T-product Tensors—Part II: Tail Bounds for Sums of Random T-product Tensors

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Abstract

This paper is the Part II of a serious work about T-product tensors focusing at establishing new probability bounds for sums of random, independent, T-product tensors. These probability bounds characterize large-deviation behavior of the extreme eigenvalue of the sums of random T-product tensors. We apply Laplace transform method and Lieb’s concavity theorem for T-product tensors obtained from our Part I paper, and apply these tools to generalize the classical bounds associated with the names Chernoff, and Bernstein from the scalar to the T-product tensor setting. Tail bounds for the norm of a sum of random rectangular T-product tensors are also derived from corollaries of random Hermitian T-product tensors cases. The proof mechanism is also applied to T-product tensor-valued martingales and T-product tensor-based Azuma, Hoeffding and McDiarmid inequalities are derived.

Index terms— random T-product tensors, T-product tensor Chernoff bound, T-product tensor Bernstein bound, T-product tensor-valued martingale, T-product tensor Azuma inequality, T-product tensor McDiarmid inequality.

1 Introduction

1.1 From Sums of Random Matrices to Sums of Random T-product Tensors

In probability theory and theoretical physics, a random matrix is a matrix-valued random variable—that is, a matrix with all entries as random variables. Many crucial physical phenomena can be modeled as random matrix problems. For example, random matrices were introduced by Eugene Wigner to model the nuclei of heavy atoms in nuclear physics [1]. Since then, random matrices have become ubiquitous in science and engineering applications. As this trend accelerates, more and more researchers have to integrate concepts from random matrices into their work. Classical random matrix theory can be difficult to apply, and it is necessary to invent new tools that are easy to use and that apply to a wide range of random matrices [2]. Tail bounds for sums of random matrices are among the most popular of these new tools. Tail bounds for sums of random matrices have already found various applications in science and engineering, including: combinatorics [3], numerical linear algebra [4], optimization [5], signal processing [6], and machine learning [7], etc.
The T-product operation between two three order tensors was introduced by Kilmer and her collaborators in \cite{8,9} to generalize the traditional matrix product. T-product operation has been demonstrated as an important mathematical framework in many fields: multilinear algebra \cite{10,13}, numerical linear algebra \cite{14}, signal processing \cite{15,16}, machine learning \cite{17}, image processing \cite{18}, computer vision \cite{19,20}, low-rank tensor approximation \cite{21,23} etc. However, all these applications assume that systems modelled by T-product tensors are deterministic and such assumption is not true and practical in solving T-product tensors associated issues. In recent years, there are more works begin to study random tensors, see \cite{24,25,26,27} and references therein.

In our Part I paper \cite{28}, we establish following inequalities about T-product tensors: (1) trace function nondecreasing/convexity; (2) Golden-Thompson inequality for T-product tensors; (3) Jensen’s T-product inequality; (4) Klein’s T-product inequality. All these inequalities are used to generalize celebrated Lieb’s concavity theorem from matrices to T-product tensors.

In this work, we will focus on establishing several new tail bounds for sums of random T-product tensors.

1.2 Tail Bounds Derived in This Paper

In this introduction section, we will highlight theorems about tail bounds for sums of random T-product tensors established in this paper. There are two categories of tail bounds discussed here: bounds for eigenvalue and bounds for eigentuples. For bounds related to eigentuples, there is a special condition to be satisfied for the T-product tensor whose eigentuple tail behavior is our interest.

Let $Y \in \mathbb{C}^{m \times m \times p}$ be a random T-positive definite (TPD) tensor and we say the tensor $Y$ satisfies Eq. (1.1) if the following inequality relation is valid for the tensor $Y$:

$$\frac{1}{p} \lambda_{\text{max}}^p(e^Y) + 1 - \frac{1}{p} \leq \text{Tr}(e^Y),$$

where $t > 0$. If we scale the random TPD tensor $Y$ as the $\lambda_{\text{max}}(e^Y) = 1$, then Eq. (1.1) always holds.

1.2.1 Tail Bounds for Sum of Hermitian T-product Tensors with Random Series

We extend normal-type tail bounds from scalers with Gaussian and Rademacher random series to T-product tensors with Gaussian and Rademacher random series. The tail bound for the maximum eigenvalue for the sum of Hermitian T-product tensors with Gaussian and Rademacher series is provided by the following Theorem \ref{thm:tail-bound}.

**Theorem 1.1 (Hermitian T-product Tensor with Gaussian and Rademacher Series Eigenvalue Version)**

Given a finite sequence of fixed T-product tensors $A_i \in \mathbb{C}^{m \times m \times p}$, and let $\{\alpha_i\}$ be a finite sequence of independent standard normal variables. We define

$$\sigma^2 \overset{def}{=} \left\| \sum_{i=1}^{n} A_i^2 \right\|,$$

then, for all $\theta \geq 0$, we have

$$\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{2\sigma^2}}.$$  \hfill (1.3)

We use $\|X\|$ for the spectral norm, which is the largest singular value for the T-product tensor $X$. Then, we have

$$\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq \theta \right) \leq 2mpe^{-\frac{\theta^2}{4\sigma^2}}.$$  \hfill (1.4)
This theorem is also valid for a finite sequence of independent Rademacher random variables \( \{ \alpha_i \} \).

The eigentuple version for T-product tensors with Gaussian and Rademacher random series is provided by the following Theorem 1.2. We use \( \| C \|_{\text{vec}} \) to represent the spectral norm of eigentuple of the tensor \( C \), which is defined as

\[
\| C \|_{\text{vec}} \defeq \max \left( \sqrt{C^H \ast C} \right) .
\]  

(1.5)

**Theorem 1.2 (Hermitian T-product Tensor with Gaussian and Rademacher Series Eigentuple Version)**

Given a finite sequence of Hermitian T-product tensors \( A_i \in \mathbb{C}^{m \times m \times p} \), and let \( \{ \alpha_i \} \) be a finite sequence of independent standard normal variables. We define

\[
\sigma^2 \defeq \left\| \sum_{i=1}^n \alpha_i A_i^2 \right\| ,
\]  

(1.6)

then, for all \( b \geq 0 \) and \( \sum_{i=1}^n t \alpha_i A_i \) satisfying Eq. (1.1) for \( t > 0 \), we have

\[
\Pr \left( \| \sum_{i=1}^n \alpha_i A_i \|_{\text{vec}} \geq b \right) \leq mpe^{-\frac{b^2}{2\sigma^2}} ,
\]

(1.7)

where \( \tilde{j} \defeq \arg\min_j \{ b_j \} \). And

\[
\Pr \left( \left\| \sum_{i=1}^n \alpha_i A_i \right\|_{\text{vec}} \geq b \right) \leq 2mpe^{-\frac{b^2}{2\sigma^2}} .
\]  

(1.8)

This theorem is also valid for a finite sequence of independent Rademacher random variables \( \{ \alpha_i \} \).

### 1.2.2 Chernoff Inequalities about T-product Tensors

Next, we will extend Chernoff bounds of random variables to random T-product tensors.

**Theorem 1.3 (T-product Tensor Chernoff Bound 1)** Consider a sequence \( \{ X_i \in \mathbb{C}^{m \times m \times p} \} \) of independent, random, Hermitian T-product tensors that satisfy

\[
X_i \succeq \mathcal{O} \quad \text{and} \quad \lambda_{\max}(X_i) \leq 1 \quad \text{almost surely}.
\]

(1.9)

Define following two quantities:

\[
\overline{\mu}_{\max} \defeq \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^n E X_i \right) \quad \text{and} \quad \overline{\mu}_{\min} \defeq \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^n E X_i \right) ,
\]

(1.10)

then, we have following two inequalities:

\[
\Pr \left( \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^n X_i \right) \geq \theta \right) \leq mpe^{-nD(\theta|\overline{\mu}_{\max})} , \quad \text{for} \quad \overline{\mu}_{\max} \leq \theta \leq 1;
\]

(1.11)

and

\[
\Pr \left( \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^n X_i \right) \leq \theta \right) \leq mpe^{-nD(\theta|\overline{\mu}_{\min})} , \quad \text{for} \quad 0 \leq \theta \leq \overline{\mu}_{\min}.
\]

(1.12)
The other version of T-product tensor Chernoff bound by changing $\mu_{\max}(\mu_{\min})$ to $\mu_{\max}(\mu_{\min})$ (without average with respect to the number of T-product tensors) is provided by the following Theorem 1.4.

**Theorem 1.4 (T-product Tensor Chernoff Bound II)** Consider a sequence $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ of independent, random, Hermitian T-product tensors that satisfy

$$X_i \succeq 0 \quad \text{and} \quad \lambda_{\max}(X_i) \leq T \quad \text{almost surely.} \quad (1.13)$$

Define following two quantities:

$$\mu_{\max} \overset{\text{def}}{=} \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i \right) \quad \text{and} \quad \mu_{\min} \overset{\text{def}}{=} \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i \right), \quad (1.14)$$

then, we have following two inequalities:

$$\Pr \left( \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \geq (1 + \theta) \mu_{\max} \right) \leq m p \left( \frac{e^\theta}{(1 + \theta)^{1+\theta}} \right)^{\mu_{\max}/T}, \quad \text{for} \ \theta \geq 0; \quad (1.15)$$

and

$$\Pr \left( \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \leq (1 - \theta) \mu_{\min} \right) \leq m p \left( \frac{e^{-\theta}}{(1 - \theta)^{1-\theta}} \right)^{\mu_{\min}/T}, \quad \text{for} \ \theta \in [0, 1]. \quad (1.16)$$

Below are theorems about Chernoff bounds for the maximum and the minimum eigentuples. Theorem 1.5 is correspond to Theorem 1.3 and Theorem 1.6 is correspond to Theorem 1.4.

**Theorem 1.5 (T-product Tensor Chernoff Bound I for Eigentuple)** Consider a sequence $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ of independent, random, Hermitian T-product tensors that satisfy

$$X_i \succeq 0 \quad \text{and} \quad \lambda_{\max}(X_i) \leq 1 \quad \text{almost surely.} \quad (1.17)$$

Define following two quantities:

$$\overline{\mu}_{\max} \overset{\text{def}}{=} \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i \right) \quad \text{and} \quad \overline{\mu}_{\min} \overset{\text{def}}{=} \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i \right), \quad (1.18)$$

then, given a real vector $b \geq 0 \in \mathbb{R}^p$ with $\bar{j} \overset{\text{def}}{=} \arg \min_j \{b_j\}$ and $\frac{1}{n} \sum_{i=1}^{n} tX_i$ satisfying Eq. (1.1), we have following two inequalities:

$$\Pr \left( \bar{d}_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \geq b \right) \leq m p e^{-n \overline{\mathcal{D}}(b_{\bar{j}}/\overline{\mu}_{\max})}, \quad \text{for} \ \overline{\mu}_{\max} \leq \frac{b_{\bar{j}}}{n} \leq 1; \quad (1.19)$$

and

$$\Pr \left( \bar{d}_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \leq b \right) \leq m p e^{-n \overline{\mathcal{D}}(b_{\bar{j}}/\overline{\mu}_{\min})}, \quad \text{for} \ 0 \leq \frac{b_{\bar{j}}}{n} \leq \overline{\mu}_{\min}. \quad (1.20)$$
Theorem 1.6 (T-product Tensor Chernoff Bound II for Eigentuple) Consider a sequence \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) of independent, random, Hermitian T-product tensors that satisfy

\[ X_i \succeq 0 \text{ and } \lambda_{\text{max}}(X_i) \leq T \text{ almost surely.} \quad (1.21) \]

Define following two quantities:

\[ \mu_{\text{max}} \overset{\text{def}}{=} \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathbb{E}X_i \right) \text{ and } \mu_{\text{min}} \overset{\text{def}}{=} \lambda_{\text{min}} \left( \sum_{i=1}^{n} \mathbb{E}X_i \right). \quad (1.22) \]

If \( \sum_{i=1}^{n} tX_i \) satisfies Eq. (1.1), we have following two inequalities:

\[ \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq (1 + \theta)\mu_{\text{max}} \right) \leq mp \left( \frac{e^{\theta}}{(1 + \theta)^{1+\theta}} \right)^{\mu_{\text{max}}/T}, \text{ for } \theta \geq 0; \quad (1.23) \]

and

\[ \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \leq (1 - \theta)\mu_{\text{min}} \right) \leq mp \left( \frac{e^{-\theta}}{(1 - \theta)^{1-\theta}} \right)^{\mu_{\text{min}}/T}, \text{ for } \theta \in [0, 1]. \quad (1.24) \]

1.2.3 Bernstein Inequaltities about T-product Tensors

For random variables, Bernstein inequalities give the upper tail of a sum of independent, zero-mean random variables that are either bounded or subexponential. In this paper, we will extend Bernstein bounds for a sum of zero-mean random T-product tensors. The bounded T-product tensor Bernstein bounds will be given by Theorem 1.7.

Theorem 1.7 (T-product Tensor Bernstein Bounds with Bounded \( \lambda_{\text{max}} \)) Given a finite sequence of independent Hermitian T-product tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) that satisfy

\[ \mathbb{E}X_i = 0 \text{ and } \lambda_{\text{max}}(X_i) \leq T \text{ almost surely.} \quad (1.25) \]

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} \mathbb{E} (X_i^2) \right\| \). Then, we have following inequalities:

\[ \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-\theta^2/2}{\sigma^2 + T\theta/3} \right); \quad (1.26) \]

and

\[ \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-3\theta^2}{8\sigma^2} \right) \text{ for } \theta \leq \sigma^2/T; \quad (1.27) \]

and

\[ \Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-3\theta}{8T} \right) \text{ for } \theta \geq \sigma^2/T. \quad (1.28) \]
Below is the subexponential T-product tensor Bernstein bounds. Different from Theorem 1.7, we relax the bounded constraint for the maximum eigenvalue for T-product tensors $X_i$ to $E(X_i^p) \preceq A_i^2$, where $p = 2, 3, 4, \ldots$.

