Beta and Gamma functions of Cayley-Dickson numbers

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1 Introduction.

This paper continues investigations of function theory over Cayley-Dickson algebras [12, 13]. Cayley-Dickson algebras $\mathbb{A}_v$ over the field of real numbers coincide with the field $\mathbb{C}$ of complex numbers for $v = 1$, with the skew field of quaternions, when $v = 2$, with the division nonassociative noncommutative algebra $\mathbb{K}$ of octonions for $v = 3$, for each $v \geq 4$ they are nonassociative and not division algebras. The algebra $\mathbb{A}_{v+1}$ is obtained from $\mathbb{A}_v$ with the help of the doubling procedure. This work provides examples of $\mathbb{A}_v$-meromorphic functions and usages of line integrals over $\mathbb{A}_v$. Here notations of previous papers [12, 13] are used. Discussions of references and results of others authors can be found in [12, 13] as well as physical applications (see also [1, 4, 6, 7, 9, 10, 11, 14, 15] and references therein). Beta and Gamma functions illustrate general theory of meromorphic functions of Cayley-Dickson numbers and also applications of line integration over $\mathbb{A}_v$.

The results below show some similarity with the complex case and as well differences caused by noncommutativity and nonassociativity of Cayley-Dickson algebras. It is necessary to mention that before works [12, 13] there was not any publication of others authors devoted to the line integration of continuous functions of Cayley-Dickson numbers or even quaternions along rectifiable paths. In works of others authors integrations over submanifolds of codimension 1 in $\mathbb{H}$ or $\mathbb{K}$ were used instead of line integral. Therefore, in this respect publications [12, 13] are the first devoted to (integral) holomorphic functions of Cayley-Dickson numbers.
If \( g \) is a complex holomorphic function on a domain \( V \) in the complex plane \( \Pi \) embedded into \( \mathcal{A}_v \) and \( g \) has a local expansion \( g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) in the ball \( B(\Pi, z_0, r^-) \) for each \( z_0 \in \text{Int}(V) \), where \( \text{Int}(V) \) is the interior of \( V \) in \( \Pi \), \( r > 0 \) and \( a_n = a_n(z_0) \in \mathbb{C} \) may depend on parameter \( z_0 \), \( B(X, a, r^-) := \{x \in X : \rho(x, a) < r\} \) for a metrizable space \( X \) with metric \( \rho \), \( f \) is a function on a domain \( U \) in \( \mathcal{A}_v \), \( v \geq 2 \), such that \( V \subset U \cap \Pi \) and \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) is the local expansion of \( f \) in \( B(\mathcal{A}_v, z_0, r^-) \) for each \( z_0 \in V \), then the line integral \( \int_\omega f(z)dz \) over \( \mathcal{A}_v \) for a rectifiable path \( \omega \) in \( V \) coincides with the classical complex Cauchy line integral, since \( \hat{f}|_V = f|_V \). Nevertheless, if \( f \) is an \( \mathcal{A}_v \)-holomorphic function on a domain \( U \) in \( \mathcal{A}_v \), \( V = U \cap \Pi \) is a domain in the complex plane \( \Pi \) embedded into \( \mathcal{A}_v \), then in general \( \int_\omega f(z)dz \) can not be reduced to Cauchy line integral for any rectifiable path \( \omega \) in \( V \), since the generalized operator \( \hat{f} \) is defined by values of \( f \) in the neighbourhood of \( \omega \) (see [12, 13]).

2 Beta and Gamma functions of Cayley-Dickson numbers.

1. Definition. The Gamma function is defined by the formula:
   \[
   \Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt,
   \]
   whenever this (Eulerian of the second kind) integral converges and defined by \( \mathcal{A}_v \)-holomorphic continuation elsewhere, where \( 0 < t \in \mathbb{R}, t^z := \exp(z\ln t) \), \( z \in \mathcal{A}_v \), \( t^z \) take its principal value, \( dt \) corresponds to the Lebesgue measure on \( \mathbb{R} \), \( \ln : (0, \infty) \to \mathbb{R} \) is the classical (natural) logarithmic function.

   Denote by \( \text{Re}(z) := (z + z^*)/2 \) the real part of \( z \in \mathcal{A}_v \), \( \mathcal{I}_v := \{z \in \mathcal{A}_v : \text{Re}(z) = 0\} \), where \( z^* \) is the conjugate of a Cayley-Dickson number \( z \).

2. Proposition. The gamma function has as singularities only simple poles at the points \( z \in \{0, -1, -2, \ldots\} \) and \( \text{res}(-n, \Gamma)M = [(-1)^n/n!]M \) for each \( M \in \mathcal{I}_v \).

