Abstract. Let $q$ be a prime. Let $G$ be a residually finite group satisfying an identity. Suppose that for every $x \in G$ there exists a $q$-power $m = m(x)$ such that the element $x^m$ is a bounded Engel element. We prove that $G$ is locally virtually nilpotent. Further, let $d, n$ be positive integers and $w$ a non-commutator word. Assume that $G$ is a $d$-generator residually finite group in which all $w$-values are $n$-Engel. We show that the verbal subgroup $w(G)$ has $\{d, n, w\}$-bounded nilpotency class.

1. Introduction

Given a group $G$, an element $g \in G$ is called a (left) Engel element if for any $x \in G$ there exists a positive integer $n = n(x, g)$ such that $[x, n g] = 1$, where the commutator $[x, n g]$ is defined inductively by the rules

$$[x, 1 g] = [x, g] = x^{-1} g^{-1} x g \quad \text{and, for } n \geq 2, \quad [x, n g] = [[x, n-1 g], g].$$

If $n$ can be chosen independently of $x$, then $g$ is called a (left) $n$-Engel element, or more generally a bounded (left) Engel element. The group $G$ is an Engel group (resp. an $n$-Engel group) if all its elements are Engel (resp. $n$-Engel).

A celebrated result due to Zelmanov [24, 25, 26] refers to the positive solution of the Restricted Burnside Problem (RBP for short): every residually finite group of bounded exponent is locally finite. The group $G$ is said to have a certain property locally if any finitely generated subgroup of $G$ possesses that property. An interesting result in this context, due to Wilson [21], states that every $n$-Engel residually finite group is locally nilpotent. Another result that was deduced following the positive solution of the RBP is that given positive integers $m, n$, if $G$ is a residually finite group in which for every $x \in G$ there exists a positive integer $q = q(x) \leq m$ such that $x^q$ is $n$-Engel, then $G$
is locally virtually nilpotent \([1]\). We recall that a group possesses a certain property virtually if it has a subgroup of finite index with that property. For more details concerning Engel elements in residually finite groups see \([1, 2, 3, 17, 18]\).

One of the goals of the present article is to study residually finite groups in which some powers are bounded Engel elements. We establish the following result.

**Theorem A.** Let \(q\) be a prime. Let \(G\) be a residually finite group satisfying an identity. Suppose that for every \(x \in G\) there exists a \(q\)-power \(m = m(x)\) such that the element \(x^m\) is a bounded Engel element. Then \(G\) is locally virtually nilpotent.

A natural question arising in the context of the above theorem is whether the theorem remains valid with \(m\) allowed to be an arbitrary natural number rather than \(q\)-power. This is related to the conjecture that if \(G\) is a residually finite periodic group satisfying an identity, then the group \(G\) is locally finite (Zelmanov, \([23\) p. 400]). Note that the hypothesis that \(G\) satisfies an identity is really needed. For instance, it is well known that there are residually finite \(p\)-groups that are not locally finite (Golod, \([5]\)). In particular, these groups cannot be locally virtually nilpotent. Similar examples have been obtained independently by Grigorchuk, Gupta-Sidki and Sushchansky and are published in \([6, 7, 20]\), respectively.

Recall that a group-word \(w = w(x_1, \ldots, x_s)\) is a nontrivial element of the free group \(F = F(x_1, \ldots, x_s)\) on free generators \(x_1, \ldots, x_s\). A word is a commutator word if it belongs to the commutator subgroup \(F'\). A non-commutator word \(u\) is a group-word such that the sum of the exponents of some variable involved in it is non-zero. A group-word \(w\) can be viewed as a function defined in any group \(G\). The subgroup of \(G\) generated by the \(w\)-values is called the verbal subgroup of \(G\) corresponding to the word \(w\). It is usually denoted by \(w(G)\). However, if \(k\) is a positive integer and \(w = x_1^k\), it is customary to write \(G^k\) rather than \(w(G)\).

There is a well-known quantitative version of Wilson’s theorem, that is, if \(G\) is a \(d\)-generator residually finite \(n\)-Engel group, then \(G\) has \(\{d, n\}\)-bounded nilpotency class. As usual, the expression “\(\{a, b, \ldots\}\)-bounded” means “bounded from above by some function which depends only on parameters \(a, b, \ldots\)”. We establish the following related result.

**Theorem B.** Let \(d, n\) be positive integers and \(w\) a non-commutator word. Assume that \(G\) is a \(d\)-generator residually finite group in which all \(w\)-values are \(n\)-Engel. Then the verbal subgroup \(w(G)\) has \(\{d, n, w\}\)-bounded nilpotency class.
A non-quantitative version of the above theorem already exists in the literature. It was obtained in [3, Theorem C].

