Quota-share and stop-loss reinsurance combination based on Value-at-Risk (VaR) optimization

A D Putri, S Nurrohmah and I Fithriani
Department of Mathematics, Faculty of Mathematics and Natural Science (FMIPA), Universitas Indonesia, Depok 16424, Indonesia
Corresponding author’s email: snurrohmah@sci.ui.ac.id

Abstract. Every insurance companies have a capacity limit related to the maximum claim that can be borne. Therefore, insurance companies need to reinsure risks to reinsurance companies. Besides of quota-share, types of reinsurance contracts that commonly used is stop-loss. The quota-share reinsurance premium is proportional based on the amount claim that is covered, but not safe against a large claim. While for stop-loss, the reinsurance premium is relatively large but safe for a large claim. So, this paper combines both types of reinsurance to cover the shortcomings with their respective strengths. After being combined, it is necessary to determine the optimal quota-share proportion and stop-loss retention. One criterion of determines optimal proportion and retention is based on Value-at-Risk (VaR) optimization. With the reinsurance premium as a constraint, this optimization problem is solved for each type of reinsurance combination, be it quota-share before stop-loss or stop-loss before quota-share. From each of these combinations, the result is optimal quota-share proportion and stop-loss retention, so as produce a minimum VaR value from the borne risk by insurance companies. by comparing the results of VaR optimization of these combinations, stop-loss before quota-share is obtained resulting in a more minimum VaR value.

Keywords: Insurer, proportion, reinsurance premium, reinsurer, retention

1. Introduction
Insurance companies provide risk management services to policyholders. When a policyholder suffered a loss, a claim will be submitted to the insurance company. In some cases, not all claims can be borne fully by insurance companies, especially for large claims that exceed the retention limit. Retention is a large risk that can be borne by the insurance company itself. Therefore, in covering the risks of the policyholders, the insurance company protects itself by using reinsurance services. Stop-loss scheme is one of the commonly used type of insurance, after the quota-share scheme.

Both for quota-share and stop-loss have pros and cons from the perspective of the insurance company. Gavranovic et al. [1] modeled the quota-share reinsurance where the reinsurance premiums paid by insurance companies is proportional to the claims they bear. While for stop-loss reinsurance as discussed by Cai et al. [2], reinsurance premiums are relatively large or expensive, depending on the retention set by the insurance company itself. In addition, quota share reinsurance is not safe against large total claims, because it can produce a loss ratio that is greater than what the insurance company can bear. However, in stop-loss reinsurance, the insurer determines the limit of claim coverage, so it is safe against large total claims. By combining the two types of reinsurance, it is expected that the
shortcomings of each type of reinsurance can be covered by the advantages of the other types of reinsurance.

After the insurance company shares the risks it has borne with the reinsurance company, as collateral for the transferred risk, the insurance company must pay the reinsurance premium. Reinsurance premiums are calculated using certain principles, among them is the expectation value principle. Because risk limitation means a reduction in profits, reinsurance premiums must be within reasonable limits. That is, the smaller the retention determined by the insurance company, the lower the insurance company’s ability to bear the risk and reinsurance premiums to be paid will be even greater, and vice versa. Therefore, it is important for insurance companies to determine the optimal retention value. One way is to measure losses that can be borne by insurance companies. Losses incurred by insurance companies are measured using certain methods, one of them is Value-at-Risk (VaR) method [3]. The aim of this study is to analyze the possible order of combination between quota-share and stop-loss that minimize VaR of the insurer.

2. Materials and method

2.1. Quota-share reinsurance policy under VaR
Quota-share reinsurance is a reinsurance agreement where the insurance company surrenders a part of the insurance value to the reinsurance company in a fixed percentage. Suppose that \( X \) is a total claim submitted by policyholders. The insurance company bears the claim in the amount of proportion \( p \) and the reinsurance company bears proportion \( 1-p \), represented by the formulation as the following:

\[
X_I = pX
\]

\[
X_R = (1-p)X
\]

where \( X_I \) is a total claim that the insurer borne and \( X_R \) denotes a total claim that the reinsurer borne.

