Gravitational Lensing By A Back Hole in Poincaré Gauge Theory of Gravity

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Using a recently found black hole solution in the framework of the Poincaré gauge theory of gravity, we study gravitational lensing for a system where the lens is a static spherically symmetric black hole. By analyzing the equations of motion for light rays in a space-time with torsion, we derive the deflection angle as the light emitted from a source pass through near the black hole and numerically solve the resulting integral. We also study the effects of torsion on the position of images. The results show that the presence of torsion slightly alters both the deflection angle and position of images in this setup.

I. INTRODUCTION

The discovery of the light bending phenomenon as it passes through a gravitational field of a celestial object was one of the first and most important observational tests of Einstein’s theory of General Relativity (GR). This phenomenon, commonly known as gravitational lensing, has various different applications in astrophysics and cosmology. More specifically, near a black hole where the gravitational field is extremely powerful, the path followed by light rays can reveal a great deal of information about the geometry and properties of the surrounding space. On the other hand, this path as well as the type and shape of any lensing effects are strongly connected to the background geometry of space-time in which the light is traversing. The theory of General Relativity, while hugely successful at scales and energies of solar system tests, is expected to be modified at extremely high energies and in very strong gravitational fields or at the scales where quantum effects become important. For these reasons, it is useful to examine the gravitational lensing in the context of alternative theories of gravity to determine the necessary corrections to the
General Relativistic results, and these corrections are expected to be more substantial, or at least observable in a very strong gravitational field of a black hole.

Gravitational lensing also provides a convenient tool to measure various cosmological parameters. Most importantly the present day rate of expansion of the universe (i.e. the Hubble parameter) can be measured by the time delay between various images of a variable source. A theory that alters the results of gravitational lensing, may also provide a way to resolve the Hubble tension problem [1–5].

One of the first comprehensive studies of the strong gravitational lensing by a Schwarzschild black hole was performed by Virbhadra and Ellis [6]. In Ref. [7] the geometric structure of the photon surfaces has been thoroughly studied. Analysis of various characteristics of gravitational lensing by black holes and naked singularities were also performed in [8–10]. Frittelli et al. found a general exact lens equation independent of the background metric in Ref. [11]. Analytical investigation of the strong black hole lensing was performed by Bozza et al. [12] and Bozza [13, 14] in the case of a Schwarzschild background and by Eiroa et al. for the Reissner-Nordström case [15]. More recently, these effects have been analyzed in the framework of various modified gravity theories [16–24].

There exist many different proposed modifications to the theory of General Relativity at high energy scales, each with their own physical justifications and characteristics. Some of the most natural and important modifications among them, are gauge theories of gravity which apply the gauge principle i.e. localization of symmetries to the gravitational interaction. One key outcome of these theories is that the space-time geometry of General Relativity, Riemannian space-time, is transformed into a non-Riemannian geometry with curvature and torsion. In these theories, the presence of torsion, which is coupled to the spin of the matter, can alter the trajectory of light rays and influence gravitational lensing effects. Also, from a quantum gravity point of view, many proposed theories for the unification of quantum mechanics and gravity, include a torsion field one way or another [25–28]. As a specific example, in string theory the effective Lagrangian at low energies has been shown to be equivalent to a Brans-Dicke generalization of a metric theory of gravity with torsion [29, 30]. Moreover, in string theory, there exists a Kalb-Ramond field whose field strength can act like a torsion field in the background geometry [31]. The presence
of this field is also well known in noncommutative field theories \(^{32}\) where torsion is also known to be present \(^{33}\).

On the other hand, the presence of torsion in many gauge theory descriptions of gravity is also well known. Applying the well-established gauging procedure (replacing global symmetries with local ones) used in the standard model of particle physics to the gravitational interaction, one arrives at a gravitational theory with a general non-symmetric connection, which naturally incorporates the spin of the matter to the gravitational interactions \(^{34}\). The gravitational interactions here are governed by two gauge potentials, which in a Riemann-Cartan geometry can be interpreted as tetrad and spin connection fields. The associated field strengths of these gauge potentials are curvature and torsion tensors \(^{35}\). This theory is usually called the Poincaré gauge theory (PGT) of gravity. There exist some well-known special cases in PGT: General Relativity (vanishing torsion), teleparallel theory (vanishing curvature) and also Einstein-Cartan theory which can be regarded as the simplest generalization of General Relativity and has the same Lagrangian as General Relativity but with a non-symmetric connection. Since its introduction, the Einstein-Cartan theory has been extensively studied in the literature \(^{36}\). Here the torsion tensor is related by an algebraic equation to the spin density of the matter and as a result is not a dynamical quantity. This means that in Einstein-Cartan theory torsion can not propagate, \(i.e.\) there are no gravitational wave modes associated with torsion \(^{37}\). However, by choosing more complicated quadratic Lagrangian in Poincaré gauge theory of gravity, propagating torsion modes can be present and there exist torsion waves in space-time \(^{38}\).

