We present a fairly simple method of obtaining a completely entangled lattice of qubits, which are relevant in quantum computing and may be important also in other uses of quantum neural networks, using a modified form of the controlled-NOT quantum gates connecting nearest neighbors for computational economy. A normal c-NOT gate, though unitary and having the simplicity of a single operation of flipping the controlled qubit, is time consuming, as it produces complex mixing coefficients. Therefore a slightly modified form of the gate, which is named $c'$ – NOT gate here, is used, which inverts the phase as well as the state of the controlled qubit when the controlling qubit is excited. It too gives a manifestly unitary transition matrix for each updating of the network while keeping all the numbers produced in the operations real. The dynamics leads to a completely entangled state of the qubits in the system with variable coefficients for the superposition of the states of the qubit nodes in different circumstances. Simulation shows a surprising property of the dynamics of the network, viz. the possibility of obtaining the initial state by a method of back-projecting the complicated entangled states that evolve after thousands of updating of the entire network involving modifications of each qubit through interactions with the neighbors through the quantum gates. We also prove that it is not possible for a sequence of unitary operators working on a net to make it move from an aperiodic regime to a periodic one, unlike some classical cases where phase-locking happens in course of evolution. However, we show that it is possible to introduce by hand periodic orbits to sets of initial states, which may be useful in forming dynamic pattern recognition systems.

PACS numbers: 03.67.Lx, 07.05.Mh, 84.35.+i
Keywords: neural network; quantum computing

I. INTRODUCTION

Quantum computing promises [1] to make a breakthrough in the capacity of computers to hold numbers and in the speed of processing on account of the entanglement of states. As neural networks can be conceived as dedicated hardware which can also process information for pattern recognition and other AI tasks, including associating specific memories with its internal dynamism, for later recovery or usage, quantum neural networks are also candidates for exhaustive study. However, this problem of transition to quantum neural nets may best be approached in stages, on account of the complications that different attributes of a quantized system bring about. The evolution of a semiclassical neural network model with an action-potential-like quantum interaction that triggers classical nodes in a stochastic manner have previously been simulated and the results agree in part with classical integrate-and-fire models while gaining some new features from the quantum characteristics of the system [2]. A quantum neural network with nearest neighbor nodes connected by c-NOT gates, as might be expected to be present in a quantum computer, have also been studied [3]. However, in both these investigations complete entanglement of all the nodes of the system have not been attempted; each qubit has been measured independent of the others, though in the latter work the interaction proceeded by unitary quantum c-NOT gates between nearest neighbor nodes. The neglect of entanglement of the whole lattice was unavoidable in these case-studies, as the concentration was on the origin and pattern of periodicity of the dynamics of the networks, which required a fairly large number of nodes, and it is virtually impossible to do simulation experiments with the corresponding huge number of completely entangled states of the system. We now present some results with a more complex model of a completely entangled quantum network on a computable scale. The simplest nontrivial net with a small number of nodes and symmetrical boundary conditions is constructed and results of computer simulations are found. It is seen that with normal c-NOT gates we need considerably more computing time than with a modified $c'$ – NOT form of the gates. We also show why a quantum system in general cannot develop any dynamic periodicity which is often seen in classical models achieving phase-locking of the nodes after pass-
ing through an aperiodic regime. We then show how ab initio periodicity may be introduced by hand.

II. AN ENTANGLED QUANTUM NETWORK MODEL

We take a $n \times n$ lattice with usual periodic boundary conditions, so that it can effectively mimic a bigger lattice in some respects. So we have $N = n \times n$ independent nodes, each a quantum qubit, e.g. a spin-1/2 object. We connect each node to its neighbor through c-NOT gates.

We are adopting this simple scheme to see how such a quantum system that treats all inputs equally and also behave under different inputs. In other words, we want a system with no pre-design for any specific purpose would operate on.

The state of the system at any instant can be represented by

$$|\psi\rangle = \sum_I a_I |\psi_I\rangle \quad (1)$$

where the complete set of unentangled product basis states includes all the possible combinations of the type

$$|q_1q_2\ldots q_N\rangle \quad (2)$$

where $N = n^2$ and each of the qubits can be either in state $|1\rangle$ or in state $|0\rangle$.

So, initially we may have a pure state with only one $a_I = 1$, and all others zero, but as the entanglement proceeds through the c-NOT gates between the nodes, we expect that all or a subclass of states may become entangled. We can of course also choose an entangled initial state, by choosing a non-factorizable superposition of the product states.

We have to consider the effect of each gate at each time step on every $a_I$. The c-NOT gate flips the controlled qubit if the controlling qubit is in state $|1\rangle$, while it does nothing if the controller is in state $|0\rangle$. The controller node is unchanged.

Hence, during simulation we can take each node in turn and consider the effect on all $a_I$ as the neighbors of this node act on this node. The procedure is similar to that described in the earlier unentangled version.

