Analysis of the chaotic maps generating different statistical distributions

M Lawnik
Faculty of Applied Mathematics, Silesian University of Technology, ul. Kaszubska 23, 44-100 Gliwice, Poland
E-Mail: marcin.lawnik@polsl.pl

Abstract. The analysis of the chaotic maps, enabling the derivation of numbers from given statistical distributions was presented. The analyzed chaotic maps are in the form

\[ x_{k+1} = F^{-1}(U(F(x_k))) \]

where \( F \) is the cumulative distribution function, \( U \) is the skew tent map and \( F^{-1} \) is the inverse function of \( F \). The analysis was presented on the example of chaotic map with the standard normal distribution in view of his computational efficiency and accuracy. On the grounds of the conducted analysis, it should be indicated that the method not always allows to generate the values from the given distribution.

1. Introduction

Pseudo-random numbers are very significant in many disciplines of science, including cryptography and simulations. In cryptography, they are essential elements of ciphers, whereas in simulations they enabling the modelling of many important physical, economic and other related processes.

Thus, many methods have been developed to make it possible to generate pseudo-random numbers from given distributions. The most popular one is the inverse cumulative distribution function method based on a specific distribution [1]. The method involves the generation of random variable \( U \in [0,1] \) derived from the uniform distribution and in the next step, transformed by means of the following dependence:

\[ X = F^{-1}(U), \]

where \( F \) is the cumulative distribution function (CDF) and \( F^{-1} \) denotes the inverse function of \( F \).

It is then that the determined random variable has a distribution determined by \( F \). The disadvantage of this method is the fact that for some distributions, for example, normal distribution, the cumulative distribution function cannot be expressed by means of elementary functions. In such cases, approximations should be used by means of other functions that are invertible.

Apart from (1), to derive numbers from the required distribution, transformations may be applied [1], which makes it possible to transform one distribution (in most cases uniform distribution) to another one. Another very popular methods for generating pseudorandom numbers from given distribution are the so called rejection and decomposition methods [1].

In [2] Lai and Chen presented an interesting way of generating numbers from given distributions, involving the construction of chaotic maps in the following form:

\[ x_{k+1} = F^{-1}(U(F(x_k))) \]

where \( F \) is the cumulative distribution function of the required distribution and \( U \) is the skew tent map. The successive iterations of (2) generate numerical values from the distribution designated by \( F \).

Skew tent map \( U \) used in (2) is designated by the equation:
For each value of parameter \( p \in (0,1) \) equation (3) is chaotic and its distribution is uniform. Lyapunov exponent for such chaotic map is calculated in accordance with:

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln f'(x_k),
\]

is equal to [3]

\[
\lambda = -p \ln p - (1-p)\ln(1-p). \tag{5}
\]

Furthermore equation (5) describes also the Lyapunov exponent of recursion (2) [2]. The graph of Lyapunov exponent (5) is shown in figure 1.

![Figure 1. Lyapunov exponent of skew tent map (3).](image)

The scope of the paper is to analyze the chaotic maps in the form of (2) in view of their computational accuracy and efficiency on the example of the constructed recursive map with the normal distribution.

1.1. Normal distribution

The normal distribution is designated by density function \( \rho(x) \) expressed by the following equation [1]:

\[
\rho(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right), \tag{6}
\]

where \( \mu \) denotes the arithmetic mean value, \( \sigma \) is the standard deviation.

The cumulative distribution function of the above equation cannot be expressed by means of elementary functions. Therefore, to take advantage of (1) or (2) it should be approximated by another function, for which an inverse function can be designated. An example of such approximation is [4]:

\[
F(x) = \frac{e^{2kx}}{1 + e^{2kx}} \left( k = \left( \frac{2}{\pi} \right) ^{1/2} \right). \tag{7}
\]

List of another functions that approximate the CDF of the normal distribution and methods that enables the derivation of variables from the normal distribution can be found in [5-8].
2. Analysis

On the grounds of (2) and (7) a chaotic map was derived. Its distribution for different values of parameter \( p \) is shown in figure 2. As seen in the graph, the derived chaotic map does not well approximate the standard normal distribution for some values of \( p \).

