CLASSIFYING FINITE 2-NILPOTENT P-GROUPS, LIE ALGEBRAS
AND GRAPHS: EQUIVALENT WILD PROBLEMS

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ABSTRACT. We reduce the graph isomorphism problem to 2-nilpotent p-groups isomorphism problem and to finite 2-nilpotent Lie algebras over the ring $\mathbb{Z}/p^3\mathbb{Z}$. Furthermore, we show that classifying problems in categories graphs, finite 2-nilpotent p-groups, and 2-nilpotent Lie algebras over $\mathbb{Z}/p^3\mathbb{Z}$ are polynomially equivalent and wild.

1. INTRODUCTION

Graph isomorphism problem is one of the central problems in graph theory. Reducing this problem to isomorphism problem of some algebraic structures, such as rings, algebras and groups has been proposed in several works, e.g., [Kim, Roush '80, Droms '87, Kayal, Saxena '05]. As an example, we describe shortly a correspondence between groups and graphs, given in [Droms '87]. For a graph $\Gamma = (V,E)$, group $G(\Gamma)$ is generated by vertices $V$ with relations $x_i \cdot x_j = x_j \cdot x_i$ for every pair of adjacent vertices $x_i$ and $x_j$ of graph $\Gamma$. In [Droms '87] it was proved that $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic if and only if the graphs $\Gamma_1$ and $\Gamma_2$ are isomorphic. Note that group $G(\Gamma)$ is infinite.

In our paper, we reduce the graph isomorphism problem to the isomorphism problem of 2-nilpotent groups finite p-group. To do this, we prove first that the latter problem can be reduced to the problem of isomorphism of finite 2-nilpotent Lie algebras over the ring $\mathbb{Z}/p^3\mathbb{Z}$. Moreover, we prove that the problems of distinguishing graphs, finite 2-nilpotent p-groups and nilpotent of class 2 Lie algebras (or H-algebras) over the ring $\mathbb{Z}/p^3\mathbb{Z}$ up to isomorphism are polynomially equivalent.

Second, we prove that the classifying problems of the above structures up to isomorphism are wild. Recall that a wild classification problem contains (in some sense) a problem of classification of pairs of matrices up to simultaneous similarity (called W-problem below). There exist different embeddings the W-problem into the problem classification up to isomorphism of the above structures. Finally, we compare the complexity of these reductions. Different embeddings the W-problem into graph isomorphism problem can play an important role in

Date: July 7, 2008.

1991 Mathematics Subject Classification. 05C60, 5C85, 05C25, 7B30, 20D15.

Key words and phrases. wild problems, nilpotent groups, nilpotent algebras, graphs.
estimation of the complexity of the last one. Note that wildness of classifying problem for
2-nilpotent p-groups was proved in [Sergeichuk ’75].

2. Defining groups on graphs

In this section we present the reduction from graphs isomorphisms problem to isomorphisms
problem of 2-nilpotent groups. Let \( \Gamma \) be an undirected loopless graph with the vertex set
\( V = \{1, ..., n\} \) and the edge set \( E \), where \(|E| = m\). Let us observe two types of variables: \( v_i \)
corresponding to the vertices of \( \Gamma \), and \( a_{ij} \) corresponding to pairs of vertices of \( \Gamma \). Variables \( a_{ij} \)
correspond to edges and non-edges of \( \Gamma \), and their number is \( l = \binom{n}{2} \). Let us index the set of
pairs \( \{(i, j) \mid 0 \leq i < j < n\} \) by natural numbers from 1 to \( l \), and denote by \( A = \{a_1, ..., a_l\} \) the set of \( a \)-variables corresponding to graph \( \Gamma \). Let us define, following Kayal and Saxena
[Kayal, Saxena ’05], a commutative algebra over \( \mathbb{Z}/p^3\mathbb{Z}, p \neq 2 \), (called a graph-algebra)
\[ R(\Gamma) := (\mathbb{Z}/p^3\mathbb{Z})[v_1, ..., v_n, a_1, ..., a_l]/I, \quad (\ast) \]
where ideal \( I \) has the following relations:

(1) \( \forall 1 \leq i \leq n, \ v_i^2 = 0, \)
(2) \( \forall 1 \leq i < j \leq n, \ v_i \times v_j = v_j \times v_i = a_{ij}, \) where \( e = (i, j), \)
(3) \( \forall i, j, \ a_j \times v_i = v_i \times a_j = 0, \ a_i \times a_j = 0, \)
(4) \( \forall 1 \leq i < j \leq n, \ pa_e = 0 \) if \( e = (i, j) \in E \) and \( p^2a_e = 0 \) if \( e = (i, j) \not\in E. \)