As in the previous case, we first do the flipping in a continuous manner by choosing for each time step $dt$ the transition submatrix for a small change

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\epsilon\sigma_1 dt) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i\epsilon dt \\ 0 & 0 & i\epsilon dt & 1 \end{bmatrix} \quad (3)$$

However, in the previous work, because the nodes remained unentangled and the only effect of the gate was simply to flip each node independently, the state space consisted of only $2n^2$ components (i.e. polynomial in $n$), in contrast to the exponential number in the entangled case. It is not possible to handle a large lattice like 40X40 in this case. In this work we have considered the smallest nontrivial lattice, i.e. a 3X3 lattice. As we have remarked earlier, because of periodic boundary conditions, this is in some limited respects equivalent to an infinite lattice. Even for a 4X4 lattice, there are now 216 product states to upgrade at each step, which is not computationally economic with a classical computer.

To optimize computing time we have first linearized the $2 - d$ label of each qubit $(i,j)$ to a single $1 - d$ label by choosing a sequence, and then we have constructed our label $I$ (stated above) by simply taking the sum:

$$I = \sum_i 2^i \quad (4)$$

where the sum is over only those qubits for which the state is $|1\rangle$ and $i$ is the linear sequential position label of the qubit $(0$ to $N)$. This permits ascertaining the state of any qubit in a particular position $i$ with a single bitwise AND ($\&$) operation and speeds up the simulation process. This also permits putting on any initial state, pure or entangled, by choosing the right combination of $I$'s.

III. PERIODIC AND APERIODIC REGIMES

Before we do our simulations, let us point out a behavior we can anticipate from purely theoretical considerations for any quantum net on which a particular sequence of unitary operators work repeatedly.

**Lemma:** It is not possible by a repeated sequence of unitary operators to move any system from an aperiodic regime to a periodic one.

**Proof:** The product of any given sequence unitary operators $U_i$ is equivalent to a single unitary operator, say $U$.

Let $|i\rangle$ be a vector in the orbit of $U$ in the periodic regime. Now if we operate on $|i\rangle$ by

$$U^\dagger = U^{-1} \quad (5)$$

$$U^{-1}|i\rangle = |j\rangle$$

where $|j\rangle$ must be on the orbit. On the other hand if the state was reached from the aperiodic regime, then we must also have, with the same inverse operation, a transition to a state in the aperiodic regime. This is not possible, because $U$ and its inverse are both linear operators and must give unique results whichever state they operate on.

Hence we would not expect any transition to a periodic system in our simulation.
IV. SIMULATION RESULTS AND A MODIFIED GATE

For a 3X3 lattice there is only one interior point. So if we initially excite the whole periphery we have initially

\[ a_{495} = 1.0 \]  

(6)

As

\[ 495 = 111101111_2 \]  

(7)

To ascertain the importance of unitarity in these simulations we first deliberately used a nonunitary series of operations with \( \epsilon \) imaginary. Our simulations show that even if we begin with a pure state, the nodes get entangled quickly after only a few steps, but after a sufficiently long time the system degenerates to a uniform state with all \( a_i = 1/\sqrt{N} \).

This seems to be because the operator \( \mathbf{A} \) above, which tries to form a continuous c-NOT gate in place of the full matrix even for the 3\( \times \)3 net must be 512\( \times \)512, we cannot check it computationally or analytically. One can argue from symmetry that the highest eigenstate must be the symmetric one, though we are not aware of any mathematical theorem that justifies this hypothesis.

To keep the computational expenses minimal, we shall adhere to real matrices. We next try to construct the infinitesimal form of a unitary matrix representation with a \( c^{\prime} - NOT \) gate which is defined as a quantum gate that reverses the phases of the flipped infinitesimal changed coefficients:

\[
C = \begin{pmatrix}
1 & 0 \\
0 & \sigma_1
\end{pmatrix}
\]  

(8)

using the imaginary parameter \( \epsilon \), becomes a nonunitary one. Hence, unlike a unitary c-NOT operator, this discrete version with an imaginary parameter does not have all eigenvalues of modulus 1. What is happening here is the emergence of the eigenstate corresponding to the highest eigenvalue, as is usually the case for multiple operations of a nonunitary operator. However, since the full matrix even for the 3X3 net must be 512\( \times \)512, we cannot check it computationally or analytically. One can argue from symmetry that the highest eigenstate must be the symmetric one, though we are not aware of any mathematical theorem that justifies this hypothesis.

