THE LARGE DIFFUSION LIMIT FOR THE HEAT EQUATION IN THE EXTERIOR OF THE UNIT BALL WITH A DYNAMICAL BOUNDARY CONDITION

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Abstract. We study the heat equation in the exterior of the unit ball with a linear dynamical boundary condition. Our main aim is to find upper and lower bounds for the rate of convergence to solutions of the Laplace equation with the same dynamical boundary condition as the diffusion coefficient tends to infinity.

1. Introduction. We consider the problem

\[
\begin{align*}
\varepsilon \partial_t u_\varepsilon - \Delta u_\varepsilon &= 0, & x \in \Omega := \{x \in \mathbb{R}^N : |x| > 1\}, & t > 0, \\
\partial_t u_\varepsilon + \partial_\nu u_\varepsilon &= 0, & x \in \partial \Omega, & t > 0, \\
u_\varepsilon(x,0) &= \varphi(x), & x \in \Omega, \\
u_\varepsilon(x,0) &= \varphi_b(x), & x \in \partial \Omega,
\end{align*}
\]

(1)

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where $N \geq 3$, $\Delta$ is the $N$-dimensional Laplacian (in $x$), $\nu$ is the exterior normal vector to $\partial \Omega$, $\partial_t := \partial/\partial t$, $\partial_\nu := \partial/\partial \nu$, and $(\varphi, \varphi_b)$ is a pair of measurable functions in $\Omega$ and $\partial \Omega$, respectively. Our aim is to study the convergence as $\varepsilon \to 0$ of the solution $u_\varepsilon$ to the solution $u$ of the problem

$$
\begin{aligned}
\Delta u &= 0, & x \in \Omega, \ t > 0, \\
\partial_t u + \partial_\nu u &= 0, & x \in \partial \Omega, \ t > 0, \\
u u(x,0) &= \varphi_b(x), & x \in \partial \Omega.
\end{aligned}
$$

(2)

For bounded domains this convergence was established in [5] and for the half-space $\Omega = \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times \mathbb{R}_+$, $N \geq 2$, in [2]. More recently, the following four theorems on the rate of this convergence have been proven in [3].

**Theorem 1.1.** Let $\Omega = \mathbb{R}^N_+$, $N \geq 2$. Let $\varphi \in L^\infty(\Omega)$, $\varphi_b \in L^\infty(\partial \Omega)$, $\mathcal{K} \subset \overline{\Omega}$ compact and $0 < \tau_1 < \tau_2 < \infty$. Then there exists $C > 0$ such that

$$
\sup_{\tau_1 < t < \tau_2} \|u_\varepsilon(t) - u(t)\|_{L^\infty(\mathcal{K})} \leq C\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, 1).
$$

**Theorem 1.2.** Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$. Let $\varphi \in L^\infty(\Omega)$ be radially symmetric such that $\sup_{\rho \geq 1} |\rho \varphi(\rho)| < \infty$. Assume further that $\varphi_b$ is constant, $\varepsilon_0 \in (0, \pi^{-1/2})$ and $0 < \tau_1 < \tau_2 < \infty$. Then there exists $C > 0$ such that

$$
\sup_{\tau_1 < t < \tau_2} \|u_\varepsilon(t) - u(t)\|_{L^\infty(\Omega)} \leq C\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, \varepsilon_0).
$$

The upper bounds from Theorems 1.1 and 1.2 are sharp.

**Theorem 1.3.** Let $\Omega = \mathbb{R}^N_+$, $N \geq 2$ and $\varphi_b \equiv 0$. Then there exist $\varphi \in L^\infty(\Omega)$ and a compact set $\mathcal{K} \subset \mathbb{R}^N_+ \times (0, \infty)$ such that

$$
u u_\varepsilon(x, t) - u(x, t) = u_\varepsilon(x, t) \geq c\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, \varepsilon_0), \quad (x, t) \in \mathcal{K},
$$

for some $\varepsilon_0 > 0$ and $c > 0$.

**Theorem 1.4.** Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$ and $\varphi_b \equiv 0$. Then there exist a radially symmetric $\varphi \in L^\infty(\Omega)$ satisfying $\sup_{\rho \geq 1} |\rho \varphi(\rho)| < \infty$ and a compact set $\mathcal{K} \subset (\mathbb{R}^3 \setminus B_1(0)) \times (0, \infty)$ such that

$$
u u_\varepsilon(x, t) - u(x, t) = u_\varepsilon(x, t) \geq c\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, \varepsilon_0), \quad (x, t) \in \mathcal{K},
$$

for some $\varepsilon_0 > 0$ and $c > 0$.

We see that for the half-space the rate does not depend on the dimension, and we obtain the same rate $\varepsilon^{1/2}$ also for the exterior of a ball in $\mathbb{R}^3$, which is a very different domain. The main motivation of this paper is the natural question whether or not other rates may occur. We show that for $\mathbb{R}^N \setminus \overline{B_1(0)}$ the rate depends on $N$.

Before we formulate our main results, we introduce some notation. Let $\Gamma_D = \Gamma_D(x, y, t)$ be the Dirichlet heat kernel on $\Omega$. Define

$$
[S_1(t)\phi](x) := \int_{\Omega} \Gamma_D(x, y, t)\phi(y) \, dy, \quad x \in \overline{\Omega}, \ t > 0,
$$

for any measurable function $\phi$ in $\Omega$. Let $P = P(x, y)$ be the Poisson kernel on $B = B(0,1) := \{x \in \mathbb{R}^N : |x| < 1\}$, that is

$$
P(x, y) := c_N \frac{1 - |x|^2}{|x - y|^N}, \quad x \in \overline{B}, \ y \in \partial B \setminus \{x\},
$$

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where \( c_N \) is a constant to be chosen such that \( \| P(x, \cdot) \|_{L^1(\partial B)} = 1 \) for \( x \in B \) (see (2.28) in [6]). Then \( P = P(x, y) \) satisfies as a function of \( x \)

\[-\Delta_x P = 0 \quad \text{in} \quad B, \quad P(x, y) = \delta_y \quad \text{on} \quad \partial B, \quad \text{(3)}\]

where \( \delta_y \) is the Dirac measure on \( \partial B = \partial \Omega \) at \( y \). We denote by \( K = K(x, y) \) the Kelvin transform of \( P \) as a function of \( x \) with respect to \( B \), that is

\[ K(x, y) := [x]^{- (N - 2)} P \left( \frac{x}{|x|^2}, y \right), \quad x \in \Omega, \quad y \in \partial \Omega \setminus \{ x \}. \quad \text{(4)} \]

Then it follows from (3) and (4) that

\[ K = K(x, y) \]

is a constant to be chosen such that

\[ ||K||_{L^1(\partial \Omega)} = 1 \]

\( \| P(x, \cdot) \|_{L^1(\partial B)} = 1 \) for \( x \in B \) (see (2.28) in [6]). Then

\[ K(x, y) := [x]^{- (N - 2)} P \left( \frac{x}{|x|^2}, y \right), \quad x \in \Omega, \quad y \in \partial \Omega \setminus \{ x \}. \quad \text{(4)} \]

Set

\[ K(x, y, t) := K(e^t x, y), \quad x \in \Omega, \quad y \in \partial \Omega, \quad t \geq 0, \quad e^t x \neq y. \quad \text{(5)} \]

Then it follows from (3) and (4) that \( K = K(x, y, t) \) as a function of \( x \) and \( t \) satisfies

\[ \left\{ \begin{array}{l} -\Delta_x K = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\partial_t K + \partial_x K = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
K(\cdot, y, 0) = \delta_y \quad \text{on} \quad \partial \Omega. 
\end{array} \right. \]

For any nonnegative measurable function \( \psi \) on \( \partial \Omega \) and \( t > 0 \), we define

\[ [S_2(t)\psi](x) := \int_{\partial \Omega} K(x, y, t) \psi(y) \, d\sigma_y \equiv \int_{\partial \Omega} K(e^t x, y) \psi(y) \, d\sigma_y, \quad x \in \Omega. \quad \text{(6)} \]

We formulate the definition of a solution of (1) by the use of the two integral kernels \( \Gamma_D \) and \( \mathcal{K} \). For simplicity, let \( \varphi_b = \varphi_b(x) \) and \( g = g(x, t) \) be continuous functions in \( \Omega \) and \( \partial \Omega \times (0, \infty) \), respectively. Then the function

\[ w(x, t) := [S_2(t)\varphi_b](x) + \int_0^t [S_2(t - s)g(s)](x) \, ds \quad \text{(7)} \]

can be defined for \( x \in \Omega \) and \( t > 0 \), and it is a classical solution of the Cauchy problem for the Laplace equation with a nonhomogeneous dynamical boundary condition

\[ \left\{ \begin{array}{l} -\Delta w = 0, \quad x \in \Omega, \quad t > 0, \\
\partial_t w + \partial_x w = g, \quad x \in \partial \Omega, \quad t > 0, \\
w(x, 0) = \varphi_b(x), \quad x \in \partial \Omega. 
\end{array} \right. \quad \text{(8)} \]