**Theorem 1.8 (Subexponential T-product Tensor Bernstein Bounds)** Given a finite sequence of independent Hermitian T-product tensors $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ that satisfy
\[
E X_i = 0 \text{ and } E(X_i^p) \preceq p!T^{p-2}A_i^2,
\]
where $p = 2, 3, 4, \ldots$.

Define the total variance $\sigma^2$ as:
\[
\sigma^2 \defeq \left\| \sum_{i=1}^{n} A_i^2 \right\|.
\]
Then, we have following inequalities:
\[
\Pr\left( \lambda_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp\left( -\frac{\theta^2}{2\sigma^2 + T\theta} \right); \tag{1.30}
\]
and
\[
\Pr\left( \lambda_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp\left( -\frac{\theta^2}{4\sigma^2} \right) \text{ for } \theta \leq \sigma^2/T; \tag{1.31}
\]
and
\[
\Pr\left( \lambda_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mp \exp\left( -\frac{\theta}{4T} \right) \text{ for } \theta \geq \sigma^2/T. \tag{1.32}
\]

Below are theorems about T-product tensor Bernstein bounds for the maximum and the minimum eigen-tuples. Theorem 1.9 is correspond to Theorem 1.7 and Theorem 1.10 is correspond to Theorem 1.8.

**Theorem 1.9 (T-product Tensor Bernstein Bounds with Bounded $\lambda_{\max}$ for Eigentuple)** Given a finite sequence of independent Hermitian T-product tensors $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ that satisfy
\[
E X_i = 0 \text{ and } \lambda_{\max}(X_i) \leq T \text{ almost surely.} \tag{1.33}
\]
Define the total variance $\sigma^2$ as: $\sigma^2 \defeq \left\| \sum_{i=1}^{n} E(X_i^2) \right\|$. Then, given a positive real vector $b \geq 0 \in \mathbb{R}^p$ with $j \defeq \arg\min_{j} \{b_j\}$ and $\sum_{i=1}^{n} tX_i$ satisfying Eq. (1.1) for any $t > 0$, we have following inequalities:
\[
\Pr\left( \mu_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq mp \exp\left( -\frac{b_j^2}{\sigma^2 + T\theta/3} \right); \tag{1.34}
\]
and
\[
\Pr\left( \mu_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq mp \exp\left( -\frac{3b_j^2}{8\sigma^2} \right) \text{ for } b_j \leq \sigma^2/T; \tag{1.35}
\]
and
\[
\Pr\left( \mu_{\max}\left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq mp \exp\left( -\frac{3b_j^2}{8T} \right) \text{ for } b_j \geq \sigma^2/T. \tag{1.36}
\]
Theorem 1.10 (Subexponential T-product Tensor Bernstein Bounds for Eigentuple) Given a finite sequence of independent Hermitian T-product tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) that satisfy

\[
\mathbb{E}X_i = 0 \text{ and } \mathbb{E}(X_i^2) \leq \frac{p!T^{p-2}}{2} A_i^2,
\]  

(1.37)

where \( p = 2, 3, 4, \ldots \).

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_i A_i^2 \right\| \). Then, given a positive real vector \( b \in \mathbb{R}^p \) with \( \tilde{j} \overset{\text{def}}{=} \arg \min \{b_j\} \) and \( \sum_{i=1}^n lX_i \) satisfying Eq. (1.1) for any \( t > 0 \), we have following inequalities:

\[
\Pr \left( \max_{i=1}^n X_i \geq b \right) \leq m \exp \left( \frac{-b_j^2}{\sigma^2 + T b_j} \right);
\]

(1.38)

and

\[
\Pr \left( \max_{i=1}^n X_i \geq b \right) \leq m \exp \left( \frac{-b_j^2}{4 \sigma^2} \right) \text{ for } b_j \leq \sigma^2 / T; \tag{1.39}
\]

and

\[
\Pr \left( \max_{i=1}^n X_i \geq b \right) \leq m \exp \left( \frac{-b_j}{4T} \right) \text{ for } b_j \geq \sigma^2 / T. \tag{1.40}
\]

1.2.4 Inequalities about T-product Tensor Martingales

T-product tensor Azuma and McDiarmid inequalities will be provided for the maximum eigenvalue and the maximum eigentuple versions.

Theorem 1.11 (T-product Tensor Azuma Inequality for Eigenvale) Given a finite adapted sequence of Hermitian tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) and a fixed sequence of Hermitian T-product tensors \( \{A_i\} \) that satisfy

\[
\mathbb{E}_{i-1}X_i = 0 \text{ and } X_i^2 \leq A_i^2 \text{ almost surely,}
\]

(1.41)

where \( i = 1, 2, 3, \ldots \).

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_i A_i^2 \right\| \). Then, we have following inequalities:

\[
\Pr \left( \max_{i=1}^n X_i \geq \theta \right) \leq m e^{-\frac{\theta^2}{8 \sigma^2}}. \tag{1.42}
\]

Theorem 1.12 (T-product Tensor McDiarmid Inequality) Given a set of \( n \) independent random variables, i.e. \( \{X_i : i = 1, 2, \ldots, n\} \), and let \( F \) be a Hermitian T-product tensor-valued function that maps these \( n \) random variables to a Hermitian T-product tensor of dimension within \( \mathbb{C}^{m \times m \times p} \). Consider a sequence of Hermitian tensors \( \{A_i\} \) that satisfy

\[
(F(x_1, \ldots, x_i, \ldots, x_n) - F(x_1, \ldots, x_i', \ldots, x_n))^2 \leq A_i^2,
\]

(1.43)

where \( x_i, x_i' \in X_i \) and \( 1 \leq i \leq n \). Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_i A_i^2 \right\| \). Then, we have following inequality:

\[
\Pr \left( \max_{i=1}^n (F(x_1, \ldots, x_n) - \mathbb{E}F(x_1, \ldots, x_n)) \geq \theta \right) \leq m e^{-\frac{\theta^2}{8 \sigma^2}}. \tag{1.44}
\]
Following two theorems are eigentuple version for T-product tensor Azuma and McDiarmid inequalities.

**Theorem 1.13 (T-product Tensor Azuma Inequality for Eigentuple)** Given a finite adapted sequence of Hermitian tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) and a fixed sequence of Hermitian T-product tensors \( \{A_i\} \) that satisfy
\[
E_{i-1}X_i = 0 \quad \text{and} \quad X_i^2 \preceq A_i^2 \quad \text{almost surely},
\]
where \( i = 1, 2, 3, \ldots \).

Define the total variance \( \sigma^2 \) as:
\[
\sigma^2 \text{def} = \sum_{i=1}^{n} A_i^2.
\]
Then, given a positive real vector \( b \in \mathbb{R}^p \) with
\[\hat{j} \text{ def}= \arg\min_{j} \{b_j\} \text{ and } \sum_{i=1}^{n} tX_i \text{ satisfying Eq. (1.1)} \text{ for any } t > 0, \]
we have following inequalities:
\[
\Pr \left( \max_{i} X_i \geq b \right) \leq mpe^{-\frac{\sigma^2}{28}}.
\]

**Theorem 1.14 (T-product Tensor McDiarmid Inequality for Eigentuple)** Given a set of \( n \) independent random variables, i.e. \( \{X_i : i = 1, 2, \cdots n\} \), and let \( F \) be a Hermitian T-product tensor-valued function that maps these \( n \) random variables to a Hermitian T-product tensor of dimension within \( \mathbb{C}^{m \times m \times p} \). Consider a sequence of Hermitian tensors \( \{A_i\} \) that satisfy
\[
(F(x_1, \cdots, x_i, \cdots, x_n) - F(x_1, \cdots, x_i', \cdots, x_n))^2 \preceq A_i^2,
\]
where \( x_i, x_i' \in X_i \) and \( 1 \leq i \leq n \). Define the total variance \( \sigma^2 \) as:
\[
\sigma^2 \text{def} = \sum_{i=1}^{n} A_i^2.
\]
Then, given a positive real vector \( b \in \mathbb{R}^p \) with \( \hat{j} \text{ def}= \arg\min_{j} \{b_j\} \) and \( t(F(x_1, \cdots, x_n) - \mathbb{E}F(x_1, \cdots, x_n)) \) satisfying Eq. (1.1) for any \( t > 0 \), we have following inequality:
\[
\Pr \left( \max_{i} (F(x_1, \cdots, x_n) - \mathbb{E}F(x_1, \cdots, x_n)) \geq b \right) \leq mpe^{-\frac{\sigma^2}{8}}.
\]

### 1.3 Paper Organization

The rest of this paper is organized as follows. In Section 2, we briefly present those important results from Part I which will be used in later sections. Section 3 utilizes Gaussian and Rademacher series as case studies to explore T-product tensor inequalities. T-product tensor Chernoff bound and its applications are discussed in Section 4. In Section 5, T-product tensor Bernstein bound and its applications are provided. Several martingale results based on random T-product tensors are discussed in Section 6. Concluding remarks are given by Section 7.

### 2 Key Results From Part I Paper

This section will review those important results obtained from Part I paper which will be used at later proofs for references conveneince. All proofs for facts listed in this section can be found at our Part I paper.

For any tensor \( C \in \mathbb{C}^{m \times n \times p} \), a dilation for the tensor \( C \), denoted as \( \mathcal{D}(C) \), will be
\[
\mathcal{D}(C) \text{def} = \begin{bmatrix} O & C \\ CH \end{bmatrix},
\]
where \( O \) is the zero tensor of compatible dimensions. Then, the following properties hold:

1. \( \mathcal{D}(A \otimes B) = \mathcal{D}(A) \otimes \mathcal{D}(B) \)
2. \( \mathcal{D}(A^\dagger) = \mathcal{D}(A)^\dagger \)
3. \( \mathcal{D}(A \odot B) = \mathcal{D}(A) \odot \mathcal{D}(B) \)
4. \( \mathcal{D}(A_1 \oplus A_2) = \mathcal{D}(A_1) \oplus \mathcal{D}(A_2) \)
5. \( \mathcal{D}(A \odot \mathcal{D}(B)) = \mathcal{D}(A \odot B) \)
6. \( \mathcal{D}(A \oplus \mathcal{D}(B)) = \mathcal{D}(A \oplus B) \)

These properties allow for efficient computation and manipulation of T-product tensors in various applications.
where \( \mathcal{D}(C) \in \mathbb{C}^{(m+n) \times (m+n) \times p} \) and we have \((\mathcal{D}(C))^H = \mathcal{D}(C)\) (Hermitian T-product tensor after dilation). From T-SVD, we have following relation for Hermitian T-product tensor:

\[
f(s) \leq g(s) \quad \text{for } s \in [a, b] \implies f(C) \leq g(C) \quad \text{when the eigenvalues of } C \text{ lie in } [a, b]. \tag{2.2}
\]

Above Eq. (2.2) is named as transfer rule.

**Corollary 1** Let \( A \) be a fixed Hermitian T-product tensor, and let \( X \) be a random Hermitian T-product tensor, then we have

\[
\mathbb{E} \text{Tr} e^{A + X} \leq \text{Tr} e^{A + \log(\mathbb{E} e^{X})}. \tag{2.3}
\]

**Corollary 2** Given a finite sequence of independent Hermitian random tensors \( \{X_i\} \in \mathbb{C}^{m \times m \times p} \). If there is a function \( f : (0, \infty) \to [0, \infty] \) and a sequence of non-random Hermitian T-product tensors \( \{A_i\} \) with following condition:

\[
f(t)A_i \geq \log \mathbb{E} e^{tX_i}, \quad \text{for } t > 0. \tag{2.4}
\]

Then, for all \( \theta \in \mathbb{R} \), we have

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq \inf_{t>0} \left\{ \exp \left( -t\theta + f(t) \lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right) \right\}. \tag{2.5}
\]

**Corollary 3** Given a finite sequence of independent random Hermitian T-product tensors \( \{X_i\} \) with dimensions in \( \mathbb{C}^{m \times m \times p} \). If there is a function \( f : (0, \infty) \to [0, \infty] \) and a sequence of non-random Hermitian T-product tensors \( \{A_i\} \) with following condition:

\[
f(t)A_i \geq \log \mathbb{E} e^{tX_i}, \quad \text{for } t > 0. \tag{2.6}
\]

Then, for all \( b \in \mathbb{R}^p \) and \( \sum_{i=1}^{n} tX_i \) satisfying Eq. (1.1), we have

\[
\Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \exp \left( \frac{f(t) \lambda_{\max} \left( \sum_{i=1}^{n} A_i \right)}{\exp(b)_j} \right) \right\}. \tag{2.7}
\]

**Corollary 4** Given a finite sequence of independent Hermitian random tensors \( \{X_i\} \in \mathbb{C}^{m \times m \times p} \). For all \( \theta \in \mathbb{R} \), we have

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq \inf_{t>0} \left\{ \exp \left( -t\theta + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} e^{tX_i} \right) \right) \right\}. \tag{2.8}
\]

**Corollary 5** Given a finite sequence of independent random Hermitian T-product tensors \( \{X_i\} \) with dimensions in \( \mathbb{C}^{m \times m \times p} \), a real vector \( b \in \mathbb{R}^p \) and \( \sum_{i=1}^{n} tX_i \) satisfying Eq. (1.1), we have

\[
\Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \exp \left( \frac{n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} e^{tX_i} \right)}{\exp(b)_j} \right) \right\}. \tag{2.9}
\]
Lemma 1 (Laplace Transform Method for T-product Tensors: Eigenvalue Version) Let $X$ be a random Hermitian T-product tensor. For $\theta \in \mathbb{R}$, we have

$$\mathbb{P}(\lambda_{\max}(X) \geq \theta) \leq \inf_{t>0} \left\{ e^{-\theta t} \mathbb{E} \text{Tr} e^{tX} \right\}$$

(2.10)

Lemma 2 (Laplace Transform Method for T-product Tensors: Eigentuple Version) Let $X \in \mathbb{C}^{m \times m \times p}$ be a random T-positive definite (TPD) tensor and an all one vector $1_p = [1, 1, \cdots, 1]^T \in \mathbb{C}^p$. If $tX$ satisfies Eq. (1.1), then, for $b \in \mathbb{R}^p$, we have

$$\mathbb{P}(d_{\max}(X) \geq b) \leq \inf_{t>0} \min_i \left\{ \mathbb{E} \left( \text{Tr} \left( e^{tX} \right) \right) \left( e^{t b} \right)_i \right\},$$

(2.11)

where $d_{\max}$ is the maximum eigentuple of the TPD tensor $X$.

