A NOTE ON INTEGRATING GROUP SCHEME ACTIONS

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Abstract. We prove a non-integrability result concerning iterative derivations on projective line, where the iterative rule is given by a non-algebraic formal group.

1. Introduction

Let \( \partial \) be a derivation on a \( \mathbb{Q} \)-algebra. Then \( \partial \) expands (uniquely) to an iterative Hasse-Schmidt derivation by the formula \( \left( \frac{\partial^{(n)}}{n!} \right)_n \). Matsumura proved that the same (without uniqueness) is still true in the case of fields of positive characteristic \( p \) for derivations satisfying the (necessary) \( \partial^{(p)} = 0 \) condition \[14\]. In Matsumura’s terminology, such a derivation \( \partial \) is \( (strongly) \ integrate\). One may wonder why to consider the iterativity condition of such a specific form (although the characteristic 0 example gives a rather strong motivation). It was noticed by Matsumura that actually the iterativity condition as above is governed by the additive group law \( X + Y \). Since in characteristic 0, any (one-dimensional) formal group is isomorphic to the additive one, it is a good choice indeed. However in the case of positive characteristic there are many more formal group laws and it is an interesting question whether the corresponding derivations are integrable. A multiplicative version of Matsumura’s theorem was proved by Tyc \[19\], where the condition \( \partial^{(p)} = 0 \) is replaced with the (necessary again) condition \( \partial^{(p)} = \partial \).

In \[9\], the authors considered a more general problem of integrating \( m \)-truncated Hasse-Schmidt derivations (i.e. sequences \( \left( \frac{\partial^{(n)}}{n!} \right)_{n<pm} \) satisfying higher Leibnitz rules and appropriate iterativity conditions). Since a 1-truncated additively iterative Hasse-Schmidt derivation is equivalent to a standard derivation \( \partial \) satisfying the condition \( \partial^{(p)} = 0 \) (similarly in the multiplicative case, where the necessary condition is \( \partial^{(p)} = \partial \)), this is a natural generalization. We extend the Matsumura’s and Tyc’s results in \[9\] to an arbitrary truncation (in the additive case, such a generalization is implicit in work of Ziegler \[21\]). A certain class of higher-dimensional commutative affine algebraic groups is treated by the first author in \[7\]. In this paper, we focus on the one-dimensional case and we comment briefly on the higher-dimensional cases in Section 3.1.

For other (than additive or multiplicative) formal group laws, the following question remains (as far as we know) open.

Question 1.1. Let \( F \) be a formal group law and \( \mathcal{D} \) be an \( F[m] \)-derivation on a field \( K \). Does \( \mathcal{D} \) expand to an \( F \)-derivation on \( K \)?

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In this paper, we proceed into the direction of the negative answer to Question 1.1. We show that for a formal group law $F$ which is non-isomorphic to the additive one or to the multiplicative one, derivations on projective line can not be “$F$-integrated” (Corollary 3.4).

We would like to point out that integrability of a derivation (with a given integrability rule) is related to the model-theoretic properties of existentially closed fields with such derivations (see [21], [11] or [8]).

This paper is organized as follows. In Section 2, we recall the connection between Hasse-Schmidt derivations and group scheme actions. In Section 3, we prove the main theorem of this paper about non-integrability of certain Hasse-Schmidt derivations on the projective line. In Section 4, we speculate about a possible approach to Question 1.1 which uses truncated Galois group schemes.

2. Group scheme actions

In this section, we interpret the notion of a (truncated) Hasse-Schmidt derivation (abbreviated as HS-derivation in the sequel) in terms of group scheme actions, recall the automorphism group functor and clarify the correspondence between the Lie algebra actions and the group scheme actions. We fix a field $k$ of positive characteristic $p$ and a $k$-algebra $R$. We also fix a complete local $k$-algebra $R$ with the maximal ideal $I$ and $m > 0$. For the notions of an $f$-derivation for a truncated formal group law $f$ and an $F$-derivations for a formal group law $F$, we refer to [9].

2.1. Definitions and bijective correspondences. The category of affine group schemes over $k$ is the category opposite to the category of Hopf algebras over $k$ [20, Section 1.4] (or it is the category of representable functors from $k$-algebras to groups, see [20, Section 1.2]). A truncated group scheme [2] over $k$ is an affine group scheme whose universe is of the form

$$\text{Spec}(k[v_1, m_1, \ldots, v_k m_k]) = \text{Spec} \left( k[X_1, \ldots, X_k]/(X_1^{p^{m_1}}, \ldots, X_k^{p^{m_k}}) \right).$$

By [20, Section 14.4] such group schemes coincide with finite connected group schemes (called infinitesimal in [3] and local in [13]). In this paper we mostly consider truncated group scheme structures on $\text{Spec}(k[v_m]) = \text{Spec} \left( k[X]/(X^{p^m}) \right)$ (the “ordinary” case). Any $m$-truncated group law $f$ naturally gives a truncated group scheme $G_f$ corresponding to the Hopf algebra with the following comultiplication map

$$\text{ev}_f(v_m \otimes 1 \otimes v_m) : k[v_m] \to k[v_m] \otimes k[v_m].$$

To go to the opposite direction, one needs to choose a generator of the $k$-algebra $k[v_m]$ (clearly, there is a natural choice).

