Partitioning an interval graph into subgraphs with small claws

Rain Jiang ● Kai Jiang ● Minghui Jiang ●
Home School, USA

Abstract

The claw number of a graph \( G \) is the largest number \( v \) such that \( K_{1,v} \) is an induced subgraph of \( G \). Interval graphs with claw number at most \( v \) are cluster graphs when \( v = 1 \), and are proper interval graphs when \( v = 2 \).

Let \( \kappa(n, v) \) be the smallest number \( k \) such that every interval graph with \( n \) vertices admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \). Let \( \bar{\kappa}(w, v) \) be the smallest number \( k \) such that every interval graph with claw number \( w \) admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \). We show that \( \kappa(n, v) = \lfloor \log_{v+1}(nv + 1) \rfloor \), and that \( \lfloor \log_{v+1}(w) \rfloor + 1 \leq \bar{\kappa}(w, v) \leq \lfloor \log_{v+1}(w) \rfloor + 3 \).

Besides the combinatorial bounds, we also present a simple approximation algorithm for partitioning an interval graph into the minimum number of induced subgraphs with claw number at most \( v \), with approximation ratio 3 when \( 1 \leq v \leq 2 \), and 2 when \( v \geq 3 \).

1 Introduction

The intersection graph \( G \) of a family \( \mathcal{F} \) of sets is a graph such that the vertices in \( G \) correspond to the sets in \( \mathcal{F} \), one vertex for each set, and an edge connects two vertices in \( G \) if and only if the corresponding two sets in \( \mathcal{F} \) intersect. A family \( \mathcal{F} \) of sets is called the representation of a graph \( G \) if \( G \) is the intersection graph of \( \mathcal{F} \).

An interval graph is the intersection graph of a family of open intervals. Subclasses of interval graphs can be defined by imposing various restrictions on the interval representation. A proper interval graph is the intersection graph of a family of open intervals in which no one properly contains another. A unit interval graph is the intersection graph of a family of open intervals of the same length. It is well-known that an interval graph is a proper interval graph if and only if it is a unit interval graph, and if and only if it is \( K_{1,3} \)-free; see [4] for a self-contained elementary proof.

The claw number \( \psi(G) \) of a graph \( G \) is the largest number \( v \geq 0 \) such that \( G \) contains the star \( K_{1,v} \) as an induced subgraph [2]. In particular, a graph with claw number 0 is an empty graph without any edges. A graph with claw number at most 1 is simply a disjoint union of cliques, and is known as a cluster graph. Clearly, every cluster graph is an interval graph, with claw number at most 1. A proper/unit interval graph is an interval graph with claw number at most 2.

In this paper, we study vertex-partitions of an interval graph into induced subgraphs with small claw numbers. A \( k \)-partition of a graph refers to a partition of its vertices into \( k \) subsets, and the resulting \( k \) vertex-disjoint induced subgraphs. For \( k \geq 1 \) and \( v \geq 1 \), let \( \mu(k, v) \) be the largest number \( n \) such that any interval graph with \( n \) vertices admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \). Correspondingly, for \( n \geq 1 \) and \( v \geq 1 \), let \( \kappa(n, v) \) be the smallest number \( k \) such that any interval graph with \( n \) vertices admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \).

* dr.minghui.jiang at gmail.com
The subchromatic number of a graph is the smallest number \( k \) such that the graph admits a \( k \)-partition into cluster graphs [3]. Broersma et al. [5, Lemma 2.4] proved that the subchromatic number of any interval graph with \( n \) vertices is at most \( \lceil \log_2(n+1) \rceil \) and this bound is best possible. Thus \( \kappa(n, 1) = \lceil \log_2(n+1) \rceil \). Gardi [9, Proposition 2.2] showed that for all \( n \geq 1 \), there exists an interval graph with \( n \) vertices that admits no vertex partition into less than \( \lceil \log_3(2n+1) \rceil \) proper interval subgraphs. Thus \( \kappa(n, 2) \geq \lceil \log_3(2n+1) \rceil \). The matching upper bound of \( \kappa(n, 2) \leq \lceil \log_3(2n+1) \rceil \) was not known.

Extending the two previous results [5, 9], we determine exact values of \( \mu(k, v) \) and \( \kappa(n, v) \) for all \( v \geq 1 \):

**Theorem 1.** For \( k \geq 1 \) and \( v \geq 1 \), \( \mu(k, v) = ((v + 1)^{k+1} - (v + 1))/v \).

**Corollary 1.** For \( n \geq 1 \) and \( v \geq 1 \), \( \kappa(n, v) = \lceil \log_{v+1}(nv + 1) \rceil \).

For \( k \geq 1 \) and \( v \geq 1 \), let \( \tilde{\mu}(k, v) \), the minimum number of induced subgraphs with claw number at most \( k \), be the largest number \( w \) such that any interval graph with claw number at most \( w \) admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \). Correspondingly, for \( w \geq v \geq 1 \), let \( \tilde{\kappa}(w, v) \) be the smallest number \( k \) such that any interval graph with claw number at most \( w \) admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \).

Clearly, \( \tilde{\mu}(k, v) = v \) for \( k = 1 \) and \( v \geq 1 \), and \( \tilde{\kappa}(w, v) = 1 \) for \( w = v \geq 1 \). Our next two results include bounds on \( \tilde{\mu}(k, v) \) for \( k \geq 2 \) and \( v \geq 1 \), and on \( \tilde{\kappa}(w, v) \) for \( w > v \geq 1 \):

**Theorem 2.** For \( k \geq 2 \) and \( v \geq 1 \), \( \tilde{\mu}(k, v) \leq (v + 1)^{k-1} \). Also, for \( k \geq 2 \), \( \tilde{\mu}(k, v) \geq (v + 1)^{k-1}/2 \) when \( v = 1 \), \( \tilde{\mu}(k, v) \geq 2(v + 1)^{k-1}/3 \) when \( v = 2 \), and \( \tilde{\mu}(k, v) \geq (v - 2)(v + 1)^{k-1} \) when \( v \geq 3 \).

**Corollary 2.** For \( w > v \geq 1 \), \( \lceil \log_{v+1} w \rceil + 1 \leq \tilde{\kappa}(w, v) \leq \lceil \log_{v+1} w \rceil + 3 \). Moreover, for \( w > v \),

\[
\begin{align*}
\tilde{\kappa}(w, v) &\leq \lceil \log_{v+1} 2(w - 1) \rceil + 2 \leq \lceil \log_{v+1} w \rceil + 3 \text{ when } v = 1, \\
\tilde{\kappa}(w, v) &\leq \lceil \log_{v+1} 3(w - 1)/2 \rceil + 2 \leq \lceil \log_{v+1} w \rceil + 3 \text{ when } v = 2, \\
\tilde{\kappa}(w, v) &\leq \lceil \log_{v+1} (w - 1)/(v - 2) \rceil + 2 \leq \lceil \log_{v+1} w \rceil + 2 \text{ when } v \geq 3.
\end{align*}
\]

