BOUNDARY SLOPES FOR THE
MARKOV ORDERING ON RELATIVELY PRIME PAIRS

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ABSTRACT. Following McShane, we employ the stable norm on the homology of the modular torus to investigate the Markov ordering on the set of relatively prime integer pairs \((q,p)\) with \(q \geq p \geq 0\). Our main theorem is a characterization of slopes along which the Markov ordering is monotone with respect to \(q\), confirming conjectures of Lee-Li-Rabideau-Schiffler that refine conjectures of Aigner. The main tool is an explicit computation of the slopes at the corners of the stable norm ball for the modular torus.

1. INTRODUCTION

The set of Markov numbers \(M = \{1,2,5,\ldots\}\) is the set of positive integers that appear in a Markov triple, i.e. a solution \((x,y,z) \in \mathbb{N}^3\) to the cubic

\[x^2 + y^2 + z^2 = 3xyz.\]

The Markov Uniqueness Conjecture (MUC), apparently due to Frobenius, asserts that each positive Markov triple is determined by its largest entry \([\text{Aig15, CF89, Fro13, Mar79}]\).

The Markov numbers admit an interpretation as a system of labels on the set \(\mathcal{H}\) of horoball regions in the complement of a rooted planar trivalent tree. That is, there is a surjective map \(\lambda_M : \mathcal{H} \to M\), whose injectivity is equivalent to MUC \([\text{Aig15, p. 54}]\). There is another well-known natural parametrization of \(\mathcal{H}\) via Farey fractions \(\lambda_F : \mathcal{H} \cong \mathbb{Q} \cap [0,1]\), so one may obtain the Markov numbers as \(m_{p/q} = \lambda_M \circ \lambda_F^{-1}(p/q)\). See Figure 1 and Figure 2 for an illustration of \(\lambda_M\) and \(\lambda_F\).

It will be convenient to record this map as follows: let \(Q \subset \mathbb{Z}^2\) indicate the set of relatively prime pairs \((q,p)\) with \(q \geq p \geq 0\). The map \((q,p) \mapsto m_{p/q}\) records the Markov label associated to the horoball with label \(p/q\). MUC predicts that this map is in fact a bijection; because \(M \subset \mathbb{N}\), one would obtain a somewhat mysterious ordering of \(Q\). We will refer to this ordering as the ‘Markov ordering’, denoted \(\prec_M\). (For the careful reader who prefers not to assume MUC, \(\prec_M\) is a priori only a strict partial order, with potentially nontrivial mutually incomparable subsets for horoballs with common Markov labels.) See Figure 3 for an illustration.

Initiating a study of \(\prec_M\), Aigner offered some conjectural statements suggested by numerical experiment \([\text{Aig15}]\): assuming the following pairs are relatively prime,

(i) \((q,p) \prec_M (q',p)\) when \(q' > q\),
(ii) \((q,p) \prec_M (q,p')\) when \(p' > p\), and
(iii) \((q,p) \prec_M (q',p')\) when \(q' > q\) and \(p + q = p' + q'\).

These three statements were subsequently proven by several authors using the theory of cluster algebras and continued fractions \([\text{Rab18, RS20, LLRS20, LPTV21}]\). Shortly thereafter, Greg McShane gave a unified proof of Aigner’s conjectures \([\text{McS21}]\) by exploiting hyperbolic geometry and the so-called stable norm, a norm on the homology of the modular torus induced by the hyperbolic length function \([\text{MR95a}]\).

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Aigner’s conjectures concern the monotonicity of $\prec_M$ along lines of slopes $\infty$, 0, and $-1$. This analysis was pushed further in the work of Lee-Li-Rabideau-Schiffler, who consider monotonicity of $\prec_M$ along lines of other slopes. They show [LLRS20, Thm. 1.2]:

(i) $\prec_M$ is monotone increasing with $q$ along lines of slope $\geq -\frac{8}{7}$,
(ii) $\prec_M$ is monotone decreasing with $q$ along lines of slope $\leq -\frac{5}{4}$, and
(iii) $\exists$ lines of slopes $-\frac{7}{6}$ and $-\frac{5}{4}$ along which $\prec_M$ is not monotone.

Moreover, the authors make precise conjectures concerning optimality of slopes for monotonicity of $\prec_M$ [LLRS20, Conj. 6.8, 6.11, 6.12]. The purpose of this note is to confirm these conjectures (though we stop just short of the full statement of [LLRS20, Conj. 6.12], which reaches beyond $M$ to an order that concerns non-relatively prime pairs as well – see Remark 4.2).

There is a well-known geometric interpretation of $M$, first discovered by Gorshkov and Cohn [Gor81, Coh55, Ser85], which we now recall: Let $X$ be the modular torus. The fundamental group of $X$ can be lifted to $\text{SL}(2, \mathbb{Z})$ so that the set of traces of simple closed geodesics on $X$ is exactly $3M$. Because traces and hyperbolic lengths are related by a monotone function (1.2), one finds that $\prec_M$ coincides with the order on $\mathbb{Q}$ induced by the hyperbolic length function (that is, with the appropriate identification of $H_1(X, \mathbb{Z}) \approx \mathbb{Z}^2$, in which the three shortest geodesics on $X$ have representatives $(1, 0), (0, 1), (-1, 1)$).

