Dynamical stability of Minkowski space in higher order gravity

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Abstract

We discuss the Minkowski stability problem in modified gravity by using dynamical system approach. The method to investigate dynamical stability of Minkowski space was proposed. This method was applied for a some modified gravity theories, such as \( f(R) \) gravity, \( f(R)+\alpha R\Box R \) gravity and scalar-tensor gravity models with non-minimal kinetic coupling.

1 Introduction

The unsolved problem of dark energy [1, 2] generate a number of so-called modified gravity theories [3]. One of the simplest from this modifications of gravity is \( f(R) \) gravity [4, 5, 6], where scalar curvature \( R \) in Einstein-Hilbert action is replaced with some function \( f(R) \), so the action take the form

\[
S = \int d^4x \sqrt{-g} f(R) + S_m.
\]

(1)

It is well known that equation of motion for this theory reads:

\[
-\frac{1}{2} f_{;i\;k} + f'R_{ik} - \nabla_i \nabla_k f' + g_{ik} \Box f' = \kappa^2 T_{ik}.
\]

(2)

This equation contain higher derivatives with respect to metric up to the forth instead of the second one in General Relativity, and this fact may be the reason of different instabilities of classical solutions. There are two most general restrictions, which may be be found by the different ways: \( f' > 0 \) to guarantee that graviton is not a ghost (or to avoid antigravity on the classical level) and \( f'' > 0 \) to guarantee that particle associated with a new degree of freedom and named scalaron [7] is not tachyon. There are may be different additional restrictions for \( f(R) \) gravity also, which associated with another solutions, for instance Jeans instability [8] or de Sitter stability condition [9]. In this sense Minkowski solution takes a special place: from the one hand its stability is very

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important for theory by the obvious reasons, from another hand the investigation of this stability is very hard task by the usual dynamical system approach (which working good in de Sitter case), because of vanish eigenvalues. In this case we need to investigate central manifold from mathematical point of view and this task is much more time-consuming. We can see the good illustration of troubles on this way in [10], where Minkowski stability was investigate in the simplest case of quadratic \( f(R) \) gravity [7], which is still under consideration [11, 12]. Some another considerations about Minkowski stability was discussed in [13]. In this paper we study stability of Minkowski solution by the dynamical system approach, but using some mathematical trick, which allow us obtain result without central manifold studying. The paper is organized as follows. In section 2 we develop our method and apply it to \( f(R) \) gravity model. In section 3 we study Minkowski stability in one of the simplest case of higher order gravity with \( R \Box R \) in the action. And in section 4 we turn to the scalar-tensor gravity model with non-minimal kinetic coupling. Some concluding remarks may be found in section 5.

\section{f(R) gravity model}

First of all let us discuss Fridman metric

\[ g_{ik} = diag(-1, a^2, a^2, a^2), \]

with \( \Lambda \)-term as the simplest non-trivial matter

\[ T_{ik} = diag(\Lambda, -\Lambda g_{11}, -\Lambda g_{22}, -\Lambda g_{33}). \]

In this case full dynamical picture is described by the 00-component of eq. (2) (for the sakes of simplicity we put \( \kappa^2 = 1 \)):

\[ \frac{1}{2} f - 3(\dot{H} + H^2) f' + 3H f' = \Lambda, \]

which can be rewrote as dynamical system of two variables \( H \) and \( R \):

\[ \begin{align*}
\dot{H} &= \frac{1}{3} R - 2H^2 \equiv F(H, R), \\
\dot{R} &= \frac{1}{3f'} (\Lambda - \frac{1}{2} f + \frac{1}{2} R f' - 3H^2 f') \equiv G(H, R).
\end{align*} \]

It is clear that all equilibrium points of this system are dS-points \((H_0, R_0)\) defined by the next relations

\[ R_0 = 12H_0^2, \quad S = \frac{1}{2} f_0 - \frac{1}{4} R_0 f'_0 - \Lambda = 0. \]

