Shape-constrained Estimation of Value Functions

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Abstract

We present a fully nonparametric method to estimate the value function, via simulation, in the context of expected infinite-horizon discounted rewards for Markov chains. Estimating such value functions plays an important role in approximate dynamic programming. We incorporate “soft information” into the estimation algorithm, such as knowledge of convexity, monotonicity, or Lipschitz constants. In the presence of such information, a nonparametric estimator for the value function can be computed that is provably consistent as the simulated time horizon tends to infinity. As an application, we implement our method on price tolling agreement contracts in energy markets.

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1 Introduction

This paper is concerned with the estimation, via simulation, of value functions in the context of expected infinite horizon discounted rewards for Markov chains. Estimating such value functions plays an important role in approximate dynamic programming and applied probability in general. In many problems of practical interest, the state space is huge or even continuous and the value function is computationally intractable. Therefore, we need to approximate the value function. In this work, we develop a fully non-parametric method to estimate the value function by incorporating shape constraints, such as knowledge of convexity, monotoncity, or Lipschitz constants.

The most common method employed to approximate the value function is parametric approximate dynamic programming; see Powell (2011) and Bertsekas (2007). In this method, the user specifies an “approximation architecture” (i.e. a set of basis functions) and the algorithm then produces an approximation in the span of this basis. Selecting the basis function is essential because an inappropriate “approximation architecture” might cause unsatisfactory results, and cannot be improved by additional sampling or computational effort.

In contrast, we are proposing a fully “non-parametric” method to avoid the difficulty of choosing a correct approximation architecture. The general idea is to take advantage of shape properties of the optimal value function in estimating the function. A variety of control problems exist on continuous state spaces for which convexity in the value function naturally arises. For instance, in a linear transition system, if the reward function in each stage is convex, then the value function is convex. Inventory models represent a well-known example of this class of problems. Singular stochastic control (Kumar and Muthuraman (2004)) and partially observed Markov processes (Smallwood and Sondik (1973)) are two other subclasses of problems for which the value function is convex. As another example, Karoui et al. (1998) show that the American-style option is convex for a generalized Black–Scholes model.

Monotonicity properties have been studied in the literature for various problems formulated as Markov decision processes. For instance, if the reward is monotone and the chain is stochastically monotone, the value function is monotone. In Papadaki and Powell (2007), the monotonicity of the value function is studied in the case of the multi-product batch dispatch problem. Stokey (1989, p. 267-268) presents general conditions that guarantee the value function will be monotonic in the underlying state variable. Discussions of monotonicity appear also in Serfozo (1979) and Topkis (1998).

In Smith and McCardle (2002) and Atakan (2003), sufficient conditions are provided on the
transition probability of a stochastic dynamic programming problem to ensure the shape properties of the value function. The goal of our work is to exploit this type of shape property to estimate the value function.

We suggest two methods for computing an approximation of the value function for a fixed policy. In the first method, we estimate the value function along a path by explicitly incorporating the shape constraint. For instance, in the case that we know the value function is convex, we consider the set of all convex functions which is a convex cone in the space of measurable functions. Having a sample path of the underlying process, one can reach a noisy observation of the value function. By projecting this noisy observation to the cone of convex functions, we achieve an estimator for the value function. Since this method requires only one sample path of the process, it can be used in reinforcement learning applications.

The second method is based on estimating the value function by taking advantage of the fixed point property of the value function in addition to the shape constraint. The value function satisfies a specific linear system of equations. Therefore, estimating the value function is possible by approximating the fixed point of this system of equations over the cone of convex functions. This fixed point can be obtained by iteratively projecting onto the cone of convex functions. The simulation results show that the second approach has reduced variance and provides more accurate estimators as compared to the first approach.

The projection onto the cone of convex functions is possible by solving a least square optimization problem. This optimization problem can be interpreted as a multi-dimensional convex regression. Convex regression is concerned with computing the best fit of a convex function to a dataset of \( n \) observations;

\[
Y_i = f(X_i) + \epsilon_i
\]

for \( i = 1, \ldots, n \). Convex regression derives a convex estimator of \( f \) by solving a least square problem. In one dimension, the theory of convex regression is well established; see Hanson and Pledger (1976) for the consistency result and Mammen (1991) and Groeneboom et al. (2001) for the rate of convergence. The consistency of convex regression has been shown in Lim and Glynn (2012), and in Seijo and Sen (2011) in the multi-dimensional case where the observations are independent.

To show the consistency of the estimator in our method, we extend the results in convex regression literature to the Markov processes. Let \( X = (X_t: t \geq 0) \) be a positive Harris recurrent and the noise sequence be a correlated sequence satisfying suitable technical assumptions. We show that the estimator is converging to the projection of \( f \) onto the cone of convex functions in the
Hilbert space of measurable functions. Lim and Glynn (2012) studied the behavior of the estimator when the model is mis-specified so that the function \( f \) is non-convex under the much stronger assumption that the function \( f \) is bounded. Our result relaxes this assumption.

Recently, Hannah and Dunson (2011, 2013) employed the notion of fitting convex functions in solving dynamic programming problems. The key differences between our work and Hannah and Dunson (2013, 2011) are as follows:

i.) Our method is fully non-parametric while their approach is semi-parametric and required adjusting several parameters before fitting a convex function or determining the prior distribution for Bayesian updating.

ii.) Hannah and Dunson (2011, 2013) used the value iteration method which involves generating many sample paths. In contrast, we are using single or two sample paths.

iii.) It is well known that value iteration type algorithms often lead to errors that grow exponentially in the problem horizon. Small local changes at each iteration can lead to a large global error of the approximation; see Section IV of Tsitsiklis and Van Roy (2001) and Ma and Powell (2009). In contrast, in our method the projection to the convex set occurs asymptotically with respect to the stationary distribution of the underlying Markov chain and has a convergence guarantee.

The literature on approximate dynamic programming (ADP) is also related to our work. Some recent works in this area suggest that the performance of parametric ADP algorithms is improved by exploiting structural properties; see Wang and Judd (2000); Cai and Judd (2010, 2012a,b,c); Cai et al. (2013). In addition, Godfrey and Powell (2001) and Powell et al. (2004) consider the cases where the value functions are known to be convex and approximate the value function by separable, piecewise linear functions of one variable. In Kunnumkal and Topaloglu (2010), the monotonicity of value functions are used to approximate the value function where the state space is finite.

In greater detail, we make the following contributions:

i.) We rigorously develop a fully non-parametric method to estimate shape constrained value functions of multi-dimensional continuous state space M.C. In the case that the value function is convex, the estimator can be represented as a piecewise linear function and evaluated at each point in linear time.
ii.) We extend the convex regression to the case in which explanatory variables are sampled along a Markov chain path. Moreover, the observations are correlated and generated along the same path.

iii.) We identify the behavior of the estimator in the case of mis-specification, where the value function is not-convex.

iv.) We show the convergence of the estimator to the solution of the projected Bellman equation as the length of the sample path goes to infinity.

v.) We extend the non-parametric method to estimate the value functions which are Lipschitz or monotone and convex.

The rest of this section is organized as follows: In Section 2, we precisely introduce the mathematical framework for our analysis. In section 3, we describe our methods. Section 4 presents the extension of multi-dimensional convex regression to the Markov processes and shows the consistency of convex regression in this general framework. In Section 5, we use the results of Section 4 to prove the convergence of our methods. In Section 6, we extend our methods to estimate the value functions by exploiting other shape structures. In Section 7, we study the efficacy of our methods by applying them to a pricing problem in energy market.

2 Formulation

Let $X = (X_t : t \geq 0)$ be a discrete time Markov chain evolving on a general continuous state space $\mathcal{X}$ embedded on $\mathbb{R}^d$. Each random variable $X_t$ is measurable with respect to the Borel $\sigma$-algebra associated with $\mathbb{R}^d$. The transition probability of the Markov chain $P(x, B)$ represents the time-homogeneous probability that the next state will be $X_{t+1} \in B$ given that the current state is $X_t = x$. Let $r(X_t)$ be the reward function received at time $t$, and $e^{-\alpha}$ be a discounting factor with $\alpha > 0$.

The value function, which is the expected infinite horizon discounted reward for the Markov chain, is given by

$$V^*(x) = \mathbb{E} \left[ \sum_{t=0}^{\infty} e^{-t\alpha} r(X_t) \bigg| X_0 = x \right].$$

According to the Markov property, we have

$$V^*(x) = \mathbb{E} \left[ r(X_t) + e^{-\alpha} V^*(X_{t+1}) \bigg| X_t = x \right].$$
Define the operator \( T : \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{R} \) by
\[
(T\phi)(x) = \mathbb{E} \left[ r(X_t) + e^{-\alpha} \phi(X_{t+1}) \big| X_t = x \right],
\]
where \( \mathcal{L}(\mathcal{X}) \) is the space of measurable functions over \( \mathcal{X} \). The operator \( T \) can be considered as the Bellman operator for a fixed policy. It is well known that \( T \) is a contraction with respect to the sup norm.

\[
\|T\phi - T\phi'\|_{\infty} \leq e^{-\alpha} \|\phi - \phi'\|_{\infty},
\]
for every \( \phi_1, \phi_2 \in \mathcal{L}(\mathcal{X}) \). Furthermore, the value function is the unique fixed point of equation \( V^* = TV^* \); see Bertsekas (2007, p.408).

Let \( \pi \) be a probability measure on \( \mathbb{R}^d \). Define
\[
\mathcal{L}_\pi^2 = \left\{ \phi : \mathbb{R}^d \rightarrow \mathbb{R} \text{ such that } \|\phi\|_{\pi} < \infty \right\}
\]
where
\[
\|\phi\|_{\pi} = \left( \int_{x \in \mathbb{R}^d} \phi^2(x) \pi(dx) \right)^{1/2}.
\]
Suppose that \( \mathcal{C} \) is the set of all convex functions over \( \mathbb{R}^d \) which are measurable with respect to \( \pi \). Note that \( \mathcal{C} \) is a closed convex cone over the space of functions \( \mathcal{L}_\pi^2 \); see Lim and Glynn (2012). The projection operator onto the cone \( \mathcal{C} \) with respect to the measure \( \pi \), represented by \( \Pi_{\mathcal{C}} \), is defined as
\[
\Pi_{\mathcal{C}}(f) = \arg \min_{\phi \in \mathcal{C}} \|f - \phi\|_{\pi}.
\]
The projection of \( f \) onto the convex cone \( \mathcal{C} \), denoted by \( \bar{\phi} = \Pi_{\mathcal{C}}(f) \), can be characterized by
\[
\langle f - \bar{\phi}, \phi - \bar{\phi} \rangle_{\pi} \leq 0
\]
for every \( \phi \in \mathcal{C} \).

