TECHNIQUES OF COMPUTATIONS OF DOLBEAULT COHOMOLOGY OF SOLVMANIFOLDS

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Abstract. We consider semi-direct products \( \mathbb{C}^n \ltimes_{\phi} N \) of Lie groups with lattices \( \Gamma \) such that \( N \) are nilpotent Lie groups with left-invariant complex structures. We compute the Dolbeault cohomology of direct sums of holomorphic line bundles over \( G/\Gamma \) by using the Dolbeault cohomology of the Lie algebras of the direct product \( \mathbb{C}^n \times N \). As a corollary of this computation, we can compute the Dolbeault cohomology \( H^{p,q}(G/\Gamma) \) of \( G/\Gamma \) by using a finite dimensional cochain complexes. Computing some examples, we observe that the Dolbeault cohomology varies for choices of lattices \( \Gamma \).

1. Introduction

Let \( G \) be a simply connected solvable Lie group and \( g \) the Lie algebra of \( G \). We assume that \( G \) admits a lattice \( \Gamma \) and a left-invariant complex structure \( J \). We consider the Dolbeault cohomology \( H^{p,\bar{q}}(G/\Gamma) \) of the complex solvmanifold \( G/\Gamma \). We also consider the cohomology \( H^{p,\bar{q}}(g) \) of the differential bigraded algebra (shortly DBA) \( \wedge^{\cdot,\cdot} g^{\cdot} \) of the complex valued left-invariant differential forms with the operator \( \bar{\partial} \).

The purpose of this paper is to prove that one can compute the Dolbeault cohomology of certain class of solvmanifolds \( G/\Gamma \) by using finite dimensional DBAs. In this paper we consider a Lie group \( G \) as in the following assumption.

Assumption 1.1. \( G \) is the semi-direct product \( \mathbb{C}^n \ltimes_{\phi} N \) so that:
(1) \( N \) is a simply connected nilpotent Lie group with a left-invariant complex structure \( J \).
(2) For any \( t \in \mathbb{C}^n \), \( \phi(t) \) is a holomorphic automorphism of \( (N, J) \).
(3) \( \phi \) induces a semi-simple action on the Lie algebra \( n \) of \( N \).
(4) \( G \) has a lattice \( \Gamma \). (Then \( \Gamma \) can be written by \( \Gamma = \Gamma' \ltimes_{\phi} \Gamma'' \) such that \( \Gamma' \) and \( \Gamma'' \) are lattices of \( \mathbb{C}^n \) and \( N \) respectively and for any \( t \in \Gamma' \) the action \( \phi(t) \) preserves \( \Gamma'' \).)
(5) The inclusion \( \wedge^{\cdot,\cdot} n^{\cdot} \subset A^{\cdot,\cdot}(N/\Gamma'') \) induces an isomorphism
\[ H^*_{\bar{\partial}}(n) \cong H^*_{\bar{\partial}}(N/\Gamma''). \]

Let \( \alpha : \mathbb{C}^n \to \mathbb{C}^* \) be a character (i.e. a representation on 1-dimensional vector space \( \mathbb{C}_\alpha \)) of \( \mathbb{C}^n \). By the projection \( \mathbb{C}^n \ltimes_{\phi} N \to \mathbb{C}^n \), we regard \( \alpha \) as a character of \( G \) as in Assumption 1.1. We consider the holomorphic line bundle \( L_\alpha = (G \times \mathbb{C}_\alpha)/\Gamma \) and the Dolbeault complex \( (A^{\cdot,*}(G/\Gamma, L_\alpha), \bar{\partial}) \) with values in the line bundle \( L_\alpha \). Let \( \mathcal{L} \) be the set of isomorphism classes of line bundles over \( G/\Gamma \) given by characters of \( \mathbb{C}^n \). We consider the direct sum
\[ \bigoplus_{L_\beta \in \mathcal{L}} A^{\cdot,*}(G/\Gamma, L_\beta) \]

of Dolbeault complexes. Then by the wedge products and the tensor products, the direct sum
\[ \bigoplus_{L_\beta \in \mathcal{L}} A^{\cdot,*}(G/\Gamma, L_\beta) \]

is a DBA.

Theorem 1.2. There exists a subDBA \( A^{\cdot,*} \) of
\[ \bigoplus_{L_\beta \in \mathcal{L}} A^{\cdot,*}(G/\Gamma, L_\beta) \]

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such that we have a DBA isomorphism \( \iota : \Lambda^*+ (a \oplus n)^* \cong A^*+ \) and the inclusion
\[
\Phi : A^*+ \to \bigoplus_{L_\beta \in \mathcal{L}} A^*+(G/\Gamma, L_\beta)
\]
induces a cohomology isomorphism.

See Section 3 for the construction of \( A^*+ \).

**Corollary 1.3.** We have the finite dimensional subDBA \( B^*+ = \Phi^{-1}(A^*+(G/\Gamma)) \) of \( A^*+(G/\Gamma) \) such that the inclusion \( \Phi : B^*+ \to A^*+(G/\Gamma) \) induces a cohomology isomorphism
\[
H^*+ (B^*+) \cong H^*+ (G/\Gamma).
\]

See Corollary 2.2 for the construction of \( B^*+ \).