The aim of the present paper is to study the effects of non-Riemannian geometry of the PGT on the deflection angle and position of images for a gravitational lensing system where the lens is a black hole. In particular, we examine the effects of torsion and spin on the lensing parameters in this framework. The structure of the paper is as follows: in Sect. \(^{11}\) we offer a brief introduction to the basic properties of PGT and review a new static spherically symmetric black hole solution, first derived in Ref. \(^{39}\) by analyzing the field equations in this theory. This solution describes a Reissner-Nordström type solution where torsion plays a similar role here to the electric charge in the usual Reissner-Nordström
II. GAUGE THEORIES OF GRAVITY WITH TORSION

As stated above, the geometric structure of PGT is a Riemann-Cartan space-time where curvature and torsion tensor are given in terms of the dynamical variables (tetrad and spin connection) by the following relations

\[ R_{\mu i}^j = 2 \left( \partial_\mu T^j_{\nu |i} + \Gamma^j_{[\mu |k} \Gamma^k_{|\nu i]} \right), \]
\[ T_{\mu \nu}^i = 2 \left( \partial_\mu e^i_{|\nu} + \Gamma^i_{|\mu j} e^j_{|\nu} \right), \quad T_{\mu} = T^\nu_{\mu \nu} \]

where \( e^i_\mu \) is the tetrad field and

\[ g_{\mu \nu} = \eta_{ij} e^i_\mu e^j_\nu, \]

is the space-time metric. The relation between the spin connection and the ordinary affine connection is given by the equation below

\[ \partial_\mu e^i_\nu + \Gamma^i_{j \mu} e^j_\nu - \Gamma^\lambda_{\mu \nu} e^i_\lambda = 0. \]

Throughout the paper, the Greek indices will refer to the holonomic coordinate bases of the manifold and the Latin indices refer to the local Lorentz frame of the tangent space. The most general Lagrangian of PGT is a quadratic function constructed by the suitable scalar combinations of the irreducible decompositions of curvature and torsion. Here following Ref. [39] we choose a Lagrangian in the form

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \mathcal{L}_m - R - \frac{1}{4} (d_1 + d_2 + 4c_1 + 2c_2) \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\nu\lambda\rho} - \frac{1}{4} (d_1 + d_2) \tilde{R}^2 + c_1 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho\mu\nu} + c_2 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\mu\nu} + d_1 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + d_2 \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} \right].
\]
where $c_1$, $c_2$, $d_1$ and $d_2$ are four constant parameters and $\tilde{R}_{\mu\nu\lambda\rho}$ is the curvature tensor constructed from the general non-symmetric connection; while $R$ refers to the Ricci scalar constructed by the Levi-Civita connection. Note that with the use of the identity $\tilde{R} = R - 2 \nabla_\lambda T^\rho_\lambda \rho + \frac{1}{4} T_{\lambda\mu\nu} T^{\lambda\mu\nu} + \frac{1}{2} T_{\lambda\mu\nu} T^{\mu\lambda\nu} - T_{\mu\lambda\nu} T^{\mu\nu\lambda}$, one can rewrite the general PGT Lagrangian with massless torsion in terms of the torsionless Einstein-Hilbert Lagrangian \cite{39}. By varying the above Lagrangian with respect to the dynamical variables, i.e. tetrad and spin connection, we get the general form of the field equations in PGT. The two field equations can be succinctly expressed in the following forms \cite{40}

\begin{align}
\nabla_\nu H^{\mu\nu}_i - E_i^\mu &= T_i^\mu, \quad (5) \\
\nabla_\nu H^{\mu\nu}_{ij} - E_{ij}^\mu &= S_{ij}^\mu, \quad (6)
\end{align}

with the following definitions

\begin{align}
H^{\mu\nu}_i &: = \frac{\partial e L_g}{\partial e^\nu^i e^\mu} = 2 \frac{\partial e L_g}{\partial T^{\nu\mu}_i}, \quad (7) \\
H^{\mu\nu}_{ij} &: = \frac{\partial e L_g}{\partial \Gamma^{\mu}_ij} = 2 \frac{\partial e L_g}{\partial R^{\nu\mu}_{ij}}, \quad (8)
\end{align}

