PARABOLIC INDUCTION VIA THE PARABOLIC PRO-$p$ IWAHORI–HECKE ALGEBRA

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ABSTRACT. Let $G$ be a connected reductive group defined over a locally compact non-archimedean field $F$, let $P$ be a parabolic subgroup with Levi $M$ and compatible with a pro-$p$ Iwahori subgroup of $G := G(F)$. Let $R$ be a commutative unital ring.

We introduce the parabolic pro-$p$ Iwahori–Hecke $R$-algebra $H_R(P)$ of $P := P(F)$ and construct two $R$-algebra morphisms $\Theta^P_M : H_R(P) \to H_R(M)$ and $\Xi^P_G : H_R(P) \to H_R(G)$ into the pro-$p$ Iwahori–Hecke $R$-algebra of $M := M(F)$ and $G$, respectively. We prove that the resulting functor $\text{Mod-}H_R(M) \to \text{Mod-}H_R(G)$ from the category of right $H_R(M)$-modules to the category of right $H_R(G)$-modules (obtained by pulling back via $\Theta^P_M$ and extension of scalars along $\Xi^P_G$) coincides with the parabolic induction due to Ollivier–Vigneras.

The maps $\Theta^P_M$ and $\Xi^P_G$ factor through a common subalgebra $H_R(M,G)$ of $H_R(G)$ which is very similar to $H_R(M)$. Studying these algebras $H_R(M,G)$ for varying $(M,G)$ we prove a transitivity property for tensor products. As an application we give a new proof of the transitivity of parabolic induction.

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1. INTRODUCTION

1.1. Motivation. Let $G$ be a connected reductive group over a locally compact non-archimedean field $F$ with residue field of characteristic $p > 0$. In this introduction all representations will be on vector spaces over a fixed field $k$. Given a parabolic subgroup $P$ of $G$ with Levi $M$, parabolic induction is a process to obtain smooth representations of $G := G(F)$ from smooth representations of $M := M(F)$.

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More precisely, it is obtained as the composite

\[
\begin{array}{c}
\text{Rep}_k(M) \longrightarrow \text{Rep}_k(P) \longrightarrow \text{Rep}_k(G), \\
V \longmapsto V \longmapsto \text{Ind}^G_P V.
\end{array}
\]

Here, \( V \) is first viewed as a smooth representation of \( P := \mathbf{P}(F) \) by letting the unipotent radical \( U_P := \mathbf{U}_P(F) \) of \( P \) act trivially. Then we form the space \( \text{Ind}^G_P V \) of locally constant functions \( f : G \to V \) satisfying \( f(\gamma g) = \gamma \cdot f(g) \) for all \( \gamma \in P \) and \( g \in G \). Parabolic induction plays a fundamental role in classifying the smooth admissible representations of \( G \).

A second important method to study smooth representations of \( G \) is via the functor of \( K \)-invariants \( V \mapsto V^K \), for \( K \) any compact open subgroup of \( G \). The ring of \( G \)-equivariant endomorphisms

\[
H_k(K,G) := \text{End}_{k[G]}(k[K\backslash G])
\]

of the space \( k[K\backslash G] \) of maps \( K\backslash G \to k \) with finite support acts on

\[
V^K \cong \text{Hom}_{k[G]}(k[K\backslash G], V)
\]

by Frobenius reciprocity. In this way one obtains a functor

\[(2) \quad \text{Rep}_k(G) \longrightarrow \text{Mod}\cdot H_k(K,G), \quad V \mapsto V^K \]

from \( \text{Rep}_k(G) \) into the category of right modules over the \( k \)-algebra \( H_k(K,G) \). If \( k \) has characteristic \( p \), it suffices to restrict attention to a pro-\( p \) Iwahori subgroup \( I_1 \) of \( G \); this is because for \( K = I_1 \) the functor \([2]\) sends non-zero representations to non-zero modules. One is therefore led to study the modules over the pro-\( p \) Iwahori–Hecke algebra \( \mathcal{H}_k(G) := H_k(I_1,G) \). A systematic study of \( \mathcal{H}_k(G) \) was carried out by Vignéras in [Vig05] (for \( F \)-split \( G \)) and [Vig16] (for arbitrary \( G \)).

Again, classifying the simple modules of \( \mathcal{H}_k(G) \) is accomplished by studying induction functors

\[
\text{Mod}\cdot \mathcal{H}_k(M) \longrightarrow \text{Mod}\cdot \mathcal{H}_k(G),
\]

for parabolic subgroups \( \mathbf{P} = \mathbf{M} \mathbf{U}_P \) of \( G \) that are compatible with \( I_1 \), which is compatible with parabolic induction via \([2]\). Such a functor was defined and studied by Ollivier [Oll10] for \( G = \text{GL}_n \), and for general \( G \) by Ollivier–Vignéras [OV18], Vignéras [Vig15], and Abe [Abe16,Abe19]. Explicitly, this functor is given by

\[(3) \quad m \mapsto m \otimes_{\mathcal{H}_k(M^+)} \mathcal{H}_k(G), \]

where \( \mathcal{H}_k(M^+) \) is a certain subalgebra of \( \mathcal{H}_k(M) \) embedding naturally into \( \mathcal{H}_k(G) \).

This algebra \( \mathcal{H}_k(M^+) \) depends on \( I_1 \cap \mathbf{U}_P(F) \), hence also on \( P \). However, the analogy between \([1]\) and \([3]\) is not as strong as one might hope for. Let us recall the reason why the parabolic \( P \) shows up in \([1]\). One could directly define an induction functor \( \text{Rep}_k(M) \to \text{Rep}_k(G) \) by \( V \mapsto \text{Ind}^G_P V \). But this functor is very difficult to handle, because the coset space \( G/M \) and hence the representation \( \text{Ind}^G_P V \) is “too large”. On the other hand, the quotient \( G/P \) is compact which makes it possible to effectively study \( \text{Ind}^G_P V \). In this light it is surprising that \([3]\) is defined using the small algebra \( \mathcal{H}_k(M^+) \). Instead, one would expect to use as big an algebra as possible to define \([3]\); the parabolic Hecke algebra \( \mathcal{H}_k(P) := H_k(I_1 \cap P,P) \) is a natural candidate although it is not at all obvious how this can be achieved.

To summarize, this article is motivated by the following questions:

(Q1) Can we replace \( \mathcal{H}_k(M^+) \) in \([3]\) by the algebra \( \mathcal{H}_k(P) \)?
(Q2) Assume the answer to (Q1) is affirmative. Is the functor
\[ \text{Mod} - \mathcal{H}_k(M) \rightarrow \text{Mod} - \mathcal{H}_k(G), \quad m \mapsto m \otimes_{\mathcal{H}_k(P)} \mathcal{H}_k(G) \]
naturally isomorphic to \( \mathfrak{X} \)?

1.2. Main results. We answer question (Q1) positively even for \( k \) an arbitrary commutative unital ring. This amounts to constructing two \( k \)-algebra homomorphisms \( \Theta^P_M : \mathcal{H}_k(P) \rightarrow \mathcal{H}_k(M) \) and \( \Xi^P_G : \mathcal{H}_k(P) \rightarrow \mathcal{H}_k(G) \).

The map \( \Theta^P_M \) (see Proposition 1.3) exists quite generally and is induced by the canonical projection map \( k[I_I \cap P\langle P \rangle] \rightarrow k[I_I \cap M\langle M \rangle] \). Its origins can be traced back to the works of Andrianov; see, e.g., [And77, (3.3)] or [AZ95, Definition of \( \Phi \) before Chapter 3, Proposition 3.28], who seems to have been the first to study “parabolic Hecke algebras”, albeit in a different context.

The main contribution of this work lies in the construction of \( \Xi^P_G \). Observing that the image of \( \Theta^P_M \) contains \( \mathcal{H}_k(M^+) \), the idea is to try to extend the embedding \( \xi^+ : \mathcal{H}_k(M^+) \rightarrow \mathcal{H}_k(G) \) to \( \text{Im}(\Theta^P_M) \) and then to define \( \Xi^P_G \) as the composite \( \mathcal{H}_k(P) \overset{\Theta^P_M}{\longrightarrow} \text{Im}(\Theta^P_M) \rightarrow \mathcal{H}_k(G) \). However, since the goal for \( \Xi^P_G : \mathcal{H}_k(P) \rightarrow \mathcal{H}_k(G) \) is to have as large image as possible, this approach will not always yield optimal results. Since in this approach \( \Xi^P_G \) factors through a subalgebra of \( \mathcal{H}_k(M) \), the best we can expect is for the image of \( \Xi^P_G \) to be canonically isomorphic to \( \mathcal{H}_k(M) \) as a \( k \)-module. If \( k \) happens to be \( p \)-torsionfree, this is indeed the case. We prove:

**Theorem** (Proposition 4.7). Assume that \( k \) is \( p \)-torsionfree. The embedding
\[ \xi^+ : \mathcal{H}_k(M^+) \rightarrow \mathcal{H}_k(G) \]
extends to an injective \( k \)-algebra morphism \( \text{Im}(\Theta^P_M) \rightarrow \mathcal{H}_k(G) \). Moreover, this extension is unique and \( \text{Im}(\Theta^P_M) \) is the maximal subalgebra of \( \mathcal{H}_k(M) \) with this property.

Further, Corollary 4.4 shows that \( \text{Im}(\Theta^P_M) \) identifies canonically with \( \mathcal{H}_k(M) \) as a \( k \)-module. If, on the other hand, \( k \) is not \( p \)-torsionfree, then \( \text{Im}(\Theta^P_M) \) is much smaller than \( \mathcal{H}_k(M) \). (For example, if \( k \) is a field of characteristic \( p \), then \( \text{Im}(\Theta^P_M) = \mathcal{H}_k(M^+) \).) Hence, even if the assertion of Proposition 4.7 could be proved without requiring that \( k \) be \( p \)-torsionfree, defining \( \Xi^P_G \) as the composite \( \mathcal{H}_k(P) \rightarrow \text{Im}(\Theta^P_M) \rightarrow \mathcal{H}_k(G) \) would yield a morphism with small image. Instead, we make the following crucial definition:

**Definition.** Let \( k \) be arbitrary. We define
\[ \Xi^P_G = \text{id}_k \otimes \Xi^P_{G,Z} : \mathcal{H}_k(P) \rightarrow \mathcal{H}_k(G). \]
Here, \( \Xi^P_{G,Z} \) is the composite \( \mathcal{H}_Z(P) \overset{\Theta^P_M}{\longrightarrow} \text{Im}(\Theta^P_M) \rightarrow \mathcal{H}_Z(G) \), where the second arrow is the map in Proposition 4.7 (for \( k = \mathbb{Z} \)).

In this way the image of \( \Xi^P_G \) will always be large, that is, it can be canonically identified with \( \mathcal{H}_k(M) \) as a \( k \)-module. The assumption on \( k \) in Proposition 4.7 is therefore inconsequential for the rest of the paper.

Question (Q2) is a consequence of the construction of \( \Theta^P_M \) and \( \Xi^P_G \), that is, we prove:

**Theorem** (Theorem 4.9). The functor \( \text{Mod} - \mathcal{H}_k(M) \rightarrow \text{Mod} - \mathcal{H}_k(G), \quad m \mapsto m \otimes_{\mathcal{H}_k(P)} \mathcal{H}_k(G) \) is naturally isomorphic to \( \mathfrak{X} \).
As an application we will give a new proof (see Corollary 5.8) of the transitivity of parabolic induction originally due to Vignéras [Vig15 Proposition 4.3].

1.3. Structure of the paper. Section 2 is devoted to setting up the notation and reviewing some parts of Bruhat–Tits theory. This part draws heavily from the original paper [BT72] by Bruhat–Tits and from the comprehensive summary in [Vig16].

In section 3 we will study the group index \( \mu_{U_P}(g) := [I_{U_P} : I_{U_P} \cap g^{-1}I_{U_P}g] \) for \( g \in P \), where \( I_{U_P} = I_1 \cap U_P \). This index appears in the double coset formula (Proposition 3.1) which is used to give an explicit description of the map \( \Theta_{P}^{M} \) (Proposition 1.3). The results of subsection 3.2 are used to prove the estimate \( \mu_{U_P}(g) \geq \mu_{U_P}(g_M) \), where \( g_M \) is the image of the projection of \( g \) in \( M \); see Proposition 3.4. It allows us to give a concrete description of a basis of \( \text{Im}(\Theta_{P}^{M}) \) which will be necessary for the construction of \( \Xi_{G}^{P} \). In subsection 3.3 we explain how \( \mu_{U_P} \) naturally gives rise to a function defined on the Iwahori–Weyl group \( W_M \) of \( M \), again denoted \( \mu_{U_P} \). Proposition 3.14 shows that \( \mu_{U_P} \) measures how far the length function on \( W_M \) deviates from the length function on the Iwahori–Weyl group \( W \) of \( G \), and that \( \mu_{U_P} \) is compatible with the Bruhat order on \( W_M \). These properties will become useful in the construction of \( \Xi_{G}^{P} \) (Proposition 4.7) and in the study of the algebras \( H_k(M, G) \) (section 5). We also obtain new and short proofs of two lemmas due to Abe in Corollaries 3.16 and 3.17.

Section 4 contains the main results. Subsection 4.1 gives a short introduction to abstract Hecke algebras as double coset algebras following [AZ95 Chapter 3]. In subsection 4.2 we describe the map \( \Theta_{P}^{M} \) explicitly; see Proposition 4.3. Together with the inequality \( \mu_{U_P}(g) \geq \mu_{U_P}(g_M) \) this yields a minimal generating system of \( \text{Im}(\Theta_{M}^{P}) \) which is even a basis provided the coefficient ring \( R \) is \( p \)-torsionfree; see Corollary 4.3. The main results on the construction of \( \Xi_{G}^{P} \) and on parabolic induction are proved in Proposition 4.7 and Theorem 4.9 respectively.

Finally, section 5 is devoted to applying our previous results to give a new proof of the transitivity of parabolic induction. The algebras \( H_k(M, G) \) are introduced and studied in subsection 5.1. In Proposition 5.3 we show that these algebras, for varying \( G \), are localizations of each other. The main result is the general Theorem 5.7 from which we deduce that parabolic induction is transitive. To finish, we construct in subsection 5.3 a natural filtration on \( H_k(M, G) \) (Proposition 5.10) which might be of independent interest.

2. Notations and preliminaries

Throughout the article we fix a locally compact non-archimedean field \( F \) with residue field \( \mathbb{F}_q \) of characteristic \( p \) and normalized valuation \( \text{val}_F : F \to \mathbb{Z} \cup \{ \infty \} \).

Given a group \( G \), a subgroup \( H \subseteq G \), and elements \( g, h \in G \), we write
\[
 h^g := g^{-1}hg, \quad H^g := \{ h^g \mid h \in H \}, \quad H_{(g)} := H \cap H^g.
\]
We also write \([g, h] := ghg^{-1}h^{-1}\) for the commutator of \( g \) and \( h \).

The symbol \( 
abla \) denotes “disjoint union”.

Given an algebraic group \( \mathbf{H} \) over \( F \), we denote by the corresponding lightface letter \( \mathbf{H} := \mathbf{H}(F) \) its group of \( F \)-rational points. The topology of \( F \) makes \( \mathbf{H} \) into a topological group. We denote
\[
 - \quad \mathbf{H}^\circ \text{ the identity component of } \mathbf{H};
\]
– $X^*(H)$ (resp. $X_*(H)$) the group of algebraic $F$-characters (resp. $F$-cocharacters) of $H$.

Let $G$ be a connected reductive group defined over $F$. Fix a maximal $F$-split torus $T$ and denote by $Z := Z_G(T)$ (resp. $N := N_G(T)$) the centralizer (resp. normalizer) of $T$ in $G$. The finite Weyl group $W_0 := N/Z$ acts on the (relative) root system $\Phi := \Phi(G, T) \subseteq X^*(T)$ associated with $T$; it identifies with the Weyl group of $\Phi$.

**Notation.** Given a subset $\Psi \subseteq \Phi$, we denote $\Psi_{\text{red}} := \{ \alpha \in \Psi \mid \frac{1}{2} \alpha \notin \Psi \}$ the set of reduced roots in $\Psi$.

Let $U_\alpha$ be the root group associated with $\alpha \in \Phi$. Then $U_{2\alpha} \subseteq U_\alpha$ whenever $\alpha, 2\alpha \in \Phi$. Fix a minimal parabolic $F$-subgroup $B$ with Levi decomposition $B = ZU$. This corresponds to a choice $\Phi^+ \subseteq \Phi$ of positive roots in $\Phi$, and we have $U = \prod_{\alpha \in \Phi^+} U_\alpha$. In this article parabolic subgroups are always standard and defined over $F$. By a Levi subgroup we mean the unique Levi $F$-subgroup $M$ of $P$ containing $Z$; this is expressed by writing $P = MU_P$, where $U_P$ denotes the unipotent radical of $P$. Conversely, a Levi subgroup $M$ in $G$ determines a unique parabolic group $P_M$ with Levi $M$ and unipotent radical $U_{P_M} = \prod_{\alpha \in (\Phi^+ \setminus \Phi_M)_{\text{red}}} U_\alpha$.

2.1. The standard apartment. Consider the finite-dimensional $\mathbb{R}$-vector space $V := \mathbb{R} \otimes X_*(T)/X_*(C)$, where $C$ denotes the connected center of $G$. The finite Weyl group $W_0$ acts on $V$ via the conjugation action on $T$, and the natural pairing $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is $W_0$-invariant, where $V^*$ denotes the $\mathbb{R}$-linear dual of $V$. Fix a $W_0$-invariant scalar product $(\cdot, \cdot)$ on $V$. The root system $\Phi$ embeds into $V^*$. Then $\Gamma = \{ \alpha \in \Phi \mid (\cdot, \cdot \rangle_{\alpha} \neq 0 \} \subseteq \Phi$.

2.1.1. **Valuations.** A valuation $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ on the root datum $(Z, (U_\alpha)_{\alpha \in \Phi})$ consists of a family of functions $\varphi_\alpha : U_\alpha \to \mathbb{R} \cup \{ \infty \}$ satisfying a list of axioms [BT72 (6.2.1)]. Then $\varphi$ is called discrete if the set of values $\Gamma_\alpha := \varphi_\alpha(U_\alpha \setminus \{ 1 \}) \subseteq \mathbb{R}$ is discrete. It is called special if $0 \in \Gamma_\alpha$ for all $\alpha \in \Phi_{\text{red}}$. We write $\Gamma'_\alpha := \{ \varphi_\alpha(u) \mid u \in U_\alpha \setminus \{ 1 \} \} \subseteq \mathbb{R}$ and $\varphi_\alpha(u) = \sup \varphi_{\alpha}(uW_{2\alpha}) \in \mathbb{R}$. Then $\Gamma'_\alpha = \Gamma_\alpha$ if $2\alpha \notin \Phi$ and $\Gamma_\alpha = \Gamma'_\alpha \cup (\frac{1}{2} \Gamma_\alpha)$ for all $\alpha \in \Phi$ [BT72 (6.2.2)].

The group $N$ acts naturally on the set of valuations via $(n.\varphi)_\alpha(u) = \varphi_{w^{-1}(\alpha)}(n^{-1}un)$, for $u \in U_\alpha$, $n \in N$, where $w := n.\nu(n) \in W_0 = N/Z$ and $\nu : N \to N/Z = W_0$ denotes the projection map.

The vector space $V$ acts faithfully on the set of valuations via $(\varphi + v)_\alpha(u) = \varphi_\alpha(u) + (\alpha, v)$, for $u \in U_\alpha$, $v \in V$.

One easily verifies $n.(\varphi + v) = n.\varphi + \nu(v)(n)$ for $n \in N$, $v \in V$ [BT72 (6.2.5)].

Given $z \in Z$, there is a unique vector $\nu(z) \in V$ satisfying $z.\varphi = \varphi + \nu(z)$ [BT72 Proposition (6.2.10), Proof of (i)]. In this way we obtain a group homomorphism

$$\nu : Z \to V, \quad z \mapsto \nu(z).$$
Restriction of characters realizes $X^*(Z)$ as a subgroup of finite index in $X^*(T)$ \hbox{[Ren10, V.2.6. Lemme]}. Therefore, given $\chi \in X^*(T)$, there exists $n \in \mathbb{Z}_{>0}$ with $n\chi \in X^*(Z)$, and one easily verifies that the definition

$$(\text{val}_F \circ \chi)(z) := \frac{1}{n} \cdot \text{val}_F((n\chi)(z)) \in \mathbb{R}, \quad \text{for } z \in Z,$$

is independent of $n$. We say that $\varphi$ is compatible with $\text{val}_F$ if

$$\langle \alpha, \nu(z) \rangle = - (\text{val}_F \circ \alpha)(z), \quad \text{for all } z \in Z, \alpha \in \Phi.$$ 

2.1.2. The apartment, hyperplanes, and affine roots. From now on we fix a discrete, special valuation $\varphi_0 = (\varphi_{0,\alpha})_{\alpha \in \Phi}$ on $(Z, (U_\alpha)_{\alpha \in \Phi})$ which is compatible with $\text{val}_F$; it exists by \hbox{[BT84, 5.1.20. Theor`eme and 5.1.23. Proposition]}. By \hbox{[Vig16, (37)]} the subgroups $U_{\alpha,r} := \{ u \in U_\alpha \mid \varphi_{0,\alpha}(u) \geq r \}$, for $r \in \mathbb{R}$, form a basis of compact open neighborhoods of the neutral element in $U_\alpha$, $\alpha \in \Phi$.

