MAXIMIZATION OF HIGHER ORDER EIGENVALUES AND APPLICATIONS

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Abstract. The present paper is a follow up of our paper [NS]. We investigate here the maximization of higher order eigenvalues in a conformal class on a smooth compact boundaryless Riemannian surface. Contrary to the case of the first nontrivial eigenvalue as shown in [NS], bubbling phenomena appear.

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1. Introduction

Let $(M, g)$ be a smooth connected compact Riemannian surface without boundary. In this paper, we extremalize higher order eigenvalues in a suitably defined conformal class. If we denote $-\Delta_g$ the Laplace-Beltrami operator on $(M, g)$, the spectrum of $-\Delta_g$ consists of the sequence $\{\lambda_k(g)\}_{k \geq 0}$ and satisfies

$$\lambda_0(g) = 0 < \lambda_1(g) \leq \lambda_2(g) \leq \ldots \leq \lambda_k(g) \leq \ldots$$

If we assume that the area $A_g(M)$ of $M$ with respect to the metric $g$ is normalized by one then by the fundamental result of Korevaar (see [Kor93] and also [YY80]), it follows that every $\lambda_k(g)$ for a given $k \geq 0$ has a universal bound depending on the topological type of $M$, over all the metrics $g$ with normalized area.
The contributions of the present paper are then related to the following extremal problem

\[ \Lambda_k(M) = \sup_g \lambda_k(g) A_g(M) \]

where the supremum is taken over all smooth Riemannian metrics \( g \) on the manifold \( M \). In [NS], we investigated this problem for \( k = 1 \), i.e. the first nontrivial eigenvalue. Here we address higher order eigenvalues \( k \geq 2 \).

We denote by \( \lambda_k(g) \) the \( k \)-eigenvalue of \( -\Delta_g \) and we have by the Courant-Hilbert formulas

\[ \lambda_k(g) = \max_{U, \dim(U) = k} \inf_{u \in U} R_{M,g}(u) \]

where \( R_{M,g}(u) \) is the so-called Rayleigh quotient given by

\[ R_{M,g}(u) = \frac{\int_M |\nabla u|^2 dA_g}{\int_M u^2 dA_g} \]

and the infimum is taken over the space

\[ U \subset \left\{ u \in H^1(M), \int_M u = 0 \right\} . \]

In the previous definition of higher order eigenvalues, the metric is assumed to be smooth. In our case, we will see that the extremal metric is not smooth. However, we will construct this metric as a limit of suitably smooth metrics and the associated higher order extremal eigenvalue will be defined in a natural way out of this limit.

As in our paper [NS], we define conformal metrics \( \bar{g} \) (belonging to the conformal class denoted \([g]\)) as metrics of the form \( \bar{g} = \mu g \) where \( \mu : M \to \mathbb{R}^+ \) is an \( L^1 \) function on \( M \) with mass 1, i.e. a probability density.

We then define

\[ \tilde{\Lambda}_k(M,[g]) = \sup_{\bar{g} \in [g]} \lambda_k(\bar{g}) . \]

We prove the following result.

**Theorem 1.1.** Let \((M,g)\) be a smooth connected compact boundaryless Riemannian surface. For any \( k \geq 2 \), there exists a sequence of metrics \((g_n)_{n \geq 1} \in [g]\) of the form \( g_n = \mu_n g \) such that

\[ \lim_{n \to \infty} \lambda_k(g_n) = \tilde{\Lambda}_k(M,[g]) \]

and a probability measure \( \mu \) such that

\[ \mu_n \rightharpoonup^* \mu \text{ weakly in measure as } n \to +\infty . \]
Moreover the following decomposition holds

\[ \mu = \mu_r + \mu_s \]

where \( \mu_r \) is a \( C^\infty \) nonnegative function and \( \mu_s \) is the singular part given by

\[ \mu_s = \sum_{i=1}^{K} c_i \delta_{x_i} \]

for some \( K \geq 1, \ c_i > 0 \) and some points \( x_i \in M \). Furthermore, the number \( K \) satisfies the bound

\[ K \leq k - 1 \]

Moreover, the weights \( c_i > 0 \) belong to the discrete set

\[ c_i \in \bigcup_{j=1}^{k} \left\{ \frac{\tilde{\Lambda}_j(S^2, [\text{round}])}{\Lambda_k(M, [g])} \right\}. \]

The regular part of the limit density \( \mu \), i.e. \( \mu_r \) is either identically zero or \( \mu_r \) is absolutely continuous with respect to the riemannian measure with a smooth positive density vanishing at most at a finite number of points on \( M \).

