THE CYCLIC INDEX OF ADJACENCY TENSOR OF
GENERALIZED POWER HYPERGRAPHS

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ABSTRACT. Let $G$ be a $t$-uniform hypergraph, and let $c(G)$ denote the cyclic index of the adjacency tensor of $G$. Let $m, s, t$ be positive integers such that $t \geq 2, s \geq 2$ and $m = st$. The generalized power $G^{m,s}$ of $G$ is obtained from $G$ by blowing up each vertex into an $s$-set and preserving the adjacency relation. It was conjectured that $c(G^{m,s}) = s \cdot c(G)$. In this paper we show that the conjecture is false by giving a counterexample, and give some sufficient conditions for the conjecture holding. Finally we give an equivalent characterization of the equality in the conjecture by using a matrix equation over $\mathbb{Z}_m$.

1. INTRODUCTION

A hypergraph $G = (V(G), E(G))$ consists of a set of vertices, say $V(G) = \{v_1, v_2, \ldots, v_n\}$, and a set of edges, say $E(G) = \{e_1, e_2, \ldots, e_k\}$, where $e_j \subseteq V(G)$ for $j \in [k] := \{1, 2, \ldots, k\}$. If $|e_j| = m$ for each $j \in [k]$, then $G$ is called an $m$-uniform hypergraph. A walk $W$ in $G$ is a sequence of alternating vertices and edges: $v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_l}, v_{i_l}$, where $\{v_{i_1}, v_{i_{j+1}}\} \subseteq e_{i_j}$ for $j = 0, 1, \ldots, l - 1$. The hypergraph $G$ is connected if every two vertices of $G$ are connected by a walk. The adjacency tensor $\mathcal{A}(G)$ of the hypergraph $G$ is defined as $\mathcal{A}(G) = (a_{i_1 i_2 \ldots i_m})$ [4], an $m$-th order $n$-dimensional tensor, where

$$
a_{i_1 i_2 \ldots i_m} = \begin{cases} \frac{1}{n(n-1)\ldots(n-m+1)} & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \in E(G); \\
0 & \text{otherwise}. \end{cases}
$$

In general, a tensor (also called hypermatrix) $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ of order $m$ and dimension $n$ over a field $\mathbb{F}$ refers to a multiarray of entries $a_{i_1 i_2 \ldots i_m} \in \mathbb{F}$ for all $i_j \in [n]$ and $j \in [m]$, which can be viewed to be the coordinates of the classical tensor (as a multilinear function) under an orthonormal basis. If $m = 2$, then $\mathcal{A}$ is a square matrix of dimension $n$.

In 2005, independently, Lim [13] and Qi [17] introduced eigenvalues for tensors $\mathcal{A}$. Denote by $\rho(\mathcal{A})$ the spectral radius of $\mathcal{A}$, and by $\mathop{\text{Spec}}(\mathcal{A})$ the spectrum of $\mathcal{A}$. If $\mathcal{A}$ is further nonnegative, then by Perron-Frobenius theorem of nonnegative tensors, $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is weakly irreducible and has $k$ eigenvalues of $\mathcal{A}$ with modulus $\rho(\mathcal{A})$, then those $k$ eigenvalues are equally distributed on the spectral circle. As for nonnegative matrices, the number $k$ is called the cyclic index of $\mathcal{A}$ [2]. The cyclic index reflects the spectral symmetry of

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nonnegative weakly irreducible tensors, which was generalized and investigated in the paper [5].

**Definition 1.1** ([5]). Let $\mathcal{A}$ be an $m$-th order $n$-dimensional tensor, and let $\ell$ be a positive integer. The tensor $\mathcal{A}$ is called *spectral $\ell$-symmetric* if

\[(1.1) \quad \text{Spec}(\mathcal{A}) = e^{i2\pi \ell} \text{Spec}(\mathcal{A}).\]

The maximum number $\ell$ such that (1.1) holds is called the *cyclic index* of $\mathcal{A}$ and denoted by $c(\mathcal{A})$, and $\mathcal{A}$ is called *spectral $c(\mathcal{A})$-cyclic*.

