SYMMETRY-FORCED RIGIDITY OF FRAMEWORKS ON SURFACES

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Abstract. A fundamental theorem of Laman characterises when a bar-joint framework realised generically in the Euclidean plane admits a non-trivial continuous deformation of its vertices. This has recently been extended in two ways. Firstly to frameworks that are symmetric with respect to some point group but are otherwise generic, and secondly to frameworks in Euclidean 3-space that are constrained to lie on 2-dimensional algebraic varieties.

We combine these two settings and consider the rigidity of symmetric frameworks realised on such surfaces. By extending the orbit matrix techniques of [32, 12], we prove necessary conditions for a framework to be symmetry-forced rigid (i.e., to have no non-trivial symmetry-preserving motion) for any group and any surface. In the cases when the surface is a sphere, a cylinder or a cone we use Henneberg-type inductive constructions on group-labeled quotient graphs to prove that these conditions are also sufficient for a number of symmetry groups, including rotation, reflection, inversion and dihedral symmetry. For the remaining groups - as well as for other types of surfaces - we provide some observations and conjectures.

1. Introduction

A finite simple graph embedded into Euclidean space $\mathbb{R}^d$ with vertices interpreted as universal joints and edges as stiff bars is known as a bar-joint framework. We are interested in establishing from the combinatorics of the graph when it is possible to deform such frameworks. A framework is rigid if there is no edge-length preserving continuous motion of the vertices which changes the distance between a pair of unconnected joints [2, 8, 39]. Deciding the rigidity of a framework is typically an NP-hard problem [1]. One way around this is to restrict attention to generic frameworks; that is, frameworks whose vertex coordinates form an algebraically independent set over $\mathbb{Q}$. A fundamental result in rigidity theory is Laman’s theorem which gives a combinatorial characterisation of generic rigid bar-joint frameworks in Euclidean 2-space.

**Theorem 1.1** (Laman, 1970 [15]). Let $G$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $(G, p)$ be a generic framework in the plane. Then $(G, p)$ is isostatic if and only if

(i) $|E(G)| = 2|V(G)| - 3$;
(ii) $|E(H)| \leq 2|V(H)| - 3$ for all $H \subseteq G$ with $|V(H)| \geq 2$.

Finding combinatorial characterisations of generic rigid bar-joint frameworks in dimensions 3 and higher remains a key open problem in discrete geometry (see [39], for example). However, very recently, Laman-type characterisations have been established for generic rigid bar-joint frameworks in 3D whose joints are constrained to concentric spheres or cylinders or to surfaces which have a one-dimensional space of tangential motions (e.g., the torus or surfaces of revolution) [24, 19, 20] (see also Theorem 2.7).

Over the last decade, a number of papers have studied when symmetry causes frameworks on a graph to become infinitesimally flexible, or stressed, and when it has no impact.
These questions not only lead to many interesting and appealing mathematical results (see [12, 17, 18, 23, 29, 32, 34, 37], for example) but they also have a number of important practical applications in biochemistry and engineering, since many natural structures such as molecules and proteins, as well as many human-built structures such as linkages and other mechanical machines, exhibit non-trivial symmetries (see [7, 30, 39], for example).

Of particular interest are symmetry-induced infinitesimal motions which are fully symmetric (in the sense that the velocity vectors exhibit the same symmetries as the framework), because for symmetry-generic configurations (i.e., configurations which are as generic as possible subject to the given symmetry constraints), the existence of a fully-symmetric infinitesimal motion guarantees the existence of a finite (i.e., continuous) motion which preserves the symmetry of the framework throughout the path [14, 28]. A symmetric framework which has no non-trivial fully symmetric motion is said to be symmetry-forced rigid [12, 17, 18, 34].

To detect fully symmetric infinitesimal motions in a symmetric framework, a symmetric analog of the rigidity matrix, called the orbit rigidity matrix, has recently been constructed in [32]. The orbit rigidity matrix of a framework with symmetry group $S$ has one row for each edge orbit, and one set of columns for each vertex orbit under the group action of $S$, and its entries can explicitly be derived in a very simple and transparent fashion (see [32] for details). The key properties of the orbit rigidity matrix are that its kernel is isomorphic to the space of $S$-symmetric infinitesimal motions of the framework, and its co-kernel is isomorphic to the space of $S$-symmetric self-stresses of the framework. Using the orbit rigidity matrix, combinatorial characterisations of symmetry-forced rigid symmetry-generic frameworks have recently been established for a number of symmetry groups in the plane [12, 18, 17].

In this paper, we extend these concepts and some of these combinatorial results to symmetric frameworks in 3D whose joints are constrained to surfaces. The combinatorial description depends on the type of the surface (see Definition 2.3), which is the dimension of the space of tangential isometries.

In Section 5, we establish an orbit rigidity matrix for such frameworks. Using this new tool, we derive necessary conditions for symmetric frameworks on surfaces to be symmetry-forced rigid for any point group which is compatible with the given surface.

Furthermore, in Sections 6–9 we use the orbit-surface rigidity matrix to derive combinatorial characterisations of symmetry-forced rigid frameworks which are embedded generically with inversive or certain improper rotational or dihedral symmetry on the sphere, with rotational, reflective or inversive symmetry on the cylinder or with rotational, reflective, inversive or certain dihedral or improper rotational symmetry on the cone. We prove the sufficiency of these combinatorial counts by first showing that a short list of Henneberg-type inductive operations is sufficient to recursively generate all of the appropriate classes of group-labeled quotient graphs (Section 7). Then we adapt results from [20, 21, 38] to show that each of these operations preserves the maximality of the rank of the orbit-surface rigidity matrix (Section 8). A summary of the results is given in Section 6.

We finish by providing a number of conjectures for some other groups and surfaces (Section 10). In particular, we briefly discuss an alternative 2-fold rotational symmetry on the cylinder: half turn symmetry with axis perpendicular to the cylinder. This situation induces a symmetry-preserving motion in a framework that counts to be minimally rigid without symmetry! For this case and some others we conjecture that the necessary counts we derived here are sufficient.
2. Frameworks on surfaces

In [20, 21] frameworks supported on surfaces were considered. In particular, attention was paid to classical surfaces such as spheres, cylinders and cones. Formally, let $M$ be a 2-dimensional irreducible algebraic variety embedded in $\mathbb{R}^3$. While the restriction that $M$ is irreducible is important in general, along similar lines to [21], our results can be extended to certain reducible varieties. However these varieties must have the special property (parallel planes, concentric cylinders, etc.) that the dimension of the space of tangential isometries of $M$ is the same as in each irreducible component.

A framework on a surface $M \subseteq \mathbb{R}^3$ is a pair $(G, p)$, where $G$ is a finite simple graph and $p : V(G) \to M$ is a map such that $p(i) \neq p(j)$ for all $\{i, j\} \in E(G)$. We also say that $(G, p)$ is a realisation of the underlying graph $G$ is $\mathbb{R}^3$ which is supported on $M$. For $i \in V(G)$, we say that $p(i)$ is the joint of $(G, p)$ corresponding to $i$, and for $e = \{i, j\} \in E(G)$, we say that the line segment between $p(i)$ and $p(j)$ is the bar of $(G, p)$ corresponding to $e$. For simplicity, we denote $p(i)$ by $p_i$ for $i \in V(G)$.

An infinitesimal motion of a framework $(G, p)$ on a surface $M$ is a sequence $u$ of velocity vectors $u_1, \ldots, u_{|V(G)|}$, considered as acting at the framework joints, which are tangential to the surface and satisfy the infinitesimal flex requirement in $\mathbb{R}^3$, $(u_i - u_j) \cdot (p_i - p_j) = 0$, for each edge $\{i, j\}$. It is elementary to show that $u$ is an infinitesimal motion if and only if $u$ lies in the nullspace (kernel) of the rigidity matrix $R_M(G, p)$ given in the following definition. The submatrix of $R_M(G, p)$ given by the first $|E(G)|$ rows provides the usual rigidity matrix, $R_3(G, p)$ say, for the unrestricted framework $(G, p)$ (see [39], for example). The tangentiality condition corresponds to $u$ lying in the nullspace of the matrix formed by the last $|V(G)|$ rows.

**Definition 2.1.** The rigidity matrix $R_M(G, p)$ of $(G, p)$ on $M$ is an $|E(G)| + |V(G)|$ by $3|V(G)|$ matrix of the form

$$
\begin{bmatrix}
R_3(G, p) \\
\mathcal{N}(p)
\end{bmatrix}.
$$

Consecutive triples of columns in $R_M(G, p)$ correspond to framework joints. $R_3(G, p)$ is the usual rigidity matrix of $(G, p)$, that is, the first $|E(G)|$ rows of $R_M(G, p)$ correspond to the bars of $(G, p)$ and the entries in row $e = \{i, j\}$ are zero except possibly in the column triples for $p_i$ and $p_j$, where the entries are the coordinates of $p_i - p_j$ and $p_j - p_i$ respectively. The final $|V(G)|$ rows of $R_M(G, p)$ (i.e. the rows of $\mathcal{N}(p)$) correspond to the joints of $(G, p)$ and the entries in the row for vertex $i$ are zero except in the columns for $i$ where the entries are the coordinates of a normal vector $N(p_i)$ to $M$ at $p_i$.

An infinitesimal motion of a framework $(G, p)$ on $M$ is called trivial if it lies in the kernel of $R_M(K_n, p)$, where $K_n$ is the complete graph on the vertex set of $G$. If every infinitesimal motion of $(G, p)$ is trivial, then $(G, p)$ is called infinitesimally rigid. Otherwise $(G, p)$ is called infinitesimally flexible.

Let $\mathbb{Q}(p)$ denote the field extension of $\mathbb{Q}$ formed by adjoining the coordinates of $p$. A framework $(G, p)$ on $M$ is said to be generic for $M$ if $\text{td}(\mathbb{Q}(p) : \mathbb{Q}) = 2|V(G)|$. This implies, [10 Corollary 3.2], that any polynomial $h(x)$ in $3|V(G)|$ variables that satisfies $h(p) = 0$ satisfies $h(q) = 0$ for all points $q \in \mathcal{M}^{3|V(G)|}$.

A framework $(G, p)$ supported on $M$ is called regular if $\text{rank} R_M(G, p) = \max\{\text{rank} R_M(G, q) : q \in \mathcal{M}^{3|V(G)|}\}$. If a framework on $M$ is generic, then it is clearly also regular. Moreover, if some realisation of a graph $G$ on $M$ is infinitesimally rigid, then the same is true for every regular (and hence every generic) realisation of $G$ on $M$. 


Theorem 2.2 (21). A regular framework \((G,p)\) on an algebraic surface \(M\) is infinitesimally rigid if and only if it is continuously rigid on \(M\).

Note that the complete graphs \(K_2\) and \(K_3\) provide curiosities when \(M\) is a cylinder in that they are continuously rigid but have non-trivial vectors in their nullspaces which are not tangential isometries. For graphs with \(|V(G)| \geq 6 - k\) on a surface of type \(M\) such worries disappear and both possible definitions of infinitesimal rigidity are equivalent.

Definition 2.3. A surface \(M\) is said to be of type \(k\) if \(\dim \ker R_M(K_n, p) \geq k\) for all complete graph frameworks \((K_n, p)\) on \(M\) and \(k\) is the largest such number.

In other words \(k\) is the dimension of the subgroup \(\Gamma\) of the Euclidean isometry group of \(\mathbb{R}^3\) consisting of isometries supported by \(M\).

Remark 2.4. Observe that every framework \((G, p)\) on the cylinder has the \(|V(G)|\) rows of \(N(p)\) linearly independent. The entries are \((x_i, y_i, 0)\) and at least one of each pair \(x_i, y_i\) is non-zero as \((x_i, y_i, z_i)\) lies on the cylinder so the block diagonal form guarantees independence. Indeed the same observation holds for almost all points on any reasonable surface \(M\). Hence, by [9, Lemma 3.4], a framework \((G, p)\) on \(M\) is regular if and only if \((G, p)\) on \(M\) is regular as a 3-dimensional framework.

A framework on a surface \(M\) is called isostatic if it is minimally infinitesimally rigid, that is, if it is infinitesimally rigid and the removal of any bar results in an infinitesimally flexible framework. The following three results concerning generic isostatic frameworks on surfaces were recently established in [21, 20].

Theorem 2.5. Let \((G, p)\) be an isostatic generic framework on the algebraic surface \(M\) of type \(k\), \(0 \leq k \leq 3\), with \(G\) not equal to \(K_1, K_2, K_3\) or \(K_4\). Then \(|E(G)| = 2|V(G)| - k\) and for every subgraph \(H\) of \(G\) with at least one edge, \(|E(H)| \leq 2|V(H)| - k\).

Lemma 2.6. Let \((G, p)\) be a generic framework on a surface \(M\) of type \(k\). Then \((G, p)\) is isostatic on \(M\) if and only if

\((1)\) \(\text{rank } R_M(G, p) = 3|V(G)| - k\) and
\((2)\) \(2|V(G)| - |E(G)| = k\).

A graph \(G\) is called (2, \(k\))-sparse if for every subgraph \(H\) of \(G\) we have \(|E(H)| \leq 2|V(H)| - k\). A (2, \(k\))-sparse graph satisfying \(|E(G)| = 2|V(G)| - k\) is called (2, \(k\))-tight.

Theorem 2.7. Let \(G = (V, E)\) be a simple graph, let \(M\) be an irreducible algebraic surface in \(\mathbb{R}^3\) of type \(k\) \(\in \{1, 2\}\) and let \((G, p)\) be a generic framework on \(M\). Then \((G, p)\) is isostatic on \(M\) if and only if \(G\) is \(K_1, K_2, K_3, K_4\) or is (2, \(k\))-tight.

We remark that for type \(k = 3\), it was shown in [25, 33] that Laman’s theorem (Theorem 1.1) applies to the sphere. It is an open problem to characterise generic isostatic frameworks on surfaces of type \(k = 0\). In particular, the natural analogue of Theorem 2.7 is known to be false. The graph formed from \(K_5\) by adding a degree 2 vertex gives an example of a (2, 0)-tight simple graph that is flexible on any such surface. Due to this complication, we will consider symmetric analogues of Laman’s theorem for surfaces of type \(k > 0\) only in this paper.

We finish this section by defining a stress for a framework on \(M\). Connelly introduced the notion of stress and stress matrix in his study of tensegrity frameworks [41] and this technique has been useful to some extent for the purposes of this paper (e.g. [6, 5, 38]). Very recently the analogous properties of stresses and stress matrices for frameworks on surfaces
have been developed [11]. We record the definition here as it will be useful for us in what follows.

Definition 2.8. A (self)-stress for \((G,p)\) on \(M\) is a pair \((\omega,\lambda)\) such that \((\omega,\lambda) \in \text{coker} R_M(G,p)\), that is, a vector \((\omega,\lambda)\) \(\in \mathbb{R}^{|E(G)|+|V(G)|}\) such that \((\omega,\lambda)^T R_M(G,p) = 0\).

Equivalently, \(\omega\) is a stress if for all \(1 \leq i \leq n\)
\[
\sum_{\{i,j\} \in E(G)} \omega_{ij} \|p_i - p_j\| + \lambda_i s_i = 0,
\]
where \(s_i\) is the triple occurring in \(R_M(G,p)\) in the row corresponding to \(p_i\) and columns corresponding to \(i\).

