Topological Invariants and Ground-State Wave Functions of Topological Insulators on a Torus

Zhong Wang\textsuperscript{1,} and Shou-Cheng Zhang\textsuperscript{1,2}

\textsuperscript{1} Institute for Advanced Study, Tsinghua University, Beijing, China, 100084
\textsuperscript{2} Department of Physics, Stanford University, Stanford, CA 94305
(Dated: May 11, 2014)

We define topological invariants in terms of the ground states wavefunctions on a torus. This approach leads to precisely defined formulas for the Hall conductance in four dimensions and the topological magneto-electric $\theta$ term in three dimensions, and their generalizations in higher dimensions. They are valid in the presence of arbitrary many-body interaction and disorder. These topological invariants systematically generalize the two-dimensional Niu-Thouless-Wu formula, and will be useful in numerical calculations of disordered topological insulators and strongly correlated topological insulators, especially fractional topological insulators.

PACS numbers: 73.43.-f,71.70.Ej,75.70.Tj

I. INTRODUCTION

Topological insulators are among the major recent developments in condensed matter physics.\textsuperscript{11-15} The physics of topological insulators started with noninteracting systems,\textsuperscript{11-15} for which simple and calculable topological invariants have been invaluable tools. More recently, it became clear that the interplay between topology and many-body interaction is a still richer field,\textsuperscript{16-20} therefore, it is highly desirable to develop topological invariants that are valid in the presence of strong interaction.

The root state of three-(spatial)-dimensional (3D) and two-(spatial)-dimensional (2D) topological insulators with time-reversal symmetry is the four-(spatial)-dimensional (4D) quantum Hall (QH) state\textsuperscript{11,66} from which the topological field theory of 3D and 2D insulators can be obtained by the procedure of “dimensional reduction”.\textsuperscript{16-20} The electromagnetic effective action of the 4D QH effect reads\textsuperscript{3,67}

\begin{equation}
S_{\text{eff}} = \frac{\sigma_{4D}}{24\pi^2} \int dt d^4x \epsilon^{i j k l} A_i \partial_j A_j \partial_k A_l \tag{1}
\end{equation}

where we have adopted the units that the electric charge $e$, the Planck constant $\hbar$, and the light velocity $c$ are all unity. The coefficient $\sigma_{4D}$ is referred to as the "4D Hall conductance" (or the 4D Hall coefficient). Physically, the 4D QH effect has the nonlinear topological electromagnetic response\textsuperscript{3,67}

\begin{equation}
\dot{\phi} = \frac{\delta S_{\text{eff}}}{\delta A_0} = \sigma_{4D} \epsilon^{i j k l} \partial_i A_j \partial_k A_l \tag{2}
\end{equation}

which is a 3D Chern-Simons term. In the presence of time-reversal symmetry, this Chern-Simons term is quantized and has been shown to be equivalent\textsuperscript{3,67} to the $Z_2$ topological invariant.\textsuperscript{12} The natural question is: Is there a formula for $\sigma_{4D}$ that is precisely defined in the presence of arbitrary interaction and disorder? Such a formula, if exists, will be especially desirable for the investigation of fractional quantum Hall states in 4D. More importantly, it may also shed light on strongly interacting topological insulators in lower dimensions.

Now a natural question arises: Can we find a formula for $\sigma_{4D}$ that is precisely defined in the presence of arbitrary interaction and disorder? Such a formula, if exists, will be especially desirable for the investigation of fractional quantum Hall states in 4D. More importantly, it may also shed light on strongly interacting topological insulators in lower dimensions.

The same question also arises for the 3D topological insulators, whose effective topological responses theory is given by\textsuperscript{11}

\begin{equation}
S_{\text{eff}} = \frac{1}{8\pi^2} \int dt d^3x \epsilon^{i j k} \partial_i A_j \partial_k A_l = \frac{1}{4\pi^2} \int dt d^3x \mathbf{E} \cdot \mathbf{B} \tag{3}
\end{equation}

This topological effective action describes the quantized topological magnetoelectric effect, in which an electric field induces a magnetization with universal constant of proportionality.\textsuperscript{11}

In the noninteracting limit, $\theta$ has a simple expression\textsuperscript{11,20,71}

\begin{equation}
\theta = \frac{1}{4\pi} \int d^3x \epsilon^{i j k} \text{Tr}[\partial_i a_j(k) + \frac{2}{3} i a_i(k) a_j(k)] a_k(k) \tag{4}
\end{equation}

which is a 3D Chern-Simons term. In the presence of time-reversal symmetry, this Chern-Simons term is quantized and has been shown to be equivalent\textsuperscript{11,20,71} to the $Z_2$ topological invariant\textsuperscript{12}. The natural question is: Is there a formula for $\theta$ that is valid in the presence of arbitrary interaction and disorder? From the experimentalist’s perspective, this question is more urgent than the 4D QH case, because many 3D topological insulators have been realized in experiments, and the electron-electron interaction has been playing more important roles.

To partially answer these questions, interacting topological invariants expressed in terms of Green’s function at zero-frequency (namely the “topological Hamiltonian”\textsuperscript{24}) for interacting insulators have been proposed\textsuperscript{23,25}, which provide an efficient approach for topological invariants of various topological insulators and superconductors [See, e.g. Ref. 41,51,57-76,80 for applications]. However, there are several shortcomings of that approach. First, it cannot be directly applied to disordered systems in which the momentum $k$ in the single-particle Green’s function is not a good quantum number.\textsuperscript{25} Second, it is unclear whether or not that approach may fail for some fractional topological states.
In Ref. Niu, Thouless and Wu found for the 2D QH a topological invariant (the first Chern number) expressed in terms of ground state wavefunction under twisted boundary condition, which is valid in the presence of arbitrary interaction and disorder. To search for the general formulas for $\sigma_{AD}$ in 4D and $\theta$ in 3D, a hopeful approach is to generalize their formula to higher dimensions. However, as we will see later, the most straightforward 4D generalization of their formula, namely the generalization of the 2D phase twisting ($\theta_1, \theta_2, \theta_3$) to the 4D phase twisting ($\theta_1, \theta_2, \theta_3, \theta_4$) [see Eq. (40)], cannot produce the 4D Hall conductance $\sigma_{AD}$. Due to this difficulty, it is unclear how this approach can be generalized to higher-dimensional topological states.

