Quantum Coherence: Reciprocity and Distribution

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Quantum coherence is the outcome of the superposition principle. Recently, it has been theorized as a quantum resource, and is the premise of quantum correlations in multipartite systems. It is therefore interesting to study the coherence content and its distribution in a multipartite quantum system. In this work, we show analytically as well as numerically the reciprocity between coherence and mixedness of a quantum state. We find that this trade-off is a general feature in the sense that it is true for large spectra of measures of coherence and of mixedness. We also study the distribution of coherence in multipartite systems by looking at monogamy-type relation—which we refer to as additivity relation—between coherences of different parts of the system. We show that for the Dicke states, while the normalized measures of coherence violate the additivity relation, the unnormalized ones satisfy the same.

I. INTRODUCTION

From everyday life experiences, we learn that arbitrary operations cannot do an assigned job. That is, specific resources—like allowed operations, “free assets” that one can use at will, and some “force” or catalyst in a prescribed amount—are needed to carry out a particular task. Therefore, to establish a quantitative theory of any physical resource, one needs to address the following fundamental issues: (i) the characterization or unambiguous definition of resource, (ii) the quantization or valid measures, and (iii) the transformation or manipulation of quantum states under the imposed constraints [1–4]. Several useful quantum resources like purity [5], entanglement [6–10], reference frames [11, 12], thermodynamics [13, 14], etc. have been identified and quantified until now. Recently, Baumgratz et al. in Ref. [15], provided a quantitative theory of coherence as a new quantum resource, borrowing the formalism already established for entanglement [6–10], thermodynamics [13, 14] and reference frames [11, 12].

Coherence arises from the superposition principle, and is defined for single as well as multipartite systems. Quantum coherence is identified by the presence of off-diagonal terms in the density matrix, and hence is a basis-dependent quantity. It being a basis-dependent quantity, local and nonlocal unitary operations can alter the amount of coherence in a quantum system. A density matrix has zero coherence with respect to a specific basis if it is diagonal in that basis. Diagonal density matrices, in the above sense, therefore represent essentially the classical mixtures. A coherent quantum state is considered as a resource in thermodynamics as it allows non-trivial transformations [16]. Quantum superposition is the most fundamental feature of quantum mechanics. Quantum coherence is a direct consequence of the superposition principle. Moreover, combined with the tensor product structure of quantum state space, it gives rise to the novel concepts such as entanglement and quantum correlations. It, being the premise of quantum correlations in multipartite systems, has attracted the attention of quantum information community significantly, and in addition to its quantification [17–19], other developments like the freezing phenomena [20], the coherence transformations under incoherent operations [21], establishment of geometric lower bound for a coherence measure [22], the complementarity between coherence and mixedness [23], its relation with other measures of quantum correlations and creation of coherence using unitary operations [24, 25], erasure of quantum coherence [26], and catalytic transformations of coherence [27] have been reported recently.

In this paper, we revisit the complementarity between coherence and mixedness of a quantum state, and the distribution of coherence in multipartite systems in considerable detail. We provide analytical and numerical results in this regard. This paper is organized as follows. In Sec. II, we briefly define the measures that quantify quantum coherence and mixedness. In Sec. III, we show that the reciprocity between coherence and mixedness in quantum systems is an extensive feature in the sense that it holds for large spectra of measures of coherence and of mixedness. In Sec. IV, we discuss the distribution of coherence in multipartite quantum systems. Numerical investigation unravels the fact that the percentage of quantum states satisfying the additivity relation of coherence increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. We provide conditions for the violation of the additivity relation of the relative entropy of coherence. In Sec. V, we investigate the distribution of coherence in a special type of quantum states called “X”-states, and provide examples. Finally, we conclude our findings in Sec. VI.

