The Einstein-Infeld-Hoffmann legacy in mathematical relativity
II. Quantum laws of motion

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September 17, 2021

Abstract

We report on recent developments towards a relativistic quantum mechanical theory of motion for a
fixed, finite number of electrons, photons, and their anti-particles, as well as its possible generalizations
to other particles and interactions.

1 Introduction

As reported in Part I, the goal of the classical part of our program is to formulate and analyze a gen-
erally covariant joint initial value problem for the motion of massive charged particles, together with the
electromagnetic and gravitational fields that they generate. We believe that proving local well-posedness,
i.e. existence and uniqueness of a solution to this initial value problem, and continuous dependence of the
solution on initial data, is a first necessary step towards the formulation of a deeper quantum theory of
motion, for which the above could serve as the classical limit.

In the second part of our report, we expand on what such a relativistic quantum theory could look like,
and present the results of our preliminary efforts towards achieving our ultimate goal, which is twofold: (1) to
provide an accurate account of empirical electromagnetism in terms of a special-relativistic \(N\)-body quantum
theory of electrons, photons, and their anti-particles; and (2) to extend that effort to general relativity by
taking gravitational effects into account.

At the same time, inspired by Einstein’s [12, 13] and Bell’s [3] writings, we want to find out whether it
is possible to formulate such a theory as a generalization of the non-relativistic theory of de Broglie [6] and
Bohm [4] in which the quantum-mechanical wave function \(\psi\) guides the actual motion of all the particles
involved.

Our approach to a relativistic quantum theory is based on a number of working hypotheses. Before listing
them, we summarize the assumptions that formed the basis of our approach to the classical theory of motion of
charged particles, as was outlined in Part I of this report:

1. All forms of matter are of a discrete (particulate) nature.

2. Particles, whether feature-less point-particles or structured ones such as ring-particles, have well-defined
world-lines or world-tubes that form the singular boundary of spacetime, in the sense that the spacetime metric
and/or classical fields permeating the spacetime develop a singularity as a point on this boundary
is approached. Moreover, these singularities must be weak enough to give rise to locally integrable field
energy-momentum densities, but not so weak as to be confined to characteristic hypersurfaces (which
are completely determined by the field equations).

*Rapport by the first author given at the 15th Marcel Grossmann Meeting on General Relativity, Univ. Roma 1 (La
Sapienza), Rome, Italy, July 3, 2018. Updated and revised: May 2019.
3. The metric of the spacetime satisfies the weak version of the twice-contracted second Bianchi identity everywhere, including in a neighborhood of any singularity of spacetime (of the type discussed in item 2 above), thus allowing the usual conservation laws of total (i.e. field+particle) energy, momentum, and angular momentum to hold in a weak sense.

In Part I, we showed how these principles may be used to derive a classical equation of motion for worldlines of matter particles (which are identified with time-like singularities of spacetime) based entirely on momentum balance laws.

Our approach to a quantum law of motion begins by replacing the third item above with another set of hypotheses:

3'. The quantum-mechanical wave function of a single particle is a section of a Clifford-algebra-valued spin bundle over configuration spacetime. Different species of particles are distinguished by their wave functions at a point in spacetime belonging to different generalized ideals (stable subspaces) of that Clifford algebra.

4'. The wave function of a system of \( N \) relativistic particles is defined on the \( N \)-body configuration spacetime, and takes its values in a tensor space generated by the tensor product of bases for the constituent 1-body wave function spaces.

5'. The wave function of a single particle satisfies a relativistic wave equation that is equivariant with respect to the action of the Poincaré group on the bundle of frames for the tangent space at any point in spacetime.

6'. As a consequence of satisfying that wave equation, the wave function of a single particle possesses a conserved, future-directed, timelike (or causal, if the particle is massless), and normalizable current, whose component in the normal direction to a given spacelike foliation of spacetime, when evaluated at a point on a leaf of that foliation gives the probability density of the particle trajectory crossing the leaf at that point (by the Born rule.) The same holds for the \( N \)-particle version of the wave function and an \( N \)-particle tensor current that is jointly conserved.