The paper is organized as follows. In the next section we describe some important ingredients of what are often called “Lie methods in group theory”. Theorems A and B are proved in Sections 3 and 4, respectively. The proofs of the main results rely of Zelmanov’s techniques that led to the solution of the RBP [24, 25, 26], Lazard’s criterion for a pro-p group to be p-adic analytic [9], and a result of Nikolov and Segal [13] on verbal width in groups.

2. Associated Lie algebras

Let $L$ be a Lie algebra over a field $\mathbb{K}$. We use the left normed notation: thus if $l_1, l_2, \ldots, l_n$ are elements of $L$, then

$$[l_1, l_2, \ldots, l_n] = \ldots [[[l_1, l_2], l_3], \ldots, l_n].$$

We recall that an element $a \in L$ is called ad-nilpotent if there exists a positive integer $n$ such that $[x_1, a] = 0$ for all $x \in L$. When $n$ is the least integer with the above property then we say that $a$ is ad-nilpotent of index $n$.

Let $X \subseteq L$ be any subset of $L$. By a commutator of elements in $X$, we mean any element of $L$ that can be obtained from elements of $X$ by means of repeated operation of commutation with an arbitrary system of brackets including the elements of $X$. Denote by $F$ the free Lie algebra over $\mathbb{K}$ on countably many free generators $x_1, x_2, \ldots$. Let $f = f(x_1, x_2, \ldots, x_n)$ be a non-zero element of $F$. The algebra $L$ is said to satisfy the identity $f \equiv 0$ if $f(l_1, l_2, \ldots, l_n) = 0$ for any $l_1, l_2, \ldots, l_n \in L$. In this case we say that $L$ is PI. Now, we recall an important theorem of Zelmanov [23, Theorem 3] that has many applications in group theory.

**Theorem 2.1.** Let $L$ be a Lie algebra over a field generated by a finite set. Assume that $L$ is PI and that each commutator in the generators is ad-nilpotent. Then $L$ is nilpotent.

2.1. On Lie Algebras Associated with Groups. Let $G$ be a group and $p$ a prime. Let us denote by $D_i = D_i(G)$ the $i$-th dimension subgroup of $G$ in characteristic $p$. These subgroups form a central series of $G$ known as the Zassenhaus-Jennings-Lazard series (see [8, p. 250] for more details). Set $L(G) = \bigoplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field $\mathbb{F}_p$ with $p$ elements.

The subalgebra of $L(G)$ generated by $D_1/D_2$ will be denoted by $L_p(G)$. The nilpotency of $L_p(G)$ has strong influence in the structure of a finitely generated group $G$. According to Lazard [10] the nilpotency
of $L_p(G)$ is equivalent to $G$ being $p$-adic analytic (for details see [10, A.1 in Appendice and Sections 3.1 and 3.4 in Ch. III] or [4, 1.(k) and 1.(o) in Interlude A]).

**Theorem 2.2.** Let $G$ be a finitely generated pro-$p$ group. If $L_p(G)$ is nilpotent, then $G$ is $p$-adic analytic.

Let $x \in G$ and let $i = i(x)$ be the largest positive integer such that $x \in D_i$ (here, $D_i$ is a term of the $p$-dimensional central series to $G$). We denote by $\bar{x}$ the element $xD_{i+1} \in L(G)$. We now quote two results providing sufficient conditions for $\bar{x}$ to be ad-nilpotent. The first lemma was established in [9, p. 131].

**Lemma 2.3.** For any $x \in G$ we have $(ad \bar{x})^p = ad (\bar{x}^p)$. Consequently, if $x$ is of finite order $t$ then $\bar{x}$ is ad-nilpotent of index at most $t$.

**Corollary 2.4.** Let $x$ be an element of a group $G$ for which there exists a positive integer $m$ such that $x^m$ is $n$-Engel. Then $\bar{x}$ is ad-nilpotent.

The following result was established by Wilson and Zelmanov in [22].

**Lemma 2.5.** Let $G$ be a group satisfying an identity. Then for each prime number $p$ the Lie algebra $L_p(G)$ is PI.

### 3. Proof of Theorem A

Recall that a group is locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. Interesting classes of groups (e.g., locally finite groups, locally nilpotent groups, residually finite groups) are locally graded (see [11,12] for more details).

It is easy to see that a quotient of a locally graded group need not be locally graded (see for instance [14, 6.19]). However, the next result gives a sufficient condition for a quotient to be locally graded [11].

