A MATHEMATICAL MODEL TO RESTORE WATER QUALITY IN URBAN LAKES USING PHOSLOCK

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ABSTRACT. Urban lakes are the life lines for the population residing in the city. Excessive amounts of phosphate entering water courses through household discharges is one of the main causes of deterioration of water quality in these lakes because of the way it drives algal productivity and undesirable changes in the balance of aquatic life. The ability to remove biologically available phosphorus in a lake is therefore a major step towards improving water quality. By removing phosphate from the water column using Phoslock essentially deprives algae and its proliferation. In view of this, we develop a mathematical model to investigate whether the application of Phoslock would significantly reduce the bio-availability of phosphate in the water column. We consider phosphorus, algae, detritus and Phoslock as dynamical variables. In the modeling process, the introduction rate of Phoslock is assumed to be proportional to the concentration of phosphorus in the lake. Further, we consider a discrete time delay which accounts for the time lag involved in the application of Phoslock. Moreover, we investigate behavior of the system by assuming the application rate

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of Phoslock as a periodic function of time. Our results evoke that Phoslock essentially reduces the concentration of phosphorus and density of algae, and plays crucial role in restoring the quality of water in urban lakes. We observe that for the gradual increase in the magnitude of the delay involved in application of Phoslock, the autonomous system develops limit cycle oscillations through a Hopf-bifurcation while the corresponding nonautonomous system shows chaotic dynamics through quasi-periodic oscillations.

1. Introduction. Urban lakes are of great importance to society in providing recreational services to citizens. The vast majority of such lakes are manmade and exposed to strong human influences, of which excessive nutrient loading, often leading to dense cyanobacterial blooms and floating scums, presents one of the most prominent impacts. Their close vicinity to urban areas means that citizens usually have easy access to those lakes, but also that they are confronted with eutrophication-related nuisances and hazards, such as turbid water, fish kills, bad smell, and toxic cyanobacterial blooms. Hence, the need for safe and aesthetically acceptable water in a modern society makes eutrophication control a key challenge for authorities.

Pollution of lakes due to excessive growth of algae and other aquatic plants has become a global issue \cite{7, 25, 27}. Overgrowth of algae in aquatic systems is the result of increase in nutrients concentration \cite{2, 3, 10, 23, 30, 33}. Algae float on the surface of water in an aquatic system thereby reduces the transfer of oxygen from air to water. After die off, algae decompose and the nutrient contained in that organic matter are converted into inorganic form by microorganisms. This decomposition process consume oxygen giving birth to hypoxic zones \cite{37, 38}. Low level of dissolved oxygen significantly affects the survival of fish and other aquatic species \cite{46}. Certain types of algae produce toxin, which are harmful not only for human population but also for some aquatic species \cite{6, 17}. Though algal bloom occurs naturally in the aquatic system but their frequency and intensity get amplified by the increase of nutrient pollution \cite{37, 38, 46}. One possible solution to this problem is curtailing the nutrient loading in aquatic system.

Any nutrient can become a limiting nutrient in an ecosystem. For algae, the most obvious limiting nutrients are nitrogen and phosphorus \cite{32}. Nitrogen removal is an expensive process, with high energy and chemical costs and specialized equipment. Algae are able to fix atmospheric nitrogen opportunistically. Phosphorus limitation is the most practical means of preventing the growth of toxic blue green algae. By using a mathematical model, Tiwari et al. \cite{50} compared the effects of nitrogen and phosphorus on the growth of algae in lakes, and showed that the sources of phosphorus provide a greater risk for bloom of algae than that of nitrogen. Several factors are responsible for the increase of phosphorus in urban lakes \cite{12, 43}, of which household discharges has major contribution \cite{11, 28, 53}. Decades of uncontrolled inputs have loaded lake sediments with phosphorus. Besides, there are some natural causes such as release of phosphate from the bottom as well as surrounding sediments of the lakes, which increases the concentration of phosphorus in urban lakes. Thus, to control the formation of algal bloom in lakes, the concentration of phosphorus should be reduced to a very low level \cite{8, 29}.

Phoslock is a patented phosphorus locking technology containing lanthanum, a naturally occurring earth element, embedded inside a clay matrix \cite{13, 45}. Lanthanum reacts with phosphate forming an insoluble and biologically inert compound. Its ability to bind phosphate makes it an effective tool in lake restoration
Phoslock is produced through a patented ion exchange process whereby lanthanum ions displace sodium ions within the clay matrix. The formulation process of Phoslock absorbs lanthanum into a non-toxic carrier such that the lanthanum retains its capacity to bind phosphate when applied to aquatic environments and results in a non-toxic mineral that becomes an inert component of the sediments. For over ten years, Phoslock has been successfully used in water resource restoration programs around the world to remove free reactive phosphorus and restore water quality. It is rapidly emerging as the most effective phosphorus inactivation and water quality restoration solution for ponds, lakes and reservoirs. Phoslock poses a negligible to very low risk to the aquatic environment while providing a high affinity to bind and remove phosphorus that results in improvements to water quality. Phoslock offers a wide range of benefits over other methods of reducing phosphorus concentrations in lakes and reservoirs. Unlike other methods that could be used to immobilize phosphorus in water and sediments, Phoslock is insensitive to the range of redox, temperature and pH conditions that are naturally found in lakes and reservoirs.

Some mathematical studies showed the effects of different control measures on the reduction of algal bloom in lakes. Misra et al. have investigated the effects of external efforts applied for the removal of algae and detritus from the lake. They found that high concentration of nutrients intensifies the occurrence of algal bloom, but the applied efforts play a pivotal role in reducing the bloom. Tiwari et al. examined the reuse of detritus generated by dead algae as fertilizer for crops. The results of their study showed that in so doing, the crop yield can be increased and simultaneously the algae in lakes reduced. Recently, Misra et al. showed that if the efficacy of budget allocation for controlling nutrients input and algae removal is suitably high, then algae can be completely mitigated from the lake. They also found that the combined effects of nutrients control and algae removal is much more beneficial for making the aquatic environment free of nutrients and algae.

From the above, we note that eutrophication of surface waters through nutrient enrichment, resulting in blooms of algae, is the most important water quality problem worldwide. Remedies against eutrophication have mainly centered on control of phosphorus in water bodies. A novel technique to control phosphorus in the lake is the application of Phoslock. Although experimental studies confirmed the potential of Phoslock to control phosphorus and algal bloom, but to the best of our knowledge, the usage of Phoslock has never been considered explicitly in modeling the control of phosphorus and algal blooms. In this paper, our objective is to assess how the use of Phoslock can be beneficial for restoring the quality of urban lakes. Further, we investigate the impacts of the periodic input of Phoslock and the time delay involved in Phoslock application on the dynamics of system.

Rest of the paper is organized as follows. In the next section, we propose our model for control of algae by using Phoslock. In the following section, the model is analyzed for boundedness, permanence, feasibility and stability of system’s equilibria. In Section 4, we modify our model by considering the effect of time delay involved in the application of Phoslock; the delay model is analyzed for existence of Hopf bifurcation by taking time delay as the bifurcation parameter. In Section 5, we further extend our model by assuming that the rate of application of Phoslock to the aquatic system as a periodic function of time; sufficient conditions are derived for the global attractivity of positive periodic solution. Numerical simulations
2. The mathematical model. Consider an urban area and a nearby lake, where the release of phosphorus has been identified as a key factor in triggering algal blooms. Let at any time $t > 0$, $N(t)$, $A(t)$ and $D(t)$, respectively denote the concentration of phosphorus, density of algae and density of detritus in the lake. Further, suppose that at the same time, $C(t)$ amount of Phoslock is present in the lake, being used to reduce the availability of phosphorus. The following assumptions are made in the modeling process.

1. It is assumed that phosphorus is continuously coming into the lake from household discharges, at a constant rate, being naturally depleted.
2. The phosphorus input accelerates algal growth; the utilization rate of phosphorus by algae is assumed in the form of a Holling type-II term, and the growth rate of algal population is proportional to the same interaction term [50].
3. Algae in lakes consist of diverse assemblage of all major taxonomic groups, which have different physiological requirements and differ in their response to light, temperature and concentration of nutrients. Here, we do not consider the effects of light, temperature and nutrients other than phosphorus.

![Figure 1. Schematic diagram for the interactions among phosphorus, algae, detritus and Phoslock. Here, cyan color represents the impact of Phoslock on reduction of phosphorus concentration; blue color stands for the periodic introduction of Phoslock in the lake; red color represents the time delay involved in the application of Phoslock. The two dashed lines headed to each other indicate that Phoslock bind with phosphorus and becomes an inert component of the sediments.](image-url)
4. Algal populations experience natural death, higher predation and intraspecific competition [44, 46]. Thus, the death rate of algae is proportional to $A$ (natural death and higher predation) as well as $A^2$ (intraspecific competition).

5. Dead algae sink to the bottom of the lake and form detritus. Bacteria and other organisms break down detritus into nutrients, a part of which contribute to the growth of phosphorus, thus returning the detritus to the ecosystem.

6. The amount of Phoslock applied to the lake is assumed to depend on the concentration of phosphorus in the lake; Phoslock depletes naturally. Phoslock react with freely reactive phosphorus in the lake and form new inert product reducing the concentrations of phosphorus and Phoslock in the lake. This reductions depend on the concentrations of phosphorus and Phoslock as well, and follow mass action law.

The schematic diagram for the interplay among all the considered dynamical variables is depicted in Fig. 1. Based on these ecological assumptions, we have the following system of nonlinear differential equations,

$$
\begin{align*}
\frac{dN}{dt} &= q - \alpha_0 N - \beta_1 N A \frac{1}{\beta_1 + \beta_11 N} - \lambda_0 NC + \pi_2 \delta D, \\
\frac{dA}{dt} &= \theta_1 \beta_1 N A \frac{1}{\beta_1 + \beta_11 N} - \alpha_1 A - \beta_10 A^2, \\
\frac{dD}{dt} &= \pi_1 (\alpha_1 A + \beta_10 A^2) - \delta D, \\
\frac{dC}{dt} &= \mu N - \mu_0 C - \phi NC.
\end{align*}
$$

Note that previous studies considered cumulative concentration of nutrients and investigated their effects on algal bloom and its consequences [37, 38, 46]. The excessive input of nutrients is found to increase algal biomass. The eutrophied lakes suffer from depleted zones of dissolved oxygen and massive fish mortality. Here, our aim is to investigate the impact of Phoslock on algal bloom reduction if the lake is overloaded due to phosphorus.

