DEFORMATION OF THE SCALAR CURVATURE AND THE MEAN CURVATURE

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Abstract. On a compact manifold $M$ with boundary $\partial M$, we study the problem of prescribing the scalar curvature in $M$ and the mean curvature on the boundary $\partial M$ simultaneously. To do this, we introduce the notion of singular metric, which is inspired by the early work of Fischer-Marsden in [15] and Lin-Yuan in [23] for closed manifold. We show that we can prescribe the scalar curvature and the mean curvature simultaneously for generic scalar-flat manifolds with minimal boundary. We also prove some rigidity results for the flat manifolds with totally geodesic boundary.

1. Introduction

Suppose that $M$ is a compact smooth manifold with boundary $\partial M$. There are two types of the Yamabe problem with boundary: Given a smooth metric $g$ in $M$, (i) find a metric conformal to $g$ such that its scalar curvature is constant in $M$ and its mean curvature is zero on $\partial M$; (ii) find a metric conformal to $g$ such that its scalar curvature is zero in $M$ and its mean curvature is constant on $\partial M$. The Yamabe problem with boundary has been studied by many authors. See [4, 15, 17, 24] and the references therein.

As a generalization of the Yamabe problem with boundary, one can consider the prescribing curvature problem on manifolds with boundary: Given a smooth metric $g$ in a compact manifold $M$ with boundary $\partial M$, (i) find a metric conformal to $g$ such that its scalar curvature is equal to a given smooth function in $M$ and its mean curvature is zero on $\partial M$; (ii) find a metric conformal to $g$ such that its scalar curvature is zero in $M$ and its mean curvature is equal to a given smooth function on $\partial M$. In particular, when the manifold is the unit ball, it is the corresponding Nirenberg's problem for manifolds with boundary. These have been studied extensively by many authors. We refer the readers to [7, 16, 22, 25] and the references therein for results in this direction.

More generally, one can consider the prescribing curvature problem on manifolds with boundary without restricting to a fixed conformal class: (i) given a smooth function $f$ in $M$, find a metric $g$ such that its scalar curvature is equal to $f$ and its mean curvature is zero, i.e. $R_g = f$ in $M$ and $H_g = 0$ on $\partial M$; (ii) given a smooth function $h$ on $\partial M$, find a metric $g$ such that its scalar curvature is zero and its mean curvature is equal to $h$, i.e. $R_g = 0$ in $M$ and $H_g = h$ on $\partial M$. This was recently studied by Cruz-Vitório in [12].

In this paper, we study the problem of prescribing the scalar curvature in $M$ and the mean curvature on the boundary $\partial M$ simultaneously. More precisely, given a
smooth function $f$ in $M$ and a smooth function $h$ on $\partial M$, we want to find a metric $g$ such that its scalar curvature is equal to $f$ and its mean curvature is equal to $h$, i.e. $R_g = f$ in $M$ and $H_g = h$ on $\partial M$. We would like to point out that there are several results in prescribing the scalar curvature in $M$ and the mean curvature on the boundary $\partial M$ simultaneously in a fixed conformal class. See [9, 10, 13, 14, 20, 21]. The flow approach was introduced to study this problem in [3, 26] for dimension 2 and in [8] for higher dimensions. However, without restricted to a fixed conformal class, there are not many results in prescribing the scalar curvature and the mean curvature on the boundary simultaneously. So our paper can be viewed as the first step to understand this problem.

In order to study the problem of prescribing the scalar curvature and the mean curvature simultaneously, we study the linearization of the scalar curvature and the mean curvature. We introduce the notion of singular space in section 2. This notion is inspired by the early work of Fischer-Marsden in [18], which studied the linearization of the scalar curvature in closed (i.e. compact without boundary) manifolds, and the work of Lin-Yuan in [23] which studied the linearization of the $Q$-curvature in closed manifolds. In section 3 we will show that some geometric properties of the manifold imply that it is singular (or not singular). We then give some examples of singular space and non-singular space in section 4. In section 5, we prove some theorems related to prescribing the scalar curvature and the mean curvature simultaneously. Finally, in section 6, we prove some rigidity results for the flat manifolds with totally geodesic boundary. See Theorem 6.2.

2. Characterization of singular spaces

For a compact $n$-dimensional manifold $M$ with boundary $\partial M$, let $\mathcal{M}$ be the moduli space of all smooth metrics defined in $M$. Denote the map

\[
\Psi: \mathcal{M} \rightarrow \mathbb{R} \times \mathbb{R}
\]

\[
g \mapsto (R_g, 2H_g)
\]

where $\gamma = g|_{\partial M}$ is the metric $g$ induced on $\partial M$, $R_g$ is the scalar curvature in $M$ and $H_g$ the mean curvature on $\partial M$ with respect to $g$.

Let $\mathcal{S}_g : S_2(M) \rightarrow C^\infty(M) \times C^\infty(M)$ be the linearization of $\Psi$ at $g$, and let $\mathcal{S}_g^* : C^\infty(M) \times C^\infty(M) \rightarrow S_2(M)$ be the $L^2$-formal adjoint of $\mathcal{S}_g$, where $S_2(M)$ is the space of symmetric 2-tensors on $M$. More precisely, for any $h \in S_2(M)$, we have

\[
\frac{d}{dt} \bigg|_{t=0} \Psi(g + th) = \mathcal{S}_g(h) = D\Psi_g \cdot h = (\delta R_g h, 2\delta H_g h).
\]

