Inference Under Convex Cone Alternatives for Correlated Data

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In this research, inferential theory for hypothesis testing under general convex cone alternatives for correlated data is developed. While there exists extensive theory for hypothesis testing under smooth cone alternatives with independent observations, extension to correlated data under general convex cone alternatives remains an open problem. This long-pending problem is addressed by (1) establishing that a generalized quasi-score statistic is asymptotically equivalent to the squared length of the projection of the standard Gaussian vector onto the convex cone and (2) showing that the asymptotic null distribution of the test statistic is a weighted chi-squared distribution, where the weights are mixed volumes of the convex cone and its polar cone. Explicit expressions for these weights are derived using the volume-of-tube formula around a convex manifold in the unit sphere. Furthermore, an asymptotic lower bound is constructed for the power of the generalized quasi-score test under a sequence of local alternatives in the convex cone. Applications to testing under order restricted alternatives for correlated data are illustrated.

1. Introduction

Correlated or longitudinal data arise in many areas of science when a response is measured at repeated instances on a set of subjects. It is assumed that the measurements on

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different subjects are independent, while those on individual subjects are correlated with an unknown correlation structure (Diggle et al., 1994). In this research, inferential theory is developed for hypothesis testing under general convex cone alternatives for correlated data using the formula for the volume of a tube around a manifold (curve, surface, etc.) on the surface of the unit sphere in an $r$-dimensional Euclidean space $\mathbb{R}^r$ (Hotelling, 1939; Weyl, 1939; Naiman, 1990). Testing for order restricted parameters in correlated data and testing for a monotone regression become special cases of this general problem. Often, interest lies in detecting an order among treatment effects, while simultaneously modeling relationships with regression parameters. There exists extensive theory for hypothesis testing under ordered alternatives with independent observations (Barlow et al., 1972; Robertson et al., 1988; Silvapulla and Sen, 2004), including smooth cone alternatives (Takemura and Kuriki, 1997). However, extension of the theory to correlated data remains an open problem.

1.1. Formulation of the testing problem

Let $Y_{ij}$ be the response measured at the $j$th ($j = 1, \ldots, n_i$) time point on the $i$th ($i = 1, \ldots, N$) subject. Let $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{in_i})^T$ be an $n_i$-dimensional vector of response variables. The mean of $Y_{ij}$ is related to the $r$-dimensional vector of covariates $\mathbf{X}_{ij}$ corresponding to the $r$-dimensional parameter vector $\gamma$ via a generalized linear model

$$
\mathbb{E}(Y_{ij}) := h(\mathbf{X}_{ij}^T \gamma),
$$

where $h(\cdot)$ is the inverse of a link function. We assume that the true distribution is unique and all expectations are taken with respect to the true probability measure $P$.

The goal is to test the general hypothesis

$$
\mathcal{H}_0: \gamma \in \mathcal{V} \quad \text{against} \quad \mathcal{H}_1: \gamma \in \mathcal{V} \oplus \mathcal{C}, \gamma \notin \mathcal{V},
$$

where $\mathcal{V}$ is an arbitrary finite dimensional vector space of $\mathbb{R}^r$, $r := \dim(\mathcal{V} \oplus \mathcal{C})$, $\mathcal{C}$ is a closed convex cone with a non-empty interior in $\mathbb{R}^r$ and $\oplus$ denotes the direct or Kronecker
sum. Without loss of generality, it is assumed that $\mathcal{C} \subset \mathcal{V}^\perp$, the orthogonal complement of $\mathcal{V}$. Under $\mathcal{H}_1$, $\dim(\gamma) = r$; whereas under $\mathcal{H}_0$, $\dim(\mathcal{V}) < r$ due to certain constraints imposed on the parameters in $\mathcal{V}$.

Seminal work of Takemura and Kuriki (1997) has established a solution for the problem of testing a simple null hypothesis regarding the multivariate Gaussian mean vector $\lambda$ against an arbitrary convex cone alternative for independent observations. In particular, they derived the asymptotic null distribution of the likelihood ratio test statistic (LRT) for testing

$$\mathcal{H}_0^1: \lambda = 0 \quad \text{against} \quad \mathcal{H}_1^1: \lambda \in \mathcal{K},$$

where $\mathcal{K}$ is a closed convex cone of dimension $d$ with a nonempty interior in $\mathbb{R}^r$ ($r \geq d$), using the techniques of convex analysis.

1.2. Main results and organization of the article

The goals of this research include the following.

1. In Section 2, we derive large-sample properties of the quadratic inference functions, extensions of the generalized method of moments (Hansen, 1982), that are required for the development of inferential theory with correlated data.

2. We derive a “generalized quasi-score” (GQS) statistic for the testing problem (2) for correlated data in Section 3. Furthermore, it is established that the asymptotic null distribution of the GQS statistic, to appropriate statistical order, is equivalent to finding the limiting distribution of the squared length of projection of a standard Gaussian vector onto the convex cone $\mathcal{K}$ (see Theorem 4).

3. In Section 3, we also establish that the asymptotic null distribution of the GQS statistic is a weighted chi-squared distribution, where the weights are mixed volumes of $\mathcal{K}$ (Takemura and Kuriki, 1997) and its polar cone (see Theorem 5). Deriving computable expressions for the weights in the asymptotic null distribution of the test statistic is a tedious and difficult process even for independent data
In this section, we present the large-sample properties of the inference functions that are required for later theoretical development of our general testing problem. See Pilla and Loader (2005a) for technical details and derivations of the results presented in this section.

Hansen (1982) proposed the generalized method of moments (GMMs) for estimating the vector of regression parameters $\beta \in B$ from a set of score functions, where the dimension of the score function exceeds that of $\beta$. He established that, under certain regularity conditions, the GMM estimator is consistent, asymptotically Gaussian, and asymptotically efficient. Qu et al. (2000) extended the GMMs to create a clever approach called the “quadratic inference function” (QIF) that implicitly estimates the underlying correlation...
structure for the analysis of longitudinal data. Their main idea was to assume that the inverse of the working correlation matrix, denoted by $R^{-1}(\alpha)$, is a linear combination of several pre-specified basis matrices. That is, $R^{-1}(\alpha) = \sum_{l=1}^{s} \alpha_l M_l$, where $\alpha_1, \ldots, \alpha_s$ are unknown constants, $M_1$ is the identity matrix of an appropriate dimension and $M_l$ ($l = 2, \ldots, s$) are pre-specified symmetric matrices with elements taking either 0 or 1 for the commonly employed working correlation structures such as exchangeable, AR-1 etc.

2.1. Properties of extended score functions

For mathematical exposition, we assume that each subject is observed at a common set of times $j = 1, \ldots, n$. Let $h_i = [h(X_{i1}^T \gamma), \ldots, h(X_{in}^T \gamma)]^T$, where $h(X_{ij}^T \gamma)$ is the inverse of a link function and the operator $\nabla$ denotes partial derivative with respect to the elements of $\gamma$; therefore, $\nabla h_i$ is the $(n \times r)$ matrix $(\partial h_i/\partial \gamma_1, \ldots, \partial h_i/\partial \gamma_r)$ for each $i = 1, \ldots, N$.

The coefficients $\alpha_1, \ldots, \alpha_s$ in $R^{-1}(\alpha)$ are treated as nuisance parameters to create the set of subject-specific basic score functions as

$$g_i(\gamma) := \begin{bmatrix} \nabla h_i^T A_i^{-1/2} M_1 A_i^{-1/2} (Y_i - h_i) \\ \vdots \\ \nabla h_i^T A_i^{-1/2} M_s A_i^{-1/2} (Y_i - h_i) \end{bmatrix} \text{ for } i = 1, \ldots, N,$$

where $A_i$ is the diagonal matrix of marginal covariance of $Y_i$ for the $i$th subject.

Define the vector of extended score functions for all subjects as $\bar{g}_N(\gamma) := N^{-1} \sum_{i=1}^{N} g_i(\gamma)$. Note that $\dim[\bar{g}_N(\gamma)] = rs > r = \dim(\gamma)$. The extended score vector $\bar{g}_N(\gamma)$ satisfies the mean zero assumption $E_\gamma[\bar{g}_N(\gamma)] = 0$, where the expectation operator is with respect to the true but unknown distribution of the response matrix $Y$.

These estimating equations can be combined optimally using the GMM (Hansen, 1982).

Let $\Sigma_{\gamma_0}(\gamma)$ be the true covariance matrix of $g_1(\gamma)$, an $s$-dimensional vector of extended score functions defined in (4). We require the following design assumptions for deriving the asymptotic theory.
A1: The pairs \((Y_i, X_i^T)\) for \(i = 1, \ldots, N\), where \(X_i = (X_{i1}, \ldots, X_{in})\) are \((r \times n)\)-dimensional matrices, are an independent sample from an \([n \times (r + 1)]\)-dimensional distribution \(F\).

A2: The number of measurements \(n_i\) on the \(i\)th subject is fixed at \(n_i = n\) for all \(i = 1, \ldots, N\).

A3: The \((rs \times rs)\)-dimensional covariance matrix \(\Sigma_{\gamma_0}(\gamma_0) := \mathbb{E}_{\gamma_0}[g_1(\gamma_0) g_1^T(\gamma_0)]\) is strictly positive definite.

