Distributed Set-Based Observers Using Diffusion Strategy

Amr Alanwar, Jagat Jyoti Rath, Hazem Said, Karl Henrik Johansson, and Matthias Althoff

Abstract—Distributed estimation is more robust against single points of failure and requires less communication overhead compared to the centralized version. Among distributed estimation techniques, set-based estimation has gained much attention as it provides estimation guarantees for safety-critical applications and copes with unknown but bounded uncertainties. We propose two distributed set-based observers using interval-based and set-membership approaches for a linear discrete-time dynamical system with bounded modeling and measurement uncertainties. Both algorithms utilize a new over-approximating zonotopes intersection step named the set-based diffusion step. We use the term diffusion since our intersection of zonotopes formula resembles the traditional diffusion step in the stochastic Kalman filter. Our new zonotopes intersection takes linear time. Our set-based diffusion step decreases the estimation errors and the size of estimated sets and can be seen as a lightweight approach to achieve partial consensus between the distributed estimated sets. Every node shares its measurement with its neighbor in the measurement update step. The neighbors intersect their estimated sets constituting our proposed set-based diffusion step. We represent sets as zonotopes since they compactly represent high-dimensional sets, and they are closed under linear mapping and Minkowski addition. The applicability of our algorithms is demonstrated by a localization example. All used data and code to recreate our findings are publicly available.

Index Terms—set-membership estimation, interval-based estimation, zonotope, zonotopes intersection, distributed estimation.

I. INTRODUCTION

State estimation algorithms either compute a single state, a probability distribution of the state, or bound the set of possible states by sets. In stochastic approaches, measurement and process noises are modeled by provided statistical distributions (e.g., Gaussian). On the other hand, set-based approaches assume noises to be unknown but bounded by known bounds. Safety-critical applications require guarantees on the state estimation during operation—such guarantees can be provided by the set-based approaches. Also, set-based approaches are traditionally used in fault detection by generating an adaptive threshold to check the consistency of the measurements with the estimated output set. Among the family of set-based approaches, interval-based and set-membership observers have been introduced separately. We focus in the subsequent literature survey on both approaches.

Interval-based observers: These observers obtain possible sets of states by combining the model and the measurements through the observer gain in order to bound state estimates by upper and lower bounds, which are obtained for instance from differential equations as in [9], [10]. Related work in [11] designs an exponentially stable interval observer for a two-dimensional time-invariant linear system. The aforementioned work is extended for arbitrary finite dimension in [12]. The previous works on linear systems have been extended to nonlinear systems in [13], [14]. Another observer was proposed based on Muller’s theorem for nonlinear uncertain systems in [15]. Also, authors in [16] introduces $H_\infty$ design into interval estimation. Interval observers for uncertain biological systems are proposed in [17]. By merging optimal and robust observer gain designs, a zonotopic Kalman Filter is proposed in [18].

Set-membership-based observers: Unlike interval observers, which are based on observer gain derivation, set-membership-based observers intersect the set of states consistent with the model and the set consistent with the measurements to obtain the corrected state set. One early example of set-membership-based observers is a recursive algorithm bounding the state by ellipsoids. Another early example based on normalized least-mean-squares (NLMS) is presented in [20]. A set-membership state estimation algorithm based on DC-programming is proposed in [21]. Authors in [22] considers linear time-varying descriptor systems for set-membership estimation. Set-membership observers for nonlinear models are investigated in [23]–[27]. They are also used in applications such as underwater robotics [28], a leader following consensus problem in networked multi-agent systems [29] and localization [30]. Authors in [31] consider a class of discrete time-varying system with an event-based communication mechanism over sensor networks. Interconnected multi-rate system is considered in [32]. Set-membership with affine-projection is considered in [33]. Finally, nonlinear kernel adaptive filtering is proposed in [34].

Different set representations have been used in set-based estimation, e.g., ellipsoids, orthotopes, and polytopes [35], [36]. Zonotopes [37] are a special class of polytopes for which one can efficiently compute linear maps and Minkowski sums—both are important operations for set-membership-based observers. A state bounding observer based on zonotopes is introduced in [38]. Set-membership for discrete-time

https://github.com/aalanwar/Distributed-Set-Based-Observers-Using-Diffusion-Strategy
piecewise affine systems using zonotopes is studied in \cite{[42]}. Another work considers discrete-time descriptor systems in \cite{[43]}. Set-based estimation of uncertain discrete-time systems using zonotopes is proposed also in \cite{[44]}. 

**Contributions:** We propose distributed set-based estimators, where a set of nodes is required to collectively estimate the set of possible states of a linear dynamical system in a distributed fashion. In traditional distributed set-based estimation, every node in a sensor-network receives the estimates based on its measurements only; then, the node intersects its set with the estimated sets of its neighbors \cite{[45]–[47]}. However, we propose to let every node shares its measurements with its neighbor for faster convergence. We also supplement our newly proposed observers with a set-based diffusion step, which intersects the shared state sets. Unlike prior efforts, we propose a new zonotopes intersection technique in the diffusion step, which reduces the over-approximation of the intersection results. We use the term diffusion since our intersection formula resembles the traditional diffusion step in stochastic Kalman filter. In set-based estimation, the center of the set is considered as a single point estimate. We show that our diffusion step enhances the single point estimate and decreases the volume of the estimated sets.

One main problem in distributed set-based estimation is the misalignment between the estimated sets by the distributed nodes, which would result in disagreements on fault detection results between nodes. This problem is usually solved by consensus methods \cite{[48]–[50]}. However, traditional consensus methods require the sensor network to perform several iterations before arriving at a consensus, which causes great overhead in set-based estimation. Our set-based diffusion step can be seen as lightweight approach to achieve partial consensus. One only obtains a partial consensus using our algorithms because every node has different neighbors with different measurements; thus, the resulting sets do not fully agree.

More specifically, we make the following contributions:

- We propose two distributed set-membership and interval-based algorithms combined with a new technique for intersection of zonotopes which is exploited in the proposed set-based diffusion step.
- We provide closed-forms for our parameter-finding optimization problems to realize faster execution times.

The rest of the paper is organized as follows: System model and preliminaries are in Section \[II\]. In Section \[III\] we present the distributed set-membership diffusion observer as our first algorithm. Our second solution is the distributed interval-based diffusion observer which is introduced in Section \[IV\]. Both algorithms are evaluated in Section \[V\]. Finally, we conclude the paper in Section \[VI\].

II. PROBLEM STATEMENT AND PRELIMINARIES

We start by stating some preliminaries before describing our proposed solution.