**Remark 1.** Let \( N \) be a simply connected nilpotent Lie group with a lattice \( \Gamma'' \) and a left-invariant complex structure \( J \). Like Nomizu’s theorem (8) for the de Rham cohomology of nilmanifolds, it is known that an isomorphism \( H^*+ (N/\Gamma'') \cong H^*+ (n) \) holds if \( (N, J, \Gamma'') \) meet one of the following conditions:

(N) The complex manifold \( N/\Gamma'' \) has the structure of an iterated principal holomorphic torus bundle (3).

(Q) \( J \) is a small deformation of a rational complex structure i.e. for the rational structure \( n_Q \subset n \) of the Lie algebra \( n \) induced by a lattice \( \Gamma'' \) (see [11, Section 2]) we have \( J(n_Q) \subset n_Q \) (2).

(C) \( (N, J) \) is a complex Lie group ([13]).

By using Corollary 1.3 we actually compute the Dolbeault cohomology of some examples in Section 5. Unlike nilmanifolds, we observe that in many cases the Dolbeault cohomology of solvmanifolds can not be completely computed by using only Lie algebras. Moreover we give examples of non-Kähler complex solvmanifolds with the Hodge symmetry.

**Remark 2.** If \( N \) has a nilpotent complex structure (see [3]), then \((\Lambda^*+ (a \oplus n)^*, \partial)\) is the minimal model of the DBA \( \bigoplus_{L_\beta \in \mathcal{L}} A^*+(G/\Gamma, L_\beta)\) (see [3]).

### 2. Holomorphic Line Bundles over Complex Tori

**Lemma 2.1.** Let \( \Gamma \) be a finitely generated free abelian group and \( \alpha : \Gamma \to \mathbb{C}^* \) a character of \( \Gamma \). If the character \( \alpha \) is non-trivial, then we have \( H^* (\Gamma, C_\alpha) = 0 \).

**Proof.** First we assume \( \Gamma \cong \mathbb{Z} \). Then we have
\[
H^0 (\mathbb{Z}, C_\alpha) = \{ m \in C_\alpha | \alpha(g)m = m, \text{ for all } g \in \mathbb{Z} \} = 0.
\]
Like the de Rham cohomology of \( S^1 \), by the Poincaré duality we have
\[
H^1 (\mathbb{Z}, C_\alpha) \cong H^0 (\mathbb{Z}, C_{\alpha^{-1}})^* = 0,
\]
and obviously \( H^p (\mathbb{Z}, C_\alpha) = 0 \) for \( p \geq 2 \). Hence the lemma holds in this case. In general case, we consider a decomposition \( \Gamma = A \oplus B \) such that \( A \) is a rank \( 1 \) subgroup and the restriction of \( \alpha \) on \( A \) is also non-trivial. Then we have the Hochshild-Serre spectral sequence \( E_r \) such that
\[
E_2^{p, q} = H^p (\Gamma / A, H^q (A, C_\alpha))
\]
and this converges to \( H^{p+q} (\Gamma, C_\alpha) \). Since \( H^q (A, C_\alpha) = 0 \) for any \( q \), we have \( E_2 = 0 \) and hence the lemma follows. \( \square \)

We consider a complex vector space \( \mathbb{C}^n \) with a lattice \( \Gamma \). Let \( \alpha : \mathbb{C}^n \to \mathbb{C}^* \) be a \( C^\infty \)-character of \( \mathbb{C}^n \). We have the holomorphic line bundle \( L_\alpha = (\mathbb{C}^n \times C_\alpha) / \Gamma \) over the complex torus \( \mathbb{C}^n / \Gamma \). We define the equivalence relation on the space of \( C^\infty \)-characters of \( \mathbb{C}^n \) such that \( \alpha \sim \beta \) if \( \alpha \beta^{-1} \) is holomorphic.

**Lemma 2.2.** Let \( \alpha : \mathbb{C}^n \to \mathbb{C}^* \) be a \( C^\infty \)-character of \( \mathbb{C}^n \). There exists a unique unitary character \( \beta \) such that \( \alpha \sim \beta \).
Proof. For a coordinate \((x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n) \in \mathbb{C}^n\), a character \(\alpha\) is written as
\[
\alpha(x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n) = \exp\left(\sum_{i=1}^n (a_i x_i + b_i y_i + \sqrt{-1}(c_i x_i + d_i y_i))\right)
\]
for some \(a_i, b_i, c_i, d_i \in \mathbb{R}^n\). Let \(\alpha'\) be the holomorphic character defined by
\[
\alpha'(x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n) = \exp\left(\sum_{i=1}^n (-a_i x_i + \sqrt{-1}y_i) + \sqrt{-1}b_i (x_i + \sqrt{-1}y_i))\right).
\]
Then the character \(\beta = \alpha \alpha'\) is unitary. If a unitary character is holomorphic, then it is trivial. Hence such \(\beta\) is unique. \(\square\)

Lemma 2.3. Let \(\beta : \mathbb{C}^n \to \mathbb{C}^*\) be a unitary \(C^\infty\)-character of \(\mathbb{C}^n\). Then the holomorphic line bundle \(L_\beta\) is trivial if and only if the restriction of \(\beta\) on \(\Gamma\) is trivial.