It was computed in [1] and [12] that

\[
\delta R_g h = \Delta_g(tr_g h) + \text{div}_g \text{div}_g h - \langle h, \text{Ric}_g \rangle,
\]

\[
2\delta H_g h = [d(tr_g h) - \text{div}_g h](\nu) - \text{div}_g X - \langle \gamma, h \rangle\gamma,
\]

where $\nu$ is the outward unit normal to $\partial M$, $II_\gamma$ is the second fundamental form of $\partial M$, $X$ is the vector field dual to the one-form $\omega(\cdot) = h(\cdot, \nu)$, $tr_h = g^{ij} h_{ij}$ is the trace of $h$ and our convention for the Laplacian is $\Delta_g f = tr_g(\text{Hess}_g f)$. Now $\mathcal{S}_g^*$,
the $L^2$-formal adjoint of $S_g$, satisfies
\[
\frac{d}{dt} \left( \int_M R_g + t h f_1 dV_g + 2 \int_{\partial M} H_{g+th} f_2 dA_{\gamma} \right) \bigg|_{t=0} = \langle S_g(h), (f_1, f_2) \rangle = \langle h, S_g^*(f_1, f_2) \rangle = \langle h, (\delta R_g h) f_1 dV_g + 2 \int_{\partial M} (\delta H_{\gamma} h) f_2 dA_{\gamma} \rangle.
\]
If we define
\[
S_g^*(f) := S_g^*(f, f) = (A_g^* f, B_g^* f) \text{ where } f \in C^\infty(M),
\]
then it follows from (2.3) in [12] that
\[
A_g^* f = - (\Delta_g f) g + \text{Hess}_g f - f R_{g},
\]
(2.4)
\[
B_g^* f = \frac{\partial f}{\partial \nu} - f II_{\gamma}.
\]

Inspired by the notion of $Q$-singular space defined in [23] (see also [18]), we have
the following:

**Definition 2.1.** The metric $g$ is called singular if $S_g^*$ defined in (2.3) is not injective, namely, $\ker(S_g^*) \neq \{0\}$. We also refer $(M, \partial M, g, f)$ as singular space, if $0 \not\equiv f \in \ker(S_g^*)$.

It follows from (2.4) that $f \in \ker(S_g^*)$ if and only if $f$ satisfies the equations
\[
\left\{ \begin{array}{l}
-(\Delta_g f) g + \text{Hess}_g f - f R_g = 0 \text{ in } M, \\
\frac{\partial f}{\partial \nu} - f II_{\gamma} = 0 \text{ on } \partial M.
\end{array} \right.
\]
(2.5)
Taking the trace of (2.5) with respect to $g$, we obtain
\[
\left\{ \begin{array}{l}
\Delta_g f + \frac{R_g}{n-1} f = 0 \text{ in } M, \\
\frac{\partial f}{\partial \nu} - \frac{H_{\gamma}}{n-1} f = 0 \text{ on } \partial M.
\end{array} \right.
\]
(2.6)
That is to say, if $f \in \ker(S_g^*)$, then $f$ must satisfy (2.6).

### 3. Singular and nonsingular spaces

In this section, we show that some geometric properties of $(M, \partial M, g)$ will imply that it is singular (or not singular).

**Proposition 3.1.** If $R_g \leq 0$ and $H_{\gamma} \leq 0$ such that one of them is not identically equal to zero, then $g$ is not singular.

**Proof.** If $f \in \ker(S_g^*)$, then (2.6) holds. Multiplying $f$ to the first equation in (2.6), integrating it over $M$ and using integration by parts, we obtain
\[
0 = \int_M \left( f \Delta_g f + \frac{R_g}{n-1} f^2 \right) dV_g
\]
(3.1)
\[
= \int_M \left( -|\nabla_g f|^2 + \frac{R_g}{n-1} f^2 \right) dV_g + \int_{\partial M} \frac{\partial f}{\partial \nu} dA_{\gamma}
\]
\[
= \int_M \left( -|\nabla_g f|^2 + \frac{R_g}{n-1} f^2 \right) dV_g + \int_{\partial M} \frac{H_{\gamma}}{n-1} f^2 dA_{\gamma}.
\]
where we have used the second equation of (2.6) in the last equality. Since $R_g \leq 0$ and $H_\gamma \leq 0$, it follows from (3.1) that $|\nabla_g f|^2 \equiv 0$ in $M$, which implies that $f \equiv c$ for some constant $c$. Hence, (2.6) reduces to

$$\frac{R_g}{n-1} c = 0 \text{ in } M \quad \text{and} \quad -\frac{H_\gamma}{n-1} c = 0 \text{ on } \partial M.$$  

Since $R_g$ or $H_\gamma$ is not identically equal to zero by assumption, we can conclude that $c = 0$, i.e. $f \equiv 0$. Therefore, we have shown that $\ker(S^*_g) = \{0\}$, as required. □

**Proposition 3.2.** If $(M, \partial M, g)$ is Ricci-flat with totally-geodesic boundary, then $g$ is singular.

*Proof.* By assumption, we have $Rig_g \equiv 0$ in $M$ and $II_\gamma \equiv 0$ on $\partial M$. If we take $f$ to be any nonzero constant function defined in $M$, then it satisfies (2.5). Thus, $\ker(S^*_g) \neq \{0\}$ and the result follows. □

In fact, we have the following:

**Proposition 3.3.** If $(M, \partial M, g)$ is Ricci-flat with totally-geodesic boundary, then

$$\ker(S^*_g) = \{\text{constant}\}.$$  

*Proof.* We have already shown that $\{\text{constant}\} \subseteq \ker(S^*_g)$ in Proposition 3.2. On the other hand, if $f \in \ker(S^*_g)$, then $f$ satisfies (2.6). This together with the assumption $Rig_g \equiv 0$ in $M$ and $II_\gamma \equiv 0$ on $\partial M$ implies that

$$\begin{cases} \Delta_g f = 0 & \text{in } M, \\ \frac{\partial f}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$  

which shows that $f$ is constant. □

The condition (3.2) gives a characterization of the Ricci-flat manifold with totally-geodesic boundary.

**Proposition 3.4.** If $f$ is a nonzero constant function lies in $\ker(S^*_g)$, then $(M, \partial M, g)$ is Ricci-flat with totally-geodesic boundary.