In the terminology of partially ordered sets [7], \( \kappa(n, v) \) is the smallest number \( k \) such that any interval order with \( n \) elements admits a partition into \( k \) weak orders (respectively, semiorders) when \( v = 1 \) (respectively, \( v = 2 \)). Similarly, \( \tilde{\kappa}(w, v) \) for \( v = 1 \) and \( v = 2 \) has alternative interpretations in terms of weak orders and semiorders in interval orders. Albertson et al. [3, Theorem 4] proved that \( \tilde{\kappa}(w, 1) \leq w \) for any \( w \geq 1 \). Although this upper bound is asymptotically weaker than the \( v = 1 \) case of our upper bound in Corollary 2, it is tight for \( w = 2 \) and \( 3 \), as we show in the next theorem, where the exact values of \( \tilde{\kappa}(w, v) \) are determined for some small values of \( w \) and \( v \):

**Theorem 3.** \( \tilde{\kappa}(2, 1) = 2, \tilde{\kappa}(3, 1) = 3, \tilde{\kappa}(3, 2) = 2, \tilde{\kappa}(4, 2) = \tilde{\kappa}(5, 2) = \tilde{\kappa}(6, 2) = 3, \tilde{\kappa}(5, 3) = 2. \)

For \( v \geq 1 \), let MIN-PARTITION\((v)\) be the problem of finding a vertex partition of a given graph into the minimum number of induced subgraphs with claw number at most \( v \). For \( k \geq 2 \) and \( v \geq 1 \), let k-PARTITION\((v)\) be the problem of deciding whether a given graph admits a vertex partition into \( k \) induced subgraphs with claw number at most \( v \). By certain generic results on vertex-partitioning [1, 6], k-PARTITION\((v)\) in general graphs is NP-hard for all \( k \geq 2 \) and \( v \geq 1 \).

Broersma et al. [5, Theorem 5.4] proved that for any fixed \( k \), deciding whether an interval graph with \( n \) vertices has subchromatic number at most \( k \) admits an algorithm running in \( O(k \cdot n^{2k+1}) \) time. In other words, they presented an \( O(k \cdot n^{2k+1}) \)-time algorithm for k-PARTITION\((1)\) in interval graphs.

Gandhi et al. [8] noted that the two results [5, Lemma 2.4 and Theorem 5.4] together imply an \( n O(\log n) \)-time exact algorithm for subcoloring interval graphs, and hence this problem is unlikely to be NP-hard. Nevertheless, they presented a 3-approximation algorithm for subcoloring interval graphs. In addition,
they presented a 6-approximation algorithm for partitioning an interval graph into the minimum number of proper interval graphs. In other words, they obtained a 3-approximation for MIN-PARTITION(1), and a 6-approximation for MIN-PARTITION(2), in interval graphs.

We present a simple approximation algorithm for MIN-PARTITION(v) in interval graphs for all v ≥ 1:

**Theorem 4.** MIN-PARTITION(v) in interval graphs admits a polynomial-time approximation algorithm with ratio 3 for 1 ≤ v ≤ 2, and with ratio 2 for v ≥ 3.

### 2 Preliminaries

Interval graphs can be recognized in linear time [10]. Henceforth when we refer to an interval graph, we assume that an interval representation of the graph is readily available when needed, and we refer to an interval graph and its interval representation interchangeably.

An *independent set* (respectively, a *clique*) is a set of pairwise non-adjacent (respectively, adjacent) vertices in a graph. Besides the claw number ψ(G) introduced earlier, there are two other common parameters for a graph G:

- the *independence number* α(G) is the maximum number of vertices in an independent set in G,
- the *vertex-clique-partition number* ϑ(G) is the minimum number of parts in a vertex partition of G into cliques.

It is easy to see that ψ(G) ≤ α(G) ≤ ϑ(G) for any graph G. Indeed, if G is an interval graph, then α(G) = ϑ(G) [10]. We next sketch a simple constructive proof of the equality α(G) = ϑ(G) for an interval graph G. Let I be a family of open intervals whose intersection graph is G. Without loss of generality, assume that all endpoints of intervals in I are integers. Then we can find an independent set and a vertex clique partition at the same time by a standard sweepline algorithm as follows:

Initialize I′ ← I and i ← 1. While I′ is not empty, let T_i = (l_i, r_i) be an interval in I′ whose right endpoint r_i is minimum, let I_i′ (respectively, I_i) be the subfamily of intervals in I′ (respectively, I) that contain the subinterval S_i = (r_i − 1, r_i) of T_i, then update I′ ← I′ \ I_i′ and i ← i + 1.

Let j be the number of rounds that the sweepline algorithm runs until I′ is empty. Then for 1 ≤ i ≤ j, S_i ⊆ T_i ∈ I_i′ ⊆ I_i. Moreover, the j intervals T_i correspond to an independent set in G, and the j subfamilies I_i′ correspond to a partition of the vertices of G into cliques. Thus α(G) = ϑ(G).

For a graph G and v ≥ 1, let κ(G, v) be the smallest number of induced subgraphs in a vertex partition of G such that each subgraph has claw number at most v.

**Lemma 1.** For k ≥ 1 and v ≥ 1, any interval graph G with ϑ(G) ≤ (v + 1)^k − 1 satisfies κ(G, v) ≤ k.

**Proof.** Fix any v ≥ 1. We prove the lemma by induction on k. For the base case when k = 1, any interval graph G with ϑ(G) ≤ (v + 1)^1 − 1 = v has ψ(G) ≤ ϑ(G) ≤ v and hence κ(G, v) ≤ 1. Now let k ≥ 2, and let G be an interval graph with ϑ(G) ≤ (v + 1)^k − 1. We next show that G admits a vertex partition into k induced subgraphs with claw number at most v.

Run the sweepline algorithm on an interval representation I of G to obtain j = ϑ(G) cliques I_i, 1 ≤ i ≤ j, such that I = ∪_i I_i. Let t = (v + 1)^k − 1 and s = (v + 1)^k−1 − 1. Then j ≤ t = (v + 1)(s + 1) − 1 = v(s + 1) + s. Let J be the union of all subfamilies I_i with i mod (s + 1) = 0. The subgraph of G represented by J has vertex-clique-partition number at most v, and hence claw number at most v too.
Each connected component in the subgraph of $G$ represented by $\mathcal{I}\setminus\mathcal{J}$ has vertex-clique-partition number at most $s = (v+1)^{k-1} - 1$, and hence admits a vertex partition into $k-1$ induced subgraphs with claw number at most $v$, by the induction hypothesis. Then by disjoint union, the subgraph of $G$ represented by $\mathcal{I}\setminus\mathcal{J}$ also admits a vertex partition into $k-1$ induced subgraphs with claw number at most $v$. Thus $
abla(G,v) \leq 1 + (k - 1) = k$.

To prove that their upper bound on subchromatic numbers of interval graphs is best possible, Broersma et al. [5, Lemma 2.4] showed that for any $k \geq 1$, there is an interval graph $G_k$ with $2^k - 1$ vertices and subchromatic number $k$. Such graphs were later used by Gandhi et al. [8] as a lower bound in their 3-approximation algorithm for subcoloring interval graphs; they refer to $G_k$ as $BC(k)$ and call them binary cliques. Similarly, Gardi [9, Lemma 2.1] showed that for any $k \geq 1$, there is an interval graph $H_k$ with $(3^k - 1)/2$ vertices that admits no vertex partition into less than $k$ proper interval graphs.

