Game Theory in Signal Processing and Communications
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Editorial

Game Theory in Signal Processing and Communications

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Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences (or utilities) of those players. Games are represented in different forms, namely, the extensive form, normal form, and characteristic and partition function forms. Noncooperative games are usually defined by normal and extensive form games, whereas cooperative games are defined by characteristic and partition function form games. In noncooperative games, the players decide without communication and coordination on their strategies by selfishly maximizing their individual payoff functions. Equilibrium concepts characterize outcomes of noncooperative games. In cooperative games, players choose their strategies by negotiation and cooperation. In general the theory provides a structured approach to many important problems arising in signal processing and communications, notably resource allocation and distributed transceiver optimization. Recent applications also occur in other emerging fields such as cognitive radio, spectrum sharing, and multihop sensor and adhoc networks. Game theory is also used in cross-layers optimization where radio resource management on physical and MAC layers is connected with economic decisions on service and application layers.

This special issue presents research in applied game theory for signal processing and communications. In order to solve conflict situations in communication and in particular wireless networks, game theory provides a systematic approach. The limited resources in wireless communications are spectrum, space, power, and time. Game theory is used to analyze spectrum sharing, resource allocation, power control, transmit strategies, and network etiquette in wireless multiuser networks. The interest in these methods is increased significantly during the last ten years in the networking and signal processing communities.

In this special issue we have been able to collect seven papers on a variety of topics. The first paper, “Saddle-point properties and nash equilibria for channel games” authored by Rudolf Mathar and Anke Schmeink studies a game between transmitter and nature represented by the channel. Equilibrium strategies are characterized for the noncooperative two-person zero-sum game in standard form.

Matthew Nokleby and A. Lee Swindlehurst analyze the cooperative bargaining on the MISO interference channel in the second paper entitled “Bargaining and the MISO interference channel”. The optimal Kalai-Smorodinsky beamformers are derived. Joint scheduling, beamforming, and power control show significant performance gains for strong interference and scenarios with more links than transmit antennas.

The third paper, “Spectrum allocation for decentralized transmission strategies: properties of nash equilibria” authored by Peter von Wyrzca et al. studies two links coexisting in the same area at the same time on the same frequency. The conflict is modeled as game in normal form and the Nash equilibrium is characterized. A modified utility function is proposed to improve the outcome in terms of sum rate efficiency.
In the fourth paper, “Stackelberg contention games in multiuser networks”, Jaeok Park and Mihaela van der Schaar propose to model a Stackelberg game for noncooperative communications in order to improve the efficiency of the equilibrium outcome. The network manager leads the game and decides on an intervention rule. The amount of additional information required is studied and it is shown that relaxing the requirements lead to small performance loss.

Zhu Han et al. in their paper “Auction-based resource allocation for cooperative video transmission protocols over wireless networks” propose wireless video transmission protocols for end-to-end distortion minimization. For multiuser networks, the share auction approach is extended into cooperative video transmission. Experimental results illustrate the gain in terms of peak signal to-noise ratio.

The sixth paper, Modeling Misbehavior in Cooperative Diversity: A Dynamic Game Approach by Sintayehu Dehnie and Nasir Memon, designs cooperative diversity protocols and analyzes misbehavior for the game in extensive form. Misbehavior turns out to be an evolutionary stable strategy. Therefore, a mechanism to detect and mitigate effects of misbehavior is developed based on a dynamic game formulation with incomplete information.

Ming-Hua Lin, Jung-Fa Tsai, and Yinyu Ye in their paper Budget Allocation in a Competitive Communication Spectrum Economy study a competitive communication spectrum market. A condition for existence of a competitive equilibrium with physical power demand requirements for the communication spectrum market with Shannon utility is derived. A centralized algorithm to reach a desired competitive equilibrium for satisfying power demands or balancing individual utilities is proposed.

We received an overwhelming response to the issue. We would like to thank all authors for their contributions to our issue, the reviewers for their help in selecting papers, and finally the editor Phillip Regalia for his support.

Holger Boche
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Research Article

Saddle-Point Properties and Nash Equilibria for Channel Games

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In this paper, transmission over a wireless channel is interpreted as a two-person zero-sum game, where the transmitter gambles against an unpredictable channel, controlled by nature. Mutual information is used as payoff function. Both discrete and continuous output channels are investigated. We use the fact that mutual information is a convex function of the channel matrix or noise distribution densities, respectively, and a concave function of the input distribution to deduce the existence of equilibrium points for certain channel strategies. The case that nature makes the channel useless with zero capacity is discussed in detail. For each, the discrete, continuous, and mixed discrete-continuous output channel, the capacity-achieving distribution is characterized by help of the Karush-Kuhn-Tucker conditions. The results cover a number of interesting examples like the binary asymmetric channel, the Z-channel, the binary asymmetric erasure channel, and the $n$-ary symmetric channel. In each case, explicit forms of the optimum input distribution and the worst channel behavior are achieved. In the mixed discrete-continuous case, all convex combinations of some noise-free and maximum-noise distributions are considered as channel strategies. Equilibrium strategies are determined by extending the concept of entropy and mutual information to general absolutely continuous measures.

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1. Introduction

Transmission over a band-limited wireless channel is often considered as a game where players compete for a scarce medium, the channel capacity. Nash bargaining solutions are determined for interference games with Gaussian additive noise. In the works [1, 2], different fairness and allocation criteria arise from this paradigm leading to useful access control policies for wireless networks.

The engineering problem of transmitting messages over a channel with varying states may also be gainfully considered from a game-theoretic point of view, particularly if the channel state is unpredictable. Here, two players are entering the scene, the transmitter and the channel state selector. The transmitter gambles against the channel state, chosen by a malicious nature, for example. Mutual information $I(X; Y)$ is considered as payoff function, the transmitter aims at maximizing, nature at minimizing $I(X; Y)$. A simple motivating example is the additive scalar channel with input $X$ and additive Gaussian noise $Z$ subject to average power constraints $E(X^2) \leq P$ and $E(Z^2) \leq \sigma^2$. By standard arguments from information theory, it follows that

$$
\max_{X: E(X^2) \leq P} \min_{Z: E(Z^2) \leq \sigma^2} I(X; X + Z) = \min_{Z: E(Z^2) \leq \sigma^2} \max_{X: E(X^2) \leq P} I(X; X + Z) = \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right)
$$

(1) is the capacity of the channel. Hence an equilibrium point exists and capacity is the value of the two-person zero-sum game. The corresponding equilibrium strategies are to increase power and noise, respectively, to their maximum values.

A similar game is considered in [3], where the coder controls the input and the jammer the noise, both from allowable sets. Saddle points, hence equilibria, and $\epsilon$-optimal strategies are determined for binary input and output quantization under power constraints for both the coder and the jammer. An extension of the mutual information game (1) to vector channels with convex covariance constraints...
is considered in [4]. Jorswieck and Boche [5] investigate a similar minimax setup for a single link in a MIMO system with different types of interference. Further extensions to vector channels and different kinds of games are considered (e.g., [6, 7]).

In this paper, we choose the approach that nature gambles against the transmitter, which aims at conveying information across the channel in an optimal way. “Nature” and “channel” are used synonymously to characterize the antagonist of the transmitter. We consider two models of the channel which yield comparable results. First, transmission is considered purely on a symbol basis. Symbols from a finite set are transmitted and decoded with certain error probabilities. The model is completely discrete, and strategies of nature are described by certain channel matrices chosen from the set of stochastic matrices. The binary asymmetric erasure channel as shown in Figure 4 may serve as a typical example.

On the other hand, continuous channel models are considered. The strategies of the channel are then given by a set of densities, each describing the conditional distribution of received values given a transmitted symbol. The finite input additive white Gaussian noise channel is a standard example thereof, and also 4-QAM with correlated noise (e.g., as shown in Figure 1) is covered by this model.

For both models, equilibrium points are sought, where the strategy of the transmitter consists of selecting the optimum input distribution against the worst-case behavior of the channel, vice versa, and both have the same game value.

The contributions of this paper are as follows. In Section 2, we demonstrate that mutual information is a convex function of the channel matrix, or the noise densities, respectively. For discrete channels, transmission is considered as a game in Section 3. Some typical binary and n-ary channels are covered by this theory, as shown in Section 5. It is demonstrated that equilibrium points exist and the according optimum strategies for both players are determined. The entropy of mixture distributions is considered in Section 6, which finally, in Section 7, leads to equilibrium points for mixed discrete-continuous channel strategies.

2. Channel Models and Mathematical Foundations

Denote the set of stochastic vectors of dimension $m$ by

\[ D^m = \left\{ \mathbf{p} = (p_1, \ldots, p_m) \mid p_i \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}. \] (2)

Each $\mathbf{p} \in D^m$ represents a discrete distribution with $m$ support points. The entropy $H$ of $\mathbf{p}$ is defined as

\[ H(\mathbf{p}) = -\sum_{i=1}^{m} p_i \log p_i. \] (3)

If $\mathbf{p}$ characterizes the distribution of some discrete random variable $X$, we synonymously write $H(X) = H(\mathbf{p})$. It is well known that the entropy $H$ is a concave function of $\mathbf{p}$, and furthermore, even Schur-concave over the set of distributions $D^m$, since it is symmetric (see [8]).

Let random variable $X$ denote the discrete channel input with symbol set $\{x_1, \ldots, x_m\}$ and distribution $\mathbf{p}$. Accordingly, random variable $Y$ denotes the output of the channel.
2.1. Discrete Output Channels. We first deal with discrete channels. If the output set consists of \( n \) symbols \( \{ y_1, \ldots, y_n \} \), then the behavior of the channel is completely characterized by the \((m \times n)\) channel matrix:
\[
W = (w_{ij})_{1 \leq i \leq m, 1 \leq j \leq n},
\]
consisting of conditional probabilities \( w_{ij} = P(Y = y_j | X = x_i) \). Matrix \( W \) is an element of the set of stochastic \((m \times n)\) matrices, denoted by \( \mathcal{S}^{m \times n} \). Its rows are stochastic vectors, denoted by \( w_1, \ldots, w_m \in \mathcal{D}^n \). The distribution of \( Y \) is then given by the stochastic vector \( q = pW \).

Mutual information for this channel model reads as
\[
I(X; Y) = H(Y) - H(Y | X) = H(pW) - \sum_{i=1}^{m} p_i H(w_i) = \sum_{i=1}^{m} p_i D(w_i || pW),
\]
where \( D(\cdot || \cdot) \) denotes the Kullback-Leibler divergence,
\[
D(p || q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}
\]
with \( p, q \in \mathcal{D}^m \).

Obviously, mutual information depends on the input distribution \( p \), controlled by the transmitter, and channel matrix \( W \), controlled by nature. To emphasize this dependence, we also write \( I(X; Y) = I(p; W) \), the following result is quoted from [9, Lemma 3.5].

**Proposition 1.** Mutual information \( I(p; W) \) is a concave function of \( p \in \mathcal{D}^m \) and a convex function of \( W \in \mathcal{S}^{m \times n} \).

The proof relies on the representation in the third line of (5), convexity of the Kullback-Leibler divergence \( D(p || q) \) as a function of the pair \((p, q)\), and concavity of the entropy \( H \).

The problem of maximizing \( I(p; W) \) over \( p \) or minimizing \( I(p; W) \) over \( W \) subject to convex constraints hence fall into the class of convex optimization problems.

2.2. General Output Channels. Entropy definition (3) generalizes to densities \( f \) of absolutely continuous distributions with respect to a \( \sigma \)-finite measure \( \mu \) as
\[
H(f) = \int f(y) \log f(y) \, d\mu(y).
\]
(see [10]). Practically relevant cases are the discrete case (3), where \( \mu \) is taken as the counting measure, densities \( f \), with respect to the Lebesgue measure \( \lambda^n \) on the \( \sigma \)-field of Borel sets over \( \mathbb{R}^n \), and mixtures hereof. These cases correspond to discrete, continuous, and mixed discrete-continuous random variables.

The approach in Section 2.1 carries over to densities of absolutely continuous distributions with respect to \( \mu \), as used in (7). The channel output \( Y \) is randomly distorted by noise, for symbol \( i \) governed by \( \mu \)-density \( f_i \). Hence, the distribution of \( Y \) given input \( X = x_i \) has \( \mu \)-density
\[
f(y | x_i) = f_i(y), \quad y \in \mathbb{R}^n.
\]

The AWGN channel \( Y = X + N \) is a special case hereof with \( f_i(y) = \varphi(y - x_i) \). Here, \( \varphi \) denotes the Lebesgue density of a Gaussian distribution \( N_0 (0, \Sigma) \).

Mutual information between channel input and output as a function of \( p = (p_1, \ldots, p_m) \) and \( (f_1, \ldots, f_m) \) may be written as
\[
I(X; Y) = I(p; f_1, \ldots, f_m) = H(Y) - H(f(Y | X)) = H(Y) - H(Y | X) = H(F) - H(F | X) = m \sum_{i=1}^{m} p_i D(f_i || pW),
\]
where \( D(f || g) = \int f \log(f/g) \, d\mu \) denotes the Kullback-Leibler divergence between \( \mu \)-densities \( f \) and \( g \).

Let \( \mathcal{F} \) denote the set of all \( \mu \)-densities. From the convexity of \( t \log t, t \geq 0 \), it is easily concluded that
\[
H \left( \sum_{i=1}^{m} p_i f_i \right) \text{ is a concave function of } p \in \mathcal{D}^m.
\]

By applying the log-sum inequality (cf. [9]), we also obtain
\[
\begin{align*}
\alpha f_1 \log \frac{f_1}{g_1} + (1 - \alpha) f_2 \log \frac{f_2}{g_2} \\
\geq (\alpha f_1 + (1 - \alpha) f_2) \log \frac{\alpha f_1 + (1 - \alpha) f_2}{\alpha g_1 + (1 - \alpha) g_2},
\end{align*}
\]
pointwise for any pairs of densities \((f_1, g_1), (f_2, g_2) \in \mathcal{F}^2 \).

Integrating both sides of the aforementioned inequality shows that
\[
D(f || g) \text{ is a convex function of the pair } (f, g) \in \mathcal{F}^2.
\]

Applying (10) and (12) to the third and forth lines of representation (9), respectively, gives the following proposition.

**Proposition 2.** Mutual information \( I(p; f_1, \ldots, f_m) \) is a concave function of \( p \in \mathcal{D}^m \) and a convex function of \( (f_1, \ldots, f_m) \in \mathcal{F}^m \).

Proposition 2 generalizes its discrete counterpart, Proposition 1. The latter is obtained from the former by identifying the rows of \( W \) as densities with respect to the counting measure with support given by the output symbol set.

In summary, determining the capacity of the channel for fixed channel noise densities \( f_1, \ldots, f_m \) leads to a concave optimization problem, namely,
\[
C = \max_{p \in \mathcal{D}^m} I(p; f_1, \ldots, f_m).
\]

Further, minimizing \( I(p; f_1, \ldots, f_m) \) over a convex set of densities \( f_1, \ldots, f_m \) for some fixed input distribution \( p \in \mathcal{D}^m \) yields a convex optimization problem.
3. Discrete Output Channel Games

In what follows, we regard transmission over a channel as a two-person zero-sum game. A malicious nature is gambling against the transmitter. If nature is controlling the channel, the transmitter wants to protect itself against a worst-case behavior of nature in the sense of maximizing the capacity of the channel by an appropriate choice of the input distribution. The question arises whether this type of channel game has an equilibrium. If the transmitter moves first and maximizes capacity under the present channel conditions, is the same game value achieved if nature deteriorates the channel against the chosen strategy of the transmitter? Hence, $I(X;Y)$ plays the role of the payoff function.

We will show that for different classes of channels equilibria exist. The basis is formed by the following minimax or saddle point theorem.

Proposition 3. Let $\mathcal{T} \subseteq \mathcal{W}^{m \times r}$ be a closed convex subset of channel matrices. Then the according channel game has an equilibrium point with value

$$\max_{p \in \mathcal{D}^m} \min_{W \in \mathcal{T}} I(p;W) = \max_{W \in \mathcal{T}} \max_{p \in \mathcal{D}^m} I(p;W).$$

The proof is an immediate consequence of von Neumann’s minimax theorem (cf. [11, page 131]). Since $\mathcal{D}^m$ and $\mathcal{T}$ are closed and convex, the main premises are concavity in $p$ and convexity in $W$, both properties assured by Proposition 1.

If $\mathcal{T} = \mathcal{W}^{m \times r}$, the value of the game is zero. Nature will make the channel useless by selecting

$$W = \begin{pmatrix} w \\ \vdots \vspace{1ex} \\ w \end{pmatrix},$$

with constant rows $w$ yielding $I(p;W) = 0$ independent of the input distribution. Obviously, (15) holds if and only if input $X$ and output $Y$ are stochastically independent.

We first consider the case that nature plays a singleton strategy, hence $\mathcal{T} = \{ W \}$, a set consisting of only one strategy. However, (14) then reduces to determining $\max_{p \in \mathcal{D}^m} I(p;W)$, the capacity $C$ of the channel for fixed channel matrix $W$. In order to characterize nonzero capacity channels, we use the variational distance between the $i$th and $j$th row of $W$, defined as

$$d(w_i, w_j) = \sum_{k=1}^{r} |w_{ik} - w_{jk}|.$$  

The condition

$$\max_{1 \leq i, j \leq m} d(w_i, w_j) = \gamma(W) > 0$$

on the channel matrix $W$ ensures that the according channel has nonzero capacity, as demonstrated in the following proposition.

Proposition 4. If $W$ satisfies (17) for some $\gamma(W) > 0$, then

$$C = \max_{p \in \mathcal{D}^m} I(p;W) \geq \frac{\gamma^2(W)}{8 \ln 2} > 0,$$

where information is measured in nats.

Proof. Let the maximum in (17) be attained at indices $i_0$ and $j_0$. Further, set $p = (1/2) (e_{i_0} + e_{j_0})$ where $e_i$ denotes the $i$th unit row vector in $\mathbb{R}^m$. The third line in (5) then gives

$$I(p;W) = \frac{1}{2} D \left( \frac{w_{i_0} + w_{j_0}}{2} \right) + \frac{1}{2} D \left( \frac{w_{i_0} + w_{j_0}}{2} \right).$$

Since

$$D(w_{i_0} || w_{j_0}) \leq \frac{1}{2 \ln 2} d^2(w_{i_0}, w_{j_0})$$

(see [9, page 58]), and

$$d \left( \frac{w_{i_0} + w_{j_0}}{2} \right) = \frac{1}{2} d(w_{i_0}, w_{j_0}),$$

it follows that

$$I(p;W) \geq \frac{1}{2 \ln 2} d^2(w_{i_0}, w_{j_0}) = \frac{\gamma^2}{8 \ln 2} > 0.$$

In summary, some channel with transition probabilities $W$ has nonzero capacity if and only if $\gamma(W) > 0$. The same condition turns out important when determining the capacity of arbitrary discrete channels.

Proposition 5. Let channel matrix $W$ satisfy condition (17). Then $C = \max_{p \in \mathcal{D}^m} I(p;W)$ is attained at $p^* = (p^*_1, \ldots, p^*_m)$ if and only if

$$D(w_i || p^*W) = \zeta$$

for some $\zeta > 0$ and all $i$ with $p^*_i > 0$. Moreover, $C = I(p^*;W) = \zeta$ holds.

Proof. Mutual information $I(p;W)$ is a concave function of $p$. Hence the KKT conditions (cf., e.g., [12]) are necessary and sufficient for optimality of some input distribution $p$. Using (5), some elementary algebra shows that

$$\frac{\partial}{\partial p_i} I(p;W) = D(w_i || pW) - 1.$$  

The full set of KKT conditions now reads as

$$p \in \mathcal{D}^m, \quad \lambda_i \geq 0, \quad i = 1, \ldots, m,$$

$$\lambda_i p_i = 0, \quad i = 1, \ldots, m,$$

$$D(w_i || pW) + \lambda_i + \nu = 0, \quad i = 1, \ldots, m,$$

which shows the assertion.

Proposition 5 has an interesting interpretation. For an input distribution $p^* = (p^*_1, \ldots, p^*_m)$ to be capacity-achieving, the Kulback-Leibler distance between the rows of $W$ and the weighted average with weights $p^*_i$ has to be the same for all $i$ with positive $p^*_i$. Hence, capacity-achieving distribution $p^*$ places the mixture distribution $p^*W$ somehow in the middle of all rows $w_i^*$. 


4. Elementary Channel Models

Discrete binary input channels are considered in this section. From the according channel games capacity-achieving distributions against worst-case channels are obtained.

4.1. The Binary Asymmetric Channel. As an example, we consider the binary asymmetric channel with channel matrix:

\[
W = W(\epsilon, \delta) = \begin{pmatrix}
1 - \epsilon & \epsilon \\
\delta & 1 - \delta
\end{pmatrix} = \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
\]

with \(0 < \epsilon, \delta < 1\) such that condition (17) is satisfied (see Figure 2). By (23), the capacity-achieving input distribution \(p = (p_0, p_1)\) satisfies

\[
D(w_1) = D(w_2).
\]

This is an equation in the variables \(p_0, p_1\) which jointly with the condition \(p_0 + p_1 = 1\) has the solution

\[
p_0^* = \frac{1}{1 + b}, \quad p_1^* = \frac{b}{1 + b},
\]

with

\[
b = \frac{a \epsilon - (1 - \epsilon)}{\delta - a(1 - \delta)}, \quad a = \exp\left(\frac{h(\delta) - h(\epsilon)}{1 - \epsilon - \delta}\right),
\]

and \(h(\epsilon) = H(\epsilon, 1 - \epsilon)\), the entropy of \((\epsilon, 1 - \epsilon)\). This result has been derived by cumbersome methods in the early paper [13].

Now assume that the strategy set of nature is given by

\[
\mathcal{T}_{\hat{\epsilon}, \hat{\delta}} = \{W(\epsilon, \delta) \mid 0 \leq \epsilon \leq \hat{\epsilon}, 0 \leq \delta \leq \hat{\delta}\},
\]

where \(0 \leq \hat{\epsilon}, \hat{\delta} < 1/2\) are given. Hence, error probabilities are bounded from the worst case by \(\hat{\epsilon}\) and \(\hat{\delta}\).

Since \(I(p; W)\) is a convex function of \(W\), \(I(p; W(\epsilon, \delta))\) is a convex function of the argument \((\epsilon, \delta) \in [0, 1]^2\). The minimum value 0 is obviously attained whenever \(\epsilon + \delta = 1\). This shows that \(I(p; W(\epsilon, \delta))\) is decreasing in \(\epsilon \in [0, \hat{\epsilon}]\) for fixed \(\delta\), and vice versa, is a decreasing function of \(\delta \in [0, \hat{\delta}]\) with \(\epsilon\) fixed. Accordingly, it holds that

\[
\min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} I(p; W) = I(p; W(\hat{\epsilon}, \hat{\delta}))
\]

for any \(p \in \mathcal{D}^2\). Further, we have

\[
\max_{p \in \mathcal{D}^2} \min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} I(p; W) = \max_{p \in \mathcal{D}^2} I(p; W(\hat{\epsilon}, \hat{\delta}))
\]

is attained at \(p^* = (p_0^*, p_1^*)\) from (28) with the replacements \(\epsilon = \hat{\epsilon}\) and \(\delta = \hat{\delta}\).

Since \(\mathcal{T}_{\hat{\epsilon}, \hat{\delta}}\) is a convex set, we obtain from Proposition 3 that a saddle point exists and the value of the game is given by

\[
\max_{p \in \mathcal{D}^2} \min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} I(p; W) = \min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} \max_{p \in \mathcal{D}^2} I(p; W)
\]

\[
= I(p^*; W(\hat{\epsilon}, \hat{\delta})).
\]

The so-called Z-channel with error probability \(\epsilon = 0\) and \(\delta \in [0, 1]\) (see Figure 3) is a special case hereof. We have

\[
\max_{p \in \mathcal{D}^2} \min_{\delta \in \mathcal{D}^2} I(p; W(0, \delta)) = \max_{p \in \mathcal{D}^2} I(p; W(0, \hat{\delta}))
\]

\[
= I(p^*; W(0, \hat{\delta})).
\]

After some algebra, from (28)

\[
p_0^* = 1 - p_1^*, \quad p_1^* = \frac{1/(1 - \hat{\delta})}{1 - 2h(\hat{\delta}/(1 - \hat{\delta})}
\]

is obtained with capacity

\[
I(p^*; W(0, \hat{\delta})) = \log_2(1 + 2^{h(\hat{\delta}/(1 - \hat{\delta}))},
\]

where information is measured in bits (cf. [14, Example 9.11]).

4.2. The Binary Asymmetric Erasure Channel. The binary asymmetric erasure channel (BEC) with bit error probabilities \(\epsilon, \delta \in [0, 1]\), and channel matrix

\[
W = W(\epsilon, \delta) = \begin{pmatrix}
1 - \epsilon & \epsilon \\
\delta & 1 - \delta
\end{pmatrix} = \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
\]

is depicted in Figure 4.

According to Proposition 4, this channel has zero capacity if and only if \(\epsilon = \delta = 1\). Excluding this case, by Proposition 5, the capacity-achieving distribution \(p^* = (p_0^*, p_1^*)\) is given by the solution of

\[
(1 - \epsilon) \log \frac{1 - \epsilon}{p_0(1 - \epsilon)} + \epsilon \log \frac{\epsilon}{p_0 \epsilon + p_1 \delta} = \delta \log \frac{\delta}{p_0 \epsilon + p_1 \delta} + (1 - \delta) \log \frac{1 - \delta}{p_0 (1 - \delta)}.
\]

Substituting \(x = p_0/p_1\), (38) reads equivalently as

\[
\epsilon \log \epsilon - \delta \log \delta = (1 - \delta) \log(\delta + \epsilon x) - (1 - \epsilon) \log \left(\epsilon + \frac{\delta}{x}\right).
\]

By differentiating with respect to \(x\), it is easy to see that the right-hand side is monotonically increasing such that exactly one solution \(p^* = (p_0^*, p_1^*)\) exists, which can be numerically computed.

If \(\epsilon = \delta\), the solution is given by \(p_0^* = p_1^* = 1/2\), as easily verified from (38).

Resembling the arguments used for the binary asymmetric channel and adopting the notation, we see that

\[
\min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} I(p; W) = I(p; W(\hat{\epsilon}, \hat{\delta}))
\]

for any \(p \in \mathcal{D}^2\). Further, we have

\[
\max_{p \in \mathcal{D}^2} \min_{W \in \mathcal{T}_{\hat{\epsilon}, \hat{\delta}}} I(p; W) = \max_{p \in \mathcal{D}^2} I(p; W(\hat{\epsilon}, \hat{\delta}))
\]
is attained at $p^* = (p_0^*, p_n^*)$, the solution of (38) with $\varepsilon$ substituted by $\hat{\varepsilon}$ and $\delta$ by $\delta$. Finally, the game value amounts to

$$\max_{p \in \mathcal{D}^n} \min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} I(p; W) = \min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} \max_{p \in \mathcal{D}^n} I(p; W)$$

$$= I(p^*; W(\hat{\varepsilon}, \delta)).$$

If $\delta = \varepsilon \leq \hat{\varepsilon}$, the result is

$$I(p^*; W(\hat{\varepsilon}, \delta)) = 1 - \varepsilon,$$

and the equilibrium strategies are $p_0^* = p_1^* = 1/2$ for the transmitter and $\varepsilon = \delta = \hat{\varepsilon}$ for nature (cf. [15, Example 8.5]).

5. The $n$-Ary Symmetric Channel

Consider the $n$-ary symmetric channel with symbol set $\{0, 1, \ldots, n-1\}$ and channel matrix

$$W(\varepsilon) = \begin{pmatrix} e_0 & e_1 & \cdots & e_{n-1} \\ e_{n-1} & e_0 & \cdots & e_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_1 & e_2 & \cdots & e_0 \end{pmatrix}$$

by cyclically shifting some error vector $\varepsilon = (e_0, e_1, \ldots, e_{n-1}) \in \mathcal{D}^n$. Let $\mathcal{E} \subseteq \mathcal{D}^n$ denote the set of strategies that nature can choose the channel state from by selecting some $\varepsilon \in \mathcal{E}$.

If $\mathcal{E} = \mathcal{D}^n$, the value of the game is zero. As mentioned earlier, nature will cripple the channel by selecting $\varepsilon = \varepsilon_n = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$,

yielding $I(X; Y) = 0$ independent of the input distribution. Note that $\varepsilon_n$ is the unique minimum element with respect to majorization, that is, $\varepsilon_n < \varepsilon$ for all $\varepsilon \in \mathcal{D}^n$. We briefly recall the corresponding definitions (see [8]). Let $p_{[i]}$ and $q_{[i]}$ denote the components of $p$ and $q$ in decreasing order, respectively. Distribution $p \in \mathcal{D}$ is said to be majorized by $q \in \mathcal{D}$ in symbols $p < q$, if $\sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} q_{[i]}$ for all $k = 1, \ldots, m$.

Hence, to avoid trivial cases, the set of strategies for nature has to be separated from this worst case.

5.1. Separation by Schur Ordering. We first investigate the set

$$\mathcal{E}_{\hat{\varepsilon}, \delta} = \{ \varepsilon = (e_0, \ldots, e_{n-1}) \in \mathcal{D}^n \mid \hat{\varepsilon} < \varepsilon, \varepsilon_{(0)} \leq \cdots \leq \varepsilon_{(n-1)} \}$$

for some fixed $\hat{\varepsilon} \neq \varepsilon_0$ and permutation $\pi$. This means that the error probabilities are at least spread out, or separated from uniformity as $\varepsilon$, with error probabilities increasing in the fixed order determined by $\pi$.

Since $\mathcal{E}_{\hat{\varepsilon}, \delta}$ is convex and closed, the set of corresponding matrices

$$\mathcal{T}_{\hat{\varepsilon}, \delta} = \{ W(\varepsilon) \mid \varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta} \}$$

is convex and closed as well.

Proposition 3 ensures the existence of an equilibrium point:

$$\max_{p \in \mathcal{D}^n} \min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} I(p; W) = \min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} \max_{p \in \mathcal{D}^n} I(p; W).$$

To determine the value $v$ of the game, we first consider $\max_{p \in \mathcal{D}^n} I(p; W(\varepsilon))$ for some fixed $\varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta}$. From (5), it follows that the maximum is attained at input distribution $p = (1/n, \ldots, 1/n)$ with value

$$\max_{p \in \mathcal{D}^n} I(p; W(\varepsilon)) = \log n - H(\varepsilon).$$

As the entropy is Schur concave, $\min_{\varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta}} (\log n - H(\varepsilon))$ is attained at $\varepsilon$ such that the value of the game is obtained as

$$\min_{\varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta}} \max_{p \in \mathcal{D}^n} I(p; W) = \log n - H(\varepsilon)$$

with according equilibrium strategies $p = (1/n, \ldots, 1/n)$ and the components of $\varepsilon$ equal to those of $\varepsilon$ rearranged according to $\pi$.

5.2. Directional Separation. In what follows, we consider channel states separated from the worst-case $\varepsilon_0$ into the direction of some prespecified $\hat{\varepsilon} \in \mathcal{D}^n$, $\hat{\varepsilon} \neq \varepsilon_0$. This set of strategies is formally described as

$$\mathcal{E}_{\hat{\varepsilon}, \delta} = \{ \varepsilon = (1 - \alpha)\varepsilon_0 + \alpha \hat{\varepsilon} \mid \alpha \leq 1 \}$$

for some given $\hat{\varepsilon} > 0$. It is obviously convex and closed. The set of corresponding channel matrices

$$\mathcal{T}_{\hat{\varepsilon}, \delta} = \{ W(\varepsilon) \mid \varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta} \}$$

is also closed and convex such that an equilibrium exists by Proposition 3. It remains to determine the game value.

Since $I(p; W)$ is a convex function of $W$, hence decreasing in $\alpha \in [\hat{\varepsilon}, 1)$:

$$\min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} I(p; W)$$

is attained at $W(\varepsilon_0)$ with $\varepsilon_0 = (1 - \hat{\varepsilon})\varepsilon_0 + \hat{\varepsilon}$. From representation (5), it can be easily seen that

$$\max_{p \in \mathcal{D}^n} I(p; W(\varepsilon_0)) = \log n - H(\varepsilon_0)$$

is attained at $p = (1/n, \ldots, 1/n)$.

Vice versa, from (5), it follows that for any $W = W(\varepsilon)$,

$$\max_{p \in \mathcal{D}^n} I(p; W(\varepsilon)) = \log n - H(\varepsilon)$$

is attained at $p = (1/n, \ldots, 1/n)$ for any $\varepsilon \in \mathcal{E}_{\hat{\varepsilon}, \delta}$. By monotonicity $\alpha \in [\hat{\varepsilon}, 1)$, it holds that

$$\min_{W \in \mathcal{T}_{\hat{\varepsilon}, \delta}} \max_{p \in \mathcal{D}^n} I(p; W) = \log n - H(\varepsilon_0),$$

which determines the game value. The equilibrium strategies are the uniform distribution for the transmitter and the extreme error vector $\varepsilon_0$ for nature.
The $n$-ary symmetric channel with error probabilities
\[
\left(1 - \delta, \frac{\delta}{n - 1}, \ldots, \frac{\delta}{n - 1}\right)
\]  
(57)
is a special case of the aforementioned by identifying $\hat{\epsilon} = (1, 0, \ldots, 0)$ and $\alpha = 1 - (n/(n - 1))\delta$.

The binary symmetric channel (BSC) with error probability $0 < \delta < 1/2$ is obtained by setting $n = 2$, $\hat{\epsilon} = (1, 0)$ and $\alpha = 1 - 2\delta$.

6. Entropy of Mixture Distributions

Let $U$ be an absolutely continuous random variable with density $g(y)$ with respect to the Lebesgue measure $\lambda^n$, and let random variable $V$ have a discrete distribution with discrete density $h(y) = p_i$, if $y = x_i$, $i = 1, \ldots, m$, and $h(y) = 0$ otherwise, $p_i \geq 0$, $\sum_{i=1}^m p_i = 1$. Furthermore, assume that $B$ is Bernoulli distributed with parameter $\alpha$, $0 \leq \alpha \leq 1$, hence $P(B = 1) = \alpha$, $P(B = 0) = 1 - \alpha$. Further, let $U$, $V$, $B$ be stochastically independent, then
\[
W = BU + (1 - B)V
\]  
(58)
has density
\[
f(y) = ag(y) + (1 - \alpha)h(y)
\]  
(59)
with respect to the measure $\mu = \lambda^n + \chi$, where $\chi$ denotes the counting measure with support $\{x_1, \ldots, x_m\}$. According to [10], the entropy of $W$ is defined as
\[
H(W) = -\int f(y) \log f(y) \, d\mu(y).
\]  
(60)

It easily follows (see [16]) that
\[
H(W) = -\alpha \int g(y) \log g(y) \, dy - \alpha \log \alpha
\]
\[-(1 - \alpha) \sum_{i=1}^m p_i \log p_i - (1 - \alpha) \log(1 - \alpha)
\]  
(61)
\[= H(B) + \alpha H(U) + (1 - \alpha)H(V).
\]

The following proposition will be useful when investigating equilibria of channel games with continuous noise densities.

**Proposition 6.** Let $p = (p_1, \ldots, p_m)$ denote some stochastic vector, and $g_1, \ldots, g_m$ be densities with respect to some measure $\mu$. It holds that
\[
H\left(\sum_{i=1}^m p_i g_i\right) - \sum_{i=1}^m p_i H(g_i) \leq H(p).
\]  
(62)

The proof is provided by the following chain of equalities and inequalities. The argument $y$ of $g_i$ is omitted for reasons of brevity:
\[
-\int \left(\sum_{i=1}^m p_i g_i\right) \log \left(\sum_{i=1}^m p_i g_i\right) \, d\mu + \sum_{i=1}^m p_i \int g_i \log g_i \, d\mu
\]
\[= -\sum_{i=1}^m p_i \int g_i \left(\log \left(\sum_{j=1}^m p_j g_j\right) - \log g_i\right) \, d\mu
\]
\[= \sum_{i=1}^m p_i \int g_i \log \frac{g_i}{\sum_{j=1}^m p_j g_j} \, d\mu
\]
\[\leq \sum_{i=1}^m p_i \int g_i \log \frac{g_i}{p_i g_i} \, d\mu
\]
\[= -\sum_{i=1}^m p_i \log p_i = h(p).
\]  
(63)

7. A Mixed Discrete-Continuous Channel Game

Let $g_1, \ldots, g_m$ be given $\lambda^n$-densities. Distribution $p^* = (p_1^*, \ldots, p_m^*)$ achieves capacity, that is, maximizes mutual information if and only if $I(X; Y)$ is maximized by $p^*$ in the set of all stochastic vectors. By representation (9), we need to solve
\[
\text{maximize } \left\{ -\int \left(\sum_{i=1}^m p_i g_i(y)\right) \log \left(\sum_{i=1}^m p_i g_i(y)\right) \, dy
\]
\[+ \sum_{i=1}^m p_i \int g_i(y) \log g_i(y) \, dy\right\}
\]  
(64)
subject to $p_i \geq 0, \quad i = 1, \ldots, m,$
\[
\sum_{i=1}^m p_i = 1.
\]

The aforementioned is a convex problem since by Proposition 2, the objective function is concave and the constraint set is convex. The Lagrangian is given by
\[
L(p, \mu, \nu) = -\int \left(\sum_{i=1}^m p_i g_i(y)\right) \log \left(\sum_{i=1}^m p_i g_i(y)\right) \, dy
\]
\[-\sum_{i=1}^m p_i \int g_i(y) \log g_i(y) \, dy
\]
\[+ \sum_{i=1}^m \mu_i p_i + \nu \left(\sum_{i=1}^m p_i - 1\right),
\]  
(65)
with the notation $\mu = (\mu_1, \ldots, \mu_m)$. The optimality conditions are (cf. [12, Chapter 5.5.3])
\[
\frac{\partial L(p, \mu, \nu)}{\partial p_i} = 0,
\]  
(66)
$\mu_i p_i \geq 0$, $\mu_i p_i = 0$,
for all $i = 1, \ldots, m$. Partial derivatives of the Lagrangian with respect to $p_i$ are easily obtained as

$$
\frac{\partial L(p, \mu, \nu)}{\partial p_i} = -(\log e) - \int g_i(y) \log \left( \sum_{j=1}^{m} p_j g_j(y) \right) \, dy
+ \int g_i(y) \log g_i(y) \, dy + \mu_i + \nu,
$$

for $i = 1, \ldots, m$. Hence (66) leads to the conditions $p_i = 0$ or

$$
\int g_i(y) \left( \log g_i(y) - \log \left( \sum_{j=1}^{m} p_j g_j(y) \right) \right) \, dy = \log e - \nu,
$$

for all $i = 1, \ldots, m$. In summary, we have demonstrated the following result.

**Proposition 7.** Let $g_1, \ldots, g_m$ be Lebesgue $\lambda^m$-densities. Input distribution $p^*$ is capacity-achieving if and only if

$$
D(g_i \| \sum_{j=1}^{m} p_j^* g_j) = \zeta,
$$

for some $\zeta > 0$, for all $i$ such that $p_i^* > 0$. Furthermore, if $H(g_i)$ is independent of $i$, then $p^*$ is capacity-achieving if and only if

$$
\int g_i(y) \log \left( \sum_{j=1}^{m} p_j^* g_j(y) \right) \, dy = \xi
$$

for some $\xi \in \mathbb{R}$, for all $i$ such that $p_i^* > 0$.

Now, assume that the strategy set of the channel consists of the densities

$$
F = \{(f_1^{(\alpha)}(y), \ldots, f_m^{(\alpha)}(y)) \mid f_i^{(\alpha)}(y) = \alpha g_i(y) + (1 - \alpha) h_i(y), \, 0 \leq \alpha \leq 1\},
$$

where $g_i$ are densities with respect to $\lambda^m$ and $h_i$ represents the singleton distribution with support point $x_i$. $f_i^{(\alpha)}$ are hence densities with respect to the measure $\lambda^m + \chi$.

$F$ represents a closed convex line segment in the space of all densities, reaching from error distribution $(g_1, \ldots, g_m)$ at $\alpha = 1$ to the error-free singleton distribution $(h_1, \ldots, h_m)$ at $\alpha = 0$. The strategy set is analogous to the $m$-ary discrete output case with directional separation in Section 5.2. In Figure 5, the mixture of a standard Gaussian and the singleton distribution in 0 is depicted for $\alpha \in \{0.0, 0.25, 0.5, 0.75, 1.0\}$. Densities are with respect to the measure $\lambda^m + \chi(0)$.

Intuitively, it seems to be clear that the channel would choose the extreme value $\alpha = 1$ as the worst case to jam the transmitter. A precise proof of this fact, however, is amazingly complicated.

**Figure 5:** Mixture density of a standard Gaussian and a singleton distribution in 0. Densities are with respect to the measure $\lambda^m + \chi$.

By (61), mutual information is given as

$$
I(X; Y) = I(p; (f_1^{(\alpha)}, \ldots, f_m^{(\alpha)}))
\begin{align*}
&= H\left(\sum_{i=1}^{m} p_i f_i^{(\alpha)}\right) - \sum_{i=1}^{m} p_i H(f_i^{(\alpha)}) \\
&= H\left(\alpha \sum_{i=1}^{m} p_i g_i + (1 - \alpha) \sum_{i=1}^{m} p_i h_i\right) \\
&\quad - \sum_{i=1}^{m} p_i H(\alpha g_i + (1 - \alpha) h_i) \\
&= H(\alpha, 1 - \alpha) + \alpha H\left(\sum_{i=1}^{m} p_i g_i\right) + (1 - \alpha) H(p) \\
&\quad - \sum_{i=1}^{m} p_i (H(\alpha, 1 - \alpha) + \alpha H(g_i)) \\
&= \alpha \left( H\left(\sum_{i=1}^{m} p_i g_i\right) - \sum_{i=1}^{m} p_i H(g_i) - H(p) \right) + H(p).
\end{align*}
$$

Since by Proposition 6, the term in curly brackets in the last line of (72) is nonpositive, for any $p$, the minimum of $I(p; (f_1^{(\alpha)}, \ldots, f_m^{(\alpha)}))$ over $\alpha \in [0, 1]$ is attained at $\alpha = 1$ with value

$$
H\left(\sum_{i=1}^{m} p_i g_i\right) - \sum_{i=1}^{m} p_i H(g_i) = \sum_{i=1}^{m} p_i D\left(g_i \| \sum_{j=1}^{m} p_j g_j\right).
$$

From Proposition 7, it follows that the right-hand side is maximized at $p^* \in D^m$ whenever

$$
D\left(g_i \| \sum_{j=1}^{m} p_j^* g_j\right) = \zeta
$$

for all $i$ with $p_i^* > 0$. 

\[\]
In summary, the channel game has an equilibrium point

\[
\max_{p \in D} \min_{(f_1^{(a)}, \ldots, f_m^{(a)}) \in F} I(p; (f_1^{(a)}, \ldots, f_m^{(a)}))
\]

\[
= \min_{(f_1^{(a)}, \ldots, f_m^{(a)}) \in F} \max_{p \in D} I(p; (f_1^{(a)}, \ldots, f_m^{(a)})).
\]

(75)

The equilibrium strategy for the channel is given by \( \alpha = 1 \). The optimum strategy \( p^* \) for the transmitter is characterized by (74). For certain error distributions \( g_i \) this condition can be explicitly evaluated (see [17]).

8. Conclusions

We have investigated Nash equilibria for a two-person zero-sum game where the channel gambles against the transmitter. The transmitter strategy set consists of all input distributions over a finite symbol set, while the channel strategy sets are formed by certain convex subsets of channel matrices or noise distributions, respectively. Mutual information is used as payoff function.

