Mean oscillation gradient estimates for elliptic systems in divergence form with VMO coefficients

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Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday

Abstract

We consider gradient estimates for \( H^1 \) solutions of linear elliptic systems in divergence form \( \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j) = 0 \). It is known that the Dini continuity of coefficient matrix \( A = (A_{ij}^{\alpha\beta}) \) is essential for the differentiability of solutions. We prove the following results:

(a) If \( A \) satisfies a condition slightly weaker than Dini continuity but stronger than belonging to VMO, namely that the \( L^2 \) mean oscillation \( \omega_{A,2} \) of \( A \) satisfies

\[
X_{A,2} := \limsup_{r \to 0} r \int_r^2 \frac{\omega_{A,2}(t)}{t^2} \exp \left( C_* \int_t^R \frac{\omega_{A,2}(s)}{s} ds \right) dt < \infty,
\]

where \( C_* \) is a positive constant depending only on the dimensions and the ellipticity, then \( \nabla u \in BMO \).

(b) If \( X_{A,2} = 0 \), then \( \nabla u \in VMO \).

(c) If \( A \in VMO \) and if \( \nabla u \in L^\infty \), then \( \nabla u \in VMO \).

(d) Finally, examples satisfying \( X_{A,2} = 0 \) are given showing that it is not possible to prove the boundedness of \( \nabla u \) in statement (b), nor the continuity of \( \nabla u \) in statement (c).

1 Introduction

Let \( n \geq 2, N \geq 1 \) and consider the elliptic system for \( u = (u^1, \ldots, u^N) \)

\[
\partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j) = 0 \quad \text{in } B_4, \quad i = 1, \ldots, N,
\]  

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where $B_4$ is the ball in $\mathbb{R}^n$ of radius four and centered at the origin, and the coefficient matrix $A = (A_{ij}^{\alpha\beta})$ is assumed to be bounded and measurable in $B_4$ and to satisfy, for some positive constants $\lambda$ and $\Lambda$,

$$|A(x)| \leq \Lambda \text{ for a.e. } x \in B_4,$$

$$\int_{B_2} A_{ij}^{\alpha\beta} \partial_{\beta} \varphi^j \partial_{\alpha} \varphi^i \, dx \geq \lambda \|
abla \varphi\|_{L^2(B_4)}^2 \text{ for all } \varphi \in H^1_0(B_4).$$

It is well known that if the coefficient matrix $A$ belongs to $C^{0,\alpha}_{\text{loc}}(B_4)$ then every solution $u \in H^1(B_4)$ of (1.1) belongs to $C^{1,\alpha}_{\text{loc}}(B_2)$; see e.g. Giaquinta [13, Theorem 3.2] where the result is attributed to Campanato [7] and Morrey [23]. It was conjectured by Serrin [24] that the assumption $u \in H^1(B_4)$ can be relaxed to $u \in W^{1,1}(B_4)$. This has been settled in the affirmative by Brezis [2,3]. (See Hager and Ross [14] for the relaxation from $u \in H^1(B_4)$ to $u \in W^{1,p}(B_4)$ for some $1 < p < 2$.) Moreover, in [2,3], it was shown that if $A$ satisfies the Dini condition

$$\int_0^2 \overline{\phi}_A(t) \, dt < \infty \quad \text{where } \overline{\phi}_A(r) := \sup_{x,y \in B_2, |x-y| < r} |A(x) - A(y)|,$$

then every solution $u \in W^{1,1}(B_4)$ of (1.1) belongs to $C^1(B_2)$. For related works on the differentiability of weak solutions under suitable conditions on $\overline{\phi}_A$, see also [15,21,22].

Differentiability of weak solutions under weaker Dini conditions involving integral mean oscillation of $A$ has also been studied. For $0 < r \leq 2$, let

$$\bar{\varphi}_A(r) := \sup_{x \in B_2} \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |A(y) - A(x)|^2 \, dy \right\}^{1/2},$$

$$\omega_A(r) := \sup_{x \in B_2} \frac{1}{|B_r(x)|} \int_{B_r(x)} |A(y) - (A)_{B_r(x)}| \, dy,$$

$$(A)_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} A(y) \, dy, \quad 0 < r \leq 2.$$ 

In Li [20] it was shown that if

$$\int_0^2 \frac{\bar{\varphi}_A(t)}{t} \, dt < \infty,$$

then every solution $u \in H^1(B_4)$ of (1.1) belongs to $C^1(B_2)$. In Dong and Kim [12] (see also [9]), this conclusion was shown to remain valid under the weaker condition that

$$\int_0^2 \frac{\omega_A(t)}{t} \, dt < \infty.$$
(Note that the finiteness of \( \int_0^2 \omega_A(t) \, dt \) or \( \int_0^2 \bar{\omega}_A(t) \, dt \) implies that \( A \) is continuous.)

The Dini condition \((1.4)\) and its integral variants \((1.5), (1.6)\) are phenomenologically sharp for the differentiability of weak solutions of \((1.1)\). In Jin, Maz'ya and van Schaftingen \([17]\), examples of continuous coefficient matrices \( A \) with moduli of continuity \( \bar{\omega}_A(t) \sim \frac{1}{\ln t} \) as \( t \to 0 \) were given showing the following phenomena:

- there exists a solution \( u \in W^{1,1}(B_4) \) of \((1.1)\) such that \( u \in W^{1,p}(B_4) \) for all \( p \in [1, \infty) \), and \( \nabla u \in BMO_{\text{loc}}(B_4) \) but \( \nabla u \notin L^\infty_{\text{loc}}(B_2) \) and \( \nabla u \notin VMO_{\text{loc}}(B_2) \);
- there exists a solution \( u \in W^{1,1}(B_4) \) of \((1.1)\) such that \( u \in W^{1,p}(B_4) \) for all \( p \in [1, \infty) \) but \( \nabla u \notin BMO_{\text{loc}}(B_2) \).