Remark 2.1. If $\text{char}(k) = 0$, then by a theorem of Cartier [20, Section 11.4] all Hopf algebras over $k$ are reduced, so there is no reasonable notion of a truncated group scheme.

The category of formal groups over $k$ is the category opposite to the category of complete Hopf algebras over $k$ (or the category of representable functors from complete $k$-algebras to groups, see [5, Chapter VII]). Similarly as in the truncated case, there is a correspondence between formal groups and formal group laws. Note that a truncated group scheme is both an affine group scheme and a formal group.

Let $f$ be a truncated group law over $k$ giving a truncated group scheme $G_f$. There is a natural correspondence between:
(1) The set of all $f$-derivations on $R$.
(2) The set of all $k$-group scheme actions of $G_f$ on $\text{Spec}(R)$.

Since a formal group law $F$ can be approximated by a direct system of truncated group laws $F[m]$ (for a more precise statement, see Section 2.2), we get a similar correspondence between:

(1) The set of all $F$-derivations on $R$.
(2) The set of all systems of compatible $k$-group scheme actions of $G_F[m]$ on $\text{Spec}(R)$ for $m > 0$.

For an arbitrary $k$-scheme $V$ it makes sense to talk about $f$-derivations or $F$-derivations on $V$ using the conditions of type (2) above.

**Remark 2.2.** Let $V$ be an arbitrary $k$-scheme.

(i) Since any HS-derivation on $R$ uniquely extends to the structure sheaf of $\text{Spec}(R)$, we get another equivalent conditions (say, in the group scheme case and for an arbitrary $k$-scheme):

(2') The set of all $f$-derivations on the structure sheaf of $V$.

(ii) By the remark below Corollary 5.2 in [12], we have yet another equivalent condition (again, in the group scheme case):

(2") The set of all sections $V \to \text{Arc}_m(V)$ of the projection $\text{Arc}_m(V) \to V$ satisfying the commutative diagram as below Corollary 5.2 in [12].

Note that $\text{Arc}_m$ is the $m$-th arc space functor (see [4]) and $\text{Arc}_1$ coincides with the tangent space.

(iii) For a more general context (including and going beyond HS-derivations), the reader is advised to consult Example 2.12 in [17].

Consider the automorphism functor $A_{V/k}$, associated to a $k$-scheme $V$, from the category of $k$-schemes to the category of groups. For a $k$-scheme $W$, we have

$$A_{V/k}(W) = \text{Aut}_W(V \times_k W)$$

(see [15] page 11]). If this functor is representable we denote the representing $k$-group scheme by $\text{Aut}_k(V)$. Assume that $A_{V/k}$ is representable. Then for the notion of an $f$-derivation in the case of a truncated group law $f$, we have a third equivalent condition, just like in the case of the usual group actions (using Yoneda Lemma), since there is always a natural map between the group functor represented by $G_f$ and the group functor $A_{V/k}$.

(3) The set of all $k$-group scheme morphisms $G_f \to \text{Aut}_k(V)$.

Similarly in the case of a formal group law $F$:

(3) The set of all compatible systems of $k$-group scheme morphisms

$$\left(G_F[m] \to \text{Aut}_k(V)\right)_{m>0}.$$

**Remark 2.3.** We point out two cases (special cases of [15] Theorem 3.7) when the functor $A_{V/k}$ is representable:

- $V$ is a projective variety over $k$;
- $V = \text{Spec}(K)$ such that the extension $k \subseteq K$ is finite (including inseparable extensions!), see [15] Example 3].
2.2. Truncations of group schemes. Let $G$ be an affine group scheme over $k$, $H$ the corresponding Hopf algebra, $\mathfrak{m}$ the kernel of the counit map $H \to k$ (the augmentation ideal) and $m > 0$. Using the base-change given by the automorphism $\text{Fr}_m : k \to k$, we get the affine group scheme over $k$

$$G^{(m)} := G \otimes_{(k, \text{Fr}_m)} k,$$

and a group scheme morphism $\text{Fr}_G^m : G \to G^{(m)}$. Let $G[m]$ be the kernel of $\text{Fr}_G^m$ which is a truncated $k$-group scheme. It corresponds to the quotient Hopf algebra $H[m] := H/\text{Fr}_G^m(H)$.

We get a direct system of truncated $k$-group schemes $(G^{(m)})_m \in \mathbb{N}$. If $G$ is an algebraic group over $k$, then $\varinjlim (G^{(m)})$ (the direct limit is taken in the category of formal groups) coincides with $\hat{G}$, the formal group which is the formalization of $G$ (it follows from [13, Lemma 1.1]).