*Proof.* By assumption, the function $f \equiv c$ satisfies (2.6). Thus, we have

$$\begin{cases} cRic_g = 0 & \text{in } M, \\ -cII_\gamma = 0 & \text{on } \partial M. \end{cases}$$  

Since $c$ is nonzero, we have $Rig_g \equiv 0$ in $M$ and $II_\gamma \equiv 0$ on $\partial M$, as required. □

**Proposition 3.5.** Suppose $R_g \equiv 0$ in $M$ and $H_\gamma \equiv 0$ on $\partial M$. If one of the following assumptions holds:

1. $Ric_g \neq 0$ in $M$, i.e. $g$ is not Ricci-flat;
2. $II_\gamma \neq 0$ on $\partial M$, i.e. $\partial M$ is not totally-geodesic,

then $g$ is not singular.

*Proof.* Let $f \in \ker(S^*_g)$. Since $R_g \equiv 0$ in $M$ and $H_\gamma \equiv 0$ on $\partial M$ by assumption, again by (2.6) we have (3.3), and thus $f \equiv c$ for some constant $c$. If $c$ is nonzero, i.e. $f$ is a nonzero constant function lies in $\ker(S^*_g)$, it follows from Proposition 3.4 that $(M, \partial M, g)$ is Ricci-flat with totally-geodesic boundary, which contradicts to the assumption. Therefore, we must have $c = 0$, i.e. $f \equiv 0$. □
We remark that a result similar to Proposition 3.5 has been obtained in [12]. See Proposition 3.3 in [12].

**Proposition 3.6.** Suppose that

\[(3.4) \quad \text{Ric}_g = \frac{R_g}{n} g = (n-1)g \text{ in } M \quad \text{and} \quad H_\gamma = n-1 \text{ on } \partial M.\]

If \(g\) is singular, then \((M, \partial M, g)\) is isometric to either a spherical cap or the standard hemisphere.

**Proof.** Since \(g\) is singular by assumption, there exists \(0 \neq f \in \ker(S^*_g)\). Note that \(f\) is not a non-constant function; otherwise, it follows from Proposition 3.4 that \(g\) is Ricci-flat, which contradicts to the assumption that the scalar curvature \(R_g\) is nonzero. Since (2.5) and (2.6) hold, we can substitute (2.6) into (2.5) and apply (3.4) to get

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{Hess}_g f + fg = 0 \quad \text{in } M, \\
\frac{\partial f}{\partial \nu} = f \quad \text{on } \partial M.
\end{array} \right.
\end{align*}
\]

Now the result follows immediately from Theorem 3 in [11]. \(\square\)

**Proposition 3.7.** Let \((M, \partial M, g)\) be an \(n\)-dimensional Einstein manifold with minimal boundary, where \(n \geq 3\). If \(g\) is singular, then \(\frac{R_g}{n-1}\) is an eigenvalue of the Laplacian with Neumann boundary condition. In this case, \(\ker(S^*_g)\) lies in the eigenspace of \(\frac{R_g}{n-1}\). In particular, \(\ker(S^*_g)\) is finite-dimensional.

**Proof.** Let \(0 \neq f \in \ker(S^*_g)\). Since \((M, \partial M, g)\) is an \(n\)-dimensional Einstein manifold with \(n \geq 3\), the scalar curvature \(R_g\) is constant. Moreover, \(H_\gamma = 0\) on \(\partial M\) by assumption. Hence, (2.6) implies that \(f\) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
\Delta_g f + \frac{R_g}{n-1} f = 0 \quad \text{in } M, \\
\frac{\partial f}{\partial \nu} = f \quad \text{on } \partial M.
\end{array} \right.
\end{align*}
\]

This implies that \(\frac{R_g}{n-1}\) is the eigenvalue of the Laplacian with Neumann boundary condition, and \(f\) is the corresponding eigenfunction. This shows that \(\ker(S^*_g)\) lies in the eigenspace of \(\frac{R_g}{n-1}\), as claimed. \(\square\)

**Proposition 3.8.** Suppose that \((M, \partial M, g)\) is scalar-flat with umbilical boundary of constant mean curvature. If \(g\) is singular, then \(\frac{H_\gamma}{n-1}\) is a Steklov eigenvalue. In this case, \(\ker(S^*_g)\) lies in the eigenspace corresponding to the Steklov eigenvalue \(\frac{H_\gamma}{n-1}\).

**Proof.** Let \(0 \neq f \in \ker(S^*_g)\). By assumption, we have

\[(3.6) \quad R_g = 0 \text{ in } M \quad \text{and} \quad H_\gamma = \frac{H_\gamma}{n-1} \gamma \text{ on } \partial M,\]
where $H_\gamma$ is constant. It follows from (3.6) that (2.6) reduces to
\[
\begin{cases}
\Delta_g f = 0 & \text{in } M, \\
\frac{\partial f}{\partial \nu} + \frac{H_\gamma}{n-1} f = 0 & \text{on } \partial M.
\end{cases}
\]
Hence, \( \frac{H_\gamma}{n-1} \) is the Steklov eigenvalue, and \( f \) is the corresponding eigenfunction. In particular, this shows that \( \ker(S^*_g) \) lies in the eigenspace corresponding to the Steklov eigenvalue \( \frac{H_\gamma}{n-1} \). This proves the assertion. \( \Box \)

We remark that a result similar to Proposition 3.8 has been obtained in [12]. See Proposition 3.1 in [12].

4. Examples

In this section, we give some examples of singular and non-singular space.

**Manifolds with negative Yamabe constant.** Suppose that \((M, \partial M, g)\) is an \( n \)-dimensional compact manifold with boundary, where \( n \geq 3 \). The Yamabe constant of \((M, \partial M, g)\) is defined as (c.f. [17])
\[
Y(M, \partial M, g) = \inf \{ E_g(u) | 0 < u \in C^\infty(M) \},
\]
where
\[
E(u) = \int_M \left( \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 \right) dV_g + 2 \int_{\partial M} H_g u^2 dA_g \\
\left( \int_M u^{2(n-1)/(n-2)} dV_g \right)^{\frac{n-2}{n-4}}.
\]
If the Yamabe constant of \((M, \partial M, g)\) is negative, then we can find a metric \( \tilde{g} \) conformal to \( g \) such that \( R_{\tilde{g}} < 0 \) in \( M \) and \( H_{\tilde{g}} = 0 \) on \( \partial M \) (c.f. Lemma 1.1. in [17]). In particular, it follows from Proposition 3.1 that \((M, \partial M, \tilde{g})\) is not singular.