In this paper, we consider mean oscillation estimates for \( \nabla u \) when \( A \) slightly fails the Dini conditions \((1.4), (1.5)\) and \((1.6)\). For \( 1 \leq p < \infty \), let \( \omega_{A,p} : (0, 2] \to [0, \infty) \) denote the \( L^p \) mean oscillation of \( A \):

\[
\omega_{A,p}(r) = \sup_{x \in B_2} \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |A(y) - (A)_{B_r(x)}|^p \, dy \right\}^{1/p}.
\]

It is clear that \( \omega_{A,1} = \omega_A \), \( \omega_{A,2} \leq \bar{\omega}_A \), \( \omega_{A,p} \) is non-decreasing in \( p \), and \( \omega_{A,p} \leq \bar{\omega}_A \) for all \( p \in [1, \infty) \).

We now state our first result.

**Theorem 1.1.** Let \( A = (A_{ij}^{\alpha\beta}) \) satisfy \((1.2)\) and \((1.3)\). There exists a constant \( C_* > 0 \), depending only on \( n, N, \Lambda \) and \( \lambda \) such that if

\[
X_{A,2} := \limsup_{r \to 0} r \int_r^2 \frac{\omega_{A,2}(t)}{t^2} \exp \left( C_* \int_t^2 \frac{\omega_{A,2}(s)}{s} \, ds \right) \, dt < \infty,
\]

then every solution \( u \in H^1(B_4) \) of \((1.1)\) satisfies \( \nabla u \in BMO_{\text{loc}}(B_2) \). Moreover, if

\[
X_{A,2} = 0,
\]

then every solution \( u \in H^1(B_4) \) of \((1.1)\) satisfies \( \nabla u \in VMO_{\text{loc}}(B_2) \).

Note that condition \((1.7)\) implies that \( \omega_{A,2}(t) \to 0 \) as \( t \to 0 \) i.e. \( A \in VMO_{\text{loc}}(B_2) \).

**Remark 1.2.** Let \( 1 < p < \infty \). Theorem \((1.1)\) remains valid if \( \omega_{A,2} \) is replaced by \( \omega_{A,p} \) and the regularity assumption \( u \in H^1(B_4) \) is replaced by \( u \in W^{1,p}(B_4) \), where the constant \( C_* \) is now allowed to depend also on \( p \). For \( p \geq 2 \), this follows from the inequality \( \omega_{A,2} \leq \omega_{A,p} \) for those \( p \). For \( 1 < p < 2 \), see Proposition \((2.3)\).

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\(^1\)The statement that \( \nabla u \notin VMO_{\text{loc}}(B_2) \) is not explicitly stated in \([17]\), but can be seen from the proof of Proposition 1.5 therein.
It is clear that if $\omega_{A,2}$ satisfies (1.5), then it satisfies (1.8) (and hence (1.7)). The following lemma gives examples which satisfy (1.8) but not necessarily (1.5).

**Lemma 1.3.** If $\limsup_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} < \frac{1}{C_*}$, then $X_{A,2} = 0$. If $\liminf_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} > \frac{1}{C_*}$, then $X_{A,2} = \infty$.

We note that, in case $\omega_{A,2}(t) \ln \frac{1}{t} \to 0$ as $t \to 0$, the BMO regularity of $\nabla u$ was proved by Acquistapace [1]. (See also [16].)

By Lemma 1.3, an explicit example of $\omega_{A,2}$ satisfying (1.8) (for any constant $C_*$) but not (1.5) is

$$\omega_{A,2}(t) \sim \frac{1}{\ln \frac{64}{|x|} \ln \ln \frac{64}{|x|} \beta}, \quad \beta \in (0, 1].$$

In addition, unlike (1.5) or (1.6), (1.8) does not imply that $A$ is continuous, e.g.

$$A_{ij}^{\alpha\beta}(x) = (2 + \sin \ln \ln \frac{64}{|x|}) \delta_{ij} \delta^{\alpha\beta}.$$

(This can be checked using the fact that the function $s \mapsto \sin s$ is Lipschitz on $\mathbb{R}$ and the fact that the function $x \mapsto \ell(x) := \ln \ln \ln \frac{64}{|x|}$ has $L^2$ mean oscillation $\omega_{\ell,2}(t) \sim \frac{1}{\ln \frac{64}{|x|} \ln \ln \frac{64}{|x|}}$.)

When $A$ is merely of vanishing mean oscillation, we have the following result.

**Theorem 1.4.** Let $A = (A_{ij}^{\alpha\beta})$ belong to $VMO(B_4)$ and satisfy (1.2) and (1.3). Then every solution $u \in W^{1,\infty}(B_4)$ of (1.1) satisfies $\nabla u \in VMO(B_4)$.

The obtained regularity in the above theorems appears sharp. As in [17], counterexamples can be produced to show that, under (1.8),

- solutions of (1.1) may not have bounded gradients (though their gradients are of vanishing mean oscillation by Theorem 1.1),
- $W^{1,\infty}$ solutions of (1.1) may not be differentiable (though their gradients are of vanishing mean oscillation by Theorem 1.4).

**Proposition 1.5.** There exist a coefficient matrix $A = (A_{ij}^{\alpha\beta}) \in C(B_4)$ satisfying (1.2), (1.3) and (1.8) and a solution $u \in H^1(B_4)$ of (1.1) such that $\nabla u \in VMO(B_4)$ but $\nabla u \notin L^\infty_{loc}(B_2)$.

