PLANAR Mappings of subexponentially integrable distortion: integrability of distortion of inverses

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Abstract. We establish the optimal regularity for the distortion of inverses of mappings of finite distortion with logarithm-iterated style subexponentially integrable distortion, which generalizes the Theorem 1. of [J. Gill, Ann. Acad. Sci. Fenn. Math. 35 (2010), no. 1, 197–207].

1. Introduction

We say that a mapping \(f: \Omega \to \mathbb{R}^n\) in a domain \(\Omega \subset \mathbb{R}^n\) is a mapping of finite distortion, if

(i) \(f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)\),

(ii) the Jacobian determinant \(J_f(z) \in L^1_{loc}(\Omega)\) and

(iii) there is a measurable function \(K(z) : \Omega \to [1, +\infty]\) with \(K(z) < \infty\) almost everywhere such that

\[
|Df(z)|^n \leq K(z) J_f(z) \quad \text{for almost all } z \in \Omega,
\]

where \(|Df(z)|\) is the operator norm of matrix \(Df(z)\). For mappings of finite distortion, we define the distortion function by

\[
K_f(z) = \begin{cases} 
\frac{|Df(z)|^n}{J_f(z)}, & \text{if } z \in \{z \in \Omega : J_f(z) > 0\} \\
1, & \text{if } z \in \{z \in \Omega : J_f(z) = 0\},
\end{cases}
\]

then the distortion inequality (1.1) becomes

\[
|Df(z)|^n = K_f(z) J_f(z).
\]

We will limit the discussion in this paper to the planar case, i.e. \(n = 2\). In this case, since \(|Df(z)| = |f_z| + |f_{\bar{z}}|\) and \(J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2\), the distortion equality (1.2) is equivalent to the Beltrami equation

\[
\frac{\partial f(z)}{\partial \bar{z}} = \mu(z) \frac{\partial f(z)}{\partial z}
\]

where \(\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)\) and \(|\mu(z)| = \frac{K_f(z) - 1}{K_f(z) + 1}\). For more details about mappings of finite distortion, we refer the reader to [10] and the references therein.

If \(||\mu||_\infty \leq k < 1\), then the classical measurable Riemann mapping theorem tells that the Beltrami equation (1.3) admits a homeomorphic solution and other solutions are represented by composing the homeomorphic solution with holomorphic functions, see [1,4].

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When $||\mu||_\infty = 1$, the Beltrami equation (1.3) becomes degenerate. David dealt with this degenerate Beltrami equation in [5], where he generalized the measurable Riemann mapping theorem when the distortion function $K$ satisfies $\exp(pK) \in L^p_{\text{loc}}(\Omega)$ for some $p > 0$. And David also noted that it is not necessary that the distortion of $f^{-1}$ is exponentially integrable when $\exp(K_f)$ is integrable. Later, Hencl and Koskela [9] proved $K_{f^{-1}} \in L^\beta_{\text{loc}}$ where $\beta = c_0p$ with absolute constant $c_0$, under the local integrability of $\exp(pK_f)$. Based on Theorem 1.1 of [3], Gill [7] ascertained the sharp inequality $c_0 < 1$. The comprehensive statement is as follows.

**Theorem A.** Suppose $f : \Omega \to \mathbb{R}^2$ is a homeomorphic mapping of finite distortion to the Beltrami equation (1.3), with the associated distortion function $K_f$. If

$$\exp(pK_f) \in L^1_{\text{loc}}(\Omega),$$

then $f^{-1}$ is a mapping of finite distortion and the distortion function $K_{f^{-1}}$ satisfies

$$K_{f^{-1}} \in L^\beta_{\text{loc}}(f(\Omega)) \text{ for } 0 < \beta < p.$$

Moreover this result is sharp in the sense that for every $p > 0$ there are functions $f$ as above such that $K_{f^{-1}} \not\in L^p_{\text{loc}}$.

