Off-shell renormalization of
the abelian Higgs model in the unitary gauge

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Abstract

We discuss the off-shell renormalization properties of the abelian Higgs model in the unitary gauge. The model is not renormalizable according to the usual power counting rules. In this paper, however, we show that with a proper choice of interpolating fields for the massive photon and the Higgs particle, their off-shell Green functions can be renormalized. An analysis of the nature of the extra singularities in the unitary gauge is given, and a recipe for the off-shell renormalization is provided.

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1 Introduction

The choice of the unitary gauge for Higgs theories is usually dismissed on account of the non-renormalizability due to power counting. This is not a happy situation since, first of all, the unitary gauge is physically a nice gauge to work with; it is devoid of gauge degrees of freedom, and only the physical degrees of freedom are retained. It is difficult to dismiss this gauge. We also believe that the unitary gauge should make sense physically, for it is formally equivalent to any other manifestly renormalizable gauges. What is most troubling is that we have no clear idea of what is causing the extra divergences which are unique to the unitary gauge. It has been shown that the unitary gauge can be used for the loop calculation of physical quantities such as the S-matrix elements [1]. Nevertheless, the situation is still not satisfying, because the cancellations of the extra UV divergences occur miraculously as we take the mass shell limit of the Green functions. What is missing is a good understanding of the nature of the extra UV divergences in the unitary gauge.

The purpose of this paper is to provide a good understanding of the renormalization properties of the unitary gauge. Our conclusion is that the unitary gauge is as renormalizable as the renormalizable gauges. There are two causes for the extra divergences in the unitary gauge. The first is an improper choice of the interpolating fields of the elementary particles. The second is the compositeness of the interpolating fields. The first implies that the simplest choice of the fields for the massive gauge boson and Higgs particle are not “good” local fields. Here, a good local field has a well defined scale dimension, and it mixes with a finite number of other local fields under the renormalization group transformations. The second implies that the off-shell Green functions are UV finite only in the coordinate space for all distinct points. The proper interpolating fields of the massive photon and Higgs particle turn out to be composite fields with scale dimensions 2 or higher, and when two or more of them coincide in space, a non-integrable singularity is generated. The Fourier transform of the Green function thus acquires additional UV singularities which cannot be removed by multiplicative renormalization of the fields. These singularities do not affect the S-matrix, since only the asymptotic limit of infinite spatial separation of the interpolating fields is relevant for the S-matrix. The same singularities would appear even in the renormalizable gauges if we chose higher dimensional composite fields as the interpolating fields.

This paper is an extension of the previous paper [2] in which the off-shell renormalizability of the massive QED was explained. The abelian Higgs
model, discussed in this paper, is more non-trivial due to the presence of the Higgs particle. A further extension to the non-abelian Higgs theories is left for future. The organization of the present paper is as follows. In sect. 2, we review the relation of the unitary gauge to the other renormalizable gauges. In sect. 3, concrete examples of the 1-loop UV divergences in the unitary gauge are given, and possible remedies for removing them are briefly indicated. In sect. 4, following the observations made in sect. 3, we discuss the proper choice of interpolating fields. In sect. 5, for completeness of the paper, we remind the reader of the renormalization conditions suitable for the unitary gauge. Then, in sect. 6, we analyze the structure of the UV divergences of the Green functions. Sects. 3, 4, and 6 constitute the main part of this paper. We briefly summarize how to calculate the S-matrix elements in the unitary gauge in sect. 7 before we conclude the paper in sect. 8. Two appendices are given to give more details of the derivations in the main text.

Throughout the paper we use the euclidean metric and dimensional regularization for the $D \equiv 4 - \epsilon$ dimensional space.

2 Review — equivalence of the unitary gauge with the renormalizable gauges

As was first emphasized in ref. [3] and more recently in ref. [4], the unitary gauge is not strictly speaking a choice of gauge. It is instead a choice of field variables (polar variables in the language of ref. [4]) which separate the gauge independent degrees of freedom from the gauge dependent ones. In this section we review the relation of the unitary gauge to the renormalizable $R_\xi$ gauge [5]. Our presentation is based upon the old analyses made in refs. [6, 3]. Our summary is somewhat simpler thanks to the use of dimensional regularization.

The gauge invariant quantities can be calculated in any gauge, either in the unitary gauge or in any renormalizable gauge such as the $R_\xi$ gauge. It is the purpose of this section to give a formal proof of this statement. (See also [4] for a more perturbative derivation.)

The lagrangian of the abelian Higgs model is given by

$$\mathcal{L}_\xi = \frac{1}{4} F^2 + (\partial_\mu \phi^* + ie A_\mu \phi^*) (\partial_\mu \phi - ie A_\mu \phi) + \frac{\lambda}{4} \left( |\phi|^2 - \frac{v^2}{2} \right)^2$$

$$+ \frac{1}{2\xi} (\partial \cdot A - \xi e v \chi)^2 + \partial_\mu \bar{c} \partial_\mu c + \xi e^2 v' \bar{c} c$$

(1)
where the real fields $\rho', \chi$ are defined by $\phi \equiv \frac{1}{\sqrt{2}}(\rho' + i\chi)$, and $c, \bar{c}$ are the FP ghosts. This lagrangian is invariant under the BRST transformation defined by

$$
\begin{align*}
\delta_\eta A_\mu &= \eta \partial_\mu c \\
\delta_\eta \phi &= i\epsilon_\eta c \phi, \quad \delta_\eta \phi^* = -i\epsilon_\eta c \phi^* \\
\delta_\eta c &= 0, \quad \delta_\eta \bar{c} = \frac{1}{\zeta} \eta (\partial \cdot A - \xi ev \chi)
\end{align*}
$$

where $\eta$ is an arbitrary Grassmann number.

