Some results on thermopiezoelectricity of nonsimple materials

Michele Ciarletta\textsuperscript{a}, Martina Nunziata\textsuperscript{b}, Francesca Passarella\textsuperscript{b}, Vincenzo Tibullo\textsuperscript{b,\ast}

\textsuperscript{a}Dipartimento di Ingegneria Civile, Università di Salerno,
\textsuperscript{b}Dipartimento di Matematica, Università di Salerno,

Abstract

In this paper, we consider the linear theory for a model of a thermopiezoelectric nonsimple material adopting the entropy production inequality proposed by Green and Laws as presented in \cite{1}. We establish reciprocity theorems and a variational principle for homogeneous and anisotropic thermopiezoelectric nonsimple materials with a center of symmetry. The proof of these theorems use the time convolution product and an alternative formulation of the field equations. Moreover, a uniqueness result is established without using the definiteness assumptions on internal energy.

Keywords: thermopiezoelectricity, nonsimple materials, Green & Laws, variational principle, reciprocity theorem, uniqueness

This paper is dedicated to prof. Brian Straughan, a great researcher but above all a good friend.

1. Introduction

In Passarella and Tibullo \cite{1}, the authors derived a theory for a thermopiezoelectric body in which the second gradient of displacement field and the second gradient of the electric potential are included in the set of independent constitutive variables. They obtained the thermodynamic restrictions and constitutive equations by using the entropy production inequality proposed by Green and Laws \cite{2}.

The theory proposed by Green and Laws is one of the theories that predict a finite velocity for the propagation of thermal signals (see the reviews of Chandrasekharaiah \cite{3, 4}). As shown by Ieşan \cite{5}, they make use of an entropy inequality in which a new constitutive function appears with the role of thermodynamic temperature (see e.g. Passarella et al. \cite{6}). In addition to the finite velocity of heat waves, this theory also results in a symmetric heat conductivity tensor.

By the other end, the origin of the theory of nonsimple elastic materials goes back to the works of Toupin \cite{7, 8}, and Mindlin \cite{9}. Toupin and Gazis \cite{10} applied the general theory of materials of grade 2 to the problem of surface deformations of a crystal. They showed that initial stress and hyperstress in a uniform crystal give rise to a deformation of a thin boundary layer near a free surface such as that observed in electron diffraction experiments.

Strain gradient theory of thermoelasticity was first presented in Ahmadi and Firoozbakhsh \cite{11}, Batra \cite{12}. The gradient theory of elasticity becomes important because it is adequate to investigate problems related to size effects and nanotechnology. In the regime of micron and nano-scales, experimental evidence and observations have suggested that classical continuum theories do not suffice for an accurate and detailed description of corresponding deformation phenomena. The theory of nonsimple thermoelastic materials has been discussed in various papers (see for example \cite{13–21}).

Furthermore, Kalpakides and Agiasofitou \cite{17} have established a theory of electroelasticity including both strain gradient and electric field gradient. They report that taking into account of the second spatial gradient of the motion makes sense especially in crack problems, moreover taking into account of second gradient of the electrical potential implies the presence of quadrupole polarization into the continuum model, of practical interest for problems concerning surface effects.

The problem of the interaction of the electromagnetic field with the motion of elastic solids was the subject of important investigations (see e.g. \cite{22–28} and the literature cited therein). Certain crystals (for example quartz) when subject to stress, become electrically polarized (piezoelectric effect). Conversely, an external electromagnetic field can produce deformation in a piezoelectric crystal.

In section 2, we begin by summarizing the fundamental equations based on the linear theory of thermopiezoelectric nonsimple materials as established in Passarella and Tibullo \cite{1} and we consider in particular the case of center-symmetric materials. In section 3, we define a mixed initial-boundary value problem under non-homogeneous initial conditions and present a characterization of the mixed initial-boundary value problem in an alternative
way, by including the initial conditions into the field equations. In section 4, starting from a reciprocity relation which involves two processes at different times, a uniqueness result is established without using the definiteness assumptions on internal energy. Moreover, a reciprocity theorem is presented. In sections 5 and 6, another reciprocity theorem based on the convolution product and a variational principle are derived (see also [22]).

2. Basic equations

We consider a body that at some instant occupies the region $B$ of the Euclidean three-dimensional space and is bounded by the smooth surface $\partial B$. The motion of the body is referred to the reference configuration $B$ and to a fixed system of rectangular Cartesian axes $Ox_i$ ($i = 1, 2, 3$).

We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In what follows we use a superposed dot to denote partial differentiation with respect to the time $t$.

As already done by Passarella and Tibullo [1], we consider the linear theory for a model of a thermopiezoelectric material, so that the constitutive equations defined on $B \times I$ (with $I = [0, \infty)$) are

$$
\begin{align*}
\tau_{ij} &= a_{ijkl}^{(11)} e_{kl} + a_{ijkl}^{(17)} V_{kl} + a_{ijkl}^{(14)} (\theta + \beta \dot{\theta}) , \\
\mu_{ijk} &= a_{ijkl}^{(22)} \kappa_{ijkl}^{(22)} + a_{ijkl}^{(23)} E_i , \\
-\sigma_i &= a_{ijkl}^{(23)} \kappa_{ijkl}^{(23)} + a_{ijkl}^{(33)} E_j , \\
Q_{ij} &= a_{ijkl}^{(17)} e_{kl} + a_{ijkl}^{(77)} V_{kl} + a_{ijkl}^{(47)} (\theta + \beta \dot{\theta}) , \\
-\rho \dot{q}_i &= a_{ijkl}^{(14)} e_{ij} + a_{ijkl}^{(47)} V_{ij} + c (\theta + \beta \dot{\theta}) ,
\end{align*}
$$

with $\beta' = \beta + \gamma/(c \beta)$, $\beta \neq 0$, $c \neq 0$. We assume that the coefficients in (7) satisfy the following symmetry relations

$$
\begin{align*}
a_{ijkl}^{(11)} &= a_{jikl}^{(11)} = a_{ijkl}^{(11)} , \\
a_{ijkl}^{(14)} &= a_{ijkl}^{(14)} , \\
a_{ijkl}^{(22)} &= a_{jikl}^{(22)} = a_{ijkl}^{(22)} , \\
a_{ijkl}^{(23)} &= a_{ijkl}^{(23)} , \\
a_{ijkl}^{(33)} &= a_{ijkl}^{(33)} , \\
a_{ijkl}^{(77)} &= a_{ijkl}^{(77)} , \\
a_{ijkl}^{(47)} &= a_{ijkl}^{(47)} .
\end{align*}
$$

Furthermore, it is

$$
e_{ij} = e_{ji} , \quad \kappa_{ijk} = \kappa_{ikj} , \quad V_{ij} = V_{ji} , \quad \tau_{ij} = \tau_{ji} , \quad \mu_{ijk} = \mu_{kji} , \quad Q_{ij} = Q_{ji} .$$

The dissipation inequality implies that the quadratic form $\mathcal{P}$ is positive semi-definite, i.e.

$$\mathcal{P}(\xi, \eta) = \frac{\gamma}{\beta} \xi^2 + k_{ij} \eta_i \eta_j \geq 0 , \quad \forall \xi, \eta .$$

The inequality (8) is equivalent to

$$\frac{\gamma}{\beta} \geq 0 , \quad k_{ij} \eta_i \eta_j \geq 0 , \quad \forall \eta_i .$$

It results that the tensor $k_{ij}$ is positive semi-definite.

