Entropy bound for a rotating system from Anti de Sitter black holes

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Abstract

General geodesic equations of the motion of spinning systems around the (3+1)-dimensional and (2+1)-dimensional rotating anti-de Sitter black holes have been obtained. Based upon these equations, we derived the entropy bound for a rotating system from Kerr-Anti de Sitter black holes and BTZ black holes, respectively. Our result coincides with that of Hod’s derived from Kerr black hole, which shows that the entropy bound of the rotating system is neither dependent on the black hole parameters, nor on spacetime dimensions. It is a universal entropy bound.

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1 Introduction

In order to rescue the generalized second law (GSL) of thermodynamics, Bekenstein conjectured, some time ago, that there exists an upper bound on the entropy of any neutral object of energy $E$ and maximal radius $R$ in the form $S \leq 2\pi ER/\hbar$ [1]. This derivation was criticized by Unruh, Wald and Pelath [2-4] for neglecting the effects of buoyancy in acceleration radiation. They concluded that no additional assumption for upper entropy bound is necessary to maintain the GSL. However their criticism was refuted by Bekenstein who showed that the buoyancy can really be negligible and does not spoil the entropy bound derivation [5,6].

Bekenstein’s entropy bound has received independent support [7-10]. Recently, extending the derivation of an upper bound on the entropy to any charged object, Bekenstein and Mayo [11], Hod[12] and Linet[13] have shown that Bekenstein’s original entropy bound can be improved. A tighter entropy bound for the nonrotating object of mass $\mu$, radius $R$ and charge $e$ is required. It has been shown that such a bound is $S \leq (2\pi R/\hbar)(E^2 - e^2/2R)$. This result agrees to an earlier finding by Zaslavskii [14] in another context. However, the fact that this entropy bound for a charged system is necessary to uphold the GSL has been challenged as well [15].

A tighter bound on entropy for objects with angular momentum has also been derived recently [16]. Refering to Hojman and Hojman’s [17] integrals of motion for a neutral object with spin $s$ moving on a Kerr black hole background, Hod obtains the entropy bound $S \leq 2\pi ER/\hbar(1 - s^2/E^2 R^2)^{1/2}$. He claimed that this bound is universal and independent of the black hole parameters which were used to derive it. In order to examine this argument, in this paper we will study the entropy bound for a spinning object falling into Anti de Sitter (AdS) black holes including (3+1)-dimensional Kerr-AdS black holes and (2+1)-dimensional BTZ black holes. We will derive geodesic equations of the motion of the spinning objects around Kerr-AdS black holes and BTZ black holes, respectively. Based upon these geodesic equations, we will show that the entropy bound for a rotating system depends neither on the black hole parameters, nor on spacetimes dimensions. It is a universal result.

2 Entropy bound from the Kerr-AdS black holes

Recently the study of the Kerr-AdS black hole model has been undertaken and shown to be important in many aspects[18-21]. The metric is

$$\text{ds}^2 = -\frac{\Delta_r}{\rho^2}[dt - \frac{a}{\Sigma} \sin^2 \theta d\phi]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\sin^2 \theta \Delta_\theta}{\rho^2}[adt - \frac{(r^2 + a^2)}{\Sigma}d\phi]^2$$ (1)
where
\[
\begin{align*}
\rho^2 &= r^2 + a^2 \cos^2 \theta \\
\Delta_r &= (r^2 + a^2)(1 + \ell^2 r^2) - 2Mr \\
\Delta_\theta &= 1 - \ell^2 a^2 \cos^2 \theta \\
\Sigma &= 1 - \ell^2 a^2
\end{align*}
\]

The parameter $M$ is the mass, $a$ the angular momentum per unit mass, and $\ell^2 = -\Lambda/3$, where $\Lambda$ is the (negative) cosmological constant. The black hole solution is valid for $a^2 < l^{-2}$ [18-21]. For $l = 0$, eq.(1) goes back to the metric for Kerr black hole. There are four roots of the polynomial $\Delta_r$, the largest root $r_+$ corresponds to the event horizon, the other positive root $r_-$ is the Cauchy horizon and another two roots $r_1, r_2$ are negative and satisfy $r_1 + r_2 = -(r_+ + r_-), r_1 r_2 = \frac{a^2}{\ell^2 r_+ r_-}$. The mass $M$, and the angular momentum per unit mass $a$, can both be expressed in terms of $r_+, r_-$ and $\ell$ as
\[
\begin{align*}
M &= \frac{(1 + \ell^2 r_+^2)(r_+ + r_-)(1 + \ell^2 r_-^2)}{2(1 - \ell^2 r_+ r_-)}, \\
a &= \sqrt{\frac{r_+ r_- (1 + \ell^2 r_+^2 + \ell^2 r_+ r_- + \ell^2 r_-^2)}{1 - \ell^2 r_+ r_-}}.
\end{align*}
\]