The apartment of $G$ is defined as the affine space

$$\mathcal{A} := \{ \varphi_0 + v \mid v \in V \}$$

under $V$. It follows from \hbox{[BT72, Proposition (6.2.10)]} that the action of $N$ on valuations induces a group homomorphism $\nu: N \rightarrow \text{Aut} \mathcal{A}$ extending (4). The fixed $W_0$-invariant scalar product on $V$ endows $\mathcal{A}$ with a Euclidean metric.

Given $\alpha \in V^*$ and $r \in \mathbb{R}$, we put

$$a_{\alpha,r} := \{ \varphi_0 + v \in \mathcal{A} \mid \langle \alpha, v \rangle + r \geq 0 \} \quad \text{and} \quad H_{\alpha,r} := \{ \varphi_0 + v \in \mathcal{A} \mid \langle \alpha, v \rangle + r = 0 \}.$$ 

We call $\Phi^{\text{aff}} := \{ a_{\alpha,r} \mid \alpha \in \Phi \text{ and } r \in \Gamma'_\alpha \}$ the set of affine roots of $\mathcal{A}$ and denote

$$\mathcal{H} := \{ H_{\alpha,r} \mid \alpha \in \Phi \text{ and } r \in \Gamma'_\alpha \} = \{ H_{\alpha,r} \mid \alpha \in \Phi_{\text{red}} \text{ and } r \in \Gamma_\alpha \}$$

the set of hyperplanes in $\mathcal{A}$. A connected component of $\mathcal{A} \smallsetminus \bigcup \mathcal{H}$ is called a chamber. Associated with $\varphi_0$ and $B$ there is a unique chamber $\mathcal{C}$ determined by $\varphi_0 \in \overline{\mathcal{C}}$ (topological closure) and

$$\mathcal{C} \subseteq \{ \varphi_0 + v \in \mathcal{A} \mid \langle \alpha, v \rangle > 0, \quad \text{for all } \alpha \in \Phi^+ \}.$$ 

We call $\mathcal{C}$ the fundamental chamber.

The action of $N$ on $\mathcal{A}$ induces natural actions on $\Phi^{\text{aff}}$ and on $\mathcal{H}$. Explicitly, we have

$$n.a_{\alpha,r} = a_{w(\alpha),r - \langle w(\alpha), n.\varphi_0 - \varphi_0 \rangle},$$

for all $a_{\alpha,r} \in \Phi^{\text{aff}}$ and $n \in N$, where $w := \nu(n) \in W_0$. A similar formula holds for $H_{\alpha,r}$. Likewise,

$$nU_{\alpha,r}n^{-1} = U_{w(\alpha),r - \langle w(\alpha), n.\varphi_0 - \varphi_0 \rangle},$$

for all $\alpha \in \Phi$, $r \in \mathbb{R}$, $n \in N$. 

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2.1.3. The affine Weyl group. Denote $s_H \in \text{Aut } \mathcal{A}$ the orthogonal reflection through $H \in \mathfrak{H}$ and put
\[ S(\mathfrak{H}) := \{ s_H \mid H \in \mathfrak{H} \} . \]

Conversely, we denote $H_s \in \mathfrak{H}$ the hyperplane fixed by $s \in S(\mathfrak{H})$. The affine Weyl group $W^{\text{aff}}$ is the subgroup of $\tilde{W} := \nu(N) \subseteq \text{Aut } \mathcal{A}$ generated by $S(\mathfrak{H})$. Notice that $ws_H w^{-1} = s_{w(H)}$ and $wH_s = H_{wsw^{-1}}$ for all $w \in \tilde{W}$, $s \in S(\mathfrak{H})$, and $H \in \mathfrak{H}$. We denote by $S^{\text{aff}}$ the set of reflections in the walls of the fundamental chamber $\mathfrak{C}$. It generates $W^{\text{aff}}$ as a group. Moreover, $W^{\text{aff}}$ acts simply transitively on the set of chambers of $\mathcal{A}$.

The stabilizer $W_{\varphi_0}$ of $\varphi_0$ in $\tilde{W}$ identifies with $W_0$, because $\varphi_0$ is special. We obtain semidirect product decompositions
\[ \tilde{W} = W_0 \ltimes (\tilde{W} \cap V) \quad \text{and} \quad W^{\text{aff}} = W_0 \ltimes (W^{\text{aff}} \cap V) , \]
and $W^{\text{aff}} \cap V$ is generated by the translations $r^{\alpha^\vee}$, for $\alpha \in \Phi_{\text{red}}$ and $r \in \Gamma_\alpha$ [BT72, Proposition (6.2.19)].

By [BT72, Proposition (6.2.22)] there exists a unique reduced root system $\Sigma$ in $V^*$ such that $W^{\text{aff}}$ is the affine Weyl group of $\Sigma$, i.e., it is the subgroup of $\text{Aut } \mathcal{A}$ generated by the reflections
\[ s_{\alpha,k} : \text{Aut } \mathcal{A} \to \text{Aut } \mathcal{A}, \quad \varphi_0 + v \mapsto \varphi_0 + v - (\langle \alpha, v \rangle + k) \cdot \alpha^\vee , \]
for $(\alpha, k) \in \Sigma^{\text{aff}} := \Sigma \times \mathbb{Z}$. By a suitable scaling we obtain a surjective map $\Phi \to \Sigma$, $\alpha \mapsto \varepsilon_\alpha \alpha$ between root systems which induces a bijection $\Phi_{\text{red}} \cong \Sigma$. By [Vig16] (39) ff we have $\varepsilon_\alpha = \varepsilon_{-\alpha} \in \mathbb{Z}_{>0}$ and $\Gamma_\alpha = \varepsilon_\alpha^{-1} \mathbb{Z}$, for $\alpha \in \Phi_{\text{red}}$, is a group.

**Notation.** In order to avoid confusion when working with the two root systems $\Phi_{\text{red}}$ and $\Sigma$ we will write $H_{(\alpha,k)}$ and $U_{(\alpha,k)}$ instead of $H_{\alpha,k}$ and $U_{\beta,\varepsilon_\beta^{-1}k}$ whenever $\beta \in \Phi_{\text{red}}$, $\alpha = \varepsilon_\beta \beta$, and $k \in \mathbb{Z}$.

Given $n \in N$ with image $w$ in $\tilde{W}$ and $(\alpha, k) \in \Sigma^{\text{aff}}$, we have $wH_{(\alpha,k)} = H_{w(\alpha,k)}$ and
\[ nU_{(\alpha,k)}n^{-1} = U_{w(\alpha,k)} . \]

2.2. Parahoric subgroups. The pointwise stabilizer of $\varphi_0$, resp. $\mathfrak{C}$, in the kernel of the Kottwitz homomorphism $\kappa_G$ [Kot97, 7.1 to 7.4] is denoted by $K$, resp. $I$. We call $I$ the Iwahori subgroup; its pro-$p$ Sylow subgroup $I_1$ is called the pro-$p$ Iwahori subgroup. Both $K$ and $I$ are examples of parahoric subgroups [HR08]; they are compact open subgroups of $G$.

We remark that $K$ is a special parahoric subgroup containing $I$, and it satisfies [Vig16] (51)
\[ K \cap U_\alpha = U_{(\alpha,0)} , \quad \text{for all } \alpha \in \Phi . \]

Put $Z_0 := Z \cap K = Z \cap I$ [HR09, Lemma 4.2.1] with pro-$p$ radical $Z_1 = Z \cap I_1$. Then $Z_0$ is the unique parahoric subgroup of $Z$, and $N$ normalizes both $Z_0$ and $Z_1$. The multiplication map
\[ \prod_{\alpha \in -\Sigma^+} U_{(\alpha,1)} \times Z_1 \times \prod_{\alpha \in \Sigma^+} U_{(\alpha,0)} \cong I_1 \]
is a homeomorphism [Vig16, Corollary 3.20] with respect to any ordering of the factors.
2.3. The Iwahori–Weyl group. We call
\[ W := N/Z_0, \quad \text{resp.} \quad W(1) := N/Z_1 \]
the Iwahori–Weyl group, resp. pro-p Iwahori–Weyl group. There are exact sequences
\[ 1 \rightarrow Z_0/Z_1 \rightarrow W(1) \rightarrow W \rightarrow 1 \]
and
\[ 0 \rightarrow \Lambda \rightarrow W \rightarrow W_0 \rightarrow 1, \]
where \( \Lambda := \frac{Z}{Z_0} \) is a finitely generated abelian group with finite torsion and the same rank as \( X_*(T) \) [HR09, Theorem 1.0.1]. It is thus written additively. When viewed as an element of \( W \) we use the exponential notation \( e^\lambda \) rather than \( \lambda \in \Lambda \) in order to avoid confusion.

Given a subset \( X \subseteq W \), we denote \( X(1) \) the preimage of \( X \) under the projection \( W(1) \rightarrow W \).

We remark that the sequence (8) splits, providing a semidirect product decomposition
\[ W = \Lambda \rtimes W_0. \]
In particular, \( W_0 \) acts on \( \Lambda \) via \( w(\lambda) = we^\lambda w^{-1} \). The group \( \Lambda(1) \) is not abelian in general. Notice that (4) factors through \( \Lambda \) (and hence \( \Lambda(1) \)).

The inclusion \( N \hookrightarrow G \) induces bijections
\[ W_0 \cong B \setminus G/B, \quad W \cong I \setminus G/I, \quad W(1) \cong I_1 \setminus G/I_1. \]

The group \( \Omega = \{ u \in W \mid u \mathfrak{C} = \mathfrak{C} \} \) is abelian and acts on \( S^{\text{aff}} \) by conjugation, and we have a decomposition
\[ W = W^{\text{aff}} \rtimes \Omega. \]
The length function \( \ell \) on the Coxeter group \( (W^{\text{aff}}, S^{\text{aff}}) \) extends to a length function \( \ell \) on \( W \) if we define \( \ell(wu) = \ell(w) \) for \( w \in W^{\text{aff}}, u \in \Omega \). By inflation we also obtain a length function \( \ell \) on \( W(1) \).

We denote by \( \leq \) the Bruhat order on \( W^{\text{aff}} \). It extends to the Bruhat order \( \leq \) on \( W \) if we put
\[ wu \leq w'u' \iff w \leq w' \quad \text{and} \quad u = u', \quad \text{for} \quad w, w' \in W^{\text{aff}}, u, u' \in \Omega. \]
We define \( v < w \) as \( v \leq w \) and \( v \neq w \). By inflation, we obtain the Bruhat order \( < \) on \( W(1) \): given \( \tilde{v}, \tilde{w} \in W(1) \) with image \( v, w \in W \), respectively, we define
\[ \tilde{v} < \tilde{w} \iff v < w. \]

2.4. The integers \( q_w \). Given \( n \in N \) with image \( w \) in \( W \) (or \( W(1) \)), one defines
\[ q_w := |I_n I/I| = |I_1 n I_1/I_1|. \]
An application of [Vig16, Proposition 3.38] shows
\[ q_w = q_{s_1} \cdots q_{s_{\ell(w)}} \]
whenever \( w = s_1 \cdots s_{\ell(w)} u \) with \( s_i \in S^{\text{aff}} \) and \( u \in \Omega \) (or \( s_i \in S^{\text{aff}}(1) \) and \( u \in \Omega(1) \)).

Given \( s \in S^{\text{aff}} \), write \( H_s = H_{\beta,r} = H_{(\alpha,k)} \) with \( \beta \in \Phi^{+}_{\text{red}}, r \in \Gamma_\beta \) and \( \alpha = \varepsilon_\beta \beta \in \Sigma^+ \), \( k = \varepsilon_\beta r \in \mathbb{Z} \). Then
\[ q_s = |U_{\beta,r}/U_{\beta,r+}| = |U_{(\alpha,k)}/U_{(\alpha,k+1)}|. \]
where \( U_{\beta,r+} := \bigcap_{r' > r} U_{\beta,r'} \). Indeed, if \( r \in \Gamma_\beta \setminus \Gamma'_\beta \), then \( 2\beta \in \Phi^+ \) and \( 2r \in \Gamma'_\beta \). In this case we have \( U_{\beta,r+} \cdot U_{\beta,2r} = U_{\beta,r} \), and [Vig16, Lemma 3.8] yields an isomorphism

\[
U_{\beta,r}/U_{\beta,r+} \cong U_{\beta,2r}/U_{\beta,2r+}.
\]

Now, the first equality in (10) follows from [Vig16, Corollary 3.3] while the second equality is clear.

Notice that \( q_s = q_{s'} \) whenever \( s, s' \in S^{\aff} \) are conjugate in \( W \) [Vig16, (67)]. As every hyperplane in \( \mathfrak{H} \) is of the form \( wH_s \) for some \( w \in W \), \( s \in S^{\aff} \), we obtain a well-defined function

\[
\mathfrak{H} \longrightarrow q^{\mathfrak{X}^{\geq 0}}, \quad q(wH_s) := q_s.
\]

For every \( w \in W \) (or \( W(1) \)) we then have [Vig16, Definition 4.14]

\[
q_w = \prod_{H \in \mathfrak{H}_w} q(H),
\]

where \( \mathfrak{H}_w \) denotes the set of hyperplanes in \( \mathfrak{H} \) separating \( \mathfrak{C} \) and \( w\mathfrak{C} \).

For \( v, w \in W \) (or \( W(1) \)) there exists a unique integer \( q_{v,w} \in q^{\mathfrak{X}^{\geq 0}} \) satisfying [Vig16, Definition 4.14]

\[
q_v q_w = q_{v,w} q_{v,w}^2.
\]

Notice that

\[
q_{v,w} = \prod_{H \in \mathfrak{H}_w \cap \mathfrak{H}_v} q(H).
\]

(In [Vig16, Lemma 4.19] this is only stated for \( v, w \in \waff \). But this follows in general from the facts that \( \mathfrak{H}_{w} = \mathfrak{H}_{w} \) and \( u\mathfrak{H}_{w} = \mathfrak{H}_{uw}^{-1} \) whenever \( w \in \waff \) and \( u \in \Omega \).)

**Remark.**

(a) If \( G \) is \( F \)-split, then \( q_w = q^{\ell(w)} \) for all \( w \in W \). For this reason the function \( w \mapsto q_w \) may be viewed as a generalized length function on \( W \).

(b) In general, \( q_{v,w} = 1 \) if and only if \( \ell(vw) = \ell(v) + \ell(w) \).

### 2.5. Levi subgroups

Let \( P = M U_P \) be a (standard) parabolic subgroup. Then \( Z \subseteq M \) and \( N := N_M(T) = N \cap M \). All the objects we have defined for \( G \) have an analogue for \( M \); and we denote them by attaching the index \( M \); for example, we write \( \Phi_M, W_{0,M}, \mathfrak{A}_M, \mathfrak{H}_M, q_{M,w}, \) etc.

Notice that \( W_{0,M} \) is contained in \( W_0 \). \( W_M \) identifies with \( \Lambda \times W_{0,M} \), and \( W_M(1) \) is the preimage of \( W_M \) under \( W(1) \rightarrow W \). The restriction \( \varphi_{0,M} \) of \( \varphi_0 \) to \( (Z, (U_\alpha)_{\alpha \in \Phi_M}) \) is again discrete, special, and compatible with \( \val_F \). The \( \mathbb{R} \)-vector space

\[
V_M := \mathbb{R} \otimes X_*(T)/X_*(C_M),
\]

where \( \mathfrak{C}_M = \left( \bigcap_{\alpha \in \Phi^+_M} \ker \alpha \right) \) is the connected center of \( M \), is a quotient of \( V \). Then \( \Phi_M = V^*_M \cap \Phi \) and \( \Sigma_M = V^*_M \cap \Sigma \). The projection \( V \rightarrow V_M \) induces a natural \( N_M \)-equivariant projection

\[
p_M: \mathfrak{A} \longrightarrow \mathfrak{A}_M
\]

sending \( \varphi_0 \mapsto \varphi_{0,M} \). Taking inverse images we obtain \( N_M \)-equivariant inclusions \( \Phi^\aff_M \subseteq \Phi^\aff \) and \( \mathfrak{H}_M \subseteq \mathfrak{H} \). Therefore, we have also \( S(\mathfrak{H}_M) \subseteq S(\mathfrak{H}) \) and hence \( W^\aff_M \subseteq W^\aff \).

**Remark.** In general we have only \( p_M(\mathfrak{C}) \subseteq \mathfrak{C}_M \) and not an equality. In this case we have \( S^\aff_M \not\subseteq S^\aff \). Therefore, the length and Bruhat order on \( W^\aff_M \) are not obtained by restricting the length and Bruhat order of \( W^\aff \).
3. Groups, double cosets, and indices

We fix a (standard) parabolic $F$-subgroup $P = M\mathbf{U}_p$ of $G$.

3.1. A double coset formula. Let $\Gamma \subseteq P$ be a compact open subgroup satisfying $\Gamma = \Gamma_M \Gamma_{U_p}$, where $\Gamma_M := \Gamma \cap M$ and $\Gamma_{U_p} := \Gamma \cap U_p$. This means that every $g \in \Gamma$ can be uniquely written as

$$g = g_M \cdot g_{U_p}, \quad \text{for some } g_M \in \Gamma_M, \; g_{U_p} \in \Gamma_{U_p}.$$  

Notice that for all $g \in P$ and $m \in M$ the indices

$$\mu(g) := [\Gamma : \langle g \rangle], \quad \mu_{U_p}(g) := [\Gamma_{U_p} : \langle \Gamma_{U_p} \rangle(g)], \quad \mu_M(m) := [\Gamma_M : \langle \Gamma_M \rangle(m)]$$

are finite, because $\Gamma \subseteq P$ is compact open. We also remark that the projection map $\text{pr}_M : P \to M$ is continuous and open, so that $\nu_M(g) := [(\Gamma_M)_{(g_M)} : \text{pr}_M(\Gamma(g))]$ is finite.

Proposition 3.1 generalizes [Gri88, Lemma 2].

**Proposition 3.1.** Given $g \in P$, consider the coset decompositions

$$\Gamma_M = \bigsqcup_{i=1}^{\mu_M(g_M)} (\Gamma_M)_{(g_M)} m_i, \quad \Gamma_{U_p} = \bigsqcup_{j=1}^{\nu_M(g)} \text{pr}_M(\Gamma(g)) h_j,$$

Then one has a decomposition of the double coset

$$\Gamma g \Gamma = \bigsqcup_{i=1}^{\mu_M(g_M)} \bigsqcup_{j=1}^{\nu_M(g)} \bigsqcup_{s=1}^{\mu_{U_p}(g)} \Gamma g u_s h_j m_i.$$  

Moreover, the union is disjoint. In particular, $\mu(g) = \mu_M(g_M) \cdot \nu_M(g) \cdot \mu_{U_p}(g)$.