System (1) is to be analyzed with the following initial conditions:

$$
N(0) > 0, \ A(0) \geq 0, \ D(0) \geq 0, \ C(0) > 0. \tag{2}
$$

Since all the parameters of system (1) are non-negative, the right hand side is a smooth function of the variables $N$, $A$, $D$ and $C$ in the positive region

$$
\Theta = \{(N, A, D, C) : N, A, D, C \in \mathbb{R}_+ \}.
$$

The biological meaning of the parameters involved in the system (1) and their values used for numerical simulations are given in Table 1.

3. Mathematical analysis. The state space of the system (1) is $\mathbb{R}_+^4$. We first show that the proposed system is positively invariant and bounded in its state space and the related dynamical properties are provided in the following theorem.

**Theorem 3.1.** The proposed system (1) is positive invariant and bounded in $\mathbb{R}_+^4$. Moreover, for any initial condition in $\mathbb{R}_+^4$, both $N$ and $C$ are persistent. Further, if

$$
\theta_1 \beta_1 - \beta_11 \alpha_1 < 0, \tag{3}
$$

then $\limsup_{t \to \infty} A(t) = 0$ while if
then \( A \) is persistent and system (1) is permanent. Furthermore, the region of attraction for all solutions of system (1) initiating in the positive orthant is given by the following set:

\[
\Omega = \{(N(t), A(t), D(t), C(t)) \in \mathbb{R}_+^4 : 0 \leq N(t) + A(t) + D(t) \leq N_m, 0 \leq C(t) \leq C_m\},
\]

which is compact and invariant with respect to system (1). The quantities \( \bar{N}, N_m \) and \( C_m \) are defined in the proof.

Proof. System (1) can be written as \( \dot{X} = f(X) \), where \( X \equiv (N, A, D, C)^T \), \( f : \mathbb{C}_+ \to \mathbb{R}_+^4 \), \( f = (f_1, f_2, f_3, f_4)^T \) with \( f_i \in \mathbb{C}^\infty(\mathbb{R}_+) \), for \( i = 1, 2, 3, 4 \);

\[
\begin{align*}
    f_1 &= q - \alpha_0 N - \frac{\beta_1 N A}{\beta_{12} + \beta_{11} N} - \lambda_0 N C + \pi_2 D, \\
    f_2 &= \frac{\theta_1 \beta_1 N A}{\beta_{12} + \beta_{11} N} - \alpha_1 A - \beta_{10} A^2, \\
    f_3 &= \pi_1 (\alpha_1 A + \beta_{10} A^2) - \delta D, \\
    f_4 &= \mu N - \mu_0 C - \phi NC.
\end{align*}
\]

Since the vector function \( f \) is locally Lipschitzian and completely continuous function of variables \( (N, A, D, C) \) in

\[
\Theta = \{(N(t), A(t), D(t), C(t)) : N > 0, A > 0, D > 0, C > 0\},
\]

any solution \( (N, A, D, C) \) of system (1) with initial conditions (2) exists and unique in some interval \([0, \kappa]\) \[4, \ 22\]. It is easy to see that if \( X(0) \in \mathbb{R}_+^4 \) with \( X_i = 0 \) for \( i = 1, 2, 3, 4 \), \( f_i(X)|_{X_i=0} \geq 0 \). Thus, any solution of system (1) with \( X_0 \in \mathbb{R}_+^4 \), say \( X(t) = X(t; X_0) \) is such that \( X(t) \in \mathbb{R}_+^4 \) for all \( t > 0 \) \[55\].

Adding the three equations of system (1), we have

\[
\frac{d(N + A + A + D)}{dt} \leq q - \delta_m (N + A + D), \quad \text{where} \quad \delta_m = \min\{\alpha_0, (1 - \pi_1) \alpha_1, (1 - \pi_2) \delta\}.
\]

Using a standard comparison theorem \[31\], we have

\[
0 \leq N(t) + A(t) + D(t) \leq \frac{q}{\delta_m} + \left( N(0) + A(0) + D(0) - \frac{q}{\delta_m} \right) e^{-\delta_m t}.
\]
Thus, as $t \to \infty$, $0 \leq N(t) + A(t) + D(t) \leq \frac{q}{\delta_m}$, we have for any $t > 0$, $0 \leq N(t) + A(t) + D(t) \leq N_m$, where $N_m = \max \left\{ \frac{q}{\delta_m}, N(0) + A(0) + D(0) \right\}$. Therefore, there exist $N', A', D' > 0$ such that $0 \leq N(t) \leq N'$, $0 \leq A(t) \leq A'$ and $0 \leq D(t) \leq D'$ for all sufficiently large values of $t$, and $0 \leq N' + A' + D' \leq N_m$. Similarly, from the last equation of system (1), we have for any $t > 0$, $0 \leq C(t) \leq C_m$, where $C_m = \max \left\{ \mu N' \mu_0 \phi N, C(0) \right\}$.

Thus, there exist positive constants $N_m$ and $C_m$ depending only on the parameters of system (1) such that $0 \leq N(t) \leq N_m$ and $0 \leq C(t) \leq C_m$ for all sufficiently large values of $t$. Hence, the solutions of system (1) and consequently all the variables appearing in the system are ultimately bounded above [4].

In the absence of algae i.e., $A = 0$, system (1) reduces to the following subsystem,

\[
\frac{dN}{dt} = q - \alpha_0 N - \lambda \phi NC, \\
\frac{dC}{dt} = \mu N - \mu_0 C - \phi NC,
\]

which has only one interior equilibrium, $\hat{E}^* = (\hat{N}^*, \hat{C}^*)$, where $\hat{C}^* = \frac{\mu \hat{N}^*}{\mu_0 + \phi \hat{N}^*}$ and $\hat{N}^*$ is positive root of the following quadratic equation

\[
\phi(\alpha_0 + \lambda \mu)N^2 - (q \phi - \alpha_0 \mu_0)N - q \mu_0 = 0. 
\]

Note that equation (6) has exactly one positive root given by

\[
\hat{N}^* = \frac{(q \phi - \alpha_0 \mu_0) + \sqrt{(q \phi - \alpha_0 \mu_0)^2 + 4q \mu_0 \phi(\alpha_0 + \lambda \mu)}}{2 \phi(\alpha_0 + \lambda \mu)}.
\]

It is checked that the equilibrium $\hat{E}^*$ is always locally as well as globally asymptotically stable.

From the first equation of system (1), we have

\[
\frac{dN}{dt} \geq q - \alpha_0 N - \frac{\beta_1 A' N}{\beta_{11}} - \lambda \phi C' N.
\]

Thus, it follows that for some $\tilde{N}$,

\[
\liminf_{t \to \infty} N(t) \geq \frac{q}{\alpha_0 + \frac{\beta_1 A'}{\beta_{11}} + \lambda \phi C'} = \tilde{N}.
\]

From the last equation of system (1), we have

\[
\frac{dC}{dt} \geq \mu \tilde{N} - \mu_0 C - \phi \tilde{N} C.
\]

Thus, it follows that for some $\tilde{C}$,

\[
\liminf_{t \to \infty} C(t) \geq \frac{\mu \tilde{N}}{\mu_0 + \phi \tilde{N}} = \tilde{C}.
\]

Thus, $N$ and $C$ are always persistent for any initial values of $N$ and $C$.

Now, we have

\[
\frac{1}{\tilde{A}} \frac{dA}{dt} \leq \frac{\theta_1 \beta_1}{\beta_{11}} - \alpha_1.
\]
Therefore, we can conclude that $A$ goes to zero if

$$\theta_1 \beta_1 - \beta_{11} \alpha_1 < 0.$$  

Also, from the second equation of system (1), we have

$$\frac{dA}{dt} \geq \frac{\theta_1 \beta_1 \tilde{N} A}{\beta_{12} + \beta_{11} N} - \alpha_1 A - \beta_{10} A^2.$$  

Thus, it follows that for some $A$,

$$\liminf_{t \to \infty} A(t) \geq \frac{\theta_1 \beta_1 \tilde{N}}{\beta_{12} + \beta_{11} N} - \frac{\alpha_1}{\beta_{10}} = \tilde{A}.$$  

Again, from the third equation of system (1), we have

$$\frac{dD}{dt} \geq \pi_1 \alpha_1 \tilde{A} - \delta D.$$  

Thus, it follows that for some $\tilde{D}$,

$$\liminf_{t \to \infty} D(t) \geq \frac{\pi_1 \alpha_1 \tilde{A}}{\delta} = \tilde{D}.$$  

Clearly, $D$ is persistent if $A$ is persistent. Note that $A$ is persistent if condition (4) holds.  

**Remark 1.** Biologically, to satisfy condition (3), the total benefit to the algal biomass on consumption of nutrients must be lesser than the natural death rate of algae. In such situation, the growth rate of algal population becomes negative and algae disappear as time progresses. On the other hand, condition (4) tells that for survival of algal population its growth due to nutrients must exceed its natural death rate. The algae-free system (5) always has unique globally stable interior equilibrium i.e., if we perturb the system around the equilibrium, the system returns back to the equilibrium in time. Therefore, we always have system with phosphorus and Phoslock.

### 3.1. Equilibrium states of system (1)

System (1) has the following two non-negative equilibria.

1. The algae-detritus-free equilibrium $E_0 = (N_0, 0, 0, C_0)$, where $N_0 = \tilde{N}^*$ and $C_0 = \tilde{C}^*$. The equilibrium $E_0$ always exists in the system.