When we say a hypergraph is *spectral $\ell$-symmetric* or *spectral $\ell$-cyclic*, this is always referring to its adjacency tensor. In particular, for a uniform hypergraph $G$, denote $c(G) := c(\mathcal{A}(G))$, called the cyclic index of $G$. For a general tensor $\mathcal{A}$, if it is spectral $\ell$-symmetric, then $\ell | c(\mathcal{A})$ by [5, Lemma 2.7]. It was also proved that if a connected $m$-uniform hypergraph is spectral $\ell$-symmetric, then $\ell | m$, and hence $c(G) | m$; see [5, Lemma 3.2, Corollary 4.3], [6, Lemma 2.5] or [21, Theorem 2.15]. In the paper [5] the authors use the construction of generalized power hypergraphs to show that for every positive integer $m$ and any positive integer $\ell$ such that $\ell | m$, there always exists an $m$-uniform hypergraph $G$ such that $G$ is spectral $\ell$-symmetric. They posed the following conjecture.

**Conjecture 1.2** ([5]). Let $G$ be a $t$-uniform hypergraph, and let $G^{m,s}$ be the generalized power of $G$, where $m = st$. Then

\[(1.2) \quad c(G^{m,s}) = s \cdot c(G).\]

The generalized power of a hypergraph is defined as follows.

**Definition 1.3** ([10]). Let $G = (V, E)$ be a $t$-uniform hypergraph. For any integers $m, s$ such that $m > t$ and $1 \leq s \leq \frac{m}{t}$, the generalized power of $G$, denoted by $G^{m,s}$, is defined as the $m$-uniform hypergraph with the vertex set $(\cup_{v \in V} v) \cup (\cup_{e \in E} e)$, and the edge set $\{u_1 \cup \cdots \cup u_t : e = \{u_1, \ldots, u_t\} \in E(G)\}$, where $v$ denotes an $s$-set corresponding to $v$ and $e$ denotes an $(m - ts)$-set corresponding to $e$, and all those sets are pairwise disjoint.

In this paper, we only consider the power hypergraphs $G^{m,s}$ with $m = st$, i.e. $G^{m,s}$ is obtained from $G$ by blowing up each vertex into an $s$-set and preserving the adjacency relation. The generalized power hypergraphs include some special cases, such as the powers of simple graphs introduced by Hu, Qi and Shao [3], the generalized powers of simple graphs introduced by Khan and Fan [11], Peng [16] introduced $s$-paths and $s$-cycles with uniformity $m$ on discussing the Ramsey number, which are exactly the generalized pows of paths and cycles (as simple graphs) respectively if $1 \leq s \leq \frac{m}{t}$. The spectral results on generalized power hypergraphs can be found in [9, 24, 11, 23, 12, 10].

For the conjecture [12], it was shown that it is true if $c(G) = 1$ [5]. In this paper we show that the conjecture is false by giving a counterexample, and give some sufficient conditions for the conjecture holding. We finally give an equivalent characterization of Eq. (1.2) by using a matrix equation over $\mathbb{Z}_m$.

2. **Preliminaries**

2.1. **Notions.** Let $\mathcal{A}$ be a real tensor of order $m$ and dimension $n$. The tensor $\mathcal{A}$ is called *symmetric* if its entries are invariant under any permutation of their indices. So, the adjacency tensor of a uniform hypergraph is symmetric.
Given a vector $x \in \mathbb{C}^n$, $Ax^m \in \mathbb{C}$ and $Ax^{m-1} \in \mathbb{C}^n$, which are defined as follows:

$$Ax^m = \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1i_2\ldots i_m}x_{i_1}x_{i_2} \cdot \cdot \cdot x_{i_m},$$

$$A(x^{m-1})_i = \sum_{i_2, \ldots, i_m \in [n]} a_{i_2i_3\ldots i_m}x_{i_2} \cdot \cdot \cdot x_{i_m}, i \in [n].$$

Let $I = (i_1i_2\ldots i_m)$ be the identity tensor of order $m$ and dimension $n$, that is, $i_1i_2\ldots i_m = 1$ if and only if $i_1 = i_2 = \cdots = i_m \in [n]$ and $i_1i_2\ldots i_m = 0$ otherwise.

