PRODUCTS OF FUNCTIONS IN HARDY AND LIPSCHITZ OR $\text{BMO}$ SPACES

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Abstract. We define as a distribution the product of a function (or distribution) $h$ in some Hardy space $H^p$ with a function $b$ in the dual space of $H^p$. Moreover, we prove that the product $b \times h$ may be written as the sum of an integrable function with a distribution that belongs to some Hardy-Orlicz space, or to the same Hardy space $H^p$, depending on the values of $p$.

1. Introduction

For $p$ and $p'$ two conjugate exponents, with $1 < p < \infty$, when we consider two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^*$, their product $fg$ is integrable, which means in particular that their pointwise product gives rise to a distribution. When $p = 1$, the right substitute to Lebesgue spaces is, for many problems, the Hardy space $H^1(\mathbb{R}^n)$, whose dual is the space $\text{BMO}(\mathbb{R}^n)$. So one may ask what is the right definition of the product of $h \in H^1(\mathbb{R}^n)$ and $b \in \text{BMO}(\mathbb{R}^n)$. In this context, the pointwise product is not integrable in general. In order to get a distribution, one has to define the product in a different way. This question has been considered by the first author in a joint work with T. Iwaniec, P. Jones and M. Zinsmeister in [BIJZ]. The present paper explores the same problem in different spaces.

The duality bracket $\langle b, h \rangle$ may be written through the almost everywhere approximation of the factor $b \in \text{BMO}(\mathbb{R}^n)$,

$$\langle b, h \rangle = \lim_{k \to \infty} \int_{\mathbb{R}^n} b_k(x) h(x) \, dx ,$$

where $b_k$ is a sequence of bounded functions, which is bounded in the space $\text{BMO}(\mathbb{R}^n)$ and converges to $b$ almost everywhere. For example, we can choose

$$b_k(x) = \begin{cases} k & \text{if } k \leq b(x) \\ b(x) & \text{if } -k \leq b(x) \leq k \\ -k & \text{if } b(x) \leq -k \end{cases}$$

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We then define the product \( b \times h \) as the distribution whose action on the test function \( \varphi \) in the Schwartz class, that is \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), is given by
\[
\langle b \times h, \varphi \rangle := \langle b\varphi, h \rangle.
\]
We use the fact that the multiplication by \( \varphi \) is a bounded operator on BMO(\( \mathbb{R}^n \)). So the right hand side makes sense in view of the duality \( \mathcal{H}^1,\text{BMO} \). Alternatively, the Schwartz class is contained in the space of multipliers of BMO(\( \mathbb{R}^n \)), which have been studied and characterized, see [S] and the discussion below. It follows from (1), used with the sequence \( b_k \) given in (2), that the distribution \( b \times h \) is given by the function \( bh \) whenever this last one is integrable.

A more precise description of products \( b \times h \) has been given in [BIJZ]. Namely, all such distributions are sums of a function in \( L^1(\mathbb{R}^n) \) and a distribution in a Hardy-Orlicz space \( \mathcal{H}^p_{\phi}, \) where \( w \) is a weight which allows a smaller decay at infinity and \( \Phi \) is given below. We will consider a slightly different situation by replacing the space BMO(\( \mathbb{R}^n \)) by the smaller space \( \text{bmo}(\mathbb{R}^n) \), defined as the space of locally integrable functions \( b \) such that
\[
\sup_{|B| \leq 1} \left( \frac{1}{|B|} \int_B |b - b_B| \, dx \right) < \infty \quad \text{and} \quad \sup_{|B| \geq 1} \left( \frac{1}{|B|} \int_B |b| \, dx \right) < \infty.
\]
Here \( B \) varies among all balls of \( \mathbb{R}^n \) and \( |B| \) denotes the measure of the ball \( B \). Also \( b_B \) is the mean of \( b \) on the ball \( B \). Recall that the BMO condition reduces to the first one, but for all balls and not only for balls \( B \) such that \( |B| < 1 \). We clearly have \( \text{bmo} \subset \text{BMO} \). We have the following, which is new compared to [BIJZ].

**Theorem 1.1.** For \( h \) a function in \( \mathcal{H}^1(\mathbb{R}^n) \) and \( b \) a function in \( \text{bmo}(\mathbb{R}^n) \), the product \( b \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion
\[
b \times h \in L^1(\mathbb{R}^n) + \mathcal{H}_{\Phi}^p(\mathbb{R}^n).
\]
\( \mathcal{H}_{\Phi}^p(\mathbb{R}^n) \) is a variant of the Hardy-Orlicz space related to the Orlicz function
\[
\Phi(t) := \frac{t}{\log(e + t)},
\]
which is defined in Section 3. It contains \( \mathcal{H}^p(\mathbb{R}^n) \) and is contained in the weighted Hardy-Orlicz space that has been considered in [BIJZ] for the general case of \( f \in \text{BMO} \).

The aim of this paper is to give some extensions of the previous situation. Indeed, (3) makes sense in other cases. First, the space \( \text{bmo} \) is the dual of the local Hardy space, as proved by Goldberg [G] who introduced it. So it is natural to extend the previous theorem to functions \( h \) in this space, which we do. Next, we can consider the Hardy space \( \mathcal{H}^p(\mathbb{R}^n) \), for \( p < 1 \), and its dual the homogeneous Lipschitz space \( \Lambda_{\gamma}(\mathbb{R}^n) \), with \( \gamma := n(\frac{1}{p} - 1) \). Indeed, a function in
the Schwartz class is also a multiplier of the Lipschitz spaces. Our statement is particularly simple when \( b \) belongs to the non homogeneous Lipschitz space \( \Lambda_\gamma(\mathbb{R}^n) \).

