OVERLAPPING DOMAIN DECOMPOSITION METHODS FOR PTYCHOGRAPHIC IMAGING

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Abstract. In ptychography experiments, redundant scanning is usually required to guarantee the stable recovery, such that a huge amount of frames are generated, and thus it poses a great demand of parallel computing in order to solve this large-scale inverse problem. In this paper, we propose the overlapping Domain Decomposition Methods (DDMs) to solve the nonconvex optimization problem in ptychographic imaging, that decouple the problem defined on the whole domain into subproblems only defined on the subdomains with synchronizing information on the interface of these subdomains, thus leading to highly parallel algorithms with good load balance. More specifically, for the nonblind recovery (with known probe in advance), by enforcing the continuity of the overlapping region for the image (sample), the nonlinear optimization model is established based on a novel smooth-truncated amplitude-Gaussian metric. Then the Alternating Direction Method of Multipliers (ADMM) is utilized to generate an efficient Overlapping Domain Decomposition based Ptychography algorithm (OD^2P) for the two-subdomain domain decomposition (DD), where all subproblems can be computed with close-form solutions. Due to the Lipschitz continuity for the gradient of the objective function, the convergence of the proposed OD^2P is derived under mild conditions. Moreover, it is extended to more general case including multiple-subdomain DD and blind recovery. Numerical experiments are further conducted to show the performance of proposed algorithms, demonstrating good convergence speed and robustness to the noise. Especially, we report the virtual wall-clock time of proposed algorithm, which shows great potential for parallel computing in massively parallel processing computers.

Key words. Overlapping domain decomposition method; Ptychography; Phase retrieval; Parallel computing; Smooth truncated amplitude-Gaussian metric; Blind recovery

AMS subject classifications. 46N10, 49N30, 49N45, 65F22, 65N21

1. Introduction. X-ray ptychography [30] enables imaging with the large field of view and nano-scale resolution, by combining the technique of coherent diffraction imaging and scanning transmission X-ray microscopy. It has been applied to vast research areas, that helps to reveal the structures and chemical contrast of nano-scale materials. Thus it is very meaningful to develop efficient algorithms for ptychographic imaging. As a special case for the phase retrieval problem (the inverse quadratic problem) arising in wide areas including optics, crystallography, materials sciences etc., various algorithms developed in [26, 31] and references therein can be exploited to tackle this problem. Especially, for ptychography, recent algorithms [9, 30, 18, 32, 25, 33, 28, 20, 5, 12] have been developed in order to further improve the performance and work in various settings, e.g. partial coherence and blind recovery for unknown probe.

As an important property of ptychography, a huge amount of the frames are usually collected such that the iterative imaging algorithms usually requires a large memory footprint and high computational cost. Therefore, it poses a great demand to develop parallel algorithms [29, 19, 27, 11]. Namely, the asynchronous parallel algorithm in [29] was performed on partitions of entire dataset asynchronously to get the sub-images, and then merge these sub-images to the whole image. However, during to possible drifts and phase ambiguities (linear phase ambiguity [17] for blind recovery) for the recovered sub-images, high computation cost was required for pairwise registration and correction of phase ambiguities. To further improve the asynchronous algorithm, the authors [19, 29] also provided synchronous parallel algorithm, where either the boundary layers of the sub-images were synchronized by averaging all local neighborhood in [29], or the alternating projection algorithm was implemented on GPU workstation [11], and similarly to [19, 27], each subdomain has the individual copy of the whole image and probe leading to high communication cost.

Domain decomposition methods (DDMs) have played a great role in solving partial differential equations numerically [24, 3, 16]. Particularly, it allows for highly parallel computing with good load balance,

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1256 × 256 × 10^4 pixels recorded for a typical dataset for 2D thin sample in a soft x-ray experiment developed in Advanced Light Source in Berkeley National Lab, while for 3D imaging, one hundred times larger dataset need be collected.
by decomposing the equations on whole domain to the problems on relatively small subdomains with information synchronization on the partition interfaces. It has also been successfully extended to large-scale image restoration and reconstruction problems, e.g. [35, 13, 7, 22, 21, 23, 10] and references therein. To the best of our knowledge, most of existing studies focused on the linear inverse problems with convex minimization modeling. In this paper, we will explore the potential using DDMs to realize parallel computing for ptychographic imaging, which essentially corresponds to a nonconvex quadratic inverse problem.

However, it seems quite difficult to design corresponding DDMs for the ptychographic imaging. On one hand, the forward operator for ptychography involves with the Fourier transform (integral operator) of the multiplication of the probe and sliding patch of the image, such that direct non-overlapping decomposition of the image into subdomains cannot split the measured frames completely, leading to both high computational and communication cost (see a related work of the DDM for nonlocal operator [8]). On the other hand, there is lack of convergence studies of DDMs for nonconvex minimization problems. Especially, the convergence for the related first-order operator splitting algorithms seems difficult to guarantee, since the convention objective function [28, 6] is nonsmooth, and the smoothed version [5] based operator-splitting algorithm cannot get rid of additional inner loop.

We aim to design more efficient DDMs for ptychographic imaging with convergence guarantee. Simply one can see that the non-overlapping domain decomposition (DD) of the image (sample) cannot reduce both the computation and communication costs compared the overlapping DD due to the Fourier transform. Therefore, we will consider the overlapping DD, such that the decomposed frames on the subdomains are completely decoupled. By enforcing the continuity of the image on joint layers of overlapping subdomains, we propose the overlapping DD based Ptychography algorithm (OD$^2$P) in the case of two-subdomain DD, utilizing the alternating direction methods of multipliers (ADMM [15, 34, 2]) in order to solve the constraint optimization problem with a modified metric. It is naturally extended to the case of the multiple-subdomain DD. As the size of the probe is much smaller than that of the image, the DDM for the blind recovery to recover the image and probe jointly is further proposed by sharing the joint probe across all subdomains, combined with the overlapping DD of the image. Comparing to the existing parallel algorithms, there is no need to correct the phase ambiguity, and moreover, a clear mathematical framework is given with fewer communication cost and strict theoretical convergence under quite mild conditions. The main contributions of this paper are summarized below:

- A novel smooth truncated metric is introduced to measure the recovered intensity and target, which allows for fast calculation with closed form of proximal mapping, and meanwhile guarantees the convergence of the first-order optimization algorithm due to its Lipschitz smoothness.
- An overlapping DD is introduced to solve ptychography reconstruction, where a narrow layer of the whole sample is shared with adjacent subdomains. We further solve the ptychography model on the overlapping DD by using ADMM, which is essentially a fully splitting algorithm on the sense that all subproblems have closed-form solutions. Furthermore, with mild condition of the stepsizes, the convergence is derived. To the best of our knowledge, it is the first time to characterize the convergence for DDMs in nonconvex problems.
- The DDMs are further extended to multiple-subdomain DD and blind recovery. For blind ptychography, we further propose a new variational model with the compact support set of Fourier transform of the illumination, to get rid of grid pathology for grid scanning. Especially, we report the virtual wall-clock time of proposed algorithm for multiple DD, which shows high speedup efficiency and great potential for parallel computing in massively parallel processing computers.

The rest of paper is given below. In section 2, the proposed OD$^2$P based on the two-subdomain DD is presented for nonblind recovery with convergence guarantee under mild conditions of the stepsizes. It is further extended to the case of multiple-subdomain DD and blind recovery in section 3. Numerous experiments are conducted in section 4. Finally, conclusions and future work are given in section 5.

2. DDM for nonblind ptychography: Two-subdomain case. In a standard ptychography experiment, a localized coherent X-ray probe (or illumination) $w$ scans through an image (or sample) $u$, while the detector collects a sequence of phaseless intensities in the far field. Throughout this paper, we consider the following discrete setting. The variable $u \in \mathbb{C}^n$ (the complex Euclidean space) corresponds to a 2D image (sample) with $\sqrt{m} \times \sqrt{m}$ pixels, and $w \in \mathbb{C}^m$ is a localized 2D probe with $\sqrt{m} \times \sqrt{m}$ pixels, where both $u$ and $w$ are written as a vector by a lexicographical order. A stack of phaseless measurements $\tilde{f}_j \in \mathbb{R}_+^n \forall 0 \leq j \leq J-1$
\( \mathbb{R}_+^n \) is the Euclidean space with non-negative elements) is collected with \( \tilde{f}_j = |\mathcal{F}(w \circ S_j u)|^2 \), where notations \(| \cdot |, (\cdot)^2\) represent the element-wise absolute, and square values of a vector respectively, \( \circ \) denotes the element-wise multiplication, and \( \mathcal{F} \) denotes the normalized discrete Fourier transformation. The operator \( S_j \in \mathbb{R}^{m \times n} \) is denoted by a binary matrix that specifies a small window with the index \( j \) and size \( m \) over the entire image \( u \). The general blind ptychography (BP) problem can then be expressed as follows:

\[
(2.1) \quad \text{To find } w \in \mathbb{C}^m \text{ and } u \in \mathbb{C}^n, \text{ such that } |\mathcal{F}(w \circ S_j u)|^2 = \tilde{f}_j \forall 0 \leq j \leq J-1.
\]

In this section, we assume that the probe \( w \) is known in advance. Denote the linear operator \( A : \mathbb{C}^n \to \mathbb{C}^m \) as \( Au := \left( (\mathcal{F}(w \circ S_0 u))^T, (\mathcal{F}(w \circ S_1 u))^T, \ldots, (\mathcal{F}(w \circ S_{J-1} u))^T \right)^T \) \( \forall u \in \mathbb{C}^n \), such that the nonblind ptychography reconstruction can be modeled below:

\[
(2.2) \quad \text{To find the image } u \in \mathbb{C}^n, \text{ s.t. } |Au|^2 = f,
\]

with the phaseless data \( f := (\tilde{f}_1^T, \tilde{f}_2^T, \ldots, \tilde{f}_{J-1}^T)^T \in \mathbb{R}^m \) and the given probe \( w \).

2.1. Overlapping domain decomposition. Denote the whole region \( \Omega := \{0, 1, 2, \ldots, n-1\} \) in the discrete setting. There exists the two-subdomain overlapping DD \( \{\Omega_d\}_{d=1}^2 \), such that \( \Omega = \bigcup_{d=1}^2 \Omega_d \) with \( \Omega_d := \{l_d^0, l_d^1, \ldots, l_d^{n_d-1}\} \), and the overlapping region is denoted as \( \Omega_{1,2} := \Omega_1 \cap \Omega_2 = \{l_0, l_1, \ldots, l_{n-1}\} \). Here we consider a special overlapping DD. As shown in Fig. 1 (b) (totally 4 \( \times \) 4 frames) for grid scan, each subdomain contains a complete scanning frame, i.e. the right (or left) boundaries of frames near the left(or right) subdomain interface lies exactly on the right(or left) boundary of subdomains. Denote the restriction operators \( R_1, R_2 \) as \( R_d u = u|_{\Omega_d} \) and \( R_{1,2} u = u|_{\Omega_{1,2}} \), i.e. \( (R_d u)(j) = u(l_d^j) \forall 0 \leq j \leq n_d - 1 \), \( (R_{1,2} u)(j) = u(l_j) \forall 0 \leq j \leq n - 1 \). Then two groups of localized shift operators can be introduced \( \{S^d_{j} \}_{j=0}^{n_d-1} \), \( d = 1, 2 \), with \( \sum_d J_d = J \). Therefore, we can denote the linear operators \( A_1, A_2 \) on the subdomains as

\[
(2.3) \quad A_d u_d := \left( (\mathcal{F}(w \circ S^d_{0} u_d))^T, (\mathcal{F}(w \circ S^d_{1} u_d))^T, \ldots, (\mathcal{F}(w \circ S^d_{n_d-1} u_d))^T \right)^T, \quad d = 1, 2.
\]

Naturally, the measurement \( f \) is also decomposed to two non-overlapping parts \( f_1, f_2 \), i.e. \( f_d := |A_d R_d u|^2 \) assuming that \( (u, f) \) follows (2.2).