**Proposition 1.6.** There exist a coefficient matrix $A = (A_{ij}^{\alpha\beta}) \in C(B_4)$ satisfying (1.2), (1.3) and (1.8) and a solution $u \in H^1(B_4)$ of (1.1) such that $\nabla u \in L^\infty(B_4) \cap VMO(B_4)$ but $\nabla u \notin C(B_2)$. 

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Theorem 1.1 and Theorem 1.4 are consequences of the following proposition on the mean oscillation of the gradient $\nabla u$ in terms of the $L^2$ mean oscillation $\omega_{A,2}$ of $A$.

**Proposition 1.7.** Let $A = (A_{ij}^{\alpha\beta})$ satisfy (1.2) and (1.3). Then there exists a constant $C_0 > 0$, depending only on $n, N, \Lambda$ and $\lambda$ such that for every $u \in H^1(B_4)$ satisfying (1.1) and for $0 < r \leq R/4 \leq 1/2$, there hold

$$\int_{B_r} |\nabla u|^2 \, dx \leq \frac{C_0 r^n}{R^n} \exp \left(2C_0 \int_{2r}^R \frac{\omega_{A,2}(t)}{t} \, dt \right) \int_{B_R} |\nabla u|^2 \, dx,$$

(1.9)

and

$$\int_{B_r} |\nabla u - (\nabla u)_r|^2 \, dx \leq \frac{C_0 r^{n+2}}{R^n} \int_{B_R} |\nabla u|^2 \, dx \times$$

$$\times \left\{ \int_{2r}^R \frac{\omega_{A,2}(t)}{t^2} \exp \left(C_0 \int_{t}^R \frac{\omega_{A,2}(s)}{s} \, ds \right) \, dt \right\}^2,$$

(1.10)

where $(\nabla u)_r = \frac{1}{|B_r|} \int_{B_r} \nabla u \, dx$ for $0 < r \leq 2$.

Moreover, if $u \in W^{1,\infty}(B_4)$, then, for $0 < r \leq R/4 \leq 1/2$,

$$\int_{B_r} |\nabla u - (\nabla u)_r|^2 \, dx \leq \frac{C_0 r^{n+2}}{R^n} \left\{ \int_{2r}^R \frac{\omega_{A,2}(t)}{t^2} \, dt \right\}^2 \sup_{B_R} |\nabla u|^2.$$ 

(1.11)

**Remark 1.8.** Let $1 < p < 2$. Under an additional assumption that $[A]_{BMO(B_4)}$ is sufficiently small, the estimates in Proposition 1.7 hold if $\omega_{A,2}$ is replaced by $\omega_{A,p}$ and the regularity assumption $u \in H^1(B_4)$ is replaced by $u \in W^{1,p}(B_4)$. We do not know if this smallness assumption can be dropped except for $p$ close to 2. See Proposition 2.3.

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## 2 Proof of the main results

**Proof of Lemma 1.3.** We claim: For $\delta \in (0, 1)$ and $a \in (0, \infty)$, the limit

$$L_a = \lim_{r \to 0} \sup_r \int_{r}^{\delta} \frac{1}{t^2} (\ln \frac{1}{t})^{a-1} \, dt$$

satisfies $L_a = \infty$ if $a > 1$, $L_a = 1$ if $a = 1$ and $L_a \leq (\ln \frac{1}{\delta})^{a-1}$ if $a < 1$.
When $a = 1$, the claim is clear. By integrating by parts, we have
\[
\int_r^\delta \frac{1}{t^2} (\ln \frac{1}{t})^{a-1} dt = -\frac{1}{t} (\ln \frac{1}{t})^{a-1}\right|_r^\delta - (a-1) \int_r^\delta \frac{1}{t^2} (\ln \frac{1}{t})^{a-2} dt.
\] (2.1)

If $a < 1$, we see from (2.1) that
\[
L_a = |a - 1| \limsup_{r \to 0} r \int_r^\delta \frac{1}{t^2} (\ln \frac{1}{t})^{a-2} dt \leq |a - 1| \limsup_{r \to 0} r \int_r^\delta \frac{1}{t} (\ln \frac{1}{t})^{a-2} dt
\]
\[
= \limsup_{r \to 0} (\ln \frac{1}{r})^{a-1}\right|_r^\delta = (\ln \frac{1}{2})^{a-1}.
\]

To prove the claim in the case $a > 1$, we may assume without loss of generality that $a < 2$. Note that (2.1) implies
\[
L_a + (a-1)L_{a-1} = \limsup_{r \to 0} r \left\{ -\frac{1}{t} (\ln \frac{1}{t})^{a-1}\right|_r^\delta \right\} = \infty.
\]

As $L_{a-1}$ is finite (as $1 < a < 2$), we thus have that $L_a = \infty$. The claim is proved.

