Solitary waves on falling liquid films in the low Reynolds number regime

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Abstract
We study the problem of a thin liquid film falling down an inclined slope. We use a simplified model to study the evolution and morphology of the solitary waves on a thin film with a periodic forcing at the inlet. In recent work by Denner et al (2016 Phys. Rev. E 93, 033121), the regime for high Re was studied and results obtained on the geometry and dispersion of the waves. We wish to establish whether similar results are observed in a regime of smaller Re and examine quantities which can be compared with experiment, such as the maximum and minimum film height as a function of a rescaled Reynolds number which accounts for the inclination of the substrate. Our results show some evidence that \( h_{\text{min}} \) collapses onto a single curve when plotted as a function of Re*, and approaches the absolute value of \( h_{\text{min}}/h_N = 0.375 \) in agreement with results obtained at higher Re. We also obtain a curve for \( d/\lambda \) which can be compared with the results reported in (Denner et al 2016 Phys. Rev. E 93, 033121).

A liquid film flowing down an inclined plane is an example of a convectively unstable open flow hydrodynamic system. As a physical system, thin liquid films have been studied for over fifty years [1]. This system can be described by the Navier–Stokes equations and for small-to-moderate Reynolds number, it can be treated as a two-dimensional problem. These liquid films have a sequence of spatiotemporal transitions which are generic to a large family of nonlinear systems. The theory we discuss here applies to thin liquid films where one can use a long-wavelength approximation, but thick films are also studied in the literature along with other types of thick flow [2]. In this note, we begin to study the dynamics and geometry of solitary waves as the Reynolds number is varied (assuming low to moderate Reynolds number). This is done at the basic level by using a low-dimensional model based on a two-field system of equations for the local flow rate \( q \) and interface position \( h \). In particular, we investigate the distances between two consecutive maxima of the waves and the distance between a maximum and the next minimum, so beginning to characterise the degree of asymmetry of the solitary waves.

The governing equations are the Navier–Stokes equations in two dimensions:

\[
\rho (\partial_t u + u \partial_x u + v \partial_y u) = -\partial_x p + \rho g \sin \beta + \mu \nabla^2 u, \quad (1a)
\]

\[
\rho (\partial_t v + u \partial_x v + v \partial_y v) = -\partial_y p - \rho g \cos \beta + \mu \nabla^2 v, \quad (1b)
\]

where \( p \) is the pressure, \( \mu \) is the dynamic viscosity and \( u \) and \( v \) are the \( x \)- and \( y \)- components of the velocity, respectively. Along with this, we have the usual continuity equation

\[
\partial_x u + \partial_y v = 0. \quad (2)
\]

These equations are subject to the kinematic stress balance or ‘momentum jump’ condition on the free surface and the normal and tangential stress boundary conditions (also both on the free surface), as well as the standard no-slip and no-penetration boundary condition at the wall [1]. By combining the long-wave expansion with a weighted residual technique based on Galerkin projection in which the velocity field is expanded onto a basis with polynomial test functions, Ruyer-Quil and Manneville [3] obtained the following second-order two-field model composed of two coupled nonlinear PDEs for the local film thickness \( h(x, t) \) and flow rate \( q(x, t) \):

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = \frac{\partial q}{\partial x},
\]

\[
\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial y} = \frac{\partial h}{\partial x}.
\]
where $\delta$ is a reduced Reynolds number defined as

$$\delta = 3 \text{Re} (3\text{Ca})^{1/3},$$

and $\eta$ is the viscous dispersion number defined as

$$\eta = (9\text{Ca}^2)^{1/3}.$$

$\delta$ and $\eta$ are derived from the Shkadov scaling \cite{4} and Re and Ca are the standard Reynolds and capillary numbers. (3b) is simply the condition of conservation of mass.

The model equations are solved numerically using a Runge-Kutta method with variable timestep and a finite difference scheme for the spatial variable. In terms of the boundary conditions, $h$ is third order in the derivative with respect to the spatial coordinate and $q$ is second order in the spatial derivative, so one must prescribe the flow rate $q$ at one fictitious node, $x_0 = 0$ upstream of the left-hand side of the computational domain, whereas the film height $h$ must be prescribed at two fictitious nodes at $x_0 = 0$ and $x_{-1} = -\Delta x$. This provides us with an inflow boundary condition corresponding to periodic forcing at the inlet.
where \( f \) is the frequency of the forcing and \( A \) is the amplitude from the mean. For the boundary condition at the outlet at the right-hand side of the computational domain, it is desirable to avoid fictitious nodes. Instead, we impose an ad hoc soft boundary condition which imitates the wave behaviour of the liquid film.

\[
q_0^{(n)} = \frac{1}{3} (1 + A \sin(2\pi ft_0)),
\]

\[
h_{-1}^{(n)} = h_0^{(n)} = 1,
\]

where \( f \) is the frequency of the forcing and \( A \) is the amplitude from the mean. For the boundary condition at the outlet at the right-hand side of the computational domain, it is desirable to avoid fictitious nodes. Instead, we impose an ad hoc soft boundary condition which imitates the wave behaviour of the liquid film.

