ON THE HEAT CONTENT FOR THE POISSON KERNEL OVER SETS OF FINITE PERIMETER.

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Abstract. This paper studies the small time behavior of the heat content for the Poisson kernel over a bounded open set \( \Omega \subset \mathbb{R}^d, d \geq 2 \), of finite perimeter by working with the set covariance function. As a result, we obtain a third order expansion of the heat content involving geometric features related to the underlying set \( \Omega \). We provide the explicit form of the third term for the unit ball when \( d = 2 \) and \( d = 3 \) and the square \([-1, 1] \times [-1, 1]\).

Keywords: covariance function, heat content, functions of bounded variation, Poisson kernel, sets of finite perimeter.

1. Introduction

Let \( d \geq 2 \) be an integer. We consider the Poisson kernel defined by

\[
p_t(x) = \frac{k_d t}{(t^2 + |x|^2)^{\frac{d+1}{2}}}, \quad x \in \mathbb{R}^d, \; t > 0,
\]

where \( |x| \) denotes the norm of \( x \in \mathbb{R}^d \) and

\[
k_d = \frac{\Gamma \left( \frac{d+1}{2} \right)}{\pi^{\frac{d+1}{2}}}.
\]

We observe that the Poisson kernel has the following properties.

i) For all \( t > 0 \), \( p_t(x) \) is a radial function. Namely, \( p_t(x) = p_t(|x| e_d) \) where \( e_d = (0, ..., 1) \in \mathbb{R}^d \). Moreover, for all \( t > 0 \), we have

\[
\int_{\mathbb{R}^d} dx \, p_t(x) = 1.
\]

ii) The heat kernel \( p_t(x) \) satisfies the following scaling property:

\[
p_t(x) = t^{-d} p_1(t^{-1} x).
\]

Before continuing, we provide some useful notations. Throughout the paper, \( \mathcal{L}(\mathbb{R}^d) \) will denote the set of all the Lebesgue measurable subsets of \( \mathbb{R}^d \). For a bounded set \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) with non-empty boundary \( \partial \Omega \), we set \( |\Omega| \) and \( H^{d-1}(\partial \Omega) \) to represent the volume of \( \Omega \) and the \((d-1)\)-Hausdorff measure of the boundary of \( \Omega \), respectively.

Henceforth, \( B_r(x) \) will stand for the ball centered at \( x \in \mathbb{R}^d \) with radius \( r \) and for simplicity \( B \) will represent the unit ball centered at zero. Also, \( S^{d-1} \) will denote the boundary of the unit ball \( B \). Moreover, the volume and surface area of the unit ball \( B \) in \( \mathbb{R}^d \) will be denoted by \( w_d \) and \( A_d \), respectively. That is,

\[
w_d = \frac{\pi^{\frac{d}{2}}}{\Gamma \left( 1 + \frac{d}{2} \right)},
\]

\[
A_d = dw_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)}.
\]

In order to establish the problem to deal with, we need to introduce the notion of perimeter for a bounded set. We say that a bounded set \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) has finite perimeter if

\[
0 \leq \sup \left\{ \int_{\Omega} dx \, \text{div} \varphi(x) : \varphi \in C_0^1(\mathbb{R}^d, \mathbb{R}^d), ||\varphi||_{\infty} \leq 1 \right\} < \infty,
\]

and we denote the last quantity by \( \text{Per}(\Omega) \).
With the appropriate geometric objects already introduced, we proceed to present the function to be investigated. Let \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) be a bounded set and define
\[
\mathbb{H}_\Omega(t) = \int_{\Omega} dx \int_{\Omega} dy p_t(x - y),
\]
which will be called the heat content of \( \Omega \) in \( \mathbb{R}^d \) with respect to the Poisson kernel. We remark that \( \Omega \) being a bounded set implies that \( \mathbb{H}_\Omega(t) \) is finite for all \( t > 0 \) since by \((1.2)\) we have
\[
0 \leq \mathbb{H}_\Omega(t) \leq \int_{\Omega} dx \int_{\mathbb{R}^d} dy p_t(x - y) = |\Omega| < \infty.
\]

The interest in studying the heat content comes from the paper \([3]\), where it was shown that the behavior of \( \mathbb{H}_\Omega(t) \) as \( t \to 0^+ \) might give an insight of the behavior of the spectral heat content for small time which is of great interest in areas such as probability and spectral theory (see for instance \([1, 4]\)) since it is expected to recover geometric quantities related to the underlying set \( \Omega \) as we will see from our main result below (see \((1.1)\)). The heat content was firstly investigated for the Gaussian heat kernel \((4\pi t)^{-d/2}e^{-t|x|^2}\) in paper as \([5, 7, 6, 12]\) for sets \( \Omega \) with piecewise smooth boundary where it was shown that
\[
\int_{\Omega} dx \int_{\Omega} dy (4\pi t)^{-d/2}e^{-t|x|^2} = |\Omega| - \frac{1}{\sqrt{\pi}} \mathcal{H}^{d-1}(\partial \Omega) t^{1/2} + o(t^{1/2}), \ t \to 0+,
\]
where we can see that the expansion involves geometric objects associated with the set \( \Omega \). This result was later extended to other heat kernels related to the stable processes in \([2, 3]\). Lately, it has been an increasing literature where similar results have been obtained for wide variety of Lévy processes and we refer the interested reader to \([8, 11]\).

In \([3]\), the following conjecture is given.

**Conjecture:** For \( \Omega \subset \mathbb{R}^d \) an open bounded set with smooth boundary, the following limit
\[
(1.6) \quad \lim_{t \to 0^+} \frac{1}{t} \left( |\Omega| - \mathbb{H}_\Omega(t) - \frac{\text{Per}(\Omega)}{\pi} t \ln \left( \frac{1}{t} \right) \right)
\]
exists. The aforementioned conjecture was deduced from the following one-dimensional result whose proof can be found in \([3]\). Let \( \Omega = (a, b) \) be an open interval with \( |\Omega| = b - a \), then
\[
\lim_{t \to 0^+} \frac{1}{t} \left( |\Omega| - \mathbb{H}_\Omega(t) - \frac{2}{\pi} t \ln \left( \frac{1}{t} \right) \right) = \frac{2}{\pi} (1 + \ln(|\Omega|)).
\]

The purpose of this paper is to show that the conjecture is true for certain class \( \mathcal{W} \) of subsets of \( \mathbb{R}^d, d \geq 2 \). To define \( \mathcal{W} \), we need to turn to the set covariance function associated with the set \( \Omega \).

**Definition 1.1 (Set covariance function).** Let \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) have finite Lebesgue measure. The covariance function of \( \Omega \) is denoted by \( g_\Omega \) and defined for each \( y \in \mathbb{R}^d \) by
\[
(1.7) \quad g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} dx \mathbbm{1}_\Omega(x) \mathbbm{1}_\Omega(x - y).
\]

We now proceed to mention some analytic properties concerning the set covariance function \( g_\Omega \) needed to enunciate our main theorem.