The independence part of the assumption A1 is between different subjects (i.e., with respect to the index \(i\)). The elements of \(X_i\) need not be independent of each other; hence, this assumption incorporates both time-dependent as well as time-independent covariates. Moreover, there exists a dependence of \(Y_i\) on \(X_i\) through the link function given in (1).

All throughout this article, \(\mathbb{E}_{\gamma_0}\) denotes an expectation operator with respect to the true parameter vector \(\gamma_0\) and all expectations are assumed to be finite. Let \(\hat{C}_N(\gamma)\) be the estimator of the second moment matrix of \(g_1(\gamma)\) so that \(\hat{C}_N(\gamma) := \frac{1}{N} \sum_{i=1}^{N} g_i(\gamma) g_i^T(\gamma)\). If the mean zero assumption for \(g_N(\gamma)\) holds, then \(N^{-1} \hat{C}_N(\gamma)\) estimates the covariance of \(g_N(\gamma)\).

2.2. Fundamental results for the quadratic inference functions

The quadratic inference function (QIF) is defined as

\[
Q_N(\gamma) := N \overline{g}_N(\gamma) \hat{C}_N^{-1}(\gamma) \overline{g}_N(\gamma).
\] (5)

If rank of \(\hat{C}_N(\gamma)\) is less than \(rs\) or is singular, then the inverse does not exist. However, any vector in the null space of \(\hat{C}_N(\gamma)\) must be orthogonal to each of the subject-specific score functions \(g_i(\gamma)\) \((i = 1, \ldots, N)\) and consequently to \(g_N(\gamma)\). Therefore, one can replace \(\hat{C}_N^{-1}(\gamma)\) by any generalized inverse such as the Moore-Penrose generalized inverse.

Our covariance estimator \(\hat{C}_N(\gamma)\) differs from that of Qu et al. (2000), who define a covariance \(C_N\) with a factor of \(N^{-2}\), and correspondingly omit the factor of \(N\) from the
QIF in (5). There are some technical flaws in their results and hence we proceed carefully without relying on those asymptotic results. Pilla and Loader (2005a) established precise large-sample results for the QIF along with rigorous proofs. The QIF in (5) is minimized under unrestricted and restricted spaces to yield the estimators \( \hat{\gamma} = \arg \min_{\gamma \in \mathbb{R}^r} Q_N(\gamma), \tilde{\gamma} = \arg \min_{\gamma \in V \oplus \mathcal{C}} Q_N(\gamma) \) and \( \gamma = \arg \min_{\gamma \in V} Q_N(\gamma) \), respectively. These estimators can be found using the iterative reweighted generalized least squares (IRGLS) algorithm developed by Loader and Pilla (2006). The IRGLS algorithm avoids the complexity of computing the second derivative matrix of \( Q_N(\gamma) \) required for employing the Newton-Raphson algorithm recommended by Qu et al. (2000).

The proof of the next lemma essentially follows from p. 26 of Lee (1996) and hence is omitted.

**Lemma 1.** Under assumptions A1–A3, the QIF estimator \( \hat{\gamma} = \arg \min_{\gamma \in \mathbb{R}^r} Q_N(\gamma) \) exists uniquely and is strongly consistent. That is, \( \hat{\gamma} \overset{p}{\rightarrow} \gamma_0 \) as \( N \rightarrow \infty \).

We require the following regularity conditions for further theoretical development.

A4: The parameter space of \( \gamma \) denoted by \( \mathcal{G} \subset \mathbb{R}^r \) is compact.

A5: The parameter space of \( \gamma \) is identifiable: \( \mathbb{E}_{\gamma_0}[g_1(\gamma)] \neq 0 \) if \( \gamma \neq \gamma_0 \).

A6: The true covariance matrix \( \Sigma_{\gamma_0}(\gamma) \) is a continuous function of \( \gamma \).

A7: The expectation \( \mathbb{E}_{\gamma_0}[\mathcal{G}_N(\gamma)] \) exists, finite for all \( \gamma \in \mathcal{G} \) and continuous in \( \gamma \).

A8: The subject-specific score functions \( g_i(\gamma) \) \((i = 1, \ldots, N)\) have uniformly continuous second-order partial derivatives with respect to the elements of \( \gamma \).

A9: The first-order partial derivatives of \( \mathcal{G}_N(\gamma) \) and \( \mathcal{C}_N(\gamma) \) have finite means and variances.

The importance of assumption A4 is that it enables us to invoke Theorem 1 of Rubin (1956); therefore, convergence statements in this article are uniform for \( \gamma \) in bounded sets.
Lemma 2. Under assumptions A1–A6, $\hat{C}_N(\gamma) \xrightarrow{p} \Sigma_{\gamma_0}(\gamma_0)$ as $N \to \infty$.

Proof. Under assumptions A1–A3 and by the strong law of large numbers, $\hat{C}_N(\gamma)$ converges to its expected value, a non-degenerate limit, for a fixed $\gamma$. That is,

$\hat{C}_N(\gamma) \xrightarrow{p} E[g_1(\gamma) g_1^T(\gamma)]$ as $N \to \infty$.

Under the stated regularity conditions and Theorem 1 of Rubin (1956), uniform convergence holds if the compactness assumption A4 holds. Hence, the claim (6) holds uniformly in $\gamma$. Lemma 1 combined with the continuity of the function $\hat{C}_N(\gamma)$ ensures that $\hat{C}_N(\hat{\gamma}) \xrightarrow{p} E[g_1(\gamma_0) g_1^T(\gamma_0)]$ as $N \to \infty$. The claim then follows from the definition of $\Sigma_{\gamma_0}(\gamma_0)$.

Let $D(\gamma) := E[\nabla g_1(\gamma)]$, where $\nabla g_1(\gamma) = \partial g_1(\gamma)/\partial \gamma$. From the strong law of large numbers, $\nabla g_N(\gamma) \xrightarrow{p} D(\gamma)$ as $N \to \infty$. This relation combined with Lemma 2, enable us to obtain the asymptotic covariance matrix of $\hat{\gamma}$. When there is no ambiguity, we drop the subscript $\gamma_0$ and write $\Sigma^{-1}(\gamma_0)$ for the true covariance matrix of $g_1(\gamma)$ evaluated at $\gamma_0$.

Let the estimated covariance of $\hat{\gamma}$ be defined as

$\hat{\text{cov}}(\hat{\gamma}) := \frac{1}{N} \left[ \nabla g_N(\hat{\gamma}) \hat{C}_N^{-1}(\hat{\gamma}) \nabla g_N(\hat{\gamma}) \right]^{-1}$ as $N \to \infty$.

Lemma 3. Under assumptions A1–A9, $N \hat{\text{cov}}(\hat{\gamma}) \xrightarrow{p} \left[ D^T(\gamma_0) \Sigma^{-1}(\gamma_0) D(\gamma_0) \right]^{-1} =: J^{-1}(\gamma_0)$ as $N \to \infty$.

The proof of the above lemma follows from the previous results. An immediate consequence of Theorems 3.1 and 3.2 of Hansen (1982) is the next result. The notation $\rightsquigarrow$ denotes convergence in distribution.

Theorem 1. Under assumptions A1–A9, $\sqrt{N} (\hat{\gamma} - \gamma_0) \rightsquigarrow N_r [0, J^{-1}(\gamma_0)]$ as $N \to \infty$. 
THEOREM 2. Under assumptions A1–A9, $(2\sqrt{N})^{-1} \nabla Q_N(\gamma_0) \rightarrow N_r[0, J(\gamma_0)]$ as $N \rightarrow \infty$.

2.3. Testing under order restricted alternatives for correlated data

In the context of correlated data, comparing several treatments, groups or populations with respect to their means, medians or location parameters often arise in many areas of scientific applications. For instance, one assumes that certain treatments are not worse than another.

The problem of testing under order restricted or constrained hypothesis in longitudinal data becomes a special case of (2). Let $Y_{ijt}$ be the measurement taken at the $j$th ($j = 1, \ldots, n_{it}$) time point on the $i$th ($i = 1, \ldots, n_t$) subject in the $t$th ($t = 1, \ldots, m$) treatment group. Let $N = \sum_t n_t$. For mathematical exposition, we assume that $n_{it} = n$ for all $(i, t)$ pairs. The mean of $Y_{ijt}$ is related to the $p$-dimensional vector of covariates $X_{ijt}$, corresponding to the $p$-dimensional parameter vector $\beta_t$ for the $t$th group, via $E(Y_{ijt}) := h(X_{ijt}^T\beta_t + \mu_t)$, where $h(\cdot)$ is the inverse of a link function, $\mu_t$ is the treatment effect for the $t$th group. Hence, $\gamma = (\mu^T, \beta^T)^T$ with $\mu = (\mu_1, \ldots, \mu_m)^T$ and $\beta = (\beta_1^T, \ldots, \beta_m^T)^T$.

The order restricted hypothesis testing problem that is of interest can be formulated as $H_0^\circ : \mu \in \mathcal{V}_0$ against $H_0^\circ : \mu \in \mathcal{C}_0, \mu \notin \mathcal{V}_0$, where $\mathcal{V}_0 = \{\mu : \mu_1 = \cdots = \mu_m\}$ and $\mathcal{C}_0 = \{\mu : \mu_1 \geq \cdots \geq \mu_m\}$ is a particular convex cone. It is clear that $r = m(p + 1), d = (m - 1)$ and $\mathcal{V}_0$ is the origin of the convex cone $\mathcal{C}_0$; hence $\mathcal{V}_0 \subset \mathcal{C}_0$. This testing problem is treated in considerable detail by Pilla et al. (2006).