In this paper we propose general topological invariants for higher dimensional topological insulators in terms of ground state wavefunctions. The boundary conditions adopted here are not the standard one used in Ref. which is a pure gauge with vanishing field strength. Using these new boundary conditions [see Sec. and Sec.], we obtain for $\sigma_{AD}$ and $\theta$ simple formulas expressed in terms of the ground state wavefunction on a torus [see Eq. (12), Eq. (29), Eq. (41), Eq. (44), etc]. We also generalize these formulas to higher dimensions [see Eq. (24), etc]. These topological invariants are valid in the presence of arbitrary interaction and disorder, thus they can be applied to topological states with strong disorders and strongly correlated topological states including fractionalized states. Unexpectedly, the generalized formula for $\sigma_{AD}$ appears not as a second Chern number, but as the difference between two first Chern numbers [Eq. (12), Eq. (29)]. Similarly, the formula for $\theta$ does not appear as a Chern-Simons form, but as the difference between two winding numbers [Eq. (41), Eq. (44)].

The rest part of this paper is organized as follows. In Sec. we study the 4D QH and define the topological invariant for integer QH in 4D. In Sec. we test this topological invariant in two noninteracting models. We then generalize the 4D topological invariant to higher dimensional QH effects in Sec. In Sec. we present the topological invariants for fractional quantum Hall effects. A different boundary condition is investigated in Sec. which leads to 4D topological invariant unrelated to the 4D Hall conductance. The next two Sections, namely Sec. and Sec. is devoted to 1D and 3D $\theta$ term respectively.

II. 4D HALL COEFFICIENTS $\sigma_{AD}$ EXPRESSED IN TERMS OF THE GROUND STATE WAVEFUNCTION

In this section we describe the topological invariant defined in terms of the ground state wave function of a 4D insulator on a torus with generalized twisted boundary conditions. For simplicity, in this section we assume that the ground state is unique, while the cases with ground state degeneracy will be studied in Sec. We take the system to be a 4D torus with circumference $L_1, L_2, L_3, L_4$ along the $x_1, x_2, x_3, x_4$ direction respectively. We take the generalized twisted boundary condition parameterized by ($\theta_1, \theta_2, \phi$) as follows. First, for $i = 1, 2,$

$$\Psi(r_1, \cdots, r_k + L_i \mathbf{x}_i, \cdots, r_N; \theta_1, \theta_2, \phi)$$

= $\exp(i\theta_1)\Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \theta_2, \phi)$ (5)

where $r_i$ is the coordinate of the $i$-th particle (other arguments such as spin are not shown here for simplicity of notation). $N$ is the total particle number, and $\mathbf{x}_i$ is the unit vector along the $x_i$ direction. This condition is the same as the one adopted in Ref. Second,

$$\Psi(r_1, \cdots, r_k + L_3 \mathbf{x}_3, \cdots, r_N; \theta_1, \theta_2, \phi)$$

= $\exp(-i\phi \frac{x_4}{L_4})\Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \theta_2, \phi)$ (6)

Since $x_4 \equiv x_4 + L_4$ on the torus, the flux $\phi$ has to be quantized as $n\phi_0$, where the unit flux $\phi_0 \equiv 2\pi$, and $n$ is an integer. Lastly

$$\Psi(r_1, \cdots, r_k + L_3 \mathbf{x}_3, \cdots, r_N; \theta_1, \theta_2, \phi)$$

= $\Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \theta_2, \phi)$ (7)

Physically, these twisted boundary conditions tell us that there is a gauge potential $A_1 = \theta_1/L_1$ along the $x_1$ ($i = 1$) direction, and a gauge potential $A_3 = -\frac{\phi x_4}{L_4}$ along the $x_3$ direction, in other words, there is a magnetic flux $\phi$ inside any 2D torus $T_4$ whose coordinates are $(x_1, x_2, x_3, x_4)$ with fixed $(x_1, x_2)$.

Before proceeding to our central results, let us briefly outline the motivations of the boundary conditions given in Eq. (5), Eq. (6), and Eq. (7). The first motivation is that the ($\theta_1, \theta_2, \theta_3$) boundary condition [see Sec. VI] does not produce the 4D Hall conductance. The second motivation is the intuitive relation between the 4D Hall effect and the 2D Hall effect. In Eq. (1), if we take $A_3, A_4$ to be independent on $x_0, x_1, x_2$, and at the same time take $A_0, A_1, A_2$ to be independent on $x_3, x_4$, then there is a "dimensional reduction" of the 4D Chern-Simons term to the 2D Chern-Simons term:

$$\sigma_{AD}e^{\text{prop}}A_0\partial_0A_2 \rightarrow \sigma_{AD}B_{34}e^{\text{prop}}A_0\partial_0A_2$$

(up to a numerical factor), where $B_{34} \equiv \partial_3A_4 - \partial_4A_3$, and the indices $\mu, \nu, \rho$ in "$e^{\text{prop}}$" take value 0, 1, 2. According to this argument, in our boundary conditions given in Eq. (5), Eq. (6), and Eq. (7), we have taken $\partial_3A_4 - \partial_4A_3 = \phi/L_3L_4$, thus we have the dimensional reduction $\sigma_{AD}e^{\text{prop}}A_0\partial_0A_2 \rightarrow \sigma_{AD}e^{\text{prop}}A_0\partial_0A_2$. Intuitively, we have the evident identity

$$\frac{\partial}{\partial \phi}(\sigma_{AD}e^{\text{prop}}A_0\partial_0A_2) = \sigma_{AD}e^{\text{prop}}A_0\partial_0A_2$$

(8)

Since the right hand side of this equation is a 2D Chern-Simons term, it seems that we can calculate $\sigma_{AD}$ using well-known results of 2D quantum Hall effects. In practice, however, it is impossible to take the derivative with respect to $\phi$ because $\phi$ is quantized, i.e. $\phi$ takes only discrete values. To resolve this difficulty, we will take a difference instead of a derivative (see below).

