Maximal graphs with respect to rank

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March 21, 2019

Abstract

The rank of a graph is defined to be the rank of its adjacency matrix. A graph is called reduced if it has no isolated vertices and no two vertices with the same set of neighbors. A reduced graph \( G \) is said to be maximal if any reduced graph containing \( G \) as a proper induced subgraph has a higher rank. In this paper, we present (1) a characterization of maximal trees (that is induced trees which are not a proper subtree of a reduced tree with the same rank); (2) a construction of two new families of maximal graphs; (3) an enumeration of all maximal graphs with rank up to 9.

Keywords: Adjacency matrix, Rank, Reduced graph, Maximal graph, Maximal tree.

AMS Mathematics Subject Classification (2010): 05C50, 05C05, 15A03.

1 Introduction

Let \( G \) be a simple graph with vertex set \( \{v_1, \ldots, v_n\} \). The adjacency matrix of \( G \) is an \( n \times n \) matrix \( A(G) \) whose \((i,j)\)-entry is 1 if \( v_i \) is adjacent to \( v_j \) and 0 otherwise. The number of vertices of \( G \) is the order of \( G \). The rank of \( G \), denoted by \( \text{rank}(G) \), is the rank of \( A(G) \). We say that \( G \) is reduced if it has no isolated vertex and no two vertices with the same set of neighbors.

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There are only finitely many reduced graphs of rank $r$ since the order of such graphs are at most $2^r - 1$ (see [1, 3]). A natural question is that what is the maximum order of a reduced graph with a given rank $r$. Kotlov and Lovász [8] answered this question asymptotically. They proved that the maximum order of such graph is $O(2^{r/2})$. Later on, Akbari, Cameron, and Khosrovshahi [1] made the following conjecture on the exact value of the maximum order.

**Conjecture 1.** For every integer $r \geq 2$, the maximum order of any reduced graph of rank $r$ is equal to

$$n(r) = \begin{cases} 
2 \cdot 2^{r/2} - 2 & \text{if } r \text{ is even}, \\
5 \cdot 2^{(r-3)/2} - 2 & \text{if } r > 1 \text{ is odd}.
\end{cases}$$

Ghorbani, Mohammadian, and Tayfeh-Rezaie [5] showed that if Conjecture 1 is not true, then there would be a counterexample of rank at most 47. They also proved that the order of every reduced graph of rank $r$ is at most $8n(r) + 14$. The maximum order of graphs with a fixed rank within the families of trees, bipartite graphs and triangle-free graphs were determined in [4, 6].

In a more general setting, in this paper we consider maximal graphs with respect to rank. A reduced graph $G$ is called maximal if it is not a proper induced subgraph of a reduced graph with the same rank as $G$. In other words, $G$ is maximal if for any reduced graph $H$ such that $G$ is obtained by removing a vertex form $H$, we have $\text{rank}(H) > \text{rank}(G)$. Note that the graphs attaining the maximum order in Conjecture 1 would be necessarily maximal.

In the classification of graphs with respect to the rank, maximal graphs are central objects, since any reduced graph of rank $r$ is a subgraph of a maximal graph with rank $r$. In [4], a characterization of maximal trees (i.e. reduced trees which are maximal within the family of trees) is reported. In Section 2, we show that the characterization of [4] is not complete. In fact, there is one more construction of such trees which was missed in [4]. Ellingham [3] presented three families of maximal graphs. In Section 3, we give a construction of two new families of maximal graphs. All maximal graphs of rank up to 7 were appeared in [3] and independently in [12, 13, 10, 9]. We continue this line of work by constructing all maximal graphs of rank 8 and 9. A report on this construction is given in Section 4.

### 2 Maximal trees

A vertex with degree one is called pendant. A vertex adjacent to a pendant vertex is said to be pre-pendant. A tree is reduced if it has no two pendant vertices with the same neighbor. A maximal tree is a tree which is maximal within the family of trees, i.e. if it is not a proper subgraph of a reduced tree with the same rank. We denote the path graph of order $n$ by $P_n$.

In [4], a characterization of maximal trees is reported as follows: every maximal tree $T$ of rank $r \geq 4$ is obtained from a maximal tree $T'$ of rank $r - 2$ in one of the following two ways:
(i) attaching a vertex of a $P_2$ to a vertex of $T'$ of rank $r - 2$ which is neither pendant nor pre-pendant;

(ii) attaching a pendant vertex of a $P_3$ to a pre-pendant vertex of $T'$ with rank $r - 2$.