It follows from (6) and (7) that

\[ \partial_t w(x, t) = \int_{\partial \Omega} \partial_t K(x, y, t) \varphi_b(y) \, d\sigma_y + \int_{\partial \Omega} K(x, y) g(y, t) \, d\sigma_y + \int_0^t \int_{\partial \Omega} \partial_t K(x, y, t - s) g(y, s) \, d\sigma_y \, ds, \quad x \in \Omega, \quad t \in (0, T). \quad \text{(9)} \]

Set

\[ \Phi(x) := \varphi(x) - [S_2(0)\varphi_b](x), \quad x \in \Omega. \quad \text{(10)} \]

Then the function

\[ v_\varepsilon(x, t) := [S_1(\varepsilon^{-1}t)\Phi](x) - \int_0^t [S_1(\varepsilon^{-1}(t - s))\partial_t w(s)](x) \, ds, \quad x \in \Omega, \quad t \geq 0, \]

satisfies

\[ \left\{ \begin{array}{l} \varepsilon \partial_t v_\varepsilon - \varepsilon \partial_x w = 0, \quad x \in \Omega, \quad t > 0, \\
v_\varepsilon = 0, \quad x \in \partial \Omega, \quad t > 0, \\
v_\varepsilon(x, 0) = \Phi(x), \quad x \in \Omega. 
\end{array} \right. \quad \text{(11)} \]
If \( g_\varepsilon(x,t) := -\partial_\nu v_\varepsilon(x,t) \) for \( x \in \partial \Omega, t > 0 \), and \( w_\varepsilon \) is defined as in (7) with \( g_\varepsilon \) instead of \( g \), then it follows from (8), (9) and (11) that

\[
\begin{aligned}
\varepsilon \partial_\nu v_\varepsilon &= \Delta v_\varepsilon - \varepsilon F_1[\varphi_\varepsilon] + \varepsilon F_2[v_\varepsilon], & x \in \Omega, \ t > 0, \\
\Delta w_\varepsilon &= 0, & x \in \Omega, \ t > 0, \\
v_\varepsilon &= 0, \quad \partial_\nu w_\varepsilon + \partial_\nu v_\varepsilon = -\partial_\nu v_\varepsilon, & x \in \partial \Omega, \ t > 0, \\
v_\varepsilon(x, 0) &= \Phi(x), & x \in \Omega, \\
w_\varepsilon(x, 0) &= \varphi_b(x), & x \in \partial \Omega,
\end{aligned}
\]

where

\[
F_1[\varphi_\varepsilon](x, t) := \int_{\partial \Omega} \partial_\nu K(x, y, t) \varphi_\varepsilon(y) \, d\sigma_y, \\
F_2[v_\varepsilon](x, t) := \int_{\partial \Omega} K(x, y) \partial_\nu v(y, t) \, d\sigma_y + \int_0^t \int_{\partial \Omega} \partial_\nu K(x, y, t-s) \partial_\nu v(y, s) \, d\sigma_y \, ds.
\]

Furthermore, the function \( u_\varepsilon := v_\varepsilon + w_\varepsilon \) is a classical solution of (1). Motivated by this observation, we formulate the definition of a solution of (1) via problem (12).

**Definition 1.5.** Let \( \varphi \) and \( \varphi_b \) be measurable functions in \( \Omega \) and \( \partial \Omega \), respectively. Let \( 0 < T \leq \infty \) and

\[
v_\varepsilon, \ w_\varepsilon \in C(\bar{\Omega} \times (0, T)), \quad \nabla v_\varepsilon \in C(\bar{\Omega} \times (0, T)).
\]

We call \((v_\varepsilon, w_\varepsilon)\) a solution of (12) in \( \Omega \times (0, T) \) if \( v_\varepsilon \) and \( w_\varepsilon \) satisfy

\[
\begin{aligned}
v_\varepsilon(x, t) &= [S_1(\varepsilon^{-1} t)\Phi](x) - \int_0^t [S_1(\varepsilon^{-1} (t-s))F_1[\varphi_\varepsilon](s)](x) \, ds \\
&\quad + \int_0^t [S_1(\varepsilon^{-1} (t-s))F_2[v_\varepsilon](s)](x) \, ds, \\
w_\varepsilon(x, t) &= [S_2(t)\varphi_b](x) - \int_0^t [S_2(t-s)\partial_\nu v_\varepsilon(s)](x) \, ds,
\end{aligned}
\]

for \( x \in \bar{\Omega} \) and \( t \in (0, T) \). Furthermore, for the solution \((v_\varepsilon, w_\varepsilon)\) of (12) in \( \Omega \times (0, T) \), we call \( u_\varepsilon := v_\varepsilon + w_\varepsilon \) a solution of (1) in \( \Omega \times (0, T) \). In the case when \( T = \infty \), we call \((v_\varepsilon, w_\varepsilon)\) a global-in-time solution of (12) and \( u_\varepsilon \) a global-in-time solution of (1).

We are ready to state the main results of this paper. For \( 1 \leq r \leq \infty \) and \( \theta \in (0, 1) \), we write \( \| \cdot \|_{L^r} := \| \cdot \|_{L^r(\partial \Omega)}, \| \cdot \|_{L^r} := \| \cdot \|_{L^r(\Omega)} \) and \( \| \cdot \|_{C^{1,\theta}} := \| \cdot \|_{C^{1,\theta}(\partial \Omega)} \) for simplicity. The latter norm is defined by

\[
\| \psi \|_{C^{1,\theta}} := \max_{k \in (0,1)} \sup_{x \in \partial \Omega} \left| \nabla^k \psi(x) \right| + \sup_{x,y \in \partial \Omega, x \neq y} \frac{\left| \nabla \psi(x) - \nabla \psi(y) \right|}{|x - y|^{\theta}}.
\]

**Theorem 1.6.** Let \( N \geq 3, \varphi \in L^\infty(\Omega) \) and \( \varphi_b \in C^{1,\theta}(\partial \Omega) \) with \( \theta \in (0, 1) \). Assume

\[
M := \sup_{x \in \Omega} |x|^{-2} |\varphi(x)| < \infty.
\]

Then for every \( \varepsilon \in (0, 1) \) the problem (12) possesses a unique global-in-time solution \((v_\varepsilon, w_\varepsilon)\) with

\[
\sup_{0 < t < T} \left[ \|v_\varepsilon(t)\|_{L^\infty} + t^\frac{1}{2} \|\nabla v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty} \right] < \infty \quad \text{for any } T > 0.
\]

These solutions have the following properties:
(i) For any $T > 0$ there exists $C_T > 0$ such that for every $\varepsilon \in (0, 1)$ and every $\varphi, \varphi_b$ as above
\[ \sup_{0 < t < T} \left[ \|v_\varepsilon(t)\|_{L^\infty} + (\varepsilon^{-1}t)^{\frac{1}{2}}\|\nabla v_\varepsilon(t)\|_{L^\infty} + \|w_\varepsilon(t)\|_{L^\infty} \right] \leq C_T(\|\varphi_b\|_{C^{1,\alpha}} + M). \] (17)

Furthermore,
\[ \nabla^j v_\varepsilon \in C^\infty(\Omega \times I) \cap BC(\overline{\Omega} \times I), \quad \partial_\tau^j \nabla^j w_\varepsilon \in C^\infty(\Omega \times I) \cap BC(\overline{\Omega} \times I) \]
for any bounded interval $I \subset (0, \infty)$ and $0 \leq \ell + j \leq 1$.

(ii) Let $T > 0$, $\tau \in (0, T)$ and $\alpha = 1$ for $N = 3$, $1 < \alpha < 2$ for $N \geq 4$.

Then there exists $C > 0$ such that for every $\varepsilon \in (0, 1)$
\[ \sup_{\tau < t < T} \|v_\varepsilon(t)\|_{L^\infty} \leq C\varepsilon^{\frac{2}{\alpha}}, \] (19)
\[ \sup_{0 < t < T} \|w_\varepsilon(t) - S_2(t)\varphi_b\|_{L^\infty} \leq C\varepsilon^{\frac{2}{\alpha}}. \] (20)

The reason why it is natural to assume (15) is explained in [3, Section 7]. As a corollary of Theorem 1.6, we see that the solution $u_\varepsilon = v_\varepsilon + w_\varepsilon$ of (1) converges to the solution $S_2(t)\varphi_b$ of (2).

**Corollary 1.** Assume the same conditions as in Theorem 1.6. Let $\alpha$ be as in (18) and $(v_\varepsilon, w_\varepsilon)$ the solution given in Theorem 1.6. Then $u_\varepsilon = v_\varepsilon + w_\varepsilon$ is a classical global-in-time solution of (1). Furthermore, for any $T > 0$ and $\tau \in (0, T)$ there exists $C > 0$ such that
\[ \sup_{\tau < t < T} \|u_\varepsilon(t) - S_2(t)\varphi_b\|_{L^\infty} \leq C\varepsilon^{\frac{2}{\alpha}} \]
for every $\varepsilon \in (0, 1)$.

