Birkhoff billiards are insecure

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Abstract

We prove that every compact plane billiard, bounded by a smooth curve, is insecure: there exist pairs of points \( A, B \) such that no finite set of points can block all billiard trajectories from \( A \) to \( B \).

Two points \( A \) and \( B \) of a Riemannian manifold \( M \) are called secure if there exists a finite set of points \( S \subset M - \{A, B\} \) such that every geodesic connecting \( A \) and \( B \) passes through a point of \( S \). One says that the set \( S \) blocks \( A \) from \( B \). A manifold is called secure (or has the finite blocking property) if any pair of its points is secure. For example, every pair of non-antipodal points of the Euclidean sphere is secure, but a pair of antipodal points is not secure, so the sphere is insecure. A flat torus of any dimension is secure.

In the recent years, the notion of security has attracted a considerable attention, see [1, 2, 3, 4, 9, 10, 11, 12]. This notion extends naturally to Riemannian manifolds with boundary, in which case one considers billiard trajectories from \( A \) to \( B \) with the billiard reflection off the boundary.

In this note we consider a compact plane billiard domain \( M \) bounded by a smooth curve and prove that \( M \) is insecure. More specifically, one has the following local insecurity result. Consider a sufficiently short outward convex arc \( \gamma \subset \partial M \) with end-points \( A \) and \( B \) (such an arc always exists).

**Theorem 1** The pair \((A, B)\) is insecure.
Proof. Denote by $T_n$ the polygonal line $A = P_0, P_1, \ldots, P_{n-1}, P_n = B$, $P_i \in \gamma$, of minimal length; this is a billiard trajectory from $A$ to $B$. If $n$ is large then $T_n$ lies in a small neighborhood of $\gamma$.

Working toward contradiction, assume that a finite set of points $S \subset M - \{A, B\}$ blocks every billiard trajectory from $A$ to $B$. Decompose $S$ as $S' \cup S''$ where the points of $S'$ lie on the boundary and the points of $S''$ lie inside the billiard table. For $n$ large enough, the trajectory $T_n$ is disjoint from $S''$. We want to show that there is a sufficiently large $n$ such that the set $P_n = \{P_1, \ldots, P_{n-1}\}$ is disjoint from $S'$.

Let $s$ be the arc-length parameter and $k(s)$ the curvature of $\gamma$. Let $\sigma$ be a new parameter on the arc $\gamma$ such that $d\sigma = (1/2)k(s)^2/3 ds$. By rescaling the arc $\gamma$, we may assume that the range of $\sigma$ is $[0, 1]$ with $\sigma(0) = A$ and $\sigma(1) = B$. Let $Q_0 = A, Q_1, \ldots, Q_{n-1}, Q_n = B$ be the points that divide the $\sigma$-measure of $\gamma$ into $n$ equal parts, that is, $Q_m = \sigma(m/n)$.

Proposition 2 One has: $|P_n - Q_n| = O(1/n^2)$.

Remark 3 This Claim is consistent with Theorem 6 (iii) of [6] which describes the limit distribution of the vertices of the inscribed polygons that best approximate a convex curve relative the deviation of the perimeter length.

To prove Proposition 2, we use the theory of interpolating Hamiltonians, see [7, 8] and especially [5]. Recall the relevant facts from this theory.

First, some generalities about plane billiards (see, e. g., [13, 14]). The phase space $X$ of the billiard ball map consists of inward unit tangent vectors $(x, v)$ to $M$ with the foot point $x$ on the boundary $\partial M$; $x$ is the position of the billiard ball and $v$ is its velocity. The billiard ball map $F$ takes $(x, v)$ to the vector obtained by moving $x$ along $v$ until it hits $\partial M$ and then elastically reflecting $v$ according to the law “angle of incidence equals angle of reflection”. Let $\phi$ be the angle made by $v$ with the positive direction of $\partial M$. Then $(s, \phi)$ are coordinates in $X$. The area form $\omega = \sin \phi \, d\phi \wedge ds$ is $F$-invariant.