It is clear that \( v_i \) representing the vertices of \( \Gamma \) are of additive order \( p^3 \), \( a_{ij} \) that represent
the edges of \( \Gamma \) are of additive order \( p \) and \( a_j \) which represent the non-edges of \( \Gamma \) are of additive
order \( p^2 \). The additive structure of the algebra \( R(\Gamma) \) is as follows:

\[ (R(\Gamma), +) = \mathbb{Z}/p^3\mathbb{Z} \oplus \bigoplus_{i=1}^{n} (\mathbb{Z}/p^3\mathbb{Z})v_i \oplus \bigoplus_{e \in E(\Gamma)} (\mathbb{Z}/p\mathbb{Z})a_e \oplus \bigoplus_{e \not\in E(\Gamma)} (\mathbb{Z}/p^2\mathbb{Z})a_e. \]

Let us denote by \( N := N(\Gamma) \) the elements of \( R := R(\Gamma) \) without the constant terms. Clearly,
\( N \) is 3-nilpotent associative algebra, i.e., \( N^3 = 0 \). More exactly, \( N \) is the nilpotent radical
of \( R \). The following theorem was proved in [Kayal, Saxena ’05]:

**Theorem 2.1.** For every two undirected graphs \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) the fol-
lowing holds:

\[ R(\Gamma_1) \approx R(\Gamma_2) \iff \Gamma_1 \approx \Gamma_2. \]

Commutative rings become too rigid for our purposes. We intend to construct a directed
graph \( \Gamma = (V, E) \) corresponding to the original graph \( \Gamma = (V, E) \). A Lie algebra \( R^0(\Gamma) \)
over \( \mathbb{Z}/p^3\mathbb{Z} \) associated with the last graph \( \Gamma = (V, E) \) has proved to be a powerful tools in
establishing isomorphism of graphs.
Denote by \( \tilde{\Gamma} = (V, \tilde{E}) \) a graph with the same set of vertices and with directed edges labeled with \( \tilde{E} = E \cup \hat{E} \) such that if there is an edge labeled by \( a_{ij} \in E \) from \( v_i \) to \( v_j \) then there is also an edge labeled by \( a_{ji} \in \hat{E} \) from \( v_j \) to \( v_i \). Denote by

\[
F(\tilde{\Gamma}) := (\mathbb{Z}/p^3\mathbb{Z}) < v_1, ..., v_n, a_1, ..., a_l >, \ v_i \in V, a_i \in \tilde{E}
\]
a free linear algebra generated by \( v_1, ..., v_n, a_1, ..., a_l \) over \( \mathbb{Z}/p^3\mathbb{Z} \)-module and as ring has two binary operations \(+, \times\).

We replace the ideal \( I \) (see the formula (*)\) of defining relations with the ideal \( I^0 \) with slightly different relations:

1. \( \forall 1 \leq i \leq n : \ u_i^2 = 0 \),
2. \( \forall 1 \leq i, j \leq n : \ a_{ij} = v_i \times v_j = -v_j \times v_i = -a_{ji} \),
3. \( \forall i, j : a_i \times v_i = v_i \times a_i = 0, \ a_i \times a_j = 0 \),
4. \( \forall i, j \leq n : p a_{ij} = 0 \) if \( (i, j) \in E \) and \( p^2 a_{ij} = 0 \) if \( (i, j) \notin E \).

The algebra \( R^0(\tilde{\Gamma}) := F(\tilde{\Gamma})/I^0 \) is a nilpotent Lie algebra of class 2 over a ring \( \mathbb{Z}/p^3\mathbb{Z} \):

\[
[[R^0, R^0], R^0] = 0,
\]

where \( [v_i, v_j] := v_i \times v_j \) and \( R^0(\tilde{\Gamma}) = R^0 \). It is called the generalized Heisenberg's algebra \( (H\text{-algebra}) \) of the graph \( \Gamma \).

It can be proved that

**Theorem 2.2.** For every two undirected graphs \( \Gamma_1 \) and \( \Gamma_2 \) holds:

\[
R^0(\tilde{\Gamma}_1) \approx R^0(\tilde{\Gamma}_2) \iff \Gamma_1 \approx \Gamma_2.
\]

\( \square \)