We now apply the claim to obtain the desired conclusions. Consider first the case that $\limsup_{r \to 0} \omega_{A,2}(t) \ln \frac{1}{t} < \frac{1}{C_*}$. Then there exist $\varepsilon \in (0, \frac{1}{C_*})$ and $\delta \in (0,1)$ so that $\omega_{A,2}(t) \leq \varepsilon (\ln \frac{1}{t})^{-1}$ in $(0, \delta)$. For $\delta \in (0, \delta)$, we compute
\[
X_{A,2} = \limsup_{r \to 0} r \int_r^\delta \frac{\omega_{A,2}(t)}{t^2} \exp \left(C_* \int_t^2 \frac{\omega_{A,2}(s)}{s} ds\right) dt
\]
\[
\leq \varepsilon (\ln \frac{1}{\delta})^{-C_* \varepsilon} \exp \left(C_* \int_\delta^2 \frac{\omega_{A,2}(s)}{s} ds\right) \limsup_{r \to 0} r \int_r^\delta \frac{1}{t^2} (\ln \frac{1}{t})^{C_* \varepsilon-1} dt.
\]

As $C_* \varepsilon < 1$, we can apply the claim to obtain
\[
X_{A,2} \leq \varepsilon (\ln \frac{1}{\delta})^{-1} \exp \left(C_* \int_\delta^2 \frac{\omega_{A,2}(s)}{s} ds\right)
\]
\[
\leq \varepsilon \exp \left(C_* \int_\delta^2 \frac{\omega_{A,2}(s)}{s} ds\right).
\]

Sending $\delta \to 0$, we obtain that $X_{A,2} = 0$.

Consider next the case that $\liminf_{t \to 0} \omega_{A,2}(t) \ln \frac{1}{t} > \frac{1}{C_*}$. Then there exist $b > \frac{1}{C_*}$ and $\delta \in (0,1)$ so that $\omega_{A,2}(t) \geq b (\ln \frac{1}{t})^{-1}$ in $(0, \delta)$. We then have
\[
X_{A,2} = \limsup_{r \to 0} r \int_r^\delta \frac{\omega_{A,2}(t)}{t^2} \exp \left(C_* \int_t^2 \frac{\omega_{A,2}(s)}{s} ds\right) dt
\]
\[
\geq b (\ln \frac{1}{\delta})^{-C_* b} \exp \left(C_* \int_\delta^2 \frac{\omega_{A,2}(s)}{s} ds\right) \limsup_{r \to 0} r \int_r^\delta \frac{1}{t^2} (\ln \frac{1}{t})^{C_* b-1} dt.
\]

As $C_* b > 1$, we deduce from the claim that $X_{A,2} = \infty$ as desired. \qed
**Proof of Theorem 1.1 and Theorem 1.4.** The results follow immediately from Proposition 1.7. □

In order to prove Proposition 1.7, we need the following estimate for harmonic replacements. (Compare [5, Lemma 3.5], [19, Lemma 3.1].)

**Lemma 2.1.** Let $A, \tilde{A}$ satisfy (1.2) and (1.3) with $\tilde{A}$ being constant in $B_4$ and $f = (f_\alpha^i) \in L^2(B_4)$. Let $R \in (0, 2)$ and suppose $u, h \in H^1(B_{2R})$ satisfy

$$
\begin{align*}
&\partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j) = \partial_\alpha f^\alpha_i \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\
&\partial_\alpha (\tilde{A}_{ij}^{\alpha\beta} \partial_\beta h^j) = 0 \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\
&u = h \quad \text{on } \partial B_{2R}.
\end{align*}
$$

Then there exists a constant $C > 0$ depending only on $n, N, \Lambda$ and $\lambda$ such that

$$
\|\nabla(u - h)\|_{L^2(B_{3R}/2)} \leq C \left[ \|f\|_{L^2(B_{2R})} + R^{-n/2}\|A - \tilde{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})} \right].
$$

**Proof.** In the proof, $C$ denotes a generic positive constant which depends only on $n$, $N$, $\Lambda$ and $\lambda$. Using that $\tilde{A}$ is constant, we have by standard elliptic estimates that

$$
\|\nabla h\|_{L^\infty(B_{7R}/4)} \leq CR^{-n/2}\|\nabla h\|_{L^2(B_{2R})} \leq CR^{-n/2}\|\nabla u\|_{L^2(B_{2R})}.
$$

Observing that

$$
\partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta(u - h)^j) = \partial_\alpha (f_\alpha^i + (A - \tilde{A})_{ij}^{\alpha\beta} \partial_\beta h^j) \quad \text{in } B_{2R}, \quad i = 1, \ldots, N,
$$

we deduce that

$$
\begin{align*}
\|\nabla(u - h)\|_{L^2(B_{3R}/2)} &\leq C \left[ \|f\|_{L^2(B_{7R}/4)} + \|A - \tilde{A}\|_{L^2(B_{7R}/4)}\|\nabla h\|_{L^\infty(B_{7R}/4)} \\
&\quad + R^{-(n+2)/2}\|u - h\|_{L^1(B_{7R}/4)} \right] \\
&\leq C \left[ \|f\|_{L^2(B_{2R})} + R^{-n/2}\|A - \tilde{A}\|_{L^2(B_{2R})}\|\nabla u\|_{L^2(B_{2R})} \\
&\quad + R^{-(n+2)/2}\|u - h\|_{L^1(B_{2R})} \right].
\end{align*}
$$

(2.2)

To estimate $\|u - h\|_{L^1(B_{2R})}$, fix some $t > 0$ and consider an auxiliary equation