3 Convex Value Exploration

In this section, we suggest two different methods to approximate the value function for a given fixed policy by incorporating the shape constraints. Here, we first focus on the convexity as a shape constraint. Next, we extend our methods to monotonicity and Lipschitz constraints. In the first
method, we estimate the value function by explicitly incorporating the shape constraints. In the second one, we improve the estimator by incorporating the shape constraint and simultaneously taking advantage of the fact that the function satisfies a specific linear system of equations. In the subsequent sections, we discuss the convergence of these methods.

3.1 Truncated Method

Let $X = (X_t: t \geq 0)$ be the underlying Markov process. Consider a single sample path of $X$. The total discounted rewards over this sample path rise to noisy observations of the value function at these sample points. Fitting a convex function to these observations gives an estimation of the value function. Since we truncate the infinite horizon discounted reward stream to get the noisy observation at each sample point, we call this method the truncated method.

Let $X_1, X_2, \ldots, X_{2N}$ be a sample path of the Markov chain with length $2N$. Also, assume that $R_1, \ldots, R_{2N}$ is the sequence of corresponding rewards at sample points $R_i = r(X_i)$ for $1 \leq i \leq 2N$. A noisy observation of $V^*(x_i)$ is thus given by

$$Y_{i}^{N} = \sum_{j=1}^{2N} e^{-(j-i)\alpha} R_j$$

for $1 \leq i \leq N$. Observe that $Y_{i}^{N} = V^*(X_i) + \epsilon_i$, where $\mathbb{E}[\epsilon_i|X_i]$ is close to zero if the number of sample points $N$ is sufficiently large. We can construct an estimator of $V^*(x)$ by projecting the noisy observations onto the cone of convex functions. The projection is possible by fitting a convex function to the points $(X_1, Y_1^{N}), \ldots, (X_N, Y_N^{N})$. Assuming $V^*(x)$ is a convex function, we use the least squares estimator (LSE) to project the noisy observations onto the cone of convex functions by solving

$$\min_{\phi \in \mathcal{C}} \sum_{i=1}^{N} (Y_i^{N} - \phi(X_i))^2. \quad (2)$$

Since $\mathcal{C}$ is an infinite-dimensional space, this minimization may appear to be computationally intractable. However, it turns out that this minimization can be formulated as a finite-dimensional
quadratic program (QP):

\[
\min_{p_i, \zeta_i} \sum_{i=1}^{N} (Y_i - p_i)^2 \\
p_i \geq p_j + \zeta_j^T (X_i - X_j) \text{ for every } 1 \leq i, j \leq N.
\] (3)

In Lim and Glynn (2012), it is shown that this least square problem has a minimizer \((p_1, \zeta_1), \ldots, (p_n, \zeta_n)\), and any minimizer \(\phi_N\) of (2) over \(C\) satisfies \(\phi_N(X_i) = p_i\). We defer more discussion of solving this optimization problem more efficiently to Chapter 5. We define our estimator \(V_N(x)\) as

\[
V_N(x) = \sup \left\{ \phi(x) : \phi \in C, \phi(X_i) = p_i, i = 1, \ldots, N \right\}.
\]

The function \(V_N\) is a convex and finite value function over the convex hull of the points \((X_1, \ldots, X_N)\). Furthermore, it is straightforward to show that \(V_N\) is a piecewise linear convex function given by

\[
V_N(X) = \max_{1 \leq i \leq N} (p_i + \zeta_i^T (X - X_i)).
\] (4)

In the next section, we will show that as the sample size \(N \to \infty\), the estimator \(V_N\) converges uniformly to \(V^*\) over every compact set.

### 3.2 Fixed Point Projection

In this section, we improve the previous method by taking advantage of the fixed point property of the value function in addition to the shape constraint. The rationale of the method is to iteratively apply the Bellman operator \(T\) and project to the cone of convex functions. This method provides an approximation of the value function as the fixed point of the operator \(T\). First, we start with the ideal case, in which we can exactly compute the expectation with respect to the stationary distribution as well as the projection to the space of convex functions. Next, we explain a numerical algorithm that approximately follows this ideal iteration procedure.

Here, we assume the value function belongs to the space of measurable functions \(L^2_\mu\) and is convex. In the next section, we study the behavior of the estimator in the general case where the value function is not convex. The value function is the fixed point of the operator \(T\), so \(V^* = TV^*\). Moreover, by the convexity assumption, \(V^*\) is a fixed point of the projection operator onto the cone of convex functions, and we have \(V^* = \Pi_C V^*\). Therefore, \(V^*\) is the fixed point of the combination
of the operators $T$ and $\Pi_C$ and satisfies

$$V^* = \Pi_C TV^*.$$ 

In the next theorem, we show the existence of such a fixed point as a result of contraction of both operators $T$ and $\Pi_C$.

**Theorem 3.1.** Let $\pi$ be the stationary distribution of the Markov chain $X$, and $r \in L^2_\pi$. Then there exists a unique fixed point $V \in L^2_\pi$ such that

$$V = \Pi_C TV.$$ 

Moreover, let $V = (V_k : k \geq 0)$ be a sequence of functions in the convex closed cone $C \subset L^2_\pi$, defined by

$$V_{k+1} = \Pi_C TV_k.$$ 

Then, we have

$$\|V_k - V\|_\pi \leq e^{-k\alpha}\|V_0 - V\|_\pi.$$ 

**Remark 3.2.** The sequence $V = (V_k : k \geq 0)$ generated in (5) does not converge for an arbitrary norm $\|\cdot\|_\pi$. The assumption that $\pi$ is the stationary distribution of the underlying Markov chain $X$ is essential to guarantee the convergence of the sequence. For instance, the projection with respect to the sup-norm is not contraction (see Example [B.1]). For a similar discussion in the context of parametric ADP, see Tsitsiklis and Van Roy (2001).

**Proof.** First, we show that if $\pi$ is the stationary distribution of the Markov chain, then the operator $T$ is a contraction with respect to the norm $\|\cdot\|_\pi$. For any two functions $\phi_1, \phi_2 \in L^2_\pi$, we have

$$E_\pi(T\phi_1 - T\phi_2)^2 = E_\pi \left( E[r(X_t) + e^{-\alpha}\phi_1(X_{t+1})|X_t] \right. 
- E[r(X_t) + e^{-\alpha}\phi_2(X_{t+1})|X_t] \bigg)^2 
\leq e^{-2\alpha}E_\pi \left( E[\phi_1(X_{t+1}) - \phi_2(X_{t+1})|X_t] \right)^2 
\leq e^{-2\alpha} \|\phi_1 - \phi_2\|_\pi^2.
Moreover, we know that $\Pi_C$, the projection operator onto the convex cone $C$, is also a contraction with respect to $\| \cdot \|_\pi$ norm; see P.26 Borwein and Lewis (2005). More precisely, if $\phi_1, \phi_2 \in L^2_\pi(X)$, then we have

$$\|\Pi_C \phi_1 - \Pi_C \phi_2\|_\pi \leq \|\phi_1 - \phi_2\|_\pi.$$ 

Note that

$$E_\pi \left( r(X_t) + e^{-\alpha} E[\phi(X_{t+1}) | X_t] \right)^2 \leq 2E_\pi r(X_t)^2 + 2e^{-2\alpha} E_\pi \phi(X_{t+1})^2 < \infty.$$ 

Thus, $T \phi \in L^2_\pi$ for every $\phi \in L^2_\pi$. Therefore,

$$\|\Pi_C T \phi_1 - \Pi_C T \phi_2\|_\pi \leq e^{-\alpha} \|\phi_1 - \phi_2\|_\pi.$$ 

The rest of the theorem follows directly from the Banach fixed point theorem. \hfill \Box

In the rest of this section, we develop a computational method to approximate the fixed point over the cone $C$ by using simulated trajectories. Exact computation of $TV$ is not generally viable. Evaluating $TV(x)$ at any $x \in X$ involves the computation of the expectation $E[V(X_{t+1}) | X_t = x]$. This expectation is over a potentially high-dimensional or infinite-dimensional space and hence can pose a computational challenge. The following proposition provides an equivalent characterization to the operator $\Pi_C T$. As a result of this proposition, it suffices to evaluate $V(\cdot)$ at two sample points rather than computing $E[V(X_{t+1}) | X_t = x]$.

**Proposition 3.3.** Let $X_{t+1}$ and $\tilde{X}_{t+1}$ be two independent samples of an M.C. at time $t + 1$ given $X_t$. Moreover, define the random variable $H_t$ such that

$$H_t = r(X_t) + \frac{e^{-\alpha}}{2} \left( V(X_{t+1}) + V(\tilde{X}_{t+1}) \right). \quad (6)$$

For every measurable function $V \in L^2_\pi$, we have

$$\Pi_C TV = \arg \min_{\phi \in C} E_\pi \left( H_t - \phi(X_t) \right)^2.$$ 

Therefore, $\Pi_C TV = \Pi_C H$.

**Proof.** Let $\bar{\phi}$ be the projection of $TV$ onto the cone of convex functions $C$, which is the minimizer
of
\[
\min_{\phi \in \mathcal{C}} \mathbb{E}_\pi (TV - \phi)^2.
\]

By using the independence of \(X_{t+1}\) and \(\tilde{X}_{t+1}\) given \(X_t\), we obtain

\[
\begin{align*}
\mathbb{E}_\pi (TV - \phi)^2 &= \mathbb{E}_\pi \left( E[r(X_t) + e^{-\alpha}V(X_{t+1})|X_t] - \phi(X_t) \right)^2 \\
&= \mathbb{E}_\pi \left( E[r(X_t) + e^{-\alpha}V(X_{t+1}) - \phi(X_t)|X_t] \right) \\
&\quad \times \left( E[r(X_t) + e^{-\alpha}V(\tilde{X}_{t+1}) - \phi(X_t)|X_t] \right) \\
&= \mathbb{E}_\pi \left( r(X_t) + e^{-\alpha}V(X_{t+1}) - \phi(X_t) \right) \\
&\quad \times \left( r(X_t) + e^{-\alpha}V(\tilde{X}_{t+1}) - \phi(X_t) \right) \\
&= \mathbb{E}_\pi \left[ \left( r(X_t) + e^{-\alpha}V(X_{t+1}) - \phi(X_t) \right)^2 \\
&\quad - \frac{e^{-2\alpha}}{4} \left( V(\tilde{X}_{t+1}) - V(X_{t+1}) \right)^2 \right]
\end{align*}
\]

for every function \(\phi \in \mathcal{L}^2_\pi\). Therefore, we can conclude that \(\tilde{\phi}\) is also the minimizer of the optimization

\[
\tilde{\phi} = \arg \min_{\phi \in \mathcal{C}} \mathbb{E}_\pi \left[ r(X_t) + \frac{e^{-\alpha}}{2} \left( V(X_{t+1}) + V(\tilde{X}_{t+1}) \right) - \phi(X_t) \right]^2.
\]

\(\square\)

By using the ergodic property of the Markov chains, it is straightforward to calculate an estimator of \(\mathbb{E}_\pi (H_t - \phi_t)^2\). At each time step \(t = 1, \ldots, N\), we generate two independent copies \(X_{t+1}\) and \(\tilde{X}_{t+1}\) given \(X_t\). We call \((X_1, X_2, \tilde{X}_2, \ldots, X_{N+1}, \tilde{X}_{N+1})\) a “two copy sample path”.

