Boundedness of solutions for Duffing equation with low regularity in time

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Abstract
It is shown that all solutions are bounded for Duffing equation \( \ddot{x} + x^2 + 1 + \sum_{j=0}^{2n} P_j(t)x^j = 0 \), provided that for each \( n+1 \leq j \leq 2n \), \( P_j(t) \in C^\gamma(T) \) with \( \gamma > 1 - \frac{1}{n} \) and for each \( 0 \leq j \leq n \), \( P_j \in L(\mathbb{T}^1) \).

1. Introduction

In 1962, Moser [6] proposed to study the boundedness of all solutions (Lagrange stability) for Duffing equation

\[ \ddot{x} + \beta x^3 + \alpha x = P(t), \quad P \in C(\mathbb{T}^1), \quad \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}, \]

where \( \beta > 0, \alpha \in \mathbb{R} \) are constants.

In 1976, Morris [5] proved the boundedness of all solutions for

\[ \ddot{x} + 2x^3 = P(t). \]

Subsequently, Morris’ boundedness results was, by Dieckerhoff-Zehnder [1] in 1987, extended to a wider class of systems

\[ \ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} P_j(t)x^j = 0, \quad n \geq 1, \quad (1.1) \]

where

\[ P_j \in C^v, \quad v \geq 1 + \frac{4}{n} + \left[ \log_2 \left( \frac{n}{2} \right) \right] \to \infty, \quad \Leftrightarrow \text{ as } n \to \infty. \]

Then they remarked that

"It is not clear whether the boundedness phenomenon is related to the smoothness in the t-variable or whether this requirement is a shortcoming of our proof."

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In 1989 and 1992, Liu \[3, 4\] proved the boundedness for
\[\ddot{x} + x^{2n+1} + a(t)x + P(t) = 0, \quad a(t) \in C^0(\mathbb{T}^1), \quad P(t) \in C^0(\mathbb{T}^1).\]

In 1991, Laederich-Levi \[2\] relaxed the smoothness requirement of \(P_j(t)\) \((j = 0, 1, \cdots, 2n)\) for (1.1) to
\[P_j \in C^{5+\varepsilon}(\mathbb{T}^1), \quad \varepsilon > 0.\]

In his PhD thesis (1995), the present author further relaxed the requirement to \(C^2\). See \[12], [13] and \[14\].

In the present paper, we will relax the smoothness requirement to \(C^1\). More exactly, we have the following theorem

**Theorem 1.1.** For Arbitrary given constant \(\gamma > 1 - \frac{1}{n}\), assume \(P_j(t) \in C^\gamma(\mathbb{T}^1)\) for \(n+1 \leq j \leq 2n\) and \(P_j(t) \in L(\mathbb{T}^1)\) for \(0 \leq j \leq n\). Then every solution \(x(t)\) of the equation (1.1)
\[\ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} P_j(t)x^j = 0, \quad n \geq 1,\]
is bounded, i.e. it exists for all \(t \in \mathbb{R}\) and
\[\sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) < C < \infty,\]
where \(C = C(x(0), \dot{x}(0))\) depends the initial data \((x(0), \dot{x}(0))\).

**Remark 1.** In \[11\], it is proved that there is a continuous periodic function \(p(t)\) such that the Duffing equation \(\ddot{x} + x^{2n+1} + p(t)x = 0\) with \(p(t) \in C^0(\mathbb{T}^1), n \geq 2, 2n + 1 > l \geq n + 2\) possesses an unbounded solution, which shows that the smoothness of the coefficients \(P_j(t)\)’s does influence the boundedness of solutions. Therefore, the result of theorem \[11\] is sharp without considering the derivative of non-integral order.

2. **Action-Angle Variable**

Replacing \(x\) by \(Ax\) in (1.1), we get
\[A\ddot{x} + A^{2n+1}x^{2n+1} + \sum_{j=0}^{2n} P_j(t)x^jA^j = 0, \quad (2.1)\]
where \(A\) is a constant large enough. That is,
\[\ddot{x} + A^{2n}x^{2n+1} + \sum_{j=0}^{2n} P_j(t)x^jA^j = 0. \quad (2.2)\]