The VaR of total cost model at a confidence level \( \alpha \), \( 0 < \alpha < 1 \), is defined as

\[
VaR_\alpha(X_T; p) = pVaR_\alpha(X) + \delta((1-p)X)
\]

where \( \delta \) denotes the reinsurance premium and \( X_T \) means total cost model i.e. total claims to be borne and reinsurance premium to be paid.

From the overall possible value of \( p \in [0,1] \), an optimal value of \( p \), i.e. \( p^* \in [0,1] \), will be sought in order that the VaR value of total cost is minimum., that is

\[
VaR_\alpha(X_T; p^*) = \min_{p\in[0,1]} VaR_\alpha(X_T; p)
\]

Assume that the reinsurance premium \( \delta(\cdot) \) satisfies \( \delta(0) = 0 \) and positive homogeneity \( \delta(pX) = p\delta(X) \) for a constant \( p > 0 \). Then the optimal quota-share coefficient, \( p^* \), depends on \( \delta(X) \) and \( VaR_\alpha(X) \) as in the following [4],

\[
p^* = \begin{cases} 
0, & \delta(X) < VaR_\alpha(X) \\
\text{any number in (0,1)}, & \delta(X) = VaR_\alpha(X) \\
1, & \delta(X) > VaR_\alpha(X) 
\end{cases}
\]

This formulation states that if the reinsurance premium is smaller than \( VaR_\alpha(X) \), to minimize \( VaR_\alpha(X_T; p^*) \), all claims should be reinsured to the reinsurer. If the reinsurance premium is equal to \( VaR_\alpha(X) \), for any value of \( p^* \in (0,1) \), \( VaR_\alpha(X_T; p^*) \) will be minimum. If the reinsurance premium is greater than \( VaR_\alpha(X) \), to minimize \( VaR_\alpha(X_T; p^*) \), the insurer must borne all claims.
2.2. Stop-Loss reinsurance policy under VaR

Stop-loss reinsurance is a reinsurance contract where the insurance company will get a compensation from the reinsurance company if the amount of the claim submitted by the policyholder exceeds the retention of a certain nominal that can be borne by the insurance company. The formulation of this policy can be written as:

\[ X_t = X \land q = \min\{X, q\} = \begin{cases} X, & X \leq q \\ q, & X \geq q \end{cases} \quad (6) \]

\[ X_R = (X - q)_+ = \max\{0, X - q\} = \begin{cases} 0, & X \leq q \\ X - q, & X > q \end{cases} \quad (7) \]

where \( q \) represents retention of the insurer and \( X \) is a total claim submitted by policyholders.

The VaR of total cost model at a confidence level \( 1 - \alpha, 0 < \alpha < 1 \), is defined as

\[ \text{VaR}_\alpha(X_T; q) = \begin{cases} \text{VaR}_\alpha(X), & \text{VaR}_\alpha(X) \leq q \\ q + \delta(X - q), & \text{VaR}_\alpha(X) > q \end{cases} \quad (8) \]

where \( X_T \) means total cost model i.e. total claims to be borne and reinsurance premium to be paid. Among possible values of \( q \in [0, \infty) \), an optimal value of \( q^* \in [0, \infty) \) will be sought in order to minimize the VaR of total cost, that is,

\[ \text{VaR}_\alpha(X_T; q^*) = \min_{q \in [0, \infty)} \text{VaR}_\alpha(X_T; q) \quad (9) \]

The optimal retention \( q^* \) in (9) exists if and only if

\[ \alpha < \frac{1}{1 + \rho} < 1 \quad (10) \]

and

\[ \text{VaR}_\alpha\left(\frac{1}{1 + \rho}\right) + \delta\left(\text{VaR}_\alpha\left(\frac{1}{1 + \rho}\right)\right) \leq \text{VaR}_\alpha(X) \quad (11) \]

hold [4].

The optimal retention \( q^* \) is given by

\[ q^* = \text{VaR}_\alpha\left(\frac{1}{1 + \rho}\right) \quad (12) \]

and the minimum VaR of \( X_T \) is

\[ \text{VaR}_\alpha(X_T; q^*) = q^* + \delta(q^*) \quad (13) \]

Hence, if equation 10 and equation 11 are satisfied, the optimal retention is \( q^* = \text{VaR}_\alpha\left(\frac{1}{1 + \rho}\right) \) and VaR of total cost model reach a minimum.