Furthermore, if we denote \( A_r \) the volume of the regular part \( \mu_r \), i.e. \( A_r = A_{\mu_r, g}(M) \), then \( A_r \) belongs to the discrete set

\[ A_r \in \bigcup_{j=0}^{k} \left\{ \frac{\tilde{\Lambda}_j(M, [g])}{\Lambda_k(M, [g])} \right\}. \]

Finally, if we denote \( \mathcal{U} \) the eigenspace of the Laplacian on \((M, \mu_r g)\) associated to the eigenvalue \( \tilde{\Lambda}_k(M, [g]) \), then there exists a family of eigenvectors \( \{u_1, \ldots, u_\ell\} \subset \mathcal{U} \) such that the map

\[ \phi : M \to \mathbb{R}^\ell \]

\[ x \to (u_1, \ldots, u_\ell) \]

is a minimizing harmonic map into the sphere \( S^{\ell-1} \).

Theorem 1.1 is a generalization of the result of our paper [NS]. In [NS] we proved Theorem 1.1 for \( k = 1 \) under the assumption that \( \tilde{\Lambda}_1(M, [g]) > 8\pi \). This last assumption was removed by Petrides in [Pet14]. He also suggested some modifications of our proof but basically following the same strategy. Note that in case \( k = 1 \) no bubbling phenomenon occurs, i.e. \( \mu_s \) is identically zero. For \( k = 2 \) the bubbling phenomenon was observed on the sphere in [Nad02].
The proof of the previous theorem relies once again on a careful analysis of a Schrödinger type operator. Indeed consider \( g' \in [g] \), by conformal invariance, the equation \( -\Delta_{g'} u = \lambda_k(g') u \) reduces to the following problem

\[
\begin{aligned}
-\Delta_g u &= \lambda_k(g') \mu u, \quad \text{on } M \\
\int_M \mu dA_g &= 1.
\end{aligned}
\]  

(5)

2. Proof of Theorem 1.1: regular part of the extremal metric

The proof follows our proof in \[NS\]. We briefly sketch the main arguments.

2.1. Step 1: Regularization. We perform a regularization by considering \( S_n \) the class of densities \( \mu \) such that \(-\frac{1}{2} \leq \mu \leq n\), \( \int_M \mu dA_g = 1 \) for \( n > 0 \). Denoting by \( \lambda_k(\mu) \) the eigenvalue problem in (5) with a density \( \mu \in S_n \), we write

\[ \tilde{\Lambda}_n = \sup_{\mu \in S_n} \lambda_k(\mu). \]

We have by a direct application of well-known bounds for Schrödinger operators (see \[LL01\]) that

**Proposition 2.1.** For any given \( n > 0 \), there exists a sequence \( \{\mu_{k,n}\}_{k \geq 0} \subset S_n \) such that as \( k \to +\infty \)

\[ \mu_{k,n} \to^* \mu_n \quad \text{weakly in measure} \]

and

\[ \lambda_k(\mu_{k,n}) \to \tilde{\Lambda}_n. \]

Furthermore, we have

\[ \int_M \mu_n dA_g = 1 \]

and

\[ -\frac{1}{2} \leq \mu_n \leq n. \]
2.2. **Step 2: Passing to the limit in** \( n \). We need to control the two sets

\[
E_n^- = \left\{ x \in M, \ -\frac{1}{2} \leq \mu_n(x) \leq 0 \right\}
\]

and

\[
E_n = \{ x \in M, \ \mu_n(x) = n \}.
\]

We have (see [NS])

**Lemma 2.2.** Let \( n > 0 \). Then there exists \( C > 0 \) such that

\[
A_g(E_n) \leq C/n.
\]

We now come to the measure estimate of the set \( E_n^- \).