$G_k$ and $H_k$ for $k \geq 1$ are recursively constructed as follows: $G_1$ (respectively, $H_1$) is just a single vertex. For $k \geq 2$, $G_k$ (respectively, $H_k$) consists of a new vertex connected to all vertices of $k$ (respectively, 3) disjoint copies of $G_{k-1}$ (respectively, $H_{k-1}$). $G_k$ and $H_k$ for $k \geq 1$ can be generalized to $A_{k,v}$ for $k \geq 1$ and $v \geq 1$: $A_{1,v}$ is just a single vertex. For $k \geq 2$, $A_{k,v}$ consists of a new vertex connected to all vertices of $v+1$ disjoint copies of $A_{k-1,v}$. Then $G_k = A_{k,1}$ and $H_k = A_{k,2}$. Refer to Figure 1 for an illustration of $H_3 = A_{3,2}$.

![Figure 1: An interval representation of $H_3 = A_{3,2}$.](image)

It is easy to see that for $k \geq 1$ and $v \geq 1$, $A_{k,v}$ is an interval graph with $\sum_{i=0}^{k-1} (v+1)^i = (v+1)^{k-1}/v$ vertices, and $\alpha(A_{k,v}) = \phi(A_{k,v}) = (v+1)^{k-1}$. Also, for $k \geq 2$ and $v \geq 1$, $\psi(A_{k,v}) = (v+1)^{k-1}$.

**Lemma 2.** For $k \geq 2$ and $v \geq 1$, $\nabla(A_{k,v},v) = k$.

**Proof.** Fix any $v \geq 1$. Since $\phi(A_{k,v}) = (v+1)^{k-1} \leq (v+1)^k - 1$, it follows by Lemma 1 that $\nabla(A_{k,v},v) \leq k$. We next show that $\nabla(A_{k,v},v) \geq k$ by induction on $k$. For the base case when $k = 2$, $A_{2,v}$ is simply $K_{1,v+1}$, and hence admits no vertex partition into less than 2 induced subgraphs with claw number at most $v$. Now proceed to the inductive step when $k \geq 3$.

Consider any vertex partition of $A_{k,v}$ into induced subgraphs with claw number at most $v$. Recall that $A_{k,v}$ consists of a new vertex connected to $v+1$ disjoint copies of $A_{k-1,v}$. To avoid a star $K_{1,v+1}$ forming around the new vertex, the subgraph that includes it can include vertices from at most $v$ copies of $A_{k-1,v}$, and hence must miss one copy of $A_{k-1,v}$ entirely. By the induction hypothesis, this copy of $A_{k-1,v}$ admits no vertex partition into less than $k-1$ induced subgraphs with claw number at most $v$. It follows that $A_{k,v}$ admits no vertex partition into less than $k$ induced subgraphs with claw number at most $v$. Thus $\nabla(A_{k,v},v) \geq k$.

Note that for $k \geq 1$ and $v \geq 1$, the interval graph $A_{k+1,v}$ satisfies $\phi(A_{k+1,v}) = (v+1)^k$ and $\nabla(A_{k+1,v},v) = k + 1$. This shows that the bound in Lemma 1 is best possible.

**3 ** $\mu(k, v)$ and $\nabla(n, v)$

In this section we prove Theorem 1 and Corollary 1.

For a family $\mathcal{I}$ of intervals, denote by $|\mathcal{I}|$ the number of intervals in $\mathcal{I}$. We first prove a technical lemma:
Lemma 3. For \( n \geq s \geq 1 \), the vertex set of any interval graph \( G \) with \( n \) vertices can be partitioned into \( 2r+1 \) subsets for some \( r \geq 0 \), including \( X_i \) and \( Y_i \) for \( 1 \leq i \leq r \), and \( Z \), such that

1. Vertices from different subsets among \( X_i \) and \( Z \) are non-adjacent in \( G \).
2. Vertices in each subset \( Y_i \) are pairwise adjacent in \( G \).
3. \( |Z| \leq |X_i| = s < |X_i| + |Y_i| \).

Proof. For any family \( Z \) of intervals, and for \( p \leq q \), denote by \( Z(p, q) \) the subfamily of intervals in \( Z \) that are contained in the interval \( (p, q) \).

Obtain an interval representation \( I \) of \( G \) with integer endpoints. Let \( a \) be the leftmost endpoint, and \( c \) the rightmost endpoint, of the intervals in \( I \).

Initialize \( Z \leftarrow I \) and \( i \leftarrow 1 \). While \( |Z| > s \), let \( b \) be the smallest integer in \((a, c]\) such that \( |Z(a, b−1)] \leq s < |Z(a, b)] \), let \( X_i \) be any subfamily of \( s \) intervals in \( Z \) such that \( Z(a, b−1) \subseteq X_i \subseteq Z(a, b) \), let \( Y_i \) be \( Z \setminus (X_i \cup Z(b, c)) \), then update \( Z \leftarrow Z \setminus (X_i \cup Y_i) \) and \( i \leftarrow i + 1 \).

Let \( r \geq 0 \) be the number of such partitioning rounds until \( |Z| \leq s \). Then \( I \) is partitioned into \( 2r+1 \) subfamilies, including \( X_i \) and \( Y_i \) for \( 1 \leq i \leq r \), and a (possibly empty) subfamily \( Z \) in the end. The following three properties can be easily verified:

1. Intervals from different subfamilies among \( X_i \) and \( Z \) do not intersect.
2. Intervals in each subfamily \( Y_i \) pairwise intersect.
3. \( |Z| \leq |X_i| = s < |X_i| + |Y_i| \).

Let \( X_i \), \( Y_i \), and \( Z \) be the subsets of vertices of \( G \) represented by intervals in \( X_i \), \( Y_i \), and \( Z \), respectively. Then the proof is complete.

We now prove Theorem 1 that \( \mu(k, v) = ((v + 1)^{k+1} - (v + 1))/v \) for \( k \geq 1 \) and \( v \geq 1 \). Fix \( v \geq 1 \), and let \( n_k = ((v + 1)^{k+1} - (v + 1))/v \).

Recall Lemma 2 that for \( k \geq 1 \) and \( v \geq 1 \), the interval graph \( A_{k+1, v} \) has \( ((v + 1)^{k+1} - 1)/v \) vertices, and satisfies \( k(A_{k+1, v}, v) = k + 1 \). This implies the upper bound of \( \mu(k, v) \leq ((v + 1)^{k+1} - 1)/v - 1 = n_k \).

In the following we prove the matching lower bound of \( \mu(k, v) \geq n_k \) by induction on \( k \).

For the base case when \( k = 1 \), we have \( n_1 = ((v + 1)^{1+1} - (v + 1))/v = v + 1 \). Any interval graph with at most \( v + 1 \) vertices obviously has claw number at most \( v \). Thus \( \mu(1, v) \geq n_1 \). Now proceed to the inductive step when \( k \geq 2 \), and let \( G \) be an interval graph with at most \( n_k \) vertices.

Apply Lemma 3 with \( s = n_{k−1} \) to partition the vertex set of \( G \) into \( 2r + 1 \) subsets, \( X_i \) and \( Y_i \) for \( 1 \leq i \leq r \), and \( Z \). Since \( n_k = (v + 1)(n_{k−1} + 1) = (v + 1)(s + 1) \), we have \( 1 \leq r \leq v + 1 \). Consider two cases:

- \( r \leq v \). Note that \( |X_i| = n_{k−1} \) for \( 1 \leq i \leq r \), and that \( |Z| \leq n_{k−1} \). By the inductive hypothesis, any interval graph with at most \( n_{k−1} \) vertices admits a vertex partition into \( k−1 \) induced subgraphs with claw number at most \( v \). By disjoint union, the subgraph of \( G \) induced by all vertices in the \( r \) subsets \( X_i \) and the subset \( Z \) admits a vertex partition into \( k−1 \) subgraphs with claw number at most \( v \). On the other hand, the subgraph of \( G \) induced by vertices in the \( r \) subsets \( Y_i \) is the union of \( r \) cliques, and has vertex-clique-partition number at most \( r \), and hence has claw number at most \( r \leq v \).

