ARITHMETIC BEHAVIOUR OF HECKE EIGENVALUES OF SIEGEL CUSP FORMS OF DEGREE TWO

SANOLI GUN, WINFRIED KOHNEN AND BIPLAB PAUL

ABSTRACT. Let $F$ and $G$ be Siegel cusp forms for $\text{Sp}_4(\mathbb{Z})$ and weights $k_1, k_2$ respectively. Also let $F$ and $G$ be Hecke eigenforms lying in distinct eigen spaces. Further suppose that neither $F$ nor $G$ is a Saito-Kurokawa lift. In this article, we study simultaneous arithmetic behaviour of Hecke eigenvalues of these Hecke eigenforms.

1. INTRODUCTION

Let $k$ be a positive integer, $\Gamma_2 := \text{Sp}_4(\mathbb{Z})$ be the full Siegel modular group of degree 2 and $S_k(\Gamma_2)$ be the space of Siegel cusp forms of weight $k$ for $\Gamma_2$. It is well known (see [17]) that when $k$ is even, $S_k(\Gamma_2)$ has a canonical subspace which is generated by the Saito-Kurokawa lift of Hecke eigenforms in the space of elliptic cusp forms of weight $2k-2$ for $\text{SL}_2(\mathbb{Z})$. This subspace is called the Maass subspace. When $k$ is odd, we shall define the zero subspace of $S_k(\Gamma_2)$ as Maass subspace. In both the cases, we shall denote these Maass subspaces by $S^*_k(\Gamma_2)$.

If $F \in S^*_k(\Gamma_2)$ is a Hecke eigenform with eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$, then one knows that $\mu_F(n) > 0$ for all $n \in \mathbb{N}$ (see [3], also see [7, Corollary 1.5]). On the contrary, if $F$ is an Hecke eigenform lying in the orthogonal complement of $S^*_k(\Gamma_2)$ in $S_k(\Gamma_2)$, then the second author [8] showed that the sequence $\{\mu_F(n)\}_{n \in \mathbb{N}}$ changes sign infinitely often.

Now suppose that $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ are Hecke eigenforms with eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$ and $\{\mu_G(n)\}_{n \in \mathbb{N}}$ respectively. In this article, we will investigate arithmetic properties of the sequence $\{\mu_F(n)\mu_G(n)\}_{n \in \mathbb{N}}$. Unlike the elliptic case, it is not known that if $F$ is not a constant multiple of $G$, then there exists a natural number $n_0$ such that $\mu_F(n_0) \neq \mu_G(n_0)$ (see [2] [13]). Henceforth, we shall assume that $F$ and $G$ lie in different eigenspaces. We shall also assume that $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ are Hecke eigenforms lying in the orthogonal complement of the Maass subspace as the arithmetic properties investigated in this article are already well understood for Hecke eigenforms inside the Maass subspace.

We start by investigating the first non-vanishing of the sequence $\{\mu_F(p^n)\mu_G(p^n)\}_{n \in \mathbb{N}}$ when $p$ is a prime. More precisely, we have the following theorem.

2010 Mathematics Subject Classification. 11F46.

Key words and phrases. Siegel modular forms, Hecke eigenvalues, multiplicity one theorem, simultaneous non-vanishing.
Theorem 1. Let $F \in S_k(\Gamma_2)$ and $G \in S_k(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace with Hecke eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$ and $\{\mu_G(n)\}_{n \in \mathbb{N}}$ respectively. Also let $F$ and $G$ lie in different eigenspaces. Then for any prime $p$, there exists an integer $n$ with $1 \leq n \leq 14$ such that

$$\mu_F(p^n)\mu_G(p^n) \neq 0.$$ 

Next, we investigate the growth of the sequence of normalized Hecke eigenvalues and prove the following theorem.

Theorem 2. Let $F \in S_k(\Gamma_2)$ and $G \in S_k(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace and having normalized Hecke eigenvalues $\{\lambda_F(n)\}_{n \in \mathbb{N}}$ and $\{\lambda_G(n)\}_{n \in \mathbb{N}}$ respectively. Also let $F$ and $G$ lie in different eigenspaces. Then for sufficiently large $x$ and any $\epsilon > 0$, one has

$$\sum_{m \leq x} \lambda_F(m)\lambda_G(m) \ll_{\epsilon} \max\{k_1, k_2\}^{3/8}x^{31/32+\epsilon},$$

where the constant in $\ll_{\epsilon}$ depends only on $\epsilon$.

As a corollary, we then derive the following.

Corollary 3. Let $F \in S_k(\Gamma_2)$ and $G \in S_k(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$ and $\{\mu_G(n)\}_{n \in \mathbb{N}}$ respectively. Also let $F$ and $G$ lie in different eigenspaces. Then for any $\epsilon > 0$, one has

$$\# \{n \leq x \mid \mu_F(n) \neq \mu_G(n)\} \gg x^{1-\epsilon},$$

where the constant $\gg$ depends on $F, G$ and $\epsilon$.

Next we investigate the question of Hecke eigenvalues which are of different sign. Here we have the following theorem;

Theorem 4. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$. Also assume that there exist $0 < c < 4$ and a Hecke eigenform $G \in S_k(\Gamma_2)$ lying in the orthogonal complement of the Maass subspace with Hecke eigenvalues $\{\mu_G(n)\}_{n \in \mathbb{N}}$ such that

$$(1) \quad \# \{p \leq x \mid p \text{ prime, } |\mu_G(p)| > cp^{k_2-\frac{3}{2}}\} \geq \frac{16}{17} \cdot \frac{x}{\log x}$$

for sufficiently large $x$. Also assume that $F$ and $G$ lie in different eigenspaces. Then half of the non-zero coefficients of the sequence $\{\mu_F(n)\mu_G(n)\}_{n \in \mathbb{N}}$ are positive and half of them are negative.

We note that the subset of primes $\{p \mid \mu_G(p) = 0\}$ has density zero (see appendix of [12]). Further the Generalized Ramanujan-Petersson conjecture proved by Weissauer ([15]) gives that for any prime $p$, $|\mu_G(p)| \leq 4p^{k_2-\frac{3}{2}}$. Thus the hypothesis in (1) is not an unreasonable one.
(especially if one also believes an analogous Sato-Tate conjecture in this setup). Now if we restrict to $p$-eigenvalues, then we can prove the following theorem;

**Theorem 5.** Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Theorem 4. Then there exists a set of primes $p$ of positive lower density such that $\mu_F(p)\mu_G(p) \gtrsim 0$.

The article is distributed as follows. In the next section, we introduce notations and preliminaries. In the last few sections, we give proofs of Theorem 1, Theorem 2, Theorem 4 and Theorem 5.

We note that proof of Theorem 1 requires intricate understanding of Hecke relations whereas the proof of Theorem 2 uses a result of the first author with R. Murty [6] and a beautiful work of Pitale, Saha and Schmidt [11] along with some elementary analytic tools. Moreover, Theorem 2 can be thought of a generalization of a work of Das, the second author and Sengupta [4]. Proof of Theorem 5 requires some standard analytic techniques and proof of Theorem 4 is rather straightforward from the works of Matomäki and Radziwiłł [9] and that we keep it here for the sake of completeness.

