Superradiance occurs when an energetic, radiating region scatters incident waves with a net amplification [1]. Paradigmatic manifestations are the Penrose process, by which particles gain momentum from a rotating black hole [2] and the Zel’dovich mechanism, where a rotating metallic cylinder increases electromagnetic oscillations [3]. Extraction of energy from rotating systems has been investigated in optics [4–5], acoustics [6], hydrodynamics [7, 8] and quantum mechanics [9–11].

Here, we consider a scenario where superradiant scattering is achieved by using a self-oscillating cavity mode. More specifically, energy is steadily supplied to the cavity, destabilizing one of its modes to small perturbations, thus leading, under the action of internal nonlinearities, to a limit cycle thus leading, under the action of internal nonlinearities, destabilizing one of its modes to small perturbations, resulting in self-sustained 2-ports-1-mode system confirm our theoretical framework.

To model this phenomenon, we derive a scattering theory of NLC which is related to the temporal coupled-mode theory (TCMT) of Fan et al. [12]. TCMT is a generic, multi-physical framework describing unitary waveguide transmission through linearly stable resonant waveguides [13–16]. In contrast, we are interested in modeling nonlinear, non-unitary scattering by cavities exhibiting a self-oscillating mode. Despite fruitful recent applications of coupled-mode theories in optics [17–18], hydrodynamics [19–20], quantum mechanics [21] and acoustics [22–24], no analogous framework exists for nonlinear or linearly unstable systems. By correspondence, any consistent attempt at such a generalization has to recover TCMT in the weakly forced limit.

For a cavity with $n$ modes and $m > n$ ports (Fig. 1), the equations governing a general coupled-mode theory take the following form:

$$\dot{a} = f(a) + K^T|s_{in}|,$$

$$|s_{out}| = G|s_{in}| + Da,$$

where $a \in \mathbb{C}^n$ are the modal amplitudes, $|s_{in}\rangle$ and $|s_{out}\rangle$ are the incoming and outgoing waves, the matrix $G \in \mathbb{C}^{m \times n}$ incorporates linear wave-to-wave direct coupling, $K, D \in \mathbb{C}^{m \times n}$ are coupling matrices and $f: \mathbb{C}^n \to \mathbb{C}^n$ describes the nonlinear modal dynamics. We focus on conditions for which phase locking or suppression of self-sustained modal dynamics is achieved due to small enough detuning and large enough forcing amplitude [27]. For forcing from the $j$th port, $|s_{in}\rangle = |s_j\rangle e^{i\omega t}$, then, we seek the forced response $\tilde{a}$ as the steady-amplitude solution of Eq. (1) at the forcing frequency $\omega$:

$$0 = -i\omega \tilde{a} + f(\tilde{a}) + K^T|s_j\rangle e^{i\omega t}.$$  

(3)

For a general model, the nonlinear scattering matrix can be obtained explicitly from the definition $|s_{out}\rangle = S|s_{in}\rangle$ and the Moore–Penrose pseudoinverse [28]:

$$S = \frac{|s_{out}\rangle \langle s_{in}|}{\langle s_{in}|s_{in}\rangle}.$$  

(4)

As sketched in Fig. 2, the NLC coupling matrices are inferred by exploring the corresponding limits of related coupled-mode theories. To relate the decay rate $\gamma$ of a forced resonance in TCMT to the growth rate $\nu$ of a linearly unstable oscillator in NLC, TCMT is interpreted as the linearization of an auxiliary linear theory (NLT) around the unforced solution $a_0 + a'$, where $a'$ is a small perturbation [29]. The scattering equations of the coupled-mode theories shown in Fig. 2 are listed in Table 1 for the case of a single mode $a \in \mathbb{C}$. Note that in both NLT and NLC, for strong forcing $||G|s_{in}||| \gg ||Da_0||$, due to the cubic terms in the equation for $a$, $|a|$ is a sublinear function of $s$ in the steady state. Therefore and because the unforced solution $a_0$ is independent of $s$, in all three considered coupled-mode theories, the terms involving $|s_{in}|$ become increasingly dominant in the equation for $|s_{out}|$ as $s$ is increased. While NLT may seem
as an obvious candidate for generalizing TCMT, by the above argument, it has the undesirable property that for strong forcing, the $S$-matrix defined by Eq. (4) converges to the background scattering matrix $C$, which describes scattering in the waveguide in the absence of a cavity [10]. A different candidate for generalizing TCMT, the NLC theory, which corresponds to TCMT both in the weakly forced limit $s \to 0$ [21] and for strong forcing, is therefore chosen to model the nonlinear scattering process.

