Chow rings of matroids are Koszul

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Received: 28 March 2022 / Revised: 27 October 2022 / Accepted: 30 October 2022 / Published online: 8 November 2022 © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
Chow rings of matroids were instrumental in the resolution of the Heron–Rota–Welsh Conjecture by Adiprasito, Huh, and Katz and in the resolution of the Top-Heavy Conjecture by Braden, Huh, Matherne, Proudfoot, and Wang. The Chow ring of a matroid is a commutative, graded, Artinian, Gorenstein algebra with linear and quadratic relations defined by the matroid. Dotsenko conjectured that the Chow ring of any matroid is Koszul. The purpose of this paper is to prove Dotsenko’s conjecture. We also show that the augmented Chow ring of a matroid is Koszul. As a corollary, we show that the Chow rings and augmented Chow rings of matroids have rational Poincaré series.

Mathematics Subject Classification Primary 16S37 · 13E10 · 05B35; Secondary 13H10 · 05E40

1 Introduction

Let $M$ be a simple matroid on a finite set $E$. The Chow ring of $M$ is the quotient ring

$$\text{CH}(M) := S_M / (I_M + J_M),$$

where

$$S_M = \mathbb{Q}[x_F \mid F \text{ is a nonempty flat of } M],$$
and $I_M$ and $J_M$ are the ideals:

$$I_M = \left( \sum_{i \in F} x_F \mid i \in E \right)$$

$$J_M = (x_F x_G \mid F, G \text{ are incomparable, nonempty flats of } M).$$

In the case of matroids associated to complex hyperplane arrangements, Chow rings of matroids first appeared in the work of de Concini and Procesi [20] as the cohomology rings of wonderful compactifications of arrangement complements. Building on this work, Feichtner and Yuzvinsky [29] later showed how to associate a smooth toric variety to any finite atomic lattice and some extra combinatorial data called a building set, and they computed a similar presentation for its Chow ring. What is now called the Chow ring of a matroid is the Chow ring of its lattice of flats with respect to its maximal building set; see Sect. 2.5 for more details.

Chow rings of matroids and their augmented variants (see Sect. 5) have garnered significant attention in recent years due to the important role they have played in the proof of the longstanding Heron–Rota–Welsh Conjecture [2] and more recently in the proof of the Top-Heavy Conjecture [7]. The proofs of these purely combinatorial conjectures rely heavily on showing that the Chow ring of a matroid has a number of nice algebraic properties; it is shown in [2] that $\text{CH}(M)$ is a graded, Artinian $\mathbb{Q}$-algebra with Poincaré duality and versions of the hard Lefschetz and Hodge-Riemann conditions. In particular, $\text{CH}(M)$ is a Gorenstein algebra admitting a presentation by quadratic relations, and so, it is natural to ask whether it is also a Koszul algebra.

In fact, recent work of Dotsenko [22] shows that the cohomology ring of $\mathcal{M}_{0,n}$, the compactification of the moduli space of $n$ marked points on $\mathbb{P}^1_C$, is Koszul; these rings are Chow rings of intersection lattices of braid arrangements (with respect to the minimal building set). Based on this work, Dotsenko explicitly conjectures that $\text{CH}(M)$ is always Koszul [22, Conjecture 5.3]. Our first main result is to confirm Dotsenko’s conjecture.

**Theorem 1.1** For any simple matroid $M$, $\text{CH}(M)$ is Koszul.

Although it is well-known that all Koszul algebras have quadratic presentations (we will call such algebras quadratic), not all quadratic algebras are Koszul. In fact, it is not even guaranteed that a quadratic algebra will be Koszul if it is also Artinian and Gorenstein; the first non-Koszul such algebra was constructed by Matsuda [34]. A thorough study of the Koszul properties of quadratic, Gorenstein algebras was conducted by Schenck, Stillman, and the first author in [36, 37]. In particular, they determined (with only 3 exceptions) the ordered pairs $(r, c)$ for which there is a non-Koszul, quadratic, Gorenstein algebra of characteristic zero with regularity $r$ and codimension $c$; see also [35].

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1 There is potential for confusion on terminology here. We follow [2] by defining the Chow ring of a matroid to be the Chow ring of the associated geometric lattice with respect to the maximal building set. Our main Theorem 1.1 shows that all such Chow rings are Koszul. As the cohomology ring of $\mathcal{M}_{0,n}$ is a Chow ring with respect to the minimal building set, our theorem does not recover Dotsenko’s result. We explain this in greater detail in Sect. 2.5. We also note that the more general statement that the Chow ring of a geometric lattice with respect to any building set is Koszul is false; see Example 6.2.
In order for a quadratic algebra to be Koszul, it is sufficient for the defining ideal of relations to have a quadratic Gröbner basis, yet there are Koszul algebras whose defining ideals have no quadratic Gröbner bases with respect to any monomial order or any linear change of variables. Previously, Feichtner and Yuzvinsky [29] gave a non-quadratic Gröbner basis for a more general construction of the Chow ring of an atomic lattice (see Sect. 2). An argument similar to [22, Proposition 3.1] shows that there is no possible quadratic Gröbner basis for $I_M + J_M$ in the given presentation.

Our approach to showing that $\text{CH}(M)$ is always Koszul involves the construction of a Koszul filtration (see Theorem 4.12). A key ingredient in the construction is the study of lattices that admit what we call a total coatom ordering. This is an ordering on the elements of the lattice satisfying certain properties similar to those of a shellable simplicial complex. This aspect of the paper was inspired by and is similar to the notion of a strongly shellable simplicial complex of Guo et al. [31]. A very similar notion is implicit in the work of Delucchi [21]. In Sect. 3, we show that the lattice of flats of a matroid always admits a total coatom ordering, thereby making it possible to always construct a Koszul filtration on $\text{CH}(M)$.

It is also natural to study the Poincaré series of $\text{CH}(M)$. The Poincaré series of a $K$-algebra $R$, denoted $P^R_K(t)$, is the generating function of the total Betti numbers of $K$ over $R$:

$$P^R_K(t) := \sum_{i \geq 0} \dim_K \text{Tor}^R_i(K, K)t^i.$$  

For many years, it was expected that all such Poincaré series were rational; this expectation became known as the Serre–Kaplansky Problem. In [3, Example 7.1], Anick gave the first example of a commutative $K$-algebra with irrational Poincaré series. Later examples of Artinian, Gorenstein algebras with irrational Poincaré series included one by Bøgvad [11]. We observe that this algebra is quadratic (see Example 6.1). Thus, not all quadratic, Artinian, Gorenstein $K$-algebras have rational Poincaré series. Our second main result, a corollary to the first, is the following.

**Theorem 1.2** For any simple matroid $M$, $\text{CH}(M)$ has a rational Poincaré series.

We also consider the augmented Chow ring $\text{CH}(M)$ of a matroid $M$. This algebra was introduced in [8] as an analog of the Chow ring of the augmented wonderful variety associated to a hyperplane arrangement in the realizable case. Like the Chow ring of a matroid, these are quadratic, Artinian, Gorenstein algebras satisfying the Kähler package. We show (Theorem 5.10) that the augmented Chow ring of a matroid is also Koszul.

The rest of the paper is organized as follows. Section 2 collects necessary definitions and background results on lattices, matroids, Chow rings, and Koszul algebras. Section 3 contains a proof that all geometric lattices admit a total coatom ordering; we also connect total coatom orderings to other lattice conditions like shellability. Section 4 contains the proofs of our main results for Chow rings of matroids, while Sect. 5 contains the analogous results for augmented Chow rings of matroids. A final Sect. 6 collects some examples and questions.
2 Background

In this section, we collect some necessary notation and background for our main results. The reader can safely skip to Sect. 3 and review this section as necessary.

Notation Throughout this section, \( R = \bigoplus_{i \geq 0} R_i \) will denote a commutative graded \( \mathbb{K} \)-algebra finitely generated as an algebra over a field \( R_0 = \mathbb{K} \) by elements of degree one. We call such an algebra standard graded and set \( R_+ = \bigoplus_{i \geq 1} R_i \) to be the graded maximal ideal.

2.1 Hilbert functions and free resolutions

Let \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) be a finitely generated, graded \( R \)-module. The Hilbert function of \( M \) is defined as \( \text{HF}_M(i) := \dim_{\mathbb{K}} M_i \), and its generating function is the Hilbert series of \( M \), denoted \( H_M(t) := \sum_{i \geq 0} \text{HF}_M(i) t^i \). The module \( M \) has a minimal graded free resolution \( F_* \) over \( R \), which is an exact sequence of the form

\[
F_* : \cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0,
\]

where \( M \cong \text{Coker}(\partial_1) \), each \( F_i = \bigoplus j R(-j)^{\beta_{i,j}^R(M)} \) is a finite-rank, graded, free \( R \)-module, and all maps are graded homomorphisms. Here, \( R(-j) \) is the rank-one free \( R \)-module with graded components \( R(-j)_i = R_{i-j} \). The numbers \( \beta_{i,j}^R(M) \) are the graded Betti numbers of \( M \) over \( R \). The total Betti numbers are denoted

\[
\beta_i^R(M) := \sum_j \beta_{i,j}^R(M),
\]

and their generating function

\[
P_M^R(t) = \sum_{i \geq 0} \beta_i^R(M) t^i
\]

is called the Poincaré series of \( M \). The regularity of \( M \) is

\[
\text{reg}_R(M) = \max\{ j - i \mid \beta_{i,j}^R(M) \neq 0 \text{ for some } i \}.
\]

When no maximum exists, we say that \( M \) has infinite regularity. When \( M \) is Artinian and \( R \) is a polynomial ring, \( \text{reg}_R(M) = \max\{i \mid M_i \neq 0\} \). See [25] for connections between regularity, free resolutions, and local cohomology.

2.2 Koszul algebras

A standard graded \( \mathbb{K} \)-algebra \( R \) is called Koszul if \( \mathbb{K} \) has a linear free resolution over \( R \); equivalently, \( R \) is Koszul if and only if \( \text{reg}_R(\mathbb{K}) = 0 \). This definition extends to the noncommutative case, but all rings we consider in this paper are commutative.
Koszul algebras are ubiquitous in algebraic geometry and topology; they appear as coordinate rings of general canonical curves [41], quotients of polynomial rings by quadratic monomial ideals [27], and all sufficiently high Veronese subalgebras of standard graded $\mathbb{K}$-algebras [5]. There are several equivalent conditions to being Koszul. Notably, Fröberg [28] showed that $R$ is Koszul if and only if
\begin{equation}
\text{P}_R^t(t)H_R(-t) = 1.
\end{equation}
Moreover, it follows from work of Avramov, Eisenbud, and Peeva [1, 4] that $R$ is Koszul if and only if all finite, graded $R$-modules have finite regularity. See [17] and [28] for surveys of the theory of Koszul algebras.

Any commutative Koszul algebra has a presentation $S/I$, where $S = \mathbb{K}[x_1, \ldots, x_n]$ is a standard graded polynomial ring over $\mathbb{K}$ and $I$ is an ideal generated by quadrics (homogeneous polynomials of degree 2); we call such rings quadratic. Not all quadratic $\mathbb{K}$-algebras are Koszul. It is sufficient for $I$ to have a quadratic Gröbner basis with respect to some monomial order, perhaps after a change of coordinates; such algebras are called $\mathbf{G}$-quadratic. Yet not all Koszul algebras are $\mathbf{G}$-quadratic; see [17, Remark 1.14]. If $\ell \in R_1$ is a nonzerodivisor, then $R$ is Koszul if and only if $R/\ell R$ is Koszul; see e.g. [15, Theorem 3.1].

When considering the Chow ring of a matroid in the introduction, there are two approaches one might take to prove the Koszul property. Since quadratic monomial ideals, like $J_M$, are Koszul, if the linear forms in $I_M$ formed a regular sequence on $S_M/I_M$, one could conclude Koszulness for $\text{CH}(M)$ immediately; however, in general, these linear forms do not form a regular sequence. Another natural approach is to look for a quadratic Gröbner basis for $I_M + J_M$. Yet, as Dotsenko has noted [22], in the given presentation there are obstructions that prevent the existence of a quadratic Gröbner basis. It is not clear if a change of basis would resolve this obstruction. We address this issue in Sect. 6.

In the absence of a quadratic Gröbner basis, one way to prove that a $\mathbb{K}$-algebra has the Koszul property is to construct a Koszul filtration, a concept first formally defined by Conca, Trung, and Valla.

**Definition 2.1** Let $R$ be a standard graded $\mathbb{K}$-algebra. A family $\mathcal{F}$ of ideals of $R$ is said to be a **Koszul filtration** of $R$ if the following conditions hold.

(i) Every ideal $I \in \mathcal{F}$ is generated by linear forms.

(ii) The ideals $(0)$ and $R_+$ are in $\mathcal{F}$.

(iii) For every nonzero $I \in \mathcal{F}$, there exists an ideal $J \in \mathcal{F}$ such that $J \subseteq I$, $I/J$ is cyclic, and $(J : I) \in \mathcal{F}$.

If $R$ has a Koszul filtration, it follows from an inductive argument on the ideals in the filtration that $R$ is Koszul [18, Proposition 1.2].

### 2.3 Lattices

Recall that a (finite) lattice is a finite partially ordered set $(\mathcal{L}, \leq)$ in which every two elements $a, b \in \mathcal{L}$ have a unique least upper bound called the **join**, denoted $a \lor b$, ...
and a unique greatest lower bound called the meet, denoted \( a \land b \). Consequently, all nonempty lattices have a unique minimum element and maximum element, denoted \( \hat{0} \) and \( \hat{1} \) respectively.

