High energy scattering and the AdS/CFT correspondence

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Abstract

We consider small angle and large impact parameter high energy scattering of colourless states in SYM using the AdS/CFT correspondence. The gauge theory scattering amplitude is linked with a correlation function of tilted Wilson loops, which can be calculated by the exchange of bulk supergravity fields between the two corresponding string worldsheets. We identify the dominant contributions, which all correspond to real phase shifts. In particular, we find a contribution of the bulk graviton which gives an unexpected ‘gravity-like’ s\textsuperscript{1} behaviour of the gauge theory phase shift in a specific range of energies and (large) impact parameters.

1 Introduction

The remarkable duality between supergravity (string theory) on AdS and supersymmetric gauge theory \cite{1} has attracted much attention. Diverse phenomenae on both sides of the correspondence were investigated (see e.g. references in \cite{2}).
In this paper we would like to consider the problem of high energy scattering of massive colourless states, in the large $s$, large impact parameter regime (near forward scattering) in YM theory, and to study it using the AdS/CFT correspondence. Although no quantitative predictions are available for high energy scattering in strongly coupled SYM, there are some qualitative expectations (unitarity, constraints on the $s$ behaviour of amplitudes related to analyticity and crossing properties) that should be satisfied, and which would be interesting to investigate from the supergravity side. Apart from that, in this regime one observes a different hierarchy of importance of various supergravity fields in comparison to e.g. ‘static’ quantities like $q\bar{q}-q\bar{q}$ potential. Another point is that here one expects the dominant contribution to come just from the ‘gluonic’ sector of the theory. The contribution of fermions and scalars is expected to be subleading, and thus the results may not be too closely tied to the $\mathcal{N} = 4$ supersymmetry (see \cite{4}).

High energy scattering amplitudes may be parameterized by phase shifts $\delta$:

$$\frac{1}{s} A(s, t) = \int d^2b \ e^{ibq} \frac{e^{i2\delta(b, s)}}{2i} - 1$$

where $b$ is the impact parameter (in the following we will denote its modulus by $L$). In field theory, there exists a spin-energy relation, the exchange of elementary scalars gives rise to a phase shift behaving like $1/s$, vectors lead to $s^0$ behaviour. The exchange of spin-2 particles (like gravitons) would on the other hand lead to a dramatic rise $s^1$. This last possibility is excluded in perturbative SYM with combinations of elementary vector, scalar and fermion exchanges\cite{1}.

Qualitatively one expects the same pattern of behaviour in the AdS/CFT correspondence. The scattering amplitude will be seen to correspond to a correlation function of two Wilson loops, which can be evaluated, on the supergravity side, as the exchange of (bulk) supergravity fields between the associated string worldsheets. We will show that the same pattern of spin-energy behaviour persists here and, in particular, that the graviton gives rise to the unexpected $s^1$ dependence in the real part of the amplitude. Finally we will show that the Drukker-Gross-Ooguri Legendre transform prescription\cite{6} for Wilson loops does not modify this result.

\footnote{Note, however, that a perturbative resummation at high energy can lead to the exchange of a compound state and a rise in the $s$ dependence\cite{5}.}
The plan of the paper is as follows. First we discuss general properties of scattering amplitudes and introduce the appropriate gauge theory observable. Then we make a passage to euclidean space and apply the AdS/CFT correspondence. In section 5 we present calculations for various fields. In section 6 we show that the Legendre transform prescription for Wilson loops \([6]\) does not change the main results. Finally we analyze the region of validity of our calculations and discuss the results.

2 Scattering amplitudes

Quark-quark scattering amplitudes in the high energy limit (and small momentum transfer) can be conveniently expressed, in the eikonal approximation, in terms of a correlator of Wilson lines \([7, 8, 9]\). The high energy limit is reached when the lines move towards the light cone. Gauge invariance is restored (see e.g. \([8]\)) by requiring that the gauge transformations at both ends of the line are the same. We will here pursue a different route \([10]\) by substituting a Wilson loop for each of the Wilson lines.

\[
-2is \int d^2x_\perp e^{iqx_\perp} \left\langle \frac{W_1W_2}{\langle W_1 \rangle \langle W_2 \rangle} - 1 \right\rangle
\]

where the Wilson loops follow classical straight lines for quark(antiquark) trajectories: \(W_1 \rightarrow x_1^\mu = p_1^\mu \tau (\pm a^\mu)\) and \(W_2 \rightarrow x_2^\mu = x_1^\perp + p_2^\mu \tau (\pm a^\mu)\) and close at infinite times. This corresponds to the scattering of colorless quark-antiquark pairs with transverse separation \(a\) (see figure 1 for the geometry rotated to euclidean space).

The rôle of the quarks in the AdS/CFT correspondence will be played, as in \([11]\), by the massive \(W\) bosons arising from breaking \(U(N + 1) \rightarrow U(N) \times U(1)\). The geometrical parameters of the configuration can be related to the energy scales by the relation

\[
cosh \chi \equiv \frac{1}{\sqrt{1 - v^2}} = \frac{s}{2m^2} - 1
\]

where \(\chi = \frac{1}{2} \log \frac{1 + v}{1 - v}\) is the Minkowski angle (rapidity) between the two lines, and \(v\) is the relative velocity. Our aim is to apply the AdS/CFT correspondence to calculate \([4]\). In order to avoid the complications of Lorentzian AdS/CFT correspondence \([12]\) we will link, following \([13]\), this observable with a related observable in euclidean space.
3 Analytical continuation to Euclidean time

In [13] a scattering amplitude with Wilson lines was linked with an analogous correlator of Wilson lines in Euclidean spacetime which form an angle $\theta$ (see figure 1). The parametrization of the Wilson lines is given by euclidean momenta

$$p_1^E = (1, 0, 0) \quad p_2^E = (\cos \theta, -\sin \theta, x_\perp)$$

After performing the calculation one analytically continues

$$\theta \longrightarrow -i \chi \sim -i \log \left( \frac{s}{2m^2} \right)$$

(5)

to obtain (2).