   Proof. Write \( \Gamma(z) \) in the form:
   \[
   (i) \ \Gamma(z) = \Phi(z) + \Psi(z),
   \]
   \[
   (ii) \ \Phi(z) = \int_0^1 e^{-t}t^{z-1}dt,
   \]
   \[
   (iii) \ \Psi(z) := \int_1^\infty e^{-t}t^{z-1}dt.
   \]
   Since \( |e^{a+M}| = e^{a} \) for each \( a \in \mathbb{R} \) and \( M \in \mathcal{I}_v \) (see Corollary 3.3 [13]), then \( |t^{z-1}| \leq t^{\delta-1} \) for each \( \text{Re}(z) \leq \delta \), where \( \delta > 0 \) is a marked number. From
\[ \lim_{t \to \infty} e^{-t}t^{z-1} = 0 \] it follows, that there exists \( C = \text{const} > 0 \) such that 
\[ |e^{-t}t^{z-1}| \leq Ce^{-t/2} \] for each \( t > 0 \) and each \( z \) with \( \text{Re}(z) \leq \delta \). Therefore, 
\( \Psi(z) \) is the \( \mathcal{A}_v \)-holomorphic function in \( \mathcal{A}_v \).

Consider change of variables \( t = 1/u \), then \( \Phi(z) = \int_1^\infty e^{-1/u}u^{-z-1}du \) for each \( \delta > 0 \) and each \( z \) with \( \text{Re}(z) \leq \delta \), hence 
\[ |e^{-1/u}u^{-z-1}| \leq u^{-\delta-1} \]. Therefore, \( \Phi(z) \) is \( \mathcal{A}_v \)-holomorphic, when \( \text{Re}(z) > 0 \). Substituting the Taylor series for \( e^{-t} \) into the integral expression (ii), we get 
\[ (iv) \quad \Phi(z) = \sum_{n=0}^\infty (-1)^n \int_0^1 t^{n+z-1}dt/n! = \sum_{n=0}^\infty (-1)^n(n+z)^{-1}/n! \]. Series 
\( (iv) \) is uniformly and abosolutely convergent in any closed domain in \( \mathcal{A}_v \setminus \{0, -1, -2, \ldots\} \) and this series gives \( \mathcal{A}_v \)-analytic continuation of \( \Phi(z) \). Thus 
\( \Gamma(z) \) has only simple poles at the points \( z \in \{0, -1, -2, \ldots\} \).

The following Tannery lemma is true for \( \mathcal{A}_v \)-valued functions (for complex valued functions see §9.2 [3]).

3. Lemma. If \( g(t) \) and \( f(t, n) \) are functions from \([a, \infty) \) to \( \mathcal{A}_v \), \( v \geq 2 \), \( \lim_{n \to \infty} f(t, n) = g(t) \), \( \lim_{n \to \infty} \lambda_n = \infty \), then \( \lim_{n \to \infty} \int_a^n t^n f(t, n)dt = \int_a^\infty g(t)dt \), provided that \( f(t, n) \) tends to \( g(t) \) uniformly on any fixed interval, and provided also that there exists a positive function \( M(t) \) such that 
\[ |f(t, n)| \leq M(T) \] for each values of \( n \) and \( t \) and such that \( \int_a^\infty M(t)dt \) converges.

Proof. The sequence \( f(t, n) \) converges uniformly to \( g(t) \) in the fixed segment \( a \leq t \leq b \), \( a < b < \infty \). Using triangle inequalities and \( |\int_a^b g(t)dt| \leq \int_a^b |g(t)|dt \) gives: 
\[ \limsup_{n \to \infty} |\int_a^n f(t, n)dt - \int_a^\infty g(t)dt| \leq 2 \int_a^b M(t)dt \] for each \( a < b < \infty \). From \( \lim_{b \to \infty} \int_a^b M(t)dt = 0 \) the statement of this lemma follows.

4. Proposition. If \( \Gamma(z, n) := n!n^z([((z(n+1))(z+n)]^{-1}, \quad n \in \mathbb{N} \), then \( \Gamma(z, n) \) tends to \( \Gamma(z) \) as \( n \to \infty \), the convergence being uniform in any bounded canonical closed subset \( U \subset \mathcal{A}_v \) which contains no any of the singularities of \( \Gamma(z) \), \( v \geq 2 \).