2. The algae-persistent equilibrium $E^* = (N^*, A^*, D^*, C^*)$, where

$$N^* = \frac{\beta_{12} (\alpha_1 + \beta_{10} A^*)}{(\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A^*}, \quad D^* = \frac{\pi_1 A^* (\alpha_1 + \beta_{10} A^*)}{\delta}, \quad C^* = \frac{\mu N^*}{\mu_0 + \phi N^*},$$

and $A^*$ is positive root of the following equation

$$F(A) = \frac{G(A)}{(\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A^*},$$

where

$$G(A) = \left[ q + \left( \frac{1}{\theta_1} - \pi_1 \pi_2 \right) \alpha_1 A - \left( \frac{1}{\theta_1} - \pi_1 \pi_2 \right) \beta_{10} A^2 \right] \times \left[ (\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A \right]$$

$$- \beta_{12}^2 \lambda \phi (\alpha_1 + \beta_{10} A^2)$$

$$- \mu_0 \{ (\theta_1 \beta_1 - \beta_{11} \alpha_1) - \beta_{11} \beta_{10} A \} + \beta_{12} \phi (\alpha_1 + \beta_{10} A)$$

$$- \beta_{12} \alpha_0 (\alpha_1 + \beta_{10} A).$$
We note the following properties of $G(A)$.

1. $G(0) > 0$ provided the following condition holds:

$$q\mu_0 + (q\phi - \alpha_0\mu_0)N_1 - (\alpha_0 + \lambda\mu)N_1^2 > 0,$$

where $N_1 = \frac{\beta_{12}\alpha_1}{\theta_1\beta_1 - \beta_{11}\alpha_1}$.

2. $G(A) < 0$ for $A = \frac{\theta_1\beta_1 - \beta_{11}\alpha_1}{\beta_{11}\beta_{10}}$.

Thus, $G(A)$ has at least one positive root which is also the root of $F(A) = 0$ in the open interval $(0, A)$. Further, we note that $F'(A) < 0$ for all $A \in (0, A)$. Therefore, $F(A)$ has unique positive root in the open interval $(0, A)$ if condition (10) holds.

3.1.1. **Stability analysis.** Regarding local stability of equilibria of the system (1), we have the following theorem.

**Theorem 3.2.** 1. The equilibrium $E_0$ is stable provided the following condition holds

$$N_0(\theta_1\beta_1 - \beta_{11}\alpha_1) - \beta_{12}\alpha_1 < 0.$$ (11)

Further, the equilibrium $E_0$ is unstable if opposite of condition (11) holds, and in that case the equilibrium $E^*$ exists.

2. The equilibrium $E^*$, if exists, is locally asymptotically stable without any condition.

**Proof.** Jacobian of system (1) is given by

$$J = \begin{pmatrix}
J_{11} & J_{12} & J_{13} & J_{14} \\
J_{21} & J_{22} & 0 & 0 \\
0 & J_{32} & J_{33} & 0 \\
J_{41} & 0 & 0 & J_{44}
\end{pmatrix},$$

where

$$J_{11} = -\left(\alpha_0 + \lambda\phi C + \frac{\beta_{12}A}{(\beta_{12} + \beta_{11}N)^2}\right), \quad J_{12} = -\frac{\beta_1N}{\beta_{12} + \beta_{11}N}, \quad J_{13} = \pi_2\delta,$$

$$J_{14} = -\lambda\phi N, \quad J_{21} = \frac{\theta_1\beta_1\beta_{12}A}{(\beta_{12} + \beta_{11}N)^2}, \quad J_{22} = -\frac{\theta_1\beta_1N}{\beta_{12} + \beta_{11}N} - \alpha_1 - 2\beta_{10}A,$$

$$J_{32} = \pi_1(\alpha_1 + 2\beta_{10}A), \quad J_{33} = -\delta, \quad J_{41} = \mu - \phi C, \quad J_{44} = -(\mu_0 + \phi N).$$

1. Evaluating the Jacobian at the equilibrium $E_0$, we get two eigenvalues as $\frac{\theta_1\beta_1N_0}{\beta_{12} + \beta_{11}N_0} - \alpha_1$ and $-\delta$, while other two are given by roots of the following equation

$$\xi^2 + \{\alpha_0 + \mu_0 + \phi(N_0 + \lambda C_0)\} \xi + \alpha_0(\mu_0 + \phi N_0) + \lambda\phi(\mu N_0 + \mu_0 C_0) = 0.$$ (12)

Clearly, roots of equation (12) are either negative or have negative real parts. Thus, the equilibrium $E_0$ is stable if $N_0 < N_1$ and unstable if $N_0 > N_1$. In the latter case, the equilibrium $E^*$ exists.

2. Characteristic equation of the Jacobian evaluated at the equilibrium $E^*$ is given by

$$P(\xi) \equiv \xi^4 + Q_1\xi^3 + Q_2\xi^2 + Q_3\xi + Q_4 = 0.$$ (13)
where

\[ Q_1 = \alpha_0 + \lambda \phi C^* + \beta_{10}A^* + \delta + \mu_0 + \phi N^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2}, \]

\[ Q_2 = \left( \alpha_0 + \lambda \phi C^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right) (\beta_{10}A^* + \delta + \mu_0 + \phi N^*) \]

\[ + (\beta_{10}A^* + \delta)(\mu_0 + \phi N^*) + \beta_{10} \delta A^* + \frac{\theta_1 \beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \beta_1 N^* \]

\[ + \lambda \phi N^*(\mu - \phi C^*), \]

\[ Q_3 = \beta_{10} \delta A^*(\mu_0 + \phi N^*) + \lambda \phi N^*(\beta_{10}A^* + \delta)(\mu - \phi C^*) \]

\[ + \left( \alpha_0 + \lambda \phi C^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right) \left\{ \beta_{10} \delta A^* + (\beta_{10}A^* + \delta)(\mu_0 + \phi N^*) \right\} \]

\[ + \frac{\theta_1 \beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \beta_1 N^* \]

\[ - \pi_1 \pi_2 \delta(\alpha_1 + 2 \beta_{10}A^*) \left( \frac{\theta_1 \beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right), \]

\[ Q_4 = \delta \beta_{10}A^*(\mu_0 + \phi N^*) \left\{ \alpha_0 + \lambda \phi C^* + (1 - \pi_1 \pi_2 \theta_1) \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right\} \]

\[ + \delta(1 - \pi_1 \pi_2 \theta_1)(\alpha_1 + \beta_{10}A^*)(\mu_0 + \phi N^*) \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \]

\[ + \lambda \phi \beta_{10} \delta N^* A^*(\mu - \phi C^*). \]

Note that \( \mu - \phi C^* > 0 \). Also, \( \pi_1 \pi_2 \theta_1 \leq 1 \). Clearly, \( Q_1 \) and \( Q_4 \) are positive.

Now,

\[ Q_1Q_2 - Q_3 = \left( \alpha_0 + \lambda \phi C^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} + \beta_{10}A^* + \delta + \mu_0 + \phi N^* \right) \]

\[ \times \left( \alpha_0 + \lambda \phi C^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right) (\beta_{10}A^* + \delta + \mu_0 + \phi N^*) \]

\[ + (\beta_{10}A^* + \delta)(\beta_{10}A^* + \delta)(\mu_0 + \phi N^*) \]

\[ + \frac{\theta_1 \beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \beta_1 N^* \left( \alpha_0 + \lambda \phi C^* + \beta_{10}A^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} \right) \]

\[ + \lambda \phi N^*(\mu - \phi C^*) \left( \alpha_0 + \lambda \phi C^* + \frac{\beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} + \mu_0 + \phi N^* \right) \]

\[ + \pi_1 \pi_2 \delta(\alpha_1 + 2 \beta_{10}A^*) \left( \frac{\theta_1 \beta_1 \beta_{12}A^*}{(\beta_{12} + \beta_{11}N^*)^2} + (\mu_0 + \phi N^*)^2(\beta_{10}A^* + \delta). \]

Note that \( Q_1Q_2 - Q_3 > 0 \). Further, we checked that \( Q_3(Q_1Q_2 - Q_3) - Q_4^2Q_4 \) is also positive. Thus, the conditions of Routh-Hurwitz criterion for the negativity of real parts of the roots of equation \( (13) \) are always satisfied.

**Remark 2.** Note that the system (1) always has an equilibrium without algae while the algae persistent equilibrium exists under some restrictions on system’s parameters. The first point of Theorem 3.2 states that the algae-detritus-free equilibrium is linked with the interior equilibrium via transcritical bifurcation. Numerically, we see that these two equilibria are related via uptake rate of phosphorus by algae, \( \beta_1 \). That is one can have system with or without algae depending on the rate of consumption of phosphorus by algae. The second point of Theorem 3.2 tells that if
the initial states of system (1) are near the equilibrium point \( E^* \), then the solution trajectories not only stay near \( E^* \) for all \( t > 0 \), but also approaches to \( E^* \) as \( t \to \infty \). Thus, if the initial values of state variables \( N, A, D \) and \( C \) are close to \( N^* \), \( A^* \), \( D^* \) and \( C^* \), respectively, then the system (1) will eventually get stabilized at the equilibrium \( E^* \). That is to say that small perturbations in system’s variables do not affect stability of the system at the interior equilibrium.

Regarding global stability of the equilibrium \( E^* \), we have the following theorem.

**Theorem 3.3.** The equilibrium \( E^* \), if feasible, is globally asymptotically stable inside the region of attraction \( \Omega \) provided the following conditions hold,

\[
\left( \frac{\beta_1 + \beta_{11}N_m}{\beta_1 + \beta_{11}N} \right)^2 N^* < \frac{\beta_{10} \alpha_0}{\theta_1}, \tag{14}
\]

\[
\frac{\pi^2 \pi^2}{\alpha_0} < \frac{\beta_{10} N^*}{\theta_1 \left[ \alpha_1 + \beta_{10}(N_m + A^*) \right]^2}. \tag{15}
\]

**Proof.** Consider the following positive definite function

\[
V = \frac{1}{2} (N - N^*)^2 + m_1 \left( A - A^* - A^* \ln \frac{A}{A^*} \right) + \frac{m_2}{2} (D - D^*)^2 + \frac{m_3}{2} (C - C^*)^2,
\]

where \( m_1, m_2 \) and \( m_3 \) are positive constants to be chosen appropriately.