Theorem 1 (Golden-Thompson inequality for T-product Tensors) Given two Hermitian T-product tensors $C, D \in \mathbb{C}^{m \times m \times p}$, we have

$$\text{Tr} (\exp(C + D)) \leq \text{Tr} (\exp(C) \ast \exp(D))$$

(2.12)

3 Hermitian T-product Tensors With Random Sequences

A Hermitian T-product tensor Gaussian series is one of the simplest cases of a sum of independent random Hermitian T-product tensors. For scalars, a Gaussian series with real coefficients satisfies a normal-type tail bound where the variance is controlled by the sum of squares coefficients. The first Section 3.1 is to extend this context to Hermitian T-product tensors. In Section 3.2, we will apply results from Section 3.1 to consider Gaussian Hermitian T-product tensor with nonuniform variances. Finally, we will provide the lower and upper bounds of random Hermitian T-product tensor expectation in Section 3.3.

3.1 Hermitian T-product Tensors with Gaussian and Rademacher Random Series

We begin with a lemma about moment-generating functions of Rademacher and Gaussian normal random variables.

Lemma 3 Suppose that the tensor $A \in \mathbb{C}^{m \times m \times p}$ is Hermitian T-product tensor. Given a Gaussian normal random variable $\alpha$ and a Rademacher random variable $\beta$, then, we have

$$\mathbb{E} e^{\alpha tA} = e^{t^2 A^2/2} \text{ and } e^{\beta^2 A^2/2} \geq \mathbb{E} e^{\beta tA},$$

(3.1)

where $t \in \mathbb{R}$.

Proof: For the standard normal random variable, because we have

$$\mathbb{E}(\alpha^{2i}) = \frac{(2i)!}{i!2^i} \text{ and } \mathbb{E}(\alpha^{2i+1}) = 0,$$

(3.2)

where $i = 0, 1, 2, \cdots$; then

$$\mathbb{E} e^{\alpha tA} = T_{mmp} + \sum_{i=1}^{\infty} \frac{\mathbb{E}(\alpha^{2i})(tA)^{2i}}{(2i)!} = T_{mmp} + \sum_{i=1}^{\infty} \frac{(t^2 A^2/2)^i}{i!} = e^{t^2 A^2/2}.$$

(3.3)
For the Rademacher random variable, we have
\[ \mathbb{E} e^{\beta t A} = \cosh(t A) \leq e^{t^2 A^2/2}. \]  
(3.4)

Therefore, this Lemma is proved. □

We are ready to present the main theorem of this section about Hermitian T-product tensors with Gaussian and Rademacher series. The eigenvalue version is provided first by Theorem 1.1.

**Theorem 1.1 (Hermitian T-product Tensor with Gaussian and Rademacher Series Eigenvalue Version)**

Given a finite sequence of fixed T-product tensors \( A_i \in \mathbb{C}^{m \times m \times p} \), and let \( \{ \alpha_i \} \) be a finite sequence of independent standard normal variables. We define
\[ \sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} A_i^2 \right\|, \]  
(1.2)

then, for all \( \theta \geq 0 \), we have
\[ \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{2\sigma^2}}. \]  
(1.3)

We use \( \|X\| \) for the spectral norm, which is the largest singular value for the T-product tensor \( X \). Then, we have
\[ \Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq \theta \right) \leq 2mpe^{-\frac{\theta^2}{2\sigma^2}}. \]  
(1.4)

This theorem is also valid for a finite sequence of independent Rademacher random variables \( \{ \alpha_i \} \).

**Proof:** Given a finite sequence of independent Gaussian or Rademacher random variables \( \{ \alpha_i \} \), from Lemma 3, we have
\[ e^{t^2 A^2} \geq \mathbb{E} e^{\alpha_i t A_i}. \]  
(3.5)

From the definition in Eq. (1.2) and Corollary 2 we have
\[ \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{2\sigma^2}}. \]  
(3.6)

This establishes Eq. (1.3). For Eq. (1.4), we have to apply the following facts about the symmetric distribution of Gaussian and Rademacher random variables to obtain
\[ \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} (-\alpha_i) A_i \right) \geq \theta \right) = \Pr \left( -\lambda_{\min} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{2\sigma^2}}. \]  
(3.7)

Then, we obtain Eq. (1.4) as follows:
\[ \Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq \theta \right) = 2\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq \theta \right) \leq 2mpe^{-\frac{\theta^2}{2\sigma^2}}. \]  
(3.8)

From the Hermitian dilation definition provided by Eq. (2.1), we can extend Theorem 1.1 from square Hermitian tensor to rectangular tensor by the following corollary.
Corollary 6 (Rectangular Tensor with Gaussian and Rademacher Series Eigenvalue Version) Given a finite sequence $A_i \in \mathbb{C}^{m \times n \times p}$ be a finite sequence of independent standard normal random variables. We define

$$\sigma^2 \overset{\text{def}}{=} \max \left\{ \left\| \sum_{i=1}^{n} A_i \right\|, \left\| \sum_{i=1}^{n} A_i^H \right\| \right\}. \quad (3.9)$$

then, for all $\theta \geq 0$, we have

$$\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq \theta \right) \leq (m + n) e^{-\frac{\theta^2}{2\sigma^2}}. \quad (3.10)$$

This corollary is also valid for a finite sequence of independent Rademacher random variables $\{\alpha_i\}$.

**Proof:** Let $\{\alpha_i\}$ be a finite sequence of independent Gaussian or Rademacher random variables. Consider a finite sequence of random Hermitian T-product tensors $\{\alpha_i \mathcal{D}(A_i)\}$ with dimensions $\mathbb{C}^{(m+n) \times (m+n) \times p}$ and the fact that the largest eigenvalue of $\mathcal{D}(A_i)$ will be the same with the largest singular of $A_i$, we have

$$\left\| \sum_{i=1}^{n} \alpha_i A_i \right\| = \lambda_{\max} \left( \mathcal{D} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \right) = \lambda_{\max} \left( \sum_{i=1}^{n} \alpha_i \mathcal{D}(A_i) \right). \quad (3.11)$$

Due to the following singular value relation

$$\sigma^2 = \left\| \sum_{i=1}^{n} \mathcal{D}(A_i)^2 \right\| = \left\| \begin{bmatrix} \sum_{i=1}^{n} A_i \star A_i^H & \mathcal{O} \\ \mathcal{O} & \sum_{i=1}^{n} A_i^H \star A_i \end{bmatrix} \right\|$$

$$= \max \left\{ \left\| \sum_{i=1}^{n} A_i \star A_i^H \right\|, \left\| \sum_{i=1}^{n} A_i^H \star A_i \right\| \right\}. \quad (3.12)$$

From Eqs. (3.11), and Theorem 1.1, this corollary is proved. $\square$

The eigentuple version for Theorem 1.1 is provided by the following Theorem 1.2.

**Theorem 1.2 (Hermitian T-product Tensor with Gaussian and Rademacher Series Eigentuple Version)**

Given a finite sequence of Hermitian T-product tensors $A_i \in \mathbb{C}^{m \times m \times p}$, and let $\{\alpha_i\}$ be a finite sequence of independent standard normal variables. We define

$$\sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} A_i^t \right\|, \quad (3.16)$$

then, for all $b \geq 0$ and $\sum_{i=1}^{n} t \alpha_i A_i$ satisfying Eq. (1.1) for $t > 0$, we have

$$\Pr \left( d_{\max} \left( \sum_{i=1}^{n} \alpha_i A_i \right) \geq b \right) \leq mpe^{-\frac{b^2}{2\sigma^2}}, \quad (1.7)$$

where $j \overset{\text{def}}{=} \arg \min_j \{b_j\}$. And

$$\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) \leq 2mpe^{-\frac{b^2}{2\sigma^2}}, \quad (1.8)$$

This theorem is also valid for a finite sequence of independent Rademacher random variables $\{\alpha_i\}$.
Proof: Given a finite sequence of independent Gaussian or Rademacher random variables \( \{\alpha_i\} \), from Lemma 3, we have
\[
E e^{t^2 A_i^2} \geq e^{E e^{\alpha_i t A_i}}.
\]
(3.13)
If \( \tilde{j} \) is determined as:
\[
\tilde{j} = \arg \min_j \{b_j\},
\]
(3.14)
where \( b_j \) are entries of the vector \( b \). Then, we have
\[
\min_{1 \leq j \leq p} \left\{ \exp \left( \frac{t^2 \lambda_{\text{max}} \left( \sum_{i=1}^{n} A_i^2 \right)}{b_j} \right) \right\} \leq \exp \left( -tb_{\tilde{j}} + \frac{t^2}{2} \lambda_{\text{max}} \left( \sum_{i=1}^{n} A_i^2 \right) \right).
\]
(3.15)
From the definition in Eq. (1.6) and Corollary 3, we have
\[
\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) \leq mp \inf_{t>0} \left\{ e^{-tb_{\tilde{j}} + \frac{t^2}{2} \lambda_{\text{max}} \left( \sum_{i=1}^{n} A_i^2 \right)} \right\} = mpe^{-\frac{b_{\tilde{j}}^2}{2\sigma^2}}.
\]
(3.16)
For Eq. (1.8), because Gaussian and Rademacher random variables are symmetric, we have
\[
\Pr \left( \left\| \sum_{i=1}^{n} -\alpha_i A_i \right\| \geq b \right) = \Pr \left( -\left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) \leq mpe^{-\frac{b_{\tilde{j}}^2}{2\sigma^2}}.
\]
(3.17)
Then, we obtain Eq. (1.8) as follows:
\[
\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) = 2\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) \leq 2mpe^{-\frac{b_{\tilde{j}}^2}{2\sigma^2}}.
\]
(3.18)
From the Hermitian dilation definition provided by Eq. (2.1), we can extend Theorem 1.2 from square Hermitian tensor to rectangular tensor by the following corollary.

Corollary 7 (Rectangular Tensor with Gaussian and Rademacher Series Eigentuple Version) Given a finite sequence \( A_i \in \mathbb{C}^{m \times n \times p} \) be a finite sequence of independent standard normal random variables. We define
\[
\sigma^2 \overset{\text{def}}{=} \max \left\{ \left\| \sum_{i=1}^{n} A_i \right\|, \left\| \sum_{i=1}^{n} A_i^H \right\| \right\}.
\]
(3.19)
then, for all \( b \geq 0 \) and \( \sum_{i=1}^{n} t\alpha_i A_i \) satisfying Eq. (1.1) for \( t > 0 \), we have
\[
\Pr \left( \left\| \sum_{i=1}^{n} \alpha_i A_i \right\| \geq b \right) \leq (m+n)pe^{-\frac{b_{\tilde{j}}^2}{2\sigma^2}},
\]
(3.20)
where \( \tilde{j} \) is defined by Eq. (3.14).
This corollary is also valid for a finite sequence of independent Rademacher random variables \( \{\alpha_i\} \).
Proof: Let \( \{\alpha_i\} \) be a finite sequence of independent Gaussian or Rademacher random variables. Consider a finite sequence of random Hermitian T-product tensors \( \{\alpha_i D(A_i)\} \) with dimensions \( \mathbb{C}^{(m+n) \times (m+n) \times p} \) and the fact that the largest eigneuple of \( D(A_i) \) will be the same with the largest eigneuple of \( A_i \), we have

\[
\left\| \sum_{i=1}^{n} \alpha_i A_i \right\|_{\text{vec}} = d_{\max} \left( D \left( \sum_{i=1}^{n} \alpha_i A_i \right) \right) = d_{\max} \left( \sum_{i=1}^{n} \alpha_i D(A_i) \right). \tag{3.21}
\]

Due to the following singular value relation

\[
\sigma^2 = \left\| \sum_i D(A_i)^2 \right\| = \left\| \begin{bmatrix} \sum_{i=1}^{n} A_i \ast A_i^H & O \\ O & \sum_{i=1}^{n} A_i^H \ast A_i \end{bmatrix} \right\| = \max \left\{ \left\| \sum_{i=1}^{n} A_i \ast A_i^H \right\|, \left\| \sum_{i=1}^{n} A_i^H \ast A_i \right\| \right\}. \tag{3.22}
\]

From Eq. (3.21) and Theorem 1.2, this corollary is proved. \( \square \)

### 3.2 A Gaussian Tensor with Nonuniform Variances

In this section, we will apply results obtained from the previous section to consider Gaussian tensor with nonuniform variances among random entries.