This claim was supported [13] by a number of explicit calculations and a general argument based on the relation between correlators of the gauge fields contracted with momenta both in Euclidean and Minkowski space$^2$. A similar link between scattering amplitudes of branes and Euclidean potentials between branes at angles [14] has already been exploited (see ref. [15]).

As an example of this formalism we will consider, following [16], the leading order perturbative QED amplitude. The Wilson line correlator

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$^2$The same analytical continuation is expected to apply also to the Wilson loop correlators in the geometry we are considering.
is (for small $e^2$)

$$\frac{\langle WW \rangle}{\langle W \rangle \langle W \rangle} \propto \left( e^{ie \int A_\mu p_\mu^E d\tau_1 - ie \int A_\nu p_\nu^E d\tau_2} \right) \sim e^{e^2 \int d\tau_1 d\tau_2 p_\mu^E G_{\mu\nu} p_\nu^E}$$

(6)

where $G_{\mu\nu}$ is the (euclidean) photon Green’s function (in the Feynman gauge for simplicity)

$$G_{\mu\nu} = \frac{1}{4\pi^2} \cdot \frac{1}{L^2 + (\tau_1 - \tau_2 \cos \theta)^2 + \tau_2^2 \sin^2 \theta g_{\mu\nu}}$$

(7)

We pass to polar coordinates, perform the integral, drop the infinite Coulomb phase and obtain

$$\exp \left( \frac{e^2}{2\pi} \cot \theta \cdot \log L \right)$$

(8)

Performing the analytical continuation (5) we obtain immediately the standard QED eikonal result (see e.g. [17], [9])

$$\delta = \frac{e^2}{2\pi} \coth \chi \cdot \log L$$

(9)

In our case we therefore have to calculate

$$\frac{\langle W_1 W_2 \rangle}{\langle W_1 \rangle \langle W_2 \rangle}$$

(10)

with the Wilson loops parametrized by (4). Since we want to perform the calculation nonperturbatively we will apply the euclidean AdS/CFT correspondence here, and compute the quantity (10) in the AdS supergravity approximation.

### 4 Wilson loop at an angle $\theta$

The string worldsheet in AdS space, corresponding to a Wilson loop $C$ is obtained [11] by minimizing the Nambu-Goto action

$$\int_{\partial \Sigma = C} d\tau d\sigma \sqrt{h}$$

(11)

where $h_{ab}$ is the induced metric $h_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$, $h = \det h_{ab}$ is its determinant and $G_{\mu\nu} = (1/z^2) \delta_{\mu\nu}$ the AdS metric. For the Wilson
loop at an angle $\theta$ (see figure 1) we choose the parametrization $\sigma \equiv x, t = \tau \cos \theta, y = \tau \sin \theta, z = z(x)$, where $z$ is the 5th coordinate of $AdS_5$. This leads to minimizing the action

$$
\int dx \frac{1}{z^2} \sqrt{1 + z_x^2}
$$

with no $\theta$ dependence. The result is given by

$$
z_x = \frac{1}{z^2} \sqrt{z_{\text{max}}^4 - z^4}
$$

with $z_{\text{max}} = a \cdot \Gamma(1/4)^2/(2\pi)^{3/2}$, where $a$ is the transverse size of the loops. The area element on the worldsheet is given by

$$
dA \equiv dt dx \sqrt{h} = dt dx z_{\text{max}}^2 = dt dz \frac{z_{\text{max}}^2}{z^2 \sqrt{z_{\text{max}}^4 - z^4}}
$$

For later reference we quote $h_{\sigma \sigma} = z_{\text{max}}^4/z^6, h_{\tau \tau} = 1/z^2$ and $\sqrt{h} = \sqrt{\det h_{ab}} = z_{\text{max}}^2/z^4$.

### 5 Wilson loop correlators

The Wilson loop correlators can be computed following [3]. One has to distinguish two cases [18]. When the transverse separation $L$ between the loops is comparable with the transverse size $a$, there may exist a connected minimal surface with the sum of the two loops as its disjoint boundary (see e.g. [19]). However when $L \gg a$ the minimal surface has two independent components and in order to calculate the correlator one has to consider the supergravity interaction between them as in [3]. This is the case we will consider here. The AdS radius is fixed by convention to 1. Then $\alpha' = 1/\sqrt{4\pi g_s N}$ and $g_Y^2 = 2\pi g_s$.