3. Mixed initial-boundary value problem

Now, we denote with $\Pi$ the mixed initial-boundary value problem defined by eqs. (1)-(5) with the restriction $\mathbf{B}$, the following initial conditions

$$u_i(0) = u_i^0 , \quad \dot{u}(0) = v_i^0 , \quad \theta(0) = \theta^0 , \quad \eta(0) = \eta^0 ,$$

in $\bar{B}$ and the following boundary conditions

$$u_i = \bar{u}_i \text{ on } S_1 \times I , \quad P_i = \bar{P}_i \text{ on } \Sigma_1 \times I ,$$
$$D u_i = \bar{D}_i \text{ on } S_2 \times I , \quad R_i = \bar{R}_i \text{ on } \Sigma_2 \times I ,$$
$$\theta = \bar{\theta} \text{ on } S_3 \times I , \quad q = \bar{q} \text{ on } \Sigma_3 \times I ,$$
$$\varphi = \bar{\varphi} \text{ on } S_4 \times I , \quad \Lambda = \bar{\Lambda} \text{ on } \Sigma_4 \times I ,$$
$$D \varphi = \bar{\varphi} \text{ on } S_5 \times I , \quad H = \bar{H} \text{ on } \Sigma_5 \times I ,$$

where $\bar{u}_i$ and $\bar{\varphi}$ are given.
where, denoted by $n_i$ the outward unit normal vector to the boundary surface $\partial B$, $q$ is the heat flux. i.e. $q = q_i n_i$, and $\{S_i, \Sigma_i\}$ are a subset of $\partial B$ such that, considering the closure relative to $\partial B$,

$$S_i \cup \Sigma_i = \partial B \quad \text{and} \quad S_i \cap \Sigma_i = \emptyset, \quad i = 1, \ldots, 5,$$

and we have

$$P_i = (\tau_{ji} - \mu_{i,jk,k})n_j - D_i(\mu_{i,jk,k}n_k) + (D_{i,n})\mu_{i,jk,k}n_j,$$

$$\Lambda = (\sigma_{ij} - Q_{ijk})n_j - D_i(Q_{ijk,n_k}) + (D_{i,n})Q_{ijk,n_k},$$

and

$$R_i = \mu_{i,jk,k}n_kn_j, \quad \Lambda = Q_{ijk,n_k},$$

where $D \equiv n_i \partial / \partial n_i$ is the normal derivative operator and $D_i \equiv (\delta_{ij} - n_i n_j) \partial / \partial n_j$ the surface gradient operator. We can prove that the functions $\mathbf{P}_i$, $R_i$, $\Lambda$ and $H$ are such that, for different times $t, s \in I$,

$$\int_{\partial B} [(\tau_{ji}(r) - \mu_{i,jk,k}(r))u_i(s) + \mu_{i,jl}(r)u_i(t)]\, n_j\, da = \int_{\partial B} [P_i(r)u_i(s) + R_i(r)\, D u_i(s)]\, da,$$

$$\int_{\partial B} \left[(\sigma_{ij}(r) - \sigma_{i,j})(r)\right] \varphi(s) + Q_{ijk}(r)\varphi_i(s)\right)n_j\, da = \int_{\partial B} [\Lambda(r)\varphi(s) + H(r)\, D \varphi(s)]\, da,$$

(12)

All right-hand terms in eqs. (10) and (11), along with $f_i$, $g$ and $h$ are the given data of the considered mixed initial-boundary value problem II and are prescribed continuous functions. We denote the given data by

$$\Gamma = \left(f_i, g, h, \vartheta_i, \eta_i, \hat{\xi}_i, \hat{\theta}_i, \hat{\varphi}_i, \hat{\omega}_i, P_i, \hat{Q}_i, \hat{\eta}, \hat{\hat{H}} \right).$$

Let us define an ordered array of functions

$$\pi = (u_i, \theta, \varphi, e_{ij}, \kappa_{ijk}, \beta_{i}, E_{ij}, \tau_{ij}, \mu_{ijk}, \sigma_{ij}, Q_{ijk}, \eta, q_{i})$$

as an admissible process on $B \times I$ with the following properties

1. $u_i \in C^{4,2}(\bar{B} \times I), \varphi \in C^{4,0}(\bar{B} \times I), \theta \in C^{2,2}(\bar{B} \times I), e_{ij}, V_{ij} \in C^{2,1}(\bar{B} \times I), \eta, \hat{\xi}_i, \hat{\theta}_i, \hat{\varphi}_i, \hat{\omega}_i, P_i, \hat{Q}_i, \hat{\eta}, \hat{\hat{H}} \in C^{4,0}(\bar{B} \times I)$;

2. $e_{ij} = e_{ji}, \kappa_{ijk} = \kappa_{jki}, V_{ij} = V_{ji}, \tau_{ij} = \tau_{ji}, \mu_{ijk} = \mu_{jki}, Q_{ijk} = Q_{jik}$ on $B \times I$.

We say that $\pi$ is a process corresponding to the supply terms $(f_i, g, h)$ if $\pi$ is an admissible process that satisfies the fundamental system of field equations (11)-(6) with the restriction (8) on $B \times [0, \infty)$.

Then, if a process $\pi$ satisfies the initial conditions (10) and the boundary conditions (11), we identify it as a solution of the mixed initial-boundary value problem II.

The set $\mathcal{V}$ of all admissible processes on $B \times I$ can be considered as a vector space. We denote by $\mathcal{K} \subseteq \mathcal{V}$ the set of all solution of the mixed initial-boundary value problem in concern.