Recently, McShane has shown how to use this viewpoint to prove Aigner’s conjectures, relying on a certain convexity induced by the hyperbolic length function. The stable norm $\|\cdot\|_s$ is a norm on $\mathbb{R}^2 \approx H_1(X, \mathbb{R})$ whose value on primitive integral points coincides with the hyperbolic length of the corresponding simple geodesic (see [McS21, MR95a, SV19, BBI01] for more detail). McShane-Rivin observed that the boundary of the $\|\cdot\|_s$-ball is strictly convex [MR95a], and Aigner’s conjectures about $\prec_M$ can then be deduced rather quickly by consideration of the unit $\|\cdot\|_s$-ball [McS21].

We follow McShane’s technique and Lee-Li-Rabideau-Schiffler’s conjectures:

Theorem 1.1. Let $r \in \mathbb{Q}$, $\sigma_- = -\frac{\log \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)}{\log \left(\frac{3}{2} + \frac{3}{10}\sqrt{5}\right)} \approx -1.2417$, and $\sigma_+ = -\frac{\log \left(\frac{3}{2} + \frac{3}{4}\sqrt{2}\right)}{\log \left(\frac{3}{2} + \frac{3}{2}\sqrt{2}\right)} \approx -1.1432$.

(i) If $r < \sigma_-$, then $\prec_M$ is monotone decreasing with $q$ along lines of slope $r$.
(ii) If $r > \sigma_+$, then $\prec_M$ is monotone increasing with $q$ along lines of slope $r$.
(iii) If $r \in (\sigma_-, \sigma_+)$, there exist lines of slope $r$ along which $\prec_M$ is not monotone.

1While an initial draft of [LLRS20] does not contain these conjectures, they can be found in a revised manuscript which was shared in private communication. This author thanks those authors for their openness.
Remark 1.3. In fact, the proof of Theorem 1.1(iii) demonstrates slightly more: Let \( \mathcal{L} \) be a line of slope \( r \in (\sigma_-, \sigma_+) \). There exist arbitrarily large \( t \in \mathbb{R} \) so that the scaling \( \mathcal{L}_t := t \cdot \mathcal{L} \) has intersection \( \mathcal{L}_t \cap Q \) which is nonempty and strictly antimodal with respect to \( q \). That is, for arbitrarily large \( t \) one has \( \mathcal{L}_t \cap Q = \{(q_1, p_1), \ldots, (q_n, p_n)\} \) with \( n \geq 3 \) and \( q_1 < \ldots < q_n \), and there exists \( j \in \{2, \ldots, n-1\} \) so that \( (q_i, p_i) \prec_M (q_j, p_j) \) for \( i < j \) and \( (q_j, p_j) \prec_M (q_i, p_i) \) for \( i > j \). See Remark 4.1 and Remark 4.2.

The proof of Theorem 1.1 amounts to a computation about (one-sided) slopes formed at two corners of the stable norm ball in the homology of the modular torus. This same computation can be performed at any point of rational slope (see Proposition 1.10 below). In fact, this allows one to make the following more precise statement about \( \prec_M \):

**Theorem 1.4.** Let \( (q, p), (q', p') \in Q \), and let \( \mu_{p/q}^+ \) and \( \mu_{p/q}^- \) be given by (1.6) and (1.7).

If \( q' < q \) and \( \frac{q' - q}{p' - p} \leq \mu_{p/q}^+ \) then \( (q, p) \prec_M (q', p') \).

If \( q' > q \) and \( \frac{q' - q}{p' - p} \geq \mu_{p/q}^- \) then \( (q, p) \prec_M (q', p') \).

**Remark 1.5.** The constants appearing in Theorem 1.1 are given by \( \sigma_- = \mu_{0/1}^+ \) and \( \sigma_+ = \mu_{1/1}^- \).

**Proof.** If either \( q' < q \) and \( \frac{q' - q}{p' - p} \leq \mu_{p/q}^+ \) or \( q' > q \) and \( \frac{q' - q}{p' - p} \geq \mu_{p/q}^- \), there is a half plane \( H \), containing \( (q', p') \), which is a support plane for the \( \|\cdot\|_s \)-ball of radius \( \|(q, p)\|_s \) centered at the origin. Strict convexity implies that \( (q', p') \) is therefore outside the ball, so \( \|(q, p)\|_s \leq \|(q', p')\|_s \). See Figure 4. \( \square \)

Our computation of the extremal slopes above is convenient when phrased in terms of Fock’s function \( \Psi \). Vladimir Fock introduced the function \( \Psi : [0, 1/2] \to \mathbb{R} \) whose value at rationals is given by

\[
\Psi(p/q) = \cosh^{-1} \left( \frac{3}{2} m_T \left( \frac{p}{q} \right) \right),
\]

where \( T \left( \frac{p}{q} \right) = \frac{p}{q-p} \). Suitably interpreted, \( \Psi(p/q) \) is the ratio of (one-half of) the hyperbolic length to an ‘arithmetic height’ for a simple closed geodesic corresponding to the homology class \( T(q, p) = (q-p, p) \). Fock showed that \( \Psi \) is a continuous, convex, and decreasing function [Foc97].

