PROBs and perverse sheaves I. Symmetric products

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Abstract

Algebraic structures involving both multiplications and comultiplications (such as, e.g., bialgebras or Hopf algebras) can be encoded using PROPs (categories with PROducts and Permutations) of Adams and MacLane. To encode such structures on objects of a braided monoidal category, we need PROBs (braided analogs of PROPs). Colored PROBs correspond to multi-sorted structures.

In particular, we have a colored PROB $\mathcal{B}$ governing $\mathbb{Z}_{\geq 0}$-graded bialgebras in braided categories. As a category, $\mathcal{B}$ splits into blocks $\mathcal{B}_n$ according to the grading. We relate $\mathcal{B}_n$ with the category $\mathcal{P}_n$ of perverse sheaves on the symmetric product $\text{Sym}^n(C)$ smooth with respect to the natural stratification by multiplicities. More precisely, we show that $\mathcal{P}_n$ is equivalent to the category of functors $\mathcal{B}_n \to \text{Vect}$. This gives a natural quiver description of $\mathcal{P}_n$.

1 Introduction. The main result

Let $k$ be a field. All vector spaces in this paper will be assumed $k$-vector spaces and all categories will be assumed $k$-linear.

A. PROPs and PROBs. The concept of a PROP was introduced by Adams and MacLane [1, 23]. It allows one to axiomatize algebraic structures on a vector space (or, more generally, on an object $A$ of a symmetric monoidal category) involving both multiplications $A \otimes^m \to A$ and comultiplications $A \to A \otimes^n$. See [27] for a modern exposition.

The term PROP is an abbreviation for “category with PROducts and Permutations”. Explicitly, a PROP is a symmetric monoidal category $(\mathcal{P}, \otimes, \mathbf{1})$ with $\text{Ob}(\mathcal{P}) = \{[m], m \in \mathbb{Z}_+\}$ identified with the set of non-negative integers so that the tensor operation $\otimes$ is, on objects, given by the addition of integers: $[m] \otimes [n] = [m+n]$ and the unit object is $\mathbf{1} = [0]$. In other words, a PROP is a strict symmetric monoidal category whose objects are tensor powers of a single object, denoted $[1]$.

Given a symmetric monoidal category $\mathcal{V}$, we can speak about algebras in $\mathcal{V}$ over a PROP $\mathcal{P}$. Such an algebra is simply a symmetric monoidal functor $F : \mathcal{P} \to \mathcal{V}$. If we denote $A = F([1]) \in \mathcal{V}$, then each space $\text{Hom}_\mathcal{P}([m], [n])$ is mapped into the space of mixed operations $A \otimes^m \to A \otimes^n$. 

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Example 1.1 (The PROP of Hopf algebras). A basic example of a structure involving multiplications and comultiplications is that of a Hopf algebra. A Hopf algebra in a symmetric monoidal category $\mathcal{V}$ is an object $A$ together with a multiplication $\mu : A \otimes A \to A$, comultiplication $c : A \to A \otimes A$, unit $e : 1 \to A$ and counit $\eta : A \to 1$ satisfying the well known relations. The corresponding PROP, denote it $\mathcal{HS}$, can be seen as the symmetric monoidal category generated by the universal Hopf algebra $a = [1]$. That is, for any symmetric monoidal category $\mathcal{V}$, we have a bijection between:

(i) Hopf algebras $A$ in $\mathcal{V}$;

(ii) Symmetric monoidal functors $F : \mathcal{HS} \to \mathcal{V}$,

given by $A = F(a)$. At a more intuitive level, $\mathcal{HS}$ contains the morphisms $\mu : a \otimes a \to a$ and similarly $c, e, \eta$ as above, together with all their iterated compositions and tensor products, which are subject to the relations of a Hopf algebra “and nothing else”.

Along with Hopf algebras, we can consider more general objects, namely bialgebras in symmetric monoidal categories. Thus a bialgebra $A$ has compatible associative $\mu : A \otimes A \to A$ and coassociative $c : A \to A \otimes A$ but the unit and counit are not required. As before, we have the PROP $\mathcal{HS}^+$ describing bialgebras. It was introduced by Markl [26].

Further, it is well known [25, 29] the concept of a Hopf algebra (or, more generally of a bialgebra) can be defined in any braided, not necessarily symmetric, monoidal category $\mathcal{V}$. To study such structures we need to modify the concept of a PROP to that of a PROB (with Braidings instead of Permutations). Thus, a PROB is a braided monoidal category whose objects are tensor powers of a single object $a = [1]$.

Example 1.2 (The PROB of braided Hopf algebras). As before, there is a PROB $\mathcal{H}$ such that Hopf algebras in a braided monoidal category $\mathcal{V}$ are in bijection with braided monoidal functors $\mathcal{H} \to \mathcal{V}$. Its objects are tensor powers of the universal braided Hopf algebra $a = [1] \in \mathcal{H}$ and morphisms are iterated compositions and tensor products of the generating morphisms $\mu, c, e, \eta$ as above subject to the relations of a braided Hopf algebra. This PROB was introduced by Habiro [13] who denoted it $\langle \mathcal{H} \rangle$.

Despite their deceptively short definitions by universal properties, the PROP $\mathcal{HS}$ and the PROB $\mathcal{H}$ are quite non-trivial objects. In particular, their structure as ordinary categories, i.e., some description of the spaces $\text{Hom}([m],[n])$ for all $m, n$, is not so easy to pin down. For the simpler PROP $\mathcal{HS}^+$ such a description was obtained by Pirashvili [28].

The goal of this paper and the one to follow [20] is to relate versions of the PROB $\mathcal{H}$ to quivers describing perverse sheaves on certain configuration spaces. Let us describe the version relevant for this paper.

B. The PROP of graded bialgebras. By a graded bialgebra in a braided monoidal category $\mathcal{V}$ we mean a bialgebra decomposed into a direct sum $A = \bigoplus_{n \geq 0} A_n$ so that $A_0 = 1$ is the unit object, the multiplication and comultiplication are homogeneous and their
A individual graded components is, strictly speaking, not necessary as all the conditions can be reformulated in terms of the individual graded components \( A_n \). Therefore we will understand a graded bialgebra \( A \) as a collection of these components: \( A = (A_n)_{n \geq 0} \). The direct sum \( \bigoplus_{n \geq 0} A_n \) can be always considered as an object of the formal direct sum completion of \( V \) but we need not require that it belongs to \( V \).

As before, there is a braided category \( \mathfrak{B} \) generated by the universal graded bialgebra \( a = (a_n)_{n \geq 0}, \quad a_0 = 1 \). Objects of \( \mathfrak{B} \) are formal tensor products \( a_\alpha = a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_r} \) associated to all the ordered partitions, i.e., sequences of positive integers \( \alpha = (\alpha_1, \cdots, \alpha_r) \). Morphisms are generated by the elementary formal morphisms

\[
\mu_{p,q} : a_p \otimes a_q \to a_{p+q}, \quad \Delta_{p,q} : a_{p+q} \to a_p \otimes a_q, \quad \mu_{p,0} = \mu_{0,p} = \Delta_{0,p} = \Delta_{0,0} = \text{Id}_{a_p},
\]
as well as the braidings, modulo the relations following from the axioms of a graded bialgebra and a braided category.

The braided category \( \mathfrak{B} \) is an example of a colored PROB in that it describes algebraic structures not on a single object but on a family of objects \( (A_n) \) of a braided category. For a discussion of colored PROPs, not PROBs see [12].

Let \( \mathcal{OP} \) be the set of all ordered partitions \( \alpha \) as above and \( \mathcal{P} \subset \mathcal{OP} \) be the set of unordered (or classical) partitions, i.e., sequences \( \alpha = (\alpha_1 \geq \cdots \geq \alpha_r > 0) \). The \( a_\alpha, \alpha \in \mathcal{P} \) form a system of representatives of isomorphism classes of objects of \( \mathfrak{B} \).

For \( \alpha \in \mathcal{OP} \) let \( |\alpha| = \sum \alpha_i \). Let \( \mathcal{OP}_n \subset \mathcal{OP} \), resp. \( \mathcal{P}_n \subset \mathcal{P} \) consist of \( \alpha \) such that \( |\alpha| = n \), i.e., that \( \alpha \) is an ordered resp. unordered partition of \( n \). Let \( \mathfrak{B}_n = \mathfrak{B} \) be the full subcategory on objects \( a_\alpha \) with \( \alpha \in \mathcal{P}_n \). It is not closed under the product \( \otimes \). The full subcategory on the \( a_\alpha, \alpha \in \mathcal{OP}_n \), is equivalent to \( \mathfrak{B}_n \).

C. Symmetric products and perverse sheaves. Let \( \text{Sym}^n(C) \) be the \( n \)th symmetric product of \( C \), i.e., the space of monic polynomials

\[
f(x) = x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in C
\]
or, equivalently, the space of effective divisors \( z = \sum_i n_i z_i \) with \( z_i \in C, \quad n_i \geq 0, \quad \sum n_i = n \).

Each such divisor \( z \) has a type which is the partition \( t(z) \in \mathcal{P}_n \) obtained by arranging the \( n_i \) in a non-increasing order, and we denote \( \text{Sym}^\alpha(C) \subset \text{Sym}^n(C) \) the subspace of \( z \) of type \( \alpha \). This gives an algebraic Whitney stratification of \( \text{Sym}^n(C) \) which we denote \( S^{(0)} \) and call the stratification by multiplicities. The open stratum of \( S^{(0)} \) is

\[
\text{Sym}_{\neq}^n(C) = \text{Sym}^{(1, \ldots, 1)}(C),
\]
the space of multiplicity-free divisors or, equivalently, of polynomials \( f(x) \) with non-zero discriminant.
Let \( \mathcal{V} \) be any abelian category (not assumed monoidal). We can then speak about \( \mathcal{V} \)-valued perverse sheaves (with respect to the middle perversity) on \( \text{Sym}^n(C) \) which are constructible with respect to the stratification \( S^{(0)} \), see [17]. They form an abelian category which we denote \( \text{Perv}(\text{Sym}^n(C); \mathcal{V}) \). For example, if \( \mathcal{V} = \text{Vect}_k \) is the category of \( k \)-vector spaces, then \( \text{Perv}(\text{Sym}^n(C); \mathcal{V}) \) is the category of perverse sheaves of \( k \)-vector spaces in the usual sense. Here is our main result, whose proof will be given at the end of Section 4.

**Theorem 1.3.** We have an equivalence of categories \( \text{Perv}(\text{Sym}^n(C); \mathcal{V}) \cong \text{Fun}(\mathcal{B}_n, \mathcal{V}) \).

In other words, we have an elementary, or quiver description of the category of perverse sheaves on \( \text{Sym}^n(C) \). The corresponding quiver (with relations) is the category \( \mathcal{B}_n \). The vertices of this quiver are the objects of \( \mathcal{B}_n \), i.e., the \( a_\alpha \), \( \alpha \in \mathcal{P}_n \). They are in bijection with the strata \( \text{Sym}^\alpha(C) \) of the stratification \( S^{(0)} \).

**Remark 1.4.** The monoidal structure \( \otimes_{m,n} : \mathcal{B}_m \times \mathcal{B}_m \to \mathcal{B}_{m+n} \) corresponds, at the level of perverse sheaves, to the functor ("comonoidal structure")

\[
\nabla_{m,n} : \text{Perv}(\text{Sym}^{m+n}(C); \mathcal{V}) \to \text{Perv}(\text{Sym}^m(C) \times \text{Sym}^n(C); \mathcal{V})
\]

defined geometrically as follows. Choose two disjoint open disks \( D, D' \subset C \) so that we have open embeddings

\[
\text{Sym}^m(C) \times \text{Sym}^n(C) \xleftarrow{i} \text{Sym}^m(D) \times \text{Sym}^n(D') \xrightarrow{j} \text{Sym}^{m+n}(C)
\]

such that the pullback functor \( i^* \) defines an equivalences on the categories of perverse sheaves with respect to the natural stratifications. Then \( \nabla_{m,n} = (i^*)^{-1} \circ j^* \).

