Algebraic Bethe Ansatz for the Zamolodchikov-Fateev and Izergin-Korepin models with open boundary conditions

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Abstract
We have considered the Zamolodchikov-Fateev and the Izergin-Korepin models with diagonal reflection boundaries. In each case the eigenspectrum of the transfer matrix is determined by application of the algebraic Bethe Ansatz.

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1 Introduction

Some one-dimensional quantum spin chain Hamiltonians and some models of classical statistical mechanics in two spatial dimensions on a lattice - the vertex models - share a common mathematical structure responsible by well-known breakthroughs in our present understanding of these models [1, 2, 3]. If the Boltzmann weights underlying the vertex models are obtained from solutions of the Yang-Baxter (YB) equation, the commutativity of a collection of transfer matrices depending on a spectral parameter, say $u$, follows, leading to their integrability. Moreover, by taking the logarithmic derivative of the transfer matrix, evaluated at a special value of this spectral parameter, one gets an associated one-dimensional quantum spin chain Hamiltonian. If fact, it might be said that the most successful approach to construct integrable two-dimensional lattice models of statistical mechanics is by solving the Yang-Baxter equation. For a given solution of this equation, one can define local Boltzmann weights to find a commuting family of transfer matrices.

The diagonalization of one-dimensional quantum spin chain Hamiltonians started with the Bethe Ansatz (BA) [4], which gradually become a powerful method in the analysis of integrable models. There are several versions: the coordinate BA [4], the algebraic BA [5], the analytical BA [6], etc. The simplest version is the coordinate BA. In this framework one can obtain the eigenfunctions and the spectrum of the Hamiltonian from its eigenvalue problem. It is really simple and clear for two-state models like the six-vertex model but becomes tricky for models with a higher number of states.

The development of the quantum inverse scattering method approach to integrability resulted in the algebraic BA [5] which is an elegant and important generalization of the coordinate BA. It is based on the idea of constructing the eigenvectors of the transfer matrix via the action of ”creation ” operators on a reference state. The creation operators are just entries of the monodromy matrix whose trace is the transfer matrix. Then by using the YB equation one can write the so-called fundamental relation which entails the generalized ”commutation relations ” of the transfer matrix with the remaining entries of the monodromy matrix ( the ”creation ” and ”annihilation ” operators). The creation operators that form a $n$-body Bethe vector are evaluated at some unknown spots, say $\{u_i\} = \{u_1, u_2, ..., u_n\}$. Finally, the action of the transfer matrix, evaluated at some spectral parameter, say $u$, will be an eingenvalue problem, if a set of equations, known as Bethe equations, not depending on $u$, determine the set $\{u_i\}$. Despite its elegance and completeness, the actual implementation of the algebraic Bethe Ansatz can become rather tricky and laborious, as will be exemplified in this article. It is usual to rely on other methods such as the coordinate or the analytical BA to gather important informations about the eingenvalues of the transfer matrix.

The integrability of open spin chains in the framework of the quantum inverse scattering method was formulated by Sklyanin relying on previous results of Cherednik [7]. In reference [8], Sklyanin used his
formalism to solve the open spin-1/2 chain with diagonal boundary terms, and his original formalism was successfully extended to others models by Mezincescu and Nepomechie in [9]. The basic idea is similar to that of the usual algebraic BA, with the important difference that now one considers the so-called double-row monodromy matrix. When considering systems on a finite interval, with independent boundary conditions at each end, one has to introduce reflection matrices to describe such boundary conditions. Integrable models with boundaries can be constructed out of a pair of reflection $K$-matrices $K^\pm(u)$ which obey the cousin equation of the YB equation - called reflection equation. $K^-(u)$ and $K^+(u)$ describe the effects of the presence of boundaries at the left and the right ends, respectively. Sklyanin has shown that the double-row monodromy matrix obeys a generalization of the fundamental relation also named reflection equation. The non-diagonal entries of the double-row monodromy matrix play the role of "creation" and "annihilation" operators and the diagonal ones are used to construct the transfer matrix, the reflection equation providing the fundamental set of "commutation relations". Similarly to the usual BA the creation operators acting on a reference state form a $n$-body Bethe vector when evaluated at some unknown spots, say $\{u_i\} = \{u_1, u_2, ..., u_n\}$, and an eigenvalue problem for the transfer will exist if a set of equations, named also Bethe equations, not depending on $u$, determine the set $\{u_i\}$. In a recent article [10] Guang-Liang Li, et al investigated the 19-vertex Izergin-Korepin (IK) model [11] with boundaries using the algebraic BA. They have found a spurious dependence of the Bethe equations on the spectral parameter $u$, such that the set $\{u_i\}$ could inconsistently depend on the spectral parameter $u$. Even more, this spurious dependence led the authors of [10] to raise the suspicion of the non-uniqueness (or inconsistency) of Sklyanin’s algebraic BA for this model. On the other hand, this model has been previously studied through the analytical BA [12, 13] and the coordinate BA [14] where no such spurious dependence was found. So, we decided to investigate a related 19-vertex model, the Zamolodchikov-Fateev (ZF) model [15], where one surely does not expect such a spurious dependence on $u$. The approach is the same as that of [10], first envisaged by Fan [16], which is the boundary version of the Tarasov work [17] for the periodic IK model. As a further check we also performed the calculations for the IK model. We do not have found such a spurious dependence for the ZF model, our results agreeing with all previous results. For our pleasant surprise we do not have found such a spurious dependence for the IK model too. Indeed, our calculations agree with those of Guang-Liang Li, et al and there is no spurious dependence on the spectral parameter at all. The point is that this spurious dependence of the Bethe equations on the spectral parameter in [10] is only apparent. A careful analysis as we perform in Section 4 bellow shows that it does not exist. It can be also checked that, in the notation of ref.[10], that their factor function $\beta(u, u_i)$ in their Eq. (97) does not depend on $u$, so that the dependence on $u$ in their Eq (102), for the non-quantum group invariant case, is non existent. The Sklyanin algebraic BA is unique and consistent for the ZF and the IK vertex models.

The paper is organized as follows: in Section 2 we define the models to be studied and introduce the
necessary notation to briefly review the basic concepts of the algebraic BA. In section 3, we present our
detailed calculations common to both models and in Section 4 the eigenspectra and the corresponding
Bethe equations are explicitly presented for each model. Section 5 is reserved for conclusions and in
the appendix we derive the fundamental commutation relations for the double-row monodromy matrix
entries.

2 The Models

To determine an integrable vertex model on a lattice it is first necessary that the bulk vertex weights be
specified by a $R$-matrix $R(u)$, where $u$ is the spectral parameter. It acts on the tensor product $V \otimes V$ for
a given vector space $V$ and satisfy a special system of functional equations, the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u),$$

in $V \otimes V \otimes V$, where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, etc.

The $R$ matrix is said to be regular if it satisfies the property $R(0) = P$, where $P$ is the permutation
matrix in $V \otimes V$: $P(|\alpha \rangle \otimes |\beta \rangle) = |\beta \rangle \otimes |\alpha \rangle$ for $|\alpha \rangle, |\beta \rangle \in V$. In addition, we will require [9] that $R(u)$ satisfies the following properties

- regularity: $R_{12}(0) = f(0)^{1/2}P_{12}$,
- unitarity: $R_{12}(u)R_{12}^{t_2}(-u) = f(u)$,
- PT-symmetry: $P_{12}R_{12}(u)P_{12} = R_{12}^{t_2}(u)$,
- crossing-symmetry: $R_{12}(u) = U_1 R_{12}^{t_2}(-u-\rho)U_1^{-1}$,

where $f(u) = x_1(u)x_1(-u)$, $x_1(u)$ being defined for each model bellow. $t_i$ denotes transposition in the
space $i$, $\rho$ is the crossing parameter and $U$ determines the crossing matrix

$$M = U^tU = M^t.$$  

(2.3)

Unitarity and crossing-symmetry together imply the useful relation

$$M_1 R_{12}^{t_2}(-u-\rho)M_1^{-1} R_{12}^{t_2}(u-\rho) = f(u).$$  

(2.4)

The boundary weights then follow from $K$-matrices which satisfy the boundary versions of the Yang-
Baxter equation [8, 9], the reflection equation

$$R_{12}(u-v)K_1^-(u)R_{12}^{t_2}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{12}^{t_2}(u-v),$$

and the dual reflection equation

$$R_{12}(-u+v)(K_1^+)^{t_2}(u)M_1^{-1} R_{12}^{t_2}(-u-v-2\rho)M_1 (K_2^+)^{t_2}(v)$$

(2.5)
\[(K_2^+)t_2(v)M_1R_{12}(-u-v-2\rho)M_1^{-1}(K_1^+)t_1(u)R_{12}^{t_1t_2}(-u+v). \tag{2.6}\]

In this case there is an isomorphism between \(K^-\) and \(K^+\):

\[K^-(u) \rightarrow K^+(u) = K^-(u - \rho)M. \tag{2.7}\]

Therefore, given a solution of the reflection equation (2.5) one can also find a solution of the dual reflection equation (2.6).

A quantum-integrable system is characterized by the monodromy matrix \(T(u)\) satisfying the fundamental relation

\[R(u-v)[T(u) \otimes T(v)] = [T(v) \otimes T(u)]R(u-v) \tag{2.8}\]

where \(R(u)\) is given by \(R(u) = PR(u)\).

In the framework of the quantum inverse scattering method, the simplest monodromies have become known as \(L\) operators, the Lax operators, here defined by \(L_{aq}(u) = R_{aq}(u)\), where the subscript \(a\) represents the auxiliary space, and \(q\) represents the quantum space. The monodromy matrix \(T(u)\) is defined as the matrix product of \(N\) Lax operators on all sites of the lattice,

\[T(u) = L_{aN}(u)L_{aN-1}(u) \cdots L_1(u). \tag{2.9}\]

We recall that the main result for integrability is that, if the boundary equations are satisfied, then the Sklyanin’s transfer matrix

\[t(u) = \text{Tr}_a \left( K^+(u)T(u)K^-(u)T^{-1}(-u) \right), \tag{2.10}\]

forms a commuting collection of operators in the quantum space

\[[t(u), t(v)] = 0, \quad \forall u, v \tag{2.11}\]

The commutativity of \(t(u)\) can be proved by using the unitarity and crossing-unitarity relations, the reflection equation and the dual reflection equation. In particular, it implies the integrability of an open quantum spin chain whose Hamiltonian (with \(K^-(0) = 1\)) is given by

\[H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \frac{dK_1^-(u)}{du}igg|_{u=0} + \frac{\text{tr}_0 K^+_0(0)H_{N,0}}{\text{tr} K^+(0)}, \tag{2.12}\]

where the two-site terms are given by

\[H_{k,k+1} = \frac{d}{du}P_{k,k+1}R_{k,k+1}(u) \bigg|_{u=0}, \tag{2.13}\]

in the standard fashion.
The three-state vertex models that we will consider are the Zamolodchikov-Fateev model and the Izergin-Korepin model. Their $R$-matrices have a common form

$$R(u) = \begin{pmatrix} x_1 & x_2 & x_3 & x_5 & x_6 & x_7 \\ y_5 & x_2 & x_4 & x_6 & x_2 & x_5 \\ y_2 & y_6 & x_2 & x_6 & x_3 & x_2 \\ y_7 & y_6 & x_3 & x_2 & x_1 & x_1 \end{pmatrix},$$

satisfying the properties (2.1–2.4).

In the context of the coordinate BA, these models were solved in [14], where an appropriate parametrization of wavefunctions was used in order to recast the coordinate BA for these three-states models in a form as simple as the coordinate BA for the two-state models [18].

### 2.1 The Zamolodchikov-Fateev model

This is the simplest three-state 19-vertex model [15]. In the Bazhanov [19] and Jimbo [20] classification, it is the $B^{(1)}_1$ model or the $A^{(1)}_1$ model in the spin-1 representation [21], due to its construction from the six-vertex model by the fusion procedure. The $R$-matrix which satisfies the YB equation (2.1) has the form (2.14) with

$$x_1(u) = \sinh(u + \eta) \sinh(u + 2\eta), \quad x_2(u) = \sinh u \sinh(u + \eta),$$

$$x_3(u) = \sinh u \sinh(u - \eta), \quad x_4(u) = \sinh u \sinh(u + \eta) + \sinh \eta \sinh 2\eta,$$

$$y_5(u) = x_5(u) = \sinh(u + \eta) \sinh 2\eta, \quad y_6(u) = x_6(u) = \sinh u \sinh 2\eta,$$

$$y_7(u) = x_7(u) = \sinh \eta \sinh 2\eta.$$  

This $R$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, P- and T-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The most general diagonal solution for $K^-(u)$ has been obtained in Ref. [22] and is given by

$$K^-(u, \beta) = \begin{pmatrix} k_{11}^-(u) \\ \beta \end{pmatrix},$$

with

$$k_{11}^-(u) = \frac{\beta \sinh u + 2 \cosh u}{\beta \sinh u - 2 \cosh u}, \quad k_{33}^-(u) = \frac{\beta \sinh(u + \eta) - 2 \cosh(u + \eta)}{\beta \sinh(u - \eta) + 2 \cosh(u - \eta)}.$$
where $\beta$ is a free parameter. By the automorphism (2.7) the solution for $K^+(u)$ follows

$$K^+(u, \alpha) = K^-(u - \rho, \alpha) = \begin{pmatrix} k_{11}^+(u) & 1 \\ k_{33}^+(u) & k_{33}^+(u) \end{pmatrix},$$

with

$$k_{11}^+(u) = -\frac{\alpha \sinh(u + \eta) - 2 \cosh(u + \eta)}{\alpha \sinh(u + \eta) + 2 \cosh(u + \eta)}, \quad k_{33}^+(u) = -\frac{\alpha \sinh(u + 2\eta) - 2 \cosh(u + 2\eta)}{\alpha \sinh(u + 2\eta) - 2 \cosh(u + 2\eta)},$$

where $\alpha$ is another free parameter.

The energy spectrum of this model was already obtained by Mezincescu et al. [22] through a generalization of the quantum inverse scattering method developed by Sklyanin [8], the so-called fusion procedure [21] and by Fireman et al. [14] through a generalization of the coordinate BA.

For a particular choice of boundary terms, the ZF spin chain has the quantum group symmetry i.e., if we choose $\beta = 2 \coth \xi_-$ and $\alpha = 2 \coth \xi_+$, with $\xi_+ \to \infty$, then the open spin chain Hamiltonian has $U_q(su(2))$-invariance [22].

The fusion procedure was also used by Yung and Batchelor [13] to solve the ZF vertex-model with inhomogeneities. The coordinate and algebraic BA with periodic boundary conditions were presented in [23].