Proof. Since \( \mathcal{A}_v \) is power-associative and \( \mathbb{R} \) is the centre of the Cayley-Dickson algebra, then \( \{n\frac{z}{2}[z(z+1)(z+2)\ldots(z+n)]^{-1}\}_{q(n+2)} \) does not depend on the order of multiplication regulated by the vector \( q(n+2) \) (see [13]). Therefore, 
\[ \Gamma(z, n) = (n/(n+1))^{z-1} \prod_{m=1}^{n} ((1 + 1/m)^{z}(1 + z/m)^{-1}) \]. Then 
\[ (1 + 1/m)^{z}(1 + z/m)^{-1} = 1 + z(z-1)/(2m^2) + O(1/m^3) \], when \( m > 0 \) is large, hence 
\[ z^{-1} \prod_{m=1}^{n} ((1 + 1/m)^{z}(1 + z/m)^{-1}) \] converges uniformly and abosolutely in any bounded canonical closed domain \( U \) in \( \mathcal{A}_v \) to an \( \mathcal{A}_v \)-holomorphic function in accordance with Theorem 3.21 [13]. In view of Formulas (3.6, 7) in [13] \( |((1 - t/n)^n t^{-1})| = (1 - t/n)^n t^{-1} \leq e^{-t/n} \), where \( a := \text{Re}(z) \). From
\[ f_n(t) = \int_0^n (1 - u)^n u^{-1} du \] and integrating by parts we get
\[ \Gamma(z, n) = \int_0^n (1 - u)^n u^{-1} du, \] hence \( \lim_{n \to \infty} \Gamma(z, n) = \int_0^\infty e^{-t} t^{-1} dt =: \Gamma(z) \] by Lemma 3.

5. Remark. From the proof of Proposition 4 it follows, that \( \Gamma(z) = z^{-1} \prod_{i=1}^\infty ((1 + 1/m)^z(1 + z/m)^{-1}) \) for each \( z \in A_v \setminus \{0, -1, -2, \ldots\}, v \geq 1. \) The latter is known as the Euler’s formula in the case of complex numbers.

6. Proposition. The gamma function satisfies identities:
(i) \( \Gamma(z + 1) = z\Gamma(z) \) and
(ii) \( \Gamma(z)\Gamma(1 - z) = \pi \csc(\pi z) \)
for each \( z \in A_v \setminus \{0, -1, -2, \ldots\}, v \geq 2. \)

Proof. In view of power associativity of \( A_v \) and that \( R \) is the centre of the Cayley-Dickson algebra we get
\[ \Gamma(z + 1) = \lim_{n \to \infty} n!z^2[(z + 1)\ldots(z + n)]^{-1} n/(z + n + 1) = z\Gamma(z), \] also
\[ \Gamma(z)\Gamma(1 - z) = \lim_{n \to \infty} \left\{ z(1 - z^2/2^2)(1 - z^2/2^2)\ldots(1 - z^2/n^2)(1 + (1 - z)/n) \right\}^{-1} = \{z \prod_{i=1}^\infty (1 - z^2/n^2) \}^{-1} \] for complex \( z \) in \( C \setminus \mathbb{Z} \) [3]. Using the \( A_v \)-holomorphic extension of this function from the complex domain onto the corresponding domain \( A_v \setminus \mathbb{Z} \) (see Proposition 3.13, Corollary 2.13 and Theorems 3.10, 3.21 [13]), we get Formula (ii).

7. Definition. A function \( F \) on an unbounded domain \( U \) in \( A_v, v \geq 2, \) is said to have an asymptotic expansion \( F \sim \sum_{|k| \leq 0}(a_k, z^k), \) if
\[ \lim_{z \to U, |z| \to \infty} z^n \{ F(z) - \sum_{|k| \leq 0}(a_k, z^k) \} = 0 \]
for each \( n \in \mathbb{N}, \) where \( k = (k_1, \ldots, k_n), \) \( |k| := k_1 + \ldots + k_n, \) \( k_j \in \mathbb{Z} \) for each \( j, n \in \mathbb{N}, \) \( (a_k, z^k) := a_k z^{k_1} \ldots a_k z^{k_n}, \) \( a_k \in A_v \) for each \( j. \)

We write \( F(z) \sim G(z) \sum_{|k| \leq 0}(a_k, z^k), \) if \( G(z)^{-1} F(z) \sim \sum_{|k| \leq 0}(a_k, z^k). \) The term \( G(z)a_0 \) is called the dominant term of the asymptotic representation of \( F(z). \)