This means that for $N \geq 4$ the convergence is faster than for $N = 3$. Moreover, we obtain an estimate from below. In the case of $N = 3$, the rates in Corollary 1 and in the following Theorem 1.7 coincide, and if $N = 4$ they can become arbitrarily close.

**Theorem 1.7.** Let $N \geq 3$. There exists a nonnegative function $\varphi \in L^\infty(\Omega)$ with (15) such that the following holds: Let $\mathcal{R}$ be a compact set in $\Omega \times (0, \infty)$. Then there exists $C > 0$ such that for any $\varepsilon \in (0, 1)$ the corresponding solution $u_\varepsilon$ of (1) with $\varphi_b \equiv 0$ satisfies
\[ u_\varepsilon(x, t) - [S_2(t)\varphi_b](x) = u_\varepsilon(x, t) \geq C\varepsilon^{\frac{N}{2} - 1}, \quad (x, t) \in \mathcal{R}. \] (21)

The rest of this paper is organized as follows. In Section 2 we recall some properties of the Dirichlet heat kernel $\Gamma_D$ and the kernel $K$. Furthermore, we prepare some useful lemmata. In Section 3, modifying the argument as in [2], we give a proof of Theorem 1.6. In Section 4 we prove Theorem 1.7.

2. **Preliminaries.** In this section we recall some properties of the Dirichlet heat kernel $\Gamma_D$ on the exterior $\Omega$ of the ball $B(0, 1)$ and obtain some estimates of integral operators $S_1(t)$, $S_2(t)$ and $F$.

We first recall some properties of the semigroup $S_1(t)$. By [9, Theorem 16.3] (see also [7, 8]) we find $C > 0$ such that
\[ |\nabla_2 \Gamma_D(x, y, t)| \leq Ct^{-\frac{N}{2}}h(t)\exp\left(-C\frac{|x - y|^2}{t}\right), \quad x, y \in \Omega, \quad t > 0, \]
where
\[ h(t) := \max\{1, t^{-1/2}\} \tag{22} \]
and \( j \in \{0, 1\} \). Then we have:

\((G_1)\) There exists \( c_1 = c_1(N) \) such that
\[
\|\nabla^j S_1(t, \phi)\|_{L^p} \leq c_1 t^{-\frac{j}{2}(\frac{2}{p} - \frac{1}{2})} h(t)^j \|\phi\|_{L^p}, \quad t > 0,
\]
for \( \phi \in L^{p,\infty}(\Omega) \), \( 1 \leq p \leq q \leq \infty \) and \( j \in \{0, 1\} \);

\((G_2)\) Let \( 0 \leq \gamma < N \). Assume that a measurable function \( f \) in \( \Omega \) satisfies
\[
|f(x)| \leq |x|^{-\gamma}
\]
for almost all \( x \in \Omega \). Then there exists \( c_2 = c_2(N, \gamma) > 0 \) such that
\[
\|\nabla^j S_1(t, f)\|_\infty \leq c_2 (1 + t)^{-\frac{j}{2}} h(t)^j, \quad t > 0,
\]
where \( j \in \{0, 1\} \) (see e.g. [4]);

\((G_3)\) Let \( \phi \in L^q(\Omega) \) with \( 1 \leq q \leq \infty \) and \( \tau > 0 \). Then \( S_1(t)\phi \) is bounded and smooth in \( \overline{\Omega} \times (\tau, \infty) \).

Next we recall some properties of the kernel \( K \) and \( S_2(t, \psi) \). By [1, Lemmata 2.1 and 2.2] we have the following two lemmata.

**Lemma 2.1.** Let \( N \geq 3 \) and \( K \) be as in (5). Then
\[
\int_{\partial\Omega} K(x, y, t) \, d\sigma_y = \int_{\partial\Omega} K(e^t x, y) \, d\sigma_y = (e^t |x|)^{-(N-2)} \tag{23}
\]
for \( (x, t) \in \overline{\Omega} \times [0, \infty) \) with \( e^t x \in \Omega \).

**Lemma 2.2.** Let \( N \geq 3 \) and \( \psi \) be a measurable function on \( \partial\Omega \) such that \( \psi \in L^\infty(\partial\Omega) \). Then
\[
S_2(t, \psi) = C^\infty(\overline{\Omega} \times (0, \infty)) \cap C^\infty(\Omega \times [0, \infty)), \tag{24}
\]
\[
-\Delta_x S_2(t, \psi) = 0 \quad \text{in} \quad \Omega \quad \text{for any} \quad t \geq 0,
\]
\[
S_2(t)[S_2(s)\psi]^b = S_2(t + s)\psi \quad \text{for} \quad s > 0 \quad \text{and} \quad t \geq 0,
\]
\[
||S_2(t)\psi||_\infty \leq e^{-(N-2)t} |x|^{-(N-2)} ||\psi||_\infty \quad \text{in} \quad \overline{\Omega} \times [0, \infty). \tag{25}
\]
Here \([S_2(t)\psi]^b\) is the restriction of \( S_2(t, \psi) \) to \( \partial\Omega \). Furthermore, for any \( \theta \in (0, 1) \), there exists \( c_3 > 0 \) such that for every \( \psi \in C^{1,\theta}(\partial\Omega) \)
\[
||S_2(t)\psi||_{C^{1,\theta}(\overline{\Omega})} \leq c_3 ||\psi||_{C^{1,\theta}}, \quad t \geq 0. \tag{26}
\]

**Proof.** Since the statement of [1, Lemma 2.2] only covers the case of nonnegative \( \psi \), we include a proof of (25). The remaining parts of the proof of [1, Lemma 2.2] did not rely on \( \psi \geq 0 \) (other than via (25)).

For \( x \in \overline{\Omega}, \ t \geq 0, \ y \in \partial\Omega \setminus \{x\} \), \( K(e^t x, y) \) is nonnegative, so that the combination of (6) with (23) proves
\[
||S_2(t)\psi||_\infty \leq \int_{\partial\Omega} K(e^t x, y) |\psi(y)| \, d\sigma_y 
\]
\[
\leq \int_{\partial\Omega} K(e^t x, y) ||\psi||_\infty \, d\sigma_y = e^{-(N-2)t} |x|^{-(N-2)} ||\psi||_\infty
\]
for \( (x, t) \in \overline{\Omega} \times [0, \infty) \) and thus (25).

Then we have:
Lemma 2.3. Let $N \geq 3$ and $\psi \in C^{1,\theta}(\partial \Omega)$ with $\theta \in (0,1)$. Let $F_1[\psi]$ be as in (13). Then

$$|F_1[\psi](x, t)| \leq c_3|\psi||\psi|_{C^{1,\theta}}, \quad x \in \overline{\Omega}, \quad t > 0,$$

and

$$|F_1[\psi](x, t)| \leq N(e^t|x|)^{(N-2)} \frac{e^t|x|}{e^t|x| - 1} |\psi|_{L^\infty}, \quad x \in \overline{\Omega}, \quad t > 0,$$

where $c_3$ is the constant given in (26).

Proof. Let $x \in \overline{\Omega}$ and $t > 0$. We prove (27). It follows from (5) that

$$\partial_t K(x, y, t) = e^t \cdot (\nabla_x K)(e^t x, y) = x \cdot \nabla_x K(x, y, t) \quad \text{for} \quad y \in \partial \Omega.$$  

Then, by (13), (29) and (6) we have

$$F_1[\psi](x, t) = \int_{\partial \Omega} \partial_t K(x, y, t) \psi(y) \, d\sigma_y$$

$$= \int_{\partial \Omega} x \cdot \nabla_x K(x, y, t) \psi(y) \, d\sigma_y = x \cdot \nabla_x \left[ S_2(t) \psi(x) \right].$$

This together with (26) implies (27).

We prove (28). Let $i, j \in \{1, \ldots, N\}$ and $y \in \partial \Omega$. By (4) we have

$$\partial_x_i K(x, y) = (2 - N)|x|^{-N} x_i P \left( \frac{x}{|x|^2}, y \right) + |x|^{-(N-2)} \partial_x_i P \left( \frac{x}{|x|^2}, y \right)$$

$$= (2 - N) \frac{x_i}{|x|^2} K(x, y) + |x|^{-(N-2)} \sum_{j=1}^{N} \partial_{x_j} \partial_{x_i} P(z, y),$$

where $z = x/|x|^2$. Since

$$\frac{\partial_{x_i} z_j}{\partial x_i} = \delta_{ij} |z|^2 - 2z_i z_j,$$

$$\partial_{x_i} P(z, y) = -c_N |z - y|^{-N-2} \left( 2z_j |z - y|^2 + N(1 - |z|^2)(z_j - y_j) \right),$$

we obtain

$$\sum_{j=1}^{N} \frac{\partial_{x_j} z_j}{\partial x_i} \partial_{x_i} P(z, y)$$

$$= -c_N |z - y|^{-N-2} \sum_{j=1}^{N} (\delta_{ij} |z|^2 - 2z_i z_j) \left( 2z_j |z - y|^2 + N(1 - |z|^2)(z_j - y_j) \right)$$

$$= c_N |z - y|^{-N-2} \left[ 2z_i |z|^2 |z - y|^2 + N(1 - |z|^2)(|z|^2 z_i + |z|^2 y_i - 2z_i (z \cdot y)) \right]$$

$$= P(z, y) \left[ 2z_i |z|^2 (1 - |z|^2)^{-1} + N|z - y|^{-2}(|z|^2 z_i + |z|^2 y_i - 2z_i (z \cdot y)) \right].$$
Thus (28) holds, and the proof of Lemma 2.3 is complete.