Basically, it is assumed that a malicious nature is controlling the channel such that equilibria are achieved when the transmitter plays the capacity-achieving distribution against worst-case attributes of the channel. In practice, however, a wireless channel is only partially controlled by nature, for example, by shadowing and attenuation effects, further, diffraction and reflection. A major contribution to the channel properties, however, is made by interference from other users. It will be a subject of future research to investigate how these effects may be combined in a single strategy set of the channel. The question arises if equilibria for the game “one transmitter against a group of others plus random effects from nature” still exist.

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Research Article

Bargaining and the MISO Interference Channel

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We examine the MISO interference channel under cooperative bargaining theory. Bargaining approaches such as the Nash and Kalai-Smorodinsky solutions have previously been used in wireless networks to strike a balance between max-sum efficiency and max-min equity in users' rates. However, cooperative bargaining for the MISO interference channel has only been studied extensively for the two-user case. We present an algorithm that finds the optimal Kalai-Smorodinsky beamformers for an arbitrary number of users. We also consider joint scheduling and beamformer selection, using gradient ascent to find a stationary point of the Kalai-Smorodinsky objective function. When interference is strong, the flexibility allowed by scheduling compensates for the performance loss due to local optimization. Finally, we explore the benefits of power control, showing that power control provides nontrivial throughput gains when the number of transmitter/receiver pairs is greater than the number of transmit antennas.

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1. Introduction

After more than a decade of intense research, multiantenna communications systems are sufficiently well understood that they now appear in current and emerging wireless standards [1, 2]. Because they offer increased spatial flexibility, multiple-antenna systems are particularly well suited to multiuser communications. Generally speaking, multiuser communication presents a complicated problem partially because performance criteria are difficult to characterize. There is no, for example, single data rate or bit-error probability to optimize. Instead, we can only maximize composite performance measures such as the network sum rate, max-min fairness, or quality-of-service requirements. Ultimately, the choice of objective function is often somewhat arbitrary.

To meet this challenge, researchers have begun to apply game theory [3], a mathematical idealization of human decision-making, to problems in multiuser communications. Game theory provides a systematic framework for the study of decision makers with potentially conflicting interests, as well as solutions for such conflicts. Accordingly, a game-theoretic analysis can provide a tractable, structured approach to resource allocation. Researchers have successfully employed game-theoretic ideas to design “fair” medium-access protocols, develop decentralized network algorithms, and otherwise solve resource-allocation problems in communications networks [4–10].

In this paper, we study the multiple-input single-output (MISO) interference channel. In the MISO interference channel, several communication links, each involving a multiantenna transmitter and a single-antenna receiver, are simultaneously active. This scenario models, for example, intercell interference in cellular systems or MIMO networks where receivers employ fixed beamformers. Multilink MISO systems have been studied in a number of previous works. For example, in [11, 12], the MISO broadcast channel is studied from the intercell interference point of view, with emphasis on maximizing the network sum rate. The same scenario is addressed in [13], but max-min fairness is used to improve the performance of weaker network links. Game-theoretic solutions for the MISO interference channel based on bargaining have been considered in [14, 15], but only for the two-user case.

Our particular focus is to maximize network performance according to the Kalai-Smorodinsky solution [16],
a cooperative bargaining approach closely related to the well-known Nash bargaining solution [17]. For our problem, the fundamental idea of the Kalai-Smorodinsky (K-S) approach is to maximize users’ rates while ensuring that users experience the same fraction of the rate they would achieve without interference. In practice, the K-S solution defines a compromise between efficiency (defined herein in terms of maximizing the sum rate) and equity (maximizing the minimum rate). Our primary contribution is an algorithm that efficiently finds the K-S solution for an arbitrary number of users, rather than just the two-user case. We transform the rate-maximization problem to a series of convex programming problems, allowing us to find the beamformers that achieve the rates defined by the K-S solution.

A drawback of the K-S solution is that when interference becomes strong for a single user, all users’ bargained rates tend toward zero. To avoid this, we also study joint scheduling and beamformer selection under K-S bargaining, which introduces a temporal degree of freedom for avoiding interference. Scheduling also convexifies the feasible region, which is an important consideration in cooperative bargaining. However, the need to jointly address scheduling and beamformer selection complicates the optimization, preventing us from easily finding the K-S solution. We therefore devise a gradient-based algorithm to find a stationary point of the K-S objective function. While we sacrifice global optimality to include scheduling, the performance advantage of employing time-division multiplexing significantly outweighs the potential loss of optimality when the interference is strong.

The paper is organized as follows: in Section 2 we present the system model, discussing the achievable rates and a few simple beamforming strategies. In Section 3 we briefly introduce the Kalai-Smorodinsky solution. In Section 4 we present algorithms for selecting beamformers and (where applicable) transmission schedules that achieve the Kalai-Smorodinsky solution. In Section 5 we examine the fairness and efficiency of our proposed algorithms and discuss the effects of power control. Finally, we give our conclusions in Section 6.

2. System Model

2.1. Signal Model. The K-user MISO interference channel, as depicted in Figure 1, is composed of K N-antenna transmitters, each of which intends to communicate with a unique single-antenna receiver. We assume a narrowband channel model where the \( i \)th transmitter sends a complex baseband vector \( x_i \). The received signal \( y_i \) contains the intended signal, cochannel interference from the other \( K-1 \) transmitters, and additive Gaussian noise:

\[
y_i = h_{i}^H x_i + \sum_{j=1, j \neq i}^{K} h_{ij}^H x_j + n_i, \tag{1}
\]

where \( h_{i,j} \) is the vector of complex channel gains between the antennas of the \( j \)th transmitter and the \( i \)th receiver, and \( \cdot^H \) denotes the Hermitian transpose. We normalize the channel gains such that—without loss of generality—\( n_i \) has unit variance. Particularly, we assume channels of the form \( h_{i,j} = \sqrt{\rho_{i,j}} h_{i,j} \), where the elements of \( h_{i,j} \) are zero-mean, unit-variance complex random variables, and \( \rho_{i,j} \) represents the expected channel gain between the \( j \)th transmitter and the \( i \)th receiver.

2.2. Achievable Rates. To define the set of achievable rates under our assumptions, we view each transmitted signal \( x_i \) as a zero-mean random vector characterized by the covariance matrix \( P_i = E[x_i x_i^H] \), where \( E[\cdot] \) denotes statistical expectation. In principle, \( P_i \) can be any positive semidefinite matrix, although we focus on the rank-one case due to the MISO setting considered here. Specifically, we assume that \( x_i \) is of the form \( x_i = w_i s_i \), where \( w_i \) is the (fixed) transmit beamformer for user \( i \), and \( s_i \) is a zero-mean, unit-variance Gaussian symbol. Thus, \( P_i = w_i w_i^H \), and the spatial characteristics of the transmitted signal are entirely characterized by the beamforming vector \( w_i \).

Each transmitter has limited peak power output, which we model by constraining the norm of each beamformer: \( \|w_i\|_2 \leq 1 \), where \( \|\cdot\|_2 \) denotes the \( \ell_2 \) norm. Let \( W_i \) denote the set of feasible beamformers:

\[
W_i = \left\{ w \in \mathbb{C}^N : \|w\|_2 \leq 1 \right\}. \tag{2}
\]

Here we have defined a general model where transmitters choose both the magnitude of the beamformer, which represents the transmit power, as well as its direction. When the beamformers are unit-norm, the channel parameter \( \rho_{i,j} \) represents the received signal-to-noise ratio (SNR) between the \( j \)th transmitter and the \( i \)th receiver. We may wonder, given the spatial freedom offered by the multiple antennas, if such power control is necessary. For example, in [18] it is shown that, when \( K \leq N \), only beamformers with \( \|w_i\|_2 = 1 \) are necessary, obviating the need for power control. However, this result does not generalize, and in Section 5 we explore the benefit of power control in a system with an arbitrary number of users.

In determining achievable rates, we assume that transmitters and receivers have full channel state information and that the receivers employ single-user detection, meaning that cochannel interference is treated as noise when decoding the incoming signal. Under these assumptions, the rate across
the ith link is bounded by the mutual information between \( x_i \) and \( y_i \), which, in terms of the beamformers, is

\[
I(x_i; y_i) = \log_2 \left( 1 + \frac{|h_{ij}^I w_j|^2}{1 + \sum_{j \neq i} |h_{ij}^I w_j|^2} \right).
\]

(3)

For notational convenience we will occasionally group the beamformers into a single \( N \times K \) matrix \( W = [w_1, w_2, \ldots, w_K] \). Then, we can denote the mutual information across the ith link as a function of the beamformers: \( I_i(W) \). The set of achievable rates is bounded by the mutual information possible under all feasible beamformers:

\[
R = \left\{ \mathbf{r} \in \mathbb{R}_+^K : r_i \leq I_i(W), \mathbf{W} \in \mathbb{W}_W^K \right\}.
\]

(4)

The feasible set \( R \) has an important property which we will exploit throughout the paper: it is comprehensive with respect to the zero vector. A set \( S \subset \mathbb{R}^K \) is comprehensive with respect to 0 provided that for every \( \mathbf{r} \in S \), 0 \( \leq \mathbf{s} \leq \mathbf{r} \) implies \( \mathbf{s} \in S \), where \( \leq \) and \( \geq \) represent element-wise vector inequalities. In our case, the rate region \( R \) is comprehensive because any user can—without altering its beamformer—freely lower its rate without impacting other users’ rates.

### 2.3. Scheduling

In general, \( R \) is not convex, suggesting that we may achieve higher rates—especially in cases of strong interference—via time-sharing. (Alternatively, the rate region may be convexified by other equivalent means such as frequency-sharing or randomized beamformer selection.) To do so, we divide each transmission into \( K \) time blocks, during each of which the transmitters may use a different beamformer. The mutual information during block \( t \) is

\[
I(x_i(t); y_i(t)) = \log_2 \left( 1 + \frac{|h_{ij}^I w_j(t)|^2}{1 + \sum_{j \neq i} |h_{ij}^I w_j(t)|^2} \right).
\]

(5)

We use \( I_i(W(t)) \) to represent the mutual information during the ith block.

The relative duration of each block is defined by the scheduling vector \( \mathbf{a} = [a_1, \ldots, a_K] \), which obeys the constraints \( a_i \geq 0 \) and \( \sum_{i=1}^K a_i = 1 \). The scheduling weights in \( \mathbf{a} \) define a convex combination of the rates achieved during each time block. With scheduling, the average achievable rate over the ith link is bounded by the average mutual information \( \sum_{t=1}^K a_t I_i(W(t)) \). The set of feasible scheduling vectors is

\[
\mathcal{A} = \left\{ \mathbf{a} \in \mathbb{R}_+^K : \sum_{t=1}^K a_t = 1 \right\}.
\]

(6)

Since time-sharing allows us to take convex combinations of rate vectors, the set of achievable rates under scheduling, denoted by \( \overline{R} \), is the convex hull of \( R \):

\[
\overline{R} = \left\{ \mathbf{r} \in \mathbb{R}_+^K : r_i \leq \sum_{t=1}^K a_t I_i(W(t)), \mathbf{a} \in \mathcal{A}, \mathbf{W}(t) \in \mathbb{W}_W^K, \forall t \right\}.
\]

(7)

To see that \( K \) time blocks are sufficient to achieve the convex hull, note that the convex hull of \( R \) can be defined as the intersection of all closed half-planes in \( \mathbb{R}^K \) that contain \( R \). So, any boundary point on the convex hull of \( R \) must lie on a convex subset of a bounding hyperplane in \( \mathbb{R}^K \) defined by at most \( K \) linearly independent boundary points of \( R \). Thus any point on the boundary of the convex hull can be achieved by taking convex combinations of at most \( K \) points in \( R \). But, since \( R \) is comprehensive, we can reach any point in the convex hull by choosing the nearest boundary point and appropriately lowering the rates of the associated \( K \) points. To see that \( K \) points are required in general, consider an extreme case where \( p_{i,j} = \infty \) for \( i \neq j \), so only a single transmitter can achieve a nonzero rate at a time. To realize the boundary of the convex hull of \( R \), each user needs its own block in which to transmit, necessitating \( K \) blocks.

### 2.4. Beamforming Strategies

A few simple strategies for choosing beamformers have previously been proposed. The first is the Nash equilibrium (NE) beamformer [14], where each transmitter maximizes its own mutual information without regard for others. The NE beamformer relies on the fact that, regardless of interference, a transmitter maximizes its mutual information simply by maximizing \( |h_{ij}^I w_i|^2 \). By the Cauchy-Schwarz inequality, the NE beamformer is

\[
\mathbf{w}_i^{NE} = \frac{h_{ij}^I}{\|h_{ij}^I\|_2}.
\]

(8)

In game-theoretic terms, this choice of beamformers is a Nash equilibrium [19], meaning that no single transmitter can improve its rate by switching to a different beamformer. While the Nash equilibrium is individually optimal from each user’s perspective, it is frequently possible for transmitters to jointly choose beamformers such that each user’s rate is higher than the NE rate. Indeed, the NE has notoriously poor performance, especially when interference is strong.

The zero-forcing strategy [14] takes the opposite approach, focusing entirely on eliminating cochannel interference in order to maximize the mutual information of other users. To specify this beamformer, let \( \mathbf{H}_{-i} \) be the \( N \times K-1 \) matrix containing all of the interference channels for the ith transmitter:

\[
\mathbf{H}_{-i} = \begin{bmatrix} h_{1,i-j} \ldots h_{i-1,i-j} h_{i+1,i-j} \ldots h_{K,i-j} \end{bmatrix}.
\]

(9)

Then, we get the zero-forcing beamformer \( \mathbf{w}_{i}^{ZF} \) by projecting the NE beamformer onto the orthogonal complement of the column space of \( \mathbf{H}_{-i} \):

\[
\mathbf{w}_{i}^{ZF} = \frac{\Pi_{\mathbf{H}_{-i}}^\perp h_{ij}^I}{\|\Pi_{\mathbf{H}_{-i}}^\perp h_{ij}^I\|_2},
\]

(10)

where \( \Pi_{\mathbf{H}_{-i}}^\perp \) represents the appropriate orthogonal projection. By choosing \( \mathbf{w}_{i}^{ZF} \), a transmitter maximizes the mutual information across the ith channel after ensuring that its signal creates no cochannel interference. However, for randomly generated channels, \( \mathbf{w}_{i}^{ZF} = 0 \) almost surely when \( K > N \). In such cases, zero-forcing trivially eliminates
interference by choosing the zero vector unless \( h_{ij} \) is outside of the column space of \( H_{-i} \), which occurs with probability zero.

Finally, we can also eliminate interference via simple time-division multiple access (TDMA) scheduling. We divide up the transmission into equally spaced blocks by setting \( a_t = 1/K \) for every \( t \), and we allow each transmitter to signal, without interference, during a single block:

\[
\mathbf{w}_{iTDMA}(t) = \begin{cases} 
\frac{\mathbf{h}_{ij}}{\|\mathbf{h}_{ij}\|_2}, & \text{if } t = i, \\
0, & \text{otherwise.}
\end{cases}
\]  

(11)

TDMA guarantees that each user has a nonzero rate regardless of interference strength as long as \( K \) is finite. However, this approach entirely ignores the possibility of interference mitigation through beamforming. So, to select beamformers more comprehensively, we must clearly define our desired performance criteria, which we discuss in the next section.

**3. Kalai-Smorodinsky Solution**

We briefly introduce the Kalai-Smorodinsky (K-S) solution in an abstract setting, which we then apply to the MISO interference channel. A K-player bargaining game is formally defined by a set of feasible payoffs \( \mathcal{U} \subseteq \mathbb{R}^K \) and a disagreement point \( \delta \in \mathcal{U} \). The disagreement point represents the utility guaranteed to each player should bargaining fail. In bargaining games, players cooperatively choose a compromise point. That is, rather than myopically maximizing individual payoff, players jointly choose a strategy that results in a mutually agreeable payoff vector. A bargaining solution is a mapping \( f(\mathcal{U}, \delta) \) to a payoff vector \( \mathbf{u}^* \in \mathcal{U} \) such that \( \mathbf{u}^* \succeq \delta \).

The K-S solution is an axiomatic bargaining solution, meaning that it is characterized abstractly by a set of (ostensibly) reasonable axioms rather than by a concrete bargaining process. First, define the ideal point \( \mathbf{b}(\mathcal{U}, \delta) \) element-wise by

\[
b_i(\mathcal{U}, \delta) = \max\{u_i : u \in \mathcal{U}, u \succeq \delta\}.
\]  

(12)

The ideal point \( \mathbf{b} \) expresses the best-case utility for each player. Then, the K-S solution is defined by the following axioms.

(1) **Pareto Efficiency.** If \( u \in \mathcal{U} \) is a vector such that \( \mathbf{u} \succeq \mathbf{u}^* \), then \( \mathbf{u} = \mathbf{u}^* \). That is, there is no point \( \mathbf{u} \in \mathcal{U} \) such that any player receives higher payoff than under \( \mathbf{u}^* \) without penalizing another player. If there is a player \( i \) for which \( u_i > u_i^* \), then there must be at least one player \( j \) for which \( u_j < u_j^* \). Pareto efficiency ensures that we do not overlook any points which improve players’ payoff without cost to other players.

(2) **Invariance to Positive Affine Transformations.** Let \( A \) be a positive affine transformation; that is, \( l(s) = (c_1 s_1 + d_1, \ldots, c_K s_K + d_K)^T \) for positive \( c_i \) and arbitrary \( d_i \). Then, if \( f(\mathcal{U}, \delta) = \mathbf{u}^* \), then \( f(l(\mathcal{U}), l(\delta)) = l(\mathbf{u}^*) \). In short, the solution must be independent to the scale and zero level of the players’ utilities.

(3) **Symmetry.** Let \( T \) be a permutation of the players. Then, \( f(T(\mathcal{U}, T(\delta))) = T(\mathbf{u}^*) \) whenever \( f(\mathcal{U}, \delta) \). Here we impose a minimal sense of fairness on the solution. Since players may be interchanged without effect, each player obtains equal utility \( (u_i^*) = u_j^* \) for all \( i, j \) if \( \mathcal{U} \) is symmetric and \( \delta_i = \delta_j \) for all \( i, j \).

(4) **Monotonicity.** Let \( (\mathcal{U}, \delta) \) and \( (\mathcal{V}, \delta) \) be bargaining games such that \( \mathcal{V} \subseteq \mathcal{U} \) and \( \mathbf{b}(\mathcal{U}, \delta) = \mathbf{b}(\mathcal{V}, \delta) \). Then, \( f(\mathcal{V}, \delta) \geq f(\mathcal{U}, \delta) \).

While Axioms 1 and 3 seem obvious for a fair and efficient bargain, Axioms 2 and 4 merit further discussion in the context of bargaining in a wireless network. Invariance to affine transformations is usually invoked because the scale (or the units) of players’ utilities may be different. The so-called interpersonal comparison of utilities is therefore undesirable, since the utilities are incommensurable. Axiom 2 solves the commensurability problem by making the solution independent to the scale level of players’ utilities; the units are abstracted away by the bargaining solution.

For our problem, we have expressed each player’s utility function in the same units (bits/sec/Hz), perhaps suggesting that Axiom 2 is unnecessary. While there is much to be said about the appropriateness of affine invariance, we note the following practical observation: *different users may regard equal rates differently.* A user with lower quality-of-service demands, for example, might assign higher utility to a particular rate than would a user with higher demands. So, users’ true utilities are arbitrary (but presumably nondecreasing) functions of the rates. In identifying the users’ utilities as the rates and invoking affine invariance, we tacitly assume that the true utilities are positive affine functions of the rates, with scale and zero level unknown. In this case, invariance to positive affine transformations is a necessary criterion for bargaining among the wireless users.

Axiom 4 prescribes a subjective notion of fairness by dictating the variation of the solution under changes in \( \mathcal{U} \). Monotonicity ensures that if we expand the set of feasible utilities, the bargained utility to each player can only increase. Indirectly, monotonicity ensures that stronger players receive higher payoff and are not unduly penalized by bargaining.

In its original presentation [16], it is shown that a unique solution \( f(\mathcal{U}, \delta) \) satisfies Axioms 1–4 for any two-player game in which \( \mathcal{U} \) is both compact and convex. In order to generalize the solution to \( K \) players, however, we need to place further restrictions on \( \mathcal{U} \) [22]. Fortunately, the generalization is straightforward when we restrict our attention to the class of bargaining games where \( \mathcal{U} \) is also comprehensive [20, 23] and satisfies the following property: if \( \mathbf{u}, \mathbf{v} \in \mathcal{U} \) satisfy \( \mathbf{u} \neq \mathbf{v} \) and \( \mathbf{u} \preceq \mathbf{v} \), then there exists \( \mathbf{w} \in \mathcal{U} \) such that \( \mathbf{w} \) strictly dominates \( \mathbf{u} \), or \( \mathbf{w} > \mathbf{u} \).

As long as \( \mathcal{U} \) is compact, convex, comprehensive, and satisfies the above criterion, the four axioms lead to a unique solution with a convenient geometric interpretation, as depicted in Figure 2 for \( \delta = 0 \). The K-S solution \( \mathbf{u}^* \) is
the largest element in $\mathcal{U}$ (with respect to any norm) that lies along the line segment connecting $\delta$ with $b$, or the maximum point $u$ such that $(u_i - \delta_i)/(b_i - \delta_i) = (u_j - \delta_j)/(b_j - \delta_j)$ for all $i, j$. Equivalently, we can express the K-S solution as an optimization over a weighted minimum objective function:

$$u^* = \arg \max_{u \in \mathcal{U}} \min \left( \frac{u_1 - \delta_1}{b_1 - \delta_1}, \ldots, \frac{u_K - \delta_K}{b_K - \delta_K} \right).$$  \hspace{1cm} (13)

The solution (13) exposes a connection between the K-S solution and max-min fairness, which focuses on improving the payoff of the weakest players. While max-min is a widely accepted criterion of fairness in both human and artificial systems [24–27], it allows weak players to limit (unfairly, one might argue) the payoff of stronger players, especially when $\mathcal{U}$ is highly asymmetric [28, 29]. “Fairness” is ultimately a subjective notion, so we refer to the max-min payoffs as equitable rather than fair, since max-min gives equal payoff to all users for convex $\mathcal{U}$.

Rather than strictly maximizing the minimum rate, the K-S solution normalizes the payoffs according to the shape of $\mathcal{U}$, placing a premium on increasing payoff to players with higher best-case payoff $b_i$. Doing so increases the sum payoff at the cost of the payoff of the weakest player. In practice, we may regard the K-S solution as a balance between strict max-min equity and max-sum efficiency, a position further justified by the results in Section 5.

3.1. Convexity. Of course, since the achievable rates for the MISO interference channel are only convex under scheduling, we should also consider K-S bargaining when $\mathcal{U}$ is not convex. Fortunately, the K-S solution has also been studied for nonconvex $\mathcal{U}$ [30, 31]. It is shown in [30] that by weakening Pareto efficiency, the solution given above extends to comprehensive, compact, but nonconvex $\mathcal{U}$. Specifically, Pareto efficiency is replaced with the following axiom:

(5) Weak Pareto Efficiency. If $u \in \mathcal{U}$ is a vector such that $u > u^*$, then $u = u^*$. That is, there exists no other $u \in \mathcal{U}$ such that every player obtains higher payoff than in $u^*$. In contrast to strong Pareto efficiency, it may indeed be possible to find a point $u \in \mathcal{U}$ that improves several players’ utilities without harming other players.

As long as $\mathcal{U}$ is compact and comprehensive, the maximal element in $\mathcal{U}$ along the line segment connecting $\delta$ and $b$ is the unique solution satisfying Axioms 2–5. Since $\mathcal{U}$ is nonconvex, the solution point $u^*$ may not be the unique weighted max-min point from (13), since there may be multiple max-min points as depicted in Figure 3. If, of course, the weak Pareto frontier of $\mathcal{U}$ coincides with its strong Pareto frontier, $u^*$ is still Pareto efficient and corresponds to the unique weighted max-min point as before. As we will see in Section 5.2, this is usually the case with the rate regions associated with the MISO interference channel.

4. K-S Bargaining for the MISO Channel

Finding the K-S solution for the MISO interference channel requires that we cast the problem in the game-theoretic framework discussed in the previous section. The recasting is straightforward. The transmitters, which choose the beamforming strategies, serve as players, and the utility function of each player is the achievable rate, which is the (average, where appropriate) mutual information. So, the set of feasible payoffs is $\mathcal{R}$, unless we allow scheduling, in which case it is $\mathcal{R}$. The simplest is to let $\delta = 0$, which tacitly assumes that if the bargaining process fails, the network simply shuts down. Another common choice [32] is the security level of each player, or the maximum payoff a player can guarantee for itself even if other players conspire against it:

$$\delta_i = \max_{w_i, w_j, j \neq i} I_i(W).$$  \hspace{1cm} (14)
In this case, each player pessimistically assumes only the worst-case rate should bargaining fail. Finally, we can choose the noncooperative Nash equilibrium rate as described in Section 2.4. Here we assume that if bargaining fails, players will simply act out of self-interest. Primarily due to simplicity, we take $\delta = 0$ for the remainder of the paper. It is possible to modify our methods to accommodate an arbitrary $\delta$, but only at the cost of increased computational complexity.

With the problem recast as a bargaining game, we can start looking for the K-S solution as defined in the previous section. Of course, in addition to finding the rates associated with the K-S solution, we need to find the beamformers (and, where appropriate, scheduling vector) that achieve the K-S rates. In this section we present algorithms that find the K-S solution by constructing the rate-achieving beamformers and scheduling vector.

4.1. Without Scheduling. First we consider the problem without scheduling, in which case we can find the optimal K-S beamformers. The first step is to find $\mathbf{b}$, the vector of best-case rates for each user. Fortunately, the best-case rates are easily computed. The best possible scenario for the $i$th transmitter is when all other transmitters shut down, and the $i$th transmitter uses the Nash equilibrium beamformer $\mathbf{w}_i^{NE} = \frac{\mathbf{h}_i}{\|\mathbf{h}_i\|_2}$, giving a best-case rate of

$$b_i = \log_2 \left( 1 + \frac{\|\mathbf{h}_i\|_2^2}{\|\mathbf{h}_i\|_2^2} \right).$$

(15)

Since we have chosen $\delta = 0$, the K-S solution forces the bargained rates $r^*$ to lie along the line segment connecting the origin and $\mathbf{b}$. In other words, they must satisfy $r^* = t\mathbf{b}$ for some scalar $0 \leq t \leq 1$. So, we can find the K-S rates and beamformers (which we gather into the matrix $\mathbf{W}$) by solving the following optimization problem:

$$\max_{\mathbf{W} \in \mathbb{C}^{M \times K}} t$$

subject to

$$tb_i = I_i(\mathbf{W}), \quad \forall i,$$

$$\|\mathbf{w}_i\|_2 \leq 1, \quad \forall i.$$ (16)

While the objective function and norm constraint in (16) are convex, the mutual information constraint is not. However, by slightly relaxing the problem, we can make the mutual information constraint convex. Instead of restricting ourselves to beamformers, we allow transmitters to choose covariance matrices $\mathbf{P}_j = E(\mathbf{x}_j \mathbf{x}_j^H)$ with arbitrary rank. We restrict the trace of the covariances to model the power constraint:

$$\text{tr}(\mathbf{P}_i) \leq 1, \quad \forall i,$$ (17)

where $\text{tr}(\cdot)$ denotes the matrix trace. In terms of covariances, the mutual information between $\mathbf{x}_i$ and $y_i$ is

$$I(\mathbf{x}_i; y_i) = \log_2 \left( 1 + \frac{\mathbf{h}_i^H \mathbf{P}_i \mathbf{h}_i}{1 + \sum_{j \neq i} \mathbf{h}_j^H \mathbf{P}_j \mathbf{h}_j} \right).$$ (18)

Exponentiating both sides and rearranging, the mutual information constraint can be written as

$$2^\rho = 1 + \frac{\mathbf{h}_i^H \mathbf{P}_i \mathbf{h}_i}{1 + \sum_{j \neq i} \mathbf{h}_j^H \mathbf{P}_j \mathbf{h}_j},$$

$$\mathbf{h}_i^H \mathbf{P}_i \mathbf{h}_i = (2^\rho - 1) \left( 1 + \sum_{j \neq i} \mathbf{h}_j^H \mathbf{P}_j \mathbf{h}_j \right).$$

(19) (20)

The equivalent constraint in (20) is affine (and therefore convex) with respect to the covariance matrices. Now, we can find the K-S solution as an optimization problem over the covariances:

$$\max_{\mathbf{P}_i \in \mathbb{C}^{M \times M}} t$$

subject to

$$\mathbf{h}_i^H \mathbf{P}_i \mathbf{h}_i = (2^{tb_i} - 1) \left( 1 + \sum_{j \neq i} \mathbf{h}_j^H \mathbf{P}_j \mathbf{h}_j \right), \quad \forall i.$$ (21)

$$\mathbf{P}_i \in S_+, \quad \forall i,$$

$$\text{tr}(\mathbf{P}_i) \leq 1, \quad \forall i,$$

where $S_+$ is the set of positive semi-definite matrices. The mutual information constraint in (21) is convex with respect to the covariances but still nonconvex with respect to $t$. The structure of (21) allows a solution by iteratively using convex optimization techniques. Our approach is to choose $t$ according to the bisection method, using a convex feasibility test to see whether or not there exist feasible covariances that achieve the associated rates $r = tb$.

Given a fixed $t$, we test for feasibility by solving the following convex feasibility problem [33]:

$$\text{find } \mathbf{P}_1, \ldots, \mathbf{P}_K$$

subject to

$$\mathbf{h}_i^H \mathbf{P}_i \mathbf{h}_i = (2^{tb_i} - 1) \left( 1 + \sum_{j \neq i} \mathbf{h}_j^H \mathbf{P}_j \mathbf{h}_j \right),$$ (22)

$$\mathbf{P}_i \in S_+, \quad \forall i,$$

$$\text{tr}(\mathbf{P}_i) \leq 1, \quad \forall i.$$ (23)

If the rates $r = tb$ are feasible, then performing the test in (22) also produces achieving covariance matrices. In our simulations, we test for feasibility using the convex programming package cvx [34].

We find the K-S covariances by combining the bisection line-search method with the feasibility test in (22), as depicted in Figure 4. We start by setting $t^{\text{min}} = 0$ and $t^{\text{max}} = 1$. At iteration $k$, we choose the test point $t(k)$ defined by $t(k) = (t^{\text{max}} + t^{\text{min}}) / 2$. We then test the rate vector $r(k) = t(k)\mathbf{b}$ for feasibility by solving the problem defined by (22). If $r(k)$ is feasible, then we set $t^{\text{min}} = t(k)$ and store the feasible covariances as the current solution. If $r(k)$ is infeasible, we set $t^{\text{max}} = t(k)$. Iterations continue until $t^{\text{max}} - t^{\text{min}} < \epsilon$ for small $\epsilon > 0$. At this point, we choose the rates $r^* = t^{\text{min}}\mathbf{b}$, which are
covariance matrix easily extracted as the sole nontrivial eigenvector of each returns rank-one covariances. The K-S beamformers are then a summary of the procedure in Algorithm 1.

Consider a fixed transmitter $j$ and set of receivers $I$ that contains at least $N$ members, but $j \not\in I$. If the vectors $\hat{h}_{ij}$ span all of $\mathbb{C}^N$, then the K-S rates $r^* \to 0$ as $\rho_{i,j} \to \infty$ for all $i \in I$.

**Proposition 1.** Consider a fixed transmitter $j$ and set of receivers $I$ that contains at least $N$ members, but $j \not\in I$. If the vectors $\hat{h}_{ij}$ span all of $\mathbb{C}^N$, then the K-S rates $r^* \to 0$ as $\rho_{i,j} \to \infty$ for all $i \in I$.

**Proof.** This result follows directly from the requirement $r^* = t b$ for scalar $0 \leq t \leq 1$. If one user’s rate approaches zero, all rates must approach zero. We argue by contradiction. Supposing users’ rates do not approach zero, $\|w_i\| \geq d$ for some fixed $d > 0$. But, since $\rho_{i,j} \to \infty$ for $i \in I$, the rates $r_i$ approach zero unless $w_i$ is orthogonal to all $\hat{h}_{ij}$, $i \in I$. Since the vectors $\hat{h}_{ij}$ span $\mathbb{C}^N$, only $w_i = 0$ is orthogonal to them all, which is a contradiction.

The requirement that the vectors $\hat{h}_{ij}$ span $\mathbb{C}^N$ is mild, since most any generating distribution will produce linearly independent channel vectors almost surely until $\mathbb{C}^N$ is spanned. The condition $\rho_{i,j} \to \infty$ for fixed $j$ and several $i \in I$ is roughly equivalent to moving a cluster of receivers $i \in I$ closer and closer to transmitter $j$. (Of course, the channel gains in a practical system will never approach infinity, but they can become large enough to induce the described asymptotic behavior.) While this scenario is somewhat unlikely, it represents a reasonable worst-case scenario. Similar statements hold when $K \to \infty$ and the gains $\rho_{i,j}$ are bounded away from zero, or when transmitter $j$ has inaccurate channel state information and $\rho_{i,j} \to \infty$ for any $i \neq j$. In a variety of asymptotic cases, the system responds to strong interference by simply shutting down.

It is perhaps unsurprising that rates go to zero when the interference gains $\rho_{i,j}$ or the number of users go to infinity. What is remarkable, however, is that all users’ rates approach zero, even though only a subset of users needs to be shut down. This occurs because of the behavior of the K-S solution for nonconvex sets. The symmetry axiom precludes our shutting down some users but not others, and we are forced instead to accept the weakly Pareto efficient point $r^* = 0$. In Section 4.2, we show how the use of scheduling alleviates this drawback.

**4.1.2. Pareto Efficiency.** If we are willing to violate symmetry, we can extend the algorithm presented above to find (strongly) Pareto efficient rates that are at least as great as the K-S rates. After finding the K-S rates, we can randomly choose a user and use the bisection/feasibility method to increase the user’s rate without decreasing other users’ rates.

**Algorithm 1:** Kalai-Smorodinsky solution.

**Input:** Channel vectors $h_{ij}$, best-case rates $b$, and tolerance $\epsilon > 0$  
**Output:** Solution rates $r^*$ and covariances $P_i^*$  
$r_{\text{max}} \leftarrow 1$  
r_{\text{min}} \leftarrow 0  
while $r_{\text{max}} - r_{\text{min}} \geq \epsilon$ do  
$t \leftarrow (r_{\text{max}} + r_{\text{min}})/2$  
Find covariances $P_i$ from feasibility test (22) using $t$  
if $t$ feasible then  
$r^* = t b$  
$P_i^* = P_i$, $\forall i$  
r_{\text{min}} \leftarrow t$  
else  
r_{\text{max}} \leftarrow t$

**Figure 4:** Depiction of bisection/feasibility algorithm for the K-S solution. The first few test points are numbered sequentially.

arbitrarily close to the K-S solution. We give a pseudocode summary of the procedure in Algorithm 1.

We emphasize that the generalization from beamformers to arbitrary-rank covariances is only an intermediate step that makes the feasibility problem convex. In [35] it is shown that any rates on the Pareto frontier (strong or weak) are achieved by rank-one covariances. Algorithm 1 therefore returns rank-one covariances except possibly for negligible numerical artifacts associated with the tolerance $\epsilon$. Experimentally, we indeed find that Algorithm 1 always returns rank-one covariances. The K-S beamformers are then easily extracted as the sole nontrivial eigenvector of each covariance matrix $P_i^*$.

Finally, we can also adapt Algorithm 1 for an arbitrary disagreement point $\delta$. The only real difficulty is to compute the best-case rates $b$ for the new disagreement point. Fortunately, the bisection/feasibility test is easily adapted to compute $b$. For each user $i$, we draw a line segment between $\delta$ and the point $q_i = (\delta_1, \ldots, \log_2(1 + \|h_{ij}\|^2), \ldots, \delta_K)$. Using the bisection/feasibility method to find the maximal point on the line segment joining $\delta$ and $q_i$, we find the maximum rate $b_i$ for user $i$ such that every other user obtains the rates given in $\delta$. Now we can straightforwardly adapt Algorithm 1 to find the K-S rates, which now lie on the line segment joining $\delta$ and $b$. However, the generality comes with a significant increase in complexity: since we have to run the bisection/feasibility algorithm for each user individually to find $b$, the computational complexity is increased by a factor of $K$.
More precisely, let \( r^* = (r^*_1, \ldots, r^*_K) \) be the K-S rates, and randomly choose a user \( i \). Then, we can test points along the line segment joining \( r^* \) and \((r^*_1, \ldots, b_i, \ldots, r^*_K)\) for feasibility as before. Thus, we maximize \( r_i \) while keeping the other rates constant. After maximizing \( r_i \), we can pick another user, maximize its rate, and continue until all users’ rates are maximized. The resulting rates are strongly Pareto efficient by construction, but they no longer conform to the K-S axioms. In fact, they do not represent a bargaining solution in any sense: while they are at least as great as the K-S rates, they do not conform to any axioms other than Pareto efficiency.

Ensuring strong Pareto efficiency increases the computational burden by approximately a factor of \( K \). In Section 5, we explore the benefits obtained, showing that, except in asymptotic cases, the K-S solution produced by Algorithm 1 is typically close to a strongly Pareto solution.

4.2. With Scheduling. Using scheduling, the K-S solution is characterized by the beamformers and scheduling vector that maximize the objective function defined by the K-S solution:

\[
J(W(1), \ldots, W(K), a) = \min_i \left\{ \frac{1}{b_i} \sum_{k=1}^{K} a_i I_i(W(t)) \right\},
\]

where we condense notation by collecting the beamformers and scheduling vector into a scheduling profile \( S = (W(1), \ldots, W(K), a) \) in the set \( \mathcal{S} = \mathcal{W}^K \times \mathcal{A} \), and we let \( r_i(S) = \sum_{k=1}^{K} a_i I_i(W(t)) \) denote user \( i \)'s average rate.

Ironically, however, taking convex combinations of mutual information prevents us from transforming (23) into a series of convex problems as in Section 4.1. Instead, we seek a locally optimal solution, which suggests a gradient-based approach. Unfortunately, \( J(S) \) is not continuously differentiable; in particular, the derivative is not continuous at the K-S point. So, instead of maximizing \( J(S) \) directly, we successively maximize smooth approximations. Define

\[
F(S, d) = \frac{1}{\lambda} \ln \frac{r_i(S)}{b_i} - d,
\]

with \( d < \min_i(r_i(S)/b_i) \). Although it may not be immediately clear, we will see that maximizing \( F(S, d) \) is nearly equivalent to maximizing \( J(S) \) for well-chosen \( d \).

To maximize \( F(S, d) \) with respect to the beamformers and scheduling vector, we use the gradient projection method [36], a well-known method used to optimize a scalar function whose argument is an element of a convex set. It has been used to optimize similar multiantenna problems in [37–39].

First, we initialize the algorithm with a randomly chosen point \( S^0 = (W^0(1), \ldots, W^0(K), a^0) \) \( \in \mathcal{X} \), and choose

\[
d^0 = \min_i \left( \frac{r_i(S^0)}{b_i} \right) - \epsilon_d,
\]

where \( \epsilon_d > 0 \) is a small constant. That is, we set \( d^0 \) close to the minimum weighted average rate under \( S^0 \).

Next, we take a step in the direction of the gradient of \( F(S^0, d^0) \). The gradient with respect to the beamformers is found by first finding the gradient of each mutual information term \( I_i(W(t)) \). Using the complex gradient

\[
\nabla_w J_i(W(t)) = \left\{ \frac{2}{\ln 2} (\frac{a_i(t) + v_i(t)}{\sigma_i(t)} - 1) h_i^H w_i(t) h_j, \quad \text{for } i = j, \right.
\]

\[
- \frac{2\sigma_i(t)}{\ln 2} \left( \frac{v_i(t)}{\sigma_i(t)} - 1 \right)^{-1} h_i^H w_i(t) h_j, \quad \text{for } i \neq j,
\]

where \( \sigma_i(t) = |h_i^H w_i(t)|^2 \) is the signal power at receiver \( i \) during block \( t \), and \( v_i(t) = 1 + \sum_{j \neq i} |h_i^H w_j(t)|^2 \) is the corresponding interference-plus-noise power.

Using the chain rule, the gradient of \( F(S; d) \) with respect to a beamformer \( w_i(t) \) is

\[
\nabla_{w_i(t)} F(S; d) = a_i \sum_{k=1}^{K} \left( \frac{r_i(S)}{b_i} - d \right)^{-1} \nabla_w J_i(W(t)). \quad (28)
\]

Since the scheduling vector is real-valued, the gradient with respect to \( a_i \) is simply a vector of partial derivatives:

\[
\nabla_{a_i} F(S; d) = a_i \sum_{k=1}^{K} \left( \frac{r_i(S)}{b_i} - d \right)^{-1} I_i(W(t)). \quad (29)
\]

Equations (28) and (29) highlight the connection between maximizing the sum of logs in \( F(S; d) \) and the minimum in \( J(S) \). By setting \( d \) close to the minimum weighted rate, \( (r_i(S)/b_i - d)^{-1} \) becomes large for the minimum-weighted-rate user \( i \). So, the mutual information terms of user \( i \) dominate the gradient of \( F(S; d) \), making it approximately proportional to the gradient of \( J(S) \).

Having computed the gradient for each element of \( S \), we take a step in the direction of steepest ascent:

\[
S^t = S^0 + s \nabla S F(S^0; d^0), \quad (30)
\]

for fixed step size \( s > 0 \). Theoretically, \( s \) can be any constant [36], but since the factor \( (r_i(S)/b_i - d)^{-1} \) may be quite large, we take \( s \) to be small, on the order of \( \epsilon_d \). Of course, following the gradient may lead to an infeasible beamformer or scheduling vector. So, we project each \( \hat{w}_i(t) \) and \( \hat{a}^t \) onto the feasible sets \( \mathcal{W}_t \) and \( \mathcal{A}_t \), respectively. It is straightforward to show that the minimum-norm projections involve normalization and zeroing out, if necessary:

\[
\text{proj}_{w_i} \{w\} = \begin{cases} w, & \text{for } ||w|| > 1, \\ \text{proj}_{\mathcal{A}_t} \{a\} = [a - \lambda]^+, \end{cases} \quad (31)
\]

\[
\text{proj}_{\mathcal{W}_t} \{w\} = \begin{cases} w, & \text{for } ||w|| \leq 1, \\ \text{proj}_{\mathcal{A}_t} \{a\} = [a - \lambda]^+, \end{cases}
\]

where \([\cdot]^+ = \max(\cdot, 0)\), and \( \lambda \geq 0 \) is a constant ensuring that the projected vector sums to unity. We can quickly solve for \( \lambda \).
using the bisection method. After taking a gradient step, we compute a new point \( S^0 \) defined by the projections onto the feasible space:

\[
\hat{w}^0_i(t) = \text{proj}_{w_i} \left\{ \omega^0(t) \right\}, \quad \forall i, t,
\]

\[
a^0 = \text{proj}_A \{ \hat{a}^0 \}.
\]

Finally, we choose a new point \( S^1 \) by stepping in the feasible direction defined by the projected vectors:

\[
S^1 = S^0 + a^0 \left(S^0 - S^0\right),
\]

for a variable step size \( 0 \leq a^0 \leq 1 \). Since (33) defines a convex combination, we always have \( S^1 \in \mathcal{S} \). We choose \( a^0 \) according to Armijo’s rule along the feasible direction, which sets \( a^0 = y^{m_0} \) for some \( 0 \leq y \leq 1 \) and \( m_0 \) the smallest nonnegative integer such that

\[
F(S^1; d^0) - F(S^1; d^0) \geq \beta y^{m_0} \left\langle \nabla S F(S^1; d^0), S^0 - S^0 \right\rangle
\]

\[
= \beta y^{m_0} \left\langle \nabla S F, a^0 - a^0 \right\rangle + \sum_{i \neq j} \left\langle \nabla w_i(t) F, \hat{w}^0_i(t) - w^0_i(t) \right\rangle.
\]

At the beginning of each subsequent iteration \( k \), we choose \( d^k \) by computing

\[
(d^k)' = \min_i \left( \frac{r_i(S^k)}{b_i} \right) - \epsilon_d.
\]

If \( (d^k)' - d^{k-1} > \epsilon_d \) we choose \( d^k = (d^k)' \), and otherwise we choose \( d^k = d^{k-1} \). Since Armijo’s rule (34) ensures \( F(S^k; d^k) > F(S^{k-1}; d^{k-1}) \), our choice of \( d^k \) guarantees \( d^k < \min_i (r_i(S^k)/b_i) \), so \( F(S^k; d^k) \) is always well defined.