We abuse the language a little bit here identifying the formal group law $\hat{G}$ (formalization of the algebraic group as in [13, p. 13]) with the corresponding formal group.

Similarly for a complete Hopf algebra $\mathcal{H}$, we have the analogous quotient $\mathcal{H}[m]$ which is again a Hopf algebra and also a complete Hopf algebra. Hence for a formal group $\mathcal{F}$, we have a direct system of truncated group schemes $\mathcal{F}[m]$ and in this case we get that $\mathcal{F} = \varinjlim \mathcal{F}[m]$ (see [13, Lemma 1.1] again).

For any $m$-truncated group law $f$ and $l \leq m$ we have $G_f[l] \cong G_f[l]$ and similarly for any formal group law $F$ and the corresponding formal group $\mathcal{F}$, we have $G_F[m] \cong \mathcal{F}[m]$.

2.3. Rational points of formal groups. Let $\mathcal{F}$ be a formal group over $k$ corresponding to a complete Hopf algebra $\mathcal{H}$. By $\mathcal{F}(\mathcal{R})$ (the set of “$\mathcal{R}$-rational points” of $\mathcal{F}$) we denote the set of all continuous $k$-algebra morphisms from $\mathcal{H}$ to $\mathcal{R}$. From general categorical reasons, $\mathcal{F}(\mathcal{R})$ is a group. If $\mathcal{F}$ comes from a formal group law $F$, then $\mathcal{F}(\mathcal{R})$ naturally corresponds to $I$, where $I$ is the maximal ideal of $\mathcal{R}$, and the operation on $\mathcal{F}(\mathcal{R})$ corresponds to $I \times I \ni (a, b) \mapsto F(a, b) \in I$ (see [3, Section 1.3]).

We note some properties of these groups.

Lemma 2.4. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of formal groups and $G$ an algebraic group over $k$. We have:

(1) The induced map $\alpha : \mathcal{F}(\mathcal{R}) \to \mathcal{F}'(\mathcal{R})$ is a homomorphism of groups.

(2) There is a natural monomorphism of groups $\hat{G}(\mathcal{R}) \to G(\mathcal{R})$.

Proof. Since the first part is clear, we comment only on (2). For simplicity, we assume that $G = \text{Spec}(H)$ is affine. Then $\hat{G}$ corresponds to $\hat{H}$, where the completion is taken with respect to the augmentation ideal of the Hopf algebra $H$. Since the map $H \to \hat{H}$ is a $k$-algebra map and any continuous homomorphism $\hat{H} \to \mathcal{R}$ is determined by its values on $H$, we get a one-to-one map $\hat{G}(\mathcal{R}) \to G(\mathcal{R})$. An easy diagram chase shows that this map is a homomorphism of groups (see also Section 2.6 in [10]).
2.4. **Truncated HS-derivations and actions of restricted Lie algebras.** In this subsection, we present the terminology from [19] and explain how does it fit into our context (see also (2.1b) in [2]). We assume that the field $k$ is perfect. Let $F$ be a formal group law. A function $f : k[X] \rightarrow k[[X]]$ is *$F$-invariant* if for any $n \in \mathbb{N}$

$$f \circ D_n = D_n \circ f,$$

where $(D_n)_n$ is the canonical $F$-derivation on $k[[X]]$ (see [9, Section 3.2]). The subset of $\text{Der}_k(k[[X]])$ consisting of $F$-invariant derivations is a restricted Lie subalgebra, which is denoted by $L(F)$. It is a one-dimensional vector space over $k$ spanned by $D_1$ [19, Lemma 2.1]. The restricted Lie algebra $L(F)$ acts on a $k$-algebra $R$, if there is a homomorphism

$$\varphi : L(F) \rightarrow \text{Der}_k(R)$$

of restricted Lie algebras.

It was noted in [19] that such an action is nothing else than a choice of $d \in \text{Der}_k(R)$ such that $d^{(p)} = c_F \cdot d$, where $c_F \in k$ is such that $D_1^{(p)} = c_F \cdot D_1$ (in this case the homomorphism $\varphi$ sends $D_1$ to $d$).

We need the following lemma which is a folklore.

**Lemma 2.5.** For any formal group law $F$ we have:

1. the restricted Lie algebra $L(F)$ is isomorphic either to $L(X + Y)$ or to $L(X + Y + XY)$,
2. the 1-truncated group law $F[1]$ is isomorphic either to $\mathbb{G}_a[1]$ or to $\mathbb{G}_m[1]$.

**Proof.** Item (1) is an easy computation and item (2) follows as in the beginning of the proof of Corollary [9,4].

The next result says that restricted Lie algebra actions correspond exactly to 1-truncated group law actions.