Now our goal is to construct a group corresponding to H-algebra \( R^0 := R^0(\tilde{\Gamma}) \). Let us observe a H-algebra \( R^0 \) as \( \mathbb{Z}/p^3\mathbb{Z} \)-module and its submodule

\[
V^0 = \bigoplus_{i=1}^n (\mathbb{Z}/p^3\mathbb{Z}) v_i
\]

with an additive basis \( B = \langle b_1, ..., b_n \rangle \), i.e., \( V^0 = \text{Span}_{\mathbb{Z}/p^3\mathbb{Z}} < b_1, ..., b_n > \). Denote by \( Z \) the center of \( R^0 \). Note that \( (R^0)^2 \subseteq Z \) and \( (R^0)^2 = Z \) if and only if the graph \( \Gamma \) has no isolated vertices. We define a group \( G \) of H-algebra \( R^0 \) (or of the graph \( \Gamma \)) in the following way. Let us denote by \( G_n = \{g_1, ..., g_n\} \) a set of \( n \) elements and define a bijection \( \chi : B \rightarrow G_n \).

Denote by \( G \) the set of all formal expressions of the form

\[
G := \{g_1^{\alpha_1} ... g_n^{\alpha_n} a_k \mid a_k \in Z, g_i \in G_n, 0 \leq \alpha_i \leq p^3 - 1\},
\]

where if \( g_i = \chi(b_i), \ i \in [1n] \).
Define the multiplication on this set (we use the multiplicative notation $b_i^{\alpha_i} \cdot b_j^{\alpha_j}$ instead of the additive notation $\alpha_i b_i + \alpha_j b_j$ for the module operation on $R^0$):

$$g_1^{\alpha_1} \ldots g_n^{\alpha_n} a g_{\beta_1}^{\beta_1} \ldots g_{n}^{\beta_n} b = g_1^{\alpha_1+\beta_1} \ldots g_n^{\alpha_n+\beta_n} \cdot a \cdot b \cdot \phi(h_2^{\alpha_2} \ldots h_n^{\alpha_n}, b_1^{\beta_1}) \phi(h_3^{\alpha_3} \ldots h_n^{\alpha_n}, b_2^{\beta_2}) \ldots \phi(h_n^{\alpha_n}, b_{n-1}^{\beta_{n-1}}),$$

where $\phi(h_i, b_j) = b_i \times b_j$ is the multiplication on $R^0$ and $\alpha_i + \beta_i = \alpha_i + \beta_i \mod p^3$. It is easy to check that $G$ is indeed a group, and $\mathbb{Z}/p^3\mathbb{Z}$-modules $G/Z$ and $V^0$ are isomorphic (as modules!). Straightforward computations show that $[G, G] = 1$, i.e., $G$ is a 2-nilpotent group. If the graph $\Gamma = (V, E)$ has no isolated vertices, then $Z = [G, G] = (R^0)^2$ is the central commutator subgroup of $G$.

Note that a similar construction of the above group $G$ corresponding to a skew-symmetric bilinear mapping on a vector space $W$ was done in [Huppert, Blackburn '82], (see also Belitski, Lipyanski, Sergeichuk, Tsurkov'08).

We ask the following natural question: how does our group $G$ change if we select another basis of $V^0$? It can be shown that, in general, we get non-isomorphic groups in this transition. Let $\hat{\Gamma} = (V, \hat{E})$ be a graph corresponding to $(\Gamma, E)$ as above and $R^0(\Gamma) = R^0$ be a H-algebra corresponding to $\hat{\Gamma}$. Let us fix the basis $St = <v_1, \ldots, v_n>$, $v_i \in V$, of $V^0$. We call this basis the standard basis of $V^0$. According to the above mentioned scheme we can construct a group $G$ corresponding to $R^0$ in the basis $St$ and a fixed bijection $\chi : St \rightarrow G$. Let us call this group $G$ by a graph group of type 1 of the H-algebra $R^0$ or of the graph $\hat{\Gamma}$ (briefly, $H$-group $G$).

Let us show that an $H$-group $G$ be a p-group of exponent $p^3$. Note that in the ring $R^0$ elements $v_i$ have order $p^3$, elements $a_i$ corresponding to the edges of $\hat{\Gamma}$ have order $p$ and elements corresponding to the non-edges of $\hat{\Gamma}$ - order $p^2$. Therefore, the orders of the elements $g_i$ and $a_i$ in representation (1) are at most $p^3$. Now assume that for $x, y \in G$ the condition $x^{p^3} = y^{p^3} = 1$ is fulfilled. Then:

$$(xy)^{p^3} = x^{p^3} y^{p^3} x y[x, y] = x^{p^3} y^{p^3} [y^{p^3}, x] [y^{p^3-1}, x] = [y, x]^{1+2+\ldots+(p^3-1)} = [y, x]^{p^3-1+p^3} = 1.$$

Therefore, exponent of the group $G$ is $p^3$.

