ASYMPTOTIC JOINT DISTRIBUTION OF THE EXTREMITIES OF A RANDOM YOUNG DIAGRAM AND ENUMERATION OF GRAPHICAL PARTITIONS

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Abstract. An integer partition of \( n \) is a decreasing sequence of positive integers that add up to \( [n] \). Back in 1979 Macdonald posed a question about the limit value of the probability that two partitions chosen uniformly at random, and independently of each other, are comparable in terms of the dominance order. In 1982 Wilf conjectured that the uniformly random partition is a size-ordered degree sequence of a simple graph with the limit probability 0. In 1997 we showed that in both, seemingly unrelated, cases the limit probabilities are indeed zero, but our method left open the problem of convergence rates. The main result in this paper is that each of the probabilities is \( e^{-0.11 \log n / \log \log n} \), at most. A key element of the argument is a local limit theorem, with convergence rate, for the joint distribution of the \( [n]^{1/4-\epsilon} \) tallest columns and the \( [n]^{1/4-\epsilon} \) longest rows of the Young diagram representing the random partition.

1. Introduction and main results

A weakly decreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_m) \), \( m = m(\lambda) \geq 1 \), of positive integers is called a partition of a positive integer \( n \) into \( m \) parts if \( \lambda_1 + \cdots + \lambda_m = n \). We will denote the set of all such partitions \( \lambda \) by \( \Omega_n \). It is customary to visualize a partition \( \lambda \) as a (Young-Ferrers) diagram formed by \( n \) unit squares, with the columns of decreasing heights \( \lambda_1, \ldots, \lambda_m \). We will use the same letter \( \lambda \) for the diagram representing the partition \( \lambda \). Introduce the positive integers

\[
\lambda'_i = \left| \{1 \leq j \leq m(\lambda) : \lambda_j \geq i \} \right|, \quad 1 \leq i \leq \lambda_1;
\]

so \( \lambda'_i \) is the number of parts in the partition \( \lambda \) that are \( i \), at least. Clearly \( \lambda'_i \) decrease and add up to \( n \); so \( \lambda' := (\lambda'_1, \ldots, \lambda'_m') \), \( (m' = \lambda_1) \), is a partition of \([n]\), usually referred to as the dual to \( \lambda \).

The dominance order on the set \( \Omega_n \) is a partial order \( \preceq \) defined as follows. For \( \lambda, \mu \in \Omega_n \), we write \( \lambda \preceq \mu \) if

\[
\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j, \quad i \geq 1;
\]
(by definition $\lambda_j = 0$ for $i > m(\lambda)$, $\mu_j = 0$ for $j > m(\mu)$). Under \( \preceq \), \( \Omega_n \) is a lattice. Brylawski [4] demonstrated how ubiquitous this lattice is. For instance, Gale-Ryser theorem (Gale [10] and Ryser [18], Brualdi and Ryser [3]) asserts: given two decreasing positive tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$, there exists a bipartite graph on a vertex set $\{X, Y\}$, $|X| = r$, $|Y| = s$, such $\alpha$ and $\beta$ are the size-ordered degree sequences of vertices in $X$ and $Y$ respectively, iff \( \sum_{t=1}^r \alpha_t = \sum_{t=1}^s \beta_t \) and $\alpha \preceq \beta'$. The lattice $\Omega_n$ is also at the core of the classic description of the irreducible representations of the symmetric group $S_n$, see Diaconis [5], Macdonald [13], Sagan [20], for instance.

Soon after [10], [18], Erdős and Gallai [6] found the necessary and sufficient conditions a partition $\lambda \in \Omega_n$, ($n$ even), has to satisfy to be graphical, i.e. to be a size-ordered degree sequence of a simple graph. According to Nash-Williams (see Sierksma and Hoogeveen [21] for the proof), the Erdős-Gallai conditions are equivalent to

\[ \sum_{j=1}^i \lambda_j' \geq \sum_{j=1}^i \lambda_j + i, \quad 1 \leq i \leq D(\lambda); \]  

here $D(\lambda)$ is the size of the Durfee square of $\lambda$, i.e. the number of rows of the largest square inscribed into the Young diagram $\lambda$. Obvious differences notwithstanding, the Gale-Ryser conditions and the Nash-Williams conditions are undeniably similar.

Macdonald [13] (Ch.1, Section 1, Example 18) posed a probabilistic question, which may be formally interpreted as follows. Let $\lambda, \mu \in \Omega_n$ be chosen uniformly at random and independently of each other; does $P(\lambda \preceq \mu)$ approach 0 as $n \to \infty$? This question had already been there in the 1979 edition of [13]. In 1982 Wilf conjectured that $\lim P(\lambda$ is graphical) = 0; apparently he wasn’t aware of Macdonald’s question.

In an attempt to prove Wilf’s conjecture, Erdős and Richmond [8] found an expression for the limiting probability that $\lambda$ satisfies the first $k$ conditions (2) as a $2k$-dimensional integral, thus reducing the problem to the question whether the integral’s value $c_k \to 0$ as $k \to \infty$. The authors also showed that $P_n := P(\lambda$ is graphical) is of order $n^{-1/2}$ at least, meaning that if $P_n \to 0$, it does so rather slowly. Rousseau and Ali [19] demonstrated that $\lim_{k \to \infty} c_k \leq 1/4$ and $c_k \geq 2^{-2k}(\frac{2k}{k})$; consequently $\lim \sup P_n \leq 1/4$, and $c_k$ cannot approach 0 faster than $k^{-1/2}$, if indeed $c_k \to 0$.

Barnes and Savage [2] discovered a recurrence-based algorithm for computing the total number of graphical partitions of $n$. They demonstrated that for $n$ ranging from 2 to 220 the fraction (probability) of graphical partitions steadily, but slowly, decreases from 0.5 to 0.3503 . . . , which is still above the Rousseau-Ali limiting bound 0.25. More recently, Kohnert [12] derived a new recursion formula that allowed to compute the fraction of graphical partitions for $n$ up to 910. For $n = 910$, the fraction is 0.3264 . . . ,
still noticeably exceeding 0.25. Thus, as \( n \) runs from 220 to 910, the fraction decreases by 0.025 only.

In [15] we proved the positive answer to Macdonald’s question. The idea of the proof, certainly inspired by [8], [19], was to show that

\[
\lim_{k \to \infty} \limsup_{n \to \infty} P(\lambda, \mu \text{ meet the first } k \text{ conditions in } (1)) = 0.
\]

A key tool was a theorem on the limiting joint distribution of the \( k \) largest parts of the random \( \lambda \) due to Fristedt [9]. We also confirmed Wilf’s conjecture by proving that, as Erdős and Richmond expected, \( \lim c_k = 0 \). We did so via a slight modification of the proof for Macdonald’s question. The proofs similarity is due to an implicit discovery in [8] that the limiting joint distribution of the \( k \) largest parts in \( \lambda \) and in its dual \( \lambda' \) is the same as that of the largest \( k \) parts in two independent partitions of \( n \).

In both cases, we found a way to use Kolmogorov’s 0−1 law for the tail events of a sequence of independent random variables to show that the limiting probability in question cannot be anything but 0 or 1. And then we used a central limit theorem to rule out the value 1. So our solution left open a fundamental question about the actual convergence rates for both Macdonald’s and Wilf’s probabilities.

Our main result in this paper is that, for \( n \) sufficiently large, each of those probabilities is

\[
\exp \left( -\frac{0.11 \log n}{\log \log n} \right),
\]

at most. Thus, the bound (3) is negligible compared to any negative power of \( \log n \), but it approaches 0 slower than any \( n^{-a} \), \( a > 0 \) being fixed.

We compared the values of this bound and the exact values of the fraction of the graphical partitions for \( n = 250, 450, 910 \), computed in [12]. For what it’s worth, replacing 0.11 with \( \approx 0.326, \approx 0.321, \approx 0.315 \), we get the actual numerical values of the fraction. Could it be that the expression (3) with a constant close to 0.25 replacing 0.11 is an asymptotic formula for the fraction?

As a direct byproduct of our proofs, (3) also bounds \( P(\lambda \preceq \lambda') \), i.e. the probability that the random partition \( \lambda \) is both an in-degree sequence and an out-degree sequence, each being size-ordered, of a directed graph.

As the first step, we prove that, for \( k = \lceil n^{1/2} \rceil \) and \( \gamma < 1/4 \), the total variation distance of the joint distribution of the \( k \) tallest columns and the \( k \) longest rows of the random diagram \( \lambda \) from the distribution of the random tuple

\[
\left( \left\lfloor \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{c} \sum_{j=1}^{k} E_j \right\rfloor, \left\lfloor \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{c} \sum_{j=1}^{k} E'_j \right\rfloor \right)_{1 \leq i \leq k}, \quad c := \frac{\pi}{\sqrt{6}}.
\]

is at most \( n^{-1/2+2\gamma}(\log n)^3 \). Here \( E_1, \ldots, E_k, E'_1, \ldots, E'_k \) are independent copies of \( E \), with \( P(E > x) = e^{-x} \).
We note that Fristedt [9] proved a convergence theorem—in terms of Prohorov distance, without an explicit convergence rate—for the $k = o(n^{1/4})$ tallest columns, or, by symmetry, for the $k$ longest rows, but not for the joint distribution. In [15] we already observed that Fristedt’s limit theorem can be reformulated in the form (4). It was this observation that led us to the argument based on Kolmogorov’s 0 – 1 law.