**Proposition 1.1.** Let \( \Omega \subset \mathbb{R}^d \) be a Lebesgue measurable set with \( |\Omega| < \infty \) and \( g_\Omega \) its corresponding covariance function. Then,
\begin{enumerate}
  \item[a)] For all \( y \in \mathbb{R}^d \), \( 0 \leq g_\Omega(y) \leq g_\Omega(0) = |\Omega| \).
  \item[b)] For all \( y \in \mathbb{R}^d \), \( g_\Omega(y) = g_\Omega(-y) \).
  \item[c)] \( \int_{\mathbb{R}^d} dy g_\Omega(y) = |\Omega|^2 \).
  \item[d)] \( g_\Omega \) is compactly supported. In fact, we have \( g_\Omega(y) = 0 \) for all \( y \in \mathbb{R}^d \) with
      \[
      |y| \geq \text{diam}(\Omega) = \sup \{|u - v|: u, v \in \Omega\}.
      \]
  \item[e)] \( g_\Omega \) is uniformly continuous over \( \mathbb{R}^d \) and \( \lim_{|y| \to \infty} g_\Omega(y) = 0 \).
\end{enumerate}
With $g_{\Omega}$ and its analytic properties properly presented, we define
\begin{equation}
\ell_{\Omega} = \inf \{ \ell > 0 : g_{\Omega}(y) = 0 \text{ for all } |y| \geq \ell \},
\end{equation}
for every $\Omega \in \mathcal{L}(\mathbb{R}^d)$ satisfying $0 < |\Omega| < \infty$. Notice that $\ell_{\Omega}$ stands for the radius of the smallest ball center at the origin that contains $\Omega$ and $0 < \ell_{\Omega} \leq \text{diam}(\Omega)$. It is not difficult to see that when $\Omega$ is a bounded convex set, then $\ell_{\Omega} = \text{diam}(\Omega)$.

Now, for each $u \in \mathbb{S}^{d-1}$, we set
\begin{equation}
V_u(\Omega) = \sup \left\{ \int_{\Omega} \langle \nabla \varphi(x), u \rangle : \varphi \in C^1_c(\Omega, \mathbb{R}), \|\varphi\|_{\infty} \leq 1 \right\},
\end{equation}
and for $0 < s$, we define
\begin{equation}
\gamma_{\Omega}(s) = \int_{\mathbb{S}^{d-1}} \mathcal{H}^{d-1}(du) \left( \frac{V_u(\Omega)}{2} - \frac{(g_{\Omega}(0) - g_{\Omega}(su))}{s} \right).
\end{equation}

We now proceed to present the class of subsets of $\mathbb{R}^d$ for which the limit given in (3.9) shall exist.

**Definition 1.2.**
\[
\mathcal{W} = \left\{ \Omega \subseteq \mathbb{R}^d : \Omega \in \mathcal{L}(\mathbb{R}^d) \text{ and bounded}, \ Per(\Omega) < \infty, \left. \gamma_{\Omega}(s) \in L((0,1), s^{-1}ds) \right\}
\]
where $L((0,1), s^{-1}ds)$ consists of the Lebesgue measurable functions $h$ with $\int_0^1 |h(s)|s^{-1}ds < \infty$ and $Per(\Omega)$, $\ell_{\Omega}$ and $\gamma_{\Omega}$ as defined in (1.4), (1.8) and (1.10), respectively.

With all the required details being introduced, we continue to present our main result.

**Theorem 1.1.** Consider $d \geq 2$ and let $\kappa_d$, $w_d$, $A_d$, and $\ell_{\Omega}$ be defined as in (1.1), (1.3) and (1.8), respectively. For $\Omega \in \mathcal{W}$, we have that
\begin{equation}
\lim_{t \to 0^+} \frac{1}{t} \left( |\Omega| - \mathcal{H}_t(\Omega) - \frac{\text{Per}(\Omega)}{\pi} t \ln \left( \frac{1}{t} \right) \right) = \\
\kappa_d \left( \frac{|\Omega|}{\ell_{\Omega}} - \int_0^1 dss^{-1}\gamma_{\Omega}(\ell_{\Omega}s) + \frac{\text{Per}(\Omega)}{\pi} \left( \ln(2\ell_{\Omega}) + \int_0^\infty d\theta(\tanh^d(\theta) - 1) \right) \right).
\end{equation}

The interesting fact about our main result is that we obtain a constant involving many geometric quantities linked to the set $\Omega$ where the computation of the term $\int_0^1 dss^{-1}\gamma_{\Omega}(\ell_{\Omega}s)$ requires as we shall see in §4 and §5 knowledge about the geometry of the support of the covariance function $g_{\Omega}$. Based on the above theorem and our results concerning the unit ball and the square $[-1,1] \times [-1,1]$, there are two open problems that arise. The first problem has to do with how rich is the class $\mathcal{W}$, we believe that every bounded convex set with piecewise smooth boundary belongs to $\mathcal{W}$ and the second problem would be if there exists a bounded set $\Omega$ with finite perimeter such that $\Omega \notin \mathcal{W}$.

The paper is organized as follows. In §2, we provide some preliminaries concerning sets of finite perimeter. In §3, we provide the proof of Theorem 1.1 with the aid of a series of propositions. Finally, in the last two sections, we compute explicitly the constant provided in our main result for the unit ball when $d \in \{2,3\}$ and the square $[-1,1] \times [-1,1]$.

**2. PRELIMINARIES: FUNCTIONS OF BOUNDED VARIATION, PERIMETER AND COVARIANCE FUNCTION.**

In this section, we introduce a couple of geometric objects associated with the set $\Omega$ under consideration which will play an important role in the proof of Theorem 1.1. The interested reader may consult [9], [12] and [13] for further details on the matter and for the proofs of the many results to be given in this section.

**Definition 2.1.** Let $G \subseteq \mathbb{R}^d$ be an open set and $f : G \to \mathbb{R}$, $f \in L^1(G)$. The total variation of $f$ in $G$ is defined by
\[
V(f, G) = \sup \left\{ \int_G dx f(x) \text{div} \varphi(x) : \varphi \in C^1_c(G, \mathbb{R}^d), \|\varphi\|_{\infty} \leq 1 \right\}.
\]
We set $BV(G) = \{ f \in L^1(G) : V(f, G) < \infty \}$ to denote the set of functions of bounded variation.
The directional derivative of \( f \) in \( G \) in the direction \( u \in \mathbb{S}^{d-1} \) is
\[
V_u(f, G) = \sup \left\{ \int_G dx \, f(x) \langle \nabla \varphi(x), u \rangle : \varphi \in C_c^1(G, \mathbb{R}), ||\varphi||_\infty \leq 1 \right\}.
\]

If \( \Omega \in \mathcal{L}(\mathbb{R}^d) \), we call \( V(1_{\Omega}, \mathbb{R}^d) \) the perimeter of \( \Omega \) and we denote this quantity by \( \text{Per}(\Omega) \). In addition, \( V_u(\Omega) \) will denote for simplicity the quantity \( V_u(1_{\Omega}, \mathbb{R}^d) \).

