REAL HYPERSURFACES IN COMPLEX TWO-PLANE
GRASSMANNIANS WITH COMMUTING RESTRICTED
JACOBI OPERATORS

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Abstract. In this paper, we have considered a new commuting condition,
that is, \((R_\xi \phi)S = S(R_\xi \phi)\) (resp. \((R_N \phi)S = S(R_N \phi)\)) between the restricted
Jacobi operator \(R_\xi \phi\) (resp. \(R_N \phi\)), and the Ricci tensor \(S\) for real hypersurfaces
\(M\) in \(G_2(\mathbb{C}^{m+2})\). In terms of this condition we give a complete classification
for Hopf hypersurfaces \(M\) in \(G_2(\mathbb{C}^{m+2})\).

Introduction

The complex two-plane Grassmannians \(G_2(\mathbb{C}^{m+2})\) are defined as the set of all
complex two-dimensional linear subspaces in \(\mathbb{C}^{m+2}\). It is a Hermitian symmetric
space of rank 2 with compact irreducible type. Remarkably, it is equipped with
both a Kähler structure \(J\) and a quaternionic Kähler structure \(\mathcal{J}\) (not containing
\(J\)) satisfying \(JJ_\nu = J_\nu J\) \((\nu = 1, 2, 3)\), where \(\{J_\nu\}_{\nu=1,2,3}\) is an orthonormal basis of
\(\mathcal{J}\). In this paper, we assume \(m \geq 3\) (see Berndt and Suh [3] and [4]).

Let \(M\) be a real hypersurface in \(G_2(\mathbb{C}^{m+2})\) and \(N\) denote a local unit normal
vector field to \(M\). By using the Kähler structure \(J\) of \(G_2(\mathbb{C}^{m+2})\), we can define a
structure vector field by \(\xi = -JN\), which is said to be a Reeb vector field. If \(\xi\) is
invariant under the shape operator \(A\), it is said to be Hopf. In addition, \(M\) is said
to be a Hopf hypersurface if every integral curve of \(M\) is totally geodesic. By the
formulas in [7, Section 2], it can be easily seen that \(\xi\) is Hopf if and only if \(M\) is
Hopf. From the quaternionic Kähler structure \(\mathcal{J}\) of \(G_2(\mathbb{C}^{m+2})\), there naturally exist
almost contact 3-structure vector fields defined by \(\xi_\nu = -J_\nu J\nu\), \(\nu = 1, 2, 3\). Next,
let us denote by \(Q^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}\) a 3-dimensional distribution in a tangent
space \(T_pM\) at \(p \in M\), where \(Q\) stands for the orthogonal complement of \(Q^\perp\) in
\(T_pM\). Thus the tangent space of \(M\) at \(p \in M\) consists of the direct sum of \(Q\) and
\(Q^\perp\), that is, \(T_pM = Q \oplus Q^\perp\).

For two distributions \([\xi]\) = \(\text{Span}\{\xi\}\) and \(Q^\perp\), we may consider two natural
invariant geometric properties under the shape operator \(A\) of \(M\), that is, \(A[\xi] \subset [\xi]\)
and \(AQ^\perp \subset Q^\perp\). By using the result of Alekseevskii [1], Berndt and Suh [2];

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have classified all real hypersurfaces with these invariant properties in \( G_2(C^{m+2}) \) as follows:

**Theorem A.** Let \( M \) be a real hypersurface in \( G_2(C^{m+2}) \), \( m \geq 3 \). Then both \( \xi \) and \( Q^\perp \) are invariant under the shape operator of \( M \) if and only if

(A) \( M \) is an open part of a tube around a totally geodesic \( G_2(C^{m+1}) \) in \( G_2(C^{m+2}) \), or

(B) \( m \) is even, say \( m = 2n \), and \( M \) is an open part of a tube around a totally geodesic \( \mathbb{H}P^n \) in \( G_2(C^{m+2}) \).

In the case of (A) in Theorem A, we want to say \( M \) is of Type (A). Similarly, in the case of (B) in Theorem A, we say \( M \) is of Type (B).