Let
\[
x \cdot \nabla K(x, y) = (2 - N)K(x, y) + |x|^{-(N-2)}P(z, y)
\]
\[
\times \left[ 2(x \cdot z) \frac{|z|^2}{1 - |z|^2} + N|z| - y^{-2}(|z|^2(x \cdot z) + |z|^2(x \cdot y) - 2(x \cdot z)(x \cdot y)) \right]
\]
\[
= K(x, y) \left[ 2 - N + \frac{2}{|x|^2 - 1} + \frac{N}{|z - y| |x|^2}(1 - x \cdot y) \right]
\]
\[
= K(x, y) \left[ 2 - N + \frac{2}{|x|^2 - 1} + \frac{N}{|x - y|^2}(1 - x \cdot y) \right].
\]

Since
\[
\frac{1}{2} \leq \frac{1 - x \cdot y}{|x - y|^2} \geq -\frac{|x| - 1}{|x|^2 - 2x \cdot y + 1} \geq -\frac{|x| - 1}{|x|^2 - 2|x| + 1} = -\frac{1}{|x| - 1},
\]
it follows from (33) that
\[
|x \cdot \nabla K(x, y)| \leq N \frac{|x|}{|x| - 1} K(x, y).
\]

This together with (30) and (23) implies
\[
|F_1[\psi](x, t)| \leq \int_{\partial \Omega} e^{t} x \cdot (\nabla_x K)(e^{t} x, y)|\psi(y)|\,d\sigma_y
\]
\[
\leq N - \frac{e^{t}|x|}{e^{t}|x| - 1} |\psi|_{L^\infty} \int_{\partial \Omega} K(e^{t} x, y, t)\,d\sigma_y
\]
\[
\leq N(e^{t}|x|)^{-(N-2)} \frac{e^{t}|x|}{e^{t}|x| - 1} |\psi|_{L^\infty}.
\]

Thus (28) holds, and the proof of Lemma 2.3 is complete. \qed

By Lemma 2.3 we obtain the following lemma.

**Lemma 2.4.** Let \(N \geq 3\) and \(\theta \in (0, 1)\). For \(\psi \in C^{1,\theta}(\partial \Omega)\) set
\[
D_{\varepsilon}[\psi](x, t) := \int_{0}^{t} [S_{1}(\varepsilon^{-1}(t - s))F_{1}[\psi](s)](x)\,ds
\]
(35)
for \(x \in \overline{\Omega}, t > 0\) and \(\varepsilon > 0\). Then \(D_{\varepsilon}[\psi]\) and \(\nabla D_{\varepsilon}[\psi]\) are bounded and smooth in \(\overline{\Omega} \times (\tau, \infty)\) for \(\tau > 0\). Furthermore, there exists \(C > 0\) such that for every \(\psi \in C^{1,\theta}(\partial \Omega)\)
\[
||D_{\varepsilon}[\psi](t)||_{L^\infty} \leq C \varepsilon^{-\frac{3}{2}t^{\frac{3}{2 \alpha}}}|\psi|_{C^{1,\alpha}},
\]
(36)
\[
||\nabla D_{\varepsilon}[\psi](t)||_{L^\infty} \leq C \varepsilon^{-\frac{3}{2}t^{\frac{3}{2 \alpha}}}|\psi|_{C^{1,\alpha}},
\]
(37)
for \(t > 0\) and \(\varepsilon > 0\), where \(\alpha\) is as in (18).

**Proof.** We first prove (36). By (27) there is \(C_1 > 0\) such that
\[
|F_1[\psi](y, s)| \leq \frac{C_1}{4}|\psi|_{C^{1,\alpha}}|y| \leq C_1 |\psi|_{C^{1,\alpha}}|y|^{-\alpha} \leq C_1 |\psi|_{C^{1,\alpha}}|y|^{-(\alpha - 1)}
\]
(38)
for \(y \in \Omega\) with \(1 \leq |y| \leq 2\), \(s > 0\) and every \(\psi \in C^{1,\theta}(\partial \Omega)\). Since
\[
e^{s}|y| - 1 \geq \frac{1}{2}e^{s}|y| \quad \text{for} \quad |y| \geq 2, \ s > 0,
\]
then
\[
|y|^{\beta} |\psi(y, s)| \leq C_1 |\psi|_{C^{1,\alpha}}|y|^{\beta - 1}
\]
\[
\leq C_1 |\psi|_{C^{1,\alpha}}|y|^{\beta - 1} 
\]
for \(1 < \beta < 2\) and \(s > 0\), and then
\[
||D_{\varepsilon}[\psi](t)||_{L^\infty} \leq C \varepsilon^{-\frac{3}{2}t^{\frac{3}{2 \alpha}}}|\psi|_{C^{1,\alpha}},
\]
(36)
by (28) and \((\alpha)\) we obtain
\[
|F_1[\psi](y, s)| \leq 2N|\psi|_{L^\infty}|y|^{-(N-2)} \leq 2N|\psi|_{L^\infty}|y|^{-\alpha} \leq 2N|\psi|_{L^\infty}|y|^{-(\alpha-1)} \tag{39}
\]
for every \(y \in \Omega\) with \(|y| \geq 2\), \(s > 0\) and \(\psi \in C^{1,\alpha}(\partial \Omega)\). Since \(0 \leq \alpha < N\), by \((\alpha)\) and \((\gamma)\) we apply property \((G_2)\) with \(\gamma = \alpha\) and \(j = 0\) to obtain \(C_2 > 0\) such that
\[
|D_\varepsilon[\psi](x, t)| \leq C_2(|\psi|_{C^{1,\alpha}} + |\psi|_{L^\infty}) \int_0^t \tau_s^{-\frac{\alpha}{2}} ds \leq \frac{2C_2}{1 - \frac{\alpha}{2}} |\psi|_{C^{1,\alpha}} \varepsilon \sqrt{t} \left(\frac{2\alpha}{\alpha - 1}\right) \tag{40}
\]
for \(\psi \in C^{1,\alpha}(\partial \Omega), \ x \in \overline{\Omega}, \ t > 0\) and \(\varepsilon > 0\), where \(\tau_s := \varepsilon^{-1}(t - s)\). Thus \((\alpha)\) holds. Furthermore, since \(h(\tau_s) = 1\) for \(t > \varepsilon\) and \(s \in (0, t - \varepsilon)\), similarly to \((40)\), by property \((G_2)\) with \(\gamma = \alpha\) and \(j = 1\) we have
\[
\left| \int_0^{t-\varepsilon} \nabla [S_1(\tau_s)F_1[\psi](s)](x) \right| ds \leq C_2(|\psi|_{C^{1,\alpha}} + |\psi|_{L^\infty}) \int_0^{t-\varepsilon} \tau_s^{-\frac{\alpha}{2}} ds \leq \frac{2C_2}{1 - \frac{\alpha}{2}} |\psi|_{C^{1,\alpha}} \varepsilon \sqrt{t} \left(\frac{2\alpha}{\alpha - 1}\right) \tag{41}
\]
for every \(\psi \in C^{1,\alpha}(\partial \Omega), \ x \in \overline{\Omega}\) and \(t \geq \varepsilon > 0\). On the other hand, since \(0 \leq \alpha - 1 < N\) and \(h(\tau_s) = \tau_s^{-1/2}\) for \(s \in [\max\{0, t - \varepsilon\}, t]\), by \((\alpha)\) and \((\gamma)\) we apply property \((G_2)\) with \(\gamma = \alpha - 1\) and \(j = 1\) to obtain \(C_3 > 0\) such that
\[
\left| \int_{\max\{0, t - \varepsilon\}}^t \nabla [S_1(\tau_s)F_1[\psi](s)](x) \right| ds \leq C_3(|\psi|_{C^{1,\alpha}} + |\psi|_{L^\infty}) \int_{\max\{0, t - \varepsilon\}}^t \tau_s^{-\frac{\alpha}{2}} \tau_s^{-\frac{1}{2}} ds \leq C_3 |\psi|_{C^{1,\alpha}} \varepsilon \sqrt{t} \left(\frac{2\alpha}{\alpha - 1}\right) \tag{42}
\]
for every \(\psi \in C^{1,\alpha}(\partial \Omega), \ x \in \overline{\Omega}\), \(t > 0\) and \(\varepsilon > 0\). By \((41)\) and \((42)\) we have \((\alpha)\).