In the simulation above we have assumed that the system acquired an entangled state as input. Rabitz et al \[ \mathcal{F} \] have presented a method of obtaining superposition of states from a ground state in a molecular system. In a similar spirit we here indicate how it may be possible to get arbitrary combinations of states in our model in a general quantum network. Let us consider a matrix, which we call an "extended unitary matrix", as given below:

\[
U(R, x') = \begin{pmatrix}
R & b|x'|\langle n + 1| \\
0 & |n + 1\rangle\langle n + 1|
\end{pmatrix}
\]  

(13)

with the normalization

\[ |a|^2 + |b|^2 = 1 \]  

(14)

This operator matrix acts on the \( (n+1) \)-dimensional basis with the last vector \( |n + 1\rangle \) an auxiliary vector not related to the \( n \)-dimensional entangled vector space.

Then, given any state \( |x\rangle \), we get the normalized new state

\[
|x''\rangle = aR|x\rangle + b|x'\rangle
\]  

(15)

where we have omitted the smaller order terms.

We notice immediately that the initial state has retained its dominance even after 1000 time steps, and also that the next to leading states are separated from it by just a single 1 in a neighboring qubit.

For another single state \( |17\rangle = |1+16\rangle \) i.e. the one with only a corner and the middle qubit of the lattice initially excited, the final state is (highest amplitude terms):

\[
|17\rangle \rightarrow 0.27(|27\rangle) + 0.23(|19 + 25\rangle)
\]  

(11)

We note that this time the original state has disappeared from the list of dominant states finally, but the terms now dominating do not have a clear choice, and with the AND operation on the bits of the dominating ones we get back the initial \( |17\rangle \)!

\[
(10001) = (11011)&(1011)&(1101)
\]  

(12)

V. CREATION AND DETECTION OF ENTANGLED INPUT STATES

(9)
which may be an additional dummy component of the system.

It is well-known that a complete set of gates, e.g. c-NOT gates, phase gates and Hadamard gates, can simulate any unitary operator. One can trivially extend the arguments to produce our extended unitary operator with such gates too. So a physical realization is not, at least in theory, an insurmountable problem.

The entangled states, despite being superpositions, are pure states. Hence, in principle, the detection of the entangled states is no more complicated than that of single states in the usual basis of the product basis of single spins, provided we rotate the basis to a one that has the chosen vector as a basis vector. One can then use Grover’s search procedure to detect the presence of any particular state. Alternatively, we can determine filtering matrices that perform the same operation directly in the original basis by adapting the sign-reversing and diffusion matrices of Grover for the superposed state taking the appropriate linear transformations. The detection process in quantum computation is of course only stochastic, as originally proposed by Deutsch.

VI. CONCLUSIONS

We see that a network of qubits which is allowed to get completely entangled through an arbitrarily constructed simple nearest neighbor interactions through quantum c′−NOT gates, does indeed do so in general, with a smearing of the excitation from the initially excited nodes to its neighbors, as expected. However, the memory of the initial state seems to be preserved in nontrivial ways depending on whether it is a pure initial state or an entangled one. The method of back-projection to the initial state is fairly simple. It would be interesting to see how entanglement with low noise can be filtered out in this system. It might also be interesting to investigate if a quantum net can serve as a filter separating entangled and separable states, by criteria similar to or different from those proposed recently by Doherty et al [3].

We have shown that unlike a classical network which may often produce phase locking among its nodes and make a transition from an aperiodic regime to a periodic one, a quantum system operated on repeatedly by the same sequence of unitary operators cannot make such a transition, but must always remain in the aperiodic or the periodic regime.

We have indicated that periodic dynamic behavior may be injected into the system at will by choosing the right operator, i.e. a suitable unitary operator that rotates some or all states with time. This may be achieved by choosing the appropriate connectivity among the nodes, which remains to be studied in detail. By giving certain subclasses of the set of states an identical period, it may become possible to use such a net for pattern identification over a huge data base created by the entire set of separable and entangled states. The advantage over a classical system remains the great expansion of the basis in using entangled qubits in place of independent classical binary registers.

The author would like to thank Todd Brun of the Institute for Advanced Study for useful discussions.

[1] M.A. Nielsen and M. Chuang, Quantum computation and quantum information (Cambridge U.P., NY, 2000)
[2] Shafee, F., "Semiclassical Neural Network", arxiv.org, quant-ph/0202015 (2002)
[3] Shafee, F., "Neural Networks with c-NOT Gated Nodes", arxiv.org, quant-ph/0202016 (2002)
[4] Doherty, A.C. et al., "Distinguishing entangled and separable states", arxiv.org, quant-ph/0112007 (2001)
[5] Schirmer, S.G., H. Rabitz et al., "Quantum control using sequence of simple control pulses", arxiv.org, quant-ph/0105155 (2001)
[6] Barenco, A. et al., Phys. Rev. A 52, 3457 (1995)
[7] Grover, L.K., " A fast quantum-mechanical algorithm for database search", Proc. 28th Annul ACM Symposium on the Theory of Computing (STOC’96) p.212 (ACM, Philadelphia, 1996).
[8] Deutsch, D., Proc. Royal Soc. Lond. A 400, 97 (1985)