2. Notations and Preliminaries

Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{N}$ and $\mathcal{P}$ denote the set of real numbers, the set of positive real numbers, the set of integers, the set of natural numbers and the set of prime numbers respectively. Also we shall use the symbol $p$ to denote a prime number.

For $f, g : \mathbb{R} \to \mathbb{C}$ with $g(x) > 0$ for all $x \in \mathbb{R}$, we say $f = o(g)$ if $|f(x)|/g(x) \to 0$ as $x \to +\infty$.

We say a subset $A$ of $\mathcal{P}$ has natural density $\alpha \in \mathbb{R}$ if

$$\lim_{x \to \infty} \frac{\# \{p \in A \mid p \leq x\}}{\# \{p \in \mathcal{P} \mid p \leq x\}}$$

exists and is equal to $\alpha$. We shall denote the natural density of $A \subset \mathcal{P}$ by $d(A)$ if it exists.

We say the density of $A \subset \mathbb{N}$ is $d(A)$ if

$$\lim_{x \to \infty} \frac{\# \{n \leq x \mid n \in A\}}{\# \{n \leq x \mid n \in \mathbb{N}\}}$$

exists and is equal to the real number $d(A)$.

Throughout the paper, we shall use definitions and basic facts about Siegel modular forms. We refer to Andrianov [1] for further details. For any integer $n \in \mathbb{N}$, the Hecke operator $T(n)$ on the space $S_k(\Gamma_2)$ is defined by

$$T(n)F := n^{2k-3} \sum_{\gamma \in \Gamma_2 \setminus \mathcal{O}_{2,n}} F | \gamma,$$

where

$$\mathcal{O}_{2,n} := \{ \gamma \in M_4(\mathbb{Z}) \mid \gamma^t J \gamma = nJ \}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
It is known that the space $S_k(\Gamma_2)$ has a basis of Hecke eigenforms. Let $F \in S_k(\Gamma_2)$ be such that $T(n)F = \mu_F(n)F$ for all $n \in \mathbb{N}$. Then one knows that $\mu_F$ is a multiplicative function. If $F \in S_k(\Gamma_2)$ is not a Saito-Kurokawa lift, by a famous work of Weissauer [15], one also knows that the generalized Ramanujan-Petersson conjecture is true, i.e. for any $\epsilon > 0$, one has

$$\mu_F(n) \ll n^{k-3/2+\epsilon}.$$  

We shall normalize these eigenvalues and define for any $n \in \mathbb{N}$

$$\lambda_F(n) := \frac{\mu_F(n)}{n^{k-3/2}}.$$  

To each Hecke eigenform $F \in S_k(\Gamma_2)$, Andrianov [1] associated a $L$-function which is now known as spinor zeta function as follows:

$$Z_F(s) := \zeta(2s+1) \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^{s+k-3/2}}.$$  

The series $Z_F(s)$ is absolutely convergent and has an Euler product in the region $\Re(s) > 1$. In fact, by the works of Andrianov [1] and Oda [10], one knows that if $F$ is not a Saito-Kurokawa lift, then the function $Z_F(s)$ is entire and that for $n \geq 3$

$$\lambda_F(p^n) = \lambda_F(p)\lambda_F(p^{n-1}) - \left[ \lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} \right] \lambda_F(p^{n-2}) + \lambda_F(p)\lambda_F(p^{n-3}) - \lambda_F(p^{n-4})$$

with the assumption that $\lambda_F(p^{n-m}) = 0$ for $n < m$. As in the elliptic case, by a work of Kowalski and Saha [12, Appendix], we have the following theorem.

**Theorem 6.** [Kowalski and Saha] Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with eigenvalues $\mu_F(n)$ for $n \in \mathbb{N}$. Also assume that $F$ lies in the orthogonal complement of Maass subspace. Then there exists $\delta > 0$ such that

$$\# \{p \leq x \mid \mu_F(p) = 0 \} \ll \frac{x}{(\log x)^{1+\delta}}.$$  

Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of Maass subspace. Also let $\{\lambda_F(n)\}_{n \in \mathbb{N}}$ and $\{\lambda_G(n)\}_{n \in \mathbb{N}}$ be the sets of normalized Hecke eigenvalues of $F$ and $G$ respectively. Further assume that $Z_F(s)$ and $Z_G(s)$ are the spinor zeta functions associated to $F$ and $G$ respectively. We then have

$$Z_F(s) := \zeta(2s+1) \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} := \prod_{p \in \mathcal{P}} \prod_{i=1}^{4} \left(1 - \alpha_{p,i}p^{-s}\right)^{-1}$$

and

$$Z_G(s) := \zeta(2s+1) \sum_{n=1}^{\infty} \frac{\lambda_G(n)}{n^s} := \prod_{p \in \mathcal{P}} \prod_{i=1}^{4} \left(1 - \beta_{p,i}p^{-s}\right)^{-1}.$$
By the work of Weissauer \[15\], we know that \(|\alpha_{p,i}| = 1 = |\beta_{p,j}|\) for all \(1 \leq i, j \leq 4\). Now define the Rankin-Selberg \(L\)-function \(L(F \times G, s)\) as follows:

\[
L(F \times G, s) := \prod_{p \mathbb{P}} \prod_{1 \leq i, j \leq 4} (1 - \alpha_{p,i} \beta_{p,j} p^{-s})^{-1}.
\]

This Euler product is absolutely convergent for \(\Re(s) > 1\). In fact, Pitale, Saha and Schmidt \[11\, Theorem C, p. 14\] proved the following theorem for Hecke eigenforms which do not belong to the Maass subspace.

**Theorem 7.** [Pitale, Saha and Schmidt] Let \(F \in S_{k_1}(\Gamma_2), G \in S_{k_2}(\Gamma_2), Z_F(s)\) and \(Z_G(s)\) be as above. Define the \(L\)-function \(L(F \times G, s)\) as in (5). Then the infinite product in (5) is absolutely convergent for \(\Re(s) > 1\) and the function \(L(F \times G, s)\) has meromorphic continuation to \(\mathbb{C}\) and is non-vanishing on the line \(\Re(s) = 1\). Moreover, the function \(L(F \times G, s)\) is entire except in the case when \(k_1 = k_2\) and \(\mu_F(n) = \mu_G(n)\) for all \(n \in \mathbb{N}\). In later case, the function \(L(F \times G, s)\) has a simple pole at \(s = 1\).

