Alternative Approaches to Solve Simple Harmonic Motion

Zhiwei Chong \(^1\) and Yajun Wei \(^2\)

\(^1\) International Division, Experimental School Affiliated with Zhuhai No.1 High School, Zhuhai, Guangdong, China

\(^2\) Zhuhai No.1 High School, Zhuhai, Guangdong, China

Abstract

This paper presents two alternative approaches to solve simple harmonic motion (SHM) without resorting to differential equations. In one approach, the distance between the equilibrium position and the maximal displacement is divided into \(N\) equal segments. In each segment, the motion is approximated as one with constant acceleration under the average of two forces at each end of the segment. Summing up the time to cover each segment and taking a large-\(N\) limit reproduce one quarter of the period for SHM. In the other approach, the time moving from the maximal displacement to the equilibrium position is divided into \(N\) equal intervals. The motion during each interval is approximated as one with constant acceleration. A second order recurrence relation for displacement is obtained. The large-\(N\) limit of its solution results in the same solution obtained from solving differential equation.

1 Introduction

Simple harmonic motion (henceforth SHM, with a spring-mass system in mind) is treated in all introductory physics textbooks[1, 2, 3, 4] and some calculus textbooks [5]. Other than the approaches in these textbooks, this paper presents two new approaches without resorting to differential equations but recurrence relations or difference equations instead. They are conceptually easy but technically slightly challenging. Nevertheless, the relevant mathematics is still within the reach of most first year undergraduate students or even good high school students.
In textbooks, mainly there are three approaches to treat the SHM problem. Firstly, a connection is established between SHM and circular motion, and the period of SHM is found [1, 2]. The merit of this approach is that it is accessible to students without knowledge in calculus. Moreover, it provides a rigorous derivation of the period for SHM. Secondly, the solution to the differential equation for SHM is presented and students are invited to verify the solution [1, 3]. This approach does not involve solving differential equation but only requires students to have some experience in differentiation. Lastly, the differential equation is solved either by integrating twice [3] or using techniques in the theory of differential equations [4, 5]. In the literature, there are a few theoretical articles on spring-mass system, however, the focus is on the effect incurred by the mass of the spring [6, 7, 8, 9]. Some experimental explorations on SHM include [10, 11, 12].

This paper originates from teaching physics to Year-10 students. The period \( T = 2\pi \sqrt{m/k} \) for the motion of a spring-mass system, where \( m \) is mass and \( k \) is spring constant, seems rather mysterious to them. By dimensional analysis, it is easy to convince them of the reasonability of the square-root part of the expression; there is no alternative combination of \( m \) and \( k \), which produces the dimension of time. However, the coefficient \( 2\pi \) remains to be a mystery.

Solving differential equations is not an option for high school students and first year undergraduate students either. Nevertheless, it is worthwhile to invite them to solve a new problem as much as they can with whatever knowledge they have at hand. For example, finding an approximate solution to SHM using average force, speed, and acceleration still constitutes a good exercise for them. Note that Year-10 students are familiar with the average approach, which is probably the only weapon at their hands for the moment. It will soon be seen that the notion of average, though seems primitive, may become quite powerful if executed appropriately.

The first exercise is easy for them: Find the time it takes the mass to move from rest at the maximum displacement \( x = A \) back to its equilibrium at \( x = 0 \) under the average of two forces exerted by the spring at these two positions. The answer is \( 2\sqrt{m/k} \), which is larger than one quarter of the period \( T_{1/4} = T/4 = \frac{1}{2} \pi \sqrt{m/k} \) and the ratio between them is 1.27. There is a byproduct here, the students found that the quantity they were after was again proportional to \( \sqrt{m/k} \).

In the second exercise, the maximum displacement is divided into two equal segments. Then the students are asked to find and sum the time to cover each segment under the average of two forces exerted by the spring at the two ends of each segment. The answer is
$4(3 - \sqrt{3})\sqrt{m/k}$, which is closer to $T_4$ and the ratio between them is $1.08 < 1.27$, where the latter is the ratio obtained in the first exercise. It can be expected that the difference from the exact value will decrease as the number of segments increases. More importantly, the exact value can be recovered as the number of segments goes to infinity. A full development along this direction is presented in Section 2.

In Section 3, another approach in which the time instead is divided into $N$ equal intervals and a system of difference equations is obtained for SHM. By solving the system of difference equations and taking large-$N$ limit, the solution obtained from solving differential equation can be reproduced. We conclude and discuss in Section 4.

2 Dividing Distance into Equal Segments: Riemann Sum

A spring-mass system consists of a mass $m$ and a massless spring with stiffness $k$. One end of the spring is fixed and the other is attached with a mass $m$ which lies on a horizontal surface without friction. It is stretched to the right by distance $A$. Its equilibrium position is denoted as $x = 0$ and the initial position $x = A$. For simplicity, we focus on the motion from its initial position back to equilibrium, that is, the first quarter of a complete cycle.