Following Gurtin [30], we will give an alternative formulation of the problem (11)-(15) in which the initial conditions (10) are incorporated into the field equations. To this aim, we introduce the product of convolution as follows

$$[f_1 * f_2](t) = \int_0^t f_1(\tau)f_2(t - \tau)\, d\tau, \quad \forall t \in I$$

for any two continuous functions $f_1$, $f_2$, on $B \times I$. It is useful to introduce

$$l(t) = 1, \quad \xi(t) = [l * l](t) = t, \quad \zeta(t) = \frac{1}{\beta} e^{-t/\beta}, \quad \chi(t) = \frac{1}{\beta'} e^{-t/\beta'},$$

with $\beta' \neq 0$, and we note in particular that

$$[l * f](t) = \int_0^t f(\tau)\, d\tau.$$

From these definitions, we easily obtain

$$\xi \ast \tilde{u}_i = u_i - v_i^0 t - v_i^0,$$

$$l \ast \hat{\theta} = \theta - \theta^0, \quad l \ast \eta = \eta - \eta^0,$$

$$\zeta \ast (\theta + \beta \tilde{\theta}) = \theta - \theta^0 e^{-t/\beta},$$

$$\chi \ast (\theta + \beta' \tilde{\theta}) = \theta - \theta^0 e^{-t/\beta'},$$

(13)

so that we can prove the following Lemma.

**Lemma 1.** Let $\pi \in \mathcal{V}$, then, $\pi$ satisfies eqs. (11), (2), (3), (6) and the initial conditions (10) if and only if

$$\xi \ast (\tau_{ij} - \mu_{i,jk,k}) - \rho u_i + F_i = 0,$$

$$\xi \ast (\sigma_{ij} - Q_{ijk} - g) = 0, \quad \rho T_0 \eta + l * q_{i} = H_i = 0,$$

$$\zeta \ast (\tau_{ij} - \beta_{i}) - a_{ij}^{(1)} \theta + L_{ij} = 0,$$

$$\chi \ast (\mu_{ijk} - \beta_{ijk}) = 0,$$

$$\zeta \ast (\sigma_{ij} - \sigma_{ij}) = 0,$$

(14)

$$\zeta \ast \left(Q_{ij} - Q_{ij}\right) - a_{ij}^{(47)} \theta + M_{ij} = 0,$$

$$\chi \ast (\rho q_i - \rho \eta) = c \theta + R_i = 0,$$

$$- \frac{1}{T_0} l \ast (q_i - q_{i}) = 0,$$

where

$$\tilde{\tau}_{ij} = a_{ij}^{(11)} E_{kl} + a_{ij}^{(17)} V_{kl},$$

$$\hat{\mu}_{ijk} = a_{ijk}^{(22)} \kappa_{ilm} + a_{ijk}^{(23)} E_{i},$$

$$\hat{\sigma}_{i} = a_{ijk}^{(23)} \kappa_{ijk} + a_{ij}^{(33)} E_{i},$$

$$Q_{ij} = a_{ij}^{(1)} E_{ij} + a_{ij}^{(77)} V_{ijkl},$$

$$- \eta \hat{\eta} = a_{ij}^{(14)} E_{ij} + a_{ij}^{(47)} V_{ij},$$

$$- \hat{q}_{i} = T_0 q_{i} \beta_{ij}$$

(15)
and
\[ F_i = \rho \xi \ast f_i + pv_i \rho t + pv_i, \quad H = \rho T_0 \psi^0 + l \ast \rho h, \]
\[ L_{ij} = a_{ij}^{(1)} \lambda^0 e^{-1/\beta}, \quad M_{ij} = a_{ij}^{(2)} \lambda^0 e^{-1/\beta}, \quad R = c e^0 e^{-1/\beta}. \]

Theorem 2. A process \( \pi \) is a solution of the mixed initial-boundary value problem in concern \( \Pi \) if and only if it satisfies eqs. (4), (5), (14) with the restriction (10) and the boundary conditions (11).

4. Uniqueness and reciprocity theorems

We consider the body \( B \) subjected to two different sets of external data
\[ \Gamma^{(\alpha)} = \left( f^{(\alpha)}_i, g^{(\alpha)}_i, h^{(\alpha)}_i, u^{(\alpha)}_i, v^{(\alpha)}_i, w^{(\alpha)}_i, \xi^{(\alpha)}_i, \nu^{(\alpha)}_i, \hat{u}^{(\alpha)}_i, \hat{v}^{(\alpha)}_i, \phi^{(\alpha)}_i, \theta^{(\alpha)}_i, \right), \]
\[ \Lambda^{(\alpha)} = \left( \hat{f}^{(\alpha)}_i, \hat{g}^{(\alpha)}_i, \hat{h}^{(\alpha)}_i, \hat{u}^{(\alpha)}_i, \hat{v}^{(\alpha)}_i, \hat{w}^{(\alpha)}_i, \hat{\xi}^{(\alpha)}_i, \hat{\nu}^{(\alpha)}_i, \hat{\phi}^{(\alpha)}_i, \hat{\theta}^{(\alpha)}_i, \right), \]
with \( \alpha = 1, 2 \), and denote the corresponding solutions of the mixed initial-boundary problem as
\[ \pi^{(\alpha)}(r, s) = \left( u^{(\alpha)}_i, v^{(\alpha)}_i, w^{(\alpha)}_i, \xi^{(\alpha)}_i, \nu^{(\alpha)}_i, \phi^{(\alpha)}_i, \theta^{(\alpha)}_i, \right), \]

Moreover, corresponding to \( \Gamma^{(\alpha)} \) and \( \pi^{(\alpha)} \), let’s define \( F^{(\alpha)}_i \), \( H^{(\alpha)}_i \), \( L^{(\alpha)} \), \( R^{(\alpha)}_i \), \( M^{(\alpha)}_i \), \( \tau^{(\alpha)}_i \), \( \mu^{(\alpha)}_i \), \( \sigma^{(\alpha)}_i \), \( \hat{Q}^{(\alpha)}_i \), \( \hat{Q}^{(\alpha)}_i \) through eqs. (12) and the divergence theorem, we arrive to
\[ \int_T \left[ \partial_t \pi^{(\alpha)}_i (r, s) - \Lambda^{(\alpha)} \phi^{(\alpha)}(t) \right] ds \]
so that, taking into account that eqs. (13) hold for \( \pi^{(1,2)} \) in \( K \), and using eqs. (12) and the divergence theorem, we arrive to
\[ \int_0^t \int_T \pi^{(\alpha)}_i (r, s) ds \]
\[ = \int_0^t \left[ \pi^{(\alpha)}_i (r, s) - \Lambda^{(\alpha)} \phi^{(\alpha)}(t) \right] ds \]