![Figure 2](image)

**Figure 2.** Distributions of constructed map for different values of parameter \( p \) (continuous line); the dotted line indicates the normal distribution.

Furthermore as seen in figure 3, the distribution numerically generated from part \( U(F(x)) \) of the constructed map in form (2), is not a uniform distribution. This is a consequence of a low value of Lyapunov exponent of (2) which is equal to (5). This means that by the application of chaotic map (2) extreme values of parameter \( p \) should not be selected. It follows that recursion in the form (2) in conjunction with (7) is not appropriate for generating numbers from standard normal distributions. Such conclusion may be generalized to each recursive map in the form of (2) with any distribution, where the value of \( p \) is close to 0 or 1.

To eliminate the above discussed disadvantage of recursion (2), the skew tent map (3) should be replaced by another chaotic map, with uniform distribution, but, at the same time, with Lyapunov exponent of a stable and positive value. Such maps can be found in [9].

Furthermore, in table 1, the mean time required for generating the numbers from the normal distribution was indicated, in comparison with (1). The results indicate that constructed map takes much more time than method (1). This is an outcome of the form of (2), which additionally requires the use of CDF (7) in the calculations.

**Table 1.** Results obtained for the analyzed function generating the normal distribution for 1000000 iterations.

| Method                      | Time in [s] |
|-----------------------------|-------------|
| method (1) with (7)         | 1.862       |
| constructed map             |             |
| \( p = 0.0001 \)            | 3.333       |
| \( p = 0.3 \)               | 3.158       |
| \( p = 0.7 \)               | 3.086       |
| \( p = 0.9999 \)            | 3.051       |
3. Conclusions
The scope of the paper was the presentation of the analysis of a chaotic map that generates numerical values from the standard normal distribution, in the form of (2). On the grounds of the conducted analysis, it should be indicated that for some values of parameter $p$ in the skew tent map, the recurrence in the form of (2) does not generate the values from the given distribution. This is due to the fact that part $U(F(x))$ of the recursion expressed by equation (2) does not render the uniform distribution. This, in turn, is a consequence of a low value of Lyapunov exponent for such values of $p$. Furthermore chaotic maps in the form of (2) takes more time to generate numbers from the given distribution in comparison with method (1).

References
[1] Devroye L 1986 Non-Uniform Random Variate Generation (New York: Springer)
[2] Lai D and Chen G 2000 Generating different statistical distributions by the chaotic skew tent map *International Journal of Bifurcation and Chaos* 10(6) 1509–12
[3] Hasler M and Maistrenko Y L 1997 An introduction to the synchronization of chaotic systems: coupled skew tent maps *IEEE Transactions on Circuits and Systems—I: Fundamental theory and Applications* 44(10) 856–66
[4] Tocher KD 1963 *The Art of Simulation* (London: English University Press)
[5] Aludaat K M and Alodat M T 2008 A note on approximating the normal distribution function *Applied Mathematical Sciences* 2(9) 425–29
[6] Bowling S R Khasawneh M T Kaewkuekool S and Cho B T 2009 A logistic approximation to the cumulative normal distribution *Journal of Industrial Engineering and Management* 2(1) 114–27
[7] Thomas D B Luk W Leong P H W and Villasenor J D 2007 Gaussian random number generators *ACM Computing Surveys* 39(4) 1–38
[8] Lawnik M 2014 The approximation of the normal distribution by means of chaotic expression *Journal of Physics: Conference Series* 490 012072
[9] Anikin V M Arkadaksky S S Kuptsov S N Remizov A S and Vasilenko L P 2008 Lyapunov exponent for chaotic 1D maps with uniform invariant distribution *Bulletin of the Russian Academy of Sciences: Physics* 72(12) 1684–88