Degenerate ground state and quantum tunneling in rotating condensates

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Quantum tunneling introduces a fundamental difference between classical and quantum mechanics. Whenever the classical ground state is non-unique (degenerate), quantum mechanics restore uniqueness thanks to tunneling. A condensate in a rotating trap with a vortex can have such a degenerate classical ground state, a degeneracy that is excluded in the absence of rotation at least when the Gross-Pitaevskii equation applies. If the rotating trap has a center of symmetry, like a figure eight (a peanut), the vortex may be on either side with the same energy yielding a degenerate ground state, a degeneracy lifted by quantum tunneling. We explain how to compute the rate of tunneling in the WKB limit by estimating the action of the trajectory in the Euclidean version of the dynamics.

INTRODUCTION

The search for physical "mesoscopic" situations where quantum tunneling could be significant is motivated by the search of quantum systems where off-diagonal correlations typical of quantum mechanics could be created and yield interferences without classical counterpart. We introduce below a novel physical system where such a tunneling exists. As well-known Bose-Einstein condensates (BEC) rotating fast enough have vortices in their ground state. We find that such vortices may have two different classical equilibrium positions mapped into each other by geometrical symmetry. Later we introduce the basic equations for describing quantum tunneling between the two degenerate states. This system is interesting because the Euclidean action allowing to estimate the tunneling frequency is complex, not purely imaginary contrary to cases discussed in textbooks, and the single estimate the tunneling frequency is complex, not purely imaginary.

The wave-function common to all atoms in a BEC trapped in a potential $V(r)$ is a solution of the Gross-Pitaevskii (GP) equation. This is a function in the ordinary sense because the quantum fluctuations can be neglected in the dilute limit, as shown long ago by Bogoliubov. Including in the GP-P equation the possibility of constant rotation of vector $\Omega$ one has:

$$
\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi - i\hbar \times \Omega \cdot \nabla \Psi + g|\Psi|^2 \Psi + \tilde{V}(r) \Psi.
$$

The trapping potential $\tilde{V}(r)$ grows sufficiently fast at infinity, $g$ is the coupling constant assumed positive and proportional to the s-wave scattering length, and $m$ is the mass of each particle. A numerical version of equation (1) is found by taking as length scale $\frac{\hbar}{\sqrt{\epsilon}}$, with $n$ being the mean density of particles over the surface $\sigma$ of the 2D condensate, $n = \frac{1}{\sigma} \int |\Psi(r)|^2 dr$, and $\epsilon$ a dimensionless parameter, arbitrary for the moment. The unit for $\Omega$ is $\frac{\hbar \Omega}{m}$, and the time unit is $\frac{\hbar}{\sqrt{\epsilon}}$. With this system of units, equation (1) becomes

$$
\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \Delta \Psi + i M \Psi + e^{-2} (|\Psi|^2 \Psi + \tilde{V}(r) \Psi),
$$

where $M$ is the real antisymmetric linear operator $M = \omega \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$, with $(x, y)$ being rectangular coordinates normal to the rotation axis and $\omega = |\Omega| \frac{\hbar}{\sqrt{2m}}$. Furthermore the potential $V(r)$ has been made dimensionless. The ground state is a solution of (1) in the form of $e^{-i\mu t} \Psi(r)$ with the smallest $\mu$, for the given $\int |\Psi|^2$. Without rotation ($\Omega = 0$), the ground state is non-degenerate [1], but no such general result exists with rotation, and we give below examples of the opposite.

Consider a potential $V(r)$ such that the condensate is constrained to stay inside a closed curve shaped like a peanut. The idea behind this choice is that the vortex created by rotation may have an equilibrium position on either side of the peanut and so yield a degenerate ground state. We get such a curve from the Cassini oval, the locus of a point at which the product, $b^2$, of the distances to two foci of coordinates $r = (\pm a, 0)$, is constant. The oval is made of two pieces for $b < a$ and of a single one for $b > a$ [2]. Physically the length $a$ (for instance) is such that the dimensionless parameter $\epsilon$ is small.

The Cartesian equation of the oval is $V(r) = 0$ with

$$
V(r) = b^4 + 4a^2x^2 - (x^2 + y^2 + a^2)^2.
$$

Besides $\epsilon$, two dimensionless parameters characterize the system, one geometrical, the ratio $\frac{b}{a}$ (assumed to be bigger than 1), and the other measuring the strength of the perturbation due to rotation. Physically, this strength is found by comparing the circulation around a vortex and the one due to a solid body rotation at angular velocity $\omega$. This yields the dimensionless ratio $\omega = \frac{m \Omega}{\hbar} a^2$, and defines $\epsilon = \frac{b}{a \sqrt{\epsilon}}$. One expects a transition for a finite value of $\omega$ from a vortex-free ground-state to a ground-state with vortex, a transition depending on the value of $\frac{b}{a}$. To find the ground state, we may simply consider the minimizers of the following non-dimensional GP-functional, or energy:

$$
\int_D \left\{ |\nabla \Psi|^2 - 2Re(iM \Psi) + \frac{1}{2\epsilon^2} |\Psi|^4 + \frac{V(r)}{\epsilon^2} |\Psi|^2 \right\} dr,
$$

where $D$ is the domain defined by $V(r) > 0$ and $\Re(\cdot)$ denotes the real part of its argument, $\Psi$ being the complex conjugate.
of $\Psi$. This minimum is constrained by the normalization condition

$$ \int_D |\Psi|^2 d\mathbf{r} = \int_D V^+(\mathbf{r}) d\mathbf{r}, $$

where $V^+(\mathbf{r})$ denotes the positive part of the Cassini oval potential $V(\mathbf{r})$. Different solution branches are found for a range of the parameter values of $a$, $b$, $\epsilon$ and the rotation speed $\omega$ using numerical methods similar to that described in [3]. We hereby present some examples.

For $a = 0.25$, $b = 0.275$, and $\epsilon = 0.001$, Fig.1 displays the energy computed using (3) for the various solution branches (i.e. the solutions without vortex, with one symmetric vortex, with one off-center vortex, with two symmetric vortices, and with three vortices) as a function of the rotation speed. The relative energy differences $(1 - E_n/E_0)$ are highlighted for $\omega$ near 360, where $E_n$ is the energy of a solution with at least one vortex, and $E_0$ the energy of the vortex free solution corresponding to the same rotation speed. Although the differences are very small, it is clear that in this range of $\omega$, the solution with a non-symmetric (off-center) vortex has the lowest energy and makes the ground state. Therefore our initial guess is correct: in this range of parameters values, the ground state with an off-center vortex is degenerate. The different solutions are shown in Fig.2.

![FIG. 1: Energy values (left) and relative energy differences (right), as functions of the rotation speed, for different solutions with the Cassini oval potential. Black-square: no vortex. Red-circle: non-symmetric vortex. Blue-triangle: two vortices. Purple-diamond: three vortices. Green-star: one symmetric vortex.](image1)

Similar energy diagrams can be found for different values of the parameters. With a smaller $b = 0.27$ but the same values of $a$ and $\epsilon$, only three different solutions branches were found near $\omega = 415$. We expect that a degenerate ground-state with a non-symmetric vortex exists in other geometries than the Cassini oval. We have replaced the Cassini oval potential $V(\mathbf{r})$ by a piecewise constant potential of value $+1$ inside two squares connected through a rectangular channel, and $(-10)$ elsewhere. When the channel width is that of the squares, $D$ is a rectangle, and the ground state solutions keep the domain symmetry. A non-symmetric vortex branch appears for a smaller channel width. We found that for a narrower channel the energy differences of the different solution branches are larger than for the Cassini oval when the ground state is degenerate (see Figs.3 and 4). Even larger energy differences between the non-symmetric ground state and the other solutions are found by decreasing the potential in the channel connecting the two sides.

![FIG. 2: Surface plots of $|\Psi|$ corresponding to the ground state without rotation (first row left, top view) and the five solutions found at the rotation speed $\omega = 360$ (upside-down view, they are respectively solutions without any vortices in the interior oval, with a single non-symmetric vortex in one half of the oval, with a vortex in the center of the oval, with two symmetric vortices, and with three vortices). At this rotation speed, the ground state is the one with a non-symmetric vortex which is a degenerate state together with its mirror image with respect to the y-axis.](image2)

![FIG. 3: Energy values (left) and relative energy differences (right), as functions of the rotation speed, for different solutions with a piecewise constant potential. Black-square: no vortex. Red-circle: non-symmetric vortex. Blue-triangle: two vortices. Purple-diamond: one symmetric vortex.](image3)

![FIG. 4: Plot of $-|\Psi(x, y)|$ for a non-symmetric degenerate ground state with the piecewise constant potential at the rotation speed $\omega = 175$.](image4)

