Estimating the second-order parameter of regular variation and bias reduction in tail index estimation under random truncation

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Abstract

In this paper, we propose an estimator of the second-order parameter of randomly right-truncated Pareto-type distributions data and establish its consistency and asymptotic normality. Moreover, we derive an asymptotically unbiased estimator of the tail index and study its asymptotic behaviour. Our considerations are based on a useful Gaussian approximation of the tail product-limit process recently given by Benchaira et al. [Tail product-limit process for truncated data with application to extreme value index estimation. Extremes, 2016; 19: 219-251] and the results of Gomes et al. [Semi-parametric estimation of the second order parameter in statistics of extremes. Extremes, 2002; 5: 387-414]. We show, by simulation, that the proposed estimators behave well, in terms of bias and mean square error.

Keywords: Bias-reduction; Extreme value index; Product-limit estimator; Random truncation; Second-order parameter.

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1. Introduction

Let \((X_i, Y_i), 1 \leq i \leq N\) be a sample of size \(N \geq 1\) from a couple \((X, Y)\) of independent random variables (rv’s) defined over some probability space \((\Omega, \mathcal{A}, P)\), with continuous marginal distribution functions (df’s) \(F\) and \(G\) respectively. Suppose that \(X\) is truncated to the right by \(Y\), in the sense that \(X_i\) is only observed when \(X_i \leq Y_i\). We assume that both survival functions \(\bar{F} := 1 - F\) and \(\bar{G} := 1 - G\) are regularly varying at infinity with negative tail indices \(-1/\gamma_1\) and \(-1/\gamma_2\) respectively, that is, for any \(x > 0\)

\[
\lim_{z \to \infty} \frac{\bar{F}(xz)}{\bar{F}(z)} = x^{-1/\gamma_1} \quad \text{and} \quad \lim_{z \to \infty} \frac{\bar{G}(xz)}{\bar{G}(z)} = x^{-1/\gamma_2}.
\]  

(1.1)

Since the weak approximations of extreme value theory based statistics are achieved in the second-order framework (see de Haan and Stadtmüller, 1996), then it seems quite natural to suppose that both df’s \(F\) and \(G\) satisfy the well-known second-order condition of regular variation specifying the rates of convergence in (1.1). That is, we assume that for any \(x > 0\)

\[
\lim_{t \to \infty} \frac{U_F(tx)/U_F(t) - x^{\gamma_1}}{A_F(t)} = x^{\gamma_1} \frac{x^{\rho_1} - 1}{\rho_1},
\]

(1.2)

and

\[
\lim_{t \to \infty} \frac{U_G(tx)/U_G(t) - x^{\gamma_2}}{A_G(t)} = x^{\gamma_2} \frac{x^{\rho_2} - 1}{\rho_2},
\]

(1.3)

where \(\rho_1, \rho_2 < 0\) are the second-order parameters and \(A_F, A_G\) are functions tending to zero and not changing signs near infinity with regularly varying absolute values at infinity with indices \(\rho_1, \rho_2\) respectively. For any df \(H, U_H(t) := H^+(1 - 1/t), t > 1\), stands for the quantile function. This class of distributions, which includes models such as Burr, Fréchet, Generalized Pareto, Student, log-gamma, stable,... takes a prominent role in extreme value theory. Also known as heavy-tailed, Pareto-type or Pareto-like distributions, these models have important practical applications and are used rather systematically in certain branches of non-life insurance, as well as in finance, telecommunications, hydrology, etc... (see, e.g., Resnick, 2006). We denote the observed observations of the truncated sample \((X_i, Y_i), i = 1, ..., N, (X_i, Y_i), i = 1, ..., n,\) as copies of a couple of rv’s \((X, Y)\), where \(n = n_N\) is a sequence of discrete rv’s for which, by of the weak law of large numbers satisfies \(n_N/N \xrightarrow{P} p := P(X \leq Y)\), as \(N \to \infty\). The usefulness of the statistical analysis under random truncation is shown in Herbst (1999) where the authors applies truncated model techniques to estimate loss reserves for incurred but not reported (IBNR) claim amounts. For a recent discussion on randomly right-truncated insurance
claims, one refers to Escudero and Ortega (2008). In reliability, a real dataset, consisting in lifetimes of automobile brake pads and already considered by Lawless (2002) in page 69, was recently analyzed in Gardes and Stupfler (2015) and Benchaira et al. (2016a) as an application of randomly truncated heavy-tailed models. The joint distribution of \( X_i \) and \( Y_i \) is \( H(x, y) := P(X \leq x, Y \leq y) = P(X \leq \min(x, Y), Y \leq y | X \leq Y) \), which equals \( p^{-1} \int_0^y F(\min(x, z))dG(z) \). The marginal distributions of the rv's \( X \) and \( Y \), respectively denoted by \( F^* \) and \( G^* \), are equal to \( p^{-1} \int_0^x \overline{G}(z)dF(z) \) and \( p^{-1} \int_0^y F(z)dG(z) \), respectively. The tail of \( F^* \) simultaneously depends on \( \overline{G} \) and \( F \) while that of \( G^* \) only relies on \( \overline{G} \). By using Proposition B.1.10 in de Haan and Ferreira (2006), to the regularly varying functions \( \overline{F} \) and \( \overline{G} \), we also show that both \( \overline{G}^* \) and \( \overline{F}^* \) are regularly varying at infinity, with respective indices \( \gamma_2 \) and \( \gamma := \gamma_1 \gamma_2 / (\gamma_1 + \gamma_2) \). Recently Gardes and Stupfler (2015) addressed the estimation of the extreme value index \( \gamma_1 \) under random right-truncation and used the definition of \( \gamma \) to derive a consistent estimator as a quotient of two Hill estimators (Hill, 1975) of tail indices \( \gamma_1 \) and \( \gamma_2 \) which are based on the upper order statistics \( X_{n-k:n} \leq ... \leq X_{n:n} \) and \( Y_{n-k:n} \leq ... \leq Y_{n:n} \) pertaining to the samples \( (X_1, ..., X_n) \) and \( (Y_1, ..., Y_n) \) respectively. The sample fraction \( k = k_n \) being a (random) sequence of integers such that, given \( n = m = m_N \), \( k_m \to \infty \) and \( k_m/m \to 0 \) as \( N \to \infty \). Under the tail dependence and the second-order regular variation conditions, Benchaira et al. (2015) established the asymptotic normality of this estimator. Recently, Worms and Worms (2016) proposed an asymptotically normal estimator for \( \gamma_1 \) by considering a Lynden-Bell integration with a deterministic threshold. The case of a random threshold, is addressed by Benchaira et al. (2016a) who propose a Hill-type estimator for randomly right-truncated data, defined by

\[
\gamma_{1}^{(BMN)} = \sum_{i=1}^{k} a_n^{(i)} \log \frac{X_{n-i+1:n}}{X_{n-k:n}}, \tag{1.4}
\]

where

\[
a_n^{(i)} := \frac{F_n(X_{n-i+1:n})/C_n(X_{n-i+1:n})}{\sum_{i=1}^{k} F_n(X_{n-i+1:n})/C_n(X_{n-i+1:n})},
\]

with \( F_n(x) := \prod_{i : X_i > x} \exp \left\{ -\frac{1}{nC_n(X_i)} \right\} \) is the well-known product-limit Woodroofer’s estimator (Woodroofer, 1985) of the underlying df \( F \) and \( C_n(x) := n^{-1} \sum_{i=1}^{n} I(X_i \leq x \leq Y_i) \). The authors show by simulation that, for small datasets, their estimator behaves better in terms of bias and root of the mean squared error (rmse), than Gardes-Supfler’s estimator.
Moreover, they establish the asymptotic normality by considering the second-order regular variation conditions (1.2) and (1.3) and the assumption $\gamma_1 < \gamma_2$. More precisely, they show that, for a sufficiently large $N$,

$$
\hat{\gamma}_1^{(BMN)} = \gamma_1 + k^{-1/2} \Lambda (W) + \frac{A_0(n/k)}{1 - \rho_1} (1 + o_p (1)),
$$

where $A_0 (t) := A_F \left( 1/\overline{F} (U_{F^*} (t)) \right) \, , \, t > 1$, and $\Lambda (W)$ is a centred Gaussian rv defined by

$$
\Lambda (W) := \frac{\gamma}{\gamma_1 + \gamma_2} \int_0^1 (\gamma_2 - \gamma_1 - \gamma \log s) \, s^{\gamma/\gamma_2 - 1} W (s) \, ds - \gamma W (1),
$$

with $\{W (s) ; s \geq 0\}$ being a standard Wiener process defined on the probability space $(\Omega, A, P)$. Thereby, they conclude that $\sqrt{k} \left( \hat{\gamma}_1^{(BMN)} - \gamma_1 \right) \overset{P}{\to} N (\lambda / (1 - \rho_1) , \sigma^2)$, as $N \to \infty$, where $\sigma^2 := \gamma^2 (1 + \gamma_1 / \gamma_2) (1 + (\gamma_1 / \gamma_2)^2) / (1 - \gamma_1 / \gamma_2)$, provided that, given $n = m$, $\sqrt{k_m} A_0 (m/k_m) \to \lambda < \infty$. Recently, Benchaira et al. (2016b) adopted the same approach to introduce a kernel estimator to the tail index $\gamma_1$ which improves the bias of $\hat{\gamma}_1^{(BMN)}$. It is worth mentioning that the assumption $\gamma_1 < \gamma_2$ is required in order to ensure that it remains enough extreme data for the inference to be accurate. In other words, they consider the situation where the tail of the rv of interest is not too contaminated by the truncation rv.