For Hecke eigenforms \(F \in S_{k_1}(\Gamma_2)\) and \(G \in S_{k_2}(\Gamma_2)\) as above with normalized eigenvalues \(\{\lambda_F(n)\}_{n \in \mathbb{N}}\) and \(\{\lambda_G(n)\}_{n \in \mathbb{N}}\) respectively, define

\[
L(F, G; s) := \sum_{n=1}^{\infty} \frac{\lambda_F(n) \lambda_G(n)}{n^s}.
\]

Note that this series \(L(F, G; s)\) is absolutely convergent for \(\Re(s) > 1\). In fact, the second author along with Das and Sengupta \[4\] proved that the function \(L(F, F; s)\) has meromorphic continuation to \(\Re(s) > 1/2\) with only a simple pole at \(s = 1\). Also they proved the following theorem.

**Theorem 8.** [Das, Kohnen and Sengupta] Let \(F \in S_k(\Gamma_2)\) be a Hecke eigenform which does not lie in the Maass subspace with normalized Hecke eigenvalues \(\{\lambda_F(n)\}_{n \in \mathbb{N}}\). Then for sufficiently large \(x\) and any \(\epsilon > 0\), we have

\[
\sum_{n \leq x} \lambda_F^2(n) = c_F x + O \left( k^{5/16} x^{31/32 + \epsilon} \right),
\]

where \(c_F > 0\) is the residue of the \(L\)-function \(L(F, F; s)\) at \(s = 1\).

To prove Corollary \[3\] we investigate analytic properties of \(L(F, G; s)\) when \(F\) and \(G\) lie in different eigenspaces. In order to do so, we need the following result on the formal power series by the first author and Ram Murty \[6, Theorem 2\].

**Theorem 9.** [Gun and Murty] Let \(P_i(T)\) and \(Q_i(T)\) be non-zero polynomials over \(\mathbb{C}\) such that degree of \(P_i\) is strictly less than the degree of \(Q_i\) for \(i = 1, 2\). Also let

\[
Q_1(T) := \prod_{i=1}^{r} (1 - \alpha_i T)^{e_i} \quad \text{and} \quad Q_2(T) := \prod_{j=1}^{t} (1 - \beta_j T)^{m_j},
\]
where $\alpha_i$'s are distinct for $1 \leq i \leq r$ and $\beta_j$'s are distinct for $1 \leq j \leq t$ and $\ell_i, m_j \in \mathbb{N}$. Let us also assume that
\[
\sum_{n \geq 0} a_n T^n = \frac{P_1(T)}{Q_1(T)} \quad \text{and} \quad \sum_{n \geq 0} b_n T^n = \frac{P_2(T)}{Q_2(T)}
\]
where $a_n, b_n \in \mathbb{C}$ for all $n \geq 0$. Then we have
\[
\sum_{n \geq 0} a_n b_n T^n = \frac{R(T)}{\prod_{i,j} (1 - \alpha_i \beta_j T)^{\ell_i m_j}},
\]
where $R(T) \in \mathbb{C}[T]$. Now if $a_0 = 1 = b_0$, then $R(0) = 1$. Further if we have $P_1'(0) = 0 = P_2'(0)$, then $R'(0) = 0$. Here $P'$ denotes the derivative of $P(T)$ with respect to $T$.

To prove Theorem 4, we shall make use of the following result on the sign changes of multiplicative functions by Matomäki and Radziwiłł [9, Lemma 2.4].

**Lemma 10.** [Matomäki and Radziwiłł] Let $K, L : \mathbb{R}_+ \to \mathbb{R}_+$ be functions such that $K(x) \to 0$ and $L(x) \to \infty$ as $x \to \infty$. Let $g : \mathbb{N} \to \mathbb{R}$ be a multiplicative function such that for every $x \geq 2$, we have
\[
\sum_{p \leq x, \ g(p) = 0} 1 \leq K(x) \quad \text{and} \quad \sum_{p \leq x, \ g(p) < 0} \frac{1}{p} \geq L(x).
\]
Then we have
\[
\# \{n \leq x \mid g(n) > 0\} = (1 + o(1)) \cdot \# \{n \leq x \mid g(n) < 0\} = \left(\frac{1}{2} + o(1)\right) x \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right) \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right),
\]
where $h$ is the characteristic function of the set $\{n \in \mathbb{N} \mid g(n) \neq 0\}$.

### 3. Proof of Theorem 11

In this section, we shall give a proof of Theorem 1. Let us recall that for any prime $p$ and any natural number $n \geq 3$, we have
\[
(7) \quad \lambda_F(p^n) = \lambda_F(p)\lambda_F(p^{n-1}) - \left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p}\right] \lambda_F(p^{n-2}) + \lambda_F(p)\lambda_F(p^{n-3}) - \lambda_F(p^{n-4}),
\]
with the assumption that $\lambda_F(p^{n-m}) = 0$ for $n < m$ are natural numbers. Similar relations hold among the Hecke eigenvalues $\lambda_G(p^n)$ for $n \geq 3$. We use these relations to derive some important consequences which will help us to prove our result. We start with a general result which might be of independent interest.
Lemma 11. Let $f_0(x) = -1$ and $f_1(x) = -x$ be polynomials over $\mathbb{Z}$. Define a family of polynomials 
\{f_n\}_{n \in \mathbb{N}} by \n\begin{equation}
f_{n+1}(x) = x f_n(x) - f_{n-1}(x).
\end{equation}

Then for any $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, we have $f_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$.

Proof. We first show by induction on $n \in \mathbb{N}$ that \n\begin{equation}
f_n(x) = -x^n + a_{n,n-1}x^{n-1} + a_{n,n-2}x^{n-2} + \cdots + a_{n,1}x + a_{n,0},
\end{equation}
where $a_{n,i} \in \mathbb{Z}$, $0 \leq i \leq n-2$. Note that this is true for $n = 0, 1$. Using (8), we have \n\begin{equation}
f_{n+1}(x) = -x^{n+1} + a_{n,n-1}x^n + (a_{n,n-2} + 1)x^{n-1} + \cdots + (a_{n,0} - a_{n-1,1})x - a_{n-1,0}.
\end{equation}

Hence by induction we have (9). Since $\mathbb{Z}$ is integrally closed, any solution in $\mathbb{Q}$ of $f_n(x) = 0$ for
any $n$ will be an integer. This completes the proof of the lemma. \hfill \Box

Lemma 12. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. Then

(1) If $\lambda_F(p^{2m}) = 0$ for some $m \geq 2$, then at least one of $\lambda_F(p), \lambda_F(p^2)$ is non-zero.

(2) There does not exist $t \in \mathbb{N}$ such that $\lambda_F(p^m) = 0$ for $t + 1 \leq m \leq t + 4$.

Proof. Suppose that $\lambda_F(p) = 0 = \lambda_F(p^2)$. Then for any $n \geq 0$, \n\begin{equation}
\lambda_F(p^{2n+4}) = f_n \left( \frac{1}{p} \right),
\end{equation}
where $f_n$’s are polynomials in $\mathbb{Z}[x]$ satisfying the hypothesis of Lemma 11. Hence by Lemma 11 we have $\lambda_F(p^{2m}) \neq 0$ for all $m \geq 2$, a contradiction to our hypothesis. This completes the proof of the first part of the lemma.