II. QUANTIFYING COHERENCE AND MIXEDNESS OF A QUANTUM STATE

In this section, we briefly review the axiomatic approach to characterize and quantify coherence, as proposed in Ref. [15], and mixedness of a quantum system.
A. Quantum coherence

In the framework of Ref. [15], all the diagonal states, in a given reference basis, constitute a set of incoherent states, denoted by \( I \). And a completely positive trace preserving (CPTP) map is an incoherent operation if it possesses a Kraus operator decomposition \( \{ K_i \} \) such that \( K_i \rho K_i^\dagger \) is incoherent for every incoherent state \( \rho \in I \). A function, \( C(\rho) \), is a valid measure of quantum coherence of the state \( \rho \) if it satisfies the following conditions [15]:

1. \( C(\rho) = 0 \) iff \( \rho \in I \).
2. Monotonicity under the incoherent operations, i.e., \( C(\Phi_\rho(\rho)) \leq C(\rho) \).
3. Convexity or nonincreasing under mixing of quantum states, i.e., \( C(\sum_k \rho_k \rho_k) \leq \sum_k p_k C(\rho_k) \).

That is, coherence cannot increase under mixing.

It is emphasized that the incoherency condition, \( M \mathcal{I} M^\dagger \in I \), places a severe constraint on the structure of the incoherent Kraus operator \( M \) [24]: there can be at most one nonzero entry in every column of \( M \). Thus, if the incoherent Kraus operator \( M \) belongs to the set of \( r \times c \) matrices \( \mathcal{M}_{r,c} \), then the maximum number of possible structures of \( M \) is \( r^c \). Note further that conditions (3) and (2b) together imply condition (2a) [15]:

\[
C(\Phi_\rho(\rho)) = \sum_n p_n C(\rho_n) \leq \sum_n p_n C(\rho_n) \leq C(\rho). \tag{1}
\]

Measures that satisfy the above conditions, include \( l_1 \) norm and relative entropy of coherence [15] and the skew information [17]. Coherence can also be quantified through entanglement. It was shown in Ref. [18] that entanglement measures which satisfy above conditions can be used to derive generic monotones of quantum coherence. Recall that quantum coherence is a basis-dependent quantity. Yao et al. in [24] asked whether a basis-independent measure of quantum coherence can be defined. They observed that the basis-free coherence is equivalent to quantum discord [24], supporting the fact that coherence is a form of quantum correlation in multipartite quantum systems. Viewing a \( d \)-dimensional quantum state \( \rho \), in the reference basis \( \{ |i\rangle \} \), as a \( d^2 \)-dimensional vector, its \( l_1 \) norm is

\[
|\rho|_{l_1} = \left( \sum_{i,j} |\rho_{ij}| \right)^{\frac{1}{2}}, \tag{2}
\]

where \( |\rho_{ij}| = \langle i|\rho|j\rangle \). The quantity \( C_{l_1}(\rho) \), which is based on \( l_1 \) norm, and given by

\[
C_{l_1}(\rho) = \sum_{i\neq j} |\rho_{ij}|, \tag{3}
\]

is a valid measure of coherence [15]. Another quantity, \( C(\rho) = \min_{\sigma \in I} S(\rho \parallel \sigma) = S(\rho) - S(\rho) \), is the relative entropy of coherence, where \( I \) is the set of incoherent states in the reference basis, \( S(\rho \parallel \sigma) = \text{Tr}(\rho \log \rho - \log \sigma) \) is the relative entropy between \( \rho \) and \( \sigma \), and \( \rho I = \sum_i (|i\rangle \langle i|) |i\rangle \langle i| \). Furthermore, a geometric measure of coherence is also proposed [15, 18, 19] which is a full coherence monotone [18]. The geometric measure is given by \( C_g(\rho) = 1 - \max_{\sigma \in I} F(\rho, \sigma) \), where \( I \) is the set of all incoherent states and \( F(\rho, \sigma) = \left( \text{Tr} \left( \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right) \right)^2 \) is the fidelity [28] of the states \( \rho \) and \( \sigma \). The maximally coherent pure state is defined by \( |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \) [15], for which \( C_{l_1}(\langle \Phi_d | \Phi_d \rangle) = d - 1 \) and \( C_g(\langle \Phi_d | \Phi_d \rangle) = \ln d \).