**Remark 1.6.** We hope the reader will allow the slightly abusive repetition of \( T \); evidently, under the natural bijection \( Q \approx \mathbb{Q} \cap [0, 1] \) these two maps are ‘the same’.

**Figure 4.** A support plane \( H \) at a corner of the stable norm ball.
Recall that, if $A \in \text{SL}(2, \mathbb{R})$ is a hyperbolic transformation with translation distance $\delta(A)$, one has
\[
2 \cosh \frac{\delta(A)}{2} = \text{tr} (A) .
\]
When $\gcd(p, q) = 1$, the translation distance of the holonomy of $(q, p)$ is the length $\|(q, p)\|_s$ of its simple geodesic representative, and the trace is three times the corresponding Markov number $m_{p/q}$, so we find
\[
\|(q, p)\|_s = 2 \cosh^{-1} \left( \frac{3}{2} m_{p/q} \right) .
\]
An equivalent description of the value of $\Psi$ at $p/q$ is thus given by
\[
\Psi \left( \frac{p}{q} \right) = \frac{\|T(q, p)\|_s}{2q} .
\]

Remark 1.7. The definition of $\Psi$ given in (1.1) and (1.3) looks cosmetically distinct from others in the literature ([SV19, eq. (3)] for example), owing to ‘$T(p/q)$’ in place of ‘$p/q$’ (or ‘$T(q, p)$’ in place of ‘$(q, p)$’). This is the result of two distinct normalizations for the Farey labelling $\lambda_T$ in the literature. In one, the root of the tree is incident to 0/1, 1/0 and 1/1, as in Figure 2 and [Aig15, LLRS20, McS21] (observe that this is natural by virtue of the integral homology $\mathbb{Z}^2$), and in the other the root sees 0/1, 1/1, and 1/2, as in [Foc97, SV17, SV19] (somewhat more natural when considering the holonomy of the modular torus in $\text{PSL}(2, \mathbb{Z})$). We’ve chosen a presentation of $z_M$ that follows [Aig15, LLRS20, McS21], but our function $\Psi$ matches that of [Foc97, SV19, SV17].

The structure of $\Psi$ is closely tied to the Markov numbers and their cousins, the Lagrange numbers [SV17, SV19]. Sorrentino-Veselov asked for an explicit computation of the left- and right-derivatives of $\Psi$ at rationals [SV19, §5]. We provide an answer below.

Proposition 1.8. Suppose that $\frac{p}{q}, \frac{r_1}{s_1},$ and $\frac{r_2}{s_2}$ span a Farey triangle with $\frac{r_1}{s_1} < \frac{p}{q} < \frac{r_2}{s_2}$, and let $n_i := m_{T(r_i/s_i)}$ and $n := m_{T(p/q)}$, so that $(n, n_1, n_2)$ is a Markov triple.

The left- and right-derivatives of $\Psi$ at $p/q$ can be computed as follows:
\[
\text{left-derivatives: } \frac{d}{dt} \bigg|_{t=\frac{p}{q}} \Psi(t) = -s_2 \cosh^{-1} \left( \frac{3n_2}{2} \right) - q \log \left( \frac{3n_2 - 6n_1}{2\sqrt{n^2 - 4}} \right)
\]
\[
\text{right-derivatives: } \frac{d}{dt} \bigg|_{t=\frac{p}{q}} \Psi(t) = s_1 \cosh^{-1} \left( \frac{3n}{2} \right) + q \log \left( \frac{3n_1 - 6n_2}{2\sqrt{n^2 - 4}} \right)
\]

Remark 1.9. It has come to the attention of the author that the calculation of the left- and right-derivatives of $\Psi$ was carried out recently, in much the same fashion, by Robert Hines [Hin20, p. 6].

One can translate this computation about $\Psi$ into a computation of the slopes at the corners of spheres with respect to the stable norm:

Proposition 1.10. Suppose $(q, p) \in \mathcal{Q}$. Let $L = \frac{d}{dt} \big|_{t=T^{-1}(\frac{p}{q})} \Psi(t)$, $R = \frac{d}{dt} \big|_{t=T^{-1}(\frac{p}{q})} \Psi(t)$, whose values are recorded in (1.4) and (1.5), and let $\ell = \frac{1}{2} \|(q, p)\|_s$ (equivalently, $\ell = \cosh^{-1}(\frac{3}{2} m_{p/q})$).

The corner of the stable norm ball at $(q, p)$ has left slope $\mu^+_{p/q}$ and right slope $\mu^-_{p/q}$ given by
\[
\mu^+_{p/q} = -\frac{\ell - Rp}{\ell + Rq}, \text{ and }
\]
\[
\mu^-_{p/q} = -\frac{\ell - Lp}{\ell + Lq} .
\]
Example 1.11. We compute $\mu_{0/1}^+$. In this case, $\ell = \cosh^{-1} \frac{3}{2} = \log \left( \frac{3 + \sqrt{5}}{2} \right)$. In order to determine the right-derivative $R$ of $\Psi$ at $0/1$, we consider the Farey trio $\left( \frac{r_1}{s_1}, \frac{r_2}{s_2} \right) = \left( \frac{1}{5}, \frac{0}{1}, \frac{1}{1} \right)$ and corresponding Markov triple $(n_1, n_2) = (1, 1, 1)$. By (1.5) we have

\[ R = 0 \cdot \cosh^{-1} \frac{3}{2} + 1 \cdot \log \left( \frac{3}{2} - \frac{3}{2 \sqrt{5}} \right) = \log \left( \frac{3}{2} - \frac{3}{10 \sqrt{5}} \right). \]

