Small $\kappa$ Asymptotics of the Almost Sure Lyapunov Exponent for the Continuum Parabolic Anderson Model

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Abstract

We prove that the almost sure Lyapunov exponent $\lambda(\kappa)$ of the continuous space Parabolic Anderson Model is bounded above by $c_u \kappa^{1/3}$ as $\kappa \downarrow 0$ under mild regularity conditions. This bound of the same order of the previously proven lower bound, $\lambda(\kappa) \geq c_l \kappa^{1/3}$.

1 Background

Let \{\(W_x : x \in \mathbb{R}^d\)\} be a Gaussian field of identically distributed copies of mean 0 Brownian Motion defined on the probability space \((\Omega, \mathcal{F}, Q)\). This field has covariance given by $\mathbb{E}[W_x(t)W_y(s)] = \Gamma(x - y)(t \wedge s)$ where $\Gamma(z) = \Gamma(\|z\|_2)$ is twice continuously differentiable, bounded by $0 \leq \Gamma(z) \leq 1$, and has the following Taylor expansion near 0:

$$\Gamma(z) = 1 - c_d \|z\|_2^2 + o(\|z\|_2^2).$$

This assumption on the Taylor expansion of $\Gamma$ can be relaxed considerably, see Remark 2.10.

We consider the following stochastic differential equation over $\mathbb{R}^d$,

$$du(x, t) = \frac{\kappa}{2} \triangle u(x, t)dt + u(x, t)\partial W_x(t), \quad x \in \mathbb{R}^d, t > 0,$$

where $\kappa > 0$ is constant, $\partial W_x$ denotes the Stratonovich differential of $W_x$, $\triangle$ is the Laplacian, and $u(x, 0) \equiv 1$. Equation (1.2) is called the Parabolic Anderson Model in $\mathbb{R}^d$, hereafter PAM.

In [4] the existence of a solution to (1.2) was established, as was the validity of the Feynman-Kac representation of the solution:

$$u(x, t) = \mathbb{E}_x^X \left[ e^{\int_0^t dW_{X(t-x)}(s)} \right].$$

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where $X(s)$ is a $\kappa$ speed $d$-dimensional Brownian Motion, i.e. the diffusion with generator $\frac{\kappa}{2} \Delta$. Throughout this paper $\mathbb{P}$ and $\mathbb{E}^X$ denote the probability measure of $X$ and expectation with respect to $\mathbb{P}$, respectively.

In studying the PAM, the Lyapunov exponent

$$\lambda(\kappa) = \lim_{t \to \infty} \frac{1}{t} u(x, t) \quad (1.4)$$

has been of primary interest. The existence of $\lambda(\kappa)$ as a deterministic limit, and its convexity were established in [3, 6, 5].

It is the purpose of this paper to improve previously derived bounds on the small $\kappa$ behavior of $\lambda(\kappa)$. In [5] it was proven that

$$\liminf_{\kappa \downarrow 0} \frac{\lambda(\kappa)}{\kappa^{1/3}} \geq c \quad (1.5)$$

and that

$$\limsup_{\kappa \downarrow 0} \frac{\lambda(\kappa)}{\kappa^{1/5}} \leq c. \quad (1.6)$$

It was conjectured that the lower bound (1.5) gave the correct asymptotics for $\lambda(\kappa)$. We prove this conjecture.

**Theorem 1.7.** Under the aforementioned conditions

$$\limsup_{\kappa \downarrow 0} \frac{\lambda(\kappa)}{\kappa^{1/3}} \leq c. \quad (1.8)$$

This result is notable in that it is one of the rare examples of differing behavior in the PAM and the discrete PAM, $x \in \mathbb{Z}^d$. In the discrete PAM,

$$\lim_{\kappa \downarrow 0} \lambda(\kappa) \ln(1/\kappa) = c$$

as proven in [2, 3, 6]. For information of the discrete PAM see [3, 6].

2 Proof of Theorem 1.7

**Remark 2.1.** This approximation approach follows from an idea of Michael Cranston’s, who used the same approximating functions in an unpublished proof of the lower bound (1.5).