To go to the unitary gauge, we make the following change of variables:

$$
B_\mu \equiv A_\mu - \partial_\mu \varphi, \quad \frac{1}{\sqrt{2}} \rho e^{i\varphi} \equiv \phi
$$

What is nice about the dimensional regularization is that the jacobian of the change of variables is unity. Hence, the lagrangian for the new fields is obtained simply by substituting (3) into the lagrangian (1):

$$
L_\xi = L_U + \Delta L
$$

where $L_U$ is the lagrangian for the unitary gauge

$$
L_U = \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2} \left( (\partial_\mu \rho)^2 + e^2 B_\mu^2 \rho^2 \right) + \frac{\lambda}{16} (\rho^2 - v^2)^2
$$

and

$$
\Delta L = \frac{1}{2\xi e^2} \left( \epsilon \partial \cdot B + \partial^2 \varphi - \xi e^2 v \rho \sin \varphi \right)^2 + \partial_\mu \bar{c} \partial_\mu c + \xi e^2 v \rho \cos \varphi \bar{c} c
$$

Since $L_U$ is gauge invariant (hence BRST invariant), the difference $\Delta L = L_\xi - L_U$ must be also BRST invariant. Since the gauge invariant quantities do not depend on $\varphi, c,$ and $\bar{c}$, we can integrate them out first. We now show that

$$
Z \equiv \int [d\varphi d\bar{c} d\rho] e^{-\int \Delta L}
$$

is independent of $\rho, B_\mu$. To show this we compute the functional derivatives $\frac{\delta Z}{\delta \rho(x)}, \frac{\delta Z}{\delta B_\mu(x)}$. First, we find

$$
\left. \eta \frac{\delta Z}{\delta \rho(x)} \right|_{\rho(x)} = \int [d\varphi d\bar{c} d\rho] \xi ev \delta_\eta (\bar{c} \sin \varphi) \cdot e^{-\int \Delta L} = 0
$$
since $\delta \eta \delta L = 0$, and the measure of integration is BRST invariant. Similarly, we find

$$\eta \frac{\delta Z}{\delta B_\mu(x)} = \int [d\varphi d\psi \bar{c}] \partial_\mu \delta \eta \bar{c} \cdot e^{-\int \Delta L} = 0 \quad (9)$$

Thus, $Z$ is independent of $\rho, B_\mu$, and we obtain the desired equality

$$\int [dB_\mu d\rho d\varphi d\psi \bar{c}] \ B_\mu ... \rho ... e^{-\int \mathcal{L}} = \int [dB_\mu d\rho] \ B_\mu ... \rho ... e^{-\int \mathcal{L}_U} \quad (10)$$

For those gauge invariant quantities that depend only on the gauge invariant fields $B_\mu, \rho$, we can use either the $R_\xi$ gauge or the unitary gauge. We get the same result. Of course, this must be true if we recall how the lagrangian of the $R_\xi$ gauge can be obtained using the method of Faddeev and Popov\footnote{Note that $(\lambda v^2)_0 \neq \lambda_0 v^2_0$. The shift $v_0$ can be chosen arbitrarily.}. Alternatively the equivalence can be shown by taking the formal limit of $\xi \to \infty$. The advantage of the above proof is that it is consistent with regularization, and hence the proof can incorporate renormalization.

3 Examples of the UV divergences in the unitary gauge

To make our discussions concrete, we show four examples of 1-loop calculations in the unitary gauge. We renormalize only the parameters of the model and leave all the fields unrenormalized:

$$\epsilon_0^2 = \epsilon^2 Z^3, \quad (\lambda v^2)_0 = Z_m \lambda v^2, \quad \lambda_0 = Z \lambda + z \quad (11)$$

where the parameters with suffix 0 are bare parameters, and $z$ depends only on $\epsilon^2$ but not on $\lambda$.

Shifting $\rho$ by an arbitrary constant $v_0 = Z_m v$, i.e., $\rho \to \rho + v_0$, we obtain the following lagrangian

$$\mathcal{L}_U = \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2} \left( (\partial_\mu \rho)^2 + \frac{\epsilon^2}{Z^3} B_\mu^2 (Z_m v + \rho)^2 \right) - \frac{1}{8} Z_m \lambda v^2 (Z_m v + \rho)^2 + \frac{1}{16} (Z \lambda + z) (Z_m v + \rho)^4 \quad (12)$$

The renormalization of the parameters is gauge invariant, and we can choose the renormalization constants the same as in any renormalizable gauge:

$$Z^3 = 1 + \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left( \frac{\epsilon^2}{3} \right) \quad (13)$$
\[ Z_m = 1 + \frac{1}{(4\pi)^2} \frac{2}{\epsilon} (\lambda - 3e^2) \]  
\[ Z_\lambda = 1 + \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left( \frac{5\lambda}{2} - 6e^2 \right), \quad z = \frac{1}{(4\pi)^2} \frac{2}{\epsilon} (12e^4) \]  
(14)  
(15)

The constant \( Z_v \) is chosen for convenience so that the tadpole vanishes, \( \langle \rho \rangle = 0 \):

\[ Z_v = 1 + \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \left( \frac{3e^2}{2} \right) + \text{(finite)} \]  
(16)

We consider four Green functions: the photon propagator \( \langle B_\mu B_\nu \rangle \), the Higgs propagator \( \langle \rho \rho \rangle \), and then two three-point functions \( \langle \rho B_\mu B_\nu \rangle, \langle \rho \rho \rho \rangle \).

The following UV divergences are found at 1-loop:

\[ \text{UV} \, \langle B_\mu (k) B_\nu \rangle = \ln \Lambda \left[ \frac{e^2}{3} \langle B_\mu (k) B_\nu \rangle + \frac{1}{e^2 v^4} k_\mu k_\nu \right] \]  
(17)

\[ \text{UV} \, \langle \rho (p) \rho \rangle = \ln \Lambda \left[ \left( 3e^2 - \frac{\lambda}{2} \right) \langle \rho (p) \rho \rangle + \frac{1}{2v^2} \right] \]  
(18)

\[ \text{UV} \, \langle \rho \, B_\mu (k_1) B_\nu (k_2) \rangle = \ln \Lambda \left[ \left( -\frac{e^2}{3} + \frac{1}{2} \left( 3e^2 - \frac{\lambda}{2} \right) \right) \langle \rho \, B_\mu B_\nu \rangle \right. 
\left. + \frac{4}{e^2 v^5} k_1 \mu k_2 \nu \langle \rho \, \rho (k_1 + k_2) \rangle \right] \]  
(19)

\[ \text{UV} \, \langle \rho (k_1) \, \rho (k_2) \, \rho \rangle = \ln \Lambda \left[ \frac{3}{2} \left( 3e^2 - \frac{\lambda}{2} \right) \langle \rho \, \rho \, \rho \rangle \right. 
\left. + \frac{\lambda}{4} \left( \langle \rho \, \rho^2 \, \rho \rangle \right) + \langle \rho \, \rho^2 \, \rho \rangle + \langle \rho \, \rho^2 \, \rho \rangle \right) 
- \frac{1}{v^3} \left( \langle \rho (k_1) \, \rho \rangle + \langle \rho (k_2) \, \rho \rangle + \langle \rho (k_1 + k_2) \, \rho \rangle \right) \right] \]  
(20)

where \( \text{UV} \) denotes that we take only the UV singular part, and \( \ln \Lambda \equiv \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \).