In this letter, we treat in detail the special case of 2-port scattering ($m = 2$) by a single self-oscillating mode ($n = 1$) whose modal dynamics are approximated by a Stuart–Landau oscillator, the normal form of a supercritical Hopf bifurcation (SHB) [31,32]. Following [10], we account for internal losses by the intrinsic decay rate $\gamma_i$:

$$D^j D = 2(\gamma - \gamma_i).$$

(5)

Throughout this work, $(\cdot)^\dagger$ denotes the Hermitian conjugate and $(\cdot)^*$ the complex conjugate. Within the TCMT framework, time-reversed perturbations grow at the effective growth rate $\gamma - \gamma_i$, indicating that the presence of losses breaks the time symmetry. Using a time-reversal argument, the following relation can be derived [29]:

$$K^* = D,$$

(6)

which is in agreement with previous studies [14,15] and which, by correspondence, holds also for the NLC coupling matrices. Consistent with the above discussion, the following set of equations is used in this letter to model the nonlinear scattering process:

$$\dot{a} = (i\omega_0 + \nu - \kappa |a|^2)a + D^\dagger |s_{in}\rangle,$$

$$|s_{out}\rangle = (C + DF^{-1} D^\dagger)|s_{in}\rangle + D a,$$

(7)

(8)

where $\omega_0 \in \mathbb{R}$ and $\nu \in \mathbb{R}$ are the eigenfrequency and linear decay or growth rate of the cavity eigenmode, $\kappa \in \mathbb{R}^+$ is the nonlinearity constant, $F = i(\omega - \omega_0) + \gamma$, and $2\gamma = |\nu| [3 + \text{sgn}(\nu)]$ [23,32,34].

By separately applying forcing from each port with $|s_{in}\rangle = s|j\rangle e^{i\omega t}$, using Eq. (1), the NLC scattering matrix is found to be

$$S = C + DF^{-1} D^\dagger + \frac{1}{s} D \sum_{j=1}^m \rho_j e^{i\varphi_j} |j\rangle,$$

(9)

where $m = 2$ is the number of ports and $\dot{a}_j = \rho_j e^{i(\omega t + \varphi_j)}$ is the forced response resulting from forcing the cavity through the $j$th port. For $|s_{in}\rangle = s|j\rangle e^{i\omega t}$, $\rho_j \in \mathbb{R}^+$ and $\varphi_j \in [0, 2\pi)$ satisfy Eq. (24), which is equivalent to

$$0 = \kappa^2 \rho^2_0 - 2\nu \kappa \rho^2_0 + (\nu^2 + \Delta^2) \rho^2_0 - |D_j|^2 s^2,$$

$$\varphi_j = -\arg D_j - \arcsin \left( \frac{\Delta \rho_j}{|D_j|s} \right),$$

(10)

(11)

where $\Delta = \omega - \omega_0$ is the detuning [29]. Equation (10) is a cubic equation for $\rho_0^2$ which, for the parameter values considered in this work, always has a single real zero corresponding to a stable forced response [35].

The background scattering matrix $C$ is a problem-specific, user-defined quantity. The coupling matrix $D$ is derived within the TCMT framework and inferred in NLC by correspondence. To generalize the classic TCMT in literature to non-unitary scattering, we specify $D$ indirectly through the ideal scattering matrix $S$, multiplied

| theory | scattering equations |
|--------|----------------------|
| TCMT $a = a_0 + a'$ | $\dot{a} = (i\omega_0 - \gamma)a + K^\dagger |s_{in}\rangle$
$|s_{out}\rangle = C|s_{in}\rangle + Da$
$= (C + DF^{-1} K^\dagger)|s_{in}\rangle + Da$
| NLT | $\dot{a} = (i\omega_0 + \nu - \kappa |a|^2)a + K^\dagger |s_{in}\rangle$
$|s_{out}\rangle = C|s_{in}\rangle + Da$
| NLC | $\dot{a} = (i\omega_0 + \nu - \kappa |a|^2)a + K^\dagger |s_{in}\rangle$
$|s_{out}\rangle = (C + DF^{-1} K^\dagger)|s_{in}\rangle + Da$