![Fig. 1. (a) Ptychography scan in the domain \( \Omega \) (grid scan): the starting scan centers at point \( O \), and then move up (or to the left) with the center point \( O' \) (or \( O'' \)); (b) Two-subdomain DD (totally 4 \( \times \) 4 frames): The subdomains \( \Omega_1, \Omega_2 \) are generated by two 4 \( \times \) 2-scans, and the overlapping region \( \Omega_{1,2} = \Omega_1 \cap \Omega_2 \).](image)

We remark that as shown in Fig. 1, the width of the overlapping region \( \Omega_{1,2} \) is \( \sqrt{m} - n_{\text{stepsize}} \), with the scan stepsize \( n_{\text{stepsize}} \) pixels (assuming the stepizes are same for \( x, y \)-directions). Therefore, letting the stepsize \( n_{\text{stepsize}} \) guarantees the sufficient overlap of the two-subdomain DD, that immediately demonstrates that with sufficient redundancy of the scan for ptychography leads to enough overlap of the DD. We also remark that one can first divide the measurements with non-overlapping partitions as shown in Fig. 1 (b), that will automatically generate the overlapping DD.
2.2. A new nonlinear optimization model. For simplicity, we consider the decomposition with only
two subdomains, and rewrite the problem (2.2) on the whole region to the following equations

\[(2.4) \quad |A_1u_1|^2 = f_1, \quad |A_2u_2|^2 = f_2,\]

with the continuity of the overlapping regions

\[(2.5) \quad \pi_{1,2}u_1 = \pi_{2,1}u_2,\]

where the operators \(\pi_{1,2}\) (restriction from \(\Omega_1\) into \(\Omega_{1,2}\)) and \(\pi_{2,1}\) (restriction from \(\Omega_2\) into \(\Omega_{1,2}\)) are denoted
as \(\pi_{1,2}u_1 := R_{1,2}R_1^Tu_1\), and \(\pi_{2,1}u_2 := R_{1,2}R_2^Tu_2\). Remarkably, the continuity condition is essential for the
perfect reconstruction of the ptychography. Without this condition, the redundancy of the boundary layer
may not be sufficient such that there will be visible artifacts in the final reconstruction results and meanwhile
phase ambiguities need to be corrected.

We will discuss how to solve the above problem efficiently. One may solve the problem using a Schwarz
alternating method [24]. However, directly solving this multivariable quadratic system is generally NP-
complete. Alternatively, we seek for the solution by nonlinear optimization. With the popular amplitude-
Gaussian metric (AGM) [33, 12] to measure the fitting errors, one can readily establish the following nonlinear
optimization model

\[
\min_{u_1, u_2} \sum_{d=1}^2 \left( ||A_du_d| - \sqrt{f_d}||^2 \right), \quad s.t. \quad \pi_{1,2}u_1 = \pi_{2,1}u_2,
\]

where \(\sqrt{\cdot}\) denotes the element-wise square root of a vector, and \(||\cdot||\) denotes the standard \(L^2\) norm in complex
Euclidean space. Due to lack of smoothness, one may introduce a modified metric [18, 5, 14] with additional
penalization factor in order to gain the convergence guarantee. However, an inner loop has to be introduced
[5], since the proximal mapping of the modified metric does not have closed form solution.

Readily one sees that the AGM is smooth except at the origin. Instead of smoothing the function by
adding a global penalization factor [18, 5, 14], a more simple but effective way is to cut off the AGM near
the origin, and then add back a smooth function which can keep the global minimizer unchanged. At the
same time, it should not increase computational cost compared with AGM. Hence, a novel smooth truncated
AGM (ST-AGM) \(G_\epsilon(\cdot; f)\) with truncated parameter \(\epsilon > 0\) is designed below:

\[(2.6) \quad G_\epsilon(z; f) := \sum_{j} g_\epsilon(z(j); f(j)),\]

where \(\forall \quad x \in \mathbb{C}, b \in \mathbb{R}^+\),

\[(2.7) \quad g_\epsilon(x; b) := \begin{cases} \frac{1 - \epsilon}{2} \left( b - \frac{1}{\epsilon}|x|^2 \right), & \text{if } |x| < \epsilon \sqrt{b}; \\ \frac{1}{2}||x| - \sqrt{b}||^2, & \text{otherwise.} \end{cases}\]

One readily sees that the new metric is a mixture of the AGM and a quadratic function (smooth
connected). Please also see Fig. 2 for the landscape of ST-AGM in 1-dimension: Among four different plots
with \(\epsilon = 0, 0.1, 0.5, 1.5\), one can observe that the minimizer does not change w.r.t. different \(\epsilon\) \((0 \leq \epsilon < 1)\);
However, if \(\epsilon > 1\), the global minimizer moves to the origin. Hence, to keep the minimizer unchanged, we
set \(\epsilon \in (0, 1)\) hereafter.

Denote the corresponding proximal mapping of \(g_\epsilon\) as \(\text{Prox}_{G_\epsilon,\lambda}(y) := \arg \min_z G_\epsilon(z; f) + \frac{\lambda}{2}\|z - y\|^2\). After
simple calculations, one derives the following closed form solution with \(\epsilon \in (0, 1)\):

\[(2.8) \quad (\text{Prox}_{G_\epsilon,\lambda}(y))(j) = \text{sign}(y(j)) \times \begin{cases} \max \left\{ 0, \frac{\lambda |y(j)|}{\sqrt{y(j)^2 + \frac{\lambda}{\epsilon}}}, \right & \text{if } |y(j)| < \epsilon - \frac{1 - \epsilon}{\lambda} \sqrt{f(j)}; \\ \sqrt{f(j) + \lambda |y(j)|}, & \text{otherwise.} \end{cases}\]

One can further show that \(G_\epsilon(z; f)\) is Lipschitz differentiable by the following lemma.
It is not difficult to see that
\[ \| \nabla \mathcal{G}_x(z; f) - \nabla \mathcal{G}_x(y; f) \| \leq L \| z - y \| \quad \forall \ z, y \in \mathbb{C}^m. \]

That concludes this lemma.

![Fig. 2. Landscapes of ST-AGM as \( g_s(x, 1) \) \( \forall x \in \mathbb{R} \) (\( \epsilon = 0, 0.1, 0.5, 1.5 \)). If \( \epsilon = 0 \), ST-AGM is exactly the same as AGM.]

Lemma 2.1. Denote \( L := \frac{2}{\epsilon} - 1 \). The function \( \mathcal{G}_x(; f) \) is Lipschitz differentiable, i.e.
\[ \| \nabla \mathcal{G}_x(z; f) - \nabla \mathcal{G}_x(y; f) \| \leq L \| z - y \| \quad \forall \ z, y \in \mathbb{C}^m. \]

Proof. We will show the Lipschitz continuity of \( \nabla_x g_{s}(; b) \). Denoting \( \mathcal{K}_1 := \{ x \in \mathbb{C} : |x| < \epsilon \sqrt{b} \} \) and \( \mathcal{K}_2 := \mathbb{C} \setminus \mathcal{K}_1 \). Readily one knows that \( g_{s}(x; b) \in \mathcal{C}^1 \) is if \( \epsilon > 0 \), where the gradient is given below
\[ \nabla_x g_{s}(x; b) = \begin{cases} (1 - \frac{1}{\epsilon})x, \text{ if } x \in \mathcal{K}_1; \\ (1 - \frac{\sqrt{\epsilon}}{|x|})x, \text{ otherwise.} \end{cases} \]

It is not difficult to see that
\[ | \nabla_x g_{s}(x_1; b) - \nabla_x g_{s}(x_2; b) | = \left| \frac{1}{\epsilon} - 1 \right| |x_1 - x_2| \quad \forall x_1, x_2 \in \mathcal{K}_1. \]

One can also get
\[ | \nabla_x g_{s}(x_1; b) - \nabla_x g_{s}(x_2; b) | \leq |(1 - \frac{\sqrt{\epsilon}}{|x_1|})(x_1 - x_2)| + |(\frac{\sqrt{\epsilon}}{|x_1|} - \frac{\sqrt{\epsilon}}{|x_2|})x_2| \]
\[ \leq |1 - \frac{\sqrt{\epsilon}}{|x_1|}|x_1 - x_2| + \frac{\sqrt{\epsilon}}{|x_1|}|x_1 - |x_2|| \leq (|\frac{1}{\epsilon} - 1| + \frac{1}{\epsilon})|x_1 - x_2| \quad \forall x_1, x_2 \in \mathcal{K}_2. \]

One can further derive
\[ | \nabla_x g_{s}(x_1; b) - \nabla_x g_{s}(x_2; b) | = |(1 - \frac{1}{\epsilon})x_1 - (1 - \frac{\sqrt{\epsilon}}{|x_2|})x_2| \]
\[ \leq |1 - \frac{1}{\epsilon}| |x_1 - x_2| + |x_2| |\frac{1}{\epsilon} - \frac{\sqrt{\epsilon}}{|x_2|}| \leq (|\frac{1}{\epsilon} - 1| + \frac{1}{\epsilon})|x_1 - x_2| \quad \forall x_1 \in \mathcal{K}_1, x_2 \in \mathcal{K}_2. \]

By (2.11)-(2.13), one gets
\[ | \nabla_x g_{s}(x_1; b) - \nabla_x g_{s}(x_2; b) | \leq (|\frac{1}{\epsilon} - 1| + \frac{1}{\epsilon})|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{C}, \]
that demonstrates the Lipschitz continuity of \( \nabla_x g_{s}(; b) \).

Finally one can derive
\[ \| \nabla_x \mathcal{G}_x(z; f) - \nabla_x \mathcal{G}_x(y; f) \| = \sqrt{\sum_j \| \nabla_x g_{s}(z(j); f(j)) - \nabla_x g_{s}(y(j); f(j)) \|^2} \]
\[ \leq (|\frac{1}{\epsilon} - 1| + \frac{1}{\epsilon}) \| z - y \| \quad \forall z, y \in \mathbb{C}^m, \]
where the last relation is based on (2.14). That concludes this lemma. \( \square \)
To solve the saddle point problem (2.18), an iterative scheme using the alternating minimization is given and then introduce the corresponding augmented Lagrangian for (2.16) as follows:

\[
\text{(2.16)}
\]

Then we are concerned with the following saddle point problem below:

\[
\text{(2.17)}
\]

where we assume that \( f > 0 \) as \([28]\) (one can always remove the zero intensity values). It does not have singularity at the origin, and is exactly the first-order optimality condition for the AGM based optimization problem in the case of \( \epsilon = 0 \). Moreover, one can notice that the stationary points of the ST-AGM defined in (2.7) keep unchanged compared with those of AGM, and new metric only differs from the AGM near the origin (local maximum) by setting the truncated parameter \( \epsilon \in (0, 1) \) in (2.7). Therefore, it should not affect the performance of new algorithm developed in the paper.