B. Mixedness

A quantum system which is not pure is mixed. A pure quantum system described by density matrix \( \rho \) is characterized by \( \text{Tr}(\rho^2) = \text{Tr}(\rho) = 1 \). \( \text{Tr}(\rho^2) \) is called the purity of \( \rho \). Noise in various forms, including inevitable interaction with environment, degrades the purity of a quantum state and renders it mixed. Mixedness characterizes disorder or loss of information, and is a complementary quantity to the purity of a quantum system. There are several ways to quantify the mixedness of a quantum state in the literature. For an arbitrary \( d \)-dimensional state, the mixedness, based on normalized linear entropy [29], is given as

\[
M_d(\rho) = \frac{d}{d-1} \left( 1 - \text{Tr}(\rho^2) \right). \tag{4}
\]

Therefore, for every quantum system, mixedness varies between zero and unity. The other operational measures of mixedness of a quantum state \( \rho \) include the von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \log \rho) \), and geometric measure of mixedness which is given by \( M_g(\rho) : = F(\rho, 1/d) = \frac{1}{d} \left( \text{Tr} \sqrt{\rho} \right)^2 \). For a \( d \)-dimensional pure quantum states \( |\phi_d\rangle \), while \( M_d(|\phi_d\rangle) \) and \( S(|\phi_d\rangle) \) vanish, \( M_g(|\phi_d\rangle) = \frac{1}{d^2} \). Thus, \( M_g(|\phi_d\rangle) \) and \( S(|\phi_d\rangle) \) lie between 0 and 1, and \( M_g(|\phi_d\rangle) \) varies between \( \frac{1}{2} \) and 1.

III. RECIPROCITY BETWEEN QUANTUM COHERENCE AND MIXEDNESS

As mixedness is complementary to purity and purity is closely related to quantum coherence, it is natural to investigate the restrictions imposed by the mixedness of a system on its quantum coherence. In this section, we show analytically and numerically that there exists a trade-off between the two quantities for different measures of coherence and mixedness.

For any arbitrary quantum system \( \rho \) in \( d \) dimensions, quantum coherence, as quantified by the \( l_1 \) norm, and mixedness, in terms of the normalized linear entropy, satisfies the following inequality

\[
\frac{C^2_{l_1}(\rho)}{(d-1)^2} + M_d(\rho) \leq 1. \tag{5}
\]

Inequality (5) dictates that for a fixed amount of mixedness, the maximal amount of coherence is limited, and vice-versa.
This important trade-off relation between quantum coherence and mixedness was obtained in Ref. [23] using the parametric form of an arbitrary $d$-dimensional density matrix, written in terms of the generators, $G_i$, of $SU(d)$ [28, 30–33], as

$$
\rho = \frac{1}{d} + \frac{1}{d} \sum_{i=1}^{d^2-1} x_i G_i,
$$

and mixedness was obtained in Ref. [23] using the parametric form of an arbitrary quantum system, $\rho = \frac{1}{d} + \frac{1}{d} \sum_{i=1}^{d^2-1} x_i G_i$. The generators $G_i$ satisfy (i) $G_i = G_i^\dagger$, (ii) $\text{Tr}(G_i) = 0$, and (iii) $\text{Tr}(G_i G_j) = 2\delta_{ij}$. In this representation, three-dimensional state is

$$
\rho = \left( \frac{1}{3} + x_1 + \frac{2x_2}{\sqrt{3}} \right) \begin{pmatrix} x_1 - ix_4 & x_2 - ix_3 \\ x_1 + ix_4 & \frac{1}{3} - x_7 + \frac{2x_3}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & ix_5 \\ x_2 + ix_5 & x_3 + ix_6 \end{pmatrix}.
$$