The aim of this paper is the estimation of the second order-parameter $\rho_1$ given in condition (1.2) which, to our knowledge, is not addressed yet in the extreme value literature. This parameter is of practical relevance in extreme value analysis due its crucial importance in selecting the optimal number of upper order statistics $k$ in tail index estimation (see, e.g., de Haan and Ferreira, 2006, page 77) and in reducing the bias of such estimation. In the case of complete data, this problem has received a lot of attention from many authors like, for instance, Peng (1998), Fraga Alves et al. (2003), Gomes et al. (2002), Peng and Qi (2004), Goegebeur et al. (2010), de Wet et al. (2012), Worms and Worms (2012), Deme et al. (2013).

Inspired by the paper of Gomes et al. (2002), we propose an estimator for $\rho_1$ adapted to the random right-truncation case. To this end, for $\alpha > 0$ and $t > 0$, we introduce the following tail functionals

$$
M^{(\alpha)} (t; F) := \frac{1}{\overline{F} (U_{F^*} (t))} \int_{U_{F^*} (t)} \log^\alpha (x / \overline{F^*} (t)) \, dF (x),
$$

$$
Q^{(\alpha)} (t; F) := \frac{M^{(\alpha)} (t; F) - \Gamma (\alpha + 1) \left( M^{(1)} (t; F) \right)^\alpha}{M^{(2)} (t; F) - 2 \left( M^{(1)} (t; F) \right)^2},
$$

and

$$
S^{(\alpha)} (t; F) := \delta (\alpha) \frac{Q^{(2\alpha)} (t; F)}{(Q^{(\alpha+1)} (t; F))^2},
$$
where \( \log^a x := (\log x)^a \) and \( \delta(\alpha) := \alpha (\alpha + 1)^2 \Gamma^2(\alpha) / (4 \Gamma(2\alpha)) \), with \( \Gamma(\cdot) \) standing for the usual Gamma function. From assertion \((ii)\) of Lemma 5.1, we have, for any \( \alpha > 0 \),

\[
M^{(\alpha)}(t; F) \rightarrow \gamma_1^a \Gamma(\alpha + 1), \quad Q^{(\alpha)}(t; F) \rightarrow q_\alpha(\rho_1) \quad \text{and} \quad S^{(\alpha)}(t; F) \rightarrow s_\alpha(\rho_1),
\]

(1.9)
as \( t \to \infty \), where

\[
q_\alpha(\rho_1) := \frac{\gamma_1^{\alpha-2} \Gamma(\alpha + 1) (1 - (1 - \rho_1)^\alpha - \alpha \rho_1 (1 - \rho_1)^{\alpha-1})}{\rho_1^2 (1 - \rho_1)^{\alpha-2}},
\]

(1.10)
and

\[
s_\alpha(\rho_1) := \frac{\rho_1^2 (1 - (1 - \rho_1)^{2\alpha} - 2 \alpha \rho_1 (1 - \rho_1)^{2\alpha-1})}{(1 - (1 - \rho_1)^{\alpha+1} - (\alpha + 1) \rho_1 (1 - \rho_1)^{\alpha})^2}.
\]

(1.11)
The three results (1.9) allow us to construct an estimator for the second-order parameter \( \rho_1 \). Indeed, by recalling that \( n = n_N \) is a random sequence of integers, let \( v = v_n \) be a subsequence of \( n \), different than \( k \), such that given \( n = m \), \( v_m \to \infty \), \( v_m/m \to 0 \) as \( N \to \infty \). The sequence \( v \) has to be chosen so that \( \sqrt{v_m} |A_0(m/v_m)| \to \infty \), which is a necessary condition to ensure the consistency of \( \rho_1 \) estimator. On the other hand, as already pointed out, the asymptotic normality of \( \tilde{\gamma}_1(BMN) \) requires that, for a given \( n = m \), \( \sqrt{k_m} A_0(m/k_m) \to \lambda < \infty \). This means that both sample fractions \( k \) and \( v \) have to be distinctly chosen. Since \( \overline{F} \) is regularly varying at infinity with index \(-1/\gamma \), then from Lemma 3.2.1 in de Haan and Ferreira (2006) page 69, we infer that, given \( n = m \), we have \( X_{m−v,m} \to \infty \) as \( N \to \infty \) almost surely. Then by using the total probability formula, we show that \( X_{n−v,n} \to \infty \), almost surely too. By letting, in (1.6), \( t = n/v \) then by replacing \( \mathbb{U}_{F^*}(n/v) \) by \( X_{n−v,n} \) and \( F \) by the product-limit estimator \( F_n \), we get an estimator \( M_n^{(\alpha)}(v) = M^{(\alpha)}(t; F_n) \) for \( M^{(\alpha)}(t; F) \) as follows:

\[
M_n^{(\alpha)}(v) = \frac{1}{F_n(X_{n−v,n})} \int_{X_{n−v,n}}^{\infty} \log^a \left( \frac{x}{X_{n−v,n}} \right) dF_n(x).
\]

(1.12)
Next, we give an explicit formula for \( M_n^{(\alpha)}(v) \) in terms of observed sample \( X_1, \ldots, X_n \). Since \( \overline{F} \) and \( \overline{G} \) are regularly varying with negative indices \(-1/\gamma_1 \) and \(-1/\gamma_2 \) respectively, then their right endpoints are infinite and thus they are equal. Hence, from Woodroofe (1985), we may write \( \int_x^\infty dF(y) / F(y) = \int_x^\infty dF^*(y) / C(y) \), where \( C(z) := P(X \leq z \leq Y) \) is the theoretical counterpart of \( C_n(z) \) given in (1.4). Differentiating the previous two integrals leads to the crucial equation \( C(x) dF(x) = F(x) dF^*(x) \), which implies that \( C_n(x) dF_n(x) = F_n(x) dF_n^*(x) \), where \( F_n^*(x) := n^{-1} \sum_{i=1}^n 1(X_i \leq x) \) is the usual empirical df based on the observed sample \( X_1, \ldots, X_n \). It follows that

\[
M_n^{(\alpha)}(v) = \frac{1}{F_n(X_{n−v,n})} \int_{X_{n−v,n}}^{\infty} \frac{F_n(x)}{C_n(x)} \log^a x dF_n^*(x),
\]
which equals
\[ M_n^{(a)}(v) = \frac{1}{n} \sum_{i=1}^{v} \frac{F_n(X_{n-1:n})}{C_n(X_{n-1:n})} \log^a \frac{X_{n-1:n}}{X_{n-1:n}}. \]
Similarly, we show that \( \overline{F}_n(X_{n-1:n}) = n^{-1} \sum_{i=1}^{n} F_n(X_{n-1:n}) / C_n(X_{n-1:n}) \). This leads to
following form of \( M_n^{(a)}(t; F) \) estimator:
\[ M_n^{(a)}(v) := \sum_{i=1}^{v} a^{(i)} \log^a \frac{X_{n-1:n}}{X_{n-1:n}}. \]