To show the convergence in terms of total variation distance we use saddle-point techniques for analysis of Cauchy integrals representing the counts of restricted partitions, which involve the generating function of restricted partitions and Freiman’s estimate of the Euler generating function of unrestricted partitions.

Using this approximation theorem we reduce, asymptotically, Nash-Williams conditions (2) to their counterpart involving the sums

$$S_i = \sum_{j=1}^{i} E_j, \quad S'_i = \sum_{j=1}^{i} E'_j,$$

A series of Chernoff-type bounds allows us essentially to embed the resulting event into an intersection of $\log \log k$ independent events, each of probability $\approx P(N \geq -1/2)$, $N$ being the standard normal variable. This yields the bound for Wilf’s $P(\lambda$ is graphical), and also for $P(\lambda \preceq \lambda')$. As for the bound of Macdonald’s $P(\lambda \preceq \mu)$, its proof is the simplified version of that for Wilf’s probability, since $\lambda$ and $\mu$ are exactly independent.

2. **Joint distribution of the $k$ largest heights and $k$ largest widths of the random diagram**

A partition of $n$ is visualized as the diagram of area $n$ with column heights decreasing from left to right. Let $p_n$ denote the total number of partitions of $n$, or equivalently the Young diagrams of area $n$, i.e. $p_n = |\Omega_n|$. It is well known since Euler that

$$p(q) := \sum_{n \geq 1} q^n p_n = \prod_{j \geq 1} (1 - q^j)^{-1}, \quad |q| < 1. \quad (5)$$

Let $p_{n,r,s}$ denote the total number of the (restricted) diagrams, those with the tallest column of height $\leq r$ and the longest (base) row of length $\leq s$. It is also known that

$$p_{r,s}(q) := \sum_{n \geq 1} q^n p_{n,r,s} = \frac{\prod_{j=1}^{r+s} (1 - q^j)}{\prod_{j=1}^{r} (1 - q^j) \prod_{k=1}^{s} (1 - q^k)} \quad \text{for } q \neq 0,$$

$$= p(q) \cdot \prod_{j>r} (1 - q^j) \cdot \prod_{k>s} (1 - q^k) \cdot \prod_{i>r+s} (1 - q^i)^{-1}, \quad (6)$$
see Andrews [4], Section 3.2. Hardy and Ramanujan [11] used the Euler formula to find a series type formula for \( p(n) \), whose simple corollary delivers

\[
p(n) = \frac{e\pi\sqrt{2n/3}}{4\sqrt{3n}} \left(1 + O(n^{-1/2})\right).
\]

This result can be obtained in a short way via a remarkably simple formula due to Freiman (see Postnikov [17]):

\[
\prod_{k \geq 1} (1 - e^{-ku})^{-1} = \exp \left( \frac{\pi^2}{6u} + \frac{1}{2} \log \frac{u}{2\pi} + O(|u|) \right),
\]

uniformly for \( u \to 0 \) within a wedge \( \{ u : \Im u \leq \varepsilon \Re u, \Re u > 0 \} \), \( \varepsilon > 0 \) being fixed, and \( \log \) standing for the main branch of the logarithmic function. (Freiman used (8) to obtain a weaker version of (7) with the remainder term \( O\left(n^{-1/4+\varepsilon}\right) \).) Our aim is a sharp asymptotic formula for \( p_{n,r,s} \) with \( r, s \) of order \( n^{1/2+\varepsilon} \).

As a warm-up preparation, let us derive (7).

**Lemma 2.1.** Let \( q = re^{i\theta}, \ 0 < r < 1, \ \theta \in (-\pi, \pi] \). Then

\[|p(q)| \leq p(r) \exp \left( -\frac{\alpha r\theta^2}{(1-r)((1-r)^2 + 2\alpha r\theta^2)} \right), \ \alpha := \frac{2}{\pi^2}.
\]

**Proof.** Using an inequality

\[\left| \frac{1}{1 - z} \right| \leq \frac{1}{1 - |z|} \exp(\Re z - |z|), \ (|z| < 1),\]

see [13], we obtain

\[|p(re^{i\theta})| \leq p(r) \exp \left( \sum_{j \geq 1} r^j (\cos(\theta j) - 1) \right).
\]

Here

\[
\sum_{j \geq 1} r^j (\cos(\theta j) - 1) = -(1 - r)^{-1} + \Re(1 - r^{i\theta})^{-1}
\]

\[
= -\frac{r}{1-r} + \Re \left( \sum_{j \geq 1} (re^{i\theta})^j \right) = -\frac{r}{1-r} + \Re \left( \frac{re^{i\theta} - 1}{1 - re^{i\theta}} \right)
\]

\[
= -\frac{1 + r}{1-r} \cdot \frac{r(1 - \cos \theta)}{(1-r)^2 + 2r(1 - \cos \theta)} \leq \frac{1 + r}{1-r} \cdot \frac{\alpha \theta^2}{(1-r)^2 + 2\alpha \theta^2},
\]

since \( 1 - \cos \theta \geq \alpha \theta^2 \) for \( |\theta| \leq \pi \). \( \square \)
This Lemma and Freiman’s formula yield (7) with a relatively little effort. First of all, by Cauchy’s integral formula,

\[ p_n = (2\pi i)^{-1} \oint_{z=\rho e^{i\theta}} z^{-(n+1)} p(z) \, dz. \]  

Predictably, we want to choose \( \rho \) close to the root of \( (\rho^n p(\rho))^' = 0 \), or setting \( \rho = e^{-\xi} \),

\[ \sum_{j \geq 1} \frac{j}{e^{j\xi} - 1} = n, \]

implying that

\[ \xi^{-2} \left[ \int_0^\infty \frac{y \, dy}{e^y - 1} + O(\xi) \right] = n. \]

Since the integral equals \( \sum_{j \geq 1} 1/j^2 = \pi^2/6 \), we select \( \xi = cn^{-1/2}, c = \frac{\pi}{\sqrt{6}} \). Break \([-\pi, \pi]\) in two parts, \([-n^{-\delta}, n^{-\delta}] \) and \([-n^{-\delta}, n^{-\delta}]^c \), where \( \delta \in (2/3, 3/4) \). Since \( \delta > 1/2 \), by Lemma 2.1 and Freiman’s formula,

\[ \left| \int_{|\theta| \geq n^{-\delta}} z^{-(n+1)} p(z) \, dz \right| \leq \rho^{-n}p(\rho) \int_{|\theta| \geq n^{-\delta}} e^{-\alpha_1 n^{3/2-2\delta}} \, d\theta \]

\[ = \exp \left( bn^{1/2} - \alpha_2 n^{3/2-2\delta} \right), \]

\( \alpha_j > 0 \) being absolute constants. Consider \( |\theta| \leq n^{-\delta} \). Since \( \delta > 1/2 \), we have \( |\theta| = o(\xi) \). So we apply Freiman’s formula for \( u = \xi - i\theta \) and, using \( \xi^2 = n^{-1} \pi^2/6 \), easily obtain

\[ \log \left( \frac{p(\rho e^{i\theta})}{(\rho e^{i\theta})^n} \right) = bn^{1/2} + \frac{1}{2} \log \frac{c}{2\pi \sqrt{n}} - \frac{1}{2\xi} i\theta - \theta^2 n^{3/2} \gamma_n \]

\[ + i\theta^3 n^2 \gamma'_n + O(\theta^4 n^{5/2} + n^{-1/2}), \]

where \( O(n^{-1/2}) \) comes from \( O(|u|) \) in Freiman’s formula, and

\[ \gamma_n = c^{-1} (1 + O(n^{-1/2})), \quad \gamma'_n = O(1). \]

(To be sure, there is also a term \( O(n^{\theta^2}) \) in (11), but it is absorbed by the big Oh-term already there.) For \( |\theta| \leq n^{-\delta} \),

\[ |\theta|^3 n^2 \leq n^{2-3\delta} \to 0, \quad \theta^4 n^{5/2} \leq n^{5/2-4\delta} \to 0, \]

as \( \delta > 2/3 \). Therefore

\[ \exp \left( -\frac{1}{2\xi} i\theta + i\theta^3 n^2 \gamma'_n + O(\theta^4 n^{5/2} + n^{-1/2}) \right) \]

\[ = 1 - \frac{1}{2\xi} i\theta + i\theta^3 n^2 \gamma'_n + O(\theta^4 n^{5/2} + n^{-1/2}). \]
Consequently, as $dz = i\,dz\,d\theta$,

$$(12) \quad \oint_{|\theta| \leq n^{-\delta}} z^{-(n+1)} p(z) \, dz = i \exp \left( bn^{1/2} + \frac{1}{2} \log \frac{c}{2\pi \sqrt{n}} \right) \times \int_{|\theta| \leq n^{-\delta}} \exp \left( -\theta^2 n^{3/2} \gamma_n \right) \left( 1 - \frac{1}{2\xi} i\theta + i\theta^3 n^2 \gamma_n' + O\left( \theta^4 n^{5/2} + n^{-1/2} \right) \right) \, d\theta.$$

