Scattering mean-free path in continuous complex media: beyond the Helmholtz equation

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We present theoretical calculations of the ensemble-averaged (a.k.a. effective or coherent) wavefield propagating in a heterogeneous medium considered as one realization of a random process. In the literature, it is usually assumed that heterogeneity can be accounted for by a random scalar function of the space coordinates, termed the potential. Physically, this amounts to replacing the constant wave speed in Helmholtz’ equation by a space-dependent speed. In the case of acoustic waves, we show that this approach leads to incorrect results for the scattering mean-free path, even when fluctuations are weak. At low frequencies, the correct value for the scattering mean-free paths is shown to be more than four times smaller (13/3, precisely), whatever the shape of the correlation function. The detailed calculation of the coherent wavefield must take into account both a scalar and an operator part in the random potential. We present two approaches to solve this problem and obtain the correct value of the mean-free path. Based on the diagrammatic approach of multiple scattering, theoretical results are obtained for the self-energy and mean-free path, within Bourret’s and on-shell approximations.

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I. INTRODUCTION

Whether quantum or classical, electromagnetic or acoustic, wave phenomena share a common theoretical ground. Hence the universality of fundamental concepts such as coherence, ballistic to diffuse transition, localization, and the observation of related experimental manifestations in all fields of mesoscopic wave physics [1–4]. In this paper, we are interested in the coherent field i.e., the statistical average of the wavefield propagating in an inhomogeneous medium whose characteristic are treated as random variables. In mesoscopic physics, determining the coherent field is the very basis of multiple scattering theory. It allows to define an effective wave speed as well as a scattering mean-free path \( \ell_s \), which is the key parameter in any multiple-scattering problem.

Here we show that the usual approach to express \( \ell_s \) as a function of the fluctuations and correlation length is incorrect. This is due to an additional term in the acoustic wave equation, which is usually overlooked. As a result, even in the most simple cases, the scattering mean-free path can be underestimated by up to a factor of 13/3, whatever the shape of the correlation function (as long as its second-order moment is finite) and no matter how weak the fluctuations are.

The theoretical framework of the present paper is the diagrammatic approach of multiple scattering [1,2]. It yields an exact equation for the coherent field known as Dyson’s equation, the essential ingredient of which is the self-energy \( \Sigma \). Unfortunately, as often in real life, one has to resort to some degree of approximation to evaluate \( \Sigma \) and obtain tractable expressions for the coherent field. The coherent wave has been extensively studied in the literature with various kinds of waves, both theoretically and experimentally [2,4].

From a theoretical point of view, in the case of continuous heterogeneous media the starting point is usually a wave equation in which disorder is introduced by a space-dependent wave velocity \( c(\vec{r}) \):

\[
\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = s(\vec{r}, t). \tag{1}
\]

Assuming linearity and time-invariance, in order to determine the wavefield \( \phi(\vec{r}, t) \) generated by any distribution of sources \( s(\vec{r}, t) \) it suffices to know the Green’s function \( g(\vec{r}, \vec{r}_s, t) \) i.e., the solution of Eq. (1) when the source term is \( s(\vec{r}, t) = \delta(\vec{r} - \vec{r}_s)\delta(t) \), with appropriate boundary conditions.

An essential point is that in Eq. (1) heterogeneity is fully characterized by a random scalar \( c(\vec{r}) \). The average Green’s function \( \langle g \rangle \), hence the coherent field \( \langle \phi \rangle \), can be calculated, provided the statistical properties of \( c(\vec{r}) \), particularly its correlation function, are known.

Here we are interested in a very simple case in which the usual approach fails. We consider an acoustic wave propagating in a lossless heterogeneous fluid. It is well known that the wave equation for the acoustic pressure does not take the form of Eq. (1): it entails an additional term with a random operator \( c(\vec{r}) \). This operator is usually neglected when dealing with multiple scattering of waves. This implies an important error in the calculation of the mean-free path at low frequencies. To our knowledge, this point has been overlooked so far.

The paper is organised as follows. In the next section we briefly recall the basics of the diagrammatic ap-
proach of multiple scattering, applied to the standard wave equation, and show why this is inappropriate in the case of acoustic waves. Section I[1] proposes a first strategy to solve the problem within the diagrammatic theory of multiple scattering; unfortunately it has a limited interest, from a physical point of view. Then section I[V] gives a complete solution of the wave equation for the average acoustic pressure. Under the Bourret approximation, we show that the self-energy $\Sigma$ exhibits three additional terms compared to the standard scalar case. Therefore section I[V] , particularly Eq. (61), is the core of the paper. In the final section, we discuss the importance of these additional terms in the simple situation of an exponentially-correlated disorder.

II. THE USUAL WAY AND ITS LIMITS

Let us consider first a homogenous and lossless medium, with a constant sound speed $c_0$. The corresponding Green’s function and its Fourier transform will be denoted $g_0$ and $G_0$ respectively. In the monochromatic regime, $G_0$ is the solution of Helmholtz’ equation:

$$\Delta G_0 - k_0^2 G_0 = \delta(\vec{r} - \vec{r}_s).$$ \hspace{1cm} (2)

$\omega$ is the angular frequency, $k_0 = \omega/c_0$ and $j = \sqrt{-1}$. With the condition of causality, in unbounded 3-D space we have:

$$G_0(\vec{r}, \vec{r}_s) = -\frac{1}{4\pi|\vec{r} - \vec{r}_s|} e^{jk_0|\vec{r} - \vec{r}_s|},$$ \hspace{1cm} (3)

$$g_0(\vec{r}, \vec{r}_s) = -\frac{1}{4\pi|\vec{r} - \vec{r}_s|} \delta(t - |\vec{r} - \vec{r}_s|/c_0).$$ \hspace{1cm} (4)