7' The motion of a system of \( N \) particles is guided by a multi-time wave function defined on the \( N \)-particle configuration spacetime, in accordance with a relativistic generalization of the de Broglie-Bohm guiding law, such as the Hypersurface Bohm-Dirac Law, in which a conserved \( N \)-particle tensor current evaluated at the actual positions of those \( N \)-particles determines their actual velocities.

8' In the classical limit, the quantum motions converge to those described in part I.

2 Least Invasive Quantization

In this section we demonstrate by way of a very simple example, how it may be possible to continuously deform a classical law into a quantum law of motion for the same particle.

Consider a particle of (bare) mass \( m \) in an external potential \( V = V(t, q), q \in \mathbb{R}^d \). The relativistic Lagrangian for the particle’s motion is

\[
\ell(t, q, \dot{q}) = m(1 - \sqrt{1 - \dot{q}^2}) - V(t, q).
\]

The corresponding action is

\[
A[q] := \int_{t_1}^{t_2} \ell(t, q, \dot{q}) dt.
\]

\(^1\)We emphasize that this is a preliminary list based on what we know so far, and is subject to modification and expansion as our investigations continue.

\(^2\)See Furey \[14, 15\] for details on this point of view.
We define the canonical momentum to be
\[ p := \frac{\partial \ell}{\partial \dot{q}} = \frac{m \dot{q}}{\sqrt{1 - \dot{q}^2}} \tag{2.3} \]
so the Hamiltonian is
\[ H(t, q, p) = p \dot{q} - \ell = \sqrt{m^2 + p^2} - m + V(t, q). \tag{2.4} \]
The Hamilton equations are therefore
\[ \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{m^2 + p^2}}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\partial_q V(t, q). \tag{2.5} \]

The Hamilton-Jacobi equation
\[ \frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0 \tag{2.6} \]
is a first-order nonlinear PDE for the Hamilton-Jacobi phase function \( S = S(t, q) \). It arises in the context of solving Hamilton’s equations by finding a canonical transformation, i.e. a change of variable in phase space \((q, p)\) such that the transformed Hamiltonian is constant. We note that \( S \) is defined on the space of generic positions of the particle, i.e. it’s a function defined on the configuration space (or configuration spacetime, in the relativistic setting.)

In the case of our particular example, it’s easy to see that the Hamilton-Jacobi equation has the form of an eikonal equation:
\[ |\partial_t \tilde{S} + V|^2 - |\partial_q \tilde{S}|^2 = m^2, \quad \tilde{S} := S - m t. \tag{2.7} \]
Moreover, this equation is derivable from an action principle: Let \( \varrho \geq 0 \) be another function defined on the configuration spacetime, set
\[ \mathcal{L} := \varrho \left( |\partial_t \tilde{S} + V|^2 - |\partial_q \tilde{S}|^2 - m^2 \right), \tag{2.8} \]
and let \( S := \int \mathcal{L} \, dq \, dt \) be the corresponding action. It is then obvious that the stationary points of \( S \) with respect to compactly supported variations in \( \varrho \) satisfy \( \text{(2.7)} \). If we now let \( \varrho \) be an active variable, and set the variation of \( S \) with respect to \( \tilde{S} \) also equal to zero, we obtain the following continuity-type equation for \( \varrho \):
\[ \partial^\mu (\varrho u_\mu) = 0. \tag{2.9} \]
Here \( \mu = 0, \ldots, d \), indices are raised and lowered using the Minkowski metric \( \eta = \text{diag}(1, -1, \ldots, -1) \) on the configuration spacetime, and we are employing the Einstein summation convention. The covectorfield \( u \) is by definition
\[ u_0 := \partial_t \tilde{S} + V, \quad u_i = \partial_q \tilde{S}, \tag{2.10} \]
so that by \( \text{(2.7)} \),
\[ u_\mu u^\mu = m^2. \tag{2.11} \]

Let us now define a complex-valued field on the configuration spacetime
\[ \psi := \sqrt{\varrho} e^{i \tilde{S}/\hbar}. \tag{2.12} \]
Evidently, using that \((\varrho, \tilde{S})\) satisfy the Euler-Lagrange equations for the stationary points of the action \( S \), one can derive the equation that \( \psi \) must satisfy, which will be a nonlinear PDE.

