Self-interaction of gravitational waves and their observability

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Abstract. Energy is not a well-defined concept in General Relativity (GR). There have been various approaches adopted, including simply discarding the concept. However, with mass-energy equivalence and the importance of mass in GR, many have thought it worth preserving. Of the various suggestions for defining energy, there is a proposal that the energy content in a time varying spacetime can be obtained by considering its second approximate Lie symmetries. Unlike others, this proposal allows one to calculate the energy in gravitational waves. When applied to cylindrical gravitational waves, it gives a self-damping of the waves. Though other waves may be expected to have different mathematical behaviour, the physics of self-damping should remain the same and one would expect a qualitatively similar result, even if it is quantitatively different in some details. If this proposal is valid, the estimates for the required sensitivity of the detectors for the various sources will have to be revised upwards. This revision is worked out here. In view of the fact that after two years of being online no signal has been seen by the detectors, it is argued that this proposal should be taken as a serious contender for a valid definition of energy in GR.

1. Introduction

In the early days of Relativity the reality of gravitational waves was questioned as they are solutions of the vacuum field equations and hence apparently have no energy. Weber & Wheeler [32] demonstrated that they would impart momentum to test particles in their path. This demonstration was extended to test particles in the path of plane gravitational waves [5]. Later, a general closed formula for the momentum imparted to test particles in an arbitrary spacetime was given [24].

Though it seemed obvious that “the waves must carry energy which is to be imparted”, no
clear measure for the energy they carried was available. This is because energy is not generally conserved in GR and hence mass and energy are not defined. One way to avoid this problem is by defining a stress-energy “pseudo-tensor” (see for example [17]), but it is observer dependent and hence not generally accepted by relativists for defining the energy (see for example the discussion of the pseudo-tensor in [21]). A deeper approach is to define some measure for the breaking of time translational symmetry as a measure for the energy density in the spacetime [3, 12, 15, 20, 25, 27, 33]. In that sense, the energy is “hidden” in the field. None of these attempts showed unambiguous success as they did not lead to any basic new physical insights or predictions. One would have hoped that a “correct” interpretation would announce itself by leading to a testable prediction. This hope implicitly regards GR as incomplete. Many would argue that it is so well tested that it should be accepted as complete, except in the quantum domain. I take the view that a problem cannot be discarded when it is inconvenient and should be faced “square-on” as Wheeler used to say. It could even be that the problems of the quantum may be elucidated by a completion of the theory at a classical level.

Following Wheeler’s approach of confronting the problem with any physical theory at its severest, nowhere is the problem of energy in GR as severe as in gravitational waves, where we have lost time translational symmetry and yet have no stress-energy tensor. Unlike electromagnetic waves (which do not undergo self-interaction because the Maxwell theory is linear), gravitational waves undergo self-interaction due to the non-linearity of GR. This self-interaction could, in principle, cause damping, like Landau damping of electromagnetic waves due to their interaction with matter [16], or cause an enhancement such as might be expected on the basis of the work on colliding gravitational waves [14, 26], where the spacetime curvature increases in some regions so that singularities are formed due to the interaction between two plane gravitational waves.

One approximate symmetry proposal is the use of broken (or approximate) Lie symmetries [1, 11] of the geodesic equations [10]. The breaking is taken to be by terms involving a small parameter whose powers, higher than some chosen value, can be neglected. It was found that though with the first order approximate symmetries no new insight is obtained [13], for the second order the timelike symmetry picks up a scaling factor [7, 8]. As this factor does not depend on the strength of the breaking it can be taken to be zero and hence the form of the perturbation should be irrelevant (much as displacements are inserted and then taken to be zero for calculations in Statics using D’Alembert’s principle). Thus, it may provide a genuine measure for the energy content of gravitational waves.