Assuming that a heterogeneous medium can be simply characterized by a space-dependence of the wave speed amounts to replacing $c_0$ by $c(\vec{r})$ in the wave equation. Then it is convenient to define the scalar potential $\alpha$:

$$\alpha(\vec{r}) = 1 - c^2/c^2(\vec{r}).$$ \hspace{1cm} (5)

The monochromatic Green’s function $G(\vec{r}, \vec{r}_s)$ in the heterogeneous medium is such that

$$\Delta G - k_0^2 G = k_0^2 \alpha G + \delta(\vec{r} - \vec{r}_s).$$ \hspace{1cm} (6)

Note that in Eq. (6) $c_0 = k_0 \omega$ is a reference speed which could be chosen arbitrarily. From a physical point of view, it make more sense to consider $\delta c = c(\vec{r}) - c_0$ as a perturbation. Therefore it is convenient to set

$$\frac{1}{c_0^2} = \frac{1}{\langle c^2(\vec{r}) \rangle},$$ \hspace{1cm} (7)

so that $\langle \alpha \rangle = 0$.

For media such that the typical speed fluctuations $\delta c$ is much smaller than the average speed $\langle c \rangle$, Eq. (6) amounts to choosing $c_0 = \langle c \rangle$, hence the reference speed is actually the average sound speed, but this is not true in the general case.

The potential $\alpha$ fully characterizes the heterogeneity, in that it measures the gap between the reference and the actual wave speed, at any point in space. The term potential comes from quantum mechanics, where the relevant wavefield is the complex amplitude of probability and obeys Schrödinger equation, in which case the heterogeneity of the medium is an actual potential energy [4]. Here, $\alpha$ is simply a dimensionless function of the space coordinate $\vec{r}$.

Eq. (6) is similar to Eq. (2), with a source term $\alpha(\vec{r}) G(\vec{r}, \vec{r}_s) + \delta(\vec{r} - \vec{r}_s)$ that entails the Green’s function itself. Hence $G$ can be expressed implicitly in a recursive manner (Lippmann-Schwinger form) as:

$$G(\vec{r}, \vec{r}_s) = G_0(\vec{r}, \vec{r}_s) + k_0^2 \int G_0(\vec{r}, \vec{r}_1) \alpha(\vec{r}_1) G(\vec{r}_1, \vec{r}_s) d^3 r_1.$$ \hspace{1cm} (8)

Substituting $G$ under the integral by the right-hand side of Eq. (8) and reiterating the process, we obtain an exact expression for $G$ as an infinite sum of multiple integrals, known as Born’s expansion:

$$G(\vec{r}, \vec{r}_s) = G_0(\vec{r}, \vec{r}_s) + k_0^2 \int G_0(\vec{r}, \vec{r}_1) \alpha(\vec{r}_1) G_0(\vec{r}_1, \vec{r}_2) d^3 r_1 + k_0^4 \int \int G_0(\vec{r}, \vec{r}_1) \alpha(\vec{r}_1) G_0(\vec{r}_1, \vec{r}_2) \alpha(\vec{r}_2) G_0(\vec{r}_2, \vec{r}_3) d^3 r_1 d^3 r_2 + \ldots$$ \hspace{1cm} (9)

The single-scattering approximation (which is commonly made in imaging of weakly heterogeneous media) consists in neglecting all terms beyond the first integral on the right-hand side of Eq. (9). In that case the Green’s function $G$ can be easily computed for any function $\alpha$, and the inverse problem (i.e. reconstructing $\alpha$ from $G$) may be solved. Naturally this approach completely fails as soon as multiple scattering is not negligible.

In this paper, we consider multiple scattering of waves but we do not aim at solving Eq. (8). Considering $\alpha$ as a random variable with known statistical parameters, we are interested in its statistical average $\langle G \rangle$. It is often termed the effective Green’s function, or the coherent Green’s function. This approach implies that we consider a given medium as one particular realization, among the infinity of possible outcomes, of the same random process. From a physical point of view, what an experimentalist would measure with an ideal point source and point receiver is $G$, not $\langle G \rangle$. But $G$ can be formally written as $\langle G \rangle + \delta G$ with $\langle \delta G \rangle = 0$ (i.e. a mean term plus statistical fluctuations, changing from one realisation to another). How $\langle G \rangle$ can be estimated in an actual experiment, and how robust the estimation is, is not our concern in the present paper. We focus on theoretical calculations for $\langle G \rangle$, derived from the statistical properties of $\alpha$. By taking the average of the Born expansion, this will obviously require to know the statistical moments of $\alpha$ with any order $n$ (i.e. quantities such as $\langle \alpha(\vec{r}_1) \alpha(\vec{r}_2) \ldots \alpha(\vec{r}_n) \rangle$).
A. The diagrammatic approach