For convenience let

$$F(f) = \int_0^t dW_{f(t-s)}(s). \quad (2.2)$$

We will approximate the Brownian paths in the Feynman-Kac representation of $u(0, t)$ using Cameron-Martin functions. In particular, we work with the families of the form

$$H_t(C) = \{ f \in C_0([0, t]; \mathbb{R}^d), \| f’ \|_2 \leq C \}. \quad (2.3)$$
We have a topology on $C_0([0, t]; \mathbb{R}^d)$ defined by the natural metric,

$$d(f, g) = (\mathbb{E}^Q (F(f) - F(g))^2)^{1/2}.$$ 

We approximate $u(x, t)$ by the contribution from successively larger balls in this topology. Defining the increasing sequences

$$C_n = Mn\kappa^{2/3}t^{1/2}, \quad \epsilon_n = n^{1/2}\kappa^{1/6}t^{1/2} \quad (2.4)$$

we let $\Gamma_n$ be a minimal $\epsilon_n/2$-net of $H_t(C_n)$. It will follow from Lemma 2.15 that

$$\lim_{n \to \infty} \mathbb{P}(d(X, \Gamma_n) > \epsilon_n) \to 0.$$ 

Adopting the shorthand

$$E_{f,n} = \{d(X, f) \leq \epsilon_n, d(X, \Gamma_{n-1}) > \epsilon_{n-1}\}, \quad (2.5)$$

we bound the Feynman-Kac formula \([1, 3]\) of $u(x, t)$ though the following decomposition

$$u(x, t) = \mathbb{E}_x^X \left[ e^{F(X)} \right]
= \sum_{n \geq 1} \mathbb{E}_x^X \left[ e^{F(X)}; d(X, \Gamma_n) \leq \epsilon_n, d(X, \Gamma_{n-1}) > \epsilon_{n-1} \right]
\leq \sum_{n \geq 1} \sum_{f \in \Gamma_n} e^{F(f)} \mathbb{E}_x^X \left[ e^{F(X) - F(f)}; E_{f,n} \right]
= \sum_{n \geq 1} \sum_{f \in \Gamma_n} e^{F(f)} \mathbb{E}_x^X \left[ e^{F(X) - F(f)} \left| E_{f,n} \right. \right] \mathbb{P}(E_{f,n})
\leq \sum_{n \geq 1} \sum_{f \in \Gamma_n} e^{F(f)} \mathbb{E}_x^X \left[ e^{F(X) - F(f)} \left| E_{f,n} \right. \right] \mathbb{P}(d(X, \Gamma_{n-1}) > \epsilon_{n-1})
\leq \sum_{n \geq 1} |\Gamma_n| \left( \sup_{f \in \Gamma_n} e^{F(f)} \right) \left( \sup_{f \in \Gamma_n} \mathbb{E}_x^X \left[ e^{F(X) - F(f)} \left| E_{f,n} \right. \right] \right)
\cdot \mathbb{P}(d(X, \Gamma_{n-1}) > \epsilon_{n-1}). \quad (2.6)$$

To show that $\lambda(\kappa) = O(\kappa^{1/3})$ we must show that $\limsup_{t \to \infty} u(x, t) \leq c' e^{\kappa^{1/3}t}$ for $\kappa \in (0, \kappa_0)$, $\kappa_0$ small. In light of the elementary fact that

$$\sum_{k \geq 1} e^{-\alpha k} = -1 + \frac{1}{1 - e^{-\alpha}} = \frac{e^{-\alpha}}{1 - e^{-\alpha}} \leq e^{-\alpha} \quad (2.7)$$

for $\alpha \geq 0$ it suffices to show that each of the summands in $2.6$ are $\leq c' e^{\kappa x^{1/3}t}$. We proceed with series of lemmata, which when taken together establish such a bound.

First is an entropy bound on $H_t(C)$. $N_{\epsilon}(H_t(C))$ denotes the number of $\epsilon$-balls under the $d$ metric needed to cover $H_t(C)$.
Lemma 2.8.

\[ N_\varepsilon(H_t(C)) \leq c_1 \exp \left\{ c_2 t \frac{Ct}{\varepsilon} \right\}. \]

Proof. From \[8\] we have that

\[ N'_\varepsilon(H_{\pi/2}(C)) \leq c_1 \exp \left\{ c'_2 \frac{Ct}{\varepsilon} \right\}. \] (2.9)

where \( N'_\varepsilon(H_t(C)) \) denotes the number of \( \varepsilon \)-balls under the \( L^2 \) metric needed to cover \( H_t(C) \). We will use scaling relations to derive the lemma.