For what follows it is useful to remark that, by eqs. (13), (16), (17), (19), the following relations hold
\[ \pi^{(\alpha)}(r, s) = \left( \mu^{(\alpha)}_i (r, s) \right) u^{(\alpha)}_i (r, s) \]
\[ + \left( \xi * \theta^{(\alpha)}_i \right) \phi^{(\alpha)}(t) \]
(21)

On the other hand, by eqs. (11) and (13) we have
\[ S_{\alpha\beta}(r, s) = - K_{\alpha\beta} \left( r, s \right) u^{(\beta)}_i (r, s) \]
and
\[ S_{\alpha\beta}(r, s) = \tau^{(\alpha)}_i (r) \xi^{(\beta)}_i (s) + \mu^{(\alpha)}_i (r) \kappa^{(\beta)}_i (s) \]
\[ - \sigma^{(\alpha)}_i (r) E^{(\beta)}_i (s) - Q^{(\alpha)}_i (r) V^{(\beta)}_i (s) \]
(17)

for different times \( r, s \in T \). We can prove
\[ \hat{S}_{\alpha\beta}(r, s) = \hat{S}_{\alpha\beta}(s, r), \]
\[ \hat{S}_{\alpha\beta}(r, s) = \hat{S}_{\alpha\beta}(r, s) - \Lambda^{(\alpha)} (r) \rho^{(\beta)}(s), \]
where \( A \) is the following differential operator
\[ A = I + \partial_t \frac{\partial}{\partial t}, \]
with \( I \) the identity operator. If we define
\[ \hat{T}_{\alpha\beta}(t) = \xi * \int_0^t \hat{S}_{\alpha\beta}(t, t - \tau) d\tau, \]
\[ T_{\alpha\beta}(t) = \xi * \int_0^t S_{\alpha\beta}(t, t - \tau) d\tau, \]
eqs. (18) imply
\[ \hat{T}_{\alpha\beta}(t) = T_{\alpha\beta}(t), \]
\[ T_{\alpha\beta}(t) = T_{\alpha\beta}(t) - \xi \ast \mu^{(\alpha)}_i \ast \kappa^{(\beta)}_i(t) \]
(20)

Now, we obtain the following reciprocity relation which involves two processes at different times

Lemma 3. Let \( \pi^{(1,2)} \in K \). Then
\[ \Gamma_{\alpha\beta}(r, s) = \Gamma_{\beta\alpha}(s, r), \quad \forall r, s \in T, \forall \alpha, \beta = 1, 2, \]
(24)
where we define

\[ \Gamma_{\alpha\beta}(r, s) = \int_B \left[ \eta f_{\alpha}(r)u_{\beta}(s) - g(\alpha)(r)\varphi(\beta)(s) \right] \, dv + \int_B \left[ \eta f_{\beta}(r)u_{\alpha}(s) - \eta f_{\alpha}(r)u_{\beta}(s) \right] \, dv \]

\[ = \frac{1}{T_0} \mathcal{H}(\alpha)(r)A\Theta(\beta)(s) \, da + \int_B \left[ \eta f_{\alpha}(r)u_{\beta}(s) - \eta f_{\beta}(r)u_{\alpha}(s) \right] \, dv \]

\[ + \frac{1}{T_0} \mathcal{H}(\alpha)(r)A\Theta(\beta)(s) \, da \]

(25)

Proof. The first step is to introduce the following function

\[ J_{\alpha\beta}(r, s) = \mathcal{S}_{\alpha\beta}(r, s) - \eta_\alpha(\alpha)(r)A\Theta(\beta)(s). \] 

(26)

Taking into account the constitutive equations (6) \( \Theta \) and eqs. (15), we have

\[ J_{\alpha\beta}(r, s) = \mathcal{S}_{\alpha\beta}(r, s) - \eta_\alpha(\alpha)(r)A\Theta(\beta)(s) \]

\[ = \mathcal{S}_{\alpha\beta}(r, s) + \eta A\Theta(\alpha)(r)A\Theta(\beta)(s) \]

\[ - \left( \rho \eta_\alpha(\alpha)(r)A\Theta(\beta)(s) + A\Theta(\alpha)(r)\rho \eta_\beta(\beta)(s) \right) \]

\[ + \gamma \Theta(\alpha)(r)\Theta(\beta)(s) \]

\[ = J_{\alpha\beta}(r, s) + \frac{1}{\beta} \gamma \Theta(\beta)(s) \dot{\Theta}(\alpha)(r) \]

(27)

On the other hand, eqs. (14), (22) and (26) lead to

\[ \int_B \left[ \eta f_{\alpha}(r)u_{\beta}(s) - \eta f_{\beta}(r)u_{\alpha}(s) \right] \, dv = \Gamma_{\alpha\beta}(r, s), \] 

(28)

and consequently, we arrive to the desired result by (27) and (25).

We will use this Lemma to establish a uniqueness theorem with no definiteness assumption on internal energy and a reciprocity theorem.

4.1. Uniqueness theorem

Let \( \pi^{(1)} \in \mathcal{K} \) and call it \( \pi \) for simplicity, we take in eq. (25) \( r = t + \tau \in \mathcal{S} = t - \tau \) with \( \alpha = \beta = 1 \) and integrating from 0 to \( t \) we obtain

\[ \int_0^t \Gamma_{11}(t + \tau, t - \tau) \, d\tau = \int_0^t E(t + \tau, t - \tau) \, d\tau \]

\[ - \int_0^t \left( \dot{\mu}(t + \tau)u(t - \tau) + \frac{1}{\beta} \gamma \Theta(t + \tau)A\theta(t - \tau) \right) \, d\tau \, dv, \] 

\[ - k_{ij} \Theta(t + \tau)A\theta_j(t - \tau) \] 

(29)

\[ \text{where} \]

\[ E(r, s) = \int_B \left[ \rho f_{\alpha}(r)u_{\alpha}(s) - g(r)\varphi(s) - \frac{1}{T_0} \mathcal{H}(r)A\theta(s) \right] \, dv \]

\[ + \int_B \left[ f_{\alpha}(r)u_{\alpha}(s) + R_i(r)\mathcal{D}u_i(s) + \mathcal{N}(r)\varphi(s) \right] \, dv \]

\[ + \mathcal{H}(r)A\varphi(s) + \frac{1}{T_0} \left[ \star q_i(r)A\theta(s) \right] \, da. \]

Obviously, eq. (24) implies

\[ \int_0^t \left[ \Gamma_{11}(t + \tau, t - \tau) - \Gamma_{11}(t - \tau, t + \tau) \right] \, d\tau = 0. \] 

