ON THE WEAK COUPLING SPECTRUM OF $N = 2$
SUPERSYMMETRIC SU($n$) GAUGE THEORY

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ABSTRACT

The weak coupling spectrum of BPS saturated states of pure $N = 2$ supersymmetric SU($n$) gauge theory is investigated. The method uses known results on the dyon spectrum of the analogous theory with $N = 4$ supersymmetry, along with the action on these states of the semi-classical monodromy transformations. For dyons whose magnetic charge is not a simple root of the Lie algebra, it is found that the weak coupling region is divided into a series of domains, for which the dyons have different electric charge, separated by walls on which the dyons decay. The proposed spectrum is shown to be consistent with the exact solution of the theory at strong coupling in the sense that the states at weak coupling can account for the singularities at strong coupling.
1. Introduction

In [1] an exact expression for the prepotential of the effective action of $N = 2$ supersymmetric SU(2) gauge theory was found. Amongst other things, the prepotential determines the masses of any BPS-saturated states in the theory, offering the hope of a complete knowledge of the spectrum of such states. To be more specific, the solution of Seiberg and Witten determines the mass of a BPS state, with magnetic charge $g$ and electric charge $q$, as

$$M_{(g,q)} = |qa(u) + ga_D(u)|$$ (1.1)

where $a(u)$ and $a_D(u)$ are functions of the gauge invariant coordinate $u$ which parametrizes the moduli space of vacua. In particular, the exact solution implies that there are points on the moduli space where a BPS state becomes massless signalling that the original effective action is no longer a good description of the long-range physics. In fact there are two of these singularities which both appear at strong coupling. They correspond to two arbitrary states, one from each of the two sets $(\pm 1, 2p)$ and $(\pm 1, 2p + 1), p \in \mathbb{Z}$, respectively, being massless. The exact state in each set which becomes massless at the singularity, depends on the path taken to the singularity [1].

Because the functions $a(u)$ and $a_D(u)$ are not single-valued around a singularity, there exists non-trivial monodromy on the moduli space. Taking a state $(g,q)$ around a singularity in the moduli space, one ends up with a state $(g,q)M$, where $M$ is a $2 \times 2$ matrix acting by multiplication to the left. It would appear, therefore, that the monodromy transformations around the two singularities, $M^+$ and $M^-$, should be symmetries of the spectrum. However, this is not the case for reasons which are explained below.

What was not determined in [1], is the actual spectrum of BPS states in the theory over the moduli space. At weak coupling, one can determine the spectrum through a semi-classical analysis. This leads to a spectrum of massive states consisting of towers of dyons with charges $(\pm 1, p), p \in \mathbb{Z}$, as well as the gauge bosons $(0, \pm 1)$. States which appear at weak coupling can then be continued into the region of strong coupling. There is subtlety, however, arising from the fact that BPS states, which are generically below threshold for decay into other states, can arrive at threshold for decay into other BPS states on a Curve of Marginal Stability (CMS). Upon crossing the curve, a BPS state can decay and disappear from the spectrum. For SU(2), there is a single CMS is given by the one-dimensional curve on which

$$\text{Im} \left( \frac{a_D(u)}{a(u)} \right) = 0.$$ (1.2)

This curve is present only in the region of strong coupling and divides the moduli space into two regions, one of which contains the weak coupling regime [2,3]. The curve passes
through the two singularities on the moduli space. The states which decay when crossing
the CMS have been determined by a simple argument in [4]. It turns out that only two
states, and their anti-particles, survive in the strong coupling region.

The monodromy transformations $\mathcal{M}^\pm$ are not individually symmetries of the BPS
spectrum because paths encircling each of these singularities pass through the CMS on
which states can decay. However, a path that encircles both singularities need not pass
through the CMS and so the combined monodromy $\mathcal{M}^+ \mathcal{M}^-$, which is the semi-classical
monodromy, or monodromy at infinity, is a symmetry of the spectrum [1].

The analysis of Seiberg and Witten has been extended to theories with an arbitrary
gauge group: [5,6,7] for $\text{SU}(n)$, [8] for $\text{SO}(2n)$, [9] for $\text{SO}(2n + 1)$, [10] for $\text{Sp}(n)$, [11] for
$G_2$ and [12] for all the other exceptional cases. For larger gauge groups, the magnetic and
electric charges are now rank($g$) vectors with respect to the unbroken $U(1)^{\text{rank}(g)}$ symmetry.

There is an analogous formula to (1.1) for the mass of BPS states

$$M_{(g,q)} = |q \cdot a(u_j) + g \cdot a_D(u_j)|,$$

(1.3)

where the vector quantities $a(u_j)$ and $a_D(u_j)$ are determined exactly in terms of the
rank($g$) coordinates $u_j$ on the moduli space. Just as in SU(2) case, there are singularities on
the moduli space and associated monodromies. However, it is now much more complicated
to determine the spectrum of BPS states, because, unlike in SU(2), there are many different
CMS, depending upon the magnetic and electric charges of the states involved, some of
which extend into the region of weak coupling. Just as in the SU(2) theory the existence
of these CMS means that the monodromies, but now even some of the semi-classical ones,
are not necessarily symmetries of the spectrum.

As a step towards a full understanding of the spectrum of BPS states we consider
the SU($n$) theory at weak coupling. In this limit, we find the positions of the CMS and
propose, with reference to the spectrum of the related $N = 4$ supersymmetric theory, a
form for the weak coupling spectrum of the theory. We find that the weak coupling region
is divided into various domains by CMS, each domain having a different spectrum of BPS
states. We then show that the weak coupling spectrum is the minimal set consistent with
the exact solution of the theory at strong coupling, in the sense that the states which
appear at weak coupling can account for the singularities that appear at strong coupling.
This suggests that like the SU(2) theory, the weak coupling spectrum includes all the states
that appear at strong coupling.

2. Semi-classical monodromies

In this section we explain how monodromies appear on the moduli space of vacua in
the semi-classical regime. Our notation follows closely that of [8,9].

