Global orbit of a complicated nonlinear system with the global dynamic frequency method

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Abstract
Global orbits connect the saddle points in an infinite period through the homoclinic and heteroclinic types of manifolds. Different from the periodic movement analysis, it requires special strategies to obtain expression of the orbit and detect the associated profound dynamic behaviors, such as chaos. In this paper, a global dynamic frequency method is applied to detect the homoclinic and heteroclinic bifurcation of the complicated nonlinear systems. The so-called dynamic frequency refers to the newly introduced frequency that varies with time, unlike the usual static variable. This new method obtains the critical bifurcation value as well as the analytic expression of the orbit by using a standard five-step hyperbolic function-balancing procedure, which represents the influence of the higher harmonic terms on the global orbit and leads to a significant reduction of calculation workload. Moreover, a new homoclinic manifold analysis maps the periodic excitation onto the target global manifold that transfers the chaos discussion of non-autonomous systems into the orbit computation of the general autonomous system. That strategy unifies the global bifurcation analysis into a standard orbit approximation procedure. The numerical simulation results are shown to compare with the predictions.

Keywords
Homoclinic bifurcation, hyperbolic function, chaos

Introduction
Nonlinear oscillations have attracted the attention of many researchers in the fields of physics, applied mathematics, and engineering fields. Accordingly, effective methods have been developed to obtain the periodic solutions of the nonlinear oscillators, such as the variational iteration method, the variational approach, and the Hamiltonian approach. Apart from the periodic movement, global bifurcation involving changes in the basins of attraction, homoclinic or heteroclinic orbits, or state space are also an important part of nonlinear analysis. Homoclinic (heteroclinic) orbits that arise from the ordinary differential equations tend towards the stationary saddle point in both forwards and backwards time.

In that case, Melnikov method is the most commonly used approach to detect the critical values of homoclinic (heteroclinic) bifurcation. Derived from the unperturbed Hamilton system, it estimates when a homoclinic tangency has occurred to indicate the onset of homoclinic points and horseshoe dynamics through an integration function. Other than that integration, Belhaq et al. constructed the periodic orbit and detected a homoclinic bifurcation based on the collision of the periodic orbit with saddle as well as the infinite period of the oscillator. Zhang et al. computed the undetermined fundamental frequency of periodic solution, so that the homoclinic bifurcation appeared in terms of the vanishing frequency. Xu et al. extended the semi-analytical perturbation-
incremental method to study global bifurcations of strongly nonlinear oscillators and determined conditions under which a limit cycle was created or destroyed.

Absence of mathematical definitions to estimate the gap between stable and unstable manifolds gives rise to problems in interpreting homoclinic solutions.\(^{17}\) Thereafter, great efforts have been changed to quantitatively obtain such homoclinic (heteroclinic) orbits. To obtain certain homoclinic orbits, Vakakis and Azeez\(^{18}\) developed an iterative technique, while Mikhlin\(^{19}\) and Feng et al.\(^{20}\) used the Padé and quasi-Padé approximation. Additionally, other analytic methods, such as the Homotopy perturbation, perturbation-incremental, hyperbolic perturbation, elliptic Lindstedt-Poincaré (LP), and hyperbolic LP methods, emerged to find the homoclinic solutions of oscillators.\(^{21-25}\) For the hyperbolic perturbation method, Chen et al.\(^{22}\) derived solutions of strongly nonlinear systems using different time scale. Li and Tang\(^{26}\) presented a generalized quadratic harmonic function perturbation method to investigate the homoclinic bifurcation of a Duffing–Harmonic–Van der Pol oscillator. Wang et al.\(^{27}\) took the series form manifolds to imitate the Shilnikov-type homoclinic orbit in a 3 D system and proposed a semi-analytical direct approach to construct the heteroclinic orbit.\(^{28}\) For those non-autonomous cases, Mikhlin and Manucharyan\(^{29}\) constructed the orbit expression of the autonomous system with the Padé approximation and found the critical value of chaos.

In this paper, a new global dynamic frequency method is proposed to consider the homoclinic (heteroclinic) bifurcation of nonlinear systems. It is the global extension of the original dynamic frequency method in dealing with the periodic solutions.\(^{30}\) The new method depends upon a dynamic frequency to epitomize the effect of complicated nonlinearity, which regards the lower-order unperturbed hyperbolic system as the foundation; thus, the remaining nonlinear components become the perturbation to form the orbit expressions. For those non-autonomous systems, a series-type homoclinic manifold approximation maps the periodic excitation onto the target global manifold, which is proved to be an efficient strategy to realize the transformation from non-autonomous systems into autonomous systems. The efficiency of the method is confirmed by comparing with the numerical simulation in the application section.

This article is structured as follows: in “Proposed approaches” section, the basic ideas of homoclinic (heteroclinic) orbits and the dynamic frequency are introduced to present how to find the orbit expression and the critical value of bifurcation. In “Applications” section, more specific cases are introduced to verify the efficiency of the proposed method. Additionally, the second-order dynamic frequency approximation greatly improves the accuracy of computation. In “Chaotic movement of the non-autonomous system” section, a global manifold approximation enables the chaos discussion from the autonomous system point of view. Finally, “Discussion” section presents the key conclusions drawn based on the studies.