Let
\[y = A^{-n}\dot{x}, \quad \dot{x} = A^n y.\]
Then
\[ \dot{y} = A^{-n} \dot{x} \]
\[ = A^{-n}(-A^{2n}x^{2n+1} - \sum_{j=0}^{2n} P_j(t)x^jA^{j-1}) \]
\[ = -A^n x^{2n+1} - \sum_{j=0}^{2n} P_j(t)x^jA^{j-1}. \]

Thus,
\[ \dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \] (2.3)

where
\[ H = A^n \left( \frac{1}{2} y^2 + \frac{1}{2(n+1)} x^{2(n+1)} \right) + \sum_{j=0}^{2n} P_j(t)x^jA^{j-1}. \] (2.4)

Let
\[ T^1_s = \{ t \in \mathbb{C}/\mathbb{Z} : |\text{Im} t| < s \} \] for any \( s > 0. \)

Consider an auxiliary Hamiltonian system
\[ \dot{x} = \frac{\partial H_0}{\partial y}, \quad \dot{y} = -\frac{\partial H_0}{\partial x}, \quad H_0 = \frac{1}{2} y^2 + \frac{1}{2(n+1)} x^{2(n+1)}. \] (2.5)

Let \((x_0(t), y_0(t))\) be the solution to (2.5) with initial \((x_0(0), y_0(0)) = (1, 0).\) Then this solution is clearly periodic. Let \(T_0\) be its minimal positive period. By Energy conservation, we have
\[ (n+1)y_0^2(t) + x_0^{2n+2}(t) \equiv 1, \quad t \in \mathbb{R}, \] (2.6)

by which, we construct the following symplectic transformation
\[ \Psi_0 : \begin{cases} x = c^\alpha I^\alpha x_0 (\theta T_0), \\
y = c^\beta I^\beta y_0 (\theta T_0), \end{cases} \]

where \(\alpha = \frac{1}{n+2}, \beta = 1 - \alpha = \frac{n+1}{n+2}, c = \frac{1}{\alpha \beta} \) and where \((I, \theta) \in \mathbb{R}^+ \times T^1_s\) is action-angle variables.

By calculation, \(\frac{\partial (x, y)}{\partial (\theta, I)} = 1.\) Thus the transformation is indeed symplectic. Clearly \(\Psi_0(I, \theta)\) is analytic in \((I, \theta) \in \mathbb{R}^+ \times T^1_{s_0}\) with some constant \(s_0 > 0.\)

Under \(\Psi_0, (2.3)\) is changed
\[ \dot{\theta} = \frac{\partial H}{\partial I}, \quad I = -\frac{\partial H}{\partial \theta}, \] (2.7)

where \(H = H_0(I) + R(I, \theta, t)\) with
\[ H_0(I) = d \cdot A^n \cdot I^{2\beta} = d \cdot A^n \cdot I^{\frac{2(n+1)}{2(n+1)}}, \quad d = \frac{c^2 \beta}{2(n+1)}. \] (2.8)

and
\[ R(I, \theta, t) = \sum_{j=0}^{2n} \frac{P_j(t)}{j+1} \left( c^{\alpha \theta} x_0 (\theta T_0) \right)^{j+1} A^{j-1} I^{\frac{j}{n+2}}. \] (2.9)

Clearly, \(R(I, \theta, t) = O(A^{\alpha - 1})\) for \(A \to \infty\) and fixed \(I\) belongs to some compact intervals.
3. Approximation Lemma

First, we cite an approximation lemma. See \[3\] and \[10\], for the detail. We start by recalling some definitions and setting some new notations. Assume $X$ is a Banach space with the norm $\| \cdot \|_X$. First recall that $C^\mu (\mathbb{R}^n; X)$ for $0 < \mu < 1$ denotes the space of bounded Hölder continuous functions $f : \mathbb{R}^n \mapsto X$ with the norm

$$\|f\|_{C^\mu (\mathbb{R}^n; X)} = \sup_{0 < |x-y| < 1} \frac{\|f(x)-f(y)\|_X}{|x-y|^{\mu}} + \sup_{x \in \mathbb{T}^n} \|f(x)\|_X.$$  

