Inverse source problem for a time-fractional heat equation with generalized impedance boundary condition

Muhammed ÇiÇEK¹ and Mansur I. ISMAILOV²
¹Department of Mathematics, Bursa Technical University, Bursa, Turkey
²Department of Mathematics, Gebze Technical University, 41400 Gebze, Kocaeli, TURKEY
E-mail: muhammed.cicek@btu.edu.tr and mismailov@gtu.edu.tr

Abstract. The paper considers an inverse source problem for a one-dimensional time-fractional heat equation with the generalized impedance boundary condition. The inverse problem is the time dependent source parameter identification together with the temperature distribution from the energy measurement. The well-posedness of the inverse problem is shown by applying the Fourier expansion in terms of eigenfunctions of a spectral problem which has the spectral parameter also in the boundary condition and by using the results on Volterra type integral equation with the kernel may have a diagonal singularity.

Keywords. Inverse source problem, Fractional diffusion equation, Generalized impedance boundary conditions, Generalized Fourier method, Weakly singular Volterra integral equation

1 Introduction

The mathematical analysis of inverse problem for the fractional diffusion equation is extensively investigated in the last decade. The first theoretical results for the inverse problem of finding the coefficients in fractional diffusion equation are obtained in [1, 3, 28, 4, 5, 8]. The mathematical literature for such kind of inverse problems are rapidly growing but without being exhaustive we refer only to [2, 6, 7, 9, 27, 19, 24, 25]. The inverse problems for time fractional diffusion equations can be obtained from the classical inverse diffusion problems by replacing the time derivative with a fractional derivative. For such a non-classical derivative, some standard methods for treating the inverse problems cannot be applied. Such a difference implies that the inverse problems for fractional diffusion equations should be more difficult. The difficulty comes from the definition of the fractional-order derivatives, which is essentially an integral with the kernel of weak singularity.

Because there is a wide mathematical literature for inverse problems for time fractional heat equation we intend to create comprehensive lists of the references which study inverse problems of finding source term and the references applying some methods of spectral analysis. In the paper [1] the uniqueness theorem is proved by using expansion in terms of eigenfunctions of suitable Sturm-Liouville problem along the Gelfand-Levitan theory. Similar eigenfunction expansion result along the analytic continuation and Laplace transform is used in determination of space-dependent source term in a fractional diffusion equation in [3]. Spectral analysis of suitable Sturm- Liouville operator is actively used in other coefficient identification problems for time-fractional diffusion problems, [4, 6, 7]. The papers [8, 25] and [9, 19] study inverse problems of finding space dependent and time-dependent source terms, respectively, in time-fractional diffusion equation by using eigenfunction expansion of the non-self adjoint spectral problem along the generalized Fourier method.

We refer to [7, 28, 1, 8, 7, 9, 19, 24, 25] for the inverse problem of finding time or space dependent source term associated with the time fractional heat equation but this list is far from the complete. As for numerical methods for such kind of inverse problems, see also [10, 11, 12, 13, 14, 26], and here we do not intend to create any comprehensive lists of the references.

The rest of the paper is organized as follows: The mathematical formulation of the problem is given in Section 2. In this section we also recall some preliminary definitions and facts on fractional calculus, on linear fractional
differential equations and necessary properties of eigenvalues and eigenfunctions of the auxiliary spectral problem. In Section 3, the existence and uniqueness of the solution of the inverse time-dependent source problem is proved. Finally, the continuous dependence upon the data of the solution of the inverse problem is also shown in this section.

2 Mathematical Preliminaries and Problem Formulation

2.1 Notes on Fractional Calculus

In this part we recall some basic definitions and facts on fractional calculus and present a necessary Lemma for further use. For details see \([15, 20, 23]\).

Consider the following initial value problem, existence and uniqueness result for such problem is given in \([15]\), for a linear fractional differential equation with order \(0 < q < 1\),

\[
\begin{cases}
D^q_{0+}[u(t) - u(0)] + \lambda u(t) = h(t), & t > 0, \\
u(0) = u_0,
\end{cases}
\]

(2.1)

where \(D^q_{0+}\) refers to the the Riemann-Liouville fractional derivative of order \(q\) \((0 < q < 1)\) in the time variable defined by

\[
D^q_{0+}u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(\tau)}{(t-\tau)^q} d\tau.
\]

The choice of the term \(D^q_{0+}(u(x,\cdot) - u(x,0))(t)\) instead of the usual term \(D^q_{0+}(u(x,\cdot))(t)\) is not only to avoid the singularity at zero, but also impose a meaningful initial condition (without fractional integral) \([15]\). By using the Laplace transform, the solution of IVP (2.1) is given in \([19]\) as

\[
u(t) = u_0 E_{q,1}(-\lambda t^q) + \int_0^t (t-\tau)^{q-1} E_{q,q}(-\lambda(t-\tau)^q)h(\tau)d\tau
\]

where \(E_{q,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+\beta)}\), \(q > 0\), \(\beta > 0\) and \(E_{q,1}(z) = E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}\), \(q > 0\) are two parameter and one parameter Mittag-Leffler function, respectively.