Until now, by using Theorem A, many geometers have investigated some characterizations of Hopf hypersurfaces in \( G_2(C^{m+2}) \) with geometric quantities like shape operator, structure (or normal) Jacobi operator, Ricci tensor, and so on. Commuting Ricci tensor means that the Ricci tensor \( S \) and the structure tensor field \( \phi \) commute each other, that is, \( S\phi = \phi S \). From such a point of view, Suh [13] has given a characterization of real hypersurfaces of Type (A) with commuting Ricci tensor

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold \((\tilde{M}, \tilde{g})\) is an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by \((\tilde{R}_X(Y))(p) = (\tilde{R}(Y,X))X)(p)\), where \( \tilde{R} \) denotes the curvature tensor of \( \tilde{M} \) and \( X, Y \) denote any vector fields on \( \tilde{M} \). It is known to be a self-adjoint endomorphism on the tangent space \( T_p\tilde{M}, p \in \tilde{M} \). Clearly, each tangent vector field \( X \) to \( \tilde{M} \) provides a Jacobi operator with respect to \( X \). Thus the Jacobi operator on a real hypersurface \( M \) of \( G_2(C^{m+2}) \) with respect to \( \xi \) (resp. \( N \)) is said to be a structure Jacobi operator (resp. normal Jacobi operator) and will be denoted by \( R_\xi \) (resp. \( R_N \)).

For a commuting problem concerned with structure Jacobi operator \( R_\xi \) and structure tensor \( \phi \) of \( M \) in \( G_2(C^{m+2}) \), that is, \( R_\xi \phi = \phi R_\xi \), Suh and Yang [14] gave a characterization of a real hypersurface of Type (A) in \( G_2(C^{m+2}) \). Also, concerned with commuting problem for the normal Jacobi operator \( \tilde{R}_N \), Pérez, Jeong and Suh [11] gave a characterization of a real hypersurface of Type (A) in \( G_2(C^{m+2}) \).

On the other hand, another commuting problem \((R_\xi \phi)A = A(R_\xi \phi)\) (resp. \((\tilde{R}_N \phi)A = A(\tilde{R}_N \phi)\)) related to the shape operator \( A \) and the restricted structure Jacobi operator \( R_\xi \phi \) (resp. the restricted normal Jacobi operator \( \tilde{R}_N \phi \)), which can be only defined in the orthogonal complement \([\xi]^\perp\) of the Reeb vector field \([\xi]\), was recently classified in [10].

Motivated by these results, let us consider the Ricci tensor \( S \) instead of the shape operator \( A \) for \( M \) in \( G_2(C^{m+2}) \). Then as a generalization, naturally, we consider a new commuting condition for the restricted structure Jacobi operator \( R_\xi \phi \) and the Ricci tensor \( S \) defined in such a way that

\[
(R_\xi \phi)S = S(R_\xi \phi).
\]

The geometric meaning of (C-1) can be explained in such a way that any eigenspace of \( R_\xi \) on the distribution \( \mathfrak{h} = \{ X \in T_xM \mid X \perp \xi \} \), \( x \in M \), is invariant by the
Ricci tensor $S$ of $M$ in $G_2(\mathbb{C}^{m+2})$. Now we want to give a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with (C-1) as follows:

**Theorem 1.** Let $M$ be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(R_\xi \phi)S = S(R_\xi \phi)$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of $\xi$, then $M$ is locally congruent with an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Next, we want to consider another commuting condition between the restricted normal Jacobi operator $\bar{R}_N\phi$ and the Ricci tensor $S$ defined by

(C-2) \hspace{1cm} (\bar{R}_N\phi)S = S(\bar{R}_N\phi),

and give a classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with (C-2) as follows:

**Theorem 2.** Let $M$ be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of $\xi$, then $M$ is locally congruent to an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Actually, according to the geometric meaning of the condition (C-1) (resp. (C-2)), we also assert that any eigenspaces of the Ricci tensor $S$ on $M$ in $G_2(\mathbb{C}^{m+2})$ are invariant under the restricted structure Jacobi operator $R_\xi \phi$ (resp. the restricted normal Jacobi operator $\bar{R}_N\phi$). In Sections 1 and 2 we give a complete proof of Theorems 1 and 2, respectively. We refer to [1], [3], [4] and [9] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

### 1. Proof of Theorem 1

In this section, by using geometric quantities in [13] and [14], we give a complete proof of Theorem 1. To prove it, we assume that $M$ is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with (C-1), that is,

(1.1) \hspace{1cm} (R_\xi \phi)SX = S(R_\xi \phi)X.

From now on, $X, Y$ and $Z$ always stand for any tangent vector fields on $M$.