### 2.2 The Izergin-Korepin model

The solution of the YB equation corresponding to $A_2^{(2)}$ in the fundamental representation was found by Izergin and Korepin [11]. The $R$-matrix has the form (2.14) with non-zero entries

$$
\begin{align*}
    x_1(u) &= \sinh(u - 5\eta) + \sinh \eta, & x_2(u) &= \sinh(u - 3\eta) + \sinh 3\eta, \\
    x_3(u) &= \sinh(u - \eta) + \sinh \eta, & x_4(u) &= \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta \\
    x_5(u) &= -2e^{-u/2} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta), & y_5(u) &= -2e^{u/2} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta) \\
    x_6(u) &= 2e^{-u/2 + 2\eta} \sinh 2\eta \sinh(\frac{u}{2}), & y_6(u) &= -2e^{u/2 - 2\eta} \sinh 2\eta \sinh(\frac{u}{2}) \\
    x_7(u) &= -2e^{-u + 2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta, \\
    y_7(u) &= 2e^{u - 2\eta} \sinh \eta \sinh 2\eta - e^{\eta} \sinh 4\eta.
\end{align*}
$$

This $R$-matrix is regular and unitary, with $f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = -6\eta - i\pi$ and

$$M = \begin{pmatrix} e^{2\eta} & 1 \\ 1 & e^{-2\eta} \end{pmatrix}. $$
Diagonal solutions for $K^-(u)$ have been obtained in [24]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$F^\pm = \begin{pmatrix} e^{-u} f^\pm(u) & g^\pm(u) \\ e^{u} f^\pm(u) & g^\pm(u) \end{pmatrix},$$

where we have defined

$$f^\pm(u) = \cosh(\frac{1}{2}u - 3\eta) \pm i \sinh(\frac{1}{2}u), \quad g^\pm(u) = \cosh(\frac{1}{2}u + 3\eta) \mp i \sinh(\frac{1}{2}u)$$

(2.23)

By the automorphism (2.7), three solutions $K^+(u)$ follow as $K^+(u) = M$, $K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$G^\pm = \begin{pmatrix} e^{u-4\eta} f^\pm(u) & h^\pm(u) \\ e^{-u+4\eta} f^\pm(u) & h^\pm(u) \end{pmatrix},$$

where we have defined

$$h^\pm(u) = \cosh(\frac{1}{2}u - 3\eta) \mp i \sinh(\frac{1}{2}u)$$

(2.25)

Here we will only consider three types of boundary solutions, one for each pair $(K^-(u), K^+(u))$ defined by the automorphism (2.7): $(1, M)$, $(F^+, G^+)$ and $(F^-, G^-)$.

The transfer matrix for the case $(1, M)$, whose corresponding spin-chain is $U_q(su(2))$-invariant [25] has been diagonalized by the analytical BA in Ref. [12]. For the other cases, $(F^+, G^+)$ and $(F^-, G^-)$, the corresponding open chain Hamiltonians are not $U_q(su(2))$-invariant. Nevertheless, their transfer matrices were diagonalized by Yung and Batchelor in [13] through the analytical BA with inhomogeneities. The energy eigenspectra for these three cases was also solved by Fireman et al [14] via the coordinate BA.

Recently has been argued by Nepomechie [26] that the transfer matrices corresponding to these solutions also have the $U_q(o(3))$ symmetry, but with a nonstandard coproduct.

It was in this model that Tarasov developed the algebraic BA for the three-state 19-vertex models with periodic boundary conditions [17]. The analytical BA with periodic boundary conditions was presented in [6] and the corresponding coordinate BA was presented in [23].

3 Boundary Algebraic Bethe Ansatz

In the previous section we have presented a common structure for the ZF and IK models. In this section we will turn to the eigenvalue problem for their double-row transfer matrix with integrable boundaries, named boundary algebraic Bethe Ansatz.
3.1 The reference state

The monodromy matrix $T(u)$ (2.9) and its reflection $T^{-1}(-u)$ can be written as 3 by 3 matrices

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}, \quad T^{-1}(-u) = \begin{pmatrix} T_{13}^*(-u) & T_{12}^*(-u) & T_{11}^*(-u) \\ T_{23}^*(-u) & T_{22}^*(-u) & T_{21}^*(-u) \\ T_{33}^*(-u) & T_{32}^*(-u) & T_{31}^*(-u) \end{pmatrix}$$ (3.1)

where

$$T_{ia}(u) = \sum_{k_1,\ldots,k_{N-1}=1}^3 \mathcal{L}_{ik_1}^{(N)}(u,\eta) \otimes \mathcal{L}_{k_1k_2}^{(N-1)}(u,\eta) \otimes \cdots \otimes \mathcal{L}_{k_{N-1}a}^{(1)}(u,\eta)$$ (3.2)

where $\mathcal{L}_{ij}^{(n)}$ are 3 x 3 matrices acting on the nth site of the lattice, defined by

$$\mathcal{L}_{11}^{(n)} = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \quad \mathcal{L}_{12}^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ x_5 & 0 & 0 \\ x_6 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}_{13}^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_7 & 0 & 0 \end{pmatrix},$$

$$\mathcal{L}_{21}^{(n)} = \begin{pmatrix} 0 & y_5 & 0 \\ 0 & 0 & y_6 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}_{22}^{(n)} = \begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, \quad \mathcal{L}_{23}^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_5 & 0 & 0 \end{pmatrix},$$

$$\mathcal{L}_{31}^{(n)} = \begin{pmatrix} 0 & 0 & y_7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}_{32}^{(n)} = \begin{pmatrix} 0 & y_6 & 0 \\ 0 & 0 & y_5 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}_{33}^{(n)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1 \end{pmatrix}.$$ (3.3)

Using the unitary relation in (2.2) we can see that the reflected monodromy matrix $T^{-1}(-u)$ has the following matrix elements

$$T_{bj}^{-1}(-u) = \frac{1}{f(u)^N} \sum_{k_1,\ldots,k_{N-1}=1}^{3} \mathcal{L}_{k_1k_2}^{(1)}(-u,-\eta) \otimes \mathcal{L}_{k_1k_2}^{(2)}(-u,-\eta) \otimes \cdots \otimes \mathcal{L}_{k_{N-1}j}^{(N)}(-u,-\eta).$$ (3.4)

Now we introduce the reference state

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(1)} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(2)} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(N)}$$ (3.5)

The actions of $T(u)$ and $T^{-1}(-u)$ on this state are

$$T(u)|0\rangle = f^N(u)T^{-1}(-u)|0\rangle = \begin{pmatrix} x_1^N(u)|0\rangle & \ast \\ 0 & x_2^N(u)|0\rangle & \ast \\ 0 & 0 & x_3^N(u)|0\rangle \end{pmatrix}$$ (3.6)

which give us, in the usual Bethe Ansatz language, the creation and annihilation operators for this reference state. Moreover, we are dealing with reflection and in this case we have a double-row monodromy matrix defined by

$$U(u) = T(u)K^-(u)T^{-1}(-u) = \begin{pmatrix} U_{11}(u) & U_{12}(u) & U_{13}(u) \\ U_{21}(u) & U_{22}(u) & U_{23}(u) \\ U_{31}(u) & U_{32}(u) & U_{33}(u) \end{pmatrix}$$ (3.7)
where \( K^{-1}(u) \) is a reflection matrix.

For the diagonal case \( K^{-1}(u) = \text{diag}(k_{11}^{-}(u), k_{22}^{-}(u), k_{33}^{-}(u)) \), the matrix elements of \( U \) have the form

\[
U_{ij}(u) = \frac{3}{\sum_{a=1}^{3} T_{ia}(u)k_{ia}^{-}(u)T_{aj}^{-1}(-u)}, \quad i, j = 1, 2, 3. \tag{3.8}
\]

It follows from (3.8) that we will need to know the commutation relations of the operators \( T(u) \) and \( T^{-1}(-u) \) in order to get the action of \( U(u) \) on the reference state (3.5). Using (2.8) with \( u = -v \) we get the matrix relation

\[
T_{21}^{-1}(-u)R_{12}(2u)T_{1}(u) = T_{1}(u)R_{12}(2u)T_{21}^{-1}(-u) \tag{3.9}
\]

Applying both sides of this relation on the reference state, we find the following relations between its matrix elements

\[
T_{21}(u)T_{12}^{-1}(-u) |0\rangle = f_{1}(u) \frac{x_{1}^{2N}(u) - x_{2}^{2N}(u)}{f^{N}(u)} |0\rangle \tag{3.10}
\]

\[
T_{31}(u)T_{13}^{-1}(-u) |0\rangle = f_{2}(u) \frac{x_{1}^{2N}(u)}{f^{N}(u)} |0\rangle - f_{3}(u)f_{1}(u) \frac{x_{2}^{2N}(u)}{f^{N}(u)} |0\rangle - f_{4}(u) \frac{x_{3}^{2N}(u)}{f^{N}(u)} |0\rangle \tag{3.11}
\]

\[
T_{32}(u)T_{23}^{-1}(-u) |0\rangle = f_{2}(u) \frac{x_{2}^{2N}(u) - x_{3}^{2N}(u)}{f^{N}(u)} |0\rangle \tag{3.12}
\]

where

\[
f_{1}(u) = \frac{y_{5}(2u)}{x_{1}(2u)}, \quad f_{2}(u) = \frac{y_{7}(2u)}{x_{1}(2u)}, \quad f_{3}(u) = \frac{x_{1}(2u)y_{5}(2u) - x_{5}(2u)y_{7}(2u)}{x_{1}(2u)x_{4}(2u) - x_{5}(2u)y_{5}(2u)}, \quad f_{4}(u) = \frac{x_{4}(2u)y_{7}(2u) - y_{5}^{2}(2u)}{x_{1}(2u)x_{4}(2u) - x_{5}(2u)y_{5}(2u)}. \tag{3.13}
\]

Using these relations we can get the action of all \( U_{ij}(u) \) on the reference state

\[
U_{11}(u) |0\rangle = k_{11}^{-}(u) \frac{x_{1}^{2N}(u)}{f^{N}(u)} |0\rangle
\]

\[
U_{22}(u) |0\rangle = f_{1}(u)U_{11}(u) |0\rangle + [k_{22}^{-}(u) - k_{11}^{-}(u)f_{1}(u)] \frac{x_{2}^{2N}(u)}{f^{N}(u)} |0\rangle
\]

\[
U_{33}(u) |0\rangle = [(f_{2}(u) - f_{1}(u)f_{3}(u))U_{11}(u) + f_{3}(u)U_{22}(u)] |0\rangle + [k_{33}^{-}(u) - k_{22}^{-}(u)f_{3}(u) - k_{11}^{-}(u)f_{4}(u)] \frac{x_{3}^{2N}(u)}{f^{N}(u)} |0\rangle \tag{3.14}
\]

and

\[
U_{ij}(u) |0\rangle = 0, \quad (i > j), \quad U_{ij}(u) |0\rangle \neq \{0, |0\rangle \}, \quad (i < j) \tag{3.15}
\]
In order to recover the usual BA structure we define new operators:

\[ D_1(u) = U_{11}(u), \quad B_1(u) = U_{12}(u), \quad B_2(u) = U_{13}(u) \]

\[ C_1(u) = U_{21}(u), \quad D_2(u) = U_{22}(u) - f_1(u)D_1(u), \quad B_3(u) = U_{23}(u) \]

\[ C_2(u) = U_{31}(u), \quad C_3(u) = U_{32}(u), \quad D_3(u) = U_{33}(u) - f_2(u)D_1(u) - f_3(u)D_2(u) \]

and then the new double-row monodromy matrix has the form

\[ U(u) \rightarrow \mathcal{U}(u) = \begin{pmatrix} D_1(u) & B_1(u) & B_2(u) \\ C_1(u) & D_2(u) & B_3(u) \\ C_2(u) & C_3(u) & D_3(u) \end{pmatrix} \]

(3.17)

The action of \( \mathcal{U}(u) \) on the reference state is now

\[ \mathcal{U}(u) |0\rangle = \begin{pmatrix} \mathcal{X}_1(u) |0\rangle \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X_1(u) |0\rangle \\ 0 \\ 0 \end{pmatrix} \]

(3.18)

where

\[ \mathcal{X}_1(u) = k_{11}(u) \frac{x_1^{2N}(u)}{f^N(u)} \]

\[ \mathcal{X}_2(u) = \left[ k_{22}(u) - k_{11}(u)f_1(u) \right] \frac{x_2^{2N}(u)}{f^N(u)} \]

\[ \mathcal{X}_3(u) = \left[ k_{33}(u) - k_{22}(u)f_3(u) - k_{11}(u)f_4(u) \right] \frac{x_3^{2N}(u)}{f^N(u)} \]

(3.19)

The transfer matrix \( t(u) \) (2.10), with diagonal left reflection \( K^{(+)} = \text{diag}(k^+_{11}, k^+_{22}, k^+_{33}) \) has the form

\[ t(u) = k^+_{11}(u)U_{11}(u) + k^+_{22}(u)U_{22}(u) + k^+_{33}(u)U_{33}(u) \]

\[ = \Omega_1(u)D_1(u) + \Omega_2(u)D_2(u) + \Omega_3(u)D_3(u) \]

(3.20)

where

\[ \Omega_1(u) = k^+_{11}(u) + f_1(u)k^+_{22}(u) + f_2(u)k^+_{33}(u) \]

\[ \Omega_2(u) = k^+_{22}(u) + f_3(u)k^+_{33}(u) \]

\[ \Omega_3(u) = k^+_{33}(u) \]

(3.21)

It is now clear that for \( \mathcal{U}(u) \) we have recovered the usual algebraic BA structure. Therefore we can look for states created by the action of the operators \( B_i(u) \) on the reference \( \Psi_0 \) which will be eigenstates of the transfer matrix (2.10). To do this we first recall the magnon number operator

\[ M = \sum_{k=1}^{N} M_k, \quad M_k = \text{diag}(0, 1, 2) \]

(3.22)
This is the analogue of the operator $S_T^+$ used in the coordinate BA construction. The relation $M \Psi_m = m \Psi_m$ where $m = N - S_T^+$, allows us to build states $\Psi_m$ such that $t(u) \Psi_m = \Lambda_m \Psi_m$. Therefore, we can start the diagonalization of $t(u)$ by considering all possible values of $m$ in a lattice with $N$ sites.