$$
\begin{align*}
\partial_\beta (\tilde{A}_{ij}^{\alpha\beta} \partial_\alpha \phi^i) &= \frac{(u - h)^j}{\sqrt{|u - h|^2 + t^2}} \quad \text{in } B_{2R}, \quad j = 1, \ldots, N, \\
\phi &= 0 \quad \text{on } \partial B_{2R}.
\end{align*}
$$
Testing the above against $u - h$, we obtain
\[
\int_{B_{2R}} \frac{|u - h|^2}{\sqrt{|u - h|^2 + t^2}} \, dx = \int_{B_{2R}} \bar{A}^{\alpha\beta}_{ij} \partial_\alpha \phi^i \partial_\beta (u - h)^j \, dx. \tag{2.3}
\]
As $u - h$ satisfies
\[
\partial_\alpha (\bar{A}^{\alpha\beta}_{ij} \partial_\beta (u - h)^j) = \partial_\alpha (f^\alpha_i + (\bar{A} - A)^{\alpha\beta}_{ij} \partial_\beta u^j) \quad \text{in} \quad B_{2R}, \quad i = 1, \ldots, N,
\]
we have
\[
\int_{B_{2R}} \bar{A}^{\alpha\beta}_{ij} \partial_\beta (u - h)^j \partial_\alpha \phi^i \, dx = \int_{B_{2R}} (f^\alpha_i + (\bar{A} - A)^{\alpha\beta}_{ij} \partial_\beta u^j) \partial_\alpha \phi^i \, dx. \tag{2.4}
\]
Inserting (2.4) into (2.3) and noting that $\|\nabla \phi\|_{L^\infty(B_2)} \leq CR$ (as $|\partial_\beta (\bar{A}^{\alpha\beta}_{ij} \partial_\alpha \phi^i)| \leq 1$), we arrive at
\[
\int_{B_{2R}} \frac{|u - h|^2}{\sqrt{|u - h|^2 + t^2}} \, dx \leq C \left[ R^{(n+2)/2} \|f\|_{L^2(B_{2R})} + R \|A - \bar{A}\|_{L^2(B_{2R})} \|\nabla u\|_{L^2(B_{2R})} \right].
\]
Noting that the constant $C$ is independent of $t$, we may send $t \to 0$ to obtain
\[
\|u - h\|_{L^1(B_{2R})} \leq C R^{(n+2)/2} \left[ \|f\|_{L^2(B_{2R})} + R^{-n/2} \|A - \bar{A}\|_{L^2(B_{2R})} \|\nabla u\|_{L^2(B_{2R})} \right]. \tag{2.5}
\]
The conclusion follows from (2.2) and (2.3). \hfill \square

**Proof of Proposition 1.7.** We only need to give the proof for a fixed $R$, say $R = 2$. Our proof is inspired by that of [20].

In the proof, $C$ denotes a generic positive constant which depends only on $n$, $N$, $\Lambda$ and $\lambda$. In particular it is independent of the parameter $k$ which will appear below. Also, we will simply write $\omega$ instead of $\omega_{A,2}$.

**Proof of (1.9):** For $k \geq 0$, let $R_k = 4^{-k}$, $\bar{A}_k = (A)_{B_{2R_k}}$ and $h_k \in H^1(B_{2R_k})$ be the solution to
\[
\partial_\alpha ((\bar{A}_k)^{\alpha\beta}_{ij} \partial_\beta h_k^j) = 0 \quad \text{in} \quad B_{2R_k}, \quad i = 1, \ldots, N,

h_k = u \quad \text{on} \quad \partial B_{2R_k}.
\]
Let $a_k = R_k^{-n/2} \|\nabla (u - h_k)\|_{L^2(B_{R_k})}$ and $b_k = \|\nabla h_k\|_{L^\infty(B_{R_k})}$.

Note that, by triangle inequality, we have
\[
\|\nabla u\|_{L^2(B_{R_k})} \leq R_k^{n/2} (a_k + b_k). \tag{2.6}
\]
By elliptic estimates for $h_k$, we have
\begin{align}
\|\nabla h_k\|_{L^2(B_{2R_k})} & \leq C\|\nabla u\|_{L^2(B_{2R_k})}, \\
\|\nabla h_k\|_{L^\infty(B_{3R_k/2})} & \leq CR_k^{-n/2}\|\nabla u\|_{L^2(B_{2R_k})}, \\
\|\nabla^2 h_k\|_{L^\infty(B_{3R_k/2})} + R_k\|\nabla^3 h_k\|_{L^\infty(B_{3R_k/2})} & \leq CR_k^{-(n+2)/2}\|\nabla u\|_{L^2(B_{2R_k})}. 
\end{align}

By Lemma 2.1,
\begin{align}
\|\nabla (u - h_k)\|_{L^2(B_{3R_k/2})} & \leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{2R_k})}.
\end{align}

By (2.7) and (2.10),
\begin{align}
R_k^{n/2}(a_k + b_k) & \leq C\|\nabla u\|_{L^2(B_{2R_k})}.
\end{align}

By (2.6) and (2.10), we have
\begin{align}
\|\nabla (u - h_{k+1})\|_{L^2(B_{R_k+1})} & \leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{R_k})} \\
& \leq C\omega(2R_k)R_k^{n/2}(a_k + b_k).
\end{align}

Hence
\begin{align}
a_{k+1} & \leq C\omega(2R_k)(a_k + b_k).
\end{align}

Next, we have by (2.10) that
\begin{align}
\|\nabla (h_{k+1} - h_k)\|_{L^2(B_{3R_{k+1}/2})} & \leq \|\nabla (u - h_{k+1})\|_{L^2(B_{3R_{k+1}/2})} + \|\nabla (u - h_k)\|_{L^2(B_{3R_{k+1}/2})} \\
& \leq C\omega(2R_k)\|\nabla u\|_{L^2(B_{R_k})} \\
& \leq C\omega(2R_k)R_k^{n/2}(a_k + b_k).
\end{align}

