Binary classification with noisy quantum circuits and noisy quantum data.

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We study the effects of single-qubit noises in the quantum circuit and the corruption in the quantum training data to the performance of binary classification problem. We find that under the presence of errors, the measurement at a qubit is affected only by the noise in the corresponding qubit, and that the errors on other qubits do not affect this outcome. Furthermore, for the task where we fit a binary classifier using a quantum training data, we show that the noise in the data can work as a regularizer, implying that we can benefit from the noise in certain cases. We support our findings with simulations.

1 Introduction

After the discovery of the intricate connection between machine learning algorithms and quantum computation, the last decade saw an explosion of work in both disciplines. The advancements in hardware technology allowed scholars to reach longer coherence times for qubits and gates with higher fidelity [5, 7, 2]. This news occasionally also comes with some preliminary breakthroughs, such as achieving a quantum advantage in random number generation problems [1]. Besides these straddles in quantum computation, the qubits and gate operations are still not perfect i.e. noise, decoherence are still significant problems that needs to be addressed [8].

The variational quantum classification (VQC) algorithms are one of the most important algorithms in quantum machine learning (QML) [6]. The performance of VQC is degraded when qubit and gate operations are imperfect. In this work, we investigate the effect of quantum noise in the binary classification task. Our findings can be summarized in three theorems: In Theorem 1, we find that as long as we only have single-qubit noises in a quantum circuit, the measurement at a qubit is affected only by the error in the corresponding qubit. This result holds even if the circuit includes entangling gates before the errors appear. In Theorem 2, we derive a closed form formula for the corrupted measurement under Krauss and coherent type errors. We find that the measurement of the qubit is robust to the errors in the sense that the binary decision is not affected by the noise for most input values. In Theorem 3, we look into the problem of fitting a classifier using a quantum data, and find that the noise in the data can work as a regularizer, implying that it can be beneficial in some cases. We add a few numerical experiments on simulated circuits in Cirq library to support our findings. The assumptions in this work are intended to reflect the operating conditions of a real device but we do not claim they fully capture all the important characteristics of the noise model in qubits. Increasing the sophistication of the assumed noise model is a subject for future investigation. We explain the proofs of our theorems in the Appendix.

1.1 General notation

In this section, we will explain the notation used in this work. For a given quantum state $\psi$, $P_\psi \{m^j = 1\}$ denotes the probability to measure the qubit number $j$ in excited state (1) defined by $\psi$. For example, we write $P_{W,\Phi(x)} \{m^1 = 1\}$ to denote the probability of observing 1 from the first qubit of the output state when the input $x$ is encoded by $\Phi$ and then is input to a quantum circuit defined by $W$. 
2 Effect of single qubit noises on the measurement of a quantum binary classifier

In this section, we discuss how single qubit noises affect the quality of quantum binary classifiers.

2.1 Problem setting

Consider a quantum circuit semantically factorized into two pieces such that the input $x$ passes through an encoder $\Phi$ (to load the classical input) and then an operational circuit $W$. We can apply such quantum circuit for a binary classification task, hereby looking at the measurements at one qubit.

To make this idea concrete, suppose we have two classes $\{-1, 1\}$, and for each $x$ in the input space $X$, its class, denoted by $c(x)$, is either $-1$ or $1$. Now suppose we are given a quantum binary classifier $\hat{c}$, which classifies $x$ by

$$\hat{c}(x) = \text{sign}(m(x)),$$

where

$$m(x) = \mathbb{P}_{W\Phi(x)}\{m^1 = 1\} - \frac{1}{2} = \langle \Phi(x)|W^\dagger M_1 W|\Phi(x)\rangle - \frac{1}{2}$$

denotes the probability of observing $1$ at the first qubit. Here, $m^1$ denotes the measurement at the first qubit, and $M_1$ denotes the corresponding measurement operator, given by

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_2 \otimes \cdots \otimes I_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix}.$$

For example, $\hat{c}$ could be a classifier fitted on some dataset - in that case, $\hat{c}(x)$ works as an estimate of the true class of $x$. We have $\hat{c} = c$ in the settings where the true class $c$ is determined by the circuit we have. In any case, we aim to investigate the effect of noise in the output of a given quantum binary classifier.