**Lemma 3.1.** Let $G$ be a locally graded group and $N$ a normal locally nilpotent subgroup of $G$. Then $G/N$ is locally graded.

In [23], Zelmanov has shown that if $G$ is a residually finite $p$-group which satisfies a nontrivial identity, then $G$ is locally finite. Next, we extend this result to the class of locally graded groups.

**Lemma 3.2.** Let $p$ be a prime. Let $G$ be a locally graded $p$-group which satisfies an identity. Then $G$ is locally finite.

**Proof.** Choose arbitrarily a finitely generated subgroup $H$ of $G$. Let $R$ be the finite residual of $H$, i.e., the intersection of all subgroups of finite index in $H$. If $R = 1$, then $H$ is a finitely generated residually finite
By Zelmanov’s result [23, Theorem 4], $H$ is finite. So it suffices to show that $H$ is residually finite. We argue by contradiction and suppose that $R \neq 1$. By the above argument, $H/R$ is finite and thus $R$ is finitely generated. As $R$ is locally graded we have that $R$ contains a proper subgroup of finite index in $H$, which gives a contradiction. Since $H$ be chosen arbitrarily, we now conclude that $G$ is locally finite, as well. The proof is complete. □

We denote by $\mathcal{N}$ the class of all finite nilpotent groups. The following result is a straightforward corollary of [21, Lemma 2.1] (see [15, Lemma 3.5] for details).

**Lemma 3.3.** Let $G$ be a finitely generated residually-$\mathcal{N}$ group. For each prime $p$, let $R_p$ be the intersection of all normal subgroups of $G$ of finite $p$-power index. If $G/R_p$ is nilpotent for each prime $p$, then $G$ is nilpotent.

We are now in a position to prove Theorem A.

**Proof of Theorem A.** Recall that $G$ is a residually finite group satisfying an identity in which for every $x \in G$ there exists a $q$-power $m = m(x)$ such that the element $x^m$ is a bounded Engel element. We need to prove that every finitely generated subgroup of $G$ is virtually nilpotent.

Firstly, we prove that all bounded Engel elements (in $G$) are contained in the Hirsch-Plotkin radical of $G$. Let $H$ be a subgroup generated by finitely many bounded Engel elements in $G$, say $H = \langle h_1, \ldots, h_t \rangle$, where $h_i$ is a bounded Engel element in $G$ for every $i = 1, \ldots, t$. Since finite groups generated by Engel elements are nilpotent [14, 12.3.7], we can conclude that $H$ is residually-$\mathcal{N}$. As a consequence of Lemma 3.3 we can assume that $G$ is residually-(finite $p$-group) for some prime $p$. Let $L = L_p(H)$ be the Lie algebra associated with the Zassenhaus-Jennings-Lazard series

$$H = D_1 \supseteq D_2 \supseteq \cdots$$

of $H$. Then $L$ is generated by $\tilde{h}_i = h_i D_2, i = 1, 2, \ldots, t$. Let $\tilde{h}$ be any Lie-commutator in $\tilde{h}_i$ and $h$ be the group-commutator in $h_i$ having the same system of brackets as $\tilde{h}$. Since for any group commutator $h$ in $h_1, \ldots, h_t$ there is a $q$-power $m = m(h)$ and a positive integer $n = n(h)$ such that $h^m$ is $n$-Engel, Corollary 2.4 shows that any Lie commutator in $\tilde{h}_1, \ldots, \tilde{h}_t$ is ad-nilpotent. On the other hand, $H$ satisfies an identity and therefore, by Lemma 2.5 $L$ satisfies some non-trivial polynomial identity. According to Theorem 2.1 $L$ is nilpotent. Let $\hat{H}$ denote the pro-$p$ completion of $H$. Then $L_p(\hat{H}) = L$ is nilpotent.
and $\hat{H}$ is a $p$-adic analytic group by Theorem 2.2. By [4, 1.(n) and 1.(o) in Interlude A)], $\hat{H}$ is linear, and so therefore is $H$. Clearly $H$ cannot have a free subgroup of rank 2 and so, by Tits’ Alternative [19], $H$ is virtually soluble. By [14, 12.3.7], $H$ is soluble. Since $h_1, \ldots, h_t$ have been chosen arbitrarily, we now conclude that all bounded Engel elements are in the Hirsch-Plotkin radical of $G$.