A subset \( \mathcal{U} \subseteq \mathcal{L} \) is called an up-set or order filter if whenever \( a \in \mathcal{U} \) and \( b \in \mathcal{L} \) such that \( a \leq b \), then \( b \in \mathcal{U} \). Given \( a, b \in \mathcal{L} \) with \( a \leq b \), the intervals \([a, b]\) and \((a, b)\) are the sets:

\[
[a, b] = \{ c \in \mathcal{L} \mid a \leq c \leq b \}
\]

\[
(a, b) = \{ c \in \mathcal{L} \mid a < c \leq b \}.
\]

Clearly, every interval \([a, b]\) is also a lattice. If \( a < b \) and there are no elements \( c \in \mathcal{L} \) with \( a < c < b \), then we say that \( b \) covers \( a \) or \( b \) is a cover of \( a \), and we write \( a \prec b \). The Hasse diagram of \( \mathcal{L} \) is a graph with vertices labeled by the elements of \( \mathcal{L} \) positioned in such a way that if \( a \prec b \), then vertex \( a \) is lower than vertex \( b \); an edge from \( a \) to \( b \) is drawn exactly when \( b \) covers \( a \). See Fig. 2 for an example.

The elements \( a \in \mathcal{L} \) that cover \( \hat{0} \) are called the atoms of \( \mathcal{L} \), collectively denoted at(\( \mathcal{L} \)). The elements \( b \in \mathcal{L} \) covered by \( \hat{1} \) are called the coatoms of \( \mathcal{L} \), collectively denoted coat(\( \mathcal{L} \)). For each element \( \hat{0} \neq a \in \mathcal{L} \), we also let coat\((a)\) denote the set of elements of \( \mathcal{L} \) covered by \( a \), which are precisely the coatoms of the lattice \([\hat{0}, a]\).

The lattice \( \mathcal{L} \) is atomic if every element of \( \mathcal{L} \) is a join of finitely many atoms. We say that \( \mathcal{L} \) is graded (or ranked) if there is a function \( \text{rk} : \mathcal{L} \to \mathbb{Z}_{\geq 0} \) such that \( \text{rk} \hat{0} = 0 \) and if \( a, b \in \mathcal{L} \) and \( a \prec b \), then \( \text{rk} b = \text{rk} a + 1 \), in which case the number \( \text{rk} \mathcal{L} = \max \{ \text{rk} a \mid a \in \mathcal{L} \} \) is called the rank of \( \mathcal{L} \). The lattice \( \mathcal{L} \) is semimodular if for all \( a, b \in \mathcal{L} \) such that \( a \land b \leq a \) and \( a \land b \leq b \), then \( a \leq a \lor b \) and \( b \leq a \lor b \); equivalently, \( \mathcal{L} \) is semimodular if it is graded and its rank function satisfies

\[
\text{rk} a + \text{rk} b \geq \text{rk}(a \land b) + \text{rk}(a \lor b),
\]

for all \( a, b \in \mathcal{L} \). Finally, a lattice is called geometric if it is finite, atomic, and semimodular. For basic results on lattices, we refer the reader to [40].

### 2.4 The lattice of flats of a matroid

A matroid \( M \) is a pair \((E, \mathcal{I})\) consisting of a finite set \( E \), called the ground set of \( M \), and a collection \( \mathcal{I} \) of subsets of \( E \) satisfying three properties:

(i) \( \emptyset \in \mathcal{I} \).

(ii) If \( I \in \mathcal{I} \) and \( I' \subseteq I \), then \( I' \in \mathcal{I} \).

(iii) If \( I_1, I_2 \in \mathcal{I} \) and \( |I_1| < |I_2| \), then there exists an element \( e \in I_2 \setminus I_1 \) such that \( I_1 \cup e \in \mathcal{I} \).

**Notation** In the above definition and the subsequent sections, we abuse notation and identify elements \( e \in E \) with the corresponding set \( \{e\} \). Thus, if \( F \subseteq E \), we will frequently write \( F \cup e \) in place of \( F \cup \{e\} \).

The members of \( \mathcal{I} \) are called independent sets of \( M \). A subset of \( E \) that is not in \( \mathcal{I} \) is called dependent. A maximal independent set is called a basis. All bases of a
matroid have the same cardinality, called the rank of $M$. Given a subset $X \subseteq E$, the rank of $X$, denoted $\text{rk}_M X$, is the cardinality of the largest independent set contained in $X$; we drop the subscript when the matroid is clear from context.

The closure of a subset $X \subseteq E$ in $M$ is

$$ \text{cl}(X) = \{ e \in E \mid \text{rk}(X \cup e) = \text{rk} X \}. $$

A subset $F$ is called a flat of $M$ if $F = \text{cl}(F)$. A hyperplane of $M$ is a flat $H$ of rank $\text{rk} H = \text{rk} M - 1$. One can give an equivalent definition of matroid as a pair $(E, \mathcal{L})$ consisting of a finite set $E$ and a collection $\mathcal{L}$ of subsets of $E$, called flats, satisfying:

(i) If $F, G \in \mathcal{L}$, then $F \cap G \in \mathcal{L}$.

(ii) If $F \in \mathcal{L}$ and $e \in E \setminus F$, then there is a unique flat $G \in \mathcal{L}$ that minimally contains $F \cup e$.

Matroids can also be characterized by their rank functions, by their bases, or by their minimal dependent sets.

The set of flats of $M$, denoted $\mathcal{L}(M)$, has the structure of a lattice; for any two flats $F, G \in \mathcal{L}(M)$, the meet is the intersection, $F \cap G = F \cap G$, and the join is the closure of the union, $F \cup G = \text{cl}(F \cup G)$. The atoms of $\mathcal{L}(M)$ are precisely the rank-one flats, and the coatoms of $\mathcal{L}(M)$ are precisely the hyperplanes of $M$. A lattice is geometric if and only if it is isomorphic to the lattice of flats of some matroid [39, Theorem 1.7.5]. However, this correspondence is not quite bijective, as different matroids can have isomorphic lattices of flats.

An element $e \in E$ is a loop of $M$ if the set $\{e\}$ is dependent. If $e, f \in E$ are not loops, then $e$ and $f$ are parallel if $\{e, f\}$ is dependent. A matroid is simple if it has no loops and no pairs of parallel elements. For any matroid $M$, there is a unique simple matroid (up to isomorphism) whose lattice of flats is isomorphic to $\mathcal{L}(M)$, called the simplification of a matroid $M$. It can be constructed as the matroid on the set of rank-one flats of $M$ such that a set of flats $\{Y_1, \ldots, Y_t\}$ is independent if and only if $\text{rk}_M(\bigvee_{i=1}^t Y_i) = t$.

**Remark 2.2** Since it is clear from the definition of the Chow ring of a matroid that any two matroids with isomorphic lattices of flats will have isomorphic Chow rings, we can always replace a matroid $M$ with its simplification, and so, we may henceforth assume that all matroids under consideration are simple without any loss of generality.

There are two constructions for producing new matroids from a given matroid that will play an important role in the subsequent sections. Given a matroid $M$ on a ground set $E$ and a subset $S \subseteq E$, the restriction of $M$ to $S$ is the matroid $M|S$ on the ground set $S$ whose flats are of the form $F \cap S$ for some flat $F$ of $M$. In particular, when $H$ is a flat of $M$, every flat of $M|H$ is also a flat of $M$ so that $M|H$ is a matroid quotient of $M$, and the lattice of flats of $M|H$ is just the interval $[\emptyset, H]$ in $\mathcal{L}(M)$. The truncation of $M$ is the matroid $T(M)$ on the ground set $E$ with rank function $\text{rk}_{T(M)} X = \min\{\text{rk}_M X, \text{rk} M - 1\}$. The lattice of flats of the truncation $T(M)$ is obtained by removing all the hyperplanes from the lattice of flats of $M$. We refer the reader to [39] for further details about these constructions as necessary.
2.5 Chow rings of atomic lattices

We now recall the more general definition of the Chow ring of an atomic lattice from [29]. Let $L$ be an atomic lattice. A subset $G \subseteq L \setminus \{\hat{0}\}$ is a building set if for any $X \in L \setminus \{\hat{0}\}$, there is a poset isomorphism

$$\varphi_X : \prod_{i=1}^{t} [\hat{0}, G_i] \to [\hat{0}, X],$$

where $\max G_{\leq X} = \{G_1, \ldots, G_t\}$ and $\varphi_X(\hat{0}, \ldots, \hat{0}, G_i, \hat{0}, \ldots, \hat{0}) = G_i$ for $1 \leq i \leq t$. Here, $\max G_{\leq X}$ denotes the set of maximal elements among all elements in $G$ that are less than or equal to $X$. Every atomic lattice $L$ has a unique maximal building set $G_{\text{max}} = L \setminus \{\hat{0}\}$ and a unique minimal building set $G_{\text{min}}$ consisting of the irreducible elements in $L \setminus \{\hat{0}\}$. (The atoms of $L$ are always irreducible, but in general, there can be more irreducibles than just the atoms; see [26, Section 2].) Given any building set $G$ of $L$, a subset $S \subseteq G$ is called nested if for any pairwise incomparable elements $G_1, \ldots, G_t \in S$, either $t \leq 1$ or $G_1 \lor \cdots \lor G_t \notin G$. The set of nested subsets of $G$ is an abstract simplicial complex denoted by $\mathcal{N}(L, G)$.

**Definition 2.3** Let $L$ be a finite, atomic lattice, and let $G$ be a building set for $L$. Then the **Chow ring of $L$ with respect to $G$** is the algebra

$$D(L, G) := \mathbb{Z}[x_G \mid G \in G]/I,$$

where $I$ is generated by

$$\prod_{i=1}^{t} x_{G_i} \text{ for } \{G_1, \ldots, G_t\} \notin \mathcal{N}(L, G)$$

and

$$\sum_{G \geq A} x_G \text{ for } A \in \text{at}(L).$$

Note that $D(L, G)$ is a quotient of the Stanley–Reisner ring associated to the simplicial complex $\mathcal{N}(L, G)$.

For any building set, Feichtner and Yuzvinsky showed that $D(L, G)$ is the Chow ring of a certain smooth, affine, toric variety [29, Theorem 3]. They also show that for any essential complex hyperplane arrangement $A$ and any building set $G$ for the intersection lattice $L(A)$ containing $\{0\}$, the cohomology ring of the De Concini and Procesi wonderful compactification $Y_{A,G}$ of the arrangement complement [19, 20] is isomorphic to $D(L(A), G)$.

At the other extreme, the Chow ring of a matroid $M$ is the special case corresponding to the maximal building set of the lattice of flats of $M$; that is,

$$\text{CH}(M) \cong D(L(M), G_{\text{max}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$
We stress the importance of which building set one chooses in the following remark.

**Remark 2.4** While reading [6] we discovered the following discrepancy: It is claimed in [6, Remark 3.2.2] that the Chow ring of the graphic matroid $M(K_n)$ of a complete graph on $n$ vertices is the cohomology ring of the Deligne-Mumford space $\overline{M}_{0,n+1}$, which was shown to be Koszul by Dotsenko. This claim may be somewhat misleading. Although Dotsenko proves the cohomology ring of $\overline{M}_{0,n+1}$ is Koszul [22] and the cohomology ring is a Chow ring associated to the lattice of flats of $M(K_n)$, it is not the Chow ring of $M(K_n)$. The lattice of flats of $M(K_n)$ is isomorphic to the lattice $\Pi_n$ of partitions of $\{1, \ldots, n\}$ ordered by refinement. The distinction between the two rings comes down to the choice of building set. The Chow ring of $M(K_n)$ is the Chow ring $D(\Pi_n, G_{\text{max}}) \otimes \mathbb{Q}$ with respect to the maximal building set $G_{\text{max}} = \Pi_n \setminus \{\hat{0}\}$. On the other hand, as pointed out in [29, Section 7], the (rational) cohomology ring of $\overline{M}_{0,n+1}$ is isomorphic to $D(\Pi_n, G_{\text{min}}) \otimes \mathbb{Q}$ with respect to the minimal building set. These two rings are not isomorphic, which can easily be seen for $n = 4$ from the fact that both rings are Artinian with nondegenerate presentations of different codimensions. Thus, to the best of our knowledge, there is not any large class of matroids whose Chow rings were previously known to be Koszul.

### 3 Lattices with total coatom orderings

As Björner writes in [10, p. 232], “A number of remarkable properties of matroids are revealed by, but not dependent on, assigning a linear order to the underlying point set.” In this section, we follow Björner’s lead and define the notion of a total coatom ordering on a lattice. The definition is related to and inspired by the definition of a strongly shellable simplicial complex of Guo et al. [31], where they showed that the independence complex of a matroid is strongly shellable. We prove a parallel result (Theorem 3.3) that the lattice of flats of a matroid admits a total coatom ordering. This result is essential to creating the Koszul filtration on the Chow ring of a matroid in Theorem 4.12.

**Definition 3.1** Let $\mathcal{L}$ be a lattice, and let $\prec$ be any total order on $\mathcal{L}$. Given an element $F \in \mathcal{L}$, a set $\mathcal{G} \subseteq \text{coat}(F)$ is called an **initial segment covered by** $F$ if for all $G, G' \in \text{coat}(F)$ such that $G' \in \mathcal{G}$ and $G \prec G'$, then $G \in \mathcal{G}$.

**Definition 3.2** We say that a finite lattice $\mathcal{L}$ admits a **total coatom ordering** if there is a total order $\prec$ on $\mathcal{L}$ with the following properties for all $F \in \mathcal{L}$:

(i) For all $G, G' \in \text{coat}(F)$, if $G \prec G'$, then there exists $G'' \in \text{coat}(F)$ such that:

(i.a) $G'' \prec G'$,

(i.b) $G' \land G'' \in \text{coat}(G')$, and

(i.c) $G \land G' \leq G''$.

(ii) If $\mathcal{G} \subseteq \mathcal{L}$ is an initial segment covered by $F$ and $G'$ is the largest element in $\mathcal{G}$ with respect to $\prec$, then the set

$$\text{coat}_\mathcal{G}(G') := \{G \land G' \mid G \in \mathcal{G} \text{ and } G \land G' \in \text{coat}(G')\}$$
Fig. 1 A total coatom ordering

is an initial segment covered by \( G' \).

The first defining property of a total coatom ordering can be visualized in the Hasse diagram of \( \mathcal{L} \) as shown in Fig. 1; the dashed lines represent relations in the lattice that are not necessarily covering relations.

First, we show that geometric lattices have the above property.