and

\begin{align}
E_i^\mu &: = e^\mu_i e L_g - T_{i\nu}^j H^{\nu\mu}_j - R_{ij\nu} H^{\nu\mu}_j, \quad (9) \\
E_{ij}^\mu &: = H_{[ij]}^\mu, \quad (10)
\end{align}

where $L_g$ is the gravitational Lagrangian included in Eq. \cite{4]. The source terms at the right hand side of Eqs. (5) and (6) are energy-momentum and spin density tensors respectively and are defined by

\begin{align}
T_i^\mu &: = \frac{\partial e L_m}{\partial e^\mu_i}, \quad S_{ij}^\mu := \frac{\partial e L_m}{\partial \Gamma^{ij}_\mu}, \quad (11)
\end{align}

where $L_m$ is the matter Lagrangian and $e$ is the determinant of the tetrad.

Black hole solutions to the PGT field equations have been studied previously by various authors \cite{41-46}. For example in Ref. \cite{47}, the authors found a solution analogous to the Kerr solution of General Relativity. More recently in Ref. \cite{39}, a new static spherically symmetric vacuum solution to the Poincaré field equations \cite{7} and \cite{8} for the Lagrangian in the form of \cite{4} has been found. This solution describes the exterior geometry for a static
spherically symmetric black hole with torsion. The solution is analogous to the Reissner-Nordström solution in General Relativity, however here there is no specific electric charge. The solution can be regarded as a modification of the Schwarzschild metric of General Relativity where torsion provides extra terms in the metric. Here, we briefly review the basic properties of this modified metric. The most general line element outside of a static, spherically symmetric black hole can be written as

\[
ds^2 = -e^{\nu(r)} \, dt^2 + e^{-\nu(r)} \, dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2.
\] (12)

In Riemann-Cartan geometry the torsion must also satisfy the intrinsic symmetries of the background space-time. This means that in addition to the metric, the torsion tensor should also satisfy the Killing equation

\[
L_\xi T_{\mu \nu} = 0
\]

where \( L_\xi \) is the Lie derivative in the direction of \( \xi \). Applying this Killing equation to static spherically symmetric space-time, the non-zero components of the torsion tensor can be explicitly written as \[39, 48, 49\]

\[
T^t_{tr} = -T^t_{rt} = a(r), \quad T^r_{t\theta} = -T^r_{\theta t} = k(r) \sin \theta \, e^{\nu(r)}
\]

\[
T^r_{tr} = -T^r_{rt} = a(r) e^{\nu(r)}, \quad T^t_{t\phi} = -T^t_{\phi t} = k(r) \sin \theta
\]

\[
T^\phi_{t\theta} = -T^\phi_{\theta t} = h(r) \frac{e^{\nu(r)}}{\sin \theta}, \quad T^\theta_{r \phi} = -T^\phi_{r \theta} = h(r) \sin \theta
\]

\[
T^\phi_{\theta r} = -T^\theta_{r \phi} = h(r) \frac{e^{\nu(r)}}{\sin \theta}, \quad T^\theta_{\phi t} = -T^\phi_{t \theta} = h(r) \sin \theta e^{\nu(r)}
\]

\[
T^\theta_{\phi t} = -T^\phi_{t \theta} = T^\phi_{r \phi} = -T^\theta_{r \phi} = g(r) e^{\nu(r)}
\]

\[
T^\theta_{r \theta} = -T^\theta_{\theta r} = T^\phi_{r \phi} = -T^\phi_{\phi r} = g(r)
\] (13)

where \( a(r), \, k(r), \, h(r) \) and \( g(r) \) are four unknown functions to be determined by solving the field equations. Following Ref. \[39\] the solution for the metric function \( \nu(r) \) is

\[
e^{\nu(r)} = \left( 1 - \frac{2m}{r} + \frac{s}{r^2} \right).
\] (14)

Substituting this relation in the metric (12), we get a solution that has the same symmetries as the Schwarzschild metric of GR. The new metric has the form of the Reissner-Nordström solution in general relativity but without any electric charge as the source \[50\].
The parameters $m$ and $s$ are some constants of integration and can be related to the field strengths of curvature and torsion, respectively. The torsion functions are also given by solving the field equations \cite{39}

\[
a(r) = \frac{e^{\nu(r)}}{2e^{\nu(r)}} = \left(\frac{2m}{r^2} - \frac{s}{r^3}\right), \quad g(r) = -\frac{1}{2r}, \quad h(r) = -\frac{\sqrt{s}}{r e^{\nu(r)}}, \quad k(r) = 0.
\]

(15)

Let us briefly analyze the properties of the black hole solution given by Eqs. \cite{12} and \cite{14}. Here, the torsion function $s$ plays a role similar to the electromagnetic charge in usual General relativistic Reissner-Nordström geometry. The position of black hole horizons are given by

\[
R_\pm = m \pm \sqrt{m^2 - s},
\]

(16)

provided that the following condition is satisfied

\[
m^2 \geq s.
\]

(17)

The outer horizon $R_+$ can be regarded as the Schwarzschild radius of the black hole.