The diagrammatic theory of multiple scattering shows that \( \langle G \rangle \) obeys an exact equation, known as Dyson’s equation \( \Box \langle \Box \). The basic ingredient in Dyson’s equation is a quantity \( \Sigma \) referred to as the self-energy or the mass operator in the literature. \( \Sigma \) can be fully determined by the statistical properties of \( \alpha \), and takes into account all orders of multiple scattering. In a nutshell, the basic idea is to rewrite the statistical average of Eq. (9) as an implicit, recursive expression for \( \langle G \rangle \). The resulting Dyson’s equation reads:

\[
\langle G(\vec{r}, \vec{r}_s) \rangle = G_0(\vec{r}, \vec{r}_s)
+ \iint G_0(\vec{r}, \vec{r}_1)\Sigma(\vec{r}_1, \vec{r}_2)G(\vec{r}_2, \vec{r}_s)d^3r_1d^3r_2. \tag{10}
\]

Assuming that the medium is statistically homogeneous (i.e. its statistical parameters are invariant under translation) \( \Sigma(\vec{r}_1, \vec{r}_2) \) only depends on \( \vec{x} = \vec{r}_1 - \vec{r}_2 \), and since so does \( G_0 \), Eq. (10) is a double convolution product on the variable \( \vec{x} \). Hence it can be simply solved by a spatial Fourier transform, which yields

\[
\langle \hat{G}(\vec{k}) \rangle = \frac{1}{k_0^2 - \Sigma(k) - k^2}, \tag{11}
\]

where \( \vec{k} \) is the dual variable for \( \vec{x} \). Assuming further that the medium is statistically isotropic (i.e., its statistical parameters are also invariant under rotation), both \( \Sigma \) and \( \hat{G} \) only depend on \( k = |\vec{k}| \).

The key issue is naturally to determine \( \Sigma \). Mathematically, \( \Sigma \) can be written as a perturbative development in \( 1/(k_0\ell_s) \), an infinite series of integrals involving statistical moments of \( \alpha \) at all orders, which can be represented by the following diagrams

\[
\Sigma = \bigcirc + \bigcirc + \bigcirc + \ldots \tag{12}
\]

Following the usual conventions, a continuous line joining two points represents the free-space Green’s function \( G_0 \) between these points; a dashed line linking \( n \) points is the \( n \)-order moment of \( \alpha \) multiplied by \( k_0^{3n} \). The inner points of a diagram are dummy variables.

The Bourret approximation (a.k.a. first-order smoothing approximation) only keeps the first two terms in the perturbative development of \( \Sigma \). This yields a simple analytical expression for \( \Sigma \) as a function of the first and second-order moments (i.e. the mean \( \langle \alpha \rangle \) and correlation function \( \langle \alpha(\vec{r}_1)\alpha(\vec{r}_2) \rangle \)). Note that the Bourret approximation does not imply at all that multiple scattering terms are neglected beyond second-order scattering, but rather than all multiple scattering events are assumed to be similar to a succession of uncorrelated single or double-scattering sequences.

Under the Bourret approximation, and having chosen \( c_0 \) such that \( \langle \alpha \rangle = 0 \), the expression for the self-energy is:

\[
\Sigma(\vec{x}) = k_0^4G_0(\vec{x})C_{\alpha\alpha}(\vec{x}). \tag{13}
\]

The last step is to determine the average Green’s function from Eq. (11). In the most general case, it is an arbitrary function of \( k \). To determine the average Green’s function \( \langle G(\vec{x}) \rangle \) in real space, one has to perform an inverse Fourier transform. This is not always possible analytically and does not always lead to a simple effective medium; \( \Sigma \) is said to be non-local. We will not discuss these issues in the present paper. Instead, we make a further approximation referred to as the on-shell approximation. It requires \( \Sigma(k) \) to be sufficiently smaller than \( k_0^2 \). Indeed, if \( \Sigma(k) \) is weak enough, we can reasonably assume that the effect it will have on the homogeneous wavenumber \( k_0 \) is small, so that when performing the inverse three-dimensional Fourier transform, the volume that essentially contributes to \( \langle G(\vec{x}_1 - \vec{x}_s) \rangle \) is the vicinity of the shell defined by \( |k| = k_0 \). In other words, this amounts to performing a zero-order development of the self-energy around \( k_0 \), hence replacing \( \Sigma(k) \) by \( \Sigma(k_0) \) in Eq. (11). In that case the expression of the average Green’s function in real space is straightforward:

\[
\langle G(\vec{x}_1 - \vec{x}_s) \rangle = \frac{1}{4\pi|\vec{x}_1 - \vec{x}_s|}e^{jk_0|\vec{x}_1 - \vec{x}_s|}. \tag{14}
\]

This means that the average Green’s function is that of a homogeneous absorbing medium with a complex-valued wavenumber \( k_{\text{eff}} \) such that

\[
k_{\text{eff}} = \sqrt{k_0^2 - \Sigma(k_0)} \tag{15}.
\]

Equation (15) is a dispersion relation from which effective phase and group velocities can be determined. Most importantly, the intensity of the average wavefield is found to decay exponentially with the distance (Beer-Lambert’s law) and the typical extinction length (termed the scattering mean-free path) is \( \ell_s = 1/2\ln(k_{\text{eff}}) \).

\( \ell_s \) is an essential parameter in all multiple scattering problems. Particularly, the range of validity of the Bourret approximation can be shown to be \( k_0\ell_s \gg 1 \). It should also be mentioned that, as a refinement of the on-shell approximation, \( k_{\text{eff}} \) can be determined more accurately with an iterative algorithm using \( k_0 \) as a first guess.

So, within the Bourret approximation, as long as the correlation function for the potential \( \alpha \) is known, the effective wave speed and scattering mean-free path can be determined quite easily and sometimes analytically.