TABLE I. Equations describing scattering at a single mode $a \in \mathbb{C}$ according to the coupled-mode theories shown in Fig. 2. The matrices $K$, $C$ and $D$ are undetermined a priori and $F = i(\omega - \omega_0) + \gamma$, $\kappa \in \mathbb{R}^+$ is the nonlinearity constant. $a_0$ is the unforced solution ($|s_{in}\rangle = 0$) of the NLC/NLT modal dynamics, and $a'$ is a small perturbation.
with the unitarity factor $\sigma \in [0, 1]$. This approach is motivated by the objective of tailoring the coupling ports to ensure ideal scattering for a given forcing intensity $s$. We begin our derivation by assuming resonant forcing ($F = \gamma$), formally approximating the TCMT scattering matrix (for $a_0 = 0$) $C + DF^{-1}D^\dagger$ with $\sigma S_*$ and rearranging to get
\[
\sigma S_* - C \approx D\gamma^{-1}D^\dagger,
\] (12)
which can be understood as a low-rank approximation problem of the $m$-by-$m$ matrix $\sigma S_* - C$ by the product of the $m$-by-$n$ matrix $D\sqrt{\gamma^{-1}}$ and its Hermitian transpose. To obtain a coupling matrix $D$ which captures the dominant spectral characteristics of $\sigma S_*$, we formulate the following two consistency conditions to determine both $D$ as well as the ideal parameter configuration, which we call "perfect matching":

(C1) The $n$ (out of $m$) eigenvalues $\mu_j$ of $D\gamma^{-1}D^\dagger$ with largest absolute value $|\mu_j| > 0$ and the corresponding eigenvectors must coincide with those of $\sigma S_* - C$.

(C2) For a perfectly matched scatterer, the $n$ (out of $m$) eigenvalues $|\lambda_j|$ of $C + D\gamma^{-1}D^\dagger$ with largest absolute value and the corresponding eigenvectors must coincide with those of $\sigma S_*$. For these, $|\lambda_j| = \sigma$. For $\sigma = 1$, $C + D\gamma^{-1}D^\dagger = S_*$ is unitary.

We recall that the definition of $S$ given in Eq. (4) implies a convention by which the diagonal elements of $S$ are reflection- and the off-diagonal elements transmission coefficients. The derivation of $C$ and $D$ is carried out here for a reflectional superradiant 2-portscattered described by
\[
S_* = \begin{pmatrix}
1 + \epsilon/\sigma & 0 \\
0 & 1 - \epsilon/\sigma
\end{pmatrix},
\] (13)
where $\epsilon$ is the asymmetry. We impose a lossless, purely transmissive background:
\[
C = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\] (14)

By the Hermiticity of the chosen $S_*$ and $C$, we can expand the 2-by-2 matrix $\sigma S_* - C$ in terms of its eigenvectors $v_j \in \mathbb{C}^2$ and $\mu_j \in \mathbb{R}$ as follows:
\[
\sigma S_* - C = \mu_1 v_1 v_1^\dagger + \mu_2 v_2 v_2^\dagger,
\] (15)
where $|\mu_1| > |\mu_2|$. To satisfy (C1), we define $D = v_1 \sqrt{\mu_1}$, so that the spectrum of the matrix $D\gamma^{-1}D^\dagger = \mu_1 v_1 v_1^\dagger$ spans the subspace corresponding to the eigenvalue $\mu_1$ of $\sigma S_* - C$ with the largest absolute value. From (C2), we deduce that the perfect matching condition is $\epsilon = 0$. Written out, the coupling matrix $D$ is given by
\[
D = \sqrt{\gamma h(\sigma, \epsilon)} \begin{pmatrix}
-g(\epsilon) \\ 1
\end{pmatrix},
\] (16)
where $\gamma = 2\nu$, $g(\epsilon) = \epsilon + \sqrt{\epsilon^2 + 1}$ and $h(\sigma, \epsilon) = (\sigma + \sqrt{\epsilon^2 + 1})/|g(\epsilon)^2 + 1|$. It is straightforward to verify using Eq. (5) that the internal decay rate $\gamma_i$ is related to $\sigma$ and $\epsilon$ by
\[
\frac{\gamma_i}{\gamma} = 1 - \frac{\sigma + \sqrt{\epsilon^2 + 1}}{2}.
\] (17)

Having derived $D$, the analytical $S$-matrix for a biased 2-ports waveguide coupled to a nonlinear self-oscillating mode with $\nu > 0$ follows directly from Eq. (9):
\[
S = \begin{pmatrix}
\frac{2\nu g(\epsilon) h(\sigma, \epsilon)}{\gamma} & 1 - \frac{2\nu g(\epsilon) h(\sigma, \epsilon)}{\gamma} \\
2\nu - i\Delta & 2\nu - i\Delta
\end{pmatrix} + \frac{\sqrt{2\nu h(\sigma, \epsilon)}}{s} \begin{pmatrix}
\rho_1 e^{i\varphi_1} g(\epsilon) & -\rho_2 e^{i\varphi_2} g(\epsilon) \\
\rho_2 e^{i\varphi_2} g(\epsilon) & \rho_1 e^{i\varphi_1} g(\epsilon)
\end{pmatrix}.
\] (18)

At each $\omega$, $\rho_{1,2}$ and $\varphi_{1,2}$ are obtained by applying Cardano’s formula to Eq. (10) with $j = 1, 2$ and substituting the results into Eq. (11).
We now focus on the distribution of energy of the scattered waves in the ports, which is determined by the absolute values of the entries of $S$, $|S_{ij}|$. We define the absorption coefficients for forcing from the $j$th port as $\alpha_j = 1 - |S_{1j}|^2 - |S_{2j}|^2$, $j = 1, 2$. Superradiant scattering is defined as $\alpha_j < 0$, implying a net amplification.