Proposition 2.4. Let \(\alpha : \mathbb{C}^n \to \mathbb{C}^*\) be a \(C^\infty\)-character of \(\mathbb{C}^n\). If \(L_\alpha\) is a non-trivial holomorphic line bundle, then the Dolbeault cohomology \(H^{\ast, 
\ast}_{\beta}(\mathbb{C}^n/\Gamma, L_\alpha)\) with values in the line bundle \(L_\alpha\) is trivial.

Proof. Let \(\beta\) be the unitary character such that \(\alpha \sim \beta\) as in Lemma 2.2. Then we have \(L_\alpha \cong L_\beta\). Let \(D\) be the flat connection on \(L_\beta\) induced by \(\beta\). We have the decomposition \(D = \bar{\partial} + \partial\) so that \(\bar{\partial}\) is the Dolbeault operator on \(L_\beta\). Since \(\beta\) is unitary, we have a Hermitian metric on \(L_\beta\) such that for a Kähler metric on \(\mathbb{C}^n/\Gamma\) we have the standard identity of the Laplacians of \(D\) and \(\bar{\partial}\) (see \(\cite{[4]}\) Section 7). Hence we have an isomorphism \(H^{\ast, 
\ast}_{\beta}(\mathbb{C}^n/\Gamma, L_\beta) \cong H^{\ast, 
\ast}_{\beta}(\mathbb{C}^n/\Gamma, L_\beta)\). If \(L_\beta\) is non-trivial, then we have \(H^{\ast, 
\ast}_{\beta}(\mathbb{C}^n/\Gamma, L_\beta) = H^{\ast}(\Gamma, \mathbb{C}_\beta) = 0\) by Lemma 2.1. Hence the proposition follows. \(\square\)

3. The construction of \(A^{\ast, 
\ast}\)

Let \(G\) be a Lie group as in Assumption 1.1. Consider the decomposition \(n_\mathbb{C} = n^{1,0} \oplus n^{0,1}\). By the condition (2), this decomposition is a direct sum of \(\mathbb{C}^n\)-modules. By the condition (3) we have a basis \(Y_1, \ldots, Y_m\) of \(n^{1,0}\) such that the action \(\phi\) on \(n^{1,0}\) is represented by \(\phi(t) = \text{diag}(\alpha_1(t), \ldots, \alpha_m(t))\). Since \(Y_j\) is a left-invariant vector field on \(N\), the vector field \(\alpha_j Y_j\) on \(\mathbb{C}^n \ltimes _\phi N\) is left-invariant. Hence we have a basis \(X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\) of \(g^{1,0}\). Let \(x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m\) be the basis of \(A^{1,0}\) which is dual to \(X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\). Then we have
\[
\bigwedge^{p,q} g^* = \bigwedge^p \langle x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m \rangle \otimes \bigwedge^q \langle \bar{x}_1, \ldots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \ldots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle.
\]

Let \(\alpha : \mathbb{C}^n \to \mathbb{C}^*\) be a character of \(\mathbb{C}^n\). Let \((A^{\ast, 
\ast}(G) \otimes \mathbb{C}_\alpha)\Gamma\) be the space of \(\mathbb{C}_\alpha\)-valued \(\Gamma\)-invariant differential forms on \(G\). Then we can identify the Dolbeault complex \(A^{\ast, 
\ast}(G/\Gamma, L_\alpha)\) with \((A^{\ast, 
\ast}(G) \otimes \mathbb{C}_\alpha)\Gamma\). Hence for \(\omega \in \bigwedge^{p,q} g^*\) and \(v_\alpha \in \mathbb{C}_\alpha\), we have
\[
\omega \otimes (\alpha^{-1} v_\alpha) \in A^{\ast, 
\ast}(G/\Gamma, L_\alpha).
\]

Let \(L\) be the set as in Introduction. By Section 2, we can regard \(L\) as the set of isomorphism classes of line bundles over \(G/\Gamma\) given by unitary characters of \(\mathbb{C}^n\). We consider the DBA \(\bigoplus_{L_\alpha \in L} A^{\ast, 
\ast}(G/\Gamma, L_\alpha)\). We define the DBA \(A^{\ast, 
\ast}\) to prove Theorem 1.2.

