Better Bounds for Online Line Chasing*

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Abstract. We study online competitive algorithms for the line chasing problem in Euclidean spaces $\mathbb{R}^d$, where the input consists of an initial point $P_0$ and a sequence of lines $X_1, X_2, ..., X_m$, revealed one at a time. At each step $t$, when the line $X_t$ is revealed, the algorithm must determine a point $P_t \in X_t$. An online algorithm is called $c$-competitive if for any input sequence the path $P_0, P_1, ..., P_m$ it computes has length at most $c$ times the optimum path. The line chasing problem is a variant of a more general convex body chasing problem, where the sets $X_t$ are arbitrary convex sets.

To date, the best competitive ratio for the line chasing problem was 28.1, even in the plane. We significantly improve this bound, by providing a 3-competitive algorithm for any dimension $d$. We also improve the lower bound on the competitive ratio, from 1.412 to 1.5358.

Keywords: Convex body chasing · Line chasing · Competitive analysis

1 Introduction

The convex body chasing is a fundamental problems in online combinatorial optimization. It asks for an incrementally-computed path, that traverses a given sequence of convex sets provided one at a time in an online fashion, and is as short as possible. Formally, the input consists of an initial point $P_0 \in \mathbb{R}^d$ and a sequence $X_1, X_2, ..., X_m \subseteq \mathbb{R}^d$ of convex sets. The objective is to find a path $P = (P_0, P_1, ..., P_m)$ with $P_t \in X_t$ for each $t = 1, 2, ..., n$ and minimum total length $\ell(P) = \sum_{t=1}^{n} \ell_{P_{t-1}P_t}$. (Throughout the paper, by $\ell_{PQ}$ we denote the Euclidean distance between points $P$ and $Q$ in $\mathbb{R}^d$.) This path $P$ must be computed online, in the following sense: the sets $X_t$ are revealed over time, one per time step. At step $t$, when set $X_t$ is revealed, we need to immediately and irrevocably identify its visit point $P_t \in X_t$. Thus the choice of $P_t$ does not depend on the future sets $X_{t+1}, ..., X_m$.

As can be easily seen, in this online scenario computing an optimal solution is not possible, and thus all we can hope for is to find a path whose length only

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approximates the optimum value. A widely accepted measure for the quality of this approximation is the competitive ratio. For a constant $c \geq 1$, we will say that an online algorithm $A$ is $c$-competitive if it computes a path whose length is at most $c$ times the optimum solution (computed offline). This constant $c$ is called the competitive ratio of $A$. Our objective is then to design an online algorithm whose competitive ratio is as close to 1 as possible.

The convex body chasing problem was originally introduced in 1993 by Friedman and Linial [9], who gave a constant-competitive algorithm for chasing convex bodies in $\mathbb{R}^2$ (the plane) and conjectured that it is possible to achieve constant competitiveness in any $d$-dimensional space $\mathbb{R}^d$. As shown in [9], this constant would have to depend on $d$; in fact it needs to be at least $\Omega(\sqrt{d})$.

The Friedman-Linial conjecture has remained open for over two decades. In the last several years this topic has experienced a sudden increase in research activity, partly motivated by connections to machine learning (see [3,7]), resulting in rapid progress. In 2016, Antoniadis et al. [1] gave a $2^{O(d)}$-competitive algorithm for chasing affine spaces of any dimension. In 2018, Bansal et al. [3] gave an algorithm with competitive ratio $2^{O(d \log d)}$ for nested families of convex sets, where the input set sequence satisfies $X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m$. Soon later their bound was improved to $O(d \log d)$ by Argue et al. [2], and then to $O(\sqrt{d \log d})$ by Bubeck et al. [6]. Finally, Bubeck et al. [7] just recently announced a proof of the Friedman-Linial conjecture, providing an algorithm with competitive ratio $2^{O(d)}$ for arbitrary convex sets.