In the unitary, perfectly matched limit, the energy reflection coefficient $|S_{11}|^2$ corresponds to the TCMT result of Fan et al., given by Eq. (17) of Ref. [12]. For comparison, set $\{\rho_1, \rho_2, \rho, \sigma\} = \{0, 0, 0, 1\}$ in Eq. (18) and $\{1/\tau, \phi, t, \tau\} = \{2\nu, -\pi/2, 1, 0\}$ in the reference.

A numerical example of the scattering matrix given by Eq. (18) is shown in Fig. 4 [36]. $S$ strongly depends on the normalized forcing amplitude $\bar{s} = s/\sqrt{|a_0|}$, undergoing a nonlinear transition from omnidirectional to purely reflectional superradiant scattering as $\bar{s}$ is increased. The absorption coefficients are each negative over a continuous range of amplitudes and frequencies, resulting from the energy production of the self-sustained mode. While the transition at low amplitudes is sharp, at higher amplitudes saturation occurs and the superradiant state persists with little change.

To confirm our theoretical analysis, a superradiant scatterer is experimentally realized by means of a self-sustained aeroacoustic mode in a side cavity which is connected to an acoustic waveguide with anechoic terminations [37]. The cavity whistling is obtained by imposing a low Mach air flow in the waveguide with a bulk velocity that exceeds the threshold of a SHB. The corresponding aeroacoustic limit cycle at $\omega_0/2\pi = 1.82$ kHz involves a longitudinal mode of the cavity which constructively interacts with the coherent vorticity fluctuations in the cavity opening [38–41]. As a byproduct of the air stream, a bias of the order of the Mach number $Ma \approx 0.17$ is imposed on the system. The measured root-mean-square (rms) of the self-sustained acoustic waves radiated in the upstream and downstream sections of the waveguide are 0.54 and 0.65 hPa. The acoustic energy of the limit cycle feeds the scattered waves, enabling superradiance in the presence of incident waves. Details of the experimental set-up are provided in the Supplemental Material [29].

To measure the scattering matrix $S$, acoustic waves produced by compression drivers placed in the anechoic ends of the waveguide are sent to the cavity. The columns $S_{1i}$ and $S_{2i}$, $i = 1, 2$ are obtained separately by applying harmonic forcing up-and downstream of the cavity. The voltage of the signal to the drivers is calibrated at each frequency to achieve a specified acoustic forcing amplitude $s$ in the waveguide at the cavity aperture. Assuming lossless transmission, the multi-microphone method is employed to reconstruct the forward- and back-propagating waves in the waveguide [42].

Similarly to the numerical example presented above, the experimentally determined $S$-matrix also exhibits a nonlinear transition with increasing $s$ from omnidirectional to purely reflectional superradiant scattering (Fig. 4). It is interesting to note that in the present biased system, superradiant reflection is stronger for incident waves in flow direction. The transition of the scattering matrix properties is characterized by a saturation at high amplitudes, confirming the robustness of the superradiant reflectional scattering predicted by theory.

Note that in the unbiased case, for $Ma = 0$ (without air flow), the scattering properties of the resonant cavity are linear, symmetric and lossy (Fig. 5). The $S$-matrix in the unbiased case is presented in detail in the Supplemental Material [29].

This work demonstrates a novel path to superradiance by exploiting the wave interaction with a self-sustained cavity mode. We focused on the nonlinear transition of the scattering matrix, which occurs when the bifurcation parameter is kept constant and the forcing amplitude is varied. An associated transition could be observed by keeping the forcing amplitude constant and varying the bifurcation parameter.

Comparing the resonance-based to the limit-cycle-
enhanced scattering of the cavity (Fig. 5), we observe that the NLC scattering matrix at high amplitudes is an enhanced version of its linearly stable counterpart, with higher maximal reflection and lower minimal transmission. We envision that the quasi-passive realization of superradiance presented in this letter will find application in flow-based acoustic metamaterials and topological insulators, which often suffer from dissipation losses. The generic nature of NLC leads us to surmise that it can be applied to the development of non-acoustic nonreciprocal scattering systems, unlocking the potential of nonlinear dynamics for design.

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