3. Results and discussion

3.1. Mean-VaR optimization for quota-share before stop-loss

This combination of quota-share and stop-loss discusses about the optimization of VaR for the borne risk of insurer when applying quota-share at the first claim and then stop-loss to optimize the VaR,
involving reinsurance premium as a constraint. The borne risk of insurer is $X_1 = pX \wedge q$, where an optimal $p$ is $p^* \in [0,1]$ and an optimal $q$ is $q^* \in [0, \infty)$ such that

$$\text{VaR}_a(p^*X \wedge q^*) = \min_{p \in [0,1], q \in [0, \infty)} \text{VaR}_a(pX \wedge q)$$

subject to

$$E(X - pX \wedge q) \leq \frac{p}{1 + \rho} \equiv P_0 < \mu$$

where $P_0$ is the reinsurance premium, $\rho > 0$ is safety loading factor depends on the company’s operating costs, $\mu$ is expectation of $X$, and $\bar{q}$ is a value of $q$ when $p = 1$ which satisfies

$$E((X - \bar{q})_+) = P_0$$

(14)

To identify optimum $p$ and $q$, a constraint graph will be made by considering $p$ as the $x$ –axis and the point where the value of $E(pX \wedge q)$ changes i.e. $\bar{q}$ as the $y$ –axis, where $p \in [0,1]$ and $q \in [0, \infty)$. For a certain value of $q$, the graphic between $p$ and $\frac{\bar{q}}{p}$ is illustrated in figure 1:

The next step to solving the optimization problem is searching for points that produce a minimum VaR. This is given by Lemma 1.

**Lemma 1**

The minimal value $\text{VaR}_a(p^*X \wedge q^*)$ is attained on the curve of

$$E(pX \wedge q) = \mu - P_0 [5].$$

Proof:

The VaR of $X_1$ is given by

$$\text{VaR}_a(X_1) = \text{VaR}_a(pX \wedge q) = \begin{cases} p\text{VaR}_a(X), & \text{VaR}_a(X) \leq \frac{q}{p} \\ q, & \text{VaR}_a(X) \geq \frac{q}{p} \end{cases}$$

(17)

There are two cases for $\text{VaR}_a(X)$ to consider:

**Case 1:** $\text{VaR}_a(X) \leq \frac{q}{p}$

For any fixed $\frac{2}{p}$, $\text{VaR}_a(pX \wedge q) = p \text{ VaR}_a(X)$.

$$\text{VaR}_a(pX \wedge q) = p \text{ VaR}_a(X)$$

$$= p \inf \{ x \in \mathbb{R}^+ | Pr(X \geq x) \leq \alpha \}$$

So, $\text{VaR}_a(pX \wedge q)$ will be minimum when $p$ is minimum.
Figure 1. Graph of $E(pX \land q) \geq \mu - P_0$

Case 2: $VaR_\alpha(X) \geq \frac{q}{p}$

Since for this case $VaR_\alpha(qX \land \alpha) = q$, so minimum $VaR_\alpha(pX \land q)$ is attained when $q$ is minimal.

From both cases, the minimum value of $VaR_\alpha(p^* X \land q^*)$ is attained when $p$ and $q$ are minimum. Since $p$ and $q$ are minimum, so does $\frac{q}{p}$. Therefore, the minimum value of $VaR_\alpha(p^* X \land q^*)$ is attained at the boundary line of $E(pX \land q) = \mu - P_0$.

After finding that the VaR value is minimum at the boundary line of $E(pX \land q) = \mu - P_0$, the next step is to find the optimal proportion $p$ and retention $q$. This is explained in the following Theorem 3.1.

