SPLITTINGS AND DISJUNCTIONS IN REVERSE MATHEMATICS

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Abstract. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics founded by Friedman and developed extensively by Simpson and others. The aim of RM is to find the minimal axioms needed to prove a theorem of ordinary, i.e. non-set-theoretic, mathematics. As suggested by the title, this paper deals with two (relatively rare) RM-phenomena, namely splittings and disjunctions. As to splittings, there are some examples in RM of theorems $A, B, C$ such that $A \leftrightarrow (B \land C)$, i.e. $A$ can be split into two independent (fairly natural) parts $B$ and $C$. As to disjunctions, there are (very few) examples in RM of theorems $D, E, F$ such that $D \leftrightarrow (E \lor F)$, i.e. $D$ can be written as the disjunction of two independent (fairly natural) parts $E$ and $F$. By contrast, we show in this paper that there is a plethora of (natural) splittings and disjunctions in Kohlenbach’s higher-order RM.

1. Introduction

Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([14, 15]) and developed extensively by Simpson ([37]) and others. We refer to [38] for a basic introduction to RM and to [36, 37] for an (updated) overview of RM. We will assume basic familiarity with RM, the associated ‘Big Five’ systems and the ‘RM zoo’ ([13]). We do introduce Kohlenbach’s higher-order RM in some detail Section 2.1.

As discussed in e.g. [18, §6.4], there are (some) theorems $A, B, C$ in the RM zoo such that $A \leftrightarrow (B \land C)$, i.e. $A$ can be split into two independent (fairly natural) parts $B$ and $C$ (over $\text{RCA}_0$). As to the possibility of $A \leftrightarrow (B \lor C)$, there is [16, Theorem 4.5] which states that a certain theorem about dynamical systems is equivalent to the disjunction of weak König’s lemma and induction for $\Sigma^0_2$-formulas; neither disjunct of course implies the other (over $\text{RCA}_0$). Similar results are in [7] for model theory, but these are more logical in nature.

It is fair to say that there are only few natural examples of splittings and disjunctions in RM, though such claims are invariably subjective in nature. Nonetheless, the aim of this paper is to establish a plethora of splittings and disjunctions in higher-order RM. In particular, we obtain splittings and disjunctions involving (higher-order) $\text{WWKL}_0$, the Big Five, and $\mathbb{Z}_2$, among others. We similarly treat the covering theorems Cousin’s lemma and Lindelöf’s lemma studied in [31]. Our main results are in Section 3 while a summary may be found in Section 4, our base theories are conservative over $\text{WKL}_0$ (or are strictly weaker).

It goes without saying that our results highlight a major difference between second- and higher-order arithmetic, and the associated development of RM. We leave it the reader to draw conclusions from this observation.

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2. Preliminaries

2.1. Higher-order Reverse Mathematics. We sketch Kohlenbach’s higher-order Reverse Mathematics as introduced in [21]. In contrast to ‘classical’ RM, higher-order RM makes use of the much richer language of higher-order arithmetic.

As suggested by its name, higher-order arithmetic extends second-order arithmetic. Indeed, while the latter is restricted to numbers and sets of numbers, higher-order arithmetic also has sets of sets of numbers, sets of sets of sets of numbers, et cetera. To formalise this idea, we introduce the collection of all finite types $\mathbf{T}$, defined by the two clauses:

(i) $0 \in \mathbf{T}$ and (ii) If $\sigma, \tau \in \mathbf{T}$ then $(\sigma \rightarrow \tau) \in \mathbf{T}$,

where $0$ is the type of natural numbers, and $\sigma \rightarrow \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \rightarrow 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \rightarrow 0$. Viewing sets as given by characteristic functions, we note that $\mathbb{Z}_2$ only includes objects of type $0$ and $1$.

The language $L_0$ includes variables $x^\rho, y^\sigma, z^\rho, \ldots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of $L_\omega$ includes the type $0$ objects $0, 1$ and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of ‘$\equiv$’ as follows: for any objects $x^\tau, y^\tau$, we have

$$[x =_{\tau} y] \equiv (\forall z_1^{\tau_1} \ldots z_k^{\tau_k})[x z_1 \ldots z_k =_0 y z_1 \ldots z_k],$$

(2.1)

if the type $\tau$ is composed as $\tau \equiv (\tau_1 \rightarrow \ldots \rightarrow \tau_k \rightarrow 0)$. Furthermore, $L_\omega$ also includes the recursor constant $R_\sigma$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type $\sigma$-objects as in the special case $\mathbf{22}$. Formulas and terms are defined as usual.

Definition 2.1. The base theory RCA$_0^\omega$ consists of the following axioms:

1. Basic axioms expressing that $0, 1, <_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.
2. Basic axioms defining the well-known $\Pi$ and $\Sigma$ combinators (aka $K$ and $S$ in [11]), which allow for the definition of $\lambda$-abstraction.
3. The defining axiom of the recursor constant $R_0$: For $m^0$ and $f^1$:

$$R_0(f, m, 0) := m \text{ and } R_0(f, m, n + 1) := f(R_0(f, m, n)).$$

(2.2)

4. The axiom of extensionality: for all $\rho, \tau \in \mathbf{T}$, we have:

$$(\forall x^\rho, y^\rho, \varphi^{\rho \rightarrow \tau})(x =_{\rho} y \rightarrow \varphi(x) =_{\tau} \varphi(y)).$$

(E$_{\rho, \tau}$)

6. QF-AC$^{1,0}$: The quantifier-free axiom of choice as in Definition 2.2

Definition 2.2. The axiom QF-AC consists of the following for all $\sigma, \tau \in \mathbf{T}$:

$$(\forall x^\sigma)(\exists y^\tau)A(x, y) \rightarrow (\exists y^{\sigma \rightarrow \tau})(\forall x^\sigma)A(x, Y(x)),$$

(QF-AC$^{\sigma, \tau}$)

for any quantifier-free formula $A$ in the language of $L_\omega$.

