Single phase and correlated phase estimation with multi-photon annihilated squeezed vacuum states.

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In the last years, several works have demonstrated the advantage of photon subtracted Gaussian states for various quantum optics and information protocols. In all these works, it was not clear about the origin of such advantages. For the first time, we have extensively studied multi photon annihilated squeezed vacuum state for single phase and correlated phase estimations. We have obtained compact expressions which have not yet reported elsewhere, showing that multi-photon subtracted states can be obtained by applying squeezing operators to a certain class of superposition states. With this tool, we have shown that for single phase estimation, albeit the use of multi-photon annihilated squeezed vacuum states at low mean photons per mode provide advantage compared to classical strategy with equivalent energy, when the total input energies is held fixed, the advantage due to photon subtraction is completely lost. However, for the correlated case in analogous scenario, some advantage appears to come from both the energy rise and improvement in photon statistics. In particular photon subtracted states conserve the advantage of about 30% even in case of realistic value of the optical losses.

I. INTRODUCTION

In the recent years, non-gaussian states have been recognized as a valuable resource for several quantum information protocols \[11, 2\]. Two basic operations that can lead to non-Gaussian states are photon addition to, or photon subtraction from a Gaussian states \[3, 4\]. The first attempt in this direction was made by Tara and Agarwal \[5\] in transferring a classical like coherent state to entirely non-classical state through photon addition and the same operation was experimentally implemented for the first time to coherent and thermal state \[8, 9\]. Furthermore, photon addition and subtraction have been reported in enhancing entanglement in two mode squeezed vacuum state (TSV) \[11,13\]. It is known that each mode of the TSV has super-poissonian photon statistics. In \[14\], it has been reported that photon subtraction makes the TSV less noisy and helps in shifting the most probable distribution to higher mean photon number, thereby it increases the mean energy of the resulting state. In the last years, photon subtracted states have been theoretically investigated reporting the advantage over TSVs for target detection in the presence of noise, the so called "quantum illumination" \[15\]. Recently, their advantage has also been demonstrated in single interferometry with parity measurements \[16, 17\].

For alleviating the contribution of vacuum noise \[18\], single mode squeezed vacuum (SSV) state mixed with a coherent pump beam is considered to be almost the best known strategy in linear interferometer, such as Mach Zehnder type, in case of large photon number and non negligible losses \[19\]. In this context, it has been demonstrated that single photon subtracted squeezed vacuum state, which is a squeezed single photon state can bring some advantage in sensitivity in certain conditions \[20\], and in terms of quantum fisher information in the low energy regime. In all these works, it was not clear whether the advantages come from energy shifts, or from the improvement in the photon noise associated with the individual mode due to photon annihilation. An obvious question on fundamental ground arises, if the advantage relies solely on increase in energy, then is it worth to go for probabilistic operation such as photon subtraction, or to increase the source energy. One of the main motivation of this paper is answering to these fundamental questions. In particular, study in detail multi-photon subtracted single mode and two mode squeezed vacuum state for single phase and correlated phase estimation respectively. On one side, we show that a multi-photon subtracted (one-) two-mode squeezed states is formally equivalent to a state obtained by a (one-) two- mode squeezing operator applied to a certain class of finite superposition states in the photon number basis. This class of states have been investigated earlier \[21, 22\] and they show quadrature squeezing their-self. One could expect that this initial squeezing could bring benefit in phase estimation. We have therefore investigated this possibility.

On the other side the equivalence between photon subtracted squeezed states and squeezing of superposition states, allows to devise alternative strategy for their generation, usually based on probabilistic post-selection conditioned by photon detection events. This is another important results shown in our work. It represent a broad generalization of what has been demonstrated in the article \[23\], i.e., that a one photon subtracted squeezed state can be generated by seeding a squeezer (non-linear parametric amplifier) by a single photon state. More in general, we show that the number of elements of the superposition state which are necessary for seeding the non-linear process is in simple relation with the order of photon subtraction.

For careful investigation about the origin of the improvement in phase measurement uncertainties if any, we have fixed the total energy by balancing the energies of the subtracted and un-subtracted states keeping the coherent pump energy constant. This choice has not been reported in literature. We shall consider this energy
balancing condition for both single and correlated phase estimations. In general we find that energy increasing of the states intrinsic to the photon subtraction operations is in most cases the origin of the advantage of the photon subtracted states. However, in case of the specific scheme of correlation measurement among two interferometer by exploiting bipartite correlated states such as TSV, the advantage of the symmetric photon subtraction cannot be explained only by energy shift.

This paper is organized as follows. In Sec. II, we shall introduce multi-photon annihilated single mode squeezed vacuum (PASSV), discussing their properties and their usefulness, for single phase estimation by the conventional measurement strategy and in the more general framework of the fisher information. We shall describe multi-photon symmetrically annihilated two mode squeezed vacuum (SPATSV) state in Sec. III. In section IIIB, we will analyze squeezing and photon statistical properties of SPATSVs. Section IIIC deals with the results of correlated phase estimation. We present results up to four and three photons subtraction for single and correlated phase estimations respectively. Finally we shall conclude with a summary and conclusion in Sec. IV.