Differentiating above equation with respect to time along the solution trajectories of the system (1) and rearranging the terms, we have

\[
d\frac{dV}{dt} = -\left[ \alpha_0 + \lambda \phi C + \frac{\beta_1 \beta_{12} A}{\beta_1 + \beta_{11} N_m} (\beta_1 + \beta_{11} N^*) \right] (N - N^*)^2
\]

\[
- \frac{m_1 \beta_{10} (A - A^*)^2 - m_2 \delta (D - D^*)^2 - m_3 (\mu + \phi N) (C - C^*)^2}{\theta_1 \left[ \alpha_1 + \beta_{10}(N_m + A^*) \right]^2} (N - N^*) (A - A^*)
\]

\[
+ \left[ \frac{m_3(\mu - \phi C^*) - \lambda \phi N^* (N - N^*)(C - C^*) + \pi_2 \delta (N - N^*)(D - D^*) + m_2 \pi_1 [\alpha_1 + \beta_{10} (A + A^*)] (A - A^*) (D - D^*)}{\beta_1 + \beta_{11} N_m} (\beta_1 + \beta_{11} N^*) \right] (N - N^*)^2
\]

Choosing \( m_1 = \frac{N^*}{\theta_1} \) and \( m_3 = \frac{\lambda \phi N^*}{\mu - \phi C^*} \), we have

\[
\frac{dV}{dt} = -\left[ \alpha_0 + \lambda \phi C + \frac{\beta_1 \beta_{12} A}{\beta_1 + \beta_{11} N_m} (\beta_1 + \beta_{11} N^*) \right] (N - N^*)^2
\]

\[
- \frac{\beta_{10} N^*}{\theta_1} (A - A^*)^2 - m_2 \delta (D - D^*)^2 - \frac{\lambda \phi N^* (\mu + \phi N)}{\mu - \phi C^*} (C - C^*)^2
\]

\[
- \beta_1 \beta_{11} N N^* \left( \frac{\beta_1 + \beta_{11} N}{\beta_1 + \beta_{11} N_m} (\beta_1 + \beta_{11} N^*) \right) (N - N^*) (A - A^*)
\]

\[
+ \left[ \pi_2 \delta (N - N^*)(D - D^*) + m_2 \pi_1 [\alpha_1 + \beta_{10} (A + A^*)] (A - A^*) (D - D^*) \right].
\]

Note that \( \frac{dV}{dt} \) can be made negative definite inside the region of attraction \( \Omega \) if the following conditions hold:

\[
\left[ \frac{\beta_1 \beta_{11} N_m N^*}{(\beta_1 + \beta_{11} N_m)(\beta_1 + \beta_{11} N^*)} \right]^2 < 4 \times \frac{\alpha_0}{2} \times \frac{\beta_{10} N^*}{2 \theta_1}, \tag{16}
\]

\[
|\pi_2 \delta|^2 < 4 \times \frac{\alpha_0}{2} \times \frac{m_2 \delta}{2}, \tag{17}
\]
\[ m_2^2 \pi_1^2 \alpha_0 \beta_{11} (N_m + A^*)^2 < 4 \times \frac{\beta_{10} N^*}{2 \theta_1} \times \frac{m_2 \delta}{2}. \] (18)

From inequalities (17) and (18), we can choose a positive value of \( m_2 \) provided,

\[ \frac{\pi_2^2 \pi_1^2}{\alpha_0} < m_2 < \frac{\beta_{10} N^*}{\theta_1 (\alpha_0 + \beta_{10} (N_m + A^*))^2}. \] (19)

Therefore, \( \frac{dV}{dt} \) is negative definite inside the region \( \Omega \), and hence \( V \) is Lyapunov function if the conditions (14) and (15) are satisfied. \( \square \)

**Remark 3.** Conditions of Theorem 3.3 indicate that for the stable coexistence, one should decrease the parameter \( \theta_1 \) or increase the parameters \( \alpha_0 \) or \( \beta_{10} \). Note that the values of \( \theta_1 \) can be decreased by reducing the concentrations of phosphorus. Therefore, by controlling the level of phosphorus may lead to stable behavior of ecosystem, which can be achieved via suitable application of Phoslock to the lake.

4. **Effect of time delay in the application of Phoslock.** In system (1), we assumed that the per-capita growth rate of Phoslock is proportional to the current concentration of phosphorus in the lake. However, it may be noted that the concentration of phosphorus in the lake reported to the policy makers may be some time old and increment in per-capita growth rate of Phoslock depends upon this data, which leads to the incorporation of intrinsic time delay in per-capita growth rate of Phoslock due to increase in concentration of phosphorus in the lake [39, 40].

In order to this, we consider that at time \( t \), the per-capita growth rate of Phoslock to control the concentration of phosphorus in the lake is in accordance with the concentration of phosphorus reported to the policy makers at time \( t - \tau \) (for some \( \tau > 0 \)). After incorporation of this time delay, system (1) becomes,

\[
\begin{align*}
\frac{dN}{dt} &= q - \alpha_0 N - \frac{\beta_1 N A}{\beta_{12} + \beta_{11} N} - \lambda \phi N C + \pi_2 \delta D, \\
\frac{dA}{dt} &= \frac{\theta_1 \beta_1 N A}{\beta_{12} + \beta_{11} N} - \alpha_1 A - \beta_{10} A^2, \\
\frac{dD}{dt} &= \pi_1 (\alpha_1 A + \beta_{10} A^2) - \delta D, \\
\frac{dC}{dt} &= \mu N (t - \tau) - \mu_0 C - \phi N C.
\end{align*}
\] (20)

Initial conditions for the system (20) take the form

\[
N(\gamma) = \psi_1(\gamma), \quad A(\gamma) = \psi_2(\gamma), \quad D(\gamma) = \psi_3(\gamma), \quad C(\gamma) = \psi_4(\gamma), \quad -\tau \leq \gamma \leq 0, \] (21)

where \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \in C_+ \) such that \( \psi_i(\gamma) \geq 0, \ i = 1, 2, 3, 4 \ \forall \ \gamma \in [-\tau, 0] \) and \( C_+ \) denotes the Banach space \( C_+([-\tau, 0], \mathbb{R}^4_{\geq 0}) \) of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^4_{\geq 0} \).

Denote the norm of an element \( \psi \) in \( C_+ \) by

\[ \| \psi \| = \sup_{-\tau \leq \gamma \leq 0} \{ | \psi_1(\gamma) |, | \psi_2(\gamma) |, | \psi_3(\gamma) |, | \psi_4(\gamma) | \}. \]

For biological feasibility, we further assume that \( \psi_i(0) \geq 0 \) for \( i = 1, 2, 3, 4 \). By the fundamental theory of functional differential equations [21], we know that there is a unique solution \((N(t), A(t), D(t), C(t))\) to system (20) with initial conditions (21).
4.1. **Hopf bifurcation analysis.** Here, we explore the possibility of occurrence of Hopf-bifurcation around the equilibria $E_0$ and $E^*$ by taking time delay $\tau$ as bifurcation parameter. We found that the presence of time delay involved in the application of Phoslock does not affect stability behavior of the equilibrium $E_0$.

To study the stability behavior of the equilibrium $E^*$ in the presence of time delay, we linearize system (20) around equilibrium $E^*$, and get

$$\frac{dY}{dt} = RY(t) + SY(t - \tau), \quad (22)$$

where

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & 0 & 0 \\ 0 & r_{32} & r_{33} & 0 \\ r_{41} & 0 & 0 & r_{44} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s_{41} & 0 & 0 & 0 \end{pmatrix}, \quad Y(\cdot) = \begin{pmatrix} n(\cdot) \\ a(\cdot) \\ d(\cdot) \\ c(\cdot) \end{pmatrix},$$

$$r_{11} = -\left(\alpha_0 + \lambda\phi C^* + \frac{\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}\right), \quad r_{12} = -\frac{\beta_1N^*}{\beta_{12} + \beta_{11}N^*}, \quad r_{13} = \pi_2\delta,$$

$$r_{14} = -\lambda\phi N^*, \quad r_{21} = \frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}, \quad r_{22} = -\beta_{10}A^*, \quad r_{32} = \pi_1(\alpha_1 + 2\beta_{10}A^*),$$

$$r_{33} = -\delta, \quad r_{41} = -\phi C^*, \quad r_{44} = -(\mu_0 + \phi N^*), \quad s_{41} = \mu.$$

Here, $n$, $a$, $d$ and $c$ are small perturbations around the equilibrium $E^*$.

After linearizing the system (20) about the equilibrium $E^*$, we get the following characteristic equation:

$$D(\xi, \tau) = \xi^4 + b_3\xi^3 + b_2\xi^2 + b_1\xi + b_0 + (c_2\xi^2 + c_1\xi + c_0)e^{-\xi\tau} = 0, \quad (23)$$

where

$$b_3 = \alpha_0 + \lambda\phi C^* + \frac{\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2} + \beta_{10}A^* + \delta + \mu_0 + \phi N^*,$$

$$b_2 = (\beta_{10}A^* + \delta + \mu_0 + \phi N^*)\left(\alpha_0 + \lambda\phi C^* + \frac{\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}\right) + \beta_{10}\delta A^*$$

$$+ (\beta_{10}A^* + \delta)(\mu_0 + \phi N^*) + \frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2} \frac{\beta_1N^*}{\beta_{12} + \beta_{11}N^*} - \lambda\phi^2N^*C^*,$$

$$b_1 = \beta_{10}\delta A^*(\mu_0 + \phi N^*) - \pi_1\pi_2\delta(\alpha_1 + 2\beta_{10}A^*)\frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}$$

$$+ \{(\beta_{10}\delta A^* + (\beta_{10}A^* + \delta)(\mu_0 + \phi N^*))\left(\alpha_0 + \lambda\phi C^* + \frac{\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}\right)$$

$$+ (\delta + \mu_0 + \phi N^*)\frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2} \frac{\beta_1N^*}{\beta_{12} + \beta_{11}N^*} - \lambda\phi^2N^*C^*(\beta_{10}A^* + \delta),$$

$$b_0 = \beta_{10}\delta A^*(\mu_0 + \phi N^*)\left(\alpha_0 + \lambda\phi C^* + \frac{\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2}\right)$$