**Theorem 3.3** Geometric lattices admit total coatom orderings.

**Proof** Let \( \mathcal{L} \) be a geometric lattice. We may assume that \( \mathcal{L} \) is the lattice of flats of some simple matroid \( M \) on the ground set \( E = [n] = \{1, 2, \ldots, n\} \). Given flats \( F' = \{i_1 > \cdots > i_r\} \) and \( F = \{j_1 > \cdots > j_s\} \) of \( M \), we will say that \( F' > F \) if \( \text{rk } F' > \text{rk } F \) or if \( \text{rk } F' = \text{rk } F \) and for some \( \ell \leq \min\{r, s\} \) we have \( i_p = j_p \) for all \( p < \ell \) and \( i_\ell < j_\ell \). It is easily checked that \( \prec \) is a total order. We claim that \( \prec \) is the desired total coatom ordering of \( \mathcal{L} \).

First, we observe \( \prec \) is compatible with restriction to flats of \( M \) in the sense that the induced order on the lattice of flats of \( M|_F \) is the analogously defined order on \( M|_F \) for any flat \( F \in \mathcal{L} \). Hence, it suffices to prove that properties (i) and (ii) of Definition 3.2 are satisfied when \( F = E \) so that \( G \) and \( G' \) are hyperplanes of \( M \). Since \( \mathcal{L} \) is a geometric lattice, we also note that saying a flat \( F \) covers a flat \( G \) is equivalent to saying that \( G \subseteq F \) and \( \text{rk } G = \text{rk } F - 1 \).

(i) Let \( H' \) and \( H \) be hyperplanes of \( M \) such that \( H < H' \). Let \( X \) be a basis for \( H \cap H' \) (in other words, a basis for the matroid \( M|(H \cap H') \)). Then \( \text{cl}(X) = H \cap H' \) so that \( |X| = \text{rk}(X) = \text{rk}(H \cap H') \). We can then find a set \( Y \subseteq E \) with \( X \cup Y = \emptyset \) such that \( X \cup Y \) is a basis for \( H' \).

Suppose that \( H' = \{i_1 > \cdots > i_r\} \) and \( H = \{j_1 > \cdots > j_s\} \). By assumption, there is an \( \ell \leq \min\{r, s\} \) such that \( i_\ell < j_\ell < j_{\ell - 1} = i_{\ell - 1} \) so that \( j_\ell \notin H' \). Hence, \( X \cup Y \cup j_\ell \) is a basis for \( M \). We note that \( |Y| = \text{rk } H' - \text{rk}(H \cap H') \geq 1 \). Consider the hyperplane \( H'' = \text{cl}(X \cup (Y \setminus y) \cup j_\ell) \) for some \( y \in Y \), and suppose that \( H'' = \{k_1 > \cdots > k_t\} \). Since \( X \cup (Y \setminus y) \subseteq H' \cap H'' \), we clearly have \( \text{rk}(H' \cap H'') = \text{rk } H' - 1 \) by construction, so that condition (i.b) holds, and \( \{j_1, \ldots, j_{\ell - 1}\} \subseteq H \cap H' = \text{cl}(X) \subseteq H'' \) so that condition (i.c) holds and, moreover, \( k_i \geq j_i \) for all \( i \leq \ell \). It follows that for some \( h \leq \ell \) we have \( k_h > i_h \) and \( k_p = i_p \) for all \( p < h \) so that \( H'' < H' \); thus, condition (i.a) holds. 

\( \square \) Springer
(ii) Let \( \mathcal{H} \) be an initial segment of hyperplanes of \( M \) with largest element \( H' = \{i_1 > i_2 > \cdots > i_r\} \). We must prove that the set coat\( _\mathcal{H}(H') \) of induced hyperplanes in \( M[H'] \) is also an initial segment. Let \( F'' < F \) be hyperplanes of \( M[H'] \) with \( F \in \text{coat}_\mathcal{H}(H') \). We must show that \( F'' \in \text{coat}_\mathcal{H}(H') \). As \( F \in \text{coat}_\mathcal{H}(H') \), we know there is a hyperplane \( H \in \mathcal{H}(H') \) with \( F = H \cap H' \). Since \( H < H' \) by assumption, we know that there is an index \( \ell \) and element \( j > i_\ell \) of the ground set of \( M \) such that

\[
H = \{i_1 > \cdots > i_{\ell - 1} > j > j_{\ell + 1} > \cdots > j_m\}.
\]

In particular, we note that \( j \notin H' \) and \( F = \{i_1 > \cdots > i_{\ell - 1} > i_h \} \) for some indices \( h_k \). Since \( F'' < F \), we similarly know that \( F'' \) contains an element \( e \) not in \( F \). As \( F'' \setminus F \subseteq H' \setminus F \), it follows that \( i_\ell \geq e \) so that \( F'' = \{i_1 > \cdots > i_{\ell - 1} > i_h > \cdots > i_{k_p - 1} > e > e_{p+1} > \cdots > e_q\} \).

Consider the hyperplane \( H'' = \text{cl}(F' \cup F) \). On the one hand, we have \( F'' \subseteq H' \cap H'' \) so that \( \text{rk } H' - 1 = \text{rk } F'' \leq \text{rk } (H' \cap H'') \). On the other hand, we note that \( H'' \neq H' \) since \( j \notin H' \), and so, it follows that \( \text{rk}(H' \cap H'') \leq \text{rk } H' - 1 \) since the lattice of flats of \( M \) is semimodular. Hence, \( \text{rk} F'' = \text{rk}(H' \cap H'') \) so that \( F'' = H' \cap H'' \) and so, we are done if we can show that \( H'' < H' \). If this is not the case, then \( H'' < H'' \) so that there is an index \( m \) such that \( H'' = \{i_1 > \cdots > i_{m - 1} > k_m > \cdots > k_b\} \), where \( i_m > k_m \). In particular, \( i_m \notin H'' \) so that \( m \geq \ell \). If \( m > \ell \), then \( j \in H'' \) and \( j > i_\ell \geq i_{m - 1} \), which is impossible. Thus, we must have \( m = \ell \) and \( k_m \geq j > i_\ell \), which implies that \( H'' < H' \), a contradiction. Hence, \( H'' < H' \) as wanted. \( \square \)

**Example 3.4** Consider the uniform matroid \( U_{5,5} \) on the ground set \( E = \{1, 2, 3, 4, 5\} \). For simplicity of notation, we will write subsets of \( E \) as strings of their elements. Since every subset of \( E \) is independent, the hyperplanes of \( U_{5,5} \) are precisely the subsets of size 4. Take \( M = T_H(U_{5,5}) \) to be the principal truncation of \( U_{5,5} \) with respect to the hyperplane \( H' = 1234 \). (See [39, p.279] for a definition of principal truncation.) Equivalently, \( M \) can be viewed as just the matroid of linearly independent columns of the matrix

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

where the columns are indexed 1 to 5. The lattice of flats of \( M \) is as shown in Fig. 2. The flats of \( M \) have been arranged in descending order according to the total coatom ordering of the preceding theorem when read from top to bottom and left to right.

In the remainder of this section, we highlight some connections between our total coatom orderings and other types of orderings and notions of shellability previously studied for lattices. The uninterested reader may safely skip this discussion and proceed directly to the next section.

Total coatom orderings owe their name to their relationship with the existing notion of a recursive coatom ordering. A finite lattice \( \mathcal{L} \) with \( \mathcal{H} \neq \mathcal{K} \) admits a recursive coatom ordering if there is a total order \( \preceq \) on the set of coatoms of \( \mathcal{L} \) such that:
Fig. 2  The lattice $L(M)$

(i) For all $H, H' \in \text{coat}(\mathcal{L})$, if $H \prec H'$ and $G \in \mathcal{L}$ such that $G < H, H'$, then there exists an $H'' \in \text{coat}(\mathcal{L})$ such that:

(i.a) $H'' \prec H'$,
(i.b) There exists $F \in \text{coat}(H')$ such that $G \leq F < H''$

(ii) For each $H' \in \text{coat}(\mathcal{L})$, the interval $[\hat{0}, H']$ admits a recursive coatom ordering $\prec_{H'}$ such that if $F \in \text{coat}(H') \cap \text{coat}(H'')$ for some $H \prec H'$ and $F' \in \text{coat}(H') \setminus \bigcup_{H'' \prec H} \text{coat}(H'')$, then $F \prec_{H'} F'$.

By convention, we also consider the lattice with $\hat{0} = \hat{1}$ to admit a recursive coatom ordering.

It is known that geometric lattices admit recursive atom orderings [42, Theorem 7.2], the dual notion to recursive coatom orderings. We could not find a reference in the literature for the following result and include a proof for completeness. The following argument was communicated to us by Vic Reiner. We refer the reader to the cited references for precise definitions of the variants of shellability, which are not needed anywhere else in the paper.

**Proposition 3.5**  If $\mathcal{L}$ is a geometric lattice, then $\mathcal{L}$ admits a recursive coatom ordering.

**Proof**  If $\mathcal{L}$ is geometric, then $\mathcal{L}$ is SL-shellable by [9, Theorem 3.7]. The dual $\mathcal{L}^*$ of the SL-shellable lattice $\mathcal{L}$ is also SL-shellable by [9, Proposition 3.5]. By definition, $\mathcal{L}^*$ is then EL-shellable [9, Definition 3.4] and hence CL-shellable by [13, Proposition 2.3]. Since $\mathcal{L}^*$ is CL-shellable, $\mathcal{L}^*$ admits a recursive atom ordering by [14, Theorem 5.11], which is equivalent to $\mathcal{L}$ admitting a recursive coatom ordering.  \(\square\)

A total coatom ordering induces a recursive coatom ordering. Conversely, that every recursive coatom ordering can be used to construct a total coatom ordering is essentially the content of a result of Delucchi [21, Lemma 2.10] restricted to the case of a graded lattice. For the convenience of the reader, we include a complete proof of these observations below translated into our notation.

**Proposition 3.6**  Let $\mathcal{L}$ be a graded lattice. Then $\mathcal{L}$ admits a recursive coatom ordering if and only if $\mathcal{L}$ admits a total coatom ordering.

**Proof**  We may clearly assume that $\hat{0} \neq \hat{1}$ in $\mathcal{L}$, since the statement follows trivially otherwise.
(\Leftrightarrow): Suppose that \(\succ\) is a total coatom ordering for \(\mathcal{L}\). We will show that the restriction of \(\succ\) to the set of coatoms of \(\mathcal{L}\) is the desired recursive coatom ordering by induction on \(\text{rk } \mathcal{L} \geq 0\). Since the case \(\text{rk } \mathcal{L} = 0\) is trivial, we may assume \(\text{rk } \mathcal{L} \geq 1\) and that every graded lattice of rank less than \(\text{rk } \mathcal{L}\) that admits a total coatom ordering has a recursive coatom ordering induced by restricting the total coatom ordering to its set of coatoms. In particular, for each \(H' \in \text{coat}(\mathcal{L})\), it is clear from the definition of a total coatom ordering that the restriction of \(\succ\) to the graded lattice \([0, H']\) is a total coatom ordering, and so, the restriction of \(\succ\) to \(\text{coat}(H')\) is a recursive coatom ordering by induction. If \(F \in \text{coat}(H) \cap \text{coat}(H')\) for some \(H \in \text{coat}(\mathcal{L})\) with \(H < H'\) and \(F' \in \text{coat}(H') \smallsetminus \bigcup_{H'' < H} \text{coat}(H'')\), then we claim that \(F < F'\). Indeed, because \(\succ\) is a total coatom ordering, we know that there exist \(H'' \in \text{coat}(\mathcal{L})\) such that \(H' \land H'' \in \text{coat}(H')\) and \(H' \land H \leq H' \land H''\). As \(F \in \text{coat}(H) \cap \text{coat}(H')\), we see that \(F \leq H' \land H \leq H' \land H''\) so that \(F = H' \land H = H' \land H''\) since \(H' \land H''\) is also a coatom of \(H'\). Let \(G\) be the initial segment of coatoms of \(\mathcal{L}\) that are less than or equal to \(H'\) with respect to the total coatom ordering. Then \(F \in \text{coat}_G(H')\), and \(\text{coat}_G(H')\) must be an initial segment covered by \(H'\) by the definition of a total coatom ordering. Hence, if \(F' < F\), we would have \(F' \in \text{coat}_G(H')\) so that \(F'' = H' \land H_0\) for some \(H_0 \in \text{coat}(\mathcal{L})\) with \(H_0 < H'\). But then since \(\mathcal{L}\) is graded and \(F' \in \text{coat}(H')\) so that \(\text{rk } F' = \text{rk } H' - 1 = \text{rk } H_0 - 1\), we see that \(F' \in \text{coat}(H_0)\), contradicting our choice of \(F'\). Thus, we must have \(F < F'\) as claimed, which establishes property (ii) of a recursive coatom ordering.

If \(H, H' \in \text{coat}(\mathcal{L})\) such that \(H < H'\) and \(G \in \mathcal{L}\) such that \(G < H, H'\), then because \(\succ\) is a total coatom ordering there exists an \(H'' \in \text{coat}(\mathcal{L})\) such that \(H'' < H', H' \land H'' \in \text{coat}(H')\), and \(H' \land H \leq H' \land H''\). Since \(G < H, H'\), it follows that \(G \leq H' \land H \leq H' \land H'' < H''\) as required for property (i) of a recursive coatom ordering to be satisfied. Therefore, \(\succ\) induces a recursive coatom ordering on \(\mathcal{L}\).

(\Rightarrow): Suppose that \(\mathcal{L}\) admits a recursive coatom ordering \(\prec_{\mathcal{T}}\), and let \(\mathcal{L}_i\) denote the set of elements of \(\mathcal{L}\) of rank \(\text{rk } \mathcal{L} - i\). Following Delucchi, we recursively construct total orders \(\succ_i\) on \(\mathcal{L}_i\) for \(0 \leq i \leq \text{rk } \mathcal{L}\) with certain properties described below. Given \(F \in \mathcal{L}_i\), we denote by \(\sigma(F)\) the smallest element of \(\mathcal{L}_{i-1}\) with respect to the order \(\succ_{i-1}\) such that \(F < \sigma(F)\). Then for each \(i \geq 1\), the orders \(\succ_i\) satisfy:

(A_i) If \(F \prec_i F'\) and \(X \prec F, F'\), then there is a \(F'' \prec_i F'\) and \(G \in \text{coat}(F')\) such that \(X \leq G \prec F''\).
(B_i) For each \(G' \in \mathcal{L}_{i+1}\), there is a recursive coatom ordering \(\sigma(G')\) on \([0, \sigma(G')]\) such that if \(G \in \text{coat}(\sigma(G'))\) and \(\sigma(G) \prec_i \sigma(G')\), then \(G \prec_{\sigma(G')} G'\).