The above equations require $\ell^2 < 1/r_+ r_-$ to ensure real and positive values of $a$ and $M$, respectively. For the extreme black hole case $r_+$ and $r_-$ degenerate and $M = M_\epsilon$, where $M_\epsilon$ is the critical mass parameter given in [18].

We consider a spinning object of rest mass $m$, intrinsic spin $s$ and proper cylindrical radius $R$, which is descending into the Kerr-AdS black hole. Following [17], the constants of motion associated with the $t$ and $\phi$ variables are
\[
\begin{align*}
E &= \pi_t - g_{t\phi} \pi_\phi \frac{s\Sigma}{2rm} + g_{tt} \pi_t \frac{s\Sigma}{2rm}, \\
J &= -\pi_\phi + g_{\phi t} \pi_t \frac{s\Sigma}{2rm} + g_{\phi\phi} \pi_\phi \frac{s\Sigma}{2rm}
\end{align*}
\]

where
\[
\begin{align*}
\pi_t &= g_t \dot{t} + g_{t\phi} \dot{\phi} \\
\pi_\phi &= g_\phi \dot{t} + g_{\phi\phi} \dot{\phi}
\end{align*}
\]

For simplicity we just consider the equatorial motions of the object. The quadratic equation for the conserved energy $E$ of the body is
\[
\ddot{\alpha} E^2 - 2\dot{\alpha} E + \dot{\gamma} = 0
\]
where the very long expressions for $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are given in the appendix. It is worth noting that taking $l \to 0$, they reproduce the expressions given in [17].

In the spirit of the analysis of Bekenstein and of Hod, we neglect the buoyancy contribution, for simplicity. Suppose the gradual approach to the black hole must stop when the proper distance from the body’s center of mass to the black hole horizon equals $R$, the body’s radius

$$\int_{r_+}^{r_+ + \delta(R)} (g_{rr})^{1/2} dr = R,$$

(10)

with $g_{rr} = \frac{r^2}{\Delta_r}$ (in equatorial plane) and $\Delta_r = l^2 (r - r_+) (r - r_-) (r - r_1) (r - r_2)$. One can get from integrating Eq.(10)

$$\delta(R) = \frac{l^2 (r_+ - r_-) (r_+ - r_1) (r_+ - r_2) R^2}{4r_+^2}.$$  

(11)

Considering the relation between $r_1, r_2$ and $r_+, r_-$ together with eqs.(3,4), it is not difficult to find that when $l \to 0$, eq.(11) reproduces eq.(6) of [16].

Using the test particle approximation $s/(mr_+) \ll 1, R \ll r_+$ together with the condition $l^2 < 1/(r_+ r_-)$, we can solve eq.(9) for $E$ to first order in the small quantities at the point of capture $r = r_+ + \delta(R)$,

$$E = u + vs + R \sqrt{w}$$

(12)

where

$$u = \frac{aJ(a^2 l^2 - 1)(2M - a^2 l^2 r_+ - l^2 r_+^3)}{-2a^2 M - a^2 r_+ + a^4 l^2 r_+ - r_+^4 + a^2 l^2 r_+^4}$$

(13)

$$v = \frac{J(a^2 l^2 - 1)}{mr_+(2a^2 M + a^2 r_+ + a^4 l^2 r_+ + r_+^2 - a^2 l^2 r_+^2)^2}$$

$$\times (2a^4 M - 6a^2 M^2 r_+ + 3a^2 Mr_+^2 + 5a^4 l^2 Mr_+^2 - a^2 r_+^3 + 2a^4 l^2 r_+^3 +$$

$$- a^6 r_+^3 + 3Mr_+^4 + 3a^2 l^2 Mr_+^4 - r_+^5 + 2a^2 l^2 r_+^5 - a^4 l^4 r_+^5)$$