Therefore by Proposition 1.10 we have

\[ \mu_{0/1}^+ = -\frac{\frac{3}{2} + \frac{\sqrt{5}}{2}}{\log \left( \frac{3 + \sqrt{5}}{2} \right) + \log \frac{15 - 3\sqrt{5}}{10}} = -\frac{\frac{3}{2} + \frac{\sqrt{5}}{2}}{\log \left( \frac{3}{2} + \frac{3}{10 \sqrt{5}} \right)}. \]

Example 1.12. We compute $\mu_{1/1}^-$. To compute the left-derivative of $\Psi$ at $1/2$ (note that $1/2 = T^{-1}(1/1)$), fix Farey trio $\left( \frac{r_1}{s_1}, \frac{r_2}{s_2} \right) = \left( 0, \frac{1}{2}, \frac{1}{1} \right)$ and corresponding Markov triple $(1, 2, 1) = (n_1, n_2)$. By (1.4),

\[ L = -1 \cdot \cosh^{-1} 3 - 2 \cdot \log \left( \frac{3}{2} - \frac{12}{8 \sqrt{2}} \right) = \log \frac{8}{9}. \]

Therefore by Proposition 1.10 we have

\[ \mu_{1/1}^- = -\frac{\cosh^{-1} 3 - \log \frac{8}{9}}{\cosh^{-1} 3 + \log \frac{8}{9}} = \frac{\frac{3}{2} + 2\sqrt{2}}{\log \frac{3}{2} (3 + 2\sqrt{2})} = \frac{\frac{3}{2} (1 + \sqrt{2})}{\log 2\sqrt{2} (1 + \sqrt{2})} = -\frac{\frac{3}{2} + \frac{3}{2}}{\log \left( \frac{4}{5} + \frac{3}{2} \sqrt{2} \right)}. \]

Remark 1.13. McShane-Rivin offer a more general version of Proposition 1.10 [MR95b, Thm. 2.1], though the proof has not appeared in the literature. Their theorem states that the exterior angle at $(q, p)$ decreases exponentially with $\max \{p, q\}$, a fact that can be deduced from the formulas above. Though Proposition 1.10 has the advantage of being exact, we emphasize that the McShane-Rivin theorem is more general: Their control on the boundary slopes applies to any hyperbolic punctured torus, and also accounts for the slope at irrational points on the boundary of the stable norm ball. It might be interesting to find analogues of (1.4) and (1.5) at points on the boundary of the stable norm ball corresponding to irrational laminations (cf. [SV19, §5]).

We conclude this discussion with a challenge: observe that the number of mutually $\prec_{\mathcal{M}}$-incomparable elements of $\mathcal{Q}$ is a bound for the potential non-injectivity of $\lambda_{\mathcal{M}}$ in relation to MUC (indeed MUC predicts that $\lambda_{\mathcal{M}}$ is injective and that $\prec_{\mathcal{M}}$ is a total order). Mutually $\| \cdot \|_{\mathcal{M}}$-incomparable points in $\mathcal{Q}$ are close to a subject that has attracted attention in convex analysis, as they can be thought of as rational points with common denominator on the $\| \cdot \|_{\mathcal{M}}$-unit sphere, a planar convex curve. By [Pet06, Thm. 1], the number of such elements can be bounded as

\[ \# \lambda_{\mathcal{M}}^1(n) = o \left( (\log n)^{2/3} \right). \]

The latter we refer to as Petrov’s bound.

Notice that Petrov’s bound is considerably stronger in the worst-case scenario than the super-logarithmic bound $\approx 2 \log \log \log n$ that arises from counting roots of $-1$ modulo $n$, as in [Aig15, Thm. 2.19], or the logarithmic bound $O(\log n)$ that follows from the analysis of Zagier [Zag82] (see also [LLRS20, Cor. 1.5(b)] for another logarithmic bound). An error-term for the count of simple closed geodesics on $X$ was obtained by McShane-Rivin in [MR95b] which provides $\# \lambda_{\mathcal{M}}^1(n) = O(\log n \log \log n)$, again super-logarithmic, although they conjecture a much stronger error term [MR95b, Conj. 4.3] which would imply the much stronger bound $O \left( (\log n)^{3/2 + \epsilon} \right)$. For any $\epsilon > 0$. As far as the author is aware, this McShane-Rivin conjecture is far beyond current methods (e.g. [EMM19, Thm. 1.1]).
Problem. Can Petrov’s bound be improved for the \( \| \cdot \|_s \)-unit sphere? Better bounds for strictly convex smooth plane curves [SD74] are evidently unapplicable here, but can one leverage McShane-Rivin’s observation that it is infinitely flat at points of irrational slope [MR95b, Thm. 2.1] to deduce improvements to Petrov’s bound? Are better bounds for \( \# \lambda_{X, \gamma}^{-1}(n) \) available by some other means?

