POLYGONAL NEGATIVE HYPERBOLIC
ROTOPULSATORS OF THE CURVED $n$-BODY PROBLEM.

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Abstract. For the $n$-body problem in spaces of negative constant Gaussian curvature, we prove for a class of negative hyperbolic rotopulsators that if that class exists, the configurations of the point masses of these rotopulsators have to be regular polygons if the rotopulsators are not relative equilibria. Additionally, we prove that if the rotopulsators are relative equilibria, there exists at most one such solution.

1. Introduction

By $n$-body problems we mean problems where we are to determine the dynamics of a number of $n$ point masses as dictated by a system of ordinary differential equations. The $n$-body problem in spaces of constant Gaussian curvature, or curved $n$-body problem for short, is described by a system of differential equations (see (2.1)) that generalises the classical, or Newtonian $n$-body problem to spaces of constant Gaussian curvature. Solutions to an $n$-body problem where the point masses describe a configuration that maintains the same shape and size over time are called relative equilibria. A rotopulsator, or rotopulsating orbit is a solution of the curved $n$-body problem for which the shape of the configuration of the point masses stays the same over time, but the size may change. An important reason to study the curved $n$-body problem, and relative equilibria and rotopulsators in particular, is to identify orbits that are unique to a particular space (see [10]). For example: Diacu, Pérez-Chavela and Santoprete (see [11], [12]) showed that rotopulsators (called homographic orbits in those papers) that have an equilateral triangle configuration and unequal masses, only exist in spaces of zero curvature. As the Sun, Jupiter and the Trojan asteroids form the vertices of an equilateral triangle, the region between these three objects likely has zero curvature. Rotopulsators were first introduced in [10], where it was proven that there are five different types of rotopulsators, two for the positive curvature case (spheres) and three for the negative curvature case (hyperboloids). For subclasses of these five different types it was proven in [16] for $n = 4$ that if the rotopulsators have rectangular configurations, they have to be squares. For two of these five subclasses it was proven in [28] for general $n$ that if the configuration forms a polygon and the rotopulsator is

1991 Mathematics Subject Classification. Primary 70F10.

Key words and phrases. $n$-body problems, curved $n$-body problem, celestial mechanics.
not a relative equilibrium, that polygon has to be a regular polygon. A logical avenue of research then is to investigate to what extent we can generalise the remaining three results from [16]. In this paper we will generalise the results for two of these three remaining classes, the so-called negative hyperbolic polygonal rotopulsators and negative elliptic hyperbolic polygonal rotopulsators (see [10], [16] and Definition 2.2). Specifically, we will prove the following result:

**Theorem 1.1.** Let \( q_1, \ldots, q_n \) be a negative hyperbolic, or negative elliptic hyperbolic polygonal rotopulsator, as in Definition 2.2. If \( \rho \) is not constant, then the \( q_i, i \in \{1, \ldots, n\} \) are the vertices of a regular polygon. If \( \rho \) is constant, then the \( q_i, i \in \{1, \ldots, n\} \) are the vertices of a unique polygonal relative equilibrium.

**Remark 1.2.** Because of the proof of Theorem 1.1 the rotopulsators in this paper are in fact the rotopulsators that were investigated in [28]. It was proven in [28] and in [4] (in the latter paper they were called ‘homographic orbits’ instead of ‘rotopulsators’) that these rotopulsators exist.

**Remark 1.3.** In [16], in Theorem 6 and Theorem 7, it was stated that for \( n = 4 \) rectangular negative hyperbolic rotopulsators and rectangular negative elliptic hyperbolic rotopulsators do not exist. These statements are not in conflict with Theorem 1.1 as the third and fourth coordinates of the point masses of the negative hyperbolic rotopulsators in [16] are constructed to be coordinates of distinct points on a hyperbola, while we do not impose that restriction in this paper.

The curved \( n \)-body problem goes back as far as the 1830s with Bolyai and Lobachevsky, who independently proposed a curved 2-body problem in hyperbolic space (see [1] and [24]). While of significant interest to great mathematicians such as Dirichlet, Schering (see [25], [26]), Killing (see [17], [18], [19]), Liebmann (see [21], [22], [23]), Kozlov, Harin (see [20]), Cariñena, Rañada and Santander (see [2]), a working model for the \( n \geq 2 \) case was not found until 2008 by Diacu, Pérez-Chavela and Santoprete (see [11], [12] and [13]). This breakthrough then gave rise to further results for the \( n \geq 2 \) case in [3], [7], [9], [10], [15], [16], [27], [30] and [32].