By the previous construction, $\Psi_0$ is our reference state $|0\rangle$, which is itself an eigenstate of $t(u)$

$$t(u) \Psi_0 = \Lambda_0(u) \Psi_0$$

with eigenvalue

$$\Lambda_0(u) = \left[k_{11}^+(u) + f_1(u)k_{22}^+(u) + f_2(u)k_{33}^+(u)\right] k_{11}^{-}(u) \frac{x^{2N}_3(u)}{f^N(u)}$$

$$+ \left[k_{22}^+(u) + f_3(u)k_{33}^+(u)\right] \left[k_{22}^{-}(u) - k_{11}^{-}(u)f_1(u)\right] \frac{x^{2N}_2(u)}{f^N(u)}$$

$$+ k_{33}^+(u) \left[k_{33}^{-}(u) - k_{22}^{-}(u)f_3(u) - k_{11}^{-}(u)f_4(u)\right] \frac{x^{2N}_3(u)}{f^N(u)}$$

(3.24)

This is the only state with $m = 0$.

### 3.2 The one-particle state

For $m = 1$ we seek a state of the form

$$\Psi_1(u_1) = B_1(u_1) |0\rangle.$$  

(3.25)

Then the action of the transfer matrix $t(u)$ on this state is

$$t(u) \Psi_1(u_1) = \Omega_1(u) D_1(u) B_1(u_1) |0\rangle + \Omega_2(u) D_2(u) B_1(u_1) |0\rangle + \Omega_3(u) D_3(u) B_1(u_1) |0\rangle.$$  

(3.26)

Since we know the action of the operators $D_i(u)$ on the reference state $|0\rangle$, we need to arrange the operator products

$$D_1(u) B_1(u_1), \quad D_2(u) B_1(u_1) \quad \text{and} \quad D_3(u) B_1(u_1)$$

(3.27)

in a normal-ordered form [17].

We anticipate that, in general, the operator-valued function $\Psi_n(u_1, \ldots, u_n)$ for a $n$-particle Bethe state will be composed by a set of normal-ordered monomials. A monomial is said to be in normal order if all elements $B_i$ are on the left, and all elements $C_i$ are on the right of the elements $D_i$.

In order to get this normal ordering we recall that the double-row monodromy matrix $U(u)$ satisfies the fundamental reflection equation

$$R_{12}(u-v) U_1(u) R_{21}(u+v) U_2(v) = U_2(v) R_{12}(u+v) U_1(u) R_{21}(u-v),$$

(3.28)

where $U_1(u) = U(u) \otimes 1$, $U_2(u) = 1 \otimes U(u)$ and $R_{21}(u) = P R_{12}(u) P$. In the appendix we show how this equation (indeed a set of 81 equations for the three-state models) can be used to recast the non-normal
ordered operator products as the above into a linear combination of normal-ordered ones, which might be called fundamental set of generalized commutation relations, or shortly commutations relations.

For the present case $t(u)\Psi_1(u_1)$ can be computed with the aid of the following commutation relations (see the appendix)

$$
D_1(u)B_1(u_1) = a_{11}(u, u_1)B_1(u_1)D_1(u) + a_{12}(u, u_1)B_1(u)D_1(u_1) + a_{13}(u, u_1)B_1(u)D_2(u_1) + a_{14}(u, u_1)B_2(u)C_1(u_1) + a_{15}(u, u_1)B_2(u)C_3(u_1) + a_{16}(u, u_1)B_2(u_1)C_1(u) (3.29)
$$

$$
D_2(u)B_1(u_1) = a_{21}(u, u_1)B_1(u_1)D_2(u) + a_{22}(u, u_1)B_1(u)D_2(u_1) + a_{23}(u, u_1)B_1(u)D_1(u_1) + a_{24}(u, u_1)B_2(u_1)D_1(u) + a_{25}(u, u_1)B_2(u)D_2(u_1) + a_{26}(u, u_1)B_2(u)C_1(u_1) + a_{27}(u, u_1)B_2(u_1)C_3(u_1) + a_{28}(u, u_1)B_2(u_1)C_1(u) + a_{29}(u, u_1)B_2(u_1)C_3(u) (3.30)
$$

$$
D_3(u)B_1(u_1) = a_{31}(u, u_1)B_1(u_1)D_3(u) + a_{32}(u, u_1)B_1(u)D_3(u_1) + a_{33}(u, u_1)B_1(u)D_2(u_1) + a_{34}(u, u_1)B_3(u_1)D_1(u) + a_{35}(u, u_1)B_3(u)D_2(u_1) + a_{36}(u, u_1)B_3(u)D_1(u_1) + a_{37}(u, u_1)B_3(u_1)D_2(u_1) + a_{38}(u, u_1)B_3(u_1)C_1(u_1) + a_{39}(u, u_1)B_3(u_1)C_3(u_1) (3.31)
$$

Introducing these relations in the eq (3.26) one gets

$$
t(u)\Psi_1(u_1) = \Omega_1(u)D_1(u)B_1(u_1)\langle 0 \rangle + \Omega_2(u)D_2(u)B_1(u_1)\langle 0 \rangle + \Omega_3(u)D_3(u)B_1(u_1)\langle 0 \rangle
$$

$$
= [a_{11}(u, u_1)\Omega_1(u)X_1(u_1) + a_{21}(u, u_1)\Omega_2(u)X_2(u_1) + a_{31}(u, u_1)\Omega_3(u)X_3(u_1)]\Psi_1(u_1) + [\sum_{j=1}^{3} \Omega_j(u)a_{j2}(u, u_1)X_2(u_1) + \sum_{j=1}^{3} \Omega_j(u)a_{j3}(u, u_1)]B_1(u)\langle 0 \rangle + [\sum_{j=2}^{3} \Omega_j(u)a_{j4}(u, u_1)X_2(u_1) + \sum_{j=2}^{3} \Omega_j(u)a_{j5}(u, u_1)]B_3(u)\langle 0 \rangle (3.32)
$$

So $\Psi_1(u_1)$ will be an eigenstate of $t(u)$ with eigenvalue

$$
\Lambda_1(u, u_1) = \sum_{j=1}^{3} \Omega_j(u)X_j(u)a_{j1}(u, u_1) (3.33)
$$

provided the following equations are satisfied

$$
\frac{X_1(u_1)}{X_2(u_1)} = \frac{\sum_{j=1}^{3} \Omega_j(u)a_{j3}(u, u_1)}{\sum_{j=1}^{3} \Omega_j(u)a_{j2}(u, u_1)} = \frac{\sum_{j=2}^{3} \Omega_j(u)a_{j5}(u, u_1)}{\sum_{j=2}^{3} \Omega_j(u)a_{j4}(u, u_1)} \equiv \Theta(u_1) (3.34)
$$

These are the Bethe equations for the one-particle state and the corresponding Bethe root $u_1$ does not depend on $u$. 

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3.3 The two-particle state

For $m = 2$ there are two linearly independent states $\mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\ket{0}$ and $\mathcal{B}_2(u_1)\ket{0}$. Therefore we seek for eigenstates of $t(u)$ in the form

$$\Psi_2(u_1, u_2) = \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\ket{0} + \mathcal{B}_2(u_1)\Gamma(u_1, u_2)\ket{0}$$  (3.35)

where $\Gamma(u_1, u_2)$ is an operator-valued function. Next we will use the condition that $\Psi_2(u_1, u_2)$ must be normal-ordered to find $\Gamma(u_1, u_2)$.

The first term in the right hand side of the eq (3.35) has its normal-ordered form given by the commutation relation:

$$\mathcal{B}_1(u_1)\mathcal{B}_1(u_2) = \omega(u_1, u_2)\left[\mathcal{B}_1(u_2)\mathcal{B}_1(u_1) + G_{d_1}(u_2, u_1)\mathcal{B}_2(u_2)\mathcal{D}_1(u_1) + G_{d_2}(u_2, u_1)\mathcal{B}_2(u_2)\mathcal{D}_2(u_1)\right]$$

$$- G_{d_1}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2) - G_{d_2}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2)$$  (3.36)

where

$$\omega(u_1, u_2) = \frac{x_3(u_1 - u_2)x_4(u_1 - u_2) - x_6(u_1 - u_2)y_6(u_1 - u_2)}{x_1(u_1 - u_2)x_3(u_1 - u_2)}$$

$$\omega(u_2, u_1)\omega(u_1, u_2) = 1$$  (3.37)

$$G_{d_1}(u_1, u_2) = -\frac{x_6(u_1 - u_2)x_2(2u_2)}{x_3(u_1 - u_2)x_1(2u_2)}$$  (3.38)

$$G_{d_2}(u_1, u_2) = \frac{x_6(u_1 + u_2)}{x_2(u_1 + u_2)}$$  (3.39)

Here we have used the following identities valid for both models,

$$\frac{y_6(-u)}{x_3(-u)} = -\frac{x_3(u)x_6(u) - x_7(u)y_6(u)}{x_3(u)x_4(u) - x_6(u)y_6(u)}$$  (3.40)

and

$$\frac{x_2(2u)}{x_1(2u)} = \frac{y_5(u - v)x_2(u + v) + x_2(u - v)x_5(u + v)f_1(u)}{y_5(u - v)x_1(u + v)}$$  (3.41)

Now we can see that (3.35) is normal ordered if it satisfies the condition

$$\Psi_2(u_2, u_1) = \omega(u_2, u_1)\Psi_2(u_1, u_2)$$  (3.42)

This condition fixes $\Gamma(u_1, u_2)$ and, by construction, the unique candidate for the eigenstate of $t(u)$ in the $m = 2$ case has the form

$$\Psi_2(u_1, u_2) = \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\ket{0} + G_{d_1}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2)\ket{0} + G_{d_2}(u_1, u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2)\ket{0}$$  (3.43)
The action of \( t(u) \) on this state reads
\[
\begin{align*}
\text{for } m = 1, v = \frac{1}{2} \text{ case (see the appendix):} \\
D_1(u) B_2(v) &= b_{11}(u, v) B_2(v) D_1(u) + b_{12}(u, v) B_2(u) D_1(v) + b_{13}(u, v) B_2(u) D_2(v) + b_{14}(u, v) B_2(u) D_3(v) + b_{15}(u, v) B_1(u) B_1(v) + b_{16}(u, v) B_1(u) B_3(v) \\
D_2(u) B_2(v) &= b_{21}(u, v) B_2(v) D_2(u) + b_{22}(u, v) B_2(u) D_1(v) + b_{23}(u, v) B_2(u) D_2(v) + b_{24}(u, v) B_2(u) D_3(v) + b_{25}(u, v) B_1(u) B_1(v) + b_{26}(u, v) B_1(u) B_3(v) + b_{27}(u, v) B_3(u) B_1(v) + b_{28}(u, v) B_3(u) B_3(v) \\
D_3(u) B_2(v) &= b_{31}(u, v) B_2(v) D_3(u) + b_{32}(u, v) B_2(u) D_1(v) + b_{33}(u, v) B_2(u) D_2(v) + b_{34}(u, v) B_2(u) D_3(v) + b_{35}(u, v) B_1(u) B_1(v) + b_{36}(u, v) B_1(u) B_3(v) + b_{37}(u, v) B_3(u) B_1(v) + b_{38}(u, v) B_3(u) B_3(v) \\
C_1(u) B_1(v) &= c_{11}(u, v) B_1(v) C_1(u) + c_{12}(u, v) B_1(v) C_3(v) + c_{13}(u, v) B_1(u) C_3(v) + c_{14}(u, v) B_2(u) C_3(v) + c_{15}(u, v) B_2(u) C_3(u) + c_{16}(u, v) D_1(u) D_1(u) + c_{17}(u, v) D_1(u) D_2(u) + c_{18}(u, v) D_1(u) D_3(v) + c_{19}(u, v) D_1(u) D_1(v) + c_{20}(u, v) D_2(u) D_2(v) + c_{21}(u, v) D_2(u) D_3(v) + c_{22}(u, v) D_3(u) D_2(v) + c_{23}(u, v) D_3(u) D_1(v) \text{ (3.47)} \\
C_3(u) B_1(v) &= c_{21}(u, v) B_1(v) C_1(u) + c_{22}(u, v) B_1(v) C_3(v) + c_{23}(u, v) B_1(u) C_3(v) + c_{24}(u, v) B_2(u) C_3(v) + c_{25}(u, v) B_2(u) C_3(u) + c_{26}(u, v) D_1(u) D_1(u) + c_{27}(u, v) D_1(u) D_2(u) + c_{28}(u, v) D_1(u) D_3(v) + c_{29}(u, v) D_1(u) D_1(v) + c_{30}(u, v) D_2(u) D_2(v) + c_{31}(u, v) D_2(u) D_3(v) + c_{32}(u, v) D_3(u) D_2(v) + c_{33}(u, v) D_3(u) D_1(v) \text{ (3.48)} \\
& \text{After a straightforward calculation we obtain } \\
t(u) \Psi_2(u_1, u_2) = \sum_{j=1}^{3} \Omega_j(u) \mathcal{X}_j(u) a_{j1}(u, u_1) a_{j1}(u, u_2) \Psi_2(u_1, u_2)
\[ + [a_{11}(u_1, u_2)X_1(u_1) \sum_{j=1}^{3} \Omega_j(u)a_{j2}(u, u_1) + a_{21}(u_1, u_2)X_1(u_1) \sum_{j=1}^{3} \Omega_j(u)a_{j3}(u, u_1)]B_1(u)B_1(u_2) \|0 \] 

\[ + [a_{11}(u_1, u_2)X_1(u_1) \sum_{j=2}^{3} \Omega_j(u)a_{j4}(u, u_1) + a_{21}(u_1, u_2)X_2(u_1) \sum_{j=2}^{3} \Omega_j(u)a_{j5}(u, u_1)]B_3(u)B_1(u_2) \|0 \]

\[ + [a_{11}(u_2, u_1)X_1(u_2) \sum_{j=1}^{3} \Omega_j(u)a_{j2}(u, u_2) + a_{21}(u_2, u_1)X_1(u_2) \sum_{j=1}^{3} \Omega_j(u)a_{j3}(u, u_2)]\omega(u_1, u_2)B_1(u)B_1(u_1) \|0 \]

\[ + [a_{11}(u_2, u_1)X_1(u_2) \sum_{j=2}^{3} \Omega_j(u)a_{j4}(u, u_2) + a_{21}(u_2, u_1)X_2(u_2) \sum_{j=2}^{3} \Omega_j(u)a_{j5}(u, u_2)]\omega(u_1, u_2)B_3(u)B_1(u_1) \|0 \]

\[ + [X_1(u_1)X_1(u_2) \sum_{j=1}^{3} \Omega_j(u)H_{j1}(u_1, u_2) + X_1(u_1)X_2(u_2) \sum_{j=1}^{3} \Omega_j(u)H_{j3}(u_1, u_2) \] 

\[ + X_2(u_1)X_1(u_2) \sum_{j=1}^{3} \Omega_j(u)H_{j2}(u_1, u_2) + X_2(u_1)X_2(u_2) \sum_{j=1}^{3} \Omega_j(u)H_{j4}(u_1, u_2)]B_2(u) \|0 \]  \text{(3.50)}