**Definition 1. (Zonotope)** A zonotope $\mathcal{Z} = \langle c, G \rangle$ consists of a center $c \in \mathbb{R}^n$ and a generator matrix $G \in \mathbb{R}^{n \times e}$. We compose $G$ of $e$ generators $g^{(i)} \in \mathbb{R}^n$, $i = 1, \ldots, e$, where $G = [g^{(1)}, \ldots, g^{(e)}]$ \cite{[51]}. 

$$Z = \left\{c + \sum_{i=1}^{e} \beta_i g^{(i)} \mid -1 \leq \beta_i \leq 1 \right\}.$$ (1)

Given two zonotopes $Z_1 = \langle c_1, G_1 \rangle$ and $Z_2 = \langle c_2, G_2 \rangle$, the following operations can be computed exactly \cite{[51]}:

1) Minkowski sum:

$$Z_1 \oplus Z_2 = \langle c_1 + c_2, [G_1, G_2] \rangle.$$ (2)

2) Linear map:

$$LZ_1 = \langle Lc_1, LG_1 \rangle.$$ (3)

Let $C \in \mathbb{R}^{n \times p}$, then $\|C\|_F = \sqrt{tr(C^T C)}$ is the Frobenius norm of $C$. The Frobenius norm of a vector $x \in \mathbb{R}^n$ equals the Euclidean norm of vector defined as $\|x\| = \sqrt{x^T x}$. The F-radius of the zonotope $Z = \langle c, G \rangle$ is the Frobenius norm of the generator matrix. We denote the reduction operator by $\downarrow_g G$ of a generator matrix $G$. It basically reduces the number of generators of a zonotope to a fixed number $g$ so that the resulting zonotope is an over-approximation \cite{[52]}. Finally, for a scalar $c$ and matrices $A$ and $B$, we define the following trace properties \cite{[53]} p.11, where $\nabla_X f(X)$ is the derivative of $f(X)$ with respect to $X$:

$$tr(cA) = c \text{ tr}(A),$$ (4)

$$tr(A + B) = \text{ tr}(A) + \text{ tr}(B),$$ (5)

$$\nabla_X \text{ tr}(AXBX^T C) = A^T C^T XB^T + CAXB,$$ (6)

$$\nabla_X \text{ tr}(B^T X^T CXB) = C^T XB^T + CXBB^T.$$ (7)

We aim to estimate the set of possible plant states in a distributed fashion by observing a basic set of physical signals through sensory devices. Consider a set of $N$ nodes indexed by $k \in \{0, \ldots, N - 1\}$ distributed geographically over some region. We denote the neighborhood of a given node $i$ by the set $\mathcal{N}_i$ containing $m_i$ nodes connected to node $i$ including the node itself. Every node is interested in estimating the set of possible states of the network state. We assume that the noise is unknown but bounded by known bound and the initial set is known. We also consider observable systems only. We consider a discrete-time, linear system model:

$$x_{k+1} = Fx_k + n_k,$$

$$y^i_k = H^i x_k + v^i_k,$$ (8)

where $x_k \in \mathbb{R}^n$ is the state at time step $k$ and $y^i_k \in \mathbb{R}^p$ is the measurement observed at node $i$. State and measurement matrices are denoted by $F$ and $H^i$, respectively. The modeling and measurement noises are denoted by $n_k$ and $v^i_k$, respectively, and are assumed to be unknown but bounded by zonotopes: $n_k \in Z_{Q, k} = (0, Q_k)$, and $v^i_k \in Z_{R, i}$. If the noises are not centered around zero, the results will be shifted to the new center. All vectors and matrices are real-valued and have proper dimensions.

III. DISTRIBUTED SET-MEMBERSHIP DIFFUSION OBSERVER

As mentioned in the introduction, there are two types of set-based observers: set-membership observers and interval-based
observers. We propose two algorithms extending related work of both observers and add the set-based diffusion step to both observers. We first show our contribution to set-membership observers. We denote the state estimated at node $i$ of the set-membership approach by $\hat{x}_{s,k}^i$ for time step $i$. The set of possible states in set-membership approaches are generally obtained from predicted, measurement, and corrected state sets, which are defined as follows:

**Definition 2. (Predicted State Set)** Given system (8) with initial set $Z_{s,0} = \langle c_{s,0}, G_{s,0} \rangle$, the predicted reachable set of states $\hat{Z}_{s,k}^i$ considering the zonotope $Z_{Q,k}$ which bounds modeling noise is defined as (9):

$$\hat{Z}_{s,k}^i = F_{s,k} \hat{Z}_{s,k-1} \oplus Z_{Q,k}. \quad (9)$$

**Definition 3. (Measurement State Set)** Given system (8), the measurement state set $S_{k}^i$ of node $i$ is defined as the set of all possible solutions $x_k$ which can be reached given $y_k^i$ and $v_k^i$ (10):

$$S_{k}^i = \left\{ x_k \mid H^i x_k - y_k^i \leq R_k^i \right\}. \quad (10)$$

When the dimension of $y_k^i \in \mathbb{R}^p$ equals one, i.e., $p = 1$, this measurement set is a strip.

**Definition 4. (Corrected State Set)** Given system (8) with initial set $Z_{s,0} = \langle c_{s,0}, G_{s,0} \rangle$, the reachable corrected state set $\hat{Z}_{s,k}^i$ of node $i$ is defined as the over approximation of the intersection between $\hat{Z}_{s,k}^i$ and $S_{k}^i$ [43] p.4:

$$\left( \hat{Z}_{s,k}^i \cap S_{k}^i \right) \subseteq \hat{Z}_{s,k}^i. \quad (11)$$

Our proposed set-membership approach consists of three steps, namely, measurement update, diffusion update, and time update. Every node in a distributed setting has access to some, not all, measurements. Therefore, we propose to share measurements and estimated sets in the measurement and diffusion update steps, respectively, in order to obtain a lightweight consensus between the distributed nodes. We first give a high-level description of the proposed algorithm in Algorithm 1 then we derive the required theory. Our approach corrects the reachable set for every node of the sensor network by determining the set of consistent states with the model and measurements received from all neighbors. More specifically, during the measurement update, every node collects measurements from neighbors, as shown in sub-figure 1.i, i.e., each node obtains a family of strips (measurements) to be intersected with the predicted zonotopic reachable set of each node (sub-figure 1.iii) to obtain the estimated zonotope $\hat{Z}_{s,k}^i$, dashed in sub-figure 1.iii. Every node collects the shared sets from its neighbors in sub-figure 1.iv. Next, each node intersects its reachable set with shared sets of the neighbors in the set-based diffusion step in sub-figure 1.v. Finally, the estimated sets evolve according to the state update model. We propose to perform the measurement update step according to the following proposition [54], which is represented graphically in Figure 1a.