Let us introduce the Ricci tensor $S$ and structure Jacobi operator $R_\xi$, briefly. The curvature tensor $R(X, Y)Z$ of $M$ in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\bar{R}(X, Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Then by contracting and using the geometric structure $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$) related to the Kähler structure $J$ and the quaternionic Kähler structure $J_\nu$ ($\nu = 1, 2, 3$), we can derive the Ricci tensor $S$ given by

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_{4m-1}\}$ denotes a basis of the tangent space $T_xM$ of $M$ at $x \in M$, in $G_2(\mathbb{C}^{m+2})$ (see [13]).
From the definition of the Ricci tensor $S$ and fundamental formulas in [13, section 2], we have
\[
SX = \sum_{i=1}^{4m-1} R(X, e_i)e_i
\]
(1.2)
\[
= (4m + 7)X - 3\eta(X)\xi + hAX - A^2 X
\]
\[
+ \sum_{\nu=1}^{3} \left\{ -3\eta_{\nu}(X)\xi_{\nu} + \eta_{\nu}(\xi)\phi_{\nu}X - \eta(\phi_{\nu}X)\phi_{\nu}\xi - \eta(X)\eta_{\nu}(\xi)\xi_{\nu} \right\},
\]
where $h$ denotes the trace of $A$, that is, $h = TrA$ (see [12, (1.4)]). By inserting $Y = Z = \xi$ into the curvature tensor $R(X,Y)Z$ and using the condition of being Hopf, the structure Jacobi operator $R_{\xi}$ becomes
\[
R_{\xi}(X) = R(X,\xi)\xi
\]
(1.3)
\[
= X - \eta(X)\xi - \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(X)\xi_{\nu} - \eta(X)\eta_{\nu}(\xi)\xi_{\nu} \right\} + 3g(\phi_{\nu}X,\xi)\phi_{\nu}\xi + \eta_{\nu}(\xi)\phi_{\nu}X + \alpha AX - \alpha^2 \eta(X)\xi
\]
(see [3, section 4]).

Using these equations (1.1), (1.2) and (1.3), we prove that the Reeb vector field $\xi$ of $M$ belongs to either $Q$ or $Q^\perp$.

**Lemma 1.1.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with (C.1). If the principal curvature $\alpha = g(A\xi,\xi)$ is constant along the direction of $\xi$, then $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

**Proof.** In order to prove this lemma, we put
\[
(1.4) \quad \xi = \eta(X_{0})X_{0} + \eta(\xi_{1})\xi_{1}
\]
for some unit vectors $X_{0} \in Q$, $\xi_{1} \in Q^\perp$ and $\eta(X_{0})\eta(\xi_{1}) \neq 0$.

In the case of $\alpha = 0$, by virtue of $Y_\alpha = \langle \xi, \eta(Y) \rangle - 4\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(\phi Y)$ in [3, Lemma 1], we obtain easily that $\xi$ belongs to either $Q$ or $Q^\perp$.

Thus, we consider the next case $\alpha \neq 0$. Putting $X = \xi$ in (1.1) and using the fact $\phi\xi = 0$, it follows that
\[
(1.5) \quad (R_{\xi}\phi)S\xi = 0.
\]

From (1.2) and (1.4), we have
\[
(1.6) \quad S\phi X_{0} = \sigma \phi X_{0},
\]
\[
(1.7) \quad SX_{0} = (4m + 7 + h\alpha - \alpha^2)X_{0} + \eta_{1}^{2}(\xi)X_{0} - \eta(X_{0})X_{0},
\]
\[
(1.8) \quad S\xi = (4m + 4 + h\alpha - \alpha^2)\xi - 4\eta_{1}(\xi)\xi_{1},
\]
where $\sigma := 4m + 8 + h\kappa + \kappa^2$.

Multiplying $\phi$ to (1.8), we have
\[
(1.9) \quad \phi S\xi = -4\eta(\xi_{1})\phi\xi_{1}.
\]

From $\phi\xi = 0$, we obtain $\phi_{1}\xi = \eta(X_{0})\phi_{1}X_{0}$ and $\phi X_{0} = -\eta(\xi_{1})\phi_{1}X_{0}$. Because of $\eta(X_{0})\eta(\xi_{1}) \neq 0$ and (1.9), (1.5) becomes
\[
(1.10) \quad 0 = R_{\xi}(\phi\xi_{1}) = R_{\xi}(\phi_{1}X_{0}) = R_{\xi}(\phi X_{0}).
\]
By substituting $X = \phi X_0$ into (1.3) and using (1.11), we get

\begin{equation}
A\phi X_0 = -\frac{4\eta^2(X_0)}{\alpha} \phi X_0. \tag{1.11}
\end{equation}

Due to \cite{5} Equation (2.10), $A\xi_1 = \alpha \xi_1$ is derived from $\alpha = 0$. This leads to

\begin{equation}
A\phi X_0 = \kappa \phi X_0, \tag{1.12}
\end{equation}

where $\kappa = a^2 + 4a^2X_0$ (see \cite{5} section 4).

