Loop Quantum Gravity
in the Momentum Representation

W. F. Chagas-Filho
Physics Department, Federal University of Sergipe, Brazil
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Abstract
We present a generalization of the first-order formalism used to describe the dynamics of a classical system. The generalization is then applied to the first-order action that describes General Relativity. As a result we obtain equations that can be interpreted as describing quantum gravity in the momentum representation.

1 Introduction
As is well-known, Quantum Mechanics can be formulated in the configuration (or position) representation or in the momentum representation. This situation emerges from the possible representations of the fundamental commutators of the quantum theory. To illustrate this, consider the simple example of the quantization of a one-dimensional system with a configuration variable $q$ and a momentum variable $p$. The corresponding quantum operators $\hat{q}$ and $\hat{p}$ must provide a representation of the fundamental commutator

$$[\hat{q}, \hat{p}] = [\hat{q}\hat{p} - \hat{p}\hat{q}] = i\hbar \quad (1)$$

The usual way to represent the commutator (1) is to choose

$$\hat{q} = q \quad \hat{p} = -i\hbar \frac{d}{dq} \quad (2)$$

In this case the wave function will be a function of $q$, that is $\psi = \psi(q)$, and we will be in the configuration representation.

Another possibility of representing the commutator (1) is to choose

$$\hat{q} = i\hbar \frac{d}{dp} \quad \hat{p} = p \quad (3)$$

In this case the wave function will be a function of $p$, that is $\psi = \psi(p)$, and we will be in the momentum representation. From a naive perspective, the operators (3) can be obtained from the operators (2) simply by substituting the letter
q by $p$ and the letter $p$ by $-q$ in equations (2). However, in a deeper level these two possibilities are related to the quantum mechanical wave-particle duality. The configuration representation is related to the particle aspect. Because of the De Broglie’s relation $\lambda = h/p$, the momentum representation is related to the wave aspect. The quantum wave-particle duality has a trace in classical mechanics in the form of a Hamiltonian duality. This duality interchanges position and momentum and leaves invariant the definition of the Poisson bracket. In this paper we will use this classical Hamiltonian duality to construct a formulation of quantum gravity in the momentum representation.

At present time, a quantum theory for the gravitational interaction, based on the canonical quantization of General Relativity (GR) is under development. It is called Loop Quantum Gravity. This theory has produced interesting results, such as the quantization of area and volume in terms of the Planck length $L_P = \sqrt{\frac{\hbar G}{c^3}} = 1.62 \times 10^{-35}m$. But with no present available way to test the theory against experimental results, the validity of LQG remains an open question [1,2,3].

In this paper we present a calculation that points in the direction of the validity of LQG. Here we present the basic equations of a momentum space formulation of quantum gravity, taking as the starting point the first-order action for GR.

This paper is organized as follows. In section two we derive the two simple classical equations that allow transitions to quantum gravity in the configuration and in the momentum representations. In section three we review the basic equations of quantum gravity in the configuration representation (Loop Quantum Gravity). In section four we present the basic equations of quantum gravity in the momentum representation. Brief concluding remarks appear in section five.

2 The first-order formalism and the transition to quantum mechanics

The first-order formalism is in the interface between Lagrangian mechanics and Hamiltonian mechanics. According to Dirac [6], a Hamiltonian formalism is a first approximation to a corresponding quantum theory. Since quantum mechanics can be formulated in the configuration or in the momentum representations, we need two first-order formalisms, one for each representation of quantum mechanics.