Let

$$A_{p,n}(x) = \frac{px}{1 + \log(1)(x) \log(2)(e^{-1} + x) \cdots \log(n)(e^{-1} + x)} - p,$$

where $\log(i)(x) = \log(\cdots(\log(x))\cdots)$ and $\exp(i)(x) = \exp(\cdots(\exp(x))\cdots)$ are $i$-iterated logarithm and exponent for $i = 1, 2, \cdots$. Gill in [8] generalized Theorem 1.1 of [3] to the solution $f$ to the Beltrami equation (1.3) with $\exp[A_{p,n}(K_f)] \in L^1_{\text{loc}}$. However there is no corresponding result analogous to Theorem A.

The aim of this article is to present a generalization of Theorem A. under the local integrability of $\exp[A_{p,n}(K_f)]$.

**Theorem 1.1.** Suppose $f : \Omega \to \mathbb{R}^2$ is a homeomorphic mapping of finite distortion to the Beltrami equation (1.3), with the associated distortion function $K_f$. If

$$\exp[A_{p,n}(K_f)] \in L^1_{\text{loc}}(\Omega),$$

then $f^{-1}$ is a mapping of finite distortion and the distortion function $K_{f^{-1}}$ satisfies

$$\log(n)(e^{-1} + K_{f^{-1}}) \in L^\beta_{\text{loc}}(f(\Omega)) \text{ for } 0 < \beta < p.$$

Moreover, for every $p > 0$ there are mappings that satisfy the assumption of theorem, yet fail (1.6) for $\beta = p$.

The $e^{-1}$ in functions $\log(i)(e^{-1} + x)$ means $\exp(i-1)(e)$ if there is not special announcement.

The rest of the paper is organized as follows. In section 2, we recall some basic facts about Legendre Transformation and obtain an inequality of Young type. The section 3 is devoted to the proof of Theorem 1.1.
Notation. $s \gg 1$ denotes $s$ is sufficiently large and $s \ll 1$ denotes $s$ is sufficiently small. $f(x) \lesssim g(x)$ and $f(y) \gtrsim g(y)$ mean that there exist constants $M$ and $m$ such that $f(x) \leq Mg(x)$ and $f(y) \geq mg(y)$ for suitable $x$ and $y$. $f(x) \sim g(x)$ means $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$. When concerned only with the convergence of improper integrals, we use notations $\int_{-\infty}^{\infty}$ and $\int_{0}^{\infty}$.

2. An inequality of Young type

We begin by recalling some basic facts about Legendre Transformation from [2]. Suppose function $\Phi(t)$ is convex and $\Phi''(t) > 0$ for $t \geq 0$, the Legendre Transformation of $\Phi(t)$ is

$$\Phi^*(s) = \max_{t \geq 0} \{st - \Phi(t)\} \text{ for } s \geq 0.$$ 

Directly from the definition, we obtain

$$(2.1) \quad ts \leq \Phi(t) + \Phi^*(s) \text{ for } t, s \geq 0.$$ 

Lemma 2.1. $(\Phi^*)'(s) = (\Phi')^{-1}(s)$.

Proof. Given $s \geq 0$, let $t(s)$ be the value such that the maximal of $st - \Phi(t)$ is obtained, i.e.

$$\Phi^*(s) = st(s) - \Phi(t(s)),$$

so $\Phi'(t(s)) = s$, then $t(s) = (\Phi')^{-1}(s)$. Consequently, we have

$$(\Phi^*)'(s) = t(s) + (s - \Phi'(t(s))) \frac{d(\Phi')^{-1}(s)}{ds} = (\Phi')^{-1}(s).$$

□

Given a strictly convex $C^2$ function $\Phi(t)$, it is not easy to compute the explicit expression of $\Phi^*(s)$ from the definition. However, by Lemma 2.1, we can obtain the asymptotical behaviour of $\Phi^*(s)$ as $s \gg 1$. The following example, coming from [6], illustrates this.