It’s obvious that there are a lot of dS points in the most general case, but in this paper we will mainly discuss one from its, which is correspond to the Minkowski point \((H_0 = 0, R_0 = 0)\) at \( \Lambda = 0 \). We denote it dS\(_M\) (it is clear that for non-vanish values of \( \Lambda \) this point is the nearest to the point \((H_0 = 0, R_0 = 0)\)). This
dS\(_M\) point always exist even in the theories without intrinsic dS points. For example in quadratic gravity \(f(R) = R + \alpha R^2\) there is no intrinsic dS points, but dS\(_M\) point exist: it mean that there is one dS point (for non-vanish value \(\Lambda\)), which is tend to the Minkowski point as \(\Lambda\) tend to zero ( for any \(f(R)\)-theory we have \(R_0 = 12H_0^2 \geq 0\) so \(R_0 \to +0\) as \(\Lambda \to +0\)). In this sense we may study Minkowski point as a limit point of dS\(_M\) points set, and interpret \(\Lambda\) as parameter of our theory.

### 2.1 Quadratic gravity

Let us apply this idea to the simplest case of quadratic gravity to demonstrate how it work. Thus we put

\[
\begin{align*}
f(R) &= R + \alpha R^2, \\
H_0 &= \sqrt{\frac{\Lambda}{3}}, \quad R_0 = 4\Lambda.
\end{align*}
\]

(8)

and find from (7)

\[
\begin{align*}
H_0 &= \sqrt{\frac{\Lambda}{3}}, \quad R_0 = 4\Lambda. \\
\end{align*}
\]

(9)

Stability of this point is governed by the characteristic equation

\[
\begin{vmatrix}
(F_H)_0 - \mu & (F_R)_0 \\
(G_H)_0 & (G_R)_0 - \mu
\end{vmatrix} = 0,
\]

(10)

where (all this expressions are true for general case of \(f(R)\)-gravity)

\[
\begin{align*}
F_H &= -4H, \\
F_R &= \frac{1}{6}, \\
G_H &= \frac{1}{6H^2 f''} (f - Rf' - 6H^2 f' - 2\Lambda), \\
G_R &= \frac{1}{6H(f'')^2} [((R - 6H^2)(f'')^2 + (6H^2 f' + f - Rf' - 2\Lambda)f''),
\end{align*}
\]

(11)(12)(13)

and index 0 mean function’s value at the studying stationary point, whereas \(H\) or \(R\) index mean partial derivative with respect to corresponding variable.

After the quite trivial calculations we find eigenvalues for quadratic gravity model

\[
\mu_{1,2} = \frac{1}{2} \left[ -\sqrt{3\Lambda} \pm \sqrt{3\Lambda - \frac{2}{3\alpha}}(1 + \alpha + 4\alpha \Lambda) \right].
\]

(15)

We can see from this expression that from \(\alpha > 0\) and \(\Lambda > 0 \Rightarrow \text{Re}(\mu_{1,2}) < 0\).

So we have \(\mu_{1,2} \to -0 \pm i\sqrt{\frac{2(\alpha+1)}{3\alpha}}\) as \(\Lambda \to +0\). It mean that any dS\(_M\) point which is arbitrarily close to Minkowski point is stable and therefore we may say that Minkowski point is stable also and this result in good agreement with [10], where were find that Minkowski point is stable in quadratic \(f(R)\) gravity in expanding universe (see below).
2.2 General case of $f(R)$ gravity

Now let us back to the general case of $f(R)$-gravity. Solution of equation (10) reads

$$\mu_{1,2} = \frac{1}{2} \left[ (G_R)_0 + (F_H)_0 \right] \pm \sqrt{(G_R)_0 + (F_H)_0}^2 + 4(F_R)_0(G_H)_0]$$, (16)

so stability conditions takes the form

$$(G_H)_0 < 0, \quad (G_R)_0 + (F_H)_0 < 0.$$ (17)

From another hand using (13), (14) and (7) we find

$$(G_H)_0 = -\frac{2}{3} f'_0, \quad (G_R)_0 + (F_H)_0 = -3H_0.$$ (18)

The most number of $f(R)$-gravity models apply $f' > 0$ and $f'' > 0$ to avoid tachyon and ghost instability. Thus we have for exact dS solution $f'_0 > 0$, $f''_0 > 0$ and therefore any dS solution is stable with respect to homogeneous isotropic metric perturbations in expanding Universe ($H > 0$).