As before, we step in the direction of the gradient, but now using the function \( F(S; d^k) \), giving \( \hat{S}^k = S^0 + s\nabla S F(S^k; d^k) \). We again take the projection \( S^{k+1} = \text{proj}_{S^k} \hat{S}^k \) onto the feasible set, and we choose a new point according to the convex combination \( S^{k+1} = S^k + a^k(S^k - S^k) \), with \( a^k \) decided by Armijo’s rule. Iterations continue until

\[
\max \left| S^{k+1} - S^k \right| < \epsilon_1,
\]

where \( \max | \cdot | \) returns the absolute value of the maximal element of its argument. At convergence, the solution point \( S^* \) is, within the specified tolerance, a stationary point of \( F(S; d_k) \). The algorithm is summarized in Algorithm 2.

Finally, we note that we cannot easily modify Algorithm 2 to use an arbitrary disagreement point \( \delta \). As before, the primary difficulty is computing the best-case rates \( b \) for the new disagreement point. Since Algorithm 2 operates on gradient ascent, we can only approximate the best-case rates. Since the best-case rates are so easily computed for \( \delta = 0 \), we focus exclusively on this case.

### Algorithm 2: Kalai-Smorodinsky solution (with scheduling).

Input: Channel vectors \( h_{ij} \), initialization point \( S^0 \), and parameters \( s, \beta, \gamma, \epsilon_d \)

Output: Stationary point \( S^* \) containing beamformers and scheduling vector.

1. \( k = 0 \)
2. \( d^0 = \min_i (r_i(S^0)/b_i) - \epsilon_d \)
3. while \( \max |S^{k+1} - S^k| \geq \epsilon_d \)
4. \( S^k \leftarrow \text{proj}_{A} \{ S^k \} \)
5. \( \beta = \min_i (r_i(S^{k+1})/b_i) - \epsilon_d \)
6. if \( (d^{k+1})' - d^k > \epsilon_d \)
7. \( d^{k+1} = (d^{k+1})' \)
8. else
9. \( d^{k+1} = d^k \)
10. \( k = k + 1 \)
11. \( S^* \leftarrow S^{k+1} \)

#### 4.2.1. Convergence. The convergence of Algorithm 2 is guaranteed by the convergence of the sequence \( \{d_k\} \). Since \( b_i \) is the best-case rate, the average rate \( r_i(S) \) cannot exceed \( b_i \). Then, by definition, \( d_k \leq \min_i (r_i(S)/b_i) \leq 1 \) for all \( k \). The sequence \( \{d_k\} \) is therefore bounded, and since it is also nondecreasing, it must converge to a limit. Furthermore, since \( d_k \) must increase by at least \( \epsilon_d \) or remain constant, \( \{d_k\} \) reaches its limit at finite \( k \). Therefore, after a finite number of iterations, we perform gradient projection on \( F(S; d) \) for fixed \( d \), which converges to a stationary point.

Of course, convergence to a stationary point of \( F(S; d) \) does not guarantee a good approximation to the K-S solution. Indeed, the result of Algorithm 2 does not, in general, satisfy the K-S axioms described in Section 3. However, if the solution point well-approximates the K-S point, then it may approximate the desirable properties of the K-S solution. So, we examine the solution point \( S^* \) in terms of the criterion for the maximum of \( F(S) \): maximizing the minimum weighted rate.

By setting \( d_k \) close to \( \min_i (r_i(S^k)/b_i) \), we give priority to increasing the minimum weighted rate. Indeed, as we let \( \epsilon_d \to 0 \), the relative benefit of increasing the minimum weighted rate becomes arbitrarily large, suggesting that the algorithm will primarily focus on maximizing \( \min_i (r_i(S^k)/b_i) \) until \( r_i(S^k)/b_i = r_j(S^k)/b_j \) for all users. However, since \( F(S; d) \) is not convex, it is always possible for gradient projection to halt at a stationary point such that \( r_i(S^k)/b_i \) and \( r_j(S^k)/b_j \) are far apart. On the other hand, since we set \( \epsilon_d \) to a fixed nonzero value, we can increase \( F(S; d) \) by increasing any one rate, even if we are at a stationary point for the minimum weighted rate. As a result, in practice, our algorithm tends to avoid such points, and \( r_i(S^k)/b_i \) and \( r_j(S^k)/b_j \) are close together. Since we cannot guarantee this
usually the case.

5. Numerical Results

5.1. Performance. To examine the performance of the proposed algorithms, we simulate on randomly generated channels. For our simulations, we choose $N = 4$ and let $K$ vary. In each simulation, we randomly place $K$ transmitter/receiver pairs on the unit square. The channel coefficients are independently drawn from the zero-mean, unit-variance, complex Gaussian distribution. The channel gains $\rho_{ij}$ are computed according to the path-loss model

$$\rho_{ij} = \frac{M}{d(i,j)^{\alpha}},$$

where $d(i,j)$ is the Euclidean distance between the $j$th transmitter and the $i$th receiver, $M$ is an arbitrary constant, and $\alpha$ is the path loss exponent. In our simulations, we set $\alpha = 4$ and choose $M = 5/8$, which forces $\rho_{ij} = 10$ dB when $d(i,j) = 1/2$. For Algorithm 1 (and related methods), we set the convergence tolerance to $\epsilon = 10^{-3}$. For Algorithm 2, we use parameters $s = 10^{-3}$, $\epsilon_d = \epsilon_t = 10^{-3}$, $\gamma = 0.5$, and $\beta = 0.05$.

In Figures 5 and 6 we examine algorithm performance in terms of efficiency and equity for $K = \{2,4,6,8,10\}$. We compare the proposed K-S algorithms with the max-min, max-sum, and TDMA rates. To compute the max-min rates, we modify Algorithm 1 to find the maximal rates such that all rates are equal. To maximize the sum rate, we employ a gradient-based method similar to [37], which returns a stationary point of the sum rate. The TDMA rates, computed easily by using the beamforming schedule from (11) provide a baseline for the scheduled K-S solutions. By definition, the TDMA rate for user $i$ is $b_i/K$. So, the rates satisfy $r_i/b_i = r_j/b_j$, making them the optimal scheduling of single-user rates in the K-S sense.

Figure 5 shows the average mutual information per user, averaged over 100 realizations for each value of $K$. Not surprisingly, the average rate is highest under sum rate maximization. Both K-S approaches degrade as we increase the number of users, but eventually the scheduling approach gives a better average rate in spite of the fact that it gives only a stationary point. In Figure 6 we examine the minimum mutual information across all links, averaged over the same 100 realizations. Max-min (again unsurprisingly) gives the highest minimum rate, followed by the K-S approaches. Max-sum gives the worst minimum rate, which drops nearly to zero beyond $K = 2$. The K-S solution allows us to maintain the sum rate while still protecting the weakest links.

Next, we focus on the performance of the scheduled K-S approach. Specifically, we examine how well the algorithm maintains the K-S constraint $r_i/b_i = r_j/b_j$. For each simulation, we compute the minimum normalized rate $c_{\text{min}} = \min \{r_i/b_i\}$. In Figure 7, we plot the empirical cumulative distribution function (CDF) the deviation of the normalized rates from $c_{\text{min}}$ for several values of $K$. An ideal CDF would form a sharp corner, meaning that all of the deviations from $c_{\text{min}}$ would be zero. The CDF for $K = 2$ approximates the ideal case, with large deviations extremely rare. As $K$ increases, the corner increasingly rounds off—the normalized rates diverge more and more from $c_{\text{min}}$. However, even for $K = 10$, most normalized rates are close to $c_{\text{min}}$.

5.2. Pareto Efficiency. Recall that since $\mathcal{R}$ is nonconvex, the K-S rates found by Algorithm 1 may be only weakly Pareto efficient. So, we compare the K-S rates to the strongly Pareto rates found in Section 4.1.2 to determine how often and how severely weakly Pareto rates occur. We set $N = 3$ and $K = 5$, and let $\rho_{ij}$ (in dB) be uniformly distributed on the interval

$$[0, 10].$$
5.3. Power Control. In [18] it is shown that, for a MISO interference channel with $K \leq N$, all strongly Pareto efficient rates can be achieved with unit-norm beamformers, making power control unnecessary. For $K > N$, however, we can easily find counter-examples in which strongly Pareto K-S rates are not achievable with unit-norm beamformers.

Since power control introduces additional complexity to a wireless system, we consider the loss associated with removing power control from the system. To do so, we slightly modify the K-S method presented in Section 4.1, changing the constraint $\text{tr}(P_i) \leq 1$ to a fixed-power constraint $\text{tr}(P_i) = 1$ for all $i$. In Figure 9 we compare the CDF of the ordinary K-S rates with the fixed-power rates, using the same 1000 realizations from Section 5.2. Figure 9 shows a measurable loss: on average, users lose 24.7% of their total throughput by giving up power control.

6. Conclusion

We have proposed a method of beamformer selection for the MISO interference channel based on the Kalai-Smorodinsky bargaining solution from cooperative game theory. Using convex optimization techniques, we can efficiently find beamformers that achieve the K-S rates. Our numerical results demonstrate that despite the nonconvexity of $\mathcal{R}$, the K-S solution is almost always strongly Pareto efficient for realistic signal-to-noise ratios. We have also shown that when $K > N$, power control is instrumental in achieving the K-S rates.

For cases of high interference, where $\mathcal{R}$ is highly nonconvex, we convexified the rate region by introducing scheduling, where transmitters may time-share among beamformers. We proposed a gradient-based method which approximates the K-S solution for this scenario. For sufficiently many users, the flexibility of time-sharing improves overall performance, even though it results in a local optimum. In both the convex and nonconvex approaches, the K-S bargaining provides a lower sum rate, but increased performance for weaker users, than maximizing the sum rate.
directly. Cooperative bargaining allows us to strike a balance between efficiency and equity for the interference channel.

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Research Article

Spectrum Allocation for Decentralized Transmission Strategies: Properties of Nash Equilibria

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The interaction of two transmit-receive pairs coexisting in the same area and communicating using the same portion of the spectrum is analyzed from a game theoretic perspective. Each pair utilizes a decentralized iterative water-filling scheme to greedily maximize the individual rate. We study the dynamics of such a game and find properties of the resulting Nash equilibria. The region of achievable operating points is characterized for both low- and high-interference systems, and the dependence on the various system parameters is explicitly shown. We derive the region of possible signal space partitioning for the iterative water-filling scheme and show how the individual utility functions can be modified to alter its range. Utilizing global system knowledge, we design a modified game encouraging better operating points in terms of sum rate compared to those obtained using the iterative water-filling algorithm and show how such a game can be imitated in a decentralized noncooperative setting. Although we restrict the analysis to a two player game, analogous concepts can be used to design decentralized algorithms for scenarios with more players. The performance of the modified decentralized game is evaluated and compared to the iterative water-filling algorithm by numerical simulations.

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1. Introduction

Over the last few years, many theoretical connections have been established between problems arising in wireless communications and those in the field of game theory [1]. One such instance is when several coexisting links consisting of transmit-receive pairs compete with an objective of maximizing their individual data rates while treating the interference as Gaussian noise [2]. Due to the wireless communication channel, the received signal at each receiver is interfered by all transmitters, and the performance of the transmission strategies is, therefore, mutually dependent. Further, since no cooperation is assumed among the links, we have an instance of the interference channel [3, 4] whose complete characterization is still an open problem. Viewed in a noncooperative game theoretic setting [5], the links can be regarded as players whose payoff functions are the individual link rates. Each player is only interested in maximizing the individual rate, without considering its action on the other players. When each player is unilaterally optimal, that is, given the strategies of the other players, a change in the own strategy will not increase the rate, a Nash equilibrium (NE) [6] is reached, and, in general, multiple equilibria are possible. It is of interest to determine these equilibria of decentralized transmission strategies since centralized control causes unnecessary signalling overhead.

A general overview of distributed algorithms for spectrum sharing based on noncooperative game theory can be found in [2]. In [7], an iterative water-filling algorithm (IWFA) for codeword updates is proposed for spectrum allocation in interfering systems. It is shown that the full-spread equilibrium is the only possible outcome of the game under weak interference situations. Such complete spectral overlap is a highly suboptimal solution over a
wide range of channels. Conditions that guarantee global convergence to such unique NE are presented in [8]. On the other hand, for strong interference channels, it is also shown in [7] that multiple NE corresponding to complete, partial, and no spectral overlap can exist. Further, it is graphically shown that these multiple NEs result in large variations in system performance. Similar game theoretic approaches to codeword adaptation can be found in [9, 10], where stability is analyzed in asynchronous CDMA systems for single and multiple cell wireless systems. Also, noncooperative games for a digital subscriber line (DSL) system have been studied in [11], where an NE is reached when each player maximizes its individual rate in a sequential manner. In [12], it is shown how different operating points, for example, the maximum weighted sum rate, the NE, and the egalitarian solution, can be obtained using an iterative algorithm. However, this scheme requires the transmitters to have different forms of channel state information. An attempt to design noncooperative spectrum sharing rules for decentralized multiuser systems with multiple antennas at both transmitters and receivers can be found in [13]. Also, in [14], a game in which transmitters compete for data rates is presented, and an efficient numerical algorithm to compute the optimal input distribution that maximizes the sum capacity of a multiaccess channel (MAC) is proposed. However, no similar optimal algorithm is known for the general interference channel.

In this paper, we consider a system consisting of two players and study the properties of NE (spectral allocation at equilibrium) obtained by the IWFA. This scenario, albeit simple, allows us to fully characterize the set of achievable operating points and shows that many of the NEs can only be attained under specific initializations. For low-interference systems, we derive conditions when the full-spread equilibrium is inferior to a separation in signal space and suggest a modification of the IWFA to increase the sum rate. For high-interference systems, we show that the operating points are almost separated in signal space and argue how the convergence properties of the IWFA can be improved. Utilizing global system knowledge, we design a modified game with desirable properties and show how it can be imitated by a decentralized noncooperative scheme corresponding to a modified IWFA. The proposed game is compared to the IWFA by numerical simulations and we illustrate how the results extend, qualitatively, to systems with more players.

The paper is organized as follows. In Section 2, the system model is presented, and the problem is formulated as a noncooperative game. Section 3 provides the analysis for the resulting Nash equilibria and derives the dependencies of the operating points on the various system parameters. An analysis of sum rate is presented in Section 4, and modified games encouraging better system performance are designed in Section 5. The proposed decentralized game is evaluated in Section 6, and finally, conclusions are drawn in Section 7.

Notation: Uppercase boldface letters denote matrices and lowercase boldface letters designate vectors. The superscripts \((\cdot)^T, (\cdot)^*\) stand for transposition and Hermitian transposition, respectively. \(I_N\) denotes the \(N \times N\) identity matrix, and \(\mathbb{1}_m\) is the \(m \times 1\) vector of ones. Further, let \(\text{diag}(\cdot)\) denote a diagonal square matrix whose main diagonal contains the elements of the vector \(\mathbf{x}\), \(E[\cdot]\) denotes the expectation operator, and \(|\cdot|\) denotes the \(l_1\)-norm.

2. Problem Formulation and Game Theoretic Approach

2.1. System Model. We consider a scenario depicted in Figure 1, where two transmit-receive pairs are sharing \(N\) orthogonal radio resources, here referred to as subcarriers. Without loss of generality, assume that the system is normalized such that the gain of the transmitted signal is unity at the dedicated receiver. The \(N \times 1\) received signal vectors are modeled as

\[
\mathbf{r}_i = \mathbf{s}_i + \sqrt{g_i}\mathbf{s}_2 + \mathbf{n}_i, \\
\mathbf{r}_2 = \sqrt{g_1}\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{n}_2, \tag{1}
\]

where \(\mathbf{r}_i\) is the received signal at the \(i\)th receiver, and \(\mathbf{s}_i\) is a complex vector corresponding to transmissions on \(N\) subcarriers by the \(i\)th transmitter. Further, \(g_i\) is the cross-gain, and \(\mathbf{n}_i\) is a zero mean Gaussian noise vector with covariance matrix \(E[\mathbf{n}_i\mathbf{n}_i^H] = \eta_i I_N\). To limit the transmit power, each transmitter obeys a long-term power constraint \(E[\mathbf{s}_i^H\mathbf{s}_i] = P_i, P_i > 0, i \in [1, 2]\). This system model may represent a multicarrier system with a frequency-flat channel or a time division multiple access (TDMA) system. Though simple, it captures the essence of the spectrum allocation problem and is amenable for a tractable analysis. Such analysis may be useful in devising decentralized spectrum sharing algorithms for more complex scenarios. Similar models have been studied in other works, like [2, 7, 8].

The individual links can correspond to different instances of the same system or to two different systems. To avoid signaling overhead and retain the dynamic nature of the scenario, we assume that each link does not have information about the parameters used by the other link. Hence, the first player is blind to \(P_2, \eta_2\) and the second player has no information about \(P_1, \eta_1\). Further, since players do not cooperate, the channels \(\{g_i\}, i \in [1, 2]\) are unknown at either end.
As we restrict the players to operate as independent units, no interference suppression techniques are devised at the receivers, and the interference is treated as noise. To maximize the mutual information, we model \( s_i, \ i \in [1, 2], \) as a zero-mean uncorrelated Gaussian vector with covariance matrix \( E[s_i s^*_i] = \text{diag}(\{p^i_j\}_{j=1}^N), \) \( \sum_{j=1}^N p^i_j = P_i, \ p^i_j \geq 0, \ i \in [1, 2], \) where \( p^i_j \) is the power of the \( i \)th link for the \( j \)th subcarrier. Letting \( R_i \) denote the rate achieved on link \( i \) under Gaussian codebook transmissions, for a given power allocation, we have [15]

\[
R_i = \sum_{j=1}^N \log \left( 1 + \frac{p^i_j}{g^i_j p^i_j + \eta_j} \right),
\]

\[
R_j = \sum_{j=1}^N \log \left( 1 + \frac{p^j_j}{g^j_j p^j_j + \eta_j} \right).
\]

Note that the individual rates are coupled by the power allocation of both players.

Each player greedily maximizes its individual rate while treating the interference as colored Gaussian noise. Although such selfish behavior may not necessarily lead to improved link rates compared to a cooperative scenario, understanding it allows us to derive various decentralized noncooperative algorithms. These schemes have the advantage of not requiring encoding/decoding by the individual links or using any interference cancellation techniques. Adopting a game theoretical framework provides useful tools to analyze the behavior of greedy systems, and the problem can be tackled in a structured way.

2.2. Game Theoretic Approach to Rate Maximization. The individual rate maximization problem can be cast as a game \( \mathcal{G} \): 

\[
\max_{\{p^i_j\}} R_i, \quad \forall i, j, \quad \text{subject to} \quad \sum_{j=1}^N p^i_j \leq P_i, \quad p^i_j \geq 0,
\]

where \( \{p^i_j\} \) is the set of power allocations \( p^i_j, \forall i, j \). It has been shown in [16] that the outcomes of such noncooperative games are always NE and hence solutions to the set of nonlinear equations highlighting simultaneous water-filling. In particular, \( \{p^i_j\} \) satisfy

\[
p^i_1 = (\mu_1 - (\eta_1 + g^i_1 p^i_1))^+,
\]

\[
p^i_2 = (\mu_2 - (\eta_2 + g^j_1 p^j_1))^+,
\]

where \( (a)^+ = \max(0, a) \), and \( \mu_1, \mu_2 \) are positive constants such that \( \sum_{j=1}^N p^i_j = P_i, \ i \in [1, 2] \). These equilibrium points are reached when players update their power using the IWFA in one of the following ways [16].

1. Sequentially: players update their individual strategies one after the other according to a fixed updating order.
2. Simultaneously: at each iteration, all players update their individual strategies simultaneously.
3. Asynchronously: all players update their individual strategies in an asynchronous way.

For the purpose of tractability, we restrict our analysis to sequential updates.

3. Properties of Nash Equilibria

The spectra used by the two players can overlap completely, partially, or be disjoint (completely separated) as illustrated in Figures 2, 3, and 4, respectively. Hence, the resulting power allocation corresponds to one of these scenarios and is likely to depend on the system parameters as well as the particular initialization. In this section, we highlight the dependence of NE on the various system parameters using analytical
methods and derive conditions under which the different power allocations are possible.

3.1. Low-Interference Systems. In communication systems with low interference, individual links generally adapt their operating point to the noise power by neglecting the interference. This is also true for the IWFA when $g_1g_2 < 1$. In fact, we have the following.

**Theorem 1.** When $g_1g_2 < 1$, a full-spread equilibrium with $p_i^1 = P_i/N$, $\forall i$, $j$ is the only possible outcome of the game $\mathcal{G}$.

**Proof.** The proof follows from [7, 17] and is omitted for brevity.

Theorem 1 shows that when $g_1g_2 < 1$, each player allocates power as if the interfering player was absent, and this behavior is independent of the total power and number of subcarriers employed by the players. However, as we show in later sections, such interference ignorant power allocation may result in suboptimal system performance. To conclude the analysis on the low-interference scenario, we have the following theorem describing the convergence properties of the IWFA.

**Theorem 2.** When $g_1g_2 < 1$, convergence of the IWFA to the full-spread equilibrium is linear with rate $g_1g_2$.

**Proof.** See Appendix A.

3.2. High-Interference Systems. When $g_1g_2 > 1$, the game admits complete, partial, or no overlap as NE [7]. In the following, we analyze the dynamics of the IWFA and study how these different NEs can be reached. We begin with the full-spread equilibrium.

**Theorem 3.** When $g_1g_2 > 1$, the full-spread equilibrium is an outcome of the IWFA if and only if it is used as an initial point.

**Proof.** See Appendix B.

Theorem 3 shows that when $g_1g_2 > 1$, players acknowledge the presence of interference and do not occupy all the subcarriers, thereby motivating the term high-interference systems. Since a full-spread equilibrium is only possible under specific initialization, the power allocation at NE generally corresponds to either partial overlap or complete separation in signal space. To study such NE, we denote the subcarrier indices in which the $i$th player allocates nonzero power by $\mathcal{K}_i$ and the set of indices corresponding to partial overlap by $\mathcal{M} = \mathcal{K}_1 \cap \mathcal{K}_2$. Further, let the cardinalities of $\mathcal{K}_i$ and $\mathcal{M}$ be $k_i$ and $m$, respectively, so that $k_1 + k_2 = N + m$. Denoting the complement of $\mathcal{M}$ in $[1, N]$ by $\mathcal{M}^c$, we have from [7] that the power allocation at NE satisfies $p_i^j = c_{i,1}$, $\forall j \in \mathcal{K}_i \cap \mathcal{M}^c$, and $p_i^j = c_{i,2}$, $\forall j \in \mathcal{M}$, where $c_{i,1}$ and $c_{i,2}$ are positive constants. Thus, each player allocates equal power at NE for the subcarriers corresponding to a partial overlap. Interestingly, such an initial allocation of power is necessary to achieve a partial overlap and is formalized in the following theorem.

**Theorem 4.** When $g_1g_2 > 1$, IWFA converges to the set of NE, where the power allocations overlap on the subcarrier indices $\mathcal{M}$ only if

1. $p_i^j(1) = c_{i,1}$, $\forall j \in \mathcal{M}$, where $c_{i,1}$ is a constant;
2. $k_iP_i > g_j(k_i - m)P_j$, $i \neq j$, $j \in [1, 2]$.

If player 1 initiates the IWFA, one has $p_1^j(1) = c_{1,1}$. The $c_{1,1}$ and $c_{1,2}$ are chosen such that the total power constraints $P_1$ and $P_2$ are satisfied.

**Proof.** See Appendix C.

Hence, we have that partial overlap with $m > 1$ can be an outcome of the game only under specific initialization. As an immediate consequence of the results derived in Appendix C, we have the following corollary.

**Corollary 1.** When condition 2 of Theorem 4 is satisfied for $m = 1$, convergence of the IWFA is linear with rate $g_1g_2(k_1 - 1)/(k_2 - 1)/k_2$.

Since the game $\mathcal{G}$ has a nonempty solution set [17], one has that when neither the conditions of Theorems 3 or 4 are satisfied, the resulting operating point must correspond to a complete separation.

These theorems provide useful insight about the structure of the outcomes of the game $\mathcal{G}$ and help us to understand the dependence on the various system parameters. However, it is also important to analyze the individual rates of the links. It has been discussed in [2, 8] that the NE often is a suboptimal operating point resulting in poor performance for low-interference systems. Therefore, it is important to compare the performance corresponding to the NE with an optimal strategy. The mathematical tractability and fact that complete and partial overlaps are not, in general, solutions provided by the IWFA motivate us to consider the optimal performance under complete separation.
4. Analysis of the Sum Rate

As a global performance measure for the system, we define the sum rate as

\[ R = R_1 + R_2, \]

where \( R_i \) is the rate achieved on link \( i \). For a separated operating point where players 1 and 2 reside in \( k \) and \( N - k \) signal space dimensions, respectively, the individual rates are

\[ R_1 = k \log \left( 1 + \frac{P_1}{k \eta_1} \right), \]
\[ R_2 = (N - k) \log \left( 1 + \frac{P_2}{(N - k) \eta_2} \right). \]

Here, we explicitly use \( k_1 = k \) and \( k_2 = N - k \) to emphasize the analysis of nonoverlapping power allocations. The optimal signal space partitioning maximizing the sum rate is given by the next theorem.

**Theorem 5.** The signal space partitioning for player 1 maximizing the sum rate is

\[ k_{\text{opt}} = \frac{P_1 N \eta_2}{P_1 \eta_2 + P_2 \eta_1}. \]

**Proof.** Note that since \( R_1 \) and \( R_2 \) are concave in \( k \), so is the sum rate \( R_1 + R_2 \). Differentiating the sum rate with respect to \( k \) and solving for the roots yield the optimal signal space partitioning \( k_{\text{opt}} \). \( \square \)

In general, the optimal partitioning is not an integer and if required, needs to be rounded. Also, since the operating points obtained by the IWFA are NE for the system, not all signal space partitioning are achievable. The following theorem provides the region of all possible signal space partitions when the IWFA is employed.

**Theorem 6.** At NE corresponding to a complete separation, the achievable region of signal space dimensions employed by player 1 satisfies

\[ \frac{N}{1 + g_2(P_2/P_1)} \leq k \leq \frac{N}{1 + (1/g_1)(P_2/P_1)}. \]

**Proof.** Let players 1 and 2 reside in separated signal spaces of dimensions \( k \) and \( N - k \), respectively, at NE. For player 1, the allocated power per dimension \( P_1/k \) satisfies \( P_1/k \leq g_1(P_2/(N - k)) \), since the water level corresponding to the allocated power must be less than the level corresponding to the interference power. Similarly, for player 2, the allocated power \( P_2/(N - k) \) satisfies \( P_2/(N - k) \leq g_2(P_1/k) \). The region containing the possible signal space partitioning for player 1 is readily obtained combining these expressions. \( \square \)

Note that the region of achievable partitioning is nonempty only when \( g_1 g_2 \geq 1 \) and expands as the channel gains are increased. For \( g_1 g_2 < 1 \), this region is empty, and only a full-spread equilibrium is possible. The optimal partitioning needs not to satisfy (8) and conditions can be derived under which the optimal signal space partitioning is a possible outcome of the IWFA.

**Theorem 7.** The optimal signal space partitioning \( k_{\text{opt}} \) is an achievable NE if and only if \( g_1 \geq \eta_2/\eta_1 \) and \( g_2 \geq \eta_1/\eta_2 \).

**Proof.** Using \( g_1 \geq \eta_2/\eta_1 \) and \( g_2 \geq \eta_1/\eta_2 \) in (8), it is straightforward to see that the optimal signal space partitioning is confined within the region of achievable separations. To prove the only if part, substitute \( k \) by \( k_{\text{opt}} \) in (8) and simplify.

**Theorem 8.** The sum rate corresponding to an operating point with optimal partitioning is higher than or equal to that of the IWFA when

\[ 1 + \frac{P_1}{N \eta_1} + \frac{P_2}{N \eta_2} \geq \left( \frac{\eta_2}{g_1 \eta_1} \right) \left( 1 - \frac{\eta_1}{g_2 \eta_2} \right). \]

**Proof.** The sum rate for a system where players reside in separated signal spaces of dimension \( k \) and \( N - k \) is

\[ R_{\text{sep}} = k \log \left( 1 + \frac{P_1}{k \eta_1} \right) + (N - k) \log \left( 1 + \frac{P_2}{(N - k) \eta_2} \right). \]

Using that \( P_1/k \eta_1 = P_2/(N - k) \eta_2 \) when \( k = k_{\text{opt}} \), we have

\[ R_{\text{sep}}^{\text{opt}} = N \log \left( 1 + \frac{P_1}{N \eta_1} + \frac{P_2}{N \eta_2} \right). \]

Further, the sum rate corresponding to a full-spread equilibrium is

\[ R_{\text{fs}} = N \log \left( \left( 1 + \frac{P_1}{g_2 P_1 + \eta_1 N} \right) \left( 1 + \frac{P_2}{g_1 P_1 + \eta_2 N} \right) \right). \]

Forming \( R_{\text{sep}}^{\text{opt}} \geq R_{\text{fs}} \) yields the desired inequality. \( \square \)

It is clear from Theorem 8 that the sum rate can be increased if the operating point corresponds to the optimal signal space partitioning. However, it follows from [7] that a complete spectral overlap is the only outcome of the IWFA when \( g_1 g_2 < 1 \). Unfortunately, the strategy based on Theorem 8 requires information about \( \{g_i\} \), \( \{P_i\} \), and \( \{\eta_i\} \), \( i \in [1, 2] \), at each player and also centralized control. This warrants a modification of the IWFA for moving the operating point from a complete spectral overlap to a separation in signal space without requiring any additional system information. The region of achievable partitioning, as defined in Theorems 6 and 7, may contain the optimal separation. However, this depends on the channel gains. By modifying the channel coefficients used in the IWFA, the region can be adjusted to close in on the optimal partitioning. Such modification is equivalent to constructing a new game whose NE has desirable properties.
5. Sum Rate Improvements

As shown in Theorem 8, the sum rate can be increased by moving to an operating point corresponding to the optimal signal space partitioning. However, such a strategy requires global system knowledge and cooperation among the players making it less attractive from a practical point of view. Using the properties of the NE, we design a game utilizing global system knowledge and show how it can be imitated in a decentralized noncooperative setting.

5.1. Generalized IWFA with Global System Knowledge. When both players have access to global system knowledge, that is, \( \{P_i\}, \{g_i\}\) and \( \{\eta_i\}, i \in [1, 2] \), a modified game can be constructed to encourage better operating points compared to those provided by the IWFA. Since a rule-based approach, that switches to another solution for certain parameter values, is extremely tailored to the system model and not easy to generalize to scenarios with more than two players, we utilize the game theoretic framework and show how the individual utility functions of the players can be modified to improve the overall system performance in terms of sum rate.

Using the analysis from Section 4, we can guide the resulting operating point toward the optimal signal space partitioning. As shown in Section 3, the IWFA is generally not globally convergent to the set of NEs with overlap on more than one subcarrier and the region of separated operating points depends highly on the channel gains. Therefore, to direct the operating point toward the optimal signal space partitioning, the interference channel coefficients \( g_1 \) and \( g_2 \) employed by the IWFA should be replaced by the modified gains \( \tilde{g}_1 = c_1g_1 = \eta_2/\eta_1 \) and \( \tilde{g}_2 = c_2g_2 = \eta_1/\eta_2 \), where \( c_1 \) and \( c_2 \) are positive scalars. This scaling is done within the algorithm, and the only possible separated operating point will be that corresponding to the optimal partitioning. For a given power allocation, these scaled channel coefficients result in virtual rates as follows:

\[
\hat{R}_1 = \sum_{j=1}^{N} \log \left( 1 + \frac{p^j_1}{\tilde{g}_2 p^j_2 + \eta_1} \right),
\]

\[
\hat{R}_2 = \sum_{j=1}^{N} \log \left( 1 + \frac{p^j_2}{\tilde{g}_1 p^j_1 + \eta_2} \right),
\]

and a modified game \( \hat{G} \) can be formulated as

\[
\begin{align*}
\text{maximize}_{\{p^j_i\}} & \quad \hat{R}_i, \\
\text{subject to} & \quad \sum_{j=1}^{N} p^j_i \leq P_i, \quad p^j_i \geq 0, \quad \forall i, j.
\end{align*}
\]

Using these channel coefficients, the region of separated NE is narrowed to one single point, namely, the optimal partitioning, and from Theorem 4, we know that the resulting operating point will, in general, not overlap on more than one subcarrier. Hence, for a large number of subcarriers, such operating points result in sum rates close to that of the optimal signal space partitioning.

However, we know from Theorem 8 that for \( g_1g_2 < 1 \), the optimal partitioning is not always the best operating point from the sum rate point of view. Since the system parameters are known, both players should determine \( k^{opt} \) and choose the modified game \( \hat{G} \) when \( R_0 < R^{opt}_0 \). The resulting sum rate will not be less than that of the IWFA, and the subcarrier allocation will differ in no more than one dimension from the optimal partitioning.

5.2. Generalized IWFA without Global System Knowledge. Since the system parameters might not be available at both players, decentralized games imitating the global game \( \hat{G} \) are of high interest. Such a game should encourage separated operating points for \( g_1g_2 > 1 \) and either move away from or move toward the optimal partitioning for \( g_1g_2 < 1 \) depending on the channel strengths. Also, the game should be such that the sum rate is increased as more system parameters become available to the players.

Instead of altering the channel coefficients gains as in the global game \( \hat{G} \), we modify the received interference plus noise power employed by the IWFA encouraging the resulting operating point to have desirable characteristics. Letting \( I^j_i \) denote the inverse of the interference plus noise power at link \( i \) for subcarrier \( j \), we have

\[
I^j_i = (g_j p^j_i + \eta_i)^{-1},
\]

Then, we propose to modify the interference plus noise power values for player \( i \) into

\[
\tilde{I}^j_i = M_i \left( \frac{I^j_i}{m(I^j_i)} \right)^{\alpha}, \quad j = 1, \ldots, N, \quad i = 1, 2,
\]

where \( \alpha \geq 1 \) is a real scalar, \( \{I^j_i\} \) is the set of all \( I^j_i, j = 1, \ldots, N \), \( m(\cdot) \) is the arithmetic mean operator, and \( M_i = \beta m(I^j_i) \), \( \beta > 0 \). The normalization by \( m(I^j_i) \) yields a threshold for the decisiveness of the exponent operation, where values above the mean are amplified and others attenuated, while the scaling by \( M_i \) controls the mean of the modified parameters and implicitly the size of the region of achievable signal space separations. The exponential operation with \( \alpha > 1 \) perturbs a possibly full-spread equilibrium and improves the convergence properties for \( g_1g_2 > 1 \), since separated operating points are encouraged. For a given power allocation, the virtual rate for player \( i \) is

\[
\hat{R}_i = \sum_{j=1}^{N} \log \left( 1 + \tilde{I}^j_i P^j_i \right),
\]

and the resulting game can be formulated as

\[
\begin{align*}
\text{maximize}_{\{p^j_i\}} & \quad \hat{R}_i, \\
\text{subject to} & \quad \sum_{j=1}^{N} p^j_i \leq P_i, \quad p^j_i \geq 0, \quad \forall i, j.
\end{align*}
\]
Note, when $\alpha = \beta = 1$, this game coincides with the IWFA. As more system information becomes available to the players, the parameters $\alpha$ and $\beta$ can be chosen such that the resulting operating point approaches the optimal partitioning. This can be achieved by altering the range of (8) by a proper choice of the scale factor $\beta$ and affecting the convergence properties with the parameter $\alpha$.

Although we designed the decentralized game for a scenario with two players, such modifications of the interference plus noise power can also be applied to a system with more players as demonstrated in [13] and in a numerical example below.

6. Numerical Examples

In this section, we evaluate the system performance in terms of sum rate for the games $g$ and $\tilde{g}$ and also study their convergence properties.

Each of the values is averaged over 50000 channel and power realizations, and two specific scenarios are considered: $g_1g_2 < 1$ and $g_1g_2 > 1$. To simplify the exercise, we let $g_1$ and $g_2$ be uniformly distributed on $[0, 1]$ when $g_1g_2 < 1$ and identically distributed according to $1 + |N(0, 1)|$ when $g_1g_2 > 1$. The total power budgets for players 1 and 2 are uniformly distributed on $[0, 6]$ and $[0, 10]$, respectively, the noise power is 1, and 10 subcarriers are shared.

The average sum rate for a system whose operating points are given by the games $g$ and $\tilde{g}$ is shown in Figures 5 and 6 for $g_1g_2 < 1$ and $g_1g_2 > 1$, respectively. In each of the two interference scenarios, the impact of the scale factor $\beta$ on the average sum rate is depicted for different values of the exponent $\alpha$. Clearly, the modified game $\tilde{g}$ yields a higher average sum rate compared to the IWFA for low-interference systems when $\beta = 1$ and $\alpha = 2$. Also, from Figure 6, we see that the resulting performance of both games is almost identical for such choice of parameters.

To study the convergence properties, we use the relative change in sum rate as a convergence criterion and set the threshold to $10^{-6}$. For $g_1g_2 < 1$ with $\beta = 1$ and $\alpha = 2$, the modified game $\tilde{g}$ requires 21 iterations on average between the players, whereas the IWFA converges in 17 iterations. This increase is due to the perturbation caused by the exponent operator in (16), where convergence toward a complete overlap is altered. However, for $g_1g_2 > 1$, the modified game requires no more than 4 iterations to converge, while 11 iterations are needed for the IWFA. From the properties of NE derived in Section 3, we know that the IWFA will provide an almost separated operating point, and here the exponent operation with $\alpha > 1$ encourages the convergence to such a separation.

From the simulation results, we observe that the individual rates at NE corresponding to a partial overlap can be increased by moving the operating point to either complete separation or overlap on one subcarrier. This leads to the conjecture that the IWFA yields Pareto optimal points under arbitrary initialization for high-interference systems.

In order to illustrate how such a decentralized game extends to a scenario with more users, we consider a system consisting of 4 players, whose power budgets are uniformly distributed on $[0, 6]$, $[0, 8]$, $[0, 10]$, and $[0, 12]$, respectively. Letting $g_{xy}$ denote the channel gain from transmitter $x$ to receiver $y$, we consider the scenarios when $g_{xy}g_{yx} < 1$ and $g_{xy}g_{yx} > 1$, $x \neq y$. When $g_{xy}g_{yx} < 1$, the gains are uniformly distributed on $[0, 1]$, and identically distributed according to $1 + |N(0, 1)|$ when $g_{xy}g_{yx} > 1$. Each value is averaged over 50000 channel and power realizations, the noise power is 1, and 10 subcarriers are shared.
The scale factor $\beta$ and the IWFA when $g_{xy} g_{yx} < 1$ and 4 players are served. The scale factor $\beta$ is varied between 1 and 5 for $\alpha = 1, 2, 4$.

Figures 7 and 8 show the average sum rate for a system whose operating points are given by the games $\hat{G}$ and $\hat{G}$ for $g_{xy} g_{yx} < 1$ and $g_{xy} g_{yx} > 1$, respectively. Similar to the game consisting of two players, the decentralized scheme yields operating points resulting in better system performance compared to the IWFA. In particular, the effect of the perturbation caused by the exponent operation is evident, where separated operating points are encouraged.

Clearly, the overall spectrum utilization benefits from a power allocation with as small overlap between the users as possible.

7. Conclusion
In this paper, we have analyzed a decentralized game, where two players compete for available spectrum by greedily maximizing the individual rates and only considering the action of the other player through the experienced interference level. When each player is allocating transmit power using the water-filling algorithm, a Nash equilibrium is reached and, in general, multiple equilibria are possible. We have studied the properties of such NE and characterized the region of achievable operating points. For high-interference systems, these equilibria correspond to almost complete separation in signal space, while for low-interference systems, a full-spread equilibrium is obtained. Further, we showed that the full-spread equilibrium is a stable operating point for the system, but often results in low overall system performance. Therefore, a decentralized algorithm should avoid an initialization with equal power on all subcarriers. We derived the region of achievable signal space partitioning and showed how it depends on the various system parameters. Altering these parameters, we constructed a decentralized noncooperative game whose NE had desirable properties. By properly modifying the value of the interference plus noise power employed by the IWFA, we showed how the overall system performance can be improved. In order to obtain quantitative results, the analysis considered a simple scenario with two links. However, many of the qualitative conclusions will remain also for scenarios with more players.

Appendices

A. Proof of Theorem 2
Without loss of generality, let the IWFA be initiated by player 2. Further, let $p_i^j(n)$ denote the power allocation of the $i$th link for the $j$th subcarrier during the $n$th iteration, and let $N_{i,n}$ be the set containing the subcarrier indices for which link $i$ allocates nonzero power during the $n$th iteration. Then, water-filling yields

$$p_i^j(n) = -g_i p_i^j(n-1) + \frac{1}{r_{2,n}} \left( P_2 + \sum_{l \in N_{i,n}} g_l p_l^j(n-1) \right)$$

(A.1)

$$p_i^j(n) = -g_i p_i^j(n) + \frac{1}{r_{1,n}} \left( P_1 + \sum_{l \in N_{i,n}} g_l p_l^j(n) \right)$$

(A.2)

where $r_{i,n}$ denotes the cardinality of $N_{i,n}$. Since the outcome of the IWFA is the full-spread equilibrium, there exists a finite $n_0$ such that $p_i^j(n) > 0$, $\forall n \geq n_0$, $\forall j$ and $i \in \{1, 2\}$.

We start by showing that the IWFA cannot converge in $n_0$ (finite) iterations under random initialization [18]. Note that the equilibrium is reached at $n_0 = 1$ only if the algorithm
is initialized with the operating point corresponding to a complete spectral overlap. Assume that the NE is reached for \( n_0 > 1 \). Then, \( p^1_i(n_0 + 1) = p^1_i(n_0) \) and \( r_{2,n} = N, \forall n \geq n_0 \). Using that \( \Sigma_{j} p^i_j(n) = M_{1}, \) (A.1) yields \( p^1_i(n_0 - 1) = p^1_i(n_0) \). This implies \( r_{1,n-1} = N, \) and (A.2) yields \( p^2_i(n_0 - 1) = p^2_i(n_0) \). By recursion, we see that \( p^i_j(n) \) is constant for all \( n \leq n_0 \). Hence, equilibrium is reached at a finite \( n_0 \) only when the IWFA is initialized with this point.