Noting that $h_{k+1} - h_k$ satisfies
\begin{align}
\partial_\alpha((\bar{A}_k)_{ij}^{\alpha\beta}\partial_\beta(h_{k+1} - h_k)^j) = \partial_\alpha((\bar{A}_k - \bar{A}_{k+1})_{ij}^{\alpha\beta}\partial_\beta h_{k+1}^j) \quad \text{in } B_{2R_k+1}, \quad i = 1, \ldots, N,
\end{align}
we thus have by elliptic estimates and (2.8) and (2.9) (applied to $h_{k+1}$) that
\begin{align}
\|\nabla (h_{k+1} - h_k)\|_{L^\infty(B_{R_k+1})} & \leq C\omega(2R_k)(a_k + b_k), \\
R_{k+1}\|\nabla^2 (h_{k+1} - h_k)\|_{L^\infty(B_{R_k+1})} & \leq C\omega(2R_k)(a_k + b_k).
\end{align}

By (2.12),
\begin{align}
b_{k+1} & \leq b_k + C\omega(2R_k)(a_k + b_k).
\end{align}

By (2.11) and (2.14), we have
\begin{align}
a_{k+1} + b_{k+1} & \leq (1 + C\omega(2R_k))(a_k + b_k).
\end{align}
We deduce that
\[ a_k + b_k \leq \prod_{j=0}^{k} (1 + C\omega(2R_j))(a_0 + b_0) \leq C \exp \left( C \sum_{j=0}^{k} \omega(2R_j) \right) \|\nabla u\|_{L^2(B_2)} \]
\[ \leq C \exp \left( C \int_{2R_k}^{2} \frac{\omega(t)}{t} \, dt \right) \|\nabla u\|_{L^2(B_2)}, \]  
(2.15)
where we have used the fact that \( \omega(t) \leq C\omega(s) \) whenever \( 0 < t \leq s \leq 4t \). We have thus shown that
\[ \int_{B_{Rk}} |\nabla u|^2 \, dx \leq CR_k^n \exp \left( C \int_{2R_k}^{2} \frac{\omega(t)}{t} \, dt \right) \int_{B_{Rk}} |\nabla u|^2 \, dx \text{ for } k \geq 0. \]
Estimate (1.9) is readily seen.
Proof of (1.10): We write
\[ h_{R_k} = \sum_{j=0}^{k} w_j \text{ where } w_0 = h_{R_0} \text{ and } w_j = h_{R_j} - h_{R_{j-1}} \text{ for } j \geq 1. \]
Using the estimate \( \|\nabla^2 h_{R_0}\|_{L^\infty(B_1)} \leq C\|\nabla u\|_{L^2(B_2)} \) together with (2.13) and (2.15), we have
\[ |\nabla h_{R_k}(x) - \nabla h_{R_k}(0)| \leq C|x| \sum_{j=0}^{k} \frac{\omega(2R_j)}{R_j} \exp \left( C \int_{2R_j}^{2} \frac{\omega(t)}{t} \, dt \right) \|\nabla u\|_{L^2(B_2)} \]
\[ \leq C|x| \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \exp \left( C \int_{t}^{2} \frac{\omega(s)}{s} \, ds \right) \, dt \|\nabla u\|_{L^2(B_2)}, \]  
(2.16)
where we have again used the fact that \( \omega(t) \leq C\omega(s) \) whenever \( 0 < t \leq s \leq 4t \). This implies
\[ \|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \]
\[ \leq CR_k^{(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \exp \left( C \int_{t}^{2} \frac{\omega(s)}{s} \, ds \right) \, dt \|\nabla u\|_{L^2(B_2)}. \]  
(2.17)
Combining (2.17) with (2.10) and (2.15), we get
\[ \|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} \leq \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \]
\[ \leq \|\nabla (u - \nabla h_{R_k})\|_{L^2(B_{R_k})} + \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \]
\[ \leq CR_k^{(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \exp \left( C \int_{t}^{2} \frac{\omega(s)}{s} \, ds \right) \, dt \|\nabla u\|_{L^2(B_2)} \]
\[ + CR_k^{n/2} \omega(2R_k) \exp \left( C \int_{2R_k}^{2} \frac{\omega(t)}{t} \, dt \right) \|\nabla u\|_{L^2(B_2)}. \]  
(2.18)
As $\omega(2R_k) \leq C\omega(t)$ whenever $2R_k \leq t \leq 4R_k$, we have

$$\int_{2R_k}^{4R_k} \frac{\omega(t)}{t^2} \exp\left( C \int_t^{2R_k} \frac{\omega(s)}{s} \, ds \right) \, dt \geq \frac{\omega(2R_k)}{CR_k} \exp\left( C \int_{2R_k}^{2} \frac{\omega(s)}{s} \, ds \right).$$

Using this in (2.18), we deduce that for $k \geq 1$ that

$$\|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} \leq CR_k^{2(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \exp\left( C \int_t^{2R_k} \frac{\omega(s)}{s} \, ds \right) \, dt \|\nabla u\|_{L^2(B_2)}.$$  

Estimate (1.10) follows.

Proof of (1.11): We adjust the proof of (1.10) exploiting the fact that $\nabla u \in L^\infty(B_2)$. First, using the fact that $a_k + b_k \leq CR_k^{n/2}\|\nabla u\|_{L^\infty(B_2)}$ in (2.13) we get instead of (2.16) the stronger estimate

$$|\nabla h_{R_k}(x) - \nabla h_{R_k}(0)| \leq C|x| \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \, dt \|\nabla u\|_{L^\infty(B_2)}, \quad (2.19)$$

and so

$$\|\nabla h_{R_k} - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})} \leq CR_k^{2(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \, dt \|\nabla u\|_{L^\infty(B_2)}. \quad (2.20)$$

Combining (2.20) with (2.10), we get for $k \geq 1$ that

$$\|\nabla u - (\nabla u)_{R_k}\|_{L^2(B_{R_k})} \leq \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})}$$

$$\leq \|\nabla (u - \nabla h_{R_k})\|_{L^2(B_{R_k})} + \|\nabla u - \nabla h_{R_k}(0)\|_{L^2(B_{R_k})}$$

$$\leq CR_k^{2(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \, dt \|\nabla u\|_{L^\infty(B_2)}$$

$$+ CR_k^{n/2}\omega(2R_k)\|\nabla u\|_{L^\infty(B_2)}$$

$$\leq CR_k^{2(n+2)/2} \int_{2R_k}^{2} \frac{\omega(t)}{t^2} \, dt \|\nabla u\|_{L^\infty(B_2)}. \quad (2.21)$$

Estimate (1.11) follows.

