Proposal for a tunable graphene-based terahertz Landau-level laser

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In the presence of strong magnetic fields the electronic bandstructure of graphene drastically changes. The Dirac cone collapses into discrete non-equidistant Landau levels, which can be externally tuned by changing the magnetic field. In contrast to conventional materials, specific Landau levels are selectively addressable using circularly polarized light. Exploiting these unique properties, we propose the design of a tunable laser operating in the technologically promising terahertz spectral range. To uncover the many-particle physics behind the emission of light, we perform a fully quantum mechanical investigation of the non-equilibrium dynamics of electrons, phonons, and photons in optically pumped Landau-quantized graphene embedded into an optical cavity. The gained microscopic insights allow us to predict optimal experimental conditions to realize a technologically promising terahertz laser.

The terahertz (THz) regime of the electromagnetic spectrum can be exploited in a wide range of applications including medical diagnostics, atmosphere and space science as well as security and information technology [1–4]. Although THz research has progressed significantly in the last 20 years, the transition from laboratory demonstration to practical environment has occurred slowly and only for some niche applications. The largest challenge is the lack of adequate, tunable THz radiation sources. In 1986, H. Aoki proposed to design Landau level (LL) lasers exploiting the discreteness of LLs in two-dimensional electron gases [5]. Here, the energetic LL spacing and thus the possible laser frequency can be externally tuned through the magnetic field. The challenge for the realization of such a laser is to obtain a stable population inversion, i.e. a larger carrier occupation within an energetically higher LL. Since conventional semiconductors exhibit an equidistant spectrum of LLs, strong Coulomb scattering acts in favor of an equilibrium Fermi-Dirac distribution and strongly counteracts the build-up of a population inversion. In contrast, graphene as a two-dimensional material with a linear dispersion exhibits a non-equidistant LL separation offering entirely different conditions for many-particle processes [6–8]. Exploiting these remarkable properties of Landau-quantized graphene, we propose an experimentally accessible scenario to achieve continuous wave lasing with tunable frequencies in the technologically promising terahertz spectral regime.

The non-equidistant arrangement of energy levels[9, 10]

$$\varepsilon_l = sgn\{l\} \hbar v_F \sqrt{\frac{2e_0 B}{\hbar} |l|},$$  \hspace{1cm} (1)

combined with selection rules for circularly polarized light, allow to selectively address individual inter-LL transitions. Here, the magnetic field $B$ is perpendicular to the graphene layer, $v_F$ denotes the Fermi velocity in graphene, and $l = ... , -2, -1, 0, 1, 2, ...$ is the LL quantum number. Left (right) circularly polarized light, denoted with $\sigma_+ (-)$, exclusively induces transitions with quantum
numbers \[11, 12\] \(|l| \rightarrow |l| + (-1)\). Thus, a linearly polarized optical pump field with an energy matching the transition \(l = -2 \rightarrow +3\) can simultaneously induce a population inversion between \(l = +2\) and \(l = +1\) as well as \(l = -1\) and \(l = -2\), cf. Fig. 1A. In contrast to conventional materials, the non-equidistant spectrum of LLs in graphene efficiently quenches Coulomb scattering due to restrictions stemming from the energy conservation.

While in a previous study \[7\], we have predicted the appearance of such a population inversion between optically coupled LLs, in this work we go a significant step forward. To achieve laser light emission, we propose to embed the graphene layer into a high quality Fabry-Perot microcavity \[13\] with a resonator mode matching the energy difference between \(l = +1\) and \(l = +2\), cf. Fig. 1C. This way, the trapped cavity photons become multiplied via stimulated emission, generating coherent terahertz intensity.

The dephasing rates \(\gamma_{\mu}\) are determined self-consistently considering many-particle and impurity-induced scattering \[23-25\]. Details of the calculation including expressions for the pump and scattering rates, as well as the self-consistent determination of the dephasing rates can be found in the supplementary material.

