DENSE IMAGES OF THE POWER MAPS FOR A DISCONNECTED REAL ALGEBRAIC GROUP

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Abstract. Let $G$ be a complex algebraic group defined over $\mathbb{R}$, which is not necessarily Zariski connected. In this article, we study the density of the images of the power maps $g \rightarrow g^k$, $k \in \mathbb{N}$, on real points of $G$, i.e., $G(\mathbb{R})$ equipped with the real topology. As a result, we extend a theorem of P. Chatterjee ([Ch1]) on surjectivity of the power map for the set of semisimple elements of $G(\mathbb{R})$. We also characterize surjectivity of the power map for a disconnected group $G(\mathbb{R})$. The results are applied in particular to describe the image of the exponential map of $G(\mathbb{R})$.

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1. Introduction

Let $G$ be a topological group. For $k \in \mathbb{N}$, let $P_k : G \rightarrow G$ be the $k$-th power map of $G$ defined by $P_k(g) = g^k$ for all $g \in G$. This article is mainly concerned with the question as to when such a map has a dense image or a surjective image for real points of an algebraic group.

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There is a vast amount of literature with regard to analogous ques-
tions in the case of exponential maps of (connected) Lie groups (see
[DH], [BM], [HL] and [DT] for example). Moreover, it is known, proved
independently by K. Hofmann, J. Lawson [HL] and M. McCrudden [Mc]
that the exponential map of a connected Lie group $G$ is surjective if
and only if $P_k(G) = G$ for all $k \in \mathbb{N}$. Recently, it was proved that the
exponential map has a dense image in $G$ if and only if the image of the
$k$-th power map is dense for all $k$ (see [BM]). This gives a motivation
to study the images of the power maps.

A well-known result of A. Borel states that the image of a word map
(and hence in particular a power map) on a semisimple algebraic group
$G$ is Zariski-dense (see [B1]). Note that the image of the power map is
not always dense for real points of an algebraic group in real topology.
For example, in case of $\text{SL}(2, \mathbb{R})$, the image of $P_2$ is not dense in real
topology but its image is Zariski-dense in $\text{SL}(2, \mathbb{R})$. Therefore our aim is
to study the density property of $P_k$ for real points of an algebraic group
in real topology. The density of $P_k$ is well understood for connected Lie
groups (see [BM], [M]). There has also been a considerable amount of
work with regard to the surjectivity of the map for both connected and
disconnected groups (cf. e.g. [Ch], [Ch1], [Ch2], [St1], [DM] and also
see references therein). However, the density of the power map is not
known for a disconnected group.

In this context, we obtain results for density of the image of $P_k$ on
real points $G(\mathbb{R})$ of a complex algebraic group $G$, which is defined over
$\mathbb{R}$ and is not necessarily Zariski connected (see Theorem 1.2). Indeed,
Theorem 1.2 provides an extension of [Ch1, Theorem 5.5], which deals
with the surjectivity of the map on the set of semisimple elements
of $G(\mathbb{R})$. As a consequence of Theorem 1.2, we get a characterization
of the surjectivity of the power map for (possibly disconnected) group
$G(\mathbb{R})$ (see Corollary 1.4).

The present paper, dealing with the power map in a disconnected
group is thus a natural continuation of previous works in [BM] and
[M]. We briefly recall the results to put things in a proper perspec-
tive. In [BM] and [M], we dealt with a connected Lie group. We gave
equivalent conditions for density of the images of $P_k$ in terms of regular
elements, Cartan subgroups and minimal parabolic subgroups. Also,
we established a relation between weak exponentiality and a dense im-
age of the power map. It was proved that the density of $P_k$ depends
only on semisimple part of the group. For a simple Lie group, we de-
termined the values of $k$ for which the power map has a dense image.
For a simply connected semisimple Lie group, the weak exponentiality
depends only on the image of the square map being dense, and the
same phenomenon occurs for any connected linear Lie group. The behaviour of a connected maximal rank subgroup is also discussed in this context. It was proved that if the power map has a dense image, then the same holds for all its maximal rank subgroups. In this set-up, we obtain many results on disconnected groups thereby extending earlier results in a broader framework.

We now describe the main results of this article.

Let $G$ be a complex algebraic group defined over $\mathbb{R}$. Let $G(\mathbb{R})$ be the set of $\mathbb{R}$-points of $G$. Note that $G(\mathbb{R})$ is equipped with the real topology and has a real manifold structure. We denote the Zariski connected component of identity of $G$ by $G^0$. Let $G(\mathbb{R})^*$ be the connected component of identity of $G(\mathbb{R})$ in the Hausdorff (viz real) topology.

The following theorem characterizes the density of the images of the power maps for a reductive group in terms of the Cartan subgroups.

**Theorem 1.1.** Let $G$ be a complex reductive algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Suppose that for any Cartan subgroup $C$ of $G$, $C \cap G^0 = C^0$. Let $k \in \mathbb{N}$. Then the following are equivalent.

1. The image of $P_k : G(\mathbb{R}) \to G(\mathbb{R})$ is dense.
2. $P_k : C(\mathbb{R}) \to C(\mathbb{R})$ is surjective for any Cartan subgroup $C$ of $G$ defined over $\mathbb{R}$.

Theorem 1.1 extends [BM, Theorem 1.1] for a reductive disconnected group. There is a class of Examples described in Example 3.4 which satisfy the hypothesis of the above theorem. An example is given to show that the conclusion of Theorem 1.1 may not hold for any arbitrary Cartan subgroups $C$ of $G$ (see Example 3.6).

We denote $(a, b) = 1$ if two integers $a$ and $b$ are co-prime. Also, the order of a finite group $F$ is denoted by $\circ(F)$. For a subset $A$ of an algebraic group $G$, we denote the set of semisimple elements of $A$ by $S(A)$.

The next theorem characterizes the density of the images of the power maps for a disconnected algebraic group which is not necessarily reductive, and provides an extension of [Ch1, Theorem 5.5]. In the context of surjectivity of the power map, an analogous result (corresponding to (1) $\iff$ (2) in Theorem 1.2) was proved by P. Chatterjee (see [Ch1, Theorem 1.8]). Also, he showed (3) $\iff$ (4) in Theorem 1.2 (see [Ch1, Theorem 5.5]).

**Theorem 1.2.** Let $G$ be a complex algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Let $A$ be a subgroup of $G$ with $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$ and $k \in \mathbb{N}$. Then the following are equivalent.
(1) $P_k(A)$ is dense in $A$,
(2) $(k, \circ (A/G(R)^*)) = 1$ and $P_k(G(R)^*)$ is dense,
(3) $(k, \circ (A/G(R)^*)) = 1$ and $P_k : S(G(R)^*) \to S(G(R)^*)$ is surjective,
(4) $P_k : S(A) \to S(A)$ is surjective.

In particular, if $G$ be a Zariski connected algebraic group defined over $\mathbb{R}$ then for an odd $k \in \mathbb{N}$, $P_k(G(R))$ is dense in $G(R)$ if and only if $P_k(G(R)^*)$ is dense in $G(R)^*$.

Corollary 4.5 implies that to determine whether the image of $P_k$ is dense in $G(R)$, it is enough to check the same for its Levi component only.

The following corollary establishes a special behaviour in the case of dense image of the power map on $G(R)$, analogous to a result of [HM, Corollary 2.1A] for the exponential map. On the other hand, it generalizes the result [HM, Corollary 2.1A] restricted to the group $G(R)$. In particular, it also generalizes [BM, Proposition 3.3] for the same.

**Corollary 1.3.** Let $G$ be a Zariski-connected complex algebraic group defined over $\mathbb{R}$. Let $N$ be a Zariski-connected, Zariski-closed algebraic normal subgroup of $G$ defined over $\mathbb{R}$. If both $P_k(G(R)/N(R))$ and $P_k(N(R))$ are dense, then $P_k(G(R))$ is dense.

Next, we characterize the surjectivity of the power map for a disconnected group $G(R)$.

**Corollary 1.4.** Let $G$ be an algebraic group defined over $\mathbb{R}$, which is not necessarily Zariski connected. Let $k \in \mathbb{N}$. Then $P_k : G(R) \to G(R)$ is surjective if and only if $P_k(Z_G(R)(u))$ is dense in $Z_G(R)(u)$ for any unipotent element $u \in G(R)$.

An application of Corollary 1.4 is given in Corollary 6.1, which characterizes the exponentiality of $G(R)$.