We observe that the above construction is not exhaustive. To see this, consider the tree $T$ of Figure 1. For any reals $\alpha, \beta$, the vector shown on the vertices of $T$ forms a null vector of $A(T)$. (Observe that the components of the given vector on the neighbors of every vertex sum up to 0.) So by Lemma 4 (below), $T$ is a maximal tree. $T$ cannot be obtained by (i). However, it can be obtained by attaching a pendant vertex of a $P_3$ to a pre-pendant vertex of some tree $T'$, but the corresponding $T'$ is not maximal. This means that $T$ cannot be constructed by (i) or (ii).

In this section, we show that there is one more construction which completes the characterization of maximal trees given in [4].

The column space and the null space of a matrix $M$ is denoted by $\text{Col}(M)$ and $\text{Nul}(M)$, respectively. A vertex $v$ of a graph $G$ is called a null vertex if for every $x \in \text{Nul}(A(G))$, the corresponding component to $v$ is zero. Note that a pre-pendant vertex is always a null vertex.

If $S$ is a subset of vertices of $G$, we denote graph obtained by removing the vertices of $S$ from $G$ by $G - S$. For simplicity, we use $G - v$ for $G - \{v\}$. We denote the degree of a vertex $v$ in a graph $G$ by $d_G(v)$, or by $d(v)$.

The following lemma is well-known and easy to verify.

**Lemma 2.** Let $G$ be a graph and $u$ be a pendant vertex of $G$ with the neighbor $v$. Then $\text{rank}(G) = \text{rank}(G - \{u, v\}) + 2$.

The following well-known lemma can be deduced from Lemma 2 by induction.

**Lemma 3.** The rank of any tree is twice its matching number.

The following lemma gives a characterization of maximal trees in terms of null vertices.

**Lemma 4.** ([4]) A reduced tree $T$ is maximal if and only if for every vertex $v$ which is not pre-pendant, $\text{rank}(T) = \text{rank}(T - v)$; or equivalently, $v$ is a null vertex if and only if it is pre-pendant.

Now, we are ready to present the main result of this section on the characterization of maximal trees.
Theorem 5. Every maximal tree \( T \) of rank \( r \geq 4 \) is obtained from a maximal tree \( T' \) of a smaller rank in one of the following three ways:

(i) attaching a vertex of a \( P_2 \) to a vertex of \( T' \) with rank \( r - 2 \) which is neither pendant nor pre-pendant;

(ii) attaching a pendant vertex of a \( P_3 \) to a pre-pendant vertex of \( T' \) with rank \( r - 2 \);

(iii) attaching a pre-pendant vertex of a \( P_3 \) to a pre-pendant vertex of \( T' \) with rank \( r - 4 \) for \( r \geq 8 \).

Proof. We first show that any tree resulting from (i)–(iii) is maximal. Let \( T' \) be a maximal tree and \( T \) is obtained by attaching a vertex \( v_1 \) of a \( P_2 \) to a vertex \( u \) of \( T' \). Let \( v_2 \) be the other vertex of \( P_2 \). In view of Lemma 2, \( \dim Nul(A(T)) = \dim Nul(A(T')) \). We see that any \( x' \in Nul(A(T')) \) can be extended to a \( x \in Nul(A(T)) \) by defining \( x(v_1) = 0 \) and \( x(v_2) = -x'(u) \). It follows that, besides \( v_1 \), all other null vertices and also pre-pendant vertices of \( T \) and of \( T' \) coincide. So by Lemma 4, \( T \) is maximal. Next, let \( T \) be obtained by (ii) from \( T' \). Suppose that \( v_1, v_2, v_3 \) are the vertices of a \( P_3 \), where \( v_1 \) is attached to a pre-pendant vertex \( u \) of \( T' \) and \( u' \) is the pendant neighbor of \( u \). From Lemma 2 it follows that \( \text{rank}(T) = \text{rank}(T') + 2 \) which means \( \dim Nul(A(T)) = \dim Nul(A(T')) + 1 \). Let \( \{x'_1, \ldots, x'_{s-1}\} \) be a basis for \( Nul(A(T')) \). We introduce a basis \( \{x_1, \ldots, x_s\} \) for \( Nul(A(T)) \) as follows. For \( 1 \leq i \leq s - 1 \), we extend \( x'_i \) to \( x_i \in Nul(A(T)) \) by defining \( x_i(v_1) = x_i(v_2) = x_i(v_3) = 0 \). Further, let \( x_s \) to be zero on \( V(T' - u') \), \( x_s(u') = -x_s(v_1) = x_s(v_3) = 1 \) and \( x_s(v_2) = 0 \). In view of Lemma 4, it turns out that \( T \) is a maximal tree. The argument for (iii) is similar to (ii).