**Theorem 1.2.** Let \( p < 1 \) and \( \gamma := n(\frac{1}{p} - 1) \). Then, for \( h \) a function in \( \mathcal{H}^p(\mathbb{R}^n) \) and \( b \) a function in \( \Lambda_\gamma(\mathbb{R}^n) \), the product \( b \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion

\[
(7) \quad b \times h \in L^1(\mathbb{R}^n) + \mathcal{H}^p(\mathbb{R}^n).
\]

Again the space \( \Lambda_\gamma(\mathbb{R}^n) \) is the dual of the local version of the Hardy space \( \mathcal{H}^p(\mathbb{R}^n) \). We will adapt the theorem to \( h \) in this space.

Let us explain the presence of two terms in the two previous theorems. The product looses the cancellation properties of the Hardy space, which explains the term in \( L^1 \). Once we have subtracted some function in \( L^1 \), we recover a distribution of a Hardy space. For \( p = 1 \), there is a loss, due to the fact that a function in \( \text{bmo} \) is not bounded, but uniformly in the exponential class on each ball of measure 1. This explains that we do not find a function in \( \mathcal{H}^1 \), but in the Hardy-Orlicz space.

As we will see, the proof uses a method that is linear in \( b \), not in \( h \). As in the case \( \mathcal{H}^1\text{-BMO} \) (see [BLJZ]), one would like to know whether the decomposition of \( b \times h \) as a sum of two terms can be obtained through linear operators, but we are very far from being able to answer this question.

All this study is reminiscent of problems related to commutators with singular integrals, or Hankel operators. In particular, such products arise when developing commutators between the multiplication by \( b \) and the Hilbert transform and looking separately at each term. It is well known that the commutator \([b, H] \) maps \( \mathcal{H}^p(\mathbb{R}) \) into \( \mathcal{H}^1_{\text{weak}}(\mathbb{R}) \) for \( b \) in the Lipschitz space \( \Lambda_\gamma(\mathbb{R}) \) (see [J]), which means that there are some cancellations between terms, compared to our statement which is the best possible for each term separately. One can also consider products of holomorphic functions in the corresponding spaces when \( \mathbb{R}^n \) is replaced by the torus, considered as the boundary of the unit disc. Statements and proofs are much simpler and there are converse statements, see [BLJZ] for the case \( p = 1 \), and also to [BG] where the problem is treated in general for holomorphic functions in Hardy-Orlicz spaces in convex domains of finite type in \( \mathbb{C}^n \). These results allow to characterize the classes of symbols for which Hankel operators are bounded from some Hardy-Orlicz space larger than \( \mathcal{H}^1 \) into \( \mathcal{H}^1 \).

Another possible generalization deals with spaces of homogeneous type instead of \( \mathbb{R}^n \). Since the seminal work of Coifman and Weiss [CW1, CW2], it has been a paradigm in harmonic analysis that this is the right setting for developing Calderon-Zygmund Theory. The contribution of Carlos Segovia, mainly in collaboration with Roberto Macías, has been fundamental to develop a general
theory of Hardy and Lipschitz spaces. We will rely on their work in the last section, when explaining how properties of products of functions in Hardy and Lipschitz or BMO spaces can generalize in this general setting. Remark that the boundary of pseudo-convex domain in $\mathbb{C}^n$, with the metric that is adapted to the complex geometry (see for instance [McN]), gives a fundamental example of such a space of homogeneous type. Calderón-Zygmund theory has been developed in this context, see [KL] for instance, in relation with the properties of holomorphic functions, reproducing formulas, Bergman and Szegő projections. Many recent contributions have been done in $H^p$ theory on spaces of homogeneous type. We refer to [GLY] and the references given there. Tools developed by Macías, Segovia and their collaborators play a fundamental role, like, for instance, for the atomic decomposition of Hardy-Orlicz spaces given by Viviani (see [V] and [BC]).

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2. **Prerequisites on Hardy and Lipschitz spaces**

We recall here the definitions and properties that we will use later on. We follow the book of Stein [St].

Let us first recall the definition of the maximal operator used for the definition of Hardy spaces. We fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ having integral 1 and support in $\{|x| < 1\}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x$ in $\mathbb{R}^n$, we put

\begin{equation}
(8) \quad f \ast \varphi(x) := \langle f, \varphi(x - \cdot) \rangle,
\end{equation}

and define the maximal function $\mathcal{M}_\varphi f$ by

\begin{equation}
(9) \quad \mathcal{M}_\varphi f(x) := \sup_{t>0} |(f \ast \varphi_t)(x)|,
\end{equation}

where $\varphi_t(x) = t^{-n} \varphi(t^{-1}x)$. We also define the truncated version of the maximal function, namely

\begin{equation}
(10) \quad \mathcal{M}_{\varphi}^{(1)} f(x) := \sup_{0<t<1} |(f \ast \varphi_t)(x)|.
\end{equation}

For $p > 0$, a tempered distribution $f$ is said to belong to the Hardy space $H^p(\mathbb{R}^n)$ if

\begin{equation}
(11) \quad \|f\|_{H^p(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \mathcal{M}_\varphi f(x)^p \, dx \right)^{\frac{1}{p}} < \infty.
\end{equation}

The localized versions of Hardy spaces are defined in the same spirit, with the truncated maximal function in place of the maximal function. Namely, a
tempered distribution \( f \) is said to belong to the space \( \mathcal{H}^p(\mathbb{R}^n) \) if
\[
\| f \|_{\mathcal{H}^p(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |\mathcal{M}_\varphi f(x)|^p dx \right)^{\frac{1}{p}} < \infty.
\]

Recall that, up to equivalence of corresponding norms, the space \( \mathcal{H}^p(\mathbb{R}^n) \) (resp. \( \mathcal{H}^p(\mathbb{R}^n) \)) does not depend on the choice of the function \( \varphi \). So, in the sequel, we shall use the notation \( \mathcal{M} f \) instead of \( \mathcal{M}_\varphi f \) (resp. \( \mathcal{M}^{(1)} f \) instead of \( \mathcal{M}_\varphi^{(1)} f \)).