The goal of a least invasive quantization procedure is to find a continuous deformation of the Lagrangian \( \mathcal{L} \), now thought of as \( \mathcal{L}[\psi] \), to a nearby Lagrangian \( \mathcal{L}^\prime[\psi] = \mathcal{L}[\psi] + \hbar^2 \mathcal{L}'[\psi] \) in such a way that the Euler-Lagrange equation of the new action corresponding to \( \mathcal{L}^\prime \) is a linear PDE in \( \psi \). This is easily accomplished by adding a Fisher-like term to \( \mathcal{L} \): Let
\[ \mathcal{L}^\prime := \mathcal{L} + \hbar^2 \frac{\partial^\mu \varrho \partial^\nu \varrho}{4 \varrho}. \tag{2.13} \]
A computation then shows that
\[
\tilde{\mathcal{L}}[\psi] = \hbar^2 |(\partial_t + \frac{i}{\hbar} V)\psi|^2 - \hbar^2 |\partial_q \psi|^2 + m^2 |\psi|^2
\]  
(2.14)
so that a stationary point \( \psi \) will satisfy the linear Klein-Gordon equation
\[
(D^\mu D_\mu + m^2) \psi = 0, \quad D_\mu := -i\hbar \partial_\mu + A_\mu,
\]  
(2.15)
which (under appropriate assumptions relating \( A_\mu \) and \( V \)), is a Lorentz-covariant equation for a massive spin-0 particle.

It should be noted that in the above admittedly over-simplified setting, \( V \) was taken to be an “external” field, but in order to connect with our program of finding equations of motions for singularities of a dynamic field, \( V \) needs to be determined by the field for which the particle in question is a singularity. For example, if we are to identify the electron with certain singularities of the electromagnetic field, then the corresponding \( V \) in the wave equation for the electron would have to be determined in terms of that electromagnetic field, evaluated at generic positions of the electron. For a detailed discussion of a proper least invasive quantization in the electromagnetic context see [21]. In any case, since the electron is a spin-1/2 particle, the appropriate wave equation is not Klein-Gordon’s but Dirac’s, so the above process needs to be suitably modified.

Finally, we should note that even though in (2.9) one may formally set \( \rho := \varrho u_0 \) and \( v_k := u_k/u_0 \) to obtain what looks like a continuity equation for a probability density
\[
\partial_t \rho + \partial_k (\rho v^k) = 0 \tag{2.16}
\]
and proceed to derive a Born Rule and guiding law for the particle from it, there is a-priori no reason why \( \rho \) would be non-negative. This suggests that even for spin-0 particles, the Klein-Gordon may not be the correct equation. Indeed, the lack of a future-directed timelike conserved current for solutions of the Klein-Gordon equation is a main reason for researchers doubting whether bosons can be treated as particles. We believe they can, and have already addressed this issue for photons [19]. The question of what the correct wave equation is for spin-0 bosons will be addressed in a forthcoming publication of ours.

3 Quantum Law of Motion for Point Test Particles

3.1 Point particle in Hoffmann-like spacetimes

We now move on to a more realistic scenario, of a single point charge, thought of as the only time-like singularity of a static, spherically symmetric 4-dimensional electrovacuum spacetime, i.e. a solution of the Einstein-Maxwell equations. As was shown in [7], the case where the electromagnetic vacuum law is the standard, linear law of Maxwell (\( D = E, H = B \)) yields a spacetime that is too singular for weak Bianchi identity to be satisfied at the singularity, so that our third working hypothesis is not satisfied. We showed however, that there are nonlinear vacuum laws, including the one proposed by Born & Infeld [5], for which the corresponding static spherically symmetric solution of the Einstein-Maxwell equations (which was fully analyzed in [23]) does satisfy the weak Bianchi identity. The spacetime that corresponds to the Born-Infeld vacuum law was discovered by Hoffmann [16] in 1935.