Situation with Minkowski solution is not so trivial in the most general case. To discuss Minkowski stability problem let us back to the dS-M point conception. We have next relation instead of (16)

$$\mu_{1,2} = \frac{1}{2} \left[ -3H_0 \pm 3H_0\sqrt{1 - B} \right]$$, (19)

where $B = \frac{4}{27} \frac{f''_0}{H^2 H_0}$ and $H_0 = H_0(\Lambda), f'_0 = f'_0(\Lambda), f''_0 = f''_0(\Lambda)$ and $H_0 \to +0$ as $\Lambda \to +0$. Since we imply $f' > 0$ and $f'' > 0$ conditions we have a few different possibilities:

- case I $B \to +\infty$ as $\Lambda \to +0$. In this case we have $\mu_{1,2} = -\frac{3}{2} H_0 \pm i\frac{3}{2} H_0 \sqrt{B}$, so $Re(\mu_{1,2}) \to -0$ and Minkowski solution is stable in the sense which was discussed above.

- case Ia $B \to B_0 > 1$ as $\Lambda \to +0$. In this case we have $\mu_{1,2} = -\frac{3}{2} H_0 \pm i\frac{3}{2} H_0 \sqrt{B - 1}$, so $Re(\mu_{1,2}) \to -0$ and Minkowski solution is stable also.

- case II $B \to B_0$, where $0 < B_0 < 1$. In this case we have $\mu_1 = -\frac{3}{2} H_0 [B + O(B^2)] \to -0$ and $\mu_2 = -\frac{3}{2} H_0 [2 - \frac{1}{2} B + O(B^2)] \to -0$ and $Im(\mu_{1,2}) = 0$, so Minkowski solution is stable.

- case IIa $B \to +0$. It's clear that in this case eigenvalues is similar to the previous one, so this case is a special case of the case II and Minkowski solution is stable.

- case IIb In the most trivial case $B \to 1$ both eigenvalues is equal to $-\frac{3}{2} H_0$, so Minkowski solution is stable also.
Thus we find that in any ghost-free and tachyon-free \( f(R) \) gravity model Minkowski solution is stable with respect to homogeneous isotropic perturbations in expanding universe \((H > 0)\). Note that main number of models relate to the class I, for instance \( R + \alpha R^n \), Hu-Sawicky [14], Battye-Aplleby [15] or Starobinsky models [16].

2.3 Anisotropic perturbations in \( f(R) \) gravity

Now let us turn to the more general case of homogeneous anisotropic perturbations. For this task we need to discuss Bianchi I metric

\[ g_{ik} = \text{diag}(-1, a^2, b^2, c^2), \] (20)

where functions \( a, b, c \) are functions of time only. (It is well known that in GR all first type Bianchi metrics can be diagonalized and conserve its form due to Einstein equations. It is also true in \( f(R) \)-gravity, so expression (20) is the most general form for Bianchi I metric in our case.) Also we introduce Hubble parameters \( H_a = \dot{a}/a, H_b = \dot{b}/b, H_c = \dot{c}/c \). Only diagonal terms of equation (2) is nontrivial and (00) component now reads:

\[ \frac{1}{2} f - (\dot{H}_a + H_a^2 + \dot{H}_b + H_b^2 + \dot{H}_c + H_c^2) f' + (H_a + H_b + H_c) \frac{\partial f'}{\partial t} = \Lambda, \] (21)