**Proof.** It is clear that the right hand side of (13) is contained in $\Gamma g \Gamma$. For the converse inclusion let $\gamma = \gamma_M \gamma_{U_p} \in \Gamma$. Write

$$\gamma_M = m m_i, \quad \text{for some } m \in (\Gamma_M)_{(g_M)} \text{ and } 1 \leq i \leq \mu_M(g_M),$$

$$\gamma_{U_p} = h h_j, \quad \text{for some } h \in \text{pr}_M(\Gamma(g)) \text{ and } 1 \leq j \leq \nu_M(g).$$

By definition of $h$ there exists $u \in \Gamma_{U_p}$ with $hu^{-1} \in \Gamma(g)$, i.e., with $ghu^{-1} = yg$ for some $y \in \Gamma$. Thus,

$$\gamma = \gamma_M \gamma_{U_p} = hh_j m_i \gamma_{U_p} = hu^{-1} \cdot u \gamma_{U_p}^{(h_j m_i)^{-1}} \cdot h_j m_i.$$  

As $\Gamma_M$ normalizes $\Gamma_{U_p}$ we may write $vu_s = u \gamma_{U_p}^{(h_j m_i)^{-1}} \in \Gamma_{U_p}$ for some $v \in (\Gamma_{U_p})_{(g)}$ and some integer $1 \leq s \leq \mu_{U_p}(g)$. Then also $u v^{g^{-1}} \in \Gamma_{U_p}$, and therefore

$$g \gamma = ghu^{-1} \cdot u \gamma_{U_p}^{(h_j m_i)^{-1}} \cdot h_j m_i = yg \cdot vu_s \cdot h_j m_i = yu^{g^{-1}} \cdot g \cdot u_s h_j m_i \in \Gamma u_s h_j m_i.$$  

This proves equality in (13). To see disjointness in (13), assume $gu_s h_j m_i = \gamma u_t h_a m_b$ for some $\gamma \in \Gamma$. Rearranging gives

$$\gamma^g = u_s h_j m_i m_b^{-1} h_t^{-1} u_t^{-1} \in \Gamma(g).$$
Applying \( \text{pr}_M \) to (14) yields
\[
\gamma_{M}^{M} = h_{j}m_{i}m_{b}^{-1}h_{a}^{-1} \in \text{pr}_M(\Gamma(g)).
\]
In particular, \( m_{i}m_{b}^{-1} = h_{j}^{-1}\gamma_{M}^{M}h_{a} \in (\Gamma_M)^{(g_M)}, \) whence \( i = b. \) Therefore, equation (15) reads \( \gamma_{M}^{M} = h_{j}h_{a}^{-1} \in \text{pr}_M(\Gamma(g)), \) and we deduce \( j = a. \) Going back to (14) gives \( \gamma^{0} = u_{s}u_{t}^{-1} \in \Gamma(g) \cap U_P. \) But notice that \( \Gamma(g) \cap U_P = (\Gamma_U)_{(g)}, \) because \( g \) normalizes \( U_P. \) Consequently, \( u_{s}u_{t}^{-1} \in (\Gamma_U)_{(g)}, \) whence \( s = t. \) This concludes the proof of the disjointness assertion.

The map \( \Gamma(g) \setminus \Gamma \rightarrow \Gamma \setminus \Gamma g \Gamma, \) sending \( \Gamma(g) \gamma \mapsto \Gamma \gamma, \) is well-defined and bijective. Hence, we have \( \mu(g) = [\Gamma : \Gamma(g)] = [\Gamma \setminus \Gamma g \Gamma] \) and the last assertion follows. \( \square \)

Remark. In general, we have \( \nu_M(g) \neq 1, \) i.e., \( \text{pr}_M(\Gamma(g)) \subsetneq (\Gamma_M)^{(g_M)}. \) As a concrete example consider the group \( P \) of upper triangular matrices inside \( \text{GL}_2(\mathbb{Q}_p). \) It contains the subgroup \( M \) of diagonal matrices and the subgroup \( U_P \) of upper triangular unipotent matrices. Let \( \Gamma = \left( \begin{smallmatrix} (1+pZ_p)^{\times} & Z_p \\ 0 & (1+pZ_p)^{\times} \end{smallmatrix} \right) \) and \( g = \left( \begin{smallmatrix} 1 & p^{-n-1} \\ 0 & 1 \end{smallmatrix} \right) \) for some integer \( n \in \mathbb{Z}_{\geq 0}. \) Then \( gM = 1, \) whence \( (\Gamma_M)^{(g_M)} = \Gamma_M. \) Given any \( \gamma = \left( \begin{smallmatrix} 1 & p^{a} \\ 0 & 1+p \end{smallmatrix} \right) \) in \( \Gamma, \) we compute \( g\gamma g^{-1} = \left( \begin{smallmatrix} 1 & p^{a+b}+p^{-n}(c-a) \\ 0 & 1+p \end{smallmatrix} \right). \) Therefore, \( g\gamma g^{-1} \in \Gamma \) if and only if \( c - a \in p^nZ_p. \) Thus, \( \Gamma(g) = \left( \begin{smallmatrix} 1+p^{a+b}+p^{-n}(c-a) \\ 0 & 1+p \end{smallmatrix} \right) \bigg| a,b,c \in \mathbb{Z}_p \right\}, \) and that
\[
\text{pr}_M(\Gamma(g)) = \left\{ \left( \begin{smallmatrix} 1+p^{a+b}+p^{-n}(c-a) \\ 0 & 1+p \end{smallmatrix} \right) \bigg| a,b,c \in \mathbb{Z}_p \right\}.
\]
From this description it is already clear that \( \nu_M(g) \neq 1 \) in general. As an exercise, and in order to illustrate the methods employed in section 3.3 we explicitly compute \( \nu_M(g). \) Consider the reduction modulo \( p^{n+1} \) map
\[
\psi: \Gamma_M = \left( \begin{smallmatrix} (1+pZ_p)^{\times} & Z_p \\ 0 & (1+pZ_p)^{\times} \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} (Z_p/p^nZ_p)^{\times} & 0 \\ 0 & (Z_p/p^nZ_p)^{\times} \end{smallmatrix} \right).
\]
Its kernel is contained in \( \text{pr}_M(\Gamma(g)), \) and we have
\[
\psi(\Gamma_M) = \left\{ \left( \begin{smallmatrix} 1+p^{a+b}+p^{n+1}Z_p \\ 0 & 1+p \end{smallmatrix} \right) \bigg| a,b \in \mathbb{Z}_p \right\} \text{ and }
\psi(\text{pr}_M(\Gamma(g))) = \left\{ \left( \begin{smallmatrix} 1+p^{a+b}+p^{n+1}Z_p \\ 0 & 1+p \end{smallmatrix} \right) \bigg| a \in \mathbb{Z}_p \right\},
\]
whence \( \psi(\Gamma_M) = p^{2n} \) and \( \psi(\text{pr}_M(\Gamma(g))) = p^n. \) Therefore,
\[
\nu_M(g) = [\Gamma_M : \text{pr}_M(\Gamma(g))] = \left| \psi(\Gamma_M) \right| / \left| \psi(\text{pr}_M(\Gamma(g))) \right| = p^n.
\]

3.2. Two technical lemmas. In this subsection we prove two technical lemmas which will be needed for the proof of the fundamental Proposition 3.4.

Recall the finite-dimensional \( \mathbb{R} \)-vector space \( V, \) the root system \( \Phi \) inside the dual vector space \( V^*, \) and the set of positive roots \( \Phi^+. \) Fix a subset \( \Psi \subseteq \Phi^+. \) We consider the partial ordering on \( \Psi \) defined by
\[
\alpha \leq \beta
\]
if there exist \( \gamma_1, \ldots, \gamma_n \in \Psi \) and \( r, s_1, \ldots, s_n \in \mathbb{Z}_{\geq 0}, \) with \( r > 0, \) such that \( \beta = r\alpha + \sum s_i \gamma_i. \) The relation is clearly reflexive and transitive. It is also antisymmetric, since there exists \( v \in V \) with \( \langle \alpha, v \rangle > 0 \) for all \( \alpha \in \Phi^+. \) We write \( \alpha < \beta \) if \( \alpha \leq \beta \) and \( \alpha \neq \beta. \)
Let $X$ be a group. For each $\alpha \in \Psi$ let $Y_\alpha$ be a subgroup of $X$. Define

$$X_\alpha := \begin{cases} Y_\alpha, & \text{if } 2\alpha \notin \Psi, \\ Y_\alpha Y_{2\alpha}, & \text{if } 2\alpha \in \Psi. \end{cases}$$

Put $X_{2\alpha} := \{1\}$ if $2\alpha \notin \Psi$. We impose the following conditions:

(i) $X$ is generated by the $Y_\alpha$, for $\alpha \in \Psi$.

(ii) For all $\alpha, \beta \in \Psi$ the commutator subgroup $[Y_\alpha, Y_\beta]$ is contained in the subgroup of $X$ generated by the $Y_{r\alpha+s\beta}$, where $r, s \in \mathbb{Z}_{>0}$ with $r\alpha+s\beta \in \Psi$.

(iii) The intersection of the groups generated by $\bigcup_{\alpha \in \Psi, \; \mu \leq 0} Y_\alpha$ and $\bigcup_{\alpha \in \Psi, \; \mu > 0} Y_\alpha$, respectively, is trivial for each $\nu \in V$.

Notice that $X_\alpha$ is a group thanks to (ii). Fix a bijection $o: \Psi_{\text{red}} \rightarrow \{1, 2, \ldots, |\Psi_{\text{red}}|\}$ and define

$$\prod_{\alpha \in \Psi_{\text{red}}} x_\alpha := x_{o^{-1}(1)} \cdot x_{o^{-1}(2)} \cdots x_{o^{-1}(|\Psi_{\text{red}}|)}, \quad \text{for } x_\alpha \in X_\alpha.$$ 

We call $o$ an ordering of the factors. It follows from \[BT72\, Lémmé (6.1.7)] that the multiplication map

$$\prod_{\alpha \in \Psi_{\text{red}}} X_\alpha \rightarrow X$$

is bijective.

**Lemma 3.2.** Let $f: X \rightarrow X$ be a group homomorphism such that

$$f(x_\alpha)x_{\alpha^{-1}} \in \langle X_\beta \mid \beta > \alpha \rangle, \quad \text{for all } x_\alpha \in X_\alpha, \; \text{all } \alpha \in \Psi_{\text{red}}.$$ 

For all $\prod_{\alpha \in \Psi_{\text{red}}} x_\alpha \in X$, with $x_\alpha \in X_\alpha$, one has

$$f\left(\prod_{\alpha \in \Psi_{\text{red}}} x_\alpha\right) = \prod_{\alpha \in \Psi_{\text{red}}} z_\alpha z_\alpha(x_\alpha)x_\alpha,$$

where $z_\alpha(x_\beta)^{(x_\beta)_\beta} \in X_\alpha$ depends only on $(x_\beta)_\beta^{<\alpha}$, and where $z_\alpha: X_\alpha \rightarrow X_{2\alpha}$ is a group homomorphism factoring through $X_\alpha/X_{2\alpha}$. Moreover,

- $z_\alpha$ and $\bar{z}_\alpha$ are uniquely determined by \[(16)\].
- $\bar{z}_\alpha$ depends only on those $x_\beta$ with $\beta < \alpha$ and $o(\beta) > o(\alpha)$.

In particular, if $o$ is such that $\beta_1 < \beta_2$ implies $o(\beta_1) < o(\beta_2)$, then $\bar{z}_\alpha$ does not depend on $(x_\beta)^{(x_\beta)_\beta^{<\alpha}}$. In this case, $\bar{z}_\alpha(x_\alpha)$ is the image of $f(x_\alpha)x_{\alpha^{-1}}$ under the projection $\prod_{\beta \in \Psi_{\text{red}}} X_\beta \rightarrow X_\alpha$.

**Remark.** (a) The homomorphism $f: X \rightarrow X$ in Lemma \[3.2\] is necessarily an automorphism.

(b) The main example to keep in mind is the case where $f: X \rightarrow X$ is conjugation by some element of $X$. (See Lemma \[3.6\] for a proof of why such $f$ satisfies the hypothesis of Lemma \[3.2\]). This is also the only morphism to which we apply Lemma \[3.2\].

**Proof of Lemma 3.2** The uniqueness assertions are immediate (for example, the uniqueness of $z_\alpha$ follows by letting $x_\alpha = 1$). Notice that (ii) above implies that $X_{2\alpha}$ is central in $X_\alpha$ and that the commutator subgroup $[X_\alpha, X_\alpha]$ is contained in $X_{2\alpha}$.

We prove \[(16)\] by induction on $|\Psi_{\text{red}}|$. Suppose $\Psi_{\text{red}} = \{\alpha\}$. By the hypothesis on $f$ the map

$$\bar{z}_\alpha: X_\alpha \rightarrow X_{2\alpha}, \quad x \mapsto f(x)x_{-1}$$

is well-defined and satisfies $\overline{z}_\alpha(x_{2\alpha}) = \{1\}$. Given $x, y \in X_\alpha$, we compute

$$
\overline{z}_\alpha(xy) = f(xy) \cdot (xy)^{-1} = f(x)f(y)y^{-1}x^{-1} = f(x)\overline{z}_\alpha(y)x^{-1} = f(x)x^{-1} \cdot \overline{z}_\alpha(y) = \overline{z}_\alpha(x) \cdot \overline{z}_\alpha(y).
$$

Hence, $\overline{z}_\alpha$ is a group homomorphism, which proves the base case (with $z_\alpha := 1$).

Suppose now $|\Psi_\text{red}| > 1$ and choose a root $\alpha_0 \in \Psi_\text{red}$ maximal with respect to the partial order. We start by proving the following useful claim:

**Claim.** Suppose $f(\prod_{\alpha \in \Psi_\text{red}} x_{\alpha}) = \prod_{\alpha \in \Psi_\text{red}} y_{\alpha}$, where $x_{\alpha}, y_{\alpha} \in X_\alpha$. Then $y_{\alpha_0}$ depends only on the $x_{\beta}$ with $\beta \leq \alpha_0$.

**Proof of the claim.** Let $\Psi'$ be the largest subset of $\Psi$ with $\Psi'_\text{red} = \{\beta \in \Psi_\text{red} \mid \beta \not\leq \alpha_0\}$, and let $Z_{\alpha_0}$ be the subgroup of $X$ generated by the groups $X_\beta$, for $\beta \in \Psi'_\text{red}$. Notice that $\Psi'$ is upwards closed in $\Psi$, that is, $\gamma \geq \beta$ with $\gamma \in \Psi$ and $\beta \in \Psi'$ implies $\gamma \in \Psi'$. Therefore, $Z_{\alpha_0}$ is a normal subgroup of $X$. The hypotheses \((i)\), \((iii)\) remain satisfied if we replace $X$, $\Psi$, and $(Y_\alpha)_{\alpha \in \Psi}$ by $Z_{\alpha_0}$, $\Psi'$, and $(Y_\alpha)_{\alpha \in \Psi'}$, respectively. Hence, the multiplication map $\prod_{\alpha \in \Psi'_\text{red}} X_\alpha \to Z_{\alpha_0}$ is bijective, and the canonical projection

$$
pr_{\leq \alpha_0} : X \cong \prod_{\beta \in \Psi_\text{red}} X_\beta \to \prod_{\beta \leq \alpha_0} X_\beta \cong X/Z_{\alpha_0},
$$

is a group homomorphism with kernel $Z_{\alpha_0}$. The hypothesis on $f$ implies $f(Z_{\alpha_0}) \subseteq Z_{\alpha_0}$, again since $\Psi'$ is upwards closed. We obtain an induced group homomorphism $\overline{f} : X/Z_{\alpha_0} \to X/Z_{\alpha_0}$ such that, after identifying $X \cong \prod_{\beta \in \Psi_\text{red}} X_\beta$ and $X/Z_{\alpha_0} \cong \prod_{\beta \leq \alpha_0} X_\beta$, the diagram

$$
\begin{array}{ccc}
\prod_{\beta \in \Psi_\text{red}} X_\beta & \xrightarrow{f} & \prod_{\beta \in \Psi_\text{red}} X_\beta \\
pr_{\leq \alpha_0} \downarrow & & \downarrow pr_{\leq \alpha_0} \\
\prod_{\beta \leq \alpha_0} X_\beta & \xrightarrow{\overline{f}} & \prod_{\beta \leq \alpha_0} X_\beta
\end{array}
$$

is commutative. As the restriction of $pr_{\leq \alpha_0}$ to $X_{\alpha_0}$ is injective, it follows immediately that $y_{\alpha_0}$ only depends on the $x_{\beta}$ with $\beta \leq \alpha_0$. The claim is proved. \(\square\)

Let $(x_{\alpha})_{\alpha} \in \prod_{\alpha \in \Psi_\text{red}}$ and write $f(\prod_{\alpha \in \Psi_\text{red}} x_{\alpha}) = \prod_{\alpha \in \Psi_\text{red}} y_{\alpha}$ as in the claim. We prove \((iv)\) in two steps.

**Step 1.** We have $y_{\alpha} = z_{\alpha}\overline{z}_\alpha(x_{\alpha})x_{\alpha}$, for $\alpha \neq \alpha_0$, with $z_{\alpha}$ and $\overline{z}_\alpha$ as in the statement of the lemma.

This follows from the induction hypothesis as follows: as $X_{\alpha_0}$ is normal in $X$, the quotient $X' := X/X_{\alpha_0}$ is a group. Put $\Psi' := \Psi \setminus \{\alpha_0, 2\alpha_0\}$. Under the projection map $X \to X'$ the subgroups $Y_{\alpha}$, $X_{\alpha}$ of $X$ embed into $X'$ for $\alpha \in \Psi'$. Denote $f' : X' \to X'$ the homomorphism induced by $f$. The hypotheses of the lemma remain satisfied if we replace $X$, $\Psi$, $(Y_{\alpha})_{\alpha \in \Psi}$, $f$ by $X'$, $\Psi'$, $(Y_{\alpha})_{\alpha \in \Psi'}$, $f'$, respectively. The diagram

$$
\begin{array}{ccc}
X \cong \prod_{\alpha \in \Psi_\text{red}} X_\alpha & \xrightarrow{f} & \prod_{\alpha \in \Psi_\text{red}} X_\alpha \cong X \\
pr \downarrow & & \downarrow pr \\
X' \cong \prod_{\alpha \in \Psi'_\text{red}} X_\alpha & \xrightarrow{f'} & \prod_{\alpha \in \Psi'_\text{red}} X_\alpha \cong X'
\end{array}
$$

is commutative. Therefore, \( f'(\prod_{\alpha \in \mathcal{Y}_r} x_{\alpha}) = \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha} \), and the induction hypothesis implies \( y_{\alpha} = z_{\alpha} \tilde{z}_{\alpha}(x_{\alpha})x_{\alpha} \) for certain elements \( z_{\alpha} \in X_{\alpha} \) and group homomorphisms \( \tilde{z}_{\alpha} : X_{\alpha} \rightarrow X_{2\alpha} \) factoring through \( X_{\alpha}/X_{2\alpha} \) and only depending on the \( x_{\beta} \) with \( \beta < \alpha \), \( \alpha \in \mathcal{Y}_r = \mathcal{Y}_r \setminus \{o_0\} \).

**Step 2.** We have \( y_{\alpha_0} = z_{\alpha_0} \tilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0} \) with \( z_{\alpha_0} \) and \( \tilde{z}_{\alpha_0} \) as in the statement of the lemma.

We introduce the following notation:

\[
x^< := \prod_{\alpha \in \mathcal{Y}_r, o(\alpha) < o(\alpha_0)} x_{\alpha}, \quad x^> := \prod_{\alpha \in \mathcal{Y}_r, o(\alpha) > o(\alpha_0)} x_{\alpha}, \quad x := x^< \cdot x_{\alpha_0} \cdot x^>, \quad x' := x^< \cdot x^>.
\]

\[
f(x^<) = \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}, \quad f(x^>) = \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}, \quad f(x) = \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}, \quad f(x') = \prod_{\alpha \in \mathcal{Y}_r} y'_{\alpha}.
\]

The claim implies that \( y_{\alpha_0}^< \) (resp. \( y_{\alpha_0}^> \), resp. \( y'_{\alpha_0} \)) depends only on the \( x_{\beta} \) with \( \beta < \alpha_0 \) and \( o(\beta) < o(\alpha_0) \) (resp. \( o(\beta) > o(\alpha_0) \), resp. \( o(\beta) \neq o(\alpha_0) \)). Using that \( X_{2\alpha_0} \) is central in \( X \) and that \( X_{\alpha_0} \) is centralized by all \( X_{\beta} \) with \( \beta \neq \alpha_0 \) we compute

\[
\prod_{\alpha \in \mathcal{Y}_r} y_{\alpha} = f(x) = f(x^<) \cdot f(x_{\alpha_0}) \cdot f(x^>) = f(x^<) \cdot f(x_{\alpha_0}) \cdot \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}^<
\]

\[
= f(x^<) \cdot \left( \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}^< \right) \cdot [f(x_{\alpha_0})] \cdot f(x^>)
\]

\[
= f(x') \cdot [f(x_{\alpha_0})] \cdot f(x_{\alpha_0}) = \left( \prod_{\alpha \in \mathcal{Y}_r} y_{\alpha}^> \right) \cdot [f(x_{\alpha_0})] \cdot f(x_{\alpha_0}).
\]

We obtain \( y_{\alpha_0} = y'_{\alpha_0} \cdot [f(x_{\alpha_0})] \cdot f(x_{\alpha_0}) \). The claim implies that the element \( z_{\alpha_0} := y'_{\alpha_0} \in X_{\alpha_0} \) only depends on the \( x_{\beta} \) with \( \beta < \alpha_0 \). Moreover, define

\[
(17) \quad \tilde{z}_{\alpha_0}(x_{\alpha_0}) := [f(x_{\alpha_0})] \cdot f(x_{\alpha_0})x_{\alpha_0}^{-1} \in X_{2\alpha_0}.
\]

Now, \( X_{2\alpha_0} \) is central in \( X_{\alpha_0} \) and \( f \) is the identity on \( X_{2\alpha_0} \), whence \( \tilde{z}_{\alpha_0}(X_{2\alpha_0}) = \{1\} \). It remains to show that \( \tilde{z}_{\alpha_0} : X_{\alpha_0} \rightarrow X_{2\alpha_0} \) is a group homomorphism. The base case shows that \( X_{\alpha_0} \rightarrow X_{2\alpha_0} \), \( x \mapsto f(x)x^{-1} \) is a homomorphism. As \( X_{2\alpha_0} \) is abelian it suffices to show that \( X_{\alpha_0} \rightarrow X_{2\alpha_0} \), \( x \mapsto [f(x), y_{\alpha_0}] \) is a homomorphism. But this is immediate from the general identity \( [uv, w] = u[v, w]u^{-1} \cdot [u, w] \) for all \( u, v, w \in X_{\alpha_0} \). Hence, \( y_{\alpha_0} = z_{\alpha_0} \tilde{z}_{\alpha_0}(x_{\alpha_0})x_{\alpha_0} \), with \( z_{\alpha_0} \) and \( \tilde{z}_{\alpha_0} \) depending only on the \( x_{\beta} \) with \( \beta < \alpha_0 \).