III. GRAVITATIONAL LENSING BY A BLACK HOLE WITH TORSION IN PGT

Gravitational lensing by a black hole with torsion has been recently studied in Ref. \cite{51} in the framework of an extension to the Einstein-Cartan-Sciama-Kibble (ECSK) theory presented in Ref. \cite{52}. In that paper, the authors obtained static vacuum solutions by including fourth-order scalar invariants constructed from curvature and torsion in the ECSK Lagrangian and found both black hole solutions and naked singularities in that setup. However, the presence of the fourth-order term in the Lagrangian will add various complications to the gauge structure of the theory. Here we choose the Poincaré gauge theory of gravity where the Lagrangian is of the quadratic type given by (4) and study the gravitational lensing by a black hole in this setup. Our aim is to determine the deflection angle of light rays near a black hole, where the geometry of the exterior space-time is described by metric (12) and (14) and torsion in the form of (13). It should be noted that in general Riemann-Cartan space-time, where the connection is not necessarily symmetric,
auto-parallel curves and metric geodesics do not coincide with each other. We begin with the equation for null geodesics in this setup. Various components of the metric geodesics, constructed by using the metric (12) and (14) are given by (we work in the equatorial plane $\theta = \pi/2$ for simplicity, without any loss of generalization)

$$\frac{d^2 t}{dp^2} + 2\left(1 - \frac{2m}{r} + \frac{s}{r^2}\right)^{-1}\left(\frac{m}{r^2} - \frac{s}{r^3}\right) \frac{dr}{dp} \frac{dt}{dp} = 0,$$

(18)

$$\frac{d^2 r}{dp^2} - \left(1 - \frac{2m}{r} + \frac{s}{r^2}\right)^{-1}\left(\frac{m}{r^2} - \frac{s}{r^3}\right) \frac{dr}{dp} \left(\frac{dt}{dp}\right)^2 + \frac{m}{r^2} - \frac{s}{r^3} \frac{dt}{dp} - r \left(1 - \frac{2m}{r} + \frac{s}{r^2}\right) \left(\frac{d\phi}{dp}\right)^2 = 0,$$

(19)

$$\frac{d^2 \phi}{dp^2} + \frac{2}{r} \frac{dr}{dp} \frac{d\phi}{dp} = 0.$$

(20)

Following the standard procedure outlined in Sect. 8.5 of Ref. [53], the deflection angle for light rays near the black hole is given by the following integral

$$\Phi = \int_{r_m}^{\infty} dr \frac{1}{r \sqrt{1 - \frac{2m}{r} + \frac{s}{r^2} \left(\frac{1 - \frac{2m}{r_m} + \frac{s}{r_m^2}}{r_m^2 \left(1 - \frac{2m}{r_m} + \frac{s}{r_m^2}\right)} - 1\right)},$$

(21)

where $r_m$ is the distance of the closest approach. Note that although these equations are constructed by the Levi-Civita connection, the resulting deflection angle is still different from that of General Relativity as the components of the metric are now depend on the torsion parameter $s$.

On the other hand, various components of the auto-parallel equation constructed by the full connection given in the A are

$$\frac{d^2 t}{dp^2} + \left(\nu'(r) + a(r)\right) \frac{dt}{dp} \frac{dr}{dp} - a(r)e^{-\nu(r)} \left(\frac{dr}{dp}\right)^2 + r^2 g(r) \left(\frac{d\theta}{dp}\right)^2 + r^2 \sin^2 \theta g(r) \left(\frac{d\phi}{dp}\right)^2 = 0,$$

(22)
FIG. 1: Deflection angle defined as the integral in Eq. (21) versus the minimum distance of the photons from the black hole $r_m$ for different values of the torsion parameter $s$. In this figure the source is assumed to be at infinity and $m = 0.5$. The horizontal axis is in terms of the Schwarzschild radius.