Here

$$\int_{|\theta| \leq n^{-\delta}} e^{-\theta^2 n^{3/2} \gamma_n} \, d\theta = \left( \pi c/n^{3/2} \right)^{1/2} \left( 1 + o(n^{-1/2}) \right),$$

$$\int_{|\theta| \leq n^{-\delta}} e^{-\theta^2 n^{3/2} \gamma_n} \left( -\frac{1}{2\xi} i\theta + i\theta^3 n^2 \gamma_n' \right) \, d\theta = 0,$$

$$\int_{|\theta| \leq n^{-\delta}} e^{-\theta^2 n^{3/2} \gamma_n} \left( \theta^4 n^{5/2} + n^{-1/2} \right) \, d\theta = O\left( n^{-3/4} n^{-1/2} \right).$$

Using these equations together with (9), (10) and (12) we finish the proof of (7).

Let us show how to modify the argument above to obtain an asymptotic formula for $p_{n,r,s}$ for $r, s \gg n^{1/2} c \log n$.

**Lemma 2.2.** Let $h = h(n) > 0$, $w = w(n) > 0$ be such that $h, w = O(n^{\beta})$, $\beta < 1/4$. If $r$ and $s$ are the integer parts of

$$\frac{n^{1/2}}{c} \log \frac{n^{1/2}}{h}, \quad \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{w}$$

respectively, then

$$p_{n,r,s} = \frac{e^{\pi \sqrt{2n/3}}}{4\sqrt{3n}} e^{-h+w} \left( 1 + O\left( n^{-1/2} (h + w + 1)^2 \right) \right).$$

**Proof.** The number $p_{n,r,s}$ is given by the Cauchy integral formula, like (9), with $p_{r,s}(q)$ instead of $p(q)$. We choose again the circle of radius $\rho = e^{-\xi}$, $\xi = c/n^{1/2}$. On this circle, the product $\prod_{i>r+s} (1-q^i)^{-1}$ in the formula (6) for $p_{r,s}(pe^{i\theta})$ is (uniformly) $\exp(O(\rho^{r+s}/(1-\rho)))$, and

$$\frac{\rho^{r+s}}{1-\rho} \sim \exp \left( -\log \frac{n}{\sqrt{c h w}} \right) = O\left( n^{-1/2} h w \right) = O\left( n^{-1/2} (h^2 + w^2) \right),$$

so

$$(13) \quad \prod_{i>r+s} (1-q^i)^{-1} = 1 + O\left( n^{-1/2} (h^2 + w^2) \right).$$
As for two other products in \((14)\),
\[
\prod_{j>r}(1-q^j) \prod_{k>s}(1-q^k) = \exp\left[-\frac{q^r + q^s}{1 - q} + O\left(\rho^r + \rho^s + \frac{\rho^{2r} + \rho^{2s}}{1 - \rho^r}\right)\right]
\]
\[
= \exp\left(-\frac{q^r + q^s}{1 - q} + O\left(n^{-1/2}(h^2 + w^2)\right)\right).
\]
We use \((13)\) and \((14)\) to bound the contribution of the set \(\{\theta : |\theta| \geq n^{-\delta}\}\) to the contour integral. To this end, evaluate first
\[
\text{Re} \left(\frac{e^{i\theta}}{1-q} - \frac{1}{1 - \rho}\right) = \frac{1}{(1 - 2\rho\cos \theta + \rho^2)(1 - \rho)} \left[(1 - \rho)(\cos \theta r - \cos((r - 1)\theta)) + (1 - \rho)^2(\cos((r - 1)\theta) - 1) + 2\rho(\cos \theta - 1)\right].
\]
Using \(|\sin x| \leq |x|\) and \(\rho = e^{-c/n^{1/2}}\), the expression within the square brackets is of order
\[
\theta^2(n^{-1/2}r + n^{-1}r^2 + 1) = O(\theta^2 \log^2 n),
\]
and the denominator exceeds a constant factor times \(n^{-1/2}(n^{-1} + \theta^2)\); so
\[
\text{Re} \left(\frac{e^{i\theta}}{1-q} - \frac{1}{1 - \rho}\right) = O\left(\frac{\theta^2 \log^2 n}{n^{-1/2}(n^{-1} + \theta^2)}\right).
\]
Consequently, as \(\rho^r \sim \frac{h}{n^{1/4}}\) and \(h = O(n^\beta)\),
\[
(15) \quad \text{Re} \left(\frac{q^r}{1-q} - \frac{\rho^r}{1 - \rho}\right) = O\left(\frac{\theta^2 n^{1+\beta} \log^2 n}{n^{-1} + \theta^2}\right),
\]
and the analogous estimate holds for \(\frac{q^s}{1-q}\).

Let
\[
|\theta| \geq n^{-\delta}, \quad \delta \in (\max\{2/3, 1/2 + \beta\}, 3/4).
\]
The remainder term in \((15)\) is \(O(\theta^2 n^{1+\beta} \log^2 n)\). So using also \((13)\), \((14)\), and then Lemma 2.1 we obtain
\[
|p_{r,s}(\rho e^{i\theta})| \leq 2|p(\rho e^{i\theta})| \exp\left[\frac{-\rho^r + \rho^s}{1 - \rho} + O\left(\theta^2 n^{1+\beta} \log^2 n\right)\right]
\]
\[
\leq 2p(\rho) \exp\left(\frac{-\rho^r + \rho^s}{1 - \rho} - \alpha n^{3/2-2\delta}\right),
\]
as \(\beta < 1/2\). Here
\[
\frac{\rho^r + \rho^s}{1 - \rho} = h + w + O\left(n^{-1/2}(h + w)\right) = h + w + O\left(n^{-1/2+\beta}\right).
\]
So, as \(-1/2 + \beta < 3/2 - 2\delta\),

\[
\left| \int_{|\theta| \geq n^{-\delta}} \frac{p_{r,s}(z)}{z^{n+1}} \, dz \right| \leq 4\pi \rho^{-n} \rho(\rho) \exp\left(-\rho^r + \rho^s - \alpha_3 n^{3/2 - 2\delta}\right)
\]

\[
\leq \exp\left( \beta n^{1/2} - h - w - \alpha_4 n^{3/2 - 2\delta} \right).
\]

(16)

Let \(|\theta| \leq n^{-\delta}\). We need a sharp estimate for \(\frac{q^r + q^s}{1 - q}\) in (14). An easy computation shows that, for \(q = e^{i\theta}\),

\[
\frac{q^r}{1 - q} = \frac{\rho^r}{1 - \rho} \left[ 1 + i\theta \left( r + \frac{\rho}{1 - \rho} \right) + O((r\theta)^2) \right] = \frac{\rho^r}{1 - \rho} + i\theta \frac{\rho^r}{1 - \rho} \left( r + \frac{\rho}{1 - \rho} \right) + O\left( \frac{(r\theta)^2 \rho^r}{1 - \rho} \right).
\]

Here

\[
\frac{\rho^r}{1 - \rho} = (1 + O(n^{-1/2})) h,
\]

and, since \(r \sim \frac{n^{1/2}}{c} \log \frac{n^{1/2}/c}{h}, h = O(n^\beta)\), we have the bounds

\[
|\theta| \max \left\{ \frac{r, n^{1/2}}{\rho} \right\} \frac{\rho^r}{1 - \rho} = O(n^{-\delta} h n^{1/2} \log n) = O(n^{1/2 + \beta - \delta} \log n) \to 0,
\]

\[
\frac{(r\theta)^2 \rho^r}{1 - \rho} = O(n^{1 - 2\delta} h \log^2 n) = O(n^{1 + \beta - 2\delta} \log^2 n) \to 0.
\]

Of course, we have the similar formulas for \(\frac{q^s}{1 - q}\). Therefore

\[
\frac{q^r + q^s}{1 - q} = h + w + i\theta \frac{\rho^r}{1 - \rho} \left( r + \frac{\rho}{1 - \rho} \right) + O\left( \frac{(r\theta)^2 \rho^r}{1 - \rho} \right) + O(n^{-1/2}(h + w)),
\]

with the \(\theta\)-dependent terms approaching zero uniformly for \(\theta\) in question. Plugging this expression into (14) and using also (13), we see that the contribution of the interval \([-n^{-\delta}, n^{-\delta}]\) to the contour integral representing \(p_{n,r,s}\) is obtained via replacing the second factor in the integrand over those \(\theta\)'s in (12) with

\[
e^{-h-w} \left[ 1 + i\theta \left( -\frac{1}{2\xi} + \frac{\rho^r}{1 - \rho} \left( r + \frac{\rho}{1 - \rho} \right) \right) + i\theta^3 n^{2\gamma} + O\left( \theta^4 n^{5/2} + n^{-1/2}(h + w + 1)^2 \right) + O\left( \frac{(r\theta)^2 \rho^r}{1 - \rho} \right) \right].
\]
As in the case of \( p_n \), the contributions of the \( i\theta \)-term and the \( i\theta^3 \)-term to the resulting integral are both zero, and we get

\[
\oint_{|\theta| \leq n^{-\delta}} \frac{P_{r,s}(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \exp \left( \frac{bn^{1/2}}{2} + \frac{1}{2} \log \frac{c}{2\pi\sqrt{n}} - h - w \right) \times \frac{(\pi c)^{1/2}}{n^{3/4}} \left( 1 + O(n^{-1}(h + w + 1)^2) \right).
\]

Combining this formula with the bound (16), and dividing by \( 2\pi i \), we complete the proof of Lemma 2.2.