As discussed in [21, §2], RCA$_0^\omega$ and RCA$_0$ prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in (2.2)

$^1$To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language $L_\omega$: only quantifiers are banned.
is called primitive recursion; the class of functionals obtained from $R_n$ for all $\rho \in T$ is called Gödel’s system $T$ of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functionals, as introduced in [21] p. 288-289.

**Definition 2.3** (Real numbers and related notions in $\text{RCA}_0^\omega$).

(1) Natural numbers correspond to type zero objects, and we use ‘$n^0$’ and ‘$n \in \mathbb{N}$’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘$q \in \mathbb{Q}$’ and ‘$<_{\mathbb{Q}}$’ have their usual meaning.

(2) Real numbers are coded by fast-converging Cauchy sequences $q(\_): \mathbb{N} \to \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)((q_{n+i} - q_n) | <_{\mathbb{Q}} \frac{1}{2^n})$. We use Kohlenbach’s ‘hat function’ from [21] p. 289 to guarantee that every $f^1$ defines a real number.

(3) We write ‘$x \in \mathbb{R}$’ to express that $x^1 := (q^1(\_))$ represents a real as in the previous item and write $[x](k) := q_k$ for the $k$-th approximation of $x$. We use the usual notations for natural, rational, and real numbers, and the $\text{RCA}_0$ proves $[\mathbb{A}]_{\text{ECF}}$, again ‘up to language’, as $\text{RCA}_0$ is formulated using sets, and $\mathbb{A}_{\text{ECF}}$ is formulated using types, namely only using type zero and one objects.

Finally, we mention the $\text{ECF}$-interpretation, of which the technical definition may be found in [11] p. 138, 2.6. Intuitively speaking, the $\text{ECF}$-interpretation $[\mathbb{A}]_{\text{ECF}}$ of a formula $\mathbb{A} \in \mathbb{L}_\omega$ is just $\mathbb{A}$ with all variables of type two and higher replaced by countable representations of continuous functionals. The $\text{ECF}$-interpretation connects $\text{RCA}_0^\omega$ and $\text{RCA}_0$ (See [21] Prop. 3.1) in that if $\text{RCA}_0^\omega$ proves $\mathbb{A}$, then $\text{RCA}_0$ proves $[\mathbb{A}]_{\text{ECF}}$, again ‘up to language’, as $\text{RCA}_0$ is formulated using sets, and $\mathbb{A}_{\text{ECF}}$ is formulated using types, namely only using type zero and one objects.

**2.2. Some axioms of higher-order arithmetic.** We introduce some functionals which constitute the counterparts of $\mathbb{Z}_2$, and some of the Big Five systems, in higher-order RM. We use the formulation of these functionals as in [21].

First of all, $\text{ACA}_0$ is readily derived from the following ‘Turing jump’ functional:

$$(\exists \varphi \leq_2 1)(\forall f^1)([\exists n](f(n) = 0) \leftrightarrow \varphi(f) = 0]. \tag{\exists^2}$$

and $\text{ACA}_0^\omega \equiv \text{RCA}_0^\omega + (\exists^2)$ proves the same $\Pi^1_2$-sentences as $\text{ACA}_0$ by [35] Theorem 2.2]. This functional is discontinuous at $f = 11\ldots$, and $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_\mathbb{R} 0$, and 0 otherwise (21 §3). Secondly, $\Pi^1_1$-$\text{CA}_0$ is readily derived from the following ‘Suslin functional’:

$$(\exists S^2 \leq_2 1)(\forall f^1)([\exists g^0](\forall x^0)(f(\overline{x}^0) = 0) \leftrightarrow S(f) = 0], \tag{S^2}$$

and $\Pi^1_1$-$\text{CA}_0^\omega \equiv \text{RCA}_0^\omega + (S^2)$ proves the same $\Pi^1_2$-sentences as $\Pi^1_1$-$\text{CA}_0$ by [35] Theorem 2.2]. By definition, the Suslin functional $S^2$ can decide whether a $\Sigma^1_2$-formula (as in the left-hand side of $(S^2)$) is true or false. Note that we allow formulas with (type one) function parameters, but not with (higher type) functional parameters. Thirdly, full second-order arithmetic $\mathbb{Z}_2$ is readily derived from the sentence:

$$(\exists E^3 \leq_3 1)(\forall Y^2)([(\exists f^1)(f(Y) = 0) \leftrightarrow E(Y) = 0], \tag{\exists^3}$$
and we define $\mathbb{Z}^2 \equiv \text{RCA}_0^\omega + (\exists^2)$. The (unique) functional from $(\exists^3)$ is also called $\bar{\exists}^3$, and we will use a similar convention for other functionals.