The desired total coatom ordering \(\succ\) of \(\mathcal{L}\) can then be constructed by defining:

\[
G \prec F \iff \begin{cases} 
\text{rk } G < \text{rk } F, \\
\text{rk } G = \text{rk } F = i \text{ and } G \prec_i F.
\end{cases}
\]  
(3.1)

The order \(\succ_0\) is trivial since \(\mathcal{L}_0 = \{1\}\), and we define \(\succ_1 = \succ_{\mathcal{T}}\) to coincide with the recursive coatom ordering on \(\mathcal{L}\). It is immediate from the definition of a recursive coatom ordering that properties (A_i) and (B_i) hold.
Assume that we have already constructed \( \succ_j \) such that properties \((A_j)\) and \((B_j)\) are satisfied for \( j \leq i \). We define a relation \( \succ_{i+1} \) on \( \mathcal{L}_{i+1} \) as follows:

\[
G \prec_{i+1} G' \Leftrightarrow \left\{ \begin{array}{ll}
\sigma(G) \prec_i \sigma(G'), & \text{or} \\
\sigma(G) = \sigma(G') & \text{and} \ G \prec_{\sigma(G')} G'
\end{array} \right.
\]

where \( \prec_{\sigma(G')} \) is a recursive coatom ordering on \([0, \sigma(G')]\) for which \((B_i)\) holds. It easily follows from the fact that \( \succ_i \) and \( \succ_{\sigma(G')} \) are total orders that \( \succ_{i+1} \) is also a total order. We must show that properties \((A_{i+1})\) and \((B_{i+1})\) are satisfied.

Suppose \( Y' \in \mathcal{L}_{i+2} \) and let \( G' = \sigma(Y') \). By \((B_i)\), there is a recursive coatom ordering \( \succ_{\sigma(G')} \) on \([0, \sigma(G')]\). The definition of a recursive coatom ordering then guarantees that there is a recursive coatom ordering \( \succ_{G'} \) on \([0, G']\) such that if \( Y \in \text{coat}(G) \cap \text{coat}(G') \) for some \( G \prec_{\sigma(G')} G' \), then \( Y <_{G'} Y' \). In particular, if \( Y \in \text{coat}(G') \) and \( G \prec_{i+1} G' \) for \( G = \sigma(Y) \), we claim that there is a \( G'' \in \text{coat}(\sigma(G')) \) such that \( Y \in \text{coat}(G'') \) and \( G'' \prec_{\sigma(G')} G' \) so that \( Y <_{G'} Y' \). Since \( G \prec_{i+1} G' \), we know that either \( \sigma(G) = \sigma(G') \) and \( G \prec_{\sigma(G')} G' \), in which case we can take \( G'' = G \), or \( \sigma(G) \prec_i \sigma(G') \). In the latter case, since \( Y < \sigma(G), \sigma(G') \), it follows from \((A_i)\) that there is an \( F'' \prec_i \sigma(G') \) and \( G'' \in \text{coat}(\sigma(G')) \) such that \( Y \leq G'' < F'' \). As \( \mathcal{L} \) is graded and \( G'' \in \text{coat}(\sigma(G')) \), we have \( \text{rk} Y = \text{rk} \sigma(G')-2 \) if \( G'' = G' \) and \( Y = \text{coat}(G'') \), and we note that \( G'' \prec_{\sigma(G')} G' \) since \( \sigma(G') \prec_i F'' \prec_i \sigma(G') \) and \((B_i)\) holds. Thus, \( Y <_{G'} Y' \) as claimed so that \((B_{i+1})\) is satisfied.

Suppose that \( G \prec_{i+1} G' \) and \( X < G, G' \). If \( \sigma(G) = \sigma(G') \), then \( G \prec_{\sigma(G')} G' \), and so, property \((i)\) in the definition of a recursive coatom ordering implies that there is a \( G'' \in \text{coat}(\sigma(G')) \) such that \( G'' \prec_{\sigma(G')} G' \) and there is a \( Y \in \text{coat}(G') \) such that \( X \leq Y < G'' \). If \( \sigma(G'') = \sigma(G') \), then \( G'' \prec_{i+1} G' \). Otherwise, we must have \( \sigma(G'') \prec_i \sigma(G') \) since \( G'' \in \text{coat}(\sigma(G')) \), which again implies that \( G'' \prec_{i+1} G' \). Hence, \((A_{i+1})\) holds when \( \sigma(G) = \sigma(G') \). On the other hand, if \( \sigma(G) \prec_i \sigma(G') \), then since \((A_i)\) holds there is an \( F'' \prec_i \sigma(G') \) and \( G'' \in \text{coat}(\sigma(G')) \) such that \( X \leq G'' < F'' \). Hence, after replacing \( G \) with \( G'' \) if necessary, we may further assume that \( G \in \text{coat}(\sigma(G')) \). In that case, \((B_i)\) implies that \( G \prec_{\sigma(G')} G' \). Hence, property \((i)\) of a recursive coatom ordering implies there exists \( G'' \in \text{coat}(\sigma(G')) \) such that \( G'' \prec_{\sigma(G')} G' \) and there exists \( Y \in \text{coat}(G') \) such that \( X \leq Y < G'' \). As already argued above, \( G'' \in \text{coat}(\sigma(G')) \) and \( G'' \prec_{\sigma(G')} G' \) implies that \( G'' \prec_{i+1} G' \). Hence, \((A_{i+1})\) also holds when \( \sigma(G) \prec_i \sigma(G') \). Thus, we can recursively construct the total orders \( \succ_i \) satisfying \((A_i)\) and \((B_i)\) for all \( i \) as claimed. Now consider the order \( \succ \) on \( \mathcal{L} \) defined in (3.1). It is easily checked that \( \succ \) is a total order since \( \succ_i \) is a total order for each \( i \). We will show that \( \succ \) is the desired total coatom ordering. Given \( F \in \mathcal{L} \) and \( G, G' \in \text{coat}(F) \) with \( G < G' \), then \( \text{rk} G = \text{rk} G' = \text{rk} \mathcal{L}_{i-1} \) for some \( i \) since \( \mathcal{L} \) is graded, and \( G \prec_i G' \). As \( G \land G' < G, G' \) and \((A_i)\) holds, there is a \( G'' \prec_i G' \) (hence \( G'' \prec_i G' \)) and a \( Y \in \text{coat}(G') \) such that \( G \land G' \leq Y < G'' \). As \( Y \leq G'' \land G' \) and \( Y \in \text{coat}(G') \), it follows that \( Y \prec G'' \land G' \) and \( \succ \) satisfies property \((i)\) of a total coatom ordering. Let \( \mathcal{G} \) be an initial segment covered by \( F \in \mathcal{L}_{i-1} \) for some \( i \geq 1 \) and \( G' \) be the largest element of \( \mathcal{G} \) with respect to \( \succ \). Suppose further that \( G \in \mathcal{G} \) such that \( G \land G' \in \text{coat}(G') \) and that \( X \in \text{coat}(G') \) with \( X < G \land G' \). Since \( \mathcal{G} \) is an initial segment covered by \( F \), we may assume \( G = \sigma(G \land G') \). If \( \sigma(X) = G' \), then \( \sigma(G \land G') = G \prec_i G' = \sigma(X) \) and \( G \prec_{\sigma(G)} G' \) as claimed.
implies $G \land G' < X$, which is a contradiction. Hence, we must have $\sigma(X) <_i G'$ as $X \in \text{coat}(G')$, whence $\sigma(X) \in \mathcal{G}$. Moreover, $X \leq \sigma(X) \land G' < G'$ and $X \in \text{coat}(G')$ implies $X = \sigma(X) \land G' \in \text{coat}_{\mathcal{G}}(G')$. This shows $\text{coat}_{\mathcal{G}}(G')$ is an initial segment covered by $G'$ so that $>_i$ satisfies property (ii) of a total coatom ordering. Therefore, $>_i$ is a total coatom ordering on $\mathcal{L}$ as wanted. $\square$

Combining Proposition 3.6 with Proposition 3.5 yields an another proof that every geometric lattice admits a total coatom ordering; however, we find Theorem 3.3 more constructive and direct.

**Corollary 3.7** A graded lattice that admits a total coatom ordering is shellable.

**Proof** By the preceding proposition, a graded lattice $\mathcal{L}$ that admits a total coatom ordering also admits a recursive coatom ordering, which is equivalent to a recursive atom ordering on $\mathcal{L}^\ast$. It then follows from [14, Theorem 5.11] that $\mathcal{L}^\ast$ is CL-shellable so that the order complex $\Delta(\mathcal{L}^\ast) = \Delta(\mathcal{L})$ is shellable by [12, Theorem 3.3]. Thus, graded lattices with a total coatom ordering are shellable. $\square$

### 4 The Koszul property of the Chow ring of a matroid

The aim of this section is to show that the Chow ring of any matroid is Koszul by building a suitable Koszul filtration. First, we describe a presentation of $\text{CH}(M)$ and a corresponding Gröbner basis better suited to our purposes.

#### 4.1 The Atom-free presentation of the Chow ring of a matroid

Let $M$ be a simple matroid with finite ground set $E$ and lattice of flats $\mathcal{L} = \mathcal{L}(M)$. Abusing notation slightly, we identify elements $i \in E$ with the rank-one flats $\{i\}$ of $M$. There are now several presentations of the Chow ring of a matroid. The Feichtner–Yuzvinsky presentation of the Chow ring of $M$ [29] is the ring

$$\text{CH}_{FY}(M) = \mathbb{Q}[x_F \mid F \in \mathcal{L} \setminus \{\emptyset\}]/I_{FY}(M),$$

where

$$I_{FY}(M) = (x_Fx_{F'} \mid F, F' \text{ incomparable}) + \left(\sum_{i \in F} x_F \mid i \in E\right).$$

In several papers [2, 7, 8], the variable $x_E$ is eliminated so that variables are associated to nonempty, proper flats. Instead, we eliminate the variables associated to the atoms of $\mathcal{L}$ to define a quadratic presentation.

**Definition 4.1** Let $\mathcal{L}_{\geq 2}$ denote the set of flats of $M$ of rank at least 2. We define the **atom-free presentation** of the Chow ring of $M$ to be the ring

$$\text{CH}_{af}(M) = \mathbb{Q}[x_F \mid F \in \mathcal{L}_{\geq 2}]/I_{af}(M),$$

where

$$I_{af}(M) = \left(x_Fx_{F'} \mid F, F' \text{ incomparable}\right) + \left(\sum_{i \in F} x_F \mid i \in E\right).$$
where
\[
I_{af}(M) = (x_F x_{F'} \mid F, F' \in L \geq 2 \text{ incomparable }) \\
+ \left( x_F \sum_{F' \supseteq F \setminus i} x_{F'} \mid F \in L \geq 2, i \in E \setminus F \right) \\
+ \left( \sum_{F \supseteq i \cup j} x_F^2 + \sum_{F' \supseteq F \supseteq i \cup j} 2x_F x_{F'} \mid i, j \in E, i \neq j \right).
\]

After a change of variables \( \varphi \) on the ring \( \mathbb{Q}[x_F \mid F \in L \setminus \{\emptyset\}] \) sending \( x_i \mapsto x_i - \sum_{F \supseteq i} x_F \) for each \( i \in E \), we see that
\[
\varphi(I_{FY}(M)) = (x_i \mid i \in E) + (x_F x_{F'} \mid F, F' \in L \geq 2 \text{ incomparable }) \\
+ \left( x_F \sum_{F' \supseteq i} x_{F'} \mid F \in L \geq 2, i \in E \setminus F \right) \\
+ \left( \sum_{F \supseteq i, F' \supseteq j} x_F x_{F'} \mid i, j \in E, i \neq j \right)
\]
(4.1)

so that \( CH_{FY}(M) \cong \mathbb{Q}[x_F \mid F \in L \setminus \{\emptyset\}]/((x_i \mid i \in E) + I_{af}(M)) \cong CH_{af}(M) \).

Remark 4.2 Although we use \( \mathbb{Q} \) as our coefficient field in this paper to keep with the terminology established in [7], our constructions work equally well over any coefficient field.

While a quadratic Gröbner basis for \( I_{FY}(M) \) or \( I_{af}(M) \) is not known, a non-quadratic Gröbner basis for \( I_{FY}(M) \) was computed by Feichtner and Yuzvinsky, which we recall here.

Theorem 4.3 [29, Theorem 2] With respect to any lexicographic order \( > \) such that \( x_F > x_G \) implies \( F \nsubseteq G \), the ideal \( I_{FY}(M) \) has a Gröbner basis consisting of the following polynomials for all \( F, F' \in L \setminus \{\emptyset\} \):
\[
x_{F'} x_F \quad F, F' \text{ incomparable}
\]
(4.2)
\[
x_{F'} \left( \sum_{G \supseteq F} x_G \right)^{rk F-rk F'} F' \subsetneq F
\]
(4.3)
\[
\left( \sum_{G \supseteq F} x_G \right)^{rk F} F \quad (4.4)
\]

A slight modification yields a non-quadratic Gröbner basis in the atom-free setting.

