PLANE MODEL-FIELDS OF DEFINITION, FIELDS OF DEFINITION, THE FIELD OF MODULI OF SMOOTH PLANE CURVES

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Abstract. Given a smooth plane curve $C$ of genus $g \geq 3$ over an algebraically closed field $\mathbb{k}$, a field $L \subseteq \mathbb{k}$ is said to be a plane model-field of definition for $C$ if $L$ is a field of definition for $C$, i.e. $\exists$ a smooth curve $C'$ defined over $L$ where $C' \times_L \mathbb{k} \cong C$, and such that $C'$ is $L$-isomorphic to a non-singular plane model $F(X,Y,Z) = 0$ in $\mathbb{P}^2_L$.

In this short note, we construct a smooth plane curve $C$ over $\mathbb{k}$, such that the field of moduli of $C$ is not a field of definition for $C$, and also fields of definition do not coincide with plane model-fields of definition for $C$. As far as we know, this is the first example in the literature with the above property, since this phenomenon does not occur for hyperelliptic curves, replacing plane model-fields of definition with the so-called hyperelliptic model-fields of definition.

1. Introduction

Consider $F$ the base field for an algebraically closed field $\mathbb{k}$. Let $F \subseteq L \subseteq \mathbb{k}$ be fields, given a smooth projective curve $C$ over $\mathbb{k}$, then $C$ is defined over $L$ if and only if there is a curve $C'$ over $L$ such that $C'$ is $\mathbb{k}$-isomorphic to $C$, i.e. $C' \times_L \mathbb{k} \cong C$. In such case, $L$ is called a field of definition of $C$. We say that $C$ is definable over $L$ if there is a curve $C'/L$ such that $C$ and $C' \times_L \mathbb{k}$ are $\mathbb{k}$-isomorphic.

Definition 1.1. The field of moduli of a smooth projective curve $C$ defined over $\mathbb{k}$, denoted by $K_{\bar{C}}$, is the intersection of all fields of definition of $C$.

It becomes very natural to ask when the field of moduli of a smooth projective curve $C$ is also a field of definition. A necessary and sufficient condition (Weil’s cocycle criterion of descent) for the field of moduli to be a field of definition was provided by Weil [12]. If $\text{Aut}(\bar{C})$ is trivial, then this condition becomes trivially true and so the field of moduli needs to be a field of definition. It is also quite well known that a smooth curve $C$ of genus $g = 0$ or $1$ can be defined over its field of moduli, where $g$ is the geometric genus of $C$. However, if $g > 1$ and $\text{Aut}(\bar{C})$ is non-trivial, then Weil’s conditions are difficult to be checked and so there is no guarantee that the field of moduli is a field of definition for $\bar{C}$. This was first pointed out by Earle [4] and Shimura [11]. More precisely, in page 177 of [11], the first examples not definable over their field of moduli are introduced, which are hyperelliptic curves over $\mathbb{k}$ with two automorphisms. There are also examples of non-hyperelliptic curves not definable over their field of moduli given in [2][3]. B. Huggins [6] studied this problem for hyperelliptic curves over a field $\mathbb{k}$ of characteristic $p \neq 2$, proving that a hyperelliptic curve $\bar{C}$ of genus $g \geq 2$ with hyperelliptic involution $i$ can be defined over $K_{\bar{C}}$ when $\text{Aut}(\bar{C})/\langle i \rangle$ is not cyclic or is cyclic of order divisible by $p$

On the other hand, one may define fields of definition of models of the same concrete type for a smooth projective curve $\bar{C}$. For example, if $\bar{C}$ is hyperelliptic, a field $M$ is called a hyperelliptic model-field of definition for $\bar{C}$ if $M$, as a field of definition for $\bar{C}$, satisfies that $\bar{C}$ is $M$-isomorphic to a hyperelliptic model of the form $y^2 = f(x)$, for some polynomial $f(x)$ of degree $2g + 1$ or $2g + 2$.

By the work of Mestre [10], Huggins [5][6], Lercier-Ritzenthaler [7], Lercier-Ritzenthaler-Sijsling [8] and Lombardo-Lorenzo in [9], one gets fair-enough characterizations for the interrelations between the three fields: the field of moduli, fields of definition and hyperelliptic model-fields of definition. For instance, if $\bar{C}$ is hyperelliptic, then there are always two of these fields, which are equal. Summing up, one obtains the next table issued from Lercier-Ritzenthaler-Sijsling [8], where $k = F$ is a perfect field of characteristic $\text{char}(F) \neq 2$:

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| $H = \text{Aut}(\overline{C})/(i)$ | Conditions | Fields of definition = | The field of moduli= |
|---|---|---|---|
| Not tamely cyclic | | Yes | Yes |
| Tamely cyclic with $\#H > 1$ | $g \text{ odd, } \#H \text{ odd}$ | No | Yes |
| | $g \text{ even or } \#H \text{ even}$ | Yes | No |
| Tamely cyclic with $\#H = 1$ | $g \text{ odd}$ | No | Yes |
| | $g \text{ even}$ | Yes | No |

By **tamely cyclic**, we mean that the group is cyclic of order not divisible by the $\text{char}(F)$.