The remainder of this paper is constructed as follows: In section 2 we will discuss needed background theory, after which we will prove Theorem 1.1 in section 3.

## 2. Background theory

**Definition 2.1.** Let \( \sigma = \pm 1 \). The \( n \)-body problem in spaces of constant Gaussian curvature is the problem of finding the dynamics of point masses

\[
q_1, \ldots, q_n \in M_\sigma^3 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + \sigma x_4^2 = \sigma \},
\]
with respective masses $m_1 > 0, ..., m_n > 0$, determined by the system of differential equations

$$\ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j(q_j - \sigma(q_i \circ q_j)q_i)}{(\sigma - \sigma(q_i \circ q_j)^2)^{\frac{3}{2}}} - \sigma(\dot{q}_i \circ \dot{q}_i)q_i, \quad i \in \{1, ..., n\}, \quad (2.1)$$

where for $x, y \in \mathbb{M}_3^2$ the product $\cdot \circ \cdot$ is defined as

$$x \circ y = x_1y_1 + x_2y_2 + x_3y_3 + \sigma x_4y_4.$$ 

Next, we will define negative hyperbolic polygonal rotopulsators and negative elliptic-hyperbolic polygonal rotopulsators: Let

$$T(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \quad \text{and} \quad S(x) = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$$

be $2 \times 2$ matrices. Then

**Definition 2.2.** If $\sigma = -1$ and there exist scalar, twice differentiable functions $x_i, y_i, i \in \{1, ..., n\}, \phi, \rho \geq 0$, for which $x_i^2 + y_i^2 - \rho^2 = -1$ and there exist constants $\beta_1, ..., \beta_n \in \mathbb{R}$, such that for a solution $q_1, ..., q_n$ of (2.1) we have that

$$q_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix}, \quad i \in \{1, ..., n\},$$

and the $q_1, ..., q_n$ lie on a polygon, then we call $q_1, ..., q_n$ a **negative hyperbolic polygonal rotopulsator**. If there exist twice differentiable functions $\theta, r \geq 0$, $r^2 - \rho^2 = -1$ and constants $\alpha_1, ..., \alpha_n \in [0, 2\pi]$, such that

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = T(\theta(t)) \begin{pmatrix} \sin(\alpha_i) \\ \cos(\alpha_i) \end{pmatrix}$$

then we call $q_1, ..., q_n$ a **negative elliptic-hyperbolic polygonal rotopulsator**.

We will need the following lemma, which was proven in a more general setting in [10], but as the proof for our particular case is not particularly long, we give a proof here as well:

**Lemma 2.3.** If $q_1, ..., q_n$ is a negative hyperbolic polygonal rotopulsator, or a negative elliptic-hyperbolic polygonal rotopulsator with functions $\rho$ and $\phi$ as in Definition 2.2, then $2\rho'\phi' + \rho\phi'' = 0$.

**Proof.** Using the wedge product, it was proven in [5] that

$$\sum_{i=1}^{n} m_i q_i \wedge \dot{q}_i = 0,$$
where 0 is the zero bivector. If \(e_1, e_2, e_3\) and \(e_4\) are the standard basis vectors in \(\mathbb{R}^4\), then

\[
0e_3 \wedge e_4 = \sum_{i=1}^{n} m_i(q_{i3}\tilde{q}_{i4} - q_{i4}\tilde{q}_{i3})e_3 \wedge e_4.
\]

As \(q_{i3} = \rho \cosh (\beta_i + \phi)\) and \(q_{i4} = \rho \sinh (\beta_i + \phi)\) by Definition 2.2, we have that

\[
(2.3) \quad q_{i3}\tilde{q}_{i4} - q_{i4}\tilde{q}_{i3} = \text{det} \begin{pmatrix} q_{i3} & \tilde{q}_{i3} \\ q_{i4} & \tilde{q}_{i4} \end{pmatrix} = \text{det} \left( \rho \begin{pmatrix} \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \\ \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \end{pmatrix} \right)''.
\]