We now fix \(\psi \in C^{1,\alpha}(\partial \Omega)\) and \(\varepsilon > 0\). It remains to prove that \(D_\varepsilon[\psi]\) and \(\nabla D_\varepsilon[\psi]\) are bounded and smooth in \(\Omega \times (\tau, \infty)\) for \(\tau > 0\). It follows from the semigroup property of \(S_1(t)\) that
\[
D_\varepsilon[\psi](x, t) = \int_0^t [S_1(\varepsilon^{-1}(t - s))F_1[\psi](s)](x) ds
\]
\[
= S_1(\varepsilon^{-1}(t - \tau/2))D_\varepsilon[\psi](x, \tau/2) + \int_{\tau/2}^t [S_1(\varepsilon^{-1}(t - s))F_1[\psi](s)](x) ds
\]
for \(x \in \overline{\Omega}\) and \(0 < \tau < t < \infty\). We observe from \((36)\) and \((G_3)\) that
\[
(x, t) \mapsto S_1(\varepsilon^{-1}(t - \tau/2))D_\varepsilon[\psi](x, \tau/2)
\]
is bounded and smooth in \(\overline{\Omega} \times (\tau, \infty)\). On the other hand, it holds from \((13)\) that
\[
(x, t) \mapsto F_1[\psi](x, t) = \partial_t [S_2(t)\psi](x).
\]
Then, by \((24)\) we apply the same argument as in \([9, \text{Section 16, Chapter 4}]\) to see that
\[
\int_{\tau/2}^t [S_1(\varepsilon^{-1}(t - s))F_1[\psi](s)](x) ds
\]
is bounded and smooth in \(\overline{\Omega} \times (\tau, \infty)\). Therefore we deduce that \(D_\varepsilon[\psi]\) and \(\nabla D_\varepsilon[\psi]\) are bounded and smooth in \(\overline{\Omega} \times (\tau, \infty)\). Thus Lemma 2.4 follows. \(\Box\)
3. Proof of Theorem 1.6. We introduce some notation. Let \( T \in (0, \infty), \varepsilon \in (0, 1) \) and \( \alpha \) be as in (18). Let \( L > 0 \). Set

\[
X_{T,L} := \left\{ v \mid v, \nabla v \in C(\Omega \times (0,T)) : \|v\|_{X_{T,L}} < \infty \right\},
\]

\[
\|v\|_{X_{T,L}} := \sup_{0 < t < T} e^{-Lt}E_L[v](t),
\]

where

\[
E_L[v](t) := \left(1 + (\varepsilon^{-1}t)^\frac{\varepsilon}{2}\right)\|v(t)\|_{L^\infty} + (\varepsilon^{-1}t)^\frac{\varepsilon}{2} \left(1 + (\varepsilon^{-1}t)^\frac{\alpha-1}{2}\right)\|\nabla v(t)\|_{L^\infty}.
\]

Then \( X_{T,L} \) is a Banach space equipped with the norm \( \| \cdot \|_{X_{T,L}} \). For the proof of assertion (i) of Theorem 1.6, we will apply the contraction mapping theorem in \( X_{T,L} \) to find a fixed point of

\[
Q_S[v](t) := S_1(\varepsilon^{-1}t)\Phi - D_\varepsilon[\varphi_\varepsilon](t) + \int_0^t S_1(\varepsilon^{-1}(t-s))F_2[v](s)ds,
\]

where \( \Phi, F_2[v] \) and \( D_\varepsilon[\varphi_\varepsilon] \) are as in (10), (14) and (35), respectively. To this end, we prepare two lemmata.

Lemma 3.1. Let \( N \geq 3 \) and \( \beta \in (0, 1) \). There exists \( C > 0 \) such that for every \( T \in (0, \infty) \) and \( L > 0 \),

\[
F_2[v](x, t) \leq C(\varepsilon^{-1}t)^{-\frac{\cdot}{2}}e^{Lt}\|v\|^{-\frac{\beta}{2} - \frac{1}{2}}\left\{1 + |x|\left(\frac{t}{|x| - 1}\right)^\beta\right\}\|v\|_{X_{T,L}}
\]

for \( x \in \Omega, 0 < t < T, \varepsilon \in (0, 1) \) and \( v \in X_{T,L} \).

Proof. Let \( T > 0, \varepsilon \in (0, 1) \) and \( v \in X_{T,L} \). It follows from (14) that

\[
F_2[v](x, t) = F_2^0[v](x, t) + F_2^\prime[v](x, t)
\]

for \( x \in \Omega \) and \( 0 < t < T \), where

\[
F_2^0[v](x, t) := \int_{\partial\Omega} K(x, y)\partial_\nu v(y, t) d\sigma_y,
\]

\[
F_2^\prime[v](x, t) := \int_0^t \int_{\partial\Omega} \partial_t K(x, y, t - s)\partial_\nu v(y, s) d\sigma_y ds.
\]

Since

\[
\sup_{0 < t < T} e^{-Lt}(\varepsilon^{-1}t)^{\frac{\varepsilon}{2}}\|\nabla v(t)\|_{L^\infty} \leq \|v\|_{X_{T,L}},
\]

by (23) we see that

\[
|F_2^0[v](x, t)| \leq \int_{\partial\Omega} K(x, y)|\partial_\nu v(y, t)| d\sigma_y
\]

\[
\leq \|\nabla v(t)\|_{L^\infty} |x|^{-\frac{\beta}{2} - \frac{1}{2}} \leq (\varepsilon^{-1}t)^{-\frac{\beta}{2}} e^{Lt} |x|^{-\frac{\beta}{2} - \frac{1}{2}} \|v\|_{X_{T,L}}.
\]

for \( x \in \Omega \) and \( t > 0 \). On the other hand, since

\[
e^{t}|x| - 1 \geq e^t - 1 \geq t, \quad e^{t}|x| - 1 \geq |x| - 1,
\]

for \( x \in \Omega \) and \( t > 0 \), for any \( \beta \in (0, 1) \) it follows from (29) and (34) that

\[
|\partial_t K(x, y, t)| \leq N \frac{e^{t}|x|}{e^{t}|x| - 1} K(x, y, t) \leq N \frac{e^{t}|x|}{t^{1-\beta}(|x| - 1)^\beta} K(x, y, t)
\]

for \( x \in \Omega \) and \( t > 0 \), for any \( \beta \in (0, 1) \).
for $x \in \Omega$ and $0 < t < T$. Then, by (23) and (46) we obtain
\[
|E_2''[v](x,t)| \leq N \frac{|x|}{(|x| - 1)^\beta} \int_0^t e^{t-s} \frac{1}{(t-s)^{1-\beta}} \int_{\partial \Omega} K(x,y,t-s) \|\nabla v(s)\|_{L^\infty} \, ds \, \nu(s) \, ds
\]
\[
\quad = N \frac{|x|}{(|x| - 1)^\beta} \int_0^t e^{t-s} \frac{1}{(t-s)^{1-\beta}} (e^{t-s}|x|)^{-1-\beta} \|\nabla v(s)\|_{L^\infty} \, ds
\]
\[
\quad \leq N \|v\|_{X_{T,L}} \int_0^t (t-s)^{-1+\beta} (\varepsilon^{-1})^{-\frac{1}{2}} e^{Ls} \, ds
\]
\[
\quad \leq C \varepsilon^{-\frac{1}{2}} (|x| - 1)^\beta t^{\frac{1}{2}} e^{Lt} \|v\|_{X_{T,L}}
\]
\[
\quad = C (\varepsilon^{-1})^{-\frac{1}{2}} e^{Lt} |x|^{-\beta} \left( \frac{t}{|x| - 1} \right)^\beta \|v\|_{X_{T,L}}
\]
(48)

for $x \in \Omega$ and $0 < t < T$ and $C = N \int_0^1 (1 - \sigma)^{\beta-1} \sigma^{-\frac{1}{2}} \, d\sigma$. Therefore, by (45), (47) and (48) we have (44). Thus Lemma 3.1 follows. \qed

**Lemma 3.2.** Let $N \geq 3$. For any $T \in (0, \infty)$, $L > 0$, $v \in X_{T,L}$ and $\varepsilon \in (0,1)$, set
\[
\tilde{D}_\varepsilon[v](t) := \int_0^t S_1(\varepsilon^{-1}(t-s)) F_2[v](s) \, ds
\]
(49)

Then for every $T > 0$ there exists $L_* > 0$ such that
\[
\|\tilde{D}_\varepsilon[v]\|_{X_{T,L}} \leq \frac{1}{2} \|v\|_{X_{T,L}}
\]
(50)

for $v \in X_{T,L}$, $\varepsilon \in (0,1)$ and $L \geq L_*$. Furthermore, for any $0 < \tau < T$ and $j \in \{0,1\}$,
\[
\nabla^j \tilde{D}_\varepsilon[v] \in C^\infty(\Omega \times (\tau,T)) \cap BC^1(\Omega \times (\tau,T)).
\]

For the proof of Lemma 3.2 we prepare the following lemma.