**Definition 2.1** [13 17]. Let $A$ be an $m$-th order $n$-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - A)x^{m-1} = 0$, or equivalently $Ax^{m-1} = \lambda x^{m-1}$, has a solution $x \in \mathbb{C}^n\backslash\{0\}$, then $\lambda$ is called an eigenvalue of $A$ and $x$ is an eigenvector of $A$ associated with $\lambda$, where $x^{m-1} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})$.

The determinant of $A$, denoted by $\det A$, is defined as the resultant of the polynomials $Ax^{m-1}$ [8], and the characteristic polynomial $\varphi_A(\lambda)$ of $A$ is defined as $\det(\lambda I - A)$ [17 3]. It is known that $\lambda$ is an eigenvalue of $A$ if and only if it is a root of $\varphi_A(\lambda)$. The spectrum of $A$ is the multi-set of the roots of $\varphi_A(\lambda)$.

The spectral symmetry of a connected hypergraph is closed related to a certain coloring of the hypergraph.

**Definition 2.2** [5]. Let $m \geq 2$ and $\ell \geq 2$ be integers such that $\ell \mid m$. An $m$-uniform hypergraph $G$ on $n$ vertices is called $(m, \ell)$-colorable if there exists a map $\phi: [n] \to [m]$ such that if $\{i_1, \ldots, i_m\} \in E(G)$, then

$$\phi(i_1) + \cdots + \phi(i_m) \equiv \frac{m}{\ell} \mod m.\tag{2.1}$$

Such $\phi$ is called an $(m, \ell)$-coloring of $G$.

If $m$ is even, an $m$-uniform hypergraph with an $(m, 2)$-coloring was called *odd-colorable* by Nikiforov [14].

**Theorem 2.3.** [5] Let $G$ be a connected $m$-uniform hypergraph. Then $G$ is spectral $\ell$-symmetric if and only if $G$ is $(m, \ell)$-colorable.

The edge-vertex incidence matrix $B_G = (b_{ev})$ of an $m$-uniform hypergraph $G$ is defined by

$$b_{ev} = \begin{cases} 1, & \text{if } v \in e \in E(G), \\ 0, & \text{otherwise}. \end{cases}$$

We may view $B_G$ as one over $\mathbb{Z}_m$, where $\mathbb{Z}_m$ is the ring of integers modulo $m$. Now Eq. (2.1) is equivalent to

$$B_G\phi = \frac{m}{\ell}\mathbf{1} \text{ over } \mathbb{Z}_m,\tag{2.2}$$

where $\phi = (\phi(1), \ldots, \phi(n))$ is considered as a column vector, and $\mathbf{1}$ is an all-ones vector of dimension $n$. So, Theorem 2.3 can be rewritten as follows.

**Corollary 2.4.** Let $G$ be a connected $m$-uniform hypergraph. Then $G$ is spectral $\ell$-symmetric if and only if the equation

$$B_Gx = \frac{m}{\ell}\mathbf{1} \text{ over } \mathbb{Z}_m\tag{2.3}$$

has a solution.
In Corollary 2.4 and other places of the paper, the number of coordinates of \( 1 \) is implicated from context, which is equal to the number of vertices of the hypergraph under discussion.

3. Cyclic index of generalized power hypergraphs

Let \( G \) be a \( t \)-uniform hypergraph, and let \( G^{m,s} \) be a generalized power hypergraph of \( G \), where \( 1 \leq s \leq \frac{m}{t} \). If \( m > st \), then each edge of \( G \) contains a vertex of degree 1, and hence \( G \) is a 1-hm bipartite hypergraph [19]. By [19, Theorem 3.2] or [2, Theorem 4.5], \( c(G^{m,s}) = m \).

So, in the following, we always assume that \( G \) is a connected \( t \)-uniform hypergraph, \( m = st \), namely, \( G^{m,s} \) is considered to be obtained from \( G \) by blowing each vertex \( v \) into an \( s \)-set \( v \) and preserving the adjacency relation. We also assume that the vertex \( v \) contains \( v \) for each \( v \in V(G) \).