The coupling of the string worldsheet to the supergravity fields is obtained by expanding the Nambu-Goto action

$$
S_{NG} = \frac{1}{2\pi \alpha'} \int dt dx e^{\Phi/2} \sqrt{\det G_{\mu \nu} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \epsilon^{ab} B_{\mu \nu} \partial_a X^\mu \partial_b X^\nu}
$$

(15)

to first order in the perturbations of the background AdS fields (denoted here generically by $\psi$):

$$
\frac{1}{2\pi \alpha'} \int dt dz \frac{\delta S_{NG}}{\delta \psi} (t, z) \psi
$$

(16)

The 4th coordinate on the boundary will be taken to be a constant.
Figure 2: Correlation function of Wilson loops is calculated through the exchange of bulk supergravity fields.

In this work we will explicitly consider the lightest relevant supergravity fields namely tachyonic scalar fields (see [20]) $s^I$, the dilaton $\Phi$, the antisymmetric tensor $B_{\mu\nu}$ and the graviton $g_{\mu\nu}$. Then the result is given by the integral

$$\langle WW \rangle \langle W \rangle \langle W \rangle = \exp\left(\frac{1}{4\pi^2\alpha'^2} \int dt_1 dt_2 \frac{dz dw}{z_x w_x} \frac{\delta S_{NG}}{\delta \psi}(t_1, z) G(x, x') \frac{\delta S_{NG}}{\delta \psi}(t_2, w) \right)$$

(17)

where $G(x, x')$ is the relevant (bulk-bulk) Green’s function, and $x$ and $x'$ are points on the two worldsheets parametrized by $t_1, z$ and $t_2, w$ (see figure 2). The factor 4 in the integrand takes into account the fact that for each value of $z$ there are two distinct points on the worldsheet, and that we are performing calculations in the leading order in $a/L$ (here there is a subtlety related to the $B_{\mu\nu}$ field which will be discussed in detail later on).

We note that the exponent in (17) can be also interpreted as an integral over one string worldsheet of the field $\psi(x')$ produced by the other worldsheet$^4$ (with the suitable coupling $\frac{\delta S_{NG}}{\delta \psi}(t_2, w)$) namely

$$\exp\left(\frac{1}{2\pi\alpha'} \int 2 dt_2 dw \frac{\delta S_{NG}}{\delta \psi}(t_2, w)\psi(x') \right)$$

(18)

A convenient change of variables is $v_+ = t_2 \sin \theta$ and $v_- = t_1 -$

$^4$It is interesting to compare formula (18) with the eikonal approximation in QCD [21], involving a one dimensional integral of the gauge field, created by one $q\bar{q}$ state, on the Wilson loop spanned by the other.
\[ t_2 \cos \theta, \text{ which gives} \]
\[
\int dt_1 dt_2 \rightarrow \int \frac{dv_+ dv_-}{\sin \theta} \rightarrow \int \frac{r dr d\beta}{\sin \theta} \quad (19)
\]

In the last equation we chose radial coordinates for \( v \)'s. The exponent in (17) corresponding to the exchange of a generic supergravity \( \psi \) field will then be given by
\[
\frac{1}{4\pi^2 \alpha'^2} \frac{4}{\sin \theta} \left\{ \int r dr d\beta \frac{dzdw}{z w_x} \frac{\delta S_{NG}}{\delta \psi}(z)G(x, x') \frac{\delta S_{NG}}{\delta \psi}(w) \right\}_\psi \quad (20)
\]
where we also used the fact that by time invariance of the individual minimal worldsheets the coupling does not depend on \( t_{1,2} \).

The \( \theta \) dependence (and thus the energy dependence) is encoded completely in the coupling and the overall Jacobian \( \frac{1}{\sin \theta} \). The tensor structure of the Green's function and its specific dependence on the AdS invariant \( u \) (see below) gives rise to the \( a/L \) dependence of the scattering amplitude. It does not influence the \( \theta \) dependence. This can be seen from the following considerations.

The Green's function \( G(x, x') \) is constructed from the invariant bitensors\(^5\) and scalar functions of the AdS invariant
\[
u = \frac{(z-w)^2 + \sum_{i=1}^4 (x_i - x'_i)^2}{2zw} \quad (21)
\]
where \( x \) and \( x' \) are the two arguments of the Green's function. In our case the 4d distance is given by
\[
\sum_{i=1}^4 (x_i - x'_i)^2 = L^2 + (t_1 - t_2 \cos \theta)^2 + t_2^2 \sin^2 \theta \quad (22)
\]

After performing the change of variables (19), we get
\[
u = \frac{L^2 + r^2}{2zw} \quad (23)
\]
with no \( \theta \) dependence.

This argument can be extended also to the tensor structure of the Green's function. Indeed apart from functions of \( \nu \) discussed above, the Green's function is constructed from the bitensors
\[
\partial_\mu u = \frac{1}{z} \left[ \frac{(x - x')_\mu}{w} - u \delta_\mu 0 \right] \quad \partial_{\nu'} u = \frac{1}{w} \left[ \frac{(x' - x)_\nu'}{z} - u \delta_{\nu'} 0 \right] \quad (24)
\]
\(^5\)A number of useful properties are listed in [23]. For explicit formulas see below.
and
\[
\partial_\mu \partial_{\nu'} u = -\frac{1}{zw} \left[ \delta_{\mu\nu'} + \frac{(x - x')_\mu}{w} \delta_{\nu'0} + \frac{(x' - x)_{\nu'}}{z} \delta_{\mu0} - u \delta_{\mu0} \delta_{\nu'0} \right]
\] (25)

where \( x \) and \( x' \) are the two (5-dimensional) arguments of the Green’s function, \( x_{\mu=0} \equiv z \) and \( x'_{\nu'=0} \equiv w \). After the change of variables (19) all these quantities do not involve \( \theta \). Therefore the energy \( (\theta) \) dependence can be read off directly from the coupling coefficients to the relevant fields. One has only to verify that the appropriate Green’s function would not give a vanishing result after integration in (20). Let us now consider the contributions of the relevant supergravity fields.