If $\mu = 0$ then $\|f\|_{C^0 (\mathbb{R}^n; X)}$ denotes the sup-norm. For $\ell = k + \mu$ with $k \in \mathbb{N}$ and $0 \leq \mu < 1$, we denote by $C^\ell (\mathbb{R}^n; X)$ the space of functions $f : \mathbb{R}^n \mapsto X$ with Hölder partial derivatives, i.e., $\partial^\alpha f \in C^\mu (\mathbb{R}^n; X_{\alpha})$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ with the assumption that $|\alpha| := |\alpha_1| + \cdots + |\alpha_n| \leq k$ and $X_{\alpha}$ is the Banach space of bounded operators $T : \prod^\alpha (\mathbb{R}^n) \mapsto X$ with the norm

$$\|T\|_{X_{\alpha}} = \sup \{|\langle T(u_1, u_2, \cdots, u_{|\alpha|}) \rangle| : \|u_i\| = 1, 1 \leq i \leq |\alpha|\}.$$ 

We define the norm

$$\|f\|_{C^\ell} = \sup_{|\alpha| \leq \ell} \|\partial^\alpha f\|_{C^\mu (\mathbb{R}^n; X_{\alpha})}.$$ 

**Theorem 3.1.** (Jackson) Let $f \in C^\ell (\mathbb{R}^n; X)$ for some $\ell > 0$ with finite $C^\ell$ norm over $\mathbb{R}^n$. Let $\phi$ be a radical-symmetric, $C^\infty$ function, having as support the closure of the unit ball centered at the origin, where $\phi$ is completely flat and takes value 1, let $K = \hat{\phi}$ be its Fourier transform. For all $\sigma > 0$ define

$$f_\sigma(x) := K_\sigma * f = \frac{1}{\sigma^n} \int_{\mathbb{T}^n} K\left(\frac{x-y}{\sigma}\right)f(y)dy.$$ 

Then there exists a constant $C \geq 1$ depending only on $\ell$ and $n$ such that the following holds: For any $\sigma > 0$, the function $f_\sigma(x)$ is a real-analytic function from $\mathbb{C}^n$ to $X$ such that if $\Delta_\sigma^\alpha$ denotes the $n$-dimensional complex strip of width $\sigma$,

$$\Delta_\sigma^\alpha := \{ x \in \mathbb{C}^n : |\text{Im} x_j| \leq \sigma, 1 \leq j \leq n \},$$ 

then for $\forall \alpha \in \mathbb{N}^n$ with $|\alpha| \leq \ell$ one has

$$\sup_{x \in \Delta_\sigma^\alpha} \|\partial^\alpha f_\sigma(x) - \sum_{|\beta| \leq \ell-|\alpha|} \frac{\partial^\beta f(x)}{\beta!} (\sqrt{-1} \text{Im} x)^\beta\|_{X_\alpha} \leq C\|f\|_{C^\ell} \sigma^{\ell-|\alpha|}, \quad (3.1)$$

and for all $0 \leq s \leq \sigma$,

$$\sup_{x \in \Delta_\sigma^\alpha} \|\partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x)\|_{X_\alpha} \leq C\|f\|_{C^\ell} \sigma^{\ell-|\alpha|}. \quad (3.2)$$

The function $f_\sigma$ preserves periodicity (i.e., if $f$ is $T$-periodic in any of its variable $x_j$, so is $f_\sigma$).
By this theorem, for each \( P_j(t) \in C(T^1) \), \( j = n + 1, \cdots, 2n \), and any \( \varepsilon > 0 \), there is a real analytic function\(^1\) \( P_{j,\varepsilon}(t) \) from \( T^1 \) to \( \mathbb{C} \) such that

\[
\sup_{t \in T^1} |P_{j,\varepsilon}(t) - P_j(t)| \leq C \varepsilon^\gamma \|P_j\|_{C^\gamma},
\]

(3.3)

and

\[
\sup_{t \in T^1} |P_{j,\varepsilon}(t)| \leq C\|P_j\|_{C^\gamma}.
\]

(3.4)

Write

\[
R(I, \theta, t) = R_\varepsilon(I, \theta, t) + R^\varepsilon(I, \theta, t),
\]

(3.5)

where

\[
R_\varepsilon(I, \theta, t) = \sum_{j=n+1}^{2n} \frac{1}{j+1} A^{j-n-1} (-i)^{j+1} x_0^{j+1}(\theta T_0)P_{j,\varepsilon}(t),
\]

\[
R^\varepsilon(I, \theta, t) = \sum_{j=n+1}^{2n} \frac{1}{j+1} A^{j-n-1} (-i)^{j+1} x_0^{j+1}(\theta T_0)P_j(t)
\]

\[+ \sum_{j=n+1}^{2n} \frac{1}{j+1} A^{j-n-1} (-i)^{j+1} x_0^{j+1}(\theta T_0)(P_j(t) - P_{j,\varepsilon}(t)).
\]