Remark 1.14. According to [Pla99], there exist strictly convex curves with rational points of denominator \( q \) growing just slower than \( q^{2/3} \) (for certain choices of \( q \)). Therefore, any improvements to Petrov’s bound must rely on more than merely strict convexity of the \( \| \cdot \|_s \)-ball.

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2. Hyperbolic geometry of Markov triples on the modular torus

The modular torus \( X \) (also called the equianharmonic torus in the literature) can be obtained as the quotient of \( \mathbb{H}^2 \) by the commutator subgroup of \( \text{PSL}(2, \mathbb{Z}) \). The holonomy group of \( X \) admits a lift to \( \text{SL}(2, \mathbb{Z}) \) (in fact, such a lift exists in much greater generality [Cul86]), so one may speak of the ‘trace of a simple closed geodesic on \( X \’). For punctured tori, any such choice of lift will give rise to a representation sending the cusp to an element of trace \(-2\), from which one can deduce using the \( \text{SL}_2 \)-trace identity that triples of simple closed curves pairwise intersecting once have traces that satisfy a Markov-like equation. A lift exists so that all simple closed curves have positive integral traces, and one finds that the set of traces of simple closed geodesics on \( X \), in the chosen lift, will be precisely \( 3 \mathcal{M} \).

In order to compute derivatives for \( \Psi \), we recall some aspects of simple closed geodesics on \( X \). The following is a topological exercise. See [McS04] for detail.

Lemma 2.1. Any collection of three simple closed geodesics pairwise intersecting once on \( X \) have two complementary triangles, exchanged by the elliptic involution, whose vertices are the three Weierstrass points of \( X \) and whose side lengths are the half-lengths of the three geodesics.

In order to compute derivatives of Fock’s function, it will be important to keep track of the behavior of hyperbolic lengths under Dehn twisting. We indicate the mapping class obtained via the right Dehn order, as in Proposition 2.2.

Proposition 2.2. Suppose that \( \alpha, \beta, \) and \( \gamma \) are a trio of simple closed geodesics on \( X \) that pairwise intersect once, and suppose that a triangular complementary component sees \( (\alpha, \beta, \gamma) \) in counterclockwise order, as in Figure 5. Then the half-length function \( \ell \) satisfies

\[
\ell \left( \tau_{\gamma}^k(\beta) \right) = (k + 1) \ell(\gamma) + \log \left( \frac{3}{2} m_\alpha - \frac{9m_\gamma^2 m_\alpha - 6m_\beta}{2\sqrt{9m_\gamma^4 - 4}} \right) + O \left( e^{-2k\ell(\gamma)} \right)
\]

as \( k \to \infty \). (The implied constant in the last term depends on \( \alpha, \beta, \) and \( \gamma \), but is independent of \( k \).)

Remark 2.3. In terms of hyperbolic geometry, Proposition 2.2 is the computation of the asymptotic behavior of half-lengths under Dehn twisting. Unsurprisingly, the leading order term looks like \( k \) times the half-length of \( \gamma \). What is important for our purposes in this calculation is the constant term.

Proof. We use the hyperbolic law of cosines on the triangle pictured in Figure 6, with lengths \( (k + 1) \ell(\gamma) \), \( \ell(\alpha) \), and \( \ell(\tau_{\gamma}^k(\beta)) \), the latter facing angle \( \theta \):

\[
\cosh \ell(\tau_{\gamma}^k(\beta)) = \cosh ((k + 1) \ell(\gamma)) \cosh \ell(\alpha) - \sinh ((k + 1) \ell(\gamma)) \sinh \ell(\alpha) \cos \theta.
\]
Because \( \cosh^{-1}(A_1 \cosh x + A_2 \sinh x) = x + \log(A_1 + A_2) + O\left(e^{-2x}\right) \) as \( x \to \infty \),
\[
\ell(\tau_\gamma^k(\beta)) = (k + 1)\ell(\gamma) + \log\left(\cosh \ell(\alpha) \cosh \ell(\gamma) - \sinh \ell(\alpha) \sinh \ell(\gamma) \cos \theta\right) + O\left(e^{-2k\ell(\gamma)}\right)
\]
\[
= (k + 1)\ell(\gamma) + \log\left(\frac{3}{2}m_\alpha - \cosh \ell(\alpha) \cosh \ell(\gamma) - \cosh \ell(\beta)\right) + O\left(e^{-2k\ell(\gamma)}\right).
\]

Finally, applying the hyperbolic law of cosines to the triangle pictured in Figure 5, we find that
\[
\cosh \ell(\beta) = \cosh \ell(\alpha) \cosh \ell(\gamma) - \sinh \ell(\alpha) \sinh \ell(\gamma) \cos \theta,
\]
whence \( \sinh \ell(\alpha) \cos \theta = \frac{\cosh \ell(\alpha) \cosh \ell(\gamma) - \cosh \ell(\beta)}{\sinh \ell(\gamma)} \). Substituting in and simplifying,
\[
\ell(\tau_\gamma^k(\beta)) = (k + 1)\ell(\gamma) + \log\left(\frac{3}{2}m_\alpha - \frac{\cosh \ell(\alpha) \cosh \ell(\gamma) - \cosh \ell(\beta)}{\sinh \ell(\gamma)}\right) + O\left(e^{-2k\ell(\gamma)}\right)
\]
\[
= (k + 1)\ell(\gamma) + \log\left(\frac{3}{2}m_\alpha - \frac{3}{2}m_\alpha \frac{3}{2}m_\gamma - \frac{3}{2}m_\beta}{\sqrt{\frac{9}{4}m_\gamma^2 - 1}}\right) + O\left(e^{-2k\ell(\gamma)}\right)
\]
\[
= (k + 1)\ell(\gamma) + \log\left(\frac{3}{2}m_\alpha - \frac{9m_\alpha m_\gamma - 6m_\beta}{2\sqrt{9m_\gamma^2 - 4}}\right) + O\left(e^{-2k\ell(\gamma)}\right). \quad \square
\]

3. Derivative calculations

Here we perform the computations for the proofs of Proposition 1.8 and Proposition 1.10.