Fourth, weak König’s lemma\(^2\) (WKL hereafter) easily follows from both the ‘intuitionistic’ and ‘classical’ fan functional, which are defined as follows:

\[(\exists^3)(\forall Y^2)(\forall f, g \in C)(\exists \Omega(Y) = \exists \Omega(Y) \rightarrow Y(f) = Y(g)),\]

\[(\exists^3)(\forall Y^2 \in \text{cont})(\forall f, g \in C)(\exists \Phi(Y) = \exists \Phi(Y) \rightarrow Y(f) = Y(g)),\]

where $\forall Y^2 \in \text{cont}$ means that $Y$ is continuous on Baire space $\mathbb{N}^\omega$. Clearly, $\exists^2, S^2$, and $\exists^3$ are a kind of comprehension axiom. As it turns out, the comprehension for Cantor space functional also yields a conservative extension of WKL:\(^3\)

\[(\exists^3)(\forall Y^2)[k_0(Y) = 0 \leftrightarrow (\exists f \in C)(Y(f) > 0)].\]

Finally, recall that the Heine-Borel theorem (aka Cousin’s lemma) states the existence of a finite sub-cover for an open cover of a compact space. Now, a functional $\Psi : \mathbb{R} \rightarrow \mathbb{R}^+$ gives rise to the canonical cover $\cup_{x \in I} I^\Psi_x$ for $I \equiv [0, 1]$, where $I^\Psi_x$ is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable cover $\cup_{x \in I} I^\Psi_x$ has a finite sub-cover by the Heine-Borel theorem; in symbols:

\[(\forall \Psi : \mathbb{R} \rightarrow \mathbb{R}^+)(\exists (y_1, \ldots, y_k))(\forall x \in I)(\exists i \leq k)(x \in I^\Psi_{y_i}).\]

By the results in \(^3\)1\(^3\), $\mathbb{Z}^2$ proves HBU, but $\Pi^1_3$-CA$^\omega_0$ cannot (for any $k$). The importance and naturalness of HBU is discussed in Section 4.

Furthermore, since Cantor space (denoted $C$ or $2^\mathbb{N}$) is homeomorphic to a closed subset of $[0, 1]$, the former inherits the same property. In particular, for any $G^2$, the corresponding ‘canonical cover’ of $2^\mathbb{N}$ is $\cup_{\sigma \in 2^\mathbb{N}} \sigma[G(f)]$ where $[\sigma^0]$ is the set of all binary extensions of $\sigma$. By compactness, there is a finite sequence $(f_0, \ldots, f_r)$ such that the set of $\cup_{i \leq r} [f_i, F(f_i)]$ still covers $2^\mathbb{N}$. By \(^3\)1 Theorem 3.3], HBU is equivalent to the same compactness property for $C$, as follows:

\[(\forall G^2)(\exists (f_1, \ldots, f_r))(\forall f^1 \leq 1)(\exists i \leq k)(f \in [f_i, G(f_i)]).\]

Finally, we need a ‘trivially uniform’ version of ATR$^0$:

\[(\exists^3)(\forall X^1, f^1)[\text{WO}(X) \rightarrow H_f(X, \Phi(X, f))],\]

where WO$(X)$ expresses that $X$ is a countable well-ordering and $H_\theta(X, Y)$ expresses that $Y$ is the result from iterating $\theta$ along $X$ (See \(^3\)7 [V] for details), and where $H_f(X, Y)$ is just $H_\theta(X, Y)$ with $\theta(n, Z)$ defined as $\exists m^0(f(n, m, Zm) = 0)$.

3 Main Results

Our motivation and starting point is the splitting $(\exists^3) \leftrightarrow ((\exists^3^0) + (\exists^2^0))$ communicated to us by Kohlenbach\(^4\) (See \(^3\)0 Rem. 6.13]). It is then a natural question if $(\exists^3)$ can be split further, as discussed in Section 6.1. We obtain similar results for MUC in Section 3.2, which yields splittings and disjunctions for $(\exists^2), (\exists^3), (Z^3)$, and FF in Section 3.3. We similarly study HBU in Section 3.4 while other covering theorems, including the original Lindelöf lemma, are discussed in Section 3.5.

\(^2\)Note that we take ‘WKL’ to be the $L_2$-sentence every infinite binary tree has a path as in \(^3\)7, while the Big Five system WKL$^0$ is $\text{RCA}_0 + \text{WKL}$, and WKL$^\omega$ is $\text{RCA}_0^\omega + \text{WKL}$.

\(^3\)The proof amounts to the observation that $\mathbb{N}^\omega$ is recursively homeomorphic to a $\Pi^1_3$-subset of Cantor space. Since this set is computable in $\exists^2$, any oracle call to $\exists^3$ can be rewritten to an equivalent oracle call to $\exists^3$ in a uniform way.
3.1. Comprehension on Cantor space. We show that $(\kappa_0)$ splits into the classical fan functional, and a functional which tests for continuity on $\mathbb{N}^N$, as follows:

$$(\exists^3)(\forall^2 Y^2)[Z(Y) = 0 \leftrightarrow (\forall f^1)(\exists N^0)(\forall g^1)((\overline{\exists} N \rightarrow Y(f) = Y(g))]. \quad (Z^3)$$

We will tacitly use $(\exists^2) \rightarrow \text{FF} \rightarrow \text{WKL}$, which holds over $\text{RCA}_0^\omega$ by [22, Prop. 4.10].

**Theorem 3.1.** The system $\text{WKL}_0^\omega + \text{QF-AC}^{2,0}$ proves $(\kappa_0^3) \leftrightarrow [(Z^3) + \text{FF}].$

**Proof.** For the forward implication, we work in $\text{WKL}_0^\omega + \text{QF-AC}^{2,0} + (\kappa_0^3).$ In case of $(\exists^3)$, we also have $(\exists^3)$, and the latter functional readily implies $(Z^3)$ and FF. In case of $(\exists^3)$, all functionals $Y^2$ are continuous on Baire space by [21, Prop. 3.7], and $Z_0 = 3$ is as required for $(Z^3).$ By WKL (and [22, Prop. 4.10]), all functionals $Y^2$ are uniformly continuous on Cantor space, i.e.

$$(\forall Y^2)(\exists N^0)(\forall f^1, g^1 \in C)((\overline{\exists} N \rightarrow Y(f) = Y(g))),$$

and the underlined formula may be treated as quantifier-free by $(\kappa_0^3).$ Applying QF-AC$^{2,0}$, we obtain FF. The law of excluded middle finishes this implication.