**Theorem 2.3.** Let $R^0_1$ and $R^0_2$ be two H-algebras and $G_1$ and $G_2$ two H-groups corresponding to them. Then the following holds:

$$R^0_1 \approx R^0_2 \iff G_1 \approx G_2$$

**Proof.** Assume that $\varphi : R^0_1 \rightarrow R^0_2$ is an isomorphism of two H-algebras $R^0_1$ and $R^0_2$. We have to show that corresponding $H$-groups $G_1$ and $G_2$ are isomorphic.
Following [Kayal, Saxena '05], we can say that an isomorphism \( \varphi : R_1^0 \to R_2^0 \) induces a bijection \( \pi_\varphi : V_1 \to V_2 \) from vertices of the graphs \( \bar{\Gamma}_1 \) to vertices of the graphs \( \bar{\Gamma}_2 \). In turn, this substitution induces a natural isomorphisms \( \pi^* : R_1^0 \to R_2^0 \) of H-algebras \( R_1^0 \) and \( R_2^0 \) which transforms a standard basis \( St_1 = \langle v_1, ..., v_n \rangle \) of \( V_1^0 \) to a standard basis \( St_2 = \langle v'_1, ..., v'_n \rangle \) of \( V_2^0 \). In other words \( \pi^*(v_i) = \psi_{\pi(i)}, \) where \( \pi_\varphi \) is a permutation on \( 1, ..., n \) corresponding to the bijection \( \pi_\varphi : V_1 \to V_2 \).

Let \( G_{n_1} = \{g_1, ..., g_n\} \) and \( G_{n_2} = \{g'_1, ..., g'_n\} \) be two set of elements. Now we define two bijections \( \chi_i : St_i \to G_{n_i} \), from \( St_i \) to \( G_{n_i} \) for \( i = 1, 2 \), respectively. Indeed we can write \( \chi_1(v_k) = g_{\tau_1(k)} \) and \( \chi_2(v_k) = g_{\tau_2(k)} \), where \( \tau_i, i = 1, 2, \) are permutations on the naturally ordered set \( 1, 2, ..., n \). According above construction we arrive at two \( H \)-groups \( G_1 \) and \( G_2 \). As consequence we get a diagram:

\[
\begin{array}{ccc}
St_1 & \xrightarrow{\pi^*} & G_{n_1} \\
\downarrow{\chi_1} & & \downarrow{\chi} \\
St_2 & \xrightarrow{\pi^*} & G_{n_2}
\end{array}
\]

where \( \chi = \chi_2 \pi^* \chi_1^{-1} : G_{n_1} \to G_{n_2} \) is a bijection from \( G_{n_1} \) to \( G_{n_2} \). We can also write \( \chi(g_k) = g'_{\hat{\pi}(k)} \), where \( \hat{\pi} = \tau_2 \tau_1^{-1} \) is a permutation on the set \( 1, 2, ..., n \). The last equality implies:

\[
(2) \quad g^{\alpha_1}_{\hat{\pi}(1)}...g^{\alpha_n}_{\hat{\pi}(n)}a'g^{\beta_1}_{\hat{\pi}(1)}...g^{\beta_n}_{\hat{\pi}(n)}b' = g^{\alpha_1+\beta_1}_{\hat{\pi}(1)}...g^{\alpha_i+\beta_i}_{\hat{\pi}(i)}a'b',
\]

\[
\varphi_2(v'^{\alpha_1}_{\hat{\pi}(2)}...v'^{\alpha_n}_{\hat{\pi}(n)}, v'^{\beta_1}_{\hat{\pi}(1)}...v'^{\beta_n}_{\hat{\pi}(n)-1}.)
\]

where \( g'_i \in G_{n_2}, v'_i \in St_2, a', b' \in Z_2 \) center of \( G_2 \) and \( \varphi_2(v'_i, v'_j) = v'_i \times v'_j \) is the multiplication of the elements \( v'_i \) and \( v'_j \) in \( R_2^0 \).

Let us define a mapping \( \Phi : G_1 \to G_2 \) from an the group \( G_1 \) to \( G_2 \) as follows:

1. If \( g_i \in G_{n_1} \), then \( \Phi(g_i) = \chi(g_i) = g'_{\hat{\pi}(i)}, \)
2. If \( a \) is a element of the center \( Z_1 \) of \( G_1 \), i.e., then \( \Phi(a) = \pi^*(a), \)
3. If \( g = g_1^{\alpha_1}...g_n^{\alpha_n}a, \) where \( g_i \in G_{n_1} \) and \( a \in Z_1 \), then \( \Phi(g_1^{\alpha_1}...g_n^{\alpha_n}a) = g_1^{\alpha_1}_{\hat{\pi}(1)}...g_n^{\alpha_n}_{\hat{\pi}(n)}\pi^*(a). \)

It is easy to show that \( \Phi : G_1 \to G_2 \) is a isomorphism of the group \( G_1 \) into \( G_2 \).