8. Lemma. Let \( f(t) \) be a function in an unbounded domain \( U \) in \( A_v, \) possibly with a branch point at 0 and such that
\[ f(z) = \sum_{m=1}^\infty a_m z^{r(m/r)} \]
when \( |z| \leq a, a > 0, r > 0, \) let also \( f \) be \( A_v \)-holomorphic in \( B(U, 0, a + \delta) \setminus \{0\}, \) where \( \delta > 0. \) Suppose, that when \( t \geq 0, |f(t)| < Ce^{bt}, \) where \( C > 0 \) and \( b > 0 \) are constants. Then
\[ F(z) = \int_0^\infty e^{-zt} f(t) dt \sim \sum_{n=1}^\infty a_n \Gamma(n/r) z^{-n/r}, \]
when \( |z| \) is large and \( |\text{Arg}(z)| \leq \pi/2 - \epsilon, \) where \( \epsilon > 0 \) is arbitrary.

Proof. For each \( n \in \mathbb{N} \) there exists a constant \( C = \text{const} > 0 \) such that
for each \( t \geq 0 \). In view of Formulas (3.2, 3) [13]

\[
|f(t) - \sum_{m=1}^{n-1} a_m t^{(m/r)-1}| \leq C t^{(n/r)-1} e^{bt}
\]

for each \( x > b \), where \( x := \text{Re}(z) \). From the condition \( |\text{Arg}(z)| \leq \pi/2 - \epsilon \) it follows, that \( x \geq |z|^2 \sin(\epsilon) \), such that \( x > b \) for \( |z| > b \csc(\epsilon) \). Therefore, for \( |\text{Arg}(z)| \leq \pi/2 - \epsilon < \pi/2 \) and \( |z| > b \csc(\epsilon) \), there is the inequality:

\[
|z^{n/r} \int_{0}^{\infty} e^{-zt} [f(t) - \sum_{m=1}^{n-1} a_m t^{(m/r)-1}] dt| \leq C \Gamma(n/r) |z|^{n/r} / (|z| \sin(\epsilon) - b)^{n/r} = O(1).
\]

9. **Proposition.** Let \( 0 < \delta < \pi/2 \), \( z \in \mathcal{A}_v \setminus \{0, -1, -2, \ldots\} \), \( |\text{Arg}(z)| \leq \pi - \delta \), \( v \geq 2 \). Then there exists the asymptotic expansion:

\[
\ln \Gamma(z) \sim (z-1/2) \ln(z) - z + (\ln(2\pi))/2 + \sum_{n=1}^{\infty} (-1)^{n-1} B_n [2n(2n-1)z^{2n-1}]^{-1},
\]

where \( B_n \) are Bernoulli numbers defined by the equation: \( (z/2) \coth(z/2) = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} B_n z^{2n}/(2n)! \).

**Proof.** If \( z > 0 \), then the substitution \( t = z u \) gives \( \Gamma(z) = \Gamma(1 + z)/z = z e^{-z} \int_0^\infty (ue^{-u}) z du \) and by analytic continuation the formula \( \Gamma(z) = z e^{-z} \int_0^\infty (ue^{-u}) z du \) is true for each copy of \( \mathbb{C} \), \( 0 \in \mathbb{C} \), embedded into \( \mathcal{A}_v \).

In view of independence of this formula from such embedding and power associativity of \( \mathcal{A}_v \) it follows, that it is true for each \( z \in \mathcal{A}_v \) with \( \text{Re}(z) > 0 \). For \( \text{Re}(z) > 0 \) and large \( |z| \) using substitutions \( e^{-t} = \eta e^{1-v} \) for \( t \in (0, \infty) \) and \( \eta \in (1, \infty) \); also the substitution \( e^{-t} = u e^{1-v} \) for \( t \) decreasing monotonously from \( \infty \) to 0 and \( u \in (0, 1) \), we get \( z^{-z} e^z \Gamma(z) = \int_0^\infty e^{-zt} (d\eta/dt - du/dt) dt \).

Consider two real solutions \( \eta \) and \( u \) of the equation \( t = u - 1 - \ln(u) \) and the equation \( \zeta^2/2 = w - \ln(1+w) \) which defines \( w = w(\zeta) \) for \( \zeta \in \mathbb{R} \oplus M \mathbb{R} \). It has two branches \( \zeta = \beta w(1-2w/3+2w^2/4-\ldots)^{-1/2} \), where \( \beta = -1 \) or \( \beta = 1 \). Each branch is the analytic function of \( w \) in the domain \( \{ w \in \mathbb{R} \oplus M \mathbb{R} : |w| < 1 \} \) with a simple zero at \( w = 0 \). For \( \beta = 1 \) there exists a unique solution \( w = \zeta + a_2 \zeta^2 + a_3 \zeta^3 + \ldots \) in \( \{ \zeta : |\zeta| < \rho \} \), \( na_n M = \text{res}(0, \zeta^{-n}) M \) for each \( n > 1 \).