It is readily observable that \( M_n^{(1)}(k) = \tilde{\gamma}_1^{(BMN)} \). Making use of (1.8) with the expression above, we get an estimator of \( S^{(a)}(t; F) \), that we denote \( S_n^{(a)} = S_n^{(a)}(v) \). This, in virtue of the third limit in (1.9), leads to estimating \( s_a(\rho_1) \). It is noteworthy that the function \( \rho_1 \to s_a(\rho_1) \), defined and continuous on the set of negative real numbers, is increasing for \( 0 < \alpha < 1/2 \) and decreasing for \( \alpha > 1/2 \), \( \alpha \neq 1 \). Then, for suitable values of \( \alpha \), we may invert \( s_a \) to get an estimator \( \hat{\rho}_1^{(a)} \) for \( \rho_1 \) as follows:
\[ \hat{\rho}_1^{(a)} := s_a^{-}(S_n^{(a)}) \), provided that \( S_n^{(a)} \in A_\alpha, \quad (1.13) \]
where \( A_\alpha \) is one of the following two regions:
\[ \{ s : (2\alpha - 1) / \alpha^2 < s \leq 4(2\alpha - 1) / (\alpha(\alpha + 1))^2, \text{ for } \alpha \in (0, 1/2) \} \],
or
\[ \{ s : 4(2\alpha - 1) / (\alpha(\alpha + 1))^2 \leq s < (2\alpha - 1) / \alpha^2, \text{ for } \alpha \in (1/2, \infty) \setminus \{1\} \} \].
For more details, regarding the construction of these two sets, one refers to Remark 2.1 and Lemma 3.1 in Gomes et al. (2002). It is worth mentioning that, for \( \alpha = 2 \), we have \( s_2(\rho_1) = (3\rho_1^2 - 8\rho_1 + 6) / (3 - 2\rho_1)^2 \) and \( s_2^{-}(s) = (6s - 4 + \sqrt{3s - 2}) / (4s - 3) \), for \( 2/3 < s < 3/4 \). Thereby, we obtain an explicit formula to the estimator of \( \rho_1 \) as follows
\[ \hat{\rho}_1^{(2)} = \frac{6S_n^{(2)} - 4 + \sqrt{3S_n^{(2)} - 2}}{4S_n^{(2)} - 3}, \text{ provided that } 2/3 < S_n^{(2)} < 3/4. \quad (1.14) \]
Next, we derive an asymptotically unbiased estimator for \( \gamma_1 \), that improves \( \tilde{\gamma}_1^{(BMN)} \) by estimating the asymptotic bias \( A_0(n/k) / (1 - \rho_1) \), given in weak approximation (4.19). Indeed, let \( v \) be equal to \( u_n := [n^{1-\epsilon}] \), for a fixed \( \epsilon > 0 \) close to zero (say \( \epsilon = 0.01 \)) so that, given \( n = m, u_m \to \infty, u_m/m \to \infty \) and \( \sqrt{u_m} |A_0(m/u_m)| \to \infty \). The validity of such a sequence is discussed in Gomes et al. (2002) (Subsection 6.1, conclusions 2 and 5). The estimator of \( \rho_1 \) pertaining to this choice of \( v \) will be denoted by \( \hat{\rho}_1^{(a)} \). We are now is proposition to define an estimator for \( A_0(n/k) \). From assertion (i) in Lemma 5.1, we have
A_0 (t) \sim (1 - \rho_1)^2 \left( M^{(2)} (t; \mathbf{F}) - 2 \left( M^{(1)} (t; \mathbf{F}) \right)^2 \right) / \left( 2 \rho_1 M^{(1)} (t; \mathbf{F}) \right), \ \text{as} \ t \to \infty. \ \text{Then, by}
\text{letting} \ t = n/k \ \text{and by replacing, in the previous quantity,} \ \mathbb{U}_{F^*} (n/k) \ \text{by} \ X_{n-k,n}, \ \mathbf{F} \ \text{by} \ \mathbf{F}_n \ \text{and} \ \rho_1 \ \text{by} \ \tilde{\rho}_1^{(s)}, \ \text{we end up with}
\hat{A}_0 (n/k) := \left( 1 - \tilde{\rho}_1^{(s)} \right)^2 \left( M_n^{(2)} (k) - 2 \left( M_n^{(1)} (k) \right)^2 \right) / \left( 2 \rho_1^{(s)} M_n^{(1)} (k) \right),
\text{as an estimator for} \ A_0 (n/k). \ \text{Thus, we obtain an asymptotically unbiased estimator}
\hat{\gamma}_1 := M_n^{(1)} (k) + \frac{M_n^{(2)} (k) - 2 \left( M_n^{(1)} (k) \right)^2}{2 M_n^{(1)} (k)} \left( 1 - \frac{1}{\tilde{\rho}_1^{(s)}} \right),
\text{for the tail index} \ \gamma_1, \ \text{as an adaptation of Peng’s estimator (Peng, 1998) to the random right-truncation case. The rest of the paper is organized as follows. In Section 2, we present our 
main results which consist in the consistency and the asymptotic normality of the estimators \ \hat{\rho}_1^{(s)} \ \text{and} \ \hat{\gamma}_1 \ \text{whose finite sample behaviours are checked by simulation in Section 3. All proofs 
are gathered in Section 4. Two instrumental lemmas are given in the Appendix.}

2. Main results

It is well known that, weak approximations of the second-order parameter estimators are 
achieved in the third-order condition of regular variation framework (see, e.g., de Haan and Stadtmüller, 
1996). Thus, it seems quite natural to suppose that df \ \mathbf{F} \ \text{satisfies}
\lim_{t \to \infty} \left\{ \mathbb{U}_{\mathbf{F}} (t x) / \mathbb{A}_{\mathbf{F}} (t) - \frac{x^{\gamma_1}}{\rho_1} - \frac{x^{\rho_1} - 1}{\rho_1} \right\} / \mathbb{B}_{\mathbf{F}} (t) = \frac{x^{\gamma_1}}{\rho_1} \left( \frac{x^{\rho_1 + \beta_1} - 1}{\rho_1 + \beta_1} - \frac{x^{\rho_1} - 1}{\rho_1} \right), \quad (2.15)
\text{where} \ \beta_1 < 0 \ \text{is the third-order parameter and} \ \mathbb{B}_{\mathbf{F}} \ \text{is a function tending to zero and not } 
\text{changing sign near infinity with regularly varying absolute value at infinity with index } \beta_1. \ \text{For convenience, we set} \ \mathbb{B}_0 (t) := \mathbb{B}_{\mathbf{F}} \left( 1 / \mathbb{F} (\mathbb{U}_{F^*} (t)) \right) \ \text{and by keeping similar notations to those used in Gomes et al. (2002), we write}
\mu_1^{(1)} := \Gamma (\alpha + 1), \ \mu_1^{(2)} (\rho_1) := \frac{\Gamma (\alpha) (1 - (1 - \rho_1)^{\alpha})}{\rho_1 (1 - \rho_1)^{\alpha}}, \quad (2.16)
\mu_1^{(3)} (\rho_1) := \begin{cases} \frac{1}{\rho_1^2} \log \left( \frac{1 - \rho_1}{1 - 2 \rho_1} \right) & \text{if} \ \alpha = 1, \\ \frac{\Gamma (\alpha)}{\rho_1^2 (\alpha - 1)} \left\{ \frac{1}{(1 - 2 \rho_1)^{\alpha-1}} - \frac{2}{(1 - \rho_1)^{\alpha-1}} + 1 \right\} & \text{if} \ \alpha \neq 1, \end{cases}
\mu_1^{(4)} (\rho_1, \beta_1) := \beta_1^{-1} \left( \mu_1^{(2)} (\rho_1 + \beta_1) - \mu_1^{(2)} (\rho_1) \right),
m_1 := \mu_1^{(2)} (\rho_1) - \mu_1^{(1)} (\rho_1) \mu_1^{(2)} (\rho_1), \ c_1 := \mu_1^{(3)} (\rho_1) - \mu_1^{(1)} (\mu_1^{(2)} (\rho_1))^2
and \( d_\alpha := \mu_\alpha^{(4)}(\rho_1, \beta_1) - \mu_\alpha^{(1)}(\rho_1, \beta_1) \). For further use, we set \( r_\alpha := 2q_\alpha \gamma_1^{2-\alpha} \Gamma(\alpha + 1) \),

\[
\eta_1 := \frac{1}{2\gamma_1 m_2 r_{\alpha+1}^2} \left( \frac{(2\alpha - 1) c_{2\alpha}}{\Gamma(2\alpha)} + c_2 r_{2\alpha} - \frac{2c_{\alpha+1} r_{2\alpha}}{\gamma_{\alpha+1} \Gamma(\alpha)} \right),
\]

\[
\eta_2 := \frac{1}{\gamma_1 m_2 r_{\alpha+1}^2} \frac{d_{2\alpha}}{\Gamma(2\alpha)} + d_2 r_{2\alpha} - \frac{2d_{\alpha+1} r_{2\alpha}}{\gamma_{\alpha+1} \Gamma(\alpha + 1)},
\]

\[
\xi := \frac{1}{\gamma_1} \left( \frac{1 - 2\alpha - 3r_{2\alpha}}{\gamma_{\alpha+1}^2 m_2} + \frac{2\alpha r_{2\alpha}}{\gamma_{\alpha+1}^2 m_2^2} \right),
\]

\[
\tau_1 := \frac{1}{\gamma_1 r_{\alpha+1}^2 \Gamma(2\alpha + 1) m_2}, \quad \tau_2 := - \frac{2r_{2\alpha}}{\gamma_1 r_{\alpha+1}^2 \Gamma(\alpha + 2) m_2},
\]

\[
\tau_3 := \frac{r_{2\alpha}}{\gamma_1 r_{\alpha+1}^2 m_2}, \quad \tau_4 := \frac{-2\alpha r_{\alpha+1} + 2(\alpha + 1) r_{2\alpha} - 4\alpha r_{\alpha+1} r_{2\alpha}}{r_{\alpha+1}^3 m_2},
\]

\[
\tau_5 := \frac{\rho_1 - 1}{2\gamma_1 \rho_1}, \quad \tau_6 := 1 + 2 \frac{1 - \rho_1}{\gamma_1 \rho_1} \quad \text{and} \quad \mu := \gamma \left( 2 + 2 \frac{1 - \rho_1}{\gamma_1 \rho_1} - \frac{1}{\rho_1} \right).
\]