- \( r = v + 1 \). Then we must have \( |X_i| = n_{k−1} \) and \( |Y_i| = 1 \) for \( 1 \leq i \leq v + 1 \), and \( |Z| = 0 \). The subgraph of \( G \) induced by all vertices in the \( r \) subsets \( X_i \) admits a vertex partition into \( k−1 \) subgraphs with claw number at most \( v \). On the other hand, the subgraph of \( G \) induced by the \( v + 1 \) vertices in the \( v + 1 \) subsets \( Y_i \) clearly has claw number at most \( v \) too.
In both cases, $G$ admits a vertex partition into $k$ induced subgraphs with claw number at most $v$. This completes the proof of Theorem 1.

We next prove Corollary 1 that $\kappa(n, v) = \lceil \log_{v+1}(nv + 1) \rceil$ for $n \geq 1$ and $v \geq 1$. Fix any $v \geq 1$.

For $1 \leq n \leq v + 1$, we have $v + 1 \leq nv + 1 < (v + 1)^2$, and hence $\lceil \log_{v+1}(nv + 1) \rceil = 1$. It is clear that $\kappa(n, v) = 1$ for this case.

Now fix $n \geq v + 2$. Then $nv + 1 \geq (v + 1)^2$ and hence $\lceil \log_{v+1}(nv + 1) \rceil \geq 2$. Let $k = \lceil \log_{v+1}(nv + 1) \rceil$. Then $n \geq ((v + 1)^k - 1)/v$. By Lemma 2, the interval graph $A_{k,v}$ with $((v + 1)^k - 1)/v$ vertices satisfies $\kappa(A_{k,v}, v) = k$. Thus $\kappa(n, v) \geq \kappa(((v + 1)^k - 1)/v, v) \geq k$.

For the other direction, let $k = \kappa(n, v)$. Then by definition of $\mu$, we have $n \geq \mu(k - 1, v) + 1$.

By Theorem 1, $\mu(k - 1, v) = ((v + 1)^k - (v + 1))/v$. Thus $n \geq ((v + 1)^k - 1)/v$. It follows that $\kappa \leq \lceil \log_{v+1}(nv + 1) \rceil = k$. This completes the proof of Corollary 1.

4 \hspace{1cm} \bar{\mu}(k, v) \text{ and } \bar{\kappa}(w, v)

In this section we prove Theorem 2 and Corollary 2.

We first prove Theorem 2. Fix $k \geq 2$ and $v \geq 1$. Recall Lemma 2 that the interval graph $A_{k+1,v}$ with claw number $(v + 1)^k$ satisfies $\kappa(A_{k+1,v}, v) = k + 1$. This implies the upper bound of $\bar{\mu}(k, v) \leq (v + 1)^k - 1$.

Write $w_1 = (v + 1)^k - 1/2$, $w_2 = 2(v + 1)^k - 1/3$, and $w_3 = (v - 2)(v + 1)^k - 1$. Let $G$ be an interval graph with claw number $w \leq w_1$ when $v = 1$, $w \leq w_2$ when $v = 2$, and $w \leq w_3$ when $v \geq 3$. In the following, we obtain a vertex partition of $G$ into $k$ induced subgraphs with claw number at most $v$, hence proving the lower bounds $\bar{\mu}(k, v) \geq w_1$ when $v = 1$, $\bar{\mu}(k, v) \geq w_2$ when $v = 2$, and $\bar{\mu}(k, v) \geq w_3$ when $v \geq 3$.

Let $\mathcal{I}$ be an interval representation of $G$ with integer endpoints. Let $j = \partial(G)$. Run the sweepline algorithm to obtain an independent set of $j$ intervals $T_i$ in $\mathcal{I}$, and a vertex covering of $\mathcal{I}$ by $j$ cliques $\mathcal{I}_i$, where $T_i \in \mathcal{I}_i$ for $1 \leq i \leq j$.

Let $s = 2w$ when $v = 1$, $s = \lceil 3w/2 \rceil$ when $v = 2$, and $s = \lceil w/(v - 2) \rceil$ when $v \geq 3$. Then for all $v \geq 1$, $s \leq (v + 1)^k - 1/2$. Let $\mathcal{J}$ be the union of all subfamilies $\mathcal{I}_i$ with $i \mod s = 0$. Then each connected component in the graph represented by $\mathcal{I} \setminus \mathcal{J}$ has vertex-clique-partition number at most $s - 1 \leq (v + 1)^k - 1 - 1$, and hence admits a vertex partition into $k - 1$ induced subgraphs with claw number at most $v$, by Lemma 1. Then by disjoint union, the graph represented $\mathcal{I} \setminus \mathcal{J}$ also admits a vertex partition into $k - 1$ induced subgraphs with claw number at most $v$. It remains to show that the graph represented by $\mathcal{J}$ has claw number at most $v$.

Suppose for contradiction that the graph represented by $\mathcal{J}$ has claw number at least $v + 1$. Let $C$ and $L_1, \ldots, L_{v+1}$ be intervals representing the center and the $v + 1$ leaves of a star $K_{1,v+1}$ in the graph, where the $v + 1$ intervals $L_1, \ldots, L_{v+1}$ are ordered from left to right. We now proceed in three different ways depending on the value of $v$:

1. $v = 1$. Either $L_1$ and $C$, or $C$ and $L_2$, are two intervals from two subfamilies $\mathcal{I}_{ps}$ and $\mathcal{I}_{qs}$ with $q - p \geq 1$. The union of the two intervals is a contiguous interval that intersects all intervals in $qs - ps + 1$ consecutive subfamilies $\mathcal{I}_i$, $ps \leq i \leq qs$, which include the $qs - ps + 1$ disjoint intervals $T_i, ps \leq i \leq qs$. Then one of these two intervals intersects at least $\lceil (qs - ps + 1)/2 \rceil \geq \lceil (s + 1)/2 \rceil = \lceil (2w + 1)/2 \rceil = w + 1$ disjoint intervals $T_i$.

2. $v = 2$. $L_1$ and $L_3$ are from two subfamilies $\mathcal{I}_{ps}$ and $\mathcal{I}_{qs}$ with $q - p \geq 2$. The union of $L_1$, $C$, and $L_3$ is a contiguous interval that intersects all intervals in $qs - ps + 1$ consecutive subfamilies $\mathcal{I}_i$, $ps \leq i \leq qs$. One of these three intervals intersects at least $\lceil (qs - ps + 1)/3 \rceil \geq \lceil (2s + 1)/3 \rceil \geq \lceil (3w + 1)/3 \rceil = w + 1$ disjoint intervals $T_i$. 


3. \( v \geq 3 \). \( L_2 \) and \( L_v \) are from two subfamilies \( I_{ps} \) and \( I_{qs} \) with \( q - p \geq v - 2 \). \( C \) intersects all intervals in \( qs - ps + 1 \) consecutive subfamilies \( I_i \), \( ps \leq i \leq qs \). Note that \( qs - ps + 1 \geq (v - 2)s + 1 \geq w + 1 \). Thus \( C \) intersects at least \( w + 1 \) disjoint intervals \( T_i \).