Thus \( w \) has two branches \( w_1 \) and \( w_2(\zeta) = w_1(-\zeta) \). Singularities of \( w(\zeta) \) are only points at which \( dw/d\zeta \) is zero or infinite, hence these are \( \zeta = 0 \), also points corresponding to \( w = 0 \) and \( w = -1 \), since \( dw/d\zeta = \zeta(1+w)/w \).

Then \( \zeta = 0 \) is not a branch-point of \( w_1 \), to \( w = -1 \) there corresponds \( \zeta = \infty \). Therefore, singularities are: \( \zeta^2 = 4n\pi M \), where \( n \in \mathbb{Z} \setminus \{0\} \). Then \( \eta \) and \( u \) are \( \mathcal{A}_v \)-holomorphic, when \( |(z+\bar{z})/2| < 2 \) possibly besides \( z = 0 \) and when \( |z| < 2\pi \), where \( \zeta^2 := 2z \) such that \( \eta = 1 + (2z)^{1/2} + a_2(2z) + a_3(2z)^{3/2} + a_4(2z)^2 + \ldots \).
u = 1 - (2z)^{1/2} + a_2(2z) - a_3(2z)^{3/2} + a_4(2z)^2 - ..., the square roots are taken positive, when z > 0. Applying Lemma 8 we get the asymptotic expansion. In view of Theorem 2.15 [13] for M ∈ ℑ_ν with |M| = 1 and α ∈ R and a loop defined by pe^{Mt} on the boundary of the sector |z| ≤ ρ and two lines Arg(z) = 0, Arg(z) = Mα, where α ∈ (-π/2, π/2), g(z) := d(η - u)/dt, provides the equality: ∫_0^∞ e^{-zt}g(t)dt = ∫_0^∞ exp(-zte^{Ma})g(te^{Ma})e^{Ma}dt, when Arg(z) ∈ M/R, Re(z) > 0 and Re(zte^{Ma}) > 0, since R ⊕ M/R is isomorphic with C which is commutative. Therefore, the latter integral converges uniformly and provides the analytic function. Two regions Re(z) > 0 and Re(zte^{Ma}) > 0 have a common area and by the analytic continuation: z^-z e^z Γ(z) = ∫_0^∞ exp(-zte^{Ma})g(te^{Ma})e^{Ma}dt, when α ∈ (-π/2, π/2). Applying Lemma 8 we get the region of validity of this asymptotic expansion, since M is arbitrary.

10. Corollary. For large |y| there is the asymptotic expansion |Γ(x + My)| ∼ (2π)^{1/2}|y|^{x-1/2} exp(-π|y|/2) uniformly by M ∈ ℑ_ν, |M| = 1, where v ≥ 2, y ∈ R.

11. Corollary. π^{1/2}Γ(2z) = 2^{2z-1}Γ(z)Γ(z + 1/2) for each z ∈ ℛ_v \ {0, −1, −2, ...}, v ≥ 2.

The proof is analogous to §§9.55, 9.56 [3], since R ⊕ M/R is isomorphic with C for each M ∈ ℑ_ν, v ≥ 2, |M| = 1.

12. Proposition. For all z ∈ ℛ_v:

1/Γ(z) = (2π)^{-1}(∫_ψ e^{ζ^{-z}}dζ)M^*

for a loop ψ and z in the plane R ⊕ M/R, M ∈ ℑ_ν, |M| = 1, ψ starts at −∞ of the real axis, encircles 0 once in the positive direction and returns to the starting point.

Proof. Consider the integral ∫_ψ e^{ζ^{-z}}dζ = ∫_ψ f(ζ)dζ, the integrand f(ζ) has a branch point at zero, but each branch is a one-valued function of ζ and each branch is ℛ_v-holomorphic in ℛ_v \ Q, where Q is a submanifold in ℛ_v of real codimension 1 such that (−∞, 0] ⊂ Q (see §3.7 [12]). Then take a branch e^{ζ^{-z}} = exp(ζ - zLn(ζ)), where Ln(ζ) takes its principal value. Consider a rectifiable loop γ in R ⊕ M/R encompassing zero in the positive direction and beginning at −ρ on the lower edge of the cut and returns to −ρ at the upper edge of the cut, where ρ > 0.