**Theorem 2.1.** Assume that both df’s \( F \) and \( G \) satisfy the second-order conditions (1.2) and (1.3) respectively with \( \gamma_1 < \gamma_2 \). Let \( \alpha \), defined in (1.3), be fixed and let \( v \) be a random sequence of integers such that, given \( n = m, v_m \rightarrow \infty, v_m/m \rightarrow 0 \) and \( \sqrt{v_m} |A_0(m/v_m)| \rightarrow \infty \), then

\[
\hat{\rho}_1^{(\alpha)} \rightarrow \rho_1, \text{ as } N \rightarrow \infty.
\]

If in addition, we assume that the third-order condition (2.15) holds, then whenever, given \( n = m, \sqrt{v_m} A_0^2(m/v_m) \) and \( \sqrt{v_m} A_0(m/v_m) B_0(m/v_m) \) are asymptotically bounded, then there exists a standard Wiener process \( \{ W(s); s \geq 0 \} \), defined on the probability space \((\Omega, A, P)\), such that

\[
s'(\rho_1) \sqrt{v} A_0(n/v) \left( \hat{\rho}_1^{(\alpha)} - \rho_1 \right) = \int_0^1 s^{-\gamma_2-1} \Delta_\alpha(s) W(s) \, ds - \xi W(1)
\]

\[
+ \eta_1 \sqrt{v} A_0^2(n/v) + \eta_2 \sqrt{v} A_0(n/v) B_0(n/v) + o_P(1),
\]

where \( s' \) is the Lebesgue derivative of \( s \) given in (1.11) and

\[
\Delta_\alpha(s) := \tau_1 \gamma_1 \log^{2\alpha - 3} s^{-\gamma} + \tau_2 \gamma_2 \log^{2\alpha-1} s^{-\gamma} + \tau_3 \gamma \log^{\alpha+1} s^{-\gamma}
\]

\[
+ \tau_4 (\alpha + 1) \gamma^2 \log^\alpha s^{-\gamma} + \tau_5 \gamma \log^2 s^{-\gamma} + \tau_6 \gamma \log s^{-\gamma} + \tau_7 \gamma^2 \log s^{-\gamma} - \frac{\gamma_1 \xi}{\gamma_1 + \gamma_2}
\]

If, in addition, we suppose that given \( n = m \),

\[
\sqrt{v_m} A_0^2(m/v_m) \rightarrow \lambda_1 < \infty \text{ and } \sqrt{v_m} A_0(m/v_m) B_0(m/v_m) \rightarrow \lambda_2 < \infty,
\]
then $\sqrt{v}A_0(n/v)\left(\hat{\rho}_1^{(a)} - \rho_1\right) \xrightarrow{D} \mathcal{N}(\eta_1\lambda_1 + \eta_2\lambda_2, \sigma_n^2)$, as $N \to \infty$, where

$$
\sigma_n^2 := \int_0^1 \int_0^1 s^{-\gamma/\tau-1}t^{-\gamma/\tau-1} \min(s, t) \Delta_\alpha(s)\Delta_\alpha(t)dsdt - 2\xi \int_0^1 s^{-\gamma/\tau}\Delta_\alpha(s)ds + \xi^2.
$$

**Theorem 2.2.** Let $k$ be a random sequence of integers, different from $v$, such that, given $n = m$, $k_m \to \infty$, $k_m/m \to 0$ and $\sqrt{k_m}A_0(m/k_m)$ is asymptotically bounded, then with the same Wiener process \{W(s); s \geq 0\}, for any $\epsilon > 0$, we have

$$
\sqrt{k}(\hat{\gamma}_1 - \gamma_1) = \int_0^1 \sqrt{s^{-\gamma/\tau-1}D(s)}W(s)ds - \mu W(1) + o_p(1),
$$

where

$$
D(s) := \frac{\gamma^3\gamma_5}{\gamma_1 + \gamma_2} \log^2 s - \left(\frac{2\gamma_5\gamma^3}{\gamma_1} + \frac{\gamma^2\gamma_6}{\gamma_1 + \gamma_2}\right) \log s + \frac{\gamma_6\gamma^2}{\gamma_1} - \frac{\gamma_1\mu}{\gamma_1 + \gamma_2}.
$$

If, in addition, we suppose that, given $n = m$, $\sqrt{k_m}A_0(m/k_m) \to \lambda < \infty$, then

$$
\sqrt{k}(\hat{\gamma}_1 - \gamma_1) \xrightarrow{D} \mathcal{N}(0, \sigma^2), \quad \text{as } N \to \infty,
$$

where $\sigma^2 := \int_0^1 \int_0^1 s^{-\gamma/\tau-1}t^{-\gamma/\tau-1} \min(s, t) D(s)D(t)dsdt - 2\mu \int_0^1 s^{-\gamma/\tau}D(s)ds + \mu^2$.

### 3. Simulation study

In this section, we study the performance of $\hat{\rho}_1^{(a)}$ (for $\alpha = 2$) and compare the newly introduced bias-reduced estimator $\hat{\gamma}_1$ with $\hat{\gamma}_1^{(BMN)}$. Let us consider two sets of truncated and truncation data respectively drawn from Burr’s models, $F(x) = (1 + x^{1/\delta})^{-\delta/\gamma_1}$ and $G(x) = (1 + x^{1/\delta})^{-\delta/\gamma_2}$, $x \geq 0$, where $\delta, \gamma_1, \gamma_2 > 0$. By elementary analysis, it is easy to verify that $F$ satisfies the third-order condition (2.15) with $\rho_1 = \beta_1 = -\gamma_1/\delta$, $A_F(x) = \gamma_1 x^{p_1}/(1 - x^{p_1})$ and $B_F(x) = (\delta/\gamma_1) A_F(x)$. We fix $\delta = 1/4$ and choose the values 0.6 and 0.8 for $\gamma_1$ and 70% and 90% for the percentage of observed data $p = \gamma_2/(\gamma_1 + \gamma_2)$. For each couple $(\gamma_1, p)$, we solve the latter equation to get the pertaining $\gamma_2$-value. We vary the common size $N$ of both samples $(X_1, \ldots, X_N)$ and $(Y_1, \ldots, Y_N)$, then for each size, we generate 1000 independent replicates. For the selection of the optimal numbers $v^*$ and $k^*$ of upper order statistics used in the computation of estimators $\hat{\rho}_1^{(a)}$, $\hat{\gamma}_1$ and $\hat{\gamma}_1^{(BMN)}$, we apply the algorithm of Reiss and Thomas (2007), page 137. Our illustration and comparison, made with respect to the absolute biases (abias) and rmse’s, are summarized in Tables 3.1 and 3.2. The obtained results, in Table 3.1, show that $\hat{\rho}_1^{(a)}$ behaves well in terms of bias and rmse and it is clear that from Table 3.2 that $\hat{\gamma}_1$ performs better $\hat{\gamma}_1^{(BMN)}$ both in bias and rmse too.
|      | $p = 0.7$ |                          | $p = 0.9$ |                          |
|------|-----------|--------------------------|-----------|--------------------------|
|      |           | $N$ | $n$ | $v^*$ | abias | rmse | $N$ | $n$ | $v^*$ | abias | rmse |
| $\gamma_1 = 0.6$ |           | 100 | 70  | 27  | 0.009 | 0.047 | 89  | 38  | 0.004 | 0.048 |
|      |           | 200 | 151 | 70  | 0.008 | 0.046 | 179 | 73  | 0.003 | 0.046 |
|      |           | 500 | 349 | 208 | 0.005 | 0.043 | 450 | 243 | 0.002 | 0.048 |
|      |           | 1000| 697 | 667 | 0.001 | 0.027 | 896 | 641 | 0.001 | 0.030 |
| $\gamma_1 = 0.8$ |           | 100 | 70  | 30  | 0.011 | 0.050 | 90  | 40  | 0.013 | 0.048 |
|      |           | 200 | 139 | 67  | 0.009 | 0.048 | 179 | 71  | 0.012 | 0.047 |
|      |           | 500 | 350 | 198 | 0.008 | 0.043 | 449 | 232 | 0.006 | 0.049 |
|      |           | 1000| 730 | 301 | 0.002 | 0.037 | 903 | 378 | 0.002 | 0.029 |