This proposal has been implemented for cylindrical gravitational waves [9]. For them the scaling factor turns out to go asymptotically as

\[ f(\rho, \omega) \sim \frac{3}{\pi^{3/2}} \times \frac{2^{11/4}}{\rho} \left[ (|\cos(\omega \rho/c)|)^{3/2} \sin(2\omega t) \right] (\omega \rho/c)^{-1/2} + O((\omega \rho/c)^{-3/2}), \]

where \( \rho \) is the radial distance from the source and \( \omega \) is the frequency of the wave. This formula does not give the energy content of the waves but only the scaling of the energy expected. The energy content estimate must come from the classical expectation, which is also recovered by first order approximate Lie symmetry analysis [10, 13]. This scaling gives the effect of the self-interaction of the waves due to their nonlinearity as a damping and not an enhancement.

The self-damping of gravitational waves will obviously have direct observational consequences. Waves that should just have been detectable on the original expectations would go below the
level of sensitivity of the detector. In view of the fact that the waves have not been detected for two years that the detectors have been online, as of the middle of 2011 [18], it seems worth taking the proposed definition of energy seriously. As will be seen, the effect of self-damping is very significant. There have been objections raised that it is unbelievable that such strong effects could occur in what is a linearizable theory. My point is that it is only an assumption that the theory is linearizable in the regime under consideration. That it be linearizable requires that the nonlinear effect be negligible compared with the linear. Since there is nothing to compare with, there is no basis for that assumption.

The plan of the paper is as follows. In the next section I will give a very brief introduction to approximate Lie symmetry analysis and review the consequences deduced from it for our purposes. In section 3 I will explore the observational significance of the proposal. In the subsequent section I will give a “poor-man’s way” (a’la Wheeler) of trying to interpret the physics of the effect. In the concluding section I will give a brief summary of the arguments and discuss them. It will be argued that the broad effect is inevitable but the specific values for the self-damping depend on the proposal and can only rely on observations for validation.

2.. Approximate Lie symmetry analysis

Given a system of \( p \) partial differential equations (PDEs) of order \( m \) for \( n \) functions, \( s^a \) of \( q \) variables, \( t^i \)

\[
E_\alpha(t^i, s^a, s^b_i, s^a_{ij}, ..., s^a_{i...m}) = 0 ,
\]

(2)
we treat it as an algebraic equation for the independent and dependent variables and their partial derivatives up to order \( m \). The equation is said to be symmetric under a point transformation \( (t^i, s^a) \rightarrow (x^i, y^a) \) if it results in the same algebraic expression in terms of the new variables. If there is only one independent variable, \( t \), it is a system of ordinary differential equations (ODEs). We shall, here, only be interested in ODEs. For these equations we shall denote the derivatives by \( s^a_b \). Lie [19] showed that if there are enough symmetries then the system of equations can be solved by transformation of variables.

Since one is dealing with differential equations one needs that the functions be differentiable of the relevant order. As such, one uses differentiable transformations and hence we can consider infinitesimal generators of symmetry. The generic generator is

\[
X = \xi(t, s^a) \frac{\partial}{\partial t} + \eta^b(t, s^a) \frac{\partial}{\partial s^b} ,
\]

(3)

where \( \xi \) and \( \eta^a \) are arbitrary functions. Of course, we need to extend the generator to incorporate the derivatives of the dependent variables. For a second order system of ODEs,

\[
X^{[1]} = \xi(t, s^a) \frac{\partial}{\partial t} + \eta^b(t, s^a) \frac{\partial}{\partial s^b} + \eta_1^b(t, s^a) \frac{\partial}{\partial s^b_1} + \eta_2^b(t, s^a) \frac{\partial}{\partial s^b_2} ,
\]

(4)
where

\[
\eta^{a[n]} = \frac{d\eta^{a[n-1]}}{dt} - s^{am}(r) \frac{d\xi}{dt} ,
\]

(5)

(\( \eta^{a[0]} = \eta^a \)), \( d/dt \) stands for the total derivative,

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + s^b \frac{\partial}{\partial s^b} + s^b_m \frac{\partial}{\partial s^b_m} .
\]

(6)
and \( s^{na}(t) \) stands for the \( n^{th} \) derivative of \( s^a \) relative to \( t \). Then

\[
XE_a|_{E_a=0} = 0 ,
\]

i.e. the symmetry generators, acting on the system of equations will be zero when evaluated for the solutions of the given system of equations.