Let us introduce the functions \(e_q(t,\lambda) := E_q(-\lambda t^q)\) and \(e_{q,q}(t,\lambda) := t^{q-1} E_{q,q}(-\lambda t^q)\) where \(\lambda \in \mathbb{R}_+\). Then following statements for the Mittag-Leffler type functions \(e_q(t,\lambda)\) and \(e_{q,q}(t,\lambda)\) holds.

**Proposition 1** \([20, 21]\)

i) For \(0 < q < 1\), \(\lambda \in \mathbb{R}_+\) the function \(e_q(t,\lambda)\) is a monotonically decreasing function.

ii) The function \(e_q(t,\lambda)\) has the estimates \(e_q(t,\lambda) \simeq e^{-\frac{\lambda t^q}{\Gamma(q+1)}}\) for \(t \ll 1\) and \(e_q(t,\lambda) \simeq \frac{1}{(1-q)\lambda t^q}\) for \(t \gg 1\).

iii) \(D^q_{0+}(e_q(t,\lambda)) = -\lambda e_{q,q}(t,\lambda)\)

\(D^q_{0+}(e_q(t,\lambda) - e_q(0,\lambda)) = -\lambda e_q(t,\lambda)\)

\(I^\gamma_{0+}(e_q(t,\lambda)) = e_q(t,\lambda)\)

where \(I^\gamma_{0+}\) is the fractional integral of order \(\gamma > 0\) for an integrable function \(f\) which is defined by \(I^\gamma_{0+}f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds\).

Taking into also account the monotonically decreasing character of \(E_{q,q}(-\lambda t^q)\) \([31]\), where \(\lambda \in \mathbb{R}_+\) the following statement holds true.
Lemma 1. ([30]) For $0 < q < 1$ Mittag-Leffler type function $E_{q,q}(-\lambda t^q)$ satisfies

$$0 \leq E_{q,q}(-\lambda t^q) \leq \frac{1}{\Gamma(q)}, \quad t \in [0, \infty), \ \lambda \geq 0.$$ 

Now, we give the following lemma which is necessary for further development.

Lemma 2. For $0 < q < 1$, $\lambda \in \mathbb{R}_+$ we have

$$\int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q) d\tau = \frac{1}{\gamma} (1 - E_q(-\gamma(t-t_0)^q)).$$

Proof. By applying change of variable $z = t-\tau$ in the above integral and using $\frac{d}{dt} E_q(-\gamma t^q) = -\gamma t^{q-1} E_{q,q}(-\gamma t^q)$, ([23]), we have

$$\int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-\gamma(t-\tau)^q) d\tau = \int_{0}^{t-t_0} z^{q-1} E_{q,q}(-\gamma z^q) dz$$

$$= -\frac{1}{\gamma} \int_{0}^{t-t_0} \frac{d}{dz} E_q(-\gamma z^q) dz = \frac{1}{\gamma} (1 - E_q(-\gamma(t-t_0)^q)).$$

We give a rule for fractional differentiation with order $0 < q < 1$ of an integral depending on a parameter, see [20].

$$D^q_{0+} \int_{0}^{t} K(t,\tau) d\tau = \int_{0}^{t} D^q_{\tau} K(t,\tau) d\tau + \lim_{\tau \to 0^+} I^1_{\tau} - q K(t,\tau).$$

We will also need to recall the following result.

Lemma 3. ([23])

Let $f_i$ be a sequence of functions defined on the interval $(a,b)$. Suppose the following conditions holds:

(i) the fractional derivative $D^q_{0+} f_i(t)$, for a given $q > 0$, exists for all $i \in \mathbb{N}, t \in (a,b]$;

(ii) both series $\sum_{i=1}^{\infty} f_i(t)$ and $\sum_{i=1}^{\infty} D^q_{0+} f_i(t)$ are uniformly convergent on the interval $[a + \epsilon, b]$ for any $\epsilon > 0$.

Then the function defined by the series $\sum_{i=1}^{\infty} f_i(t)$ is $q$ differentiable and satisfies $D^q_{0+} \sum_{i=1}^{\infty} f_i(t) = \sum_{i=1}^{\infty} D^q_{0+} f_i(t)$.