The following propositions reveal the link among functions of bounded variation, directional variation and sets of finite perimeter. The proof of these results can be found in [10].

**Proposition 2.1.** Let \( G \) be an open subset of \( \mathbb{R}^d \) and consider \( f \in L^1(\mathbb{R}^d) \). Then, \( V(f, G) \) is finite if and only if the directional variation \( V_u(f, G) \) is finite for every direction \( u \in \mathbb{S}^{d-1} \) and
\[
V(f, G) = \frac{1}{2w_{d-1}} \int_{\mathbb{S}^{d-1}} \mathcal{H}^{d-1}(du) V_u(f, G).
\]

In particular, for any \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) with finite perimeter, we have
\[
\text{Per}(\Omega) = \frac{1}{2w_{d-1}} \int_{\mathbb{S}^{d-1}} \mathcal{H}^{d-1}(du) V_u(\Omega).
\]

In addition, if \( g : \Omega \subseteq \mathbb{R}^d \to \mathbb{R} \) is a Lipschitz function, we denote
\[
\text{Lip}_\Omega(g) = \sup \left\{ \frac{|g(y) - g(x)|}{|y - x|} : x, y \in \Omega, x \neq y \right\}.
\]

**Proposition 2.2.** Let \( \Omega \in \mathcal{L}(\mathbb{R}^d) \) be such that \( |\Omega| \) is finite and consider \( g_\Omega \) its corresponding covariance function and \( u \in \mathbb{S}^{d-1} \). The following assertions are equivalent.

(i) \( V_u(\Omega) \) is finite.

(ii) \( \lim_{r \to 0} \frac{g_\Omega(0) - g_\Omega(ru)}{|r|} \) exists and is finite.

(iii) The real valued function \( g^*_\Omega(r) = g_\Omega(ru) \) is Lipschitz. Moreover,
\[
\text{Lip}_\Omega(g^*_\Omega) = \lim_{r \to 0} \frac{g_\Omega(0) - g_\Omega(ru)}{|r|} = \frac{V_u(\Omega)}{2}.
\]

3. **Proof of Theorem 1.1**

The key step to proving Theorem 1.1 consists on expressing the heat content \( \mathbb{H}_\Omega(t) \) in terms of the set covariance function \( g_\Omega \) whose support is contained in a ball centered at the origin and appeal to spherical coordinates to obtain a suitable decomposition.

To begin with, by applying Fubini's Theorem and performing a simple change of variable, we have based on (1.5) and (1.7) that
\[
\mathbb{H}_\Omega(t) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, p_t(x - y) \mathbb{1}_\Omega(y) \mathbb{1}_\Omega(x) = \int_{\mathbb{R}^d} dz \, p_t(z) g_\Omega(z).
\]

By using the scaling property of the Poisson kernel, namely, \( p_t(x) = t^{-d} p_1(t^{-1} x) \) and the change of variables \( w = t^{-1} z \), we arrive at
\[
\mathbb{H}_\Omega(t) = \int_{\mathbb{R}^d} dz \, p_t(z) g_\Omega(z) = \int_{\mathbb{R}^d} dz \, t^{-d} p_1(t^{-1} z) g_\Omega(z) = \int_{\mathbb{R}^d} dw \, p_1(w) g_\Omega(tw).
\]

Next, the facts that \( g_\Omega(tw) = 0 \) only when \( |tw| \geq \ell_\Omega \) (see (1.8)), \( g_\Omega(0) = |\Omega| \) and (4.10) lead to the following decomposition of \( \mathbb{H}_\Omega(t) \).
\[
\mathbb{H}_\Omega(t) = \int_{|tw| < \ell_\Omega} dw \, p_1(w) g_\Omega(tw) = |\Omega| \int_{|tw| < \ell_\Omega} dw \, p_1(w) + \int_{|tw| < \ell_\Omega} dw \, p_1(w) (g_\Omega(tw) - g_\Omega(0)) = |\Omega| - |\Omega| \int_{|tw| \geq \ell_\Omega} dw \, p_1(w) + \int_{|tw| < \ell_\Omega} dw \, p_1(w) (g_\Omega(tw) - g_\Omega(0)).
\]
Therefore, we conclude from the above identity that
\( \Omega | - \mathbb{H}_\Omega (t) = | \Omega | \phi_\Omega (t) + I_\Omega (t) , \)
where we have defined
\[
\phi_\Omega (t) = \int_{|tw| \geq \ell_\Omega} dw \, p_1 (w) , \\
I_\Omega (t) = \int_{|tw| < \ell_\Omega} dw \, p_1 (w) \, (g_\Omega (0) - g_\Omega (tw)) .
\]

We now keep decomposing the function \( I_\Omega (t) \) previously defined.

**Proposition 3.1.** Consider \( I_\Omega (t) \) as defined in (3.2) and set
\[
\Psi_\Omega (t) = \int_0^{t_\Omega t^{-1}} \frac{dr \, r^d}{(1 + r^2)^{\frac{d+1}{2}}}.
\]

Then,
\[
I_\Omega (t) = \frac{\text{Per} (\Omega)}{\pi} t \Psi_\Omega (t) - t R_\Omega (t)
\]
where
\[
R_\Omega (t) = \int_0^{t_\Omega t^{-1}} dr \, p_1 (r e_d) \gamma_\Omega (tr) \geq 0,
\]
with \( \gamma_\Omega \) as defined in (1.10).

**Proof.** We start by noticing that for \( t > 0 \) and \( r > 0 \) the integral
\[
\int_{S^{d-1}} H^{d-1} (du) \left( \frac{g_\Omega (0) - g_\Omega (tr)}{tr} \right)
\]
can be reexpressed using identities (2.2) and (1.10) as
\[
\frac{1}{2} \int_{S^{d-1}} H^{d-1} (du) V_u (\Omega) - \gamma_\Omega (tr) = \text{Per} (\Omega) w_{d-1} - \gamma_\Omega (tr).
\]

Now, with the aid of spherical coordinates and the fact that the Poisson kernel is a radial function, we can rewrite \( I_\Omega (t) \) as follows.
\[
I_\Omega (t) = t \int_0^{t_\Omega t^{-1}} dr \, r^d \, p_1 (r e_d) \int_{S^{d-1}} H^{d-1} (du) \left( \frac{g_\Omega (0) - g_\Omega (tr)}{tr} \right) .
\]

Therefore, the desired result follows by applying the identity (3.5) to (3.6). We have also used the fact that (1.3) and (1.1) imply that
\[
\kappa_d w_{d-1} = \Gamma \left( \frac{d + 1}{2} \right) \pi^{-\frac{d+1}{2}} \frac{\pi^{\frac{d-1}{2}}}{\Gamma (1 + \frac{d-1}{2})} = \frac{1}{\pi}.
\]

Finally, \( R_\Omega (t) \geq 0 \) is deduced from Proposition 2.2 part (iii) and this finishes the proof. \( \square \)

Before we carry on, we observe that we have shown by combining the previous result with identity (3.1) that
\[
| \Omega | - \mathbb{H}_\Omega (t) = | \Omega | \phi_\Omega (t) + \frac{\text{Per} (\Omega)}{\pi} t \Psi_\Omega (t) - t R_\Omega (t).
\]

Now, we proceed to study the small time behavior of the functions \( \phi_\Omega (t) \), \( \Psi_\Omega (t) \) and \( R_\Omega (t) \).