**Proposition 1.** Given are the zonotope $\hat{Z}_{s,k-1}^i = \langle \hat{c}_{s,k-1}^i, \hat{G}_{s,k-1}^i \rangle$, the family of $m_i$ measurement sets $S_{k}^i$ in (11) and the design parameters $\lambda_{s,k}^{i,j} \in \mathbb{R}^{n \times p}$, $\forall j \in N_i$. The intersection between the zonotope and measurement sets can be over-approximated by a zonotope $\bar{Z}_{s,k}^i = \langle \bar{c}_{s,k}^i, \bar{G}_{s,k}^i \rangle$, where

$$\bar{c}_{s,k}^i = \hat{c}_{s,k-1}^i + \sum_{j \in N_i} \lambda_{s,k}^{i,j} (y_k^i - H_k^i \hat{c}_{s,k-1}^i), \quad (12)$$

$$\bar{G}_{s,k}^i = \left[ (I - \sum_{j \in N_i} \lambda_{s,k}^{i,j} H_k^i) \hat{G}_{s,k-1}^i, \lambda_{s,k}^{i,1} R_k^1, \ldots, \lambda_{s,k}^{i,m_i} R_k^{m_i} \right]. \quad (13)$$

Fig. 1: Intersections for the set-membership approach in Algorithm 1. Figure 1a represents the measurement update step which consists of intersecting strips with zonotope. The resulting over-approximated zonotope (dashed) using Proposition 1 is presented in sub-figure iii. Figure 1b illustrates the diffusion update step, where sub-figure v. shows the proposed over-approximated zonotope (bold) which is the intersection of two zonotopes using Theorem 1.
Proof. We aim to find the zonotope that over-approximates the intersection. Let \( x \in (\tilde{Z}_{s,k-1}^i \cap S_k^1 \cap \ldots \cap S_k^{m_i}) \), then there is a \( z \), where
\[
x = \bar{c}_{s,k-1} + \tilde{G}_{s,k-1}^i z.
\]
Adding and subtracting \( \sum_{j \in N_i} \lambda_{i,j} H_k^j \tilde{G}_{s,k-1}^i z \) to (14) results in
\[
x = \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} H_k^j \tilde{G}_{s,k-1}^i z + (I - \sum_{j \in N_i} \lambda_{i,j} H_k^j) \tilde{G}_{s,k-1}^i z.
\]
Given that \( x \) is inside the intersection of the zonotope \( \tilde{Z}_{s,k-1}^i \) and the family of strips, then \( x \in S_k^i, \forall j \in N_i \), i.e., there exists \( b_j \in [-1_{p \times 1}, 1_{p \times 1}] \) for the \( j^{th} \) strip in (10) so that:
\[
H_k^j x = y_k^j + R_k^j b_j.
\]
Inserting (14) in (16) results in
\[
H_k^j \bar{c}_{s,k-1} = y_k^j - H_k^j \bar{c}_{s,k-1} + R_k^j b_j.
\]
Inserting (17) in (15) results in
\[
x = \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} (y_k^j - H_k^j \bar{c}_{s,k-1} + R_k^j b_j) + (I - \sum_{j \in N_i} \lambda_{i,j} H_k^j) \tilde{G}_{s,k-1}^i z
\]
\[
= \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} (y_k^j - H_k^j \bar{c}_{s,k-1}) + (I - \sum_{j \in N_i} \lambda_{i,j} H_k^j) \tilde{G}_{s,k-1}^i z + \sum_{j \in N_i} \lambda_{i,j} R_k^j b_j
\]
\[
= \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} (y_k^j - H_k^j \bar{c}_{s,k-1}) + (I - \sum_{j \in N_i} \lambda_{i,j} H_k^j) \tilde{G}_{s,k-1}^i z + \sum_{j \in N_i} \lambda_{i,j} R_k^j b_j
\]
\[
= \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} (y_k^j - H_k^j \bar{c}_{s,k-1}) + (I - \sum_{j \in N_i} \lambda_{i,j} H_k^j) \tilde{G}_{s,k-1}^i z + \sum_{j \in N_i} \lambda_{i,j} R_k^j b_j
\]
\[
= \bar{c}_{s,k-1} + \sum_{j \in N_i} \lambda_{i,j} (y_k^j - H_k^j \bar{c}_{s,k-1}) + \tilde{G}_{s,k-1}^i z + \sum_{j \in N_i} \lambda_{i,j} R_k^j b_j
\]
\[
= \bar{c}_{s,k-1} + \tilde{G}_{s,k-1}^i z + \sum_{j \in N_i} \lambda_{i,j} R_k^j b_j.
\]
Note that \( z_b \in [-1, 1] \) as \( b \in [-1_{p \times 1}, 1_{p \times 1}] \), as mentioned before, and \( z \in [-1, 1] \) because of the zonotope definition in (1). Thus, the center and the generator of the over-approximating zonotope are \( \bar{c}_{s,k} \) and \( \tilde{G}_{s,k}^i \), respectively.

Let \( \Lambda_{s,k}^i = \begin{bmatrix} \lambda_{i,1} & \lambda_{i,2} & \ldots & \lambda_{i,m_i} \end{bmatrix} \). Proposition 1 extends [55] for multi-strips case. However, we should note that considering all the strips at once for the intersection with the zonotope is better than considering strip by strip as we end up with one optimization function for calculating the design parameter \( \Lambda_{s,k}^i \) representative of the size of the final set. This appear clearly if the design parameter is \( \Lambda_{s,k}^i \) is based on the radius of the resultant zonotope as shown in Figure 2 for one example where we show the intersection of the zonotope with all strips at once and the intersection of strip by strip with the zonotope. However, if the design parameter \( \Lambda_{s,k}^i \) is chosen to minimize the Frobenius norm of the generator of the resultant zonotope, the resultant over-approximation would be the same. We chose to using the F-radius/Frobenius norm as a light weight approach indicator to the size of the zonotope, thus the closed form solution for finding the design parameter \( \Lambda_{s,k}^i \) gives the same result of considering strip by strip in [55].

As previously mentioned, every node shares its corrected zonotope \( \tilde{Z}_{s,k}^i = (\bar{c}_{s,k}^i, \tilde{G}_{s,k}^i) \) with its neighbours during the set-based diffusion step. We find the intersection between the shared zonotopes using the following theorem:

**Theorem 1.** The intersection between \( m_i \) zonotopes \( \tilde{Z}_{s,k}^i = \langle \bar{c}_{s,k}^i, \tilde{G}_{s,k}^i \rangle \) can be over-approximated using the zonotope

![Fig. 2: The difference between considering strip by strip and all at once in intersecting with a initial zonotopes. The three strips were omitted from the intersecting for clarity.](image)
\[ \hat{z}_{s,k}^i = \langle \hat{c}_{s,k}^i, \hat{x}_{s,k}^i \rangle \]

as follows:

\[
\hat{c}_{s,k}^i = \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} c_{s,k}^j, \tag{19}
\]

\[
\hat{G}_{s,k}^i = \frac{1}{\sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j}} \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} G_{s,k}^{i,m_j}, \tag{20}
\]

where \( w_{k,j}^{i,j} \) is a weight such that \( \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} \neq 0 \).