3. Hypothesis Testing Under Convex Cone Alternatives for Correlated Data

In this section, we first derive a statistic for the general testing problem (2) using the decomposition of $\gamma \in \mathbb{R}^r$ and next define a new co-ordinate system to transform the null
space \( \mathcal{V} \). Lastly, we derive the asymptotic distribution of the our test statistic under the model hypothesis.

3.1. Generalized quasi-score statistic for correlated data and canonical formulation of the testing problem

Define the \textit{generalized quasi-score} (GQS) statistic as

\[
S_N := Q_N(\overline{\gamma}) - Q_N(\tilde{\gamma})
\]

for testing the hypothesis (2), where \( \overline{\gamma} \) and \( \tilde{\gamma} \) are defined in Section 2.2.

It is more convenient to define a co-ordinate system to transform the null space \( \mathcal{V} \). If an appropriate transformation is found, we can reduce the general problem to a standardized form involving projections of independently and identically distributed standard Gaussian random variables as described in the next section.

Let \( P \) be a basis matrix for the space \( \mathcal{V}^\perp \) whose columns correspond to the constraints imposed by \( H_0 \). For example, if \( d \) constraints are imposed on \( \mathcal{G} \) under \( H_0 \), then \( P \) is an \((r \times d)\)-dimensional matrix. The choice of the matrix \( P \) is problem dependent as shown next.

\textbf{Lemma 4.} The hypothesis testing problem (2) can equivalently be represented in terms of the canonical space as testing for

\[
\mathcal{H}_0^2 : P^T \gamma = 0 \quad \text{against} \quad \mathcal{H}_1^2 : P^T \gamma \in \mathcal{C} := P^T \mathcal{C}.
\]

\textbf{Proposition 1.} Under assumptions A1–A8 and when \( \mathcal{H}_0^2 : P^T \gamma = 0 \) holds, 

\[
\sqrt{N} \left( P^T \tilde{\gamma} - P^T \gamma \right) \rightsquigarrow \mathbf{Z}^* \sim N_d(0, \Omega(\gamma_0)) \quad \text{as} \quad N \to \infty,
\]

where \( \Omega(\gamma_0) := P^T J^{-1}(\gamma_0) P \).

\textbf{Example 1.} (Order-restricted testing with three treatment groups). In the case of order-restricted testing with three groups, \( \gamma = (\mu_1, \mu_2, \mu_3, \beta_1, \beta_2, \beta_3)^T \) and \( \mathcal{V} \) consists of
vectors of the form $(\mu, \mu, \mu, \beta_1, \beta_2, \beta_3)^T$ which has dimension 4. A basis matrix for $V^\perp$ is

$$P^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$P^T \gamma = \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 - 2\mu_3 \end{pmatrix}. \tag{10}$$

Therefore, the rows of the matrix $P$ span the space $V^\perp$. Fig. 1 demonstrates that (i) if $P^T \hat{\gamma}$ lies in the interior of the convex cone $C_1$, then $P^T \hat{\gamma} = P^T \tilde{\gamma}$ and (ii) if $P^T \hat{\gamma}$ lies outside $C_1$, then $P^T \tilde{\gamma}$ is a projection of $P^T \hat{\gamma}$ onto $C_1$ (the orthogonal projection if $\Omega(\gamma_0) = I_d$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Projection of $P^T \hat{\gamma}$ onto the cone $C_1$ for $d = 2$, the number of constraints imposed under $H_0^2$.}
\end{figure}

Remark 1. Every $\gamma \in \mathcal{G} \subset \mathbb{R}^r$ admits a unique orthogonal decomposition of the form $\gamma = \gamma_1 + \gamma_2$ such that $\gamma_1 \in \mathcal{V}$ and $\gamma_2 \in \mathcal{V}^\perp$ and $\mathcal{C} \subset \mathcal{V}^\perp$. It is clear that $P^T \gamma = P^T \gamma_2$; therefore, $P^T \gamma \in P^T \mathcal{C}$ since $P^T \mathcal{V} = 0$ under $H_0^2$.

Let $P_K Z$ be the projection of $Z$ onto $K$ and let $\| \cdot \|$ denote the vector norm.
3.2. Asymptotic equivalence between \( S_N \) and \( \|P_K Z\|^2 \)

Owing to Lemma 4, the hypothesis testing problem (2) is equivalent to that of (9). It will be established that finding the limiting distribution of \( S_N \) for the model hypothesis, to appropriate statistical order, is equivalent to finding the limiting distribution of a length of a certain projection onto \( K \).

We present the two main theorems of this section. The first is based on the quadratic approximation to the inference function \( Q_N(\gamma) \) in an \( N^{-1/2} \)-neighborhood of \( \gamma_0 \) and the second is based on the transformed null space.

**Theorem 3.** Under assumptions A1–A8, \( Q_N(\gamma) - Q_N(\tilde{\gamma}) = N (\gamma - \tilde{\gamma})^T J(\gamma_0) (\gamma - \tilde{\gamma}) + o_P(1) \) as \( N \to \infty \).

**Proof.** Let \( \hat{\xi} := \sqrt{N} (\tilde{\gamma} - \gamma_0) \) so that \( \tilde{\gamma} = (\gamma_0 + N^{-1/2} \hat{\xi}) \). From the quadratic approximation of the QIF [Theorem 5 of Pilla and Loader (2005a)] the following result holds

\[
Q_N\left(\gamma_0 + N^{-1/2} \xi\right) = Q_N(\gamma_0) + 2 \langle \xi, W_N \rangle + \xi^T J(\gamma_0) \xi + o_P(1),
\]

where \( \xi \in \mathbb{R}^r \) is a fixed vector, \( \langle , \rangle \) is the inner product and \( W_N = (2\sqrt{N})^{-1} \nabla Q_N(\gamma_0) \). Equivalently,

\[
Q_N(\gamma) - Q_N(\gamma_0) = 2 \langle \xi, W_N \rangle + \xi^T J(\gamma_0) \xi + o_P(1).
\]

The minimizer \( \xi^* \) of the quadratic approximation in (11) is given by \( \xi^* = -J^{-1}(\gamma_0)W_N \).

From Theorem 2, \( W_N \) has a limiting distribution and hence it follows that \( \xi^* \) lies in the ball of radius \( r_N \) with probability converging to 1. This fact, combined with the uniformity of the error term in (11) yields the next result. If \( \hat{\xi} \) is the minimizer of the quadratic approximation in (11), then the QIF estimator becomes \( \hat{\gamma} = (\gamma_0 + N^{-1/2} \hat{\xi}) \).

Equivalently, \( \hat{\xi} = -J^{-1}(\gamma_0)W_N + o_P(1) \). Therefore, it follows that

\[
Q_N(\hat{\gamma}) - Q_N(\gamma_0) = 2 \langle \hat{\xi}, W_N \rangle + \hat{\xi}^T J(\gamma_0) \hat{\xi} + o_P(1).
\]
Since \( W_N = -J(\gamma_0) \tilde{\xi} + o_P(1) \), equations (12) and (13) simplify to \( Q_N(\gamma) - Q_N(\hat{\gamma}) = (\xi - \tilde{\xi})^T J(\gamma_0) (\xi - \tilde{\xi}) + o_P(1) = N (\gamma - \hat{\gamma})^T J(\gamma_0) (\gamma - \hat{\gamma}) + o_P(1) \).

In the next theorem, we establish the relation between the GQS statistic for correlated data and the squared length of projection of the standard Gaussian vector \( Z \) onto \( K \) for independent data. Consequently, we can derive a result for the asymptotic null distribution of \( S_N \) by employing the seminal work of Takemura and Kuriki (1997).

**Theorem 4.** Under assumptions A1–A9, the GQS statistic \( S_N \) for testing the hypothesis under a general convex cone alternative (2) is asymptotically equivalent to the statistic \( \| P_K Z \|^2 \) for testing the hypothesis (9), where \( Z \sim N_r(0, I_r) \). That is,

\[
S_N \approx \| P_K Z \|^2 \quad \text{as} \quad N \to \infty. 
\]

**Proof.** The QIF estimators obtained by minimizing \( Q_N(\gamma) \) under the spaces \( V, (V \oplus \mathcal{E}) \) and \( G \subset \mathbb{R}^r \) are ordered as \( Q_N(\gamma) \geq Q_N(\tilde{\gamma}) \geq Q_N(\hat{\gamma}) \). From Theorem 3, in an \( N^{-1/2} \)-neighborhood of the true parameter vector \( \gamma_0 \), \( Q_N(\gamma) - Q_N(\hat{\gamma}) = N (\gamma - \hat{\gamma})^T J(\gamma_0) (\gamma - \hat{\gamma}) + o_P(1) \). From Proposition 1, it suffices to consider the transformed hypothesis (9). There exists an \((r \times r)\)-dimensional matrix \( L \) such that \( L^T L = J(\gamma_0) \); for example, Cholesky factorization of \( J(\gamma_0) \).