\[
\begin{align*}
\frac{d^2r}{dp^2} & - \frac{1}{2} \nu'(r) \left( \frac{dr}{dp} \right)^2 - \left( a(r)e^{\nu(r)} \right) \frac{dt}{dp} \frac{dr}{dp} + \left( \frac{1}{2} \nu'(r) + a(r) \right) e^{2\nu(r)} \left( \frac{dt}{dp} \right)^2 \\
& + r e^{\nu(r)} (rg(r) - 1) \left( \frac{d\theta}{dp} \right)^2 + r e^{\nu(r)} \sin^2 \theta (rg(r) - 1) \left( \frac{d\phi}{dp} \right)^2 = 0, \\
\frac{d^2\theta}{dp^2} & - \sin \theta e^{\nu(r)} \left( \frac{k(r)}{r^2} + h(r) \right) \frac{dt}{dp} \frac{d\phi}{dp} + \left( \frac{2}{r} - g(r) \right) \frac{dr}{dp} \frac{d\theta}{dp} \\
& + \sin \theta \left( \frac{k(r)}{r^2} + h(r) \right) \frac{d\phi}{dp} \frac{dr}{dp} - \sin \theta \cos \theta \left( \frac{d\phi}{dp} \right)^2 + \left( g(r)e^{\nu(r)} \right) \frac{dt}{dp} \frac{d\theta}{dp} = 0, \\
\frac{d^2\phi}{dp^2} & + \left( \frac{2}{r} - g(r) \right) \frac{dr}{dp} \frac{d\phi}{dp} + \left( g(r)e^{\nu(r)} \right) \frac{dt}{dp} \frac{d\phi}{dp} \\
& + \frac{e^{\nu(r)}}{\sin \theta} \left( \frac{k(r)}{r^2} + h(r) \right) \frac{dt}{dp} \frac{d\theta}{dp} + 2 \left( \cot \theta \right) \frac{d\phi}{dp} \frac{d\theta}{dp} - \frac{1}{\sin \theta} \left( k(r) + h(r) \right) \frac{dr}{dp} \frac{d\theta}{dp} = 0.
\end{align*}
\]
Again, we work in the equatorial plane $\theta = \pi/2$. In this case, Eq. (24) gives

$$e^{\nu(r)} \frac{dt}{dp} = \frac{dr}{dp}.$$  \hspace{1cm} (26)

Using the above, Eqs. (22), (23) and (25) can be written as

$$\frac{d^2 t}{dp^2} + \left( \nu'(r)e^{\nu(r)} \right) \left( \frac{dt}{dp} \right)^2 + \left( r^2 g(r) \right) \left( \frac{d\phi}{dp} \right)^2 = 0 ,$$  \hspace{1cm} (27)

$$\frac{d^2 r}{dp^2} + \left( r e^{\nu(r)} \left( rg(r) - 1 \right) \right) \left( \frac{d\phi}{dp} \right)^2 = 0 ,$$  \hspace{1cm} (28)

$$\frac{d^2 \phi}{dp^2} + \frac{2}{r} \frac{d\phi}{dp} \frac{d\phi}{dp} = 0 .$$  \hspace{1cm} (29)

Using torsion functions (15), the Eqs. (27) and (28) become

$$\frac{d^2 t}{dp^2} + 2 \left( \frac{m}{r^2} - \frac{s}{r^3} \right) \left( \frac{dt}{dp} \right)^2 - \frac{r}{2} \left( \frac{d\phi}{dp} \right)^2 = 0 ,$$  \hspace{1cm} (30)

$$\frac{d^2 r}{dp^2} - \frac{3}{2} r \left( 1 - \frac{2m}{r} + \frac{s}{r^2} \right) \left( \frac{d\phi}{dp} \right)^2 = 0 .$$  \hspace{1cm} (31)

Eq. (29) can be rewritten as

$$\frac{d}{dp} \left\{ \ln \left( \frac{d\phi}{dp} \right) + \ln r^2 \right\} = 0 ,$$  \hspace{1cm} (32)

which immediately gives

$$r^2 \frac{d\phi}{dp} = J ,$$  \hspace{1cm} (33)

where $J$ is a constant of motion interpreted as the angular momentum. We also have

$$ds^2 = E \, dp^2 .$$  \hspace{1cm} (34)

Using Eqs. (26), (30) and (33) to find $d\phi/dp$, $(dt/dp)^2$ and $d^2 t/dp^2$ respectively; noting that $E = 0$ for photons, Eq. (31) takes the following form after some simplification

$$\frac{d^2 r}{dp^2} + 2 \left( \frac{J}{r} \right)^2 \left[ \left( \frac{m}{r^2} - \frac{s}{r^3} \right) - \frac{1}{4r} \left( 1 - \frac{2m}{r} + \frac{s}{r^2} \right) \right] = 0 .$$  \hspace{1cm} (35)
We are interested in the trajectory of the photons in the $r - \phi$ plane. To find $r(\phi)$, we can eliminate $dp$ from Eq. (35) by noting that

$$\frac{d^2 r}{dp^2} = \frac{d}{dp} \left( \frac{dr}{d\phi} \right) = \frac{d\phi}{dp} \frac{d}{d\phi} \left( \frac{d\phi}{dp} \frac{dr}{d\phi} \right) = \frac{J}{r^2} \left[ \frac{d^2 r}{d\phi^2} - \frac{2}{r} \left( \frac{dr}{d\phi} \right)^2 \right]. \quad (36)$$