Similarly, we can define (c.f. [15])
\[
Q(M, \partial M, g) = \inf \{ Q_g(u) | 0 < u \in C^\infty(M) \},
\]
where
\[
Q_g(u) = \int_M \left( \frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 \right) dV_g + 2 \int_{\partial M} H_g u^2 dA_g \\
\left( \int_{\partial M} u^{\frac{2(n-1)}{n-2}} dA_g \right)^{\frac{n-2}{n-4}}.
\]
If \( Q(M, \partial M, g) < 0 \), then we can find a metric \( \tilde{g} \) conformal to \( g \) such that \( R_{\tilde{g}} = 0 \) in \( M \) and \( H_{\tilde{g}} < 0 \) on \( \partial M \) (c.f. Proposition 1.4 in [15]). In particular, it follows from Proposition 3.1 that \((M, \partial M, \tilde{g})\) is not singular.

**Ricci-flat manifolds with totally geodesic boundary.** Suppose that \((M, g)\) is a closed (i.e. compact without boundary) manifold which is Ricci-flat. Consider the product manifold \( \tilde{M} = [0,1] \times M \) equipped with the product metric \( \tilde{g} = dt^2 + g \). Then \( \tilde{g} \) is still Ricci-flat, and its boundary \( \partial \tilde{M} = \{0\} \times M \cup \{1\} \times M \) is totally geodesic. Therefore, it follows from Proposition 3.2 that \((M, \partial M, \tilde{g})\) is singular. For example, we can take \((M, g)\) to be any compact Calabi-Yau manifold. It is Ricci-flat. Then \( \tilde{M} = [0,1] \times M \) equipped with the product metric \( \tilde{g} = dt^2 + g \) is singular.
Now suppose that \((M_0, g_0)\) is a closed manifold such that \(g_0\) is flat. Then the product manifold \(\tilde{M} = [0, 1] \times M\) equipped with the product metric \(\tilde{g} = dt^2 + g\) is still flat and has totally geodesic boundary. Therefore, it follows from Proposition \ref{proposition:flat_product} that \((\tilde{M}, \partial \tilde{M}, \tilde{g})\) is singular. For example, if we take \((M_0, g_0)\) to be the \(n\)-dimensional torus \(T^n\) equipped with the flat metric \(g_0\), then \([0, 1] \times T^n\) equipped with the product metric \(dt^2 + g_0\) is flat and has geodesic boundary, and hence is singular.

**Product manifolds.** Suppose that \((M, g)\) is a closed Riemannian manifold which is scalar-flat but not Ricci-flat. Consider the product manifold \(\tilde{M} = [0, 1] \times M\) equipped with the product metric \(\tilde{g} = dt^2 + g\). Then \((\tilde{M}, \partial \tilde{M}, \tilde{g})\) is still scalar-flat but not Ricci-flat. Its boundary \(\tilde{M} = \{(0) \times M\} \cup \{(1) \times M\}\) is totally geodesic, and thus, its mean curvature is zero. It follows from Proposition \ref{proposition:product_scalar} that \((\tilde{M}, \partial \tilde{M}, \tilde{g})\) is not singular.

For example, let \(S^2\) be the 2-dimensional unit sphere equipped with the standard metric \(g_1\), and \(\Sigma\) be a 2-dimensional compact manifold with genus at least 2 equipped with the hyperbolic metric \(g_{-1}\). Then the product manifold \(M = S^2 \times \Sigma\) is a closed manifold, and the product metric \(g = g_1 + g_{-1}\) has zero scalar curvature and is not Ricci-flat. From the above discussion, we can conclude that \(M = [0, 1] \times S^2 \times \Sigma\) equipped with the metric \(dt^2 + g_1 + g_{-1}\) is not singular.

**The upper hemisphere.** Let

\[
S^*_n = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_n^2 = 1, x_{n+1} \geq 0\}
\]

be the \(n\)-dimensional upper hemisphere. We have the following:

**Proposition 4.1.** Let \((S^*_n, \partial S^*_n)\) be the \(n\)-dimensional upper hemisphere equipped with the standard metric \(g_c\), i.e. the sectional curvature of \(g_c\) is 1, where \(n \geq 3\). Then \(g_c\) is singular. Moreover,

\[
\ker(S^*_c) = \text{span}\{x_1, \ldots, x_n\},
\]

where \((x_1, \ldots, x_n, x_{n+1})\) are the coordinates of \(S^*_n \subset \mathbb{R}^{n+1}\).

**Proof.** Note that \(g_c\) is Einstein and the boundary \(\partial S^*_n\) is totally-geodesic, i.e.

\[
\text{Ric}_{g_c} = \frac{R_{g_c}}{n} g_c \quad \text{in } S^*_n \quad \text{and} \quad II_{g_c} = 0 \quad \text{on } \partial S^*_n.
\]

where \(R_{g_c} \equiv n(n-1)\). Note also that the coordinate functions \(x_i, 1 \leq i \leq n\), satisfy the following Obata-type equation: (see \[11\] for example)

\[
\text{Hess}_{g_c} x_i + x_i g_c = 0 \quad \text{in } S^*_n \quad \text{and} \quad \frac{\partial x_i}{\partial \nu} = 0 \quad \text{on } \partial S^*_n.
\]

Combining \[11\] and \[12\], we can conclude that the coordinate functions \(x_i, 1 \leq i \leq n\), satisfy \[22\]. Thus, \(\text{span}\{x_1, \ldots, x_n\}\) is contained in \(\ker(S^*_c)\). In particular, \(g_c\) is singular.