Now, consider a smooth plane curve $\overline{C}$, i.e. $\overline{C}$ viewed as a smooth curve over $\mathbb{k}$ admits a non-singular plane model defined by an equation of the form $F(X,Y,Z) = 0$ in $\mathbb{P}^2_{\mathbb{k}}$, where $F(X,Y,Z)$ is a homogenous polynomial of degree $d \geq 4$ over $\mathbb{k}$ with $g = \frac{1}{2}(d-1)(d-2) \geq 3$. Similarly, we define a so-called **plane model-fields of definition for** $C$:

**Definition 1.2.** Given a smooth plane curve $\overline{C}$ over $\overline{\mathbb{k}}$, a subfield $\mathbb{M} \subset \overline{\mathbb{k}}$ is said to be a **plane model-field of definition for** $C$ if and only if the following conditions holds

(i) $\mathbb{M}$ is a field of definition for $\overline{C}$.

(ii) $\exists$ a smooth curve $C'$ defined over $\mathbb{M}$, which is $\overline{\mathbb{k}}$-isomorphic to $\overline{C}$, and $\overline{\mathbb{k}}$-isomorphic to a non-singular plane model $F(X,Y,Z) = 0$, for some homogenous polynomial $F(X,Y,Z) \in M[X,Y,Z]$ of degree $d \geq 3$.

In this short note, we start with a smooth plane curve $\overline{C}$ over $\overline{\mathbb{k}}$ where the field of moduli is not a field of definition by the work of B. Huggins in [5]. Next, we go further, following the techniques developed in [1], to construct a twist of $\overline{C}$, for which there is a field of definition for $\overline{C}$, which is not a plane model-field of definition.

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2. The example

Consider the **Hessian group of order** 18, denoted by $\text{Hess}_{18}$, which is $\text{PGL}_3(\overline{\mathbb{k}})$-conjugate to the group generated by

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad R := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. $$

First, we reproduce an example, by B. Huggins in [5] Chp. 7, §2], of a smooth $\overline{\mathbb{k}}$-plane curve of genus 10 not definable over its field of moduli, and with full automorphism groups $\text{Hess}_{18}$.

**Definition 2.1.** A quaternion extension of a field $K$ is a Galois extension $K'/K$ such that $\text{Gal}(K'/K)$ is isomorphic to the quaternion group to order 8.

**Definition 2.2.** ([5] Lemma 7.2.3]) A field $K$ is of level 2 if $-1$ is not a square in $K$, but it is a sum of two squares in $K$.

**Lemma 2.3.** ([5] Lemma 7.2.3]) Let $K$ be a field of level 2. Then, for $u, v \in K^* \setminus (K^*)^2$ such that $uv \notin (K^*)^2$, $K(\sqrt{u}, \sqrt{v})$ is embeddable into a quaternion extension of $K$ if and only if $-u$ is a norm from $K(\sqrt{-v})$ to $K$ (i.e. $-u = x^2 + vy^2$ for some $x, y \in K$).

For instance, the field $K := \mathbb{Q}(\zeta_3)$ is of level 2, since $(\zeta_3^2)^2 + \zeta_3^2 = -1$ and $\sqrt{-1} \notin K$. It is easily shown that $\pm 2$ are not norms from $K(\sqrt{-13})$ to $K$. So neither $K(\sqrt{2}, \sqrt{13})$ nor $K(\sqrt{-2}, \sqrt{13})$ are embeddable into a quaternion extension of $K$.

Now fix $K$ to be the field $\mathbb{Q}(\zeta_3)$, and define the following:

$$\phi := XYZ, $$

$$\psi := X^3 + Y^3 + Z^3, $$

$$\chi := (XY)^3 + (YZ)^3 + (XZ)^3.$$
Suppose that \( u, v \in \mathbb{Q}^* \), such that \( L := K(\sqrt{u}, \sqrt{v}) \) is a \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) extension of \( K \) that can not be embedded into a quaternion extension of \( K \). Let

\[
\begin{align*}
\phi_\varphi & := \zeta_3 \sqrt{u} + \sqrt{v} + c_\varphi \sqrt{uv}, \\
\phi_\psi & := \zeta_3 \sqrt{u} + \sqrt{v} + \zeta_3 \sqrt{uv}, \\
\psi_\varphi & := \sqrt{u} + \sqrt{v} + \frac{c_\psi}{12}.
\end{align*}
\]

Fix an algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \) containing \( L \) as above.