**Proposition 3.2.** Consider \( \phi_\Omega (t) \) for \( t > 0 \) as defined in (3.2). Then,
\( i \) \( 0 \leq \phi (t) \leq 1 \) is a nondecreasing function on \([0, \infty)\) with \( \phi_\Omega (0) = \lim_{t \to 0^+} \phi_\Omega (t) = 0. \)
\( ii \)
\[
\lim_{t \to 0^+} \frac{\phi_\Omega (t)}{t} = \frac{A_d \kappa_d}{\ell_\Omega},
\]
with \( \ell_\Omega, A_d \) and \( \kappa_d \) as defined in (1.8), (1.3) and (1.1), respectively.
Proof. To begin with, by appealing to spherical coordinates and the definition of the Poisson kernel, we have for \( t > 0 \) that
\[
\phi_{\Omega}(t) = \kappa_d \int_{|w| \geq \ell_\Omega} \frac{dw}{(1 + |w|^2)^{d/2}} = A_d \kappa_d \int_{\ell_\Omega}^\infty \frac{dr \, r^{d-1}}{t^2 (1 + r^2)^{d/2}}.
\]
It is clear that the above identity and (4.10) imply (i). We also note that \( \phi_{\Omega}(t) \) is differentiable over \((0, \infty)\) with
\[
\phi'_{\Omega}(t) = \frac{A_d \kappa_d \ell_\Omega^d}{(t^2 + \ell_\Omega^2)^{d/2}}.
\]
Then, by part (i), we can apply L'Hôpital's rule to obtain
\[
\lim_{t \to 0^+} \frac{\phi_{\Omega}(t)}{t} = \lim_{t \to 0^+} \phi'_{\Omega}(t) = \frac{A_d \kappa_d}{\ell_\Omega},
\]
which proves (ii) which in turn finishes the proof. \( \Box \)

**Proposition 3.3.** Let \( \Psi_{\Omega}(t) \) be the function defined in (3.3) for \( t > 0 \). Then,
\[
\Psi_{\Omega}(t) = \ln \left( \frac{1}{t} \right) + F_{\Omega}(t)
\]
where
\[
(3.8) \quad F_{\Omega}(t) = \ln \left( \ell_\Omega + \sqrt{\ell_\Omega^2 + t^2} \right) + \int_0^{\arcsinh(\ell_\Omega \, t^{-1})} d\theta \tanh^d(\theta) - 1.
\]
Furthermore,
\[
(3.9) \quad \lim_{t \to 0^+} F_{\Omega}(t) = \ln(2\ell_\Omega) + \int_0^\infty d\theta \left( \tanh^d(\theta) - 1 \right).
\]

**Proof.** The change of variable \( r = \sinh(\theta) \) turns the function \( \Psi_{\Omega}(t) \) defined in (3.3) into
\[
\int_0^{\arcsinh(\ell_\Omega \, t^{-1})} d\theta \tanh^d(\theta)
\]
that together with the identity
\[
\arcsinh(\ell_\Omega \, t^{-1}) = \ln \left( \frac{1}{t} \right) + \ln \left( \ell_\Omega + \sqrt{\ell_\Omega^2 + t^2} \right)
\]
produces the desired identity (3.8) since
\[
\Psi_{\Omega}(t) = \int_0^{\arcsinh(\ell_\Omega \, t^{-1})} d\theta \left( 1 + (\tanh^d(\theta) - 1) \right)
\]
\[
= \arcsinh(\ell_\Omega \, t^{-1}) + \int_0^{\arcsinh(\ell_\Omega \, t^{-1})} d\theta \left( \tanh^d(\theta) - 1 \right).
\]

On the other hand, the following equality
\[
\tanh^d(\theta) = \left( 1 - \frac{2}{e^{2\theta} + 1} \right)^d = 1 + \sum_{j=1}^d \binom{d}{j} \frac{(-2)^j}{(e^{2\theta} + 1)^j}
\]
and the fact that
\[
\int_0^\infty \frac{d\theta}{(e^{2\theta} + 1)^j} \leq \int_0^\infty d\theta \, e^{-2\theta \, j} = \frac{1}{2j}
\]
show that
\[
\int_0^\infty d\theta \left| \tanh^d(\theta) - 1 \right| \leq \sum_{j=1}^d \binom{d}{j} \frac{2^j}{2j}.
\]
Thus, an application of the Lebesgue dominated converge theorem shows that (3.9) holds and this finishes the proof. \( \Box \)
We now continue with the study of \( R_\Omega(t) \) which is the function that contains more geometric information about \( \Omega \) since it involves the geometric objects \( \ell_\Omega \) and \( g_\Omega \) as we can see from its definition in (3.4).

**Proposition 3.4.** Let \( R_\Omega(t) \) be the function defined in (3.4) and consider \( \gamma_\Omega \) as defined in (1.10). Then,

\[
(3.10) \quad \lim_{t \to 0^+} R_\Omega(t) = \kappa_d \int_0^1 ds s^{-1} \gamma_\Omega(\ell_\Omega s),
\]

with \( \kappa_d \) as defined in (1.1).