II. PASSVS

PASSV states are defined as:

\[ |\text{PASSV}\rangle_m = N_m^m(\lambda) a^m \hat{S}(\lambda e^{i\chi}) |0\rangle, \quad (1) \]

where \( \hat{S}(\lambda e^{i\chi}) \) is the single mode squeezing operator with \( \lambda = \sinh^2 r, r \) being the squeezing parameter and \( \chi \) is the squeezing angle. The squeezing operator applied to the vacuum state originates SSV with energy (mean number of photon) equal to \( \lambda \). The number of subtracted photon is \( m \), obtained by \( m \) consecutive action of the power annihilation operator on \( a \). Since the photon subtraction is not an unitary operation, it is necessary to introduce the normalization constant \( N_m^m(\lambda) \). Its explicit form can be found in \([24]\) as \( N_m^m(\lambda) = m! (-i \sqrt{\lambda})^m P_m(i \sqrt{\lambda}) \), with \( P_m \) the \( m \)th order Legendre’s polynomial. A known effect of the photon subtraction is the increasing of the mean energy of the state. This is intuitively explained because it is most probable to subtract a photon from a high populated state, so the operation corresponds to a selection of more energetic part of the state. In particular the mean photon number of PASSV state, for \( m = 0 \rightarrow 4 \) which correspond to zero, one, two,three and four photon subtraction from the SSV state, becomes \( \lambda, 3\lambda - 1, 3(3 + 5\lambda)/(1 + 3\lambda), (3 + 30\lambda + 35\lambda^2)/(3 + 5\lambda) \) respectively.

Indeed, incorporating integration within an ordered product (IWOP) technique \([25]\), we have found a new writing form for PASSV states, which is equivalent to seeding squeezing operator with photon number superposition state \( |\rho^s(\lambda, \chi)\rangle_m \) in input as follows:

\[ |\text{PASSV}\rangle_m = \hat{S}(\lambda e^{i\chi}) |\rho^s(\lambda, \chi)\rangle_m, \quad (2) \]

Here and in the following we set the squeezing angle to \( \chi = 0 \) without loss of generality. For \( m = 0 \), the input simplifies to vacuum state as expected. For \( m = 1 \), it becomes a single photon state as reported in \([20]\). Note that, for other values of \( m \), it becomes a \((m + 1)/2\) components superposition state for odd \( m \), and a \((m + 2)/2\) components superposition state for even \( m \). Given the PASSV states in the form of Eq \((2)\) the energy increasing with \( m \) can be straightforwardly understood because of the growing in mean value of photon number of the seeding superposition states. From fundamental perspective, squeezing effects are associated with photon number superposition state \([21, 22]\), even though this class of states can not be obtained by any unitary transformation on vacuum state, like standard squeezed state. However these superposition states do not always have lower quadrature noise compared to vacuum state, thus these initial seeding states do not necessarily means a stronger squeezing of the final state. For instance, for \( m = 1 \), the seeding state is a single photon state having 33% of more quadrature noise than the vacuum state. We checked for subsequent odd values of \( m \), although the quadrature noise of the seeding states decrease with respect to single photon state, its value still remain above the vacuum noise. Variance of \( \gamma = (a - \hat{a}) / i \sqrt{2} \) quadrature of the PASSV states has been plotted in Fig \([2]\). It is clear that the \( \gamma \) quadrature squeezing for PASSVS is worse compared to SSV for odd numbers of photon subtraction (anti-squeezing is observed for \( \lambda < 1 \) ), while for even numbers the quadrature squeezing is better in the same range of \( \lambda < 1 \).

In the next subsection, we shall discuss the performance of PASSV states in phase estimation in connection to the quadrature squeezing.

A. Single phase estimation with PASSV states

Let us consider the Mach Zehnder interferometer (MZI) sketched in Fig \([1]\), where one ports of the first beam splitter is injected with coherent light and the other port with a PASSV state. Thus, the total input state is \( |\Psi\rangle_{1,2} = |\text{PASSV}\rangle_1 \otimes |\alpha| e^{i\psi}\rangle_2 \), where \( |\alpha| \), \( \mu = |\alpha|^2 \), and \( \psi \) are the amplitude, mean photon number and phase of the coherent pump respectively.

The uncertainty in measuring the phase \( \phi \), in the configuration of Fig \([1]\) is expressed by

\[ U(\phi) = \frac{\sqrt{\Delta^2 \dot{\phi}}}{|\partial \bar{a} / \partial \phi|}, \quad (3) \]

where \( \dot{\phi} \) is the photon number difference operator at the output port of the interferometer and \( \Delta^2 \dot{\phi} \) is its
variance. For zero-mean quadrature field such as SSVs, \( \langle \hat{\phi} \rangle = (\mu - \lambda) \cos(\phi) \). For SSV, it can be shown that the lowest uncertainty is reached for \( \phi = \pi/2 \) and in the limit of \( \lambda \gg \mu \), the uncertainty is shot-noise limited, scaling as \( \lambda^{-1/2} \). Whereas, in case of \( \lambda \ll \mu \), it is given by \( (\Delta^2 X_{\theta=\psi+\pi/2})^{1/2} = (\hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2} \) is the rotated quadrature at the generic angle \( \psi \) of the state at the input port “1”. In our case and for the choice of \( \psi = 0 \), the sub-shot noise sensitivity is related to the squeezing of the \( X_{\pi/2} \equiv Y \) quadrature.