In the following result, we show that the density of the image of $P_k$ on $G(R)$ is equivalent to that for any maximal rank subgroup, which is not necessarily connected in Hausdorff topology. In [BM, Theorem 1.6], authors deal with the same issue for a connected maximal rank subgroup. Analogous results were proved in [Ch1] and [Ch2] by P. Chatterjee with regard to surjectivity of $P_k$.

**Corollary 1.5.** Let $G$ be a Zariski connected complex algebraic group defined over $\mathbb{R}$, and let $H$ be a maximal rank $\mathbb{R}$-algebraic subgroup of $G$. Let $k \in \mathbb{N}$. Suppose that $P_k(G(R))$ is dense in $G(R)$. Then the following hold:
Let $k$ be an odd integer. If $(k, \circ(H(\mathbb{R})/H(\mathbb{R})^*)) = 1$, then $P_k(H(\mathbb{R}))$ is dense in $H(\mathbb{R})$. In particular, this holds if $H$ is Zariski connected.

(2) Let $k$ be an even integer. If $(k, \circ(H(\mathbb{R})/H(\mathbb{R})^*)) = 1$, then $P_k(H(\mathbb{R}))$ is dense in $H(\mathbb{R})$. In particular, if $H$ is Zariski connected, then $P_k(H(\mathbb{R}))$ is dense in $H(\mathbb{R})$ if and only if $H(\mathbb{R}) = H(\mathbb{R})^*$.

Next we give a measure theoretic description of the dense image of the power map.

**Proposition 1.6.** Let $G$ be a Zariski connected complex algebraic group defined over $\mathbb{R}$. Then the following are equivalent.

1. $P_k(G(\mathbb{R}))$ is dense in $G(\mathbb{R})$.
2. The complement of $P_k(G(\mathbb{R}))$ in $G(\mathbb{R})$ has zero Haar measure.

The paper is organized as follows. In §2, we recall some results and deduce Proposition 2.5. In §3, we prove some preliminary results and deduce Theorem 1.2. In §4 we derive Theorem 1.1. In §5, we prove Corollary 1.3. Corollary 1.4 is proved in §6. In §7, we deduce Theorem 1.5. Finally in §8, we discuss the issue about the complement of the image of the $k$-th power map having zero Haar measure.

### 2. Preliminaries

Let $G$ be an algebraic group over a field $\mathbb{F}$ of characteristic zero, which is not necessarily connected. Let $G^0$ denote the Zariski-connected component of the identity of $G$. The maximal, Zariski connected, Zariski closed, normal unipotent subgroup of $G$ is called the unipotent radical of $G$ and is denoted by $R_u(G)$. Note that $R_u(G) = R_u(G^0)$.

The following theorem is contained in [Mos].

**Theorem 2.1.** Let $G$ be a linear algebraic group, not necessarily Zariski connected, over a field $\mathbb{F}$ of characteristic 0. Let $G^0$ denote the connected component of the identity of $G$, and let $R_u(G^0)$ denote the unipotent radical of $G^0$. Then the extension

$$1 \to R_u(G^0) \to G \to G/R_u(G^0) \to 1$$

splits, i.e., $G = L \ltimes R_u(G^0)$ (a semidirect product), where $L \subseteq G$ is an $\mathbb{F}$-subgroup of $G$ isomorphic to $G/R_u(G)$.

This ensures that if $G$ is a complex algebraic group (possibly disconnected) defined over $\mathbb{R}$, then $G(\mathbb{R}) = L(\mathbb{R}) \times R_u(G^0)(\mathbb{R})$. Indeed, let $g \in G(\mathbb{R})$. Since $G = L \ltimes R_u(G^0)$, there exist unique elements
l ∈ L and u ∈ R_u(G^0) such that g = lu. Let σ : G → G be the anti-
homomorphism automorphism of G so that G(ℝ) is precisely the fixed
point of this automorphism. Thus we have

\[ lu = g = \sigma(g) = \sigma(l)\sigma(u). \]

This gives \( \sigma(l)^{-1}l = u\sigma(u)^{-1} \). Since \( L \cap R_u(G^0) \) is trivial, we get
\( \sigma(l) = l \) and \( \sigma(u) = u \).

We haven’t found any suitable reference for Theorem 2.4 given below
though it may be known to experts. So we include a proof for com-
pleteness. It essentially follows from the proof of [Mo1, Proposition 2.6]
and is similar to https://mathoverflow.net/questions/280874/are-the-
semi-simple-elements-in-a-non-connected-reductive-algebraic-group-
dense. The proof of Theorem 2.4 and Proposition 2.5 are due to P.
Chatterjee.

We need some results from [GOV].

**Definition 1.** (See p. 105, §3.2. of [GOV].) A commutative algebraic
group whose Zariski connected component is a torus is called a quasi
torus. It is clear that if \( G \) is an algebraic group and \( s \in G \) is a semisim-
ple element then the the Zariski closure of the group generated by \( s \) is
a quasi torus.

We now state a result which follows from a general result from [GOV].

**Theorem 2.2.** Let \( G \) be a complex algebraic group which is not nec-
essarily connected. Assume that the Zariski connected component \( G^0 \)
is reductive. Let \( s \in G \) be a semisimple element. Then \( Z_G(s)^0 \) is a
reductive group.

**Proof.** This follows from Proposition 3.6, p. 107, of [GOV], as the
Zariski closure of the group generated by \( s \) is a quasi torus. \( \square \)

Notation: For an algebraic group the set of semisimple elements in
\( G \) is denoted by \( S(G) \). For an element \( a \in G \) let \( C_a : G \to G \) denote
the conjugation defined by \( C_a(x) := axa^{-1} \) for all \( x \in G \).

We will assume the following result.

**Theorem 2.3.** Let \( G \) be a connected reductive group over any alge-
braically closed field. Then the set of semisimple elements \( S(G) \) con-
tains a non-empty open set and hence \( S(G) \) is Zariski dense in \( G \).

**Proof.** See Theorem in §2.5., p. 28 of [H]. \( \square \)

**Theorem 2.4.** Let \( G \) be a complex algebraic group, which not nec-
essarily connected. Suppose the connected component \( G^0 \) is reductive.
Then for each \( a \in G \) the the set of semisimple elements \( S(G) \cap G^o a \) is a non-empty Zariski open subset of the coset \( G^o a \). In particular \( S(G) \) is Zariski dense in \( G \).

**Proof.** We will prove first the Zariski density of semisimple elements in \( G^o a \), and deduce the stronger statement that the set of semisimple elements of \( G^o a \) contains a non-empty open set of \( G^o a \) (for this we will show that this set coincides with image of a variety under a morphism).

As the underlying field is of characteristic zero for all \( b \in G \) there is an \( b_s \in S(G) \) such that \( G^o b = G^o b_s \) (here \( b_s \) is the semisimple part in the Jordan decomposition of \( b \)) we may assume without loss of generality that \( a \in S(G) \). Since all elements of \( S(Z_G(a)^o) \) commutes with \( a \), and \( a \) is semisimple so are all the elements of \( [S(Z_G(a)^o)] a \). Moreover by considering relations modulo the normal subgroup \( G^o \), it is immediate that for all \( g \in G^o \) the set \( g[S(Z_G(a)^o)] a g^{-1} \subset G^o a \). Thus

\[
\bigcup_{g \in G^o} g[S(Z_G(a)^o)] a g^{-1} \subset G^o a.
\]

Moreover, all the elements of \( \bigcup_{g \in G^o} g[S(Z_G(a)^o)] a g^{-1} \) are semisimple elements. We claim that \( \bigcup_{g \in G^o} g[S(Z_G(a)^o)] a g^{-1} \) contains an open set of the irreducible variety \( G^o a \).