Let $H$ be a finitely generated subgroup of $G$, and $K$ be the subgroup generated by all bounded Engel elements (in $G$) contained in $H$. Now, we need to prove that $K$ is a nilpotent subgroup of finite index in $H$. By the previous paragraph, $K$ is locally nilpotent. By Lemma 3.1, $H/K$ is a locally graded group. Since $G$ satisfies a nontrivial identity, by Lemma 3.2, $H/K$ is finite and so, $K$ is finitely generated. From this we deduce that $K$ is nilpotent. The proof is complete.

4. Proof of Theorem B

Combining the positive solution of the RBP with the result [3, Theorem C] one can show that if $u$ is a non-commutator word and $G$ is a finitely generated residually finite group in which all $u$-values are $n$-Engel, then the verbal subgroup $u(G)$ is nilpotent. This section is devoted to obtain a quantitative version of the aforementioned result.

The proof of Theorem B require the following lemmas.

**Lemma 4.1.** Let $d, m, n$ positive integers. Let $G$ be a $d$-generator residually finite group in which $x^m$ is $n$-Engel for every $x \in G$. Then the subgroup $G^m$ has $\{d, m, n\}$-bounded nilpotency class.

**Proof.** Let $H = G^m$. By [3, Theorem C], $H$ is locally nilpotent. Moreover, Lemma 3.1 ensures us that the quotient group $G/H$ is locally graded. By Zelmanov’s solution of the RBP, locally graded groups of finite exponent are locally finite (see for example [12, Theorem 1]), and so $G/H$ is finite of $\{d, m\}$-bounded order. We can deduce from [14, Theorem 6.1.8(ii)] that $H$ has $\{d, m\}$-boundedly many generators. In particular, $H$ is nilpotent. In order to complete the proof, we need to show that $H$ has $\{d, m, n\}$-bounded class yet.

Note that there exists a family of normal and finite index subgroups $\{N_i\}_{i \in \mathbb{Z}}$ in $G$ which are all contained in $H$ such that $H$ is isomorphic to a subgroup of the Cartesian product of the finite quotients $H/N_i$. We show that all quotients have $\{d, m, n\}$-bounded class. Indeed, we have $H/N_i = (G/N_i)^m$. Note that $H$ is $\{d, m\}$-boundedly generated. Thus, by [13, Theorem 1], $H/N_i$ is $\{d, m\}$-boundedly generated where any generator is an $m$-th power which is an $n$-Engel element. By [17, Lemma 2.2], there exists a number $c$ depending only on $\{d, m, n\}$ such
that each factor $H/N_i$ has nilpotency class at most $c$. So $H$ is of nilpotency class at most $c$, as well. The proof is complete. 

A well known theorem of Gruenberg says that a soluble group generated by finitely many Engel elements is nilpotent (see [14, 12.3.3]). We will require a quantitative version of this theorem whose proof can be found in [16, Lemma 4.1].

**Lemma 4.2.** Let $G$ be a group generated by $m$ elements which are $n$-Engel and suppose that $G$ is soluble with derived length $d$. Then $G$ is nilpotent of $\{d, m, n\}$-bounded class.

For the reader’s convenience we restate Theorem B.

**Theorem B.** Let $d, n$ be positive integers and $w$ a non-commutator word. Assume that $G$ is a $d$-generator residually finite group in which all $w$-values are $n$-Engel. Then the verbal subgroup $w(G)$ has $\{d, m, n\}$-bounded nilpotency class.

**Proof.** Let $w = w(x_1, ..., x_r)$ be a non-commutator word. We may assume that the sum of the exponents of $x_1$ is $k \neq 0$. Substitute 1 for $x_2, ..., x_r$ and an arbitrary element $g \in G$ for $x_1$. We see that $g^k$ is a $w$-value for every $g \in G$. Thus every $k$-th power is $n$-Engel in $G$. Lemma 4.1 ensures that $G^k$ has $\{d, n, w, \}$-bounded nilpotency class.

Following an argument similar to that used in the proof of Lemma 4.1 we can deduce that the verbal subgroup $w(G)$ is nilpotent. By Zelmanov’s solution of the RBP, locally graded groups of finite exponent are locally finite (see for example [12, Theorem 1]), and so $G/G^k$ is finite of $\{d, w\}$-bounded order. Thus, the verbal subgroup $w(G)$ has $\{d, m, w\}$-bounded derived length.