**Lemma 3.3.** Let $0 \leq a < 1$ and $0 \leq b < 1$ be such that $0 \leq a + b < 1$. Let $\gamma \geq 0$ and $T > 0$. Then, for any $\delta > 0$, there exists $L_* \geq 1$ such that
\[
\sup_{0 < t < T} e^{-Lt} \gamma \int_0^t e^{Ls} s^{-a} (t-s)^{-b} \, ds \leq \delta \quad \text{for } L \geq L_*.
\]

**Proof.** Let $T > 0$, $\gamma \geq 0$ and $\delta > 0$. For any $\mu \in (0,1)$ and $L > 0$, we have
\[
\int_0^t e^{Ls} s^{-a} (t-s)^{-b} \, ds = \left( \int_0^\mu + \int_{\mu t}^{(1-\mu)t} + \int_{(1-\mu)t}^t \right) e^{Ls} s^{-a} (t-s)^{-b} \, ds
\]
\[
\quad \leq (1 - \mu)^{-b} t^{-b} e^{Lt} \int_0^\mu s^{-a} \, ds + \mu^{-a-b} t^{-a-b} \int_0^t e^{Ls} \, ds
\]
\[
\quad + (1 - \mu)^{-a} e^{Lt} \int_{(1-\mu)t}^t (t-s)^{-b} \, ds
\]
\[
\quad = \frac{1}{1-a} (1 - \mu)^{-b} \mu^{-a} t^{1-a-b} e^{Lt} + \mu^{-a-b} e^{Lt} \left( e^{Lt} - 1 \right)
\]
\[
\quad + \frac{1}{1-b} (1 - \mu)^{-a} \mu^{-a-b} e^{Lt}
\]
for \(0 < t < T\) and \(L > 0\). Then, since \(a + b < 1\) and \(\gamma \geq 0\), taking a sufficiently small \(\mu \in (0, 1/2)\) if necessary, we obtain
\[
e^{-Lt} \int_0^t e^{Ls}s^{-a}(t-s)^{-b}\,ds
\leq C(\mu^{1-a} + \mu^{1-b})e^{T(1-a-b+\gamma)} + \mu^{-a-b-t-a-b+\gamma}e^{1 - \frac{e^{-Lt}}{L}}
\leq C(\mu^{1-a} + \mu^{1-b})T(1-a-b+\gamma) + \mu^{-a-b-t-a-b+\gamma}e^{1 - \frac{e^{-Lt}}{L}}
\leq \frac{\delta}{2} + \mu^{-a-b-t-a-b+\gamma}e^{1 - \frac{e^{-Lt}}{L}}
\]
for \(0 < t < T\) and \(L > 0\), where \(C = 2^b(1-a)^{-1} + 2^a(1-b)^{-1}\). Let
\[
f(t, L) := t^{\gamma-a-b}e^{\frac{e^{-Lt}}{L}}, \quad t \in (0, T), L > 0,
\]
Then we see that in the case of \(\gamma \geq a+b\), \(0 \leq f(t, L) \leq T^{\gamma-a-b}L^{-1}\) for all \(t \in (0, T)\) and \(L > 0\), and the choice \(L_e = 2\mu^{-a-b-t-a-b}T^{\gamma-a-b}\) verifies the lemma. If, on the other hand, \(\gamma < a+b\), then for every \(L > 0\), \(\limsup_{t \to 0^+} f(t, L) = 0\), and thus
\[
t_L := \arg\max_{t \in (0, T)} f(t, L)
\]
exists and satisfies \(t_L \in (0, T)\) and
\[
\frac{d}{dt} f(t, L)|_{t=t_L} = 0
\]
and hence \(1 - e^{-Lt_L} = \frac{L}{a+b-\gamma}t_L e^{-Lt_L}\,
so that
\[
f(t, L) \leq f(t_L, L) = \frac{t_L^{\gamma+1-a-b}}{a+b-\gamma}e^{-Lt_L}
\]
for all \(t \in (0, T), L > 0\).
As, for any \(L > 0\),
\[
\sup_{s>0} \gamma^{\gamma+1-a-b}e^{-Lt} = L^{-\gamma(1-a-b)}(\gamma + 1 - a - b)^{\gamma+1-a-b}e^{-(\gamma+1-a-b)},
\]
we may conclude that also in the case \(\gamma < a+b\), \(\sup_{t \in (0, T)} f(t, L) \to 0\) as \(L \to \infty\), which together with (51) completes the proof of Lemma 3.3.

We prove Lemma 3.2.

**Proof of Lemma 3.2.** Let \(0 < T < \infty\) and let \(a\) be as in (18). Let
\[
0 < \beta < \min\left\{\frac{\alpha-1}{N}, 2-\alpha\right\} \quad \text{if} \quad N \geq 4, \quad 0 < \beta < \frac{1}{4} \quad \text{if} \quad N = 3.
\]
It follows from (44) and (18) that with \(C_1 > 0\) as in (44)
\[
|F_2[y](y,s)| \leq C_1 e^{L^s} \varepsilon^{-\frac{1}{2}} \|v\|_{X_{T,L}} \left[|y|^{-(N-2)} + s^\beta \eta_\beta(y) + s^\beta |y|^{-(N-3+\beta)}\chi(|y|>2)\right]
\leq C_1 e^{L^s} \varepsilon^{-\frac{1}{2}} \|v\|_{X_{T,L}} \left[|y|^{-(\alpha-1)} + s^\beta \eta_\beta(y) + s^\beta |y|^{-(\alpha-1+\beta)}\chi(|y|>2)\right]
\]
for \(y \in \Omega, L > 0\) and \(0 < s < T\), where \(\eta_\beta(y) := (|y|-1)^{-\beta}\chi_{1 \leq |y| \leq 2}\). By (52) we have
\[
\eta_\beta \in L^{\frac{N}{N-1}}(\Omega) \quad \text{if} \quad N \geq 4, \quad \eta_\beta \in L^4(\Omega) \quad \text{if} \quad N \geq 3.
\]
For \( s > 0 \) and \( \tau > 0 \), set

\[
I(s, \tau) := s^{-\frac{1}{2}} \left[ \tau^{-\frac{\alpha+1}{2}} + s^\beta \tau^{-d} + s^\beta (1 + \tau)^{-\frac{\alpha+1}{2}} \right] \tag{55}
\]

where \( d = (\alpha - 1)/2 \) if \( N \geq 4 \) and \( d = 3/8 \) if \( N = 3 \). Then it follows that

\[
I(s, \varepsilon^{-1} \tau) \leq \varepsilon^{\frac{\alpha+1}{2}} I(s, \tau) \quad \text{for} \quad s > 0, \; \tau > 0, \; \varepsilon \in (0, 1). \tag{56}
\]

By (53) and (54) we apply properties (G2) and (G1) to obtain \( C_2 > 0 \) such that

\[
\| \nabla^2 S_1(\tau) F_2[v](s) \|_{L^\infty} \leq C_2 e^{-Lt} \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, \tau) \, ds \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, t-s) \, ds \tag{57}
\]

for \( L > 0, \; 0 < s < t \), where \( \tau \varepsilon = e^{-1}(t-s) \) and \( h \) is as in (22). By (49), (57) and (56) we have

\[
\| \tilde{D}_c[v](t) \|_{L^\infty} \leq \int_0^t \| S_1(\tau) F_2[v](s) \|_{L^\infty} \, ds \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, t-s) \, ds \tag{58}
\]

for \( t > 0, \; L > 0 \) and \( \varepsilon \in (0, 1) \). This implies that

\[
e^{-Lt} \left( 1 + (e^{-1}t)^{\frac{1}{2}} \right) \| \tilde{D}_c[v](t) \|_{L^\infty} \leq C_2 e^{-Lt} \left( 1 + (e^{-1}t)^{\frac{1}{2}} \right) \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, t-s) \, ds \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, t-s) \, ds \tag{59}
\]

for \( t > 0, \; L > 0 \) and \( \varepsilon \in (0, 1) \). Then, by Lemma 3.3 with (55), taking a sufficiently large \( L \geq 1 \) if necessary, we obtain

\[
\sup_{0 < t < T} e^{-Lt} \left( 1 + (e^{-1}t)^{\frac{1}{2}} \right) \| \tilde{D}_c[v](t) \|_{L^\infty} \leq \frac{1}{4} \| v \|_{X_{T,L}} \tag{59}
\]

for \( 0 < \varepsilon < 1 \). Similarly to (58), it follows from (56) and (57) that

\[
\| \nabla \tilde{D}_c[v](t) \|_{L^\infty} \leq \left( \int_0^{\max \{ t-\varepsilon, 0 \}} + \int_0^t \right) \| \nabla S_1(\tau) F_2[v](s) \|_{L^\infty} \, ds \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \int_0^t e^{Ls} I(s, \tau) \, ds + \int_0^t e^{Ls} \tau^{-\frac{1}{2}} I(s, \tau) \, ds \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} \left( \int_0^t e^{Ls} I(s, t-s) \, ds + \varepsilon^{\frac{\alpha+1}{2}} \int_0^t e^{Ls} (t-s)^{-\frac{1}{2}} I(s, t-s) \, ds \right) \tag{60}
\]