Because

\[
\left( \rho \begin{pmatrix} \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \\ \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \end{pmatrix} \right)'' = \rho'' \begin{pmatrix} \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \\ \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \end{pmatrix} + 2\rho' \phi' \begin{pmatrix} \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \\ \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \end{pmatrix} + \rho \phi'' \begin{pmatrix} \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \\ \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \end{pmatrix},
\]

using that the determinant of a matrix with two identical rows is zero, we can rewrite (2.4) as

\[
q_{i3}\tilde{q}_{i4} - q_{i4}\tilde{q}_{i3} = 0 + \rho \left(2\rho' \phi' + \phi''\right) \text{det} \begin{pmatrix} \cosh (\beta_i + \phi) & \sinh (\beta_i + \phi) \\ \sinh (\beta_i + \phi) & \cosh (\beta_i + \phi) \end{pmatrix} = \rho \left(2\rho' \phi' + \phi''\right) \cdot (-1).
\]

So combined with (2.3), we get

\[
0e_3 \wedge e_4 = -\rho \sum_{i=1}^{n} m_i(2\rho' \phi' + \rho \phi'')e_3 \wedge e_4 = -\rho(2\rho' \phi' + \rho \phi'') \sum_{i=1}^{n} m_i e_3 \wedge e_4,
\]

giving that indeed \(2\rho' \phi' + \rho \phi'' = 0\). \(\square\)

3. Proof of Theorem 1.1

Let \(q_1, \ldots, q_n\) be a negative hyperbolic polygonal rotopulsator as in Definition 2.2. Let \(I\) be the \(2 \times 2\) identity matrix. Then inserting (2.2) into (2.1) and multiplying both sides of the resulting system of equations for the third and fourth coordinates of \(q_i\) from the left with \(S(\phi + \beta_i)^{-1}\) gives, as now \(\sigma = -1\),

\[
\left( \rho''I + (2\rho' \phi' + \rho \phi'') \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \rho(\phi')^2I \right) = \rho \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
= \sum_{j=1,j \neq i}^{n} m_j \rho \left( \frac{\sinh(\beta_j - \beta_i)}{\cosh(\beta_j - \beta_i)} + (q_i \odot q_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)
\]

\[
+ ((x_i)^2 + (y_i)^2 - (\rho')^2 + \rho^2(\phi')^2) \rho \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \(\odot\) denotes the standard basis and \(\mathbb{R}^4\).
which can be rewritten as
\[
\begin{pmatrix}
\rho'' + (2\rho'\phi' + \rho\phi'') \\
(0 & 1)
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
- ((x_i')^2 + (y_i')^2 - ((\rho')^2 + \rho^2(\phi')^2)I)
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\]
\[
= \sum_{j=1, j\neq i} m_j \rho \left( \frac{\sinh(\beta_j - \beta_i)}{\cosh(\beta_j - \beta_i)} \right) \left( (q_i \odot q_j)^2 - 1 \right)^{\frac{3}{2}}.
\]

Collecting terms for the first coordinate on both sides of (3.1) gives
\[
2\rho'\phi' + \rho\phi'' = \sum_{j=1, j\neq i} \frac{m_j \rho \sinh(\beta_j - \beta_i)}{((q_i \odot q_j)^2 - 1)^{\frac{3}{2}}},
\]

By Lemma 2.3, \(2\rho'\phi' + \rho\phi'' = 0\), so
\[
0 = \sum_{j=1, j\neq i} \frac{m_j \rho \sinh(\beta_j - \beta_i)}{((q_i \odot q_j)^2 - 1)^{\frac{3}{2}}}.
\]

Now let \(\beta_1 = \min\{\beta_j | j \in \{1, \ldots, n\}\}\). Then \(\sinh(\beta_j - \beta_1) \geq 0\) and \(\sinh(\beta_j - \beta_1) = 0\) if and only if \(\beta_j = \beta_1\), so as \((\sigma - \sigma(q_i \odot q_j)^2)^{\frac{3}{2}} > 0\), for (3.2) to hold, all \(\beta_j\) have to be equal to \(\beta_1\). This means that \(q_{i3}\) and \(q_{i4}\) are independent of \(i\). Therefore, as the \(q_i\) are vertices of a polygon, we have that \(q_{i1}\) and \(q_{i2}\) are coordinates of vertices of a polygon that lie on a circle of radius \(r = \sqrt{\rho^2 - 1}\), which means that by Theorem 1.1 of [28] that if \(\rho\) and therefore \(r\) is not constant, the \(q_i\) are the vertices of a regular polygon. If \(\rho\) is constant, then \(q_1, \ldots, q_n\) is a polygonal relative equilibrium solution and by Theorem 1.2 of [31] there exists at most one such solution. This completes the proof.

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