The interaction strength between electrons and cavity photons is determined by the coupling element \(g_{\mu}^0\). Furthermore, the photon generation rate is influenced by the number of emitters that is given by the magnetic field dependent LL degeneracy \(N_B = BA/\Phi_0\) corresponding to the number of magnetic flux quanta \(\Phi_0 = h/e\) within the graphene layer of area \(A\). Since the electron-photon coupling is relatively weak in graphene, spontaneous emission into non-lasing modes is negligibly small. We consider a finite cavity photon lifetime \((2\kappa)^{-1} = Q/\omega_{\sigma \pm}\) that is given by the quality factor \(Q\) and the photon frequency \(\omega_{\sigma \pm}\). Therewith we account for photon losses due to cavity imperfections and laser light out-coupling, which leads to a decay of the photon number towards a thermal occupation \(n_0\). To prove whether coherent laser light is emitted from graphene, we also track the temporal evolution of the photon statistics via the second-order correlation function \(g^{(2)}(t, \tau)\).

The solution of the luminescence Bloch equations reveals the non-equilibrium dynamics of the electronic configuration and the number of photons within the cavity, which provides a microscopic understanding of the
Figure 2. **Laser dynamics.** (A) Time development of the occupation probabilities of the two laser levels in the conduction band $n_{\pm}$ at $B = 4\, T$. The thermal electron population at $t = 0$ is inverted (blue shaded area) after the pump field is turned on (yellow line). Without stimulated emission, the inversion would stay at the indicated value $\Delta_{\text{pump}}$ (dashed lines). (B) Evolution of the right circularly polarized photon number $n_{\pm}$ (logarithmic) and the second order correlation $g^{(2)}_{\sigma^-}$ (right axis). The population inversion induces an exponential increase of the photon number by stimulated emission. Due to the finite pump and relaxation rates, the population inversion is depleted with rising photon number, resulting in a quasi-stationary threshold inversion $\Delta_{\text{th}}$. During the stable laser equilibrium the system emits coherent laser light characterized by the second-order correlation function $g^{(2)}_{\sigma^-} = 1$.

switch-on characteristics of the Landau level laser. In the following, we investigate the dynamics at room temperature and the following experimentally accessible conditions: the cavity cross-section area $A = 100\, \mu m^2$ (also size of the graphene sheet, cf. Fig. 1C), the cavity length is fixed due to the resonance condition $L = \lambda_p/2$, a quality factor $Q = 5000$, a background screening $\varepsilon_{bg} = 3.3$ (corresponding to a SiC substrate), and a reasonable impurity strength [31] determined by an impurity-induced LL broadening of $2.5\, meV$ at $B = 4\, T$.

**Laser dynamics.** At first we study the laser dynamics at the fixed magnetic field $B = 4\, T$ and the pump intensity $I = 10\, kW/cm^2$. Since in undoped graphene, the electron and hole populations within conduction and valence band are fully symmetric, we focus on the electron dynamics in the following. Figure 2A shows the temporal evolution of the electron occupation probability of the two laser levels $l = +2$ and $l = -1$. Initially both occupations are in thermal equilibrium characterized by a Fermi-Dirac distribution with $\rho_1 > \rho_2$. At 100 ps, the constant optical pump field is turned on transferring carriers from $l = -3$ to $l = +2$ giving rise to a population inversion with $\rho_2 > \rho_1$, cf. the blue-shaded region in Fig. 1A.

Phonon-induced relaxation of excited carriers counteracts the optical excitation mainly through transitions $l = +2 \rightarrow -2$ and $l = +2 \rightarrow -3$, cf. red arrows in Fig. 1B. Coulomb-induced scattering is strongly suppressed due to the non-equidistant nature of the optically excited Landau levels. Besides the increase of $\rho_2$, the pump process indirectly leads to a decrease of $\rho_1$, since $\rho_{-3}$ is optically depopulated opening up the channel for phonon scattering via $l = +1 \rightarrow -3$, which is sketched as the inner green arrow in Fig. 1B. Shortly after the pulse is switched on, a quasi-equilibrium between optical excitation and relaxation due to the emission of phonons is reached resulting in the pump-induced population inversion $\Delta_{\text{pump}}$.