**Proof.** We know from (3.4) that

\[
R_\Omega(t) = \int_0^{\ell_\Omega t^{-1}} dr p_1(r \kappa_d \gamma_\Omega(tr)),
\]

for \( t > 0 \). Next, the change of variables \( r = \ell_\Omega t^{-1} s \) turns \( R_\Omega(t) \) into

\[
\ell_\Omega^{d+1} \kappa_d \int_0^1 ds \frac{s^d \gamma_\Omega(\ell_\Omega s)}{(t^2 + \ell_\Omega^2 s^2)^{d+1}}.
\]

Now, the basic inequality \( \ell_\Omega^2 s^2 \leq t^2 + \ell_\Omega^2 s^2 \) shows that

\[
0 \leq R_\Omega(t) \leq \kappa_d \int_0^1 ds s^{-1} \gamma_\Omega(\ell_\Omega s).
\]

In particular, we conclude

\[
(3.11) \quad \lim_{t \to 0^+} R_\Omega(t) \leq \kappa_d \int_0^1 ds s^{-1} \gamma_\Omega(\ell_\Omega s).
\]

On the other hand, since \( \gamma_\Omega \geq 0 \) due to Proposition 2.2, part (iii), we obtain by Fatou’s lemma that

\[
(3.12) \quad \lim_{t \to 0^+} R_\Omega(t) \geq \ell_\Omega^{d+1} \kappa_d \int_0^1 ds \lim_{t \to 0^+} \frac{s^d \gamma_\Omega(\ell_\Omega s)}{(t^2 + \ell_\Omega^2 s^2)^{d+1}} = \kappa_d \int_0^1 ds s^{-1} \gamma_\Omega(\ell_\Omega s).
\]

Hence, the desired result follows from (3.11) and (3.12). \( \square \)

**Remark 3.1.** Observe that in the proof of Proposition 3.4, the condition \( \gamma_\Omega(\ell_\Omega s) \in L^1((0, 1), s^{-1}ds) \) is not required, which makes even more interesting the open problem concerning the existence of a bounded set \( \Omega \) with finite perimeter such that \( \Omega \notin \mathcal{W} \) where the class \( \mathcal{W} \) is given in definition 1.2.

Now that we have all the necessary facts at hand, we proceed to give the proof of Theorem 1.1.

**Proof of Theorem 1.1:** Because of the identity (3.7) and Proposition 3.3, we obtain

\[
(3.13) \quad \frac{1}{t} \left( \mid \Omega \mid - \mathcal{H}_\Omega(t) - \frac{\text{Per} \Omega}{\pi} t \ln \left( \frac{1}{t} \right) \right) = \mid \Omega \mid \frac{\phi_\Omega(t)}{t} + \frac{\text{Per} \Omega}{\pi} F_\Omega(t) - R_\Omega(t).
\]

Therefore, we note that the required limit (1.11) comes from combining together the results given in propositions 3.2, 3.3 and 3.4.

The following two sections are not only introduced to compute some limits but also to show how relevant is the geometry of the underlying set \( \Omega \) under consideration when performing calculations.

### 4. Computation of the Third Term for the Unit Ball

In order to compute the limit given in Theorem 1.1 for the unit ball \( B = B_1(0) \), we require to find the explicit expression for \( g_B(z) \) which can be done because of the symmetry of the ball.

Observe that \( B + z = B_1(z) = \{x \in \mathbb{R}^d : |x - z| \leq 1\} \) which implies that \( g_B(z) = |B \cap (B + z)| \) represents the volume of the intersection of two balls of radii one. Consequently, it is clear that

\[
\text{supp}(g_B) = \{z \in \mathbb{R}^d : g_B(z) \neq 0\} = B_2(0),
\]

which in turn entails that \( \ell_B = \inf \{\ell > 0 : g_B(z) = 0\} \) for all \(|z| \geq \ell\) = 2.
It is also geometrically clear that $g_B(z) = g_B(Tz)$ for any orthonormal linear transformation $T$ on $\mathbb{R}^d$, which implies that $g_B$ is a radial function so that
\begin{equation}
(4.1) \quad g_B(z) = g_B(|z|e_d),
\end{equation}
with $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$.

The following lemma provides a formula for the volume of the intersection of two unit balls in $\mathbb{R}^d$ whose proof can be found in [2].

**Lemma 4.1.** Let $d \geq 2$ be an integer and $0 \leq s \leq 1$. Let $B = B_1(0)$ and $B(2s) = B_1(2se_d)$. Then, we have
\begin{equation}
(4.2) \quad g_B(2se_d) = |B \cap B(2s)| = 2 A_{d-1} \Theta \left( \sqrt{1 - s^2} \right) - 2s w_{d-1} \left( 1 - s^2 \right)^{\frac{d-1}{2}},
\end{equation}
where
\begin{equation}
\Theta(z) = \int_0^{\arcsin(z)} d\theta \sin^{d-2}(\theta) \cos^2(\theta),
\end{equation}
for $0 \leq z \leq 1$ and $w_{d-1}$ as defined in (1.3). In particular, we obtain
\begin{enumerate}[i)]
\item $\Theta(1) = \frac{|B|}{2A_{d-1}} = \frac{w_d}{2A_{d-1}},$
\item $\Theta(z) = \begin{cases} \frac{1}{4} \left( \arcsin(z) + z\sqrt{1 - z^2} \right) & \text{if } d = 2, \\ \frac{1}{3} \left( 1 - (1 - z^2)^{\frac{3}{2}} \right) & \text{if } d = 3. \end{cases}$
\end{enumerate}

(4.5) \quad g_B(2se_d) \begin{cases} 2 \arcsin(\sqrt{1 - s^2}) - 2s \sqrt{1 - s^2}, & \text{if } d = 2, \\ \frac{4\pi}{3}(1 - s^3) - 2\pi s(1 - s^2), & \text{if } d = 3. \end{cases}

With the previous estimates at hand, we proceed to show the finiteness of $\int_0^1 ds s^{-1} \gamma_B(2s)$ for all $d \geq 2$ integer.

**Theorem 4.1.** For all $d \geq 2$ integer, we have that
\begin{equation}
(4.6) \quad \gamma_B(2s) = A_d \left( w_{d-1} \left\{ 1 - (1 - s^2)^{\frac{d-1}{2}} \right\} - A_{d-1} \left\{ \frac{\Theta(1) - \Theta(\sqrt{1 - s^2})}{s} \right\} \right)
\end{equation}
where $w_{d-1}$, $A_d$, $A_{d-1}$ and $\Theta$ as defined in (1.3) and (4.3), respectively. As a result,
\begin{equation}
(4.7) \quad 0 \leq \int_0^1 ds s^{-1} \gamma_B(2s) \leq \frac{A_d w_{d-1} \sigma_d}{2},
\end{equation}
with $\sigma_d = 1_{\{2\}}(d) + \frac{d-1}{2} \cdot 1_{\{3, \infty\}}(d)$.