Now our task is to formulate these intuitive arguments as a precise mathematical framework. We can define the Berry connection

$$a_i(\theta_1, \theta_2, \phi) = -i(\Psi(\theta_1, \theta_2, \phi)|\partial_0|\Psi(\theta_1, \theta_2, \phi))$$

(9)

and the Berry curvature

$$F_{ij}(\theta_1, \theta_2, \phi) = \frac{\partial a_i}{\partial \theta_j} - \frac{\partial a_j}{\partial \theta_i}$$

(10)
from which we can define a first Chern number

\[
C(\phi) = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 F_{12}(\theta_1, \theta_2, \phi)
\]  

(11)

where we have chosen the notation “C” instead of “C₁” to distinguish C with the first Chern number appearing in the 2D quantum Hall effect.²⁴

With these preparations, the general formula for \( \sigma_{4D} \) appearing in Eq. (1) is proposed as

\[
\sigma_{4D} = C(\phi_0) - C(0)
\]  

(12)

This is the difference between two first Chern numbers, the first of which is the Chern number with a unit flux \( \phi_0 \equiv 2\pi \) in \( T_{34} \), and the second is the Chern number without this flux, in other words, Eq. (12) measures the jump of the first Chern number after inserting a flux \( \phi_0 \) in \( T_{34} \). The necessity of the second term \( C(0) \) in Eq. (12) can be easily appreciated in a noninteracting model [see Eq. (19)] to be presented in Sec. III. It is also useful to note that \( C(0) \) may be zero if the ground state has certain symmetries. For instance, if there is time reversal symmetry, we have \( C(0) = 0 \) and \( \sigma_{4D} = C(\phi_0) \).

Eq. (12) is expressed in terms of the Berry phase of ground states wavefunctions on a torus, which is well-defined in the presence of arbitrary interaction and disorder.²² Eq. (12) can also be written equivalently as

\[
\sigma_{4D} = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 [F_{12}(\theta_1, \theta_2, \phi_0) - F_{12}(\theta_1, \theta_2, 0)]
\]  

(13)

Eq. (12) and Eq. (13) are among the central equations of the present paper.

Several remarks about Eq. (12) are in order. The noninteracting topological invariance for the 2D quantum Hall effect, namely the TKNN invariant,²² is expressed as the first Chern number in the Brillouin zone. The Niu- Thouless-Wu formula,²⁶ as a generalization of the TKNN invariant, is again a first Chern number. Given the second Chern number in Eq. (2) for the 4D noninteracting quantum Hall effect, we may try to express the 4D Hall coefficient \( \sigma_{4D} \) as a second Chern number on certain parameter space, for an interacting system. However, this attempt turns out to be unfruitful. Instead, the topological invariant defined in Eq. (12), which gives \( \sigma_{4D} \), is the difference between two first Chern numbers.

Let us conclude this section with a side remark that the Laughlin’s gauge argument²⁸ can also be generalized to 4D QH. The boundary condition in the \( x_3, x_4 \) direction are the same as given by Eq. (6) and Eq. (7), but the system is open along the \( x_2 \) direction. When we do the adiabatic evolution \( \theta_1 \to \theta_1 + 2\pi \), the charge transferred from the boundary \( x_3 = 0 \) to \( x_3 = L_2 \) is denoted as \( \Delta Q(\phi) \). The Hall conductance is given as \( \sigma_{4D} = \Delta Q(\phi_0) - \Delta Q(0) \).

III. THE NONINTERACTING LIMIT: TWO SIMPLE MODELS

In this section we will check in two simple noninteracting models [Eq. (14) and Eq. (19)] that Eq. (12) gives the same result as Eq. (2), as it should do in the noninteracting limit. Incorporating well-known results of topological classification of noninteracting insulators, we will show that Eq. (12) reduces to Eq. (2) for all noninteracting 4D insulators.

First let us consider a noninteracting Hamiltonian for 4D QH⁴¹

\[
h(k) = i \sum_{i=1}^{4} \sin k_i \Gamma^i + M(k) \Gamma^0
\]  

(14)

where \( M(k) = m + 4 - \sum_{i=1}^{12} \cos k_i \), \( v \) and \( m \) being parameters of the Hamiltonian, and \( k_i \in [0, 2\pi] \) is the \( i \)-th momentum of the free Bloch state (the lattice constant has been taken as unity). The Gamma matrices here satisfy the identities \( \{\Gamma^i, \Gamma^j\} = 2\delta^{ij} \). For our convenience we choose the representation \( \Gamma^1 = \tau^1, \Gamma^2 = \tau^2, \Gamma^3 = \tau^3\sigma^1, \Gamma^4 = \tau^1\sigma^2, \Gamma^5 = \tau^1\sigma^3 \).

Instead of solving the model numerically in the real space, which is less illuminating for our purpose, let us do calculation in the limit that \( |m| \) is significantly smaller than unity. In this limit we can keep only the \( k \)-linear terms near \( k = 0 \), and the Dirac Hamiltonian reads

\[
h(k) \approx i(p_1 \tau^1 + p_2 \tau^2) + \tau^3(vk_0 \sigma^1 + vk_4 \sigma^2 + m\sigma^3)
\]  

(15)

In the presence of twisted boundary conditions, the momenta should be replaced by \( k_1 \to -i\delta_1, k_2 \to -i\delta_2, k_3 \to k_3, k_4 \to k_4 \). Let us calculate the first term \( C(\phi_0) \) of Eq. (12) for the Dirac Hamiltonian in Eq. (13). In this linear-\( k \) limit, we can first solve the Hamiltonian \( h(k_3, k_4) = vk_3 \sigma^1 + vk_4 \sigma^2 + m\sigma^3 \), whose eigenvalues read²⁸

\[
E_0 = m; \quad E_{nk} = \pm \sqrt{m^2 + 2nBv^2} \quad (n = 1, 2, \cdots)
\]  

(16)

where \( B = \phi_0/L_2L_4 \). The corresponding eigen-wavefunctions are \( (\psi_0, 0)^T \) and \( (\psi_n, \pm \psi_{n-1})^T \), where \( \psi_0 \) is the wavefunction of the \( n \)-th Landau level of Schrodinger particles,²⁸ whose precise forms do not concern us for our purpose. It is useful to note that when \( m = 0 \), the existence of the zero mode \( E_0 \) is guaranteed by the Atiyah-Singer index theorem. Inputting the eigenvalues given in Eq. (16) into the second parenthesis in Eq. (15), we have a series of 2D Hamiltonians

\[
h_0 = v(k_1 \tau^1 + k_2 \tau^2) + m\tau^3;
\]

\[
h_{nk} = v(k_1 \tau^1 + k_2 \tau^2) + E_{nk} \tau^3 \quad (n = 1, 2, \cdots)
\]  