If the above equation is not satisfied but if

\[
E_a = E_a^0 + \epsilon E_a^1 + \epsilon^2 E_a^2 + ... \tag{8}
\]

and there exist an infinitesimal symmetry generator, \( X_0 \), of \( E_a^0 \), and \( X_1 \) such that

\[
X = X_0 + \epsilon X_1 \tag{9}
\]

and

\[
XE_a|_{E_a=0} = O(\epsilon^2) ,
\]

then we say that \( X \) is an approximate symmetry generator of the system of equations \( E_a = 0 \). If \( X_0 = kX_1 \), with \( k \) allowed to be zero or a non-zero constant then \( X \) is said to be a trivial approximate symmetry. If that is not the case then it is said to be non-trivial.

Higher order approximate symmetries can be obtained similarly. In particular we can define second approximate symmetries by asking for

\[
X = X_0 + \epsilon X_1 \epsilon^2 X_2 ,
\]

now with the requirement that

\[
XE_a|_{E_a=0} = O(\epsilon^3) .
\]

While for (first) approximate symmetries it makes no difference whether we use solutions of \( E_a = 0 \) or \( E_a^0 = 0 \), for higher order symmetries in general it does. This fact will be important.

The infinitesimal symmetry generators for the system of ODEs form a Lie algebra. The symmetries for a system of ODEs for a free particle in \( n \)-dimensional space admit the Lie algebra \( sl(n+2, \mathbb{R}) \). For Minkowski spacetime there are, then, 10 isometries and 35 generators. As soon as the isometries of the space are reduced the Lie symmetries are also (drastically) reduced. For Schwarzschild spacetime there are only 4 isometries (corresponding to angular momentum and energy conservation) and only 6 Lie symmetries (the 4 isometries and the generators for geodetic re-parametrization). The extra symmetries are \( \partial/\partial s \) and \( s\partial/\partial s \), where \( s \) is the geodetic parameter, i.e. the proper time. The extra isometries, corresponding to momentum and spin conservation, are recovered as (first) approximate symmetries [13]. (The radial equation of motion also has two non-trivial approximate symmetries, but that is not relevant for our present purpose.) However, the Reissner-Nordström spacetime does not possess any (first) approximate symmetries (or any non-trivial approximate symmetries for the radial equation). To further extend the analysis used for the Schwarzschild spacetime, we tried using second approximate symmetries [7, 8]. We found that we could recover the lost conservation laws as (second) approximate symmetries provided that we re-scaled the proper time translation symmetry generator by a position dependent factor. Since time translation corresponds to energy, the factor gives a position-dependent re-scaling of energy. The results for the Reissner-Nordström [7] and the Kerr spacetimes [8] gave physically reasonable results.
To follow through the programme we then applied the procedure with second approximate symmetries to the exact cylindrical gravitational wave spacetime. The result was a position dependent re-scaling of the energy of the cylindrical wave [9]. This was interpreted as giving the change of energy due to self-interaction. The factor involves a Bessel function of order zero. The asymptotic behaviour of this scaling factor, given in Eq.(1), is to reduce the energy by a factor of a square-root of the distance from the axis.

3.. Observational implications

The main purpose of this paper is to examine the observational implications of this attenuation of gravitational waves implicit in the approximate symmetry proposal and look for predictions that can be tested.