**Proof.** We aim to find the zonotope which over-approximates the intersection. Let \( x \in (Z^1_{s,k} \cap Z^2_{s,k} \cap \ldots \cap Z^m_{s,k}) \) then \( x \) is within the zonotope defined in \( \hat{z}_{s,k}^i \), i.e., we have \( z^i \in [-1, 1] \) for each zonotope \( j \) such that

\[ x = \hat{c}_{s,k}^i + \hat{G}_{s,k}^i z. \tag{21} \]

By multiplying \( \hat{G}_{s,k}^i \) with \( w_{k,j}^{i,j} \) and summing for all \( m_i \) zonotopes, we obtain

\[
x = \frac{1}{\sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j}} \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} \hat{G}_{s,k}^i \hat{x}_{s,k}^j + \frac{1}{\sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j}} \sum_{j \in \mathcal{N}_i} w_{k,j}^{i,j} \hat{G}_{s,k}^i z \]

\[
= \hat{c}_{s,k}^i + \hat{G}_{s,k}^i z. \tag{22} \]

Note that \( z \in [-1, 1] \) as \( z^1, \ldots, z^{m_i} \) come form the participating zonotopes within the same interval. Thus, the center and the generator of the over-approximating zonotope are \( \hat{c}_{s,k}^i \) and \( \hat{G}_{s,k}^i \), respectively.

Our zonotopes intersection takes linear time \( (O(n)) \). We introduce \( w_k^i = [w_k^{i,1}, \ldots, w_k^{i,m_i}] \). The optimal design of the parameters \( w_k^i \) can be chosen such that the size of the zonotope \( \hat{z}_{s,k}^i = \langle \hat{c}_{s,k}^i, \hat{G}_{s,k}^i \rangle \) is minimal. Using the F-radius/Frobenius norm as an indicator of zonotopic size, we choose \( w_k^i \), which satisfies

\[
w_k^i = \arg \min_{w_k^i} \| \hat{G}_{s,k}^i \|^2_F. \tag{23} \]

The following proposition is proposed to compute the optimal weights \( w_k^{i,j} \).

**Proposition 2.** For the estimated zonotopic set \( \hat{z}_{s,k}^i = \langle \hat{c}_{s,k}^i, \hat{G}_{s,k}^i \rangle \) in \( \hat{z}_{s,k}^i \) and \( \hat{G}_{s,k}^i \), the optimal design parameters \( w_k^{i,j}, \forall j \in \mathcal{N}_i \) where \( \sum_{j \in \mathcal{N}_i} w_k^{i,j} = 1 \), can be obtained as:

\[
w_k^{i,j} = \frac{1}{\text{tr}(\hat{G}_{s,k}^i \hat{G}_{s,k}^i^T)} \sum_{r=1}^{m_i} \frac{1}{\text{tr}(\hat{G}_{s,k}^i \hat{G}_{s,k}^i^T)} \tag{24} \]

**Proof.** The Frobenius norm of the generator matrix can be computed as follows:

\[
\| \hat{G}_{s,k}^i \|^2_F = \text{tr}(\hat{G}_{s,k}^i \hat{G}_{s,k}^i^T) \]

Let \( \beta_r = \text{tr}(\hat{G}_{s,k}^i \hat{G}_{s,k}^i^T) \), therefore we obtain the following constrained optimization problem:

\[
w_k^{i,j} = \arg \min_{w_k^{i,j}} \beta_r(w_k^{i,r})^2,
\]

subject to \( f(w_k^i) = \sum_{r=1}^{m_i} w_k^{i,r} - 1 = 0. \tag{26} \]

This can be solved by introducing Lagrange multiplier \( s \). The Lagrangian function for \( f(w_k^i) \) is

\[
\mathcal{L} = \arg \min_{w_k^{i,j}} \beta_r(w_k^{i,r})^2 - s \sum_{r=1}^{m_i} w_k^{i,r} - 1). \tag{27} \]

The necessary condition \( \forall j \in \mathcal{N}_i \) for an extremum point is

\[
\nabla_{w_k^{i,j}} \mathcal{L} = 2w_k^{i,j} \beta_r - s = 0. \tag{28} \]

The constraint provides the last condition:

\[
\sum_{r=1}^{m_i} w_k^{i,r} - 1 = 0. \tag{29} \]

Inserting \( (28) \) in \( (29) \) results in:

\[
s = \frac{2}{2 \beta_r} \sum_{r=1}^{m_i} \frac{1}{\beta_r}. \tag{30} \]

Inserting \( (30) \) into \( (28) \) results in:

\[
w_k^{i,j} = \frac{s}{2 \beta_j} \frac{1}{\beta_j} \frac{1}{\beta_j}. \tag{31} \]

It remains to check if the extremum point is a minimum. First, we find the Jacobian of \( f(w_k^i) \) with respect to \( w_k^{i,j} \):

\[
\nabla_{w_k^{i,j}} f(w_k^i) = 1, \forall j \in \mathcal{N}_i. \tag{32} \]

Then, we compute the bordered Hessian matrix \( H_b \), while denoting \( \nabla_{w_k^i} X(w_k^i) \) by \( X_{w^i} \) and \( \nabla_{w_k^i \cdot w_k^i} X(w_k^i) \) by \( X_{w^i, w^i} \) for simplicity:

\[
H_b = \begin{bmatrix} 0 & -f_{w^1} & \cdots & -f_{w^{m}} \\
-f_{w^1} & \mathcal{L}_{w^1,1} - s f_{w^1,1} & \cdots & \mathcal{L}_{w^1,m} - s f_{w^1,m} \\
\vdots & \vdots & \ddots & \vdots \\
-f_{w^{m}} & \mathcal{L}_{w^m,1} - s f_{w^m,1} & \cdots & \mathcal{L}_{w^m,m} - s f_{w^m,m} \end{bmatrix}. \tag{33} \]

The determinants of the \( m_i - 1 \) largest principal minors of \( (33) \) are negatives. Thus, the extremum in \( (31) \) is a minimum.
After presenting our distributed set-membership approach using a diffusion strategy. We present our interval-based diffusion observer.

IV. DISTRIBUTED INTERVAL-BASED DIFFUSION OBSERVER

Unlike the set-membership observer developed in the previous section, which was based on geometric intersection, we propose the following Luenberger-type interval-based observer

\[ \dot{x}_{v,k}^i = F \dot{x}_{v,k}^i - n_k + \sum_{j \in N_i} \lambda_{v,k}^{i,j} (y_j^i - H_j^i \dot{x}_{v,k}^i - v_j^i) \tag{34} \]

where \( \dot{x}_{v,k}^i \) is the state estimated by interval-based observer, and \( \sum_{j \in N_i} \lambda_{v,k}^{i,j} \) are the time-varying observer gains. The design of the observer makes use of the bounds of the noises. For the distributed system in [3], the proposed design consists of two steps: Luenberger update and diffusion update. During the Luenberger update, every node shares its measurement with its neighbour, while in the diffusion step, every node shares the estimated information with its neighbours. We first discuss the Luenberger update step.