Let \( \Lambda = (\Lambda_1, \Lambda_2, \ldots) \) denote the uniformly random partition of \( n \), visualized as the Ferrers diagram \( \Lambda \) with left-to-right ordered columns of decreasing heights \( \Lambda_1 \geq \Lambda_2 \geq \ldots \). Let \( \Lambda' = (\Lambda'_1, \Lambda'_2, \ldots) \) denote the partition (Ferrers diagram) dual to \( \Lambda \); so \( \Lambda'_j \) is the length of the \( j \)-th longest row of \( \Lambda \).

Lemma 2.2 yields an asymptotic formula for the joint distribution of \( \Lambda_1, \Lambda'_1 \) together with a convergence rate.

**Corollary 2.3.** Introduce \( H_1, W_1 \) by setting

\[
\Lambda_1 = \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{H_1}, \quad \Lambda'_1 = \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{W_1}.
\]

If \( h, w = O(n^\beta), \beta < 1/4 \), then

\[
P(H_1 \geq h, W_1 \geq w) = e^{-h-w} \left( 1 + O(n^{-1/2}(h + w + 1)^2) \right).
\]

Informally, \( H_1 \) and \( W_1 \) are asymptotically independent, each exponentially distributed with parameter 1.

**Proof.** Immediate from \( P(H_1 \geq h, W_1 \geq w) = \frac{p_{r,s}}{p_n} \), with

\[
r := \left\lceil \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{h} \right\rceil, \quad s := \left\lceil \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{w} \right\rceil,
\]

and Lemma 2.2.

The limit marginal distributions of \( \Lambda_1 \) and \( \Lambda'_1 \) were known since the pioneering work of Erdős and Lehner [7]. The novelty here is the asymptotic independence of \( \Lambda_1 \) and \( \Lambda'_1 \) and the explicit convergence rate. Fristedt [9] extended the Erdős-Lehner result considerably, by establishing the limit joint distribution of the first \( o(n^{1/4}) \) largest parts of \( \Lambda \), thence, separately, the first \( o(n^{1/4}) \) largest parts of \( \Lambda' \), but without an explicit convergence rate. Our goal is to prove a counterpart of the Fristedt result for the distribution of the first \( k \) parts of \( \Lambda \) and jointly the first \( k \) parts of \( \Lambda' \), together with an explicit convergence rate, for \( k = \lceil n^\gamma \rceil, \gamma < 1/4 \). We will use some of the techniques from our studies of the random Young diagram [14] and the random solid diagram [16], with the added emphasis on the convergence rates.
as related to \( k \), the number of the largest parts of \( \Lambda \) and \( \Lambda' \) that we focus on.

Let \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_k), \vec{\lambda'} = (\lambda'_1, \ldots, \lambda'_k) \) be such that \( \lambda_j \) and \( \lambda'_j \) decrease, \( \lambda_k > k, \lambda'_k > k \), and \( \sum_j \lambda_j, \sum_j \lambda'_j \leq n \). A diagram \( \lambda \) of area \( n \), with the leftmost \( k \) tallest columns of height \( \lambda_1, \ldots, \lambda_k \), and the “bottom-most” \( k \) longest rows of length \( \lambda'_1, \ldots, \lambda'_k \), exists iff

\[
(17) \quad \nu := n - \sum_{j=1}^{k} \lambda_j - \sum_{j=1}^{k} \lambda'_j + k^2 > 0.
\]

If we delete these \( k \) columns and \( k \) rows, we end up with a diagram of area \( \nu \), with the tallest column of height \( r = \lambda_k - k \) at most, and the longest row of length \( s = \lambda'_k - k \) at most. So introducing \( p_n(\vec{\lambda}, \vec{\lambda'}) \), the total number of the diagrams of area \( n \) with parameters \( \vec{\lambda}, \vec{\lambda'} \), we see that \( p_n(\vec{\lambda}, \vec{\lambda'}) = p_{\nu,r,s} \). To apply Lemma 2.2 to \( p_{\nu,r,s} \), the parameters \( \nu, r = \lambda_k - k, s = \lambda'_k - k \) need to meet the conditions of this Lemma, with \( \nu \) playing the role of \( n \), of course. This observation coupled with the statement of Corollary 2.3, and Fristedt’s result for the first \( o(n^{1/4}) \) parts of the random partition is our motivation for focusing on \( k = [n^\gamma], \gamma < 1/4 \), and the integers \( \lambda_j, \lambda'_j \), such that

\[
(18) \quad \lambda_j = \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{h_j}, \quad \lambda'_j = \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{w_j}, \quad 1 \leq j \leq k,
\]

\( h_j, w_j \) (weakly) increase with \( j \) increasing, and

\[
(19) \quad h_k, w_k = O(n^{\gamma} \log n); \quad h_1, w_1 \geq n^{-1/2+2\gamma}.
\]

With \( \nu \) instead of \( n \) in Lemma 2.2 we need to determine \( h^* \) and \( w^* \) such that

\[
\begin{align*}
\nu &= \lambda_k - k = \left[ \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{h^*} \right], \\
\nu &= \lambda'_k - k = \left[ \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{w^*} \right],
\end{align*}
\]

i.e

\[
\begin{align*}
\frac{\nu^{1/2}}{c} \log \frac{n^{1/2}}{h^*} &= \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{h_k} - k + O(1), \\
\frac{\nu^{1/2}}{c} \log \frac{n^{1/2}}{w^*} &= \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{w_k} - k + O(1).
\end{align*}
\]

Here, by the definition (17),

\[
(20) \quad \nu = n + k^2 - \sum_{j=1}^{k} \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{h_j w_j} + O(k) = n - \theta(n^{\gamma+1/2} \log n),
\]

\( \theta(A) \) denoting a reminder term of order \( A \) exactly. Using the second line in (20), and the fact that \( |\log h_k|, |\log w_k| \) are of order \( \log n \) exactly, we easily
show that $h^*, w^*$ exist, and are given by
\begin{equation}
(21) \quad h^* = (1 + O(n^{\gamma-1/2} \log^2 n))h_k, \quad w^* = (1 + O(n^{\gamma-1/2} \log^2 n))w_k,
\end{equation}
implying that
\[
h^* + w^* = h_k + w_k + O((h_k + w_k)n^{\gamma-1/2} \log^2 n) = h_k + w_k + O(n^{2\gamma-1/2} \log^3 n).
\]

Another simple evaluation, based on the first line in (20), yields
\begin{equation}
(22) \quad \pi \sqrt{\frac{2\nu}{3}} = \pi \sqrt{\frac{2n}{3} - \sum_{j=1}^{k} \log \left( \frac{n^{1/2}}{h_j w_j} \right)^2 + O(n^{2\gamma-1/2} \log n)}.
\end{equation}

It follows then from Lemma 2.2 and (21), (22) that
\begin{equation}
(23) \quad p_n(\vec{\lambda}, \vec{\lambda}') = \frac{\pi \sqrt{2n/3}}{4 \sqrt{n}} \prod_{j=1}^{k} \frac{h_j w_j}{n^{1/2}} \times \exp(-h_k - w_k + O(n^{2\gamma-1/2} \log^3 n)).
\end{equation}

**Note.** The RHS of (23) is obviously positive, whence it had better be 1, at least. And indeed, the RHS approaches infinity for $h_j, w_j$ meeting the constraints (19).