**Theorem 1** If an insurer takes quota-share before stop-loss to minimize the VaR at a confident level $1 - \alpha$, then

- For $VaR_\alpha(X) \leq \bar{q} \frac{\mu}{\mu - P_0}$, the optimal proportion and retention are $p^* = 1 - \frac{P_0}{\mu}$ and $q^* = \infty$, with minimal value $VaR_\alpha(p^* X \land q^*) = \left(1 - \frac{P_0}{\mu}\right) VaR_\alpha(X)$.
- For $VaR_\alpha(X) \geq \bar{q} \frac{\mu}{\mu - P_0}$ the optimal proportion and retention are $p^* = 1$ and $q^* = \bar{q}$, with minimal value $VaR_\alpha(p^* X \land q^*) = \bar{q}$.

**Proof:**

For $VaR_\alpha(X) \leq \bar{q} \frac{\mu}{\mu - P_0}$ or $VaR_\alpha(X) \leq \frac{q}{p}$, based on Lemma 1, $VaR_\alpha(pX \land q)$ will be minimum at the minimal $p$ which is the vertical asymptote of $E(pX \land q) = \mu - P_0$ in Figure 1, i.e. $p^* = 1 - \frac{P_0}{\mu}$ and $q^* = \infty$. Therefore, the minimal value of $VaR_\alpha(pX \land q)$ is

$$VaR_\alpha(p^* X \land q^*) = \begin{cases} p^* \frac{q}{p}, &VaR_\alpha(X) \leq \frac{q^*}{p^*} \\ q^*, &VaR_\alpha(X) \geq \frac{q^*}{p^*} \end{cases}$$
For \( VaR_\alpha(X) \geq \bar{q} \frac{\mu}{\mu - P_0} \) or \( VaR_\alpha(X) \geq \frac{q}{p} \), based on Lemma 1, \( VaR_\alpha(pX \wedge q) \) will be minimum at the minimal \( q \) or minimal \( \frac{2}{p} \), that is, \( p^* = 1 \) and \( q^* = \bar{q} \). Hence, the minimal value of \( VaR_\alpha(pX \wedge q) \) is

\[
VaR_\alpha(p^*X \wedge q^*) = \begin{cases} 
   p^* VaR_\alpha(X), & VaR_\alpha(X) \leq \frac{q^*}{p^*} \\
   q^*, & VaR_\alpha(X) \geq \frac{q^*}{p^*} 
\end{cases}
\]

\[
= \begin{cases} 
   VaR_\alpha(X), & VaR_\alpha(X) \leq \bar{q} \\
   \bar{q}, & VaR_\alpha(X) \geq \bar{q} 
\end{cases}
\]

Therefore, it is obtained that the minimum value of VaR depends on the proportion of quota-share and retention of stop-loss that varies according to the condition of \( VaR_\alpha(X) \).

3.2. Mean-VaR optimization for stop-loss before quota-share

Similar with the first combination of quota-share and stop-loss, in this section we will discuss the optimization of VaR for the borne risk of insurer when applying quota-share first for the combination and then stop-loss, involving reinsurance premium as a constraint. The retained risk of the insurer is \( X_t = pX + (1-p)(X \wedge q) \), where an optimal \( p \) is \( p^* \in [0,1] \) and an optimal \( q \) is \( q^* \in [0, \infty) \) such that

\[
VaR_\alpha(p^*X + (1-p^*)(X \wedge q^*)) = \min_{p \in [0,1], q \in [0, \infty)} VaR_\alpha(pX + (1-p)(X \wedge q)) \tag{18}
\]

subject to

\[
(1 - p)E((X - q)_+) \leq \frac{P_0}{1 + \mu} \tag{19}
\]

\[
(1 - p)E((X - q)_+) = P_0 \iff \mu - P_0 = E(aX + (1-a)(X \wedge b)) = \begin{cases} 
   E(X), & E(X) \leq b \\
   aE(X) + (1-a)b, & E(X) \geq b 
\end{cases}
\]

where \( P_0 \) is the reinsurance premium and \( \bar{q} \) is a value of \( q \) when \( p = 1 \) which satisfies

\[
E((X - \bar{q})_+) = P_0
\]
To identify an optimum $p$ and $q$, a constraint graph will be made by considering $p$ as the $x$–axis and the point where the value of $E(aX + (1 - a)(X \wedge b))$ changes i.e. $q$ as the $y$–axis, where $p \in [0,1]$ and $q \in [0, \infty)$. The graphic between $p$ and $q$ is illustrated in figure 2.