In each of the three cases, we can find a star \( K_{1,w+1} \) in \( G \) represented by some interval in \( J \) intersecting \( w + 1 \) disjoint intervals \( T_i \in I_i \), a contradiction to our assumption that \( G \) has claw number \( w \). Thus the graph represented by \( J \) must have claw number at most \( v \). This completes the proof of Theorem 2.

We next prove the bounds on \( \tilde{\kappa}(w, v) \) in Corollary 2 for \( w > v \geq 1 \). Let \( k = \lceil \log_{v+1} w \rceil + 1 \). Then \( w \geq (v + 1)^{k-1} \) and \( k \geq 2 \). By Lemma 2, the interval graph \( A_{k,v} \) with claw number \((v + 1)^{k-1}\) satisfies \( \kappa(A_{k,v}, v) = k \). Thus we have the lower bound \( \tilde{\kappa}(w, v) \geq \tilde{\kappa}(v + 1)^{k-1}, v) \geq k \).

For the other direction, let \( \tilde{\kappa} = \tilde{\kappa}(w, v) \), where \( \tilde{\kappa} \geq 2 \). Then by definition of \( \mu \), we have \( w \geq \mu(\tilde{\kappa} - 1, v) + 1 \). By the lower bounds in Theorem 2, we have \( w \geq (v + 1)^{\tilde{\kappa}-2}/2 + 1 \) when \( v = 1 \), \( w \geq 2(v + 1)^{\tilde{\kappa}-2}/3 + 1 \) when \( v = 2 \), and \( w \geq (v - 2)(v + 1)^{\tilde{\kappa}-2} + 1 \) when \( v \geq 3 \). Correspondingly,

- when \( v = 1 \), \( \tilde{\kappa}(w, v) \leq \lceil \log_{v+1} 2(w - 1) \rceil + 2 = \lceil \log_{v+1} (w - 1) \rceil + 3 \leq \lceil \log_{v+1} w \rceil + 3 \),
- when \( v = 2 \), \( \tilde{\kappa}(w, v) \leq \lceil \log_{v+1} (w - 1)/2 \rceil + 2 = \lceil \log_{v+1} (w - 1)/2 \rceil + 3 \leq \lceil \log_{v+1} w \rceil + 3 \),
- when \( v \geq 3 \), \( \tilde{\kappa}(w, v) \leq \lceil \log_{v+1} (w - 1)/(v - 2) \rceil + 2 \leq \lceil \log_{v+1} w \rceil + 2 \).

Thus we have the upper bounds on \( \tilde{\kappa}(w, v) \). This completes the proof of Corollary 2.

5. \( \tilde{\kappa}(w, v) \) for small \( w \) and \( v \)

In this section we prove Theorem 3 that \( \tilde{\kappa}(2, 1) = 2 \), \( \tilde{\kappa}(3, 1) = 3 \), \( \tilde{\kappa}(3, 2) = 2 \), \( \tilde{\kappa}(4, 2) = \tilde{\kappa}(5, 2) = \tilde{\kappa}(6, 2) = 3 \), and \( \tilde{\kappa}(5, 3) = 2 \).

We first prove that \( \tilde{\kappa}(2, 1) = 2 \) and \( \tilde{\kappa}(3, 1) = 3 \). Recall [3, Theorem 4] that \( \tilde{\kappa}(w, 1) \leq w \) for any \( w \geq 1 \). In particular, \( \tilde{\kappa}(2, 1) \leq 2 \) and \( \tilde{\kappa}(3, 1) \leq 3 \). From the other direction, it is clear that \( \tilde{\kappa}(w, v) \geq 2 \) for any \( w > v \geq 1 \). Thus \( \tilde{\kappa}(2, 1) = 2 \). The following lemma implies that \( \tilde{\kappa}(3, 1) \geq 3 \), and hence we have \( \tilde{\kappa}(3, 1) = 3 \).

**Lemma 4.** Let \( J_3 \) be the set of 12 open intervals with lengths 1, 2, and 3, and with integer endpoints between 0 and 5. Then the interval graph represented by \( J_3 \) has claw number 3 and admits a vertex partition into three subgraphs with claw number at most 1, but it admits no vertex partition into two subgraphs with claw number at most 1.

**Proof.** The interval graph represented by \( J_3 \), with maximum interval length 3, clearly has claw number 3. Moreover, it admits a vertex partition, by left endpoint modulo 3, into three subgraphs with claw number at most 1 (that is, three cluster graphs). To show that the graph admits no vertex partition into two subgraphs with claw number at most 1, we suppose the contrary and color each interval in \( J_3 \) either white or black according to such a partition, then find three intervals of the same color that represent \( K_{1,2} \) to reach a contradiction.

![Figure 2: Intervals from \( J_3 \) for the three cases in the proof of Lemma 4.](image_url)

Refer to Figure 2. Consider three cases:
1. \( J_3 \) contains three consecutive unit intervals among which the left and the right have opposite colors. Let \((a, a + 1), (a + 1, a + 2), (a + 2, a + 3)\) be the three unit intervals. Consider also the two length-2 intervals \((a, a + 2)\) and \((a + 1, a + 3)\), and the length-3 interval \((a, a + 3)\). By symmetry, we can assume without loss of generality that \((a, a + 1)\) and \((a + 1, a + 2)\) are white, and \((a + 2, a + 3)\) is black. Then both \((a, a + 2)\) and \((a, a + 3)\) must be black to avoid forming a white star \(K_{1,2}\) with \((a, a + 1)\) and \((a + 1, a + 2)\). But then \((a, a + 3), (a, a + 2), (a + 2, a + 3)\) form a black star \(K_{1,2}\).

2. The five unit intervals in \( J_3 \) all have the same color. Consider any four consecutive unit intervals, say, \((a, a + 1), (a + 1, a + 2), (a + 2, a + 3), (a + 3, a + 4)\). Assume without loss of generality that they are all white. Consider also the three length-2 intervals \((a, a + 2), (a + 1, a + 3), (a + 2, a + 4)\). Either one of them is white, forming a white star \(K_{1,2}\) with two unit intervals, or all of them are black, forming a black star \(K_{1,2}\) by themselves.

3. The five unit intervals in \( J_3 \) have alternating colors. Let \((a, a + 1), (a + 1, a + 2), (a + 2, a + 3), (a + 3, a + 4), (a + 4, a + 5)\) be the five unit intervals. Assume without loss of generality that \((a, a + 1), (a + 2, a + 3), (a + 4, a + 5)\) are white, \((a + 1, a + 2)\) and \((a + 3, a + 4)\) are black. Then the length-3 interval \((a, a + 3)\) must be black to avoid forming a white star \(K_{1,2}\) with \((a, a + 1)\) and \((a + 2, a + 3)\). Similarly, the length-3 interval \((a + 2, a + 5)\) must be black to avoid forming a white star \(K_{1,2}\) with \((a + 2, a + 3)\) and \((a + 4, a + 5)\). Then the two intervals \((a, a + 3)\) and \((a + 2, a + 5)\), together with either \((a + 1, a + 2)\) or \((a + 3, a + 4)\), form a black star \(K_{1,2}\).