Including an optical cavity, the number of photons increases exponentially via stimulated emission, once a population inversion is established. Figure 2B shows the time evolution of the right circularly polarized photon number $n_{\pm}$. The chain reaction of stimulated emissions requires more than 100 ps to generate a significant number of photons reflecting the weak electron-light interaction in graphene and the finite cavity photon lifetime.

The growing photon avalanche is accompanied by a decrease of the population inversion, due to the finite pump and relaxation rates. The occupation of the upper laser level $\rho_2$ decreases, since the stimulated emission of photons via $l = +2 \rightarrow +1$ breaks the balance between pumping and phonon relaxation. Similarly, the finite lifetime of $l = +1$ leads to an accumulation of carriers resulting in an enhanced $\rho_1$, cf. Fig. 2A. After approximately 500 ps, a new quasi-equilibrium is reached that is characterized by a reduced threshold population inversion $\Delta_{\text{th}}$. At that point, gain and cavity losses compensate each other and the number of photons remains constant. Hence, to enter the laser regime, the pumped inversion $\Delta_{\text{pump}}$ has to be larger than $\Delta_{\text{th}}$. An analytic expression for the threshold population inversion can be extracted from the static limit of (3) and (4) by only considering the resonant polarization $S_{\pm}^{(2)}$ and by neglecting spontaneous emission $\propto \rho_2 (1 - \rho_1)$ and higher-order photon correlations yielding

$$\Delta_{\text{th}} = \frac{\kappa (\kappa + \gamma_{\text{th}})}{4N_B |g^{(2)}_{\sigma^+}|^2}.$$  

The larger the cavity losses ($\propto \kappa$) and the faster the decay of the polarizations ($\propto (\kappa + \gamma_{\text{th}})$), the higher the threshold inversion. On the other hand, a large number of emitters ($\propto N_B$) and a strong carrier-photon coupling $g^{(2)}_{\sigma^+}$ result in a higher photon generation rate and therefore act in favor of a low threshold.

Finally, to describe the statistics of the emitted photons, we calculate the second-order correlation function $g^{(2)}_{\sigma^+}$, cf. the right y-axis in Fig. 2B. Initially before the optical pumping, we find $g^{(2)} = 2$ characterizing photons in thermal equilibrium. Once a population inversion is reached, the number of photons increases due to stimulated emissions, and $g^{(2)}$ approaches the value 1 characterizing coherent laser light.
Figure 3. **Laser tunability.** Time development of the (A) population inversion, (B) photon number, and (C) the second order correlation function for technologically relevant magnetic fields at constant pump intensity of $I = 10 \text{kW/cm}^2$. The multiple number of optical phonon modes, which can either support or counteract the laser operation lead to a remarkable magnetic field dependence of the laser dynamics. There are three regimes (marked with I, II, and III), where the population inversion is large enough to generate a significant number of photons. Within these magnetic field regimes the system produces coherent THz radiation characterized by $g^{(2)} = 1$.

Tunability of the laser frequency. A crucial advantage, of the LL laser is its tunability, since the spacing between LLs is adjustable through the magnetic field. However, to allow carriers to perform cycles within the three-level laser system, a non-radiative decay $l = 1 \rightarrow -3$ (and $l = 3 \rightarrow -1$) via the emission of optical phonons is required, which have discrete energies. The non-trivial interplay of the multiple phonon modes gives rise to a very interesting magnetic field dependence of the laser dynamics. Figure 3 shows the temporal evolution of (A) the population inversion, (B) the photon number $n_\mu$, and (C) second-order correlation function $g^{(2)}$ for the technologically relevant magnetic fields $B$ at a constant pump intensity of $I = 10 \text{kW/cm}^2$.