Example 2.2. Put $\Phi(t) = \exp\left(\frac{t}{\log(1 + t)}\right)$. After differentiating and taking the logarithm, we have

$$\log(\Phi'(t)) \sim \frac{t}{\log(t)} \text{ as } t \gg 1.$$ 

Let $\frac{t}{\log(t)} = \log(s)$, then

$$(2.2) \quad t \sim \log(s) \log_2(s) \quad \text{as } s \gg 1.$$ 

In other words, when $t$ satisfies (2.2), we have

$$\log(\Phi'(t)) \sim \log(s) = \log[\Phi'(\Phi^*(s))] = \log(\Phi'(\Phi^*(\log(s))))].$$

By the monotonicity of $\log(\cdot)$ and $\Phi'(\cdot)$, we have

$$(\Phi^*)'(s) \sim \log(s) \log_2(s).$$

Hence, by the Newton-Leibniz formula, we show

$$\Phi^*(s) \sim s \log(s) \log_2(s) \quad \text{as } s \gg 1.$$ 

By the method analogous to Example 2.2, we present an inequality of Young type, which plays the crucial role in the proof of Theorem 1.1.
Lemma 2.3. Given $\beta > 0$, there exist constants $C_1, C_2 > 0$ such that

$$ts \leq C_1 \Phi(t) + C_2 \Psi(s) \quad \text{for } t, s \geq 0,$$

where $\Phi(t) = \exp[A_{p,n}(\exp(n)(t^\beta))]$ and $\Psi(s) = s \left[\log(n+1) \left( e^{e^{\cdots e^s}} + s \right) \right]^\beta$.

Proof. We divide the proof into two cases.

Case 1: $0 \leq s \leq C_1$ for some $C_1 > 0$.

Since $t \leq \Phi(t)$ for $t \geq 0$ and $\Psi(s) \geq 0$ for $s \geq 0$, we obtain

$$st \leq C_1 \Phi(t) + \Psi(s) \quad \text{for } t \geq 0 \text{ and } 0 \leq s \leq C_1.$$

Case 2: $s \gg 1$.

Since

$$\log \Phi'(t) \sim \frac{\exp(n)(t^\beta)}{\exp(n-1)(t^\beta) \cdots \exp(1)(t^\beta)t^\beta} \quad \text{for } t \gg 1,$$

and

$$\exp(n-2)(t^\beta) \cdots \exp(1)(t^\beta)t^\beta < \exp(n-1)(t^\beta) \quad \text{for } t \geq 0,$$

we have

$$\frac{\exp(n)(t^\beta)}{\exp(n-1)(t^\beta)} > \log \Phi'(t) > \frac{\exp(n)(t^\beta)}{\left[\exp(n-1)(t^\beta)\right]^2} \quad \text{for } t \gg 1.$$

Next consider RHS of (2.4). Let $b = \exp(n)(t^\beta)$, we consider

$$\frac{b}{[\log(b)]^2} = \log(s), \quad \text{i.e.} \quad \frac{b^\frac{1}{2}}{\log(b^\frac{1}{2})} = \sqrt{4 \log(s)}.$$

By Example 2.2, we have

$$b^\frac{1}{2} \sim \sqrt{\log(s) \log(2)(s)} \quad \text{as } s \gg 1.$$

Taking $n$ successive logarithms, we have

$$t \sim \left[\log(n+1)(s)\right]^\beta \quad \text{as } s \gg 1.$$

In other words, when $t$ satisfies (2.6), it follows from (2.5) and RHS of (2.4) that

$$\log(\Phi'(t)) > \log(s) = \log[\Phi'((\Phi^*)(s))] \quad \text{as } s \gg 1.$$

So, by the monotonicity of $\log(\cdot)$ and $\Phi'(\cdot)$, we have

$$\left(\Phi^*\right)'(s) \lesssim \left[\log(n+1)(s)\right]^\beta \quad \text{as } s \gg 1.$$

