SPLITTING MULTIDIMENSIONAL NECKLACES

MARK DE LONGUEVILLE AND RADE T. ŽIVALJEVIĆ

Abstract. The well-known “splitting necklace theorem” of Alon [1] says that each necklace with $k \cdot a_i$ beads of color $i = 1, \ldots, n$ can be fairly divided between $k$ “thieves” by at most $n(k - 1)$ cuts. Alon deduced this result from the fact that such a division is possible also in the case of a continuous necklace $[0,1]$ where beads of given color are interpreted as measurable sets $A_i \subset [0,1]$ (or more generally as continuous measures $\mu_i$). We demonstrate that Alon’s result is a special case of a multidimensional, consensus division theorem of $n$ continuous probability measures $\mu_1, \ldots, \mu_n$ on a $d$-cube $[0,1]^d$. The dissection is performed by $m_1 + \ldots + m_d = n(k - 1)$ hyperplanes parallel to the sides of $[0,1]^d$ dividing the cube into $m_1 \cdot \ldots \cdot m_d$ elementary parallelepipeds where the integers $m_i$ are prescribed in advance.

1. Introduction

The problem of consensus division arises when two or more competitive or cooperative parties, each guided by their individual objective functions, divide an object according to some notion of fairness. There are many different mathematical reformulations of this problem depending on what kind of divisions are allowed, what kind of object is divided, whether the parties involved are cooperative or not, etc. Early examples of problems and results of this type are the "ham sandwich theorem" of Steinhaus and Banach, the envy-free "cake-division problem" of Steinhaus, the equipartition of measurable sets by hyperplanes of Grünbaum and Hadwiger, and more recently the “splitting necklace theorem” of Alon, [11, 12, 13, 14, 17]. A model example of a fair-division theorem when two parties are involved is the Hobby-Rice theorem.

Theorem 1. [12] Let $\mu_1, \mu_2, \ldots, \mu_n$ be a collection of continuous probability measures on $[0,1]$. Then there exists a partition of $[0,1]$ by $n$ cut points into $n + 1$ intervals $I_0, I_1, \ldots, I_n$ and the corresponding signs $\epsilon_0, \epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ such that for each measure $\mu_i$,

$$\sum_{j=0}^{n} \epsilon_j \cdot \mu_i(I_j) = 0.$$

A well known consequence of this result is the “necklace theorem”, proved by Goldberg and West [9], which says that every open necklace with $d$ kind of stones (an even number of each kind) can be divided between two thieves using no more than $d$ cuts.

A celebrated generalization of Theorem 1 is the following “splitting necklace theorem” of Alon, which extends the result of Goldberg and West to the case of $q$ “thieves”. We formulate the continuous version which includes Theorem 1 as a special case and which can be used to deduce the corresponding discrete version.

Date: October 2006.
Theorem 2. Let $\mu_1, \mu_2, \ldots, \mu_n$ be a collection of $n$ continuous probability measures on $[0,1]$. Let $k \geq 2$ and $N := n(k - 1)$. Then there exists a partition of $[0,1]$ by $N$ cut points into $N$ intervals $I_0, I_1, \ldots, I_N$ and a function $f : \{0,1,\ldots,N\} \to \{1,\ldots,k\}$ such that for each $\mu_i$ and each $j \in \{1,2,\ldots,k\}$,

$$\sum_{f(p) = j} \mu_i(I_p) = 1/k.$$ 

Our main objective in this paper is to show that there exist higher dimensional analogs (Theorems 4 and 5) of the splitting necklace theorem which include Theorems 1 and 2 as special cases. This may sound as a surprise in light of Theorem 5.2 claiming that, given $l \geq 0$, for every $d \geq 2$ there exist 2-colorings of $[0,1]^d$ which do not admit “bisections” of size at most $l$. This ambiguity is immediately resolved by the observation that the “bisections” allowed in [3] were of quite special nature ($d$-dimensional checkerboards) while in our approach there are no restrictions on the coloring (labelling) of elementary parallelepipeds.

An important step leading to the generalization of the “splitting necklace theorem” was the recognition of the role of “rainbow complexes” $\Omega(Q)$ where $Q$ is an arbitrary $d$-dimensional, convex polytope and $S$ a finite set of “colors” used for labelling the vertices of $Q$. These complexes turn out to be (topologically) shellable (Theorem 5) and to have other interesting properties reflecting the geometry and combinatorics of the base polytope $Q$, Section 6.

2. Two-dimensional necklaces and the configuration space $\Omega(m,n)$

As a preliminary step, before we address the general case of a $d$-dimensional necklace ($d$-dimensional carpet) $I^d = [0,1]^d$, with $n$ measures $\mu_1, \mu_2, \ldots, \mu_n$ on $I^d$, and $k$ parties (thieves) interested in a fair division, we focus our attention on the case $d = k = 2$. This case exhibits all the main features of the general $d$-dimensional problem and provides a motivation for the introduction of configuration spaces $\Omega(m,n)$ and their generalizations.

A “splitting” of a square $I^2 = [0,1] \times [0,1]$ is a partition of $I^2$ into smaller rectangles by lines parallel to the sides of the square. Assuming that the square is positioned in the coordinate system so that the diagonally opposite vertices are $(0,0)$ and $(1,1)$, a $(m \times n)$-partition is determined by a choice of $m$ points $0 = x_0 \leq x_1 \leq x_2 \ldots \leq x_m \leq x_{m+1} = 1$ on the $x$-axes and $n$ points $0 = y_0 \leq y_1 \leq y_2 \ldots \leq y_n \leq y_{n+1} = 1$ on the $y$-axes.