As a first step towards a full-blown two-body quantum-mechanical problem, one may study the dynamics of a single electron, thought of as a test point charge, placed in the electrostatic and gravitational field of another, much more massive point charge such as a nucleus, the latter thought of as the central singularity of a static spherically symmetric spacetime, while the contribution of the former to the geometry of the spacetime is ignored. At the same time, the Born-Oppenheimer approximation allows one to reduce the 2-body quantum mechanical situation to a single body one. The dynamics of the electron is thus determined by its wave function that, due to the electron being a massive spin-1/2 particle, satisfies the Dirac equation on the background spacetime of the nucleus, with the electrostatic field of the nucleus appearing in the Dirac operator as a minimal coupling term.
This program was carried out successfully in [1], where it was shown that the Dirac Hamiltonian on Hoffmann-like spacetimes is essentially self-adjoint regardless of the nuclear charge, its essential spectrum is the same as that of the free Dirac operator on Minkowski space, its point spectrum is non-empty and consists of infinitely many eigenvalues in the gap, accumulating at the right endpoint. Numerical investigations of this problem are in the works, including comparison of the eigenvalues with those of the Dirac+Coulomb potential and the Dirac+Born electrostatic potential Hamiltonians on a Minkowski background.

3.2 Point particle in zero-gravity Kerr-Newmann spacetime

As mentioned above, the main obstacle on the path to a well-defined quantum theory that could encompass both gravity and electromagnetism is that when Maxwell-Lorentz electrodynamics is coupled to Einstein's gravity, the infinities inherent in ML cause the spacetime to have curvature singularities that are too strong for even a weak notion of energy-momentum conservation to hold. In our quest to find a remedy for the strong curvature singularities of well-known solutions of Einstein-Maxwell equations such as the Kerr-Newmann solution, another avenue that we have pursued is to work with the zero-gravity limit of such singular spacetimes. This is a (geometrically well-defined) limit when Newton’s gravitational constant $G$, which appears as a parameter in these metrics, is set to zero. The limiting spacetime and the electromagnetic field on it were analyzed in [22]. It is axially symmetric, static, and locally isometric to Minkowski space, but is topologically non-trivial. These facts were already known to Carter [8], who discovered the maximal analytical extension of the Kerr-Newmann solution. It was also known that the constant time slices of this manifold are double-sheeted, and have the topology of $\mathbb{R}^3$ branched over the un-knot (See Fig. 1).

![Figure 1: Visualizing the double-sheeted spacelike slices of zero-G Kerr-Newmann spacetime as a branched cover of the un-knot. Arrows indicate that the top of the disk in one copy of $\mathbb{R}^3$ is to be identified with the bottom of the disk in the other copy.](image)

The spacetime is indeed singular on a 2-dim. timelike cylinder $S^1 \times \mathbb{R}$, which is the world-tube of a ring-like singularity. The most intriguing aspect of this solution is that at any instant of time the ring singularity, when viewed from one of the two sheets of space, appears to be positively charged, and from the other sheet, negatively charged.

As a first step in studying the quantum problem, one may identify the zero-gravity Kerr-Newmann (zGKN) solution with the spacetime outside a ring-like singularity representing a positively charged particle such as a proton. Since the metric of zGKN and the electromagnetic field corresponding to it are well known, it is easy to formulate the quantum dynamics of a test electron placed in the vicinity of the ring singularity by studying the Dirac equation on the zGKN background. This task was carried out in [17], where it was shown, using techniques developed in [2, 26], that the pertinent Dirac Hamiltonian is essentially self-adjoint and its essential spectrum is the same as the standard Dirac operator on Minkowski space. We further showed that its discrete spectrum is nonempty, provided the radius of the ring is small enough. The ground state of the Dirac Hamiltonian for the single test electron has support in both sheets of the spacetime, but is mainly concentrated in the sheet where the ring singularity appears to be positively charged, see Fig. 2, where the horizontal axis is a radial coordinate that runs from $-\infty$ to 0 in one sheet and 0 to $\infty$ in the other sheet, while the vertical axis is $|\psi|^2$.