Lemma 3.1. If \( G \) is spectral \( \ell \)-symmetric, then \( G^{m,s} \) is also spectral \( \ell \)-symmetric. In particular, \( G^{m,s} \) is spectral \( c(G) \)-symmetric and hence \( c(G) \mid c(G^{m,s}) \).

Proof. Suppose that \( G \) is spectral \( \ell \)-symmetric. By Corollary 2.4, the equation

\[
B_G x = \ell \cdot 1
\]

has a solution \( \phi \) over \( \mathbb{Z}_t \). Now define a map \( \Phi \) on \( G^{m,s} \) such that

\[
\Phi|_{v} = \phi(v)
\]

for each vertex \( v \in V(G) \). Then

\[
B_{G^{m,s}} \Phi = s \cdot B_G \phi = \frac{st}{\ell} \cdot 1 = \frac{m}{\ell} \cdot 1 \text{ over } \mathbb{Z}_m,
\]

which implies that \( G^{m,s} \) is spectral \( \ell \)-symmetric also by Corollary 2.4.

Lemma 3.2. \( G^{m,s} \) is spectral \( s \)-symmetric.

Proof. For each vertex \( v \in V(G) \), \( v \) is blowing into an \( s \)-set \( v \) of vertices of \( G^{m,s} \), and is assumed to be contained in \( v \). Define a map \( \Phi \) on \( G^{m,s} \) such that \( \Phi|_{v} = 1 \) and \( \Phi|_{v \setminus \{v\}} = 0 \) for each vertex \( v \in V(G) \). Then

\[
B_{G^{m,s}} \Phi = t \cdot 1 = \frac{m}{s} \cdot 1 \text{ over } \mathbb{Z}_m,
\]

which implies that \( G^{m,s} \) is spectral \( s \)-symmetric by Corollary 2.4.

Lemma 3.3. If \( G^{m,s} \) is spectral \( s \cdot \ell' \)-symmetric, then \( G \) is spectral \( \ell' \)-symmetric.

Proof. By Corollary 2.4 there exists a map \( \Phi \) defined on \( G^{m,s} \) such that

\[
B_{G^{m,s}} \Phi = \frac{m}{s \cdot \ell'} \cdot 1 = \frac{t}{\ell'} \cdot 1 \text{ over } \mathbb{Z}_m.
\]

Now define a map \( \phi \) on \( G \) such that \( \phi(v) = \sum_{u \in v} \Phi(u) \) for each \( v \in V(G) \). So we have

\[
B_G \phi = B_{G^{m,s}} \Phi = \frac{t}{\ell'} \cdot 1 \text{ over } \mathbb{Z}_m.
\]

As \( m \) is a multiple of \( t \),

\[
B_G \phi = \frac{t}{\ell'} \cdot 1 \text{ over } \mathbb{Z}_t,
\]

which implies that \( G \) is spectral \( \ell' \)-symmetric by Corollary 2.4.

By Lemma 3.2, we may assume \( c(G^{m,s}) = s \cdot \ell' \), where \( \ell' \) is a positive integer. By Lemma 3.3, we know that \( G \) is spectral \( \ell' \)-symmetric and hence \( \ell' \mid c(G) \) by [2, Lemma 2.7]. So we get the following result immediately.

Corollary 3.4. \( c(G^{m,s}) \mid s \cdot c(G) \).
Corollary 3.5. $G^{m,s}$ is spectral $\frac{s(c(G))}{(s,c(G))}$-symmetric. In particular, if $(s,c(G)) = 1$ or $(s,t) = 1$, then $c(G^{m,s}) = s \cdot c(G)$.

Proof. By Lemma 3.1 and Lemma 3.2 we know that $c(G)|c(G^{m,s})$ and $s|c(G^{m,s})$, implying that $\frac{s(c(G))}{(s,c(G))}c(G^{m,s})$. So, $G^{m,s}$ is spectral $\frac{s(c(G))}{(s,c(G))}$-symmetric. As $c(G)|t$, if $(s,t) = 1$, then $(s,c(G)) = 1$. If $(s,c(G)) = 1$, then $s \cdot c(G)|c(G^{m,s})$. Then result follows by Corollary 3.4.