The mass is released from rest. The goal is to find the position $x$ of the mass at instant $t$, that is, the function $x(t)$. Moreover, we denote the time it takes the mass to move from $x$ back to its equilibrium as $\tau$. Clearly, $t + \tau = T_4$. We will first find $\tau$, then $t$ readily results. Figure 1 helps to illustrate the idea.

![Figure 1: Illustration of the idea.](image)

The following is the key to our approach: The distance $x$ covered by the mass during the time interval $\tau$ is divided into $N$ equal segments. See Figure 2 for details. The motion in each segment is considered as one moving with constant acceleration under the average of two forces exerted by the spring at each end of the segment under consideration. The quantity $\tau$ can be obtained by calculating and summing over the time to cover each segment. The
sum appears to be the Riemann sum of a familiar integral which can be readily integrated, and the same result as that obtained from solving differential equation can be recovered.

![Figure 2: Divide the distance into N equal segments.](image)

The size of each segment is denoted as $\Delta x = x/N$. The equilibrium is denoted as $x_0 = 0$ and the right end of the $j$-th segment is $x_j = j \Delta x$. In particularly, $x_N = x$. The time it takes the mass to cover the $j$-th segment is denoted as $\tau_j$ and $\tau = \sum_{j=1}^{N} \tau_j$.

By mechanical energy conservation, the speed $v_j$ at position $x_j$ satisfies

$$\frac{1}{2} m v_j^2 + \frac{1}{2} k x_j^2 = \frac{1}{2} k A^2, \quad j = 0, 1, \ldots, N,$$

then $v_j$ is obtained as

$$v_j = \omega A \left( 1 - \delta^2 j^2 \right)^{\frac{1}{2}}, \quad \omega^2 \equiv k/m, \quad \delta \equiv \Delta x/A. \tag{1}$$

Recursively,

$$v_{j-1} = \omega A \left( 1 - \delta^2 (j - 1)^2 \right)^{\frac{1}{2}}. \tag{2}$$

Both $v_j$ and $v_{j-1}$ together with the average acceleration found below will be used to find the time taken to cover the $j$-th segment.

In the $j$-th segment, the mass moves with an average acceleration under the average force on it. Applying Newton’s second law and recalling $x_j = j \Delta x, \Delta x = x/N$ and $\delta = \Delta x/A$ we have

$$m \bar{a}_j = \frac{1}{2} k (x_{j-1} + x_j) = \frac{1}{2} k \Delta x (2j - 1) = \frac{1}{2} k A \delta (2j - 1),$$

or

$$\bar{a}_j = \frac{1}{2} \omega^2 A \delta (2j - 1). \tag{3}$$

Note that $\omega A$ in (1) and $\omega^2 A$ in (3) are maximum speed and acceleration for SHM, respectively.
With (1), (2), and (3), we are ready to calculate the time it takes the mass to cover the
$j$-th segment, that is,
\[ \tau_j = \frac{v_{j-1} - v_j}{a_j} = \frac{2}{\omega \delta} \frac{(1 - \delta^2 (j-1)^2)^{\frac{1}{2}} - (1 - \delta^2 j^2)^{\frac{1}{2}}}{2j-1} \]
\[ = \frac{2 \delta}{\omega} \left( \frac{1}{(1 - \delta^2 (j-1)^2)^{\frac{1}{2}} + (1 - \delta^2 j^2)^{\frac{1}{2}}} \right). \]
Equivalently,
\[ \omega A \tau_j = \frac{2 \Delta x}{[1 - (x_{j-1}/A)^2]^{\frac{1}{2}} + [1 - (x_j/A)^2]^{\frac{1}{2}}}. \quad (4) \]
Note the fact that $x_{j-1} < x_j$, and the inequalities below follow from (4).
\[ \frac{\Delta x}{[1 - (x_{j-1}/A)^2]^{\frac{1}{2}}} < \omega A \tau_j < \frac{\Delta x}{[1 - (x_j/A)^2]^{\frac{1}{2}}}. \]
Summing up $j$ from 1 to $N$ and recalling $\tau = \sum_{j=1}^{N} \tau_j$ gives
\[ \sum_{j=1}^{N} \frac{\Delta x}{[1 - (x_{j-1}/A)^2]^{\frac{1}{2}}} < \omega A \tau < \sum_{j=1}^{N} \frac{\Delta x}{[1 - (x_j/A)^2]^{\frac{1}{2}}}. \]
Note the left sum is the so-called lower sum and the right the upper sum in Riemann integral
[5, 13]. Then
\[ \omega A \tau = \int_{0}^{x} \frac{dx}{\sqrt{1 - (x/A)^2}} = A \arcsin(x/A). \]
Equivalently,
\[ x = A \sin \omega \tau. \quad (5) \]