for \( 0 < t < T \) and \( L > 0 \). Then we have

\[
e^{-Lt} (e^{-1}t)^{\frac{1}{2}} \left( 1 + (e^{-1}t)^{\frac{\alpha+1}{2}} \right) \| \nabla \tilde{D}_c[v](t) \|_{L^\infty} \leq C_2 \varepsilon^{\frac{1}{2}} \| v \|_{X_{T,L}} e^{-Lt} (t^{\frac{1}{2}} + t^{\frac{\alpha+1}{2}}) \left( \int_0^t e^{Ls} I(s, t-s) \, ds + \int_0^t e^{Ls} (t-s)^{-\frac{1}{2}} I(s, t-s) \, ds \right) \tag{61}
\]
for \( t > 0, \ L > 0 \) and \( 0 < \varepsilon < 1 \). Similarly to (59), by Lemma 3.3, we obtain
\[
\sup_{0 < t < T} e^{-Lt}(\varepsilon^{-1}t)^{\frac{1}{2}} \left( 1 + (\varepsilon^{-1}t)^{\frac{\alpha}{2}} \right) \|\nabla \tilde{D}_x[v](t)\|_{L^\infty} \leq \frac{1}{4} \|v\|_{X_{T,L}} \tag{60}
\]
for \( 0 < \varepsilon < 1 \) and sufficiently large \( L \geq 1 \). Combining (59) and (60), we deduce that in this case
\[
\|\tilde{D}_x[v]\|_{X_{T,L}} \leq \frac{1}{2} \|v\|_{X_{T,L}}
\]
for \( 0 < \varepsilon < 1 \). Thus (50) holds. On the other hand, for \( v \in X_{T,L} \), it follows from (14) that \( F_2 \in C(\bar{\Omega} \times (0, T)) \). Then, applying the parabolic regularity theorem (see e.g. [9], cf. proof of Lemma 2.4), we deduce that
\[
\nabla^j \tilde{D}_x[v] \in C^\infty(\Omega \times (\tau, T)) \cap BC(\bar{\Omega} \times (\tau, T))
\]
for any \( 0 < \tau < T \) and \( j \in \{0, 1\} \). Therefore we complete the proof of Lemma 3.2. \( \square \)

Now we are ready to prove Theorem 1.6.

**Proof of Theorem 1.6.** It follows from (10), (15), (25) and (18) that
\[
|\Phi(x)| \leq |\varphi(x)| + ||S_2(0)|\varphi_b(x)|| \leq |x|^{-(N-2)}(M + |\varphi_b|_{L^\infty}) \leq |x|^{-\alpha}(M + |\varphi_b|_{L^\infty})
\]
for all \( x \in \bar{\Omega} \). Then, by \((G_2)\) we find \( c_\ast > 0 \) such that
\[
|\nabla^j S_1(t)|_{L^\infty} \leq c_\ast(M + |\varphi_b|_{L^\infty})(1 + t)^{-\frac{\alpha}{2}} h(t)^j
\]
for \( t > 0 \) and \( j \in \{0, 1\} \) and with \( h \) from (22). Let \( 0 < T < \infty \) and \( L \geq 1 \). Then
\[
E_x[S_1(\varepsilon^{-1}t)\Phi](t)
\leq c_\ast(M + |\varphi_b|_{L^\infty}) \left( 1 + (\varepsilon^{-1}t)^{\frac{\alpha}{2}} \right) (1 + \varepsilon^{-1}t)^{-\frac{\alpha}{2}}
+ c_\ast(M + |\varphi_b|_{L^\infty})(\varepsilon^{-1}t)^{\frac{1}{2}} \left( 1 + (\varepsilon^{-1}t)^{\frac{2\alpha}{2-\alpha}} \right) (1 + \varepsilon^{-1}t)^{-\frac{\alpha}{2}} h(\varepsilon^{-1}t)
\leq 4c_\ast(M + |\varphi_b|_{L^\infty})
\]
for \( t > 0 \) and \( 0 < \varepsilon < 1 \). Furthermore, by Lemma 3.2, taking a sufficiently large \( L \geq 1 \) if necessary, we see that
\[
\|\tilde{D}_x[v]\|_{X_{T,L}} \leq \frac{1}{2} \|v\|_{X_{T,L}}, \quad v \in X_{T,L}, \tag{62}
\]
for \( 0 < t < T \) and \( 0 < \varepsilon < 1 \). For this choice of \( L \), on the other hand, by Lemma 2.4 we find \( C_L > 0 \) such that
\[
e^{-Lt}E_x[D_x[\varphi_b]](t)
\leq C|\Phi(x)|_{C^{1,\alpha}} e^{-Lt}(1 + (\varepsilon^{-1}t)^{\frac{\alpha}{2}}) \varepsilon^{\frac{\alpha}{2}} t^{\frac{2-\alpha}{2}}
+ C|\Phi(x)|_{C^{1,\alpha}} e^{-Lt}(\varepsilon^{-1}t)^{\frac{1}{2}} \left( 1 + (\varepsilon^{-1}t)^{\frac{2\alpha}{2-\alpha}} \right) \varepsilon^{\frac{\alpha}{2}} t^{\frac{2-\alpha}{2}} h(\varepsilon^{-1}t) \leq C_L|\Phi(x)|_{C^{1,\alpha}}
\]
for \( t > 0 \) and \( 0 < \varepsilon < 1 \). Set
\[
m := 2 \left\{ 4c_\ast(M + |\varphi_b|_{L^\infty}) + C_L|\varphi_b|_{C^{1,\alpha}} \right\}. \tag{64}
\]
We deduce from (43), (61), (63), (62) and (64) that
\[
\|Q_\varepsilon[v]\|_{X_{T,L}} \\
\leq \sup_{0 < t < T} e^{-Lt}E_{c}[S_{1}(\varepsilon^{-1}t)\Phi](t) + \sup_{0 < t < T} e^{-Lt}E_{c}[D_{c}[\varphi_{b}]](t) + \|\tilde{D}_{\varepsilon}[v]\|_{X_{T,L}}
\]
\[
\leq 4c_{s}(M + |\varphi_{b}|_{L^{\infty}}) + C_{L}|\varphi_{b}|_{C^{1,\theta}} + \frac{1}{2}\|v\|_{X_{T,L}} \leq m
\]
for \(v \in X_{T,L}\) with \(\|v\|_{X_{T,L}} \leq m\) and \(0 < \varepsilon < 1\). Similarly, we deduce from (62) that
\[
\|Q_\varepsilon[v_1] - Q_\varepsilon[v_2]\|_{X_{T,L}} = \|\tilde{D}_{\varepsilon}[v_1] - \tilde{D}_{\varepsilon}[v_2]\|_{X_{T,L}} \leq \frac{1}{2}\|v_1 - v_2\|_{X_{T,L}}
\]
(66)
for \(v_1, v_2 \in X_{T,L}\). By (65) and (66) applying the contraction mapping theorem, for every \(\varepsilon \in (0,1)\) we find a unique \(v_\varepsilon \in X_{T,L}\) with \(\|v_\varepsilon\|_{X_{T,L}} \leq m\) such that
\[
v_\varepsilon = Q_\varepsilon[v_\varepsilon] = S_{1}(\varepsilon^{-1}t)\Phi - D_{c}[\varphi_{b}](t) + \tilde{D}_{\varepsilon}[v_\varepsilon](t) \quad \text{in} \quad X_{T,L}.
\]
In particular, it follows from (65) and (64) that with some \(C > 0\)
\[
\|v_\varepsilon\|_{X_{T,L}} \leq C(|\varphi_{b}|_{C^{1,\theta}} + M) \quad \text{for every} \quad \varepsilon \in (0,1).
\]
Furthermore, by (G3) and Lemmata 2.4, 3.2 we see that
\[
\nabla^{j}v_\varepsilon \in C^{\infty}(\Omega \times (T_{1},T_{2})) \cap BC(\Omega \times (T_{1},T_{2}))
\]
for any \(0 < T_{1} < T_{2}\) and \(j \in \{0,1\}\).