We have derived analytically the uncertainty according to Eq. 3 when injecting PASSV states, although it is cumbersome to be reported. The results are shown graphically in Fig. 2b, compared with the SNL at equivalent total energy (dotted lines). It is easy to check that for PASSV the uncertainty always approach asymptotically the SNL when \( \lambda \gg \mu \). For \( \lambda \ll \mu \), the uncertainty is basically determined by the variance of the \( Y \) quadrature, reported in Fig. 2a, as expected. Indeed the advantage over the SNL is present only in the region of quadrature squeezing, and PASSV \((m > 0)\) performs better than SSV only for even \( m \). However we are going to show that this apparent improvement is due only ascribed to the energy increasing of the state due to photon subtraction. For that purpose, we have renormalized the energy of the initial SSV state before the photon subtraction, so that the mean number of photon of the subtracted states \((m = 0 - 4)\) are all equal to \( \lambda \). In this way, also the total input energies to the interferometer is fixed to \( N_{\text{tot}} = \mu + \lambda \). With the energy balancing, SSV outperforms SPASSV regardless of the values of \( \lambda \), as represented in Fig. 2. Incidentally, we have observed that far from the optimal working point \( \pi/2 \), PASSV can still provide some advantage, even under energy balancing conditions as shown in Fig. 3. Typically, this happens from value of \( \lambda \) in a middle range (namely from \( \mu/100 \approx \lambda < \mu/10 \)).

Next we shall see the advantage if any in QFI perspective.

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\[
\begin{align*}
\text{Figure 1: Schematic of MachZehnder interferometer for phase estimation } \phi.
\end{align*}
\]

\[
\begin{align*}
\text{Figure 2: Quadrature squeezing and phase measurement uncertainty at working point } \phi = \pi/2 \text{ (for } \mu = 100, \text{ detection efficiency } \eta = 0.98, \text{ and } \psi = 0). \text{ Different colours correspond to different number of photon subtraction, dashed lines represent the coherent state: (a.) Quadrature squeezing, (b.) Phase uncertainty, and (c.) Phase measurement as per balancing condition.}
\end{align*}
\]

\[
\begin{align*}
\text{Figure 3: Phase measurement uncertainty for } \phi \neq \pi/2 \approx \pi/2 - 1 \text{ at } \mu=10000, \eta = 0.98 \text{ and } \psi = 0. \text{ Different colours correspond to different number of photon subtraction,dashed line is the classical strategy.}
\end{align*}
\]

B. Quantum fisher information

The quantum parameter estimation theory, establishes that the lower bound the the uncertainty is given by the following expression
where $\hat{F}$ is the generator of the unitary transformation associated with the parameter $\phi$, i.e $\hat{U}(\phi) = e^{i\hat{H}\phi}$ and $|\psi\rangle_{1,2}$ being the total input state entering to the interferometer. In the case of the MZI, the generator is the photon number operator i.e $\hat{n}_3 = \hat{a}_3^\dagger \hat{a}_3$ where $\hat{a}_3 = (\hat{a}_1 + \hat{a}_2)/\sqrt{2}$. As per Eq.\ref{eq:5} we shall evaluate QFI by considering PASSVs and coherent state as inputs to the interferometer. The complete expressions for QFI for these cases are cumbersome to present here, so we present the corresponding results graphically in Fig.\ref{fig4}(a) and \ref{fig4}(b).

This confirms that the advantage in phase parameter estimation in a MZI provided by photon subtracted of squeezed state is exclusively due to the increasing in the energy of the field. Using a simple SSV state with the same energy provides the similar sensitivity.

\section{III. SPATSV States}

Starting from the definition of the TSV as the two mode squeezing operator $S_{1,2}(\lambda e^{i\chi})$ applied to the vacuum, the SPATSV can be obtained by the non Hermitian operation represented as

$$|SPATSV\rangle = |\Psi(\lambda, \chi)\rangle_m,$$

where $N_m^-$ is the normalization constant, $\lambda$ is the mean energy (mean photon number) per mode for the TSV, $\chi$ is squeezing angle and $m$ is the number of subtracted photons. It is possible to express the state in Fock basis of infinite dimensional Hilbert space as follows

$$|\Psi(\lambda, \chi)\rangle_m = \frac{N_m^- (\lambda)}{\sqrt{1 + \lambda}} \sum_{n=0}^{\infty} \left(\frac{\lambda e^{i\chi}}{1 + \lambda}\right)^{\frac{n + m}{2}} \frac{(n + m)!}{n!} |n, n\rangle_{1,2}.$$