**Theorem 2.** Given are the system (3), the measurements \( y_j^i \), several zonotopes bounding \( x_0 \in \langle c_0, G_0 \rangle, n_k \in Z_Q,k = \langle 0, Q_k \rangle, v_k \in Z^i = \langle 0, R_k \rangle \), and the state \( \dot{x}_{v,k}^i \in \langle \dot{c}_{v,k}^i, \dot{G}_{v,k}^i \rangle \). The zonotope bounding the uncertain states can be iteratively obtained as \( \dot{x}_{v,k}^i \in \langle \dot{c}_{v,k}^i, \dot{G}_{v,k}^i \rangle \), where,

\[ \dot{c}_{v,k}^i = (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \dot{c}_{v,k}^i - 1 + \sum_{j \in N_i} \lambda_{v,k}^{i,j} y_j^i \]

\[ \dot{G}_{v,k}^i = (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \dot{G}_{v,k}^i - 1, \]

where \( \lambda_{v,k}^{i,j} \) are the time-varying observer gains. We propose

\[ (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \dot{G}_{v,k}^i - 1 + \sum_{j \in N_i} \lambda_{v,k}^{i,j} y_j^i \]

\[ (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \dot{G}_{v,k}^i - 1, \]

Proof. Given \( \dot{x}_{v,k}^i \in \langle \dot{c}_{v,k}^i, \dot{G}_{v,k}^i \rangle \), one obtains:

\[ \dot{x}_{v,k}^i \in \langle \dot{c}_{v,k}^i, \dot{G}_{v,k}^i \rangle \]

**Algorithm 2 Distributed Interval-based Diffusion Observer**

Start with \( \dot{x}_0 = x_0 \), zonotope \( Z_0 = (c_0, G_0) \) for all \( k \), and at every time instant \( i \), compute at every node \( i \):

**Step 1:** Luenberger update:

\[ \lambda_{v,k}^{i,j} = \arg \min \| \dot{G}_{v,k}^i \| F \]

\[ \bar{c}_{v,k}^i = (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \dot{c}_{v,k}^i - 1 + \sum_{j \in N_i} \lambda_{v,k}^{i,j} y_j^i \]

\[ \bar{G}_{v,k}^i = (F - \sum_{j \in N_i} \lambda_{v,k}^{i,j} H_j^i) \bar{G}_{v,k}^i - 1, \]

Step 2: Diffusion update:

\[ w_i = \arg \min \| \bar{G}_{v,k}^i \| F \]

\[ \bar{c}_{v,k}^i = \frac{1}{\sum_{j \in N_i}} \sum_{j \in N_i} w_{j,i} \bar{c}_{v,k}^i \]

\[ \bar{G}_{v,k}^i = \frac{1}{\sum_{j \in N_i}} \sum_{j \in N_i} w_{j,i} \bar{G}_{v,k}^i \]

Let us introduce \( \Lambda_{v,k}^i = \left[ \lambda_{v,k}^{i,1}, \ldots, \lambda_{v,k}^{i,m} \right] \). We propose to compute the design vectors \( \Lambda_{v,k}^i \) such that:

\[ \Lambda_{v,k}^i = \arg \min \| \bar{G}_{v,k}^i \| F \]

The following proposition is provided to compute the optimal parameters \( \Lambda_{v,k}^i \).

**Proposition 3.** For the estimated zonotopic set \( \hat{Z}_{v,k}^i = \langle \hat{c}_{v,k}^i, \hat{G}_{v,k}^i \rangle \) corresponding to node \( i \), the optimal design parameters \( \Lambda^i \) can be obtained as:

\[ \Lambda_{v,k}^i = \frac{FG_{v,k}^i - \hat{G}_{v,k}^i \Gamma_k^T}{\Gamma_k^T \hat{G}_{v,k}^i R_{v,k}^i \Gamma_k + \sum_{r=1}^{m} \Omega_r R_{v,k}^i \Gamma_k^T \Omega_r^T} \]

Proof. Let \( \Lambda_{v,k}^i = \Lambda_{v,k}^i \Omega_r \). We rewrite [36] as

\[ \bar{G}_{v,k}^i = \frac{FG_{v,k}^i \hat{G}_{v,k}^i - \Gamma_k^T}{\Gamma_k^T \hat{G}_{v,k}^i R_{v,k}^i \Gamma_k + \sum_{r=1}^{m} \Omega_r R_{v,k}^i \Gamma_k^T \Omega_r^T} \]

The Frobenious norm of [40] can be computed as

\[ \| \bar{G}_{v,k}^i \|_F^2 = \text{tr}(\bar{G}_{v,k}^i \bar{G}_{v,k}^i) \]

\[ = \text{tr}\left( (F - \Lambda_{v,k}^i \Omega_r \Gamma_k^T) (F - \Lambda_{v,k}^i \Omega_r \Gamma_k^T)^T \right) \]

\[ + \sum_{r=1}^{m} \Omega_r R_{v,k}^i \Gamma_k^T \Omega_r^T \Lambda_{v,k}^i \]

\[ + \text{tr}(FG_{v,k}^i - FG_{v,k}^i \Gamma_k^T \Lambda_{v,k}^i + \Lambda_{v,k}^i \Omega_r \Gamma_k^T \Lambda_{v,k}^i)^T) \]

\[ + \text{tr}(\Lambda_{v,k}^i \Omega_r \Gamma_k^T \Lambda_{v,k}^i)^T) \]

(41)
where $\mathcal{R}^r = R_k^T R_k^r$, and $\mathcal{G} = \mathcal{G}_{i-1}^r \mathcal{G}_{i}^r$. The optimal value of $\Lambda_{v,k}$ can be obtained by solving

$$\nabla \Lambda_{v,k} \|\hat{G}_{v,k}\|^2_F = 0,$$

where the Jacobian in (42) can be computed by applying matrix properties in (39) and (40) to (41):

$$\nabla \Lambda_{v,k} \|\hat{G}_{v,k}\|^2_F = -2F\Gamma_k^T + 2\Lambda_{v,k}\Gamma_k \Gamma_k^T + \sum_{r=1}^{m_i} 2\Lambda_{v,k} \Omega_r \mathcal{R}^r \Omega_r^T = 0$$

(43)

By inserting the optimal $\Lambda_{v,k}$ from (39) in (43), one can see that (43) is fulfilled. $\square$

Following the Luenberger update step, in the diffusion step, each node shares the information of the estimated zonotope $(\hat{c}_{v,k}, \hat{G}_{v,k})$ with its neighbors. The intersection between the shared zonotopes is then computed as discussed earlier in Theorem 1. The iterative design of the above two-step Luenberger observer is provided in Algorithm 2.