**Corollary 2.4.** Let $\gamma < 1/4$. Then, uniformly for $(\lambda_j, \lambda'_j)_{1 \leq j \leq k}$ defined in (18), with $h_j, w_j$, $(1 \leq j \leq \lceil n^{\gamma} \rceil)$ satisfying (19),
\[
P\left( \bigcap_{1 \leq j \leq k} \{ \Lambda_j = \lambda_j, \Lambda'_j = \lambda'_j \} \right) = \exp(-h_k - w_k + O(n^{2\gamma-1/2} \log^3 n)) \prod_{j=1}^{k} \frac{h_j w_j}{n^{1/2}}^2.
\]

To interpret this result gainfully, introduce the “slanted” $A_j$ as follows. Let $E = (E_1, \ldots, E_k)$, $E' = (E'_1, \ldots, E'_k)$ be such that all $E_i, E'_i$ are independent copies of a random variable $\bar{E}$, with $P(\bar{E} > x) = e^{-x}$, $x \geq 0$. Define
\begin{equation}
(24) \quad A_j = \left[ \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{S_j} \right], \quad A'_j = \left[ \frac{n^{1/2}}{e} \log \frac{n^{1/2}}{S'_j} \right],
\end{equation}
\begin{equation}
(25) \quad S_j = \sum_{i=1}^{j} E_i, \quad S'_j = \sum_{i=1}^{j} E'_i.
\end{equation}

For the generic values $u_1, \ldots u_k, v_1, \ldots, v_k$ of $S_1, \ldots, S_k, S'_1, \ldots, S'_k$, the joint density is $e^{-u_k-v_k}$, provided that $u_1 \leq \cdots \leq u_k$, $v_1 \leq \cdots \leq v_k$; the density is zero otherwise.
Let us compute \( P(\cap_{1 \leq j \leq k} \{ A_j = \lambda_j, A'_j = \lambda'_j \}) \). Using (18) and (24), we obtain that

\[
A_j = \lambda_j \quad \text{iff} \quad S_j \in I(\lambda_j) := \frac{n^{1/2}}{c} e^{-c\lambda_j n^{-1/2}} \left( e^{-c n^{-1/2}}, 1 \right),
\]

\[
A'_j = \lambda'_j \quad \text{iff} \quad S'_j \in I(\lambda'_j) := \frac{n^{1/2}}{c} e^{-c\lambda'_j n^{-1/2}} \left( e^{-c n^{-1/2}}, 1 \right).
\]

The intervals \( I(\lambda_j) (I(\lambda'_j) \text{ resp.}) \) do not overlap, with \( \inf\{x : x \in I(\lambda_j)\} \geq \max\{x : x \in I(\lambda_j)\} \) (\( \inf\{x : x \in I(\lambda'_j)\} \geq \max\{x : x \in I(\lambda'_j)\} \) resp.), if \( \lambda_{j-1} - \lambda_j \geq 1 \left( \lambda'_{j-1} - \lambda'_j \geq 1 \right. \) resp.). For this choice of \( \lambda_1, \ldots, \lambda_k, \lambda'_1, \ldots, \lambda'_k \), we have

\[
(26) \quad P \left( \bigcap_{1 \leq j \leq k} \{ A_j = \lambda_j, A'_j = \lambda'_j \} \right) = \prod_{j=1}^{k-1} |I(\lambda_j)||I(\lambda'_j)| \int_{u \in I(\lambda_k)} \int_{v \in I(\lambda'_k)} e^{-u-v} du dv.
\]

For \( 1 \leq j \leq k \),

\[
|I(\lambda_j)| = \frac{n^{1/2}}{c} e^{-c\lambda_j n^{-1/2}} \left( 1 - e^{-c n^{-1/2}} \right)
\]

\[
= \left( 1 + O(n^{-1/2}) \right) \exp \left( - \log \frac{n^{1/2}}{h_j} \right) = \left( 1 + O(n^{-1/2}) \right) \frac{h_j}{n^{1/2}}.
\]

and likewise

\[
|I(\lambda'_j)| = \left( 1 + O(n^{-1/2}) \right) \frac{w_j}{n^{1/2}}.
\]

So

\[
(27) \quad \prod_{j=1}^{k-1} |I(\lambda_j)||I(\lambda'_j)| = \left( 1 + O(n^{\gamma-1/2}) \right) \prod_{j=1}^{k-1} \frac{h_j w_j}{n^{1/2}} \frac{n^{1/2}}{c}.
\]

Next, the leftmost, i.e. the infimum, points of \( I(\lambda_k) \) and \( I(\lambda'_k) \) are respectively

\[
h_k (1 + O(n^{-1/2})) = h_k + O(n^{\gamma-1/2}), \quad w_k (1 + O(n^{-1/2})) = w_k + O(n^{\gamma-1/2}).
\]

Consequently

\[
(28) \quad \int_{u \in I(\lambda_k)} \int_{v \in I(\lambda'_k)} e^{-u-v} du dv = \left( 1 + O(n^{\gamma-1/2}) \right) \frac{h_k w_k}{n^{1/2}} \frac{n^{1/2}}{c} e^{-h_k - w_k}.
\]

Combining (26), (27) and (28), we conclude

\[
P \left( \bigcap_{1 \leq j \leq k} \{ A_j = \lambda_j, A'_j = \lambda'_j \} \right) = \left( 1 + O(n^{\gamma-1/2}) \right) \prod_{j=1}^{k} \frac{h_j w_j}{n^{1/2}} \frac{n^{1/2}}{c} e^{-h_k - w_k};
\]
Thus, with probability 1

\[ z \in \Lambda \]  

so does \( \Lambda \). A bound matching the bound in the item (1) imply

\[ P \left( \bigcap_{1 \leq j \leq k} \{ A_j = \lambda_j, A_j' = \lambda_j' \} \right) = \left( 1 + O(n^{2\gamma-1/2} \log^3 n) \right) \]

(29)

\[ \times P \left( \bigcap_{1 \leq j \leq k} \{ A_j = \lambda_j, A_j' = \lambda_j' \} \right). \]

Lemma 2.5. Let \( \mathcal{G} \) denote the set of all tuples \( (\lambda_j, \lambda_j')_{1 \leq j \leq k} \) defined in (18), with \( h_j, w_j, (1 \leq j \leq [n^\gamma]) \) satisfying (19), and such that \( \lambda_j, \lambda_j' \) strictly decrease with \( j \). Uniformly for \( (\lambda_j, \lambda_j')_{1 \leq j \leq k} \in \mathcal{G} \),

Let us show that whp \( (A_j, A_j')_{1 \leq j \leq k} \in \mathcal{G} \). (1) Recalling the definition of \( A_j \),

\[ P \left( \min_{2 \leq j \leq k} (A_{j-1} - A_j) = 0 \right) \leq \sum_{j=2}^k P(A_{j-1} - A_j = 0) \]

\[ = \sum_{j=2}^k P \left( \frac{S_j}{S_{j-1}} \leq e^{cn^{-1/2}} \right) = \sum_{j=2}^k \iint_{u+v \leq \exp \left( \frac{c}{n^{1/2}} \right)} \frac{u^{j-2}}{(j-2)!} e^{-u-v} \, du \, dv \]

\[ = \sum_{j=2}^k \int_0^\infty \frac{u^{j-2}}{(j-2)!} e^{-u(1-e^{-cn^{-1/2}})} \, du = \sum_{j=2}^k (1-e^{-cn^{-1/2}})^{j-1} \]

\[ = O(k^2 n^{-1/2}) = O(n^{2\gamma-1/2}) \to 0. \]

Thus, with probability \( 1 - O(n^{2\gamma-1/2}) \), \( A_j \) strictly decreases, and similarly so does \( A_j' \).

(2) Next, for \( S_k \) in the formula for \( A_k \) we have \( E[e^{zS_k}] = (1-z)^{-k} \) if \( z \in (0,1) \). So, using Chernoff-type bound,

\[ P(S_k \geq n^\gamma \log n) \leq \frac{E[e^{zS_k}]}{\exp \left( zn^\gamma \log n \right)} \leq \frac{(1-z)^{-n^\gamma}}{\exp \left( zn^\gamma \log n \right)}. \]

The last fraction attains its minimum at \( z = 1 - \frac{\log n}{n^\gamma} \), and so

(30)  

\[ P(S_k \geq n^\gamma \log n) \leq \exp \left( -0.5n^\gamma \log n \right). \]

Analogous bound holds for \( S_k' \) in the formula for \( A_k' \).

(3) For \( S_1 \) in the formula for \( A_1 \), and \( S_1' \) in the formula for \( A_1' \),

(31)  

\[ P(S_1 \leq n^{-1/2+2\gamma}) = P(S_1' \leq n^{-1/2+2\gamma}) = 1 - e^{-n^{-1/2+2\gamma}} \leq n^{-1/2+2\gamma}, \]

a bound matching the bound in the item (1).

Summarizing the bounds in the items (1), (2), (3), we obtain:

(32)  

\[ P((A_j, A_j')_{1 \leq j \leq k} \in \mathcal{G}) \geq 1 - O(n^{2\gamma-1/2}). \]
Theorem 2.6. According to Nash-Williams conditions (2), $\Lambda$ is graphical iff immediate, based on

where $D$ the largest square inscribed into the Young diagram $\Lambda$.

Consequently, for two generic subsets $B_1, B_2$ of all, weakly decreasing, tuples $(\lambda_j, \gamma_j)_{1 \leq j \leq k}$, with $\sum_{1 \leq j \leq k} \lambda_j \leq n, \sum_{1 \leq j \leq k} \gamma_j \leq n$.