2.2 Weak singular Volterra integral equations

In this part we recall some basic results on Volterra type integral equation with the kernel may have a diagonal singularity. For details see ([29, 32]).

Consider the Volterra integral equation

$$u(t) = \int_{0}^{t} Q(t,\tau) u(\tau) d\tau + f(t), \quad 0 \leq t \leq 1. \quad (2.2)$$

Denote $\Delta = \{(t,\tau) : 0 \leq \tau < t \leq 1\}$ and introduce the class $S'$ of kernels $Q(t,\tau)$ that are defined and continuous on $\Delta$ and satisfy for $(t,\tau) \in \Delta$ the inequality

$$|Q(t,\tau)| \leq c (t-\tau)^{-\nu}, \quad \nu > 0, \quad c = \text{const} > 0.$$
where $f$ is weakly singular if $\nu < 1$. A weak singularity of the kernel implies that the corresponding integral operator is compact in the space $C[0, 1]$. More precisely, the following statement holds true.

**Lemma 4.** ([29]) Let $Q(t, \tau) \in S^\nu$ and $\nu < 1$. Then the Volterra integral operator $T$ defined by $(Tr)(t) = \int_0^t Q(t, \tau)r(\tau)d\tau$ maps $C[0, 1]$ into itself and $T : C[0, 1] \rightarrow C[0, 1]$ is compact.

The proof of Lemma 2 is standard a detailed argument can be found in [29]. A consequence of Lemma 2 is the following result.

**Lemma 5.** ([29]) Let $f \in C[0, 1]$ and $Q(t, \tau) \in S^\nu$ with $\nu < 1$. Then Eq. (2.2) has a unique solution $u \in C[0, 1]$.

We will also need to recall the results on weakly singular version of the Gronwall’s inequality.

**Lemma 6.** ([29]) Let $T, \varepsilon, M \in \mathbb{R}_+$ and $0 < q < 1$. Moreover assume that $\delta : [0, T] \rightarrow \mathbb{R}$ is a continuous function satisfying the inequality

$$|\delta(t)| \leq \varepsilon + \frac{M}{\Gamma(q)} \int_0^t (t-\tau)^{-\nu}|\delta(\tau)|d\tau, \text{ with } \nu = 1-q$$

for all $t \in [0, T]$. Then

$$|\delta(t)| \leq \varepsilon E_q(Mt^q)$$

for $t \in [0, T]$.

### 2.3 Problem Formulation and Associated Spectral Problem

Let $T > 0$ be a fixed number. In the rectangle $\Omega_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$, we will consider the following time-dependent heat conduction equation

$$D^\nu_{t+}(u(x, t) - u(x, 0)) = u_{xx} + r(t)f(x, t), \quad (x, t) \in \Omega_T, \quad (2.3)$$

supplemented with the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \quad (2.4)$$

and the boundary condition

$$u(0, t) = 0, \quad 0 < t \leq T, \quad (2.5)$$

$$au_{xx}(1, t) + du_x(1, t) + bu(1, t) = 0, \quad 0 < t \leq T, \quad (2.6)$$

where $f(x, t), \varphi(x)$ are given functions and $a, b, d$ are given numbers not simultaneously equal to zero.

The choice of the term $D^\nu_{t+}(u(x, .) - u(x, 0))(t)$ instead of the usual term $D^\nu_{t+}(u(x, .))(t)$ is not only to avoid the singularity at zero, but also impose a meaningful initial condition (without fractional integral) [15].

If the function $r(t)$ is known, the problem of finding $u(x, t)$ from (2.3)-(2.6) is called the direct problem. However, the problem here is that the source function $r(t)$ is unknown, which needs to be determined by energy condition

$$\int_0^1 u(x, t)dx = E(t), \quad 0 \leq t \leq T, \quad (2.7)$$

where $E(t)$ are given functions. This problem is called the inverse problem.

On the other hand, use of integral condition (1.4) arises when the data on the boundary cannot be measured directly, but only the average value of the solution can be measured along the boundary. More precisely classical boundary conditions (Neumann, Dirichlet and Robin type) are not always adequate as it depends on the physical context which data can be measure at the boundary of the physical domain. The classical boundary conditions cases, one can have a selection of such large noise local space measurement, but which on average produce a less noisy non-local measurement (2.7).

The inverse problem of finding $r(t)$ in classical heat conduction equation $u_t = u_{xx} + r(t)f(x, t)$ with the conditions (2.3)-(2.7) was already studied in [18] by using eigenfunction expansion in terms of auxiliary spectral
problem along the generalized Fourier method. Our aim in present paper is to transfer this method to fractional cases.