On the other hand, \(x_i, 1 \leq i \leq n\), is an eigenfunction corresponding to the eigenvalue \(n\) of the Laplacian with Neumann boundary condition (this follows from taking trace of \[12\]). In fact, it is well-known that the eigenspace is spanned by \(x_i\) where \(1 \leq i \leq n\). Hence, it follows from Proposition \[5.7\] that

\[
\ker(S^*_c) \subseteq \text{the eigenspace of } n = \text{span}\{x_1, \ldots, x_n\}.
\]
This proves the assertion. □

The unit ball. Let
\[ D^n = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1 \} \]
be the n-dimensional unit ball equipped with flat metric \( g_0 \). We have the following:

**Proposition 4.2.** The n-dimensional unit ball \((D^n, \partial D^n, g_0)\) equipped with the flat metric is a singular space. Moreover, we have
\[
\ker(S^*_{g_0}) = \text{span}\{x_1, \cdots, x_n\},
\]
where \((x_1, \ldots, x_n)\) are the coordinates of \(D^n\).

**Proof.** There holds
\[
\text{(4.3) } \text{Ric}_{g_0} \equiv 0 \text{ in } D^n \text{ and } II_{g_0} = \frac{H_{g_0}}{n - 1} \text{ on } \partial D^n,
\]
where \(H_{g_0} = n - 1\). We can easily check that \(x_i, 1 \leq i \leq n\), satisfies
\[
\text{(4.4) } \text{Hess}_{g_0}x_i = 0 \text{ in } D^n \text{ and } \frac{\partial x_i}{\partial \nu} = x_i \text{ on } \partial D^n.
\]
Combining (4.3) and (4.4), we can see that \(x_i, 1 \leq i \leq n\), satisfies (2.5). Therefore,
\[
\text{span}\{x_1, \cdots, x_n\} \subseteq \ker(S^*_{g_0}),
\]
which implies that \(g_0\) is singular.

On the other hand, the eigenspace of the Steklov eigenvalue 1 is spanned by \(x_i\), where \(1 \leq i \leq n\) (see Example 1.3.2 in [19] for example). This together with Proposition 3.8 implies that \(\ker(S^*_{g_0}) \subseteq \text{span}\{x_1, \cdots, x_n\}\) and the proof is completed. □

5. Prescribing scalar curvature and mean curvature simultaneously

Given a Riemannian manifold with boundary \((M, \partial M, \bar{g})\), we have the following theorem was proved by Cruz and Vitório in [12].

**Theorem 5.1** (Theorem 3.5 in [12]). Let \(f = (f_1, f_2) \in L^p(M) \oplus W^{\frac{1}{2}, p}(\partial M)\) where \(p > n\). Suppose that \(S^*_{\bar{g}}\) is injective. Then there exists \(\eta > 0\) such that if
\[
\|f_1 - R_{\bar{g}}\|_{L^p(M)} + \|f_2 - H_{\bar{g}}\|_{W^{\frac{1}{2}, p}(\partial M)} < \eta,
\]
then there is a metric \(g \in \mathcal{M}^{2,p}\) such that \(\Psi(g) = f\). Moreover, \(g\) is smooth in any open set whenever \(f\) is smooth.

More generally, we have the following:

**Theorem 5.2.** Let \(f = (f_1, f_2) \in L^p(M) \oplus W^{\frac{1}{2}, p}(\partial M)\) where \(p > n\). Define
\[
\Phi := \left\{ (f_1, f_2) \mid \int_M f_1fdV_{\bar{g}} + \int_{\partial M} f_2f dA_{\bar{g}} = 0 \text{ for all } f \in \ker(S^*_{\bar{g}}) \right\}.
\]
There exists \(\eta > 0\) such that if \((f_1, f_2) \in \Phi\) and
\[
\|f_1 - R_{\bar{g}}\|_{L^p(M)} + \|f_2 - H_{\bar{g}}\|_{W^{\frac{1}{2}, p}(\partial M)} < \eta,
\]
then there is a metric \(g \in \mathcal{M}^{2,p}\) such that \(\Psi(g) = f\). Moreover, \(g\) is smooth in any open set whenever \(f\) is smooth.
Proof. It was proved in P.5 of [12] that $A^*_g$ is elliptic in $M$, and properly elliptic, and $B^*_g$ satisfies the Shapiro- Lopatinski\'j condition at any point of the boundary. Thus, $S^*_g$ defined in (2.3) has injective symbol. Hence, we have the following decomposition: (see [2] and [18]; see also Theorem 4.1 in [23])

\[
C^\infty(M) \times C^\infty(\partial M) = \text{Im } S_g \oplus \ker S^*_g.
\]

Combining (5.1) and (5.2), we have $\text{Im } S_g = \Phi$. By identifying $\Phi$ with its tangent space, we can see that the map $\Psi$ defined in (2.1) is a submersion at $g$ with respect to $\Phi$. We can now apply the Generalized Inverse Function Theorem (c.f. Theorem 4.3 in [23]) and conclude the local subjectivity of $\Phi$ at $g$. This proves the assertion.

The following theorem shows that we can prescribe the scalar curvature in $M$ and the mean curvature on the boundary $\partial M$ simultaneously for a generic scalar-flat manifold with minimal boundary.

**Theorem 5.3.** Suppose that $(M, \partial M, \bar{g})$ is not a singular space such that $R_{\bar{g}} = 0$ in $M$ and $H_{\bar{g}} = 0$ on $\partial M$. Then, for any given functions $f_1 \in C^\infty(M)$ and $f_2 \in C^\infty(\partial M)$, there exists a metric $g$ such that $R_g = f_1$ in $M$ and $H_g = f_2$ on $\partial M$.

