Chapter 1

The QM/NCG Correspondence

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In Honor of Bal on the Occasion of His 85th Birthday

The correspondence between quantum mechanics and noncommutative geometry is illustrated in the context of the noncommutative AdS$_2$/$\text{CFT}_1$ duality where CFT$_1$ is identified as conformal quantum mechanics. This model is conjectured here to describe the gauge/gravity correspondence in one dimension.

1. A QM/NCG Correspondence

In this article we will focus on the case of two dimensions with Euclidean signature where the positive curvature space is given by a sphere $S^2$ with isometry group $SO(3)$ and the negative curvature space is given by a pseudo-sphere $\mathbb{H}^2$ with isometry group $SO(1,2)$ (which is Euclidean AdS$^2$). The space $S^2 \times \text{AdS}^2$ is in fact the near-horizon geometry of extremal black holes in general relativity and string theory. The information loss problem in four dimensions on $S^2 \times \text{AdS}^2$ is then reduced to the information loss problem in two dimensions on AdS$^2$.

The quantization of these two spaces yields the fuzzy sphere $S^2_N$ and the noncommutative pseudo-sphere $\text{AdS}_{\theta}^2$ respectively which enjoy the same isometry groups $SO(3)$ and $SO(1,2)$ as their commutative counterparts. The fuzzy sphere is unstable and suffers collapse in a phase transition to Yang-Mills matrix models (topology change or geometric transition) whereas the noncommutative pseudo-sphere can sustain black hole configurations (by including a dilaton field) and also suffers collapse in the form of the information loss process (quantum gravity transition).

Explicitly, the noncommutative AdS$_2^\theta$ is given by the following embed-
The $K^a$ are the generators of the Lie group $SO(1, 2) = SU(1, 1)/\mathbb{Z}_2$ in the irreducible representations of the Lie algebra $[K^a, K^b] = i\epsilon^{abc}K^c$ given by the discrete series $D_±^k$ with $k = \{1/2, 1, 2/3, 2, 3/2, ...\}$.\footnote{12}

By substituting the solution (2) in the constraint equation in (1) the value of the deformation parameter $\kappa$ is found to be quantized in terms of the $su(1, 1)$ pseudo-spin quantum number $j \equiv k - 1$ as follows:

$$R^2 = k(k - 1).$$ \hspace{1cm} (3)

The commutative limit is then defined by $\kappa \to 0$ or $k \to \infty$.

The fact that we have for $AdS^2$ two disconnected one-dimensional boundaries makes this case very different from higher dimensional anti-de Sitter spacetimes and is probably what makes the $AdS^2/CFT_1$ correspondence the most mysterious case among all examples of the $AdS/CFT$ correspondence. For example, see\footnote{11}

In\footnote{11} the difficulty of the $AdS^2/CFT_1$ correspondence is traced instead to the fact that the conformal quantum mechanics residing at the boundary is only quasi-conformal and as a consequence the theory in the bulk is only required to be quasi-$AdS$. In other words, we will take the opposite view and start or assume that the $CFT_1$ theory on the boundary is really given by the dAFF conformal quantum mechanics.\footnote{9,10}

In the dAFF conformal quantum mechanics the $so(2, 1) = su(1, 1)$ Lie algebra generators $P$ (translations), $D$ (scale transformations) and $K$ (special conformal transformations) are realized in terms of a single degree of freedom $q(t)$ with scaling dimension $\Delta$. The corresponding conjugate momentum is given by $p(t) = \dot{q}(t)$ and we start from the standard Heisenberg algebra

$$[q, p] = i.$$ \hspace{1cm} (4)

The $so(2, 1)$ Lie algebra is given explicitly by

$$[D, P] = -iP, \quad [D, K] = iK, \quad [K, P] = -2iD.$$ \hspace{1cm} (5)

A straightforward calculation gives then

$$P = \frac{p^2}{2} + V(q), \quad V(q) = \frac{q}{q^2}.$$ \hspace{1cm} (6)
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\[ D = tP + \Delta (qp + pq), \quad \Delta = -\frac{1}{2}. \]  
(7)

\[ K = -t^2 P + 2tD + \frac{1}{16} q^2. \]  
(8)