In view of Theorem 2.15 [13] the value of the integral is not changed by the deformation to a contour γ consisting of the lower edge of the cut intersected with [−ρ, −δ], where 0 < δ < ρ, the circle |z| = δ in the plane R ⊕ M/R, and the upper edge of the cut intersected with [−ρ, −δ]. On
the upper edge of the cut in \( \gamma \): \( \zeta = ue^{\pi M} \), where \( u > 0 \), \( u \in \mathbb{R} \), and

\[ f(\zeta) = \exp(-u - zln(u) - z\pi M) = e^{-u}e^{-z\pi M}. \]

On the lower edge of the cut in \( \gamma \): \( \zeta = ue^{-\pi M} \) and \( f(\zeta) = e^{-u}e^{z\pi M} \). Therefore, \( f_\gamma f(\zeta) d\zeta = (e^{z\pi M} - e^{-z\pi M}) \int_0^\gamma e^{-u}e^{-z}du + J \), where \( J := \int_{-\pi}^{\pi} \exp(\delta e^{\theta M})(\delta(1-z)e^{(1-z)\theta M}d\theta, \)

since \( \mathbb{R} \oplus M\mathbb{R} \) is isomorphic with \( \mathbb{C} \) and \( f(\zeta)h = f(z)h \) for each \( h \) and \( z \in \mathbb{R} \oplus M\mathbb{R} \) (see Theorem 2.7 [13]).

If \( z = x + yM \), where \( x \) and \( y \in \mathbb{R} \), then \( |J| \leq \int_{-\pi}^{\pi} \delta^{1-x} \exp(\delta \cos(\theta) + y\theta)d\theta \leq 2\pi \delta^{1-x}e^{\delta + \pi |y|} \), consequently, \( \lim_{\delta \to 0} J = 0 \), when \( x < 1 \). Hence \( f_\gamma e^{-\zeta}e^{-z}\zeta = 2\sin(\pi z)(f_\gamma e^{-u}e^{-z}du)M \) for \( Re(1-z) > 0 \). Suppose \( \psi \) is the loop obtained from \( \gamma \) by tending \( \rho \) to the infinity, then \( f_\psi e^{\zeta}e^{-z}\zeta = 2\sin(\pi z)(f_\psi e^{-u}e^{-z}du)M = 2\sin(\pi z)\Gamma(1-z)M \), since \( \Gamma(z)\Gamma(1-z) = \pi \csc(\pi z) \), hence \( 1/\Gamma(z) = (2\pi)^{-1}(f_\psi e^{\zeta}e^{-z}\zeta)M^* \). Since \( M \in \mathcal{I}_v \) with \( |M| = 1 \) is arbitrary, then this formula is true in \( \mathcal{A}_v \setminus Q \) with \( Re(1-z) > 0 \). By the complex holomorphic continuation this formula is true for all values of \( z \) in \( \mathbb{R} \oplus M\mathbb{R} \).

**13. Corollary.** Let \( M \) and \( \psi \) be as in Proposition 12, then \( \Gamma(z) = (2\sin(\pi z))^{-1}(f_\psi e^{\zeta}e^{-z}\zeta)M^* \) for each \( z \in \mathbb{R} \oplus M\mathbb{R} \setminus \mathbb{Z} \).

**14. Definition.** The Beta function \( B(p, q) \) of Cayley-Dickson numbers \( p, q \in \mathcal{A}_v, v \geq 2 \), is defined by the equation:

\[ B(p, q) := \int_0^1 \zeta^{p-1}(1-\zeta)q^{-1}d\zeta, \]

whenever this integral (Eulerian of the first kind) converges, where \( \zeta^{p-1} := e^{(p-1)ln(\zeta)} \) and the logarithm has its principal value. This equation defines \( B(p, q) \) for each \( Re(p) > 0 \) and \( Re(q) > 0 \). For others values of \( p \) and \( q \) it is defined by the complex holomorphic continuation by \( p \) and \( q \) separately and subsequently in each complex plane \( \mathbb{R} \oplus M\mathbb{R} \) and \( \mathbb{R} \oplus S\mathbb{R} \), \( M, S \in \mathcal{I}_v \), \( |M| = 1 \) and \( |S| = 1 \).

**15. Proposition.** Let \( p, q \in \mathcal{A}_v, v \geq 2 \), such that the minimal subalgebra \( \mathcal{Y}_{p,q} \) containing \( p \) and \( q \) has embedding into \( \mathbb{K} \), then

\[ B(p, q) - B(p, p_0 - q') = B(p_0 - p', q) + B(p_0 - p', q_0 - q')/(q')^2q_2/2, \]

where \( p_0 := Re(p) \), \( p' := p - Re(p) \), \( q_2 \perp p' \), \( q_1 \parallel p' \) relative to the scalar product \( (z, \eta) := Re(\eta z^*) \), \( q' = q_1 + q_2 \).