Table 3.1. Absolute bias and rmse of the second-order parameter estimator based on 1000 right-truncated samples of Burr’s models.

|      | $p = 0.7$ |                          | $p = 0.9$ |                          |
|------|-----------|--------------------------|-----------|--------------------------|
|      |           | $N$ | $n$ | $k^*$ | abias | rmse | $N$ | $n$ | $k^*$ | abias | rmse |
| $\gamma_1 = 0.6$ |           | 100 | 70  | 12  | 0.068 | 0.263 | 11  | 0.127 | 0.259 | 89  | 16  | 0.013 | 0.152 |
|      |           | 200 | 140 | 26  | 0.048 | 0.200 | 24  | 0.090 | 0.223 | 180 | 34  | 0.006 | 0.116 |
|      |           | 500 | 349 | 63  | 0.020 | 0.123 | 58  | 0.072 | 0.173 | 449 | 83  | 0.002 | 0.078 |
|      |           | 1000| 703 | 115 | 0.007 | 0.097 | 112 | 0.011 | 0.121 | 898 | 176 | 0.001 | 0.037 |
| $\gamma_1 = 0.8$ |           | 100 | 70  | 13  | 0.067 | 0.311 | 12  | 0.222 | 0.217 | 89  | 16  | 0.063 | 0.220 |
|      |           | 200 | 140 | 25  | 0.014 | 0.219 | 24  | 0.163 | 0.282 | 179 | 33  | 0.033 | 0.150 |
|      |           | 500 | 349 | 64  | 0.011 | 0.152 | 59  | 0.033 | 0.222 | 449 | 85  | 0.021 | 0.097 |
|      |           | 1000| 707 | 145 | 0.007 | 0.054 | 125 | 0.017 | 0.133 | 897 | 179 | 0.013 | 0.058 |

Table 3.2. Absolute biases and rmse’s of the tail index estimators based on 1000 right-truncated samples of Burr’s models.
4. Proofs

4.1. Proof of Theorem 2.1. We begin by proving the consistency of \( \hat{N}_n^\alpha \) defined in (1.13). We let

\[ L_n(x; v) := \frac{F_n(X_{n-v,n})}{F_n(X_{n-v})} - \frac{F(X_{n-v,n})}{F(X_{n-v})}, \]

and we show that for any \( \alpha > 0 \)

\[ M_n(\alpha) = \gamma_1^\alpha \mu_1(1) + \int_1^\infty L_n(x; v) d\log^\alpha x + (1 + o_P(1)) \alpha \gamma_1^{\alpha-1} \mu_2(\rho_1) A_0(n/v), \quad (4.17) \]

where \( \mu_1(1) \) and \( \mu_2(\rho_1) \) are as in (2.16). It is clear that from formula (1.12), \( M_n(\alpha) \) may be rewritten into \(- \int_1^\infty \log^\alpha x d\overline{F}_n(X_{n-v,n})/\overline{F}_n(X_{n-v})\), which by an integration by parts equals \( \int_1^\infty \overline{F}_n(X_{n-v,n})/\overline{F}_n(X_{n-v}) d\log^\alpha x \). The latter may be decomposed into

\[ \int_1^\infty L_n(x; v) d\log^\alpha x + \int_1^\infty \left( \frac{F_n(X_{n-v,n})}{F_n(X_{n-v})} - x^{-1/\gamma_1} \right) d\log^\alpha x + \int_1^\infty x^{-1/\gamma_1} d\log^\alpha x. \]

It is easy to verify that \( \int_1^\infty x^{-1/\gamma_1} d\log^\alpha x \) equals \( \gamma_1^\alpha \mu_1(1) \). Since, \( X_{n-v,n} \to \infty \), almost surely, then by making use of the uniform inequality of the second-order regularly varying functions, to \( \overline{F} \), given in Proposition 4 of Hua and Joe (2011), we write: with probability one, for any \( 0 < \epsilon < 1 \) and large \( N \)

\[ \left| \frac{F(X_{n-v,n})}{\overline{F}(X_{n-v,n})} - x^{-1/\gamma_1} \right| - x^{-1/\gamma_1} \frac{F(x_{n-v,n})}{\overline{F}(X_{n-v,n})} - 1 \leq \epsilon x^{-1/\gamma_1 + \epsilon}, \quad \text{for any } x \geq 1, \quad (4.18) \]

where \( \tilde{A}_{F}(t) \sim A_F(t) \), as \( t \to \infty \). This implies, almost surely, that

\[ \int_1^\infty \left( \frac{F_n(X_{n-v,n})}{F_n(X_{n-v})} - x^{-1/\gamma_1} \right) d\log^\alpha x \]

\[ = \tilde{A}_F(1/\overline{F}(X_{n-v,n})) \left\{ \int_1^\infty x^{-1/\gamma_1} \frac{F(x_{n-v,n})}{\overline{F}(X_{n-v,n})} - 1 \right\} d\log^\alpha x + o_P \left( \int_1^\infty x^{-1/\gamma_1 + \epsilon} d\log^\alpha x \right). \]

We check that \( \int_1^\infty x^{-1/\gamma_1} \frac{F(x_{n-v,n})}{\overline{F}(X_{n-v,n})} - 1 \gamma_1 \rho_1 d\log^\alpha x = \alpha \gamma_1^{\alpha-1} \mu_2(\rho_1) \) and \( \int_1^\infty x^{-1/\gamma_1 + \epsilon} d\log^\alpha x \) is finite. From Lemma 7.4 in Benchaira et al. (2016a), \( X_{n-v,n}/U_F^* (n/v) \) \( \overrightarrow{P} \to 1 \), as \( N \to \infty \), then by using the regular variation property of \( |A_F(1/\overline{F}(\cdot))| \) and the corresponding Potter’s inequalities (see, for instance, Proposition B.1.10 in de Haan and Ferreira (2006)), we get

\[ \tilde{A}_F(1/\overline{F}(X_{n-v,n})) = (1 + o_P(1)) A_F(1/\overline{F}(U_F^* (n/v))) = (1 + o_P(1)) A_0(n/v), \]

therefore

\[ M_n(\alpha) = \gamma_1^\alpha \mu_1(1) + \int_1^\infty L_n(x; v) d\log^\alpha x + \alpha \gamma_1^{\alpha-1} \mu_2(\rho_1) A_0(n/v)(1 + o_P(1)). \]
In the second step, we use the Gaussian approximation of $L_n(x)$ recently given by Benchaira et al. (2016a) (assertion (6.26)), saying that: for any $0 < \epsilon < 1/2 - \gamma/\gamma_2$, there exists a standard Wiener process $\{W(s); s \geq 0\}$, defined on the probability space $(\Omega, \mathcal{A}, P)$ such that

$$\sup_{x \geq 1} x^{(1/2 - \epsilon)/\gamma_1} \left| \sqrt{v} L_n(x; v) - \mathcal{L}(x; W) \right| \xrightarrow{P} 0, \text{ as } N \to \infty, \quad (4.19)$$

where $\{\mathcal{L}(x; W); x > 0\}$ is a Gaussian process defined by

$$\frac{\gamma}{\gamma_1} x^{-1/\gamma_1} \left\{ x^{1/\gamma} W(x^{-1/\gamma}) - W(1) \right\} + \frac{\gamma}{\gamma_1 + \gamma_2} \int_0^1 s^{-\gamma/\gamma_2 - 1} \left\{ x^{1/\gamma} W(x^{-1/\gamma} s) - W(s) \right\} ds.$$

Let us decompose $\sqrt{v} \int_1^\infty L_n(x; v) d\log^a x$ into

$$\int_1^\infty \mathcal{L}(x; W) d\log^a x + \int_1^\infty \{ \sqrt{v} L_n(x; v) - \mathcal{L}(x; W) \} d\log^a x.$$

By using approximation (4.19), we obtain $\int_1^\infty \{ \sqrt{v} L_n(x; v) - \mathcal{L}(x; W) \} d\log^a x = o_P(1).$

We showed in Lemma 5.2 that $\int_1^\infty \mathcal{L}(x; W) d\log^a x = O_P(1),$ therefore $\int_1^\infty L_n(x; v) d\log^a x = O_P(v^{-1/2}),$ it follows that

$$M_n(a) (v) = \gamma_n^a \mu_1^{(1)} + v^{-1/2} \int_1^\infty \mathcal{L}(x; W) d\log^a x$$

$$+ \alpha_1 \mu_1^{(2)}(\rho_1) A_0(n/v) (1 + o_P(1)) + o_P(v^{-1/2}). \quad (4.20)$$