The obvious implication of the predicted effect is that our expectation of how the energy in the wave decreases with the distance of the source and the frequency of the wave (and hence the length of the antenna) will be very seriously overestimated. However, one can not simply take the attenuation factor to be $\sqrt{c/\rho \omega}$, as that is found for cylindrical waves. There are no infinite axially symmetric sources in Nature to emit genuine cylindrical waves. Nor is it a priori clear that a spherical, or slightly aspherical, source will give the same attenuation factor. Since a cylinder is “less curved” than a sphere (in that the cylinder can be constructed from a plane sheet, while the sphere cannot) the curvature effect of a spherical wave should not be less than that of a cylindrical wave. The classical spherical wave energy density $E_{cs} \sim 1/\rho^2$. We would thus expect that $E_{sph} \sim \rho^{-5/2-\alpha}, \alpha > 0$. The cylindrical waves depend on the distance, via a Bessel function of order zero. The asymptotic approximation of this function gives the scaling factor mentioned above. One would expect that the energy in spherical waves will depend on the distance via a spherical Bessel function, which has the same asymptotic behaviour. As such, the scaling factor can be expected to be the same for both types of waves. Of course, there are no exact spherical gravitational waves either. However, we could approximate the effect of the spherical waves by patching together two Nutku “spherical waves with strings attached” [22]. Ideally, one would like to know exactly how much the attenuation for realistic models of the source. This analysis is vital for a more precise test of the proposal, but we will use the weaker condition for our present purposes. Here we will just take the least attenuation that there can be, namely that of the cylindrical waves.

The attenuation factor needs to be considered more carefully from the observational viewpoint. The frequencies observed will depend on the length of the detector, in that the waves must “fit into them”. Thus the wavelengths must be simple fractions of the detector length. The series for $1/\sqrt{n}$ has very slow convergence. However, the spectrum of the waves may give faster convergence. Unfortunately, we cannot use the Planck spectrum for the harmonics as it derives from the quantum nature of the electromagnetic field, which is not directly applicable to gravitational waves. Since quantum gravity only applies at ultra-high energies, the relevant spectrum can be estimated by an exponentially damped tail. Roughly estimating the attenuation factor for this exponential damping, gives 1.5 times the value for the basic mode, which has a wavelength half of the detector length.

An interesting source to consider for our purpose is SN1987A, which had a 0.1 asphericity in
its explosion [28]. One could regard the resulting waves as, in some sense, combined “spherical” and cylindrical waves. Crudely, then, one could take a cylindrical component of 0.1 of the total and assume that it has the predicted attenuation factor. The axis of the explosion is estimated to make an angle of $\pi/4 = 45^\circ$ with our line of sight. It seems reasonable to estimate the warped geometry attenuation factor $f \sim \sqrt{c/\rho^3 + \omega}$. This reduction is enormous for SN1987A, which is at a distance of $\sim 51.4$ kpc. For an interferometer with an arm of $\sim 3$ km the attenuation factor comes out to be about a million! For a bar detector it would be greater by a factor of about twenty. For the [6] binary pulsar $E$ would be about 300,000 for the interferometer and a few millions for the bar detector.

4.. Physical interpretation of the scaling

Notice that $\omega \rho/c$ is the number of waves, $n$, that fit into the distance $\rho$. Thus $f \sim 1/\sqrt{n}$. It is worth trying to see this in Wheeler’s “poor man’s way” to fully understand what is happening. Whereas the Newtonian space and time can be thought of as a rigid spatial grid and an orthogonal rigid line of time, and the special relativistic spacetime as a rigid grid of space and time the general relativistic spacetime is a flexible grid of space and time. The former can be thought of as made of steel girders and the latter of rubber girders. The wave would then travel through the warped spacetime. As such, the distance traveled would not be the classically estimated distance but distance along the “troughs” and “peaks” of the wave. Another way of visualizing it is an oscillating membrane (like a trampoline). The length along the membrane at maximum stretching will be more than the diameter. Hence we should anyhow expect the effective distance that the wave travels to be greater than the classical distance, and hence the wave to be attenuated relative to the classical expectation. Further, the attenuation should depend on $n$. The question remains, exactly how it depends on $n$. The proposed definition of energy gives the factor noted above.