$$w = \frac{r_+ - r_-)^2 (-1 - l^2 r_+^3 - 2l^2 r_+ - 3l^2 r_+^2 + 3l^2 r_+^2 + 2l^2 r_+^2 + 2l^2 r_+^3)^2}{4(1 + l^2 r_+^2)^4 r_+^4 (r_- + r_+)^4}$$

$$\times (J^2 + m^2 r_+^3 + 2l^2 m^2 r_+^4 + l^4 m^2 r_+^6 - 4J^2 l^2 r_+ - 2m^2 r_+ - 2l^2 r_+^3 r_+ -$$

$$+ 4l^2 m^2 r_+^3 + 2l^4 m^2 r_+^4 + l^6 m^2 r_+^6 - 2J^2 l^2 r_+^2 r_+ + l^4 m^2 r_+^2 + 2l^2 m^2 r_+^2 r_+^2 + 2l^2 m^2 r_+^2 r_+^2 + 4J^2 l^6 r_+^2 r_+^2$$

$$+ 4l^2 m^2 r_+^3 r_+^3 + 3J^2 l^8 r_+^3 r_+^3 - 2J^2 l^4 r_+^3 r_+^3 + 4J^2 l^6 r_+^3 r_+^3 + 2J^2 l^8 r_+^3 r_+^3 + 4J^2 l^6 r_+^3 r_+^3$$

$$+ 3J^2 l^8 r_+^3 r_+^3 + 2J^2 l^8 r_+^3 r_+^3 + J^2 l^8 r_+^3 r_+^3 + J^2 l^8 r_+^3 r_+^3).$$

Eq.(12) reduces to (7) of [16] after taking the limit $l \to 0$ and substitution of Eqs.(3,4).

After the assimilation of the spinning body, the change of the black hole mass and angular momentum are $dM = E$ and $dL = J$, respectively. Taking cognizance of eq.(12) and using the first-law of black hole
thermodynamics,

\[ dM = \frac{\kappa}{8\pi} dA + \Omega dL \]  

(16)

where \( \kappa = \frac{r_+ (1 + a^2 l^2 + 3 l^2 r_+^2 - a^2 / r_+^2)}{2 (r_+^2 + a^2)} \) and \( \Omega = \frac{a (1 - l^2 a^2)}{r_+^2 + a^2} \) are the surface gravity and rotational angular frequency of the black hole respectively, we find

\[ dA = \frac{8\pi}{\kappa} (u + vs + R \sqrt{w} - \Omega J) \]  

(17)

Substituting eqs.(3,4), it is easy to see that \( u - \Omega J = 0 \). Carefully choosing the total angular momentum of the body at the critical value

\[ J = J^* = \sqrt{\frac{m^2 (1 + l^2 r_+^2)^2 (r_+ + r_-)^2 s^2}{(-1 + 2 l^2 r_+ r_- + l^4 r_+^3 r_- + l^4 r_+^2 r_-^2 + l^4 r_-^3 r_+^2)(m^2 R^2 - s^2)}}, \]  

(18)

the minimum value of the increase in the black hole surface area caused by an assimilation of a spinning body with given parameters \( m, s, R \) is

\[ dA_{\text{min}} = \frac{8\pi}{\kappa} \sqrt{m^2 R^2 - s^2} \]  

(19)

The minimum exists only for \( s \leq mR \).

By virtue of the GSL, we derived an upper bound to the entropy \( S \) of an arbitrary system of proper energy \( E \), intrinsic angular momentum \( s \) and proper radius \( R \) falling into the Kerr-AdS black hole

\[ S \leq 2 \pi \sqrt{(RE)^2 - s^2} \]  

(20)

It is evident that the entropy \( S \) of the rotating system should be bounded and this bound is more stringent than the original Bekenstein bound [1]. It is worth noting that although we used a different black hole model to derive the entropy bound, the final result is the same as in [16] and independent of the black hole parameters which are used to derive it.