(17)

The value of \( C(\phi_0) \) can be obtained as the summation of the first Chern number of \( h_0 \) and \( h_{nk} \), namely \( \frac{1}{2} [\text{sgn}(E_0) + \sum_n \sum_{n=1}^{\infty} \text{sgn}(E_{nk})] = \frac{1}{2} \text{sgn}(m) \), thanks to the fact that the ground state wavefunctions is a Slater determinant of Bloch states in the noninteracting cases. In this calculation we have not been careful about the high energy regularization, thus we can only assert that \( C(\phi_0) = \frac{1}{2} \text{sgn}(m) \) + constant. Since we require \( C(\phi_0) = 0 \) as \( m \to +\infty \), we have \( C(\phi_0) = \frac{1}{2} \text{sgn}(m) - 1 \). Similarly we can obtain that \( C(0) = 0 \), therefore we have

\[
\sigma_{4D} = C(\phi_0) - C(0) = \frac{1}{2} \text{sgn}(m) - 1
\]  

(18)

which is the same as \( c_2 \) obtained from Eq. (2) [see also Ref. ²⁸ for calculations for a different model using charge pumping.]
Let us move to the second noninteracting model, which will explain the reason why we must include the second term $C(0)$ in Eq. (12). The simple model has the free Hamiltonian

$$h(k) = v(sin k_1 \tau^1 + sin k_2 \tau^2) + (m + 2 - cos k_1 - cos k_2) \tau^1\tau^2$$

which is independent on $k_3$ and $k_4$. If we take $m = -0.1$, then it is obvious that both $C(0)$ and $C(0)$ are nonzero, however, they are equal, therefore $\sigma_{2D} = C(0) - C(0) = 0$. From Eq. (2), it is obvious that we have $\sigma_{2D} = c_2 = 0$, therefore, Eq. (12) and Eq. (2) produce the same result in this example.

Although we have only explicitly checked that Eq. (12) reduces to Eq. (2) in Dirac models, it is possible to make a more general statement that Eq. (12) is always equivalent to Eq. (2) in the noninteracting limit. In fact, as has been shown in Ref. 15, 66, there is a Dirac-Hamiltonian representative in each class of the 4D QH insulators, which means that any noninteracting Hamiltonian for 4D insulator can always be smoothly connected to a Dirac Hamiltonian. Therefore, equivalence between Eq. (12) and Eq. (2) in Dirac model implies their equivalence for all noninteracting Hamiltonians. In the presence of interaction, however, Eq. (2) loses definition, while Eq. (12) remains useful.

IV. QUANTUM HALL EFFECT IN $d = 2l + 2$ SPATIAL DIMENSIONS

Eq. (12) can be generalized to $d = 2l + 2$ spatial dimensions. The boundary conditions for the $(x_1, x_2)$ direction given in Eq. (5) are unchanged, while the boundary conditions for other directions are defined as

$$\Psi(r_1, \cdots, r_k + L_{2j+1} \tilde{x}_{2j+1}, \cdots, r_N; \theta_1, \theta_2, \phi_1, \cdots, \phi_l) = \exp(-i\frac{\pi}{L_{2j+2}}) \Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \theta_2, \phi_1, \cdots, \phi_l)$$

and

$$\Psi(r_1, \cdots, r_k + L_{2j+2} \tilde{x}_{2j+2}, \cdots, r_N; \theta_1, \theta_2, \phi_1, \cdots, \phi_l) = \Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \theta_2, \phi_1, \cdots, \phi_l)$$

for $j = 1, 2, \cdots l$. Physically, these conditions mean that there is a flux $\phi_j$ in the 2D torus $T_{2j+1,2j+2}$. We can define the Berry connection

$$a_i(\theta_1, \theta_2, \phi_1, \cdots, \phi_l) = -i(\delta_i^k \Psi(\theta_1, \theta_2, \phi_1, \cdots, \phi_l))$$

for $i = 1, 2$, and a first Chern number

$$C(\phi_1, \cdots, \phi_l) = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 F_{12}(\theta_1, \theta_2, \phi_1, \cdots, \phi_l)$$

Now the $d$-dimensional Hall conductance is given by

$$\sigma_d = \sum_{\phi_i < \phi_j < \phi_0, \phi_0} (-1)^{\delta(\phi_0, 0)} C(\phi_1, \cdots, \phi_l) + \cdots - C(0, 0, 0)$$

where the delta function satisfies $\delta(\phi, 0) = 1$ when $\phi_i = 0$, and $\delta(\phi, 0) = 0$ when $\phi_i = \phi_0 \equiv 2\pi$. When $d = 4$ (i.e. $l = 1$), Eq. (24) reduces to Eq. (12). The original Niu-Thouless-Wu formula is also a special case of Eq. (24) with $d = 2$ (i.e. $l = 0$).

V. FRACTIONAL QUANTUM HALL EFFECTS

One of the main motivations for introducing the topological invariant in Eq. (12) is its potential applications in fractional quantum Hall states. Before moving to higher dimensions, let us first present a review of the Niu-Thouless-Wu formula of 2D fractional QH. As has been known from the Ref. 31, fractional quantization of 2D Hall conductance is possible if the ground states are degenerate on a 2D torus.