2.1 The first-order formalism for the configuration space formulation of quantum mechanics

A first-order formalism which can be considered as the classical limit of a configuration space formulation of quantum mechanics is the usual first-order formalism. It is based on the action functional
\[ S = \int_{t_1}^{t_2} dt[p\dot{q} - H(q, p)] \] (4)

where \( H(q, p) \) is the Hamilton’s function. Varying action (4) we find

\[ \delta S = \int_{t_1}^{t_2} dt[-\frac{\partial H}{\partial q}\delta q + p\delta \dot{q} + (\dot{q} - \frac{\partial H}{\partial p})\delta p] \] (5)

Integrating by parts the second term we have

\[ \int_{t_1}^{t_2} dt p\delta \dot{q} = p\delta q \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \dot{p}\delta q \]

Inserting this result into the variation (5) we are left with

\[ \delta S = \int_{t_1}^{t_2} dt[-(\dot{p} + \frac{\partial H}{\partial q})\delta q + (\dot{q} - \frac{\partial H}{\partial p})\delta p + p\delta q \bigg|_{t_1}^{t_2}] \]

The above integral vanishes if Hamilton’s equations

\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \] (6)

are satisfied. In this case the variation (5) reduces to the surface term

\[ \delta S = p\delta q \bigg|_{t_1}^{t_2} \]

Now we require that \( \delta q(t_1) = 0 \) and leave \( \delta q \) arbitrary at \( t = t_2 \). We therefore see that, as a function of the final point of the trajectory, action (4) satisfies

\[ p = \frac{\delta S}{\delta q} \] (7)

As we shall see below, equation (7) plays a central role in the transition to quantum gravity in configuration space.

2.2 The first-order formalism for the momentum space formulation of quantum mechanics

We now introduce a first-order formalism which can be considered as the classical limit for a momentum space formulation of quantum mechanics. This formalism can be constructed using the Hamiltonian duality transformation

\[ q \rightarrow p, \quad p \rightarrow -q \] (8)

which leaves invariant the structure of the Hamilton’s equations (6) and the definition of the Poisson bracket

\[ \{A, B\} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \]
which defines the algebraic structure in the phase space \((q, p)\).

Applying the duality transformation (8) to action (4) we obtain the new action

\[
S = \int_{t_1}^{t_2} dt [-q\dot{p} - \dot{H}(q, p)] \tag{9}
\]

Varying action (9) we have

\[
\delta S = \int_{t_1}^{t_2} dt [-\dot{p} + \frac{\partial \tilde{H}}{\partial q} \delta q - \frac{\partial \tilde{H}}{\partial p} \delta p - q \delta \dot{p}] \tag{10}
\]

Integrating by parts the last term gives

\[-\int_{t_1}^{t_2} dt q \delta \dot{p} = -q \delta p \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \dot{q} \delta p\]

Inserting this result into the variation (10) we find that

\[
\delta S = \int_{t_1}^{t_2} dt [-\dot{p} + \frac{\partial \tilde{H}}{\partial q} \delta q + (\dot{q} - \frac{\partial \tilde{H}}{\partial p}) \delta p - q \delta p \bigg|_{t_1}^{t_2} \tag{11}
\]

Now, if Hamilton´s equations

\[
\dot{q} = \frac{\partial \tilde{H}}{\partial p} \quad \dot{p} = -\frac{\partial \tilde{H}}{\partial q}
\]

are valid, the variation (11) reduces to the surface term

\[
\delta S = -q \delta p \bigg|_{t_1}^{t_2}
\]

We now impose that \(\delta p(t_1) = 0\) and leave \(\delta p\) arbitrary at \(t = t_2\). We then see that, as a function of the end point, action (9) satisfies

\[
- q = \frac{\delta S}{\delta p} \tag{12}
\]

Equation (12) is the central equation in this paper. As we will see below, it allows the the transition to quantum gravity in momentum space.