The next step to solving the optimization problem is searching for points that minimize the VaR. This is given by Lemma 2.

**Lemma 2**

The minimal value $VaR_{\alpha}(p^*X + (1 - p^*)(X \wedge q^*))$ is attained on the curve of $(1 - p)E((X - q)_+) = P_0$ [5].

Proof:

The VaR of $X_i$ is given by

$$VaR_{\alpha}(X_i) = VaR_{\alpha}(px + (1 - p)(X \wedge q)) = \begin{cases} VaR_{\alpha}(X), & VaR_{\alpha}(X) \leq q \\ p VaR_{\alpha}(X) + (1 - p)q, & VaR_{\alpha}(X) \geq q \end{cases}$$

(20)

Case 1: $VaR_{\alpha}(X) \leq q$

In this case, $VaR_{\alpha}(pX + (1 - p)(X \wedge q)) = VaR_{\alpha}(X)$ so the minimum $VaR_{\alpha}(pX + (1 - p)(X \wedge q))$ is attained at minimal $VaR_{\alpha}(X)$.

Case 2: $VaR_{\alpha}(X) \geq q$

In this case, $VaR_{\alpha}(pX + (1 - p)(X \wedge q)) = p VaR_{\alpha}(X) + (1 - p)q$. For any fixed $q$ and $0 < p \leq 1$, the minimum $p VaR_{\alpha}(X) + (1 - p)q$ is attained at minimal $p$. For any fixed $q$ and $p = 0$, the minimum $p VaR_{\alpha}(X) + (1 - p)q$ is attained at minimal $q$.

From both cases, the minimal value $VaR_{\alpha}(p^*X + (1 - p^*)(X \wedge q^*))$ is attained at the minimal $p$ and $q$. Which is the boundary line of $(1 - p)E((X - q)_+) = P_0$.

**Theorem 2**

If an insurer takes stop-loss before quota-share to minimize the VaR at a confident level $1 - \alpha$.

- For $VaR_{\alpha}(X) \leq \mu$, the optimal proportion and retention are $p^* = 1 - \frac{P_0}{\mu}$ and $q^* = 0$, with minimal value $VaR_{\alpha}(p^*X + (1 - p^*)(X \wedge q^*)) = \left(1 - \frac{P_0}{\mu}\right) VaR_{\alpha}(X)$.
For $VaR_d(X) \geq \bar{q} + \frac{P_0}{s(\bar{q})}$ the optimal proportion and retention are $p^* = 0$ and $q^* = \bar{q}$, with minimal value $VaR_a(p^*X + (1 - p^*)(X \land q^*)) = \bar{q}$.

Otherwise, the optimal reinsurance form is the combination stop-loss before quota-share, where $p^*$ and $q^*$ are determined by

$$ (VaR_a(X) - q^*)S(q^*) - E((X - q^*)_+) = 0; $$

$$ (1 - p^*)E((X - q^*)_+) = P_0. $$

Proof:

From equation 20,

$$ VaR_a(pX + (1 - p)(X \land q)) = \begin{cases} VaR_a(X), & VaR_a(X) \leq q \\ pVaR_a(X) + (1 - p)q, & VaR_a(X) \geq q \end{cases} $$

For $VaR_a(X) \geq q$,

$$ VaR_a(pX + (1 - p)(X \land q)) = pVaR_a(X) + (1 - p)q $$

$$ = VaR_a(X) - \frac{VaR_a(X) - q}{\int_q^\infty S_X(x)dx}P_0 $$

Define

$$ f(q) = \frac{VaR_a(X) - q}{\int_q^\infty S_X(x)dx}, \quad q \in [0, VaR_a(X)] $$

then,

$$ f'(q) = \frac{(VaR_a(X) - q)S(q) - \int_q^\infty S_X(x)dx}{(\int_q^\infty S_X(x)dx)^2} $$

In addition, define

$$ g(q) = (VaR_a(X) - q)S(q) - \int_q^\infty S_X(x)dx $$

If $VaR_a(X) \leq \mu$, then $g(q) \leq (E(X) - q)S(q) - E[(X - q)_+] \leq 0$ and $f'(q) = \frac{g(q)}{(\int_q^\infty s_X(x)dx)} \leq 0$.