Conversely, assume that \( \psi : G_1 \to G_2 \) is an isomorphism of two \( H \)-groups \( G_1 \) and \( G_2 \). We have to show that the corresponding H-algebras \( R_1^0 \) and \( R_2^0 \) are also isomorphic.

Let us examine renewal process permitting to construct an H-algebra \( L \) isomorphic to \( R_0^0 \) from the above group \( G := \{g_1^{\alpha_1}...g_n^{\alpha_n}a_k | a_k \in Z, g_i \in G_n \} \). The group \( G/Z \) decomposes into a direct product of \( n \) cyclic groups of order \( p^3 \) and \( u_i = g_i Z \) are their generators. Denote by \( U^0 = \text{Span}_{u/Z} < u_1, ..., u_n > \) a module generated by \( u_1, ..., u_n \) over \( Z/p^3Z \). Denote by \( L = U \bigoplus Z \) a direct sum of two \( Z/p^3Z \)-modules \( U \) and \( Z \). Next we want to equip this
module with a structure of H-algebra. We set:

\[(3) \quad u_j \times u_i = -u_i \times u_j = [g_i, g_j], \quad u_i \times z_j = z_i \times z_j = 0, \text{ for all } u_i \in L, z_j \in Z\]

and extend this rules of multiplication on L by the distributivity. The equalities

\[ [gh, x] = [g, x][h, x] \text{ and } [g, h]^{-1} = [h, g], \text{ for all } g, h, x \in G, \]

guarantee correctness of definition \((3)\). Using formulas \((3)\), it is also easy to check that \(L\) is an H-algebra. Let us show that the algebra \(L\) is isomorphic to \(R^0\). It is clear that \(L\) and \(R^0\) are isomorphic as \(\mathbb{Z}/p^3\mathbb{Z}\)-modules. Since \(g_ig_jg_k^{\circ} = g_i^2g_j^2g_k\varphi(v_j, v_i), g_i, g_j \in G, b_i \in St_1,\) we obtain \(\varphi(v_j, v_i) = [g_i, g_j],\) where \(\varphi(v_j, v_i) = v_j \times v_i\) is the product of elements \(v_i\) and \(v_j\) in \(R^0\). Therefore, a mapping \(\varphi : R^0 \to L\) which is determined by rules:

\[(4) \quad \varphi(v_i) = u_i, i \in [1n], \text{ and } \varphi(z) = z \text{ for all } z \in Z, \]

is an isomorphism from the algebra \(R^0\) into \(L\). We are now ready to prove that isomorphism \(\psi : G_1 \to G_2\) of the groups \(G_1\) and \(G_2\) implies isomorphism \(\hat{\psi} : R^0_1 \to R^0_2\) of corresponding algebras \(R^0_1\) and \(R^0_2\). Let \(L_i = U_i \oplus Z_i,\) where \(U_i = \text{Span}_{\mathbb{Z}/p^3\mathbb{Z}}(u_i, \ldots, u_{i_n}), u_i = g_iZ_i,\) and \(Z_i, i = 1, 2,\) are centers of the groups \(G_1\) and \(G_2,\) respectively. As above (see the formulae \((3)\)), we have \(\varphi_i : R^0_i \to L_i\) the isomorphisms from \(R^0_i\) into \(L_i\) for \(i = 1, 2.\) It is sufficient for the proof of this part of Theorem to construct an isomorphism \(\psi_1 : L_1 \to L_2\) of the algebras \(L_1\) and \(L_2.\) Setting \(\psi_1(u) = \psi(g)Z_1\) for \(u = gZ_1,\) and \(\psi_1(z) = \psi(z), z \in Z_1,\) we obtain \(\psi_1(u_i \times u_j) = \psi[g_i, g_j] = [\psi g_i, \psi g_j]\) for all \(i, j \in [1n].\) Note that correctness of the definition of the mapping \(\psi_1\) follows from equality: \(\psi_1(Z_1) = \psi(Z_1) = Z_2.\)

On the other hand, \(\psi_1(u_i) \times \psi_1(u_j) = (\psi g_i)Z_1 \times (\psi g_j)Z_1 = [\psi g_i, \psi g_j],\) i.e., \(\psi_1\) preserves multiplication of elements from algebra \(L_1.\) It is also easy to show that \(\psi_1\) preserves other operations of algebra \(L_1.\) Since \(\psi : G_1 \to G_2\) is an isomorphism of \(H\)-groups, \(\psi : L_1 \to L_2\) is an isomorphism of \(H\)-algebras \(L_1\) and \(L_2.\) Therefore, \(\varphi_2^{-1}\psi \varphi_1 = \psi : R^0_1 \to R^0_2\) is an isomorphism of \(H\)-algebras \(R^0_1\) and \(R^0_2.\) The proof is complete.