To prove the second part of the lemma, let us assume that $\lambda_F(p^m) = 0$ for $t + 1 \leq m \leq t + 4$. Using (7), we have \n\begin{equation}
\lambda_F(p^t) = -\lambda_F(p^{t+4}) = 0.
\end{equation}

Using induction and the identity (7), we get that $\lambda_F(p^m) = 0$ for $1 \leq m \leq t + 4$. This implies that $\lambda_F(p) = 0 = \lambda_F(p^2)$, a contradiction to the first part of the lemma. \hfill \Box

Lemma 13. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. Then

(1) For some $m \geq 0$, $\lambda_F(p^{2m+1}) \neq 0$ implies that $\lambda_F(p) \neq 0$.

(2) If $\lambda_F(p) \neq 0$, then for any $m \in \mathbb{N}$, there exists $0 \leq i \leq 3$ such that $\lambda_F(p^{2i(m+i)+1}) \neq 0$. 
Proof. We shall show by induction on \( m \) that \( \lambda_F(p) = 0 \) implies that \( \lambda_F(p^{2m+1}) = 0 \) for all \( m \geq 0 \). It is clearly true for \( m = 0, 1 \). Using (7), we have

\[
\lambda_F(p^{2m+1}) = \left[ \lambda_F(p^2) + \frac{1}{p} \right] \lambda_F(p^{2m-1}) - \lambda_F(p^{2m-3}).
\]

By induction hypothesis, one knows that

\[
\lambda_F(p^{2m-1}) = 0 = \lambda_F(p^{2m-3})
\]

and hence \( \lambda_F(p^{2m+1}) = 0 \). This completes the proof of the first part.

To prove the second part, assume that there exists \( m_0 \in \mathbb{N} \) such that

\[
(10) \quad \lambda_F(p^{2(m_0+1)+1}) = 0
\]

for all \( 0 \leq i \leq 3 \). Using (7) and (10) for \( i = 2, 3 \), we have

\[
\lambda_F(p^{2m_0+6}) = -\lambda_F(p^{2m_0+4}) = \lambda_F(p^{2m_0+2})
\]

as \( \lambda_F(p) \neq 0 \). Again using (7) and (10), we get

\[
\lambda_F(p^{2m_0+6}) = -\left[ \lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} \right] \lambda_F(p^{2m_0+4}) - \lambda_F(p^{2m_0+2}).
\]

Hence

\[
0 = \lambda_F(p^{2m_0+6}) + \lambda_F(p^{2m_0+4}) = -\left[ \lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} \right] \lambda_F(p^{2m_0+4}) - \lambda_F(p^{2m_0+2})
\]

\[
= -\left[ \lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} \right] - 2\lambda_F(p^{2m_0+2}).
\]

This implies that

\[
\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} = 2
\]

as \( \lambda_F(p^{2m_0+2}) \neq 0 \) by second part of Lemma 12. Replacing

\[
\lambda_F(p^{2m_0+4}) = -2\lambda_F(p^{2m_0+2}) - \lambda_F(p^{2m_0})
\]

in the relation

\[
0 = \lambda_F(p^{2m_0+5}) = \lambda_F(p)\left[ \lambda_F(p^{2m_0+4}) + \lambda_F(p^{2m_0+2}) \right],
\]

we get \( \lambda_F(p^{2m_0+2}) + \lambda_F(p^{2m_0}) = 0 \) as \( \lambda_F(p) \neq 0 \). Then

\[
0 = \lambda_F(p^{2m_0+3}) = \lambda_F(p)\left[ \lambda_F(p^{2m_0+2}) + \lambda_F(p^{2m_0}) \right] - \lambda_F(p^{2m_0-1}) = -\lambda_F(p^{2m_0-1}).
\]

This shows that if \( \lambda_F(p) \neq 0 \) and \( \lambda_F(p^{2(m_0+i)+1}) = 0 \) for all \( 0 \leq i \leq 3 \) and for some \( m_0 \in \mathbb{N} \), then \( \lambda_F(p^{2m_0-1}) = 0 \). Arguing similarly and using induction, we can now show that \( \lambda_F(p^{2m+1}) = 0 \)
for all $1 \leq m \leq m_0 + 3$. Note that

$$0 = \lambda_F(p^5) = \lambda_F(p)[\lambda_F(p^4) + \lambda_F(p^2) - 1]$$

$$= \lambda_F(p)[-\lambda_F(p^2) + \lambda_F^2(p) - 2]$$

$$= \frac{1}{p}\lambda_F(p),$$

a contradiction to our hypothesis. This completes the proof of the second part of Lemma 13. □

**Remark 3.1.** Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. If $\lambda_F(p) \neq 0$, then there does not exist $m \in \mathbb{N}$ such that $\lambda_F(p^{2(m+i)}) = 0$ for all $0 \leq i \leq 3$.

**Proof.** Suppose that there exists $m_0 \in \mathbb{N}$ such that

$$\lambda_F(p^{2(m_0+i)}) = 0, \text{ for } 0 \leq i \leq 3.$$ 

Arguing as in Lemma 13, then we have $2 + 1/p + \lambda_F(p^2) - \lambda_F^2(p) = 0$ and $\lambda_F(p^{2m}) = 0$ for $1 \leq m \leq m_0 + 3$ as $\lambda_F(p) \neq 0$. This implies that $\lambda_F^2(p) = 2 + 1/p$ and hence $\lambda_F(p^4) = -1$, a contradiction.

We now complete the proof of Theorem 1.

**Proof.** Without loss of generality, we can assume that $\lambda_F(p)\lambda_G(p) = 0$ and $\lambda_F(p^2)\lambda_G(p^2) = 0$, otherwise we are done.

First suppose that $\lambda_F(p) = \lambda_G(p) = \lambda_F(p^2) = \lambda_G(p^2) = 0$. Then using the identity (7), we see that $\lambda_F(p^4)\lambda_G(p^4) = 1$. Hence we are done.

Now we assume that $\lambda_F(p) = \lambda_G(p) = \lambda_F(p^2) = 0$ but $\lambda_G(p^2) \neq 0$. Then

$$\lambda_G(p^6) = \left[\lambda_G(p^2) + \frac{1}{p}\right]\lambda_G(p^4) - \lambda_G(p^2)$$

implies that either $\lambda_G(p^4) \neq 0$ or $\lambda_G(p^6) \neq 0$. Now using Lemma 12, we are done.

Next assume that $\lambda_F(p) = 0 = \lambda_F(p^2)$ and $\lambda_G(p) \neq 0$. Using Lemma 12, we know that $\lambda_F(p^{2n}) \neq 0$ for all $n \geq 2$. Since $\lambda_G(p) \neq 0$, by Remark 3.1, we have at least one of

$$\lambda_G(p^4), \lambda_G(p^6), \lambda_G(p^8), \lambda_G(p^{10})$$

is non-zero. Hence we are done in this case.

Finally, we assume that $\lambda_F(p) = 0, \lambda_F(p^2) \neq 0$ and $\lambda_G(p) \neq 0, \lambda_G(p^2) = 0$. Since $\lambda_F(p) = 0$ we know by Lemma 13 that $\lambda_F(p^{2n-1}) = 0$ for all $n \in \mathbb{N}$.