And for (11) component we have:

\[ -\frac{1}{2} f + (\dot{H}_a + H_a^2 + H_a H_b + H_a H_c) f' + H_a \frac{\partial f'}{\partial t} - \frac{\partial^2 f'}{\partial t^2} - (H_a + H_b + H_c) \frac{\partial f'}{\partial t} = -\Lambda, \] (22)

for (22):

\[ -\frac{1}{2} f + (\dot{H}_b + H_b^2 + H_a H_b + H_b H_c) f' + H_b \frac{\partial f'}{\partial t} - \frac{\partial^2 f'}{\partial t^2} - (H_a + H_b + H_c) \frac{\partial f'}{\partial t} = -\Lambda, \] (23)

for (33):

\[ -\frac{1}{2} f + (\dot{H}_c + H_c^2 + H_c H_b + H_a H_c) f' + H_c \frac{\partial f'}{\partial t} - \frac{\partial^2 f'}{\partial t^2} - (H_a + H_b + H_c) \frac{\partial f'}{\partial t} = -\Lambda, \] (24)

So we have some strange situation: the highest derivative is contained in all three equations by the similar way: by the term \( \dot{f}' \) and therefore \( \dot{R} \) (note here that including of matter in r.h.s. do not change situation). So the system of differential equations is degenerated with respect to highest derivatives.

The interpretation of this fact may be the next. Actually the number of independent variables less then 3. For illustration of this proposition let us try to transform system (21)-(24). Introducing new variable \( H = H_a + H_b + H_c \) we find for expression in (21):

5
\[ H_a + H_b^2 + \dot{H}_b + H_c + H_e^2 = R - H^2 - \dot{H}, \]
so the equation (21) take the form:
\[ \frac{1}{2} f - (R - \dot{H} - H^2)f' + Hf''\dot{R} = 0. \] (25)

From another hand summing equations (22)+(23)+(24) we find
\[ -\frac{3}{2} f + (\dot{H} + H^2)f' - 2Hf''\dot{R} - 3f'''\dot{R}^2 - 3f''\dot{R} = 0. \] (26)
So we have actually system (25)-(26) of two differential equations with two variables instead of system (22)-(24) of three equations with three variables. Actually it do not mean that this is final result, because there may be further simplifications and this procedure is well known from the literature [17], where was shown how system (22)-(24) can be transform to unique equation with one variable, but for our special task it is comfortable to use two variables: \( H \) and \( R \). Note one more time: this result is the general one for any \( f(R) \)-theories in Bianchi I ansatz (20).

Let us consider this result in GR limit \( f = R, \Lambda = 0 \). We have from equations (25), (26):
\[ \frac{1}{2} R - (R - \dot{H} - H^2) = 0, \]
\[ -\frac{3}{2} R + (\dot{H} + H^2) = 0. \] (27)
We can see that situation is absolutely similar to the previous one: system is degenerated. It mean that there is only one independent variable – and this is true result, as we know from Kasner solution. System (27) tell us that actually in this case there is only one equation \( \dot{R} = 0 \). Substituting this back to the system (27) we find \( \dot{H} + H^2 = 0 \), which have solution \( H = 1/t \). From this relation we immediately reproduce one of the Kasner expressions \( \sum p_i = 1 \).

Let us now study solutions of equations (25), (26) using dynamical system approach. First of all we rewrite it as dynamical system:
\[
\begin{align*}
\dot{H} &= -\frac{1}{f} \left( \frac{1}{2} f + Hf''D - \Lambda \right) + R - H^2 \equiv F(H, R, D), \\
\dot{R} &= D, \\
\dot{D} &= \frac{1}{fR} \left( -2f - 3Hf''D - 3f'''D^2 + Rf' + 4\Lambda \right) \equiv G(H, R, D).
\end{align*}
\] (28)
To investigate stability of some solution \( (H_0, R_0, D_0) \) of the system (28) we linearize it near this solution:
\[
\begin{align*}
\dot{H} &= (F_H)_{0} H + (F_R)_{0} R + (F_D)_{0} D, \\
\dot{R} &= D, \\
\dot{D} &= (G_H)_{0} H + (G_R)_{0} R + (G_D)_{0} D,
\end{align*}
\] (29)
where \((F_H)_0\) denote the value of partial derivative of function \(F\) with respect to \(H\) at the point \((H_0, R_0, D_0)\) etc. Characteristic equation for the linearized system (29) is:

\[
\left| \begin{array}{ccc}
(F_H)_0 - \mu & (F_R)_0 & (F_D)_0 \\
0 & -\mu & 1 \\
(G_H)_0 & (G_R)_0 & (G_D)_0 - \mu
\end{array} \right| = 0,
\]

which give us equation for eigenvalues \(\mu\). It is easy to see that all equilibrium points are determined by the expression \(D_0 = 0\). From another hand we have \((G_H)_0 = -D_0 = 0\) for any equilibrium points of the system (28), so actually we have instead of (30):

\[
\left| \begin{array}{ccc}
(F_H)_0 - \mu & 1 & 0 \\
(G_R)_0 & (G_D)_0 - \mu & 0
\end{array} \right| = 0,
\]

which give us eigenvalues:

\[
\mu_1 = (F_H)_0, \quad \mu_{2,3} = \frac{1}{2} \left( (G_D)_0 \pm \sqrt{(G_D)_0^2 + 4(G_R)_0} \right),
\]

so stability conditions takes the form

\[
(F_H)_0 = -2H_0 < 0, \quad (G_D)_0 = -H_0 < 0, \quad (G_R)_0 < 0,
\]

It is easy to find that all equilibrium points are determined by the next expressions:

\[
D_0 = 0, \quad R_0 = \frac{2f(R_0) - 4\Lambda}{f'(R_0)}, \quad H_0^2 = \frac{3}{4} R_0.
\]

This expressions are totally identical to (7) with replacing \(H \to 3H\), so we can see that all equilibrium points are some kind of dS-points. The physical meaning of this points is not quite clear, since we have produce a some manipulations with initial variables, so first of all let us discuss it on the example of power-law function \(f\), for which equations may be solved exactly:

\[
f(R) = R + \alpha R^n.
\]

In the case \(n = 2\) there is only equilibrium point \((0, 0, 0)\). For \(\alpha > 0\) and \(n > 2\) there are three equilibrium points \((H_0, R_0, D_0)\): \((0, 0, 0)\), \((\pm \frac{4\sqrt{3}}{3} \alpha [(n - 2)]^{\frac{1}{n-2}}, \alpha (n - 2)]^{\frac{1}{n-2}}, 0\)\). Let us discuss the physical meaning of this points. Expression for \(R\) may be rewrote as

\[
R = 2(\dot{H} + H^2 - H_aH_b - H_aH_c - H_bH_c),
\]
thus we have
\[ H = 0, \quad H_a + H_b + H_c = 0, \]
\[ R = 0, \quad H_a H_b + H_a H_c + H_b H_c = 0, \quad (37) \]
which give us
\[ H_b^2 + H_b H_c + H_c^2 = 0. \quad (38) \]
The only possibility to satisfy last expression (for non-complex values of Hubble parameters) is \( H_a = H_b = H_c = 0 \), which is correspond to the Minkowski space.

Note also that this point exist for any \( f(R) \) gravity model with \( f(0) = 0, \Lambda = 0 \).