Since \( r_{1,n} = r_{2,n} = N, \forall n \geq n_0 \), (A.1) and (A.2) yield

\[
\begin{align*}
p^1_i(n + 1) - p^1_i(n) &= g_1 g_2 (p^1_i(n) - p^1_i(n - 1)), \quad (A.3) \\
p^2_i(n + 1) - p^2_i(n) &= g_1 g_2 (p^2_i(n) - p^2_i(n - 1)).
\end{align*}
\]

It follows from (A.3) that the convergence of the IWFA is linear with rate \( g_1 g_2 \).

**B. Proof of Theorem 3**

Assuming that the full-spread equilibrium is the outcome of the game \( g \), it follows from Appendix A that the IWFA cannot converge in \( n_0 \) iterations under random initialization [18]. Further, (A.3) hold for \( n \geq n_0 \). We now show that a full-spread equilibrium is not attained for \( n > n_0 \). By the Cauchy criterion, \( p^i_j(n) \) converges if and only if \( |p^i_j(n + 1) - p^i_j(n)| \) converges to 0 as \( n \rightarrow \infty \). However, since \( g_1 g_2 > 1 \), it is clear from (A.3) that \( |p^i_j(n + 1) - p^i_j(n)| \) cannot converge to 0, unless \( p^i_j(n_0 + 1) - p^i_j(n_0) = 0, \forall j \). From Appendix A, we see that such a scenario is not possible for a random initialization, thereby proving the theorem.

**C. Proof of Theorem 4**

Assuming partial overlap at convergence, there exists a finite \( n_0 \) such that \( p^i_j(n) > 0, \forall n \geq n_0, j \in \mathcal{K}_i, i \in [1, 2] \). The following lemma is necessary to prove the theorem.

**Lemma 1.** Defining \( n_0 \) as above, one has \( p^1_i(\hat{n}) > 0, \forall j \in \mathcal{M}, i \in [1, 2] \) and \( 1 < \hat{n} \leq n_0 \).

**Proof.** If, for some \( \hat{n} < n_0 \), we have \( p^1_i(\hat{n}) = 0, j \in \mathcal{M} \), then \( p^1_j(n) < M \), which is an upper bound for all \( j \in [1, N] \). However, \( p^1_j(n) \) has the largest value for all \( j \in \mathcal{M} \) and \( n \). This leads to a contradiction and thereby proves the lemma for \( i = 1 \). Similar arguments hold for \( i = 2 \).

To simplify the analysis, we consider two cases: (1) \( k_i > m, \forall i \) and (2) \( k_i = m \) for some \( i \).

**Case 1** \( k_i > m \). Stack the powers corresponding to the subcarriers with spectral overlap in the vector \( \mathcal{P}(n) = \{ p^i_j(n) \}_{j \in \mathcal{M}} \), \( i \in [1, 2] \), and denote the difference in power for two consecutive updates by \( \delta^i(n) = \mathcal{P}(n) - \mathcal{P}(n - 1), i \in [1, 2] \). Then, for \( n \geq n_0 \), we can write (A.1) and (A.2) as

\[
\begin{align*}
p^1_i(n) &= g_1 M_{1,1}^i p^1_i(n - 1) + \frac{p^1_{k_2}}{k_2} \mathcal{I}_m, \quad (C.1) \\
p^2_i(n) &= g_2 g_1 M_{1,1}^i p^2_i(n) + \frac{p^2_{k_1}}{k_1} \mathcal{I}_m, \quad (C.2)
\end{align*}
\]

where \( M_i = -\mathcal{I}_m + (1/k_i) \mathcal{I}_m \mathcal{I}_m^T, i \in [1, 2] \), \( \mathcal{I}_m \) is an \( m \times m \) identity matrix, and \( \mathcal{I}_m \) is an \( m \times 1 \) vector of ones. The following properties of \( M_i \) are useful in the subsequent steps.

(i) \( M_i \) is Hermitian with eigenvalue \(-1 \) with multiplicity \( m - 1 \) and \((1 + m/k_i) \) with multiplicity 1. Further, when \( k_i > m \), all eigenvalues of \( M_i \) are nonzero. Thus, \( M_i \) is invertible for \( k_i > m \).

(ii) The eigenvector corresponding to the eigenvalue \((-1 + m/k_i) \) is \( \mathcal{I}_m \) and is orthogonal to the eigenvectors corresponding to the eigenvalue \(-1 \). Since the eigenvectors of \( M_1 \) and \( M_2 \) are identical, they commute [19]. Further, the matrix \( M_1, M_2, i_1, i_2 \in [1, 2] \) has eigenvalue 1 with multiplicity \( m - 1 \) and \((-1 + m/k_i) \) with multiplicity 1. Thus, \( M_1, M_2 \) is invertible for \( k_0 > m, l \in [1, 2] \).

We first show that an appropriate initialization satisfying \( p^1_i(n_0) = c^1_i(n_0), \forall j \in \mathcal{M} \) is necessary for the IWFA to converge in \( n_0 \) iterations. Assuming an equilibrium at \( n = n_0 \), it follows from [7] that \( p^1_j(n_0) = c^1_i(n_0), \forall j \in \mathcal{M}, i \in [1, 2] \). Evaluating (A.1) for \( n = n_0 \) and \( n = n_0 + 1 \) and noting that \( p^1_j(n_0) = p^1_j(n_0 + 1), \) we have \( p^1_j(n_0 - 1) = p^1_j(n_0) = c^1_i(n_0) \mathcal{I}_m \) (this can also be argued using (C.1) and the invertibility properties of \( M_i \)). Otherwise there exists an index \( j \) such that \( p^1_j(n_0 - 1) = p^1_j(n_0) = 0 \), which is not possible using water-filling. Then, we have that \( p^1_j(n_0 - 1) = c^1_i(n_0 - 1), j \in \mathcal{M} \), that is, \( p^1_j(n - 1) \) is constant for \( j \in \mathcal{M} \). Applying this repeatedly yields equal power allocation for \( p^1_j(1), j \in \mathcal{M} \), if \( p^1_j(\hat{n}) \neq 0 \) for \( j \in \mathcal{M} \) and all \( \hat{n} < n_0 \). Lemma 1 eliminates such a possibility and, therefore, equilibrium can be reached in \( n_0 \) iterations only under specific initialization.

Using (C.1) and (C.2), for all \( n \geq n_0 \), we have

\[
\begin{align*}
\delta^1(n + 1) &= g_1 M_{1,1}^i \delta^1(n), \quad (C.3) \\
\delta^1(n + 1) &= g_1 M_{1,1}^i \delta^1(n + 1). \quad (C.4)
\end{align*}
\]

Further, substituting (C.3) in (C.4) and vice versa, we obtain

\[
\begin{align*}
\delta^1(n) &= g_1 g_2 M_{1,1}^i M_{2,1}^i \delta^1(n - 1), \quad n \geq n_0 + 2, \\
\delta^1(n) &= g_1 g_2 M_{1,1}^i M_{2,1}^i \delta^1(n - 1), \quad n \geq n_0 + 1.
\end{align*}
\]

Let \( M_i = \mathcal{V}_i \mathcal{V}_i^* \) be the eigenvalue decomposition of \( M_i \) and \( \delta^i_n = \mathcal{V}^* \delta^i(n) \). Then, (C.5) can be written as

\[
\delta^i_n = g_1 g_2 A_n \delta^i_{n-1}, \quad i \in [1, 2],
\]

where \( A = \text{diag}(1, 1, \ldots, 1, (k_1 - m)(k_2 - m)/k_1 k_2) \). Equations (C.5) and (C.6) suggest that the IWFA converges if and only
if \( \phi_i^j \) converges to a vector with all components equal to zero. Using \( \Lambda \), we have that \( \phi_i^j \rightarrow 0 \) only if

\[
\phi_i^j(k) = 0, \quad k \in [1, m - 1],
\]

\[
(\frac{(k_1 - m)(k_2 - m)}{k_1k_2}) < \frac{1}{g_1g_2},
\]

where we used (C.4) and (C.6) to show that \( \phi_i^j(k) = 0 \) implies \( \phi_i^j(k) = 0, \forall n > n_0 + 1 \). Thus, (C.7) shows that partial overlap is an outcome of the game \( g_i \) only under judicious initialization. Further, (C.8) gives a condition on system parameters for convergence.

We now explore condition (C.7) in more detail. Combining (C.1) and (C.2), we get

\[
p_i^1(n) = g_1g_2M_1M_2P_i^1(n - 1) + (-1 + m/k_1)g_2P_i^1(n) + P_i^1(n),
\]

\[
\forall n \geq n_0.
\]

Recall that the eigenvector matrix of \( M_1 \) has the form \( V = [Q, (1/\sqrt{m})I_m] \), with \( Q^t M_1 M_2 = Q^* \) and \( Q^* \perp_m = 0 \). Using this in (C.9) yields

\[
Q^* p_i^1(n) = g_1g_2Q^* p_i^1(n - 1), \quad \forall n \geq n_0.
\]

We then have \( \phi_i^j(k) = 0, \quad k \in [1, m - 1] \), and if only if \( Q^* g_i^1(n_0) = 0 \). Further, from (C.10), we have \( Q^* g_i^1(n_0) = Q^* p_i^1(n_0) - Q^* p_i^1(n_0 - 1) = (g_1g_2 - 1)Q^* p_i^1(n_0 - 1) \). Thus, \( Q^* g_i^1(n_0) = 0 \) implies \( Q^* p_i^1(n_0 - 1) = 0 \) as \( g_1g_2 > 1 \). Hence, \( Q^* p_i^1(n_0 - 1) = 0 \) and \( p_i^1(n_0 - 1) \) is constant for all \( j \in M \). As in the discussion preceding (C.3), it can be shown that (C.7) holds only under specific initialization. Hence, condition (1) of Theorem 4 is shown.

To show (C.8), let \( p_i^1 = p_i^{1,ol} \), \( j \in M \) and \( p_i^j = p_i^{j,ol}, j \in K \cap M \) denote the power levels of player \( i \) for the subcarriers with and without spectral overlap, respectively. Then, for player 1, we have

\[
(k_1 - m)p_i^{1,ol} + mp_i^{1,ol} = P_1,
\]

\[
p_i^{1,ol} + g_2p_i^{2,ol} = p_i^{1,ol},
\]

where (C.11) follows from the power constraint of player 1 and (C.12) is due to the water-filling. Similarly, for player 2, we have

\[
(k_2 - m)p_i^{2,ol} + mp_i^{2,ol} = P_2,
\]

\[
p_i^{2,ol} + g_1p_i^{1,ol} = p_i^{2,ol}.
\]

Solving these equations for \( p_i^{1,ol} \) and \( p_i^{2,ol} \), we get

\[
p_i^{1,ol} = \frac{k_2P_1 - g_1(k_2 - m)P_2}{k_1k_2 - g_1g_2(k_1 - m)(k_2 - m)}.
\]

\[
p_i^{2,ol} = \frac{k_1P_2 - g_1(k_1 - m)P_1}{k_1k_2 - g_1g_2(k_1 - m)(k_2 - m)}.
\]

From (C.8), we have that the denominator is positive and, therefore, the overlapping power allocations are nonzero only when \( k_1P_2 > g_1(k_2 - m)P_1 \) and \( k_2P_1 > g_1(k_1 - m)P_2 \).

Case 2 (\( k_i = m \) for some \( i \)). As in the earlier case, it can be shown that the IWFA converges in \( n_0 \) iterations only under specific initialization. For random initialization, it can be shown that

\[
p_i^j(n + 1) - p_i^j(n) = -g_1(p_i^1(n) - p_i^j(n - 1)), \quad \forall j \in M,
\]

\[
p_i^j(n + 1) - p_i^1(n) = -g_1(p_i^j(n + 1) - p_i^j(n)), \quad \forall j \in M,
\]

when \( k_i = m \) for some \( i \). Then, it immediately follows that an equilibrium is not reached for \( g_1g_2 > 1 \).

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Research Article

Stackelberg Contention Games in Multiuser Networks

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Interactions among selfish users sharing a common transmission channel can be modeled as a noncooperative game using the game theory framework. When selfish users choose their transmission probabilities independently without any coordination mechanism, Nash equilibria usually result in a network collapse. We propose a methodology that transforms the noncooperative game into a Stackelberg game. Stackelberg equilibria of the Stackelberg game can overcome the deficiency of the Nash equilibria of the original game. A particular type of Stackelberg intervention is constructed to show that any positive payoff profile feasible with independent transmission probabilities can be achieved as a Stackelberg equilibrium payoff profile. We discuss criteria to select an operating point of the network and informational requirements for the Stackelberg game. We relax the requirements and examine the effects of relaxation on performance.

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1. Introduction

In wireless communication networks, multiple users often share a common channel and contend for access. To resolve the contention problem, many different medium access control (MAC) protocols have been devised and used. Recently, the selfish behavior of users in MAC protocols has been studied using game theory. There have been attempts to understand the existing MAC protocols as the local utility maximizing behavior of selfish users by reverse engineering the current protocols (e.g., [1]). It has also been investigated whether existing protocols are vulnerable to the existence of selfish users who pursue their self-interest in a noncooperative manner. Noncooperative behavior often leads to inefficient outcomes. For example, in the 802.11 distributed MAC protocol, DCF, and its enhanced version, EDCF, competition among selfish users can lead to an inefficient use of the shared channel in Nash equilibria [2]. Similarly, a prisoner’s dilemma phenomenon arises in a noncooperative game for a generalized version of slotted-Aloha protocols [3].

In general, if a game has Nash equilibria yielding low payoffs for the players, it will be desirable for them to transform the game to extend the set of equilibria to include better outcomes [4]. The same idea can be applied to the game played by selfish users who compete for access to a common medium. If competition among selfish users brings about a network collapse, then it is beneficial for them to design a device which provides incentives to behave cooperatively. Game theory [4] discusses three types of transformation: (1) games with contracts, (2) games with communication, and (3) repeated games.

A game is said to be with contracts if the players of the game can communicate and bargain with each other, and enforce the agreement with a binding contract. The main obstacle to apply this approach to wireless networking is the distributed nature of wireless networks. To reach an agreement, users should know the network system and be able to communicate with each other. They should also be able to enforce the agreed plan.

A game with communication is the one in which players can communicate with each other through a mediator but they cannot write a binding contract. In this case, a correlated equilibrium is predicted to be played. Altman et al. [5] study correlated equilibria using a coordination mechanism in a slotted Aloha-type scenario. Unlike the first approach, this
does not require that the actions of players be enforceable. However, to apply this approach to the medium access problem, signals need to be conveyed from a mediator to all users, and users need to know the correct meanings of the signals.

A repeated game is a dynamic game in which the same game is played repeatedly by the same players over finite or infinite periods. Repeated interactions among the same players enable them to sustain cooperation by punishing deviations in subsequent periods. A main challenge of applying the idea of repeated games to wireless networks is that the users should keep track of their past observations and be able to detect deviations and to coordinate their actions in order to punish deviating users.

Besides the three approaches above, another approach widely applied to communication networks is pricing [6]. A central entity charges prices to users in order to control their utilization of the network. Nash equilibria with pricing schemes in an Aloha network are analyzed in [7, 8]. Implementing a pricing scheme requires the central entity to have relevant system information as well as users’ benefits and costs, which are often their private information. Eliciting private information often results in an efficiency loss in the presence of the strategic behavior of users as shown in [9]. Even in the case where the entity has all the relevant information, prices need to be computed and communicated to the users.

In this paper, we propose yet another approach using a Stackelberg game. We introduce a network manager as an additional user and make him access the medium according to a certain rule. Unlike the Stackelberg game of [10] in which the manager (the leader) chooses a certain strategy before users (followers) make their decisions, in the proposed Stackelberg game he sets an intervention rule first and then implements his intervention after users choose their strategies. Alternatively, the proposed Stackelberg game can be considered as a generalized Stackelberg game in which there are multiple leaders (users) and a single follower (the manager) and the leaders know the response of the follower to their decisions correctly. With appropriate choices of intervention rules, the manager can shape the incentives of users in such a way that their selfish behavior results in cooperative outcomes.

In the context of cognitive radio networks, [11] proposes a related Stackelberg game in which the owner of a licensed frequency band (the leader) can charge a virtual price for using the frequency band to cognitive radios (followers). The virtual price signals the extent to which cognitive radios can exploit the licensed frequency band. However, since prices are virtual, selfish users may ignore prices when they make decisions if they can gain by doing so. On the contrary, in the Stackelberg game of this paper, the intervention of the manager is not virtual but it results in the reduction of throughput, which selfish users care about for sure. Hence, the intervention method provides better grounds for the network manager to deal with the selfish behavior of users.

Chen et al. [12, 13] use game theoretic models to study random access. Their approach is to capture the information and implementation constraints using the game theoretic framework and to specify utility functions so that a desired operating point is achieved at a Nash equilibrium. If conditions under which a certain type of dynamic adjustment play converges to the Nash equilibrium are met, such a strategy update mechanism can be used to derive a distributed algorithm that converges to the desired operating point. However, this control-theoretic approach to game theory assumes that users are obedient. In this paper, our main concern is about the selfish behavior of users who have innate objectives. Because we start from natural utility functions and affect them by devising an intervention scheme, we are in a better position to deal with selfish users. Furthermore, the idea of intervention can potentially lead to a distributed algorithm to achieve a desired operating point.

By formulating the medium access problem as a noncooperative game, we show the following main results.

1. Because the Nash equilibria of the noncooperative game are inefficient and/or unfair, we transform the original game into a Stackelberg game, in which any feasible outcome with independent transmission probabilities can be achieved as a Stackelberg equilibrium.

2. A particular form of a Stackelberg intervention strategy, called total relative deviation (TRD)-based intervention, is constructed and used to achieve any feasible outcome with independent transmission probabilities.

3. The additional amount of information flows required for the transformation is relatively moderate, and it can be further reduced without large efficiency losses.

The rest of this paper is organized as follows. Section 2 introduces the model and formulates it as a noncooperative game called the contention game. Nash equilibria of the contention game are characterized, and it is shown that they typically yield suboptimal performance. In Section 3, we transform the contention game into another related game called the Stackelberg contention game by introducing an intervening manager. We show that the manager can implement any transmission probability profile as a Stackelberg equilibrium using a class of intervention functions. Section 4 discusses natural candidates for the target transmission probability profile selected by the manager. In Section 5, we discuss the flows of information required for our results and examine the implications of some relaxations of the requirements on performance. Section 6 provides numerical results, and Section 7 concludes the paper.

2. Contention Game Model

We consider a simple contention model in which multiple users share a communication channel as in [14]. A user represents a transmitter-receiver pair. Time is divided into slots of the same duration. Every user has a packet to transmit and can send the packet or wait. If there is only one transmission, the packet is successfully transmitted within the time slot. If more than one user transmits a packet...
simultaneously in a slot, a collision occurs and no packet is transmitted.

We summarize the assumptions of our contention model.

(1) A fixed set of users interacts over a given period of time (or a session).
(2) Time is divided into multiple slots, and slots are synchronized.
(3) A user always has a packet to transmit in every slot.
(4) The transmission of a packet is completed within a slot.
(5) A user transmits its packet with the same probability in every slot. There is no adjustment in the transmission probabilities during the session. This excludes coordination among users, for example, using time division multiplexing.
(6) There is no cost of transmitting a packet.

We formulate the medium access problem as a noncooperative game to analyze the behavior of selfish users. We denote the set of users by $N = \{1, \ldots, n\}$. Because we assume that a user uses the same transmission probability over the entire session, the strategy of a user is its transmission probability, and we denote the strategy of user $i$ by $p_i$, and the strategy space of user $i$ by $P_i = [0, 1]$ for all $i \in N$.

Once the users decide their transmission probabilities, a strategy profile can be constructed. The users transmit their packets independently according to their transmission probabilities, and thus the strategy profile determines the probability of a successful transmission by user $i$ in a slot. A strategy profile can be written as a vector $p = (p_1, \ldots, p_n)$ in $P = P_1 \times \cdots \times P_n$, the set of strategy profiles. The payoff function of user $i$, $u_i : P \rightarrow \mathbb{R}$, is defined as

$$u_i(p) = k_i p_i \prod_{j \neq i} (1 - p_j),$$

where $k_i > 0$ measures the value of transmission of user $i$ and $p_i \prod_{j \neq i} (1 - p_j)$ is the probability of successful transmission by user $i$.

We define the contention game by the tuple $\Gamma = (N, (P_i), (u_i))$. If the users choose their transmission probabilities taking others’ transmission probabilities as given, then the resulting outcome can be described by the solution concept of Nash equilibrium [4]. We first characterize the Nash equilibria of the contention game.

**Proposition 1.** A strategy profile $p \in P$ is a Nash equilibrium of the contention game $\Gamma$ if and only if $p_i = 1$ for at least one $i$.

**Proof.** In the contention game, the best response correspondence of user $i$ assumes two sets: $b_i(p_{-i}) = \{1\}$ if $\prod_{j \neq i} (1 - p_j) > 0$ and $b_i(p_{-i}) = [0, 1]$ if $\prod_{j \neq i} (1 - p_j) = 0$. Suppose that user $i$ chooses $p_i = 1$. Then it is playing its best response while other users are also playing their best responses, which establishes the sufficiency part. To prove the necessity part, suppose that $p$ is a Nash equilibrium and $p_i < 1$ for all $i \in N$. Since $\prod_{j \neq i} (1 - p_j) > 0$, $p_i$ is not a best response to $p_{-i}$, which is a contradiction. \(\square\)

If a Nash equilibrium $p$ has only one user $i$ such that $p_i = 1$, then $u_i(p) > 0$ and $u_j(p) = 0$ for all $j \neq i$ where $u_i(p)$ can be as large as $k_i$. If there are at least two users with the transmission probability equal to 1, then we have $u_i(p) = 0$ for all $i \in N$. Let $\mathcal{U}_i = \{u \in \mathbb{R}^n : u_i \in [0, k_i], u_j = 0$ for all $j \neq i\}$. Then, the set of Nash equilibrium payoffs is given by

$$\mathcal{U}(NE) = \bigcup_{i=1}^n \mathcal{U}_i. \quad (2)$$

Given the game $\Gamma$, we can define the set of feasible payoffs by

$$\mathcal{U} = \{(u_1(p), \ldots, u_n(p)) : p \in P\}. \quad (3)$$

A payoff profile $u \in \mathcal{U}$ is Pareto efficient if there is no other element $v$ in $\mathcal{U}$ such that $v \succeq u$ and $v_i > u_i$ for at least one user $i$. We also call a strategy profile $p$ Pareto efficient if $u(p) = (u_1(p), \ldots, u_n(p))$ is a Pareto efficient payoff profile. Let $\mathcal{U}(PE)$ be the set of Pareto efficient payoffs.

There are $n$ points in $\mathcal{U}(NE) \cap \mathcal{U}(PE)$, namely, $u$ such that $u_i = k_i$ and $u_j = 0$ for all $j \neq i$, for $i = 1, \ldots, n$. These are the corner points of $\mathcal{U}(PE)$ in which only one user receives a positive payoff. Therefore, Nash equilibrium payoff profiles are either inefficient or unfair. Moreover, since $p_i = 1$ is a weakly dominant strategy for every user $i$, in a sense that $u_i(1, p_{-i}) \succeq u_i(p)$ for all $p \in P$, the most likely Nash equilibrium is the one in which $p_i = 1$ for all $i \in N$. At the most likely Nash equilibrium, every user always transmits its packet, and as a result no packet is successfully transmitted. Hence, the selfish behavior of the users is likely to lead to a network collapse, which gives zero payoff to every user, as argued also in [15].

Figure 1 presents the payoff spaces of two homogeneous users with $k_1 = k_2 = 1$. If coordination between the two users is possible, they can achieve any payoff profile in the dark area of Figure 1(a). For example, $(1/2, 1/2)$ can be achieved by arranging user 1 to transmit only in odd-numbered slots and user 2 only in even-numbered slots. This kind of coordination can be supported through direct communications among the users or mediated communications. However, if such coordination is not possible and each user has to choose one transmission probability, Nash equilibria yield the payoff profiles in Figure 2(b). The set of feasible payoffs of the contention game is shown as the dark area of Figure 1(c). The set of Pareto-efficient payoff profiles is the frontier of that area. The lack of coordination makes the set of feasible payoffs smaller reducing the area of Figure 1(a) to that of Figure 1(c). Because the typical Nash equilibrium payoff is $(0, 0)$, the next section develops a transformation of the contention game, and the set of equilibria of the resulting Stackelberg game is shown to expand to the entire area of Figure 1(c).

### 3. Stackelberg Contention Game

We introduce a network manager as a special kind of user in the contention game and call him user 0. As a user, the
manager can access the channel with a certain transmission probability. However, the manager is different from the users in that he can choose his transmission probability depending on the transmission probabilities of the users. This ability of the manager enables him to act as the police. If the users access the channel excessively, the manager can intervene and punish them by choosing a high transmission probability, thus reducing the success rates of the users.

Formally, the strategy of the manager is an intervention function \( g : P \rightarrow [0,1] \), which gives his transmission probability \( p_0 = g(p) \) when the strategy profile of the users is \( p \). \( g(p) \) can be interpreted as the level of intervention or punishment by the manager when the users choose \( p \). Note that the level of intervention by the manager is the same for every user. We assume that the manager has a specific “target” strategy profile \( \tilde{p} \), that his transmission has no value to him (as well as to others), and that he is benevolent. One representation of his objective is the payoff function of the following form:

\[
u_0(g,p) = \begin{cases} 1 - g(p) & \text{if } p = \tilde{p}, \\ 0 & \text{otherwise} \end{cases}
\] (4)

This payoff function means that the manager wants the users to operate at the target strategy profile \( \tilde{p} \) with the minimum level of intervention.

We call the transformed game the Stackelberg contention game because the manager chooses his strategy \( g \) before the users make their decisions on the transmission probabilities. In this sense, the manager can be thought of as a Stackelberg leader and the users as followers. The specific timing of the Stackelberg contention game can be outlined as follows.

1. The network manager determines his intervention function.
2. Knowing the intervention function of the manager, the users choose their transmission probabilities simultaneously.
3. Observing the strategy profile of the users, the manager determines the level of intervention using his intervention function.
4. The transmission probabilities of the manager and the users determine their payoffs.

Timing 1 happens before the session starts. Timing 2 occurs at the beginning of the session whereas timing 3 occurs when the manager knows the transmission probabilities of all the users. Therefore, there is a time lag between the time when the session begins and when the manager starts to intervene. Payoffs can be calculated as the probability of successful transmission averaged over the entire session, multiplied by valuation. If the interval between timing 2 and timing 3 is short relative to the duration of the session, the payoff of user \( i \) can be approximated as the payoff during the intervention using the following payoff function:

\[
u_i(g,p) = k_i p_i (1 - g(p)) \prod_{j \neq i} (1 - p_j). \] (5)

The transformation of the contention game into the Stackelberg contention game is schematically shown in Figure 2. The figure shows that the main role of the manager is to set the intervention rule and to implement it. The users still behave noncooperatively maximizing their payoffs, and the intervention of the manager affects their selfish behavior even though the manager does neither directly control their behavior nor continuously communicate with the users to convey coordination or price signals.

In the Stackelberg routing game of [10], the strategy spaces of the manager and a user coincide. If that is the case in the Stackelberg contention game, that is, if the manager chooses a single transmission probability before the users choose theirs, then this intervention only makes the channel lossy but it does not provide incentives for users not to choose the maximum possible transmission probability. Hence, in order to provide an incentive to choose a smaller transmission probability, the manager needs to vary...
his transmission probability depending on the transmission probabilities of the users.

A Stackelberg game is analyzed using a backward induction argument. The leader predicts the Nash equilibrium behavior of the followers given his strategy and chooses the best strategy for him. The same argument can be applied to the Stackelberg contention game. Once the manager decides his strategy $g$ and commits to implement his transmission probability according to $g$, the rest of the Stackelberg contention game (timing 2–4) can be viewed as a noncooperative game played by the users. Given the intervention function $g$, the payoff function of user $i$ can be written as

$$
\tilde{u}_i(p; g) = k_i p_i (1 - g(p)) \prod_{j \neq i} (1 - p_j).
$$

In essence, the role of the manager is to change the noncooperative game that the users play from the contention game $T$ to a new game $T^g = (N, (P_i), (u_i(\cdot; g)))$, which we call the content game with intervention $g$. Understanding the noncooperative behavior of the users given the intervention function $g$, the manager will choose $g$ that maximizes his payoff.

We now define an equilibrium concept for the Stackelberg contention game.

**Definition 1.** An intervention function of the manager $g$ and a profile of the transmission probabilities of the users $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n)$ constitutes a Stackelberg equilibrium if (i) $\tilde{p}$ is a Nash equilibrium of the contention game with intervention $g$ and (ii) $\tilde{p} = \hat{p}$ and $g(\tilde{p}) = 0$.

Combining (i) and (ii), an equivalent definition is that $(g, \tilde{p})$ is a Stackelberg equilibrium if $\tilde{p}$ is a Nash equilibrium of $T^g$ and $g(\tilde{p}) = 0$. Condition (i) says that once the manager chooses his strategy, the users will play a Nash equilibrium strategy profile in the resulting game, and condition (ii) says that expecting the Nash equilibrium strategy profile of the users, the manager chooses his strategy that achieves his objective.

### 3.1. Stackelberg Equilibrium with TRD-Based Intervention

As we have mentioned earlier, the manager can choose only one level of intervention that affects the users equally. A question that arises is which strategy profile the manager can implement as a Stackelberg equilibrium with one level of intervention for every user. We answer this question constructively. We propose a specific form of an intervention function with which the manager can attain any strategy profile $\tilde{p}$ with $0 < \tilde{p}_i < 1$ for all $i$. The basic idea of this result is that because the strategy of the manager is not a single intervention level but a function whose value depends on the strategies of the users, he can discriminate the users by reacting differently to their transmission probabilities in choosing the level of intervention. Therefore, even though the realized level of intervention is the same for every user, the manager can induce the users to choose different transmission probabilities.

To construct such an intervention function, we first define the TRD of $p$ from $\tilde{p}$ by

$$
h(p) = \sum_{i=1}^{n} \frac{p_i - \tilde{p}_i}{\tilde{p}_i} = \frac{p_1}{\tilde{p}_1} + \cdots + \frac{p_n}{\tilde{p}_n} - n.
$$

Since $g$ determines the transmission probability of the manager, its range should lie in $[0, 1]$. To satisfy this constraint, we define the TRD-based intervention function by

$$
g^*(p) = [h(p)]_0^1,
$$

where the operator $[x]_0^1 = \min\{\max\{x, a, b\}\}$ is used to obtain the “trimmed” value of TRD between 0 and 1.

The TRD-based intervention can be interpreted in the following way. The manager sets the target at $\tilde{p}$. As long as the users choose small transmission probabilities so that the TRD of $p$ from $\tilde{p}$ does not exceed zero, the manager does not...
Proof. We need to check two things. First, $\tilde{p}$ is a Nash Equilibrium of $\Gamma^*$. Second, $g^*(\tilde{p}) = 0$. It is straightforward to confirm the second. To show the first, the payoff function of user $i$ given others’ strategies $\tilde{p}_{-i}$ is

$$
\tilde{u}_i(p_i, \tilde{p}_{-i}; g^*) = k_i p_i (1 - g^*(p_i, \tilde{p}_{-i})) \prod_{j \neq i} (1 - \tilde{p}_j)
$$

$$
= \begin{cases}
0 & \text{if } p_i > 2\tilde{p}_i, \\
k_i p_i \left(2 - \frac{p_i}{\tilde{p}_i}\right) \prod_{j \neq i} (1 - \tilde{p}_j) & \text{if } \tilde{p}_i \leq p_i \leq 2\tilde{p}_i, \\
k_i p_i \prod_{j \neq i} (1 - \tilde{p}_j) & \text{if } p_i < \tilde{p}_i.
\end{cases} 
$$

\hspace{1cm} (9)

It can be seen from the above expression that $\tilde{u}_i(p_i, \tilde{p}_{-i}; g^*)$ is increasing on $p_i < \tilde{p}_i$, reaches a peak at $p_i = \tilde{p}_i$, and then stays at 0 on $p_i > 2\tilde{p}_i$. Therefore, user $i$’s best response to $\tilde{p}_i$ is $\tilde{p}_i$ for all $i$, and thus $\tilde{p}$ constitutes a Nash equilibrium of the contention game with TRD-based intervention, $\Gamma^*_g$. \hfill \Box

Corollary 1. Any feasible payoff profile $u \in U$ of the contention game with $u_i > 0$ for all $i \in N$ can be achieved by a Stackelberg equilibrium.

Corollary 1 resembles the Folk theorem of repeated games [4] in that it claims that any feasible outcome can be attained as an equilibrium. Incentives not to deviate from a certain operating point are provided by the manager’s intervention in the Stackelberg contention game, while in a repeated game players do not deviate since a deviation is followed by punishment from other players.

3.2. Nash Equilibria of the Contention Game with TRD-Based Intervention. In Proposition 2, we have seen that $\tilde{p}$ is a Nash equilibrium of the contention game with TRD-based intervention. However, if other Nash equilibria exist, the outcome may be different from the one that the manager intends. In fact, any strategy profile $p$ with $p_i = 1$ for at least one $i$ is still a Nash equilibrium of $\Gamma^*_g$. The following proposition characterizes the set of Nash equilibria of $\Gamma^*_g$ that are different from those of $\Gamma$.

Proposition 3. Consider a strategy profile $\tilde{p}$ with $\tilde{p}_i < 1$ for all $i \in N$. $\tilde{p}$ is a Nash equilibrium of the contention game with TRD-based intervention if and only if either

\hspace{1cm} (i) $\tilde{p} = \hat{p}$, 

\hspace{1cm} or

\hspace{1cm} (ii) $\sum_{j \neq i} \frac{\tilde{p}_j - \hat{p}_j}{\tilde{p}_j} \geq 2 \quad \forall \ i = 1, \ldots, n$. \hspace{1cm} (11)

Proof. See Appendix A. \hfill \Box

Transforming $\Gamma$ to $\Gamma^*_g$ does not eliminate the Nash equilibria of the contention game. Rather, the set of Nash equilibria expands to include two classes of new equilibria. The first Nash equilibrium of Proposition 3 is the one that the manager intends the users to play. The second class of Nash equilibria are those in which the sum of relative deviations of other users is already too large that no matter how small transmission probability user $i$ chooses, the level of intervention stays the same at 1.

Since $\tilde{p}$ is chosen to satisfy $0 < \tilde{p}_i < 1$ for all $i$ and $g^*$ satisfies $g^*(\tilde{p}) = 0$, it follows that $\tilde{u}_i(\tilde{p}) > 0$ for all $i$. (Since we mostly consider the TRD-based intervention function $g^*$, we will use $\tilde{u}_i(\tilde{p})$ instead of $\tilde{u}_i(\tilde{p}; g^*)$ when there is no confusion.) For the second class of Nash equilibria in Proposition 3, $\tilde{u}_i(\hat{p}) = 0$ for all $i$ because $g^*(\hat{p}) = 1$. Therefore, the payoff profile of the second class of Nash equilibria is Pareto dominated by that of the intended Nash equilibrium in that the intended Nash equilibrium yields a higher payoff for every user compared to the second class of Nash equilibrium.

The same conclusion holds for Nash equilibria with more than one user with transmission probability 1 because every user gets zero payoff. Finally, the remaining Nash equilibria are those with exactly one user with transmission probability 1. Suppose that $p_i = 1$. Then the highest payoff for user $i$ is achieved when $p_j = 0$ for all $j \neq i$. Denoting this strategy profile by $e_i$, the payoff profile of $e_i$ is Pareto dominated by that of $\tilde{p}$ if $1 - g^*(e_i) = 1 + n - (1/\tilde{p}_i) < \tilde{p}_i \prod_{j \neq i} (1 - \tilde{p}_j)$.

3.3. Reaching the Stackelberg Equilibrium. We have seen that there are multiple Nash equilibria of the contention game with TRD-based intervention and that the Nash equilibrium $\tilde{p}$ in general yields higher payoffs to the users than other Nash equilibria. If the users are aware of the welfare properties of different Nash equilibria, they will tend to select $\tilde{p}$.

Suppose that the users play the second class of Nash equilibria in Proposition 3 for some reason. If the Stackelberg contention game is played repeatedly and the users anticipate that the strategy profile of the other users will be the same as that of the last period, then it can be shown that under certain conditions there is a sequence of intervention functions convergent to $g^*$ that the manager can employ to have the users reach the intended Nash equilibrium $\tilde{p}$, thus approaching the Stackelberg equilibrium.

Proposition 4. Suppose that at $t = 0$ the manager chooses the intervention function $g^*$ and that the users play a Nash equilibrium $\tilde{p}^0$ of the second class.

Without loss of generality, the users are enumerated so that the following holds:

$$
\frac{\tilde{p}_{i}^0}{p_i} = \frac{\tilde{p}_2}{p_2} \leq \cdots \leq \frac{\tilde{p}_{n-1}}{p_{n-1}} \leq \frac{\tilde{p}_n}{p_n}. 
$$

\hspace{1cm} (12)
Suppose further that for each \( i \), either \((\hat{p}_i^0/\hat{p}_n) - (\hat{p}_i^1/\hat{p}_n) < 2 \) or \((\hat{p}_i^0/\hat{p}_n) \leq 1 \) holds. 

At \( t \geq 1 \); Define
\[
c^t = \sum_{j \neq n} \frac{\hat{p}_j^{t-1}}{\hat{p}_j} + 1. \tag{13}
\]

Assume that the manager employs the intervention function \( g^t(p) = [h^t(p)]_0 \) where
\[
h^t(p) = \frac{p_1}{p_1} + \cdots + \frac{p_n}{p_n} - c^t, \tag{14}
\]

and that user \( i \) chooses \( \hat{p}_i \) as a best response to \( \hat{p}^{t-1}_i \) given \( g^t \).

Then \( \lim_{t \to \infty} \hat{p}_i = \hat{p}_i \) for all \( i = 1, \ldots, n \) and \( \lim_{t \to \infty} c^t = 1 \).

**Proof.** See Appendix B. \( \square \)

The reason that no user has an incentive to deviate from the second class of Nash equilibria is that since others use high transmission probabilities, the TRD is over 1 no matter what transmission probability a user chooses. Since the punishment level is always 1, a reduction of the transmission probability by a user is not rewarded by a decreased level of intervention. If the relative deviations of \( p_i \) from \( \hat{p}_i \) are not too disperse, the manager can successively adjust down the effective range of punishment so that he can react to the changes in the strategies of the users. Proposition 4 shows that this procedure succeeds to have the strategy profile of the users converge to the intended Nash equilibrium.

**4. Target Selection Criteria of the Manager**

So far we have assumed that the manager has a target strategy profile \( \tilde{p} \) and examined whether he can find an intervention function that implements it as a Stackelberg equilibrium. This section discusses selection criteria that the manager can use to choose the target strategy profile. To address this issue, we rely on cooperative game theory because a reasonable choice of the manager should have a close relationship to the likely outcome of bargaining among the users if bargaining were possible for them [4]. The absence of communication opportunities among the users prevents them from engaging in bargaining or from directly coordinating with each other.

**4.1. Nash Bargaining Solution.** The pair \((F, \nu)\) is an \( n \)-person bargaining problem where \( F \) is a closed and convex subset of \( \mathbb{R}^n \), representing the set of feasible payoff allocations and \( \nu = (v_1, \ldots, v_n) \) is the disagreement payoff allocation. Suppose that there exists \( y \in F \) such that \( y_i > v_i \) for every \( i \).

**Definition 2.** \( x \) is the Nash bargaining solution for an \( n \)-person bargaining problem \((F, \nu)\) if it is the unique Pareto efficient vector that solves
\[
\max_{x \in F, x \geq \nu} \prod_{i=1}^n (x_i - v_i). \tag{15}
\]

Consider the contention game \( \Gamma. (\mathcal{U}, 0) \) can be regarded as an \( n \)-person bargaining problem where \( \mathcal{U} \) is defined in (3) and \( 0 \) is the disagreement point. The vector \( 0 \) is the natural disagreement point because it is a Nash equilibrium payoff as well as the minimax value for each user. The only departure from the standard theory is that the set of feasible payoffs \( \mathcal{U} \) is not convex. (We do not allow public randomization among users, which requires coordination among them.) However, we can carry the definition of the Nash bargaining solution to our setting as in [15].

Since the manager knows the structure of the contention game, he can calculate the Nash bargaining solution \( u \) for \((\mathcal{U}, 0)\) and find the strategy profile \( \tilde{p} \) that yields \( u \). Then the manager can implement \( \tilde{p} \) by choosing \( g^* \) based on \( \tilde{p} \). Notice that the presence of the manager does not decrease the payoffs of the users because \( g^*(\tilde{p}) = 0 \). The Nash bargaining solution for \((\mathcal{U}, 0)\) has the following simple form.

**Proposition 5.** \((n - 1)^{n-1}/n^n (k_1, \ldots, k_n)\) is the Nash bargaining solution for \((\mathcal{U}, 0)\), and it is attained by \( p_i = 1/n \) for all \( i = 1, \ldots, n \).

**Proof.** The maximand in the definition of the Nash bargaining solution can be written as
\[
\max_{u \in \mathcal{U}, u \geq 0} \prod_{i=1}^n u_i. \tag{16}
\]

Since any \( u \in \mathcal{U} \) satisfies \( u \geq 0 \), the above problem can be expressed in terms of \( p \):
\[
\max_{p \in \mathcal{P}} \left( \prod_{i=1}^n k_i \right) \prod_{i=1}^n p_i (1 - p_i)^{n-1}. \tag{17}
\]

The logarithm of the objective function is strictly concave in \( p \), and the first-order optimality condition gives \( p_i = 1/n \) for all \( i = 1, \ldots, n \). \( \square \)

The Nash bargaining solution for \((\mathcal{U}, 0)\) treats every user equally in that it specifies the same transmission probability for every user. Therefore, the manager does not need to know the vector of the values of transmission \( k = (k_1, \ldots, k_n) \) to implement the Nash bargaining solution. The Nash bargaining solution coincides with the Kalai-Smorodinsky solution [16] because the maximum payoff for user \( i \) is \( k_i \) and the Nash bargaining solution is the unique efficient payoff profile in which each user receives a payoff proportional to its maximum feasible payoff.

If the manager wants to treat the users with discrimination, he can use the generalized Nash product
\[
\prod_{i=1}^n (x_i - v_i)^{w_i}, \tag{18}
\]
as the maximand to find a nonsymmetric Nash bargaining solution, where \( w_i > 0 \) represents the weight for user \( i \). One example of the weights is the valuation of the users. (If \( k_i \) is private information, it would be interesting to construct a mechanism that induces users to reveal their true values \( k_i \).)
The nonsymmetric Nash bargaining solution for \((\mathcal{U}, \emptyset)\) can be shown to be achieved by \(p_i = (\omega_i / \sum_j \omega_j)\) for all \(i\) using the similar method to the proof of Proposition 5.

4.2. Coalition-Proof Strategy Profile. If some of the users can communicate and collude effectively, the network manager may want to choose a strategy profile which is self-enforcing even in the existence of coalitions. Since we define a user as a transmitter-receiver pair, a collusion may occur when a single transmitter sends packets to several destinations and controls the transmission probabilities of several users.

Given the set of users \(N = \{1, \ldots, n\}\), a coalition is any nonempty subset \(S \subseteq N\). Let \(\mathbf{p}_S\) be the strategy profile of the users in \(S\).

**Definition 3.** \(\bar{\mathbf{p}}\) is coalition-proof with respect to a coalition \(S\) in a noncooperative game \((\mathcal{N}, [0, 1]^N, (u_i))\) if there does not exist \(\mathbf{p}_S \in [0, 1]^S\) such that \(u_i(\mathbf{p}_S, \bar{\mathbf{p}} - S) \geq u_i(\bar{\mathbf{p}})\) for all \(i \in S\) and \(u_i(\bar{\mathbf{p}}_S, \bar{\mathbf{p}} - S) > u_i(\bar{\mathbf{p}})\) for at least one user \(i \in S\).