Now we introduce noise model. We denote the corrupted circuit and the encoder by $\tilde{W}$ and $\tilde{\Phi}$, respectively. We assume the following:

$$\hat{W} = UW,$$

$$\tilde{\Phi}(x) = V\Phi(x),$$

where

$$U = U_1 \otimes U_2 \otimes \cdots \otimes U_n,$$

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_n,$$

are unitaries with single qubit gates, which can be random. In other words, we assume that an additional random gate appears for each of the qubits and no additional entanglement occurs through the noise. For example, coherent error in a set of parameterized single-qubit rotation gates would satisfy this condition.

2.2 Main results

Our first finding is that the output distribution of the measurement at a qubit is not affected by the noises in the other qubits that occur after entangling gates in the circuit.

**Theorem 1.** Assume noise model (1), and let $U' = U_1 \otimes I_2 \otimes \cdots \otimes I_2$ and $V' = V_1 \otimes I_2 \otimes \cdots \otimes I_2$.

1. If there’s no encoder noise, i.e., $V \equiv I_{2^n}$, then the measurement of the first qubit does not depend on $\{U_k : 2 \leq k \leq n\}$. In other words,

$$P_{UW\Phi(x)}\{m^1 = 1\} = P_{U'W\Phi(x)}\{m^1 = 1\}$$

holds for any input $x$ and any set of random unitary matrices $U_2, U_3, \ldots, U_n \in \mathbb{C}^{2 \times 2}$, where $m^1$ is a shorthand notation for the measurement of the first qubit in the circuit.

2. Moreover, if $W$ have no entangling gates, i.e., it has the form of

$$W = W_1 \otimes W_2 \otimes \cdots \otimes W_n,$$

then

$$P_{UWV\Phi(x)}\{m^1 = 1\} = P_{U'W'V'\Phi(x)}\{m^1 = 1\}$$

for any input $x$ and any set of random unitary matrices $U_2, U_3, \ldots, U_n \in \mathbb{C}^{2 \times 2}$ and $V_2, V_3, \ldots, V_n \in \mathbb{C}^{2 \times 2}$.

**Remark 1.** In the proof of Theorem 1, we actually prove a more general result - the statements hold for $U = U_1 \otimes U_{2^n}$ and $V = V_1 \otimes V_{2^n}$, where $U_{2^n}$ and $V_{2^n}$ are any unitaries with $n - 1$ qubits. In other words, for the invariance of the distribution of the measurement at the first qubit, it is only required that there is no entangling noise that includes the first qubit.
Figure 1: The schematic for our setup in section 2 for Theorem 1. The state $|\Phi(x)\rangle$ is prepared with some input $x$. (a) Then an arbitrary unitary $W$ is applied and then qubit 1 is measured, under the presence of noise in $W$. Each $U_i$ is a random unitary which represents the noise applied to qubit $i$. (b) The setting where $W$ has no entangling gates and we additionally have the noise $V = V_1 \otimes \cdots \otimes V_n$ in the encoder.

Theorem 1 implies that no matter how many qubits we have, as long as we have only the single-qubit noises, the performance of the quantum classifier is affected only by the noise in the qubit we are measuring. In fact, this statement implies that we can simply remove all the single-qubit gates in the other qubits that appear after entanglements - it will change the quantum circuit, but the resulting classifier remains the same.

In order to test the validity of this statement, we constructed the circuit in Figure 1 (a) and (b) on the simulator model at Cirq and Pennylane [3]. We ran a circuit as in Figure 1 with 4 to 16 qubits with simulated error models and the error between the measurement of the first qubit without noise and the measurement in the presence of the noise is always not larger than the machine precision, $10^{-16}$.

Now we look further into the second setting of Theorem 1 and set up a specific model for the individual noise terms to investigate the range of values that

$$\hat{m}(x) = P_{UWV}\Phi(x)m^1 = 1 - \frac{1}{2}$$

can have.