### 5.1 Scalar exchange

Using our conventions (16), the coupling of scalar modes \( s^I \) to the string worldsheet was derived [3] to be
\[
\frac{\delta S_{NG}}{\delta s^I_k} = -\frac{2k}{z^2}
\] (26)

where the integer \( k \geq 2 \). The scalar Green’s function has the form [3]
\[
\frac{\alpha_{0,\Delta}}{B_{\Delta}} \cdot \frac{1}{(2u)^{\Delta-3}} {}_2F_1(\Delta, \Delta - \frac{3}{2}; 2\Delta - 3; -\frac{2}{u})
\] (27)

(for the \( s^I \) modes \( \Delta = k \)). The leading dependence in \( a/L \) of (24) is governed by the asymptotic large \( u \) behaviour of the Green’s function (in this case \( \text{const} \cdot u^{-k} \) for large \( u \)). Here and in the following we will always restrict ourselves to calculations using this asymptotic part. Its contribution to (20) is
\[
\left\{ \ldots \right\} = k^2 \left[ \frac{\sqrt{\pi} \epsilon_{\text{max}}^{k-1} \Gamma \left( 1 + \frac{k}{4} \right)}{2 \Gamma \left( \frac{3+k}{4} \right)} \right]^2 \cdot \frac{\alpha_{0,k}}{B_k} \cdot \left( \sum_A Y_A^2 \right) \cdot \frac{\pi}{k - 1} \left( \frac{a}{L} \right)^{2(k-1)}
\] (28)

where
\[
\alpha_{0,k} = \frac{k - 1}{2\pi^2}
\] (29)
\[
B_k = \frac{2^{3-k} N^2 k(k - 1)}{\pi^2 (k + 1)^2}
\] (30)

9
\[ c_{\text{max}} = \frac{z_{\text{max}}}{a} = \frac{\Gamma(1/4)^2}{(2\pi)^{3/2}} \]  

\[ \sum_A Y_A^2 = \frac{(k + 3)(k + 2)}{\pi^2(k + 1)^2} \]  

where \( \sum_A Y_A^2 \) denotes a summation over the spherical harmonics on \( S^5 \). After continuation to Minkowski space the whole dependence on \( s \) is

\[ \frac{1}{\sin \theta} = i \frac{1}{\sinh \chi} \sim i \frac{2m^2}{s} \]  

Hence the phase shift \( \delta \) (see (31)) for the leading \((k = 2)\) mode is given by

\[ \delta_{KK} = (\frac{g_s}{2N}) \cdot \left( \frac{a}{L} \right)^2 \cdot \frac{2m^2}{s} \]  

The 1/s behaviour of \( \delta_{KK} \) is consistent with (flat space) field theory expectations for scalar exchange. The \( L \) dependence, on the other hand, does not follow from known symmetry arguments.

### 5.1.1 Dilaton exchange

For the dilaton

\[ \frac{\delta S_{NG}}{\delta \Phi} = \frac{1}{2} \frac{z_{\text{max}}^2}{z^4} \]  

The Green’s function is analogous to (27) but with \( \Delta = 4 + k \), \( k \geq 0 \) and the normalization \( B_\Delta \) substituted by \( B_k' \) (see below). The leading contribution to (20) is

\[ \{ \ldots \} = \left[ \sqrt{\pi^4 \Delta^{-1}} \Gamma \left( \frac{\Delta - 1}{4} \right) \right]^2 \frac{\alpha_0 \Delta}{B_k'} \cdot \sum_A Y_A^2 \cdot \frac{\pi}{\Delta - 1} \left( \frac{a}{L} \right)^{2(\Delta - 1)} \]  

where

\[ B_k' = \frac{N^2}{2^{k-1} \pi^2 (k + 1)(k + 2)} \]  

This leads to the phase shift (for \( k = 0 \))

\[ \delta_{\Phi} = \frac{g_s}{2N} \left( \frac{192 \Gamma(5/4)^8}{\pi^8} \right) \cdot \left( \frac{a}{L} \right)^4 \cdot \frac{2m^2}{s} \]
5.2 Antisymmetric tensor exchange

The coupling of the antisymmetric $B$ tensor field follows from (15). It is therefore given by

$$\frac{\delta S_{NG}}{\delta B_{\mu\nu}} \delta B_{\mu\nu} = \cos \theta (B_{zt} + z_x B_{zt}) + \sin \theta (B_{zy} + z_x B_{zy})$$  (39)

However the calculation of the phase shift is in this case more involved \[24\]. In the $AdS_5 \times S^5$ background the perturbations of the NS-NS $B_{\mu\nu}$ field mix with fluctuations of the Ramond-Ramond 2-form $A_{\mu\nu}$. Therefore we have to project the coupling (39) on to a coupling with the (lowest) mass eigenstate $A_{\mu\nu}^{\text{eig}}$, which is (a real part of a) linear combination of a 2-form $B$ and $*dB$, where $*$ is the Hodge dual. Therefore this state should couple \[24\] to the string through a linear combination of the coupling (39) and the derivative coupling:

$$\frac{\delta S_{NG}}{\delta B_{\mu\nu}} \cdot \sqrt{G} \cdot \epsilon_{\mu
u\rho\sigma} \partial_{\rho} A_{\sigma\delta}^{\text{eig}}$$  (40)

We will not determine here the appropriate coefficients and normalization but rather concentrate on isolating the $a/L$ and $s$ dependence which is relevant for our analysis.