(3.6)

(3.7)

Now let us restrict \( I \) belongs to some compact intervals, \([1, 4]\), say. Let

\[
A^{-1} < \varepsilon_0.
\]

For a sufficiently small \( \varepsilon_0 > 0 \), letting

\[
\varepsilon = (\varepsilon_0/A^{n-1})^{1/\gamma},
\]

(3.8)

by Theorem 3.1, we have the following facts:

(i) \( R^\varepsilon(I, \theta, t) \) is real analytic in \( (I, \theta) \in [1, 4] \times T^1_{\theta_0} \) for fixed \( t \in T^1 \) and \( R^\varepsilon(I, \theta, \cdot) \in L^1(T^1) \) for fixed \( (I, \theta) \in [1, 4] \times T^1_{\theta_0} \), and

\[
\sup_{(I, \theta,t) \in [1, 4] \times T^1_{\theta_0} \times T^1} |R^\varepsilon(I, \theta, t)| \leq C\varepsilon_0,
\]

(3.9)

where \( C \) is a constant\(^2\) depending on only \( \|P_j\|_{C^\gamma} \).

(ii) \( R_\varepsilon(I, \theta, t) \) is real analytic in \( (I, \theta, t) \in [1, 4] \times T^1_{\theta_0} \times T^1_\varepsilon \) and

\[
\sup_{(I, \theta,t) \in [1, 4] \times T^1_{\theta_0} \times T^1_\varepsilon} |R_\varepsilon(I, \theta, t)| \leq CA^{n-1},
\]

(3.10)

where \( C \) is a constant depends on only \( \|P_j\|_{C^\gamma} \). Therefore, we have

\[
H(I, \theta, t) = H_0(I) + R_\varepsilon(I, \theta, t) + R^\varepsilon(I, \theta, t).
\]

(3.11)

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\(^1\)A complex value function \( f(t) \) of complex variable \( t \) in some domain in \( \mathbb{C} \) is called real analytic if it is analytic in the domain and is real for real argument \( t \).

\(^2\)Denote by \( C \) a universal constant which may be different in different place.
4. Symplectic transformations

We will look for a series of symplectic transformations \( \Psi_1, \cdots, \Psi_N \) such that

\[
H^{(N)} = H \circ \Psi_1 \circ \cdots \circ \Psi_N = H_0^N + O(t_0),
\]

where \( H_0^N(\mu) \approx \mathcal{A}^n \mu^\frac{2n+1}{n+2} \) and that Moser’s twist works for \( H^{(N)} \).

To this end, let \( \Psi_1 : (\mu, \phi) \mapsto (I, \theta) \) is implicitly defined by

\[
\Psi_1 : \left\{ \begin{array}{l}
I = \mu + \frac{\partial S_1}{\partial \mu} \\
\phi = \theta + \frac{\partial S_1}{\partial \phi}
\end{array} \right.
\]

with \( S_1 = S_1(\mu, \theta, t) \) to be specified later. If \( \Psi_1 \) is well-defined, then it is symplectic, since

\[
dI \wedge d\theta = (1 + \frac{\partial^2 S_1}{\partial \mu \partial \phi}) d\mu \wedge d\theta = d\mu \wedge d\phi.
\]

The transformed Hamiltonian function \( H^{(1)}(\mu, \phi, t) = H \circ \Psi_1(\mu, \phi, t) \). We express temporarily in the variable \((\mu, \theta)\) instead of \((\mu, \phi)\):

\[
H^{(1)}(\mu, \theta, t) = H(\mu + \frac{\partial S_1}{\partial \theta}(\theta, t) + \frac{\partial S_1}{\partial t}(\theta, t), \theta + \frac{\partial S_1}{\partial \phi}(\theta, t) + \frac{\partial S_1}{\partial t}(\theta, t))
\]