It is more convenient to consider \( \theta \)-parametrization under the transformed null space \( L V =: V^* \), where \( V \) is the null space under the \( \gamma \)-parametrization. Let \( \theta = \sqrt{N} L \gamma \) so that \( \hat{\theta} = \sqrt{N} L \tilde{\gamma} \), \( \bar{\theta} = \sqrt{N} L \bar{\gamma} \) and \( \tilde{\theta} = \sqrt{N} L \tilde{\gamma} \). By Theorem 1,

\[
(\hat{\theta} - \theta_0) \sim Z \sim N_r(0, L_r) \quad \text{as} \quad N \to \infty. 
\]

Furthermore, Theorem 3 yields

\[
Q_N(\gamma) = Q_N(\hat{\gamma}) + \| \theta - \tilde{\theta} \|^2 + o_P(1). 
\]

A given \( \theta \in \mathbb{R}^r \) admits an orthogonal decomposition as \( \theta = \theta_1 + \theta_2 \) such that \( \theta_1 \in V^* \) and \( \theta_2 \in (V^*)^\perp \). However, orthogonality is not preserved by \( L \); hence \( (V^*)^\perp \neq L V^\perp \).
The hypothesis (9) can be re-expressed as

\[ \mathcal{H}_0^3: \theta_1 \in V^*, \theta_2 = 0 \quad \text{against} \quad \mathcal{H}_1^3: \theta_1 \in V^*, \theta_2 \in \mathcal{K} \quad \text{and} \quad \theta_2 \neq 0, \]

where \( \mathcal{K} = \{L(C \oplus V)\} \cap \{(V^*)^\perp\} \) is a \( d \)-dimensional cone in \( \mathbb{R}^r \) since \( \dim((V^*)^\perp) = d \).

Similarly, the estimator \( \hat{\theta} = \sqrt{N} L \tilde{\gamma} \) admits an orthogonal decomposition as \( \hat{\theta}_1 + \hat{\theta}_2 \) such that \( \hat{\theta}_1 \in V^* \) and \( \hat{\theta}_2 \in (V^*)^\perp \). By orthogonality, equation (16) can be expressed as

\[ Q_N \left[ N^{-1/2} L^{-1} (\theta_1 + \theta_2) \right] - Q_N(\tilde{\gamma}) = \|\theta_1 - \hat{\theta}_1\|^2 + \|\theta_2 - \hat{\theta}_2\|^2 + o_P(1). \]  

Under \( \mathcal{H}_0^3 \), (18) simplifies to

\[ Q_N \left[ N^{-1/2} L^{-1} (\theta_1 + \theta_2) \right] - Q_N(\tilde{\gamma}) = \|\theta_1 - \hat{\theta}_1\|^2 + \|\theta_2 - \hat{\theta}_2\|^2 + o_P(1). \]

Under \( \mathcal{H}_0^3 \), \( \theta_1 \in V^* \) and \( \theta_2 = 0 \), whereas under \( \mathcal{H}_1^3 \), \( \theta_1 \in V^* \) and \( \theta_2 \in \mathcal{K} \).

At first, consider minimizing over \( \mathcal{H}_0^3: \theta_2 = 0 \). The right-hand side of (19) is uniquely minimized at \( \theta_1 = \hat{\theta}_1 \) and \( \theta_2 = 0 \). By definition, the left hand side is minimized at \( \theta_1 = \bar{\theta}_1 \). By uniformity of the error term and uniqueness of the minimum, it follows that \( \bar{\theta}_1 = \hat{\theta}_1 + o_P(1) \). Under \( \mathcal{H}_1^3 \), the right-hand side of (18) is minimized at \( \theta_1 = \hat{\theta}_1 \) and \( \theta_2 = \mathcal{P}_\mathcal{K} \hat{\theta}_2 \).

**Fig. 2.** Projections of estimators of \( \theta \) onto the cone \( \mathcal{K} \). Asymptotically, the relation \( \tilde{\theta} = \mathcal{P}_\mathcal{K} \hat{\theta} + o_P(1) \) holds.

\( \theta_2 = \mathcal{P}_\mathcal{K} \hat{\theta}_2 \). Therefore, the left hand side of (18) is minimized at \( \bar{\theta}_1 = \hat{\theta}_1 + o_P(1) \) and \( \bar{\theta}_2 = \mathcal{P}_\mathcal{K} \hat{\theta}_2 + o_P(1) \). In effect, minimizing \( Q_N(\cdot) \) in (18) under \( \mathcal{H}_0^3 \) and \( \mathcal{H}_1^3 \) yield respectively,
\[ Q_N(\gamma) - Q_N(\bar{\gamma}) = \|\bar{\theta}_2\|^2 + o_P(1) \] and \[ Q_N(\gamma) - Q_N(\bar{\gamma}) = \|\bar{\theta}_2 - \bar{\hat{\theta}}_2\|^2 + o_P(1) \] since \[ \|\bar{\theta}_1 - \bar{\theta}_1\|^2 = o_P(1) \] and \[ \|\bar{\theta}_1 - \bar{\theta}_1\|^2 = o_P(1). \] Consequently, \[ S_N = [Q_N(\gamma) - Q_N(\bar{\gamma})] = \|\bar{\theta}_2\|^2 - \|\bar{\theta}_2 - \bar{\hat{\theta}}_2\|^2 + o_P(1) = \|\bar{\hat{\theta}}_2\|^2 + o_P(1) \] since \[ \left(\bar{\mathcal{P}}_\mathcal{K} \bar{\theta}_2 - \bar{\hat{\theta}}_2\right) \] and \[ \bar{\mathcal{P}}_\mathcal{K} \bar{\theta}_2 \] are orthogonal as shown in Fig. 2. Furthermore, since \[ \bar{\mathcal{P}}_\mathcal{K} \bar{\theta}_1 = 0 \] under \( \mathcal{H}_0^3 \) given that \( \hat{\theta} \in \mathcal{V}^r \) and \( \mathcal{K} \subset (\mathcal{V}^r)^\perp \), it follows that \[ S_N = \|\bar{\mathcal{P}}_\mathcal{K} \hat{\theta}\|^2 + o_P(1). \] Moreover, \[ \bar{\mathcal{P}}_\mathcal{K} \theta_0 = 0 \] yielding \[ \bar{\mathcal{P}}_\mathcal{K} \hat{\theta} = \bar{\mathcal{P}}_\mathcal{K} (\hat{\theta} - \theta_0). \] The result follows from (15).

3.3. Asymptotic null distribution of the generalized quasi-score test

We derive the asymptotic distribution of the GQS statistic \( S_N \) when \( \mathcal{H}_0^3 \) holds. Until now, we established that (1) the testing problems (2) and (9) are equivalent (Lemma 4, Section 3.1) and (2) there exists an asymptotic relation between the statistic \( S_N \) based on correlated data for the testing problem (9) and the statistic \( \|\mathcal{P}_\mathcal{K} \mathcal{Z}\|^2 \) based on independent data for the testing problem (3) (Theorem 4, Section 3.2). These main results, in conjunction with Theorem 2.1 of Takemura and Kuriki (1997), yield the asymptotic null distribution of \( S_N \) for the testing problem (9). The weights of this asymptotic null distribution are mixed volumes of \( \mathcal{K} \) and its polar or dual cone \( \mathcal{K}^0 \) (Webster, 1994).

Let \( \mathcal{S}^{(d-1)} \) be the closed \((d-1)\)-dimensional unit sphere in \( \mathbb{R}^r \), \( \mathcal{M} = \mathcal{K} \cap \mathcal{S}^{(d-1)} \) be the \((d-1)\)-dimensional convex manifold and \( \mathcal{M}^0 = \mathcal{K}^0 \cap \mathcal{S}^{(d-1)} \). Let \( \partial_{d-k,k}(\mathcal{M}, \mathcal{M}^0) \) be the mixed volumes of \( \mathcal{M} \) and \( \mathcal{M}^0 \) for \( k = 0, \ldots, d \).

Theorem 5. Under assumptions A1–A9 and when \( \mathcal{H}_0 : \gamma \in \mathcal{V} \) holds, the asymptotic distribution of the GQS statistic \( S_N \) for any \( c > 0 \) is

\[
\lim_{N \to \infty} \mathbb{P} (S_N \leq c) = \mathbb{P} \left( \|\mathcal{P}_\mathcal{K} \mathcal{Z}\|^2 \leq c \right)
\]

\[
= \sum_{k=0}^{d} \left( \begin{array}{c} d \\ k \end{array} \right) \frac{\partial_{d-k,k}(\mathcal{M}, \mathcal{M}^0)}{\omega_k \omega_{d-k}} \mathbb{P} \left( \chi^2_{d-k} \leq c \right),
\]

where \( d := \text{dim}(\mathcal{K}) \), \( \omega_{k-1} = 2 \pi^{k/2}/\Gamma(k/2) \) is the volume of the unit sphere \( \mathcal{S}^{(k-1)} \) embedded in \( \mathbb{R}^r \) and \( \chi^2_{k} \) is the chi-squared distribution with \( k \) degrees of freedom. The \( \chi^2_{k} \) for \( k = 0 \) is simply a point mass at the origin so that \( \mathbb{P}(\chi^2_{0} \leq c) = 1 \) for all \( c > 0 \).
The right-hand side of (20) is a weighted mean of several tail probabilities of chi-squared distributions; hence, it is often referred to as \(\text{chi-bar-squared}\) distribution and denoted by \(\chi^2\) (Shapiro, 1988). In general, it is very difficult to derive explicit expressions for the weights for the asymptotic null distribution in (20). In certain special cases of \(\mathcal{K}\), weights are known explicitly or can be evaluated numerically. For instance, for polyhedral cones (i.e., the cones defined by a finite number of linear constraints) one can calculate the weights. For the general case of non-polyhedral cones, Section 3.5 of Silvapulla and Sen (2004) provide a simulation-based approach.