Let \( \frac{p_1}{s_1} < \frac{p}{q} < \frac{p_2}{s_2} \) span a Farey triangle, with corresponding Markov triple \((n_1, n, n_2)\), and so that \( n = m_{T(p/q)} \) and \( n_i = m_{T(r_i/s_i)} \). (Recall that \( T(p/q) = \frac{p}{p-q} \).)
The left-derivative of $\Psi(t)$ at $t = \frac{p}{q}$ may be computed as

$$\left. \frac{d}{dt} \right|_{t=\frac{p}{q}} \Psi(t) = \lim_{\frac{u}{v} \to \frac{p}{q}} \left( \frac{\Psi\left(\frac{p}{q}\right) - \Psi\left(\frac{u}{v}\right)}{\frac{p}{q} - \frac{u}{v}} \right) = \lim_{k \to \infty} \frac{\Psi\left(\frac{p}{q}\right) - \Psi\left(\frac{u_k}{v_k}\right)}{\frac{p}{q} - \frac{u_k}{v_k}} ,$$

where the last expression involves any sequence $\left(\frac{u_k}{v_k}\right)_{k=1}^\infty$ that approaches $\frac{p}{q}$ from the left. Since $\frac{r_1}{s_1} < \frac{p}{q}$, such a sequence can be obtained by Dehn twisting: let

$$\frac{u_k}{v_k} = \frac{r_1 + k p}{s_1 + k q}$$

for each $k \in \mathbb{N}$. Now we have

$$\frac{p}{q} - \frac{u_k}{v_k} = \frac{p}{q} - \frac{r_1 + k p}{s_1 + k q} = \frac{1}{(s_1 + k q)q} .$$

In order to compute $\Psi\left(\frac{u_k}{v_k}\right)$, we need to understand the hyperbolic length $\|T(v_k, u_k)\|_s$. Fortunately, we have in (2.1) a calculation of the hyperbolic length of the curve represented by

$$T(v_k, u_k) = T((s_1, r_1) + k(q, p)) = \tau^{k}_{T(q, p)} T(s_1, r_1) .$$

By Proposition 2.2 we find:

$$\frac{\Psi\left(\frac{p}{q}\right) - \Psi\left(\frac{u_k}{v_k}\right)}{\frac{p}{q} - \frac{u_k}{v_k}} = (s_1 + kq)q \left( \frac{\|T(q, p)\|_s}{2q} - \frac{\|\tau^{k}_{T(p, q)} T(s_1, r_1)\|_s}{2(s_1 + kq)} \right)$$

$$= \frac{1}{2} (s_1 + kq) \|T(q, p)\|_s - \frac{1}{2} q \|\tau^{k}_{T(p, q)} T(s_1, r_1)\|_s$$

$$= \frac{1}{2} s_1 \|T(q, p)\|_s + \frac{1}{2} kq \|T(p, q)\|_s$$

$$- q \left[ (k + 1) \frac{1}{2} \|T(q, p)\|_s + \log \left( \frac{3n_2}{2} - \frac{9mn_2 - 6n_1}{2\sqrt{9n^2_2 - 4}} \right) + O(e^{-k\|T(q, p)\|_s}) \right]$$

$$= (s_1 - q) \cosh^{-1} \left( \frac{3n_2}{2} \right) - q \log \left( \frac{3n_2}{2} - \frac{9mn_2 - 6n_1}{2\sqrt{9n^2_2 - 4}} \right) + O(e^{-k\|T(q, p)\|_s})$$

$$= -q \cosh^{-1} \left( \frac{3n_2}{2} \right) - q \log \left( \frac{3n_2}{2} - \frac{9mn_2 - 6n_1}{2\sqrt{9n^2_2 - 4}} \right) + O(e^{-k\|T(q, p)\|_s})$$

The limit is evident as $k \to \infty$, and the computation is complete.

Of course, the exact same computation can be carried out for the right-derivative as well. We leave the details to the reader.

The transformation $f : (x, y) \mapsto \left( \frac{1-x}{2y}, \frac{x}{2y} \right)$ maps the graph of $\Psi$ to the portion of the unit sphere, with respect to the stable norm, that lies inside $\{(q, p) : q \geq p \geq 0\}$. Indeed,

$$\|f(t, \Psi(t))\|_s = \frac{1}{2\Psi(t)} \left( \begin{array}{c} 1-t \\ t \\ 2\Psi(t) \end{array} \right) \|_{s} = \frac{1}{2\Psi(t)} \|(1-t, t, s)\|_s .$$

When $t = p/q \in \mathbb{Q}$, by (1.3) we find that

$$\frac{1}{2\Psi(t)} \|(1-t, t)\|_s = \frac{1}{2\Psi(p/q)} \left( \left\| \begin{array}{c} 1-t \\ t \end{array} \right\|_{2\Psi(t)} \right) \|_{s} = \frac{1}{\Psi(p/q)} \cdot \frac{\|T(q, p)\|_s}{q} = 1 .$$

By continuity, $\|f(t, \Psi(t))\|_s = 1$ for all $t \in [0, 1/2]$.