**D. The simplest example.** (1) Let \( n = 2 \). The category \( \mathcal{B}_2 \) has two objects, \( a_1 \otimes a_1 \) and \( a_2 \), and its morphisms are generated by:

\[
\begin{array}{ccc}
R &\begin{array}{c}
\rho
\end{array} &a_1 \otimes a_1 \begin{array}{c}
\mu_{1,1}
\end{array} \begin{array}{c}
\Delta_{1,1}
\end{array} a_2 \\
\end{array}
\]

with \( R \) being the braiding, therefore invertible. Note that \( a_1 \) is primitive, i.e.,

\[
\Delta|_{a_1} = \Delta_{1,0} + \Delta_{0,1} : a_1 \longrightarrow (a_1 \otimes 1) \oplus (1 \otimes a_1)
\]

is the sum of two copies of the identity morphism of \( a_1 \). This implies that \( \Delta_{1,1}\mu_{1,1} = \text{Id} + R \), cf. [17], §5.2. In particular, \( \text{Id} - \Delta_{1,1}\mu_{1,1} = -R \) is invertible. This means that \( \text{Fun}(\mathcal{B}_2, \mathcal{V}) \) is the category of diagrams in \( \mathcal{V} \)

\[
\Psi \xleftarrow{b} \Phi
\]

such that \( T_\Psi = \text{Id}_\Psi - ab \) is invertible (which implies that \( T_\Phi = \text{Id}_\Phi - ba \) is invertible, see Eq. (1.1.6) of [17]).
(2) On the other hand, \( \text{Sym}^2(\mathbb{C}) = \mathbb{C}^2 \) is the space of quadratic polynomials \( x^2 + a_1x + a_2 \), and the only non-open stratum of \( S^{(0)} \) is the parabola given by \( a_1^2 = 4a_2 \) formed by polynomials with a double root. Factoring out by translational symmetry, we find that

\[
\text{Perv}(\text{Sym}^2(\mathbb{C}); \mathcal{V}) \simeq \text{Perv}(\mathbb{C}, 0; \mathcal{V})
\]

is identified with the category of \( \mathcal{V} \)-valued perverse sheaves on \( \mathbb{C} \) with the only possible singularity at 0.

The \( n = 2 \) instance of Theorem 1.3 reduces therefore to (a \( \mathcal{V} \)-valued version of) the classical result of [3, 10] describing \( \text{Perv}(\mathbb{C}, 0; \text{Vect}_k) \) in terms of \( (\Phi, \Psi) \)-diagrams of vector spaces as above.

E. Discussion and further plans. Theorem 1.3 can be seen as a refinement of two previous results:

(1) The main result of [17] which identifies the category of graded bialgebras \( A = (A_n)_{n \geq 0} \) in a braided monoidal \( \mathcal{V} \) with that of factorizable systems \( (F_n)_{n \geq 0} \) of perverse sheaves on all the symmetric products \( \text{Sym}^n(\mathbb{C}) \). The refinement consists in passing from such factorizable systems to individual perverse sheaves on an individual symmetric product and in allowing \( \mathcal{V} \) to be an arbitrary abelian (not necessarily monoidal) category.

(2) The special case \( \mathfrak{g} = \mathfrak{gl}_n \) of the main result of [19] which describes perverse sheaves on any quotient \( W \backslash \mathfrak{h} \) where \( \mathfrak{h} \) is the Cartan subalgebra of a complex reductive Lie algebra \( \mathfrak{g} \) and \( W \) is the Weyl group. If \( \mathfrak{g} = \mathfrak{gl}_n \), then \( W \backslash \mathfrak{h} = \text{Sym}^n(\mathbb{C}) \), and we get a description of \( \text{Perv}(\text{Sym}^n(\mathbb{C})) \). This description, however, is more cumbersome than Theorem 1.3 so the present refinement consists in giving a neater one-shot description.

Our proof of Theorem 1.3 uses the results (1) and (2) above by assembling the categories constructed in (2) into a single braided category \( \mathcal{CM} \) and constructing a graded bialgebra in this category so that application of (1) leads to an identification \( \mathcal{CM} \sim \mathcal{B} \).

It is also interesting to note the similarity between the description of the PROP \( \mathcal{HS}^+ \) given by Pirashvili [28] and the description (2) above proceeding in terms of so-called mixed Bruhat sheaves [19]. Both descriptions involve natural “bivariant” objects: covariant in one direction, contravariant in the other with some base change-type relations relating the two variances.

In a sequel to this paper [20] we plan to describe the category of perverse sheaves on the Ran space \( \text{Ran}(\mathbb{C}) \) in terms of a category related to the PROB \( \mathcal{H} \) governing braided Hopf algebras.

Both papers can be seen as developing the observation, going back to Lurie, that bialgebras are Koszul dual to \( E_2 \)-algebras and thus [6, 22] to locally constant factorization algebras on \( \mathbb{R}^2 = \mathbb{C} \) i.e., to factorizable (complexes of) sheaves on the Ran space. Informally, an \( E_2 \)-algebra can be seen as a cochain complex \( E \) with two (homotopy compatible (homotopy) associative multiplications. Now, Koszul duality gives an equivalence between associative dg-algebras and coalgebras. Applying it to one of the two multiplications on \( E \), we get a
structure consisting of (homotopy) compatible multiplication and comultiplication, i.e., a homotopy version of a dg-bialgebra $E^i$. Koszul duality being a derived equivalence, for $E^i$ to be an honest (non-dg) bialgebra, $E$ must be a nontrivial complex. What makes our approach work is a remarkable match between this type of complexes and the Cousin complexes playing an essential role in our earlier descriptions of perverse sheaves [17, 19].

F. Outline of the paper. Apart from the present introductory §1, the paper has three more sections.

In §2 we, first, give background material on braided categories and graded bialgebras in such categories. In particular, we give a self-contained treatment of Deligne's interpretation of braidings in terms of “2-dimensional tensor products” of objects labelled by points in the plane. We recall the concept of contingency matrices and their vertical and horizontal contractions from [18]. We further associate to a contingency matrix $M = \{m_{ij}\}$ and a graded bialgebra $A$ an object $A_M$ which is the 2-dimensional tensor product of the components $A_{m_{ij}}$. We use the multiplication and comultiplication in $A$ to connect the objects $A_M$, and $A_N$ whenever $M$ is obtained from $N$ by a vertical or horizontal contractions and establish (Proposition 2.8) a system of relations for such connecting morphisms.

In §3 we define a category $\mathcal{CM}$ whose objects are symbols $[M]$ associated to contingency matrices $M$ and the relations of Proposition 2.8 are promoted into into a system of defining relations for the morphisms of the category. Thus any graded bialgebra $A$ in any braided category $\mathcal{V}$ gives rise to a functor $\xi_A : \mathcal{CM} \rightarrow \mathcal{V}$ (Corollary 3.2) sending $[M]$ to $A_M$. We further make $\mathcal{CM}$ into a braided monoidal category so that the functor $\xi_A$ above is in fact braided monoidal (Proposition 3.9).

In §4 we notice that $\mathcal{CM}$ carries a braided bialgebra $a$ with components $[n]$ associated to $1 \times 1$ matrices $n$. This allows us to connect $\mathcal{B}$ and $\mathcal{CM}$ by a braided monoidal functor which we show to be an equivalence (Theorem 4.3). Finally, for each $n$ we compare the degree $n$ block $\mathcal{CM}_n$ with the specialization, for $g = gl_n$, of the concept of mixed Bruhat sheaf which was introduced in [19] for description of perverse sheaves on the adjoint quotient $W\backslash g$ of a reductive Lie algebra $g$. This comparison yields the first in the two identifications below

$$\text{Perv}(\text{Sym}^n(\mathcal{C}); \mathcal{V}) \simeq \text{Fun}(\mathcal{CM}_n, \mathcal{V}) \overset{\text{Thm. 4.3}}{\simeq} \text{Fun}(\mathcal{B}_n, \mathcal{V})$$

thus proving Theorem 1.3.

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2 Graded bialgebras and contingency matrices

A. Braids, braided categories and bialgebras. Let $\text{Br}_n$ be the braid group on $n$ strands, with the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

\[(2.1) \quad \sigma_p\sigma_{p+1}\sigma_p = \sigma_{p+1}\sigma_p\sigma_{p+1}, \quad \sigma_p\sigma_q = \sigma_q\sigma_p, \quad |p-q| \geq 2.\]

Let also $S_n$ be the symmetric group with the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the same relations as in (2.1) together with $\sigma_i^2 = 1$. Thus we have the surjective morphism $p : \text{Br}_n \to S_n$.

By a monoidal category we will mean a strictly associative monoidal category, with a strict unit object denoted by $1$. Let $V$ be a braided monoidal category with braiding denoted by $R : V \otimes W \to W \otimes V$.

Given objects $V_1, \ldots, V_n$ of $V$, and an element $b \in \text{Br}_n$, we have the permutation $t = p(b) \in S_n$, and the braiding isomorphism

$$R_b : V_1 \otimes \cdots \otimes V_n \longrightarrow V_{t(1)} \otimes \cdots \otimes V_{t(n)}.$$

A bialgebra in $V$ is an object $A$ of $V$ equipped with an associative multiplication $\mu : A \otimes A \to A$ and a coassociative comultiplication $\Delta : A \to A \otimes A$ satisfying the following compatibility condition: $\Delta$ is a morphism of algebras, if the multiplication on $A \otimes A$ is defined using the braiding:

$$A \otimes A \otimes A \otimes A \xrightarrow{\text{Id} \otimes R \otimes \text{Id}} A \otimes A \otimes A \otimes A \xrightarrow{\mu \otimes \mu} A \otimes A.$$

B. Geometric interpretation of braided categories. Let $V$ be a category, $I$ be a finite set and $(V_i)_{i \in I}$ be a family of objects of $V$ labelled by $I$. If $V$ is symmetric monoidal, then we can speak about the object $\bigotimes_{i \in I} V_i$ without specifying an order on $I$: the ordered tensor products in different orders are canonically identified with each other.

If $V$ is braided monoidal, then the notation $\bigotimes_{i \in I} V_i$ does not make sense, as there is no single canonical identification between two given ordered products. It was pointed out by Deligne that the correct structure on $I$ to make the product canonical is not an ordering (which of course suffices) but an embedding $I \hookrightarrow \mathbb{C}$. In other words, once we assign to each $i \in I$ a complex number $z_i \in \mathbb{C}$ such that $z_i \neq z_j$ for $i \neq j$, there is a “well-defined object” $\bigotimes_{i \in I} V_i(z_i)$ (the 2-dimensional tensor product with respect to $V_i$ positioned as $z_i$). When the $z_i$ move, these objects unite into a local system on $\mathbb{C}^I_\neq$ whose monodromy gives the braiding. For convenience of the reader we recall a precise elementary construction.

**Definition 2.2.** Let $V$ be a category.

(a) A pseudo-object (or an object defined up to a unique isomorphism) of $V$ is a datum $V$ of a set $K$, of objects $V_k \in V$ for each $k \in K$ and of isomorphisms $\varphi_{k,k'} : V_k \to V_{k'}$ given for each $k, k'$ and satisfying

$$\varphi_{kk} = \text{Id}, \quad \varphi_{k',k''} \varphi_{k,k'} = \varphi_{k,k''}, \quad \forall k, k', k''.$$
The objects $V_k$ are called the determinations of the pseudo-object $V$, and the morphisms $\varphi_{kk'}$ are called the transition maps of $V$.

(b) A morphism $u$ from a pseudo-object $V = (V_k, \varphi_{kk'})_{k,k' \in K}$ to a pseudo-object $W = (W_l, \psi_{l,l'})_{l,l' \in L}$ of $\mathcal{V}$ is a datum of morphisms $u_{k,l}: V_k \to W_l$ for all $k \in K$, $l \in L$ such that

$$u_{k,l} = u_{k',l} \varphi_{k,k'}, \quad u_{k,l'} = \psi_{l,l'} u_{k,l}, \quad \forall k, k' \in K, \ l, l' \in L.$$ 

With this definition, pseudo-objects in $\mathcal{V}$ form a category $Ps(\mathcal{V})$. Any actual object of $\mathcal{V}$ can be considered as a pseudo-object with $|K| = 1$. This gives a functor $\mathcal{V} \to Ps(\mathcal{V})$ which is easily seen to be an equivalence. In this way a pseudo-object can be seen to be “as good as an actual object” of $\mathcal{V}$.