Putting together Steps 1 and 2 finishes the proof of (16). For the last statement we may assume that \( \alpha = \alpha_0 \) is maximal. Then the claim follows from (17), because \( y_{\alpha_0}^> \) depends only on those \( x_{\beta} \) with \( \beta < \alpha_0 \) and \( o(\beta) > o(\alpha_0) \).

For Lemma 3.3 we choose the ordering \( o \) of the factors such that \( \alpha < \beta \) implies \( o(\alpha) < o(\beta) \).
Lemma 3.3. Assume that $X$ is finite. Let $Y \subseteq X$ and $Z_\alpha \subseteq X_\alpha$, for each $\alpha \in \Psi_{\text{red}}$, be subgroups. Assume further that the following condition is satisfied:

For all $1 \leq i \leq |\Psi_{\text{red}}|$ and all $(x_1, \ldots, x_{i-1}) \in \prod_{j=1}^{i-1} X_{o^{-1}(j)}$, there exists $z \in X_{o^{-1}(i)}$ depending only on $(x_1, \ldots, x_{i-1})$ and satisfying the following property: whenever $\prod_{\alpha \in \Psi_{\text{red}}} y_\alpha \in Y$ is such that $y_{o^{-1}(j)} = x_j$, for all $1 \leq j \leq i-1$, we have $y_{o^{-1}(i)} \in zZ_{o^{-1}(i)}$.

Then $|Y| \leq \prod_{\alpha \in \Psi_{\text{red}}} |Z_\alpha|$.

Proof. Given $0 \leq i \leq |\Psi_{\text{red}}|$ and $(x_1, \ldots, x_i) \in \prod_{j=i+1}^{i} X_{o^{-1}(j)}$, we put

$$Y(x_1, \ldots, x_i) := \left\{ \prod_{\alpha \in \Psi_{\text{red}}} y_\alpha \in Y \mid y_{o^{-1}(j)} = x_j \text{ for all } 1 \leq j \leq i \right\}.$$  

For $i = 0$ we define $(x_1, \ldots, x_0)$ to be the empty tuple $(\)$. In this case we have $Y(\) = Y$. Whenever $i < |\Psi_{\text{red}}|$ we say that $x_{i+1} \in X_{o^{-1}(i+1)}$ extends $(x_1, \ldots, x_i)$ if $Y(x_1, \ldots, x_{i+1})$ is not the empty set.

With $i$ as above we will prove

(18) $|Y(x_1, \ldots, x_i)| \leq \prod_{j=i+1}^{|\Psi_{\text{red}}|} |Z_{o^{-1}(j)}|$, for all $(x_1, \ldots, x_i) \in \prod_{j=1}^{i} X_{o^{-1}(j)}$

by descending induction on the length $i$ of the tuple $(x_1, \ldots, x_i)$. Then (18) for $i = 0$ is precisely the assertion of the lemma.

The base case $i = |\Psi_{\text{red}}|$ is satisfied, because the right hand side equals 1 (being the empty product) and since $|Y(x_1, \ldots, x_{|\Psi_{\text{red}}|})|$ is 1 or 0 depending on whether $x_1 \cdots x_{|\Psi_{\text{red}}|}$ lies in $Y$ or not. Assume now that (18) is satisfied for some $1 \leq i \leq |\Psi_{\text{red}}|$. Take $(x_1, \ldots, x_{i-1}) \in \prod_{j=1}^{i-1} X_{o^{-1}(j)}$ and put

$$J(x_1, \ldots, x_{i-1}) := \left\{ y \in X_{o^{-1}(i)} \mid y \text{ extends } (x_1, \ldots, x_{i-1}) \right\}.$$

We may assume that $Y(x_1, \ldots, x_{i-1})$ is non-empty. In this case $J(x_1, \ldots, x_{i-1})$ is also non-empty and condition (3.3) implies $|J(x_1, \ldots, x_{i-1})| = |Z_{o^{-1}(i)}|$. Observe that

$$Y(x_1, \ldots, x_{i-1}) = \bigcup_{y \in J(x_1, \ldots, x_{i-1})} Y(x_1, \ldots, x_{i-1}, y).$$

We now compute

$$|Y(x_1, \ldots, x_{i-1})| \leq \sum_{y \in J(x_1, \ldots, x_{i-1})} |Y(x_1, \ldots, x_{i-1}, y)|$$

$$\leq \sum_{y \in J(x_1, \ldots, x_{i-1})} \prod_{j=i+1}^{|\Psi_{\text{red}}|} |Z_{o^{-1}(j)}|$$

$$= |Z_{o^{-1}(i)}| \cdot \prod_{j=i+1}^{|\Psi_{\text{red}}|} |Z_{o^{-1}(j)}| = \prod_{j=i}^{|\Psi_{\text{red}}|} |Z_{o^{-1}(j)}|,$$

where the second estimate uses the induction hypothesis. This finishes the induction step and proves the lemma. □
3.3. An inequality of indices. Let \( \Gamma \subseteq P \) be an open compact subgroup with \( \Gamma = \Gamma_M I_{U_P} \), where \( \Gamma_M = \Gamma \cap M \) and \( I_{U_P} = I \cap U_P \). Here \( I \) is the (fixed) Iwahori subgroup of \( G \). For example, \( \Gamma \) could be \( K \cap P \) or \( I \cap P \) or even \( I_1 \cap P \).

Remark. Since the function \( \mu_{U_P} \) takes values in \( q^{Z_{>0}} \), there is an equivalence

\[ \mu_{U_P}(g) \leq \mu_{U_P}(g') \iff \mu_{U_P}(g) \text{ divides } \mu_{U_P}(g'). \]

Therefore, we use “\( \leq \)" in this context to implicitly mean “divides”.

The main goal of this section is to prove the following fundamental result.

**Proposition 3.4.** Each \( g \in P \) with image \( g_M \) in \( M \) satisfies

\[ \mu_{U_P}(g) \geq \mu_{U_P}(g_M). \]

**Example.** The ensuing proof is long and technical notwithstanding the lemmas in subsection [3.2]. We will therefore discuss first an example in order to fix ideas. Let \( G = \text{GL}_3(\mathbb{Q}_p) \), \( P \) the Borel subgroup of upper triangular matrices, and \( M \) the torus of diagonal matrices. Let \( I \) be the standard Iwahori determined by \( P \).

Let \( g = g_{U_M} \) with \( g = \begin{pmatrix} \frac{1}{p} x & z & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in U_P \) and \( g_M = \text{diag}(p^{m+n}, p^n, 1) \in M \) with \( n, m \in \mathbb{Z} \). The inequality in Proposition 3.4 is equivalent to

\[ |(I_{U_P})(g)/H| \leq |(I_{U_P})(g_M)/H|, \]

where \( H \subseteq I_{U_P} \) is any sufficiently small open normal subgroup. For a general element \( \begin{pmatrix} 1 & w \\ 0 & v \end{pmatrix} \) of \( I_{U_P} \) we compute

\[ (20) \quad g_M^{-1} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} g_M = \begin{pmatrix} 1 & -m \cdot n \cdot w \\ 0 & 1 \end{pmatrix}. \]

For (20) to lie in \( (I_{U_P})(g_M) \) it is necessary and sufficient that \( u, v, w \) satisfy the following conditions:

(i) \( \text{val}_{\mathbb{Q}_p}(p^{-m}u) \geq 0 \); (ii) \( \text{val}_{\mathbb{Q}_p}(p^{-n}v) \geq 0 \); (iii) \( \text{val}_{\mathbb{Q}_p}(p^{-m-n}w) \geq 0 \).

We deduce

\[ (I_{U_P})(g_M) = \begin{pmatrix} 1 & p^{\max(0, -m)} \cdot Z_p \cdot Z_p \\ 0 & 1 \end{pmatrix}. \]

Now, compute that \( g^{-1} \begin{pmatrix} 1 & u w \\ 0 & 1 \end{pmatrix} g \)

\[ (21) \quad \begin{pmatrix} 1 & p^{-m}u \cdot y \cdot p^{-m}u - x \cdot p^{-n}v + p^{-m-n}w \\ 0 & 1 \end{pmatrix}. \]

Observe that (21) and the right hand side of (20) differ only in the upper right entry, and their difference is \( z(u, v) := y \cdot p^{-m}u - x \cdot p^{-n}v \) which depends only on the terms coming from root groups of smaller roots. Lemma 3.2 shows that this is the general behavior. For (21) to lie in \( (I_{U_P})(g) \) it is necessary that \( u \) and \( v \) satisfy \[ (21) \quad \text{val}_{\mathbb{Q}_p}(z(u, v)) \geq \min\{0, -m - n\}. \]

The important observation to make here is that (iii') may fail even if \( u \) and \( v \) satisfy (i) and (ii). This further restriction on \( u \) and \( v \) is the main reason why there is an inequality in (19). Now, we assume that \( u \) and \( v \) do satisfy (iii') and we determine the possible upper right entries in (21). If \( \text{val}_{\mathbb{Q}_p}(z(u, v)) \geq -m - n \),
then the possible entries lie exactly in $p^{\max(0,-m-n)}\mathbb{Z}_p$. If, on the other hand, we have $-m-n > \text{val}_{\mathbb{Q}_p}(z(u,v)) \geq 0$, then they lie exactly in the proper coset $z(u,v) + p^{\max(0,-m-n)}\mathbb{Z}_p$. Note that in both cases $p^{\max(0,-m-n)}\mathbb{Z}_p$ does not depend on $u$ and $v$. If $H = \begin{pmatrix} 1 & p^r\mathbb{Z}_p & p^r\mathbb{Z}_p \\ 0 & 1 & p^r\mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix}$, for $r \gg 0$ large enough, this discussion shows

$$| (I_{U_p}(g)/H) | \leq \left| \frac{p^{\max(0,-m-n)}\mathbb{Z}_p}{p^r\mathbb{Z}_p} \right| \cdot \left| \frac{p^{\max(0,-n)}\mathbb{Z}_p}{p^r\mathbb{Z}_p} \right| \cdot \left| \frac{p^{\max(0,-m-n)}\mathbb{Z}_p}{p^r\mathbb{Z}_p} \right| = | (I_{U_p}(g_{M})/H) |.$$

The role of Lemma 3.3 is to show that we can make this estimate in general.

We now turn to the proof of Proposition 3.3. As the example above illustrates, it will be necessary to analyse $(I_{U_p}(g)) = I_{U_p} \cap g^{-1}I_{U_p}g$. We will make extensive use of the identification

$$I_{U_p} \cong \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M} U_{(\alpha,0)},$$

which follows from (7). As $G$ is not assumed $F$-split, the root system $\Phi$ need not be reduced. If $\alpha, 2\alpha \in \Phi$, the root group $U_{\alpha}$ is not abelian in general; it contains the non-trivial abelian subgroup $U_{2\alpha}$, and the quotient $U_{\alpha}/U_{2\alpha}$ is abelian. This phenomenon motivates the next definition.

**Definition.** Let $\mathbf{FGrp}$ be the category whose

- objects are pairs $(X_0, X)$ of groups such that $X_0$ is a normal subgroup of $X$;
- morphisms $(X_0, X) \rightarrow (X'_0, X')$ are group homomorphisms $f: X \rightarrow X'$ satisfying $f(X_0) \subseteq f(X'_0)$.

We write $(X_0, X) \subseteq (X'_0, X')$ if $X \subseteq X'$ and $X_0 = X'_0 \cap X$.

There is a canonical functor $\text{gr}: \mathbf{FGrp} \rightarrow \mathbf{Grp}$ into the category of graded groups given by

$$\text{gr}(X_0, X) := \text{gr}_0(X_0, X) \times \text{gr}_1(X_0, X),$$

where $\text{gr}_0(X_0, X) := X_0$ and $\text{gr}_1(X_0, X) := X/X_0$. We will need the following elementary lemma, the proof of which will be left as an exercise for the reader.

**Lemma 3.5.** Let $(X_0, X) \subseteq (X'_0, X')$ be objects in $\mathbf{FGrp}$.

(a) One has $\text{gr}(X_0, X) \subseteq \text{gr}(X'_0, X')$.

(b) Assume $[X' : X] < \infty$. Then $[X' : X] = [\text{gr}(X'_0, X') : \text{gr}(X_0, X)]$.

3.3.1. **Proof of Proposition 3.3**. Let $g \in P$. Recall that $g_M$ denotes the image of $g$ in $M$ and that $g_U := g_M^{-1}g \in U_P$. The group $K_M = K \cap M$ normalizes $I_{U_p}$ and hence the function $\mu_{U_p}: P \rightarrow q^{Z_{\geq 0}}$ is constant on $K_MgK_M$. By [HR09] Lemma 4.1.1 the group $K_M$ is a maximal parahoric subgroup of $M$. Hence, the Cartan decomposition [HR09] Theorem 1.0.3] implies that the intersection $K_MgK_M \cap Z$ is non-empty. Thus, we may assume $g_M \in Z$.

For each $\alpha \in \Sigma^+ \setminus \Sigma_M$ we have $g_M^{-1}U_{(\alpha,0)}g_M = U_{(\alpha,0,\nu(g_M))} g_M$ by (5) and hence

$$U_{(\alpha,0)}(g_M) = \begin{cases} U_{(\alpha,0)}, & \text{if } \langle \alpha, \nu(g_M) \rangle \leq 0, \\ U_{(\alpha,0)}g_M & \text{otherwise}. \end{cases}$$

(22)
Write $\Psi := \Phi^+ \setminus \Phi_M$ and choose an ordering $o$ of the factors of $\prod_{\alpha \in \Psi_{\text{red}}} U_\alpha$ in such a way that $\beta < \alpha$ implies $o(\beta) < o(\alpha)$.

Consider the automorphism $f : U_P \to U_P$, $x \mapsto g_U^{-1} x g_U$. We proceed with a series of lemmas.

**Lemma 3.6.** For each $x_\alpha \in U_\alpha$, $\alpha \in \Psi_{\text{red}}$, the element $f(x_\alpha)x_\alpha^{-1}$ is contained in the subgroup $\langle U_\beta \mid \beta > \alpha \rangle$ of $U_P$ generated by the $U_\beta$ for $\beta \in \Psi_{\text{red}}$ with $\beta > \alpha$.

**Proof.** Write $g_U = u_{\alpha_1} \cdots u_{\alpha_r}$ for certain $u_{\alpha_i} \in U_{\alpha_i}$ and $\alpha_1, \ldots, \alpha_r \in \Psi_{\text{red}}$. We prove the assertion by induction on $r$.

As $(Z, (U_\alpha)_{\alpha \in \Phi})$ is a root group datum, it satisfies (BT72) (6.1.1), (DR 2):

**DR 2** For each $\alpha, \beta \in \Phi$ the commutator subgroup $[U_\alpha, U_\beta]$ is contained in the group generated by the $U_{r\alpha+s\beta}$, for $r, s \in \mathbb{Z}_{>0}$ with $r\alpha + s\beta \in \Phi$.

The base case $r = 1$ is clear from (DR 2). Now, assume $r > 1$ and $y := (u_{\alpha_1} \cdots u_{\alpha_r})^{-1} \cdot x_\alpha \cdot (u_{\alpha_1} \cdots u_{\alpha_r})^{-1} \in \langle U_\beta \mid \beta > \alpha \rangle$.

Again from (DR 2) we know $y^{u_{\alpha r}} \in \langle U_\beta \mid \beta > \alpha \rangle$. Therefore, $g_U^{-1} x_\alpha g_U x_\alpha^{-1} = y^{u_{\alpha r}} \cdot [u_{\alpha r}, x_\alpha]$ is contained in $\langle U_\beta \mid \beta > \alpha \rangle$, proving the assertion. \hfill $\square$

Lemma 3.6 allows us to apply Lemma 3.2 for each $(x_\alpha)_\alpha \in \prod_{\alpha \in \Psi_{\text{red}}} U_\alpha$ there exist an element $z_\alpha(x_\beta)_{\beta < \alpha} \in U_\alpha$ (depending only on $(x_\beta)_{\beta < \alpha}$) and a group homomorphism $\tilde{z}_\alpha : U_\alpha \to U_{2\alpha}$, factoring through $U_\alpha / U_{2\alpha}$ and independent of $(x_\beta)_{\beta < \alpha}$, such that

$$f\left( \prod_{\alpha \in \Psi_{\text{red}}} x_\alpha \right) = \prod_{\alpha \in \Psi_{\text{red}}} z_\alpha(x_\beta)_{\beta < \alpha} \cdot \tilde{z}_\alpha(x_\alpha) \cdot x_\alpha. \quad (23)$$

Identify $\Psi_{\text{red}}$ with $\Sigma^+ \setminus \Sigma_M$. For $\alpha \in \Psi_{\text{red}}$ we consider the homomorphism $f_\alpha : U_{(\alpha,0)}^M \to U_\alpha$, $x \mapsto \tilde{z}_\alpha(x) \cdot x$.

**Remark.** Observe that $f_\alpha$ is the identity if the root system $\Phi$ is reduced (e.g., if $G$ is $F$-split). In this case Lemmas 3.7 and 3.8 are trivial.

**Lemma 3.7.** The image of $f_\alpha$ is open in $U_\alpha$.

**Proof.** Put $\Psi' := \{ \beta \in \Psi_{\text{red}} \mid \beta \not< \alpha \}$ and let $Z_\alpha \subseteq U_P$ be the subgroup generated by the $U_\beta$ with $\beta \in \Psi'$. Then $Z_\alpha$ is normal in $U_P$ by (DR 2) and the multiplication map induces a homeomorphism $U_\beta \cong Z_\alpha$. The projection map $\text{pr}_\alpha : Z_\alpha \to U_\alpha$ and the automorphism $f' := f|_{Z_\alpha}$, induced by the inner automorphism $f$, are open. Hence, the subset $f_\alpha(U_{(\alpha,0)}^M) = (\text{pr}_\alpha \circ f')(U_{(\alpha,0)}^M \times \prod_{\beta \in \Psi' \setminus \{\alpha\}} U_{(\beta,0)}) \subseteq U_\alpha$ is open. \hfill $\square$

**Lemma 3.8.** $[U_{(\alpha,0)} : f_\alpha(U_{(\alpha,0)}^M) \cap U_{(\alpha,0)}] \geq [U_{(\alpha,0)} : (U_{(\alpha,0)})(g_M)]$

**Proof.** For each subgroup $X$ of $U_\alpha$ we have $(U_{2\alpha} \cap X, X) \subseteq (U_{2\alpha}, U_\alpha)$ in $\text{FGrp}$. In order to simplify the notation we write $X$ instead of $(U_{2\alpha} \cap X, X)$.

**Step 1.** We show $\text{gr}_0(f_\alpha(U_{(\alpha,0)}^M) \cap U_{(\alpha,0)}) = \text{gr}_0((U_{(\alpha,0)})(g_M))$, i.e.,

$$f_\alpha(U_{(\alpha,0)}^M) \cap U_{(\alpha,0)} \cap U_{2\alpha} = (U_{(\alpha,0)})(g_M) \cap U_{2\alpha}. \quad (24)$$
Take $x \in U(\alpha,0)$ with $f_{\alpha}(x^{gM}) = \tilde{z}_\alpha(x^{gM}) \cdot x^{gM} \in U(\alpha,0) \cap U_{2\alpha}$. As $\tilde{z}_\alpha$ takes values in $U_{2\alpha}$, we must have $x^{gM} \in U_{2\alpha}$. As $\tilde{z}_\alpha$ vanishes on $U_{2\alpha}$ we deduce $f_{\alpha}(x^{gM}) = x^{gM}$ which is contained in the right hand side of (24). Conversely, given $x \in U(\alpha,0)$ with $x^{gM} \in U(\alpha,0) \cap U_{2\alpha}$, we have $x^{gM} = f_{\alpha}(x^{gM})$, which is contained in the left hand side of (24).