**Proposition 2.6.** Let $d \in \text{Der}_k(R)$. Then for any formal group law $F$ the following are equivalent:

1. the derivation $d$ defines an action of $L(F)$ on $R$,
2. we have $d^{(p)} = c_F \cdot d$,
3. there exists an $F[1]$-derivation $(d_n : R \rightarrow R)_{n<p}$ such that $d_1 = d$.

**Proof.** By Lemma 2.5 we may assume that $F = X + Y$ or $F = X + Y + XY$. Equivalence of (1) and (2) was already explained above. Implication from (3) to (2) is clear.

For the remaining implication, the case of $F = X + Y$ is dealt with in [9, Remark 2.9]. For the case $F = X + Y + XY$, we notice that if $\partial = (\partial_n)_{n<p}$ is an $F[1]$-derivation, then it satisfies (for $n \leq p - 2$)

$$\partial_1 \circ \partial_n = (n + 1)\partial_{n+1} + n\partial_n.$$

Thus we can inductively define

$$d_{n+1} := \frac{1}{n + 1}(d \circ d_n - nd_n).$$

The straightforward computation that $(d_n)_{n<p}$ is an $F[1]$-derivation is left to the reader.
3. Iterative derivations on projective line

Assume that $k$ is a perfect field of characteristic $p > 0$ and $m$ is a positive integer. In this section we discuss integrability of truncated HS-derivations on projective line $\mathbb{P}^1 = \mathbb{P}^1_k$. A (truncated) HS-derivation on $\mathbb{P}^1$ is by definition a (truncated) HS-derivation on the structure sheaf of $\mathbb{P}^1$ (see [12]).

It is very easy to describe (truncated) HS-derivations on $\mathbb{P}^1$: they correspond to (truncated) HS-derivations $\partial$ over $k$ on the polynomial algebra $k[t]$ such that $\partial$ preserves (after taking the unique extension to $k[t]$) the subalgebra $k[1/t]$. For a usual derivation $D$ the above condition means that $\deg(D(t)) \leq 2$ which geometrically corresponds to the fact that the tangent bundle of projective line coincides with the line bundle $\mathcal{O}(2)$ (see Remark 2.2 for a more general geometric picture). Such an (truncated) HS-derivation on $\mathbb{P}^1$ satisfies a given iterativity condition if and only if the corresponding (truncated) HS-derivation on $k[t]$ does.

We note a projective version of our integrability results from [9].

**Theorem 3.1.** Let $\partial$ be a $\mathbb{G}_a[m]$-derivation (resp. $\mathbb{G}_m[m]$-derivation) on $\mathbb{P}^1$. Then $\partial$ can be expanded to a $\mathbb{G}_a$-derivation (resp. $\mathbb{G}_m$-derivation) on $\mathbb{P}^1$.

**Proof.** Let us fix $\partial$, a $\mathbb{G}_a[m]$-derivation or a $\mathbb{G}_m[m]$-derivation on $\mathbb{P}^1$ and we also denote by $\partial$ the corresponding $m$-truncated HS-derivation on $k(t)$. By [9] Prop. 4.5 and [9] Prop. 4.10, there is a canonical element $a \in k(t)$ for $\partial$. To finish the proof as in [9] Thm. 4.7, it is enough to notice that the canonical $\mathbb{G}_a$-derivation and the canonical $\mathbb{G}_m$-derivation preserve $k[1/t]$. It is well-known in the additive case and in any case it follows by induction from the formula below ($n > 0$)

$$0 = \partial_n(t^{-1}) = t\partial_n(t^{-1}) + \partial_1(t)\partial_{n-1}(t^{-1}),$$

since $\partial_1(t) = 1$ or $\partial_1(t) = t + 1$ and $\partial_i(t) = 0$ for $i > 1$. \qed

We will show that for most of the other iterativity rules, truncated HS-derivations on $\mathbb{P}^1$ can not be integrated. It will follow from the result below saying that the existence of a non-trivial $F$-derivation on $\mathbb{P}^1$ is quite a restrictive condition on a formal group law $F$. For the notion of the height of a formal group law we refer the reader to [5] Def. 18.3.3).

**Theorem 3.2.** Let $F$ be a formal group law over $k$. If there is a non-trivial $F$-derivation on $\mathbb{P}^1$, then $\text{ht}(F) = 1$ or $\text{ht}(F) = \infty$.

**Proof.** Assume that there is a non-trivial $F$-derivation on $\mathbb{P}^1$. Since the height of a formal group law does not change after a base-change, we can assume that $k$ is algebraically closed. Let $\mathcal{F}$ be the formal group corresponding to $F$. By [15] page 21], the automorphism functor $A_{\mathbb{P}^1/k}$ is representable by the linear algebraic group $\text{PGL}(2, k)$. Let $H$ be the Hopf algebra of $\text{PGL}(2, k)$ and $\mathcal{H}$ the complete Hopf algebra corresponding to $\mathcal{F}$ (so $\mathcal{H}$ is isomorphic to the power series ring in one variable $t$ and the complete Hopf algebra structure is given by $t \mapsto F(t \otimes 1 + 1 \otimes t)$).