3. Symmetric graphs

In this section we review some basic properties of symmetric graphs. In particular, we introduce the notion of a ‘gain graph’ which is a useful tool to describe the underlying combinatorics of symmetric frameworks (see also [12, 24, 31], for example).

3.1. Quotient gain graphs. Given a group \(S\), an \(S\)-gain graph is a pair \((H,\psi)\), where \(H\) is a directed multi-graph (which may contain \(|S| - 1\) loops at each vertex and up to \(|S|\) multiple edges between any pair of vertices) and \(\psi : E(H) \to S\) is a map which assigns an element of \(S\) to each edge of \(H\). The map \(\psi\) is also called the gain function of \((H,\psi)\) (see Figures 1 (b) and (d) for examples of \(C_2\)-gain graphs). A gain graph is a directed graph, but its orientation is only used as a reference orientation, and may be changed, provided that we also modify \(\psi\) so that if an edge has gain \(x\) in one orientation, then it has gain \(x^{-1}\) in the other direction. Note that if \(S\) is a group of order 2, then the orientation is irrelevant. For simplicity, we omit the labels of edges with identity gain in the figures.

![Figure 1](image-url)  
**Figure 1.** \(C_2\)-symmetric graphs ((a), (c)) and their quotient gain graphs ((b), (d)), where \(C_2\) denotes half-turn symmetry.

Let \(G\) be a finite simple graph. An automorphism of \(G\) is a permutation \(\pi\) of the vertex set \(V(G)\) of \(G\) such that \(\{i,j\} \in E(G)\) if and only if \(\{\pi(i),\pi(j)\} \in E(G)\). The set of all automorphisms of \(G\) forms a group, called the automorphism group \(\text{Aut}(G)\) of \(G\). An action of a group \(S\) on \(G\) is a group homomorphism \(\theta : S \to \text{Aut}(G)\). If \(\theta(x)(i) \neq i\) for all \(i \in V(G)\) and all non-trivial elements \(x\) of the group \(S\), then the action \(\theta\) is called free. If \(S\) acts on \(G\) by \(\theta\), then we say that the graph \(G\) is \(S\)-symmetric (with respect to \(\theta\)). Throughout this paper, we only consider free actions, and we will omit to specify the action \(\theta\) if it is clear from the context. In that case, we write \(xv\) instead of \(\theta(x)(v)\).

For an \(S\)-symmetric graph \(G\), the quotient graph \(G/S\) is the multi-graph which has the set \(V(G)/S\) of vertex orbits as its vertex set and the set \(E(G)/S\) of edge orbits as its edge set. Note that an edge orbit may be represented by a loop in \(G/S\).
While several distinct graphs may have the same quotient graph, a gain labeling makes the relation one-to-one, provided that the underlying action is free. To see this, choose an arbitrary representative vertex \( i \) for each vertex orbit, so that each vertex orbit has the form \( S_i = \{ xi | x \in S \} \). If the action is free, an edge orbit connecting \( S_i \) and \( S_j \) can be written as \( \{ xi, xx'j | x \in S \} \) for a unique \( x' \) in \( S \). We then orient the edge orbit from \( S_i \) to \( S_j \) in \( G/S \) and assign it the gain \( x' \). This gives the quotient \( S \)-gain graph \(( G/S, \psi) \).

Conversely, let \(( H, \psi) \) be an \( S \)-gain graph. For \( x \in S \) and \( i \in V(H) \), we denote the pair \(( x, i) \) by \( xi \). The covering graph (or lifted graph) of \(( H, \psi) \) is the simple graph with vertex set \( S \times V(H) = \{ xi | x \in S, i \in V(H) \} \) and the edge set \( \{ \{ xi, x\psi(e)j \} | x \in S, e = (i, j) \in V(H) \} \).

Clearly, \( S \) acts freely on the covering graph with the action \( \theta \) defined by \( \theta(x) : i \mapsto xi \) for \( x \in S \) under which the quotient comes back to \(( H, \psi) \). So there is a one-to-one correspondence between \( S \)-gain graphs and \( S \)-symmetric graphs with free actions.

The map \( c : G \to H \) defined by \( c(xi) = i \) and \( c(\{ xi, x\psi(e)j \}) = (i, j) \) is called a covering map. The fiber \( c^{-1}(i) \) of a vertex \( i \in V(H) \) and the fiber \( c^{-1}(e) \) of an edge \( e \in E(H) \) coincide with a vertex orbit and edge orbit of \( G \), respectively.

### 3.2. Balanced gain graphs and the switching operation.

Let \(( H, \psi) \) be an \( S \)-gain graph, and let \( C = e_1, e_2, \ldots, e_k, e_1 \) be a cycle in \(( H, \psi) \), where \( e_i \in E(H) \) for all \( i \). We define the gain of \( C \) as \( \psi(C) = \psi(e_1) \cdot \psi(e_2) \cdot \ldots \cdot \psi(e_k) \) if each edge is oriented in the forward direction, and if an edge \( e_i \) is directed backwards, then we replace \( \psi(e_i) \) by \( \psi(e_i)^{-1} \) in the product. (If \( S \) is an additive group, then we replace the product by the sum.) The cycle \( C \) is called balanced if \( \psi(C) \) is equal to the identity element \( id \) in \( S \). Otherwise the cycle is called unbalanced. For any \( F \subseteq E(H) \), we say that \( F \) is unbalanced if \( F \) contains an unbalanced cycle. In that case, we also say that the subgraph of \(( H, \psi) \) which is induced by \( F \) is unbalanced.

If \( S \) is of order 2, then we will think of \( S \) as the group \( \mathbb{Z}_2 = \{ 0, 1 \} \) with addition as the group operation. So a subgraph of a \( \mathbb{Z}_2 \)-gain graph \(( H, \psi) \) will be unbalanced if and only if it contains a cycle with gain 1. Note that a \( \mathbb{Z}_2 \)-gain graph is commonly known as a signed graph in the literature [11] [40]. Let \( v \) be a vertex in an \( S \)-gain graph \(( H, \psi) \). To switch \( v \) with \( x \in S \) means to change the gain function \( \psi \) on \( E(H) \) as follows:

\[
\psi'(e) = \begin{cases} 
  x \cdot \psi(e) \cdot x^{-1} & \text{if } e \text{ is a loop incident with } v \\
  x \cdot \psi(e) & \text{if } e \text{ is a non-loop incident from } v \\
  \psi(e) \cdot x^{-1} & \text{if } e \text{ is a non-loop incident to } v \\
  \psi(e) & \text{otherwise}
\end{cases}
\]

In particular, if we switch a vertex \( v \) in a \( \mathbb{Z}_2 \)-gain graph \(( H, \psi) \) with 0, then the gain function \( \psi \) remains unchanged, and if we switch \( v \) with 1, then the gain of every non-loop edge that is incident with \( v \) changes its gain from 0 to 1 or vice versa, and the gains of all other edges remain the same.

We say that a gain function \( \psi' \) is equivalent to another gain function \( \psi \) on the same edge set if \( \psi' \) can be obtained from \( \psi \) by a sequence of switching operations.

In the following, we summarise some key properties of the switching operation. Detailed proofs of these results for an arbitrary discrete symmetry group \( S \) can be found in [12]. For the special case of signed graphs, these theorems were first proved by Zaslavsky in the 1980s [11].

**Proposition 3.1.** ([12] Prop. 2.2) Switching a vertex of an \( S \)-gain graph \(( H, \psi) \) does not alter the balance of \(( H, \psi) \).
Proposition 3.2. ([12] Prop. 2.3 and Lemma 2.4) An $S$-gain graph $(H, \psi)$ is balanced if and only if the vertices in $V(H)$ can be switched so that every edge in the resulting $S$-gain graph $(H, \psi')$ has the identity element of $S$ as its gain.

Lemma 3.3. ([12] Lemma 2.5) Let $(G, \psi)$ be an $S$-gain graph, and let $U \subseteq V(G)$ and $W \subseteq V(G)$ be subsets of $V(G)$. Further, let $H$ be the signed subgraph of $(G, \psi)$ induced by $U$, and let $K$ be the signed subgraph of $(G, \psi)$ induced by $W$, and suppose that $H$, $K$ and $H \cap K$ is connected. If $H$ and $K$ are balanced, then $H \cup K$ is also balanced.

4. Symmetric frameworks on surfaces

Let $M \subseteq \mathbb{R}^3$ be a surface, let $G$ be a finite simple graph, and let $p : V(G) \to M$. A symmetry operation of the framework $(G, p)$ on $M$ is an isometry $x$ of $\mathbb{R}^3$ which maps $M$ onto itself (i.e., $x$ is a symmetry of $M$) such that for some $\alpha_x \in \text{Aut}(G)$, we have

$$x(p_i) = p_{\alpha_x(i)} \quad \text{for all } i \in V(G).$$

The set of all symmetry operations of a framework $(G, p)$ on $M$ forms a group under composition, called the point group of $(G, p)$. Clearly, we may assume wlog that the point group of a framework is always a symmetry group, i.e., a subgroup of the orthogonal group $O(\mathbb{R}^d)$.

We use the Schoenflies notation for the symmetry operations and symmetry groups considered in this paper, as this is one of the standard notations in the literature about symmetric structures (see [3, 7, 14, 13, 28, 31], for example). The relevant groups in this paper are $\mathcal{C}_s$, $\mathcal{C}_m$, $\mathcal{C}_i$, $\mathcal{C}_{mv}$, $\mathcal{C}_{mh}$, $\mathcal{D}_m$, $\mathcal{D}_{mh}$, $\mathcal{D}_{md}$ and $S_{2m}$. $\mathcal{C}_s$ is a group of order 2 generated by a single reflection, $\mathcal{C}_m$, $m \geq 1$, is a cyclic group generated by a rotation $C_m$ about an axis through the origin by an angle of $\frac{2\pi}{m}$, $\mathcal{C}_i$ is the group generated by an inversion, $\mathcal{C}_{mv}$ is a dihedral group that is generated by a rotation $C_m$ and a reflection whose reflectional plane contains the rotational axis of $C_m$, and $\mathcal{C}_{mh}$ is generated by a rotation $C_m$ and the reflection whose reflectional plane is perpendicular to the axis of $C_m$. Further, $\mathcal{D}_m$ denotes a symmetry group that is generated by a rotation $C_m$ and another 2-fold rotation $C_2$ whose rotational axis is perpendicular to the one of $C_m$. $\mathcal{D}_{mh}$ and $\mathcal{D}_{md}$ are generated by the generators $C_m$ and $C_2$ of a group $\mathcal{D}_m$ and by a reflection $s$. In the case of $\mathcal{D}_{mh}$, the mirror of $s$ is the plane that is perpendicular to the $C_m$ axis and contains the origin (and hence contains the rotational axis of $C_2$), whereas in the case of $\mathcal{D}_{md}$, the mirror of $s$ is a plane that contains the $C_m$ axis and forms an angle of $\frac{\pi}{m}$ with the $C_2$ axis. Finally, $S_{2m}$ is a symmetry group which is generated by a 2$m$-fold improper rotation (i.e., a rotation by $\frac{\pi}{m}$ followed by a reflection in the plane which is perpendicular to the rotational axis).

Given a surface $M$, a symmetry group $S$ and a graph $G$, we let $\mathcal{R}^M_{(G,S)}$ denote the set of all realisations of $G$ on $M$ whose point group is either equal to $S$ or contains $S$ as a subgroup [27] [26]. In other words, the set $\mathcal{R}^M_{(G,S)}$ consists of all realisations $(G, p)$ of $G$ in $\mathbb{R}^3$ which are supported on $M$ and for which there exists an action $\theta : S \to \text{Aut}(G)$ so that

$$x(p(i)) = p(\theta(x)(i)) \quad \text{for all } i \in V(G) \text{ and all } x \in S.$$

A framework $(G, p) \in \mathcal{R}^M_{(G,S)}$ satisfying the equations in (1.1) for the map $\theta : S \to \text{Aut}(G)$ is said to be of type $\theta$, and the set of all realisations in $\mathcal{R}^M_{(G,S)}$ which are of type $\theta$ is denoted by $\mathcal{R}^M_{(G,S,\theta)}$ (see again [27] [26] and Figure 2). It is shown in [27] that if $p$ is injective, then $(G, p)$ is of a unique type $\theta$ and $\theta$ is necessarily also a homomorphism.

Let $S$ be an abstract group, and $G$ be a $S$-symmetric graph with respect to a free action $\theta : S \to \text{Aut}(G)$. Suppose also that $S$ acts on $\mathbb{R}^d$ via a homomorphism $\tau : S \to O(\mathbb{R}^d)$. 


Figure 2. Realisations of the cycle graph $C_4$ in $\mathcal{R}^M_{(C_4,\mathcal{E}_s)}$ of different types, where $M$ is the Euclidean plane and $\mathcal{E}_s = \{id, s\}$ is the reflection group. The framework in (a) is of type $\theta_a$, where $\theta_a : \mathcal{E}_s \to \text{Aut}(C_4)$ is the homomorphism defined by $\theta_a(s) = (14)(23)$, and the framework in (b) is of type $\theta_b$, where $\theta_b : \mathcal{E}_s \to \text{Aut}(C_4)$ is the homomorphism defined by $\theta_b(s) = (13)(24)$.

Then we say that a framework $(G,p)$ on a surface $M$ is $S$-symmetric (with respect to $\theta$ and $\tau$) if $(G,p) \in \mathcal{R}^M_{(G,\tau(S),\theta)}$, that is, if

$$\tau(x)(p(i)) = p(\theta(x)i) \quad \text{for all } x \in S \text{ and all } i \in V(G).$$

Recall that the type $k$ of a surface $M$ is the dimension of the group of isometries of $\mathbb{R}^3$ acting tangentially to $M$. Relevant to our purposes will be the subgroup of symmetric isometries and we define the symmetric type $k_S$ of $M$ to be the dimension of this subgroup.

For simplicity, we will assume throughout this paper that a framework $(G,p) \in \mathcal{R}^M_{(G,S,\theta)}$ has no joint that is ‘fixed’ by a non-trivial symmetry operation in $S$ (i.e., $(G,p)$ has no joint $p_i$ with $x(p_i) = p_i$ for some $x \in S$, $x \neq id$). In particular, this will simplify the construction of the orbit-surface rigidity matrix in the next section, since in this case this matrix has a set of 3 columns for each orbit of vertices under the action $\theta$.

Let $Q_S$ denote the field extension of $\mathbb{Q}$ formed by adjoining the entries of all the matrices in $S$ to $\mathbb{Q}$. We say that a framework $(G,p)$ in $\mathcal{R}^M_{(G,S,\theta)}$ with quotient $S$-gain graph $(G_0,\psi)$.
is $S$-generic if $\text{td} [Q_S(p) : Q_S] = 2|V(G_0)|$. This implies that the only polynomial equations in $3|V(G)|$ variables that evaluate to zero at $p$ are those that define $S$ or $M$. This is the natural extension of the definitions of generic seen in the literature [21, 10, 12].