Step 2. We prove $\text{gr}_1(f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)) \subseteq \text{gr}_1((U(\alpha,0))(gM))$. We first show

$$ (f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)) \cdot U_{2\alpha} \subseteq (U_{gM}(\alpha,0) \cap U(\alpha,0))U_{2\alpha} = (U(\alpha,0))(gM) \cdot U_{2\alpha}. $$

This inclusion is a consequence of $f_{\alpha}((U_{gM}(\alpha,0)) \subseteq (U(\alpha,0))(gM)$. The inclusion is a consequence of $f_{\alpha}(U_{gM}(\alpha,0))U_{2\alpha} = U_{gM}(\alpha,0)U_{2\alpha}$ and the equality follows from (22). We compute

$$ \text{gr}_1(f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)) = \frac{f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)}{f_{\alpha}(U_{gM}(\alpha,0)) \cap U_{2\alpha}} \subseteq \frac{U(\alpha,0))(gM) \cdot U_{2\alpha}}{U_{2\alpha}} = \text{gr}_1((U(\alpha,0))(gM)). $$

Step 3. Proof of the assertion. Steps 1 and 2 imply $\text{gr}(f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)) \subseteq \text{gr}((U(\alpha,0))(gM))$. The index of $f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)$ in $U(\alpha,0)$ is finite by Lemma 3.7. Applying Lemma 3.5(b) twice finally shows

$$ [U(\alpha,0) : f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)] = [\text{gr}(U(\alpha,0)) : \text{gr}(f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0))] \geq [\text{gr}(U(\alpha,0)) : \text{gr}((U(\alpha,0))(gM))] = [U(\alpha,0) : (U(\alpha,0))(gM)]. $$

Given $\alpha \in \Psi_{\text{red}}$ and $(x_{\beta})_{\beta < \alpha} \in \prod_{\beta \in \Psi_{\text{red}}} U_{\beta}$, we consider the subset

$$ X_{(x_{\beta})_{\beta < \alpha}} := U(\alpha,0) \cap \{ \tilde{z}_\alpha(x_{\beta}^{gM})_{\beta < \alpha} \cdot z_\alpha(x_{\alpha}^{gM}) \cdot x_{\alpha}^{gM} \mid x_{\alpha} \in U(\alpha,0) \}. $$

As $(I_U)_g = I_U \cap f(I_{U_P})$ it is immediate from (23) that

$$ (I_U)_g = \left\{ \prod_{\alpha \in \Psi_{\text{red}}} y_{\alpha} = f\left( \prod_{\alpha \in \Psi_{\text{red}}} x_{\alpha}^{gM} \right) \in \prod_{\alpha \in \Psi_{\text{red}}} U(\alpha,0) \mid x_{\alpha} \in U(\alpha,0) \text{ and } y_{\alpha} \in X_{(x_{\beta})_{\beta < \alpha}} \text{ for all } \alpha \in \Psi_{\text{red}} \right\}. $$

Lemma 3.9. The set $X_{(x_{\beta})_{\beta < \alpha}}$ is either empty or a left coset of $f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)$.

Proof. Clearly, $X_{(x_{\beta})_{\beta < \alpha}}$ is stable under right multiplication by elements of $f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)$. Now, take two elements

$$ \gamma_i := z_\alpha(x_{\beta}^{gM})_{\beta < \alpha} \cdot z_\alpha(x_{i}^{gM}) \cdot x_{i}^{gM} \in X_{(x_{\beta})_{\beta < \alpha}}, \text{ with } x_{i} \in U(\alpha,0), i = 1, 2. $$

Then $\gamma_2^{-1}\gamma_1 = f_{\alpha}(x_{i}^{-1}x_{i}^{gM}) \in f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)$ proving the claim. □

Choose $r \in \mathbb{Z}_{>0}$ big enough so that $U(\alpha,0)$ is contained in $f_{\alpha}(U_{gM}(\alpha,0)) \cap U(\alpha,0)$ for all $\alpha \in \Psi_{\text{red}}$ and such the subgroup $H := \prod_{\alpha \in \Psi_{\text{red}}} U(\alpha,0)$ of $I_U$ is contained in $(I_U)_g \cap (I_U)_g$. Observe that $H$ is normal, since the valuation $\varphi_0$ satisfies [BT72] (6.2.1), (V 3)].
We will apply Lemma 3.3 next with $X$, $Y$, $Z_{\alpha}$ equal to $I_{U_{P}}/H$ and $(I_{U_{P}})_{(g)}/H$ and $f_{a}(U^{g_{M}}_{(\alpha,0)}) \cap U_{(\alpha,0)}/U_{(\alpha,r)}$, respectively. If $\prod_{\alpha \in \Psi_{\text{red}}} x_{\alpha} \in \prod_{\alpha \in \Psi_{\text{red}}} U_{(\alpha,0)}$ and $\prod_{\alpha \in \Psi_{\text{red}}} y_{\alpha} = f(\prod_{\alpha \in \Psi_{\text{red}}} x_{\alpha}^{g_{M}})$, then each $x_{\alpha}$, and hence $(x_{\beta})_{\beta<\alpha}$, depends only on $(y_{\beta})_{\beta<\alpha}$ (apply the Lemma 3.2 to $f^{-1}$). Therefore, $X_{(x_{\beta})_{\beta<\alpha}}$ depends only on $(y_{\beta})_{\beta<\alpha}$ and, in particular, on $(y_{1}, \ldots, y_{(\alpha-1)})$ by our choice of $\alpha$. In fact, $X_{(x_{\beta})_{\beta<\alpha}}$ depends only on the cosets $y_{\beta}U_{(\beta,r)}$, for $\beta<\alpha$, since we assumed $U_{(\beta,r)} \subseteq f_{\beta}(U^{g_{M}}_{(\beta,0)}) \cap U_{(\beta,0)}$. Now, (26) and Lemma 3.9 show that condition (3.3) is satisfied. Lemma 3.3 implies

$$\left| (I_{U_{P}})(g)/H \right| \leq \prod_{\alpha \in \Psi_{\text{red}}} \left| f_{a}(U^{g_{M}}_{(\alpha,0)}) \cap U_{(\alpha,0)}/U_{(\alpha,r)} \right|. \tag{27}$$

Further, using Lemma 3.8 we estimate

$$\left| f_{a}(U^{g_{M}}_{(\alpha,0)}) \cap U_{(\alpha,0)}/U_{(\alpha,r)} \right| = \frac{|U_{(\alpha,0)}/U_{(\alpha,r)}|}{|U_{(\alpha,0)}/(U_{(\alpha,0)}(g_{M})) / U_{(\alpha,r)}|} \leq \frac{|U_{(\alpha,0)}/U_{(\alpha,r)}|}{|U_{(\alpha,0)}/(U_{(\alpha,0)}(g_{M}))|} = \left| (U_{(\alpha,0)}(g_{M})/U_{(\alpha,r)} \right|. \tag{28}$$

Putting (27) and (28) together yields

$$\left| (I_{U_{P}})(g)/H \right| \leq \prod_{\alpha \in \Psi_{\text{red}}} \left| (U_{(\alpha,0)}(g_{M})/U_{(\alpha,r)} \right| = \left| (I_{U_{P}})(g)/H \right|. \tag{29}$$

Finally, we conclude $\mu_{U_{P}}(g) = \frac{|U_{(\alpha,0)}/H|}{|I_{U_{P}}(g)/H|} \geq \frac{|I_{U_{P}}/H|}{|I_{U_{P}}(g)/H|} = \mu_{U_{P}}(g)$, finishing the proof of Proposition 3.4.

3.4. Properties of $\mu_{U_{P}}(w)$.

Notation. As $\mu_{U_{P}}: M \rightarrow q^{\mathbb{Z}_{\geq 0}}$ is constant on double cosets with respect to $K_{M}$ (hence also $I_{M}$), we obtain from the Bruhat decomposition (9) an induced function

$$\mu_{U_{P}}: W_{M} \rightarrow q^{\mathbb{Z}_{\geq 0}}. \tag{29}$$

As $K_{M}$ contains representatives of $W_{0,M}$, it follows that $\mu_{U_{P}}$ is constant on the double cosets with respect to $W_{0,M}$.

The analogous map $\mu_{U_{P}}: W_{M}(1) \rightarrow q^{\mathbb{Z}_{\geq 0}}$ is obtained from (29) by inflation. It is clear that all results in this section are still true if we replace $W_{M}$ by $W_{M}(1)$.

Our goal in this section will be to study the properties of the function $\mu_{U_{P}}$.

Lemma 3.10. Let $\lambda \in \Lambda$ and $w_{0} \in W_{0,M}$. Then

$$\mu_{U_{P}}(e^{\lambda}w_{0}) = \prod_{\alpha \in \Sigma^{+} \cap \Sigma_{M} \langle \alpha, \nu(\lambda) \rangle > 0} \left| U_{(\alpha,0)}/U_{(\alpha,\langle \alpha,\nu(\lambda) \rangle)} \right|. \tag{30}$$

Proof. Let $m \in Z$ be a representative of $\lambda \in \Lambda$. Then the multiplication map induces an $m$-equivariant bijection $I_{U_{P}} \cong \prod_{\alpha \in \Sigma^{+} \cap \Sigma_{M}} U_{(\alpha,0)}$, and we compute

$$(I_{U_{P}})(m) = I_{U_{P}} \cap m^{-1}I_{U_{P}}m \cong \prod_{\alpha \in \Sigma^{+} \cap \Sigma_{M}} U_{(\alpha,0)} \cap m^{-1}U_{(\alpha,0)m}$$

$$= \prod_{\alpha \in \Sigma^{+} \cap \Sigma_{M} \langle \alpha, \nu(\lambda) \rangle \leq 0} U_{(\alpha,0)} \times \prod_{\alpha \in \Sigma^{+} \cap \Sigma_{M} \langle \alpha, \nu(\lambda) \rangle > 0} U_{(\alpha,\langle \alpha,\nu(\lambda) \rangle)}. \tag{31}$$
But then
\[ \mu_{U^p}(e^\lambda w_0) = \mu_{U^p}(e^\lambda) = [I_{U^p} : (I_{U^p})(m)] = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M \langle \alpha, \nu(\lambda) \rangle > 0} |U_{(\alpha,0)}/U_{(\alpha,\langle \alpha,\nu(\lambda) \rangle)}|. \]

**Definition.** Given \((\alpha, k) \in \Sigma^{aff} = \Sigma \times \mathbb{Z}\), we put (cf. (11))
\[ q(\alpha, k) := q(H_{(\alpha,k)}) = |U_{(\alpha,k)}/U_{(\alpha,k+1)}| \in \mathbb{Q}^\geq 0. \]

Notice that \(q(\alpha, k) = q(w \cdot (\alpha, k))\) by (3). In particular, \(q(\alpha, k) = q(-\alpha, k)\) (take \(w = s_{\alpha}\)) and \(q(\alpha, k) = q(\alpha, k + \langle \alpha, \nu(\lambda) \rangle)\) (take \(w = e^{-\lambda} \in \Lambda\)).

3.4.1. The opposite parabolic. Let \(P^{op}\) be the parabolic subgroup opposite \(P\) with Levi \(M\) and unipotent radical \(U^{op}\). Writing \(I_{U^{op}} = I \cap U^{op}\), we define
\[ \mu_{U^{op}}(g) := [I_{U^{op}} : (I_{U^{op}})(g)], \quad \text{for } g \in P. \]

It is constant on double cosets with respect to \(K_M\), hence restricts to a function \(\mu_{U^{op}} : W_M \to \mathbb{Q}^\geq 0\). Proposition 3.11 explains how \(\mu_{U^{op}}\) is related with \(\mu_{U^p}\).

**Proposition 3.11.** \(\mu_{U^{op}}(w) = \mu_{U^p}(w^{-1})\) for all \(w \in W_M\).

**Proof.** We may assume \(w = e^\lambda \in \Lambda\). Lemma 3.10 implies
\[ \mu_{U^p}(-\lambda) = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M \langle \alpha, \nu(-\lambda) \rangle > 0} \prod_{k=0}^{\langle \alpha, \nu(-\lambda) \rangle - 1} q(\alpha, k). \]

A similar argument shows
\[ \mu_{U^{op}}(\lambda) = \prod_{\alpha \in (-\Sigma^+) \setminus \Sigma_M \langle \alpha, \nu(\lambda) \rangle > 0} |U_{(\alpha,1)}/U_{(\alpha,\langle \alpha,\nu(\lambda) \rangle + 1)}| = \prod_{\alpha \in (-\Sigma^+) \setminus \Sigma_M \langle \alpha, \nu(\lambda) \rangle > 0} \prod_{k=1}^{\langle \alpha, \nu(\lambda) \rangle} q(\alpha, k). \]

Using \(\langle -\alpha, \nu(-\lambda) \rangle = \langle \alpha, \nu(\lambda) \rangle\), \(q(-\alpha, k) = q(\alpha, k)\), and \(q(\alpha, 0) = q(\alpha, \langle \alpha, \nu(\lambda) \rangle)\), we obtain \(\mu_{U^p}(-\lambda) = \mu_{U^{op}}(\lambda)\).

3.4.2. Changing the parabolic subgroup. Let \(Q = LU_Q\) be a parabolic subgroup of \(G\) with \(M \subseteq L\). Then \(P \cap L\) is a parabolic subgroup of \(L\) with Levi \(M\) and unipotent radical \(U_P \cap L\).

**Proposition 3.12.** \(\mu_{U^p}(w) = \mu_{U^{p \cap L}}(w) \cdot \mu_{U^Q}(w)\) for all \(w \in W_M\).

**Proof.** It suffices to prove the assertion for \(w \in \Lambda\). Using Lemma 3.10 we obtain
\[ \mu_{U^p}(w) = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M \langle \alpha, \nu(w) \rangle > 0} |U_{(\alpha,0)}/U_{(\alpha,\langle \alpha,\nu(w) \rangle)}| \]
\[ = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M \langle \alpha, \nu(w) \rangle > 0} |U_{(\alpha,0)}/U_{(\alpha,\langle \alpha,\nu(w) \rangle)}| \cdot \prod_{\alpha \in \Sigma^+ \setminus \Sigma_L \langle \alpha, \nu(w) \rangle > 0} |U_{(\alpha,0)}/U_{(\alpha,\langle \alpha,\nu(w) \rangle)}| \]
\[ = \mu_{U^{p \cap L}}(w) \cdot \mu_{U^Q}(w). \]

□
3.4.3. Relating $\mu_{U,p}$ with the structure of $W_M$. The fact that $\mu_{U,p}$ can be defined on $W_M$ may seem rather coincidental. There is, however, a strong relationship between the two. To begin with, the function

\begin{equation}
\delta_M: M \rightarrow \mathbb{Q}^\times, \quad m \mapsto \mu_{U,p}(m)/\mu_{U,p}(m^{-1})
\end{equation}

is easily seen to be a group homomorphism by showing $\delta_M(m) = [mI_{U,p}m^{-1} : I_{U,p}]$ (generalized index; see [Vig96] I.2.7 for various properties), and this clearly induces a group homomorphism $\delta_M: W_M \rightarrow \mathbb{Q}^\times$. One can say more:

**Lemma 3.13.** $\delta_M$ is trivial on $W^{\text{aff}}$, hence factors through a character $\Omega_M \rightarrow \mathbb{Q}^\times$.

*Proof.* Given $s \in S^{\text{aff}}$, we have $s = s^{-1}$. As $\delta_M$ is a group homomorphism taking only positive values, this implies $\delta_M(s) = 1$. As $W^{\text{aff}}$ is generated by $S^{\text{aff}}$, the assertion follows.

Thus, $\mu_{U,p}$ carries (at least some) information about the group structure of $W_M$. Our main result in this section shows that $\mu_{U,p}$ measures the deviation between the length functions $\ell$ on $W$ and $\ell_M$ on $W_M$, and that it is monotone with respect to the Bruhat order $\leq_M$ on $W_M$.

**Proposition 3.14.** Let $v, w \in W_M$. Then:

(a) $q_w = \mu_{U,p}(w)\mu_{U,p}(w^{-1}) \cdot q_{M,w}$;

(b) $\mu_{U,p}(v) \leq \mu_{U,p}(w)$ whenever $v \leq_M w$.

Before we give the proof, we deduce a simple corollary:

**Corollary 3.15.** $\mu_{U,p}(vw) \leq \mu_{U,p}(v)\mu_{U,p}(w)$ and $q_{v,w} = \frac{\mu_{U,p}(v)\mu_{U,p}(w)}{\mu_{U,p}(vw)} \cdot q_{M,v,w}$ for all $v, w \in W_M$.

*Proof.* As $\delta_M: W_M \rightarrow \mathbb{Q}^\times$ is a group homomorphism, we have $\frac{\mu_{U,p}(v)\mu_{U,p}(w)}{\mu_{U,p}(vw)} = \frac{\mu_{U,p}(v^{-1})\mu_{U,p}(w^{-1})}{\mu_{U,p}(v^{-1}w^{-1})}$. Hence, Proposition 3.14(a) implies

\[
q_{v,w}^2 = \frac{q_{wv}}{q_{vw}} = \frac{\mu_{U,p}(v)\mu_{U,p}(v^{-1}) \cdot \mu_{U,p}(w)\mu_{U,p}(w^{-1})}{\mu_{U,p}(vw)\mu_{U,p}(v^{-1}w^{-1})} \cdot \frac{q_{M,w} \cdot q_{M,w}}{q_{M,vw}} \cdot \left(\frac{\mu_{U,p}(v)}{\mu_{U,p}(vw)} \cdot q_{M,v,w}\right)^2.
\]

Taking the (positive) square root implies the second assertion. Since $\mathcal{H}_{M,w} = \mathcal{H}_w \cap \mathcal{H}_M$ for all $w \in W$ (see [12] for the definition of $\mathcal{H}_w$), we have $\mathcal{H}_{M,w} \cap v\mathcal{H}_{M,w} \subseteq \mathcal{H}_v \cap v\mathcal{H}_w$. Therefore $q_{M,v,w}$ divides $q_{v,w}$, and the first assertion follows from the second.

*Proof of Proposition 3.14.*

(a) Under the inclusion $\mathcal{H}_M \subseteq \mathcal{H}$ we have

\[
\mathcal{H}_M = \{H(\alpha,k) \in \mathcal{H} \mid (\alpha,k) \in \Sigma^{\text{aff}}_M\}.
\]

Take $w = e^\lambda w_0$, with $\lambda \in \Lambda$ and $w_0 \in W_{0,M}$. By [Vig16] Lemma 5.6 and Proposition 5.9, 2] (using $w_0(\Sigma^+ \setminus \Sigma_M) = \Sigma^+ \setminus \Sigma_M$) there is a decomposition

\[
\mathcal{H}_w = \mathcal{H}_w^+ \sqcup \mathcal{H}_w^- \sqcup \mathcal{H}_{M,w}.
\]
where

\[ \mathcal{H}_w^+ := \left\{ H_{(\alpha, k)} \mid \alpha \in \Sigma^+ \setminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda) \rangle > 0 \right\} \quad \text{and} \]
\[ \mathcal{H}_w^- := \left\{ H_{(\alpha, k)} \mid \alpha \in \Sigma^+ \setminus \Sigma_M \text{ with } \langle \alpha, \nu(\lambda) \rangle < 0 \right\}. \]

Thanks to Lemma 3.10 we compute
\[ \prod_{H \in \mathcal{H}_w^+} q(H) = \prod_{\alpha \in \Sigma^+ \setminus \Sigma_M} q(\alpha, k) = \mu_U(-\lambda) = \mu_U(w^{-1}). \]

Likewise, a short computation reveals \( \prod_{H \in \mathcal{H}_w^-} q(H) = \mu_U(w) \) keeping in mind \( q(\alpha, k) = q(\alpha, k + \langle \alpha, \nu(\lambda) \rangle) \). Taking everything together, the assertion follows from (12) applied to both \( q_w \) and \( q_{M,w} \).

(b) Let \( v, w \in W_M \) with \( v \leq_M w \). By definition of the Bruhat order, e.g., [BB06, Definition 2.1.1], there exists a chain \( v = v_0, v_1, \ldots, v_k = w \) in \( W_M \) with \( v_i v_{i-1}^{-1} \in S(\mathcal{H}_M) \) (\( \iff i \in S(\mathcal{H}_M) \)) and \( \ell_M(v_{i-1}) < \ell_M(v_i) \) for all \( 1 \leq i \leq k \).

We are therefore reduced to the case \( w = tv \) and \( \ell_M(v) < \ell_M(w) \) for some \( t \in S(\mathcal{H}_M) \). In fact, by [BB06, Theorem 2.2.6], we may even assume \( \ell_M(w) = \ell_M(v) + 1 \).