**Theorem 2.6.** If $k = [n^\gamma]$ and $\gamma < 1/4$, then

$$d_{TV}(\mu_{(\tilde{A}, \tilde{A}')} \mu_{(\tilde{A}, \tilde{A}')} ) = O(n^{2\gamma - 1/2} \log^3 n).$$

Consequently, for two generic subsets $B_1, B_2$ of all, weakly decreasing, tuples $(\lambda_j, \lambda_j')_{1 \leq j \leq k}$, with $\sum_{1 \leq j \leq k} \lambda_j \leq n$,

$$P(\tilde{A} \in B_1, \tilde{A}' \in B_2) = P(\tilde{A} \in B_1)P(\tilde{A}' \in B_2) + O(n^{2\gamma - 1/2} \log^3 n).$$

**Proof.** Immediate, based on $A = A \cap G + A \cap G^c$ and (29), (32), (33). □

3. Graphical Partitions

Our task is to bound Wilf’s probability $P(n) := P(\Lambda$ is graphical) using Theorem 2.6. According to Nash-Williams conditions (2), $\Lambda$ is graphical iff

$$\sum_{j=1}^{i} A'_j \geq \sum_{j=1}^{i} A_j + i, \quad 1 \leq i \leq D(\Lambda),$$

where $D(\Lambda)$ is the size of the Durfee square of $\Lambda$, i.e. the number of rows of the largest square inscribed into the Young diagram $\Lambda$.

Combining (30), (31) and Theorem 2.6 we see that, with probability $1 - O(n^{2\gamma - 1/2} \log^3 n)$, all $A_j, A'_j$ are between $an^{1/2} \log n$ and $bn^{1/2} \log n$, for some constants $a, b$. In particular, with probability this high, $A_k/k, A'_k/k \gg 1$, whence $D(\Lambda) \gg k$, and

$$\sum_{j=1}^{i} A_j + i = (1 + O(n^{-1/2})) \sum_{j=1}^{i} A_j.$$ 

Using this fact and applying Theorem 2.6 yet again, we see that

$$P(n) \leq O(n^{2\gamma - 1/2} \log^3 n) + P \left( \bigcap_{1 \leq i \leq k} \left\{ \frac{A'_i}{n^{1/2} \log n} \in [a, b] \right\} \bigcap \left\{ \sum_{j=1}^{i} A_j \geq (1 + O(n^{-1/2})) \sum_{j=1}^{i} A'_j \right\} \right),$$

(35)
where, as we recall,
\[
A_j = \left[ \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{S_j} \right], \quad S_j = \sum_{\ell=1}^j E_\ell,
\]
\[
A_j' = \left[ \frac{n^{1/2}}{c} \log \frac{n^{1/2}}{S_j'} \right], \quad S_j' = \sum_{\ell=1}^j E_\ell',
\]
with \(E_\ell, E_\ell'\) being independent copies of \(E\), \(\Pr(E > x) = e^{-x}\). The event on the RHS of (35) looks more complex than the Nash-Williams condition (34), but crucially that event is expressed in terms of eminently tractable sums of the i.i.d. random variables.

Now, for \(A_j' u n^{1/2} \log n \in [a, b]\), \(1 \leq u \leq k\), we have
\[
\left\{ \sum_{j=1}^i A_j \geq (1 + O(n^{-1/2})) \sum_{j=1}^i A_j' \right\} \subseteq \left\{ \prod_{j=1}^i \frac{S_j'}{S_j} \geq 1 + O\left(n^{\gamma - 1/2} \log n\right) \right\},
\]
uniformly for \(i \leq k\). Thus, for \(k = [n^\gamma]\),
\[
\Pr(n) \leq O\left(n^{2\gamma - 1/2} \log^3 n\right) + P_k(n),
\]
(36)
\[
P_k(n) := P\left( \min_{1 \leq i \leq k} \prod_{j=1}^i \frac{S_j'}{S_j} \geq \frac{1}{2} \right).
\]

That \(\lim_{n \to \infty} \Pr(n) = 0\) was already proved in [15]. We will show that in fact \(\Pr(n) \to 0\) faster than any negative power of \(\log n\). To this end, we need to analyze the likely behavior of the products \(\prod_{j=1}^i S_j' / S_j\).

Let \(\kappa = \kappa(k) < k\) be such that \(\kappa \to \infty\), \(\kappa = o(k)\) as \(k \to \infty\). Using \(S_j - j = \sum_{i=1}^j (E_i - 1)\), we have: for \(z \in (0, 1)\),
\[
E[\exp(z(S_j - j))] = (E[\exp(z(E - 1))])^j = \left( \frac{e^{-z}}{1 - z} \right)^j.
\]
So, given \(d > 0\), denoting \(R_j := S_j - j\) and using Chernoff-type bound, we estimate
\[
\Pr(|R_j| \geq d) = \Pr(|S_j - j| \geq j d) \leq \left( \frac{e^{-z}}{e^{zd}} \right)^j = \exp\left[ j \left( \log \frac{e^{-z}}{1 - z} - zd \right) \right].
\]
The last function attains its minimum at \(z = \frac{d}{1 + d}\), and so
(37) \[
\Pr(|R_j| \geq d) \leq \exp\left[ j (\log (1 + d) - d) \right] \leq \exp\left( -jd^2 / 2 \right).
\]
Therefore, picking $\alpha > 1/2$ and denoting $y_1 = \log(\kappa - 1)$, $y_2 = \log k$,

$$P \left( \bigcup_{\kappa \leq j \leq k} \{|R_j| \geq j^{-1/2}(\log j)^\alpha\} \right) \leq \sum_{j=\kappa}^{k} P(|R_j| \geq j^{-1/2}(\log j)^\alpha)$$

$$\leq \sum_{j=\kappa}^{k} \exp(- (\log j)^{2\alpha}) \leq \int_{\kappa-1}^{y_2} \exp(- (\log x)^{2\alpha}) \, dx$$

$$= \int_{y_1}^{y_2} \exp(- y^{2\alpha} + y) \, dy \leq \exp(- y_1^{2\alpha} + y_1) \int_{y_1}^{\infty} \exp[(y - y_1)(1 - 2\alpha y_1^{2\alpha - 1})] \, dy$$

$$\leq 2 \exp(- y_1^{2\alpha} + y_1) = 2\kappa \exp(- (\log \kappa)^{2\alpha})$$,

so that

(38) $$P \left( \bigcup_{\kappa \leq j \leq k} \{|R_j| \geq j^{-1/2}(\log j)^\alpha\} \right) \leq 2\kappa \exp(- (\log \kappa)^{2\alpha}),$$

Consider $j < \kappa$. First of all, denoting $R_j' := S_j' - j$,

$$R_j' - R_j = \frac{1}{j} \sum_{t=1}^{j} Y_j, \quad Y_j := E_j' - E_j,$$

So

$$\sum_{j<\kappa} (R_j' - R_j) = \sum_{j<\kappa} \frac{1}{j} \sum_{t=1}^{j} Y_t = \sum_{t=1}^{\kappa-1} Y_t \sum_{j=t}^{\kappa-1} \frac{1}{j},$$

and, picking $\omega = \omega(\kappa) \to \infty$ as $\kappa \to \infty$, by Chebychev’s inequality,

$$P \left( \sum_{j<\kappa} (R_j' - R_j) > \sqrt{\omega \kappa} \right) \leq \frac{1}{(\omega \kappa)^{-1}} \text{Var} \left( \sum_{j=1}^{\kappa-1} (R_j' - R_j) \right)$$

(39) $$= 2(\omega \kappa)^{-1} \sum_{t=1}^{\kappa-1} \left( \sum_{j=t}^{\kappa-1} \frac{1}{j} \right)^2 = O(\omega^{-1}),$$

since

$$2 \sum_{t=1}^{\kappa-1} \left( \sum_{j=t}^{\kappa-1} \frac{1}{j} \right)^2 \leq \sum_{t=1}^{\kappa-1} \frac{1}{t} \log \frac{\kappa - 1}{t} \leq \sum_{t=1}^{\kappa-1} \left( \frac{1}{t^2} + \log^2 \frac{\kappa - 1}{t} \right)$$

$$\leq \sum_{t=1}^{\infty} \frac{1}{t^2} + (\kappa - 1) \int_{0}^{1} \log^2 x \, dx = O(\kappa).$$
Further, for $\beta > 1$, by independence of $S'_j$ and $S_j$ we have: for $0 < z < 1$,

$$P\left(\frac{S'_j}{S_j} \geq \beta\right) = P(e^{z(S'_j - \beta S_j)} \geq 1)$$

$$\leq E[e^{z(S'_j - \beta S_j)}] = \left(\frac{1}{1-z}\right)^j \left(\frac{1}{1+z\beta}\right)^j.$$  