Definition 3.1. Let \(x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m\) be the basis of \(A^{1,0}\) as above. By Lemma 2.2, we have the unitary character \(\beta_j\) such that \(\alpha_j \sim \beta_j\). We consider the holomorphic line bundles \(L_{\beta_j^{-1}}\) over \(G/\Gamma\). By \(A^{\ast, 
\ast}(G/\Gamma, L_{\beta_j^{-1}}) = (A^{\ast, 
\ast}(G) \otimes \mathbb{C}_{\beta_j^{-1}})\Gamma\), for \(\mathbb{C}_{\beta_j^{-1}} \ni v_{\beta_j^{-1}} \neq 0\) we consider
\[
\alpha_j^{-1} y_j \otimes (\beta_j v_{\beta_j^{-1}}) \in A^{\ast, 
\ast}(G/\Gamma, L_{\beta_j^{-1}}).
\]
Let $A^{*,*}$ be the subDBA of $\bigoplus_{m \in \mathbb{L}} A^{*,*}(G/\Gamma, L_\alpha)$ defined by

$$A^{p,q} = \bigwedge^p \langle x_1, \ldots, x_n, \alpha_1^{-1} y_1 \otimes (\beta_1 v_{\beta_1}), \ldots, \alpha_m^{-1} y_m \otimes (\beta_m v_{\beta_m}) \rangle$$

$$\bigotimes \bigwedge^q \langle \bar{x}_1, \ldots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1 \otimes (\gamma_1 v_{\gamma_1}), \ldots, \bar{\alpha}_m^{-1} \bar{y}_m \otimes (\gamma_m v_{\gamma_m}) \rangle.$$

**Lemma 3.2.** Let $\iota : \bigwedge^{*,*}(a \oplus n)^* \rightarrow A^{*,*}$ be the algebra homomorphism defined by

$$\iota(x_i) = x_i,$$

$$\iota(\alpha_j^{-1} y_j) = \alpha_j^{-1} y_j \otimes \beta_j v_{\beta_j^{-1}}.$$

Then we have a DBA isomorphism

$$\iota : \bigwedge^{*,*}(a \oplus n)^* \cong A^{*,*}.$$

**Proof.** Since $\alpha_j^{-1} \beta_j$ is holomorphic, we have

$$\bar{\partial} (\alpha_j^{-1} y_j \otimes \beta_j v_{\beta_j^{-1}}) = \alpha_j^{-1} (\bar{\partial} y_j) \otimes \beta_j v_{\beta_j^{-1}}.$$

This implies $\bar{\partial} \circ \iota = \iota \circ \bar{\partial}$. Hence the lemma follows. $\square$

Let $g$ be the left-invariant Hermitian metric on $G$ defined by

$$g = x_1 \bar{x}_1 + \cdots + x_n \bar{x}_n + \alpha_1^{-1} \alpha_1^{-1} y_1 \bar{y}_1 + \cdots + \alpha_m^{-1} \alpha_m^{-1} y_m \bar{y}_m.$$

Let $\beta : \mathbb{C}^n \rightarrow \mathbb{C}^*$ be a unitary $C^\infty$-character of $\mathbb{C}^n$. Take $C_\beta \ni v_\beta \neq 0$. Then $\beta^{-1} v_\beta$ is a $C^\infty$-frame of the line bundle $L_\beta = (G \times \mathbb{C}^n)/\Gamma$. We define the Hermitian metric $h_\beta$ on $L_\beta$ such that $h_\beta(\beta^{-1} v_\beta, \beta^{-1} v_\beta) = 1$. Let $\bar{g}_{h_\beta} : A^{p,q}(G/\Gamma, L_\beta) \rightarrow A^{n+p, n+q}(G/\Gamma, L_\beta)$ be the $\mathbb{C}$-anti-linear Hodge star operator of $g \otimes h_\beta$ on $A^{*,*}(G/\Gamma, L_\beta)$

$$\bar{\delta} = \bar{g}_{h_\beta} \circ \bar{\partial} \circ \bar{g}_{h_\beta}, \quad \square_{g_{h_\beta}} = \bar{\delta} \bar{\delta} + \bar{\delta} \bar{\delta}$$

and

$$\mathcal{H}^{p,q}(G/\Gamma, L_\beta) = \{ \omega \in A^{*,*}(G/\Gamma, L_\beta) | \square_{g_{h_\beta}} \omega = 0 \}.$$

We consider the $\bar{\delta}$-Laplace operator $\square_{g_{h_\beta}}$ on the direct sum $\bigoplus_{L_\beta \in \mathbb{L}} A^{*,*}(G/\Gamma, L_\beta)$.