In each case, there are three intervals in \( J_3 \) representing a monochromatic star \(K_{1,2}\). □

We next prove that \(\bar{\kappa}(3, 2) = 2\) and \(\bar{\kappa}(4, 2) = \bar{\kappa}(5, 2) = \bar{\kappa}(6, 2) = 3\). By our upper bound of \(\bar{\kappa}(w, v) \leq \lceil\log_{v+1} 3(w - 1)/2\rceil + 2\) for \(w > v = 2\) in Corollary 2, we have \(\bar{\kappa}(6, 2) \leq 3\). On the other hand, it is clear that \(\bar{\kappa}(3, 2) \geq 2\). Thus

\[
2 \leq \bar{\kappa}(3, 2) \leq \bar{\kappa}(4, 2) \leq \bar{\kappa}(5, 2) \leq \bar{\kappa}(6, 2) \leq 3.
\]

As a warm-up exercise, we first prove the following lemma which implies that \(\bar{\kappa}(6, 2) \geq 3\), and hence \(\bar{\kappa}(6, 2) = 3\).

**Lemma 5.** Let \( J_6 \) be the family of 24 open intervals with lengths 1, 3, 5, and 6 as illustrated in Figure 3. Then the interval graph represented by \( J_6 \) has claw number 6 and admits a vertex partition into three subgraphs with claw number at most 2, but it admits no vertex partition into two subgraphs with claw number at most 2.

![Figure 3: The family \( J_6 \) of 24 intervals includes 13, 1, 2, and 8 intervals with lengths 1, 3, 5, and 6, respectively.](image)

**Proof:** The interval graph represented by \( J_6 \) clearly has claw number 6, and admits a vertex partition, by lengths \(\{1\} \cup \{3\} \cup \{5, 6\}\), into three subgraphs with claw number at most 2. Suppose for contradiction that it admits a vertex partition into two subgraphs with claw number at most 2. Then any two intervals in \( J_6 \) containing five unit intervals in their intersection must have the same color, because otherwise one of them would have the same color as three of the five unit intervals, forming a monochromatic star \(K_{1,3}\). Consequently all length-5 and length-6 intervals must have the same color.
Consider the length-3 interval in the middle and the three unit intervals contained in it. If any of these four intervals has the same color as the length-5 and length-6 intervals, then it would form a monochromatic star together with the two length-5 intervals and a length-6 interval that intersects all of them. Otherwise, these four intervals in the middle would have the same color, and would form a monochromatic star \( \bar{K}_{1,3} \) by themselves. In both cases, we reach a contradiction.

The next lemma implies that \( \bar{\kappa}(5, 2) \geq 3 \), and hence \( \bar{\kappa}(5, 2) = 3 \).

**Lemma 6.** Let \( J_5 \) be the family of open intervals with lengths 1, 3, and 5, and with integer endpoints between 0 and 79. The interval graph represented by \( J_5 \) has claw number 5 and admits a vertex partition into three subgraphs with claw number at most 2, but it admits no vertex partition into two subgraphs with claw number at most 2.

*Proof.* The interval graph represented by \( J_5 \) clearly has claw number 5, and admits a vertex partition (by lengths) into three subgraphs with claw number at most 2. To show that the graph admits no vertex partition into two subgraphs with claw number at most 2, we suppose the contrary and color each interval in \( J_5 \) either white or black according to such a partition, then find four intervals of the same color that represent a star \( \bar{K}_{1,3} \) to reach a contradiction.

Consider any subset of nine intervals in \( J_5 \) whose lengths and relative positions are as illustrated in Figure 4. Focus on the length-5 interval in the middle, and consider the other eight intervals as its neighbors. We refer to the length-5 interval on the left and the leftmost unit interval as the *left neighbors*, the length-5 interval on the right and the rightmost unit interval as the *right neighbors*, the length-3 interval and the three unit intervals in the middle as the *middle neighbors*, respectively, of the length-5 interval in the middle. If the length-5 interval in the middle has the same color as both a left neighbor and a right neighbor, then to avoid a monochromatic star \( \bar{K}_{1,3} \) it cannot have the same color as any of its middle neighbors. But then the four middle neighbors would have the same color, and form a monochromatic star \( \bar{K}_{1,3} \) by themselves. Thus we have the following property:

No length-5 interval in \( J_5 \) can have both a left neighbor and a right neighbor of the same color as itself.

Partition the unit intervals in \( J_5 \) into four *groups* according to their left endpoints modulo 4. We say that two unit intervals in the same group are *4-adjacent* if their left endpoints differ by exactly 4. For any two 4-adjacent unit intervals \( A \) and \( B \) in \( J_5 \), denote by \( AB \) the unique length-5 interval in \( J_5 \) that contains them.

Consider a chain of 4-adjacent unit intervals from the same group in \( J_5 \), in the middle row of the three rows of intervals illustrated in Figure 5. Suppose that both \( R \) and \( S \) are white. Then \( RS \) must be black. Then \( QR \) and \( ST \) cannot be both black. Without loss of generality, assume that \( QR \) is white. Then \( Q \) is black,
and so is $PQ$. Then the argument repeats as the chain extends further to the left with alternating colors: $P$ is white, the unit interval that is 4-adjacent to $P$ on the left, if any, is black, and so on. Now consider $ST$. If $ST$ is white, then the situation is symmetric to the case that $QR$ is white, with respect to the starting interval $RS$. If $ST$ is black, then $T$ is white and so is $TU$. Then the argument repeats as the chain extends further to the right with alternating colors, for $U$ and $V$ and so on.

In summary, in the chain of 4-adjacent unit intervals from each group, there is at most one monochromatic subchain of two or more consecutive 4-adjacent unit intervals of the same color, and moreover the length of such a monochromatic subchain, if it exists, is either two or three. For each such monochromatic subchain, if its length is two, mark any one of the two unit intervals as a hole, or else if its length is three, mark the unit interval in the middle (for example, $S$ among $R,S,T$ as in Figure 5) as a hole. Then there are at most $h \leq 4$ holes from the four groups, and they separate the other $79 - h$ unit intervals in $J_5$ into at most $h + 1$ contiguous blocks. The longest block has length at least $(79 - 4)/(4 + 1) = 15$. Within each block, every two 4-adjacent unit intervals have different colors.

Encode white as 0 and black as 1. Then the colors of all unit intervals in each block form a periodic binary string of period 8, whose periodic pattern is the concatenation of a 4-bit binary string and its complement. There are 16 periodic patterns:

$$
\begin{align*}
00001111 & \ 00011110 & \ 00101101 & \ 00111100 & \ 01001011 & \ 01011010 & \ 01101001 & \ 01110000 \\
10000111 & \ 10010110 & \ 10100101 & \ 10110100 & \ 11000011 & \ 11010010 & \ 11101001 & \ 11110000 \\
\end{align*}
$$

It is easy to verify that each of the 16 patterns is a rotation of either $10100101$ or $11000011$. Since the longest block has length at least $15 = 2 \cdot 8 - 1$, the binary string corresponding to this block must contain either $10100101$ or $11000011$ as a substring.

Refer to Figure 6 for the two cases. Consider the two length-5 intervals containing the five unit intervals corresponding to the first five bits and the last five bits, respectively, of this substring. Since each of them contains three white unit intervals, they must both be black to avoid forming a white star $K_{1,3}$. But then the two black length-5 intervals, and the two black unit intervals contained by either of them, would form a black star $K_{1,3}$.

The next lemma implies that $\kappa(4, 2) \geq 3$, and hence $\kappa(4, 2) = 3$.

