Noisy non-transitive quantum games

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Abstract

We study the effect of quantum noise in 3 × 3 entangled quantum games. By taking into account different noisy quantum channels, we analyze how a two-player, three-strategy Rock–Scissor–Paper game is influenced by the quantum noise. We consider the winning non-transitive strategies R, S and P such that R beats S, S beats P and P beats R. The game behaves as a noiseless game for the maximum value of the quantum noise. It is seen that Alice’s payoff is heavily influenced by the depolarizing noise as compared to the amplitude damping noise. A depolarizing channel causes a monotonic decrease in players’ payoffs as we increase the amount of quantum noise. In the case of the amplitude damping channel, Alice’s payoff function reaches its minimum for $\alpha = 0.5$ and is symmetrical. This means that larger values of quantum noise influence the game weakly. On the other hand, the phase damping channel does not influence the game. Furthermore, the Nash equilibrium and non-transitive character of the game are not affected under the influence of quantum noise.

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1. Introduction

Recently, rapid interest has been developed in the discipline of quantum information [1] which has led to the creation of quantum game theory [2–5]. In the last few years, tremendous efforts have been made toward the development of quantum game theory [6–12]. In this area, much work has been devoted to convert classical games into the quantum domain such as the prisoners’ dilemma game [13–15] and many other games [16–20] involving two and three players. Quantum games with 3 × 3 payoff matrices and larger have been discussed by Wang et al [21]. Recently, Sousa et al [22] have proposed that quantum games can be used to control the access of processes to the CPU in a quantum computer. More recently, Iqbal and Abbott [23] have reported that quantum games can directly be constructed from a system of Bell’s inequalities using Arthur Fine’s analysis.

In quantum information processing, the major problem is to faithfully transmit unknown quantum states through a noisy quantum channel. When quantum information is sent through
Table 1. In the ‘rock, scissors, paper’ game, a player can win regardless of the strategy chosen by an opponent. The first number in each entry corresponds to Alice’s payoff and the second number corresponds to Bob. Winning strategies are non-transitive in that \( R > S > P > R \). A payoff of +1 has been assigned to winning, −1 to losing and 0 for both in the case of a tie.

|       | Bob       |         |       |
|-------|-----------|---------|-------|
|       | \( R \)   | \( S \) | \( P \) |
| Alice | \( (0, 0)\) | \( (−1, 1)\) | \( (−1, 1)\) |
| Alice | \( (−1, 1)\) | \( (0, 0)\) | \( (1, −1)\) |
| Alice | \( (1, −1)\) | \( (−1, 1)\) | \( (0, 0)\) |

In this paper, we study the effect of quantum noise in a two-player, three-strategy entangled quantum game (RSP game). We consider different noisy channels parameterized by a quantum noise parameter \( \alpha \) such that \( \alpha \in [0, 1] \). The lower and upper limits of the quantum noise parameter represent the fully coherent and fully decohered system, respectively. It is seen that for the maximum value of the quantum noise parameter the game behaves as a noiseless game. The depolarizing channel influences the game’s payoff more heavily as compared to the amplitude damping channel. The payoff function for the amplitude damping channel is symmetrical with its minimum at \( \alpha = 0.5 \). Furthermore, the phase damping channel does not influence the game.

2. Noisy quantum RSP game

The rock, scissors and paper game is a game for two players typically played using hands. It is a simple two-player, three-strategy game, the children’s chosen game ‘rock (\( R \)), scissors (\( S \)) and paper (\( P \))’ denoted by RSP, in which rock beats scissors, scissors beats paper, and paper beats rock (\( R > S > P > R \)). The classical payoff matrix for this game is given in table 1. The RSP game is a zero-sum game and it has no pure strategy Nash equilibrium. Since this game has the feature that no single choice is best, instead, the best strategy for rational players is to make the choices randomly and with equal probability, which gives it a mixed strategy Nash equilibrium. The mixed strategy Nash equilibrium produces zero expected payoff for the classical game when both players use each of \( R \), \( S \) and \( P \) with probability 1/3.