In 2D, the standard boundary condition is given by Eq. (5) except that the argument $\phi$ is absent. Suppose that a 2D fractional quantum Hall system has $p$-fold degenerate ground states $|\Psi_1(\theta_1, \theta_2)\rangle, \cdots, |\Psi_p(\theta_1, \theta_2)\rangle$. The Hall conductance is given by an average over these degenerate ground states $\bar{\sigma}_{2D}$ (recall that we have taken the units $\hbar = c = 1$)

$$\sigma_{2D} = \frac{1}{p} \int_0^{2\pi} d\theta_1 d\theta_2 \sum_{\alpha=1}^{p} (\langle \partial_{\theta_1} \Psi_\alpha | \partial_{\theta_2} \Psi_\alpha \rangle - \langle \partial_{\theta_2} \Psi_\alpha | \partial_{\theta_1} \Psi_\alpha \rangle)$$

$$= \frac{1}{p} \int_0^{2\pi} d\theta_1 d\theta_2 TrF_{12}(\theta_1, \theta_2) = \frac{1}{C_1}$$

(25)

where the matrix elements of the non-Abelian Berry curvature $F_{ij}$ read $F_{ii} = \partial a_{i}^{\alpha \beta} - \partial a_{i}^{\beta \alpha} + [a_{i}, a_{j}]^{\alpha \beta}$, in which $a_{i}^{\alpha \beta} = -i(\langle \Psi_{\alpha} | \partial_{\theta_{i}} | \Psi_{\beta} \rangle - \langle \Psi_{\beta} | \partial_{\theta_{i}} | \Psi_{\alpha} \rangle)$ is the non-Abelian Berry connection. The average Chern number $C_1 = \frac{1}{p} C_1 = \frac{1}{p} \int_0^{2\pi} d\theta_1 d\theta_2 TrF_{12}(\theta_1, \theta_2)$, where $C_1$ is the standard definition of the first Chern number of the $U(p)$ fiber bundle. Note that the $[a_{i}, a_{j}]$ term in $F_{ij}$ vanishes after the tracing. It is a mathematical fact that the first Chern number $C_1$ is quantized as an integer, therefore, the Hall conductance is quantized as a rational number with denominator $p$.

Eq. (25) can be rewritten as

$$\sigma_{2D} = \frac{1}{p} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 (\partial_{\theta_{1}} \Psi_{1} | \partial_{\theta_{2}} \Psi_{1} \rangle - \langle \partial_{\theta_{2}} \Psi_{1} | \partial_{\theta_{1}} \Psi_{1} \rangle)$$

(26)

where we have picked up a ground state $\Psi_1$ from the degenerate ground state $\Psi_1, \cdots, \Psi_p$. The parameter space has been enlarged to $(0 < \theta_1 < 2\pi, 0 < \theta_2 < 2\pi)$.

Now let us move to higher dimensions. For a 4D fractional QH system, suppose that the ground states are $p$-fold degenerate on the 4D torus $T^4$ with boundary conditions described in Sec. III in other words, the ground states form a $U(p)$ bundle over the 2D torus with coordinates $(\theta_1, \theta_2)$ (Note that $\phi$ is fixed). We can define the Berry connection $a_{i}^{\alpha \beta}(\theta_1, \theta_2, \phi) = -i(\langle \Psi_{\alpha} | \partial_{\theta_{i}} | \Psi_{\beta} \rangle - \langle \Psi_{\beta} | \partial_{\theta_{i}} | \Psi_{\alpha} \rangle)$ and the Berry curvature $F_{ij}^{\alpha \beta} = \partial a_{i}^{\alpha \beta} - \partial a_{i}^{\beta \alpha} + [a_{i}, a_{j}]^{\alpha \beta}$, Eq. (11) can be straightforwardly generalized as

$$C(\phi) = \frac{1}{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 TrF_{12}(\theta_1, \theta_2, \phi)$$

(27)
Note that in Sec[III] we considered non-degenerate ground state, therefore, the symbol “Tr” in Eq.(27) is absent Eq.(11). We can also define the average (first) Chern number for 4D QH as
\[ C(\phi) = C(\phi)/p \] (28)
By analogy with Eq.(25), the 4D Hall conductance \( \sigma_{4D} \) for fractional quantum Hall effects is obtained as
\[ \sigma_{4D} = \overline{C}(\phi_0) - \overline{C}(0) \] (29)
Eq.(29) is among the central results of this paper. In the presence of time reversal symmetry, the second term vanishes. Eq.(29) reduces to Eq.(12) when \( p = 1 \), namely the case without ground state degeneracy.

To conclude this section, we mention that the generalization of Eq.(24) for \( d = 2l + 2 \) dimensional fractional states read
\[ C_d = \sum \theta_i \cdots \theta_{i+\varphi_0} (-1)^{\varphi_0} \overline{C}(\phi_1, \ldots, \phi_l). \]

VI. MORE TOPOLOGICAL INVARIANT FOR 4D FRACTIONAL QH

Having studied the 4D fractional Hall conductance using the \((\theta_1, \theta_2, \phi)\) boundary conditions we have chosen, let us investigate other choices of boundary conditions. The simplest choice is
\[
\Psi(r_1, \ldots, r_k + L \hat{x}_i, \ldots, r_N; \theta_1, \theta_2, \theta_3, \theta_4) = \exp(i\theta_1) \Psi(r_1, \ldots, r_k, \ldots, r_N; \theta_1, \theta_2, \theta_3, \theta_4) \] (30)
for \( i = 1, 2, 3, 4 \). Suppose that the ground states are \( p\)-fold degenerate, then these ground states form an \( U(p) \) fiber bundle on the 4D torus parameterized by \((\theta_1, \theta_2, \theta_3, \theta_4)\) with \( 0 \leq \theta_i < 2\pi \). We can define a natural invariant
\[
C_2 = \frac{1}{32\pi^2} \int d^4\theta e^{i\beta \theta} \text{Tr} F_{ij} F_{kl} \] (31)
where the matrix elements of non-Abelian Berry curvature \( F_{ij} \) are defined as \( F_{ij}^{\alpha\beta} (\theta_1, \theta_2, \theta_3, \theta_4) = \partial_i a_j^{\alpha\beta} - \partial_j a_i^{\alpha\beta} + i[a_i, a_j]^{\alpha\beta} \), where \( i, j = 1, 2, 3, 4 \). Eq.(31) is a second Chern number defined for fractional QH states in 4D. It should be noted that there are many correct forms of the (lower-case) \( C_2 \) in Eq.(24), which is defined in terms of the free Bloch states of noninteracting systems.

For \( 2n \)-dimensional quantum Hall effects, we can straightforwardly generalize \( C_2 \) to \( C_n \) as
\[
C_n = \frac{1}{n!} \int \text{Tr}(F_{ij}^{\alpha\beta})^n = \frac{1}{2^n n!(2\pi)^n} \int d^{2n} \theta e^{i\sum_{\alpha\beta} \theta_{ij} \text{Tr} F_{ij}^{\alpha\beta} \cdots F_{ij}^{\alpha\beta}} \] (32)
which are topological invariants for higher-dimensional fractional QH states.