Combining (1.11) and (1.12), we obtain

\begin{equation*}
\{\alpha^2 + 8\eta^2(X_0)\} \phi X_0 = 0.
\end{equation*}

This means $\phi X_0 = 0$ which gives rise to a contradiction. Thus this lemma is proved. \hfill \Box

Now, we shall divide our consideration into two cases that $\xi$ belongs to either $Q^\perp$ or $Q$, respectively. Next, we further study the case $\xi \in Q^\perp$. We may put $\xi = \xi_1 \in Q^\perp$ for our convenience sake.

**Lemma 1.2.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field $\xi$ belongs to $Q^\perp$, then the Ricci tensor $\kappa$ commutes with the shape operator $A$, that is, $SA = AS$.

**Proof.** Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using \cite{8} section 2, (2.2) and (2.3), it gives us

\begin{equation}
\phi AX = \nabla_X \xi = \nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 A\xi. \tag{1.13}
\end{equation}

Taking the inner product with $\xi_2$ and $\xi_3$ in (1.13), respectively gives $q_3(X) = 2\eta_3(A\xi)$ and $q_2(X) = 2\eta_2(A\xi)$. Then (1.13) can be revised:

\begin{equation}
\phi AX = 2\eta_3(A\xi)\xi_2 - 2\eta_2(A\xi)\xi_3 + \phi_1 A\xi. \tag{1.14}
\end{equation}

From this, by applying the inner product with any tangent vector $Y$, we have

\begin{equation*}
g(\phi AX, Y) = 2\eta_3(A\xi)g(\xi_2, Y) - 2\eta_2(A\xi)g(\xi_3, Y) + g(\phi_1 A\xi, Y).
\end{equation*}

Then, by using the symmetric (resp. skew-symmetric) property of the shape operator $A$ (resp. the structure tensor field $\phi$), we have

\begin{equation}
-g(\xi, A\phi Y) = 2g(\xi, A\xi_3)g(\xi_2, Y) - 2g(\xi, A\xi_2)g(\xi_3, Y) - g(Y, A\phi_1 X) \tag{1.15}
\end{equation}

for any tangent vector fields $X$ and $Y$ on $M$. Then it can be rewritten as below:

\begin{equation}
A\phi X = 2\eta_3(A\xi)\xi_2 - 2\eta_2(A\xi)\xi_3 + A\phi_1 X. \tag{1.16}
\end{equation}

**Note.** Hereafter, the process used from (1.14) to (1.15) will be expressed as “taking a symmetric part of (1.14)”. Bearing in mind that $\xi = \xi_1 \in Q^\perp$, (1.2) is simplified:

\begin{equation}
SX = (4m + 7)X - 7\eta(X)\xi - 2\eta_2(X)\xi_2 - 2\eta_3(X)\xi_3 + \phi_1 \phi X + hAX - \alpha^2 X. \tag{1.16}
\end{equation}

Multiplying $\phi_1$ to (1.16) and using basic formulas in \cite{7} Section 2, we have

\begin{equation}
\phi_1 \phi AX = 2\eta_3(A\xi)\xi_3 + 2\eta_2(A\xi)\xi_2 - AX + \alpha\eta(X)\xi. \tag{1.17}
\end{equation}

By replacing $X$ as $AX$ into (1.16) and using (1.17), we obtain

\begin{equation}
SAX = (4m + 6)AX - 6\alpha\eta(X)\xi + hA^2X - A^3 X \tag{1.18}
\end{equation}
and taking a symmetric part of (1.18) again, we get
(1.19) \[ ASX = (4m + 6)AX - 6\alpha \eta(X)\xi + hA^2X - A^3X. \]

Comparing (1.18) and (1.19), we conclude that
\[ SAX = ASX \]
for any tangent \( X \).

By the way, we have equations (1.13) and (1.15) for the Ricci tensor likewise related to the shape operator. We may consider similar ones about the Ricci tensor as below:

**Lemma 1.3.** Let \( M \) be a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \). If the Reeb vector field \( \xi \) belongs to \( Q^\perp \), we have the following formulas

(i) \( \phi SX = 2n_3(SX)\xi_2 - 2n_2(SX)\xi_3 + \phi_1 SX + \text{Rem}(X) \)

(ii) \( S\phi X = 2n_3(SX)\xi_2 - 2n_2(SX)\xi_3 + S\phi_1 X + \text{Rem}(X) \),

where the remainder term \( \text{Rem}(X) \) is denoted by
\[
\text{Rem}(X) = 4(m + 2)\{2n_2(X)\xi_3 - 2n_3(X)\xi_2 + \phi X - \phi_1 X\}.
\]