V. Evaluation

Our proposed algorithms are implemented in Matlab 2019 on an example similar to the one presented in [54, 57], where a network of eight nodes attempts to track the position of a rotating object. All computations run on a single thread of an Intel(R) Core(TM) i7-8750 with 16 GB RAM. We made use of CORA [58–60] for zonotope operations. Our example is quite representative for set-based state estimation, since it includes modeling noise and measurements noise. The state of each node consists of the unknown 2-dimensional position of the rotating object. The state matrix in (39) is as follows:

$$F = \begin{bmatrix} 0.992 & -0.1247 \\ 0.1247 & 0.992 \end{bmatrix},$$

(44)

and the measurement matrix $H$ is [0 1] or [1 0] in the sequence of the taken measurements. We run our proposed algorithms in comparison with the one proposed by Garcia et al. [46] on the same generated data set. We should note that related work does not consider sharing the measurements between the neighbors like our approach and this affects the estimation results but comes at cost of extra communication. Figure 3 shows the true values, upper bound, and lower bound for each dimension of the estimated state using the set-membership approach in Algorithm 1 while each node is connected to four neighbors. We start by a set $(160 \times 160 m^2)$ covering the whole localization area at the initial point (time step 0), then it gets smaller due to receiving measurements and performing geometric intersection to correct the estimated state. In addition, we repeat the same experiments using Algorithm 2 and present the results in Figure 4.

The effect of the diffusion step is analyzed graphically over a network with low connectivity, where every node is connected to two nodes only. Snapshots of the estimated zonotopes by the distributed nodes in Algorithm 2 are shown in Figure 5. The triangles are the true positions of the monitoring nodes. The estimates are the centers of the zonotopes, which are represented by red pluses. Figure 5a and 5b show the results without and with the diffusion step, respectively. As shown in aforementioned figures, diffusion step allows the estimated zonotopes by the distributed nodes to partially consense on a set, which is one of the advantages of adding the diffusion step. The Hausdorff distance measures how far two subsets of a metric space are from each other. Thus, as another measure of the estimated zonotopes consistency for all the distributed nodes, we calculate the Hausdorff distance between the set of vertices of each zonotope at different time step. We analyze the Hausdorff distance over different network connectivities. The results are reported in Table II for Garcia et al. [46], Algorithm 1 and 2 with and without the diffusion step. The diffusion step enhances the alignment between the estimated zonotopes which has a significant effect on a network with a low connectivity. For the aforementioned network, every node has access to a lower number of measurements, and thus the diffusion step provides more information to the distributed nodes and enhances the estimated zonotopes alignment, estimation results, and radiuses of the estimated zonotopes.

One important aspect of the performance of the set-based estimation algorithm is reducing the resulting radius of the over-approximating estimated set. Therefore, we analyze the radiuses of the estimated zonotopes of the proposed algorithms in comparison to the previous work in [46]. Table II shows the mean and standard deviation of the radiuses with and without the diffusion step of the proposed algorithms. The diffusion decreases the radiuses due to the proposed intersection criteria. Moreover, our proposed algorithms with and without the diffusion step are much better than the previous work [46]. We note that the network with higher connectivity has a lower radius as the intersection with more strips decreases the estimated set. The center of the estimated zonotope is considered as a single point estimate of the proposed algorithms. Therefore, we report the localization error of the estimated centers by the proposed algorithms in Table III. The diffusion significantly enhances the center estimate of the proposed algorithms. Again, the diffusion step is more effective in a network with low connectivity.

Table IV shows the execution time of each step in the proposed algorithms while again changing the number of neighbors. The time update step does not depend on the number of the neighbors. To measure the execution time, we run each step $10^5$ times with random generated zonotopes, having 20 generators, and take the average of the execution time.

VI. Conclusion

We propose distributed set-based and interval-based observers using diffusion strategy. Our algorithms remove the need for a fusion center. They only requires every node to communicate with its neighbors: first to share the data, and second to share the estimates. The diffusion step ensures that information is propagated throughout the network in order to converge on the best estimate and provide consistency between the estimated sets. We propose new over-approximation for zonotopes intersection to compose the diffusion step. We evaluate our algorithms in a localization example of a rotating object.
Fig. 3: True values, upper bounds and lower bounds of the two-dimensional estimated states using set-membership approach in Algorithm 1, where every nodes has four neighbors.

Fig. 4: True values, upper bounds and lower bounds of the two-dimensional estimated states using interval-based approach in Algorithm 2, where every nodes has four neighbors.

TABLE I: The mean and standard deviation of the Hausdorff distance (m) between the estimated zonotopes where every node has two, four, or six neighbors.

| Algorithm | Diffusion | Six neighbors | Four neighbors | Two neighbors |
|-----------|-----------|---------------|---------------|--------------|
|           | mean      | std           | mean          | std          | mean         | std          |
| Alg. 2    | ✓         | 0.242 0.160   | 0.824 0.409   | 2.813 2.163  |
| Alg. 2    | X         | 1.517 1.393   | 2.703 2.118   | 3.829 2.360  |
| Alg. 1    | ✓         | 0.333 0.242   | 1.897 1.332   | 3.871 2.362  |
| Alg. 1    | X         | 1.813 1.553   | 3.405 2.460   | 4.855 2.464  |
| Garcia    | -         | 32.482 19.578 | 29.523 17.387 | 25.443 14.517 |

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REFERENCES

[1] G. Welch, G. Bishop, et al., “An introduction to the Kalman filter,” 1995.
[2] M. Pourasghar, V. Puig, and C. Ocampo-Martinez, “Interval observer versus set-membership approaches for fault detection in uncertain systems using zonotopes,” International Journal of Robust and Nonlinear Control, vol. 29, no. 10, pp. 2819–2843, 2019.
[3] V. Puig, “Fault diagnosis and fault tolerant control using set-membership approaches: Application to real case studies,” International Journal of Applied Mathematics and Computer Science, vol. 20, no. 4, pp. 619–635, 2010.
[4] C. Combastel, “Merging Kalman filtering and zonotopic state bounding for robust fault detection under noisy environment,” IFAC-PapersOnLine, vol. 48, no. 21, pp. 289–295, 2015.
[5] Y. Wang, Z. Wang, V. Puig, and G. Cembrano, “Zonotopic fault estimation filter design for discrete-time descriptor systems,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 5055–5060, 2017.
[6] W. Zhang, Z. Wang, Y. Shen, S. Guo, and F. Zhu, “Interval estimation of actuator fault by interval analysis,” IET Control Theory Applications, vol. 13, no. 16, pp. 2717–2724, 2019.
[7] C. Combastel and A. Zolghadri, “FDI in cyber physical systems: A distributed zonotopic and gaussian kalman filter with bit-level reduction,” IFAC-PapersOnLine, vol. 51, no. 24, pp. 776–783, 2018.
[8] M. Pourasghar, V. Puig, and C. Ocampo-Martinez, “Comparison of set-membership and interval observer approaches for state estimation of uncertain systems,” in European Control Conference, pp. 1111–1116, IEEE, 2016.
[9] D. Efimov, W. Perruquetti, T. Raïssi, and A. Zolghadri, “Interval observers for time-varying discrete-time systems,” IEEE Transactions on Automatic Control, vol. 58, no. 12, pp. 3218–3224, 2013.
[10] D. Efimov, T. Raissi, and A. Zolghadri, “Control of nonlinear and lpv systems: interval observer-based framework,” IEEE Transactions on Automatic Control, vol. 58, no. 3, pp. 773–778, 2013.
[11] F. Mazenc and O. Bernard, “Asymptotically stable interval observers for planar systems with complex poles,” IEEE Transactions on Automatic
Fig. 5: Snapshots of the estimated zonotopes with the red pluses as centers. The triangles are the true positions for the observing nodes. The blue rectangular is the true position of rotating target. Figure 5a shows the distributed estimated zonotopes using Algorithm 2 without the diffusion step. On the other hand, Figure 5b shows Algorithm 2 with the diffusion step.