The length $L$ of the cavity is adjusted to the $B$-dependent resonance condition $L = \lambda_\mu/2 = \pi c h/(\varepsilon_2 - \varepsilon_1)$. Moreover, the pump frequency is changed to match the transition $l = -3 \rightarrow 2$. Since, the distance between LLs and also their broadening increases with the magnetic field, the pump rate $P \propto I/(\omega^2 \gamma)$ (cf. supplementary material) decreases. Thus, at magnetic fields $B > 5 \text{T}$ the pump intensity is not sufficient to exceed the threshold population inversion $\Delta_{th}$. At very low magnetic fields $B < 0.5 \text{T}$, the separation of LLs becomes too small to selectively pump a single LL transition, so that neighboring LLs are also pumped. As a result, the population inversion completely vanishes within the first 200ps, cf. Fig. 3(A). Between 0.5T and 5T, we find three distinguished zones (marked with I, II, and III), where lasing takes place. The thermal equilibrium at $t = 0$ is pumped to an intermediate quasi-equilibrium at $t = 100 \text{ps}$. Only if the achieved population inversion $\Delta_{pump}$ significantly exceeds $\Delta_{th}$ (blue areas in Fig. 3A), stimulated emission can induce a photon avalanche and the photon number exponentially increases, cf. yellow areas in Fig. 3B. The time scale of that process strongly depends on how large $\Delta_{pump}$ is and how well phonon-induced processes assist the laser cycle. The green zones in Fig. 3C further illustrate that the three regimes of Fig. 3A coincide with the emission of coherent laser light.

The appearance of the three distinct magnetic field zones can be well understood by examining the $B$-dependence of the phonon relaxation rates. In particular, we distinguish between laser supporting channels and processes which deplete the population inversion, cf. green and red arrows in Fig. 1B. The corresponding net phonon relaxation rates (sum over all phonon modes) during the quasi-equilibrium are illustrated in Fig. 4. A significant number of photons (white curve) is only generated in $B$ regimes, where the laser cycle is effectively sup-
Here, we vary these experimentally accessible quantities aiming at optimal conditions for lasing. Figure 5A illustrates the number of photons within quasi-equilibrium as a function of the pump intensity and the magnetic field. Within the black areas, the pump intensity is too low to establish lasing. The pronounced line between dark and bright areas denotes the threshold intensity as a function of the magnetic field. Above the threshold intensity, \( \Delta_{\text{pump}} \) exceeds \( \Delta_{\text{th}} \) and the emission of coherent light occurs. The general upward trend of the threshold intensity is owed to the decrease of the pump transition rate with the magnetic field. Moreover, the peaks within the threshold curve are caused by the phonon resonances counteracting the population inversion and are equivalent to the peaks of the red filled curve in Fig. 4. We observe that also for \( B > 5 \) T lasing can occur at high pump intensities of above 10 kW/cm\(^2\). For very high magnetic fields with \( B > 7 \) T, the threshold intensity strongly increases, since the pump efficiency decreases and a new phonon-induced counteracting relaxation process with \( l = +2 \rightarrow 0 \rightarrow -2 \) becomes resonant. This investigation shows that - provided sufficient pump power - the proposed laser design is in principle tunable over a broad spectral range. At a pump intensity of 20 kW/cm\(^2\), the Landau-level laser can be continuously tuned in a range 3–9.5 THz by applying magnetic fields of 0.7–7 T.

Furthermore, the laser threshold can be lowered by improving the experimental conditions. Figure 5B shows the magnetic field dependence of the threshold pump intensity for different cavity quality factors \( Q \) and different temperatures \( T \). In general, high quality factors and low temperatures lead to an overall decrease of the laser threshold. The influence of the \( Q \) factor can be explained by the dependence \( \Delta_{\text{th}} \propto \kappa^2 \propto Q^{-2} \), cf. Eq. 5. That means, the higher the photon lifetime, the lower the minimum gain to compensate cavity losses. However, for \( Q \rightarrow \infty \) the minimum pump intensity still has to be sufficient to invert the initial thermal LL occupation resulting in a saturation behavior for higher \( Q \) values. Thus, cooling the system has a much higher impact on the threshold, cf. the dashed lines in Fig. 5B.