In 2D, the first Chern number \( C_1 \) of the \( U(p) \) bundle is proportional to the Hall conductance \( \sigma_{2D} \). In fact, Eq.(23) tells us that \( C_1 = \rho \sigma_{2D} \), thus \( C_1 \) does not give us new topological invariant other than \( \sigma_{2D} \) and \( \rho \). However, the 4D case is quite different. The key difference between 2D and 4D is as follows. For the 2D QH, both \( \sigma_{2D} \) and \( C_1 \) are defined under the same boundary condition parameterized by \((\theta_1, \theta_2)\). For 4D quantum Hall insulators, the topological invariants \( C_2 \) and \( \sigma_{4D} \) is defined using different boundary conditions [ Eq.(5), Eq.(6), and Eq.(7) for \( \sigma_{2D} \), but Eq.(30) for \( C_2 \)], therefore, there is no direct relation between \( C_2 \) and \( \sigma_{4D} \). In principle, \( C_2 \) can take different values given the same value of ground state degeneracy \( p \) and Hall coefficient \( \sigma_{4D} \). The topological invariants \( C_2 \) suggests that there are rich structures in 4D quantum Hall effects. Higher dimensional QHS are similar: Higher Chern numbers \( C_n (n = 2, 3, \ldots) \) are not directly related to the Hall coefficient \( \sigma_{4D} \) because they are defined under different boundary conditions.

VII. TOPOLOGICAL INSULATORS IN ONE-DIMENSION

In this section we will briefly discuss 1D topological insulators to prepare us for the investigation of 3D topological insulator in Sec[VIII]. One-dimensional topological insulators can be characterized by a \( \theta \) term, \( S_{\text{eff}} = \frac{1}{2\pi} \int dt dx e^{i\theta} \partial_\mu A_\mu \) (33)
Let us study the 1D insulator on a torus \( T^1 \), which is just a circle. We take the boundary condition as
\[
\Psi(r_1, \ldots, r_k + L \hat{x}_i, \ldots, r_N; \theta_1) = \exp(i\theta_1) \Psi(r_1, \ldots, r_k, \ldots, r_N; \theta_1) \] (34)
namely that there is a gauge potential \( A_0 = \theta_1/L_1 \).

Now there exists a simple topological invariant \( \Gamma \)
\[
\Gamma = \int_0^{2\pi} d\theta_1 a_1(\theta_1) \] (35)
where the Berry connection is defined as \( a_1(\theta_1) = -i(\Psi(\theta_1) \partial_{\theta_1} \Psi(\theta_1)) \). Eq.(35) is an interacting generalization of the Zak phase [89]. It has been applied to 1D models [30,31], though its relation to \( \theta \) term was not discussed. Eq.(35) is defined modulo \( 2\pi \) because a local gauge transformation of the wavefunction can change it by \( 2\pi \).

When the ground state \( |\Psi(\theta_1)\rangle \) is not degenerate, the \( \theta \) value is given by \( \theta = \Gamma \). Since we are mainly concerned with higher dimensional topological insulators, we will not study applications of this 1D formula in details. It is useful to mention that the quantity \( \partial_\theta \langle\Psi|\partial_{\theta} |\Psi\rangle \), where \( \lambda \) is a tuning parameter of the many-body Hamiltonian, is usually more useful than \( \theta \) itself, because \( \partial_\theta \langle\Psi|\partial_{\theta} |\Psi\rangle \) does not have any ambiguity under local gauge transformation of wavefunction [88].

When the ground states are \( p\)-fold degenerate, the natural generalization of Eq.(35) is
\[
\Gamma = \int d\theta_1 \text{Tr} a_1(\theta_1) \] (36)
where the non-Abelian gauge potential is defined as $a^{\alpha \beta} = -i(\Psi_{\alpha}(\theta_1)|\partial_\beta|\Psi_{\beta}(\theta_1))$. The $\theta$ angle in Eq. (33) is given by

$$\theta = \Gamma$$

(37)

where the average $\Gamma$ is defined as $\Gamma = \frac{1}{2}\Gamma$.

VIII. TOPOLOGICAL INSULATORS IN THREE-DIMENSIONS: INTEGER AND FRACTIONAL

The approach we applied to 4D QH states can be naturally generalized to 3D. The 3D boundary conditions are chosen as follows. First,

$$\Psi(r_1, \cdots, r_k + L_1 \hat{x}_1, \cdots, r_N; \theta_1, \phi) = \exp(i\theta_1)\Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \phi)$$

(38)

where $r_i$ is the coordinate of the $k$-th particle (other variables such as spin are not shown for simplicity of notation), and $\hat{x}_1$ is the unit vector along the $x_1$ direction. Second,

$$\Psi(r_1, \cdots, r_k + L_2 \hat{x}_2, \cdots, r_N; \theta_1, \phi) = \exp(-i\frac{\theta_1}{L_2})\Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \phi)$$

(39)

and

$$\Psi(r_1, \cdots, r_k + L_3 \hat{x}_3, \cdots, r_N; \theta_1, \phi) = \Psi(r_1, \cdots, r_k, \cdots, r_N; \theta_1, \phi)$$

(40)

where $\phi$ satisfies the same quantization condition as discussed in Sec. II. Now the $\theta$ angle in Eq. (3) is proposed (for the cases without ground state degeneracy) as

$$\theta = \Gamma(\phi_0) - \Gamma(0)$$

(41)

where $\phi_0 \equiv 2\pi$, and

$$\Gamma(\phi) = \int_0^{2\pi} d\theta_1 a_1(\theta_1, \phi)$$

(42)

$a_1(\theta_1, \phi) = -i(\Psi(\theta_1, \phi)|\partial_\phi|\Psi(\theta_1, \phi))$ being the Berry connection defined in terms of the ground state wavefunction. One can derive Eq. (41) by calculating the Berry phase gained by the adiabatic evolution $A_1 \rightarrow A_1 + 2\pi/L_1$. Due to the topological terms $\frac{1}{2}\partial_\phi a_1(\partial_2 A_1 - \partial_3 A_2)$ contained in the $\theta$ term, when a flux $\phi$ exists in $\mathcal{T}_{23}$, as Eq. (39) and Eq. (40) indicate, the adiabatic evolution of $A_1 \rightarrow A_1 + 2\pi/L_1$ generates a topological phase $\phi/2\pi$, which should be identified as the Berry phase accumulated by the adiabatic evolution of ground state wavefunction, namely $\int d\theta_1 a_1(\theta_1, \phi)$. It follows that Eq. (41) is the formula for $\theta$. Note that potentially there is another term $\partial_\theta a_0 A_1$ that can contribute to the Berry phase in the evolution $A_1 \rightarrow A_1 + 2\pi/L_1$, which is the reason why the second term in Eq. (41) appears.