First we show that the morphism \( \Psi : G^o \times Z_G(a)^o \to G^o \) defined by

\[
\Psi(g, x) := gx C_a(g^{-1}), \text{ for all } (g, x) \in G^o \times Z_G(a)^o
\]

is dominant. As the underlying field is of characteristic zero it is enough to show that the map is a submersion at the point \( (e, e) \) of the domain. This is done in Mathoverflow (reference mentioned as above) and hence we omit it. Now let \( R_a : G \to G \) be the right translation defined by \( R_a(z) := za \) for all \( z \in G \). Thus \( R_a : G^o \to G^o a \) is an isomorphism of varieties. In particular dominance of \( \Psi \) is equivalent to that of \( R_a \circ \Psi : G^o \times Z_G(a)^o \to G^o a \). Observe that

\[
R_a \circ \Psi(g, x) := g x a g^{-1}, \text{ for all } (g, x) \in G^o \times Z_G(a)^o.
\]

Let \( T \subset Z_G(a)^o \) be a maximal torus. Since \( a \) is semisimple in \( G \) the group \( Z_G(a)^o \) is reductive, and hence the set of semisimple elements \( S(Z_G(a)^o) = \bigcup_{z \in Z_G(a)^o} z T z^{-1} \) is Zariski dense in \( Z_G(a)^o \). Thus

\[
L_a \circ \Psi(G^o \times S(Z_G(a)^o)) = L_a \circ \Psi(G^o \times \bigcup_{z \in Z_G(a)^o} z T z^{-1})
\]

\[
= \bigcup_{g \in G^o, z \in Z_G(a)^o} g z t z^{-1} a g^{-1}
\]
is Zariski dense in $G^0a$. Now one may consider the morphism $\Phi : G^\circ \times Z_G(a)^\circ \times T \to G^0a$ defined by

$$\Psi(g, z, t) := gztz^{-1}ag^{-1},$$

for all $(g, t, z) \in G^\circ \times Z_G(a)^\circ \times T$.

It is clear that $\Psi$ is dominant and hence its image contains a non-empty open set of $G^0a$, and further it is clear that image of $\Psi$ coincides with $\bigcup_{g \in G^0} g[S(Z_G(a)^\circ)]ag^{-1}$. This proves the claim, and the theorem. □

Proposition 2.5. Let $G$ be a complex algebraic group, not necessarily Zariski connected, defined over $\mathbb{R}$ such that $G_0$ is reductive. Then $S(G(\mathbb{R}))$ contains an open dense subset of $G(\mathbb{R})$ in the Hausdorff topology of $G(\mathbb{R})$.

Proof. Since $G(\mathbb{R})/G_0(\mathbb{R})$ embeds in $G/G_0$, it follows that $G(\mathbb{R})/G_0(\mathbb{R})$ is a finite group. Let $g_1, \ldots, g_p \in G(\mathbb{R})$ such that $G(\mathbb{R}) = g_1G_0(\mathbb{R}) \cup \cdots \cup g_pG_0(\mathbb{R})$.

It is enough to prove that for all $i = 1, \ldots, p$, the set $S(g_iG_0(\mathbb{R}))$ contains an open dense subset of $g_iG_0(\mathbb{R})$ in the Hausdorff topology of $G(\mathbb{R})$. By Lemma 2.4, there is a Zariski open set $W$ of $g_iG_0(\mathbb{R})$ consisting of semisimple elements of $G$.

Let $X_{sm}$ be the set of smooth points of an irreducible affine variety $X$ defined over $\mathbb{R}$, and $M := X_{sm}(\mathbb{R}) \neq \emptyset$. Let $W \neq \emptyset$ be a Zariski open set. Then $W \cap X(\mathbb{R})$ is an open dense subset of $X(\mathbb{R})$ in the Hausdorff topology of $X(\mathbb{R})$. Now observe that $g_iG_0_{sm}(\mathbb{R}) = g_iG_0(\mathbb{R})$. Hence by the above fact, it follows that $W \cap g_iG_0(\mathbb{R})$ is open dense in $g_iG_0(\mathbb{R})$. □

3. Cartan subgroups and power maps on reductive groups

In this section, we recall the definition of a Cartan subgroup for a disconnected group and prove Theorem 1.1.

The following proposition characterizes the density of the power map for a disconnected reductive group.

Proposition 3.1. Let $G$ be a complex reductive algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Let $k \in \mathbb{N}$. Then the following are equivalent.

(1) The image of $P_k : G(\mathbb{R}) \to G(\mathbb{R})$ is dense.
(2) $P_k : G(\mathbb{R})/G(\mathbb{R})^* \to G(\mathbb{R})/G(\mathbb{R})^*$ is surjective and the image of $P_k : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is dense.
(3) $P_k : G(\mathbb{R})/G_0(\mathbb{R}) \to G(\mathbb{R})/G_0(\mathbb{R})$ is surjective and the image of $P_k : G_0(\mathbb{R}) \to G_0(\mathbb{R})$ is dense.

Proof. We first prove (1) $\Rightarrow$ (2). Let $k$ be an integer. If $H$ is a Lie group with finitely many connected components and $H^*$ is the connected
component containing identity, then the density of \( P_k(H) \) in \( H \) implies that \( P_k \) on \( H/H^\ast \) is surjective and that \( P_k(H^\ast) \) is dense in \( H^\ast \). To see the latter part, let \( U \) be any open set in \( H^\ast \). Since, \( P_k : H \to H \) is dense, there exists \( h \in U \) which has \( k \)-th root in \( H \). Let \( g \in H \) such that \( g^k = h \). Now \( g^kH^\ast = H^\ast \), as \( g \in H^\ast \). Since \( o(H/H^\ast) \) and \( k \) are co-prime, we have \( g \in H^\ast \). This shows that \( P_k : H^\ast \to H^\ast \) is dense.

For (2) \( \Rightarrow \) (1), we see \( P_k(G(\mathbb{R})^\ast) \) is dense implies that \( P_k : S(G(\mathbb{R})^\ast) \to S(G(\mathbb{R})^\ast) \) is surjective by [BM, Theorem 1.1]. Indeed, any semisimple element is contained in a Cartan subgroup. Since \( P_k : G(\mathbb{R})/G(\mathbb{R})^\ast \to G(\mathbb{R})/G(\mathbb{R})^\ast \) is surjective, we have \( P_k : S(G(\mathbb{R})) \to S(G(\mathbb{R})) \) is also surjective by [Ch1, Theorem 5.5]. Now the result follows from Proposition [2,5].

Next we will prove (1) \( \Leftrightarrow \) (3). Since \( G(\mathbb{R})^\ast \subset G^0 \) we have \( G(\mathbb{R})^\ast \subset G^0(\mathbb{R}) \subset G(\mathbb{R}) \). This implies \( G^0(\mathbb{R})^\ast = G(\mathbb{R})^\ast \). We also observe that

\[
o(G(\mathbb{R})/G(\mathbb{R})^\ast) = o(G(\mathbb{R})/G^0(\mathbb{R})) \circ (G^0(\mathbb{R})/G(\mathbb{R})^\ast).
\]

Hence, \( k \) is co-prime to \( o(G(\mathbb{R})/G(\mathbb{R})^\ast) \) if and only if \( k \) is co-prime to both \( o(G(\mathbb{R})/G^0(\mathbb{R})) \) and \( o(G^0(\mathbb{R})/G(\mathbb{R})^\ast) \).

Now (1) \( \Leftrightarrow \) (3) follows from the above observation and the arguments of (1) \( \Leftrightarrow \) (2) applied to \( G^0(\mathbb{R}) \) in stead of \( G(\mathbb{R}) \). \( \square \)

Now we recall the definition of a Cartan subgroup of a Zariski disconnected complex reductive algebraic group \( G \) from [Mo1].

**Definition 2.** An algebraic subgroup \( C \) of \( G \) is called a Cartan subgroup if all the following properties hold:

1. \( C \) is diagonalizable,
2. \( C \) has finite index in its normalizer (in \( G \)),
3. \( C \) contains an element \( z \) generating \( C \) as an algebraic group.

It is noted in [Mo1] that Cartan subgroups are abelian and all elements of a Cartan subgroup are semisimple.

**Lemma 3.2.** Let \( G \) be a complex reductive algebraic group, not necessarily Zariski-connected, defined over \( \mathbb{R} \). Let \( g \in G(\mathbb{R}) \) be a semisimple element. Then there exists a Cartan subgroup \( C \) of \( G \) defined over \( \mathbb{R} \) such that \( g \in C \).

**Proof.** For any given semisimple element \( g \in G \), Mohrdieck constructed a Cartan subgroup \( C \) of \( G \) such that \( g \in C \) [Mo2, Proposition 2.1]. The proof of the lemma essentially follows from the construction of the Cartan subgroup \( C \). Now, if we take \( g \in G(\mathbb{R}) \) then it is enough to prove that the Cartan subgroup \( C \) containing \( g \) is defined over \( \mathbb{R} \).
This proof is in light of Mohrdieck’s proof. First assume that $G$ is semisimple and defined over $\mathbb{R}$. Let $g \in G(\mathbb{R})$ be a semisimple element. Then $Z_G(g)^0$ is reductive (see [GOV]) and it is defined over $\mathbb{R}$ (see [S, Corollary 12.1.4]). Since $Z_G(g)^0$ is connected algebraic group defined over $\mathbb{R}$, by [3, Theorem 18.2(i)] there exists a maximal torus $S$ of $Z_G(g)^0$ defined over $\mathbb{R}$. Let $H$ be the algebraic subgroup generated by $S$ and $g$. Note that $H$ is defined over $\mathbb{R}$. Recall from the proof of [Mo2, Proposition 2.1] that $H$ is a Cartan subgroup. This proves the lemma when $G$ is semisimple.