A quantum analog of the RSP game was formulated by Iqbal et al [12] and Stohler et al [31]. In the quantum version of this game, the strategies \( R \), \( S \) and \( P \) are represented by three matrices \( U_1 \), \( U_2 \) and \( U_3 \) given as [31]

\[
U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (1)
\]
In this paper, we adopt the quantization scheme as introduced in [8], by considering a general entangled state parameterized by an entanglement parameter $\gamma$. We assume that initially Alice and Bob share a general entangled qutrit state of the form

$$|\Psi_{in}\rangle = \cos \gamma |00\rangle + \sin \gamma \sin^2 \theta |11\rangle + \sin \gamma \cos^2 \theta |22\rangle,$$

with the normalized coefficient

$$N = \sqrt{\sin^2 \gamma (\cos^4 \theta + \sin^4 \theta) + \cos^2 \gamma}.$$  

If we set $\theta = \pi/4$ and $\tan \gamma = 2$ in the above equation, the two-qutrit state becomes the maximally entangled state of the form

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle),$$

and for $\gamma = 0$, the state becomes an unentangled state $|00\rangle$ producing the results for the classical game. We can define the strategies of the players by the unitary operator $U(x, y)$ of the form [31]. The unitary matrix $U(x, y)$, is used to transform the initial qutrit state of the game and is given by

$$U(x, y) = \begin{bmatrix}
  e^{i x} \cos y & i e^{i x} \sin y & 0 \\
  i \sin y \cos x & \cos x \cos y & i e^{i y} \sin x \\
  -\sin y \sin x & i \sin x \cos y & e^{i y} \cos x
  \end{bmatrix},$$

where $0 \leq \{x, y\} \leq \pi/2$ and the choice of $x$ and $y$ define a player’s mixed strategy in a $3 \times 3$ game, just as the choice of $\theta$ does in the $2 \times 2$ game [3]. Here the set of strategies $(x_1, y_1)$ and $(x_2, y_2)$ define Alice’s and Bob’s mixed strategies.

The interaction between the system and its environment introduces the decoherence to the system, which is a process of the undesired correlation between the system and the environment. The evolution of a state of a quantum system in a noisy environment can be described by the super-operator $\Phi$ in the Kraus operator representation as [1]

$$\rho_f = \Phi \rho_i = \sum_k E_k \rho_i E_k^\dagger,$$

where the Kraus operators $E_i$ satisfy the following completeness relation:

$$\sum_k E_k^\dagger E_k = I.$$

We have constructed Kraus operators for the game from single qutrit Kraus operators (as given in equations (9)–(11) below) by taking their tensor product over all $n^2$ combinations of $\pi (i)$ indices

$$E_k = \otimes_{\pi} e^{\pi (i)},$$

where $n$ is the number of Kraus operators for a single qutrit channel. The single qutrit Kraus operators for the amplitude damping channel are given by [32]

$$E_0 = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \sqrt{1-\alpha} & 0 \\
  0 & 0 & \sqrt{1-\alpha}
  \end{pmatrix}, \quad E_1 = \begin{pmatrix}
  0 & \sqrt{\alpha} & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{pmatrix}, \quad E_2 = \begin{pmatrix}
  0 & 0 & \sqrt{\alpha} \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{pmatrix}. \quad (9)$$

Similarly, the single qutrit Kraus operators for the phase damping channel are given as [32]

$$E_0 = \sqrt{1-\alpha} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{pmatrix}, \quad E_1 = \sqrt{\alpha} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \omega & 0 \\
  0 & 0 & \omega^2
  \end{pmatrix}. \quad (10)$$
The single qutrit Kraus operators for the depolarizing channel are given by [33]

\[ E_0 = \sqrt{1 - \alpha} I, \quad E_1 = \sqrt{\frac{\alpha}{8}} Y, \quad E_2 = \sqrt{\frac{\alpha}{8}} Z, \quad E_3 = \sqrt{\frac{\alpha}{8}} Y^2, \quad E_4 = \sqrt{\frac{\alpha}{8}}YZ, \]

\[ E_5 = \sqrt{\frac{\alpha}{8}} Y^2Z, \quad E_6 = \sqrt{\frac{\alpha}{8}}YZ^2, \quad E_7 = \sqrt{\frac{\alpha}{8}} Y^2Z^2, \quad E_8 = \sqrt{\frac{\alpha}{8}} Z^2, \]

\[ (11) \]

where

\[ Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \]

\[ (12) \]