Takemura and Kuriki (1997) clarify the geometric meaning of the weights when the boundary of the cone is smooth or piecewise smooth. However, as the following example demonstrates, in the case of order restricted testing problem, one does not have a smooth cone or a smooth manifold and hence a more general approach to calculate the weights is warranted which is derived in Section 4.

**Example 2.** (Asymptotic null distribution of \(S_N\) for three treatment groups). Consider the problem of order restricted testing with three treatment groups. Under \(H_0^0: \mu_1 = \mu_2 = \mu_3\), the asymptotic distribution in (20) has an explicit expression. The convex cone \(\mathcal{K}\) is the region between the two vectors defining the constraints in (10). Let \(\phi\) be the angle of the cone \(\mathcal{K} \subset \mathbb{R}^2\) at the vertex. Derivation of \(\phi\) will be presented in Example 4 in Section 4.1. Note that the angles (in radians) of \(\mathcal{K}\) and \(\mathcal{K}^0\) at their vertices sum to \(\pi\).

Divide the plane into the following four regions:

1. The cone \(\mathcal{K}\) such that \(Z \in \mathcal{K}\) with a probability of \(\phi/(2\pi)\) and conditional on \(Z \in \mathcal{K}\), it follows that \(\mathcal{P}_\mathcal{K}Z = Z\) with \(\|\mathcal{P}_\mathcal{K}Z\|^2 \sim \chi^2_2\).
2. The dual cone \(\mathcal{K}^0\) such that \(\mathcal{P}_\mathcal{K}Z = 0\) for all \(Z \in \mathcal{K}^0\) with a probability of \([1/2 - \phi/(2\pi)]\) and conditional on \(Z \in \mathcal{K}^0\), it follows that \(\|\mathcal{P}_\mathcal{K}Z\|^2 \sim \chi^2_0\).
3. The two regions \(\mathcal{K}^\dagger\) and \(\mathcal{K}^*\), where \(\mathcal{P}_\mathcal{K}Z\) is a multiple of one of the two vectors defining the constraints in (10); conditional on \(Z \in \mathcal{K}^\dagger\) or \(Z \in \mathcal{K}^*\) with a total probability of \(1/2\), it follows that \(\|\mathcal{P}_\mathcal{K}Z\|^2 \sim \chi^2_1\).
For $d = 2$, the asymptotic distribution in (20) simplifies to a $\chi^2$ distribution:

$$S_N \sim \|P_K Z\|^2 \sim \left(1 - \frac{\phi}{2 \pi}\right) \chi^2_0 + \frac{1}{2} \chi^2_1 + \frac{\phi^2}{4 \pi} \chi^2_2$$

as $N \to \infty$.

Remark 2. The right hand side expression of (21) also occurs in the context of the asymptotic null distribution of the LRT statistic for testing for two-component mixture model (Lindsay, 1995; Lin, 1997; Pilla and Loader, 2005b).

Example 3. (Asymptotic null distribution of $S_N$ under order restricted alternatives). We consider the order restricted testing problem discussed in Section 2.3. For convenience, we reorder the elements of the parameter vector as $\gamma = (\mu^T, \beta^T)^T$, with corresponding permutations of the rows and columns of $J(\gamma_0)$ [see Lemma 3 for the definition of $J(\gamma_0)$].

We partition $J(\gamma_0)$ as

$$J(\gamma_0) := \begin{pmatrix} J_{\mu\mu} & J_{\mu\beta} \\ J_{\beta\mu} & J_{\beta\beta} \end{pmatrix}.$$ 

Let $J_{\mu\mu}$ be the appropriate submatrix of $J^{-1}(\gamma_0)$. From the formula for an inverse of a partitioned matrix, it follows that $J_{\mu\mu} = \left(J_{\mu\mu} - J_{\mu\beta} J_{\beta\beta}^{-1} J_{\beta\mu}\right)^{-1}$. Since subjects in different groups are independent, the variance matrix $J_{\mu\mu}$ is diagonal. Let $Z^\dagger \sim N_m(0, Q)$, where $Q$ is a pre-specified diagonal variance matrix.

In the order restricted testing problem, the asymptotic null distribution has an explicit expression. Under $H_0^\circ$, for any $c > 0$, the result (20) reduces to $\lim_{N \to \infty} P(S_N \leq c) = \sum_{k=0}^d p(d-k+1, d; Q) P(\chi^2_{d-k} \leq c)$, where $Q = J_{\mu\mu}$ and $p(k, d; Q)$ is the level probability that the projection of $Z^\dagger$ onto $C_0$ with a weight vector $Q$ consists of exactly $k$ distinct points [Section 2.4, Robertson et al. (1988)]. The unknown weight vector is replaced with $\hat{Q}$. This problem is developed and treated in considerable detail by Pilla et al. (2006).

The weights $p(k, d; Q)$ are also referred to as $\chi^2$ weights (Robertson et al., 1988). Such weights appear in the null asymptotic or exact distribution of several test statistics when there are inequality constraints on parameters. In certain cases, exact expressions for
these weights are available and in other cases, one may obtain approximations or bounds (Silvapulla and Sen, 2004).

4. Asymptotic Null Distribution of $S_N$: The Volume-of-Tube Formula

In this section, we derive explicit expressions for the weights in the asymptotic null distribution of $S_N$ in (20) by representing $\mathcal{M}$ and $\mathcal{K}$ in parametric form and in turn using the Hotelling-Weyl-Naiman volume-of-tube formula.

4.1. Parametric representation of $\mathcal{M}$ and $\mathcal{K}$

From a geometrical perspective, the rows of the $(r \times d)$-dimensional matrix $P$ span the space $V^\perp$. From the orthogonal decomposition in Section 3, it follows that $\gamma = \gamma_1 + \gamma_2 = \gamma_1 + Pu$ such that $\gamma_1 \in V$ and for some $u \in \mathbb{R}^d$. Let $\mathcal{N} := \{u : Pu \in \mathcal{C}\}$. Given that $\gamma \in \mathcal{C} = \{Pu : u \in \mathcal{N}\}$, the hypothesis (2) can be expressed as

$$H_0^4 : u = 0 \quad \text{against} \quad H_1^4 : u \in \mathcal{N}.$$  \hspace{4pt} (22)

In order to represent $\mathcal{K}$ and $\mathcal{M}$ in a parametric form, we require the following result.

**Proposition 2.** The matrix $P^* := (L^{-1})^T P$ forms a basis for $(V^*)^\perp$, where $V^* = L V$.

**Proof.** Let $y \in V^*$ so that $y = Lz$ for some $z \in V$. Let $x \in C(P^*)$, the column space of the matrix $P^*$, so that $x \in (L^{-1})^T Pu$ for some vector $u = (u_1, \ldots, u_d)^T \in \mathbb{R}^d$. It follows that, $\langle x, y \rangle = (u^T P^T L^{-1})(Lz) = \langle Pu, z \rangle = 0$ since $z \in V$ and $Pu \in V^\perp$. Therefore, the space spanned by $P^*$ is a subspace of $(V^*)^\perp$. However, this subspace and $(V^*)^\perp$ have the same dimension $d$, therefore they must be equal. \hfill \blacksquare

We return to the orthogonal decomposition of $\theta = \theta_1 + \theta_2$ such that $\theta_1 \in V^*$ and $\theta_2 \in (V^*)^\perp$. Proposition 2 ensures the following representation: $\theta_2 = \mathcal{P}_{(V^*)^\perp} \theta = \mathcal{P}_{V^*} \theta = P^* \left[(P^*)^T P^*\right]^{-1} (P^*)^T \theta = P^* \left[P^T J(\gamma_0) P\right]^{-1} P^T L^{-1} \theta$ since $L^T L = J(\gamma_0)$. There-
fore,
\[
\theta_2 = \sqrt{N} P^* \left[ P^T J(\gamma_0) P \right]^{-1} P^T (\gamma_1 + Pu) = \sqrt{N} Hu \text{ since } P^T \gamma_1 = 0,
\]

where
\[
H := P^* \left[ P^T J(\gamma_0) P \right]^{-1} P^T P.
\]

Next, the \(d\)-dimensional cone \(K = \{ L(C \oplus V) \} \cap \{(V^*)^\perp\} \) and the \((d-1)\)-dimensional manifold \(M = K \cap S^{(d-1)}\) are represented in the parametric form by considering a vector function \(T(u) : N \subset \mathbb{R}^d \rightarrow M \subset \mathbb{R}^{(d-1)}\), where \(u \in N\). That is,
\[
T(u) := \{ Hu : u \in N \subset \mathbb{R}^d \text{ and } \|Hu\| = 1\},
\]

where \(H\) is an \((r \times d)\) matrix defined in (24). In effect, we can redefine \(M := \{T(u) \in S^{(d-1)} : u \in N \subset \mathbb{R}^d\} \) and \(K := \{Hu : u \in N \subset \mathbb{R}^d\}\).