Because the function $r$ is space independent, $a$, $b$ and $d$ are constants and the boundary conditions are linear and homogeneous, the method of separation of variables is suitable for studying the inverse problem (2.3)-(2.7). It is well-known that the main difficulty in applying the Fourier method is its basisness, i.e. the expansion in terms of eigenfunctions of the auxiliary spectral problem [22]

\[
\begin{cases}
X''(x) + \mu X(x) = 0, \quad 0 \leq x \leq 1, \\
X(0) = 0 \\
(\mu a - b)X(1) = dX'(1).
\end{cases}
\] (2.8)

In contrast to the classical Sturm–Liouville problem, this problem has the spectral parameter also in the boundary condition. It makes it impossible to apply the classical results on expansion in terms of eigenfunctions.

Consider the spectral problem (2.8) with $ad > 0$. It is known that its eigenvalues $\mu_n$, $n = 0, 1, 2, \ldots$ are real and simple. They form an unbounded increasing sequence and the eigenfunction $X_n(x)$ corresponding to $\mu_n$ has exactly $n$ simple zeros in the interval $(0, 1)$. The sign of the first eigenvalue $\mu_0$ is given as

\[
\begin{align*}
\mu_0 &< 0 < \mu_1 < \mu_2 < \ldots, \text{ if } \frac{b}{d} > 1, \\
\mu_0 &= 0 < \mu_1 < \mu_2 < \ldots, \text{ if } \frac{b}{d} = 1, \\
0 &< \mu_0 < \mu_1 < \mu_2 < \ldots, \text{ if } \frac{b}{d} < 1,
\end{align*}
\]

The eigenvalues and eigenfunctions have the following asymptotic behaviour: $\sqrt{\mu_n} = \pi n + O\left(\frac{1}{n}\right)$, $X_n(x) = \sin(\pi n x) + O\left(\frac{1}{n}\right)$, for sufficiently large $n$. Let $n_0$ be arbitrary fixed non-negative integer. The system of eigenfunctions $\{X_n(x)\}$ $(n = 0, 1, 2, \ldots; n \neq n_0)$ is a Riesz basis for $L_2[0, 1]$. The system $\{u_n(x)\}$ $(n = 0, 1, 2, \ldots; n \neq n_0)$ which is biorthogonal to the system $\{X_n(x)\}$ $(n = 0, 1, 2, \ldots; n \neq n_0)$ has the form

\[
u_n(x) = \frac{X_n(x) - \frac{X_n(1)}{X_n(1)} X_n(0)}{\|X_n\|^2_{L_2[0,1]} + \frac{a}{d} X_n^2(1)}.
\]

The following Bessel-type inequalities are valid for the systems $\{X_n(x)\}$ and $\{u_n(x)\}$ $(n = 0, 1, 2, \ldots; n \neq n_0)$.

**Lemma 7 (Lg)** If $\psi(x) \in L_2[0,1]$, then the estimates

\[
\sum_{n=0}^{\infty} |(\psi, X_n)|^2 \leq c_1 \|\psi\|^2_{L_2[0,1]}, \quad \sum_{n=0}^{\infty} |(\psi, u_n)|^2 \leq c_2 \|\psi\|^2_{L_2[0,1]}
\]

hold for some positive constants $c_1$ and $c_2$, where

\[
(\psi, X_n) = \int_0^1 \psi(x) X_n(x) dx \quad \text{and} \quad (\psi, u_n) = \int_0^1 \psi(x) u_n(x) dx.
\]

Let us denote $\Phi_{n_0}^4[0, 1] := \{\psi(x) \in C^4[0,1]; \psi(0) = \psi''(0), \psi(1) = \psi'(1) = \psi''(1) = \psi'''(1) = 0, \int_0^1 \psi(x) X_{n_0}(x) dx = 0\}$. 

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Lemma 8 ([16]) If \( \psi(x) \in \Phi^4_{n_0}[0,1] \), then we have:

\[
\mu^2_n(\psi, X_n) = (\psi^4, X_n), \quad \mu^2_n(\psi, u_n) = (\psi^4, u_n), \quad n \geq 0, \\
\sum_{n=0 \atop n \neq n_0}^{\infty} |\mu_n(\psi, X_n)| \leq c_3 \|\psi\|_{C^4[0,1]}, \quad \sum_{n=0 \atop n \neq n_0}^{\infty} |\mu_n(\psi, u_n)| \leq c_4 \|\psi\|_{C^4[0,1]}, \\
\sum_{n=0 \atop n \neq n_0}^{\infty} |(\psi, X_n)| \leq c_5 \|\psi\|_{C^4[0,1]}, \quad \sum_{n=0 \atop n \neq n_0}^{\infty} |(\psi, u_n)| \leq c_6 \|\psi\|_{C^4[0,1]},
\]

where \( c_3, c_4, c_5 \) and \( c_6 \) are some positive constants.