The $l_1$ norm of coherence of a $d$-dimensional system, given by Eq. (6), can be written as [23]

$$
C_{l_1}(\rho) = \sum_{i=1}^{(d^2-d)/2} \sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}.
$$

And, the mixedness is given by

$$
M_f(\rho) = 1 - \frac{d}{2(d-1)} \sum_{i=1}^{d^2-1} x_i^2.
$$

Eq. (5) ensures that the normalized coherence $\frac{C_{l_1}(\rho)}{d_{l_1}(\rho)}$, of a quantum system with mixedness $M_f(\rho)$, is bounded to a region below the ellipse $\frac{C^{2}_{l_1}(\rho)}{d_{l_1}(\rho)} + \left( \sqrt{M_f(\rho)} \right)^2 = 1$. The quantum states with (normalized) quantum coherence that lie on the conic section are the maximally coherent states corresponding to a fixed mixedness and vice-versa [23].

It is interesting to note that provided $C_{l_1}(\rho) = \left( \sum_{ij} |\psi_{ij}|^2 \right)^{1/2}$ were a valid coherence measure, one could easily show that a complementarity relation, analogous to Eq. (5), holds:

$$
\frac{C^{2}_{l_1}(\rho)}{\left(1 - \frac{1}{d}\right)^2} + M_f(\rho) \leq 1.
$$

In Ref. [15], it was shown that the quantity $\tilde{C}_{l_1}(\rho) = C_{l_1}^2(\rho) = \sum_{ij} |\psi_{ij}|^2$ satisfies conditions (1) and (3). However, it fails to satisfy the condition (2b) in general. Thus it is not clear whether $C_{l_1}^2(\rho)$ is a valid coherence measure in the framework of above resource theory.

A natural question that arises is whether the reciprocity between quantum coherence and mixedness is measure specific? Put differently, does complementarity between coherence and mixedness hold for other measures of coherence and mixedness? It is trivial to note that

$$
\frac{C_f(\rho)}{\ln d} + \frac{S(\rho)}{\ln d} \leq 1,
$$

and

$$
C_g(\rho) + M_g(\rho) = 1 - \left( \max_{\sigma \in I} F(\rho, \sigma) - F(\rho, \tilde{\rho}/d) \right) \leq 1.
$$

We observe from Eqs. (5), (11) and (12) that for valid coherence measures, there is trade-off between functions of normalized coherence and normalized mixedness. This complementarity between coherence and mixedness appears to be a general feature. It would be an interesting exercise to investigate whether a given measure of coherence respects reciprocity with different measures of mixedness. In particular, we are interested in whether the following relations hold:

$$
\frac{C_{l_1}(\rho)}{(d-1)^2} + \frac{S(\rho)}{\ln d} \leq 1
$$

$$
C_{l_1}(\rho) + M_f(\rho) \leq 1, \text{ etc.}
$$

For rank-1 (pure) states, the above complementarity relations are trivially satisfied since mixedness for pure states is, by definition, zero (for geometric measure of coherence, one will have to set $M_f(\rho) \equiv 0$ by hand). Interestingly, for higher rank quantum states also the above reciprocity relations hold.

We provide numerical evidences which suggest that trade-off between coherence and mixedness is indeed an extensive feature of quantum systems (see Figs. 1 and 2). Though the reciprocity relation, $\frac{C_{l_1}(\rho)}{\ln d} + M_f(\rho) \leq 1$, is in conflict, it is well below the trivial value 2, for all states. We observe that higher is the rank of quantum states and number of qubits, more is the violation. We found numerically that the reciprocity relation $\frac{C_{l_1}(\rho)}{\ln d} + M_f(\rho) \leq 1$, is violated by two-qubit states also. An example of a two-qubit state which violates this relation is given in Eq. (14).