In a nutshell, the theory of interpolating Hamiltonians asserts that the billiard ball map equals an integrable symplectic map, modulo smooth symplectic maps that fix the boundary of the phase space $X$ to all orders. More specifically, one can choose new symplectic coordinates $H$ and $Z$ near the boundary $\phi = 0$ such that $\omega = dH \wedge dZ$, $H$ is an integral of the map $F$, up to all orders in $\phi$, and

$$F^*(Z) = Z + H^{1/2}, \quad (1)$$
also up to all orders in $\phi$. The function $H$ is given by a series in even powers of $\phi$, namely,

$$H = k^{-2/3} \phi^2 + O(\phi^4),$$

and this series is uniquely determined by the above conditions on $H$ and $Z$.

**Lemma 4** One may choose the coordinate $Z$ in such a way that $Z = \sigma + O(\phi^2)$.

**Proof.** Let $Z = f(s) + g(s)\phi + O(\phi^2)$. We have: $\omega = dH \wedge dZ$. Equating the coefficients of $\phi \, d\phi \wedge ds$ and of $\phi^2 \, d\phi \wedge ds$ and using (2) we obtain the equations:

$$2k^{-2/3}(s)f'(s) = 1, \quad 2k^{-2/3}(s)g'(s) + \frac{2}{3}k^{-5/3}(s)k'(s)g(s) = 0.$$

The first equation implies that $df = d\sigma$ and the second that $g = Ck^{-1/3}$ where $C$ is a constant. We can choose $f(0) = 0$. Since $Z$ is defined up to summation with functions of $H$, it follows from (2) that the term $g(s)\phi$ can be eliminated by subtracting $CH^{1/2}$. \hfill $\Box$

Now we can prove Proposition 2. The billiard trajectory $T_n$ corresponds to a phase orbit $x_0, \ldots, x_n$, $F(x_i) = x_{i+1}$. Since $H$ is an integral of the map $F$, the orbit $x_0, \ldots, x_n$ lies on a level curve $H = c_n$. Due to (1), we have: $n\sqrt{c_n} = O(1)$, and hence $c_n = O(1/n^2)$ which, in view of (2), implies that

$$\phi = O \left( \frac{1}{n} \right).$$

Consider $\sigma$ and $Z$ as functions on the phase space $X$. Since $\sigma(x_m) = P_m$ and the $\sigma$-coordinate of $Q_m$ is $m/n$, we need to show that

$$\sigma(x_m) = \frac{m}{n} + O \left( \frac{1}{n^2} \right).$$

Since $F$ is a shift in $Z$-coordinate, see (1), one has:

$$Z(x_m) = \frac{m}{n} (Z(x_n) - Z(x_0)) = \frac{m}{n} (\sigma(x_n) - \sigma(x_0)) + O \left( \frac{1}{n^2} \right) = \frac{m}{n} + O \left( \frac{1}{n^2} \right),$$

the second equality due to Lemma 4 and (3). This proves Proposition 2. \hfill $\Box$
From now on, we identify the arc $\gamma$ with the segment $[0,1]$ using the parameter $\sigma$; the points $P_1, \ldots, P_{n-1}$ are considered as reals between 0 and 1. Assume that a finite set $S' = \{t_1, \ldots, t_k\} \subset (0,1)$ is blocking, that is, for all sufficiently large $n$, one has $P_n \cap S' \neq \emptyset$.

Some of the numbers $t_i \in S'$ may be rational; denote them by $p_i/q_i$, $i = 1, \ldots, l$ (fractions in lowest terms), and let $Q = q_1 \cdot \cdots \cdot q_l$. Set $n_i = 1 + (N + i)Q$, $i = 0, \ldots, k$.