One other natural variant of convex body chasing that also attracted attention in the literature is line chasing, where all sets $X_t$ are lines. Friedman and Linial [9] gave an online algorithm for line chasing in $\mathbb{R}^2$ with ratio 28.53. Their algorithm was simplified by Antoniadis et al. [1], who also slightly improved the ratio, to 28.1. Earlier, in 2014, Sitters [13] showed that a generalized work function algorithm has constant competitive ratio for line chasing, but he did not determine the value of the constant.

**Our results.** We study the line chasing problem discussed above. Our main results is a 3-competitive algorithm for line chasing in $\mathbb{R}^d$, for any dimension $d \geq 2$, significantly improving the competitive ratios from [9,1,13]. Our algorithm is very simple and essentially memoryless, as it only needs to keep track of the last line in the request sequence. Its amortized analysis is based on a simple potential function. We start by providing the algorithm for line chasing in the plane, in Section 2, and later in Section 3 we extend it to an arbitrary dimension. We also provide a lower bound (see Section 4), showing that no online algorithm can achieve competitive ratio better than 1.5358, even in the plane. This improves the lower bound of $\sqrt{2} \approx 1.412$ for line chasing established in [9]. To the best of our knowledge, this is also the best lower bound for convex body chasing in the plane.

**Other related work.** A very general model for online optimization and competitive analysis, called Metrical Task Systems (MTS), was introduced in [5]. An instance of MTS specifies a metric space $M$, an initial point $P_0 \in M$, and a sequence of non-negative functions $\tau_1, \tau_2, \ldots, \tau_m$ over $M$ called tasks. These
tasks arrive online, one at a time. At each step \( t \), the algorithm needs to choose a point \( P_t \in M \) where it moves to “process” the current task \( \tau_t \). The goal is to minimize the total cost defined by \( \sum_{t=1}^{m} (\mu(P_{t-1}, P_t) + \tau_t(P_t)) \), where \( \mu() \) is the metric in \( M \). Thus in MTS, in addition to movement cost, at each step we also pay the cost of “processing” \( \tau_t \). For any metric space \( M \) with \( n \) points, if we allow arbitrary non-negative task functions then the competitive ratio of \( 2n - 1 \) can be achieved and is optimal. This general bound is not particularly useful, because in many online optimization problems that can be modelled as an MTS, the metric space \( M \) has additional structure and only tasks of some special form are allowed, which makes it possible to design online algorithms with constant competitive ratios, independent of the size of \( M \).

An MTS where \( M = \mathbb{R}^d \) and all functions \( \tau_t \) are convex is referred to as convex function chasing, and was studied in [11,14]. For the special case of convex functions on the real line, a 2-competitive algorithm was given in [4].

An MTS where each task function \( \tau_t \) is a characteristic function of a subset \( X_t \subseteq M \) is called a Metrical Service Systems (MSS) [8]. In other words, in an MSS, in each step \( t \) the algorithm needs to move to a point in \( X_t \). One variant of MSS’s that has been particularly well studied is the famous \( k \)-server problem (see, for example, [12,10]), in which one needs to schedule movement of \( k \) servers in response to requests arriving online in a metric space, where each request must be covered by one server. (In the MSS representation of the \( k \)-server problem, each set \( X_t \) consists of all \( k \)-tuples of points that include the request point at step \( t \).) Naturally, convex body chasing can be thought of as an MSS where \( M = \mathbb{R}^d \) and the request sets are arbitrary convex subsets of \( \mathbb{R}^d \).

2 A 3-Competitive Algorithm in the Plane

In this section, we present our online algorithm for line chasing in \( \mathbb{R}^2 \) with competitive ratio 3. The intuition is this: suppose that the last requested line is \( L \) and that the algorithm moved to point \( P \in L \). Let \( L' \) be the new request line, \( S \) the intersection point of \( L \) and \( L' \), and \( r = \ell_{S,P} \). A naïve greedy algorithm would move to the point \( \bar{P} \) on \( L' \) nearest to \( P \) (see Figure 1) at cost \( h = \ell_{P,\bar{P}} \).
If $h$ is small, then $r - \ell_{SP} = o(h)$, that is the distance between the greedy algorithm’s point and $S$ decreases only by a negligible amount. But the adversary can move to $S$, paying cost $r$, and then alternate requests on $L$ and $L'$. On this sequence the overall cost of this algorithm would be $\omega(r)$, so it would not be constant-competitive. This example shows that if the angle between $L$ and $L'$ is small then the drift distance towards $S$ needs to be roughly proportional to $h$.