$$
\rho = \begin{pmatrix}
0.2501 & 0.0490 - 0.0090i & -0.1392 - 0.1148i & -0.2141 - 0.0515i \\
0.0490 + 0.0090i & 0.2064 & 0.1588 - 0.0438i & 0.0137 + 0.0650i \\
-0.1392 + 0.1148i & 0.1588 + 0.0438i & 0.3001 & 0.1858 + 0.0115i \\
-0.2141 + 0.0515i & 0.0137 - 0.0650i & 0.1858 - 0.0115i & 0.2434
\end{pmatrix}.
$$

Note that $\rho = \rho^\dagger$, $Tr \rho = 1$, $Tr \rho^2 = 0.5539$, and eigenvalues of $\rho$ are $[0.664, 0.336, 0, 0]$. Hence, $\rho$ is a valid rank-2 density matrix. For this density matrix, $\frac{C_{l_1}(\rho)}{\ln d} + M_f(\rho) = 0.5334 + 0.5948 = 1.1282 > 1$.
IV. DISTRIBUTION OF QUANTUM COHERENCE

Quantum coherence is a resource. Coherence of a multiparty quantum system \( \rho_{AB_1B_2\ldots B_n} \) is seen as a quantum correlation amongst the subsystems. We wish to study the distribution of coherence among the constituent subsystems. In particular, we are interested in the following monogamy-type relation \[ C(\rho_{AB_1B_2\ldots B_n}) - \sum_{k=1}^{n} C(\rho_{AB_k}) \geq 0, \] where \( C \) is some valid coherence measure and \( \rho_{AB_k} \) is the two-party reduced density matrix obtained after partial tracing all subsystems but subsystems \( A \) and \( B_k \). If the above relation is satisfied for arbitrary quantum system \( \rho_{AB_1B_2\ldots B_n} \), we say that the distribution of coherence is “faithful” with respect to subsystem \( A \). In the language of monogamy [34], \( C \) is monogamous with respect to pivot \( A \). If the above relation does not hold for any \( \rho_{AB_1B_2\ldots B_n} \), \( C \) is unfaithful or non-monogamous. Below we provide several interesting results on the distribution of quantum coherence in multipartite quantum systems. Remember that coherence is a basis-dependent quantity. In considering following results and theorems, we assume that quantum system under investigation is described in a fixed reference basis. Let \( \{|a_i\rangle\} \) and \( \{|b_i^{(k)}\rangle\} \) be the bases of subsystems \( A \) and \( B_k \) respectively, such that \( \rho_{AB} \equiv \sum_{i} |i \rangle \langle i|_{AB} \), \( |i \rangle_{AB} = \sum_k c_{k,i}|a_k^{(1)} \cdots b_i^{(n)} \rangle \), \( \rho_{AB} = \sum_i |i \rangle \langle i|_{AB}, \rho_{AB_j} = \text{Tr}_{AB_{\bar{j}}} \rho_{AB}, \) etc.

A. Numerical results

From numerical findings listed in Table I, we observe a number of important results. The percentage of quantum states satisfying the coherence additivity relation increases with increasing number of parties, the rank of quantum states and raising the power of coherence measures under investigation. Furthermore, for fixed rank and fixed number of qubits, the number of quantum states which satisfy the monogamy condition is larger for entropy-based coherence measure than distance-based coherence measure.

B. Analytical results

In this section, we provide conditions for the violation of the additivity relation of the relative entropy of coherence \( C_r \) [37].

**Theorem 1.** The relative measure of coherence violates the additivity relation for quantum states \( \rho_{AB} \equiv \rho_{AB_1B_2\ldots B_n} \) which satisfy \( S(\rho_{AB}) + S(\rho_A) = \sum_{k=1}^{n} S(\rho_{AB_k}) \).