In the semi-classical regime, we can parameterize the moduli space in terms of the
VEV of the classical Higgs field $\phi$. We can use the freedom to perform global gauge trans-
formations to take $\phi$ to be in the Cartan subalgebra of the Lie algebra of the gauge group, so $\phi = a \cdot H$, which defines the $n - 1$ component vector $a$. This does not entirely fix the global gauge transformations since it leaves the freedom to perform discrete transformations in the Weyl group. This discrete degree-of-freedom can be fixed, for example, by demanding that $\text{Re}(a)$ lies in the fundamental Weyl chamber, i.e. with respect to some choice of simple roots $\alpha_i$ of the Lie algebra

$$\text{Re}(\alpha_i \cdot a) \geq 0, \quad i = 1, 2, \ldots, n - 1. \quad (2.1)$$

We will denote this region $W$. The semi-classical regime is defined as the region for which $|\alpha_i \cdot a| \gg \Lambda$, where $\Lambda$ is the dynamically generated mass scale in the theory, and it is only in this region that $W$ coincides with the quantum moduli space.

The boundary of $W$ is made up of the walls $\partial W_i$ on which $\text{Re}(\alpha_i \cdot a) = 0$. The wall $\partial W_i$ is an invariant subspace under the action of the Weyl reflection in the simple root $\alpha_i$, which we denote as $r_i$. Moreover, a point $a \in \partial W_i$ is identified with $r_i(a)$. The subspace $S_i$ of $W$, of co-dimension two, defined by $\alpha_i \cdot a = 0$, is the $i$th semi-classical singularity. Obviously $S_i$ lies in the wall $\partial W_i$ and is left invariant by the Weyl reflection $r_i$.

The prepotential can be calculated exactly in perturbation theory since it only receives contributions at the tree and one-loop level. This leads to the following expression for the quantity $a_D$ in the semi-classical regime

$$a_D = \frac{i}{2\pi} \sum_{\alpha \in \Phi^+} \alpha(\alpha \cdot a) \left[ \ln \left( \frac{\alpha \cdot a}{\Lambda} \right)^2 + 1 \right], \quad (2.2)$$

where the sum is over $\Phi^+$ the set of positive roots of the Lie algebra. The pre-potential also receives non-perturbative corrections, however these do not affect the following weak coupling analysis. The logarithm gives rise to singularities if $\alpha \cdot a = 0$, for some root $\alpha$ of the Lie algebra. These singularities occur precisely at the points where classically one would expect the residual gauge symmetry to be enhanced beyond the maximally abelian subgroup $U(1)^{n-1}$, due to the fact that the gauge bosons associated to the root $\alpha$, and its negative, become massless. In the full quantum theory, these semi-classical singularities are not in fact present, rather they are split into pairs of singularities which coalesce in the limit $\Lambda \rightarrow 0$. Furthermore, they are not caused by gauge bosons becoming massless, which physically we would not expect because of strong coupling effects, rather they are a signal that certain magnetically charged dyons become massless. At weak coupling, however, these pairs of quantum singularities are not resolved and they give rise to the effective semi-classical singularities which appear in (2.2). We will have more to say about the quantum singularities in section 5.

Notice that $a_D$ is not a single-valued function of $a$ and hence there are non-trivial monodromies as one encircles a singularity. The origin of the monodromy is the fact that

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1 The choice of the real part in (2.1) is not unique, it would be just as convenient to choose the region $\text{Re}(e^{-i\phi} \alpha_i \cdot a) \geq 0$, where $\phi$ is some phase angle.
going around the singularity $S_i$ involves identifying points on $\partial W_i$ by the Weyl reflection $r_i$. This is illustrated in figure 1.

\[ r_i \]

\[ \partial W_i \quad S_i \quad \text{Im}(\alpha_i \cdot a) \]

\[ \text{Re}(\alpha_i \cdot a) \]

\[ 1 \quad 2 \]

Figure 1. Path leading to monodromy $M_i$ around $S_i$.

Under this identification, the vector $(a_D, a)$ undergoes a monodromy transformation. Defining a path around $S_i$ to be in a positive sense if on $\partial W_i$ the Weyl reflection $r_i$ takes a point with $\text{Im}(\alpha \cdot a) < 0$, labelled as 1 in figure 1, to a point with $\text{Im}(\alpha \cdot a) > 0$, labelled as 2 in figure 1. The monodromy can be derived by considering the change of $a_D$ along the path

\[ a(t) = a - \left(1 - e^{i\pi t}\right) \alpha_i (\alpha_i \cdot a)/2, \quad 0 \leq t \leq 1, \]  

(2.3)

giving

\[ \begin{pmatrix} a_D \\ a \end{pmatrix}_2 = M_i \begin{pmatrix} a_D \\ a \end{pmatrix}_1 = \begin{pmatrix} r_i & -\alpha_i \otimes \alpha_i \\ 0 & r_i \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}_1. \]  

(2.4)

In a negative sense, the monodromy around $S_i$ is given by $M_i^{-1}$. All other monodromies are generated by conjugation from the monodromies $M_i^{\pm 1}$ associated to the simple roots.

The existence of non-trivial monodromies implies that if we follow the mass of a BPS state along some closed path in moduli space then we do not necessarily get a state of the same mass at the end of the journey. If the path encircles the singularity $S_i$ in a positive sense, starting with a dyon of charge $(g, q)$ of mass $M_{(g,q)}$, as we cross through the wall of the Weyl chamber at point 1 to arrive at point 2, the quantities $(a_D, a)$ change discontinuously as in (2.4). Since the mass of a state must be continuous along a continuous path in moduli space, this implies that the charges of the state must be transformed. From

\[ \left| (g, q) M_i \begin{pmatrix} a_D \\ a \end{pmatrix} \right| = M_{(g,q)M_i}, \]  

(2.5)
it follows that we end up with a state of charge \((g, q)M_i\), where by definition \(M_i\) acts by matrix multiplication to the left.