\[
\partial_t q = \nu_f \partial_x q,
\]

\[
\partial_t h = \nu_f \partial_x h.
\]

\( \nu_f > 0 \) is a parameter which is fine-tuned empirically to minimise fictitious reflections at the outlet. For more technical details on this such as convergence and stability of the scheme and the formulation of the soft boundary condition, see appendix F3 of [1]. The initial condition for the solitary wave is the Nusselt flat film solution, which is
after normalisation by the Nusselt film height and Nusselt velocity. If we consider a flat, laminar, falling liquid film flowing down an inclined plane with no perturbations, the Nusselt flat film solution is a unique equilibrium solution to the two-dimensional Navier–Stokes equations, with an associated flat film height known as the Nusselt film height

\[ \frac{h(x, 0)}{h_N} = 1, \quad (8a) \]

\[ q(x, 0) = 1, \quad (8b) \]

where \( h \) and \( q \) are the film height and flow rate per unit span, respectively. The associated average film velocity (the Nusselt velocity) is defined as

\[ u_N = \frac{g \sin \beta h_N^2}{3 \mu}, \quad (9) \]

where \( \mu \) and \( \rho \) are the dynamic viscosity and the density of the liquid film, \( q = u_N h_N \) is the flow rate per unit span, and \( g \) is the acceleration due to gravity. The associated average film velocity (the Nusselt velocity) is defined as

\[ u_N = \frac{g \sin \beta \rho h_N^2}{3 \mu}, \quad (10) \]

The Nusselt solution takes the form of a steady uniform parallel flow with a parabolic velocity profile: the word ‘equilibrium’ refers to the fact that it is a constant, steady-state solution to the Navier–Stokes equations [5]. Following the recent work by Denner et al [6], we consider an inclination-corrected scaling so that the Reynolds number \( \text{Re} \) is scaled as:

\[ \text{Re}^* = \text{Re} \sin \beta. \quad (11) \]

In figures 1–3 we provide an example of a solution to the model equations plotted at various times to demonstrate the spatiotemporal evolution of solitary waves on a falling liquid film. The parameter values are \( \text{Re} = 20 \) and \( \beta = 60^\circ \). The times are \( t = 100 \), \( t = 200 \) and \( t = 1000 \).

We solve the model equations numerically to plot the maximum and minimum film height (figures 4 and 5, respectively) for the inclination angles \( 60^\circ \), \( 75^\circ \) and \( 90^\circ \) and for a range of Reynolds numbers between 10 and 40. The maxima and minima are then normalized by the Nusselt film height \( h_N \) and plotted against the driving Reynolds number \( \text{Re}^* \).

We next investigate the distance between two consecutive maxima and the distance between a maximum and the next minimum, as this allows us to characterise the degree of asymmetry of the solitary waves.

In particular, we have attempted to quantify the asymmetry by dividing the distance between the global maximum and the next minimum (denoted by \( d \) and plotted in figure 6) by the distance between the global maximum and the next maximum (denoted by \( \lambda \) and plotted in figure 7) and plotting the ratio of the two distances against the driving Reynolds number in figure 8. We point out that \( d \) and \( \lambda \) are both invariants between upstream and downstream at a distance of 3 or 4 wavelengths from the inlet provided that the forcing frequency \( f \)
is low. At low frequencies, the geometry and shape of the solitary waves is quasisteady with slight fluctuations due to interactions of a pulse with capillary ripples on pulse tails and only begins to distort downstream from the inlet at intermediate forcing frequencies [7]. The solitary waves can also be distorted at the outlet of the computational domain due to spurious reflections from the boundary condition scheme: this is partly the motivation for the soft boundary condition, which can be tuned to minimise these reflections.

Our results show some evidence that the properties observed for solitary waves in the high Reynolds number regime in [6] also appear in the regime of smaller Reynolds numbers. In particular, the minimum film height normalized by the Nusselt film height forms a curve when plotted against the driving Reynolds number which approaches an absolute value which agrees with the results obtained at higher Reynolds number in [6]. Similarly, when we plot the ratio of the distance between the global maximum and the next consecutive minimum and the distance to the next consecutive maximum as a function of the driving Reynolds number, the ratio collapses

Figure 6. Plot of $d$ against driving Reynolds number $Re^*$. Legend shows the angle in degrees.

Figure 7. Plot of $\lambda$ against driving Reynolds number $Re^*$. Legend shows the angle in degrees.
onto a single curve, although some further work would be needed to strengthen these conclusions and compare
the low-dimensional results to the ones reported in [6]: this would provide further evidence that solitary waves
have the same characterisation in the low Reynolds number regime (at least in terms of geometry and shape).
Some further analysis of the kind we have discussed is carried out in [8].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary
information files).

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Figure 8. Plot of $d/\lambda$ against driving Reynolds number $Re^*$. Legend shows the angle in degrees.