We first consider the case when $\lambda_F(p^4) = 0$. Then using (7), we have $\lambda_F(p^n) \neq 0$ for $n = 6, 8, 10, 12$. Since $\lambda_G(p) \neq 0$, using Remark 3.1, we are done.
Now assume that $\lambda_F(p^4) \neq 0$ and $\lambda_G(p^4) = 0$, otherwise we are done. We will show in this case that $\lambda_G(p^6) \neq 0$ except when $p = 2$. Since $\lambda_G(p^4) = 0$, we get

\begin{equation}
[2 + 1/p - \lambda_G^2(p)]\lambda_G^2(p) = 1.
\end{equation}

Using (11) and (7), we have

\begin{equation}
\lambda_G(p^6) = -\lambda_G(p^4) + \lambda_G(p^2)\lambda_G(p^3) = 1/p - \lambda_G^2(p).
\end{equation}

Again using (11), we see that $1/p - \lambda_G^2(p) = 0$ only when $p = 2$. If $\lambda_F(p^6) \neq 0$, we are done except when $p = 2$. So without loss of generality, we can assume that $\lambda_F(p^6) = 0$ when $p \neq 2$.

Then

\begin{equation}
1 + \lambda_F(p^4) = \left[\lambda_F(p^2) + \frac{1}{p}\right]\lambda_F(p^2), \quad \lambda_F(p^2) = \left[\lambda_F(p^2) + \frac{1}{p}\right]\lambda_F(p^4)
\end{equation}

and hence

\begin{align*}
\lambda_F(p^8) &= -\lambda_F(p^4), \\
\lambda_F(p^{10}) &= -\lambda_F(p^2), \\
\lambda_F(p^{12}) &= -1, \\
\lambda_F(p^{14}) &= -\frac{1}{p}.
\end{align*}

We are now done by Remark (3.1).

It only remains to prove the case when $p = 2$ and $\lambda_G^2(2) = 1/2$. In this case,

\begin{align*}
\lambda_G(2^8) &= -1, \\
\lambda_G(2^{10}) &= -1/2.
\end{align*}

Now note that either $\lambda_F(2^8) \neq 0$ or $\lambda_F(2^8) = 0$ and $\lambda_F(2^{10}) = -\lambda_F(2^6) \neq 0$. This completes the proof of Theorem III. \hfill \Box

4. PROOF OF THEOREM II AND COROLLARY III

In this section, we shall complete the proof of Theorem II and Corollary III. In order to prove Theorem II we first establish a relation between the functions $L(F, G; s)$ and $L(F \times G, s)$. More precisely, we show the following.

**Lemma 14.** Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Theorem II. Then for $\Re(s) > 1$, one has

\begin{equation}
L(F, G; s) = g(s)L(F \times G; s),
\end{equation}

where

\begin{equation}
g(s) := \prod_{p \in \mathcal{P}} g_p(p^{-s}).
\end{equation}

Here $g_p(X)$’s are polynomials of degree $\leq 15$ and the Euler product on the right hand side of (13) is absolutely convergent for $\Re(s) > 1/2$. Further, there exists an absolute constant $A > 0$ such that

\begin{equation}
g(s) \ll \sigma^A \left(\sigma - \frac{1}{2}\right)^{-A}.
\end{equation}
holds uniformly for any $\sigma := \Re(s) > 1/2$.

Proof. Consider the $L$-functions

$$L(F, s) := \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} \quad \text{and} \quad L(G, s) := \sum_{n=1}^{\infty} \frac{\lambda_G(n)}{n^s}.$$  

These $L$-functions are absolutely convergent for $\Re(s) > 1$ and by (4), we have

$$L(F, s) = \frac{Z_F(s)}{\zeta(2s + 1)} \quad \text{and} \quad L(G, s) = \frac{Z_G(s)}{\zeta(2s + 1)}.$$  

Here $Z_F(s), Z_G(s)$ are the spinor zeta functions associated to $F$ and $G$ respectively. Since $\lambda_F(n)$ and $\lambda_G(n)$ are multiplicative, again using (4), we can write

$$\sum_{n=0}^{\infty} \lambda_F(p^n)T^n = \frac{1 - \frac{1}{p}T^2}{\prod_{1 \leq i \leq 4}(1 - \alpha_{p,i}T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \lambda_G(p^n)T^n = \frac{1 - \frac{1}{p}T^2}{\prod_{1 \leq i \leq 4}(1 - \beta_{p,i}T)}.$$  

Now by Theorem 9 one has

$$\sum_{n=0}^{\infty} \lambda_F(p^n)\lambda_G(p^n)T^n = \frac{g_p(T)}{\prod_{1 \leq i,j \leq 4}(1 - \alpha_{p,i}\beta_{p,j}T)},$$  

where $g_p(T) \in \mathbb{C}[T]$ is a polynomial of degree at most 15. Also $g_p(0) = 1$ and $g'_p(0) = 0$, where $g'_p$ is the derivative of $g_p$. The fact $|\alpha_{p,i}| = |\beta_{p,j}| = 1$ for $1 \leq i, j \leq 4$, implies that the coefficients of $g_p(T)$ are bounded by an absolute constant. Since $g_p(0) = 1$, the coefficients of $T$ in the polynomial $g_p(T)$ is zero and other coefficients are bounded by an absolute constant, it is easy to conclude that

$$\prod_{p \in \mathcal{P}} g_p(p^{-s})$$  

is absolutely convergent for $\Re(s) > 1/2$. This shows that for $\sigma > 1$, we have

$$L(F, G; s) = L(F \times G; s)g(s).$$  

It remains to show that $g(s)$ has the required bound. Let

$$g_p(T) := 1 + a(p^2)T^2 + \cdots + a(p^{15})T^{15},$$  

where $a(p^i) \in \mathbb{C}$ and $a(p^i)$ are bounded by an absolute constant for all $2 \leq i \leq 15$ and for all $p$. Let $A > 0$ be an integer such that $|a(p^2)| \leq A$ for all $p \in \mathcal{P}$. Thus

$$|g_p(p^{-s})| = \left| 1 + \sum_{2 \leq n \leq 15} a(p^n)p^{-ns} \right| \leq h_p(\sigma),$$  

where

$$h_p(s) := 1 + Ap^{-2s} + |a(p^3)|p^{-3s} + \cdots + |a(p^{15})|p^{-15s}.$$  

Now note that
\[(14) \quad (1 - p^{-2s})^A h_p(s) = 1 + O(p^{-3\sigma}).\]
The left hand side of (14) is nothing but the $p$-th Euler factor of the Dirichlet series
\[\zeta(2s)^{-A} h(s), \quad \text{where} \quad h(s) := \prod_{p \in \mathcal{P}} h_p(s).\]
Hence for all $\sigma > 1/2$, we have
\[g(s) \ll \left( \frac{\sigma}{\sigma - 1/2} \right)^A.\]
This completes the proof of Lemma 14.