For the noise in $W$, it is enough to specify the model for $U_1$, which acts only on qubit 1, by the result of Theorem 1. We assume

$$U_1 = e^{-i\frac{\mu+\epsilon^2}{2}X} \text{ where } \epsilon \sim N(0, \sigma^2). \quad (2)$$

Here, $N(0, \sigma^2)$ denotes the normal distribution with mean 0 and variance $\sigma^2$. For the encoder, again it is sufficient to set up a model for the noise in the first qubit.

$$V_1 = \sigma_C, \text{ where }$$

$$P\{C = j\} = \begin{cases} 1 - 3p & \text{if } j = 0 \\ p & \text{if } j = 1, 2, 3 \end{cases}. \quad (3)$$

Here, we let $\sigma_0 = I_2$(i.e., no noise). In other words, we assume that an additional $\sigma_1$, $\sigma_2$, or $\sigma_3$ gate randomly appears each with probability $p$ where $\sigma_j$ represents the standard Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Under this noise, we have the following distribution of the measurement.
Theorem 2. If \( W \) has no entangling gate, then under noise models of (2) and (3), it holds that

\[
\hat{m}(x) \in (1 - 4p) \cdot e^{-\frac{1}{2}\sigma^2} \cos \mu \cdot \left[ m(x) - \frac{1}{2} \cdot |\tan \mu|, m(x) + \frac{1}{2} \cdot |\tan \mu| \right].
\]

Theorem 2 implies that as long as \( p < 1/4 \) and \( m(x) \geq \frac{1}{2} \cdot |\tan \mu| \), the sign of \( m(x) \) remains the same. In other words, \( \text{sign} (\hat{m}(x)) = \text{sign}(m(x)) \) for \( x \) values that are not very close to the separating boundary, and the predicted label for such \( x \) is theoretically not affected by the noises, i.e., the binary classification task is robust to the noise.

However, Theorem 2 also implies that the noise can shrink \( m(x) \). For example, let us look into the setting where \( \mu = 0 \). In this case, we have \( \hat{m}(x) = (1 - 4p) e^{-\frac{1}{2}\sigma^2} \cdot m(x) \) and the shrinkage occurs by the terms \( 1 - 4p \) and \( e^{-\frac{1}{2}\sigma^2} \). This shrinkage of \( m(x) \) implies that a larger sample of measurements would be required for an accurate decision.

2.3 Numerical Experiments

Here we check the validity of Theorem 2 through simulations. We compute \( m(x) \), which is the cost function when no noise is present, and we compute \( \hat{m}(x) \) when the circuit is subject to the error models defined in (2) and (3). We simulate the setting where each \( W_k \) is a rotation gate. We randomly generate \( \theta = (\theta_1, \cdots, \theta_n) \) 100 times and measure the output from \( W(\theta) = W_1(\theta_1) \otimes \cdots \otimes W_n(\theta_n) \). For multiple values of \( p \) that determines the rate of occurrence of errors, we sketch \( \frac{\hat{m}(x)}{m(x)} \) for fixed \( \sigma \), the variance of the coherent error. The state \( \Phi(x) \) we prepare is \( \Phi(x) = [0101] \), and then we apply \( W \) single-qubit gates. Recall that in the setting where \( \mu = 0 \), we are expected to have \( \frac{\hat{m}(x)}{m(x)} \) proportional to \( 1 - 4p \), i.e., the ratio is a linear function of \( p \). This is illustrated in Figure 2.

3 Effect of single qubit noises in the training data on the fitted classifier

In this section, we look into the problem of training a binary classifier with noisy quantum data.

Figure 2: Empirical results for Theorem 2. \( \hat{m}(x)/m(x) \) vs. the Kraus probability \( p \). \( \sigma = 1 \) (blue), \( \sigma = 2 \) (orange), \( \sigma = 5 \) (green), \( \sigma = 7.5 \) (red), \( \sigma = 10 \) (purple). We set \( \Phi(x) = [0101] \) and \( \mu = 0 \).
and the estimator of $\theta$ is obtained by minimizing the empirical risk:

$$\hat{\theta}_N = \arg\min_{\theta \in \mathbb{R}^k} \tilde{R}_N(\theta) = \arg\min_{\theta \in \mathbb{R}^k} \frac{1}{N} \sum_{i=1}^{N} \ell(m_\theta(\psi_i) \cdot Y_i).$$

Now suppose we observe a noisy data $\{(\psi_i, Y_i)\}_{1 \leq i \leq N}$ instead of $\{(\psi_i)\}_{1 \leq i \leq N}$. Then what we obtain by applying the above procedure is the corrupted empirical risk

$$\tilde{R}_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} \ell(m_\theta(\tilde{\psi}_i) \cdot Y_i)$$

where

$$\tilde{m}_\theta(\psi_i) = \langle \psi_i | V_i W(\theta)^\dagger M_1 W(\theta) V_i | \psi_i \rangle - \frac{1}{2}.$$

and the corrupted estimator

$$\hat{\theta}_N = \arg\min_{\theta \in \mathbb{R}^k} \tilde{R}_N(\theta).$$

We will look into the performance of $\hat{\theta}_N$ and compare it to the performance of the true estimator $\hat{\theta}_N$.