Corollary 4.4 With respect to any lexicographic order \( > \) such that \( x_F > x_G \) implies \( F \nsubseteq G \), the ideal \( I_{af}(M) \) has a Gröbner basis consisting of the following polynomials for all \( F, F' \in L \geq 2 \):
\[
x_{F'} x_F \quad F, F' \text{ incomparable}
\]
(4.5)
\[
x_{F'} \left( \sum_{G \supseteq F} x_G \right)^{rk F-rk F'} F' \subsetneq F
\]
(4.6)
\[
\left( \sum_{G \supseteq F} x_G \right)^{rk F}
\]
(4.7)
Proof} As in the proof of [6, Proposition 3.3.2], we note that the automorphism \( \varphi \) of \( \mathbb{Q}[x_F \mid F \in \mathcal{L} \setminus \{\emptyset\}] \) of (4.1) satisfies \( \text{in}_\succ \varphi(x_F) = x_F \) for all flats \( F \); hence, \( \text{in}_\succ \varphi(f) = \text{in}_\succ f \) for all polynomials \( f \) so that \( I_{FY}(M) \) and \( I_{af}(M) \) have the same initial ideal and the images under \( \varphi \) of the Gröbner basis for \( I_{FY}(M) \) form a Gröbner basis for \( \varphi(I_{FY}(M)) = (x_i \mid i \in E) + I_{af}(M) \). This Gröbner basis includes the polynomials listed above and the polynomials:

\[
(x_i - \sum_{F_i \supseteq i} x_F) (x_j - \sum_{F_j \supsetneq j} x_{F'}) \quad i, j \in E, i \neq j
\]

(4.8)

\[
x_F (x_i - \sum_{F_i \supseteq i} x_{F'}) \quad F \in \mathcal{L}_{\geq 2}, i \in E, F \not\supsetneq i
\]

(4.9)

\[
(x_i - \sum_{F_i \supseteq i} x_{F'}) (\sum_{G \supseteq F} x_G)^{\text{rk} \quad F - 1} \quad i \in E, i \subsetneq F
\]

(4.10)

\[
x_i \quad i \in E
\]

(4.11)

From this, we see that

\[
\text{in}_\succ (\varphi(I_{FY}(M))) = (x_i \mid i \in E) + (x_F x_{F'} \mid F, F' \in \mathcal{L}_{\geq 2} \text{ incomparable})
\]

\[
+ \left( x_{F'} x_F^{\text{rk} \quad F - \text{rk} \quad F'} \mid F, F' \in \mathcal{L}_{\geq 2}, F' \subsetneq F \right) + \left( x_F^{\text{rk} \quad F} \mid F \in \mathcal{L}_{\geq 2} \right),
\]

(4.12)

which shows that omitting the above polynomials in (4.8)–(4.10) still leaves a Gröbner basis for \( \varphi(I_{FY}(M)) \).

Since \( \prec \) is an elimination order for the variables \( x_i \) with \( i \in E \), it follows that the polynomials listed in the statement of the corollary are a Gröbner basis for the elimination ideal \( \varphi(I_{FY}(M)) \cap \mathbb{Q}[x_F \mid F \in \mathcal{L}_{\geq 2}] \) by [23, Theorem 3.3]. It then suffices to note that \( \varphi(I_{FY}(M)) \cap \mathbb{Q}[x_F \mid F \in \mathcal{L}_{\geq 2}] = \pi((x_i \mid i \in E) + I_{af}(M)) = I_{af}(M) \), where \( \pi : \mathbb{Q}[x_F \mid F \in \mathcal{L} \setminus \{\emptyset\}] \to \mathbb{Q}[x_F \mid F \in \mathcal{L}_{\geq 2}] \) is the natural algebra retract sending \( x_i \mapsto 0 \) for all \( i \in E \).

Combining (4.12) with Macaulay’s Theorem yields the following.

**Corollary 4.5** The ring \( \text{CH}_{af}(M) \) has a \( \mathbb{Q} \)-basis consisting of all monomials

\[
x_F^\alpha = x_{F_1}^{\alpha_1} \cdots x_{F_r}^{\alpha_r}
\]

for all chains of flats \( F = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_r \supseteq F_{r+1} = \emptyset\} \) for some \( r \geq 0 \) with \( F_i \in \mathcal{L}_{\geq 2} \) for all \( i \leq r \) and all \( \alpha = (\alpha_1, \ldots, \alpha_{r+1}) \in \mathbb{Z}_{\geq 0}^{r+1} \) such that \( \sum_{i=1}^{r+1} \alpha_i = \text{rk} \quad F_1 \) and \( 1 \leq \alpha_i < \text{rk} \quad F_i - \text{rk} \quad F_{i+1} \) if \( i \leq r \).

Following [6, 29], we call the monomials of Corollary 4.5 the **nested monomials** of \( \text{CH}_{af}(M) \). In this notation, we note that \( x_{\emptyset}^0 = 1 \).

Adiprasito et al. [2] proved that \( \text{CH}(M) \) has the **Kähler package**; that is, \( \text{CH}(M) \) is an Artinian, Gorenstein \( \mathbb{Q} \)-algebra of regularity \( \text{reg}(\text{CH}(M)) = \text{rk} \quad M - 1 \) with versions of the hard Lefschetz and Hodge–Riemann conditions. Other proofs of these properties have now been given in [6, 8]. It is well known that an Artinian, graded
$K$-algebra $A$ is a Poincaré duality algebra if and only if $A$ is Gorenstein; see e.g. [38, Proposition 2.1].

**Remark 4.6** As a consequence of Corollary 4.4, we note that $I_{af}(M)$ contains the polynomials

$$x_{H}x_{E}, \quad x_{F}x_{H}^{rk_{H} - rk_{F}}, \quad x_{F}x_{E}^{rk_{M} - rk_{F} - 1}, \quad x_{H}^{rk_{H}}, \quad x_{E}^{rk_{M} - 1}$$

(4.13)

for any $F \in \mathcal{L}_{\geq 2}$ and hyperplane $H \supseteq F$. Note that the exponents of the terms of each binomial in (4.13) are the same since $rk_{H} = rk_{M} - 1$ if $H$ is a hyperplane. However, we have intentionally written these relations as such to indicate how they can be used to convert non-nested monomials of $CH_{af}(M)$ into nested monomials.

Additionally, we have $x_{F}x_{E}^{rk_{M} - 1} = 0$ in $CH_{af}(M)$ for all $F \in \mathcal{L}_{\geq 2}$. Since $CH_{af}(M)$ is an Artinian Gorenstein ring, it follows that $x_{E}^{rk_{M} - 1}$ is a socle generator of $CH_{af}(M)$. These observations will be useful in the computations below.

### 4.2 A Koszul filtration for the Chow ring of a matroid

We are now in position to describe the Koszul filtration. When $M$ is a matroid of rank 1 or 2, we note that $CH(M) \cong Q$ or $CH(M) \cong Q[x_{E}]/(x_{E}^{2})$, respectively. In either case, the Chow ring is clearly Koszul, and moreover, it has a Koszul filtration. In the former case, the filtration consists of only the zero ideal, and in the latter case, the filtration consists of the zero ideal and the graded maximal ideal. Hence, in what follows, we may assume the following without any loss of generality.

**Notation** Throughout the remainder of this section, $M$ denotes a simple matroid with $rk_{M} \geq 3$ and $A = CH(M)$ with respect to the atom-free presentation.

Although we will not exactly present the proof this way, the idea behind the construction of the desired Koszul filtration is that we can use natural matroid operations such as truncation and restriction to reduce the rank of our matroid down to the low-rank cases mentioned above and inductively lift a Koszul filtration to the Chow ring of our arbitrary matroid $M$. Recall that a flat $H$ of $M$ is called a hyperplane if $rk_{H} = rk_{M} - 1$, and the hyperplanes are precisely the coatoms in the lattice of flats $\mathcal{L} = \mathcal{L}(M)$. Since the hyperplanes are exactly the flats omitted when truncating the matroid $M$, we are naturally led to study ideals generated by hyperplane variables.

**Lemma 4.7** Let $\mathcal{H}$ be a set of hyperplanes of $M$. Then the ideal $(x_{H} \mid H \in \mathcal{H})$ has a $Q$-basis consisting of all nested monomials $x_{F}^{\alpha}$ with $F = \{F_{1} \supseteq F_{2} \supseteq \cdots \supseteq F_{r} \supseteq F_{r+1} = \emptyset\}$ for some $r \geq 0$ such that either:

(i) $F_{1} \in \mathcal{H}$, or

(ii) $F_{1} = E$, $H \supseteq F_{2}$ for some $H \in \mathcal{H}$, and $\alpha_{1} = rk_{M} - rk_{F_{2}} - 1$.

**Proof** Consider the case of a single hyperplane $H$. The general case follows easily from the principal case. If $F_{1} = H$, then clearly $x_{F_{1}}^{\alpha_{1}} \in (x_{H})$. If we have $F_{1} = E$, $H \supseteq F_{2}$, and $\alpha_{1} = rk_{M} - rk_{F_{2}} - 1$, then by (4.13) we have $x_{E}^{\alpha_{1}}x_{F_{2}} = -x_{H}^{\alpha_{1}}x_{F_{2}}$.
provided $F_2 \neq \emptyset$ or $x_{E}^{\alpha_1} = -x_{H}^{\alpha_1}$ if $F_2 = \emptyset$; either way, it follows that $x_{E}^{\alpha} \in (x_H)$. Hence, $(x_H)$ contains all monomials in the statement of the lemma. Conversely, if $f \in (x_H)$, then $f = x_H g$ for some $g \in \text{CH}(M)$. Write $g = \sum c_{F} \cdot x_{F}^{\alpha}$ as a $\mathbb{Q}$-linear combination of nested monomials. For any monomial $x_{F}^{\alpha}$ in the support of $g$, we see that $x_{H} x_{F}^{\alpha}$ is nonzero only if $F_1 \subseteq H$. Indeed, if $F_1 \nsubseteq H$, then either $F_1$ and $H$ are incomparable or $F_1 = E$. Either way, the defining relations of $\text{CH}(M)$ or (4.13) imply that $x_{F_1} x_{H} = 0$. If $F_1 = H$, then $x_{H} x_{F}^{\alpha}$ is a scalar multiple of a nested monomial, even if $\alpha_1 = \text{rk} H - \text{rk} F_2 - 1 = \text{rk} M - \text{rk} F_2 - 2$ since then $x_{H} x_{F}^{\alpha} = -x_{E}^{rkM-rkF_2-1} x_{F_2}^{\alpha_2} \ldots x_{F_r}^{\alpha_r}$. If $H \nsubseteq F_1$ and $\text{rk} H - \text{rk} F_1 \geq 2$, then $x_{H} x_{F}^{\alpha}$ is also a nested monomial. If $H \nsubseteq F_1$ and $\text{rk} H - \text{rk} F_1 = 1$, then $x_{H} x_{F}^{\alpha} = -x_{E} x_{F_1}^{\alpha_1} \ldots x_{F_r}^{\alpha_r}$ is also a scalar multiple of a nested monomial. Thus, we can write

$$f = \sum_{G_1 \supseteq H} c_{G} \cdot x_{G}^{\beta} + \sum_{H \nsubseteq G_2} c_{G} \cdot x_{E}^{rkM-rkG_2-1} x_{G_2}^{\beta_2} \ldots x_{G_r}^{\beta_r}.$$ 

Since $A = \text{CH}(M)$ is an Artinian, Gorenstein ring, it follows by linkage [33, Remark 2.7] that the quotient ring $A/(0 : m)$ is also Gorenstein for any monomial $m$, and one might hope that this ring is the Chow ring of a matroid quotient of $M$. We identify the matroids associated to such quotients in some simple but important cases.

**Proposition 4.8** For a matroid $M$, we have $(0 : x_E) = (x_H \mid H \in \text{coat}(E))$ and $A/(0 : x_E) \cong \text{CH}(T(M))$.

**Proof** Clearly, the latter ideal is contained in the former by (4.13). Suppose that $f \in A$ with $x_E f = 0$, and write $f = \sum c_{F} \cdot x_{F}^{\alpha}$ as a $\mathbb{Q}$-linear combination of nested monomials. For each nested monomial $x_{F}^{\alpha}$ in $A$, we note that $x_{E} x_{F}^{\alpha} = 0$ only if $F_1$ is a hyperplane or $F_1 = E$ and $\alpha_1 = \text{rk} M - \text{rk} F_2 - 1$ (including the possibility that $F_2 = \emptyset$). Otherwise, $x_{E} x_{F}^{\alpha}$ is still a nested monomial. It then follows from the equality $x_{E} f = 0$ that

$$f = \sum_{F_1 \in \text{coat}(E)} c_{F} \cdot x_{F}^{\alpha} + \sum_{F_1 = E, \alpha_1 = \text{rk} M - \text{rk} F_2 - 1} c_{F} \cdot x_{F}^{\alpha}.$$ 

For each monomial in the latter sum, if $F_2 \neq \emptyset$, we note that $\text{rk} F_2 \leq \text{rk} M - 2$ so that there is a hyperplane $H \nsubseteq F_2$ as the lattice of flats of $M$ is graded, and $x_{E}^{\text{rk} M - \text{rk} F_2 - 1} x_{F_2} = -x_{H}^{\text{rk} M - \text{rk} F_2} x_{F_2}$. Otherwise, if $F_2 = \emptyset$, we have $x_{E}^{\text{rk} M - 1} = -x_{H}^{\text{rk} M}$ for any hyperplane $H$. This shows that $f \subseteq (x_H \mid H \in \text{coat}(E))$.