A typical example is that of a disordered random medium with an exponentially decaying correlation function \( C_{\alpha\alpha}(\vec{x}) = \sigma^2 e^{-(x/\ell_c)} \). \( \sigma^2 \) denotes the variance of \( \alpha \) and \( \ell_c \) its correlation length. In that case, under the Bourret approximation the self-energy is:

\[
\tilde{\Sigma}(k) = \frac{\sigma^2\ell_c^2 k_0}{k^2 + (jk_0 - 1/\ell_c)^2}. \tag{16}
\]
The on-shell approximation yields a simple analytic expression for the scattering mean-free path as a function of frequency and correlation length:
\[ \ell_s = \frac{1}{\sigma^2 \omega} \frac{1 + 4k_0^2 \ell_c^2}{2k_0^3 \ell_c^2} \] (17)

### B. The acoustic case

Whatever the shape of the correlation function and whatever the nature of the wave (acoustic, electromagnetic, ...), the same formalism will hold as soon as we deal with a wavefield propagating in a heterogeneous medium in which heterogeneity is fully described by a scalar function such as \( \chi \). Such is the case when the local wavespeed \( c(\mathbf{r}) \) suffices to capture the heterogeneity, which is usually assumed as a starting point in multiple scattering theories. However this description of heterogeneity may sometimes be completely misleading, even in very simple situations.

Let us take the example of acoustic waves in a lossless fluid. The medium is characterized by its mass density \( \rho \) and compressibility \( \chi \) at rest. With no sources, the linearized equations of elastodynamics read
\[ \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p, \] (18)
\[ \nabla \cdot \mathbf{u} = -\chi p. \] (19)

\( \mathbf{u}(\mathbf{r}, t) \) is the displacement undergone by the particle initially at rest at point \( \mathbf{r} \), \( \mathbf{v} = \partial \mathbf{u}/\partial t \) is the particle velocity, and \( p(\mathbf{r}, t) \) is the acoustic pressure. \( \mathbf{u}, \mathbf{v} \) and \( p \) are first-order infinitesimal quantities. To establish Eqs. (18) and (19) all second-order non-linear terms have been neglected whatever their physical origin (convective or thermodynamic). From a physical point of view, Eqs. (18) and (19) are an expression of Newton’s second law and Hooke’s law (the relative dilation \( \nabla \cdot \mathbf{u} \) undergone by an infinitesimal volume of fluid is proportional and opposed to the acoustic pressure). If neither \( \rho \) nor \( \chi \) depend on space coordinate \( \mathbf{r} \), then Eqs. (18) and (19) lead to the usual wave equation with a constant sound velocity \( c_0 = 1/\sqrt{\rho \chi} \), which applies to all quantities describing the sound wave (acoustic pressure, velocity, displacement, dilatation, etc. ...).

In a heterogeneous fluid, the local sound velocity naturally depends on the space coordinate \( \mathbf{r} \) as \( c(\mathbf{r}) = 1/\sqrt{\rho(\mathbf{r}) \chi(\mathbf{r})} \). It is therefore tempting to replace \( c_0 \) by \( c(\mathbf{r}) \) in the wave equation, but this is not always correct.

Combining Eqs. (18) and (19), we obtain the following equations for the acoustic pressure and velocity:
\[ \Delta p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\nabla \cdot \nabla p}{\rho} = 0, \] (20)
\[ \Delta \mathbf{v} - \frac{1}{c^2} \frac{\partial^2 \mathbf{v}}{\partial t^2} - \frac{\nabla \chi \cdot \nabla \cdot \mathbf{v}}{\chi} = 0. \] (21)

If \( \rho \) does not vary in space, Eq. (20) yields the usual wave equation for the acoustic pressure \( p \), with a space-dependent velocity \( c \). And if \( \chi \) does not vary in space, Eq. (21) yields the usual wave equation for the velocity \( \mathbf{v} \). But in the general case where both \( \rho \) and \( \chi \) vary with \( \mathbf{r} \), none of the acoustic variables satisfy the usual wave equation [15,16]. However the resulting equation for the acoustic pressure (Eq. (20)) can be Fourier-transformed over time, then rearranged in order to define a potential, as we did before. We obtain an equation similar to Helmholtz’:
\[ \Delta P - k_0^2 P = k_0^2 \gamma P. \] (22)

The potential \( \gamma \) is such that:
\[ \gamma(\mathbf{r}) = \alpha(\mathbf{r}) + \frac{1}{k_0} \nabla \beta(\mathbf{r}) \cdot \nabla. \] (23)

\( \alpha \) is defined in Eq. (5) and
\[ \beta(\mathbf{r}) = \ln \left[ \frac{\rho_0}{\rho(\mathbf{r})} \right]. \] (24)

\( \rho_0 \) is an arbitrary constant with the dimensions of a mass density.

The first term in the expression of \( \gamma \) is the usual potential \( \alpha \), related to the spatial fluctuations of sound speed. In addition, there is an other term, which unlike \( \alpha \) is not a simple scalar but an operator acting on the field it is applied to. It should be noticed that though we are interested in acoustic waves, the same problem could arise for different kinds of waves e.g., electromagnetic waves propagating in a heterogeneous medium showing fluctuations of both relative permeability \( \mu(\mathbf{r}, \omega) \) and permittivity \( \epsilon(\mathbf{r}, \omega) \). In that case, Maxwell’s equations yield the following wave equation for the monochromatic electric field \( \mathbf{E}(\mathbf{r}, \omega) \), with a potential that contains an operator part [18]:
\[ \nabla \times \nabla \times \mathbf{E} - k_0^2 \epsilon \mathbf{E} - \nabla \left[ \ln(\mu) \right] \times \nabla \times \mathbf{E} = \mathbf{0}. \] (25)

Whatever the physical nature of the wave, the applicability of the diagrammatic techniques when the relevant potential \( \gamma \) has both a scalar part and an operator part as well as the impact of the operator part on the final result are not trivial. In the next two sections, we propose two approaches to deal with this problem and obtain the average Green’s function, in the case of acoustic waves.