Our algorithm is designed so that this distance is roughly $h/\sqrt{2}$ if $h$ is small (with the coefficient chosen to optimize the competitive ratio), and that it becomes 0 when $L'$ is perpendicular to $L$.

Algorithm **Drift.** Suppose that the last request is line $L$ and that the algorithm is on point $P \in L$. Let the new request be $L'$ and for any point $X \in L$, let $X$ be the orthogonal projection of $X$ onto $L'$. If $L'$ does not intersect $L$, move to $P' = P$. Otherwise, let $S = L \cap L'$ be the intersection point of $L$ and $L'$. Let also $r = \ell_{SP}$, $h = \ell_{P\bar{P}}$, and $s = \ell_{SP}$ (see Figure 1).

Move to point $P' \in L'$ such that $\ell_{SP'} = s - x$, where $x = \frac{1}{\sqrt{2}}(h + s - r)$.

**Theorem 1.** Algorithm **Drift** is 3-competitive for the line chasing problem in $\mathbb{R}^2$.

**Proof.** We establish an upper bound on the competitive ratio via amortized analysis, based on a potential function. The (always non-negative) value of this potential function, $\Phi(P, A)$, depends on locations $P, A \in L$ of the algorithm’s and the adversary’s point on the current line $L$. If $L'$ is the new request line, and $P', A' \in L'$ are the new locations of the algorithm’s and adversary’s points, we want this function to satisfy

$$\ell_{PP'} + \Phi(P', A') - \Phi(P, A) \leq 3 \ell_{AA'}.$$  \hfill (1)

Since initially the potential is 0 and is always non-negative, adding inequality [1] for all moves will establish 3-competitiveness of Algorithm **Drift**.

The potential function we use in our proof is $\Phi(P, A) = \sqrt{3}\ell_{AP}$. Substituting this formula, inequality [1] reduces to

$$\ell_{PP'} + \sqrt{3}(\ell_{A'A'} - \ell_{AP}) \leq 3 \ell_{AA'}.$$  \hfill (2)

It thus remains to prove inequality [2]. Let $g = \ell_{AA}$, $z = \ell_{A'\bar{A}}$, and $v = \ell_{\bar{A}P}$.

We first discuss the trivial case of non-intersecting $L$ and $L'$. Keeping with the general notation, here we have $x = 0$ and thus $\ell_{PP'} = h$. Moreover, $g = \ell_{AA} = h$ as well. For fixed $z$, we have $\ell_{AA'} = \sqrt{h^2 + z^2}$, i.e., the right hand side of [2] is fixed, whereas the left hand side is maximized if $A'$ is on the other side of $A$ than $\bar{P}$. The left hand side is thus at most

$$h + \sqrt{3}z \leq \sqrt{2}\sqrt{h^2 + 3z^2} \leq \sqrt{2}\sqrt{3(h^2 + z^2)} = \sqrt{6}\ell_{AA'} < 3 \ell_{AA'},$$

where the first inequality follows from the power mean inequality (for powers 1 and 2), proving this easy case.
The situation when $L'$ and $L$ do intersect is illustrated in Figure 2 (The figure shows only the case when $\bar{A}$ is between $S$ and $P'$.) Orient $L'$ from left to right (with $P$ being to the right of $S$), as shown in this figure. We want to express the distances in the above inequality in terms of $s$, $h$, $v$, and $z$ (keeping in mind that $x$ and $r$ are functions of $h$ and $s$):

$$\ell_{PP'} = \sqrt{x^2 + h^2}$$
$$\ell_{AP} = vr/s = (v\sqrt{s^2 + h^2})/s$$
$$\ell_{AA'} = \sqrt{z^2 + g^2}$$

The values of $g$ and $\ell_{A'P'}$ depend on some cases, that we consider below.