It is easily checked that there is a well-defined surjective $\mathbb{Q}$-algebra map $\pi : A \to \text{CH}(T(M))$ defined by $x_F \mapsto x_E$ for $F \notin \text{coat}(E)$ and $x_F \mapsto 0$ otherwise. We will show that $\text{Ker} \pi = (x_H \mid H \in \text{coat}(E))$. Again, it is clear that the latter ideal is contained in the former. If $f \in \text{Ker} \pi$ and we write $f = \sum c_{F} \cdot x_{F}^{\alpha}$ as a $\mathbb{Q}$-linear combination of nested monomials, then

$$0 = \pi(f) = \sum_{F_1 \notin \text{coat}(E)} c_{F} \cdot x_{F}^{\alpha} = \sum_{\text{rk} F_1 \leq \text{rk} M - 2} c_{F} \cdot x_{F}^{\alpha} + \sum_{F_1 = E, \alpha_1 < \text{rk} M - \text{rk} F_2 - 1} c_{F} \cdot x_{F}^{\alpha}.$$ 

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expresses $\pi(f)$ as a linear combination of nested monomials in $\text{CH}(T(M))$. In particular, note that the latter sum does not include any nested monomials with $\text{rk } F_2 = \text{rk } M - 2$ since $F_2$ is a hyperplane in $T(M)$ so that $x_{F_2}x_E = 0$ by (4.13). Hence, the coefficients of all the above monomials in $f$ must be zero so that

$$f = \sum_{F_1 \in \text{coat}(E)} c_{F, \alpha} x_{F}^{\alpha} + \sum_{F_1 = E, \alpha_1 = \text{rk } M - \text{rk } F_2 - 1} c_{F, \alpha} x_{F}^{\alpha}.$$  

However, for every nonempty flat $F_2$ we note that $x_{F_2}x_E^{\text{rk } M - \text{rk } F_2 - 1} = -x_{F_2}x_H^{\text{rk } H - \text{rk } F_2}$ for any hyperplane $H \supseteq F_2$, and $x_E^{\text{rk } M - 1} = -x_H^{\text{rk } H}$ for any hyperplane $H$. Thus, we see that $f \in (x_H \mid H \in \text{coat}(E))$ as wanted, and combining this with the preceding paragraph yields $A/(0 : x_E) \cong \text{CH}(T(M))$. 

**Corollary 4.9** Let $F$ be a flat of $M$ with $\text{rk } F \geq 2$.

(a) The kernel of the natural surjective homomorphism $\pi_{M,F} : A \rightarrow \text{CH}(M|F)$ sending $x_G \mapsto x_G$ if $G \subseteq F$ and $x_G \mapsto 0$ otherwise is $(x_G \mid G \not\subseteq F)$.

(b) Suppose that $G$ is a set of flats of $M$ such that $(F, E) \subseteq G$ and $G \cap \{\varnothing, F\} = \varnothing$. If $\text{rk } F \geq 3$, then

$$(x_G \mid G \in G) : x_F = (x_G \mid G \not\subseteq F) + (x_G \mid G \in \text{coat}(F)).$$

(4.14)

Otherwise, $(x_G \mid G \in G) : x_F = A_+.$

**Proof** (a) It is easily checked that there is a well-defined surjective $\mathbb{Q}$-algebra map $\pi : A \rightarrow \text{CH}(M|F)$ defined by the given assignments. We will show that $\text{Ker } \pi = (x_G \mid G \not\subseteq F).$ Again, it is clear that the latter ideal is contained in the former. If $f \in \text{Ker } \pi$ and we write $f = \sum c_{G, \alpha} x_{G}^{\alpha}$ as a $\mathbb{Q}$-linear combination of nested monomials, then $0 = \pi(f) = \sum_{G \subseteq F} c_{G, \alpha} x_{G}^{\alpha}$ expresses $\pi(f)$ as a linear combination of nested monomials in $\text{CH}(M|F)$. Hence, the coefficients of all the preceding monomials in $f$ must be zero so that $f = \sum_{G \not\subseteq F} c_{G, \alpha} x_{G}^{\alpha} \in (x_G \mid G \not\subseteq F)$ as wanted.

(b) First, we note that the colon ideal contains $(x_G \mid G \not\subseteq F)$. Indeed, if $G'$ is a flat such that $G' \not\subseteq F$, then either $F \not\subseteq G'$ so that $x_{G'} \in (x_G \mid G \in G)$, or $G'$ is incomparable with $F$ so that $x_{G'}x_F = 0$. Since both ideals in (4.14) contain $(x_G \mid G \not\subseteq F)$, it suffices by part (a) to show that their images agree in $\text{CH}(M|F)$. Hence, it suffices to assume $F = E$ and to prove that $(0 : x_E) = (x_H \mid H \in \text{coat}(E))$ when $\text{rk } M \geq 3$ and $(0 : x_E) = A_+$ when $\text{rk } M = 2$, which follow from the preceding proposition and the discussion at the beginning of this section respectively. 

**Proposition 4.10** Let $\mathcal{H}$ be a nonempty set of hyperplanes of $M$ and $H'$ be any hyperplane. Then:

(a) $(0 : x_{H'}) = (x_F \mid F \not\subseteq H')$ and $A/(0 : x_{H'}) \cong \text{CH}(M|H').$

(b) $(0 : (x_H \mid H \in \mathcal{H})) = (x_F \mid F \not\subseteq H$ for all $H \in \mathcal{H})$.

(c) Suppose that $H' \not\in \mathcal{H}$. If $\text{rk } H' \geq 3$ and for every $H \in \mathcal{H}$ there exists an $F \in \text{coat}_H(H')$ such that $H \cap H' \subseteq F$, then

$$(x_H \mid H \in \mathcal{H}) : x_{H'} = (0 : x_{H'}) + (x_F \mid F \in \text{coat}_H(H'))$$
Otherwise, if $\operatorname{rk} H' = 2$, then $(x_H \mid H \in \mathcal{H}) : x_{H'} = A_+.$

**Proof** (a) We have already seen in the proof of Lemma 4.7 that for any nested monomial $x_F^\alpha$ in $A$ we have $x_{H'} x_F^\alpha = 0$ if $F_1 \not\subseteq H'$ and $x_{H'} x_F^\alpha$ is a scalar multiple of a nested monomial otherwise. Furthermore, multiplication by $x_{H'}$ takes distinct nested monomials with $F_1 \subseteq H'$ to scalar multiples of distinct nested monomials. In particular, it is immediate that the latter ideal is contained in the former. Moreover, if $f \in A$ with $x_{H'} f = 0$ and we write $f = \sum c_F, \alpha x_F^\alpha$ as a $\mathbb{Q}$-linear combination of nested monomials, it follows from the equality $x_{H'} f = 0$ that $f = \sum_{F_1 \not\subseteq H'} c_F, \alpha x_F^\alpha$, which shows that $f \in (x_F \mid F \not\subseteq H')$. The isomorphism $A/(0 : x_{H'}) \cong \mathcal{CH}(M|H')$ then follows from part (a) of the preceding corollary.

(b) Consider the homomorphism $\eta : A \to \bigoplus_{H \in \mathcal{H}} \mathcal{CH}(M|H)$ sending each $f \in A$ to the tuple of its images under the natural maps $A \to \mathcal{CH}(M|H)$. By part (a), we know that $\operatorname{Ker} \eta = \bigcap_{H \in \mathcal{H}} (0 : x_H) = (0 : (x_H \mid H \in \mathcal{H}))$. Clearly, this ideal contains $x_F$ for every flat $F$ such that $F \not\subseteq H$ for all $H \in \mathcal{H}$. Conversely, if $f \in \operatorname{Ker} \eta$, then the proof of the previous part shows that $f$ has an expansion in terms of nested monomials of the form $f = \sum_{F_1 \not\subseteq H, \ all \ H \in \mathcal{H}} c_F, \alpha x_F^\alpha$. Hence, $f \in (x_F \mid F \not\subseteq H$ for all $H \in \mathcal{H}$) as wanted.

(c) The statement is clear if $\operatorname{rk} H' = 2$, since $x_{H'} x_F = 0$ for every $F \neq H'$ and $x_{H'}^2 = -x_F^2 = x_H^2 \in (x_H \mid H \in \mathcal{H})$ for any $H \in \mathcal{H}$ by (4.13). So, we may assume that $\operatorname{rk} H' \geq 3$. In that case, the latter ideal is again clearly contained in the former since $x_F x_{H'} = -x_F x_E = x_F x_H$ if $F = H \cap H'$ and $\operatorname{rk} F = \operatorname{rk} M - 2$. Suppose that $f \in A$ with $x_{H'} f \in (x_H \mid H \in \mathcal{H})$, and write $f = \sum c_F, \alpha x_F^\alpha$ as a $\mathbb{Q}$-linear combination of nested monomials. Then:

$$x_{H'} f = \sum_{F_1 \subseteq H'} c_{F_1, \alpha} x_{H'} x_F^\alpha = \sum_{\operatorname{rk} H' - \operatorname{rk} F_1 \geq 2} c_{F_1, \alpha} x_{H'} x_F^\alpha + \sum_{F_1 = H'} c_{F_1, \alpha} x_{H'} x_F^\alpha + \sum_{\operatorname{rk} H' - \operatorname{rk} F_1 = 1} c_{F_1, \alpha} x_{E} x_F^\alpha - \sum_{\operatorname{rk} H' - \operatorname{rk} F_1 = 1} c_{F_1, \alpha} x_{E} x_F^\alpha - \sum_{\operatorname{rk} H' - \operatorname{rk} F_1 = 1} c_{F_1, \alpha} x_{E} x_F^\alpha$$

By Lemma 4.7, the coefficients of the monomials in the first two sums must all be zero, and the coefficient of a monomial in the last two sums is also zero unless $F_1 \not\subseteq H$ for some $H \in \mathcal{H}$ or $F_1 = H'$ and $F_2 \not\subseteq H$ for some $H \in \mathcal{H}$. Hence, we have

$$f = \sum_{F_1 \not\subseteq H'} c_{F_1, \alpha} x_F^\alpha + \sum_{\operatorname{rk} H' - \operatorname{rk} F_1 = 1} c_{F_1, \alpha} x_F^\alpha + \sum_{F_2 \subseteq H \in \mathcal{H}} c_{F_2, \alpha} x_{H'} x_F^\alpha.$$
It follows that \( f \in (0 : x_{H'}) + (x_F^r x_{H'}^{rH' - rF - 1}, x_{H'}^{rH' - 1} \mid F \subseteq H \cap H' \text{ for some } H \in \mathcal{H}) \). It remains to show that

\[
(0 : x_{H'}) + (x_F^r x_{H'}^{rH' - rF - 1}, x_{H'}^{rH' - 1} \mid F \subseteq H \cap H' \text{ for some } H \in \mathcal{H})
\]
is contained in the ideal \( (0 : x_{H'}) + (x_F \mid F \in \text{coat}_{\mathcal{H}}(H')) \). After quotienting by \((0 : x_{H'})\), it suffices by part (a) to show that

\[
(x_F^r x_{H'}^{rH' - rF - 1}, x_{H'}^{rH' - 1} \mid F \subseteq H \cap H' \text{ for some } H \in \mathcal{H})
\]
is contained in the ideal \( J = (x_F \mid F \in \text{coat}_{\mathcal{H}}(H')) \) in \( \text{CH}(M|H') \).

Since \( \mathcal{H} \neq \emptyset \), we have \( \text{coat}_{\mathcal{H}}(H') \neq \emptyset \) by assumption so that \( J \neq 0 \). By Remark 4.6, we know \( x_{H'}^{rH' - 1} \) generates the socle of \( \text{CH}(M|H') \) and is, therefore, contained in \( J \). We must also show that \( x_F x_{H'}^{rH' - rF - 1} \in J \) if \( F \subseteq H \cap H' \) for some \( H \in \mathcal{H} \) and \( r_{H'} \geq r_{F} + 2 \). For this, since the elements \( \text{coat}_{\mathcal{H}}(H') \) are hyperplanes of the matroid \( M|H' \), we note that

\[
(0 : J) = (x_F \mid G \subseteq H', G \nsubseteq F' \text{ for all } F' \in \text{coat}_{\mathcal{H}}(H'))
\]

by part (b). Because \( \text{CH}(M|H') \) is Gorenstein, it follows by linkage [24, Theorem 21.23] that \( (0 : (0 : J)) = J \), and so, it suffices to note that each such \( x_G \) annihilates \( x_F x_{H'}^{rH' - rF - 1} \) to prove that \( x_F x_{H'}^{rH' - rF - 1} \in J \). Given a flat \( G \) with \( G \subseteq H' \) and \( G \nsubseteq F' \) for all \( F' \in \text{coat}_{\mathcal{H}}(H') \), we note that \( G \nsubseteq F \) since otherwise we would have \( G \subseteq F \subseteq H \cap H' \subseteq F' \) for some \( F' \in \text{coat}_{\mathcal{H}}(H') \) by assumption. Consequently, either \( G \) and \( F \) are incomparable so that \( x_F x_G = 0 \) or \( F \subseteq G \). If further \( G \subseteq H' \), then \( x_G x_{H'}^{rH' - rF - 1} = 0 \) by Corollary 4.4 as \( r_{H'} - r_{F} - 1 \geq r_{H'} - r_{G} \), whereas if \( G = H' \), then \( x_F x_G x_{H'}^{rH' - rF - 1} = x_F x_{H'}^{rH' - rF} = 0 \) again by Corollary 4.4. Either way, we have \( x_F x_G x_{H'}^{rH' - rF - 1} = 0 \) as wanted. \( \square \)

The condition in part (c) of the above proposition that for every \( H \in \mathcal{H} \) there exists an \( F \in \text{coat}_{\mathcal{H}}(H') \) such that \( H \cap H' \subseteq F \) is essential for the colon ideal to be generated by linear forms. We give some examples below showing how the colon ideal can fail to be generated by linear forms if this condition is not satisfied.

**Example 4.11** Consider the uniform matroid \( U_{5, 6} \) on the ground set \( E = \{1, 2, 3, 4, 5, 6\} \). Since every subset of \( E \) of size at most 5 is independent, the hyperplanes of \( U_{5, 6} \) are precisely the subsets of size 4. For \( H' = 1256 \), the proof of the proposition shows that in \( \text{CH}(U_{5, 6}) \) we have

\[
(x_{1234}) : x_{1256} = (0 : x_{1256}) + (x_{12} x_{1256}, x_{12}^3 x_{1256}) = (0 : x_{1256}) + (x_{12} x_{1256}),
\]

where \( x_{12} x_{1256} \notin (0 : x_{1256}) \) since it is a nested monomial of \( \text{CH}(U_{5, 6}|H') \).

If we take \( M = T_{H'}(U_{5, 6}) \) to be the principal truncation \( U_{5, 6} \) with respect to \( H' \), then \( M \) is a rank 4 matroid whose hyperplanes are \( H' \) and all subsets of \( E \) of size 3.
that are not contained in \(H'\). Even in this case, we still have

\[
(x_{234}) : x_{1256} = (0 : x_{1256}) + (x_{1256}^2),
\]

where \(x_{1256}^2 \notin (0 : x_{1256})\) since it is a nested monomial of \(\text{CH}(M|H')\).