Suppose that $G$ is reductive algebraic group defined over $\mathbb{R}$. Then the center $Z(G)$ is also defined over $\mathbb{R}$ (see §12.1.7 of [S]) and hence $G/Z(G)$ is defined over $\mathbb{R}$ (see [S, Corollary 12.2.2]). Rest of the proof follows from the proof of [Mo2, Proposition 2.1].

**Corollary 3.3.** Let $G$ be a complex reductive algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Then there exists a Cartan subgroup, which is defined over $\mathbb{R}$.

**Proof.** The proof is immediate from Lemma 3.2 and proposition 2.5. □

Let $G$ be a complex reductive algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. We call a subgroup $H$ of $G(\mathbb{R})$ a Cartan subgroup if there exists a Cartan subgroup $C$ of $G$ defined over $\mathbb{R}$ such that $H = C \cap G(\mathbb{R}) = C(\mathbb{R})$.

Corollary 3.3 says that Cartan subgroup exists in $G(\mathbb{R})$.

**Proof of Theorem 1.1.** (1) ⇒ (2) In view of Proposition 3.1, $k$ is co-prime to $\circ(G(\mathbb{R})/G(\mathbb{R})^*)$ and hence $k$ is co-prime to both $\circ(G(\mathbb{R})/G^0(\mathbb{R}))$ and $\circ(G^0(\mathbb{R})/G(\mathbb{R})^*)$.

Let $C$ be a Cartan subgroup of $G$. Since $C \cap G^0 = C^0$, it implies that $C/C^0$ embeds in $G/G^0$. As $C^0(\mathbb{R}) = C(\mathbb{R}) \cap G^0$, it follows that $C(\mathbb{R})/C^0(\mathbb{R})$ embeds in $G(\mathbb{R})/G^0(\mathbb{R})$. Since $(k, \circ(G(\mathbb{R})/G^0(\mathbb{R}))) = 1$, we have $(k, \circ(C(\mathbb{R})/C^0(\mathbb{R}))) = 1$.

One may assume $k$ to be prime. Let $k$ be odd. As $C^0(\mathbb{R})^* = C(\mathbb{R})^*$, the order of $C^0(\mathbb{R})/C(\mathbb{R})^*$ is a power of 2. It follows that $P_k : C^0(\mathbb{R}) \rightarrow C^0(\mathbb{R})$ is surjective. If $k = 2$, then $G^0(\mathbb{R}) = G^0(\mathbb{R})^*$ and $P_2 : G(\mathbb{R})^* \rightarrow G(\mathbb{R})^*$ is surjective. So $G(\mathbb{R})^*$ is weakly exponential by [Ch1, Theorem 1.6 (4)]. Hence Cartan subgroups of $G(\mathbb{R})^*$ are connected. As $C^0(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})^*$, we have $P_2 : C^0(\mathbb{R}) \rightarrow C^0(\mathbb{R})$ is surjective. Therefore $P_k : C^0(\mathbb{R}) \rightarrow C^0(\mathbb{R})$ is surjective for any $k$.

Since $C(\mathbb{R})$ is abelian, $(k, \circ(C(\mathbb{R})/C^0(\mathbb{R}))) = 1$ and $P_k : C^0(\mathbb{R}) \rightarrow C^0(\mathbb{R})$ is surjective, we have $P_k : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ is surjective.
by Theorem 1.2, it is enough to show that \( P_k : S(G(\mathbb{R})) \to S(G(\mathbb{R})) \) is surjective. Let \( g \in S(G(\mathbb{R})) \). Then by Lemma 3.2 there exists a Cartan subgroup \( C \) of \( G \) defined over \( \mathbb{R} \) such that \( g \in C(\mathbb{R}) \). By hypothesis, there exists an element \( h \in C(\mathbb{R}) \) such that \( h^k = g \). Since all elements of \( C(\mathbb{R}) \) are semisimple, the assertion follows. \( \square \)

Now we provide some examples satisfying the hypothesis of Theorem 1.1.

**Example 3.4.** If \( G \) is a Zariski disconnected complex algebraic group such that \( G^0 \) is simply connected semisimple, then \( G \) satisfies the hypothesis of Theorem 1.1 (see [Mo1, Lemma 2.4]).

If \( G = \Gamma \ltimes G^0 \), where \( G \) is a semisimple algebraic group and \( \Gamma \) is a subgroup of the group of diagram automorphisms of the Dynkin diagram, then any Cartan subgroup \( C \) of \( G \) satisfies \( C \cap G^0 = C^0 \) (see [Mo1, Corollary 2.2]).

The following lemma says that if \( C \cap G^0 = C^0 \) and \( G/G^0 \) is cyclic, then it is of the form \( G = \Gamma \ltimes G^0 \) for \( \Gamma = G/G^0 \).

**Lemma 3.5.** Let \( G \) be an algebraic group such that \( G^0 \) is semisimple. Assume that \( G/G^0 \) is cyclic. Then any Cartan subgroup \( C \) of \( G \) satisfies \( C \cap G^0 = C^0 \) if and only if the exact sequence \( 1 \to G^0 \to G \to G/G^0 \to 1 \) splits.

**Proof.** Let \( n := o(G/G^0) \), and let \( g \in G \) be a semisimple element such that \( G/G^0 \) is generated by the coset \( gG^0 \). It is enough to prove that there is a semisimple element \( h \in G \) such that \( h^n = e \) and \( gG^0 = hC^0 \).

Since \( G^0 \) is reductive, there is a Cartan subgroup \( C \) of \( G \) such that \( g \in C \). As \( C^0 = C \cap G^0 \), \( C/C^0 \) embeds in \( G/G^0 \) which implies \( \text{ord}(gC^0) = n = \text{ord}(gG^0) \). Hence \( g^n \in C^0 \). Since \( C^0 \) is a torus, we may choose \( s \in C^0 \) such that \( s^n = g^n \). Clearly as \( s \in C \), it follows that \( s \) commutes with \( g \). Now set \( h := gs^{-1} \) to obtain the result. \( \square \)

The following example shows that \( P_k(G(\mathbb{R})) \) being dense does not necessarily imply the surjectivity of \( P_k \) on Cartan subgroup \( C(\mathbb{R}) \), where \( C(\mathbb{R})^* \) is proper subgroup of \( C \cap G(\mathbb{R})^* \).

**Example 3.6.** Let \( H = S^1 \ltimes \mathbb{Z}/2q\mathbb{Z} \) be a compact disconnected group defined as before \( (e^{ia}, \bar{n})(e^{ib}, \bar{m}) := (e^{ia}(e^{ib})^{(-1)^n}, \bar{n} + \bar{m}) \) for \( \alpha, \beta \in \mathbb{R}, \bar{n}, \bar{m} \in \mathbb{Z}/2q\mathbb{Z} \). We take the subgroup \( \Gamma \) generated by \( (e^{i\pi}, \bar{q}) \) in \( H \). Then \( \Gamma \) is a subgroup of order 2 and it is normal in \( H \). Now consider \( G = H/\Gamma \). Then \( G/G^* \) is of order \( q \). Since \( P_k : G^* \to G^* \) is surjective, by Theorem 1.2, \( P_k(G) \) is dense in \( G \) if and only if \( (k, q) = 1 \). Note that any Cartan subgroup is either conjugate to \( S^1 \) or \( \mathbb{Z}/2q\mathbb{Z} \). (see
Exercise 5, page-181 for a particular case). If we take $C$ to be the subgroup generated by $(1, \bar{1})$ in $G$. Then $C$ is a Cartan subgroup of order $q$, $C \cap G^*$ is of order 2 and $C^*$ is trivial. Hence $P_k : C \to C$ is surjective if and only if $(k, 2) = 1$ and $(k, q) = 1$.

4. Characterization of the dense image of $P_k$ for a disconnected group

In this section, we recall some definitions and deduce Theorem 1.2.

We recall that for a connected Lie group $G$, an element $g \in G$ is said to be regular if the nilspace $N(\text{Ad}_g - I)$ is of minimal dimension. An element $g$ is $P_k$-regular if $(dP_k)_g$ is non-singular.