Case 1: $\bar{A}$ is between $S$ and $P'$, as in Figure 2. Then $g = h(s - v)/s$. Our goal is first to find $A'$ for which the bound in (2) is tightest. For a given $z$, among the two locations of $A'$ at distance $z$ from $\bar{A}$, the one on the left gives a larger value of the left-hand side of (2), while the right-hand side is the same for both. Thus we can assume that $A'$ is to the left of $\bar{A}$, so $\ell_{A'P'} = z + v - x$. Then we can rewrite (2) as follows:

$$\frac{1}{\sqrt{3}} \ell_{PP'} - \ell_{AP} + v - x \leq \sqrt{\ell_{AA'}} - z$$

(3)

By elementary calculus, the right-hand side is minimized for $z = \frac{1}{\sqrt{2}} g$, so we can assume that $z$ has this value. Then inequality (3) reduces to

$$\frac{1}{\sqrt{3}} \ell_{PP'} - \ell_{AP} + v - x \leq \sqrt{2} g.$$  

(4)

After substituting $g = h(s - v)/s$ and $\ell_{AP} = vr/s$, inequality (4) reduces further to

$$s(\frac{1}{\sqrt{3}} \ell_{PP'} - x - \sqrt{2}h) \leq v(r - s - \sqrt{2}h).$$

(5)

The expression in the parenthesis on the right-hand side of (5) is non-positive by triangle inequality, so the right-hand side is minimized when $v$ is maximized, that is $v = s$, and then it reduces to

$$x^2 + h^2 \leq 3(r - s + x)^2.$$  

(6)
Recall that $x = \frac{1}{\sqrt{2}}(h + s - r)$. Since $r - h \leq s \leq r$, we have

\[
x^2 + h^2 = \frac{1}{2}(h + s - r)^2 + h^2 \leq \frac{1}{2} h^2 + h^2 = \frac{3}{2} h^2 \leq \frac{3}{2} \left[ (\sqrt{2} - 1) (r - s) \right]^2 = 3(r - s + x)^2,
\]

proving (6).

**Case 2**: $\bar{A}$ is before $S$. In this case we have $g = h(v - s)/s$. Just as in Case 1, we can assume that $A'$ is to the left of $\bar{A}$, so that $\ell_{A'P'} = z + v - x$, and (2) reduces to

\[
\frac{1}{\sqrt{3}} \ell_{PP'} - \ell_{AP} + v - x \leq \sqrt{2} g.
\]

After substituting $g = h(v - s)/s$ and $\ell_{AP} = v r/s$, inequality (4) reduces further to

\[
s \left( \frac{1}{\sqrt{3}} \ell_{PP'} - x + \sqrt{2} h \right) \leq v (r - s + \sqrt{2} h).
\]

The expression in the parenthesis on the right-hand side of (8) is non-negative, so the right-hand side is minimized when $v = s$ (because in this case $v \geq s$), so (8) reduces to the same inequality (6) as in Case 1, completing the argument for Case 2.

**Case 3**: $\bar{A}$ is after $\bar{P}$. In this case we have $g = h(v + s)/s$. Symmetrically to Case 1, we can now assume that $A'$ is to the right of $\bar{A}$, so that now $\ell_{A'P'} = z + v + x$, and that $z = \frac{1}{\sqrt{2}} g$. Then, analogously to (4), we can rewrite (2) as follows:

\[
\frac{1}{\sqrt{3}} \ell_{PP'} - \ell_{AP} + v + x \leq \sqrt{2} g
\]

After substituting $g = h(v + s)/s$ and $\ell_{AP} = v r/s$, inequality (9) reduces further to

\[
s \left( \frac{1}{\sqrt{3}} \ell_{PP'} + x - \sqrt{2} h \right) \leq v (r - s - \sqrt{2} h).
\]

The expression in the parenthesis on the right-hand side of (10) is non-negative, so the right-hand side is minimized when $v = 0$, and then it reduces to

\[
x^2 + h^2 \leq 3(\sqrt{2}h - x)^2.
\]

To prove this, we proceed similarly as in Case 1:

\[
x^2 + h^2 \leq \frac{3}{2} h^2 \leq \frac{3}{2} (h + r - s)^2 = 3(\sqrt{2}h - x)^2,
\]

proving (11).