Figure 1: A two copy sample path of length 4
Under appropriate conditions over the process $X$, we have

$$
\frac{1}{N} \sum_{t=1}^{N} (H_t - \phi(X_t))^2 \to \mathbb{E}_\pi (H_t - \phi(X_t))^2
$$
as $N \to \infty$.

Here, we discuss a potential but unsuccessful Monte Carlo approach to approximate $V^*$. By the convexity assumption, the value function is the fixed point of $V^* = \Pi_C TV^*$, and therefore is the minimizer of the optimization problem

$$
\min_{\phi \in C} \|\phi - \Pi_C T \phi\|_\pi^2.
$$

Similar to Proposition 3.3, it is possible to show that the fixed point $V^*$ is also the minimizer of

$$
\min_{\phi \in C} \mathbb{E}_\pi \left[ (r(X_t) + e^{-\alpha} \phi(X_{t+1}) - \phi(X_t)) \left( r(X_t) + e^{-\alpha} \phi(X_{t+1}) - \phi(X_t) \right) \right].
$$

One might solve the optimization problem

$$
\min_{\phi \in C} \frac{1}{N} \sum_{t=1}^{N} \left( r(X_t) + e^{-\alpha} \phi(X_{t+1}) - \phi(X_t) \right) \left( r(X_t) + e^{-\alpha} \phi(X_{t+1}) - \phi(X_t) \right).
$$

(9)

However, it can be easily shown that this optimization problem is non-convex and unbounded for any finite sample path of length $N$; see Example B.2. We can solve this difficulty by employing an iterative projection procedure. Before discussing this method, we impose an additional shape constraint to bound the value function. This assumption helps to restrict the cone of convex functions and make the projection more tractable.

**Assumption 3.4.** Let the state space $X$ be bounded. Moreover, assume that for every $x \in X$, the sub-gradient of $V^*$ is bounded by a constant $K$:

$$
\|\nabla V(x)\|_\infty < K,
$$

and $V(0) > -K$.

**Example 3.5.** Suppose that there exists a constant $K$ such that for every state $x \in X$ we have

$$
\left| r(x) - E[r(X_1)|X_0 = x] \right| \leq K.
$$
It is straightforward to show that
\[
|V^*(x) - \frac{r(x)}{1 - e^{-\alpha}}| \leq \frac{K}{(1 - e^{-\alpha})^2}.
\]

Therefore, if the reward function is bounded over the state space, then Assumption (3.4) holds.

Now, we present an alternative method to estimate the value function by using convexity and the fixed point property. The method is similar to the ideal procedure in Theorem 3.1. The main difference is using the random vector \( \hat{H}^k = (\hat{H}_1^k, \ldots, \hat{H}_N^k) \) for a piecewise linear function \( \hat{V}_k(\cdot) \) instead of \( T\hat{V}_k \). We first generate a two copy sample path of length \( N \). This sample path does not change throughout the procedure. We iteratively compute the random vector \( \hat{H}^k = (\hat{H}_1^k, \ldots, \hat{H}_N^k) \) for a piecewise linear function \( \hat{V}(\cdot) \). Next, we project \( \hat{H}^k \) onto the convex cone \( C \) to achieve \( \hat{V}_{k+1} \).

Each convex projection is a least square finite-dimensional optimization problem. By following this procedure iteratively, an estimation of the fixed point over the cone \( C \) is obtained. The details of the method are as follows:

Algorithm Fixed Point Projection

**Input:** \( N \) and \( \epsilon \)

**Output:** The estimator of the value function \( \hat{V}_k \).

**Initialize:** Select a piecewise-linear function \( V_0(x) \), and set \( k = 0 \), \( V_{-1}(x) = 0 \).

**Generating Sample Path:** Generate a “two copy sample path” of length \( N + 1 \).

while \( \| \hat{V}_k - \hat{V}_{k-1} \|_N > \epsilon \) do
  i.) Compute \( (H_t^k)_{t=1}^N \) from (10).
  ii.) Project \( (H_t^k)_{t=1}^N \) by solving the optimization problem (11), and find \( (p_i, \zeta_i) \) for \( i = 1, \ldots, N \).
  iii.) Update \( \hat{V}_{k+1}(\cdot) \) thorough (12), and \( k \leftarrow k + 1 \).
end while

return The piecewise-linear function \( \hat{V}_k \).

**Generating Sample Path:** Generate a “two copy sample path” of length \( N + 1 \). At each time step \( t = 1, \ldots, N \), generate two independent copies \( X_{t+1} \) and \( \tilde{X}_{t+1} \) given \( X_t \).

**Updating Step:** Evaluate

\[
\hat{H}_t^k = r(X_t) + \frac{e^{-\alpha}}{2}(\hat{V}_k(X_{t+1}) + \hat{V}_k(\tilde{X}_{t+1}))
\] (10)

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for every $t = 1, \ldots, N$. The sequence $\hat{H}^K = (\hat{H}_1^k, \ldots, \hat{H}_N^k)$ is a noisy observation of $TV_k(X_t)$.

**Projection:** Project $\hat{H}^k$ onto the cone of convex functions by solving the finite-dimensional convex program

$$
\min \frac{1}{N} \sum_{t=0}^{N-1} (\hat{H}_t^k - p_t)^2
$$

$$
p_i \geq p_j + \zeta_j^T (X_i - X_j) \quad \text{for every } 1 \leq i, j \leq N
$$

$$
-K \leq \zeta_j^l \leq K \quad \text{for every } i = 1, \ldots, N, \text{ and } l = 1, \ldots, d
$$

$$
-K \leq p_j + \zeta_j^T X_i.
$$

Given the optimal solution $(p_t, \zeta_t, X_t)_{t=1}^N$ to this optimization, we can construct a piecewise linear convex function. Define

$$
\hat{V}_{k+1}(x) = \max_{0 \leq i \leq N} (p_i + \zeta_i^T (x - X_i)).
$$

The updating and projection stages for a fixed “two copy sample path” should be continued until a desired level of accuracy is reached. We can consider $\hat{V}_k(x)$ as an estimator for the value function. In the next section, we will show that for sufficiently large sample size $N$ and a large number of iterations $k$, the estimator $\hat{V}_k(x)$ converges uniformly to the value function $V^*(x)$ over every compact set.

## 4 Empirical Projection Consistency

In this section we describe a generalization of the consistency result of convex regression in Lim and Glynn (2012) to the positive Harris chains. Our result includes the model mis-specification case without any extra assumption to bound the function. In the next section, we use this result to show that the estimators in truncated method and fixed point projection method converge to the value function as the sample size grows to infinity.

Let $X = (X_t : t \geq 1)$ be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $\mathcal{F}$-measurable random variable $Y$, we can define the projection onto the cone $\mathcal{C}$ with respect to the norm $\pi$ as the solution of

$$
\min_{\phi \in \mathcal{C}} \mathbb{E}_\pi (Y - \phi(X))^2.
$$

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Let \(Y^N_i: 1 \leq i \leq N, 1 \leq N\) be a sequence of random vectors in which \(Y^N = (Y^N_1, \ldots, Y^N_N)\) for every \(N \geq 1\). We show that if \((Y^N: N \geq 1)\) converges on average to \(Y\), then the empirical projection of this sequence onto the cone of convex functions gives a consistent estimate of projecting \(Y\) onto this cone. For ease of exposition, we define a sequence of random vectors as strongly ergodic in the following way:

**Definition 4.1.** Suppose that \(X = (X_t: t \geq 1)\) is a positive Harris chain with stationary distribution \(\pi\), and \((Y^N_t: 1 \leq N, 1 \leq t \leq N)\) be a sequence of random variables. We call this sequence “strongly ergodic” if there exists a \(\mathcal{F}\)-measurable random variable \(Y\) such that \(\|
abla Y\|^2 < \infty\),

\[
\frac{1}{N} \sum_{t=1}^{N} (Y^N_t - g(X_t))^2 I(\|X_t\| \leq c) \rightarrow E_{\pi}(Y - g(X_t))^2 I(\|X\| \leq c) \quad \text{a.s.}
\]

for every function \(g \in L^2_{\pi}\), and \(c \leq \infty\).

To illustrate this definition, we provide several examples.

**Example 4.2.** Let the \(Y^N_t = f(X_t) + \nu_t\) be such that \(\nu_t\) is a sequence of i.i.d noise terms with respect to \(X\) such that \(E\nu_t^2 < \infty\), and \(f\) is a convex function in \(C\). Then by the strong law of large numbers, we obtain that \((Y^N: N \geq 1)\) is “strongly ergodic”.

**Example 4.3.** In Lemma A.2, we show that if \(E_{\pi}r(X_t)^2 < \infty\), then the two following random sequences are “strongly ergodic”:

\[
Y^N_t = \sum_{j=t}^{\infty} e^{-(j-t)\alpha} r(X_j),
\]

\[
\overline{Y}^N_t = \sum_{j=t}^{2N} e^{-(j-t)\alpha} r(X_j).
\]

**Example 4.4.** Assume that \((X_t, \tilde{X}_t)\) is a “two copy sample path” of a Harris recurrent chain. Let \(V \in L^2_{\pi}\) and as we defined in (10),

\[
H^N_t = r(X_t) + \frac{1}{2}(V(X_{t+1}) + V(\tilde{X}_{t+1})).
\]

Then, \((H^N: N \geq 1)\) is a “strongly ergodic” sequence; see Lemma A.3.
Let $g_N$ be the optimizer of the convex optimization problem

$$\min_{g \in C} \frac{1}{N} \sum_{t=1}^{N} (Y_t^N - g(X_t)^2)$$

for $N \geq 1$. Note that similar to (11), we can convert this optimization problem to a finite quadratic convex problem. In the following theorem, we show that $g_N$ is an estimator for $g^*$, the projection of $Y$ onto the space of convex functions.