By the bijective correspondences from Section 2.1 we get a compatible system of $k$-group scheme morphisms corresponding to the non-trivial $F$-derivation on $\mathbb{P}^1$

$$(\mathcal{F}[m] \to \text{Aut}_k(\mathbb{P}^1) = \text{PGL}(2, k))_{m \geq 1}.$$

For each $m$, the above morphism factors through a morphism

$$\mathcal{F}[m] \to \text{PGL}(2, k)[m],$$
since the morphism $F[m] \to \text{PGL}(2, k)$ composed with $\text{Fr}^m_{\text{PGL}(2, k)}$ is the trivial morphism and
\[ \text{PGL}(2, k)[m] = \ker(\text{Fr}^m_{\text{PGL}(2, k)}) \]
(see Section 2.2). Since we have
\[ \lim_{\to}(F[m]) \cong F, \quad \lim_{\to}(\text{PGL}(2, k)[m]) \cong \widehat{\text{PGL}(2, k)} \]
(see Section 2.2), we get a non-trivial morphism of formal groups
\[ \Psi : F \to \widehat{\text{PGL}(2, k)}. \]

We proceed to show that the “Zariski closure of the image of $\Psi$” is a commutative algebraic subgroup $V$ of $\text{PGL}(2, k)$ and that $\Psi$ factors through a formal group morphism $F \to \widehat{V}$.

Let $\Phi$ be dual to $\Psi$, $I$ the kernel of $\Phi$, $I = I \cap H$ and $R$ the quotient of $H$ by $I$.

We have a commutative diagram:

\[
\begin{array}{ccc}
I & \to & I \\
\downarrow & & \downarrow \\
H & \to & \hat{H} \\
\downarrow & & \downarrow \\
R & \to & \hat{H} \\
\downarrow & & \downarrow \\
\hat{R}, & \to & \hat{H}, \\
\end{array}
\]

where the completion of $R$ is taken with respect to the image of the maximal ideal in the local ring of $I \in \text{PGL}(2, k)$. Since $\Phi$ is non-trivial (i.e. $\Phi(\hat{H}) \neq k$), the induced map $\alpha : \hat{R} \to \hat{H}$ is non-trivial as well.

Let $V$ be the closed subvariety of $\text{PGL}(2, k)$ corresponding to $\text{Spec}(R)$. The above diagram means exactly that $\Phi$ factors through a morphism $F \to \hat{V}$ whose image is “Zariski dense in $V$” (since $R \to \hat{H}$ is one-to-one). We need to show that $V$ is a commutative algebraic subgroup of $\text{PGL}(2, k)$.

Let $\mathcal{R}$ be a complete $k$-algebra which is DVR and $K$ the algebraic closure of its field of fractions. Clearly, $\mathcal{H} \otimes_k \mathcal{R} \cong \mathcal{R}[t]$ as complete $\mathcal{R}$-algebras. After base-change, we get the following commutative diagram

\[
\begin{array}{ccc}
H \otimes_k \mathcal{R} & \to & \hat{H} \otimes_k \mathcal{R} \\
\downarrow & & \downarrow \\
R \otimes_k \mathcal{R} & \to & \mathcal{H} \otimes_k \mathcal{R} \\
\downarrow & & \downarrow \\
\mathcal{R}, & \to & \mathcal{R}, \\
\end{array}
\]

where $\gamma$ is still an embedding, since $\mathcal{R}$ is flat over $k$. For any $a$ belonging to the maximal ideal of $\mathcal{R}$, we have the evaluation map $\text{ev}_a : \mathcal{R}[t] \to \mathcal{R}$ and we denote the composition of this map with $\gamma$ by $\gamma_a \in V(\mathcal{R})$. 

The set of all evaluation maps as above forms a commutative group \( \mathcal{F}(\mathcal{R}) \) (see Section 2.3). By Lemma 2.4
\[
\Gamma := \beta^*(\Psi^*(\mathcal{F}(\mathcal{R})))
\]
is a commutative subgroup of PGL(2, \( \mathcal{R} \)). Using the natural embedding of \( V(\mathcal{R}) \) into PGL(2, \( \mathcal{R} \)) we see that \( \Gamma \subseteq V(\mathcal{R}) \). However \( \gamma \) is an embedding and for any \( T \in \mathcal{R}[t] \) there is \( a \) in the maximal ideal of \( \mathcal{R} \) such that \( T(a) \neq 0 \) (either \( a = 0 \) or \( a \) in the maximal ideal of \( \mathcal{R} \) works). Hence for any \( r \in \mathcal{R} \) there is \( \gamma_a \in \Gamma \) such that \( \gamma_a(r) \neq 0 \). It means that \( \Gamma \) is Zariski dense in \( V(K) \). It is well-known (see e.g. [1, Chapter 1 §2]) that the Zariski closure of a commutative subgroup of an algebraic group is a commutative subgroup. Hence \( V \) is a commutative algebraic subgroup of PGL(2, \( k \)) and \( \Psi : \mathcal{F} \to \text{PGL}(2, k) \) factors through a formal group morphism \( \mathcal{F} \to \mathcal{V} \).