**Case 4**: $\bar{A}$ is between $P'$ and $\bar{P}$. Then $g = h(s - v)/s$ (as in Case 1). Similar to Case 3, we can assume that $A'$ is to the right of $\bar{A}$, so that now $\ell_{A'P'} = z - v + x$, and that $z = \frac{1}{\sqrt{2}} g$. Then, analogously to (4), we can rewrite (2) for this case as follows:

\[
\frac{1}{\sqrt{3}} \ell_{PP'} - \ell_{AP} - v + x \leq \sqrt{2} g
\]
After substituting $g = h(s-v)/s$ and $\ell_{AP} = vr/s$, inequality (12) reduces further to
\[
s\left(\frac{1}{\sqrt{3}} \ell_{PP'} + x - \sqrt{2}h\right) \leq v \left(r + s - \sqrt{2}h\right).
\] (13)

We now have two sub-cases. If the expression in the parenthesis on the right-hand side of (13) is non-negative then the right-hand side is minimized when $v = 0$, so inequality (13) reduces to inequality (11) from Case 3. If this expression is negative (that is when $r + s < \sqrt{2}h$), then it is sufficient to prove (13) with $v$ on the right-hand side replaced by $s$ (because $v \leq s$). This reduces it to
\[
\frac{1}{\sqrt{3}} \ell_{PP'} + x \leq r + s.
\] This last inequality follows from $\ell_{PP'} \leq r$ and $x \leq s$. 

\[\square\]

**Tightness of the analysis.** We now show that our analysis of Algorithm Drift is tight; that is, the algorithm is no better than 3-competitive. To see this, note that (assuming that $h$ is very small compared to $r$) there are two moves that make inequality (2) tight:
- One move is when $A = P, A'$ is to the right of $\bar{A}$ with $\ell_{\bar{A}A'} = \frac{1}{\sqrt{2}} h \approx x$.
- The second move is when $A = S$ and $A' = S$.

The adversary can use the first move to move away from our server, and from then on he can use moves of the second type until our server converges. This sequence can be repeated arbitrarily many times, thus proving that the competitive ratio of Algorithm Drift is not better than 3.

### 3 An Algorithm for Arbitrary Dimension

In this section, we show how to extend Algorithm Drift to Euclidean spaces $\mathbb{R}^d$ for arbitrary dimension $d \geq 2$. This extension, that we call ExtDrift, is quite simple, and consists of projecting the whole space onto an appropriately chosen plane that contains the new request line. While such approach was suggested already by Friedman and Linial [9], their choice of plane may lose a constant factor in the competitive ratio. We choose the particular plane carefully, so that ExtDrift is also 3-competitive.

Let $P$ be the current ExtDrift position and $L'$ the new request line. If $P \in L'$, ExtDrift makes no move. Otherwise, let $U$ be the uniquely determined plane which contains both $L'$ and $P$. ExtDrift makes the move prescribed by Drift in the plane $U$ for $P, L'$ and the projection of $L$ onto $U$.

**Theorem 2.** Algorithm ExtDrift is 3-competitive for the line chasing problem in $\mathbb{R}^d$, for arbitrary dimension $d \geq 2$.

**Proof.** We prove that (1) holds in arbitrary dimension. If $P \in L'$ then $L$ and $L'$ are co-planar, so the analysis from the previous section works directly.

So assume that $P \notin L'$. We first allow the adversary to perform a free move from its current position $A$ to point $\bar{A}$ defined as the orthogonal projection of $A$ onto $U$, and then we analyze the move within $U$ (that is, in a two-dimensional setting), as if the adversary started from point $\bar{A}$. 

We note that $\ell_{AX} \leq \ell_{A\tilde{X}}$ for any point $X \in U$, as $(\ell_{AX})^2 = (\ell_{A\tilde{X}})^2 + (\ell_{A\tilde{A}})^2$ by definition of $\tilde{A}$. It follows that:

- In the free adversary move from $A$ to $\tilde{A}$ the potential function decreases (by taking $X = P$ in the above inequality) and both costs are 0. Further, in the move within $U$, with the adversary starting from $\tilde{A}$, Algorithm ExtDrift makes the same move as Drift, which implies that (1) is satisfied. Thus the complete move (combining the free adversary move and the move inside $U$) satisfies inequality (1) as well.