The associated splitting (partition) is the division of $I^2$ into (possibly degenerate) rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $i = 0, \ldots, m$ and $j = 0, \ldots, n$. Recall an elementary fact that the space of all $m$-partitions $0 = x_0 \leq x_1 \leq x_2 \ldots \leq x_m \leq x_{m+1} = 1$ of the unit interval $I$ is naturally identified as the simplex $\Delta_m$ where $t_i := x_{i+1} - x_i$ are the associated barycentric coordinates. Similarly, the barycentric coordinates associated to a $y$-partition are $s_j = y_{j+1} - y_j$. It follows that the space of all $(m \times n)$-partitions of the square $I^2$ is naturally parameterized by points of the product $\Delta_m \times \Delta_n$.

The basic cell $C_{(m,n)} = \Delta_m \times \Delta_n$ should play in the case of 2-dimensional partitions the role analogous to the role the cell $C_m = \Delta_m$ plays in the case of 1-dimensional partitions. The next step is to introduce two “thieves” or players who want to divide among themselves elementary rectangles $R_{(i,j)} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ arising from the subdivision. By construction, the degenerate elementary
rectangles, i.e. the rectangles such that either \( x_i = x_{i+1} \) or \( y_j = y_{j+1} \) are allowed. However, it is instructive to keep in mind that the “thieves” are primarily interested in non-degenerated rectangles.

In the 1-dimensional case, a division of intervals between two thieves was described by a function \( \omega : \tilde{m} \to \{+, -\} \) where \( \tilde{m} := \{0, 1, \ldots, m\} \) and \( \omega(i) = + \) (alternatively \( \omega(i) = - \)) means that the interval \([x_i, x_{i+1}]\) was allocated to the first (respectively second) player.

Similarly, in 2-dimensions a function \( \omega : \tilde{m} \times \tilde{n} \to \{+, -\} \) completely describes an allocation of elementary rectangles to the two players.

As in the 1-dimensional case, a natural configuration space \( \Omega(m, n) \) for the 2-dimensional problem should take into account all \((m \times n)\)-partitions of \( I^2 \) together with all possible allocation functions \( \omega \in \{+, -\}^{\tilde{m} \times \tilde{n}} \). In other words a typical element in \( \Omega(m, n) \) is a triple \((t, s; \omega) \in C_{(m,n)} \times \{+, -\}^{\tilde{m} \times \tilde{n}} \). Collecting together all triples \((t, s; \omega)\) corresponding to a fixed \( \omega \in \{+, -\}^{\tilde{m} \times \tilde{n}} \) we observe that \( \Omega(m, n) \) ought to be the union of cells \( C_{(m,n)} \) := \{(t, s; \omega) \mid (t, s) \in \Delta_m \times \Delta_n \}. Two cells \( C_{(m,n)} \) and \( C_{(m,n)}' \) can have a point in common. This happens precisely if whenever \( \omega(i, j) \neq \omega'(i, j) \), the corresponding rectangle \( R_{(i,j)} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) is degenerate. This leads us to the definition of the following space

\[
\Omega(m, n) = \coprod_{\omega \in \{+, -\}^{\tilde{m} \times \tilde{n}}} C_{(m,n)}/\approx
\]

where \((t, s; \omega) \approx (t', s'; \omega')\) if and only if \( t = t' \) and \( s = s' \) and

\[
(\omega(i, j) \neq \omega'(i, j)) \Rightarrow (t_i = t'_i = 0 \text{ or } s_j = s'_j = 0).
\]

Here is a convenient way to “visualize” the configuration space \( \Omega(m, n) \). An element \( x = (t, s; \omega) \in \Omega(m, n) \) is visualized as a \((m + 1) \times (n + 1)-“chessboard”\) where the pair \((t, s) \in \Delta_m \times \Delta_n \) determines the size and the shape of each of the elementary parallelepipeds while the coloring (labelling) is described by the function \( \omega \) (Figure 1).

| + | + | - | + | - |
|---|---|---|---|---|
| - | - | + | + | - |
| + | - | + | - | + |
| - | + | + | - | - |

**Figure 1.** An element of \( \Omega(m, n) \).

Each cell \( C_{(m,n)} \) is visualized as the polytope \( C_{(m,n)} := \Delta_m \times \Delta_n \) with vertices colored (labelled) by \(+\) or \(-\), according to the prescription given by \( \omega \), while the total configuration space \( \Omega(m, n) \) is the union of cells \( C_{(m,n)} \) (Figure 2). Note that the elementary parallelepipeds from Figure 1 are in one-to-one correspondence with the vertices of \( C_{(m,n)} \) so one can read off the labelling function \( \omega \) both from the coloring of the elementary parallelepipeds and the coloring of the vertices of the cell \( C_{(m,n)} \).
The proof of the two–dimensional analogue of Alon’s theorem relies on the following important property of the configuration space $\Omega(m, n)$.

**Theorem 3.** The configuration space $\Omega(m, n)$ is a $(m+n)$-dimensional, $(m+n-1)$-connected, free $\mathbb{Z}_2$-complex.

In subsequent sections we will obtain a stronger and much more general result. However, here we present an outline of a direct proof of this theorem which provides additional insight into the structure of complexes $\Omega(m, n)$.