**Lemma 7.** Let $J_4$ be the family of open intervals with integer lengths between 2 and 7, and with integer endpoints between 0 and 21. The interval graph represented by $J_4$ has claw number 4 and admits a vertex partition into three subgraphs with claw number at most 2, but it admits no vertex partition into two subgraphs with claw number at most 2.

**Proof.** The interval graph represented by $J_4$ clearly has claw number 4, since the maximum length 7 is only one plus three times the minimum length 2. By the upper bound for the $v = 2$ case of Corollary 2, we have $\kappa(4, 2) \leq 3$, and hence the graph represented by $J_4$ admits a vertex partition into three subgraphs with claw number at most 2. We say that a 2-partition of a family of intervals is bad if there are four intervals representing a star $K_{1,3}$ in the same part, and say that it is good otherwise. Assisted by a computer program, we next show that $J_4$ admits no good 2-partition.
Note that \( J_4 \) contains exactly \( 22 - \ell \) intervals of each length \( \ell, \, 2 \leq \ell \leq 7 \). The total number of intervals in \( J_4 \) is \( n = 20 + 19 + 18 + 17 + 16 + 15 = 105 \). A brute-force algorithm that enumerates all \( 2^{105} \) 2-partitions of \( J_4 \) is obviously too slow, but we can speed it up by a standard branch-and-bound technique.

The algorithm works as follows. Sort the intervals in \( J_4 \) by increasing right endpoint, breaking ties by decreasing left endpoint. Then call the following recursive function with a partial 2-partition of \( J_4 \) with the first interval in the first part.

Given a good partial 2-partition of the first \( i \) intervals in \( J_4 \), the recursive function reports the 2-partition and returns if \( i = n \). Otherwise, it checks whether the next interval in \( J_4 \) can be added to either part without forming a star \( K_{1,3} \), then recurses on the resulting at most two good partial 2-partitions of the first \( i + 1 \) intervals.

We wrote a C program implementing this simple branch-and-bound algorithm; see the source code in the appendix. On a typical laptop computer, it took less than two minutes for the program to verify that \( J_4 \) admits no good 2-partition.

Recall (1) earlier. By Lemma 5, Lemma 6, and Lemma 7, we have \( \bar{\kappa}(4, 2) = \bar{\kappa}(5, 2) = \bar{\kappa}(6, 2) = 3 \). The previous three lemmas also imply that interval graphs admitting no vertex partition into less than three proper interval graphs do not always contain \( H_3 \) (which has claw number 9) as an induced subgraph. This partially answers an open question of Gardi [9, page 53] in the negative.

The following lemma shows that \( \bar{\kappa}(3, 2) \leq 2 \), and hence \( \bar{\kappa}(3, 2) = 2 \).

**Lemma 8.** Any interval graph with claw number at most 3 admits a vertex partition into two subgraphs with claw number at most 2.

**Proof.** Let \( G \) be an interval graph, and let \( \mathcal{I} \) be an interval representation of \( G \) with integer endpoints. Let \( j = \varnothing(G) \). Run the sweepline algorithm to obtain an independent set of \( j \) intervals \( T_i \) in \( \mathcal{I} \), and a vertex partition of \( \mathcal{I} \) into \( j \) cliques \( \mathcal{I}'_i \), where \( T_i \in \mathcal{I}'_i \) for \( 1 \leq i \leq j \). For \( h \in \{0, 1, 2, 3\} \), let \( J_h \) be the union of the subfamilies \( \mathcal{I}'_i \) with \( i \mod 4 = h \). We next show that the subgraph of \( G \) induced by \( J_0 \cup J_1 \) has claw number at most 2. The argument for \( J_2 \cup J_3 \) is similar.

Suppose for contradiction that \( J_0 \cup J_1 \) contains four intervals \( O, A, B, C \) representing a star \( K_{1,3} \), where \( O \) is the center, and \( A, B, C \) are the three leaves. Suppose that \( O \in \mathcal{I}'_o, \, A \in \mathcal{I}'_a, \, B \in \mathcal{I}'_b, \, \) and \( C \in \mathcal{I}'_c \), where \( a < b < c \). Note there are only two different values 0 and 1 for \( a, b, c \) modulo 4. We claim that if \( a > o - 3 \), then \( c \geq o + 3 \). Consider two cases. If \( o \mod 4 = 0 \), then condition \( a > o - 3 \) implies that \( a \geq o \), and hence \( b \geq o + 1 \), and hence \( c \geq o + 4 \). If \( o \mod 4 = 1 \), then condition \( a > o - 3 \) implies that \( a \geq o - 1 \), and hence \( b \geq o \), and hence \( c \geq o + 3 \).

In summary, we must have either \( a \leq o - 3 \) or \( c \geq o + 3 \):

- If \( a \leq o - 3 \), then the \( o - a \) intervals \( T_i \) for \( a \leq i \leq o - 1 \) and the interval \( O \) would together be \( o - a + 1 \geq 4 \) pairwise-disjoint intervals intersecting \( A \).
- If \( c \geq o + 3 \), then the \( c - o \) intervals \( T_i \) for \( o \leq i \leq c - 1 \) and the interval \( C \) would together be \( c - o + 1 \geq 4 \) pairwise-disjoint intervals intersecting \( O \).

In both cases, we reach a contradiction to our assumption that \( G \) has claw number at most 3. \( \square \)

We finally prove that \( \bar{\kappa}(5, 3) = 2 \). The lower bound of \( \bar{\kappa}(5, 3) \geq 2 \) is obvious. The following lemma gives the tight upper bound of \( \bar{\kappa}(5, 3) \leq 2 \):

**Lemma 9.** Any interval graph with claw number at most 5 admits a vertex partition into two subgraphs with claw number at most 3.
Proof. Let $G$ be an interval graph with claw number at most 5, and let $\mathcal{I}$ be an interval representation of $G$. Partition $\mathcal{I}$ into two subfamilies $\mathcal{I}_{<2}$ and $\mathcal{I}_{\geq 2}$ of intervals properly containing less than two and at least two, respectively, disjoint intervals in $\mathcal{I}$. Since any interval that intersects more than three disjoint intervals must properly contain at least two of them, the subgraph of $G$ induced by intervals in $\mathcal{I}_{<2}$ clearly has claw number at most 3. We claim that the subgraph of $G$ induced by intervals in $\mathcal{I}_{\geq 2}$ has claw number at most 3 too.

Suppose the contrary. Let $C$ and $L_1, L_2, L_3, L_4$ be intervals in $\mathcal{I}_{\geq 2}$ representing the center and the four leaves of a star $K_{1,4}$, where $L_1, L_2, L_3, L_4$ are disjoint and ordered from left to right. Then $C$ properly contains $L_2$ and $L_3$. Since each of $L_2$ and $L_3$ also properly contains at least two disjoint intervals in $\mathcal{I}$, $C$ intersects at least six disjoint intervals in $\mathcal{I}$, contradicting our assumption that $G$ has claw number at most 5.

The proof of Theorem 3 is now complete.

6 Approximation algorithm

In this section we prove Theorem 4.