3 Well-Posedness of the inverse problem

The main result on existence and uniqueness of the solution of the inverse problem (2.3)-(2.7) is presented. We prove the existence and uniqueness of the solution of inverse problem (2.3)-(2.7) by means of construction of a contraction mapping from energy condition (2.7). We call classical solution as a pair of functions \( \{u(x, t), r(t)\} \) satisfying \( u(x, t) \in C^2([0, 1], \mathbb{R}), \quad D_{0, +}^q u(x, t) = f(x, t) \in C([0, T], \mathbb{R}) \) and \( r(t) \in C[0, T] \).

3.1 Formal construction of the solution

Theorem 1. Suppose that the following conditions hold:

(A1) \( \varphi(x) \in \Phi^4_{n_0}[0,1] \).

(A2) \( E(t) \in C^1[0, T]; \quad E(0) = \frac{1}{0} \varphi(x) dx \).

(A3) \( f(x, t) \in C(\Omega_T); \quad f(x, t) \in \Phi^4_{n_0}[0,1], \forall t \in [0, T]; \)

Then there exists a unique classical solution of the inverse problem (2.3)-(2.7) in \( \Omega_T \).

Proof For arbitrary \( r(t) \in C[0, T] \), the solution of (2.3)-(2.6) can be written in the form

\[
u(x, t) = \sum_{n=0 \atop n \neq n_0}^{\infty} v_n(t) X_n(x), \quad (3.1)\]

where the functions \( v_n(t), n = 0, 1, 2, \ldots; n \neq n_0 \), are to be determined. By using the Fourier’s method, we can easily see that \( v_n(t), n = 0, 1, 2, \ldots; n \neq n_0 \), satisfy the following system of countably many linear fractional differential equations:

\[
\begin{cases}
D_{0, +}^q (v_n(t) - v_n(0)) + \mu_n v_n(t) = r(t) f_n(t) \\
v_n(0) = \varphi_n
\end{cases}, \quad (3.2)
\]

where \( \mu_n \) are eigenvalues of (2.8)

According to IVP (2.1), it can easily be seen that the solutions of (3.2) is of the form

\[
v_n(t) = \varphi_n e_q(t, \mu_n) + \int_0^t e_q(t - \tau, \mu_n) r(\tau) f_n(\tau) d\tau
\]

with \( \varphi_n = \int_0^1 \varphi(x) u_n(x) dx \), \( f_n(t) = \int_0^1 f(x, t) u_n(x) dx \) and the solution is given formally

\[
u(x, t) = \sum_{n=0 \atop n \neq n_0}^{\infty} [\varphi_n e_q(t, \mu_n) + \int_0^t e_q(t - \tau, \mu_n) r(\tau) f_n(\tau) d\tau] X_n(x), \quad (3.3)
\]

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The formulas (3.3) and (2.8) yield a following Volterra integral equation of the first kind with respect to $r(t)$:

$$\int_{0}^{t} K(t, \tau)r(\tau)d\tau + F(t) = E(t) \quad (3.4)$$

where

$$K(t, \tau) = \sum_{n=0}^{\infty} \left[ f_n(\tau)\tilde{c}_{q,q}(t - \tau, \mu_n) \left( \frac{1}{0} X_n(x)dx \right) \right]$$

and

$$F(t) = \sum_{n=0}^{\infty} \left[ \varphi_n e_q(t, \mu_n) \left( \frac{1}{0} X_n(x)dx \right) \right].$$

Further, the equation (3.4) yields the following Volterra integral equation of the second kind by taking fractional derivative $D^q_{0+}$:

$$\int_{0}^{t} D^q_{0+}K(t, \tau)r(\tau)d\tau + r(t), \lim_{\tau \to t-} I^{q-1}_{\tau}K(t, \tau) + D^q_{0+}(F(t) - F(0)) = D^q_{0+}(E(t) - E(0)). \quad (3.5)$$