**Theorem 2.4.** (B. Huggins, [Lemma 7.2.5 and Proposition 7.2.6]) Following the above notations, let

\[
F_{\sqrt{u}, \sqrt{v}}(X, Y, Z) := c_{\varphi \varphi} \phi^2 - 6c_{\varphi \psi} \phi \psi - 18c_{\psi \psi} \psi^2 + \chi.
\]

Then the equation \( F_{\sqrt{u}, \sqrt{v}}(X, Y, Z) = 0 \) such that \( F_{\sqrt{u}, \sqrt{v}}(X, 1, 1) \) is square free, defines a smooth \( \overline{\mathbb{Q}} \)-plane curve \( \overline{C} \) over \( \overline{\mathbb{Q}} \), with automorphism group \( \text{Hess}_{18} \). The field of moduli \( K_{\overline{C}} \) is \( K = \mathbb{Q}(\zeta_3) \), but it is not a field of definition.

**Remark 2.5.** The condition that \( F_{\sqrt{u}, \sqrt{v}}(X, 1, 1) \) is square free is possible. For example, with \( u = 2 \) and \( v = 13 \), the resultant of \( F_{\sqrt{u}, \sqrt{v}}(X, 1, 1) \) and \( \frac{F_{\sqrt{u}, \sqrt{v}}}{\sqrt{u}}(X, 1, 1) \) is not zero.

**Lemma 2.6.** Let \( \overline{C} \) be a smooth curve defined over an algebraically closed field \( \overline{\mathbb{F}} \), with \( F = k \) and \( k \) perfect. An \( \mathbb{F} \)-isomorphism \( \phi : \overline{C} \to \overline{C} \) does not change the field of moduli or fields of definition, that is both \( \overline{C} \) and \( \overline{C}' \) have the same fields of moduli and fields of definitions.

**Proof.** A field \( L \subseteq \overline{\mathbb{F}} \) is a field of definition for \( \overline{C} \) if and only if there exists a smooth curve \( C'' \) over \( L \), such that \( C'' \times_L \overline{\mathbb{F}} \) is \( \mathbb{F} \)-isomorphic to \( \overline{C} \) through some \( \psi : C'' \times_L \overline{\mathbb{F}} \to \overline{C} \). Hence \( \phi^{-1} \circ \psi : C'' \times_L \overline{\mathbb{F}} \) is a \( \mathbb{F} \)-isomorphism, and \( L \) is a field of definition for \( \overline{C} \). The converse is true by a similar discussion. Consequently, the field of moduli for \( \overline{C} \) and \( \overline{C}' \) coincides, being the intersection of all fields of definition. \( \square \)

**Corollary 2.7.** Consider a smooth \( \overline{\mathbb{Q}} \)-plane curve \( \overline{C} \) defined by an equation of the form

\[
\frac{c_{\varphi \varphi}}{p^2}(XYZ)^2 - \frac{6c_{\varphi \psi}}{p}(XYZ)(X^3 + \frac{1}{p} Y^3 + \frac{1}{p^2} Z^3) - 18c_{\psi \psi}(X^3 + \frac{1}{p} Y^3 + \frac{1}{p^2} Z^3)^2 + \frac{1}{p} X^3 Y^3 + \frac{1}{p^3} Y^3 Z^3 + \frac{1}{p^3} X^3 Z^3 = 0,
\]

where \( p \in \mathbb{Q} \), in particular \( \overline{C} \) admits \( \mathbb{Q}(\sqrt{u}, \sqrt{v}, \zeta_3) \) as a plane model-field of definition for \( \overline{C} \). Then \( \text{Aut}(\overline{C}) \) is isomorphic to \( \text{Hess}_{18} \). Moreover, the field of moduli \( K_{\overline{C}} \) is \( K = \mathbb{Q}(\zeta_3) \), but it is not a field of definition. \( \square \)