The normalization constant is of the form $N_m^- (\lambda) = \frac{1}{(m!)^2 \lambda^m P_m (2\lambda + 1)^{-1/2}}$, where $P_m$ is the $m$-th order Legendre’s polynomial. Furthermore, using squeezed transformation of mode operators $a_j$, it is possible to generate the state in eq (7) by applying squeezing operator to a $m + 1$ components finite superposition of photon number states, $|\rho(\lambda, \chi)\rangle_m$, as it follows:

$$|\Psi(\lambda, \chi)\rangle_m = S_{1,2}(\lambda e^{i\chi})|\rho(\lambda, \chi)\rangle_m,$$

$$|\rho(\lambda, \chi)\rangle_m = \sum_{k=0}^{m} C_k^m (\lambda, \chi)|k, k\rangle_{1,2},$$

shows a general increasing of the QFI for PASSVs at the increasing of $m$, for both low and high values of $\lambda$ and photon subtraction is always advantageous with respect to best classical strategies (dotted line). Nevertheless, for energy balancing condition, the advantage is completely lost as evident from Fig.\ref{fig4}(b) and also from the expression of QFI in the limit $N_{tot} \rightarrow \infty$ (at a finite fixed coherent energy) reported here:

$$F_Q = \begin{cases} 
2N^2_{Tot} |m=0, \\
\frac{2N^2_{Tot}}{3} |m=1, \\
\frac{2N^2_{Tot}}{5} |m=2, \\
\frac{2N^2_{Tot}}{7} |m=3, \\
\frac{2N^2_{Tot}}{9} |m=4
\end{cases},$$

Figure 4: Quantum fisher information, $\mu = 100$. Different colours correspond to different number of photon subtraction, dashed lines present classical strategies: (a.) without energy balance (b.) balancing condition.
\[ C_k^m(\lambda, \chi) = e^{i\chi_m} \sqrt{\frac{(1 + \lambda)^m}{P_m(2\lambda + 1)}} \times e^{i\chi_k} \left( \begin{array}{c} m \\ k \end{array} \right) \left( \sqrt{\frac{\lambda}{\lambda + 1}} \right)^k \] (11)

with \( \sum_k |C_k^m(\lambda, \chi)|^2 = 1 \). Interestingly, by an examination of the coefficient of \(|\rho(\lambda, \chi)\rangle_m\), reported in Eq. (11), it comes out that superposition state \(|\rho(\lambda, \chi)\rangle_m\) is similar to a truncated TSV up to the components with \( k \leq m \). They differ only by a binomial coefficient and a normalization factor. Thus, for \( m = 0 \) the superposition state collapses to the vacuum and the final state \(|\Psi(\lambda, \chi)\rangle_0\) coincides obviously with TSV. For \( m = 1 \), namely one photon subtraction, the corresponding normalized two components photon number superposition state is

\[ |\rho(\lambda, \chi)\rangle_1 = \frac{1}{\sqrt{2\lambda + 1}} \left( \sqrt{1 + \lambda}|0, 0\rangle + e^{i\chi} \sqrt{\lambda}|1, 1\rangle \right). \] (12)

This superposition state is an entangled state for non-zero value of \( \lambda \) and in the high \( \lambda \) limit, it become a maximal entangled. In this case it is equivalent to a truncated TVS state up to \( m = 1 \).

In general, Eqs. (9), (10) and (11) suggest that a SPATSV state can be generated by seeding the input modes of a non-linear two-mode-squeezing interaction, by an opportune superposition state in the photon number basis, see Fig. 5b. This, represent an alternative way to generate the photon subtracted states in contrast to common approach depicted Fig. 5a, consisting in a post selection of the state, conditioned to double click events at the detectors placed in the two arms, experimentally realized through unbalanced beam splitter (BS).

![Figure 5](image1)

**Figure 5**: Generation scheme for \( m = 1 \): (a) Probabilistic process by two beam splitter of high transmittance placed in two arms of the PDC source. Simultaneous clicks on the two single photon detectors confirms the generation of SPATSV. (b) Alternative approach to the generation of SPATSV consists in injecting entangled super position state into the non-linear crystal (NL).

![Figure 6](image2)

**Figure 6**: Non-classical amplitude quadrature correlation. Different colors correspond to different number of photon subtraction: a) For superposition state except the dotted curve which corresponds to TSV. b) For the photon subtracted state.

A. Squeezing and photon statistics of the SPATSV state

Analogously to what has been already discussed in Sec. II for single mode superposition states, also two mode superposition states \(|\rho(\lambda, \chi)\rangle_m\) are squeezed in the quadrature difference even though they do not minimize the uncertainty principle. The maximum squeezing is reached for the quadrature at the angle \( \chi \), for example for \( \chi = 0 \) it the amplitude quadrature \( X^- = X_1 - X_2 \) reported in Fig. 6(a), where \( X_j = (a_j + a_j^\dagger)/\sqrt{2} \) is the quadratures of individual input modes (\( j = 1, 2 \)). Note that, for \( m \geq 1 \) non-classical correlations are always present, becoming stronger at the increasing of \( m \), especially in the region of small \( \lambda \). When \( m \geq 2 \) the squeezing level overpass the TSV limit (dotted purple line). The quadrature noise behaviour of the seeding superposition state has a direct effect on the squeezing properties of the SPATSV as shown in Fig. 6(b), basically leading to a further noise reduction especially for \( \lambda < 1 \) with respect to the TSV state. This effect is not trivially related to an energy shift and it can bring beneficial when using SPATSV state for specific interferometric schemes, as studied in Sec. IIIIB.