3. Curves of marginal stability at weak coupling

To probe the region of weak coupling, we take \(\Lambda \to 0\) with \(\alpha_i \cdot a \neq 0\). In this limit the mass of a state \((g, q)\) with non-vanishing magnetic charge is dominated by the magnetic charge: \(M(g, q) \to C|a \cdot g|\). To see this notice that that as \(\Lambda \to 0\)

\[
a_D \to -\left\{ \frac{\ln \Lambda}{\pi} \right\} a.
\]

(3.1)

In section 4, we shall argue that dyons can only have a magnetic charge which is a root of the Lie algebra. For such states, we will need to know where the CMS are located. A dyon with magnetic charge \(\alpha + \beta\) is at threshold for decay to two dyons with magnetic charges \(\alpha\) and \(\beta\), where \(\alpha, \beta, \alpha + \beta \in \Phi\), the root system of \(su(n)\), when

\[
C_{\alpha, \beta} : |a \cdot (\alpha + \beta)| = |a \cdot \alpha| + |a \cdot \beta|, \quad \text{i.e. } \alpha \cdot a/\beta \cdot a \in \mathbb{R} \geq 0.
\]

(3.2)

This defines the CMS that we denote \(C_{\alpha, \beta}\) which is a hyperplane of co-dimension one in moduli space. Notice that in the weak coupling limit the CMS for states with non-vanishing magnetic charge in the weak coupling limit, do not depend on the electric charge of state. Obviously this will cease to be true away from the region of weak coupling.

The following properties of \(C_{\alpha, \beta}\) will be useful. (i) The curve only has a non-trivial overlap with \(W\) if \(\alpha\) and \(\beta\) are either both positive roots or both negative roots. (ii) \(C_{\alpha, \beta}\) divides \(W\) into two distinct domains which we denote \(D_{\alpha, \beta}^{\pm}\).

In order to demonstrate these two properties, consider the smooth map \(x : W \to \mathbb{C}\) where

\[
x(a) = \frac{a \cdot \alpha}{a \cdot \beta}.
\]

(3.3)

Suppose, first of all, that \(\alpha\) is a positive root and \(\beta\) is a negative root. Writing the real and imaginary parts \(a \cdot \alpha = a + ib\) and \(a \cdot \beta = p + iq\), then in \(W\), by definition, \(a \geq 0\) and \(p \leq 0\). Consider the subspace of \(W\) which maps to \(x(a) \in \mathbb{R}\). For real \(x(a)\) it follows that \(x(a) = a/p \leq 0\), so that the image of \(W\) under the map can never be real and positive. From (3.2) we see that the CMS \(C_{\alpha, \beta}\) is precisely the portion of \(W\) on which \(x(a)\) is real and positive. This demonstrates the first advertised property of \(C_{\alpha, \beta}\).

The second property follows immediately. Suppose that \(\alpha\) and \(\beta\) are both positive roots. Then under \(x\), the image of \(W\) can never be the negative real axis. Furthermore the image of \(C_{\alpha, \beta}\) is precisely the positive real axis. Hence \(W\) is separated by \(C_{\alpha, \beta}\) into two distinct domains \(D_{\alpha, \beta}^{\pm}\) and \(D_{\alpha, \beta}^{-}\) according to whether \(\text{Im}(x(a)) > 0\) or \(\text{Im}(x(a)) < 0\). Obviously an identical argument can be followed if \(\alpha\) and \(\beta\) are both negative roots.
Notice that $C_{\alpha,\beta}$ passes through $a \cdot \alpha = 0$ and $a \cdot \beta = 0$, which are on the walls of $W$, corresponding to $x = 0$ and $x = \infty$, respectively.

From this information it is possible to deduce all the CMS for a state with magnetic charge given by the root $\gamma$, which we suppose is a positive root. For each pair of positive roots $\alpha$ and $\beta$ such that $\gamma = \alpha + \beta$, then $C_{\alpha,\beta}$ is a CMS for the state with magnetic charge $\gamma$. At this point it will prove convenient to write the roots in terms of the weight vectors of the $n$ dimensional representation of $su(n)$, $e_i$, $i = 1, \ldots, n$, with inner products $e_i \cdot e_j = \delta_{ij} - (1/n)$. The positive roots are then $e_i - e_j$, with $i < j$, and the simple roots are $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n - 1$. So if $\gamma = e_i - e_j$ then its CMS are $C_{e_i-e_{k},e_{k}-e_{j}}$ with $i < k < j$ corresponding to the decay

$$ (e_i - e_j) \to (e_i - e_k) + (e_k - e_j). \quad (3.4) $$

So for this state of magnetic charge $e_i - e_j$, $W$ is divided into $2^{j-i-1}$ distinct regions by its $j - i - 1$ CMS given by the intersections

$$ D_{e_i-e_{i+1},e_{i+1}-e_j}^\epsilon \cap \cdots \cap D_{e_i-e_k,e_k-e_j}^\epsilon \cap \cdots \cap D_{e_i-e_{j-1},e_{j-1}-e_j}^\epsilon, \quad (3.5) $$

where $\epsilon_k = \pm$ for $i < k < j$. Notice that there are no CMS for a state whose magnetic charge is a simple root. Obviously an identical picture holds for the CMS of the negative roots. In fact $C_{\alpha,\beta} \equiv C_{-\alpha,-\beta}$ and $D_{\alpha,\beta}^\pm \equiv D_{-\alpha,-\beta}^\pm$.

For SU(3) there is one positive root $\alpha_1 + \alpha_2$ which is not simple. So for the state of magnetic charge $\alpha_1 + \alpha_2$, $W$ is divided into two regions separated by the CMS $C_{\alpha_1,\alpha_2}$. Figure 2 shows a two-dimensional cross-section of $W$. Notice, in this case, that $C_{\alpha_1,\alpha_2}$ intersects $\partial W_1$ in $S_1$ and $\partial W_2$ in $S_2$.

![Figure 2. Two-dimensional cross-section of $W$ for SU(3).](image-url)
4. The weak coupling dyon spectrum

In this section, we use the knowledge of the dyon spectrum of the SU($n$) $N = 4$ supersymmetric theory and the positions of the CMS to deduce the weak coupling dyon spectrum of the $N = 2$ theory.