Clearly, multiplicative renormalization of the fields removes the first term of singularity for each Green function. But after the wave function renormalization we are still left with the following UV divergences:

\[ \text{UV} \, \langle B_\mu (k) B_\nu \rangle = \ln \Lambda \frac{1}{e^2 v^4} k_\mu k_\nu \]  
(21)

\[ \text{UV} \, \langle \rho (p) \rho \rangle = \ln \Lambda \frac{1}{2v^2} \]  
(22)

\footnote{Note \( \langle \rho \, \rho \, B_\alpha \rangle = 0 \) and \( \langle B_\alpha B_\beta B_\gamma \rangle = 0 \) due to the charge conjugation invariance.}
\[ UV \langle \rho \, B_\mu (k_1) B_\nu (k_2) \rangle = \ln \Lambda \frac{4}{e^2 v^5} k_1 k_2 \langle \rho \, \rho (k_1 + k_2) \rangle \] (23)

\[ UV \langle \rho (k_1) \, \rho (k_2) \, \rho \rangle = \ln \Lambda \left[ \frac{\lambda}{4} \left( \langle \rho^2/2v \, \rho \rangle + \langle \rho \, \rho^2/2v \rangle \right) \right] - \frac{1}{v^3} \left( \langle \rho (k_1) \, \rho \rangle + \langle \rho (k_2) \, \rho \rangle + \langle \rho (k_1 + k_2) \, \rho \rangle \right) \] (24)

The local divergences in Eqs. (21,22) imply that the scale dimensions of \( B_\mu \), \( \rho \) are actually 3 and 2, respectively, to the contrary to the naively expected scale dimension 1 for both fields.\footnote{The tree-level propagator of \( B_\mu \) already implies 2 for its scale dimension. This is the very cause of non-renormalizability by power counting.}

The non-local divergence given by Eq. (23) is at first sight difficult to remove. From the scale dimension 3 of \( B_\mu \) found above, we expect the following operator product expansion (OPE) in coordinate space

\[ B_\mu (x) B_\nu (0) \sim \frac{1}{x^4} \left( \delta_{\mu \nu} - 4 \frac{x_\mu x_\nu}{x^2} \right) \frac{1}{e^2 v^3} \rho (0) \] (25)

but this gives a singularity of the form

\[ \ln \Lambda \, \delta_{\mu \nu} \frac{1}{e^2 v^3} \langle \rho \, \rho (k_1 + k_2) \rangle \] (26)

which is different from the singularity of Eq. (23). The same singularity is actually contained in the Green function with \( B_\mu \) replaced by \( B_\mu \rho^2 \):

\[ UV \left[ \langle \rho \left( B_\mu \rho^2 \right) (k_1) B_\nu (k_2) \rangle + \rho \langle B_\mu (k_1) \left( B_\nu \rho^2 \right) (k_2) \rangle \right] = \ln \Lambda \left[ - \frac{20e^2}{3} \langle \rho \, B_\mu \, B_\nu \rangle - \frac{11e^2}{3} \left( \langle \rho \left( B_\mu \rho^2 \right) B_\nu \rangle + \rho \langle B_\mu \left( B_\nu \rho^2 \right) \rangle \right) \right] - \frac{8}{e^2 v^5} k_1 k_2 \langle \rho \, \rho (k_1 + k_2) \rangle \] (27)

This suggests that we must consider a linear combination of \( B_\mu \) and \( B_\mu \rho^2 \) for renormalization.

Out of the two singularities given by Eq. (24), the first implies that \( \rho \) mixes with \( \frac{1}{v^2} \) under renormalization. The second implies also the necessity of considering a linear combination of \( \rho \) and \( \frac{\rho^2}{2v} \). This is because the Green
function of $\frac{\rho^2}{2v}$ has the singularity

$$UV \left[ \langle \rho^2 \rho \rangle + \langle \rho^2 \rho \rangle + \langle \rho^2 \rho \rangle \right]$$

$$= \ln \Lambda \left[ -\frac{9\lambda}{4} \langle \rho \rho \rangle - \frac{3\lambda}{4} \left( \langle \rho^2 \rho \rangle + \langle \rho^2 \rho \rangle + \langle \rho^2 \rho \rangle \right) ight.$$  
$$+ \frac{1}{v^3} (\langle \rho(k_1) \rangle + \langle \rho(k_2) \rangle + \langle \rho(k_1 + k_2) \rangle) \right] \tag{28}$$

and the last singularity can cancel the second singularity of Eq. (24).

We are not quite done yet with renormalizing the fields $B_\mu$ and $\rho$, but we have obtained enough clues to the right way of doing renormalization. To summarize the preliminary consideration in this section, we have found that the scale dimensions of $B_\mu$ and $\rho$ are 3 and 2, respectively, and that for renormalization we are compelled to consider a linear combination of $B_\mu$ and $B_\mu \frac{\rho}{v}$ for the massive photon and that of $\rho$ and $\frac{\rho^2}{2v}$ for the Higgs.

## 4 Choice of interpolating fields

Naively, both $B_\mu$ and $\rho$ are elementary fields of scale dimension 1. But our observations in the previous section indicate to the contrary: the correct scale dimensions of $B_\mu$ and $\rho$ are 3 and 2, respectively. Furthermore, the fields $B_\mu$ and $\rho$ cannot be renormalized multiplicatively, and we must consider a linear combination of $B_\mu$ and $B_\mu \frac{\rho}{v}$ for the massive photon and that of $\rho$ and $\frac{\rho^2}{2v}$ for the Higgs. To understand the necessity of considering such linear combinations, let us consider $\rho$ further. In terms of the original field variables $\phi$ and $\phi^*$, $\rho$ is given as

$$\rho = -v_0 + \sqrt{2\phi^* \phi} \tag{29}$$

Taking $\phi^* \phi - \frac{v_0^2}{2}$ as a small field, we can expand

$$\rho = v_0 \left( \frac{1}{2} \left( \frac{2\phi^* \phi}{v_0^2} - 1 \right) - \frac{1}{8} \left( \frac{2\phi^* \phi}{v_0^2} - 1 \right)^2 + ... \right) \tag{30}$$

Hence, in terms of $\phi^* \phi$, $\rho$ is obtained as an infinite series. The field $\phi^* \phi$ translates to a well defined local field of scale dimension 2 upon quantization, but $\rho$ does not. $\rho$ is a linear combination of an infinite number of local fields.
\( \phi^* \phi \), and thus it will not have a well defined scale dimension. It is simply impossible to quantize \( \rho \).