We need some assumptions over the structure of the Markov chain.

**Assumption 4.5.** For the Markov chain $X$, we have:

i.) It is positive Harris recurrent with unique stationary distribution $\pi$.

ii.) $\pi(B) > 0$ for every positive radius ball $B$ which is a subset of state space $X$.

iii.) $E_\pi X_t^2 < \infty$.

**Theorem 4.6.** Assume that $(Y^N : N \geq 1)$ is “strongly ergodic”, and Assumption (4.5) holds. Let $g_N$ be the solution of (13) and $g^* \in C$ be the unique minimizer of

$$\min_{g \in C} \mathbb{E}_\pi (Y - g(X))^2.$$ 

Then,

$$\frac{1}{N} \sum_{t=1}^{N} (g_N(X_t) - g^*(X_t))^2 \to 0 \quad \text{a.s.}$$

as $N \to \infty$. Moreover,

$$\sup_{\|x\| \leq c} |\hat{g}_n(x) - g^*(x)| \to 0 \quad \text{a.s.}$$

as $N \to \infty$, for every $c > 0$.

The proof follows the same steps as the convergence proof in Lim and Glynn (2012). The main difference is the use of the ergodic property of Harris chains instead of the strong law of large number for i.i.d random variables. Moreover, we continue to allow the model mis-specification in which $f(X) = E(Y|X)$ is not a convex function.

We first start by showing the consistency of the projection onto the compact disk $H_c = \{X : \|X\| \leq c\}$ for every $c > 0$. Then, by expanding this projection over the whole space, we conclude the theorem.
For every $c > 0$, define $C_c$ as the set of all functions $g \in L^2_\pi$ such that $g$ is a convex function over the disc $\{X : \|X\| \leq c\}$. Similar to Proposition 3 in Lim and Glynn (2012), we can show that $C_c$ is a closed subset of $g \in L^2_\pi$. Therefore, there exists a unique function $g^*_c \in C_c$ which is the projection of $Y$ onto $C_c$.

$$\min_{g \in C_c} \mathbb{E}_\pi(Y - g)^2$$

It is clear that $g_c(x) = E[Y | X = x]$ for almost every $x \notin H_c$. In Lemma A.1, we show $g^*_c$ converges to $g^*$ as $c$ goes to infinity.

**Proof of Theorem 4.6.** Similar to the steps 1, 2, and 3 in Lim and Glynn (2012), we have

$$\frac{1}{N} \sum_{i=1}^{N} (g_N(X_i) - g^*(X_i))^2 \leq \frac{2}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i)), \quad (14)$$

and for sufficiently large $N$

$$\frac{1}{N} \sum_{i=1}^{N} (g_N(X_i))^2 \leq \frac{8}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))^2 + \frac{2}{N} \sum_{i=1}^{N} (g^*(X_i))^2$$

$$\leq 9\mathbb{E}_\pi(Y - g^*(X))^2 + 3\mathbb{E}_\pi g^*(X)^2 = \beta. \quad (15)$$

We conclude the last inequality from the “strongly ergodic” property of $(Y^N : N \geq 1)$. By the Cauchy–Schwarz inequality, the tail of the empirical inner product can be uniformly bounded for every $c > 0$ and sufficiently large $N$. Observe that

$$\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X_i\| > c)$$

$$\leq \left(\frac{1}{N} \sum_{i=1}^{N} (g_N(X_i) - g^*(X_i))^2\right)^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))^2I(\|X_i\| > c)\right)^{1/2}$$

$$\leq (\beta + 2\mathbb{E}_\pi g^*(X))^2)^{1/2} \left(\mathbb{E}_\pi(Y - g^*(X))I(\|X\| > c)\right)^{1/2}. \quad (16)$$

In the last line, we used the “strongly ergodic” assumption and the triangle inequality. Since $\mathbb{E}_\pi Y^2 < \infty$ and $\mathbb{E}_\pi g^*(X)^2 < \infty$, the right hand side can be smaller than any $\epsilon > 0$ for large enough $c$. Thus,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X_i\| > c) \leq \epsilon$$

for sufficiently large $c$. Therefore, the terms in (14), which correspond to the samples outside the
disk $\mathcal{H}_c$ can be made arbitrarily small. In the next lemma, we show that $g_N$ converges to $g^*_c$ inside the disk.

**Lemma 4.7.** Then for every $c > 0$,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i))(g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c) \leq 0.$$  

See the Appendix for the proof. This lemma ensures that there exists a sequence $\delta_N$ converging to zero such that

$$\delta_N \geq \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i))(g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c)$$

$$= \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) \left( (g_N(X_i) - g_c^*(X_i)) + (g^*(X_i) - g_c^*(X_i)) \right) I(\|X_i\| < c)$$

$$= \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i))(g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c)$$

$$+ \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i))(g^*(X_i) - g_c^*(X_i)) I(\|X_i\| < c)$$

$$+ \frac{1}{N} \sum_{i=1}^{N} (g^*(X_i) - g_c^*(X_i))(g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c)$$

$$+ \frac{1}{N} \sum_{i=1}^{N} (g^*(X_i) - g_c^*(X_i))^2 I(\|X_i\| < c).$$

We can get a lower bound for (15) by the Cauchy–Schwarz inequality.

$$\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i))(g^*(X_i) - g_c^*(X_i)) I(\|X_i\| < c) \geq$$

$$-2(E_\pi(Y - g^*(X))^2)^{1/2}(E_\pi(g^*(X) - g_c^*(X))^2 I(\|X\| < c))^{1/2}$$

for sufficiently large $N$. Similarly, we can get a lower bound for (19) by using (15) and Cauchy-Schwarz inequality

$$\frac{1}{N} \sum_{i=1}^{N} (g^*(X_i) - g_c^*(X_i))(g_N(X_i) - g^*(X_i)) I(\|X_i\| < c)$$

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≥ - \left( \left( \frac{1}{N} \sum_{i=1}^{N} (g_N(X_i))^2 \right) + \left( \sum_{i=1}^{N} (g^*(X_i))^2 \right) \right)^{1/2} \\
\times \left( \frac{1}{N} \sum_{i=1}^{N} (g^*(X_i) - g^*_c(X_i))^2 I(\|X_i\| < c) \right)^{1/2} \\
≥ -2\left( \beta + E_\pi(g^*(X))^2 \right)^{1/2} \left( E_\pi(g^*(X) - g^*_c(X))^2 I(\|X\| < c) \right)^{1/2}

By combining these inequalities and using Lemma 4.7, we obtain

\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X_i\| < c)

≤ \delta(N) + \beta_1 \min\{(E_\pi(g^*(X) - g^*_c(X))^2 I(\|X\| < c))^{1/2}, 1\}. 

According to Lemma A.1, there exists a sufficiently large c for every \( \epsilon > 0 \) such that

\[ E_\pi(g^*(X) - g^*_c(X))^2 I(\|X\| < c) \leq \epsilon. \]

Therefore, there exists a sufficiently large c for any \( \epsilon > 0 \) such that

\[ \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X_i\| < c) \leq \epsilon. \]

Now, we can use (14) and (16) to conclude the theorem.

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (g_N(X_i) - g^*(X_i))^2 \]

\[ \leq \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X\| < c) \]

\[ + \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g^*(X_i))(g_N(X_i) - g^*(X_i))I(\|X\| > c) \leq 2\epsilon. \]

The second part of the theorem is similar to Step 8 in Lim and Glynn (2012).
5 Convergence of the Value Function Estimator

In this section we show that the estimators given by the truncated method and the fixed point projection method converge to the value function as the sample size grows to infinity. The convergence of the truncated method holds for a general setting. However, we show the convergence of the estimator given by the fixed point projection method under Assumption (3.4) over the value function.

**Theorem 5.1.** Let

\[ Y^N_t = \sum_{j=t}^{2N} e^{-(i-t)\alpha} R_j, \]

and \( V_N(x) \) be the estimator of truncated method defined by (4). Assume that (4.5) holds. Then, we have

\[ \sup_{\|x\|<c} \|\Pi_C V^*(x) - V_N(x)\| \to 0, \]

where \( V_N \) is computed from (3).

**Proof.** According to Lemma (A.2), \( Y^N_t \) is strongly ergodic. Therefore, the result is a direct conclusion of Theorem 4.6.

Now, we are ready to prove the convergence result of the convex iterative projection method. Let

\[ \hat{C}_f = \{ \phi \in C : \|\nabla \phi(x)\|_\infty \leq K \text{ for every } x \in \mathcal{X}, \phi(0) \geq -K \}. \]

In the next theorem, we show that the the estimator (12) converges to the value function if the value function belongs to \( \hat{C}_f \). In the case that the value function is not convex, the estimators converge to the fixed point of \( \bar{V} = \Pi_C T \bar{V} \). The existence of this fixed point is shown in Theorem 3.1.

**Theorem 5.2.** Consider a “two copy sample path” \( X_0, (X_1, \tilde{X}_1), \ldots, (X_N, \tilde{X}_N) \), and assume that (4.4) holds. Let \( \hat{V}_k : k \geq 1 \) be a sequence of convex functions generated by (10, 11, 12), and \( \|\hat{V}_0\|_\infty < \infty \). Let

\[ \bar{V} = \Pi_C (T \bar{V}). \]
If $V \in \tilde{C}_f$, then there exists a sequence $\beta_N$ converging to zero such that
\[
\frac{1}{N} \sum_{t=0}^{N-1} (\tilde{V}_k(X_t) - V(X_t))^2 \leq e^{-k\alpha} \gamma_N + \beta_N
\]
for sufficiently large $N$, where $\gamma_N = \frac{1}{N} \sum_{t=0}^{N-1} (\tilde{V}_0(X_t) - V^*(X_t))^2$.

**Proof.** First, we give some motivation for the definition of $H_t$ in (10). Next, by using the contraction property of projection, we show that the distance between $\tilde{V}_k$ and $V$ is approximately shrinking by a factor of $e^{-\alpha}$ at each stage of the iteration.