Hardy-Orlicz spaces are defined in a similar way. Given a continuous function \( P : [0, \infty) \to [0, \infty) \) increasing from zero to infinity (but not necessarily convex, \( P \) is called the Orlicz function), the Orlicz space \( L^P \) consists of measurable functions \( f \) such that
\[
\| f \|_{L^P} := \inf \left\{ k > 0 : \int_{\mathbb{R}^n} P \left( k^{-1} |f| \right) dx \leq 1 \right\} < \infty.
\]

Then \( \mathcal{H}^P \) (resp. \( \mathcal{H}^p \)) is the space of tempered distributions \( f \) such that \( \mathcal{M} f \) is in \( L^P \) (resp. \( \mathcal{M}^{(1)} f \) is in \( L^p \)). We will be particularly interested by the choice of the function \( \Phi \) given in (6) as the Orlicz function. It is easily seen that the function \( \Phi \) is equivalent to a concave function (take \( t/ (\log(c + t)) \), for \( c \) large enough). So there is no norm on the space \( L^\Phi \). In general, \( \| \cdot \|_{L^P} \) is homogeneous, but is not sub-additive. Nevertheless (see [BJZ]),
\[
\| f + g \|_{L^\Phi} \leq 4 \left( \| f \|_{L^\Phi} + \| g \|_{L^\Phi} \right).
\]

**Definition 2.1.** \( L^\Phi_* \) is the space of functions \( f \) such that
\[
\| f \|_{L^\Phi_*} := \sum_{j \in \mathbb{Z}^n} \| f \|_{L^\Phi_*(j + Q)} < \infty,
\]
where \( Q \) is the unit cube centered at 0.

We accordingly define \( \mathcal{H}^\Phi_* \) (resp. \( \mathcal{H}^\Phi_* \)). Using the concavity described above, we have \( \Phi(st) \leq C s \Phi(t) \) for \( s > 1 \). It follows that \( L^\Phi \) is contained in \( L^\Phi_* \) as a consequence of the fact that \( \| f \|_{L^\Phi_*(j + Q)} \leq \int_{j + Q} \Phi(|f|) dx \). The converse inclusion is not true.

We will restrict to \( p \leq 1 \), since otherwise Hardy spaces are just Lebesgue spaces. We will need the atomic decompositions of the spaces \( \mathcal{H}^p(\mathbb{R}^n) \) (resp. \( \mathcal{H}^p(\mathbb{R}^n) \)), which we recall now.

**Definition 2.2.** Let \( 0 < p \leq 1 < q \leq \infty \), \( p < q \), and \( s \) an integer. A \( (p, q, s) \)-atom related to the ball \( B \) is a function \( a \in L^q(\mathbb{R}^n) \) which satisfies the following conditions:
\[
support(a) \subset B \quad \text{and} \quad \| a \|_q \leq |B|^{\frac{1}{q'} - \frac{1}{p}},
\]
\[ \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0, \quad \text{for } 0 \leq |\alpha| \leq s. \]

Here \( \alpha \) varies among multi-indices, \( x^\alpha \) denotes the product \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). Condition (16) is called the moment condition.

The atomic decomposition of \( \mathcal{H}^p(\mathbb{R}^n) \) is as follows. Let us fix \( q > p \) and \( s > n \left( \frac{1}{p} - 1 \right) \). Then a tempered distribution \( f \) is in \( \mathcal{H}^p(\mathbb{R}^n) \) if and only if there exists a sequence of \( (p,q,s) \)-atoms \( a_j \) and constants \( \lambda_j \) such that

\[ f := \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{and} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty, \]

where the first sum is assumed to converge in the sense of distributions. Moreover, \( f \) is the limit of partial sums in \( \mathcal{H}^p(\mathbb{R}^n) \), and \( \|f\|_{\mathcal{H}^p(\mathbb{R}^n)} \) is equivalent to the infimum, taken on all such decompositions of \( f \), of the quantities

\[ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}}. \]

For the local version, we consider other kinds of atoms when the balls \( B \) are large. We have the following, where we have fixed \( q > p \) and \( s > n \left( \frac{1}{p} - 1 \right) \).

A tempered distribution \( f \) is in \( \mathcal{H}^p(\mathbb{R}^n) \) if and only if there exists a sequence of functions \( a_j \), constants \( \lambda_j \) and balls \( B_j \) for which (17) holds, and such that

(i) when \( |B_j| \leq 1 \), then \( a_j \) is a \( (p,q,s) \)-atom related to \( B_j \);

(ii) when \( |B_j| > 1 \), then \( a_j \) is supported in \( B_j \) and

\[ \|a_j\|_q \leq |B_j|^{\frac{1}{q} - \frac{1}{p}}. \]

In other words, one still has the atomic decomposition, except that for large balls one does not ask for any moment condition on atoms.