**Proof.** Let $f_1 \in C^\infty(M)$ and $f_2 \in C^\infty(\partial M)$. Since $M$ is compact, we can choose $L > 0$ large enough such that

\[
\frac{\|f_1\|_\infty}{L} + \frac{\|f_2\|_\infty}{L^{1/2}} < \eta,
\]

where $\eta > 0$ is given as in Theorem 5.1. Since $R_{\bar{g}} = 0$ in $M$ and $H_{\bar{g}} = 0$ on $\partial M$ by assumption, the inequality (5.3) can be written as

\[
\left\| \frac{f_1}{L} - R_g \right\|_\infty + \left\| \frac{f_2}{L^{1/2}} - H_{\bar{g}} \right\|_\infty < \eta.
\]

We can now apply Theorem 5.1 to conclude that $R_g = \frac{f_1}{L}$ in $M$ and $H_g = \frac{f_2}{L^{1/2}}$ on $\partial M$ for some smooth metric $g$. Thus the metric $L^{-1}g$ satisfies

\[
R_{L^{-1}g} = LR_g = \frac{f_1}{L} = f_1 \quad \text{in } M,
\]

and

\[
H_{L^{-1}g} = L^{1/2}H_g = L^{1/2}(\frac{f_2}{L^{1/2}}) = f_2 \quad \text{on } \partial M,
\]

as required. □

As we have seen in section 4, the product manifold $M = [0, 1] \times S^2 \times \Sigma$ equipped with the metric $dt^2 + g_1 + g_{-1}$ is scalar-flat, has totally geodesic boundary, and is not singular, where $S^2$ is the 2-dimensional unit sphere equipped with the standard metric $g_1$, and $\Sigma$ be a 2-dimensional compact manifold with genus at least 2 equipped with the hyperbolic metric $g_{-1}$. Combining this with Theorem 5.3 we have the following:

**Corollary 5.4.** Let $M = [0, 1] \times S^2 \times \Sigma$. For any $f_1 \in C^\infty(M)$ and $f_2 \in C^\infty(\partial M)$, there exists a metric $g$ such that $R_g = f_1$ in $M$ and $H_g = f_2$ on $\partial M$.

We also have the following:
Theorem 5.5. Suppose $(M, \partial M, \bar{g})$ is Ricci-flat with totally-geodesic boundary. For any $(f_1, f_2) \in \Phi_0$ where

$$\Phi_0 := \left\{ (f_1, f_2) \in C^\infty(M) \times C^\infty(\partial M) \left| \int_M f_1 dV_{\bar{g}} = \int_{\partial M} f_2 dA_{\bar{g}} = 0 \right. \right\},$$

there exists a metric $g$ such that $R_g = f_1$ in $M$ and $H_\gamma = f_2$ on $\partial M$.

Proof. If $(M, \partial M, \bar{g})$ is Ricci-flat with totally-geodesic boundary, it follows from Proposition 3.3 that

$$\ker(S_{\bar{g}}^*) = \{\text{constant}\}.$$ 

Hence, $\Phi_0$ defined in (5.5) is contained in $\Phi$ defined in (5.1). Let $(f_1, f_2) \in \Phi_0$. We can choose $L > 0$ sufficiently large such that

$$\left\| \frac{f_1}{L} - R_g \right\|_\infty + \left\| \frac{f_2}{L^{1/2}} - H_\gamma \right\|_\infty < \eta$$

where $\eta$ is the given as in Theorem 5.2. Since $(f_1/L, f_2/L^{1/2}) \in \Phi_0 \subset \Phi$, it follows from Theorem 5.2 that $R_g = \frac{f_1}{L}$ in $M$ and $H_\gamma = \frac{f_2}{L^{1/2}}$ on $\partial M$ for some smooth metric $g$ closed to $\bar{g}$. Thus the metric $L^{-1}g$ satisfies

$$R_{L^{-1}g} = LR_g = L\left(\frac{f_1}{L}\right) = f_1 \text{ in } M,$$

and

$$H_{L^{-1}g} = L^{1/2}H_\gamma = L^{1/2}\left(\frac{f_2}{L^{1/2}}\right) = f_2 \text{ on } M,$$

as required. \qed

As we have seen in section 4 for any closed Ricci-flat $(M, g)$, the product manifold $\bar{M} = [0, 1] \times M$ equipped with the product metric $\bar{g} = dt^2 + g$ is Ricci-flat with totally-geodesic boundary. Therefore, from Theorem 5.5 we immediately have the following

Corollary 5.6. Suppose $(M, g)$ is a closed Ricci-flat manifold. Let $\bar{M} = [0, 1] \times M$ be the product manifold equipped with the product metric $\bar{g} = dt^2 + g$. Then, for any $(f_1, f_2) \in C^\infty(M, \partial M)$ such that

$$\int_{\bar{M}} f_1 dV_{\bar{g}} = 0 \text{ and } \int_{\partial \bar{M}} f_2 dA_{\bar{g}} = 0,$$

there exists a metric $g$ such that $R_g = f_1$ in $\bar{M}$ and $H_\gamma = f_2$ on $\partial \bar{M}$.

Next we have the following theorem of prescribing the scalar curvature and the mean curvature simultaneously on the upper hemisphere.

Theorem 5.7. Let $f_1 \in C^\infty(S^n_+)$ and $f_2 \in C^\infty(\partial S^n_+)$ such that

$$\int_{S^n_+} x_i f_1 dV_{g_c} = \int_{\partial S^n_+} x_i f_2 dA_{g_c} = 0 \text{ for } 1 \leq i \leq n.$$ 

Then there exists a metric $g$ such that $R_g = f_1$ in $S^n_+$ and $H_\gamma = f_2$ on $\partial S^n_+$.

Proof. It follows from Proposition 4.11 that $\ker(S_{g_c}^*) = \text{span}\{x_1, \cdots, x_n\}$. Hence, the space

$$\left\{ (f_1, f_2) \in C^\infty(S^n_+) \times C^\infty(\partial S^n_+) \left| \int_{S^n_+} x_i f_1 dV_{g_c} = \int_{\partial S^n_+} x_i f_2 dA_{g_c} = 0 \text{ for } 1 \leq i \leq n \right. \right\}$$
Deformation of the scalar curvature and the mean curvature

lies in Φ defined in (5.1). Using this, we can follow the same argument as in the proof of Theorem 5.5 to finish the proof.