We will construct $\bigotimes_{i \in I} V_i(z_i)$ as a pseudo-object and start with describing its indexing set $K$.

**Definition 2.3.** Let $Z \subset \mathbb{C}$ be a finite subset, $|Z| = n > 0$. A $Z$-snake is a simple curve $S \subset \mathbb{C}$ which is a finite perturbation of $\mathbb{R}$ and passes through each element of $Z$ once. See the center and right of Fig. 1.

![Snakes as the images of the interval $(-1, 1)$ in the disk.](image)

Denote by $\text{Sn}(Z)$ the set of isotopy classes of $Z$-snakes. A snake being oriented from $(-\infty)$ to $(+\infty)$, each $S \in \text{Sn}(Z)$ gives an ordering $i_1 = i_1(S), \ldots, i_n = i_n(S)$ of the set $I$. It is classical that $\text{Sn}(Z)$ is a left torsor over $\text{Br}_n$, with $\sigma_p S$ being obtained from $S$ by an “upper twist” reversing the path between $z_{ip(S)}$ and $z_{ip+1(S)}$, see the right of Fig. 1. Conceptually, this follows from the interpretation of $\text{Br}_n$ as the mapping class group of the $n$-pointed disk $[7, 8]$. More precisely, let $D = \{|z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$ with $\partial D = S^1$ the unit circle and let $-1 < t_1 < \cdots < t_n < 1$ be real numbers. Then $\text{Br}_n \cong \pi_0(\mathcal{H})$, where

$$\mathcal{H} = \text{Homeo}(D, \text{Id}_{\partial D}, \{t_1, \cdots, t_n\})$$

is the group of homeomorphisms of $D$, identical on $\partial D$ and preserving $\{t_1, \cdots, t_n\}$ as a set.

At the same time, let $\overline{\mathbb{C}} = \mathbb{C} \cup S^1_{\infty}$ be the disk compactification of $\mathbb{C}$ by $S^1_{\infty}$, the circle of directions at $\infty$.

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Proposition 2.4. We have an identification $\text{Sn}(Z) \cong \pi_0(\mathcal{M})$, where $\mathcal{M}$ is the space of homeomorphisms $f : D \to \mathbb{C}$ which restrict to the standard identification $\partial D = S^1 \to S^1_\infty$ on the boundary and take $\{t_1, \cdots, t_n\}$ to $Z$. The snake corresponding to $f \in \mathcal{M}$ is the image $f((−1, 1))$ (flattened near $\pm \infty$ to coincide with $\mathbb{R}$ there). See Fig. 1.

Proof: This is equivalent to the classical encoding of braids by curve diagrams, see [7] §1.3.3, esp. Fig. 2 there.

As $\mathcal{M}$ is a torsor over $\mathcal{H}$, we have that $\text{Sn}(Z) = \pi_0(\mathcal{M})$ is a torsor over $\text{Br}_n = \pi_0(\mathcal{H})$.

Definition 2.5. Let $\mathcal{V}$ be a braided category, $(V_i)_{i \in I}, |I| = n$ be a family of objects and $z_i \in \mathbb{C}, i \in I$ be distinct complex numbers. Let $Z = \{z_i\}_{i \in I} \subset \mathbb{C}$. We define a pseudo-object $\bigotimes_{i \in I} V_i(z_i)$ of $\mathcal{V}$ with the indexing set $K = \text{Sn}(Z)$. For $S \in \text{Sn}(Z)$ the corresponding determination is defined as

$$\left( \bigotimes_{i \in I} V_i(z_i) \right)_S = V_{i_1(S)} \otimes \cdots \otimes V_{i_n(S)},$$

the ordered tensor product along the snake. Given two snakes $S, S' \in \text{Sn}(Z)$ with $S' = bS$, $b \in \text{Br}_n$, we define the transition map

$$\varphi_{S,S'} := R_b : V_{i_1(S)} \otimes \cdots \otimes V_{i_n(S)} \to V_{i_1(S')} \otimes \cdots \otimes V_{i_n(S')}$$

to be the braiding isomorphism associated to $\beta$.

C. Factorization algebra point of view on a braided category. It is convenient to extend Definition 2.5 slightly, bringing it close to the formalism of factorization algebras [6].

By a closed disk in $\mathbb{C}$ we mean a subset $D \subset \mathbb{C}$ which is either homeomorphic to the standard disk $\{|z| \leq 1\} \subset \mathbb{C}$ or is a single point (a disk of radius 0). Let $(V_i)_{i \in I}$ be as above and $Z_i \subset \mathbb{C}$ be disjoint closed disks. We define

$$\bigotimes_{i \in I} V_i(z_i) := \bigotimes_{i \in I} V_i(z_i), \quad \forall z_i \in Z_i, \ i \in I.$$

Formally, we view the objects in the RHS of this definition as the stalks of a $\mathcal{V}$-valued local system on the contractible space $\prod_{i \in I} Z_i$ and we define the LHS as the object of global sections of this local system.

Alternatively, let $Z = \bigcup_{i \in I} Z_i$. Then we can speak about $Z$-snakes that is, simple curves $S \subset \mathbb{C}$ which are finite perturbations of $\mathbb{R}$ which intersect each $Z_i$ along a closed interval (possibly reducing to a single point). A $Z$-snake defines an ordering on $I = \pi_0(Z)$. As before, the set $\text{Sn}(Z)$ of isotopy classes of $Z$-snakes is a $\text{Br}_n$-torsor, and we can define $\bigotimes_{i \in I} V_i(Z_i)$ as a pseudo-object with indexing set $\text{Sn}(Z)$ consisting of ordered tensor products along the snakes.
Proposition 2.6. (1) Let \( V \) be a braided category, \( f : J \to I \) a surjection of finite sets and \((V_j)_{j \in J}\) a family of objects of \( V \). Let \((Z_i)_{i \in I}\) and \((Y_j)_{j \in J}\) be two families of closed disks in \( \mathbb{C} \), each consisting of disjoint disks and such that \( Y_j \subset Z_{f(j)} \), \( j \in J \). In each such situation we have a canonical associativity isomorphism

\[
\alpha_f : \bigotimes_{i \in I} \left( \bigotimes_{j \in f^{-1}(i)} V_j(Y_j) \right)(Z_i) \to \bigotimes_{j \in J} V_j(Y_j).
\]

These isomorphisms satisfy the following compatibility.

(2) Let \( K \to J \to I \) be two composable surjections of finite sets, and \((V_k)_{k \in K}\) be a family of objects of \( V \). Let \((X_k)_{k \in K}\), \((Y_j)_{j \in J}\) and \((Z_i)_{i \in I}\) be three families of closed disks in \( \mathbb{C} \), each consisting of disjoint disks and such that \( X_k \subset Y_{g(k)} \), \( k \in K \) and \( Y_j \subset Z_{f(j)} \), \( j \in J \). For each \( i \in I \) let \( g_i : (fg)^{-1}(i) \to f^{-1}(i) \) be the restriction of \( g \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\bigotimes_{i \in I} \left( \bigotimes_{j \in f^{-1}(i)} V_k(X_k) \right)(Z_i) & & \bigotimes_{j \in J} \left( \bigotimes_{k \in g^{-1}(j)} V_k(X_k) \right)(Z_j) \\
\downarrow \alpha_{f} & & \downarrow \alpha_{g} \\
\bigotimes_{i \in I} \left( \bigotimes_{k \in (fg)^{-1}(i)} V_k(X_k) \right)(Z_i) & & \bigotimes_{k \in K} V_k(X_k)
\end{array}
\]

Proof (sketch): (1) Let \( n = |I|, m = |J|, m_i = |f^{-1}(i)| \), so \( m = \sum_{i \in I} m_i \). Let also

\[
Z = \bigcup_{i \in I} Z_i, \quad Y = \bigcup_{j \in J} Y_j, \quad Y/i = \bigcup_{j \in f^{-1}(i)} Y_j, \quad i \in I.
\]

Call a \((Y,Z)\)-snake a curve \( S \) which is both a \( Z \)-snake and a \( Y/i \)-snake for each \( i \). A \((Y,Z)\)-snake is also a \( Y \)-snake. Let \( \text{Sn}(Y,Z) \) be the set of isotopy classes of \((Y,Z)\)-snakes. It is a torsor over the subgroup \( B \) in \( \text{Br}_m \) which is the wreath product of the \( \text{Br}_{m_i} \). We can view both the source and target of the desired \( \alpha_f \) as pseudo-objects with the indexing set \( \text{Sn}(Y,Z) \), after which the map \( \alpha_f \) is defined as the identity on each determination (corresponding to each \((Y,Z)\)-snake).

To prove (2), we introduce, similarly to (1), the concept of \((X,Y,Z)\)-snakes and the set \( \text{Sn}(X,Y,Z) \) of their isotopy classes. After this, all the arrows in the diagrams can be seen as morphisms of pseudo-objects indexed by \( \text{Sn}(X,Y,Z) \) and the statement becomes obvious.

\[\Box\]

D. Contingency matrices. We recall some constructions and terminology of [18]. By a contingency matrix we mean a rectangular matrix \( M = [m_{ij}]_{i=1}^{n} \) with \( m_{ij} \in \mathbb{Z}_{\geq 0} \) such that each row and each column contain at least one non-zero entry. We will also formally include the contingency matrix \( \emptyset \) of size \( 0 \times 0 \). The weight of \( M \) is the number

\[
\Sigma M = \sum_{i,j} m_{ij} \in \mathbb{Z}_{\geq 0}, \quad M \neq \emptyset, \quad \Sigma \emptyset = 0.
\]

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We denote by CM the set of all contingency matrices, by CM_n the set of all contingency matrices of weight n, by CM(r, s) the set of all contingency matrices of size r \times s and put CM_n(r, s) := CM_n \cap CM(r, s). We have the horizontal and vertical contractions
\[
\begin{align*}
\hat{\partial}_j^i : CM_n(r, s) &\rightarrow CM_n(r, s - 1), \quad i = 0, \ldots, s - 2, \\
\hat{\partial}_i^m : CM_n(r, s) &\rightarrow CM_n(r - 1, s), \quad j = 0, \ldots, r - 2,
\end{align*}
\]
which add up the (j+1)st and the (j+2)nd column (resp. (i+1)st and (i+2)nd row). These contractions make the collection of CM_n(r, s) for all r, s into an augmented bi-semisimplicial set, see [18], Prop. 1.4.

For M, N \in CM_n we put M \lessdot N, if M can be obtained from N by a series of horizontal contractions \hat{\partial}_j^i and M \lessdot" N, if M can be obtained from N by a series of vertical contractions \hat{\partial}_i^m. This defines two partial orders \lessdot', \lessdot" on CM_n. Thus we have the elementary inequalities \hat{\partial}_j^i N \lessdot' N, resp. \hat{\partial}_i^m N \lessdot N and the partial order \lessdot', resp. \lessdot" is generated by such elementary inequalities via transitive closure. For any M, L \in CM_n we write
\[
\text{Sup}(M, L) = \{ O \in CM_n | M \lessdot O \lessdot" P \}.
\]
An elementary inequality \hat{\partial}_j^i' N \lessdot' N is called anodyne if, for any j, among the two entries \(n_{i,j+1}\) and \(n_{i,j+2}\) of the (j + 1)st and (j + 2)nd column of N that are added in \(\hat{\partial}_j^i' N\), there is at least one zero. For example,
\[
\hat{\partial}_i^0 N = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \lessdot' \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 3 \end{pmatrix} = N
\]
is anodyne. A general inequality M \lessdot' N is called anodyne, if there is a chain of elementary anodyne inequalities M = M_1 \lessdot' \cdots \lessdot' M_k = N connecting M and N.