**Proposed approaches**

Consider the following nonlinear system

$$\ddot{u} + ku = f_1(u) + f_2(u, \dot{u})$$  \hspace{1cm} (1)

where \(f_1, f_2\) are polynomials about \(u\) and its first-order derivative \(\dot{u}\), as \(f_1(u) = \sum_{i=0}^{m} a_i u^i\) and \(f_2(u, \dot{u}) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \beta_{j,i} \dot{u}^i u^j\); \(m\) and \(n\) are integers; \(\beta_{0,1}\) can be normally regarded as the bifurcation governing parameter.

To illustrate the new approach, we need to construct the energy equation of equation (1)

$$\frac{1}{2} \ddot{u}^2 = E - \frac{1}{2} ku^2 + \int f_1(u) \dot{u} \, dt + \int f_2(u, \dot{u}) \dot{u} \, dt$$  \hspace{1cm} (2)

where \(E = E_0 + E_1(t)\) represents the total mechanical energy.\(^{31}\) For the conservative case \(f_2 = 0\), \(E = E_0\) is a constant number. As for the non-conservative case \(f_2 \neq 0\), it gives

$$\frac{dE}{dt} = \frac{dE_1(t)}{dt} = f_2(u, \dot{u}) \dot{u}$$  \hspace{1cm} (3)

and \(E_1(t)\) can be expanded in terms of \(u, \dot{u}\).
According to the constant variation method, we renew the frequency component $e$ where $e$ should be unfolded with different types of hyperbolic solution $(u, \dot{u})$, as shown in Figure 1(a) and equation (1), has a heteroclinic orbit that connects the saddles $S_{\pm}$, as shown in Figure 1(a); additionally, equation (1) also has two types of homoclinic orbits that connect the single saddle point $S_0$, as shown in Figure 1(b) and (c). We will discuss those three cases in the following sections separately.

### Heteroclinic bifurcation of the system

As shown in Figure 1(a) and equation (1), $\kappa = \omega_0^2$ and $\omega_0$ represent the fundamental frequency, and the polynomial expression $f_1(u)$ is an odd function of $u$

$$f_1(u) = \alpha_3 u^3 + e \sum_{i=2}^{m} \alpha_{2i+1} u^{2i+1}, \quad m \in 2, \ldots, N$$  

where $e$ is regarded as the perturbation and also regarded as a book-keeping parameter. Then, equation (4) becomes

$$\frac{1}{2} \dot{U}^2 = E_0 - \frac{1}{2} u^2 \omega_0^2 + \frac{1}{4} \alpha_3 u^4 + e \sum_{i=2}^{m} \frac{1}{2i+2} \alpha_{2i+1} u^{2i+2} + e \int f_2(u, \dot{u}) \dot{u} \, dt$$  

Considering $e = 0$, the system has an unperturbed hyperbolic form heteroclinic solution

$$\begin{cases} u = \pm \omega_0 \tanh \omega_{1.0} t, \\ \dot{u} = \pm \omega_0 \omega_{1.0} \text{sech}^2 \omega_{1.0} t \end{cases}$$

where the amplitude $\omega_0$ and static frequency $\omega_{1.0}$ can be obtained from numerical integration. The perturbation terms in equation (6) include $f_2(u, \dot{u})$ and also part of $f_1(u)$ for $m > 1$.

Substitute equation (7) into equation (6) and take the positive sign of $u, \dot{u}$ for brevity in the discussion. According to the constant variation method, we renew the frequency component $\omega_{1.0}^2$ on the left with a dynamic variable $\omega(t)$ to unfold $\dot{U}$ and fit the perturbed nonlinear terms signed by $e$

$$\begin{cases} \frac{1}{2} (\omega_0 \text{sech}^2 \omega_{1.0} t)^2 \dot{\omega}_{1.0}^2 = E_0 - \frac{1}{2} \omega_0^2 u^2 + \frac{1}{4} \alpha_3 u^4, \\ \frac{1}{2} (\omega_0 \text{sech}^2 \omega_{1.0} t)^2 \omega_{1.0}^2(t) = E_0 - \frac{1}{2} \omega_0^2 u^2 + \frac{1}{4} \alpha_3 u^4 + e \left( \sum_{i=2}^{m} \frac{\alpha_{2i+1} u^{2i+2}}{2(i+1)} + \int f_2(u, \dot{u}) \dot{u} \, dt \right) \end{cases}$$

which leads to the definition of the global dynamic frequency, like

$$\omega(t) = \omega_{1.0} + \sum_{i=1}^{k} \dot{\omega}_{1,i}(t), \quad k \in 1, 2, \ldots, N$$
where $\omega_{1,0}$ and $\omega_{1,k}(t)$ represent separately the static and $k$th-order dynamic frequency, so that the unperturbed hyperbolic solution in equation (7) changes to

$$
\begin{align*}
\frac{1}{2} (a_0 \text{sech}^2 \omega_{1,0} t)^2 \left[\omega_{1,0} + \sum_{j=1}^{k} \epsilon j \omega_{1,j}(t)\right]^2 &= E_0 - \frac{1}{2} \alpha_0^2 (a_0 \tan \omega_{1,0} t)^2 + \frac{1}{4} \alpha_0 (a_0 \tan \omega_{1,0} t)^4 \\
&+ \epsilon \sum_{i=2}^{m} \frac{\alpha_{ij+1,0}}{2i+2} (a_0 \tan \omega_{1,0} t)^{2i+2} + \epsilon \sum_{i=0}^{n} \frac{\beta_{ij}}{2j+2} (a_0 \tan \omega_{1,0} t)^{2j+2} + O(\epsilon^{k+1})
\end{align*}
$$

Perform the integration and collect the power series of $\epsilon$ lower than $k$ in equation (11). Then, expand and balance the hyperbolic function terms using the limit scale $(\text{sech}^2 \omega_{1,0} t)$ to eliminate such terms $(E_0, \text{sech}^2 \omega_{1,0} t, \tan \omega_{1,0} t$ or $\text{sech}^2 \omega_{1,0} t \tan \omega_{1,0} t, \text{sech}^4 \omega_{1,0} t)$ in the resulted equation with $\tan^2 \omega_{1,0} t = 1 - \text{sech}^2 \omega_{1,0} t$.