By Corollary 3.5 Conjecture 1.2 holds in some special cases, including the case of $c(G) = 1$. However, Conjecture 1.2 does not hold in general. Now we give a counterexample to show the negative answer to the conjecture.

Definition 3.6 ([14]). Let $n \geq 16k$ and let partition $[n]$ into three sets $A, B, C$ such that $|A| \geq 6k$, $|B| \geq 6k$ and $|C| \geq 4k$. Define the four families of $4k$-subsets of $[n]$.

$$E_1 := \{e : e \subset [n], |e \cap A| = 2k, |e \cap C| = 2k\},$$
$$E_2 := \{e : e \subset [n], |e \cap B| = 2k, |e \cap C| = 2k\},$$
$$E_3 := \{e : e \subset [n], |e \cap A| = k, |e \cap B| = 3k\},$$
$$E_4 := \{e : e \subset [n], |e \cap A| = 3k, |e \cap B| = k\}.$$

Now define a $4k$-uniform hypergraph $G$ by setting $V(G) = [n]$ and $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$. We call $G$ a Nikiforov’s hypergraph as it is introduced by Nikiforov.

Nikiforov [14] showed that Nikiforov’s hypergraphs $G$ are odd-colorable, or $(4k, 2)$-colorable in terms our definition, by defining a function $\phi$ on $G$ such that $\phi|A = 1$, $\phi|B = 4k - 1$ and $\phi|C = 0$. By Theorem 2.3 $G$ is spectral $2$-symmetric.

By the following result, if $G$ is a Nikiforov’s hypergraph and $s$ is even, then

$$c(G^{m,s}) \neq s \cdot c(G).$$

So we give a negative answer to Conjecture 1.2

Theorem 3.7. Let $G$ be a $4k$-uniform Nikiforov’s hypergraph. Then the following results hold.

1. $c(G) = 2$.
2. If $s$ is even, then $c(G^{m,s}) = s$.

Proof. (1) We first show that $c(G) = 2$. Suppose that $G$ is spectral $\ell$-symmetric. Then there exists a $\phi : [n] \rightarrow [4k]$ such that $B_G\phi = \frac{4k}{\ell}$ over $Z_{4k}$. It is easily seen that $\phi$ is constant on each of $A, B, C$ by the equation. So, let $\phi|A := a$, $\phi|B := b$ and $\phi|C := c$. Then, by considering the edges in $E_1$, we have

$$2ka + 2kc = \frac{4k}{\ell} \mod 4k,$$

which implies that $\ell$ equals 1 or 2, and hence $c(G) = 2$ as $G$ is spectral $2$-symmetric.

2. By Corollary 3.4, $c(G^{m,s}) = 2s$, where $m = 4ks$. By Lemma 3.2 $G^{m,s}$ is spectral $s$-symmetric, and hence $s|c(G^{m,s})$. We will show that if $s$ is even, then $G^{m,s}$ is not spectral $2s$-symmetric so that $c(G^{m,s}) = s$.

Assume to the contrary that $G^{m,s}$ is spectral $2s$-symmetric. Then there exists a $\Phi : V(G^{m,s}) \rightarrow [4ks]$ such that

$$B_{G^{m,s}}\Phi = \frac{4ks}{2s} = 2k \mod Z_{4ks}.$$
For each \( v \in V(G) \), define \( \phi(v) := \sum_{u \in v} \Phi(u) \). So we have
\[
B_{G^{m,s}} \Phi = B_G \phi = 2k \text{ over } \mathbb{Z}_{4k}. 
\]
It is also easily seen that \( \phi|_A := \alpha, \phi|_B := \beta \) and \( \phi|_C := \iota \). By considering the edges in \( E_3 \) and \( E_4 \) respectively, we have
\[
\alpha + 3\beta = 2 \mod 4s, \quad 3\alpha + \beta = 2 \mod 4s.
\]
So
\[
\alpha - \beta = 0 \mod 2s, \quad \alpha + \beta = 1 \mod s,
\]
which yields a contradiction as \( s \) is an even number. \( \square \)

Finally we give an equivalent characterization of Eq. (1.2) in Conjecture 1.2.