Finally we have the following theorem of prescribing the scalar curvature and the mean curvature simultaneously on the unit ball.

**Theorem 5.8.** Given any $f_1 \in C^\infty(D^n)$ and $f_2 \in C^\infty(\partial D^n)$ such that

\[
\int_{D^n} f_1 \, dV_{g_0} = \int_{\partial D^n} f_2 \, dA_{\gamma_0} = 0 \quad \text{for any } 1 \leq i \leq n.
\]

Then there exists a metric $g$ such that $R_g = f_1$ in $D^n$ and $H_\gamma = f_2$ on $\partial D^n$.

Proof. It follows from Proposition 4.2 that $\ker(S^*_{g_0}) = \text{span}\{x_1, \ldots, x_n\}$. Hence, the space of all $(f_1, f_2)$ satisfying (5.6) lies in Φ defined in (5.1). Hence, we can follow the same argument as in the proof of Theorem 5.5 to finish the proof.

6. Rigidity results

Suppose that $(M, \partial M, \bar{g}, f)$ is a singular space such that

\[
R_{\bar{g}} = 0 \text{ in } M \quad \text{and} \quad H_\gamma = 0 \text{ on } \partial M.
\]

We define the following functional:

\[
\mathcal{F}(g) = \int_M R_g \, dV_g + 2 \int_{\partial M} H_\gamma \, dA_\gamma.
\]

for $g \in \mathcal{M}$. We have the following:

**Lemma 6.1.** The metric $\bar{g}$ is a critical point of $\mathcal{F}$ defined in (6.2).

Proof. We compute

\[
\frac{d}{dt} \mathcal{F}(\bar{g} + th) \bigg|_{t=0} = \int_M (\delta R_{\bar{g}}h) \, dV_{\bar{g}} + 2 \int_{\partial M} (\delta H_\gamma h) \, dA_\gamma + \int_M R_{\bar{g}} \frac{\partial}{\partial t} dV_{\bar{g} + th} \bigg|_{t=0} + 2 \int_{\partial M} H_\gamma \frac{\partial}{\partial t} dA_\gamma + th \bigg|_{t=0} = \langle S_\gamma(h), (f, f) \rangle = \langle h, S_\gamma(f, f) \rangle = 0,
\]

where we have used (6.1) and the fact that $S_\gamma(f, f) = 0$. This proves the assertion.

From now on, we suppose that $(M, \partial M, \bar{g}, f)$ is a singular space which is flat (hence is Ricci-flat) and has totally geodesic boundary. It follows from Proposition 3.2 and Proposition 3.3 that $\bar{g}$ is singular and we can take $f \equiv 1$. Then the functional $\mathcal{F}$ defined in (6.2) becomes

\[
\mathcal{F}(g) = \int_M R_g \, dV_g + 2 \int_{\partial M} H_\gamma \, dA_\gamma.
\]

We will prove the following rigidity theorem.

**Theorem 6.2.** Let $(M, \partial M, \bar{g})$ be a compact $n$-dimensional manifold which is flat and has totally geodesic boundary. If $g$ is sufficiently closed to $\bar{g}$ such that

(i) $R_g \geq 0$ in $M$ and $H_\gamma \geq 0$ on $\partial M$,

(ii) $g$ and $\bar{g}$ induce the same metric on $\partial M$,

then $(M, \partial M, g)$ is also flat and has totally geodesic boundary.

To prove Theorem 6.2, we need have the following proposition from [5]:
Proposition 6.3 (Proposition 11 in [5]). Let $M$ be a compact $n$-dimensional manifold with boundary $\partial M$. Fix a real number $p > n$. If $\| g - \bar{g} \|_{W^{2,p}(M, \bar{g})}$ is sufficiently small such that $g$ and $\bar{g}$ induce the same metric on $\partial M$, then we can find a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \text{id}$ and $h = \varphi^*(g) - \bar{g}$ is divergence-free with respect to $\bar{g}$. Moreover,

\begin{equation}
\| h \|_{W^{2,p}(M, \bar{g})} \leq N \| g - \bar{g} \|_{W^{2,p}(M, \bar{g})}
\end{equation}

where $N$ is a positive constant that depends only on $M$.

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Suppose that $g$ and $\bar{g}$ are given as in Theorem 6.2. We can apply Proposition 6.3 to get a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \text{id}$, $h = \varphi^*(g) - \bar{g}$ is divergence-free with respect to $\bar{g}$ and satisfies (6.4). Note that

\begin{equation}
h = \varphi^*(g) - \bar{g} = 0 \quad \text{on} \quad \partial M,
\end{equation}

since $g$ and $\bar{g}$ induce the same metric on $\partial M$ and $\varphi|_{\partial M} = \text{id}$. We compute

\begin{equation}
\mathcal{F}(\varphi^*g) = \mathcal{F}(\bar{g}) + D\mathcal{F}(h) + \frac{1}{2}D^2\mathcal{F}(h, h) + E_3,
\end{equation}

where $E_3$ is bounded by (see (7.11) in [6])

\begin{equation}
|E_3| \leq C \| h \|_{C^0(M, \bar{g})} \int_M |\nabla \bar{g} h|^2 dV_{\bar{g}}
\end{equation}

for some constant $C$ depending only on $(M, \partial M, \bar{g})$, thanks to (6.3). It follows from the assumption and Lemma 6.1 that

\begin{equation}
\mathcal{F}(\bar{g}) = 0 \quad \text{and} \quad D\mathcal{F}(h) = 0.
\end{equation}