It follows a five-step balancing algorithm

$$
\begin{align*}
\text{step 1} : & \text{ constant term } \rightarrow E_0, \\
\text{step 2} : & \text{ sech}^2 \omega_{1,0} t \rightarrow a_0, \\
\text{step 3} : & \{ \tan \omega_{1,0} t, or \text{sech}^2 \omega_{1,0} t \tan \omega_{1,0} t \} \rightarrow \beta_{0,1}, \\
\text{step 4} : & \text{ sech}^4 \omega_{1,0} t \rightarrow \omega_{1,0}, \\
\text{step 5} : & \text{ remaining terms } \rightarrow \omega_{1,k}(t),
\end{align*}
$$

(12)

to decide those unknown parameters: $E_0, a_0, \beta_{0,1}, \omega_{1,0}, \omega_{1,k}(t)$. Distinguishing from the Fourier expansion in hyperbolic HB, the new approach does not terminate the balancing procedure at any $k$th-order hyperbolic term, like $(\tan k \omega_{1,0} t, \text{sech}^2 k \omega_{1,0} t)$, but uses $\omega_{1,k}(t)$ to represent those remaining terms. It highlights a specific advantage of this energy-balance approach. Meanwhile, no matter what the specific values $m$ and $n$ are in the governing equation, the algorithm is always the same as shown in equation (12). The advantage of the strategy can be found in “Applications” section.

It is clear that equation (10) only gives the implicit solutions on the phase diagram for theoretical studies. The explicit drivetrain relation between $(u, \ddot{u})$ can be permitted through the integration

$$
\theta(t) = \int \omega(t) \, dt
$$

(13)
and finally gives the standard heteroclinic solution
\[ u = a_0 \tanh \theta(t) \]  
(14)

which involves \( \theta(t) \) as a new dynamic phase component.

For more details, the first and second-order approximation solutions are taken as examples. For the first-order global dynamic frequency approximation, \( k = 1 \), equation (10) changes to
\[
\begin{cases}
  u = a_0 \tanh \omega_{1,0} t, \\
  \dot{u} = a_0 \omega_{1,0} \text{sech}^2 \omega_{1,0} t, \\
  \ddot{U} = a_0 [\omega_{1,0} + \varepsilon \omega_{1,1}(t)] \text{sech}^2 \omega_{1,0} t
\end{cases}
\]  
(15)

Substituting equation (15) into equation (11) and neglecting all those \( \varepsilon^2 \) components, it gives
\[
\begin{align*}
\frac{1}{2} \left[ (a_0 \text{sech}^2 \omega_{1,0} t)^2 [\omega_{1,0}^2 + 2 \varepsilon \omega_{0,0} \omega_{1,0}(t)] \right] &= E_0 - \frac{1}{2} \omega_{0,0}^2 (a_0 \tanh \omega_{1,0} t)^2 \\
+ \frac{1}{4} \omega_{3,0} (a_0 \tanh \omega_{1,0} t)^4 + \varepsilon \sum_{i=2}^{m} \frac{\omega_{2i+1,0}}{2i+2} (a_0 \tanh \omega_{1,0} t)^{2i+2} \\
+ \varepsilon \sum_{i=0}^{n} \sum_{j=1}^{n-i} \beta_{ij} (a_0 \tanh \omega_{1,0} t)^i (a_0 \text{sech}^2 \omega_{1,0} t)^{j+1} dt + O(\varepsilon^3)
\end{align*}
\]  
(16)

which leads to the unknown parameters \( (E_0, a_0, \beta_{0,1}, \omega_{1,0}, \omega_{1,1}(t)) \) and the first orbit expression according to equations (12) and (14).

For the second-order global dynamic frequency approximation, \( k = 2 \), equation (10) changes to
\[
\begin{cases}
  u = a_0 \tanh \omega_{1,0} t, \\
  \dot{u} = a_0 [\omega_{1,0} + \varepsilon \omega_{1,1}(t) + \varepsilon^2 \omega_{1,2}(t)] \text{sech}^2 \omega_{1,0} t
\end{cases}
\]  
(17)

Substituting equation (17) into equation (11) and neglecting all those \( \varepsilon^3 \) components, it produces the unknown variables and also the heteroclinic orbit expressions by following the similar strategies.