Hereafter, we consider the nonlinear optimization problem with ST-AGM. In order to enable the parallel computing of \( u_1 \) and \( u_2 \), we introduce an auxiliary variable \( v \) which is only defined on the interface of two overlapping subdomains, and then are concerned with the following model:

\[
\text{(2.15)}
\]

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\[
\text{(2.16)}
\]

and then introduce the corresponding augmented Lagrangian for (2.16) as follows:

\[
\text{(2.17)}
\]

Then we are concerned with the following saddle point problem below:

\[
\text{(2.18)}
\]

We remark that when considering the following optimization problem w.r.t. the solution on the entire region

\[
\min_u G_c(Au; f),
\]

one can readily get the first-order optimality condition by (2.10)

\[
A^*(1 - \frac{\sqrt{V}}{\max(\epsilon \sqrt{V}, |Au|)}) \circ Au = 0,
\]

where we assume that \( f > 0 \) as \([28]\) (one can always remove the zero intensity values). It does not have singularity at the origin, and is exactly the first-order optimality condition for the AGM based optimization problem in the case of \( \epsilon = 0 \). Moreover, one can notice that the stationary points of the ST-AGM defined in (2.7) keep unchanged compared with those of AGM, and new metric only differs from the AGM near the origin (local maximum) by setting the truncated parameter \( \epsilon \in (0, 1) \) in (2.7). Therefore, it should not affect the performance of new algorithm developed in the paper.

Hereafter, we consider the nonlinear optimization problem with ST-AGM. In order to enable the parallel computing of \( u_1 \) and \( u_2 \), we introduce an auxiliary variable \( v \) which is only defined on the interface of two overlapping subdomains, and then are concerned with the following model:

\[
\text{(2.15)}
\]

where we assume that \( f > 0 \) as \([28]\) (one can always remove the zero intensity values). It does not have singularity at the origin, and is exactly the first-order optimality condition for the AGM based optimization problem in the case of \( \epsilon = 0 \). Moreover, one can notice that the stationary points of the ST-AGM defined in (2.7) keep unchanged compared with those of AGM, and new metric only differs from the AGM near the origin (local maximum) by setting the truncated parameter \( \epsilon \in (0, 1) \) in (2.7). Therefore, it should not affect the performance of new algorithm developed in the paper.

2.3. Overlapping DD based ptychography algorithm (OD\(^2\)P). As one of the most popular first-order operator-splitting algorithm, the ADMM [15, 34, 2] has been applied to ptychography reconstruction, showing competitive performances for Poisson noisy measurements as well as large scan stepizes [33, 5, 12]. We will also solve the proposed model (2.15) using ADMM.

In order to develop an iterative scheme without inner loop as well as with fast convergence for large-step scan, two auxiliary variables \( z_1, z_2 \) are introduced below:

\[
\text{(2.16)}
\]

and then introduce the corresponding augmented Lagrangian for (2.16) as follows:

\[
\text{(2.17)}
\]

Then we are concerned with the following saddle point problem below:

\[
\text{(2.18)}
\]

To solve the saddle point problem (2.18), an iterative scheme using the alternating minimization is given below: To get \((u_1^{n+1}, u_2^{n+1}, v^{n+1}, z_1^{n+1}, z_2^{n+1}, \Lambda_1^{n+1}, \Lambda_2^{n+1}, \Lambda_1^{n+1}, \Lambda_2^{n+1})\) by the following steps:

\[
\text{(2.19)}
\]

\[
\text{Step 1.} \quad (v^{n+1}, z_1^{n+1}, z_2^{n+1}) = \arg\min_{u_1, u_2, v, z_1, z_2} \mathcal{L}(u_1^n, u_2^n, v, \Lambda_1^n, \Lambda_2^n, \Lambda_1^n, \Lambda_2^n, \Gamma_1^n, \Gamma_2^n)
\]

\[
\text{Step 2.} \quad \Gamma_1^{n+1} = \Gamma_1^n + A_d u_1^n - z_1^{n+1}, d = 1, 2;
\]

\[
\text{Step 3.} \quad (u_1^{n+1}, u_2^{n+1}) = \arg\min_{u_1, u_2} \mathcal{L}(u_1, u_2, v^{n+1}, z_1^{n+1}, z_2^{n+1}, \Lambda_1^n, \Lambda_2^n, \Lambda_1^{n+1}, \Lambda_2^{n+1}, \Gamma_1^n, \Gamma_2^n)
\]

\[
\text{Step 4.} \quad \Lambda_1^{n+1} = \Lambda_1^{n+1} - \Lambda_2^{n+1} + \pi_d, d = 1, 2;
\]

\[
\text{Step 4.} \quad \Lambda_1^{n+1} = \Lambda_1^{n+1} - \Lambda_2^{n+1} + \pi_d, d = 1, 2;
\]
with the approximation solutions \((u^n_d, u^n_0, v^n, z^n_1, z^n_2, \Gamma^n_1, \Gamma^n_0, \Lambda^n_{1,2}, \Lambda^n_{2,1})\) in the \(n^{th}\) step.

One readily knows that the objective functions for all subproblems (sub-optimization problem) of proposed ADMMs are differentiable, such that we have
\[
\begin{align*}
0 &= \nabla z_d G_d (z_d^{n+1}; f_d) - \eta \Gamma^n_d, \quad d = 1, 2; \\
0 &= v^{n+1} - \frac{1}{2} \sum_{d=1}^{2} (\pi_{d,3-d} u^n_d + \Lambda^n_{d,3-d}); \\
0 &= (r \pi_{d,3-d} \pi_{d,3-d} + \eta A^n_d) z_d^{n+1} - (r \pi_{d,3-d} (v^{n+1} - \Lambda^n_{d,3-d}) + \eta A^n_d (z_d^{n+1} - \Gamma^n_d)), \quad d = 1, 2;
\end{align*}
\]
where the third equation is derived by taking partial gradient of the augmented Lagrangian and the multiplier update in (2.19). By (2.8) and the first two equations in (2.20), one can get closed forms of \(z_d^{n+1}\) and \(v^{n+1}\). In order to get the solution for \(u^n_d, d = 1, 2\), we further discuss how to solve the linear systems in the last two equations in (2.20). With the definition of \(A_d\) in (2.3), one can get
\[
A_d^* A_d = \text{diag}(\sum_{j=0}^{J_d-1} (S_d^j)^T |w|^2), \quad d = 1, 2.
\]
One can also get \(\pi_{d,3-d} = R_d R^T_{d,3-d} R^T_{d,3-d} = \text{diag}(\sigma_d), \quad d = 1, 2\), with \(\sigma_d \in \mathbb{R}^{n_d}\) denoted as
\[
\sigma_d(j) = \begin{cases} 1, & \text{if } l_d^j \in \Omega_{1,2}; \\ 0, & \text{otherwise}, \end{cases} \quad 0 \leq j \leq n_d - 1.
\]
Therefore, \(u^n_d, d = 1, 2\) can be determined explicitly as below: \(0 \leq j_d \leq n_d - 1, \quad j_d \in \mathbb{Z}\),
\[
u^{n+1}_d(j_d) = \begin{cases} \frac{(r \pi_{d,3-d} (v^{n+1} - \Lambda^n_{d,3-d}) + \eta A^n_d (z_d^{n+1} - \Gamma^n_d))(j_d)}{r + \sum_{j_d=0}^{J_d} ((S_d^j)^T |w|^2)(j_d)}, & \text{if } l_d^j \in \Omega_{1,2}; \\ \frac{(A^n_d (z_d^{n+1} - \Gamma^n_d))(j_d)}{\sum_{j_d=0}^{J_d} ((S_d^j)^T |w|^2)(j_d)}, & \text{otherwise}. \end{cases}
\]

The overall overlapping DD based ptychography algorithm (OD\(^2\)P) is given as below:

\[\text{Algorithm 1: Overlapping DD based Ptychography algorithm (OD}\^2\text{P)}\]

\begin{itemize}
  \item \textbf{Step 0.} Initialize \(u^0_1 = 1, u^0_2 = 1, v^0, z^0 = A_1 u^0_1, z^0_2 = A_2 u^0_2\), and multipliers \(\Gamma^0_1 = 0, \Gamma^0_2 = 0, \Lambda^0_{1,2} = \Lambda^0_{2,1} = 0\). Set \(n := 0\).
  \item \textbf{Step 1.} Update \(z^n_1\) and \(z^n_2\) in parallel by
    \begin{align*}
    z_{d}^{n+1} &= \text{Prox}_{G_d, \eta} (\Gamma^n_d + A_d u^n_d), \quad d = 1, 2; \\
    \text{and update } v^{n+1} \text{ by}
    
    v^{n+1} &= \frac{1}{2} \sum_{d=1}^{2} (\pi_{d,3-d} u^n_d + \Lambda^n_{d,3-d});
    \end{align*}
  \item \textbf{Step 2.} Update multipliers \(\Gamma^n_1, \Gamma^n_2\) in parallel by
    \begin{align*}
    \Gamma^n_d &= \Gamma^n_d + A_d u^n_d - z^{n+1}_d, \quad d = 1, 2; \\
    \text{Step 3.} \text{ Update } u^{n+1}_1, u^{n+1}_2 \text{ in parallel by (2.23).}
    \text{Step 4.} \text{ Update multipliers } \Lambda^{n+1}_{1,2}, \Lambda^{n+1}_{2,1} \text{ by}
    \begin{align*}
    \Lambda^{n+1}_{d,3-d} &= \Lambda^{n+1}_{d,3-d} + \pi_{d,3-d} u^{n+1}_d - v^{n+1}, \quad d = 1, 2;
    \end{align*}
    \item \textbf{Step 5.} If satisfying the stopping condition, then stop and output \(u^{n+1}_1, u^{n+1}_2\) as the final solution; otherwise set \(n := n + 1\), and go to Step 1.
\end{itemize}
2.4. Convergence analysis. First by the last equation in (2.20) and the multipliers update in (2.19), one can derive
\begin{equation}
0 = \eta A_d^*(A_d u_{d+1}^n + \Gamma_{d+1}^n - z_{d+1}^n) + r A_{d-3}^T \Lambda_{d-3}^{n+1}, d = 1, 2.
\end{equation}

**Assumption 1.** The matrices $A_d^T A_d$ $(d = 1, 2)$ are non-singular, i.e. there exists strictly positive constant $\bar{c}_d$ $(d = 1, 2)$, s.t. $|A_d h_d| \geq \bar{c}_d |h_d|$ \quad $\forall h_d \in \mathbb{C}^n$.

The assumption is to guarantee the boundedness of the iterative sequence \{$u^n$\} if \{$A_d u^n$\} is bounded. By (2.21), the assumption holds with sufficiently small stepsize, i.e. each scan position and its neighbours have sufficient overlapping (One can also refer to Remark 4.1 of [5]).