**Sketch of proof.** We proceed by induction on $\nu = m + n$. The complexes $\Omega(m, 0)$ and $\Omega(0, n)$ are isomorphic to $[2]^*\mathbb{Z}_2 \cong \partial \Delta_m$ respectively. The complex $[2]^*(m+1) = [2] \ast \ldots \ast [2]$ is naturally isomorphic to the boundary $\partial \Delta_m$ of the crosspolytope $\Delta_m := \text{conv} \{ e_i, -e_i \}_{i=1}^m \subset \mathbb{R}^m$. This holds also in the case $m = n = 0$, i.e. $\Omega(0, 0) \cong \mathbb{Z}_2$ is the boundary of $\partial \Delta_1 \cong [-1, +1]$.

Surprisingly enough, the complex $\Omega = \Omega(m, n)$ exhibits formal structure similar to the complex $\partial \Delta_m$ in the sense that it can be associated a “north and south pole” and the “upper and lower hemisphere” $\Omega^+ \Omega^-$ with all the usual consequences including the associated Mayer-Vietoris exact sequence of the triple $(\Omega; \Omega^+, \Omega^-)$.

In order to define a “north and south pole” in $\Omega(m, n)$, let us start with a maximally degenerated partition $(\bar{t}, \bar{s}) \in C(m, n)$ where $\bar{t}_0 = \bar{s}_0 = 1$ and $\bar{t}_i = \bar{s}_j = 0$ for $i \geq 1$ and $j \geq 1$. In this partition there is only one non-degenerated elementary rectangle, consequently there are only two associated elements $c_+ := (\bar{t}, \bar{s}; +)$ and $c_- := (\bar{t}, \bar{s}; -)$ in $\Omega(m, n)$.

The “upper hemisphere” $\Omega^+(m, n)$ is the set of all points $x = (t, s; \omega) \in \Omega(m, n)$ which are visible from $c_+$, i.e. such that both $x$ and $c_+$ belong to the same maximal cell in $\Omega(m, n)$. In other words $x \in \Omega^+(m, n)$ has a representative $x = (t, s; \omega)$ such that $\omega(0, 0) = +$. The “lower hemisphere” $\Omega^-(m, n)$ is defined similarly as the set of all points $x = (t, s; \omega)$ which allow a representation such that $\omega(0, 0) = -$.

Both $\Omega^+(m, n)$ and $\Omega^-(m, n)$ are contractible. Indeed, both spaces are star-shaped, with centers $c_+$ and $c_-$ respectively, and a contraction is defined by the linear homotopy.

Let us focus on the structure of the “equatorial set” $E(m, n) := \Omega^+(m, n) \cap \Omega^-(m, n)$. By definition $x = (t, s; \omega) \in E(m, n)$ if either $t_0 = 0$ or $s_0 = 0$. From here it follows that $E(m, n) = A \cup B$ where $A \cong \Omega(m-1, n)$ and $B \cong \Omega(m, n-1)$. Since $x \in A \cap B$ if and only if $t_0 = s_0 = 0$, we observe that $A \cap B \cong \Omega(m-1, n-1)$. A twofold application of the Mayer-Vietoris sequence to the

![Figure 2. A part of $\Omega(1, 1)$ and a labelled cell of $\Omega(2, 1)$](image)
triads \((\Omega(m, n); \Omega^+(m, n), \Omega^-(m, n))\) and \((E(m, n); A, B)\) together with a Seifert–van Kampen argument for determining the fundamental group of \(\Omega(m, n)\) yields the desired connectivity.

Theorem 4 following the usual Configuration space/Test map scheme \([12]\), is the basis for the following version of the two-dimensional splitting necklace theorem.

**Theorem 4** (Two-dimensional necklace for two thieves). Let \(\mu_1, \ldots, \mu_n\) be a collection of \(n\) continuous probability measures on the unit square \(I^2 = [0, 1]^2\). Then for any choice of \(m_1, m_2 \geq 0\) of integers such that \(m_1 + m_2 = n\), there exist \(m_1\) vertical and \(m_2\) horizontal cuts of the square, and a coloring of the elementary rectangles obtained this way by two colors “+” and “−” such that \(\mu_i(A_+) = \mu_i(A_-) = \frac{1}{2}\) for all \(i\) where \(A_+\) (respectively \(A_-\)) is the union of all elementary parallelepipeds colored by “+” (respectively “−”).

We omit the proof of Theorem 4 since it will be subsumed by a more general argument used in the proof of Theorem 4 and instead turn our attention to the general case of a necklace in \(d\) dimensions for an arbitrary number of thieves.

### 3. The complex \(\Omega(Q; G)\) of \(G\)-labelled polytopes

The 2-dimensional splitting necklace theorem presented in Section 2 especially the construction of the configuration space \(\Omega(m, n)\) with favorable properties (Theorem 3), reveal that higher dimensional analogs and extensions should be within reach by similar methods. Apparently the most natural generalization that comes to mind is the splitting of a \(d\)-dimensional cube by hyperplanes parallel to its sides. Moreover, in order to extend the 1-dimensional “splitting necklace theorem”, we should replace \(\{+, −\}\) by an arbitrary set \(G\) of labels (colors) corresponding to different “thieves”. The letter \(G\) should indicate that the labels are often elements of a given finite group, e.g. \(G \cong \mathbb{Z}_2 \cong \{+, −\}\) in the case of two thieves.