Where

\[ H_{11}(u_1, u_2) = a_{14}(u, u_1) (c_{16}(u_1, u_2) + c_{18}(u_1, u_2)) + a_{15}(u, u_1) (c_{26}(u_1, u_2) + c_{29}(u_1, u_2)) \]

\[ + b_{12}(u, u_1)G_{d_1}(u_1, u_2) + \omega(u_1, u)a_{11}(u, u_1)a_{12}(u, u_2)G_{d_1}(u, u_1) \]

\[ H_{12}(u_1, u_2) = a_{14}(u, u_1) (c_{17}(u_1, u_2) + c_{110}(u_1, u_2)) + a_{15}(u, u_1) (c_{27}(u_1, u_2) + c_{211}(u_1, u_2)) \]

\[ + b_{13}(u, u_1)G_{d_1}(u_1, u_2) + \omega(u_1, u)a_{11}(u, u_1)a_{12}(u, u_2)G_{d_2}(u, u_1) \]

\[ H_{13}(u_1, u_2) = a_{14}(u, u_1)c_{19}(u_1, u_2) + a_{15}(u, u_1)c_{210}(u_1, u_2) + b_{12}(u, u_1)G_{d_2}(u_1, u_2) \]

\[ + \omega(u_1, u)a_{11}(u, u_1)a_{13}(u, u_2)G_{d_1}(u, u_1) \]

\[ H_{14}(u_1, u_2) = a_{14}(u, u_1)c_{111}(u_1, u_2) + a_{15}(u, u_1)c_{212}(u_1, u_2) + b_{13}(u, u_1)G_{d_2}(u_1, u_2) \]

\[ + \omega(u_1, u)a_{11}(u, u_1)a_{13}(u, u_2)G_{d_2}(u, u_1) \]  \text{(3.51)}
and

\[ H_{j1}(u_1, u_2) = a_{j6}(u, u_1)(c_{16}(u_1, u_2) + c_{18}(u_1, u_2)) + a_{j7}(u, u_1)(c_{26}(u_1, u_2) + c_{28}(u_1, u_2)) + b_{j2}(u, u_1)G_d(u_1, u_2) + \omega(u_1, u_1)a_{j1}(u, u_1)a_{j2}(u, u_2)G_d(u, u_1) + a_{j1}(u, u_1)a_{j4}(u, u_2)d_1(u, u) \]

\[ H_{j2}(u_1, u_2) = a_{j6}(u, u_1)(c_{17}(u_1, u_2) + c_{19}(u_1, u_2)) + a_{j7}(u, u_1)(c_{27}(u_1, u_2) + c_{29}(u_1, u_2)) + b_{j3}(u, u_1)G_d(u_1, u_2) + \omega(u_1, u_1)a_{j1}(u, u_1)a_{j2}(u, u_2)G_d(u, u_1) + a_{j1}(u, u_1)a_{j4}(u, u_2)d_1(u, u) \]

\[ H_{j3}(u_1, u_2) = a_{j6}(u, u_1) + a_{j7}(u, u_1)c_{210}(u_1, u_2) + b_{j2}(u, u_1)G_d(u_1, u_2) + \omega(u_1, u_1)a_{j1}(u, u_1)a_{j3}(u, u_2)G_d(u, u_1) + a_{j1}(u, u_1)a_{j5}(u, u_2)d_1(u, u) \]

\[ H_{j4}(u_1, u_2) = a_{j6}(u, u_1) + a_{j7}(u, u_1)c_{212}(u_1, u_2) + b_{j3}(u, u_1)G_d(u_1, u_2) + \omega(u_1, u_1)a_{j1}(u, u_1)a_{j3}(u, u_2)G_d(u, u_1) + a_{j1}(u, u_1)a_{j5}(u, u_2)d_1(u, u) \] (3.52)

for \( j = 2, 3 \). Again, \( \Psi(u, u_1, u_2) \) will be an eigenstate of \( t(u) \) with eigenvalue

\[ \Lambda_2(u, u_1, u_2) = \sum_{j=1}^{3} \Omega_j(u)\mathcal{X}_j(u)a_{j1}(u, u_1)a_{j1}(u, u_2) \] (3.53)

provided the following Bethe equations are satisfied

\[ \frac{\mathcal{X}_1(u_1)}{\mathcal{X}_2(u_1)} = \Theta(u_1)\frac{a_{21}(u_1, u_2)}{a_{11}(u_1, u_2)}, \quad \frac{\mathcal{X}_1(u_2)}{\mathcal{X}_2(u_2)} = \Theta(u_2)\frac{a_{21}(u_2, u_1)}{a_{11}(u_2, u_1)} \] (3.54)

where \( \Theta(u_i), i = 1, 2 \) are given by (3.34).

### 3.4 The three-particle state

In this case we have to consider combinations and permutations of the states of the type \( B_1B_1B_1 | 0 \) and \( B_2B_1 | 0 \).

Let us consider the normal-ordered operator

\[ \Phi_3(u_1, u_2, u_3) = B_1(u_1)\Phi_2(u_2, u_3) + B_2(u_1)\Phi_1(u_2)\Gamma_1(u_1, u_2, u_3) + B_2(u_1)\Phi_1(u_3)\Gamma_2(u_1, u_2, u_3) \] (3.55)

with two exchange proprieties

\[ \Phi_3(u_1, u_2, u_3) = \omega(u_1, u_2)\Phi_3(u_2, u_1, u_3) = \omega(u_2, u_3)\Phi_3(u_1, u_3, u_2), \] (3.56)

which can be used, together the commutations relations, to find the operator valued functions

\[ \Gamma_1(u_1, u_2, u_3) = \omega(u_2, u_3)[a_{11}(u_2, u_3)G_{d_1}(u_1, u_3)D_1(u_3) + a_{21}(u_3, u_2)G_{d_2}(u_1, u_3)D_2(u_3)] \]

\[ \Gamma_2(u_1, u_2, u_3) = a_{11}(u_1, u_3)G_{d_1}(u_1, u_2)D_1(u_2) + a_{21}(u_2, u_3)G_{d_2}(u_1, u_2)D_2(u_2) \] (3.57)
Note that
\[
\Gamma_1(u_1, u_2, u_3) = \omega(u_2, u_3)\Gamma_2(u_1, u_3, u_2) \quad (3.58)
\]

The eigenvalue problem for the 3-particle state is now reduced to the problem of finding the action of the operators \(D_\alpha(u)\), \(\alpha = 1, 2, 3\) on the state
\[
\Psi_3(u_1, u_2, u_3) = \Phi_3(u_1, u_2, u_3) |0\rangle \quad (3.59)
\]

To do this we will need recall the appendix to get more five commutation relations:
\[
\begin{align*}
D_1(u)B_3(v) &= x_{11}(u, v)B_3(v)D_1(u) + x_{12}(u, v)B_1(v)D_1(u) + x_{13}(u, v)B_1(u)D_1(v) \\
&\quad + x_{14}(u, v)B_1(u)D_2(v) + x_{15}(u, v)B_1(u)D_3(v) + x_{16}(u, v)B_2(u)C_1(v) \\
&\quad + x_{17}(u, v)B_2(u)C_3(v) + x_{18}(u, v)B_2(v)C_1(u) \\
D_2(u)B_3(v) &= x_{21}(u, v)B_3(v)D_2(u) + x_{22}(u, v)B_1(v)D_2(u) + x_{23}(u, v)B_1(u)D_1(v) \\
&\quad + x_{24}(u, v)B_1(u)D_2(v) + x_{25}(u, v)B_1(u)D_3(v) + x_{26}(u, v)B_3(u)D_1(v) \\
&\quad + x_{27}(u, v)B_3(u)D_2(v) + x_{28}(u, v)B_3(u)D_3(v) + x_{29}(u, v)B_2(u)C_1(v) \\
&\quad + x_{210}(u, v)B_2(u)C_3(v) + x_{211}(u, v)B_2(v)C_1(u) + x_{212}(u, v)B_2(v)C_3(u) \quad (3.60)
\end{align*}
\]
\[
\begin{align*}
D_3(u)B_3(v) &= x_{31}(u, v)B_3(v)D_3(u) + x_{32}(u, v)B_1(v)D_3(u) + x_{33}(u, v)B_1(u)D_1(v) \\
&\quad + x_{34}(u, v)B_1(u)D_2(v) + x_{35}(u, v)B_1(u)D_3(v) + x_{36}(u, v)B_3(u)D_1(v) \\
&\quad + x_{37}(u, v)B_3(u)D_2(v) + x_{38}(u, v)B_3(u)D_3(v) + x_{39}(u, v)B_2(u)C_1(v) \\
&\quad + x_{310}(u, v)B_2(u)C_3(v) + x_{311}(u, v)B_2(v)C_1(u) + x_{312}(u, v)B_2(v)C_3(u) \quad (3.61)
\end{align*}
\]
\[
\begin{align*}
C_1(u)B_2(v) &= y_{11}(u, v)B_2(v)C_1(u) + y_{12}(u, v)B_2(v)C_3(u) + y_{13}(u, v)B_2(u)C_1(v) \\
&\quad + y_{14}(u, v)B_2(u)C_3(v) + y_{15}(u, v)B_1(v)D_1(u) + y_{16}(u, v)B_1(v)D_2(u) \\
&\quad + y_{17}(u, v)B_3(v)D_1(u) + y_{18}(u, v)B_3(v)D_2(u) + y_{19}(u, v)B_1(u)D_1(v) \\
&\quad + y_{110}(u, v)B_1(u)D_2(v) + y_{111}(u, v)B_1(u)D_3(v) + y_{112}(u, v)B_3(u)D_1(v) \\
&\quad + y_{113}(u, v)B_3(u)D_2(v) + y_{114}(u, v)B_3(u)D_3(v) \quad (3.62)
\end{align*}
\]
\[
\begin{align*}
C_3(u)B_2(v) &= y_{21}(u, v)B_2(v)C_1(u) + y_{22}(u, v)B_2(v)C_3(u) + y_{23}(u, v)B_2(u)C_1(v) \\
&\quad + y_{24}(u, v)B_2(u)C_3(v) + y_{25}(u, v)B_1(v)D_1(u) + y_{26}(u, v)B_1(v)D_2(u) \\
&\quad + y_{27}(u, v)B_1(v)D_3(u) + y_{28}(u, v)B_3(v)D_1(u) + y_{29}(u, v)B_3(v)D_2(u) \\
&\quad + y_{210}(u, v)B_3(v)D_3(u) + y_{211}(u, v)B_1(u)D_1(v) + y_{212}(u, v)B_1(u)D_2(v) \\
&\quad + y_{213}(u, v)B_1(u)D_3(v) + y_{214}(u, v)B_3(u)D_1(v) + y_{215}(u, v)B_3(u)D_2(v) \\
&\quad + y_{216}(u, v)B_3(u)D_3(v) \quad (3.63)
\end{align*}
\]
After a straightforward but (the reader should be advised) quite lengthy computation we obtain the following simplified expressions

\[
D_\alpha(u)\Psi_3(u_1, u_2, u_3) = X_\alpha(u) \prod_{i=1}^{3} a_{\alpha 1}(u, u_i) \Psi_3(u_1, u_2, u_3)
\]

\[
+ \sum_{i=1}^{3} \prod_{j=1}^{i-1} \omega(u_j, u_i) \left[ X_1(u_i) a_{\alpha 2}(u, u_i) \prod_{k \neq i}^{3} a_{11}(u, u_k) + X_2(u_i) a_{\alpha 3}(u, u_i) \prod_{k \neq i}^{3} a_{21}(u, u_k) \right] \mathcal{B}_1(u) \Psi_2(\hat{u}_i)
\]

\[
+ (1 - \delta_{\alpha,1}) \sum_{i=1}^{3} \prod_{j=1}^{i-1} \omega(u_j, u_i) \left[ X_1(u_i) a_{\alpha 4}(u, u_i) \prod_{k \neq i}^{3} a_{11}(u, u_k) + X_2(u_i) a_{\alpha 5}(u, u_i) \prod_{k \neq i}^{3} a_{21}(u, u_k) \right] \mathcal{B}_3(u) \Psi_2(\hat{u}_i)
\]

\[
+ \sum_{i=1}^{2} \sum_{j=i+1}^{3} (X_1(u_i) X_1(u_j) a_{11}(u_i, u_m) a_{11}(u_j, u_m) H_{\alpha 1}(u_i, u_j)
\]

\[
+ X_2(u_i) X_1(u_j) a_{21}(u_i, u_m) a_{11}(u_j, u_m) H_{\alpha 2}(u_i, u_j) + X_1(u_i) X_2(u_j) a_{11}(u_i, u_m) a_{21}(u_j, u_m) H_{\alpha 3}(u_i, u_j)
\]

\[
+ X_2(u_i) X_2(u_j) a_{21}(u_i, u_m) a_{21}(u_j, u_m) H_{\alpha 4}(u_i, u_j) \right) \prod_{k=1}^{i-1} \omega(u_k, u_i) \prod_{l=1}^{j-1} \omega(u_l, u_j) \mathcal{B}_2(u) \Psi_1(u_m)
\]

(3.65)

for \( \alpha = 1, 2, 3 \) and \( m \neq \{i, j\} \). In the above expression we have introduced the symbol \( \hat{u}_i \), which as usual means that the state \( \Psi_2 \) has to be evaluated at spots different form \( u_i \).