It is routine to compute the derivative $df$ at $(t, \Psi(t))$, and Proposition 1.10 follows easily.
4. Proof of Theorem 1.1

We show how Theorem 1.1 can be deduced from Theorem 1.4.

Proof of Theorem 1.1. Let \((q, p), (q', p') \in \mathcal{Q}\) and \(r = \frac{q'- q}{p'- p}\), and suppose without loss of generality that \(q' < q\). Because \(\sigma_- = \mu_0^+\) and \(\sigma_+ = \mu_{1/1}^-\) (see Example 1.11 and Example 1.12), convexity of the stable norm sphere implies that \(\mu_{p/q}' \mu_{p'/q'}' \mu_{p'/q}' \mu_{p/q}' \in [\sigma_-, \sigma_+]\). If \(r < \sigma_- \leq \mu_{p/q}'\) then Theorem 1.4 implies that \((q, p) \prec_M (q', p')\), so \(\prec_M\) is decreasing with \(q\), demonstrating item (i). If \(r > \sigma_+ \geq \mu_{p/q}'\) then Theorem 1.4 implies that \((q', p') \prec_M (q, p)\), so \(\prec_M\) is increasing with \(q\), demonstrating item (ii).

Now suppose that \(r \in (\sigma_-, \sigma_+)\), and let \(\mathcal{L}\) be a line of slope \(r\). Let \(P\) and \(Q\), respectively, be the intersections of \(\mathcal{L}\) with the lines through the origin of slopes 0 and 1. Because \(r > \sigma_- = \mu_{0/1}^+\), near \(P\) the line \(\mathcal{L}\) is inside the ball of radius \(\|P\|_s\) about the origin; because \(r < \sigma_+ = \mu_{1/1}^-\), near \(Q\) it is inside the ball of radius \(\|Q\|_s\) about the origin. It follows that, restricted to the line \(\mathcal{L}\), as a function of \(q\) the norm \(\|\cdot\|_s\) is decreasing near \(Q\) and increasing near \(P\). By convexity, there is a point \(O = (q_o, p_o)\) on the line segment \(QP\) so that, as a function of \(q\), \(\|\cdot\|_s\) is decreasing on \(QO\) and increasing on \(OP\).

Recall that \(\mathcal{L}_t\) indicates \(t \cdot \mathcal{L}\), the scaling of \(\mathcal{L}\) by \(t\). It remains to explore \(\prec_M\) on \(\mathcal{L}_t\), or, equivalently, \(\|\cdot\|_s\) on \(\mathcal{L}_t \cap \mathcal{Q}\). Let \(\mathcal{Q}' = \{(q, p) \in \mathbb{Z}^2 : q \geq p \geq 0\}\). Suppose that \(\mathcal{L}_t \cap \mathcal{Q}' = \{(q_1, p_1), \ldots, (q_n, p_n)\}\) with \(q_1 < \ldots < q_n\). Strict convexity of \(\|\cdot\|_s\) implies that there is some \(j \in \{1, \ldots, n\}\) so that

\[
\|(q_1, p_1)\|_s > \ldots > \|(q_j, p_j)\|_s < \ldots < \|(q_n, p_n)\|_s .
\]

Moreover, for \(t\) large enough \((4.1)\) holds with \(2 \leq j \leq n-1\): The points \(\frac{1}{t}(q_i, p_i)\) are dense in \(PQ\), so by continuity of \(\|\cdot\|_s\) there is a \(t\) large enough so that there are points \(\frac{1}{t}(q_i, p_i)\) in both \(QO\) and \(OP\), i.e. \(\|\frac{1}{t}(q_1, p_1)\|_s > \|\frac{1}{t}(q_2, p_2)\|_s\) and \(\|\frac{1}{t}(q_{n-1}, p_{n-1})\|_s < \|\frac{1}{t}(q_n, p_n)\|_s\). This demonstrates non-monotonicity with respect to \(q\) of \(\|\cdot\|_s\) on \(\mathcal{L}_t \cap \mathcal{Q}'\) (and, in fact, strict antimodality of \(\|\cdot\|_s\) on \(\mathcal{L}_t \cap \mathcal{Q}'\)).

In order to deduce strict antimodality (and non-monotonicity) for \(\prec_M\), it remains to go from \(\mathcal{L}_t \cap \mathcal{Q}'\) to \(\mathcal{L}_t \cap \mathcal{Q}\), i.e. one needs choices for \(t\) so that there are many relatively prime points \((q_i, p_i)\) in \(\mathcal{L}_t \cap \mathcal{Q}\).