In conclusion, we predict a strategy to achieve coherent terahertz laser emission exploiting the unique properties of graphene in magnetic fields. Based on a microscopic and fully quantum-mechanical study of the coupled electron, phonon, and photon dynamics in optically pumped Landau-quantized graphene coupled to an optical cavity, we show that the emission of coherent terahertz radiation can be obtained under feasible experimental conditions. Provided an adequate cavity and sufficient pump power, the laser frequency can be externally tuned in the range of 3 – 9.5 THz by applying magnetic fields of 0.7 – 7 T. The presented work provides a concrete recipe for the experimental realization of tunable graphene-based terahertz laser systems.
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Proposal for a tunable graphene-based terahertz Landau-level laser

–SUPPLEMENTARY MATERIAL–

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MANY-PARTICLE HAMILTON OPERATOR

The temporal evolution of electrons in Landau-quantized graphene coupled to a set of photon and phonon modes is determined by the many-particle Hamilton operator

\[ H = H_{el} + H_{ph} + H_{pt}. \]  

(6)

The electronic part reads

\[ H_{el} = H_{el,0} + H_{el,el} + H_{el,l} = \sum_i \varepsilon_i a^\dagger_i a_i + \frac{1}{2} \sum_{ijkl} V_{ijkl} a^\dagger_i a^\dagger_j a_k a_l - i\hbar \frac{e_0}{m_0} \sum_{ij} \mathbf{M}_{ij} \cdot \mathbf{A}(t)a^\dagger_i a_j, \]  

(7)

and is constituted by the electronic creation and annihilation operators \( a^\dagger \) and \( a \). Here the compound index \( i = (l, m, s, \xi) \) determines the electronic state\[9, 19\], containing the Landau level index \( l = \ldots, -2, -1, 0, 1, 2, \ldots \), the quantum number \( m = 0, 1, \ldots, N_B - 1 \), which can be associated with the position of the cyclotron orbits in the graphene plane of surface \( A (N_B = Ae_0 B/(2\pi \hbar)) \) is the number of magnetic flux quanta within the graphene plane, the spin \( s = \pm 1/2 \) and valley index \( \xi = \pm 1 \). We include the free contribution of carriers with eigenenergies \( \varepsilon_i \) (cf. the manuscript), the carrier-carrier interaction determined by the Coulomb matrix element \( V_{ijkl} \), and a semi-classical carrier-light coupling, which is given by the optical matrix element \( \mathbf{M}_{ij} = \langle i | \nabla | j \rangle \) and the local vector potential \( \mathbf{A}(t) \).

The elementary charge and the electron mass are denoted by \( e_0 \) and \( m_0 \), respectively. The tight-binding expressions of the electronic eigenenergies, eigenfunctions and all matrix elements can be found in our review article about Landau-quantized graphene\[19\]. The semi-classical carrier-light coupling is used to describe the interaction with the optical pump field, whereas the light of the laser mode is treated fully quantum mechanically.

The phonon (photon) part of the Hamiltonian denoted with the subscript ‘ph’ (‘pt’) is given by

\[ H_{ph} = H_{0,ph} + H_{el-ph} = \sum_{iq} \hbar \Omega_{\nu q} b^\dagger_{\nu q} b_{\nu q} + \sum_{ij\nu q} G^{\nu q}_{ij} a^\dagger_i a_j (b_{\nu q} + b^\dagger_{\nu q}) \]  

(8)

\[ H_{pt} = H_{0,pt} + H_{el-pt} = \sum_{\mu} \hbar \omega_\mu c^\dagger_\mu c_\mu - i\hbar \sum_{ij\mu} (g^{\mu}_{ij} a^\dagger_i a_j c_\mu - g^{\mu*}_{ij} c^\dagger_\mu a_i a_j) \]  

(9)

and includes phononic (photonic) creation operators \( b^\dagger_{\nu q} \) (\( c^\dagger_\mu \)) corresponding to the mode \( \nu \) (\( \mu \)) and the phonon momentum \( \mathbf{q} \). It consists of a free part given by the phonon (photon) frequency \( \Omega_{\nu q} \) (\( \omega_\mu \)) and an interaction part including the carrier-phonon (photon) matrix element \( G^{\nu q}_{ij} \) (\( g^{\mu}_{ij} \)).