**Theorem 2.8.** Consider the family \( C_p \) of smooth plane curves over the plane model-field of definition \( L = \mathbb{Q}(\sqrt{u}, \sqrt{v}, \zeta_3) \) given by an equation of the form

\[
\frac{c_{\varphi \varphi}}{p^2}(XYZ)^2 - \frac{6c_{\varphi \psi}}{p}(XYZ)(X^3 + \frac{1}{p} Y^3 + \frac{1}{p^2} Z^3) - 18c_{\psi \psi}(X^3 + \frac{1}{p} Y^3 + \frac{1}{p^2} Z^3)^2 + \frac{1}{p} X^3 Y^3 + \frac{1}{p^3} Y^3 Z^3 + \frac{1}{p^3} X^3 Z^3 = 0,
\]

where \( p \) is a prime integer such that \( p \equiv 3 \) or \( 5 \mod 7 \). Given a smooth plane curve \( C' \) over \( L \) in \( C_p \), then there exists a twist \( C'' \) of \( C \) over \( L \) which does not have \( L \) as a plane model-field of definition. Moreover, the field of moduli of \( C'' \) is \( \mathbb{Q}(\zeta_3) \), and it is not a field of definition for \( C'' \).

**Proof.** Consider the Galois extension \( M'/L \) with \( M' = L(\cos(2\pi/7), \sqrt[p]{u}) \), where all the automorphisms of \( \overline{C} := C \times_L \overline{\mathbb{Q}} \) are defined. Let \( \sigma \) be a generator of the cyclic Galois group \( \text{Gal}(L(\cos(2\pi/7)))/L) \). We define a 1-cocycle on \( \text{Gal}(M'/L) \cong \text{Gal}(L(\cos(2\pi/7))/L) \times \text{Gal}(L(\sqrt[u]{u})/L) \) by mapping \( (\sigma, id) \mapsto [Y : pX] \) and \( (id, \tau) \mapsto id \). This defines an element of \( \text{H}^1(\text{Gal}(M'/L), \text{Aut}(\overline{C})) \), coming from the inflation of an element in \( \text{H}^1(\text{Gal}(L(\cos(2\pi/7))/L), \text{Aut}(\overline{C})) \).

This 1-cocycle is trivial if and only if \( p \) is a norm of an element of \( L(\cos(2\pi/7)) \) over \( L \). However, this is not the case, since \( \mathbb{Q}(\cos(2\pi/7)) \) and \( L \) are disjoint with \( [L : \mathbb{Q}] = 2 \) and \( [\mathbb{Q}(\cos(2\pi/7)) : \mathbb{Q}] = 7 \) coprime, and moreover \( p \) is...
not a norm of an element of $\mathbb{Q}(\cos(2\pi/7))$ over $\mathbb{Q}$ being inert by our assumption. Consequently, the twist $C'$ is not $L$-isomorphic to a non-singular plane model in $\mathbb{P}^2_L$ by [1] Theorem 4.1. That is, $L$ is not a plane model-field of definition for $C'$. The last sentence in the theorem follows by Lemma 2.6 and Corollary 2.7. □

**Remark 2.9.** By our work in [7], we know that a non-singular plane model of $C'$ exists over at least a degree 3 extension of $L$.

**References**

[1] E. Badr, F. Bars, E. Lorenzo García, *On twists of smooth plane curves*, arXiv:1603.08711v1.
[2] R. Hidalgo, *Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals*, Arch. Math. 93 (2009), 219-224.
[3] B. Huggins; *Fields of moduli and fields of definition of curves*. PhD thesis, Berkeley (2005), see [http://arxiv.org/abs/math/0610247v1](http://arxiv.org/abs/math/0610247v1).
[4] C. J. Earle, *On the moduli of closed Riemann surfaces with symmetries*, Advances in the Theory of Riemann Surfaces. Ann. Math. Studies 66 (1971), 119-130.
[5] B. Huggins, *Fields of moduli and fields of definition of curves*. PhD thesis, Berkeley (2005), arxiv.org/abs/math/0610247v1.
[6] B. Huggins; *Fields of moduli of hyperelliptic curves*. Math. Res. Lett. 14 (2007), 249-262.
[7] R. Lercier and C. Ritzenthaler, *Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects*. J. Algebra, 372:595636, 2012.
[8] R. Lercier, C. Ritzenthaler, and J. Sijsling, *Explicit galois obstruction and descent for hyperelliptic curves with tamely cyclic reduced automorphism group*. Math. Comp. To appear.
[9] D. Lombardo, E. Lorenzo García; *Computing twists of hyperelliptic curves*, arXiv:1611.04866 November 2016.
[10] J.-F. Mestre. *Construction de courbes de genre 2 a partir de leurs modules*. In Effective methods in algebraic geometry (Castiglioncello, 1990) , volume 94 of Progr. Math. , pages 313-334. Birkhäuser Boston, Boston, MA, 1991.
[11] G. Shimura, *On the field of rationality for an abelian variety*, Nagoya Math. J. 45 (1971), 167-178.
[12] A. Weil, *The field of definition of a variety*, American J. of Math. vol. 78, n17 (1956), 509-524.

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