**Homoclinic bifurcation of the Z\textsubscript{2} symmetry system**

As shown in Figure 1(b) and equation (1), \( \kappa = -\omega_{0}^2 \) and the polynomial expression \( f_1 \) in equation (5) are still an odd function of \( u \). Then, equation (4) changes to
\[
\frac{1}{2} \dot{U}^2 = E_0 + \frac{1}{2} u^2 \omega_{0,0}^2 + \frac{1}{4} \omega_{3,0} u^4 + \varepsilon \sum_{i=2}^{m} \frac{\omega_{2i+1,0}}{2i+2} u^{2i+2} + \varepsilon \int f_2(u, \dot{u}) \dot{u} \, dt
\]  
(18)

Considering \( \varepsilon = 0 \), the system has an unperturbed hyperbolic form homoclinic solution
\[
\begin{cases}
  u = \pm a_0 \text{sech} \omega_{1,0} t, \\
  \dot{u} = \mp a_0 \omega_{1,0} \tanh \omega_{1,0} t \text{sech} \omega_{1,0} t
\end{cases}
\]  
(19)

where \( a_0 \) and \( \omega_{1,0} \) can be obtained from numerical integration. Introducing the global dynamic frequency \( \omega(t) \) as shown in equation (9), equation (19) changes to
\[
\begin{cases}
  u = \pm a_0 \text{sech} \omega_{1,0} t, \\
  \dot{u} = \mp a_0 \omega_{1,0} \tanh \omega_{1,0} t \text{sech} \omega_{1,0} t \\
  \ddot{U} = \mp a_0 \omega(t) \tanh \omega_{1,0} t \text{sech} \omega_{1,0} t
\end{cases}
\]  
(20)
Substituting equation (20) into equation (18) and only taking the positive sign of \( u \), it leads to

\[
\frac{1}{2} \left( a_0 \tanh \omega_{1,0}t \, \text{sech} \omega_{1,0}t \right)^2 \left[ \omega_{1,0} + \sum_{j=1}^{k} \epsilon \omega_{j}(t) \right]^2 = E_0 + \frac{1}{2} \omega_0^2 (a_0 \text{sech} \omega_{1,0}t)^2 + \frac{1}{4} x_{3,0} (a_0 \text{sech} \omega_{1,0}t)^4 + \epsilon \sum_{j=2}^{m} \frac{x_{2j+1,0}}{2j+2} (a_0 \text{sech} \omega_{1,0}t)^{2j+2} + \epsilon \sum_{i=0}^{n} \sum_{j=1}^{m-i} \int \beta_{ij} (a_0 \text{sech} \omega_{1,0}t)^i (-a_0 \omega_{1,0} \tanh \omega_{1,0}t \, \text{sech} \omega_{1,0}t)^{i+1} dt + O(\epsilon^{k+1})
\]

Perform the integration and collect the power series of \( \epsilon \) lower than \( k \) in equation (21). Then, balance the hyperbolic function terms according to the limit scale \( (\tanh \omega_{1,0}t \, \text{sech} \omega_{1,0}t)^j \), so that there are no such terms \( (E_0, \text{sech}^2 \omega_{1,0}t, \tanh \omega_{1,0}t, \text{tanh} \omega_{1,0}t \, \text{sech} \omega_{1,0}t, \text{tanh}^2 \omega_{1,0}t \, \text{sech}^2 \omega_{1,0}t) \) in the resulted equation. It follows a five-step algorithm

\[
\begin{align*}
\text{step 1} & : \text{constant term} \rightarrow E_0, \\
\text{step 2} & : \text{sech}^2 \omega_{1,0}t \rightarrow a_0, \\
\text{step 3} & : \{\tanh \omega_{1,0}t \text{ or } \tanh \omega_{1,0}t \, \text{sech} \omega_{1,0}t \} \rightarrow \beta_0, \\
\text{step 4} & : \text{tanh}^2 \omega_{1,0}t \, \text{sech} \omega_{1,0}t \rightarrow \omega_1, \\
\text{step 5} & : \text{remaining terms} \rightarrow \omega_{1, k}(t),
\end{align*}
\]

(22)

to directly decide the unknown parameters: \( E_0, a_0, \beta_0, \omega_1, \omega_{1, k}(t) \) and also the perturbed solution in equation (20). Moreover, a standard homoclinic solution can be written as

\[
u = \pm a_0 \text{sech} \theta(t)
\]

(23)

where \( \theta(t) \) is a dynamic phase as shown in equation (13).

**Homoclinic bifurcation of the non-Z\( _2 \) symmetry system**

As shown in Figure 1(c) and equation (1), \( \kappa = -\omega_0^2 \), but the polynomial expression \( f_1 \) has both odd and even-order nonlinear components of \( u \) in its expression, like

\[
f_1(u) = x_{2,0} u + \epsilon \sum_{i=3}^{m} x_{i,0} u^i
\]

(24)

Then, the energy equation, equation (4) changes to

\[
\frac{1}{2} \dot{U}^2 = E_0 + \frac{1}{2} u^2 \omega_0^2 + \frac{1}{3} x_{2,0} u^3 + \epsilon \sum_{i=3}^{m} \frac{x_{i,0}}{i+1} u^{i+1} + \epsilon \int f_2(u, \dot{u}) \, dt
\]

(25)

In terms of the dynamic frequency \( \omega(t) \) in equation (9), the implicit homoclinic solution gives

\[
\begin{align*}
\dot{u} &= a_0 \text{sech}^2 \omega_{1,0}t, \\
\dot{\theta} &= -2a_0 \omega_{1,0} \tanh \omega_{1,0}t \, \text{sech} \omega_{1,0}t, \\
\dot{U} &= -2a_0 \omega(t) \, \text{tanh} \omega_{1,0}t \, \text{sech}^2 \omega_{1,0}t
\end{align*}
\]