In the above equations, \( \alpha \) represents the quantum noise parameter and \( \omega = e^{\frac{2\pi i}{3}} \). The final state of the game after the action of the channel can be written as

\[ \rho_f = \Phi_\alpha(|\Psi_{in}\rangle \langle \Psi_{in}|), \]

\[ (13) \]

where \( \Phi_\alpha \) is the super-operator realizing the quantum channel parametrized by the real number \( \alpha \) (quantum noise parameter). After the action of the players’ unitary operations, the game’s final state transforms into

\[ \rho_j = (U_A(x, y) \otimes U_B(x, y))(\Phi_\alpha(|\Psi_{in}\rangle \langle \Psi_{in}|)(U_A^\dagger(x, y) \otimes U_B^\dagger(x, y)). \]

\[ (14) \]

The payoff operators for Alice and Bob can be written as

\[ (P_{A,B})_{\text{Oper.}} = \sum_{i,j=0}^{2} S^A_{ij} P_{ij}, \]

\[ (15) \]

where \( S^A_{ij} \) are the elements of the payoff matrix in the \( i \)th row and \( j \)th column of the classical game as given in table 1 and

\[ P_{ij} = \sum_{i,j=0}^{2} |ij\rangle \langle ij|. \]

\[ (16) \]

The players’ payoffs can be calculated by using the relation

\[ S^{A,B}(x_i, y_j, \alpha) = \text{Tr}[(P^{A,B})_{\text{Oper.}} \rho_f^{A,B}], \]

\[ (17) \]

where Tr represents the trace of the matrix. The payoffs of Alice and Bob, when product state \( |00\rangle \) is used as initial state of the game, becomes

\[ S^{A,B}_{cl}(x_i, y_j) = \cos^2 y_1 \cos^2 y_2 s_{00} + \cos^2 x_2 \cos^2 y_1 \sin^2 y_2 s_{01} + \cos^2 y_1 \sin^2 x_2 \sin^2 y_2 s_{02} + \cos^2 x_1 \cos^2 y_2 \sin^2 y_1 s_{10} + \cos^2 x_1 \sin^2 x_2 \sin^2 y_1 \sin^2 y_2 s_{11} + \cos^2 x_1 \sin^2 x_2 \sin^2 y_1 \sin^2 y_2 s_{12} + \cos^2 y_2 \sin^2 x_1 \sin^2 y_1 s_{20} + \sin^2 x_1 \cos^2 x_2 \sin^2 y_1 \sin^2 y_2 s_{21} + \sin^2 x_1 \sin^2 x_2 \sin^2 y_1 \sin^2 y_2 s_{22}, \]

\[ (18) \]

which gives 0 expected payoff if we set \( x_1 = y_1 = 0 \) and \( x_2 = y_2 = 0 \) (i.e. when both the players play their classical strategies) and put the values of the elements of payoff matrix \( S^A_{ij} \) as given in table 1. The payoffs of Alice and Bob for the general entangled initial state (equation (2)) are obtained as

\[ S^{A,B}_{gen}(x_i, y_j) = \frac{1}{4\cos^2(y) + (3 + \cos(4\theta))\sin^2(y)}[2(2\cos^4(\theta)\sin(y))\sin^2(x_1) \times (S_{12}\cos^2(x_2) + S_{11}\sin^2(x_2)) + \cos^2(x_1)(S_{22}\cos^2(x_2) + S_{21}\sin^2(x_2)) - (S_{11} - S_{12} - S_{21} + S_{22})\cos^2(\theta)\cos(y_1 + y_2)], \]

\[ (19) \]
where the subscripts AD, Dep and PD in equations (20)–(22) represent the amplitude damping, depolarizing and phase damping channels, respectively. The coefficients $A_{ij}$, $B_{ij}$ and $C_{ij}$ are given in the appendix. It can be easily checked from equations (20)–(22) that by setting $\alpha = 0$ and $\gamma = 0$ which reduces to the result obtained in equation (18).

2.1. Results using different noise models

By using equations (2)–(6), (9) and (13)–(17), the players’ payoffs in the case of a maximally entangled initial state for the amplitude damping channel are obtained as

$$S_{AD}^{A,B}(x_i, y_i, \alpha) = \frac{1}{12} \left( A_{00}S_{00} + A_{01}S_{01} + A_{02}S_{02} + A_{10}S_{10} + A_{11}S_{11} + A_{12}S_{12} + A_{20}S_{20} + A_{21}S_{21} + 2A_{22}S_{22} \right).$$