**Proof.** Multiplying \( \phi \) to (1.16), we get the equivalent equation of the Left side of (i) as follows:
\[
\phi SX = (4m + 7)\phi X - \phi_1 X + 2n_2(X)\xi_3 - 2n_3(X)\xi_2 + h\phi AX - \phi_1 AX.
\]

Using (1.14), and (1.15), the right side of (i) is can be replaced by
\[
\begin{align*}
\phi SX &= 2n_3(SX)\xi_2 - 2n_2(SX)\xi_3 + \phi_1 SX + \text{Rem}(X) \\
&= 2n_3((4m + 7)X - 2n_3(X)\xi_3 + \phi_1 \phi X + hAX - A^2X)\xi_2 \\
&\quad - 2n_2((4m + 7)X - 2n_2(X)\xi_2 + \phi_1 \phi X + hAX - A^2X)\xi_3 \\
&\quad + (4m + 7)\phi_1 X - 2n_2(X)\xi_2 + 2n_3(X)\xi_3 - \phi X + h\phi_1 AX - \phi_1 A^2X \\
&\quad + \text{Rem}(X) \\
&= (4m + 7)\phi X - \phi_1 X + 2n_2(X)\xi_3 - 2n_3(X)\xi_2 + h\phi AX - \phi_1 AX.
\end{align*}
\]

Combining (1.20) and (1.21), we get the equation (i). In addition, (ii) can be obtained by taking a symmetric part of (i).

By virtue of Lemmas 1.2 and 1.3, we assert the following:

**Lemma 1.4.** Let \( M \) be a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with (C-1). If \( \xi \in Q^\perp \), we have \( A(\phi S - S\phi) = (\phi S - S\phi)A \).

**Proof.** By (i) (resp. (ii)) in Lemma 1.3, we have the left side of (1.1) (the right side of (1.1)) as follows:
\[
\begin{align*}
R_\xi \phi SX &= 2\phi SX + \alpha A\phi SX + \text{Rem}(X), \\
SR_\xi \phi X &= 2S\phi X + \alpha SA\phi X + \text{Rem}(X).
\end{align*}
\]

Combining equations in (1.22), we have
(1.23) \[ R_\xi \phi SX - SR_\xi \phi X = 2\phi SX + \alpha A\phi SX - 2S\phi X - \alpha SA\phi X = 0. \]
Case 1: $\alpha = 0$. Equation (1.23) becomes $S\phi X = \phi SX$. By virtue of Theorem [13], we conclude that if $M$ is a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with (1.1), then $M$ satisfies the condition of Type (A).

Thus, we may assume the following case.

Case 2: $\alpha \neq 0$.

Using Lemma 1.2 becomes

\[(1.24) \quad 2(\phi S - S\phi) + \alpha(A\phi S - AS\phi) = 0.\]

Taking a symmetric part of (1.24), we have

\[(1.25) \quad 2(\phi S - S\phi) - \alpha(S\phi A - \phi SA) = 0.\]

Combining (1.24) and (1.25), we know

\[(1.26) \quad A(\phi S - S\phi) = (\phi S - S\phi)A.\]

Lemma 1.5. Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If $M$ satisfies $A(\phi S - S\phi) = (\phi S - S\phi)A$ and $\xi \in Q^\perp$, then we have $S\phi X = \phi SX$.

Proof. Since the shape operator $A$ and the tensor $\phi S - S\phi$ are both symmetric operators and commute with each other, they are diagonalizable. So there exists a common basis $\{E_1, E_2, ..., E_{4m-1}\}$ such that the shape operator $A$ and the tensor $\phi S - S\phi$ both can be diagonalizable. In other words, $AE_i = \lambda_i E_i$ and $(\phi S - S\phi)E_i = \beta_i E_i$, where $\lambda_i$ and $\beta_i$ are scalars for all $i \in 1, 2, ..., 4m-1$.

Here replacing $X$ by $\phi X$ in (1.16) (resp. multiplying $\phi$ to (1.16)), we have

\[(1.27) \quad \begin{cases} S\phi X = (4m + 7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + hA\phi X - A^2\phi X, \\
\phi SX = (4m + 7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + h\phi AX - \phi A^2 X. \end{cases}\]

Combining equations in (1.27), we get

\[(1.28) \quad S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X.\]

Putting $X = E_i$ into (1.28) and using $AE_i = \lambda_i E_i$, we obtain

\[(1.29) \quad (S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i\phi E_i + \phi \lambda_i^2 E_i.\]

Taking the inner product with $E_i$ into (1.29), we have

\[\beta_ig(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) = 0.\]

Since $g(E_i, E_i) \neq 0$, $\beta_i = 0$ for all $i \in 1, 2, ..., 4m-1$. This is equivalent to $(S\phi - \phi S)E_i = 0$ for all $i \in 1, 2, ..., 4m-1$. It follows that $S\phi X = \phi SX$ for any tangent vector field $X$ on $M$.