Using this, Eq. (35) takes the following form

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r} \left( \frac{dr}{d\phi} \right)^2 + 2r^2 \left( \frac{m}{r^2} - \frac{s}{r^3} \right) - \frac{r}{2} \left( 1 - \frac{2m}{r} + \frac{s}{r^2} \right) = 0. \quad (37)$$

This equation immediately gives the following integral for the deflection angle

$$\Phi = \pm \int_{r_m}^{r_S} \frac{2 \, dr}{\sqrt{4r^4C_1 - 2r^2 + 8mr - 5s}} - C_2, \quad (38)$$

where $C_1$ and $C_2$ are constants of integration and $r_m$ and $r_S$ are the closest approach distance and position of the source, respectively. The integral in Eq. (38) is of the elliptic type which has no analytic solutions and should be solved numerically. The numerical analysis was performed using the Monte Carlo method to find the deflection angle when the geometry outside of the black hole lens is given by the metric of the Eqs. (12) and (14) where also a torsion function in the form of Eq. (13) is present. We also employed the Nintegrate package in the Mathematica software system to cross-check our results.

A. Deflection angle

The deflection angle when light rays follow the metric geodesics is given by Eq. (21). Figure [1] shows the deflection angle in terms of the minimum distance $r_m$ for different values of the torsion parameter $s$. The value of the mass parameter $m$ is chosen to be $m = 0.5$ in this figure and the minimum distance $r_m$ in the horizontal axis is in terms of the Schwarzschild radius of the black hole. The plot is obtained by solving the integral of Eq. (21) numerically using the Monte Carlo method and shows a slight change in the deflection angle when the torsion parameter $s$ is increased. This in turn leads to a slight change to the position of the observed images compared to what is expected from GR.
It is obvious from the figure that the value of $\Phi$ in Eq. (21) decreases when the torsion parameter $s$ increases.

In the case of the auto-parallel curves, first let us assume that the light ray comes from infinity ($r_S = \infty$) and reaches a minimum distance $r_m$ from the lens, before reaching the observer. Figure 2 shows the deflection angle in terms of arcseconds with respect to $r_m$ for different values of the torsion parameter $s$. The values of other constants are set to $C_1 = 1, m = 0.5$. As expected, the deflection angle decreases with increasing minimum distance $r_m$. Also, the effects of torsion only become noticeable for small values of $r_m$ and generally the deviation from GR increases with increasing the absolute value of the torsion parameter $s$.

For a more realistic system, we next assume that $r_S$ has a finite value. In Figure 3, we plot the deflection angle versus the minimum distance $r_m$ for $r_S = 100$ and different values of torsion parameter $s$. The values of other constants are as before. The figure again shows the effects of torsion on the deflection angle for light rays lensed by a black hole in non-Riemannian space-times. For comparison, the case $s = 0$ (General relativistic result) is also included in all of the figures. Generally, in the case of the auto-parallel curves, positive values of the torsion parameter $s$ tend to increase the deflection angle compared to the General relativistic case ($s = 0$) while negative values of $s$ will decrease the deflection angle. Figure 4 shows the difference between general relativistic deflection angle and PGT black hole described by metric (14) for the case $s = 0.1$. The left and right figures are for metric geodesics deflection angle given by (21) and auto-parallel case given by (38) respectively. The difference, generally increases when the minimum distance $r_m$ approaches the black hole horizon. The expected correction from the effects of torsion is in the order of microarcseconds. Similar estimations for the correction of deflection angle were obtained in Refs. 54 55 in the weak field limit of fourth order $f(R)$ gravity.
FIG. 2: Deflection angle defined as the integral in Eq. (38) versus the minimum distance of the photons from the black hole $r_m$ for different values of the torsion parameter $s$ in Eq. (14). In this figure the source is assumed to be at infinity i.e. very far from the lens. The values of other constants are $m = 0.5$ and $C_1 = 1$. The horizontal axis is in terms of the Schwarzschild radius.