The electron-photon Hamiltonian can be deduced from the semi-classical electron-light coupling by quantizing the vector potential \( \mathbf{A} \) and expanding it in normal modes. Hence, the electron-photon matrix element is given by

\[ g^{\mu}_{ij} = \frac{e_0}{m_0} \sqrt{\frac{\hbar}{2e_0 V \omega_\mu}} \mathbf{M}_{ij} \cdot \mathbf{e}_\mu, \]  

(10)

with the normalized polarization vector of the photon mode \( \mathbf{e}_\mu \) and the quantization volume \( V \), which in case of a laser is equal to the volume of the cavity.

EQUATIONS OF MOTION

We evaluate the Heisenberg equation of motion \( i\hbar \partial_t \langle \mathcal{O} \rangle = \langle [\mathcal{O}, H] \rangle \) to determine the temporal evolution of the occupation probabilities of electronic eigenstates \( \rho_i = \langle a_i^\dagger a_i \rangle \) and the photon numbers \( n_\mu = \langle c_\mu^\dagger c_\mu \rangle \). To prove whether
coherent laser light is emitted from graphene, we also track the temporal evolution of the photon statistics via the second-order correlation function $g^{(2)}$, which for zero delay time is given by

$$g^{(2)}(t) = \frac{\langle c^\dagger_c(t)c^\dagger_c(t)c_{\mu}(t)c^\dagger_{\mu}(t) \rangle}{\langle c^\dagger_c(t)c_{\mu}(t) \rangle^2} = 2 + \frac{h_\mu(t)}{n_\mu(t)^2}. \quad (11)$$

Coherent laser light (Poisson statistics) is characterized by $g^{(2)}(t) = 1$, whereas $g^{(2)}(t) > 1$ holds for thermal and $g^{(2)}(t) < 1$ for non-classical light [26]. To calculate $g^{(2)}$ we need to consider the evolution of the photon-photon correlation $h_\mu(t) = \langle c^\dagger_c(t)c_{\mu}(t) \rangle$. To this end, we calculate all relevant electron-photon-correlations up to the quadruplet level [27, 28], thus including equations for $T_\mu^\nu(t) = \langle c^\dagger_c(t)a^\dagger_{\mu}(t)c_{\mu}(t) \rangle$ and $U_\mu^\nu(t) = \langle c^\dagger_c(t)a^\dagger_{\mu}(t)c_{\mu}(t) \rangle$. Carrier-carrier and carrier-phonon correlations beyond doublets are neglected. We obtain the following set of coupled differential equations:

$$\frac{d}{dt}\rho_{\mu} = 2 \sum_{i,j} \mathcal{R}\{|g^{\mu}_{ij}|^2 S_{ij}^{\mu} - |g^{\mu}_{ij}|^2 S_{ij}^{\mu}\} + \sum_{j} P_{ij}(\rho_j - \rho_i) + \Gamma_{i}^{in}(1 - \rho_i) - \Gamma_{i}^{out} \rho_i \quad (12)$$

$$\frac{d}{dt}n_\mu = 2 \sum_{i,j} |g^{\mu}_{ij}|^2 \mathcal{R}\{S_{ij}^{\mu}\} - 2\kappa_\mu(n_\mu - n_\mu^0) \quad (13)$$

$$\frac{d}{dt}S_{ij}^{\mu} = i(\omega_{ij} + \omega_\mu + i\kappa_\mu + i\gamma_{ij})S_{ij}^{\mu} + \rho_j(1 - \rho_i) - n_\mu(\rho_i - \rho_j) - T_i^{\mu} + T_j^{\mu} \quad (14)$$

$$\frac{d}{dt}T_i^{\mu} = -(2\kappa_\mu + \gamma_{ii})T_i^{\mu} + 2 \sum_j \mathcal{R}\{|g^{\mu}_{ij}|^2 U_{ij}^{\mu} - |g^{\mu}_{ij}|^2 U_{ij}^{\mu}\}$$