Now, let \( T \) be a maximal tree of rank \( r \geq 4 \) which is not obtained by (i). We prove that \( T \) is obtained by (ii) or (iii). Note that the only reduced tree of rank \( \geq 4 \) and diameter \( \leq 3 \) is \( P_4 \) which is not maximal. So the diameter of \( T \) is at least 4. Consider a longest path \( P \) in \( T \) and call its first five vertices from one end \( u, v, w, y, z \), respectively. So \( u \) is a pendant vertex and \( d(v) = 2 \). We claim that \( w \) is not a pre-pendant vertex. Otherwise, for any vector \( x \in Nul(A(T)) \), we have \( x(w) = 0 \). Also, since the sum of the components of \( x \) corresponding to the neighbors of \( v \) is zero, we have \( x(u) = 0 \) which is impossible by Lemma 4. This proves the claim. Furthermore, if \( d(w) \geq 3 \), then by Lemmas 2 and 4, \( T - \{u, v\} \) would be a maximal tree of rank \( r - 2 \) (because \( Nul(A(T - \{u, v\})) \) can be obtained by the restriction of the vectors of \( Nul(A(T)) \) to \( T - \{u, v\} \)) which contradicts our assumption on \( T \). Thus \( d(w) = 2 \). We show that \( T' = T - \{u, v, w\} \) is a reduced tree of rank \( r - 2 \). Applying Lemmas 2 and 4, we find that \( \text{rank}(T') = \text{rank}(T - u) - 2 = r - 2 \). In order to prove that \( T' \) is reduced, it suffices to show that \( y \) is a pre-pendant vertex in \( T \). Let \( M \) be a maximum matching of \( T \). If \( y \) is not covered by \( M \), then \( wy \notin M \). It turns out that \( (M \setminus \{uw\}) \cup \{uw, wy\} \) is a matching of \( T \) with larger size than \( M \) which in turn implies that \( y \) is covered by every maximum matching of \( T \), and so by Lemma 8, \( \text{rank}(T - y) = r - 2 \). From Lemma 4, it follows that \( y \) is a pre-pendant vertex of \( T \), as desired. Hence \( T' \) is reduced. If \( T' \) is a maximal tree, then \( T \) is obtained by (ii). Now,
suppose that $T'$ is not a maximal tree. Let $p$ be the pendant neighbor of $y$. Recall that $z$ is also a neighbor of $y$. We show that:

(a) $p$ is the only null vertex of $T'$ which is not pre-pendant;

(b) $z$ is a pre-pendant vertex of $T'$;

(c) $d_{T'}(y) = 2$;

(d) $T'' = T' - \{y, p\}$ is a maximal tree of rank $r - 4$.

The claimed situation is demonstrated in Figure 2. From (a)–(d) it follows that $T$ is obtained by (iii). So the proof will be completed by verifying (a)–(d) as follows.

(a) As $T'$ is not maximal, in view of Lemma 4, $T'$ has at least one non-pre-pendant null vertex. Suppose that $q \neq p$ is a null vertex of $T'$ which is not pre-pendant. Let $\{x_1', \ldots, x_{s-1}'\}$ be a basis for the null space of $A(T')$. We introduce a basis $\{x_1, \ldots, x_s\}$ for the null space of $A(T)$ as follows. For $1 \leq i \leq s - 1$, we let $x_i(u) = x_i(v) = x_i(w) = 0$. Moreover, let $x_s$ to be zero on $V(T' - p)$, $x_s(u) = -x_s(w) = x_s(p) = 1$, and $x_s(v) = 0$. All $x_1, \ldots, x_s$ are zero on $q$ which means that $q$ is a non-pre-pendant null vertex for $T$ which is a contradiction by Lemma 4. Therefore, $p$ is a unique non-pre-pendant null vertex of $T'$.