4.1. Symmetry-forced rigidity and the orbit-surface rigidity matrix. Given an $S$-symmetric framework $(G, p)$ on a surface $M$, we are interested in non-trivial motions of $(G, p)$ on $M$ which preserve the symmetry group $S$ of $(G, p)$ throughout the path. Infinitesimal motions corresponding to such symmetry-preserving continuous motions are ‘$S$-symmetric infinitesimal motions’ (see also [12, 31, 32]):

An infinitesimal motion $u$ of a framework $(G, p)$ in $\mathbb{R}^M_{(G, S, \theta)}$ is $S$-symmetric if

$$x(u_i) = u_\theta(x(i))$$

for all $i \in V(G)$ and all $x \in S$,

i.e., if $u$ is unchanged under all symmetry operations in $S$. Note that all the velocity vectors $u_i$, considered as acting at the framework joints, are of course tangential to the surface $M$.

We say that $(G, p) \in \mathcal{R}^M_{(G, S, \theta)}$ is $S$-symmetric infinitesimally rigid if every $S$-symmetric infinitesimal motion is trivial. Note that the dimension of the space of trivial $S$-symmetric infinitesimal motions, $k_S$, can easily be read off from the character table for $S$ (see [3], for example).

A self-stress $(\omega, \lambda) \in \mathbb{R}^{|E(G)|+|V(G)|}$ of $(G, p)$ is $S$-symmetric if $\omega_e = \omega_f$ whenever $e$ and $f$ belong to the same edge orbit $Se = \{xe | x \in S\}$ of $G$, and $\lambda_i = \lambda_j$ whenever $i$ and $j$ belong to the same vertex orbit $Si = \{xi | x \in S\}$ of $G$.

![Figure 4](image_url)  
Figure 4. Infinitesimal motions of frameworks in the Euclidean plane: (a) a $C_2$-symmetric non-trivial infinitesimal motion; (b) a $C_s$-symmetric trivial infinitesimal motion; (c) a non-trivial infinitesimal motion which is not $C_s$-symmetric.

In Euclidean space, a key tool to study symmetric infinitesimal motions is the orbit rigidity matrix. This matrix is defined as follows (see also [32]):

**Definition 4.1.** Let $(G, p)$ be an $S$-symmetric framework (with respect to $\theta$ and $\tau$) in Euclidean 3-space which has no joint that is ‘fixed’ by a non-trivial symmetry operation in $S$. Further, let $(G_0, \psi)$ be the quotient $S$-gain graph of $(G, p)$. For each edge $e \in E(G_0)$, the orbit rigidity matrix $O(G, p, S)$ of $(G, p)$ has the following corresponding $(3|V(G_0)|$-dimensional) row vector:

**Case 1:** Suppose $e = (i, j)$, where $i \neq j$. Then the corresponding row in $O(G, p, S)$ is:

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 0 \cdots 0 & (p_i - \tau(\psi(e))(p_j)) \\
0 \cdots 0 & (p_j - \tau(\psi(e))^{-1}(p_i))\end{pmatrix} \begin{pmatrix} 0 \cdots 0 \end{pmatrix}.$$

Case 2: Suppose \( e = (i, i) \) is a loop in \((G_0, \psi)\). Then \( \psi(e) \neq \text{id} \) and the corresponding row in \( O(G, p, S) \) is:

\[
\begin{pmatrix}
0 \\
\vdots \\
0 \\
(2p_i - \tau(\psi(e))(p_i) - \tau(\psi(e))^{-1}(p_i)) \\
0 \ldots 0
\end{pmatrix}.
\]

Using the above definition of the orbit rigidity matrix for frameworks in Euclidean 3-space, we can easily set up the orbit-surface rigidity matrix as follows:

Definition 4.2. Let \((G, p)\) be a framework in \( \mathcal{R}^M_{(G, S, \theta)} \) with quotient \( S\)-gain graph \((G_0, \psi)\).

The orbit-surface rigidity matrix \( O_M(G, p, S) \) of \((G, p)\) is the \((|E(G_0)| + |V(G_0)|) \times 3|V(G_0)| \) block matrix

\[
\begin{bmatrix}
O(G, p, S) \\
\mathcal{N}_0(p_0)
\end{bmatrix}
\]

where \( O(G, p, S) \) is the standard orbit rigidity matrix for the framework and symmetry group considered in \( \mathbb{R}^3 \) (see Definition 4.1) and \( \mathcal{N}_0(p_0) \) represents the surface normals to the framework joints corresponding to the vertices of \( G_0 \).

A framework \((G, p) \in \mathcal{R}^M_{(G, S, \theta)}\) is \( S\)-regular if \( O_M(G, p, S) \) has maximal rank among all realisations in \( \mathcal{R}^M_{(G, S, \theta)} \). By Remark 2.4 we see that an \( S\)-symmetric framework on \( M \) is \( S\)-regular if and only if the framework is \( S\)-regular in Euclidean 3-space. That is, to check whether \((G, p) \in \mathcal{R}^M_{(G, S, \theta)}\) is \( S\)-regular, we need only check that \( O(G, p, S) \) has maximal rank. Note that if a framework on \( M \) is \( S\)-generic, then it is clearly also \( S\)-regular, and if some \( S\)-symmetric realisation of a graph \( G \) is \( S\)-symmetric infinitesimally rigid, then the same is true for every \( S\)-regular realisation of \( G \).

An \( S\)-gain graph \((G_0, \psi)\) is \( S\)-independent if \( O_M(G, p, S) \) has linearly independent rows and \( S\)-dependent otherwise. Clearly, if \((G, p)\) is \( S\)-isostatic (i.e., minimally \( S\)-symmetric infinitesimally rigid) then \((G_0, \psi)\) is \( S\)-independent. The following lemma is an easy exercise.

Lemma 4.3. Let \( M \) be a surface with symmetric type \( k_S \) with respect to a symmetry group \( S \). Let \( \mathcal{N}_M(G, p, S) \) be the nullspace of \( O_M(G, p, S) \). Then \( \dim \mathcal{N}_M(G, p, S) \geq k_S \).

The following result summarises the key properties of the orbit-surface rigidity matrix:

Theorem 4.4. Let \((G, p)\) be a framework in \( \mathcal{R}^M_{(G, S, \theta)} \). Then the solutions to \( O_M(G, p, S)u = 0 \) are isomorphic to the space of \( S\)-symmetric infinitesimal motions of \((G, p)\). Moreover, the solutions to \((\omega, \lambda)^T O_M(G, p, S) = 0 \) are isomorphic to the space of \( S\)-symmetric self-stresses of \((G, p)\).

Proof. This follows immediately from the corresponding result for the orbit rigidity matrix in Euclidean 3-space [32, Theorem 6.1 and Theorem 8.3].

For example, it was shown in [32, Theorem 6.1] that a vector \( u \) lies in the kernel of the orbit rigidity matrix \( O(G, p, S) \) if and only if \( u \) is the restriction of an \( S\)-symmetric infinitesimal motion of \((G, p)\) to the joints corresponding to the vertices of the quotient \( S\)-gain graph of \( G \). Therefore, a vector \( u \) lies in the kernel of the orbit-surface rigidity matrix \( O_M(G, p, S) \) if and only if \( u \) is in the kernel of \( O(G, p, S) \) and the velocity vectors of \( u \) are tangential to the surface \( M \). Moreover this holds if and only if \( u \) is the restriction of an \( S\)-symmetric infinitesimal motion of \((G, p)\) on \( M \) to the joints corresponding to the vertices of the quotient \( S\)-gain graph of \( G \).

Similarly, the proof of [32, Theorem 8.3] can easily be adapted to show that the solutions to \((\omega, \lambda)^T O_M(G, p, S) = 0 \) are isomorphic to the space of \( S\)-symmetric self-stresses of \((G, p)\).
An $S$-symmetric framework $(G,p)$ is $S$-symmetric rigid if every $S$-symmetric continuous motion is a rigid motion of $M$.

It is an easy extension of [2] (see also [21] [28]) to show that for $S$-regular realisations, a symmetric infinitesimal motion implies a continuous symmetry preserving motion. Thus we have the following.

**Theorem 4.5.** Let $(G,p)$ be an $S$-regular framework in $\mathcal{R}^M_{(G,S,\theta)}$. Then $(G,p)$ is $S$-symmetric infinitesimally rigid on $\mathcal{M}$ if and only if $(G,p)$ is $S$-symmetric rigid on $\mathcal{M}$.

### 5. Necessary conditions for symmetry-forced rigidity

We now establish analogues of Maxwell’s theorem, showing exact combinatorial counts that must be satisfied by any $S$-generic $S$-isostatic framework in $\mathcal{R}^M_{(G,S,\theta)}$. We need the following definition (see also [12] [31]).

**Definition 5.1.** Let $(H,\psi)$ be an $S$-gain graph and let $k,\ell,m$ be nonnegative integers with $m \leq \ell$. Then $(H,\psi)$ is called $(k,\ell,m)$-gain-sparse if

- $|F| \leq k|V(F)| - \ell$ for any nonempty balanced $F \subseteq E(H)$;
- $|F| \leq k|V(F)| - m$ for any nonempty $F \subseteq E(H)$.

A $(k,\ell,m)$-gain-sparse graph $(H,\psi)$ satisfying $|F| = k|V(F)| - m$ is called $(k,\ell,m)$-gain-tight. Similarly, an edge set $E$ is called $(k,\ell,m)$-gain-sparse ($(k,\ell,m)$-gain-tight) if it induces a $(k,\ell,m)$-gain-sparse ($(k,\ell,m)$-gain-tight) graph.

We first establish a necessary condition for a framework in $\mathcal{R}^M_{(G,S,\theta)}$ to be $S$-isostatic, where $S$ is a cyclic group.

**Theorem 5.2.** Let $\mathcal{M}$ be an irreducible algebraic surface of type $k$. Let $S$ be a cyclic symmetry group of $\mathbb{R}^3$ acting on $\mathcal{M}$ such that under $S$, $\mathcal{M}$ has type $k_S$. Let $(G,p)$ be an $S$-generic $S$-isostatic framework in $\mathcal{R}^M_{(G,S,\theta)}$ with quotient $S$-gain graph $(G_0,\psi)$. Then $(G_0,\psi)$ is $(2,k,k_S)$-gain-tight.

**Proof.** First observe that if there exists an unbalanced subgraph $(H,\psi)$ of $(G_0,\psi)$ with $|E(H)| > 2|V(H)| - k_S$, then Lemma [4.3] implies that there is a row dependence in the orbit-surface matrix for that subgraph. Similarly, we must have $|E(G_0)| = 2|V(G_0)| - k_S$. So it remains to check that if $F \subseteq E(G_0)$ is balanced, then $|F| \leq 2|V(F)| - k$.

**Claim 5.3.** Switching a vertex does not change the rank of the orbit-surface matrix.

**Proof of Claim 5.3** By the first step in the proof of [12] Lemma 5.2 switching does not effect the rank of the 3-dimensional orbit matrix. Since $p$ is $S$-regular the claim follows using the argument in Remark 2.4. 

Now suppose there exists a balanced edge set $F \subseteq E(G_0)$ with $|F| > 2|V(F)| - k$. Then Claim 5.3 allows us to switch the vertices of the graph induced by $F$ so that every edge gain in this subgraph is the identity element of $S$. The submatrix of $O_M(G,p,S)$ consisting of all those rows which correspond to the edges and vertices of the subgraph induced by $F$ is a standard surface rigidity matrix. Since $|F| > 2|V(F)| - k$, it follows from Lemma 2.6 that this matrix has a row dependence, a contradiction.

We remark that the sparsity condition for unbalanced subgraphs is simpler than the reader may have anticipated. This is because of the restriction to cyclic groups. In general we can take greater care to deal with the different possible ‘subgroups induced by edge...
sets’, which are defined as follows (see also [12, 31]), and hence derive stronger necessary conditions.

Recall from Section 3.2 that for a cycle \( C \) of the form \( e_1, e_2, \ldots, e_k, e_1 \) in an \( S \)-gain graph \((H, \psi)\), the gain \( \psi(C) \) of \( C \) is defined as \( \psi(C) = \prod_{i=1}^{k} \psi(e_i)^{\text{sign}(e_i)} \). For \( F \subseteq E(H) \), we define \( \langle F \rangle \) to be the subgroup of \( S \) generated by the elements in the set
\[
\{ \psi(C) \mid \text{C is a cycle in the subgraph induced by} \ F \}.
\]
In particular, note that \( F \) is balanced if and only if \( \langle F \rangle \) is the trivial group.

So if \( M \) is the unit sphere and \( S \) is a dihedral group \( \mathcal{D}_m \), for example, then it is possible for some subset of edges \( F \) of the \( \mathcal{D}_m \)-gain graph, the group \( \langle F \rangle \) is neither trivial nor the entire group \( \mathcal{D}_m \), but the cyclic subgroup \( \mathcal{C}_m \) of \( \mathcal{D}_m \). In that case, we need to adjust the number \( k_{\mathcal{D}_m} = 0 \) to \( k_{\langle F \rangle} = 1 \) in the sparsity count for \( F \), where \( k_{\langle F \rangle} \) is the dimension of the space of isometries of \( \mathbb{R}^3 \) which act tangentially on the unit sphere and are symmetric with respect to the group \( \langle F \rangle = \mathcal{C}_m \). With this in mind, the following is proved similarly to Theorem 5.2 (see, also, [12] Lemma 5.2).

**Theorem 5.4.** Let \( M \) be an irreducible algebraic surface of type \( k \). Let \( S \) be a non-cyclic symmetry group of \( \mathbb{R}^3 \) acting on \( M \) such that \( M \) has type \( k_S \) under \( S \). Let \((G, p)\) be an \( S \)-generic \( S \)-isostatic framework in \( R^M_{(G,S,\theta)} \) with quotient \( S \)-gain graph \((G_0, \psi)\). Then \((G_0, \psi)\) satisfies

\[
\begin{align*}
(1) \quad |E(G_0)| &= 2|V(G_0)| - k_S, \\
(2) \quad |F| &\leq 2|V(F)| - k_{\langle F \rangle} \quad \text{for all} \ F \subseteq E(G_0).
\end{align*}
\]

6. **Combinatorial characterisations of generic rigidity**

In the rest of this paper we will consider the more substantial problem of proving these counts are sufficient to guarantee that a symmetric framework supported on a surface is symmetry-forced isostatic. We will focus on three classical surfaces in 3-space, namely the sphere, the cylinder, and the cone. That is

- the unit sphere \( S \) centered at the origin, defined by the equation \( x^2 + y^2 + z^2 = 1 \);
- the unit cylinder \( y = S^1 \times \mathbb{R} \) about the \( z \)-axis, defined by the equation \( x^2 + y^2 = 1 \);
- the unit cone \( C \) about the \( z \)-axis, defined by the equation \( x^2 + y^2 = z^2 \);

6.1. **The Sphere.** Although it has not previously been stated, by combining results of [33] and [12], the following theorem is immediate.

**Theorem 6.1 (Rotation, reflection or dihedral symmetry on the sphere).** Let \( S \) be the group \( \mathcal{C}_m \) representing \( m \)-fold rotational symmetry or the group \( \mathcal{C}_s \) representing reflectional symmetry about a plane through the origin. Let \((G, p)\) be a framework in \( R^S_{(G,S,\theta)} \) with quotient \( S \)-gain graph \((G_0, \psi)\). Then \((G, p)\) is \( S \)-isostatic if and only if \((G_0, \psi)\) is \((2, 3, 1)\)-gain-tight.