- The free move is only beneficial for the adversary: taking $X = A'$ shows that the cost of moving to $A'$ from $\tilde{A}$ is no more costly for the adversary than moving to $A'$ from $A$.

\qed

\section{A Lower Bound of 1.5358}

Finally, in this section, we show how to improve an existing lower bound of $\sqrt{2} \approx 1.41$ to 1.5358. Our bound holds even in two dimensions.

\textbf{Theorem 3.} The competitive ratio of any deterministic online algorithm $A$ for the line chasing problem is at least 1.5358.

\textit{Proof.} We describe our adversarial strategy below. On the created input, we will compare the cost of $A$ to the cost of an offline optimum $Opt$. We assume that both $A$ and $Opt$ start at origin point $P_0 = A_0 = (0,0)$.

Our construction is parameterized with real positive numbers $c_1 = 0.5535$, $c_2 = 0.4965$, $c_3 = 0.8743$, $a_1 = 1.3012$, $a_2 = 0.6663$, $p_2 = 0.5612$, and $p_3 = 0.1696$.

We fix points $P_1 = (0, c_1)$, $C_2 = (0, c_1 + c_2)$, $C_3 = (0, c_1 + c_2 + c_3)$ and $A_3 = (1, c_1)$, see Figure 3 for illustration. For succinctness, we use notation $\triangle(x, y) = \sqrt{x^2 + y^2}$.

\textbf{Initial part: Line $L_1$.} The first request line is the line $P_1A_3$, denoted $L_1$. Without loss of generality, we can assume that $A$ moves to $P_1$. This is because the adversary can either play the strategy described below or its mirror image (flipped against the line $P_0P_1$), so any deviation from $P_1$, either to the left or right, can only increase the cost of $A$.

From now on, for any point $Q$ we denote its projection on line $L_1$ by $Q^x$.

\textbf{Middle part: Line $L_2$.} Next, the adversary issues the request line $C_2A_3$, denoted $L_2$. Let $P_2 \in L_2$ and $A_1 \in L_2$ be the points to the left of $A_3$, such that $\ell_{P_2A_2} = p_2$ and $\ell_{A_1A_3} = a_1$.

Let $\bar{A}$ be the point on $L_2$ chosen by $A$. If $\bar{A}$ lies to the right of point $P_2$, then the adversary forces $A$ to move to $A_1$ (by giving sufficiently many different lines that go through $A_1$ at different angles). $Opt$ may then serve the whole sequence by going from $A_0$ to $A_1$ at cost

$\ell_{A_0A_1} = \triangle(c_1 + c_2 \cdot a_1, a_1 - 1) \leq 1.23679$
Fig. 3. Visual description of our lower bound. Lines $L_1$, $L_2$ and $L_3$ are presented to an online algorithm. Blue arrows describe possible movements of $\text{Opt}$, while gray thick arrows describe a path of an algorithm that minimizes the competitive ratio for this adversarial construction. Red thick half-line denotes the forbidden region.

while the cost of $\mathcal{A}$ is then at least

$$\ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{P_2A_1} = \ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{A_3A_1} - \ell_{A_3P_2}$$
$$= c_1 + \Delta(1 - p_2, c_2 \cdot p_2) + \Delta(a_1, c_2 \cdot a_1) - \Delta(p_2, c_2 \cdot p_2)$$
$$\geq 1.89948$$

Hence, the competitive ratio in this case is at least 1.5358.

We call the half-line of $L_2$ to the right of point $P_2$ forbidden region. From now on, we assume that the point chosen by $\mathcal{A}$ in $L_2$ does not lie in this region.