Further analysis showed that the radius of the ring should be of the order of the anomalous magnetic moment of the electron, and that given this choice, the full spectrum will be close to the standard
Dirac-Coulomb problem for Hydrogenic atoms. Furthermore, the broken symmetry due to nonzero ring radius causes the well-known degeneracy in the Dirac spectrum of Hydrogen to be broken, resulting in effects that are qualitatively similar to Lamb shift and hyperfine splitting, without the need to appeal to QED methods, see Fig. 3 (axes have the same meaning as in Fig. 2.)

Figure 2: Ground state of single-electron Dirac Hamiltonian on a zGKN background

Figure 3: Numerical evidence of a qualitative Lamb shift for Dirac's Hamiltonian on zGKN. The parameter $\alpha$ is the ring radius, which here is set equal to electron's anomalous magnetic moment, measured in units of the Compton wavelength of electron.

4 Towards a Quantum Law of Motion for Ring-Like Singularities of Spacetime

Going back to the intriguing feature of the ring singularity of zGKN, i.e. that the same ring is found to have a positive charge when seen from one of the two sheets of that spacetime, and a negative charge when seen from the other sheet, one may wonder whether this ring singularity is representing both a particle and its anti-particle. In [18], extrapolating from the existence of such “bi-particle” structures in General Relativity, we proposed a novel interpretation of Dirac’s “wave equation for the relativistic electron” in which the electron and the positron are merely the two different “topological spin” states of a single more fundamental particle, not distinct particles in their own right. This novel interpretation resolves the dilemma that Dirac’s wave equation seems to be capable of describing both the electron and the positron in external fields in many relevant situations, while the bi-spinorial wave function has only a single position variable in its argument, not two –as it should if it were a quantum-mechanical two-particle wave equation. We formulated a Dirac equation for such a ring-like bi-particle which interacts with a static point charge located elsewhere in the topologically non-trivial physical space associated with the moving ring particle. Indeed, for quasi-static motions of the ring, this equation is nothing but the one we have been discussing in the previous section, namely Dirac’s equation on the zGKN background! The difference is in the interpretation, since the wave function in the present case is defined on the configuration space of the center of the ring, which is now moving with respect to a fixed point charge located on one of the two sheets. See Fig. 4 for a schematic...
illustration of the two interpretations. Furthermore, we showed that the motion of the ring can be governed by a de Broglie-Bohm type law extracted from the Dirac equation.

Figure 4: Left: The physical spacetime with a single ring singularity in straight line motion relative to the rest frame of a designated origin, which in the depicted scenario gets “swept over” by the ring singularity; Right: Trajectory of a test particle in straight line motion in the static zGKN spacetime having the center of its ring singularity as the origin (marked 0). The test particle transits through the ring from one sheet to the other.

Our work in this direction is so far limited to quasi-static motions of the ring, which result in the ring remaining circular to the leading order, so that its motion can be specified solely in terms of the trajectory of the center of the circle, together with the evolving unit normal to the plane of the circle. Clearly, much work remains to be done in order to extend these results to the general case, which would presumably involve deformations of the ring as it moves.

The next natural step in this direction would be to study the full 2-body quantum problem, either in the context of nonlinear electrodynamics, or in the zero-gravity context by studying the two-rings analog of zGKN. Due to the strong nonlinearity of the BI vacuum law however, there is currently no explicit solution known to the classical 2-body problem, even in the absence of gravity, thus making any progress in that direction quite difficult. As to the two rings, that analysis is already under way. We know for example that the space should be four-sheeted (in general, $2^N$ sheets are needed for $N$ ring singularities), and that it is possible to come up with a Dirac equation for the 2-body wave function (with synchronized times), equipped with a minimal coupling term that models the electromagnetic interaction of the two rings. It is important to note however, that there is no chance of “turning gravity back on” in these models, since doing so will bring back the strong singularities of the type present in the Kerr-Newman spacetime, together with all the causal pathologies that they entail.