The function $(1-z)(1+z\beta)$ attains its maximum at $z = \frac{\beta-1}{2\beta} \in (0, 1)$, and the maximum value is $1 + \frac{(\beta-1)^2}{4\beta}$. Therefore

$$P\left(\frac{S'_j}{S_j} \geq \beta\right) \leq \left(1 + \frac{(\beta-1)^2}{4\beta}\right)^{-j}.$$  

For $\beta = \beta_j = 1 + \sqrt{\omega}$,

$$1 + \frac{(\beta-1)^2}{4\beta} = \frac{1 + \frac{\omega}{j} + \frac{\omega}{j^2}}{1 + \sqrt{\frac{\omega}{j}}} = f(\omega/j),$$

$$f(x) := 1 + \frac{x}{4 + \sqrt{x}}.$$  

Consequently

$$P\left(\bigcup_{1 \leq j < \kappa} \left\{\frac{S'_j}{S_j} \geq \beta_j\right\}\right) \leq \sum_{j \geq 1} P\left(\frac{S'_j}{S_j} \geq \beta_j\right) \leq \sum_{j \geq 1} f(\omega/j)^{-j}.$$  

Let us bound the sum. Consider $j \geq \omega$. Using $1 + x \geq \exp(x - \frac{x^2}{2(1-x)})$ for $x < 1$,

$$f(\omega/j) \geq 1 + \frac{\omega/j}{8} \geq \exp\left(\frac{\omega}{8j}\right),$$

so

$$\sum_{j \geq \omega} f(\omega/j)^{-j} \leq \kappa e^{-\omega/\omega}.$$  

For $j \leq \omega$, introducing $y_j = j/\omega$,

$$f(\omega/j)^{-j} \leq [g(y_j)]^\omega, \quad g(y) := \left(1 + \frac{1}{8\sqrt{y}}\right)^{-y}.$$  

Since $(\log g(y))^\omega > 0$,

$$\max_{y \in [1/\omega, 1]} g(y) = \max\{g(1/\omega), g(1)\}$$

$$= \max \left\{(1 + \frac{\sqrt{\omega}}{8})^{-1/\omega}, \frac{8}{9}\right\} = \left(1 + \frac{\sqrt{\omega}}{8}\right)^{-1/\omega},$$

$$= \left(1 + \frac{\sqrt{\omega}}{8}\right)^{-1/\omega},$$

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$$= \left(1 + \frac{\sqrt{\omega}}{8}\right)^{-1/\omega}. $$
for all $\omega$ large enough. Therefore

$$f(\omega/j)^{-j} \leq f(\omega) = \left(1 + \sqrt[8]{\omega}\right)^{-1},$$

and it can be shown that

$$\sum_{j \leq 1, \omega} f(\omega/j) \leq 2f(\omega) = \frac{2}{1 + \sqrt[8]{\omega}} \leq 16 \omega^{-1/2}.$$

We conclude that

$$P \left( \bigcup_{1 \leq j < \kappa} \left\{ \frac{S_j'}{S_j} \geq 1 + \sqrt[8]{\omega/j} \right\} \right) \leq \kappa e^{-\omega/9} + 16 \omega^{-1/2}.$$

For $P_k(n)$ defined in (36) we obviously have

$$P_k \leq P_{k,1}(n) := P \left\{ \min_{\kappa \leq \ell \leq k} \prod_{j=\kappa}^{\ell} \frac{S_j'}{S_j} \geq \frac{1}{2} \prod_{j=1}^{\kappa-1} S_j \right\}.$$

Observe that on the event $E_1 := \bigcap_{j < \kappa} \left\{ \frac{S_j'}{S_j} < 1 + \sqrt[8]{\omega/j} \right\}$ we have

$$\prod_{j < \kappa} \frac{S_j'}{S_j} \geq \prod_{j < \kappa} \left(1 + \sqrt[8]{\omega/j}\right)^{-1} \geq \exp \left( - \sum_{j < \kappa} \sqrt[8]{\omega/j} \right) \geq \exp(-3\sqrt{\omega\kappa}).$$

Further, on the event

$$E_2 := \bigcup_{\kappa \leq j \leq k} \left\{ |R_j| + |R_j'| \leq 2j^{-1/2}(\log j)^\alpha \right\}$$

we have: by $\log(1+x) \leq x$,

$$\prod_{j=\kappa}^{\ell} \frac{S_j'}{S_j} = \prod_{j=\kappa}^{\ell} \frac{1 + R_j'}{1 + R_j} \leq \exp \left( \sum_{j=\kappa}^{\ell} \log \frac{1 + R_j'}{1 + R_j} \right) \leq \exp \left( \sum_{j=\kappa}^{\ell} \frac{R_j' - R_j}{1 + R_j} \right)$$

$$\leq \exp \left( \sum_{j=\kappa}^{\ell} (R_j' - R_j) + 4 \sum_{j=\kappa}^{\ell} j^{-1}(\log j)^{2\alpha} \right)$$

$$\leq \exp \left( \sum_{j=\kappa}^{\ell} (R_j' - R_j) + 2(\log k)^{2\alpha+1} \right).$$
Combining (38), (39), (40) and (41), we arrive at

\[ P_k(n) \leq P_{k,2}(n) + U, \]

(42)

\[ P_{k,2}(n) := P \left\{ \min_{\kappa \leq \ell \leq k} \sum_{j=1}^{\ell} (R'_j - R_j) \geq -V \right\}, \]

\[ U = U(\kappa, \omega) := 4\kappa \exp\left( -(\log \kappa)^{2\alpha} \right) + \kappa e^{-\omega/9} + 17\omega^{-1/2}, \]

\[ V = V(\kappa, \omega, k) := 4\sqrt{\omega \kappa} + 2(\log k)^{2\alpha + 1}; \]

here \( \alpha > 1/2 \) is fixed. We will use (42) for

\[ \kappa = \left\lfloor \exp\left( (\log k)^{1/2} \right) \right\rfloor, \quad \omega = \exp\left( \frac{2 \log k}{\log \log k} \right), \]

in which case

\[ U \leq U^* := 18 \exp\left( -\frac{\log k}{\log \log k} \right), \]

(44)

\[ V \leq V^* := \exp\left( \frac{2 \log k}{\log \log k} \right). \]

Usefulness of this choice of \( \kappa \) and \( \omega \) will become clearer later.

Let us use independence of \( Y_j = E'_j - E_j \) to show that the event in (42) is almost contained in the intersection of a sequence of independent events such that the product of their individual probabilities goes to zero at a certain explicit rate. Introduce the sequence \( \{\ell_r\} \):

(45)

\[ \ell_0 = \kappa, \quad \ell_r = \ell_0 \zeta^r, \quad \zeta = \zeta(k) = [a(\log \log k) \log k], \]

\( a > 0 \) being fixed. So

\[ |\{r > 0 : \ell_r \leq k\}| \geq \rho = \rho(k) := \left\lfloor \frac{\log \frac{k}{2}}{\log [a(\log \log k) \log k]} \right\rfloor \sim \frac{\log k}{\log \log k}, \]

since by (43) \( \log \kappa = o(\log k) \). Introducing

\[ Z_r := \sum_{j=1}^{\ell_r} (R'_j - R_j) = \sum_{j=1}^{\ell_r} \frac{1}{j} \sum_{t=1}^{j} Y_t, \quad \ Y_t = E'_t - E_t, \]

we have: for \( r \in [1, \rho] \),

\[ Z_r = \sum_{j=1}^{\ell_r} \sum_{t=1}^{j} Y_t = \sum_{t=1}^{\ell_r} Y_t \sum_{j=t}^{\ell_r} \frac{1}{j} \]

\[ = \sum_{t=1}^{\ell_r} Y_t \sum_{j=t}^{\ell_r} \frac{1}{j} + \sum_{t=\ell_r-1+1}^{\ell_r} Y_t \sum_{j=t}^{\ell_r} \frac{1}{j} =: Z_{r,1} + Z_{r,2}. \]

(47)

Here \( Z_{r,1} \) is measurable with respect to \( \mathcal{F}_{r-1} \), the \( \sigma \)-field generated by \( Y_1, \ldots, Y_{\ell_r-1} \), and \( Z_{r,2} \) is independent of \( \mathcal{F}_{r-1} \). Crucially as well, \( Z_{r,2} \) are mutually independent.
To see, *semi-formally*, the reason behind our choice of $\ell_r$, observe that, since $Y_t$ are independent and $E[Y_t] = 0$, $\text{Var}(Y_t) = 2$,

$$\begin{align*}
\text{Var}(Z_{r,1}) &= 2 \sum_{t=1}^{\ell_r-1} \left( \sum_{j=t}^{\ell_r} \frac{1}{j} \right)^2 \leq \sum_{t=1}^{\ell_r-1} \left( \frac{1}{t^2} + \log^2 \frac{\ell_r}{t} \right) \\
&\leq \frac{\pi^2}{6} + \ell_r \int_0^{\ell_r} (\log x)^2 \, dx \leq (2 + \varepsilon_r)\ell_{r-1} \left( \log \frac{\ell_r}{\ell_{r-1}} \right)^2, \quad (\varepsilon_r \to 0),
\end{align*}$$