Next, let us define Lipschitz spaces. For \( \delta \in \mathbb{R}^n \) we note \( D^1_\delta = D_\delta \) the difference operator, defined by setting \( D_\delta f(x) = f(x + \delta) - f(x) \) for \( f \) a continuous function (see \[Gr\] for instance). Then, by induction, we define

\[ D^{k+1}_\delta f = D_\delta \left( D^k_\delta f \right) \]

for \( k \) a non negative integer, so that

\[ D^k_\delta f(x) = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(x + s\delta). \]

For \( \gamma > 0 \) and \( k = \lfloor \gamma \rfloor \) the integer part of \( \gamma \), we set

\[ \|f\|_{\Lambda_\gamma} = \|f\|_{L^\infty} + \sup_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^n \setminus \{0\}} \frac{|D^{k+1}_\delta f(x)|}{|\delta|^{\gamma}}. \]

\( \Lambda_\gamma(\mathbb{R}^n) \), the inhomogeneous Lipschitz space of order \( \gamma \), is defined as the space of continuous functions \( f \) such that \( \|f\|_{\Lambda_\gamma} < \infty \). It is well known that \( f \in \Lambda_\gamma(\mathbb{R}^n) \)
is of class $C^k(\mathbb{R}^n)$, with $k := \lfloor \gamma \rfloor$. Moreover, for $\alpha$ a multi-index with $|\alpha| \leq k$,

$$\|\partial^\alpha f\|_{\Lambda_{\gamma-|\alpha|}} \leq C(n, \gamma) \|f\|_{\Lambda_{\gamma}}.$$  

Similarly, we define the homogeneous Lipschitz $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ with

$$\|f\|_{\dot{\Lambda}_\gamma} = \sup_{x \in \mathbb{R}^n} \sup_{\delta \in \mathbb{R}^n \setminus \{0\}} \frac{|D_{\delta}^{k+1}f(x)|}{|\delta|^{\gamma}},$$

3. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 To simplify notations, we will write $H^p$ in place of $H^p(\mathbb{R}^n)$, $\text{BMO}$ in place of $\text{BMO}(\mathbb{R}^n)$, etc.. The proof is inspired by the one given in [BIJZ] for the product $b \times h$ when $b$ is in $\text{BMO}$. Recall that we assume that $b \in \text{bmo}$. The function $h \in H^1$ admits an atomic decomposition with bounded atoms,

$$h := \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|h\|_{H^1}.$$  

When the sequence $h_\ell$ tends to $h$ in $H^1$, the product $b \times h_\ell$ tends to $b \times h$ as a distribution. So we can write

$$b \times h = \sum_j \lambda_j (b \times a_j),$$

where the limit is taken in the distribution sense. Since the $a_j$ are bounded functions with compact support, the product $b \times a_j$ is given by the ordinary product. We want to write $b \times h = h^{(1)} + h^{(2)}$, with $h^{(1)} \in L^1$ and $h^{(2)} \in H^1$. Let us write, for each term $a_j$, which is assumed to be adapted to $B_j$,

$$b \times a_j = (b - b_{B_j})a_j + b_{B_j}a_j.$$  

By the $\text{BMO}$ property as well as the fact that $|a_j| \leq |B_j|^{-1}$, we have the inequality

$$\sum_j |\lambda_j| \int_{\mathbb{R}^n} |b - b_{B_j}| |a_j| dx \leq C\|b\|_{\text{bmo}} \|h\|_{H^1}.$$  

Here $\|b\|_{\text{bmo}}$ is the sum of the two finite quantities that appear in the definition of $\text{bmo}$ given by (4). We call

$$h^{(1)} := \sum_j \lambda_j (b - b_{B_j})a_j,$$

which is the sum of a normally convergent series in $L^1$. Since convergence in $L^1$ implies convergence in the distribution sense, it follows that $h^{(2)}$ is

$$h^{(2)} := \sum_j \lambda_j b_{B_j}a_j,$$
which is well defined in the distribution sense. Moreover
\[
Mh^{(2)} \leq \sum_j |\lambda_j|\|b_{B_j}\| Ma_j
\]
\[
\leq \sum_j |\lambda_j|b - b_{B_j}\|Ma_j + \|b\| \sum_j |\lambda_j| Ma_j.
\]
The first term is in \(L^1\) since \(Ma_j \leq |B_j|^{-1}\). In order to conclude, we have to prove that the second term is in \(L^\Phi\). We first use the fact that \(\|Ma_j\|_1 \leq C\) for some uniform constant \(C\), which is classical and may be found in [St] for instance. Then we have to prove that, for \(\psi \in L^1\), the product \(b\psi\) is in \(L^\Phi\).

We claim that \(b\) belongs uniformly to the exponential class on each ball of measure 1. Indeed, by John-Nirenberg Inequality which is valid for \(b\), for some constant \(C\), which depends only on the dimension, and for each ball \(B\) such that \(|B| = 1\),
\[
(22) \quad \int_B \exp \left( \frac{|b(x) - b_B|}{C\|b\|_{bmo}} \right) \, dx \leq 2.
\]
Moreover, since \(b \in bmo\), we have the inequality \(|b_B| \leq \|b\|_{bmo}\). To prove that \(b\psi\) is in \(L^1\), we first consider each such ball separately. We use the following lemma, which is an adaptation of lemmas given in [BJZ].