Similarly, an elementary inequality \hat{\partial}_i^m N \lessdot" N is called anodyne if, for each j, among the two entries \(n_{i+1,j}\) and \(n_{i+2,j}\) there is at least one zero. A general inequality M \lessdot" N is called anodyne, if there is a chain of elementary anodyne inequalities M = M_1 \lessdot" \cdots \lessdot" M_k = N.

E. Components of a graded bialgebra associated to contingency matrices. Let \(A = (A_n)_{n \geq 0}\) be a graded bialgebra in a braided monoidal category \(\mathcal{V}\). Let \(M \in CM(r, s)\) be a contingency matrix of size \(r \times s\). We put
\[
A_M = \bigotimes_{(p, q) \in \{1, \ldots, r\} \times \{1, \ldots, s\}} A_{m_{p, q}}(p, q),
\]
the 2-dimensional tensor product corresponding to \(A_{m_{p, q}}\) put in the position \((p, q) \in \mathbb{R}^2\), i.e., \(p + q\sqrt{-1} \in \mathbb{C}\). We call \(A_M\) the component of \(A\) associated to \(M\) (even though it is, strictly speaking, a tensor product of graded components).
By definition, $A_M$ is a pseudo-object with indexing set $\text{Sn}(Z)$ consisting of (isotopy classes of) snakes passing through $Z = \{1, \cdots, r\} \times \{1, \cdots, s\}$. Among such snakes we distinguish the lexicographic snake $\text{Lex}$ which reads the elements of $Z$ one by one horizontally and the antilexicographic snake $\text{Alex}$ which reads them one by one vertically, see Fig. 2. The corresponding determinations of $A_M$ are

\begin{align}
(A_M)_{\text{Lex}} &= (A_{m_{1,1}} \otimes A_{m_{2,1}} \otimes \cdots \otimes A_{m_{r,1}}) \otimes \cdots \otimes (A_{m_{1,s}} \otimes A_{m_{2,s}} \otimes \cdots \otimes A_{m_{r,s}}), \\
(A_M)_{\text{Alex}} &= (A_{m_{1,1}} \otimes A_{m_{1,2}} \otimes \cdots \otimes A_{m_{1,s}}) \otimes \cdots \otimes (A_{m_{r,1}} \otimes A_{m_{r,2}} \otimes \cdots \otimes A_{m_{r,s}}),
\end{align}

i.e., the ordered tensor products of the $A_{m_{ij}}$ read along the columns, resp. rows of $M$.

**F. Horizontal comultiplication and vertical multiplication.** Our next goal is to define, for any $M \leq N$, the horizontal comultiplication map $\Delta_{M,N} : A_M \to A_N$, and for any $M \leq N$, the vertical multiplication map $\mu_{N,M} : A_N \to A_M$, using the comultiplication and multiplication in $A$.

Let first

$$\text{CM}(r, s) \ni M = c_j'N \leq' N \in \text{CM}(r, s + 1), \quad j \in \{1, \cdots, s - 1\}$$

be an elementary inequality, so that for each $p$ we have:

$$m_{p,j+1} = n_{p,j+1} + n_{p,j+2}, \quad m_{p,q} = n_{p,q}, \quad q \leq j, \quad m_{p,q} = n_{p,q+1}, \quad q \geq j + 2.$$ 

Introduce closed disks $Z_{p,q}' \subset \mathbb{C}$, $p = 1, \cdots, r$, $q = 1, \cdots, s$ by:

$$Z_{p,q}' = \begin{cases} 
\text{A thin ellipse encircling the points } (p, j + 1) \text{ and } (p, j + 2), & \text{ if } q = j + 1; \\
\text{the point } (p, q), & \text{ if } q \leq j; \\
\text{the point } (p, q + 1), & \text{ if } q \geq j + 2.
\end{cases}$$

Then, on one hand, we have a canonical identification

$$A_M = \bigotimes_{(p,q) \in \{1, \cdots, r\} \times \{1, \cdots, s\}} A_{m_{p,q}}(p,q) \longrightarrow \bigotimes_{(p,q) \in \{1, \cdots, r\} \times \{1, \cdots, s\}} A_{m_{p,q}}(Z_{p,q}')$$

obtained by moving each $Z_{p,q}'$ to the point $(p, q)$ along the shortest (straight) path and then contracting it to that point if needed. On the other hand, Proposition 2.6 gives an identification

$$A_N = \bigotimes_{(p,q) \in \{1, \cdots, r\} \times \{1, \cdots, s\}} A_{p,q}'(Z_{p,q}').$$
where

\[ A'_{p,q} = \bigotimes_{(a,b)\in Z'_{p,q}} A_{n_{a,b}} = \begin{cases} A_{n_{p,j+1}} \otimes A_{n_{p,j+2}}, & \text{if } q = j + 1; \\ A_{n_{p,q}}, & \text{if } q \leq j; \\ A_{n_{p,q+1}}, & \text{if } q \geq j + 2. \end{cases} \]

Using these identifications, we define a morphism

\[ \Delta_{M,N} : A_M \longrightarrow A_N \]

to be the 2-dimensional tensor product, over \((p, q) \in \{1, \cdots, r\} \times \{1, \cdots, s\}\), of the morphisms \(\Delta^{(p,q)} : A_{m_{p,q}} \rightarrow A'_{p,q}\) given by

\[ \Delta^{(p,q)} = \begin{cases} \Delta_{n_{p,j+1},n_{p,j+2}} : A_{m_{p,j+1}} \longrightarrow A_{n_{p,j+1}} \otimes A_{n_{p,j+2}}, & \text{if } q = j + 1; \\ \text{Id} : A_{m_{p,q}} \longrightarrow A_{n_{p,q}}, & \text{if } q \leq j; \\ \text{Id} : A_{m_{p,q}} \longrightarrow A_{n_{p,q+1}}, & \text{if } q \geq j + 2 \end{cases} \]

and positioned at the \(Z'_{p,q}\). In order to write \(\Delta_{M,N}\) as a morphism of ordered tensor products without the use of the braiding, we can use the antilexicographic determinations.

Similarly, let

\[ \text{CM}(r, s) \ni M = \hat{\bigotimes}_i^{r-1} N \leq^{\bigotimes} N \in \text{CM}(r+1, s), \quad i \in \{1, \cdots, r-1\} \]

be an elementary inequality, so that for each \(q\) we have

\[ m_{i+1,q} = n_{i+1,q} + n_{i+2,q}, \quad m_{p,q} = n_{p,q}, \quad p \leq i, \quad m_{p,q} = n_{p+1,q}, \quad p \geq i + 1. \]

Introduce closed disks \(Z''_{p,q} \subset \mathbb{C}, \ p = 1, \cdots, r, \ q = 1, \cdots, s\) by:

\[ Z''_{p,q} = \begin{cases} \text{A thin ellipse encircling the points } (i+1,q) \text{ and } (i+2,q), & \text{if } p = i + 1; \\ \text{the point } (p,q), & \text{if } p \leq i; \\ \text{the point } (p+1,q), & \text{if } p \geq i + 2. \end{cases} \]

Then, on one hand, we have a canonical identification

\[ A_M = \bigotimes_{(p,q)\in\{1,\cdots,r\}\times\{1,\cdots,s\}} A_{m_{p,q}}(p,q) \longrightarrow \bigotimes_{(p,q)\in\{1,\cdots,r\}\times\{1,\cdots,s\}} A_{m_{p,q}}(Z''_{p,q}) \]

obtained by moving each \(Z''_{p,q}\) to the point \((p,q)\) along the shortest (straight) path and then contracting it to that point if needed. On the other hand, Proposition 2.6 gives an identification

\[ A_N = \bigotimes_{(p,q)\in\{1,\cdots,r\}\times\{1,\cdots,s\}} A''_{p,q}(Z''_{p,q}); \]

where

\[ A''_{p,q} = \bigotimes_{(a,b)\in Z''_{p,q}} A_{n_{a,b}} = \begin{cases} A_{n_{i+1,q}} \otimes A_{n_{i+2,q}}, & \text{if } p = i + 1; \\ A_{n_{p,q}}, & \text{if } p \leq i; \\ A_{n_{p+1,q}}, & \text{if } p \geq i + 2. \end{cases} \]
Using these identifications, we define a morphism

\[ \mu_{N,M} : A_N \longrightarrow A_M \]

to be the 2-dimensional tensor product, over \((p,q) \in \{1, \cdots, r\} \times \{1, \cdots, s\}\), of the morphisms \(\mu^{(p,q)} : A''_{p,q} \to A_{m_{p,q}}\) given by

\[ \mu^{(p,q)} = \begin{cases} 
\mu_{n_{i+1,q},n_{i+2,q}} : A_{n_{i+1,q}} \otimes A_{n_{i+2,q}} \to A_{m_{i+1,q}}, &\text{if } p = i + 1; \\
\text{Id} : A_{m_{p,q}} \to A_{n_{p,q}}, &\text{if } p \leq i; \\
\text{Id} : A_{m_{p,q}} \to A_{n_{p+1,q}}, &\text{if } p \geq i + 2
\end{cases} \]

and positioned at the \(Z''_{p,q}\). In order to write \(\mu_{N,M}\) as a morphism of ordered tensor products without the use of the braiding, we can use the lexicographic determinations.

**Proposition 2.8.** (a') Let \(M \leq' N\). For all chains of elementary inequalities \(M = M_1 \leq' \cdots \leq' M_k = N\), the composition

\[ \Delta_{M,N} = \Delta_{M_{k-1},M_k} \Delta_{M_{k-2},M_{k-1}} \cdots \Delta_{M_1,M_2} : A_M \longrightarrow A_N \]

has the same value.

(a'') Let \(M \leq'' N\). For all chains of elementary inequalities \(M = M_1 \leq'' \cdots \leq'' M_k = N\), the composition

\[ \mu_{N,M} = \mu_{M_2,M_1} \mu_{M_3,M_2} \cdots \mu_{M_k,M_{k-1}} : A_N \longrightarrow A_M \]

has the same value.

(b) The morphisms \(\Delta_{M,N}, M \leq' N\) and \(\mu_{N,M}, M \leq'' N\) thus defined satisfy the following properties:

(b1') If \(L \leq' M \leq' N\), then \(\Delta_{L,N} = \Delta_{M,N} \Delta_{L,M}\).

(b1'') If \(L \leq'' M \leq'' N\), then \(\mu_{N,L} = \mu_{M,L} \mu_{N,M}\).

(b2) If \(M \geq'' N \leq' L\), then

\[ \Delta_{N,L} \mu_{N,M} = \sum_{O \in \text{Sup}(M,L)} \mu_{O,L} \Delta_{M,O} : A_M \longrightarrow A_L. \]

(b3') If \(M \leq' N\) is an anodyne inequality, then \(\Delta_{M,N}\) is an isomorphism.

(b3'') If \(M \leq'' N\) is an anodyne inequality, then \(\mu_{N,M}\) is an isomorphism.

**Proof:** Parts (a') and (b1') follow from coassociativity of the comultiplication. Parts (a'') and (b1'') follow from associativity of the multiplication. Parts (b3') and (b3'') are obvious from the definitions. It remains to prove (b2). We do it in three steps.
Consider first the simplest case when $L = (l_1, l_2)$ is a $1 \times 2$ matrix, $M = (m_1, m_2)^t$ is a $2 \times 1$ matrix and $N = (n)$ is a $1 \times 1$ matrix so that

$$l_1 + l_2 = m_1 + m_2 = n.$$ 

In this case $\text{Sup}(M, L)$ consists of $2 \times 2$ contingency matrices $O = \begin{pmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{pmatrix}$ such that

$$o_{11} + o_{21} = l_1, \quad o_{12} + o_{22} = l_2, \quad o_{11} + o_{12} = m_1, \quad o_{21} + o_{22} = m_2.$$ 

The claim (b2) has then the form

$$(2.9) \quad \Delta_{l_1,l_2} \mu_{m_1,m_2} = \sum_{O \in \text{Sup}(M, L)} (\mu_{o_{11},o_{21}} \otimes \mu_{o_{12},o_{22}}) \circ (\text{Id} \otimes R_{A_{o_{12}}}, A_{o_{21}} \otimes \text{Id}) \circ (\Delta_{o_{11},o_{12}} \otimes \Delta_{o_{21},o_{22}}),$$

the appearance of the braiding in the middle coming from comparing the Alex and Lex determinations of $A_O$. But this equality is simply the reformulation, at the level of graded components, of the compatibility between multiplication and comultiplication in $A$. Cf. [17] Eq. (4.2.4).