As described in Part I of this report, there are other electromagnetic theories, such as the Bopp-Lande-Thomas-Podolsky (BLTP) theory, which may allow us to find electromagnetic spacetimes whose singularities are mild enough so that they can represent charged particles. Once such solutions are known, a least invasive quantization procedure as outlined in Section 2 could conceivably lead to a quantum law of motion for those particles. Investigations are currently under way on various aspects of this program.

5 Quantum Laws of Motion for Interacting Point-Particle Systems

As mentioned at the outset, our goal is the eventual formulation of a relativistic quantum-mechanical theory of motion for a fixed number $N$ of electrons, photons, and their anti-particles in 3+1-dimensional Minkowski spacetime, and its generalization to other particles and interactions. We are certainly not the first researchers to state this as our goal, as can be attested to by the following passage in [9]:

“The Compton effect, at its discovery, was regarded as a simple collision of two bodies, and yet the detailed discussion at the present time involves the idea of the annihilation of one photon and the simultaneous creation of one among an infinity of other possible ones. We would like to be able to treat the effect as a two-body problem, with the scattered photon regarded as the same individual as the incident, in just the way we treat the collisions of electrons.”
Such a goal however, of treating electrons and photons on an equal footing, within a quantum-mechanical framework of a fixed number of particles, has so far remained elusive. Progress has been obstructed in particular by the lack of a viable candidate for the quantum-mechanical photon wave function and its pertinent relativistic wave equation which furnishes a conserved probability current for the photon position, obeying Born’s rule. Recently such a photon wave function and wave equation have been constructed in [19]. In a forthcoming joint work with M. Lienert [20], we show how the photon wave equation of [19] can be coupled with Dirac’s well-known relativistic wave equation for the electron in a Lorentz-covariant manner to accomplish what Darwin has asked us to do: to “treat the effect as a two-body problem, with the scattered photon regarded as the same individual as the incident, in just the way we treat the collisions of electrons.”

We work with Dirac’s manifestly Lorentz-covariant formalism of multi-time wave functions [10]. For our \( N = 2 \) body problem the wave function \( \Psi(x_{el}, x_{ph}) \) depends on the two generic spacetime events \( x_{el} \) and \( x_{ph} \) of the electron and the photon, respectively, which must be space-like separated. Both the Dirac operator of a free electron [25], and the Dirac-type operator of a free photon constructed in [19], act on \( \Psi(x_{ph}, x_{el}) \). Unique solvability of this system of evolution equations requires imposing a suitable boundary condition at the subset of co-incident events, \( \{x_{el} = x_{ph}\} \). Conservation of the particle currents dictates the boundary condition, up to a choice in a phase. We prove that the resulting initial-boundary-value problem is well-posed.

It is intuitively obvious that a boundary condition at co-incident events \( \{x_{el} = x_{ph}\} \) amounts to a local pair interaction between electron and photon in a Lorentz-covariant manner. Our boundary condition is compatible with the kind of interaction expected for an electron and a photon in Compton scattering. In particular, when we plot the joint 2-body probability density \( \rho \) as a function of generic positions of the two particles, we observe that, as it evolves in time, the density forms four peaks, corresponding to the wave function being mostly supported near four distinct configurations: (1) both particles going to the left, (2) both particles going to the right, (3) particles going away from each other for all time; and (4) particles initially moving towards each other and “bouncing off” of one another. Indeed the fourth peek appears to “hit” the boundary and get reflected, see Fig. 5 for snapshots of the evolving density.

![Figure 5: Contour plot of the density \( \rho \) in photon-electron configuration space at six consecutive snapshots of common time \( t = t_{el} = t_{ph} \). The photon axis is horizontal, the electron axis vertical.](http://sites.math.rutgers.edu/~shadi/phel1dint_slow.gif)

We have also carried out a limited number of numerical experiments with our system of equations, which indeed demonstrate the process of Compton scattering, but which also have revealed an unexpected novel phenomenon: photon capture and subsequent release by the electron. For all practical purposes this scenario seems indistinguishable from the scenario of an annihilation of a photon, followed by a subsequent emission of another one. In our quantum-mechanical \( N = 2 \) body model the photon of course never gets destroyed or created, precisely as envisioned by Darwin.