By using equations (2)–(6), (11)–(12) and (13)–(17), the players’ payoffs for the maximally entangled case for depolarizing channel become

$$S_{Dep}^{A,B}(x_i, y_i, \alpha) = \frac{1}{3072} \left( 8B_{00}S_{00} + B_{01}S_{01} + B_{02}S_{02} + B_{10}S_{10} + B_{11}S_{11} + B_{12}S_{12} + B_{20}S_{20} + B_{21}S_{21} + B_{22}S_{22} \right).$$

Similarly, using equations (2)–(6), (7), (10) and (13)–(17), the players’ payoffs for the maximally entangled initial state for the phase damping channel are given by

$$S_{PD}^{A,B}(x_i, y_i, \alpha) = \frac{1}{172} \left( 16C_{00}S_{00} + 8C_{01}S_{01} + 8C_{02}S_{02} + 16C_{10}S_{10} + 8C_{11}S_{11} + 8C_{12}S_{12} + 4C_{20}S_{20} + 4C_{21}S_{21} + 4C_{22}S_{22} \right).$$

The above equation reduces to the classical results for $\gamma = 0$ as given in equation (18). Therefore, it is clear that the underlying classical game is a subset of the quantum game. In the next section we present our results for different noise channels. We have made our calculations by considering the general entangled initial qutrit state. For the sake of simplicity, we present only the results for a maximally entangled state. However, we have checked our results by setting $\alpha = 0$ and $\gamma = 0$ which reduces to the result obtained in equation (18).

3. Discussions

In this work, we analyze the non-transitive two-player, three-strategy entangled quantum game usually termed as the RSP game influenced by the quantum noise. We consider different...
noisy quantum channels and show that how the game’s payoff is influenced by these quantum channels. We consider the winning non-transitive strategies $R$, $S$ and $P$ such that $R$ beats $S$, $S$ beats $P$, and $P$ beats $R$ ($R > S > P > R$). It is seen that the game’s payoff is differently influenced by different quantum channels.

In figure 1, we plot Alice’s payoff as a function of the quantum noise parameter $\alpha$ for $x_1 = y_1 = \pi/2$, $x_2 = y_2 = 0$ for amplitude damping (solid line), depolarizing (dashed line) and phase damping (dotted line) channels. It is seen that Alice’s payoff is heavily influenced by depolarizing noise as compared to the amplitude damping noise. It causes a monotonic decrease in players’ payoffs as the amount of quantum noise is increased. It is evident from the figure that in the case of amplitude damping channel, Alice’s payoff reaches its minimum for $\alpha = 0.5$ and it is symmetrical. This implies that the larger amount of quantum noise influences the game weakly. However, the phase damping channel has no effect on the players’ payoffs.

In figure 2, we plot Alice’s payoff as a function of her strategy $x_1$ for $y_1 = \pi/2$, $x_2 = y_2 = 0$ and $\alpha = 0.5$ for amplitude damping (solid line), depolarizing (dashed line) and phase damping (dotted line) channels. It can be seen from the figure that the phase damping channel does not influence Alice’s payoff. On the other hand, Alice’s payoff is strongly affected by amplitude damping and depolarizing channels.

In figure 3, we plot Alice’s payoff as a function of her strategy $y_1$ for $x_1 = \pi/2$, $x_2 = y_2 = 0$ and $\alpha = 0.5$ for amplitude damping (solid line), depolarizing (dashed line) and phase damping (dotted line) channels. It can be seen that Alice’s payoff is decreased due to the presence of quantum noise for both the amplitude damping and depolarizing channels. On the other hand, in the case of phase damping channel, Alice’s payoff remains unaffected in the presence of quantum noise. In addition, it can be easily checked that for the maximum value of the quantum noise parameter (i.e. at $\alpha = 1$) the game behaves as a noiseless game. For $\alpha = 1$, we obtain a single curve for all the three channels (the phase damping channel curve in figures 2 and 3). Hence, we can say that the game becomes a noiseless game for the maximum value of quantum noise.
In figures 4–6, we present the 3D graphs of Alice’s payoff as a function of $\alpha$ and her strategy $x_1$ for $y_1 = \pi/2$, $x_2 = y_2 = 0$ for amplitude damping, depolarizing and phase damping channels, respectively. One can easily see that the depolarizing channel influences the game more strongly as compared to the amplitude damping channel. In figure 7, we plot Alice’s payoff as a function of her strategies $x_1$ and $y_1$ for $\alpha = 0.3$ and $x_2 = y_2 = 0$ for the amplitude damping channel. It is important to note that similar graphs (as shown in figure 7)
Figure 4. Alice’s payoff plotted as a function of her strategy $x_1$ and $\alpha$ for $y_1 = \pi/2, x_2 = y_2 = 0$ for the amplitude damping channel.