(b) We claim that all the neighbors of $y$ excluding $p$ are pre-pendant. To obtain a contradiction, let $h$ be a non-pre-pendant neighbor of $y$. Since $p$ is the only non-pre-pendant null vertex of $T'$, $h$ is not a null vertex and thus there is a vector $x \in \text{Nul}(A(T'))$ such that $x(h) \neq 0$. Let $T''$ be the connected component of $T' - y$ containing $h$. We define the vector $y$ on $V(T)$ such that $y(a) = 2x(a)$ for $a \in V(T'')$, $y(p) = -x(h)$, and $y(b) = x(b)$ for the remaining vertices $b$ of $T'$. Clearly, $y$ belongs to $\text{Nul}(A(T'))$ with $y(p) \neq 0$. So $p$ is not a null vertex which is a contradiction. Therefore, excluding $p$ all the neighbors of $y$ (including $z$) are pre-pendant.

(c) We establish this claim by a contradiction. Assume $d_{T'}(y) = k \geq 3$, and $T'_1, \ldots, T'_k$ are the components of $T' - y$. If for at least two $j$’s, $T'_j$ contains a vertex in distance $\geq 4$ from $y$, then we have a path longer than $P$ in $T$ which is a contradiction. So, for some $j$, any pendant vertex $q$ of $T'_j$ have distance $\ell \leq 3$ from $y$. If $\ell = 3$, let $Q = qq_1q_2y$ be the path
between $q$ and $y$. The vertex $q_1$ is pre-pendant and thus a null vertex. The vertex $q_2$ is a neighbor of $y$ and by (b), it is pre-pendant and hence a null vertex. Now, since $Q$ is a longest path between a vertex of $T_j'$ and $y$, we have $d_T(q_1) = 2$. As the two neighbors of $q$ are null, it follows that $q$ is also null which is a contradiction. If $\ell = 2$, then we consider $Q = qy$. Since $y$ is a pre-pendant vertex, $y$ is a null vertex. Similarly, we have $d_T(q_1) = 2$. Thus $q$ is a null vertex which is a contradiction. It turns out that $k = 2$.

(d) Lemma 2 implies that rank($T''$) = $r - 4$. As $y$ and $p$ are null vertices of $T'$, Nul($A(T'')$) can be obtained by the restriction of any vector of Nul($A(T')$) to $T''$. From (a), it follows that every non-pre-pendant vertex of $T''$ is not a null vertex and so by Lemma 4, $T''$ is a maximal tree.

The proof is now complete.

See Table 2 for an illustration of how maximal trees with rank up to 8 can be constructed by Theorem 5.

| Rank | Maximal trees |
|------|--------------|
| 2    | ![Tree 2]    |
| 4    | ![Tree 4]    |
| 6    | ![Tree 6]    |
| 8    | ![Tree 8]    |

Table 1: Maximal trees up to rank 8 and their recursive constructions by Theorem 5. The paths $P_2$, $P_3$ and $P_5$ are shown with white vertices.

3 Constructions of maximal graphs

Ellingham constructed three families of maximal graphs. In this section, we first describe his constructions and then we present two more families of maximal graphs.
Let $F = F(n)$ denote a graph with

$$
V(F) = \{a, b_1, \ldots, b_n, c_1, \ldots, c_n\},
$$

$$
E(F) = \{ab_i, ac_i, b_ic_i \mid 1 \leq i \leq n\}.
$$

This graph is called a *friendship graph*. Ellingham proved that the graph $F(n)$ is maximal if and only if $n$ is a square-free integer.

The second family consists of graphs $L = L(m, n)$ defined as follows:

$$
V(L) = A \cup B \cup C = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\} \cup \{c_1, \ldots, c_n\},
$$

$$
E(L) = K(A) \cup K(B) \cup K(C) \cup P(A, B) \cup K(A, C) \cup K(B, C),
$$

where $K(V)$ denotes the edge set of the complete graph on $V$, $K(U, V)$ denotes the set of edges joining every vertex in $U$ to every vertex in $V$, and for two sets $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$, the set $P(U, V) = \{u_i v_i \mid 1 \leq i \leq k\}$ forms a perfect matching between $U$ and $V$. If $m \geq 3$ and $n \geq 0$, then $L(m, n)$ is a maximal graph with the exceptions: $L(4, 5)$, $L(3, 7)$, $L(5, 5)$, $L(3, 8)$, $L(4, 7)$, $L(7, 4)$.