As an application of the above lemma, one can derive the following analytic properties of the $L$-function $L(F, G; s)$.

**Lemma 15.** Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Theorem 2. Then the function $L(F, G; s)$ admits an analytic continuation to $\Re(s) > 1/2$.

**Proof.** We know from Lemma 14 that for $\sigma > 1$
\[L(F, G; s) = g(s)L(F \times G, s).\]
Now holomorphicity of $g(s)$ to $\Re(s) > 1/2$ along with the fact that $L(F \times G, s)$ has analytic continuation to $\mathbb{C}$ (see Theorem 2) implies that $L(F, G; s)$ can be continued analytically upto $\sigma > 1/2$. 

To prove Theorem 2 we also need the following convexity bound.

**Lemma 16.** Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Theorem 2. Then for any $\epsilon > 0$ and $0 < \delta < 1$, one has
\[(15) \quad L(F \times G, \delta + it) \ll \epsilon \max\{k_1, k_2\}^{6(1-\delta+\epsilon)}|3 + it|^{8(1-\delta+\epsilon)}.
\]
To prove Lemma 16 we shall use the following strong convexity principle due to Rademacher.

**Proposition 17.** [Rademacher] Let $g(s)$ be holomorphic and of finite order in $a < \Re(s) < b$, and continuous on the closed strip $a \leq \Re(s) \leq b$. Also let
\[|g(a + it)| \leq E|P + a + it|^\alpha \quad \text{and} \quad |g(b + it)| \leq F|P + b + it|^\beta,
\]
where $E, F$ are positive constants and $P, \alpha, \beta$ are real constants satisfying
\[P + a > 0, \quad \alpha \geq \beta.
\]
Then for $a < \sigma < b$, we have
\[|g(s)| \leq (E|P + s|^\alpha)^{\frac{\beta - \alpha}{\beta}} (F|P + s|^\beta)^{\frac{\alpha}{\beta}}.
\]
We now complete the proof of Lemma 16.

Proof. Without loss of generality, let us assume that \( k_1 \geq k_2 > 2 \). It is known by [11, sec. 5.1] that \( F \) (also \( G \)) can be associated to a cuspidal, automorphic representation \( \pi \) (resp. \( \pi' \)) of \( \text{GSp}_4(\mathbb{A}) \) such that \( \pi \) (resp. \( \pi' \)) has trivial central character, the archimedean component \( \pi_{\infty} \) (resp. \( \pi'_{\infty} \)) is a holomorphic discrete series representation with scalar minimal \( K \)-type \( (k_1, k_2) \) and for each finite place \( p \), the local representation \( \pi_p \) [resp. \( \pi'_p \)] is unramified. Here \( \mathbb{A} \) is the ring of adeles of \( \mathbb{Q} \). The real Weil group \( W_{\mathbb{R}} \) is given by \( \mathbb{C}^\times \sqcup j\mathbb{C}^\times \) such that \( j^2 = -1 \) and \( jzj^{-1} = \bar{z} \) for \( z \in \mathbb{C}^\times \). Then the real Weil group representations underlying Siegel modular forms \( F \) and \( G \) of weights \( k_1 \) and \( k_2 \) respectively are given by (see page 90 of [11] and page 2397 of [14]) \( \varphi_{2k_1-3} \oplus \varphi_1 \) and \( \varphi_{2k_2-3} \oplus \varphi_1 \), where for \( k \in \mathbb{N} \), \( \varphi_k \) is defined by

\[
\varphi_k : \mathbb{C}^\times \ni re^{i\theta} \mapsto \begin{bmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} (1)^k \\ 1 \end{bmatrix}.
\]

Then the parameter of \( \pi_{\infty} \times \pi'_k \) is

\[
(\varphi_{2k_1-3} \oplus \varphi_1) \otimes (\varphi_{2k_2-3} \oplus \varphi_1) = \begin{cases} \varphi_{2k_1+2k_2-6} \oplus \varphi_{2(k_1-k_2)} \oplus \varphi_{2k_1-2} \oplus \varphi_{2k_1-4} & \text{if } k_1 > k_2 \\ \varphi_{4k_1-6} \oplus \varphi_+ \oplus \varphi_- \oplus \varphi_{2k_1-2} \oplus \varphi_{2k_1-4} & \text{if } k_1 = k_2. \end{cases}
\]

Here \( \varphi_+ \) and \( \varphi_- \) are given by

\[
\varphi_+ : re^{i\theta} \mapsto 1, \quad j \mapsto 1;
\]
\[
\varphi_- : re^{i\theta} \mapsto 1, \quad j \mapsto -1.
\]

Now from [14, Table 2], one can easily see that the gamma factors of \( L(F \times G, s) \) are as follows:

\[
L_\infty(F \times G, s) := \begin{cases} \Gamma_\mathbb{C}(s+k_1+k_2-3)\Gamma_\mathbb{C}(s+k_1-k_2)\Gamma_\mathbb{C}(s+k_1-1)\Gamma_\mathbb{C}(s+k_1-2) & \text{if } k_1 > k_2, \\
\Gamma_\mathbb{C}(s+k_2-1)\Gamma_\mathbb{C}(s+k_2-2)\Gamma_\mathbb{C}(s+1)\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1) & \text{if } k_1 = k_2, \end{cases}
\]

where \( \Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma(s/2) \) and \( \Gamma_\mathbb{C}(s) := \frac{1}{2}(2\pi)^{-s}\Gamma(s) \). Again by [11, Theorem 5.2.3], we know that the completed \( L \)-function

\[
L^*(F \times G, s) := L_\infty(F \times G, s) L(F \times G, s)
\]

satisfies the functional equation

\[
L^*(F \times G, 1-s) = \epsilon(F \times G, s)L^*(F \times G, s),
\]
where $\epsilon(F \times G, s) \in \mathbb{C}$ and has absolute value $1$. Thus for any $s \in \mathbb{C}$ with $\sigma > 1$, we have

$$|L(F \times G, 1-s)| = \left| \frac{L_\infty(F \times G, s)}{L_\infty(F \times G, 1-s)} \right| \cdot |L(F \times G, s)|.$$  

Note that for $s = c + it$ with $1 < c < 3/2$, we have

$$\left| \frac{L_\infty(F \times G, c + it)}{L_\infty(F \times G, 1 - c - it)} \right| \ll k_1^{6(2c-1)} |1 + it|^{8(2c-1)}.$$  

Let $c = 1 + \epsilon$ with $0 < \epsilon < 1/2$. Since $|L(F \times G, 1 + \epsilon + it)| \ll \epsilon$, for any $0 < \delta < 1$, using Proposition 17, we have

$$|L(F \times G, \delta + it)| \ll k_1^{6(1-\delta+\epsilon)} |3 + it|^{8(1-\delta+\epsilon)}.$$  

This completes the proof of the lemma.

□

Now we are ready to prove Theorem 2.