An extension and a multidimensional analogue of the configuration space \(\Omega(m, n) = \Omega(m; n; \mathbb{Z})\) is the space \(\Omega(m; G) = \Omega(m_1, m_2, \ldots, m_d; G)\) defined as follows. A typical element in \(\Omega(m, G)\) is a pair \((t; \omega) \in Q_m \times G^{m_1 \times \ldots \times m_d}\) where \(Q_m := \Delta_{m_1} \times \ldots \times \Delta_{m_d}\) is the space of all \(m\)-partitions of the hypercube \(I^d\). More precisely each of the coordinates \(t_i\) of \(t = (t_1, \ldots, t_d)\) is a partition \(0 = x^0_1 \leq x^1_1 \leq \ldots \leq x^i_{m_1} \leq x^{i+1}_{m_1} \leq 1\) of the interval \([0, 1]\) so an elementary (possibly degenerate) \(d\)-parallelepiped associated to \(t_i\), indexed by \(j = (j_1, \ldots, j_d) \in \bar{m}_1 \times \ldots \times \bar{m}_d\), is

\[
R_i(t) = [x^1_{j_1}, x^1_{j_1+1}] \times \ldots \times [x^d_{j_d}, x^d_{j_d+1}].
\]

For each labelling function \(\omega : \bar{m}_1 \times \ldots \times \bar{m}_d \rightarrow G\) there is an associated cell \(C^\omega_m = \Delta_{m_1} \times \ldots \times \Delta_{m_d}\). It is convenient to visualize the cell \(C^\omega_m\) as the polytope \(Q = \Delta_{m_1} \times \ldots \times \Delta_{m_d}\) with all vertices labelled by elements from \(G\). This leads us to the following definition.

**Definition 1.** The configuration space \(\Omega(m; G)\) is defined as the quotient space:

\[
\coprod_{\omega \in G^{m_1 \times \ldots \times m_d}} C^\omega_m / \approx
\]

where \((t; \omega) \approx (s; \nu)\) if and only if \(t = s\) and 

\[
\omega(j) \neq \nu(j) \Rightarrow R_i(t) = R_i(s)\] is a degenerated \(d\)-parallelepiped.
A natural extension of the configuration space $\Omega(m;G)$ is the cell complex $\Omega(Q;G)$ where $Q$ is an arbitrary convex polytope $Q \subset \mathbb{R}^d$. Given a function $\omega : \text{vert}(Q) \to G$, the associated cell $Q^\omega$ is described as the polytope with each vertex $v$ decorated (labelled) by the corresponding element $\omega(v)$. In particular $Q^\omega = C_m^\omega$ if $Q = \Delta_{m_1} \times \ldots \times \Delta_{m_d}$. Given $t \in Q$, the associated element in $Q^\omega$ will be denoted by $(t, \omega)$. The cell $Q^\omega$ is sometimes referred to as a vertex-colored polytope and $\Omega(Q;G)$ is the associated rainbow complex.

**Definition 2.** The configuration space $\Omega(Q;G)$ is defined as the quotient space: $$\prod_{\omega \in Q^{\text{vert}(Q)}} Q^\omega / \approx$$ where $(t, \omega) \approx (s, \nu)$ if and only if $t = s$ and if $F \subset Q$ is the minimal face such that $t \in F$, then $\omega|_{\text{vert}(F)} = \nu|_{\text{vert}(F)}$.

4. Shellability of $\Omega(Q;G)$

One of the key ingredients in the proof of the higher dimensional splitting necklace theorem is the proof that the complex $\Omega(m;G)$ is $(|m| - 1)$-connected where $|m| := m_1 + \ldots + m_d$. This could be proved along the lines of the proof of Theorem 3. In this section we offer a different proof of a more general fact that $\Omega(Q;G)$ is always a $d$-dimensional, $(d - 1)$-connected regular cell complex.

4.1. **Topological shellability.** A convenient way to prove that a (regular, polyhedral, simplicial) $d$-dimensional cell complex is $(d - 1)$-connected is to show that it is shellable [4, 10]. There are many different concepts of shellability. Here, as a variation on a theme, we introduce a form of shellability which will be referred to as topological shelling.

**Definition 3.** Suppose that $K$ is a finite, regular cell complex. A total ordering $C_1, C_2, \ldots, C_k$ of its maximal cells is a topological shelling of $K$ if

$$\dim(C_1) \geq \dim(C_2) \geq \ldots \geq \dim(C_k)$$

and for each $j > 1$ either (a) or (b) is satisfied where

(a) $(\cup_{i<j} C_i) \cap C_j$ is a (non-empty) contractible subset of $\partial(C_j)$,

(b) $(\cup_{i<j} C_i) \cap C_j = \partial(C_j)$ where $\partial(C_j) \cong S^{\dim(C_j) - 1}$.

The following result is easily established by induction on the number of maximal cells in $K$.

**Proposition 1.** Suppose that $K$ is a finite cell complex which admits a topological shelling $C_1, C_2, \ldots, C_k$. Let $n_i := \dim(C_i)$. Then $K$ is homotopic to the wedge $\bigvee_{j \in S} S^{n_j}$ where $S := \{ j \mid (\cup_{i<j} C_i) \cap C_j = \partial(C_j) \}$.