In the unitary gauge, one might naively expect that any local polynomials of \( \rho \) and \( B_\mu \) are “good” local fields with definite scale dimensions. But we have found this is not the case. “Good” fields are local polynomials in terms of the original field variables \( A_\mu, \phi, \) and \( \phi^* \). Neither \( \rho \) nor \( B_\mu \) meets this criterion. To avoid inconvenient mixing with lower dimensional fields, we must choose interpolating fields of the smallest scale dimensions. Hence, the interpolating field of the Higgs particle must be a gauge invariant scalar of the smallest dimension which is invariant under charge conjugation. Similarly, the interpolating field of the massive photon must be a gauge invariant vector of the smallest dimension that flips sign under charge conjugation. Such fields are unique, and are readily given by \( \phi^* \phi \) for the Higgs and the conserved charge current \( J_\mu \equiv \phi^* i (D_\mu - \bar{D}_\mu) \phi \) for the massive photon, where \( D_\mu \) is the covariant derivative. These interpolating fields have been considered to explain the absence of any qualitative difference between the confining phase and Higgs phase of the standard model [8].

To summarize the above discussion, we have found that the interpolating field of the massive photon is given by \( A_\mu \), where

\[
e_0 v_0^2 A_\mu \equiv J_\mu \equiv \phi^* i (D_\mu - \bar{D}_\mu) \phi = e_0 v_0^2 \cdot B_\mu \left( 1 + \frac{\rho}{v_0} \right)^2
\]

(31)

The interpolating field of the Higgs is given by \( \Phi \), where

\[
v_0 \Phi \equiv \phi^* \phi - \frac{v_0^2}{2} = v_0 \cdot \rho \left( 1 + \frac{\rho}{2v_0} \right)
\]

(32)

Already at this point it is obvious why the off-shell Green functions of \( A_\mu \) and \( \Phi \) should be renormalizable. According to the equivalence of the unitary gauge to any renormalizable gauge (reviewed in sect. 2), the Green functions of \( B_\mu \) and \( \rho \) are the same in the unitary and renormalizable gauges. In the renormalizable gauges, \( A_\mu \) and \( \Phi \) are renormalizable multiplicatively.\(^4\)

Hence, also in the unitary gauge, \( A_\mu \) and \( \Phi \) are renormalizable.

Whenever fields of dimension 2 or higher are involved, the UV singularities of the Green functions are not completely UV finite for a good reason.

\(^4\)The existence of these gauge invariant interpolating fields is essential for understanding the physical meaning of the Coleman-Weinberg effect \([9]\) or equivalently the first-order nature of the BCS transition \([10]\).

\(^5\)Strictly speaking \( \Phi \) mixes with the identity field. In the renormalizable gauges \( A_\mu \) and \( \Phi \) will also mix with BRST trivial composite fields.
All the Green functions are UV finite in coordinate space for distinct space points, but when two or more fields coincide in space, we find unintegrable singularities which give rise to UV divergences in the Fourier transform of the Green functions. The situation is the same in any renormalizable gauge. In sect. 6 we determine the structure of the expected UV divergences.

5 Renormalization conditions

This section is meant only for completeness. The renormalization constants for the parameters are introduced as in Eqs. (11):

\[
e_0^2 = \frac{e^2}{Z_3}, \quad (\lambda v^2)_0 = Z_m \lambda v^2, \quad \lambda_0 = Z_\lambda \lambda + z
\]  

where \( e^2, \lambda v^2, \) and \( \lambda \) are renormalized parameters.

There are many ways of renormalizing the theory. Let us describe only two examples here. First, in the MS scheme the renormalization constants \( Z_3 - 1, Z_m - 1, Z_\lambda - 1, \) and \( z \) have only the pole parts in \( \epsilon. \) Second, in the physical renormalization scheme, the renormalization constants are defined by the following three conditions:

- \( m^2 \equiv e^2 v^2 \) is the physical photon mass squared. Namely, the propagator \( \langle A_\mu(k)A_\nu \rangle \) has a pole at \( k^2 = -m^2. \)
- \( m_H^2 \equiv \frac{1}{2} \lambda v^2 \) is the physical Higgs mass squared. Namely, the propagator \( \langle \Phi(p)\Phi \rangle \) has a pole at \( p^2 = -m_H^2. \) (If \( m_H^2 > 4m^2 \) and the Higgs is unstable, we choose \( m_H^2 \) as the center of a smeared pole.)
- The forward scattering amplitude of two Higgs bosons at zero kinetic energy is \( 6\lambda. \)

There are many substitutes for the third condition. We may alternatively specify a value of the forward scattering amplitude of two photons at zero energy.

6 General analysis of the structure of the Green functions

In this section we first study the multiplicative renormalization properties of the interpolating fields \( A_\mu \) and \( \Phi. \) Then, we analyze the transverse nature of the Green functions involving \( A_\mu. \) Finally, using scaling arguments, we
determine the UV divergences of the Green functions of $A_\mu$ and $\Phi$ that are left after multiplicative renormalization.

6.1 Renormalization of the interpolating fields

We first consider the renormalized conserved current $J_{r,\mu}$ and the renormalized composite field $(\phi^*\phi)_r$. Once they are given, the renormalized $A_{r,\mu}$ and $\Phi_r$ are defined by

$$ ev^2 A_{r,\mu} = J_{r,\mu}, \quad v\Phi_r = (\phi^*\phi)_r - \frac{v^2}{2} $$

(34)

The renormalization of the conserved current $J_\mu$ is well known. In terms of the bare fields the equation of motion is given by

$$ \partial_\nu F_{\nu\mu} = e_0 J_\mu $$

(35)

where $F_{\nu\mu} \equiv \partial_\nu B_\mu - \partial_\mu B_\nu$ is the usual field strength. Since the field strength is renormalized by $\frac{1}{\sqrt{Z_3}} F_{\nu\mu}$, the current is renormalized by

$$ J_{r,\mu} \equiv \frac{1}{Z_3} J_\mu $$

(36)

To renormalize $\phi^*\phi$, we recall it is conjugate to the mass parameter $(\lambda v^2)_0$, i.e., it is given by the derivative of the lagrangian with respect to the parameter:

$$ \phi^*\phi = -4 \frac{\partial L_U}{\partial (\lambda v^2)_0} $$

(37)

Therefore, the renormalized $\phi^*\phi$ is obtained by replacing the bare parameter $(\lambda v^2)_0$ by the renormalized $\lambda v^2$. Hence, the renormalization is done by

$$ (\phi^*\phi)_r = Z_m \phi^*\phi $$

(38)

Actually this is not quite correct, since $\phi^*\phi$ mixes with the identity field under renormalization. The correct prescription is given by

$$ (\phi^*\phi)_r = Z_m \phi^*\phi + z_S m_H^2 $$

(39)

where $z_S$ is a UV divergent constant.