We will begin by considering the nonderivative coupling (39) with the substitution $B_{\mu\nu} \rightarrow A_{\mu\nu}^{\text{eig}}$.

The Green’s function $G_{\mu\nu\mu'\nu'}$ should be constructed from the invariant bitensors and should be antisymmetric in both pairs of indices. These requirements lead to the following tensor structures:

$$T_1^B = \partial_\mu \partial_{\mu'} u \partial_\nu \partial_{\nu'} u - \partial_\nu \partial_{\nu'} u \partial_\mu \partial_{\mu'} u$$  (41)

$$T_2^B = \partial_\mu \partial_{\nu'} u \partial_\nu \partial_{\mu'} u - \partial_\nu \partial_{\nu'} u \partial_\mu \partial_{\mu'} u - \partial_\mu \partial_{\mu'} u \partial_\nu \partial_{\nu'} u + \partial_\nu \partial_{\nu'} u \partial_\mu \partial_{\mu'} u$$  (42)

The Green’s function is then given\[6\] for large $u$ \[24\] to be

$$G^B = (\text{Normalization}) \left( -\frac{1}{u^3} T_1^B + \frac{1}{u^4} T_2^B \right)$$  (43)

Here we encounter the subtlety mentioned after formula (17). The contraction of the Green’s function with the coupling (39) has to be done with care, as there are a number of terms which are linear in the

\[6\text{Here we consider the lowest mode } k = 1 \text{ of the } m^2 = ek^2 \text{ family of } \textbf{21} \text{.} \]
derivatives $z_x$. When we integrate the resulting expression over the string worldsheet, and we change variables to $z$, we have to recall that there are two values of $x$ corresponding to each $z$ so we have in effect:

$$\int_0^a dx I \to \int_0^{z_{\text{max}}} \frac{dz}{z_x} (I + I_{z \to z_{\text{max}}, x \to a-x}) \quad (44)$$

Therefore the contribution would vanish unless we also keep the full expression \[21\] for $u$ i.e.

$$u = \frac{(L + x_1 - x_2)^2 + (z - w)^2 + (t_1 - t_2)^2}{2zw} \quad (45)$$

After some computer algebra the result (leading in $a/L$) is

$$\left\langle \frac{\delta S_{\text{NG}}}{\delta B_{\mu \nu}} \frac{\delta S_{\text{NG}}}{\delta B_{\mu \nu}} \right\rangle = (\text{number}) \frac{32\pi wz}{3L^4} \cos \theta \quad (46)$$

Performing the remaining integration leads to

$$\delta B = \frac{g_s}{2N} \cdot (\text{number}') \cdot \left(\frac{a}{L}\right)^4 \quad (47)$$

The leading energy dependence follows from the angular behaviour

$$\frac{\cos \theta}{\sin \theta} = i \coth \chi \sim i s^0 \quad (48)$$

This behaviour $i \coth \chi$ is exactly the same which appears in field theory with vector exchange \[9\].

Let us now go back and consider the derivative coupling \[40\]. By the arguments in section \[5\] the leading $\theta$ dependence will not change. Moreover an explicit calculation shows that the leading contribution to the phase shift has exactly the same $(a/L)^4 \coth \chi$ dependence as for the coupling \[33\]. In the high energy limit we have therefore a contribution of the 2-form field:

$$\delta B = \frac{g_s}{2N} \cdot (\text{number}'') \cdot \left(\frac{a}{L}\right)^4 \quad (49)$$

### 5.3 Graviton exchange

The graviton coupling can be obtained by expanding the Nambu-Goto action and retaining the first order:

$$\sqrt{h} \left[ \frac{\delta h_{\tau \tau}}{2h_{\tau \tau}} + \frac{\delta h_{\sigma \sigma}}{2h_{\sigma \sigma}} \right] = \frac{z_{\text{max}}^2}{2z^2} \delta h_{\tau \tau} + \frac{z^2}{2z_{\text{max}}^2} \delta h_{\sigma \sigma} \quad (50)$$
and the expression for the variation of the induced metric $h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ in terms of the bulk metric:

$$\delta h_\theta^{\tau\tau} = \cos^2 \theta \delta G_{tt} + \sin^2 \theta \delta G_{yy} + \sin 2\theta \delta G_{ty}$$  \hspace{1cm} (51)

$$\delta h_\sigma^{\sigma\sigma} = \delta G_{xx} + z_x^2 \delta G_{zz} + 2z_x \delta G_{zx}$$  \hspace{1cm} (52)

The graviton field $g_{\mu\nu} \equiv \delta G_{\mu\nu}$ should be decomposed into a scalar part $g^\alpha_\alpha$ that mixes with the RR field strength and a ‘pure’ graviton field $g'_{\mu\nu}$:

$$g_{\mu\nu} = g'_{\mu\nu} - \frac{1}{3} G_{\mu\nu} g^\alpha_\alpha$$  \hspace{1cm} (53)

The physical graviton propagator (for $g'_{\mu\nu}$) is (see [25])

$$\kappa^2 \left[ (\partial_\mu \partial_{\mu'} u \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \partial_\nu \partial_{\mu'} u) G(u) + g_{\mu\nu} g'_{\mu'\nu'} H(u) \right]$$  \hspace{1cm} (54)

with $G(u) \sim (3/32\pi^2) \cdot 1/u^4$, $H(u) \sim (-1/48\pi^2) \cdot 1/u^2$ for large $u$. $\kappa^2$ is the gravitational constant equal in our units $\kappa^2 = 15\pi^3/(2N^2)$. Explicit expressions for $G(u)$ and $H(u)$ are given in [25].