For the noise $V_i$, we consider the bitflip noise which we considered in (3).

**Assumption 1.**

$$V_i = \sigma_{C_i}, i = 1, 2, \cdots, N,$$

where $\{C_1, C_2, \cdots, C_N\}$ is an i.i.d sample from the distribution of $C$ defined as the following.

$$\mathbb{P}\{C = j\} = \begin{cases} 1 - 3p & \text{if } j = 0 \\ p & \text{if } j = 1, 2, 3 \end{cases}.$$

We assume the following for the loss function $\ell$.

**Assumption 2.** The loss function $\ell : (-\frac{1}{2}, \frac{1}{2}) \mapsto (0, \infty)$ satisfies

(a) $\ell$ is convex and monotone decreasing,

(b) $\ell(0) = 1$.

Assumption 2 is satisfied by most well known loss functions, such as hinge loss $\ell(t) = (1 - t)_+$ and logistic loss $\log(1 + \exp(-t))/\log(2)$.

3.2 Main results

Our main finding is that the noise in the data can work as a regularizer, hence we can benefit from the noise in some cases. The idea follows the argument of [4] which proves a similar result in the setting where we fit a classifier with a non-quantized data $\{(X_i, Y_i)\}_{1 \leq i \leq N}$.

**Theorem 3.** Under Assumption 1 and 2, the conditional expectation of the corrupted empirical risk given the training data $\{(\psi_i, Y_i)\}_{1 \leq i \leq N}$ is given by

$$E[\tilde{R}_N(\theta) | \psi_{1:N}, Y_{1:N}] = (1 - 4p) \cdot \left[ \tilde{R}_N(\theta) + \lambda \cdot \tilde{P}_N(\theta) \right],$$

where $\tilde{P}_N(\theta)$ satisfies

$$\tilde{P}_N(\theta) \geq \frac{1}{N} \sum_{i=1}^{N} \left( \ell\left(\frac{1}{2} | m_\theta(\psi_i) | \right) + \ell\left(-\frac{1}{2} | m_\theta(\psi_i) | \right) \right).$$

and $\lambda = \frac{4p}{1 - 4p}$.

Therefore, the corrupted estimator $\hat{\theta}_N$ can be thought as an approximation of the regularized estimator

$$\theta_N^{\text{regularized}} = \arg\min_{\theta \in \mathbb{R}^k} \left[ \tilde{R}_N(\theta) + \lambda \cdot \tilde{P}_N(\theta) \right]$$

where $\lambda = 4p/(1 - 4p)$. Note that under Assumption 2, $t \mapsto \frac{\ell\left(\frac{1}{2} t \right) + \ell\left(-\frac{1}{2} t \right)}{2}$ is nondecreasing at $t > 0$, implying that having $\tilde{P}_N(\theta)$ as a penalty term in (4) results in a shrinkage on $|m_\theta(\psi_i)|$ values. This prevents the resulting estimator $\hat{\theta}_N$ from excessively overfitting the training data.

4 Conclusions

In this work, we studied the effect of the single-qubit errors on the classification performance.

The most unexpected result from our finding is that even in the case of entangling gates in a quantum circuit, the noise on a gate or qubit is only going to affect the measurement from the specific qubit i.e. the noise from other qubits is not going to corrupt the measurement at the qubit of interest.

Our work also shows that the noise in the training data can even be beneficial for the goal of finding an optimal quantum classifier. These properties depend on the structure of the problem, e.g.,
binary classification with the measurement at the first qubit, which we discuss in this work.

Many questions are remaining. Can we still observe such robustness of the measurement to the noise even in the case where we can have entangling gates as the noise, or in the setting where our goal is beyond binary classification? We aim to explore such questions in the future works.