Summing up Lemmas [1.2, 1.3, 1.4, 1.5] and [13, Theorem], we conclude that if $M$ is a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying (C-1), then $M$ satisfies the condition of Type (A).

Hereafter, let us check whether the Ricci tensor of a model space of Type (A) satisfies the commuting condition (C-1).
From (1.2) and [3, Proposition 3], we obtain the following equations:

\[
SX = \begin{cases}
(4m + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\
(4m + 6 + h\beta - \beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\
(4m + 6 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\
(4m + 8)X & \text{if } X \in T_\mu 
\end{cases}
\]

\[
R_\xi(X) = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
(\alpha\beta + 2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\
(\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\
0 & \text{if } X \in T_\mu 
\end{cases}
\]

\[
(R_\xi\phi)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
(\alpha\beta + 2)\phi_\nu & \text{if } X = \xi_\nu \in T_\beta \\
(\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\
0 & \text{if } X \in T_\mu 
\end{cases}
\]

Combining above three formulas, it follows that

\[
(R_\xi\phi)SX - S(R_\xi\phi)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
0 & \text{if } X = \xi_\nu \in T_\beta \\
0 & \text{if } X \in T_\lambda \\
0 & \text{if } X \in T_\mu.
\end{cases}
\]

Remark 1.6. When \( \xi \in Q^{-1} \), a Hopf hypersurface \( M \) in \( G_2(C^{m+2}) \) with (C-1) is locally congruent to of Type (A) by virtue of [13, Theorem].

When \( \xi \in Q \), a Hopf hypersurface \( M \) in \( G_2(C^{m+2}) \) with (C-1) is locally congruent to of Type (B) by virtue of [13, Main Theorem].

Now let us consider our problem for a model space of Type (B) which will be denoted by \( M_B \). In order to do this, let us calculate \( (R_\xi\phi)S = SR_\xi\phi \) related to the \( M_B \). On \( T_xM_B, x \in M_B \), the equations (1.2) and (1.3) are reduced to the following equations, respectively:

\[
(1.30) \quad SX = (4m + 7)X - 3\eta(X)\xi + hAX - A^2X - \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu \xi\}
\]

\[
(1.31) \quad R_\xi(X) = X - \eta(X)\xi + \alpha AX - \alpha^2\eta(X)\xi - \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu \xi\}.
\]

From (1.30) and (1.31) and [3, Proposition 2], we obtain the following

\[
(1.32) \quad SX = \begin{cases}
(4m + 4 + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\
(4m + 4 + h\beta - \beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\
(4m + 8)\phi_\nu & \text{if } X = \phi_\nu \in T_\gamma \\
(4m + 7 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\
(4m + 7 + h\mu - \mu^2)X & \text{if } X \in T_\mu
\end{cases}
\]
\[ R_\xi(X) = \begin{cases} 
0 & \text{if } X = \xi \in T_\alpha \\
\alpha \beta \xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\
4\phi_\xi & \text{if } X = \phi_\xi \in T_\gamma \\
(1 + \alpha \lambda)\phi X & \text{if } X \in T_\lambda \\
(1 + \alpha \mu)\phi X & \text{if } X \in T_\mu 
\end{cases} \]

\[ (R_\xi \phi)X = \begin{cases} 
0 & \text{if } X = \xi \in T_\alpha \\
4\phi_\xi & \text{if } X = \xi_\ell \in T_\beta \\
-\alpha \beta \xi_\ell & \text{if } X = \phi_\xi \in T_\gamma \\
(1 + \alpha \mu)(\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\
(1 + \alpha \lambda)(\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. 
\end{cases} \]

From (1.32), (1.33) and (1.34), it follows that

\[ (R_\xi \phi)SX - SR_\xi \phi X = \begin{cases} 
0 & \text{if } X = \xi \in T_\alpha \\
4(h\beta - \beta^2 - 4)\phi_\xi & \text{if } X = \xi_\ell \in T_\beta \\
\alpha \beta(4h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi_\xi \in T_\gamma \\
(1 + \alpha \mu)(\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\
(1 + \alpha \lambda)(\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. 
\end{cases} \]

By calculation, we have \( \lambda + \mu = \beta \) on \( M_B \). From (1.35), we see that \( M_B \) satisfies (C-1), only when \( h = \beta \) and \( h\beta - \beta^2 - 4 = 0 \). This gives us to a contradiction.