Let us consider the basis $x_1, \ldots, x_n, y_1, \ldots, y_m$ of $\bigwedge^{1,*}(a \oplus n)^*$. Let $g'$ be the Hermitian metric on $a \oplus n$ defined by

$$g' = x_1 \bar{x}_1 + \cdots + x_n \bar{x}_n + y_1 \bar{y}_1 + \cdots + y_m \bar{y}_m.$$

Let $\bar{g}' : \bigwedge^{p,q}(a \oplus n)^* \rightarrow \bigwedge^{n+p, n+q}(a \oplus n)^*$ be the $\mathbb{C}$-anti-linear Hodge star operator of $g'$ on $\bigwedge^{*,*}(a \oplus n)^*$ and let

$$\bar{\delta} = \bar{g}' \circ \bar{\partial} \circ \bar{g}'', \quad \square_g = \bar{\delta} \bar{\delta} + \bar{\delta} \bar{\delta}$$

and

$$\mathcal{H}^{p,q}(a \oplus n) = \{ \omega \in \bigwedge^{*,*}(a \oplus n)^* | \square_g \omega = 0 \}.$$

**Lemma 3.3.** We consider the isomorphism $\iota : \bigwedge^{*,*}(a \oplus n)^* \cong A^{*,*}$ as in Lemma 3.2. Then we have

$$\iota \circ \square_g = (\square_{g_{h_\beta}}) \circ \iota.$$

**Proof.** Let $\oplus_{g_{h_\beta}}$ be the Hodge star operator on $\bigoplus_{L_\beta \in \mathbb{L}} A^{*,*}(G/\Gamma, L_\beta)$. It is sufficient to show

$$\iota \circ \bar{g}' = (\oplus_{g_{h_\beta}}) \circ \iota.$$

For a multi-index $I = \{i_1, \ldots, i_r\}$, we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_r}$, $y_I = y_{i_1} \wedge \cdots \wedge y_{i_r}$, $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_r}$, and $\beta_I = \beta_{i_1} \cdots \beta_{i_r}$. For multi-indices $I, K \subset \{1, \ldots, n\}$ and $J, L \subset \{1, \ldots, m\}$, we have

$$\bar{g}'(x_I \wedge y_J \wedge \bar{x}_K \wedge \bar{y}_L) = \epsilon_{x_I} \wedge y_{I'} \wedge \bar{x}_{K'} \wedge \bar{y}_{L'}$$
where $I', J', K'$ and $L'$ are complements and $\epsilon$ is the sign of a permutation. We also have

\[ \oplus \, \hat{g} \otimes h_n(x_I \wedge \alpha_j^{-1} y_J \wedge \bar{x}_K \wedge \bar{\alpha}_L^{-1} \bar{y}_L \otimes \beta_J \gamma_L v_{\beta_j^{-1} \gamma_L^j}) \]

\[ = \epsilon x_{I'} \wedge \alpha_j^{-1} y_{J'} \wedge \bar{x}_{K'} \wedge \bar{\alpha}_L^{-1} \bar{y}_{L'} \otimes \beta_{J'} \gamma_{L'} v_{\beta_j \gamma_L}. \]

Hence we only need to show

\[ \beta_j^{-1} \gamma_L^{-1} = \beta_{J'} \gamma_{L'}. \]

Since a Lie group with a lattice is unimodular (see [11, Remark 1.9]), the action $\phi$ on $n$ is represented by unimodular matrices. Hence we have $\alpha_j \bar{\alpha}_L \alpha_{J'} \bar{\alpha}_{L'} = 1$. This implies $\beta_j^{-1} \gamma_L^{-1} = \beta_{J'} \gamma_{L'}$. Hence the lemma follows. \qed

**Corollary 3.4.** The inclusion

\[ \Phi : A^{*,*} \to {} \bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta) \]

induces an injection

\[ H^p_q(a \oplus n) \cong H^p_q(A^{*,*}) \to H^p_q(G/\Gamma, L_\beta). \]

**Proof.** We have isomorphisms $H^p_q(G/\Gamma, L_\beta) \cong H^p_q(G/\Gamma, L_\beta)$ and $H^p_q(a \oplus n) \cong H^p_q(a \oplus n)$ (see [12]). By Lemma 3.3 we have

\[ \iota(H^p_q(a \oplus n)) \subset \bigoplus_{L_\beta \in \mathcal{L}} H^p_q(G/\Gamma, L_\beta). \]

Hence the corollary follows. \qed

4. PROOF OF THE MAIN THEOREM

**Proposition 4.1.** Let $G$ be a Lie group as in Assumption 1.1. $G/\Gamma$ is a holomorphic fiber bundle over a torus with a nilmanifold as a fiber,

\[ N/\Gamma'' \to G/\Gamma \to \mathbb{C}^n/\Gamma' \]

such that the structure group of this fibration is discrete.

**Proof.** Consider the covering $\mathbb{C}^n \times (N/\Gamma'') \to G/\Gamma$ such that the covering transformation is the action of $\Gamma'$ on $\mathbb{C}^n \times (N/\Gamma'')$ given by $g \cdot (a, b) = (a + g, \phi(g)b)$. Hence we have the fiber bundle $G/\Gamma \to \mathbb{C}^n/\Gamma'$ with the fiber $N/\Gamma''$ and the discrete structure group $\phi(\Gamma') \subset \text{Aut}(N)$. Since $\phi(g)$ is a holomorphic automorphism, this fiber bundle is holomorphic. \qed