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A  Proof of Theorem 1

Let us write

\[ U_{2:n} = U_2 \otimes \cdots \otimes U_n \quad \text{and} \quad U_1 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \]

so that \( U \) can be written as

\[ U = U_1 \otimes U_{2:n} = \begin{bmatrix} u_{11} \cdot U_{2:n} & u_{12} \cdot U_{2:n} \\ u_{21} \cdot U_{2:n} & u_{22} \cdot U_{2:n} \end{bmatrix}, \]

and write

\[ \Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix}. \]

Then we have

\[ \mathbb{P}_{UW \Phi(x)} \{ m^1 = 1 \} \]
\[ = \mathbb{E} \left[ \mathbb{P}_{UW \Phi(x)} \{ m^1 = 1 \mid U \} \right] \]
\[ = \mathbb{E} \left[ (\Phi(x)|W^\dagger U^\dagger M_1 UW \Phi(x)) \right] \]
\[ = \mathbb{E} \left[ \begin{bmatrix} 0 & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix} \begin{bmatrix} u_{11} \cdot U_{2:n} & u_{12} \cdot U_{2:n} \\ u_{21} \cdot U_{2:n} & u_{22} \cdot U_{2:n} \end{bmatrix} \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix} \right] \]
\[ = \mathbb{E} \left[ \left\| u_{21} \cdot U_{2:n} \Phi_1(x) + u_{22} \cdot U_{2:n} \Phi_2(x) \right\|^2 \right] \]
\[ = \mathbb{E} \left[ \left\| u_{21} \cdot \Phi_1(x) + u_{22} \cdot \Phi_2(x) \right\|^2 \right], \quad \text{since} \; U_{2:n} \; \text{is unitary} \]

which does not depend on the distribution of \( U_{2:n} \). Hence, the first claim is proved.

Similarly, for the second claim we write

\[ W_{2:n} = W_2 \otimes \cdots \otimes W_n \quad \text{and} \quad W_1 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \]

and

\[ V_{2:n} = V_2 \otimes \cdots \otimes V_n \quad \text{and} \quad V_1 = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \]

then it holds that

\[ \mathbb{P}_{UWV \Phi(x)} \{ m^1 = 1 \} \]
\[ = \mathbb{E} \left[ \mathbb{P}_{UWV \Phi(x)} \{ m^1 = 1 \mid U, V \} \right] \]
\[ = \mathbb{E} \left[ (\Phi(x)|W^\dagger WV^\dagger U^\dagger M_1 UWV \Phi(x)) \right] \]
\[ = \mathbb{E} \left[ \left\| \begin{bmatrix} 0 & 0 \\ 0 & I_{2^{n-1}} \end{bmatrix} \begin{bmatrix} u_{11} \cdot U_{2:n} & u_{12} \cdot U_{2:n} \\ u_{21} \cdot U_{2:n} & u_{22} \cdot U_{2:n} \end{bmatrix} \begin{bmatrix} w_{11} \cdot W_{2:n} & w_{12} \cdot W_{2:n} \\ w_{21} \cdot W_{2:n} & w_{22} \cdot W_{2:n} \end{bmatrix} \begin{bmatrix} v_{11} \cdot V_{2:n} & v_{12} \cdot V_{2:n} \\ v_{21} \cdot V_{2:n} & v_{22} \cdot V_{2:n} \end{bmatrix} \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix} \right\|^2 \right] \]
\[ = \mathbb{E} \left[ \left\| (u_{21} w_{11} v_{11} + u_{22} w_{21} v_{11} + u_{21} w_{12} v_{21} + u_{22} w_{22} v_{21}) \cdot U_{2:n} W_{2:n} V_{2:n} \Phi_1(x) \right\|^2 \right] \]
\[ + \left\| (u_{21} w_{11} v_{12} + u_{22} w_{21} v_{12} + u_{21} w_{12} v_{22} + u_{22} w_{22} v_{22}) \cdot U_{2:n} W_{2:n} V_{2:n} \Phi_2(x) \right\|^2 \]
\[ = \mathbb{E} \left[ \left\| (u_{21} w_{11} v_{11} + u_{22} w_{21} v_{11} + u_{21} w_{12} v_{21} + u_{22} w_{22} v_{21}) \Phi_1(x) \right\|^2 \right] \]
\[ + \left\| (u_{21} w_{11} v_{12} + u_{22} w_{21} v_{12} + u_{21} w_{12} v_{22} + u_{22} w_{22} v_{22}) \Phi_2(x) \right\|^2 \],
where the last equality holds since $U_{2n}W_{2n}V_{2n}$ is unitary. Therefore, $\tilde{m}_\theta(x)$ does not depend on $U_{2n}$, $W_{2n}$ and $V_{2n}$ and thus the second claim is proved.