By performing the action of transfer matrix \( t(u) \) on the state \( \Psi_3(u_1, u_2, u_3) \), we get

\[
t(u) \Psi_3(u_1, u_2, u_3) = \left( \sum_{\alpha=1}^{3} \Omega_\alpha(u) X_\alpha(u) \prod_{i=1}^{2} a_{\alpha 1}(u, u_i) \right) \Psi_3(u_1, u_2, u_3)
\]

(3.66)

all the unwanted terms vanishing provided that the Bethe equations are satisfied:

\[
\frac{X_1(u_i)}{X_2(u_i)} = \Theta(u_i) \prod_{j \neq i=1}^{3} \frac{a_{21}(u_i, u_j)}{a_{11}(u_i, u_j)} \quad (i = 1, 2, 3)
\]

(3.67)

### 3.5 The n-particle state

From the previous results one can seek for operator valued functions with a recurrence relation of the form

\[
\Phi_n(u, \ldots, u_n) = \mathcal{B}_1(u_1) \Phi_{n-1}(u_2, \ldots, u_n)
\]

\[
+ \mathcal{B}_2(u_1) \sum_{i=2}^{n} f^{(i)}_1(u_1, \ldots, u_n) \Phi_{n-2}(u_2, \ldots, \hat{u}_i, \ldots, u_n) D_1(u_i)
\]

\[
+ \mathcal{B}_2(u_1) \sum_{i=2}^{n} f^{(i)}_2(u_1, \ldots, u_n) \Phi_{n-2}(u_2, \ldots, \hat{u}_i, \ldots, u_n) D_2(u_i)
\]

(3.68)
It was shown in ([10]) that the above operator will be normal ordered satisfying \( n - 1 \) exchange conditions

\[
\Phi_n(u_1, \ldots, u_i, u_{i+1}, \ldots, u_n) = \omega(u_i, u_{i+1})\Phi_n(u_1, \ldots, u_{i+1}, u_{i}, \ldots, u_n)
\]  

(3.69)

provided that the functions \( F^{(i)}_\alpha(u_1, \ldots, u_n) \) are given by

\[
F^{(i)}_\alpha(u_1, \ldots, u_n) = \prod_{j=2}^{i-1} \omega(u_j, u_i) \prod_{k=2, k \neq i}^{n} a_{\alpha_1}(u_i, u_k)G_{\alpha}(u_1, u_i), \quad (\alpha = 1, 2)
\]  

(3.70)

Therefore the \( n \)-particle state will be given by

\[
\Psi_n(u_1, \ldots, u_n) = \Phi_n(u_1, \ldots, u_n)|0\rangle
\]  

(3.71)

and the action of the operators \( D_\alpha(u) \), \( \alpha = 1, 2, 3 \), on this state will be represented by

\[
D_\alpha(u)\Psi_n(u_1, \ldots, u_n) = \mathcal{X}_\alpha(u)\prod_{i=1}^{n} a_{\alpha_1}(u_i)\Psi_n(u_1, \ldots, u_n)
\]

\[
+ \sum_{i=1}^{n-1} \prod_{j=1}^{i-1} \omega(u_j, u_i) [\mathcal{X}_1(u_i) a_{\alpha_2}(u_i, u_{i+1}) \prod_{k \neq i} a_{\alpha_1}(u_i, u_k) + \mathcal{X}_2(u_i) a_{\alpha_3}(u_i, u_{i+1}) \prod_{k \neq i} a_{\alpha_1}(u_i, u_k)] B_1(u)\Psi_{n-1}(u_i^{\dagger})
\]

\[
+ (1-\delta_{0,1}) \sum_{i=1}^{n-1} \prod_{j=1}^{i-1} \omega(u_j, u_i) [\mathcal{X}_1(u_i) a_{\alpha_4}(u_i, u_{i+1}) \prod_{k \neq i} a_{\alpha_1}(u_i, u_k) + \mathcal{X}_2(u_i) a_{\alpha_5}(u_i, u_{i+1}) \prod_{k \neq i} a_{\alpha_1}(u_i, u_k)] B_3(u)\Psi_{n-1}(u_i^{\dagger})
\]

\[
\times \prod_{k=1}^{i-1} \omega(u_k, u_i) \prod_{l=1}^{j-1} \omega(u_l, u_j) B_2(u)\Psi_{n-2}(u_i^{\dagger}, u_j^{\dagger})
\]  

(3.72)

Finally, the corresponding \( n \)-particle eigenvalue problem will be

\[
t(u)\Psi_n(u_1, \ldots, u_n) = \left( \sum_{\alpha=1}^{3} \Omega_\alpha(u)\mathcal{X}_\alpha(u) \prod_{i=1}^{n} a_{\alpha_1}(u_i) \right) \Psi_n(u_1, \ldots, u_n)
\]  

(3.73)

provided that the Bethe equations are satisfied:

\[
\frac{\mathcal{X}_1(u_k)}{\mathcal{X}_2(u_k)} = \Theta(u_k) \prod_{j=1}^{n} a_{\alpha_1}(u_k, u_j) a_{\alpha_1}(u_k, u_j), \quad (k = 1, 2, \ldots, n)
\]  

(3.74)
4 Explicit Solutions

In this section explicit expressions of the eigenvalue problem are presented for both models. First we recall the appendix to get the coefficients $a_{ij}(u, v)$ which appear effectively in the Bethe Ansatz expressions (3.73) and (3.74):

$$a_{11}(u, v) = \frac{x_1(v - u) x_2(u + v)}{x_2(v - u) x_1(u + v)}$$

$$a_{21}(u, v) = \omega(u, v)[\frac{x_1(u + v)x_4(u + v) - x_5(u + v)y_5(u + v)}{x_1(u + v)x_2(u + v)}]$$

$$a_{31}(u, v) = \frac{x_2(u - v) x_2(u + v)^2 - x_6(u + v)y_6(u + v)}{x_2(u + v)x_3(u + v)} \quad (4.1)$$

For the factor with the boundary contributions $\Theta(u_i)$, we will consider only the simplest expression

$$\Theta(u_i) = -\frac{\Omega_2(u_25(u, u_i) + \Omega_3(u)\beta_{34}(u, u_i)}{\Omega_2(u_{24}(u, u_i) + \Omega_3(u)\beta_{34}(u, u_i)} \quad (4.2)$$

where the coefficients are given by

$$a_{24}(u, v) = \frac{x_6(u - v) x_3(u + v)}{x_3(u - v) x_2(u + v)} - f_1(v) \frac{x_6(u + v)}{x_2(u + v)}$$

$$a_{25}(u, v) = -\frac{x_6(u + v)}{x_2(u + v)} \quad (4.3)$$

$$a_{34}(u, v) = f_3(u)[f_1(v) \frac{x_6(u + v)}{x_2(u + v)} - \frac{x_6(u - v) x_3(u + v)}{x_3(u - v) x_2(u + v)}]$$

$$+ f_1(v) \frac{y_6(u - v)}{x_3(u - v)} \left[\frac{x_6(u + v)y_6(u + v) - x_2^2(u + v)}{x_2(u + v)x_3(u + v)}\right]$$

$$- \frac{x_6(u - v)y_6(u - v) - x_2^2(u - v)}{x_2(u + v)} \frac{y_6(u + v)}{x_3(u - v)}$$

$$a_{35}(u, v) = f_3(u) \frac{x_6(u + v)}{x_2(u + v)} + \frac{y_6(u - v)}{x_3(u - v)} \left[\frac{x_6(u + v)y_6(u + v) - x_2^2(u + v)}{x_2(u + v)x_3(u + v)}\right] \quad (4.4)$$

4.1 ZF Model

Substituting the matrix elements of the $R$ matrix and of the $K$ matrices for this model we get the following expressions for the terms with boundary contributions

$$X_1(u) = -\frac{\beta \sinh u + 2 \cosh u \alpha_1^{2N}(u)}{\beta \sinh u - 2 \cosh u \rho^N(u)}$$

$$X_2(u) = \frac{\sinh 2u}{\beta \sinh(u + 2\eta)} - \frac{2 \cosh(u + 2\eta)}{\beta \sinh u - 2 \cosh u \rho^N(u)}$$

$$X_3(u) = \frac{\sinh(2u - \eta) \beta \sinh(u + \eta) - 2 \cosh(u + \eta)}{\beta \sinh(u + 2\eta) - 2 \cosh(u + 2\eta) \rho^N(u)}$$

$$X_3(u) = -\frac{\sinh(2u - \eta) \beta \sinh(u + \eta) + 2 \cosh(u - \eta)}{\beta \sinh(u - 2 \cosh u) \rho^N(u)}$$

(4.5)
\[ \Omega_1(u) = \frac{\sinh(2u + 3\eta)}{\sinh(2u + \eta)} \cdot \frac{\alpha \sinh u - 2 \cosh u}{\alpha \sinh(u + \eta) - 2 \cosh(u + \eta)} \]
\[ \Omega_2(u) = \frac{\sinh(2u + 2\eta)}{\sinh 2u} \cdot \frac{\alpha \sinh u - 2 \cosh u}{\alpha \sinh(u + 2\eta) - 2 \cosh(u + 2\eta)}, \]
\[ \Omega_3(u) = \frac{\alpha \sinh u + 2 \cosh u}{\alpha \sinh(u + 2\eta) - 2 \cosh(u + 2\eta)}, \]

and
\[ \Theta(u) = \frac{\sinh(2u_i + 2\eta) \alpha \sinh(u_i + \eta) + 2 \cosh(u_i + \eta)}{\alpha \sinh(u_i + \eta) + 2 \cosh(u_i + \eta)}. \]

The coefficients \( a_{11}(u, v) \) are given by
\[ a_{11}(u, v) = \frac{\sinh(u + v)}{\sinh(u + v + 2\eta)} \cdot \frac{\sinh(u - v - 2\eta)}{\sinh(u - v)}. \]
\[ a_{21}(u, v) = \frac{\sinh(u + v)}{\sinh(u + v + 2\eta)} \cdot \frac{\sinh(u + v + 3\eta)}{\sinh(u + v + \eta)} \cdot \frac{\sinh(u - v - 2\eta)}{\sinh(u - v)}. \]
\[ a_{31}(u, v) = \frac{\sinh(u - v + \eta)}{\sinh(u - v - \eta)} \cdot \frac{\sinh(u + v + \eta)}{\sinh(u + v + \eta)}. \]
\[ a_{21}(u, v) = \frac{\sinh(u + v + 3\eta)}{\sinh(u + v + \eta)} \cdot \frac{\sinh(u - v + \eta)}{\sinh(u - v + \eta)}. \]
\[ a_{11}(u, v) = \frac{\sinh(u + v)}{\sinh(u + v + \eta)} \cdot \frac{\sinh(u - v - \eta)}{\sinh(u - v - \eta)}. \]

It means that the \( n \)-particle state \( \Psi_n(\{u_i\}) \) (3.71) is an eigenfunction of the ZF transfer matrix with the eigenvalue
\[ \Lambda_n(u, \{u_i\}) = \chi_1(u) \Omega_1(u) \prod_{i=1}^{n} \frac{\sinh(u + u_i)}{\sinh(u + u_i + 2\eta)} \cdot \sinh(u - u_i - 2\eta) \]
\[ + \chi_2(u) \Omega_2(u) \prod_{i=1}^{n} \frac{\sinh(u + u_i)}{\sinh(u + u_i + 2\eta)} \cdot \sinh(u + u_i + 3\eta) \]
\[ \times \prod_{i=1}^{n} \frac{\sinh(u - u_i + \eta)}{\sinh(u - u_i - \eta)} \cdot \sinh(u - u_i). \]
\[ + \chi_1(u) \Omega_1(u) \prod_{i=1}^{n} \frac{\sinh(u - u_i + \eta)}{\sinh(u - u_i - \eta)} \cdot \sinh(u + u_i + 3\eta) \]

provided that the parameters \( u_i \) satisfy the Bethe equations
\[ \left( \frac{\sinh(u_i + 2\eta)}{\sinh(u_i)} \right)^{2N} = \frac{\alpha \sinh(u_i + \eta) + 2 \cosh(u_i + \eta)}{\alpha \sinh(u_i + \eta) - 2 \cosh(u_i + \eta)} \cdot \frac{\beta \sinh(u_i + 2\eta) - 2 \cosh(u_i + 2\eta)}{\beta \sinh(u_i + 2\eta) + 2 \cosh(u_i)}. \]
\[ \prod_{j=1, j \neq i}^{n} \frac{\sinh(u_i + u_j + 3\eta) \sinh(u_i - u_j + \eta)}{\sinh(u_i + u_j + \eta) \sinh(u_i - u_j - \eta)} \]
\[ i = 1, 2, ..., n \]

Putting these expressions in a symmetric for \( (u_i \rightarrow u_i - \eta) \) one can see that we have recovered the previous results obtained by the fusion procedure [22, 13].
4.2 IK Model

In this model the $K$ matrices have no free parameters and we have to consider two cases: the quantum group invariant and the non-quantum group invariant. For both cases the $a_{ij}(u, v)$ coefficients are given by

$$
a_{11}(u, v) = \frac{\sinh \left( \frac{1}{2} (u + v) \right)}{\sinh \left( \frac{1}{2} (u + v) - 2\eta \right)} \frac{\sinh \left( \frac{1}{2} (u - v) + 2\eta \right)}{\sinh \left( \frac{1}{2} (u - v) \right)},
$$

$$
a_{21}(u, v) = \frac{\sinh \left( \frac{1}{2} (u + v) - 4\eta \right)}{\sinh \left( \frac{1}{2} (u + v) - 2\eta \right)} \frac{\sinh \left( \frac{1}{2} (u - v) - 2\eta \right)}{\sinh \left( \frac{1}{2} (u - v) \right)} \frac{\cosh \left( \frac{1}{2} (u + v) - \eta \right) \cosh \left( \frac{1}{2} (u + v) - \eta \right)}{\cosh \left( \frac{1}{2} (u + v) - 3\eta \right)} \frac{\cosh \left( \frac{1}{2} (u - v) + \eta \right) \cosh \left( \frac{1}{2} (u + v) - \eta \right)}{\cosh \left( \frac{1}{2} (u - v) - \eta \right) \cosh \left( \frac{1}{2} (u + v) - 3\eta \right)},
$$

$$
a_{31}(u, v) = \frac{\cosh \left( \frac{1}{2} (u - v) - 3\eta \right) \cosh \left( \frac{1}{2} (u + v) - 5\eta \right)}{\cosh \left( \frac{1}{2} (u - v) - \eta \right) \cosh \left( \frac{1}{2} (u + v) - 3\eta \right)},
$$

$$
a_{21}(u, v) = \frac{\sinh \left( \frac{1}{2} (u + v) - 4\eta \right) \sinh \left( \frac{1}{2} (u - v) - 2\eta \right)}{\sinh \left( \frac{1}{2} (u + v) \right) \sinh \left( \frac{1}{2} (u - v) + 2\eta \right) \cosh \left( \frac{1}{2} (u - v) - \eta \right) \cosh \left( \frac{1}{2} (u + v) - 3\eta \right)}. \quad (4.11)
$$

Though, the boundary contributions are different for each case:

4.2.1 The quantum group invariant case

In this case we have

$$
\mathcal{X}_1(u) = \frac{x_1^N(u)}{\rho^N(u)}
$$

$$
\mathcal{X}_2(u) = e^{2\eta} \frac{\sinh u \ x_2^N(u)}{\sinh(u - 2\eta) \ \rho^N(u)}
$$

$$
\mathcal{X}_3(u) = e^{2\eta} \frac{\sinh u \ \cosh(u - 5\eta) \ x_3^N(u)}{\sinh(u - 4\eta) \ \cosh(u - 3\eta) \ \rho^N(u)} \quad (4.12)
$$

$$
\Omega_1(u) = \frac{\sinh(u - 6\eta) \ \cosh(u - \eta)}{\sinh(u - 2\eta) \ \cosh(u - 3\eta)}
$$

$$
\Omega_2(u) = e^{-2\eta} \frac{\sinh(u - 6\eta)}{\sinh(u - 4\eta)}
$$

$$
\Omega_3(u) = e^{-2\eta} \quad (4.13)
$$

and

$$
\Theta(u_i) = e^{-2\eta} \frac{\sinh(u_i - 2\eta)}{\sinh u_i}, \quad i = 1, 2, ..., n \quad (4.14)
$$