If the Hamiltonian and the ground state depend on a tuning parameter, which we denote as $\theta_2$, then $\theta$ is a function of $\theta_2$.

The derivative of $\theta$ with respect to $\theta_2$ is given by the gauge-invariant formula

$$\frac{\partial \theta}{\partial \theta_2} = \int_0^{2\pi} d\theta_1 [F_{21}(\theta_1, \theta_2, \phi_0) - F_{21}(\theta_1, \theta_2, 0)]$$

(43)

where $F_{21}(\theta_1, \theta_2, \phi) = \partial_\phi a_1 - \partial_\phi a_2$, and $a_1(\theta_1, \theta_2, \phi) = -i(\Psi(\theta_1, \theta_2, \phi)|\partial_\phi|\Psi(\theta_1, \theta_2, \phi))$. Similar to the 1D case discussed in Sec. VII, the quantity $\partial \theta/\partial \theta_2$ is usually more useful than $\theta$ itself, because $\partial \theta/\partial \theta_2$ is invariant under any local gauge transformation of the wavefunction.

We will apply Eq. (41) to a noninteracting Dirac model in Appendix A, which gives the same result as obtained from Eq. (41).

In the above calculations we have assumed that the $\theta$ term is isotropic, which is always satisfied if there is time reversal symmetry (though the Maxwell terms are generally still anisotropic). If the $\theta$ term is anisotropic, namely that we have $\chi_i E_i B_j = \chi_j E_j e_{ij}(\partial_1 A_1 - \partial_2 A_2)$, we should calculate each coefficient $\chi_{ij}$ separately, which is also given by Eq. (41) except that the twisted phase $\theta_i$ in Eq. (38) is added in the $x_i$ direction instead of the $x_1$ direction, and the flux $\phi$ [see Eq. (39) and Eq. (40)] is added in the $(x_i, x_1)$ plane.

For 3D fractional states with $p$-fold ground state degeneracy, we can generalize Eq. (41) as

$$\theta = \Gamma(\phi_0) - \Gamma(0)$$

(44)

where $\Gamma(\phi) \equiv \frac{1}{p} \int_0^{2\pi} d\theta_1 \text{Tr}(\hat{a}_1(\theta_1, \phi))$. The logic is similar to Sec. VII. An important feature is notable here. We have the transformation rule $a_1 = U a_1 U^\dagger + iU \partial U^\dagger$ under a local gauge transformation of the basis of ground state wavefunction, where $U = U(\theta_1, \phi)$ is a $p \times p$ unitary matrix. This may change $\Gamma(\phi)$ by multiples of $2\pi/p$, therefore, the $\theta$ angle of fractional topological insulators is determined modulo $2\pi/p$.

As a digression, let us briefly mention the generalization for $d = 2l + 1$ (spatial dimensional (isotropic)) $\theta$ term if the system does not have ground state degeneracy on a $d$ dimensional torus. The formula reads

$$\theta_d = \sum_{\phi_1, \cdots, \phi_l} (-1)^l \Sigma_{\phi_1, \cdots, \phi_l} \Gamma(\phi_1, \cdots, \phi_l)$$

(45)

where $\Sigma_{\phi_1, \cdots, \phi_l} \equiv \sum_{\phi_1, \cdots, \phi_l} (-1)^l \Sigma_{\phi_1, \cdots, \phi_l} \Gamma(\phi_1, \cdots, \phi_l) = \Gamma(\phi_1, \cdots, \phi_l)/p$, the integer $p$ being the ground state degeneracy.

IX. CONCLUSIONS

In this paper we have defined precise topological invariants in terms of the ground state wavefunctions on a torus. This approach provides a conceptual framework in which many
topological invariants and topological-field-theoretical coefficients, such as $\sigma_{4D}$ (in 4D) and $\theta$ (in 3D), acquire precise definitions even in the presence of arbitrary interaction and disorder.

Numerically, we do not expect that the wavefunction (on a torus) approach followed in the present paper will be as efficient as the topological Hamiltonian approach mentioned in Sec.II. However, the present approach has a wider range of validity because it is applicable in the presence of arbitrary interaction and disorder, therefore, the present approach is highly desirable for certain purposes, especially when both interaction and disorder are present, or when the interaction is so strong that exotic fractional states are generated. It is also useful to note that the topological invariants in the present paper can also be applied to bosonic topological insulators, for which other topological invariants are hard to define.

X. ACKNOWLEDGEMENTS

ZW would like to thank Liang Kong and Yong-Shi Wu for helpful discussions. ZW is supported by NSFC under Grant No. 11304175. SCZ is supported by the Department of Energy, Office of Basic Energy Sciences, Division of Materials Sciences and Engineering, under contract DE-AC02-76SF00515.

Appendix A: Application in a three-dimensional noninteracting model

In the noninteracting limit, Eq. (41) should give the same $\theta$ as the noninteracting formula\footnote{ grinder@uci.edu }. In this appendix we will check this in a simple noninteracting model. This appendix follows similar calculations of Sec.III.