For LHS of (2.4), by the argument similar to the one used in RHS of (2.4), we obtain

$$\left(\Phi^*\right)'(s) \gtrsim \left[\log(n+1)(s)\right]^\beta \quad \text{as } s \gg 1.$$

Combining (2.7) and (2.8), we obtain

$$\left(\Phi^*\right)'(s) \sim \left[\log(n+1)(s)\right]^\beta \quad \text{as } s \gg 1.$$

Hence, by the Newton-Leibniz formula, we get

$$\Phi^*(s) \sim s[\log(n+1)(s)]^\beta < \Psi(s) \quad \text{as } s \gg 1.$$
It follows from (2.1) and (2.9) that there exists constant $C_2 > 0$ such that
\begin{equation}
(2.10) \quad ts \leq \Phi(t) + C_2 \Psi(s) \quad \text{for } t \geq 0 \text{ and } s \gg 1.
\end{equation}

Combining (2.3) and (2.10), we complete the proof. \hfill \Box

3. Proof of Theorem 1.1

We begin with four lemmas.

**Lemma 3.1** ([12], Theorem 1.1.). Suppose that $\Psi$ is a strictly increasing, differentiable function and satisfies
\begin{enumerate}[(A–1)]
  \item \[ \int_1^\infty \frac{\Psi'(t)}{t} \, dt = \infty, \]
  \item \[ \lim_{t \to \infty} t \Psi'(t) = \infty. \]
\end{enumerate}

Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion and the distortion function $K_f$ satisfies $\exp(\Psi(K_f)) \in L^1_{\text{loc}}(\Omega)$. Then $f$ satisfies the Lusin’s condition (N), i.e. $f(E)$ has Lebesgue measure zero if $E$ has Lebesgue measure zero.

Given a mapping $f : \Omega \to \mathbb{R}^n$, we denote $N(f, \Omega, y)$ by the number of preimages of point $y$ in $\Omega$ under $f$. We say $f$ has essentially bounded multiplicity, if $N(f, \Omega, y)$ is bounded for a.e. $y \in \mathbb{R}^n$.

From the proof of Theorem 1.2 in [13], we know the assertion of Theorem 1.2 in [13] remains valid if both the mapping and its distortion function lie in local Sobolev spaces. So, we have the following result.

**Lemma 3.2.** Let $f : \Omega \to \mathbb{R}^2$ be a mapping of finite distortion and the distortion function $K_f$ satisfies $K_f \in L^1_{\text{loc}}(\Omega)$. If $f$ has essentially bounded multiplicity and $f$ is not a constant, then $J_f > 0$ almost everywhere in $\Omega$.

Suppose that a function $A$ has the properties:

\begin{enumerate}[(A–1)]
  \item $A : [1, \infty) \to [0, \infty)$ is a smooth increasing function with $A(1) = 0$.
  \item \[ \int_1^\infty \frac{A(t)}{t^2} \, dt = \infty. \]
\end{enumerate}

The associated function of $A$ is denoted by
\begin{equation}
(P(3.1)) \quad P(t) = \begin{cases}
  t^2, & 0 \leq t \leq 1 \\
  \frac{t^2}{A^{-1}(\log t^2)}, & t \geq 1.
\end{cases}
\end{equation}

Let us recall the notation
\begin{equation}
W^{1, Q}_{\text{loc}} = \left\{ f \in W^{1, 1}_{\text{loc}}(\Omega) : Q(|Df|) \in L^1_{\text{loc}}(\Omega) \right\}.
\end{equation}

**Lemma 3.3** ([4], Theorem 20.5.1.). Given a function $A$ satisfying (A–1) and (A–2) and the associated function $P$ is defined by (3.1). Let $f : \Omega \to \mathbb{R}^2$ be a mapping of finite distortion and the distortion function $K_f$ satisfies $\exp(A(K_f)) \in L^1_{\text{loc}}(\Omega)$, then
\begin{equation}
\mathcal{A}_{p,n} \in W^{1, P}_{\text{loc}}(\Omega).
\end{equation}