Therefore, the $n$-particle state $\Psi_n(\{u_i\})$ is an eigenfunction of the IK transfer matrix $t(u)$ with eigenvalue

$$
\Lambda_n(u, \{u_i\}) = \frac{\sinh(u - 6\eta) \ \cosh(u - \eta) \ x_1^N(u)}{\sinh(u - 2\eta) \ \cosh(u - 3\eta) \ \rho^N(u)} \prod_{i=1}^{n} a_{11}(u, u_i)
$$

$$
+ \frac{\sinh(u - 6\eta) \ \sinh u \ x_2^N(u)}{\sinh(u - 4\eta) \ \sinh(u - 2\eta) \ \rho^N(u)} \prod_{i=1}^{n} a_{21}(u, u_i)
$$

$$
+ \frac{\sinh u \ \cosh(u - 5\eta) \ x_3^N(u)}{\sinh(u - 4\eta) \ \cosh(u - 3\eta) \ \rho^N(u)} \prod_{i=1}^{n} a_{31}(u, u_i) \quad (4.15)
$$
provided that the parameters $u_i$ are solutions of the Bethe equations

$$
\left( \frac{\sinh(\frac{1}{2}u_i - 2\eta)}{\sinh(\frac{1}{2}u_1)} \right)^{2N} = \prod_{j=1, j\neq i}^{n} \frac{\sinh(\frac{1}{2}(u_i + u_j) - 4\eta) \sinh(\frac{1}{2}(u_i - u_j) - 2\eta)}{\sinh(\frac{1}{2}(u_i + u_j)) \sinh(\frac{1}{2}(u_i - u_j) + 2\eta)} \times \frac{\cosh(\frac{1}{2}(u_i - u_j) + \eta) \cosh(\frac{1}{2}(u_i + u_j) - \eta)}{\cosh(\frac{1}{2}(u_i - u_j) - \eta) \cosh(\frac{1}{2}(u_i + u_j) - 3\eta)} 
$$

Putting these relation in a symmetric form ($u_i \rightarrow u_i + 2\eta$) one can see that we have the results obtained by ([12, 13]) using the analytical Bethe Ansatz.

### 4.2.2 The non-quantum group invariant cases

These were the cases considered by Guang-Liang Li, et al [10], where a spurious dependence on the spectral parameter could exist. Here we shall see that this dependence of the Bethe equations on the spectral parameter is inexistent.

As the correspondent $K$-matrix solutions are complex conjugated, we will consider here only the case of $(F^+, G^+)$, defined in Section 2.

A way to turn more direct the comparison of our results with those previously known is to borrow the notation presented in [13]: first, the contributions from the boundaries have the form

$$
X_1(u)\Omega_1(u) = \alpha(u) \frac{x^{2N}(u)}{\rho^N(u)}, \quad X_2(u)\Omega_2(u) = \beta(u) \frac{x^{2N}(u)}{\rho^N(u)}, \quad X_3(u)\Omega_3(u) = \gamma(u) \frac{x^{2N}(u)}{\rho^N(u)}
$$

(4.17)

where

$$
\alpha(u) = \frac{(f^+(u))^2}{x_1(2u)} \xi^+(u) \sinh(u - 6\eta) \\
\beta(u) = \alpha(u) \frac{\sinh u}{\sinh(u - 4\eta)} \frac{\cosh \eta - i \sinh(u - 2\eta)}{\cosh \eta + i \sinh(u - 2\eta)} \\
\gamma(u) = \alpha(u) \frac{\xi^-(u)}{\xi^+(u)} \frac{\sinh u}{\sinh(u - 2\eta)} \frac{\sinh(u - 2\eta)}{\sinh(u - 6\eta)}
$$

(4.18)

and

$$
\xi^\pm(u) = 2 \cosh(u - 3\eta) \pm 2i \sinh 2\eta
$$

(4.19)

Second, the boundary factors $\Theta(u_i)$ are given by

$$
\Theta(u_i) = \frac{e^{-u_i} \sinh(u_i - 2\eta) \cosh(\frac{1}{2}(u + u_i) - 3\eta) + i \sinh(\frac{1}{2}(u - u_i) - 2\eta)}{\sinh u_i \cosh(\frac{1}{2}(u - u_i) - \eta) + i \sinh(\frac{1}{2}(u + u_i) - 4\eta)}.
$$

(4.20)

Here we note that (4.20) is the expression obtained in [10] for their $\beta(u, u_i)$ factors. From this result it follows the dependence on $u$ in their Bethe equations. However, analyzing the (4.20) expression carefully,
more precisely, using the identity

$$
\frac{\cosh(\frac{1}{2}(u + u_i) - 3\eta) + i \sinh(\frac{1}{2}(u - u_i) - 2\eta)}{\cosh(\frac{1}{2}(u - u_i) - \eta) + i \sinh(\frac{1}{2}(u + u_i) - 4\eta)} = \frac{\cosh(\eta - i \sinh(u_i - 2\eta)}{\cosh(u_i - 3\eta)}
$$

(4.21)

one we can see that this \( u \) dependency is only apparent. Therefore the correct expressions for \( \Theta(u_i) \) are given by

$$
\Theta(u_i) = e^{-u_i} \frac{\sinh(2u_i - 2\eta)}{\sinh(u_i)} \frac{\cosh(\eta - i \sinh(u_i - 2\eta)}{\cosh(u_i - 3\eta)}, \quad i = 1, \ldots, n.
$$

(4.22)

Next, to recover our notation we will need of the following identity

$$
\frac{\Omega_2(u)}{\Omega_1(u)} = \Theta(u) \frac{\sinh u}{\sinh(u - 4\eta)} \frac{\cosh(u - 3\eta)}{\cosh(u - \eta)}
$$

(4.23)

Therefore the \( n \)-particle state \( \Psi_n(\{u_i\}) \) is an eigenfunction of the \( \text{IK} \) transfer matrix with the boundary contribution \( (F^+, G^+) \), and has eigenvalue

$$
\Lambda_n(u, \{u_i\}) = \alpha(u) \frac{\rho^N(\{u\})}{\rho^N(u)} \prod_{i=1}^n a_{11}(u, u_i) + \beta(u) \frac{\rho^N(\{u\})}{\rho^N(u)} \prod_{i=1}^n a_{21}(u, u_i) + \gamma(u) \frac{\rho^N(\{u\})}{\rho^N(u)} \prod_{i=1}^n a_{31}(u, u_i)
$$

(4.24)

provided that the \( u_i \) parameters satisfy the Bethe equations

$$
\left( \frac{\sinh(\frac{1}{2}u_i - 2\eta)}{\sinh(\frac{1}{2}u_i)} \right)^{2n} = \frac{\beta(u_i) \sinh(u_i - 4\eta)}{\alpha(u)} \frac{\cosh(u_i - \eta)}{\cosh(u_i - 3\eta)} \times \prod_{j=1, j \neq i}^n \frac{\sinh(\frac{1}{2}(u_i + u_j) - 4\eta) \sinh(\frac{1}{2}(u_i - u_j) - 2\eta)}{\sinh(\frac{1}{2}(u_i + u_j) - 3\eta) \sinh(\frac{1}{2}(u_i - u_j) - \eta)}
$$

(4.25)

Writing the above Bethe equations in a symmetric form \((u_i \rightarrow u_i - \eta)\) one can see that is exactly the result obtained in [13] using the analytical BA.

5 Conclusion

The main result of this paper may be summarized by saying that the Sklyanin algebraic Bethe Ansatz is unique and consistent for the ZF and the IK vertex models with boundaries, corroborating all previous investigations in these models. Thus the Bethe vectors here obtained can be used for instance to obtain the eigenvalue problem for the corresponding Gaudin models [27] with boundaries and their associated systems of Knizhnik-Zamolodchikov equations [28]. Also, our results encourage the believe that other 19-vertex models, such as the \( \text{osp}(2|1) \) model and the \( \text{sl}(2|1) \) model in the presence of boundaries can also be solved by the algebraic Bethe Ansatz. In fact, preliminary results [29] indicate that this is indeed true.
It was in the IK model that Tarasov developed the algebraic BA for the three-state models with periodic boundary conditions [17]. Here, with the aid of previous works [16, 10], two of the three-state 19-vertex models have their algebraic Bethe Ansatz derived using a generalization of the Tarasov’s approach. The algebraic BA for $n$-state models with periodic boundary conditions was developed by Martins in [30]. Therefore we believe that the Martins’s approach can be generalized to include the diagonal open boundary conditions.

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A The Commutation Relations

The equation (3.26) gives us the commutation relations for the matrix elements of the double-row monodromy matrix which play a fundamental role in the algebraic Bethe Ansatz. Here we present the commutations relations and their coefficients using a compact notation. Recall that the operator

\[ U(u) = \begin{pmatrix} D_1(u) & B_1(u) & B_2(u) \\ C_1(u) & D_2(u) & B_3(u) \\ C_2(u) & C_3(u) & D_3(u) \end{pmatrix}, \quad (A.1) \]

satisfy the fundamental reflection equation

\[ R_{12}(u-v)U_1(u)R_{21}(u+v)U_2(v) = U_2(v)R_{12}(u+v)U_1(u)R_{21}(u-v). \quad (A.2) \]

Substituting (2.14) and (A.1) into (A.2) we get 81 equations involving products of two matrix elements of \( U(u) \). These equations can be manipulated in order to put the product of pairs of operators in the normal ordered form. To do this we shall proceed in the following way. First we denote by \( E[i, j] = 0 \) the \((i, j)\) component of the matrix equation (A.2) and collect them in blocks \( B[i, j], i = 1, \ldots, 5, j = i, \ldots, 10 - i, \) defined by

\[ B[i, j] = \{ F_{ij} = E[i, j], \quad f_{ij} = E[j, i], \quad FF_{ij} = E[10 - i, 10 - j], \quad ff_{ij} = E[10 - j, 10 - i] \} \quad (A.3) \]

From these blocks we can see that the pair \((F_{ij}, f_{ij})\) as well as \((FF_{ij}, ff_{ij})\) can be solved simultaneously.

We introduce the notation

\[ D_i = D_i(u), \quad d_i = D_i(v), \quad B_i = B_i(u), \quad b_i = B_i(v), \quad C_i = C_i(u), \quad c_i = C_i(v) \quad (A.4) \]

for the operators of the double-row monodromy matrix and

\[ X_i = x_i(u + v), \quad Y_i = y_i(u + v), \quad x_i = x_i(u - v), \quad y_i = y_i(u - v), \quad (A.5) \]

for the Boltzmann weights and

\[ \{ Z \}_{ij} = \{ z \}_{ij}(v, u), \quad \{ z \}_{ij} = \{ z \}_{ij}(u, v), \quad F_i = f_i(u), \quad f_i = f_i(v). \quad (A.6) \]

for the coefficients of the commutations relations, where

\[ \{ Z \} = A, B, C, D, E, X, Y \quad \text{and} \quad \{ z \} = a, b, c, d, e, x, y. \quad (A.7) \]

Taking into account these simplifications, we will indicate the pair \((F_{ij}, f_{ij})\) or \((FF_{ij}, ff_{ij})\) for which the corresponding normal ordered relations were obtained:
\[ (F_{14}, f_{14}) \]

\[
\begin{align*}
D_{1}b_1 &= a_{11}b_1D_1 + a_{12}B_1d_1 + a_{13}B_1d_2 + a_{14}B_2c_1 + a_{15}B_2c_2 + a_{16}b_2C_1 \\
C_{1}d_1 &= A_{11}d_1C_1 + A_{12}D_1c_1 + A_{13}D_1c_2 + A_{14}B_1c_2 + A_{15}B_3c_2 + A_{16}b_1C_2
\end{align*}
\]  

(A.8)

where the coefficients are

\[
\begin{align*}
A_{11} &= \frac{x_1 X_2}{x_2 X_1}, & A_{12} &= -F_1 \frac{X_5}{X_1} - \frac{y_5 X_2}{x_2 X_1}, & A_{13} &= \frac{X_5}{X_1}, \\
A_{14} &= -\frac{y_5 X_6}{x_2 X_1}, & A_{15} &= \frac{X_7}{X_1}, & A_{16} &= \frac{x_1 X_6}{x_2 X_1}.
\end{align*}
\]  

(A.9)

\[ (F_{17}, f_{17}) \]

\[
\begin{align*}
D_{1}b_2 &= b_{11}b_2D_1 + b_{12}B_2d_1 + b_{13}B_2d_2 + b_{14}B_2d_3 + b_{15}B_1b_1 + b_{16}b_1B_3 \\
C_{2}d_1 &= B_{11}d_1C_2 + B_{12}D_1c_2 + B_{13}D_2c_2 + B_{14}D_3c_2 + B_{15}C_1c_1 + B_{16}C_3c_1
\end{align*}
\]  

(A.10)

where

\[
\begin{align*}
B_{11} &= \frac{x_1 X_3}{x_3 X_1}, & B_{12} &= -F_1 \frac{y_6 X_6}{x_3 X_1} - F_2 \frac{X_7}{X_1} - \frac{y_7 X_3}{x_3 X_1}, & B_{13} &= \frac{y_6 X_6}{x_3 X_1} - F_3 \frac{X_7}{X_1}, \\
B_{14} &= -\frac{X_7}{X_1}, & B_{15} &= -\frac{y_6 X_2}{x_3 X_1}, & B_{16} &= -\frac{X_5}{X_1}.
\end{align*}
\]  

(A.11)

\[ (FF_{36}, ff_{36}) \]

\[
\begin{align*}
D_{1}b_3 &= x_{11}b_1D_1 + x_{12}b_1D_1 + x_{13}B_1d_1 + x_{14}B_1d_2 + x_{15}B_1d_3 + x_{16}B_2c_1 \\
&\quad + x_{17}B_2c_3 + x_{18}b_2C_1 \\
C_{3}d_1 &= X_{11}d_1C_3 + X_{12}d_1C_1 + X_{13}D_1c_1 + X_{14}D_2c_1 + X_{15}D_3c_1 + X_{16}B_1c_2 \\
&\quad + X_{17}B_3c_2 + X_{18}b_1C_2
\end{align*}
\]  