### III. FIRST STRATEGY

The first solution is to perform a change of variable. Unfortunately all relevant physical quantities at stake (dilation, velocity, displacement, velocity potential, etc. ...) exhibit the same problem as the pressure field: the wave equations they obey entail a potential with both a scalar and an operator term. However, there is a mathematical trick to overcome this difficulty. Let us introduce
as a new relevant field the ratio of the acoustic pressure to the square-root of the mass density at rest:

$$F(\vec{r}) = \frac{P(\vec{r})}{\sqrt{\rho(\vec{r})}}.$$  \hfill (26)

After some straightforward algebra, $F$ can be shown to obey a wave equation similar to Eq. (10), with a scalar potential

$$\eta(\vec{r}) = \alpha(\vec{r}) + \frac{3}{4k_0^2} \left[ \vec{\nabla} \beta \right]^2 - \frac{\Delta \rho}{2k_0^2 \rho}.$$  \hfill (27)

Hence the usual methods (Bourret, on-shell approximations, etc.) can be applied to determine the average field, except that the relevant correlation function to describe the heterogeneity is now that of $\eta$ instead of $\alpha$.

From a mathematical point of view, the problem is solved. Yet from a physical point of view, things are different. Experimentally, one can measure the acoustic pressure, possibly the velocity field or at least one of its components, but not a quantity as strange as the ratio of pressure to the square-root of mass density. And it would be most certainly wrong to assume that

$$\langle F(\vec{r}) \rangle \approx \langle P(\vec{r}) \rangle \left( \frac{1}{\rho(\vec{r})} \right)$$  \hfill (28)

in order to retrieve the average pressure field $\langle P \rangle$. Therefore, despite its simplicity the practical application of this strategy is probably limited to comparisons with numerical simulations of wave propagation, where $\langle F \rangle$ can be easily evaluated.

IV. SECOND STRATEGY

In this section, the acoustic pressure $P$ is kept as the relevant variable, and the issue is to determine the average Green’s function of Eq. (22). The reference velocity is still chosen according to Eq. (7). Since the medium is assumed to be statistically invariant under translation $\langle \beta(\vec{r}) \rangle$ does not depend on the space coordinate $\vec{r}$. Hence, despite the presence of $\beta$, the average value of the potential $\gamma$ will still be zero, since

$$\langle \vec{\nabla} \beta \rangle = \vec{\nabla} \langle \beta \rangle = \vec{0}.$$  \hfill (29)

As usual, the Green’s function for Eq. (22) can be written as a Lippmann-Schwinger equation

$$G(\vec{r}, \vec{r}_s) = G_0(\vec{r}, \vec{r}_s) + k_0^2 \int G_0(\vec{r}, \vec{r}_1) \gamma(\vec{r}_1) G(\vec{r}_1, \vec{r}_s) d^3r_1.$$  \hfill (30)

$\gamma$ is not a simple scalar function, which precludes the usual definition of its autocorrelation function and that of the self-energy. To overcome this difficulty, we start by introducing a two-variable potential $V$ such that

$$V(\vec{r}_1, \vec{r}_2) = \gamma(\vec{r}_1) \delta(\vec{r}_1 - \vec{r}_2) = \alpha(\vec{r}_1) \delta(\vec{r}_1 - \vec{r}_2) + \frac{1}{k_0^2} \vec{\nabla}_1 \beta(\vec{r}_1) \cdot \vec{\nabla}_2 \delta(\vec{r}_1 - \vec{r}_2).$$  \hfill (31)

Eq. (30) becomes

$$G(\vec{r}, \vec{r}_s) = G_0(\vec{r}, \vec{r}_s) + k_0^2 \int G_0(\vec{r}, \vec{r}_1) V(\vec{r}_1, \vec{r}_2) G(\vec{r}_2, \vec{r}_s) d^3r_1 d^3r_2.$$  \hfill (33)

The next steps are as usual to develop Eq. (33) into a Born expansion by iteration, then to take its statistical average and write it under a recursive form (Dyson’s equation). Under the Bourret approximation, only the first two terms in the self-energy are kept. The first one vanishes since $c_0$ is set so that $\langle \gamma \rangle = 0$. The second term reads:

$$\Sigma(\vec{r}_a, \vec{r}_b) = k_0^4 \int \int (V(\vec{r}_a, \vec{r}_1) G_0(\vec{r}_1, \vec{r}_2) V(\vec{r}_2, \vec{r}_b) d^3r_1 d^3r_2.$$  \hfill (34)

As a consequence, the self-energy Eq. (34) gives rise to four terms, involving the following dimensionless correlation functions and their derivatives:

$$C_{\alpha\alpha}(\vec{r}_1, \vec{r}_2) = \langle \alpha(\vec{r}_1) \alpha(\vec{r}_2) \rangle,$$
$$C_{\beta\beta}(\vec{r}_1, \vec{r}_2) = \langle \beta(\vec{r}_1) \beta(\vec{r}_2) \rangle,$$
$$C_{\alpha\beta}(\vec{r}_1, \vec{r}_2) = \langle \alpha(\vec{r}_1) \beta(\vec{r}_2) \rangle,$$
$$C_{\beta\alpha}(\vec{r}_1, \vec{r}_2) = \langle \beta(\vec{r}_1) \alpha(\vec{r}_2) \rangle.$$  \hfill (35)