For the purpose to express the solution in terms of $t$, we note the following fact. For
$x = A$, the time it takes the mass to arrive at its equilibrium is just $T_{\frac{1}{4}}$ which is determined
by $\sin \omega T_{\frac{1}{4}} = 1$, or $\omega T_{\frac{1}{4}} = \pi/2$ and $T_{\frac{1}{4}} = \frac{\pi}{2 \omega}$. Therefore, $t = T_{\frac{1}{4}} - \tau$, and (5) becomes
\[ x = A \sin \omega \left( T_{\frac{1}{4}} - t \right) = A \sin \omega \left( \frac{\pi}{2 \omega} - t \right) = A \cos \omega t, \quad (6) \]
which is exactly the same as that obtained from solving the differential equation for SHM
[4, 5].

3 Dividing Time into Equal Intervals: Difference Equations

Another approach is to divide the time $t$ after the mass is released into $N$ equal intervals.
See Figure 3 for details. A recurrence relation or a system of difference equations will be
obtained. In the end, a large $N$ limit is taken to recover the solution obtained from solving differential equations.

To be specific, let $x_i$ and $v_i$ denote the position and speed of the mass at the end of the $i$-th time interval, respectively. The size of each time interval is denoted as $\Delta t = t/N$, and the distance covered in the $i$-th time interval is denoted as $\Delta x_i = x_{i-1} - x_i$, where $x_i$ is the position of the mass at the end of the $i$-th time interval. Note that $x_0 = A$ and $x_i < x_{i-1}$.

The average speed during the $i$-th interval is

$$\frac{1}{2} \left( v_i + v_{i-1} \right) = \frac{\Delta x_i}{\Delta t}. \quad (7)$$

The key step in our approach is the following. Within each time interval, the motion is considered as one with constant acceleration under the average of two elastic forces with spring extensions corresponding to the two ends of the time interval. That is,

$$v_i - v_{i-1} = \bar{a}_i \Delta t, \quad (8)$$

where $\bar{a}_i$ is the average acceleration in the $i$-th time interval which can be obtained by applying Newton’s second law.

$$m \bar{a}_i = \frac{1}{2} k \left( x_i + x_{i-1} \right),$$

or

$$\bar{a}_i = \frac{1}{2} \omega^2 \left( x_i + x_{i-1} \right), \quad \omega^2 \equiv k/m. \quad (9)$$

Solving $v_i$ and $v_{i-1}$ from (7) and (8) gives

$$2v_i = 2 \frac{\Delta x_i}{\Delta t} + \bar{a}_i \Delta t, \quad (10)$$

$$2v_{i-1} = 2 \frac{\Delta x_i}{\Delta t} - \bar{a}_i \Delta t. \quad (11)$$
Recursively, an alternative expression for \( v_i \) can be obtained from (11)

\[
2v_i = 2 \frac{\Delta x_{i+1}}{\Delta t} - a_{i+1} \Delta t. \tag{12}
\]

Equating (10) with (12) gives

\[
\Delta x_{i+1} - \Delta x_i = \frac{1}{2} (\bar{a}_i + \bar{a}_{i+1}) \Delta t^2 \tag{13}
\]

With the expression for average acceleration in (9), a system of difference equations for \( x_i \) is obtained.

\[
(1 + \alpha^2) x_{i+1} - 2 (1 - \alpha^2) x_i + (1 + \alpha^2) x_{i-1} = 0, \quad \alpha \equiv \frac{1}{2} \omega \Delta t. \tag{14}
\]

To solve (14) we need two initial conditions. One is obviously the initial position of the mass, that is, \( x_0 = A \). The other can be found as follows. The distance covered in the first time interval is \( A - x_1 = \frac{1}{2} \bar{a}_1 \Delta t^2 = \frac{1}{4} \omega^2 (A + x_1) \Delta t^2 = \alpha^2 (A + x_1) \), where the second equality follows from (9) and the third from the definition of \( \alpha \) in (14). Then the two initial conditions are

\[
x_0 = A, \quad x_1 = \frac{1 - \alpha^2}{1 + \alpha^2} A \tag{15}
\]

The system of difference equations (14) with initial conditions (15) can be solved by following a standard procedure which resembles the technique in solving second order ordinary differential equations. The characteristic equation is

\[
(1 + \alpha^2) r^2 - 2 (1 - \alpha^2) r + (1 + \alpha^2) = 0. \tag{16}
\]

Its two roots are

\[
r_+ = \frac{1 + i \alpha}{1 - i \alpha}, \quad r_- = \frac{1 - i \alpha}{1 + i \alpha}, \quad i^2 = -1. \tag{17}
\]

