On the Quantum Levels
of Isolated Spherically Symmetric
Gravitational Systems

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Abstract
The known canonical quantum theory of a spherically symmetric pure (Schwarzschild) gravitational system describes isolated black holes by plane waves $\exp(-iMc^2\tau/\hbar)$ with respect to their continuous masses $M$ and the proper time $\tau$ of observers at spatial infinity.

On the other hand Bekenstein and Mukhanov postulated discrete mass levels for such black holes in the spirit of the Bohr-Sommerfeld quantisation in atomic physics.

The two approaches can be related by postulating periodic boundary conditions in time for the plane waves and by identifying the period $\Delta$ in real time with the period $\Delta_H = 8\pi G M/c^3$ in Euclidean time. This yields the mass spectrum $M_n = (1/2)\sqrt{n} m_P$, $n = 1, 2, \cdots$.

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1 Introduction

The quantum theory of black holes coupled to matter has been studied intensively during the last years. Most of the time one starts with a quantised matter field coupled to the classical geometry of a black hole and then asks for the backreaction of this coupling to the geometry (for two recent introductions see e.g. the reviews [1] and [2]). The geometry itself is generally not quantised, because an analysis of the totally quantised coupled systems meets so severe difficulties that such an approach up to now has resisted a breakthrough, at least in 4 spacetime dimensions (for a recent review of the state of the art see ref. [3]).

There is, however, another possible approach to the problem, which starts from the quantum theory of an isolated spherically symmetric gravitational system and then studies the coupling of such a quantum system to (quantised) matter, at least approximately. This approach might have the advantage that a considerable part of the vast gravitational gauge structure has already been dealt with. It is still in its infancy and only the first step, namely the canonical quantisation of spherically symmetric pure gravity has been achieved [4] (Thiemann and myself used Ashtekar’s framework, Kuchař [5] the usual geometrodynamical one).

One the other hand, long before this systematically derived quantum theory of isolated Schwarzschild black holes the quantisation of its energy (mass) levels had been postulated heuristically first by Bekenstein and later by Mukhanov and others in the spirit of the Bohr-Sommerfeld quantisation rules in atomic physics. The idea and the results of this approach have been discussed again in a recent letter by Bekenstein and Mukhanov [6] leading to a lively response by Ashtekar and his school [7] for which ”area” is one of the basic observables in their quantisation program in terms of loop variables.

In the following I want to indicate how the main features of Bekenstein’s and Mukhanov’s approach may be related to our (and Kuchař’s) ”canonical” results. The arguments are quite straightforward and perhaps too simple, but probably worth discussing!
2 Summary of the canonical and the Bekenstein-Mukhanov quantisation of black holes

In the following I recall only the most essential results needed for discussing the relation between the two quantum theoretical treatments of black holes mentioned above. All the details can be found in refs. [4], [5] and [6] and the literature quoted there.

The rotational symmetric line element is

\[ ds^2 = -(N(r, t) dt)^2 + q_r(r, t) (dr + N^r(r, t) dt)^2 + R^2(r, t) d\Omega^2 . \]  

(1)

The velocity of light \( c \), here put equal to 1, will be restored explicitly below. The Schwarzschild variables \( r, t \) are meant to represent their extensions to the Kruskal manifold, too. So \( r \) may run from \(-\infty \) to \( +\infty \) etc.. \( N \) is the lapse and \( N^r \) the radial shift function, \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). The manifolds are assumed to be asymptotically flat.

It is essential not to start with the Schwarzschild gauge \( R(r, t) = r, N^r = 0 \), because otherwise one misses essential elements of the canonical structure [4,5]. Solving the classical constraints leads to just one canonical pair of observables - in the sense of Dirac -, namely the mass \( M(\ldots) \) (the dots indicate canonical variables in the original phase space) and a canonically conjugate time functional \( T[\ldots] \) which may be represented as

\[ T[q_r, R] = 2 \int_{\Sigma} (1 - 2MG/R)^{-1} w(q_r, R) , \]  

\[ w(q_r, R) = [(R')^2 - q_r(1 - 2MG/R)]^{1/2} , \]

where \( R' = dR/dr \) and where \( \Sigma \) means a 1-dimensional "radial" (Cauchy) surface which extends from \( r \to -\infty \) to \( r \to +\infty \). \( T \) and \( M \) obey the Poisson brackets

\[ \{T, M\} = 1 \]  

among themselves and the following ones with the total (unreduced!) Hamiltonian \( H_{tot} \)

\[ \dot{T} = \{T, H_{tot}\} = N_+(t) + N_-(t) , \]  

\[ \dot{M} = \{M, H_{tot}\} = 0 , \]