$$+ 2 \sum_j \mathcal{R}\{|g^{\mu}_{ij}|^2 S_{ij}^{\mu}(n_\mu + 1 - \rho_i) - |g^{\mu}_{ij}|^2 S_{ij}^{\mu}(n_\mu + \rho_i)\} \quad (15)$$

$$\frac{d}{dt}U_{ij}^{\mu} = i(\omega_{ij} + \omega_\mu + 3i\kappa_\mu + i\gamma_{ij})U_{ij}^{\mu} - 2|g^{\mu}_{ij}|^2(S_{ij}^{\mu})^2$$

$$- h_\mu(\rho_i - \rho_j) - 2n_\mu(T_i^{\mu} - T_j^{\mu}) + 2(1 - \rho_i)T_j^{\mu} - 2\rho_j T_i^{\mu} \quad (16)$$

$$\frac{d}{dt}h_\mu = 4 \sum_{i,j} |g^{\mu}_{ij}|^2 \mathcal{R}\{U_{ij}^{\mu}\} - 4\kappa_\mu h_\mu, \quad (17)$$

where we have rescaled $S_{ij}^{\mu} \rightarrow g^{\mu}_{ij}S_{ij}^{\mu}$ and $U_{ij}^{\mu} \rightarrow g^{\mu}_{ij}U_{ij}^{\mu}$ for simplicity. Further, $\omega_{ij} = (\varepsilon_i - \varepsilon_j)/\hbar$ denotes the electronic transition frequency and the finite photon lifetime $(2\kappa_\mu)^{-1} = Q/\omega_\mu$ accounts for cavity losses, which are determined by the cavity quality factor $Q$. The Coulomb and phonon interactions are treated within second order Born-Markov approximation [17], which gives rise to the scattering rates $\Gamma_i^{in/out,el} = \Gamma_i^{in/out,el} + \Gamma_i^{in/out,ph}$ with

$$\Gamma_i^{in,el} = \frac{2\pi}{h^2} \sum_{abc} V_{abc}(V_{ciab} - V_{icab})\rho_a\rho_b(1 - \rho_c)\mathcal{L}(\gamma_{ac} + \gamma_{bi}, \omega_{ac} + \omega_{bi}) \quad (18)$$

$$\Gamma_i^{out,el} = \frac{2\pi}{h^2} \sum_{abc} V_{abc}(V_{ciab} - V_{icab})(1 - \rho_a)(1 - \rho_b)\rho_c\mathcal{L}(\gamma_{ac} + \gamma_{bi}, \omega_{ac} + \omega_{bi}) \quad (19)$$

$$\Gamma_i^{in,ph} = \frac{2\pi}{h^2} \sum_{j \neq q} |G^{\mu q}_{ij}|^2 \rho_j \left( N_{\nu q} \mathcal{L}(\gamma_{ij}, \omega_{ji} + \Omega_{\nu q}) + (N_{\nu q} + 1)\mathcal{L}(\gamma_{ij}, \omega_{ji} - \Omega_{\nu q}) \right) \quad (20)$$

$$\Gamma_i^{out,ph} = \frac{2\pi}{h^2} \sum_{j \neq q} |G^{\mu q}_{ij}|^2 (1 - \rho_j) \left( N_{\nu q} \mathcal{L}(\gamma_{ij}, \omega_{ji} + \Omega_{\nu q}) + (N_{\nu q} + 1)\mathcal{L}(\gamma_{ij}, \omega_{ji} - \Omega_{\nu q}) \right). \quad (21)$$

Applying a bath approximation, the phonon number $N_{\nu q} = \langle b_{\nu q}^\dagger b_{\nu q} \rangle$ can be fixed to the thermal occupation (Bose-Einstein statistics). This is a good approximation in the considered laser regime. Phonon scattering is considered only for the dominant optical phonon modes GTO, GLO, KTO and KLO [21, 22], with $\epsilon_{KLO} = 151$ meV, $\epsilon_{KTO} = 162$ meV, $\epsilon_{TL} = 198$ meV and $\epsilon_{GTO} = 192$ meV (Einstein approximation).

