac{1}{2} \frac{d^2}{d\alpha^2} \text{tr}(\psi(v + \alpha dx) + \psi(v + \alpha dx)) \leq \frac{1}{2}(\omega_1 ||dx||^2 + \omega_2 ||ds||^2), \]

where \( \omega_1 \) and \( \omega_2 \) depend on the choice of the kernel function. Furthermore, we get
\[ f_{1}'(0) = -2\delta(v)^2 < 0. \]

Let \( \delta := \delta(v) \). The following lemma yields an upper bound of \( f_{1}'(\alpha) \).

**Lemma 5.1** One has
\[ f_{1}'(\alpha) \leq 2\delta^2 \psi''(\lambda_{\min}(v) - 2\alpha \delta). \]

Similar to the LO analogue [2], we can conclude that
\[ \tilde{\alpha} \geq \tilde{\alpha} := 1/1320 p\delta^{(p+2)/(p+1)}. \]

**Lemma 5.2 (Lemma 3.7 in [3])** Let \( \alpha \leq \tilde{\alpha} \). Then
\[ f(\alpha) \leq -\alpha \delta^2. \]

It follows from Corollary 3.5, Lemma 5.2 and Lemma 5.1, we have the following result.

**Theorem 5.3** One has
\[ f(\tilde{\alpha}) \leq -\Psi(v) \frac{\frac{p}{2(p+1)}}{7920 p}. \]

Let \( \Psi_{0} \) be the value of \( \Psi(v) \) after the \( \mu \)-update and \( \Psi_{k}, k = 1, \cdots, K \) be the subsequent values in the same outer iteration, where \( K \) denotes the total number of inner iterations in the outer iteration. From Theorem 5.3, we have
\[ \Psi_{k+1} \leq \Psi_{k} - \beta(\Psi_{k})^{1/\gamma}, k = 0, \cdots, K-1, \]

where \( \beta = \frac{1}{7920 p} \), and \( \gamma = \frac{p+2}{2(p+1)}. \)

**Theorem 5.4** One has
\[ K \leq 7920 p\Psi_{0}^{\frac{p+2}{2(p+1)}}. \]

It follows from Theorem 5.4 that the upper bound for the total number of iterations is given by
\[ 7920 p\Psi_{0}^{\frac{p+2}{2(p+1)}} \log(2N / \epsilon)/ \theta. \]

Note that \( t = \rho(s) \leq 1 + \sqrt{2s} \). From Corollary 3.7, Lemma 5.1, and \( \rho(t) \leq (t^2 - 1)/2 \) when \( t \geq 1 \), we obtain
\[ \Psi_0 \leq \frac{\theta N + 2\sqrt{\tau N} + \tau}{1 - \theta}. \]  

(14)

Combining the results of (13) and (14), we have the following result.

**Theorem 5.5** Let \( \theta = \Theta(1) \) and \( \tau = O(N) \). Then the iteration bound of large-update IPMs for SOCO is given by

\[ O\left( pN^{(p+2)/(2(p+1))} \log(N/\varepsilon) \right). \]

**Corollary 5.6** Let \( p = O(\log N) \). Then the iteration bound reduces to

\[ O(\sqrt{N} \log N \log(N/\varepsilon)), \]

which matches the currently best known complexity bound for large-update IPMs for SOCO.

From Corollary 3.7, Lemma 3.3 and \( 1 - \sqrt{1 - \theta} \leq \theta \), we get

\[ \Psi_0 \leq \frac{p\pi + 8}{8(1 - \theta)} \left( \theta \sqrt{2N} + \sqrt{2\tau} \right)^2. \]

(15)

Combining the results of (13) and (15), we have the following lemma, which gives the complexity bound for small-update method.

**Theorem 5.7** Let \( \theta = \Theta(1/\sqrt{N}) \) and \( \tau = O(1) \). Then the iteration bound of small-update IPMs for SOCO is given by

\[ O\left( \sqrt{N} \log(N/\varepsilon) \right), \]

which matches the currently best known complexity bound for small-update methods.

### 6. Conclusions

In this paper, we have analyzed primal-dual IPMs for SOCO based on the trigonometric kernel function considered in [1] for LO. By means of Jordan algebra, we obtained the complexity bounds for large- and small-update IPMs for SOCO, which are the currently best known complexity bounds for the kernel function based IPMs.

The numerical experiment is an interesting work for comparing the behavior of the algorithm in this article with the existing methods. Furthermore, we can extend the obtained results to the symmetric optimization and the Cartesian \( P_1(\kappa) \)-LCPs over symmetric cones.

### References

[1] El Ghami M, Guennounb Z A, et al. 2012 *J. Comput. Appl. Math.* **236** 3613-3623

[2] Bouafia M, Benterki D and Yassine A 2016 *J. Optim. Theory Appl.* **170** 528-545

[3] Peyghami M R and Hafshejani S F 2014 *Numer. Algorithms* **67** 33-48

[4] El Ghami M 2017 *J. Pure Appl. Math.* **114** 797-818

[5] Li X, Zhang M W and Chen Y 2017 *J. Optim. Theory Appl.* **53** 487-506

[6] Cai X Z, Li L, et al. 2017 *Pac. J. Optim.* **13** 547-570

[7] Wang G Q and Zhu D T 2010 *Nonlinear Anal.* **73** 3705-3722

[8] El Ghami M and Wang G Q 2017 *Int. J. Appl. Math.* **30** 11-33

[9] Bai Y Q, Wang G Q and Roos C 2009, *Nonlinear Anal.* **70** 3584-3602

[10] Wang G Q and Bai Y Q 2009 *Appl. Math. Comput.* **215** 1047-1061

[11] Tang J Y, He G Q, Fang L 2012 *Pac. J. Optim.* **8** 321-346

[12] Xie P Y, Luo K Y and Hu L F 2018 *J. Phys. Conf. Series* **1053** (2018) 012031

[13] Hafshejani S F, Jahromi A F and Peyghami M R 2018 *Optim.* **67** 113-137

[14] Schmieta S H and Alizadeh F 2001 *Math. Oper. Res.* **26** 543-564

[15] Faybusovich L 1997 *J. Comput. Appl. Math.* **86** 149-175