The coupling constant \( g \) is completely fixed by conformal invariance. Indeed, we compute the Casimir operator

\[ -C = \frac{1}{2} (PK + KP) - D^2 = \frac{g}{2} - \frac{3}{16} \equiv \mp k(k-1). \]  
(9)

The minus sign corresponds to Lorentzian AdS\(^2\) (relevant to the dAFF quantum mechanics) whereas the plus sign corresponds to Euclidean AdS\(^2\) (relevant to the noncommutative AdS\(^2\)).

It is clearly observed that the Lorentz group SO\((1, 2)\) is the fundamental unifying symmetry structure underlying the following commutative, noncommutative and quantum spaces:

(1) The AdS\(^2\) spacetime (classical geometry and gravity).

(2) The noncommutative AdS\(^2\) space (or first quantization of the geometry).

(3) The quantum theory on the boundary (conformal quantum mechanics).

(4) The IKKT-type Yang-Mills matrix models defining quantum gravity (or second quantization of the geometry).

In particular, the algebra of quasi-primary operators on the boundary is seen to be a subalgebra of the operator algebra of noncommutative AdS\(^2\). This leads us to the conclusion/conjecture that the theory in the bulk must be given by noncommutative geometry and not by classical gravity, i.e. it is given by AdS\(^2\) and not by AdS\(^2\). We end up therefore with an AdS\(^2\)/CFT\(_1\) correspondence where the CFT\(_1\) is given by the dAFF conformal quantum mechanics. See \(^\text{[1]}\) for more detail.

This is then a correspondence or duality between quantum mechanics (QM) on the boundary and noncommutative geometry (NCG) in the bulk which provides a concrete model for the AdS\(^{d+1}\)/CFT\(_d\) correspondence\(^\text{[3]}\) in one dimension (see figure 1).

In summary, the main argument underlying this QM/NCG duality consists in the following main observations:

(1) Both noncommutative AdS\(^2\) and the dAFF conformal quantum mechanics enjoy the same symmetry structure given by the group SO\((1, 2)\).

However, dAFF conformal quantum mechanics is only quasi-conformal...
in the sense that there is neither an invariant vacuum state nor strictly
speaking primary operators.\cite{9,10} Analogously, noncommutative \( \text{AdS}_2^\theta \) is
only quasi-\( \text{AdS} \) as it approaches \( \text{AdS}_2^2 \) only at large distances (commu-
tative limit).

(2) Asymptotically \( \text{AdS}_2^\theta \) is an \( \text{AdS}_2^2 \) spacetime, i.e. it has the same bound-
ary\cite{8} and furthermore the algebra of quasi-primary operators on the
boundary (which defines in the same time the geometry of the bound-
ary and the dAFF quantum mechanics) is in some sense a subalgebra
of the operator algebra of noncommutative \( \text{AdS}_2^\theta \).\cite{1}

(3) Metrically the Laplacian operator on the noncommutative \( \text{AdS}_2^\theta \) shares
the same spectrum as the Laplacian operator on the commutative \( \text{AdS}_2^2 \)
spacetime.\cite{7} The boundary correlation functions computed using the
quasi-primary operators reproduces the bulk \( \text{AdS}_2^2 \) correlation func-
tions.\cite{10}

2. Noncommutative black holes and matrix models

The matrix model describing quantum gravitational fluctuations about the
noncommutative \( \text{AdS}_2^\theta \) black hole is given by a 4–dimensional Yang-Mills
matrix model with mass deformations which are invariant only under the
subgroup of \( \text{SO}(1,2) \) pseudo-rotations around \( D_4 \equiv \Phi \) (dilaton field). This
model is given explicitly by the action\cite{2}

\[
S[D] = NT\text{Tr}
\left( -\frac{1}{4}[D_\mu, D_\nu][D^\mu, D^\nu] + 2i\kappa D^2[D^1, D^3] + \beta D_a D^a
+ 2i\kappa_1 \hat{\Phi}[D^1, D^3] - \frac{1}{2}\kappa_1^2 D_1 D^1 - \frac{1}{2}\kappa_2^2 D_3 D^3 \right).
\]  