**Theorem 3.8.** \( c(G^{m,s}) = s \cdot c(G) \) if and only if the equation
\[
(3.1) \quad B_G x = \frac{t}{c(G)} 1 \text{ over } \mathbb{Z}_m
\]
has a solution.

**Proof.** Suppose that \( c(G^{m,s}) = s \cdot c(G) \). Then \( G^{m,s} \) is spectral \( s \cdot c(G) \)-symmetric, and by Corollary 2.4 there exists a map \( \Phi : V(G^{m,s}) \rightarrow [m] \) such that
\[
B_{G^{m,s}} \Phi = \frac{m}{s \cdot c(G)} 1 = \frac{t}{c(G)} 1 \text{ over } \mathbb{Z}_m.
\]
For each \( v \in V(G) \), define \( \phi(v) := \sum_{u \in v} \Phi(u) \). So we have \( B_{G^{m,s}} \Phi = B_G \phi \), and get the necessity.

On the other hand, if \( B_G x = \frac{t}{c(G)} 1 \) has a solution \( \phi \) over \( \mathbb{Z}_m \). Define a map \( \Psi : V(G^{m,s}) \rightarrow [m] \) such that
\[
\sum_{u \in v} \Phi(u) = \phi(v), \text{ for each } v \in V(G).
\]
There are \( |V(G)| \) independent linear equations; such \( \Phi \) is easily got (e.g. for each \( v \in V(G) \), take \( \Phi(v) = \phi(v) \) and \( \Phi(u) = 0 \) for each \( u \in v \setminus \{v\} \)). So we have
\[
B_{G^{m,s}} \Phi = B_G \phi = \frac{t}{c(G)} 1 = \frac{m}{s \cdot c(G)} 1 \text{ over } \mathbb{Z}_m.
\]
So \( G^{m,s} \) is spectral \( s \cdot c(G) \)-symmetric. The sufficiency follows by Corollary 3.4. \( \square \)

As \( G \) is spectral \( c(G) \)-symmetric, by Corollary 2.4, the equation
\[
(3.2) \quad B_G x = \frac{t}{c(G)} 1 \text{ over } \mathbb{Z}_t
\]
has a solution. Obviously, if the equation (3.1) has a solution, then the equation (3.2) has a solution as \( m \) is a multiple of \( t \). However, the converse does not hold in general; see the previous counterexample.
4. Remark

For a nonnegative weakly irreducible tensor \( A \), its cyclic index \( c(A) \) is exactly the number of eigenvalues with modulus \( \rho(A) \). The is implied by Perron-Frobenius theorem for nonnegative tensors, where an eigenvalue of \( A \) is called \( H^+ \)-eigenvalue (respectively \( H^{++} \)-eigenvalue) if it is associated with a nonnegative (respectively positive) eigenvector. For the notion of irreducible or weakly irreducible tensors, one can refer to \([1]\) and \([7]\). It is known that the adjacency tensor of a uniform hypergraph \( G \) is weakly irreducible if and only if \( G \) is connected \([15, 22]\).

**Theorem 4.1** (The Perron-Frobenius Theorem for nonnegative tensors).

1. (Yang and Yang \([22]\)) If \( A \) is a nonnegative tensor, then \( \rho(A) \) is an \( H^+ \)-eigenvalue of \( A \).

2. (Friedland, Gaubert and Han \([7]\)) If furthermore \( A \) is weakly irreducible, then \( \rho(A) \) is the unique \( H^{++} \)-eigenvalue of \( A \), with a unique positive eigenvector, up to a positive scalar.

3. (Chang, Pearson and Zhang \([1]\)) If moreover \( A \) is irreducible, then \( \rho(A) \) is the unique \( H^+ \)-eigenvalue of \( A \), with a unique nonnegative eigenvector, up to a positive scalar.