Once again, by using the fact that $\int_1^\infty \mathcal{L}(x; W) d\log^a x = O_P(1),$ we get

$$M_n(a) (v) = \gamma_n^a \mu_1^{(1)} + \alpha_1 \mu_1^{(2)}(\rho_1) A_0(n/v) (1 + o_P(1)) + o_P(v^{-1/2}).$$

It particular, for $\alpha = 1$, we have $\mu_1^{(1)} = 1$, this means that

$$M_n^{(1)} (v) = \gamma_1 + \mu_1^{(2)}(\rho_1) A_0(n/v) (1 + o_P(1)) + o_P(v^{-1/2}),$$

which implies that

$$\left( M_n^{(1)} (v) \right)^2 = \gamma_1^2 + 2\gamma_1 \mu_1^{(2)}(\rho_1) A_0(n/v) (1 + o_P(1)) + o_P(v^{-1/2}). \quad (4.21)$$

Likewise, for $\alpha = 2$, we have $\mu_2^{(1)} = 2$, then

$$M_n^{(2)} (v) = 2\gamma_1^2 + 2\gamma_1 \mu_2^{(2)}(\rho_1) A_0(n/v) (1 + o_P(1)) + o_P(v^{-1/2}). \quad (4.22)$$
Similar to the definition of \( M_n^{(a)} (v) \), let \( Q_n^{(a)} (v) \) be \( Q^a (t; F) \) with \( \mathbb{U} F^* (t) \) and \( F \) respectively replaced by by \( X_{n-v:n} \) and \( F_n \). From (1.7), we may write

\[
Q_n^{(a)} (v) = \frac{M_n^{(a)} (v) - \Gamma (\alpha + 1) \left( M_n^{(1)} (v) \right)^{\alpha}}{M_n^{(2)} (v) - 2 \left( M_n^{(1)} (v) \right)^2}.
\]

Then, by using the approximations above, we end up with

\[
Q_n^{(a)} (v) = (1 + o_P (1)) \frac{\alpha \gamma_1^{a-1} \left( \mu_2 (2) (\rho_1) - \mu_2 (1) (\rho_1) \right)}{2 \gamma_1 \left( \mu_2 (2) (\rho_1) - \mu_2 (1) (\rho_1) \right)}.
\]

By replacing \( \mu_2 (1) \), \( \mu_2 (2) (\rho_1) \) and \( \mu_2 (1) (\rho_1) \) by their corresponding expressions, given in (2.16), with the fact that \( \alpha \Gamma (\alpha) = \Gamma (\alpha + 1) \), we show that the previous quotient equals \( q_a (\rho_1) \) given in (1.10). This implies that \( Q_n^{(a)} (v) \overset{P}{\to} q_a (\rho_1) \) and therefore \( S_n^{(a)} (v) \overset{P}{\to} s_a (\rho_1) \), as \( N \to \infty \), as well. By using the mean value theorem, we infer that \( \tilde{\rho}_1^{(a)} = s_a \left( S_n^{(a)} (v) \right) \overset{P}{\to} \rho_1 \), as sought. Let us now focus on the asymptotic representation of \( \tilde{\rho}_1^{(a)} \). We begin by denoting \( \tilde{M}_n^{(a)} (v) \), \( \tilde{S}_n^{(a)} (v) \) and \( \tilde{Q}_n^{(a)} (v) \) the respective values of \( M^{(a)} (t; F) \), \( S^{(a)} (t; F) \) and \( Q^{(a)} (t; F) \) when replacing \( \mathbb{U} F^* (t) \) by \( X_{n-v:n} \). It is clear that the quantity \( S_n^{(a)} (v) - S_a (\rho_1) \) may be decomposed into the sum of

\[
T_{n1} := -\delta (\alpha) \frac{Q_n^{(a+1)} (v)^2 - \tilde{Q}_n^{(a+1)} (v)^2}{Q_n^{(a+1)} (v) \tilde{Q}_n^{(a+1)} (v)} Q_n^{(2a)} (v; F_n),
\]

\[
T_{n2} := \delta (\alpha) \frac{Q_n^{(2a)} (v) - \tilde{Q}_n^{(2a)} (v) \tilde{Q}_n^{(a+1)} (v)}{\tilde{Q}_n^{(a+1)} (v)} \quad \text{and} \quad T_{n3} := \tilde{S}_n^{(a)} (v) - S_a (\rho_1).
\]

Since \( Q_n^{(a)} (v) \overset{P}{\to} q_a (\rho_1) \), then by using the mean value theorem, we get

\[
T_{n1} = -\left( 1 + o_P (1) \right) 2 \delta (\alpha) q_{2a} q_{a+1}^{-3} \left( Q_n^{(a+1)} (v) - \tilde{Q}_n^{(a+1)} (v) \right).
\]

Making use of the third-order condition (2.15), with analogy of the weak approximation given in Gomes et al. (2002) (page 411), we write

\[
M_n^{(a)} (v) = \gamma_1^{a} \mu_1^{(1)} + v^{-1/2} \int_1^\infty L (x; W) \, d \log^a x + \alpha \gamma_1^{a-1} \mu_2^{(2)} (\rho_1) A_0 (n/v) \Rightarrow \frac{d}{n/v \rho_1 (1 + o_P (1)) + o_P (v^{-1/2})}.
\]
Likewise, by similar arguments, we also get
\[ M_n^{(o)}(v) = \gamma_{1}^{(1)} \mu_{1}^{(1)} + \alpha \gamma_{1}^{(4)} \mu_{1}^{(4)} (\rho_1) A_0 (n/v) \]
\[ + \alpha \gamma_{1}^{(4)} \mu_{1}^{(4)} (\rho_1, \beta_1) A_0 (n/v) B_0 (n/v) (1 + o_p (1)) + o_p (v^{-1/2}). \]

Let us write
\[ Q_n^{(o)} (v) = Q_n^{(o)} (v) \]
\[ = \frac{M_n^{(o)} (v) - \Gamma (\alpha + 1) M_n^{(1)} (v) \right)^2}{M_n^{(2)} (v) - 2 \left( M_n^{(1)} (v) \right)^2} - \frac{M_n^{(o)} (v) - \Gamma (\alpha + 1) M_n^{(1)} (v)}{M_n^{(2)} (v) - 2 \left( M_n^{(1)} (v) \right)^2}. \]

By reducing to the common denominator and by using the weak approximations (4.23) and (4.24) with the fact that \( A_0 (n/v) \to P \), \( \sqrt{v} A_0 (n/v) \) and \( \sqrt{\alpha} A_0 (n/v) B_0 (n/v) \) are stochastically bounded, we get
\[ \sqrt{v} A_0 (n/v) (Q_n^{(o)} (v) - \tilde{Q}_n^{(o)} (v)) \]
\[ = \int_1^\infty \mathcal{L} (x; W) d g_1 (x; \alpha) + \theta_1 (\alpha) \sqrt{v} A_0 (n/v) B_0 (n/v) + o_p (1), \]

where
\[ g_1 (x; \alpha) := \gamma_{1}^{(1)} \log^2 x - \frac{\alpha \Gamma (\alpha) + 1}{2} r_\alpha \gamma_{1}^{(2)} \log^2 x - \left( \alpha \mu_{1}^{(1)} - 2 \alpha \Gamma (\alpha) \right) r_\alpha \gamma_{1}^{(1)} \log x, \]
\[ \theta_1 (\alpha) := \alpha \gamma_{1}^{(2)} \left\{ d_\alpha - \Gamma (\alpha) r_\alpha d_2 \right\} / (2m_2) \]

and \( d_\alpha, r_\alpha \) and \( m_2 \) being those defined in the beginning of Section 2. It follows that
\[ \sqrt{v} A_0 (n/v) T_{n1} \]
\[ = -2 \delta (\alpha) q_{2a} q_{\alpha+1}^{-3} \left\{ \int_1^\infty \mathcal{L} (x; W) d g_1 (x; \alpha + 1) + \theta_1 (\alpha + 1) \sqrt{v} A_0 (n/v) B_0 (n/v) + o_p (1) \right\}. \]

Likewise, by similar arguments, we also get
\[ \sqrt{v} A_0 (n/v) T_{n2} \]
\[ = \delta (\alpha) q_{\alpha+1}^2 \left\{ \int_1^\infty \mathcal{L} (x; W) d g_1 (x; 2\alpha) + \theta_1 (2\alpha) \sqrt{v} A_0 (n/v) B_0 (n/v) + o_p (1) \right\}. \]