By Taylor’s formula and (3.11)

\[
H^{(1)}(\mu, \theta, t) = H_0(\mu) + \frac{\partial S_1}{\partial \theta}(\theta, t) + R_e(\mu, \theta, t) + \frac{\partial S_1}{\partial \phi}(\theta, t) + \frac{\partial S_1}{\partial t}(\theta, t)
\]

where

\[
R_e^1(\mu, \theta, t) = \int_0^1 (1 - \tau) \frac{\partial^2 S_1}{\partial \theta^2} H_0(\mu + \frac{\partial S_1}{\partial \theta}(\tau, \theta, t), \theta + \frac{\partial S_1}{\partial \phi}(\tau, \theta, t)) d\tau
\]

\[+ \int_0^1 \frac{\partial^3 S_1}{\partial \theta^3} H_0(\mu + \frac{\partial S_1}{\partial \theta}(\tau, \theta, t), \theta + \frac{\partial S_1}{\partial \phi}(\tau, \theta, t)) d\tau + \frac{\partial S_1}{\partial t}.
\]

Let

\[
\frac{\partial \mu H_0}{\partial \theta} \frac{\partial S_1}{\partial \theta} + R_e(\mu, \theta, t) = [R_e](\mu, t), \quad [R_e](\mu, t) = \int_0^1 R_e(\mu, \theta, t) d\theta.
\]

Then

\[
H^{(1)}(\mu, \theta, t) = H_0(\mu) + [R_e](\mu, t) + R_e^1(\mu, \theta, t) + \frac{\partial S_1}{\partial \phi}(\theta, t) + \frac{\partial S_1}{\partial t}(\theta, t)
\]

where

\[
H_0^1(\mu, t) = H_0(\mu) + [R_e](\mu, t).
\]

We are now in position to solve (3.4).

\[
S_1(\mu, \theta, t) = \int_0^\theta \frac{[R_e](\mu, t) - R_e(\mu, \theta, t)}{6 \partial \mu H_0(\mu)} d\theta.
\]
Thus, by the implicit function theorem, $\Psi$ and $S_1$, $\theta$, $\epsilon$, and $\phi$ are well-defined in $(\mu, \theta, t) \in [1,4] \times T_{\theta_0} \times T_{\epsilon}$, and analytic in the domain, and

$$\sup_{(\mu, \theta, t) \in [1,4] \times T_{\theta_0} \times T_{\epsilon}} |S_1(\mu, \theta, t)| \leq CA^{-1}. \quad (4.8)$$

Thus, by the implicit function theorem, $\Psi_1(\mu, \phi, \theta) : [1 + O(A^{-1}), 4 - O(A^{-1})] \times T_{\theta_0}^1 \times T_{\epsilon}^1 \rightarrow [1,4] \times T_{\theta_0}^1 \times T_{\epsilon}^1$.

- **Estimate of $H_0^1(\mu, t)$**.

  By (3.10), we have that $H_0^1(\mu, t)$ is analytic in $[1,4] \times T_{\epsilon}$, and

  $$CA^0 \geq |\partial^2_{\mu} H_0^1(\mu, t)| \geq \frac{A^n}{C}, \quad t \in T_{\epsilon/2}, \quad (4.9)$$

  and by Cauchy’s estimate

  $$\sup_{(\mu, \theta, t) \in [1,4] \times T_{\theta_0}^1 \times T_{\epsilon}^1} |\partial \partial^{1/2} H_0^1(\mu, t)| \leq \frac{2CA^{-1}}{\epsilon} \leq C \frac{A^{-1}n}{\epsilon} \leq C \frac{A^{-1}n}{\epsilon} = C \frac{A^{-1}n}{\epsilon} \leq C \frac{A^{-1}(1 + 1/\epsilon)}{\epsilon}. \quad (4.10)$$

- **Estimate of $R_0^1(\mu, \theta, t)$**.

  By (4.8) and the Cauchy estimate,

  $$\sup_{(\mu, \theta, t) \in [1,4] \times T_{\theta_0} \times T_{\epsilon}} |\partial S_1(\mu, \theta, t)| \leq \frac{2CA^{-1}}{\epsilon} \leq C \frac{A^{-1}n}{\epsilon} \leq C \frac{A^{-1}n}{\epsilon} \leq C \frac{A^{-1}n}{\epsilon} \leq C \frac{A^{-1}(1 + 1/\epsilon)}{\epsilon} = C \frac{A^{-1}(1 + 1/\epsilon)}{\epsilon}. \quad (4.11)$$

  where

  $$\sigma := n - \frac{1}{\gamma} = n \left(\gamma - (1 - \frac{1}{n})\right). \quad (4.12)$$