According to the definition of tensor product in \([18]\), for a tensor \( A \) of order \( m \) and dimension \( n \), and two diagonal matrices \( P, Q \) both of dimension \( n \), the product \( PAQ \) has the same order and dimension as \( A \), whose entries are defined by

\[
(PAQ)_{i_1i_2\ldots i_m} = p_{i_1}a_{i_1i_2\ldots i_m}q_{i_2i_2\ldots q_{i_1i_1}}.
\]

If \( P = Q^{-1} \), then \( A \) and \( P^{m-1}AQ \) are called diagonal similar. It is proved that two diagonal similar tensors have the same spectrum \([18]\).

**Theorem 4.2** (\([22]\)). Let \( A \) and \( B \) be two \( m \)-th order \( n \)-dimensional real tensors with \( |B| \leq A \), namely, \( |b_{i_1i_2\ldots i_m}| \leq a_{i_1i_2\ldots i_m} \) for each \( i_j \in [n] \) and \( j \in [m] \). Then

1. \( \rho(B) \leq \rho(A) \).

2. Furthermore, if \( A \) is weakly irreducible and \( \rho(B) = \rho(A) \), where \( \lambda = \rho(A)e^{i\theta} \) is an eigenvalue of \( B \) corresponding to an eigenvector \( y \), then \( y \) contains no zero entries, and \( B = e^{-i\theta}D^{-(m-1)}AD \), where \( D = \text{diag}\{\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|}\} \).

**Theorem 4.3** (\([22]\)). Let \( A \) be an \( m \)-th order \( n \)-dimensional weakly irreducible nonnegative tensor. Suppose \( A \) has \( k \) distinct eigenvalues with modulus \( \rho(A) \) in total. Then these eigenvalues are \( \rho(A)e^{i\frac{2\pi j}{k}} \), \( j = 0, 1, \ldots, k - 1 \). Furthermore,

\[
A = e^{-i\frac{2\pi j}{k}}D^{-(m-1)}AD,
\]

and the spectrum of \( A \) remains invariant under a rotation of angle \( \frac{2\pi j}{k} \) (but not a smaller positive angle) of the complex plane.

Suppose \( A \) be as in Theorem 4.3. If \( \text{Spec}(A) \) is invariant under a rotation of angle \( \theta \) of the complex plane, i.e. \( \text{Spec}(A) = e^{i\theta}\text{Spec}(A) \), then \( \rho(A)e^{i\theta} \) is an eigenvalue of \( A \) by Theorem 4.1. By Theorem 4.3 \( \theta = \frac{2\pi j}{k} \) for some \( j \in [k] \), and hence by Theorem 4.2 (and taking \( B = A \)), \( \text{Spec}(A) = e^{i\frac{2\pi j}{k}}\text{Spec}(A) \). So, for some positive integer \( \ell, \ell | k \),

\[
\text{Spec}(A) = e^{i\frac{2\pi j}{k}}\text{Spec}(A).
\]

The number \( k \) in Theorem 4.3 is exactly the cyclic index of \( A \). In addition, if \( A \) is spectral \( \ell \)-symmetric, Then \( \ell \mid c(A) \) by Theorem 4.3.
Now return to a connected $t$-uniform hypergraph $G$ and its power $G_{m,s}$, where $m = st$. By Lemma 3.1, $G_{m,s}$ is spectral $c(G)$-symmetric; and by Lemma 3.2, $G_{m,s}$ is also spectral $s$-symmetric. So $G_{m,s}$ has eigenvalues

$$\rho(G_{m,s})e^{i\frac{2\pi ji}{s \cdot c(G)}}, \ i \in [c(G)], \ j \in [s].$$

In particular, $\rho(G_{m,s})e^{i\frac{2\pi}{d}}$ is an eigenvalue of $G_{m,s}$, where $d = \frac{s \cdot c(G)}{s \cdot c(G)}$. So by Theorem 4.2, $G_{m,s}$ is spectral $d$-symmetric, which is consistent with Corollary 3.5.

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