Aside quadrature squeezing, other statistical properties of the field can be improved in terms of noise reduc-
Figure 7: 3D plots showing joint photon number distribution in SPASTV state. \( j \) and \( k \) are the photon number in the mode "1" and "2", respectively. The parameters value chosen are \( \lambda = 0.6 \), (a) \( m = 0 \), (b) \( m = 1 \), (c) \( m = 3 \).

Figure 8: Mandels \( Q \) of the SPATSV state in function of the mean photon number per mode (\( \lambda \)). Here, we have chosen \( \eta = 0.98 \). The \( Q \) parameter is computed from the variance of the photon number \( N \) and its mean value, given by:

\[
Q = \frac{\text{Var}(N) - \langle N \rangle}{\langle N \rangle},
\]

where \( N = \hat{a}^\dagger \hat{a} \) is the photon number operator. For classical light Mandel parameter is bounded by \( Q \geq 0 \). It is worth noting that even if the individual modes of TSV have thermal statistics, after the application of subtraction operation with \( m > 1 \) become non classical for low value of mean photons per mode, \( \lambda \), as evident from the negative value of Mandel’s parameter reported in Fig. 8. Moreover we have found that when the energies of subtracted states are balanced to the energy of TSV state, Mandel parameter becomes more negative and the negative region shifts to higher mean number of photons \( \lambda \). So, this non-classical behaviour is clearly not related to an energy shift due to the photon subtraction, rather it is a more fundamental shrinkage of the photon number distribution.

B. Correlated phase estimation with SPASTV states

Figure 9: Correlated interferometric scheme: The modes of the bipartite input state \(| \psi \rangle \) are mixed with two identical coherent states \(| \alpha \rangle = |\mu e^{i\psi} \rangle \) in two interferometers \( I_1(\phi_1) \) and \( I_2(\phi_2) \). A joint detection is performed and the observable \( \hat{C}(\phi_1, \phi_2) \) is measured. The losses are accounted by considering two identical detectors in both channels with the same quantum efficiency, i.e, \( \eta_5 = \eta_7 = \eta \).

The interferometry system we will consider in this section is presented in Fig. 9. It is composed by two linear interferometers, for instance a pair of MZIs in the figure, whose photo-currents at the read out ports are jointly measured. This is an elegant and powerful scheme in the detection of extremely faint phase signals whose magnitude can be much smaller than other sources of noise, including the shot noise. The advantage of this scheme comes from the fact that the same signal shared by the two interferometers, even if hidden in the noise in the single device, can emerges by correlating their outputs. This strategy has been already considered in several highly demanding applications, in general related to the research of stochastic fundamental backgrounds, such as gravitational wave background \([28, 29]\), primordial sources \([30, 31]\) and quantum gravity effects at the Plank scale.
The advantage of using quantum state of light in such a configuration has been analysed in ref. [34, 35]. It has been shown that injection of quantum state of light in the classically unused input ports (1 and 2 in the picture), either in the form a pair of independent squeezed state or of in the form of a TSV can deliver better sensitivity. In case of TSV, for specific working conditions, i.e. very close to the dark fringes and for high quantum efficiency, the quantum advantage is dramatic even with respect to the double squeezing. Here our purpose is to investigate if and to what extent photon subtracted TVS could performs in virtue of their improved non-classical properties discussed in Sec. IIIA

1. Noise reduction factor at the read-out ports

Let start considering the correlation properties of the read out signals at the ports (5 and 7) which are the basis of the sensitivity enhancement. In particular we are interested into photo-current subtraction, proportional to the photon number difference \( N_5 - N_7 \). Here we consider the noise reduction factor, a standard measure of non-classical correlation for a bipartite state defined as [36]

\[
\text{NRF} = \frac{\langle \Delta^2 (N_5 - N_7) \rangle}{\langle N_5 \rangle + \langle N_7 \rangle} \tag{14}
\]

The numerator is the variance of photon number difference and the denominator represents the standard quantum limit. Thus, NRF < 1 indicates non-classical correlation.

In the following we shall use the equivalence between the input-output relation of a MZI carrying a phase shift \( \phi \) and a beam splitter with transmittance \( \tau = \cos^2(\phi/2) \). In the scheme of Fig.3 assuming \( \phi_1 = \phi_2 \), \( \tau \) corresponds to the transmission from the input port 1(2), where quantum states are injected, to the read-out port 5(7). On the other side, the fraction of coherent power injected in ports 3(4) and transmitted to the same output port 5(7) is \( 1 - \tau \).