Therefore
\[ \sqrt{v} A_0 (n/v) (T_{n1} + T_{n2}) = \int_1^\infty \mathcal{L} (x; W) d g(x; \alpha) + K (\alpha) \sqrt{v} A_0 (n/v) B_0 (n/v) + o_p (1), \]

where \( K (\alpha) := \delta (\alpha) (q_{\alpha+1}^{-2} \theta_1 (2\alpha) - 2q_{2a} q_{\alpha+1}^{-3} \theta_1 (\alpha + 1)) \) and
\[ g(x; \alpha) := \delta (\alpha) (q_{\alpha+1}^{-2} g_1 (x; 2\alpha) - 2q_{2a} q_{\alpha+1}^{-3} g_1 (x; \alpha + 1)). \]
Once again by using the third-order condition (2.15) with the fact that \( A_0 (n/v) \xrightarrow{P} 0 \) and 
\[ \sqrt{v} A_0 (n/v) B_0 (n/v) = O_P (1), \]
we show that 
\[ \sqrt{v} A_0 (n/v) T_{n3} = \eta_1 \sqrt{v} A_0^2 (n/v) + o_P (1). \]
It is easy to check that 
\[ K(\alpha) \equiv \eta_2, \]
hence we have
\[ \sqrt{v} A_0 (n/v) \left( S_n^{(\alpha)} (v) - s_\alpha (\rho_1) \right) \]
\[ = \int_1^\infty \mathcal{L} (x; W) d g(x; \alpha) + \eta_1 \sqrt{v} A_0^2 (n/v) + \eta_2 \sqrt{v} A_0 (n/v) B_0 (n/v) + o_P (1), \]
where \( \eta_1 \) and \( \eta_2 \) are those defined in the beginning of Section 2. Recall that 
\[ S_n^{(\alpha)} (v) = s_\alpha \left( \hat{\rho}_1^{(\alpha)} \right), \]
then in view of the mean value theorem and the consistency of \( \hat{\rho}_1^{(\alpha)} \), we end up with
\[ s'_\alpha (\rho_1) \sqrt{v} A_0 (n/v) (\hat{\rho}_1^{(\alpha)} - \rho_1) \]
\[ = \int_1^\infty \mathcal{L} (x; W) d g(x; \alpha) + \eta_1 \sqrt{v} A_0^2 (n/v) + \eta_2 \sqrt{v} A_0 (n/v) B_0 (n/v) + o_P (1). \]
Finally, integrating by parts with elementary calculations complete the proof of the second part of the theorem, namely the Gaussian approximation of \( \hat{\rho}_1^{(\alpha)} \). For the third assertion, it suffices to calculate
\[ E \left[ \int_0^1 s^{-\gamma_2 - 1} \Delta_\alpha (s) W (s) d s - \xi W (1) \right]^2 \]
to get the asymptotic variance \( \sigma_{\alpha}^2 \), therefore we omit details.

4.2. Proof of Theorem 2.2. Let us write
\[ \sqrt{k} \left( \hat{\gamma}_1 - \gamma_1 \right) = \sqrt{k} \left( M_n^{(1)} (k) - \gamma_1 \right) + \frac{\hat{\rho}_1^{(\star)}}{2 \hat{\rho}_1^{(\star)} M_n^{(1)} (k)} \sqrt{k} \left( M_n^{(2)} (k) - 2 (\gamma_1^{(1)} (k))^2 \right). \]
From, Theorem 3.1 in Benchaira et al. (2016a) and Theorem 2.1 above both \( M_n^{(1)} (k) = \hat{\gamma}_1^{(BMN)} \) and \( \hat{\rho}_1^{(\star)} \) are consistent for \( \gamma_1 \) and \( \rho_1 \) respectively. It follows that
\[ \sqrt{k} \left( \hat{\gamma}_1 - \gamma_1 \right) \]
\[ = \sqrt{k} \left( M_n^{(1)} (k) - \gamma_1 \right) + \frac{\rho_1 - 1}{2 \gamma_1 \rho_1} \sqrt{k} \left( M_n^{(2)} (k) - 2 (\gamma_1^{(1)} (k))^2 \right) (1 + o_P (1)). \]
By applying the weak approximation (4.20), for \( \alpha = 1 \), we get
\[ \sqrt{k} \left( M_n^{(1)} (k) - \gamma_1 \right) = \int_1^\infty \mathcal{L} (x; W) d \log x + \frac{\sqrt{v} A_0 (n/k)}{1 - \rho_1} + o_P (1) \quad (4.25) \]
Using the mean value theorem and the consistency of $M_n^{(1)}(k)$ yield

\[
\sqrt{k} \left( (M_n^{(1)}(k))^2 - \gamma_1^2 \right) = 2\gamma_1 \left\{ \int_1^{\infty} L(x; W) \, d \log x + \frac{\sqrt{k} A_0(n/k)}{1 - \rho_1} + o_P(1) \right\} (1 + o_P(1)).
\]

From Lemma 5.2 and the assumption $\sqrt{k} A_0(n/k) = O_P(1)$ as $N \to \infty$ we have

\[
\sqrt{k} \left( (M_n^{(1)}(k))^2 - \gamma_1^2 \right) = \int_1^{\infty} L(x; W) \, d (2\gamma_1 \log x) + \frac{2\gamma_1}{1 - \rho_1} \sqrt{k} A_0(n/k) + o_P(1).
\]

Once again, by applying the weak approximation (4.20), for $\alpha = 2$, we write

\[
\sqrt{k} \left( (M_n^{(2)}(k) - 2\gamma_1^2) = \int_1^{\infty} L(x; W) \, d (2\gamma_1 \log x + 2\gamma_1 \mu_2^{(2)}(\rho_1) \sqrt{k} A_0(n/k) + o_P(1),
\]

where $\mu_2^{(2)}(\rho_1) = (1 - (1 - \rho_1)^2) / (\rho_1 (1 - \rho_1)^2)$. It follows that

\[
\sqrt{k} \left( (M_n^{(2)}(k) - 2 (M_n^{(1)}(k))^2) = \int_1^{\infty} L(x; W) \, d (\log^2 x - 4\gamma_1 \log x) + \frac{2\gamma_1 \rho_1}{(1 - \rho_1)^2} \sqrt{k} A_0(n/k) + o_P(1).
\]

By combining approximations (4.25) and (4.26), we obtain

\[
\sqrt{k} (\hat{\gamma}_1 - \gamma_1) = \int_1^{\infty} L(x; W) \, d \Psi(x) + o_P \left( \sqrt{k} A_0(n/k) \right), \text{ as } N \to \infty,
\]

where $\Psi(x) := \tau_6 \log x + \tau_5 \log^2 x$. Finally, making an integration by parts and a change of variables, with elementary calculations, achieve the proof of the first assertion of the theorem. The second part is straightforward.

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5. Appendix

Lemma 5.1. Assume that the second-order regular variation condition (1.2) holds, then for any $\alpha > 0$

\[
\begin{align*}
(i) \lim_{t \to \infty} \frac{M^{(a)} (t; F) - \mu^{(1)}_a \left( M^{(1)} (t; F) \right)^{2 \alpha}}{(M^{(1)} (t; F))^{2 \alpha-1} A_0 (t)} &= \alpha \left( \mu^{(2)}_a (\rho_1) - \mu^{(1)}_a \mu^{(2)}_a (\rho_1) \right), \\
(ii) \lim_{t \to \infty} Q^{(a)} (t; F) &= q_a (\rho_1), \\
(iii) \lim_{t \to \infty} S^{(a)} (t; F) &= s_a (\rho_1).
\end{align*}
\]

Proof. Let us consider assertion (i). We begin by letting

\[
U^{(a)} := - \int_{1}^{\infty} \log^a s^a \, ds s^{-1/\gamma_1} \text{ and } \ell (t) := \frac{M^{(a)} (t; F) - \mu^{(1)}_a \left( M^{(1)} (t; F) \right)^{\alpha}}{A_0 (t)},
\]

to show that, for any $\alpha > 0$

\[
\lim_{t \to \infty} \ell (t) = \alpha \gamma_1^{-1} \left( \mu^{(2)}_a (\rho_1) - \mu^{(1)}_a \mu^{(2)}_a (\rho_1) \right). \tag{5.27}
\]

Indeed, let us first decompose $\ell (t)$ into

\[
\frac{M^{(a)} (t; F) - U^{(a)}}{A_0 (t)} - \mu^{(1)}_a \left( M^{(1)} (t; F) \right)^{\alpha} \left( U^{(1)} \right)^{\alpha} + \frac{U^{(a)} (t) - \mu^{(1)}_a \left( U^{(1)} \right)^{\alpha}}{A_0 (t)},
\]

and note that $\mu^{(1)}_a = \Gamma (\alpha + 1)$ where $\Gamma (a) = \int_0^\infty e^{-x} x^{a-1} \, dx = \int_1^\infty t^{-2} \log^a \, t \, dt, \ a > 0$. It is easy to verify that $U^{(a)} - \mu^{(1)}_a \left( U^{(1)} \right)^{\alpha} = 0$, therefore

\[
\ell (t) = \frac{M^{(a)} (t; F) - U^{(a)}}{A_0 (t)} - \mu^{(1)}_a \left( M^{(1)} (t; F) \right)^{\alpha} \left( U^{(1)} \right)^{\alpha}.
\tag{5.28}
\]