On the other hand, set
\[
w_\varepsilon(x,t) = [S_{2}(t)\varphi_{0}](x) + \int_{0}^{t} [S_{2}(t-s)\partial_{\varepsilon} v_\varepsilon(s)](x) ds
\]
for \(x \in \Omega, t \in (0,T)\) and \(\varepsilon \in (0,1)\). By (25) and (64) we have that with some \(C > 0\)
\[
\|w_\varepsilon(t)\|_{L^{\infty}} \leq \|S_{2}(t)\varphi_{0}\|_{L^{\infty}} + \int_{0}^{t} \|S_{2}(t-s)\partial_{\varepsilon} v_\varepsilon(s)\|_{L^{\infty}} ds
\]
\[
\leq |\varphi_{b}|_{L^{\infty}} + \int_{0}^{t} |\nabla v_\varepsilon(s)|_{L^{\infty}} ds
\]
\[
\leq \frac{m}{8c_{s}} + \int_{0}^{t} (\varepsilon^{-1}s)^{-\frac{1}{2}} \left(1 + (\varepsilon^{-1}s)^{-\frac{1}{2}}\right)^{-1} e^{Ls}\|v_{\varepsilon}\|_{X_{T,L}} ds
\]
\[
\leq \frac{m}{8c_{s}} + \varepsilon \frac{1}{2} e^{LT} m \int_{0}^{t} s^{-\frac{1}{2}} ds \leq C(|\varphi_{b}|_{C^{1,\theta}} + M) < \infty
\]
(68)
for \(t \in (0,T)\) and \(0 < \varepsilon < 1\). Furthermore, it follows from (9), (13) and (14) that
\[
\partial_{\ell} w_{\varepsilon}(x,t) = F_{1}[\varphi_{b}](x,t) + F_{2}[v](x,t).
\]
Then, applying similar arguments as in Lemmata 2.3 and 3.1, we see that
\[
\partial_{\ell} w_{\varepsilon} \in BC(\Omega \times (T_{1},T))
\]
for \(0 < T_{1} < T\). This together with (24) and (68) implies that
\[
\partial_{\ell}^{\ell} \nabla^{j} w_{\varepsilon} \in C^{\infty}(\Omega \times (T_{1},T)) \cap BC(\Omega \times (T_{1},T))
\]
for \(0 < T_{1} < T\) and \(0 \leq \ell + j \leq 1\). Therefore we deduce that \((v_{\varepsilon},w_{\varepsilon})\) is a solution of (12) in \(\Omega \times (0,T)\) which, furthermore, by (67) and (68) satisfies (17).

Let \(\varepsilon \in (0,1)\) and \((\tilde{v}_{\varepsilon},\tilde{w}_{\varepsilon})\) be a solution of (12) in \(\Omega \times (0,T_{*})\) for any \(T_{*} > T\) and such that \(\tilde{v}_{\varepsilon} \in X_{T_{*},L}\) with some \(L > 0\). Then \(\tilde{v}_{\varepsilon} \in X_{T_{*},L}\) and since
\[
v_{\varepsilon} - \tilde{v}_{\varepsilon} = Q_{\varepsilon}[v_{\varepsilon}] - Q_{\varepsilon}[\tilde{v}_{\varepsilon}] = \tilde{D}_{\varepsilon}[v_{\varepsilon} - \tilde{v}_{\varepsilon}] \quad \text{in} \quad X_{T_{*},L},
\]
by (50) we have
\[ \|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T,L}} \leq \frac{1}{2}\|v_\varepsilon - \tilde{v}_\varepsilon\|_{X_{T,L}}. \]
This implies that \( v_\varepsilon = \tilde{v}_\varepsilon \) in \( X_{T,L} \). Therefore we deduce that \((v_\varepsilon, w_\varepsilon)\) is a global-in-time solution of (12) satisfying (16) and is unique among these, because for every global-in-time solution \((\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)\) satisfying (16), \( \tilde{v}_\varepsilon \) belongs to \( X_{T,L} \) for every \( T > 0 \) and \( L > 0 \).

It remains to prove assertions (i) and (ii). Assertion (i) and (19) immediately follow from Theorem 1.6 and Definition 1.5. On the other hand, by (68) we have
\[
\|w_\varepsilon(t) - S_2(t)\varphi_b\|_{L^\infty} \leq \int_0^t \|S_2(t - s)\partial_x z(s)\|_{L^\infty} \, ds
\leq \int_0^t |\nabla v_\varepsilon(s)|_{L^\infty} \, ds \leq C\|v_\varepsilon\|_{X_{T,L}} \varepsilon^{\frac{\alpha}{2}} (1 + T^{\frac{2}{\alpha} - 1})
\]
for all \( t \in (0, T) \). This implies (20). Thus assertion (ii) follows, and the proof of Theorem 1.6 is complete.

**Proof of Corollary 1.** Corollary 1 immediately follows from Theorem 1.6 and Definition 1.5.

4. Estimates from below.

**Lemma 4.1.** Let \( \varepsilon \in (0, 1) \), \( \varphi_b \equiv 0 \) and \( \varphi \) be nonnegative and satisfy (15). Let \( z \) be a solution of
\[
\partial_t z - \Delta z = 0 \quad \text{in} \quad \Omega \times (0, \infty), \quad z = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \quad z(\cdot, 0) = \varphi \quad \text{in} \quad \Omega.
\]
Set \( u_\varepsilon(x,t) = z(x, \varepsilon^{-1}t) \) for \( (x,t) \in \Omega \times [0, \infty) \). Then the solution \( u_\varepsilon \) of (1) satisfies
\[
u_\varepsilon(x,t) \geq u_\varepsilon(x,t), \quad (x,t) \in \Omega \times (0, \infty).
\]

**Proof.** By nonnegativity of \( z \) in \( \Omega \) and the homogeneous Dirichlet boundary condition in (69), we see that \( \partial_\nu z \leq 0 \) on \( \partial \Omega \). From (69) we conclude that \( u_\varepsilon \) solves
\[
\varepsilon \partial_\nu u_\varepsilon - \Delta u_\varepsilon = 0 \quad \text{in} \quad \Omega \times (0, \infty), \quad u_\varepsilon(\cdot, 0) = \varphi \quad \text{in} \quad \Omega
\]
and
\[
\partial_t u_\varepsilon(x,t) + \partial_\nu u_\varepsilon(x,t) = \varepsilon^{-1} \partial_t z(x, \varepsilon^{-1}t) + \partial_\nu z(x, \varepsilon^{-1}t) - \partial_\nu z(x, \varepsilon^{-1}t) \leq 0
\]
on \( \partial \Omega \times (0, \infty) \). Therefore, \( u_\varepsilon \) is a subsolution of (1), while \( u_\varepsilon \) is a supersolution, and applying the comparison principle (see [10, Theorem 2.2]), we obtain (70).

**Lemma 4.2.** Let \( b > 1 \) and put
\[
\varphi(x) = |x|^{2-N} \chi_{\{|x| > b\}}, \quad x \in \Omega.
\]
Then for any compact set \( \mathcal{F}_a \subset \Omega \) and \( \tau > 0 \) there exists \( C > 0 \) such that the solution \( z \) of (69) satisfies
\[
z(x,t) \geq C(t^{-\frac{\alpha}{2}})^{1+b} \quad \text{for} \quad x \in \mathcal{F}_a, \quad t > \tau.
\]

**Proof.** Let \( \mathcal{F}_a \) be a compact set in \( \Omega \). Then we can take \( a \in (1, b) \) and \( \beta > 0 \) such that \( a < |x| \leq \beta b \) for all \( x \in \mathcal{F}_a \). Since it follows from [7, Theorem 1.1] that there are \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[
\Gamma_D(x,y,t) \geq C_1 t^{\frac{\alpha}{2}} \exp\left(-C_2 \frac{|x-y|^2}{t}\right)
\]
for all $x, y \in \Omega$ with $|x| > a$, $|y| > a$ and $t > 0$, by (71) we have $C_3 > 0$ satisfying

$$z(x, t) = \int_{\Omega} \Gamma_D(x, y, t) \varphi(y) \, dy = \int_{S^{N-1}} \int_{b}^{\infty} r^{2-N} \Gamma_D(x, r\omega, t) r^{N-1} \, d\omega \, dr$$

$$\geq C_1 \int_{b}^{\infty} \int_{S^{N-1}} rt^{-\frac{N}{2}} \exp\left(-C_2 \frac{|x-r\omega|}{t}\right) \, d\omega \, dr$$

$$\geq C_1 |S^{N-1}| t^{-\frac{N}{2}} \int_{b}^{\infty} r \exp\left(-C_2 \frac{(1+\beta)^2 r^2}{t}\right) \, dr$$

$$= C_3 t^{-\frac{N}{2}+1} (1+\beta)^{-2} \exp\left(-C_2 \frac{(1+\beta)^2 b^2}{t}\right)$$

for all $x \in \Omega$ with $a < |x| \leq \beta b$ and $t > 0$. This implies (72), thus Lemma 4.2 follows.

Now we are ready to prove Theorem 1.7.

**Proof of Theorem 1.7.** Let $\mathcal{K}$ be a compact set in $\Omega$ such that $\mathcal{K} \subset \mathcal{K}_* \times [t_1, t_2]$ for some compact set $\mathcal{K}_* \subset \Omega$ and $0 < t_1 < t_2 < \infty$, and let $\varphi$ be as in Lemma 4.2. Then, applying Lemma 4.1 and Lemma 4.2 to $\tau := t_1$, we see that there exists $C_* > 0$ such that

$$u_\varepsilon(x, t) \geq z(x, \varepsilon^{-1} t) \geq C_* (\varepsilon^{-1} t)^{-\frac{N}{2}+1} \geq C_* t_2^{-\frac{N}{2}+1} \varepsilon^{\frac{N}{2}-1}$$

for all $(x, t) \in \mathcal{K}$. This implies (21), and the proof of Theorem 1.7 is complete.

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