For \( f \in H_{\pi/2}(C) \) let \( g(s) = f\left(\frac{\pi}{2t}s\right), \ g \in C_0([0, t]). \) Then

\[ \|g'\|_2^2 = \int_0^t \left\| \frac{\pi}{2t}f'\left(\frac{\pi}{2t}s\right) \right\|_2^2 ds = \frac{\pi}{2t} \int_0^{\pi/2} \|f'(r)\|_2^2 dr = \frac{\pi}{2t} \|f'\|_2^2. \]

Thus \( g \in H_t\left(\sqrt{\frac{Ct}{\pi}}\right) \) and we have a bijection \( H_{\pi/2}(C) \leftrightarrow H_t\left(\sqrt{\frac{Ct}{\pi}}\right). \) This mapping also affects the radii of \( L^2 \)-balls. Letting \( h(s) = k\left(\frac{\pi}{2t}s\right) \) where \( k \in H_{\pi/2}(C) \) we see that

\[ \int_0^t \|g(s) - h(s)\|_2^2 ds = \int_0^t \left\| f\left(\frac{\pi}{2t}s\right) - k\left(\frac{\pi}{2t}s\right) \right\|_2^2 ds = \frac{2t}{\pi} \int_0^t \|f(r) - k(r)\|_2^2 dr. \]

So a \( L^2 \)-ball of radius \( \varepsilon \) in \( H_{\pi/2}(C) \) maps to a \( L^2 \)-ball of radius \( \sqrt{\frac{2t}{\pi}} \varepsilon \) in \( H_t\left(\sqrt{\frac{C}{\pi}}\right). \) From (2.9) and these scaling arguments that

\[ N'_\varepsilon(H_t(C)) = N'_\varepsilon\left(H_{\pi/2}\left(\sqrt{\frac{2t}{\pi}}C\right)\right) \]

\[ \leq c_1 \exp \left\{ c'_2 \frac{\sqrt{2t}C}{\varepsilon} \right\} = c_1 \exp \left\{ c_2 t \frac{Ct}{\varepsilon} \right\}. \]

This bound has so far been proven for the \( L^2 \) metric, we need to show that it applies to the \( d \) metric. It follows from (1.1) that

\[ d^2(f, g) = \mathbb{E}_Q(F(f) - F(g))^2 \]

\[ = 2 \int_0^t \left( 1 - \Gamma(f(s) - g(s)) \right) ds \]

\[ = 2 \int_0^t c_d \|f(s) - g(s)\|^2_2 - o \left( \|f(s) - g(s)\|^2_2 \right) ds \]

\[ \leq 2c_d \|f - g\|^2_2. \]

Thus every radius \( \varepsilon \) \( d \)-ball is contained in a radius \( \varepsilon\sqrt{2c_d} \) \( L^2 \)-ball and, allowing for changes to the constant \( c_2, \) we have proven the bound. \( \square \)
Remark 2.10. The domination of the $d$ metric by the $L^2$ metric in the final step of the proof of Theorem 2.8 is the sole reason for the assumption (1.1). This assumption can be weakened, so long as the metric domination is preserved.

Corollary 2.11.

\[ |\Gamma_n| \leq c_1 e^{2c_2 M n^{1/3} t}. \]

Proof.

\[ |\Gamma_n| = N_{c_n/2}(H_t(C_n)) \]
\[ \leq c_1 \exp \left\{ 2c_2 \frac{C_t}{c_n} \right\} \]
\[ = c_1 e^{2c_2 M n^{1/2} \kappa^{1/2} t} \]
\[ \leq c_1 e^{2c_2 M n^{1/3} t}. \]

Note that the last inequality follows from $\kappa < 1$. \qed

Lemma 2.12.

\[ \mathbb{E}_Q \sup_{f \in H_t(C)} F(f) \leq 2c_4 (c_3 C)^{1/2} \eta^{3/4} + O \left( \eta^{1/2} \right) \]

Proof. First we apply Fernique-Talagrand (Theorem 4.1 of [1]) and then we use of Lemma 2.8 and the elementary fact that $d(f, g) \leq (2t)^{1/2}$ to obtain

\[ \mathbb{E}_Q \sup_{f \in H_t(C)} F(f) \leq K \int_0^{\text{diam}(H_t(C))} \left( \ln N_{\delta}(H_t(C)) \right)^{1/2} d\delta \]
\[ \leq K \int_0^{(2t)^{1/2}} \left( \ln c_1 + c_2 \frac{C_t}{\delta} \right)^{1/2} d\delta \]
\[ = K (\ln c_1)^{1/2} \int_0^{(2t)^{1/2}} \left( 1 + \frac{c_2 C_t \ln c_1}{\ln c_1 \delta} \right)^{1/2} d\delta. \]