Remark 4.1. Let $G$ be a complex algebraic group defined over $\mathbb{R}$. Then any element $g \in G(\mathbb{R})$ can be written as $g = g_s g_u$ where $g_s, g_u \in G(\mathbb{R})$ and $g_s$ is semisimple, $g_u$ is unipotent, and $g, g_s, g_u$ commute with each other. Then we have $\text{Ad}_g = \text{Ad}_{g_s} \text{Ad}_{g_u}$. Since all the eigenvalues of $\text{Ad}_{g_u}$ are 1, the generalized eigenspace for eigenvalue 1 of $\text{Ad}_{g_s}$ is the same as that of $\text{Ad}_{g_u}$. Note that $N(\text{Ad}_{g_s} - I) = \ker(\text{Ad}_{g_s} - 1)$ as $\text{Ad}_{g_u}$ is semisimple. The dimension of $\ker(\text{Ad}_{g_s} - 1)$ is equal to the dimension of the centralizer $Z_{G(\mathbb{R})}(g_s)$. So $g$ is regular in $G(\mathbb{R})$ if and only if $Z_{G(\mathbb{R})}(g_s)$ has minimal dimension. In particular, $g$ is regular in $G(\mathbb{R})$ if and only if $g_s$ is regular in $G(\mathbb{R})$.

To prove Theorem 1.2 we use the following lemma, which is itself an equivalent criterion for the dense image of $P_k$ for a connected linear Lie group. We shall also use it later.

Lemma 4.2. Let $G$ be a complex algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Let $k \in \mathbb{N}$. Then the following are equivalent.

1. $P_k : S(\text{Reg}(G(\mathbb{R}))) \to S(\text{Reg}(G(\mathbb{R})))$ is surjective.
2. $P_k : S(G(\mathbb{R})) \to S(G(\mathbb{R}))$ is surjective.
3. The image of the map $P_k : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is dense.

Proof. First, we will show that each of (1) and (2) implies (3). In view of [BM, Theorem 1.1], it is enough to show that $\text{Reg}(G(\mathbb{R})^*) \subset P_k(G(\mathbb{R})^*)$. Let $g \in \text{Reg}(G(\mathbb{R})^*)$. Let $g = g_s g_u$ where $g_s$ and $g_u$ are respectively the semisimple part and unipotent part of the Jordan decomposition of $g$. By Remark 4.1, $g$ is regular if and only if $g_s$ is regular. Hence, either of the hypothesis (1) or (2) imply that there exists $h_s \in S(\text{Reg}(G(\mathbb{R})))$ such that $h_s^k = g_s$. Since $g_u$ is unipotent, there exists a unique unipotent element $h_u$ such that
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$h^k_u = g_u$. As the Zariski closure of the cyclic subgroups generated by $g_u$ and $h_u$ are the same, $g_s$ commutes with $h_u$.

Since $h_u$ is unipotent, there exists a unique nilpotent element $X \in \text{Lie}(G(\mathbb{R})^*)$ such that $h_u = \exp X$. As $g_s$ commutes with $h_u$, we have $g_s \exp X g_s^{-1} = \exp X$, which gives

$$\exp(Ad_g(X)) = \exp X.$$ 

Now by the uniqueness of the element $X \in \text{Lie}(G(\mathbb{R})^*)$, we get $Ad_{g_s}(X) = X$. Therefore, $Ad_{h^k_s}(X) = X$. By [BM, Lemma 2.1], $h_s$ is regular and $P_k$-regular. This implies $Ad_{h_s}(X) = X$, which in turn shows that $h_s$ commutes with $h_u$. Hence, we have

$$g = g_s g_u = h^k_s h^k_u = (h_s h_u)^k.$$ 

Now suppose that (3) holds. Let $g \in S(G(\mathbb{R})^*)$ (or $S(\text{Reg}(G(\mathbb{R})^*))$). Then there exists a (or a unique) Cartan subgroup $C \subset G(\mathbb{R})^*$ such that $g \in C$. By hypothesis, we have $P_k(C) = C$ by [BM, Theorem 1.1]. Thus there exists $h \in C$ such that $h^k = g$. Since $g$ is semisimple, $h$ is also semisimple, which proves the lemma.

We need the next lemma to prove Theorem 1.2.

**Lemma 4.3.** Let $G$ be a complex reductive algebraic group, not necessarily Zariski-connected, defined over $\mathbb{R}$. Let $k \in \mathbb{N}$. Let $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$. Suppose that $P_k(A)$ is dense in $A$. Then for any $s \in S(A)$ there exists a closed abelian subgroup $B_s$ of $A$ with finitely many connected components and such that $P_k : B_s \to B_s$ is surjective.

**Proof.** In view of Propositions 2.5 and 3.1, we note that $P_k(A)$ is dense in $A$ if and only if $P_k : S(A) \to S(A)$ is surjective. By [Ch1, Theorem 5.5], $P_k : S(A) \to S(A)$ is surjective if and only if $k$ is co-prime to the order of $A/G(\mathbb{R})^*$ and $P_k : S(G(\mathbb{R})^*) \to S(G(\mathbb{R})^*)$ is surjective. We conclude our assertion by following the path of the proof of [Ch1, Theorem 5.5(1)].

Let $s \in S(A)$. If the order of $A/G(\mathbb{R})^*$ is $m$ than $s^m \in G(\mathbb{R})^*$. On page-230 of [Ch1, Theorem 5.5], it is shown that there exists a maximal $\mathbb{R}$-torus $T$ of $G' = Z_G(s^m)$ such that $s T s^{-1} = T$ and $P_k : Z_{T(\mathbb{R})}^*(s) \to Z_{T(\mathbb{R})}^*(s)$ is surjective. Moreover, it is shown in Case-1 and Case-2 of [Ch1, Theorem 5.5] that $s = (t^c s^d)^k$ for some $c, d \in \mathbb{Z}$ (one can choose $d$ to be positive), where $t \in Z_{T(\mathbb{R})}^*(s)$. Also, $s^n \in Z_{T(\mathbb{R})}^*(s)$ for some positive integer $n$.

Now we consider the subgroup $H$ of $A$ generated by $Z_{T(\mathbb{R})}^*(s)$ and the element $s$. Then $H$ is a closed abelian subgroup of $A$ and $H/H^*$ is
finite. Further, $P_k : H \to H$ is surjective as $P_k : Z_{T_{(\mathbb{R})}}(s) \to Z_{T_{(\mathbb{R})}}(s)$ is surjective and $s = (i^c s^d)^k \in H$. Now set $B_s := H$. □

Recall that (see [Ch1]) a subgroup $A$ of an algebraic group $G$ is said to be a splittable group if it is closed under Jordan decomposition. viz, if $g \in A$, then both the semisimple part $g_s$ and the unipotent part $g_u$ of the Jordan decomposition lie in $A$. Note that if $G$ is an algebraic group over $\mathbb{R}$, then $G(\mathbb{R})$ is closed under Jordan decomposition. Indeed, let $g \in G(\mathbb{R})$, and let $\sigma$ be an anti-holomorphic automorphism on $G$. Then

$$\sigma(g_s)\sigma(g_u) = \sigma(g_s g_u) = \sigma(g) = g = g_s g_u$$

gives $g_s^{-1}\sigma(g_s) = g_u\sigma(g_u)^{-1}$, which shows that $\sigma(g_s) = g_s$ and $\sigma(g_u) = g_u$. Moreover, since any unipotent element $u \in G(\mathbb{R})$ lies in $G(\mathbb{R})^*$ for any subgroup $A$ with $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$, $A$ is closed under Jordan decomposition.

We now give a proof of Theorem 1.2.

**Proof of Theorem 1.2**

(1) $\Rightarrow$ (2) : follows along the lines of the proof of (2) $\Rightarrow$ (1) for Proposition 3.1.