**Proof.** Let \( \rho_{AB} = \rho_{AB_1B_2\ldots B_n} \) be the density matrix of an \((n+1)\)-party quantum system, and \( \rho_{AB_k} = \text{Tr}_{AB_{\bar{k}}} \rho_{AB_1B_2\ldots B_n} \) be the reduced density matrix obtained after partial tracing all subsystems.
Table I. Percentage of quantum states, of varying ranks, satisfying the additivity relation for the “normalized” coherence measures $C_r(\rho)$ and $C_r(\sigma)$ for 3, 4 and 5 qubits in the computational basis [35]. The percentage of quantum states satisfying the additivity relation increases with increasing number of parties [36], with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. For every rank, $2 \times 10^3$ three, four and five qubit states are generated Haar uniformly.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{rank} & \text{no. of qubits} & \delta_{C_{r_1}} & \delta_{C_{r_2}} & \delta_{C_{r_3}} & \delta_{C_{r_5}} \\
\hline
1 & 3 & 0.185 & 32.045 & 62.915 & 5.14 & 84.56 \\
& 4 & 0.015 & 64.765 & 94.445 & 64.225 & 99.92 \\
& 5 & 0.035 & 96.07 & 99.95 & 99.02 & 100. \\
\hline
2 & 3 & 0.445 & 38.245 & 70.935 & 75.425 & 99.685 \\
& 4 & 0.095 & 73.705 & 97.74 & 99.245 & 100. \\
& 5 & 0.145 & 98.713 & 99.995 & 99.995 & 100. \\
\hline
3 & 3 & 0.615 & 41.77 & 73.885 & 93.595 & 99.98 \\
& 4 & 0.14 & 79.475 & 98.395 & 99.975 & 100. \\
& 5 & 0.185 & 99.205 & 100. & 100. & 100. \\
\hline
4 & 3 & 0.72 & 42.385 & 75.155 & 97. & 99.985 \\
& 4 & 0.18 & 80.845 & 98.825 & 100. & 100. \\
& 5 & 0.265 & 99.385 & 99.995 & 100. & 100. \\
\hline
\end{array}
\]

It can be easily shown that for $\sigma_{AB} = \sigma_{AB_1B_2...B_k}$, $\sum_k S(\sigma_{AB}) = S(\sigma_{AB}) \geq (n - 1)S(\sigma_A) \geq 0$. This bound is a simple consequence of the strong subadditivity relation, $S(\rho_{ABC}) + S(\rho_A) \leq S(\rho_{AB}) + S(\rho_{AC})$, of von Neumann entropy. Hence $\Delta_1$ and $\Delta_2$ are non-negative. When $\sum_k S(\rho_{AB_k}) = S(\rho_{AB}) + S(\rho_A)$, i.e., $\Delta_1$ vanishes, $C_r(\rho_{AB}) \leq \sum_k C_r(\rho_{AB_k})$. Thus $C_r$ violates the additivity relation.

Very recently, a special case of above result was obtained in Ref. [24]. It was shown that $C_r$ violates the additivity relation for an arbitrary tripartite state $\rho_{ABC}$ which satisfies the strong subadditivity relation of von Neumann entropy. Tripartite states satisfying the strong subadditivity relation are reported in Ref. [38].

Analogous result can be obtained for other distributions as well. For instance, the following distribution yields

\[
C_r(\rho_{AB}) - \sum_{k=1}^{n} C_r(\rho_{AB_k}) = S(\rho_{AB}) - \sum_{k=1}^{n} [S(\rho_{AB_k}) - S(\rho_{AB_k})] = \Delta_1 - \Delta_2 - C_r(\rho_A).
\]

where $\rho_{AB_k} = Tr_{B_k}\rho_{AB_1B_2...B_k}$ be the reduced density matrix obtained after partial tracing subsystem $B_k$. Again since $\sum_{k=1}^{n} S(\sigma_{AB_k}) = (n - 1)S(\sigma_{AB}) \geq 0$ for $\sigma_{AB} = \sigma_{AB_1B_2...B_k}$ [39, 40], $\Delta_1$ and $\Delta_2$ are non-negative. When either $\Delta_1 \leq \Delta_1 + (n - 2)C_r(\rho_{AB})$ or $\Delta_2 = 0$, $C_r(\rho_{AB}) \leq \sum_{k=1}^{n} C_r(\rho_{AB_k})$. Thus $C_r$ violates the additivity relation.