To proceed we first make the substitution $r = \frac{\delta \ln c_1}{c_2 C_t}$ so that

\[ \mathbb{E}_Q \sup_{f \in H_t(C)} F(f) \leq K \frac{c_2 C_t}{(\ln c_1)^{1/2}} \int_0^{\sqrt{2 \ln c_1}} \left( 1 + \frac{1}{r} \right)^{1/2} dr. \]

For brevity, we define the constants

\[ c_3 = \frac{\sqrt{2 \ln c_1}}{c_2}, \quad c_4 = \frac{Kc_2}{(\ln c_1)^{1/2}}. \]
Then we make the trigonometric substitution \( \tan \theta = r^{1/2} \). Thus

\[
E_Q \sup_{f \in H_t(C)} F(f) \leq c_4 C t \int 2 \sec^3 \theta d\theta
\]

\[
= c_4 C t [\tan \theta \sec \theta + \ln |\tan \theta + \sec \theta|]_0^{\sqrt{r} + 1}
\]

\[
= c_4 C t \left[ \frac{c_3}{C t^{1/2}} \sqrt{\frac{c_3}{C t^{1/2}} + 1}
  + \ln \left| \sqrt{\frac{c_3}{C t^{1/2}}} + \sqrt{\frac{c_3}{C t^{1/2}} + 1} \right| \right].
\]

Now we make use of Taylor’s Theorem. First for \( \sqrt{z + 1} = 1 + \frac{z}{2} + O(z^2) \) and then for \( \ln(1 + z) = z + O(z^2) \).

\[
E_Q \sup_{f \in H_t(C)} F(f) \leq c_4 C t \left[ \frac{c_3}{C t^{1/2}} \left( 1 + \frac{c_3}{2 C t^{1/2}} + O \left( \frac{1}{C^2 t} \right) \right)
  + \ln \left| \sqrt{\frac{c_3}{C t^{1/2}}} + 1 + \frac{c_3}{2 C t^{1/2}} + O \left( \frac{1}{C^2 t} \right) \right| \right]
\]

\[
= c_4 C t \left[ \frac{c_3}{C t^{1/2}} \left( 1 + \frac{c_3}{2 C t^{1/2}} + O \left( \frac{1}{C^2 t} \right) \right)
  + \sqrt{\frac{c_3}{C t^{1/2}}} + \frac{c_3}{2 C t^{1/2}} + O \left( \frac{1}{C t^{1/2}} \right) \right]
\]

\[
= 2 c_4 (c_3 C)^{1/2} t^{3/4} + O \left( t^{1/2} \right).
\]

**Corollary 2.13.** For all \( n \in \mathbb{N} \),

\[
\sup_{f \in \Gamma_n} e^{F(f)} \leq e^{4 c_4 c_3^{1/2} M^{1/2} n^{1/3} t} \quad \text{Q.a.s. as } t \to \infty
\]

for \( t \in \mathbb{N} \).

**Proof.** We again note that \( d(f, g) \leq (2t)^{1/2} \) and then applying Borell’s Inequality (Theorem 2.1 in [1]),

\[
Q \left( \sup_{f \in \Gamma_n} e^{F(f)} > e^{4 c_4 c_3^{1/2} M^{1/2} n^{1/3} t} \right)
\]

\[
\leq Q \left( \sup_{f \in H_t(C_n)} e^{F(f)} > e^{4 c_4 c_3^{1/2} M^{1/2} n^{1/3} t} \right)
\]

\[
\leq Q \left( \sup_{f \in H_t(C_n)} F(f) > 4 c_4 c_3^{1/2} M^{1/2} n^{1/3} t \right)
\]

\[
\leq 2 \exp \left\{ - \frac{1}{2 t} \left( 4 c_4 c_3^{1/2} M^{1/2} n^{1/3} t - 2 c_4 (c_3 C_n)^{1/2} t^{3/4} - O \left( t^{1/2} \right) \right)^2 \right\}
\]