(30)

Let’s use eqs. (24), (30) and the following relations

\[ \int_0^t \left[ \dot{\mu}_i(t + \tau)u_i(t - \tau) - \dot{\mu}_i(t - \tau)u_i(t + \tau) \right] \, d\tau = \]

\[ = \dot{\mu}_i(2t)u_i^0 + u_i(2t)v_i^0 - 2u_i(t)\dot{\mu}_i(t), \]

\[ \int_0^t \left[ \dot{\Theta}(t + \tau)\Theta(t - \tau) - \dot{\Theta}(t - \tau)\Theta(t + \tau) \right] \, d\tau = \]

\[ = \Theta(2t)\dot{\Theta}(t) - \Theta(t)\dot{\Theta}(t), \]

\[ \int_0^t \left[ \Theta_s(t + \tau)\dot{\Theta}_s(t + \tau) - \Theta_s(t - \tau)\dot{\Theta}_s(t - \tau) \right] \, d\tau = \]

\[ = -\Theta_s(t)\Theta_{ss}(t) + \frac{1}{\beta} \left[ \Theta_s(t)\Theta_{ss}(t) - 2\Theta_s(t)\Theta_{ss}(t) \right], \]

to arrive to

\[ \int_0^t \left[ E(t + \tau, t - \tau) - E(t - \tau, t + \tau) \right] \, d\tau \]

\[ - \int_B \left( \dot{\mu}_i(2t)u_i^0 + u_i(2t)v_i^0 - 2u_i(t)\dot{\mu}_i(t) \right) \]

\[ + \frac{1}{\beta} \gamma \left[ \Theta(2t)\dot{\Theta}(t) - \Theta(t)\dot{\Theta}(t) \right] + k_{ij} \left[ -\Theta_s(t)\Theta_{ss}(t) \right] \]

\[ + \beta \left[ \Theta_s(t)\dot{\Theta}_s(t) - 2\Theta_s(t)\dot{\Theta}_s(t) \right] \] 

(31)

Eq. (31) implies

\[ \dot{G}(t) = \int_0^t \left[ E(t + \tau, t - \tau) - E(t - \tau, t + \tau) \right] \, d\tau \]

\[ \int_B \left[ \rho \left[ \dot{\mu}_i(2t)u_i^0 + u_i(2t)v_i^0 \right] + \frac{1}{\beta} \gamma \Theta(2t)\dot{\Theta}(t) \right] \, dv, \] 

(32)
where
\[ G(t) = \int_0^t \int_B \mathcal{P}[\theta(\tau), \Theta_i(\tau)] \, dv \, d\tau + \int [\rho u_i(t) u_i(t) + \beta k_{ij} \Theta_j(t) \Theta_i(t)] \, dv, \]  
with \( \mathcal{P} \) defined by eq. 3.

Now, we can prove the following uniqueness theorem

**Theorem 4.** Assume that,

1. \( \beta, \gamma \) are strictly positive,
2. the following quadratic form is definite
   \[ F = a_{ij}^{(33)} E_j E_i + a_{ijkl}^{(77)} V_{kl} V_{ji}. \]

If \( S_4 \) is nonempty, the initial-boundary values problem 2 has as at most one solution.

**Proof.** Clearly, the difference \( \pi \) of any two solutions of II corresponds to null data. For this solution, the function \( G(t) \) defined by eq. 33 vanishes initially and its derivative 32 is identically zero, then \( G(t) = 0 \) for all \( t \in I \).

Since \( \rho, \beta, \gamma > 0 \) and \( \mathcal{P} \) is positive semi-definite, then for all \( t \in I \)
\[ u_i = 0, \quad \theta = 0 \quad \text{on} \quad B \times I. \]  
(34)

From eqs. 12, it follows
\[ e_{ij} = 0, \quad \kappa_{ijk} = 0, \quad \text{on} \quad B \times I. \]  
(35)

Moreover, the constitutive equations 30 and the equation of energy 3 with homogeneous initial conditions imply
\[ q_i = 0, \quad \rho \eta = 0 \quad \text{on} \quad B \times I. \]  
(36)

On the other hand, it follows from eqs. 33, 3a, 3b, 3c, 3d,
\[ \int_B F(t) \, dv = \int_B \left[ \tau(t) \varphi_j(t) + Q_{ij}(t) \varphi,ij(t) \right] \, dv. \]  
(37)

Taking into account eqs. 2, 12 and 33, the divergence theorem and the null data, we have
\[ \int_B F(t) \, dv = - \int_B g(t) \varphi(t) \, dv + \int_B \left[ \Lambda(t) \varphi(t) + H(t) \varphi(t) \right] \, da = 0. \]  
Consequently, given that \( F \) is definite, we arrive to
\[ F(t) = 0 \quad \Rightarrow \quad E_j = 0, \quad V_{ji} = 0, \quad \text{on} \quad B \times I. \]  
(38)

4.2. Reciprocity theorem

In this subsection we derive a reciprocity theorem based on Lemma 3 and following the method shown by Ieșan 2.

**Lemma 5.** Let be \( \pi^{(1,2)} \in \mathcal{K} \). Then we have
\[ I_{\alpha \beta}(t) = I_{\beta \alpha}(t), \quad \forall t \in I, \quad \forall \alpha, \beta = 1, 2, \]  
(39)

with
\[ I_{\alpha \beta}(t) = \int_B \left[ \mathcal{F}_i^{(a)}(t) * u_i^{(3)}(t) - \xi * g^{(a)} * \varphi^{(3)}(t) \right] \, dv + \frac{1}{T_0} \int \left[ \mathcal{P}_i^{(a)}(t) * u_i^{(3)}(t) + R_i^{(a)}(t) * \mathcal{D} u_i^{(3)}(t) \right] \, da + \int_B \left[ \mathcal{L}^{(a)}(t) * \theta^{(3)}(t) \right] \, dv, \]
where we define \( \mathcal{F}_i^{(a)}, \mathcal{H}_i^{(a)}, \mathcal{R}_i^{(a)} \) by eq. 13 and
\[ \mathcal{L}^{(a)} = \frac{1}{\beta} \Gamma^{(a)}(t) \xi, \quad \mathcal{L}_j^{(a)} = \frac{\beta}{\xi} s_{ij} \theta^{(a)} l. \]