Here we introduce new notation \( A = [\alpha(n - 2)]^{\frac{1}{n - 3}} \). From (36) we have:
\[ H_a H_b + H_a H_c + H_b H_c = \frac{1}{4} A^2, \quad (39) \]
and using definition of \( H \)
\[ H_a = \pm \frac{\sqrt{3}}{2} A - H_b - H_c, \quad (40) \]
we find the next expression
\[ H_b^2 + H_b (H_c + \frac{\sqrt{3}}{2} A) + H_c^2 + \frac{1}{4} A^2 + \frac{\sqrt{3}}{2} AH_c = 0, \quad (41) \]
solving this equation with respect to \( H_b \) we find discriminant
\[ -3H_c^2 \pm \sqrt{3}AH_c - \frac{1}{4} A^2, \quad (42) \]
which can not be positive, but vanish at the point \( H_c = \pm \frac{\sqrt{3}}{6} A \). Thus we have
\[ (+\frac{\sqrt{3}}{2} A, A^2, 0) \iff H_a = H_b = H_c = \frac{\sqrt{3}}{6} A, \quad (43) \]
which correspond to the usual dS-point in expanding universe, and
\[ (-\frac{\sqrt{3}}{2} A, A^2, 0) \iff H_a = H_b = H_c = -\frac{\sqrt{3}}{6} A, \quad (44) \]
which correspond to the dS-point in collapsing universe.

Now let us discuss stability conditions (33). It’s clear that \( \mu_1 < 0 \) for any \( f(R) \) model in expanding universe. Expression for \( \mu_{2,3} \) may be rewrote as
\[ \mu_{2,3} = -\frac{1}{2} H_0 \pm \frac{1}{2} H_0 \sqrt{1 - B}, \quad (45) \]
where \( \tilde{B} = \frac{4 R_0 f''_0 - f'_0}{H_0^2 f'_0} \) and index 0 mean function’s value at dS point. Thus we reproduce well known condition \( \tilde{B} > 0 \) for stability of dS point [18, 19]. As for dS_M point we have situation absolutely similar to the previous one (isotropic perturbations). So for stability of Minkowski space in expanding universe it is enough one of the next conditions in any possible combinations:
• $f' \to +\infty$ or $f' \to A > 0$ or $f' \to +0$ as $\Lambda \to +0$

• $f'' \to +\infty$ or $f'' \to B > 0$ or $f'' \to +0$ as $\Lambda \to +0$

Thus our main conclusion is: Minkowski space is stable in any tachyon-free ($f'' > 0$) and ghost-free ($f' > 0$) $f(R)$ gravity model in expanding universe with respect to isotropic and basic anisotropic (homogeneous) perturbations.

3 $f(R) + \alpha R\Box R$ gravity model

Now let us discuss possible influence of higher derivative terms on Minkowski stability problem. As the simplest example of such kind of theory we study action in the next form

$$S = \int d^4x \sqrt{-g} [f(R) + \alpha R\Box R] + S_m.$$  \hspace{1cm} (46)

This theory is more complicated than usual $f(R)$ gravity model, so we discuss the simplest case of isotropic perturbations only. We have next additional terms in the left hand side of Fridman equation (5) for FRW metric (3) (for more details see [20])

$$\dot{\alpha}(2R\ddot{R} + 36H^2\dot{R} - \dot{R}^2 - 48H^2\ddot{R} - 12H\dot{R}),$$  \hspace{1cm} (47)

so instead of system (6) we have now

$$\begin{align*}
\dot{H} &= \frac{1}{3} R - 2H^2, \\
\dot{\dot{R}} &= C, \\
\dot{C} &= D, \\
\dot{D} &= \frac{1}{12\alpha^2} \left( A - \Lambda + 2\alpha RD + 36\alpha H^3C - \alpha C^2 - 48\alpha H^2D \right) \equiv M,
\end{align*}$$  \hspace{1cm} (48)

where $A = A(H, R, C) = \frac{1}{3} f + (3H^2 - \frac{1}{2}R)f' + 3Hf''C$ is the left hand side of equation (4). First of all note, that there is no additional dS-point due to (47)-terms, but this terms may change the stability conditions for dS-point (including dS-$M$) arising from $f(R)$-part. Nevertheless in such kind of theory we have $R_0 \to +0$ as $\Lambda \to +0$) as in the previous one. The linearized equation governing stability at equilibrium point take the form