Step 2. Next, suppose that $N$ is a contingency matrix of arbitrary size $r \times s$ and both inequalities $M \geq^n N \leq^l L$ are elementary, so $N = \partial^i_j L = \partial^i_j M$ for some $i$ and $j$.

The set $\text{Sup}(M, L)$ consists then of $(r + 1) \times (s + 1)$ contingency matrices $O$ such that $\partial^i_j O = M, \partial^i_j O = L$. The only part of $O$ not fixed by these conditions, is the $2 \times 2$ submatrix $\overline{O}$ on rows $i + 1, i + 2$ and columns $j + 1, j + 2$. Therefore the situation is combinatorially similar to Step 1. More precisely, $\text{Sup}(M, L)$ is in bijection with the set of $2 \times 2$ contingency matrices $\overline{O} = \begin{pmatrix} o_{i+1,j+1} & o_{i+1,j+2} \\ o_{i+2,j+1} & o_{i+2,j+2} \end{pmatrix}$ such that

$$o_{i+1,j+1} + o_{i+1,j+2} = m_{i+1,j+1}, \quad o_{i+2,j+1} + o_{i+2,j+2} = m_{i+2,j+1},$$

$$o_{i+1,j+1} + o_{i+2,j+1} = l_{i+1,j+1}, \quad o_{i+1,j+2} + o_{i+2,j+2} = l_{i+1,j+2}.$$ 

Let

$$\overline{\lambda}_\overline{p} : A_{m_{i+1,j+1}} \otimes A_{m_{i+2,j+1}} \longrightarrow A_{l_{i+1,j+1}} \otimes A_{l_{i+1,j+2}}$$

be the adaptations to our case of the LHS and RHS of (2.9), i.e.,

$$\overline{\lambda} = \Delta_{l_{i+1,j+1},l_{i+1,j+2}} \mu_{m_{i+1,j+1},m_{i+2,j+1}},$$

and $\overline{p}$ is the sum over the $\overline{O}$ as above. Thus $\overline{\lambda} = \overline{p}$ by Step 1.

We claim that the equality (b2) in our situation reduces to that in Step 1, i.e., to the equality (2.9). Indeed, let $\lambda, \rho : A_M \rightarrow A_L$ be the LHS and RHS of (b2). Note that each of these morphisms is decomposed as a 2-dimensional tensor product of “elementary” morphisms of the following 4 types:

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- The identity morphism from some $A_{m_{pq}}$ to some $A_{l_{p'q'}}$ for $p \neq j + 1$, $q \neq i + 1, i + 2$, the same morphism for both $\lambda$ and $\rho$.

- The multiplication $A_{m_{i+1,q}} \otimes A_{m_{i+2,q}} \rightarrow A_{l_{i+1,q}}$ for $q \neq i + 1$, the same morphism for both $\lambda$ and $\rho$.

- The comultiplication $A_{m_{i+1,q}} \rightarrow A_{l_{i+1,q}} \otimes A_{l_{i+2,q}}$ for $q \neq i + 1, j + 2$, the same morphism for both $\lambda$ and $\rho$.

- The morphism $\lambda$ for $\lambda$ and $\rho$ for $\rho$.

This implies that $\lambda = \rho$, thus establishing Step 2.

**Step 3.** Let now $M \geq^r N \leq^l L$ be arbitrary. Let us represent both inequalities as chains of elementary ones:

\[(2.10) \quad M = M_0 \geq^r M_1 \geq^r \cdots \geq^r M_a = N \leq^l M_{a+1} \leq^l \cdots \leq^l M_{a+b} = L.\]

Note that there is a unique chain as above with given $M$ and $L$; in particular, $N$ with $M \geq^r N \leq^l L$ is unique (if it exists, which is our assumption). We deduce the equality (b2) by applying Step 2 several times. For this, consider *taxicab paths*

$$\gamma = [x_0, x_1, \cdots, x_{a+b}]$$

in the rectangle $[0, a] \times [0, b]$. Such a path consists of segments $[x_{i-1}, x_i]$, $i = 1, \cdots, a + b$ of length 1 which can be either horizontal or vertical, with $x_0 = (0, 0)$ and $x_{a+b} = (a, b)$, see Fig. 3.

![Figure 3: A taxicab path $\gamma$ in a rectangle $[0, a] \times [0, b]$.](image_url)

Given such $\gamma = [x_0, \cdots, x_{a+b}]$, we call a $\gamma$-*chain* a sequence of contingency matrices $L_\bullet = (L_0 = M, L_1, \cdots L_{a+b} = L)$ such that:

1. $L_{i-1} \geq^r L_i$, if the interval $[x_{i-1}, x_i]$ is horizontal.

2. $L_{i-1} \leq^l L_i$, if the interval $[x_{i-1}, x_i]$ is vertical.
Note that the equalities in a $\gamma$-chain must be elementary. Let $\text{Ch}(\gamma)$ be the set of $\gamma$-chains. For $L_\bullet \in \text{Ch}(\gamma)$ we have the morphism $T_{L_\bullet} : A_M \to A_L$ defined as the composition

$$A_M = A_M L_0 \xrightarrow{T_{L_\bullet, 1}} A_M L_1 \xrightarrow{T_{L_\bullet, 2}} \cdots \xrightarrow{T_{L_\bullet, a+b}} A_M L_{a+b} = A_L,$$

where $T_{L_\bullet, i} = \mu_{L_{i-1}, L_i}$, if the interval $[x_{i-1}, x_i]$ is horizontal and $T_{L_\bullet, i} = \Delta_{L_{i-1}, L_i}$, if the interval $[x_{i-1}, x_i]$ is vertical. The following is straightforward.

**Lemma 2.11.** (a) For $\gamma = \gamma_{\min} := [(0,0), (1,0), \cdots, (a,0), (a,1), \cdots, (a,b)]$ being the minimal (bottom right) path, there is a unique $\gamma$-chain, namely (2.10).

(b) For $\gamma = \gamma_{\max} := [(0,0), (0,1), \cdots, (0,b), (1,b), \cdots, (a,b)]$ being the maximal (left top) path, the set $\text{Ch}(\gamma)$ is in bijection with $\text{Sup}(M, L)$. More precisely, for each $O \in \text{Sup}(M, L)$ there is a unique $\gamma$-chain $L_\bullet$ such that $L_b = O$. \qed

The lemma implies that

$$\Delta_{N,L,M,N} = \sum_{L_\bullet \in \text{Ch}(\gamma_{\min})} T_{L_\bullet}, \quad \sum_{O \in \text{Sup}(M, L)} \mu_{O,L} \Delta_{M,O} = \sum_{L_\bullet \in \text{Ch}(\gamma_{\max})} T_{L_\bullet},$$

the sum in the RHS of the first equality consisting of one summand. So our statement reduces to the following:

**Lemma 2.12.** The sum $\sum_{L_\bullet \in \text{Ch}(\gamma)} T_{L_\bullet}$ is independent on the taxicab path $\gamma$ in $[0,a] \times [0,b]$.

**Proof:** It is enough to show the invariance of the sum under an elementary modification of a path along a $1 \times 1$ square which changes a horizontal-then-vertical pair of unit intervals to the vertical-then-horizontal pair completing the square. But such invariance is a consequence of Step 2, because the inequalities corresponding to unit intervals are elementary ones. \qed

This establishes Step 3 and Proposition 2.8 is proved.
3 The category of contingency matrices as a braided monoidal category

A. Contingency matrices as objects of a category. We introduce a category \( \mathcal{CM} \) to have, as objects, formal symbols \( [M] \) for all contingency matrices \( M \in \mathcal{CM} \). Morphisms in \( \mathcal{CM} \) are generated by the generating morphisms

\[
\delta'_{M,N} : [M] \to [N], \quad M \preceq' N, \quad \delta''_{N,M} : [N] \to [M], \quad M \preceq'' N
\]

subject to the relations

(\( \text{CM}1' \)) If \( L \preceq' M \preceq' N \), then \( \delta'_{L,N} = \delta'_{M,N} \delta'_{L,M} \).

(\( \text{CM}1'' \)) If \( L \preceq'' M \preceq'' N \), then \( \delta''_{N,L} = \delta''_{M,L} \delta''_{N,M} \).

(\( \text{CM}2 \)) If \( M \geq'' N \preceq' L \), then

\[
\delta''_{N,L} \delta''_{M,N} = \sum_{O \in \text{Sup}(M,L)} \delta''_{O,L} \delta''_{M,O}.
\]

(\( \text{CM}3' \)) If \( M \preceq' N \) is an anodyne inequality, then \( \delta'_{M,N} \) is invertible.

(\( \text{CM}3'' \)) If \( M \preceq'' N \) is an anodyne inequality, then \( \delta''_{N,M} \) is invertible.

More precisely, let \( \mathcal{CM}^+ \) be the category with the objects and generating morphisms as above which are subject to the relations (\( \text{CM}1' \)), (\( \text{CM}1'' \)) and (\( \text{CM}2 \)). Let \( E \subset \text{Mor}_{\mathcal{CM}^+} \) be the set of the \( \delta'_{M,N} \), \( \delta''_{N,M} \) corresponding to anodyne inequalities \( M \preceq' N, M \preceq'' N \). Then \( \mathcal{CM} = \mathcal{CM}^+[E^{-1}] \) is the localization of \( \mathcal{CM}^+ \) with respect to \( \Sigma \). The set \( E \) satisfies the Ore condition, as follows from the next proposition which we leave to the reader.

Proposition 3.1. Let \( M \preceq' N \preceq'' L \) be inequalities in \( \mathcal{CM} \).

(a) If one of these inequalities is anodyne, then \( \text{Sup}(M,L) \) consists of one element, i.e., there exists a unique diagram of inequalities in \( \mathcal{CM} \)

\[
\begin{array}{ccc}
O & \xrightarrow{\geq''} & L \\
\downarrow^{\preceq'} & & \downarrow^{\preceq'} \\
M & \xrightarrow{\geq''} & N.
\end{array}
\]

(b) Moreover, if \( L \preceq' N \) is anodyne, then \( O \preceq' M \) is anodyne. If \( M \geq'' N \) is anodyne, then \( O \geq'' L \) is anodyne.

Let \( \mathcal{CM}_n \subset \mathcal{CM} \) be the full subcategory on objects \( [M] \), \( M \in \mathcal{CM}_n \). Since any inequality \( M \preceq' N, M \preceq'' N \) implies equality of the weights \( \Sigma M = \Sigma N \), the \( \mathcal{CM}_n \) for different \( n \) are mutually orthogonal:

\[
\text{Hom}_{\mathcal{CM}}(\mathcal{CM}_n, \mathcal{CM}_{n'}) = 0, \quad n \neq n'.
\]

Proposition 2.8 can be reformulated as follows.
Corollary 3.2. Let $A$ be a graded bialgebra in a monoidal category $\mathcal{V}$. Then the correspondence

$$[M] \mapsto A_M, \quad \delta_{M,N}^\prime \mapsto \Delta_{M,N}, \quad \delta_{N,M}^{\prime \prime} \mapsto \mu_{N,M}$$

defines a functor $\xi_A : \mathcal{CM} \to \mathcal{V}$.

B. Row and column exchange isomorphisms. Two row vectors $(m_1, \cdots, m_r)$ and $(n_1, \cdots, n_r)$ of the same size $r$ will be called disjoint, if, for each $i = 1, \cdots, r$, at least one of the two numbers $m_i, n_i$ is equal to 0, i.e., $m_i n_i = 0$. Similarly for column vectors.