$$+ \delta(\mu_0 + \phi N^*)\frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2} \frac{\beta_1N^*}{\beta_{12} + \beta_{11}N^*}$$

$$- \lambda\phi^2\beta_{10}N^*A^*C^* - \pi_1\pi_2\delta(\alpha_1 + 2\beta_{10}A^*)(\mu_0 + \phi N^*)\frac{\theta_1\beta_1\beta_12A^*}{(\beta_{12} + \beta_{11}N^*)^2},$$

$$c_2 = \lambda\phi\mu N^*, \quad c_1 = \lambda\phi\mu N^*(\beta_{10}A^* + \delta), \quad c_0 = \lambda\phi\mu\beta_{10}\delta N^*A^*.$$
For \( \tau = 0 \), equations (13) and (23) are same, and in that case the equilibrium \( E^* \) is always locally asymptotically stable. Thus, for \( \tau = 0 \), all the roots of equation (23) are either negative or have negative real parts while for \( \tau > 0 \), it has infinitely many roots. By Rouche’s Theorem and continuity in \( \tau \), sign of roots of equation (23) will change if it crosses imaginary axis i.e., if equation (23) has purely imaginary roots. Hence, putting \( \xi = i \omega (\omega > 0) \) in equation (23), and separating real and imaginary parts, we get

\[
\omega^4 - b_2 \omega^2 + b_0 = -(c_0 - c_2 \omega^2) \cos(\omega \tau) - c_1 \omega \sin(\omega \tau), \tag{24}
\]

\[
-b_3 \omega^3 + b_1 \omega = (c_0 - c_2 \omega^2) \sin(\omega \tau) - c_1 \omega \cos(\omega \tau). \tag{25}
\]

Squaring and adding equations (24) and (25), we get

\[
(\omega^4 - b_2 \omega^2 + b_0)^2 + (-b_3 \omega^3 + b_1 \omega)^2 = (c_0 - c_2 \omega^2)^2 + c_1^2 \omega^2. \tag{26}
\]

Simplifying equation (26) and substituting \( \omega^2 = \psi \), we have

\[
\Psi(\psi) = \psi^4 + a_3 \psi^3 + a_2 \psi^2 + a_1 \psi + a_0 = 0, \tag{27}
\]

where

\[
a_3 = -2b_2 + b_0^2, \quad a_2 = b_2^2 + 2b_0 - 2b_1 b_3 - c_2^2, \quad a_1 = -2b_2 b_0 + b_1^2 - c_1^2 + 2c_2 c_0, \quad a_0 = b_0^2 - c_0^2.
\]

The following lemma state the conditions on \( a_i \)'s for the existence of positive root (roots) of the equation (27).

**Lemma 4.1.** The polynomial equation (27) has

(1) at least one positive root if

\[
\begin{align*}
(a) \quad & a_3 > 0, \quad a_2 < 0, \quad a_1 > 0, \quad a_0 < 0. \\
(b) \quad & a_3 < 0, \quad a_2 < 0, \quad a_1 > 0, \quad a_0 < 0. \\
(c) \quad & a_3 < 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 < 0. \\
(d) \quad & a_3 < 0, \quad a_2 > 0, \quad a_1 < 0, \quad a_0 < 0.
\end{align*}
\]

(2) exactly one positive root if

\[
\begin{align*}
(a) \quad & a_3 < 0, \quad a_2 < 0, \quad a_1 < 0, \quad a_0 < 0. \\
(b) \quad & a_3 > 0, \quad a_2 < 0, \quad a_1 < 0, \quad a_0 < 0. \\
(c) \quad & a_3 > 0, \quad a_2 > 0, \quad a_1 < 0, \quad a_0 < 0. \\
(d) \quad & a_3 > 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 < 0.
\end{align*}
\]

For other choices of \( a_i \)'s, the existence of positive root cannot be guaranteed.

Now, we have the following theorem.

**Theorem 4.2.** Let \( \psi_0 = \omega_0^2 \) be a positive root of equation (27), then there exists \( \tau = \tau^0 \) such that the equilibrium \( E^* \) is locally asymptotically stable when \( 0 \leq \tau < \tau^0 \) and unstable for \( \tau > \tau^0 \), where

\[
\tau^k = \frac{1}{\omega_0} \tan^{-1} \left[ \frac{c_1 \omega_0 (\omega_0^4 - b_2 \omega_0^2 + b_0) + (c_0 - c_2 \omega_0^2) (b_3 \omega_0^3 - b_1 \omega_0)}{(c_0 - c_2 \omega_0^2) (\omega_0^4 - b_2 \omega_0^2 + b_0) - c_1 \omega_0 (b_3 \omega_0^3 - b_1 \omega_0)} \right] + \frac{k \pi}{\omega_0}
\]

for \( k = 0, 1, 2, 3 \cdots \). Further, the system (20) undergoes a Hopf-bifurcation through the equilibrium \( E^* \) at \( \tau = \tau^0 \) provided \( \Psi'(\omega_0^2) > 0 \).
Proof. Since $\psi = \omega_0^2$ is a solution of equation (27), the characteristic equation (23) has pair of purely imaginary roots $\pm i\omega_0$. It follows from equations (24) and (25) that $\tau^k$ is a function of $\omega_0$ for $k = 0, 1, 2, \ldots$. The system will be locally asymptotically stable around the equilibrium $E^*$ for $\tau = 0$. In that case by Butler’s lemma, the equilibrium $E^*$ will remain stable for $\tau \leq \tau_0$, such that $\tau_0 = \min_{k \geq 0} \tau^k$ and the equilibrium $E^*$ will be unstable for $\tau \geq \tau_0$ provided,

$$\text{sgn} \left[ \frac{d(Re(\xi))}{d\tau} \right]_{\tau=\tau_0} \neq 0.$$  

Differentiating equation (23) with respect to $\tau$, we get

$$\frac{d\xi}{d\tau} = \xi (c_2 \xi^2 + c_1 \xi + c_0) e^{-\xi \tau}$$

This gives,

$$\left( \frac{d\xi}{d\tau} \right)^{-1} = \frac{4 \xi^3 + 3 b_3 \xi^2 + 2 b_2 \xi + b_1 + (2 c_2 \xi + c_1) e^{-\xi \tau} - \tau (c_2 \xi^2 + c_1 \xi + c_0) e^{-\xi \tau}}{\xi (c_2 \xi^2 + c_1 \xi + c_0) e^{-\xi \tau} - \tau \xi}.$$

Now,

$$\text{sgn} \left[ \frac{d(Re(\xi))}{d\tau} \right]_{\tau=\tau_0} = \text{sgn} \left[ \frac{d(Re(\xi))}{d\tau} \right]_{\tau=\tau_0}^{-1} = \text{sgn} \left[ Re \left( \frac{d\xi}{d\tau} \right)^{-1} \right]_{\xi=i\omega_0} = \text{sgn} \left[ \frac{4 \omega_0^6 + 3 b_3 \omega_0^4 + 2 b_2 \omega_0^2 + a_1}{c_1^2 \omega_0^2 + (c_0 - c_2 \omega_0^2)^2} \right] = \text{sgn} \left[ \frac{\Psi'(\omega_0^2)}{c_1^2 \omega_0^2 + (c_0 - c_2 \omega_0^2)^2} \right].$$

It may be noted that $\Psi'(\omega_0^2) \neq 0$ if any condition stated in Lemma 4.1 holds. Thus, the transversality condition is satisfied and a Hopf-bifurcation occurs at $\tau = \tau_0^0$ i.e., a family of periodic solutions bifurcate from the equilibrium $E^*$ as $\tau$ passes through its critical value, $\tau_0^0$ [18].

**Remark 4.** If equation (27) has two positive real roots $\psi_+$ (corresponds to $\omega_+^2$) and $\psi_-$ (corresponds to $\omega_-^2$), where $\psi_+ > \psi_-$ i.e., the characteristic equation (23) has two pair of purely imaginary roots $\pm i\omega_\pm$. In this case, we have

$$\tau^k = \frac{1}{\omega_\pm} \tan^{-1} \left[ \frac{c_1 \omega_\pm (\omega_+^2 - b_2 \omega_+^2 + b_0) + (c_0 - c_2 \omega_+^2)(b_3 \omega_+^2 - b_1 \omega_+^2)}{(c_0 - c_2 \omega_+^2)(\omega_+^2 - b_2 \omega_+^2 + b_0) - c_1 \omega_+ (b_3 \omega_+^2 - b_1 \omega_+^2)} \right] + \frac{k \pi}{\omega_\pm}, \quad (28)$$

for $k = 0, 1, 2, 3, \ldots$. If the following transversality conditions are satisfied

$$\text{sgn} \left[ \frac{d(Re(\xi))}{d\tau} \right]_{\tau=\tau_+^0} > 0, \quad \text{sgn} \left[ \frac{d(Re(\xi))}{d\tau} \right]_{\tau=\tau_-^0} < 0,$$

then there exists a positive integer $m$ such that there are $m$ switches from stability to instability and back to stability. In other words, when $\tau \in [0, \tau_0^m), (\tau_0^m, \tau_1^m), \ldots, (\tau_0^{m-1}, \tau_1^m)$, the equilibrium $E^*$ is stable, and when $\tau \in (\tau_0^m, \tau_0^0), (\tau_1^m, \tau_1^0), \ldots, (\tau_0^{m-1}, \tau_1^{m-1})$, the equilibrium $E^*$ is unstable. Therefore, Hopf-bifurcations occur around the equilibrium $E^*$ at $\tau_k^\pm$ for $k = 0, 1, 2, 3, \ldots$. 

\[\square\]
5. Impact of periodic input of Phoslock. Phoslock is increasingly used worldwide to control the release of phosphorus in the sediment and the cyanobacterial blooms [36, 51]. Therefore, Phoslock may be a viable option to reduce the availability of phosphorus in the lakes. Since the continuous uncontrolled inputs of phosphorus have loaded the lake sediments, the concentration of phosphorus and density of algae in the lake change around the year, so we assume that the rate of application of Phoslock vary with time instead of constant input. To incorporate the periodic input of Phoslock in the lake, we further extend our previous delayed application of Phoslock vary with time instead of constant input. To incorporate the effect of seasonal changes by considering the periodic rate parameter, we neglect phase shift and simply incorporate the periodic input of Phoslock in the lake, we further extend our previous delayed system (20) as follows,

\[
\begin{align*}
\frac{dN}{dt} &= q - \alpha_0 N - \frac{\beta_1 NA}{\beta_2 + \beta_1 N} - \lambda \phi NC + \pi_2 \delta D, \\
\frac{dA}{dt} &= \theta_1 \frac{\beta_1 NA}{\beta_2 + \beta_1 N} - \alpha_1 A - \beta_{10} A^2, \\
\frac{dD}{dt} &= \pi_1 (\alpha_1 A + \beta_{10} A^2) - \delta D, \\
\frac{dC}{dt} &= \mu(t) N(t - \tau) - \mu_0 C - \phi NC. \\
\end{align*}
\]

(29)

Here, we assume that the rate parameter \( \mu(t) \) is positive, continuous and bounded with positive lower bound. For simplicity, we neglect phase shift and simply incorporate the rate parameter \( \mu(t) \), with a period of one year.