(26)
Substituting equation (26) into equation (25) leads to

\[
\frac{1}{2} (2a_0 \tanh \omega_{1,0} t \sech^2 \omega_{1,0} t) \left[ \omega_{1,0} + \sum_{i=1}^{n} \varepsilon^{i} \omega_{1,i}(t) \right]^2 = E_0 + \frac{1}{2} \omega_0^2 (a_0 \sech^2 \omega_{1,0} t)^2 \\
+ \frac{1}{3} \omega_0 (a_0 \sech^2 \omega_{1,0} t)^3 + \varepsilon \sum_{i=3}^{n} \omega_i (a_0 \sech^2 \omega_{1,0} t)^{i+1} \\
+ \varepsilon \sum_{i=0}^{n} \sum_{j=1}^{n-i} \beta_{i,j} (a_0 \sech^2 \omega_{1,0} t)^j \left( -2a_0 \omega_{1,0} \tanh \omega_{1,0} t \sech^2 \omega_{1,0} t \right)^{i+1} dt + O(\varepsilon^{k+1})
\]

(27)

Perform the integration and collect the power series of \( \varepsilon \) lower than \( k \) in equation (27). Then, expand and balance the hyperbolic function terms according to the scale \( \tanh \omega_{1,0} t \sech^2 \omega_{1,0} t \), so that there are no such terms \( E_0, \sech^2 \omega_{1,0} t, \tanh \omega_{1,0} t \) or \( \sech^2 \omega_{1,0} t \tanh \omega_{1,0} t \) or \( \tanh \omega_{1,0} t \sech^4 \omega_{1,0} t \) or \( \tanh^2 \omega_{1,0} t \sech^4 \omega_{1,0} t \) in the resulted equation. It is a five-step algorithm

\[
\begin{align*}
\text{step 1:} & \quad \text{constant term} \to E_0, \\
\text{step 2:} & \quad \sech^4 \omega_{1,0} t \to a_0, \\
\text{step 3:} & \quad \{ \tanh \omega_{1,0} t \text{ or } \tanh \omega_{1,0} t \sech^2 \omega_{1,0} t \text{ or } \tanh \omega_{1,0} t \sech^4 \omega_{1,0} t \} \to \beta_{0,1}, \\
\text{step 4:} & \quad \tanh^2 \omega_{1,0} t \sech^4 \omega_{1,0} t \to \omega_{1,0}, \\
\text{step 5:} & \quad \text{remaining terms} \to \omega_{1,k}(t)
\end{align*}
\]

(28)

to decide those unknown variables \( E_0, a_0, \beta_{0,1}, \omega_{1,0}, \omega_{1,k}(t) \). Moreover, the homoclinic solution for the system without \( Z_2 \) symmetry can be written as

\[ u = a_0 \sech^2 \theta(t) \]

(29)

where \( \theta(t) \) is a dynamic phase as shown in equation (13).

According to the discussion, the main differences between the global dynamic frequency and other analytic approaches are: (1) perform the discussion through energy-based equation and not the original differential equation; (2) balance limited hyperbolic terms to find the orbit expressions up to any order approximation without the usual termination, while the increase in the computation error from the HB method is attributed to the truncation of the higher-order nonlinear terms. Those points will be demonstrated by applying the proposed approaches to more specific cases to show the efficiency of the method.

**Applications**

In this section, different representative nonlinear cases will be examined by using the proposed approaches.

**Samples of the heteroclinic bifurcation**

Consider the heteroclinic orbit in the following ninth-order nonlinear system

\[ \ddot{u} + \omega_0^2 u = \alpha_5 u^3 + \varepsilon (\alpha_5 u^3 + \alpha_7 u^5 + \alpha_9 u^7) + \varepsilon (\beta_{0,1} - \beta_{2,1} u^2 - \beta_{0,3} u^2) \dot{u} \]

(30)
According to equation (12) and the procedure in “Heteroclinic bifurcation” section, the first-order approximation algorithm is

\[
\begin{align*}
\text{step 1:} & \quad E_0 = \frac{1}{2} a_0^2 (\omega_0^2 - \frac{1}{2} a^2 x_{3,0}) - a_0^6 \left( \frac{1}{10} a_0^4 x_{9,0} - \frac{1}{8} a_0^2 x_{7,0} - \frac{1}{6} a x_{5,0} \right), \\
\text{step 2:} & \quad \omega_0^2 = \frac{a_0^6}{c} (x_{3,0} - a_0^4 x_{5,0} - a_0^8 x_{9,0}) = 0, \\
\text{step 3:} & \quad \beta_{0,1} = \frac{a_0^2}{35} (7 \beta_{2,1} + 24 \omega_{1,0}^2), \\
\text{step 4:} & \quad \omega_{1,0}^2 = \frac{1}{2} a_0^6 (x_{3,0} + 4 a_0^4 x_{9,0} + 3 a_0^8 x_{7,0} + 2 a_0^6 x_{5,0}), \\
\text{step 5:} & \quad \omega_{1,1}(t) = -\frac{a_0^2}{60 \omega_{1,0}} (3 a_0^4 x_{7,0} + a_0^8 x_{5,0}) \text{sech}^2 \omega_{1,0} t + \frac{a_0^6}{80 \omega_{1,0}} (x_{7,0} + 4 a_0^4 x_{9,0}) \text{sech}^4 \omega_{1,0} t - \frac{a_0^8}{30 \omega_{1,0}} (6 a_0^2 \beta_{0,3} - 7 \beta_{2,1}) \text{tanh} \omega_{1,0} t - \frac{a_0^6}{10 \omega_{1,0}} x_{9,0} \text{sech}^6 \omega_{1,0} t - \frac{e}{7} a_0^2 \beta_{1,0} \beta_{0,3} \text{tanh} \omega_{1,0} \text{sech}^2 \omega_{1,0} t, \\
\end{align*}
\]

(31)

which produces the unknown variables and the heteroclinic orbit expression in equation (14). Then, in terms of \(\omega_{1,1}(t)\) and equation (16), the second-order orbit approximation result can be similarly obtained. In Appendix 1, the first-order heteroclinic solutions are presented according to different parameter groups from Table 1. The critical values of bifurcation and phase diagrams are also shown in Table 1 and Figure 2, which are compared with that of numerical simulation. It shows that the global dynamic frequency results coincide very well with the numerical simulation and performs the second approximation that may further improve the accuracy of computation.