In the $N = 4$ supersymmetric theory the Higgs field transforms as a vector of the SO(6) $R$-symmetry. The vacuum expectation value of the Higgs field is therefore specified by the $6 (n - 1)$-dimensional vectors $a^I$, $I = 1, \ldots, 6$. On a special subspace of the moduli space for which

$$a^I = \xi^I b,$$  \hspace{1cm} (4.1)

an analysis of the semi-classical spectrum of monopole states has recently been performed [13,14]. The monopole solutions are obtained by embedding the ’t Hooft Polyakov monopole of the SU(2) theory into the SU($n$) theory by specifying an $su(2)$ subalgebra of $su(n)$ associated to a particular root of $su(n)$ [15]. These solutions are therefore spherically symmetric.

The vector $b$ defines a set of simple roots via $b \cdot \alpha_i > 0$. The moduli space of a particular solution depends upon whether the root $\alpha$ is a simple root, or not. If the root is simple, then the moduli space is identical to that of the SU(2) monopole, i.e. it is of the form $\mathbb{R}^3 \times S^1$, corresponding to three translational and one charge degrees of freedom. Semi-classical quantization proceeds exactly as in the SU(2) theory and leads to a tower of dyons with magnetic charge $\alpha_i$ and electric charge $p\alpha_i$, for $p \in \mathbb{Z}$. The embedding of the anti-monopole gives a similar tower of dyons with opposite magnetic charge.

For solutions corresponding to non-simple roots, the moduli space carries an additional internal component $\mathcal{M}_0$ [13,14,16,17,18]. In a certain asymptotic limit, this internal part of the moduli space describes the fact that the solution for $\alpha = \sum n_i \alpha_i$ can be thought of a superposition of $\sum n_i$ fundamental monopoles whose magnetic charges are simple roots. The exact form of the moduli space is $\mathbb{R}^3 \times (\mathbb{R} \times \mathcal{M}_0)/\mathbb{Z}$, where the $\mathbb{R}^3$ factor corresponds to the translational degrees of freedom and the $\mathbb{R}$ factor to the overall charge degree of freedom. The factor $\mathcal{M}_0$ in the asymptotic regime, encodes the relative positions and charge angles of the fundamental monopoles. On proceeding to a semi-classical quantization, it turns out that there is unique harmonic form on $\mathcal{M}_0$ and hence there exists a bound-state of the $\sum n_i$ fundamental monopoles at threshold carrying magnetic charge $\alpha$ [13,19]. The resulting spectrum of states consists of a tower of dyons with magnetic charge $\alpha$ and electric charge $p\alpha$, for $p \in \mathbb{Z}$, where $\alpha$ is any root of the algebra.

To summarize, on the special subspace (4.1) of the moduli space the spectrum of

\footnote{We are assuming here that $b$ is not orthogonal to any root so that the gauge symmetry is maximally broken to U(1)$^{n-1}$.}
dyons, whose magnetic charge is a root, is the same for all roots:

\[ N = 4 : \quad (\alpha, p\alpha), \quad \alpha \in \Phi, \quad p \in \mathbb{Z}. \quad (4.2) \]

This result is a rather compelling piece of evidence for the old duality conjecture of Goddard, Nuyts and Olive [20] in the context an \( N = 4 \) supersymmetric gauge theory. Of course, one would like to know the full spectrum of dyon states in the theory. In particular, requiring the theory to have a full \( S \)-duality implies that there exist dyons with a magnetic charge which are not roots of the algebra (but are in the root lattice) [21], as in the SU(2) theory. To prove this will require an analysis of the appropriate multi-monopole moduli spaces. However, the \( S \)-duality conjecture on the special subspace (4.1), does not require the existence of dyon states whose electric and magnetic charges are not proportional.

The analysis of the \( N = 4 \) theory has important implications for the weak coupling spectrum of the \( N = 2 \) theory. For the case of SU(2), the complete spectrum of the \( N = 4 \) supersymmetric theory consists of dyons with charges \((g, q)\), where \( g \) and \( q \) are co-prime integers. The states of magnetic charge \( g \) can be thought of as a bound-state of \( g \) fundamental monopoles of unit magnetic charge. The fact that such states appear in the spectrum of the \( N = 4 \) theory, is a consequence of the fact that there exists a bound-state of \( g \) fundamental monopoles. In semi-classical quantization the bound-state is manifested by the existence of a unique harmonic form on the appropriate multi-monopole moduli space [22,23]. In the context of the \( N = 2 \) theory, these bound-states would not exist for the simple reason that they would require the existence of a holomorphic harmonic form on the multi-monopole moduli space. But the existence of such a form is ruled out because it would inevitably have an anti-holomorphic partner, in contradiction to the uniqueness of the harmonic form following from the analysis in the \( N = 4 \) theory. Hence in the SU(2) theory the weak coupling spectrum of dyons is only a subset of the states in the \( N = 4 \) theory consisting of the dyons with unit magnetic charge, i.e. the states \((\pm 1, p), \quad p \in \mathbb{Z}\).

We can think of these two towers of dyon states as being associated to the single root of \( su(2) \), and its negative.

Now consider the SU(\( n \)) theory with \( N = 2 \) supersymmetry. The analogue of the special subspace (4.1) is the subspace for which

\[ a = e^{i\phi}b, \quad (4.3) \]

for some phase \( \phi \) and real vector \( b \). Notice that the special subspace (4.3) has dimension \( n \) and corresponds to the intersection of all the CMSs \( C_{\alpha,\beta} \). Just as in the \( N = 4 \) case we define a set of simple roots with respect to \( b \). Notice that these simple roots are the same simple roots as were defined in (2.1) with respect to \( \text{Re}(e^{-i\phi}a) \) (see the footnote after (2.1)). On the special subspace (4.3) we can determine the spectrum of dyons with magnetic charges which are roots of the algebra by analogy with the preceding argument for SU(2). As we have already remarked, in the \( N = 4 \) theory the dyons with magnetic
charge which are not simple roots exist as bound states by virtue of there being a unique harmonic form on the internal part of the multi-monopole moduli space $\mathcal{M}_0$. Since the harmonic form is unique, such bound-states will not exist in the $N = 2$ theory. Hence we expect that the spectrum of dyon states on the subspace only consists of states with magnetic charge being a simple root, or its negative:

$$N = 2: \quad (\pm \alpha_i, p\alpha_i), \quad p \in \mathbb{Z}.$$ (4.4)

We can immediately deduce from this analysis that, since in the weak coupling regime there are no CMS for a state whose magnetic charge is a simple root, the dyons $(\pm \alpha_i, p\alpha_i)$ must exist throughout the region of weak coupling. A consistent picture of the weak coupling spectrum of dyons can now be built up by transporting the basic dyons around the semi-classical singularities.