### 2.3 The transition to quantum mechanics

It is important to stress that the first-order formalism of section 2.2 was introduced to be used as the classical limit of a momentum space formulation of quantum mechanics. Since classical mechanics can not be formulated in momentum space because the quantum wave-particle duality practically disappears at the classical level, the classical Hamilton equations for the dynamic variables \(q\) and \(p\) derived from the Hamiltonian \(\tilde{H}(q, p)\) will in general appear to be inconsistent. However, the quantum Schrödinger equation obtained from the quantum operator corresponding to \(\tilde{H}(q, p)\) will be consistent.
The simplest example of the above situation is a free non-relativistic particle, described by the Hamiltonian

\[ H = \frac{p^2}{2m} \]

The Hamilton equations for this system are

\[ \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = 0 \]

Quantization of this system using the operators (2) leads to the Schrödinger equation

\[ -\hbar^2 \frac{\partial^2 \psi(q, t)}{\partial q^2} = i\hbar \frac{\partial \psi(q, t)}{\partial t} \]

Now, from the duality transformation (8), we obtain the Hamiltonian

\[ \tilde{H} = \frac{q^2}{2m} \]

The Hamilton equations now are

\[ \dot{q} = \frac{\partial \tilde{H}}{\partial p} = 0, \quad \dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\frac{q}{m} \]

which look inconsistent because the particle does not move while its momentum varies. However, quantization of this system using the operators (3) leads to the Schrödinger equation

\[ -\hbar^2 \frac{\partial^2 \psi(p, t)}{\partial p^2} = i\hbar \frac{\partial \psi(p, t)}{\partial t} \]

which is perfectly consistent.

## 3 Loop Quantum Gravity

In this section we review the basic equations that define quantum gravity in the configuration representation (LQG).

In 1986 Ashtekar [4,5] introduced a new set of variables to describe General Relativity. In this new set of variables GR can be described by the first-order action (for details on this construction see ref. [1], chapter four)

\[ S = \frac{1}{8\pi G} \int d^4x(E^a_i A^i_a - \lambda^i D_a E^a_i - \lambda^a E^i_a E^b_i - \lambda F^{ij}_{ab} E^a_i E^b_j) \]  

where

\[ D_a V^i = \partial_a V^i + \epsilon^i_{jk} A^j_a V^k \]

is the covariant derivative,

\[ F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b \]
is the curvature and $F^{ij}_{ab} = \epsilon^i_k E^k_{ab}$. The variables $\lambda^i$, $\lambda^a$ and $\lambda$ are Lagrange multipliers without dynamics.

Indices $i, j, ... = 1, 2, 3$ are internal $SU(2)$ indices and $a, b = 1, 2, 3$ are space indices. Comparing action (13) with action (4) we see that

a) the configuration variable is $A^i_1(\vec{x})$

b) the canonical momentum is $E^i_a(\vec{x})$

c) the total [6] Hamiltonian density is given by $H_T = \lambda^i D_a E^a_i + \lambda^a F^a_{ab} E^b_i + \lambda F^{ij}_{ab} E^a_i E^b_j$

Varying action (13) in relation to the variables $\lambda^i$, $\lambda^a$ and $\lambda$ we obtain the first-class [6] constraints

\begin{align*}
D_a E^a_i &= 0 \quad (14a) \\
F^a_{ab} E^b_i &= 0 \quad (14b) \\
F^{ij}_{ab} E^a_i E^b_j &= 0 \quad (14c)
\end{align*}

Equation (14a) is the requirement of invariance of the theory under internal $SU(2)$ transformations. Equation (14b) is the requirement of invariance of the theory under space diffeomorphisms. Equation (14c) is the canonical Hamiltonian. Equations (14) are equivalent to the Einstein equations [1]

Now, using the analog of equation (7), that is

$$E^a_i = \frac{\delta S}{\delta A^a_i}$$

we make a transition to the Hamilton-Jacobi formalism as an intermediate step to the quantum theory. Equations (14) then become [1]

\begin{align*}
D_a \frac{\delta S}{\delta A^a_i} &= 0 \quad (15a) \\
F^{ai}_{ab} \frac{\delta S}{\delta A^a_b} &= 0 \quad (15b) \\
F^{ij}_{ab} \frac{\delta S}{\delta A^a_i} \frac{\delta S}{\delta A^b_j} &= 0 \quad (15c)
\end{align*}