**Remark 2.4.** This group is defined by generators \(g_1, \ldots, g_n, a_1, \ldots, a_l,\) where \(l = \binom{n}{2},\) with defining relations:

\[(5) \quad [g_i, g_j] = a_e, \quad \text{where } e = (i, j), \quad a_ea_{e'} = a_{e'}a_e, a_e g_i = g_i a_e,\]

\[g_i^{a^3} = 1, \quad 1 \leq i \leq n, a_e^a = 1, \text{ if } e \in E, \text{ else } a_e^a = 1.\]

### 3. Defining graphs on groups

In this section, we construct graphs corresponding to finite groups so that the groups are isomorphic if and only if the corresponding graphs are isomorphic. Let \(G = (A, \circ)\) be a finite group. We construct a directed multigraph \(\Gamma(G) = (V, E)\) that corresponds to \(G.\) We denote by \(m(e)\) the multiplicity of an edge \(e\) in this graph, and by \(d(v)\) the degree of a node \(v.\) \(V\) consists of \(A\) and all the ordered triples from \(A \times A \times A.\) For every \(u, v, w \in A\) such that \(u \circ v = w,\) we add edges \((u, (u, v, w)), (v, (u, v, w)), ((u, v, w), w)\) to \(E\) with the
Proof. Suppose now that for \( u, v, w \in A \) such that \( u \circ v \neq w \), we add edges \((u, (u, v, w)), (v, (u, v, w))\) to \( E \) of multiplicities \( m(u, (u, v, w)) = m((u, v, w), v)) = 1 \) and \( m((v, (u, v, w))) = 2 \). For every \( u, v, w \in A \) such that \( u \circ v = w \), we add edges \((u, (u, v, w)), (v, (u, v, w))\) to \( E \) of multiplicities \( m(u, (u, v, w)) = m((u, v, w), v)) = 1 \). A \( d((u, v, w)) \) for \((u, v, w) \in A \times A \times A \) is in between \( 2n \) and \( 4n \). Since \( u \in A \) participates in \( |A|^2 \) triples from \( A \times A \times A \), \( d(u) \geq |A|^2 \). Further, we only speak of finite groups of size 3 and more, thus \( d(u) > d(v, w, t) \) for every \( u \in A \) and every \((v, w, t) \in A \times A \times A \). To be complete, we prove here the following theorem (see, e.g., \[\text{Hoffman '81}\]).

**Theorem 3.1.** Let \( G = (A, \circ) \) and \( H = (B, \cdot) \) be finite groups. Then \( G \cong H \) if and only if \( \Gamma(G) \cong \Gamma(H) \).

Proof. Let us denote \( \Gamma(G) = (V, E) \) and \( \Gamma(H) = (V', E') \). The only if direction is trivial, since every isomorphism \( h \) from \( G \) to \( H \) can be extended to a mapping \( f : V \rightarrow V' \) so that \( f(a) = h(a) \) for \( a \in A \) and \( f(a, b, c) = (h(a), h(b), h(c)) \) for \( (a, b, c) \in A \times A \times A \). \( f \) is a bijection since \( h \) is a bijection, and it preserves the group operation since the edges are preserved by \( f \). Likewise, \( f^{-1} \) is a bijection because \( h^{-1} \) is a bijection. Thus \( f \) and \( f^{-1} \) are edge-preserving bijections, and \( f \) is a graph isomorphism.

Suppose now that \( f \) is an isomorphism from \( \Gamma(G) \) to \( \Gamma(H) \). Since \( f \) preserves node degrees, it maps \( A \) to \( B \) and \( A \times A \times A \) to \( B \times B \times B \). Therefore, \( f \) restricted to \( A \), denoted \( f_A \), is a bijection from \( A \) to \( B \). If remains to show that \( f_A \) the group operation. Let \( u, v, w \in A \) so that \( u \circ v = w \). By construction, \((u, (u, v, w)), (v, (u, v, w)), ((u, v, w), w) \in E \) while \( m((u, (u, v, w))) = m((u, v, w), v)) = 1 \) and \( m((v, (u, v, w))) = 2 \). As \( f \) is a graph isomorphism,

\[
(f(u), f((u, v, w))), (f(v), f((u, v, w))), (f(u, v, w), f(w)) \in E'
\]

with multiplicities \( m(f(u), f((u, v, w))) = m(f((u, v, w)), f(w)) = 1 \) and \( m((f(v), f((u, v, w))) = 2 \). Then by construction of \( \Gamma(H) \), \( f(u) \cdot f(v) = f(w) \) and \( f_A \) is a group isomorphism.