Due to (2.9), one can prove that the function $G_{2}(\cdot; f)$ has the descent property [5]:
\begin{equation}
G_{2}(z; f) - G_{2}(\hat{z}; f) - \Re(\langle \nabla_{z} G_{2}(\hat{z}; f), z - \hat{z} \rangle) \leq \frac{L}{2} \|z - \hat{z}\|^2 \quad \forall z, \hat{z} \in \mathbb{C}^m.
\end{equation}

**Lemma 2.2.** Denote $E(z) = \frac{1}{2} \|Tz - f\|^2 + M(z)$, with convex function $M$ and linear mapping $T$. Letting $z^*$ be the global minimizer, i.e. $E(z^*) = \arg \min_z E(z)$, then we have
\begin{equation}
E(z) - E(z^*) \geq \frac{L}{2} \|T(z - z^*)\|^2.
\end{equation}

The proof of above lemma is trivial, and therefore we omit the details. Denote $X^n := (u^n, u_2^n, v^n, z_1^n, z_2^n, \Lambda_{1,2}^n, \Lambda_{2,1}^n, \Gamma_1^n, \Gamma_2^n)$. Then we can prove the non-increasing of the sequence \{$\mathcal{L}(X^n)$\}.

**Lemma 2.3.**
\begin{equation}
\mathcal{L}(X^n) - \mathcal{L}(X^{n+1}) \geq \left( \frac{\eta}{2} - \frac{2L^2}{r} \|A_1^* A_1\|^2 \right) \|\pi_{1,2}(u_1^n - u_{1}^{n+1})\|^2 + \left( \frac{\eta}{2} - \frac{2L^2}{r} \|A_2^* A_2\|^2 \right) \|\pi_{2,1}(u_2^n - u_{2}^{n+1})\|^2
\end{equation}
\begin{equation}
+ \left( \frac{\eta - 3L \eta}{2} - \frac{L^2}{r} \|\pi_{1,2} A_1^* A_1\|^2 \right) \|z_1^n - z_{1}^{n+1}\|^2 + \left( \frac{\eta - 3L \eta}{2} - \frac{L^2}{r} \|\pi_{2,1} A_2^* A_2\|^2 \right) \|z_2^n - z_{2}^{n+1}\|^2
\end{equation}
\begin{equation}
+ r \|v^n - v_{n}^{n+1}\|^2 + \frac{\eta}{2} \sum_{d=1}^2 \|A_d(u_d^n - u_{d}^{n+1})\|^2
\end{equation}
\begin{equation}
\geq \Re(\langle \nabla_{z} \mathcal{L}(X^n; f) - \eta \Gamma_{d+1}^n, z_d^n - z_{d}^{n+1} \rangle) + \frac{\eta - L}{2} \|z_d^n - z_{d}^{n+1}\|^2.
\end{equation}

**Proof.** In order to estimate the decay of the augmented Lagrangian, the following relation [5] is given below:
\begin{equation}
\|z\|^2 - \|y\|^2 = \|z - y\|^2 + 2 \Re(\langle z, y - y \rangle) \quad \forall z, y \in \mathbb{C}^m.
\end{equation}

First for the $z_d$--subproblems, by (2.30), one readily has
\begin{equation}
\mathcal{L}(X^n) - \mathcal{L}(X^{n+1}) \geq \sum_{d=1}^2 G_{2}(z_d^n; f_d) + \frac{\eta}{2} \|z_d^n - (\Gamma_d^n + A_d u_d^n)\|^2 - \left( G_{2}(z_d^{n+1}; f_d) + \frac{\eta}{2} \|z_d^{n+1} - (\Gamma_d^n + A_d u_d^n)\|^2 \right)
\end{equation}
\begin{equation}
\geq \Re(\langle \nabla_{z} \mathcal{L}(X^n; f_d) - \eta \Gamma_d^n, z_d^n - z_{d}^{n+1} \rangle) + \frac{\eta - L}{2} \|z_d^n - z_{d}^{n+1}\|^2.
\end{equation}

Further by the first equation in (2.20), (2.9) and Cauchy’s inequality $\Re(\langle z, y \rangle) \geq -\frac{1}{2} \|z\|^2 - \frac{1}{2L} \|y\|^2$, one gets
\begin{equation}
\mathcal{L}(X^n) - \mathcal{L}(X^{n+1}) \geq \sum_{d=1}^2 \|z_d^n - z_{d}^{n+1}\|^2.
\end{equation}

For $v$--subproblem, it is quite standard, and we omit the details. Hence combined with (2.31), one gets
\begin{equation}
\mathcal{L}(X^n) - \mathcal{L}(X^{n+1}) \geq \sum_{d=1}^2 \|z_d^n - z_{d}^{n+1}\|^2 + r \|v^n - v_{n}^{n+1}\|^2.
\end{equation}

For the multipliers update \{$\Gamma_d^n$\}, one gets
\begin{equation}
\mathcal{L}(u_1^n, u_2^n, v^n, z_1^{n+1}, z_2^{n+1}, \Lambda_{1,2}^n, \Lambda_{2,1}^n, \Gamma_1^n, \Gamma_2^n) - \mathcal{L}(u_1^n, u_2^n, v^n, z_1^{n+1}, z_2^{n+1}, \Lambda_{1,2}^n, \Lambda_{2,1}^n, \Gamma_1^n, \Gamma_2^n)
\end{equation}
\begin{equation}
= -\eta \sum_{d=1}^2 \|\Gamma_d^n - \Gamma_d^{n+1}\|^2 \geq \frac{\eta - L}{2} \|v^n - v_{n}^{n+1}\|^2.
\end{equation}
where the last equation is derived by the first equation in (2.20) and (2.9).

For the sequences \( \{u_d^n\} \), based on Lemma 2.2, one gets

\[
(2.34) \quad L(u_1^n, u_2^n, v^{n+1}, z_1^{n+1}, z_2^{n+1}) - L(u_1^{n+1}, u_2^{n+1}, v^{n+1}, z_1^{n+1}, z_2^{n+1}) \geq \frac{\eta}{2} \sum_d \|A_d(u_d^{n+1} - u_d^n)^2\| + \frac{\eta}{2} \left( \|\pi_{1,2}(u_1^n - u_1^{n+1})\|^2 + \|\pi_{2,1}(u_2^n - u_2^{n+1})\|^2 \right).
\]

For the multipliers \( \{\Lambda_{1,2}^n\} \) and \( \{\Lambda_{2,1}^n\} \), update, by (2.28), one has

\[
\Lambda_{1,2}^{n+1} = \frac{2}{\eta} \Lambda_{1,2}^n A_1^T \left( A_1 u_1^{n+1} + \Gamma_1 + z_1^n \right) + \frac{2}{\eta} \Lambda_{1,2}^n A_1 \nabla_x G(z_1^n; f_1) + \frac{2}{\eta} \pi_{1,2} A_1^T z_1^{n+1},
\]

by the first equation in (2.20) and \( A_1^T A_1 \) being diagonal matrix. Then one can estimate the successive error of the multiplier \( \{\Lambda_{1,2}^n\} \) as \( \|\Lambda_{1,2}^n - \Lambda_{1,2}^{n+1}\| \leq \frac{\eta}{\eta - 2} \|\Lambda_{1,2}^n A_1\| \|\pi_{1,2}(u_1^n - u_1^{n+1})\| + \frac{L + \eta}{\eta} \|\pi_{1,2} A_1\| \|z_1^n - z_1^{n+1}\|, \) by (2.9). By the inequality \((a + b)^2 \leq 2(a^2 + b^2) \) \( \forall a, b \in \mathbb{R} \), one immediately obtains

\[
(2.35) \quad \|\Lambda_{1,2}^n - \Lambda_{1,2}^{n+1}\|^2 \leq \frac{2\eta}{\eta} \|\Lambda_{1,2}^n A_1\|^2 \|\pi_{1,2}(u_1^n - u_1^{n+1})\|^2 + \frac{2(L + \eta)^2}{\eta} \|\pi_{1,2} A_1\|^2 \|z_1^n - z_1^{n+1}\|^2.
\]

Similarly, one can get

\[
(2.36) \quad \|\Lambda_{2,1}^n - \Lambda_{2,1}^{n+1}\|^2 \leq \frac{2\eta}{\eta} \|\Lambda_{2,1}^n A_2\|^2 \|\pi_{2,1}(u_2^n - u_2^{n+1})\|^2 + \frac{2(L + \eta)^2}{\eta} \|\pi_{2,1} A_2\|^2 \|z_2^n - z_2^{n+1}\|^2.
\]

Combined with (2.35) and (2.36), one gets

\[
L(u_1^n, u_2^n, v^{n+1}, z_1^{n+1}, z_2^{n+1}, \Lambda_{1,2}^n, \Lambda_{2,1}^n, \Lambda_{1,2}^{n+1}, \Lambda_{2,1}^{n+1}, \Gamma_1^{n+1}, \Gamma_2^{n+1}) - L(X^{n+1})
= -r\|\Lambda_{1,2}^n - \Lambda_{1,2}^{n+1}\|^2 - r\|\Lambda_{2,1}^n - \Lambda_{2,1}^{n+1}\|^2
\geq - \frac{2\eta}{\eta} \|\Lambda_{1,2}^n A_1\|^2 \|\pi_{1,2}(u_1^n - u_1^{n+1})\|^2 + \frac{2\eta}{\eta} \|\Lambda_{2,1}^n A_2\|^2 \|\pi_{2,1}(u_2^n - u_2^{n+1})\|^2
\]

\[
- \frac{2(L + \eta)^2}{\eta} \|\pi_{1,2} A_1\|^2 \|z_1^n - z_1^{n+1}\|^2 - \frac{2(L + \eta)^2}{\eta} \|\pi_{2,1} A_2\|^2 \|z_2^n - z_2^{n+1}\|^2.
\]

Summing up (2.32), (2.33), (2.34), and (2.37) concludes this lemma. □

**Lemma 2.4.** The augmented Lagrangian is lower bounded, i.e. \( L(X^n) > -\infty \), if

\((\eta, r) \in \{ (\eta, r) \in \mathbb{R}_+^2 : \eta - 1 \geq 0, r - \eta \max\{\|\pi_{1,2} A_1\|, \|\pi_{2,1} A_2\|\} \geq 0 \} \). \]

**Proof.** First we have

\[
L(X^{n+1}) := \sum_d G_e(z_d^{n+1}; f_d)
+ \frac{\eta}{2} \|\pi_{1,2} u_1^{n+1} - v^{n+1} + \Lambda_{1,2}^{n+1}\|^2 - \frac{\eta}{2} \|\Lambda_{1,2}^{n+1}\|^2
+ \frac{\eta}{2} \|\pi_{2,1} u_2^{n+1} - v^{n+1} + \Lambda_{2,1}^{n+1}\|^2 - \frac{\eta}{2} \|\Lambda_{2,1}^{n+1}\|^2
+ \eta \sum_d \left( \frac{1}{2} \|A_d u_d^{n+1} - z_d^{n+1} + \Gamma_d^{n+1}\|^2 - \frac{\eta}{2} \|\Gamma_d^{n+1}\|^2 \right)
\]