We have then two parameters \( \kappa \) and \( \kappa_1 = 2\pi T\kappa \) or equivalently \( \kappa \) and
\( T \). We will fix the value of the parameter \( \kappa_1 \) so that an increase in the
actual temperature \( T \) of the black hole corresponds to a decrease in the
noncommutativity parameter (or inverse gauge coupling constant) \( \kappa \). We
are interested in the values \( \kappa \neq 0 \) and \( \beta = 0 \).

The pure noncommutative \( \text{AdS}_2^\theta \) geometry appears as a classical (global)
minimum of the matrix model\cite{10} with \( \kappa \neq 0 \) and \( \kappa_1 = 0 \), i.e. \( T = 0 \).

Yang-Mills matrix models are generally characterized by two phases: 1) A
Yang-Mills phase and 2) A geometric phase due to the Myers terms, i.e.
the cubic terms in\cite{10}.
Fig. 1. The QM/NCG correspondence: The bulk is given by the algebra associated with noncommutative AdS$^2$ while the boundary is given by a subalgebra thereof. The behavior near-boundary is precisely that of commutative AdS$^2$.

The first cubic term in (10) is responsible for the condensation of the geometry (AdS$^2$ space) whereas the second cubic term in (10) is responsible for the stability of the noncommutative black hole solution in the AdS$^2$ space. There should then be two critical points $\kappa_{1cr}$ and $\kappa_{2cr}$ corresponding to the vanishing of the two cubic expectation values $\langle i\text{Tr}D^2[D^1,D^3]\rangle$ and $\langle i\text{Tr}\Phi[D^1,D^3]\rangle$ respectively. By assuming now that $\kappa_1 > \kappa$ we have $\kappa_{2cr} > \kappa_{1cr}$ and we have the following three phases (a gravitational phase, a geometric phase and a Yang-Mills phase):

- For $\kappa > \kappa_{2cr}$ we have a black hole phase where $\hat{\Phi} \propto \hat{X}_2$. This is the gravitational phase in which we have a stable noncommutative AdS$^2$ black hole.
- For $\kappa_{1} < \kappa < \kappa_{2}$ the second cubic term effectively vanishes and condensation is now only driven by the first cubic term. In other words,
although we have in this phase $\hat{\Phi} \sim 0$ the three matrices $D_{\alpha}$ still fluctuate about the noncommutative AdS$_2^\theta$ background given by the matrices $\hat{X}_\alpha$. This is the geometric phase.

- As we keep decreasing the temperature, i.e. increasing $\kappa$ below $\kappa_1$ the AdS$_2^\theta$ background evaporates in a transition to a pure Yang-Mills matrix model. In this Yang-Mills phase, all four matrices become centered around zero.

This is the Hawking process as it presents itself in the Yang-Mills matrix model, i.e. as an exotic line of discontinuous transitions with a jump in the entropy, characteristic of a 1st order transition, yet with divergent critical fluctuations and a divergent specific heat characteristic of a 2nd order transition.

An alternative way of recasting Hawking process as a phase transition starts from dilaton gravity in two dimensions which can be rewritten as a non-linear sigma model. For constant dilaton configurations, i.e. for $\Phi = \text{constant}$ this action reduces to the potential term

$$V = m^2 \int d^2x \sqrt{\text{det} g} \left( M - N(\Phi) \right)^2, \quad N(\Phi) = \int_{\Phi}^{\Phi'} \text{d}\Phi' V(\Phi').$$

The limit $m^2 \to \infty$ is understood. The mass functional $M$ is constant on the classical orbit given by the ADM mass of the AdS$_2^\theta$ black hole, viz $M = \frac{2\pi^2}{\Lambda}$. For the case of the Jackiw-Teitelboim action we have $V = 2\Lambda^2 \Phi$ and $N = \Lambda^2 \Phi^2$. The noncommutative analogue of this Jackiw-Teitelboim potential term is a real quartic matrix model

$$V = N_H \text{Tr}(\mu \hat{\Phi}^2 + g \hat{\Phi}^4), \quad N_H = 2k - 1.$$  

The regularization issues are discussed in the next section. The parameters $\mu$ and $g$ are given explicitly by

$$\mu = -2\pi M_r \left( \frac{m^2}{k^2} \right), \quad g = \pi \Lambda^2 \left( \frac{m^2}{k^2} \right).$$