The third family consists of graphs $M(m, n)$, with $m \geq 1$ and $n \geq 2$, where $M = M(m, n)$ has vertex set and edge set

$$
V(M) = \{a\} \cup B \cup C \cup D = \{a\} \cup \{b_1, \ldots, b_m\} \cup \{c_1, \ldots, c_n\} \cup \{d_1, \ldots, d_n\},
$$

$$
E(M) = K(B) \cup K(C) \cup K(D) \cup K(\{a\}, B) \cup K(B, C) \cup K(C, D) \setminus P(C, D).
$$

Below we present two more constructions of infinite families of maximal graphs. The following lemma is useful.

**Lemma 6.** ([2]) Let $B$ be a symmetric matrix and

$$
A = \begin{pmatrix}
B & y \\
y^\top & b
\end{pmatrix}.
$$

(i) If $y \not\in \text{Col}(B)$, then $\text{rank}(A) = \text{rank}(B) + 2$.

(ii) If $y \in \text{Col}(B)$ with $Bx = y$ and $b \neq y^\top x$, then $\text{rank}(A) = \text{rank}(B) + 1$.

(iii) If $y \in \text{Col}(B)$ with $Bx = y$ and $b = y^\top x$, then $\text{rank}(A) = \text{rank}(B)$.

**Theorem 7.** Let $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$, and $G$ be the graph with

$$
V(G) = U \cup V \cup \{u, v\},
$$

$$
E(G) = \{uv, uv_1, \ldots, uv_n\} \cup K(U \cup V) \setminus P(U, V).
$$

Then $G$ is a maximal graph.
Proof. The adjacency matrix of $G$ is as follows:

$$A = \begin{pmatrix} J - I & J - I & 0 & 0 \\ J - I & J - I & 1 & 0 \\ 0^\top & 1^\top & 0 & 1 \\ 0^\top & 0^\top & 1 & 0 \end{pmatrix},$$

where $J$ is the $n \times n$ matrix of all 1 and $1$ is the all 1 vector of length $n$. We see that $\text{rank}(A) = n + 2$ and the matrix

$$B = \begin{pmatrix} J - I & 1 & 0 \\ 1^\top & 0 & 1 \\ 0^\top & 1 & 0 \end{pmatrix}, \quad (1)$$

is a full rank submatrix of $A$. In view of Lemma 6, in order to show that $G$ is a maximal graph, it is sufficient to prove that if $y \in \text{Col}(A)$ is a $(0, 1)$-vector with $Ax = y$ and $x^\top Ax = 0$, then $y = 0$ or $y$ is a column of $A$. So we let $x^\top Ax = 0$ and

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad (2)$$

where $x_1, x_2, y_1, y_2$ are vectors of length $n$. As the last $n + 2$ columns of $A$ span $\text{Col}(A)$, with no loss of generality, we may assume that $x_1 = 0$. Hence

$$B \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \end{pmatrix}. \quad (3)$$

It turns out that

$$\begin{pmatrix} x_2^\top \\ x_3 \\ x_4 \end{pmatrix} B \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = x^\top Ax = 0,$$

and thus

$$\begin{pmatrix} y_2^\top \\ y_3 \\ y_4 \end{pmatrix} B^{-1} \begin{pmatrix} y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0. \quad (4)$$

Let $\gamma$ be the number of non-zero entries of $y_2$. Since

$$B^{-1} = \begin{pmatrix} \frac{1}{n-1}J - I & 0 & \frac{-1}{n-1}1 \\ 0 & 0 & 1 \\ \frac{-1}{n-1}1^\top & 1 & \frac{n}{n-1} \end{pmatrix},$$

by a straightforward computation we come up with the following equation:

$$\frac{1}{n-1} \gamma^2 - \frac{1}{n-1} (n - 1 + 2y_4) \gamma + \frac{1}{n-1} \left(2(n - 1)y_3y_4 + ny_4\right) = 0,$$

8
or equivalently
\[ \gamma^2 - (n - 1 + 2y_4)\gamma + 2(n - 1)y_3y_4 + ny_4 = 0. \]  
(3)

On the other hand, from \( Ax = y \) it follows that
\[ y_1 = (J - I)x_2, \]  
(4)
\[ y_2 = (J - I)x_2 + x_31, \]  
(5)
\[ y_3 = 1^\top x_2 + x_4, \]  
(6)
\[ y_4 = x_3. \]  
(7)

We now consider the following four cases based on the values of \( y_3 \) and \( y_4 \).