  By assuming $1 \geq \gamma > 1 - \frac{1}{n}$,

  $$0 < \sigma \leq 1. \quad (4.12)$$

  By (3.10) and noting $H_0(\mu) = dA^{-1} - \frac{2n+2}{\mu + n + 2}$, we have

  $$\sup_{(\mu, \theta, t) \in \Psi_1} |R_0^1(\mu, \theta, t)| \leq CA^0A^{-2} + C \frac{A^{-1}n}{\epsilon} \leq \frac{CA^0}{\gamma} A^{-1 + \sigma}, \quad (4.13)$$
where

\[ \mathcal{D}_1 = [1 + O(A^{-1}), 4 - O(A^{-1})] \times \mathbb{T}^{1}_{\varepsilon/2} \times \mathbb{T}^{1}_{\varepsilon/2}. \]

By (4.8) and the implicit function theorem, there exist \( U_1(\mu, \phi, t) \), \( V_1(\mu, \phi, t) \) analytic in \( \mathcal{D}_1 \) such that

\[ \sup_{\mathcal{D}_1} |U_1| \leq CA^{-1}, \hspace{1em} \sup_{\mathcal{D}_1} |V_1| \leq CA^{-1}, \]  

(4.14)

and

\[ \Psi_1 : \begin{cases} \lambda = \mu + U_1(\mu, \phi, t) \\ \phi = \phi + V_1(\mu, \phi, t) \end{cases} \]  

(4.15)

By (4.8) and the implicit function theorem, there exist \( U \) and \( \lambda \) such that

\[ H^1(\mu, \phi, t) = H^1_0(\mu, t) + R^1_0(\mu, \phi, t) + R^c \circ \Psi(\mu, \phi, t), \]  

(4.16)

where

\[ \tilde{R}^1_0(\mu, \phi, t) = R^1_0(\mu, \phi + V_1(\mu, \phi, t), t) \]  

(4.17)

and

\[ \sup_{\mathcal{D}_1} |\tilde{R}^1_0(\mu, \phi, t)| \leq C \varepsilon_0^{-\frac{1}{7}} A^{n-1-\sigma}. \]  

(4.18)

Similarly, let

\[ \Psi_2 : \begin{cases} \lambda = \lambda + \frac{\partial S_2}{\partial \phi} \\ \phi = \phi + \frac{\partial S_2}{\partial \lambda} \end{cases} \]  

(4.19)

with \( S_2 = S_2(\lambda, \phi, t) \) is defined by

\[ S_2(\lambda, \phi, t) = \int_0^1 \frac{\phi}{\partial \lambda} \frac{\tilde{R}^1_0(\lambda, t) - \tilde{R}^1_0(\lambda, \phi, t)}{\partial \mu H^1_0(\mu, t)} \, dt, \hspace{1em} [\tilde{R}^1_0](\lambda, t) = \int_0^1 \tilde{R}^1_0(\lambda, \phi, t) \, d\phi. \]  

(4.20)

By (4.9) and (4.13),

\[ \sup_{\mathcal{D}_2} |S_2(\lambda, \phi, t)| \leq CA^{-n}(\frac{1}{\varepsilon_0})^{\frac{1}{7}} A^{n-1-\sigma} \leq C(\frac{1}{\varepsilon_0})^{\frac{1}{7}} A^{n-1-\sigma}. \]  

(4.21)

It follows from the implicit function theorem that \( \Psi_2 : (\lambda, \tilde{\phi}) \in \mathcal{D}_2 \to \mathcal{D}_1 \) is well-defined, where \( \mathcal{D}_2 = [1 + O(A^{-1}), 4 - O(A^{-1})] \times \mathbb{T}_{\varepsilon/4} \times \mathbb{T}_{\varepsilon/4} \). By Cauchy estimate,

\[ \sup_{\mathcal{D}_2} |\partial S_2(\lambda, \phi, t)| \leq C \varepsilon A^{-\sigma}(\frac{1}{\varepsilon_0})^{\frac{1}{7}} \leq CA^{-\sigma-1}(\frac{1}{\varepsilon_0})^{\frac{1}{7}} A^{n-\frac{6}{7}} \leq C(\frac{1}{\varepsilon_0})^{\frac{1}{7}} A^{n-2\sigma-1}. \]  