The one-cut solution (disordered phase) in which the dilaton field $\Phi$ fluctuates around zero occurs for all values $M \leq M_*$ where

$$M_* = \frac{\Lambda}{\sqrt{\pi} m}.$$  

The large mass limit $m^2 \to \infty$ is then required to be correlated with the commutative limit $k \to \infty$. This critical value $M_*$ is consistent with the black hole mass $M = \frac{2\pi^2}{\Lambda}$. In other words, the dilaton field is zero
when the mass parameter $M$ of the matrix model is lower than the black hole mass, i.e. the black hole has already evaporated and we have only pure $\text{AdS}_2^\theta$ space in this phase.

Above the line $M = M_*$ we have a two-cut solution (non-uniform ordered phase) where the dilaton field $\hat{\Phi}$ is proportional to the idempotent matrix $\gamma$ (which satisfies $\gamma^2 = 1$). In this two-cut phase we have therefore a non-trivial dilaton field corresponding to a stable $\text{AdS}_2^\theta$ black hole. The transition at $M = M_*$ between the one-cut ($\text{AdS}_2^\theta$ space) and two-cut phases ($\text{AdS}_2^\theta$ black hole) is third order.

3. Phase Structure of the Noncommutative $\text{AdS}_2^\theta \times S^2_N$

The quantum gravitational fluctuations about the noncommutative geometry of the fuzzy sphere $S^2_N$, the noncommutative pseudo-sphere $H^2_\theta$ and the noncommutative near-horizon geometry $\text{AdS}_2^\theta \times S^2_N$ are given by IKKT-type Yang-Mills matrix models with additional cubic Myers terms which are given explicitly by the actions

$$S_S[C] = N_S \text{Tr}_S L_S[C], \quad S^2. \tag{15}$$
$$S_H[D] = N_H \text{Tr}_H L_H[D], \quad \mathbb{H}^2. \tag{16}$$
$$S_H[S, C] = N_H N_S \text{Tr}_S \text{Tr}_S \left( L_S[C] + L_H[D] - \frac{1}{4} [D_a, C_b] [D^a, C_b] \right), \quad H^2 \times S^2. \tag{17}$$

The Lagrangian terms are given in terms of three $(2l+1) \times (2l+1)$ Hermitian matrices $C_a$ (where $N_S = 2l + 1 \equiv N$) and three Hermitian operators $D_a$ by the equations

$$L_S[C] = -\frac{1}{4} [C_a, C_b]^2 + \frac{2i}{3} \alpha_{abc} C_a C_b C_c + \beta_S C_a^2. \tag{18}$$
$$L_H[D] = -\frac{1}{4} [D_a, D_b] [D^a, D^b] + \frac{2i}{3} \kappa f_{abc} D^a D^b D^c + \beta_H D_a D^a. \tag{19}$$

The coefficients $\alpha_{abc}$ and $f_{abc}$ provide the structure constants of the rotation group $SO(3) = SU(2)/\mathbb{Z}_2$ and the pseudo-rotation group $SO(1, 2) = SU(1, 1)/\mathbb{Z}_2$ respectively.

The solutions of the equations of motion which follow from the actions (15), (16) and (17) are precisely given by the fuzzy sphere $S^2_N$, the noncommutative pseudo-sphere $\mathbb{H}^2_\theta$ and the noncommutative near-horizon geometry.
AdS$_{2} \times S_{N}^{2}$ configurations. These noncommutative configurations are given explicitly by

\[ C_a = \phi_S L_a, \quad \phi_S = \alpha \varphi_S \]  
\[ D_a = \phi_H K_a, \quad \phi_H = \kappa \varphi_H \]  
\[ C_a = \phi_S L_a \otimes 1_H, \quad D_a = 1_S \times \phi_H K_a. \]

These matrix models (15), (16) and (17) exhibit emergent geometry transitions from a geometric phase (the sphere, the pseudo-sphere and the near-horizon geometry of a Reissner-Nordstrom black hole) to a pure Yang-Mills matrix phase with no background geometrical structure.