Let \( C \) denote the Banach space of continuous functions \( \psi : [-\tau, 0] \to \mathbb{R}^4 \) with norm

\[
\|\psi\| = \sup_{-\tau \leq \theta \leq 0} \{|\psi_1(\theta)|, |\psi_2(\theta)|, |\psi_3(\theta)|, |\psi_4(\theta)|\},
\]

(30)

where \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4) \). Initial conditions of system (29) are given as

\[
N(\theta) = \psi_1(\theta), \quad A(\theta) = \psi_2(\theta), \quad D(\theta) = \psi_3(\theta), \quad C(\theta) = \psi_4(\theta), \quad \theta \in [-\tau, 0],
\]

(31)

where the initial function \( \psi = (\psi_1, \psi_2, \psi_3, \psi_4) \) belongs to the Banach space \( C = C([-\tau, 0], \mathbb{R}^4) \) of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^4 \). For biological reasons, we must have \( \psi_i(\theta) \geq 0 \) for all \( \theta \in [-\tau, 0] \) \((i = 1 - 4)\). Let \( \mu^l = \min \mu(t) \) and \( \mu^m = \max \mu(t) \) for all \( t > 0 \).

**Lemma 5.1.** Let \( \kappa \) be a real number and \( f \) be a nonnegative function defined on \([\kappa, +\infty)\) such that \( f \) is integrable on \([\kappa, +\infty)\) and is uniformly continuous on \([\kappa, +\infty)\), then \( \lim_{t \to +\infty} f(t) = 0 \) [18].

Boundedness and persistence of the variables \( N(t), A(t) \) and \( D(t) \) in system (29) will be same as in Theorem 3.1. From the last equation of system (29), we have

\[
\frac{dC}{dt} = \mu(t) N(t - \tau) - \mu_0 C - \phi NC \leq \mu^m N' - \mu_0 C.
\]

Thus, for any \( t > 0, 0 \leq C(t) \leq C_m' \), where \( C_m' = \max \left\{ \frac{\mu^m N'}{\mu_0}, C(0) \right\} \).

Also, from the last equation of system (1), we have

\[
\frac{dC}{dt} \geq \mu^l N - \mu_0 C - \phi N'C.
\]
Thus, it follows that for some $\tilde{C}'$,

$$
\liminf_{t \to \infty} C(t) \geq \frac{\mu' \tilde{N}}{\mu_0 + \tilde{\phi} N'} = \tilde{C}'.
$$

Hence, the system (29) is uniformly persistent if condition (4) is satisfied.

5.1. Global attractivity. Global asymptotic stability of solutions of system (29) is defined as follows.

**Definition 5.2.** System (29) with initial conditions (31) is said to be globally asymptotically stable if for any two solutions $(N(t), A(t), D(t), C(t))$ and $(\tilde{N}(t), \tilde{A}(t), \tilde{D}(t), \tilde{C}(t))$ of the system (29), the following conditions are satisfied,

$$
\lim_{t \to \infty} |N(t) - \tilde{N}(t)| = 0, \quad \lim_{t \to \infty} |A(t) - \tilde{A}(t)| = 0,
$$
$$
\lim_{t \to \infty} |D(t) - \tilde{D}(t)| = 0, \quad \lim_{t \to \infty} |C(t) - \tilde{C}(t)| = 0.
$$

**Theorem 5.3.** If there exist $\tilde{a}_i > 0 \ (i = 1-4)$ such that

$$
\liminf_{t \to \infty} \left\{ \tilde{a}_1 e^{-2H_1(t + \pi_2 \delta e^{L_3})} + \tilde{a}_4 \left( \phi - \mu(t + \tau) \right) - \frac{\beta_1 (\tilde{a}_1 \beta_{11} e^{H_2} + \tilde{a}_2 \beta_{11} \beta_{12})}{(\beta_{12} + \beta_{11} e^{L_1})^2} \right\} > 0,
$$
$$
\liminf_{t \to \infty} \left\{ \tilde{a}_2 \beta_{10} e^{-2H_1(t + \pi_2 \delta e^{L_3})} - \tilde{a}_3 \pi_1 e^{-L_3} (\alpha_1 + 2 \beta_{10} e^{H_2}) \right\} > 0,
$$
$$
\liminf_{t \to \infty} \left\{ \tilde{a}_3 \pi_1 e^{-2H_1(t + \pi_2 \delta e^{L_3})} - \tilde{a}_1 \pi_1 e^{-L_3} \right\} > 0. \tag{32}
$$

Then, the system (29) with initial conditions (31) is globally asymptotically stable. The quantities $L_i$’s and $H_i$’s are defined in the proof.

**Proof.** Let system (29) has at least one positive periodic solution $(\tilde{N}(t), \tilde{A}(t), \tilde{D}(t), \tilde{C}(t))$ and also assume,

$$
e^{L_1} \leq N(t) \leq e^{H_1}, \quad e^{L_2} \leq A(t) \leq e^{H_2}, \quad e^{L_3} \leq D(t) \leq e^{H_3}, \quad e^{L_4} \leq C(t) \leq e^{H_4}.
$$

Suppose $(N(t), A(t), D(t), C(t))$ is any other positive periodic solution of system (29) with initial conditions (31). Define

$$
V_1(t) = |\ln N(t) - \ln \tilde{N}(t)|, \quad V_2(t) = |\ln A(t) - \tilde{A}(t)|,
$$
$$
V_3(t) = |\ln D(t) - \ln \tilde{D}(t)|, \quad V_4(t) = |C(t) - \tilde{C}(t)|. \tag{33}
$$

The right-upper derivatives of $V_1(t)$, $V_2(t)$, $V_3(t)$ and $V_4(t)$ along the solutions of system (29) with initial conditions (31) are given by

$$
D^+ V_1(t) = \text{sgn}(N(t) - \tilde{N}(t)) \left( \frac{\dot{N}(t)}{N(t)} - \frac{\dot{\tilde{N}}(t)}{\tilde{N}(t)} \right) + \frac{\beta_{11} A(t)}{(\beta_{12} + \beta_{11} N(t))(\beta_{12} + \beta_{11} \tilde{N}(t))} |N(t) - \tilde{N}(t)|,
$$
$$
- \frac{\beta_{11} A(t)}{(\beta_{12} + \beta_{11} N(t))} |A(t) - \tilde{A}(t)| + \frac{\pi_2 \delta}{N(t)} |D(t) - \tilde{D}(t)| - \frac{\beta_{11} A(t)}{(\beta_{12} + \beta_{11} N(t))} \lambda \phi (C(t) - \tilde{C}(t)). \tag{34}
$$
\[ D^+ V_2(t) = \text{sgn}(A(t) - \overline{A}(t)) \left( \frac{\dot{A}(t)}{A(t)} - \frac{\dot{\overline{A}}(t)}{\overline{A}(t)} \right) \]
\[ \leq \frac{\theta_1 \overline{\beta}_1 \beta_{12}}{(\beta_{12} + \beta_{11} N(t))(\beta_{12} + \beta_{11} \overline{N}(t))} |N(t) - \overline{N}(t)| - \beta_{10}|A(t) - \overline{A}(t)|; \]  
(35)
\[ D^+ V_3(t) = \text{sgn}(D(t) - \overline{D}(t)) \left( \frac{\dot{D}(t)}{D(t)} - \frac{\dot{\overline{D}}(t)}{\overline{D}(t)} \right) \]
\[ \leq -\frac{\pi_1 A(t)}{D(t)\overline{D}(t)} \{\alpha_1 + \beta_{10}A(t)\}|D(t) - \overline{D}(t)| + \frac{\pi_1 A(t)}{D(t)} \{\alpha_1 + \beta_{10}(A(t) + \overline{A}(t))\}|A(t) - \overline{A}(t)|; \]  
(36)
\[ D^+ V_4(t) = \text{sgn}(C(t) - \overline{C}(t))(\dot{C}(t) - \dot{\overline{N}}(t)) \leq \mu(t)|N(t - \tau) - \overline{N}(t - \tau)| - \phi|N(t) - \overline{N}(t)|. \]  
(37)
Define
\[ V_* = \int_{t-\tau}^{t} \mu(s + \tau)|N(s) - \overline{N}(s)|ds. \]  
(38)
Calculating the right-upper derivative of \( V_* \) along the solutions of system (29) with initial conditions (31), we get
\[ D^+ V_*(t) = \mu(t + \tau)|N(t) - \overline{N}(t)| - \mu(t)|N(t - \tau) - \overline{N}(t - \tau)|. \]  
(39)
Let
\[ V(t) = \bar{a}_1 V_1(t) + \bar{a}_2 V_2(t) + \bar{a}_3 V_3(t) + \bar{a}_4(V_4(t) + V_*(t)). \]  
(40)
Using equations (34)–(40), the right-upper derivative of \( V(t) \) is given by
\[ D^+ V(t) \leq -\bar{b}_1(t)|N(t) - \overline{N}(t)| - \bar{b}_2(t)|A(t) - \overline{A}(t)| - \bar{b}_3(t)|D(t) - \overline{D}(t)| - \bar{b}_4(t)|C(t) - \overline{C}(t)|, \quad \forall \ t \geq T, \]  
(41)
where
\[ \bar{b}_1(t) = \bar{a}_1 e^{-2H_1}(\phi + \pi_2 e^{L_1}) + \bar{a}_4 \{\phi - \mu(t + \tau)\} - \frac{\beta_1(\bar{a}_1 \beta_{11} e^{H_2} + \bar{a}_2 \theta_1 \beta_{12})}{(\beta_{12} + \beta_{11} e^{L_1})^2}, \]
\[ \bar{b}_2(t) = \bar{a}_2 \beta_{10} + \frac{\bar{a}_1 \beta_1}{\beta_{12} + \beta_{11} e^{H_1}} - \bar{a}_3 \pi_1 e^{-L_2}(\varphi + 2 \beta_{10} e^{H_2}), \]
\[ \bar{b}_3(t) = \bar{a}_3 \pi_1 e^{-L_2} - \bar{a}_1 \pi_2 e^{-L_1}, \]
\[ \bar{b}_4(t) = \lambda \bar{a}_4 \phi. \]  
(42)
Now, integrating the inequality (41) from \( T \) to \( t \), we have
\[ \int_{T}^{t} \{\bar{b}_1(t)|N(t) - \overline{N}(t)| + \bar{b}_2(t)|A(t) - \overline{A}(t)| + \bar{b}_3(t)|D(t) - \overline{D}(t)| + \bar{b}_4(t)|C(t) - \overline{C}(t)|\} \, dt \]
\[ \leq V(T) - V(t). \]  
(43)
Implying,
\[ \int_{T}^{t} \{\bar{b}_1(t)|N(t) - \overline{N}(t)| + \bar{b}_2(t)|A(t) - \overline{A}(t)| + \bar{b}_3(t)|D(t) - \overline{D}(t)| + \bar{b}_4(t)|C(t) - \overline{C}(t)|\} \, dt < \infty. \]  
(44)
Since the solutions \((N(t), A(t), D(t), C(t))\) and \((\overline{N}(t), \overline{A}(t), \overline{D}(t), \overline{C}(t))\) are bounded on \([0, \infty)\), it follows from conditions in (32) that \(|N(t) - \overline{N}(t)|, |A(t) - \overline{A}(t)|, |D(t) - \overline{D}(t)|, |C(t) - \overline{C}(t)|\) are bounded on \([0, \infty)\).
Figure 2. Bifurcation diagram of the system (1) with respect to the uptake rate of phosphorus by algae, $\beta_1$. Rest of the parameters are at the same values as in Table 1 except $\mu = 0.25$. The algae-detritus-free equilibrium $E_0$ exchanges its stability with the interior equilibrium $E^*$ when the parameter $\beta_1$ crosses its critical value from below.