Obviously, $\mathcal{A}_{p,n}$ satisfies (A–1) and (A–2). We denote the associated function of $\mathcal{A}_{p,n}$ by $P_n$. Next we present a lemma essentially due to Gill [8].
Lemma 3.4. Suppose \( f \in W^{1,p_n}_\text{loc}(\Omega) \) is a solution to the Beltrami equation (1.3) in a domain \( \Omega \subset \mathbb{R}^2 \) and the distortion function \( K_f(z) \) satisfies \( \exp[A_{p,n}(K_f(z))] \in L^1_{\text{loc}}(\Omega) \), then for all \( 0 < \beta < p \), we have
\[
J_f \left[ \log(n+1) \left( e^{e^{-e}} + J_f \right) \right] \beta \in L^1_{\text{loc}}(\Omega).
\]

We now prove Theorem 1.1.

Proof. Since \( A'_{p,n}(x) \gtrsim \frac{1}{\log(1)(x) \log(2)(x) \cdots \log(n)(x)} \) as \( x \gg 1 \), we know \( A_{p,n}(x) \) satisfies (C - 1) and (C - 2). It follows from Lemma 3.1 that \( f \) satisfies the Lusin's condition (N).

Since \( x \lesssim \exp(A_{p,n}(x)) \) for \( x \geq 1 \), it follows from (1.5) that
\[
K_f \in L^1_{\text{loc}}(\Omega).
\]

So, Lemma 3.2 tells us \( J_f > 0 \) almost everywhere in \( \Omega \).

Given compact set \( \widetilde{M} \subset f(\Omega) \), we have \( M = f^{-1}(\widetilde{M}) \subset \Omega \) is a compact set. By Corollary 3.3.3 in [4], we obtain \( f \) is differentiable almost everywhere in \( \Omega \). So, we can divide the set \( M \) into two subsets \( M' \) and \( M'' \), where \( M' \) is the subset in which \( f \) is differentiable and \( J_f(z) > 0 \) and \( M'' = M \setminus M' \) has Lebesgue measure zero. For any \( z \in M' \), by Lemma A.29 of [10], we have
\[
Df^{-1}(f(z)) = (Df(z))^{-1},
\]
so \( |Df^{-1}(f(z))|^2 J_f(z) = K_f(z) \) and \( K_{f^{-1}}(f(z)) = K_f(z) \). So, it follows from Corollary A.36 (c) of [10] and the Lusin's condition (N) of \( f \) that

\[
\int_M |Df^{-1}(w)|^2 \, dw = \int_M K_f(z) \, dz
\]
and
\[
\int_M \left[ \log(n) \left( e^{e^{-e}} + K_{f^{-1}} \right) \right] \beta \, dw = \int_M \left[ \log(n) \left( e^{e^{-e}} + K_f \right) \right] \beta J_f \, dz.
\]

By (3.2) and \( J_{f^{-1}} \leq |Df^{-1}|^2 \), it follows from (3.3) that \( J_{f^{-1}} \in L^1_{\text{loc}}(f(\Omega)) \). So by Theorem 3.3 of [9], we have \( f^{-1} \) is a mapping of finite distortion.

Next we prove (1.6). Because of (3.4), it suffices to prove
\[
\int_M \left[ \log(n) \left( e^{e^{-e}} + K_f(z) \right) \right] \beta J_f(z) \, dz < \infty
\]
for any compact set \( M \subset \Omega \). Let
\[
s = J_f(z) \quad \text{and} \quad t = \left[ \log(n) \left( e^{e^{-e}} + K_f(z) \right) \right] \beta.
\]
Since

\[ A_{p,n}(\exp_n(t^{1/2})) \leq A_{p,n}(K_f(z)) + p \left( e^{e^{-e}} - 1 \right), \]

it follows from Lemma 2.3 that there exist constants \( C' \) and \( C'' \) such that

\[ (3.6) \quad ts \leq C' \exp[A_{p,n}(K_f)] + C'' J_f \left[ \log_{(n+1)} \left( e^{e^{-e}} + J_f \right) \right]^\beta. \]