\EQ (2.20), (2.28)

\[
\geq \frac{\eta}{2} \|\pi_{1,2} u_1^{n+1} - v^{n+1} + \Lambda_{1,2}^{n+1}\|^2 - \frac{\eta}{2} \|\pi_{1,2} A_1^T \left( A_1 u_1^{n+1} + \Gamma_1 + z_1^n \right) + \frac{\eta}{2} \|\pi_{1,2} A_1 \nabla_x G(z_1^n; f_1) + \frac{\eta}{2} \pi_{1,2} A_1^T z_1^{n+1} \|^2
+ \sum_d \left( \frac{1}{2} \|A_d u_d^{n+1} - z_d^{n+1} + \Gamma_d^{n+1}\|^2 + G_e(z_d^{n+1}; f_d) - \frac{\eta}{2} \|\nabla_x G_e(z_d^{n+1}; f_d)\|^2 \right)
\geq \frac{\eta}{2} \|\pi_{1,2} u_1^{n+1} - v^{n+1} + \Lambda_{1,2}^{n+1}\|^2 + \frac{1}{2} \|\pi_{2,1} u_2^{n+1} - v^{n+1} + \Lambda_{2,1}^{n+1}\|^2
+ \sum_d \left( \frac{1}{2} \|A_d u_d^{n+1} - z_d^{n+1} + \Gamma_d^{n+1}\|^2 \right)
+ \sum_d \left( \frac{1}{2} \|A_d u_d^{n+1} - z_d^{n+1} + \Gamma_d^{n+1}\|^2 \right)
\]
By the definition of (2.7) and its gradient (2.10), one gets

\[
\sum_{d,j} \left( g_d(z_d^{n+1}(j), f_d(j)) - \frac{1}{2\eta} \| \nabla_x g_d(z_d^{n+1}(j), f_d(j)) \|^2 \right)
\]
\[
= \sum_d \frac{1}{2} \left( 1 - \frac{1}{\eta} \right) \sum_{j_1 \in \{j : |z_d^{n+1}(j)| > \epsilon \sqrt{f_d(j)} \}} \| z_d^{n+1}(j_1) \| - \sqrt{f_d(j_1)} \|^2
\]
\[
+ \sum_d \sum_{j_2 \in \{j : |z_d^{n+1}(j)| < \epsilon \sqrt{f_d(j)} \}} \left( \frac{1 - \eta}{2} f_d(j_2) - \frac{1 - \eta}{2} (1 + \frac{1 - \epsilon}{2\eta}) |z_d^{n+1}(j_2)|^2 \right).
\]

Hence one gets

\[
\mathcal{L}(X^{n+1}) \geq \frac{\eta}{2} \| \pi_{1,2} u_1^{n+1} - v^{n+1} + A_{1,2}^{n+1} \|^2 + \frac{\eta}{2} \| \pi_{2,1} u_2^{n+1} - v^{n+1} + A_{2,1}^{n+1} \|^2
\]
\[
+ \sum_d \left( \frac{\eta}{2} (1 - \frac{1}{r} \max \{ \| \pi_{1,2} A_1^i \|, \| \pi_{2,1} A_2^i \| \} ) \| A_d u_d^{n+1} - z_d^{n+1} + \Gamma_d^{n+1} \|^2 \right)
\]
\[
+ \sum_d \frac{1}{2} \left( 1 - \frac{1}{\eta} \right) \sum_{j_1 \in \{j : |z_d^{n+1}(j)| > \epsilon \sqrt{f_d(j)} \}} |z_d^{n+1}(j_1) | - \sqrt{f_d(j_1)} \|^2
\]
\[
+ \sum_d \sum_{j_2 \in \{j : |z_d^{n+1}(j)| < \epsilon \sqrt{f_d(j)} \}} \left( \frac{1 - \eta}{2} f_d(j_2) - \frac{1 - \eta}{2} (1 + \frac{1 - \epsilon}{2\eta}) |z_d^{n+1}(j_2)|^2 \right).
\]

(2.38)

Readily one knows the last term is bounded since \( |z_d^{n+1}(j_2)| < \epsilon \sqrt{f_d(j_2)} \). By setting \( \eta \geq 1 \) and \( r \geq \eta \max \{ \| \pi_{1,2} A_1^i \|, \| \pi_{2,1} A_2^i \| \} \), one can immediately conclude this lemma.

Lemma 2.5. Letting \((r, \eta) \in \mathcal{K}\) with

\[
\mathcal{K} := \left\{ (r, \eta) \in \mathbb{R}^2_+ : \eta \geq 1, r - 2\eta \max \{ \| A_1^i \|, \| A_2^i \| \} > 0, \frac{n-3L}{2} - \frac{L^2}{\eta} - \frac{2(L+n)^2}{r^2} \max \{ \| \pi_{1,2} A_1^i \|^2, \| \pi_{2,1} A_2^i \|^2 \} > 0 \right\},
\]

the following assertions hold:

1. The sequence \( \{ \mathcal{L}(X^n) \} \) is non-increasing with the finite lower bound.
2. The following successive error converges to zero, i.e.

\[
\lim_{n \to +\infty} X^n - X^{n+1} = 0,
\]

3. The iterative sequence \( \{ X^n \} \) is bounded under Assumption 1.

Before giving the detailed proof, we will show that the set \( \mathcal{K} \) is nonempty. By denoting \( c_0 := 2 \max \{ \| A_1^i \|, \| A_2^i \| \} \), \( c_1 := 2 \max \{ \| \pi_{1,2} A_1^i \|^2, \| \pi_{2,1} A_2^i \|^2 \} \), and letting \( \eta \geq 1, r > c_0 \eta \), we further have

\[
c_1 \frac{(L+n)^2}{r^2} < \frac{c_2}{c_0} \frac{(L+n)^2}{r} = \frac{c_2}{c_0} (L/n + 1)^2 \leq \frac{c_2}{c_0} (L + 1)^2 \quad \text{the first inequality is based on } r > c_0 \eta, \text{ and the second is based on } \eta \geq 1.
\]

Therefore, one just needs to guarantee the following inequality \( \frac{n-3L}{2} - \frac{L^2}{\eta} - \frac{2(L+n)^2}{r^2} \max \{ \| \pi_{1,2} A_1^i \|^2, \| \pi_{2,1} A_2^i \|^2 \} > 0 \), with choosing proper \( \eta \). By letting \( \eta > 6L + 2L^2 + \frac{2L^2}{c_0} (L + 1)^2 \), the above inequality holds. Finally setting \( (r, \eta) \) in the nonempty set \( \{ (r, \eta) : \frac{n-3L}{2} - \frac{L^2}{\eta} - \frac{2(L+n)^2}{r^2} > 0 \} \), it also belongs to the set \( \mathcal{K} \), that immediately implies it is nonempty. Roughly to say, with big enough parameters \( \eta \) and \( r \) (\( r \) is no less than \( \eta \) with a constant factor), the above lemma holds.

Proof. The first two items (Items (1)-(2)) can be easily proved by Lemmas 2.3-2.4, (2.20), and (2.28). Notice that the last inequality is required in order to guarantee the positivity of the coefficients for the terms related with \( \| z_d^n - z_d^{n+1} \|, d = 1, 2 \).

One can readily get the boundedness of \( \{ z_d^n \} \) by (2.38). By the first equation of (2.20) and continuity of \( \nabla_d G_d(z_d; f_d) \), the sequences \( \{ \Gamma_d^n \} \) are also bounded. Since \( \{ A_d u_d^n - z_d^n + \Gamma_d^n \} \) are bounded by (2.38) and Assumption 1, one can further conclude the boundedness of \( \{ u_d^n \} \). By (2.28), one can get the boundedness of \( \{ \Lambda_1^{n+1} \} \) and \( \{ \Lambda_2^{n+1} \} \), as well as the boundedness of \( \{ u^n \} \). That finishes the proof of Item (3). □

Theorem 1 (subsequence convergence). Letting \((r, \eta) \in \mathcal{K}, \) any limit point of \( \{ X^n \} \) is the stationary point of the saddle point problem (2.18) under Assumption 1.
Proof. By Lemma 2.5 and (2.20)-(2.28), one can readily prove that any limit point \(X^* := (u^*_1, u^*_2, v^*, z^*_1, z^*_2, \Lambda^*_1, \Lambda^*_2, \Lambda^*_3, \Gamma^*_1, \Gamma^*_2)\) of the iterative sequence satisfies the following equations

\[
\begin{align*}
0 &= \nabla_{z_d} G(z^*_d; f_d) - \eta \Gamma^*_d; \\
0 &= v^* - \frac{1}{2} \sum_{d=1}^{2} (\pi_{d,3-d} u^*_d + \Lambda^*_d); \\
0 &= (r \pi_{d,3-d} u^*_d + \eta A^*_d A_d) u^*_d - (r \pi_{d,3-d} (v^* - \Lambda^*_d)) + \eta A_d(z^*_d - \Gamma^*_d), \quad d = 1, 2; \\
0 &= A_d u^*_d - z^*_d; \\
0 &= \pi_{d,3-d} u^*_d - v^*, \quad d = 1, 2;
\end{align*}
\]

(2.41)

By the last two equations of the above relation, it can be simplified below:

\[
\begin{align*}
0 &= \nabla_{z_d} G(z^*_d; f_d) - \eta \Gamma^*_d, \quad d = 1, 2; \\
0 &= \Lambda^*_1 + \Lambda^*_2; \\
0 &= r \pi_{d,3-d} \Lambda^*_d + \eta A^*_d \Gamma^*_d, \quad d = 1, 2; \\
0 &= A_d u^*_d - z^*_d, \quad d = 1, 2; \\
0 &= \pi_{d,3-d} u^*_d - v^*, \quad d = 1, 2,
\end{align*}
\]

(2.42)

that is exactly the stationary point of (2.18), and therefore it finishes the proof. \(\Box\)

**Theorem 2 (global convergence).** Letting \((r, \eta) \in \mathcal{K}\), the iterative sequence \(\{X^n\}\) converges to the stationary point of the saddle point problem (2.18) under Assumption 1.