This shows that the system (29) with initial conditions (31) is globally asymptotically stable.

Corollary 1. Assume that the application rate of Phoslock, $\mu(t)$, is $\omega$–periodic function, then the system (29) has a positive $\omega$–periodic solution if it is uniformly persistent. Further, if the conditions in Theorem 5.3 are satisfied, then there exists a unique positive $\omega$–periodic solution which is globally asymptotically stable [47].

6. Numerical simulation. First, we conduct some numerical simulations to investigate the behaviors of system (1) using MatLab solver ode45. Unless it is mentioned, the set of parameter values used for numerical simulations is the same as in Table 1.

We see how the algae-detritus-free equilibrium, $E_0$, and the interior equilibrium, $E^*$ are linked with the uptake rate of phosphorus by algae, $\beta_1$. For this, we vary the parameter $\beta_1$ and plot the variables of system (1), Fig. 2. We get that there exists a critical value $\beta_1^c = 0.85$, below which the algal population disappears from the system, but above this critical value, the algae population persists in the system. Thus, the equilibrium $E_0$ disappears and the equilibrium $E^*$ emanates as the parameter $\beta_1$ crosses its critical value from below. Hence, it is imperative to decrease the uptake rate of phosphorus to achieve the algae-free system. Therefore, for complete eradication of algal bloom, attention must be paid on the control of

\[ D(t) \] and \[ |C(t) - C(t)| \] are bounded and uniformly continuous on $[0, \infty)$. Hence, by Lemma 5.1, we have from inequality (44) that,

\[ \lim_{t \to \infty} |N(t) - \bar{N}(t)| = 0, \quad \lim_{t \to \infty} |A(t) - \bar{A}(t)| = 0, \]

\[ \lim_{t \to \infty} |D(t) - \bar{D}(t)| = 0, \quad \lim_{t \to \infty} |C(t) - \bar{C}(t)| = 0. \]

This shows that the system (29) with initial conditions (31) is globally asymptotically stable. \[ \square \]
Figure 3. The equilibrium values of phosphorus (first row) and algae (second row) as functions of $\phi$ and $q$ (first column), $\mu$ and $\beta_1$ (second column), and $\lambda$ and $\pi_2$ (third column). Parameters are at the same values as in Table 1 except $\mu = 0.25$, and the initial conditions are chosen as $(0.3543, 0.8983, 0.6913, 0.1825)$.

Now, we see the impacts of some important parameters, $q$, $\beta_1$, $\pi_2$, $\mu$, $\phi$ and $\lambda$, on the equilibrium abundances of phosphorus and algae in the lake. We vary two parameters at a time viz. $(\phi, q)$, $(\mu, \beta_1)$ and $(\lambda, \pi_2)$ on the rectangular grid of $[0,1] \times [0,1]$, and plot the concentration of phosphorus and density of algae, Fig. 3. We see that with the increase in input rate of phosphorus, the concentration of phosphorus and density of algae increase at a very fast rate. But, if the reaction rate of Phoslock with phosphorus increases, the phosphorus and algae population decrease. For very low uptake rate of phosphorus by algae and introduction rate of Phoslock, the phosphorus concentration is very high but decreases diagonally and ultimately reaches zero value on increasing these two parameters. The algae population disappear from the lake for very low values of their uptake on phosphorus and high rate of Phoslock introduction. Increasing the uptake rate of phosphorus by algae, $\beta_1$, proliferate the growth of algae and algal population reaches a plateau. The concentration of phosphorus and density of algae increase with the increase in formation of phosphorus after bacterial decomposition of dead algal cells. After bonding with Phoslock, phosphorus settles to the bottom of the lake forming an inert compound. This reduction in phosphorus concentration declines the algal biomass. Thus, we find that the parameters $q$, $\beta_1$ and $\pi_2$ contribute significantly to the growth of algal population while the parameters $\mu$, $\phi$ and $\lambda$ are responsible for decrement in algae density; the formers are linked to the level of phosphorus.
whereas the latters are related to the application of Phoslock and its impact on phosphorus reduction. Thus, we may conclude that by applying more Phoslock in the lake suffering from phosphorus overload and eutrophication, one can get rid of phosphorus and algal overgrowth.

6.1. Sensitivity analysis. Here, we perform sensitivity analysis to test which parameters influence the most algal population in agreement with the model assumptions. Following [15, 16], we determine sensitivity graph by using the code
myAD (automatic differentiation), Fig 4. To measure the sensitivity of the system parameters from Fig. 4, we compute the sensitivity coefficient by normalizing the sensitivity functions and deriving the $L^2$ norm of the resulting functions, obtained by

$$C_{ij} = \left\| \frac{\partial x_i}{\partial q_j} \frac{q_j}{\max x_i} \right\|_2^2 = \int_{t_0}^{t_f} \left\| \frac{\partial x_i}{\partial q_j} \frac{q_j}{\max x_i} \right\|^2 dt.$$  \hspace{1cm} (45)

Among many identifiability strategies investigated to quantify the sensitivity coefficients, the most easiest strategies imply that the larger the coefficients, the more mastery the model outcome with respect to that parameter. In this case, a parameter is more likely to be identifiable when there is a greater sensitivity. Since the sensitivity functions are evaluated at some nominal values for the parameters, the analysis is only local. After ranking the sensitivity functions, we categorize the most influential or most sensitive parameters (in descending order) to the least ones, Fig. 5. Our case yields the ordering $[\theta_1, \beta_1, \alpha_1, \beta_{12}, \tau_1, \beta_{10}, \lambda, \phi, \mu_0]^T$, that is, out of the 15 parameters, only the first eleven ranked parameters are the most identifiable and sensitive parameters. Since other parameters, $\alpha_0, \tau_2, \delta$ and $\pi_1$, are not sensitive, they will not affect the dynamics of the system much.

6.2. Effect of time delay in the application of Phoslock. Here, we report the simulations performed to investigate the behavior of delay differential equations (20) using the Matlab solver dde23. We see that for the set of parameter values in Table 1, equation (27) has exactly one positive root. Thus, the characteristic equation (23) has a pair of purely imaginary roots and the corresponding critical value of the time delay $\tau$ is obtained as $\tau^0 = 53$ days. Also, the transversality condition is satisfied. Therefore, the system will undergo a Hopf-bifurcation at the equilibrium $E^*$ at $\tau = \tau^0$. To see the effect of time delay on the dynamics of system (20), we draw bifurcation diagram of the system (20) by varying the delay parameter $\tau$ in the interval $[0, 150]$, Fig. 6. In the figure, the blue line represents the
Figure 6. Bifurcation diagram of the system (20) with respect to \( \tau \). Here, the blue line represents the upper limit of the oscillation cycle and the red line represents the lower limit of the oscillation cycle. Parameters are at the same values as in Table 1, and the initial conditions are chosen as \((0.3543, 0.8983, 0.6913, 0.1825)\). The system exhibits Hopf-bifurcation through limit cycle oscillations for gradual increase in the delay parameter.