\[6\]
\begin{align*}
&= 2 \exp \left\{ \frac{-1}{2t} \left( 4c_4 c_3^{1/2} M^{1/2} n \kappa^{1/3} t - 2c_4 (c_3 M n)^{1/2} \kappa^{1/3} t - O \left( t^{1/2} \right) \right)^2 \right\} \\
&\leq \exp \left\{ \frac{-nt}{2} \left( 2c_4 c_3^{1/2} M^{1/2} \kappa^{1/3} - O \left( t^{-1/2} \right) \right)^2 \right\} \\
&\leq \exp \left\{ \frac{-nt}{2} \left( c_4 c_3^{1/2} M^{1/2} \kappa^{1/3} \right)^2 \right\} \text{ for } t > T,
\end{align*}

where the constant $T$ is taken to be large enough that this holds for all $n \in \mathbb{N}$ and $\kappa \in (0, \kappa_0)$. Summing over $n$ and then $t \in \{i \in \mathbb{N}, i > T\}$ using (2.7) we see that this quantity is summable. An application of the Borell-Cantelli Lemma competes the proof.

**Lemma 2.14.** For all $n \in \mathbb{N}$,

\[
\sup_{f \in \Gamma_n} \mathbb{E}^X \left[ e^{F(X) - F(f)} \middle| E_{f,n} \right] \leq e^{(1+4c_2 M) n \kappa^{1/3} t} \text{ Q-a.s. as } t \to \infty.
\]

for $t \in \mathbb{N}$.

**Proof.**

\[
\begin{align*}
Q \left( \sup_{f \in \Gamma_n} \mathbb{E}^X \left[ e^{F(X) - F(f)} \middle| E_{f,n} \right] > e^{(1+4c_2 M) n \kappa^{1/3} t} \right) \\
&\leq \sum_{f \in \Gamma_n} Q \left( \mathbb{E}^X \left[ e^{F(X) - F(f)} \middle| E_{f,n} \right] > e^{(1+4c_2 M) n \kappa^{1/3} t} \right) \\
&\leq \sum_{f \in \Gamma_n} e^{-(1+4c_2 M) n \kappa^{1/3} t} \mathbb{E}^Q \mathbb{E}^X \left[ e^{F(X) - F(f)} \middle| E_{f,n} \right] \\
&\leq \sum_{f \in \Gamma_n} e^{-(1+4c_2 M) n \kappa^{1/3} t} \mathbb{E}^X \left[ e^{d(X,f)^2/2} \middle| E_{f,n} \right] \\
&\leq \sum_{f \in \Gamma_n} e^{-(1+4c_2 M) n \kappa^{1/3} t} e^{r_n^2/2}
\end{align*}
\]

As $F(X) - F(f)$ is a centered Gaussian, $\mathbb{E}^Q \left[ e^{F(X) - F(f)} \right] = e^{\mathbb{E}^Q (F(X) - F(f))^2/2} = e^{d(X,f)^2/2}$.

\[
\begin{align*}
&= \sum_{f \in \Gamma_n} e^{-(1+4c_2 M) n \kappa^{1/3} t} e^{d(X,f)^2/2} \\
&\leq \sum_{f \in \Gamma_n} e^{-(1+4c_2 M) n \kappa^{1/3} t} e^{r_n^2/2}
\end{align*}
\]

By the definition of $E_{f,n}$ (2.5). We recall Corollary 2.11 and the definition of $\epsilon_n$ (2.4) to finish the proof.

\[
\begin{align*}
&\leq |\Gamma_n| e^{-(1+4c_2 M) n \kappa^{1/3} t} e^{r_n^2/2} \\
&\leq c_1 e^{2c_2 M n \kappa^{1/3} t} e^{-(1+4c_2 M) n \kappa^{1/3} t} e^{\frac{1}{2} n \kappa^{1/3} t} \\
&= c_1 e^{-(\frac{1}{2}+2c_2 M) n \kappa^{1/3} t}.
\end{align*}
\]
Again using (2.7) this is summable over $n \in \mathbb{N}$ and then over $t \in \mathbb{N}$ so that the Borell-Catelli Lemma completes the proof.