Two different regimes, depending on the parameters value, can be distinguished corresponding to different type of correlation between \( N_5 \) and \( N_7 \).

It can be shown that when the output signal is dominated by photon coming from coherent beam, i.e. \( \lambda \tau \ll \mu (1-\tau) \), each interferometer acts similarly to a homodyne detectors and the difference photon number becomes proportional to the difference among the quadrature of the input modes at the ports 1 and 2:

\[
\langle N_5 - N_7 \rangle_\alpha \propto \sqrt{\frac{\mu}{2}} \sin(\phi)X^-_{\theta=\psi+\pi/2} \tag{15}
\]

Here \( X^-_{\theta=\psi+\pi/2} \) is the quadrature difference of the input state. For \( \psi = \pi/2 \), it reduces to the difference of amplitude quadratures \( X^- = X_1 - X_2 \). Therefore, quadrature non-classical correlation of the input state studied in Sec. IIIA immediately traduces in a non classical correlation between the photon numbers at the readout ports. The numerical NRF is plotted in Fig. 10. In particular it happens that for \( \lambda \gg 1 \) the NRF for different \( m \) values reduce to NRF of TSV in the form \( \text{NRF}_{m} \approx 1 - \tau + \tau/(4\lambda) \), a tendency that one can appreciate already from \( \lambda = 2 \) as reported by the dotted line in Fig. 10. This is explained by the behavior of the quadrature squeezing in the same energy limit shown in Fig. 6(b). Conversely, for low value of the mean photon number \( \langle \lambda \ll 1 \rangle \), in which quadrature squeezing increases with the order of photon subtraction, the expressions of NRF for different \( m \) values are as follows

\[
\text{NRF}_{m=0} \approx 1 - 2\tau \sqrt{\lambda - \lambda} \tag{16}
\]

\[
\text{NRF}_{m=1} \approx 1 - 4\tau \sqrt{\lambda - 2\lambda} \tag{17}
\]

\[
\text{NRF}_{m=2} \approx 1 - 6\tau \sqrt{\lambda - 3\lambda} \tag{18}
\]

From these sets of equations, NRFs are linear in terms of BS transmittance \( \tau \) and their respective slopes increase...
with the subtraction number $m$, demonstrating the advantage in non-classical correlation. This scaling is represented in Fig. 10 by the solid lines.

On the other side, in the opposite situation of $\lambda \tau \gg \mu (1 - \tau)$ basically the coherent beam does not contribute significantly to the outputs and the two interferometers can be seen as attenuators with transmission $\tau$ of the input state. Here, the photon number entanglement among the modes of the TVS or the photon subtracted TVS are preserved for $\tau \sim 1$ and high detection efficiency. In this case one expects a perfect photon number correlation at the output ports which does not depend on the energy $\lambda$ of the input quantum state. This explains the sudden dropping down of the NRF observed in Fig. 10 for $\tau \sim 1$. However this opposite limiting situation is reached earlier at $\tau \approx 1$ for SPATSV than TSV because of increasing in the mean energy. Thus in the transient between the conditions $\lambda \tau \ll \mu (1 - \tau)$ and $\lambda \tau \gg \mu (1 - \tau)$, SPATSV shows higher non-classical correlation over TSV as evident from Fig. 10 b. In this transient, it turns out that the non-classical correlation for SPATSV is affected differently by the detection loss $\eta$. For instance, for $\lambda = 2$, and close to $\tau \approx 1$, there exists a value of detection efficiency at which NRF for SPATSV drops with respect to TSV. In the next section we shall show that the characteristics of the NRF are strictly related with the sensitivity of the double interferometric set up.

2. Phase correlation estimation

The setup in Fig. 9 is devoted to the comparison among phase signals, in particular faint stochastic phase noises, which might be correlated (or not) in the two interferometers. Here, rather than the magnitude of phase noise in the single MZI, the quantity under estimation is the covariance between the phase fluctuation. This estimate can be somehow related to a joint measurement of the read-out signals $N_5$ and $N_7$. Whatever joint observable $\hat{C}(\phi_1, \phi_2) = \hat{C}(N_5(\phi_1), N_7(\phi_2))$ with local non-null double partial derivative $\partial^2_{\phi_1, \phi_2} \hat{C}(\phi_1, \phi_2)$, leads to phase noise covariance estimation $\tilde{U}$ [35]. Here, the goal is to investigate whether photon subtracted TSV can lead to some sensitivity advantage. The uncertainty of measurement in case of a stochastic signals much fainter than photon shot noise is

$$U = \sqrt{\frac{2 \text{Var} \left[ \hat{C}(\phi_1, \phi_2) \right]}{\partial^2_{\phi_1, \phi_2} \hat{C}(\phi_1, \phi_2)}}. \quad (19)$$

Given that TVS and SPATVS leads to noise reduction in the read-out signal subtraction as discussed in Sec. III-B-1, it is natural to chose the joint observable $\hat{C}$ as $\hat{C}(\phi_1, \phi_2) = (N_5(\phi_1) - N_7(\phi_2))^2 = N_5^2 + N_7^2 - 2N_5N_7$. The classical bound, obtained with coherent states at ports "3" and "4" and vacuum at the ports "1" and "2",

is given by $U_{cl} = \sqrt{\frac{2}{\eta \mu \cos^2[\phi/2]} 35}$. Hereinafter, we present the uncertainties $U_m$ for $m$-th SPATSV state, as normalized to the coherent classical limit, namely $U_m = U_m/U_{cl}$.