**Proof.** Making the substitution \( \eta \mapsto 1 - \eta \) of the variable, we get

\[ \int_0^1 \eta^{q-1}(1-\eta)^{p-1}d\eta = \int_0^1 (1-\eta)^{q-1}p^{p-1}d\eta, \]

but in general \( p \) and \( q \) do not commute. In view of Formulas (3.2, 3.3) [13] the commutator of two terms in the integral is:

\[ [t^{p-1}, (1-t)^{q-1}] = 2t^{p-1}(1-t)^{q-1}[(\sin |p'lnt|)/|p'lnt|)(\sin |q'lnt|)(1-1)]/|q'lnt|1-1. \]
\( t \rangle \rangle p' \text{Int}(q'_2 \text{ln}(1 - t)). \) 

On the other hand, \(((\sin |M|)/|M|)M = [e^M - e^{-M}]/2\) for each \( M \in \mathcal{I}_v \), hence 

\[
\int_0^1 [t^{p-1}, (1-t)^{q-1}] dt = (\int_0^1 t^{p-1}(1-t)^{q-1} \frac{t^{1-p} - t^{1-q}}{1-t} \frac{(1-t)^q - (1-t)^p}{1-t} dt) (q')^* q'_2 / 2
\]

where \( K \) is alternative and 

\[
B(p, q) - B(p, q_0 - q') - B(p_0 - p', q) + B(p_0 - p', q_0 - q') (q')^* q'_2 / 2, \]

since \( K \) is alternative and 

\[
p' q'_2' = p'(q'q'^*) q'_2 = p'(q'q'^*) q'_2. \]

16. **Remark.** Let \( G \) be a classical Lie group over \( \mathbb{R} \) and \( g = T_e G \) be its Lie algebra (finite dimensional over \( \mathbb{R} \)). Suppose that \( e : V \to U \) is the exponential mapping of the neighbourhood \( V \) of zero in \( g \) into a neighbourhood \( U \) of the unit element \( e \in G \). \( \ln : U \to V \) is the logarithmic mapping. Then, \( w = \ln(e^u \circ e^v) \), \( w = w(u, v) \), is given by the Campbell-Hausdorff formula in terms of the adjoint representation \((ad \ u)(v) := [u, v]::

\[
w = \sum_{n=1}^{\infty} n^{-1} \sum_{r+s=n, r \geq 0, s \geq 0} (w'_{r,s} + w''_{r,s}), \text{ where } w'_{r,s} = \sum_{m=1}^{\infty} (-1)^{m-1} m^{-1} \sum_{s} ^{\star} ((\prod_{i=1}^{m-1} (ad \ u))^{r_i} (ad \ v)^{s_i} (r_i!)^{-1}(s_i!)^{-1}) (ad \ u)^{m} (r_m!)^{-1}(v),
\]

\[
w''_{r,s} = \sum_{m=1}^{\infty} (-1)^{m-1} m^{-1} \sum_{s} ^{**} ((\prod_{i=1}^{m-1} (ad \ u))^{r_i} (ad \ v)^{s_i} (r_i!)^{-1}(s_i!)^{-1})(u),
\]

\( \sum' \) means the sum by \( r_1 + \ldots + r_m = r, s_1 + \ldots + s_{m-1} = s - 1, r_1 + s_1 \geq 1, \ldots, r_{m-1} + s_{m-1} \geq 1 \), \( \sum'' \) means the sum by \( r_1 + \ldots + r_{m-1} = r - 1, s_1 + \ldots + s_{m-1} = s, r_1 + s_1 \geq 1, \ldots, r_{m-1} + s_{m-1} \geq 1 \). In particular, this formula can be applied to the multiplicative group \( G = \mathbf{H} \setminus \{0\} \) with \( U = G \) and \( V = q \), since each quaternion can be represented as a \( 2 \times 2 \) complex matrix, where generators of \( \mathbf{H} \) are Pauli matrices [2].

17. **Theorem.** Let \( p, q \in \mathbb{A}_v \), \( v \geq 2 \), such that the minimal subalgebra \( \mathcal{Y}_{p,q} \) generated by \( p \) and \( q \) has embedding into \( \mathbf{H} \), then \( \Gamma(p) \Gamma(q) = \Gamma(w(p, q))B(p, q) - \{ [\Gamma(w(p, q)) - \Gamma(w(p, q_0 - q'))]q'^* q'_2 [B(p, q) - B(p_0 - p', q)]/2 \}

where \( p_0 := Re(p), p' := p - Re(p), q'_2 \perp p', q'_1 \parallel p' \) relative to the scalar product \((z, \eta) := Re(z \eta^*)\), \( q' = q'_1 + q'_2 \), \( w(p, q) \) is given in Remark 16.