**Lemma 3.1.** If the integral on \(B\) of \(\exp|b|\) is bounded by 2, then, for some constant \(C\),
\[
\|b\psi\|_{L^\Phi(B)} \leq C \int_B |\psi| \, dx.
\]

**Proof.** By homogeneity it is sufficient to find some constant \(c\) such that, for \(\int_B |\psi| \, dx = c\) we have
\[
\int_B \frac{|b\psi|}{\log(e + |b\psi|)} \, dx \leq 1.
\]
If we cut the integral into two parts depending on the fact that \(|b| < 1\) or not, we conclude directly that the first part is bounded by \(c\), since we have a majorant by suppressing the denominator. For the second part, we can suppress \(b\) in the denominator. Then, we use the duality between the \(L \log L\) class, and the Exponential class. It is sufficient to prove that the Luxembourg norm of \(\frac{|\psi|}{\log(e + |\psi|)}\) in the class \(L \log L\) is bounded by \(1/2\) for \(c\) small enough, which is elementary.

We have an estimate for each cube \(j + Q\), and sum up. This finishes the proof of Theorem 1.1.

\[\square\]
Since \( \text{bmo} \) is the dual of \( h^1 \), it is natural to see what is valid for \( h \in h^1 \). We can state the following.

**Theorem 3.2.** For \( h \) a function in \( h^1(\mathbb{R}^n) \) and \( b \) a function in \( \text{bmo}(\mathbb{R}^n) \), the product \( b \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion

\[
(23) \quad b \times h \in L^1(\mathbb{R}^n) + h^\Phi(\mathbb{R}^n).
\]

**Proof.** Again, we start from the atomic decomposition of \( h \). In view of (14), it is sufficient to consider only those atoms \( a_j \) that are adapted to balls \( B \) such that \( |B| \geq 1 \). Remember that they do not satisfy the moment condition (16). This one was only used to insure that \( \|M a_j\|_1 \leq C \) for some independent constant. We now have \( \|M^{(1)} a_j\|_1 \leq C \) since \( M^{(1)} a_j \), which is bounded by \( |B_j|^{-1} \) is supported in the ball of same center as \( B_j \) and radius twice the radius of \( B_j \). Except for this point, the proof is identical. □

Before leaving the case \( p = 1 \), let us add some remarks. Multipliers of the space \( \text{BMO} \) have been characterized by Stegenga in [S] (see also [CL]) when \( \mathbb{R}^n \) is replaced by the torus. It is easy to extend this characterization to \( \text{bmo} \). Let us first define the space \( \text{imo} \) as the space of locally integrable functions \( b \) such that

\[
(24) \quad \sup_{|B| \leq 1} \left( \log(e + 1/|B|) \right) \int_B |b - b_B| dx < \infty \quad \text{and} \quad \sup_{|B| \geq 1} \left( \frac{1}{|B|} \int_B |b| dx \right) < \infty.
\]

\( \text{imo} \) stands for logarithmic mean oscillation.

**Proposition 3.3.** Let \( \phi \) a locally integrable function. Then the following properties are equivalent.

(i) The function \( \phi \) is bounded and belongs to the space \( \text{imo} \).

(ii) For every \( b \in \text{bmo} \), the function \( b \phi \) is in \( \text{bmo} \).

**Proof.** We give a direct proof, which is standard, for completeness. The proof of (i)⇒(ii) is straightforward. Indeed, let us first consider balls \( B \) such that \( |B| \leq 1 \). Writing \( b = (b - b_B) + b_B \), we conclude directly for the first term, and have to prove that

\[
|b_B| \times \frac{1}{|B|} \int_B |\phi - \phi_B| dx \leq C \|b\|_{\text{bmo}} \|\phi\|_{\text{imo}}.
\]

Let \( B' \) the ball of same center as \( B \) and radius 1. It is well known that the fact that \( b \) is in \( \text{BMO} \) implies that

\[
|b_B - b_{B'}| \leq C \log(e + 1/|B|) \|b\|_{\text{BMO}}.
\]

We conclude, using the fact that \( |b_B| \leq C \|b\|_{\text{bmo}} \). The proof is even simpler for balls \( B \) such that \( |B| \geq 1 \).
Conversely, assume that we have (ii). Taking \( b = 1 \), we already know that \( \phi \) is in \( \text{bmo} \). Also, by the closed graph theorem, we know that there exists some constant \( C \) such that, for every \( b \in \text{bmo} \), the function

\[
\|b\phi\|_{\text{bmo}} \leq C\|b\|_{\text{bmo}}.
\]

We first claim that \( \phi \) is bounded. By the Lebesgue differentiation theorem, it is sufficient to prove that, for each ball \( B \), the mean \( \phi_B \) is bounded. But \( \phi_B = \langle \phi, |B|^{-1} \rangle \), \( \chi_B = \langle b\phi, a \rangle \), where \( a \) is some atom of \( h^1 \) and \( b \) is bounded by 1. Indeed, the characteristic function \( \chi_B \) may be written as the square of a function of mean zero, taking values \( \pm 1 \) on \( B \). So \( \phi_B \) is bounded. Now, since \( \phi \) is bounded, the assumption implies that, for a ball \( B \) such that \( |B| \leq 1 \),

\[
|b_B| \times \frac{1}{|B|} \int_B |\phi - \phi_B| dx \leq C\|b\|_{\text{bmo}}.
\]

It is sufficient to find a function \( b \) with norm bounded independently of \( B \) and such that \( |b_B| \geq c \log(e + 1/|B|) \). The function \( \log(|x - x_B|^{-1}) \), with \( x_B \) the center of \( B \), has this property.  