We note that the renormalized $A_{r,\mu}$ and $\Phi_r$ are independent of the constant shift $v_0 = Z_v v$ of the Higgs field $\rho$. However, both in practical loop calculations and in formal diagrammatic studies, it is extremely convenient to have the tadpole vanishing. Therefore, we choose $Z_v$ so that $\langle \rho \rangle = 0$. 10
To summarize, the renormalized $A_{r,\mu}$ and $\Phi_r$ are defined by

$$A_{r,\mu} \equiv \frac{1}{ev^2} \frac{1}{Z_3} J_\mu = \frac{1}{Z_3} \frac{1}{v^2} B_\mu (v_0 + \rho)^2 \quad (40)$$

$$\Phi_r \equiv \frac{1}{v} \left( Z_m \phi^* \phi + z_S m_H^2 - \frac{v^2}{2} \right)$$

$$= \frac{1}{v} \left( Z_m \frac{1}{2} (v_0 + \rho)^2 + z_S m_H^2 - \frac{v^2}{2} \right) \quad (41)$$

6.2 Transverse nature of the Green functions

We consider the Green functions of $\partial_\mu J_\mu$ to understand the transverse nature of the Green functions. This is nothing but the Ward identities except that in renormalizable gauges they are derived for the Green functions of charged fields. Here, the Ward identities are derived for the charge neutral (gauge invariant) fields. In the unitary gauge the Feynman rules are simple, and the Ward identities are most easily derived using Feynman diagrams. Here only the results are summarized, leaving a sketch of derivations in Appendix A. All the Ward identities are given for the bare fields.

We obtain the following Ward identities:

$$k_\mu \langle J_\mu (k) J_\nu \rangle = 2k_\nu \langle \phi^* \phi \rangle \quad (42)$$

$$k_{1\mu} \langle J_\mu (k_1) J_\nu (k_2) (\phi^* \phi) \rangle = 2k_{1\nu} \langle (\phi^* \phi)(k_1 + k_1)(\phi^* \phi) \rangle \quad (43)$$

$$k_{1\mu} \langle J_\mu (k_1) J_\nu (k_2) (\phi^* \phi) \rangle = 2k_{1\nu} \langle (\phi^* \phi)(k_1 + k_2) (\phi^* \phi) (\phi^* \phi) \rangle \quad (44)$$

$$k_{1\alpha} \langle J_\alpha (k_1) J_\beta (k_2) J_\gamma (k_3) J_\delta \rangle$$

$$= 2 \left[ k_{1\beta} \langle (\phi^* \phi)(k_1 + k_2) J_\gamma (k_3) J_\delta \rangle + \text{permutations} \right] \quad (45)$$

These imply the following structure:

$$\langle J_\mu (k) J_\nu \rangle = T^{(1)}_{\mu\nu} (k) + 2\delta_{\mu\nu} \langle \phi^* \phi \rangle \quad (46)$$

$$\langle J_\mu (k_1) J_\nu (k_2) (\phi^* \phi) \rangle = T^{(2)}_{\mu\nu} (k_1, k_2) + 2\delta_{\mu\nu} \langle (\phi^* \phi)(k_1 + k_2)(\phi^* \phi) \rangle \quad (47)$$

$$\langle J_\mu (k_1) J_\nu (k_2) (\phi^* \phi)(p)(\phi^* \phi) \rangle = T^{(3)}_{\mu\nu} (k_1, k_2, p)$$

$$+ 2\delta_{\mu\nu} \langle (\phi^* \phi)(k_1 + k_2)(\phi^* \phi)(p)(\phi^* \phi) \rangle \quad (48)$$

$$\langle J_\alpha (k_1) J_\beta (k_2) J_\gamma (k_3) J_\delta \rangle = T^{(4)}_{\alpha\beta\gamma\delta} (k_1, k_2, k_3)$$

$$+ 2 \left[ \delta_{\alpha\beta} \langle (\phi^* \phi)(k_1 + k_2) J_\gamma (k_3) J_\delta \rangle + \delta_{\gamma\delta} \langle J_\alpha J_\beta (\phi^* \phi)(-k_1 - k_2) \rangle + \ldots \right]$$

$$- 4 \left[ \delta_{\alpha\beta} \delta_{\gamma\delta} \langle (\phi^* \phi)(k_1 + k_2)(\phi^* \phi) \rangle + \ldots \right] \quad (50)$$
where $T^{(i)}$ tensors are all transverse:

$$k_\mu T^{(1)}_\mu (k) = 0$$

$$k_{1,\mu} T^{(2)}_{\mu \nu} (k_1, k_2) = 0, \quad k_{2,\nu} T^{(2)}_{\mu \nu} (k_1, k_2) = 0$$

$$k_{1,\mu} T^{(3)}_{\mu \nu \alpha} (k_1, k_2, p) = 0, \quad k_{2,\nu} T^{(3)}_{\mu \nu \alpha} (k_1, k_2, p) = 0$$

$$k_{1,\alpha} T^{(4)}_{\alpha \beta \gamma \delta} (k_1, k_2, k_3) = 0, \ldots$$

The above results can be easily generalized to an arbitrary Green function of $J$'s and $(\phi^* \phi)$'s. In general we find

$$\langle J_\mu \ldots (\phi^* \phi) \ldots \rangle = \text{(transverse part)} + \ldots$$

where the terms denoted by dots are Green functions in which pairs of $J$'s are replaced by $(\phi^* \phi)$'s.