(4.22)

Let

\[ H^{(2)}(\lambda, \phi, t) := H^1(\lambda + \frac{\partial S_2}{\partial \phi}, \phi, t) + \frac{\partial S_2}{\partial t} \]

\[ = H^1_0(\lambda, t) + \partial_\lambda H^1_0(\lambda, t) \frac{\partial S_2}{\partial \phi} + R^1_0(\lambda, \phi, t) + R^c(\lambda, \phi, t) \]

\[ + R^c \circ \Psi_1 \circ \Psi_2(\lambda, \tilde{\phi}, t), \]  

(4.23)
where

\[ R^2_e(\lambda, \phi, t) = \int_0^1 (1 - \tau) \partial_2^2 H_0^e(\lambda + \partial S_2 \tau, \phi, t) \left( \frac{\partial S_2}{\partial \phi} \right)^2 d\tau \]

\[ + \int_0^1 \partial_1 R^1_e(\lambda + \partial S_2 \phi, \phi, t) \frac{\partial S_2}{\partial \phi} d\tau + \frac{\partial S_2}{\partial t} \]  \quad (4.24)

By (4.20),

\[ \partial_1 H_0^e(\lambda, \phi, t) \frac{\partial S_2}{\partial \phi} + \bar{R}^1_e(\lambda, \phi, t) = [\bar{R}^1_e](\lambda, t). \]

Let

\[ H^2_0(\lambda, t) = H_0^e(\lambda, t) + [\bar{R}^1_e](\lambda, t). \]  \quad (4.25)

It follows that

\[ H^2(\lambda, \phi, t) = H^2_0(\lambda, t) + R^2_e(\lambda, \phi, t) + R^e \circ \Psi_1 \circ \Psi_2(\lambda, \phi, t). \]  \quad (4.26)

- **Estimate of** \( H^2_0(\lambda, t). \)
  By (4.9), (4.10) and (4.13),

\[ CA^n \geq |\partial_1 H_0^2(\lambda, t)| \geq \frac{A^n}{C}, \quad \lambda \in [1, 4], \quad t \in T_{\varepsilon/2}. \] \quad (4.27)

\[ \sup_{(\lambda, t) \in [1, 4] \times T_{\varepsilon/4}} |\partial_1 H_0^2(\lambda, t)| \leq C \varepsilon_0 \frac{\lambda}{A} A^{(n-1)(1+\frac{1}{n})}. \] \quad (4.28)

- **Estimate of** \( R^2_e(\lambda, \phi, t). \)
  By (4.13), (4.21), (4.22) and (4.23), (4.24).

\[ \sup_{\lambda, \phi} |R^2_e(\lambda, \phi, t)| \leq C A^n \left( \frac{1}{\varepsilon_{01}^2} \right)^{2/\gamma} A^{1+\sigma} A^{1-\sigma} + C \left( \frac{1}{\varepsilon_{01}^3} \right)^{2/\gamma} A^{1+2\sigma} A^{1-2\sigma} \]  \quad (4.29)

Take \( N \in \mathbb{N} \) with \( n - \sigma N \leq -1 \). Repeating the above procedure \( N \) times, we get a series of symplectic transformations \( \Psi_1, \cdots, \Psi_N \) such that

\[ H^N(\rho, \xi, t) = H \circ \Psi_1 \circ \cdots \circ \Psi_N \]

\[ = H^0(\rho, \xi, t) + R^N_e(\rho, \xi, t) + R^e \circ \Psi_1 \circ \Psi_2(\rho, \xi, t), \]

where \((\rho, \xi, t) \in [1 + O(A^{-1}), 4 - O(A^{-1})] \times T_{\varepsilon/2N}^1 \times T_{\varepsilon/2N}^1 \times T_{\varepsilon/2N}^1 \times T_{\varepsilon/2N}^1 \times T_{\varepsilon/2N}^1 \), and

\[ \Phi \triangleq \Psi_1 \circ \cdots \circ \Psi_N : [1 + O(A^{-1}), 4 - O(A^{-1})] \times T_1 \times T_1 \rightarrow [1, 4] \times T_1 \times T_1, \]  \quad (4.30)
\[ \Phi = id + O(A^{-1}). \]  

and \( H_N^0(p, t) \) satisfies

\[ CA^n \geq |\partial^2 H_N^0(p, t)| \geq \frac{A^n}{r}, \quad \rho \in [2, 3], \quad t \in T, \]  

\[ \sup_{(p, t) \in [2, 3] \times T} |\partial^2 H_N^0(p, t)| \leq C \varepsilon_0^{-\frac{1}{N'}} A^{(a-1)(1+\frac{1}{N'})} \]  

and \( R_N^0(p, \xi, t) \) satisfies that for \( 0 \leq p + q \leq 6 \).