(4)

(5)

where \( \dot{X} \equiv dX/dt \). Here \( N_-(t) = N(r \to -\infty, t), N_+(t) = N(r \to +\infty, t) \)

(I assume that there are 2 spatial asymptotic "ends" of the manifold like for
the Kruskal one. Compared to ref. [4] I have divided the Hamiltonian by 2).
The first of the last two equations shows the meaning of $T$: If $\tau_+, \tau_-$ are the
proper times at the two spatial infinities, then $\dot{\tau}_+ = N_{+}(t), \dot{\tau}_- = -N_{-}(t)$. Thus the functional $T$ represents an observable "time" and the quantity

$$\delta = T - (\tau_+ - \tau_-)$$  \hspace{1cm} (6)

has a vanishing Poisson bracket with $H_{tot}$ and therefore it is a constant!
The functional $T$ is nonvanishing only if the shift $N^r$ does not vanish [4]. So it
represents a measure for the slicing of spacetime into space and time relative
to the static (Schwarzschild) slicing for which $T = 0$! If one integrates in eq.
(2) not till $r = \infty$ but only till $r$ at the upper limit, then the resulting $T(t, r)$
can be interpreted as the Killing time associated with the system [5].
If we restrict $H_{tot}$ to the surface of the phase space where the constraints
vanish we get the reduced Hamiltonian

$$H_{red} = M(N_{+} + N_{-}) ,$$  \hspace{1cm} (7)

which results from the nonvanishing (ADM) surface terms. Because of the
Poisson brackets (3) we have $\{T, H_{red}\} = N_{+} + N_{-}$ in agreement with the
relation (4).
All this shows that in the case of spherically symmetric pure gravity the
complete elimination of the gauge degrees of freedom leads to a 1+1 dimen-
sional integrable mechanical system with the canonical coordinate $T$ and the
conjugate momentum $M$ both of which form the reduced phase space!
Quantisation of the system conveniently starts from this reduced phase space
(one could also adopt Dirac's approach, because the functional $T[\ldots]$ is an-
nihilated by the constraints if one choses an appropriate operator ordering
[4]):
First we "promote" the quantities $E = Mc^2$ and $T$ to operators $\hat{E}$ and $\hat{T}$
and the Poisson bracket (3) to the commutator

$$[\hat{E}, \hat{T}] = \hbar/i .$$  \hspace{1cm} (8)

Here we encounter a (physical) problem [4]: If $\hat{E}$ and $\hat{T}$ both are represented
by selfadjoint operators then the spectrum $\{E\}$ of $\hat{E}$ has to be the whole real
axis because the unitary operator $\exp(iT\mu/\hbar), \mu$ real, generates translations
$E \rightarrow E + \mu$. This would lead to negative masses, naked singularities and
unwanted instabilities. If, therefore, one wants the spectrum of \( \hat{E} \) to be bounded from below, \( E \geq 0 \), then \( \hat{T} \) cannot be selfadjoint. A possible way out is to define
\[
\hat{S} = \frac{1}{2}(\hat{E}\hat{T} + \hat{T}\hat{E})
\]
with \([\hat{S}, \hat{E}] = i\hbar\), (9)

because \( \exp(i\hat{S}\beta/\hbar) \) just rescales \( E \), \( E \rightarrow e^\beta E \).

Important is the Schroedinger equation: Consider an observer at \( r \rightarrow +\infty \) with the reduced Hamilton operator
\[
\hat{H}_{\text{red}} = N_+\hat{E}
\]
and with the proper time \( \tau \equiv \tau_+ \), \( \dot{\tau} = N_+ \). In the \( M \)-representation the wave function \( \varphi(M, t) \) obeys the simple Schroedinger equation
\[
\begin{align*}
&i\hbar \partial_t \varphi = \hat{H}_{\text{red}} \varphi = N_+ Mc^2 \varphi, \text{ or} \\
&i\hbar \partial_\tau \varphi(M, \tau) = Mc^2 \varphi(M, \tau),
\end{align*}
\]
with the solutions
\[
\varphi(M, \tau) = \chi(M) e^{-i\hbar Mc^2 \tau}
\]
and the scalar product
\[
(\varphi_1, \varphi_2) = \int_0^\infty dM \varphi_1^* \varphi_2.
\]

Obviously the mass spectrum is continuous and if nothing happens the isolated gravitational sytem (black hole) just ”sits” there, described by a plane wave in time!