![Figure 11: Normalized uncertainty vs $\phi$ with $\mu = 10^{12}$, different colors correspond to different number of photon subtraction $m$, pink solid line represents two independent squeezed state; (a.) for $\lambda = 2$, $\eta = 0.98$ (b.) $\lambda = 0.05$, $\eta = 0.98$ (c.) energy balancing scenario for $\lambda = 2$, $\eta = 0.96$](image)

Analytical results of uncertainties in function of the working central phase $\phi_1 = \phi_2 = \phi$ are plotted in Fig. 11. One can clearly discern two different regions; one laying in the range $10^{-5} < \phi < 1$ and the other for smaller value of phase, $\phi << 10^{-6}$, separated by a short transient.

The first limit depicts the situation in which coherent photon transmitted to the read out ports, quantified by $(1 - \tau)\mu$, is much larger than the transmitted TSV photons, $\tau \lambda$. In this case, quadrature non-classical correlation in the input modes are responsible for the read-out signal correlation. In order to provide compact expression we report analytic results in relevant regimes. In the limit of high coherent power and $\mu \gg 1$ and $\lambda \ll 1$
one gets:

\[
U_{m=0} \approx \sqrt{2} \left[ 1 - \tau \eta \left( 2\sqrt{\lambda} - 2 \lambda \right) \right]
\]

(20)

\[
U_{m=1} \approx \sqrt{2} \left[ 1 - \tau \eta \left( 4\sqrt{\lambda} + \frac{1}{2} \lambda (3\eta \tau - 16) \right) \right]
\]

(21)

\[
U_{m=2} \approx \sqrt{2} \left[ 1 - \tau \eta \left( 6\sqrt{\lambda} + \frac{9}{2} \lambda (\eta \tau - 4) \right) \right]
\]

(22)

\[
U_{m=3} \approx \sqrt{2} \left[ 1 - \tau \eta \left( 8\sqrt{\lambda} + \lambda (9\eta \tau - 32) \right) \right].
\]

(23)

Note that, unless overall numerical constant and the introduction of the detection efficiency \( \eta \), these expressions follow the NRF ones reported in Eqs. (16,17,18) up to the terms in \( \sqrt{\lambda} \), which are the most significant in the limit discussed. It is clear that in general there is an advantage provided by the photon subtraction, especially for \( \tau \) close to 1 (\( \phi \) close to zero), which is the one represented in Fig.11, for \( 10^{-5} < \phi < 1 \).

In the case of strong squeezing (\( \mu \gg \lambda \gg 1 \)), it turns out that respective expressions for different \( m \) do not differ much from each other reducing to:

\[
U_{m=0,1,2,3} \approx \sqrt{2} \left( 1 - \tau \eta - \frac{\tau \eta}{4\lambda} \right)
\]

(24)

Even in this case the analogy with the behavior of the NRF in the same limit is clear.

Let us discuss now the second limit, in which the coherent photons do not reach the read-out ports "5" and "7", while the TSV is totally transmitted, i.e. when \( 1 - \tau \mu \ll \tau \lambda \). In this case perfect photon number correlation of the input entangled state are directly found in \( N_5 \) and \( N_7 \). For \( \mu \gg 1 \) (and \( \phi \to 0 \) ) we obtaining the following asymptotic behavior:

\[
U_{m=0,1,2,3} \approx \sqrt{2} \sqrt{(1 - \eta)/\eta} \quad \lambda \ll 1
\]

(25)

\[
U_{m=0} \approx 2 \sqrt{5}(1 - \eta) \quad \lambda \gg 1
\]

\[
U_{m=1} \approx 2 \sqrt{3}(1 - \eta)
\]

\[
U_{m=2} \approx 2 \sqrt{13/5}(1 - \eta))
\]

\[
U_{m=3} \approx 2 \sqrt{17/7}(1 - \eta).
\]

(26)

Eq.s (25,26) shows that in this regime of perfect photon number correlations the uncertainty reduction is mainly limited by the detection efficiency. This means that there exists always a value of the efficiency high enough to make this regime more advantageous with respect to the one exploiting quadrature correlation. For example in Fig.11 \( \eta = 0.98 \) guarantees a stronger advantage for \( \phi \ll 10^{-6} \). Even in this region the subtraction of \( m \) photons from the TSV state brings a further improvement which increases with \( m \). Furthermore, Fig.11 also follows similar behaviour to the NRF in the transient regime, providing 50% advantage for SPATSV (\( m = 3 \)) over TSV in the high \( \lambda \) which further increases to three times in the low value of mean energy per mode.