**Proof.** Let $K_{\leq j}$ and $K_{< j}$ be the subcomplexes of $K$ defined by $K_{\leq j} := \cup_{i \leq j} C_i$ and $K_{< j} := \cup_{i < j} C_i$. Suppose that by induction hypothesis the statement is true for all $j < j_0$. If $K_{< j_0} \cap C_{j_0}$ is a contractible subset of $\partial(C_{j_0})$ then $K_{< j_0}$ and $K_{< j_00}$ have the same homotopy type, consequently $K_{< j_0}$ is also a wedge of spheres.

Suppose $K_{< j_0} \cap C_{j_0} = \partial(C_{j_0})$. By the induction hypothesis $K_{< j_0}$ is a wedge of spheres, $K_{< j_0} \simeq \bigvee_{s=1}^t S^{p_s}$ where

$$p := \min\{ p_s \}_{s=1}^t \geq \dim(C_{j_0}) > \dim(\partial(C_{j_0}))$$.
It follows that $\partial(C_{j_0})$ is contractible in $K_{<j_0}$, hence
\[ K_{j_0} \simeq S^{\dim(C_{j_0})} \vee \bigvee_{s=1}^{t} S^{p_s}. \]

\[ \square \]

**Corollary 1.** A cell complex $K$ admitting a topological shelling is $(n-1)$-connected, provided it is pure $n$-dimensional, i.e. if all its maximal cells have the same dimension $n$. \[ \square \]

### 4.2. Topological shellability of $\Omega(Q; [k])$.

**Theorem 5.** The complex $\Omega(Q; G)$ admits a topological shelling for each convex $d$-polytope $Q \subset \mathbb{R}^d$ and each finite set $G$ of labels (colors).

**Proof.** Suppose that $k := |G|$ is the cardinality of the set $G$. If the polytope $Q = \Delta = \Delta_\nu$ is a $\nu$-dimensional simplex then
\[ \Omega(Q; G) \cong \Omega(\Delta_\nu; [k]) = [k] \ast \ldots \ast [k] = [k]^*(\nu+1) \]
is a simplicial complex which is well known to be (lexicographically) shellable. Indeed, each $\nu$-dimensional simplex in $[k] = [k]^*(\nu+1)$ is obtained from the simplex $\Delta$ by coloring its vertices with colors from $[k]$. In other words each of these simplexes is a vertex-colored polytope $\Delta^f$ where $f : \{0,1,\ldots,\nu\} \rightarrow [k]$ is the associated coloring function. Given functions $f, g \in [k]^\nu$, the lexicographical ordering defined by
\[ f \prec g \iff f(i) < g(i) \text{ where } i := \min\{j \mid f(j) \neq g(j)\} \]
induces a shelling $\{\Delta^f\}_{f \in [k]^{\nu}}$ of the rainbow complex $\Omega(\Delta; [k])$. Indeed, for each $g \in [k]^\nu$ the complex $(\cup_{f \prec_g} \Delta^f) \cap \Delta^g$ is easily shown to be a union of facets of the simplex $\Delta^g$.

A convex polytope $Q \subset \mathbb{R}^d$ with vertices $\text{vert}(Q) = \{v_0, v_1, \ldots, v_\nu\}$ is the image $\text{Im}(h)$ of an associated affine map $h : \Delta_\nu \rightarrow Q$, $j \mapsto v_j$. Given a function $f : \tilde{m} \rightarrow [k]$, there is an induced map $h_f^f : \Delta^f \rightarrow Q^f$ of vertex-colored polytopes. This map however does not extend to a cellular map of complexes $\Omega(\Delta; [k])$ and $\Omega(Q; [k])$ since the intersection $\Delta^f \cap \Delta^g$ is not necessarily mapped to $Q^f \cap Q^g$. Nevertheless, the following claim shows that both complexes admit formally the same shelling order.

**Claim:** The ordering $\{Q^f\}_{f \in [k]^\nu}$ arising from the lexicographical ordering of functions $\Omega(Q; [k])$ is a topological shelling of the complex $\Omega(Q; [k])$.

**Proof of the Claim:** Given a function $g \in [k]^\nu$, let $L_g$ be the complex
\[ L_g := \Omega_{=g} \cap Q^g = (\cup_{f < g} Q^f) \cap Q^g = \cup_{f < g} (Q^f \cap Q^g). \]

According to Definition 3, we have to demonstrate that $L_g$ is either contractible or $L_g = \partial(Q^g) \cong S^{d-1}$. The intersection $Q^f \cap Q^g$ is the union of all vertex-colored polytopes $F^h$ where $F$ is a face of $Q$ and $h : \text{vert}(F) \rightarrow [k]$ agrees with both $f$ and $g$ on $\text{vert}(F)$, i.e., $h = f|_{\text{vert}(F)} = g|_{\text{vert}(F)}$. In the special case when $f(j) = g(j)$ for all but one element $j_0 \in \tilde{\nu}$, i.e., if $\{j \in \tilde{\nu} \mid f(j) = g(j)\} = \tilde{\nu} \setminus \{j_0\}$, we observe that $Q^f \cap Q^g$ is essentially the “anti-star” $\text{a-Star}(v_{j_0})$ of the vertex $v_{j_0}$ in $\partial(Q^g) \cong \partial(Q)$, i.e., the union of all facets in $Q$ that do not contain the vertex $v_{j_0}$. The anti-star corresponds to the facet $\Delta^f \cap \Delta^g$ of $\Delta^f$, resp. $\Delta^g$, in the original shelling.
Given a face $F$ of $Q$, let $\text{open-Star}(F)$ be the union of all relative interiors of all proper faces of $Q$ which contain $F$ as a face