Hence, we give a complete proof of Theorem 1.

2. Proof of Theorem 2

For a commuting problem in quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator \( \tilde{R}(X,N)N \in T_xM, \ x \in M \) for real hypersurfaces \( M \) in quaternionic projective space \( QP^m \) or in quaternionic hyperbolic space \( QH^m \), where \( \tilde{R} \) denotes the curvature tensor of \( QP^m \) or of \( QH^m \). He [2] has also shown that the curvature adaptedness, when the normal Jacobi operator commutes the shape operator \( A \), is equivalent to the fact that the distributions \( Q \) and \( Q^A = \text{Span}\{\xi_1, \xi_2, \xi_3\} \) are invariant by the shape operator \( A \) of \( M \), where \( T_xM = Q \oplus Q^\perp, \ x \in M \). In this section, by using the notion of normal Jacobi operator \( \tilde{R}(X,N)N \in T_xM, \ x \in M \) for real hypersurfaces \( M \) in \( G_2(\mathbb{C}^{m+2}) \) and geometric quantities in [11] and [13], we give a complete proof of Theorem 2.

From now on, let \( M \) be a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) with

\[ \tilde{R}(X,N)X = S(\tilde{R}(X,N)X) \]

for any tangent vector field \( X \) on \( M \). The normal Jacobi operator \( \tilde{R}_N \) of \( M \) is defined by \( \tilde{R}_N(X) = \tilde{R}(X,N)N \) for any tangent vector \( X \in T_xM, \ x \in M \). In [11] Introduction, we obtain the following equation

\[ \tilde{R}_N(X) = X + 3\eta(X)\xi + 3 \sum_{\nu=1}^3 \eta_{\nu}(X)\xi_{\nu} \]

\[ - \sum_{\nu=1}^3 \{ \eta_{\nu}(\xi)\phi_{\nu}X - \eta_{\nu}(\xi)\eta(X)\xi_{\nu} - \eta_{\nu}(\phi_{\nu}X)\phi_{\nu}\xi \}. \]
Lemma 2.1. Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\sigma(\xi) = g(A\xi, \xi)$ is constant along the direction of $\xi$, then $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

Proof. In order to prove this lemma, we assume (1.4) again, for some unit vectors $X_0 \in Q, \xi_1 \in Q^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

On the other hand, from (2.2) and (1.4), we have
\begin{align}
(2.3) & \quad \hat{R}_N X_0 = 4\eta^2(X_0)X_0 + 4\eta(\xi)\eta(X_0)\xi_1 \\
(2.4) & \quad \hat{R}_N \xi = 4\xi + 4\eta(\xi)\xi_1.
\end{align}

Using (1.7), (1.8), (2.3), (2.4) and inserting $X = \phi X_0$ into (2.1), we have the following equations:

the left side of (2.1) = $(\hat{R}_N \phi)S\phi X_0 = \sigma \hat{R}_N \phi^2 X_0$
\begin{align}
&= -\sigma \hat{R}_N X_0 + \eta(X_0)\hat{R}_N \xi \\
&= -\sigma \{4\eta^2(X_0)X_0 + 4\eta(\xi)\eta(X_0)\xi_1\} \\
&\quad + \sigma \{4\eta(X_0)\xi + 4\eta(X_0)\eta(\xi)\xi_1\} \\
&= 4\sigma\eta(X_0)\eta_1(\xi)\xi_1
\end{align}

the right side of (2.1) = $S\hat{R}_N(\phi^2 X_0) = -S\hat{R}_N X_0 + \eta(X_0)S\hat{R}_N \xi$
\begin{align}
&= -4\eta^2(X_0)SX_0 - 4\eta(\xi)\eta(X_0)S\xi_1 \\
&\quad + 4\eta(X_0)S\xi + 4\eta(X_0)\eta(\xi)S\xi_1 \\
&= -4\eta^2(X_0)\{4m + 7 + \alpha h - \alpha^2\}X_0 - 3\eta(X_0)\xi \\
&\quad + \eta_1(\xi)X_0 - \eta(X_0)\eta_1(\xi)\xi_1 \\
&\quad + 4\eta(X_0)\{4m + 4 + \alpha h - \alpha^2\}\xi - 4\eta_1(\xi)\xi_1,
\end{align}