$$\begin{vmatrix}
-4H_0 - \mu & \frac{1}{6} & 0 & 0 \\
0 & -\mu & 1 & 0 \\
0 & 0 & -\mu & 1 \\
(M_H)_0 & (M_R)_0 & (M_C)_0 & (M_D)_0 - \mu
\end{vmatrix} = 0,$$  \hspace{1cm} (49)
or
\[ a_0 \mu^4 + a_1 \mu^3 + a_2 \mu^2 + a_3 \mu + a_4 = 0, \tag{50} \]
with
\[ a_0 = 1, \ a_1 = 4 H_0 - (M_D)_0, \ a_2 = -4 H_0 (M_D)_0 - (M_C)_0, \]
\[ a_3 = -(M_R)_0 - 4 H_0 (M_C)_0, \ a_4 = -4 H_0 (M_R)_0 - (M_H)_0 / 6. \tag{51} \]

Since the finding of general solution of (50) is a hard task we use Routh-Hurwitz theorem [21] which tell us that all solutions of (50) have a negative real parts (and therefore equilibrium point 0 is stable) if and only if satisfy next relations:
\[ T_0 = a_0 > 0, \tag{52} \]
\[ T_1 = a_1 > 0, \tag{53} \]
\[ T_2 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0, \tag{54} \]
\[ T_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0, \tag{55} \]
\[ T_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = a_4 T_3 > 0. \tag{56} \]

Now let us calculate partial derivatives of \( M \). By using (7) relations we find
\[ (M_H)_0 = \frac{f_0''}{2 \alpha}, \tag{57} \]
\[ (M_R)_0 = -\frac{H_0 f_0''}{2 \alpha}, \tag{58} \]
\[ (M_C)_0 = \frac{f_0'''}{4 \alpha} + 3 H_0^2, \tag{59} \]
\[ (M_D)_0 = -2 H_0, \tag{60} \]

and therefore
\[ a_1 = 6 H_0, \ a_2 = 5 H_0^2 - \frac{f_0''}{4 \alpha}, \ a_3 = -12 H_0^3 - \frac{H_0 f_0''}{2 \alpha}, \ a_4 = \frac{2 H_0^2 f_0''}{\alpha} - \frac{f_0'''}{12 \alpha}. \tag{61} \]

We can see that (52) and (53) satisfied automatically in expanding Universe \((H_0 > 0)\), (54) give us
\[ 42 H_0^2 - \frac{f_0''}{\alpha} > 0, \tag{62} \]

from (55) we find
\[ -504 H_0^4 - 81 \frac{H_0^3 f_0''}{\alpha} + \frac{f_0'''}{\alpha^2} + \frac{3 f_0'}{\alpha} > 0, \tag{63} \]
and (56) give us
\[ a_4 = \frac{2H^2 f''_0}{\alpha} - \frac{f'_0}{12\alpha} > 0. \] (64)

In principle by using this inequalities we may verify stability of any dS point for any shape of function \( f(R) \), but mainly we are interested in dS \(_M\) point. For this point as we already mentioned we have \( H^2 \to +0 \) as \( \Lambda \to +0 \). From another hand it is well known [22] that only positive values of \( \alpha \) give us a ghost free theory. Thus from (62) we have \( f''_0 < 0 \), from (64) we have \( f'_0 < 0 \) and from (63) we find \( 3f''_0 + 2f'_0^2/\alpha > 0 \) (note also that for negative \( \alpha \) the last inequality is impossible). This three conditions guarantee us stability of Minkowski solution with respect to isotropic perturbations. We can see that taking into account higher derivative terms may significantly change stability conditions for \( f(R) \) gravity.