**Samples of the \(Z_2\) symmetry homoclinic bifurcation**

Consider the homoclinic orbit in the following ninth-order nonlinear system

\[
\begin{align*}
\ddot{u} - \omega_0^2 u = x_{3,0} u^3 + \varepsilon (x_{5,0} u^5 + x_{7,0} u^7 + x_{9,0} u^9) + \varepsilon (\beta_{0,1} - \beta_{2,1} u^2 - \beta_{0,3} u^2) \dot{u},
\end{align*}
\]

(32)

According to equation (22) and the procedure in “Homoclinic bifurcation of the \(Z_2\) symmetry system” section, the first-order approximation algorithm is

\[
\begin{align*}
\text{step 1:} & \quad E_0 = 0, \\
\text{step 2:} & \quad 60 a_0^2 + 30 a_0^4 x_{3,0} + \varepsilon (20 a_0^4 x_{5,0} + 15 a_0^8 x_{7,0} + 12 a_0^8 x_{9,0}) = 0, \\
\text{step 3:} & \quad \beta_{0,1} = \frac{a_0^2}{35} (14 \beta_{2,1} + 60 a_0^2 \beta_{0,3}), \\
\text{step 4:} & \quad 30 a_0^2 x_{3,0} + \varepsilon (40 a_0^4 x_{5,0} + 45 a_0^8 x_{7,0} + 48 a_0^8 x_{9,0}) + 60 a_0^2 = 0, \\
\text{step 5:} & \quad \omega_{1,1}(t) = a_0^2 \left( \frac{2 \omega_{1,0}^2 \beta_{0,3} - \beta_{2,1}}{35} \right) \text{tanh} \omega_{1,0} t + a_0^2 \left( \frac{3 a_0^4 \beta_{0,3}}{5 \omega_{1,0}} + \frac{3 a_0^8 x_{7,0}}{8 \omega_{1,0}} + \frac{a_0^6 x_{9,0}}{6 \omega_{1,0}} \right) \text{tanh}^2 \omega_{1,0} t - \frac{1}{7} a_0^2 \beta_{0,1} \beta_{0,3} \text{tanh} \omega_{1,0} t - a_0^2 \left( \frac{2 a_0^4 \beta_{0,3}}{5 \omega_{1,0}} + \frac{a_0^6 x_{9,0}}{8 \omega_{1,0}} \right) \text{tanh}^4 \omega_{1,0} t + \frac{a_0^6 x_{9,0}}{10 \omega_{1,0}} \text{tanh}^6 \omega_{1,0} t, \\
\end{align*}
\]

(33)

which produces the unknown parameters and the first-order homoclinic orbit expression according to equation (23). Then, in terms of \(\omega_{1,1}(t)\) and equation (16), the second-order approximation results can be obtained similarly. In Appendix 2, the first-order homoclinic solutions are presented according to different initial values from Table 2.
To verify the efficiency of the method, in Table 2 and Figure 3, the critical values of bifurcation and phase diagram are presented as compared with the numerical simulation.

Samples of the non-$Z_2$ symmetry homoclinic bifurcation

Consider the homoclinic orbit in the following fifth-order nonlinear system

$$\ddot{u} - \omega_0^2 u = x_{2,0}u^2 + \varepsilon(x_{3,0}u^3 + x_{4,0}u^4 + x_{5,0}u^5) + \varepsilon(\beta_{0,1} - \beta_{2,1}u^2 - \beta_{0,3}u^2)\dot{u}$$

(34)

According to equation (28) and the procedure in the “Homoclinic bifurcation of the non-$Z_2$ symmetry system” section (under the ‘Proposed approaches’ section), the first-order approximation algorithm is

$$\begin{cases}
\text{step 1:} & E_0 = 0, \\
\text{step 2:} & 30\omega_0^2 + 20a_0x_{2,0} + \varepsilon(10a_0^3x_{5,0} + 12a_0^2x_{4,0} + 15a_0^2x_{3,0}) = 0, \\
\text{step 3:} & \beta_{0,1} = \frac{32}{77}a_0^2\omega_{1,0}^2\beta_{0,3} + \frac{8}{21}a_0^2\beta_{2,1}, \\
\text{step 4:} & 60a_0^2x_{1,0}^2 + 10a_0x_{2,0} + \varepsilon(20a_0^2x_{5,0} + 18a_0^2x_{4,0} + 15a_0^2x_{3,0}) = 0, \\
\text{step 5:} & \omega_{1,1}(t) = \frac{a_0^2}{693}(96a_0^2\beta_{0,3} - 143\beta_{2,1}) \tanh\omega_{1,0}t + \frac{4}{11}a_0^2\omega_{1,0}^5\beta_{0,3} \tanh^3\omega_{1,0}t \\
& + \frac{a_0^2}{80\omega_{1,0}}[4(5a_0^2x_{5,0} + 3a_0x_{4,0}) + 5x_{3,0}] \tanh^2\omega_{1,0}t \\
& - \frac{a_0^2}{99}(60a_0^2\beta_{0,3} - 11\beta_{2,1}) \tanh^3\omega_{1,0}t + \frac{a_0^2}{24\omega_{1,0}}x_{5,0} \tanh^6\omega_{1,0}t \\
& - \frac{a_0^2}{60\omega_{1,0}}(10a_0^2x_{5,0} + 3a_0x_{4,0}) \tanh^4\omega_{1,0}t
\end{cases}$$