(i) \( y_3 = y_4 = 0 \). So \( \gamma^2 - (n - 1)\gamma = 0 \). Since \( y_4 = 0 \), by (7) we have \( x_3 = 0 \) and so by (1) and (5), \( y_1 = y_2 \). Therefore, if \( \gamma = 0 \), then \( y = 0 \), otherwise \( \gamma = n - 1 \) and then \( y_1 = y_2 \) is one of the columns of \( J - I \). This implies that \( y \) is \( i \)-th column of \( A \) for some \( 1 \leq i \leq n \).

(ii) \( y_3 = 0, y_4 = 1 \). From (7), we have \( x_3 = 1 \) and by (4) and (5), \( y_2 = y_1 + 1 \). Since \( y_1 \) and \( y_2 \) are \((0, 1)\)-vectors, the last equality is only possible for \( y_1 = 0 \) and \( y_2 = 1 \). It turns out that \( y \) is \((2n + 1)\)-th column of \( A \).

(iii) \( y_3 = 1, y_4 = 0 \). As in Case (i), from \( y_4 = 0 \) it follows that \( y_1 = y_2 \). On the other hand, by (3), we have \( \gamma^2 - (n - 1)\gamma = 0 \). This shows that \( \gamma = 0 \) or \( \gamma = n - 1 \). If \( \gamma = 0 \), then \( y_1 = y_2 = 0 \) and so \( y \) is the last column of \( A \). If \( \gamma = n - 1 \), then \( y_1 = y_2 \) is one of the columns of \( J - I \) which implies that \( y \) is \( i \)-th column of \( A \) for some \( n + 1 \leq i \leq 2n \).

(iv) \( y_3 = y_4 = 1 \). As in Case (ii), from \( y_4 = 1 \) it follows that \( y_2 = 1 \) which means that \( \gamma = n \). But \( \gamma = n \) does not satisfy (3). This shows that this case is not possible.

The result now follows. \( \square \)

Theorem 8 as below embodies our second construction of maximal graphs.

**Theorem 8.** Let \( U = \{u_1, \ldots, u_n\} \), \( V = \{v_1, \ldots, v_n\} \), and \( G \) be the graph with \( V(G) = U \cup V \cup \{u, v\} \), \( E(G) = K(U) \cup K(V) \cup P(U, V) \cup \{uv, uu_1, \ldots, uu_n, vv_1, \ldots, vv_n\} \).

*Then \( G \) is a maximal graph.*

**Proof.** We have
\[
A = A(G) = \begin{pmatrix}
J - I & I & 0 & 1 \\
I & J - I & 1 & 0 \\
0^\top & 1^\top & 0 & 1 \\
1^\top & 0^\top & 1 & 0
\end{pmatrix}.
\]
We see that rank($A$) = $n + 2$ and the same matrix $B$ as given in (11) is a full rank submatrix of $A$. Let $y$ be a $(0,1)$-vector in Col($A$) with $Ax = y$ and that $x^T Ax = 0$. As the last $n + 2$ columns of $A$ span Col($A$), with no loss of generality, we assume that $x_1 = 0$. So we have

\begin{align*}
y_1 &= x_2 + x_4 1, \\
y_2 &= (J - I)x_2 + x_3 1, \\
y_3 &= 1^T x_2 + x_4, \\
y_4 &= x_3.
\end{align*}

Let $\gamma$ be the number of non-zero entries of $y_2$. Then $\gamma$ satisfies Equation (3). We now consider the following four cases based on the values of $y_3$ and $y_4$.

(i) $y_3 = y_4 = 0$. By (11), we have $x_3 = 0$ and by (10), $1^T x_2 = -x_4$ and so $J x_2 = -x_4 1$. From (9) it follows that $y_2 = -x_2 - x_4 1 = -y_1$. Since $y_1$ and $y_2$ are $(0,1)$-vectors, this is only possible when $y_1 = y_2 = 0$ and so $y = 0$.