B Proof of Theorem 2

We first introduce a lemma.

**Lemma 1.** Assume noise model (1), (2) and (3), and suppose $W$ has the form of

$$W = W_1 \otimes \cdots \otimes W_n.$$ Define

$$m^j(x) = \mathbb{P}_{UW^j\Phi(x)} \{ m^1 = 1 \mid C = j \} - \frac{1}{2}$$

for $j \in \{0, 1, 2, 3\}$. Then the following statements hold.

(a) $m^0(x) + m^1(x) + m^2(x) + m^3(x) = 0$,

(b) $|m^0(x) - \cos \mu \cdot e^{-\frac{1}{2}\sigma^2} \cdot m(x)| \leq \frac{1}{2} |\sin \mu| \cdot e^{-\frac{1}{2}\sigma^2}$.

Applying Lemma 1, we have

$$\hat{m}(x) = \mathbb{P}\{ m^1 = 1 \} - \frac{1}{2}$$

$$= \mathbb{E}[\mathbb{P}\{ m^1 = 1 \mid C \}] - \frac{1}{2}$$

$$= (1 - 3p) \cdot m^0(x) + p \cdot m^1(x) + p \cdot m^2(x) + p \cdot m^3(x)$$

$$= (1 - 4p) \cdot m^0(x) + p \cdot (m^0(x) + m^1(x) + m^2(x) + m^3(x))$$

$$= (1 - 4p) \cdot m^0(x),$$

and the desired statement immediately follows from (b) of Lemma 1.

C Proof of Theorem 3

We apply the following lemma:

**Lemma 2.** Under Assumption 2, the following inequality holds for any $a, b, c, d \in \mathbb{R}$ with $a+b+c+d = 0$.

$$\frac{\ell(a) + \ell(b) + \ell(c) + \ell(d)}{4} \geq \frac{\ell\left(\frac{1}{2}|a|\right) + \ell\left(-\frac{1}{2}|a|\right)}{2}.$$ The proof is given in the next section. Now we prove Theorem 3. The arguments apply the idea of [4]. We first compute

$$\mathbb{E}[\tilde{R}_N(\theta) \mid \psi_{1:N}, Y_{1:N}]$$

$$= \mathbb{E}[\tilde{R}_N(\theta) \mid \psi_{1:N}, Y_{1:N}, C]$$

$$= (1 - 3p) \cdot \frac{1}{N} \sum_{i=1}^{N} \ell(m^0_\theta(\psi_i) \cdot Y_i) + p \cdot \frac{1}{N} \sum_{i=1}^{N} \ell(m^1_\theta(\psi_i) \cdot Y_i) + p \cdot \frac{1}{N} \sum_{i=1}^{N} \ell(m^2_\theta(\psi_i) \cdot Y_i) + p \cdot \frac{1}{N} \sum_{i=1}^{N} \ell(m^3_\theta(\psi_i) \cdot Y_i)$$

$$= (1 - 4p) \tilde{R}_N(\theta) + 4p \cdot \frac{1}{N} \sum_{i=1}^{N} \ell(m_\theta(\psi_i) \cdot Y_i) \ell(m^1_\theta(\psi_i) \cdot Y_i) + \ell(m^2_\theta(\psi_i) \cdot Y_i) + \ell(m^3_\theta(\psi_i) \cdot Y_i)$$

where

$$m^j_\theta(x) = \langle \Phi(x) \mid (\sigma_j \otimes I_{2^{n-1}})W(\theta)^\dagger M_1 W(\theta)(\sigma_j \otimes I_{2^{n-1}})\Phi(x) \rangle - \frac{1}{2}$$
Applying Lemma 1 with $U = I_{2n}$, we have

$$m_\theta(x) + m_\delta^1(x) + m_\delta^2(x) + m_\delta^3(x) = 0$$

Therefore, by lemma 2, it holds that

$$\frac{1}{N} \sum_{i=1}^{N} \ell(m_\theta(\psi_i) \cdot Y_i) + \ell(m_\delta^1(\psi_i) \cdot Y_i) + \ell(m_\delta^2(\psi_i) \cdot Y_i) + \ell(m_\delta^3(\psi_i) \cdot Y_i) \geq \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\ell(\frac{1}{3}|m_\theta(\psi_i)|)}{2} + \ell\left( - \frac{1}{3}|m_\theta(\psi_i)| \right) \right),$$

and this proves the claim.