**Proposition 5** For $N$ sufficiently large, at least one of the sets $P_{n_i}$ is disjoint from $S'$.

**Proof.** Assume not. Then, by the Pigeonhole Principle, there exist $l, i, j$ such that $t_l \in P_{n_i} \cap P_{n_j}$. According to Proposition 2 there is a constant $C$ (independent of $n$) such that, for $P_m \in P_n$, one has:

$$\left| P_m - \frac{m}{n} \right| < \frac{C}{n^2}.$$  

Therefore

$$\left| t_l - \frac{m_1}{n_i} \right| < \frac{C}{n_i^2}, \quad \left| t_l - \frac{m_2}{n_j} \right| < \frac{C}{n_j^2} \quad (5)$$

for some $m_1, m_2$.

**Lemma 6** If $N$ sufficiently large then $t_i \notin \mathbb{Q}$.

**Proof.** First, we claim that, given a fraction $p/q$ and a constant $C$, if

$$\left| \frac{p}{q} - \frac{m}{n} \right| < \frac{C}{n^2}$$

for all sufficiently large $n$ then $m/n = p/q$.

Indeed, if $m/n \neq p/q$ then $1 \leq |pn - qm|$, hence

$$\frac{1}{qn} \leq \left| \frac{p}{q} - \frac{m}{n} \right| < \frac{C}{n^2},$$

which cannot hold for $n > Cq$.

Next, we claim that, for all $M, N \in \mathbb{Z}$ and each $i = 1, \ldots, l$,

$$\frac{M}{1 + NQ} \neq \frac{p_i}{q_i}.$$
Indeed, if the equality holds then $Mq_i = p_i(1 + NQ)$. The right hand side is divisible by $q_i$ but $1 + NQ$ is coprime with $q_i$; this contradicts the assumption that $q_i$ and $p_i$ are coprime.

The two claims combined imply the lemma. \hfill \Box

Next, (5) and the triangle inequality imply that

$$\left| \frac{m_1}{n_i} - \frac{m_2}{n_j} \right| < C \left( \frac{1}{n_i^2} + \frac{1}{n_j^2} \right)$$

for some $m_1, m_2$. It follows that

$$|m_1n_j - m_2n_i| < C \left( \frac{n_j}{n_i} + \frac{n_i}{n_j} \right).$$

The expression in the parentheses on the right hand side has limit 2, as $N \to \infty$, hence one has, for sufficiently great $N$,

$$|m_1n_j - m_2n_i| < 3C. \quad (6)$$

Denote by $\mathcal{M}$ the (finite) set of fractions with the denominators $jQ$, $j \in \{1, 2, \ldots, k\}$, and let $\delta > 0$ be the distance between the sets $S' - \mathbb{Q}$ and $\mathcal{M}$.

**Lemma 7** For sufficiently large $N$, one has:

$$|m_1n_j - m_2n_i| > \delta Q^2 N/2.$$

**Proof.** For $N$ large enough, it follows from (5) that

$$\left| t_l - \frac{m_1}{n_i} \right| < \frac{\delta}{2}.$$

Since $t_l \notin \mathbb{Q}$, it follows that the distance from $m_1/n_i$ to $\mathcal{M}$ is greater than $\delta/2$. One has:

$$|m_1n_j - m_2n_i| = |n_j - n_i| \ n_i \left| \frac{m_1}{n_i} - \frac{m_2 - m_1}{n_j - n_i} \right| > Q \cdot QN \cdot \frac{\delta}{2},$$

as claimed. \hfill \Box

Finally, for large $N$, Lemma 7 contradicts inequality (6), and Proposition 5 follows. \hfill \Box

This Proposition implies Theorem 1 and we are done. \hfill \Box
Remark 8. Theorem II along with its proof, can be extended to billiards in higher dimensional Euclidean spaces: the role of the curve $\gamma$ is played by the shortest geodesic on the boundary of the billiard table connecting $A$ and $B$.

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