Fig. 12(a) shows the dependence of the normalized uncertainty from the detection efficiency in the condition of \( (1 - \tau)\mu \ll \tau \lambda \). Note that, while for high efficiency all the curves corresponding to different \( m \) collapse to the same value, the SPATSV state seems more robust, compared to the TVS when \( \eta \) is relatively small.

However, we have to observe that when detection efficiency is small, the opposite regime dominated by quadrature correlation of the input state, namely for \( (1 - \tau)\mu \gg \tau \lambda \), allows reaching better absolute uncertainty.

\[
\begin{align*}
U_{\mu=0,1,2,3} & \approx 2 \sqrt{2} \left( 1 - \tau \eta - \frac{\tau \eta}{4\lambda} \right) \\
& \approx 2 \sqrt{2} \left( 1 - \tau \eta - \frac{\tau \eta}{4\lambda} \right)
\end{align*}
\]

Figure 12: Normalized uncertainty vs the detection efficiency \( \eta \), for \( \mu = 10^{12} \), \( \lambda = 2 \), \( \phi = 10^{-8} \). (a) Energy of SPATVS increases with \( m \). (b) Balanced energy: the parameters of SPATVS are renormalized to have the same mean number of photons for all \( m \)

In the spirit of this letter, we are interested analyzing whether the uncertainty improvement presented in Fig.11 in Eq.s (26) and shown in Fig. 12 could be solely explained by the increasing of the mean energy of the state after photon subtraction operation. Here we follow the similar approach of energy balancing considered in the Sec. 11 where the energies of two mode photon sub-
tracted states \((m = 0, 1, 2, 3)\) are made equivalent to the energy of TSV, i.e., \(\lambda\). It comes out that the advantages for high quantum efficiency reported in Fig. 11(a-b) are almost washed away (see Fig. 11(c)). However, in case of realistic value of the optical losses the results are plotted in Fig. 12.

From the figure it is evident that as long as the condition of photon number correlation is fulfilled, the improvement due to annihilated states in terms of uncertainty reduction at higher losses is not compromised. For instance, in this scenario SPATSV \((n = 3)\) gives around 26 \% of uncertainty reduction advantage compared to TSV at 80 \% of detection efficiency. Thus, the improvement in uncertainty reduction by subtraction scheme is in general not only related to the energy shifts, but it also comes from the enhancement in mode correlation and statistics.

IV. SUMMARY AND CONCLUSIONS

We have studied in detail multi photon subtracted one- and two-mode squeezed vacuum state, in relation to phase estimation in both single and correlated interferometry. We have obtained new compact form expressions which show photon subtraction is an operation that results equivalent to the squeezing of a certain finite superposition states in the photon number basis. The number of components in the finite superposition increases with the number of subtracted photons. Quadrature squeezing is always associated with superposition states, and more is the component of superposition, the stronger is the quadrature squeezing. However, the squeezing of the final single mode PASSV state, after the application of the \(S\) transformation, not necessarily improves with the number of subtracted photons. In the case of odd number of photon subtraction it is definitely worse than SSV, while for even photon subtraction, it is better than the SSV only for relatively small brightness, basically due to the mean energy enhancement of the state. This behaviour is completely mapped in the uncertainty in the phase estimation when conventional measurement photon number difference is considered at the output ports of the second beam splitter of a MZI. Moreover, by comparing the phase sensitivity after re-adjusting the energy of the PASSVs to match the one of SSV, the advantage of the photon subtraction disappears, at least considering the optimal working point of \(\phi = \pi/2\). For other values of the central phase we have found different behaviour and in some cases, as shown in Fig. 5, where the advantage of photon subtraction is preserved also when energies are balanced. In terms of QFI, we found improvements by the number of photon subtraction, but for energy balancing condition this advantage is lost. However, Heisenberg limit is reached asymptotically for large total number of photons \(N_{tot}\rightarrow\infty\) entering to the interferometer and in loss less scenario.

Looking at these features of photon subtraction in single interferometry, we were motivated to test symmetrically multi-photon annihilated two mode squeezed vacuum state for correlated phase estimation \([34, 35]\). Usually such states are generated by probabilistic events with low success rate. We showed analytically how symmetric photon subtraction from two mode squeezed vacuum is equivalent to the squeezing of a finite component superposition state, suggesting an alternative way for the deterministic generation of SPATSV states. We found that, these superposition states always show quadrature squeezing and their strength increases with the number of symmetrical photon subtraction. Various statistical properties including photon number distribution, Mandel’s \(Q\) function, and noise reduction factor manifested higher non-classical features of SPATSV with respect to TSV suggesting its importance for correlated phase estimation. Concerning the detection of phase correlations among two MZIs, in general we found SPATSV achieving a smaller uncertainty than TSV for \(\phi \approx 0\). In this bipartite case, without energy balancing, SPATSV provides more advantage compared to TSV for low losses and such advantages at low losses are almost washed away for energy balancing scenario. For the last condition, SPATSV conserve the advantage of about 30\% with respect to TVS against high losses because of improved statistical properties.

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