Corollary 8 Given a tensor \( \mathcal{A} \in \mathbb{C}^{m \times n \times p} \) and a random tensor \( \mathcal{X} \in \mathbb{C}^{m \times n \times p} \) whose entries are independent standard Gaussian normal random variables. Let \( \circ \) represent the Hadamard product (entrywise) between two T-product tensors with the same dimensions. Then, we have

\[
\Pr \left( \| \mathcal{X} \circ \mathcal{A} \| \geq \theta \right) \leq (m + n)pe^{-\frac{\theta^2}{2\sigma^2}}, \tag{3.23}
\]

where

\[
\sigma^2 = \max \left\{ \sum_{j=1}^{m} |a_{1,j,1}|^2, \sum_{j=1}^{m} |a_{2,j,1}|^2, \cdots, \sum_{j=1}^{m} |a_{m,j,1}|^2, \sum_{i=1}^{m} |a_{i,1,1}|^2, \sum_{i=1}^{m} |a_{i,2,1}|^2, \cdots, \sum_{i=1}^{m} |a_{i,n,1}|^2 \right\}. \tag{3.24}
\]

where \( a_{i,j,k} \) are entries of the tensor \( \mathcal{A} \).

Proof: Since we can decompose the tensor \( \mathcal{X} \circ \mathcal{A} \) as:

\[
\mathcal{X} \circ \mathcal{A} = \sum_{i=j=k=1}^{m,n,p} x_{i,j,k} a_{i,j,k} \mathcal{E}_{i,j,k}, \tag{3.25}
\]

where \( \mathcal{E}_{i,j,k} \in \mathbb{C}^{m \times n \times p} \) is the tensor with all zero entries except unity at the position \( i, j, k \); then, we have

\[
\sum_{i=j=k=1}^{m,n,p} (a_{i,j,k} \mathcal{E}_{i,j,k}) \ast (a_{i,j,k} \mathcal{E}_{i,j,k})^H = \sum_{i=k=1}^{m,p} \left( \sum_{j=1}^{n} |a_{i,j,k}|^2 \right) \mathcal{E}_{i,i,1}
\]

\[
= \text{fdiag} \left( \sum_{j=1}^{n} |a_{1,j,1}|^2, \sum_{j=1}^{n} |a_{2,j,1}|^2, \cdots, \sum_{j=1}^{n} |a_{m,j,1}|^2 \right). \tag{3.26}
\]
where fdiag is the tensor with dimensions in \( \mathbb{C}^{m \times m \times p} \) such that the frontal diagonal matrix is a diagonal matrix and zero matrices at the other matrices parallel to the frontal matrix. Similarly, we also have

\[
\sum_{i=j=k=1}^{m,n,p} (a_{i,j,k}E_{i,j,k})^H \ast (a_{i,j,k}E_{i,j,k}) = \sum_{j=k=1}^{n,p} \left( \sum_{i=1}^{m} \left| a_{i,j,k} \right|^2 \right) E_{j,j,1} = \text{fdiag} \left( \sum_{i=1}^{m} \left| a_{i,1,1} \right|^2, \sum_{i=1}^{m} \left| a_{i,2,1} \right|^2, \cdots, \sum_{i=1}^{m} \left| a_{i,n,1} \right|^2 \right) \quad (3.27)
\]

Therefore, we have

\[
\sigma^2 = \max \left\{ \lambda_{\text{max}} \left( \text{fdiag} \left( \sum_{j=1}^{n} \left| a_{1,j,1} \right|^2, \sum_{j=1}^{n} \left| a_{2,j,1} \right|^2, \cdots, \sum_{j=1}^{n} \left| a_{m,j,1} \right|^2 \right) \right), \right. \\
\quad \left. \lambda_{\text{max}} \left( \text{fdiag} \left( \sum_{i=1}^{m} \left| a_{i,1,1} \right|^2, \sum_{i=1}^{m} \left| a_{i,2,1} \right|^2, \cdots, \sum_{i=1}^{m} \left| a_{i,n,1} \right|^2 \right) \right) \right\} \\
= \max \left\{ \sum_{j=1}^{n} \left| a_{1,j,1} \right|^2, \sum_{j=1}^{n} \left| a_{2,j,1} \right|^2, \cdots, \sum_{j=1}^{n} \left| a_{m,j,1} \right|^2, \right. \\
\quad \left. \sum_{i=1}^{m} \left| a_{i,1,1} \right|^2, \sum_{i=1}^{m} \left| a_{i,2,1} \right|^2, \cdots, \sum_{i=1}^{m} \left| a_{i,n,1} \right|^2 \right\} \quad (3.28)
\]

Finally, from Corollary 6, this Corollary is proved.

Following corollary is the eigentuple version for Corollary 8

**Corollary 9** Given a tensor \( \mathcal{A} \in \mathbb{C}^{m \times n \times p} \) and a random tensor \( \mathcal{X} \in \mathbb{C}^{m \times n \times p} \) whose entries are independent standard Gaussian normal random variables. Let \( \circ \) be used to represent a Hadamard product (entrywise) between two T-product tensors with the same dimensions. Then, for all \( b \geq 0 \) with \( j \) defined by Eq. (3.14), and \( t \mathcal{X} \circ \mathcal{A} \) satisfying Eq. (1.1) for \( t > 0 \), we have

\[
\Pr \left( \|\mathcal{X} \circ \mathcal{A}\|_{\text{vec}} \geq b \right) \leq (m + n)pe^{-\frac{\kappa^2}{2\sigma^4}}, \quad (3.29)
\]

where

\[
\sigma^2 = \max \left\{ \sum_{j=1}^{n} \left| \alpha_{1,j,1} \right|^2, \sum_{j=1}^{n} \left| \alpha_{2,j,1} \right|^2, \cdots, \sum_{j=1}^{n} \left| \alpha_{m,j,1} \right|^2, \right. \\
\quad \left. \sum_{i=1}^{m} \left| \alpha_{i,1,1} \right|^2, \sum_{i=1}^{m} \left| \alpha_{i,2,1} \right|^2, \cdots, \sum_{i=1}^{m} \left| \alpha_{i,n,1} \right|^2 \right\} \quad (3.30)
\]

The terms \( \alpha_{i,j,k} \) are entries of the tensor \( \mathcal{A} \).
Proof: From the proof from Corollary [8] we also have

\[
\sigma^2 = \max \left\{ \lambda_{\max} \left( \text{fdiag} \left( \sum_{j=1}^{n} |a_{1,j,1}|^2, \sum_{j=1}^{n} |a_{2,j,1}|^2, \cdots, \sum_{j=1}^{n} |a_{m,j,1}|^2 \right) \right), \right. \\
\lambda_{\max} \left( \text{fdiag} \left( \sum_{i=1}^{m} |a_{i,1,1}|^2, \sum_{i=1}^{m} |a_{i,2,1}|^2, \cdots, \sum_{i=1}^{m} |a_{i,n,1}|^2 \right) \right) \left. \right\}
\]

\[
= \max \left\{ \sum_{j=1}^{n} |a_{1,j,1}|^2, \sum_{j=1}^{n} |a_{2,j,1}|^2, \cdots, \sum_{j=1}^{n} |a_{m,j,1}|^2, \right. \\
\sum_{i=1}^{m} |a_{i,1,1}|^2, \sum_{i=1}^{m} |a_{i,2,1}|^2, \cdots, \sum_{i=1}^{m} |a_{i,n,1}|^2 \left. \right\}
\]

Finally, from Corollary [7] this Corollary is proved.

3.3 Lower and Upper Bounds of Spectral Norm Expectation

Given a finite sequence \( A_i \in \mathbb{C}^{m \times m \times p} \), and let \( \{\alpha_i\} \) be a finite sequence of independent standard normal variables. We define following random tensor

\[
\mathcal{X} = \sum_{i=1}^{n} \alpha_i A_i.
\]

From Theorem [1.1] we have

\[
\mathbb{E} \left( \|\mathcal{X}\|^2 \right) = \int_{0}^{\infty} \text{Pr} \left( \|\mathcal{X}\| > \sqrt{t} \right) dt \leq \int_{0}^{\infty} 2mp e^{-\frac{t}{2\sigma^2}} dt = 4mp\sigma^2
\]

(3.33)

where \( \sigma^2 = \left\| \sum_{i=1}^{n} A_i^2 \right\| \). On the other hand, from Jensen’s inequality, we have

\[
\mathbb{E} \left( \|\mathcal{X}\|^2 \right) = \mathbb{E} \|\mathcal{X}^2\| \geq \|\mathbb{E}(\mathcal{X}^2)\| = \left\| \sum_{i=1}^{n} A_i^2 \right\| = \sigma^2.
\]

(3.34)

From both Eqs. (3.33) and (3.34), we have following relation:

\[
c\sigma \leq \mathbb{E} \|\mathcal{X}\| \leq 2\sigma \sqrt{mp}
\]

(3.35)

This shows that the tensor variance parameter \( \sigma^2 \) controls the expected norm \( \mathbb{E} \|\mathcal{X}\| \) with square root of logarithmic function for the tensor dimensions.

4 Chernoff Bounds for T-product Tensors

In this section, we will extend Chernoff bounds of random variables to random T-product tensors.
4.1 T-product Tensor Chernoff Bounds Derivations

We begin to present a lemma about the semidefinite relation for the tensor moment-generating function of a random TPSD T-product tensor.

**Lemma 4** Given a random TPSD T-product tensor with $\lambda_{\text{max}}(\mathcal{X}) \leq 1$, then, for any $t \in \mathbb{R}$, we have

$$I + (e^{t} - 1)E\mathcal{X} \succeq e^{t}\mathcal{X}.$$ (4.1)

**Proof:** Consider a convex function $f(x) = e^{tx}$, we have

$$1 + (e^{t} - 1)x \geq f(x),$$ (4.2)

where $x \in [0, 1]$. Since the eigenvalues of the random tensor $\mathcal{X}$ lie in the interval $[0, 1]$, from Eq. (2.2), we obtain

$$I + (e^{t} - 1)\mathcal{X} \succeq e^{t}\mathcal{X}.$$ (4.3)

Then, this Lemma is proved by taking the expectation with respect to the random T-product tensor $\mathcal{X}$. \hfill \Box

Given two real values $c, d \in [0, 1]$, we define *binary information divergence* of $c$ and $d$, expressed by $\mathcal{D}(c||d)$, as

$$\mathcal{D}(c||d) \equiv c \log \frac{c}{d} + (1 - c) \frac{1 - c}{1 - d}.$$ (4.4)

We are ready to present T-product tensor Chernoff inequality by theorem 1.3.

**Theorem 1.3 (T-product Tensor Chernoff Bound I)** Consider a sequence $\{\mathcal{X}_i \in \mathbb{C}^{m \times m \times p}\}$ of independent, random, Hermitian T-product tensors that satisfy

$$\mathcal{X}_i \succeq \mathcal{O} \text{ and } \lambda_{\text{max}}(\mathcal{X}_i) \leq 1 \text{ almost surely.}$$ (1.9)

Define following two quantities:

$$\overline{\mu}_{\text{max}} \equiv \lambda_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} EX_i \right) \text{ and } \overline{\mu}_{\text{min}} \equiv \lambda_{\text{min}} \left( \frac{1}{n} \sum_{i=1}^{n} EX_i \right),$$ (1.10)

then, we have following two inequalities:

$$\Pr \left( \lambda_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{X}_i \right) \geq \theta \right) \leq mpe^{-n\mathcal{D}(\theta||\overline{\mu}_{\text{max}})}, \text{ for } \overline{\mu}_{\text{max}} \leq \theta \leq 1;$$ (1.11)

and

$$\Pr \left( \lambda_{\text{min}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{X}_i \right) \leq \theta \right) \leq mpe^{-n\mathcal{D}(\theta||\overline{\mu}_{\text{min}})}, \text{ for } 0 \leq \theta \leq \overline{\mu}_{\text{min}}.$$ (1.12)

**Proof:** From Lemma 4, we have

$$I + f(t)E\mathcal{X} \succeq Ee^{t\mathcal{X}},$$ (4.5)
where \( f(t) \equiv e^t - 1 \) for \( t > 0 \). By applying Corollary 4, we obtain

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \alpha \right) \leq m p \inf_{t>0} \exp \left( -t\alpha + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} (I + f(t)E X_i) \right) \right)
\]

\[
= m p \inf_{t>0} \exp \left( -t\alpha + n \log \lambda_{\max} (I + f(t) \frac{1}{n} \sum_{i=1}^{n} E X_i) \right)
\]

\[
= m p \inf_{t>0} \exp \left( -t\alpha + n \log (1 + f(t) \mu_{\max}) \right).
\] (4.6)

The last equality follows from the definition of \( \mu_{\max} \) and the eigenvalue map properties. When the value \( t \) at the right-hand side of Eq. (4.6) is

\[
t = \log \frac{\alpha}{n - \alpha} - \log \frac{\bar{\mu}_{\max}}{1 - \mu_{\max}},
\] (4.7)

we can achieve the tightest upper bound at Eq. (4.6). By substituting the value \( t \) in Eq. (4.7) into Eq. (4.6) and change the variable \( \alpha \rightarrow n\theta \), Eq. (1.11) is proved. The next goal is to prove Eq. (1.12).

If we apply Lemma 4 to the sequence \( \{ -X_i \} \), we have

\[
I - g(t) E X_i \preceq E e^{t(-X_i)},
\] (4.8)

where \( g(t) \equiv 1 - e^t \) for \( t > 0 \). By applying Corollary 4 again, we obtain

\[
\Pr \left( \lambda_{\min} \left( \sum_{i=1}^{n} X_i \right) \leq \alpha \right) = \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} (-X_i) \right) \geq \alpha \right)
\]

\[
\leq m p \inf_{t>0} \exp \left( t\alpha + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} (I - g(t)E X_i) \right) \right)
\]

\[
= m p \inf_{t>0} \exp \left( t\alpha + n \log \left( 1 - f(t) \lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} E X_i \right) \right) \right)
\]

\[
= m p \inf_{t>0} \exp \left( t\alpha + n \log (1 - g(t) \mu_{\min}) \right),
\] (4.9)

where we apply the relation \( \lambda_{\min}(-\frac{1}{n} \sum_{i=1}^{n} E X_i) = -\lambda_{\max}(\frac{1}{n} \sum_{i=1}^{n} E X_i) \) at the equality =1. When the value \( t \) at the right-hand side of Eq. (4.9) is

\[
t = \log \frac{\bar{\mu}_{\min}}{1 - \mu_{\min}} - \log \frac{\alpha}{n - \alpha},
\] (4.10)

we can achieve the tightest upper bound at Eq. (4.9). By substituting the value \( t \) in Eq. (4.10) into Eq. (4.9) and change the variable \( \alpha \rightarrow n\theta \), Eq. (1.12) is proved also.