**Lemma 2.3.** For any \( n > 0 \), we have

\[
A_g(E_n^-) = 0.
\]

**Remark 2.4.** The previous lemma and particularly its proof is instrumental in our paper with Grigor’yan [GNS15] on bounds on the number of negative eigenvalues for Schrödinger operators with sign-changing potentials.

By the previous Lemmata, one can prove the following convergence result:

\[
\mu_n \rightharpoonup^* \mu, \text{ weakly in measure as } n \to \infty
\]

and furthermore

\[
\mu > 0 \text{ a.e. in } M.
\]

By Lebesgue decomposition theorem, we have

\[
(6) \quad \mu = \mu_r + \mu_s,
\]

where \( \mu_s \), the singular part of the measure, can be decomposed into an absolutely singular part and a discrete part.

The regularity theory developed in [NS] (and invoking the result of Petrides [Pet14]) shows that the absolutely singular part of the measure \( \mu \) vanishes identically. Hence the previous decomposition only involves a regular part, absolutely continuous with respect to the Riemannian measure, and a purely discrete singular part.

Finally, by the results in [NS], the statements of Theorem 1.1 on the regular part follow. We refer the reader to our paper (see also [Pet14]). We postpone the proof of (3) to the next section since the proof is very similar to the one of (2).
3. Proof of Theorem 1.1: the singular part

We decompose the spectrum of $-\Delta$ on $M$ endowed with the metric $\mu g$ into a regular part and a singular part. We give the definitions below.

**Definition 3.1.** We denote $\Lambda_r$ the discrete set of eigenvalues of $-\Delta$ on the manifold $(M, \mu_r g)$ where $\mu_r$ is the regular part of the measure $\mu$.

**Definition 3.2.** We denote $\Lambda_s$ the finite discrete set of eigenvalues defined by: let $i = 1, \ldots, K$ where $K$ is the number of bubbles in Theorem 1.1. Fix $k \geq 2$ and denote

\[
\lambda^x_{i,k} = \lim_{\epsilon \to 0} \lambda^x_{i,\epsilon,k}
\]

where $\lambda^x_{i,\epsilon,k}$ is a (nontrivial) eigenvalue in the flat metric for $\epsilon$ sufficiently small of

\[
\begin{aligned}
-\Delta u &= \lambda u & \text{on } B(x_i, \epsilon), \\
 u &= 0 & \text{on } \partial B(x_i, \epsilon).
\end{aligned}
\]

Notice that the previous limit exists by monotonicity of the eigenvalues with respect to the domain. Then we have

\[
\Lambda_s = \bigcup_{k \geq 0} \bigcup_{i=1}^K \lambda^x_{i,k},
\]

and $\lambda^x_{i,k}$ is a "singular" eigenvalue associated to the Dirac mass $\delta_{x_i}$ where $x_i \in M$.

We now split the spectrum in the following way

\[
Spec(-\Delta) = \Lambda_r \cup \Lambda_s.
\]

By definition, if $\mu_r \equiv 0$ then $\Lambda_r$ is empty and the spectrum is purely singular.

We first prove the following result on the singular spectrum.

**Lemma 3.3.** If $\mu_r$ is not identically zero, then for any $i = 1, \ldots, K$

\[
\lambda^x_{i,k} = 0,
\]

i.e.

\[
K \leq k.
\]

**Proof.** There exists $\delta > 0$ such that one can find $K$ $\delta$–neighborhoods of each $x_i \in M$ such that they do not intersect. For any $\epsilon > 0$, one
constructs functions $\psi_{i,\epsilon}$ for any $i = 1, \ldots, K$ supported in a ball of center $x_i$ and radius $\delta$ with value 1 in this ball and satisfying
\[ \int_{B(x_i, \delta)} |\nabla \psi_{i,\epsilon}|^2 < \epsilon. \]
Then modifying the metric $g_n$ into $g'_n = \psi_{i,\epsilon} g_n$ such that
\[ \lim_{n \to \infty} A_{g'_n} = c_i, \]
one has for sufficiently large $n$ that
\[ \frac{\int_{B(x_i, \delta)} |\nabla \psi_{i,\epsilon}|^2}{\int_{B(x_i, \delta)} |\psi_{i,\epsilon}|^2} < \frac{2\epsilon}{c_i}. \]
Since the functions $\psi_{i,\epsilon}$ have mutually disjoint supports, it follows that
\[ \lambda_{x_i}^K < 2 \frac{\epsilon}{\inf_i c_i}, \]
hence the result since $\epsilon$ can be taken arbitrary small. \hfill \Box