### 6.3 Photon propagator

Eq. (46) gives the longitudinal part of $\langle J_\mu J_\nu \rangle$ explicitly, but we can analyze its structure in much more detail. Especially we can express $\langle J_\mu J_\nu \rangle$ in terms of the full propagator $\langle B_\mu B_\nu \rangle$. (See Appendix B for a sketch of the derivation.) Let us first denote the self-energy of $B_\mu$ by

$$\Sigma_{\mu \nu} (k) = \delta_{\mu \nu} \Sigma_1 (k^2) + \frac{k_\mu k_\nu}{m_0^2} \Sigma_2 (k^2)$$

where $m_0^2 \equiv e_0^2 v_0^2$, so that the full propagator is given by

$$\langle B_\mu (k) B_\nu \rangle = \frac{1}{k^2 + \Sigma_1} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{k_\mu k_\nu}{k^2} \frac{m_0^2}{m_0^2 \Sigma_1 + k^2 \Sigma_2}$$

Now we can show diagrammatically the following relation:

$$\langle J_\mu J_\nu \rangle = \frac{1}{e_0^2} \Sigma_{\mu \alpha} \langle B_\alpha B_\beta \rangle \Sigma_{\beta \nu} - \frac{1}{e_0^2} \Sigma_{\mu \nu} + 2 \delta_{\mu \nu} \langle \phi^* \phi \rangle$$

The two-point function of the current is then found to have the following structure:

$$\langle J_\mu (k) J_\nu \rangle = \frac{1}{e_0^2} \left[ \frac{(k^2)^2}{k^2 + \Sigma_1} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) - k^2 \delta_{\mu \nu} + k_\mu k_\nu \right] + 2 \delta_{\mu \nu} \langle \phi^* \phi \rangle$$

Note that this depends only on the transverse part $\Sigma_1$ but not on the longitudinal part $\Sigma_2$. 
Since $\frac{1}{\sqrt{Z_3}} F_{\mu \nu}$ is renormalized, the transverse part of $\langle B_\mu B_\nu \rangle$ can be renormalized by the factor $\frac{1}{Z_3}$. Hence,

$$\frac{1}{Z_3} \frac{1}{k^2 + \Sigma_1}$$

is free of UV divergences. Therefore, for the renormalized current $J_{r,\mu}$, we obtain

$$\langle J_{r,\mu}(k) J_{r,\nu} \rangle = \frac{1}{e^2} \frac{1}{Z_3(k^2 + \Sigma_1)} (k^2)^2 \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)
- \frac{1}{e^2} \frac{1}{Z_3} \left( k^2 \delta_{\mu\nu} - k_\mu k_\nu \right) + 2\delta_{\mu\nu} \left[ \frac{1}{Z_3 Z_m} \langle (\phi^* \phi)_r \rangle - \frac{z S}{Z_3 Z_m} m_H^2 \right]$$

(61)

### 6.4 Renormalization of the Green functions

In coordinate space the Green functions of the renormalized $J_{r,\mu}$’s and $(\phi^* \phi)_r$’s are UV finite as long as we keep all the space points distinct. When two or more points coincide, a non-integrable singularity can be generated, and this gives rise to a UV singularity in the Fourier transform.

The renormalized current $J_{r,\mu}$ has scale dimension 3, and we expect its two-point function to have a singularity which is a polynomial in momentum up to second order. The formula (61) we derived above gives a precise structure of the UV divergence. Hence, a renormalized two-point function is defined by

$$\langle J_{r,\mu}(k) J_{r,\nu} \rangle_{\text{ren}} \equiv \langle J_{r,\mu}(k) J_{r,\nu} \rangle + \frac{1}{e^2} \left( \frac{1}{Z_3} - 1 \right) (k^2)^2 \delta_{\mu\nu} - k_\mu k_\nu
- 2\delta_{\mu\nu} \left[ \frac{1}{Z_3 Z_m} \langle (\phi^* \phi)_r \rangle - \frac{z S}{Z_3 Z_m} m_H^2 \right]
= \frac{1}{e^2} \left( \frac{1}{Z_3(k^2 + \Sigma_1)} (k^2)^2 - k^2 \right) \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)
+ 2\delta_{\mu\nu} \langle (\phi^* \phi)_r \rangle$$

(62)

The scale dimension of $(\phi^* \phi)_r$ is 2, and its two-point function has a constant singularity independent of momentum. Thus, we can define

$$\langle (\phi^* \phi)_r(p) (\phi^* \phi)_r \rangle_{\text{ren}} \equiv \langle (\phi^* \phi)_r(p) (\phi^* \phi)_r \rangle - D$$

(63)

where $D$ is a UV divergent constant.\footnote{Actually one can show $D = -2zS$ to all orders in perturbation theory.}
Let us now consider the three-point function \( \langle J_{r,\mu} J_{r,\nu} (\phi^* \phi)_r \rangle \). In coordinate space the product of two currents gives a dimension four singularity proportional to the scalar \( (\phi^* \phi)_r \), and this gives rise to the only non-integrable singularity. Hence, the transverse part is multiplicatively renormalizable, and we obtain

\[
\langle J_{r,\mu}(k_1) J_{r,\nu}(k_2) (\phi^* \phi)_r \rangle_{\text{ren}} \equiv \langle J_{r,\mu} J_{r,\nu} (\phi^* \phi)_r \rangle
- 2\delta_{\mu\nu} \left( \frac{1}{Z_3^2 Z_m} - 1 \right) \langle (\phi^* \phi)_r (k_1 + k_2)(\phi^* \phi)_r \rangle - 2\delta_{\mu\nu} D
\]

\[
= \frac{Z_m^2}{Z_3^2} T^{(2)}_{\mu\nu} + 2\delta_{\mu\nu} \langle (\phi^* \phi)_r (k_1 + k_2)(\phi^* \phi)_r \rangle_{\text{ren}}
\]

The product of two \( (\phi^* \phi)'s \) contains a constant singularity \( D \) as we have seen above, but this singularity does not affect the three-point function of \( (\phi^* \phi) \) for generic external momenta. Hence, we do not need any subtraction, and we obtain

\[
\langle (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r \rangle_{\text{ren}} = \langle (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r \rangle
\]

Similarly, the four-point (or higher-point) function of \( (\phi^* \phi)'s \) does not require further subtractions:

\[
\langle (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r \rangle_{\text{ren}} = \langle (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r \rangle
\]

In the Green function of two \( J \)'s and two \( (\phi^* \phi)'s \), the only singularity comes from two \( J \)'s coincident in space, producing a scalar \( (\phi^* \phi) \). Hence, the transverse part is multiplicatively renormalizable, and we obtain

\[
\langle J_{r,\mu}(k_1) J_{r,\nu}(k_2) (\phi^* \phi)_r (\phi^* \phi)_r \rangle_{\text{ren}} \equiv \langle J_{r,\mu} J_{r,\nu} (\phi^* \phi)_r (\phi^* \phi)_r \rangle
- 2\delta_{\mu\nu} \left( \frac{1}{Z_3^2 Z_m} - 1 \right) \langle (\phi^* \phi)_r (\phi^* \phi)_r (\phi^* \phi)_r \rangle
\]

\[
= \frac{Z_m^2}{Z_3^2} T^{(3)}_{\mu\nu} + 2\delta_{\mu\nu} \langle (\phi^* \phi)_r (k_1 + k_2)(\phi^* \phi)_r \rangle_{\text{ren}}
\]