**Final part: Line $L_3$.** Finally, the adversary issues the request line $C_3A_3$, denoted $L_3$. Let $P'_2$ be the intersection of line $P_1P_2$ with line $L_3$. Next, let $A_2$ and $P_3$ be the points on the line $L_3$ to the left of $A_3$, such that $\ell_{A_2A_3} = a_2$ and $\ell_{P_3A_3} = p_3$. Note that $P_3$ belongs to the interval $P'_2A_3$.

Let $P_3$ be the point on $L_3$ chosen by $\mathcal{A}$. We consider two cases.

**Case 1:** $P_3$ lies at point $P_3$ or to its left. In this case, the adversary forces $\mathcal{A}$ to move to $A_3$. $\text{Opt}$ may serve the whole sequence by going from $A_0$ to $A_3$ paying

$$\ell_{A_0A_3} = \Delta(1, c_1) \leq 1.142963.$$
We may now argue that the cost of $A$ is minimized if $\overline{P}_3$ is equal to $P_3$: If $\overline{P}_3$ is to the left of point $P_3'$, then the cost of $A$ is at least $\ell_{P_0P_1} + \ell_{P_1\overline{P}_3} + \ell_{\overline{P}_3A_3}$. Both the second and the third summand decrease when we move $\overline{P}_3$ towards $P_3'$. Hence, now we may assume that $\overline{P}_3$ belongs to the interval $P_3' P_3$. As the path of $A$ must avoid forbidden region, its cost is at least $\ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{P_2\overline{P}_3} + \ell_{\overline{P}_3A_3}$. The sum of the last two summands decreases when we move $\overline{P}_3$ towards $P_3$. Therefore, we obtain that the cost of $A$ is at least

$$\ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{P_2\overline{P}_3} + \ell_{\overline{P}_3A_3} = c_1 + \Delta(1 - p_2, c_2 \cdot p_2) + \Delta((c_2 + c_3) \cdot p_3 - c_2 \cdot p_2, p_2 - p_3) + \Delta(p_3, (c_2 + c_3) \cdot p_3) \geq 1.75537.$$ 

Thus, in this case the competitive ratio is at least 1.5358.

Case 2: If $\overline{P}_3$ lies to the right of point $P_3$, then the adversary forces $A$ to move to $A_2$. $\text{OPT}$ may serve the whole sequence by going from $A_0$ to $A_2$ at cost

$$\ell_{A_0A_2} = \Delta(c_1 + (c_2 + c_3) \cdot a_2, 1 - a_2) \leq 1.50435.$$ 

To go from $P_1$ to $\overline{P}_3$ and avoid the forbidden region, $A$ has to pay at least $\ell_{P_1P_2} + \ell_{P_2\overline{P}_3}$. Therefore, its cost is at least

$$\ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{P_2\overline{P}_3} + \ell_{\overline{P}_3A_2} \geq \ell_{P_0P_1} + \ell_{P_1P_2} + \ell_{P_2\overline{P}_3} + \ell_{\overline{P}_3A_3} - \ell_{P_3A_3} = c_1 + \Delta(1 - p_2, c_2 \cdot p_2) + \Delta((c_2 + c_3) \cdot p_3 - c_2 \cdot p_2, p_2 - p_3) + \Delta(a_2, (c_2 + c_3) \cdot a_2) - \Delta(p_3, (c_2 + c_3) \cdot p_3) \geq 2.31039.$$ 

Thus, in this case the ratio is also at least 1.5358. \qed

5 Final Comments

Establishing the optimal competitive ratio for line chasing remains an open problem. We believe that none of our bounds is tight.

For instance, it should be possible to improve the upper bound using an algorithm with memory, for example by storing the actual work function at each step. The intuition is that in the first move, if $L$ and $P$ are the initial line and position and $L'$ is the new request line, then the algorithm should move to the nearest point $\hat{P}$ on $L'$. More generally, if the requests on $L$ and $L'$ alternate (and their angle is small), the algorithm should initially drift towards $S = L \cap L'$ slowly and only gradually accelerate to a rate that is proportional to the distance to the other line.

It appears also that our lower bound strategy can be improved by introducing additional steps, although this gives only very small improvements and leads to a very involved analysis. It is possible that an approach fundamentally different from ours may give a better bound with simpler analysis.
Better Bounds for Online Line Chasing

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