For the reverse implication, we work in $\text{RCA}_0^\omega + (Z^3) + \text{FF}$. In case of $(\exists^3)$, all functionals $Y^2$ are continuous on Baire space by [21, Prop. 3.7], and FF readily implies $(\kappa_0^3)$ by noting that the latter restricted to $Y^2$ uniformly continuous on $C$ is trivial. In case of $(\exists^3)$, let $Y_0$ be $Y$ on $C$, and zero otherwise. Now define $\kappa_0$ as follows: in case $Z(Y_0) = 0$, $Y_0$ is continuous on Cantor space, and use FF to decide whether $(\exists f \in C)(Y(f) > 0)$; in case $Z(Y_0) \neq 0$, then $(\exists f \in C)(Y(f) > 0)$, and $\kappa_0(Y) := 0.$ The law of excluded middle finishes this implication. \qed

**Corollary 3.2.** The system $\text{RCA}_0^\omega + \text{QF-AC}^{2,0}$ proves $[(\kappa_0^3) + \text{WKL}] \leftrightarrow [(Z^3) + \text{FF}]$ and the system $\text{RCA}_0^\omega$ proves $(\exists^3) \leftrightarrow [(\exists^2) + (Z^3)].$

3.2. The intuitionistic fan functional. A hallmark of intuitionistic mathematics is Brouwer’s continuity theorem which expresses that all functions on the unit interval are (uniformly) continuous (S). In the same vein, the intuitionistic fan functional $\Omega^3$ as in MUC provides a modulus of uniform continuity on Cantor space. This axiom can be split nicely into classical and non-classical parts as follows.

**Theorem 3.3.** The system $\text{RCA}_0^\omega + \text{QF-AC}^{2,0}$ proves

$\text{MUC} \leftrightarrow [(\kappa_0^3) + \text{WKL} + \neg(\exists^3)] \leftrightarrow [(\kappa_0^3) + \text{WKL} + \neg(\exists^2)] \leftrightarrow [(\kappa_0^3) + \text{WKL} + \neg(\exists^3)].$

**Proof.** For the first equivalence, assume MUC and note that the latter reduces the decision procedure for $(\exists f \in C)(Y(f) > 0)$ to a finite search involving only $2^{\Omega(Y)}$ sequences. Furthermore, $(\exists^2)$ clearly implies the existence of a discontinuous function on Cantor space, i.e. $\text{MUC} \rightarrow \neg(\exists^2)$ follows, while $\text{MUC} \rightarrow \text{WKL}$ follows from [37, IV.2.3]. Now assume $(\kappa_0^3) + \text{WKL} + \neg(\exists^2)$ and recall that by the latter all functionals $Y^2$ are continuous on Baire space by [21, Prop. 3.7]. By WKL (and [22, Prop. 4.10]), all functionals $Y^2$ are uniformly continuous on Cantor space, i.e.

$$(\forall Y^2)(\exists N^0)(\forall f^1, g^1 \in C)((\overline{\exists} N \rightarrow Y(f) = Y(g))), \quad (3.1)$$

and the underlined formula may be treated as quantifier-free by $(\kappa_0^3).$ Applying QF-AC$^{2,0}$, we obtain MUC. For the remaining equivalences, since $(\exists^3) \leftrightarrow [(\kappa_0^3) + (\exists^2)]$, we have that $\neg(\exists^3) 

\rightarrow (\exists^2), \text{ and the same for } \neg(\exists^2). \text{ Finally, note that } (\exists^3) \rightarrow (S^2) \rightarrow (\exists^2) \text{ implies } \neg(\exists^2) \rightarrow \neg(\exists^2) \rightarrow \neg(\exists^3). \quad \Box
Recall the ECF-interpretation introduced at the end of Section 2.1. By [21] §9.5, we have \([\text{MUC}]_{\text{ECF}} \leftrightarrow \text{WKL}\) and \(\text{WKL} \rightarrow [(\kappa_0^3)]_{\text{ECF}}\), while \([(\exists^2)]_{\text{ECF}} \leftrightarrow (0 = 1)\) as \(\exists^2\) is discontinuous (and therefore has no countable representation). Hence, \(\neg(\exists^2)\) cannot be replaced by \(\neg\text{ACA}_0\) in the theorem, as \([A]_{\text{ECF}} \leftrightarrow A\) for \(A \in L_2\).

Furthermore, the axiom MUC can also be split as follows.

**Corollary 3.4.** The system \(\text{RCA}_0^\omega + \text{QF-AC}^{2.0}\) proves

\[
\text{MUC} \leftrightarrow [(\exists^2) + \neg(\exists^2)] \leftrightarrow [(\exists^2) + \neg(\exists^2)] \leftrightarrow [(\exists^2) + \neg(\exists^2)].
\]

**Proof.** By Corollary 3.3 and the theorem, we have \(\text{MUC} \leftrightarrow [(\exists^2) + \neg(\exists^2)]\), and we may omit \((\exists^2)\) because all functionals on \(\mathbb{N}\) are continuous given \(\neg(\exists^2)\).

By the same corollary, \([(\exists^2) + \neg(\exists^2)] \leftrightarrow [(\kappa_0^3)] + \text{WKL} + \neg(\exists^2)]\), and the latter is equivalent to MUC by the theorem. The same reasoning applies to \(\neg(\exists^2)\). \(\square\)

As a result of the previous, the RM of \((\kappa_0^3)\) is pretty robust. Indeed, for a sentence \(W\) implying \((\kappa_0^3)\), if the former implies the existence of a discontinuous functional, we obtain \((\exists^2)\) by [21] §8. What happens when \(W\) does not imply this existence, is captured (in part) by the following theorem.