**Lemma 2.15.** For $n \geq 2$ we can choose $M > 1$ arbitrarily large such that for $\kappa \in (0, \kappa_0(M))$, $\kappa_0(M)$ a nonnegative decreasing function, we have that

\[ P(d(X, \Gamma_{n-1}) > \epsilon_{n-1}) \leq 4e^{-\frac{\pi^2}{2} \frac{1}{n-1} \epsilon_{n-1}^{1/3}}. \]

**Proof.** For each path $X$ we define the path $g_X$ as the linear interpolation between the points $(0, X(0)), (\frac{1}{\kappa}, X(\frac{1}{\kappa})), (\frac{2}{\kappa}, X(\frac{2}{\kappa})), (\frac{3}{\kappa}, X(\frac{3}{\kappa})), \ldots$. The we have

\[ P(d(X, \Gamma_{n-1}) > \epsilon_{n-1}) \leq P\left(d(X, g_X) > \frac{\epsilon_{n-1}}{2}\right) + P\left(g_X, \Gamma_{n-1} > \frac{\epsilon_{n-1}}{2}\right) \leq P\left(d(X, g_X) > \frac{\epsilon_{n-1}}{2}\right) + P (g_X \notin H_t(C_{n-1})) \quad (2.16) \]

as $\Gamma_{n-1}$ is an $\epsilon_{n-1}/2$-net of $H_t(C_{n-1})$. We bound each of these terms in turn.

\[ P\left(d(X, g_X) > \frac{\epsilon_{n-1}}{2}\right) = P\left(d(X, g_X)^2 > \frac{\epsilon_{n-1}^2}{4}\right) \]

\[ = P\left(\sum_{i=1}^{\kappa t} d(Y_i, 0)^2 > \frac{\epsilon_{n-1}^2}{4}\right) \]

where $Y_i(s) = X(s + (i - 1)\kappa) - g_X(s + (i - 1)\kappa), s \in (0, \frac{1}{\kappa})$. The $Y_i$ are iid rate $\kappa$ Brownian bridges on $(0, \kappa)$ and that

\[ d(Y_i, 0)^2 = E^Q \left[ \int_0^{1/\kappa} dW_{Y_i(1/\kappa-s)}(s) - \int_0^{1/\kappa} dW_0(s) \right]^2 \]

\[ = 2 \left( \frac{1}{\kappa} - \int_0^{1/\kappa} \Gamma(Y_i(1/\kappa-s)) ds \right) \leq \frac{2}{\kappa}. \]

It follows that $d(Y_i, 0)^2$ has a logarithmic moment generating function bounded on $\mathbb{R}$,

\[ \Lambda(\lambda) \leq \max \left\{ \frac{2\lambda}{\kappa}, 1 \right\}, \]

and therefore has a good rate function \[\exists\] such that

\[ \lim_{|x| \to \infty} \frac{\Lambda^*(x)}{|x|} = \infty. \quad (2.17) \]
Applying Cramér’s Theorem we have
\[
P \left( d(X, g_X) > \frac{c_{n-1}}{2} \right) = P \left( \sum_{i=1}^{n^t} d(Y_i, 0)^2 > \frac{c_{n-1}^2}{4n^t} \right)
\]
\[
= P \left( \frac{1}{\kappa t} \sum_{i=1}^{n^t} d(Y_i, 0)^2 > \frac{c_{n-1}^2}{4n^t} \right)
\]
\[
\leq 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{c_{n-1}^2}{4n^t} \right) \right\}
\]
\[
= 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{n - 1}{4\kappa^2} \right) \right\}
\]

Restricting \( \kappa \) to be small, \( \frac{c_{n-1}^2}{4n^t} \) is ensured to be large for \( n \geq 2 \) so that by (2.17) we have
\[
\leq 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{n - 1}{4\kappa^2} \right) \right\}
\]
\[
= 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{n - 1}{4\kappa^2} \right) \right\}
\]

Turning to the second term of (2.16),
\[
P \left( g_X \not\in H_t(C_{n-1}) \right) = P \left( \|g_X\|_2 > C_{n-1} \right)
\]
\[
= P \left( \|g_X\|_2^2 > C_{n-1}^2 \right)
\]
\[
= P \left( \sum_{i=1}^{n^t} \frac{1}{\kappa} \left\| X(i/\kappa) - X((i-1)/\kappa) \right\|_2^2 > C_{n-1}^2 \right)
\]
\[
= P \left( \frac{1}{\kappa t} \sum_{i=1}^{n^t} \left\| X(i/\kappa) - X((i-1)/\kappa) \right\|_2^2 > \frac{C_{n-1}^2}{\kappa^2 t} \right)
\]