By using the properties (iii) in Proposition 1, it is easy to show that

$$\lim_{\tau \to t-} I^{q-1}_{\tau}K(t, \tau) = \lim_{\tau \to t-} \sum_{n=0}^{\infty} \sum_{n \neq n_0} f_n(\tau)\tilde{c}_{q,q}(t - \tau, \mu_n) \left( \frac{1}{0} X_n(x)dx \right)$$

$$= \frac{1}{0} f(x,t)dx,$$

$$D^q_{0+}(F(t) - F(0)) = -\sum_{n=0}^{\infty} \sum_{n \neq n_0} \varphi_n e_q(t, \mu_n) \left( \frac{1}{0} X_n(x)dx \right) \quad (3.6)$$

and

$$D^q_{0+}K(t, \tau) = -\sum_{n=0}^{\infty} \sum_{n \neq n_0} \mu_n\tilde{c}_{q,q}(t - \tau, \mu_n)f_n(\tau) \left( \frac{1}{0} X_n(x)dx \right) \quad (3.7).$$

We obtain the Volterra integral equation of the second kind with respect to $r(t)$ in the form

$$r(t) = P(t) + \int_{0}^{t} Q(t, \tau)r(\tau)d\tau \quad (3.8)$$

with the free term $P(t) = \frac{D^q_{0+}(E(t) - E(0)) - D^q_{0+}(F(t) - F(0))}{\frac{1}{0} f(x,t)dx}$ and kernel $Q(t, \tau) = -\frac{D^q_{0+}K(t, \tau)}{\frac{1}{0} f(x,t)dx}$. 

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According to the Lemma 1, we estimate the kernel of (3.8) in the following form:

\[ |Q(t, \tau)| = \frac{|D_q K(t, \tau)|}{\int_0^1 |f(x, t)| dx} \leq \frac{C}{(t - \tau)\mu}, \]

where \( C = \frac{M_{\epsilon_4} \max_{t \in [0, T]} \| f(\cdot, t) \|_{C^1([0, 1])}}{\Gamma(q) \min_{t \in [0, T]} \int_0^1 |f(x, t)| dx} \) with \( M \geq |X_n(x)|, \forall n \in \mathbb{N}, \forall x \in [0, 1] \) and \( \mu = 1 - q \).

Because the kernel \( Q(t, \tau) \) belongs to the class \( S^\mu \) with \( 0 < \mu < 1 \) the Volterra integral equation (3.8) is weakly singular. Then it has a unique solution \( r \in C[0, 1] \) according by Lemma 4 and 5.

### 3.2 Existence and Uniqueness of the solution of the inverse problem

First, let us show that the solution of inverse problem (2.3)-(2.7) is unique. Suppose that there were two solutions pair \((r, u)\) and \((a, v)\) of the inverse problem (2.3)-(2.7). Then from form of solution (3.3) and (3.8), we have

\[ u(x, t) - v(x, t) = \sum_{n=0}^{\infty} \left( \int_0^t e_{q, q}(t - \tau, \mu_n) f_n(\tau)[|r(\tau) - a(\tau)|] d\tau \right) X_n(x) \]  

(3.10)

and

\[ r(t) - a(t) = \int_0^t Q(t, \tau) [|r(\tau) - a(\tau)|] d\tau. \]  

(3.11)

Then (3.11) yields \( r = a \). After inserting \( r = a \) in (3.10), we have \( u = v \).

So far we have proved the uniqueness of the solution of the inverse problem. Because the solution \( u(x, t) \)
is formally given by the series form (3.3), we need to show that the series corresponding to \( u(x, t), u_x(x, t), u_{xx}(x, t) \) and \( D_q^{n+}(u(x, .) - u(x, 0)) \) represent continues functions. Under the assumptions \((A_1)-(A_3)\) and Lemma 7, for all \((x, t) \in \Omega_T\), the series corresponding to \( u(x, t) \) is bounded above by the series

\[ \sum_{n=0}^{\infty} \left[ |v_n| + \frac{1}{\mu_n} \max_{t \in [0, T]} |r(t)| \max_{t \in [0, T]} |f_n(t)| \right] \]  

(3.12)

The majorizing series (3.12) is convergent by using Lemmas 7 and 8. This implies that by the Weierstrass M-test, the series (3.3) is uniformly convergent in the rectangle \( \Omega_T \) and therefore, the solution \( u(x, t) \) is continuous in the rectangle \( \Omega_T \).