Moreover, if \( S \) is a dihedral group \( \mathcal{C}_{mv} \), where \( m \) is odd, then \((G, p)\) is \( S \)-isostatic if and only if \( G_0 \) is ‘maximum \( \mathcal{D} \)-tight’ (as defined in [12] Def. 7.1]).

**Proof.** Let \( S \) be one of the groups listed above. We may think of an \( S \)-symmetric framework \((G, p)\) supported on the unit sphere as the ‘coned framework’ \((G * 0, p^*)\), where \((G * 0, p^*)\) is the framework obtained from \((G, p)\) by adding a new joint \( p_0 \) at the origin (i.e., the centre of the sphere) which is linked to every joint of \((G, p)\) by a bar. We may now invert vertex orbits (under the symmetry group \( S \) of \((G * 0, p^*)\)) so that we obtain a framework on the upper half-sphere, and then project (gnomonically) the resulting framework from the origin to the plane \( z = 1 \). Note that this yields a framework \((G, q)\) in the plane which
also has symmetry $S$. Moreover, as shown in \cite{32}, the $S$-symmetric infinitesimal rigidity properties of $(G * 0, p^*)$ and $(G, q)$ are the same. Therefore, the result follows directly from \cite{12} Theorems 6.3 and 8.2 and \cite{33} Theorems 3.7 and 6.2.

We note that the proof techniques we employ below can easily be adapted to give direct inductive proofs of these results (see also \cite{31}). We leave the details to the reader.

Note that for dihedral groups of the form $C_{mv}$, where $m$ is even, there does not exist a combinatorial characterisation of symmetry-generic symmetry-forced rigid frameworks in the plane. For example, it was shown in \cite{12} that Bottema’s mechanism (a realisation of inversion symmetry on the sphere) is falsely predicted to be $C_{2v}$-symmetric infinitesimally rigid by the matroidal counts for the orbit rigidity matrix. Thus, the corresponding results for the sphere also remain open.

In general, for any point group $S$ of a framework on the sphere, except for the groups $C_m$, $C_s$, $C_i$, $C_{mh}$ and $S_{2m}$ there are no tangential isometries (i.e., no rotations) which are $S$-symmetric. Thus, for those groups, we need to cope with the $(2, 3, 0)$-gain-sparsity count to establish characterisations for symmetry-forced rigidity on the sphere. This is a significant obstacle, as it was recently observed that this gain-sparsity count is in general not matroidal \cite{12}.

For the group $C_i$, we will prove the following theorem in the subsequent sections.

**Theorem 6.2** (Inversion symmetry on the sphere). Let $S$ be the group $C_i$ representing inversion symmetry. Let $(G, p)$ be a framework in $\mathcal{R}_{(G,S,\theta)}^S$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2, 3, 3)$-gain-tight.

For the groups $C_{mh}$, where $m$ is odd, and $S_{2m}$, where $m$ is even, we will prove the following theorem in the subsequent sections.

**Theorem 6.3** (Dihedral and improper rotational symmetry on the sphere). Let $S$ be the group $C_{mh}$, where $m$ is odd, or $S_{2m}$, where $m$ is even. Let $(G, p)$ be a framework in $\mathcal{R}_{(G,S,\theta)}^S$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2, 3, 1)$-gain-tight.

For the remaining groups $C_{mh}$, where $m$ is even, and $S_{2m}$, where $m$ is odd, we will provide the corresponding conjectures in Section \cite{10}. Note that these groups contain the group $C_i$ as a subgroup (whereas $C_{mh}$, where $m$ is odd, and $S_{2m}$, where $m$ is even, do not). Thus, we need to consider gain-sparsity counts which depend on the groups $(F)$ induced by edge subsets $F$ of the gain graph $G_0$, since there is only a 1-dimensional space of rotations which is symmetric with respect to the subgroups $C_s$, $C_m$, $C_{mh}$, or $S_{2m}$, but there is a 3-dimensional space of rotations which is symmetric with respect to the inversion subgroup $C_i$.

6.2. The cylinder. We now consider the case of the surface being a cylinder. Note that the cylinder has point group symmetry $\mathcal{D}_{\infty h}$, and hence the possible point groups of frameworks on the cylinder are $C_s$, $C_m$ (with the rotational axis being the axis of the cylinder for $m > 2$), $C_i$, $C_{mv}$, $C_{mh}$, $D_m$, $D_{mh}$, $D_{md}$ and $S_{2m}$ (recall Section \cite{1}).

We will prove the following two main theorems for the cylinder in the subsequent sections.

**Theorem 6.4** (Rotation symmetry on the cylinder). Let $S$ be the symmetry group $C_m$ representing $m$-fold rotational symmetry around the z-axis. Let $(G, p)$ be a framework in $\mathcal{R}_{(G,S,\theta)}^S$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2, 2, 2)$-gain-tight.
Theorem 6.5 (Reflection or inversion symmetry on the cylinder). Let $S$ be the group $\mathcal{E}_s$ (where the mirror plane of the reflection either contains the $z$-axis or is equal to the plane $z = 0$) or the inversion group $\mathcal{E}_i$. Let $(G, p)$ be a framework in $\mathcal{R}^{\mathcal{E}_s}_{(G, S, \theta)}$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $G_0$ is $(2, 2, 1)$-gain-tight.

We will discuss the remaining groups and provide some conjectures in Section 10.

6.3. The cone. Note that the cone defined by the polynomial $x^2 + y^2 = z^2$ has the same point group symmetry $D_{\infty h}$ as the cylinder, and hence the possible point groups of frameworks on the cone are the same as the ones for the cylinder.

We will prove the following main theorem for the cone in the subsequent sections.

Theorem 6.6 (Reflection, rotation, inversion, dihedral or improper rotational symmetry on the cone). Let $S$ be the group $\mathcal{E}_m$ representing $m$-fold rotation around the $z$-axis or the group $\mathcal{E}_s$, where the mirror plane of the reflection is perpendicular to the $z$-axis, or the inversion group $\mathcal{E}_i$, or the dihedral group $\mathcal{E}_{mh}$, or the improper rotational group $S_{2m}$. Let $(G, p)$ be a framework in $\mathcal{R}^{\mathcal{E}_m}_{(G, S, \theta)}$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2, 1, 1)$-gain-tight.

Again we will return to the remaining groups in Section 10.

We finish this section by observing a corollary to Theorem 6.5 which points out that for certain types of graphs, groups and surfaces, generic rigidity (without symmetry) is equivalent to symmetry-generic symmetry-forced rigidity.

Remark 6.7. If $S$ is a group of order 2, then an $S$-symmetric graph $G$, where the action $\theta : S \to \text{Aut}(G)$ is free on both $V(G)$ and $E(G)$, is $(2, 2)$-tight if and only if its quotient $S$-gain graph $G_0$ is $(2, 2, 1)$-gain-tight. To see this, note that an unbalanced subset $F$ of edges of $G_0$ satisfies $|F| = 2|V(F)| - 1$ if and only if $|c^{-1}(F)| = 2|c^{-1}(V(F))| - 2$, where $c : G \to G_0$ is the covering map. Thus, combining Theorem 2.7 with Theorem 6.6, it follows that a generic realisation of $G$ on the cylinder $\mathcal{Y}$ is isostatic if and only if an $S$-generic realisation of $G$ on $\mathcal{Y}$ is $S$-isostatic, where $S = \mathcal{E}_s$ or $\mathcal{E}_i$ (see also Conjecture 10.9).

7. Inductive Constructions

To prove Theorems 6.2, 6.6 we will make use of a celebrated proof technique in rigidity theory: an inductive proof using Henneberg-type graph constructions [12, 20, 22, 24, 31, 59]. This comes in two steps. First we prove a characterisation of $(2, k, k_S)$-gain-tight graphs, for the relevant choices of $k$ and $k_S$, showing that all such graphs can be generated from the smallest such graph by simple operations. Then we apply these operations to frameworks and show that they preserve the dimension of the nullspace of the orbit-surface rigidity matrix.

7.1. Admissible Operations. Let $(H, \psi)$ be an $S$-gain graph. The Henneberg 1 move is the addition of a new vertex $v$ and two edges $e_1$ and $e_2$ to $H$ such that the new edges are incident with $v$ and are not both loops at $v$. The three possible ways this can be done are illustrated in Figure 5. Note that if $e_1$ and $e_2$ are not loops and not parallel edges, then their labels can be arbitrary. This move is called $\text{H1a}$ and is depicted in Figure 5(a). If they are non-loops, but parallel edges, then the labels are assigned so that $\psi(e_1) \neq \psi(e_2)$, assuming that $e_1$ and $e_2$ are directed to $v$. This move is called $\text{H1b}$ and is depicted in Figure 5(b). Finally, if one of the edges, say $e_1$, is a loop at $v$, then we set $\psi(e_1) \neq \text{id}$. This move is called $\text{H1c}$ and is depicted in Figure 5(c).
The Henneberg 2 move deletes an edge of \((H, \psi)\) and adds a new vertex and three new edges to \((H, \psi)\). First, one chooses an edge \(e\) of \(H\) (which will be deleted) and a vertex \(z\) of \(H\) which may be an end-vertex of \(e\). Then one subdivides \(e\), with a new vertex \(v\) and new edges \(e_1\) and \(e_2\), such that the tail of \(e_1\) is the tail of \(e\) and the tail of \(e_2\) is the head of \(e\). The gains of the new edges are assigned so that \(\psi(e_1) \cdot \psi(e_2) = \psi(e)\). Finally, we add a third new edge, \(e_3\), to \(H\). This edge is oriented from \(z\) to \(v\) and its gain is such that every 2-cycle \(e_i e_j\), if it exists, is unbalanced. There are four possible ways this can be done, as illustrated in Figure 6.

Suppose first that the edge \(e\) is not a loop. If none of the edges \(e_i\) is a loop or a parallel edge, then the move is called H2a (see Figure 6(a)). If none of the edges \(e_i\) is a loop, but exactly two of the edges are parallel edges (i.e., \(z\) is an end-vertex of \(e\)), then the move is called H2b (see Figure 6(b)). If the edge \(e\) is a loop, then the moves corresponding to H2a and H2b are called H2c and H2d, respectively (see Figures 6(c) and (d)).

For a \((2, k, k_S)\)-gain-tight \(S\)-gain graph \((H, \psi)\), an inverse Henneberg 1 or 2 move on \(v \in V(G_0)\) is admissible if the resulting \(S\)-gain graph \(H'\) is \((2, k, k_S)\)-gain-tight and the covering graph of \(H'\) is simple.

A vertex-to-\(K_4\) operation on an \(S\)-gain graph \((H, \psi)\) removes a vertex \(v\) (of arbitrary degree) and all the edges incident with \(v\), and adds in a copy of \(K_4\) with only trivial gains (see also Figure 7). Without loss of generality we may assume that all edges incident with \(v\) are directed to \(v\), i.e., are of the form \((x, v)\) for some \(x \in V(H)\). Each removed edge \((x, v)\) is replaced by an edge \((x, y)\) for some \(y\) in the new \(K_4\), where the gain is preserved, that is, \(\psi((x, v)) = \psi((x, y))\). If the deleted vertex \(v\) is incident to a loop \((v, v)\) in \(H\) (this may be the case if \(H\) is \((2, k, k_S)\)-gain tight for \(k_S = 1\), for example), then this loop is replaced by...
an edge \((y, z)\) with the same gain as \((v, v)\), where \(y\) and \(z\) are two (not necessarily distinct) vertices of \(K_4\).

For a \((2, k, k_S)\)-gain-tight \(S\)-gain graph \((H, \psi)\), the inverse move, a \(K_4\)-contraction on a copy of \(K_4\) with only trivial gains is admissible if the resulting graph \(H'\) is \((2, k, k_S)\)-gain-tight and the covering graph is simple. Note that we only apply the \(K_4\)-contraction when there are no additional edges induced by the vertices of the \(K_4\).

**Figure 7.** The vertex-to-\(K_4\) operation (in this case expanding a degree 4 vertex which is not incident to any loop). Directions and gain labels of the edges are omitted.

In the following, an edge \(e = (x, v)\) with gain \(\psi(e)\) in \((H, \psi)\) will be denoted by \((x, v)_{\psi(e)}\). A vertex-to-4-cycle operation on an \(S\)-gain graph \((H, \psi)\) removes a vertex \(v\) and all the edges incident with \(v\), adds in two new vertices \(v_1, v_2\), and chooses two neighbours \(a, b\) of \(v\) (wlog with edges \((a, v)_{\alpha}\) and \((b, v)_{\beta}\)) and creates a 4-cycle \(a, v_1, b, v_2\) with edges \((a, v_1)_{\alpha}, (a, v_2)_{\alpha}, (b, v_1)_{\beta}, (b, v_2)_{\beta}\) (see also Figure 8). Each of the removed edges \((x, v)\), \(x \neq a, b\), is replaced by an edge \((x, v_i)\) for some \(i = 1, 2\), where the gain is preserved, that is, \(\psi((x, v)) = \psi((x, v_i))\). If the deleted vertex \(v\) is incident to a loop \((v, v)\) in \(H\) (this may be the case if \(H\) is \((2, k, k_S)\)-gain tight for \(k_S = 1\), for example), then this loop is replaced by a loop \((v_i, v_i)\) (with the same gain) for some \(i = 1, 2\).

The inverse operation, a 4-cycle contraction, on a \((2, k, k_S)\)-gain-tight \(S\)-gain graph is admissible if the resulting graph \(H'\) is \((2, k, k_S)\)-gain-tight and the covering graph of \(H'\) is simple. It will suffice for our purposes to contract only 4-cycles in which each edge has trivial gain. Thus, we may restrict to \(\alpha\) and \(\beta\) above both equaling the identity, so it is easy to see that if \(H\) is \((2, k, k_S)\)-gain-tight, then applying a vertex-to-4-cycle operation to \(H\) results in a \((2, k, k_S)\)-gain-tight graph.

**Figure 8.** The vertex-to-4-cycle operation. Directions and gain labels of the edges are omitted.

An edge joining operation takes two \((2, k, 1)\)-gain-tight \(S\)-gain graphs \(H_0, H_1\) and creates the graph \(H_0 \oplus H_1\) which has vertex set the disjoint union of \(V(H_0)\) and \(V(H_1)\) and edge set \(E(H_0) \cup E(H_1) \cup \{(a, b)\}\) where \(a \in V(H_0)\) and \(b \in V(H_1)\) is an edge with arbitrary gain. It is clear that \(H_0 \oplus H_1\) is \((2, k, 1)\)-gain-tight if and only if \(H_0\) and \(H_1\) are \((2, k, 1)\)-gain-tight.
Note that in the covering graph, each of the above operations is a graph operation that preserves the underlying symmetry. Some of them can be recognised as performing standard - non-symmetric - Henneberg operations \[39\] simultaneously.

7.2. Recursive Characterisations. We now derive inductive constructions for \((2, i, i)\)-gain-tight graphs, where \(i = 1, 2, 3\), and for \((2, 2, 1)\)-gain-tight graphs. Inductive constructions of \((2, 3, 1)\)-gain tight graphs were established in \[12\] Theorem 4.4] (using Henneberg 1 and Henneberg 2 moves only). Note that as the balanced and unbalanced subgraph conditions are the same for \((2, i, i)\)-gain-tight graphs, \(i = 1, 2, 3\), we do not need to worry about preserving cycles with non-trivial gain in the first three theorems.