D Proof of lemmas

D.1 Proof of Lemma 1

By Theorem 1, we may assume $U = U_1 \otimes I_{2n-1}$ and $V = V_1 \otimes I_{2n-1}$. First observe

$$m^j(x) = \mathbb{P}\{m^1 = 1 \mid C = j\} - \frac{1}{2}$$

$$= \mathbb{E}\{m^1 = 1 \mid C = j, \varepsilon\} - \frac{1}{2}$$

$$= \mathbb{E}\left[ \langle \Phi(x) | (\sigma_j \otimes I_{2n-1})^\dagger W^\dagger (U_1 \otimes I_{2n-1})^\dagger \mathcal{M}_1(U_1 \otimes I_{2n-1}) W (\sigma_j \otimes I_{2n-1}) | \Phi(x) \rangle \right] - \frac{1}{2}.$$  

Next, we write

$$U_1 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad W_1 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

and write $\Phi(x)$ in a block matrix form.

$$\Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix}, \quad \Phi_1(x), \Phi_2(x) \in \mathcal{C}^{2n-1}.$$  

Now if $j = 0$,

$$m^0(x) = \| (u_{21} w_{11} + u_{22} w_{21}) W_{2n} \Phi_1(x) + (u_{21} w_{12} + u_{22} w_{22}) W_{2n} \Phi_2(x) \|^2 - \frac{1}{2}$$

$$= \| (u_{21} w_{11} + u_{22} w_{21}) \Phi_1(x) + (u_{21} w_{12} + u_{22} w_{22}) \Phi_2(x) \|^2 - \frac{1}{2}.$$  

If $j = 1$,

$$m^1(x) = \| (u_{21} w_{11} + u_{22} w_{21}) W_{2n} \Phi_2(x) + (u_{21} w_{12} + u_{22} w_{22}) W_{2n} \Phi_1(x) \|^2 - \frac{1}{2}$$

$$= \| (u_{21} w_{11} + u_{22} w_{21}) \Phi_2(x) + (u_{21} w_{12} + u_{22} w_{22}) \Phi_1(x) \|^2 - \frac{1}{2}.$$
Let us define

\[ m^2(x) = \|-(u_{21}w_{11} + u_{22}w_{21})\Phi_2(x) + (u_{21}w_{12} + u_{22}w_{22})\Phi_1(x)\|^2 - \frac{1}{2} \]

and

\[ m^3(x) = \|(u_{21}w_{11} + u_{22}w_{21})\Phi_1(x) - (u_{21}w_{12} + u_{22}w_{22})\Phi_2(x)\|^2 - \frac{1}{2}. \]

Now we define \( a_1 = u_{21}w_{11} + u_{22}w_{21} \) and \( a_2 = u_{21}w_{12} + u_{22}w_{22} \), then we can write

\[
m^0(x) + m^1(x) + m^2(x) + m^3(x) \\
= \|a_1\Phi_1(x) + a_2\Phi_2(x)\|^2 + \|a_1\Phi_2(x) + a_2\Phi_1(x)\|^2 + \|-a_1\Phi_2(x) + a_1\Phi_1(x)\|^2 + \|a_1\Phi_1(x) - a_2\Phi_2(x)\|^2 - 2 \\
= 2(\|a_1\|^2 + |a_2|^2)(\|\Phi_1(x)\|^2 + \|\Phi_2(x)\|^2) - 2 \\
= 2 \cdot 1 \cdot \|\Phi(x)\|^2 - 2 \\
= 0,
\]

where the third equality holds since the vector \((a_1, a_2)\) is the second row of the matrix \(U_1W_1\) which is unitary. Hence, (a) is proved.