**Proof.** Let \( S_R := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq R, 0 \leq y \leq R\} \). Then 

\[
\Gamma(p) \Gamma(q) = \int_0^R \int_0^R e^{-x-y} x^p y^q dxdy = \lim_{R \to \infty} \int_0^R \int_0^R e^{-x-y} x^p y^q dxdy = \lim_{R \to \infty} \int_{S_R} e^{-x-y} x^p y^q dxdy
\]
in accordance with the Fubini theorem, since each function \( f : U \to \mathbf{H} \) has
the form \( f(z) = f_1(z) + f_2(z)i + f_3(z)j + f_4(z)k \) for each \( z \) in a domain \( U \) in \( \mathbb{H} \), \( f_1, f_2, f_3, f_4 \) are real-valued functions, \( \{1, i, j, k\} \) are generators of \( \mathbb{H} \).

Consider a triangle \( T_R := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y, x + y \leq R\} \) and put
\[
f(x, y) := e^{-x-y}x^{p-1}y^{q-1},
\]
where \( p, q \in \mathbb{A} \) are marked, then
\[
\int \int_{S_R} f(x, y) \, dx \, dy - \int \int_{S_R} f(x, y) \, dx \, dy \leq \int \int_{S_R \setminus T_R} f(x, y) \, dx \, dy.
\]
We have
\[
\lim_{R \to \infty} \int \int_{S_R} |f(x, y)| \, dx \, dy = \Gamma(p_0)\Gamma(q_0),
\]
hence
\[
\lim_{R \to \infty} \int \int_{S_R \setminus S_{R/2}} |f(x, y)| \, dx \, dy = 0.
\]
Therefore,
\[
\Gamma(p)\Gamma(q) = \lim_{R \to \infty} \int \int_{T_R} e^{-x-y}x^{p-1}y^{q-1} \, dx \, dy.
\]
The substitution \( x + y = \xi, y = \xi \eta \) and application of the Fubini theorem gives
\[
\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^1 e^{-\xi \eta p} (1 - \eta)^{p-1} \eta^{q-1} \, d\xi \, d\eta,
\]
since \( H \) is associative, \( \xi^{p-1} \) commutes with \((1 - \eta)^{p-1}, \xi^{q-1} \) commutes with \( \eta^{q-1} \).
Therefore,
\[
\Gamma(p)\Gamma(q) = \Gamma(w(p, q))B(p, q) + \int_0^\infty \int_0^1 e^{-\xi \eta p}[(1 - \eta)^{p-1}, \xi^{q-1}] \eta^{q-1} \, d\xi \, d\eta.
\]
Let \( M \) and \( N \) be in \( \mathbb{I}_v \), then \( e^Ne^M = (\cos |M|)e^N + [(\sin |N|)/|N|]Me^{N_1 - N_2} \),
where \( M \perp N_2, M \parallel N_1 \) relative to the scalar product \((z, \eta) := Re(z\eta^*)\),
\( N_1, N_2 \in \mathbb{I}_v, N = N_1 + N_2 \) (see Formulas (3.2, 3.3) [13]). Therefore,
\[
\int_0^\infty \int_0^1 e^{-\xi \eta p}[(1 - \eta)^{p-1}, \xi^{q-1}] \eta^{q-1} \, d\xi \, d\eta = \\
- \int_0^\infty \int_0^1 e^{-\xi \eta p}[(\xi^{q-1} - \xi^{q_0 - q'}) (q^* q') [[(1 - \eta)^{p-1} - (1 - \eta)^{p_0 - p'} - (1 - \eta)^{p_0 - p'} - 1] \eta^{q-1} \, d\xi \, d\eta/2.
\]
18. **Note.** Proposition 15 and Theorem 17 show differences in identities for Beta and Gamma functions between commutative case of \( C \) and noncommutative cases of \( \mathbb{A}_v, v \geq 2, \) and \( H \) particularly. Certainly, in the particular case if \( \Upsilon_{p,q} \) has embedding into \( C \), then \( q^* q' = 0 \) and Proposition 15 and Theorem 17 give classical results, but for general \( p \) and \( q \) the subalgebra \( \Upsilon_{p,q} \) can have no any embedding into \( C \).
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