Finally, we consider the four-point function of \( J \)'s. This has two causes of UV singularities: one coming from a pair of coincident \( J \)'s producing a \( \phi^* \phi \), and another coming from four coincident \( J \)'s producing an identity operator. On the other hand Eq. (60) gives

\[
\langle J_{r,\alpha}(k_1) J_{r,\beta}(k_2) J_{r,\gamma}(k_3) J_{r,\delta} \rangle = \frac{1}{Z_3^4} T^{(4)}_{\alpha\beta\gamma\delta}
+ \frac{2}{Z_3^2 Z_m} (\delta_{\alpha\beta} \langle (\phi^* \phi)_r (k_1 + k_2)(\phi^* \phi)_r \rangle + \ldots)
- \frac{4}{Z_3^2 Z_m^2} (\langle (\phi^* \phi)_r (k_1 + k_2)(\phi^* \phi)_r \rangle + \ldots)
\]

\[\text{(68)}\]
Therefore, the transverse part is multiplicatively renormalizable, and we can renormalize the four-point function as

\[ \langle J_{r,\alpha}(k_1) J_{r,\beta}(k_2) J_{r,\gamma}(k_3) J_{r,\delta}\rangle_{\text{ren}} \equiv \langle J_{r,\alpha} J_{r,\beta} J_{r,\gamma} J_{r,\delta}\rangle - 2 \left( \frac{1}{Z_3^2 Z_m} - 1 \right) (\delta_{\alpha\beta} (\langle \phi^* \phi \rangle_r J_{r,\gamma} J_{r,\delta} + \ldots) + \ldots) - 4 \left( \frac{1}{Z_3^4 Z_m^2} - 1 \right) - 2 \left( \frac{1}{Z_3^2 Z_m} - 1 \right) ) (\delta_{\alpha\beta} \delta_{\gamma\delta} (\langle (\phi^* \phi)_r (\phi^* \phi)_r \rangle + \ldots) + \ldots) - 4D (\delta_{\alpha\beta} \delta_{\gamma\delta} + \ldots) \]

\[ = \frac{1}{Z_3^4} T^{(4)}_{\alpha\beta\gamma\delta} + 2 (\delta_{\alpha\beta} (\langle \phi^* \phi_r J_{r,\gamma} J_{r,\delta}\rangle_{\text{ren}} + \ldots) + \ldots) - 4 (\delta_{\alpha\beta} \delta_{\gamma\delta} (\langle (\phi^* \phi)_r(k_1 + k_2) (\phi^* \phi)_r\rangle_{\text{ren}} + \ldots) + \ldots) \]

(69)

The above results generalize easily for an arbitrary renormalized Green function of \( J \)'s and \( (\phi^* \phi) \)'s. From Eq. (55), we find

\[ \langle J_{r,\mu} \ldots (\phi^* \phi)_r \ldots \rangle_{\text{ren}} = (\text{renormalized transverse part}) + \ldots \]

(70)

where the transverse part is multiplicatively renormalized, and the dotted part is a linear combination of lower point renormalized Green functions.

### 6.5 1-loop results

We have explicitly computed the UV divergent part of all the two-point and three-point functions and the four-point function \( \langle AAAA \rangle \), and we have verified the structure of UV divergences found in the previous subsection. Besides the renormalization constants \( Z_3, Z_m, Z_\lambda, z \) given in Eqs. (13,14,15), we have obtained

\[ z_S = -\frac{1}{2} \ln \Lambda, \quad D = \ln \Lambda \]

(71)

No other renormalization constants are necessary to renormalize higher point Green functions.

### 7 S-matrix elements

Before concluding this paper, let us briefly discuss the implications of our results to the calculation of the S-matrix elements in the unitary gauge.

\footnote{Note that the relation \( D = -2z_S \) alluded to in the previous footnote is satisfied at 1-loop.}
The first step is to calculate the off-shell Green function

\[ \langle A_{r,\mu} \ldots \Phi_r \ldots \rangle \quad (72) \]

using \( A_{r,\mu} = \frac{1}{\epsilon v^2} J_{r,\mu} \) and \( \Phi_r = \frac{1}{v} \left( (\phi^* \phi)_r - \frac{v^2}{2} \right) \) as interpolating fields. This is given as the sum of a UV finite transverse part and a linear combination of lower point Green functions with UV divergent coefficients. The UV divergent part is irrelevant to the S-matrix, since it does not have the desired pole structure as we take the external momenta to the mass shell. Hence, only the transverse part contributes to the S-matrix element, and it is UV finite.

In practice we do not need to use \( A_r \) and \( \Phi_r \) as interpolating fields; according to the general theory of S-matrix, we should be able to use any interpolating fields. The simplest choice of the interpolating fields is definitely \( B_\mu \) for the photon and \( \rho \) for the Higgs. Their Green functions are not multiplicatively renormalizable, but by normalizing \( B_\mu \) and \( \rho \) properly we can obtain the same S-matrix elements as from the Green functions of \( A_r \)'s and \( \Phi_r \)'s.

The propagator of \( B_\mu \) is given by Eq. \((57)\), and only its transverse part is renormalizable:

\[
\frac{1}{Z_3} \langle B_\mu(k)B_\nu \rangle = \frac{1}{Z_3(k^2 + \Sigma_1)} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) + k_\mu k_\nu \text{(UV divergent)} \quad (73)
\]

In perturbation theory the longitudinal part has no single particle pole, and we can use \( \frac{1}{\sqrt{Z_3}} B_\mu \) as an interpolating field of the massive photon. As for \( \rho \), its two-point function is not renormalizable, and the residue \( Z_\rho \) at the single particle pole is UV divergent:

\[
\langle \rho(p)\rho \rangle \to Z_\rho \frac{1}{p^2 + m_H^2} \quad \text{as} \quad p^2 \to -m_H^2 \quad (74)
\]

Hence, we must use \( \frac{\rho}{\sqrt{Z_\rho}} \) as the interpolating field of the Higgs.

Therefore, for the Green function with \( n_B B \)'s and \( n_\rho \rho \)'s,

\[
\frac{1}{Z_3^n Z_\rho^n} \langle B_\mu \ldots \rho \ldots \rangle \quad (75)
\]

has the same UV finite pole part as \( \langle A_{r,\mu} \ldots \Phi_r \ldots \rangle \) as we take the mass shell limit of the external momenta. The possibility of calculating the S-matrix elements from the Green functions of \( B \)'s and \( \rho \)'s is thus assured.