Figure 5. Alice’s payoff plotted as a function of her strategy $x_1$ and $\alpha$ for $y_1 = \pi/2, x_2 = y_2 = 0$ for the depolarizing channel.

are obtained for the amplitude damping and depolarizing channels for the entire range of the quantum noise parameter (01), except that Alice’s payoff is heavily influenced by the depolarizing channel in comparison to the amplitude damping channel. In other words, the payoffs are reduced from 1 to 0.5 as we increase the value of the quantum noise parameter from 0 to 0.99 and for $\alpha = 1$, the game becomes noiseless (in the phase damping channel.
Figure 6. Alice’s payoff plotted as a function of her strategy $x_1$ and $\alpha$ for $y_1 = \pi/2$, $x_2 = y_2 = 0$ for the phase damping channel.

Figure 7. Alice’s payoff plotted as a function of her strategies $x_1$ and $y_1$ for $\alpha = 0.3$ and $x_2 = y_2 = 0$ for amplitude damping.

In case, payoff reaches to 1 for Alice’s strategies $x_1 = y_1 = \pi/2$). However, the phase damping channel does not influence Alice’s payoff; it is also clear from figure 6 that the payoff remains 1 for the entire range of $\alpha$ at $x_1 = \pi/2$. Furthermore, a similar graph as in figure 7 is seen for the payoff function for the phase damping channel with its maximum value 1 at $x_1 = y_1 = \pi/2$. 
It is therefore clear from figure 7 that due to the presence of noise, the Nash equilibrium strategy of Alice is not changed for the entire range of her strategies $x_1$ and $y_1$. Therefore, the Nash equilibrium of the game does not change under the influence of quantum noise and the non-transitive character of the classical game is not affected.

4. Conclusions

We study the influence of quantum noise on the Rock–Scissor–Paper (RSP) game under different noise models. We consider the winning non-transitive strategies $R$, $S$ and $P$ such that $R$ beats $S$, $S$ beats $P$, and $P$ beats $R$. Our investigations show that for the maximum value of the quantum noise parameter the game behaves as a noiseless game. It is seen that the game’s payoff is strongly influenced by the depolarizing noise as compared to the amplitude damping noise. It is shown that under the influence of depolarizing channel the players’ payoffs decrease monotonically as a function of the quantum noise parameter. However, in the case of amplitude damping channel, the payoff function reaches its minimum for $\alpha = 0.5$ and is symmetrical. Therefore, the amplitude damping channel influences the game weakly for higher values of quantum noise. Furthermore, the phase damping channel does not influence the game’s payoff. Therefore, the game deserves a careful study during its implementation.

It is also seen that the non-transitive character of the classical game and the game’s Nash equilibrium are not affected by the quantum noise.

Appendix

The coefficients $A_{ij}$ in equation (20) are given as

\[ A_{00} = (2 + p^2 + p(2 + p) \cos(2y_2) + \cos(2y_1)) \times (p(2 + p) + (2 - 4p + 5p^2) \cos(2y_2)) \times 2 \sin(2y_1) \sin(2y_2) + 2p \sin(2y_1) \sin(2y_2) \] (A.1)

\[ A_{01} = (-4(-1 + p))p \cos^2(y_1) \sin^2(x_2) + \cos^2(x_2) \times (2 + p^2 - p(2 + p) \cos(2y_2) + \cos(2y_1)(p(2 + p)) + (-2 + 4p - 5p^2) \cos(2y_2) \times 2 \sin(2y_1) \sin(2y_2) - 2p \sin(2y_1) \sin(2y_2)) \] (A.2)

\[ A_{02} = (-4(-1 + p))p \cos^2(x_2) \cos^2(y_1) + \sin^2(x_2)(2 + p^2 - p(2 + p) \cos(2y_2) + \cos(2y_1)(p(2 + p)) + (-2 + 4p - 5p^2) \cos(2y_2) \times 2 \sin(2y_1) \sin(2y_2) - 2p \sin(2y_1) \sin(2y_2)) \] (A.3)