We assume that the random processes $\alpha$ and $\beta$ are jointly stationary and invariant under rotation. Then all correlation functions only depend on $x = |\vec{r}_1 - \vec{r}_2|$ and $C_{\alpha\beta}(x) = C_{\beta\alpha}(x)$. Only three correlation functions suffice to characterize the disorder. They can be rewritten as

$$C_{\alpha\alpha}(x) = \sigma_{\alpha}^2 c_{\alpha\alpha}(x),$$
$$C_{\beta\beta}(x) = \sigma_{\beta}^2 c_{\beta\beta}(x),$$
$$C_{\alpha\beta}(x) = C_{\beta\alpha}(x) = \sigma_{\alpha} \sigma_{\beta} c_{\alpha\beta}(x).$$  \hfill (36)

$\sigma_{\alpha}^2$, $\sigma_{\beta}^2$ are the variances of $\alpha$ and $\beta$ respectively. $c_{\alpha\alpha}(x)$ is the correlation coefficient between $\alpha(\vec{r})$ and $\alpha(\vec{r} + \vec{x})$. $c_{\beta\beta}(x)$ is the correlation coefficient between $\beta(\vec{r})$ and $\beta(\vec{r} + \vec{x})$. $c_{\alpha\beta}(x)$ is the correlation coefficient between $\alpha(\vec{r})$ and $\beta(\vec{r} + \vec{x})$. Replacing $V$ in Eq. (34) by Eq. (35), we can write the self-energy $\Sigma$ as a sum of four contributions:

$$\Sigma = \Sigma_{\alpha\alpha} + \Sigma_{\beta\alpha} + \Sigma_{\alpha\beta} + \Sigma_{\beta\beta}.$$  \hfill (37)

A. First term

The first term is:

$$\Sigma_{\alpha\alpha}(\vec{r}_a - \vec{r}_b) = k_0^4 \langle \alpha(\vec{r}_a) G_0(\vec{r}_a - \vec{r}_b) \alpha(\vec{r}_b) \rangle$$  \hfill (38)

$$= k_0^4 G_0(\vec{r}_a - \vec{r}_b) C_{\alpha\alpha}(\vec{r}_a - \vec{r}_b).$$  \hfill (39)

With $\vec{x} = \vec{r}_a - \vec{r}_b$ we find the usual contribution to the self-energy Eq. (13).

The three additional terms are more complicated, since they involve combinations of $\alpha$ and $\beta$ as well as spatial derivatives.
B. Second term

The second term mixes contributions from $\beta$ and $\alpha$:

$$ \Sigma_{\beta\alpha}(\vec{r}_a - \vec{r}_b) = k_0^2 \times \int \left( \langle x \rangle G_0(\vec{r}_a - \vec{r}_1) \nabla \cdot \nabla \{ \delta(\vec{r}_a - \vec{r}_1) \} \right) d^3 r_1. $$

Performing two integrations by parts and using the properties of the Dirac distribution yields:

$$ \int \left( \nabla \cdot \nabla \{ \delta(\vec{r}_a - \vec{r}_1) \} \right) d^3 r_1. $$

Taking advantage of the assumed radial symmetry, in three dimensions we have:

$$ \Sigma_{\beta\alpha}(x) = k_0^2 \frac{\partial G_0}{\partial x} \frac{\partial C_{\alpha\beta}}{\partial x}. $$

C. Third term

Similarly to the second term, the third one implies both $\beta$ and $\alpha$:

$$ \int \left( \alpha \langle \vec{r}_a \rangle G_0(\vec{r}_a - \vec{r}_2) \nabla \cdot \nabla \{ \delta(\vec{r}_2 - \vec{r}_b) \} \right) d^3 r_2. $$

Again, performing two integrations by parts and using the properties of the Dirac distribution, we obtain:

$$ \Sigma_{\alpha\beta}(\vec{r}_a - \vec{r}_b) = -k_0^2 \nabla \cdot \left( G_0(\vec{r}_a - \vec{r}_b) \nabla \{ C_{\alpha\beta}(\vec{r}_a - \vec{r}_b) \} \right). $$

Since all functions involved here have radial symmetry, in three dimensions the expression above simplifies into:

$$ \Sigma_{\alpha\beta}(x) = \frac{k_0^2}{x^2} \frac{\partial}{\partial x} \left[ x^2 G_0(x) \frac{\partial C_{\alpha\beta}}{\partial x} \right]. $$

D. Fourth term

The last term is the most complicated one:

$$ \Sigma_{\beta\beta}(\vec{r}_a - \vec{r}_b) = \int \int \left( \nabla \cdot \nabla \{ \delta(\vec{r}_a - \vec{r}_1) \} \right) \times G_0(\vec{r}_1 - \vec{r}_2) \nabla \cdot \{ C_{\beta\beta}(\vec{r}_2 - \vec{r}_b) \} d^3 r_1 d^3 r_2. $$

Given the radial symmetry, in three dimensions we have:

$$ \Sigma_{\beta\beta}(x) = -\nabla \cdot \left( \nabla \nabla \{ C_{\beta\beta}(x) \} \nabla G_0(x) \right). $$