The general solution to the system of difference equations is

\[
x_k = \lambda_+ r_+^k + \lambda_- r_-^k, \quad k = 0, 1, \cdots, N, \tag{18}
\]

where \( \lambda_+ \) and \( \lambda_- \) are constants to be determined by initial conditions (15). They are

\[
\lambda_+ = \lambda_- = \frac{1}{2} A. \tag{19}
\]

Finally, we obtain the solution to (14) and (15)

\[
x_k = \frac{1}{2} A (r_+^k + r_-^k), \quad k = 0, \cdots, N. \tag{20}
\]
In particular, the position of the mass at instant \( t \), or equivalently, at the end of the \( N \)-th time interval is

\[
x_N = \frac{1}{2} A (r_+^N + r_-^N).
\] (21)

The solution to the SHM can be recovered by taking the large \( N \) limit of (21). For this purpose, we write

\[
r_+^N = \left( \frac{1 + i \alpha}{1 - i \alpha} \right)^N = \frac{\left( 1 + \frac{1}{2} i \omega \Delta t \right)^N}{\left( 1 - \frac{1}{2} i \omega \Delta t \right)^N} = \left( \frac{1 + \frac{1}{2} i \omega t \frac{1}{N}}{1 - \frac{1}{2} i \omega t \frac{1}{N}} \right)^N.
\] (22)

In the large-\( N \) limit, the numerator becomes \( e^{i\omega t/2} \) and the denominator \( e^{-i\omega t/2} \). Thereby, the limit for \( r_+^N \) is \( \lim_{N \to \infty} r_+^N = e^{i\omega t} \) and similarly \( \lim_{N \to \infty} r_-^N = e^{-i\omega t} \). Finally, the large-\( N \) limit of (21) is

\[
x(t) = \lim_{N \to \infty} x_N = \frac{1}{2} A \left( e^{i\omega t} + e^{-i\omega t} \right) = A \cos \omega t,
\] (23)

which is exactly the same result obtained from solving differential equation for SHM [4, 5].

4 Summary

This paper presents two alternative approaches to solve the problem of SHM. Both use mathematics within the reach of undergraduate students taking an introductory physics course or even good high school students. The key idea is dividing the distance or time into \( N \) segments or intervals. After obtaining relevant physical quantities, a large-\( N \) limit is taken. Then, exactly the same result can be obtained as that from solving differential equations, which is beyond the reach of most students taking the course.

The value of our approaches lies in two aspects. On one hand, they enable students to attack a problem at an early stage using elementary physics and mathematics rather than after they have learned enough calculus or even theory of differential equations at a much later stage. On the other hand, even this paper is presented to students after they are able to solve differential equations, they can still learn quite some mathematics and problem solving skills from this paper. Moreover, by comparing various approaches, they have a good chance to further appreciate the power, simplicity, and elegance of calculus.

References

[1] D.C. Giancoli, Physics for Scientists and Engineers with Modern Physics. (Pearson Education, 2008).
[2] H.D. Young, R.A. Freedman, and A.L. Ford, *Sears and Zemansky’s University Physics with Modern Physics*. (Pearson Education, 2012).

[3] D. Kleppner, and R. Kolenkow, *An Introduction to Mechanics*. (Cambridge University Press, 2014).

[4] D. Morin, *Introduction to Classical Mechanics: with Problems and Solutions*. (Cambridge University Press, 2008).

[5] J. Hass, C. Heil, and M. D. Weir, *Thomas’ Calculus*. (Pearson Education, 2018).

[6] Eduardo E. Rodriguez, and Gabriel A. Gesnouin. “Effective mass of an oscillating spring.” *Phys. Teach.* 45, no. 2 (2007): 100-103.

[7] Ruby Lawrence. “Equivalent mass of a coil spring.” *Phys. Teach.* 38, no. 3 (2000): 140-141.

[8] James T. Cushing, “The spring-mass system revisited.” *Am. J. of Phys.* 52, no. 10 (1984): 925-933.

[9] Robert Weinstock, “Oscillations of a particle attached to a heavy spring: An application of the Stieltjes integral.” *Am. J. of Phys.* 47, no. 6 (1979): 508-514.

[10] Calin Galeriu, Scott Edwards, and Geoffrey Esper. “An Arduino investigation of simple harmonic motion.” *Phys. Teach.* 52, no. 3 (2014): 157-159.

[11] John Kinchin, “Using Tracker to prove the simple harmonic motion equation.” *Phys. Edu.* 51, no. 5 (2016): 053003.

[12] Unofre Pili, and Renante Violanda. “Measuring a spring constant with a magnetic spring-mass oscillator and a telephone pickup.” *Phys. Edu.* 54, no. 4 (2019): 043001.

[13] M. Spivak, *Calculus*. (Publish or Perish, 2008).