(35)
which produces the unknown parameters and the first-order homoclinic orbit expression according to equation (29). Then, in terms of $x_1(t)$ and equation (28), the second-order approximation results can be obtained similarly. Appendix 3 provides the first-order homoclinic solutions according to different groups of parameter values from Table 3.

To verify the efficiency of the method, in Table 3 and Figure 4, the critical values of bifurcation and phase diagrams are presented as compared with the numerical simulation.

### Chaotic movement of the non-autonomous system

The non-autonomous systems normally have a period excitation in the form of sine or cosine functions. They belong to the local manifolds surrounding a single equilibrium, different from the global manifolds addressed in this paper. As it is known that the global bifurcation of a non-autonomous system normally leads to chaotic movement. Melnikov method is the frequently taken approach by integrating the unperturbed homoclinic (heteroclinic) orbits over infinite time.\(^6,25\) However, there are drawbacks in that method, such as the weakly nonlinear assumption and limited accuracy of computation. Therefore, some modified methods try to use the perturbed homoclinic (heteroclinic) orbit to finish the Melnikov-type integration.\(^15,24\)

Districting from the general strategies, we apparently transfer the non-autonomous system into an autonomous one, so that it minimizes the residuals through a globally optimal fit. That unifies the procedure of critical value

| Groups | System parameter values | Bifurcation value $\beta_{0,1}$ |
|--------|-------------------------|-------------------------------|
| $G_i$  | $\omega_0$  | $x_{3,0}$  | $x_{5,0}$  | $x_{7,0}$  | $x_{9,0}$  | $\varepsilon$ | $\beta_{2,1}$ | $\beta_{0,3}$ | $M_1$ | $M_2$ | $M_3$ |
| 3      | 1                      | -5                         | -5                       | -5                       | -5                       | 1                       | 2                       | 2                  | 0.376 | 0.388 | 0.376 |
| 4      | 1                      | -2                         | -2                       | -2                       | -2                       | 1                       | 3                       | 3                  | 1.110 | 1.217 | 1.141 |

$M_1$: numerical simulation; $M_2$: first-order dynamic frequency; $M_3$: second-order dynamic frequency.
computation in autonomous and non-autonomous systems and breaks the weak perturbation assumption during Melnikov integration.

Consider the homoclinic bifurcation problem in the following non-autonomous $Z_2$ system

$$\ddot{u} - \omega_0^2 u = 3_0 u^3 + \varepsilon (3_5 u^5 + 5_7 u^7 + 9_5 u^9) + \varepsilon (b_{0,1} - b_{2,1} u^2 - b_{0,3} \dot{u}^2) \dot{u} + F \cos \Omega t$$  \hspace{1cm} (36)

which includes a periodic excitation $F \cos \Omega t$ and perturbed homoclinic orbit connects the saddle point at $F = 0$ as shown in “Homoclinic bifurcation of the $Z_2$ symmetry system” section. In the case of primary (main) resonance, the excitation frequency $\Omega$ and static frequency $\omega_{1,0}$ of the system have the resonance relationship $\Omega \approx \omega_{1,0} \cdot \lambda = 1$ ($\lambda \neq 1$ for other resonance ratios).

The homoclinic bifurcation of equation (36) has a typical application in the bistable nonlinear behavior. The so-called bistable oscillator has a unique double-well restoring force potential $V(x)$, as depicted in Figure 5. It can be excited to exhibit aperiodic or chaotic vibrations between different wells (positions II and III) versus single well of monostable oscillator (position I). Accordingly, these solutions are associated with the oscillations within a single potential well and those that cross the center potential well barrier.

Similar to equation (18), the energy equation of equation (36) is

$$\frac{1}{2} \dot{U}^2 = E_0 + \frac{1}{2} \omega_0^2 \dot{u}^2 + \frac{1}{4} \omega_{1,0} u^4 + \varepsilon \sum_{i=2}^{4} \frac{\omega_{2i+1,0}}{2i+2} u^{2i+2} + F \int (\cos \omega_{1,0} t) \dot{u} \, dt$$

\hspace{1cm} + \varepsilon \int (b_{0,1} - b_{2,1} u^2 - b_{0,3} \dot{u}^2) \dot{u} \, dt$$  \hspace{1cm} (37)
To transfer the non-autonomous system into autonomous system, it requires to project the effect of underlined periodic excitation onto the target global orbit. We construct the following equivalence manifold relationship in terms of \( (u, \dot{u}) \) in equation (19), like

\[
(\cos\theta) \text{sech}\theta \tanh\theta = \left( \sum_{i=0}^{n} \Gamma_i \text{sech}^i\theta \right) \text{sech}\theta \tanh\theta, \quad n \in \mathbb{Z} \tag{38}
\]

where \( \theta = \omega_1 t \) and \( \Gamma_i \) is the unknown polynomial coefficient obtained with the Mathematica code-FindFit according to equation (38), which takes a function of any form, and then automatically searches for the parameters that yield the best global fit to the target data. Figure 6 is the fitted manifold at \( n = 9 \), which coheres with the nonlinearities in equation (36) and permits enough accuracy of computation. Then the following discussion can be performed in the same type of global manifold.