We also show that this graph construction is suitable for defining a functor from the category of groups into the category of graphs as it preserves homomorphisms.

**Theorem 3.2.** Let \( G = (A, \circ) \) and \( H = (B, \cdot) \) be finite groups. A homomorphism \( h \) from \( G \) to \( H \) can be extended to a homomorphism from \( \Gamma(G) \) to \( \Gamma(H) \).

Proof. Let us denote \( \Gamma(G) = (V, E) \) and \( \Gamma(H) = (V', E') \). We extend \( h \) to a mapping \( f \) from \( V \) to \( V' \) by setting \( f(u) = h(u) \) for all \( u \in A \) and \( f((u, v, w)) = (h(u), h(v), h(w)) \) for all \((u, v, w) \in A \times A \times A \). It remains to show that \( f \) is a graph homomorphism from \( \Gamma(G) \) to \( \Gamma(H) \), i.e. it maps edges to edges (but a non-edge can be mapped onto an edge). Let \((u, (u, v, w))\) be an edge of multiplicity 1 in \( \Gamma(G) \). As \( h(u) \) is a member of triple \((h(u), h(v), h(w))\), \((h(u), h(v), h(w)) \in E' \). The same is true for every edge \((v, (u, v, w))\) of multiplicity 1. Suppose now that for \( u, v, w \in A \) such that \( u \circ v = w \), \( \Gamma(G) \) contains an edge \((v, (u, v, w))\) of
multiplicity 2 or an edge \(((u, v, w), w)\). As \(h\) is a homomorphism, \(h(u) \cdot h(v) = h(w)\), meaning that edge \(((h(v), h(u), h(v)), h(w))\) has multiplicity 2 and edge \(((h(u), h(v), h(w)), h(w))\) has multiplicity 1 and in \(\Gamma(H)\), as required. \(\square\)

Even though we are working with directed multigraphs, a graph isomorphism problem is the same as for simple undirected graphs in the sense that, given two directed multigraphs, we can always construct (in polynomial time) a pair of simple undirected graphs that will be isomorphic if and only if the original pair is isomorphic.

4. Wildness

A matrix problem given by a set \(A_1\) is a set of \(a\)-tuples of matrices from \(M_{n \times m}\) and \(A_2\) a set of admissible matrix transformations. This and following definition have first appeared in [Belitskii, Sergeichuk '03]. Given two matrix problems \(A = (A_1, A_2)\) and \(B = (B_1, B_2)\), \(A\) is contained in \(B\) if there exists a \(b\)-tuple \(T(x) = T(x_1, ..., x_a)\) of matrices, whose entries are non-commutative polynomials in \(x_1, ..., x_a\), such that

1. \(T(A) = T(A_1, ..., A_a) \in B_1\) if \(A = (A_1, ..., A_a) \in A_1\);
2. for every \(A, A' \in A_1\), \(A\) reduces to \(A'\) by transformations \(A_2\) if and only if \(T(A)\) reduces to \(T(A')\) by transformations \(B_2\).

A pair of matrices matrix problem, denoted \(W = (W_1, W_2)\), is defined as

\[
W_1 = \{A, B \mid A, B \in M_{n \times n}\}
\]

and

\[
W_2 = \{S(A, B)S^{-1} \mid S \in M_{n \times n} \text{ non-singular}\}.
\]

A matrix problem is called wild if it contains \(W\), and tame otherwise.

Now we need the fact on wildness of some class of finite p-groups. The classifying problem for the above groups contains a problem of reducing skew-symmetric matrices over \(\mathbb{Z}/p\mathbb{Z}\) by congruence transformations to block-triangle matrices. The latter problem is a matrix problem and it contains \(W\) in the above sense. We formulate this theorem here for completeness.

**Theorem 4.1.** [Sergeichuk '75] Let \(G\) be a 2-nilpotent finite p-group which is an extension of an abelian group \(A\) by an abelian group \(B\):

\[
1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1.
\]

Problem of classifying of such groups \(G\) with group \(A\) of the order \(p\) is tame. However, if the order of \(A\) is more than \(p\), the above problem is wild.

We are now ready to prove
**Theorem 4.2.** The problems of classifying graphs, $H$-algebras over $\mathbb{Z}/p^3\mathbb{Z}$ and $H$-groups up to isomorphism are wild.