**Proof.** Consider $u \in \mathbb{S}^{d-1}$ and observe by using (4.1) that $g_B(2su) = g_B(2se_d)$ for $s > 0$ and $g_B(0) = |B| = 2A_{d-1}\Theta(1)$ due to (4.4). Thus, it follows from by identity (4.2) that
\begin{equation}
(4.8) \quad \frac{g_B(0) - g_B(2su)}{2s} = \frac{g_B(0) - g_B(2se_d)}{2s} = A_{d-1} \left( \frac{\Theta(1) - \Theta(\sqrt{1 - s^2})}{s} \right) + w_{d-1}(1 - s^2)^{\frac{d-1}{2}},
\end{equation}
for $0 < s \leq 1$. We note that the above identity implies based on Proposition 2.2, part (iii) that $V_u(B) = V_{e_d}(B)$ for all $u \in \mathbb{S}^{d-1}$, which in turn implies by (2.2) that
\begin{equation}
Per(B) = \frac{1}{2w_{d-1}} \int_{\mathbb{S}^{d-1}} \mathcal{H}^{d-1}(du) V_u(B) = \frac{V_{e_d}(B)}{2w_{d-1}} Per(B).
\end{equation}
Therefore, we conclude that $V_u(B) = 2w_{d-1}$ for all $u \in \mathbb{S}^{d-1}$.
We remark that the aforementioned facts also imply identity (4.6) since by using (1.10) and the fact that \( H^{d-1}(\mathbb{S}^{d-1}) = A_d \), we arrive at

\[
\gamma_B(2s) = \int_{\mathbb{S}^{d-1}} H^{d-1}(du) \left( \frac{V_a(B)}{2} - \frac{(g_B(0) - g_B(2su))}{2s} \right) = A_d \left( \frac{V_a(B)}{2} - \frac{(g_B(0) - g_B(2sc_d))}{2s} \right) = A_d \left( \frac{w_d(1 - (1 - s^2)^{\frac{d+1}{2}})}{s} - A_d - \left( \frac{\Theta(1) - \Theta(\sqrt{1 - s^2})}{s} \right) \right).
\]

On the other hand, by appealing to the basic inequality \( 1 - \sigma dx \leq (1 - x)^{\frac{d-2}{2}} \) for \( 0 \leq x \leq 1 \), we conclude that \( 1 - (1 - s^2)^{\frac{d+1}{2}} \leq \sigma ds^2 \) for \( 0 < s \leq 1 \). Hence, due to (4.6) and the fact that \( \Theta(z) \) is an increasing function as long as \( z \in [0, 1] \), we arrive at

\[
0 \leq \gamma_B(2s) \leq A_d w_d(1 - (1 - s^2)^{\frac{d+1}{2}}) \leq A_d w_d(1) \sigma ds^2
\]

for all \( 0 < s \leq 1 \) which in turn implies inequality (4.7) and this finishes the proof. \( \square \)

Now, we have all the necessary tools to compute explicitly the limit in Theorem 1.1 for the unit ball in the plane and space.

**Corollary 4.1.** Let \( B = B_1(0) \) be the unit ball in \( \mathbb{R}^d \). Then,

\[
(4.9) \quad \lim_{t \to 0+} \frac{1}{t} \left( |B| - \mathbb{H}_B(t) - \frac{\text{Per}(B)}{\pi} t \ln \left( \frac{1}{t} \right) \right) = \begin{cases} 6 \ln(2) - 2 & \text{if } d = 2, \\ 4 \ln(2) & \text{if } d = 3. \end{cases}
\]

**Proof.** By using identity (4.6) and (4.4), we have that

\[
\gamma_B(2s) = \begin{cases} 2\pi \left( 2 - \sqrt{1 - s^2} + \frac{1}{\pi s} \{ 2 \arcsin(\sqrt{1 - s^2}) - \pi \} \right) & \text{if } d = 2, \\ \frac{4\pi^2}{s^2} & \text{if } d = 3, \end{cases}
\]

which in turn implies by using an integral calculator that

\[
(4.10) \quad \int_0^1 ds s^{-1} \gamma_B(2s) = \begin{cases} \pi(\pi - 4 \ln(2)) & \text{if } d = 2, \\ \frac{2\pi^2}{s^2} & \text{if } d = 3. \end{cases}
\]

Therefore, by combining together the values given in the table below together with (4.10), we deduce the desired result. \( \square \)

| Values for the unit ball |
|--------------------------|
| \( d \) | \( A_d = \text{Per}(B) \) | \( w_d = |B| \) | \( \kappa_d \) | \( \int_0^\infty d\theta(\tanh^d(\theta) - 1) \) |
|---|---|---|---|
| 2 | \( 2\pi \) | \( \pi \) | \( \frac{1}{\pi} \) | \( -1 \) |
| 3 | \( 4\pi \) | \( \frac{4\pi}{3} \) | \( \frac{1}{\pi^2} \) | \( -\ln(2) - \frac{1}{2} \) |

5. Results Related to the Square \([-1, 1] \times [-1, 1]\). 

In this section, \( Q \) will denote the square \([-1, 1] \times [-1, 1] \) in \( \mathbb{R}^2 \) so that \( \ell_Q = \text{diam}(Q) = 2\sqrt{2} \). The purpose in this section is to provide an alternative expression for \( \int_0^1 ds s^{-1} \gamma_Q(2\sqrt{2}s) \) which allows us to show its finiteness and find its explicit expression.

**Proposition 5.1.** Consider \( (a, b) \in \mathbb{R}^2 \) and \( g_Q(a, b) \) the covariance function of \( Q \). Then,

\[
(5.1) \quad g_Q(a, b) = (2 - |a|)(2 - |b|)1_{[-2,2] \times [-2,2]}(a, b) \quad (4 - 2 \{ |a| + |b| \} + |ab|) 1_{[-2,2] \times [-2,2]}(a, b).
\]

As a consequence, \( \text{supp}(g_Q) = \{ z \in \mathbb{R}^2 : g_Q(z) \neq 0 \} = [-2, 2] \times [-2, 2] \).
ii) If \( u = (\cos(\theta), \sin(\theta)) \in S^1, \theta \in (0, 2\pi) \), we have that

\[
V_u(Q) = 2 \{ |\cos(\theta)| + |\sin(\theta)| \} = 2 \{ |u \cdot e_1| + |u \cdot e_2| \},
\]

where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \).

**Proof.** To begin with, we remark that for any \((a, b) \in \mathbb{R}^2\), we have that \( Q + (a, b) \) is the square with sides of length 2 centered at \((a, b)\) with vertices \( v_1 = (a + 1, b + 1), v_2 = (a - 1, b + 1), v_3 = (a - 1, b - 1) \) and \( v_4 = (a + 1, b - 1) \). Now, when \( Q + (a, b) \) intersects \( Q \), one of the vertices \( v_1, v_2, v_3 \) or \( v_4 \) will belong to one of the sub-squares on each quadrant conforming \( Q \) and this vertex determines the area of the rectangle \((Q + (a, b)) \cap Q\). For example, if we consider the vertex \( v_1 \) such that \( v_1 \) will belong to one of the sub-squares, we obtain that \( g_Q(a, b) \) will be equal to \((2 + a)(2 + b)\) as long as \((a, b) \in [-2, 0] \times [-2, 0]\). Now, by using the symmetric property of the covariance function, namely, \( g_Q(w) = g_Q(-w) \), we obtain that \( g_Q(a, b) = (2 - a)(2 - b) \) whenever \((a, b) \in [0, 2] \times [0, 2]\), which corresponds to the case of considering the vertex \( v_3 \) belonging to one of the sub–squares conforming \( Q \). Following the same reasoning with the vertex \( v_2 \) and \( v_4 \), we conclude identity (5.1).