However, coherence measures are not normalized in general. Thus, they do not lie between zero and unity for arbitrary quantum systems. But in investigating monogamy relations for quantum correlation measures, we consider that value of all quantities in the monogamy inequality lies in the same range. Therefore, it is reasonable to consider the normalized coherence. Suppose that $\rho_{AB} = \rho_{AB_1B_2...B_k} \in (\mathbb{C}^d)^{n+1}$ be a multipartite density operator. Considering the normalized relative entropy of coherence, we have

\[
\frac{C_r(\rho_{AB})}{\ln d^{n+1}} - \frac{\sum_{k=1}^{n} C_r(\rho_{AB_k})}{\ln d^n} = \frac{2(\Delta_1 - \Delta_2) - (n - 1) \sum_{k=1}^{n} C_r(\rho_{AB_k})}{2(n + 1)\ln d}.
\]

Again, when $\Delta_1$ vanishes, the normalized $C_r$ does not satisfy the additivity relation. Similarly, for the other distribution, we can obtain

\[
\frac{C_r(\rho_{AB})}{\ln d^{n+1}} - \frac{\sum_{k=1}^{n} C_r(\rho_{AB_k})}{\ln d^n} = \frac{n(\Delta_3 - \Delta_4) - \sum_{k=1}^{n} C_r(\rho_{AB_k}) - n(n - 2)C_r(\rho_{AB})}{n(n + 1)\ln d}.
\]

Thus, when $\Delta_3 = 0$, the normalized $C_r$ violates the additivity relation.

V. COHERENCE IN X STATES

Quantum states having \(\text{“}X\text{”}\)-structure are referred to as X states. Consider an $n$-qubit X state given by

\[
\rho = p|gGHZ\rangle\langle gGHZ| + (1 - p)I_d, \quad (20)
\]
where $|g_{GHZ}⟩ = (α|0⟩^{⊗n} + β|1⟩^{⊗n})$ with $|α|^2 + |β|^2$, $I_d$ is $d \times d$ identity matrix, $d = 2^n$ and $0 \leq p \leq 1$. It is easy to show that for this state, $C_l(ρ) = 2p|αβ|$ and $C_r(ρ) = -\left(p|α|^2 + \frac{1-p}{d}\right)\log_2\left(p|α|^2 + \frac{1-p}{d}\right) - \left(p|β|^2 + \frac{1-p}{d}\right)\log_2\left(p|β|^2 + \frac{1-p}{d}\right) - (p + \frac{1-p}{d})\log_2\left(p + \frac{1-p}{d}\right).

Theorem 2. For an $(n+1)$-party state $ρ_{AB_1B_2⋯B_n}^{X}$ in a given basis, any measure of coherence $C$ satisfies the additivity relation. That is, $X$ states satisfy the relation $C(ρ_{AB_1B_2⋯B_n}^{X}) - \sum_{k=1}^{n} C(ρ_{AB_k}^{X}) \geq 0$, where $ρ_{AB_k}^{X} = Tr_{AB_{k+1}⋯B_n} ρ^{X}$ is a two-qubit reduced density matrix.

Proof: This is because all the two-party reduced density matrices of $(n+1)$-party X states in the given basis are diagonal and any valid measure of quantum coherence vanishes for diagonal states. □

\[\begin{align*}
\delta_{C_1} & \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \\
\delta_{C_2} & \quad n=6 \quad n=10 \quad n=15 \quad n=20 \quad n=30
\end{align*}\]

Figure 3. Coherence score (y-axis) versus the number of excitations $r \leq \frac{n}{2}$ (x-axis) of Dicke states using the “normalized” coherence measures $C_{l1}$ (top panel) and $C_{r}$ (bottom panel). All quantities are dimensionless. The normalized measures of coherence do not satisfy the additivity relation of coherence for the Dicke states.