By definition, \(\bar{\mathbf{p}}\) is coalition-proof with respect to the grand coalition \(S = N\) if and only if \(u_i(\bar{\mathbf{p}}) = (u_1(\bar{\mathbf{p}}), \ldots, u_n(\bar{\mathbf{p}}))\) is Pareto efficient. If \(\bar{\mathbf{p}}\) is a Nash equilibrium, then it is coalition-proof with respect to any one-person “coalition.” The noncooperative game of our interest is the contention game with TRD-based intervention \(g^*\).

**Proposition 6.** \(\bar{\mathbf{p}}\) is coalition-proof with respect to a two-person coalition \(S = \{i, j\}\) in the contention game with TRD-based intervention \(g^*\) if and only if \(p_i + \tilde{p}_j \leq 1\).

**Proof.** See Appendix C.

The proof of Proposition 6 shows that if \(p_i + \tilde{p}_j > 1\) then users \(i\) and \(j\) can jointly reduce their transmission probabilities to increase their payoffs at the same time. For example, suppose that users 1 and 2 are controlled by the same transmitter and that the manager selects the target \(\bar{\mathbf{p}}\) with \(p_1 = 0.3\) and \(p_2 = 0.8\). Then \(\bar{u}_1(\bar{\mathbf{p}}) = 0.06k_1[\prod_{j \neq 1, 2}(1 - \tilde{p}_j)]\) and \(\bar{u}_2(\bar{\mathbf{p}}) = 0.56k_2[\prod_{j \neq 1, 2}(1 - \tilde{p}_j)]\). Suppose that the two users jointly deviate to \((p_1, p_2) = (0.25, 0.75)\). Then the new payoffs are \(\bar{u}_1(p_1, p_2, \tilde{p}_{N\setminus\{1, 2\}}) = 0.0625k_1[\prod_{j \neq 1, 2}(1 - \tilde{p}_j)]\) and \(\bar{u}_2(p_1, p_2, \tilde{p}_{N\setminus\{1, 2\}}) = 0.5625k_2[\prod_{j \neq 1, 2}(1 - \tilde{p}_j)]\), which is strictly better for both users. A decrease in \(p_i\) and \(p_j\) at the same time also increases the payoffs of all the users not belonging to the coalition, which implies that a target \(\bar{\mathbf{p}}\) with \(p_i + \tilde{p}_j > 1\) is not Pareto efficient. This observation leads to the following corollary.

**Corollary 2.** If \(\bar{\mathbf{p}}\) is Pareto efficient in the contention game with TRD-based intervention \(g^*\), then it is coalition-proof with respect to any two-person coalition.

In fact, we can generalize the above corollary and provide a stronger statement.

**Proposition 7.** \(\bar{\mathbf{p}}\) is Pareto efficient in the contention game with TRD-based intervention \(g^*\) if and only if it is coalition-proof with respect to any coalition.

\[\text{Proof: \ See Appendix D.}\]

5. Informational Requirement and Its Relaxation

We have introduced and analyzed the contention game and the Stackelberg contention game with TRD-based intervention. In this section, we discuss what the players of each game need to know in order to play the corresponding equilibrium.

5.1. Contention Game and Nash Equilibrium. In a general noncooperative game, each user needs to know, or predict correctly, the strategy profile of others in order to find its best response strategy. In the contention game with the payoff function \(u_i(\mathbf{p}) = k_ip_1\prod_{j \neq i}(1 - p_j)\), it suffices for user \(i\) to know the sign of \(\prod_{j \neq i}(1 - p_j)\), that is, whether it is positive or zero, to calculate its best response. On the other hand, \(p_i = 1\) is a weakly dominant strategy for any user \(i\), which means setting \(p_i = 1\) is weakly better no matter what strategies other users choose. Hence, the Nash equilibrium \(\mathbf{p} = (1, \ldots, 1)\) does not require any knowledge on others’ strategies.

5.2. Stackelberg Contention Game with TRD-Based Intervention and Stackelberg Equilibrium. Considering the timing of the Stackelberg contention game outlined in Section 3, we can list the following requirements on the manager and the users for the Stackelberg equilibrium to be played.

**Requirement M.** Once the users choose the transmission probabilities, the manager observes the strategy profile of the users.

The manager needs to decide the level of intervention as a function of the transmission probabilities of the users. If the manager can distinguish the access of each user and have sufficiently many observations to determine the transmission probability of each user, then this requirement will be satisfied. If the manager can observe the channel state (idle, success, and collision) and identify the users of successfully transmitted packets, he can estimate the transmission probability of each user in the following way. First, he can obtain an estimate of \(\prod_{i \in N}(1 - p_i)\) by calculating the frequency of idle slots, called \(q_{idle}\). Second, he can obtain an estimate of \(p_1\prod_{j \neq i}(1 - p_j)\) by calculating the frequency of slots in which user \(i\) succeeds to transmit its packet, called \(q_i\). Finally, an estimate of \(p_i\) can be obtained by solving \((p_i/1 - p_i) = (q_i/q_{idle})\) for \(p_i\).

**Requirement U.** User \(i\) knows \(g^*\) (and thus \(\bar{\mathbf{p}}\)) and \(p_{-i}\) when it chooses its transmission probability.

Requirement \(U\) is sufficient for the Nash equilibrium of the contention game with TRD-based intervention to be played by the users. User \(i\) can find its best response strategy by maximizing \(\bar{u}_i\) given \(g^*\) and \(p_{-i}\). In fact, a weaker requirement is compatible with the Nash equilibrium of the contention game with TRD-based intervention. Suppose that user \(i\) knows the form of intervention function \(g^*\) and
the value of \( \tilde{p}_i \), and observes the intervention level \( p_0 \). The embedded in the TRD-based intervention function \( g^* \) can be thought of as a recommended strategy profile by the manager (thus the communication from the manager to the users occurs indirectly through the function \( g^* \)). Even though user \( i \) does not know the recommended strategies to others, that is, the values of \( \tilde{p}_j, j \neq i \), it knows its recommended transmission probability. From the form of the intervention function, user \( i \) can derive that it is of its best interest to follow the recommendation as long as all the other users follow their recommended strategies. Observing \( p_0 = 0 \) confirms its belief that other users play the recommended strategies, and it has no reason to deviate.

The users can acquire knowledge on the intervention function \( g^* \) through one of three ways: (i) known protocol, (ii) announcement, and (iii) learning. The first method is effective in the case where a certain network manager operates in a certain channel (e.g., a frequency band). The community of users will know the protocol (or intervention function) used by the manager. This method does not require any information exchange between the manager and the users. Neither teaching of the manager nor learning of the users will know the protocol (or intervention function). The manager to the users, which is sometimes costly or may even be impossible in practice.

Finally, if the Stackelberg contention game is played repeatedly with the same intervention function, the users may be able to recover the form of the intervention function chosen by the manager based on their observations on \( (p_0, \tilde{p}) \), for example, using learning techniques developed in [17–19]. However, this process may take long and the users may not be able to collect enough data to find out the true functional form if there is limited experimentation of the users.

**Remark.** If users are obedient, the manager can use centralized control by communicating \( \tilde{p}_i \) to user \( i \). Additional communication and estimation overhead required for the Stackelberg equilibrium can be considered as a cost incurred to deal with the selfish behavior of users, or to provide incentives for users to follow \( \tilde{p} \).

5.3. Limited Observability of the Manager. The construction of the TRD-based intervention function assumes that the manager can observe or estimate the transmission probabilities of the users correctly. In real applications, the manager may not be able to observe the exact choice made by each user. We consider several scenarios under which the manager has limited observability and examine how the TRD-based intervention function can be modified in those scenarios.

5.3.1. Quantized Observation. Let \( \mathcal{I} = \{I_0, I_1, \ldots, I_m\} \) be a set of intervals which partition \([0, 1]\). We assume that each interval contains its right end point. For simplicity, we will consider intervals of the same length. That is, \( \mathcal{I} = \{\{0\}, \{0, 1/m\}, \{1/m, 2/m\}, \ldots, \{(m-1)/m, 1\}\} \), and we call \( I_0 = \{0\} \) and \( I_m = \{(r-1)/m, (r/m)\} \) for all \( r = 1, \ldots, m \).

Suppose that the manager only observes which interval in \( \mathcal{I} \) each \( p_i \) belongs to. In other words, the manager observes \( r_i \) instead of \( p_i \) such that \( p_i \in I_{r_i} \). In this case, the level of intervention is calculated based on \( r = (r_1, \ldots, r_n) \) rather than \( p \). It means that given \( p \), \( p_0 \) would be the same for any \( p_i, p_j \) if \( r_i \) and \( r_j \) belong to the same \( I_r \). Since any \( p_i \in \{(r-1)/m, (r/m)\} \) is weakly dominated by \( p_i = (r/m) \), the users will choose their transmission probabilities at the right end points of the intervals in \( \mathcal{I} \). This in turn will affect the choice of a target by the manager. The manager will be restricted to choose \( \tilde{p} \) such that \( \tilde{p}_i \in \{(1/m), \ldots, ((m-1)/m)\} \) for all \( i \in N \). Then the manager can implement \( \tilde{p} \) with the intervention function \( g(r) = g^*(p) \), where \( p_i \) is set equal to \((r_i/m)\). In summary, the quantized observation on \( p \) restricts the choice of \( \tilde{p} \) by the manager from \((0, 1)^N \) to \( \{(1/m), \ldots, ((m-1)/m)\}^N \).

Figure 3 shows the payoff profiles that can be achieved by the manager with quantized observation. When the number of intervals is moderately large, the manager has many options near or on the Pareto efficiency boundary.

5.3.2. Noisy Observation. We modify the Stackelberg contention game to analyze the case where the manager observes noisy signals of the transmission probabilities of the users. Let \( P_i = [\epsilon, 1 - \epsilon] \) be the strategy space of user \( i \), where \( \epsilon \) is a small positive number. We assume that the users can observe the strategy profile \( p \), but the manager observes a noisy signal of \( p \). The manager observes \( p_i' \) instead of \( p_i \) where \( p_i' \) is uniformly distributed on \([p_i - \epsilon, p_i + \epsilon]\), independently over \( i \in N \). Suppose that the manager chooses a target \( \tilde{p} \) such that \( \tilde{p}_i \in [2\epsilon, 1 - 2\epsilon] \). The expected payoff of user \( i \) when the manager uses an intervention function \( g \) is

\[
E[\tilde{u}_i(p;g) \mid p] = k_i p^\top \left[ \prod_{j \neq i} (1 - p_j) \{1 - E[g(p') \mid p]\} \right]. \tag{19}
\]

Hence, the intervention function is effectively \( E[g(p') \mid p] \) instead of \( g(p) \) when the manager observes \( p' \). If \( \tilde{p} \) is a Nash equilibrium of the contention game with intervention \( g \) when \( p \) is perfectly observable to the manager and \( E[g(p') \mid p] = g(p) \) for all \( p \) such that \( \max_{p' \in P} |p_i - \hat{p}_i| \leq \epsilon \), then \( \tilde{p} \) will still be a Nash equilibrium of the contention game with intervention \( g \) when the manager observes a noisy signal of the strategy profile of the users.

Consider the TRD-based intervention function \( g^* \). Since \( g^*(p) \geq 0 \) for all \( p \in P \) and \( h(p') > 0 \) with a positive probability when \( p = \tilde{p} \), \( E[g^*(p') \mid p] > 0 \) whereas \( g^*(\tilde{p}) = 0 \). Since \( g^* \) is kinked at \( \tilde{p} \), the noise in \( p' \) will distort the incentives of the users to choose \( \tilde{p} \).
The manager can implement his target \( \hat{p} \) at the expense of intervention with a positive probability. If the manager adopts the following intervention function:

\[
g(\mathbf{p}) = \sum_{i \in N} \frac{1}{\hat{p}_i} \left( \frac{1 + \epsilon q}{\hat{p}_i} - \hat{p}_i \right) + \frac{(n + 1) \epsilon q}{1 + \epsilon q},
\]

where \( q = \sum_{i \in N} (1/\hat{p}_i) \), then \( \hat{p} \) is a Nash equilibrium of the contention game with intervention \( g \), but the average level of intervention at \( \hat{p} \) is

\[
E[g(\mathbf{p}) \mid \hat{p}] = g(\hat{p}) = \frac{\epsilon q}{1 + \epsilon q} > 0,
\]

which can be thought of as the efficiency loss due to the noise in observations.

Figure 4 illustrates the set of payoff profiles that can be achieved with the intervention function given by (20). As the size of the noise gets smaller, the set expands to approach the Pareto efficiency boundary.

5.3.3. Observation on the Aggregate Probability. We consider the case where the manager can observe only the frequency of the slots that are not accessed by any user. If the users transmit their packets according to \( \mathbf{p} \), then manager observes only the aggregate probability \( \prod_{i \in N} (1 - p_i) \). In this scenario, the intervention function that the manager chooses has to be a function of \( \prod_{i \in N} (1 - p_i) \), and this implies that the manager cannot discriminate among the users.

The TRD-based intervention function \( g^* \) allows the manager to use different reactions to each user’s deviation. In the effective region where the TRD is between 0 and 1, one unit increase in \( p_i \) results in \( (1/\hat{p}_i) \) units increase in \( p_0 \). However, this kind of discrimination through the structure of the intervention function is impossible when the manager cannot observe individual transmission probabilities.

This limitation forces the manager to treat the users equally, and the target has to be chosen such that \( \hat{p}_i = \hat{p} \).
are aware of the dependence of $p_0$ on their transmission probabilities and try to model this dependence based on their observations $(p_i(1 - p_0)\prod_{j \neq i}(1 - p_j))$. Specifically, user $i$ builds a conjecture function $f_i : [0, 1] \rightarrow [0, 1]$, which means that user $i$ conjectures that the value of $(1 - p_0)\prod_{j \neq i}(1 - p_j)$ will be $f_i(p_i)$ if he chooses $p_i$. The equilibrium concept appropriate in this context is conjectural equilibrium first introduced by Hahn [20].

**Definition 4.** A strategy profile $\hat{\mathbf{p}}$ and a profile of conjectures $(f_1, \ldots, f_n)$ constitutes a conjectural equilibrium of the contention game with intervention $g$ if

$$k_i \hat{p}_i f_i(\hat{p}_i) \geq k_i p_i f_i(p_i) \quad \forall p_i \in P_i$$

$$f_i(\hat{p}_i) = (1 - g(\hat{p})) \prod_{j \neq i} (1 - \hat{p}_j)$$

for all $i \in N$.

The first condition states that $\hat{p}_i$ is optimal given user $i$’s conjecture $f_i$, and the second condition says that its conjecture is consistent with its observation. It can be seen from this definition that the conjectural equilibrium is a generalization of Nash equilibrium in that any Nash equilibrium is a conjectural equilibrium with every user holding the correct conjecture given others’ strategies. On the other hand, it is quite general in some cases, and in the game we consider, for any strategy profile $\hat{\mathbf{p}} \in P$, there exists a conjecture profile $(f_1, \ldots, f_n)$ that constitutes a conjectural equilibrium. For example, we can set $f_i(p_i) = (1 - g(\hat{p})) \prod_{j \neq i} (1 - \hat{p}_j)$ if $p_i = \hat{p}_i$ and 0 otherwise.

Since the TRD-based intervention function $g^*$ is linear in each $p_i$, it is natural for the users to adopt a conjecture function of the linear form. Let us assume that conjecture functions are of the following trimmed linear form:

$$f_i(p_i) = \left[a_i - b_i p_i\right]_0^1$$

for some $a_i, b_i > 0$.

We say that a conjecture function $f_i$ is linearly consistent at $\hat{\mathbf{p}}$ if it is locally correct up to the first derivative at $\hat{\mathbf{p}}$, that is, $f_i(\hat{p}_i) = (1 - g(\hat{p})) \prod_{j \neq i} (1 - \hat{p}_j)$ and $f'_i(\hat{p}_i) = -(\partial g(\hat{p})/\partial p_i) \prod_{j \neq i} (1 - \hat{p}_j)$. Since the TRD-based intervention function $g^*$ is linear in each $p_i$, the conjecture function $f^*_i(p_i) \triangleq g^*(p_i, \hat{\mathbf{p}} - \hat{\mathbf{p}})$ is linearly consistent at $\hat{\mathbf{p}}$, and $\hat{\mathbf{p}}$ and $(f^*_1, \ldots, f^*_n)$ constitutes a conjectural equilibrium. Therefore, as long as the users use linearly consistent conjectures, limited observability of the users does not affect the final outcome. To build linearly consistent conjectures, however, the users need to experiment and collect data using

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**Table 1: Individual payoffs and system utilization ratios with homogeneous users.**

| $n$ | Individual payoff | System utilization ratio |
|-----|------------------|-------------------------|
| 3   | 0.14815          | 0.44444                 |
| 10  | 0.03874          | 0.38742                 |
| 100 | 0.00370          | 0.36973                 |

---

**Figure 5:** Payoffs that can be achieved by the manager who observes only the aggregate probability. (a) Homogeneous users with $k_1 = k_2 = 1$. (b) Heterogeneous users with $k_1 = 1$ and $k_2 = 2$. 

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5.4. **Limited Observability of the Users and Conjectural Equilibrium.** We now relax Requirement U and assume that user $i$ can observe only the aggregate probability $(1 - p_0) \prod_{j \neq i}(1 - p_j)$. Even though the users do not know the exact form of the intervention function of the manager, they
local deviations from the equilibrium point in a repeated play of the Stackelberg contention game. A loss in performance may result during this learning phase.

6. Illustrative Results

6.1. Homogeneous Users. We assume that the users are homogeneous with \( k_i = 1 \) for all \( i \in N \). Given a transmission probability profile \( p \), the system utilization ratio can be defined as the probability of successful transmission in a given slot

\[
\tau(p) = \sum_{i \in N} p_i \prod_{j \neq i} (1 - p_j).
\]

Note that the maximum system utilization ratio is 1, which occurs when only one user transmits with probability 1 while others never transmit. Table 1 shows the individual payoffs and the system utilization ratios for the number of users 3, 10, and 100 when the manager implements the target at the symmetric efficient strategy profile \( \tilde{p}^* = (1/n, \ldots, 1/n) \).

We can see that packets are transmitted in approximately 37% of the slots with a large number of users even if there is no explicit coordination among the users. The system utilization of our model converges to \( 1/e \approx 36.8\% \) as \( n \) goes to infinity, which coincides with the maximal throughput of a slotted Aloha system with Poisson arrivals and an infinite number of users [21], but in our model users maintain their selfish behavior, and we do not use any feedback information on the channel state.

6.2. Heterogeneous Users. We now consider users with difference valuations. Specifically, we assume that \( k_i = i \) for \( i = 1, \ldots, n \). We will consider three targets: \( \tilde{p}^1 = (1, \ldots, n)/\sum_{i=1}^{n} i \), \( \tilde{p}^2 = (1/n, \ldots, 1/n) \), and \( \tilde{p}^3 \) with which \( u_i(\tilde{p}^3; g^*) = u_i(\tilde{p}^1; g^*) \) for all \( i, j \). \( \tilde{p}^1 \) assigns a higher transmission probability to a user with a higher valuation. \( \tilde{p}^2 \) treats all the users equally regardless of their valuations. \( \tilde{p}^3 \) is egalitarian in that it yields the same individual payoff to every user, which implies that a user with a low valuation is assigned a higher transmission probability.

Table 2 shows that a tradeoff between efficiency (measured by the sum of payoffs) and equity exists when users are heterogeneous. A higher aggregate payoff is achieved when users with high valuations are given priority. At the same time, it limits access by users with low valuations, which increases variations in individual payoffs. Also, the results in Table 2 are consistent with the \( \tilde{p}^2 \) being a Nash bargaining solution and that \( \tilde{p}^1 \) is a nonsymmetric Nash bargaining solution with weights equal to valuations.

7. Conclusion

We have analyzed the problem of multiple users who share a common communication channel. Using the game theory framework, we have shown that selfish behavior is likely to lead to a network collapse. However, full system utilization requires coordination among users using explicit message exchanges, which may be impractical given the distributed nature of wireless networks. To achieve a better performance without coordination schemes, users need to sustain cooperation. We provide incentives for selfish users to limit their access to the channel by introducing an intervention function of the network manager. With TRD-based intervention functions, the manager can implement any outcome of the contention game as a Stackelberg equilibrium. We have discussed the amount of information required for implementation, and how the various kinds of relaxations of the requirements affect the outcome of the Stackelberg contention game.

Our approach of using an intervention function to improve network performance can be applied to other situations in wireless communications. Potential applications of the idea include sustaining cooperation in multihop networks and limiting the attack of adversary users. An intervention function may be designed to serve as a coordination device in addition to providing selfish users with incentives to cooperate. Finally, designing a protocol that enables users to play the role of the manager in a distributed manner will be critical to ensure that our approach can be adopted in completely decentralized communication scenarios, where no manager is present.
Appendices

A. Proof of Proposition 3

Recall \( h(p) = (p_1 / \hat{p}_1) + \cdots + (p_n / \hat{p}_n) - n \) used to define \( g^*(p) \). We examine whether a strategy profile \( \hat{p} \) with \( \hat{p}_i < 1 \) for all \( i \in N \) constitutes a Nash equilibrium of \( \Gamma_x \) by considering four cases on the value of \( h(\hat{p}) \).

**Case 1** \( (h(\hat{p}) < 0) \). Let \( \epsilon = -h(\hat{p}) > 0 \). If user \( i \) changes its transmission profile from \( \hat{p}_i \) to \( \hat{p}_i + \epsilon \), then its payoff increases because \( p_0 \) is still zero. Hence \( \hat{p} \) cannot be a Nash equilibrium if \( h(\hat{p}) < 0 \).

**Case 2** \( (h(\hat{p}) = 0) \). Consider arbitrary user \( i \). If it deviates to \( p_i < \hat{p}_i \), then \( \hat{u}_i \) decreases. \( \hat{u}_i(p, \hat{p} - i) \) is differentiable and strictly concave on \( p_i > \hat{p}_i \). Since \( (d\hat{u}_i / dp_i) = (k_i \prod_{j \neq i} (1 - \hat{p}_j)) (1 - \hat{p}_i) - 2 \hat{p}_i / \hat{p}_j \), \( k_i > 0 \) and \( \hat{p}_i < 1 \) for all \( i \),

\[
\text{sign} \left( \frac{d\hat{u}_i}{dp_i} \bigg|_{p_i=\hat{p}_i} \right) = \text{sign} \left( 1 + n - \sum_{j \neq i} \frac{\hat{p}_j}{\hat{p}_i} - 2 \frac{\hat{p}_i}{\hat{p}_j} \right).
\]

There is no gain for user \( i \) from deviating to any \( p_i > \hat{p}_i \) if and only if \( (d\hat{u}_i / dp_i) \big|_{p_i=\hat{p}_i} \leq 0 \), which is equivalent to \( \hat{p}_i \geq \bar{p}_i \). For \( \hat{p} \) to be a Nash equilibrium, we need \( \hat{p}_i \geq \bar{p}_i \) for all \( i = 1, \ldots, n \). To satisfy \( h(\hat{p}) = 0 \), all inequalities should be equalities. Hence, only \( \hat{p} = \hat{\bar{p}} \) is a Nash equilibrium among \( \hat{p} \) such that \( h(\hat{p}) = 0 \).

**Case 3** \( (0 < h(\hat{p}) < 1) \). Since \( \hat{u}_i \geq 0 \), there is no gain for user \( i \) to deviate to \( p_i \) such that \( h(p_i, \hat{p} - i) \geq 1 \). If there is a gain from deviation to \( p_i \) such that \( h(p_i, \hat{p} - i) < 0 \), then there is another profitable deviation \( p'_i \), such that \( h(p'_i, \hat{p} - i) = 0 \) by using the argument of Case 1. Therefore, we can restrict our attention to deviations \( p_i \) that lead to \( 0 < h(p_i, \hat{p} - i) < 1 \). At such a deviation by user \( i \),

\[
\tilde{u}_i(p, \hat{p} - i) = k_i \prod_{j \neq i} (1 - \hat{p}_j) p_i \left( 1 + n - \sum_{j \neq i} \frac{\hat{p}_j}{\hat{p}_i} - \frac{\hat{p}_i}{\hat{p}_j} \right). \tag{A.2}
\]

\( \hat{p}_i \) is best response to \( \hat{p} - i \) if and only if \( (d\tilde{u}_i / dp_i) \big|_{p_i=\hat{p}_i} = 0 \). Using the first derivative given in Case 2, we obtain

\[
\frac{\hat{p}_i}{\hat{p}_i} = 1 + n - \sum_{j \neq i} \frac{\hat{p}_j}{\hat{p}_i} = 1 - h(\hat{p}) < 1. \tag{A.3}
\]

For \( \hat{p} \) to be a Nash equilibrium, the above inequality should be satisfied for every \( i \), which in turn implies

\[
\sum_{i=1}^{n} \frac{\hat{p}_i}{\hat{p}_i} < n. \tag{A.4}
\]

and this contradicts the initial assumption \( h(\hat{p}) > 0 \). Therefore, there is no \( \hat{p} \) with \( 0 < h(\hat{p}) < 1 \) that constitutes a Nash equilibrium.

**Case 4** \( (h(\hat{p}) \geq 1) \). Since \( \tilde{u}_i(\hat{p}) = 0 \) for every \( i \), there is a profitable deviation of user \( i \) only if there exists \( p_i \in (0, \hat{p}_i) \) such that \( h(p_i, \hat{p} - i) < 1 \). Equivalently, if setting \( p_i = 0 \) yields \( h(p_i, \hat{p} - i) \geq 1 \), then there is no profitable deviation of user \( i \) from \( \hat{p}_i \). Since

\[
h(0, \hat{p} - i) = \sum_{j \neq i} \frac{\hat{p}_j}{\hat{p}_j} - n, \tag{A.5}
\]

\( \hat{p} \) with \( h(\hat{p}) \geq 1 \) is a Nash equilibrium if and only if

\[
\sum_{j \neq i} \frac{\hat{p}_j}{\hat{p}_j} - n \geq 1 \quad \forall \ i = 1, \ldots, n. \tag{A.6}
\]

B. Proof of Proposition 4

Consider \( t = 1 \). User \( i \) chooses \( \hat{p}_i^1 \) to maximize

\[
\tilde{u}_i^1(p, \hat{p} - i) = k_i p_i (1 - g^1(p, \hat{p} - i)) \prod_{j \neq i} (1 - \hat{p}_j^1) \]

\[
= \begin{cases} 
\prod_{j \neq i} (1 - \hat{p}_j^1) & \text{if } p_i < \bar{p}_i 
\frac{\hat{p}_n^1 + \hat{p}_i^1}{\hat{p}_n^1} 
\left( 1 - \frac{\hat{p}_n^1 + \hat{p}_i^1}{\hat{p}_n^1} \right) & \text{if } p_i < \bar{p}_i 
\frac{\hat{p}_n^1 + \hat{p}_i^1}{\hat{p}_n^1} 
\left( 1 - \frac{\hat{p}_n^1 + \hat{p}_i^1}{\hat{p}_n^1} \right) & \text{if } p_i > \bar{p}_i 
\end{cases} \tag{B.1}
\]

If \( 0 \leq (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_0^0 / \hat{p}_0) < 2 \), the maximum is attained at \( \hat{p}_1^1 \) that satisfies

\[
\frac{\hat{p}_1^1}{\hat{p}_1} = 1 - \frac{1}{2} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_1^0}{\hat{p}_1} \right). \tag{B.2}
\]

Notice that \( \hat{p}_1^1 = \hat{p}_n^0 \). If \( (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_1^0 / \hat{p}_1) \geq 2 \), then \( \tilde{u}_i^1(p, \hat{p} - i) = 0 \) for all \( p_i \geq 1 \). Since any \( p_i > 1 \) is a best response in this case, we assume that \( \hat{p}_1^1 = \hat{p}_n^0 \). If we assume that \( \hat{p}_1^1 \) is chosen according to (B.2), we do not need the assumption that for each \( i \) either \( (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_i^0 / \hat{p}_i) < 2 \) or \( (\hat{p}_1^0 / \hat{p}_1) \leq 1 \) in the proposition.

Consider \( t = 2 \). First, consider user \( i \) such that \( (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_1^0 / \hat{p}_1) < 2 \). Since \( (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_1^0 / \hat{p}_1) = \frac{1}{2} \left( (\hat{p}_n^0 / \hat{p}_n) - (\hat{p}_1^0 / \hat{p}_1) \right) \),
0 \leq (\hat{p}_n^0/\hat{p}_n) - (\hat{p}_i^0/\hat{p}_i) < 2. Using an analogous argument, we get
\[
\frac{\hat{p}_i^2}{\hat{p}_i} = 1 - \frac{1}{2} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right) = 1 - \frac{1}{2} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right).
\]
(B.3)

Next consider user \( i \) such that \((\hat{p}_n^0/\hat{p}_n) \leq 1 \). Since \((\hat{p}_n^0/\hat{p}_n) = 1 \), we again have \( 0 \leq (\hat{p}_i^0/\hat{p}_i) - (\hat{p}_i^0/\hat{p}_i) < 2 \) and the best response is given by
\[
\frac{\hat{p}_i^2}{\hat{p}_i} = 1 - \frac{1}{2} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right) = 1 - \frac{1}{2} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right).
\]
(B.4)

Considering a general \( t \geq 2 \), we get
\[
\frac{\hat{p}_i^t}{\hat{p}_i} = 1 - \frac{1}{2^{t-1}} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right)
\]
for user \( i \) such that \((\hat{p}_n^0/\hat{p}_n) - (\hat{p}_i^0/\hat{p}_i) < 2 \) and
\[
\frac{\hat{p}_i^t}{\hat{p}_i} = 1 - \frac{1}{2^{t-1}} \left( \frac{\hat{p}_n^0}{\hat{p}_n} - \frac{\hat{p}_i^0}{\hat{p}_i} \right)
\]
(B.5)

for user \( i \) such that \((\hat{p}_n^0/\hat{p}_n) \leq 1 \). Taking limits as \( t \to \infty \), we obtain the conclusions of the proposition.

**C. Proof of Proposition 6**

Suppose that the users in the coalition \( S = \{i, j\} \) choose \((p_i, p_j)\) instead of \((\hat{p}_i, \hat{p}_j)\). Then
\[
h(p_i, p_j, \hat{p}_s) = \frac{p_i}{\hat{p}_i} + \frac{p_j}{\hat{p}_j} + (n-2) - n = \frac{p_i}{\hat{p}_i} + \frac{p_j}{\hat{p}_j} - 2,
\]
\[
\tilde{u}_i(p_i, p_j, \hat{p}_s) = k_{S} \prod_{k \not\in S} (1 - \hat{p}_k) p_i(1-p_j)(1-g^*(p_i,p_j,\hat{p}_s)),
\]
\[
\tilde{u}_j(p_i, p_j, \hat{p}_s) = k_{S} \prod_{k \not\in S} (1 - \hat{p}_k) p_j(1-p_i)(1-g^*(p_i,p_j,\hat{p}_s)).
\]
(C.1)

Hence, \( \tilde{p} \) is coalition-proof with respect to \( S \) if and only if there does not exist \((p_i, p_j) \in [0, 1]^2 \) such that
\[
p_i(1-p_j)(1-g^*(p_i,p_j,\hat{p}_s)) \geq \tilde{p}_i(1-\tilde{p}_j),
\]
(C.2)
\[
p_j(1-p_i)(1-g^*(p_i,p_j,\hat{p}_s)) \geq \tilde{p}_j(1-\tilde{p}_i),
\]
(C.3)
with at least one inequality strict.

First, notice that setting \( p_i = \tilde{p}_i \) and \( p_j \neq \tilde{p}_j \) will violate one of the two inequalities. The inequality for user \( i \) will not hold if \( p_j > \tilde{p}_j \), and the one for user \( j \) will not hold if \( p_j < \tilde{p}_j \). Hence, both \( p_i \neq \tilde{p}_i \) and \( p_j \neq \tilde{p}_j \) are necessary to have both inequalities satisfied at the same time. We consider four possible cases.

**Case 1** \((p_i < \tilde{p}_i \text{ and } p_j > \tilde{p}_j)\). Since \( g^*(\cdot) \geq 0 \), (C.2) is violated.

**Case 2** \((p_i > \tilde{p}_i \text{ and } p_j < \tilde{p}_j)\). Equation (C.3) is violated.

**Case 3** \((p_i < \tilde{p}_i \text{ and } p_j < \tilde{p}_j)\). Since \( h(p_i, p_j, \hat{p}_s) < 0 \), \( g^*(p_i, p_j, \hat{p}_s) = 0 \). Hence, (C.2) and (C.3) become
\[
p_i(1-p_j) \geq \tilde{p}_i (1-\tilde{p}_j),
\]
\[
p_j(1-p_i) \geq \tilde{p}_j (1-\tilde{p}_i).
\]
(C.4)

We consider the contour curves of \( p_i(1-p_j) \) and \( p_j(1-p_i) \) going through \((\tilde{p}_i, \tilde{p}_j)\) in the \((p_i, p_j)\)-plane. The slope of the contour curve of \( p_i(1-p_j) \) at \((\tilde{p}_i, \tilde{p}_j)\) is \((1-\tilde{p}_j)/\tilde{p}_i\) and that of \( p_j(1-p_i) \) is \((\tilde{p}_i/1-\tilde{p}_i)\). There is no area of mutual improvement if and only if
\[
\frac{1-\tilde{p}_j}{\tilde{p}_i} \geq \frac{\tilde{p}_i}{1-\tilde{p}_i},
\]
(C.5)
which is equivalent to \( \tilde{p}_i + \tilde{p}_j \leq 1 \).

**Case 4** \((p_i > \tilde{p}_i \text{ and } p_j > \tilde{p}_j)\). Since \( h(p_i, p_j, \hat{p}_s) > 0 \), \( g^*(p_i, p_j, \hat{p}_s) = h(p_i, p_j, \hat{p}_s) \) as long as \((p_i/\tilde{p}_i) + (p_j/\tilde{p}_j) \leq 3 \). Hence, (C.2) and (C.3) become
\[
p_i(1-p_j) \left( 3 - \frac{p_i}{\tilde{p}_i} - \frac{p_j}{\tilde{p}_j} \right) \geq \tilde{p}_i (1-\tilde{p}_j),
\]
\[
p_j(1-p_i) \left( 3 - \frac{p_i}{\tilde{p}_i} - \frac{p_j}{\tilde{p}_j} \right) \geq \tilde{p}_j (1-\tilde{p}_i).
\]
(C.6)

The slope of the contour curve of \( p_i(1-p_j)(3 - (p_i/\tilde{p}_i) - (p_j/\tilde{p}_j)) \) at \((\tilde{p}_i, \tilde{p}_j)\) is
\[
\frac{(1-\tilde{p}_j)(3 - 2(\tilde{p}_i/\tilde{p}_j) - (\tilde{p}_j/\tilde{p}_j))}{\tilde{p}_i(3 + (1/\tilde{p}_j) - (\tilde{p}_j/\tilde{p}_j)) - 2(\tilde{p}_j/\tilde{p}_j)) = 0,
\]
(C.7)
and that of \( p_j(1-p_i)(3 - (p_i/\tilde{p}_i) - (p_j/\tilde{p}_j)) \) is
\[
\frac{\tilde{p}_j(3 + (1/\tilde{p}_j) - 2(\tilde{p}_j/\tilde{p}_j)) - (\tilde{p}_j/\tilde{p}_j))}{(1-\tilde{p}_j)(3 - (p_i/\tilde{p}_i) - 2(\tilde{p}_j/\tilde{p}_j))} = +\infty.
\]
(C.8)

Therefore, there is no \((p_i, p_j) > (\tilde{p}_i, \tilde{p}_j)\) that satisfies (C.2) and (C.3) at the same time.

**D. Proof of Proposition 7**

The “if” part is trivial because a strategy profile that is coalition-proof with respect to the grand coalition is Pareto efficient. To establish the “only if” part, we will prove that if for a given strategy profile there exists a coalition that can improve the payoffs of its members then its deviation will not hurt other users outside of the coalition, which shows that the original strategy profile is not Pareto efficient.

Consider a strategy profile \( \hat{p} \) and a coalition \( S \subseteq N \) that can improve upon \( \hat{p} \) by deviating from \( \hat{p}_s \) to \( p_s \). Let \( p_0 = g^*(p_s, \hat{p}_s) \) the transmission probability of the manager after the deviation by coalition \( S \). Since choosing \( p_s \) instead of \( \hat{p}_s \) yields higher payoffs to the members of \( S \), we have
\[
p_i(1-p_0) \prod_{j \in S \setminus \{i\}} (1-p_j) \geq \tilde{p}_i \prod_{j \in S \setminus \{i\}} (1-\tilde{p}_j)
\]
(D.1)
for all $i \in S$ with at least one inequality strict. We want to show that the members not in the coalition $S$ do not get lower payoffs as a result of the deviation by $S$, that is,

$$(1 - p_0) \prod_{j \in S} (1 - p_j) \geq \prod_{j \in S} (1 - \tilde{p}_j). \tag{D.2}$$

Suppose $(1 - p_0) \prod_{j \in S} (1 - p_j) < \prod_{j \in S} (1 - \tilde{p}_j)$. We can see that $p_0 < 1$ and $0 < p_i < 1$ for all $i \in S$ because the right-hand side of (D.1) is strictly positive. Combining this inequality with (D.1) yields $p_i > \tilde{p}_i$ for all $i \in S$, which implies $p_0 > 0$. We can write $p_i = \tilde{p}_i + \epsilon_i$ for some $\epsilon_i > 0$ for $i \in S$. Then $p_0 = g^*(p_S, \tilde{p} - S) = \sum_{i \in S} (\epsilon_i / \tilde{p}_i)$. (D.1) can be rewritten as

$$\tilde{p}_i \prod_{j \in S \setminus \{i\}} (1 - \tilde{p}_j) \leq (\tilde{p}_i + \epsilon_i) (1 - p_0) \prod_{j \in S \setminus \{i\}} (1 - \tilde{p}_j - \epsilon_j)$$

$$< (\tilde{p}_i + \epsilon_i) (1 - p_0) \prod_{j \in S \setminus \{i\}} (1 - \tilde{p}_j) \tag{D.3}$$

for all $i \in S$. Simplifying this gives

$$\frac{\epsilon_i}{\tilde{p}_i} > \frac{p_0}{1 - p_0} \tag{D.4}$$

for all $i \in S$. Summing these inequalities up over $i \in S$, we get

$$p_0 = \sum_{i \in S} \frac{\epsilon_i}{\tilde{p}_i} > |S| \frac{p_0}{1 - p_0}, \tag{D.5}$$

where $|S|$ is the number of the members in $S$. This inequality simplifies to $p_0 < 1 - |S| \leq 0$, which is a contradiction.

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Research Article

Auction-Based Resource Allocation for Cooperative Video Transmission Protocols over Wireless Networks

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Cooperative transmission has been proposed as a novel transmission strategy that takes advantage of broadcast nature of wireless networks, forms virtual MIMO system, and provides diversity gains. In this paper, wireless video transmission protocols are proposed, in which the spectrum resources are first allocated for the source side to broadcast video packets to the relay and destination, and then for the relay side to transmit side information generated from the received packets. The proposed protocols are optimized to minimize the end-to-end expected distortion via choosing bandwidth/power allocation, configuration of side information, subject to bandwidth and power constraints. For multiuser cases, most of current resource allocation approaches cannot be naturally extended and applied to the networks with relay nodes for video transmission. This paper extends the share auction approach into the cooperative video communication scenarios and provides a near-optimal solution for resource allocation. Experimental results have demonstrated that the proposed approach has significant advantage of up to 4 dB gain in single user case and 1.3 dB gain in multiuser case over the reference systems in terms of peak-to-signal-noise ratio. In addition, it reduces the formidable computational complexity of the optimal solution to linear complexity with performance degradation of less than 0.3 dB.

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1. Introduction

Over the past few decades, wireless communications and networking have experienced unprecedented growth. With the advancement in video coding technology, transmitting real-time encoded video programs over wireless networks has become a promising service for applications such as video-on-demand and interactive video telephony. Recently, Distributed Source Coding (DSC) and Wyner-Ziv (WZ) coding have been proposed for video transmission [1, 2]. In DSC, the relay transcodes video packets to form the multiple descriptions of the video contents [1, 2]. The reencoded video is coded by WZ coding, that is, instead of sending the original video, the relay sends side information to the destination to improve the decoded video quality. Combining the directed transmitted video and relay’s side information, the destination can explore the source diversity to improve the reconstructed video quality.

On the other hand, cooperative communication [3–15] has recently attracted significant attention as an effective transmission strategy, which efficiently takes advantage of the broadcast nature of wireless networks. The basic idea is to let nodes in a wireless network share information and transmit data cooperatively as a virtual antenna array. This collaboration significantly improves the system performance. With the fast growth of the video streaming technology, the concept of cooperative video communication can enable the relay nodes to play more intelligent and active roles in processing, transcoding, or re-adapting the media information received before transmitting to the next node. As a consequence, the advantage of flexibility simultaneously increases the complexity of the network management and
resource allocation. So far there is little work in studying the full-fledged cooperative video network due to the high complexity.

Due to the distributed nature of the relay nodes, it is nature to connect the idea of DSC to video cooperative communication. In this paper, we propose a wireless video transmission protocol that leverages the benefits of both cooperative transmission and the idea of DSC. There are two phases involved in the transmission. In the first phase, the source broadcasts the video to the relay and destination. The relay conducts transcoding for the received video content to coarse quality and transmits it as side information. In the second phase, the relay chooses one of two options, that is, either to transmit the coarse-quality video packets using the amplify-and-forward or decode-and-forward protocols, or to encode the coarse-quality video using forward error coding (FEC), and then transmit the parity data to the destination. We assume a control channel is available to let the destination know the processing settings chosen by the relay node. The destination decodes the video transmitted in the first phase and transcodes it using the same encoding parameters deployed in the relay node. The coarse-quality video generated at destination will be combined with the side information sent from the relay to improve the reconstructed coarse-quality video. (To achieve real-time streaming, when the source collects one new group of pictures ( GOPs), the encoded bitstream of previous GOP can be transmitted by the proposed scheme.) In this paper, an optimization problem is formulated to minimize the end-to-end expected distortion by dynamically choosing the protocol mode, bandwidth/power allocation, FEC coding parameter, subject to the bandwidth and power constraints. From the analysis and simulation results using 3D-Set Partitioning in Hierarchical Trees (3D-SPIHT) video coding [16], the proposed cooperative video transmission protocol has a significant PSNR gain over the traditional direct video transmission. Especially by employing the cooperative video transmission with side information, we have obtained up to 4 dB gain over the amplify-and-forward or decode-and-forward protocols.

For the multiuser case, we concentrate on the resource allocation method using auction theory, which is a subfield of the game theory which attempts to mathematically capture behavior in strategic situations, in which an individual’s success in making choices depends on the choices of others. In the auction scenarios, there is a central spectrum moderator that masters the resources and there are autonomous users that request resources in the network. Very recently, researchers start to explore the auction-theory-based solutions for resource allocation for video communications [17, 18] based on a Vickrey-Clarke-Groves VCG auction. General cooperative data communications based on share auction is also studied in [7, 19].