The previous proposition allows an interpretation in view of Theorem 1.1. The duals of Hardy-Orlicz spaces have been studied by S. Jansen [J], see also the work of Viviani [V] where duality is deduced from their atomic decomposition. In particular, the dual of the space \( h^\Phi \) is the space \( \text{mo} \). It follows that the dual of the space \( L^1 + h_\Phi \) is the space \( L^\infty \cap \text{mo} \). So if a duality argument was possible, which is not the case since we are not dealing with Banach spaces, we would conclude that multiplication by \( \text{bmo} \) maps \( h^1 \) into \( L^1 + h_\Phi \). Recall that we have a weaker statement.

Proof of Theorem 1.2. When \( p > \frac{\gamma}{n+1} \), the proof is an easy adaptation of the previous one. We start again from an atomic decomposition of \( h \) and define \( h^{(1)} \) and \( h^{(2)} \) as before. To conclude for \( h^{(1)} \in L^1 \), it is sufficient to prove that, for all balls \( B \), one has

\[
\int_B |b - b_B| dx \leq C|B|^{\frac{1}{p}}\|b\|_{\Lambda_\gamma}.
\]

If \( B \) has center \( x_B \) and radius \( r \), it follows at once from the inequality \( |b(x) - b(x_B)| \leq r^\gamma\|b\|_{\Lambda_\gamma} \leq |B|^{\gamma/n}\|b\|_{\Lambda_\gamma} \), and the choice \( \gamma = n(1/p - 1) \).

Next we conclude directly for \( h^{(2)} \), using the fact that \( b \) is bounded, so that \( \mathcal{M}h^{(2)} \leq \|b\|_\infty \sum_j |\lambda_j| \mathcal{M}a_j \). This last quantity is in \( L^p \) since

\[
\int |\mathcal{M}h^{(2)}| \leq \|b\|_\infty \sum_j |\lambda_j|^p \int |\mathcal{M}a_j|^p \leq \|b\|_\infty \sum_j |\lambda_j|^p \int |\mathcal{M}a_j|^p
\]

and \( \mathcal{M}a_j \)'s are uniformly in \( L^p \).
For smaller values of $p$, we start again from an atomic decomposition of $h$, but choose the atoms $a_j$ to be $(p, \infty, s)$ for $s$ to be chosen later, that is, to satisfy the moment condition \([\text{16}]\) up to order $s$. We then have to modify the choice of $h^{(1)}$ and $h^{(2)}$ in order to be able to treat the first term as above. We use the following definition.

**Definition 3.4.** For $f$ a locally square integrable function and $B$ a ball in $\mathbb{R}^n$, we define $P^k_B f$ as the orthogonal projection in $L^2(B)$ of $f$ onto the space of polynomials of degree $\leq k$.

The next lemma is classical. It is the easy part of the identification of Lipschitz spaces with spaces of Morrey-Campanato, see [C]. We give its proof for completeness.

**Lemma 3.5.** Let $\gamma > 0$ and $k \geq \gamma$. There exists a constant $C$ such that, for $f$ a function in $\Lambda_\gamma(\mathbb{R}^n)$ and $B$ a ball in $\mathbb{R}^n$, then

\[
\frac{1}{|B|} \int_B |f(x) - P^k_B f(x)| \, dx \leq C \|f\|_{\Lambda_\gamma} |B|^\gamma/n.
\]

**Proof.** In fact we prove an $L^2$ inequality instead of an $L^1$, which is better. In this case, it is sufficient to prove the same inequality with $P^k_B f$ replaced by some polynomial $P$ of degree $\leq k$. This allows to conclude for $\gamma$ not an even integer. Indeed, take for $P$ the Taylor polynomial at point $x_B$ (assuming that $B$ has center $x_B$ and radius $r$) and order $\lfloor \gamma \rfloor$, using the fact that it makes sense by \([20]\). Then, by Taylor’s formula, $|f - P|$ is bounded on $B$ by $C r^{\gamma} \leq C |B|^\gamma/n$. For $\gamma$ an integer, we conclude for \([25]\) by interpolation. \[\square\]

Let us come back to the proof of Theorem 1.2. We start again from an atomic decomposition of $f$. We fix $k \geq \gamma$ and pose

\[
h^{(1)} := \sum_j \lambda_j (b - P^k_{B_j} b) a_j.
\]

Using the previous lemma, we conclude as before that $h^{(1)}$ is in $L^1$. In order to have $h^{(2)} := b \times h - h^{(1)}$ in $H^p$, it is sufficient that each term $(P^k_{B_j} b) a_j$ be, up to the multiplication by a uniform constant, a $(p, \infty, s')$-atom with $s' \geq \gamma$. The moment condition is clearly satisfied if we have $s \geq k + s'$. We can in particular choose $k = s' = \lfloor \gamma \rfloor$ and $s = 2 \lfloor \gamma \rfloor$. It remains to prove that $P^k_{B_j}$ is uniformly bounded. This follows from the following lemma.

**Lemma 3.6.** Let $k$ be a positive integer. There exists a constant $C > 0$ such that for every ball $B$ in $\mathbb{R}^n$,

\[
\|P^k_B f\|_{L^\infty(B)} \leq C \|f\|_{L^\infty(B)},
\]

for all functions $f$ which are bounded on the ball $B$. 
Proof. We remark first that, by invariance by translation we can assume that \( B \) is centered at 0. Next, by invariance by dilation, we can also assume that \( |B| = 1 \). So we have to prove it for just one fixed ball. Now, since the projection is done on a finite dimensional space,

\[
\| P^k_B f \|_{L^\infty(B)} \leq C_k \| P^k_B f \|_{L^2(B)} \leq C_k \| f \|_{L^2(B)} \leq C_k \| f \|_{L^\infty(B)}.
\]

\[\square\]

This allows to conclude for the proof of the theorem. \[\square\]

As for the case \( p = 1 \), we can take \( h \) in the local Hardy space.