The energy conservation is softened due to the Lorentzian broadening

$$\mathcal{L}(\gamma, \omega) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + \omega^2}, \quad (22)$$
whose width is given by the dephasing $\gamma_{ij}$, which is self-consistently determined [17] considering impurity and many-particle scattering. It reads

$$\gamma_{ij} = \gamma_{\text{imp}} + \gamma_{ij}^{\text{el}} + \gamma_{ij}^{\text{ph}} \quad \text{with} \quad \gamma_{ij}^{\text{el/ph}} = \frac{1}{2} \sum_{k=i,j} (\Gamma_{k}^{\text{in,el/ph}} + \Gamma_{k}^{\text{out,el/ph}}).$$

Since the scattering rates $\Gamma_i$ themselves depend on the dephasing, they are determined iteratively starting with $\gamma_{ij} = \gamma_{\text{imp}}$.

The disorder contribution to the equation of motion is derived within a selfconsistent Born approximation, following the approach of Shon and Ando [23, 24]. We assume [19],

$$\gamma_{\text{imp}} = \frac{v_F}{l_B \sqrt{A_{\text{imp}}}} = \frac{e_0 B}{\hbar A_{\text{imp}}},$$

where $A_{\text{imp}}$ denotes a dimensionless parameter characterising the scattering strenght of the impurity potential [23, 24].

To obtain the optical pump rate $P_{ij}$, the equation of motion for the microscopic polarization $p_{ij} = \langle a_i^\dagger a_j \rangle$ is solved within the Markow and rotating wave approximation. For a constant optical pump field with the frequency $\omega_P$, intensity $I_P$, and polarization $e_P$ one obtains:

$$P_{ij} = \left( \frac{e_0}{m_0} \right)^2 \frac{\pi I_P}{\epsilon_0 \omega_P} |M_{ij} \cdot e_P|^2 \left( \mathcal{L}(\gamma_{ij}, \omega_{ij} + \omega_P) + \mathcal{L}(\gamma_{ij}, \omega_{ij} - \omega_P) \right)$$

(25)

The degeneracy of Landau levels in spin $s = \pm 1/2$, valley $\xi = \pm 1$ and quantum number $m = 0, 1, \ldots, N_B - 1$ gives rise to a total amount of $4N_B$ LLs with the same energy. Our numerical calculations show that for $N_B \gg 1$ the electronic dynamics only depend on the Landau level index $l$, i.e. all degenerated levels behave equally. Thus, we define averaged quantities,

$$\rho_l = \frac{1}{4N_B} \sum_{m,s,\xi} \rho(l,m,s,\xi)$$

(26)

$$S_{l,l'}^\mu = \frac{1}{4N_B} \sum_{m,s,\xi} S_{l,m,s,\xi}^\mu(l',m,s,\xi)$$

(27)

where we only have to consider $s$-, $\xi$- and $m$-diagonal polarizations, since other polarizations are forbidden by selection rules [19]. $T_{l,l'}^\mu$ and $U_{l,l'}^\mu$ are treated in analogous manner. As we assume that all observables are in good approximation independent of $m, s$ and $\xi$, we set $\rho_l \approx \rho(l,m,s,\xi)$, $S_{l,l'}^\mu \approx S_{l,m,s,\xi}^\mu(l',m,s,\xi)$ and so forth. Hence, the photon generation rate in Eq. 13 can be written as,

$$\sum_{ij} |g_{ij}^\mu|^2 S_{jl}^\mu \approx 4N_B \sum_{l_i,l_j} |g_{l_i,l_j}^\mu|^2 S_{l_i,l_j}^\mu,$$

(28)

where [19] $g_{ij}^\mu = g_{l_i,l_j}^\mu \delta_{m_i,m_j} \delta_{s_i,s_j} \delta_{\xi_i,\xi_j}$. The same procedure applies for the sums in Eq. 17.