Let me now turn to the quantisation scheme for black holes as discussed by Bekenstein and Mukhanov [6]: Here the area \( A \) enclosed by the horizon of the black hole is the starting point,
\[
A = 4\pi R_S^2, \quad R_S = 2M G/c^2.
\]

According to the old Bohr-Sommerfeld quantisation rules in atomic physics finite 2-dimensional regions \( B \) with boundary \( \partial B \) in phase should be quantised by the prescription
\[
\frac{1}{2\pi} \oint_{\partial B} p \, dq = \frac{1}{2\pi} \int_B dp \wedge dq = n \hbar, \quad n = 1, 2, \ldots
\]

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This and related arguments lead to the assumption that the area $A$ (and therefore the mass $M$) are quantised accordingly:

$$A_n = 16 \pi G^2 M_n^2/c^4 = \text{const.} \ n \ h = \alpha n l_P^2 , \ n = 1, 2, \ldots ,$$  \hspace{1cm} (16)

where $l_P = (\hbar G/c^3)^{1/2} = 1.6 \cdot 10^{-35}$ m is Planck’s length and $\alpha$ a dimensionless constant to be determined. We thus get the following quantisation of the mass:

$$M_n = \frac{1}{4} \sqrt{\frac{\alpha}{\pi}} \sqrt{n} m_P \ , \ m_P = \sqrt{c h/G} = 2.2 \cdot 10^{-8} \text{ kg} = 1.2 \cdot 10^{19} \text{ GeV}/c^2 \ .$$  \hspace{1cm} (17)

In order to determine the constant $\alpha$ the relation

$$S = \frac{k_B c^3}{4G h} A + k_B a = \frac{k_B}{4} \frac{A}{l_P^2} + k_B a$$  \hspace{1cm} (18)

between the entropy $S$ of a black hole and its area enclosed by the horizon can be used ($a$ is a normalisation constant for the entropy and $k_B$ is Boltzmann’s constant). The constants $\alpha$ and $a$ can be specified as follows: Suppose $g(n)$ is the degeneracy of the $n$th level. Then we can associate the (statistical) entropy $S = k_B \ln g(n)$ with it, or

$$g(n) = e^{S/k_B} = e^{\alpha n/4} + a .$$  \hspace{1cm} (19)

Requiring $g(n = 1) = 1$ yields $a = -\alpha/4$ and since $g(n)$ has to be an integer one concludes $\alpha = 4 \ln k, k = 2, 3, \ldots$. If one now identifies $g(n)$ with the number of possible ways one can build up the $n$th level when one starts from ”nothing” ($n=0$), then $k=2$, and therefore

$$g(n) = 2^{n-1} \ , \ \alpha = 4 \ln 2 ,$$  \hspace{1cm} (20)

so that finally

$$A_n^{BM} = 4 \ln 2 n l_P^2 \ , \ n = 1, 2, \ldots$$  \hspace{1cm} (21)

$$S_n^{BM} = k_B \ln 2 (n - 1) \ ,$$  \hspace{1cm} (22)

$$M_n^{BM} = \frac{1}{2} \sqrt{\frac{\ln 2}{\pi}} \sqrt{n} m_P \ .$$  \hspace{1cm} (23)

As the Hawking temperature at spatial infinity is given by

$$\beta_H = \frac{1}{k_B T_H} = \frac{8\pi G M}{\hbar c^3} ,$$  \hspace{1cm} (24)

the application of this formula to the quantised mass levels gives the rather strange result that the temperature is quantised, too!
3 Relating the canonical wave functions to quantised levels of the black hole

If one wants to relate the continuous mass values $M$ in the wave function (12) to the discrete mass levels (23) one has to impose additional conditions on $\varphi(M, \tau)$. The following two assumptions will achieve this:

1. Suppose the plane wave (12) represents the gravitational system only for a finite time interval $\Delta \tau \equiv \Delta > 0$, because a collapse has terminated this state more or less abruptly. Formally this property can be implemented by the requirement that the wave function $\varphi(M, \tau)$ obeys periodic boundary conditions, $\varphi(M, \Delta) = \varphi(M, 0)$, which leads to

$$c^2 M \Delta = 2\pi \hbar n, \quad n = 1, 2, \ldots ,$$

where we have assumed that the mass is positive. The last relation makes the mass spectrum discontinuous in the same way as a spatial box with the corresponding boundary conditions makes the momentum for free particles discrete.