Since (by e.g. [1, Section 10.2]) any commutative linear algebraic group over \( k \) has an algebraic composition series whose factors are isomorphic either to \( \mathbb{G}_a \) or to \( \mathbb{G}_m \) (it is true even for solvable groups using the Lie-Kolchin theorem [1, Cor. 10.5]), we get a non-trivial formal group morphism of the form \( \mathcal{F} \to \hat{\mathbb{G}}_a \) or of the form \( \mathcal{F} \to \hat{\mathbb{G}}_m \). The classification of commutative formal groups up to isogeny \([13, Chapter II §4]\) gives that \( \mathcal{F} \cong \hat{\mathbb{G}}_a \) or \( \mathcal{F} \cong \hat{\mathbb{G}}_m \), hence (see [3, Theorem 18.5.1]) \( \text{ht}(F) = 1 \) or \( \text{ht}(F) = \infty \).

**Remark 3.3.** (1) Using properties of commutative linear groups (see e.g. [1]), one can generalize Theorem 3.2 from \( \mathbb{P}^1 \) to any \( k \)-scheme \( W \) such that \( A_{W/k} \) is representable by a linear algebraic group over \( k \) (e.g. \( W = \mathbb{P}^n \)).

(2) One can further generalize Theorem 3.2 from \( \mathbb{P}^1 \) to any \( k \)-scheme \( W \) such that \( A_{W/k} \) is representable by an algebraic group, if we also allow that \( \text{ht}(F) = 2 \) (to cover the case of supersingular elliptic curves, see [18, Cor. 7.5]).

**Corollary 3.4.** Let \( F \) be a formal group law over \( k \). If \( \text{ht}(F) \neq 1 \) and \( \text{ht}(F) \neq \infty \), then there is an \( F[1] \)-derivation on \( \mathbb{P}^1 \) which does not expand to an \( F \)-derivation.

**Proof.** By [3, Theorem p. 69], there are (up to isomorphism) only two \( k \)-group scheme structures on \( \text{Spec}(k[X]/(X^p)) \):
\[
\mathbb{G}_a[1] := \ker(Fr : \mathbb{G}_a \to \mathbb{G}_a), \quad \mathbb{G}_m[1] := \ker(Fr : \mathbb{G}_m \to \mathbb{G}_m),
\]
since the Dieudonné module of such a group scheme coincides with \( k \) (length 1), the Frobenius map is 0 and the Verschiebung map is either 0 or 1. If \( F \) is a formal group law over \( k \) and \( \text{ht}(F) = 1 \), then \( F[1] \cong \mathbb{G}_m[1] \) (Verschiebung is 1), otherwise \( F[1] \cong \mathbb{G}_a[1] \) (Verschiebung is 0).

Let us assume that \( \text{ht}(F) \neq 1 \) and \( \text{ht}(F) \neq \infty \). Therefore \( F[1] \cong \mathbb{G}_a[1] \) (since \( \text{ht}(F) \neq 1 \)). Since the standard derivation on \( k[t] \) gives a non-zero derivation on \( \mathbb{P}^1 \), we have a non-zero \( \mathbb{G}_a[1] \)-derivation \( \partial \) on \( \mathbb{P}^1 \). Since \( F[1] \cong \mathbb{G}_a[1] \), \( \partial \) is also an \( F[1] \)-derivation. By Theorem 3.2 this derivation does not expand to an \( F \)-derivation.

**Example 3.5.** It is easy to see (reasoning as at the end of the proof of Theorem 3.2) that for \( F \) as in Corollary 3.3 there is \( m \) such that a non-zero \( F[1] \)-derivation on \( \mathbb{P}^1 \) does not expand to an \( F[m] \)-derivation on \( \mathbb{P}^1 \). One could wonder about
the particular value of $m$. Considering $F$ and $D$ from Example 3.3 it is easy to compute that

$$\partial_4(1/t) = t^{10} + t^4 + t^{-2} + t^{-5},$$

so $k[1/t]$ is not preserved by $D[2]$, hence $D[2]$ does not extend from $\mathbb{A}^1$ to $\mathbb{P}^1$. This calculation suggests that $m = 2$ (if $F^2 \not\cong \mathbb{G}_a[2]$ and $F^2 \not\cong \mathbb{G}_m[2]$).

**Remark 3.6.** Results of this section may look related to the isotriviality theorems from [12]. However, there is a fundamental difference. In [12], a different type of HS-derivations on projective varieties is considered – their restriction to the base field $k$ is *generic* (i.e. this restriction gives an *existentially closed structure*, see e.g. [6]). In this paper such a restriction is the 0-derivation, which is very non-generic. Clearly, the proofs are related since they both use the automorphism functor.