We are going to compute $D^2\mathcal{F}(h, h)$. To this end, we have the following formula: (see the last equation in P.124 of [12])

\begin{align*}
\int_M f\delta R_{\bar{g}} dV_{\bar{g}} + 2 \int_{\partial M} f\delta H_{\bar{g}} h dA_{\bar{g}} \\
= \int_M \left( -\Delta_{\bar{g}} f(\text{tr}_{\bar{g}} h) + \langle \text{Hess}_{\bar{g}} f, h \rangle - f(\text{Ric}_{\bar{g}}) \right) dV_{\bar{g}} \\
+ \int_M \left( \text{tr}_{\bar{g}} h \frac{\partial f}{\partial \nu} - f(\text{II}_{\bar{g}} , h) \right) dA_{\bar{g}}
\end{align*}

for any metric $\bar{g}$ and any smooth function $f$. In particular, if we take $f \equiv 1$ and $\bar{g} = \bar{g} + t h$, we have

\begin{align*}
\int_M (\delta R_{\bar{g} + th}) dV_{\bar{g} + th} + 2 \int_{\partial M} (\delta H_{\bar{g} + th} h) dA_{\bar{g} + th} \\
= - \int_M (h, \text{Ric}_{\bar{g} + th}) dV_{\bar{g} + th} - \int_M (\text{II}_{\bar{g} + th} , h) dA_{\bar{g} + th}.
\end{align*}
Differentiating it with respect to \( t \), evaluating it at \( t = 0 \) and using the fact that \( \bar{g} \) is flat with totally geodesic boundary, we obtain

\[
D^2 F_{\bar{g}}(h, h) = \left. \frac{d}{dt} \left( \int_M (\delta R_{\bar{g} + th} h) dV_{\bar{g} + th} + 2 \int_{\partial M} (\delta H_{\bar{g} + th} h) dA_{\bar{g} + th} \right) \right|_{t=0} \\
= - \int_M \left. \frac{d}{dt} (Ric_{\bar{g} + th}) \right|_{t=0} \langle h, \cdot \rangle dV_{\bar{g}} - \int_M \left. \frac{d}{dt} (II_{\bar{g} + th}) \right|_{t=0} \langle h, \cdot \rangle dA_{\bar{g}} \\
= - \int_M \left. \frac{d}{dt} (Ric_{\bar{g} + th}) \right|_{t=0} \langle h, \cdot \rangle dV_{\bar{g}}
\]

where the last equality follows from (6.5). There holds (see (3.2) in [23] for example)

\[
\frac{d}{dt} (Ric_{\bar{g} + th})|_{t=0} = - \frac{1}{2} (\Delta_L h_{jk} + \nabla_j \nabla_k (tr_\bar{g} h) + \nabla_j (div_\bar{g} h)_k + \nabla_k (div_\bar{g} h)_j).
\]

Here the Licherowicz Laplacian acting on \( h \) is defined as

\[
\Delta_L h_{jk} = \Delta h_{jk} + 2(Ric^\bar{g} \cdot h)_{jk} - Ric_{ij} h^i_k - Ric_{ki} h^i_j,
\]

where the geometric quantities on the right hand side is with respect to \( \bar{g} \). Since \( \bar{g} \) is flat and \( div_\bar{g} h = 0 \), it follows from (6.9)-(6.11) that

\[
D^2 F_{\bar{g}}(h, h) = \frac{1}{2} \int_M \left( h_{jk} \Delta_\bar{g} h_{jk} + h_{jk} \nabla_j (tr_\bar{g} h) \right) dV_{\bar{g}}.
\]

By integration by parts, (6.5) and the fact that \( h \) is divergence-free with respect to \( \bar{g} \), we can rewrite (6.12) as

\[
D^2 F_{\bar{g}}(h, h) = \frac{1}{2} \int_M \left( h_{jk} \Delta_\bar{g} h_{jk} - \nabla_j h_{jk} \nabla_k (tr_\bar{g} h) \right) dV_{\bar{g}} = - \frac{1}{2} \int_M |\nabla_\bar{g} h|^2 dV_{\bar{g}}.
\]

Now, we can combine (6.10), (6.8) and (6.13) to obtain

\[
\mathcal{F}(\varphi^* g) = - \frac{1}{2} \int_M |\nabla_\bar{g} h|^2 dV_{\bar{g}} + E_3
\]

where \( E_3 \) satisfies (6.7). By assumption (i) in Theorem 6.2 and the fact that \( \varphi \) is a diffeomorphism, we have

\[
\mathcal{F}(\varphi^* g) = \mathcal{F}(g) \geq 0.
\]

Combining (6.14) and (6.15), we get

\[
0 \leq - \frac{1}{2} \int_M |\nabla_\bar{g} h|^2 dV_{\bar{g}} + E_3.
\]

In view of (6.4), (6.7) and (6.16), we can conclude that \( \nabla_\bar{g} h = 0 \) when \( g \) is sufficiently close to \( \bar{g} \). In particular, \( h_{jk} \) is constant for each pair of \( j, k \). Since \( h = 0 \) on \( \partial M \) by (6.4), we must have \( h = 0 \) in \( M \). That is to say, \( \varphi^*(g) = \bar{g} \). Hence, \( (M, \partial M, g) \) is also flat and has totally geodesic boundary. This finishes the proof of Theorem 6.2.

We remark that the second variation of the functional defined in (6.3) has been computed in [1] in general, without assuming that \( (M, \partial M, \bar{g}) \) is Ricci-flat with totally geodesic boundary.

As we have seen in section 4, if \( T^n \) is the \( n \)-dimensional torus equipped with the flat metric \( g_0 \), then \([0, 1] \times T^n \) equipped with the product metric \( dt^2 + g_0 \) is flat and
has geodesic boundary. Combining this with Theorem 6.2, we have the following rigidity result:

**Theorem 6.4.** Consider $\tilde{M} = [0, 1] \times T^n$ equipped with the product metric $\tilde{g} = dt^2 + g_0$, where $T^n$ is the $n$-dimensional torus equipped with the flat metric $g_0$. If $g$ is sufficiently closed to $\tilde{g}$ such that

(i) $R_g \geq 0$ in $\tilde{M}$ and $H_{\gamma} \geq 0$ on $\partial \tilde{M}$,

(ii) $g$ and $\tilde{g}$ induce the same metric on $\partial \tilde{M}$,

then $g$ is also flat and has totally geodesic boundary.

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