Regarding ii), we know by Proposition 2.2 that

\[
\frac{V_u(Q)}{2} = \lim_{r \to 0} \frac{g_Q(0, 0) - g_Q(ru)}{r},
\]

for every \( u \in S^1 \). Next, by assuming that \( 0 < r \leq 1 \), we have that

\( ru \in B_1((0, 0)) \subseteq [-2, 2] \times [-2, 2] \).

Thus, by (5.1), we have for every \( 0 < r \leq 1 \) and \( u = (\cos(\theta), \sin(\theta)) \) that

\[
\frac{g_Q(0, 0) - g_Q(ru)}{r} = 2 \{ |\cos(\theta)| + |\sin(\theta)| \} - r \cos(\theta) \sin(\theta),
\]

which in turn implies (5.2) due to (5.3). \( \square \)

The following lemma corresponds to the representation of the square \([-2, 2] \times [-2, 2]\) in polar coordinates whose proof is omitted since it is a basic problem of multi–variable calculus.

**Lemma 5.1.** Consider \( \theta_i = \frac{2\pi}{7} i \) for \( i \in \{0, ..., 8\} \). Define for \( i \in \{0, ..., 7\} \), \( \eta_i : [\theta_i, \theta_{i+1}] \to \mathbb{R} \) by

\[
\eta_i(\theta) = \frac{2}{\{ |\cos(\theta)| 1_{(0,3,4,7)}(i) + |\sin(\theta)| 1_{(1,2,5,6)}(i) \}}.
\]

Then,

i) for all \( i \in \{0, 1, ..., 7\} \), we have that

\[
2 \leq \eta_i(\theta) \leq 2\sqrt{2},
\]

and each \( \eta_i \) is a continuous function over \([\theta_i, \theta_{i+1}]\).

ii) for all \( r \geq 0 \) and \( u = (\cos(\theta), \sin(\theta)) \) with \( \theta \in (0, 2\pi] \), we have

\[
\mathbb{I}_{[-2,2] \times [-2,2]}(ru) = \sum_{i=0}^{7} \mathbb{I}_{[\theta_i, \theta_{i+1}]}(\theta) \mathbb{I}_{[\eta_i, \eta_i]}(r).
\]

**Remark 5.1.** Observe that every point \( \sigma \) in the boundary of \([-2, 2] \times [-2, 2]\) determines a unique angle \( \theta \in (0, 2\pi] \). The distance from \( \sigma \) to \((0, 0)\) is given by the \( \eta_i(\theta) \)’s defined in the previous lemma. Each \( \sigma \) belongs to the side of a right triangle where this side coincides in part with one of the sides of the square \([-2, 2] \times [-2, 2]\) and one of the vertices of that triangle is located at \((0, 0)\) from which we can see that inequality (5.4) holds true.

Now, we proceed to show that finiteness of \( \int_0^1 ds s^{-1} \gamma_Q(2\sqrt{2}s) \).
Theorem 5.1. Let \( \theta_i = \frac{\pi}{8} i \) for \( i \in \{0, ..., 8\} \) and consider \( \gamma_Q \) as defined in (1.10). Then,

\[
\int_0^1 dss^{-1} \gamma_Q(2\sqrt{s}) = \sum_{i=0}^{7} I_i,
\]

where

\[
I_i = \int_{\theta_i}^{\theta_{i+1}} d\theta \left( |\cos(\theta)| \sin(\theta) |\eta_i(\theta)| + 2 \{ |\cos(\theta)| + |\sin(\theta)| \} \ln \left( \frac{2\sqrt{s}}{\eta_i(\theta)} \right) + \sqrt{2} \left( 1 - \frac{2\sqrt{s}}{\eta_i(\theta)} \right) \right).
\]

In particular, \( \int_0^1 dss^{-1} \gamma_Q(2\sqrt{s}) \) is finite since each integrand involved in the term \( I_i \) is a continuous function over the compact set \([\theta_i, \theta_{i+1}]\) due to Lemma 5.1.

Proof. Let \( u = (\cos(\theta), \sin(\theta)) \in S^1 \) with \( \theta \in (0, 2\pi) \). Define for \( 0 < s \leq 1 \),

\[
\Lambda_Q(u, s) = \frac{V_u(Q)}{2} - \frac{1}{2\sqrt{s}} \left( g_Q(0, 0) - g_Q(2\sqrt{2}su) \right),
\]

and notice that \( \Lambda_Q(u, s) \geq 0 \) because of Proposition 2.2, part (iii). Now, by combining (5.1) and (5.5), we arrive at

\[
g_Q(2\sqrt{2}su) = \begin{cases} 
4 - 4\sqrt{2}s \{ |\cos(\theta)| + |\sin(\theta)| \} + 8s^2 |\cos(\theta)| \sin(\theta) | & \text{if } 0 < s \leq \frac{\eta(\theta)}{2\sqrt{2}}, \\
0 & \text{if } \frac{\eta(\theta)}{2\sqrt{2}} < s \leq 1.
\end{cases}
\]

which in turn implies by using (5.2) that \( \Lambda_Q(u, s) \) is equal to

\[
\sum_{i=0}^{7} \mathbb{1}_{(\theta_i, \theta_{i+1})} \left( 2\sqrt{2}s |\cos(\theta)| \sin(\theta)| \mathbb{1}_{[0, \eta(\theta)]}(s) + \left( 2(|\cos(\theta)| + |\sin(\theta)|) - \frac{\sqrt{2}}{s} \right) \mathbb{1}_{[\frac{\eta(\theta)}{2\sqrt{2}}, 1]}(s) \right).
\]

Next, observe that by (1.10), we have that

\[
\gamma_Q(2\sqrt{2}) = \int_0^{2\pi} d\theta \Lambda_Q(u, s),
\]

which implies by Fubini’s theorem that

\[
\int_0^1 dss^{-1} \gamma_Q(2\sqrt{s}) = \int_0^{2\pi} d\theta \int_0^1 dss^{-1} \Lambda_Q(u, s) = \sum_{i=0}^{7} \int_{\theta_i}^{\theta_{i+1}} d\theta \int_0^1 dss^{-1} \Lambda_Q(u, s).
\]

On the other hand, notice that for \( \theta \in (\theta_i, \theta_{i+1}] \), we obtain due to (5.7) that

\[
\int_0^1 dss^{-1} \Lambda_Q(u, s) = \int_0^{\eta(\theta)} dss^{-1} \Lambda_Q(u, s) + \int_{\eta(\theta)}^{1} dss^{-1} \Lambda_Q(u, s) = |\cos(\theta)| \sin(\theta)| \mathbb{1}_{[0, \eta(\theta)]}(s) + \sqrt{2} \left( 1 - \frac{2\sqrt{s}}{\eta_i(\theta)} \right),
\]

and thus the desired result follows. \( \square \)