(48)
i.e. $\text{Var}(Z_{r,1}) = O(\ell_{r-1}(\log \log k)^2)$. Thus, *typically*, $|Z_{r,1}|$ is roughly of order $O(\sqrt{\ell_{r-1} \log \log k})$. As for $Z_{r,2}$,

$$\begin{align*}
\text{Var}(Z_{r,2}) &= 2 \sum_{t=\ell_{r-1}+1}^{\ell_r} \left( \sum_{j=t}^{\ell_r} \frac{1}{j} \right)^2 \geq 2 \sum_{t=\ell_{r-1}+1}^{\ell_r} \log^2 \left( \frac{\ell_r}{t} \right) \\
&\sim 2\ell_r \int_0^1 (\log x)^2 \, dx = 4\ell_r,
\end{align*}$$

(49)
meaning, *hopefully*, that $|Z_{r,2}|$ assumes values of order $\sqrt{\ell_r}$ with not to too small probability. And the interval $[-\sqrt{\ell_r}, \sqrt{\ell_r}]$ contains the interval $[-\sqrt{\ell_{r-1} \log \log k}, \sqrt{\ell_{r-1} \log \log k}]$ with room to spare, because

$$\frac{\sqrt{\ell_{r-1} \log \log k}}{\sqrt{\ell_r}} = O\left( \sqrt{\frac{\log \log k}{\log k}} \right).$$

Notice also that $\sqrt{\ell_r} \gg V^*$, the bound for $V$ defined in (44), if

$$r \geq r^* := \frac{3 \log k}{(\log \log k)^2} \ll \rho(k) \sim \frac{\log k}{\log \log k}.$$ 

So we should expect that for those $r$, conditioned on $\mathcal{F}_{r-1}$, the probability that $Z_r > -V^*$ is close to the unconditional $P(Z_{r,2} > -V^*) \sim \frac{1}{2}$, the latter holding because $Z_{r,2}$ and $-Z_{r,2}$ are equidistributed. Once these steps are justified, we will get a bound analogous to, but weaker than the naive bound

$$P_{k,2}(n) \leq \left( \frac{1}{2} + o(1) \right)^{\rho(k)} = \exp\left( -(\log 2)\rho(k) \right),$$

see (46) for the definition of $\rho(k)$. 

To start, since $E[e^{\xi Y}] = (1 - \xi^2)^{-1}$ for $|\xi| < 1$, we see that, by (48),

$$E[\exp(\xi Z_r, 1)] = \prod_{t=1}^{\ell_r-1} \left[ 1 - \xi^2 \left( \sum_{j=t}^{\ell_r} \frac{1}{j} \right)^2 \right]^{-1} \leq \exp \left[ \xi^2 (1 + o(1)) \sum_{t=1}^{\ell_r-1} \left( \sum_{j=t}^{\ell_r} \frac{1}{j} \right)^2 \right] \leq \exp \left[ (2 + o(1)) \ell_{r-1} \left( \log \frac{\ell_r}{\ell_{r-1}} \right)^2 \xi^2 \right].$$

for $|\xi| = o\left((\log \ell_r)^{-1}\right)$. Consequently, for every such $\xi > 0$,

$$P(Z_{r,1} \geq \sqrt{\ell_r}) \leq \exp \left[ (2 + o(1)) \ell_{r-1} \left( \log \frac{\ell_r}{\ell_{r-1}} \right)^2 \xi^2 - \xi \sqrt{\ell_r} \right].$$

The exponent attains its absolute minimum at

$$\xi = \frac{\sqrt{\ell_r}}{(2 + o(1)) \ell_{r-1} \left( \log \frac{\ell_r}{\ell_{r-1}} \right)^2} \ll \zeta^{-r/2} \ll (\log k)^{-1} = O\left((\log \ell_r)^{-1}\right),$$

for $r \geq 4$, as $\ell_r \leq k$, meaning that this $\xi$ is (easily) admissible for $r \geq r^*$. For these $r$ and $\xi$ we obtain

$$P(Z_{r,1} \geq \sqrt{\ell_r}) \leq \exp \left( -\frac{\ell_r}{(4 + o(1)) \ell_{r-1} \left( \log \frac{\ell_r}{\ell_{r-1}} \right)^2} \right) \leq \exp \left( -\frac{a \log k}{(4 + o(1)) \log \log k} \right),$$

if $k$ is large enough; see (45) for the ratio $\ell_r/\ell_{r-1}$. Consequently, with $\rho = \rho(k)$ defined in (46),

$$P \left( \bigcup_{r=r^*}^{\ell_m^*} \{ Z_{r,1} \geq \sqrt{\ell_r} \} \right) \leq \rho \exp \left( -\frac{a \log k}{(4 + o(1)) \log \log k} \right) = \exp \left( -\frac{a \log k}{(4 + o(1)) \log \log k} \right).$$

(50)

Notice at once that the last RHS dwarfs $U^*$, see (44), the upper bound for the term $U$ in the inequality (42) for $P_k$. 
The rest is short. Since the random variables $Z_{r,1}$ are mutually independent, we use (50) to bound

$$P_{k,2}(n) \leq P \left( \bigcap_{r=r^*} \{ Z_r \geq -V^* \} \right)$$

$$\leq P \left( \bigcup_{r=r^*} \{ Z_{r,1} \geq \sqrt{\ell_r} \} \right) + P \left( \bigcap_{r=r^*} \{ Z_r \geq -V^*, Z_{r,1} \leq \sqrt{\ell_r} \} \right)$$

$$\leq \exp \left( -\frac{a \log k}{(4 + o(1)) \log \log k} \right) + \prod_{r=r^*} P(Z_{r,2} \geq -V^* - \sqrt{\ell_r}),$$

Now, $r^*$ was chosen to make $\sqrt{\ell_r} \gg V^*$, and, by (49), we also know that $\text{Var}(Z_{r,2}) \sim 4\ell_r$. Since $Z_{r,2}$ is asymptotically normal with mean 0 and variance $4\ell_r$, we see that

$$P(Z_{r,2} \geq -V^* - \sqrt{\ell_r}) = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{\infty} e^{-u^2/2} \, du + o(1).$$

Since $\rho = \rho(k) \sim \frac{\log k}{\log \log k}$, and $r^* = o(\rho)$, we obtain

$$P_{k,2}(n) \leq \exp \left( -\frac{a \log k}{(4 + o(1)) \log \log k} \right)$$

$$+ \left( \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{\infty} e^{-u^2/2} \, du + o(1) \right)^{\frac{\log k}{\log \log k}} \leq \exp \left( -\frac{b \log k}{\log \log k} \right),$$

if

$$b < \log \left( \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{\infty} e^{-u^2/2} \, du \right)^{-1} = 0.445...,$$

and $a > 4b$. Observe also that

$$\exp \left( -\frac{b \log k}{\log \log k} \right) \gg U^* := 18 \exp \left( -\frac{\log k}{\log \log k} \right),$$

see (44). So, by (42),

$$P_k(n) \leq P_{k,2}(n) + U^* \leq \exp \left( -\frac{0.445.. \log k}{\log \log k} \right).$$

Using this bound and (36), we conclude that, for every $\gamma < 1/4$,

$$P(n) \leq O(n^{2\gamma - 1/2} \log^3 n) + \exp \left( -\frac{0.445.. \log k}{\log \log k} \right) \bigg|_{k=\lfloor n^\gamma \rfloor}.$$

**Theorem 3.1.** For $n$ large enough,

$$P(n) \leq \exp \left( -\frac{0.11 \log n}{\log \log n} \right).$$
4. Pairs of comparable partitions

Almost as an afterthought, let \( \Lambda, \Delta \) be two uniformly random, mutually independent, partitions of \( n \). Our task is to bound Macdonald’s probability, i.e.

\[
Q(n) := P(\Lambda \succeq \Delta) = P \left( \bigcap_i \left\{ \sum_{j=1}^{\Delta_i} \leq \sum_{j=1}^{\Lambda_i} \right\} \right).
\]

(Notice that the Durfee squares do not enter the comparability conditions at all.) This time the tuples \((\Delta_j)_{j \geq 1}\) and \((\Lambda_j)_{j \geq 1}\) are mutually independent. Furthermore, the distributions of the sub-tuples, \((\Delta_j)_{1 \leq i \leq [n]}\) and \((\Lambda_j)_{1 \leq i \leq [n]}\), are each within the total variation distance of order \(O((n^{-1/2+2\gamma}\log^3 n))\) from the distribution of \((\Lambda_j)_{1 \leq i \leq [n]}\); see Theorem 2.6. Thus a simplified version of the proof of Theorem 3.1 establishes

**Theorem 4.1.** For \( n \) large enough,

\[
Q(n) \leq \exp \left( -0.11 \log n \right).
\]

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