B. Position of the images

The position of images can be obtained by utilizing the black hole lens equation which can be written as

$$\Phi(u, r_O, r_S) = \Delta \phi \equiv \phi_O - \phi_S + 2n\pi .$$

Here $n$ is an integer number, $\Phi$ is given by Eq. (21) for the metric geodesics and Eq. (38) for the auto-parallel curves; $\Delta \phi$ is the relative azimuthal position between the source and the observer and $\phi_O$ and $\phi_S$ are the azimuthal position of the observer and the source, respectively. Generally, the position of the images can be obtained by finding the impact parameter $u$ from the above lens equation and use it to find the angle in which the observer detects the photon; however, there is a situation where this equation can be simplified greatly. When the source and the observer are both very far from the lens, we
FIG. 3: Deflection angle defined as the integral in Eq. (38) versus the minimum distance of the photons from the black hole $r_m$ for different values of the torsion parameter $s$ in Eq. (14). In this figure the source is assumed to be at $r_S = 100$. The values of other constants are $m = 0.5$ and $C_1 = 1$. The horizontal axis is in terms of the Schwarzschild radius.

can use the following approximation

$$\sin \theta \simeq \frac{u}{r_O}, \quad \sin \theta_S = \frac{u}{r_S}$$

(40)

to eliminate $r_O$ and $r_S$ from Eq. (39) and get the Ohanian lens equation

$$\alpha - \theta - \theta_S = -\gamma \equiv \Delta \phi - \pi.$$  \hspace{1cm} (41)

Here $\alpha$ is the deflection angle between asymptotic incoming and outgoing trajectories of the photon i.e. $\alpha = \Phi(u, \infty, \infty) - \pi$ and $\theta$ is the detection angle of the photon by the observer. $\theta_S$ can be related to $\theta$ by the relation

$$\theta_S = \sin^{-1} \left( \frac{d_{OL}}{d_{LS}} \sin \theta \right),$$ \hspace{1cm} (42)

where $d_{OL}$ and $d_{LS}$ are distances between observer and the lens, and lens and the source respectively.
Figure 4: The difference between general relativistic deflection angle and the black hole with torsion for metric geodesics (left) and auto-parallel curves (right) for the case of the torsion parameter $s = 0.1$. The difference is in the order of microarcseconds and becomes greater when the minimum distance $r_m$ gets closer to the black hole horizon. The vertical axes in the figures are in terms of milliarcseconds (mas).

Figure 5 shows the position of images for different values of the source position. Here we assumed the source and the observer to be very far from the lens, so the Ohanian lens equation would be a reasonable approximation. The vertical axis is $\log \epsilon$ where $\epsilon$ is defined as

$$
\epsilon \equiv \left( \frac{\theta}{\bar{\theta}} - 1 \right),
$$

(43)

Where $\bar{\theta}$ is the position of the photon sphere (the so-called shadow of the black hole). The solid line in Figure 5 shows the position of images for the value of $s = 0.1$ for the torsion parameter in Eq. (14). For comparison again we plot the position of images for the torsion-less case $s = 0$. The figure shows that the effects of torsion slightly alter the position of images compared to the General relativistic case, specially when the source and the lens are slightly misaligned. Note that $\Delta \phi = 0$ and $\Delta \phi = \pi$ indicate the source in front of the black hole and directly behind the black hole, respectively. For positive values of $\Delta \phi$ we have primary images where the image is on the same side of the source while for negative values of $\Delta \phi$ (secondary images), the image is located at the opposite side of the source. Note that for both of these images we have $n = 0$ in Eq. (39). Higher order
FIG. 5: Logarithmic behaviour of the position of images versus the position of the source $\Delta \phi$ for torsion parameter $s = 0.1$ (solid line) and $s = 0$ (dashed line). Vertical axis shows $\log \epsilon$ with $\epsilon \equiv (\theta - 1)$ where $\theta$ is the position of the image and $\bar{\theta}$ is the position of the shadow of the black hole. Both $\theta$ and $\bar{\theta}$ are in microarcseconds ($\mu$as) while $\Delta \phi$ is in milliarcseconds ($\text{mas}$). The source and the observer assumed to be very far from the lens.

images can be obtained by choosing different values of $n$ in the lens equation. Generally, as can be seen from the figure, by increasing the value of $\Delta \phi$, primary images move to the position of the source, while the secondary image moves closer to the shadow of the black hole. In case of the perfect alignment between source, lens and the observers, the primary and secondary images merge to produce an Einstein’s ring.

IV. CONCLUSION

In this paper, we study gravitational lensing in the framework of a new static spherically symmetric black hole solution recently found in the framework of the Poincaré gauge theory of gravity. In this solution, the effects of torsion appear as a single parameter in the line
element. The resulting metric is of the Reissner-Nordström type, but with a dynamical torsion instead of an electric or magnetic source.