We now proceed to calculate the term \( I_i \) provided in the last theorem. The values to be given have been calculated by using a computational method.
Lemma 5.2.  
\[ I_0 = \int_0^{\pi/4} d\theta \left( 2 \sin(\theta) + 2 \{\cos(\theta) + \sin(\theta)\} \ln \left( \sqrt{2} \cos(\theta) \right) + \sqrt{2}(1 - \sqrt{2} \cos(\theta)) \right) \]
\[ = 2 \ln(2 + \sqrt{2}) + \frac{\sqrt{2}}{4}(\pi - 8). \]
\[ I_1 = \int_{\pi/4}^{\pi/2} d\theta \left( 2 \cos(\theta) + 2 \{\cos(\theta) + \sin(\theta)\} \ln \left( \sqrt{2} \sin(\theta) \right) + \sqrt{2}(1 - \sqrt{2} \sin(\theta)) \right) \]
\[ = 2 \ln(2) - 2 \ln \left( 2 + \sqrt{2} \right) - 4 \ln(\sqrt{2} - 1) + \frac{\sqrt{2}}{4}(\pi - 8). \]

Proof. To begin with, for \( 0 \leq \theta \leq \frac{\pi}{2} \) we have that \( \cos(\theta) \geq 0 \) and \( \sin(\theta) \geq 0 \). By appealing to the definitions of the \( \eta_i \)’s given in Lemma 5.1, we have that
\[ \eta_i(\theta) = \begin{cases} 
\frac{-2}{\cos(\theta)} & \text{if } i = 0, \\
\frac{-2}{\sin(\theta)} & \text{if } i = 1.
\end{cases} \]
Thus, the desired results are obtained by turning to the definition of \( I_0 \) and \( I_1 \) given in Theorem 5.1. □

Lemma 5.3.  
\[ I_2 = \int_{\pi/2}^{3\pi/4} d\theta \left( -2 \cos(\theta) + 2 \{\sin(\theta) - \cos(\theta)\} \ln \left( \sqrt{2} \sin(\theta) \right) + \sqrt{2}(1 - \sqrt{2} \sin(\theta)) \right) \]
\[ = 2 \ln(2) - 2 \ln(2 + \sqrt{2}) + 4 \ln(\sqrt{2} + 1) + \frac{\sqrt{2}}{4}(\pi - 8). \]
\[ I_3 = \int_{3\pi/4}^{\pi} d\theta \left( 2 \sin(\theta) + 2 \{\sin(\theta) - \cos(\theta)\} \ln \left( -\sqrt{2} \cos(\theta) \right) + \sqrt{2}(1 + \sqrt{2} \cos(\theta)) \right) \]
\[ = 2 \ln(2 + \sqrt{2}) + \frac{\sqrt{2}}{4}(\pi - 8). \]

Proof. For \( \frac{\pi}{2} \leq \theta \leq \pi \), we have that \( |\cos(\theta)| = -\cos(\theta) \) and \( |\sin(\theta)| = \sin(\theta) \). Also, we obtain that
\[ \eta_i(\theta) = \begin{cases} 
\frac{-2}{\sin(\theta)} & \text{if } i = 2, \\
\frac{-2}{\cos(\theta)} & \text{if } i = 3.
\end{cases} \]
The proof is complete by using the definition of \( I_2 \) and \( I_3 \) given in Theorem 5.1. □

Lemma 5.4. We have that
a) \( I_0 = I_4, \quad I_1 = I_5 \).
b) \( I_2 = I_6, \quad I_3 = I_7 \).

Proof. By using the identities \( \sin(\theta_0 - \pi) = -\sin(\theta_0) \) and \( \cos(\theta_0 - \pi) = -\cos(\theta_0) \), we observe by performing the change of variable \( \theta = \theta_0 - \pi \) in the integral expressions of \( I_0 \) and \( I_1 \) provided in Lemma 5.2 that
\[ I_0 = \int_{\pi}^{5\pi/4} d\theta_0 \left( -2 \sin(\theta_0) - 2 \{\cos(\theta_0) + \sin(\theta_0)\} \ln \left( -\sqrt{2} \cos(\theta_0) \right) + \sqrt{2} \left( 1 + \sqrt{2} \cos(\theta_0) \right) \right), \]
\[ I_1 = \int_{5\pi/4}^{3\pi/2} d\theta_0 \left( -2 \cos(\theta_0) - 2 \{\cos(\theta_0) + \sin(\theta_0)\} \ln \left( -\sqrt{2} \sin(\theta_0) \right) + \sqrt{2} \left( 1 + \sqrt{2} \sin(\theta_0) \right) \right). \]
It follows from Theorem 5.1 that the last expressions are equal to \( I_4 \) and \( I_5 \), respectively, since when \( \pi \leq \theta_0 \leq \frac{3\pi}{4} \), then \( |\cos(\theta_0)| = -\cos(\theta_0), \ |\sin(\theta_0)| = -\sin(\theta_0) \) and
\[ \eta_i(\theta_0) = \begin{cases} 
\frac{-2}{\cos(\theta_0)} & \text{if } i = 4, \\
\frac{-2}{\sin(\theta_0)} & \text{if } i = 5.
\end{cases} \]
On the other hand, a similar argument shows that
\[ I_2 = \int_{\pi/2}^{\pi/4} d\theta_0 \left( 2 \cos(\theta_0) + 2(\cos(\theta_0) - \sin(\theta_0)) \ln \left( -\sqrt{2} \sin(\theta_0) \right) + \sqrt{2} \left( 1 + \sqrt{2} \sin(\theta_0) \right) \right), \]
\[ I_3 = \int_{\pi/4}^{2\pi} d\theta_0 \left( -2 \sin(\theta_0) + 2(\cos(\theta_0) - \sin(\theta_0)) \ln \left( \sqrt{2} \cos(\theta_0) \right) + \sqrt{2} \left( 1 - \sqrt{2} \cos(\theta_0) \right) \right), \]
which are equivalent according to Theorem 5.1 to the integral expressions of \( I_6 \) and \( I_7 \), respectively.

Finally, based on the previous estimates and Theorem 1.1, we arrive at

**Corollary 5.1.** Let \( Q \) be the square \([-1, 1] \times [-1, 1]\). Then,

(i) \[
\int_0^1 ds \ s^{-1} \gamma_Q(2\sqrt{2}s) = 2\sqrt{2}(\pi - 8) + 8 \ln \left( 2\left( 3 + 2\sqrt{2} \right) \right). 
\]

(ii) \[
\lim_{t \to 0^+} \frac{1}{t} \left( |Q| - H_Q(t) - \frac{\text{Per}(Q)}{\pi} t \ln \left( \frac{1}{t} \right) \right) = \frac{4}{\pi} \left( 2\sqrt{2} - 1 \right) + \ln \left( \frac{16}{3 + 2\sqrt{2}} \right). 
\]

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