2. The next problem is to determine the time interval $\Delta$. Intuitively one expects that it should be larger than the time the light needs to travel across the horizon, $\Delta > 2R_S/c$. Now there is a natural time period $\Delta_H$ associated with a black hole, namely that of its (imaginary) Euclidean time axis (see the reviews [1,2] and [3]):

$$\Delta_H = \hbar / \beta_H = 8\pi M G/c^3 = 4\pi R_S/c = 8\pi M \frac{m}{m_P} t_P ,$$

$$t_P = \sqrt{\hbar G/c^5} = 5.4 \cdot 10^{-44} s .$$

It is tempting to identify $\Delta$ with $\Delta_H$. By doing so we obtain

$$M_n = \frac{1}{2} \sqrt{n} m_P ,$$

$$A_n = 4\pi n l_P^2 .$$

Notice that the mass values (27) differ from those of the eq.(23) by the factor $\sqrt{\ln 2/\pi} \approx 0.47$ only!

Let me give a - not very satisfactory - plausibility argument why one may identify the period $\Delta$ in real (i.e. Lorentzian) time $t$ with the period $\Delta_H$ in
Euclidean time: A very simple derivation [9] of $\Delta_H$ starts from the Euclidean Schwarzschild line element

$$ds^2_E = (1 - R_S/r)d(ict)^2 + (1 - R_S/r)^{-1}dr^2 + r^2d\Omega^2 ,$$

which again has a coordinate singularity at $r = R_S$. But now $r \geq R_S$ always. Introducing the new coordinate

$$\rho = 2R_S(1 - R_S/r)^{1/2}$$

the Euclidean line element (29) becomes

$$ds^2_E = \rho^2d(ict/2R_S)^2 + (r/R_S)^4 d\rho^2 + r^2d\Omega^2 .$$

The first term has the form of polar coordinates in the plane with the angular variable $ict/2R_S$. If this has the period $2\pi$ then it has the period $\Delta_H = 4\pi R_S/c$. The argument no longer applies in the Lorentzian case. However, if $t$ is *periodic from the start*, then it appears plausible to attribute to $t$ the period $\Delta_H$, too! It may be worth trying!

Suppose now that $n \gg 1$. Then we have $M_{n+1} - M_n = m_P^2/(8M_n)$ and can define the frequencies $\omega_n$ by

$$\hbar\omega_n = E_{n+1} - E_n = \frac{m_P^2c^2}{8M_n} = \frac{E_P}{4\sqrt{n}} , \quad E_P = 1.2 \cdot 10^{22} \text{MeV} .$$

If we assume the frequency $\omega_n$ to be that of a gravitational or an electromagnetic wave emitted in a transition of a system with mass $M = M_{n+1}$ then this wave has the length

$$\lambda = \frac{2\pi c}{\omega_n} = 16\pi \frac{M}{m_P}l_P \approx 80 \frac{M}{M_\odot} \text{km} ,$$

or the frequency

$$\nu = c/\lambda \approx 3.8 \frac{M_\odot}{M} \text{kHz} ,$$

where $M_\odot$ is the mass of the sun ($\approx 2 \cdot 10^{30} \text{kg}$). Thus, a black hole with a mass 10 times that of the sun would emit such waves with a wave length of about 800 km or a frequency of about 380 Hz!
If we define $\beta_n$ by $\beta_n = 8\pi G M_n/(hc^3)$ (see eq. (24)) then we have the relation

$$\hbar \omega_n \beta_n = \pi ,$$

(34)

which has a superficial similarity to Wien’s displacement law $\hbar \omega_{\text{max}} \beta = 2, 82 \ldots$, with $\omega_{\text{max}}$ as the frequency where Planck’s distribution has its maximum for a given temperature $T = 1/(k_B \beta)$. Whereas Wien’s law has an essential physical content in connection with Planck’s distribution for a given temperature $T$, the relation (34) appears mainly as a consequence of the definition of $\beta_n$!