The majorizing series for \( x \)-partial derivative is

\[ \sum_{n=0}^{\infty} \left[ |v_n| \sqrt{\mu_n} + \frac{1}{\sqrt{\mu_n}} \max_{t \in [0, T]} |r(t)| \max_{t \in [0, T]} |f_n(t)| \right] \]  

(3.13)

for \( xx \)-partial derivative is

\[ \sum_{n=0}^{\infty} \left[ |v_n| \mu_n + \max_{t \in [0, T]} |r(t)| \max_{t \in [0, T]} |f_n(t)| \right]. \]  

(3.14)
It can easily be seen that the series (3.13)–(3.14) are convergent by employing Lemma 7 and 8, Schwarz inequality. Hence by the Weierstrass M-test, the series obtained for \(x\)-partial and \(xx\)-partial derivatives of (3.3) is uniformly convergent in the rectangle \(\Omega_T\). Therefore, their sums \(u_x(x,t)\) and \(u_{xx}(x,t)\) are continuous in \(\Omega_T\).

Now it remains to show that \(q\)-fractional derivative of the series \(u(x,t) - u(x,0)\) represents continuous function on \(\Omega_T\). We will show that for any \(\epsilon > 0\) and \(t \in [\epsilon, T]\), the following series

\[
\sum_{n=0}^{\infty} [D_{0+}^q(v_n(t) - v_n(0))]X_n(x)
\]

corresponding to \(q\)-fractional derivative of the function \(u(x,t) - u(x,0)\) is uniformly convergent. Before it we need to recall the Lemma 3. Now we can see that equation (3.2) yields

\[
D_{0+}^q(v_n(t) - v_n(0)) = -\mu_n\varphi_n e_q(t,\mu_n) - \mu_n \int_0^t e_{q,q}(t-\tau,\mu_n)r(\tau) f_n(\tau)d\tau + r(t)f_n(t).
\]

We have the following estimates

\[
|D_{0+}^q(v_n(t) - v_n(0))| \leq |\varphi_n| \mu_n e_q(\epsilon,\mu_n) + 2 \max_{t \in [0,T]} |r(t)| \max_{t \in [0,T]} |f_n(t)|
\]

and we obtain a majorant series as follow

\[
\sum_{n=0}^{\infty} \left[ |\varphi_n| \mu_n e^{-\frac{\epsilon}{1+q}\tau} + 2 \max_{t \in [0,T]} |r(t)| \max_{t \in [0,T]} |f_n(t)| \right] .
\]

Consequently \(D_{0+}^q(u(x,t) - u(x,0))\) is uniformly convergent in the rectangle \(\Omega_T\).

4 Lipschitz stability of the solution of the inverse problem

The following result on continuously dependence on the data of solution of the inverse problem (1.1)-(1.4) holds.

**Theorem 2** Let \(\mathcal{S}\) be the class of triples in the form of \(\Phi = \{f, \varphi, E\}\) which satisfy the assumptions \((A_1)-(A_3)\) of Theorem 1 and

\[
0 < N_0 \leq \min_{t \in [0,T]} \int_0^1 f(x,t)dx , \quad \|f\|_{C^4(0,T)} \leq N_1, \quad \|\varphi\|_{C^1([0,1])} \leq N_2, \quad \|E\|_{C^1([0,1])} \leq N_3,
\]

for some positive constants \(N_i, i = 0, 1, 2, 3\).

Then the solution pair \((r, u)\) of the inverse problem (2.3)-(2.7) depends continuously upon the data in \(\mathcal{S}\).

**Proof.** Let \(\Phi = \{f, \varphi, E\}\) and \(\Phi = \{\tilde{f}, \tilde{\varphi}, \tilde{E}\} \in \mathcal{S}\) be two sets of data. Let us denote \(\|\Phi\| = \|f\|_{C^4(0,T)} + \|\varphi\|_{C^1([0,1])} + \|E\|_{C^1([0,1])}\).

Let \((r, u)\) and \((\tilde{r}, \tilde{u})\) be the solutions of the inverse problems (2.3)-(2.7) corresponding to the data \(\Phi\) and \(\Phi\), respectively.

According to (3.8) we have
\[ r(t) = P(t) + \int_0^t Q(t, \tau) r(\tau) d\tau \quad (3.16) \]

and

\[ \tilde{r}(t) = \tilde{P}(t) + \int_0^t \tilde{Q}(t, \tau) \tilde{r}(\tau) d\tau. \quad (3.17) \]

Firstly, from equations (3.5)-(3.7) and using \( E(t) \in C^1[0, T] \), Lemma 2, it is easy to compute that

\[ |D^0_{q+}(F(t) - F(0))| \leq N_4, \quad |D^0_{q+}(E(t) - E(0))| \leq N_5 \quad (3.18) \]

where \( N_4 = c_4 N_2 M, N_5 = \frac{T^q}{q^q(1-q)} N_3. \)

Let us estimate the difference \( r - \tilde{r} \).