According to the fixed point assumption, $\bar{V}$ is the minimizer of
\[
\min_{\phi \in C} \mathbb{E}_\pi (T \bar{V} - \phi)^2.
\]
Therefore, due to Proposition 3.3 we can conclude that $\bar{V}$ is also the minimizer of the optimization
\[
\bar{V} = \arg \min_{\phi \in C} \mathbb{E}_\pi (r(X_t) + \frac{e^{-\alpha}}{2} (\bar{V}(X_{t+1}) + \bar{V}(\tilde{X}_{t+1})) - \phi(X_t))^2.
\]

Observe that $\tilde{C}_f$ is a closed convex subset of $C$. Let
\[
\tilde{V}_N = \arg \min_{\phi \in \tilde{C}_f} \frac{1}{N} \sum_{t=0}^{N-1} (\tilde{H}_t - \phi(X_t))^2,
\]
where
\[
\tilde{H}_t = r(X_t) + \frac{e^{-\alpha}}{2} (\bar{V}(X_{t+1}) + \bar{V}(\tilde{X}_{t+1})).
\]
Suppose $\tilde{\pi}_N$ is the empirical semi-norm induced by sample path $X_0, X_1, \ldots, X_N$. The distance between the two functions $f, g$ under $\tilde{\pi}_N$ is
\[
\|f - g\|_{\tilde{\pi}_N} = \left( \frac{1}{N} \sum_{t=0}^{N-1} (f(X_t) - g(X_t))^2 \right)^{1/2}.
\]
The following lemma asserts that the empirical norm asymptotically converges to $L^2_\pi$ norm over $\tilde{C}_f$. 

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Lemma 5.3. We have
\[
\sup_{\phi_1, \phi_2 \in \tilde{C}_f} \left| \frac{1}{N} \sum_{t=0}^{N-1} (\phi_1(X_t) - \phi_2(X_t))^2 - \mathbb{E}_\pi(\phi_1(X_t) - \phi_2(X_t))^2 \right| \to 0
\] (22)
as \(N\) goes to infinity.

See the Appendix for the proof.

Note that \(\tilde{V}\) and \(V_{k+1}\) are the projection of \((\tilde{H}_t)\) and \((H_k^t)\) onto the convex set \(\tilde{C}_f\) with respect to the semi-norm \(\tilde{\pi}\). Since the projection to the convex set is a contraction, we obtain
\[
\|\hat{V}_{k+1} - \tilde{V}_N\|_{\tilde{\pi}_N}^2 \leq \|H_k - \tilde{H}\|_{\tilde{\pi}_N}
\leq e^{-\alpha/2} \|V_k(X_{t+1}) - \nabla(X_{t+1})\|_{\tilde{\pi}_N}
+ e^{-\alpha/2} \|V_k(\tilde{X}_{t+1}) - \nabla(\tilde{X}_{t+1})\|_{\tilde{\pi}_N}.
\]

By Lemma 5.3, we can bound the right hand side by its expectation and an error term less than \(\delta(N)\) in which \(\delta(N) \downarrow 0\) as \(N \to \infty\). Thus,
\[
\|\hat{V}_{k+1} - \tilde{V}_N\|_{\tilde{\pi}_N} \leq e^{-\alpha/2} \left( \mathbb{E}(V_k(X_{t+1}) - \nabla(X_{t+1}))^2 \right)^{1/2}
+ e^{-\alpha/2} \left( \mathbb{E}(V_k(\tilde{X}_{t+1}) - \nabla(\tilde{X}_{t+1}))^2 \right)^{1/2} + \delta_N
= e^{-\alpha} \left( \mathbb{E}(V_k(X_t) - \nabla(X_t))^2 \right)^{1/2} + \delta_N
\leq e^{-\alpha} \|V_k - \nabla\|_{\pi} + 2\delta_N.
\]

By the triangle inequality, we obtain
\[
\|\hat{V}_{k+1} - V^*\|_{\tilde{\pi}_N} \leq \|\hat{V}_{k+1} - \tilde{V}_N\|_{\tilde{\pi}_N} + \|\tilde{V}_N - \nabla\|_{\tilde{\pi}_N}
\leq e^{-\alpha} \|V_k - \nabla\|_{\pi} + 2\delta_N.
\]

From the last inequality, it can be inductively observed that
\[
\|\hat{V}_k - \nabla\|_{\tilde{\pi}_N} \leq e^{-\alpha/2} \|\hat{V}_0 - \nabla\|_{\tilde{\pi}_N} + \frac{1 - e^{-(k+1)\alpha}}{1 - e^{-\alpha}} (\|\hat{V}_N - \nabla\|_{\pi} + \delta_N)
\lim_{k \to \infty} \|\hat{V}_k - \nabla\|_{\pi} \leq \beta_N = \frac{1}{1 - e^{-\alpha}} (\|\hat{V}_N - \nabla\|_{\pi} + \delta_N).
\]

As a result of Theorem 4.6, we can show that \(\|\hat{V}_N - \nabla\|_{\tilde{\pi}_N}\) converges to zero as \(N \to \infty\); see Lemma
for more details. Therefore, $\beta_N$ is converging to zero.

6 Extensions

Here, we consider two extensions to estimate the value functions by exploiting other shape structures. First, we consider the case in which the value function is Lipschitz for a known constant $K$. The second extension employs the property that the value function is non-decreasing and convex. The main difference between exploiting different shape properties is the projection/regression step. In both truncated and fixed point projection methods, we can replace the projection onto the set of convex functions with projection onto the set of Lipschitz or non-decreasing and convex functions. The other steps of the methods are similar.

Lipschitz Assume that we know the value function $V^*$ is Lipschitz for a known constant $K$. We exploit this property to estimate the value function. In particular, we assume that the value function belongs to the set

$$\text{Lip}_K = \left\{ \phi : \phi \in \mathcal{L}_\pi, \left| \phi(0) \right| < \tilde{K}, \left| \phi(x) - \phi(y) \right| \leq K \left\| x - y \right\|_2 \text{ for all } x, y \in \mathbb{R}^d \right\}.$$ 

We can easily show that $\text{Lip}_K$ is a closed convex set in the Hilbert space $\mathcal{L}_\pi$. Projection onto the $\text{Lip}_K$ can be defined as

$$\Pi_{\text{Lip}_K}(f) = \arg\min_{\phi \in \text{Lip}_K} \|f - \phi\|_\pi.$$ 

for every $f \in \mathcal{L}_\pi$. Let $(X_1, X_2, \ldots, X_{2N})$ be a sample path of length $2N$. Let $Y^N_i$ be the random variable defined in Equation (1) which represents a noisy observation of the value function at the sample point $X_i$. Similar to the truncated method for convex value functions, the estimator of $V^*$ can be achieved by projecting the random vector $Y^N = (Y^N_1, \ldots, Y^N_N)$ onto the convex set $\text{Lip}_K$. The projection is possible by solving the following QP:

$$\min_{p_i} \sum_{i=1}^{N} (Y^N_i - p_i)^2$$

$$p_i - p_j \leq K \left\| X_i - X_j \right\| \quad \text{for every } 0 \leq i, j \leq N$$

$$-\tilde{K} \leq p_0 \leq \tilde{K},$$

where $X_0 = 0$. 23
Having the optimal solution $p_1, \ldots, p_N$ to this optimization, we can construct an estimator belonging to the set $\text{Lip}_K$. Define

$$V_N(x) = \min_{0 \leq i \leq N} \left( p_i + K \|x - X_i\| \right).$$

Note that the estimator can be evaluated at each point $x$ in linear time. Similar to Theorem 5.1, it is possible to show that the estimator $V_N(x)$ uniformly converges to $V^*(x)$ over every compact set as $N \to \infty$.

Similarly, we can extend the fixed point projection to the Lipschitz case. For a fixed two copy sample path of length $N$, one can find $H^N = (H^N_1, \ldots, H^N_N)$ from [10] and next project the vector $H^N$ to the convex closed set $\text{Lip}_K$ by solving a similar QP to (23).

**Convex and Monotonic.** As another extension, we consider the case that the value function is both convex and non-decreasing. There is a variety of Markov decision problems in the queue admission, batch service, marketing, and aging and replacement settings, where the value function is monotone. We say that a function $\phi : \mathbb{R}^d \to \mathbb{R}$ is non-decreasing if $\phi(x) \leq \phi(y)$ whenever $x \leq y$ (so that $x_i \leq y_i$ for $1 \leq i \leq d$). We now adjust the definition of the cone of functions $\mathcal{C}$ to

$$\mathcal{CM} = \{ \phi : \phi \in \mathcal{L}_\pi, \text{and} \phi \text{ is a convex and non-decreasing function.} \}.$$

One can easily show that $\mathcal{CM}$ is a closed and convex cone of the Hilbert space $\mathcal{L}_\pi$. Here, we have to project the noisy observations $Y^N$ computed in [11] for truncated method, or $H^N$ computed in [10] for fixed point projection onto the cone $\mathcal{CM}$. The projection is again obtained by solving a QP:

$$\min_{p_i, \zeta_i} \sum_{i=1}^N (Y^N_i - p_i)^2$$

subject to

$$p_i \geq p_j + \zeta_j^T (X_i - X_j) \quad \text{for every} \ 1 \leq i, j \leq N$$

$$\zeta_i \geq 0 \quad \text{for all} \ 1 \leq i \leq N.$$

In addition, the estimator $V_N(x)$ can be evaluated by

$$V_N(x) = \max_{1 \leq i \leq N} (p_i + \zeta_i^T (x - X_i)).$$
Note that for every $x \in \mathbb{R}^d$, we have $\partial V_N(x) = \zeta_i \geq 0$ for some $i$. Therefore, the estimator is convex and non-decreasing.

7 Case Study: Pricing Tolling Contracts

This section considers the problem of scheduling dual-fuel power stations in the presence of switching costs. One of the fundamental problems encountered in the energy markets is the pricing of tolling agreement contracts. By signing a “tolling contract”, power plant owners can reduce their exposure to fuel prices by transferring control of the plant to a third party. This third party is then responsible for any costs, fuel or otherwise, involved in meeting power plant obligations. The complexity of pricing such contracts arises as a result of interplay between limited flexibility and uncertainty.

Consider a renter who has leased a dual-fuel power plant in a de-regulated market. The agent dynamically determines the operating mode of the power plant as the fuel and electricity prices fluctuate. Our goal is to evaluate the expected total profit for given fixed scheduling policies.

Note that by using the policy iteration method, it is straightforward to update the policy iteratively and achieve the optimal scheduling policies as well.