**Theorem 3.5.** If MUC \(\rightarrow W\) and \(\neg(\exists^2) \rightarrow W \rightarrow (\kappa_0^3)\) over \(\text{RCA}_0^\omega\), then \(\text{WKL}_0^\omega + \text{QF-AC}^{2.0}\) proves \(W \leftrightarrow (\kappa_0^3)\).

**Proof.** The forward implication is immediate. For the reverse implication, consider \((\exists^2) \lor \neg(\exists^2)\); in the former case, we obtain \((\exists^2)\) and hence \(W\), while in the latter case, we may use the proof of Theorem 3.3 to the continuity of all functionals on Baire space and WKL imply [3.1], which yields MUC thanks to \((\kappa_0^3)\) and QF-AC^{2.0}, and \(W\) follows by assumption. \(\square\)

### 3.3. More splittings and disjunctions.

The results regarding the non-classical axiom MUC also yield splittings for the classical axioms FF, \((\exists^2)\), \((\exists^3)\), and \((Z^3)\).

**Theorem 3.6.** The system \(\text{RCA}_0^\omega + \text{QF-AC}^{2.0}\) proves

\[
[(\kappa_0^3) + \text{WKL}] \leftrightarrow [(\exists^2) \lor \text{MUC}] \leftrightarrow [(\kappa_0^3) + \text{FF}];
\]

\(\text{RCA}_0^\omega\) proves FF \(\leftrightarrow [(\exists^2) \lor \text{MUC}]\), while \(\text{WKL}_0^\omega\) proves FF \(\leftrightarrow [(\exists^2) \lor (\kappa_0^3)]\).

**Proof.** For the first equivalence in (3.2), the reverse implication is immediate if \((\exists^2)\) holds, while it follows from Theorem 3.3 if MUC holds. For the forward implication, if \((\exists^2)\), we have \((\exists^3)\), while if \(\neg(\exists^2)\), we follow the proof of Theorem 3.3 to obtain MUC. The second equivalence in (3.2) follows in the same way. For the third equivalence, the reverse implication is immediate, while the forward implication follows by considering \((\exists^2) \lor \neg(\exists^2)\), noting that all functionals on \(C\) are continuous in the latter case. For the final equivalence, we only need to prove \((\kappa_0^3) \rightarrow \text{FF}\) given \(\text{WKL}\). The implication is immediate if \((\exists^2)\), while it follows in the same way as in the proof of Theorem 3.2 in case \(\neg(\exists^2)\). \(\square\)

**Theorem 3.7.** The system \(\text{RCA}_0^\omega + \text{QF-AC}^{2.0}\) proves \((\exists^2) \leftrightarrow [\text{FF} \lor \neg\text{MUC}]\) and \((\exists^3) \leftrightarrow [\text{FF} + (Z^3) \lor \neg\text{MUC}]\) and \((Z^3) \leftrightarrow [(\exists^2) \lor (\exists^3)] \leftrightarrow [(\exists^2) \lor (\exists^3) \lor \neg\text{FF} \lor \text{MUC}]\).

**Proof.** The second equivalence follows from the first one by Corollary 3.2. For the first equivalence, the forward implication is immediate, and for the reverse implication, Corollary 3.4 implies \(\neg\text{MUC} \leftrightarrow (\neg\text{FF} \lor (\exists^3))\). Since FF is assumed, we obtain \((\exists^2)\). For the third equivalence, the reverse implication is immediate in case
(3.3), while $Z=\exists \exists 0$ works if $\neg(3^2)$ as all functionals on Baire space are continuous then; for the forward implication, consider $(3^2) \lor \neg(3^2)$ and use Corollary 3.2 in the former case. The final equivalence now follows from the first equivalence. \hfill \Box

### 3.4. Heine-Borel compactness.

We discuss the rich world of splittings and disjunctions associated to WKL. First of all, we provide a disjunction for WKL.

**Theorem 3.8.** The system RCA$_0^+$ proves that

$$WKL \leftrightarrow [(3^2) \lor HBU] \leftrightarrow [X \lor HBU] \leftrightarrow [\exists x \lor HBU] \leftrightarrow [\forall x \lor HBU].$$

(3.3)

for any $X \in L_2$ such that ACA$_0 \rightarrow X \rightarrow WKL_0$.

**Proof.** We prove the first equivalence and note that the other equivalences in (3.3) follow in the same way. The reverse implication follows from HBU$_c \rightarrow WKL$ and $(3^2) \rightarrow ACA_0 \rightarrow WKL_0$. For the forward implication, note that all functionals on $\mathbb{N}$ are continuous given $\neg(3^2)$, and hence uniformly continuous on $C$ by WKL. Hence, all functionals on $C$ have an upper bound, which immediately implies HBU$_c$. The law of excluded middle $(3^2) \lor \neg(3^2)$ finishes the proof. \hfill \Box

Secondly, let $T_1$ be [10] Theorem 4.5.2 i.e. the $L_2$-sentence: *For all $k \in \mathbb{N}$ and all compact metric spaces $X$ and continuous functions $F : X \rightarrow X$, $F^k$ is a continuous function from $X$ into $X.* Note that over RCA$_0$, the statement $T_1$ is equivalent to $WKL \lor \Sigma_2^0 \text{-IND}$, where the latter is the induction schema restricted to $\Sigma_2^0$-formulas.