Remark 2.2. If the Dini condition (1.4) or (1.5) holds, it can be seen from (2.12) that $\{\nabla h_k(0)\}$ converges to some $P \in \mathbb{R}^{N \times n}$, from which it follows that

$$\lim_{r \to 0} r^{-n/2}\|\nabla u - P\|_{L^2(B_r)} = 0,$$

yielding the continuity of $\nabla u$ at the origin. We have thus recovered the results on the differentiability of $H^1$ solutions of Brezis [2, 3] and Li [20].
Proof of Proposition 1.5. We take \( N = 1 \) and drop the indices \( i, j \) in the expression of \( A \) (so that \( A = (A^{\alpha\beta}) \)). Following [17, Lemma 2.1], we make the ansatz that

\[
A^{\alpha\beta}(x) = \delta^{\alpha\beta} + a(|x|) \left( \delta^{\alpha\beta} - \frac{x^{\alpha} x^{\beta}}{|x|^2} \right),
\]

\[u(x) = x^1 v(|x|).\]

Then

\[
\partial_\alpha(A^{\alpha\beta} \partial_\beta u) = x^1 \left( v''(|x|) + \frac{n + 1}{|x|} v'(|x|) - \frac{n - 1}{|x|^2} a(|x|) v(|x|) \right).
\]

Selecting now

\[
a(r) = -\frac{1 + n \ln \frac{64}{r}}{(n - 1)(\ln \frac{64}{r})^2 \ln \ln \frac{64}{r}},
\]

\[v(r) = \ln \ln \frac{64}{r},\]

we see that \( A \) is continuous in \( \bar{B}_4 \), satisfies (1.2), (1.3) and \( u \) is an \( H^1 \) solution of (1.1). The matrix \( A \) admits a modulus of continuity \( \bar{\omega} A(t) \sim \frac{1}{\ln \frac{64}{t} \ln \ln \frac{64}{t}} \) as \( t \to 0 \) and so (1.8) holds. It is readily seen that \( u \in W^{1,p}(B_4) \) for all \( p \in [1, \infty) \), \( \nabla u \in VMO(B_4) \) but \( \nabla u \not\in L^{\infty}_{\text{loc}}(B_2) \).

Proof of Proposition 1.6. Instead of the choice in the proof of Proposition 1.5, we now choose

\[
a(r) = -\frac{\sin \ln \ln \frac{64}{r} + \cos \ln \ln \frac{64}{r} (1 + \ln \frac{64}{r} + n \ln \frac{64}{r} \ln \ln \frac{64}{r})}{(n - 1)(\ln \frac{64}{r})^2 (\ln \ln \frac{64}{r})^2 (2 + \sin \ln \ln \frac{64}{r})},
\]

\[v(r) = 2 + \sin \ln \ln \ln \frac{64}{r} - \frac{1}{\ln \frac{64}{r} \ln \ln \frac{64}{r}}\]

It is readily checked that \( A \) is continuous in \( \bar{B}_4 \), satisfies (1.2), (1.3), (1.8) and \( u \) is an \( H^1 \) solution of (1.1), \( \nabla u \in L^{\infty}(B_4) \cap VMO(B_4) \) but \( \nabla u \not\in C(B_2) \).

Finally, we briefly touch on the validity of Theorem 1.1 when \( \omega_{A,2} \) is replaced by \( \omega_{A,p} \) for \( 1 < p < 2 \). For this, we only need the following \( L^p \) version of Proposition 2.3.

Proposition 2.3. Let \( A = (A^{\alpha\beta}_{ij}) \) satisfy (1.2) and (1.3). Let \( 1 < p < 2 \). Then there exist constants \( \gamma > 0 \) and \( C_* > 0 \) depending only on \( n, N, p, \Lambda \) and \( \lambda \) such that, provided \( [A]_{\text{BMO}}(B_4) < \gamma \), there hold for every \( u \in W^{1,p}(B_4) \) satisfying (1.1) and for \( 0 < r \leq R/4 \leq 1/2 \) that

\[
\int_{B_r} |\nabla u|^p \, dx \leq \frac{C_* n^2}{R^n} \exp \left( 2C_* \int_{2r}^R \frac{\omega_{A,p}(t)}{t} \, dt \right) \int_{B_R} |\nabla u|^p \, dx,
\]

(2.22)
and
\[
\int_{B_r} |\nabla u - (\nabla u)_r|^p \, dx \leq \frac{C r^{n+2}}{R^n} \int_{B_R} |\nabla u|^p \, dx \times \left\{ \int_{2r}^R \frac{\omega_{A,p}(t)}{t^2} \exp \left( C_1 \int_{t}^{R} \frac{\omega_{A,p}(s)}{s} \, ds \right) \, dt \right\}^2, \tag{2.23}
\]

where \((\nabla u)_r = \frac{1}{|B_r|} \int_{B_r} \nabla u \, dx\) for \(0 < r \leq 2\).