Proof. It is quite standard to prove that the gradient of the augmented Lagrangian is bounded by the \(\|X^n - X^{n+1}\|\), which simply follows Lemma 4.3 of [5]. One just needs to show that the augmented Lagrangian with the ST-AGM metric is semi-algebraic (We study this property by separating the real and imaginary parts of the complex-valued variables following [5]), since it can be expressed by

\[
\begin{align*}
\left\{(z_{d,r}(j), z_{d,i}(j), F(j)) \in \mathbb{R}^3 : \quad &F(j) - \frac{1}{2}(f_d(j)) - \frac{1}{2}((z_{d,r}(j))^2 - (z_{d,i}(j))^2) = 0 \\
\cap &\{(z_{d,r}(j), z_{d,i}(j), F(j)) \in \mathbb{R}^3 : \quad (z_{d,r}(j))^2 + (z_{d,i}(j))^2 - \epsilon^2 f_d(j) > 0\}\right\}
\end{align*}
\]

\[
\bigcup \left\{(z_{d,r}(j), z_{d,i}(j), F(j)) \in \mathbb{R}^3 : \quad \frac{1}{2}((z_{d,r}(j))^2 + (z_{d,i}(j))^2 + f_d(j)) - F(j) \geq 0 \\
\cap &\{(z_{d,r}(j), z_{d,i}(j), F(j)) \in \mathbb{R}^3 : \quad \frac{1}{2}((z_{d,r}(j))^2 + (z_{d,i}(j))^2 + f_d(j)) - F(j)^2 - f_d(j)((z_{d,r}(j))^2 + (z_{d,i}(j))^2) = 0 \\
\cap &\{(z_{d,r}(j), z_{d,i}(j), F(j)) \in \mathbb{R}^3 : \quad (z_{d,r}(j))^2 + (z_{d,i}(j))^2 - \epsilon^2 f_d(j) \leq 0\}\right\},
\]

with \(z_{d,r}(j) := z_{d,r}(j) + i z_{d,i}(j)\) and \(i^2 = -1\). Immediately one sees that the function \(D_j(z_d; f_d) := q_j(z_d(j); f_d(j))\) is semi-algebraic. Then ST-AGM \(G_j(z_d; f_d) = \sum_j D_j(z_d; f_d)\) is also semi-algebraic since the finite sum of the algebraic functions are also semi-algebraic [1]. Since other terms other than ST-AGM are also semi-algebraic, the augmented Lagrangian function \(L(X)\) is semi-algebraic. Thus one concludes this theorem based on Theorem 1, the gradient of augmented Lagrangian bounded by \(\|X^n - X^{n+1}\|\) and semi-algebraic property following [1]. \(\Box\)

We remark that Theorem 1 demonstrates that limit point of iterative sequence satisfies the constraint \(\pi_{1,2} u_1 = \pi_{2,1} u_2\). Denote

\[
\forall 0 \leq j \leq n - 1, \quad u^*(j) := \begin{cases} (R^T_1 u^*_1)(j), & \text{if } j \in \Omega_1 \setminus \Omega_{1,2}; \\
(R^T_2 v^*)(j), & \text{if } j \in \Omega_{1,2}; \\
(R^T_2 u^*_2)(j), & \text{otherwise};\end{cases}
\]

\(z^* := Au^*, \quad \Gamma^* := (\Gamma^*_1, \Gamma^*_2)^T\). Further with the given stationary point system (2.42), we have

\[
\begin{align*}
0 &= \nabla_{z} G(z^*; f) - \eta \Gamma^*, \\
0 &= Au^* - z^*, \\
0 &= A^* \Gamma^*.
\end{align*}
\]
Then we get $A^* \nabla_2 G_d(Au; f) = 0$, that is exactly the first-order optimality condition for the optimization problem on the whole domain as $\min_u G_d(Au; f)$. Hence, by merging these two sub-solutions, Theorem 2 actually demonstrates the proposed algorithm is able to produce the stationary point for the original optimization problem defined on the whole domain.

3. Extensions. In this part, we mainly discuss further extensions for multiple-subdomain DD (more than two) and blind recovery.

3.1. Multiple-subdomain DD. We consider the DD with $D$ subdomains $(D \geq 2)$. Denote the overlapping DD as $\Omega := \bigcup_{d=1}^{D} \Omega_d$, and there exists a non-empty index set $\mathcal{N} := \{(i_1, i_2) : \Omega_{i_1} \cap \Omega_{i_2} \neq \emptyset \land 1 \leq i_1, i_2 \leq D\}$ denoting the index pairs for overlapped subdomains, and $\mathcal{N} := \{\Omega_{i_1, i_2} \in \mathcal{N} : i_1 < i_2\}$. In a similar manner to the two-subdomain case in the last section, one can denote $A_d, f_d (1 \leq d \leq D)$ and the restriction operator $\pi_{i_1, i_2} \forall (i_1, i_2) \in \mathcal{N}$ from the subdomain $\Omega_{i_1}$ to the overlapping region $\Omega_{i_1, i_2}$ between the subdomains $\Omega_{i_1}$ and $\Omega_{i_2}$, such that to get the solution $u_d$ on the subdomain $\Omega_d \forall 1 \leq d \leq D$, one has to solve the following equations:

$$|A_d u_d|^2 = f_d \forall 1 \leq d \leq D; \quad \pi_{i_1, i_2} u_{i_1} = \pi_{i_2, i_1} u_{i_2} \forall (i_1, i_2) \in \mathcal{N},$$

that exactly extends (2.4) and (2.5) to multiple subdomains DD. The following constraint optimization problem is given by introducing auxiliary variables $\{z_d\}_{d=1}^{D}$ and $\{v_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}$:

$$\begin{align*}
\min_{\{u_d\}_{d=1}^{D}, \{v_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}, \{z_d\}_{d=1}^{D}, \{\Lambda_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}, \{\Gamma_d\}_{d=1}^{D}} & \sum_{d=1}^{D} G_d(z_d; f_d) \\
\text{s.t.} \quad & \pi_{i_1, i_2} u_{i_1} - v_{i_1, i_2} = 0, \pi_{i_2, i_1} u_{i_2} - v_{i_1, i_2} = 0 \quad (i_1, i_2) \in \mathcal{N}, \\
& z_d - A_d u_d = 0, \quad 1 \leq d \leq D.
\end{align*}$$

Similar to the two-subdomain case, the corresponding augmented Lagrangian is given below:

$$\begin{align*}
\mathcal{L}_m(\{u_d\}, \{v_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}, \{z_d\}_{d=1}^{D}, \{\Lambda_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}, \{\Gamma_d\}_{d=1}^{D}) := \sum_{d=1}^{D} G_d(z_d; f_d) \\
+ \sum_{(i_1, i_2) \in \mathcal{N}} \left( r \mathcal{R}(\pi_{i_1, i_2} u_{i_1} - v_{i_1, i_2}) + \frac{\eta_d}{2} \|\pi_{i_1, i_2} u_{i_1} - v_{i_1, i_2}\|^2 \right) \\
+ \sum_{(i_1, i_2) \in \mathcal{N}} \left( r \mathcal{R}(\pi_{i_2, i_1} u_{i_2} - v_{i_1, i_2}) + \frac{\eta_d}{2} \|\pi_{i_2, i_1} u_{i_2} - v_{i_1, i_2}\|^2 \right) \\
+ \eta \sum_{d=1}^{D} \left( \mathcal{R}(\Gamma_d, A_d u_d - z_d) + \frac{1}{2} \|A_d u_d - z_d\|^2 \right),
\end{align*}$$

with multipliers $\{\Lambda_{i_1, i_2}\}_{(i_1, i_2) \in \mathcal{N}}$ and $\{\Gamma_d\}_{d=1}^{D}$.

We only analyze the $u_d$-subproblem which is different from the two-subdomain case in Algorithm 1. One has

$$\begin{align*}
\min_{u_d} & \left( \sum_{(i_1, d) \in \mathcal{N}} \frac{r}{2} \|\Lambda_{d, i_1} + \pi_{d, i_1} u_d - v_{d, i_1}\|^2 + \sum_{(i_1, d) \in \mathcal{N}} \frac{\eta_d}{2} \|\Lambda_{d, i_1} + \pi_{d, i_1} u_d - v_{d, i_1}\|^2 \right) + \frac{1}{2} \|A_d u_d - z_d\|^2,
\end{align*}$$

such that the first order optimality condition is given as

$$\begin{align*}
\left( \eta A^*_d u_d + r \sum_{(i_1, d) \in \mathcal{N}} \pi^T_{d, i_1} (\pi_{d, i_1} u_d - v_{d, i_1}) \right)_{u_d} \\
= & \eta A^*_d (z_d - \Gamma_d) + r \sum_{(i_1, i_2, d) \in \mathcal{N}} \pi^T_{d, i_1} u_{i_1, i_2} + r \sum_{(i_1, d) \in \mathcal{N}} \pi^T_{d, i_1} (v_{i_1, d} - \Lambda_{d, i_1}).
\end{align*}$$

Similar to the case of two-subdomain, this subproblem has closed-form solution as (2.23), such that it can also be very efficiently solved.

Similar to the two subdomain decomposition, the overall overlapping DD based Ptychography algorithm for multiple subdomains (OD$^2$P$_m$) is given as below:

Algorithm 2: Overlapping DD based Ptychography algorithm for Multiple Subdomains (OD$^2$P$_m$)
Step 0. Initialize \( u_d^0 = 1 \), \( \{v_{i_1,i_2}^0\} \), \( z_d^0 = A_d u_d^0 \), and multipliers \( \Gamma_d^0 = 0, \Lambda_{i_1,i_2}^0 = 0 \). Set \( n := 0 \).

Step 1. Update \( \{z_d^{n+1}\}_{d=1}^D \) by (2.24) and update \( \{v_{i_1,i_2}^{n+1}\}_{(i_1,i_2) \in \mathcal{N}} \) by

\[
v_{i_1,i_2}^{n+1} = \frac{1}{2} (\pi_{i_1,i_2} u_{i_1}^n + \pi_{i_2,i_1} u_{i_2}^n + \Lambda_{i_1,i_2}^n + \Lambda_{i_2,i_1}^n);
\]

Step 2. Update multipliers in parallel by

\[
\Gamma_d^{n+1} = \Gamma_d^n + A_d u_d^n - z_d^{n+1} \quad \forall \ 1 \leq d \leq D;
\]

Step 3. Update \( \{u_d^{n+1}\}_{d=1}^D \) in parallel by

\[
u_d = \left( \eta A_d^* A_d + r \sum_{(d,i_1), (i_1,i_2) \in \mathcal{N}} (\pi_{d,i_1}^T \pi_{d,i_1} + \pi_{d,i_2}^T \pi_{d,i_2}) \right)^{-1} \times \left( \eta A_d^* (z_d^{n+1} - \Gamma_d^n) + r \sum_{(d,i_2) \in \mathcal{N}} \pi_{d,i_2}^T (u_{i_1}^{n+1} - \Lambda_{i_1,i_2}^n) + r \sum_{(i_1,d) \in \mathcal{N}} \pi_{d,i_1}^T (v_{i_1,i_2}^{n+1} - \Lambda_{i_2,i_1}^n) \right);
\]

Step 4. Update multipliers by

\[
\Lambda_{i_1,i_2}^{n+1} = \Lambda_{i_1,i_2}^n + \pi_{i_1,i_2}^T u_{i_1}^{n+1} - v_{i_1,i_2}^{n+1},
\]

\[
\Lambda_{i_2,i_1}^{n+1} = \Lambda_{i_2,i_1}^n + \pi_{i_2,i_1}^T u_{i_2}^{n+1} - v_{i_2,i_1}^{n+1},
\]

for all \((i_1,i_2) \in \mathcal{N}\).

Step 5. If satisfying the stopping condition, then stop and output \( u_d^{n+1} \) as the final solution; otherwise set \( n := n + 1 \), and go to Step 1.

Among all the variables update in Algorithm 2, most variables (except \( v_{i_1,i_2}^{n+1} \) and the multipliers \( \Lambda_{i_1,i_2}^{n+1} \)) can be computed independent to other variables defined on another subdomain. The information of overlapping regions \( v_{i_1,i_2}^{n+1} \) is synchronized by the interfaces of two sub-solutions \( u_{i_1}^n, u_{i_2}^n \), as well as the multiplier \( \Lambda_{i_1,i_2}^n \). Hence, in principle the proposed OD^2P_m can be implemented highly parallel, since the communications only happen at the boundary layers (much smaller sizes compared with the entire image).