Using this expression one can explicitly check that the whole $\theta$ dependence arises from the correlation function $\langle \delta h_\theta^{\tau\tau} \delta h_\theta^{\tau\tau} = 0 \rangle$, defined by contracting (51) with the Green’s function (54). Residual $\theta$ dependence in $\langle \delta h_\theta^{\tau\tau} \delta h_\sigma \sigma \rangle$ cancels after performing the angular $\beta$ integral in (20). Explicitly we get

$$\langle \delta h_\theta^{\tau\tau} \delta h_\theta^{\tau\tau} = 0 \rangle = \frac{1}{z_w^2 w^2} \left[ (2G(u) \cos^2 \theta + H(u) \right]$$

$$\langle \delta h_\theta^{\tau\tau} \delta h_\sigma \sigma \rangle = \frac{1}{z_w^2 w^2} \left[ \frac{2r^2 z_x^2 \cos^2 (\beta + \theta)}{z^2} G(u) + (1 + z_x^2) H(u) \right]$$

$$\langle \delta h_\sigma \sigma \delta h_\sigma \sigma \rangle = \frac{1}{z_w^2 w^2} \frac{\left[ 2(w^2 w_x z_x + w_x z(x_1 - x_2 + z z_x) \right]}{w^2} \right.$$  \hspace{1cm} (55)

$$+ \frac{w(z - (x_1 - x_2 + (1 + u) w z) z_x))^2}{w^2} \cdot G(u) +$$

$$+ (1 + w_x^2) (1 + z_x^2) H(u) \right]$$

Inserting this into (20) and using (50) we get the result

$$\frac{1}{4\pi^2 \alpha^2 \Gamma^2 \gamma^2} \frac{a^6}{\Gamma^2 \left( \frac{3}{4} \right)} \left( \frac{a}{L} \right)^6 \frac{\cos^2 \theta}{\sin \theta} + \frac{1}{\sin \theta} \cdot (\theta\text{-independent pieces})$$  \hspace{1cm} (56)
The leading energy behaviour now follows from:

$$\frac{\cos^2 \theta}{\sin \theta} = i \coth \chi \cosh \chi \sim i \frac{s}{2m^2}$$  \hspace{1cm} (57)

So the phase shift is

$$\delta_{\text{grav}} = \frac{g_s}{2N} \frac{15\pi^3 c_{\text{max}}^6 \Gamma^2 \left(\frac{3}{4}\right)}{2 \Gamma^2 \left(\frac{1}{4}\right)} \left(\frac{a}{L}\right)^6 \frac{s}{2m^2}$$  \hspace{1cm} (58)

This is a rather unexpected prediction for a field theory scattering amplitude. Before we discuss in more detail the range of validity of the above expression we will first analyze whether the counterterms introduced by Drukker, Gross and Ooguri [6], which are necessary for the finiteness of the Wilson loop expectation values, can change the above result.

### 6 The Legendre transform prescription

In [6] the Wilson loop prescription was modified by taking the Legendre transform:

$$\hat{A} \longrightarrow A - \int d\tau \sqrt{h} h^\sigma G_{ij} Y^i \partial_x Y^j$$  \hspace{1cm} (59)

where $A$ is the Nambu-Goto action while $Y^i = z \Theta^i$ and the $\Theta^i$ are coordinates on $S^5$ satisfying $\sum_{i=1}^6 \Theta^i \Theta^i = 1$. The boundary contribution in (59) cancels exactly the $1/z$ divergence arising from integrating (14). Since the additional term does depend on the metric, it might, in principle, change (56). It suffices to calculate it’s behaviour under variations of $h_{\tau \tau}$. We get

$$\int d\tau \sqrt{h} \frac{1}{2h_{\tau \tau}} h^\sigma G_{ij} Y^i \partial_x Y^j \delta h_{\tau \tau} = \int d\tau \frac{z^4}{2z_{\text{max}}^2} \frac{z}{z} \delta h_{\tau \tau} \sim \int d\tau \delta h_{\tau \tau}$$  \hspace{1cm} (60)

which vanishes when $z \longrightarrow 0$ (note that the graviton Green’s functions are nonsingular at the boundary — they vanish).