(ii) $y_3 = 0$, $y_4 = 1$. By (11), we have $x_3 = 1$ and by (10), $1^T x_2 + x_4 = 0$ and so $J x_2 + x_4 1 = 0$. So $y_1 + y_2 = J x_2 + x_4 1 + 1 = 1$. On the other hand, from (8) it is clear that $\gamma = 1$ or $\gamma = n$. If $\gamma = 1$, then $y_2 = e_i$, the $i$-th column of the identity matrix, for some $1 \leq i \leq n$. Therefore, $y_1 = 1 - e_i$. It turns out that $y$ is $i$-th column of $A$ for some $1 \leq i \leq n$. If $\gamma = n$, then $y_2 = 1$ and $y_1 = 0$ and thus $y$ is the $(2n + 1)$-th column of $A$.

(iii) $y_3 = 1$, $y_4 = 0$. From (3), we have $\gamma = 0$ or $\gamma = n - 1$. Also, from (10) and (11), we have $x_3 = 0$ and $J x_2 + x_4 1 = 1$. Therefore, by (8) and (9), we see that $y_1 + y_2 = 1$. If $\gamma = 0$, then $y_2 = 0$ and $y_1 = 1$ and so $y$ is the last column of $A$. If $\gamma = n - 1$, then $y$ is the $i$-th column of $A$ for some $n + 1 \leq i \leq 2n$.

(iv) $y_3 = y_4 = 1$. As before we have $x_3 = 1$ and $J x_2 + x_4 1 = 1$. It follows that $y_1 + y_2 = J x_2 + x_4 1 + 1 = 1 + 1$. This is only possible when $y_1 = y_2 = 1$. Therefore, $\gamma = n$. But $\gamma = n$ does not satisfy (3). This shows that this case is not possible.

The proof is now complete. □

4 Maximal graphs with small rank

In this section we give some statistics of maximal graphs with small rank. We start by Table 2 in which all the maximal graphs with rank at most 5 are depicted.

The maximal graphs up to rank 7 were enumerated in [3] and independently in the series of the papers [12, 13, 10, 9]. More information on maximal graphs up to rank 7 was given in [9] from which we quote Tables 3 and 4 containing the distribution of maximal graphs with ranks 6 and 7 based on their orders.
We continue this line of work for ranks 8 and 9. This is done by implementing an algorithm for constructing all maximal graphs with a given rank from [3] (see also [1]). For a given integer \( r \), the input of the algorithm is the set of reduced graphs with both order and rank equal to \( r \) and the output of the algorithm is the set of all maximal graphs of rank \( r \). The input of the algorithm was generated by using Mckay’s database of small graphs [11]. As an outcome, we construct all maximal graphs with rank 8 and 9. We found that there are exactly 2807 maximal graphs with rank 8. Their orders run over from 8 to 30. Also, there are exactly 122511 maximal graphs with rank 9. Their orders run over from 9 to 38 with except for 33, 35, 36. In Table 5 for the sake of completion, a summary of the number of maximal graphs of rank up to 9 is given. Moreover, the distributions of maximal graphs with rank 8 and 9 based on their orders are given in Tables 6 and 7. In Table 8 we report more detailed information based on the orders and sizes (the number of edges) of maximal graphs with rank 8.
| Rank | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|------|----|----|----|----|----|----|----|----|
| # Maximal graphs | 1  | 1  | 3  | 8  | 27 | 183| 2807| 122511 |

Table 5: The number of maximal graphs up to rank 9.

| Order | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|
| # Maximal graphs | 38 | 52 | 80 | 78 | 117| 98 | 90 | 254| 137| 81 | 115| 243|

Table 6: The distribution of maximal graphs with rank 8.

| Order | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-------|----|----|----|----|----|----|----|----|----|----|
| # Maximal graphs | 192| 472| 1014| 786| 1402| 1562| 2198| 1963| 3509| 2824|

| Order | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|-------|----|----|----|----|----|----|----|----|----|----|
| # Maximal graphs | 3660| 17229| 51315| 20069| 8663| 2941| 1622| 528| 266| 136|

| Order | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 |
|-------|----|----|----|----|----|----|----|----|----|----|
| # Maximal graphs | 39 | 42 | 42 | 24 | 0  | 7  | 0  | 0  | 2  | 4  |

Table 7: The distribution of maximal graphs with rank 9.

**Acknowledgements**

The research of the authors was in part supported by a grant from IPM.

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Table 8: The distribution of maximal graphs with rank 8 in terms of order \( n \) and size \( m \).