(2) $\Rightarrow$ (1) : We first give a brief outline of the proof and later explain the steps. In step 1, we assume $G(\mathbb{R}) = L(\mathbb{R}) \ltimes R_u(G)(\mathbb{R})$ and consider a dense set $D = S(A \cap L(\mathbb{R})) R_u(G)(\mathbb{R})$ of $A$. In step 2, by using the hypothesis, we see that $P_k : S(A \cap L(\mathbb{R})) \to S(A \cap L(\mathbb{R}))$ is surjective, i.e., $P_k(A \cap L(\mathbb{R}))$ is dense. In step 3, to each element $x_0 \in S(A \cap L(\mathbb{R}))$ we associate a subgroup $B_{x_0}$ of $A$ such that $B_{x_0}$ is abelian with finitely many components and $P_k : B_{x_0} \to B_{x_0}$ is surjective. Then we construct a solvable subgroup $G_{x_0}$ of $A$ given by $G_{x_0} = B_{x_0} \ltimes R_u(G)(\mathbb{R})$. We show there exists a dense open set $W_{x_0}$ of $G_{x_0}$ such that each element of $W_{x_0}$ has $k$-th root in $G_{x_0}$ (and hence in $A$). In step 4, we give an existence of a dense set $W$ (of $A$) whose elements have $k$-th roots in $G$ as required.

**Step 1:** Since $S(L(\mathbb{R}))$ is dense in $L(\mathbb{R})$ by Proposition 2.1 and $A$ is open in $L(\mathbb{R})$, $S(A \cap L(\mathbb{R}))$ is dense in $A \cap L(\mathbb{R})$.

**Step 2:** From Theorem 2.1 we obtain $G(\mathbb{R}) = L(\mathbb{R}) \ltimes R_u(G^0)(\mathbb{R})$. Since $G(\mathbb{R})^* \subset A \subset G(\mathbb{R})$, we have $L(\mathbb{R})^* \subset A \cap L(\mathbb{R}) \subset L(\mathbb{R})$. Note that if $(k, o(A/G(\mathbb{R})^*)) = 1$, then $(k, o((A \cap L(\mathbb{R}))/L(\mathbb{R})^*)) = 1$. Also $P_k(G(\mathbb{R})^*)$ is dense in $G(\mathbb{R})^*$, which implies that $P_k(L(\mathbb{R})^*)$ is dense in $L(\mathbb{R})^*$. Now by Lemma 4.2 applied to the reductive group $L(\mathbb{R})^*$, we have $P_k : S(L(\mathbb{R})^*) \to S(L(\mathbb{R})^*)$ is surjective. So $P_k : S(A \cap L(\mathbb{R})) \to S(A \cap L(\mathbb{R}))$ is surjective by [Ch1, Theorem 5.5]. Hence $P_k(A \cap L(\mathbb{R}))$ is dense in $A \cap L(\mathbb{R})$. 

Step 3: For a given \( x_0 \in S(A \cap L(\mathbb{R})) \), by Lemma [1.3] there exists a closed abelian subgroup \( B_{x_0} \) of \( A \cap L(\mathbb{R}) \) such that \( P_k(B_{x_0}) = B_{x_0} \). Also, \( B_{x_0} \) has finitely many connected components in real topology. Now consider the subgroup \( G_{x_0} = B_{x_0} \rtimes R_u(G)(\mathbb{R}) \) of \( A \). Note that the unipotent radical \( N = R_u(G)(\mathbb{R}) \) is a simply connected nilpotent Lie group. Let \( N = N_0 \supset N_1 \supset \cdots \supset N_r = \{ e \} \) be the central series of \( N \). Let \( V_j := N_j/N_{j+1} \) for \( j = 0, 1, \ldots, r - 1 \). As \( N \) is simply connected, \( V_j \)'s are all finite dimensional real vector spaces. This \( N \) is \( \mathbb{R} \)-nilpotent in the sense of [DM]. Now for \( j = 0, 1, \ldots, r - 1 \), let \( F_j \) be the set of elements of \( B_{x_0} \) which acts trivially on \( V_j \) under conjugation action. Note that \( F_j \) is a closed subgroup of \( B_{x_0} \) for all \( j \). We shall consider the following two cases separately.

i) \( \dim(F_j) < \dim(B_{x_0}) \) for all \( j \),

ii) for some \( j \), \( \dim(F_j) = \dim(B_{x_0}) \).

Suppose (i) holds. Then \( U' := B_{x_0} - \bigcup_j F_j \) is a dense open set in \( B_{x_0} \). It is easy to see that all elements in \( U' \) act non trivially on all \( V_j \)'s. Therefore by [DM Theorem 1.1(i)], \( gN \subset P_k(G_{x_0}) \) for all \( g \in U' \). If we take \( W_{x_0} = U' \times N \), then \( W_{x_0} \) is a dense open subset of \( G_{x_0} \) such that \( W_{x_0} \subset P_k(G_{x_0}) \).

Now suppose (ii) holds. Let \( I \subset \{ 0, 1, \ldots, r - 1 \} \) be the set of indices such that \( \dim(F_j) = \dim(B_{x_0}) \) for all \( j \in I \). Then \( \bigcup_{i \in I} F_i \) is a proper closed analytic subset of \( B_{x_0} \) of smaller dimension. We will show that for all \( x \in (B_{x_0} \setminus \bigcup_{i \in I} F_i) \), \( xN \subset P_k(G_{x_0}) \).

Let \( j \in \{ 0, 1, \ldots, r - 1 \} \) be such that \( \dim(F_j) = \dim(B_{x_0}) \). Then \( F_j \) is the union of some connected components of \( B_{x_0} \) (containing the identity component of \( B_{x_0} \)).

Let \( x \in \bigcap_{i \in I} F_i \) such that \( x \in (B_{x_0} \setminus \bigcup_{i \notin I} F_i) \). Since \( P_k(B_{x_0}) = B_{x_0} \), we get \( P_k : B_{x_0}/B^*_{x_0} \to B_{x_0}/B^*_{x_0} \) is surjective, and hence \( k \) is co-prime to the order of the component group \( \Gamma = B_{x_0}/B^*_{x_0} \). Let the cardinality of \( \Gamma \) be \( n \). Then there exist integers \( a \) and \( b \) such that \( ak + bn = 1 \). Moreover, we can choose \( a \) to be a positive integer.

Now, \( x = x^{ak+bn} = (x^a)^k(x^n)^b \). Note that \( x^n \in B^*_{x_0} \) and so does \( (x^n)^b \). As \( B^*_{x_0} \) is a connected abelian group, it is divisible. So there exists \( x' \in B^*_{x_0} \) such that \( x'^k = (x^n)^b \). This gives \( x = (x^a)^k x'^k = (x^a x')^k \) as both \( x^a \) and \( x' \) commute with each other. Since \( \bigcap_{i \in I} F_i \) is a subgroup, \( x^a x' \in \bigcap_{i \in I} F_i \). This means that for any \( j \), if \( v \in V_j \) is fixed by the action of \( x \), then there exists a \( k \)-th root of \( x \) (namely \( x^a x' \)), which also fixes \( v \). By applying [DM Theorem 1.1(i)], we conclude that \( xn \in P_k(G_{x_0}) \) for all \( n \in N \).

Hence, there exists a dense open set \( W' \) in \( B_{x_0} \) such that \( W'_{x_0} \subset P_k(G_{x_0}) \), where \( W'_{x_0} = W' \times N \). Thus both cases we have a dense open
set \( W_{x_0} \) in \( G_{x_0} \) such that each elements of \( W_{x_0} \) has \( k \)-th roots in \( G_{x_0} \), and hence \( k \)-th roots in \( A \).

**Step 4:** We note that \( S(A \cap L(\mathbb{R}))N = \cup_{x_0 \in S(A \cap L(\mathbb{R}))}G_{x_0} \). Now set \( W = \cup_{x_0 \in S(A \cap L(\mathbb{R}))}W_{x_0} \). Then \( W \) is a dense set in \( A \) such that each element of \( W \) has a \( k \)-th root in \( A \). This proves (2) implies (1).

(2) \( \iff \) (3) This follows from Lemma 4.2.

(3) \( \iff \) (4) This follows from [Ch1, Theorem 5.5].

A. Borel and Tits showed that if \( G \) is a Zariski connected algebraic group defined over \( \mathbb{R} \), then either \( G(\mathbb{R}) = G(\mathbb{R})^* \) or \( G(\mathbb{R})/G(\mathbb{R})^* \) is a direct product of cyclic groups of order two (see [BT, Theorem 14.4]). Hence, the statement follow immediately from the above.

\[ \square \]

**Remark 4.4.**

1. Theorem 1.2 ensures that there exists an integer \( m_G \) such that \( P_k(G(\mathbb{R})) \) is dense in \( G(\mathbb{R}) \) for all \( k \) co-prime to \( m_G \) as the cardinality of \( G(\mathbb{R})/G(\mathbb{R})^* \) is finite.