This specific pricing/control problem is widely considered to be a challenging control problem. In mathematical finance literature, several authors, including Dixit (1989); Brekke and Oksendal (1994); Johnson and Zervos (2010), have focused on obtaining closed-form solutions by making simplified assumptions. The problem also is also studied by Deng and Xia (2006); Carmona and Ludkovski (2008); Djehiche et al. (2009); Bardou et al. (2008) in parametric ADP literature.

7.1 Modeling

We adopt the model of Carmona and Ludkovski (2008) in our study. Consider a dual-mode power plant that can use either natural gas or oil. Due to increased development of natural gas infrastructure in coastal US regions in recent years, these power plants have become popular. To run the plant, the operator buys natural gas or oil, converts it into electricity and sells the output on the market.

The fluctuation of prices can be modeled by the gas/oil spark-spread. The spark-spread is the difference between the price of electricity (output) and the prices of its primary fuels (inputs). Specifically, let $P_t$ and $G_t$ be the prices of electricity and gas at time $t$. The heat rate, denoted by $H_R G$, is the amount of fuel needed by a power plant to produce one kilowatt-hour (kWh) of
electricity. The gas spark spread is given by

\[ X_t^1 = (P_t - \overline{PR}_G \cdot G_t). \]

Similarly, the oil spark spread is represented by \( X_t^2 = (P_t - \overline{PR}_O \cdot O_t) \), where \( O_t \) is the price of oil at time \( t \); see Eydeland and Wolyniec (2003, p. 49-51) for more details. Empirical studies (Eydeland and Wolyniec (2003)) have suggested that the spark spread is indeed stationary. We model the driving process \( X_t = (X_t^1, X_t^2) \) as a 2-dimensional Ornstein–Uhlenbeck process with jump, namely

\[ dX^n_t = \kappa(\theta - X^n_t)dt + \Sigma^n \cdot dW_t + Y^n dN_t, \quad n = 1, 2, \]

where \( W = (t \geq 0; W_t) \) is a 2-dimensional standard Wiener process, \( N_t \) is an independent Poisson processes with intensity \( \lambda \), \( Y^n \) is an independent exponential random variable, and \( \Sigma \in \mathbb{R}^{2 \times 2} \) is a constant non-degenerate volatility matrix.

The mode of operation at each time step \( t \) is represented by \( s_t \in \{oil, gas\} \). Also, let \( s_{t-1} \) be the operation mode immediately before the starting time. Switching is allowed only at the beginning of each time slot. Moreover, changing the operation mode is costly, requiring extra fuel and various overhead costs. Let \( C^t_{i,j} = c_{i,j}X^n_t \) be the cost of switching from mode \( i \) to mode \( j \) if \( i \neq j \). Clearly, if there is no switching, the switching cost is \( C_{i,i} = 0 \) for \( i \in \{oil, gas\} \).

The profit function \( \psi(X_t, s_t) \) is considered as a linear function of spark spread. For instance, if the plant is fueled by natural gas, we define

\[ \psi(X_t, \text{gas}) \triangleq \overline{Cap}_G \cdot (X_t^1 - K_G), \]

where \( K_G \) is the operating cost and \( \overline{Cap}_G \) is the capacity of the plant in gas mode. Therefore, the value function is

\[ V_u(x^1, x^2, i) = \mathbb{E} \left[ \sum_{t=0}^{\infty} e^{-\alpha t} \left( \psi(X_t, s_t) - C^t_{s_{t-1}, s_t} \right) \left| X_0 = (x^1, x^2), s_{-1} = i \right. \right], \]

where \( u \) is the switching policy. In Theorem 3.5.4 of Ludkovski (2005), It is shown that the optimal value function \( V_{opt}(X, i) \) is convex for this model. Furthermore, it is straightforward to show that \( V_{myopic}(X, i) \) is a convex function for myopic policy.

Let the current operation mode of the power plant be \( s_{t-1} = i \) at the beginning of the time slot
Under the myopic policy, the operation mode is switched from $i$ to $j$ in the case that

$$
\psi(X_t, j) - C_{t,i,j}^t \geq \psi(X_t, i).
$$

We numerically compute the value function for the the myopic policy by using the convexity property of the myopic policy.

### 7.2 Numerical Results

In this section, we report our numerical results to estimate the value function for the switching problem for a fixed policy. The value function is computed for the myopic policy. The value function $J(x,s)$ is estimated at the point $x = (10,10)$, $s = \text{gas}$ by the truncated method and fixed point projection.

The results are compared with the parametric recursive least squares method developed for policy evaluation (RLSAPI) in [Ma and Powell (2009)](Ma and Powell (2009)). Most of the other methods suggested for solving this problem are based on value iteration, and can not compute the performance by a single sample path.

We need to specify an appropriate approximation architecture for the parametric method. Approximation architectures that span polynomials are known to work well for switching problems. We use all monomials with degree at most three which we call the cubic basis as our approximation architectures. To have a fair comparison, we compare the result of a two copy sample path of length $N$ with a single sample path of length $2N$ used in RLSAPI.

For solving the optimization (3), we use the cutting plane algorithm as a more efficient approach for solving this optimization problem.

Table 7.2 reports the averages (Mean) and the standard deviation (Std) of the estimators computed by the truncated method, the fixed point projection, and the RLSAPI. We wish to compute $V_{\text{myopic}}(x,s)$ at $x = (10,10)$ and $s = \text{gas}$. The results of truncated method and the RLSAPI method are based on 2000 replications for each value of $N$. In the fixed point projection, we estimate based on 100 replications for each value of $N$. Since we approximate the value function at a single point, we can use Monte Carlo simulation to compute the value at this point as a benchmark. We compute the value of $V_{\text{myopic}}(10,10, \text{gas})$ in the last row of Table 7.2 by averaging the discounted reward of $M = 200,000$ sample paths with length $N = 200$. 

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### Table 1: Performance of the Truncated Method, the Fixed Point Projection, and the RLSAPI.

| N   | Truncated Method | Fixed Point Projection | RLSAPI |
|-----|------------------|------------------------|--------|
|     | Mean  | Std  | Mean  | Std  | Mean  | Std  |
| 2000| 718.22| 33.94| 714.16| 21.88| 713.81| 22.66|
| 2500| 717.33| 32.11| 715.96| 16.92| 713.61| 17.62|
| 3000| 717.70| 31.95| 717.48| 14.81| 713.53| 17.36|
| 4000| 716.84| 31.56| 717.12| 11.17| 713.96| 16.32|

\[ V_{\text{myopic}}(10, 10, \text{gas}) = 716.47 \]

The parameters of the O-U process are set as \( \Sigma = [1, 0.2; 0.2, 1], \kappa = 2, e^{-\alpha} = .9, \theta = 10, dt = .1, \lambda = 2. \) The switching cost coefficients are \( c_{\text{gas,oil}} = 1 \) and \( c_{\text{oil,gas}} = 2, \) and the profit functions are

\[
\psi(x_1, x_2, \text{gas}) = 10 \cdot (x^1 - 5),
\]

\[
\psi(x_1, x_2, \text{oil}) = 15 \cdot (x^2 - 6.66).
\]

Table 1 shows that the performance of the truncated method and fixed point projection are better compared to the RLSAPI. Furthermore, the fixed point projection has less variance compared to the Truncated method. Finally, it is clear that larger sample sets yield a significant performance improvement.

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A Proof Details

Lemma A.1. Suppose that $f \in L^2_\pi$. Let $g^*$ be the projection of $f$ onto $C$, and $g^*_n$ be the projection of $f$ onto $C_n$. Then, there exists a subsequence $c_k \uparrow \infty$ such that

$$\|g^* - g^*_c\|_\pi \to 0$$

as $k$ goes to infinity.

Proof. For this lemma, we first observe that the sequence of projected functions are converging to a convex function. Then, we show this limit is the projection with respect to $C$.

First, observe that $C_1 \supset C_2 \supset \cdots \supset C$. Let $f(X) = E[Y|X]$, then $f \in L^2_\pi$. The projection of $f$ onto $C_n$ is $g^*_n$ and $g^*_m \in C_n$ for every $m \geq n$. So we have

$$\langle f - g^*_n, g^*_m - g^*_n \rangle_\pi \leq 0,$$

$$\|f - g^*_m\|^2_\pi + \|g^*_m\|^2_\pi = \|f\|^2_\pi < \infty.$$

It is easy to show that

$$\|f\|^2_\pi \geq \|f - g^*_m\|^2_\pi \geq \|f - g^*_n\|^2_\pi + \|g^*_m - g^*_n\|^2_\pi,$$

for every $m > n$. Therefore, $\|f - g^*_m\|^2$ is an increasing bounded sequence. It follows that $W = \lim_{m \to \infty} \|f - g^*_m\|^2$ for some $W < \|f\|^2$. Since $\|f - g^*_m\|^2_\pi - \|f - g^*_n\|^2_\pi \geq \|g^*_m - g^*_n\|^2_\pi$ for $m \geq n$, $(g^*_n)$ is a Cauchy sequence in $L^2_\pi$. Therefore, $g^*_n$ converges to $g_\infty$ in $L^2$ norm. This implies that there is a sub-sequence $g^*_{n_k}$ converging almost surely to $g_\infty$.

Let $x, y \in \mathbb{R}^d$, and $0 < \theta < 1$. Then,

$$g^*_{n_k}(\theta x + (1 - \theta)y) \leq \theta g^*_n(x) + (1 - \theta)g^*_n(y)$$

for every $n_k > \max(x, y)$. Since, $g^*_{n_k}$ converges almost surely to $g_\infty$ as $n_k$ goes to infinity, $g_\infty$ is a convex function in $L^2_\pi$, from which we conclude that $g_\infty \in C$.