Let $M = \|m_{ij}\| \in \text{CM}(r,s)$ be a contingency matrix of size $r \times s$. We denote by

$$M_i = (m_{i1}, \cdots, m_{is}), \quad M_j = (m_{1j}, \cdots, m_{rj})^t, \quad i = 1, \cdots, r, \quad j = 1, \cdots, s,$$

the $i$th row and the $j$th column of $M$. For $i = 1, \cdots, r - 1$ let $\sigma_{i,i+1}'' M$ be the matrix obtained from $M$ by interchanging the $i$th and $(i+1)$st rows. For $j = 1, \cdots, s - 1$ let $\sigma_{j,j+1}' M$ be the matrix obtained from $M$ by interchanging the $j$th and $(j+1)$st columns.

Recall that the vertical contraction $\partial_{i-1}''$, $i = 0, \cdots, r - 1$, adds together the $i$th and $(i + 1)$st rows of a contingency matrix. Suppose that our $M \in \text{CM}(r,s)$ is such that $M_i$ and $M_{i+1}$ are disjoint. Then the inequalities

$$M \geq'' \partial_{i+1}'' M = \partial_{i-1}'' (\sigma_{i,i+1}'' M) \leq'' \sigma_{i,i+1}'' M,$$

are anodyne, and we define the row exchange isomorphism

$$\varepsilon_{i,i+1}'' = (\delta_{i,i+1}'' M, \partial_{i-1}'' M)^{-1} \circ \delta_{M,\partial_{i-1}'' M} : [M] \longrightarrow [\sigma_{i,i+1}'' M]$$
in the category $\mathcal{CM}$.

Similarly, suppose that $M_j$ and $M_{j+1}$ are disjoint. Then we have anodyne inequalities

$$M \geq' \partial_{j-1}' M = \partial_{j-1}' (\sigma_{j,j+1}' M) \leq' \sigma_{j,j+1}' M,$$

and we define the column exchange isomorphism in $\mathcal{CM}$

$$\varepsilon_{j,j+1}' = \delta_{j-1}' M, \sigma_{j,j+1}' M \circ (\delta_{M,\partial_{j-1}' M})^{-1} : [M] \longrightarrow [\sigma_{j,j+1}' M].$$

Proposition 3.3. (a) Let $M \in \text{CM}(r,s)$ and $i = 1, \cdots, r - 2$ be such that $M_i, M_{i+1}, M_{i+2}$ are mutually disjoint. For any permutation $\tau = (\tau(1), \tau(2), \tau(3))$ of $\{1, 2, 3\}$ let $M_\tau$ be the matrix obtained from $M$ by permuting the $i$th, $(i + 1)$st and $(i + 2)$nd rows of $M$ according to $\tau$, e.g., $M_{(123)} = M$, $M_{(213)} = \sigma_{i,i+1}'' M$ etc. Then the hexagon of row exchange isomorphisms

$$\begin{array}{ccc}
\varepsilon_{i,i+1}'' [M_{(213)}] & \xrightarrow{\varepsilon_{i+1,i+2}''} & [M_{(321)}] \\
[M_{(123)}] & \xrightarrow{\varepsilon_{i,i+1}''} & [M_{(312)}] \\
\varepsilon_{i+1,i+2}'' [M_{(132)}] & \xrightarrow{\varepsilon_{i,i+1}''} & [M_{(321)}]
\end{array}$$

is commutative (braid relation).

(b) A similar braid relation for column exchange isomorphisms in the case when $M_j$, $M_{j+1}$ and $M_{j+2}$ are mutually disjoint.
Proof: We show (a), since (b) is similar. By construction, each arrow in the hexagon is the composition of two isomorphisms going through an intermediate object: one isomorphism is of the form \( \delta'' \) corresponding to an anodyne inequality \( \succeq'' \), the other an inverse of a \( \delta'' \) of this kind. Let us restore these intermediate objects and draw the corresponding morphisms \( \delta'' \) (without inverting them). We get a diagram with 12 vertices. Let also

\[
\overline{M} = \partial''_{i-1} \partial''_i M = \partial''_{i-1} \partial''_i M
\]

be the \((r-2) \times s\) matrix obtained by summing all three rows, the \(i\)th, the \((i+1)\)st and \((i+2)\)nd, of \(M\). Then we have an anodyne inequality \( \overline{M} \succeq'' N \), where \(N\) is any of the 12 matrices corresponding to the 12 vertices of the extended diagram above. Therefore, putting the object \( \overline{M} \) inside that diagram, we decompose it into 12 triangles which commute because of the relation \( (\mathcal{CM}1'') \) (transitivity of the maps \( \delta'' \)). In this way we get a diagram whose shape is the barycentric subdivision of the original hexagon (considered as a 2-dimensional cell complex) and which consists of commuting triangles. This implies the commutativity of the (1-dimensional boundary of the) hexagon, which is the claim.

Proposition 3.4. (a) Let \( M \in \text{CM}(r, s) \) and \( i = 1, \ldots, r - 3 \) have the following property: any vector from the set \( \{ M_i, M_{i+1} \} \) and any vector from the set \( \{ M_s, M_{i+3} \} \) are disjoint. For any permutation \( \alpha = (\tau(1), \tau(2), \tau(3), \tau(4)) \) of \( \{1, 2, 3, 4\} \) let \( M_\alpha \) be the matrix obtained from \( M \) by permuting the \(i\)th, \((i+1)\)st, \((i+2)\)nd and \((i+3)\)rd rows according to \( \tau \), e.g.,

\[
M_{(1324)} = \sigma''_{i+1,i+2} M.
\]

Then the diagram of row exchange isomorphisms

\[
[M_{(1324)}] \xrightarrow{\varepsilon''_{i+1,i+1}} [M_{(1324)}] \xrightarrow{\varepsilon''_{i+2,i+3}} [M_{(1342)}] \xrightarrow{\varepsilon''_{i+2,i+3}} [M_{(1342)}] \xrightarrow{\varepsilon''_{i+1,i+1}} [M_{(3412)}]
\]

commutes.

(b) A similar statement for column exchange isomorphisms in the case when any vector from \( \{ M^j, M^{j+1} \} \) and any vector from \( \{ M^{j+2}, M^{j+3} \} \) are disjoint.

Proof: We prove (a), since (b) is similar. It suffices to prove the commutativity of the central diamond. The argument is similar to that of Proposition 3.3. That is, we expand the diamond (a 4-gon) to an 8-gon by restoring the intermediate objects and drawing the \( \delta'' \)-morphisms without inverting anything. Let \( \widetilde{M} = \partial''_i \partial''_{i-1} M_{(1324)} \) be the matrix obtained by summing the \(i\)th and \((i+1)\)st rows and separately summing the \((i+1)\)nd and \((i+3)\)rd rows of \(M_{(1324)}\). Then we have an anodyne inequality \( \widetilde{M} \succeq'' N \) where \( N \) is any of the 8 matrices from the 8-gon above. So putting \( \widetilde{M} \) inside the 8-gon, we fill the 8-gon by commutative triangles which implies that the original diamond commutes as well.

C. The monoidal structure on \( \mathcal{CM} \). We make \( \mathcal{CM} \) into a monoidal category by putting, on the level of objects,

\[
[M] \otimes [N] = [M \oplus N], \quad M \oplus N = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.
\]
On the level of morphisms, if $M_1 \leq' M_2$ and $N_1 \leq' N_2$, then $M_1 \oplus N_1 \leq' M_2 \oplus N_2$, and we put
\[
\delta'_{M_1,M_2} \otimes \delta'_{N_1,N_2} = \delta'_{M_1 \oplus N_1,M_2 \oplus N_2}.
\]
Similarly, if $M_1 \leq' M_2$ and $N_1 \leq' N_2$, then $M_1 \oplus N_1 \leq' M_2 \oplus N_2$, and we put
\[
\delta''_{M_2,M_1} \otimes \delta''_{N_2,N_1} = \delta''_{M_2 \oplus N_2,M_1 \oplus N_1}.
\]
Further, let $M_1 \leq' N_1$ and $M_2 \leq'' N_2$. Then we have the diagram of inequalities
\[
\begin{array}{ccc}
N_1 \oplus N_2 & \xrightarrow{\geq''} & N_1 \oplus M_2 \\
\downarrow & & \downarrow \\
M_1 \oplus N_2 & \xrightarrow{\geq''} & M_1 \oplus M_2
\end{array}
\]
and $\text{Sup}(M_1 \oplus N_2, N_1 \oplus M_2) = \{N_1 \oplus N_2\}$ consists of one element. Therefore
\[
\delta'_{M_1 \oplus M_2,N_1 \oplus M_2} \otimes \delta''_{M_1 \oplus N_1, M_1 \oplus M_2} = \delta''_{N_1 \oplus N_2,N_1 \oplus M_2} \delta'_{M_1 \oplus N_1,N_1 \oplus N_2}
\]
and we define $\delta'_{M_1,N_1} \otimes \delta''_{N_2,M_2}$ to be equal to this common value.

Similarly, let $M_1 \leq'' N_1$ and $M_2 \leq' N_2$. We have the diagram of inequalities
\[
\begin{array}{ccc}
N_1 \oplus N_2 & \xrightarrow{\geq''} & M_1 \oplus N_2 \\
\downarrow & & \downarrow \\
N_1 \oplus M_2 & \xrightarrow{\geq''} & M_1 \oplus M_2
\end{array}
\]
and put
\[
\delta''_{N_1,M_1} \otimes \delta'_{M_2,N_2} \coloneqq \delta'_{M_1 \oplus M_2,M_1 \oplus N_2} \delta''_{N_1 \oplus M_2,M_1 \oplus M_2} = \delta''_{N_1 \oplus N_2,M_1 \oplus N_2} \delta'_{M_1 \oplus N_2,M_1 \oplus N_2},
\]
the second inequality following from $\text{Sup}(N_1 \oplus M_2, M_1 \oplus N_2) = \{N_1 \oplus N_2\}$.

**Proposition 3.5.** The above data on objects and generating morphisms define a monoidal structure $\otimes$ on $\text{CM}$ with unit object $1 = [\varnothing]$.

**Proof:** By construction, the operation $\otimes$ is strictly associative on objects. What remains to prove is that $\otimes$ extends to a functor in each argument, i.e., that our definitions are compatible with the relations in $\text{CM}$. For this, we proceed as follows.

First, our definitions imply that for any two generating morphisms $f : [M_1] \rightarrow [M_2]$, $g : [N_1] \rightarrow [N_2]$ we have
\[
f \otimes g = (f \otimes \text{Id}_{[N_2]})(\text{Id}_{[M_1]} \otimes g) = (\text{Id}_{[M_2]} \otimes g)(f \otimes \text{Id}_{[N_1]}).
\]
So it suffices to show that for any $M, N \in \text{CM}$ the operations $(- \otimes \text{Id}_{[N]})$, $(\text{Id}_{[M]} \otimes -)$ on generating morphisms preserve the relations in $\text{CM}$. We consider $(- \otimes \text{Id}_{[N]})$, the case of $(\text{Id}_{[M]} \otimes -)$ being similar.
For (CM1'), (CM1') such preservation is obvious. For (CM2) it follows from the identification
\[
\Sup(L, M) \rightarrow \Sup(L \oplus N, M \oplus N), \quad O \mapsto O \oplus N.
\]
For (CM3'), (CM3'\') it follows from the following obvious fact: if \(M_1 \leq' M_2\) is anodyne, then \(M_1 \oplus N \leq' M_2 \oplus N\) is anodyne also, and similarly for \(M_1 \leq'' M_2\).

D. Braiding on CM. Let \(M \in \CM(p, q)\) and \(N \in \CM(r, s)\). We define the braiding isomorphism
\[
R_{\{M\},\{N\}} : [M] \otimes [N] = [M \oplus N] \rightarrow [N \oplus M] = [N] \otimes [M]
\]
by mimicking the standard Eckmann-Hilton procedure in topology ("jeu de taquin" proving the commutativity of \(\pi_2\)). More precisely, we define \(R_{\{M\},\{N\}}\) as the composition
\[
[M \oplus N] = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \xrightarrow{R'_{M,N}} \begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix} \xrightarrow{R_{M,N}} \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} = [N \oplus M],
\]
where:
- \(R''_{M,N}\) is the composition of \(pr\) row exchange isomorphisms moving \(r\) rows of \((0N)\) past the \(p\) rows of \((M0)\). This can be done in several ways but Proposition 3.4(a) implies that all of them lead to the same result, which is denoted \(R''_{M,N}\).
- \(R'_{M,N}\) is the composition of \(qs\) column exchange isomorphisms moving \(s\) columns of \((N)\) past \(q\) columns of \((0M)\). Again, this can be done in several ways but Proposition 3.4(b) implies that all of them lead to the same result, which is denoted \(R'_{M,N}\).