For the convergence analysis, it is almost the same as the case of two subdomains and therefore we omit the details and give the convergence theorem directly as below:

**Theorem 3.** Letting \((r, \eta) \in \mathcal{K}_D\) with

\[
\mathcal{K}_D := \left\{ (r, \eta) \in \mathbb{R}_+^2 : \eta \geq 1, \ r - 2 \eta \max \{ \|A_d^* A_d\| \} D_{d=1}^D > 0, \right. \\
\left. \eta \frac{r}{\eta} - \frac{r^2}{\eta^2} - \frac{2(\eta + \eta^2)}{\eta} \max_{(i_1,i_2) \in \mathcal{N}} \max \{ \|\pi_{i_1,i_2}^T \pi_{i_1,i_2}\|, \|\pi_{i_1,i_2}^T \pi_{i_2,i_1}\|, \|\pi_{i_2,i_1}^T \pi_{i_2,i_1}\| \} > 0 \right\},
\]

then under Assumption 1 for D-subdomains, the iterative sequence \( X^n := \{u^n_d\}, \{v_{i_1,i_2}^n\}_{(i_1,i_2) \in \mathcal{N}}, \{z^n_d\}, \{\Lambda_{i_1,i_2}^n\}_{(i_1,i_2) \in \mathcal{N}}, \{\Gamma_d^n\}_{d=1}^D \) generated by OD^2P_m converges to the stationary point of the saddle point problem of the augmented Lagrangian denoted in (2.32).

Similar to the 2-subdomain case, the set (3.6) of the parameters \( r, \eta \) is nonempty.

### 3.2. Blind Ptychography

To reduce the grid pathology [5] (ambiguity derived by the multiplication of any periodical function and the true solution) due to grid scan, we introduce the support set constraint of the probe, i.e. \( \mathcal{O} := \{w : (\mathcal{F}w)(j) = 0, \ j \in \mathcal{J}\} \), with the support set \( \mathcal{J} \) of the Fourier transform of the probe \( w \). We will explain why it would help to remove the grid pathology intuitively. Assume that one gets a solution multiplied by a non-constant vector \( e \), such that one gets \( \mathcal{F}(w \circ S_j(u \circ e)) = (\mathcal{F}S_j e) * \mathcal{F}w = \mathcal{F}(S_j u) \). Obviously, it demonstrates that \( \mathcal{F}S_j e * \mathcal{F}w \) will enlarge the support set, that may disobey the compact support constraint.

For simplicity, we consider the blind ptychography problem for two-subdomain DD:

\[
\min_{\{w,u_1,u_2,v,z_1,z_2\}} \sum_{d=1}^2 G_d(B_d(w,u_d); f_d) + \mathbb{I}_{\mathcal{O}}(w), \quad s.t. \pi_{d,3-d} u_d - v = 0, \ d = 1, 2,
\]
where the bilinear mapping $B_d(w, u_d)$ is denoted as $(B_d)_{jd}(\omega, u_d) := F(w \circ (S^d_j u_d)) \forall 0 \leq j_d \leq J_d - 1$ ($\sum_{d=1}^{2} J_d = J$), and the indicator function $I_\mathcal{O}$ is denoted as

$$I_\mathcal{O}(w) := \begin{cases} 0, & \text{if } w \in \mathcal{O}; \\ +\infty, & \text{otherwise}. \end{cases}$$

To enable parallel computing, we consider the following constraint optimization problems

$$\min_{\{w, w_1, w_2, u_1, u_2, v, z_1, z_2\}} \sum_{d=1}^{2} G_d(B_d(w_d, u_d); f_d) + I_\mathcal{O}(w)$$

s.t. $\pi_{1,2} u_1 - v = 0$, $\pi_{2,1} u_2 - v = 0$, $w_d = w$, $z_d = B_d(w_d, u_d), d = 1, 2$.

To get rid of heavy notations, we introduce the following definitions. By denoting

$$\pi = \begin{pmatrix} \pi_{1,2} & 0 \\ 0 & \pi_{2,1} \end{pmatrix}, \quad I_0 = \begin{pmatrix} I_0 \\ I_0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} I_1 \\ I_1 \end{pmatrix}, \quad B(W, U) = \begin{pmatrix} B_1(w_1, u_1) \\ B_2(w_2, u_2) \end{pmatrix}$$

with identity matrices $I_0, I_1$, we rewrite the constrained condition below:

$$\pi U - I_0 v = 0, \quad W - I_1 w = 0, \quad B(W, U) - Z = 0,$$

where $U = (u_1^T, u_2^T)^T$, $W = (w_1^T, w_2^T)^T$, and $Z = (z_1^T, z_2^T)^T$. Hence we consider the following model:

$$\min_{w, W, U, v, Z} G_d(Z; f) + I_\mathcal{O}(w)$$

s.t. $\pi U - I_0 v = 0, \quad W - I_1 w = 0, \quad B(W, U) - Z = 0,$

with $f := (f_1^T, f_2^T)^T$.

Similarly, the augmented Lagrangian can be formulated below:

$$\mathcal{L}_b(w, W, U, v, Z, \Lambda, \Gamma, \Delta) = G_d(Z; f) + I_\mathcal{O}(w)$$

$$+ r R((\pi U - I_0 v, \Lambda)) + \eta R((B(W, U) - Z, \Gamma)) + \mu R((W - I_1 w, \Delta))$$

$$+ \frac{\tau}{2} \|\pi U - I_0 v\|^2 + \frac{\tau}{2} \|B(W, U) - Z\|^2 + \frac{\tau}{2} \|W - I_1 w\|^2,$$

with multipliers $\Lambda, \Gamma, \Delta$ and positive parameters $r, \eta, \mu$. Then alternatively one needs to solve the following saddle point problem below

$$\max_{\Lambda, \Gamma, \Delta} \min_{w, W, U, v, Z} \mathcal{L}_b(w, W, U, v, Z, \Lambda, \Gamma, \Delta).$$

One can solve the above saddle point problem step by step using a variant ADMM with additional proximal term for the $U$–subproblem below:

$$w^{n+1} = \arg \min_{w, v, Z} \mathcal{L}_b(w, W^n, U^n, v, Z, \Lambda^n, \Gamma^n, \Delta^n)$$

$$\Gamma^{n+1} = \Gamma^n + B(W^n, U^n) - Z^{n+1};$$

$$W^{n+1} = \arg \min_{W} \mathcal{L}_b(w^{n+1}, W, U^n, v^{n+1}, Z^{n+1}, \Lambda^n, \Gamma^{n+1}, \Delta^n);$$

$$U^{n+1} = \arg \min_{U} \mathcal{L}_b(w^{n+1}, W^{n+1}, U, v^{n+1}, Z^{n+1}, \Lambda^n, \Gamma^{n+1}, \Delta^n) + \frac{\tau}{2} \|U - U^n\|^2;$$

$$\Lambda^{n+1} = \Lambda^n + \pi U^{n+1} - I_0 v^{n+1};$$

$$\Delta^{n+1} = \Delta^n + W^{n+1} - I_1 w^{n+1};$$

For the $w$–subproblem, $w^{n+1} = \arg \min_{w} I_\mathcal{O}(w) + \frac{\nu}{2} \|\Delta^n + W^n - I_1 w\|^2$. Readily one has

$$w^{n+1} = \frac{1}{2} \mathcal{F}^* M \mathcal{F} \mathcal{I}^T_2 (\Delta^n + W^n),$$
where the matrix $M$ is diagonal with diagonal elements defined as

$$M(j,j) = \begin{cases} 1, & \text{if } j \in \mathcal{J}; \\ 0, & \text{otherwise}. \end{cases}$$

For the $v$–subproblem,

$$v^{n+1} = \arg \min_v \frac{\mu}{2} \| \Lambda^n + \pi U^n - I_0 v \|^2 = \frac{1}{2} I_0^T (\Lambda^n + \pi U^n). \tag{3.16}$$

For the $z$–subproblem,

$$Z^{n+1} = \arg \min_Z \mathcal{G}_v(Z; f) + \frac{\eta}{2} \| \Gamma^n + B(W^n, U^n) - Z \|^2$$

$$= \text{Prox}_{\mathcal{G}_v; \eta}(\Gamma^n + B(W^n, U^n)), \tag{3.17}$$

where $\text{Prox}_{\mathcal{G}_v; \eta}(Z) = (\text{Prox}_{\mathcal{G}_v; \eta}(z_1), \text{Prox}_{\mathcal{G}_v; \eta}(z_2))^T \forall Z \equiv (z_1^T, z_2^T)^T$.

Denote two related operators by fixing one variable for the bilinear operator $B(\cdot, \cdot)$ as $D_W(U) := B(W, U)$, $D_U(W) := B(W, U) \forall W, U$. Then $D_W$ and $D_U$ are linear, and moreover, they are linear (also bounded) w.r.t. the subscripts $W, U$ respectively.

For $W$–subproblem, one has

$$W^{n+1} = \arg \min_W \frac{\eta}{2} \| B(W, U^n) - Z^{n+1} + \Gamma^{n+1} \|^2 + \frac{\mu}{2} \| W - I_1 v^{n+1} + \Delta^n \|^2$$

$$= (\eta D_{U^n} + \mu I_1)^{-1} (\eta D_{U^n} (Z^{n+1} - \Gamma^{n+1}) + \mu (I_1 v^{n+1} - \Delta^n)), \tag{3.18}$$

with $D_{U^n} = \text{diag} \left( (\sum_{j_1=0}^{J_1-1} s^1_{j_1}|u_{1}^{n+1}|^2)^T, (\sum_{j_2=0}^{J_2-1} s^2_{j_2}|u_{2}^{n+1}|^2)^T \right)^T$.

For $U$–subproblem, one has to solve the following problem

$$U^{n+1} = \arg \min_U \frac{\eta}{2} \| B(W^{n+1}, U) - Z^{n+1} + \Gamma^{n+1} \|^2 + \frac{\mu}{2} \| U - I_0 v^{n+1} + \Lambda^n \|^2 + \frac{\gamma}{2} \| U - U^n \|^2$$

$$= (\eta D_{W^{n+1}} + r \pi^T \pi + \gamma I)^{-1} (\eta D_{W^{n+1}} (Z^{n+1} - \Gamma^{n+1}) + r \pi^T (I_0 v^{n+1} - \Lambda^n) + \gamma U^n), \tag{3.19}$$

where $D_{W^{n+1}} = \text{diag} \left( (\sum_{j_1=0}^{J_1-1} (s^1_{j_1}|u_{1}^{n+1}|^2)^T, (\sum_{j_2=0}^{J_2-1} (s^2_{j_2}|u_{2}^{n+1}|^2)^T \right)^T$ and $I$ denotes the identity matrix, i.e. $IU = U$.

Finally the overlapping DD based blind ptychography (OD²BP) is given as below:

---

**Algorithm 3:** Overlapping DD based Blind Ptychography (OD²BP)

1. **Step 0.** Set $n := 0$.
2. **Step 1.** Update $w^{n+1}, v^{n+1}, Z^{n+1}$ in parallel by (3.15), (3.16), (3.17).
3. **Step 2.** Update the multipliers $\Gamma^{n+1}$ by (3.10).
4. **Step 3.** Update $W^{n+1}$ by (3.18).
5. **Step 4.** Update $U^{n+1}$ by (3.19).
6. **Step 5.** Update $\Delta^{n+1}$ and $\Lambda^{n+1}$ by (3.13) and (3.14).
7. **Step 6.** If satisfying the stopping condition, then stop and output $U^{n+1}$ as the final output; otherwise, set $n := n + 1$, and go to Step 1.