This fundamental result is confirmed in the case of the sphere by Monte Carlo simulations of the Euclidean Yang-Mills matrix model (15). But in the other cases we have only at our disposal the effective potential. Indeed, the Monte Carlo method can not be applied to the matrix models (16) and (17) in their current form for two main reasons. First, the Lorentzian signature of the embedding spacetime (these noncommutative spaces are in fact branes residing in a larger spacetime). Second, the Hilbert spaces $\mathcal{H}_k$ corresponding to the noncommutative operator algebras are actually infinite-dimensional (which is required by the underlying $SO(1,2)$ symmetry).

However, the matrix models (16) and (17) can be regularized by means of a simple cutoff which consists in keeping only the $2^k - 1$ states in the Hilbert spaces turning therefore the infinite-dimensional operator algebras into finite-dimensional matrix algebras. By assuming also that the AdS spacetime has the same radius as the sphere we have the natural identification

\[ N_H = 2k - 1 \equiv N. \]  

The one-loop effective potential around the pseudo-sphere background (21) is computed from the action (16) using the background field method. We obtain the result (with $\kappa^4 = 4k^3(k-1)$ and $\tau_H = -\beta_H/\kappa^2$)

\[ \frac{2V_H}{N^2} = \kappa^4 \left[ \frac{1}{4} \varphi_H^4 - \frac{1}{3} \varphi_H^3 + \frac{1}{2} \tau_H \varphi_H^2 \right] + \log \varphi_H^2, \quad T_H = \frac{1}{\kappa^4} = g_H^2. \]  

The one-loop effective potential around the sphere background (20) is obtained from this result with the substitutions $k \rightarrow -l$, $\varphi_H \rightarrow \varphi_S$, $\kappa \rightarrow \alpha$, $\beta_H \rightarrow -\beta_S$, $\tau_H \rightarrow \tau_S$, $g_H \rightarrow g_S$ and $T_H \rightarrow T_S$. 

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The description of the phase structure of the noncommutative pseudo-
sphere $H^2_\theta$ using the effective potential (24) is therefore formally identical
to the fuzzy sphere case treated in detail in [13][14].

The sphere/pseudo-sphere solution $\varphi_{S,H} = \varphi_\beta (\tau_{S,H}) \neq 0$ is found to
be stable only in the regime $0 < \tau_{S,H} < 2/9$. This is the fuzzy or non-
commutative geometry phase. The Yang-Mills solution $\varphi_{S,H} = \varphi_0 = 0$
(matrix phase) is found to be the ground state or global minimum of the
system in the regime $2/9 < \tau_{S,H} < 1/4$. In fact, the coexistence curve
between the geometric phase (low temperature $T_H$) and the matrix phase
(high temperature $T_H$) asymptotes to the line $\tau_{S,H} = 2/9$.

We turn now to the case of the noncommutative near-horizon geometry
$AdS^2_\theta \times S^2_N$. We can immediately state the possible phases of the noncom-
mutative near-horizon geometry $AdS^2_\theta \times S^2_N$ in the space $(\tau_S, \tau_H, \bar{\alpha}, \bar{\kappa})$ as
follows.

- A Yang-Mills matrix phase with no background geometrical structure
which is expected at high temperature (both $\bar{\alpha}$ and $\bar{\kappa}$ approach zero)
or $2/9 < \tau_{S,H} < 1/4$.
- A 2−dimensional geometric fuzzy sphere phase ($\bar{\alpha}$ approaches infinity,
$\bar{\kappa}$ approaches zero and $0 < \tau_{S,H} < 2/9$).
- A 2−dimensional geometric noncommutative pseudo-sphere phase ($\bar{\alpha}$
approaches zero, $\bar{\kappa}$ approaches infinity and $0 < \tau_{S,H} < 2/9$).
- A 4−dimensional geometric noncommutative near-horizon geometry
$AdS^2_\theta \times S^2_N$ phase at low temperature (both $\bar{\alpha}$ and $\bar{\kappa}$ approach infinity
and $0 < \tau_{S,H} < 2/9$).