**Proposition 3.6.** The isomorphisms \(R_{\{M\},\{N\}}\) make \(\CM\) into a braided monoidal category.

**Proof:** We first show that the \(R_{\{M\},\{N\}}\) are natural in each variable. Naturality in the first variable means that for any morphism \(\varphi : [M] \rightarrow [L]\) and any object \([N]\) in \(\CM\) the diagram (the naturality square)
\[
\begin{array}{ccc}
[M] \otimes [N] & \xrightarrow{R_{\{M\},\{N\}}} & [N] \otimes [M] \\
\varphi \otimes [N] & & [N] \otimes [\varphi] \\
[L] \otimes [N] & \xrightarrow{R_{\{L\},\{N\}}} & [N] \otimes [L]
\end{array}
\]
is commutative. To show this, it suffices to assume that \(\varphi\) is one of the elementary generating morphisms, i.e., we are in either of the two cases:

(i) \(\varphi = \delta'_{\{M\},\{L\}}\), where \(M = \partial'_{\{L\}} L\) is obtained from \(L\) by a horizontal contraction (adding two adjacent columns);

(ii) \(\varphi = \delta''_{\{M\},\{L\}}\), where \(L = \partial''_{\{M\}} M\) is obtained from \(M\) by a vertical contraction (adding two adjacent rows).
Consider the case (i). The naturality square whose commutativity we need to prove, decomposes into two:

\[(\begin{array}{c}
\left(\begin{array}{cc}
M & 0 \\
0 & N
\end{array}\right) & R_{MN}^p \\
\phi \otimes [N] & 0 \\
\end{array}\right) \xrightarrow{\psi} \left(\begin{array}{c}
\left(\begin{array}{cc}
0 & N \\
M & 0
\end{array}\right) & R_{MN} \\
\psi & [N] \otimes \phi \\
\end{array}\right) \xrightarrow{R_{LN}} \left(\begin{array}{c}
\left(\begin{array}{cc}
N & 0 \\
0 & M
\end{array}\right) \\
\end{array}\right),
\]

where \(\psi\) is induced by the inequality \(\left(\begin{array}{cc}0 & N \\
M & 0\end{array}\right) \leq' \left(\begin{array}{cc}0 & N \\
L & 0\end{array}\right)\). Note that the other two vertical arrows are, by construction, also induced by the corresponding inequalities. We prove the commutativity of each of the two squares separately.

Left square: The morphisms \(R_{MN}^p\) and \(R_{LN}^p\) are compositions of row exchange isomorphisms, i.e., of \(\delta^p\)-isomorphisms induced by anodyne vertical contractions and of the inverses of such isomorphisms. These isomorphisms go through intermediate objects corresponding to matrices obtained from \(\left(\begin{array}{cc}M & 0 \\
0 & N\end{array}\right)\) and \(\left(\begin{array}{cc}L & 0 \\
0 & N\end{array}\right)\) by some number of row exchanges and then, possibly, summation of two disjoint adjacent rows. Let us restore these intermediate objects and the \(\delta^p\)-isomorphisms connecting them, without inverting these isomorphisms. In this way we replace the square by a diagram of the form

\[(\begin{array}{c}
\left(\begin{array}{cc}
M & 0 \\
0 & N
\end{array}\right) & \delta^p \\
\phi \otimes [N] & \delta^p \\
\end{array}\right) \xrightarrow{\delta^p} \left(\begin{array}{c}
\left(\begin{array}{cc}
0 & N \\
M & 0
\end{array}\right) \\
\end{array}\right),
\]

where the horizontal maps are \(\delta^p\)-isomorphisms. Note further that we have morphisms between the corresponding intermediate objects indicated by the dotted vertical arrows. They correspond to the horizontal contractions of the intermediate matrices. We obtain a ladder diagram consisting of many squares, with horizontal maps being \(\delta^p\)-isomorphisms and vertical maps being \(\delta^p\)-morphisms. We claim that each of these squares is commutative. Indeed, such a square corresponds to a square of inequalities of the type discussed in Proposition 3.1: two of the inequalities of the same type (in our case, \(\leq^p\)) are anodyne. In this situation Proposition 3.1 and the relation \((\text{CM}2)\) give that the square is commutative, as the sum in \((\text{CM}2)\) consists of one summand. This implies that the boundary of the entire diagram, formed by inverting the isomorphisms oriented \(\leftarrow\), i.e., the left square in (3.7), is commutative.

Right square: The morphisms \(R_{MN}^l\) and \(R_{LN}^l\) are compositions of column exchange isomorphisms, i.e., of \(\delta^p\)-isomorphisms induced by anodyne horizontal contractions and of the in-
verses of such isomorphisms. Restoring the intermediate objects involved in these isomorphisms, we obtain a diagram somewhat similar to (3.8):

\[
\begin{array}{c}
\left[ \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix} \right] \xrightarrow{\delta'} \cdots \xrightarrow{\delta''} \left[ \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \right]
\end{array}
\]

This diagram consists entirely of \(\delta'\)-morphisms. Further, unlike (3.8), the bottom row here is longer than the top one, since \(M = \partial_j L\) has one fewer column than \(L\), being obtained from \(L\) by adding the \((j + 1)\)st and \((j + 2)\)nd columns. Let us denote these columns for short by \(l = L^{j+1}\) and \(l' = L^{j+2}\).

Now, some objects in the bottom row can be assigned "matches" in the top one, from which they receive \(\delta'\)-maps which we add to the diagram as vertical arrows. These objects correspond to matrices which contain the columns \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ l' \end{pmatrix}\) situated next to each other, and the corresponding matching matrix in the top row is obtained by adding these columns. In this way we get several vertical arrows which decompose our diagram into fragments of two types.

A fragment of the first type is a square obtained when two vertical arrows are positioned next to each other. Each such square is commutative by transitivity of \(\delta'\)-maps.

A fragment of the second type is obtained when a column of \(N\), denote it \(n\), is moved past \(l\) and \(l'\). Such a fragment has the form (we do not depict any other columns that are unchanged throughout the procedure):

\[
\begin{array}{c}
\left[ \begin{pmatrix} 0 & n \\ 1 + l' & 0 \end{pmatrix} \right] \xrightarrow{\delta'} \left[ \begin{pmatrix} n \\ 1 + l' \end{pmatrix} \right] \xrightarrow{\delta'} \left[ \begin{pmatrix} n & 0 \\ 0 & 1 + l' \end{pmatrix} \right]
\end{array}
\]

To show that this fragment becomes commutative after inverting the arrows oriented \(\leftarrow\), we decompose it by the dotted arrows (which are likewise \(\delta'\)-morphisms) into two 4-gons and two triangles. Each of them is commutative by transitivity of \(\delta'\)-morphisms.

This proves that the right square in (3.7) is commutative in the situation of Case (i) above, i.e., under the assumption that \(\varphi = \delta''_{ML}\), where \(M = \partial_j L\). In this way we show the naturality of the \(R_{[M],[N]}\) in the first variable in Case (i).

Naturality in the first argument in Case (ii) when \(\varphi = \delta''_{ML}\), \(L = \partial_i M\), is analyzed completely analogously except the roles of the left and right squares in (3.7) will be interchanged.

Further, the naturality in the second argument is also completely analogous. This proves that the \(R_{[M],[N]}\) are natural in both arguments.
To prove that $R$ is a braiding, it remains to show the commutativity of the braiding triangles [2, 14]. These triangles are of two classes. The triangles of the first class have the form

$$\begin{align*}
[M] \otimes [M'] \otimes [N] &\xrightarrow{[M] \otimes R_{[M'] \otimes [N]}} [N] \otimes [M] \otimes [M'] \\
&\xrightarrow{R_{[M] \otimes [N] \otimes [M']}} [N] \otimes [M'] \otimes [M]
\end{align*}$$

for any three objects $[M], [M'], [N]$. The triangles of the second class are similarly associated to any $[M], [N], [N']$ and express two ways of passing from $[M] \otimes [N] \otimes [N']$ to $[N] \otimes [N'] \otimes [M]$. The commutativity of such triangles follows straightforwardly from Propositions 3.3 (braid relation for row or column exchanges) and 3.4. Proposition 3.6 is proved.

We now notice the following refinement of Corollary 3.2.

**Proposition 3.9.** In the situation of Corollary 3.2, the functor $\xi_A : \mathcal{M} \rightarrow V$ is braided monoidal.

**Proof:** We first construct isomorphisms

$$\xi_A([M]) \otimes \xi_A([N]) = A_M \otimes A_N \xrightarrow{\sigma_{M,N}} A_{M \oplus N} = \xi_A([M] \otimes [N]).$$

Suppose $M$ is of size $p \times q$ and $N$ is of size $r \times s$. By definition, the component $A_M$, being a 2-dimensional tensor product, is a pseudo-object and as such, is given in terms of determinations corresponding to snakes. In particular (2.7), the determination $(A_M)_{\text{Lex}}$ corresponding to the Lex snake, is the ordered tensor product of the $A_{m_{ij}}$ along the columns of $M$. Similarly, $(A_M)_{\text{Alex}}$, the determination corresponding to the Alex snake, is the ordered tensor product of the $A_{m_{ij}}$ along the rows of $M$. They are identified by the braiding $R_{b_M}$, where $b_M \in \text{Br}_{pq}$ is the braid (depending only on $p$ and $q$) connecting Lex and Alex snakes for $M$ (in fact, for any $p \times q$ matrix). Similarly for $(A_N)_{\text{Lex}}$ and $(A_N)_{\text{Alex}}$ and $R_b$ for $b_N \in \text{Br}_{rs}$.

Now note that reading $M \oplus N$ along the columns (and ignoring the 0’s in the off-diagonal blocks) is the same as first reading $M$ along the columns and then reading $N$ in the same way. This gives an isomorphism $\varphi_{M,N}^{\text{Lex}} : (A_M)_{\text{Lex}} \otimes (A_N)_{\text{Lex}} \xrightarrow{\sigma_{M,N}} (A_{M \oplus N})_{\text{Lex}}$. Similarly, reading $M \oplus N$ along the rows (and ignoring the 0s as above) is the same as first reading $M$ and then reading $N$ in this way. This gives an isomorphism $\varphi_{M,N}^{\text{Alex}} : (A_M)_{\text{Lex}} \otimes (A_N)_{\text{Alex}} \xrightarrow{\sigma_{M,N}} (A_{M \oplus N})_{\text{Alex}}$. We claim that $\varphi_{M,N}^{\text{Lex}}$ and $\varphi_{M,N}^{\text{Alex}}$ give the same morphism of pseudo-objects $\varphi_{M,N} : A_M \otimes A_N \xrightarrow{\oplus} A_{M \oplus N}$. Indeed, consider the juxtaposition (direct sum) homomorphism

$$\oplus : \text{Br}_{pq} \times \text{Br}_{rs} \rightarrow \text{Br}_{pq + rs}.$$
(q + s). We notice that $b_{M \oplus N} = b_M \oplus b_N$ and therefore we have a commutative square

$$
\begin{array}{ccc}
(A_M)_\text{Lex} \otimes (A_N)_\text{Lex} & \xrightarrow{\varphi_{M,N}^{\text{Lex}}} & (A_{M \oplus N})_\text{Lex} \\
R_b M \otimes R_b N & & R_b M \otimes N \\
(A_M)_\text{Alex} \otimes (A_N)_\text{Alex} & \xrightarrow{\varphi_{M,N}^{\text{Alex}}} & (A_{M \oplus N})_\text{Alex}
\end{array}
$$

which implies that the resulting morphism of pseudo-objects is the same for both Lex and Alex determinations. This defines $\varphi_{M,N}$.