**Proof.** Taking into account eqs. 13, 17, 39 and that
\[ k_{ij} \Theta_j^{(a)} * \mathcal{A}^{(3)}(t) = k_i \Theta_i^{(a)} * l * \theta_j^{(3)}(t) \]
\[ + \beta k_{ij} \Theta_j^{(a)} * \theta_i^{(3)}(t) - \beta k_{ij} \theta_i^{(a)} l * \theta_j^{(3)}(t) \]
\[ \text{eq. 25 leads to} \]
\[ \xi * \int_0^t \Gamma_{\alpha \beta}(t, t - \tau) \, d\tau = \int_B \left[ - \rho u_i^{(a)} * u_i^{(3)}(t) \right] \]
\[ - \frac{1}{\beta} \gamma l * \theta^{(a)} * \theta^{(3)}(t) + l \, k_{ij} \Theta_j^{(a)} * \Theta_i^{(3)}(t) \]
\[ \beta k_{ij} \Theta_j^{(a)} * \Theta_i^{(3)}(t) + \mathcal{F}_i^{(a)} * u_i^{(3)}(t) \]
\[ - \xi * g^{(a)} * \varphi^{(3)}(t) - \mathcal{L}^{(a)}(t) \]
\[ + \frac{1}{T_0} \xi * \mathcal{H}_i^{(a)} * \mathcal{A}^{(3)}(t) - \mathcal{L}_j^{(a)}(t) \]
\[ + \frac{1}{T_0} \mathcal{L}^{(a)}(t) \]
\[ \text{from this expression and Lemma 3 it is easy to prove that the following relation holds} \]
\[ \xi * \int_0^t \Gamma_{\alpha \beta}(t, t - \tau) \, d\tau + \int_B \left[ \mathcal{L}_j^{(a)}(t) * \theta_j^{(3)}(t) + \mathcal{L}_j^{(a)}(t) * \theta_j^{(3)}(t) \right] \, d\tau, \]
and this is equivalent to eq. 39.
5. Alternative reciprocity theorem

We now prove an alternative reciprocity theorem in which the operator \( A \) defined in eq. (19) is not used.

**Theorem 6.** If we define

\[
\mathcal{J}_{\alpha\beta}(t) = \chi \ast \xi \ast \int_B \left[ \mathcal{F}^{(\alpha)}_t \ast u^{(\beta)}_t(t) - \xi \ast g^{(\alpha)} \ast \varphi^{(\beta)}(t) \right] dv + \chi \ast \xi \ast \int_B \left\{ \alpha \ast \beta \right\} \left[ \mathcal{P}^{(\alpha)}_t \ast u^{(\beta)}_t(t) + \mathcal{R}^{(\alpha)} \ast \mathcal{D}u^{(\beta)}_t(t) \right] dv + \Lambda^{(\alpha)} \ast \varphi^{(\beta)}(t) + H^{(\alpha)} \ast \mathcal{D} \varphi^{(\beta)}(t) \right] + \frac{1}{L_0} \ast q^{(\alpha)} \ast \theta^{(\beta)}(t) \right\} da + \mathcal{R}^{(\alpha)} \ast \theta^{(\beta)}(t) - \chi \ast \frac{1}{L_0} H^{(\alpha)} \ast \theta^{(\beta)}(t) \right\} dv,
\]

we have

\[
\mathcal{J}_{\alpha\beta}(t) = \mathcal{J}_{\beta\alpha}(t) \quad \forall t \in I, \forall \alpha, \beta = 1, 2.
\] (41)

**Proof.** We introduce the following function

\[
\mathcal{J}_{\alpha\beta}(t) = \chi \ast \xi \ast \mathcal{J}_{\alpha\beta}(t) - \xi \ast \mathcal{R}^{(\alpha)} \ast \theta^{(\beta)}(t).
\] (42)

From (21) and (14), we obtain, with help of eq. (20),

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \chi \ast \xi \ast \mathcal{J}_{\alpha\beta}(t) - \frac{1}{L_0} H^{(\alpha)} \ast \theta^{(\beta)}(t) \right\} dv =
\]

\[
\int_B \left\{ \chi \ast \xi \ast \mathcal{T}_{\alpha\beta}(t) + \epsilon \ast \xi \ast \theta^{(\alpha)} \ast \theta^{(\beta)}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \chi \ast \xi \ast \mathcal{J}_{\alpha\beta}(t) + \theta^{(\alpha)} \ast \mathcal{R}^{(\beta)} \ast \mathcal{J}_{\alpha\beta}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\beta\alpha}(t) - \chi \ast \xi \ast \mathcal{J}_{\beta\alpha}(t) + \theta^{(\alpha)} \ast \mathcal{R}^{(\beta)} \ast \mathcal{J}_{\beta\alpha}(t) \right\} dv.
\] (43)

Using (12), (23), (14), (6) and theorem of divergence, we have

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \chi \ast \xi \ast \mathcal{J}_{\alpha\beta}(t) + \theta^{(\alpha)} \ast \mathcal{R}^{(\beta)} \ast \mathcal{J}_{\alpha\beta}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \mathcal{J}_{\beta\alpha}(t) \right\} + \mathcal{R}^{(\alpha)} \ast \theta^{(\beta)}(t)dv =
\]

\[
\int_B \left\{ k_{ij} \mathcal{J}_{\alpha\beta}(t) - k_{ij} \mathcal{J}_{\beta\alpha}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \mathcal{J}_{\beta\alpha}(t) \right\} dv.
\]

From this equation and eqs. (40), (41) we have

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \chi \ast \xi \ast \mathcal{J}_{\alpha\beta}(t) + \theta^{(\alpha)} \ast \mathcal{R}^{(\beta)} \ast \mathcal{J}_{\alpha\beta}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \mathcal{J}_{\beta\alpha}(t) \right\} + \mathcal{R}^{(\alpha)} \ast \theta^{(\beta)}(t)dv =
\]

\[
\int_B \left\{ k_{ij} \mathcal{J}_{\alpha\beta}(t) - k_{ij} \mathcal{J}_{\beta\alpha}(t) \right\} dv =
\]

\[
\int_B \left\{ \mathcal{J}_{\alpha\beta}(t) - \mathcal{J}_{\beta\alpha}(t) \right\} dv.
\]

so that, with the help of eq. (43), we arrive to eq. (11). \( \square \)