The following demonstrates that such choices exist for arbitrarily large \(t\): Choose \(t\) and \(k\) so that \((k, k-1) \in \mathcal{L}_t \cap \mathcal{Q}\), for a large \(k \in \mathbb{N}\) relatively prime to \(v\), and let

\[
(q_j, p_j) = (k, k-1) + (j-1)(v, -u) \text{ for } j = 1, \ldots, K ,
\]

where \(K = \left\lfloor \frac{k-1}{v} \right\rfloor\). The Siegel-Walfisz Theorem provides an asymptotic count of \(\pi(x; v, k)\), the number of primes \(\leq x\) which are \(\equiv k\pmod{v}\). Provided \(x\) is large relative to \(v\) (indeed, for us \(v\) is fixed while \(x \to \infty\)) and \(h\) is large relative to \(x\) (for us, \(h \gtrsim x\)), one has

\[
\pi(x; v, k) - \pi(x - h; v, k) \sim \frac{h}{\phi(v) \log x} .
\]

(See [MV07, Cor. 11.19] for a precise statement of the Siegel-Walfisz Theorem.)

We find that for any \(\delta > 0\) and \(h = \delta x\) there exists \(N\) so that

\[
\pi((1 + \delta)x; v, k) - \pi(x; v, k) \geq 2
\]

for all \(x \geq N\). As \(t\) and \(k\) satisfy a linear relationship (that is \(C_1 k \leq t \leq C_2 k\) for some constants \(C_1\) and \(C_2\)), we may apply this estimate to both the intervals \([k, tq_0]\) and \([tq_0, qK]\), and one obtains several points with prime \(q\)-coordinate inside \(t \cdot QO\) and \(t \cdot OP\). This demonstrates that the points \(\{(q_i, p_i)\}\) defined in \((4.2)\) contain points from \(\mathcal{Q}\) for which \(\prec_M\) both decreases and increases with respect to \(q\). Hence we have strict antimodality with respect to \(q\) for \(\mathcal{L}_t \cap \mathcal{Q}\). \(\square\)

Remark 4.1. The reader may observe that \((4.1)\) implies that there exists \(T > 0\) so that, for all \(t \geq T\), the stable norm \(\|\cdot\|_s\) is strictly antimodal with respect to \(q\) on the intersection \(\mathcal{L}_t \cap \mathcal{Q}'\).
Remark 4.2. Lee-Li-Rabideau-Schiffler claim more than the statement of Theorem 1.1(iii). Namely, they extend \( \prec_M \) from \( Q \) to \( Q' \) (see also [LPTV21]). The latter is induced by something they call the ‘Markov distance’ – though one should note that the Markov distance is not a metric, as it does not obey the triangle inequality [LLRS20, Rem.3.7]. With the notation that \( m_{q,p} \) stands for the Markov distance from \((0,0)\) to \((q,p)\), one has \( m_{q,p} = m_{p/q} \) when \( \gcd(q,p) = 1 \). A formula computing \( m_{q,p} \) in terms of \( m_{q',p'} \), where \((q',p') = \frac{1}{\gcd(q,p)}(q,p)\), can be found in [LLRS20, Lem.6.2]. One finds that

\[
m_{q,p} = \frac{1}{L_{q',p'}} \cdot 2 \sinh \left( \frac{1}{2} \| (q,p) \|_s \right),
\]

where \( L_{s,r} = \sqrt{9 - \frac{4}{m}} \) for \( m = m_{r/s} \) (that is, \( L_{s,r} \) is the Lagrange number associated to Markov number \( m_{r/s} \) – the Lagrange numbers are fundamental invariants in Diophantine approximation [Aig15, CF89]).

In [LLRS20, Conj.6.12], it is conjectured that there is a \( T \) so that, for all \( t \geq T \), the Markov distance is strictly antimodal with \( q \) on the intersection \( L_t \cap Q' \). As \( \sinh \) is an increasing function, it is tempting to prove this claim by combining Remark 4.1 and (4.3). However, the Lagrange number \( L_{q',p'} \) provides a significant obstacle. Indeed, we point out that the orders on \( Q' \) induced by the Markov distance (in which \((q,p) < (r,s)\) whenever \( m_{q,p} < m_{r,s} \)) and by the stable norm \( \| \cdot \|_s \) (in which \((q,p) < (r,s)\)) whenever \( \| (q,p) \|_s < \| (r,s) \|_s \) do not coincide in general: one has

\[
m_{9,0} = 2584 > 2378 = m_{5,5} \quad \text{while} \quad \| (9,0) \|_s = 8.66 \ldots < 8.81 \ldots = \| (5,5) \|_s.
\]

This is in stark contrast to the situation on \( Q \), in which \( m_{q,p} \) and \( \| (q,p) \|_s \) are related by an increasing function (1.2). Because the orders induced by the Markov distance and the stable norm do not coincide, the proof above of strict antimodality of the stable norm on \( L_t \cap Q' \) for \( t \) sufficiently large does not translate to the same for the Markov distance. Thus our proof stops short of demonstrating the full statement of [LLRS20, Conj.6.12].

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\[2\] This formula corrects the false claim that \( m_{q,p} = \frac{2}{3} \cosh \frac{1}{2} \| (q,p) \|_s \) found in a previous version of this manuscript. The previous ‘Corollary 4.2’, linking Markov distances to traces of non-primitive curves, is likewise withdrawn.
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