For our proposed cooperative video transmission, we study the video communications over the full-fledged cooperative network, and we focus on how to use relay nodes to improve the overall system performance, and especially on how to conduct resource allocation for relays. Each relay helps to connect a group of transmitters with a number of receivers. During the resource allocation process, the spectrum resources are first allocated for the transmitters who broadcast video packets to the relay and destination, and then for the relay nodes that transmit side information generated from the received packets to the destination, clearly to balance the resource allocation among source and relay nodes, and the resources used by the relays for each source are very critical for the overall network performance. We propose a quasishare auction-based approach, which explores the concept of share auction into this new domain. (In general the share auction concept cannot be naturally extended to video communications, due to the complexity to express the cooperative video end-to-end distortion and to obtain the close form update function.) Experimental results have demonstrated that the proposed approach has significant advantage of up to 1.3 dB gain over the reference system. In addition, it reduces the formidable computational complexity of the optimal solution to linear complexity with performance degradation of less than 0.3 dB.

This paper is organized as follows. In Section 2, the basics of cooperative transmission are studied, and the channel model, modulation, and coding scheme are discussed. In Section 3, the cooperative video transmission protocol is proposed and analyzed. In Section 4, the proposed resource allocation using quasishare auction is demonstrated and analyzed for multiuser case. A performance upper bound is also proposed. Simulations’ results are shown in Section 5, and conclusions are drawn in Section 6.

2. Traditional Cooperative Communication Protocols and Channel Model

For the cooperative transmission system, we first consider a single source-destination case, in which there are source node $s$, relay node $r$, and destination node $d$. A more general multiuser case will be discussed in Section 4. The cooperative transmission consists of two phases. In Phase 1, source $s$ broadcasts its information to both destination node $d$ and relay node $r$. The received signals $Y_{s,d}$ and $Y_{s,r}$ at destination $d$ and relay $r$ can be expressed as

$$Y_{s,d} = \sqrt{P_s G_{s,d}} X_{s,d} + n_d,$$

$$Y_{s,r} = \sqrt{P_s G_{s,r}} X_{s,r} + n_r,$$

respectively, where $P_s$ represents the transmit power to the destination from the source, $X_{s,d}$ is the transmitted information symbol with unit energy at Phase 1 at the source, $G_{s,d}$ and $G_{s,r}$ are the channel gains from $s$ to $d$ and $r$, respectively, and $n_d$ and $n_r$ are the additive white Gaussian noises (AWGNs). Without loss of generality, we assume that the noise power is the same for all the links, denoted by $\sigma^2$. We also assume that the channels are stable over each transmission frame.

For direct transmission, without the relay node’s help, the signal-to-noise ratio (SNR) that results from $s$ to $d$ can be expressed by

$$\Gamma_{DT} = \frac{P_s G_{s,d}}{\sigma^2}. $$
For the amplify-and-forward (AF) cooperation transmission, in Phase 2, the relay amplifies $Y_{s,r}$ and forwards it to the destination with transmitted power $P_r$. The received signal at the destination is

$$Y_{r,d} = \sqrt{P_r} Y_{s,r} + n'_{d}$$  \hspace{1cm} (4)

where

$$X_{r,d} = Y_{s,r} \frac{1}{| Y_{s,r} |}$$ \hspace{1cm} (5)

is the energy-normalized transmitted signal from the source to the destination at Phase 1, $G_{s,r}$ is the channel gain from the relay to the destination, and $n'_{d}$ is the received noise at Phase 2. Substituting (2) into (5), we can rewrite (4) as

$$Y_{r,d} = \sqrt{P_r} Y_{s,r} \frac{X_{r,d} + n_{r}}{\sqrt{P_r} G_{s,r} + \sigma^2} + n'_{d}$$ \hspace{1cm} (6)

Using (6), the relayed SNR at the destination for the source can be obtained by

$$\Gamma_{s,r,d}^{AF} = \frac{P_r G_{s,r} G_{r,d}}{\sigma^2 (P_r G_{s,r} + P_r G_{r,d} + \sigma^2)}$$ \hspace{1cm} (7)

Therefore, by (3) and (7), we have the combined SNR at the output of maximal ratio combining (MRC) as

$$\Gamma_{r,d}^{AF} = \Gamma_{s,r,d}^{DF} + \Gamma_{s,r,d}^{AF}$$ \hspace{1cm} (8)

Notice that even though the SNR is improved, the bandwidth efficiency is reduced to half due to the half duplex of source transmission and relay transmission.

In the decode-and-forward (DF) cooperation transmission protocol, the relay decodes the source information in Phase 1 and retransmits to the destination in Phase 2. The destination combines the direct transmission information and relayed information together. The achievable rate can be calculated as follows:

$$R_{DF} = \max_{0 < r < 1} \min \{R_1, R_2\} = \log_2 \left( 1 + \Gamma_{r,d}^{DF} \right)$$ \hspace{1cm} (9)

where

$$R_1 = \log_2 \left[ 1 + \left( 1 - \rho^2 \right) \frac{P_r G_{r,d}}{\sigma^2} \right]$$ \hspace{1cm} (10)

$$R_2 = \log_2 \left[ 1 + \left( 1 - \frac{P_r G_{r,d}}{\sigma^2} + \frac{P_r G_{s,r} G_{r,d}}{\sigma^2} + \frac{2 P_r G_{s,r} G_{r,d} P_r}{\sigma^2} \right) \right]$$ \hspace{1cm} (11)

In this paper, we assume Rayleigh fading scenario. The bit error rate for a packet can be written as [20]

$$P_e = \frac{1}{2} - \frac{1}{2} \sqrt{\Gamma \frac{1+\Gamma}{1+1}},$$ \hspace{1cm} (12)

where $\Gamma$ is either $\Gamma_{s,r,d}^{DF}$ in (3), $\Gamma_{r,d}^{AF}$ in (7), or $\Gamma_{r,d}^{DF}$ in (9), depending on the transmission protocol. If each packet has the length of $L$ bits, the packet dropping rate is $1 - (1 - P_e)^L$.

Reed-Solomon (RS) Code is an important subclass of the nonbinary BCH error-correcting codes in which the encoder operates on multiple bits rather than individual bits. An RS code is specified as RS($N, M$). This means that the encoder takes $M$ data symbols and adds parity symbols to make an $N$-symbol codeword. There are $N - M$ parity symbols. An RS decoder can correct up to $t$ symbols that contain errors in a codeword, where $2t = N - M$. So by adapting $t$, we can have different level of channel protections. The coded BER can be closely bounded by [20]

$$P_{r}^{RS} \leq \frac{1}{2} \left[ 1 - \sum_{i=0}^{t} \binom{N}{i} (P_e)^i (1 - P_e)^{(N-i)} \right] .$$ \hspace{1cm} (13)

(Notice that BER and SER for RS code have the relation $BER/SER = 2^{(m-1)}/(2^m - 1)$. (13) is the performance bound which is accurate when $m$ is large.) Here we assume that the BER is equal to 0.5 if the number of errors is greater than $t$. RS codes can also be shortened to fit different coding length requirements.

### 3. Proposed Cooperative Video Transmission

In this section, we first propose our protocol in Section 3.1. Then an optimization problem is formulated to achieve the best performance in Section 3.2. We analyze the algorithm and discuss the implementation issues in Section 3.3 and Section 3.4, respectively.

#### 3.1. Proposed Cooperative Video Transmission Protocols

Currently, most of researches on cooperative transmission focus on data transmissions. However, video has different characteristics from generic data, such as decoding dependency and delay constraint. We propose cooperative protocols for video transmission to better utilize system resources for performance improvement. Moreover, because of the broadcast nature of the phase 1 in cooperative transmission, the source information is distributed over the relays without any cost. We can further improve end-to-end video quality via exploring source diversity using the idea of DSC. In Figure 1, we propose a cooperative video transmission protocol that can leverage the benefits of both cooperative transmission and the idea of DSC. Specifically, in the first phase, the source broadcasts the video to the relay and destination. In the second phase, the relay has two choices as follows.

1. The relay can use AF or DF to relay the packets of coarse contents of the video. The destination combines the direct transmission and relay transmission to improve the quality of the received video with error concealment.

2. The relay can transcode the received video to a coarse-quality video and then encode the coarse-quality video using a systematic Reed-Solomon code. Only the parity is transmitted to the destination. The destination decodes the video transmitted in the first stage and transcodes it using the same coding parameters used by the relay node to construct the coarse-quality video. This coarse-quality video will be combined with the parity check bits sent from the
relay to ensure the reconstructed coarse video quality which will be utilized for error concealment. (Notice that the relay might receive corrupted video packet. As a result, the relay might generate wrong parity bits and the performance at the destination can be impaired. To overcome this, source can use sufficient level of FEC to protect the video stream to transmit to relay and destination; so both relay and destination can have similar video quality. In other words, by carefully joint optimizing source coding and channel level along two paths, the final reconstructed video quality can be further improved.)

We can see that the proposed protocol explores not only the inherited spatial diversity and multipath diversity from cooperative transmission, but also the source diversity from the idea of DSC. Moreover, the proposed scheme is backward compatible, in the sense that the source-to-destination link is not modified. The current existing direct transmission scheme can coexist with the proposed scheme. This compatibility facilitates the deployment of the proposed cooperative video transmission.

In the protocol, we assume that a control channel is available for the destination to know the processing procedures used in the relay node. However, the exchanging information, such as the RS coding rate, requires minimal of communication cost and update frequency.

3.2. Optimization of Proposed Protocols. In this paper, we use 3D-SPIHT [16] as the video encoder, due to its advantage that SPIHT produces an embedded bitstream. Notice that if embedded video codecs are employed, the head segment of successful received packets serves as the coarse-quality version of the original video. Other video encoders can be implemented in a similar way.

Let us define $D_{\text{max}}$ as the distortion without receiving any packets, $\Delta D_k$ as the distortion reduction when receiving packet $k$ after successfully receiving packet 1, 2, ..., $k-1$, and $P_k^{(X)}$ as the probability that receiving all packets from packet 1 to $k$ successfully using protocol $X$. The estimated distortion can be written as

$$ED^{(X)} = D_{\text{max}} - \sum_{k=1}^{K} \Delta D_k P_k^{(X)}, \quad (14)$$

where $K$ is the maximal number of packets constrained by the bandwidth. Notice that in order to decode the $k$th packet, packet 1 to packet $k-1$ must be correctly decoded.

The problem is to optimize the power and bandwidth usage at the relay node under the system bandwidth and overall power constraints. For the power constraint, we assume the total overall power is bounded by $P_0$. For the bandwidth constraint, we suppose the source and relays share the same channel, the total number of packets transmitted from source and relay is $K$, and the packet length is $L$. So the total bandwidth is $W$, which is the constraint for both the direct source-destination transmission and relay-destination transmission. For any cooperative protocol, we suppose the relay sends a total of $K < K$ packets to the destination. Due to the bandwidth constraint, the direct transmission has...
only $K - \bar{k}$ packets for transmission instead. We define the bandwidth parameter as

$$\theta = \frac{\bar{k}}{K}. \quad (15)$$

Notice that the constraints are the sum of bandwidth and the sum of power, which are fair compared to the direct transmission without cooperation. If we consider individual constraints (such as $P_t \leq P_0$ and $P_r \leq P_0$), the performance of the proposed scheme would be better since the constraints are looser.

Moreover, if we also optimize the RS coding rate $\eta = M/N$, the problem can be formulated as

$$\min_{\theta,P_r,P_t,\eta} E[D], \quad (16)$$

s.t. \begin{align*}
\text{bandwidth constraint:} & \quad 0 \leq \theta < 1, \\
\text{power constraint:} & \quad P_t + P_r \leq P_0. \quad (17)
\end{align*}

The problem in (16) is a constrained optimization problem. The objective function $E[D]$ will be explained in the following subsection. The constraints are the bandwidth and power constraints which are linear. The objective functions for different protocols might not be linear. Some standard nonlinear approaches such as interior-point-method [21] can be employed to solve the problem.

### 3.3. Performance Analysis

In this subsection, we study the performance of different transmission protocols. We define $p_{sr}$ as the packet loss rate for sending a packet from source node to relay node, $p_{r,d}$ as the packet loss rate for sending a packet from the relay node to destination node, $p_{sd}$ as the packet loss rate for sending a packet from the source node to destination node, $p_{comb}$ as the packet loss rate for sending a packet from source node to destination node using combined decoding, and $p_{DSC}$ as the packet loss rate for sending parity check bits from the relay. For direct transmission, relay transmission without combined decoding and relay transmission with combined decoding using AF/DF, we suppose that all transmissions are protected by RS($L,M_1$), where $L$ is the packet length and $M_1$ is the message length.

#### 3.3.1. Direction Transmission

The power for the source is $P_0$. The successful transmission probability for receiving all correct packet 1 to packet $k$ can be written as

$$p_{k}^{(DT)} = (1 - p_{sd})^k, \quad (18)$$

where $p_{sd}$ can be calculated from (3), (12), and (13). The distortion is

$$E[D^{(DT)}] = D_{\max} - \sum_{k=1}^{K} \Delta D_k p_{k}^{(DT)}. \quad (19)$$

Notice that all bandwidth is used for direct transmission.

#### 3.3.2. Relay Transmission without Combined Decoding

We use equal power for the source and relay in this scenario. Using this protocol, the packet is lost if both the direct transmission and relay transmission fail. Thus,

$$p_{k}^{(RT)} = \begin{cases} 
1 - p_{sd}(1 - (1 - p_{sr})(1 - p_{r,d})), & k \leq \bar{k} = \frac{\theta W}{L} \\
\frac{p_{r}^{(RT)}}{\bar{k}} (1 - p_{sd})^{k - \bar{k}}, & K - \bar{k} \geq k > \bar{k}, \end{cases} \quad (20)$$

where the first case represents the situation where the relay retransmits the packets, while the second case represents the direct transmission only. The total transmitted packets from the source is reduced to $K - \bar{k}$, due to the relay transmission.

Then, we need to solve the following problem to achieve the minimal expected distortion:

$$E[D^{(RT)}] = \min_{(0 < \theta < 1)} E[D^{(RT)}(\theta)] = D_{\max} - \sum_{k=1}^{K - \bar{k}} \Delta D_k p_{k}^{(RT)}. \quad (21)$$

Clearly it can be solved by line-search over $\theta$.

#### 3.3.3. Relay Transmission with Combined Decoding Using AF/DF

For AF, $p_{comb}$ can be calculated by (8), (12), and (13). For DF, $p_{comb}$ can be calculated by (9), (12), and (13). It can be proved that the power constraint and bandwidth constraint in (16) can be decoupled without loss of optimality. Due to the page limitation, we omit the proof. We assume that the power is optimally allocated in this case. Similarly to the previous case, we can write

$$p_{k}^{(CD)} = \begin{cases} 
(1 - p_{comb})^k, & k \leq \bar{k} = \frac{\theta W}{L} \\
\frac{p_{r}^{(CD)}}{\bar{k}} (1 - p_{sd})^{k - \bar{k}}, & K - \bar{k} \geq k > \bar{k}. \end{cases} \quad (22)$$

The first case and second case have the same physical meaning as (20). Similar to (21), we can also write

$$E[D^{(CD)}] = \min_{(0 < \theta < 1)} E[D^{(CD)}(\theta)] = D_{\max} - \sum_{k=1}^{K - \bar{k}} \Delta D_k p_{k}^{(CD)}. \quad (23)$$

#### 3.3.4. Relay Transmission with Parity Check

In our proposed protocol, instead of sending the original packets from the source, the relay encodes using another RS code RS($L,M_2$), and sends the parity bits with length of $L - M_2$ only. The destination combines the direct transmission part of $M_2$ bits and the relay transmission bits to improve the link quality. In this case, $\theta = (L - M_2)/L$. Here we assume the equal power allocation for the source and relay. We can write

$$p_{k}^{(DSC)} = (1 - p_{DSC})^k, \quad (24)$$
where the packet error rate is the product of the successful packet transmission rate of source-to-relay path and the successful packet transmission rate after RS$(L,M_2)$ decoding from the source to the destination, that is,

$$p_{DSC} = 1 - (1 - P_{s,r})(1 - P_{RS}^{L,M_2})^t. \quad (25)$$

Define $t' = (L - M_2)/2$. We have the BER after the decoding of RS$(L,M_2)$ code for both direct transmission and relay transmission as

$$P_{RS}^{L,M_2} \leq \frac{1}{2} \left( 1 - \sum_{j=0}^{t'} \sum_{i=0}^{\frac{t'}{2}} \left( \frac{M_2}{i} \right) \left( P_{s,d} \right)^i \left( 1 - P_{s,d} \right)^{L-M_2-i} \right) \times \frac{L-M_2}{i} \left( P_{r,d} \right)^i \left( 1 - P_{r,d} \right)^{L-M_2-i}. \quad (26)$$

Here $P_{s,d}$ is the BER of direct transmission calculated from (3) and (12), and $P_{r,d}$ is the BER of transmission from the relay to the destination calculated from

$$\Gamma_{r,d} = \frac{P_r G_{r,d}}{\sigma^2} \quad (27)$$

and (12). We use the fact that the RS code can decode up to $t'$ errors in either direct transmission part or the relay transmission part in (26). Notice that in order to have the fair comparison with the other schemes, the direct transmission of this scheme is also protected by an RS code RS$(M_2,M_1)$, where $M_1$ is the length of original source bits per packet.

3.4. Implementation Consideration. It is possible to incorporate other video transcoding/processing algorithms into the proposed system. For example, we can use a transcoder to convert the received video into a lower-resolution and lower-quality version after the following processing.

1. Use a down-sampling algorithm to change the resolution of the image, for example, the QCIF image (176×144) can be converted to a 96×80 resolution image using 6/11 horizontal scalar and 5/9 vertical scalar. The scaling ratio is adjustable.

2. Use a QP for quantizing the DCT coefficients.

3. Use a truncation tool to adjust the SNR quality. The transcoded version is packed in a single packet and transmitted to the destination. A certain time of retransmission is allowed for this packet. Therefore, the scalar, QP, and truncation parameters in the transcoder side can be jointly optimized with the source coding parameters at the transmitter side to achieve the best performance. In the receiver side, the received packets from the main channel and the relay channel are used together to recover the original videos. The information sent by relay channel can help to recover the lost packets sent via the main channel. Of course, the uppersampling is needed for the lower-resolution image to get back to the original size.

The other issue is the complexity for coordination for resource allocation. The optimization in (16) is performed, and resource allocation parameters (such as bandwidth and power) are sent back to the source and relay. The size of the information (a few bytes in our case) is relatively trivial compared with the video packets.

4. Proposed Quasishare Auction Schemes for MultiUser Case

In the previous section, we study the single source-relay-destination case in which one relay tries to help one source-destination pair for the received PSNR. In this section, based on the proposed video cooperative protocol, we investigate multiuser case in which one relay tries to help a group of source-destination pairs to achieve the social optimum, that is, the overall PSNR. We first formulate the multiuser resource allocation problem for the relay node in Section 4.1. Then, the quasishare auction is proposed and analyzed in Section 4.2. Finally, we employ one approach in the literature to our problem as the performance bound in Section 4.3.

4.1. Multiuser Resource Allocation for Relay Node. We consider the full-fledged cooperative network, in which each node can serve as transmitter, relay, or receiver. To make problem a bit simpler, we assume that the nodes that play relay functions have been predetermined (so the relay node determination problem is not in the scope of this paper), so that each relay helps to connect a group of transmitters with a number of receivers as shown in Figure 2. In this paper, we suppose the cooperative transmission system has source node $s_i$, one relay node $r$ and destination node $d_i$.

Denote set $I$ as $I$ source-destination pairs accessing one particular relay node in the network. To achieve real-time transmission, the overall allocated transmission time slots for all nodes to transmit $I$ GOPs is set to the required time to playback one GOP. We reserve $\%$ of time slots for relay node. The rest of time slots are allocated to each source-destination pair equally. Due to the distributed location of the nodes, each source-destination pair experiences different channel condition in both direct and cooperative transmission path. Besides, since different sources transmit different video sequences and the contents are changing over time, the relay needs to dynamically adjust rate allocation to provide optimal video quality. The main issue is how to assign relay’s time slots to each source-destination pair for delivering side information to achieve overall optimal video quality. Define
\( \theta_i \) as the fraction of relay’s time slots assigned to source-destination pair \( i \) and \( \eta_i \) as the channel coding rate selected by the source \( i \). We can formulate the considered network within each GOP time scale as

\[
\min_{\theta_i, \eta_i} \sum_{i \in I} E[D_i(\theta_i, \eta_i)],
\]

subject to

\[
\sum_{i \in I} \theta_i \leq 1,
\]

\[
0 < \eta_i \leq 1, \quad \forall i \in I.
\]

By given one \( \theta_i \), the minimal achievable distortion for received video at destination \( i \) can be calculated as follows:

\[
ED_{n,r,d_i}(\theta_i) = \min_{\eta_i} E[D_i(\theta_i, \eta_i)].
\]

The problem in (30) can be solved locally in each source.

For the relay, the resource allocation problem is to optimize the overall distortion by dividing the relay’s resources, which are the time slots. The problem can be formulated as

\[
\min_{\theta_i} \sum_{i \in I} ED_{n,r,d_i}(\theta_i),
\]

subject to

\[
\sum_{i \in I} \theta_i \leq 1.
\]

In the next two subsections, we discuss two solutions to solve the problem in (31).

4.2. Proposed Quasishare Auction. In this subsection, we find a distributed solution to solve (28). Due to the distributed nature, different source-destination pairs try to optimize their own performances in a noncooperative way. Notice that this noncooperation is between the different source-destination pairs, and cooperative transmission is employed for the relay transmission. Based on the idea of share auction, we propose a quasishare auction that takes advantage of setting for cooperative video transmission. The rules of the quasishare auctions are described below:

(i) Information. Public available information includes noise density \( \sigma^2 \) and bandwidth \( W \). The relay also announces a positive reserve bid \( \beta \) (or reserve price in some literature) and a price \( \pi > 0 \) to all sources. Each source \( i \) also knows the channel gains along direct and cooperative transmission path, namely, \( G_{s_i,d_i}, G_{n,s_i}, \) and \( G_{r,d_i} \).

(ii) Bids. Source \( i \) submits \( b_i \geq 0 \) to the relay.

(iii) Allocation. Relay allocates proportions of time slot for source-destination pair \( i \) according to

\[
\theta_i = \frac{b_i}{\sum_{j \in I} b_j + \beta}.
\]

(iv) Payments. In our case, source \( i \) pays the relay \( C_i = \pi \theta_i \).

A bidding profile is defined as the vector containing the sources’ bids, \( \mathbf{b} = (b_1, \ldots, b_I) \). The bidding profile of source \( i \)’s opponents is defined as \( b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_I) \), so that \( \mathbf{b} = (b_i, b_{-i}) \). Each source \( i \) chooses bid \( b_i \) to maximize its payoff

\[
S_i(b_i; b_{-i}, \pi) = \Delta E[D_i(\theta_i(b_i; b_{-i}))] - C_i(b_i; b_{-i}, \pi),
\]

where

\[
\Delta E[D_i(\theta_i(b_i; b_{-i}))] = E[D_{n,r,d_i}(0)] - E[D_{n,r,d_i}(\theta_i(b_i; b_{-i}))].
\]

Each source chooses its price to maximize its payoff function in (34). If the price is increased, then from (33), the relay allocates more time slots to this user. As a result, the distortion is reduced. However, the cost \( C_i \) also increases. Consequently, if the other users do not change their prices, there is an optimal point to set the price.

Although video’s rate-distortion (RD) curve is often a convex decreasing function; however, (35) is generally not a concave increasing function owing to applying optimization over all possible channel coding rate for each \( \theta_i \) in (30). Notice that above payoff function for the quasishare auction is similar to the soul of “Pricing Anarchy,” in which the users pay the tax for their usage for the system resources.

Here, we omit the dependency on \( \beta \). If the reserve bid \( \beta = 0 \), then the resource allocation in (33) only depends on the ratio of the bids. In other words, a bidding profile \( \mathbf{b} \) for any \( k > 0 \) leads to the same resource allocation, which is not desirable in practice. That is why we need a positive reserve bid. However, the value of \( \beta \) is not important as long as it is positive. For example, if we increase \( \beta \) to \( k\beta \), then sources can just scale \( \mathbf{b} \) to \( k\mathbf{b} \), which results in the same resource allocation. For simplicity, we can simply choose \( \beta = 1 \) in the practice.

In (34), if the others’ bids \( b_{-i} \) are fixed, source \( i \) can increase its time slot \( \theta_i \) in (33) by increasing \( b_i \). As a result, the distortion is reduced and \( \Delta E[D_i] \) is improved. However, the payoff faction needs to pay the price for \( \theta_i \). Depending one different price per unit \( \pi \) announced by the relay, there are three different scenarios:

1. If \( \pi \) is too small, the payoff function \( S_i \) in (34) is still an increasing function. As a result, the source tries to maximize its own benefit by setting price high. Consequently, \( b_i \to \infty \).
2. If \( \pi \) is too large, the payoff function \( S_i \) is a decreasing function. As a result, the source would not participate in the bidding by setting \( b_i = 0 \).
3. If \( \pi \) is set to the right value, the payoff function \( S_i \) is a quasi-concave shape function, that is, it increases first and then decreases within the feasible region. Consequently, there is an optimal \( b_i \) for the source to optimize its performance.

Based on the observation above, the quasishare auction algorithm is shown as follows. The relay conducts line search for \( \pi \) from the situation in which \( b_i = 0, \forall i \) to the situation in which \( b_i = \infty, \forall i \). For each \( \pi \), different sources set bids
to optimize their own performances and report the expected distortion to the relay. By doing so, the computation is distributed to each source node. Among all \( \pi s' \), the relay selects the solution with the best overall system performance and announces the final \( \theta_i \) to each source \( i \).

Compared with the share auction and the proposed quasishare auction, the final results are the same if the bid update for share auction can be obtained and \( \Delta E[D_i] \) is a concave increasing function. For data communication, the bids can be updated in a close form. However, due to the complexity to express the cooperative video end-to-end distortion, the close form update function cannot be obtained. As a result, we can only apply the quasishare auction for the video cooperative transmission.

4.3. Performance Upper Bound. In this subsection, we investigate a performance upper bound similar to the VCG auction proposed in the literature and compared with our proposed approach. (Notice that the VCG auction [22–24] is not the contribution of this paper. Moreover, we do not claim any efficiency result in a repeated dynamic setting, where more sophisticated strategies can be adopted.) In the performance upper bound, the relay asks all sources to reveal their evaluations of the relay's time slots, upon which the relay calculates the optimal resource allocation and allocates accordingly. A source pays the "performance loss" of other sources induced by its own participation of the auction. In the context of cooperative video transmissions, the performance upper bound can be described as follows.

(i) Information. Public available information includes noise density \( \sigma^2 \) and bandwidth \( W \). Source \( s_i \) knows channel gain \( G_{s_i,d} \). The relay knows channel gains \( G_{r,d} \) for all \( i \) and can estimate the channel gains \( G_{s_i,r} \) for all \( i \) when it receives bids from the sources.

(ii) Bids. Source \( s_i \) submits the function \( \Delta Q_i(\theta_i, G_{s_i,r}, G_{r,d}) \) to the relay, which represents the distortion decrease as a function of the relay parameter \( \theta_i \) and channel gains \( G_{s_i,r} \) and \( G_{r,d} \):

(iii) Allocation. The relay determines the time slot allocation by solving the following problem (for notational simplicity we omit the dependence on \( G_{s_i,r} \) and \( G_{r,d} \)),

\[
\theta^* = \arg\max_{\theta} \sum_{j \in I} Q_i(\theta_j) \quad (36)
\]

(iv) Payments. For each source \( i \), the relay solves the following problem.

\[
\theta^{*i} = \arg\max_{\theta, \theta_j = 0} \sum_{j \in I} Q_j(\theta_j), \quad (37)
\]

that is, the total distortion decreases without allocating resource to source \( i \). The payment of source \( i \) is then

\[
C_i = \sum_{j \neq i \in I} Q_j(\theta^{*i}_j) - \sum_{j \neq i \in I} Q_j(\theta^{*}_j), \quad (38)
\]

which is the performance loss of all other sources because of including source \( i \) in the allocation. Source \( i \) in the performance upper bound obtains the payoff function as

\[
Y_i = \Delta D_i(\theta_i) - C_i. \quad (39)
\]

Although a source can submit any function it wants, it has been shown [22] that it is a (weakly) dominant strategy to bid truthfully, that is, revealing the true function form of its distortion decrease

\[
Q_i(\theta_i) = \max \{D_{s_i,r,d}(0) - D_{s_i,r,d}(\theta_i), 0\}. \quad (40)
\]

As a result, the resource allocation of the performance upper bound as calculated in (36) achieves the efficient allocation [22]. Note that the sources do not need to know global network information, that is, no need of knowing the channel gains related to other sources in the network. The auction can achieve the efficient allocation in one shot, by allowing the relay to gather a lot of information and perform heavy but local computation.

Although the performance upper bound has the desirable social optimal, it is usually computationally expensive for the relay to solve \( I + 1 \) nonconvex optimization problems. To solve a nonconvex optimization, the common solution like interior point method needs a complexity of \( O(I^2) \). As the result, the overall complexity for the performance upper bound is \( O(I^3) \), while the proposed quasishare auction has linear complexity. Furthermore, there is a significant communication overhead to submit \( Q_i(\theta_i) \) for each source \( i \), which is proportional to the number of source nodes and reserved time slot for relay node. In the proposed scheme, the bids and the corresponding resource allocation are iteratively updated. This is similar to the distributed power control case, where the signal-to-interference-noise ratio and power update are iteratively obtained. As a result, the overall signalling can be reduced.

5. Simulation Results

In order to test the proposed scheme, we set up two sets of simulations. First, we study the single source-relay-destination case for our proposed cooperative video transmission. Then, we investigate the multiuser case for the proposed resource allocation using auction theory.

5.1. Single Source-Relay-Destination Case. The overall power is \( P_0 = 0.2 \) W, the noise power is \(-100 \) dbm, and the propagation factor is 3. The source is located at the origin and the destination is located at (1000 m, 0 m). The relay is moved from the (100 m, 400 m) to (900 m, 400 m). The packet length is \( L = 255 \). Two tested video streams are Foreman and News in QCIF resolution (176 × 144) with refresh rate 30 frames per second.

In Figures 3 and 4, we show the PSNR as a function of the relay location for video News and video Foreman, respectively. Here we normalize the relay location in x-axis over the distance from the source to the destination. From the figures, we can see that the direct transmission has the worst performance and generates the unacceptable
reconstructed video quality. The cooperative transmission without combining of SNR at the receiver has the best performance when the relay is located at the middle of the source and the destination. For the AF protocol, the best performance is achieved when the relay is relatively close to the destination; for the DF protocol, the optimal relay location is toward the source, and the DF protocol has better performance than the AF protocol when the relay is close to the source. These facts are very different from the data domain cooperative transmission. Finally, the relay transmission with parity check (shown as coded coop) has superior performance (about 4 dB gain) than the other protocols when the source and relay are close. When the relay is far away from the source, its performance degrades fast. This is because the performance is impaired by the source-relay link. On the whole, the proposed cooperative protocols achieve better performances than those of direct transmission, and the characteristics of the performance improvement are very different from the data domain cooperative protocols.

In Figure 5, we show the RS code rates for different videos of inner code (RS code for direct transmission) and outer code (RS code for relay transmission), respectively. We can see that the inner coding rate is reduced when the relay is far away from the source. This is because the source-relay channel needs more protection. On the other hand, when the relay is close to the source, the relay-destination link is protected more by the lower outer RS code rate. This two-level RS codes provide the cooperative video transmission scheme with additional 4 dB gain in video quality.

Notice that the proposed cooperative system will not always perform well in every relay location. The location of relay needs to be close to the source-destination link. Otherwise, the cooperative transmission will not work, in the sense that the optimization in (16) degrades to traditional direct transmission with $\theta = 0$.

The other concern is to study which protocol fits a certain situation best. For the AF/DF protocol, the received SNR can be significantly increased. This is especially true for low SNR case. However, the signal needs to be stored in the receiver for combining at the second stage. This increases the implementation cost. For the relay transmission without combined decoding, the implementation cost is minor, but it has inferior performance when the SNR is low. The proposed scheme with parity check provides an improvement over the relay transmission without combined decoding in a cost effective manner. However, the relay needs to be close to the source to ensure a the good source-relay channel.
5.2. Multiple User Case. For multiple user case, the simulations are setup as follows. The power for all source nodes and relay node is 0.1 W, the noise power is $5 \times 10^{-10}$ W, and the propagation factor is 3. Source node 1 to node 3 are located at $(-400 \text{ m}, 0 \text{ m})$, $(-300 \text{ m}, 50 \text{ m})$, and $-200 \text{ m}, -20 \text{ m}$, respectively. The corresponding destination node 1 to 3 are located at $(200 \text{ m}, 0 \text{ m})$, $(400 \text{ m}, 100 \text{ m})$, and $(300 \text{ m}, 30 \text{ m})$, respectively. The relay is located at the origin. We reserve 30% of bandwidth for relay to transmit the parity check bits. The packet length is $L = 255$. We use a 3D-SPHIT [16] codec to compress video sequence in QCIF resolution $(176 \times 144)$ with refresh rate 30 frames per second. The GOP is set to 16, and each source node will transmit 10 GOPs to its corresponding destination node. To evaluate the performance under different video content and different level of motion activity in the video sequence, we compare three different sets of video sequences. The first set consists of low motion video sequences: news, grandma, and akiyo. The second set contains high motion video sequences stefan, foreman, and coastguard. The third set contains mixed level of motion video sequences, including silent, foreman, and news.

To demonstrate that the proposed scheme can utilize the relay’s bandwidth effectively to achieve better perceptual quality, we compare the constant bit rate (CBR) scheme which allocates equal amount of time slots for relay to transmit parity bits for each video source. In Figure 6, we show the average PSNR gain when we compare the proposed scheme and the CBR scheme for all three video sets. As we can see, the proposed scheme can have PSNR gain between 0.8 dB and 1.3 dB when the received video quality is between 30 dB and 40 dB, which is a noticeable quality improvement. The performance gain achieved by the proposed scheme is mainly contributed by jointly leveraging the diversity of different video source RD characteristics and nodes’ channel conditions, and dynamically allocating suitable amount of time slots to each video source. To further assess the impact of different level of motion activities, we show the PSNR performance for three different video sets in Figure 7. As revealed, the performance gain is consistent for all levels of motion activities owing to the dynamic resource allocation.

To evaluate how close the performance of the proposed scheme can approach to the optimal solution, we list the
PSNR difference between the proposed scheme and the optimal solution in Tables 1, 2, and 3. As shown in these three tables, the performance loss is only between 0.1 dB and 0.3 dB. Note that the computation complexity and the communication overhead to obtain the optimal solution are extremely high. The proposed distributed scheme can achieve similar video quality by requiring much lower computation and communication overhead.

6. Conclusions

In this paper, we propose the cooperative video transmission protocols using auction theory. The source broadcasts its information in the first stage. The relay can either retransmit the low quality video packets or transmit the coded parity bits instead. The destination uses this relay information as side information to improve the quality of video transmission. We formulate the problem as to minimize the estimated distortion under the power and bandwidth constraints. Four different cooperative schemes are compared for the performance improvement over different scenarios and for the implementation concerns. The proposed video cooperative scheme has the best performance among all schemes, if the source and relay are closely located together. For multiuser case, we further propose the resource allocation for the relay to improve the multiuser cooperative video transmission using quasishare auction. Specifically, based on the observation of available information, we propose a quasishare auction for the relay to allocate the transmission time slots with improved signaling and convergence. Compared to the performance upper bound which is complicated and unpractical, the proposed quasishare auction can reduce the complexity, while the performance gap is only 0.1 dB to 0.3 dB.

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| Bandwidth (kbps) | Optimal (dB) | Proposed (dB) | Gap (dB) | Optimal (dB) | Proposed (dB) | Gap (dB) |
|-----------------|-------------|--------------|---------|-------------|--------------|---------|
| 191.25          | 30.65       | 30.49        | 0.16    | 478.12      | 34.34        | 0.17    |
| 573.75          | 36.89       | 36.61        | 0.28    | 765.00      | 38.62        | 0.27    |
| 956.25          | 40.54       | 40.30        | 0.24    |             |              |         |
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Research Article

Modeling Misbehavior in Cooperative Diversity: A Dynamic Game Approach

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Cooperative diversity protocols are designed with the assumption that terminals always help each other in a socially efficient manner. This assumption may not be valid in commercial wireless networks where terminals may misbehave for selfish or malicious intentions. The presence of misbehaving terminals creates a social-dilemma where terminals exhibit uncertainty about the cooperative behavior of other terminals in the network. Cooperation in social-dilemma is characterized by a suboptimal Nash equilibrium where wireless terminals opt out of cooperation. Hence, without establishing a mechanism to detect and mitigate effects of misbehavior, it is difficult to maintain a socially optimal cooperation. In this paper, we first examine effects of misbehavior assuming static game model and show that cooperation under existing cooperative protocols is characterized by a noncooperative Nash equilibrium. Using evolutionary game dynamics we show that a small number of mutants can successfully invade a population of cooperators, which indicates that misbehavior is an evolutionary stable strategy (ESS). Our main goal is to design a mechanism that would enable wireless terminals to select reliable partners in the presence of uncertainty. To this end, we formulate cooperative diversity as a dynamic game with incomplete information. We show that the proposed dynamic game formulation satisfied the conditions for the existence of perfect Bayesian equilibrium.

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1. Introduction

Cooperative wireless communications is based on the principle of direct reciprocity where wireless terminals attain some of the benefits of multiple input multiple output (MIMO) systems through cooperative relaying, that is, by helping each other. Since direct reciprocity is “help me and I help you” kind of protocol, a terminal will be motivated to help others attain cooperative diversity gain with the anticipation to reap those same benefits when the helped terminals reciprocate. When all terminals obey rules of cooperation, a stable and socially efficient cooperation is realizable, which may be true in wireless networks under the control of a single entity wherein terminals cooperate to achieve a common objective, as in military tactical networks. On the other hand, in commercial wireless networks where terminals are individually motivated to cooperate, the assumption that terminals will always obey rules of cooperation may not hold: (1) terminals may misbehave and violate rules of cooperation to reap the benefits without bearing the cost, (2) well-behaved terminals may refuse to relay for their potential partners without the assurance that the partners will reciprocate. While the first reason is motivated by a selfish intention to save energy, the second reason is motivated by the absence of mechanisms to incentivize cooperation in existing cooperative protocols. Hence, in commercial wireless networks, it is difficult to ensure a stable and socially efficient cooperation without implementing a mechanism to detect and mitigate misbehavior.

Game theoretic approaches have been proposed to design mechanisms that incentivize cooperation in commercial wireless networks. The proposed mechanisms belong to either price-based or reputation-based schemes. In price-based cooperation [1, 2], terminals are charged for channel
use when transmitting their own data and get reimbursed when forwarding for other terminals. It is shown that the pricing scheme leads to a Nash equilibrium that is Pareto-superior. In reputation-based schemes [3, 4], the authors proposed Generous Tit for Tat (GTFT) algorithm which conditions the behavior of nodes based on their past history. The authors showed that if the game is played long enough, GTFT leads to an equilibrium point that is Pareto-optimal. The game theoretic models in the aforementioned works in particular and in literature in general, consider a static game model where players are assumed to make decisions simultaneously. Since simultaneous decision making implies that players are unable to observe each other’s actions, static game models do not capture well dynamics of cooperative interactions. Recently a dynamic Bayesian game framework has been proposed to model routing in energy constrained wireless ad hoc networks [5], which provides the motivation for our work.

Motivated by the inadequacy of static game models to fully characterize cooperative communications, we formulate interactions of terminals in cooperative diversity as a dynamic game with incomplete information. The dynamic game formulation captures temporal and information structure of cooperative interactions. Temporal structure of a dynamic game defines the order of play: cooperative transmissions occur in sequential manner wherein a source terminal transmits first and then potential cooperators decide to either cooperate or deviate from cooperation. The sequential nature of cooperative transmissions is dictated by the half-duplex constraint of wireless devices, that is, a relay terminal cannot receive and transmit at the same time in the same frequency band. The information structure of dynamic games characterizes what each player knows when it makes a decision: in commercial wireless networks, intention of each user is not known a priori, hence, incomplete information specification of the game represents the uncertainty each user has about the intention of other users in the network. In this paper, we present a general dynamic game framework that may fit any of the existing cooperative diversity protocols. We show that the proposed model captures important aspects of existing cooperative diversity protocols. We also show that the proposed dynamic game formulation satisfies the requirements for the existence of perfect Bayesian equilibrium.

This paper is organized as follows. In Section 2, the system model is described. In Section 3, game theoretic analysis of cooperative diversity is presented. Background of dynamic games is presented in Section 4. In Section 5, a dynamic game framework is presented. Finally, in Section 6, concluding remarks are given.

2. System Model

We consider $N$-user TDMA-based cooperative diversity system wherein terminals forward information for each other using any one of the existing cooperative schemes. We assume that a source terminal randomly selects utmost one potential cooperator (relay) among all its neighboring terminals. It is important to note that random selection of potential cooperators indicates the assumption held by all terminals that their relay terminals are always willing to help. A source terminal and its potential partner establish a possible cooperative partnership prior to data transmission by exchanging control frames. Through the established cooperative partnership, terminals enter into a nonbinding agreement to forward information for each other (see Figure 1). Details of the mechanism by which cooperative partnerships are formed is beyond the scope of this work as our primary focus is on examining the sustainability of this partnership.

The interterminal channels are characterized by Rayleigh fading. We denote by $y_{sd}, y_{sr}, y_{rd}$ instantaneous signal-to-noise ratio (SNR) of source-destination, source-relay, and relay-destination channels. Information is transmitted at a rate of $R$ b/s in a frame length of $M$-bits. We assume that all users transmit at the same power level and modulation/rate.

3. Game Theoretic Analysis of Cooperative Diversity

3.1. Two-User Cooperation. In this section, we examine the cooperative interaction between terminals within the framework of noncooperative game theory. We assume that the benefits of cooperation and the cost it incurs are common knowledge. That is, terminals are willing to expend their own resources to help other terminals achieve reliable communication with the expectation to achieve those same benefits when the helped terminals reciprocate. We assume that terminals are individually rational in that terminals behave in a manner to maximize their individual benefits from cooperation. We assume rational behavior of terminals is common knowledge, that is, terminals know that other terminals are rational. Individuality rationality is crucial for the evolution of cooperation as it states that well-behaved terminals have strong preference for partners that conform to rules of cooperation. On the other hand, individual rationality may lead to selfish behavior where a terminal is tempted to economize on cost of cooperation (energy) while reaping the benefits. We show that in the presence of selfish users, individual rationality dominates cooperation which would consequently lead to a noncooperative Nash equilibrium that is suboptimal in the Pareto sense.

We denote the strategy available to all terminals by $\Theta$ where $\Theta \in \{\theta_0 = \text{cooperate}, \theta_1 = \text{misbehave}\}$, that is, $\Theta$ is
the strategy space of the game. Source terminal \( S_i \) transmits to the network whenever it has information to send. Thus, its strategy space is a singleton and is denoted by \( \Theta_i \). On the other hand, relay terminal \( R_j \) may either obey the rules of cooperation or deviate from it. Thus, the strategy space of \( R_j \) is a nonsingleton set which is defined as \( \Theta_j = \{ \theta_0 = \text{cooperate}, \theta_1 = \text{misbehave} \} \), where \( \theta_j \in \Theta_j \) is pure strategy of \( R_j \). We assume that a misbehaving relay node \( R_j \) adopts mixed strategy where it plays pure strategy \( \theta_j \) with probability \( p_j(\Theta_j) \). It is obvious that mixed strategy incurs uncertainty in the game since source terminal \( S_i \) has no knowledge whether \( R_j \) conforms to cooperation or violates it. Terminal \( R_j \) being a rational player will adopt this strategy to confuse its partner by mimicking the unpredictable nature of the wireless channel. From a game-theoretic viewpoint, mixed strategy ensures that the game has Nash equilibrium.