Moreover, if \(u \in W^{1,\infty}(B_4)\), then, for \(0 < r \leq R/4 \leq 1/2\),
\[
\int_{B_r} |\nabla u - (\nabla u)_r|^p \, dx \leq \frac{C r^{n+2}}{R^n} \left\{ \int_{2r}^R \frac{\omega_{A,p}(t)}{t^2} \, dt \right\}^2 \sup_{B_R} |\nabla u|^p. \tag{2.24}
\]

The proof of Proposition 2.3 is the same as that of Proposition 1.7 but now using the following harmonic replacement estimate:

**Lemma 2.4.** Let \(1 < p < 2\). Let \(A, \bar{A}\) satisfy (1.2) and (1.3) with \(\bar{A}\) being constant in \(B_4\) and \(f = (f_i^{A}) \in L^p(B_4)\). Let \(R \in (0, 1)\) and suppose \(u, h \in W^{1,p}(B_{4R})\) satisfy
\[
\begin{aligned}
\partial_\alpha \left( A_{ij}^{\alpha \beta} \partial_\beta u^j \right) &= \partial_\alpha f_i \quad \text{in } B_{3R}, \quad i = 1, \ldots, N, \\
\partial_\alpha \left( \bar{A}_{ij}^{\alpha \beta} \partial_\beta h^j \right) &= 0 \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \\
u &= h \quad \text{on } \partial B_{2R}.
\end{aligned}
\]

Then there exist constants \(\gamma > 0\) and \(C > 0\) depending only on \(n, N, p, \Lambda\) and \(\lambda\) such that, provided \([A]_{BMO(B_{4R})} \leq \gamma\),
\[
\|\nabla (u - h)\|_{L^p(B_{3R}/2)} \leq C \left[ R^{n(1/p - 1/p')} \|f\|_{L^{p'}(B_{3R})} + R^{-n/p} \|A - \bar{A}\|_{L^p(B_{3R})} \|\nabla u\|_{L^p(B_{3R})} \right].
\]

**Proof.** We amend the proof of Lemma 2.1 using \(L^p\) theories for elliptic systems whose leading coefficients have small \(BMO\) semi-norm.\(^2\) In the proof, \(C\) denotes a generic positive constant which depends only on \(n, N, p, \Lambda\) and \(\lambda\).

It is known that (see e.g. Dong and Kim \([10, 11]\))\(^3\) provided \([A]_{BMO(B_{4R})} \leq \gamma\) for some small enough \(\gamma\) depending only on \(n, N, p, \Lambda\) and \(\lambda\), one has
\[
\|\nabla u\|_{L^{p'}(B_{2R})} \leq C \left[ \|f\|_{L^{p'}(B_{3R})} + R^{n(1/p' - 1/p')} \|\nabla u\|_{L^p(B_{3R})} \right]. \tag{2.25}
\]

Using that \(\bar{A}\) is constant, we have by standard elliptic estimates that
\[
\|\nabla h\|_{L^\infty(B_{7R}/4)} \leq C R^{-n/p} \|\nabla h\|_{L^p(B_{2R})} \leq C R^{-n/p} \|\nabla u\|_{L^p(B_{2R})}.
\]

\(^2\)When \(p\) is close to 2 such smallness assumption is not needed, see e.g. \([6, 25]\).

\(^3\)For further references, see \([4, 5, 8, 18, 25]\).
Using
\[ \partial_\alpha (A^\alpha_{ij} \partial_j (u - h)^j) = \partial_\alpha (f^\alpha_i + (\bar{A} - A)^\alpha_{ij} \partial_j h^j) \quad \text{in } B_{2R}, \quad i = 1, \ldots, N, \]
and once again the fact that \([A]_{BMO(B_{4R})} \leq \gamma\), we have
\[ \| \nabla (u - h) \|_{L^p(B_{3R/2})} \leq C \left[ \| f \|_{L^p(B_{7R/4})} + \| A - \bar{A} \|_{L^p(B_{7R/4})} \| \nabla h \|_{L^\infty(B_{7R/4})} + R^{-(n+p')/p'} \| u - h \|_{L^1(B_{7R/4})} \right]. \] (2.26)

To estimate \( \| u - h \|_{L^1(B_{2R})} \), recall from the proof of Lemma 2.1 the chain of identities
\[ \int_{B_{2R}} \frac{|u - h|^2}{\sqrt{|u - h|^2 + t^2}} \, dx = \int_{B_{2R}} \bar{A}^\alpha_{ij} \partial_\alpha \phi^i \partial_j (u - h)^j \, dx = \int_{B_{2R}} (f^\alpha_i + (\bar{A} - A)^\alpha_{ij} \partial_j u^j) \partial_\alpha \phi^i \, dx, \]
which imply
\[ \int_{B_{2R}} \frac{|u - h|^2}{\sqrt{|u - h|^2 + t^2}} \, dx \leq C \left[ R^{(n+p')/p'} \| f \|_{L^{p'}(B_{2R})} + R \| A - \bar{A} \|_{L^p(B_{2R})} \| \nabla u \|_{L^{p'}(B_{2R})} \right]. \]

Noting that the constant \( C \) is independent of \( t \), we may send \( t \to 0 \) to obtain
\[ \| u - h \|_{L^1(B_{2R})} \leq C R^{(n+p')/p'} \left[ \| f \|_{L^{p'}(B_{2R})} + R^{-n/p} \| A - \bar{A} \|_{L^p(B_{2R})} \| \nabla u \|_{L^{p'}(B_{2R})} \right]. \] (2.27)

The conclusion follows from (2.25), (2.26) and (2.27). \( \square \)

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