As a preliminary to the general case, let us first consider SU(3). A two-dimensional cross-section of $W$ appears in figure 2. Suppose we take the dyons states $(\alpha_1, p\alpha_1)$ along $S_1$ in a positive sense. The states are transformed by the semi-classical monodromy transformation $M_1$, giving

$$(\alpha_1, p\alpha_1)M_1 = (-\alpha_1, -(p + 2)\alpha_1).$$ (4.5)

These states are in the tower of anti-dyons associated to the simple root $\alpha_1$. Now consider taking the dyons $(\alpha_1, p\alpha_1)$ along a path that winds around the singularity $S_2$ in a positive sense. The resulting states are

$$(\alpha_1, p\alpha_1)M_2 = (\alpha_1 + \alpha_2, p(\alpha_1 + \alpha_2) + \alpha_2).$$ (4.6)

Similarly the same states taken along a path which encircles $S_2$ in a negative sense, end up as

$$(\alpha_1, p\alpha_1)M_2^{-1} = (\alpha_1 + \alpha_2, p(\alpha_1 + \alpha_2) - \alpha_2).$$ (4.7)

One might now think that by taking the same states twice around $S_2$ one would end up with more dyon states. For example

$$(\alpha_1, p\alpha_1)M_2M_2 = (\alpha_1, p\alpha_1 - 2\alpha_2).$$ (4.8)

However, such states would be inconsistent with the fact that on the special subspace (4.3) we only expect dyons whose magnetic charge is a simple root to have an electric charge which is proportional to the magnetic charge. The way to resolve this problem, is to notice that in order to wind around the singularity $S_2$ twice would necessarily entail crossing the CMS $C_{\alpha_1, \alpha_2}$. So we have to assume that the states in (4.7) and (4.6) decay on crossing $C_{\alpha_1, \alpha_2}$. This is consistent with the fact already noted, that on on the special subspace (4.3), which coincides with $C_{\alpha_1, \alpha_2}$, there are no dyon states whose magnetic charge is a non-simple root.

The two sets of states in (4.6) and (4.7) are therefore present in the disjoint regions of $W$, illustrated in figure 2, that we have defined in section 3 to be $D^\pm_{\alpha_1, \alpha_2}$, respectively.
Hence the spectrum of dyon states at weak coupling can be summarized as follows. The
dyon states \((\alpha_i, p\alpha_i)\), \(i = 1, 2\), are present throughout \(W\). For the non-simple root \(\alpha_1 + \alpha_2\),
there is a tower of dyon states either (4.6) or (4.7), present in the two regions \(D_{\alpha_1, \alpha_2}^\pm\),
respectively, separated by \(C_{\alpha_1, \alpha_2}\) on which the states decay. In each of these regions there
is a corresponding set of anti-dyons with opposite magnetic charge.

Suppose we had started with the dyons \((\alpha_2, p\alpha_2)\), what states are created on taking
these states around the singularity \(S_1^+\)? In fact, no new states are created since
\[
(\alpha_2, p\alpha_2)M_1^{\pm 1} = (\alpha_1, (p \pm 1)\alpha_1)M_2^{\mp 1},
\]
and furthermore the states \((\alpha_2, p\alpha_2)M_1^{\pm 1}\) are present in \(D_{\alpha_1, \alpha_2}^{\pm}\), as they should for con-
sistency.

Consider now the SU(4) case. It will be convenient to label the tower of dyons associ-
ted to the simple roots as \(Q_i = (\alpha_i, p\alpha_i)\). Just as in the SU(3) case, starting from the
dyons \(Q_1\), the dyons \(Q_1M_2^{\pm 1}\), of magnetic charge \(\alpha_1 + \alpha_2\), will be generated in the disjoint
regions \(D_{\alpha_1, \alpha_2}^\pm\), by winding around \(S_2\) in either a positive or negative sense. These states
are identical to \(Q_2M_1^{\mp 1}\). In a similar way, dyons \(Q_2M_3^{\pm 1}\), of magnetic charge \(\alpha_2 + \alpha_3\),
will be generated in the disjoint regions \(D_{\alpha_2, \alpha_3}^\pm\), by winding around the singularity \(S_3\) in
either a positive or negative sense. These states are identical to \(Q_3M_2^{\mp 1}\).

Now consider the affect of taking the states \(Q_1M_2^{\pm 1}\) around the singularity \(S_3\). A
two-dimensional cross-section of the wall \(\partial W_3\) is illustrated in figure 3 showing how the
regions \(D_{\alpha_1, \alpha_2}^\pm\) and the CMS \(C_{\alpha_1, \alpha_2}\) intersect it.

![Figure 3. Cross-section of the wall \(\partial W_3\) showing the two
distinct regions for dyons of magnetic charge \(\alpha_1 + \alpha_2\).

Suppose we take the states \(Q_1M_2^{\pm 1}\) around the singularity \(S_3\) in a positive or negative
sense. We end up with four regions each containing a different set of states:

\[
\begin{align*}
D^-_{\alpha_1,\alpha_2+\alpha_3} \cap D^-_{\alpha_1+\alpha_2,\alpha_3} & : Q_1 M_2 M_3 \\
D^-_{\alpha_1,\alpha_2+\alpha_3} \cap D^+_{\alpha_1+\alpha_2,\alpha_3} & : Q_1 M_2 M_2^{-1} \\
D^+_{\alpha_1,\alpha_2+\alpha_3} \cap D^-_{\alpha_1+\alpha_2,\alpha_3} & : Q_1 M_2^{-1} M_3 \\
D^+_{\alpha_1,\alpha_2+\alpha_3} \cap D^+_{\alpha_1+\alpha_2,\alpha_3} & : Q_1 M_2^{-1} M_3^{-1}.
\end{align*}
\] (4.10)

This is illustrated in figure 4.