**Example 4.** (Explicit Expressions for the convex cone \(N \subset \mathbb{R}^2\) and the angle \(\phi\) of \(K\)).

We return to the problem of testing under order restricted hypothesis for correlated data. If \(H_1^0 : \mu_1 > \mu_2 > \mu_3\), then the choice of \(P\) is given in Example 1. Therefore, \(\gamma_2 = Pu\) corresponds to \(\mu_1 = (u_0 + u_1 + u_2), \mu_2 = (u_0 - u_1 + u_2)\) and \(\mu_3 = (u_0 - 2u_2)\), where \(u_0 = (\mu_1 + \mu_2 + \mu_3)/3\). The constraints \(\mu_1 > \mu_2\) and \(\mu_2 > \mu_3\) under \(H_1^0\) yield respectively, \(u_1 > 0\) and \(u_2 > u_1/3\). Consequently, the convex cone \(N = \{u : u_1 > 0, u_2 > u_1/3\}\). It is clear that \(u\) lies in the cone bounded by the vectors \(v_1 = (1, 1/3)^T\) and \(v_2 = (0, 1)^T\). The cone \(K\) is then bounded by the \(\theta_2\)-component of \(LP v_1\) and \(LP v_2\). Hence
\[
\cos(\phi) = \frac{\langle \frac{Hv_1}{\|Hv_1\|}, \frac{Hv_2}{\|Hv_2\|} \rangle}{\|Hv_1\| \|Hv_2\|}
\]
yields an explicit expression for the angle \(\phi\) defined in Example 2.

4.2. Asymptotic null distribution of \(S_N\) in terms of the geometry of \(M\)

As a first step, we establish the connection between the distribution of a squared length of projection of \(Z\) onto \(K\) and the volume-of-tube problem. We take a different approach
from Lin and Lindsay (1997) in order to cast the problem in the general framework of this article.

The geodesic (or angular) distance between two points on any manifold is defined as the shortest measured distance between the points within the manifold itself. Let $\mathfrak{T}(\rho, \mathcal{M})$ or $\mathfrak{T}(\phi, \mathcal{M})$ be the spherical tube around the topological $(d-1)$-dimensional manifold $\mathcal{M}$ of Euclidean radius $\rho$ or geodesic radius $\phi$ embedded in $S^{(d-1)}$, where $\rho = \sqrt{2[1 - \cos(\phi)]}$, $u \in \mathcal{N}$ and $\dim(\mathcal{N}) = d$. Since $S^{(d-1)}$ is also a manifold and $\mathfrak{T} \subset S^{(d-1)}$, the geodesic distance between two points on $\mathfrak{T}$ is the length of the segment of the great circle (arc) connecting the two points. We view each ray $\{\zeta \eta : \zeta \geq 0\}$ as a cone on which to make a projection, yielding $\hat{Z}_\eta$ that depends on $\eta$. We redefine the cone as $\mathcal{K} := \{\zeta \eta : \zeta > 0, ||\eta|| = 1\}$ to yield $\mathcal{P}_\mathcal{K} Z = \arg \min_\zeta \langle \eta, Z \rangle^2 = \sup_\eta \langle \eta, Z \rangle_+$, where $\langle \cdot, \cdot \rangle_+$ denotes the positive part of the inner product. In effect, we have

$$\|\mathcal{P}_\mathcal{K} Z\|^2 = \left[\sup_{\eta \in \mathcal{K}} \langle \eta, Z \rangle_+ \right]^2 . \tag{26}$$

In order to reduce the problem to that of a uniform process, we condition on $||Z||^2$ and integrate over the conditional distribution. Consequently, from (26) we can express

$$\mathbb{P} \left( \|\mathcal{P}_\mathcal{K} Z\|^2 \leq c \right) = \mathbb{P} \left[ \left( \sup_{\eta \in \mathcal{K}} \langle \eta, Z \rangle_+ \right)^2 \leq \frac{c}{||Z||^2} \right]$$

$$= \int_c^\infty \mathbb{P} \left[ \left( \sup_{\eta \in \mathcal{K}} \langle \eta, U \rangle_+ \right)^2 \leq \frac{c}{z} \right] f_r(z) \, dz \tag{27}$$

where $U = (\frac{Z_1}{||Z||}, \ldots, \frac{Z_r}{||Z||})$ is uniformly distributed on $S^{r}$, an $r$-dimensional unit sphere embedded in $\mathbb{R}^{(r+1)}$, and $f_r(z)$ is a $\chi^2$ density with $r$ degrees of freedom. Therefore, the right-hand side of (20) can be determined from (27), provided the probability in the integrand can be found, at least approximately.

The uniformity property of $U$ reduces the problem of finding $\mathbb{P}(\sup_\eta \langle \eta, U \rangle_+ \leq \sqrt{c/z})$ to that of determining the volume of the tube $\mathfrak{T}(\rho, \mathcal{M})$ including the end points correc-
tions proposed by Naiman (1990). Consequently,

$$\mathbb{P}\left( \sup_{\eta \in \mathcal{K}} \langle \eta, U \rangle_+ \leq \sqrt{\frac{c}{z}} \right) = \mathbb{P}\left[ U \in \mathcal{T}(\phi, \mathcal{M}) \right] = \frac{\vartheta_{\mathcal{M}}(\phi)}{\omega_{r-1}},$$

where $\omega_{r-1} = 2 \pi^{r/2}/\Gamma(r/2)$ is the volume of $S^r$ and $\vartheta_{\mathcal{M}}(\phi)$ is the volume of $\mathcal{T}(\phi, \mathcal{M})$. Therefore, (27) and (28) establish a connection between $\|P_{\mathcal{K}Z}\|^2$ and volume of the tube $\mathcal{T}(\phi, \mathcal{M})$ around $\mathcal{M}$ embedded in $S^r$. Essentially, we established that the distribution of $\|P_{\mathcal{K}Z}\|^2$ can be determined explicitly by finding $\vartheta_{\mathcal{M}}(\phi)$ which equals $\cos^{-1}(\sqrt{c/z})$ for any $0 \leq \phi \leq \pi/2$.

The asymptotic expansion of the tail probability of the $\sup (\eta, Z)_+$ can also be obtained using the Euler-Poincaré characteristic $\mathcal{E}$ method, developed by Adler (1981) and Worsley (1995a,b), where the expectation of the $\mathcal{E}$ of an excursion set is evaluated. Takemura and Kuriki (2002) establish the equivalence between the tube and Euler characteristic methods under the assumption that $\mathcal{M}$ is a manifold with piecewise smooth boundary.

We motivate the geometric concepts through the order restricted alternatives for correlated data. We define a *corner* to mean a point where two faces of the boundary of $\mathcal{M}$ meet.

**Example 5.** (Geometry of $\mathcal{M}$ for the order restricted alternatives). We assume that $n_i = n$ ($i = 1, \ldots, N$) and the number of subjects in each group is equal so that we have a balanced design. In this case, $\omega_0 = 2$ and $\omega_1 = 2 \pi$. First consider three treatment groups (i.e., $m = 3$), then the number of restrictions $d$ equals two corresponding to $\mu_1 < \mu_2 < \mu_3$. Therefore, $\dim(\mathcal{K}) = 2$ and $\mathcal{M}$ is just an arc with two end points. Suppose $m = 4$ corresponding to three constraints, then $\mathcal{M}$ is a spherical triangle. The interior corresponds to $\mu_1 < \mu_2 < \mu_3 < \mu_4$ with three corners $\mu_1 = \mu_2 = \mu_3$, $\mu_2 = \mu_3 = \mu_4$, $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$ and three edges $\mu_1 = \mu_2$, $\mu_2 = \mu_3$, $\mu_3 = \mu_4$.

Example 5 demonstrates that determining $\vartheta_{\mathcal{M}}(\phi)$ for $d \geq 3$ depends on the geometry of $\mathcal{M}$. Naiman (1990) derived expressions for the volume of a tube by decomposing the
tube into different sections, corresponding to the main part of the manifold, hemispherical caps along boundaries of the manifold, circular wedges at the boundaries and so on. Adding up these terms yields a series involving partial beta functions [Lemma 3.6 of Naiman (1990)]. Substituting these terms into (27) yields a series involving partial gamma functions; the first four terms of which are given in the next theorem whose proof essentially follows from Pilla and Loader (2005b) and hence is omitted.

**Theorem 6.** Under assumptions A1–A9 and when $H_0: \gamma \in \mathcal{V}$ holds, the asymptotic distribution of $S_N$ for a $d$-dimensional manifold $\mathcal{M}$ and for any $c > 0$ is given by

$$
\lim_{N \to \infty} P(S_N \geq c) = P(\|\mathcal{P}_K Z\|^2 \geq c) \\
= \frac{\kappa_0}{\omega_{d-1}} P(\chi^2_{d-1} \geq c) + \frac{\ell_0}{2 \omega_{d-2}} P(\chi^2_{d-1} \geq c) + \frac{\kappa_2 + \ell_1 + v_a}{2 \pi \omega_{d-3}} P(\chi^2_{d-2} \geq c) \\
+ \frac{(\ell_2 + v_1 + \tau)}{4 \pi \omega_{d-4}} P(\chi^2_{d-3} \geq c) + o \left(e^{(d-5)/2} e^{-c/2}\right) \quad \text{as} \quad c \to \infty,
$$

(29)

where $\kappa_0$ is the $(d-1)$-dimensional volume of the manifold $\mathcal{M}$, $\kappa_2$ is the measure of curvature of $\mathcal{M}$, $\ell_0$ is the $(d-2)$-dimensional volume of the boundaries of $\mathcal{M}$, $\ell_1$ is the measure of rotation of the boundary, $\ell_2$ is the measure of curvature similar to $\kappa_2$, $v_a$ is the measure of rotation angles at points (or along edges) where two boundary faces meet, $v_1$ is the combination of these rotation angles with the rotation of the edges, and $\tau$ is the measure of the size of wedges at corners where three boundary faces of $\mathcal{M}$ meet.