Recall that $M^{(a)} (t; F) = \int_u^\infty \log^a \left( x/u \right) \, dF (x) / F (u), \ d\log^a x = : M^{(a)}_u (F),$

Making use, once again, of Proposition 4 of Hua and Joe (2011), we write: for possibly different function $\tilde{A}_F$, with $\tilde{A}_F (y) \sim A_F (y), \ y \to \infty$, for any $0 < \epsilon < 1$ and $x \geq 1$, we have

\[
\left| \frac{\tilde{A}_F (\gamma_1^{-2} A_F (1/F (u))) - x^{-1/\gamma_1} x^{\gamma_1/\gamma_1 - 1}}{x^{-1/\gamma_1} x^{\gamma_1/\gamma_1 - 1} - 1} \right| \leq \epsilon x^{-1/\gamma_1 + \epsilon}, \ \text{as } u \to \infty.
\]
By using elementary analysis, we easily show that this inequality implies that
\[
\frac{M_u^{(\alpha)}(F) - U^{(\alpha)}}{A_F(1/F(u))} \to \alpha \gamma_1^{\alpha - 1} \mu_1^{(2)}(\rho_1), \text{ as } u \to \infty.
\]
Hence, since \(1/F(u) \to \infty\) as \(u \to \infty\), then \(\tilde{A}_F(1/F(u)) \sim A_F(1/F(u)) = A_0(t).\) This means that
\[
\frac{M^{(\alpha)}(t, F) - U^{(\alpha)}}{A_0(t)} \to \alpha \gamma_1^{\alpha - 1} \mu_1^{(2)}(\rho_1), \text{ as } t \to \infty. \tag{5.29}
\]
Note that for \(\alpha = 1\), we have \(U^{(1)} = \gamma_1\) and therefore \((M^{(1)}(t, F) - \gamma_1)/A_0(t) \to \mu_1^{(2)}(\rho_1)\), which implies that \(M^{(1)}(t, F) \to \gamma_1\). By using the mean value theorem and the previous two results we get
\[
\frac{(M^{(1)}(t, F))^{\alpha} - \gamma_1^{\alpha}}{A_0(t)} \to \alpha \gamma_1^{\alpha - 1} \mu_1^{(2)}(\rho_1), \text{ as } t \to \infty. \tag{5.30}
\]
Combining (5.28), (5.29) and (5.30) leads to (5.27). Finally, we use the fact that \(M^{(1)}(t, F) \to \gamma_1\) to achieve the proof of assertion \((i)\). To show assertion \((ii)\), we apply assertion \((i)\) twice, for \(\alpha > 0\) and for \(\alpha = 2\), then we divide the respective results to get
\[
Q^{(\alpha)}(t; F) = \frac{M^{(\alpha)}(t; F) - \mu^{(1)}_{\alpha} [M^{(1)}(t; F)]^{\alpha}}{M^{(2)}(t; F) - 2 [M^{(1)}(t; F)]^{2}}
\sim \frac{(M^{(1)}(t; F))^{\alpha - 1} \alpha \left(\mu^{(2)}_{\alpha}(\rho_1) - \mu^{(1)}_{\alpha} \mu^{(2)}_{1}(\rho_1)\right)}{2 \left(\mu^{(2)}_{2}(\rho_1) - \mu^{(1)}_{2} \mu^{(2)}_{1}(\rho_1)\right)}.
\]
By replacing \(\mu^{(1)}_{\alpha}\) and \(\mu^{(2)}_{\alpha}(\rho_1)\) by their expressions, given in (2.16), we get
\[
\alpha \left(\mu^{(2)}_{\alpha}(\rho_1) - \mu^{(1)}_{\alpha} \mu^{(2)}_{1}(\rho_1)\right)
= \alpha \left\{\frac{\Gamma(\alpha)(1 - (1 - \rho_1)^{\alpha})}{\rho_1 (1 - \rho_1)^{\alpha}} - \frac{\Gamma(\alpha + 1)(1 - (1 - \rho_1))}{\rho_1 (1 - \rho_1)}\right\}
= \alpha \left\{\frac{\Gamma(\alpha)(1 - (1 - \rho_1)^{\alpha})}{\rho_1 (1 - \rho_1)^{\alpha}} - \frac{\Gamma(\alpha + 1)}{1 - \rho_1}\right\}.
\]
Since \(M^{(1)}(t, F) \to \gamma_1\), then
\[
Q^{(\alpha)}(t; F) \to \frac{\gamma_1^{\alpha - 2} \Gamma(\alpha + 1)(1 - (1 - \rho_1)^{\alpha} - \alpha \rho_1 (1 - \rho_1)^{\alpha - 1})}{\rho_1^2 (1 - \rho_1)^{\alpha - 2}}, \text{ as } t \to \infty,
\]
which is \(q_\alpha(\rho_1)\) given in (1.10). For assertion \((iii)\), it is clear that
\[
\delta(\alpha) = \frac{Q_t^{(2\alpha)}}{(Q_t^{(\alpha + 1)})^2} \to \frac{\rho_1^2 (1 - (1 - \rho_1)^{2\alpha} - 2 \alpha \rho_1 (1 - \rho_1)^{2\alpha - 1})}{(1 - (1 - \rho_1)^{\alpha + 1} - (\alpha + 1) \rho_1 (1 - \rho_1)^{\alpha})^2},
\]
which meets the expression of \(s_\alpha(\rho_1)\) given in (1.11). \(\square\)
Lemma 5.2. For any $\alpha > 0$, we have $\int_{1}^{\infty} \mathcal{L}(x; W) d \log^\alpha x = O_p(1)$.

Proof. Observe that $\int_{1}^{\infty} \mathcal{L}(x; W) d \log^\alpha x$ may be decomposed into the sum of

$$I_1 := \frac{\gamma}{\gamma_1} \int_{1}^{\infty} x^{1/\gamma_2} W \left( x^{-1/\gamma} \right) d \log^\alpha x, \quad I_2 := -\frac{\gamma}{\gamma_1} W(1) \int_{1}^{\infty} x^{-1/\gamma_1} d \log^\alpha x,$$

$$I_3 := \frac{\gamma}{\gamma_1 + \gamma_2} \int_{1}^{\infty} x^{1/\gamma_2} \left\{ \int_{0}^{1} s^{-\gamma/\gamma_2 - 1} W \left( x^{-1/\gamma} s \right) ds \right\} d \log^\alpha x,$$

and

$$I_4 := -\frac{\gamma}{\gamma_1 + \gamma_2} \left\{ \int_{0}^{1} s^{-\gamma/\gamma_2 - 1} W(s) ds \right\} \int_{1}^{\infty} x^{-1/\gamma_1} d \log^\alpha x.$$

Next we show that $I_i = O_p(1), \ i = 1, \ldots, 4$. To this end, we will show that $E |I_i|$ is finite for $i = 1, \ldots, 4$. Indeed, we have

$$E |I_1| \leq \frac{\gamma}{\gamma_1} \int_{1}^{\infty} x^{-1/\gamma_1} x^{1/\gamma} E \left| W \left( x^{-1/\gamma} \right) \right| d \log^\alpha x.$$

By elementary calculations, we get

$$E |I_3| \leq \frac{\gamma_2}{\gamma_1 + \gamma_2} \int_{1}^{\infty} x^{1/\gamma_2} \left\{ \int_{0}^{1} s^{-\gamma/\gamma_2 - 1} E \left| W \left( x^{-1/\gamma} s \right) \right| ds \right\} d \log^\alpha x.$$

By similar arguments we also show that $E |I_2| \leq \gamma_1^{-1} \Gamma (\alpha + 1)$ which is finite as well. For the third term $I_3$, we have

$$E |I_3| \leq \frac{\gamma_2}{\gamma_1 + \gamma_2} \int_{1}^{\infty} x^{1/\gamma_2} \left\{ \int_{0}^{1} s^{-\gamma/\gamma_2 - 1} E \left| W \left( x^{-1/\gamma} s \right) \right| ds \right\} d \log^\alpha x.$$

By elementary calculations, we get

$$E |I_3| \leq \frac{\gamma_2}{(\gamma_2 - 2\gamma)(\gamma_1 + \gamma_2)} \left( \frac{\gamma_1}{2\gamma - \gamma_1} \right)^\alpha \Gamma (\alpha + 1),$$

which is also finite. By using similar arguments, we get

$$E |I_4| \leq \frac{2\gamma_2 \gamma_1 \Gamma (\alpha + 1)}{\gamma_1 + \gamma_2 (\gamma_2 - 2\gamma)} < \infty,$$

as sought. □