In order to simplify the calculation of the one-loop effective potential around
the near-horizon geometry background (22) we will consider the following
case:

- First, in order to have a single unified temperature $T_{HS}$ we must have
a single unified gauge coupling constant $g^2_{HS} = 1/T_{HS}$, viz
$$\alpha = \kappa.$$

- The geometric quantum entanglement between the sphere and the
pseudo-sphere sectors (through the third term in the action (17)) is
essential for the emergence of a four-dimensional space. However, there
exists a special case where this geometric quantum entanglement can
be partially removed while keeping the background emergent geometry
four-dimensional. This is given by the case
$$\tau_S = \tau_H \iff \beta_S = -\beta_H.$$
• With this choice (26) one can check that the order parameters $\varphi_S$ and $\varphi_H$ must solve identical equations of motion. A simplified model is then obtained by setting

$$\varphi_S = \varphi_H \equiv \varphi.$$  

(27)

The one-loop effective potential around the near-horizon geometry background (22) computed from the action (17) using the background field method is then given by

$$V_{HS} = \frac{2}{N^2 \Lambda_H^2} \left[ \frac{1}{4} \varphi^4 - \frac{1}{3} \varphi^3 + \frac{1}{2} \tau_H \varphi^2 \right] + \log \varphi^2, \quad \bar{\kappa}^4 = \frac{\tilde{\kappa}^4}{4N^2}.$$  

(28)

This effective potential is of the same form as the pseudo-sphere effective potential (24) with the substitution $\tilde{\kappa}^4 \rightarrow 2 \bar{\kappa}^4$.

In the special case given by (25), (26) and (27) the 2-dimensional geometric phases disappear and we end up with a single phase transition from a noncommutative near-horizon geometry $\text{AdS}_2 \times S^2_N$ phase to a Yang-Mills matrix phase as the temperature $T_{HS} = 1/\bar{\kappa}^4 = g_{\text{HS}}^2$ is increased.

References

1. B. Ydri, [arXiv:2108.13982 [hep-th]].
2. B. Ydri, [arXiv:2109.00380 [hep-th]].
3. B. Ydri and L. Bouraiou, [arXiv:2109.01010 [hep-th]].
4. J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231-252 (1998).
5. J. Hoppe, Ph.D. Thesis, (1982). J. Madore, Class. Quant. Grav. 9, 69 (1992).
6. P. M. Ho and M. Li, Nucl. Phys. B 590, 198 (2000).
7. D. Jurman and H. Steinacker, JHEP 1401, 100 (2014).
8. A. Pinzul and A. Stern, Phys. Rev. D 96, no. 6, 066019 (2017).
9. V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A 34, 569 (1976).
10. C. Chamon, R. Jackiw, S.Y. Pi, L. Santos, Phys. Lett. B 701, 503-507 (2011).
11. A. Strominger, JHEP 01, 007 (1999).
12. V. Bargmann, Ann. Math.48 (1947) 568.
13. P. Castro-Villarreal, R. Delgadillo-Blando, B. Ydri, Nucl. Phys. B 704, 111 (2005).
14. T. Azuma, S. Bal, K. Nagao, J. Nishimura, JHEP 05, 005 (2004).
15. R. Delgadillo-Blando, D. O’Connor, B. Ydri, Phys. Rev. Lett. 100, 201601 (2008).
16. R. Delgadillo-Blando, D. O’Connor, JHEP 11, 057 (2012).
17. M. Cavaglia, Phys. Rev. D 59, 084011 (1999) [hep-th/9811059].
18. M. Cadoni, P. Carta, Mod. Phys. Lett. A 16, 171-178 (2001).
19. Y. Shimamune, Phys. Lett. B 108, 407 (1982).