**Proof of Theorem 1.2** For $L_\beta \in \mathcal{L}$, by Borel’s results in [7, Appendix 2], we have the spectral sequence $(E_r, d_r)$ of the filtration of $A^{p,q}(G/\Gamma, L_\beta)$ induced by the holomorphic fiber bundle $p : G/\Gamma \to \mathbb{C}^n/\Gamma'$ as in Proposition 1.1 such that:

1. $E_r$ is $4$-graded, by the fiber-degree, the base-degree and the type. Let $p,q E_{r}^{s,t}$ be the subspace of elements of $E_r$ of type $(p, q)$, fiber-degree $s$ and base-degree $t$. We have $p,q E_{r}^{s,t} = 0$ if $p + q = s + t$ or if one of $p, q, s, t$ is negative.
2. If $p + q = s + t$, then we have

\[ p,q E_{2}^{s,t} \cong \bigoplus_{i \geq 0} H_\beta^{i,i-s}(\mathbb{C}^n/\Gamma', L_\beta \otimes H^{p-i,q-s+i}(N/\Gamma'')) \]

where $H^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic fiber bundle $\bigcup_{b \in \mathbb{C}^n/\Gamma'} \text{H}^{p-i,q-s+i}(p^{-1}(b))$.
3. The spectral sequence converges to $H_\beta(G/\Gamma, L_\beta)$.

By the assumption $H_\beta^{*,*}(n) \cong H_\beta^{*,*}(N/\Gamma'')$, the fiber bundle $H^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic vector bundle with the fiber $H_\beta^{p-i,q-s+i}(n)$ induced by the action $\phi$ of $\Gamma$ on $H_\beta^{p-i,q-s+i}(n)$.  

Since the action $\phi$ on $n$ is semi-simple, the action of $\mathbb{C}^n$ on $H^{p-i,q-s+i}_\delta(n)$ induced by $\phi$ is diagonalizable. The fiber bundle splits as $H^{p-i,q-s+i}(N/\Gamma') = \oplus \delta_j$ for some $\delta_j \in \mathcal{L}$. Hence we have

$$H^{i,i-s}_\delta(\mathbb{C}^n/\Gamma', \bigoplus_{\delta_j \in \mathcal{L}} \mathbb{C} \otimes L_{\delta_j} \otimes L_{\delta_j}).$$

By Proposition 2.4, we have $\bigoplus_{\delta_j \in \mathcal{L}}\mathbb{C} \otimes H^{p-i,q-s+i}(N/\Gamma') = H^{i,i-s}_\delta(\mathbb{C}^n/\Gamma', \bigoplus_{\delta_j \in \mathcal{L}} \mathbb{C} \otimes L_{\delta_j} \otimes L_{\delta_j}).$

For the direct sum $\bigoplus_{\delta_j \in \mathcal{L}} A^* \ast (G/\Gamma, L_{\delta_j})$, we consider this spectral sequence $E_r$. Then we have

$$t \geq 0 \sum_{i \geq 0} H^{i,i-s}_\delta(\mathbb{C}^n/\Gamma', \bigoplus_{\delta_j \in \mathcal{L}} \mathbb{C} \otimes H^{p-i,q-s+i}(N/\Gamma')) \cong \sum_{i \geq 0} H^{i,i-s}_\delta(\mathbb{C}^n/\Gamma') \otimes H^{p-i,q-s+i}_\delta(n).$$

This implies an isomorphism $E_2 \cong \bigoplus_{p,q} H^{p,q}_\delta(a \oplus n)$. On the other hand, by Corollary 3.4, we have an injection

$$H^{p,q}_\delta(a \oplus n) \to H^{p,q}_\delta\left( \bigoplus_{\delta_j \in \mathcal{L}} A^* \ast (G/\Gamma, L_{\delta_j}) \cong E_\infty. $$

Hence the spectral sequence degenerates at $E_2$ and the theorem follows. \qed

**Corollary 4.2.** Let $B^* \ast A^* \ast (G/\Gamma)$ be the subDBA of $A^* \ast (G/\Gamma)$ given by

$$B^{p,q} = \left\{ x_I \wedge \alpha_I^{-1} \beta_J y_J \wedge \bar{x}_K \wedge \alpha_L^{-1} y_L \bar{y}_L | |I| + |K| = p, |J| + |L| = q \right\}$$

The restriction of $\beta_J \gamma_L$ on $\Gamma$ is trivial. Then the inclusion $B^* \ast A^* \ast (G/\Gamma)$ induces a cohomology isomorphism

$$H^{* \ast}_\delta(B^* \ast) \cong H^{* \ast}_\delta(G/\Gamma).$$

**Proof.** By Lemma 2.3

$$\Phi(x_I \wedge \alpha_I^{-1} y_J \wedge \bar{x}_K \wedge \alpha_L^{-1} y_L \bar{y}_L \otimes \beta_J \gamma_L y_L \bar{y}_L) \in A^{* \ast}(G/\Gamma)$$

if and only if the restriction of $\beta_J \gamma_L$ on $\Gamma$ is trivial. Hence we have $\Phi^{-1}(A^{* \ast}(G/\Gamma)) = B^* \ast. \qed$