If the manifold $\mathcal{M}$ is a single point (i.e., $d = 1$), the result (29) simplifies to $0.5 P(\chi^2_{1} \geq c)$. If $\mathcal{M}$ is one-dimensional (i.e., $d = 2$), the result reduces to $(\kappa_0/2 \pi) P(\chi^2_{2} \geq c) + (\ell_0/4) P(\chi^2_{1} \geq c)$, where $\kappa_0$ is the length of $\mathcal{M}$ and $\ell_0$ is the number of boundary caps which equals 2. This last result is same as that obtained by Lin and Lindsay (1997); however, we provide an explicit formula for $\kappa_0$ based on the parametric representation of $\mathcal{M}$ which is derived in the next section.

**Remark 3.** For convex manifold $\mathcal{M}$, $\kappa_2 = 0$ in the asymptotic expansion (29). In the case of order restricted alternatives, the manifold $\mathcal{M}$ is a high-dimensional tetrahedron.
whose corners correspond to the constraints imposed on \( \gamma \) under \( \mathcal{H}_1 \); hence \( v_0 \neq 0 \). Also, \( \ell_2 = 0 \) in the case of order restricted testing problem.

Owing to Theorem 6, the weights in Theorem 2.1 of Takemura and Kuriki (1997) for the independent data case can be determined using the volume-of-tube formula. When the critical geodesic radius of the manifold (Naiman, 1990) is greater than or equal to \( \pi/2 \), all of the coefficients in (29) are nonnegative leading to a finite mixture of chi-squared distribution or \( \chi^2 \) distribution. The critical radius is greater than equal to \( \pi/2 \) if and only if the smallest cone containing the manifold is convex. Lemma 2.1 of Takemura and Kuriki (2002) provides a formula for computing the critical radius.

4.3. Explicit expressions for the geometric constants

We derive explicit expressions (in suitable forms for computation) for the geometric constants in (29) using the representation of \( T(u) \), defined in (25), and its derivatives. Our main goal is to reduce the evaluation of the constants to integrals over appropriate parts of \( \mathcal{M} \).

The profound result of Gauss-Bonnet theorem (Do Carmo, 1976; Milman and Parker, 1977) connecting curvatures of manifolds with the Euler-Poincaré characteristic (Worsley, 1995a,b; Adler and Taylor, 2004) can be employed to find some of the geometric constants appearing in Theorem 6. When \( \mathcal{M} \) is two-dimensional (i.e., \( d = 3 \)), the number of pieces contributing to \( \mathcal{M} \) minus the number of holes equals \( E \). In particular, \( \kappa_2 + \ell_1 + v_0 = 2 \pi E - \kappa_0 \), which eliminates the need to compute \( \kappa_2, \ell_1 \) and \( v_0 \) directly.

Remark 4. Lin and Lindsay (1997) assume that the cone is convex and smooth; hence no corners (i.e., \( v_0 = 0 \)). Both their Theorem 3.1 and the result for \( d = 3 \) presented in Section 4 of Lin and Lindsay (1997) become special cases of our general result established in Theorem 6.

In the parametric representation of \( \mathcal{M} \), the function \( T(u) \) has an embedded constraint \( ||H u|| = 1 \) for a \( d \)-dimensional vector \( u \in \mathcal{N} \). This embedded constraint means
that the manifold $\mathcal{M}$ is of dimension $(d - 1)$. Hence, we can represent in terms of a $(d - 1)$-dimensional parameter vector $\rho$ with $u \equiv u(\rho)$ and express $T(\rho) = T[u(\rho)]$. For example, such a transformation can be carried out using the polar co-ordinates. Denote the domain of $\rho$ by $N^*$. We express $T(\rho) = [T^1(\rho_1, \ldots, \rho_{d-1}), \ldots, T^r(\rho_1, \ldots, \rho_{d-1})]$ for the parametric representation of $\mathcal{M}$.

**A10:** The transformation $T(\rho)$ is one-to-one and each $T_l (l = 1, \ldots, r)$ is twice continuously differentiable on $N^* \subset \mathbb{R}^{d-1}$.

The following expressions are derived under A10. Define an $[r \times (d - 1)]$ matrix $S(\rho) := [T^1(\rho) \cdots T_{d-1}(\rho)]$, where $T_k(\rho) = \partial T(\rho)/\partial \rho_k$ for $k = 1, \ldots, (d - 1)$ and $T_1(\rho), \ldots, T_{d-1}(\rho)$ are the column vectors.

The volume of the manifold $\mathcal{M}$ is expressed as

$$\kappa_0 = \int_{\rho \in N^*} \det [S^T(\rho) S(\rho)]^{1/2} d\rho.$$  

Finding $\ell_2$ is essentially similar to that of finding $\kappa_2$ by simply treating each face of the boundary as a new manifold. Therefore, we describe the method to find $\kappa_2$, although it is zero for convex manifolds. In the case of testing under order restricted alternatives, the constant $\ell_2 = 0$. The measure of curvature of $\mathcal{M}$ is expressed as

$$\kappa_2 = \int_{\rho \in N^*} \frac{1}{2} [\Upsilon(\rho) - (d - 1)(d - 2)] \det [S^T(\rho) S(\rho)]^{1/2} d\rho,$$

where $\Upsilon(\rho) = 2 \sum_{k=2}^{d-1} \sum_{l=1}^{k-1} (\varpi_{kl}^T \varpi_{kl} - \varpi_{lk}^T \varpi_{lk})$ with

$$\varpi_{lk}^T = e_l^T [S^T(\rho) S(\rho)]^{-1} \left[ \frac{\partial}{\partial \rho_k} S(\rho) \right] \left[ I_{d-1} - S(\rho) \left\{ S^T(\rho) S(\rho) \right\}^{-1} S^T(\rho) \right]$$

and $e_l$ as the basis vector for $\mathbb{R}^{d-1}$.

The volume of $\partial \mathcal{M}$, the boundary of $\mathcal{M}$, is $\ell_0$ while $\ell_1$ measures the curvature of $\partial \mathcal{M}$; both of these need to be determined for each face of the boundary. For instance, on the face where $\rho_{d-1} = 1$, define $S_{d-1}(\rho) := [T_1(\rho) \cdots T_{d-2}(\rho)]$ and $B_{d-1}(\rho) := \varphi \left[ I_{d-1} - S(\rho) \left\{ S^T(\rho) S(\rho) \right\}^{-1} S^T(\rho) \right] T_{d-1}(\rho)$, where $\varphi$ is a normalizing constant.
The geometric constants $\ell_0$ and $\ell_1$ can be determined via

$$\ell_0 = \int_{\rho \in \partial M} \det \left[ S_1^T(\rho) S_1(\rho) \right]^{1/2} d\rho$$

and

$$\ell_1 = \int_{\rho \in \partial M} \ell_1(\rho) \det \left[ S_1^T(\rho) S_1(\rho) \right]^{1/2} d\rho,$$

where

$$\ell_1(\rho) = -\sum_{k=1}^{d-2} e_k^T \left[ S_1^T(\rho) S_1(\rho) \right]^{-1} \left[ \frac{\partial}{\partial \rho_k} S_1^T(\rho) \right] \mathbf{B}_{d-1}(\rho).$$

Similarly,

$$\upsilon_i = \int_{\rho \in \partial^2 M} \upsilon_i(\rho) \det \left[ S_1^T(\rho) S_1(\rho) \right]^{1/2} d\rho,$$

where $\partial^2 M$ is the region or corner at which two boundary faces of $M$ meet,

$$\upsilon_i(\rho) = -\sum_{k=1}^{d-3} e_k^T \left[ S_1^T(\rho) S_1(\rho) \right]^{-1} \left[ \frac{\partial}{\partial \rho_k} S_1^T(\rho) \right] \left[ \mathbf{B}_{d-2}(\rho) + \mathbf{B}_{d-1}(\rho) \right] \tan \left[ \frac{\phi(\rho)}{2} \right]$$

and $\phi(\rho)$ is the angle between $\mathbf{B}_{d-2}(\rho)$ and $\mathbf{B}_{d-1}(\rho)$. Since $M$ is of dimension $(d - 1)$, $\partial M$ and $\partial^2 M$ are of dimensions $(d - 2)$ and $(d - 3)$, respectively. Lastly, we consider the edges where two boundary faces meet. If we consider the edge where $\rho_{d-2} = \rho_{d-1} = 1$ and define $S_1(\rho) := [T_1(\rho) \cdots T_{d-3}(\rho)]$, then

$$v_o = \int_{\rho \in \partial^2 M} v_o(\rho) \det \left[ S_1^T(\rho) S_1(\rho) \right]^{1/2} d\rho,$$

where $v_o(\rho) = \arccos([\mathbf{B}_{d-2}(\rho), \mathbf{B}_{d-1}(\rho)])$.