6. Variational principle

In this section, we formulate a variational principle for the considered model. To this aim, we define for each \( t \in I \)
the functional $\Lambda_t$ defined on $V$ as follows

$$
\begin{align*}
\Lambda_t\{\pi\} &= \int_B \left\{ \chi \ast \left[ \zeta \ast \left( \tau_{ij,j} - \mu_{kji,kj} \right) + \mathcal{F}_i \right. \\
&\quad - \frac{1}{2} \rho u_i \right. \ast u_i + \zeta \ast \zeta \ast \left( \sigma_{i,i} - Q_{ji,ji} - g \right) \ast \varphi \\
&\quad - \frac{1}{T_0} \left( -l \ast q_{i,i} + \mathcal{H} - \rho \mathcal{T}_0 \varphi \right) \ast \theta \\
&\quad + \zeta \ast \left[ \zeta \ast \left( \tau_{ij} - \frac{1}{2} \tau_{ij} \right) + \mathcal{L}_{ij} \right] \ast e_{ij} \\
&\quad + \zeta \ast \xi \ast \left( \mu_{ij,k} - \frac{1}{2} \hat{\mu}_{ij,k} \right) \ast \kappa_{ijk} \\
&\quad - \zeta \ast \xi \ast \left( \sigma_i - \frac{1}{2} \hat{\sigma}_i \right) \ast e_i \\
&\left. \right\} \, dv \\
&\quad - \chi \ast \zeta \ast \xi \ast \int \left[ P_t \ast u_i \, da + \int_{\Sigma_3} \left( P_t \ast \hat{P}_t \right) \ast u_i \, da \right] \\
&\quad - \chi \ast \zeta \ast \xi \ast \int \left[ R_i \ast \hat{a}_i \, da + \int_{\Sigma_2} \left( R_i \ast \hat{R}_i \right) \ast D u_i \, da \right] \\
&\quad - \frac{1}{T_0} \chi \ast \xi \ast l \ast \int \left[ q \ast \hat{\theta} \, da + \int_{\Sigma_3} \left( q - \hat{q} \right) \ast \theta \, da \right] \\
&\quad - \chi \ast \zeta \ast \xi \ast \int \left[ \Lambda \ast \hat{\varphi} \, da + \int_{\Sigma_4} \left( \Lambda - \hat{\Lambda} \right) \ast \varphi \, da \right] \\
&\quad - \chi \ast \zeta \ast \xi \ast \int \left[ H \ast \complement \varphi \, da + \int_{\Sigma_5} \left( H - \hat{H} \right) \ast D \varphi \, da \right].
\end{align*}
$$

We say that the variation of $\Lambda_t$ is zero at $\pi$ over $V$ if and only if $\frac{d}{d\lambda} \Lambda_t \{ \pi + \lambda \pi' \}$ exists and is zero for any $\pi' \in V$, i.e.

$$
\delta \Lambda_t \{ \pi \} = 0 \iff \left. \frac{d}{d\lambda} \Lambda_t \{ \pi + \lambda \pi' \} \right|_{\lambda=0} = 0.
$$

**Theorem 7.** Fixed $t \in I$, the variation $\delta \Lambda_t \{ \pi \}$ of functional $\Lambda_t$ corresponding to $\pi \in V$ is null if and only if $\pi$ is a solution of the considered mixed initial-boundary value problem $\Pi$, i.e. $\delta \Lambda_t \{ \pi \} = 0$ if and only if $\pi \in \mathcal{K}$.

**Proof.** To begin, we point out that for any $\pi, \pi' \in V$

$$
\begin{align*}
\int_B \left[ \left( \tau_{ji} - \mu_{kji,kj} \right) \ast u_i + \mu_{ij,k} \ast u_i \right] \, da \\
&= \int_B \left[ P_t \ast u_i + R_t \ast D u_i \right] \, da, \\
\int_B \left[ \zeta \ast \varphi \ast Q_{ji,ji} \ast \varphi + Q_{ji,ji} \ast \varphi \right] \, da \\
&= \int_B \left[ \Lambda \ast \varphi + H \ast D \varphi \right] \, da,
\end{align*}
$$

where we take into account eqs. (17) and (18). Now, by means of the well-known properties of the convolution product, the definition of variation, eqs. (17), (18), (19) and the divergence theorem, we arrive to

$$
\delta \Lambda_t \{ \pi \} = \int_B \left\{ \chi \ast \zeta \ast \left[ \zeta \ast \left( \tau_{ji,j} - \mu_{kji,kj} \right) + \mathcal{F}_i \ast \rho u_i \right] \ast u_i' \\
+ \chi \ast \zeta \ast \left[ \zeta \ast \left( \sigma_{i,i} - Q_{ji,ji} - g \right) \ast \varphi \right] \\
- \frac{1}{T_0} \left\{ \chi \ast \left( \rho \mathcal{T}_0 \varphi \right) \ast \theta \right\} \\
+ \chi \ast \zeta \ast \left[ \left( \mu_{ij,k} - \frac{1}{2} \hat{\mu}_{ij,k} \right) \ast \kappa_{ijk} \right] \\
- \chi \ast \xi \ast \left( \sigma_i - \frac{1}{2} \hat{\sigma}_i \right) \ast e_i \\
- \int \left[ \left( q - \hat{q} \right) \ast \theta \right] \, da \right\} \\
- \chi \ast \xi \ast \left\{ \int \left[ P_t \ast \left( u_i - \hat{u}_i \right) \, da - \int \left( P_t \ast \hat{P}_t \right) \ast u_i' \, da \right] \\
+ \chi \ast \xi \ast \left\{ \int \left[ R_t \ast \left( D u_i - \hat{D} u_i \right) \, da - \int \left( R_t \ast \hat{R}_t \right) \ast D u_i' \, da \right] \\
+ \frac{1}{T_0} \chi \ast \xi \ast l \ast \int \left[ \left( q - \hat{q} \right) \ast \theta \right] \, da \right\} \\
+ \chi \ast \xi \ast \left\{ \int \left[ \Lambda \ast \left( \varphi - \hat{\varphi} \right) \, da - \int \left( \Lambda - \hat{\Lambda} \right) \ast \varphi \right] \, da \right\} \\
+ \chi \ast \xi \ast \left\{ \int \left( H \ast \complement \varphi - \hat{\varphi} \right) \, da - \int \left( H - \hat{H} \right) \ast D \varphi \right\}
$$

where $\tau_{ji,j}, \mu_{ij,k}, \sigma_i, \kappa_{ijk}, \varphi, \Lambda, H$ are defined by eqs. (18) corresponding to $\pi'$. Then, for any
\( \pi' \in V \) we have that \( \delta \Lambda; \{ \pi \} = 0 \) if and only if eqs. (14), (15), (16), (17) hold.

7. Conclusions

In this paper we considered the linear theory of ther-mopiezoelectric nonsimple materials as established in Passarella and Tibullo and in particular the case of center-symmetric materials.