Beyond demonstrating that relativistic quantum mechanics with a fixed finite number \( N \) of interacting particles is feasible, we also formulate the de Broglie–Bohm-type foundations of this Lorentz-covariant quantum model [11]. This is accomplished by adapting to our interacting 2-body model the so-called “hypersurface Bohm–Dirac”-type formulation for non-interacting particles [11]. This formulation requires one to specify a foliation of spacetime by spacelike hypersurfaces. In our model these hypersurfaces are given by a time-like Killing vector field that is determined self-consistently by the initial data of the wave function. The guiding law for the particles is furnished by the conserved current of our quantum-mechanical multi-time wave function.

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3 You can see an animation of the density plot of \( \rho \) as a function of the two positions by going to [http://sites.math.rutgers.edu/~shadi/phel1dint_slow.gif](http://sites.math.rutgers.edu/~shadi/phel1dint_slow.gif)

4 This should put to rest the often-voiced claim that the de Broglie–Bohm theory could not be made relativistic.
function. We extend a theorem of Teufel–Tumulka [24], which implies that unique particle motions typically exist globally in time.

![Figure 6](image1.png)

**Figure 6**: Interacting (solid) versus non-interacting (dotted) electron-photon trajectories.

The numerically computed trajectories confirm that our boundary condition causes the particles to scatter off of one another. See Fig. 6 for example of a two pairs of particles with identical initial conditions, one evolved according to the free evolution (i.e. without any boundary condition on the coincidence set) and the other using our boundary condition.

The intriguing phenomenon of photon “capture and release” by the electron that we alluded to in the above can also be illustrated by the numerically computed electron and photon trajectories, see Fig. 7.

![Figure 7](image2.png)

**Figure 7**: World-lines corresponding to possible capture-and-release phenomena (Different colors correspond to the photon-electron pair starting at different initial positions that are successively closer to each other.)

Finally, by sampling a large ensemble of random positions distributed according to the initial wave function, we can also illustrate that the empirical statistics over the possible actual trajectories reproduces Born’s rule in our model, a consequence of the equivariance of the evolution of the probability densities. See Fig. 8 and note the four clusters of trajectories that correspond to the four peaks of the joint probability density that was mentioned in the above.

![Figure 8](image3.png)

**Figure 8**: Electron & photon world-lines corresponding to 100 pairs of typical initial positions.

More numerical investigation is currently under way, to explore various regions in the parameter space.
for the dynamics (e.g. mass and energy of incident electron, frequency profile of incident photon, their initial
distance, etc.) and to quantify the energy and momentum transfer between the two particles.

Acknowledgements

We gratefully acknowledge helpful discussions with: M. K. Balasubramanian, E. Carlen, O. Darrigol, C.
Furey, S. Goldstein, J.-M. Graf, F. Hehl, M. Jansen, H. Jauslin, J. Lebowitz, M. Lein, N. Leopold, M.
Lienert, T. Newman, B. Simon, A. Soffer, H. Spohn, W. Struyve, R. Tumulka, and M. Winklmeier. We
thank the organizers, R. Ruffini and R. Jantzen, for inviting us to MG15.

References

[1] Balasubramanian, M. K., “Scalar fields and spin-half fields on mildly singular spacetimes”, Thesis
(Ph.D.)–Rutgers, The State University of New Jersey, 77 pages, (2015).

[2] D. Batic, H. Schmid, and M. Winklmeier. “On the eigenvalues of the Chandrasekhar-Page angular
equation,” J. Math. Phys., 46(1):012504, 35, (2005).

[3] Bell, J.S., “Speakable and Unspeakable in Quantum Mechanics,” 2nd ed., Cambridge Univ. Press,
Cambridge (2004).

[4] Bohm, D., “A suggested interpretation of the quantum theory in terms of ‘hidden’ variables. Part I,”
Phys. Rev. 85:166–179 (1952); “Part II,” ibid., 180–193 (1952).

[5] Born, M., and Infeld, L., “Foundations of the New Field Theory”, Nature, 132, p. 1004, (1933).