**Theorem 3.7.** For \( h \) a function in \( \mathcal{H}^p(\mathbb{R}^n) \) and \( b \) a function in \( \Lambda_\gamma(\mathbb{R}^n) \), the product \( b \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion

\[
(27) \quad b \times h \in L^1(\mathbb{R}^n) + \mathcal{H}^p(\mathbb{R}^n).
\]

**Proof.** The adaptation of the previous proof is done in the same way as we have done for Theorem 3.2 compared to Theorem 1.1. We leave it to the reader. \[\square\]

We did not give estimates of the norms, but it follows from the proof of Theorem 1.2 that we have the inequality

\[
\| h^{(1)} \|_1 + \| h^{(2)} \|_{\mathcal{H}^p} \leq C \| h \|_{\mathcal{H}^p} \times \| b \|_{\Lambda_\gamma(\mathbb{R}^n)}.
\]

So the bilinear operator

\[
\mathcal{P} : \Lambda_\gamma(\mathbb{R}^n) \times \mathcal{H}^p(\mathbb{R}^n) \to L^1(\mathbb{R}^n) + \mathcal{H}^p(\mathbb{R}^n)
\]

\[
(b, h) \mapsto b \times h
\]

is continuous. It is easy to see that the term in \( L^1 \) is present in general: for instance take an example in which the product is positive. The same remarks are valid for all three other cases.

**Remark 3.8.** The product of \( b \in \dot{\Lambda}_\gamma(\mathbb{R}^n) \) and \( h \in \mathcal{H}^p(\mathbb{R}^n) \) is also well defined. It belongs to some \( L^1(\mathbb{R}^n) + \mathcal{H}^p_0(\mathbb{R}^n) \) with a weight \( w \) conveniently chosen. Now \( b \) is no more bounded, but can increase as \( |x|^{-\alpha} \) at infinity. We can take any weight \( (1 + |x|)^{-\alpha} \), with \( \alpha > \gamma p \).

4. **Generalization to spaces of homogeneous type**

All proofs generalize easily to spaces of homogeneous type once one has been able to define correctly the product \( b \times h \). We will not give into details of terminology and proofs when the generalization may be done without any difficulty, but will essentially concentrate on the definition of the product.

Let us first recall some definitions. We assume that we are given a locally compact Hausdorff space \( X \), endowed with a quasi-metric \( d \) and a positive regular measure \( \mu \) such that the doubling condition

\[
(28) \quad 0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < +\infty
\]
holds, for all $x$ in $X$ and $r > 0$. Here, by a quasi-metric $d$, we mean a function $d : X \times X \to [0; +\infty]$ which satisfies

( i) $d(x, y) = 0$ if and only if $x = y$ ;
( ii) $d(x, y) = d(y, x)$ for all $x, y$ in $X$;
( iii) there exists a finite constant $\kappa \geq 1$ such that

\begin{equation}
\tag{29}
d(x, y) \leq \kappa (d(x, z) + d(z, y))
\end{equation}

for all $x, y, z$ in $X$.

Given $x \in X$ and $r > 0$, we note $B_{(x, r)} = \{ y \in X : d(x, y) < r \}$ the ball with center $x$ and radius $r$.

**Definition 4.1.** We call space of homogeneous type $(X, d, \mu)$ such a locally compact space $X$, given together with the quasi-metric $d$ and the nonnegative Borel measure $\mu$ on $X$ that satisfies the doubling condition.

On such a space of homogeneous type we can define the $\text{BMO}(X)$ and $\text{bmo}(X)$ spaces and have the John-Nirenberg inequality. Just replace Euclidean balls by the balls on $X$, and the Lebesgue measure $dx$ by the measure $d\mu$. The Hardy space $\mathcal{H}_1(X)$ can be defined by the atomic decomposition and we have the duality $\mathcal{H}_1-\text{BMO}$. But we want also to define the product $b \times h$ for $b \in \text{bmo}(X)$ and $h \in \mathcal{H}_1(X)$, while we can no more speak of distributions. In view of Theorem 3.3 which generalizes easily in this context, we can take $C \cap L^\infty \cap \text{bmo}$ as a space of test functions, but we need to have density theorems of such functions in the space of continuous compactly supported functions to recover the pointwise product when it is integrable. We encounter a fortiori this difficulty when dealing with Lipschitz spaces.

Macías and Segovia have overcome this kind of difficulty for being able to develop the theory of $\mathcal{H}_p$ spaces for $p < 1$ (see also [U]). We will assume that the measure $\mu$ does not charge points for simplification. They have proved [MS1] that, without loss of generality, $X$ may be assumed a normal space of homogeneous type and of order $\alpha > 0$ when, eventually, the quasi-distance is replaced by an equivalent one. That is, the quasi-metric $d$ and the measure $\mu$ are assumed to satisfy the following properties.