**Proof.** Let $\Gamma = (V,E)$ be a graph with $3$ or more vertices ($|V| \geq 3$) and $G$ be the group corresponding to a ring $R^0$ and, as consequence, to graph $\Gamma$ as in Theorem 3.1. It is easy to see that for the center $Z$ of $G$ holds:

$$\bigoplus_{e \in E(\Gamma)} (\mathbb{Z}/p\mathbb{Z})a_e \oplus \left( \bigoplus_{e \notin E(\Gamma)} (\mathbb{Z}/p^2\mathbb{Z})a_e \right) \subseteq Z$$

Therefore, the center $Z$ has the order $\geq p^3$. Since $H$-group $G$ is a 2-nilpotent $p$-group (of exponent $p^3$), by Theorem 4.1 the problem classifying such groups $G$ up to isomorphism is wild.

Another way to view this is that we use correspondence between graphs and groups as in Theorem 3.1. We can also take a wild class 2-nilpotent groups (see Theorem 4.1) and construct an embedding of groups from this class into graphs according to Section 3.

\[\square\]

5. **Complexity**

It will be show below that the classifying problems for graphs, $H$-groups and $H$-algebras over the ring $\mathbb{Z}/p^3\mathbb{Z}$ up to isomorphism are polynomially equivalent. We assume that $H$-algebras are given by specifying the product of its basis elements over $\mathbb{Z}/p^3\mathbb{Z}$; $H$-groups are given by systems of generators and defining relations and graphs are given by their adjacency matrices.

Indeed, the order of $H$-group and the size of a basis of an $H$-algebra corresponding to a graph are polynomial in the size of this graph by construction. From Theorem 5.1 a size of a graph $\Gamma(G)$ corresponding to a finite group $G$ with $m$ elements is $O(m^3)$. The size of a basis of algebras $R(\Gamma)$ and $R^0(\Gamma)$ corresponding to graph $\Gamma$ with $n$ vertices is $O(n^2)$. The size of a basis of an $H$-algebra $R^0$ and the corresponding $H$-group $G$ is also $O(n^2)$. Therefore, using the notation $\leq_T^P$ for Turing reducibility in polynomial time, we can state that

\[(6)\]

$$GI \leq_T^P HAI \leq_T^P HGI \leq_T^P GI,$$

where $GI$, $HAI$ and $HGI$ denote the problems of distinguishing graphs, $H$-algebras over the ring $\mathbb{Z}/p^3\mathbb{Z}$ and $H$-groups up to isomorphism. Therefore, the above problems are polynomially equivalent.

6. **Acknowledgments**

The authors are grateful to Professor G. Belitskii for his fruitful discussions and interest to this work. The first author was partly supported by Israeli Ministry of Absorption.
REFERENCES

[Belitski, Lipyanski, Sergeichuk, Ts urgkov’08] Genrich Belitski, Ruvim Lipyanski, Vladimir V. Sergeichuk, Arkadii Tsurgkov, Problem of classifying associative algebras over a field of characteristic not 2 are wild, to appear in J. of Linear Algebra and its Applications.
[Belitskii, Sergeichuk ’03] G. Belitskii and V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361, 2003, pp. 203-222.
[Dr ons ’87] Carl Droms, Isomorphism of graph groups, Proc. of the American Mathematical Society, Vol. 100, Number 3, July 1987, pp. 407-409.
[Hoffman ’81] C. Hoffman, Group-theoretic algorithms and graph isomorphism, Lecture Notes in Computer Science 136, Springer-Verlag, 1981.
[Huppert, Blackburn ’82] Bertram Huppert, Norman Blackburn, Finite group II, Springer-Verlag, Berlin Heidelberg New York, 1982.
[Kayal, Saxena ’05] Neeraj Kayal, Nitin Saxena, On the Ring Isomorphism and Automorphism Problems, Proceedings of IEEE Conference on Computational Complexity 2005, pp. 2-12.
[Kim, Roush ’80] K. H. Kim and F. W. Roush, Homology of certain algebras defined by graphs, J. Pure Appl. Algebra 17, 1980, pp. 179-186.
[Pierce ’82] R. Pierce, Associative algebras, 1982.
[Saxena, Agrawal ’05] Nitin Saxena and Manindra Agrawal, Automorphisms of Finite Rings and Applications to Complexity of Problems, 22nd STACS, Springer LNCS 3404, pp. 1-17, 2005.
[Sergeichuk ’75] V. Sergeichuk, The classification of metabelian p-groups (Russian), Matrix problems, Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977, pp. 150-161.

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