4 Scalar-tensor gravity model with non-minimal kinetic coupling

Now let us try to apply developed technic to the scalar-tensor gravity model with non-minimal kinetic coupling [23, 24]

\[ S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) - 2\Lambda \right], \] (65)

where we incorporate \( \Lambda \)-term in the action. Equations of motion for FRW metric [25] takes the form [26]

\[ 3H^2 - \frac{1}{2} \dot{\phi}^2 + \frac{9}{2} \kappa H^2 \dot{\phi}^2 = V + \Lambda, \] (66)

\[ 2\dot{H} + 3H^2 + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \kappa (2\dot{H} \dot{\phi}^2 + 3H^2 \ddot{\phi}^2 + 4H \dot{\phi} \dddot{\phi}) = V + \Lambda, \] (67)

\[ (\dddot{\phi} + 3H \dddot{\phi}) - 3\kappa (H^2 \dddot{\phi} + 2H \dot{H} \ddot{\phi} + 3H \ddot{\phi} \dot{\phi} - 3V') = -V', \] (68)

where equation (66) is the first integral of (67) and (68). We can see that this theory have second order equations, it mean that we may exclude \( \dot{H} \) from the system without loss of generality. Indeed, multiplying equation (67) on \( 6\kappa H \dot{\phi} \), equation (68) on \( (2 + \kappa \dot{\phi}^2) \), summing and resolving with respect to highest derivative term, we gain the next dynamical system

\[ \dot{\phi} = \Phi, \] (69)

\[ \ddot{\phi} = -\frac{3H(2+3\kappa \Phi^2-6\kappa H^2-9\kappa^2 H^2 \Phi^2)-(2+\kappa \Phi^2)V'}{2+\kappa \Phi^2-6\kappa H^2+9\kappa^2 H^2 \Phi^2} \equiv f(H, \phi, \Phi), \]

where \( H \) now is not dynamical variable but parameter depending on \( \Lambda \) and combination \( V + \Lambda \) was excluded by using (66). Equilibrium points of system (69) are defined by the next relations

\[ \dot{\phi}_0 = 0, \quad \ddot{\phi}_0 = 0, \quad V'(\phi_0) = 0, \quad 3H_0 = \Lambda + V(\phi_0), \] (70)
where $V'(\phi_0) = 0$ is the consequence of (68). Since we are interested in Minkowski solution ($H_0 = 0$), we need to put also $V(\phi_0) = 0$. Eigenvalues $\mu_i$ of system (69) may be find from the next equation

$$\begin{vmatrix}
-\mu & 1 \\
 f_\phi & f_\phi - \mu
\end{vmatrix} = 0,$$

which have solution

$$\mu_{1,2} = \frac{1}{2} \left[ (f_\phi)_0 \pm (f_\phi)_0 \sqrt{1 + 4 \frac{(f_\phi)_0}{(f_\phi)_0}} \right],$$

so the necessary and sufficient condition of equilibrium point’s stability are

$$(f_\phi)_0 = \frac{-2V''(\phi_0)}{2 - 6\kappa H_0^2} < 0,$$

and

$$(f_\phi)_0 = -3H_0 < 0.$$  

Thus we can see that Minkowski stability condition in expanding Universe ($H_0 > 0$) is $V''(\phi_0) > 0$ (which is quite natural for any true vacuum solution), whereas stability of any nontrivial dS solution is governed by (73) relation, where $H_0$ defined by $3H_0^2 = \Lambda + V(\phi_0)$. Note also that Minkowski stability is not depend on sign of $\kappa$ parameter and this is the most unexpected result.

### 5 Conclusion

In this paper we propose a some universal asymptotic method for investigation stability of Minkowski solution in a wide class of modified gravity theories. The main idea quite simple: we introduce lambda term as parameter and find eigenvalues of dS point. After that we investigate limit of eigenvalues at $\Lambda \to +0$. This allow us find Minkowski stability conditions. In some cases our method may be much more simple than any another one. So we hope it will be useful for a number researchers working in this field. Also we have applied our method to a some modified gravity theories and have found new original results (at least in Sec. 3).

### 6 Acknowledgments

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