Next, we define

\[
\Psi(x) = \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix} = W_1\Phi(x)
\]

and compute

\[
m^0(x) = \mathbb{E}[\{\text{measurement} = 1 \mid \epsilon, C = 0\}] - \frac{1}{2} \\
= \mathbb{E}[\Phi(x)|W_1^\dagger M_1 W_1|\Phi(x)] - \frac{1}{2} \\
= \mathbb{E}\left[\begin{bmatrix} 0 & 0 \\ I_{2n-1} & 0 \end{bmatrix} \begin{bmatrix} \cos \frac{\mu_1 + \epsilon_1}{2} \cdot I_{2n-1} & -i \sin \frac{\mu_1 + \epsilon_1}{2} \cdot I_{2n-1} \\ -i \sin \frac{\mu_1 + \epsilon_1}{2} \cdot I_{2n-1} & \cos \frac{\mu_1 + \epsilon_1}{2} \cdot I_{2n-1} \end{bmatrix} \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix} \right]^2 - \frac{1}{2} \\
= \mathbb{E}\left[\left(-i \sin \frac{\mu_1 + \epsilon_1}{2} \cdot \Psi_1(x) + \cos \frac{\mu_1 + \epsilon_1}{2} \cdot \Psi_2(x)\right)^2\right] - \frac{1}{2} \\
= \mathbb{E}\left[\frac{1}{2} + \cos(\mu_1 + \epsilon_1) \cdot \left(\|\Psi_2(x)\|^2 - \frac{1}{2}\right) - \sin(\mu_1 + \epsilon_1) \cdot \text{Re}(i\Psi_1(x)^\dagger \Psi_2(x))\right] - \frac{1}{2} \\
= \cos \mu_1 \cdot m(x) \cdot e^{-\frac{1}{2}\sigma^2} + \sin \mu_1 \cdot e^{-\frac{1}{2}\sigma^2} \cdot \text{Im}(\Psi_1(x)^\dagger \Psi_2(x)).
\]

Note that

\[ |\text{Im}(\Psi_1(x)^\dagger \Psi_2(x))| \leq |\Psi_1(x)^\dagger \Psi_2(x)| \leq \frac{|\Psi_1(x)|^2 + |\Psi_2(x)|^2}{2} = \frac{1}{2}. \]

This proves (b).

**D.2 Proof of Lemma 2**

Let us define

\[ s = \frac{1}{2} \cdot \max \{ |a + b|, |a + c|, |a + d| \}. \]

Then we have

\[
\frac{\ell(a) + \ell(b) + \ell(c) + \ell(d)}{4} \geq \frac{\ell(s) + \ell(-s)}{2},
\]

\[
\ell(s) + \ell(-s) = \ell(s) + \ell(s) = 2 \ell(s).
\]
since
\[
\frac{\ell(a) + \ell(b) + \ell(c) + \ell(d)}{4} \geq \frac{1}{2} \cdot \left[ \ell \left( \frac{a + b}{2} \right) + \ell \left( \frac{c + d}{2} \right) \right]
\]
by Jensen’s inequality
\[
= \frac{1}{2} \cdot \left[ \ell \left( \frac{a + b}{2} \right) + \ell \left( -\frac{a + b}{2} \right) \right]
\]
since \(a + b + c + d = 0\)
\[
= \frac{1}{2} \cdot \left[ \ell \left( \frac{|a + b|}{2} \right) + \ell \left( -\frac{|a + b|}{2} \right) \right]
\]
and the above procedure also holds when \((a + b, c + d)\) is replace by \((a + c, b + d)\) or \((a + d, b + c)\). Note that \(t \mapsto \frac{\ell(t) + \ell(-t)}{2}\) is nondecreasing at \(t > 0\) because of the convexity of \(\ell\). Now by definition of \(s\), we have
\[
|a| = |b + c + d| = \frac{1}{2}(b + c) + (c + d) + (d + c) = \frac{1}{2}(|a + b) + (a + c) + (a + d)| \leq 3s,
\]
and therefore
\[
\frac{\ell(s) + \ell(-s)}{2} \geq \frac{\ell(\frac{1}{3}|a|) + \ell(-\frac{1}{3}|a|)}{2}.
\]
The desired inequality immediately follows from these observations.