\(^8\)The two pole parts can differ by a finite multiplicative constant since the single particle residue of \( \langle \Phi_r \Phi_r \rangle \) is UV finite but not necessarily 1.
8 Concluding remarks

We hope to have convinced the reader of the off-shell renormalizability of the unitary gauge for the abelian Higgs model. In perturbative calculations in the unitary gauge, we certainly encounter unfamiliar UV divergences, but our analysis in this paper clarifies the origin of these divergences: mixing and high scale dimensions of the fields. We have also seen how these divergences disappear in the physical limit. Now that we understand the nature and the reason of these divergences, the unitary gauge should be promoted to the same rank as the covariant and $R_\xi$ gauges.

The extension of the present work to non-abelian Higgs theories should be straightforward. Especially for those non-abelian Higgs theories with no massless gauge boson (for example, the SU(2) theory with a Higgs doublet), we expect no qualitative change in the analysis. For those theories with massless gauge bosons, however, a gauge fixing is necessary, and we will need to modify the analysis to accommodate the gauge fixing.

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A Ward identities

We sketch a diagrammatic derivation of the Ward identities given in the main text. We consider bare quantities, and $v_0$ is determined so that $\langle \rho \rangle = 0$, and $m_0^2 \equiv e_0^2 v_0^2$. The choice of the vanishing tadpole is not essential, but it simplifies the diagrammatic analysis considerably. The propagator of the photon is given by

$$D_{\mu\nu}(k) \equiv \langle B_\mu(k)B_\nu \rangle_{tree} = \frac{\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m_0^2}}{k^2 + m_0^2}$$

(76)

This satisfies

$$k_\mu D_{\mu\nu}(k) = \frac{k_\nu}{m_0^2}$$

(77)

and we write this relation diagrammatically as
Fig. 1 Divergence of the photon propagator

where a double line represents a derivative factor $k_\mu$, and a dot represents a Kronecker delta $\delta_{\mu\nu}$. The Feynman rules of the vertices involving photons are given by

\[
\begin{align*}
\text{\ldots} & \quad -2 e^2 \nu \delta_{\mu\nu} \\
\text{\ldots} & \quad -2 e^2 \delta_{\mu\nu}
\end{align*}
\]

Fig. 2 Feynman rules

where a broken line represents a Higgs propagator.

We first consider the two-point function of $A_\mu$. We obtain the diagrammatic expression given below, in which a cross denotes amputation of the external propagator:

\[
\left\langle A_\mu A_\nu \right\rangle = \begin{align*}
&= \quad \begin{array}{c}
\text{\ldots} \\
\text{\ldots} \\
\text{\ldots}
\end{array} \\
&= \quad \begin{array}{c}
\text{\ldots} \\
\text{\ldots} \
\text{\ldots}
\end{array}
\end{align*}
\]

Taking into account the vanishing tadpole, we now make the following observation:
Now, using Eq. (77), we find a lot of cancellations, and we eventually obtain

\[ k_\mu \left\langle A_\mu A_\nu \right\rangle = \frac{k_\nu}{m_0^2} - \frac{1}{m_0^4} \]

which implies

\[ k_\mu \left\langle A_\mu(k) A_\nu(k') \right\rangle = \frac{2k_\nu}{m_0^2 v_0^2} \langle \phi^* \phi \rangle \] (78)

For the Green function \( \langle A_\mu(k) A_\nu(k') \Phi \ldots \Phi \rangle \) with an arbitrary number of \( \Phi \)'s, the derivation of the Ward identity goes the same way as above. The extra insertions of \( \Phi \)'s does not change the nature of derivation. We only need to pay attention to the two \( A \)'s:

Using Eq. (77) and diagrammatic identities similar to those for the propagator, we obtain
which implies
\[ k_\mu \left< A_\mu (k) A_\nu (k') \Phi \ldots \right> = - \frac{1}{m_0^4} \begin{array}{c} \ldots \end{array}_k - \frac{1}{m_0^4} \begin{array}{c} \ldots \end{array}_{k'} \]

(79)

With more insertions of $A$’s, the derivation goes similarly. In general we obtain
\[ k_{1\mu_1} \langle A_{\mu_1} (k_1) A_{\mu_2} (k_2) \ldots A_{\mu_M} (k_M) \Phi \ldots \rangle \]
\[ = \frac{2}{m_0^4 v_0} \sum_{m=2}^{M} k_{1\mu_m} \langle A_{\mu_2} (k_2) \ldots \Phi (k_1 + k_m) \ldots A_{\mu_M} (k_M) \Phi \ldots \rangle \]  
(80)

In the main text, the Ward identities are rewritten for $J$’s and $(\phi^* \phi)$’s instead of $A$’s and $\Phi$’s.

### B The relation between $\langle B_\mu B_\nu \rangle$ and $\langle A_\mu A_\nu \rangle$

With a more detailed diagrammatic analysis, we can derive an explicit relation between $\langle B_\mu B_\nu \rangle$ and $\langle A_\mu A_\nu \rangle$. Let us denote the self-energy correction to the propagator by $\Pi_{\mu\nu}$ so that the full propagator $\langle B_\mu B_\nu \rangle$ can be written recursively as
\[ \langle B_\mu B_\nu \rangle = D_{\mu\nu} - D_{\mu\alpha} \Pi_{\alpha\beta} \langle B_\beta B_\nu \rangle \]  
(81)

The self-energy correction can be written in terms of 1PI diagrams as follows:
We now observe the following diagrammatic identities:

\[ + \]

\[ = \]

\[ = -\Pi_{\mu\nu} + (\Pi_1 + \Pi_2)_{\mu\alpha} \beta \]

\[ = -\Pi_{\mu\nu} + (\Pi_1 + \Pi_2)_{\mu\alpha} \beta \]

Using these and the diagrammatic expression of \( \langle AA \rangle \) given in Appendix A, we obtain

\[ \langle A_\mu A_\nu \rangle = \frac{1}{m_0^2} \left[ \sum_{\mu\alpha} \langle B_\alpha B_\beta \rangle \Sigma_{\beta\nu} - \Sigma_{\mu\nu} + 2e_0^2 \delta_{\mu\nu} \langle \phi^* \phi \rangle \right] \quad (82) \]

where

\[ \Sigma_{\mu\nu} \equiv m_0^2 \delta_{\mu\nu} + \Pi_{\mu\nu} \quad (83) \]

Replacing \( e_0^2 \nu_0^2 A \) by \( J \), we obtain the relation (58) in the main text.
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