Step 1. We show \( \mu_U(tw) \mu_U((tw)^{-1}) \leq \mu_U(v) \mu_U(w^{-1}) \). The hypothesis \( \ell_M(tw) = \ell_M(w) - 1 \) implies \( H_t \in \mathcal{H}_{M,w} \). But since \( \mathcal{H}_{M,w'} = \mathcal{H}_{w'} \cap \mathcal{H}_M \), for all \( w' \in W_M \), we also have \( H_t \in \mathcal{H}_{w} \). Let \( \Gamma_w \) be a minimal gallery in \( \mathcal{A} \) connecting \( v \) and \( w \). Denote \( (H_1, H_2, \ldots, H_{\ell(w)}) \) the sequence of hyperplanes crossed by \( \Gamma_w \). Then \( H_t = H_i \) for some \( 1 \leq i \leq \ell(w) \). By folding \( \Gamma_w \) along \( H_t \) and deleting the repeated chamber we obtain a gallery \( \Gamma_{tw} \) of length \( \ell(w) - 1 \) in \( \mathcal{A} \) connecting \( v \) and \( tw \).

Of course, \( \Gamma_{tw} \) crosses all hyperplanes in \( \mathcal{H}_{tw} \) at least once. Applying (a) twice, we compute
\[ \mu_U(w) \mu_U(w^{-1}) q_{M,w} = q_w \]
\[ = \prod_{j=1}^{\ell(w)} q(H_j) \]
\[ = \prod_{j=1}^{i-1} q(H_j) \cdot \prod_{j=i+1}^{\ell(w)} q(H_j) \cdot q(H_t) \]
\[ \geq q_{tw} \cdot q(H_t) \]
\[ = \mu_U(tw) \mu_U((tw)^{-1}) \cdot q_{M,tw} \cdot q(H_t). \]

Since \( \ell_M(w) = \ell_M(tw) + 1 \), we have \( q_{M,w} = q_{M,tw} \cdot q(H_t) \), finishing the proof of Step 1.
Step 2. We show $\mu_{U_P}(tw) \leq \mu_{U_P}(w)$. By Lemma 3.13 we have $\delta_M(tw) = \delta_M(w)$. Hence,

$$\mu_{U_P}(tw)^2 = \mu_{U_P}(tw)\mu_{U_P}((tw)^{-1}) \cdot \delta_M(tw) \leq \mu_{U_P}(w)\mu_{U_P}(w^{-1}) \cdot \delta_M(w) = \mu_{U_P}(w)^2$$

using Step 1. Taking square roots, the assertion follows. □

3.5. Positive elements. We collect here some results on positive elements in the Levi subgroup $M$ which will become important for the definition of parabolic induction. The results in this section are not new, but we will give new proofs of two results obtained by Abe in [Abe16]

Definition. An element $m \in M$ is called $M$-positive (or just positive if no confusion arises) if $\mu_{U_P}(m) = 1$. The monoid of positive elements is denoted $M^+$ (or even $M^{+,G}$ if we want to stress that $M$ is considered as a Levi subgroup in $G$). Notice that $K_M \subseteq M^+$.

The elements in the monoid $M^- := (M^+)^{-1}$ are called $M$-negative (or negative).

Remark. In view of $\mu_{U_{pop}}(m^{-1}) = \mu_{U_P}(m)$ (Proposition 3.11), $m$ being positive is equivalent to

$$mI_{U_P}m^{-1} \subseteq I_{U_P} \quad \text{and} \quad I_{U_{pop}} \subseteq mI_{U_{pop}}m^{-1},$$

where $I_{U_{pop}} = I \cap U_{pop}$. This recovers the classical definition, cf. [BK98, (6.5)] or [Vig98, II.4].

Definition. (a) An element $w \in W_M$ (or in $W_M(1)$) is called $M$-positive (or positive) if $\mu_{U_P}(w) = 1$. The monoid of positive elements in $W_M$ is denoted $W_{M^+}$.

The elements in $W_{M^-} := (W_{M^+})^{-1}$ are called $M$-negative (or negative).

(b) An element $\lambda \in \Lambda$ (or in $\Lambda(1)$) is called strictly $M$-positive (or strictly positive) if there exists a central element $a \in M$ with $\lambda = aZ_0$ (or $\lambda = aZ_1$) and if

$$\langle \alpha, \nu(\lambda) \rangle < 0 \quad \text{for all } \alpha \in \Sigma^+ \setminus \Sigma_M.$$

(c) A central element $a \in M$ is called strictly $M$-positive (or strictly positive) if $aZ_0 \in \Lambda$ is.

Remark. (a) If $\lambda \in \Lambda$ is strictly positive, then $\langle \alpha, \nu(\lambda) \rangle = 0$ for all $\alpha \in \Sigma_M$, because $\lambda = aZ_0$ for some element $a$ in the center of $M$. The strictly positive elements are contained in $\Lambda_{M^+} := \Lambda \cap W_{M^+}$ (e.g., by Lemma 3.10).

(b) An element $\lambda \in \Lambda$ is strictly positive if and only if it lies in the image of a strongly positive element $a$ in the sense of [BK98, (6.16)]. In particular, [BK98, (6.14)], strictly positive elements exist. Moreover, given any $m \in M$ we have $a^n m \in M^+$ for $n \gg 0$.

(c) The Bruhat decomposition of $M$ [10] induces bijections $W_{M^+} \cong I_M \setminus M^+/I_M$ and $W_{M^+}(1) \cong I_{1,M} \setminus M^+/I_{1,M}$ [OV15, Remark 2.11(2)].

(d) We have $W_{M^+} \cong \Lambda_{M^+} \rtimes W_{0,M}$ and [Vig15, Lemma 2.2]

$$\Lambda_{M^+} = \{ \lambda \in \Lambda \mid \langle \alpha, \nu(\lambda) \rangle \leq 0 \quad \text{for all } \alpha \in \Sigma^+ \setminus \Sigma_M \}.$$
Corollary 3.16 ([Abe16 Lemma 4.5]). Let either \( v, w \in W_{M}^{+} \) or \( v, w \in W_{M}^{-} \). Then we have \( q_{v,w} = q_{M,v,w} \) and, in particular,
\[
\ell_{M}(v) + \ell_{M}(w) - \ell_{M}(vw) = \ell(v) + \ell(w) - \ell(vw).
\]

Proof. If \( v, w \in W_{M}^{+} \) then Corollary 3.15 yields \( q_{v,w} = q_{M,v,w} \) (and also \( vw \in W_{M}^{+} \)). If, on the other hand, \( v, w \in W_{M}^{-} \) then \( v^{-1}, w^{-1}, (vw)^{-1} \in W_{M}^{+} \) and \( \mu_{U_{p}}(vw) = \mu_{U_{p}}(v) \cdot \mu_{U_{p}}(w) \), because \( (30) \) is a group homomorphism. Again, Corollary 3.15 implies \( q_{v,w} = q_{M,v,w} \).

Notice that \( q_{v,w} = q_{M,v,w} \) implies \( \mathcal{H}_{v} \cap v\mathcal{H}_{w} = \mathcal{H}_{M,v} \cap v\mathcal{H}_{M,w} \). Hence, the second assertion follows from \( \ell(vw) = \ell(v) + \ell(w) - 2 \cdot |\mathcal{H}_{v} \cap v\mathcal{H}_{w}| \), [Vig16 Remark 4.18], and a similar formula for \( \ell_{M}(vw) \).

Corollary 3.17 ([Abe16 Lemma 4.1]). Let \( w \in W_{M}^{+} \) (resp. \( w \in W_{M}^{-} \)) and \( v \in W_{M} \) with \( v \leq M \). Then \( v \in W_{M}^{+} \) (resp. \( v \in W_{M}^{-} \)).

Proof. Replacing \( w \) by \( w^{-1} \) if necessary, we may assume \( w \in W_{M}^{+} \). Then Proposition 3.14(b) implies \( v \in W_{M}^{+} \). \( \square \)

4. PARABOLIC INDUCTION

4.1. Reminder on abstract Hecke algebras. We recall briefly the basics of Hecke algebras. A thorough introduction with complete proofs can be found in [AZ95 Chapter 3, §1.2].

Let \( X \) be a topological group and \( S \subseteq X \) a submonoid containing a compact open subgroup \( \Gamma \). Then \( S \) acts on the right on the free \( \mathbb{Z} \)-module on the set of right cosets \( \Gamma \backslash S \)
\[
\mathbb{Z}[\Gamma \backslash S] := \bigoplus_{\gamma \in \Gamma \backslash S} \mathbb{Z}.(\Gamma \gamma).
\]
The \( \mathbb{Z} \)-module of \( \Gamma \)-invariants
\[
H(\Gamma, S) = \mathbb{Z}[\Gamma \backslash S]^\Gamma = \{ t \in \mathbb{Z}[\Gamma \backslash S] \mid t\gamma = t \quad \text{for all} \ \gamma \in \Gamma \}
\]
is free. The element
\[
T_{g} := T_{g}^{S} := \sum_{\Gamma h} (\Gamma h) \in H(\Gamma, S),
\]
where the sum ranges over all right cosets contained in the double coset \( \Gamma g \Gamma \), depends only on \( \Gamma g \Gamma \). Moreover, \( (T_{g})_{\Gamma g \Gamma \in \Gamma \backslash S/\Gamma} \) forms a \( \mathbb{Z} \)-basis of \( H(\Gamma, S) \).

The isomorphism
\[
\text{End}_{S}(\mathbb{Z}[\Gamma \backslash S]) \xrightarrow{\cong} H(\Gamma, S), \quad T \mapsto T((\Gamma))
\]
endows \( H(\Gamma, S) \) with the structure of an associative ring with unit \( T_{1} \). Concretely, given elements \( t = \sum_{i} a_{i} \cdot (\Gamma g_{i}) \) and \( t' = \sum_{j} a'_{j} \cdot (\Gamma g'_{j}) \) in \( H(\Gamma, S) \), one has
\[
t \cdot t' = \sum_{i,j} a_{i}a'_{j} \cdot (\Gamma g_{i}g'_{j}) \in H(\Gamma, S).
\]

Given a commutative unital ring \( R \), the \( R \)-algebra
\[
H_{R}(\Gamma, S) := R \otimes H(\Gamma, S)
\]
is called the Hecke algebra over \( R \) associated with \( (\Gamma, S) \).
4.2. Parabolic induction. Let $P = MU_P$ be a parabolic $F$-subgroup of $G$. Recall the pro-$p$ Iwahori subgroup $I_1$ of $G$. Then $I_{1,M} = I_1 \cap M$ is a pro-$p$ Iwahori subgroup of $M$ \cite{HR09} Lemma 4.1.1. Fix a commutative unital ring $R$. The algebras
\[
\mathcal{H}_R(G) := H_R(I_1, G) \quad \text{and} \quad \mathcal{H}_R(M) := H_R(I_{1,M}, M)
\]
are the usual pro-$p$ Iwahori–Hecke $R$-algebras \cite{Vig16} (1) of $G$ and $M$, respectively. We write $T_w := T_g$ whenever $w \in W(1)$ corresponds to $I_1gI_1$ under \cite{9}; a similar convention applies to $\mathcal{H}_R(M)$. Then $(T_w)_{w \in W(1)}$ and $(T^M_w)_{w \in W_M(1)}$ are $R$-bases of $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$, respectively.

Definition. The $R$-algebra $\mathcal{H}_R(P) := H_R(I_1 \cap P, P)$ is called the parabolic pro-$p$ Iwahori–Hecke $R$-algebra.

The main goal of this section will be to construct two $R$-algebra morphisms
\[
\Theta^P_M : \mathcal{H}_R(P) \longrightarrow \mathcal{H}_R(M) \quad \text{and} \quad \Xi^G_P : \mathcal{H}_R(M) \longrightarrow \mathcal{H}_R(G).
\]
Pulling back along $\Theta^P_M$ and extending scalars along $\Xi^P_G$ then defines a functor
\[
(32) \quad \text{Mod-} \mathcal{H}_R(M) \longrightarrow \text{Mod-} \mathcal{H}_R(G), \quad m \mapsto m \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R(G)
\]
from the category of right $\mathcal{H}_R(M)$-modules to the category of right $\mathcal{H}_R(G)$-modules. We then go on to prove that \cite{32} is naturally isomorphic to the parabolic induction functor studied in \cite{OVT18} (4.2) and \cite{Vig15}.

4.2.1. The positive subalgebra. Recall the monoid $M^+$ of $M$-positive elements. The algebra
\[
\mathcal{H}_R(M^+) := H_R(I_{1,M}, M^+)
\]
is called the positive subalgebra of $\mathcal{H}_R(M)$. (The fact that it is indeed a subalgebra of $\mathcal{H}_R(M)$ is clear from the explicit definition of the multiplication.)

We collect two well-known and fundamental properties of $\mathcal{H}_R(M^+)$. 

Proposition 4.1 \cite{Vig98} II.5. Consider the injective $R$-linear map
\[
\xi : \mathcal{H}_R(M) \longrightarrow \mathcal{H}_R(G), \quad T^M_m \mapsto T_m.
\]
The restriction $\xi^+ : \mathcal{H}_R(M^+) \rightarrow \mathcal{H}_R(G)$ respects the product.

Remark. Proposition 4.1 relies solely on the Iwahori decomposition of $I_1$, i.e., we could replace $I_1$ by any compact open subgroup $\Gamma$ satisfying $\Gamma = (\Gamma \cap U_{P,w}) \cdot (\Gamma \cap M) \cdot (\Gamma \cap U_P)$. However, Proposition 4.1 fails for groups like $K$ that do not admit an Iwahori decomposition.

Proposition 4.2 \cite{Vig98} II.6. The following assertions are equivalent:

(a) $\xi^+ : \mathcal{H}_R(M^+) \rightarrow \mathcal{H}_R(G)$ extends to a morphism $\tilde{\xi}^+ : \mathcal{H}_R(M) \rightarrow \mathcal{H}_R(G)$ of $R$-algebras.

(b) There exists a strictly positive element $a \in M$ such that $T_a$ is invertible in $\mathcal{H}_R(G)$.

If one of the assertions holds then $T_a$ is invertible for all strictly positive elements $a \in M$ and $\xi^+$ is unique.
Proof. If \( a \in M \) is strictly positive then \( T_a^M \) is central and invertible in \( \mathcal{H}_R(M) \). Since \( (T_a^M)^n \cdot T_a^M = T_a^m \), for any \( m \in M \) and \( n \gg 0 \), it follows that \( \mathcal{H}_R(M) \) is the localization of \( \mathcal{H}_R(M^+) \) at \( T_a^M \). The proposition is a formal consequence of this fact. \( \square \)

4.2.2. The morphism \( \Theta^P_M \). Let \( \Gamma \subseteq P \) be a compact open subgroup satisfying \( \Gamma = \Gamma_M \cap U_P \), where \( \Gamma_M = \Gamma \cap M \) and \( \Gamma_U = \Gamma \cap U_P \). For example, \( \Gamma \) could be the intersection of \( P \) with one of the groups \( K, I, I_1 \). Recall the notation \( g_M := \text{pr}_M(g) \), where \( \text{pr}_M : P \to M \) is the projection.

**Proposition 4.3.** The \( R \)-linear map

\[
\Theta^P_M := \Theta^P_{M,R} : H_R(\Gamma, P) \rightarrow H_R(\Gamma_M, M),
\]

\[
T_g^P \mapsto \nu_M(g)\mu_{U_P}(g) \cdot T_g^M
\]

is a homomorphism of \( R \)-algebras.

**Proof.** The projection \( \text{pr}_M : P \to M \) induces an \( R \)-linear map \( \vartheta : R[\Gamma \setminus P] \rightarrow R[\Gamma_M \setminus M] \), given explicitly by \( (\Gamma g) \mapsto (\Gamma_M g_M) \). Since \( \text{pr}_M(\Gamma) \subseteq \Gamma_M \) and \( \vartheta \) is right \( P \)-linear if we let \( P \) act by inflation on \( R[\Gamma_M \setminus M] \), it follows that \( \vartheta \) maps \( H_R(\Gamma, P) = R[\Gamma \setminus P]^P \) into \( H_R(\Gamma_M, M) = R[\Gamma_M \setminus M]^{\Gamma_M} \). By restriction we obtain an \( R \)-linear map \( \theta : H_R(\Gamma, P) \rightarrow H_R(\Gamma_M, M) \). It is obvious from the explicit description of multiplication \( 3.1 \) that \( \theta \) respects the product. Therefore, it remains to prove \( \theta = \Theta^P_M \). Let \( g \in P \). By Proposition 3.1 we have

\[
T_g^P = \sum_{i=1}^{\mu_M(g_M)} \sum_{j=1}^{\nu_M(g)} \sum_{s=1}^{\mu_{U_P}(g)} (\Gamma gu_jh_jm_i)
\]

in \( R[\Gamma \setminus P] \), for certain \( u_s \in \Gamma_U \), \( h_j \in (\Gamma_M)_M \), and \( m_i \in \Gamma_M \) with \( \Gamma_M = \bigcup_{i=1}^{\mu_M(g_M)} (\Gamma_M)_M m_i \). Since \( g_M h_j \in \Gamma_M \), by definition of \( h_j \), we obtain

\[
\theta(T_g^P) = \sum_{i,j,s} (\Gamma_M g_M h_j m_i) = \sum_{i,j,s} (\Gamma_M g_M m_i) = \nu_M(g)\mu_{U_P}(g) \cdot T_g^M
\]

where the last equality uses \( \Gamma_M g_M = \bigcup_{i=1}^{\mu_M(g_M)} \Gamma_M g_M m_i \), cf. [AZ95] Chapter 3, Lemma 1.2. Hence, \( \theta \) and \( \Theta^P_M \) coincide. \( \square \)

For the next two consequences we assume \( \Gamma_U = I_{U_P} \).

**Corollary 4.4.** The system \( (\mu_{U_P}(m)T_m^M) \), where \( m \) runs through a system of representatives of \( \Gamma_M \setminus M / \Gamma_M \) with \( \mu_{U_P}(m) \neq 0 \) in \( R \), generates \( \text{Im}(\Theta^P_{M,R}) \) over \( R \). If \( R \) is \( p \)-torsionfree, then it is an \( R \)-basis.

**Proof.** Notice that \( \nu_M(m) = 1 \) whenever \( m \in M \). Moreover, we have \( \mu_{U_P}(g) \geq \mu_{U_P}(g_M) \), for all \( g \in P \), by Proposition 3.1. The assertion now follows from Proposition 4.3 \( \square \)

**Corollary 4.5.** The algebra \( H_R(\Gamma_M, M^+) \) is contained in \( \text{Im}(\Theta^P_M) \) with equality whenever \( qR = 0 \).

**Proof.** Since \( \Gamma_M \) normalizes \( U_P = I_{U_P} \), we have \( \Gamma_M \subseteq M^+ \) and \( H_R(\Gamma_M, M^+) \) is defined. Keeping in mind that \( \mu_{U_P} \) takes values in \( q^{\mathbb{Z}_{>0}} \), the assertion follows from Corollary 4.3 \( \square \)
4.2.3. The morphism $\Xi^P_G$. Lacking a good reference, we formulate Vignéras' "fundamental lemma", a proof of which appears in [Vig05] Lemma 13 (or [Vig06] 1.2) for $F$-split $G$, and which for general $G$ is known to the experts.

**Lemma 4.6 (Fundamental Lemma).** Let $v, w \in W(1)$. Then

$$q_{v, w} \cdot T_v^{-1} T_{vw} - T_w \in \bigoplus_{w' < w} \mathbb{Z} T_{w'} \quad \text{in } \mathcal{H}_\mathbb{Z}(G),$$

where $<$ denotes the Bruhat order in $W(1)$.

**Proof.** Let $o$ be the orientation of $(\mathfrak{s}, \mathfrak{f})$ (cf. [Vig16] 5.2) which places $\mathfrak{c}$ in the negative half-space of each hyperplane $H \in \mathfrak{f}$. Then $E_o(w) = T_w$ in the notation of [Vig16] Definition 5.22. Computing inside $\mathcal{H}_{\mathbb{Z}[P^{-1}]}(G)$, we have

$$q_{v, w} T_v^{-1} T_{vw} = q_{v, w} E_o(v)^{-1} E_o(vw) = E_o(w) \in \mathcal{H}_\mathbb{Z}(G),$$

by [Vig16] Theorem 5.25, and the assertion follows from [Vig16] Corollary 5.26. \(\square\)

**Proposition 4.7.** Assume that $R$ is $p$-torsionfree. Then $\xi^+ : \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$ (see Proposition 4.1) extends uniquely to an injective $R$-algebra morphism

$$\tilde{\xi}^+ : \text{Im}(\Theta^P_M) \to \mathcal{H}_R(G),$$

and $\text{Im}(\Theta^P_M)$ is the maximal subalgebra of $\mathcal{H}_R(M)$ with this property.