The tensorial product between the two gradients is a Hessian matrix. In spherical coordinates and for a function with radial symmetry, we have [10]:

$$ \nabla \otimes \nabla C_{\beta\beta} = \begin{bmatrix} \frac{\partial^2 C_{\beta\beta}}{\partial x^2} & 0 & 0 \\ 0 & \frac{1}{x} \frac{\partial C_{\beta\beta}}{\partial x} & 0 \\ 0 & 0 & \frac{1}{x} \frac{\partial C_{\beta\beta}}{\partial x} \end{bmatrix}. $$

Hence:

$$ \Sigma_{\beta\beta}(x) = -\frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^2 \frac{\partial^2 C_{\beta\beta}}{\partial x^2} \frac{\partial G_0}{\partial x} \right]. $$

V. APPLICATION AND DISCUSSION

In real space, the four contributions to the self-energy add up to give:

$$ \Sigma_{\alpha\alpha}(x) = k_0^2 \sigma_{\alpha\alpha}^2 G_0(x) - k_0^2 \sigma_{\alpha\beta} G_0(x) \left[ c_{\alpha\beta}'' + 2c_{\alpha\beta}' / x \right] $$

$$ -\sigma_{\beta\beta}^2 c_{\beta\beta}'' G_0(x) + c_{\beta\beta}'' G_0(x) + 2c_{\alpha\beta}' G_0(x)/x. $$

For simplicity the $x$-dependence of $G_0$ and of the correlation coefficients have been omitted, and the prime means derivation with respect to $x$.

If $\sigma_{\beta\beta} = 0$, the self energy Eq. (51) is reduced to the standard term $\Sigma_{\alpha\alpha}$. In the general case where all four terms coexist it is not obvious to determine the orders of magnitude of the additional terms, since they involve five physical parameters: two variances and three correlation lengths. In order to highlight the importance of the additional terms relatively to the first one, we studied the most simple case where $\sigma_{\alpha\alpha} = \sigma_{\beta\beta} = \sigma$ and $c_{\alpha\alpha} = c_{\alpha\beta} = c_{\beta\beta} = c$. The variance $\sigma^2$ appears as a mere multiplicative term, and from a physical point of view everything will depend on the typical correlation length $\ell_c$. Exponential and gaussian correlation functions were tested, and similar trends were obtained. We only give the result for the exponential case, which entails simpler analytical expressions.

In the case of an exponentially-correlated disorder, we have

$$ c(x) = \exp \left[ -\frac{x}{\ell_c} \right]. $$

In 3-D, the calculation of $\Sigma$ yields:

$$ \tilde{\Sigma}(k_0) = -\frac{\sigma^2}{\ell_c^2} \left( \frac{(k_0 \ell_c)^4 + j k_0 \ell_c}{1 - 2 j k_0 \ell_c} \right) $$

$$ -\frac{1}{k_0 \ell_c} \arctan \left( \frac{k_0 \ell_c}{1 - j k_0 \ell_c} \right). $$
Note that \( \sigma^2 \) can be chosen arbitrarily small, in order to always comply with the on-shell approximation \( \tilde{\Sigma}(k_0) \ll k_0^2 \).

As long as \( k_0 \ell_c \gg 1 \) (i.e. the correlation length is much larger than the wavelength, the first term dominates:

\[
\tilde{\Sigma}(k_0) \sim \tilde{\Sigma}_{\alpha\alpha}(k_0) = \frac{\sigma^2}{\ell_c^2} \frac{(k_0 \ell_c)^4}{1 - 2j k_0 \ell_c}. \tag{54}
\]

Therefore at high frequencies, considering the usual wave equation with a space-dependent wave speed is legitimate to determine the coherent pressure field. However it becomes completely wrong as soon as \( k_0 \ell_c \) is comparable to unity.

On the contrary, the impact of the three additional terms (\( \tilde{\Sigma}_{\alpha\beta}, \tilde{\Sigma}_{\beta\alpha}, \text{and} \tilde{\Sigma}_{\beta\beta} \)) on the scattering mean-free path \( \ell_s = -k_0/\text{Im}(\tilde{\Sigma}(k_0)) \) is far from negligible. As an illustration, Figs. 1 and 2 compare the scattering mean-free paths obtained with \( \ell_s \) and without \( (\ell_s^{\alpha\alpha}) \) the additional terms.

In the high-frequency regime, the difference is negligible (less than 6% for \( k_0 \ell_c > 10 \)). But below \( k_0 \ell_c \sim 1.5 \), the three additional terms in the self-energy are larger than the first one. As a result, at low frequencies the actual mean-free path can be nearly five times smaller than expected! The exact ratio is 13/3; the same behavior was obtained in the case of a gaussian-correlated disorder. Interestingly, it can be shown that the 13/3 ratio is independent of the correlation function (as long as it is integrable, see Appendix). Note that care should be taken when taking the low-frequency limit; the on-shell approximation requires \( \tilde{\Sigma}(k_0) \) to be much smaller than \( k_0^2 \), hence when \( k_0 \ell_c \to 0 \) the results are consistent only if the variance is kept such that \( \sigma^2 \ll (k_0 \ell_c)^2 \).

It is also interesting to plot the exponent \( n = -\omega/\ell_s d\omega/d\ell_s \) as a function of frequency (Fig. 3). Indeed, since a power-law dependence of the attenuation length is often assumed, \( n \) commonly serves as an indicator of the scattering regime. In both cases, \( 1/\ell_s \) is found to be proportional to \( \omega^4 \) at low frequency and \( \omega^2 \) at high frequency. These two trends are usually referred to the Rayleigh and stochastic regimes and are used to characterize scattering media based on the measured dependence of acoustic attenuation with frequency. Fig. 3 shows how misleading the omission the additional terms in the wave equation can be, especially at intermediate frequencies \( (k_0 \ell_c \sim 1) \): the exponent can be 35% lower than expected.