[6] de Broglie, L.V.P.R., La nouvelle dynamique des quanta, in “Cinquième Conseil de Physique Solvay”
(Bruxelles 1927), ed. J. Bordet, (Gauthier-Villars, Paris, 1928); English transl.: “The new dynamics
of quanta”, p.374-406 in: G. Bacciagaluppi and A. Valentini, “Quantum Theory at the Crossroads,”
(Cambridge Univ. Press, 2009).

[7] Burtscher, A. Y., Kiessling, M. K.-H., and Tahvildar-Zadeh, A. S., “Weak second Bianchi identity for
spacetimes with timelike singularities,” preprint (20 pages), [https://arxiv.org/abs/1901.00813],
(2019).

[8] Carter, B., “Global structure of the Kerr family of gravitational fields,” Phys. Rev., 174:1559–1571,
(1968.)

[9] Darwin, C. G., “Notes on the Theory of Radiation,” Proc. Roy. Soc. London A, 136 (829), pp. 36-52,
(1932)

[10] Dirac, P. A. M., “Relativistic Quantum Mechanics,” Proc. R. Soc. Lond. A, 136:453–464 (1932).

[11] Dühr, D., Goldstein, S., Münch-Berndl, K., and Zanghì, N., “Hypersurface Bohm–Dirac models” Phys.
Rev. A, 60: 2729–2736 (1999).

[12] Einstein, A., “Zum gegenwärtigen Stand des Strahlungsproblems,” Phys. Zeitschr. 10, 185–193 (1909).

[13] Einstein, A., “Über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der
Strahlung,” Verh. Deutsch. Phys. Ges. 7, 482–500 (1909).

[14] Furey, C., “Charge quantization from a number operator,” Physics Letters B, 742, pp. 195-199, (2015).

[15] Furey, C., “SU(3)c x SU(2)l x U(1)y (xU(1)x)” as a symmetry of division algebraic ladder operators”,
The European Physical Journal C 78(5), p. 375, (2018).
[16] Hoffmann, B., “On the new field theory,” *Proc. R. Soc. Lond. A*, 148, pp. 353–364, (1935)

[17] Kiessling, M. K.-H. & Tahvildar-Zadeh, A. S., “The Dirac point electron in zero-gravity Kerr–Newman spacetime,” *Journal of Mathematical Physics* 56 042303 (2015).

[18] Kiessling, M. K.-H. & Tahvildar-Zadeh, A. S., “A novel quantum-mechanical interpretation of the Dirac equation,” *Journal of Physics A* 49 135301 (2016).

[19] Kiessling, M. K.-H. & Tahvildar-Zadeh, A. S., “On the Quantum Mechanics of a Single Photon,” *Jour. Math Phys.* 59, 112302, 34 pages, (2018).

[20] Kiessling, M. K.-H., Lienert, M., & Tahvildar-Zadeh, A. S., “Relativistic Interacting Photon-Electron System in One Space Dimension,” in preparation, 34 pp., (2019).

[21] Kiessling, M. K.-H., “Electromagnetic field theory without divergence problems. II. A least invasively quantized theory”, *J. Statist. Phys.*, 116, 1-4, pp. 1123–1159, (2004)

[22] Tahvildar-Zadeh, A. S., “On a zero-gravity limit of Kerr–Newman spacetimes and their electromagnetic fields,” *Journal of Mathematical Physics*. 56 042501 (2015).

[23] Tahvildar-Zadeh, A. S., “On the static spacetime of a single point charge,” *Rev. Math. Phys.*, 23 (3), pp. 309–346, (2011).

[24] Teufel, S. and Tumulka, R., “Simple Proof for Global Existence of Bohmian Trajectories,” *Commun. Math. Phys.*, 258:349–365 (2005).

[25] Thaller, B., *The Dirac Equation*, Springer Verlag, Berlin (1992).

[26] Winklmeier, M., and Yamada, O., “Spectral analysis of radial Dirac operators in the Kerr-Newman metric and its applications to time-periodic solutions,” *J. Math. Phys.*, 47(10):102503, 17, (2006.)