The utility function of terminal \( S_i \) is defined in terms of cooperative diversity gain which is denoted by \( u_i(p_i(\Theta_i), p_j(\Theta_j)) \), where \( p_i(\Theta_j) \) captures behavior of its partner. In the next section, we formally define the utility function for cooperative diversity in terms of achievable performance gains at the physical layer. For the purpose of simplifying the discussion in this section, achievable cooperative diversity gain when all terminals obey the rules of cooperation is denoted by \( \rho_c \). On the other hand, when all terminals opt out of cooperation, each terminal derives a degraded cooperative diversity gain compared to the attainable benefit; this utility is denoted by \( \rho_{nc} \) where obviously \( \rho_{nc} < \rho_c \). We assume that each terminal expends a fraction of its available power for cooperation, which defines the cost of cooperation and is denoted by \( \rho_c \). We assume that the cost of cooperation is strictly less than the attainable cooperation benefit, that is, \( \rho_c < \rho_c \). The utility matrix of the game is then

\[
U = \begin{pmatrix}
\rho_c - \rho_c & \rho_{nc} - \rho_c \\
\rho_c & \rho_{nc}
\end{pmatrix},
\]

where \( \rho_c - \rho_c \) is the net utility when all terminals cooperate, \( \rho_{nc} - \rho_c \) is the utility to a well-behaved terminal when its partner deviates from cooperation. The terminal that deviates from cooperation derives utility \( \rho_c \) at no cost and \( \rho_{nc} \) is the noncooperative utility.

Suppose terminals \( i \) and \( j \) form cooperative partnership where each terminal affirms its willingness to cooperate via a protocol handshake. A willingness to cooperate may indicate that a terminal has enough available power to expend for cooperation. It may also indicate a terminal’s intention to economize on the other terminal’s cooperative behavior. We assume that both terminals \( i \) and \( j \) play mixed strategies when each terminal acts as a relay to help the other terminal. Their mixed strategies, respectively, are

\[
P_i = \left[ p_i(\theta_0) p_j(\theta_1) \right], \quad P_j = \left[ p_j(\theta_0) p_j(\theta_1) \right],
\]

where \( p_j(\theta_0) \) is the probability with which relay terminal \( R_j \) cooperates with source terminal \( S_i \), and \( p_j(\theta_1) \) is probability of misbehavior. Similarly \( p_j(\theta_0) p_j(\theta_1) \) capture probabilities of cooperation and misbehavior when terminal \( i \) acts as a relay to terminal \( j \). The expected net utility function of each terminal can be shown as

\[
u_i(p_i(\Theta_i), p_j(\Theta_j)) = P_i U P_j^T = p_i(\theta_0) \rho_c + (1 - p_i(\theta_0)) \rho_{nc} - p_i(\theta_0) \rho_c,
\]

(3)

where \( [ \cdot ]^T \) is the transpose operator.

When both terminals obey the rules of cooperation \( (p_i(\theta_0) = 1, p_j(\theta_0) = 1) \), each derives a net utility of \( \rho_c - \rho_c \). We examine next steady-state behavior of the game when either player deviates from cooperation by adopting mixed strategy. Let us consider the case where terminal \( j \) is a potential cooperator that plays mixed strategy \( P_j \). The goal of an individually rational and mixed strategy playing terminal \( j \) is as follows: (1) maximize its net expected utility by minimizing the cost of cooperation and (2) behave in a manner that make it difficult for terminal \( i \) to distinguish between effects of channel dynamics and misbehavior. Thus, terminal \( j \) strategically selects \( P_j \) (mimicking inherent uncertainty of the wireless channel) in such a way that player \( i \) is indifferent in expected net utility. That is, player \( j \) chooses a mixed strategy where player \( i \) would achieve the same expected utility irrespective of the strategy terminal \( j \) plays. If such a mixed strategy exists, it means that in the long-run terminal \( i \) may be unable to learn about the behavior of its partner.

However, terminal \( i \) is a rational player and will learn in the long run about the behavior of its potential partner by observing its utility. In wireless communications, quality of service metrics such as target frame error rate (FER) helps terminals determine degradation in achievable cooperative diversity performance gain. Thus, there is no \( P_j \) that will make terminal \( i \) indifferent in expected utility. Due to the lack of indifferent strategy that could confuse its partner, rational player \( j \) will reason that it can forgo the cooperation cost (i.e., \( p_j(\theta_0) = 0 \)) in order to maximize its expected net utility. It is obvious to see from (3) that if player \( i \) is well behaved \( (p_i(\theta_0) = 1) \) and player \( j \) misbehaves \( (p_j(\theta_0) = 0) \), player \( i \) would derive net expected utility of \( u_i(p_i(\Theta_i), p_j(\Theta_j)) = \rho_{nc} - \rho_c \). On the other hand, the misbehaving partner \( j \) would achieve expected utility \( u_j(p_i(\Theta_i), p_j(\Theta_j)) = \rho_{nc} \). Note that \( (1 - p_j(\theta_0)) \rho_c \) is an amount of energy terminal \( j \) saves by misbehaving.

Similarly, for the case of mixed strategy play by terminal \( i \), the same arguments can be applied to show that there is no \( P_i \) that will make player \( j \) indifferent in expected net utility, which indicates that a selfishly rational player \( i \) will also be tempted to forgo the cooperation cost (i.e., \( p_i(\theta_0) = 0 \)) to derive a net expected utility \( u_i(p_i(\Theta_i), p_j(\Theta_j)) = \rho_c \). Thus, an individually rational terminal \( i \) will play \( p_i(\theta_0) = 0 \) to achieve the highest utility irrespective of the strategy adopted.
3.2. Evolution of Selfish Behavior. We consider a cooperative diversity system comprised of a population of terminals that interact randomly to attain cooperative diversity gain. We assume that at any given time a terminal can interact only with utmost one partner in the population. Due to mobility, we assume that every terminal $i$ interacts at least once with every other terminal $j, i \neq j$.

Suppose that initially the population conforms to cooperation. Now assume that a small group of selfish terminals (mutants) enter the cooperative diversity system. The question we would like to answer is if the mutants can successfully invade the cooperative diversity system.

Let $nC$ denote the initial number of cooperators and $nM$ denote the number of mutants, note $nM \ll nC$. The rationale behind the presence of very few mutants is to show vulnerability of cooperative diversity to misbehavior (see Figure 3). We denote by $pC$ and $pM$ the fraction of cooperating and misbehaving terminals, respectively. In other words, $nC$ terminals cooperate with probability $pC$ while the rest of the terminals deviate from cooperation with probability $pM$. We assume that the population of cooperators and mutants play pure strategy. Although cooperators and mutants adopt pure strategy, the entire population plays mixed strategy. The mixed strategy probability vector of the population is

$$
P = \begin{bmatrix} pC & pM \end{bmatrix}^T.
$$

(4)

The utility matrix of the game is defined in (1).

We examine the interaction within the population within evolutionary game theory framework to characterize dynamics of the spread of misbehavior in multiuser cooperative diversity. Evolutionary game theory deals with constantly interacting players that adapt their behavior by observing their utilities. The evolution of strategies into higher utility yielding strategies is characterized by using replicator dynamics [7]. Replicator dynamics predicts the rate at which strategies that yield higher utilities spread through the network. Thus, for multiterminal cooperative diversity system with utility matrix $U$ and mixed strategy $P$ that varies continuously with time, the evolution of cooperation and misbehavior is given by the replication equation:

$$
\dot{pC} = pC \left[ (UP)_1 - p^T UP \right],
$$

(5)

$$
\dot{pM} = pM \left[ (UP)_2 - p^T UP \right],
$$

(4b)

where $\dot{x}$ denotes the derivative, $(UP)_1$ and $(UP)_2$ are expected utilities of cooperators and mutants, respectively:

$$
(UP)_1 = pCpC + pMnC - c_c,
$$

(6a)

$$
(UP)_2 = pCpC + pMnC.
$$

(6b)

The first term on the right-hand side in (6a) is utility derived from cooperating-cooperator cooperation while the second term on the right-hand side in (6a) is utility derived from cooperating-mutant cooperation; the third term is the cost incurred by cooperators. Similarly, the first term on the right
hand side in (6b) is utility derived from mutant-cooperator cooperation while the second term is from mutant-mutant cooperation which actually results in noncooperation. The average utility of the population is

$$P^T\text{UP} = p_c(p_c - c_c) + p_m\rho_m.$$

It is evident that cooperators derive utility that is strictly less than the average utility of the population, that is, $(\text{UP})_1 < P^T\text{UP}$. On the other hand, mutants reap utility that is well above the average, that is, $(\text{UP})_2 > P^T\text{UP}$. Dynamics of the game dictates that nodes observe their utilities and adapt to strategies that provide higher utilities. In other words, low-utility cooperators will start imitating strategy of mutants (their misbehaving partners) and forgo the cooperation cost in an attempt to achieve a higher utility. That is, low-utility cooperators will learn that they can do better at the expense of other nodes. Due to the absence of techniques to deter misbehavior, the number of misbehaving nodes (mutants) increases monotonically while the number of cooperators grows at a negative rate. This indicates that the mutants successfully invade a relatively larger population of well-behaved cooperators. A decrease in number of cooperators indicates a reduction in the number of nodes that selfish nodes will cheat on. The population will reach a steady state where there is no cooperator left to exploit. The network evolves to a noncooperative state where each node opts out of cooperation as shown in Figure 4. Thus, noncooperation is an evolutionary stable strategy (ESS) which means that the presence of a few misbehaving nodes can drive away cooperators from the Pareto optimal cooperative strategy. ESS is robust against coalition of cooperators that attempt to shift the equilibrium point toward cooperation. That is, a small number of cooperators cannot invade a population of misbehaving nodes. Thus, cooperation is an evolutionary unstable strategy. Hence, we have shown that the presence of misbehaving nodes impedes evolution of socially efficient and stable cooperation.

Hence, without establishing a mechanism to detect and mitigate effects of misbehavior, cooperative diversity will not evolve into a stable system in which users interact in a socially efficient manner to attain a Pareto efficient equilibrium. The game theoretic analysis presented in this section assumes a static game model where the order in which terminals make decisions has not been taken into account. Indeed, the order of play has no significance in the outcome of the analysis since the goal has been to give insight into effects of selfish behavior in existing cooperative schemes. While the static game model proves useful in the analysis, due to its simplicity it may not capture the underlying dynamics of cooperative schemes. Even though evolutionary game theory enables us to analyze dynamics of interaction of a population of nodes, it does not provide a framework to capture the complex structure of cooperative interactions. In the next section, we characterize cooperative communications within the dynamic Bayesian game framework which would enable us to develop mechanisms that ensure evolution of stable cooperation. The Bayesian dynamic game model fully captures relevant details of cooperative interactions between source and relay nodes. First we present background material on dynamic games.

4. Dynamic Games: Background

Dynamic games model a decision-making problem where the order of play and information available to each player are very significant to understanding the decision of each player [8, 9]. While order of play characterizes sequential interactions, information available to each player describes what each player knows when making decisions. For instance, cooperative interactions occur sequentially, that is, source terminals always transmit first and then relay terminals decide to either forward or drop the transmission. A dynamic game is represented in extensive-form [10].

In extensive form, a game is represented in a tree structure which describes the sequential interactions and evolution of the game. The root of the tree where the game begins is the initial decision node and is denoted by $\mathcal{D}$. A noninitial node $\mathcal{D}$
that has branches leading to and away from it is a decision node which may indicate end of a stage game and represent the sequence relation of the decision of the players [11]. A decision node with no outgoing branches is referred to as a terminal node and it is where the game ends.

A dynamic game is a multistage game, where a stage game is represented by one level on the tree. In the temporal domain, stages of the game are defined by time periods where the kth stage is denoted by tk [12]. A dynamic game with finite number of stages is referred to as a finite-horizon game where tk ∈ {0, 1, ..., K}; otherwise, it is an infinite horizon game, that is, tk ∈ [0, 1, ...].

4.1. Information Sets. The edges of the tree represent actions available at decision nodes that would lead to other decision nodes. The sequence of actions defines the path that connects decision nodes to each other (within a stage) or decision nodes to terminal nodes. The path for each stage game tk identifies history h(tk) of play during time period tk. Players may have uncertainty about history of the game which is referred to as a game of imperfect information. That is, when it is its turn to move, a player has no knowledge about the node the game has reached. This uncertainty is captured in a set of decision nodes the game can possibly reach. We refer to this set of decision nodes as information set and is denoted as h. Information sets identify information possessed by players [9]. For instance, in a game of perfect information where players have exact knowledge about history of the game, the information set is a singleton set, that is, for all h ∈ H, |h| = 1, where H is information set of the game. On the other hand, in a game of incomplete information where some players have private information, the information set is a non-singleton set for at least one of the players, that is, ∃h ∈ H, such that |h| > 1. An elliptic curve is drawn around a player to show its uncertainty about which node in the information set is reached, as shown in Figure 5.

In a game of incomplete information, the action taken by a player is a function of which decision node in its information set has been reached. We denote by A(h) the set of actions available to a player with information set h. The action taken by the player at stage game tk is denoted by a(tk) and it is a mapping from h to A(h), that is, a(tk) : h → A(h). In extensive form games, players may adopt random strategies at each information set. This is called behavior strategy wherein players assign probability measure over actions available at each information set. Behavior strategy is denoted by σ(a(tk) | h) where σ(a(tk) | h) ∈ Δ(A(h)), Δ(A(h)) probability distribution over A(h). For instance, in a cooperative network wherein every one obeys the rules of cooperation σ(a(tk) | h) = 1, which is pure strategy. Nature is usually introduced as a nonstrategic player that randomly informs players which decision node D in h has been reached. Figure 5 shows cooperative communications as a dynamic game. The initial node is a source terminal that transmits to the network. The two decision nodes represent potential cooperators where behavior of D1 is known perfectly as shown by its singleton information set, whereas D2 maintains private information that is not common knowledge in the network. Nature N randomly assigns decision nodes for player D2.

5. Cooperative Diversity as a Dynamic Game with Incomplete Information

We have shown that cooperation in wireless networks is characterized by social-dilemmas which ultimately impede the evolution of a socially efficient cooperation. It is evident that social-dilemmas are prevalent in commercial wireless networks where terminals violate rules of cooperation for selfish reasons. In the presence of heterogeneously behaving terminals, cooperators exhibit uncertainty about the intention of their potential partners which makes selection of a reliable partner challenging. Our goal is to develop a mechanism that would enable terminals strategically select reliable partners in the presence of uncertainty. To this end, we develop a framework in which cooperative communications is formulated as a dynamic game with incomplete information. Note that a dynamic game with incomplete information is a dynamic Bayesian game.

We consider a wireless communications system with a population of N terminals wherein terminals that are within transmission ranges of each other form a cooperative diversity system. We assume that benefits of cooperation and the cost it incurs are common knowledge. That is, terminals are willing to expend their own resources to help others achieve reliable communication with the expectation to achieve those same benefits when their partners reciprocate. Terminals are rational in that they behave in a manner to maximize their individual benefit of cooperation. We assume that terminals maintain private information pertaining to their behavior (i.e., to either cooperate or misbehave). Note that the problem formulation is general in that it is not tailored toward one particular cooperative diversity protocol. However, we may present examples based on a specific protocol for purposes of simplifying discussions.

We formulate cooperative communications as a finite-horizon discrete-time dynamic game. The game is discrete-time since each player is assumed to have a finite number of strategies [8]. Within each stage tk, k = 0, 1, ..., K, a source terminal and its potential cooperators (relay) interact repeatedly for a duration of T seconds. The assumption of multiple cooperative interactions within a stage game is intuitively valid since cooperative transmissions span

![Figure 5: Extensive form representation of a cooperative network.](image-url)
multiple time slots. The period $T$ for each stage game $t_k$ may be defined as the time it takes a cooperatively transmitted signal to reach its intended destination. We assume that duration of a stage game $T$ is long enough to average out effects of channel variation. It is obvious that a new stage game starts when a source terminal $i$ (i.e. $i \in \{1, 2, \ldots, N\}$) that has data to send begins transmitting to the network. We characterize next the behavior of every potential relay terminal $j$ and source terminal $i$ within a dynamic Bayesian game framework. Note that we use the terms relay and potential cooperator interchangeably. We next model selfish behavior of relay terminals within a dynamic Bayesian game framework. We then present a framework in which source terminals make optimal decisions.

5.1. Modeling Selfish Behavior. We assume each relay terminal $j$ maintains private information which corresponds to the notion of type in Bayesian games. The set of types available to relay terminal $j$ constitutes relay terminal’s type space defined as $\Theta_j = \{\theta_0 = \text{Coop}, \theta_1 = \text{Misbehave}\}$. Since every terminal $j$ either conforms to cooperation or deviates from it, $\Theta_j$ is also the global type space of the game. Following the notation of Bayesian games, type of player $j$ is denoted by $\theta_j$ while other players’ type is denoted by $\theta_{-j}$, where $\theta_j, \theta_{-j} \in \Theta_j$. We assume that types associated with each relay terminal are independent.

Type space of every relay terminal $j$ maps to an action space $A_j$ which defines a set of actions $a_j(t_k)$ available to player $j$ of type $\theta_j$. The set of actions $A_j$ defines information set $h_j$ of relay terminal $j$; in other words, $h_j$ maps to action space $A_j(h_j)$, that is, $a_j(t_k | h_j) : h_j \rightarrow A_j(h_j)$. Note that the change in notation is to show that the action selected by the relay is a function of the information set. We assume that type of terminal $j$ and the associated action $a(t_k | h_j)$ do not change within a stage game. Indeed, a relay that obeys rules of cooperation do not change its type at each stage game. On the other hand, a misbehaving relay may strategically change its type at the beginning of each stage game. In this paper we assume that a misbehaving relay adopts behavior strategy wherein it randomly changes its behavior from cooperation to misbehavior at each stage game. Behavior strategy assigns a conditional probability over $A_j$, that is, $\sigma_j = p(a_j(t_k | h_j)$). For completeness, we define history of the game at the beginning of stage game $t_k$ as $h(t_k) = (a(t_0), a(t_1), \ldots, a(t_{k-1}))$. It is intuitive to assume that a relay which violates rules of cooperation may not need to observe history of the game when it chooses its actions. The utility function of relay terminal of type $\theta_j$ is denoted as $u_j(\theta_j, \theta_{-j})$ where $\theta_{-j}$ is type of other terminals. Later in this section we give a formal definition of the utility function.

We present examples to elucidate the game theoretic framework we just introduced. Let us consider Amplify-and-Forward (AF) [13] cooperation protocol where a potential cooperator $j$ amplifies faded and noisy version of signal received from source terminal $i$ and forwards it to a destination. Suppose that an amplification factor that depends on the potential cooperator’s type and dynamics of the channel is defined as

$$\mathcal{B}(h_{i,j}, h_j) = \beta a_j(t_k | h_j),$$

where $\beta$ is amplification subject to power constraint at the relay and dynamics of the interuser channel denoted as $h_{i,j}$ [13]. On the other hand, $a_j(t_k | h_j)$ captures action taken by relay $j$ when one of the decision nodes in its information set is reached. We describe below various types of relay terminal $j$ which will give a significant insight into the dynamic game framework.

(1) First, we consider a cooperative network where every relay node $j$ obeys the rules of cooperation. This is a network where nodes cooperate for a common objective, that is, type of each relay node $j$ is $\theta_j = 0$. Consequently, the information set of each relay $j$ is a singleton set, that is, $|h_j| = 1$ and the corresponding action space is $A_j(h_j) = \{1\}$. Since relay node $j$ has deterministic behavior, it would play $a_j(t_k | h_j) = 1$ with probability $\sigma_j(t_k) = 1$, that is, it plays pure strategy (it always forwards). History at the end of stage game is $h_{i,j} = (a(t_0) = 1, a(t_1) = 1, \ldots, a(t_k) = 1)$. The amplification $\mathcal{B}(h_{i,j}, h_{i,j})$ is then a function of channel dynamics and power constraint at the relay, that is, $\mathcal{B}(h_{i,j}, h_{i,j}) = \beta$. The extensive form representation of this game is straightforward. We would like to point out that the dynamic game framework can be used to design a resource management for a cooperative network such as this one (see Figure 6).

(2) In the second example, we consider a cooperative network where relay nodes violate rule of cooperation in probabilistic manner. That is, relay node $j$ plays behavior strategy where it exhibits mixed behaviors of cooperation and selfishness. This is a network where nodes have uncertainty about the behavior of other nodes. In other words, relay node $j$ has private information, that is, type of relay node $j$ is $\theta_j = 1$. The relay has two strategies that it selects randomly, that is, it decides to either forward or refuse cooperation which means that it has two decision nodes

![Figure 6: Example 1. Extensive form representation of a cooperative network with perfect information; $R_j$, $R_i$, and $S_i$ denote cooperative relay nodes and $S_i$ denotes source node $i$. Note the absence of Nature in this network.](image-url)

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in its information set $h_j$, that is $|h_j| = 2$. Since the relay adopts behavior strategy, the action space is captured in random variable $\mathcal{A}_j(h_j)$ where $\mathcal{A}_j(h_j) = \{0, 1\}$. The adopted behavior strategy is defined as $\sigma_j(t_k) = p(a_j(t_k \mid h_j)$ where $p(a_j(t_k \mid h_j) \in \Delta(\mathcal{A}_j(h_j))$. $\Delta(\mathcal{A}_j(h_j))$ is probability measure over set of actions $\mathcal{A}_j(h_j)$. Randomly behaving relay either cooperates (i.e., $a_j(t_k \mid h_j) = 1$) with probability $\sigma_j$ or it deviates from cooperation (i.e., $a_j(t_k \mid h_j) = 0$) while with probability $1 - \sigma_j(t_k)$. Consequently, the amplification is a function of relay behavior and dynamics of the channel, that is, $\mathcal{B}(h_j, h_{Si}, R_j) \in \{0, \beta\}$. Note that in the special case where a relay always refuses to forward, that is, $\Theta_j = (0_1)$, $|h_j| = 1$, and $a_j(t_k \mid h_j) \in \mathcal{A}_j = \{0\}$ deterministically, thus $\mathcal{B}(h_j, h_{Si}, R_j) = 0$ (see Figure 7).

3 The third example is a continuation of the second example. Here we consider an intelligent and selfish relay $j$ of type $\theta_j = 1$. The relay is intelligent in the sense that it always forwards for its partner but at a randomly selected reduced power level. Obviously the relay has selfish intentions, that is, minimizing its cost-to-benefit ratio. We assume that selfish relay $R_j$ random selects a normalized power level $l$ from a finite set of power levels $L$, where $0 < l < 1$. Thus, information set of the relay is defined by the set of normalized power levels $L$, that is, $|h_j| = |L|$. The action space of the selfish relay is the set of power levels, that is, $\mathcal{A}_j(h_j) = (0, \ldots, 1)$. The behavior strategy is $\sigma_j(t_k) = p(a_j(t_k \mid h_j))$ where $a_j(t_k \mid h_j) = l$, $l \in L$. The amplification $\mathcal{B}(h_j, h_{Si}, R_j)$ is obviously determined by behavior of the relay and channel dynamics, where $\mathcal{B}(h_j, h_{Si}, R_j) = \{0, \ldots, \beta\}$. Note that a terminal which exhibits such ambiguous behavior may exploit dynamics of the channel to evade detection (see Figure 8).

The extensive form representation of Example 1 is straightforward since all information sets are singleton sets. On the other hand, for Examples 2 and 3 Nature $\mathcal{N}$ will assign decision nodes to relay $j$. The probability with which decision nodes are assigned is determined by the behavior strategy of the relay. The role of Nature can be justified within the context of behavior strategy. Since relay node $j$ plays behavior strategy, it requires a device that will randomly select a strategy from the possible set of strategies. Nature will play the role of this randomizing device and assign strategies at each stage of the game. We assume the amount of power relay expends for randomization is negligible compared to cost it would have incurred by cooperating. Although it is customary to put Nature at the beginning of a game, Kreps and Wilson [9] noted that moves of Nature may also be put anywhere on the game tree.

5.2. Behavior of Source Terminals. While introducing the model for selfish behavior in the previous subsection, we said that each relay maintains private information pertaining to its behavior. The private information and the sequential nature of cooperative interactions gives relay terminals a dominant position in deciding to either cooperate or misbehave. In other words, source terminals are vulnerable to defection by their partners. In this subsection, we present a framework for designing a technique where source terminals make optimal decisions in the presence of uncertainty.

It is evident that a stage game begins when a source terminal starts transmitting to the network. In the language of game theory, this means a source terminal makes the decision to transmit whenever it has information to transmit. In the extensive-form representation, a source terminal has only a single decision node which characterizes the decision to transmit. Thus, any source terminal $i$ has an information set that is a singleton. In other words, its decision node maps to an action space that is also a singleton, that is, $\mathcal{A}_i = \{1\}$, which implies that if a source terminal has data to send, it will transmit to the network with probability 1. Note $a(t_k \mid h_i) = 1$ captures the decision to transmit. It follows from the singleton information set that the type space of source
terminal \(i\) is also a singleton set. In the subsequent paragraphs we describe a framework for selecting reliable partners. We introduce the concept of belief which characterizes each source terminal’s level of uncertainty about the behavior of its potential partners.

**Definition 1.** Belief of source terminal \(i\) \(\mu_i^k(t_k)\) is a subjective probability measure over the possible types of relay terminal \(j\) given \(\theta_i\) and history \(h^i(t_k)\) at the beginning of stage game \(t_k\), that is,

\[
\mu_i^k(t_k) = p(\theta_j | \theta_i, h^i(t_k)).
\]  

(9)

We would like to point out that by maintaining belief, source terminals deviate from the assumption (as in existing cooperative protocols) that their partners are always willing to cooperate. Indeed, belief is a security parameter that characterizes the level of trust each terminal maintains on its potential partners. We assume that beliefs are independent across the network which is intuitively valid since beliefs are subjective measures of terminal behavior. We assume that every source terminal \(i\) maintains a strictly positive belief, that is, \(\mu_i^k(t_k) > 0\). This is intuitively valid in commercial wireless networks that are characterized by dynamic user population where it is difficult to have definite prior knowledge about the behavior of every user. We assume that the belief structure of the dynamic game is common knowledge which means that relay terminals (which are also potentially source terminals) are aware that cooperation is belief based. We argue that individual rationality together with knowledge of game structure motivates relays to adopt behavior strategy.

The obvious questions are (1) since \(\mu_i^k(t_k)\) is conditioned on how relay \(j\) behaves in the previous stage \(t_{k-1}\) \((h^i(t_k))\), how would source \(i\) learn about the history since it does not perfectly observe what Nature assigned to the relay (game of imperfect information)?, (2) how is belief at the first stage of the game \(\mu_i^k(t_0)\) initialized? Before addressing the questions, we would like to point out that each source terminal \(i\) determines behavior of its partners using any of the misbehavior detection techniques proposed in [14–17]. Although actions of relay terminal \(j\) are not perfectly observable, the effects of relay’s actions are captured by the detection techniques which will provide a probabilistic measure of the history. This probability measure will be used to update belief of source terminal \(i\) at the end of stage game \(t_k\). Before we discuss how prior beliefs are assigned, we introduce belief system that describes the belief updating procedure.

5.3. Belief System. The belief system defines belief updating procedure for each source terminal \(i\) using Bayes’ rule at the end of each stage game \(t_k\). The posterior belief at the end of stage game \(t_k\) is

\[
\mu_i^k(t_k) = \frac{p(h^i(t_k), \theta_i | \theta_j) p(\theta_j)}{\sum_{\theta_j \in \Theta} p(h^i(t_k), \theta_i | \theta_j) p(\theta_j)},
\]  

(10)

where \(p(h^i(t_k), \theta_i | \theta_j)\) is probability measure on the history of the game at the end of stage game \(t_k\), which is obtained from a detection technique; \(p(\theta_j)\) is prior belief at the beginning of stage game \(t_k\). At the end of each stage game, source terminals obtain new information about behavior of their partners. The belief at the end of stage game \(t_k\) will be used as prior belief for the next stage game \(t_{k+1}\). The belief at the end of the last stage of the game \(t_K\) reveals reputation of relay terminal \(j\) which is a measure of the relay’s trustworthiness.

It is important to note that detection techniques are designed to tolerate certain levels of false alarm and miss detection. While false alarm events result in degradation of belief probability, miss detection events wrongly elevate belief probability of misbehaving terminals. Thus, it is obvious that accuracy of the belief system is determined by the robustness of the detection technique implemented.

5.3.1. Initializing Beliefs. At stage game \(t_0\), source terminal \(i\) may assign prior belief \(\mu_i^k(t_0)\) in any of the following ways.

(1) Nondistributed. If source terminal \(i\) has no prior interaction with relay terminal \(j\), it will assign equal prior probabilities for all possible types of relay terminal \(j\), that is,

\[
\mu_i^k(t_0) = \begin{cases} 
\frac{1}{2} & \text{if } \Theta_j = \{\theta_0\}, \\
\frac{1}{2} & \text{if } \Theta_j = \{\theta_1\}.
\end{cases}
\]  

(11)

(2) Direct Reciprocity. This is also a nondistributed approach in which source terminals initialize their beliefs based on what they know about the relay. Thus, if source terminal \(i\) and relay terminal \(j\) have prior history of cooperation, source terminal \(i\) will condition future cooperation based on past history. That is, the prior belief for the new cooperative interaction will be set to the reputation of the relay in the previous cooperation, that is, \(\mu_i^k(t_0) = p(\theta_j | \theta_i, h^i(t_K))\), where \(h^i(t_K)\) history at the last stage game of the previous cooperative interaction.

(3) Distributed (Indirect Reciprocity). Indirect reciprocity is a mechanism where terminals obtain information on their potential partners from other terminals in the network. It is a distributed mechanism which is enabled by exchanging of reputation information. At the end of each cooperative interaction, source terminals reveal reputation information of their partners to the rest of the network. By exchanging reputation information, each terminal gains a global view of the network. Note that indirect reciprocity is a robust mechanism which ensures stable and socially efficient cooperation [18] if adopted by all nodes.

It is important to note that detection techniques are designed to tolerate a certain level of false alarm and miss detection, which means that accuracy of the belief system is determined by the performance of the detection technique implemented.

5.4. Partner Selection. Partner selection is the mechanism by which source terminals select reliable relays based on their
past history. We assume that each source terminal $i$ stores belief information on each potential relay in a trust vector,

$$
\mu_i = [\mu_i^1, \ldots, \mu_i^l], \quad \sum_j \mu_j^i = 1, \quad j \in N \setminus i,
$$

where $\mu_i$ is normalized trust vector. It is clear that relay terminals with relatively higher normalized belief will be more likely selected as partners. It is important to note that a selected potential relay may refuse cooperation based on its belief about source terminal $i$. Source terminal $i$ may share its trust vector with other terminals in the network. For instance, terminal $i$ may inform terminal $l$ about behavior of terminal $k$. Terminal $l$ then forms a weighted belief about $k$ based on its belief about $i$, that is,

$$
\mu_l^k = \mu_i^l \mu_i^k, \quad \mu_i^j : l’s \ belief \ about \ i.
$$

5.5. Utility Function. The utility function of the game is a measure of the net cooperation gain of each individual node. It is defined in terms of the attainable benefit of cooperation and the cost incurred. The attainable benefit of cooperation is measured by the average frame success rate (FSR)

$$
\text{FSR} = [1 - \text{BER}]^M,
$$

where BER is average bit error. For instance, for cooperative AF BER is given by

$$
\text{BER} = \int_0^\infty Q\left(\sqrt{1 + \rho} (y_s, d + f(y_r, d))\right) \times p(y_s) p(y_r) d y_s d y_r d y_{rd},
$$

$\rho$ is modulation parameter, $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-z^2/2} dz$.

The cost of cooperation $\mathcal{E}_R$ which is incurred by a relay terminal $R$ is sum of (1) energy expended to establish cooperative partnership; (2) energy expended to forward information bearing signals to help a partner. The total energy a relay terminal expends for cooperation,

$$
\mathcal{E}_R = \mathcal{E}_{R,\text{data}} + \mathcal{E}_{R,\text{handshake}},
$$

where $\mathcal{E}_{R,\text{data}}$ energy expended to forward data and $\mathcal{E}_{R,\text{handshake}}$ energy expended to establish cooperative partnership. The source terminal also expends $\mathcal{E}_{S,\text{handshake}}$ for protocol handshake. Total energy expended for cooperative transmission of information bearing signal is given by $\mathcal{E}_t = \mathcal{E}_{R,\text{data}} + \mathcal{E}_{S,\text{data}}$, where $\mathcal{E}_{S,\text{data}}$ is energy expended by source terminal assuming the presence of direct transmission from source to destination. Note $\mathcal{E}_{R,\text{handshake}} \ll (\mathcal{E}_{S,\text{data}} + \mathcal{E}_{R,\text{data}}).

In [19] utility function of a wireless network is defined as a measure of the number of information bits received per joule of total energy expended,

$$
\mu_i = \frac{T_i(\mathcal{E})}{\mathcal{E}} \text{ bits/joule},
$$

where $T_i(\mathcal{E}) = W \times \text{FSR}$ is throughput of user $u_i$, $W$ is the bandwidth, and $\mathcal{E} = \mathcal{E}_t + \mathcal{E}_{\text{handshake}}$ is total cost of cooperation. Note that $\mathcal{E}_{\text{handshake}} = \mathcal{E}_{R,\text{handshake}} + \mathcal{E}_{S,\text{handshake}}$ contributes zero utility since no information bits are transmitted during the protocol handshake. Thus, (17) defines a well-behaved utility function where $\mathcal{E}_t \to 0, u_i \to 0$, and $\mathcal{E}_t \to \infty, u_i \to 0$. We verify behavior of the utility function as shown in Figure 9. Note that the utility function is inverse of the cost-to-benefit ratio (see Figures 10 and 11).

5.6. Formal Definition of the Game. Cooperative communications is a 6-tuple dynamic Bayesian game $\mathcal{G} = (N, \Theta, \mathcal{A}, \mu, u)$, where $N$ is the number of nodes in the cooperative network. $\Theta$ is the type space of relay nodes, $\mathcal{A}$ is action space profile of the nodes, $\mu$ is system of beliefs of source nodes, and $u$ is a vector of utility functions.

5.7. Perfect Bayesian Equilibrium (PBE). PBE is a belief-based solution concept for dynamic games of incomplete information [9]. Unlike static games where equilibrium points are comprised of strategies, PBE incorporates belief in the equilibrium definition [20]. In [20], the author noted the importance of beliefs in the equilibrium definition. Thus, PBE defines a solution concept where players make optimal decisions at each stage of the game given their beliefs. We show that the proposed dynamic Bayesian game model for cooperative communications satisfies the requirements for the existence of PBE [9],

(1) Requirement 1: at each information set the player with the move has some beliefs about which node in its information set has been reached.

(2) Requirement 2: given its belief a player must be sequentially rational, that is, whenever it is its turn to move, the player must choose an optimal strategy from that point on.

(3) Requirement 3: beliefs are determined using Bayes’ rule.
Proof. Requirement 1 is trivially satisfied since the information sets of source nodes are singleton sets which indicate that whenever a source node has information to send, it transmits to the network. Thus, we can assign probability one to each decision node in the singleton set at each stage game $t_k$. Requirement 2 is met by the problem this thesis set out to solve, that is, we would like to design a mechanism where source nodes make optimal decisions given their belief. Requirement 3 is satisfied by the belief system in (10). Thus, the proposed dynamic game model satisfies the conditions for the existence of PBE and that it admits PBE. It also admits sequential equilibrium since for every extensive form game, there exists at least one sequential equilibrium [9, Proposition 1]. We argue based on evolutionary game theoretic arguments that if (1) a significant fraction of the nodes adopts sequential rationality (obey Requirement 2) and (2) they share reputation information with other nodes, an evolutionary stable cooperation is attainable.

6. Conclusion

In this paper we develop a dynamic Bayesian game theoretic framework for cooperative diversity. We showed that the proposed game theoretic framework captures vital aspects of cooperative communications. We showed that the dynamic game framework admits perfect Bayesian equilibrium. The framework presented in this paper would provide a foundation to develop a reputation-based cooperative diversity system where source terminals exchange belief information to confine cooperation to terminals whose behavior is known a priori.

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1. Introduction

The competitive economy equilibrium problem of a communication system consists of finding a set of prices and distributions of frequency or tone power spectra to users such that each user maximizes his/her utility, subject to his/her budget constraints, and the limited power bandwidth resource is efficiently utilized. Although the study of the competitive equilibrium can date back to Walras [1] work in 1874, the concepts applied to a communication system just emerged few years ago because of the great advances in communication technology recently. In a modern communication system such as cognitive radio or digital subscriber lines (DSL), users share the same frequency band and how to mitigate interference is a major design and management concern. The Frequency Division Multiple Access (FDMA) mechanism is a standard approach to eliminate interference by dividing the spectrum into multiple tones and preassigning them to the users on a nonoverlapping basis. However, this approach may lead to high system overhead and low bandwidth utilization. Therefore, how to optimize users’ utilities without sacrificing the bandwidth utilization through spectrum management becomes an important issue. That is why the spectrum management problem has recently become a topic of intensive research in the signal processing and digital communication community.

From the optimization perspective, the problem can be formulated either as a noncooperative Nash game [2–5]; or as a cooperative utility maximization problem [6, 7]. Several algorithms were proposed to compute a Nash equilibrium solution (Iterative Waterfilling Algorithm (IWFA) [2, 4]; Linear Complementarity Problem (LCP) [3]) or globally optimal power allocations (Dual decomposition method, [8–10]) for the cooperative game. Due to the problem’s nonconvex nature, these algorithms either lack global convergence or may converge to a poor spectrum sharing strategy. Moreover, the Nash equilibrium solution may not achieve social communication economic efficiency; and, on
the other hand, an aggregate social utility (i.e., the sum of all users’ utilities) maximization model may not simultaneously optimize each user’s individual utility.

Recently, Ye [11] proposed a competitive economy equilibrium solution that may achieve both social economic efficiency and individual optimality in dynamic spectrum management. He proved that a competitive equilibrium always exists for the communication spectrum market with Shannon utility for spectrum users, and under a weak-interference condition the equilibrium can be computed in a polynomial time. In [11], Ye assumes that the budget is fixed, but this paper deals how adjusting the budget can further improve the social utility and/or meet each individual physical demand. This adds another level of resource control to improve spectrum utilization.

This study investigates how to allocate budget between users to meet each user’s physical power demand or balance all individual utilities in the competitive communication spectrum economy. We prove what follows.

1. A competitive equilibrium that satisfies each user’s physical power demand always exists for the communication spectrum market with Shannon utilities if the total power demand is less than or equal to the available total power supply.

2. A competitive equilibrium where all users have identical utility value always exists for the communication spectrum market with Shannon utilities.

Computational results and comparisons between the competitive equilibrium and Nash equilibrium solutions are also presented. The simulation results indicate that the competitive economy equilibrium solution provides more efficient power distribution to achieve a higher social utility in most cases. Besides, the competitive economy equilibrium solution can make more users to obtain higher individual utilities than the Nash equilibrium solution does in most cases. Moreover, the competitive economy equilibrium takes the power supply capacity of each channel into account, while the Nash equilibrium model assumes the supply unlimited where each user just needs to satisfy its power demand.

The remainder of this paper is organized as follows. The mathematical notations are illustrated in Section 2. Section 3 describes the competitive communication spectrum market considered in this study. Section 4 formulates two competitive equilibrium models that address budget allocation on satisfying power demands and budget allocation on balancing individual utilities. Section 5 demonstrates a toy example of two users and two channels. Section 6 describes how to solve the market equilibrium and presents the computational results. Finally, conclusions are made in the last section.

2. Mathematical Notations

First, a few mathematical notations. Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space; $\mathbb{R}^+_n$ denote the subset of $\mathbb{R}^n$ where each coordinate is nonnegative. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of nonnegative real numbers, respectively.

Let $X \in \mathbb{R}^{m \times n}$ denote the set of ordered $m$-tuples $X = (x_{11}, \ldots, x_{m1})$ and let $\mathbb{X}_i \in (\mathbb{R}^{m-1 \times n}$ denote the set of ordered $(m-1)$-tuples $\mathbb{X}_i = (x_{i1}, x_{i2}, \ldots, x_{im})$, where $x_{ij} = (x_{ij1}, \ldots, x_{ijn}) \in \mathbb{X}_i$ for $i = 1, \ldots, m$. For each $i$, suppose there is a real utility function $u_i$ defined over $X$. Let $A_i(\mathbb{X}_i)$ be a subset of $\mathbb{X}_i$ defining for each point $\mathbb{X}_i \in \mathbb{X}_i$, then the sequence $\{X_1, \ldots, X_m, u_1, \ldots, u_m, A_1(\mathbb{X}_1), \ldots, A_m(\mathbb{X}_m)\}$ will be termed an abstract economy. Here $A_i(\mathbb{X}_i)$ represents the feasible action set of agent $i$ that is possibly restricted by the actions of others, such as the budget restraint that the cost of the goods chosen at current prices does not exceed his income, and the prices and possibly some or all of the components of his income are determined by choices made by other agents. Similarly, utility function $u_i(\mathbb{X}_i, \bar{x}_i)$ for agent $i$ depends on his or her actions $\mathbb{X}_i$ as well as actions $\bar{x}_i$ made by all other agents. Also, denote $x_j = (x_{j1}, \ldots, x_{jm}) \in \mathbb{R}^m$ for a given $x \in X$.

A function $u : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be concave if for any $x, y \in \mathbb{R}^n$ and any $0 \leq \alpha \leq 1$, we have $u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y)$; and it is strictly concave if $u(\alpha x + (1 - \alpha)y) > \alpha u(x) + (1 - \alpha)u(y)$ for $0 < \alpha < 1$. It is monotone increasing if for any $x, y \in \mathbb{R}^n$, $x \geq y$ implies that $u(x) \geq u(y)$.

3. Competitive Communication Spectrum Market

Let the multiuser communication system consist of $m$ transmitter-receiver pairs sharing a common frequency band discretized by $n$ tones. For simplicity, we will call each of such transmitter-receiver pair a “User.” Each user $i$ will be endowed a “monetary” budget $w_i > 0$ and use it to “purchase” powers, $x_{ij}$, across tones $j = 1, \ldots, n$, from an open market so as to maximize its own utility $u_i(\mathbb{X}_i, \bar{x}_i)$:

$$\text{maximize}_{\mathbb{X}_i} u_i(\mathbb{X}_i, \bar{x}_i)$$

subject to

$$p^T \mathbb{x}_i = \sum_j p_j x_{ij} \leq w_i,$$  \hspace{1cm} (1)

$$\mathbb{x}_i \geq 0,$$

where $\mathbb{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ and $\bar{x}_i \in (\mathbb{R}^{m-1 \times n}$ are power units purchased by all other users, and $p_j$ is the unit price for tone $j$ in the market.

A commonly recognized utility for user $i$, $i = 1, \ldots, m$, in communication is the Shannon utility [12]:

$$u_i(\mathbb{x}_i, \bar{x}_i) = \sum_{j=1}^n \log \left( 1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq j} \sigma_{kj} x_{kj}} \right),$$  \hspace{1cm} (2)

where parameter $\sigma_{ij}$ denotes the normalized background noise power for user $i$ at tone $j$, and parameter $d_{ij}$ is the normalized crosstalk ratio from user $k$ to user $i$ at tone $j$. Due to normalization we have $d_{ij} = 1$ for all $i, j$. Clearly, $u_i(\mathbb{x}_i, \bar{x}_i)$ is a continuous concave and monotone increasing function in $\mathbb{x}_i \in \mathbb{R}^n$ for every $\bar{x}_i \in (\mathbb{R}^{m-1 \times n}$. 