The tensor Chernoff bounds discussed at Theorem 1.3 is not related to \( \mu_{\max} \) and \( \mu_{\min} \) directly. Following theorem is another version of tensor Chernoff bounds to associate the probability range in terms of \( \mu_{\max} \) and \( \mu_{\min} \) directly and this format of tensor Chernoff bounds is easier to be applied.

**Theorem 1.4 (T-product Tensor Chernoff Bound II)** Consider a sequence \( \{ X_i \in \mathbb{C}^{m \times m \times p} \} \) of independent, random, Hermitian tensors that satisfy

\[
X_i \succeq O \text{ and } \lambda_{\max}(X_i) \leq T \text{ almost surely.}
\] (1.13)
Define following two quantities:

\[ \mu_{\text{max}} \overset{\text{def}}{=} \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathbb{E} \mathcal{X}_i \right) \quad \text{and} \quad \mu_{\text{min}} \overset{\text{def}}{=} \lambda_{\text{min}} \left( \sum_{i=1}^{n} \mathbb{E} \mathcal{X}_i \right), \tag{1.14} \]

then, we have following two inequalities:

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \geq (1 + \theta) \mu_{\text{max}} \right) \leq mp \left( \frac{e^\theta}{(1 + \theta)_{1+\theta}} \right)^{\mu_{\text{max}}/T}, \quad \text{for} \ \theta \geq 0; \tag{1.15}\]

and

\[
\Pr \left( \lambda_{\text{min}} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \leq (1 - \theta) \mu_{\text{min}} \right) \leq mp \left( \frac{e^{-\theta}}{(1 - \theta)_{1-\theta}} \right)^{\mu_{\text{min}}/T}, \quad \text{for} \ \theta \in [0, 1]. \tag{1.16}\]

**Proof:** Without loss of generality, we can assume \( T = 1 \) in our proof. From Eq. (4.6) and the inequality \( \log(1 + x) \leq x \) for \( x > -1 \), we have

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \geq \alpha \right) \leq mp \inf_{t > 0} \exp\left( -t \alpha + (e^t - 1) \mu_{\text{max}} \right) \tag{4.11}\]

By selecting \( t = \log(1 + \theta) \) and \( \alpha \rightarrow (1 + \theta) \mu_{\text{max}} \), we can establish Eq. (1.15).

From Eq. (4.9) and the inequality \( \log(1 + x) \leq x \) for \( x > -1 \), we have

\[
\Pr \left( \lambda_{\text{min}} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \leq \alpha \right) \leq mp \inf_{t > 0} \exp\left( -t \alpha - (e^t - 1) \mu_{\text{min}} \right) \tag{4.12}\]

By selecting \( t = -\log(1 - \theta) \) and \( \alpha \rightarrow (1 - \theta) \mu_{\text{min}} \), we can establish Eq. (1.16). Therefore, this theorem is proved. \( \square \)

### 4.2 T-product tensor Chernoff Inequalities for Eigentuple

In this section, we will present T-product tensor Chernoff inequalities about the maximum of eigentuple.

**Theorem 1.5 (T-product Tensor Chernoff Bound I for Eigentuple)** Consider a sequence \( \{\mathcal{X}_i \in \mathbb{C}^{m \times m \times p}\} \) of independent, random, Hermitian T-product tensors that satisfy

\[ \mathcal{X}_i \succeq \mathcal{O} \quad \text{and} \quad \lambda_{\text{max}}(\mathcal{X}_i) \leq 1 \quad \text{almost surely.} \tag{1.17} \]

Define following two quantities:

\[ \overline{\mu}_{\text{max}} \overset{\text{def}}{=} \lambda_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \mathcal{X}_i \right) \quad \text{and} \quad \overline{\mu}_{\text{min}} \overset{\text{def}}{=} \lambda_{\text{min}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \mathcal{X}_i \right), \tag{1.18} \]

then, given a real vector \( b \geq 0 \in \mathbb{R}^p \) with \( \hat{j} \overset{\text{def}}{=} \arg \min_{j} \{b_j\} \) and \( \frac{1}{n} \sum_{i=1}^{n} t \mathcal{X}_i \) satisfying Eq. (1.1), we have following two inequalities:

\[
\Pr \left( d_{\text{max}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{X}_i \right) \geq b \right) \leq mp e^{-n \mathbb{D}(\frac{b_{\hat{j}}}{n} || \overline{\mu}_{\text{max}})}, \quad \text{for} \ \frac{b_{\hat{j}}}{n} \leq 1; \tag{1.19}\]

and

\[
\Pr \left( d_{\text{min}} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{X}_i \right) \leq b \right) \leq mp e^{-n \mathbb{D}(\frac{b_{\hat{j}}}{n} || \overline{\mu}_{\text{min}})}, \quad \text{for} \ 0 \leq \frac{b_{\hat{j}}}{n} \leq \overline{\mu}_{\text{min}}. \tag{1.20}\]
Proof: From Lemma 4, we have
\[ I + f(t)E\mathcal{X}_i \geq Ee^{t\mathcal{X}_i}, \] (4.13)
where \( f(t) \overset{\text{def}}{=} e^t - 1 \) for \( t > 0 \). By applying Corollary 5, we obtain
\[
\Pr \left( d_{\max} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \geq b \right) \leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} Ee^{t\mathcal{X}_i} \right) \right)}{e^{tb_j}} \right\}
\leq mp \inf_{t>0} \exp \left( -tb_j + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} Ee^{t\mathcal{X}_i} \right) \right)
\leq mp \inf_{t>0} \exp \left( -tb_j + n \log \lambda_{\max} \left( I + f(t) \frac{1}{n} \sum_{i=1}^{n} E\mathcal{X}_i \right) \right)
= mp \inf_{t>0} \exp \left( -tb_j + n \log (1 + f(t)\overline{\mu}_{\max}) \right), \tag{4.14}
\]
The last equality follows from the definition of \( \overline{\mu}_{\max} \) and spectral mapping theorem. When the value \( t \) at the right-hand side of Eq. (4.14) is
\[ t = \log \frac{b_j}{n - b_j} - \log \frac{\overline{\mu}_{\max}}{1 - \overline{\mu}_{\max}}, \tag{4.15}\]
we can achieve the tightest upper bound at Eq. (4.14). By substituting the value \( t \) in Eq. (4.15) into Eq. (4.14), Eq. (1.19) is proved. The next goal is to prove Eq. (1.20).

If we apply Lemma 4 to the sequence \( \{-\mathcal{X}_i\} \), we have
\[ I - g(t)E\mathcal{X}_i \geq Ee^{t(-\mathcal{X}_i)}, \tag{4.16}\]
where \( g(t) \overset{\text{def}}{=} 1 - e^t \) for \( t > 0 \). By applying Corollary 5 again, we obtain
\[
\Pr \left( \lambda_{\min} \left( \sum_{i=1}^{n} \mathcal{X}_i \right) \leq b \right) = \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} (-\mathcal{X}_i) \right) \geq b \right)
\leq mp \inf_{t>0} \min_{1 \leq j \leq p} \left\{ \frac{\exp \left( n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} Ee^{-t\mathcal{X}_i} \right) \right)}{e^{tb_j}} \right\}
\leq mp \exp \left( -tb_j + n \log \lambda_{\max} \left( \frac{1}{n} \sum_{i=1}^{n} (I - g(t)E\mathcal{X}_i) \right) \right)
= mp \exp \left( -tb_j + n \log (1 - f(t)\lambda_{\min} \left( \frac{1}{n} \sum_{i=1}^{n} E\mathcal{X}_i \right)) \right)
= mp \exp \left( -tb_j + n \log (1 - g(t)\overline{\mu}_{\min}) \right), \tag{4.17}
\]
where we apply the relation \( \lambda_{\min}(-\frac{1}{n} \sum_{i=1}^{n} E\mathcal{X}_i) = -\lambda_{\max}(\frac{1}{n} \sum_{i=1}^{n} E\mathcal{X}_i) \) at the equality =. When the value \( t \) at the right-hand side of Eq. (4.17) is
\[ t = \log \frac{\overline{\mu}_{\min}}{1 - \overline{\mu}_{\min}} - \log \frac{b_j}{n - b_j}, \tag{4.18}\]
we can achieve the tightest upper bound at Eq. (4.17). By substituting the value $t$ in Eq. (4.18) into Eq. (4.17), therefore, Eq. (1.20) is proved also.

The tensor Chernoff bounds discussed at Theorem 1.5 is not related to $\mu_{\text{max}}$ and $\mu_{\text{min}}$ directly. Following theorem is another version of tensor Chernoff bounds to associate the probability range in terms of $\mu_{\text{max}}$ and $\mu_{\text{min}}$ directly and these formats of tensor Chernoff bounds are easier to be applied.

**Theorem 1.6 (T-product Tensor Chernoff Bound II for Eigentuple)** Consider a sequence $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ of independent, random, Hermitian T-product tensors that satisfy

$$X_i \succeq 0 \text{ and } \lambda_{\text{max}}(X_i) \leq T \text{ almost surely.}$$

(1.21)

Define following two quantities:

$$\mu_{\text{max}} \overset{\text{def}}{=} \lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathbb{E}X_i \right) \text{ and } \mu_{\text{min}} \overset{\text{def}}{=} \lambda_{\text{min}} \left( \sum_{i=1}^{n} \mathbb{E}X_i \right).$$

(1.22)

If $\sum_{i=1}^{n} tX_i$ satisfies Eq. (1.1), we have following two inequalities:

$$\Pr \left( d_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq (1 + \theta)\mu_{\text{max}}1 \right) \leq mp \left( \frac{e^{\theta}}{(1 + \theta)^{1+\theta}} \right)^{\mu_{\text{max}}/T}, \text{ for } \theta \geq 0;$$

(1.23)

and

$$\Pr \left( d_{\text{min}} \left( \sum_{i=1}^{n} X_i \right) \leq (1 - \theta)\mu_{\text{min}}1 \right) \leq mp \left( \frac{e^{-\theta}}{(1 - \theta)^{1-\theta}} \right)^{\mu_{\text{min}}/T}, \text{ for } \theta \in [0, 1].$$

(1.24)

**Proof:** Without loss of generality, we can assume $T = 1$ in our proof. From Eq. (4.14) and the inequality $\log(1 + x) \leq x$ for $x > -1$, we have

$$\Pr \left( d_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq (1 + \theta)\mu_{\text{max}}1 \right) \leq mp \inf_{t>0} \exp(-tb_j^* + (e^t - 1)\mu_{\text{max}})$$

(4.19)

By selecting $t = \log(1 + \theta)$ and $b_j^* \to (1 + \theta)\mu_{\text{max}}$, we can establish Eq. (1.23).

From Eq. (4.17) and the inequality $\log(1 + x) \leq x$ for $x > -1$, we have

$$\Pr \left( \lambda_{\text{min}} \left( \sum_{i=1}^{n} X_i \right) \leq (1 - \theta)\mu_{\text{min}}1 \right) \leq mp \inf_{t>0} \exp(-tb_j^* - (e^t - 1)\mu_{\text{min}})$$

(4.20)

By selecting $t = -\log(1 - \theta)$ and $b_j^* \to (1 - \theta)\mu_{\text{min}}$, we can establish Eq. (1.24). Therefore, this theorem is proved.

**4.3 Application of T-product Tensor Chernoff Bounds**

One application of T-product tensor Chernoff bounds is to estimate the expectation of the maximum eigenvalue of independent sum of random T-product tensors.
Corollary 10 (Upper and Lower Bounds for the Maximum Eigenvalue) Consider a sequence \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) of independent, random, Hermitian T-product tensors that satisfy
\[
X_i \succeq O \quad \text{and} \quad \lambda_{\max}(X_i) \leq T \quad \text{almost surely.} \tag{4.21}
\]
Then, we have
\[
\mu_{\max} \leq \mathbb{E}\lambda_{\max}\left(\sum_{i=1}^{n} X_i\right) \leq C m p e^{-\mu_{\max}/T}, \tag{4.22}
\]
where the constant value of \( C \) is about 10.28.

Proof: The lower bound at Eq. (4.22) is true from the convexity of the function \( A \rightarrow \lambda_{\max}(A) \) and the Jensen’s inequality.

For the upper bound, we have
\[
\mathbb{E}\lambda_{\max}\left(\sum_{i=1}^{n} X_i\right) = \int_{0}^{\infty} \Pr\left(\lambda_{\max}\left(\sum_{i=1}^{n} X_i\right) \geq t\right) dt \\
\leq 1 \int_{0}^{\infty} m p e^{-\delta t + (e^{\delta} - 1)\mu_{\max}/T} dt \\
= e^{e^{\delta} \delta m p e^{-\mu_{\max}/T}} \\
\leq e^{\delta_{\opt} e^{\delta_{\opt}} \delta_{\opt} m p e^{-\mu_{\max}/T}} = C m p e^{-\mu_{\max}/T}, \tag{4.23}
\]
where the inequality \( \leq 1 \) comes from Eq. (4.11) with the scaling factor \( T \). If we select \( \theta \) as the solution of the following relation \( e^{\delta_{\opt} \theta} = \frac{1}{\delta_{\opt}} \) to minimize the right-hand side of Eq. (4.23), we have the desired upper bound when \( \delta_{\opt} \approx 0.56699 \). This corollary is proved. \( \square \)

5 Bernstein Bounds for T-product Tensors

For random variables, Bernstein inequalities give the upper tail of a sum of independent, zero-mean random variables that are either bounded or subexponential. In this section, we wish to extend Bernstein bounds for a sum of zero-mean random T-product tensors.