where $\sigma := 4m + 8 + h\kappa + \kappa^2$. Recalling that $\eta(X_0) \neq 0$ and combining (2.4) and (2.6), we have
\begin{align}
4\sigma\eta(X_0)\eta_1(\xi)\xi_1 &= -4\eta^2(X_0)\{4m + 7 + \alpha h - \alpha^2\}X_0 - 3\eta(X_0)\xi \\
&\quad + \eta_1^2(\xi)X_0 - \eta(X_0)\eta_1(\xi)\xi_1 \\
&\quad + 4\eta(X_0)\{4m + 4 + \alpha h - \alpha^2\}\xi - 4\eta_1(\xi)\xi_1.
\end{align}

Taking the inner product of above equation with $X_0$, we get
\begin{align}
0 &= -4\eta(X_0)\{4m + 7 + \alpha h - \alpha^2\} - 3\eta^2(X_0) + \eta_1^2(\xi) \\
&\quad + 4\{4m + 4 + \alpha h - \alpha^2\}\eta(X_0) \\
&= -4\eta(X_0)\{3 - 3\eta^2(\xi) + \eta_1^2(\xi)\} \\
&= -16\eta(X_0)\eta_1^2(\xi).
\end{align}

This gives a contradiction. Thus, we give a complete proof of this lemma. \( \Box \)

Now this case implies that $\xi$ belongs to the distribution $Q^\perp$.

Lemma 2.2. Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with (2.1). If $\xi \in Q^\perp$, we have $S\phi = \phi S$. 

Proof. Putting $\xi = \xi_1 \in Q^\perp$ for our convenience sake, (2.2) becomes
\[ \bar{R}_N(X) = X + 7\eta(X)\xi + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 - \phi_1 \phi X. \]

Because of (i) and (ii) in lemma 1.3 we have the following equations:
\begin{equation}
\begin{cases}
\bar{R}_N \phi SX = 2\phi SX - \text{Rem}(X), \\
S\bar{R}_N \phi X = 2S\phi X - \text{Rem}(X),
\end{cases}
\tag{2.7}
\end{equation}

where \( \text{Rem}(X) = 4(m + 2)\{2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + \phi X - \phi_1 X\} \).

Combining equations in (2.7), we conclude that (2.1) is equivalent to \( S\phi X = \phi SX \). \( \square \)

In the case of $\xi \in Q^\perp$, by using (i) and (ii) in Lemma 1.3, and Lemma 2.2, we can be easily seen that the commuting condition \( S\phi = \phi S \) is equivalent to \( (\bar{R}_N \phi)S = S(\bar{R}_N \phi) \).

Therefore, by Lemma 2.2 and [13, Theorem], we can assert that:

**Remark 2.3.** Real hypersurfaces of Type (A) in $G_2(C^{m+2})$ satisfies the condition (C-2).

When $\xi \in Q$, a Hopf hypersurface $M$ in $G_2(C^{m+2})$ with (C-2) is locally congruent to of Type (B) by virtue of [9, Main Theorem].

Let us consider our problem for a model space of Type (B) which will be denoted by $M_B$. In order to do this, let us calculate \( (\bar{R}_N \phi)S = S(\bar{R}_N \phi) \) of $M_B$. From [3, Proposition 2], we obtain
\begin{equation}
\bar{R}_N(X) = \begin{cases}
4\xi & \text{if } X = \xi \in T_\alpha \\
4\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\
0 & \text{if } X = \phi \xi_\ell \in T_\gamma \\
X & \text{if } X \in T_\lambda, \\
X & \text{if } X \in T_\mu,
\end{cases}
\tag{2.8}
\end{equation}

\begin{equation}
(\bar{R}_N \phi)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
0 & \text{if } X = \xi_\ell \in T_\beta \\
-4\xi_\ell & \text{if } X = \phi \xi_\ell \in T_\gamma \\
\phi X & \text{if } X \in T_\lambda \\
\phi X & \text{if } X \in T_\mu.
\end{cases}
\tag{2.9}
\end{equation}

From (2.8) and (2.9), it follows that
\begin{equation}
(\bar{R}_N \phi)SX - S(\bar{R}_N \phi)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
0 & \text{if } X = \xi_\ell \in T_\beta \\
4(h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi \xi_\ell \in T_\gamma \\
(\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\
(\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu.
\end{cases}
\end{equation}

We see that $M_B$ satisfies (C-2), only when $h = \beta$ and $h\beta - \beta^2 - 4 = 0$. This gives us to a contradiction.

Thus, we can give a complete proof of Theorem 2 in the introduction.
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