For a vertex \(v\) of a directed multi-graph \(G_0\), we will denote the set of vertices which are adjacent to \(v\) (in the underlying undirected multi-graph) by \(N(v)\). Each of the vertices in \(N(v)\) is called a neighbour of \(v\) in \(G_0\).

In the following 4 theorems it is easy to establish that the operations under consideration preserve \((2,\ell,m)\)-gain-sparsity. We concentrate on the converse where in all 4 cases it is clear that the minimum degree is 2 or 3.

**Theorem 7.1.** Let \(S\) be a group of order 2, \(G\) be a simple graph and \((G_0, \psi)\) be its quotient \(S\)-gain graph. Then \((G_0, \psi)\) is \((2,3,3)\)-gain-tight if and only if \((G_0, \psi)\) can be constructed sequentially from \(K_2\) by \(H1a, H1b, H2a, H2b\), and \(H2b\) operations.

**Proof.** Note that if \(G_0\) is \((2,3,3)\)-gain-tight, then it cannot contain any loop. Suppose \(G_0\) is a counterexample to the theorem. Then it is easy to see that there does not exist a vertex \(v \in V(G_0)\) of degree 2. Since \(G_0\) is \((2,3,3)\)-gain-tight, every vertex of degree 3 must have three distinct neighbors. But now, standard arguments can be used to show that an inverse \(H2a\) operation (performed on one of the degree 3 vertices and one pair of its neighbors) preserves \((2,3,3)\)-gain-tightness \[15\] \[29\].

**Theorem 7.2.** Let \(S\) be a cyclic group, \(G\) be a simple graph and \((G_0, \psi)\) be its quotient \(S\)-gain graph. Then \((G_0, \psi)\) is \((2,2,2)\)-gain-tight if and only if \((G_0, \psi)\) can be constructed sequentially from \(K_1\) by \(H1a, H1b, H2a, H2b\), vertex-to-\(K_4\) and vertex-to-4-cycle operations.

**Proof.** Note that if \(G_0\) is \((2,2,2)\)-gain-tight, then it cannot contain any loop. Standard arguments (see, for example, \[19\] \[24\] \[35\]) show that the Henneberg operations listed in the statement above suffice to construct \(G_0\) from \(K_1\), unless all vertices of \(G_0\) of degree less than 4 are of degree 3, each of these has 3 distinct neighbours, and each pair of these has an edge with gain equal to the gain we must add during an inverse Henneberg 2 operation. Thus, if \(G_0\) is a counterexample to the theorem, then there exists a vertex \(v \in V(G_0)\) contained in a copy of \(K_4\). This \(K_4\) has a gain labeling which is equal to or, by Proposition 3.1, is equivalent to the trivial gain labelling. Denote this copy of \(K_4\) as \(K\).

\(K\) is admissible for a \(K_4\)-to-vertex contraction unless there is a vertex \(x \notin K\) and edges \((x,a),(x,b)\) for \(a,b \in K\) with equal gains. Since the final vertex \(c \in K\) is not adjacent to \(x\), simple counting \[20\] shows there is a 4-cycle contraction merging \(v\) and \(x\) which results in a \((2,2,2)\)-gain-tight graph contrary to our assumption. \(\square\)

**Theorem 7.3.** Let \(S\) be a cyclic group, \(G\) be a simple graph, and \((G_0, \psi)\) be its quotient \(S\)-gain graph. Then \((G_0, \psi)\) is \((2,1,1)\)-gain-tight if and only if \((G_0, \psi)\) can be constructed sequentially from a single vertex with an unbalanced loop by \(H1a, H1b, H1c, H2a, H2b, H2c, H2d\), vertex-to-\(K_4\), vertex-to-4-cycle and edge joining operations.

In the proof we use the terminology \(2G\) to refer to the graph with vertex set \(V(G)\) and 2 copies of each edge in \(E(G)\). Of course we will be referring to gain graphs, so
some appropriate gain assignment is assumed. It will also be convenient to define \( f(G) := 2|V(G)| - |E(G)| \).

**Proof.** Note that \( G_0 \) may contain loops. Using standard arguments, it is easy to see that if \( G_0 \) is a counterexample to the theorem, then there exists a vertex \( v \in V(G_0) \) of degree 3 and either \( v \) has exactly two neighbours and is contained in \( 2K_3 - e \) or \( v \) is contained in a copy of \( K_4 \) and this copy has a gain labelling which is equal to or, by Proposition \([3.1]\) is equivalent to the trivial gain labelling. Denote this copy as \( K \).

\( K \) (or the subgraph induced by the vertices of \( K \)) admits an admissible \( K_4 \)-to-vertex contraction unless there is a vertex \( x \notin K \) and edges \((x,a),(x,b)\) for \( a,b \in K \) with equal gains. In such case there is a 4-cycle contraction merging \( v \) and \( x \) which results in a \((2,1,1)\)-gain-tight graph unless for the final vertex \( c \in K \) the edge \((x,c) \in E(G_0)\) with \( \psi((x,c)) = id \). (Note that if \( \psi((x,c)) = id \), then \( \psi((x,c)) = \psi((v,c)) = id \) so that a 4-cycle-contraction would not result in a simple covering graph.)

The graph induced by \( v,a,b,c \) and \( x \) is a copy of \( K_5 - e \). Repeat the whole process for every degree 3 vertex. We see that either \( K \) is induced and we have a copy of \( K_5 - e \) or \( K \) was not induced and we have a copy of \( 2K_3 - e \).

We now argue as in \([19]\) Lemma 4.10. Let \( Y = \{Y_1,\ldots,Y_n\} \) be the subgraphs which are copies of \( K_5 - e \) or \( 2K_3 - e \). They are necessarily vertex disjoint since \( f(Y_i \cup Y_j) = 2 - f(Y_i \cap Y_j) \) and every proper subgraph \( X \) of \( K_5 \) has \( f(X) \geq 2 \). Let \( V_0 \) and \( E_0 \) be the sets of vertices and edges of \( G_0 \) which are in none of the \( Y_i \). Then

\[
f(G_0) = \sum_{i=1}^{n} f(Y_i) + 2|V_0| - |E_0|
\]

so \( |E_0| = 2|V_0| + n - 1 \). Each vertex in \( V_0 \) is incident to at least 4 edges. If every \( Y_i \) is incident to at least 2 edges in \( E_0 \), then there are at least \( 4|V_0| + 2n \) edge/vertex incidences in \( E_0 \). This implies \( |E_0| \geq 2|V_0| + n \), a contradiction. Thus, either there is a copy \( Y_i \) with no incidences, which would imply \( G_0 = Y_i \), since \( G_0 \) is connected, contrary to our assumption, or there is a copy with one incidence, i.e., \( G_0 \) contains a bridge and there is an edge separation move on this bridge contrary to our assumption. \( \square \)

For the following we have to be more careful to preserve the gain-sparsity of subgraphs.

**Theorem 7.4.** Let \( S \) be a group of order 2, \( G \) be a simple graph and \((G_0,\psi)\) be its quotient \( S \)-gain graph. Then \((G_0,\psi)\) is \((2,2,1)\)-gain-tight if and only if \((G_0,\psi)\) can be constructed sequentially from a single vertex with an unbalanced loop by \( H_1a, H_1b, H_1c, H_2a, H_2b, H_2c, H_2d, vertex-to-K_4, vertex-to-4-cycle \) and edge joining operations.

**Proof.** We will think of \( S \) as the group \( Z_2 = \{0,1\} \) with addition as the group operation. Suppose \( G_0 \) is a counterexample to the theorem. Note that \( S \) has order 2, so \( v \) has at least two neighbours. First, suppose \( N(v) = \{a,b\} \).

**Claim 7.5.** \( v \) is contained in a copy of \( 2K_3 - e \) (with arbitrary, simplicity of the covering graph preserving gain labels).

**Proof of Claim 7.5** Suppose that \( v \) is not contained in a copy of \( 2K_3 - e \). It suffices to check the case when \( a \) and \( b \) are not joined by an edge in \( G_0 \). Suppose there are distinct subgraphs \( H_1, H_2 \) of \( G_0 - v \) with \( a,b \in V(H_i) \), \( f(H_i) = 2 \) for \( i = 1,2 \) and all paths in \( H_i \) from \( a \) to \( b \) have gain \( \alpha_i \) where \( \alpha_1 \neq \alpha_2 \). (The gain of a path in a gain graph is defined analogously to the gain of a cycle (recall Section \([3.2]\)).) Then \( H_1 \cap H_2 \) is connected since \( f(H_1 \cap H_2) = 2 \), which implies that all paths from \( a \) to \( b \) in \( H_1 \cap H_2 \) have 2 distinct gains,
serve that there must exist balanced subgraphs $H$. However, we must have $c / \not c$ and $(b, c)$ gives a graph $H$ is balanced. Using Proposition 3.1 we may assume that the additional edges in $H$ have 0 gains, and hence we find that $a$ contradiction. Thus, there is a choice of gain for the edge $(a, b)$ so that the corresponding inverse H2b move is admissible. 

Now let $N(v) = \{a, b, c\}$.

**Claim 7.6.** $v$ is contained in a copy of $K_4$ in which every edge has gain 0.

**Proof of Claim 7.6.** We will show that if $v$ is not contained in a copy of $K_4$ with gain 0 on every edge, then there is an admissible H2a move.

Let $\alpha_{av}, \alpha_{bv}, \alpha_{cv}$ be the gains on the edges $(a, v), (b, v), (c, v)$. Suppose first that $(a, b)_{\alpha_{av} + \alpha_{bv}}$ and $(b, c)_{\alpha_{bv} + \alpha_{cv}} \notin E(G_0)$ $((a, c)_{\alpha_{av} + \alpha_{cv}}$ may or may not be in $E(G_0))$. See Figure 9. Observe that there must exist balanced subgraphs $H_{ab}$ of $G_0 - v$ such that $a, b \in V(H_{ab})$, $c \notin V(H_{ab})$ and $f(H_{ab}) = 2$ and $H_{bc}$ such that $b, c \in V(H_{bc})$, $a \notin V(H_{bc})$ and $f(H_{bc}) = 2$. However, we must have $f(H_{ab} \cup H_{bc}) = 2$, thus adding $v$ and its 3 incident edges to $H_{ab} \cup H_{bc}$ gives a graph $H^*$ with $f(H^*) = 1$. $H_{ab} \cap H_{bc}$ is connected, so Lemma 3.3 implies $H_{ab} \cup H_{bc}$ is balanced. Using Proposition 3.1 we may assume that the 3 additional edges in $H^*$ all have 0 gains, and hence we find that $H^*$ is balanced. Since $H^*$ is a subgraph of $G_0$ this gives a contradiction.

![Figure 9](image_url)

**Figure 9.** The first two cases in the proof of Claim 7.6. Directions and labels omitted.

Now suppose that $(a, b)_{\alpha_{av} + \alpha_{bv}} \notin E(G_0)$ but $(a, c)_{\alpha_{av} + \alpha_{cv}}$ and $(b, c)_{\alpha_{bv} + \alpha_{cv}} \in E(G_0)$. As $S$ has order 2, we use Proposition 3.1 to show that we may assume the gains of $(v, a), (v, b), (v, c), (a, c), (b, c)$ are all 0. See Figure 9. Now if $H_{ab}$ is a subgraph of $G_0 - v$ containing $a, b$ but not $c$, then $f(H_{ab}) = 2$. We are done unless equality holds and $H_{ab}$ is balanced. Now consider paths from $a$ to $b$ in $H_{ab}$. If all such paths have 0 gain then all cycles in $H_{ab} \cup v \cup c$ (along with the relevant edges) have gain 0, but $f(H_{ab} \cup v \cup c) = 1$. Thus, there must be a path from $a$ to $b$ with gain 1 and adding $(a, b)$ with gain 0 during the inverse H2a move gives an unbalanced subgraph, as required.

Finally, suppose $(a, b)_{\alpha_{av} + \alpha_{bv}}, (a, c)_{\alpha_{av} + \alpha_{cv}}$ and $(b, c)_{\alpha_{bv} + \alpha_{cv}} \in E(G_0)$. By Proposition 3.1 we may switch vertices to make a copy of $K_4$ with gain 0 on each edge. □

Let $K$ denote the copy of $K_4$ containing $v$. As in the previous proof, we suppose first that $v$ belongs to a $K_4$ whose vertices induce no additional edge. Since we cannot apply a $K_4$-contraction, there are vertices $a, b \in K$ and a vertex $x \notin K$ such that $(a, x)_{\alpha}, (b, x)_{\alpha} \in E(G_0)$. To see this, note that any unbalanced subgraph containing $K$ is still unbalanced as a subgraph of the contracted graph. Now let $c$ be the final vertex in $K$. 

Claim 7.7. \((c, x) \in E(G_0)\).

Proof of Claim 7.7. By applying Proposition 8.1 to \(x\) (if necessary) we may assume that the 4-cycle \(C\) induced by \(v, a, b, x\) has label 0 on each edge. Apply 4-cycle contraction to \(C\) merging \(v\) and \(x\). It is routine, as in the previous proofs, to check that the counts hold for unbalanced subgraphs. The sparsity conditions for balanced subgraphs hold since every edge of \(C\) and \((K)\) has gain 0. \qed

We have shown that the subgraph induced by \(K\) and \(x\) is \((2, 1)\)-tight. The same is true, for \(K\), when the vertices of \(K\) induce an additional edge. Finally, we may show, exactly as in the proof of the previous theorem, that \(G_0\) contains a bridge. \qed

We will briefly discuss extensions to \((k, l, m)\)-gain-tight graphs for other triples in the final section.

8. Operations on Frameworks Supported on Surfaces

We now consider the geometric question of how these inductive operations behave as operations on frameworks rather than on graphs. We pursue this by mostly linear algebraic techniques using the orbit-surface rigidity matrix and making extensive use of arguments from \([21, 20]\) and \([38]\).

8.1. Henneberg moves. Our first lemma is simple linear algebra.

Lemma 8.1. Let \(\mathcal{M} \in \{S, Y, \mathcal{E}\}\) and let \(S\) be any possible point group. Let \((G, p)\) be an \(S\)-regular \(S\)-isostatic framework in \(\mathbb{R}^{\mathcal{M}}_{(G, S, \theta)}\) with quotient \(S\)-gain graph \((G_0, \psi)\). Let \((G_0', \psi')\) be formed from \((G_0, \psi)\) by a Henneberg 1 move and let \(G'\) be the corresponding covering graph. Then any \(S\)-regular realisation of \(G'\) is \(S\)-isostatic.

Proof. Let \((G', p')\) be an \(S\)-regular realisation of \(G'\). Then simply note that by the block structure of \(O_M(G', p', S)\) and the regularity of \((G', p')\), we have

\[
\text{rank } O_M(G', p', S) = \text{rank } O_M(G, p, S) + 3.
\]

\(\Box\)

Our second lemma is more involved but by utilising the proof technique of \([20]\) Lemma 4.2 we can still argue for any group and surface simultaneously.