By assuming that $VaR_a(X)$ and $P_0$ are fixed, the minimal value of $VaR_a(pX + (1 - p)(X \land q)) = p^* VaR_a(X) + (1 - p^*)q^* = VaR_a(X) - f(q)P_0$ is attained at maximum $f(q)$, which is $q = 0$.

So, $VaR_a(X) - f(q)P_0 = \left(1 - \frac{P_0}{\mu}\right)VaR_a(X)$.

If $VaR_a(X) > \mu$, implies that $g(q_0) = 0$ for any $q_0 \in (0, VaR_a(X))$. If $\bar{q} \leq q_0$, so $VaR_a(pX + (1 - p)(X \land q)) = pVaR_a(X) + (1 - p)q \geq p \left(\bar{q} + \frac{P_0}{s(\bar{q})}\right) + (1 - p)q$. The minimal value of $VaR_a(pX + (1 - p)(X \land q))$ is attained at minimal $q$ i.e. $\bar{q}$.

Otherwise, if $q_0 < \bar{q}$ i.e. $\mu < VaR_a(X) < \bar{q} + \frac{P_0}{s(\bar{q})}$, the function $VaR_a(pX + (1 - p)(X \land q)) = pVaR_a(X) + (1 - p)q \leq p \left(\bar{q} + \frac{P_0}{s(\bar{q})}\right) + (1 - p)q$ is minimum at minimal $q$ i.e. $q_0$.

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1725 (2021) 012097 doi:10.1088/1742-6596/1725/1/012097
3.3. Case study

Assume that the insurer’s total loss $X$ in one period follows a lognormal distribution with mean $\mu$ of 5, and the variance $\sigma^2$ of 4. The expectation of total loss $X$ is 1096.63 and the VaR of $X$ at different confidence level are presented in table 1.

Given several different levels of $P_0$ less than $\mu$, or $E(X)$ as reinsurance premium that must be paid by an insurer to the reinsurance company, then the retention $\bar{q}$ can be found such that satisfy $E((X - \bar{q}^+) = P_0$.

In table 2, we present the calculated critical values of $\bar{q}$ and $\bar{q} - \frac{\mu}{\mu - P_0}$ based on Theorem 1.

While table 3 displays the calculated critical values of $\bar{q}$ and $\bar{q} + \frac{P_0}{S(\bar{q})}$ based on Theorem 2.

Table 2 and 3 show that for increasing reinsurance premium, the critical values of $\bar{q}$ and $\bar{q} - \frac{\mu}{\mu - P_0}$ will decrease. For each pair of reinsurance premium $P_0$ and confidence level $\alpha$, we calculated an optimal reinsurance be it quota-share (QS), stop-loss (SL), quota-share before stop-loss (QSbSL), or stop-loss before quota-share (SLbQS); and the minimal VaR value of the retained risk of the insurer. The result is presented in Table 4 for the combination of stop-loss after quota-share, and Table 5 presents the result for the combination of quota-share after stop-loss.

According to table 4, for the confidence level of $\alpha = 5\%$ and constraint reinsurance premium $P_0$ of 800, the optimal reinsurance is a pure stop-loss policy, with the retention $q = 806.82$ and the minimal VaR of retained risk is 806.82. Keeping $P_0$ the same and increasing $\alpha$ to $10\%$, the optimal reinsurance turned to quota-share with the proportion $p = 0.27$ and the minimal VaR reduces to 520.92.