Let us study a simple 3D noninteracting Dirac model given as

$$h(k) = v \sin k_1 \tau^1 + [v \sin k_2 \sigma^1 + v \sin k_3 \sigma^2 + M(k) \sigma^3] \tau^3$$  \hspace{1cm} (A1)

where $M(k) = m + 3 - \sum_{i=1}^3 \cos k_i$. In the limit that $|m| << 1$, the low energy physics is dominated by the $k = 0$ region, and we can linearly expand $h(k)$ as

$$h(k) \approx v k_1 \tau^1 + (v k_2 \sigma^1 + v k_3 \sigma^2 + m \sigma^3) \tau^3$$  \hspace{1cm} (A2)

The boundary conditions are given in Eq. (38), Eq. (39), and Eq. (40), which mean that there is a flux $\phi$ inside the 2D torus $T^2$. First let us take $\phi = 0 \equiv 2\pi$. By a calculation similar to Sec.III we can first solve $h(k_1, k_3) = v k_2 \sigma^1 + v k_3 \sigma^2 + m \sigma^3$ after replacing $k_i \rightarrow -i(\partial_i - A_i)$, whose eigenvalues read

$$E_0 = m; \ E_{n\pm} = \pm \sqrt{m^2 + 2nBv^2} (n = 1, 2, \cdots)$$  \hspace{1cm} (A3)

and the corresponding wavefunctions are $(\psi_0, 0)^T$ and $(\psi_0, \pm \psi_{n-1})^T$. Now we put these eigenvalues back into the parenthesis of Eq. (A2), then we have a serial of 1D Hamiltonians

$$h_0 = v k_1 \tau^1 + m \tau^3; \ h_{n\pm} = v k_1 \tau^1 + E_{n\pm} \tau^3 (n = 1, 2, \cdots)$$  \hspace{1cm} (A4)

Now the $\Gamma(\phi_0)$ in Eq. (41) can be found as $\frac{\pi}{2} [\text{sgn}(E_0) + \sum_n \sum_{n=0} \text{sgn}(E_{n\pm})] = \frac{\pi}{2} \text{sgn}(m)$, which is similar to Sec.III. Again, due to the high-energy regularization, we can only assert that $\Gamma(\phi_0) = \frac{\pi}{2} \text{sgn}(m) + \text{constant}$. Consideration similar to Sec.III leads to $\Gamma(\phi_0) = \frac{\pi}{2} (\text{sgn}(m) - 1)$. Similarly we have $\Gamma(0) = 0$, therefore, from Eq. (41) it follows that

$$\theta = \Gamma(\phi_0) - \Gamma(0) = \frac{\pi}{2} (\text{sgn}(m) - 1)$$  \hspace{1cm} (A5)

which means that $\theta = -\pi$ when $m < 0$. This is consistent with the result obtained from the noninteracting Chern-Simons term\footnote{ wangzhongemail@gmail.com }.

\begin{itemize}
  \item[1] X.-L. Qi and S.-C. Zhang, Physics Today \textbf{63}, 33 (2010).
  \item[2] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. \textbf{82}, 3045 (2010).
  \item[3] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. \textbf{83}, 1057 (2011).
  \item[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. \textbf{95}, 226801 (2005).
  \item[5] B.A. Bernevig and S.C. Zhang, Phys. Rev. Lett. \textbf{96}, 106802 (2006).
  \item[6] C. L. Kane and E. J. Mele, Phys. Rev. Lett. \textbf{95}, 146802 (2005).
  \item[7] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science \textbf{314}, 1757 (2006).
  \item[8] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science \textbf{318}, 766 (2007).
  \item[9] L. Fu and C. L. Kane, Phys. Rev. B \textbf{74}, 195312 (2006).
  \item[10] J. E. Moore and L. Balents, Phys. Rev. B \textbf{75}, 121306 (2007).
  \item[11] X.-L. Qi, T. Hughes, and S.-C. Zhang, Phys. Rev. B \textbf{78}, 195424 (2008).
  \item[12] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. \textbf{98}, 106803 (2007).
  \item[13] R. Roy, Phys. Rev. B \textbf{79}, 195322 (2009).
  \item[14] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B \textbf{78}, 195125 (2008).
  \item[15] A. Kitaev, Proceedings of the L.D.Landau Memorial Conference "Advances in Theoretical Physics". Arxiv preprint 0901.2686 (2009).
  \item[16] S. Raghu, X.-L. Qi, C. Honerkamp, and S.-C. Zhang, Phys. Rev. Lett. \textbf{100}, 156401 (2008).
  \item[17] A. Shitade, H. Katsura, J. Kune, X.-L. Qi, S.-C. Zhang, and N. Nagaosa, Phys. Rev. Lett. \textbf{102}, 256403 (2009).
  \item[18] Y. Zhang, Y. Ran, and A. Vishwanath, Phys. Rev. B \textbf{79}, 245331 (2009).
  \item[19] B. Seradjeh, J. E. Moore, and M. Franz, Phys. Rev. Lett. \textbf{103}, 066402 (2009).
  \item[20] D. A. Pesin and L. Balents, Nat. Phys. \textbf{6}, 376 (2010).
\end{itemize}
The lower-case letter $c_2$ refers to the noninteracting limit, which should not be confused with the upper-case $C_2$ defined in Sec. VI. Enlarging the unit cell can partially overcome this difficulty, which enables an extension of this approach to disordered systems.

Twisted boundary conditions have also found applications elsewhere, see e.g. Refs. 92, 93. Note that there are two ways to define the twisted boundary condition. The first is to put the twisted phase factor $(\theta_1, \theta_2, \phi)$ in the wavefunction, as we did in Eq. (5), Eq. (6), and Eq. (7). The second way is to add phase parameters in the Hamiltonian instead of the wavefunction. These two ways are equivalent and can be translated into each other. In fact, a gauge transformation of the wavefunction changes the twisted boundary condition to the periodic boundary condition, at the price of adding the twisted phase factor to the Hamiltonian.

This is analogous to the Kaluza-Klein compactification.

We can also write
$$\sigma_4 D = \left[ C(n\phi_0) - C(0) \right]/n$$
for any integer $n$. For simplicity, this will not be pursued in the present paper.

It is worth noting that such a basis $\{|\Psi_1\rangle, \cdots, |\Psi_p\rangle\}$ can be found only locally, namely that they can be defined only in a topologically trivial patch on the 2D torus parameterized by $(\theta_1, \theta_2)$. The quantities we actually need, such as $\text{Tr} F_{12}(\theta_1, \theta_2)$, are independent on the basis choices.

For simplicity, we assume that the degenerate ground states cannot be divided into smaller subspaces, with each subspace forming a fiber bundle over the torus parameterized by $(\theta_1, \theta_2)$. Otherwise we can just pick up one subspace that cannot be reduced further into smaller subspaces, and everything discussed in this section will be unchanged.