**Corollary 3.9.** The system RCA$_0^+$ proves

$$T_1 \leftrightarrow [HBU \lor \Sigma_2^0 \text{-IND}] \leftrightarrow [WKL \lor \Sigma_2^0 \text{-IND}].$$

(3.4)

**Proof.** We only need to prove the first equivalence. The reverse direction is immediate as $HBU \rightarrow WKL \rightarrow T_1$ and $\Sigma_2^0 \text{-IND} \rightarrow T_1$. For the forward direction,

$$T_1 \rightarrow [WKL \lor \Sigma_2^0 \text{-IND}] \rightarrow [ACA_0 \lor HBU \lor \Sigma_2^0 \text{-IND}],$$

since ACA$_0$ implies $\Sigma_2^0 \text{-IND}$ (for any $k$), and we obtain the equivalence in (3.4). \hfill \Box

Thirdly, while (3.3) and (3.4) may come across as *spielerei*, WKL $\leftrightarrow [ACA_0 \lor HBU]$ is actually of great conceptual importance, as follows.

**Template 3.10.** To prove a theorem $T$ in WKL$_0^+$, proceed as follows:

(a) Prove $T$ in ACA$_0$ (or even using $3^2$), which is *much* easier than in WKL$_0$.

(b) Prove $T$ in RCA$_0^+$ + HBU using the existing ‘uniform’ proof from the literature based on Cousin’s lemma (See e.g. [3, 4, 6, 17, 19, 33, 40]).

(c) Conclude from (a) and (b) that $T$ can be proved in WKL$_0^+$.

Hence, even though the goal of RM is to find the *minimal* axioms needed to prove a theorem, one can nonetheless achieve this goal by (only) using non-minimal axioms. We leave it to the reader to ponder how much time and effort could have been (and will be) saved using the previous three steps (for WKL or other axioms). As an exercise, the reader should try to prove *Pincherle’s theorem* ([25, p. 97]) via Template 3.10 using realisers for the antecedent as in the original [34].

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4 For instance, the functional $3^2$ uniformly converts between binary-represented reals and reals-as-Cauchy-sequences. In this way, one need not worry about representations and the associated extensionality like in Definition 3.3. By the proof of [23] Prop. 4.7, $3^2$ also uniformly converts a continuous function into an RM-code, i.e. we may ‘recycle’ proofs in second-order arithmetic.
Fourth, in [7 Theorem 2.28], an equivalence between \( \neg \text{WKL}_0 \lor \text{ACA}_0 \) and the following theorem is established: *there is a complete theory with a non-principal type and only finitely many models up to isomorphism.* The contraposition of the latter, which we shall denote \( T_0 \) and satisfies \( T_0 \leftrightarrow \text{WKL}_0 + \neg \text{ACA}_0 \), is described in [7] as a peculiar but natural statement about some pre-ordering.

**Corollary 3.11.** The system RCA^0_0 + T_0 proves HBU. The system RCA^\omega_0 proves \( T_0 \leftrightarrow [\text{WKL}_0 \land \neg \text{ACA}_0] \leftrightarrow [\text{HBU} \land \neg \text{ACA}_0] \).

*Proof.* In \( \neg T_0 \leftrightarrow [\neg \text{WKL}_0 \lor \text{ACA}_0] \), use (3.3) to replace WKL by \( \text{ACA}_0 \lor \text{HBU} \), i.e.

\[
\neg T_0 \leftrightarrow [\neg \text{ACA}_0 \land \neg \text{HBU}] \lor \text{ACA}_0 \leftrightarrow [\neg \text{ACA}_0 \lor \text{ACA}_0] \land [\text{ACA}_0 \lor \neg \text{HBU}].
\]

Omitting the underlined formula, the second (and first) part follows. \( \square \)

By the previous, the negation of WKL_0 or ACA_0 implies axioms of Brouwer’s intuitionistic mathematics, i.e. strange (as in ‘non-classical’) behaviour is almost guaranteed. The equivalence involving \( \neg \text{WKL}_0 \lor \text{ACA}_0 \) remains surprising. By contrast, \( T_0 \) seems fairly normal, relative to e.g. \( T_1 \), by the following result.

**Corollary 3.12.** RCA_0 proves \( T_1 \leftrightarrow (T_0 \lor \Sigma^0_2\text{-IND}) \) and \( \text{WKL}_0 \leftrightarrow [\text{ACA}_0 \lor T_0] \).

*Proof.* The second forward implication follows from \( \text{ACA}_0 \lor \neg \text{ACA}_0 \), while the second reverse implication is immediate. The first reverse implication is immediate, while the first forward implication follows from:

\[
T_1 \to [\text{WKL}_0 \lor \Sigma^0_2\text{-IND}] \to [\text{ACA}_0 \lor T_0 \lor \Sigma^0_2\text{-IND}] \to [T_0 \lor \Sigma^0_2\text{-IND}],
\]

since \( \text{ACA}_0 \) proves induction for any arithmetical formula. \( \square \)

Finally, the negation of HBU also occurs naturally as follows.

**Theorem 3.13.** The system RCA^\omega_0 + \text{FF} + \text{QF-AC}^{2.1} proves \((\exists^2) \leftrightarrow [\text{UATR} \lor \neg \text{HBU}]\).

*Proof.* For the forward implication, consider \( \text{HBU} \lor \neg \text{HBU} \). In the former case, we obtain UATR by [30, Cor. 6.6] and [31, Theorem 3.3]. For the reverse implication, \( [\neg \text{HBU} \lor \text{FF}] \to (\exists^2) \), which follows from MUC \( \to \text{HBU} \) and Theorem 3.7. \( \square \)

It is a natural RM-question, posed previously by Hirschfeldt (see [26, §6.1]), whether the extra axioms are needed in the base theory of Theorem 3.13. The answer is positive in this case: the ECF-translation converts the equivalence in the theorem to \((0 = 1) \leftrightarrow [(0 = 1) \lor \neg \text{WKL}]\), which is only true if we have WKL (which is exactly \( [\text{FF}]_{\text{ECF}} \)) in the base theory. Hence, the base theory needs (at least) WKL.