5.1 T-product Tensor Bernstein Bounds Derivation

We will consider bounded T-product tensor Bernstein bounds first by considering the bounded Bernstein moment-generating function with the following Lemma.

Lemma 5 Given a random Hermitian T-product tensor \( \mathcal{X} \in \mathbb{C}^{m \times m \times p} \) that satisfies:
\[
\mathbb{E}\mathcal{X} = 0 \quad \text{and} \quad \lambda_{\max}(\mathcal{X}) \leq 1 \quad \text{almost surely.} \tag{5.1}
\]
Then, we have
\[
e^{(e^{t} - t - 1)\mathbb{E}(\mathcal{X}^2)} \geq \mathbb{E} e^{t\mathcal{X}} \tag{5.2}
\]
where \( t > 0 \).
**Proof:** If we define a real function \( g(x) \equiv \frac{e^{tx} - 1}{x^2} \), it is easy to see that this function \( g(x) \) is an increasing function for \( 0 < x \leq 1 \). From Eq (2.2), we have

\[
g(\lambda) \leq g(1)\lambda.
\]  

(5.3)

Moreover, we also have

\[
e^{t\lambda} = \mathcal{I} + t\lambda + g(\lambda) \star \lambda^2 \\
\leq \mathcal{I} + t\lambda + g(1)\lambda^2,
\]

(5.4)

where the \( \leq \) comes from Eq. (5.3). By taking the expectation for both sides of Eq. (5.4), we then obtain

\[
\mathbb{E}e^{t\lambda} \leq \mathcal{I} + g(1)\mathbb{E}(\lambda^2) \leq e^{g(1)\mathbb{E}(\lambda^2)} = e^{(e^t - t - 1)\mathbb{E}(\lambda^2)}.
\]

(5.5)

This lemma is established. \( \Box \)

We are ready to present Bernstein inequalities for random T-product tensors with bounded \( \lambda_{\text{max}} \).

**Theorem 1.7 (T-product Tensor Bernstein Bounds with Bounded \( \lambda_{\text{max}} \)).** Given a finite sequence of independent Hermitian T-product tensors \( \{\mathcal{X}_i \in \mathbb{C}^{m \times m \times p}\} \) that satisfy

\[
\mathbb{E}\mathcal{X}_i = 0 \text{ and } \lambda_{\text{max}}(\mathcal{X}_i) \leq T \text{ almost surely.}
\]

(1.25)

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \equiv \left\| \sum \mathbb{E}(\mathcal{X}_i^2) \right\| \). Then, we have following inequalities:

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^n \mathcal{X}_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-\theta^2/2}{\sigma^2 + T\theta/3} \right);
\]

(1.26)

and

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^n \mathcal{X}_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-3\theta^2}{8\sigma^2} \right) \text{ for } \theta \leq \sigma^2/T;
\]

(1.27)

and

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^n \mathcal{X}_i \right) \geq \theta \right) \leq mp \exp \left( \frac{-3\theta}{8T} \right) \text{ for } \theta \geq \sigma^2/T.
\]

(1.28)

**Proof:** Without loss of generality, we can assume that \( T = 1 \) since the summands are 1-homogeneous and the variance is 2-homogeneous. From Lemma 5, we have

\[
\mathbb{E}e^{t\lambda_i} \leq e^{(e^t - t - 1)\mathbb{E}(\lambda_i^2)} \text{ for } t > 0.
\]

(5.6)

By applying Corollary 2, we then have

\[
\Pr \left( \lambda_{\text{max}} \left( \sum_{i=1}^n \mathcal{X}_i \right) \geq \theta \right) \leq mp \exp \left( -t\theta + (e^t - t - 1)\lambda_{\text{max}} \left( \sum_{i=1}^n \mathbb{E}(\mathcal{X}_i^2) \right) \right)
\]

(5.7)
The right-hand side of Eq. (5.7) can be minimized by setting \( t = \log(1 + \theta/\sigma^2) \). Substitute such \( t \) and simplify the right-hand side of Eq. (5.7), we obtain Eq. (1.26).

For \( \theta \leq \sigma^2/T \), we have
\[
\frac{1}{\sigma^2 + T\theta/3} \geq \frac{1}{\sigma^2 + T(\sigma^2/T)/3} = \frac{3}{4\sigma^2},
\]
then, we obtain Eq. (1.27). Correspondingly, for \( \theta \geq \sigma^2/T \), we have
\[
\frac{\theta}{\sigma^2 + T\theta/3} \geq \frac{\sigma^2/T}{\sigma^2 + T(\sigma^2/T)/3} = \frac{3}{4T},
\]
and, we obtain Eq. (1.28) also.

The following theorem 1.8 is the extension of the theorem 1.7 by allowing the moments of the random \( T \)-product tensors to grow at a controlled rate. We have to prepare subexponential Bernstein moment-generating function Lemma first for later proof of Theorem 1.8.

**Lemma 6** Suppose that \( \mathcal{X} \) is a random Hermitian \( T \)-product tensor that satisfies
\[
\mathbb{E}[\mathcal{X}] = 0 \text{ and } \mathbb{E}(\mathcal{X}^p) \preceq \frac{p!A^2}{2} \text{ for } p = 2, 3, 4, \ldots.
\]
Then, we have
\[
\exp\left(\frac{t^2A^2}{2(1-t)}\right) \succeq \mathbb{E}e^{tx},
\]
where \( 0 < t < 1 \).

**Proof:** From Taylor series of the tensor exponential expansion, we have
\[
\mathbb{E}e^{tx} = \mathcal{I} + t\mathbb{E}[\mathcal{X}] + \sum_{p=2}^{\infty} \frac{t^p\mathbb{E}(\mathcal{X}^p)}{p!} \leq \mathcal{I} + \sum_{p=2}^{\infty} \frac{t^pA^2}{2} = \mathcal{I} + \frac{t^2A^2}{2(1-t)} \succeq \exp\left(\frac{t^2A^2}{2(1-t)}\right),
\]
therefore, this Lemma is proved.

**Theorem 1.8 (Subexponential T-product Tensor Bernstein Bounds)** Given a finite sequence of independent Hermitian \( T \)-product tensors \( \{\mathcal{X}_i \in \mathbb{C}^{m \times m \times p}\} \) that satisfy
\[
\mathbb{E}\mathcal{X}_i = 0 \text{ and } \mathbb{E}(\mathcal{X}_i^p) \preceq \frac{p!T^{p-2}}{2}A^2_i,
\]
where \( p = 2, 3, 4, \ldots \).

Define the total variance \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_i^{n} A^2_i \right\| \). Then, we have following inequalities:
\[
\Pr\left(\lambda_{\max}\left(\sum_{i=1}^{n} \mathcal{X}_i\right) \geq \theta\right) \leq mp \exp\left(\frac{-\theta^2/2}{\sigma^2 + T\theta}\right);
\]

\[24\]
and
\[ Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq m_p \exp \left( -\frac{\theta^2}{4\sigma^2} \right) \text{ for } \theta \leq \sigma^2 / T; \] (1.31)

and
\[ Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq m_p \exp \left( -\frac{\theta^2}{4\sigma^2 T} \right) \text{ for } \theta \geq \sigma^2 / T. \] (1.32)

**Proof:** Without loss of generality, we can assume that \( T = 1 \). From Lemma 6, we have
\[ E \exp (t X_i) \preceq \exp \left( \frac{t^2 A_i^2}{2(1-t)} \right), \] (5.13)
where \( 0 < t < 1 \).

By applying Corollary 2, we then have
\[ Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq m_p \exp \left( -t \theta + \frac{t^2 \lambda_{\max} \left( \sum_{i=1}^{n} A_i^2 \right)}{2(1-t)} \right) = m_p \exp \left( -t \theta + \frac{\sigma^2 t^2}{2(1-t)} \right). \] (5.14)

The right-hand side of Eq. (5.14) can be minimized by setting \( t = \frac{\theta}{\sigma^2 + \sigma^2} \). Substitute such \( t \) and simplify the right-hand side of Eq. (5.14), we obtain Eq. (1.30).

For \( \theta \leq \sigma^2 / T \), we have
\[ \frac{1}{\sigma^2 + T \theta} \geq \frac{1}{\sigma^2 + T(\sigma^2 / T)} = \frac{1}{2\sigma^2}, \] (5.15)
then, we obtain Eq. (1.31). Similarly, for \( \theta \geq \sigma^2 / T \), we have
\[ \frac{\theta}{\sigma^2 + T \theta} \geq \frac{\sigma^2 / T}{\sigma^2 + T(\sigma^2 / T)} = \frac{1}{2T}, \] (5.16)
therefore, we also obtain Eq. (1.32). \( \Box \)

5.2 T-product Tensor Bernstein Bounds for Eigentuple

In this section, we will extend T-product tensor bernstein bounds from the maximum eigenvalue discussed at previous section to the maximum eigentuple.

**Theorem 1.9 (T-product Tensor Bernstein Bounds with Bounded \( \lambda_{\max} \) for Eigentuple)**

Given a finite sequence of independent Hermitian T-product tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) that satisfy
\[ E X_i = 0 \text{ and } \lambda_{\max}(X_i) \leq T \text{ almost surely.} \] (1.33)

Define the total varaince \( \sigma^2 \) as: \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_{i} E (X_i^2) \right\| \). Then, given a positive real vector \( b \geq 0 \in \mathbb{R}^p \) with \( j \overset{\text{def}}{=} \arg \min_j \{b_j\} \) and \( \sum_{i=1}^{n} t X_i \) satisfying Eq. (1.1) for any \( t > 0 \), we have following inequalities:
\[ Pr \left( d_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq m_p \exp \left( -\frac{b_j^2 / 2}{\sigma^2 + T \theta / 3} \right); \] (1.34)
and
\[
\Pr \left( \mathbf{d}_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq m p \exp \left( \frac{-3b_j^2}{8\sigma^2} \right) \text{ for } b_j \leq \sigma^2 / T; \quad (1.35)
\]
and
\[
\Pr \left( \mathbf{d}_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq m p \exp \left( \frac{-3b_j^2}{8T} \right) \text{ for } b_j \geq \sigma^2 / T. \quad (1.36)
\]

**Proof:** Without loss of generality, we can assume that \( T = 1 \) since the summands are 1-homogeneous and the variance is 2-homogeneous. From Lemma 5, we have
\[
\mathbb{E} e^{tX_i} \preceq e^{(e^t - t - 1)\mathbb{E}(X_i^2)} \text{ for } t > 0. \quad (5.17)
\]
By applying Corollary 3, we then have
\[
\Pr \left( \mathbf{d}_{\text{max}} \left( \sum_{i=1}^{n} X_i \right) \geq b \right) \leq m p \min_{t > 0} \inf_{1 \leq j \leq p} \left\{ \exp \left( f(t) \lambda_{\text{max}} \left( \sum_{i=1}^{n} A_i \right) \right) \right\}
\leq m p \exp \left( -tb_j + (e^t - t - 1)\lambda_{\text{max}} \left( \sum_{i=1}^{n} \mathbb{E}(X_i^2) \right) \right)
= m p \exp \left( -tb_j + \sigma^2(e^t - t - 1) \right). \quad (5.18)
\]
The right-hand side of Eq. (5.18) can be minimized by setting \( t = \log(1 + b_j / \sigma^2) \). Substitute such \( t \) and simplify the right-hand side of Eq. (5.18), we obtain Eq. (1.34).

For \( b_j \leq \sigma^2 / T \), we have
\[
\frac{1}{\sigma^2 + Tb_j / 3} \geq \frac{1}{\sigma^2 + T(\sigma^2 / T) / 3} = \frac{3}{4\sigma^2}, \quad (5.19)
\]
then, we obtain Eq. (1.35). Correspondingly, for \( b_j \geq \sigma^2 / T \), we have
\[
\frac{\theta}{\sigma^2 + Tb_j / 3} \geq \frac{\sigma^2 / T}{\sigma^2 + T(\sigma^2 / T) / 3} = \frac{3}{4T}, \quad (5.20)
\]
and, we obtain Eq. (1.36) also. \( \square \)

Below theorem is another variation of T-product tensor Bernstein bounds by subexponential constraints of \( \mathbb{E}(X_i^p) \).