$$\text{open-Star}(F) = \bigcup_{F \subseteq G \neq Q} \text{rel-int}(G).$$

One easily checks that

$$a\text{-Star}(v) = \partial(Q) \setminus \text{open-Star}(v).$$

In light of the fact that “$<$” is a shelling order of the simplicial complex $[k]^{s(\tilde{v})}$, i.e., $(\cup_{f \prec g} \Delta^f) \cap \Delta^g$ is a union of facets, we observe that there exists a non-empty set $S \subset \tilde{v}$ such that

$$L_g = \bigcup_{j \in S} a\text{-Star}(v_j) = \partial(Q) \setminus \bigcap_{j \in S} \text{open-Star}(v_j).$$

Finally,

$$\bigcap_{j \in S} \text{open-Star}(v_j) = \text{open-Star}(F)$$

where $F := \text{supp}\{\{v_j\}\}_{j \in S}$ is the minimal face of $Q$ containing all vertices $v_j$. It follows that

$$L_g = \partial(Q) \setminus \text{open-Star}(F)$$

which completes the proof since if the open star of $F$ is non-empty, its complement is homeomorphic to a $(d-1)$-dimensional cell.

5. The necklace theorem in arbitrary dimension

The following result of Borsuk-Ulam type is a key tool for many applications of equivariant topological methods in combinatorics and discrete geometry, [13, 17].

**Theorem 6** (Bárány, Schlosman, Szücs [6]; Dold [8]). Let $G = \mathbb{Z}_p$ be the cyclic group of prime order $p$. Suppose that $\Omega$ is a finite, $(N-1)$-connected, free $G$-cell complex where $N = n(p-1)$ for some integer $n \geq 1$. Assume that $E$ is a real, linear $G$-representation of dimension $N$, having no trivial subrepresentations, i.e. such that $E^G = \{0\}$. Then every continuous $G$-equivariant map $f : \Omega \to E$ has a zero.

Although the proofs of this result are nowadays readily available [13], for the reader’s convenience and self containment of the paper we outline a short proof of this fact.

**Proof.** Assume that there is a map $f : \Omega \to E$ without a zero. This yields a $G$-equivariant map $\bar{f} : \Omega \to S(E)$ to the $(N-1)$-sphere $S(E)$ in $E$. As $p$ is prime and 0 is the only element in $E$ fixed by all elements in $G$, it follows that the induced action on $S(E)$ is free. Hence by the $(N-1)$-connectedness of $\Omega$ there exists a $G$-equivariant map $g : S(E) \to \Omega$. Now consider the map $(g \circ f)_\# : C_*(\Omega) \to C_*(\Omega)$ for the cellular chain complex with respect to a finite $G$-invariant cell structure. As every orbit of a cell consists of $p$ elements, the Lefshetz trace $\Lambda(g \circ f) = \sum (-1)^i \text{tr}(g \circ f)_\#$ will be divisible by $p$. If we compute the Lefshetz trace now on the homology level, we obtain $\Lambda(g \circ f) = \sum (-1)^i \text{tr}(g \circ f)_* = 1$ as the map factors through the homology of an $(N-1)$-sphere. A contradiction!  

□
Given a set $X$ of hyperplanes in $I^d$, let $C(X)$ be the set of cells (connected components) of $I^d \setminus \bigcup X$. For any coloring (labelling) map $\omega : C(X) \to [k]$, let $A_i := \bigcup \{c \in C(X) \mid \omega(c) = i\} = \bigcup \omega^{-1}(i)$ be the union of all cells colored by the same color $i$.

**Theorem 7** (Higher dimensional necklace theorem). Assume $n, d \geq 1$ and $k \geq 2$, and let $\mu_1, \ldots, \mu_n$ be a collection of $n$ continuous probability measures on the $d$-dimensional cube $I^d \subset \mathbb{R}^d$. For any selection of non-negative integers $m_1, \ldots, m_d$ such that $m_1 + \cdots + m_d = n(k-1)$ there exists a fair division with $m_i$ hyperplane cuts parallel to $i$-th coordinate hyperplane. In other words there exists a set $\mathcal{X} = \bigcup_{i=1}^d X_i$ of $n(k-1)$ hyperplanes such that $|X_i| = m_i$, each $H \in X_i$ is perpendicular to $e_i$, and for some coloring function $\omega : C(X) \to [k]$

$$
\mu_i(A_j) = \frac{1}{k} \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, k
$$

where $A_i := \bigcup \omega^{-1}(i)$ are the unions of all cells colored by the same color.

We reduce the proof of the theorem to the case $k = p, p$ prime.

**Lemma 1.** If the previous theorem holds for parameters $k_1, k_2 \geq 2$ (in place of $k$) then it also holds for $k = k_1 k_2$.

**Proof.** Before commencing the proof, the reader is referred to Figure 3 for a rough idea how the reduction claimed in the lemma is achieved. This is an example with $n(k-1) = 6$ cutting hyperplanes where $n = 2, d = 2, k = k_1 \cdot k_2 = 2 \cdot 2 = 4, m_1 = 2,$ and $m_2 = 4$. The densities of the two measures $\mu_1$ and $\mu_2$ are indicated by the light and dark grey regions. The cube will be divided in the first step into $k_1 = 2$ pieces. Then the two pieces will be treated separately.