In the first example above, the issue is that the two hyperplanes in question have rank 4 but their meet \(1234 \land 1256 = 12\) has rank 2. One might hope that by choosing only sets of hyperplanes that are locally connected (in the sense that between any two hyperplanes there is a sequence of hyperplanes in the set such that each pair of consecutive hyperplanes has a meet of rank \(\text{rk } M - 2\)), we might avoid quadratic terms in these colon ideals. The following example shows even this is not enough.

Again, consider \(\text{CH}(U_{5,6})\). Although the set of hyperplanes \(\{1234, 2345, 3456, 1456, 1256, 1236\}\) in \(U_{5,6}\) is locally connected (the corresponding graph is a 6-cycle), one computes that

\[
(x_{1234}, x_{2345}, x_{3456}, x_{1456}, x_{1236}) : x_{1256} = (0 : x_{1256}) + (x_{126}, x_{156}, x_{25}x_{1256}),
\]

where the quadratic monomial \(x_{25}x_{1256}\) is a minimal generator of this colon ideal.

**Theorem 4.12** The Chow ring of a matroid has a Koszul filtration.

**Proof** Let \(M\) be a matroid with lattice of flats \(L\). As previously noted at the beginning of this subsection, we may assume without loss of generality that \(M\) is simple and \(\text{rk } M \geq 3\). By Theorem 3.3, we also know that the lattice \(L\) admits a total coatom ordering \(\prec\). In what follows, all ideals are considered with respect to the atom-free presentation of \(A = \text{CH}(M)\), and all initial segments of flats are with respect to the chosen total coatom ordering. However, all up-sets refer to the natural partial order of \(L\). Consider the collections of ideals:

\[
\mathcal{F}_0 = \{(x_G | G \in \mathcal{G}) \mid \mathcal{G} \subseteq L_{\geq 2} \text{ an up-set}\}
\]

\[
\mathcal{F}_1 = \{(x_G | G \not\subseteq F) + (x_G | G \in \mathcal{G}) \mid F \in L_{\geq 2}, \ \mathcal{G} \text{ an initial segment covered by } F\},
\]

and set \(\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1\). Note that \((0) \in \mathcal{F}_0\) when \(\mathcal{G} = \emptyset\) and that \(A_+ \in \mathcal{F}_0\) when \(\mathcal{G} = L_{\geq 2}\). We claim that \(\mathcal{F}\) is the desired Koszul filtration.

Let \(0 \neq I \in \mathcal{F}_0\) so that \(I = (x_G | G \in \mathcal{G})\) for some nonempty up-set \(\mathcal{G}\). We then choose any flat \(G' \in \mathcal{G} \text{ minimal in } \mathcal{G}\) with respect to the partial order of \(L\) and set \(G' = \mathcal{G} \setminus \{G'\}\). It is easy to see that \(G'\) is also an up-set since we are removing a minimal element of \(\mathcal{G}\). Consider the ideal \(J = (x_G | G \in G') \in \mathcal{F}_0\). Clearly \(J \subseteq I\), and \(I/J\) is cyclic. Since \(G'\) contains all flats properly containing \(G'\) and none of the flats contained in \(G'\), it follows from Corollary 4.9 that

\[
(J : I) = (J : x_{G'}) = (x_G | G \not\subseteq G') + (x_G | G \in \text{coat}(G'))
\]

if \(\text{rk } G' \geq 3\) and \((J : I) = A_+\) if \(\text{rk } G' = 2\). In the former case, it follows that \((J : I) \in \mathcal{F}_1\), and we have \((J : I) \in \mathcal{F}_0\) in the latter case.
Let $0 \neq I \in \mathcal{F}_1$ so that $I = (x_G \mid G \nsubseteq F) + (x_G \mid G \in \mathcal{G})$ for some flat $F$ and $\mathcal{G}$ an initial segment of flats covered by $F$. If $\mathcal{G} = \emptyset$, then $I = (x_G \mid G \nsubseteq F) \in \mathcal{F}_0$, and so, we have already seen in the preceding paragraph how to choose an ideal contained in $I$ with the desired properties. Hence, we may assume that $\mathcal{G} \neq \emptyset$ so that $rk F \geq 3$, and we can take $G'$ to be the largest element of $\mathcal{G}$ with respect to the total coatom ordering and $G' = G \setminus \{G'\}$. Then $J = (x_G \mid G \nsubseteq F) + (x_G \mid G \in \mathcal{G}') \in \mathcal{F}_1$. Again, $J \subseteq I$ with $I/J$ cyclic. If $rk G' = 2$, then the set of flats $\{G \in \mathcal{L}_{\geq 2} \mid G \nsubseteq G'\} \cup G'$ contains all flats properly containing $G'$ and none of the flats contained in $G'$ so that $(J : I) = (J : x_{G'}) = A_+ \in \mathcal{F}_0$ by Corollary 4.9. On the other hand, if $rk G' \geq 3$, we claim that $(J : I) = (J : x_G) = (x_G \mid G \nsubseteq G') + (x_G \mid G \in \text{coat}_G(G'))$. (4.15)

Indeed, since $(J : I) \supseteq J \supseteq (x_G \mid G \nsubseteq F)$ and the latter ideal is the kernel of the natural surjection from $A \to \text{CH}(M|F)$ by Corollary 4.9, it suffices to show that

$$(x_G \mid G \in \mathcal{G}) : x_{G'} = (x_G \mid G \nsubseteq G', G \subseteq F) + (x_G \mid G \in \text{coat}_G(G'))$$

in $\text{CH}(M|F)$. As $\mathcal{G}$ is a set of hyperplanes in $M|F$, this follows from part (c) of Proposition 4.10 so that (4.15) holds as claimed. Note that the conditions on $\mathcal{G}$ necessary to apply Proposition 4.10 are satisfied since $\mathcal{G}$ is an initial segment with respect to a total coatom ordering. Furthermore, the definition of a total coatom ordering implies that $\text{coat}_G(G')$ is an initial segment of flats covered by $G'$ so that $(J : I) \in \mathcal{F}_1$. Thus, $\mathcal{F}$ is a Koszul filtration for $A$.

We get the following corollary to our main theorem.

**Corollary 4.13** The Chow ring of a matroid has a rational Poincaré series.

**Proof** It is well-known that any commutative Koszul $K$-algebra $A$ has a rational Poincaré series; indeed, the Hilbert series $H_A(t)$ is always rational so that the Poincaré series $P^A_K(t) = \frac{1}{H_A(-t)}$ is as well by (2.1). \qed

### 5 The Koszul property of the augmented Chow ring of a matroid

#### 5.1 The augmented Chow ring of a matroid

The **augmented Chow ring** of a simple matroid $M$ with lattice of flats $\mathcal{L} = \mathcal{L}(M)$ is the ring

$$\text{CH}(M) := S_M/(I_M + J_M),$$

where

$$S_M = \mathbb{Q}[y_i, x_F \mid i \in E, F \in \mathcal{L} \setminus \{E\}],$$
and $I_M$ and $J_M$ are the ideals

\[
I_M = (y_i - \sum_{i \notin F} x_F \mid i \in E), \\
J_M = (x_{FG} \mid F, G \text{ incomparable}) + (y_i x_F \mid i \in E, i \notin F).
\]

Augmented Chow rings of matroids were introduced in [8] and employed in [7] as a key ingredient in the resolution of the Top-Heavy Conjecture that interpolates between the combinatorics of the lattice of flats of a matroid (as encoded by its graded Möbius algebra) and the good algebraic properties of the Chow ring of the matroid. In particular, it is shown [7, Theorem 2.19] that the augmented Chow ring of a matroid is also a quadratic Artinian Gorenstein graded algebra. It was pointed out to us by Chris Eur that the augmented Chow ring of $M$ could be viewed as the Feichtner–Yuzvinsky Chow ring $D(\mathcal{L}, \mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$ of the lattice of flats of a related matroid, the free coextension of $M$, with respect to a certain building set. Using this observation, we show how to obtain an atom-free presentation for the augmented Chow ring of a matroid and that all of the results of the preceding section carry over with only minor modifications to show that the augmented Chow ring of a matroid is also Koszul.

We begin by sketching the details of Eur’s observation about the augmented Chow ring. Let $M$ be a simple matroid on the ground set $E$. The free coextension of $M$ is the matroid $(M^* + e)^*$; it is the dual of the free single element extension of the dual matroid of $M$. We refer the interested reader to [39, Sections 2.1, 7.2] for further details on duality and free extensions of matroids. For our purposes, it suffices to note that the free coextension $(M^* + e)^*$ is a matroid on the ground set $E \cup \{e\}$, where $e$ is any element not in $E$, with rank function defined as follows for any set $F \subseteq E$:

\[
\begin{align*}
\text{rk}_{(M^* + e)^*} F &= \begin{cases} 
\text{rk}_M F + 1, & \text{if } \text{cl}_{M^*}(E \setminus F) \neq E \\
\text{rk}_M F, & \text{if } \text{cl}_{M^*}(E \setminus F) = E
\end{cases} \\
\text{rk}_{(M^* + e)^*}(F \cup e) &= \text{rk}_M F + 1
\end{align*}
\]

Using the description of the rank function of the dual of $M$, it is easily checked that the condition $\text{cl}_{M^*}(E \setminus F) = E$ is satisfied if and only if $\text{rk}_M F = |F|$, which is equivalent to $F$ being an independent set of $M$. From this, it can be seen that a set of the form $F \cup e$ is a flat of the free coextension of $M$ if and only if $F$ is a flat of $M$, and a set $F$ is a flat of the free coextension if and only if $F$ is an independent set of $M$. Thus, in a way, the free coextension treats independent sets and flats of $M$ on equal footing. By identifying the flat $F$ of $M$ with the flat $F \cup e$ of the free coextension, we see that the lattice of flats of the free coextension contains an isomorphic copy of the lattice of flats of $M$, just shifted upward in rank by one, in addition to all independent sets of $M$ with their usual rank.

Consider the subset $\mathcal{G}_\text{aug} \subseteq \mathcal{L}((M^* + e)^*)$ defined by

\[
\mathcal{G}_\text{aug} = \{i \mid i \in E\} \cup \{F \cup e \mid F \in \mathcal{L}(M)\}.
\]

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We claim that $G_{\text{aug}}$ is a building set for the lattice of flats of the free coextension. By the preceding paragraph, we know that every non-minimal element of the lattice of flats of the free coextensions is either a nonempty independent set of $M$ or of the form $F \cup e$ for some flat $F$ of $M$. When $F$ is a flat of $M$, $F \cup e \in G_{\text{aug}}$ so that the required poset isomorphism for $G_{\text{aug}}$ to be building is just the identity map $[\emptyset, F \cup e] \to [\emptyset, F \cup e]$. On the other hand, when $F$ is an independent set of $M$, $\max(G_{\text{aug}})_{\leq F} = \{i \mid i \in F\}$. Since the interval $[\emptyset, F]$ is just the Boolean lattice of all subsets of $F$, it is easily seen that the map $\varphi_F : \prod_{i \in F}[\emptyset, i] \to [\emptyset, F]$ given by $\varphi_F(A_1, \ldots, A_r) = \bigcup_{j=1}^r A_j$ is the required poset isomorphism for $G_{\text{aug}}$ to be building.

The **Feichtner–Yuzvinsky presentation** of the augmented Chow ring of a simple matroid $M$ is the ring

$$
\text{CH}_F(M) = \mathbb{Q}[y_i, y_{F \cup e} \mid i \in E, F \in \mathcal{L}] / I_F(M),
$$

where

$$
I_F(M) = (y_{F \cup e}y_{F' \cup e} \mid F, F' \text{ incomparable}) + (y_i y_{F \cup e} \mid i \in E, i \notin F)
+ (y_i + \sum_{i \in F} y_{F \cup e}, \sum_F y_{F \cup e} \mid i \in E).
$$

Setting $x_F = y_{F \cup e}$ and using the last linear form above to eliminate $x_E$ recovers the usual presentation of the augmented Chow ring of $M$.

**Definition 5.1** The **atom-free presentation** of the augmented Chow ring of $M$ is the ring

$$
\text{CH}_{\text{af}}(M) = \mathbb{Q}[x_F \mid F \in \mathcal{L} \setminus \{\emptyset\}] / I_{\text{af}}(M),
$$

where

$$
I_{\text{af}}(M) = (x_F x_{F'} \mid F, F' \text{ incomparable}) + (x_F \sum_{i \in F'} x_{F'} \mid i \in E, i \notin F)
+ ((\sum_{i \in F'} x_{F'})^2 \mid i \in E)
$$

After a change of variables $\varphi$ on the ring $\mathbb{Q}[y_i, x_F \mid i \in E, F \in \mathcal{L}]$ sending $y_i \mapsto y_i - \sum_{i \in F} x_F$ for each $i \in E$ and sending $x_\emptyset \mapsto x_\emptyset - \sum_{F \neq \emptyset} x_F$, we see that

$$
\varphi(I_F(M)) = (x_\emptyset, y_i \mid i \in E) + (x_F x_{F'} \mid F, F' \in \mathcal{L} \setminus \{\emptyset\} \text{ incomparable})
+ (x_F \sum_{i \in F'} x_{F'} \mid F \in \mathcal{L} \setminus \{\emptyset\}, i \in E, i \notin F)
+ ((\sum_{i \in F'} x_{F'}) (\sum_{F \neq \emptyset} x_F) \mid i \in E)
= (x_\emptyset, y_i \mid i \in E) + I_{\text{af}}(M),
$$

(5.1)

where the last equality follows after a suitable change of generators from the fact that

$$
(\sum_{i \in F'} x_{F'}) (\sum_{F \neq \emptyset} x_F) = (\sum_{i \in F'} x_{F'})^2 + \sum_{F \neq \emptyset, i \notin F} x_F (\sum_{i \in F'} x_{F'}).$$
And so, we see that
\[
\text{CH}(M) \cong \text{CH}_{FY}(M) \cong \frac{\mathbb{Q}[y_i, x_F \mid i \in E, F \in \mathcal{L}]}{(x_{\varnothing}, y_i \mid i \in E) + I_{af}(M)} \cong \text{CH}_{af}(M).
\]

**Theorem 5.2** [29, Theorem 2] With respect to any lexicographic order \( > \) such that \( x_F > x_G \) implies \( F \nsubseteq G \) and \( y_i > x_F \) for all \( i \in E \) and all flats \( F \), the defining ideal \( I_{FY}(M) \) of the augmented Chow ring of \( M \) has a Gröbner basis consisting of the following polynomials for all \( i \in E \) and \( F, F' \in \mathcal{L} \):

\[
\begin{align*}
&x_F x_{F'} \quad F, F' \text{ incomparable} \quad (5.2) \\
y_i x_F \quad i \notin F \quad (5.3) \\
x_F' \left( \sum_{G \supseteq F} x_G \right)^{\text{rk} F - \text{rk} F'} \quad F' \subsetneq F \quad (5.4) \\
y_i \left( \sum_{G \supseteq F} x_G \right)^{\text{rk} F} \quad i \in F \quad (5.5) \\
\left( \sum_{G \supseteq F} x_G \right)^{\text{rk} F + 1} \quad (5.6) \\
y_i + \sum_{i \in F} x_F \quad i \in E \quad (5.7)
\end{align*}
\]

A slight modification yields a Gröbner basis in the atom-free setting. We omit the proof since it is completely analogous to the proof of Corollary 4.4.