\[ A_{10} = (-4(-1 + p))p \cos^2(y_2) \sin^2(x_1) + \cos^2(x_1)(2 + p^2 + p(2 + p) \cos(2y_2) - \cos(2y_1)(p(2 + p)) + (2 - 4p + 5p^2) \cos(2y_2) \times 2 \sin(2y_1) \sin(2y_2) - 2p \sin(2y_1) \sin(2y_2)) \] (A.4)

\[ A_{11} = (2(-1 + p)(- \cos(y_1 + y_2)) \sin(2x_1) \sin(2y_2) \times ((-1 + p) \cos(y_1) \cos(y_2) + \sin(y_1) \sin(y_2)) \]
\[ A_{12} \] = \(-\frac{1}{2} \cos^2(x_1)(-4 - 2p^2 + 2p(2 + p) \cos(2y_1))
- p(-2 + 5p) \cos(2y_1 - y_2) + 4p \cos(2y_2)
+ 2p^2 \cos(2y_2) - 4 \cos(2y_1 + y_2))
+ 6p \cos(2y_1 + y_2) - 5p^2 \cos(2(y_1 + y_2)) \sin^2(x_2)
+ 4(-1 + p) \cos^2(x_2)((-1 + p) \sin(x_1)^2 - p \cos^2(x_1)
\sin^2(y_2)) + 2(-1 + p)((-1 + p) \cos(y_1) \cos(y_2)
\times \cos(y_1 + y_2) \sin(2x_1) \sin(2x_2) + \sin(y_2)(\cos(y_1 + y_2)
\times \sin(2x_1) \sin(2x_2) \sin(y_1) - 2p \sin^2(x_1) \sin^2(x_2) \sin(y_2)))

\[ A_{20} \] = \(-4(-1 + p) \cos^2(x_1) \cos^2(y_2)
+ \sin^2(x_1)(2 + p^2 + p(2 + p) \cos(2y_2)
- \cos(2y_1)(p(2 + p) + (2 - 4p + 5p^2) \cos(2y_2))
+ 2 \sin(2y_1) \sin(2y_2) - 2p \sin(2y_1) \sin(2y_2))

\[ A_{21} \] = \(-2(-1 + p) \cos^2(x_2) \cos^2(y_1)(1 + (1 - 2p) \cos(2y_2)) \sin^2(x_1)
+ 4 \cos^2(x_1) \sin^2(x_2) - 8p \cos^2(x_1) \sin^2(x_2)
+ 4p^2 \cos^2(x_1) \sin^2(x_2)
- 4(-1 + p) \cos^2(x_1) \cos^2(y_2) \sin^2(x_1) \sin^2(y_1)
+ 4p \sin^2(x_1) \sin^2(x_2) \sin^2(y_1) - 4p^2 \sin^2(x_1) \sin^2(x_2) \sin^2(y_1)
- 4(-1 + p) \cos^2(x_1) \cos^2(x_2) \sin^2(y_2)
+ 4(1 + 2p^2) \cos^2(x_2) \sin^2(x_1) \sin^2(y_1) \sin^2(y_2)
+ 2(-1 + p) \cos(y_1 + y_2) \sin(2x_1) \sin(2x_2)((-1 + p) \cos(y_1) \cos(y_2)
+ \sin(y_1) \sin(y_2)) + 2(-1 + p) \cos^2(x_2) \sin^2(x_1) \sin(2y_1) \sin(2y_2))

\[ A_{22} \] = \((-(-1 + p) \cos^2(y_1)(-1 + (-1 + 2p) \cos(2y_2)) \sin^2(x_1) \sin^2(x_2)
- (-1 + p^2) \cos(y_1) \cos(y_2) \cos(y_1 + y_2) \sin(2x_1) \sin(2x_2)
+ 2p \cos(x_2) \sin^2(x_1) \sin^2(y_1) - 2p^2 \cos^2(x_2) \sin^2(x_1) \sin^2(y_1)
+ 2p \cos^2(y_2) \sin^2(x_1) \sin^2(y_1) \sin^2(y_1)
- 2p^2 \cos^2(y_2) \sin^2(x_1) \sin^2(x_2) \sin^2(y_1)
+ \cos(y_1 + y_2) \sin(2x_1) \sin(2x_2) \sin(y_1) \sin(y_2)
- p \cos(y_1 + y_2) \sin(2x_1) \sin(2x_2)
\times \sin(y_1) \sin(y_2) + 2 \sin^2(x_1) \sin^2(x_2) \sin^2(y_1) \sin^2(y_2)
+ 4p^2 \sin^2(x_1) \sin^2(x_2) \sin^2(y_1) \sin^2(y_2)
+ 2(-1 + p) \cos^2(x_2)((-1 + p) \cos^2(x_2)
- p \sin^2(x_2) \sin^2(y_2)) - \sin^2(x_1) \sin^2(x_2) \sin(2y_1) \sin(2y_2)
+ p \sin^2(x_1) \sin^2(x_2) \sin(2y_1) \sin(2y_2))