\[ \sup_{(p, \xi, t) \in [2, 3] \times T} \int_0^1 |\partial_p^p \partial_\xi^q R_N^0(p, \xi, t)| dt \leq CA^\gamma (1) N' \]  

\[ \leq CA^{-1} \left( \frac{1}{\varepsilon_0} \right)^{N'/N} \]  

\[ \leq C \varepsilon_0 \]  

where \( C \) depends on \( N \) and we have assumed that \( A \) is large enough such that

\[ A^{-1} \left( \frac{1}{\varepsilon_0} \right)^{N'/N} < \varepsilon_0. \]

Let

\[ \mathcal{H}(p, \xi, t) = R_N^0(p, \xi, t) + R \circ \Psi^1 \circ \cdots \circ \Psi^N. \]  

Then by (3.9), (4.34), (4.30) and (4.31)

\[ \sup_{(p, \xi, t) \in [2, 3] \times T} \int_0^1 |\partial_p^p \partial_\xi^q \mathcal{H}(p, \xi, t)| dt \leq C \varepsilon_0, \quad 0 \leq p + q \leq 6. \]

Now,

\[ H_N(p, \xi, t) = H_N^0(p, t) + \mathcal{H}(p, \xi, t). \]  

5. Proof of theorem

For \( H_N^0 \) the Hamiltonian equation is

\[
\begin{align*}
\dot{\rho} &= \frac{\partial H_N^0}{\partial \rho}, \\
\dot{\xi} &= \frac{\partial H_N^0}{\partial \xi} = \frac{\partial \mathcal{H}(\rho, \xi)}{\partial \rho} + \frac{\partial \mathcal{H}(\rho, \xi)}{\partial \xi} + O(\varepsilon_0),
\end{align*}
\]  

\[ (5.1) \]

Note

\[ H_N^0 = d \cdot A^n \cdot \rho^2 + O(A^{n-1}). \]

By using Picard iteration and Gronwall’s inequality and noting (4.36), we get that the time-1 map of (5.1) is of the form

\[
\mathcal{H}: \begin{cases} 
\rho_1 = \rho(t)|_{t=1} = \rho_0 + F(\rho_0, \xi_0), \\
\xi_1 = \xi(t)|_{t=1} = \xi_0 + \alpha(\rho_0) + G(\rho_0, \xi_0), \quad (\rho_0, \xi_0) \in [2, 3] \times T^1
\end{cases}
\]
with
\[ \alpha(\rho_0) = \int_0^1 \frac{\partial H_0^N}{\partial \rho} (\rho_0, t) dt, \quad |\partial \rho_0 \alpha(\rho_0)| \geq CA^n > 0, \]
and
\[ |\partial^p \rho \partial^q \xi F| \leq Ce_0, \quad |\partial^p \rho \partial^q \xi G| \leq Ce_0, \quad p + q \leq 5. \]

Since (5.1) is Hamiltonian, the map \( P \) is symplectic. By Moser’s twist theorem at pp.50-54 of [7], \( \mathcal{P} \) has an invariant curve \( \Gamma \) in the annulus \([2, 3] \times T^1\). Since \( A \) can be arbitrarily large, it follows that the time-1 map of the original system has an invariant curve \( \Gamma_A \) in the annulus \([2A + C, 3A - C] \times T^1\) with \( C \) is a constant independent of \( A \). Choosing a sequence \( A = A_k \to \infty \) as \( k \to \infty \), we have that there are countable many invariant curves \( \Gamma_{A_k} \), clustering at \( \infty \). Therefore any solution of the original system is bounded. This completes the proof of Theorem.

**Remark 2.** Any solutions starting from the invariant curves \( \Gamma_{A_k} (k = 1, 2, \ldots) \) are quasi-periodic with frequencies \((1, \omega_k) \) in time \( t \), where \((1, \omega_k) \) satisfies Diophantine conditions and \( \omega > CA^k \).

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