4 Discussion

We have seen that, starting from the canonical quantisation of pure spherically symmetric gravity, two assumptions, namely periodicity in time and identification of this period with the associated Euclidean time period $\Delta_H$, lead to a mass spectrum which is very similar to those postulated or derived in other approaches.

Let me comment on the periodicity assumption first: As far as I can see it is not in contradiction to the basic elements which entered into the derivation of the Schroedinger eq. (11), because the second canonical variable $T$ ("time") does not appear at all in this equation. The properties of the time functional $T$ are, of course, affected and the details of its modification still have to be worked out. They are not essential for the main arguments above, because only the form of the reduced Hamiltonian (10) is important!

As the mass spectrum is no longer continuous, the scale transformations resulting from the commutation relations (9) cannot hold anymore and the scalar product (13) has to be changed.

In this connection there is another new feature: If we have free particles in a spatial box of length $L$ then this length is fixed for all wave functions, whereas the period $\Delta$ depends on the state characterized by $n$:

$$\Delta = \Delta_n = 8\pi G M_n/c^3 = 8\pi \frac{M_n}{m_p} t_p = 4\pi \sqrt{n} t_p \approx 10^{-4} \frac{M}{M_{\odot}} s .$$

(35)
(For $M_n \approx M_\odot$ one has $\sqrt{n} \approx 10^{38}$!) Thus, the wave functions (12) take the form
\[ c_n e^{-\frac{i}{\hbar}E_P \sqrt{n} \tau}, \quad 0 \leq \tau \leq 4\pi \sqrt{n} t_P. \] (36)

Furthermore, postulating periodicity in time means that time translation invariance gets broken and formally energy is no longer conserved for the system. This, however, is intuitively obvious: Terminating a given state of an isolated system is only possible if some interactions with other systems are involved. We did not introduce such systems explicitly, but they are in the background of our time cutoff for the given system.

Next let me comment on the notion of entropy in this context: Contrary to the approach by Bekenstein and Mukhanov no properties of the entropy have to be used in order to derive the levels (27). Looking at the wave functions (36) the states they describe do not appear to be degenerate. However, this does not affect the formula (22), because the degeneracy $g(n)$ means something else, namely it counts the number of ways the level $n$ can be built up from $n = 0!$ So we still can introduce the entropy (22). However, we now have the strict inequality
\[ S_{BM}^{n} / k_B < \frac{1}{4} \frac{A_n}{l_P^4} - \frac{\alpha}{4} \] (37)
for all $n$, because of the factor $\ln 2/\pi \approx 0.22$ by which the relations (22) and (28) differ. This inequality is compatible with the general Bekenstein hypothesis [10]
\[ S / k_B \leq \frac{2\pi}{c\hbar} R E, \] (38)
where $R$ is the radius of the smallest sphere that surrounds the system with energy $E$.

If we relax the assumption $\Delta = \Delta_H$ and merely require
\[ \Delta = \gamma R_S / c, \] (39)
where $\gamma$ is of order 1 or larger, then we get instead of eq. (27)
\[ M_n = \sqrt{\frac{\pi}{\gamma}} \sqrt{n} m_P. \] (40)
For $\gamma = 2$ the interval $\Delta$ coincides with Wald’s\[11\] ”formation time” $h/\kappa = \Delta_H/2\pi$, $\kappa$: surface gravity; and for $\gamma = 3\sqrt{3}/2$ the interval $\Delta$ coincides with the characteristic luminosity attenuation time of a collapsing star\[12\]. These examples show again that one should expect $\gamma$ to be not much larger than 1!

Notice that the value of $\gamma$ affects the wavelength (33) where the factor $16\pi$ after the second equality sign has to be replaced by $4\gamma$.

Finally I would like to compare the minimal area

$$a_0 = A_1 = 4\pi l_P^2,$$  \hspace{1cm} (41)

resulting from eq. (28) with those of other authors: Bekenstein and Mukhanov have $a_0 = 4\ln 2 l_P^2$. Ashtekar and Lewandowski\[13\] derive $a_0 = \sqrt{3} l_P^2/4$.

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