The above results, (3.3) and (3.4), in particular, suggests that mathematical naturalness does not inherit to disjuncts, which is in accordance with our intuitions.

**3.5. Other covering theorems.** We study two covering lemmas related to HBU, namely the Lindelöf lemma and a weak version of HBU.

**3.5.1. The Lindelöf lemma.** We study splittings and disjunctions for the Lindelöf lemma LIN from [31]. We stress that our formulation of HBU and LIN is faithful to the original theorems from 1895 and 1903 by Cousin ([11]) and Lindelöf ([23]).

**Definition 3.14.** [LIN] For every \( \Psi : \mathbb{R} \to \mathbb{R}^+ \), there is a sequence of open intervals \( \cup_{n \in \mathbb{N}} (a_n, b_n) \) covering \( \mathbb{R} \) such that \((\forall n \in \mathbb{N}) (\exists x \in \mathbb{R}) [(a_n, b_n) \cap I^x_x = \emptyset] \).

**Theorem 3.15.** Let \( X \in L_2 \) be such that \( \text{ACA}_0 \to X \to \text{WKL}_0 \).
(1) The system $\text{RCA}_0^e + \text{QF-AC}^{0,1}$ proves $\text{LIN} \leftrightarrow (\exists \bar{Y}) \vee \neg \text{WWKL} \leftrightarrow (\exists \bar{Y}) \vee \neg X$.

(2) If $Y \in L_2$ is provable in $\text{ACA}_0$ but not in $\text{RCA}_0$, then $\text{RCA}_0^e$ proves $Y \vee \neg \text{LIN}$.

Proof. For the first item, $\text{RCA}_0^e + \text{QF-AC}^{0,1}$ proves $[\text{LIN} \vee \text{WWKL}] \leftrightarrow \text{HBU}$ by \cite{31} Theorem 3.13. Hence, the first forward implication follows from $\text{WWKL} \vee \neg \text{WWKL}$. For the first reverse implication, $\text{LIN}$ follows from $\text{HBU}$ by the aforementioned equivalence. In case $\neg \text{WWKL}$ holds, we also have $(\exists \bar{Y}) \vee \neg (\exists \bar{Y}) \leftrightarrow \text{WWKL}$. Hence, all functionals on $\mathbb{R}$ are continuous by \cite{21} Prop. 3.12, and the countable sub-cover provided by the rationals suffices for the conclusion of $\text{LIN}$. The second equivalence follows in the same way by considering $X \vee \neg X$. For the second item, consider $(\exists \bar{Y}) \vee \neg (\exists \bar{Y})$. \hfill $\square$

Finally, we discuss foundational implications of our results. Now, \cite{33} implies:

$$\neg \text{LIN} \leftrightarrow (\exists \bar{Y}) \vee \neg \text{HBU} \text{ and } \neg \text{LIN} \rightarrow (\exists \bar{Y}).$$

(3.5)

On one hand, thanks to the ECF-translation, $\text{WWKL}^e + \text{HBU}$ is a conservative extension of $\text{WWKL}_0$, which in turn is a $\Pi^1_2$-conservative extension of primitive recursive arithmetic $\text{PRA}$. The latter is generally believed to correspond to Hilbert’s finitistic mathematics (\cite{39}). Hence, following Simpson’s remarks on finitistic mathematics (\cite{37} IX.3.18), $\text{WWKL}^e + \text{HBU}$ also contributes to the partial realisation of Hilbert’s program for the foundations of mathematics. On the other hand, $\text{RCA}_0^e + \text{WWKL} + \neg \text{HBU}$ and $\text{RCA}_0^e + \text{QF-AC}^{0,1} + \neg \text{LIN}$ imply $(\exists \bar{Y})$, i.e. these systems do not contribute to Hilbert’s program in the aforementioned way.

Hence, if one values partial realisations of Hilbert program (which are called ‘very important’ by Simpson in \cite{37} IX.3.18), then $\text{HBU}$ and $\text{LIN}$ are practically forced upon one, in light of the previous. However, these covering lemmas require full second-order arithmetic as in $\mathbb{Z}^2$ for a proof, i.e. they fall far outside of the Big Five classification of RM.

3.5.2 Weak Heine-Borel compactness. We study WHBU, a weak version of HBU based on weak weak König’s lemma (WWKL hereafter; see \cite{37} X.1). Note that WWKL is exceptional in that it is a theorem from the RM zoo that does sport a number of equivalences involving natural/mathematical statements.

In particular, by \cite{37} X.1.9, WWKL is equivalent to the statement that any cover $\cup_{n \in \mathbb{N}} (a_n, b_n) \subset [0, 1]$ is such that $\sum_{i=0}^{\infty} |a_n - b_n| \geq 1$, which is of independent historical interest. We define the higher-order version of this covering theorem as:

$$(\forall \Psi : \mathbb{R} \rightarrow \mathbb{R}^+, k \in \mathbb{N})(\exists \langle y_1, \ldots, y_n \rangle)\left(\sum_{i=1}^{n} |I_{y_i}^\Psi| \geq 1 - \frac{1}{2^k}\right) \quad \text{(WHBU)}$$

We could also use the statement $\text{HBU}_{\text{mil}}$ from \cite{33} §3.3 instead of WHBU, but the latter is more elegant, and does not depend on the notion of randomness.