(A.12)

with the following coefficients

\[
\begin{align*}
X_{11} &= \frac{x_2 X_3}{x_3 X_2}, & X_{12} &= \frac{y_5 Y_6}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} A_{11}, \\
X_{13} &= -F_1 \frac{y_6 X_4}{x_3 X_2} - F_2 \frac{X_6}{X_2} - \frac{y_7 Y_6}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} A_{12}, \\
X_{14} &= -F_3 \frac{X_6}{X_2} - \frac{y_6 Y_5}{x_3 X_2} A_{13}, \\
X_{15} &= -\frac{X_6}{X_2}, & X_{16} &= -\frac{y_7 Y_5}{x_3 X_2} A_{14}, \\
X_{17} &= -\frac{y_6 X_5}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} A_{15}, & X_{18} &= \frac{y_5}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} A_{16}.
\end{align*}
\]  

(A.13)

Note that for each pair of equations the corresponding commutation relations are related by interchanging

\[
\begin{align*}
u \leftrightarrow v, & \quad D_i \leftrightarrow d_i, & B_i \leftrightarrow c_i, & \quad C_i \leftrightarrow b_i
\end{align*}
\]  

(A.14)
• \((F_{24}, f_{24})\)

\[
C_1 b_1 = c_{11} b_1 C_1 + c_{12} b_1 C_3 + c_{13} B_1 c_3 + c_{14} B_3 c_3 + c_{15} b_2 C_2 + c_{16} d_1 D_1 + c_{17} d_1 D_2
\]
\[
+ c_{18} D_1 d_1 + c_{19} D_1 d_2 + c_{20} D_2 d_1 + c_{21} D_2 d_2
\]
(A.15)

with the following coefficients

\[
c_{11} = \frac{X_1}{x_1}, \quad c_{12} = \frac{x_5}{x_2} X_6, \quad c_{13} = -\frac{y_5}{x_2} X_6, \quad c_{14} = -\frac{X_7}{x_2}, \quad c_{15} = \frac{X_5}{x_1},
\]
\[
c_{16} = \frac{Y_5}{x_2} + F_1 \frac{x_5}{x_2} X_2, \quad c_{17} = \frac{x_5}{x_2} X_2, \quad c_{18} = -f_1 \left(\frac{y_5}{x_2} X_2 + F_1 \frac{X_5}{x_1}\right),
\]
\[
c_{19} = -\left(\frac{y_5}{x_2} X_2 + F_1 \frac{X_5}{x_1}\right), \quad c_{110} = -f_1 \frac{X_5}{x_1}, \quad c_{111} = -\frac{X_5}{x_1}.
\]  
(A.16)

The pairs \((F_{16}, f_{16}), (F_{18}, f_{18}), (FF_{16}, f_{f_{16}}), (FF_{18}, f_{f_{18}})\) and \(F_{19}\) form a closed set of equations from which we have derived the following commutation relations

\[
B_{2b_1} = e_{11} b_1 B_2 + e_{12} b_2 B_1 + e_{13} b_2 B_3,
\]
\[
C_1 c_2 = E_{11} c_2 C_1 + E_{12} c_1 C_2 + E_{13} c_3 C_2
\]
\[
B_{3b_2} = e_{41} b_2 B_3 + e_{42} b_1 B_2 + e_{43} b_2 B_2
\]
\[
C_2 c_3 = E_{41} c_3 C_2 + E_{42} c_2 C_1 + E_{43} c_2 C_3
\]
\[
B_{1b_2} = e_{21} b_2 B_1 + e_{22} b_2 B_3 + e_{23} b_1 B_2 + e_{24} b_3 B_2
\]
\[
C_2 c_1 = E_{21} c_1 C_2 + E_{22} c_2 C_2 + E_{23} c_2 C_1 + E_{24} c_2 C_3
\]
\[
B_{2b_3} = e_{31} b_3 B_2 + e_{32} b_1 B_2 + e_{33} b_3 B_1 + e_{34} b_2 B_3
\]
\[
C_3 c_2 = E_{31} c_2 C_3 + E_{32} c_2 C_1 + E_{33} c_1 C_2 + E_{34} c_3 C_2
\]
\[
B_{2b_2} = b_2 B_2, \quad C_2 c_2 = c_2 C_2
\]  
(A.17)

where

\[
e_{11} = \frac{x_2}{x_1} X_2, \quad e_{12} = \frac{y_5}{x_1} X_6, \quad e_{13} = \frac{x_2}{x_1} X_6
\]  
(A.18)
\[
E_{41} = \frac{x_2}{x_1} X_2, \quad E_{42} = \frac{x_2}{x_1} Y_6, \quad E_{43} = \frac{x_5}{x_1}
\]  
(A.19)
\[
e_{21} = \frac{X_5 X_1}{x_1 x_2 - X_6 Y_6}, \quad e_{22} = \frac{x_5 (x_2^2 + x_3^2 - x_5 y_5)}{x_1 (x_2^2 - x_5 y_5)} \frac{X_2 X_6}{X_2 - X_6 Y_6}
\]
\[
e_{23} = \frac{x_1 x_5}{x_2 - x_5 y_5} \frac{X_6 Y_6}{X_2 - X_6 Y_6} + \frac{x_5}{x_1} \frac{X_2^2}{X_2^2 - X_6 Y_6}, \quad e_{24} = -\frac{x_1 x_2}{x_2^2 - x_5 y_5} \frac{X_3 X_6}{X_3 X_6 - X_6 Y_6}
\]  
(A.20)
\[
e_{31} = \frac{x_1 x_2}{x_2^2 - x_5 y_5} \frac{X_3 X_2}{X_2^2 - X_6 Y_6}, \quad e_{32} = -\frac{x_5 (x_2^2 + x_3^2 - x_5 y_5)}{x_1 (x_2^2 - x_5 y_5)} \frac{X_2 Y_6}{X_2^2 - X_6 Y_6}
\]
\[
e_{33} = \frac{x_2}{x_1} \frac{X_3 Y_6}{X_2^2 - X_6 Y_6}, \quad e_{34} = \frac{x_5}{x_1} \frac{X_6 Y_6}{X_2^2 - X_6 Y_6} - \frac{x_1 x_5}{x_2^2 - x_5 y_5} \frac{X_3 Y_6}{X_3 Y_6 - X_6 Y_6}
\]  
(A.21)
\( D_2 b_1 = a_{21} b_1 D_2 + a_{22} B_1 d_1 + a_{23} B_1 b_2 + a_{24} B_3 d_1 + a_{25} B_3 d_2 + a_{26} B_2 c_1 + a_{27} B_2 c_1 + a_{28} B_1 C_3 + a_{29} B_2 c_3 \)

\( C_1 d_2 = A_{21} d_1 C_1 + A_{22} D_1 c_1 + A_{23} D_2 c_1 + A_{24} D_1 c_1 + A_{25} D_2 c_3 + A_{26} B_1 c_2 + A_{27} B_3 c_2 + A_{28} b_1 C_2 + A_{29} b_3 C_2 \)  \( \text{(A.22)} \)

with

\[
\begin{align*}
    a_{21} &= \frac{x_4 X_4}{x_2 X_2} + \frac{x_6 X_6}{x_2 X_2} + \frac{x_4 Y_5}{x_2 X_2} A_{13} + \frac{x_6 X_{14}}{x_2} \\
    a_{22} &= -\frac{y_5 Y_4}{x_2 X_2} - (F_1 + \frac{y_5 Y_5}{x_2 X_2}) a_{12} + \frac{x_4 Y_5}{x_2 X_2} A_{11} + \frac{x_6 X_{12}}{x_2} \\
    a_{23} &= -\frac{y_5 Y_4}{x_2 X_2} - (F_1 + \frac{y_5 Y_5}{x_2 X_2}) a_{13}, \quad a_{24} = -\frac{y_6 X_6}{x_2 X_2} + \frac{x_6 X_{11}}{x_2}, \quad a_{25} = \frac{X_6 X_2}{X_2} \\
    a_{26} &= -(F_1 + \frac{y_5 Y_5}{x_2 X_2}) a_{14} + \frac{x_4 Y_5}{x_2 X_2} A_{16} + \frac{x_6 X_{18}}{x_2} \\
    a_{27} &= -(F_1 + \frac{y_5 Y_5}{x_2 X_2}) a_{15}, \quad a_{28} = \frac{y_6 X_6}{x_2 X_2} - (F_1 + \frac{y_5 Y_5}{x_2 X_2}) a_{16} + \frac{x_4 Y_5}{x_2 X_2} A_{14} + \frac{x_6 X_{16}}{x_2} \\
    a_{29} &= \frac{x_4 X_5}{x_2 X_6} + \frac{x_4 Y_5}{x_2 X_6} A_{15} + \frac{x_6 X_{17}}{x_2} \quad \text{(A.23)}
\end{align*}
\]

\( B_1 b_1 = e_{01} b_1 B_1 + e_{02} b_2 D_2 + e_{03} b_3 D_1 + e_{04} B_2 d_1 + e_{05} B_2 d_2 \)

\( C_1 c_1 = E_{01} c_1 C_1 + E_{02} d_2 C_2 + E_{03} d_1 C_2 + E_{04} D_1 c_2 + E_{05} D_2 c_2 \)  \( \text{(A.24)} \)

with

\[
\begin{align*}
    e_{01} &= \frac{x_3 x_4 - x_6 y_6}{x_1 x_3}, \quad e_{02} = \frac{x_3 x_4 - x_6 y_6 X_6}{x_1 x_3 X_2} \\
    e_{03} &= \frac{x_3 x_4 - x_6 y_6 X_3}{x_1 x_3 X_2} + F_1 \frac{x_3 x_4 - x_6 y_6}{x_1 x_3} \frac{X_6}{X_2} \\
    e_{04} &= \frac{x_6 X_3}{x_3 X_2} - f_1 \frac{x_6 X_6}{X_2}, \quad e_{05} = \frac{X_6}{X_2} \quad \text{(A.25)}
\end{align*}
\]

These commutation relations play a special role in the construction of the \( n \)-particle states, as mentioned above.

\( B_1 b_3 = d_{11} b_3 B_1 + d_{12} b_1 B_1 + d_{13} b_2 D_1 + d_{14} b_2 D_2 + b_1 b_2 d_1 + d_{16} B_2 d_2 + d_{17} B_2 d_3 \)

\( C_3 c_1 = D_{11} c_1 C_3 + D_{12} c_1 C_1 + D_{13} d_1 C_2 + D_{14} d_2 C_2 + D_{15} D_1 c_2 + D_{16} D_2 c_2 + D_{17} D_3 c_2 \)  \( \text{(A.26)} \)
where

\[
\begin{align*}
D_{11} &= \frac{X_3}{X_4 + Y_5 B_{16}}, & D_{12} &= \frac{(y_5 Y_6 + x_2 Y_5 B_{16}) E_{01} - y_5 Y_6}{x_2 (X_4 + Y_5 B_{16})}, \\
D_{13} &= -\frac{(y_5 Y_6 + x_2 Y_5 B_{16}) E_{01} - f_1 y_5 X_2 + x_2 Y_5 B_{11}}{x_2 (X_4 + Y_5 B_{16})}, \\
D_{14} &= -\frac{(y_5 Y_5 + x_2 Y_5 B_{16}) E_{02} - y_5 Y_2}{x_2 (X_4 + Y_5 B_{16})}, \\
D_{15} &= -\frac{(y_5 Y_6 + x_2 Y_5 B_{16}) E_{04} + F_1 y_5 X_2 + F_2 x_2 X_5 + x_2 Y_5 B_{12}}{x_2 (X_4 + Y_5 B_{16})}, \\
D_{16} &= -\frac{(y_5 Y_6 + x_2 Y_5 B_{16}) E_{05} + y_5 X_2 + F_3 x_2 X_5 + x_2 Y_5 B_{13}}{x_2 (X_4 + Y_5 B_{16})}, \\
D_{17} &= -\frac{X_5 + Y_5 B_{14}}{X_4 + Y_5 B_{16}}
\end{align*}
\]

and

\[
\begin{align*}
B_3 b_1 &= d_{21} b_1 B_3 + d_{22} b_1 B_1 + d_{23} b_2 D_1 + d_{24} b_2 D_2 + d_{25} b_2 D_3 + d_{26} B_2 d_1 + d_{27} B_2 d_2, \\
C_1 c_3 &= D_{21} c_3 C_1 + D_{22} c_1 C_1 + D_{23} d_1 C_2 + D_{24} d_2 C_2 + D_{25} d_3 C_2 + D_{26} D_1 c_2 + D_{27} D_2 c_2
\end{align*}
\]

where

\[
\begin{align*}
d_{21} &= \frac{X_4}{X_3} + \frac{Y_5}{X_3} B_{16}, & d_{22} &= \frac{y_5 Y_6}{x_2 X_3} + \frac{y_5 Y_6}{x_2 X_3} e_{01} + \frac{Y_5}{X_3} B_{15}, \\
d_{23} &= F_1 \frac{y_5 X_2}{x_2 X_3} + F_2 \frac{X_5}{X_3} - \frac{y_5 Y_6}{x_2 X_3} e_{03} + \frac{Y_5}{X_3} B_{12}, \\
d_{24} &= \frac{y_5 X_2}{x_2 X_3} + F_3 \frac{X_5}{x_2 X_3} - \frac{y_5 Y_6}{x_2 X_3} e_{02} + \frac{Y_5}{X_3} B_{13}, & d_{25} &= \frac{X_5}{X_3} + \frac{Y_5}{X_3} B_{14}, \\
d_{26} &= f_1 \frac{y_5 X_2}{x_2 X_3} - \frac{y_5 Y_6}{x_2 X_3} e_{04} + \frac{Y_5}{X_3} B_{11}, \\
d_{27} &= -\frac{y_5 X_2}{x_2 X_3} - \frac{y_5 Y_6}{x_2 X_3} e_{05}
\end{align*}
\]

\[\text{• (} F_{28}, f_{28} \text{)} \]

\[
\begin{align*}
D_{2b_2} &= b_{21} b_2 D_2 + b_{22} B_2 d_1 + b_{23} B_2 d_2 + b_{24} B_2 d_3 + b_{25} B_1 b_1 + b_{26} B_1 b_3 \\
&\quad + b_{27} B_3 b_1 + b_{28} B_3 b_3, \\
C_{2c_2} &= B_{21} d_2 C_2 + B_{22} D_1 c_2 + B_{23} D_2 c_2 + B_{24} D_3 c_2 + B_{25} C_1 c_1 + B_{26} C_3 c_1 \\
&\quad + B_{27} C_1 c_3 + B_{28} C_3 c_3
\end{align*}
\]
where

\[ B_{21} = 1 + \frac{x_5 X_3}{x_2 X_2} d_{27} + \frac{Y_6 Y_5}{X_2 X_2} c_{05}, \quad B_{22} = -f_1 B_{12} + \frac{x_5 X_3}{x_2 X_2} d_{23} + \frac{Y_6 Y_5}{X_2 X_2} c_{03} \]

\[ B_{23} = -f_1 B_{13} + \frac{x_5 X_3}{x_2 X_2} d_{24} + \frac{Y_6 Y_5}{X_2 X_2} c_{02}, \quad B_{24} = -f_1 B_{14} + \frac{x_5 X_3}{x_2 X_2} d_{25} \]

\[ B_{25} = -f_1 B_{15} + \frac{x_5 X_3}{x_2 X_2} d_{22} + \frac{Y_6 Y_5}{X_2 X_2} c_{01}, \quad B_{26} = -f_1 B_{16} + \frac{x_5 X_3}{x_2 X_2} d_{21} \]

\[ B_{27} = -\frac{y_5 X_3}{x_2 X_2}, \quad B_{28} = \frac{X_6}{X_2} \]