![Figure 3. The reduction in action](image-url)
• \( m_0 + \cdots + m_d = n(k_1 - 1) \),

• \( \sum_{i=1}^d m_i = n(k_2 - 1) \) for all \( j = 1, \ldots, k_1 \), and

• \( \sum_{j=0}^{k_1} m_i = m_i \) for all \( j = 1, \ldots, d \).

This is certainly possible as

\[
    n(k_1 - 1) + k_1 n(k_2 - 1) = n(k - 1) = \sum_{i=1}^d m_i.
\]

In the Figure we chose \( m_0 = m_1 = 1 \), \( m_1 = m_2 = 1 \), \( m_2 = 0 \), and \( m_2 = 2 \).

By assumption there exists a set \( A_0 \) of \( n(k_1 - 1) \) hyperplanes of which \( m_i \) are perpendicular to \( e_i \) and \( \omega : C(\mathcal{X}) \to [k] \) such that \( \mu_i(A_j) = \frac{1}{k_i} \) for all \( i = 1, \ldots, n, j = 1, \ldots, k_1 \), where \( A_i \) are the unions of cells associated to \( \omega \).

For each \( j = 1, \ldots, k_1 \), consider the rescaled restrictions of the measures to the regions \( A_j \), i.e.,

\[
    \mu_j(S) = k_1 \mu_i(S \cap A_j^0).
\]

In other words for each \( j \), \( \mu_1, \ldots, \mu_n \) is a set of \( n \) probability measures on \( I^d \) which have support only in \( A_j \). Now for each \( j \) let by assumption \( \mathcal{X} \) be a set of \( n(k_2 - 1) \) hyperplanes of which \( m_i \) are perpendicular to \( e_i \) and \( \omega : C(\mathcal{X}) \to [k_2] \) such that for the associated \( A_j \) we have

\[
    \mu_j(A_i) = \frac{1}{k_2}
\]

for all \( j = 1, \ldots, k_1 \), and \( i, i' = 1, \ldots, k_2 \).

We will now construct the desired pair \( (\mathcal{X}, \omega) \) as follows. Let \( \mathcal{X} = X^0 \cup X^1 \cup \cdots \cup X^{k_1} \). To define the map \( \omega : C(\mathcal{X}) \to [k_1] \) consider a cell \( c \subseteq C(\mathcal{X}) \). Let \( j_1 \subseteq \{k_1\} \) be the unique element with \( c \subseteq A_j \) and \( j_2 \) be the unique element in \( [k_2] \) with \( c \subseteq A_j \). Then let

\[
    \omega(c) = \left( \omega^0(A_j^0), \omega^{j_2}(A_j^{j_2}) \right).
\]

\[
    \square
\]

Applying the previous lemma we will now prove Theorem \ref{thm:main}

\textbf{Proof.} As we may assume \( k \) to be prime, let \( G = \mathbb{Z}_p \) be the cyclic group of prime order \( p \). Let \( \mathbb{E} \) be the space of all \( n \times p \)-matrices with row sums equal to zero. \( G \) acts on \( \mathbb{E} \) by cyclic column permutations. Let us construct a continuous \( G \)-equivariant map \( f : \Omega(m_1, \ldots, m_d; G) \to \mathbb{E} \) such that each zero of this map corresponds to a desired solution. Let \( (t, \omega) = (t_1, \ldots, t_d, \omega) \in \Omega(m_1, \ldots, m_d; G) \). Following the notation from Section \ref{sec:construction} each \( t_i \in \Delta_m \) is a partition \( 0 = x_0 = x_1 = \ldots = x_m = x_{m+1} = 1 \) of the unit interval \([0, 1]\). Let \( \mathcal{X} := \bigcup_{i=1}^d \mathcal{X}_i \) where \( \mathcal{X}_i := \{H_i^{j} \}_{j=1}^{m} \) is the collection of hyperplanes orthogonal to the unit vector \( e_i \) defined by \( H_i^{j} := \{y \in \mathbb{R}^d \mid y_i = x_j\} \). The collection \( \mathcal{X} \) dissects \( I^d \) into \( m_1 \ldots m_d \) elementary \( d \)-parallelepipeds while the coloring function \( \omega : C(\mathcal{X}) \to [p] \) colors these \( d \)-parallelepipeds by \( p \) colors. Let \( A_i := \bigcup_{i=1}^d \omega^{-1}(i) \) be the union of all \( d \)-parallelepipeds colored by the color \( i \). By construction, an element \( (t, \omega) \) corresponds to a fair division if \( \mu_j(A_i) = 1/p \) for each \( i \) and \( j \). Consequently the vector \( v_i = v_i(t, \omega) := (\mu_j(A_i) - 1/p)_{j=1}^{n} \in \mathbb{R}^n \),
which continuously depends on the input data \((t, \omega)\), is equal to 0 if and only if the division is fair from the point of view of \(i\)-th player ("thief"). By definition let
\[
f((t, \omega)) := [v_1, v_2, \ldots, v_p]
\]
be the map \(f : \Omega(m;G) \rightarrow \mathbb{E}\) obtained by writing \(v_i\) as column vectors of a matrix in \(\mathbb{E}\). The map \(f\) is obviously \(G\)-equivariant. By Theorem 4 and Corollary 1 \(\Omega = \Omega(m;G)\) is a \(n(p-1)\)-connected, free \(G\)-cell complex. Hence, \(\Omega, \mathbb{E}\) and \(f\) together satisfy the conditions of Theorem 6. Consequently \(f\) must have a zero which completes the proof of the theorem. □