There exist four positive constants $A_1, A_2, K_1$ and $K_2$, such that

\begin{equation}
\tag{30}
A_1 r \leq \mu(B_{(x, r)}) \leq A_2 r \quad \text{if} \quad 0 \leq r \leq K_1 \mu(X)
\end{equation}

\begin{equation}
\tag{31}
B_{(x, r)} = X \quad \text{if} \quad r > K_1 \mu(X).
\end{equation}

\begin{equation}
\tag{32}
|d(x, z) - d(y, z)| \leq K_3 r^{1-\alpha} d(x, y)^\alpha,
\end{equation}

for every $x, y$ and $z$ in $X$, whenever $d(x, z) < r$ and $d(y, z) < r$.

In the Euclidean case, a normal quasi-distance is given by $|x - y|^n$. The assumption \((32)\) is satisfied with $\alpha = 1/n$.

Let us then define the Lipschitz spaces, as it is natural, by the following.
Definition 4.2. Let $\gamma > 0$. The Lipschitz space $\Lambda_\gamma(X,d,\mu)$ consists of bounded continuous functions $f$ on $X$ for which, for some constant $C$ and all $x,y$,

$$|f(x) - f(y)| \leq Cd(x,y)^\gamma.$$  

Note that there is a change of parameter in the Euclidean space, $\gamma$ has been changed into $\gamma/n$.

Remark that with these definitions Lipschitz spaces are contained in $lmo$. We know that the space is not reduced to 0 when $\gamma$ is not larger than $\alpha$ because of the fact that the distance itself satisfies this kind of condition. Macías and Segovia have proved in [MS2] that one can build approximate identities in order to approach continuous functions with compact support by Lipschitz functions of order $\gamma$, for any $\gamma < \alpha$. Moreover, they define the space of distributions $(E^\alpha)^*$ as the dual of the space $E^\alpha$, consisting of all functions with bounded support, belonging to $\Lambda_\beta$ for every $0 < \beta < \alpha$. From this point, they can use distributions to define $\mathcal{H}^p$ spaces when $p > (1+\alpha)^{-1}$. We recover in the Euclidean case the condition $p > n/(n+1)$, which is the range where atoms are assumed to satisfy only the moment condition of order zero, and where the dual is a Lipschitz space defined by a condition implying only one difference operator.

This notion of distribution is exactly what we need for the definition of products. With the conditions above and in the corresponding range of $p$, products may be defined in the distribution sense and the four theorems are valid.

Remark that, for $X$ the boundary of a bounded smooth pseudo-convex domain of finite type, Lipschitz spaces can be defined for all values of $\gamma$ and $\mathcal{H}^p$ spaces can be defined for $p$ arbitrarily small. The moment conditions of higher order rely on the use of vector fields related to the geometric structure of the boundary. We refer to [McN] for the geometrical aspects, and to [BG] for the detailed statements related to products of holomorphic functions in $\mathcal{H}^p$ and $\text{BMO}$ or Lipschitz spaces.

References

[BG] A. Bonami and S. Grellier, Decomposition theorems for Hardy-Orlicz spaces and weak factorization, preprint.

[BIJZ] A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister, On the product of functions in $\text{BMO}$ and $\mathcal{H}^1$, Ann. Inst. Fourier, Grenoble 57, 5 (2007) 1405-1439.

[C] S. Campanato, Proprietà di hölderianità di alcune classi di funzioni, Ann. Scuola Norm. Sup. Cl. Sci. 17 (1963), 175-188.

[CL] D. C. Chang and S. Y. Li, On the boundedness of multipliers, commutators and the second derivatives of Green’s operators on $H^1$ and $\text{BMO}$, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 28 n°2 (1999), 341-356.

[CW1] R. Coifman, G. Weiss, Analyse Harmonique Non-commutative sur Certains Espaces Homogènes, (1977), 569-645. Lecture Notes in Math. 242, Springer, Berlin, 1971.
[CW2] R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. J. Math. **83** (1977), 569-645.

[G] D. Goldberg, *A local version of Hardy spaces*, Duke J. Math. **46** (1979), 27-42.

[GLY] L. Grafakos, L. Liu and D. Yang, *Maximal function characterizations of Hardy spaces on RD-spaces and their applications*, preprint.

[Gr] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc. Upper Saddle River, New Jersey 07458.

[J] S. Janson, *Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation*, Duke Math. J. **47** (1980), no. 4, 959–982.

[JPS] S. Janson, J. Peetre and S. Semmes, *On the action of Hankel and Toeplitz operators on some function spaces*, Duke Math. J. **51** (1984), 937–958.

[KL] S. G. Krantz and S.-Y. Li, *Boundedness and Compactness of Integral Operators on Spaces of Homogeneous Type and Applications*, J. Math. Anal. Appl. **258** No.2 (2001), 642-657.

[McN] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. in Math **109** (1994), 108-139.

[MS1] R. A. Maciás and C. Segovia, *Lipschitz function on Spaces of Homogeneous type*, Advances in Math. **33** (1979), 257-270.

[MS2] R. A. Maciás and C. Segovia, *A decomposition into Atoms of Distributions on Spaces of Homogeneous type*, Advances in Math. **33** (1979), 271-309.

[S] D. A. Stegenga, *Bounded Toeplitz operators on $H^1$ and applications of the duality between $H^1$ and the functions of bounded mean oscillation*, Amer. J. Math. **98** (1976), no. 3, 573–589.

[St] E. M. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series 43, Princeton University Press, Princeton 1993.

[U] A. Uchiyama *The factorization of $H^p$ on the space of homogeneous type*, Pacific J. Math. **92** (1981), 453–468.

[V] B. E. Viviani, *An Atomic Decomposition of the Predual of BMO(p)*, Revista Matemática Iberoamericana **3** (1987), 401-425.

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