Lemma 8.2. Let \(\mathcal{M} \in \{S, Y, \mathcal{E}\}\) and let \(S\) be any possible point group. Let \((G, p)\) be an \(S\)-generic \(S\)-isostatic framework in \(\mathbb{R}^{\mathcal{M}}_{(G, S, \theta)}\) with quotient \(S\)-gain graph \((G_0, \psi)\). Let \((G_0', \psi')\) be formed from \((G_0, \psi)\) by a Henneberg 2 move and let \(G'\) be the corresponding covering graph. Then any \(S\)-generic realisation of \(G'\) is \(S\)-isostatic.

Proof. Let \(S = \{x_1 = \text{id}, x_2, \ldots, x_{|S|}\}\). For an edge \(e_0 = (1, 2)\) (with gain \(\alpha \in S\)) in \(G_0\) we apply the argument in \([20]\) Lemma 4.2 simultaneously to each edge in the edge orbit \(e^{-1}(e_0)\), where \(c : G \to G_0\) is the covering map. (Note that if \(e_0\) is a loop, then the proof follows analogously.) Suppose \(V(G_0') = V(G_0) \cup \{0\}\) and let \(p' = (p_{x_1(0)}, p_{x_2(0)}, \ldots, p_{x_{|S|}(0)}, p)\), where \((G', p')\) is \(S\)-generic. Suppose that \((G', p')\) is not \(S\)-symmetric infinitesimally rigid. Then it follows that every specialised framework in \(\mathbb{R}^{\mathcal{M}}_{(G', S, \theta)}\) is \(S\)-symmetric infinitesimally flexible. Consider a sequence of specialisations \((G', p^k)\) in which only the joints \(p_{x_1(0)}, \ldots, p_{x_{|S|}(0)}\) are specialised to the joints \(p^k_{x_1(0)}, \ldots, p^k_{x_{|S|}(0)}\), respectively, and for each \(i = 1, \ldots, |S|\), \(p^k_{x_i(0)}\) tends to \(p_{x_i(0)}(2)\) in the direction \(x_i(a)\) where \(x_i(a)\) is a tangent vector at \(p_{x_i(0)}(2)\) which is orthogonal to a tangent vector \(x_i(b)\) at \(p_{x_i(0)}(2)\) where \(x_i(b)\) is orthogonal to \(p_{x_i(0)}(2) - p_{x_i(1)}\).
See Figure 10. More precisely, the normalised vector \((p_{x_i \alpha(2)} - p_{x_i(0)})/\|p_{x_i \alpha(2)} - p_{x_i(0)}\|\) converges to \(x_i(a)\), as \(k \to \infty\).

![Diagram](image.png)

**Figure 10.** An illustration of the specialisation in the case \(S = \mathcal{C}_3\).

Each of the \(S\)-symmetric frameworks \((G', p^k)\) has a unit norm non-trivial \(S\)-symmetric infinitesimal flex \(u^k\) which is orthogonal to the space of \(S\)-symmetric trivial infinitesimal motions of its framework, \((G', p^k)\). By the Bolzano-Weierstrass theorem there is a subsequence of the sequence \(u^k\) which converges to a vector, \(u^\infty\), say, of unit norm. Discarding framework points and relabeling we may assume this holds for the original sequence. The \(S\)-symmetric limit motion of the degenerate \(S\)-symmetric framework \((G', p^\infty)\) is denoted by \(u^\infty = (u^\infty_{x_1(0)}, u^\infty_{x_2(0)}, \ldots, u^\infty_{x_3(0)}, u)\). Also, we have \(p^\infty = (p_{x_1 \alpha(2)}, p_{x_2 \alpha(2)}, \ldots, p_{x_3 \alpha(2)}, p)\).

We claim that the velocities \(u_{x_1(1)}, u_{x_2 \alpha(2)}\) give an \(S\)-symmetric infinitesimal motion of the framework on \(\mathcal{M}\) consisting of the bars joining \(p_{x_i(1)}\) and \(p_{x_i \alpha(2)}, i = 1, \ldots, |S|\). To see this note that in view of the well-behaved convergence of \(p^k_{x_i(0)}\) to \(p_{x_i \alpha(2)}\) (in the \(x_i(a)\) direction) it follows that the velocities \(u_{x_i \alpha(2)}\) and \(u^\infty_{x_i(0)}\) have the same component in the \(x_i(a)\) direction, and so \((u_{x_i \alpha(2)} - u^\infty_{x_i(0)}) \cdot x_i(a) = 0\). Since \(u_{x_i \alpha(2)} - u^\infty_{x_i(0)}\) is tangential to \(\mathcal{M}\) it follows from the choice of \(x_i(a)\) that \(u_{x_i \alpha(2)} - u^\infty_{x_i(0)}\) is orthogonal to \(p_{x_i \alpha(2)} - p_{x_i(1)}\). On the other hand \(u_{x_i(1)} - u^\infty_{x_i(0)}\) is orthogonal to \(p_{x_i \alpha(2)} - p_{x_i(1)}\) and so taking differences \(u_{x_i \alpha(2)} - u_{x_i(1)}\) is orthogonal to \(p_{x_i \alpha(2)} - p_{x_i(1)}\), as desired.

It now follows, by the symmetry-forced rigidity of \((G, p)\) and (hence) the symmetry-forced rigidity of the degenerate framework \((G', p^\infty)\), that the restriction motion \(u^\infty_{\text{res}} = u\), and hence \(u^\infty\) itself, is a trivial \(S\)-symmetric infinitesimal motion. This is a contradiction since the motion has unit norm and is orthogonal to the space of trivial \(S\)-symmetric infinitesimal motions.

**8.2. Vertex Surgery moves.** The following proof depends on there being at least one \(S\)-symmetric trivial infinitesimal motion.
Lemma 8.3. Let \( M \in \{ \mathcal{Y}, \mathcal{E} \} \) and

- if \( M = \mathcal{Y} \) let \( S \) be \( \mathcal{C}_m \) (with the \( z \)-axis as the rotational axis), \( \mathcal{C}_s \) (with mirror orthogonal to the \( z \)-axis or a plane containing the \( z \)-axis) or \( \mathcal{G}_4 \), and
- if \( M = \mathcal{E} \) let \( S \) be \( \mathcal{C}_m \) (with the \( z \)-axis as the rotational axis), \( \mathcal{C}_s \) (with mirror orthogonal to the \( z \)-axis), \( \mathcal{C}_t \), \( \mathcal{G}_{mh} \) (with the \( z \)-axis as the rotational axis) or \( S_{2m} \) (with the \( z \)-axis as the rotational axis).

Let \( (G, p) \) be an \( S \)-generic \( S \)-isostatic framework in \( \mathcal{R}^M_{(G, S, \vartheta)} \) with quotient \( S \)-gain graph \( (G_0, \psi) \) and let \( (G'_0, \psi') \) be formed from \( (G_0, \psi) \) by a vertex-to-\( K_4 \) move. Then any \( S \)-generic realisation of the covering graph \( G' \) of \( (G'_0, \psi') \) is \( S \)-isostatic.

Proof. First let \( M = \mathcal{Y} \). The proof is similar to [21] Lemma 5.2 with \( K_4 \) replacing \( H \). Let \( n = |V(G_0)| \). Let \( v_s \) be a fixed vertex of \( K_4 \). Consider the orbit-surface rigidity matrix \( O_y(G', p', S) \) with column triples in the order of \( v_1, v_2, v_3, v_4, v_5, \ldots, v_n \) where \( v_1, v_2, v_3, v_4 = v_s \) are the vertices of \( K_4 \). Order the rows of \( O_y(G', p', S) \) in the order of the edges \( e_1, \ldots, e_6 \) for \( K_4 \) followed by the \( n \) rows of the block diagonal matrix whose diagonal entries are the respective normal vectors to \( M \) at \( p_1, \ldots, p_n \), followed by the remaining rows for the edges of \( E(G'_0) \setminus E(K_4) \). Note that the submatrix formed by the first 10 rows is the 1 by 2 block matrix

\[
\begin{bmatrix}
O_y(c^{-1}(K_4), p', S) & 0 \\
X & X
\end{bmatrix}
\]

where \( c : G' \rightarrow G'_0 \) is the covering map. Suppose, by way of contradiction, that \( G'_0 \) is not \( S \)-isostatic. Since \( 2n - |E(G'_0)| = 2 \) there is a vector \( u \) in the kernel of \( O_y(G', p', S) \) which corresponds to an \( S \)-symmetric non-trivial infinitesimal motion.

Claim 8.4. We have \( u_{K_4} = 0 \).

Proof. In the following, we will identify an \( S \)-symmetric infinitesimal motion of a framework \( (G, p) \) with its restriction to the vertices of the quotient \( S \)-gain graph \( G_0 \) of \( G \).

First, suppose \( S = \mathcal{C}_m \). By adding to \( u \) some trivial \( S \)-symmetric infinitesimal motion, we may assume that \( u_4 = 0 \). Write \( u = (u_{K_4}, u_{G'_0 \setminus K_4}) \) where \( u_{K_4} = (u_1^4, u_2^4, u_3^4, u_4^4) \). The matrix \( O_y(G', p', S) \) has the block form

\[
O_y(G', p', S) = \begin{bmatrix}
O_y(c^{-1}(K_4), p', S) & 0 \\
X & X
\end{bmatrix}
\]

where \( X = [X_1 \ 0 \ 0 \ 0] \) is the matrix formed by the last \( |E(G')| - 6 + n - 4 \) rows. Since \( K_4 \) is \( S \)-isostatic on \( \mathcal{Y} \) and \( O_y(c^{-1}(K_4), p', S)u_{K_4} = 0 \) it follows that \( u_{K_4} \) is an \( S \)-symmetric trivial infinitesimal motion. But \( u_4 = 0 \) and so \( u_{K_4} = 0 \).

Now suppose \( S = \mathcal{C}_s \). There are two sub-cases. If \( S \) is generated by a reflection whose mirror is orthogonal to the \( z \)-axis then we may assume that \( u_4 = (0, 0, u_4^4) \) (since in this case the space of \( S \)-symmetric trivial infinitesimal motions is the one-dimensional space of infinitesimal rotations about the \( z \)-axis). However, \( u_{K_4} \) is a rigid motion, so we must have \( u_{K_4} = (u_1^4, u_2^4, 0, \ldots, u_4^4, u_4^4, 0) \), and hence as above \( u_{K_4} = 0 \). Similarly, if \( S \) is generated by a reflection through a plane containing the \( z \)-axis, then we may assume that \( u_4 = (u_1^4, u_2^4, 0) \) (since in this case the space of \( S \)-symmetric trivial infinitesimal motions is the one-dimensional space of infinitesimal translations along the \( z \)-axis), and \( u_{K_4} \) is a rigid motion, so \( u_{K_4} = (0, 0, u_1^4, \ldots, 0, 0, u_4^4) \), and by the above \( u_{K_4} = 0 \).

The final case, when \( S = \mathcal{G}_4 \), is similar. \( \square \)

Consider now the framework vector \( \tilde{p} = (p_4, p_4, p_4, p_4, p_5, \ldots, p_n) \) in which the first 4 framework joints are specialised to \( p_4 \) and let \( p_s = (p_4, p_5, \ldots, p_n) \) be the restricted vector.
with associated $S$-generic framework $(G, p_s)$. By the hypotheses, this framework is $S$-symmetric infinitesimally rigid.

The matrix $X_2 = X_2(p')$ is square with nonzero vector $u_{G\setminus K_4}$ in the kernel and so the determinant as a polynomial in the coordinates of the $p_i'$ vanishes identically. It follows that $\det X_2(p)$ vanishes identically and that there is a nonzero vector, $w_{G'_0\setminus K_4}$ say, in the kernel. But now we obtain the contradiction
\[
\left[ O_2(G, p_s, S) \right] \left[ \begin{array}{cc} \mathbf{0} \\
\mathbf{0} \end{array} \right] = \left[ \begin{array}{cc} a & \mathbf{0} \\
\mathbf{0} \end{array} \right] \left[ \begin{array}{c} \mathbf{0} \\
w_{G'_0\setminus K_4} \end{array} \right] = \mathbf{0}
\]
where $a = (0, 1, 0)$.

The case $M = \emptyset$ follows by the same argument with minor changes to the proof of Claim [2.4] for each group $S$ and with $a = (0, y, -y)$.

**Lemma 8.5.** Let $M \in \{\emptyset, \emptyset\}$ and
\begin{itemize}
  \item if $M = \emptyset$, let $S$ be $C_2$ (with the $z$-axis as the rotational axis), $C_s$ (with mirror orthogonal to the $z$-axis or a plane containing the $z$-axis) or $C_1$ and
  \item if $M = \emptyset$, let $S$ be $C_2$ (with the $z$-axis as the rotational axis), $C_s$ (with mirror orthogonal to the $z$-axis), $C_1$, $C_{mh}$ (with the $z$-axis as the rotational axis) or $S_{2m}$ (with the $z$-axis as the rotational axis).
\end{itemize}

Let $(G, p)$ be an $S$-generic $S$-isostatic framework in $\mathcal{R}_{(G, S, \theta)}$ with quotient $S$-gain graph $(G_0, \psi)$ and let $(G'_0, \psi')$ be formed from $(G_0, \psi)$ by a vertex-to-$4$-cycle move. Then any $S$-generic realisation of the covering graph $G'$ of $(G'_0, \psi')$ is $S$-isostatic.

The proof is similar to [3.5] Proposition 1 with minor modifications due to the different definition of a stress in the surface context (recall Definition 2.8) and to using the orbit-surface rigidity matrix rather than the rigidity matrix of a $3$-frame.

**Proof.** Choose a vertex $v_1$ of $G_0$ to be split and suppose $v_2, v_3, \ldots, v_k$ are the neighbors of $v_1$. Without loss of generality, we may assume that all edges joining $v_1$ and $v_i$, $i = 1, \ldots, k$, are directed away from $v_1$ and that the edge $(v_1, v_i)$ has gain $\alpha_i$. (If the edge $(v_1, v_i)$ appears $l > 1$ times, then we denote the corresponding edge gains by $\alpha_i = \alpha_{i1}, \ldots, \alpha_{il}$, for a fixed numbering of the edges; in this case, the gains $\alpha_{i1}, \ldots, \alpha_{il}$ are of course all distinct.) Let $G'_0$ be the $S$-gain graph obtained from $G_0$ by adding a new vertex $v_0$ and two edges $(v_0, v_2)$ and $(v_0, v_3)$ with respective gains $\alpha_2$ and $\alpha_3$. The covering graph of $G'_0$ is denoted by $G'$.

Further, we let $(G^*, p^*)$ be the framework obtained from $(G, p)$ by setting $p^*(v_0) = p(v_1)$ and $p^*(v_i) = p(v_i)$ for all other vertices $v_i, i \neq 0$, of $G^*$.

Consider the $S$-gain graph $G'_0$ which is obtained from $G^*_0$ by swapping some number of edges $(v_1, v_j)$ for $j \in \{4, 5, \ldots\}$ to edges $(v_0, v_j)$ (keeping the same gains). (If there are multiple edges joining $v_1$ with $v_2$ and $v_3$, then we may also swap edges of the form $(v_1, v_2)$ and $(v_1, v_3)$ to edges $(v_0, v_2)$ and $(v_0, v_3)$, provided that their gains are not equal to $\alpha_2$ or $\alpha_3$.) Let $G'$ be the covering graph of $G'_0$.