Next, we show that the $\varphi_{M,N}$ are natural in $[M]$ and $[N]$. It suffice to check the naturality on generating morphisms $\delta'$ or $\delta''$ for $M$ or $N$. Naturality for $\delta'$ (horizontal contractions, adding some adjacent columns) is immediate in the Alex determination. Indeed, when reading the $A_{m_{ij}}$ along the rows, the action of $\delta'$, i.e., horizontal comultiplication, will respect the order of the product, i.e., will produce new tensor factors in positions which are adjacent with respect to the order. Similarly, naturality for $\delta''$ is immediate in the Lex determination, reading the $A_{m_{ij}}$ along the columns.

This naturality makes $\xi_A$ into a monoidal functor. It remains to show that $\xi_A$ is in fact a braided monoidal functor, i.e., preserves the braiding. This verification is straightforward and left to the reader. Proposition 3.9 is proved.
4 The category of contingency matrices and the PROB \( \mathcal{B} \) of graded bialgebras

A. The graded bialgebra \( \mathfrak{a} \) in \( \mathcal{CM} \). We now define a graded bialgebra \( \mathfrak{a} = (a_n)_{n \geq 0} \) in \( \mathcal{CM} \) with components \( a_n = [n] \) (the object corresponding to the \( 1 \times 1 \) contingency matrix \( (n) \)) for \( n > 0 \) and \( a_0 = 1 = [\emptyset] \). The multiplication and comultiplication are given by

\[
\mu_{m,n} : [m] \otimes [n] = \left[ \begin{pmatrix} m & 0 \\ n & 0 \\ 0 & p \end{pmatrix} \right] \xrightarrow{\delta'} \left[ \begin{pmatrix} (m + n) & 0 \\ 0 & p \end{pmatrix} \right] \xrightarrow{\delta''} [m + n],
\]

\[
\Delta_{m,n} : [m + n] \xrightarrow{\delta'} [m, n] \xrightarrow{\delta'} \left[ \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \right] = [m] \otimes [n].
\]

Proposition 4.1. The morphisms \( \mu_{m,n}, \Delta_{m,n} \) make \( \mathfrak{a} \) into a graded bialgebra in \( \mathcal{CM} \).

Proof: We first prove associativity. For this, we must compare two morphisms \( [m] \otimes [n] \otimes [p] \to [m + n + p] \) corresponding to two bracketing of the triple product. There morphisms are the compositions of the upper and lower paths in the boundary of the following diagram, the paths obtained by inverting the \( \delta' \)-isomorphisms:

To prove that these two paths have the same composition, we decompose the diagram into four 4-gons by the dotted arrows as shown and notice that each of these 4-gons is commutative.

Indeed, the leftmost 4-gon commutes by transitivity of \( \delta' \)-morphisms. The rightmost 4-gon commutes by transitivity of \( \delta'' \)-morphisms. The remaining two 4-gons commute by the relation \((\mathcal{CM}^2)\) since the sum in that relation consists of one summand by Proposition 3.1.

This proves associativity of \( \mu \). The proof of coassociativity of \( \Delta \) is similar.

Finally, we prove compatibility of \( \Delta \) and \( \mu \). This is expressed by Eq. (2.9), we we assume that we are in the situation of (2.9). The composition \( \Delta_{t_1,t_2} \mu_{m_1,m_2} \) is, in our case, given by
the border (top horizontal followed by the right vertical) path in the following diagram:

\[
\begin{bmatrix}
(m_1 & 0 \\
0 & m_2)
\end{bmatrix}
\xrightarrow{(\delta')^{-1}}
\begin{bmatrix}
(m_1 & 0 \\
0 & m_2)
\end{bmatrix}
\xrightarrow{\delta''}
\begin{bmatrix}
(n_1 & 0 \\
0 & n_2)
\end{bmatrix}
\xrightarrow{\alpha (\delta'')^{-1}}
\begin{bmatrix}
(l_1 & 0 \\
0 & l_2)
\end{bmatrix}
\]

Here the matrix \( O = \begin{bmatrix} o_{11} & o_{12} \\
o_{21} & o_{22} \end{bmatrix} \) is a (so far arbitrary) element of \( \text{Sup}(M, L) \). We denoted for short by \( \psi, \psi_O, \varphi, \varphi_O \) the \( \delta' \) and \( \delta'' \)-morphisms in the square and by \( \alpha \) and \( \beta \) the inverted isomorphisms at the end and the beginning of the border path. By the relation (\( \mathcal{CM}2 \)) we have

\[
\psi \varphi = \sum_{O \in \text{Sup}(M, L)} \varphi_O \psi_O.
\]

So it suffices to show that for each \( O \in \text{Sup}(M, L) \) we have

\[
(4.2) \quad \alpha \varphi_O \psi_O \beta = (\mu_{o_{11}, o_{21}} \otimes \mu_{o_{12}, o_{22}}) \circ (\text{Id} \otimes R_{A_{o_{12}, A_{o_{21}}} \otimes \text{Id}}) \circ (\Delta_{o_{11}, o_{12}} \otimes \Delta_{o_{21}, o_{22}}).
\]

so that the summands in the RHS of (2.9) match those in (\( \mathcal{CM}2 \)). We represent the two sides of (4.2) by the upper and lower path in the boundary of the following diagram (more precisely, the paths, going from left to right, are obtained by inverting the isomorphisms oriented the other way):

To show the equality of the compositions of these paths, we decompose the diagram into two 4-gons and a pentagon by the dotted arrows \( \kappa \) and \( \rho \), where:
\( \kappa \) is the composition
\[
\begin{pmatrix}
o_{11} & o_{12} \\
o_{21} & o_{22}
\end{pmatrix} \xrightarrow{\delta'} \begin{pmatrix}
o_{11} & 0 & o_{12} & 0 \\
0 & o_{21} & 0 & o_{22}
\end{pmatrix}
\]
Column exchange \[
\begin{pmatrix}
o_{11} & 0 & o_{12} & 0 \\
0 & 0 & o_{21} & o_{22}
\end{pmatrix},
\]
so it composed entirely of \( \delta' \)-morphisms and their inverses.

\( \rho \) is the composition
\[
\begin{pmatrix}
o_{11} & o_{12} \\
o_{21} & o_{22}
\end{pmatrix} \xrightarrow{(\delta' )^{-1}} \begin{pmatrix}
o_{11} & 0 & o_{12} \\
0 & o_{21} & 0 \\
0 & 0 & o_{22}
\end{pmatrix}
\]
Row exchange \[
\begin{pmatrix}
o_{11} & 0 \\
0 & o_{21} & 0 \\
0 & 0 & o_{22}
\end{pmatrix},
\]
so it is composed entirely of \( \delta'' \)-morphisms and their inverses. The left 4-gon in the decomposed diagram is commutative by transitivity of \( \delta' \)-morphisms. The right 4-gon is commutative by transitivity of \( \delta'' \)-morphisms. Finally, the pentagon consists entirely of anodyne \( \delta' \) or \( \delta'' \)-isomorphisms and their inverses which move the \( o_{ij} \) around in the plane. We can view them as moving 4 points in the plane. After we go around the pentagon, we return to the same position. Moreover, the braid on 4 strands representing this move, is trivial. This triviality of the braid implies the commutativity of the pentagon. We leave further details to the reader. Proposition 4.1 is proved.

**B. The category \( \mathfrak{CM} \) and the PROB \( \mathcal{B} \).**
Recall the PROB \( \mathcal{B} \) governing graded bialgebras, see §1 B.

**Theorem 4.3.** We have an equivalence of braided monoidal categories \( \xi : \mathfrak{CM} \rightarrow \mathcal{B} \). In particular, for any \( n > 0 \) we have an equivalence of ordinary (non-monoidal) categories \( \xi_n : \mathfrak{CM}_n \rightarrow \mathcal{B}_n \).

**Proof:** Recall that \( \mathcal{B} \) has a graded bialgebra \( \mathfrak{a} \). By definition, the components \( \mathfrak{a}_n \) are generating objects for \( \mathcal{B} \), i.e., any other object is isomorphic to a tensor product of several of the \( \mathfrak{a}_n \). Similarly, the objects \( [n] \) associated to \( 1 \times 1 \) matrices, are generating objects for \( \mathfrak{CM} \). Indeed, any object \( [M] \) associated to any \( r \times s \) contingency matrix \( M = \| m_{ij} \| \), is isomorphic to the tensor product (in any order) of the individual \( [m_{ij}] \), the latter product being represented by the diagonal \( r s \times r s \) matrix with entries \( m_{ij} \) in the corresponding order. This can be easily seen by moving the \( m_{ij} \) around in the matrix by using anodyne \( \delta' \) and \( \delta'' \)-isomorphisms and their inverses.

Next, the graded bialgebra \( \mathfrak{a} \), Corollary 3.2 and Proposition 3.9 give a braided monoidal functor
\[
\xi = \xi_{\mathfrak{a}} : \mathfrak{CM} \rightarrow \mathcal{B}, \quad [M] \mapsto \mathfrak{a}_M.
\]
We prove that \( \xi \) is an equivalence. For this, we use the graded bialgebra \( \mathfrak{a} \) in \( \mathfrak{CM} \) constructed in Proposition 4.1. As \( \mathfrak{a} \in \mathcal{B} \) is the universal graded bialgebra, we get a braided monoidal functor
\[
F = F_{\mathfrak{a}} : \mathcal{B} \rightarrow \mathfrak{CM}, \quad \mathfrak{a}_n \mapsto [n].
\]
We claim that the functors $\xi$ and $F$ are quasi-inverse to each other. Indeed, look at the composition $F\xi: \CM \to \CM$, a braided monoidal functor. It takes any generating object $[n]$ to itself. Therefore $F\xi$ is isomorphic to $\text{Id}$. Similarly, look at $\xi F: \BM \to \BM$. It is a braided monoidal functor which takes any generating object $a_n$ to itself. Therefore $\xi F$ is isomorphic to $\text{Id}$. 

\[ \square \]

C. Proof of Theorem 1.3. Because of Theorem 4.3, Theorem 1.3 can be reformulated as follows.

Reformulation 4.4. For any abelian category $\mathcal{V}$ we have an equivalence of categories
\[ \text{Perv}(\text{Sym}^n(\mathbb{C}); \mathcal{V}) \cong \text{Fun}(\CM, \mathcal{V}). \]

This statement is a consequence (particular case) of the main result of [19] (Theorem 2.6) which describes perverse sheaves on $W\backslash \mathfrak{h}$ where $\mathfrak{h}$ is the Cartan subalgebra of a reductive complex Lie algebra $\mathfrak{g}$ and $W$ is the Weyl group of $\mathfrak{g}$. More precisely, [19] deals with $\text{Vect}_k$-valued perverse sheaves, but extension to perverse sheaves with values in an arbitrary abelian category $\mathcal{V}$ is trivial. Our case corresponds to $\mathfrak{g} = \mathfrak{gl}_n$, when $\mathfrak{h} = \mathbb{C}^n$ and $W = S_n$, so $W\backslash \mathfrak{h} = \text{Sym}^n(\mathbb{C})$. The description of [19] is in terms of mixed Bruhat sheaves (Definition 2.1 there) which are certain diagrams with objects labelled by the set
\[ \Xi = \Xi_{g} = \bigsqcup_{I \subset \Delta_{\text{sim}}} W\backslash (W/W_I \times W/W_J) \]

where $\Delta_{\text{sim}}$ is the set of simple roots of $\mathfrak{g}$ and $W_I, I \subset \Delta_{\text{sim}}$ is the subgroup in $W$ generated by the simple reflections $s_\alpha, \alpha \in I$. For $\mathfrak{g} = \mathfrak{gl}_n$, the set $\Xi_{\mathfrak{g}}$ is identified with $\text{CM}_n$, see [18], and the axioms of a mixed Bruhat sheaves become identical to the relations in the category $\CM$. This finishes the proof.
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