2. Let \( G \) be an algebraic group over \( \mathbb{C} \), which is not necessarily Zariski-connected. Let \( G(\mathbb{C}) \) denote the complex point of \( G \). Let \( k \in \mathbb{N} \). Then it is immediate that, \( P_k(G(\mathbb{C})) \) is dense in \( G(\mathbb{C}) \) if and only if \( k \) is co-prime to the order of \( G/G^0 \).

Theorem 1.2 asserts the following corollary.

**Corollary 4.5.** Let \( G \) be as in Theorem 1.2. Let \( G(\mathbb{R}) = L(\mathbb{R}) \rtimes R_u(G)(\mathbb{R}) \) and \( k \in \mathbb{N} \). Then \( P_k(G(\mathbb{R})) \) is dense if and only if \( P_k(L(\mathbb{R})) \) is dense.

**Proof.** The proof follows using the same procedure as in the proof of Theorem 1.2 applied to the group \( A = G(\mathbb{R}) \). \[ \square \]

## 5. Application to weak exponentiality

In this section, we begin with two remarks and prove Corollary 1.3.

For a Lie group \( G \) with Lie algebra \( \text{Lie}(G) \), let \( \exp : \text{Lie}(G) \to G \) be the exponential map of \( G \). We recall that \( G \) is weakly exponential if \( \exp(\text{Lie}(G)) \) is dense in \( G \). The group \( G \) is said to exponential if \( \exp(\text{Lie}(G)) = G \).

**Remark 5.1.** We use the same notations as in Theorem 1.2. In view of (1) \( \Rightarrow \) (2) in Theorem 1.2, we see that \( P_k(A) \) is dense implies \( k \) is co-prime to the order of \( A/G(\mathbb{R})^* \) and \( P_k(G(\mathbb{R})^*) \) is dense. Then by [BM, Corollary 1.3], it follows that \( P_k(A) \) is dense for all \( k \) if and only if \( A \) is weakly exponential.
The concluding statement of Remark 5.1 can be thought of as the analogous result of Hofmann and Lawson [HL Proposition 1], which states the following: For a closed subgroup $H$ (possibly disconnected) of a connected Lie group, $H$ is exponential if and only if $H$ is divisible, i.e., $P_k(H) = H$ for all $k \in \mathbb{N}$.

**Remark 5.2.** Let $G$ be a Zariski-connected complex algebraic group defined over $\mathbb{R}$. Then the following are equivalent: (i) the map $P_2 : S(G(\mathbb{R})) \to S(G(\mathbb{R}))$ is surjective, (ii) the image of the map $P_2 : G(\mathbb{R}) \to G(\mathbb{R})$ is dense, (iii) $G(\mathbb{R})$ is weakly exponential. This can be seen from [Ch1 Theorem 1.6] and the fact that $G(\mathbb{R})/G(\mathbb{R})^*$ is a group of order $2^m$ for some $m$. It can also be deduced from Theorem 1.2. Indeed, by [Ch1 Theorem 5.5] and Lemma 4.2, statement (i) implies 2 is co-prime to the order of the group $G(\mathbb{R})/G(\mathbb{R})^*$ and $P_2(G(\mathbb{R})^*)$ is dense. Thus by Theorem 1.2, $P_2(G(\mathbb{R}))$ is dense. Now (ii) $\iff$ (iii) follows from Remark 5.1.

Note that if we replace $G(\mathbb{R})$ by $G(\mathbb{R})^*$ in statements (i)-(iii) of Remark 5.2 then the equivalence follows from [Ch1 Theorem 1.6]. Also, Remark 5.2 generalizes the result for a connected linear group (see [BM Corollary 1.5]). Note that [BM Corollary 1.5] (see Proposition 5.3 below) can be proved using results different from theorems in [BM]. For example, we now provide a proof which is due to P. Chatterjee.

**Proposition 5.3.** Let $G$ be a connected linear Lie group. Then $G$ is weakly exponential if and only if $P_2(G)$ is dense.

**Proof.** We assume that $P_2(G)$ is dense in $G$. Since $G$ is linear so is $G/Rad(G)$ (see p. 26. [OV Proposition 5.2]). Now as $G/Rad(G)$ is a connected linear semisimple Lie group, it is isomorphic to $A(\mathbb{R})^*$ for some Zariski connected (semisimple) algebraic group $A$ defined over $\mathbb{R}$. Since $P_2(G)$ is dense in $G$, it follows that $P_2(A(\mathbb{R})^*)$ is dense in $A(\mathbb{R})^*$. Using Theorem 1.2, we see that $P_2 : S(A(\mathbb{R})^*) \to S(A(\mathbb{R})^*)$ is surjective. Now apply [Ch1 Theorem 1.6] to see that $A(\mathbb{R})^*$ is weakly exponential. Thus $G/Rad(G)$ is weakly exponential. As $Rad(G)$ is connected solvable, it is weakly exponential and hence by [HM Lemma 3.5] $G$ is weakly exponential. The other part is obvious. □

Next, we prove Corollary 1.3

**Proof of Corollary 1.3** By Theorem 1.2(a), as $G$ is Zariski connected for any odd $k \in \mathbb{N}$, $P_k(G(\mathbb{R})/N(\mathbb{R}))$, $P_k(N(\mathbb{R}))$ and $P_k(G(\mathbb{R}))$ are dense.

Suppose both the images of $P_2 : G(\mathbb{R})/N(\mathbb{R}) \to G(\mathbb{R})/N(\mathbb{R})$ and $P_2 : N(\mathbb{R}) \to N(\mathbb{R})$ are dense. Then by Remark 5.2 $G(\mathbb{R})/N(\mathbb{R})$ and $N(\mathbb{R})$ are weakly exponential and hence connected. Recall that for any
closed subgroup $L$ of a topological group $H$, $H$ is connected if $H/L$ and $L$ are connected. This implies $G(\mathbb{R}) = G(\mathbb{R})^*$. Since both the groups $G(\mathbb{R})^*/N(\mathbb{R})^*$ and $N(\mathbb{R})^*$ are weakly exponential, by [HM, Corollary 2.1A], we have $G(\mathbb{R})^*$ is weakly exponential. Therefore $P_2(G(\mathbb{R}))$ is dense by Remark 5.2. □

6. Characterization of the surjectivity of $P_k$

In this section, we prove Corollary 1.4 and consequently deduce Corollary 6.1, which is a well known result for the case of the exponential map.

Proof of Corollary 1.4: By [Ch1, Lemma 5.6], $P_k : G(\mathbb{R}) \to G(\mathbb{R})$ is surjective if and only if for every unipotent element $u \in G(\mathbb{R})^*$, the map $P_k : S(Z_G(\mathbb{R}))(u) \to S(Z_G(\mathbb{R}))(u)$ is surjective. Note that $Z_G(\mathbb{R})(u) = Z_G(u)(\mathbb{R})$. For each unipotent element $u \in G(\mathbb{R})^*$, let $H_u = Z_G(u)$. Since $P_k$ is surjective on $S(H_u(\mathbb{R}))$, by [Ch1, Theorem 5.5], we have $P_k : H_u(\mathbb{R})/H_u(\mathbb{R})^* \to H_u(\mathbb{R})/H_u(\mathbb{R})^*$ and $P_k : S(H_u(\mathbb{R})) \to S(H_u(\mathbb{R}))$ are surjective. By Lemma 4.2, $P_k(H_u(\mathbb{R}))$ is dense in $H_u(\mathbb{R})^*$, which implies $P_k(H_u(\mathbb{R}))$ is dense in $H_u(\mathbb{R})$ by Theorem 1.2. This completes the proof. □

The following result is well known (see [DT, Theorem 2.2]) and was later deduced by P. Chatterjee (see [Ch1, Corollary 5.7]). Here we prove it as an application of Corollary 1.4.

Corollary 6.1. Let $G$ be a connected real algebraic group. Then $G(\mathbb{R})^*$ is exponential if and only if $Z_{G(\mathbb{R})^*}(u)$ is weakly exponential for all unipotent elements $u \in G(\mathbb{R})^*$.

Proof. By McCrudden’s criterion, $G(\mathbb{R})^*$ is exponential if and only if $P_k : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is surjective for all $k \in \mathbb{N}$. Further, by Corollary 1.4, $P_k : G(\mathbb{R})^* \to G(\mathbb{R})^*$ is surjective if and only if $P_k(Z_{G(\mathbb{R})^*}(u))$ is dense in $Z_{G(\mathbb{R})^*}(u)$ for all unipotent elements $u \in G(\mathbb{R})^*$. Now by Remark 5.1 we conclude that $Z_{G(\mathbb{R})^*}(u)$ is weakly exponential. □

7. Maximal rank subgroup

In this section, we prove Corollary 1.5.