Now, we show that $g_\infty$ is the projection of $f$ over $C$. Every convex function $\phi \in C$ is convex over $H_\phi$, so $\phi \in C_m$ for every $m$. Thus

$$\langle f - g^*_n, \phi - g^*_n \rangle_\pi \leq 0.$$
Since $\|g_n^* - g_\infty\|_\pi \to 0$, it is easy to show that
\[ \langle f - g_\infty, \phi - g_\infty \rangle_\pi \leq 0. \]

This equality holds for every $\phi \in \mathcal{C}$. As a result, we can conclude that $g_\infty$ is the projection of $f$ over $\mathcal{C}$. According to the convexity of $\mathcal{C}$, the projection of $f$ is unique and equal to $g^*$. Thus, $g_\infty = g^*$ almost everywhere, and we have
\[ \|g_n^* - g^*\|_\pi \to 0. \]

Lemma 4.7. Then for every $c > 0$,
\[ \lim \sup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (Y_i - g_c^*(X_i))(g_N(X_i) - g_c^*(X_i))I(\|X_i\| < c) \leq 0. \]

Proof of Lemma 4.7. Replacing the $g_N$ with a fixed convex function $\tilde{g}$ not dependent on $N$, showing the lemma is straightforward. The right-hand average was converging to the inner product $\langle Y - g_c, \tilde{g} - g_c \rangle_\pi$. Since, $g_c$ is projection of $Y$ to the close convex set $\mathcal{C}_c$, this inner product is negative. However, this argument fails for $g_N$ since it depends on $N$. To fix this difficulty, we show that it is possible to approximate every function $g_N$ by a member of a finite set of convex functions over $\mathcal{H}_c$. So, the limit is bounded with a corresponding limit for a fixed convex function which is asymptotically negative according to the projection property.

By Assumption 4.5, we have $\pi(X \in B) > 0$ for every compact disc $B$, and (15) ensures that
\[ \frac{1}{N} \sum_{i=1}^{N} g_N(X_i)^2 \leq \beta. \]

Similar to Proposition 4 in Lim and Glynn (2012), it is possible to show that for each $c > 0$, there exists a deterministic $\gamma(c)$, such that $g_N$ is Lipschitz over $\mathcal{H}_c$ with factor $\gamma(c)$ for sufficiently large $N$. It follows that for every $\epsilon > 0$, there exists a finite collection of convex functions $h_1, h_2, \ldots, h_m$ which is $\epsilon$-net for $\mathcal{C}_{c(1+\delta)}$; for every large $N$ there exists some $h_k$ such that
\[ \sup_{x \in \mathcal{H}_c} |g_N(x) - h_k(x)| \leq \epsilon; \]
see Theorem 6 of Bronshtein (1976). If \( h_k \) and \( g_N \) satisfy this property, observe that

\[
\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c)
\]

\[
\leq \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (g_N(X_i) - h_k(X_i)) I(\|X_i\| < c)
\]

\[
+ \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (h_k(X_i) - g_c^*(X_i)) I(\|X_i\| < c)
\]

\[
\leq \sup_{x \in \mathcal{H}_c} |g_N(x) - h_k(x)| \frac{1}{N} \sum_{i=1}^{N} |Y_i^N - g_c^*(X_i)| I(\|X_i\| < c)
\]

\[
+ \max_{1 \leq k \leq m} \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (h_k(X_i) - g_c^*(X_i)) I(\|X_i\| < c)
\]

\[
\leq \epsilon \cdot \left( \frac{1}{N} \sum_{i=1}^{N} |Y_i^N - g_c^*(X_i)|^2 I(\|X_i\| < c) \right)^{1/2}
\]

\[
+ \max_{1 \leq k \leq m} \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (h_k(X_i) - g_c^*(X_i)) I(\|X_i\| < c)
\]

Because \( h_k \)'s are bounded over \( \mathcal{H}_c \), and \((Y_i^N)\) is "strongly ergodic",

\[
\frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) ((h_k(X_i) - g_c^*(X_i))) I(\|X_i\| < c) \rightarrow 
\]

\[
\mathbb{E}_\pi (Y - g_c^*(X)) (h_k(X) - g_c^*(X)) I(\|X\| < c) \leq 0
\]

as \( N \to \infty \). Here, we use the fact that \( g_c^* \) is the projection of \( Y \) to \( \mathcal{C}_c \), and \( h_k \) is also a bounded convex function over \( \mathcal{H}_c \). As a result,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (Y_i^N - g_c^*(X_i)) (g_N(X_i) - g_c^*(X_i)) I(\|X_i\| < c) \leq 
\]

\[
\epsilon \left( \mathbb{E}_\pi |Y_i^N - g_c^*(X_i)|^2 I(\|X_i\| < c) \right)^{1/2}
\]

for every \( \epsilon > 0 \). This ensures the lemma.

\[\blacksquare\]

**Lemma A.2.** Let \( X = (X_t: t \geq 0) \) be a Harris ergodic chain. Assume that \( \mathbb{E}_\pi r(X)^2 < \infty \). Then
the following two random sequences are “strongly ergodic”:

\[ Y_t = \sum_{j=t}^{\infty} e^{-(j-t)\alpha_r} r(X_t) \]
\[ \bar{Y}^N_t = \sum_{j=t}^{2N} e^{-(j-t)\alpha_r} r(X_t). \]

**Proof.** Since \( X \) is a Harris ergodic chain, there exists a stationary process \( \hat{X} \) initialized by invariant measure \( \pi \) and a finite coupling time \( T \) such that \( X_t = \hat{X}_t \) for all \( t \geq T \); see Proposition 3.13 in Asmussen (2003, Chapter VII). Let

\[ \hat{Y}_t = \sum_{j=t}^{\infty} e^{-(j-t)\alpha_r} r(\hat{X}_t) \]

for \( t \geq 0 \). For every \( g \in L^2_\pi \), we have

\[
\frac{1}{N} \sum_{t=1}^{N} (\hat{Y}_t - g(\hat{X}_t))^2 I(\|\hat{X}_t\| \leq c) \to E_\pi (\hat{Y} - g(\hat{X}))^2 I(\|X\| \leq c) \quad \text{a.s.}
\]

by the Birkhoff–Khinchin theorem; see Corollary 6.23 in Breiman (1992, p.115). Moreover, we can easily show that

\[
\frac{1}{N} \sum_{t=1}^{T} \left| (\hat{Y}_t - g(\hat{X}_t))^2 - (Y_t - g(X_t))^2 \right| I(\|\hat{X}_t\| \leq c) \to 0
\]

as \( N \to \infty \). Therefore, by using the fact that \( X_t = \hat{X}_t \) for all \( t \geq T \), we conclude that

\[
\frac{1}{N} \sum_{t=1}^{N} (Y_t - g(X_t))^2 I(\|X_t\| \leq c) \to E_\pi [(Y - g(X))^2 I(\|X\| \leq c)] \quad \text{a.s.}
\]

as \( N \to \infty \). Therefore, the sequence \( (Y_t) \) is strongly ergodic.

We can also use this property to show \( (\bar{Y}^N_t) \) is strongly ergodic. By applying the triangle inequality, we have

\[
\sqrt{\frac{1}{N} \sum_{t=1}^{N} (Y_t - g(X_t))^2 I(\|X_t\| \leq c)} - \sqrt{\frac{1}{N} \sum_{t=1}^{N} (\bar{Y}^N_t - g(X_t))^2 I(\|X_t\| \leq c)} \]

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\[
\sqrt{\frac{1}{N} \sum_{t=1}^{N} (Y_t - \overline{Y}_N)^2} = \sqrt{\frac{1}{N} \sum_{t=1}^{N} \left( \sum_{j=2N+1}^{\infty} e^{-(j-t)\alpha}r(X_j) \right)^2}
\]

\[
eq e^{-(N+1)\alpha} \sqrt{\frac{1}{N} \sum_{t=1}^{N} e^{-2(N-t)\alpha} \left( \sum_{j=2N+1}^{\infty} e^{-(j-(2N+1))\alpha}r(X_j) \right)}.
\]

We can show that the right hand side of (25) converges to zero. It is clear that

\[
\sqrt{\frac{1}{N} \sum_{t=1}^{N} e^{-2(N-t)\alpha}} \to 0
\]
as \(N \to \infty\). Let

\[
U_N \triangleq \sum_{j=2N+1}^{\infty} e^{-(j-(2N+1))\alpha}r(X_j).
\]

According to the assumption that \(X\) is a Harris ergodic chain and Theorem 3.6 of Asmussen (2003, Chapter VII), we have

\[
\mathbb{E}[U_N | X_0 = x] \to \mathbb{E}_\pi[V(X_t)] < \infty
\]
as \(N \to \infty\). By using this fact and applying the Borel–Cantelli Lemma, it is straightforward to show that \(e^{-(N+1)\alpha}U_N\) goes to zero almost surely. Observe that

\[
\sum_{N=1}^{\infty} \mathbb{P}_x(e^{-(N+1)\alpha}U_N \geq \epsilon) \leq \sum_{N=1}^{\infty} \frac{e^{-(N+1)\alpha}}{\epsilon} \mathbb{E}_x[U_N] < \infty
\]

for every \(\epsilon > 0\). Therefore, \(\mathbb{P}_x(e^{-(N+1)\alpha}U_N > \epsilon ; \ i.o.) = 0\), and the right hand side of (25) converges to zero almost surely as \(N \to \infty\). Hence, \((\overline{Y}_N)\) is strongly ergodic.

**Lemma A.3.** Assume that \((X_t, \tilde{X}_t)\) is a two copy sample path of a positive Harris recurrent chain, and \(E_\pi(r(X_t)^2 + V(X_t)^2) < \infty\). Let

\[
H_t = r(X_t) + \frac{1}{2}(V(X_{t+1}) + V(\tilde{X}_{t+1})�
\]

Then, \(H = (H_t : t \geq 0)\) is a strongly ergodic sequence.

**Proof.** The proof is based on constructing a positive Harris recurrent chain for the two copy sample path and using the ergodic property. By the assumption, the Markov chain \(X\) is positive Harris recurrent. Therefore, there exists a regeneration set \(R\) such that:
i.) Letting $\tau_R = \inf\{ t \geq 0 : X_t \in R \}$, we have $P_x(\tau_R < \infty) = 1$ for all $x \in \mathcal{X}$.

ii.) For some $r > 0$, $1 > \epsilon > 0$ and some probability measure $\lambda$ on $\mathcal{X}$,

$$P^r(x,B) \geq \epsilon \lambda(B), \quad x \in R$$

for all $B \in \mathcal{F}$; see Asmussen (2003, p.198).