In order to find $\tau$, we need to calculate the area of the spherical triangle which is achieved by the Euler’s formula: area of the triangle equals $(\phi_1 + \phi_2 + \phi_3 - \pi)$, where $\phi_1, \phi_2$ and $\phi_3$ are the three internal angles of the triangle. Loader and Pilla (2007) describe the method of determining these angles by first finding the vectors defining the corners of the triangles.
5. Power Under a Sequence of Local Alternatives

In this section, we derive an asymptotic lower bound for the power of the GQS statistic under a sequence of local alternatives in $K$. This plays an important role in comparing the result with a test against the unrestricted alternative. To the best of the author’s knowledge, a lower bound has not been established in the literature even for independent data; hence it would be an interesting one to derive.

From the parameterization defined in Section 4.1, we can express $\gamma = \gamma_1 + Pu$ such that $\gamma_1 \in \mathcal{V}$ and $u \in \mathcal{N}$. Following the hypothesis (22), we consider a sequence of local alternatives of the form

$$u_N = \frac{u^*}{\sqrt{N}} \quad \text{for a fixed vector \quad} u^* \in \mathcal{N} \subset \mathbb{R}^d \quad \text{and \quad} N = 1, 2, \ldots.$$  

From the derivation of $\theta_2$ in (23), the relation $\theta_2 = \sqrt{N} H u_N = H u^*$ holds under the sequence of local alternatives (30).

As a first step, we define a statistic for testing $H_0 : \gamma \in \mathcal{V}$ against the unrestricted alternative $H_2 : \gamma \in \mathcal{G}$ as $S_N := [Q_N(\gamma) - Q_N(\hat{\gamma})]$, where $\gamma$ and $\hat{\gamma}$ are defined in Section 2.2. Using the arguments similar to Theorem 1 and the result (15), one can establish that $\hat{\theta} \sim Z^\dagger \sim N_r(H u^*, I_r)$ under the sequence of local alternatives.

The arguments given in Robertson et al. (1988) and Pilla and Loader (2005a) yield the following result.

**Theorem 7.** The asymptotic local power of the unrestricted test statistic $S_N^*$ for a sequence of alternatives (30) is

$$\lim_{N \to \infty} \mathbb{P}(S_N^* \geq b_1) = \mathbb{P}\left[\chi^2_r(\delta^2) \geq b_1\right],$$

where $b_1 > 0$ is a constant, $\delta = \|H u^*\|$ and $\chi^2_r(\delta^2)$ is the chi-square distribution with a non-centrality parameter $\delta^2$ and with $r$ degrees of freedom.
The above result is equivalent to Theorem 7 of Pilla and Loader (2005a); however, here the non-centrality parameter $\delta$ is represented in terms of $H$. Theorem 7 yields an exact local power and the next one establishes a lower bound for $S_N^*$. 

**Theorem 8.** A lower bound for the asymptotic power of $S_N^*$, under a sequence of alternatives defined in (30), is

$$\lim_{N \to \infty} P(S_N^* \geq b_1) \geq 1 - \Phi\left(\sqrt{b_1 - \delta}\right).$$

It is worth noting that finding the asymptotic power for $S_N$ under the sequence of local alternatives in $K$ is hard and it does not have a simple weighted non-central chi-squared distribution with a prespecified non-centrality parameter. The following result gives an asymptotic lower bound for the power of $S_N$ under a sequence of local alternatives (30). It demonstrates that $S_N$ under restricted alternatives (i.e., testing for $H_0$ against $H_1$) is locally more powerful than $S_N^*$ under no restriction (i.e., testing for $H_0$ against $H_2$).

**Theorem 9.** A lower bound for the asymptotic power of $S_N$ for a sequence of alternatives (30) is

$$\lim_{N \to \infty} P(S_N \geq b_2) \geq P\left(N(\delta, 1) \geq \sqrt{b_2}\right) = 1 - \Phi\left(\sqrt{b_2 - \delta}\right),$$

where $b_2 > 0$ is a constant, $\delta := \|H u^*\|$ and $\Phi$ is the standard Gaussian cumulative distribution.

**Proof.** Let $d = \arg\min_{b \in K} \|Z^i - b\|^2$, where $Z^i \sim N_{r}(H u^*, I_r)$. Consequently, $d = P_K Z^i$.

Further let, $L := \{c H u^*: c > 0\}$ and $d^* = \arg\min_{a \in L} \|Z^i - a\|^2$, then $d^* = P_L Z^i$. Since $L \subset K$, it follows that $\|d\| \geq \|d^*\|$. Equivalently, $\|P_K Z^i\| \geq \|P_L Z^i\|$.

Let $a = H u^*/\|H u^*\|$ so that $\|a\| = 1$. It is clear that $\|d^*\| = \langle a, Z^i \rangle_+$. The proof of $\langle a, Z^i \rangle \sim N(\delta, 1)$ is presented next. We have $E \left[\langle a, Z^i \rangle\right] = \langle H u^*, H u^* \rangle / \|H u^*\| = \delta$ and variance $V(\|d^*\|) = \|H u^*\|^{-2} V[(H u^*)^T Z^i] = 1$, since $Z^i \sim N_{r}(H u^*, I_r)$. Therefore, the relation $\langle a, Z^i \rangle \sim N(\delta, 1)$ holds. Furthermore, $\|P_K Z^i\| \geq \langle a, Z^i \rangle$, since the right-hand side is the length of the projection of $Z^i$ onto $H u^* \in K$ and $\langle a, Z^i \rangle$ is distributed as $N(\delta, 1)$. The theorem claim follows from (14).

**Example 6.** (Comparison of the local power of $S_N^*$ and $S_N$ for the order restricted testing problem). Define $\gamma_N := (\mu_{1,N}, \mu_{2,N}, \ldots, \mu_{m,N}, \beta_1, \ldots, \beta_m)^T$ as a sequence of
local parameter vectors for $N = 1, 2, \ldots$. The local alternatives take the form $\mu_{k,N} = \mu_{k-1,N} + \epsilon_{k-1}/\sqrt{N}$ for $k = 2, \ldots, m$, where $\epsilon_1, \ldots, \epsilon_{m-1}$ are fixed negative constants. As $N \to \infty$, the sequence of local alternatives approach the null hypothesis $H_0: \gamma \in \mathcal{V}$.

Consider three treatment groups (i.e., $m = 3$) leading to six possible orderings, with each order corresponding to an arc on the unit circle. Union of these six arcs comprises the unit circle $S^1$. Due to the balanced design assumption, each of these arcs is of the same length; therefore, the angle of the cone $\mathcal{K}$ is $\phi = \pi/3 = 60^\circ$. At the level of significance $\alpha = 0.05$, the critical values corresponding to the two tests $S_N$ and $S^*_N$ are $b_2 = 3.820$ and $b_1 = 5.991$, respectively. Table 1 presents the asymptotic lower bounds on the local power for the two tests. The table also presents the exact asymptotic local power for $S^*_N$ obtained using the asymptotic formula (31). It is clear that except for $\delta = 0$, the asymptotic local power of $S_N$ is better than that of $S^*_N$, in terms of both the lower bound and the exact power.

| Test          | $\delta$ |
|---------------|-----------|
|               | 0   | 1   | 2   | 3   | 4   | 5   |
| $S_N$ Lower bound | 0.025 | 0.170 | 0.518 | 0.852 | 0.980 | 0.999 |
| $S_N^*$ Exact  | 0.050 | 0.133 | 0.416 | 0.771 | 0.957 | 0.996 |
| $S_N^*$ Lower bound | 0.007 | 0.074 | 0.327 | 0.710 | 0.940 | 0.995 |

6. Discussion

In this research, inferential theory is developed for the problem of testing under convex cone alternatives for correlated data. Such a problem occurs when interest lies in detecting ordering of treatment effects, while simultaneously modeling relationships with other covariates. The testing problem (2) is also applicable to the analysis of clustered multi-
categorical data. In this framework, \( Y_{it} = (Y_{i1t}, \ldots, Y_{iKt})^T \) denotes the \( K \)-categorical response on the \( i \)th observation in the cluster \( t \), where \( Y_{ijt} = 1 \) if category \( j \) (\( j = 1, \ldots, K \)) is observed and 0 otherwise.

We established that the GQS statistic is asymptotically equivalent to the squared length of the projection of the standard Gaussian vector onto an arbitrary convex cone with a nonempty interior. We further derived the asymptotic null distribution of the GQS statistic under convex cone alternatives for correlated data as a weighted chi-squared distribution. The weights in the asymptotic distribution are the mixed volumes of the convex cone and its polar cone which do not have explicit expressions except in special cases. For non-polyhedral cones, closed-form expressions for the weights are very complicated and therefore; often a simulation approach is employed for computing them [Section 3.5, Silvapulla and Sen (2004)]. In this article, explicit formulas are derived for the calculation of these weights using the Hotelling-Weyl-Naiman volume-of-tube formula.

Furthermore, an asymptotic lower bound is derived for the power of the test under a sequence of local alternatives in \( K \) for correlated data which establishes that the test under restricted alternatives is more powerful than the test under no restriction. Note that Barlow et al. (1972) and Robertson et al. (1988) derive the asymptotic power only under specified alternative hypothesis. The current theory is applicable to many practical problems of interest including testing for a monotone regression function and for the analysis of clustered multi-categorical data.

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