The transition to the quantum theory in the configuration space is then obtained by substituting the classical action $S$ by the wave functional $\Psi(A)$ in equations (15). The final result is

\begin{align*}
D_a \frac{\delta}{\delta A^a_i} \Psi(A) &= 0 \quad (16a) \\
F^{ai}_{ab} \frac{\delta}{\delta A^a_b} \Psi(A) &= 0 \quad (16b) \\
F^{ij}_{ab} \frac{\delta}{\delta A^a_i} \frac{\delta}{\delta A^b_j} \Psi(A) &= 0 \quad (16c)
\end{align*}

Equations (16) are the quantum gravity equations [1] in the configuration representation.
4 Quantum gravity in the momentum representation

In this section we derive the basic equations that we interpret to define quantum gravity in the momentum representation. First we apply the duality transformation

$$A^i_a \rightarrow E^a_i \quad E^a_i \rightarrow -A^i_a$$ (17)

to the first-order action (13) for GR. We obtain the action

$$S = \frac{1}{8\pi G} \int d^4x (-A^i_a \dot{E}^a_i - \lambda_i \nabla^a A^a_i - \lambda_b R^{ab}_i A^i_a - R^{ab}_i A^i_a A^j_b)$$ (18)

where the covariant derivative $\nabla^a$ is defined by

$$\nabla^a V^i = \partial^a V^i + \epsilon^{ijk} E^a_j V^k$$

and the curvature $R^{ab}_i$ is given by

$$R^{ab}_i = \partial^a E^b_i - \partial^b E^a_i + \epsilon^{ijk} E^a_j E^b_k$$

with $R^{ab}_i = \epsilon_{ijk} R^{ab}_k$. It is important to mention here that we should not interpret action (18) as describing a classical physical system. Rather, it should be interpreted as a formal equation describing the classical limit of a quantum physical system.

The equations of motion for the variables $\lambda_i$, $\lambda_a$ and $\lambda$ give the first-class [6] constraints

$$\nabla^a A^a_i = 0 \quad R^{ab}_i A^i_a = 0 \quad R^{ab}_i A^i_a A^j_b = 0$$ (19a,b,c)

The next step towards the quantum theory is to use the general equation (9) we derived in section two. In the present case equation (12) becomes

$$A^i_a = -\frac{\delta S}{\delta E^a_i}$$ (20)

Substituting equation (20) into equations (19) we have

$$\nabla^a \frac{\delta S}{\delta E^a_i} = 0 \quad R^{ab}_i \frac{\delta S}{\delta E^b_i} = 0 \quad R^{ab}_i \frac{\delta S}{\delta E^a_i} \frac{\delta S}{\delta E^b_j} = 0$$ (21a,b,c)
Finally, the transition to the quantum theory is completed by substituting the classical action $S$ by the wave functional $\Psi(E)$ in momentum space. This gives

$$\nabla^a \frac{\delta}{\delta E_i^a} \Psi(E) = 0 \quad (22a)$$

$$R_{i}^{ab} \frac{\delta}{\delta E_i^b} \Psi(E) = 0 \quad (22b)$$

$$R_{ij}^{ab} \frac{\delta}{\delta E_i^a} \frac{\delta}{\delta E_j^b} \Psi(E) = 0 \quad (22c)$$

We interpret equations (22) as the quantum gravity equations in the momentum representation.

5 Conclusions

In this paper we presented a generalization of the first-order formalism used to describe the dynamics of a classical system. This generalization is based on the Hamiltonian duality that interchanges the configuration and the momentum variables. The generalization is then applied to the first-order action that describes General Relativity. As a result of this, we obtain equations that can be interpreted as describing quantum gravity in the momentum representation. The conclusion of this paper is that, as quantum mechanics, quantum gravity can be formulated in the configuration or in the momentum representation.

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