Define the Markov chain $\mathbf{X} = \left( (X_t, X_{t+1}, \tilde{X}_{t+1}) : t \geq 0 \right)$ over the state space $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$. The transition probability of $\mathbf{X}$ is induced by the structure of the two copy sample path:

$$\mathbb{P}(\mathbf{X}_{t+1} \in B_1 \times B_2 \times B_3 | \mathbf{X}_t = (x,y,z)) = \begin{cases} \mathbb{P}(y,B_2)\mathbb{P}(y,B_3) & \text{if } y \in B_1, \\ 0 & \text{if } y \notin B_1. \end{cases}$$

We show that this Markov chain is positive Harris recurrent. Let

$$\mathcal{R} \overset{\Delta}{=} \mathcal{X} \times R \times \mathcal{X}.$$  

It is clear that the first hitting time of $\mathcal{R}$ is almost surely finite. Moreover, we have

$$\mathbb{P}^r+1((x,y,z), B_1 \times B_2 \times B_3) = \int_{w \in B_1} \mathbb{P}^r(y, dw)\mathbb{P}(w,B_2)\mathbb{P}(w,B_3)$$

$$\geq \epsilon \int_{w \in B_1} \lambda(dw)\mathbb{P}(w,B_2)\mathbb{P}(w,B_3)$$

$$= \epsilon \bar{\lambda}(B_1, B_2, B_3)$$

for the measurable sets $B_1, B_2, B_3$ and $(x,y,z) \in \mathcal{R}$. So the measure $\lambda(\cdot)$ is a common component for the regenerative set $\mathcal{R}$. Therefore, the Markov chain $\mathbf{X}$ is positive Harris recurrent. Moreover, it is easy to show that $\bar{\pi}(dx) = \pi(dx)\mathbb{P}(x,dy)\mathbb{P}(x,dz)$ is the invariance probability for $X$.

Thus, $H = (H_t : t \geq 0)$ is a strongly ergodic sequence as a direct result of the strong law of large numbers for positive Harris recurrent chains; see Meyn and Tweedie (2009, Chapter 17, p.416).

**Lemma 5.3.** We have

$$\sup_{\phi_1, \phi_2 \in \mathcal{C}_f} \left| \frac{1}{N} \sum_{t=0}^{N-1} (\phi_1(X_t) - \phi_2(X_t))^2 - \mathbb{E}_x(\phi_1(X_0) - \phi_2(X_0))^2 \right| \rightarrow 0 \quad (22)$$

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as $N$ goes to infinity.

Proof. The proof is based on covering $\tilde{C}_f$ by a finite set of functions. Then, we can apply the ergodic property of the Markov chain over each member of that set.

For every constant $c$, and $\delta$, By Assumption (3.4), the functions $\phi \in \tilde{C}_f$ are convex bounded and Lipschitz over the set $\mathcal{H}_c$; see, for example, Van Der Vaart and Wellner (1996, Page. 165). It follows that for every $\epsilon > 0$, there exists a finite collection of bounded functions $\Gamma = \{h_1, h_2, \ldots, h_m\}$ that is an $\epsilon$-net for $\tilde{C}_f$. This means for every $\phi \in \tilde{C}_f$ there exits some $h_k$ such that $\sup_{x \in \mathcal{H}} |\phi(x) - h_k(x)| \leq \epsilon$; see Theorem 6 of Bronshtein (1976), and for a recent result, see Guntuboyina and Sen (2012). For every constant $c$, and $\delta$, By Assumption (3.4), the functions $\phi \in \tilde{C}_f$ are convex bounded and Lipschitz over the set $\mathcal{H}_c$; see, for example, Van Der Vaart and Wellner (1996, Page. 165). It follows that for every $\epsilon > 0$, there exists a finite collection of bounded functions $\Gamma = \{h_1, h_2, \ldots, h_m\}$ that is an $\epsilon$-net for $\tilde{C}_f$. This means for every $\phi \in \tilde{C}_f$ there exits some $h_k$ such that $\sup_{x \in \mathcal{H}} |\phi(x) - h_k(x)| \leq \epsilon$; see Theorem 6 of Bronshtein (1976), and for a recent result, see Guntuboyina and Sen (2012). For every $\phi_1, \phi_2 \in \tilde{C}_f$, suppose $h_i, h_j \in \Gamma$ are in their $\epsilon$ neighborhoods, correspondingly. Let $\Delta \phi = \phi_1 - \phi_2$, and $\Delta h = h_i - h_j$. Then, we have

$$\|\Delta \phi \|_{\tilde{\pi}_N} \leq \| (\Delta \phi - \Delta h) I(\|X\| < c) \|_{\tilde{\pi}_N} + \| \Delta h I(\|X\| < c) \|_{\tilde{\pi}_N} + \| \Delta \phi I(\|X\| > c) \|_{\tilde{\pi}_N} \leq 2\epsilon + \| \Delta h I(\|X\| < c) \|_{\tilde{\pi}_N} + 2\tilde{K} \left( \frac{1}{N} \sum_{t=0}^{N-1} I(\|X\| > c) \right)^{1/2}.$$

Here, we are using the Assumption (3.4) that $|\phi_1(x) - \phi_2(x)| \leq \tilde{K}$ and $|\Delta \phi(x) - \Delta h(x)| < 2\epsilon$ for all $\|x\| < c$. Similarly, we have

$$\|\Delta \phi \|_\pi \geq \| (\Delta \phi - \Delta h) I(\|X\| < c) \|_\pi - \| \Delta h I(\|X\| < c) \|_\pi - \| \Delta \phi I(\|X\| > c) \|_\pi \geq \| \Delta h I(\|X\| < c) \|_\pi - 2\epsilon - 2\tilde{K} \left( \mathbb{E}_\pi I(\|X\| > c) \right)^{1/2}.$$

For sufficiently large $c$ and $N$, we get

$$2\tilde{K} \left( \frac{1}{N} \sum_{t=0}^{N-1} I(\|X\| > c) \right)^{1/2} + 2\tilde{K} \left( \mathbb{E}_\pi I(\|X\| > c) \right)^{1/2} \leq \epsilon.$$

Therefore,

$$\|\Delta \phi \|_{\tilde{\pi}_N} - \|\Delta \phi \|_\pi \leq \| \Delta h I(\|X\| < c) \|_{\tilde{\pi}_N} - \| \Delta h I(\|X\| < c) \|_\pi \leq 5\epsilon.$$

A similar argument shows that the right-hand side is also an upper for $\|\Delta \phi \|_\pi - \|\Delta \phi \|_{\tilde{\pi}_N}$. To summarize, we obtain

$$\|\Delta \phi \|_{\tilde{\pi}_N} - \|\Delta \phi \|_\pi \leq \max_{1 \leq i, j \leq m} \left| \frac{1}{N} \sum_{t=0}^{N-1} (h_i(X_t) - h_j(X_t))^2 I(\|X_t\| \leq c) \right| \leq 39.$$
\[-\mathbb{E}_\pi(h_i(X_t) - h_j(X_t))^2 I(||X_t|| \leq c)^{1/2} + 5\epsilon\]

for every \(\epsilon > 0\), and sufficiently large \(c\) and \(N\). Positive Harris recurrent assumption of the Markov chain ensures that the second term converges to zero almost surely. Thus,

\[
\limsup_{N \to \infty} |\|\Delta \phi\|_{\bar{\pi}_N} - \|\Delta \phi\|_\pi| \leq 5\epsilon
\]

for any arbitrarily small \(\epsilon\). Thus, (5.3) holds. \(\square\)

**Lemma A.4.** We have

\[
\lim_{N \to \infty} \|\bar{V} - \bar{V}_N\|_{\bar{\pi}_N} = 0.
\]

**Proof.** Let \(W_N\) be the projection of \(\bar{H}\) onto the convex cone \(\mathcal{C}\). Recall that \(\tilde{V}_N\) is the projection of \(\bar{H}\) to the convex set \(\bar{\mathcal{C}}_f\). Since, \(\bar{\mathcal{C}}_f\) is a subset of \(\mathcal{C}\), we have

\[
\|\bar{H} - W_N\|_{\bar{\pi}_N} \leq \|\bar{H} - \tilde{V}_N\|_{\bar{\pi}_N}.
\]

By Assumption (3.4), \(\bar{V}\) belongs to \(\bar{\mathcal{C}}_f\). This ensures that

\[
\|\bar{V} - \bar{V}_N\|_{\bar{\pi}_N}^2 \leq \|\bar{H} - \bar{V}\|_{\bar{\pi}_N}^2 - \|\bar{H} - \tilde{V}_N\|_{\bar{\pi}_N}^2
\]

\[
\leq \|\bar{H} - \bar{V}\|_{\bar{\pi}_N}^2 - \|\bar{H} - W_N\|_{\bar{\pi}_N}^2
\]

\[
= \|\bar{V} - W_N\|_{\bar{\pi}_N}^2 + 2(\bar{V} - W_N, W_N - \bar{H})_{\bar{\pi}_N}
\]

\[
\leq \|\bar{V} - W_N\|_{\bar{\pi}_N}^2 + 2\|\bar{V} - W_N\|_{\bar{\pi}_N} \|W_N - \bar{H}\|_{\bar{\pi}_N}
\]

\[
\leq 3\|\bar{V} - W_N\|_{\bar{\pi}_N}^2 + 2\|\bar{V} - W_N\|_{\bar{\pi}_N} \|\bar{V} - \bar{H}\|_{\bar{\pi}_N}.
\]

Theorem [4.6] implies that \(\|\bar{V} - W_N\|_{\bar{\pi}_N}^2\) converges to zero. Therefore, \(\|\bar{V} - \bar{V}_N\|_{\bar{\pi}_N}^2\) also converges to zero as \(N \to \infty\). \(\square\)

**B Examples**

**Example B.1.** Consider the line \(L = \{(x, y) \mid 16y = x\}\) as a convex set, and project the points \(x_1 = (1, 2)\) and \(x_2 = (-2, -2)\) onto this convex set with respect to the sup-norm. It can be easily shown that

\[
\|\Pi_L x_1 - \Pi_L x_2\|_\infty = 6.58 > \|x_1 - x_2\|_\infty = 4.
\]
**Example B.2.** Let $S = (X_1, X_2, \tilde{X}_2, \ldots, X_N, \tilde{X}_N)$ be a sequence of $2N - 1$ normal random variables. Suppose that $\tilde{X}_m, X_k$ are the largest and the second largest numbers in this sequence. For any large enough $M$, let

$$g_M(x) \triangleq \begin{cases} 
1 & x \leq X_k \\
(M - 1) \frac{Y - X_k}{\tilde{X}_m - X_k} + 1 & x > X_k.
\end{cases}$$

Clearly, the function $g_M$ is convex, and we have

$$\frac{1}{N} \sum_{i=1}^{N-1} (g_M(X_i) - \alpha g_M(\tilde{X}_{i+1}))(g_M(X_i) - \alpha g(X_{i+1}))$$

$$= \frac{(N-1)(1-\alpha)^2}{N} + \frac{1-\alpha}{N} (1-\alpha M) \to -\infty$$

as $M \to \infty$. Therefore, minimization problem (9) is unbounded with a probability of at least $1/4$ for any large $N$. 

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