6. Concluding remarks

It is customary to formulate consensus division theorems for (vector-valued) measures \(\mu\) that are continuous i.e. defined by density functions \(d\mu = f \cdot dm\), where \(m\) is the Lebesgue measure. It is not difficult to see that majority of these results (including our Theorems 4 and 7) hold for much more general classes of measures. For a broader perspective on this problem and other examples of consensus division theorems the reader is referred to [17, 14]. Here we restrict ourselves to the observation that the measures used in multidimensional splitting necklace theorems do not have to be positive. Moreover, the continuity condition can be replaced by a much weaker condition that \(\mu(\partial(Q)) = 0\) where \(Q \subset I^d\) is an arbitrary parallelepiped and \(\partial(Q)\) its boundary.

The “rainbow complexes” \(\Omega(Q;[k])\), introduced in Section 3, appear to have some independent interest as topological/geometric objects which capture some of the combinatorial properties of the underlying polytope \(Q\). For example if \(Q \subset \mathbb{R}^d\) is a simplicial polytope, then the Euler characteristic \(\chi(\Omega(Q;[k]))\) is given by the formula
\[
\chi(\Omega(Q;[k])) = k \cdot F_Q(-k) := f_0 k - f_1 k^2 + \ldots + (-1)^{d-1} f_{d-1} k^d + (-1)^d k^{d+1},
\]
where \((f_0, f_1, \ldots, f_{d-1}, f_d)\) is the \(f\)-vector of \(Q\). A broader outlook should place rainbow complexes \(\Omega(Q;[k])\) and their generalizations into the category of combinatorially defined configuration spaces associated to polytopes, (Eulerian) posets, simplicial complexes etc. In this generality they could be seen as relatives of toric varieties and their combinatorial counterparts (extensions) such as moment-angle complexes \(\mathcal{Z}_K\) [13, 5], homotopy colimits over posets [15] etc.

Acknowledgement: The authors would like to thank the organizers of the special program “Computational Applications of Algebraic Topology”, hosted by the Mathematical Sciences Research Institute (MSRI, Berkeley, Fall 2006), for the support, excellent working conditions and stimulating research atmosphere. The second author acknowledges the support by the Serbian Ministry of Science (projects 144014 and 144026).

References

[1] N. Alon. Splitting necklaces. Advances in Math., 63:247–253, 1987.
[2] N. Alon. Non-constructive proofs in combinatorics. Proc. Int. Congr. Math., Kyoto/Japan 1990, Vol. II (1991), 1421–1429.
[3] N. Alon and D.B. West. The Borsuk–Ulam theorem and bisection of necklaces. Proc. Amer. Math. Soc., 98:623–628, 1986.
[4] V. Bukhshtaber, T. Panov. Actions of tori, combinatorial topology and homological algebra. Russian Math. Surveys 55 (2000), no. 5, 825–921.
[5] M.W. Davis, T. Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.* 62 (1991), no. 2, 417–451.

[6] I. Bárány, S. B. Schlosman, A. Szűcs. On a topological generalization of a theorem of Tverberg. *J. Lond. Math. Soc.*, 23 (2), 1981, 158–164.

[7] A. Björner. Topological methods. In: *Handbook of Combinatorics*, R. Graham, M. Grötschel, and L. Lovász (Eds.), North-Holland, Amsterdam, 1995.

[8] A. Dold. Simple proofs of some Borsuk–Ulam results. In: “Northwestern Homotopy Conference” (H. R. Miller, S. B. Priddy, eds.), Contemp. Math. 19, 1983, 65–69.

[9] C.H. Goldberg, D.B. West. Bisection of circle colorings. *SIAM J. Algebraic Discrete Methods* 6 (1985), 93–106.

[10] B. Grünbaum. Partitions of mass–distributions and convex bodies by hyperplanes, *Pacific J. Math.*, 10 (1960), 1257–1261.

[11] H. Hadwiger. Simultane Vierteilung zweier Körper, *Arch. Math. (Basel)*, 17 (1966), 274–278.

[12] C.R. Hobby, J.R. Rice. A moment problem in $L_1$ approximation. *Proc. Amer. Math. Soc.*, 16:665–670, 1965.

[13] J. Matoušek. *Using the Borsuk-Ulam Theorem*; Lectures on Topological Methods in Combinatorics and Geometry. Springer 2003.

[14] P. Mani-Levitska, S. Vrećica, and R. Živaljević. Topology and combinatorics of partitions of masses by hyperplanes. arXiv:math.CO/0310377 v1 23 Oct 2003.

[15] V. Welker, G. Ziegler, R. Živaljević. Comparison lemmas and applications for diagrams of spaces. *J. Reine Angew. Math.*, 500 (1999), 117–149.

[16] G.M. Ziegler. Lectures on Polytopes. *Graduate Texts in Mathematics*, Springer 1995.

[17] R. Živaljević, *Topological methods*. In: CRC Handbook of Discrete and Computational Geometry (new edition), J.E. Goodman, J. O’Rourke (eds.), Boca Raton 2004.

Freie Universität Berlin, Fachbereich Mathematik, Arnimallee 3, 14195 Berlin, Germany

Mathematical Institute SANU, Knez Mihailova 35/1, p.f. 367, 11001 Belgrade, Serbia