Note that there exists a family of normal and finite index subgroups $\{N_i\}_{i \in \mathbb{Z}}$ in $G$ that are all contained in $w(G)$ such that $w(G)$ is isomorphic to a subgroup of the Cartesian product of the finite quotients $w(G)/N_i$. We show that all quotients $w(G)/N_i$ have $\{d, n, w\}$-bounded class. Indeed, we have $w(G)/N_i = w(G/N_i)$. We also have $w(G)$ is $\{d, w\}$-boundedly generated. By [13, Theorem 3] each quotient $w(G)/N_i$ is $\{d, w\}$-boundedly generated by $w$-values which are $n$-Engel elements. Since $w(G)$ has $\{d, m, w\}$-bounded derived length, according to Lemma 4.2 we can deduce that $w(G)/N_i$ has $\{d, n, w\}$-bounded nilpotency class. Thus, $w(G)$ has $\{d, n, w\}$-bounded nilpotency class, as well. This completes the proof.

**References**

[1] R. Bastos, *On residually finite groups with Engel-like conditions*, Comm. Algebra, 44 (2016) 4177–4184.
[2] R. Bastos, N. Mansuroğlu, A. Tortora, M. Tota, *Bounded Engel elements in groups satisfying an identity*, Arch. Math., 110 (2018) 311–318.

[3] R. Bastos, P. Shumyatsky, A. Tortora, M. Tota, *On groups admitting a word whose values are Engel*, Int. J. Algebra Comput., 23 (2013) 81–89.

[4] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, *Analytic Pro-p Groups*, Cambridge University Press, Cambridge, (1991).

[5] E.S. Golod, *On nil-algebras and finitely approximable p-groups*, Izv. Akad. Nauk SSSR Ser. Mat., 28 (1964) 273–276.

[6] R.I. Grigorchuk, *On Burnside's problem on periodic groups*, Functional Anal. Appl., 14 (1980) 41–43.

[7] N. Gupta, S. Sidki, *On the Burnside problem for periodic groups*, Math. Z., 182 (1983) 385–386.

[8] B. Huppert, N. Blackburn, *Finite Groups II*, Springer-Verlag, Berlin, (1982).

[9] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. École Norm. Sup., 71 (1954) 101–190.

[10] M. Lazard, *Groupes analytiques p-adiques*, IHES Publ. Math., 26 (1965) 389–603.

[11] P. Longobardi, M. Maj and H. Smith, *A note on locally graded groups*, Rend. Sem. Mat. Univ. Padova, 94 (1995) 275–277.

[12] O. Macedońska, *On difficult problems and locally graded groups*, J. Math. Sci. (N.Y.), 142 (2007) 1949–1953.

[13] N. Nikolov, D. Segal, *Powers in finite groups*, Groups Geom. Dyn., 5 (2011) 501–507.

[14] D.J.S. Robinson, *A course in the theory of groups*, 2nd edition, Springer-Verlag, New York, (1996).

[15] P. Shumyatsky, *Applications of Lie ring methods to group theory*, in Nonassociative Algebra and Its Applications, (Eds R. Costa et al.), Marcel Dekker, New York, (2000) 373–395.

[16] P. Shumyatsky, D.S. Silveira, *On finite groups with automorphisms whose fixed points are Engel*, Arch. Math., 106 (2016) 209–218.

[17] P. Shumyatsky, A. Tortora, M. Tota, *On varieties of groups satisfying an Engel type identity*, J. Algebra, (2016), 447 (2016) 479–489.

[18] P. Shumyatsky, A. Tortora, M. Tota, *Engel groups with an identity*, Int. J. Algebra Comput., (2018), preprint available at [arXiv:1805.12411](http://arxiv.org/abs/1805.12411) [math.GR].

[19] J. Tits, *Free subgroups in linear groups*, J. Algebra, 20 (1972) 250–270.

[20] V.I. Sushchansky, *Periodic p-elements of permutations and the general Burnside problem*, Dokl. Akad. Nauk SSSR, 247 (1979) 447–461.

[21] J.S. Wilson, *Two-generator conditions for residually finite groups*, Bull. London Math. Soc., 23 (1991), 239–248.

[22] J.S. Wilson, E.I. Zelmanov, *Identities for Lie algebras of pro-p groups*, J. Pure Appl. Algebra, 81 (1992) 103–109.

[23] E.I. Zelmanov, *On the restricted Burnside problem*, In: Proceedings of the International Congress of Mathematicians (1990) 395–402.

[24] E. Zelmanov, *The solution of the restricted Burnside problem for groups of odd exponent*, Math. USSR Izv., 36 (1991) 41–60.

[25] E. Zelmanov, *The solution of the restricted Burnside problem for 2-groups*, Math. Sb., 182 (1991) 568–592.
[26] E. I. Zelmanov, *Lie algebras and torsion groups with identity*, J. Comb. Algebra, 1, (2017) 289–340.

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