For classical cylindrical waves the energy density $E_{cc} \sim 1/\rho$. (In terms of the proposal, this energy is recovered by the first approximate symmetries.) Thus the total energy density of the waves is asymptotically $E_{cyl} \sim \sqrt{c/\rho^3 \omega}$. This result runs counter to our usual intuition based on a linear field like electromagnetism. In that case the loss of energy would have to be compensated by a gain of energy in the intervening region, causing heating of that region. Here that energy has gone into warping, and thus enlarging the spacetime. Again, the dependence on the frequency of the wave seems counter-intuitive from the linear view-point but is easily seen to be reasonable from the warped spacetime viewpoint.

5.. Conclusion

Weber claimed to have observed a signal on his non-cryogenic bar detector synchronous with the light and neutrino burst from SN1987A [29]. Since the standard theory of the bar detector developed by him [30] leads to energy requirements incompatible with the energy output of SN1987A ($\sim 10^{53}$ ergs), this claim was generally regarded as untenable. Indeed, Weber tried to revamp his theory of the detector to provide much greater efficiency of detection, using a claimed quantum coherence effect [23, 31]. While Bassan did not believe the claim, he admitted to having seen the signal claimed by Weber, as he was visiting Weber’s laboratory at the time [2]. If our proposal is borne out, then Weber’s claim of a sensitivity a billion times greater than
that provided by his original theory would get reduced to an effective thousand — still far from sufficient to provide observability.

A detailed analysis of the sensitivity requirements for various sources according to their strengths and distances will be presented subsequently [4]. Of course, the approximate symmetry definition is only a proposal. The general prediction that the waves will be attenuated according to the number of waves that fit into the distance traversed is robust, but the explicit dependence as $1/\sqrt{n}$ is not. This is a gap in the theory that only experiments can fill. If the expected signals are not seen despite the design accuracy being sufficient, our proposal could be tentatively accepted till such time as the sensitivity of observation is adequate to test it. Of course, since we know the output of waves from the Hulse-Taylor binary pulsar, a good test would be a sufficiently accurate observation of signals from it.

Consider the Hulse-Taylor binary pulsar as the source of the waves. It has been running for a long time and the attenuation factor given should apply. However, an exploding supernova sends its waves into a space that is nearly flat. The point is that the fabric of the spacetime is being stretched by the waves but the full stretching will not occur right at the beginning. Consequently, the waves should not be attenuated that much in the beginning. After a while the space around it has been stretched as much as it can be and the waves are merely maintaining the stretching. Consequently, the attenuation should be greater later. It would be very interesting to see this effect. However, the build-up of the attenuation factor must be extremely rapid and it does not seem likely that this effect will be observable.

The proposal may seem to be so outlandish as to be dismissed out of hand. The reason is that one assumes that for gravitational waves linearized gravity would provide good guidelines and nonlinear effects would appear as perturbations on it. Indeed, that is how the theory of gravitational waves is normally developed. However, there are two points to note. For nonlinear systems there is no guarantee that one has not entered into a “chaotic” regime. The parable of the butterfly (let us call it Osama) flapping its wings at one place in the World (say Abbottabad) causing a hurricane three days later at a place nearly antipodal to it (let us say New York), is worth bearing in mind. The other point is that a “perturbation”, by definition, is negligible compared to the main effect. For a zero energy, it is not clear what would be a sufficiently small change to qualify as a “perturbation”.

The modern laser interferometer gravitational wave detectors came on line in June-July 2009. At the time there had even been hopes that the waves might have been detected by mid-July 2009. However, in two years of running no signal has been seen. There are now limits being put on the strengths of sources on the basis of the lack of a signal [18]. Since the proposal discussed in this paper pre-dates even the start of the observations, leave alone the lack of observations [9], it is worth considering this proposal as a contender for a valid definition of energy in GR.

It is interesting to note that Weber’s claimed “quantum coherence” effect leads to much greater efficiency of the bar detector than the interferometer (by a factor of about a billion). As is intuitively clear, the efficiency of a longer detector should be greater. According to our proposal, the efficiency increases with the square root of the length. By this proposal the efficiency of the bar detector of $3 \text{ m}$ length is about 30 times less than the interferometer of $3 \text{ km}$ arm. We urgently need that both the bar detector and interferometer come on-line to provide the data so desperately needed.
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