However one could envisage a modification of the bulk action which would regularize the action upon imposing equations of motion. Suppose, for instance, that we modify the action to

$$\sqrt{h}(1 - c_1) + \partial_x (\sqrt{h})c_2$$  \hspace{1cm} (61)
where the counterterms $c_1$ and $c_2$ do not depend explicitly on $h_{\tau\tau}$ (e.g. $g^{ab}G_{ij}\partial_a Y^i \partial_b Y^j$ is such a term). This is equivalent to
\[
\sqrt{h}(1 + c_1 - \partial_x c_2) + \partial_x(\sqrt{h}c_2) \tag{62}
\]
For the second term to cancel the Nambu-Goto singularity arising from $\sqrt{h} \sim z^{-4}$, $c_2$ should behave like $z^3$ and hence by similar reasoning as in (61) it would lead to a vanishing coupling to the graviton and so would not give a contribution. Any ‘softer’ behaviour like $z^2$ would, on the other hand, introduce additional singularities which would have to be cancelled by additional counterterms etc. Therefore the cancelation of the $\delta h_{\tau\tau}$ coupling should come just from the first term in (62). But now all the coupling to $\delta h_{\tau\tau}$ comes from $\sqrt{h}$ so the term in parentheses should give 0 upon inserting equations of motion.

But clearly this is impossible, since this would lead to wrong results for $\langle W \rangle$. The above argument does not prove, of course, that some more complicated counterterms would not help to cure this behaviour, but it does not seem to be very likely.

\section{Range of validity of the weak field approximation}

In this section we will analyze the range of validity of the results obtained above. The general assumption in these calculations is that the impact parameter $L$ is much greater than the transverse size $a$ of the $W$-boson pair (which plays the role of a quark-antiquark pair). A second constraint comes from the fact that we are staying within the linearized gravitational regime. We will concentrate on the graviton contribution which leads to the strongest constraint. The calculations should be valid when the field $\delta G_{tt}$ produced by one of the moving (tilted) worldsheets, evaluated at points on second worldsheet, should be much smaller than the background AdS metric $G_{tt}$.

\[
\delta G_{tt} \ll G_{tt} = \frac{1}{w^2} \tag{63}
\]

The field produced at the point $t_1 = 0$ by the second (tilted) worldsheet is given by (compare (17) and (18))
\[
\frac{1}{2\pi \alpha'} \int dx dt \frac{z_{\text{max}}^2}{2z^2} \langle \delta h^a_{\tau\tau} \delta h^b_{\tau\tau} = 0 \rangle \tag{64}
\]

\footnote{We are grateful to J. Maldacena for pointing this out.}
Keeping only the $\theta$-dependent term which is dominant after making the transition to Minkowski space, and using $u = (L^2 + t^2)/(2zw)$ one gets

$$\frac{1}{2\pi\alpha'} 2 \int \frac{dz}{z} \frac{z_{\text{max}}^2}{2z^2} \frac{1}{z^2 w^2} 2 \cos^2 \theta \frac{3}{32\pi^2} \int_{-\infty}^{\infty} \frac{2^4 s^4 w^4}{(L^2 + t^2)^4} dt \quad (65)$$

Which gives finally

$$(\text{number}) \frac{w^2 a^3}{L^7} \cos^2 \theta \ll \frac{1}{w^2} \quad (66)$$

The constraint is most restrictive when evaluating at $w \sim a$, which is as far as the string worldsheet extends into the 5th dimension of the $AdS_5$, so we get finally (dropping factors of order 1, and taking some unit mass $m \sim 1$):

$$\cos^2 \theta \sim s^2 \ll \left( \frac{L}{a} \right)^7 \quad (67)$$

We see that when we fix the impact parameter we cannot go to arbitrarily high energies. At some point the gravitational field becomes so strong that one would have to consider multigraviton exchanges and presumably perform resummation using the full picture of strings propagating in AdS space. The analysis of this regime is beyond the scope of this paper. We note that this constraint is specific to the analytic continuation from Euclidean to Minkowski geometry.

It is to be noticed that when (67) is satisfied the graviton phase shift $\delta_{\text{grav}}$ is indeed small as it should be

$$\delta_{\text{grav}} \sim \left( \frac{a}{L} \right)^6 s \ll \left( \frac{a}{L} \right)^{6-7/2=2.5} \quad (68)$$

Other fields considered by us lead to less stringent constraints.

8 Analysis of the high energy scattering amplitudes at large impact parameter

The leading $L$ and $s$ dependence of the phase shifts for the relevant supergravity fields is $\delta_{KK} = \propto (a/L)^2 s^{-1}$ for the tachyonic $s^I$ scalar, $\delta_B = \propto (a/L)^4$ for the $B_{\mu\nu}$ field and $\delta_{\text{grav}} = \propto (a/L)^6 s$ for the graviton.
The proportionality constants, obtained explicitly for the scalars and the graviton, are real.

The fact that we obtained real amplitudes is linked with the expectation that for large impact parameters the scattering should be predominantly elastic.

Standard crossing relations and analyticity properties of amplitudes relate the phase of the amplitude with the energy behaviour and signature of the exchanged state. For amplitudes behaving like $1/s$ and $s$, a real amplitude implies positive signature $\xi = +1$, while for the constant amplitude of the antisymmetric tensor we should have negative signature $\xi = -1$. This is indeed consistent with the behaviour of the coupling of these fields to the string worldsheet under an exchange of the quark with the antiquark in one of the Wilson loops. This exchange translates into a change of orientation of the associated string worldsheet. The coupling to the scalars and the graviton stays invariant ($\xi = +1$) while the coupling to the $B_{\mu\nu}$ field changes sign. In this way we may also separate off the $B_{\mu\nu}$ contribution from the others.

Now we would like to analyze which supergravity fields are dominant in different regions of the $(a/L, s)$ parameter plane. We stay within the weak field approximation (67) where all the phase shifts are small. We will consider two regimes.