**Definition.** For arbitrary $R$ we obtain an $R$-algebra morphism

$$\Xi^P_G := \Xi^P_{G,R} := \text{id}_R \otimes (\tilde{\xi}^+ \circ \Theta^P_{M,\mathbb{Z}}) : \mathcal{H}_R(P) \to \mathcal{H}_R(G).$$

**Proof of Proposition 4.7.** We view $\mathcal{H}_R(M)$ (resp. $\mathcal{H}_R(G)$) as a subalgebra of $\mathcal{H}_{R[\mathfrak{p}^{-1}]}(M)$ (resp. $\mathcal{H}_{R[\mathfrak{p}^{-1}]}(G)$), which is possible by our assumption on $R$. Let $a \in M^+$ be a strictly positive element. Then $T_a$ is invertible in $\mathcal{H}_{R[\mathfrak{p}^{-1}]}(G)$ by [Vig16] Proposition 4.13 1). By Proposition 4.2 the map $\xi^+$ extends uniquely to an $R[\mathfrak{p}^{-1}]$-algebra morphism

$$\tilde{\xi}^+ : \mathcal{H}_{R[\mathfrak{p}^{-1}]}(M) \to \mathcal{H}_{R[\mathfrak{p}^{-1}]}(G).$$

Explicitly, it is given as follows: let $m \in M$, and choose $n \in \mathbb{Z}_{>0}$ such that $a^n m$ is $M$-positive. Then

$$\tilde{\xi}^+ (T_m^M) = T_a^{-n} \cdot \xi^+ (T_a^M) = T_a^{-n} T_a^{n m}.$$ (33)

It suffices to prove the following:

**Claim.** The preimage of $\mathcal{H}_R(G)$ under $\tilde{\xi}^+$ coincides with $\text{Im}(\Theta^P_M)$.

The claim implies the assertion of the proposition: given any extension $\xi' : \mathcal{A} \to \mathcal{H}_R(G)$ of $\xi^+ : \mathcal{H}_R(M^+) \to \mathcal{H}_R(G)$, we have $\xi' = \xi^+|_{\mathcal{A}}$ by the uniqueness of $\xi^+$, and then $\mathcal{A} \subseteq \text{Im}(\Theta^P_M)$ by the claim.

We now prove the claim. By Corollary 3.4 the family $(\mu_U (w) T_M^w)^{w \in W_M(1)}$ is an $R$-basis of $\text{Im}(\Theta^P_M)$. As $a$ is strictly positive, so is the element $\lambda := a \mathfrak{Z}_1 \in \Lambda(1)$.

Given any $w \in W_M(1)$, there exists $n \in \mathbb{Z}_{>0}$ such that $e^n \lambda w \in W_{M^+}(1)$. Now, Corollary 3.15 shows

$$q_{n \lambda, w} = \frac{\mu_{U_p} (n \lambda) \mu_{U_p} (w)}{\mu_{U_p} (e^n \lambda w)} \cdot q_{M, n \lambda, w} = \mu_{U_p} (w).$$
The second equality uses \( q_{M,n} \cdot q_{M,w} = q_{M,e^{n\lambda}w} \), that means, \( q_{M,n\lambda,w} = 1 \), which holds because \( a \) lies in the center of \( M \). By Lemma 4.6 we have

\[
\tilde{\xi}^+(\mu_{U^p}(w)T^M_w) = \mu_{U^p}(w) \cdot T^M_{\lambda^{-n}w} = q_{n\lambda,w} \cdot T^M_{\lambda^{-1}w}
\]

(34)

for certain \( c_{w'} \in \mathbb{Z} \), viewed as elements of \( R \). This shows that \( \tilde{\xi}^+ \) is injective and that \( \text{Im}(\Theta^P_M) \) is contained in \( (\tilde{\xi}^+)^{-1}(\mathcal{H}_R(G)) \).

Conversely, let \( T = \sum_{i=1}^k x_i \cdot T^M_{w_i} \in \mathcal{H}_{R[p^{-1}]}(M) \) with \( x_i \in R[p^{-1}] \setminus \{0\} \) and \( \tilde{\xi}^+(T) \in \mathcal{H}_R(G) \). We prove \( T \in \text{Im}(\Theta^P_M) \) by induction on \( k \). The case \( k = 0 \) is trivial. Assume \( k > 0 \). Rearranging if necessary, we may assume that \( w_k \in \mathcal{W}_M(1) \) is maximal among \( \{w_1, \ldots, w_k\} \) with respect to the Bruhat order in \( W(1) \). Let \( n \in \mathbb{Z}_{>0} \) with \( e^{n\lambda}w_i \in \mathcal{W}_{M+1}(1) \) for all \( i \). Then \( T^M_{n\lambda} \cdot T = \sum_{i=1}^k x_i \cdot T^M_{e^{n\lambda}w_i} \) lies in \( \mathcal{H}_{R[p^{-1}]}(M^+) \), and hence

\[
\tilde{\xi}^+(T) = \sum_{i=1}^k x_i \cdot T^M_{n\lambda} \cdot T^M_{e^{n\lambda}w_i} = \sum_{i=1}^k x_i q_{n\lambda,w_i}^{-1} \left( T_{w_i} + \sum_{w'_i < w_i} c_{w'} T_{w'} \right) \in \mathcal{H}_R(G).
\]

Again, we have \( q_{n\lambda,w_i} = \mu_{U^p}(w_i) \). Hence, maximality of \( w_k \) implies \( x_k R, \mu_{U^p}(w_k) \), whence \( x_k T_{w_k} \in \text{Im}(\Theta^P_M) \). By the induction hypothesis we have \( T - x_k T_{w_k}^M \in \text{Im}(\Theta^P_M) \). We conclude \( T \in \text{Im}(\Theta^P_M) \), finishing the proof.

We note the following useful consequence of the proof of Proposition 4.7.

**Corollary 4.8.** Let \( a \in M \) be strictly positive and \( g \in P \) arbitrary. Then

\[
T^a \cdot \Xi^P_G(T^P_g) = \nu_M(g) \mu_{U^p}(g) \cdot T^{an\lambda g}, \quad \text{in } \mathcal{H}_R(G)
\]

whenever \( n \in \mathbb{Z}_{>0} \) is such that \( a^n g_m \in M^+ \).

**Proof.** The assertion follows by extension of scalars from the case \( R = \mathbb{Z} \). Thus, it suffices to prove

\[
T^a \cdot \tilde{\xi}^+(\Theta^P_{M,Z}(T^P_g)) = \nu_M(g) \mu_{U^p}(g) \cdot T^{an\lambda g}, \quad \text{in } \mathcal{H}_Z(G),
\]

where the computation takes place in \( \mathcal{H}_{[p^{-1}]}(G) \). But this is clear from (33) and the fact that \( \Theta^P_{M,Z}(T^P_g) = \nu_M(g) \mu_{U^p}(g) \cdot T^M_{g_m} \). \( \square \)

4.2.4. **Equivalence of parabolic inductions.** Having constructed two morphisms \( \Theta^P_M : \mathcal{H}_R(P) \to \mathcal{H}_R(M) \) and \( \Xi^P_G : \mathcal{H}_R(P) \to \mathcal{H}_R(G) \), we obtain a functor

(35)

\[
\text{Mod-} \mathcal{H}_R(M) \to \text{Mod-} \mathcal{H}_R(G), \quad m \mapsto m \otimes_{\mathcal{H}_R(P)} \mathcal{H}_R(G)
\]

from the category of right \( \mathcal{H}_R(M) \)-modules to the category of right \( \mathcal{H}_R(G) \)-modules by viewing \( m \) via \( \Theta^P_{Mg} \) as a right \( \mathcal{H}_R(P) \)-module and then extending scalars along \( \Xi^P_G \). There is also the parabolic induction, due to [OV18 (4.2)],

(36)

\[
\text{Mod-} \mathcal{H}_R(M) \to \text{Mod-} \mathcal{H}_R(G), \quad m \mapsto m \otimes_{\mathcal{H}_R(M^+)} \mathcal{H}_R(G),
\]

given by viewing \( m \) as a right \( \mathcal{H}_R(M^+) \)-module and extending scalars along the \( R \)-algebra morphism \( \xi^+ : \mathcal{H}_R(M^+) \to \mathcal{H}_R(G) \) (Proposition 4.1). Theorem 4.9 is an easy consequence of the construction of \( \Xi^P_G \).

**Theorem 4.9.** The functors (35) and (36) are canonically isomorphic.
Proof. Let \( n \) be a right \( \mathcal{H}_R(G) \)-module and let
\[
\rho: m \times \mathcal{H}_R(G) \longrightarrow n
\]
be an \( R \)-bilinear map satisfying \( \rho(m, TT') = \rho(m, T)T' \) for all \( m \in m \) and \( T, T' \in \mathcal{H}_R(G) \). The assertion of the theorem is then tantamount with the equivalence of the following two properties:
\[
\begin{align*}
& (i) \quad \rho(mT^M, T) = \rho(m, \xi^+(T^M)T) \quad \text{for all} \quad m \in m, \quad T^M \in \mathcal{H}_R(M^+), \quad \text{and} \quad T \in \mathcal{H}_R(G). \\
& (ii) \quad \rho(m\Theta_M^P(T^P), T) = \rho(m, \Xi_G^P(T^P)T) \quad \text{for all} \quad m \in m, \quad T^P \in \mathcal{H}_R(P), \quad \text{and} \quad T \in \mathcal{H}_R(G).
\end{align*}
\]
Given \( m \in M^+ \), we have \( \Theta_M^P(T^P_m) = T^M_m \) and \( \Xi_G^P(T^P_m) = \xi^+(T^M_m) \). Thus, (ii) implies (i).

Conversely, assume (i) and fix a strictly positive element \( a \in M \). Let \( g \in P \) and choose \( n \in \mathbb{Z}_{>0} \) such that \( a^ngM \in M^+ \). Then
\[
\begin{align*}
\rho(m \cdot \Theta_M^P(T^P_g), T) &= \rho(m \cdot \nu_M(g)\mu_{U_P}(g) \cdot T^M_{gM}, T) \\
&= \rho(m \cdot (T^M_{a^n})^{-1} \cdot \nu_M(g)\mu_{U_P}(g)T^M_{a^n gM}, T) \\
&= \rho(m \cdot (T^M_{a^n})^{-1}, \nu_M(g)\mu_{U_P}(g)T^M_{a^n gM} \cdot T) \quad \text{(by (i))} \\
&= \rho(m \cdot (T^M_{a^n})^{-1}, T^M_{a^n} \cdot \Xi_G^P(T^P_g) \cdot T) \quad \text{(by Corollary 4.8)} \\
&= \rho(m, \Xi_G^P(T^P_g) \cdot T) \quad \text{(by (i))}
\end{align*}
\]
keeping in mind \( \xi^+(T^M_m) = T_m \), for all \( m \in M^+ \). Hence, (i) implies (ii). \qed

5. Transitivity of parabolic induction

We observe that only a proper quotient of the parabolic pro-\( p \) Iwahori–Hecke algebra \( \mathcal{H}_R(P) \) affects the parabolic induction functor: both morphisms \( \Theta_M^P \) and \( \Xi_G^P \) factor through
\[
R \otimes \text{Im}(\Theta_M^P, \Xi_G^P).
\]
This suggests to study this algebra.

5.1. Definitions and compatibilities.

**Definition.** We put \( \mathcal{H}_R(M, G) := R \otimes \text{Im}(\Theta_M^P, \Xi_G^P) \). Given \( w \in W_M(1) \), we define
\[
\tau^M_w := 1 \otimes \mu_{U_P}(w)T^M_w \in \mathcal{H}_R(M, G).
\]
From Corollary 4.3 it follows that \( (\tau^M_w)_{w \in W_M(1)} \) is an \( R \)-basis of \( \mathcal{H}_R(M, G) \).

Finally, write
\[
\begin{align*}
\Theta_M^M, G: \mathcal{H}_R(M, G) \longrightarrow \mathcal{H}_R(M), \quad \text{and} \\
\Xi_G^G, M: \mathcal{H}_R(M, G) \longrightarrow \mathcal{H}_R(G)
\end{align*}
\]
for the maps induced by \( \Theta_M^P \) and \( \Xi_G^P \), respectively.

**Remark.** (a) Although not explicit in the notation, the algebra \( \mathcal{H}_R(M, G) \) depends on \( P \). However, in our context \( M \) and \( P \) determine each other so that no confusion will arise.

(b) Notice that \( \mathcal{H}_R(G, G) = \mathcal{H}_R(G) \).
(c) The computation \(34\) actually shows
\[
\xi_{G,M}^G (\tau_{w}^M G) = T_{w} + \sum_{w' < w} c_{w'} T_{w'}' \in \mathcal{H}_R(G),
\]
for all \(w \in W_M(1)\). In particular, \(\xi_{G,M}^G\) is injective.

**Lemma 5.1.** Let \(v, w \in W_M(1)\) with \(q_{v,w} = 1\). Then \(\tau_{v}^M G \cdot \tau_{w}^M G = \tau_{vw}^M G\).

**Proof.** We may assume \(R = \mathbb{Z}\). Corollary \(3.15\) shows \(\mu_{U_P} (v) \cdot \mu_{U_P} (w) = \mu_{U_P} (vw)\) and \(q_{M,v,w} = 1\). Hence, \(\tau_{v}^M G \cdot \tau_{w}^M G = \mu_{U_P} (v) \cdot \mu_{U_P} (w) \cdot T_{v}^M T_{w}^M = \mu_{U_P} (vw) \cdot T_{vw}^M = \tau_{vw}^M G\). \(\square\)

5.1.1. The morphisms \(\theta_{M}^{L,G}\).

**Lemma 5.2.** Let \(M \subseteq L\) be Levi subgroups in \(G\). The map
\[
\theta_{M}^{L,G} : \mathcal{H}_R(M,G) \rightarrow \mathcal{H}_R(M,L), \quad \tau_{v}^M G \mapsto \mu_{U_{PL}} (w) \cdot \tau_{w}^M L
\]
is a morphism of \(R\)-algebras. Given another Levi subgroup \(L'\) containing \(L\), the diagram
\[
\mathcal{H}_R(M,G) \xrightarrow{\theta_{M}^{L,G}} \mathcal{H}_R(M,L') \xrightarrow{\theta_{M}^{L',G}} \mathcal{H}_R(M,L)
\]
commutes, i.e., \(\theta_{M}^{L,G} = \theta_{M}^{L,L'} \circ \theta_{M}^{L',G}\).

**Proof.** We may assume \(R = \mathbb{Z}\). By Proposition \(3.12\) we compute
\[
\tau_{w}^M G = \mu_{U_P} (w) T_{w}^M = \mu_{U_{PL}} (w) \cdot \mu_{U_P \cap L} (w) T_{w}^M = \mu_{U_{PL}} (w) \cdot \tau_{w}^M L,
\]
for all \(w \in W_M(1)\). Hence, \(\theta_{M}^{L,G}\) is the inclusion map and, in particular, a morphism of \(Z\)-algebras. If \(L'\) is another Levi subgroup containing \(L\), then
\[
(\theta_{M}^{L,L'} \circ \theta_{M}^{L',G})(\tau_{w}^M G) = \theta_{M}^{L,L'} (\mu_{U_{PL}} (w) \tau_{w}^M G') = \mu_{U_{PL}} (w) \mu_{U_P \cap L'} (w) \cdot \tau_{w}^M L
\]
\[
= \mu_{U_{PL}} (w) \cdot \tau_{w}^M L = \theta_{M}^{L,G} (\tau_{w}^M G),
\]
for all \(w \in W_M(1)\), again by Proposition \(3.12\). \(\square\)

**Proposition 5.3.** Let \(M \subseteq L\) be Levi subgroups in \(G\). Let \(\lambda \in \Lambda(1)\) be a strictly \(L\)-positive element. Then
\[
\mathcal{H}_R(M,L) \cong \mathcal{H}_R(M,G) [\left( \tau_{\lambda}^M G \right)^{-1}]
\]
and \(\theta_{M}^{L,G} : \mathcal{H}_R(M,G) \rightarrow \mathcal{H}_R(M,L)\) is the localization morphism.

**Proof.** Notice that \(\tau_{\lambda}^M G\) lies in the center of \(\mathcal{H}_R(M,G)\), since \(\lambda\) is lifted by a central element in \(L\) (and hence in \(M\)). Also \(\theta_{M}^{L,G} (\tau_{\lambda}^M G) = \tau_{\lambda}^M L\) is central and invertible in \(\mathcal{H}_R(M,L)\) for each \(n \in \mathbb{Z}_{>0}\). Hence, \(\theta_{M}^{L,G}\) induces a well-defined \(R\)-algebra morphism
\[
\tilde{\theta}_{M}^{L,G} : \mathcal{H}_R(M,G) [\left( \tau_{\lambda}^M G \right)^{-1}] \rightarrow \mathcal{H}_R(M,L).
\]
It suffices to construct an \(R\)-linear inverse. Let \(w \in W_M(1)\). Choose \(n \in \mathbb{Z}_{>0}\) such that \(e^{n\lambda} w \in W_{L^\Sigma}(1)\). As \(n\lambda\) is lifted by a central element of \(L\), we have \(q_{L,n\lambda,w} = 1\). Hence, Lemma \(5.7\) shows
\[
\tau_{w}^M L = \tau_{-n\lambda}^M L \cdot \tau_{n\lambda}^M L \cdot \tau_{w}^M L = \tau_{-n\lambda}^M L \cdot \tau_{e^{n\lambda} w}^M L = \tau_{-n\lambda}^M L \cdot \theta_{M}^{L,G} (\tau_{e^{n\lambda} w}^M G).
\]
Hence, we obtain an $R$-linear map
\[ \gamma: \mathcal{H}_R(M, L) \longrightarrow \mathcal{H}_R(M, G) \]
\[ \left( (\tau^M_{\lambda, G})^{-1} \right), \quad \frac{\tau_M L}{\tau_M G}, \]

which does not depend on the choice of $n$. By (38) we have $\tilde{\theta}^{L,G}_M \circ \gamma = \text{id}_{\mathcal{H}_R(M, L)}$. Conversely, let $w \in W_M(1)$ and $n \in \mathbb{Z}_{>0}$. Take $m \in \mathbb{Z}_{>0}$ with $e^{m \lambda} w \in W_L^+(1)$. As $m \lambda$ is lifted by a central element in $L$, we have $\mu_{U \cap L}(e^{m \lambda} w) = \mu_{U \cap L}(w)$. Applying Proposition 3.12 twice, we compute
\[ \mu_{U \cap L}(w) \cdot \mu_{U \cap L}(e^{-n \lambda} w) = \mu_{U \cap L}(w) \cdot \mu_{U \cap L}(e^{-n \lambda} w) \]
\[ = \mu_{U \cap L}(w) \cdot \mu_{U \cap L}(w) = \mu_{U \cap L}(w). \]

This shows $\tau_{M,G}^{M,G} \cdot \tau_{M,G}^{M,G} = \mu_{U \cap L}(w) \cdot \tau_{M,G}^{M,G}$. Now,
\[ \left( \gamma \circ \tilde{\theta}^{L,G}_M \right) \left( \frac{\tau^M_{\lambda, G}}{\tau^L_{\lambda, G}} \right) = \gamma \left( \tau^M_{-n \lambda} \cdot \tilde{\theta}^{L,G}_M(\tau^M_{\lambda, G}) \right) = \frac{\mu_{U \cap L}(w) \cdot \gamma(\tau^M_{-n \lambda} \cdot \tau^M_{\lambda, G})}{\tau^M_{\lambda, G} \cdot \tau^M_{\lambda, G}} = \frac{\tau^M_{\lambda, G} \cdot \tau^M_{\lambda, G}}{\tau^M_{\lambda, G} \cdot \tau^M_{\lambda, G}}. \]

Hence, $\gamma \circ \tilde{\theta}^{L,G}_M = \text{id}_{\mathcal{H}_R(M, G) \circ (\tau^M_{\lambda, G})^{-1}}$, finishing the proof. \qed

5.1.2. The morphisms $\xi^G_{L,M}$.

**Lemma 5.4.** Let $M \subseteq L$ be Levi subgroups in $G$. There exists a unique $R$-algebra morphism
\[ \xi^G_{L,M}: \mathcal{H}_R(M, G) \longrightarrow \mathcal{H}_R(L, G) \]

which is natural in $R$ and satisfies the following property: for all Levi subgroups $M \subseteq L \subseteq L'$ in $G$, the diagram
\[ \begin{array}{ccc}
\mathcal{H}_R(M, G) & \xrightarrow{\xi^G_{L,M}} & \mathcal{H}_R(L, G) \\
\theta^{L',G}_M & \downarrow & \theta^{L',G}_L \\
\mathcal{H}_R(M, L') & \xrightarrow{\xi^G_{L,M}} & \mathcal{H}_R(L, L')
\end{array} \]

commutes, i.e., $\theta^{L',G}_L \circ \xi^G_{L,M} = \xi^G_{L,M} \circ \theta^{L',G}_M$. Moreover, $\xi^G_{L,M}$ is injective.