Let $G$ be a Zariski connected complex algebraic group. The rank of $G$ is the dimension of any of its maximal tori. An algebraic subgroup $H$ of $G$ is said to be of maximal rank if rank of $H$ is the same as the rank of $G$. 
**Proof of Corollary 1.5** Note that $H(\mathbb{R})^*$ is a maximal rank subgroup of $G(\mathbb{R})^*$. We first prove that $P_k(G(\mathbb{R})^*)$ is dense in $G(\mathbb{R})^*$ implies that $P_k(H(\mathbb{R})^*)$ is dense in $H(\mathbb{R})^*$. Indeed, note that a maximal torus $T$ of $H$ is also a maximal torus of $G$. Let $T$ be defined over $\mathbb{R}$. For any odd $k \in \mathbb{N}$, $P_k(H(\mathbb{R})^*)$ is dense in $H(\mathbb{R})^*$ by Theorem 1.2. Thus in order to prove the assertion for any $k$, we only need to check for $k = 2$. Now $P_2(G(\mathbb{R})^*)$ is dense in $G(\mathbb{R})^*$ if and only if $T(\mathbb{R})^* = G(\mathbb{R})^* \cap T(\mathbb{R})$ (see [Ch1, Theorem 1.6]). Since $T(\mathbb{R})$ is contained in $H(\mathbb{R})$, we have $T(\mathbb{R})^* \subset H(\mathbb{R})^*$. This implies that $T(\mathbb{R})^* \subset H(\mathbb{R})^* \cap T(\mathbb{R})$. Also, $H(\mathbb{R})^* \cap T(\mathbb{R}) \subset G(\mathbb{R})^* \cap T(\mathbb{R})$ implies $T(\mathbb{R})^* = H(\mathbb{R})^* \cap T(\mathbb{R})$. So $P_2(H(\mathbb{R})^*)$ is dense in $H(\mathbb{R})^*$ by [Ch1, Theorem 1.6].

Using the above arguments, we prove statements (i) and (ii) of Corollary 1.5 as shown below.

(1) : For odd $k$, the hypothesis together with Theorem 1.2 implies the first part of statement (i) of Corollary 1.5. For the Zariski connected case, the assertion follows from the fact that $H(\mathbb{R})/H(\mathbb{R})^*$ is of order $2^m$ for some $m > 0$.

(2) : For this case, we only need to show that $P_2(H(\mathbb{R}))$ is dense in $H(\mathbb{R})$. By hypothesis, $P_k(G(\mathbb{R})^*)$ is dense in $G(\mathbb{R})^*$ and as shown earlier $P_k(H(\mathbb{R})^*)$ is dense in $H(\mathbb{R})^*$. The result now follows from Theorem 1.2.

For the Zariski connected case, $P_2$ is surjective on $H(\mathbb{R})/H(\mathbb{R})^*$ implies $H(\mathbb{R}) = H(\mathbb{R})^*$. Again, by using Theorem 1.2 it follows that $P_k(H(\mathbb{R}))$ is dense in $H(\mathbb{R})$ if and only if $H(\mathbb{R})$ is connected in real topology.

\[ \square \]

8. **A Remark on Complement of the Image of $P_k$**

Recall that for any locally compact topological group $G$ (not necessarily connected), there exists a unique (up to scalar) non-zero left invariant measure called Haar measure, which is finite on compact sets.

Let $G$ be a connected Lie group. Then the following two statements are equivalent:

(a) $P_k(G)$ is dense in $G$.

(b) The Haar measure of the complement of $P_k(G)$ in $G$ is zero.

Similarly, the statements given below are also equivalent.

(a') exp($g$) is dense in $G$.

(b') The Haar measure of the complement of exp($g$) in $G$ is zero.

Since the complement of Reg($G$) in $G$ has zero Haar measure, both (a) $\Leftrightarrow$ (b) and (a') $\Leftrightarrow$ (b') follow.

Let $G$ be a Zariski connected complex algebraic group defined over $\mathbb{R}$. Recall that $G(\mathbb{R})$ may not be connected in real topology. To prove
We say that an element \( g \in G \) is said to be regular if \( Z_G(g_s) \) is of minimal dimension.

Let \( \text{Reg}(G(\mathbb{R})) := \text{Reg}(G) \cap G(\mathbb{R}) \) and \( \text{Reg}(G(\mathbb{R})^*) := \text{Reg}(G) \cap G(\mathbb{R})^* \). It is known that \( \text{Reg}(G(\mathbb{R})) \) is an open dense set in \( G(\mathbb{R}) \).

**Lemma 8.1.** Let \( k \in \mathbb{N} \) and \( g \in G(\mathbb{R}) \) be such that \( h^k = g \). Then \( g \) is regular if and only if \( h \) is regular and \( P_k \)-regular. Moreover, if \( k \) is odd for a regular element \( g \in G(\mathbb{R}) \), then there exists a \( P_k \)-regular element \( h \in G(\mathbb{R}) \) such that \( h^k = g \).

**Proof.** Let \( g \in G(\mathbb{R}) \) be such that \( h^k = g \). Let \( g = g_s g_u \) and \( h = h_s h_u \) be the Jordan decompositions of \( g \) and \( h \) respectively. Thus we get \( h^k_s = g_s \) and \( h^k_u = g_u \). Since \( g \) is regular, \( g_s \) is regular. Note that \( Z_G(h_s) \subset Z_G(g_s) \). Since \( Z_G(g_s) \) is of minimal dimension, we have \( Z_G(h_s)^0 = Z_G(g_s)^0 \), which shows that \( h_s \) is regular. In particular, \( h \) is regular. The fact that \( h \) is \( P_k \)-regular follows from the same argument as in the proof of [BM, Lemma 2.1]. Conversely, since \( h \) is regular and \( P_k \)-regular, \( g \) is regular.

Let \( g \in G(\mathbb{R}) \) be regular. Then by definition, \( g_s \) is regular. Since \( G \) is a Zariski connected complex algebraic group, we have \( \text{Reg}(G) \subset P_k(G) \) for any \( k \). Therefore, by [Ch2, Theorem 4.4] for an odd \( k \), there is a \( P_k \)-regular \( h_s \in G(\mathbb{R}) \) such that \( g_s = h_s^k \). Using the proof of Lemma 1.2 we can conclude that \( h \) is \( P_k \)-regular. \( \square \)

**Lemma 8.2.** Let \( G \) be a Zariski connected complex algebraic group defined over \( \mathbb{R} \). Let \( k \in \mathbb{N} \). Then \( P_k : G(\mathbb{R})/G(\mathbb{R})^* \to G(\mathbb{R})/G(\mathbb{R})^* \) and \( P_k : \text{Reg}(G(\mathbb{R})^*) \to \text{Reg}(G(\mathbb{R})^*) \) are surjective together imply \( P_k : \text{Reg}(G(\mathbb{R})) \to \text{Reg}(G(\mathbb{R})) \) is surjective.

**Proof.** It is immediate that \( P_k : \text{Reg}(G(\mathbb{R})^*) \to \text{Reg}(G(\mathbb{R})^*) \) is surjective implies \( P_k : S(\text{Reg}(G(\mathbb{R})^*)) \to S(\text{Reg}(G(\mathbb{R})^*)) \) is surjective. By Lemma 1.2, \( P_k : S(G(\mathbb{R})^*) \to S(G(\mathbb{R})^*) \) is surjective. Again by applying [Ch1, Theorem 5.5], we obtain \( P_k : S(G(\mathbb{R})) \to S(G(\mathbb{R})) \) is surjective. Now using Lemma 8.1 we deduce that \( P_k : \text{Reg}(G(\mathbb{R})) \to \text{Reg}(G(\mathbb{R})) \) is surjective. \( \square \)

The above lemma leads to the following result.

**Proposition 8.3.** Let \( G \) be a Zariski connected complex algebraic group defined over \( \mathbb{R} \). Let \( k \in \mathbb{N} \). Then \( P_k : G(\mathbb{R}) \to G(\mathbb{R}) \) is dense if and only if \( P_k : \text{Reg}(G(\mathbb{R})) \to \text{Reg}(G(\mathbb{R})) \) is surjective.

The above proposition proves (1) \( \iff \) (2) of Proposition 1.6.
DENSE IMAGES OF THE POWER MAPS FOR A DISCONNECTED REAL ALGEBRAIC GROUP

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