Albertson et al. [3, Theorem 4] showed that $\kappa(w, 1) \leq w$ for any $w \geq 1$, which implies that $\kappa(v+2, v) = \kappa(3, 1) \leq 3$ when $v = 1$. When $v = 2$, by our upper bound of $\kappa(w, v) \leq \left\lceil \log_{v+1} 3(w-1)/2 \right\rceil + 2$ for $w > v = 2$ in Corollary 2, we have $\kappa(v+2, v) = \kappa(4, 2) \leq \lceil \log_3 9/2 \rceil + 2 = 3$. When $v = 3$, we have $\kappa(v+2, v) = \kappa(5, 3) \leq 2$ by Lemma 9. When $v > 3$, we have $\kappa(v+2, v) \leq \lceil \log_{v+1} (v+1)/(v-2) \rceil + 2 = 2$ by Corollary 2. In summary, we have $\kappa(v+2, v) \leq 3$ when $1 \leq v \leq 2$, and $\kappa(v+2, v) \leq 2$ when $v \geq 3$. Since the proof of Albertson et al. [3, Theorem 4] and our proofs of these upper bounds are constructive, we have the following proposition:

Proposition 1. For any $v \geq 1$, there is a polynomial-time algorithm that partitions any interval graph with claw number at most $v+2$ into $t$ induced subgraphs with claw number at most $v$, with $t = 3$ for $1 \leq v \leq 2$, and $t = 2$ for $v \geq 3$.

Our approximation algorithm for MIN-PARTITION($v$) works as follows. Given an interval graph $G$, first obtain an interval representation $\mathcal{I}$ of $G$. Initialize $\mathcal{J} \leftarrow \mathcal{I}$ and $i \leftarrow 1$. While $\mathcal{J}$ is not empty, let $\mathcal{J}'$ be the subfamily of intervals in $\mathcal{J}$ each properly containing at least $v+1$ disjoint intervals in $\mathcal{J}$, let $\mathcal{I}_i \leftarrow \mathcal{J} \setminus \mathcal{J}'$, then update $\mathcal{J} \leftarrow \mathcal{J}'$ and $i \leftarrow i+1$.

Let $k$ be the maximum round $i$ in which $\mathcal{J}$ is not empty. Then $(\mathcal{I}_1, \ldots, \mathcal{I}_k)$ is a $k$-partition of $\mathcal{I}$. For $1 \leq i \leq k$, since each interval in $\mathcal{I}_i$ properly contains at most $v$ disjoint intervals in $\mathcal{I}_i$, the subgraph of $G$ induced by each subfamily $\mathcal{I}_i$ has claw number at most $v+2$. Apply the algorithm in Proposition 1 to partition the subgraph of $G$ induced by each subfamily $\mathcal{I}_i$ into $t$ subgraphs, with $t = 3$ for $1 \leq v \leq 2$, and $t = 2$ for $v \geq 3$. Then $G$ is partitioned into $kt$ subgraphs with claw number at most $v$.

Observe that for $2 \leq i \leq k$, each interval in $\mathcal{I}_i$ properly contains at least $v+1$ disjoint intervals in $\mathcal{I}_{i-1}$. Thus $G$ contains $A_{k,v}$ as an induced subgraph, and it follows by Lemma 2 that $G$ admits no vertex partition into less than $k$ induced subgraphs with claw number at most $v$. Thus our algorithm for MIN-PARTITION($v$) achieves an approximation ratio of $t$, with $t = 3$ for $1 \leq v \leq 2$, and $t = 2$ for $v \geq 3$. This completes the proof of Theorem 4.

7 Open questions

For $k \geq 2$ and $v \geq 2$, is there a polynomial-time algorithm for $k$-PARTITION($v$) in interval graphs? In particular, for $k = v = 2$, is there a polynomial-time algorithm that decides whether an interval graph...
admits a vertex partition into two proper interval graphs? Can the gaps between lower and upper bounds in Theorem 2 and Corollary 2 be reduced?

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A counterexample  Gardi reported [9, Lemma 2.3] that for $t > 1$, any $K_{1,t}$-free interval graph admits a vertex partition into $\lceil \log_3 3t/2 \rceil$ proper interval graphs. By substituting $t$ with $w + 1$, this would imply that $\tilde{\kappa}(w, 2) \leq \lceil \log_3 3w/2 \rceil$ for $w \geq 1$, in particular, $\tilde{\kappa}(w, 2) \leq 2$ for $w \leq 6$. But this cannot hold, since we proved in Theorem 3 that $\tilde{\kappa}(4, 2) = \tilde{\kappa}(5, 2) = \tilde{\kappa}(6, 2) = 3$; see our Lemmas 5, 6, and 7. We refer to Figure 7 for a counterexample that identifies a flaw in the proof of [9, Lemma 2.3]. This flaw also affects several other results, including an upper bound on $\kappa(n, 2)$ [9, Proposition 2.7].

Figure 7: A counterexample to the correctness of the algorithm COLOR-CLIQUESS-LOGARITHMIC in Gardi’s constructive proof of [9, Lemma 2.3] with $t = 6$. The nine intervals are first partitioned into seven cliques $C_0 = \{1, 2\}$, $C_1 = \{3\}$, $C_2 = \{4\}$, $C_3 = \{5, 6\}$ $C_4 = \{7\}$, $C_5 = \{8\}$, $C_6 = \{9\}$ by the algorithm CANONICAL-PARTITION-INTO-CLIQUESS, and then grouped into two parts $C_0 = \{3, 4, 7, 8\}$ and $C_1 = \{1, 2, 5, 6, 9\}$ by the algorithm COLOR-CLIQUESS-LOGARITHMIC. The subgraph represented by $C_1$ contains an induced star $K_{1,3}$ with center 5 and three leaves 1, 6, 9, and hence is not a proper interval graph as claimed.
Source code for Lemma 7

#include <stdio.h>
#define N 105
#define K 512

int ll[N], rr[N], xx[N][K], yy[N][K], zz[N][K], kk[N];
char part[N];

int min(int a, int b) { return a < b ? a : b; }
int max(int a, int b) { return a > b ? a : b; }

int a(int i, int j) { return max(ll[i], ll[j]) < min(rr[i], rr[j]); }

int claw(int i, int j, int k, int l) {
    return a(i, j) && a(i, k) && a(i, l) && !a(j, k) && !a(j, l) && !a(k, l);
}

int bad(int i, int j, int k, int l) {
    return claw(i, j, k, l) || claw(j, k, l, i) || claw(k, l, i, j) || claw(l, i, j, k);
}

int good(int i) {
    int h, p = part[i];

    for (h = 0; h < kk[i]; h++)
        if (part[xx[i][h]] == p && part[yy[i][h]] == p && part[zz[i][h]] == p)
            return 0;
    return 1;
}

int solve(int i) {
    if (i == N) {
        printf("good\n");
        return 1;
    }
    for (part[i] = 0; part[i] <= 1; part[i]++)
        if (good(i) && solve(i + 1))
            return 1;
    return 0;
}

int main() {
    int n, l, r, i;

    n = 0;
    for (r = 2; r <= 21; r++)
        for (l = r - 2; l >= 0 && l >= r - 7; l--)
            ll[n] = l, rr[n] = r, n++;
    printf("n = %d\n", n);
    for (i = 0; i < n; i++) {
        int x, y, z, k = 0;

        for (x = 0; x < i; x++)
            for (y = x + 1; y < i; y++)
                for (z = y + 1; z < i; z++)
if (bad(i, x, y, z)) {
    if (k == K) {
        printf("K is too small\n");
        return 0;
    }
    xx[i][k] = x, yy[i][k] = y, zz[i][k] = z, k++;
}
    kk[i] = k;
}

if (!solve(1))
    printf("no good\n");
return 0;
}