Figure 4. Cross-section of the wall \(\partial W_3\) showing the four distinct regions for dyons of magnetic charge \(\alpha_1 + \alpha_2 + \alpha_3\).

Notice that under the Weyl reflection \(r_3\) the CMS \(C_{\alpha_1,\alpha_2}\) is mapped into the CMS \(C_{\alpha_1,\alpha_2+\alpha_3}\). Moreover, the CMS \(C_{\alpha_1+\alpha_2,\alpha_3}\) intersects \(\partial W_3\) in \(S_3\). No new states are generated by considering other possibilities, for example \(Q_1 M_2 M_3 \simeq Q_2 M_3 M_2^{-1} \simeq Q_3 M_2^{-1} M_1^{-1}\).

The generalization to \(SU(n)\) follows in an obvious way. We can generate a complete set of states by starting with the dyons \(Q_i\), whose magnetic charges are simple roots. First of all, \(Q_i M_i^{\pm 1}\) generates the tower of anti-dyons \((-\alpha_i, -(p \pm 2)\alpha_i)\). Dyons with magnetic charges which are not simple roots are generated by

\[
Q_i M_i^{\varepsilon_i+1} M_i^{\varepsilon_{i+2}} \cdots M_j^{\varepsilon_{j-1}},
\] (4.11)

where \(\varepsilon_k = \pm 1\) and we take \(j > i + 1\). Using the fact that \(\alpha_i + \cdots + \alpha_{j-1} = e_i - e_j\), these dyons have charges

\[
\left( e_i - e_j, p' (e_i - e_j) - \sum_{k=i+1}^{j-1} \varepsilon_k (e_i - e_k) \right),
\] (4.12)
for some integer $p'$ related to $p$. Each of the $2^{j-i-1}$ sets in (4.11) is present in a different region of $W$ given by

$$D_{e_i-e_{i+1},e_{i+1}-e_j} \cap \cdots \cap D_{e_{k+1}-e_k,e_k-e_j} \cap \cdots \cap D_{e_{j-1}-e_j,e_j-e_j}. \quad (4.13)$$

Each of these regions is separated by a CMS on which the dyons, which all have magnetic charge $e_i - e_j$, decay.

It is straightforward to see that the same set of dyons can be generated by starting with $Q_k$, where $i \leq k \leq j$. In fact the sets

$$Q_k \left( M_{k+1}^{e_{k+1}} \cdots M_{j-1}^{e_{j-1}} \right) \left( M_{k-1}^{-e_{k-1}} \cdots M_{i+1}^{-e_{i+1}} \right), \quad (4.14)$$

coincide exactly with those in (4.11).

In the next section we shall require the semi-classical monodromy associated to a tower of dyon states. If the tower of dyons have magnetic charge $\alpha$, this is defined to be the monodromy around the singularity $\alpha \cdot a = 0$. So for the tower of dyons associated to the simple roots $Q_i$ the semi-classical monodromy is simply $M_i$. For dyons whose magnetic charges are not simple roots, we can find the monodromy by writing the tower of states as in (4.14), i.e. as $Q_i M$, where $M$ is some product of monodromies of the simple roots. In order to encircle the semi-classical singularity for these dyons, we must first of all follow a path with monodromy $M^{-1}$, which transforms the states to $Q_i$, i.e. with magnetic charge $\alpha_i$. Then we loop the singularity $S_i$ with monodromy $M_i$ and finally follow a path with monodromy $M$ to return to the starting point. So the total semi-classical monodromy for the dyons $Q_i M$ is

$$M^{-1} M_i M. \quad (4.15)$$

Since a given tower can be formed in different ways (4.14), there follow a series of identities. For example $Q_1 M_2 \simeq Q_2 M_1^{-1}$ implies that

$$M_2^{-1} M_1 M_2 = M_1 M_2 M_1^{-1}. \quad (4.16)$$

Such identities can be proved by explicit computation.

There exists a symmetry of the spectrum which follows from the fact that the theory has a discrete global symmetry $\mathbb{Z}_{2n}$ on the moduli space, being the discrete remnant left over from the the classical $U(1)_R$ symmetry which is broken by quantum effects. This symmetry acts as

$$u_j \rightarrow e^{i\pi j/n} u_j. \quad (4.17)$$

In the semi-classical regime this means $a \rightarrow e^{i\pi /n} a$ and it is a simple matter to work out the associated transformation on the spectrum at weak coupling. First of all,

$$\left( \begin{array}{c} a_D \\ a \end{array} \right) \rightarrow e^{i\pi /n} \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right), \quad (4.18)$$

so that on the states the symmetry acts as

$$\left( g, q \right) \rightarrow \left( g, q \right) \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right). \quad (4.19)$$

Since this transformation relates dyon states in the same tower, it is clearly a symmetry of the weak coupling spectrum that we have proposed.
5. Strong coupling singularities

In this section, we consider the relation between our proposed weak coupling spectrum and the behaviour of the theory at strong coupling. Underlying the exact solution of the model is a hyperelliptic curve \[1,5,6\] which appears in the following way. In order to parameterize the whole moduli space it is necessary to introduce gauge invariant coordinates \(u_j\), \(j = 2, 3, \ldots, n\). At weak coupling, these are related to the variables \(a\) in the following way:

\[
\prod_{i=1}^{n} (x - e_i \cdot a) = x^n - \sum_{j=2}^{n} u_j x^{n-j} \equiv W_n(x, u_j),
\]

(5.1)

where \(x\) is some auxiliary variable and we have defined the polynomial \(W_n(x, u_j)\) in \(x\). The quantum moduli space of the \(SU(n)\) theory is then precisely the moduli space of the curve \(C\) defined by:

\[
C : \quad y^2 = (W_n(x, u_j))^2 - \Lambda^{2n}.
\]

(5.2)

Notice that the polynomial on the right-hand-side factors into \(W_n(x, u_j) \pm \Lambda^n\). Let us suppose that the zeros of \(W_n(x, u_j)\) are located at \(z_i(u_j)\), \(i = 1, \ldots, n\). It follows that the zeros of the right-hand-side are located at

\[
z_i^\pm(u_j, \Lambda) \equiv z_i(u_2, \ldots, u_{n-1}, u_n \pm \Lambda^n).
\]

(5.3)

The curve \(C\) can be represented as the two-sheeted \(x\) plane with cuts running between pairs \(z_i^+\) and \(z_i^-\).