**Theorem 1.10 (Subexponential T-product Tensor Bernstein Bounds for Eigentuple)** Given a finite sequence of independent Hermitian T-product tensors \( \{ X_i \in \mathbb{C}^{m \times m \times p} \} \) that satisfy
\[
\mathbb{E} X_i = 0 \text{ and } \mathbb{E}(X_i^p) \preceq \frac{p!T^{p-2}}{2} A_i^2, \quad (1.37)
\]
where \( p = 2, 3, 4, \ldots \).
Define the total variance $\sigma^2$ as: $\sigma^2 \overset{\text{def}}{=} \| \sum_{i=1}^{n} A_i^2 \|$. Then, given a positive real vector $\mathbf{b} \in \mathbb{R}^p$ with $\tilde{j} \overset{\text{def}}{=} \arg \min_j \{ b_j \}$ and $\sum_{i=1}^{n} t A_i$ satisfying Eq. (1.1) for any $t > 0$, we have following inequalities:

\[
\Pr \left( \left| \sum_{i=1}^{n} X_i \right| \geq \mathbf{b} \right) \leq m_p \exp \left( \frac{-b^2_j / 2}{\sigma^2 + Tb_j} \right); \quad (1.38)
\]

and

\[
\Pr \left( \left| \sum_{i=1}^{n} X_i \right| \geq \mathbf{b} \right) \leq m_p \exp \left( \frac{-b^2_j}{4\sigma^2} \right) \quad \text{for} \quad b_j \leq \sigma^2 / T; \quad (1.39)
\]

and

\[
\Pr \left( \left| \sum_{i=1}^{n} X_i \right| \geq \mathbf{b} \right) \leq m_p \exp \left( \frac{-b^2_j}{4T} \right) \quad \text{for} \quad b_j \geq \sigma^2 / T. \quad (1.40)
\]

**Proof:** Without loss of generality, we can assume that $T = 1$. From Lemma 6, we have

\[
\mathbb{E} \exp \left( t \lambda_{\max} \left( \sum_{i=1}^{n} A_i^2 \right) \right) \leq \exp \left( \frac{t^2 A_i^2}{2(1 - t)} \right), \quad (5.21)
\]

where $0 < t < 1$.

By applying Corollary 3, we then have

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} A_i^2 \right) \geq \theta \right) \leq m_p \inf_{t > 0} \min_i \left\{ \exp \left( f(t) \lambda_{\max} \left( \sum_{i=1}^{n} A_i \right) \right) \right\} \left( \frac{e^{tb_j}}{c} \right),
\]

\[
\leq m_p \exp \left( -tb_j + \frac{t^2}{2(1 - t)} \lambda_{\max} \left( \sum_{i=1}^{n} A_i^2 \right) \right) = m_p \exp \left( -tb_j + \frac{\sigma^2 t^2}{2(1 - t)} \right). \quad (5.22)
\]

The right-hand side of Eq. (5.22) can be minimized by setting $t = \frac{b_j}{b_j + \sigma^2}$. Substitute such $t$ and simplify the right-hand side of Eq. (5.22), we obtain Eq. (1.38).

For $b_j \leq \sigma^2 / T$, we have

\[
\frac{1}{\sigma^2 + Tb_j} \geq \frac{1}{\sigma^2 + T(\sigma^2 / T)} = \frac{1}{2\sigma^2}, \quad (5.23)
\]

then, we obtain Eq. (1.39). Similarly, for $b_j \geq \sigma^2 / T$, we have

\[
\frac{\theta}{\sigma^2 + Tb_j} \geq \frac{\sigma^2 / T}{\sigma^2 + T(\sigma^2 / T)} = \frac{1}{2T}, \quad (5.24)
\]

, therefore, we also obtain Eq. (1.40).
5.3 Application of Tensor Bernstein Bounds

The tensor Bernstein bounds can also be extended to rectangular tensors by dilation. Consider a sequence of tensors \( \{Y_i\} \in \mathbb{C}^{m \times n \times p} \) satisfying following:

\[
\mathbb{E}Y_i = 0 \quad \text{and} \quad \|Y_i\| \leq T \quad \text{almost surely.} \tag{5.25}
\]

If the variance \( \sigma^2 \) is expressed as:

\[
\sigma^2 \overset{\text{def}}{=} \max \left\{ \left\| \sum_{i=1}^{n} Y_i \right\|, \left\| \sum_{i=1}^{n} Y_i^H \right\| \right\}, \tag{5.26}
\]

we have

\[
\Pr \left( \left\| \sum_{i=1}^{n} Y_i \right\| \geq \theta \right) \leq (m + n)p \exp \left( \frac{-\theta^2/2}{\sigma^2 + T \theta/3} \right) \tag{5.27}
\]

from Theorem 1.7.

Another application of tensor Bernstein bounds is to get upper and lower bounds for the maximum eigenvalue with subexponential tensors. This application can relax the corollary 10 conditions by allowing the moments of the random tensors to grow at a controlled rate.

**Corollary 11 (Upper and Lower Bounds for the Maximum Eigenvalue for Subexponential Tensors)** Consider a sequence \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) of independent, random, Hermitian T-product tensors that satisfy

\[
X_i \succeq O \quad \text{and} \quad \mathbb{E}(X_i^p) \leq \frac{m^{p-2}}{2} \lambda_i^2, \tag{5.28}
\]

and \( \sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} A_i^2 \right\| \). Then, we have

\[
\mu_{\max} \leq \mathbb{E} \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \leq 2mp \left( \sigma J \left( \frac{\sigma}{2T} \right) + 2Te^{-\frac{\sigma^2}{4T}} \right), \tag{5.29}
\]

where \( J(\sigma/2T) \overset{\text{def}}{=} \int_{0}^{\sigma/2T} e^{-s^2} \, ds \).

**Proof:** The lower bound at Eq. (5.29) is true from the convexity of the function \( A \to \lambda_{\max}(A) \) and the Jensen’s inequality.

For the upper bound, we have

\[
\mathbb{E} \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) = \int_{0}^{\infty} \Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq t \right) \, dt
\]

\[\leq 1 \int_{0}^{\frac{\sigma^2}{4T}} \exp \left( -\frac{t^2}{4\sigma^2} \right) \, dt + mp \int_{\frac{\sigma^2}{4T}}^{\infty} \exp \left( -\frac{t}{4T} \right) \, dt
\]

\[= 2mp \left( \sigma \Theta \left( \frac{\sigma}{2T} \right) + 2Te^{-\frac{\sigma^2}{4T}} \right), \tag{5.30}
\]

where the inequality \( \leq 1 \) comes from the Eqs. (1.31) and (1.32). This corollary is proved by introducing Gaussian integral function \( \Theta(x) \overset{\text{def}}{=} \int_{0}^{x} e^{-s^2} \, ds \).
# 6 T-product Tensor Martingales Inequalities

In this section, we introduce concepts about T-product tensor martingales in Section 6.1 and extend Hoeffding, Azuma, and McDiarmid inequalities to random T-product tensors context in Section 6.2. These bounds are extended to the eigentuple version in Section 6.3.

## 6.1 T-product Tensor Martingales

Several basic definitions about T-product tensor martingales will be provided here for later T-product tensor martingales related bounds. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a master probability space. Consider a filtration $\{F_i\}$ contained in the master sigma algebra as:

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\infty \subset \mathcal{F}. \quad (6.1)$$

Given such a filtration, we define the conditional expectation $E_i[\cdot] \triangleq E_i[\cdot | F_i]$. A sequence $\{Y_i\}$ of random tensors is called adapted to the filtration when each tensor $Y_i$ is measurable with respect to $F_i$. We can think that an adapted sequence is one where the present depends only on the past.

An adapted sequence $\{X_i\}$ of Hermitian T-product tensors is named as a tensor martingale when

$$E_{i-1}X_i = X_{i-1} \quad \text{and} \quad E\|X_i\| < \infty, \quad (6.2)$$

where $i = 1, 2, 3, \cdots$. We obtain a scalar martingale if we track any fixed entry of a tensor martingale $\{X_i\}$.

Given a tensor martingale $\{X_i\}$, we can construct the following new sequence of tensors

$$Y_i \triangleq X_i - X_{i-1} \quad \text{for} \ i = 1, 2, 3, \cdots \quad (6.3)$$

We then have $E_{i-1}Y_i = 0$.

## 6.2 Tensor Martingale Deviation Bounds for Eigenvalues

Two Lemmas should be presented first before presenting tensor martingale deviation bounds and their proofs.

**Lemma 7 (Tensor Symmetrization)** Let $A \in \mathbb{C}^{m \times m \times p}$ be a fixed Hermitian T-product tensor, and let $X$ be a random Hermitian T-product tensor with $E X = O$. Then

$$E \text{Tr} e^{A + X} \leq E \text{Tr} e^{A + 2\beta X}, \quad (6.4)$$

where $\beta$ is a Rademacher random variable.

**Proof:** Build an independent copy random tensor $Y$ from $X$, and let $E_Y$ denote the expectation with respect to the new random tensor $Y$. Then, we have

$$E \text{Tr} e^{A + X} = E \text{Tr} e^{A + X - E_Y Y} \leq E \text{Tr} e^{A + (X - Y)} = E \text{Tr} e^{A + \beta (X - Y)}, \quad (6.5)$$

where the first equality uses $E_Y Y = O$; the inequality uses the convexity of the trace exponential with Jensen’s inequality; finally, the last equality comes from that the random tensor $X - Y$ is a symmetric random tensor and Rademacher is also a symmetric random variable.

This Lemma is established by the following:

$$E \text{Tr} e^{A + X} \leq E \text{Tr} \left( e^{A/2 + \beta X} e^{A/2 - \beta Y} \right) \leq \left( E \text{Tr} e^{A + 2\beta X} \right)^{1/2} \left( E \text{Tr} e^{-2\beta Y} \right)^{1/2} = E \text{Tr} e^{A + 2\beta X}. \quad (6.6)$$
where the first inequality comes from T-product tensor Golden-Thompson inequality by Theorem 1, the second inequality comes from the Cauchy-Schwarz inequality, and the last identity follows from that the two factors are identically distributed.

Following lemma is introduced to provide the tensor cumulant-generating function of a symmetrized random tensor.

**Lemma 8 (Cumulant-Generating Function of Symetrized Random T-product Tensor)** Given that $X \in \mathbb{C}^{m \times m \times p}$ is a random Hermitian T-product tensor and $A \in \mathbb{C}^{m \times m \times p}$ is a fixed Hermitian T-product tensor that satisfies $X^2 \preceq A^2$. Then, we have

$$\log \mathbb{E} \left[ e^{2\beta tX} \right] \preceq 2t^2 A^2,$$

where $\beta$ is a Rademacher random variable.

**Proof:** From Lemma 3 we have

$$\mathbb{E} \left[ e^{2\beta tX} \right] \preceq e^{2t^2 X^2}.$$ (6.8)

And, from the monotone property of logarithm, we also have

$$\log \mathbb{E} \left[ e^{2\beta tX} \right] \preceq 2t^2 X^2 \preceq 2t^2 A^2 \text{ for } t \in \mathbb{R}.$$ (6.9)

Therefore, this Lemma is proved. $\square$

In probability theory, the Azuma inequality for a scaler martingale gives normal concentration about its mean value, and the deviation is controlled by the total maximum squared of the difference sequence. Following theorem is the T-product tensor version for Azuma inequality.

**Theorem 1.11 (T-product Tensor Azuma Inequality for Eigenvalue)** Given a finite adapted sequence of Hermitian tensors $\{X_i \in \mathbb{C}^{m \times m \times p}\}$ and a fixed sequence of Hermitian T-product tensors $\{A_i\}$ that satisfy

$$\mathbb{E} X_i = 0 \text{ and } X_i^2 \preceq A_i^2 \text{ almost surely},$$

where $i = 1, 2, 3, \ldots$.

Define the total variance $\sigma^2$ as:

$$\sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} A_i^2 \right\|.$$ Then, we have following inequalities:

$$\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{n\sigma^2}}.$$ (1.42)

**Proof:** Define the filtration $\mathcal{F}_i \overset{\text{def}}{=} \mathcal{F}(X_1, \ldots, X_i)$ for the process $\{X_i\}$. Then, we have

$$\mathbb{E} \text{Tr} \exp \left( \sum_{i=1}^{n} tX_i \right) = \mathbb{E} \left( \mathbb{E} \left( \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + tX_n \right) \mid \mathcal{F}_n \right) \mid \mathcal{F}_{n-1} \right) \leq \mathbb{E} \left( \mathbb{E} \left( \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + 2\beta tX_n \right) \mid \mathcal{F}_n \right) \mid \mathcal{F}_n \right) \leq \mathbb{E} \left( \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + \log \mathbb{E} \left( e^{2\beta tX_n} \mid \mathcal{F}_n \right) \right) \right) \leq \mathbb{E} \text{Tr} \exp \left( \sum_{i=1}^{n-1} tX_i + 2t^2 A_n^2 \right),$$

(6.10)
where the first equality comes from the total expectation property of conditional expectation; the first inequality comes from Lemma 7; the second inequality comes from Corollary 1 and the relaxation for the conditioning on the inner expectation to the larger algebra \( \mathcal{F}_n \); finally, the last inequality requires Lemma 8.

If we continue the iteration procedure based on Eq. (6.10), we have

\[
\mathbb{E} \text{Tr} \exp \left( \sum_{i=1}^{n} tX_i \right) \leq \text{Tr} \exp \left( 2t^2 \sum_{i=1}^{n} A_i^2 \right),
\]  

(6.11)

then apply Eq. (6.11) into Lemma 1, we obtain

\[
\Pr \left( \lambda_{\max} \left( \sum_{i=1}^{n} X_i \right) \geq \theta \right) \leq \inf_{t > 0} \left\{ e^{-t \mathbb{E} \text{Tr} \exp \left( \sum_{i=1}^{n} tX_i \right)} \right\}
\[
\leq \inf_{t > 0} \left\{ e^{-t \mathbb{E} \text{Tr} \exp \left( 2t^2 \sum_{i=1}^{n} A_i^2 \right)} \right\}
\[
\leq \inf_{t > 0} \left\{ e^{-t \mathbb{E} \exp \left( 2t^2 \sigma^2 \right)} \right\}
\[
\leq \inf_{t > 0} \left\{ e^{-t \mathbb{E} \exp \left( 2t^2 \sigma^2 \right)} \right\}
\[
\leq \inf_{t > 0} \left\{ e^{-t \mathbb{E} \exp \left( 2t^2 \sigma^2 \right)} \right\}
\[
= \inf_{t > 0} \left\{ e^{-t \mathbb{E} \exp \left( 2t^2 \sigma^2 \right)} \right\}
\[
\leq mpe^{-\frac{\theta^2}{8\sigma^2}},
\]  

(6.12)

where the third inequality utilizes \( \lambda_{\max} \) to bound trace, the equality applies the definition of \( \sigma^2 \) and spectral mapping theorem, finally, we select \( t = \frac{\theta}{4\sigma^2} \) to minimize the upper bound to obtain this theorem. \( \square \)

If we add extra assumption that the summands are independent, Theorem 1.11 gives a T-product tensor extension of Hoeffding’s inequality. If we apply Theorem 1.11 to a Hermitian T-product tensor martingale, we will have following corollary.