Proof of Theorem 2. From the work of Weissauer [15] one knows that the generalized Ramanujan-Petersson conjecture is true for $F$ and $G$ and so for any $\epsilon > 0$, one has

$$\lambda_F(n) \lambda_G(n) \ll n^\epsilon.$$  

Hence by the Perron’s summation formula, we have

$$\sum_{n \leq x} \lambda_F(n) \lambda_G(n) = \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} L(F, G; s) \frac{x^s}{s} ds + O \left( \frac{x^{1+2\epsilon}}{T} \right).$$

Now we shift the line of integration to $1/2 < \Re(s) := \delta < 1$ (to be chosen later). Since there are no singularities of the function $L(F, G; s)x^s/s$ in the region bounded by the lines joining the points $1 + \epsilon - iT, 1 + \epsilon + iT, \delta + iT$ and $\delta - iT$, we have

$$\sum_{n \leq x} \lambda_F(n) \lambda_G(n) = I_1 + I_2 + I_3 + O \left( \frac{x^{1+2\epsilon}}{T} \right),$$

where

$$I_1 := \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} L(F, G; s) \frac{x^s}{s} ds, \quad I_2 := \frac{1}{2\pi i} \int_{\delta+iT}^{1+\epsilon+iT} L(F, G; s) \frac{x^s}{s} ds$$

and

$$I_3 := \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{\delta-iT} L(F, G; s) \frac{x^s}{s} ds.$$  

Using Lemma 14 and Lemma 16, one can easily get

$$I_1 \ll \epsilon (\delta - 1/2)^{-A_k} k_1^{6(1-\delta+\epsilon)} x^{\delta} T^{8(1-\delta+\epsilon)},$$  

where $k = \max(k_1, k_2)$. Similarly, one can get

$$I_2, I_3 \ll \epsilon (\delta - 1/2)^{-A_k} k_1^{6(1-\delta+\epsilon)} x^{1+\epsilon} T^{8(1-\delta+\epsilon)-1}.$$
We shall put $T = x^\alpha$, where $\alpha > 0$ is a real number to be chosen later. Thus we have
\[
\sum_{n \leq x} \lambda_F(n) \lambda_G(n) \ll \epsilon (\delta - 1/2) A k^{6(1-\delta+\epsilon)} \left( x^{8\alpha(1-\delta+\epsilon)+\delta} + x^{1+8\alpha(1-\delta+\epsilon)\alpha+\epsilon} + x^{1-\alpha+\epsilon} \right).
\]
Choosing $\alpha = 1/16$ and $\delta = 15/16$, one has
\[
\sum_{n \leq x} \lambda_F(n) \lambda_G(n) \ll \epsilon k^{3/8+\epsilon} x^{31/32+\epsilon}.
\]
This completes the proof of Theorem 2.

Proof of Corollary 3. We know from Theorem 8 and Theorem 2 that
\[
\sum_{n \leq x} \lambda_F^2(n) = c_F x + O(x^{31/32}) \quad \text{and} \quad \sum_{n \leq x} \lambda_F(n) \lambda_G(n) = O(x^{31/32}),
\]
where $c_F > 0$. Suppose that $k_2 \leq k_1$. Using partial summation, we get
\[
\sum_{n \leq x} \mu_F^2(n) = cx^{2k_1-2} + O(x^{2k_1-2-1/32}) \quad \text{and} \quad \sum_{n \leq x} \mu_F(n) \mu_G(n) = O(x^{k_1+k_2-2-1/32}),
\]
where $c = \frac{c_F}{2k_1-2}$. Now let
\[
S(x) := \sum_{n \leq x} [\mu_F(n) - \mu_G(n)] \mu_F(n).
\]
Note that for any $\epsilon > 0$, we have
\[
S(x) \leq c(\epsilon) \cdot \#\{ n \leq x \mid n \in \mathbb{N}, \mu_F(n) \neq \mu_G(n) \} x^{2k_1-3+\epsilon},
\]
where $c(\epsilon) > 0$ is a constant depending only on $\epsilon > 0$. Now by applying (16), we conclude that
\[
\#\{ n \leq x \mid n \in \mathbb{N}, \mu_F(n) \neq \mu_G(n) \} \gg_{F,G,\epsilon} x^{1-\epsilon}.
\]
When $k_1 \leq k_2$, we consider the sum $\sum_{n \leq x} |\mu_G(n) - \mu_F(n)| \mu_G(n)$ and proceed as above to get the result. This completes the proof of Corollary 3.

Remark 4.1. To prove Corollary 3, we have only used the property
\[
\sum_{n \leq x} \lambda_F(n) \lambda_G(n) = o(x),
\]
as $x \to \infty$ but Theorem 2 gives an explicit upper bound and hence is also of independent interest. We also note that the Corollary 3 is weaker than the optimal one. In fact, using identities (17), (18) and the Weissauer bound and proceeding along the same line of the proof of Corollary 3, we get
\[
\#\{ p \leq x \mid p \text{ prime, } \mu_F(p) \neq \mu_G(p) \} \geq \frac{1}{32} \cdot \frac{x}{\log x}.
\]
However our proof follows without appealing to prime number theorem.
5. PROOFS OF THEOREM 4 AND THEOREM 5

In this section, we complete the proofs of Theorem 4 and Theorem 5. Let us start with the following lemma.

Lemma 18. Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms in the orthogonal complement of the Maass subspace and having normalized eigenvalues $\{\lambda_F(n)\}_{n \in \mathbb{N}}$ and $\{\lambda_G(n)\}_{n \in \mathbb{N}}$ respectively. Also assume that $F$ and $G$ lie in different eigenspaces and there exists $0 < c < 4$ such that

$$\#\{p \leq x \mid |\lambda_G(p)| > c\} \geq \frac{16}{17} \cdot \frac{x}{\log x}$$

for sufficiently large $x$. Then we have

$$\sum_{p \leq x} \lambda_F^2(p) \lambda_G^2(p) \gg \frac{x}{\log x}.$$

Proof. Note that by [11, Theorem 5.1.2], one knows that the transfers of $F$ and $G$ are irreducible unitary cuspidal and self-contragredient automorphic representations of $GL_4(\mathbb{A})$. Hence by [16, Theorem 3], we have

$$\sum_{p \leq x} \lambda_F^2(p) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad \text{and} \quad \sum_{p \leq x} \lambda_G^2(p) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

as $x \to \infty$. Let $S$ be the set of primes $p$ such that $|\lambda_G(p)| > c$. Thus for sufficiently large $x$, we have

$$\sum_{p \leq x} \lambda_F^2(p) \lambda_G^2(p) > c^2 \sum_{p \leq x, p \notin S} \lambda_F^2(p).$$