Observe that \( \left\| X(i/\kappa) - X((i-1)/\kappa) \right\|_2^2 \sim \chi^2_{d} \) iid and that chi-squared random variables so they have the logarithmic moment generating function
\[
\Lambda(\lambda) = \ln \mathbb{E} e^{\lambda X^2} = \begin{cases} 
-\frac{4}{\kappa^2} \ln(1 - 2\lambda) & 1 - 2\lambda > 0 \\
\infty & \text{otherwise}
\end{cases}
\]
which has Fenchel-Legendre transform
\[
\Lambda^*(x) = \sup_{\lambda} \{\lambda x - \Lambda(\lambda)\} = \frac{1}{2} (x + d \ln x - d - d \ln d).
\]

Taking \( x \) large we get that \( \Lambda^*(x) \geq \frac{4}{\kappa^2} \). Using Cramér’s Theorem again,
\[
P \left( g_X \not\in H_t(C_{n-1}) \right) \leq 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{C_{n-1}^2}{\kappa^2 t} \right) \right\}
\]
\[
= 2 \exp \left\{ -\kappa t \Lambda^* \left( \frac{M^2(n - 1)^2}{\kappa^2 / 3} \right) \right\}
\]
Restricting \( \kappa \) to again be small ensures that \( \frac{M^2(n-1)^2}{\kappa^{2/3}} \) is large for \( n \geq 2 \) and \( M \geq 1 \) so that

\[
\leq 2 \exp \left\{ -\kappa t \left( \frac{M^2(n-1)^2}{2\kappa^{2/3}} \right) \right\} \\
= 2e^{-\frac{M^2}{2}(n-1)^2\kappa^{1/3} t} \\
\leq 2e^{-\frac{M^2}{2}(n-1)^2\kappa^{1/3} t}.
\]

Combining these bounds using (2.10) completes the lemma.

Supporting lemmata complete, we return to the proof of Theorem 1.7. Beginning with (2.6) we apply Corollaries 2.11, 2.13 and Lemmata 2.14, 2.15.

\[
\limsup_{t \to \infty} u(x, t) \\
\leq \limsup_{t \to \infty} \sum_{n \geq 1} |\Gamma_n| \left( \sup_{f \in \Gamma_n} e^{F(f)} \right) \left( \sup_{f \in \Gamma_n} \mathbb{P}^X \left[ e^{F(X)-F(f)} \mid E_{f,n} \right] \right) \\
\cdot \mathbb{P}(d(X, \Gamma_{n+1}) > \epsilon_{n+1}) \\
\leq \limsup_{t \to \infty} \sum_{n \geq 1} c_1 e^{2c_2 M \kappa^{1/3} t} e^{4c_4 c_3^{1/2} M^{1/2} n \kappa^{1/3} t} e^{(1+4c_2 M) n \kappa^{1/3} t} e^{(d(X, \Gamma_{n-1}) > \epsilon_{n-1})} \\
= \limsup_{t \to \infty} \sum_{n \geq 1} c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t} e^{(d(X, \Gamma_{n-1}) > \epsilon_{n-1})} \\
\leq \limsup_{t \to \infty} c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t} \\
+ \sum_{n \geq 2} c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t} 4e^{-\frac{M^2}{2}(n-1)^2\kappa^{1/3} t} \\
= \limsup_{t \to \infty} c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t} \\
+ 4c_1 e^{-\frac{M^2}{2}\kappa^{1/3} t} \sum_{n \geq 2} e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2} - \frac{M^2}{2}) n \kappa^{1/3} t} \\
\leq \limsup_{t \to \infty} c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t} \\
+ 4c_1 e^{-\frac{M^2}{2}\kappa^{1/3} t} e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2} - \frac{M^2}{2}) n \kappa^{1/3} t} \\
\leq \limsup_{t \to \infty} 5c_1 e^{(1+6c_2 M+4c_4 c_3^{1/2} M^{1/2}) n \kappa^{1/3} t}.
\]
Returning to the definition of $\lambda(\kappa)$ we then have for $\kappa \in (0, \kappa_0(M))$

$$
\lambda(\kappa) \leq \limsup_{t \to \infty} \frac{1}{t} \ln u(x, t)
\leq \limsup_{t \to \infty} \frac{1}{t} \ln 5c_1 e^{(1+6c_2 M + 4c_4 c^1_3 M^{1/2}) \kappa^{1/3} t}
= (1 + 6c_2 M + 4c_4 c^1_3 M^{1/2}) \kappa^{1/3}
$$

which completes the proof of Theorem 1.7.

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