In the Poincaré gauge theory of gravity, torsion couples to the spin of the matter and can alter the path of particles moving through the gravitational field. This effect on the orbit of particles can be more pronounced in the extremely strong gravitational field of a black hole, or when the spin density of the matter (usually defined as the expectation value of the square of the spin density tensor) is very high. The effect of dynamical torsion on the equation of motion of particles is two-fold. For particles with non-vanishing spin, the coupling of torsion to the spin density tensor results in the equation of motion different from the geodesic equations of GR. However, for spinless particles, the equations of motion reduce to that of GR. On the other hand, even for particles with vanishing spin, torsion can alter the solution to the field equations, i.e. the tetrad field and as a result, the path of particles and light rays will be different from their corresponding trajectories in General Relativity. It has been shown previously that in the absence of an explicit coupling between the electromagnetic field and torsion in the Lagrangian, consistency with the Maxwell’s equations compels photons to follow the metric geodesics of the spacetime [57–59]. However, even in this case the deflection angle and the position of images will not be the same as in GR, as the metric around the black hole will be modified.

The effects of the torsion parameter in Eq. (14) on the gravitational lensing near a black hole are summarized in Figs. [1-4]. Here, both deflection angle and position of images slightly differ from the General relativistic case, specially when the minimum distance of the incoming photon from the black hole is small i.e. near the photon sphere. This may provide a way to detect dynamical torsion and test various modified gravity theories by observing the gravitational lensing by an extremely strong gravitational field of black holes or other extremely dense objects. This is also may change the estimation of the Hubble parameter from such a lensing system and may help in the resolution of the so-called Hubble tension problem.

Finally, we briefly discuss the possibility of observing the effects of torsion on the black hole lensing parameters. As the magnitude of corrections is directly related to the ratio of the black hole mass and its distance to the observer, the best candidates for observing
the effects and constraining the torsion parameter seems to be the supermassive black holes at the center of the milky way and nearby galaxies (specially M87). Gravitational lensing of the stars rotating the black hole at the center of the milky way with eccentric orbits is quite common \[60\]. One of the most interesting lensing candidates is the star S6, because the eccentricity of its orbit brings it quite close to the black hole horizon \[60\]. In this range any possible effects of torsion are expected to be the largest. Note that separation of images for this and similar stars are at the limits of current observational capabilities; nonetheless, these systems provide a convenient way to test various modified gravity theories via gravitational lensing.
Appendix A: Components of Affine Connection

Here, we present the explicit components of the affine connection satisfying the sym-
metries of a static spherically symmetric space-time with torsion

\[
\begin{align*}
\Gamma^0_{01} &= -\Gamma^1_{10} = \frac{1}{2} \nu'(r), \\
\Gamma^0_{11} &= -a(r)e^{-\nu(r)}, \\
\Gamma^0_{32} &= -\Gamma^0_{23} = \frac{1}{2} k(r) \sin \theta, \\
\Gamma^0_{00} &= \left( \frac{1}{2} \nu'(r) + a(r) \right) e^{2\nu(r)}, \\
\Gamma^1_{00} &= \left( \frac{1}{2} \nu'(r) + a(r) \right) e^{2\nu(r)}, \\
\Gamma^1_{32} &= -\Gamma^3_{12} = \frac{1}{2} k(r) \sin \theta e^{\nu(r)}, \\
\Gamma^2_{02} &= g(r) e^{\nu(r)}, \\
\Gamma^2_{21} &= \Gamma^3_{31} = \frac{1}{r}, \\
\Gamma^2_{30} &= -\frac{1}{2r^2} (2r^2 h(r) + k(r)) \sin \theta e^{\nu(r)}, \\
\Gamma^3_{30} &= -\sin \theta \cos \theta, \\
\Gamma^3_{02} &= \frac{1}{2} \frac{k(r) e^{\nu(r)}}{r^2 \sin \theta}, \\
\Gamma^3_{20} &= \frac{1}{2} \frac{(2r^2 h(r) + k(r)) e^{\nu(r)}}{r^2 \sin \theta}, \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot \theta,
\end{align*}
\]

Where functions \( a(r) \), \( k(r) \), \( h(r) \) and \( g(r) \) are given by Eq. [15] and we have

\[
b(r) = \frac{e^{\nu(r)}}{2}, \quad c(r) = \frac{e^{\nu(r)}}{2r}, \quad d(r) = \frac{\sqrt{3}}{r}, \quad l(r) = 0. \tag{A1}
\]

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