By the given hypothesis, the set

$$\sum_{\substack{p \leq x \atop p \notin S}} \lambda_F^2(p) \leq 16 \cdot \#\{p \leq x \mid p \notin S\} \leq \frac{16}{17} \cdot \frac{x}{\log x}$$

for sufficiently large $x$. This implies that

$$\sum_{p \leq x} \lambda_F^2(p) \lambda_G^2(p) \gg \frac{x}{\log x}$$

for sufficiently large $x$. This completes the proof of Lemma 18. □

We now complete the proof of Theorem 5 and then use Theorem 5 to complete the proof of Theorem 4.
5.1. **Proof of Theorem 5.** Using [11, Theorem 5.1.2], we know that the transfers of $F$ and $G$ are irreducible unitary cuspidal and self-contragredient automorphic representations of $GL_4(\mathbb{A})$. Hence by [16, Theorem 3], we have

$$\sum_{p \leq x} \lambda_F(p)\lambda_G(p) = o\left(\frac{x}{\log x}\right),$$

as $x \to \infty$. Consider the sum

$$S^+(x) := \sum_{p \leq x} [\lambda_F(p)\lambda_G(p) + 16]\lambda_F(p)\lambda_G(p).$$

Observe that

$$S^+(x) \leq \sum_{\lambda_F(p)\lambda_G(p) > 0} [\lambda_F(p)\lambda_G(p) + 16]\lambda_F(p)\lambda_G(p) \leq 512 \cdot \#\{p \leq x \mid \lambda_F(p)\lambda_G(p) > 0\}.$$

On the other hand, using Lemma 18 and (18), we have for sufficiently large $x$

$$S^+(x) = \sum_{p \leq x} \lambda^2_F(p)\lambda^2_G(p) + 16 \sum_{p \leq x} \lambda_F(p)\lambda_G(p) \gg \frac{x}{\log x}.$$

Thus by (19) and (20), we conclude that there exists a set of primes having positive density such that $\lambda_F(p)\lambda_G(p) > 0$. Similarly, by considering the sum

$$S^-(x) := \sum_{p \leq x} [\lambda_F(p)\lambda_G(p) - 16]\lambda_F(p)\lambda_G(p)$$

and arguing as above one can conclude that there exists a set of primes having positive density such that $\lambda_F(p)\lambda_G(p) < 0$. □

5.2. **Proof of Theorem 4.** It follows from Theorem 5 that there exists $\delta > 0$ such that

$$\#\{p \leq x \mid \lambda_F(p)\lambda_G(p) = 0\} \leq \#\{p \leq x \mid \lambda_F(p) = 0\} + \#\{p \leq x \mid \lambda_G(p) = 0\} = O\left(\frac{x}{(\log x)^{1+\delta}}\right)$$

for sufficiently large $x$. Also by Theorem 5 we know that the set $\{p \in \mathcal{P} \mid \lambda_F(p)\lambda_G(p) < 0\}$ has positive lower density. Hence the multiplicative function $\lambda_F(n)\lambda_G(n)$ satisfies the hypothesis of Lemma 10. We now apply Lemma 10 to complete the proof of Theorem 4. □

**Remark 5.1.** Let $F, G$ be elliptic non-CM cusp forms of weights $k_1, k_2$ and levels $N_1, N_2$ respectively. Also let $F$ and $G$ be distinct Hecke eigenforms with eigenvalues $\{\mu_F(n)\}_{n \in \mathbb{N}}$ and $\{\mu_G(n)\}_{n \in \mathbb{N}}$ respectively. Then the method adopted here for Theorem 4 can be applied to prove unconditionally that half of the non-zero coefficients of the sequence $\{\mu_F(n)\mu_G(n)\}_{n \in \mathbb{N}}$ are positive and half of them are negative. One can also show unconditionally that there exists a set of primes $p$ of positive lower density such that $\mu_F(p)\mu_G(p) \geq 0$. 


Acknowledgment: We would like to thank Ralf Schmidt for sending us his paper and useful comments.

REFERENCES

[1] A. N. Andrianov, Euler products that correspond to Siegel modular forms of genus 2, Russian Math. Surveys 29 (1974), no. 3, 45–116.
[2] S. Böcherer, Bemerkungen über die Dirichletreihen von Koecher und Maass, Mathematica Gottingensis 68 (1986), 36 pp.
[3] S. Breulmann, On Hecke eigenforms in the Maass space, Math. Z. 232 (1999), no. 3, 527–530.
[4] S. Das, W. Kohnen and J. Sengupta, On a convolution series attached to a Siegel Hecke cusp form of degree 2, Ramanujan J. 33 (2014), no. 3, 367–378.
[5] S. Gun, B. Kumar and B. Paul, The first simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms to appear in J. Number Theory.
[6] S. Gun and M. R. Murty, Generalization of an identity of Ramanujan, J. Ramanujan Math. Soc. 31 (2016), no. 2, 125–135.
[7] S. Gun, B. Paul and J. Sengupta, On Hecke eigenvalues of Siegel modular forms in the Maass space, Forum Math. 30 (2018), no. 3, 775–783.
[8] W. Kohnen, A note on eigenvalues of Hecke operators on the Siegel modular forms of degree two, Proc. AMS 113 (1991), no. 3, 639–642.
[9] K. Matomäki and M. Radziwill, Sign changes of Hecke eigenvalues, Geom. Funct. Anal. 25 (2015), 1937–1955.
[10] T. Oda, On the poles of Andrianov L-functions, Math. Ann. 256 (1981), 323–340.
[11] A. Pitale, A. Saha and R. Schmidt, Transfer of Siegel cusp forms of degree 2, Mem. Amer. Math. Soc. 232 (2014), no. 1090, vi+107.
[12] E. Royer, J. Sengupta and J. Wu, Non-vanishing and sign changes of Hecke eigenvalues for Siegel cusp forms of genus two. With an appendix by E. Kowalski and A. Saha, Ramanujan J. 39 (2016), no. 1, 179–199.
[13] A. Saha, A relation between multiplicity one and Böcherer’s conjecture, Ramanujan J. 33 (2014), no. 2, 263–268.
[14] R. Schmidt, Archimedean aspects of Siegel modular forms of degree 2, Rocky Mountain J. Math. 47 (2017), 2381–2422.
[15] R. Weissauer, Endoscopy for GSp(4) and the cohomology of Siegel modular threefolds, Lecture Notes in Math. 1968, Springer, Berlin, 2009.
[16] J. Wu and Y. Ye, Hypothesis H and the prime number theorem for automorphic representations, Funct. Approx. Comment. Math. XXXVII.2 (2007), 461–471.
[17] D. Zagier, Sur la conjecture de Saito-Kurokawa (d’après H. Maass), Seminar on Number Theory, Paris 1979–80, pp. 371–394, Progr. Math. 12, Birkhäuser, 1981.

(Sanoli Gun, Biplab Paul) INSTITUTE OF MATHEMATICAL SCIENCES, HOMI BHABHA NATIONAL INSTITUTE, C.I.T CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.
E-mail address: sanoli@imsc.res.in
E-mail address: biplabpaul@imsc.res.in

(Winfried Kohnen) MATHEMATISCHES INSTITUT DER UNIVERSITÄT, INF 288, D-69120, HEIDELBERG, GERMANY.
E-mail address: winfried@mathi.uni-heidelberg.de