(9)
The coefficients $B_{ij}$ in equation (21) are given as

$$B_{00} = (64 + 3p(-16 + 9p) + (8 - 9p)^2 \cos(2(y_1 + y_2)))$$

(A.10)

$$B_{20} = (256 - 12p(-16 + 9p) - 4(8 - 9p)^2 \cos(2(y_1 + y_2)) - 8(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2))$$

(A.11)

$$B_{10} = (256 - 12p(-16 + 9p) - 4(8 - 9p)^2 \cos(2(y_1 + y_2)) + 8(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2))$$

(A.12)

$$B_{02} = (256 - 12p(-16 + 9p) - 4(8 - 9p)^2 \cos(2(y_1 + y_2)) - 8(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2))$$

(A.13)

$$B_{21} = (384 + 6p(-16 + 9p) + 2(8 - 9p)^2 \cos(2(y_1 + y_2))$$

$$- (8 - 9p)^2 \cos(2(x_1 + x_2))(5 + 3 \cos(2(y_1 + y_2))) + 4(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2)$$

$$- 2(8 - 9p)^2 \cos(2(x_1 - x_2)) \sin^2(y_1 + y_2) - 4(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2))$$

(A.14)

$$B_{11} = (384 + 6p(-16 + 9p) + 2(8 - 9p)^2 \cos(2(y_1 + y_2))$$

$$+ (8 - 9p)^2 \cos(2(x_1 + x_2))(5 + 3 \cos(2(y_1 + y_2))) - 4(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2)$$

$$+ 2(8 - 9p)^2 \cos(2(x_1 - x_2)) \sin^2(y_1 + y_2) - 4(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2))$$

(A.15)

$$B_{12} = (384 + 6p(-16 + 9p) + 2(8 - 9p)^2 \cos(2(y_1 + y_2))$$

$$- (8 - 9p)^2 \cos(2(x_1 + x_2))(5 + 3 \cos(2(y_1 + y_2)))$$

$$- 4(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2)$$

$$- 2(8 - 9p)^2 \cos(2(x_1 - x_2)) \sin^2(y_1 + y_2) + 4(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2))$$

(A.16)

$$B_{22} = (384 + 6p(-16 + 9p) + 2(8 - 9p)^2 \cos(2(y_1 + y_2))$$

$$+ (8 - 9p)^2 \cos(2(x_1 + x_2))(5 + 3 \cos(2(y_1 + y_2)))$$

$$+ 4(8 - 9p)^2 \cos(2x_1) \sin^2(y_1 + y_2)$$

$$+ 2(8 - 9p)^2 \cos(2(x_1 - x_2)) \sin^2(y_1 + y_2) + 4(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2))$$

(A.17)

$$B_{01} = (256 - 12p(-16 + 9p) - 4(8 - 9p)^2 \cos(2(y_1 + y_2))$$

$$+ 8(8 - 9p)^2 \cos(2x_2) \sin^2(y_1 + y_2)).$$

(A.18)

The coefficients $C_{ij}$ in equation (22) are given as

$$C_{00} = 2 + 2 \cos(2y_1) \cos(2y_2) + (-2 - 3(-2 + p)p) \sin(2y_1) \sin(2y_2))$$

(A.19)

$$C_{01} = \cos^2(x_2)^2(4 + 5(-2 + p)p \cos(2(y_1 - y_2))$$

$$+ (-4 + 6p - 3p^2) \cos(2(y_1 + y_2)))$$

(A.20)
\[ C_{02} = (4 + 3(-2 + p)p \cos(2(y_1 - y_2)) + (-4 + 6p - 3p^2) \cos(2(y_1 + y_2))) \sin^2(y_2) \] (A.21)

\[ C_{10} = \cos^2(x_1)(2 - 2 \cos(2y_1) \cos(2y_2)) + (2 + 3(-2 + p)p) \sin(2y_1) \sin(2y_2) \] (A.22)