Now consider the Dicke states [41], which is symmetric with respect to interchange of qubits, given by

\[|D_{n,r}⟩ = \binom{n}{r}^{-\frac{1}{2}} \sum_{\sigma} ρ(0)^{⊗(n-r)} \otimes |1⟩^{⊗r},\]

where $\sum_{\sigma}$ represents sum over all $\binom{n}{r}$ permutations of $(n-r)$ $|0⟩$s and $r$ $|1⟩$s. Note that the Dicke state itself, in Eq. (21), is not an $X$ state but its all two-qubit reduced density matrices, in the computational basis, are same and has $X$ structure. Again, one can show that for the normalized measures of coherence

\[\begin{align*}
\delta_{C_{1\text{ normalized}}}(|D_{n,r}⟩) & = C_{l1}(|D_{n,r}⟩) - (n-1)C_{l1}(ρ_{D_{n,r}}^{(2)}) \\
& = \left(\frac{n}{r} - 1\right) - \frac{2r(n-r)}{n},
\end{align*}\]

and

\[\begin{align*}
\delta_{C_{r\text{ normalized}}}(|D_{n,r}⟩) & = C_{r}(|D_{n,r}⟩) - (n-1)C_{r}(ρ_{D_{n,r}}^{(2)}) \\
& = \log_2\left(\binom{n}{r}\right) - \frac{2r(n-r)}{n},
\end{align*}\]

where $ρ_{D_{n,r}}^{(2)}$ is the two-qubit reduced density matrix of the Dicke state. For $n \geq 3$ and $1 \leq r \leq n$, $\delta_{C_{l1}}(|D_{n,r}⟩)$ and $\delta_{C_{r}}(|D_{n,r}⟩)$ are non-positive. Thus, quantum coherence measures violate the additivity relation for the Dicke states in the computational basis (see Fig. 3).

Analogous result was obtained in Ref. [42], that the Dicke state is always non-monogamous with respect to quantum discord [43] and quantum work-deficit [44], and the Dicke state with more number of parties is more non-monogamous to that with a smaller number of parties.

However, if one considers the unnormalized measures of coherence, then

\[\begin{align*}
\delta_{C_{1}}(|D_{n,r}⟩) & = C_{l1}(|D_{n,r}⟩) - (n-1)C_{l1}(ρ_{D_{n,r}}^{(2)}) \\
& = \left(\frac{n}{r} - 1\right) - \frac{2r(n-r)}{n},
\end{align*}\]

and

\[\begin{align*}
\delta_{C_{r}}(|D_{n,r}⟩) & = C_{r}(|D_{n,r}⟩) - (n-1)C_{r}(ρ_{D_{n,r}}^{(2)}) \\
& = \log_2\left(\binom{n}{r}\right) - \frac{2r(n-r)}{n},
\end{align*}\]

In this case, when $n \geq 3$ and $1 \leq r \leq n$, $\delta_{C_{l1}}(|D_{n,r}⟩)$ and $\delta_{C_{r}}(|D_{n,r}⟩)$ are non-negative. Thus, quantum (unnormalized) coherence measures satisfy the additivity relation for the Dicke states in the computational basis.

VI. CONCLUSION

In this paper, we have shown that the reciprocity between coherence and mixedness of quantum states is a general feature as this complementarity holds for large spectra of measures of coherence and of mixedness. The numerical investigation of the distribution of coherence in multiparticle systems reveals that the percentage of quantum states satisfying the additivity relation increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under
We have provided conditions for the violation of the additivity relation of the relative entropy of coherence. We have further shown that the normalized measures of coherence violate the additivity relation for the Dicke states.

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