Let $\omega, \lambda$ be an $S$-symmetric self-stress on $(G', p^*)$. In the following, we will identify an $S$-symmetric self-stress of a framework with its restriction to the edges and vertices of the corresponding quotient $S$-gain graph. Let $\omega_{ij} = \omega((v_i, \alpha_j(v_j)))$ for $i = 0, 1$ and $j = 2, 3$, and let $\lambda_i = \lambda(v_i)$. Then note that if we set $\tilde{\omega}_{12} = \omega_{02} + \omega_{12}, \tilde{\omega}_{13} = \omega_{03} + \omega_{13}, \tilde{\lambda}_1 = \lambda_0 + \lambda_1$, and $\tilde{\omega}(e) = \omega(e)$ for all other edges $e$ and $\tilde{\lambda}_i = \lambda_i$ for all other vertices $v_i$, then $(\tilde{\omega}, \tilde{\lambda})$ is an $S$-symmetric self-stress of $(G, p)$. Thus, since $(G, p)$ is $S$-isostatic, we must have $\tilde{\omega}_{12} = 0$ for each $j$ and $\tilde{\lambda}_i = 0$ for each $i$.

It follows that around $p_1$ we have $\omega_{12}(p_1 - p_{\alpha_{2}}(2)) + \omega_{13}(p_1 - p_{\alpha_{3}}(3)) + \lambda_1 a = 0$. Since $(G, p)$ is $S$-generic, and $p_1 - p_{\alpha_{2}}(2), p_1 - p_{\alpha_{3}}(3)$ and the normal $a$ to $M$ at $p_1$ are not coplanar,
we have \( \omega_{12} = \omega_{13} = \lambda_1 = 0 \). Similarly, we deduce that \( \omega_{02} = \omega_{03} = \lambda_0 = 0 \). Thus, the rows of \( O_M(G', p', S) \) are linearly independent. Now we perturb \((G', p')\) within a neighbourhood \( B(p', \epsilon) \cap M \), for sufficiently small \( \epsilon \) to find an \( S \)-generic position \((G', p')\) which is guaranteed to be \( S \)-isostatic since \((G', p')\) is.

\[ \square \]

**Lemma 8.6.** Let \( M \in \{y, \ell\} \) and

- if \( M = y \) let \( S \) be \( \mathcal{C}_m \) (with the z-axis as the rotational axis), \( \mathcal{C}_s \) (with mirror orthogonal to the z-axis or a plane containing the z-axis) or \( \mathcal{C}_t \) and
- if \( M = \ell \) let \( S \) be \( \mathcal{C}_m \) (with the z-axis as the rotational axis), \( \mathcal{C}_s \) (with mirror orthogonal to the z-axis), \( \mathcal{C}_t \), \( \mathcal{C}_{mh} \) (with the z-axis as the rotational axis) or \( S_{2m} \) (with the z-axis as the rotational axis).

Let \((G_1, p_1)\) and \((G_2, p_2)\) be two \( S \)-generic \( S \)-isostatic frameworks in \( \mathcal{R}_{(G,S,\theta)}^3 \) with quotient \( S \)-gain graphs \((G_1, \psi_1)\) and \((G_2, \psi_2)\). Then an \( S \)-generic framework \((G_1 \oplus G_2, p)\) corresponding to the edge join of \( G_0\) and \( G_2\) with joining edge given arbitrary gain is \( S \)-isostatic on \( M \).

**Proof.** The orbit-surface matrices \( O_M(G_1, p_1, S) \) and \( O_M(G_2, p_2, S) \) have maximal rank. Hence the block matrix

\[
\begin{pmatrix}
O_M(G_1, p_1, S) & 0 \\
0 & O_M(G_2, p_2, S)
\end{pmatrix}
\]

has a 2-dimensional nullspace. The final joining edge, \( S \)-generically, eliminates the additional \( S \)-symmetric infinitesimal motion.

\[ \square \]

Note that the final lemma clearly fails for settings where there are more or less than one isometry. This is in contrast to the vertex-to-\( K_4 \) move where we expect it to be applicable to any group/surface context despite requiring a symmetric isometry for our proof technique.

## 9. Laman type theorems

We have now put together enough results to prove our main theorems. For convenience let us say that, for an \( S \)-gain graph \( G_0 \), \( V(G_0) = \{1, \ldots, n\} \) and \( p_i = (x_i, y_i, z_i) \) for each \( i \).

**Proof of Theorem 6.2.** Theorem 5.2 proves the necessity.

For the sufficiency we use induction and Theorem 4.1. Let the edge of \( K_2 \) be \( e = (1, 2) \) with gain \( \alpha \in \mathcal{C}_t \), and let \( p_{\alpha(i)} = (x'_i, y'_i, z'_i) \), \( i = 1, 2 \). Then note that \( O_S(K_2, p, \mathcal{C}_t) \) is the 3 × 6 matrix

\[
\begin{pmatrix}
x_{1} - x'_{2} & y_{1} - y'_{2} & z_{1} - z'_{2} & x_{2} - x'_{1} & y_{2} - y'_{1} & z_{2} - z'_{1} \\
x_{1} & y_{1} & z_{1} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{2} & y_{2} & z_{2}
\end{pmatrix}
\]

which is easily checked to have rank 3. Theorem 4.1 gives us a short list of operations that generate all \((2, 3, 3)\)-gain-tight graphs. For the inductive step suppose \((G, p)\) is \( \mathcal{C}_t \)-isostatic and suppose \( G' \) is formed from \( G \) by any one of these operations. Then Lemmas 8.1 and 8.2 confirm that any \( \mathcal{C}_t \)-generic realisation of \( G' \) is \( \mathcal{C}_t \)-isostatic.

\[ \square \]

**Proof of Theorem 6.3.** Theorem 5.2 proves the necessity.

For the sufficiency we use induction and [12, Theorem 4.4]. Let \( S \in \{\mathcal{C}_{mh}, S_{2m}\} \) and let \( K_4^i \) denote a loop at vertex 1 with non-trivial gain \( \alpha \). Further, let \( p_{\alpha(1)} = (x'_1, y'_1, z'_1) \) and \( p_{\alpha^{-1}(1)} = (x''_1, y''_1, z''_1) \). Then note that \( O_S(K_4^i, p, S) \) is the 2 × 3 matrix

\[
\begin{pmatrix}
2x_{1} - x'_{1} - x''_{1} & 2y_{1} - y'_{1} - y''_{1} & 2z_{1} - z'_{1} - z''_{1} \\
x_{1} & y_{1} & z_{1}
\end{pmatrix}
\]
which is easily checked to have rank 2 in each case. [12] Theorem 4.4 gives us a short list of operations that generate all \((2,3,1)\)-gain-tight graphs. For the inductive step suppose \((G, p)\) is \(S\)-isostatic and suppose \(G'\) is formed from \(G\) by any one of these operations. Then Lemmas 8.1 and 8.2 confirm that any \(S\)-generic realisation of \(G'\) is \(S\)-isostatic.

\(\square\)

**Proof of Theorem 6.4** Theorem 5.2 proves the necessity.

For the sufficiency we use induction and Theorem 7.2 First note that \(O_y(K_1, p, C_m)\), where \(K_1\) is the vertex \(1\), is the \(1 \times 3\) matrix

\[
\begin{bmatrix}
x_1 & y_1 & 0 \\
\end{bmatrix}
\]

with rank 1. Theorem 7.2 gives us a short list of operations that generate all \((2,2,2)\)-gain-tight graphs. For the inductive step suppose \((G, p)\) is \(C_m\)-isostatic and suppose \(G'\) is formed from \(G\) by any one of these operations. Then Lemmas 8.1, 8.2, 8.3 and 8.5 confirm that any \(C_m\)-generic realisation of \(G'\) is \(C_m\)-isostatic.

\(\square\)

**Proof of Theorem 6.5** Theorem 5.2 proves the necessity.

For the sufficiency we use induction and Theorem 7.3 Let \(S \in \{C_s, C_i, C_{m_{hi}}, S_{2m}\}\) and \(K^*_1\) denote a loop at vertex \(1\) with non-trivial gain \(\alpha\). Further, let \(p_{\alpha(1)} = (x_1'1, y_1', z_1')\). Clearly, \(p_{\alpha(1)} = p_{\alpha^{-1}(1)}\), since \(S\) is a group of order 2. Note that \(O_y(K^*_1, p, S)\) is the \(2 \times 3\) matrix

\[
\begin{bmatrix}
2(x_1 - x_1') & 2(y_1 - y_1') & 2(z_1 - z_1') \\
x_1 & y_1 & 0
\end{bmatrix}
\]

which has rank 2 in each case. Theorem 7.3 gives us a short list of operations that generate all \((2,1,1)\)-gain-tight graphs. For the inductive step suppose \((G, p)\) is \(S\)-isostatic and suppose \(G'\) is formed from \(G\) by any one of these operations. Then Lemmas 8.1, 8.2, 8.3 and 8.5 confirm that any \(S\)-generic realisation of \(G'\) is \(S\)-isostatic.

\(\square\)

**Proof of Theorem 6.6** Theorem 5.2 proves the necessity.

For the sufficiency we use induction and Theorem 7.3 Let \(S \in \{C_s, C_i, C_{m_{hi}}, S_{2m}\}\) and \(K^*_1\) denote a loop at vertex \(1\) with non-trivial gain \(\alpha\). Further, let \(p_{\alpha(1)} = (x_1', y_1', z_1')\) and \(p_{\alpha^{-1}(1)} = (x_1'', y_1'', z_1'')\). Note that \(O_y(K^*_1, p, S)\) is a \(2 \times 3\) matrix

\[
\begin{bmatrix}
2x_1 - x_1' & 2y_1 - y_1' & 2z_1 - z_1' \\
x_1 & y_1 & -z_1
\end{bmatrix}
\]

which has rank 2 in each case. Theorem 7.3 gives us a short list of operations that generate all \((2,1,1)\)-gain-tight graphs. For the inductive step suppose \((G, p)\) is \(S\)-isostatic and suppose \(G'\) is formed from \(G\) by any one of these operations. Then Lemmas 8.1, 8.2, 8.3 and 8.5 confirm that any \(S\)-generic realisation of \(G'\) is \(S\)-isostatic.

\(\square\)

10. **Further Work**

We finish by outlining a number of avenues of further developments. We start with a slight diversion into matroid theory.

10.1. **Matroids and inductive constructions.** Let \(J_{k,\ell,m}\) be the family of \((k, \ell, m)\)-gain-sparse edge sets in \((H, \psi)\). Then, as noted in [12] [31], \(J_{k,\ell,m}\) forms the family of independent sets of a matroid on \(E(H)\) for certain \((k, \ell, m)\). Let \(M(k, \ell, m) := (E(H), J_{k,\ell,m})\) whether or not the triple \(k, \ell, m\) induces a matroid. Using 3 basic matroids as ‘building blocks’ we can use standard matroid techniques to see that \(M(k, \ell, m)\) is a matroid for a large range of triples \(k, l, m \in \mathbb{N}\).
First note that when \( \ell = m \), \((k, \ell, m)\)-gain-sparsity is exactly \((k, \ell)\)-sparsity (on a multigraph) and \( M(k, \ell) \) is known to be a matroid for all \( 0 \leq \ell < 2k \). We also assume in this paper that \( m \leq \ell \).

Our 3 basic matroids are the frame matroid \( M(1, 1, 0) \), the cycle matroid \( M(1, 1, 1) \) and the bicircular matroid \( M(1, 0, 0) \). Then the previous sentence tells us that each possible option for \( M(1, \ell, m) \) is a matroid. For \( k > 1 \), using matroid union and Dilworth truncation we know the following:

1. \( M(k, k + t, m + t) \) is a matroid for \( 0 \leq t < k \) and \( m \leq k \) (take \( m \) copies of the cycle matroid and \( k - m \) copies of the frame matroid and then apply \( t \) Dilworth truncations),

2. \( M(k, k - m + n + t, n + t) \) with \( 0 \leq n \leq m \leq k \) and \( n + t < k + m \) is a matroid (take \( n \) copies of the cycle matroid, \( m - n \) copies of the bicircular matroid and \( k - m \) copies of the frame matroid and then apply \( t \) Dilworth truncations).

By the above remarks we know that: \( M(2, 3, 3) \) is a matroid (see also Theorem 6.2); \( M(2, 3, 2) \) is a matroid (this count does not appear in symmetry-forced rigidity analyses of frameworks in the plane or on surfaces, as the surface must be a sphere and there does not exist a symmetry group with two fully symmetric rotations. However it does occur for periodic frameworks or in ‘anti-symmetric’ rigidity analyses (see Section 10.7); \( M(2, 3, 1) \) is a matroid (see also Theorem 6.1); \( M(2, 3, 0) \) is not a matroid in general; \( M(2, 2, 2) \) is a matroid (see also Theorem 6.3); \( M(2, 2, 1) \) is a matroid (see also Theorem 6.5); \( M(2, 2, 0) \) is a matroid (see also Conjecture 10.3 below); \( M(2, 1, 1) \) is a matroid (see also Theorem 6.6); \( M(2, 1, 0) \) is a matroid (see also Conjecture 10.7 below); \( M(2, 0, 0) \) is a matroid (this count this count appears for frameworks on surfaces of type \( k = 0 \) which we do not consider in this paper (recall Section 2)).

For \( k \geq 3 \) these observations still give us a lot of information; however they do not tell us anything about the case when \( \ell - m > k \). We do not know if \( M(2, 3, 0) \) is typical or atypical for such triples.

Returning to the subject of the paper, we note that combining the results of Section 7 with [12, Theorem 4.4] and [24, Theorem 4.7] gives inductive constructions for \((2, \ell, m)\)-gain-tight graphs for all \( 1 \leq m \leq \ell < 4 \). However, the case when \( m = 0 \) is completely open. One indication of the potential difficulty to overcome here is that the minimum vertex degree in the graph may be 4. The analogue of the Henneberg moves for degree 4 vertices are known as X and V-replacement [8, 12, 22, 36]. V-replacement is known to not preserve \((2, \ell)\)-sparsity. X-replacement has been used to some effect in [12] so it is plausible that it could be used for the cases in question here. However, while X-replacement (as an operation on frameworks) is easy to understand in the plane (the generic argument is based on the simple fact that, generically, two lines intersect (see also [12]), the question whether the corresponding operation in 3-dimensions preserves generic rigidity is still open [8, 22] (two generic lines in 3D need not intersect!). This difficulty also arises for X-replacements on frameworks supported on surfaces since two lines will typically not intersect on a point on the surface. However, the X-replacement operation on frameworks supported on surfaces may still be more accessible than the X-replacement operation in the general 3-dimensional case.

10.2. The sphere. As indicated in Section 6 it seems difficult to establish characterisations for symmetry-forced rigidity on the sphere for groups other than \( e_m, e_4, e_5, e_{m/2}, s_{2m} \), as there are no tangential isometries (i.e., rotations) which are symmetric with respect to these groups.
An exception are the groups $\mathcal{C}_{m even}$, where $m$ is odd, as for these groups, we may combine results in [12] and [33] to obtain a characterisation for symmetry-forced rigidity on the sphere even though there are no rotations which are symmetric with respect to $\mathcal{C}_{m even}$. Theorems 6.2 and 6.3 provide characterisations for the groups $\mathcal{C}_m$, $\mathcal{C}_s$, $\mathcal{C}_t$, $\mathcal{C}_{m even}$ (with $m$ odd), $\mathcal{C}_{m h}$ (with $m$ odd) and $\mathcal{S}_{2m}$ (with $m$ even). The obstacles for $\mathcal{C}_{m even}$, where $m$ is even, were described in Section 6 (see also [12]). This leaves the groups $\mathcal{C}_{m h}$, where $m$ is even, and $\mathcal{S}_{2m}$, where $m$ is odd.