\[ C_{11} = (2 + 2 \cos^2(x_1) - 2 \cos(2x_1) - 2 \cos(2x_2) + 2 \cos^2(x_1) \cos(2x_2)) + 2 \cos(2x_1) \cos(2x_2) + 2 \cos^2(x_1) \cos(2y_1) \cos(2y_2) + 2 \cos^2(x_1) \cos(2x_2) \cos(2y_1) \cos(2y_2) - (2 + 3(-2 + p)p) \cos^2(x_1) \sin(2y_1) \sin(2y_2) - (2 + 3(-2 + p)p) \cos^2(x_1) \cos(2x_2) \sin(2y_1) \sin(2y_2) - \sin(2x_1) \sin(2x_2)(2 + 3(-2 + p)p + (2 + 3(-2 + p)p) \cos(2y_1 + y_2)) + \sqrt{3}p(-2 + 3p)(\sin(2y_1) + \sin(2y_2))) \] (A.23)

\[ C_{12} = (2 + 2 \cos^2(x_1) - 2 \cos(2x_1) + 2 \cos(2x_2) - 2 \cos^2(x_1) \cos(2x_2)) - 2 \cos(2x_1) \cos(2x_2) + 2 \cos^2(x_1) \cos(2y_1) \cos(2y_2) - 2 \cos^2(x_1) \cos(2x_2) \cos(2y_1) \cos(2y_2) - (2 + 3(-2 + p)p) \cos^2(x_1) \times \sin(2y_1) \sin(2y_2) + (2 + 3(-2 + p)p) \cos^2(x_1) \cos(2x_2) \sin(2y_1) \sin(2y_2) + \sin(2x_1) \sin(2x_2)(2 + 3(-2 + p)p + (2 + 3(-2 + p)p) \cos(2y_1 + y_2)) + \sqrt{3}p(-2 + 3p)(\sin(2y_1) + \sin(2y_2))) \] (A.24)

\[ C_{20} = (16 - 16 \cos(2x_1) - 16 \sin^2(x_1)(2 \cos(2y_1) \cos(2y_2)) + (-2 - 3(-2 + p)p) \sin(2y_1) \sin(2y_2)) \] (A.25)

\[ C_{21} = (2 + 8 \cos^2(x_1) - 2 \cos(2x_1) + 2 \cos(2x_2)) - 8 \cos^2(x_1) \cos(2x_2) - 2 \cos(2x_1) \cos(2x_2) + 2 \cos^2(x_2) \cos(2y_1) \cos(2y_2) - 2 \cos^2(x_2) \cos(2x_2) \cos(2y_1) \cos(2y_2) - (2 + 3(-2 + p)p) \cos(2x_2) \sin(2y_1) \sin(2y_2) + (2 + 3(-2 + p)p) \cos(2x_1) \cos(2x_2) \sin(2y_1) \sin(2y_2) + 2 \sin^2(x_1)(2 \cos(2y_1) \cos(2y_2) + (-2 - 3(-2 + p)p) \sin(2y_1) \sin(2y_2)) + 2 \sin(2x_1) \sin(2x_2)(2 + 3(-2 + p)p + (2 + 3(-2 + p)p) \cos(2y_1 + y_2)) + \sqrt{3}p(-2 + 3p)(\sin(2y_1) + \sin(2y_2))) \] (A.26)

\[ C_{22} = (2 + 8 \cos^2(x_1) - 2 \cos(2x_1) - 2 \cos(2x_2)) + 8 \cos^2(x_1) \cos(2x_2) + 2 \cos(2x_1) \cos(2x_2) - 2 \cos(2x_2) \cos(2y_1) \cos(2y_2) + 2 \cos(2x_1) \cos(2x_2) \cos(2y_1) \cos(2y_2) + (2 + 3(-2 + p)p) \cos(2x_2) \sin(2y_1) \sin(2y_2) - (2 + 3(-2 + p)p) \cos(2x_1) \cos(2x_2) \sin(2y_1) \sin(2y_2) + 2 \sin^2(x_1)(2 \cos(2y_1) \cos(2y_2) + (-2 - 3(-2 + p)p) \sin(2y_1) \sin(2y_2)) - 2 \sin(2x_1) \sin(2x_2)(2 + 3(-2 + p)p + (2 + 3(-2 + p)p) \cos(2y_1 + y_2)) + \sqrt{3}p(-2 + 3p)(\sin(2y_1) + \sin(2y_2))) \] (A.27)
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