**Fixed** $n \equiv \log \frac{L}{a} / \log s$

Because of the form of the constraint (67) it will turn out to be convenient to parameterize

$$\frac{L}{a} = s^n$$

where $n$ is a (real parameter), and look for dominant contributions when increasing $s$, while keeping $n$ fixed. The constraint now is just $n > 2/7$. The $n$ dependence of the phase shifts for the tachyonic scalar, antisymmetric tensor field and the graviton is in this parameterization $\delta_{KK} \sim s^{-1-2n}$, $\delta_B \sim s^{-4n}$ and $\delta_{grav} \sim s^{-6n}$ respectively. There are 3 cases. For $n$ between 2/7 and 1/2, the graviton contribution dominates, followed by $B_{\mu\nu}$ and the KK $s^I$ scalar, at $n = 1/2$ all 3 contributions are comparable, the precise values of the numerical coefficients (ignored in this analysis) would eventually differentiate between the 3 contributions which all behave here like $(a/L)^4$, while for $n > 1/2$ the KK $s^I$ scalar dominates. Notably enough, the above
shows that in a certain region of (large impact parameter) phase space
the bulk graviton exchange gives a dominant contribution to the SYM
scattering amplitude.

**Fixed $L$**

When we keep $L/a$ fixed and large and increase $s$ (but staying below
the limit $(L/a)^{7/2}$) we find that the graviton contribution gives a linear
rise of the phase shift with $s$. Indeed although $s$ is not arbitrarily large
it dominates over the $B_{\mu\nu}$ contribution:

$$\delta_{\text{grav}} \sim \left(\frac{a}{L}\right)^6 s \sim \left(\frac{a}{L}\right)^2 s \cdot \delta_B \sim \left(\frac{L}{a}\right)^{7/2-2} = 1.5 \cdot \delta_B \quad (70)$$

So for sufficiently large $L/a$ we see a linear rise of the phase shift with
$s$ in gauge theory. This behaviour is rather unexpected in SYM (see
the discussion). Note however, that within the constraints, we are in
the region of applicability of the eikonal approximation as the graviton
phase shift remains small.

## 9 Discussion

At this point we may qualitatively contrast the situation with the
case of perturbative (non-supersymmetric) QCD. The leading large
log resummation give rise to amplitudes rising like

$$s^{\frac{4}{3} \alpha_s N \log 2} \quad (71)$$

The exponent depends on the coupling constant, and increases with
$\alpha_s$ — this is a dynamical effect coming from enhanced gluon radiation
at small $x$. In the strongly coupled phase (our calculation) we get
the exponent 1 which is purely ‘kinematical’ — it reflects just the
spin-2 nature of the graviton. This may perhaps be thought of as a
nonperturbative strong coupling limit of some dynamically generated
enhancement, but if it were so there still remain some questions.

On the perturbative side one expects unitarity effects to set in at
large $s$ thus leading to a constant $s^0$ or at most logarithmic $\log s$ behaviour of the phase shift. We may postulate that the same kind of

\[8\] However there may be subtleties in explicitly carrying over of the Froissart bound to the $\mathcal{N} = 4$ SYM, nonconfining CFT case.
unitarization on the supergravity side, might bring down the behaviour from $s^1$ to $s^0$. However this would involve some very subtle behaviour specific to AdS. Qualitatively such a strong suppression would call for some cancelation of gravity in the high energy limit. The RR force that cancels the static gravitational interaction between D-branes is negligible when one goes to the high relative velocity limit. One can check that indeed in the large impact parameter limit the dominant contribution to the D-brane scattering phase shift comes also from single graviton exchange leading to the phase shift proportional to $s$. In the same regime the contribution of the excited string modes would be suppressed. We expect the same pattern of behaviour here.

For the case of 4-point Virasoro-Shapiro amplitude, which contains the contribution of higher string modes, the large $s$, fixed $t$ regime gives just a modification of the gravity tree level result due to reggeization:

$$\frac{s^2}{t} \rightarrow \frac{s^2}{t} \cdot \exp \left[ \frac{t\alpha'}{2} \log(\alpha' s) \right] \quad (72)$$

This is a slightly wider regime than the one considered by us, as fixing even small $t$ involves integration over all impact parameters. The phase shift for small $t$ is again proportional to $s$.

Also analysis of perturbative resummation in gravity and string theory in flat space (see e.g. [28]) does not seem to soften the energy behaviour.

For the case of $AdS_5$ one expects some differences, in particular one obtains a different $t$ dependence without the $1/t$ pole (which is regulated by the finite radius of $AdS$). It would be extremely interesting to investigate these issues in the AdS context, however as we saw, the tree level energy behaviour had exactly the same origin as in flat space.

The $s$ behaviour obtained by us seems to be quite a robust and generic feature of gravity mediated scattering in various contexts. However the novel feature of the AdS/CFT correspondence allows to translate this kind of behaviour to gauge theory scattering amplitudes. This makes the full understanding of the high energy behaviour for AdS string theory into an interesting and subtle problem.

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9 In the large radius limit the dominant contribution to the Green’s functions should come from the region of small $u$. All the above Green’s functions behave like $u^{-3/2}$, which translates into $1/R^3$ behaviour characteristic of a free scalar propagator in flat 5D space.
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Note added: As the final version of this paper was being typed a preprint [29] appeared which addresses a related problem of high energy scattering of (coloured) quarks.

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