**Corollary 5.3** With respect to any lexicographic order \( > \) such that \( x_F > x_G \) implies \( F \nsubseteq G \), the defining ideal \( I_{af}(M) \) of the augmented Chow ring of \( M \) has a Gröbner basis consisting of the following polynomials for all \( F, F' \in \mathcal{L} \setminus \{\varnothing\} \):

\[
\begin{align*}
x_F x_{F'} & \quad F, F' \text{ incomparable} \quad (5.8) \\
x_F' \left( \sum_{G \supseteq F} x_G \right)^{\text{rk} F - \text{rk} F'} \quad F' \subsetneq F \quad (5.9) \\
\left( \sum_{G \supseteq F} x_G \right)^{\text{rk} F + 1} & \quad (5.10)
\end{align*}
\]

**Corollary 5.4** The ring \( \text{CH}_{af}(M) \) has a \( \mathbb{Q} \)-basis consisting of all monomials

\[
x_F^\alpha = x_{F_1}^{\alpha_1} \cdots x_{F_r}^{\alpha_r}
\]

for all chains of flats \( \mathbf{F} = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_r \supseteq F_{r+1} = \varnothing\} \) for some \( r \geq 0 \) and all \( \alpha = (\alpha_1, \ldots, \alpha_{r+1}) \in \mathbb{Z}_{\geq 0}^{r+1} \) such that \( \sum_i \alpha_i = \text{rk} F_1 + 1, 1 \leq \alpha_i < \text{rk} F_i - \text{rk} F_i+1 \) if \( i < r \), and \( \alpha_r \leq \text{rk} F_r \).

As before, we call the monomials of Corollary 5.4 the **nested monomials** of \( \text{CH}_{af}(M) \).

### 5.2 A Koszul filtration for the augmented Chow ring of a matroid

When \( M \) is a matroid of rank 1, we note that \( \text{CH}(M) \cong \mathbb{Q}[x_E]/(x_E^2) \), which we have previously noted has a Koszul filtration. We may, therefore, assume the following without any loss of generality.
**Notation** Throughout the remainder of this section, $M$ denotes a simple matroid with $\text{rk } M \geq 2$ and $A = \text{CH}(M)$ with respect to the atom-free presentation.

**Remark 5.5** As a consequence of Corollary 5.3, we note that $I_{af}(M)$ contains the polynomials

$$x_H x_E, \quad x_F x_H^{\text{rk } H - \text{rk } F} + x_F x_E^{\text{rk } M - \text{rk } F - 1}, \quad x_H^{\text{rk } H + 1} + x_E^{\text{rk } M}$$

(5.11)

for any $F \in \mathcal{L} \setminus \{\emptyset\}$ and hyperplane $H \supseteq F$. In particular, we note that the last type of relation is the primary difference when compared with the ordinary Chow ring of $M$. Since $x_F x_E^{\text{rk } M} = 0$ in $A$ for all $F \in \mathcal{L} \setminus \{\emptyset\}$ and $A$ is an Artinian Gorenstein ring, it follows that $x_E^{\text{rk } M}$ is a socle generator of $A$.

The proofs of the following results are completely analogous to those of Sect. 4 aside from some minor modifications based on the above remark.

**Lemma 5.6** Let $\mathcal{H}$ be a set of hyperplanes of $M$. Then the ideal $(x_H \mid H \in \mathcal{H})$ in $A = \text{CH}(M)$ has a $\mathbb{Q}$-basis consisting of all nested monomials $x_F^\alpha$ with $F = \{F_1 \supseteq F_2 \supseteq \cdots \supseteq F_r \supseteq F_{r+1} = \emptyset\}$ for some $r \geq 0$ such that either:

(i) $F_1 \in \mathcal{H}$, or

(ii) $F_1 = E$, $H \supseteq F_2 \neq \emptyset$ for some $H \in \mathcal{H}$ and $\alpha_1 = \text{rk } M - \text{rk } F_2 - 1$, or

(iii) $F_1 = E$, $F_2 = \emptyset$, and $\alpha_1 = \text{rk } M$.

**Proposition 5.7** For any matroid $M$, we have $(0 : x_E) = (x_H \mid H \in \text{coat}(E))$ and $A/(0 : x_E) \cong \text{CH}(T(M))$.

**Corollary 5.8** Let $F$ be a flat of $M$ with $\text{rk } F \geq 1$.

(a) The kernel of the natural surjective homomorphism $\pi_{M,F} : A \rightarrow \text{CH}(M|F)$ sending $x_G \mapsto x_G$ if $G \subseteq F$ and $x_G \mapsto 0$ otherwise is $(x_G \mid G \not\subseteq F)$.

(b) Suppose that $G$ is a set of flats of $M$ such that $(F, E) \subseteq G$ and $G \cap [\emptyset, F] = \emptyset$. If $\text{rk } F \geq 2$, then

$$(x_G \mid G \in \mathcal{G}) : x_F = (x_G \mid G \not\subseteq F) + (x_G \mid G \in \text{coat}(F)).$$

Otherwise, $(x_G \mid G \in \mathcal{G}) : x_F = A_+.$

**Proposition 5.9** Let $\mathcal{H}$ be a nonempty set of hyperplanes of $M$ and $H'$ be any hyperplane. Then:

(a) $(0 : x_{H'}) = (x_F \mid F \not\subseteq H')$ and $A/(0 : x_{H'}) \cong \text{CH}(M|H')$.

(b) $(0 : (x_H \mid H \in \mathcal{H})) = (x_F \mid F \not\subseteq H$ for all $H \in \mathcal{H}$).

(c) Suppose that $H' \notin \mathcal{H}$. If $\text{rk } H' \geq 2$ and for every $H \in \mathcal{H}$ there exists an $F \in \text{coat}_\mathcal{H}(H')$ such that $H \cap H' \subseteq F$, then

$$(x_H \mid H \in \mathcal{H}) : x_{H'} = (0 : x_{H'}) + (x_F \mid F \in \text{coat}_\mathcal{H}(H')).$$ 

Otherwise, if $\text{rk } H' = 1$, then $(x_H \mid H \in \mathcal{H}) : x_{H'} = A_+.$
Theorem 5.10  The augmented Chow ring of a matroid has a Koszul filtration. In particular, such rings have rational Poincaré series.

It is tempting to seek a common framework for the Koszul property of Chow rings and augmented Chow rings of matroids. The first natural approach is to consider Chow rings of geometric lattices with respect to arbitrary building sets. Examples 6.2–6.3 show that there is no such result in this level of generality.

6 Examples and questions

In this section, we collect some relevant examples and questions. We begin with Bøgvad’s algebra with irrational Poincaré series.

Example 6.1  (A quadratic Artinian Gorenstein algebra with irrational Poincaré series) This example is due to Bøgvad [11] and is based on Anick’s [3] construction; however, it is not obvious that this construction leads to a quadratic algebra. Let $K$ denote a field of characteristic 2. (Computation suggests that any field will do.) Set

$$R = K[x_1, x_2, x_3, x_4, x_5]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5).$$

Anick proved that $R$ has irrational Poincaré series. One can easily check that $R$ is superlevel in the language of Mastroeni et al. [36]. Therefore, the idealization $\tilde{R} = R \ltimes \omega_R(-3)$ is quadratic and Gorenstein; here $\omega_R$ denotes the canonical module of $R$. The Poincaré series of $\tilde{R}$ is rationally related to that of $R$ [32], and so $\tilde{R}$ also has irrational Poincaré series. (In both papers, the ideal $(x_1, x_2, x_3, x_4, x_5)^3$ is included among the relations but is redundant.)

One might hope for a more general result about the Koszul property of Chow rings of atomic lattices and arbitrary building sets. Even for geometric lattices, the following examples show that this is not possible.

Example 6.2  (A geometric lattice with Gorenstein but not Koszul Chow ring) In general, the Chow rings $D(\mathcal{L}, \mathcal{G}) \otimes_\mathbb{Z} \mathbb{Q}$ of Feichtner and Yuzvinsky need not be Koszul for all building sets $\mathcal{G}$ even when $\mathcal{L}$ is a geometric lattice. For example, suppose $\mathcal{L} = B_3$ is the Boolean lattice on 3 elements, which is the lattice of flats of the uniform matroid $U_{3,3}$ on the ground set $\{1, 2, 3\}$. Then $\mathcal{G} = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$ is easily checked to be a building set for $\mathcal{L}$, but (after suppressing commas and braces in the variable indices) the Chow ring is

$$D(\mathcal{L}, \mathcal{G}) \otimes_\mathbb{Z} \mathbb{Q} = \frac{\mathbb{Q}[x_1, x_2, x_3, x_123]}{(x_1x_2x_3, x_1 + x_123, x_2 + x_123, x_3 + x_123)} \cong \frac{\mathbb{Q}[x_{123}]}{(x_{123}^3)},$$

which is evidently not Koszul. Interestingly, one can check using Macaulay2 that this is the only building set for the lattice $B_3$ for which the Chow ring is not Koszul.

Example 6.3  (A geometric lattice with non-Koszul Chow ring for the minimal building set) Even when $\mathcal{G} = \mathcal{G}_{\min}$ is the minimal building set of a geometric lattice $\mathcal{L}$,
Fig. 3 An atomic lattice $\mathcal{L}$ with non-Gorenstein Chow ring

$D(\mathcal{L}, \mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$ can fail to be Koszul. Let $M$ be the uniform matroid $U_{3,4}$. The minimal building set of $\mathcal{L}(M)$ is $\mathcal{G} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}\}$. The Chow ring is

$$\mathbb{Q}[x_1, x_2, x_3, x_4, x_{1234}] \quad (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_1 + x_{1234}, x_2 + x_{1234}, x_3 + x_{1234}, x_4 + x_{1234})$$

$$\cong \frac{\mathbb{Q}[x_{1234}]}{(x_{1234}^3)},$$

which is also not Koszul.

The Gorenstein property of the (augmented) Chow ring of a matroid seems to play an important role in our methods. It was already clear from the work of Feichtner and Yuzvinsky that not all Chow rings of atomic lattices were Gorenstein; see [29, pp. 22–23]. We present a simple example.

**Example 6.4** (A non-geometric lattice with Koszul but not Gorenstein Chow ring)
Consider the lattice $\mathcal{L}$ with Hasse diagram in Fig. 3. It is easy to check that $\mathcal{L}$ is atomic but not semimodular. The Chow ring of $\mathcal{L}$ with respect to the maximal building set $\mathcal{G}_{\text{max}} = \mathcal{L} \setminus \{\hat{0}\}$ is

$$D(\mathcal{L}, \mathcal{G}_{\text{max}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \frac{\mathbb{Q}[x_e, x_f, x_\hat{1}]}{(x_e, x_f, x_\hat{1})^2},$$

which is Koszul but not Gorenstein.

The preceding examples naturally lead to the following questions.

**Question 6.5** Under what combinatorial conditions on the building set $\mathcal{G}$ and the lattice $\mathcal{L}$ is the Chow ring $D(\mathcal{L}, \mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$ Koszul? Is there a Chow ring of a geometric lattice that fails to be Koszul with respect to some building set while still being quadratic?

**Question 6.6** Is the Chow ring of any graded, atomic lattice with respect to its maximal building set always Koszul?
While Dotsenko showed that \(D(\Pi_n, G_{\text{min}})\) has a quadratic Gröbner basis, a similar monomial order on \(\text{CH}(M) = D(\mathcal{L}(M), G_{\text{max}}) \otimes \mathbb{Z} \mathbb{Q}\) does not seem to produce a quadratic Gröbner basis. On the other hand, there does not appear to be a Hilbert function obstruction to one, leaving open the following question.

**Question 6.7** Is the Chow ring of a matroid G-quadratic?

We close with a potential generalization of our results.

**Question 6.8** Does every quadratic, Artinian, Gorenstein \(\mathbb{K}\)-algebra with the Kähler package have the Koszul property?

It follows from [35, Theorem 4.3] that not all quadratic, Gorenstein \(\mathbb{K}\)-algebras have the Kähler package; in particular, there are quadratic Gorenstein \(\mathbb{K}\)-algebras with non-unimodal \(h\)-vectors.

**Acknowledgements** The authors thank Emanuele Delucchi, Vladimir Dotsenko, Chris Eur, Vic Reiner, and Botong Wang for many helpful conversations. We also thank the anonymous referee for their careful reading of this paper and their many comments which helped to improve the paper’s exposition. Computations with Macaulay2 [30], especially Justin Chen’s matroid package [16], were very helpful while working on this project. Mastroeni was supported by an AMS-Simons Travel Grant. McCullough was supported by National Science Foundation Grant DMS-1900792.

**Declaration**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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