### 3.1. Higher dimensional iterativity conditions

In this subsection, we will briefly discuss the case of an action of higher-dimensional groups (different actions than those which were discussed in Remark 3.3). Let $e \geq 2$ and we fix an $e$-dimensional algebraic group $G$ such that $k(G) \sim_k k[X_1, \ldots, X_e]$.

We would like to check (as a natural starting point) whether an analogue of Theorem 3.1 is true in this context, i.e. whether the canonical $G$-derivation on $k[X_1, \ldots, X_e]$ extends to a $G$-derivation on $\mathbb{P}^e$. For simplicity, we will assume that $G = U$ is unipotent of dimension 2. This case is analyzed in [7] and the integrability results (for actions on field extensions) for such groups are proved there.

In the case of $\mathbb{P}^1$, the condition for an HS-derivation $D$ on $k[X]$ to extend to $\mathbb{P}^1$ was very easy: it was enough that $D$ (after extending to $k(X)$) preserves the subring $k[1/X]$. In the case of $\mathbb{P}^2$ we have two preservation conditions: an HS-derivation $D$ on $k[X, Y]$ extends to $\mathbb{P}^2$ if and only if $D$ (after extending to $k(X, Y)$) preserves the subrings $k[X/Y, 1/Y]$ and $k[Y/X, 1/Y]$. In principle, there is also one more “cocycle condition”, but it is trivially satisfied, in the case of $\mathbb{P}^2$.

It is an easy computation to verify that both the subrings $k[X/Y, 1/Y]$ and $k[Y/X, 1/Y]$ are HS-subrings of $k(X, Y)$ with the canonical $\mathbb{G}_a^2$-derivation. We will see below that for $U$ which is a non-split extension of $\mathbb{G}_a$ by $\mathbb{G}_a$, the preservation condition fails.

**Example 3.7.** We take the unipotent group $U$ where the group law on $U$ is given by

$$(X, Y) \ast (W, Z) = \left( X + W + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} Y^i Z^{p-i}, Y + Z \right).$$

We want to check whether $k[Y/X, 1/X]$ is an HS-subring of $k(X, Y)$ equipped with the canonical $U$-derivation. However, for $p > 3$ we have

$$D_{(0,1)}(1/X) = - \frac{D_{(0,1)}(X)}{X^2} = - \frac{Y^{p-1}}{X^2} \not\in k[Y/X, 1/X].$$

### 4. Truncated Galois groups

We have hoped that using truncated Galois groups from [2] one could answer Question 1.1 in the negative for certain formal group laws, but we could not quite accomplish it. In this section we describe this idea and the problems related to it.
4.1. General strategy. Let $K = k(X^m)$ and $L = k(X)$, where $k$ is a perfect field of positive characteristic $p$ (for simplicity, we can assume that $k = \mathbb{F}_p$). Let $G_r(L/K)$ be the truncated group scheme over $K$ representing the automorphism functor $A_{\text{Spec}(L)/\text{Spec}(K)}$ on the category of truncated $K$-algebras. By [2] Section 2, this functor is indeed representable by a truncated group scheme over $K$.

Remark 4.1. It is good to point out here that $G_r(L/K)$ does not coincide with $\text{Gal}(L/K)$, the (scheme-theoretic) Galois group of $L$ over $K$, which represents the functor $A_{\text{Spec}(L)/\text{Spec}(K)}$ on the entire category of $K$-algebras (see e.g. Examples 2 and 3 in [14]), but there is a group scheme morphism $G_r(L/K) \to \text{Gal}(L/K)$.

Since $G_r(L/K) = G_r(L/K)[m]$ (see [2] (2.1c)) and by using properties of the $m$-truncation functor, it is easy to see that the induced morphism

$$G_r(L/K) \to \text{Gal}(L/K)[m]$$

is an isomorphism.

By [2] 2.1(a), for a $K$-group scheme $G$, the group scheme actions of $G$ on $L/K$ (i.e. on the $K$-scheme $\text{Spec}(L)$) correspond to the $K$-group scheme morphisms $G \to G_r(L/K)$. Let us fix a truncated group law $f$ over $k$. We get yet another bijective correspondence fitting to Section 2.1 (for the group scheme case) with the following set.

\[(3') \text{ The set of all } K\text{-group scheme morphisms } \]

$$G_f \otimes_k K \to G_r(L/K).$$

Let $D$ be an $f$-derivation on $L$ corresponding to a $K$-group scheme action of $G_f \otimes_k K$ on $L/K$. By [2] Def. and Remarks 5.1, $L/K$ is a PHS for this action if and only if the field of absolute constants of $D$ coincides with $K$ (since, by our assumptions, the rank of $G_f$ coincides with $[L : K]$). By [2] Prop. 5.2, $L/K$ is a PHS if and only if the corresponding morphism $G_f \otimes_k K \to G_r(L/K)$ is an embedding. Therefore, the standard $G_a[m]$-derivation on $L$ gives rise to an embedding

$$G_a[m] \otimes_k K \to G_r(L/K).$$

By [2] Thm. 5.3], this embedding induces an isomorphism

$$\left(G_a[m] \otimes_k K\right)^\sharp \cong G_r(L/K),$$

where the $\sharp$-functor (from the category of $K$-group schemes to itself) is described in Section 4 of [2].