It will be convenient to consider a two-dimensional slice of moduli space defined by \(u_j = 0\), \(j = 1, \ldots, n-2\), and \(v \equiv u_n\), a fixed number which we take to be real and greater than \(\Lambda^n\). The slice is parameterized by \(u \equiv u_{n-1}\). The slice cuts across \(2n\) singular points \(S_j^\pm\) given by the \(n\) solutions of each of the two equations:

\[
\left( \frac{-u}{n} \right)^n + \left( \frac{v \pm \Lambda^n}{n-1} \right)^{(n-1)} = 0,
\]

(5.4)

for fixed \(v\). The slice is illustrated in figure 5.
At the origin, labelled as $u(0)$ in figure 5, the polynomial is simply $W_n(x) = x^n - v$, so that the branch points of $C$ are arranged in pairs at each $n^{th}$ root of unity:

$$z_j^\pm = \omega^{j-1} (v \pm \Lambda^n)^{1/n},$$

(5.5)

where $\omega = e^{2\pi i/n}$. The positions of the branch points are illustrated in figure 6 which also shows a set of homology cycles $\gamma_j, \delta_j$, with $j = 1, \cdots, n$. In the figure only the parts of the cycles on the upper sheet are shown.

Figure 5. Singularities and paths on the slice $u_j = (0, \ldots, 0, u, v)$, for fixed $v$.

Figure 6. Basis of homology cycles of $C$. 
A singularity corresponds to a situation when two zeros, either \( z_i^+ \) with \( z_i^+ \), or \( z_i^- \) with \( z_j^- \), coalesce, which can also be described as the vanishing of a certain homology cycle of \( C \). By carefully tracing out the paths figure 5, one finds that the singularities \( S_j^\pm \) correspond to following vanishing cycles:

\[
S_j^+ \quad \text{i.e.} \quad z_j^+ \sim z_{j+1}^+ \quad \text{with} \quad \gamma_j
\]
\[
S_j^- \quad \text{i.e.} \quad z_j^- \sim z_{j+1}^- \quad \text{with} \quad \delta_j.
\]

The other possible singularities can be reached by taking more complicated paths, for example the dotted path in figure 5 corresponds to \( z_1^+ \) coalescing with \( z_3^+ \) along the vanishing cycle \( \gamma_1 + \gamma_2 \).

In the semi-classical regime, the origin of the slice, \( u(0) = (0, \ldots, 0, v) \), corresponds to

\[
e_{\sigma(j)} \cdot a = v^{1/n} \omega^{j-1},
\]  

where \( \sigma(j) \) is the following permutation of \( \{1, 2, \ldots, n\} \):

\[
\sigma(j) = \begin{cases} 
2j - 1 & j \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\
2(n-j + 1) & j > \left\lfloor \frac{n+1}{2} \right\rfloor.
\end{cases}
\]  

(In the above \( [x] \) is the integer part of \( x \).) The purpose of the permutation is to ensure that \( u(0) \) lies in the fundamental Weyl chamber \( W \) according to (2.1) which requires \( \text{Re}(e_i \cdot a) \geq \text{Re}(e_j \cdot a) \), for \( i < j \). In the weak coupling limit \( \Lambda \to 0 \), each of the pairs of singularities \( S_j^\pm \) coalesce to become the semi-classical singularity

\[
(e_{\sigma(j)} - e_{\sigma(j+1)}) \cdot a = 0.
\]

It is straightforward to show in this limit that \( u(0) \) lies in the region

\[
(D_{\alpha_1, \alpha_2}^+ \cap D_{\alpha_3, \alpha_4}^+ \cap \cdots) \cap (D_{\alpha_2, \alpha_3}^- \cap D_{\alpha_4, \alpha_5}^- \cap \cdots).
\]

One also finds through a numerical analysis that, in the weak coupling limit, the paths to the singularities from \( u(0) \), given by the vanishing of the cycles \( \gamma_j \) and \( \delta_j \), do not cross any CMS.

There is a concrete relation between the curve \( C \) and the quantities \( a(u_j) \) and \( a_D(u_j) \), appearing in the mass formula of BPS states, provided by a very particular one-form \( \lambda \) [6,7]:

\[
g_\eta \cdot a_D + q_\eta \cdot a = \oint \eta, \lambda,
\]  

where \( \eta \) is some homology cycle on \( C \). This implies a mapping between homology cycles \( \eta \) and charges \((g_\eta, q_\eta)\) whose explicit form we shall write down below. As explained in [7], the intersection number of two cycles is given by

\[
\eta \cap \rho = g_\eta \cdot q_\rho - q_\eta \cdot g_\rho.
\]
At the singularity described by the vanishing of a cycle $\eta$, the dyons of charge $\pm(g\eta, q\eta)$ become massless since $\oint \eta = g\eta \cdot a_D + q\eta \cdot a = 0$. In fact, it is the existence of these massless states that causes the breakdown of the effective action. Furthermore, if a singularity corresponds to the states $\pm(g, q)$ becoming massless, then the monodromy around the singularity can be calculated using perturbation theory in dual variables. On finds [7]

$$M(g, q) = \left(\frac{1 - q \otimes g}{g \otimes g} + \frac{-q \otimes q}{1 + g \otimes q}\right) = 1 + \left(\frac{-q}{g}\right) \otimes (g, q), \quad (5.13)$$

with the property that

$$(g, q)M(g, q) = (g, q). \quad (5.14)$$

We now determine the mapping between vanishing cycles and dyon charges. First of all, the vanishing cycles $\gamma_j$ and $\delta_j$ correspond to massless dyons of magnetic charge $\pm(e_{\sigma(j)} - e_{\sigma(j) + 1})$. Secondly, the cycles $\gamma_j$ and $\delta_j$ have the following intersection numbers

$$\gamma_j \cap \gamma_{j+1} = \delta_j \cap \delta_{j+1} = 1,$$
$$\gamma_j \cap \delta_j = 2, \quad \gamma_j \cap \delta_{j-1} = -2, \quad \gamma_j \cap \delta_{j+1} = 0. \quad (5.15)$$