We defined a mixed initial-boundary value problem under non-homogeneous initial conditions and presented a characterization of the mixed initial-boundary value problem in an alternative way, by including the initial conditions into the field equations. Starting from a reciprocity relation which involves two processes at different times, a reciprocity theorem has been presented. Moreover, a uniqueness result was established without using the definiteness assumptions on internal energy. Finally, another reciprocity theorem based on the convolution product and a variational principle have been derived.

Further developments of this theory could be related to general, non center-symmetric, materials. Another possible application is to the study of wave propagation in isotropic materials.

References

[1] F. Passarella and V. Tibullo. Uniqueness of solutions in thermopiezoelectricity of nonsimple materials. submitted to ZAMM, 2022.
[2] A. E. Green and N. Laws. On the entropy production inequality. Arch. Rat. Mech. An., 45:47–53, 1972. ISSN 0003-9527.
[3] D. S. Chandrasekharaiah. Hyperbolic thermoelasticity: A review of recent literature. Appl. Mech. Rev., 51:705–729, 1998.
[4] D. S. Chandrasekharaiah. Some theorems in generalized micropolar thermoelasticity. Arch. Mech., 38(3):319–328, 1986. ISSN 0373-2029.
[5] D. Iesan. Thermoelasticity of continua, volume 118 of Solid Mechanics and Its Applications. Springer, Dordrecht, 2004. ISBN 978-90-481-6634-3. doi: 10.1007/978-1-4020-2310-1.
[6] F. Passarella, V. Tibullo, and V. Zampoli. On microstretch ther-moviscoelastic composite materials. Eur. J. Mech. A. Solids, 37:294–303, 2013. doi: 10.1016/j.euromechsol.2012.07.002.
[7] R. A. Toupin. Elastic materials with couple-stresses. Arch. Rat. Mech. An., 11(1):385–414, 1962. doi: 10.1007/BF00253945.
[8] R. A. Toupin. Theories of elasticity with couple-stress. Arch. Rat. Mech. An., 17(2):85–112, 1964. doi: 10.1007/BF00253505.
[9] R. D. Mindlin. Micro-structure in linear elasticity. Arch. Rat. Mech. An., 16(1):51–78, 1964. doi: 10.1007/BF00248400.
[10] R. A. Toupin and D. C. Gazis. Surface effects and initial stress in continuum and lattice models of elastic crystals. In Lattice Dynamics, pages 597–605. Elsevier, 1965.
[11] G. Ahmad and K. Firoozbakht. First strain gradient theory of ther-moviscoelasticity. Int J Solids Struct., 11(3):339–345, 1975. doi: 10.1016/0020-7683(75)90073-6.
[12] R. C. Batra. Thermodynamics of non-simple elastic materials. J Elasticity, 6(4):451–456, 1976. doi: 10.1007/BF00040904.
[13] R. D. Mindlin and N. N. Eshel. On first strain-gradient theories in linear elasticity. Int J Solids Struct., 4(1):109–124, 1968. doi: 10.1016/0020-7683(68)90036-X.
[14] G. Ahmad. Thermelastic stability of first strain gradient solids. Int. J. Non Linear Mechn., 12(1):23–32, 1977. doi: 10.1016/0020-7462(77)90013-0.
[15] D. Iesan. Thermoelectricity of nonsimple materials. J Therm Stresses, 6(2-4):167–188, 1983. doi: 10.1080/0149538308942176.
[16] M. Ciariella and D. Iesan. On the nonlinear theory of nonsimple thermoelastic bodies. J Therm Stresses, 12(4):545–557, 1989.
[17] V. K. Kalpakides and E. K. Agiasifitou. On material equations in second gradient electroweakelasticity. J Elasticity, 67(3):205–227, 2002. doi: 10.1023/A:1024926609803.
[18] M. Aouadi, M. Ciariella, and V. Tibullo. Analytical aspects in strain gradient theory for chiral Cosserat thermoelastic materials within three Green-Naghdi models. J Therm Stresses, 42(6):681–697, 2019. doi: 10.1080/01495379.2019.1571974.
[19] M. Aouadi, F. Passarella, and V. Tibullo. Exponential stability in Mindlin’s Form II gradient thermoelasticity with microtemperatures of type III: Mindlin’s II gradient thermoelastic. Proc. R. Soc. London, Ser. A, 476(2241), 2020. doi: 10.1098/rspa.2020.0459.
[20] M. Aouadi, A. Amendola, and V. Tibullo. Asymptotic behavior in Form II Mindlin’s strain gradient theory for porous thermoelastic diffusion materials. J Therm Stresses, 43(2):191–209, 2020. doi: 10.1080/01495379.2019.1653802.
[21] V. Bartilomo and F. Passarella. Basic theorems for nonsimple thermoelastic solids. Boll. Inst. Polithech. Iagi. Sér. I. Mat. Mecc. Teor. Fiz., 43(1):1-2:59–70, 1997.
[22] A. C. Eringen. Electromagnetic theory of microstretch elasticity and bone modeling. Int. J Eng Sci, 42(3-4):231–242, 2004. doi: 10.1016/S0020-7225(03)00288-X.
[23] C. Truesdell and R. Toupin. The classical field theories. In S. Flügge, editor, Handbuch der physique, volume III. Springer-Verlag, Berlin - Heidelberg - New York, 1960.
[24] H. Parkus. Magneto-thermoelasticity, volume 118. Springer, 1972.
[25] R. A. Grot. Relativistic continuum physics: electromagnetic interactions. In A. C. Eringen, editor, Continuum physics, volume III - Mixtures and EM Field Theories, pages 129–219. Elsevier, 1976.
[26] W. Nowacki. Mathematical models of phenomenological piezo-electricity. In R. H. O. Brulin, editor, New Problems in Mechanics of Continua, pages 30–50. University of Waterloo Press, Ontario, 1983.
[27] G. A. Maugin. Continuum mechanics of electromagnetic solids, volume 33 of Applied Mathematics and Mechanics. North Holland, Amsterdam, New York, Oxford, Tokyo, 1988.
[28] A. Morro and B. Straughan. A uniqueness theorem in the dynamical theory of piezoelectricity. Math. Methods Appl. Sci., 14(5):295–299, 1991. doi: 10.1002/mma.1670140502.
[29] F. Passarella, V. Tibullo, and V. Zampoli. On the heat-flux dependent thermoelasticity for micropolar porous media. J Therm Stresses, 34(8):778–794, 2011. doi: 10.1080/01495379.2011.564041.
[30] M. E. Gurtin. Variational principles for linear elastodynamics. Archive for Rational Mechanics and Analysis, 16(1):34–50, 1964. doi: 10.1007/BF00248489.