For any $l \in \{1, \ldots, m\}$, let $K_l := K(X^l)$, so we have a tower of fields $K \subseteq K_l \subseteq L$. By $\sharp$, we denote the functor from the category of $K_l$-group schemes to itself, where $K_l$ is playing the role of $K$. The hope is to find $l < m$ and a formal group law $F$ such that:

- $(F[l] \otimes_k K_l)^\sharp$ embeds into $(G_a[l] \otimes_k K_l)^\sharp$,
- $(F[m] \otimes_k K)^\sharp$ does not embed into $(G_a[m] \otimes_k K)^\sharp$.

That would give an $F[l]$-derivation on $L$ which can not be integrated (even to an $F[m]$-derivation) using the following observation.

Lemma 4.2. Assume $L$ is an $f$-field. Then $L/K$ is an PHS for $G_f \otimes_k K$ if and only if for any $1 \leq l \leq m$, $L/K_l$ is an PHS for $G_f[l] \otimes_k K_l$. 
Proof. Let $D$ be an $f$-derivation on $L$ giving the group scheme action of $G_f \otimes_k K$ on $L/K$. We define

$$C_D := \{ x \in L \mid (\forall 0 < i < p) D_i(x) = 0 \}, \quad C_D^{\text{abs}} := \{ x \in L \mid (\forall 0 < i < p^m) D_i(x) = 0 \}$$

(see [8, Def. 3.5]). It is enough to show that $C_D^{\text{abs}} = (C_D)^{p^m-1}$, which follows from [8, Lemma 3.31]. □

Remark 4.3. In fact, if the hope above turns out to be correct, we would get a strong negative answer to Question 1.1, since then there would be a non-zero $F[l]$-derivation on $L$ and no non-zero $F[m]$-derivations on $L$.

Unfortunately, there are no particular examples of truncated Galois groups in [2] and it is not clear to us how to determine such group schemes. These group schemes may be quite complicated, since (besides the case of $p = m = 2$), $G_r(L/K)$ is not commutative. In the next subsection we note a possible way of obtaining some information about the group scheme $G_r(L/K)$.

4.2. Towards computing truncated Galois groups. Trying to gather information about the group scheme $G_r(L/K)$, it is useful to identify the $f$-derivations on $L$, which give rise to the PHS group scheme actions on $L/K$. To find such derivations, it is natural to ask the following question.

Question 4.4. For which formal group laws $F$, the canonical $F$-derivation on $k[t]$ restricts to $k[t]$ or (after extending the canonical $F$-derivation to the field of Laurent series) restricts to $k(t)$?

The (possible) answer of the form “for all formal group laws $F$” would imply that the ideas from the previous section can not be used to answer Question 1.1 in the negative, since then for any $m$, a non-trivial $F[m]$-derivations would exist on $k(t)$ for an arbitrary formal group law $F$, and each $F[m] \otimes_k K$ would embed into $G_r(L/K)$. Such an answer to Question 4.4 would also suggest a possibility of the positive answer to Question 1.1 for any formal group law $F$ and would also provide some information about the structure of the group scheme $G_r(L/K)$.

Obviously, the canonical $F$-derivation restricts to $k[t]$ if and only if $F \in k[[X]][Y]$, but it is unclear to us for which formal group laws the latter condition occurs. The next example comes from trying to answer Question 4.4 for a specific $F$. It shows that up to the truncation level $m = 3$ (this was as far as we were patient enough to check), there are non-trivial $F[3]$-derivations on $k(t)$ for $F$ of height 2.

Example 4.5. It can be checked that the formulas below

$$\partial_1(t) = 1, \quad \partial_2(t) = t^2, \quad \partial_3(t) = 0, \quad \partial_4(t) = t^6 + t^{12},$$

$$\partial_5(t) = 0, \quad \partial_6(t) = t^4, \quad \partial_7(t) = 0$$

give an $F[3]$-derivation on $F_2[t]$ for $F = \overline{F}_{\Delta_2}(X, Y)$ from [5] (3.2.3)]. Therefore, the truncated group scheme $F[3]$ embeds into $G_r(L/K)$ (at least for $p = 2$ and $m = 3$).

Remark 4.6. It may be interesting to compare the group scheme embeddings

$$F[m] \otimes_k K \to G_r(L/K)$$

with the embeddings from Section 12 of [16]:

$$\text{Gal}(L/K) \otimes_k K \to G_r(L/K),$$
which are coming from the Galois group of a given Picard-Vessiot structure on the extension $L/K$.

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