Theorem 3.16. Let $X \in L_2$ be such that $\text{ACA}_0 \rightarrow X \rightarrow \text{WWKL}_0$. $\text{RCA}_0^e$ proves

$$\text{WWKL} \leftrightarrow [(\exists \bar{Y}) \vee \text{WHBU}] \leftrightarrow [X \vee \text{WHBU}].$$

(3.6)

Proof. This theorem is proved in the same way as Theorem 3.15. Indeed, for the first forward implication, consider $(\exists \bar{Y}) \vee \neg (\exists \bar{Y})$ and note that in the latter case $\cup_{y \in [0, 1]} \bigcup_{i=1}^{\infty} I_{y_i}^\Psi$ is a countable sub-cover of the canonical cover since all functions are continuous. The first reverse implication is trivial in light of \cite{37} X.1.9, and the other equivalences are proved similarly (and using Theorem 3.15). \hfill $\square$

\footnote{It is an interesting historical tidbit that a two-dimensional version of [37] X.1.9.3 was Borel’s motivation for formulating and proving the (countable) Heine-Borel theorem (See [9] p. 50, Note).}
The following version of Corollary 3.17 for WHBU is readily proved based on WWKL ∨ Σ^0_2-IND and (3.6). We can prove similar results for the strong bounding principles and bounded comprehension principles instead of induction (37, p. 72).

**Corollary 3.17.** The system RCA_0^ω proves

\[ \text{WHBU} \lor \Sigma^0_2-\text{IND} \iff \text{WWKL} \lor \Sigma^0_2-\text{IND}. \]

We can also obtain a version of Theorem 3.15 for WHBU.

**Corollary 3.18.** Let \( X \in L_2 \) be such that ACA_0 \( \rightarrow X \rightarrow \text{WWKL}_0 \); the system RCA_0^ω + QF-AC^0,1 proves LIN \( \leftrightarrow \text{WHBU} \lor \neg\text{WWKL} \iff \text{WHBU} \lor \neg X \).

Finally, let \( (n + 1)\)-WWKL be the generalisation of WWKL to trees computable in the \( n \)-th Turing jump, as formulated in [2]. Note that ACA_0 \( \rightarrow (n + 2)\)-WWKL \( \rightarrow (n + 1)\)-WWKL \( \not\rightarrow \text{WKL} \) over RCA_0. While (3.18) applies to \( X = (n + 1)\)-WWKL, we also have the following corollary.

**Corollary 3.19.** For \( n \geq 1 \), RCA_0^ω proves \( \text{HBU} \lor n\text{-WWKL} \iff \text{WKL} \lor n\text{-WWKL} \).

**Proof.** The forward implication is immediate, while for the reverse implication follows from (3.18) as WWKL \( \rightarrow [\text{ACA}_0 \lor \text{HBU}] \rightarrow [n\text{-WWKL} \lor \text{HBU}] \).

Finally, it is a natural question if there are other theorems in the RM zoo for which we can find results like (3.13) and (3.16). We will provide a positive answer for (fragments of) Ramsey’s theorem in a future publication.

4. Conclusion

The following table summarises some of our results, without mentioning the base theory; the latter is however always conservative over WKL_0 (or is weaker). In light of this, we may conclude that the higher-order framework yields plenty of equivalences for disjunctions and splittings, in contrast to the second-order framework.

| MUC \( \leftrightarrow \text{WWKL} + (\kappa_0^3) + \neg (\exists^2) \) | (\kappa_0^3) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\kappa_0^3) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| MUC \( \leftrightarrow \text{WWKL} + (\kappa_3^3) + \neg (\exists^2) \) | (\kappa_0^3) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\kappa_0^3) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| MUC \( \leftrightarrow \text{WWKL} + (\kappa_3^3) + \neg (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + \neg \text{MUC} \) | (\exists^2) \( \leftrightarrow (\exists^2) + \neg \text{MUC} \) |
| MUC \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| \( T_1 \leftrightarrow T_0 \lor \Sigma^0_2\text{-IND} \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| WWKL \( \leftrightarrow (\exists^2) + \text{HBU} \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| WWKL \( \leftrightarrow (\exists^2) + \text{HBU} \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) |
| \( \text{LIN} \leftrightarrow (\exists^2) + \text{HBU} \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) | (\exists^2) \( \leftrightarrow (\exists^2) + (\exists^2) \) |

**Figure 1.** Summary of our results

Finally, Simpson describes the ‘mathematical naturalness’ of logical systems as:

From the above it is clear that the [Big Five] five basic systems \( \text{RCA}_0, \text{WK}_0, \text{ACA}_0, \text{ATR}_0, \Pi^1_1-\text{CA}_0 \) arise naturally from investigations of the Main Question. The proof that these systems are mathematically natural is provided by Reverse Mathematics. (37, I.12)

We leave it to the reader to decide if the aforementioned results bestow naturalness onto the theorems involved in the equivalences. We do wish to point out that some of the theorems in Figure 1 are natural, well-established, and date back more than a century already; we provide a motivation for (3.2) and HBU as follows.

Dirichlet already discusses the characteristic function of the rationals, which is essentially \( \exists^2 \), around 1829 in [12], while Riemann defines a function with countably
many discontinuities via a series in his Habilitationsschrift ( [20] p. 115). Furthermore, the Cousin lemma from [11] p. 22, which is essentially HBU, dates back about 135 years. As shown in [31], HBU and HBU are essential for the development of the gauge integral ( [5] ), which in turn provides a formalisation of the Feynman path integral ( [10], [27], [29] ) and financial mathematics ( [28], [29] ).

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6The collected works of Pincherle contain a footnote by the editors (See [34] p. 67) which states that the associated Teorema (published in 1882) corresponds to the Heine-Borel theorem.
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