(A.31)

\[ C_3 b_1 = c_{21} b_1 C_1 + c_{22} b_1 C_3 + c_{23} B_1 c_3 + c_{24} B_3 c_3 + c_{25} b_2 C_2 + c_{26} d_1 D_1 + c_{27} d_1 D_2 + c_{28} d_1 D_3 + c_{29} D_1 d_1 \]

\[ + c_{30} D_1 d_2 + c_{31} D_1 d_3 + c_{32} D_2 d_1 + c_{33} D_2 d_2 + c_{34} D_2 d_3 \]

(A.32)

where

\[ c_{21} = -\frac{y_6 Y_5}{x_3 X_2} (c_{11} - 1), \quad c_{22} = \frac{x_5}{x_3} - \frac{y_6 Y_5}{x_3 X_2} c_{12}, \quad c_{23} = -\frac{y_7 Y_5}{x_3} \frac{y_6 Y_5}{x_3 X_2} c_{13} \]

\[ c_{24} = -\frac{d_6 Y_5}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} c_{14}, \quad c_{25} = \frac{y_6 X_1}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} c_{15} \]

\[ c_{26} = F_{14} X_6 + \frac{x_6 X_3}{x_3 X_2} + \frac{y_6 Y_7}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} c_{16} \]

\[ c_{27} = F_{14} X_6 + \frac{x_6 X_3}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} c_{17}, \quad c_{28} = \frac{x_6 X_3}{x_3 X_2} \]

\[ c_{29} = -f_1 \frac{y_6}{x_3 X_2} - f_1 F_1 X_4 + \frac{y_6 X_4}{x_3 X_2} - f_2 \frac{y_6}{x_3 X_2} c_{18} \]

\[ c_{30} = -f_1 \frac{y_6}{x_3 X_2} - f_1 F_1 X_4 - \frac{y_6 X_4}{x_3 X_2} - \frac{y_6 Y_5}{x_3 X_2} c_{19}, \]

\[ c_{31} = -f_1 \frac{y_6 X_4}{x_3 X_2} - f_1 F_1 X_4 - \frac{y_6 Y_5}{x_3 X_2} c_{110} \]

\[ c_{32} = -\frac{y_6 X_4}{x_3 X_2} - F_{14} X_6 + \frac{y_6 Y_5}{x_3 X_2} c_{111}, \quad c_{213} = -f_1 \frac{X_6}{X_2}, \quad c_{214} = -\frac{X_6}{X_2} \]  

(A.33)

\[ D_3 b_1 = a_{31} b_1 D_3 + a_{32} B_1 d_1 + a_{33} B_1 d_2 + a_{34} B_3 d_1 + a_{35} B_3 d_2 + a_{36} B_2 c_1 + a_{37} B_2 c_3 + a_{28} B_2 C_1 + a_{29} b_2 C_3 \]

\[ C_1 d_3 = A_{31} d_3 C_1 + A_{32} D_1 c_1 + A_{33} D_1 c_3 + A_{34} D_1 c_3 + A_{35} D_2 c_3 + A_{36} B_1 c_2 + A_{37} B_2 C_2 + A_{28} d_1 C_2 + A_{29} b_3 C_2 \]  

(A.34)
where

\begin{align*}
a_{31} &= \frac{x_2 X_2}{x_3 X_3} + \frac{x_2 Y_6}{x_3 X_3}X_{15} \\
a_{32} &= -\frac{f_1 y_7 Y_5}{x_3 X_3} + \frac{y_5 Y_7}{x_3 X_3}A_{11} - (Q_1)a_{22} - (Q_2)a_{12} + \frac{x_2 Y_6}{x_3 X_3}X_{12} \\
a_{33} &= -\frac{y_f Y_5}{x_3 X_3} - (Q_2)a_{13} - (Q_1)a_{28} \\
a_{34} &= -\frac{f_1 y_6 X_2}{x_3 X_3} - (Q_1)a_{24} + \frac{x_2 Y_6}{x_3 X_3}X_{11}, \quad a_{35} = -\frac{y_6 X_2}{x_3 X_3} - (Q_1)a_{25} \\
a_{36} &= \frac{y_5 Y_7}{x_3 X_3}A_{16} - (Q_1)a_{26} - (Q_2)a_{14} + \frac{x_2 Y_6}{x_3 X_3}X_{18} \\
a_{37} &= -\frac{y_f X_1}{x_3 X_3} - (Q_2)a_{15} - (Q_1)a_{27} \\
a_{38} &= \frac{y_6 X_2}{x_3 X_3}A_{14} - (Q_2)a_{16} - (Q_1)a_{28} + \frac{x_2 Y_6}{x_3 X_3}X_{16} \\
a_{39} &= \frac{y_5 X_1}{x_3 X_3} + \frac{y_6 Y_7}{x_3 X_3}A_{15} - (Q_1)a_{29} + \frac{x_2 Y_6}{x_3 X_3}X_{17} \\
\end{align*}

(A.35)

\begin{align*}
D_3 b_2 &= b_{31}B_2d_3 + b_{32}B_2d_1 + b_{33}B_2d_2 + b_{34}B_2d_3 + b_{35}B_1b_1 + b_{36}B_1b_3 \\
&\quad + b_{37}B_3b_1 + b_{38}B_3b_3 \\
C_2 d_3 &= B_{31}D_3c_2 + B_{32}D_1c_2 + B_{33}D_2c_2 + B_{34}d_3c_2 + B_{35}C_1c_1 + B_{36}C_3c_1 \\
&\quad + B_{37}C_1c_3 + B_{38}C_3c_3
\end{align*}

(A.36)

where

\begin{align*}
b_{31} &= -\frac{y_f X_1}{x_3 X_3} - (Q_2)b_{14} - (Q_1)b_{24} \\
b_{32} &= -\frac{f_2 y_7 X_1}{x_3 X_3} + \frac{x_1 Y_5}{x_3 X_3}D_{13} + \frac{x_1 Y_7}{x_3 X_3}(B_{11} + D_{13}B_{16} + E_{03}B_{15}) \\
&\quad - (Q_2)b_{12} - (Q_1)b_{22} \\
b_{33} &= -\frac{f_3 y_7 X_1}{x_3 X_3} + \frac{x_1 Y_5}{x_3 X_3}D_{14} + \frac{x_1 Y_7}{x_3 X_3}(D_{14}B_{16} + E_{02}B_{15}) - (Q_2)b_{13} - (Q_1)b_{23} \\
b_{34} &= \frac{x_1 X_1}{x_3 X_3} + \frac{x_1 Y_7}{x_3 X_3}(B_{14} + D_{17}B_{16}) + \frac{x_1 Y_5}{x_3 X_3}D_{17} \\
b_{35} &= \frac{x_1 Y_5}{x_3 X_3}D_{12} + \frac{x_1 Y_7}{x_3 X_3}(D_{12}B_{16} + E_{01}B_{15}) - (Q_2)b_{15} - (Q_1)b_{25} \\
b_{36} &= -\frac{y_f Y_5}{x_3 X_3} - (Q_2)b_{16} - (Q_1)b_{26} \\
b_{37} &= \frac{x_1 X_1}{x_3 X_3} + \frac{x_1 Y_7}{x_3 X_3}D_{11} + \frac{x_1 Y_5}{x_3 X_3}D_{11}B_{16} - (Q_1)b_{27} \\
b_{28} &= -\frac{y_6 X_2}{x_3 X_3} - (Q_1)b_{28}
\end{align*}

(A.37)
Here we have used the notation

\[ Q_1 = F_3 + \frac{y_6}{x_3} Y_6 \quad Q_2 = F_2 + F_1 \frac{y_6}{x_3} Y_6 + \frac{y_7}{x_3} Y_7 \]  \tag{A.38}

The remaining commutation relations do not participate effectively for the algebraic BA. Nevertheless, we will write them below, but without their coefficients which are very cumbersome.

- \((FF_{25}, ff_{25})\) and \((F_{27}, f_{27})\)

\[
D_2 b_3 = X_{21} b_3 D_2 + X_{22} b_1 D_2 + X_{23} B_1 d_1 + X_{24} B_1 d_2 + X_{25} B_1 d_3 \\
+X_{26} B_3 d_1 + X_{27} B_3 d_2 + X_{28} B_3 d_3 + X_{29} B_2 c_1 + X_{210} B_2 c_3 \\
+X_{211} b_2 C_1 + X_{212} b_2 C_3
\]

\[
C_3 d_2 = x_{21} d_2 C_3 + x_{22} d_2 C_1 + x_{23} D_1 c_1 + x_{24} D_2 c_1 + x_{25} D_3 c_1 \\
+x_{26} D_1 c_3 + x_{27} D_2 c_3 + x_{28} D_3 c_3 + x_{29} B_1 c_2 + x_{210} B_1 c_2 \\
+x_{211} b_1 C_2 + x_{212} b_3 C_2 \]  \tag{A.39}

and

\[
C_1 b_2 = Y_{11} b_2 C_1 + Y_{12} b_2 C_3 + Y_{13} B_2 c_1 + Y_{14} B_2 c_3 + Y_{15} b_1 D_1 \\
+Y_{16} b_1 D_2 + Y_{17} b_3 D_1 + Y_{18} b_3 D_2 + Y_{19} B_1 d_1 + Y_{110} B_1 d_2 \\
+Y_{111} B_1 d_3 + Y_{112} B_3 d_1 + Y_{113} B_3 d_2 + Y_{114} B_3 d_3
\]

\[
C_2 b_1 = y_{11} b_1 C_2 + y_{12} b_3 C_2 + y_{13} B_1 c_2 + y_{14} B_3 c_2 + y_{15} d_1 C_1 \\
+y_{16} d_2 C_1 + y_{17} d_1 C_3 + y_{18} d_2 C_3 + y_{19} D_1 c_1 + y_{110} D_2 c_1 \\
+y_{111} D_3 c_1 + y_{112} D_1 c_3 + y_{113} D_2 c_3 + y_{114} D_3 c_3 \]  \tag{A.40}

- \((FF_{24}, ff_{24})\) and \((F_{37}, f_{37})\)

\[
C_2 b_2 = c_{31} D_2 d_3 + c_{32} B_2 c_2 + c_{33} D_3 d_3 + c_{34} D_2 d_2 + c_{35} B_3 c_1 \\
+c_{36} D_2 d_3 + c_{37} D_1 d_2 + c_{38} d_1 D_3 + c_{39} d_1 D_2 + c_{310} b_3 C_1 \\
+c_{311} B_1 c_1 + c_{312} D_3 d_1 + c_{313} d_3 D_1 + c_{314} d_3 D_2 + c_{315} D_3 d_2 \\
+c_{316} D_1 d_3 + c_{317} b_2 C_2 + c_{318} d_1 D_1 + c_{319} D_1 d_1 + c_{320} D_2 d_1 \\
+c_{321} b_3 C_3 + c_{322} b_1 C_1 + c_{323} d_2 D_1 + c_{324} b_1 C_3 + c_{325} B_3 c_3 \]  \tag{A.41}
and

\[
C_3b_3 = c_{41}d_1D_2 + c_{42}D_1d_2 + c_{43}D_2d_1 + c_{44}B_1c_1 + c_{45}d_2D_3 + c_{46}b_3C_1 + c_{47}b_2C_2 + c_{48}D_3d_1 + c_{49}d_2D_1 + c_{410}D_3d_2 + c_{411}b_1C_3 + c_{412}d_2D_2 + c_{413}b_1C_1 + c_{414}D_1d_3 + c_{415}d_1D_3 + c_{416}D_2d_2 + c_{417}d_1D_1 + c_{418}D_1d_1 + c_{419}B_2c_2 + c_{420}d_3D_1 + c_{421}B_3c_3 + c_{422}b_3C_3 + c_{423}B_3c_1 + c_{424}D_3d_3 + c_{425}D_2d_3 \quad (A.42)
\]

• (\textit{FF}_{12}, ff_{12}) and (\textit{FF}_{45}, ff_{45})

\[
D_3b_3 = X_{31}b_3D_3 + X_{32}b_1D_3 + X_{33}B_1d_1 + X_{34}B_1d_2 + X_{35}B_1d_3 + X_{36}B_3d_1 + X_{37}B_3d_2 + X_{38}B_3d_3 + X_{39}B_2c_1 + X_{310}B_2c_3 + X_{311}b_2C_1 + X_{312}b_2C_3
\]

\[
C_3d_3 = x_{31}d_3C_3 + x_{32}d_3C_1 + x_{33}D_1c_1 + x_{34}D_2c_1 + x_{35}D_3c_1 + x_{36}D_1c_3 + x_{37}D_2c_3 + x_{38}D_3c_3 + x_{39}B_1c_2 + x_{310}B_3c_2 + x_{311}b_1C_2 + x_{312}b_3C_2 \quad (A.43)
\]

and

\[
C_3b_2 = Y_{21}b_2C_3 + Y_{22}b_2C_1 + Y_{23}B_2c_1 + Y_{24}B_2c_3 + Y_{25}b_1D_1 + Y_{26}b_1D_2 + Y_{27}b_1D_3 + Y_{28}b_3D_1 + Y_{29}b_3D_2 + Y_{210}b_3D_3 + Y_{211}B_1d_1 + Y_{212}B_1d_2 + Y_{213}B_1d_3 + Y_{214}B_3d_1 + Y_{215}B_3d_2 + Y_{216}B_3d_3
\]

\[
C_2b_3 = y_{21}b_3C_2 + y_{22}b_1C_2 + y_{23}B_1c_2 + y_{24}B_3c_2 + y_{25}d_1C_1 + y_{26}d_2C_1 + y_{27}d_3C_1 + y_{28}d_1C_3 + y_{29}d_2C_3 + y_{210}d_3C_3 + y_{211}D_1c_1 + y_{212}D_2c_1 + y_{213}D_3c_1 + y_{214}D_1c_3 + y_{215}D_2c_3 + y_{216}D_3c_3 \quad (A.44)
\]