### Table 4. Comparison of the critical value obtained by different methods.

| Groups | System parameter values | Bifurcation value \( \beta_{0,1} \) |
|--------|-------------------------|-----------------------------------|
| \( G_i \) | \( \omega_0 \) \( x_{3,0} \) \( x_{5,0} \) \( x_{7,0} \) \( x_{9,0} \) \( \beta_{2,1} \) \( \beta_{0,3} \) \( F \) | \( M_1 \) \( M_2 \) \( M_3 \) \( M_4 \) |
| 7 | 1 -5 -5 -5 -5 2 2 0.04 | 0.373 0.403 0.380 0.256 |

\( M_1 \): numerical simulation; \( M_2 \): first-order dynamic frequency; \( M_3 \): second-order dynamic frequency; \( M_4 \): Melnikov method.
Equation (37) gives those unknown variables \( \left( E_0, a_0, \beta_{0,1}, \omega_{1,0}, \omega_{1,k}(t) \right) \). Meanwhile, the accuracy of the computation can be further improved by using the second-order dynamic frequency approximation, as \( \beta_{0,1} = 0.380 \) shown in Table 4. Figure 7 presents the bifurcation diagram about the critical parameter at 0.378 and the phase diagram of chaos onset from the numerical simulation.

**Discussion**

This paper mainly studies the global orbits concerning monostable and bistable systems. It can be easily extended to more complicated tristable nonlinear systems. Actually, the hyperbolic type of triple-well orbits has been analytically obtained to find the critical values of chaos through Melnikov integrations.\(^{33,34}\) Thus, people may directly implement the global dynamic frequency method to solve such tristable or even more complicated systems.

**Conclusion**

In this paper, a new global dynamic frequency approach is presented to find the homoclinic (heteroclinic) orbit and the critical value of bifurcation. With the computation of the dynamic frequency, the orbit expressions and bifurcation parameters are progressively obtained through a standard five-step hyperbolic function analysis. A series-type global manifold analysis also unifies the critical value computation between autonomous and non-autonomous systems.

The key merits of this approach can be summarized as follows: (1) the available fields of orbit computation broadly cover autonomous and non-autonomous, \( Z_2 \) and non-\( Z_2 \) symmetry nonlinear systems; (2) the orbit expression and the critical value computation concentrate on a standard hyperbolic function analysis enable the major methodology to be easily interpreted and programmed by other researchers; and (3) this presented approach also gives insight into the global dynamic analysis in other complicated engineering fields, such as the multi-stability, electronic force, and fractional-order nonlinear problems.

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Appendix 1

\[ G_1 : u(t) = 0.447214 \tanh[0.707107t + 0.282843\log(\cosh 0.707107t)] \]

and

\[ G_2 : u(t) = 0.220862 \tanh[0.71994t - 0.00815719 \tanh(0.71994t) + 0.0125423 \sech^2(0.71994t) + 0.105409 \log(\cosh(0.71994t)) ] \]
Appendix 2

\[ G_3 : u(t) = 0.538422 \tanh(0.86472t) - 0.104319 \tanh(0.86472t) \]
\[ - 0.000347974 \text{sech}^3(0.86472t) + 0.00198202 \text{sech}^6(0.86472t) \]
\[ - 0.00574704 \text{sech}^4(0.86472t) + 0.0432435 \text{sech}^2(0.86472t) \]
\[ + 0.2251 \log(\cosh(0.86472t)) + 0.0007 \tanh(0.86472t) \text{sech}^8(0.86472t) \]
\[ - 0.005 \tanh(0.86472t) \text{sech}^4(0.86472t) + 0.0098 \tanh(0.86472t) \text{sech}^2(0.86472t) \]
\[ + 0.000000347974 \tanh(0.86472t) \text{sech}^6(0.86472t) \]

and

\[ G_4 : u(t) = 0.218218 \tanh(0.724655t) - 0.022003 \tanh(0.724655t) \]
\[ - 0.000195152 \text{sech}^4(0.724655t) + 0.000121285 \text{sech}^6(0.724655t) \]
\[ - 0.000496074 \text{sech}^8(0.724655t) + 0.0479207 \text{sech}^2(0.724655t) \]
\[ + 0.0163 \log(\cosh(0.724655t)) - 0.003 \tanh(0.724655t) \text{sech}^4(0.724655t) \]
\[ - 0.00211169 \tanh(0.724655t) \text{sech}^2(0.724655t) \]

Appendix 3

\[ G_5 : u = 0.561 \text{sech}(1.007t) + 0.099 \tanh(1.128t) - 0.051 \text{sech}^2(1.128t) \]
\[ - 0.172 \log(\cosh(1.128t)) + 0.007 \tanh(1.128t) \text{sech}^2(1.128t) \]

and

\[ G_6 : u = 0.775 \text{sech}(1.029t) + 0.166 \tanh(1.27t) - 0.163 \text{sech}^2(1.27t) \]
\[ - 0.479 \log(\cosh(1.27t)) + 0.003 \tanh(1.27t) \text{sech}^4(1.27t) \]
\[ + 0.021 \tanh(1.27t) \text{sech}^2(1.27t) \]