Note that \( A_{p,n}(x) \) satisfies (A–1) and (A–2) conditions, then Lemma 3.3 implies

\[ f \in W^{1,p}_{loc}(\Omega), \]

where \( P_n \) is the associated function of \( A_{p,n} \). So, it follows from Lemma 3.4 that

\[ (3.7) \quad J_f \left[ \log_{(n+1)} \left( e^{e^{-e}} + J_f \right) \right]^\beta \in L^{1}_{loc}(\Omega). \]

Hence, according to (3.6), (1.5) and (3.7), (3.5) is proved.

To show Theorem 1.1 is sharp, as in Theorem 4 of [8], we consider Kovalev–type function \( h \) in \( \Omega = \mathbb{D} \) as

\[ (3.8) \quad h(z) = \frac{z}{|z|} \rho(|z|) \]

where \( \rho(t) = \left[ \log_{(n+1)} \left( e^{e^{-e}} + \frac{1}{t} \right) \right]^{-\frac{1}{2}} \left[ \log_{(n+2)} \left( e^{e^{-e}} + \frac{1}{t} \right) \right]^{-\frac{1}{2}} \) and both of \( e^{e^{-e}} \) mean \( \exp((n+1)(e)) \). For the reader’s convenience, we carry out the main computation. By (3.4), it’s enough to check

\[ (3.9) \quad J_h \left[ \log_{(n)} \left( e^{e^{-e}} + K_h \right) \right]^{\rho} \notin L^{1}_{loc}(\mathbb{D}). \]

From the definition of \( h \), it’s sufficient to consider \( h \) in the small enough neighbourhood of 0. So with the formulas in section 6.5.1 of [11], when \( |z| \ll 1 \), we have

\[ (3.10) \quad J_h(z) \sim \frac{1}{|z|^2} \log_{(1)} \left( \frac{1}{|z|} \right) \cdots \log_{(n)} \left( \frac{1}{|z|} \right) \left[ \log_{(n+1)} \left( \frac{1}{|z|} \right) \right]^{-p-1} \left[ \log_{(n+2)} \left( \frac{1}{|z|} \right) \right]^{-1} \]

and

\[ K_h(z) = \frac{\rho(|z|)}{|z| \rho'(|z|)} \sim \log_{(1)} \left( \frac{1}{|z|} \right) \log_{(2)} \left( \frac{1}{|z|} \right) \cdots \log_{(n+1)} \left( \frac{1}{|z|} \right). \]

Since

\[ \log \left( e^{e^{-e}} + K_h(z) \right) \sim \log(K_h(z)) \sim \log_{(2)} \left( \frac{1}{|z|} \right) \text{ as } |z| \ll 1, \]

we get

\[ (3.11) \quad \left[ \log_{(n)} \left( e^{e^{-e}} + K_h \right) \right]^{\rho} \sim \left[ \log_{(n+1)} \left( \frac{1}{|z|} \right) \right]^{\rho} \text{ as } |z| \ll 1. \]
Combining (3.10) and (3.11), we obtain

\[ J_h \left[ \log(n) \left( e^{e^{\cdots e} + K_h} \right) \right]^p \sim \frac{1}{|z|^2 \log(1)(\frac{1}{|z|})} \cdots \frac{1}{\log(n+2)(\frac{1}{|z|})}. \]

Now, (3.9) is obtained from

\[
\int_0^1 \frac{1}{t \log(1)(\frac{1}{t})} \cdots \frac{1}{\log(n+2)(\frac{1}{t})} \, dt \\
= \int_1^{+\infty} \frac{1}{s \log(1)(s)} \cdots \frac{1}{\log(n+2)(s)} \, ds \\
= \cdots = \int_1^{+\infty} \frac{1}{\log(x)} \, dx = \infty.
\]

The proof is complete. \[\Box\]

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