The vanishing of the cycles $\gamma_j$ correspond to dyons of charge

$$\left(\alpha_{\sigma(j)} + \alpha_{\sigma(j)+1}, \alpha_{\sigma(j)+1}\right), \quad 1 \leq j \leq \left[\frac{n-1}{2}\right]$$
$$(-)^n \left(\alpha_{n-1}, \frac{1}{2}(1 - (-)^n)\alpha_{n-1}\right), \quad j = \left[\frac{n+1}{2}\right]$$
$$-\left(\alpha_{\sigma(j)-2} + \alpha_{\sigma(j)-1}, \alpha_{\sigma(j)-2}\right), \quad \left[\frac{n+3}{2}\right] \leq j \leq n - 1$$
$$-\langle\alpha_1, 0\rangle, \quad j = n, \quad (5.16)$$

along with

$$(g_{\delta_j}, q_{\delta_j}) = (g_{\gamma_j}, q_{\gamma_j} + g_{\gamma_j}). \quad (5.17)$$

This assignment is the unique choice consistent with (5.12) and (5.15), up to the freedom to perform the overall transformation

$$(g, q) \to (g, q)\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad p \in \mathbb{Z}. \quad (5.18)$$

A more revealing way to write the charges of the dyons in (5.16), whose magnetic charges are not simple roots, is

$$\left(\alpha_{\sigma(j)} \alpha_{\sigma(j)+1}, 1 \leq j \leq \left[\frac{n-1}{2}\right]\right)$$
$$-\left(\alpha_{\sigma(j)-2} \alpha_{\sigma(j)-2}, \left[\frac{n+3}{2}\right] \leq j \leq n - 1\right) \quad (5.19)$$

16
Recall, the electric charges of dyons whose magnetic charges are not simple roots, is different in each of the domains in (4.13). Furthermore, as we have already remarked, the paths to the singularities along the vanishing cycles $\gamma_j$ and $\delta_j$, in the weak coupling limit, do not pass through any CMS, therefore the singularities $S_j^{\pm}$ must be caused by dyons which are actually in the spectrum at $u(0)$. From (5.10), we see that the dyons with charges (5.19) are indeed present at $u(0)$. It is an important and highly non-trivial check of our proposal for the weak coupling spectrum that the dyons (5.19), which are responsible for the singularities $S_j^{\pm}$ in the slice, are actually present in the spectrum at $u(0)$.

The monodromies around the singularities follow from the charges of the states and the formula (5.13). However, they can also be deduced directly from the curve $C$. For example, a loop around the singularity at which $\gamma_i$ vanishes, makes $z_i^+$ interchange with $z_{i+1}^+$. The way that the homology cycles are transformed can be determined from the Picard-Lefshetz formula [24]. This states that the action of the monodromy around a singularity corresponding to a vanishing cycle $\nu$ on a cycle $\eta$ is

$$M_\nu(\eta) = \eta - (\eta \cap \nu)\eta. \quad (5.20)$$

It is easy to verify that the monodromy computed from (5.20) is consistent with that computed from (5.13) by virtue of the relation

$$(g_{M_\nu(\eta)}, q_{M_\nu(\eta)}) = (g_\eta, q_\eta)M_{(g_{\nu}, q_\nu)}, \quad (5.21)$$

and (5.12).

As a non-trivial check of these assignments (5.16) we now show that they lead to the correct semi-classical monodromies in (2.4). In the full quantum theory, the semi-classical singularities split into two singularities. With reference to figure 5, we see that the product of the two monodromies corresponding to the vanishing of the cycles $\gamma_j$ and $\delta_j$, taken in that order, should give the appropriate semi-classical monodromy. So if the pair of quantum singularities correspond to the massless dyons $(\alpha_j, p\alpha_j)M$ and $(\alpha_j, (p+1)\alpha_j)M$, respectively, where $M$ is some semi-classical monodromy transformation and $p$ is some integer, then, taking into account that monodromy transformations act on charge vectors to the left, we should have

$$M^{-1}M_jM = M_{(\alpha_j, p\alpha_j)M}M_{(\alpha_j, (p+1)\alpha_j)M}, \quad (5.22)$$

where the left-hand-side is the semi-classical monodromy associated to the tower of dyons $Q_jM$ in (4.15). To verify (5.22), firstly one shows

$$M_j = M_{(\alpha_j, p\alpha_j)M}(\alpha_j, (p+1)\alpha_j), \quad (5.23)$$

which follows from the explicit expressions in (2.4) and (5.13). Then from (5.13) it is easy to see that for any semi-classical monodromy $M$

$$M_{(g, q)M} = M^{-1}M_{(g, q)M}. \quad (5.24)$$

Both (5.23) and (5.24) together imply (5.22).
6. Final comments

The problem of finding the spectrum of BPS saturated states in these theories is complicated by the existence of CMS. At weak coupling we have found the CMS, but in order to make progress across the whole of the moduli space it would be desirable to have a description of the regions where particular dyon states are stable in terms of the auxiliary hyper-elliptic curve $C$ which underpins the exact solution of the theory. Some recent progress in this direction has been made in [25].

Our more modest aim has been to propose a form for the semi-classical spectrum of BPS saturated states in $N=2$ supersymmetric SU($n$) gauge theory at weak coupling based on the known spectrum of the associated $N=4$ supersymmetric theory and the positions of the CMS. The spectrum was shown to be consistent with the exact solution of the theory at strong coupling, however, as we have mentioned, the dyon spectrum at strong coupling is still an open question. In particular, one would like to known the geometry of the CMS at strong coupling.

Lastly, it would be an interesting challenge to understand the picture of the spectrum at weak coupling directly in terms of semi-classical quantization. In fact it is possible to show directly how monopoles decay on a CMS at weak coupling, since on these subspace the moduli space of a monopole changes discontinuously.

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