**Definition 10.1.** Let $(H, \psi)$ be a $\mathcal{C}_{m h}$-gain graph, where $m$ is even, or a $\mathcal{S}_{2m}$-gain graph, where $m$ is odd. Then $(H, \psi)$ is called $(2,3,1)^i$-gain-sparse if

- $|F| \leq 2|V(F)| - 3$ for any nonempty $F \subseteq E(H)$ with $\langle F \rangle = \mathcal{C}_0$ or $\langle F \rangle = \mathcal{C}_i$;
- $|F| \leq 2|V(F)| - 1$ otherwise.

A $(2,3,1)^i$-gain-sparse graph $(H, \psi)$ satisfying $|F| = 2|V(F)| - 1$ is called $(2,3,1)^i$-gain-tight.

**Conjecture 10.2.** Let $S$ be the group $\mathcal{C}_{m h}$, where $m$ is even, or the group $\mathcal{S}_{2m}$, where $m$ is odd. Let $(G, p)$ be an $S$-generic realisation on $S$ and let $(G_0, \psi)$ be the quotient $S$-gain graph of $G$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2,3,1)^i$-gain-tight.

### 10.3 The Cylinder

**Conjecture 10.3.** Let $S$ be the cyclic group $\mathcal{C}_2$ representing 2-fold rotation around an axis which is orthogonal to the $z$-axis. Let $(G, p)$ be an $S$-generic realisation on $\mathcal{T}$ and let $(G_0, \psi)$ be the quotient $S$-gain graph of $G$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2,2,0)$-gain-tight.

This conjecture is of particular interest because it implies that there is a symmetry-preserving motion in a framework that counts to be generically minimally rigid without symmetry. We illustrate such a motion in the following example.

**Example 10.4.** Let $G_0$ be the gain graph consisting of a 0-gain $K_4$ on vertices $a, b, c, d$, together with an additional edge $(c, d)$ with gain 1. Then, with symmetry group $\mathcal{C}_2$ as
Figure 11. A $\mathcal{C}_2$-symmetric framework $(G,p)$ on the cylinder $Y$ which has a non-trivial symmetry-preserving motion, but whose underlying graph $G$ is generically isostatic on $Y$ (without symmetry). The grey joints are at the ‘back’ of the cylinder.

defined in Conjecture 10.3, the covering graph $G$ of $G_0$ consists of two vertex disjoint copies of $K_4$ joined by two edges. Theorem 2.7 implies that generic realizations of $G$ (without symmetry) are rigid on the cylinder $Y$. However, the quotient $\mathbb{C}_2$-gain graph $G_0$ of $G$ satisfies $|E(G_0)| = 7 < 8 = 2|V(G_0)| - 0$. Thus, embedded $\mathbb{C}_2$-generically on $Y$, as in Figure 11, Theorem 5.2 implies the existence of a non-trivial continuous motion on $Y$.

Definition 10.5. Let $(H,\psi)$ be a $\mathcal{C}_{mh}$-gain graph, a $\mathcal{C}_{mv}$-gain graph or a $S_{2m}$-gain graph. Then $(H,\psi)$ is called $(2,2,1)^r$-gain-sparse if

- $|F| \leq 2|V(F)| - 2$ for any nonempty $F \subseteq E(H)$ with $\langle F \rangle = \mathcal{C}_1$ or $\langle F \rangle = \mathcal{C}_{m'}$, $m' \leq m$;
- $|F| \leq 2|V(F)| - 1$ otherwise.

A $(2,2,1)^r$-gain-sparse graph $(H,\psi)$ satisfying $|F| = 2|V(F)| - 1$ is called $(2,2,1)^r$-gain-tight.

Conjecture 10.6. Let $S$ be the group $\mathcal{C}_{mh}$, $\mathcal{C}_{mv}$ or $S_{2m}$, where the rotational axis is the cylinder axis. Let $(G,p)$ be an $S$-generic framework in $\mathcal{S}_{(G,S,\theta)}^{(H)}$ with quotient $S$-gain graph $(G_0,\psi)$. Then $(G,p)$ is $S$-isostatic if and only if $(G_0,\psi)$ is $(2,2,1)^r$-gain tight.

We remark that $(2,2,1)^r$-gain-tight graphs are a special class of $(2,2,1)$-gain-tight graphs.

For the groups $D_m$, $D_{mh}$ and $D_{md}$ it is plausible that the necessary conditions given in Theorem 5.4 are also sufficient, but difficult to prove due to the $(2,2,0)$-gain-sparsity count (recall Section 10.1). Moreover, the remarks in Section 6.1 give a warning that unexpected behaviour may arise, and hence we do not provide explicit conjectures for these groups.

10.4. The cone. For the cone, note that there is no symmetry group which turns a generically rigid framework with a free action on the vertex set on the cone into a flexible one. However we do suggest the following conjecture.

Conjecture 10.7. Let $S$ be the group $\mathcal{C}_2$ representing 2-fold rotation about an axis perpendicular to the $z$-axis (i.e., perpendicular to the axis of the cone) or the group $\mathcal{C}_s$, where the mirror plane of the reflection contains the $z$-axis. Let $(G,p)$ be an $S$-generic framework in
Table 2. Summary of counts for the various symmetry groups on the cylinder $Y$. We use ‘containing’ as short hand for a plane containing the $z$-axis and ‘perpendicular’ for a line perpendicular to the $z$-axis.

| Group | rotation axis | reflection plane | Necessary count | Sufficient? |
|-------|---------------|------------------|-----------------|-------------|
| $\mathcal{C}_s$ | - | containing $(2, 2, 1)$-gain-tight | Theorem 6.5 |
| $\mathcal{C}_s$ | - | $z = 0$ $(2, 2, 1)$-gain-tight | Theorem 6.5 |
| $\mathcal{C}_m$ | $z$-axis | - $(2, 2, 2)$-gain-tight | Theorem 6.4 |
| $\mathcal{C}_2$ | perpendicular | - $(2, 2, 0)$-gain-tight | Conjecture 10.3 |
| $\mathcal{C}_i$ | - | - $(2, 2, 1)$-gain-tight | Theorem 6.5 |
| $\mathcal{C}_{mv}$ | $z$-axis | containing $(2, 2, 1)^r$-gain-tight | Conjecture 10.6 |
| $\mathcal{C}_{mh}$ | $z$-axis | $z = 0$ $(2, 2, 1)^r$-gain-tight | Conjecture 10.6 |
| $S_{2m}$ | $z$-axis | $z = 0$ $(2, 2, 1)^r$-gain-tight | Conjecture 10.6 |
| $\mathcal{D}_m$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |
| $\mathcal{D}_{mh}$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |
| $\mathcal{D}_{md}$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |

Table 3. Summary of counts for the various symmetry groups on the cone $C$. We use ‘containing’ as short hand for a plane containing the $z$-axis and ‘perpendicular’ for a line perpendicular to the $z$-axis.

| Group | rotation axis | reflection plane | Necessary count | Sufficient? |
|-------|---------------|------------------|-----------------|-------------|
| $\mathcal{C}_s$ | - | containing $(2, 1, 0)$-gain-tight | Conjecture 10.7 |
| $\mathcal{C}_s$ | - | $z = 0$ $(2, 1, 1)$-gain-tight | Theorem 6.6 |
| $\mathcal{C}_m$ | $z$-axis | - $(2, 1, 1)$-gain-tight | Theorem 6.6 |
| $\mathcal{C}_2$ | perpendicular | - $(2, 1, 0)$-gain-tight | Conjecture 10.7 |
| $\mathcal{C}_i$ | - | - $(2, 1, 1)$-gain-tight | Theorem 6.6 |
| $\mathcal{C}_{mv}$ | $z$-axis | containing $(2, 1, 0)^r$-gain-tight | ? |
| $\mathcal{C}_{mh}$ | $z$-axis | $z = 0$ $(2, 1, 1)^r$-gain-tight | Theorem 6.6 |
| $S_{2m}$ | $z$-axis | $z = 0$ $(2, 1, 1)^r$-gain-tight | Theorem 6.6 |
| $\mathcal{D}_m$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |
| $\mathcal{D}_{mh}$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |
| $\mathcal{D}_{md}$ | $z$-axis | $z = 0$ Theorem 5.4 | ? |

$\mathcal{B}^C_{G,S,\theta}$ with quotient $S$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $S$-isostatic if and only if $(G_0, \psi)$ is $(2, 1, 0)$-gain-tight.

It is also plausible that the remaining groups ($\mathcal{C}_{mv}$, $\mathcal{D}_m$, $\mathcal{D}_{mh}$ and $\mathcal{D}_{md}$) can be understood similarly. For $\mathcal{C}_{mv}$ and $\mathcal{D}_m$, for example, the conjecture would be that $(2, 1, 0)^r$-gain tightness characterises the symmetry-forced isostatic frameworks (where $(2, 1, 0)^r$-gain-tightness is defined analogously to Def. 10.5), since $k_{i(F)} = 1$ if $\langle F \rangle$ is a purely rotational group (with the rotational axis being the axis of the cone) and $k_{i(F)} = 0$ otherwise.

10.5. **Non-free actions.** For the cylinder, rotational symmetry about the $z$-axis is necessarily a free group action. However, for reflection symmetry $s$ about a plane containing the $z$-axis, for example, this plane intersects the cylinder in 2 disjoint lines. If we allow symmetry-generic realisations to include joints lying on these lines then we must adapt the counts and the orbit-surface rigidity matrix accordingly. Any such joint which is ‘fixed’ by the reflection $s$ will have only 1 degree of freedom, as it has to stay on the reflection plane
of $s$ and on the cylinder. While we do not expect any new complication to arise in this more general situation, the proofs will become significantly more messy due to the reduced number of columns in the orbit-surface matrix for fixed vertices [32]. Similar observations apply to the cone.

10.6. **Surfaces of Revolution.** In [20] Laman type theorems were developed for any surface with exactly 1 isometry. Here we have concentrated on the cone. We expect that our methods are adaptable to any other surface with one isometry (such as tori, hyperboloids and paraboloids).

However, we highlight that the same group acting on two different surfaces (that a priori have the same number of trivial motions) can give different numbers of symmetric trivial motions and hence different combinatorial counts with an example theorem.

**Theorem 10.8** (Reflection symmetry on the elliptical cylinder). Let $\mathcal{F}$ be an elliptical cylinder about the $z$-axis and let $\mathcal{E}_s$ be generated by a reflection whose mirror plane contains the $z$-axis. Let $(G, p)$ be a framework in $\mathcal{E}^3_{(G,\mathcal{E},\delta)}$ with quotient $\mathcal{E}_s$-gain graph $(G_0, \psi)$. Then $(G, p)$ is $\mathcal{E}_s$-isostatic if and only if $(G_0, \psi)$ is $(2, 1, 1)$-gain-tight.

This is in contrast with the same group for the cone, where Theorem 5.2 implies that the $\mathcal{E}_s$-gain graph must be $(2, 1, 0)$-gain-tight.

10.7. **Incidental symmetry.** In this paper, we focused on the symmetry-forced rigidity of symmetric frameworks on surfaces. More generally, one may ask when an $S$-symmetric framework $(G, p)$ on a surface $M$ is not only $S$-symmetric infinitesimally rigid (i.e., $(G, p)$ has no non-trivial $S$-symmetric infinitesimal motion), but also infinitesimally rigid (i.e., $(G, p)$ has no non-trivial infinitesimal motion at all).

A fundamental result in the rigidity analysis of (‘incidentally’) symmetric frameworks in Euclidean $d$-space is that the rigidity matrix $R(G, p)$ of an $S$-symmetric framework $(G, p)$ can be transformed into a block-decomposed form, where each block $R_i(G, p)$ corresponds to an irreducible representation $\rho_i$ of the group $S$ [13, 26]. This breaks up the rigidity analysis of $(G, p)$ into a number of independent subproblems. In fact, the symmetry-forced rigidity properties of $(G, p)$ are described by the block matrix $R_1(G, p)$ corresponding to the trivial irreducible representation $\rho_1$ of $S$. In [32] the orbit rigidity matrix was derived to simplify the symmetry-forced rigidity analysis of Euclidean frameworks (the orbit rigidity matrix $O(G, p, S)$ is equivalent to the block matrix $R_1(G, p)$, but it can be constructed without using any methods from group representation theory). Moreover, in the very recent paper [31], an ‘orbit rigidity matrix’ was established for each of the blocks $R_i(G, p)$, and these new tools were successfully used to characterise $S$-generic infinitesimally rigid frameworks for a number of point groups $S$.

These methods can clearly be extended to analyse the infinitesimal rigidity of $S$-generic frameworks on surfaces. However, note that for each surface $M$ and each symmetry group $S$ considered in this paper (except for the groups $\mathcal{E}_s$ and $\mathcal{E}_i$ on the cylinder $Y$), there always exists an irreducible representation $\rho_i$ of $S$ with the property that there is no trivial $\rho_i$-symmetric infinitesimal motion (i.e., there is no trivial motion in the kernel of the corresponding block matrix $R_i(G, p)$). Therefore, we need to deal with a $(2, k, 0)$-gain-sparsity count in each of these cases (more precisely, $(2, 3, 0)$-gain-sparsity for the sphere, $(2, 2, 0)$-gain-sparsity for the cylinder, and $(2, 1, 0)$-gain-sparsity for the cone), which gives rise to the difficulties outlined in Sections 6.1 and 10.1.

For the groups $\mathcal{E}_s$ and $\mathcal{E}_i$ on the cylinder $Y$, however, the block-decomposed orbit-surface rigidity matrix consists of two blocks, one corresponding to the trivial representation $\rho_1$ of
the group (this block is equivalent to the orbit-surface rigidity matrix) and one corresponding to the other irreducible representation $\rho_2$ (this block is equivalent to an ‘anti-symmetric’ orbit-surface rigidity matrix), and for both of these blocks, one needs to consider the same $(2,2,1)$-gain-sparsity count to test whether the block has maximal rank. Therefore, we propose the following conjecture.

**Conjecture 10.9.** Let $S$ be the group $\mathcal{C}_s$ or the group $\mathcal{C}_i$, and let $(G,p)$ be an $S$-generic framework on the cylinder $Y$. Then $(G,p)$ is isostatic if and only if $(G,p)$ is $S$-isostatic.

### 10.8. Algorithmic implications.

We expect that $(k,\ell,m)$-gain-sparsity can be checked deterministically in polynomial time whenever $(k,\ell,m)$-gain-sparsity is a matroidal property. This would confirm that our theorems provide efficient combinatorial descriptions of symmetry-forced rigidity. We leave the exact details to the reader, but remark that: the quotient gain graph of a given symmetric simple graph can clearly be obtained in polynomial time; $(2,\ell,\ell)$-gain-sparsity for $\ell = 0, 1, 2, 3$ is known to be polynomial time computable [16]; the case corresponding to Theorem 6.1 has been considered [12, Section 10]; and the remaining cases considered in this paper can be checked using similar arguments.

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