**Corollary 12** Given a Hermitian T-product tensor martingale \( \{ Y_i : i = 1, 2, \cdots, n \} \in \mathbb{C}^{m \times m \times p} \), and let \( X_i \) be the difference sequence of \( \{ Y_i \} \), i.e., \( X_i \overset{\text{def}}{=} Y_i - Y_{i-1} \) for \( i = 1, 2, 3, \cdots \). If the difference sequence satisfies

\[
\mathbb{E} X_i = 0 \quad \text{and} \quad X_i^2 \preceq A_i \quad \text{almost surely},
\]  

(6.13)

where \( i = 1, 2, 3, \cdots \) and the total variance \( \sigma^2 \) is defined as:

\[
\sigma^2 \overset{\text{def}}{=} \left\| \sum_{i} A_i^2 \right\|.
\]

Then, we have

\[
\Pr \left( \lambda_{\max} \left( Y_n - \mathbb{E} Y_n \right) \geq \theta \right) \leq mpe^{-\frac{\theta^2}{8\sigma^2}}.
\]  

(6.14)

In the scalar setting, McDiarmid inequality can be treated as a corollary of Azuma’s inequality. McDiarmid inequality states that a function of independent random variables exhibits normal concentration about its mean, and the variance depends on the function value sensitivity with respect to the input. Following theorem is the McDiarmid inequality for the T-product tensor.

**Theorem 1.12 (T-product Tensor McDiarmid Inequality)** Given a set of \( n \) independent random variables, i.e. \( \{ X_i : i = 1, 2, \cdots, n \} \), and let \( F \) be a Hermitian T-product tensor-valued function that maps these \( n \) random variables to a Hermitian T-product tensor of dimension within \( \mathbb{C}^{m \times m \times p} \). Consider a sequence of Hermitian tensors \( \{ A_i \} \) that satisfy

\[
\left( F(x_1, \cdots, x_i, \cdots, x_n) - F(x_1, \cdots, x_i', \cdots, x_n) \right)^2 \preceq A_i^2,
\]  

(1.43)
where \( x_i, x'_i \in X_i \) and \( 1 \leq i \leq n \). Define the total variance \( \sigma^2 \) as: 
\[
\sigma^2 \overset{\text{def}}{=} \left\| \sum_{i=1}^{n} A_i^2 \right\|
\]
Then, we have following inequality:
\[
\Pr (\lambda_{\max} (F(x_1, \cdots, x_n) - \mathbb{E} F(x_1, \cdots, x_n)) \geq \theta) \leq m \text{pe} \frac{\sigma^2}{8n^2}.
\] (1.44)

**Proof:** We define following random tensors \( \mathcal{Y}_i \) for \( 0 \leq i \leq n \) as:
\[
\mathcal{Y}_i \overset{\text{def}}{=} \mathbb{E} (F(x_1, \cdots, x_n)|X_1, \cdots, X_i) = \mathbb{E}_{X_{i+1}} \mathbb{E}_{X_{i+2}} \cdots \mathbb{E}_{X_n} F(x_1, \cdots, x_n),
\]
where \( \mathbb{E}_{X_{i+1}} \) is the expectation with respect to the random variable \( X_{i+1} \). The constructed sequence \( \mathcal{Y}_i \) forms a martingale. The associated difference sequence with respect to \( \mathcal{Y} \), denoted as \( \{Z_i\} \), can be stated as:
\[
Z_i \overset{\text{def}}{=} \mathcal{Y}_i - \mathcal{Y}_{i-1} = \mathbb{E}_{X_{i+1}} \mathbb{E}_{X_{i+2}} \cdots \mathbb{E}_{X_n} (F(x_1, \cdots, x_n) - \mathbb{E}_{X_i} F(x_1, \cdots, x_n)) \quad (6.16)
\]
Because \( (x_1, \cdots, x_i) \) forms a filtration with respect to \( i \), we have
\[
\mathbb{E}_{X_{i-1}} \mathcal{Y}_i = \mathbb{E}_{X_{i-1}} \mathbb{E}_{X_{i+1}} (\cdots) = \mathbb{E}_{X_{i-1}} \mathbb{E}_{X_i} (\cdots) = \mathbb{E}_{X_{i-1}} \mathcal{Y}_{i-1}, \quad (6.17)
\]
then,
\[
\mathbb{E}_{X_{i-1}} Z_i = \mathbb{E}_{X_{i-1}} \mathcal{Y}_i - \mathbb{E}_{X_{i-1}} \mathcal{Y}_{i-1} = 0 \quad (6.18)
\]
Let \( X'_i \) be an independent copy of \( X_i \), and construct the following two random vectors:
\[
\begin{align*}
x' &= (X_1, \cdots, X_{i-1}, X'_i, X_{i+1}, \cdots, X_n), \\
x &= (X_1, \cdots, X_{i-1}, X_i, X_{i+1}, \cdots, X_n). \quad (6.19)
\end{align*}
\]
Note that \( \mathbb{E}_{X_i} F(x) = \mathbb{E}_{X'_i} F(x') \) and \( F(x) \) does not depend on \( X'_i \), we can express \( Z_i \) from Eq. \( 6.16 \) as
\[
Z_i = \mathbb{E}_{X_{i+1}} \mathbb{E}_{X_{i+2}} \cdots \mathbb{E}_{X_n} \mathbb{E}_{X'_i} (F(x) - F(x')) \quad (6.20)
\]
Since two vectors \( x \) and \( x' \) are differ only at the \( i \)-th position, we have
\[
(F(x) - F(x'))^2 \leq A_i^2 \quad (6.21)
\]
from requirement provided by Eq. \( 1.43 \). Then, we have following upper bound
\[
\mathbb{E}_{X_{i+1}} \mathbb{E}_{X_{i+2}} \cdots \mathbb{E}_{X_n} \mathbb{E}_{X'_i} (F(x) - F(x'))^2 \leq A_i^2 \quad (6.22)
\]
Therefore, from conditions provided by Eq. \( 6.18 \) and Eq. \( 6.22 \), this theorem is proved by applying Corollary \( 12 \) to the martingale \( \{\mathcal{Y}_i\} \). \( \square \)

### 6.3 Tensor Martingale Deviation Bounds for Eigentuple

In this section, we will extend results about martingale deviation bounds for eigenvalues from Section 6.2 to martingale deviation bounds for eigentuple.
Theorem 1.13 (T-product Tensor Azuma Inequality for Eigentuple) Given a finite adapted sequence of Hermitian tensors \( \{X_i \in \mathbb{C}^{m \times m \times p}\} \) and a fixed sequence of Hermitian T-product tensors \( \{A_i\} \) that satisfy
\[
\mathbb{E} X_{i-1} = 0 \quad \text{and} \quad X_i^2 \preceq A_i^2 \text{ almost surely},
\] (1.45)
where \( i = 1, 2, 3, \ldots \).

Define the total variance \( \sigma^2 \) as:
\[
\sigma^2 \overset{\text{def}}{=} \left| \sum_{i=1}^{n} A_i^2 \right|.
\]
Then, given a positive real vector \( b \in \mathbb{R}^p \) with \( \tilde{j} \overset{\text{def}}{=} \arg \min_j \{b_j\} \) and \( t X_i \) satisfying Eq. (1.1) for any \( t > 0 \), we have following inequalities:
\[
\Pr \left( \max_{i=1}^{n} X_i \geq b \right) \leq mpe^{-\frac{b^2}{8\sigma^2}}.
\] (1.46)

**Proof:** From Eq. (6.10), we have
\[
\mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{n} t X_i \right) \leq \mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{n-1} t X_i + 2t^2 A_i^2 \right).
\] (6.23)
If we continue the iteration procedure based on Eq. (6.23), we have
\[
\mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{n} t X_i \right) \leq \operatorname{Tr} \exp \left( 2t^2 \sum_{i=1}^{n} A_i^2 \right),
\] (6.24)
then apply Eq. (6.24) into Lemma 2, we obtain
\[
\Pr \left( \max_{i=1}^{n} X_i \geq b \right) \leq \inf_{t > 0} \min_{1 \leq j \leq p} \left\{ \frac{\mathbb{E} \left( \operatorname{Tr} \left( \exp \left( \sum_{i=1}^{n} t X_i \right) \right) \right) }{e^{b \tilde{j}}} \right\}
\]
\[
\leq \inf_{t > 0} \left\{ e^{-tb} \mathbb{E} \operatorname{Tr} \exp \left( \sum_{i=1}^{n} t X_i \right) \right\}
\]
\[
\leq \inf_{t > 0} \left\{ e^{-tb} \mathbb{E} \operatorname{Tr} \exp \left( 2t^2 \sum_{i=1}^{n} A_i^2 \right) \right\}
\]
\[
\leq \inf_{t > 0} \left\{ e^{-tb} m \lambda_{\max} \left( \operatorname{exp} \left( 2t^2 \sum_{i=1}^{n} A_i^2 \right) \right) \right\}
\]
\[
= \inf_{t > 0} \left\{ e^{-tb} m \exp \left( 2t^2 \sigma^2 \right) \right\}
\]
\[
\leq mpe^{-\frac{b^2}{8\sigma^2}},
\] (6.25)
where the third inequality utilizes \( \lambda_{\max} \) to bound trace, the equality applies the definition of \( \sigma^2 \) and spectral mapping theorem, finally, we select \( t = \frac{b_{\tilde{j}}}{2\sigma^2} \) to minimize the upper bound to obtain this theorem. \( \square \)

If we add an extra assumption that the summands are independent, Theorem 1.13 gives a T-product tensor extension of Hoeffding’s inequality. If we apply Theorem 1.13 to a Hermitian T-product tensor martingale, we will have the following corollary.
Corollary 13 Given a Hermitian $T$-product tensor martingale $\{Y_i : i = 1, 2, \cdots, n\} \in \mathbb{C}^{m \times m \times p}$, and let $\mathcal{X}_i$ be the difference sequence of $\{Y_i\}$, i.e., $\mathcal{X}_i \triangleq Y_i - Y_{i-1}$ for $i = 1, 2, 3, \cdots$. If the difference sequence satisfies

$$
\mathbb{E}_{i-1} \mathcal{X}_i = 0 \text{ and } \mathcal{X}_i^2 \preceq A_i \text{ almost surely},
$$

(6.26)

where $i = 1, 2, 3, \cdots$ and the total variance $\sigma^2$ is defined as: $\sigma^2 \triangleq \left\| \sum_i A_i^2 \right\|$. Then, given a positive real vector $b \in \mathbb{R}^p$ with $\tilde{j} \triangleq \arg \min_j \{b_j\}$ and $t (Y_n - \mathbb{E}Y_n)$ satisfying Eq. (1.1) for any $t > 0$, we have

$$
\Pr (d_{\max} (Y_n - \mathbb{E}Y_n) \geq b) \leq mpe^{-\frac{\tilde{j}^2}{8\sigma^2}}.
$$

(6.27)

Following theorem is the McDiarmid inequality of the maximum eigentuple for the $T$-product tensor.

Theorem 1.14 (T-product Tensor McDiarmid Inequality for Eigentuple) Given a set of $n$ independent random variables, i.e. $\{X_i : i = 1, 2, \cdots, n\}$, and let $F$ be a Hermitian $T$-product tensor-valued function that maps these $n$ random variables to a Hermitian $T$-product tensor of dimension within $\mathbb{C}^{m \times m \times p}$. Consider a sequence of Hermitian tensors $\{A_i\}$ that satisfy

$$
(F(x_1, \cdots, x_i, \cdots, x_n) - F(x_1, \cdots, x'_i, \cdots, x_n))^2 \preceq A_i^2,
$$

(1.47)

where $x_i, x'_i \in X_i$ and $1 \leq i \leq n$. Define the total variance $\sigma^2$ as: $\sigma^2 \triangleq \left\| \sum_i A_i^2 \right\|$. Then, given a positive real vector $b \in \mathbb{R}^p$ with $\tilde{j} \triangleq \arg \min_j \{b_j\}$ and $t (F(x_1, \cdots, x_n) - \mathbb{E}F(x_1, \cdots, x_n))$ satisfying Eq. (1.1) for any $t > 0$, we have following inequality:

$$
\Pr (d_{\max} (F(x_1, \cdots, x_n) - \mathbb{E}F(x_1, \cdots, x_n)) \geq b) \leq mpe^{-\frac{\tilde{j}^2}{8\sigma^2}}.
$$

(1.48)

Proof: By the same argument from the proof in Theorem 1.12 this theorem is proved by applying Corollary 13 to the martingale $\{Y_i\}$.

7 Conclusion

In Part I paper of this serious work about $T$-product tensors, we generalize Laplace transform method and Lieb’s concavity theorem from matrices to $T$-product tensors. In this Part II paper, we apply these techniques to extend the following classical bounds from the scalar to the $T$-product tensor settings: Chernoff and Bernstein inequalities. The purpose of these probability inequalities tries to identify large-deviation behavior of the extreme eigenvalue and eigentuple of the sums of random $T$-product tensors. Finally, we also apply these proof techniques developed at this work to study $T$-product tensor-valued martingales by proving Azuma, Hoeffding, and McDiarmid inequalities under $T$-product.
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