**Proof.** We first construct a unique morphism $\xi^G_{L,M}$, natural in $R$, making the diagram
\[ \begin{array}{ccc}
\mathcal{H}_R(M, G) & \xrightarrow{\xi^G_{L,M}} & \mathcal{H}_R(L, G) \\
\theta^{G}_M & \downarrow & \theta^{G}_L \\
\mathcal{H}_R(M, L) & \xrightarrow{\xi^G_{L,M}} & \mathcal{H}_R(L)
\end{array} \]

commutative. Afterwards, we check injectivity and that $\mathcal{B}$ commutes.
Step 1. We prove unique existence provided $R$ is $p$-torsionfree. In this case, $\theta_{M,G}^L$ and $\theta_{M,G}^{L,L'}$ are the canonical inclusions, hence uniqueness is clear. We have to show that $\xi_{L,M}^L$ maps $\mathcal{H}_R(M,G)$ into $\mathcal{H}_R(L,G)$. Let $w \in W_M(1)$. Recall that by (37) we have

$$\xi_{L,M}^L(\tau_w^{M,L}) = T_w^L + \sum_{w' <_L w} c_{w'} T_{w'}^L$$

in $\mathcal{H}_R(L)$, where $<_L$ denotes the Bruhat order in $W_M(1)$. By Proposition 3.14(b) we have $\mu_{U,F_L}(w') \leq \mu_{U,F_L}(w)$ for all $w' \in W_L(1)$ with $w' <_L w$. We deduce that

$$\xi_{L,M}^L(\tau_w^{M,G}) = \mu_{U,F_L}(w) \cdot \xi_{L,M}^L(\tau_w^{M,L}) = \mu_{U,F_L}(w) T_w^L + \sum_{w' <_L w} c_{w'} \mu_{U,F_L}(w) T_{w'}^L$$

lies in $\mathcal{H}_R(L,G)$. This proves existence of an embedding $\xi_{L,M}^G : \mathcal{H}_R(M,G) \to \mathcal{H}_R(L,G)$ making (40) commutative.

Step 2. We prove unique existence for general $R$. Existence follows from Step 1 by extension of scalars from $\mathbb{Z}$ to $R$. (If $R$ is $p$-torsionfree this construction coincides with the one in Step 1 by the uniqueness assertion.) We have to prove uniqueness for general $R$. Take a surjection $f : R' \to R$ for some $p$-torsionfree ring $R'$ (e.g., the large polynomial ring $\mathbb{Z}[X_r \mid r \in R]$). By naturality of the diagram

$$\begin{array}{ccc}
\mathcal{H}_{R'}(M,G) & \xrightarrow{\xi_{E,M}^G} & \mathcal{H}_{R'}(L,G) \\
f \otimes \text{id} & \downarrow & f \otimes \text{id} \\
\mathcal{H}_R(M,G) & \xrightarrow{\xi_{E,M}^G} & \mathcal{H}_R(L,G)
\end{array}$$

it follows that $\xi_{E,M}^G$ is uniquely determined by the naturality requirement.

Step 3. Injectivity of $\xi_{E,M}^G$. By construction we have

$$\xi_{E,M}^G(\tau_w^{M,G}) = \tau_w^{L,G} + \sum_{w' <_L w} c_{w'} \tau_{w'}^{L,G},$$

for all $w \in W_M(1)$, for certain $c_{w'} \in R$. In particular, $\xi_{E,M}^G$ is injective.

Step 4. Commutativity of (39). By naturality we may assume $R = \mathbb{Z}$. The outer and lower square in

$$\begin{array}{ccc}
\mathcal{H}_R(M,G) & \xrightarrow{\xi_{E,M}^G} & \mathcal{H}_R(L,G) \\
\mathcal{H}_R(M,L') & \xrightarrow{\theta_{E,M}^{L,L'}} & \mathcal{H}_R(L,L') \\
\mathcal{H}_R(M,L) & \xrightarrow{\xi_{E,M}^L} & \mathcal{H}_R(L)
\end{array}$$

commute by construction. By Lemma 5.2 and since $\theta_{E,M}^{L,L'}$ is injective, the upper square commutes.
Lemma 5.5. Let \( M \subseteq L \subseteq L' \) be Levi subgroups in \( G \). Then the diagram
\[
\begin{array}{c}
\mathcal{H}_R(M, G) \xrightarrow{\xi_{L,M}^G} \mathcal{H}_R(L, G) \\
\downarrow \quad \downarrow \\
\mathcal{H}_R(L', G) \xrightarrow{\xi_{L',M}^G} \mathcal{H}_R(L', L')
\end{array}
\]
commutes, i.e., \( \xi_{L',M}^G = \xi_{L',L}^G \circ \xi_{L,M}^G \).

Proof. By naturality it suffices to prove the assertion for \( R = \mathbb{Z} \). By naturality, and since \( \mathcal{H}_{\mathbb{Z}}(M', G) \subseteq \mathcal{H}_{\mathbb{Z}^{[p-1]}}(M', G) = \mathcal{H}_{\mathbb{Z}^{[p-1]}}(M') \) for all Levi subgroups \( M' \) in \( G \), we may even assume \( R = \mathbb{Z}^{[p-1]} \). Hence, we need to prove commutativity of
\[
\begin{array}{c}
\mathcal{H}_{\mathbb{Z}^{[p-1]}}(M) \xrightarrow{\xi_{L,m}^G} \mathcal{H}_{\mathbb{Z}^{[p-1]}}(L) \\
\downarrow \quad \downarrow \\
\mathcal{H}_{\mathbb{Z}^{[p-1]}}(L') \xrightarrow{\xi_{L',M}^G} \mathcal{H}_{\mathbb{Z}^{[p-1]}}(L').
\end{array}
\]
(Notice that \( \xi_{L,M}^G = \xi_{L,M}^L \) for all Levi subgroups \( M \subseteq L \) in \( G \) whenever \( p \) is invertible, because then \( \theta_M^{L,G} \) and \( \theta_L^{L,G} \) are the identity morphisms in \([40]\).)

Let \( w \in W_M(1) \) and take a strictly \( M \)-positive element \( \lambda \in \Lambda(1) \). Let \( n \in \mathbb{Z}_{>0} \) with \( e^{n\lambda}w \in W_{M'}(1) \). Then both \( n\lambda \) and \( e^{n\lambda}w \) are \( M^+, L^-, M^+L'^-, \) and \( L^+, L'^- \) positive. Hence,
\[
\begin{align*}
\xi_{L,M}^L(T_w^M) &= (T_{n\lambda}^L)^{-1} \cdot T_{e^{n\lambda}w}^L, \\
\xi_{L',M}^L(T_{n\lambda}^L) &= T_{n\lambda}^L,
\end{align*}
\]
Therefore, we compute
\[
(\xi_{L',L}^L \circ \xi_{L,M}^L)(T_w^M) = \xi_{L',L}^L((T_{n\lambda}^L)^{-1} \cdot T_{e^{n\lambda}w}^L) = (T_{n\lambda}^L)^{-1} \cdot T_{e^{n\lambda}w}^L = \xi_{L',M}^L(T_w^M). \tag{41}
\]

5.2. Transitivity of parabolic induction.

Proposition 5.6. Let \( M \subseteq L \subseteq L' \) be Levi subgroups in \( G \). The canonical map
\[
\begin{align*}
\mathcal{H}_R(M, L') \otimes_{\mathcal{H}_R(M, G)} \mathcal{H}_R(L, G) &\longrightarrow \mathcal{H}_R(L, L'), \\
x \otimes y &\longmapsto \xi_{L,M}^L(x) \cdot \theta_L^{L',G}(y)
\end{align*}
\]
is an isomorphism of \( \mathcal{H}_R(M, L') \cdot \mathcal{H}_R(L, G) \)-bimodules.\(^1\)

Proof. The map is well-defined, since \([39]\) commutes, and preserves the bimodule structure by definition. Let \( \lambda \in \Lambda(1) \) be a strictly \( L' \)-positive element. By Proposition 5.3 the map \((41)\) identifies with the canonical map
\[
\mathcal{H}_R(M, G)\left[\left(\tau_{\lambda}^{M,G}\right)^{-1}\right] \otimes_{\mathcal{H}_R(M, G)} \mathcal{H}_R(L, G) \longrightarrow \mathcal{H}_R(L, G)\left[\left(\tau_{\lambda}^{L,G}\right)^{-1}\right],
\]
which is clearly an \( R \)-linear isomorphism. \( \square \)

\(^1\)The maps \( \mathcal{H}_R(M, G) \to \mathcal{H}_R(M, L') \) and \( \mathcal{H}_R(M, G) \to \mathcal{H}_R(L, G) \) are the obvious ones, namely \( \theta_M^{L',G} \) and \( \xi_{L,M}^G \), respectively. Likewise, the bimodule structure is the obvious one.
Let \( \mathcal{L} \subseteq \mathcal{L}' \) and \( \mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}'' \) be Levi subgroups in \( \mathbf{G} \) with \( \mathcal{M} \subseteq \mathcal{L} \) and \( \mathcal{M}' \subseteq \mathcal{L}' \). Then the canonical map
\[
\mathcal{H}_R(M, L) \otimes \mathcal{H}_R(M'', G) \to \mathcal{H}_R(M, L) \otimes \mathcal{H}_R(M', L') \otimes \mathcal{H}_R(M'', G),
\]
\( x \otimes y \mapsto x \otimes 1 \otimes y \)
is an isomorphism of \( \mathcal{H}_R(M, L)\)-\( \mathcal{H}_R(M'', G) \)-bimodules.

**Proof.** There are natural isomorphisms
\[
\mathcal{H}_R(M, L) \otimes \mathcal{H}_R(M'', G)
\]
\( \cong \mathcal{H}_R(M, L) \otimes \mathcal{H}_R(M', L') \otimes \mathcal{H}_R(M'', G) \)
\( \cong \mathcal{H}_R(M, L) \otimes \mathcal{H}_R(M', L') \otimes \mathcal{H}_R(M'', G) \)
the second isomorphism being given by Proposition 5.6. The composite sends \( x \otimes y \mapsto x \otimes 1 \otimes y \).

As an application we give another proof of the transitivity of parabolic induction, which is originally due to Vignéras [Vig15, Proposition 4.3].

**Corollary 5.8.** Let \( \mathcal{M} \subseteq \mathcal{L} \) be Levi subgroups in \( \mathbf{G} \). Let \( \mathfrak{m} \) be a right \( \mathcal{H}_R(M) \)-module. Then there is a natural isomorphism of right \( \mathcal{H}_R(G) \)-modules
\[
\mathfrak{m} \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(L) \otimes \mathcal{H}_R(L) \mathcal{H}_R(G) \cong \mathfrak{m} \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(G).
\]

**Proof.** Since there is a natural right \( \mathcal{H}_R(G) \)-linear isomorphism
\[
\mathfrak{m} \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(G) \cong \mathfrak{m} \otimes \mathcal{H}_R(M) \mathcal{H}_R(M) \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(G)
\]
(and similarly with \( (M \cdot L) \) replaced by \( (M \cdot L) \)), we are reduced to proving the assertion for \( \mathfrak{m} = \mathcal{H}_R(M) \). Now, by Theorems 4.9 and 5.7 there are \( \mathcal{H}_R(M) \)-\( \mathcal{H}_R(G) \)-bimodule isomorphisms
\[
\mathcal{H}_R(M) \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(L) \otimes \mathcal{H}_R(L) \mathcal{H}_R(G) \cong \mathcal{H}_R(M) \otimes \mathcal{H}_R(M, G) \mathcal{H}_R(G)
\]
\( \cong \mathcal{H}_R(M) \otimes \mathcal{H}_R(M, L) \mathcal{H}_R(L) \otimes \mathcal{H}_R(L, G) \mathcal{H}_R(G)
\]
\( \cong \mathcal{H}_R(M) \otimes \mathcal{H}_R(M \cdot L) \mathcal{H}_R(L) \otimes \mathcal{H}_R(L) \mathcal{H}_R(G) \). \( \square \)

### 5.3. Alcove walk bases and a filtration

We finish by describing a natural \( \mathbb{Z}_{\geq 0} \)-filtration on the \( R \)-algebra \( \mathcal{H}_R(M, G) \) coming from \( \mu_{U_P} : W_M(1) \to \mathbb{Z}_{\geq 0} \). To do this we need to describe alcove walk bases for \( \mathcal{H}_R(M, G) \).

**Definition.** Let \( o \) be an orientation of \( \{ \mathcal{S}_M, \mathcal{F}_M \} \) [Vig16, 5.2]. Let \( (E_o(w))_{w \in W_M(1)} \) be the associated alcove walk basis in \( \mathcal{H}_R(M) \) [Vig16, Definition 5.22]. We define
\[
E_o^{M, G}(w) := 1 \otimes \mu_{U_P}(w) \cdot E_o(w) \in \mathcal{H}_R(M, G), \quad \text{for all } w \in W_M(1).
\]

**Remark.** The element \( E_o^{M, G}(w) \) is indeed well-defined: since \( E_o(w) = T_w^M + \sum_{w' \leq w} c_{w'} T_w^M \), for certain \( c_{w'} \in \mathbb{Z} \), [Vig16, Corollary 5.26] and by Proposition 3.14(b) we even have
\[
E_o^{M, G}(w) = T_w^M + \sum_{w' \leq w} c_{w'}' \cdot T_w^M \in \mathcal{H}_R(M, G),
\]
where \( c_{w'}' \) is the image of \( \frac{\mu_{U_P}(w)}{\mu_{U_P}(w')} \cdot c_{w'} \) in \( R \). Hence, \( (E_o^{M, G}(w))_{w \in W_M(1)} \) is an \( R \)-basis of \( \mathcal{H}_R(M, G) \).
Lemma 5.9. Let $o$ be an orientation of $(\mathcal{A}_M, \mathcal{B}_M)$. Let $v, w \in W_M(1)$. Then

$$E_o^{M,G}(v) \cdot E_{o\bullet w}^{M,G}(w) = q_{v,w} \cdot E_o^{M,G}(vw).$$

Proof. We may assume $R = \mathbb{Z}$. Then $E_o(v) \cdot E_{o\bullet w}(w) = q_{M,v,w} \cdot E_o(vw)$ by Theorem 5.25. Corollary 3.15 shows

$$E_o^{M,G}(v) \cdot E_{o\bullet w}^{M,G}(w) = \frac{\mu_{i,p}(v)\mu_{j,p}(w)}{\mu_{i,p}(vw)} \cdot q_{M,v,w}E_o^{M,G}(vw) = q_{v,w} \cdot E_o^{M,G}(vw). \quad \square$$

Definition. A $\mathbb{Z}_{\geq 0}$-filtration of an $R$-algebra $A$ is a family $(\mathcal{F}_i A)_{i \in \mathbb{Z}_{\geq 0}}$ of $R$-submodules satisfying

- $\mathcal{F}_i A \subseteq \mathcal{F}_{i+1} A$ for all $i \geq 0$;
- $\mathcal{F}_i A \cdot \mathcal{F}_j A \subseteq \mathcal{F}_{i+j} A$ for all $i, j \geq 0$;
- $1 \in \mathcal{F}_0 A$;
- $A = \bigcup_{i \geq 0} \mathcal{F}_i A$.

Proposition 5.10. The free $R$-submodules $\mathcal{F}_n^{M,G}$ of $\mathcal{H}_R(M,G)$ generated by \(\{o_{w}\}_{w \in W_M(1)}\), define a $\mathbb{Z}_{\geq 0}$-filtration on $\mathcal{H}_R(M,G)$.

Moreover, $\mathcal{F}_0^{M,G} \cong \mathcal{H}_R(M^+) \text{ via } \sigma_M^{M,G}$.

Proof. The only thing that is not immediately clear is $\mathcal{F}_i^{M,G} \cdot \mathcal{F}_j^{M,G} \subseteq \mathcal{F}_{i+j}^{M,G}$ for $i, j \geq 0$. Given any orientation $o$ of $(\mathcal{A}_M, \mathcal{B}_M)$, the set $\{E_o^{M,G}(w)\}_{w \in W_M(1)}$ is an $R$-basis of $\mathcal{F}_n^{M,G}$ by [12] and Proposition 3.14(b). Hence, the claim follows from Lemma 5.9 and Corollary 3.15. \quad \square

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References

[Abe16] Noriyuki Abe, Modulo $p$ parabolic induction of pro-$p$-Iwahori Hecke algebra, J. Reine Angew. Math. 749 (2019), 1–64, DOI 10.1515/crelle-2016-0043. MR3935898

[Abe19] Noriyuki Abe, Parabolic inductions for pro-$p$-Iwahori Hecke algebras, Adv. Math. 355 (2019), 106776, 63, DOI 10.1016/j.aim.2019.106776. MR3996728

[And77] A. N. Andrianov, The expansion of Hecke polynomials for the symplectic group of genus $n$ (Russian), Mat. Sb. (N.S.) 104(146) (1977), no. 3, 390–427, 511. MR0562510

[AZ95] A. N. Andrianov and V. G. Zhuravlev, Modular forms and Hecke operators, Translations of Mathematical Monographs, vol. 145, American Mathematical Society, Providence, RI, 1995. Translated from the 1990 Russian original by Neal Koblitz, DOI 10.1090/mmono/145. MR1349824

[BB06] Anders Björner and Francesco Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2238669

[BK98] Colin J. Bushnell and Philip C. Kutzko, Smooth representations of reductive $p$-adic groups: structure theory via types, Proc. London Math. Soc. (3) 77 (1998), no. 3, 582–634, DOI 10.1112/S0024611598000574. MR1643417

[BT72] F. Bruhat and J. Tits, Groupes réductifs sur un corps local (French), Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5–251. MR0327923
[BT84] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée* (French), Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376. MR756316

[Gri88] V. A. Gritsenko, *Parabolic extensions of the Hecke ring of the general linear group* (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 154 (1986), no. Anal. Teor. Chisel i Teor. Funktsiĭ. 7, 36–45, 176, DOI 10.1007/BF01374983; English transl., J. Soviet Math. 43 (1988), no. 4, 2533–2540. MR859521

[Hey19] Claudius Heyer. *Applications of parabolic Hecke algebras: parabolic induction and Hecke polynomials*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2019. doi:http://dx.doi.org/10.18452/20137

[HR08] Thomas Haines and Michael Rapoport. On parahoric subgroups. *Advances in Mathematics*, 219(1):188–198, 2008. doi:10.1016/j.aim.2008.04.020

[HR09] Thomas J. Haines and Sean Rostami, *The Satake isomorphism for special maximal parahoric Hecke algebras*, Represent. Theory 14 (2010), 264–284, DOI 10.1090/S1088-4165-10-00370-5. MR2602034

[Kot97] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. 109 (1997), no. 3, 255–339, DOI 10.1023/A:1000102604688. MR1485921

[Oll10] Rachel Ollivier, *Parabolic induction and Hecke modules in characteristic $p$ for $p$-adic $GL_n$*. Algebra Number Theory 4 (2010), no. 6, 701–742, DOI 10.2140/ant.2010.4.701. MR2728487

[OV18] Rachel Ollivier and Marie-France Vigneras, *Parabolic induction in characteristic $p$*, Selecta Math. (N.S.) 24 (2018), no. 5, 3973–4039, DOI 10.1007/s00029-018-0440-0. MR3874689

[Ren10] David Renard, *Représentations des groupes réductifs $p$-adiques* (French), Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR2567785

[Vig96] Marie-France Vignéras, *Représentations $l$-modulaires d’un groupe réductif $p$-adique avec $l \neq p$* (French, with English summary), Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1395151

[Vig98] Marie-France Vignéras, *Induced $R$-representations of $p$-adic reductive groups*, Selecta Math. (N.S.) 4 (1998), no. 4, 549–623, DOI 10.1007/s000290050040. MR1668044

[Vig05] Marie-France Vignéras, *Pro-$p$-Iwahori Hecke ring and supersingular $\mathbf{F}_p$-representations*, Math. Ann. 331 (2005), no. 3, 523–556, DOI 10.1007/s00208-004-0592-4. MR2122539

[Vig06] Marie-France Vignéras, *Algèbres de Hecke affines génériques* (French, with French summary), Represent. Theory 10 (2006), 1–20, DOI 10.1090/S1088-4165-06-00185-3. MR2192484

[Vig15] Marie-France Vignéras, *The pro-$p$ Iwahori Hecke algebra of a reductive $p$-adic group, V (parabolic induction)*, Pacific J. Math. 279 (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Vig16] Marie-France Vigneras, *The pro-$p$-Iwahori Hecke algebra of a reductive $p$-adic group I*, Compos. Math. 152 (2016), no. 4, 693–753, DOI 10.1112/S0010437X15007666. MR3484112

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