There are four types of agents in this market. The first-type agents are users. Each user aims to maximize its own utility under its budget constraint and the decisions by all other users. The second-type agent, “Producer or Provider,” who installs power capacity supply $s_j \geq 0$ to the market from a convex and compact set $S$ to maximize his or her utility. We assume that they are fixed as $S$ in this paper, and $\sum_i d_i \leq \sum_j s_j$, that is, the total power demand is less than or equal to the available total power supply.

The third agent, “Market,” sets tone power unit “price” $p_j \geq 0$, which can be interpreted as a “preference or ranking” of tones $j = 1, \ldots, n$. For example, $p_1 = 1$ and $p_2 = 2$ simply mean that users may use one unit of $S_j$ to trade for two units of $S_j$.

The fourth agent, “Budgeting,” allocates “monetary budget” $w_i > 0$ to user $i$ from a bounded total budget, say $\sum_i w_i = m$.

Figure 1 illustrates the interaction among four types of agents in the proposed competitive spectrum market. Each user $i$ determines its power allocation $x_i$ under its budget $w_i$, power spectra unit price density $p$ and the decisions by all other users $x_i$. The producer installs power capacity spectra density $s$ based on power spectra unit price density $p$ to maximize his or her utility. The market sets power spectra unit price density $p$ based on tone power distribution $x$ and power capacity spectra density $s$ to make market clear. The budgeting agent allocates budget $w_i$ to user $i$ from a bounded total budget according to tone power distribution $x$ for satisfying power demands or balancing individual utilities. In this study, we assume power capacity spectra density $s$ is fixed. Therefore, $s$ is determined first in the system.

4. **Budget Allocation in Competitive Communication Spectrum Market**

In this section, we discuss how to adjust “monetary” budget to satisfy each user’s prespecified physical power demand or to balance all individual utilities in a competitive spectrum market.

4.1. **Budget Allocation on Satisfying Individual Power Demands**. The first question is whether or not the “Budgeting” agent can adjust “monetary budget” for each user to meet each user’s desired total physical power demand $d_i$ that may be composed from any tone combination. We give an affirmative answer in this section.

A competitive market equilibrium is a power distribution $[x_i^*, \ldots, x_m^*, p^*, w^*]$ such that

(i) (user optimality) $x_i^*$ is a maximizer of (1) given $x_j^*$, $p^*$ and $w_j^*$ for each $i$;

(ii) (market efficiency) $p^* \geq 0$, $\sum_{i=1}^m x_i^* \leq s_j$, $p_j^*(s_j - \sum_{i=1}^m x_{ij}^*) = 0$ for all $j$. This condition says that if tone power capacity $s_j$ is greater than or equal to the total power consumption for tone $j$, $\sum_{i=1}^m x_{ij}^*$, then its equilibrium price $p_j^*$ is 0;

(iii) (budgeting according to demands) given $x^*$, $w^*$ is a maximizer of

$$\max_w \sum_i \left( \max \left\{ 0, d_i - \sum_j x_{ij}^* \right\} \right) w_i,$$

subject to $\sum_i w_i = m$, $w \geq 0$.

This condition says that if user $i$’s power demand is not met, that is, $d_i - \sum_j x_{ij} > 0$, then one should allocate more or all “money budget” to user $i$. Any budget allocation is optimal if $d_i - \sum_j x_{ij} \leq 0$ for all $i$, that is, if every user’s physical power demand is met.

Since the “Budgeting” agent’s problem is a bounded linear maximization, and all other agents’ problems are identical to those in Ye [11], we have the following corollary.

**Corollary 4.1.** The communication spectrum market with Shannon utilities has a competitive equilibrium that satisfies each user’s tone power demand, if the total power demand is less than or equal to the available total power supply.

Now consider the KKT conditions of (1):

$$\lambda p^* - \nabla_x u_i(x_i^*, x_j^*) \geq 0,$$

$$\lambda \geq 0,$$

$$\lambda \cdot \left( (p^*)^T x_i^* - w_i^* \right) = 0,$$

$$\left( x_i^* \right)^T \cdot \left( \lambda p^* - \nabla_x u_i(x_i^*, x_j^*) \right) = 0,$$

$$(p^*)^T x_i^* - w_i^* \leq 0,$$

$$x_i^* \geq 0,$$

where $\nabla_x u_i(x_i, x_j) \in \mathbb{R}^n$ denotes any subgradient vector of $u(x_i, x_j) \in \mathbb{R}^n$ with respect to $x_i$.

Since $\lambda p^* \geq \nabla_x u_i(x_i^*, x_j^*)$ and $(\nabla_x u_i(x_i, x_j))_j = 1/\alpha_{ij} + \sum_{k \neq i} d_{kj} x_{ij} + x_{ij} > 0$, for all $j$, we have $p^* > 0$ and $\lambda > 0$.

Then, $(p^*)^T x_i^* = w_i^*$. The optimality conditions in (4) can be simplified to

$$w_i^* \cdot \nabla_x u_i(x_i^*, x_j^*) \leq \left( \nabla_x u_i(x_j^*, x_j^*)^T x_i^* \right) \cdot p^*,$$

$$(p^*)^T x_i^* = w_i^*,$$

$$x_i^* \geq 0.$$
The complete necessary and sufficient conditions for a competitive equilibrium with satisfied power demands can be summarized as

\[ w_i^* \cdot \nabla u_i(x_i^*, x_i^+) \leq \left( \nabla u_i(x_i^*, x_i^+) \right)^T x_i^+ \cdot p^*, \quad \forall i, \]

\[ \sum_i x_{ij}^* \leq s_j, \quad \forall j, \]

\[ s^T p^* \leq \sum_i w_i^*, \]

\[ \max\left\{ 0, d_i - \sum_j x_{ij}^* \right\} - \lambda \leq 0, \quad \forall i, \]

\[ w_i^* \left( \max\left\{ 0, d_i - \sum_j x_{ij}^* \right\} - \lambda \right) = 0, \quad \forall i, \]

\[ \sum_i w_i^* = m, \]

\[ x_i^*, p^*, w^* \geq 0, \quad \forall i. \]

(6)

Note that the conditions \((p^*)^T x_i^* = w_i^*\) for all \(i\) are implied by the conditions in (6): multiplying \(x_i^* \geq 0\) to both sides of the first inequality, we have \((p^*)^T x_i^* = w_i^*\) for all \(i\), which, together with other inequality conditions in (6), imply

\[ \sum_i w_i^* \geq s^T p^* \geq (p^*)^T \sum_i x_i^* = \sum_i (p^*)^T x_i^* \geq \sum_i w_i^*, \]

(7)

that is, every inequality in the sequence must be tight, which implies \((p^*)^T x_i^* = w_i^*\) for all \(i\).

On the other hand, the 4–6th conditions in (6) are optimality conditions of budget allocator’s linear program, where \(\lambda\) is the dual variable. Then, we have a characterization theorem of a competitive equilibrium that satisfies power demands.

**Theorem 4.2.** Every equilibrium of the discretized communication spectrum utility with the Shannon utility that satisfies power demands has the following properties:

1. \(p^* > 0\) (every tone power has a price);
2. \(\sum s_i^* = s\) (all powers are allocated);
3. \((p^*)^T s = \sum w_i^*\) (all user budgets are spent);
4. \(\sum i x_{ij}^* \geq d_i\) for all \(i\) (all user demands are met);
5. If \(x_{ij}^* > 0\) then \(\left( \nabla u_i(x_i^+, x_i^+) \right)^T x_i^+ \cdot p_i^* - w_i^* - \left( \nabla u_i(x_i^+, x_i^+) \right)_j = 0\) for all \(i, j\) (every user only purchases most valuable tone power).

**Proof.** Note that

\[ \left( \nabla u_i(x_i, x_i^+) \right)_j = \frac{1}{\sigma_{ij} + \sum_k \sigma_{k,j} x_{kj} + x_{ij}} > 0, \quad \forall x \geq 0. \]

(8)

Since \(w_i^*\) cannot be zero for all \(i\), there is at least one \(i\) such that

\[ w_i^* \cdot \nabla u_i(x_i^*, x_i^+) > 0, \]

so that the first inequality of (6) implies that \(p^* > 0\).

The second property is from \((p^*)^T (\sum_i x_i^*) = (p^*)^T s, \sum_i x_i^* \leq s\) and \(p^* > 0\).

The third is from \((p^*)^T x_i^* = w_i^*\) for all \(i\) and \(\sum x_i^* = s\).

We prove the fourth property by contradiction. Suppose, \(d_i - \sum_j x_{ij}^* > 0\) for \(i \in I\) for a nonempty index set \(I\). Then, \(w_i = 0\) for all \(i \notin I\). Then,

\[ \sum_{i \in I} d_i > \sum_{i \in I} x_{ij}^* = \sum_{i \in I} x_i^* = s, \]

(10)

which is a contradiction to the assumption \(\sum d_i \leq \sum s_j\).

The last one is from the complementarity condition of user optimality.

The fourth property of Theorem 4.2 implies that equilibrium conditions (6) can be simplified to

\[ w_i^* \cdot \nabla u_i(x_i^*, x_i^+) \leq \left( \nabla u_i(x_i^*, x_i^+) \right)^T x_i^+ \cdot p_i^*, \quad \forall i, \]

\[ \sum_i x_{ij}^* \geq d_i, \quad \forall i, \]

\[ \sum_i x_i^* \leq s_j, \quad \forall j, \]

\[ s^T p^* \leq \sum_i w_i^*, \]

\[ \sum_i w_i^* = m, \]

\[ x_i^*, p^*, w^* \geq 0, \quad \forall i. \]

(11)

Note that the constraint \(\sum_i w_i^* = m\) is merely a normalizing constraint and it can be replaced by another type of normalizing constraint such as \(\prod_i w_i^* \geq 1\). Moreover, multiple competitive equilibria may exist due to the nonconvexity of the optimality conditions of the spectrum management problem with minimal user power demands.

### 4.2. Budget Allocation on Balancing Individual Utilities.

The second question is whether or not the “Budgeting” agent can adjust “monetary budget” for each user such that a certain fairness is achieved in the spectrum market; for example, every user obtains the same utility value, which is also a critical issue in spectrum management. We again give an affirmative answer in this section.

Here, a competitive market equilibrium is a density point \([x_i^*, \ldots, x_m^*, p^*, w^*]\) such that

1. (user optimality) \(x_i^*\) is a maximizer of (1) given \(x_i^+, p^*\) and \(w_i^*\) for every \(i\);
2. (market efficiency) \(p^* \geq 0, \sum_{i=1}^m x_i^* \leq s_j, p_j^*(s_j - \sum_{i=1}^m x_{ij}^*) = 0\) for all \(j\).
(iii) (budgeting according to individual utilities) Given \(x^*, w^*\) is a minimizer of
\[
\min_w \sum_i u_i(x^*_i, x^*_j) w^*_i, \quad \text{s.t.} \sum_i w_i = m, \quad w \geq 0. \tag{12}
\]

This condition says that if user \(i\)'s utility is higher than any others', that is, \(u_i(x^*_i, x^*_j) > u_j(x^*_j, x^*_j)\), then one should shift "money budget" from user \(i\) to user \(j\). Any budget allocation is optimal if \(u_i(x^*_i, x^*_j)\) are identical for all \(i\), that is, if every user has the same utility value.

Since the "Budgeting" agent's problem is again a bounded linear maximization, and all other agents' problems are identical to those in Ye [11], we have the following corollary.

**Corollary 4.3.** The communication spectrum market with Shannon utilities has a competitive equilibrium that balances each user’s utility value.

The complete necessary and sufficient conditions for a competitive equilibrium with balanced utilities can be summarized as
\[
w_i^* \cdot \nabla_x u_i(x^*_i, x^*_j) \leq (\nabla_x u_i(x^*_i, x^*_j)^T x^*_j) \cdot p^*, \quad \forall i,
\]
\[
\sum_i x^*_{ij} \leq s_j, \quad \forall j,
\]
\[
s^T p^* \leq \sum_i w_i^*,
\]
\[
u_i(x^*_i, x^*_j) - \lambda \geq 0, \quad \forall i,
\]
\[
w_i^*(u_i(x^*_i, x^*_j) - \lambda) = 0, \quad \forall i,
\]
\[
\sum_i w_i^* = m,
\]
\[
x^*_i, p^*, w^* \geq 0, \quad \forall i. \tag{13}
\]

Note that the conditions \((p^*)^T x^*_j = w_i\) for all \(i\) are implied by the conditions in (13). On the other hand, the 4–6 conditions in (13) are optimality conditions of budget allocator’s linear program for balancing utilities, where \(\lambda\) is the dual variable.

Again, we have a characterization theorem of a competitive equilibrium that balances individual utilities.

**Theorem 4.4.** Every equilibrium of the discretized communication spectrum market with the Shannon utility that balances individual utilities has the following properties:

1. \(p^* > 0\) (every tone power has a price);
2. \(\sum_i x^*_i = s\) (all powers are allocated);
3. \((p^*)^T s = \sum_i w_i^*\) (all user budgets are spent);
4. \(u_i(x^*_i, x^*_j)\) are identical for all \(i\) (all user utilities are the same);
5. If \(x^*_ij > 0\) then \((\nabla_x u_i(x^*_i, x^*_j)^T x^*_j) \cdot p^*_j - w_i \cdot (\nabla_x u_i(x^*_i, x^*_j)))_j = 0\) for all \(i, j\) (every user only purchases most valuable tone power).

**Proof.** The proof of properties 1, 2, 3, and 5 are the same as Theorem 4.2. The fourth property is from the 5th condition of (13). If \(w_i = 0\), then the user cannot participate in the game. Therefore, \(w_i > 0\) and \(u_i(x^*_i, x^*_j) = \lambda\), \(\forall i\) by the 5th condition of (13), which implies all user utilities are identical.

The fourth property of Theorem 4.4 implies that equilibrium conditions (13) can be simplified to
\[
w_i^* \cdot \nabla_x u_i(x^*_i, x^*_j) \leq (\nabla_x u_i(x^*_i, x^*_j)^T x^*_j) \cdot p^*, \quad \forall i,
\]
\[
u_i(x^*_i, x^*_j) = \lambda, \quad \forall i,
\]
\[
\sum_i x^*_{ij} \leq s_j, \quad \forall j,
\]
\[
s^T p^* \leq \sum_i w_i^*,
\]
\[
\sum_i w_i^* = m,
\]
\[
x^*_i, p^*, w^* \geq 0, \quad \forall i. \tag{14}
\]

**5. An Illustration Example**

Consider two channels \(f_1\) and \(f_2\), and two users \(x\) and \(y\). Let the Shannon utility function for user \(x\) be
\[
\log\left(1 + \frac{x_1}{1 + y_1}\right) + \log\left(1 + \frac{x_2}{4 + y_2}\right), \quad \text{(15)}
\]
and one for user \(y\) be
\[
\log\left(1 + \frac{y_1}{2 + x_1}\right) + \log\left(1 + \frac{y_2}{4 + x_2}\right), \quad \text{(16)}
\]
and let the aggregate social utility be the sum of the two individual user utilities.

Assume a competitive spectrum market with power supply for two channels is \(s_1 = s_2 = 2\) and the initial endowments for two users is \(w_x = w_y = 1\). Then the competitive solution is
\[
p_1 = \frac{3}{5}, \quad p_2 = \frac{2}{5},
\]
\[
x_1 = \frac{5}{3}, \quad x_2 = 0, \quad \text{(17)}
\]
\[
y_1 = \frac{1}{3}, \quad y_2 = 2,
\]
where the utility of user \(x\) is 0.3522, the utility of user \(y\) is 0.2139, and the social utility has value 0.5661.

Now consider each of them has a physical power demand \(d_x = d_y = 2\). From above example we find \(x_1 + x_2 = 5/3\) can not satisfy user \(x\)'s power demand \(d_x = 2\) if \(w_x = w_y = 1\). By the proposed method, we can adjust the initial budget endowments to \(w_x = 6/5\) and \(w_y = 4/5\), then the equilibrium price will remain the same and the equilibrium allocation will be
\[
x_1 = 2, \quad x_2 = 0,
\]
\[
y_1 = 0, \quad y_2 = 2, \quad \text{(18)}
\]
where the utility of user $x$ is 0.4771, the utility of user $y$ is 0.1761, and the social utility has value 0.6532.

Since the Nash equilibrium model only considers each user’s power demand, we set the power constraints of user $x$ and user $y$ as 2 and get a Nash equilibrium $x_1 = 2$, $x_2 = 0$, $y_1 = 1$, $y_2 = 1$, where the utility of user $x$ is 0.3010, the utility of user $y$ is 0.1938, and the social utility has value 0.4771. Comparing the competitive equilibrium and Nash equilibrium solutions, one can see that the competitive equilibrium provides a power distribution that not only makes both users with an identical utility value but also achieves a higher social utility than the Nash equilibrium does.

Now consider user $x$ and user $y$ need to have more balanced individual utilities. By the proposed method, we can adjust the initial endowments to $w_x = 4/5$ and $w_y = 6/5$, then the equilibrium price will remain the same and the equilibrium power distribution will be

$$x_1 = \frac{4}{3}, \quad x_2 = 0,$$
$$y_1 = \frac{2}{3}, \quad y_2 = 2,$$

where the utilities of user $x$ and user $y$ are both 0.25527, and the social utility is 0.31054.

If the power constraints of user $x$ and user $y$ are set as $4/3$ and $8/3$, respectively, then the Nash equilibrium will be $x_1 = 4/3$, $x_2 = 0$, $y_1 = 5/3$, $y_2 = 1$, where the utility of user $x$ is 0.1761, the utility of user $y$ is 0.2730, and the social utility has value 0.4491. Comparing the competitive equilibrium and Nash equilibrium solutions again, one can see that the competitive equilibrium provides a power distribution that not only makes both users with an identical utility value but also achieves a higher social utility than the Nash equilibrium does.

### 6. Numerical Simulations

This section presents some computer simulation results on using two different approaches to achieve budget allocation for satisfying each user’s power demand or balancing individual utilities. We compare the competitive equilibrium solution with Nash equilibrium solution in social utility and individual utilities under various number of channels and number of users in a weak-interference communication environment. In a weak-interference communication channel, the Shannon utility function is approximated by

$$u_i(x_i, x_j) = \sum_{j=1}^{n} \log \left( 1 + \frac{x_{ij}}{a_{ij} + a_i^j \left( \sum_{k \neq i} x_{kj} \right) } \right),$$

where $a_i^j$ represent the average of normalized crosstalk ratios for $k \neq i$. Furthermore, we assume $0 \leq a_i^j \leq 1$, that is,
Table 1: Number of iterations required to achieve the budget allocation where the competitive equilibrium satisfies power demands \( d_i = 0.5(\sum_j x_j/m) \) and \( d_i = \sum_j x_j/m \) by the iterative algorithm, error tolerance = 0.01, average of 10 simulation runs.

| No. of channels | No. of users | \( d_i = 0.5(\sum_j x_j/m) \) | \( d_i = \sum_j x_j/m \) |
|----------------|-------------|-----------------------------|-----------------------------|
| 2              | 1            | 1                           | 1                           |
| 4              | 1            | 1                           | 1                           |
| 6              | 1            | 1                           | 1                           |
| 8              | 1            | 1                           | 1                           |
| 10             | 1            | 1                           | 1                           |
| 12             | 1            | 1                           | 1                           |
| 14             | 1            | 1                           | 1                           |
| 16             | 1            | 1                           | 1                           |
| 18             | 1            | 1                           | 1                           |
| 20             | 1            | 1                           | 1                           |

Table 2: Comparisons of CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium satisfies power demands \( d_i = \sum_j x_j/m \) between two approaches, error tolerance = 0.01 and average of 10 simulation runs.

| No. of channels | No. of users | M1* | M2† | M1 | M2 | M1 | M2 | M1 | M2 |
|----------------|-------------|-----|-----|----|----|----|----|----|----|
| 2              | 2            | 0.033 | 1.085 | 0.049 | 1.228 | 0.069 | 1.358 | 0.088 | 1.624 | 0.136 | 1.882 |
| 4              | 4            | 0.022 | 1.164 | 0.080 | 1.479 | 0.267 | 2.255 | 0.463 | 3.465 | 1.011 | 6.450 |
| 6              | 6            | 0.028 | 1.270 | 0.106 | 2.207 | 0.312 | 5.129 | 0.639 | 10.545 | 1.947 | 19.406 |
| 8              | 8            | 0.025 | 1.516 | 0.103 | 3.788 | 0.510 | 10.305 | 0.875 | 25.697 | 2.592 | 51.210 |
| 10             | 10           | 0.035 | 1.889 | 0.130 | 7.222 | 0.525 | 27.027 | 0.938 | 44.909 | 2.455 | 111.270 |
| 12             | 12           | 0.028 | 2.482 | 0.158 | 12.558 | 0.603 | 41.747 | 1.816 | 93.028 | 3.164 | 190.489 |
| 14             | 14           | 0.028 | 3.231 | 0.161 | 20.454 | 0.528 | 66.719 | 2.464 | 150.099 | 2.708 | 322.979 |
| 16             | 16           | 0.039 | 4.793 | 0.184 | 33.251 | 0.684 | 102.846 | 1.260 | 263.820 | 6.006 | 519.137 |
| 18             | 18           | 0.041 | 6.529 | 0.250 | 46.043 | 0.627 | 150.047 | 2.181 | 385.401 | 5.781 | 773.646 |
| 20             | 20           | 0.042 | 9.322 | 0.247 | 66.839 | 0.703 | 215.038 | 2.645 | 553.401 | 4.689 | 1179.129 |

* M1: iterative algorithm
† M2: solving the entire optimal conditions.
the average cross-interference ratio is not above 1 or it is less than the self-interference ratio (always normalized to 1). In all simulated cases, the channel background noise levels are chosen randomly from the interval $[0, m]$, and the normalized crosstalk ratios $d_i$ are chosen randomly from the interval $[0, 1]$. The power supply of each channel $j$ is $\bar{z}_j = m$, $j = 1, \ldots, n$. The total budget is $\sum_i w_i = m$. All simulations are run on a Genuine Intel CPU 1.66 GHz Notebook.

### 6.1. Budget Allocation on Satisfying Individual Power Demands.
In this section, we compute the budget allocation where the competitive equilibrium meets power demands $d_i = 0.95(\sum_j \bar{z}_j/m)$ for all users under various number of channels and number of users. Two approaches are adopted to find out the budget allocation strategy: one is solving the entire optimality conditions in (11) by optimization solver LINGO; the other is iteratively adjusting the total budget $m$ among different users based on whether their power demands are satisfied or not. In the iterative algorithm, all user budgets $w_i$ are set as 1 initially, then the competitive equilibrium can be derived from given channel capacity and user budget. If some user’s power demand is not satisfied in the resulting competitive equilibrium, the budgeting agent reallocates budget to users and computes a new competitive equilibrium. The procedure reiterates until a desired competitive equilibrium is reached for satisfying power demands. The iterative algorithm that allocates more budget to the users with more power shortage and keeps the total budget as $m$ is summarized in Algorithm 1.

In each iteration, given channel capacity $\bar{z}_j$ and user budget $w_i$, the competitive equilibrium is derived by an iterative waterfilling method [13]. Since the competitive equilibrium in each iteration satisfies $\sum_i x_{ij}^* = \bar{z}_j = m$ and $\sum_i w_i x_{ij}^* = m$, and each user optimizes his own utility under his budget constraint and the equilibrium prices, relatively increasing one user’s budget makes him obtain more powers and others obtain fewer powers. In Algorithm 1, the user budget is reassigned according to the power shortage of each user in the equilibrium solution. The idea of comparing the user’s power shortage with average shortage makes more budget be allocated to the users with higher power shortage and the total budget remains $m$. The term $\min_i w_i$ aims to keep new $w_i$ not less than 0. The power demand value and the error tolerance have a significant impact on the number of iterations required to converge to the budget allocation where the competitive equilibrium meets the power demands. Figure 2 indicates the convergence behavior of the iterative algorithm for satisfying power demands for the case of 2 users and 2 channels illustrated in Section 5. Each user has a physical power demand $d_i = d_j = 2$. The error tolerance is set as 0.01. As the figure shows, at first, user $x$ has power shortage and user $y$ has power surplus, then the algorithm converges after eight iterations and the errors

### Table 3: Number of iterations and CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium satisfies power demands $d_i = 0.95(\sum_j \bar{z}_j/m)$ in large-scale problems by the iterative method, error tolerance = 0.05 and average of 10 simulation runs.

| No. of channels | 2 users | 10 users | 50 users | 100 users |
|-----------------|---------|----------|----------|-----------|
|                 | Iterations | Time     | Iterations | Time     |
| 256             | 1        | 0.072    | 1         | 2.162     |
| 512             | 1        | 0.255    | 1         | 5.656     |
| 1024            | 1        | 0.388    | 1         | 15.978    |

### Table 4: Comparisons of social utility and individual utility between competitive equilibrium (CE) with power demands $d_i = \sum_j \bar{z}_j/m$ and Nash equilibrium (NE), error tolerance = 0.01 and average of 100 simulation runs.

| No. of channels | 2 users | 4 users | 6 users | 8 users | 10 users |
|-----------------|---------|---------|---------|---------|---------|
|                 | Social\(\dagger\) | Indiv\(\dagger\) | Social | Indiv | Social | Indiv | Social | Indiv | Social | Indiv |
| 2               | 9.20%   | 83%     | 8.51%   | 58%    | 7.85%   | 53%   | 8.42%   | 51%   | 9.38%   | 51%   |
| 4               | 6.78%   | 87%     | 6.21%   | 70%    | 6.21%   | 62%   | 6.40%   | 58%   | 6.11%   | 57%   |
| 6               | 5.73%   | 70%     | 6.52%   | 58%    | 6.27%   | 62%   | 6.33%   | 58%   | 6.15%   | 56%   |
| 8               | 5.73%   | 70%     | 6.49%   | 58%    | 6.27%   | 62%   | 6.33%   | 58%   | 6.15%   | 56%   |
| 12              | 5.61%   | 82%     | 5.31%   | 72%    | 5.58%   | 72%   | 5.27%   | 72%   | 5.15%   | 71%   |
| 14              | 5.62%   | 87%     | 5.31%   | 72%    | 5.58%   | 72%   | 5.27%   | 72%   | 5.15%   | 71%   |
| 16              | 5.59%   | 90%     | 5.26%   | 72%    | 5.56%   | 72%   | 5.27%   | 72%   | 5.15%   | 71%   |
| 18              | 5.52%   | 97%     | 5.26%   | 88%    | 5.34%   | 85%   | 5.25%   | 81%   | 5.02%   | 74%   |
| 20              | 5.47%   | 97%     | 5.26%   | 88%    | 5.34%   | 85%   | 5.25%   | 81%   | 5.02%   | 74%   |

\(\dagger\) Social: (average social utility in CE – average social utility in NE)/average social utility in NE
Indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.
Table 5: Number of iterations required to achieve the budget allocation where the competitive equilibrium has balanced individual utilities by the iterative algorithm, difference tolerance = 0.01 and average of 10 simulation runs.

| No. of channels | No. of users | 2 | 4 | 6 | 8 | 10 |
|----------------|--------------|---|---|---|---|----|
|                | Iter* | Diff* | Iter | Diff | Iter | Diff | Iter | Diff | Iter | Diff |
| 2              | 5     | 0.0076 | 10   | 0.0078 | 14   | 0.0079 | 18   | 0.0080 | 97   | 0.0080 |
| 4              | 4     | 0.0075 | 20   | 0.0079 | 27   | 0.0080 | 21   | 0.0080 | 74   | 0.0080 |
| 6              | 4     | 0.0077 | 9    | 0.0079 | 18   | 0.0080 | 22   | 0.0081 | 46   | 0.0081 |
| 8              | 4     | 0.0080 | 8    | 0.0078 | 40   | 0.0079 | 149  | 0.0080 | 33   | 0.0080 |
| 10             | 4     | 0.0080 | 13   | 0.0081 | 17   | 0.0079 | 57   | 0.0081 | 24   | 0.0080 |
| 12             | 4     | 0.0078 | 16   | 0.0080 | 35   | 0.0079 | 31   | 0.0080 | 29   | 0.0080 |
| 14             | 5     | 0.0078 | 8    | 0.0080 | 13   | 0.0079 | 21   | 0.0080 | 67   | 0.0080 |
| 16             | 5     | 0.0078 | 9    | 0.0080 | 12   | 0.0079 | 27   | 0.0080 | 48   | 0.0080 |
| 18             | 4     | 0.0076 | 6    | 0.0079 | 10   | 0.0078 | 18   | 0.0079 | 26   | 0.0080 |
| 20             | 4     | 0.0077 | 7    | 0.0078 | 8    | 0.0079 | 11   | 0.0079 | 20   | 0.0079 |

*Iter: number of iterations
*Diff: (max; ui − min; ui)/min; ui.

Table 6: Comparisons of CPU time (seconds) required to achieve the budget allocation where competitive equilibrium has balanced individual utilities between two approaches, difference tolerance = 0.01 and average of 10 simulation runs.

| No. of channels | No. of users | M1* | M2† | M1 | M2 | M1 | M2 | M1 | M2 | M1 | M2 |
|----------------|--------------|-----|-----|---|----|---|----|---|----|---|----|
| 2              |              | 0.048 | 0.330 | 0.056 | 0.364 | 0.061 | 0.447 | 0.060 | 0.575 | 0.239 | 0.837 |
| 4              |              | 0.046 | 0.377 | 0.088 | 0.647 | 0.127 | 1.100 | 0.148 | 1.892 | 0.738 | 3.606 |
| 6              |              | 0.048 | 0.467 | 0.075 | 1.005 | 0.116 | 2.469 | 0.353 | 5.425 | 1.550 | 11.273 |
| 8              |              | 0.041 | 0.641 | 0.069 | 2.052 | 0.319 | 5.555 | 1.663 | 13.305 | 1.422 | 26.173 |
| 10             |              | 0.070 | 0.872 | 0.113 | 3.366 | 0.214 | 10.264 | 1.759 | 27.294 | 1.056 | 54.902 |
| 12             |              | 0.063 | 1.247 | 0.139 | 6.345 | 0.397 | 19.048 | 0.919 | 47.069 | 1.428 | 101.013 |
| 14             |              | 0.064 | 1.822 | 0.095 | 9.692 | 0.217 | 32.551 | 0.577 | 81.780 | 2.633 | 168.536 |
| 16             |              | 0.056 | 2.542 | 0.119 | 14.928 | 0.216 | 52.972 | 0.953 | 123.817 | 3.320 | 274.966 |
| 18             |              | 0.058 | 3.328 | 0.103 | 22.686 | 0.261 | 74.310 | 1.117 | 191.992 | 1.733 | 401.128 |
| 20             |              | 0.057 | 4.333 | 0.098 | 31.805 | 0.192 | 102.436 | 0.506 | 272.994 | 1.674 | 557.339 |

*M1: iterative algorithm
†M2: solving entire optimal conditions.

\((d_i - \sum_j x_{ij}^*)/d_i\) for user x and user y are both below error tolerance 0.01.

Table 1 lists the number of iterations required to find out the budget allocation with \(d_i = 0.5(\sum_j z_j/m)\) and \(d_i = \sum_j z_j/m\) by the above iterative algorithm. The cases of \(d_i = \sum_j z_j/m\) need more iterations since the total power demand \(\sum d_i\) is equal to the total channel capacity \(\sum z_j\). This requirement is tight and the budget allocation makes each user get the same physical power in the competitive equilibrium, that is, \(\sum j x_{ij}^* = n\), for all \(i\). Table 2 compares the CPU time used by two different approaches under power demands \(d_i = \sum_j z_j/m\). The iterative algorithm spends much less time than the method of solving entire optimal conditions on finding out the budget allocation and the competitive equilibrium. We can also use the iterative method to solve large scale problems. The number of iterations and the CPU time required to solve large-scale problems are listed in Table 3.

We observe that more iterations and CPU time spending for 100 users and 256 channels than those spending for 100 users and 1024 channels because the stop condition of the iterative algorithm is “\((d_i - \sum_j x_{ij}^*)/d_i \leq \text{error tolerance.}" In our simulations in Table 3, \(d_i = 0.95 \times 256\) for 100 users and 256 channels and \(d_i = 0.95 \times 1024\) for 100 users and 1024 channels, therefore the case of 100 users and 1024 channels requires fewer iterations and less total CPU time to reach the error tolerance 0.05 than the case of 100 users and 256 channels does. However the CPU time spending for one iteration in the case of 100 users and 256 channels is less than that in the case of 100 users and 1024 channels.

In comparing competitive equilibrium with Nash equilibrium, the total power allocated to user \(i\), \(\sum_j x_{ij}^*\), in competitive equilibrium is used as the power constraint for user \(i\) in Nash equilibrium model to derive a Nash equilibrium. The simulation results averaged over 100 independent runs indicates that the average social utility of competitive equilibrium is higher than that of Nash equilibrium in...
all cases with \( d_i = 0.5(\sum_j \bar{\sigma}_{ij}/m) \) and in most cases with \( d_i = \sum_j \bar{\sigma}_{ij}/m \), even though the difference is not significant. However, in certain type of problems, for instance, the channels being divided into two categories: high quality and low quality, the competitive equilibrium solution performs much better than the Nash equilibrium solution does. Table 4 compares social utility and individual utility between the competitive equilibrium and the Nash equilibrium when one channel being divided into two categories: high quality and low quality. The term \( \min_i \bar{\sigma}_{ij} \) aims to keep the budget allocation problem by solving the entire optimal conditions. The idea of using the reciprocal of individual utility makes some budget be transferred from the high-utility users to low-utility users. Since relatively increasing one user’s budget makes him obtain more powers and others obtain fewer powers, this will decrease the difference between highest individual utility and lowest individual utility. The term \( \min_i \bar{\sigma}_{ij} \) aims to keep the budget allocation problem by solving the entire optimal conditions. The idea of using the reciprocal of individual utility makes some budget be transferred from the high-utility users to low-utility users. Since relatively increasing one user’s budget makes him obtain more powers and others obtain fewer powers, this will decrease the difference between highest individual utility and lowest individual utility. The term \( \min_i \bar{\sigma}_{ij} \) aims to keep

Algorithm 2 is similar to Algorithm 1 for budget allocation on satisfying power demands. For balancing individual utilities, herein the user budget is adjusted based on the individual utility in the equilibrium solution. The idea of using the reciprocal of individual utility makes some budget be transferred from the high-utility users to low-utility users. Since relatively increasing one user’s budget makes him obtain more powers and others obtain fewer powers, this will decrease the difference between highest individual utility and lowest individual utility. The term \( \min_i \bar{\sigma}_{ij} \) aims to keep the budget allocation problem by solving the entire optimal conditions. The idea of using the reciprocal of individual utility makes some budget be transferred from the high-utility users to low-utility users. Since relatively increasing one user’s budget makes him obtain more powers and others obtain fewer powers, this will decrease the difference between highest individual utility and lowest individual utility. The term \( \min_i \bar{\sigma}_{ij} \) aims to keep

6.2. Budget Allocation on Balancing Individual Utilities. To consider fairness, we adjust each user’s endowed monetary budget \( w_i \) to reach a competitive equilibrium where the individual utilities are balanced. Herein we also adopt two approaches to find out the budget allocation: one is solving the entire optimality conditions in (14) by optimization solver LINGO; the other is iteratively adjusting total budget \( m \) among different users based on their individual utilities. The iterative algorithm that shifts some budget from high-utility users to low-utility users and keeps the total budget as \( m \) is summarized in Algorithm 2.

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6.2. Budget Allocation on Balancing Individual Utilities. To consider fairness, we adjust each user’s endowed monetary budget \( w_i \) to reach a competitive equilibrium where the individual utilities are balanced. Herein we also adopt two approaches to find out the budget allocation: one is solving the entire optimality conditions in (14) by optimization solver LINGO; the other is iteratively adjusting total budget \( m \) among different users based on their individual utilities. The iterative algorithm that shifts some budget from high-utility users to low-utility users and keeps the total budget as \( m \) is summarized in Algorithm 2.

### Table 7: Number of iterations and CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium has balanced individual utilities in large-scale problems by the iterative method, difference tolerance = 0.01 and average of 10 simulation runs.

| No. of channels | 2 users | 10 users | 50 users | 100 users |
|----------------|---------|----------|----------|-----------|
|                | iterations | time    | iterations | time    |
| 256            | 1       | 0.119    | 4        | 6.775     |
| 512            | 1       | 0.211    | 3        | 14.309    |
| 1024           | 1       | 0.452    | 3        | 35.631    |

| No. of users | 2 users | 4 users | 6 users | 8 users | 10 users |
|--------------|---------|---------|---------|---------|----------|
|              | social(*) | indiv†  | social | indiv | social | indiv | social | indiv | social | indiv |
| 2             | 0.96%    | 46%     | −0.59% | 45%   | −0.41% | 47%   | −0.34% | 48%   | −0.44% | 48%   |
| 4             | 1.05%    | 55%     | 0.83%  | 53%   | 0.42%  | 50%   | 0.01%  | 48%   | −0.42% | 48%   |
| 6             | 1.14%    | 56%     | 0.08%  | 50%   | −0.15% | 48%   | −0.08% | 48%   | 0.01%  | 49%   |
| 8             | 1.27%    | 58%     | 0.66%  | 57%   | 0.17%  | 51%   | 0.00%  | 49%   | 0.19%  | 52%   |
| 10            | 1.15%    | 61%     | 0.88%  | 58%   | 0.52%  | 54%   | 0.22%  | 52%   | −0.08% | 53%   |
| 12            | 1.35%    | 66%     | 0.78%  | 57%   | 0.50%  | 56%   | 0.39%  | 54%   | 0.16%  | 52%   |
| 14            | 1.43%    | 67%     | 0.85%  | 60%   | 0.28%  | 55%   | 0.10%  | 52%   | 0.16%  | 54%   |
| 16            | 1.60%    | 75%     | 0.88%  | 60%   | 0.42%  | 55%   | 0.30%  | 54%   | 0.29%  | 55%   |
| 18            | 1.63%    | 72%     | 0.71%  | 59%   | 0.34%  | 54%   | 0.14%  | 55%   | 0.16%  | 52%   |
| 20            | 1.50%    | 73%     | 0.80%  | 59%   | 0.30%  | 56%   | 0.11%  | 54%   | 0.17%  | 54%   |

*Social: (average social utility in CE − average social utility in NE)/average social utility in NE
†Indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.
Table 9: Comparisons of social utility and individual utility between competitive equilibrium (CE) with balanced individual utilities and Nash equilibrium (NE) under two tiers of channels, difference tolerance = 0.01 and average of 100 simulation runs.

| No. of channels | 2     | 4     | 6     | 8     | 10    | 12    | 14    | 16    | 18    | 20    |
|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Social          | 9.02% | 81%   | 9.86% | 73%   | 10.01%| 69%   | 8.85% | 67%   | 9.62% | 67%   |
| Div social      | 9.68% | 82%   | 7.67% | 78%   | 8.30% | 77%   | 8.54% | 71%   | 8.23% | 71%   |
| Div indiv       | 6.06% | 87%   | 6.55% | 81%   | 6.87% | 77%   | 7.43% | 76%   | 7.16% | 75%   |
| Div social      | 5.56% | 88%   | 6.24% | 81%   | 6.41% | 78%   | 6.80% | 76%   | 6.75% | 75%   |
| Div indiv       | 5.46% | 87%   | 6.12% | 84%   | 6.18% | 80%   | 6.47% | 77%   | 6.38% | 75%   |
| Div social      | 5.66% | 88%   | 5.75% | 84%   | 5.88% | 80%   | 5.94% | 79%   | 6.38% | 77%   |
| Div indiv       | 5.65% | 91%   | 6.01% | 85%   | 5.65% | 82%   | 5.80% | 81%   | 5.86% | 77%   |
| Div social      | 5.63% | 92%   | 5.84% | 89%   | 5.75% | 83%   | 5.86% | 82%   | 5.70% | 79%   |
| Div indiv       | 5.78% | 94%   | 5.84% | 88%   | 5.80% | 83%   | 5.81% | 83%   | 5.54% | 80%   |
| Div social      | 5.66% | 94%   | 5.58% | 88%   | 5.77% | 86%   | 5.72% | 81%   | 5.63% | 80%   |

∗ Social: (average social utility in CE − average social utility in NE)/average social utility in NE
† Div indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.

and the CPU time required to solve large-scale problems for balancing utilities by the iterative method. We observe that more iterations are required for 100 users and 256 channels than those required for 100 users and 1024 channels because the stop condition of the proposed algorithm is “(max uj − min uj)/min uj ≤ difference tolerance.” In our simulations in Table 7, the balanced individual utilities for 100 users and 1024 channels are higher than those for 100 users and 256 channels, therefore the case of 100 users and 1024 channels requires fewer iterations to reach the difference tolerance 0.05 than the case of 100 users and 256 channels does. However the CPU time spending for one iteration in the case of 100 users and 256 channels is less than that in the case of 100 users and 1024 channels.

In comparing competitive equilibrium with Nash equilibrium, the total power allocated to each user in competitive equilibrium is also used as the power constraint to derive a Nash equilibrium. The simulation results averaged over 100 independent runs are displayed in Table 8. We find that, in most cases, more users get higher individual utilities in competitive equilibrium than those in Nash equilibrium and the social utility of competitive equilibrium remains higher than that of Nash equilibrium. Table 9 lists the comparisons in the communication environment involving two tiers of channels, one half of channels with σij, j = 1, ..., n/2, chosen randomly from the interval [0, 0.1] and the other half of channels with σij, j = n/2 + 1, ..., n, chosen randomly from the interval [1, m]. We can observe that the
competitive equilibrium not only makes more users obtain higher individual utilities but also significantly enhances the social utility. In other words, using budget allocation we can derive a competitive equilibrium that provides a power allocation strategy to balance individual utilities without sacrificing the social utility. Moreover, in the competitive equilibrium model with balanced individual utilities, all users have identical utility value. However, in the Nash equilibrium model the average difference between maximal individual utility and minimal individual utility is over 15%.

7. Conclusions

This study proposes two competitive equilibrium models: (1) to satisfy each user’s physical power demand and (2) to balance all individual utilities in a competitive communication spectrum economy. Theoretically, we prove that a competitive equilibrium with physical power demand requirements always exists for the communication spectrum market with Shannon utility if the total power demand is less than or equal to the available total power supply. A competitive equilibrium with identical individual utilities also exists for the communication spectrum market with Shannon utility. Computationally, we use two approaches to find out the budget allocation where the competitive equilibrium satisfies power demand or balances individual utilities: one solves the characteristic equilibrium conditions and the other employs an iterative tatonament-type method by adjusting budget to each user. The iterative method performs significantly faster and can efficiently solve large-scale problems, which makes the competitive economy equilibrium model applicable in real-time spectrum management.

In comparing with the Nash equilibrium solution under the identical power usage of each user obtained from the competitive equilibrium model, our computational results show that the social utility of the competitive equilibrium solution is better than that of the Nash equilibrium solution in most cases. Under the equilibrium condition with balanced individual utilities, the competitive economy equilibrium solution makes more users obtain higher individual utilities than Nash equilibrium solution does without sacrificing the social utility.

In this study, we propose a centralized algorithm to reach a desired competitive equilibrium for satisfying power demands or balancing individual utilities. In the future, a distributed algorithm should be developed especially when a centralized controller is not available in the network. Besides, although the iterative method works well in our computational experiments, its convergence is unproven. We plan to do so in future work. We would also consider further study in how to adjust another exogenous factor’s (power supply) to achieve a better social solution while maintaining individual satisfaction. That is, how to set the power supply capacity for each channel to make spectrum power allocation more efficient under the competitive equilibrium market model.

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