Based on our simulations, the following observations are in order: by increasing rotation speed with a standard symmetric harmonic potential, the ground state is given first [1] by the
vortex free solution, then the single centrally symmetric vortex solution, then the solution with two and/or more vortices. It can be argued that, if in the range of the rotation speed where the ground state has only one vortex, there are substantial energy differences between this ground state and other solutions (such as the vortex-free and the multi-vortex solutions), by properly perturbing such potentials in a non-symmetric fashion, non-symmetric ground state can be created and some can be degenerate. For instance, by lowering the potential away from the center (but not too far), symmetrically with respect to the $y$-axis, a couple of local pinning sites can be formed which would attract the central vortex to the vicinity of one of these sites, and lead to a solution whose energy is lower than the central vortex state. Yet if the perturbation is small, then the changes to the energies of the vortex free or multi-vortex solutions are also small, hence leaving the non-symmetric vortex as the new ground state. Such a solution can have the vortex situated at either one of the two pinning sites, thus becomes degenerate. This scenario has been confirmed in our numerical experiments as well. Other configurations of pinning sites, with three or four-fold symmetry for instance, could produce solutions with even higher degeneracy. The keys to realize this experimentally, aside for being able to produce the right geometric pattern, are to increase enough the pinning strength to differentiate the energy between the central vortex state and the pinned vortex state and to, at the same time, maintain substantial energy differences between the single vortex solution and the other solution branches. Our simulations provide strong evidence to such possibilities. It did not escape our attention that a superconducting thin film in a perpendicular magnetic field yields a closely related situation, mathematically speaking at least [4]. Therefore we expect that in certain range of parameters too the same kind of peanut like geometry yields there a degenerate ground state with a non-symmetric vortex.

Whenever the classical ground-state is degenerate, one expects quantum tunneling to restore the symmetry and make one single quantum ground-state out of the set of different classical ground-states derived from each other by symmetry. In the semi-classical approximation, the tunneling time $T_Q$ is proportional to $\exp(S_E/\hbar)$, where $S_E$ is the Euclidean classical action, supposed to be much larger than $\hbar$. If the system is very large, $T_Q$ may be so large that quantum tunneling is never observed. Nevertheless one may imagine intermediate cases, outside of the realm of atomic or nuclear phenomena, where the Euclidean action is not that large compared to $\hbar$, leading to a non-negligible tunneling probability. The tunneling of an optical soliton between two coupled fibers [8] yields such an example.

Let us discuss the relevance of the various possible tunneling effects [6]. We mean now by tunneling an effect allowing a physical system in a double well potential to go from one well to the other either by purely quantum effects or by a mixture of thermal and quantum fluctuations or by thermal fluctuations only. This defines an effective tunneling time. The first typical time is the quantum tunneling time, $T_Q$, then the coherence time $T_c$. The latter time is the maximum time during which the quantum state inside one of the wells remains coherent, supposing that the condensate is at finite temperature, since at zero temperature this coherence time is infinite. In the case of BEC, $T_c$ is the mean-free flight time of a particle in the ground state colliding with a thermal particle divided by the number of particles in the ground state, the collision process being Poissonian. If there are many particles in the ground state, this coherence time may become very short because of this division. The third typical time is the Arrhenius-time $T_A$ for the passage over the energy barrier separating the two classical ground states under the effect of the thermal fluctuations. If $T_A$ is very large, two possibilities are met. If $T_Q \ll T_c$, the effective tunneling time is $T_Q$. If $T_c \ll T_Q$, the effective tunneling time is of order $T^2_c$. If the Arrhenius time is less than $T_Q$, the transition will always be by activated barrier crossing. Otherwise, the shortest of the two times, $T^2_Q/T_c$ or $T_A$, will be the effective transition time.

In the present problem the tunneling time can be calculated by quantizing the dynamical GP equation [7]. If one does not attempt to compute the prefactor of the exponential $\exp(S_E/\hbar)$, the derivation of $T_Q$, that amounts to calculate the Euclidean trajectory, is usually fairly standard. However, because the present problem has a peculiarity compared to the ones usually treated, we shall explain the general idea in some details. The equation (2) is the Euler-Lagrange condition making stationary the action

$$S = \int \text{d}r \text{d}t \mathcal{L},$$

associated to the Lagrangian $\mathcal{L}$, which is the following functional of $\Psi$ and $\bar{\Psi}$,

$$\mathcal{L} = \frac{i}{2} \left( \Psi \frac{\partial \bar{\Psi}}{\partial t} - \bar{\Psi} \frac{\partial \Psi}{\partial t} \right) + \frac{1}{2} |\nabla \Psi|^2 + \frac{1}{2\epsilon^2} |\Psi|^4 + \frac{1}{\epsilon^2} (V(r) - \mu) |\Psi|^2 + \frac{i}{2} \bar{\Psi} \mathcal{M} \Psi - \Psi \mathcal{M} \bar{\Psi}. \quad (4)$$

We include the ground state energy $\mu$ in (4) to make it clearer the possibility of real or complex solutions. To perform the analytic continuation of the dynamical equations, one has to write first the relevant quantities with real functions of $r$ and $t$. The Lagrangian (4) becomes:

$$\mathcal{L} = (u \frac{\partial v}{\partial t} - v \frac{\partial u}{\partial t}) + \mathcal{L}_\infty,$$

where

$$\mathcal{L}_\infty = \frac{1}{2} \left( (\nabla u)^2 + (\nabla v)^2 \right) + [v\mathcal{M}(u) - u\mathcal{M}(v)] + \frac{1}{\epsilon^2} (V(r) - \mu)(u^2 + v^2) + \frac{1}{2\epsilon^2} (u^2 + v^2)^2. \quad (5)$$