### 3 Entropy bound from the BTZ black hole

The entropy bound for the rotating system derived from the (3+1)-dimensional Kerr-AdS black hole is the same as that from Kerr black hole [16], which shows that this bound does not depend on the black hole parameters. Now it is of interest to ask the question whether this bound only exists for (3+1)-dimensional cases and whether it will be changed if it is derived from a different dimensional black hole. In this section we will concentrate our attention on the (2+1)-dimensional BTZ black holes [23,24]. The metric of this black hole reads

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \]  

(21)
where the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$$  \hspace{1cm} (22)

$$N^\phi = -\frac{J}{2r^2}$$

where $-\infty < t < +\infty, 0 < r < \infty$, and $0 \leq \phi \leq 2\pi$. $M, J$ appearing in (22) are two constants of integration which can be interpreted as the black hole mass and angular momentum. The lapse function $N^2(r)$ vanishes when

$$r_\pm = l \left( \frac{M}{2} \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right) \right)^{1/2}$$  \hspace{1cm} (23)

Here $r_+$ is the event horizon and $r_-$ is the Cauchy horizon for $M > 0$ and $|J| < Ml$. In the extreme case $|J| = Ml$, $r_+$ and $r_-$ degenerate.

We proceed to devise a gedanken experiment in which a spinning object of mass $m$, intrinsic spin $s$ and proper cylindrical radius $R$ is decending into the BTZ black hole. Using constants of motion associated with the $t$ and $\phi$ definition

$$E = \pi_t - g_{t\phi,r} \pi_r \frac{s}{2mr} + g_{tt,r} \pi_r \frac{s}{2mr}$$  \hspace{1cm} (24)

$$L = -\pi_\phi - g_{\phi,r} \pi_r \frac{s}{2mr} + g_{\phi\phi,r} \frac{s}{2mr}$$  \hspace{1cm} (25)

We obtained the quadratic equation for the conserved energy $E$ of the spinning body in the form of (9), here

$$\alpha = r^2 - Js/m + (M - r^2/l^2)s^2/m^2$$  \hspace{1cm} (26)

$$\beta = JL/2 - LM s/m + J L s^2 / (2l^2 m^2)$$  \hspace{1cm} (27)

$$\gamma = L^2 M - J^2 m^2 / 4 - L^2 r^2 / l^2 + M m^2 r^2 - m^2 r^4 / l^2 - J L^2 s / (l^2 m)$$

$$+ [J^2 / 2l^2 - 2Mr^2 / l^2 + L^2 r^2 / (l^4 m^2) + 2r^4 / l^4] s^2 - (J^2 / 4 - Mr^2 + r^2 / l^2) s^4 / (l^4 m^2)$$  \hspace{1cm} (28)

Taking $s \to 0$ and adopting re-scalings given in [25], the quadratic equation reproduces the special case given in [25].

Neglecting the buoyancy contribution, we suppose the gradual approach of the spinning object to the black hole must stop when the proper distance from the body’s center of mass to the black hole horizon equals $R$. Considering $N^2 = \frac{(r_+^2 - r_+^2)(r^2 - r_+^2)}{l^2 r^2}$, from Eq.(10), we have

$$\delta(R) = \frac{(r_+^2 - r^2) R^2}{2r_+^2 l^2}.$$  \hspace{1cm} (29)

Using conditions of approximation for test particle $s/(mr_+) \ll 1, R \ll r_+$, we can get the energy expression in the same form as (12), where $u, v, w$ here are

$$u = \frac{JL}{2r_+^2}$$  \hspace{1cm} (30)
After the infall of the spinning object, the change of the black hole mass and angular momentum are \( dM = E \) and \( dL = L \), respectively. Employing the first-law of black hole thermodynamics Eq. (16), where the surface gravity and angular velocity here are

\[
\kappa = \sqrt{\frac{M^2 - J^2}{r_+^2}} = \frac{r_+^2 - r_-^2}{l^2 r_+}, \quad \Omega = \frac{J}{2 r_+^2}.
\]

When the total angular momentum attains the critical value

\[
L^* = \frac{m r_+ s}{\sqrt{m^2 R^2 - s^2}},
\]

there is a minimum increase in the black hole surface area caused by an assimilation of the spinning body in the form

\[
dA_{\text{min}} = 8 \pi \sqrt{m^2 R^2 - s^2}
\]

This minimum exists only for \( s \leq mR \).

Arguing from the GSL, we derive an upper bound to the entropy \( S \) of an arbitrary system of proper energy \( E \), intrinsic angular momentum \( s \) and proper radius \( R \) from the BTZ black hole

\[
S \leq 2 \pi \sqrt{(RE)^2 - s^2}
\]

This result is the same as obtained from the (3+1)-dimensional cases.