As an improvement one has

**Lemma 3.4.** We have actually
\[ K \leq k - 1. \]

*Proof.* Assume by contradiction $K = k$. This implies that all the masses satisfy for $i = 1, \ldots, K$
\[ c_i = \frac{1}{K} \]
and furthermore,
\[ \mu_r \equiv 0. \]
Indeed if $\mu_r$ is not identically zero, then it implies by the previous lemma that $\lambda_K^{x_i} = 0$. Denote $\bar{\mu}$ the measure on $M$ maximising the first non trivial eigenvalue in the conformal class $[g]$. By [NS], $\bar{\mu}$ has no singular part. As a consequence one has
\[ \lambda_k = \frac{8\pi}{k}. \]
Assume that $M$ is not topologically a sphere. Then we have
\[ \lambda_1 (M, \bar{\mu} g) > 8\pi. \]
We then modify the approximating sequence $g_n = \mu_n g$ into $\bar{g}_n = \bar{\mu}_n g$ such that
\[ \bar{\mu}_n \rightharpoonup^{*} \bar{\mu} \]
and
\[ \bar{\mu} = \bar{\mu}_s + \bar{\mu}_r. \]
where
\[ \bar{\mu}_r = \bar{\mu}/K \]
and \( \bar{\mu}_s \) has point singularities at \( x_1, \ldots, x_{K-1} \) with weights \( 1/K \). Moreover one can choose the sequence \( \bar{\mu}_n \) such that the first eigenvalue at singular points will be equal to \( \frac{8\pi}{K} \).

Since the first eigenvalue of the regular part \( \bar{\mu}_r \) is strictly larger than \( \frac{8\pi}{K} \), it follows that
\[
\lim_{n \to \infty} \lambda_K(M, \bar{\mu}_n g) > \frac{8\pi}{K},
\]
hence a contradiction.

Assume now that \( M \) is a sphere. In that case, one can always assume without loss of generality, that the regular part \( \mu_r \) is not identically zero. This can be done by composing the approximating measures \( \mu_n \) with a suitable Möbius transformation. Hence we are done. \( \blacksquare \)

We now prove the relations (2) and (3). Actually the proofs are completely parallel and we just prove (2). This completes the proof of Theorem 1.1. We then prove

Lemma 3.5. Using the notations of Theorem 1.1, one has

\[
(8) \quad c_i \in \bigcup_{j=1}^k \left\{ \frac{\Lambda_j(S^2, [g_{\text{round}}])}{\Lambda_k(M, [g])} \right\},
\]

The proof of Lemma 3.5 is done in several steps. We first have

Lemma 3.6. For any \( i = 1, \ldots, K \), one has

\[
\Lambda_k(M, [g]) \in \bigcup_{m=1}^{\infty} \{ \lambda_{m,i}^x \},
\]

where \( \lambda_{m,i}^x \) are defined as above.

Proof. Assume the contrary. Then there exists \( i \) such that

\[
\Lambda_k(M, [g]) \notin \bigcup_{m=1}^{\infty} \{ \lambda_{m,i}^x \}.
\]