### Table 1. VaR of total risk $X$

| $\alpha$ | 1% | 5% | 10% | 15% | 20% | 25% | 30% | 35% |
|----------|----|----|-----|-----|-----|-----|-----|-----|
| VaR      | 15563.69 | 3982.67 | 1925.81 | 1179.52 | 798.91 | 571.91 | 423.60 | 320.74 |

### Table 2. Critical values of $\bar{q}$ and $\bar{q} - \frac{\mu}{\mu - P_0}$ for different constraint premium.

| $P_0$ | 600 | 800 | 1000 |
|-------|-----|-----|------|
| $\bar{q}$ | 2425.58 | 806.82 | 143.40 |
| $\bar{q} - \frac{\mu}{\mu - P_0}$ | 5356.00 | 2982.76 | 1627.40 |

### Table 3. Critical values of $\bar{q}$ and $\bar{q} + \frac{P_0}{S(\bar{q})}$ for different constraint premium.

| $P_0$ | 600 | 800 | 1000 |
|-------|-----|-----|------|
| $\bar{q}$ | 2425.58 | 806.82 | 143.40 |
| $\bar{q} + \frac{P_0}{S(\bar{q})}$ | 9832.98 | 4847.22 | 2111.90 |
Table 4. Type of reinsurance and optimal retention with minimal VaR calculated based on Theorem 1 for four values of confidence level (\(\alpha\)), reinsurance premium (\(P_0\)), SL means stop-loss, and QS means quota-share.

| \(P_0\)   | \(\alpha = 1\%\)       | \(\alpha = 5\%\)       | \(\alpha = 10\%\)      |
|-----------|-------------------------|-------------------------|-------------------------|
| 600       | SL (\(q^* = 2425.58\)) | QS (\(p^* = 0.45\))    | QS (\(p^* = 0.45\))    |
| VaR       | 2425.58                 | 1803.63                 | 872.14                  |
| 800       | SL (\(q^* = 806.82\))  | SL (\(q^* = 806.82\))  | QS (\(p^* = 0.27\))    |
| VaR       | 806.82                  | 806.82                  | 520.92                  |
| 1000      | SL (\(q^* = 143.40\))  | SL (\(q^* = 143.40\))  | SL (\(q^* = 143.40\))  |
| VaR       | 143.40                  | 143.40                  | 143.40                  |

Table 5. Type of reinsurance and optimal retention with minimal VaR calculated based on Theorem 2 for four values of confidence level (\(\alpha\)), reinsurance premium (\(P_0\)), SL means stop-loss, and SLbQS means stop-loss before quota-share.

| \(P_0\)   | \(1\%\)   | \(5\%\)   | \(10\%\)   |
|-----------|------------|------------|-------------|
| 600       | SL (\(q^* = 2425.58\)) | SLbQS (\(p^* = 0.296; q^* = 572.41\)) | SLbQS (\(p^* = 0.411; q^* = 109.45\)) |
| VaR       | 2425.58    | 1581.84    | 855.35      |
| 800       | SL (\(q^* = 806.82\))  | SLbQS (\(p^* = 0.061; q^* = 572.41\)) | SLbQS (\(p^* = 0.214; q^* = 109.45\)) |
| VaR       | 806.82     | 780.89     | 498.53      |
| 1000      | SL (\(q^* = 143.40\))  | SL (\(q^* = 143.40\))  | SLbQS (\(p^* = 0.018; q^* = 109.45\)) |
| VaR       | 143.40     | 143.40     | 141.71      |

As for stop-loss before quota-share combination, if the insurer set the confidence level \(\alpha\) to 1 \% and \(P_0 = 600\), the optimal reinsurance is pure stop-loss with retention \(q = 2425.58\) and the minimal VaR of retained risk is 2425.58 (table 5). Increasing \(\alpha\) to 10 \% while keeping \(P_0\) the same, the optimal reinsurance is quota-share after stop-loss with the proportion \(p = 0.411\) and retention \(q = 109.45\), and the minimal VaR reduces to 855.35.

4. Conclusion
The minimal VaR of the insurer depends on the proportion of quota-share and retention of stop-loss that varies according to the condition of \(VaR_q(X)\). For \(VaR_{\alpha}(X)\) which is greater or smaller than function of the expected total loss, reinsurance premium, and stop-loss retention when quota-share is not applied, the minimal VaR are the same in both quota-share before stop-loss and stop-loss before quota-share. However, if \(VaR_{\alpha}(X)\) does not meet these conditions, the minimal VaR of retained risk can yield a smaller value in stop-loss after quota-share policy. Therefore, it is recommended for an insurer to apply quota-share after stop-loss to minimize the VaR of the insurer.
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