**Remark 3.** Suppose $\phi : \mathbb{C}^n \to \text{Aut}(n^{1,0})$ is a holomorphic map. Since each $\alpha_j$ is holomorphic, $\beta_j$ is trivial. Hence we have $B^{n,0} \ast = \bigwedge^{p,0} \mathfrak{g}^*$. Moreover if $N$ is a complex Lie group, then $G = \mathbb{C}^n \ltimes \phi N$ is also a complex Lie group and any element of $B^{1,0} = \mathfrak{g}^{1,0}$ is holomorphic and hence $\bar{\partial} B^{0,0} = 0$. Hence we have an isomorphism

$$H^{p,q}(G/\Gamma) \cong \bigwedge^p \mathfrak{g}^{1,0} \otimes H^q_\delta(B^{0,q}).$$

**Remark 4.** We suppose the following condition:

(*) For multi-indices $J, L$, if the restriction of $\beta_J \gamma_L$ on $\Gamma$ is trivial, then $\beta_J \gamma_L$ itself is trivial. Then we have $B^* \ast \subset \bigwedge^* \mathfrak{g}^*$ and hence we have an isomorphism

$$H^{* \ast}_\delta(\mathfrak{g}) \cong H^{* \ast}_\delta(G/\Gamma).$$
5. Examples

5.1. Example 1. Let $G = \mathbb{C} \ltimes \mathbb{C}^2$ such that $\phi(x + \sqrt{-1}y) = \left( \begin{array}{cc} e^x & 0 \\ 0 & e^{-x} \end{array} \right)$. Then for some $a \in \mathbb{R}$ the matrix

$$
\begin{pmatrix}
e^a & 0 \\
0 & e^{-a}
\end{pmatrix}
$$

is conjugate to an element of $SL(2, \mathbb{Z})$. Hence for any $0 \neq b \in \mathbb{R}$ we have a lattice $\Gamma = (a\mathbb{Z} + b\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ such that $\Gamma''$ is a lattice of $\mathbb{C}^2$. Then for a coordinate $(z_1 = x + \sqrt{-1}y, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$ we have

$$
\bigwedge^{p,q} B^* = \bigwedge^{(d\bar{z}_1, e^{-x}d\bar{z}_2, e^x d\bar{z}_3) \otimes (d\bar{z}_1, e^{-x}d\bar{z}_2, e^x d\bar{z}_3)}.
$$

Since we have $e^x \sim e^{-\sqrt{-1}y}$, the subDBA

$$A^{*,*} \subset \bigoplus_{L_\alpha \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

as in Definition 3.1 is given by

$$A^{p,q} = \bigwedge^{(d\bar{z}_1, e^{-x}d\bar{z}_2, e^{-x}d\bar{z}_3) \otimes (d\bar{z}_1, e^{-x}d\bar{z}_2, e^x d\bar{z}_3)} \bigotimes (d\bar{z}_1, e^{-x}d\bar{z}_2, e^{-x}d\bar{z}_3 \otimes e^x d\bar{z}_3 \otimes e^{-x}d\bar{z}_3).
$$

$B^{p,q} \subset A^{p,q}(G/\Gamma)$ varies for a choice of $b \in \mathbb{R}$ as the following.

(A) If $b = 2n\pi$ for $n \in \mathbb{Z}$, then we have:

$$B^{p,q} = \bigwedge^{(d\bar{z}_1, e^{-x}d\bar{z}_2, e^{-x}d\bar{z}_3) \otimes (d\bar{z}_1, e^{-x}d\bar{z}_2, e^x d\bar{z}_3)}.
$$

(B) If $b = (2n - 1)\pi$ for $n \in \mathbb{Z}$, then we have:

$$B^{1,0} = \langle d\bar{z}_1 \rangle, \quad B^{0,1} = \langle d\bar{z}_1 \rangle,
$$

$$B^{2,0} = \langle dz_2 \wedge dz_3 \rangle, \quad B^{0,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3 \rangle,
$$

$$B^{1,1} = \langle d\bar{z}_1 \wedge d\bar{z}_1, e^{-2x-2\sqrt{-1}y}d\bar{z}_2 \wedge d\bar{z}_2, e^{2x+2\sqrt{-1}y}d\bar{z}_3 \wedge d\bar{z}_3, dz_2 \wedge d\bar{z}_3, dz_2 \wedge d\bar{z}_3, dz_3 \wedge d\bar{z}_3 \rangle,
$$

$$B^{3,0} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 \rangle,
$$

$$B^{2,1} = \langle d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_1, e^{-2x-2\sqrt{-1}y}d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2, e^{2x+2\sqrt{-1}y}d\bar{z}_3 \wedge d\bar{z}_3, dz_2 \wedge d\bar{z}_3, dz_2 \wedge d\bar{z}_3, dz_3 \wedge d\bar{z}_3 \rangle,
$$

$$B^{1,2} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, e^{-2x+2\sqrt{-1}y}d\bar{z}_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2, e^{2x+2\sqrt{-1}y}d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_1, d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_1 \rangle,
$$