In Feynman’s formulation of quantum mechanics [5], any observable is given by a formal integral over all paths with
The tunneling corresponds to situations where the end-points of the trajectory \(X(\cdot)\) are not linked by a classical trajectory, yielding the dominant saddle-point contribution to Feynman’s integral at \(\hbar\) small. In this limit, and if there is no such classical trajectory, one extends the range of possible values of \(t\) to the complex plane. The Euclidean action associated to tunneling is found by taking \(t\) purely imaginary in the equations of motion, \(t = i\tau\), \(\tau\) real, leading to the expression

\[
\mathcal{L}_E = i \left( \frac{\partial u}{\partial \tau} - \frac{\partial v}{\partial \tau} \right) + \mathcal{L}_\infty. \tag{6}
\]

The Euclidean equations of motion for \(u\) and \(v\) are derived by variation of \(\int dt \int d\mathbf{r} \mathcal{L}_E\), leading to the two coupled equations:

\[
i \frac{\partial u}{\partial \tau} = -\frac{1}{2} \nabla^2 v + \mathcal{M}(u) + \frac{v}{\epsilon^2}(u^2 + v^2 + V - \mu), \tag{7}
\]

and

\[
i \frac{\partial v}{\partial \tau} = \frac{1}{2} \nabla^2 u - \mathcal{M}(v) - \frac{u}{\epsilon^2}(u^2 + v^2 + V - \mu). \tag{8}
\]

If the contribution proportional to \(\mathcal{M}\) is absent (that is without rotation), the Lagrangian in equation (5), and the dynamical equations (7)-(8) become real by changing \(v\) into \(iv\), yielding a formally real Euclidean dynamics. However this cannot be done with a non-zero \(\mathcal{M}\) because the transformed Lagrangian is complex and cannot be made real by any simple transformation. Although this is rarely considered, a fully complex Euclidean action does not hurt any general principle.

In the present problem, the tunneling probability is proportional to \(\exp(\Re(S_E)/\hbar)\), and the end states, at times \(\tau = \pm \infty\), are the classical ground states, with real \(u_\infty\) and \(v_\infty\), because either in real or in Euclidean dynamics, the equilibrium state is the state which minimizes \(\int d\mathbf{r} \mathcal{L}_\infty\) (i.e. the part of the action not proportional to time derivatives). Note that by substituting \(t\) for \(-i\tau\) in the pair of equations (7) and (8), one recovers the classical GP equation, where \(u\) and \(v\) are purely real. In the case considered here, where the asymmetrical one-vortex state is the ground state, the GP equation leads to stable equilibrium states, \((u_\infty, v_\infty)\), that forbids any tunneling. While the tunneling occurs when using the Euclidean equations (7)-(8) with real \(\tau\), where \((u_\infty, v_\infty)\) are unstable equilibrium states. For arbitrary \(\tau\), the functions \(u\) and \(v\) are both complex, without any simple relation in between, like being a pair of complex conjugates. Therefore the present Euclidean problem involves two complex fields, each having their own set of different vortices, a rather unusual fact. The numerical integration of the Eulerian system requires some caution, and will be treated in a future publication. Actually, while the mathematical problem is well posed, as a Dirichlet problem with boundary conditions at infinite time, the numerical search of solutions should avoid to solve an initial value Cauchy problem that is ill-defined because of the instability of this Euclidean dynamics at very short wavelength [9]. Furthermore, a change of variable \(\tau\) into, for example \(\tanh(\tau)\), would lead to a finite integration time. Then the trajectory joining the two equilibrium states, could be found by starting slightly off one equilibrium state and by finding the pair \(u, v\) that minimizes \(\int d\mathbf{r} d\tau |\mathcal{L}(u, v)|^2\). This procedure should avoid the instability effects for large wave numbers. Along this trajectory, the transient vortices formed on each field, will merge at the end, into a single one.

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[9] The short-wave (large \(k\)) approximation yields solution of the coupled set (7)-(8), proportional to \(e^{ik\cdot r + ik^2r}\) (when considering the first term in the r.h.s.). Because of the very fast growth in “time” \(\tau\) of the amplitude of the exponential, the set (7)-(8) makes an ill-posed initial value problem. However the unlimited growth in time of the Fourier mode becomes irrelevant when one considers, as here, this set as defining an elliptic problem in time, with boundary condition at plus and minus infinity.