### 4 Conclusions and discussions

Using the method proposed by Hojman and Hojman [17], we have derived the geodesic equations for the spinning object moving around the (3+1)-dimensional Kerr-AdS black holes and (2+1)-dimensional BTZ black holes, respectively. These geodesic equations are general compared to those obtained in the models without cosmological constant in [17] for Kerr black hole case and without considering the spinning of the object for BTZ black hole [25]. Based upon these geodesic equations, we derived the entropy bound for the rotating system to maintain the GSL. These results coincide with that obtained from Kerr black hole [16], which supports Hod’s argument that the entropy bound of the rotating system is independent of the black hole parameters. Besides it is worth noting that our result from the three-dimensional BTZ black holes indicates that this entropy bound is also independent of the dimensions of spacetimes. Therefore the entropy bound for the rotating system is universal.

However at the first sight, the universality of the entropy bound for the rotating system cannot be extended to a charged system. Because at least the electric potential will change the form when we study it...
for the (2+1)-dimensional case. The exact dependence of the entropy bound on the system’s parameters for the charged system still needs further exploration.

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5 Appendix

Here we give the expressions of \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \) in eq.(9) in the main text.

\[
\tilde{\alpha} = \alpha + k_1 l^2 + k_2 l^4 \tag{36}
\]
\[
\tilde{\beta} = \beta + p_1 l^2 + p_2 l^4 + p_3 l^6
\]
\[
\tilde{\gamma} = \gamma + q_1 l^2 + q_2 l^4 + q_3 l^6 + q_4 l^8
\]

where

\[
k_1 = -a^4 - a^2 r^2 + (a^2 + 2a^2 M/r - r^2)s^2/m^2 \tag{37}
\]
\[
k_2 = a^2 r^2 s^2/m^2
\]
\[
p_1 = -a^3 J - 2a^3 JM/r - aJ^2 + (-2a^2 J + 2a^4 JM/r^3 + 3a^2 JM/r)s/m
\]
\[
+ (aJ - a^3 JM^2/r^4 - a^3 JM/r^3 + 2aJM/r)s^2/m^2 \tag{38}
\]
\[
p_2 = a^5 J + a^3 J r^2 + a^4 J s/m + (-a^3 J - 2a^3 JM/r + aJ^2)s^2/m^2
\]
\[
p_3 = -a^3 J r^2 s^2/m^2
\]
\[
q_1 = a^2 J^2 - 4a^2 J^2 M/r - J^2 r^2 - a^2 m^2 r^2 - m^2 r^4 + (-2aJ^2 + 4a^3 J^2 M/r^3)s/m
\]
\[
+ [2a^2 - 2a^2 J^2 M^2/(m^2 r^4) + 2a^2 M/r + 2J^2 M/(m^2 r) - 2Mr + 2r^2]s^2
\]
\[
+ [-a^2 M^2/r^4 - 2a^2 M/r^3 + 3M^2/r^2 - 2M/r]s^2/m^2 \tag{39}
\]
\[
q_2 = a^4 J^2 + 2a^4 J^2 M/r + 2a^2 J^2 r^2 + (4a^3 J^2 - 2a^5 J^2 M/r^3)s/m
\]
\[
+ [a^4 J^2 M^2/(m^2 r^4) - 4a^2 J^2 M/(m^2 r) + 2a^2 r^2 + J^2 r^2/m^2 + 2r^4]s^2
\]
\[
+ (-a^2 - 2a^2 M/r - r^2)s^4/m^2
\]
\[
q_3 = -a^6 J^2 - a^4 J^2 r^2 - 2a^5 J^2 s/m
\]
\[
+ (2a^4 J^2 M/r - 2a^2 J^2 r^2) s^2/m^2 + (-a^2 r^2 - r^4)s^4/m^2
\]
\[
q_4 = a^4 J^2 r^2 s^2/m^2
\]
\(\alpha, \beta\) have the same expressions given in [17]. There is a sign mistake for the expression \(\gamma\) in [17], where the last term should be negative. The correct formula is

\[
\gamma = -e^2\phi^2k_1 + 2e\phi(j - eh)k_3 - (j - eh)^2k_2 - \delta^2\Delta M^2
\]

(40)

so that the requirement

\[
\beta^2 - \alpha\gamma = \Delta\delta^2[(j - eh)^2 - k_1 M^2]
\]

(41)

can be satisfied.

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