We then modify the metric in the following way: consider a smooth cut-off function for \( \delta, \epsilon > 0 \) defined by

\[
\psi(\delta, \epsilon) = \begin{cases} 1 & \text{on } M \setminus B(x_i, 2\delta), \\ 1 - \epsilon & \text{on } B(x_i, \delta), \end{cases}
\]
Then for sufficiently small $\epsilon$, there exists a sequence $\delta_n \to 0$ such that if we denote $\bar{g}_n = \psi(\delta_n, \epsilon)g_n$ where $g_n$ is the metric constructed in the existence part of Theorem 1.1, one has
\[
\lim_{n \to \infty} \lambda_k(\bar{g}_n) = \bar{\Lambda}_k(M, [g]).
\]
By properly choosing the sequence $\delta_n$, one has
\[
A_{\bar{g}_n}(M) \leq \frac{1}{2} c_i \epsilon.
\]
Therefore, the normalized metric
\[
h_n = \frac{\bar{g}_n}{A_{\bar{g}_n}(M)} \in [g_n]
\]
is such that
\[
\lim_{n \to \infty} \lambda_k(h_n) > \bar{\Lambda}_k(M, [g]),
\]
hence contradicting the definition of $\bar{\Lambda}_k(M, [g])$.
\[\square\]

One has similarly

**Lemma 3.7.** For any $i = 1, \ldots, K$, one has
\[
\bar{\Lambda}_k(M, [g]) \in \Lambda_r.
\]

We are then in position to prove Lemma 3.5.

**Proof of Lemma 3.5.** Assume by contradiction that there exists an index $i$ such that
\[
c_i \notin \bigcup_{j=1}^{k-1} \left\{ \frac{\Lambda_j(S^2, [\text{ground}])}{\bar{\Lambda}_k(M, [g])} \right\}.
\]

By Lemmata 3.6 and 3.7, there exists an $m \geq 1$ such that
\[
\lambda_{m,i} = \bar{\Lambda}_k(M, [g])
\]
For $\epsilon > 0$ and $i = 1, \ldots, K$, we introduce a sequence of metrics on $S^2$. Let $\Omega \subset S^2$ be an open domain and $\psi : \Omega \to M$ a conformal map such that $x_i \in \psi(\Omega)$. Set
\[
\tilde{g}^{\epsilon,i}_n = \psi^* g'_n|_{B(x_i, \epsilon)},
\]
such that $A_{\tilde{g}^{\epsilon,i}_n} \to c_i$. If $\lambda_{j,n,i}^{\epsilon}$ denote the eigenvalue of the Laplace-Beltrami operator on $(S^2, \tilde{g}^{\epsilon,i}_n)$ then one can define the following limits:
\[
\tilde{\lambda}_j^{\epsilon,i} = \lim_{n \to \infty} \lambda_{j,n,i}^{\epsilon},
\]
\[
\tilde{\lambda}_j^{i} = \lim_{\epsilon \to 0} \tilde{\lambda}_j^{\epsilon,i}
\]
we prove
\[ \tilde{\lambda}_m^i = \lambda_n^i = \frac{\Lambda_m(S^2, [g_{\text{round}}])}{c_i}. \]

Assume not, i.e.,
\[ \lambda_n^i \neq \frac{\Lambda_m(S^2, [g_{\text{round}}])}{c_i}. \]

Then we can modify metrics \( \tilde{g}_n^{i,i} \) into the metrics \( \tilde{g}_n^{i,i} \) on the sphere
having the same area such that the area of \( S^2 \setminus \Omega \) in the new metrics
tends to 0 and the corresponding eigenvalues \( \Lambda_j^{c,n,i} \) satisfy
\[ \lim_{\epsilon \to 0} \Lambda_j^{c,n,i} > \tilde{\lambda}_m^i = \lambda_n^i \]

One can modify the sequence \( g_n \), the one obtained in the existence
part of Theorem 1.1, in a neighborhood of \( x_i \) into a metric \( g'_n \) by trans-
planting the problem to the sphere \( S^2 \) in the following way. We set on \( \Omega \)
\[ \tilde{g}_n^{i,i} = \psi^* g'_n|_{B(x_i,\epsilon)}, \]

Then we have
\[ \lim_{n \to \infty} \lambda_k(g'_n) > \tilde{\lambda}_k(M, [g]). \]

On the other hand,
\[ \tilde{\lambda}_k(M, [g]) \notin \bigcup_{m=1}^{\infty} \{ \lambda_m^i \}, \]

hence a contradiction. This finishes the proof of Theorem 1.1.

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