$$B^{3,1} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \rangle, \quad B^{1,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \rangle,
$$

$$B^{2,2} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, e^{-2x-2\sqrt{-1}y}d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2, e^{2x+2\sqrt{-1}y}d\bar{z}_1 \wedge d\bar{z}_3 \wedge d\bar{z}_3, d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_1, d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_1 \rangle,
$$

$$B^{3,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_3 \wedge d\bar{z}_3 \rangle, \quad B^{2,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,
$$

$$B^{3,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.
$$

(C) If $b \neq n\pi$ for any $n \in \mathbb{Z}$, then we have:

$$B^{1,0} = \langle d\bar{z}_1 \rangle, \quad B^{0,1} = \langle d\bar{z}_1 \rangle,
$$

$$B^{2,0} = \langle d\bar{z}_2 \wedge dz_3 \rangle, \quad B^{0,2} = \langle d\bar{z}_2 \wedge dz_3 \rangle,
$$

$$B^{1,1} = \langle d\bar{z}_1 \wedge d\bar{z}_1, d\bar{z}_2 \wedge d\bar{z}_3, dz_2 \wedge dz_3 \rangle,
$$

$$B^{3,0} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 \rangle, \quad B^{2,1} = \langle d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_1, d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.
cases, this isomorphism does not hold.

By Corollary 4.2 for each case we have an isomorphism $H^p_q(G/\Gamma) \cong B^{p,q}$. Moreover considering the left-invariant Hermitian metric $g = d\bar{z}_1 d\bar{z}_1 + e^{-2z} d\bar{z}_2 d\bar{z}_2 + e^{2z} d\bar{z}_3 d\bar{z}_3$, we have $\mathcal{H}^{p,q}(G/\Gamma) \cong B^{p,q}$.

**Remark 5.** In the case (A), the Dolbeault cohomology $H^{p,q}_\partial(G/\Gamma)$ is isomorphic to the Dolbeault cohomology of complex 3-torus. But $G/\Gamma$ is not homeomorphic to a complex 3-torus. Moreover considering the metric $\varphi$, the space of the harmonic forms does not satisfy Hodge symmetry (i.e. $\mathcal{H}^{p,q}(G/\Gamma) \neq \mathcal{H}^{q,p}(G/\Gamma)$).

**Remark 6.** By Hattori’s result in [6], we have an isomorphism $H^*(G/\Gamma) \cong H^*(g)$ of the de Rham cohomology of $G/\Gamma$ and the Lie algebra cohomology. Hence considering the space $H^\text{L}(G/\Gamma)$ of left-invariant $d$-harmonic forms of the left-invariant Hermitian metric $\varphi$, we have $\mathcal{H}^\text{L}_\partial(G/\Gamma) \cong \mathcal{H}^\text{L}_\partial(G/\Gamma)$. By simple computations, in the case (C) we have the Hodge decomposition $\mathcal{H}^\text{L}_\partial(G/\Gamma) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(G/\Gamma)$. Hence $G/\Gamma$ has cohomological properties (for example the Fröhlicher spectral sequence degenerates at $E_1$) of compact Kähler manifolds. But by Arapura’s result (solving Benson-Gordon’s conjecture) in [1], $G/\Gamma$ admits no Kähler structure.

**Remark 7.** In the case (C), an isomorphism $H^{p,q}_\partial(G/\Gamma) \cong H^{p,q}_\partial(G/\Gamma)$ holds. But in the other cases, this isomorphism does not hold.

**5.2. Example 2.** Let $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^2$ such that

$$\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}.$$ 

Then we have $a + \sqrt{-1}b, c + \sqrt{-1}d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}b)$ and $\phi(c + \sqrt{-1}d)$ are conjugate to elements of $SL(4, \mathbb{Z})$ where we regard $SL(2, \mathbb{C}) \subset SL(4, \mathbb{R})$ (see [5]). Hence we have a lattice $\Gamma = (\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)) \ltimes_\varphi \Gamma''$ such that $\Gamma''$ is a lattice of $\mathbb{C}^2$. For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$, we have

$$\bigwedge_{p,q} g^* = \bigwedge_{p,q} (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3, d\bar{z}_2, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3).$$

We have

$$A^{p,q} = \bigwedge_{p,q} (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3) \otimes (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3).$$

for $z_1 = x_1 + \sqrt{-1}y_1$.

If $b, d \in \pi \mathbb{Z}$, then we have

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge_{p,q} (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3) \otimes (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3).$$

If $b \notin \pi \mathbb{Z}$ or $c \notin \pi \mathbb{Z}$, then we have

$$B^{0,1} = (d\bar{z}_1), B^{0,2} = (d\bar{z}_2 \wedge d\bar{z}_3), B^{0,3} = (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3)$$

and

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge_{p} (d\bar{z}_1, e^{-z_1} d\bar{z}_2, e^{z_1} d\bar{z}_3) \otimes B^{0,q}.$$
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