TABLE II: The mean and standard of the radius (m) of the estimated zonotopes by the proposed algorithms in comparison to [46]. We present the result with and without the diffusion step over different network connectivities where every node has two, four, or six neighbors.

| Algorithm | Diffusion | Six neighbors | Four neighbors | Two neighbors |
|-----------|-----------|---------------|----------------|---------------|
|           |           | mean | std   | mean | std   | mean | std   |
| Alg. 1    | ✓         | 11.877 | 0.057 | 12.626 | 0.221 | 15.104 | 0.442 |
| Alg. 1    | ✗         | 13.312 | 0.419 | 13.084 | 0.272 | 12.920 | 0.210 |
| Alg. 2    | ✓         | 13.266 | 1.235 | 13.515 | 0.432 | 15.257 | 0.393 |
| Alg. 2    | ✗         | 16.690 | 1.443 | 17.174 | 0.918 | 18.531 | 1.330 |
| Garcia [46]| -         | 53.681 | 21.992 | 49.932 | 19.373 | 44.671 | 16.040 |

Control, vol. 55, no. 2, pp. 523–527, 2010.

[12] F. Mazenc and O. Bernard, “Interval observers for linear time-invariant systems with disturbances,” Automatica, vol. 47, no. 1, pp. 140–147, 2011.

[13] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri, “Interval state observer for nonlinear time varying systems,” Automatica, vol. 49, no. 1, pp. 200–205, 2013.

[14] Z. He and W. Xie, “Control of non-linear switched systems with average dwell time: interval observer-based framework,” IET Control Theory & Applications, vol. 10, no. 1, pp. 10–16, 2016.

[15] N. Meslem, N. Ramdani, and Y. Candau, “Interval observers for uncertain nonlinear systems. application to bioreactors,” IFAC Proc. Volumes, vol. 41, no. 2, pp. 9667–9672, 2008.

[16] W. Tang, Z. Wang, Y. Wang, T. Raïssi, and Y. Shen, “Interval estimation methods for discrete-time linear time-invariant systems,” IEEE Transactions on Automatic Control, vol. 64, no. 11, pp. 4717–4724, 2019.

[17] J.-L. Gouzé, A. Rapaport, and M. Z. Hadj-Sadok, “Interval observers for uncertain biological systems,” Ecological Modelling, vol. 133, no. 1-2, pp. 45–56, 2000.

[18] C. Combastel, “Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence,” Automatica, vol. 55, pp. 265–273, 2015.

[19] F. Schewpepe, “Recursive state estimation: Unknown but bounded errors and system inputs,” IEEE Transactions on Automatic Control, vol. 13, no. 1, pp. 22–28, 1968.

[20] S. Gollamudi, S. Nagaraj, S. Kapoor, and Y.-F. Huang, “Set-membership filtering and a set-membership normalized LMS algorithm with an adaptive step size,” IEEE Signal Processing Letters, vol. 5, no. 5, pp. 111–114, 1998.

[21] T. Alamo, J. M. Bravo, M. J. Redondo, and E. F. Camacho, “A set-membership state estimation algorithm based on DC programming,” Automatica, vol. 44, no. 1, pp. 216–224, 2008.

[22] W. Tang, Z. Wang, Q. Zhang, and Y. Shen, “Set-membership estimation for linear time-varying descriptor systems,” Automatica, vol. 115, p. 108867, 2020.

[23] T. Rassi, N. Ramdani, and Y. Candau, “Set membership state and parameter estimation for systems described by nonlinear differential equations,” Automatica, vol. 40, no. 10, pp. 1771–1777, 2004.

[24] M. Milanese and A. Vicino, “Estimation theory for nonlinear models and set membership uncertainty,” Automatica, vol. 27, no. 2, pp. 403–408, 1991.

[25] E. Scholte and M. E. Campbell, “A nonlinear set-membership filter for on-line applications,” International Journal of Robust and Nonlinear Control, vol. 13, no. 15, pp. 1337–1358, 2003.
TABLE III: The mean and standard deviation of the localization error (m) of the center of the estimated zonotopes using the proposed algorithms in comparison to \cite{46} over different network connectivities, where every node has two, four, or six neighbors.

| Algorithm | Diffusion | Six neighbors | Four neighbors | Two neighbors |
|-----------|-----------|---------------|----------------|--------------|
|           |           | mean | std | mean | std | mean | std |
| Alg. 1    | ✓         | 2.227 | 0.490 | 2.419 | 0.485 | 3.485 | 0.486 |
| Alg. 2    | ✓         | 2.463 | 0.409 | 3.528 | 0.315 | 5.605 | 0.477 |
| Alg. 3    | ✓         | 1.495 | 0.513 | 0.892 | 0.415 | 4.882 | 0.871 |
| Alg. 4    | ✓         | 2.489 | 0.489 | 3.550 | 0.459 | 5.853 | 0.761 |
| Garcia   | ✓         | 11.729 | 1.573 | 11.877 | 1.571 | 12.059 | 1.717 |

TABLE IV: Execution Time in \(\mu\) seconds of the proposed measurement, diffusion, Luenberger, and time updates in Algorithm 1 and 2 with different number of neighbors.

| Step           | Six neighbors | Four neighbors | Two neighbors |
|----------------|---------------|----------------|--------------|
| Measurement update | 117 109 95     |                |              |
| Diffusion update     | 101 93 72      |                |              |
| Time update            | 33 33 33       |                |              |
| Luenberger update      | 111 104 92     |                |              |

\[26\] H. Lahanier, E. Walter, and R. Gomeni, “OMNE: a new robust membership-set estimator for the parameters of nonlinear models,” *Journal of Pharmacokinetics and Biopharmaceutics*, vol. 15, no. 2, pp. 203–219, 1987.

\[27\] D. Ding, Z. Wang, and Q. Han, “A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks,” *IEEE Transactions on Automatic Control*, pp. 1–1, 2019.