From (3.16) and (3.17) we obtain

\[ r(t) - \tilde{r}(t) = P(t) - \tilde{P}(t) + \int_0^t \left[ Q(t, \tau) - \tilde{Q}(t, \tau) \right] r(\tau) d\tau + \int_0^t \tilde{Q}(t, \tau) [r(\tau) - \tilde{r}(\tau)] d\tau. \quad (3.19) \]

Let \( \epsilon_1 = \left\| P - \tilde{P} \right\|_{C([0, T])} + \frac{T^q}{q^q(q - 1)} \frac{2N_1Mc_4}{N_6^2} \left\| f - \tilde{f} \right\|_{C^{q,0}(\Omega_T)} \left\| \tilde{r} \right\|_{C([0, T])}. \) Then denoting \( R(t) = \left| r(t) - \tilde{r}(t) \right| \), equation (3.19) implies the inequality

\[ R(t) \leq \epsilon_1 + \frac{\epsilon_2}{1(q)} \int_0^t (t - \tau)^{q-1} R(\tau) d\tau \]

where \( \epsilon_2 = \frac{N_1Mc_4}{N_6} \). Then, a weakly singular Gronwall’s inequality, see Lemma 6, implies that

\[ R(t) \leq \epsilon_1 E_\eta(\epsilon_2 t^q), \quad t \in [0, T]. \]

Finally using (1) and 3.19 we obtain It follows from (3.8) that

\[ \left\| r - \tilde{r} \right\|_{C([0, T])} \leq \epsilon_3 \left( \left\| P - \tilde{P} \right\|_{C([0, T])} + \left\| r \right\|_{C([0, T])} \frac{T^q}{q^q(q - 1)} \frac{N_1Mc_4}{N_6^2} \left\| f - \tilde{f} \right\|_{C^{q,0}(\Omega_T)} \right) \quad (3.20) \]

where \( \epsilon_3 = E_\eta(\epsilon_2 T^q). \) Also one can estimate that

\[ \left\| P - \tilde{P} \right\|_{C([0, T])} \leq M_1 \left\| f - \tilde{f} \right\|_{C^{q,0}(\Omega_T)} + M_2 \left\| \varphi - \tilde{\varphi} \right\|_{C^1([0, 1])} + M_3 \left\| E - \tilde{E} \right\|_{C^1([0, T])}. \quad (3.21) \]

where \( M_1 = \frac{N_1N_3}{N_6^2}, \quad M_2 = \frac{c_3 M c_4}{N_6^2}, \quad M_3 = \frac{T^q}{q^q(q - 1)} \frac{N_1Mc_4}{N_6^2}. \)

By using the inequality (3.21), from (3.20) we obtain

\[ \left\| r - \tilde{r} \right\|_{C([0, T])} \leq M_4 \left( \left\| f - \tilde{f} \right\|_{C^{q,0}(\Omega_T)} + \left\| \varphi - \tilde{\varphi} \right\|_{C^1([0, 1])} + \left\| E - \tilde{E} \right\|_{C^1([0, T])} \right) \]

where \( M_4 = \max(\epsilon_3 M_2, \epsilon_3 M_3, \epsilon_3 M_1 + \epsilon_3 \left\| r \right\|_{C([0, T])} \frac{T^q}{q^q(q - 1)} \frac{N_1Mc_4}{N_6^2}). \)
This shows that $r$ depends continuously upon the input data. From (3.3), a similar estimate is also obtained for the difference $u - \tilde{u}$ as
\[
\|u - \tilde{u}\|_{C^1(\Omega_T)} \leq M_5 \|\Phi - \tilde{\Phi}\|.
\]
This completes the proof of Theorem 2.

5 Conclusion

The paper considers an inverse source problem of identification the time dependent source parameter together with the temperature distribution from the energy measurement for a one-dimensional time-fractional heat equation with the generalized impedance boundary condition. The well-posedness of the inverse problem is shown by using the Fourier expansion in terms of eigenfunctions of a spectral problem which has the spectral parameter also in the boundary condition. To contrast to the fact that the problem under consideration in present paper is obtained from the classical diffusion problem [18] by replacing the time derivative with a fractional derivative, the fractional inverse problem should be more difficult. The difficulty comes from the definition of the fractional-order derivatives, which is essentially an integral with the kernel of weak singularity. For such a non-classical derivative, some standard methods on Volterra type integral equations for treating the inverse problems cannot be applied. It needs more difficult informations on Volterra operators with the kernel of weak singularity at diagonal. This approach can be extended to the time-fractional analogues of the classical inverse initial boundary value problems for the heat equation with different boundary and overdetermination conditions, which are the line of future investigations.

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