---

For the convergence analysis, the blind case becomes more sophisticated. Limited to the current technique, we have to make the assumption of the boundedness for the iterative sequence. Then with proper $(r, \eta, \mu, \gamma)$, similarly to the nonblind case, one can prove that any limit point of $\{\hat{X}^n\}$ converges to the stationary point of the saddle point problem (3.8).
4. Numerical Experiments. To get perfect reconstruction results, periodical boundary condition is usually enforced as [5], that produces almost same redundancy of each pixel of the sample. To make the simulation more realistic, instead we assume that the image to be recovered has boundary with pixel values being all ones, because the sample during the experiments are usually in the vacuum and the region without sample occupying has constant contrast. In the implementation of all algorithms, we enforce the region to be ones.

We consider the zone-plate probe with 64 × 64 pixels (See Fig. 3 (c)-(d)), and the complex-valued sample is generated by taking two real-valued images as its absolute (See Fig. 3 (a) ) and phase (See Fig. 3 (b) ) parts respectively. The raster grid is adopted for ptychography scan, i.e. all scanning points lie in isometric square grid.

For noisy free measurements, the proposed algorithms stop at the \( n \)th iteration, if the R-factors \( \text{RE}^n := \frac{\sum_d \| |d u_r^g| - \sqrt{J} f_d |\|}{\sum_d \| \sqrt{J} f_d \|} \leq 1.0 \times 10^{-5} \) or \( \frac{\sum_d \| |d u_g^g| - \sqrt{J} f_d |\|}{\sum_d \| \sqrt{J} f_d \|} \leq 1.0 \times 10^{-5} \) for non-blind and blind recovery, respectively, or the maximum iteration number reaches 1000. For the noisy measurements, the proposed algorithms stop at \( n \)th iteration, if the relative error \( \text{RE}^n := \max_d \frac{\| |u_r^d - u_r^g| - |d f_d |\|}{\| |d f_d |\|} \leq 1.0 \times 10^{-3} \) or the maximum iteration number reaches 200. In order to measure the quality of recovery from noisy measurements, the signal-to-noise ratio (SNR) in dB is used, which is denoted below: \( \text{SNR}(u_r, u_g) = -10 \log_{10} \frac{\| u_r - u_g \|^2}{\| u_r \|^2} \), where \( u_r \) and \( u_g \) corresponds to the recovery image and ground truth respectively.

We initialize all variables as shown in Algorithms 1-3. As mentioned in subsection 2.2, the parameter \( \epsilon \) used for the smooth truncated metric (2.7) sets to 0.5. The other parameters existed in the proposed algorithms are chosen heuristically in order to either get faster convergence for noiseless measurements, or get better recovery quality for noisy measurements, which will be specified in the following tests. Finally, the final solution over the entire domain is obtained by merging the solutions on the subdomains derived by proposed algorithms, i.e. keeping the pixel values of non-overlapping region unchanged, while averaging the overlapping regions.

4.1. Performances of OD²P. We first show the performance of the proposed OD²P algorithm for the nonblind case with two-subdomain decomposition. Set the parameters \( \eta = 0.1, \tau = 4.0 \times 10^5 \). The resolution of the sample is of 256 × 256 pixels. Set the scan stepsize to 8 pixels for both \( x \) and \( y \) directions. Therefore the entire measurement \( f \) consists of \( J = 25 \times 25 \) frames, and its partitions \( f_1 \) and \( f_2 \) consist of \( 7 \) and \( 25 \times 25 \) frames respectively. The recovery results are put in Fig. 4, where the recovery results (absolute parts in the first row of Fig. 4 and phase parts in the second row of Fig. 4 ) at 1st, 2nd, 5th, 50th, and the final iterations demonstrate how the proposed algorithm improve the results gradually. Especially, one may notice that the mismatch between the overlapping regions of two decomposed parts disappears as iteration goes, by inferring from the first three columns of Fig. 4. That again shows the efficiency of the proposed algorithm. The absolute values of residual between the recovery images and the truth are put in Fig. 5, where one can obviously observe the differences between two decomposed parts get weak as iteration goes. The errors changes including the relative errors and R-factors w.r.t. iteration numbers are put in Fig. 6, that implies the convergence of the proposed algorithm.

To further show the performance the proposed algorithm, the test on the noisy data is given as well. Noisy data is generated by contamination of Poisson noises with different levels (using a scaling factor to control the noise level), such that two different noisy cases with \( \text{SNR}_{\text{intensity}} = 39.8, 29.9\text{dBs} \) (SNR\(_{\text{intensity}}\) denotes the SNR between the noisy and clean measurements) are considered. The parameter \( \eta \) sets to
Fig. 4. Recovery results at 1st, 2nd, 5th, 50th, 168th (final iteration) iterations. Recovered absolute (top and down parts for decomposed samples) in 1-2 rows and phase in 3-4 rows respectively (top and down parts for decomposed samples), shown in the range of [0, 1] and [0, π] for the absolute and phase respectively.

Fig. 5. Residuals. (a)-(e): absolute values of the differences between the recovery results at 1st, 2nd, 5th, 50th, 168th (final iteration) iterations and the truth; (f)-(j): errors between the overlapping parts of two subdomains at different iterations.

Fig. 6. RF (R-factor), and RE (Relative error) changes v.s. iterations.
the same as the noiseless case, while $r$ sets to 90, and 150 for two different noise levels respectively. We put the recovery results in Fig. 7 (the SNRs of final recovered images are 18.1 dB and 12.2 dB). That demonstrates the proposed algorithm can work well for the Poisson noisy data, showing great potential for real experimental data analysis (usually contaminated by Poisson noise).

4.2. Extended Tests. To further test the performances of proposed DDMs, we will test more cases, including blind recovery with two subdomains and the nonblind recovery with multiple subdomains.

(1) Blind recovery

Initial guess of the probe is generated by $w^0 := \frac{1}{J} \sum_j F^* \sqrt{F_j}$ as $[5]$. In the part we will show the performance of OD$^2$BP for the blind case with two-subdomain decomposition. We put the recovery results including the absolute parts of recovered probe, the absolute and phase parts of recovered images to Fig. 8 at the 1st, 2nd, 5th, 50th, and the final iterations. Similarly to the nonblind case, the mismatch between the iterative solutions on the two subdomains gradually disappears and furthermore, the probe is well recovered meanwhile, demonstrating that the proposed algorithm also works well for the blind case.
Fig. 8. Blind recovery results at 1st, 2nd, 5th, 50th, 211th (final iteration) iterations. The first and second rows: Recovered absolute (top and down parts for decomposed samples) and phase respectively (top and down parts for decomposed samples), shown in the range of \([0, 1000]\), \([0, 1]\) and \([0, \pi]\) for the first, 2nd-3rd, and last two rows, respectively. The parameters set to \(\eta = 0.1, r = 5.0 \times 10^3, \mu = 2.0 \times 10^2\).

(2) Multiple subdomains
In the part we will show the performance of OD\(^2\)P\(_m\). For simplicity, the stripe-type multiple DD is implemented. To get a relative large problem, we consider the image with 512 \times 512 pixels (Interpolated of the image shown in Fig. 3), and the scan step size sets to 16 pixels. The parameters set to the same as the case of two-subdomain. We first put the recovery results in the case of 4 subdomains to Fig. 9, immediately showing the effectiveness of proposed algorithm in the case of multiple subdomains. More tests are conducted with \(D = 6, 8, 10\) subdomains DD, and the related convergence curves are put to Fig. 10. On one hand, one can readily infer that the proposed algorithm converges pretty well as numbers of subdomains vary. On the other hand, its convergence becomes slower as the number of subdomains increase. However, it seems not very sensitive to the number of subdomains, since the maximum iteration number to reach the given accuracy changes a little bit (around 10% more iterations for 10-subdomains than that for 2-subdomain) as reported in the following Table 1.

We also measure the virtual wall-clock times (The proposed OD\(^2\)P\(_m\) algorithm is assumed to be fully parallelized at the subdomain regardless of the communication time as [22]), and a single processor handles a single subdomain in order to show the potential of parallel computing, and report them in Table 1, that show high speedup ratios and the speedup efficiency (speedup ratios divided by the number of subdomains) is above 80%. Although we only report the virtual wall-clock time regardless of the communication cost, the proposed algorithm should be suitable for parallel computing, due to the low cost of information exchange happening on the narrow overlapping regions of adjacent subdomains. Hence, we will further evaluate the
Fig. 9. Recovery results with $D = 4$ subdomains at 1st, 2nd, 5th, 50th, 200th iterations: Recovered absolute (1st-4th rows) and phase (5th-8th rows) of the iterative solutions on the four subdomains, respectively. They are shown in the range of $[0,1]$ and $[0,\pi]$ for the absolute and phase parts respectively.

performance of proposed algorithms on large-scale computer clusters, and leave it as future work.

Table 1

Performances for the case of multiple subdomains. The speedup ratio is denoted by the ratio between the virtual wall-clock time and the runtime of ADMM algorithm [5] without DD. The 2nd-3rd columns present the RF (R-factor), and RE (Relative error) for the proposed algorithms and 4th column presents the iter. no. (short for iteration number) when satisfying the stopping conditions.

| $D$ | RF(1E-5) | RE(1E-5) | Iter no. | virtual wall-clock time in seconds (Speedup ratio) |
|-----|----------|----------|----------|-------------------------------------------------|
| 1   | 0.997    | 0.482    | 587      | 601(1.00)                                       |
| 2   | 0.995    | 0.458    | 611      | 346(1.74)                                       |
| 4   | 0.998    | 0.426    | 626      | 167(3.60)                                       |
| 6   | 0.993    | 0.434    | 637      | 109(5.51)                                       |
| 8   | 0.999    | 0.409    | 651      | 88.0(6.83)                                      |
| 10  | 0.999    | 0.416    | 670      | 72.0(8.35)                                      |

4.3. Impact by parameters. In order to evaluate the robustness of proposed OD$^2$P w.r.t. the parameters, we conduct the experiments with noiseless data, where one parameter varies and meanwhile other parameters are kept unchanged. Set $\eta_0 = 0.1$, $r_0 = 4.0 \times 10^3$. The scan stepsize sets to 8 pixels. We show the performance impacted by setting $\eta \in \{5^{-2} \times \eta_0, 5^{-1} \times \eta_0, \eta_0, 5 \times \eta_0, 5^2 \times \eta_0\}$ (keeping $r = r_0$), and put the
5. Conclusions. Overlapping DDMs have been successfully applied to ptychography reconstruction, that lead to OD$^2$P for the two-subdomain nonblind recovery, OD$^2$P$_m$ for multiple subdomains and OD$^2$BP for the blind recovery. With the newly-designed smooth truncated metric, these proposed algorithms are efficiently computed, since all subproblems have closed form solutions, and their convergences are well guaranteed under some mild conditions. Numerical experiments are further conducted to show the performance of proposed algorithms, demonstrating good convergence speed, robust to the noise and parameters. In the future, we will optimize the current algorithms on massively parallel processing computers, and explore more other applications including Fourier ptychography [37], partial coherence analysis [4], and high dimensional ptychographic imaging problems [36].

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