\[28\] L. Jaulin, “Robust set-membership state estimation; application to underwater robotics,” *Automatica*, vol. 45, no. 1, pp. 202–206, 2009.

\[29\] X. Ge, Q.-L. Han, and F. Yang, “Event-based set-membership leader-following consensus of networked multi-agent systems subject to limited communication resources and unknown-but-bounded noise,” *IEEE Transactions on Industrial Electronics*, vol. 64, no. 6, pp. 5045–5054, 2017.

\[30\] P. Bouron, D. Meizel, and P. Bonnifait, “Set-membership non-linear observers with application to vehicle localisation,” in *European Control Conference*, pp. 1255–1260, IEEE, 2001.

\[31\] L. Ma, Z. Wang, H.-K. Lam, and N. Kyriakou, “Distributed event-based set-membership filtering for a class of nonlinear systems with sensor saturations over sensor networks,” *IEEE Transactions on Cybernetics*, vol. 47, no. 11, pp. 3772–3783, 2016.

\[32\] L. Orihuela, S. Roshany-Yamchi, R. A. García, and P. Millán, “Distributed set-membership observers for interconnected multi-rate systems,” *Automatica*, vol. 85, pp. 221–226, 2017.

\[33\] M. Z. A. Bhotto and A. Antoniou, “Robust set-membership affine-projection adaptive-filtering algorithm,” *IEEE Transactions on Signal Processing*, vol. 60, no. 1, pp. 73–81, 2011.

\[34\] K. Chen, S. Werner, A. Kuh, and Y.-F. Huang, “Nonlinear adaptive filtering with kernel set-membership approach,” *IEEE Transactions on Signal Processing*, vol. 68, pp. 1515–1528, 2020.

\[35\] C. Durieu, E. Walter, and B. Polyak, “Multi-input multi-output ellipsoidal state bounding,” *Journal of optimization theory and applications*, vol. 111, no. 2, pp. 273–303, 2001.

\[36\] N. Xia, F. Yang, and Q.-L. Han, “Distributed networked set-membership filtering with ellipsoidal state estimations,” *Information Sciences*, vol. 432, pp. 52 – 62, 2018.

\[37\] S. Liu, Z. Wang, G. Wei, and M. Li, “Distributed set-membership filtering for multirate systems under the round-robin scheduling over sensor networks,” *IEEE Transactions on Cybernetics*, pp. 1–11, 2019.

\[38\] G. Belforte, B. Bona, and V. Cerone, “Parameter estimation algorithms for a set-membership description of uncertainty,” *Automatica*, vol. 26, no. 5, pp. 887–898, 1990.

\[39\] J. Blesa, V. Puig, and J. Saludes, “Robust fault detection using polytope-based set-membership consistency test,” *IET Control Theory & Applications*, vol. 6, no. 12, pp. 1767–1777, 2012.

\[40\] W. Kühn, “Zonotope dynamics in numerical quality control,” in *Mathematical Visualization*, pp. 125–134, 1998.

\[41\] C. Combastel, “A state bounding observer based on zonotopes,” in *European Control Conference*, pp. 2589–2594, IEEE, 2003.

\[42\] S. M. Tabatabaeipour and J. Stoistrup, “Set-membership state estimation for discrete time piecewise affine systems using zonotopes,” in *European Control Conference*, pp. 3143–3148, IEEE, 2013.

\[43\] Y. Wang, V. Puig, and G. Cembrano, “Set-membership approach and Kalman observer based on zonotopes for discrete-time descriptor systems,” *Automatica*, vol. 93, pp. 435–443, 2018.

\[44\] V. Puig, P. Cugueró, and J. Quevedo, “Worst-case state estimation and simulation of uncertain discrete-time systems using zonotopes,” in *European Control Conference*, pp. 1691–1697, IEEE, 2001.

\[45\] Y. Wang, T. Alamo, V. Puig, and G. Cembrano, “A distributed set-membership approach based on zonotopes for interconnected systems,” in *IEEE Conference on Decision and Control*, pp. 668–673, 2018.

\[46\] R. A. García, L. Orihuela, P. Millán, M. G. Ortega, and F. R. Rubio, “Kalman-inspired distributed set-membership observers,” in *European Control Conference*, pp. 2515–2520, IEEE, 2016.

\[47\] Y. Wang, T. Alamo, V. Puig, and G. Cembrano, “Distributed zonotopic set-membership state estimation based on optimization methods with partial projection,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 4039–4044, 2017.

\[48\] S. Zheng, X. Zhang, and Q. Lu, “Distributed set-membership observer-based consensus of nonlinear delayed multi-agent systems under round-robin protocols,” in *Chinese Control And Decision Conference*, pp. 118–123, 2018.

\[49\] A. Garulli and A. Giannitrapani, “A set-membership approach to consensus problems with bounded measurement errors,” in *IEEE Conference on Decision and Control*, pp. 2276–2281, 2008.

\[50\] M. Zhou, J. He, P. Cheng, and J. Chen, “Discrete average consensus with bounded noise,” in *IEEE Conference on Decision and Control*, pp. 5270–5275, 2013.

\[51\] W. Kühn, “Rigorously computed orbits of dynamical systems without the wrapping effect,” *Computing*, vol. 61, no. 1, pp. 47–67, 1998.
International Workshop on Hybrid Systems: Computation and Control, pp. 291–305, 2005.

[53] K. Petersen and M. Pedersen, “The matrix cookbook,” Technical University of Denmark, vol. 15, 2008.

[54] A. Alanwar, H. Said, and M. Althoff, “Distributed secure state estimation using diffusion Kalman filters and reachability analysis,” in IEEE Conference on Decision and Control, pp. 4133–4139, 2019.

[55] T. Alamo, J. M. Bravo, and E. F. Camacho, “Guaranteed state estimation by zonotopes,” Automatica, vol. 41, no. 6, pp. 1035–1043, 2005.

[56] R. T. Rockafellar, “Lagrange multipliers and optimality,” SIAM review, vol. 35, no. 2, pp. 183–238, 1993.

[57] F. S. Cattivelli, C. G. Lopes, and A. H. Sayed, “Diffusion strategies for distributed Kalman filtering: formulation and performance analysis,” Proc. of Cognitive Information Processing, pp. 36–41, 2008.

[58] M. Althoff, “An introduction to CORA 2015,” in Proc. of the Workshop on Applied Verification for Continuous and Hybrid Systems, pp. 120–151.

[59] M. Althoff and D. Grebenyuk, “Implementation of interval arithmetic in CORA 2016,” in Proc. of the 3rd International Workshop on Applied Verification for Continuous and Hybrid Systems, pp. 145–173.

[60] M. Althoff, D. Grebenyuk, and N. Kochdumper, “Implementation of taylor models in CORA 2018,” in Proc. of the 5th International Workshop on Applied Verification for Continuous and Hybrid Systems, pp. 91–105.