The upper limit of the oscillation cycle and the red line represents the lower limit of the oscillation cycle. From the figure, it can be inferred that the system (20) is stable for low values of time delay and undergoes Hopf bifurcation through limit cycle oscillations on increasing the values of time delay. The critical value of time delay at which stability change occurs is obtained as \( \tau^0 = 53 \) days. Below the threshold value of time delay, the delayed system shows stable dynamics, whereas above the threshold value, the system shows limit cycle oscillations. That is, large delay in the application of Phoslock may induce limit cycle oscillations in the system, which should be prevent. Further, we see how the input rate of nutrients and introduction rate of Phoslock affects the critical value of time delay at which stability behavior of the system (20) changed. We draw two parameters bifurcation diagrams in the \((q, \tau)\) and \((\mu, \tau)\) planes, Fig. 7. In the figure, \( \ast \) represents the combinations of these parameters at which the equilibrium \( E^\ast \) is stable and \( \ast \) otherwise. This figure shows the approximate region of stability in these parametric spaces. From Fig. 7(a), we see that for very low values of \( q \), the system remains stable for all values of \( \tau \). As the values of \( q \) increases, the critical values of \( \tau \) at which stability
Figure 7. Stability region for the system (20) in (a) $q-\tau$ and (b) $\mu-\tau$ planes. Here, $\ast$ represents the stable equilibrium for corresponding values of the parameters and $\ast$ represents otherwise. Parameters are at the same values as in Table 1, and the initial conditions are chosen as $(0.3543,0.8983,0.6913,0.1825)$.

switch occurs decreases. But, after a certain value of $q$, the critical values of $\tau$ does not change. Similar behavior is observed on varying the parameters $\mu$ and $\tau$, Fig. 7(b). We note that the region of stability decreases in the latter case. If the input rate of phosphorus in the lake is very high, we must apply Phoslock within the critical value of time delay at which the system looses its stability. Once system enters into unstable region, it would be very difficult to measure exact concentration of phosphorus and density of algae, and hence one can not predict how much Phoslock needed for controlling the overgrowth of algae. Similarly, for high rate of introduction of Phoslock, we need to apply it before the threshold value of time delay in order to achieve stable equilibrium density of algae.

6.3. Impact of periodicity in the application of Phoslock. We now perform numerical simulations to investigate the behavior of the nonautonomous delay system (29). The time dependent parameter $\mu(t)$ is assumed to be a sinusoidal function

$$\mu(t) = \mu + \mu_{11} \sin(\omega t)$$

with period of 365 days. We set $\mu_{11} = 0.03$, and investigate the behavior of the system (29) by varying the values of time delay. We compare the dynamics of the delayed nonautonomous system with the autonomous counterpart. Our aim is to explore different dynamical behaviors, including chaos. In the absence of time delay, the nonautonomous system (29) exhibits periodic solution, Fig. 8. Next, we increase the values of time delay to $\tau = 10$ days, and get that the system shows periodic solution for this low value of time delay, Fig. 9. Thus, for the same set of parameter values, the non-delayed (delayed) nonautonomous system (29) exhibits positive periodic solution while the corresponding autonomous system without delay (with delay) shows stable dynamics. Now, we show that the periodic solution is globally attractive. For this, we plot the solution trajectories starting from three different initial values at $\tau = 10$ days, Fig. 10. It is apparent from
the figure that all the periodic solutions initiating from three different initial values $(1.2, 0.6, 0.2, 0.18)$, $(2, 0.7, 0.3, 0.08)$ and $(2.8, 0.5, 0.1, 0.28)$ converge to a single periodic solution i.e., the positive periodic solution is globally attractive. Next, we choose $\tau = 85$ days and plot the solution trajectories of the system (29), Fig. 11. It is evident from the figure that the system shows chaotic dynamics for $\tau = 85$ days. Therefore, the global stability of the delayed nonautonomous system is affected due to enhancement of the delay parameter, whereas the persistence of the system is not affected by the magnitude of the delay.

To investigate the combined effects of the time delay and the seasonal forcing, we draw the bifurcation diagram of the system (29) with respect to time delay by varying the values of $\tau$ in the interval $[20, 140]$, Fig. 12. It is clear from the figure that the delayed nonautonomous system shows chaotic behavior through quasi-periodic oscillations for increasing the values of time delay. We obtain the threshold value of the time delay as $\tau^c = 68.5$ days at which the stability behavior of the delayed nonautonomous system changes. Below the threshold value of time delay, the delayed nonautonomous system shows positive periodic solution with period 365 days. However, above the threshold value of the delay parameter, the nonautonomous system enters into the chaotic regime through quasi-periodic oscillations. The occurrence of chaotic oscillation may be explained through incommensurate
Figure 9. Simulation results of the nonautonomous system (29) at \( \tau = 10 \) days. Parameters are at the same values as in Table 1, and \( \mu_{11} = 0.03 \). For the periodic input of Phoslock, the system exhibits positive periodic solutions for lower values of time delay.

limit cycles [19, 24]. Since these oscillations are not commensurate, i.e., the frequency of the first type of oscillation is not multiple of the frequency of the second type of oscillation, the combined delayed nonautonomous system produces chaos.

Further, we draw the Poincaré map of the system (29) on \( A-D-C \) space at \( N = 0.8 \) \( \mu_g/L \) for \( \tau = 85 \) days, Fig. 13. In the figure, scattered distribution of the sampling points implies the chaotic behavior of the system. We also draw the maximum Lyapunov exponent of the system (29) for \( \tau = 85 \) days, Fig. 14. To draw the maximum Lyapunov exponent, we first simulate the delayed nonautonomous system (29), then considering the time series solutions of each component, we compute the Lyapunov exponents following [41, 54]. In Fig. 14, positive values of the maximum Lyapunov exponent indicate the chaotic regime of the system. It is to be noted here that the nonautonomous system can produce positive periodic solution with a period of 365 days. The delayed autonomous system produces limit cycle oscillations. None of the nonautonomous and delay autonomous system produces chaotic oscillation. However, the combined delayed nonautonomous system produces chaotic oscillation.

7. Results and discussion. As many lakes are suffering from nutrient pollution and thus eutrophication, water quality managers are challenged to find ways to achieve the good ecological state of lakes. The technique of using modified clay
Phoslock seems very promising in tackling the problem of phosphorus loadings from sediments that has been identified as one of the major causes of failure in previous lake restoration attempts [20]. The present study explored the application of Phoslock as an effective tool to achieve a sustained reduction in phosphate levels. Our study concerned with urban lakes which is experiencing excess of phosphorus level in the water. Our model comprised of phosphorus, algae, detritus and Phoslock as dynamical variables. We observed that the system has two equilibria; the equilibrium without algae and the interior equilibrium. The algae-free equilibrium disappears while the interior equilibrium arises as the uptake rate of phosphorus by algae crosses its critical value from below. Further, the algae persistent equilibrium is always locally asymptotically stable. Moreover, we derived sufficient conditions under which algae-persistent equilibrium is globally asymptotically stable. We performed semi-relative sensitivity analysis to test which parameters influenced the most algal growth in the lake with the model assumptions. We computed the sensitivity coefficient by normalizing the sensitivity functions and deriving the $L^2$ norm of the resulting functions. The sensitivity results showed that $\theta_1$, the algal growth due to uptake of phosphorus, is the most sensitive parameter. Thus, to lessen the problem of algal bloom, we should decrease the level of phosphorus in the lake, which can
be achieved by suitable application of Phoslock; that is increasing the application rate of Phoslock, $\mu$.

Further, we have extended our model by considering periodic input of Phoslock and the time lag involved in measuring the concentration of phosphorus and applying Phoslock. We consider periodic function (sinusoidal function) with a period of one year to incorporate the seasonal input of Phoslock. In the absence of time delay, the autonomous system (1) shows stable dynamics, whereas the nonautonomous system (29) shows a unique positive periodic solution with a period of one year. Further, we have explored the effect of time delay on the nonautonomous system (29) and the corresponding autonomous system (20). We have observed that the autonomous system undergoes a Hopf-bifurcation through limit cycle oscillations, whereas the corresponding nonautonomous system showed chaotic dynamics for increasing values of time delay. We have observed that delay and seasonal forcing have the synergism effect for producing chaotic oscillations. We observed that for a range of the time delay, the positive periodic solution is globally asymptotically stable. However, if the time delay is increased above a threshold value, then the system (29) exhibits chaotic oscillations. Thus, we can conclude that the time delay together with seasonal forcing is responsible for the chaotic dynamics, which was absent in the autonomous system. Previously, Misra et al. [40] showed that large
values of time delay in the application of budget induces limit cycle oscillations. However, if the growth of budget is logistic in addition to its growth due to increased density of algae, the system enters into chaotic regime for larger values of time lag in budget application [39].

As human populations increase and land-use intensifies, toxic and unsightly nuisance blooms of algae are becoming larger and more frequent in freshwater lakes. In most cases, the blooms are predominantly blue-green algae, which are favored by low ratios of nitrogen to phosphorus. To overcome this serious issue, aquatic scientists have devoted much effort to understand the causes of such blooms and how they can be prevented or reduced. Several experimental studies revealed that Phoslock has the strongest impact on eutrophication reduction [51, 52], and is suitable for field applications. Epe et al. [14] found that use of Phoslock decreases the intensity of mineralization processes and prevented algal and cyanobacterial blooms. Meis et al. [36] found that Phoslock control sediment phosphorus release and cyanobacterial blooms. Bishop et al. [5] reported the phosphorus binding efficiency of Phoslock in aqueous and sediment matrices and the consequent impact on algae assemblage
composition and water quality parameters in Laguna Niguel Lake, California. Our study also showed that Phoslock is able to rapidly and significantly decrease phosphorus concentration and control algal population in the lake. However, delay in the application of Phoslock induced limit cycle oscillations whereas delay together with periodic input of Phoslock resulted in chaotic dynamics. Thus, one should apply Phoslock for the control of phosphorus within the threshold value of time.
delay at which the system showed limit cycle oscillations to prevent the crash of the aquatic ecosystem. The results provide water resource managers an information on operational implementation and efficacy of Phoslock in binding phosphorus. There are many directions to extend the current work to have more biological insights. For example, it would be interesting to include the role of space in our model as the space could affect the stability properties of ecosystems and it has been identified as an important factor in regulating spatial patterns which are ubiquitous in nature. In nature, spatial variability of algae is very common [9, 35, 49]. Diffusion and advection play a crucial role in the spatial movement of algae in water bodies. Thus, extending the model studied here by incorporating the role of space might be worth investigating. Moreover, there exists many random factors that cause the change of the algae densities [26]. The results of present investigation can be further extrapolated with stochastic excitations on the algae densities.

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