However, as to the effective velocity, the effect of the additional terms is very limited since the real part of the wave vector is

\[
\text{Re}(k_{\text{eff}}) \approx k_0 - \frac{\text{Re}[\tilde{\Sigma}(k_0)]}{2k_0}, \tag{55}
\]

which will always remain close to \( k_0 \) within the on-shell approximation.

\[\text{FIG. 1: (Color online) Scattering mean-free path as a function of } k_0 \ell_c, \text{ with (solid line) and without (dashed line) the additional terms in the self-energy.}\]

\[\text{FIG. 2: (Color online) Ratio of the scattering mean-free paths obtained with and without the additional terms in the self-energy as a function of } k_0 \ell_c.\]

VI. CONCLUSION

Starting from the wave equation for the acoustic pressure in an heterogeneous and non-dissipative fluid, we have calculated the coherent wave, taking into account spatial variations of both density and compressibility. The calculation is based on the diagrammatic approach of multiple scattering, within Bourret and on-shell approximations. Interestingly, the results show that discarding the random operator term, as is usually done, amounts to overestimate the scattering mean-free path by up to a factor of five. The error is particularly large at low frequencies, when the correlation length is comparable
FIG. 3: (Color online) Characteristic exponent \( n \) obtained with (solid line) and without (dashed line) the additional terms in the self-energy as a function of dimensionless frequency \( k_0 \ell_c \).

to or smaller than the wavelength. Though the results presented here are theoretical and rather academic, we believe they are of importance for all practical applications involving multiple scattering of acoustic waves e.g., characterisation of colloids and suspensions. Moreover, from a theoretical point of view, the scattering mean-free path is the basic ingredient to describe the ballistic-to-diffuse transition in a complex medium. It is therefore crucial to determine it properly.

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### VII. APPENDIX

#### A. Low-frequency regime

Assuming that the correlation functions \( C_{\alpha\alpha} \), \( C_{\alpha\beta} \) and \( C_{\beta\beta} \) are identical, we have

\[
\frac{\Sigma(x)}{\sigma^2} = k_0^4 G_0 c - k_0^2 G_0 c + k_0^2 G_0 c + 2 c/x [c'' G_0 + c'' G_0 + 2 c'' G_0'/x]
\]

Given the radial symmetry, the 3-D Fourier transform of \( \Sigma \) is

\[
\tilde{\Sigma}(k_0) = \frac{4\pi}{k_0} \int_0^\infty \Sigma(x) \sin(k_0 x) dx.
\]  

Hence the imaginary part:

\[
\text{Im} \tilde{\Sigma}(k_0) = -\frac{4\pi \sigma^2}{k_0} \int_0^\infty \left[ k_0^4 c - 2 k_0^2 c'/x + c''/x \right] \sin^2(k_0 x) dx
\]

\[
+ \frac{1}{2} \int_0^\infty \sin(2k_0 x) dx. \tag{58}
\]

For \( k_0 x \to 0 \), a Taylor expansion of the sines up to the sixth order followed by integrations by parts yield

\[
\text{Im} \tilde{\Sigma}(k_0) \to -4\pi \sigma^2 k_0^5 \int_0^\infty \frac{13}{3} x^2 c(x) dx. \tag{59}
\]

If the additional terms due to the random operator are neglected, Equation (58) reduces to

\[
\text{Im} \tilde{\Sigma}(k_0) = -\frac{4\pi \sigma^2}{k_0} \int_0^\infty k_0^4 c \sin^2(k_0 x) dx \tag{60}
\]

\[
\to -4\pi \sigma^2 k_0^5 \int_0^\infty x^2 c(x) dx. \tag{61}
\]

As a consequence, in the low frequency limit \( k_0 \ell_c \to 0 \), the ratio of the mean-free path calculated with \( (\ell_s) \) or without \( (\ell_s^{(\alpha)}) \) the additional terms is 13/3. This ratio does not depend on the precise shape of the correlation function \( c(x) \), as long as its second-order moment is finite.

#### B. Gaussian-correlated disorder

In the case where \( c(x) = \exp(-x^2/\ell_c^2) \), we obtain

\[
\frac{\tilde{\Sigma}(k_0)}{k_0^2} = \sqrt{\frac{\pi \sigma^2}{4}} \left[ jk_0 \ell_c (9 E - 1) + \frac{8}{\sqrt{\pi}} \right.
\]

\[
+ \frac{1}{k_0 \ell_c} 4j(3E - 1) + \frac{1}{(k_0 \ell_c)^2} \frac{8}{\sqrt{\pi}} + \frac{8j(E - 1)}{(k_0 \ell_c)^3} \right]. \tag{62}
\]

In the expression above, we have introduced a dimensionless constant \( E \):

\[
E = (1 + \text{erf}(j k_0 \ell_c)) e^{-k_0^2 \ell_c^2}. \tag{63}
\]

If the additional terms are neglected, we have

\[
\tilde{\Sigma}(k_0) = \sqrt{\frac{\pi \sigma^2}{4}} \left[ jk_0 \ell_c (E - 1) \right]. \tag{64}
\]

The general behaviour of the ratio \( (\ell_s/\ell_s^{(\alpha)}) \) is very similar to the exponential case.
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