Mean Temperature Profiles in Turbulent Thermal Convection

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To predict the mean temperature profiles in turbulent thermal convection, the thermal boundary layer (BL) equation including the effects of fluctuations has to be solved. In [Shishkina et al., Phys. Rev. Lett. 114 (2015)], the thermal BL equation with the fluctuations taken into account as an eddy thermal diffusivity has been solved for large Prandtl-number fluids for which the eddy thermal diffusivity and the velocity field can be approximated respectively as a cubic and a linear function of the distance from the plate. In the present work we make use of the idea of Prandtl’s mixing length model and relate the eddy thermal diffusivity to the stream function. With this proposed relation, we can solve the thermal BL equation and obtain a closed-form expression for the dimensionless mean temperature profile in terms of two independent parameters for fluids with a general Prandtl number. With a proper choice of the parameters, our predictions of the temperature profiles are in excellent agreement with the results of our direct numerical simulations for a wide range of Prandtl numbers from 0.01 to 2547.9 and Rayleigh numbers from $10^7$ to $10^9$.

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I. INTRODUCTION

Turbulent thermal convection is a major topic in geophysical and astrophysical fluid dynamics and an important problem in engineering and technological applications. The classical systems to study turbulent thermal convection are Rayleigh–Bénard convection (RBC) and horizontal convection (HC), where a fluid is confined between a heated bottom plate and a cooled top plate and a fluid is heated at one end of the bottom plate and cooled at the other end of the bottom plate.

One important and well-studied question in turbulent thermal convection research is how the mean convective heat and momentum transport, represented by the Nusselt number (Nu) and Reynolds number (Re) respectively, depend on the main input parameters of the system, which are the Rayleigh number $Ra \equiv \alpha g \Delta H^3 / (\kappa \nu)$ and the Prandtl number $Pr \equiv \nu / \kappa$. Here $\nu$ denotes the kinematic viscosity, $\kappa$ the thermal diffusivity, $\alpha$ the isobaric thermal expansion coefficient of the fluid, $g$ the acceleration due to gravity, $H$ the distance between the heated plate (part) and the cooled plate (part) for RBC (HC), and $\Delta \equiv T_h - T_c > 0$ with $T_h$ and $T_c$, respectively the temperatures of the heated plate (part) and the cooled plate (part) for RBC (HC). The dependence of Nu and Re on Ra and Pr is influenced significantly by the imposed boundary conditions [11–13]. Grossmann and Lohse developed a scaling theory (GL) for RBC, which nowadays allows one to predict Nu and Re if the pre-factors fitted with the latest experimental and numerical data are used. The GL theory was later extended to the case of HC and magnetoconvection [20].

Closely related to the scaling problem of the heat and momentum transport in different convective systems is the problem to predict the spatial profiles of the mean flow characteristics. Among which, the time- and horizontally area-averaged profile of the temperature as a function of the vertical distance $z$ from the heated bottom plate is of particular research interest. In the Oberbeck–Boussinesq approximation of RBC, the mean temperature depends only weakly on $z$ in the core part of the domain. There exists a certain region in the bulk in which the mean temperature behaves as a logarithmic function of $z$ [21], and this logarithmic region is expected to almost fill the entire bulk for very large $Ra$ [22]. Near to the bottom and top plates, the mean temperature changes much more rapidly with $z$ than in the bulk. The knowledge of these boundary layer (BL) profiles of the mean temperature near the bottom and top plates is important for many engineering applications as well as for the development of reliable turbulence models for thermal...
convection. It remains one of the most challenging unsolved problems to predict the mean temperature boundary layer profiles.

In its derivation of the heat and momentum transport scalings in BL-dominated regimes in RBC, the GL theory [4, 9] assumes that the viscous BL thickness is proportional to Re\(^{-1/2}\). This scaling relation holds, in particular, in the classical Prandtl-Blasius (PB) boundary-layer theory [23, 24] for steady flows. The mean temperature profiles obtained in experimental and numerical studies have been compared against the profiles obtained from the PB theory, and systematic deviations were reported [25–30]. The deviations are generally larger for larger Ra and smaller Pr, and they remain even after an application of a dynamical rescaling procedure [31] that takes into account the time variation of the BL thickness.

In the PB theory, the pressure gradient vanishes and fluctuations do not exist. The effect of a non-zero pressure gradient within the BLs, or equivalently the effect of a large-scale mean circulating flow that is not parallel to the isothermal plate, was studied in Shishkina et al. [32] and led to the BL equations, which are similar to those of Falkner and Skan (FS) [33]. With the FS approach one calculates the ratio of the thermal to viscous BL thicknesses more accurately compared to PB but the FS approximation does not lead to a significantly better prediction of the mean temperature profiles, as the limits of the PB and FS profiles for infinitesimal Pr are the same [34]. For large Pr and a flow with a constant shear rate, Shraiman and Siggia [35] derived the mean temperature profile and the relation between the heat flux and shear rate in thermal convection. Their mean temperature profile also coincides with the PB prediction for infinitely large Pr [23]. Ching [36] generalized their approach to the case of a position-dependent shear rate and derived the temperature profile as a function of two parameters which are associated with the local thermal BL thickness and the shear rate. Good agreement of these derived profiles with the actual ones can be obtained only when the two parameters are taken as free fitting parameters.

In Shishkina et al. [37] we derived a new thermal BL equation for turbulent RBC in large-Pr fluids. The equation takes into account the effect of fluctuations, which are neglected in the PB or FS BL equations, using an eddy thermal diffusivity. In the case of large Pr, the thermal BL is nested within the viscous BL, thus the eddy thermal diffusivity and the horizontal mean velocity can be approximated respectively as a cubic and a linear function of the distance from the plate. For the limits Pr \(\gtrsim 1\) and Pr \(\to \infty\) of such simplification of the BL equations for large Pr, the mean temperature profiles were analytically obtained and shown to be in very good agreement with the profiles obtained in Direct Numerical Simulations (DNS) of RBC for, respectively, Pr = 4.38 (water) and Pr = 2547.9 (glycerol).

In the present paper we shall derive a thermal BL equation for fluids with a general Pr, including very small Pr. We extend the approximation of the eddy thermal diffusivity to larger z and propose an approximate relation between the eddy thermal diffusivity and the stream function within the thermal BL. Then we can solve the resulting thermal BL equation to obtain the mean temperature profiles in terms of two independent parameters. With a proper choice of the parameters, our theoretical predictions are in perfect agreement with the mean temperature profiles obtained in the DNS for Pr down to 0.01. Our present approach can be reduced to that of [37] in the case of large Pr.

II. BASIC EQUATION

Following [37], we consider the quasi two-dimensional fluid flow along a semi-infinite horizontal heated plate and assume that far away from the plate, there exists a constant mean velocity, the wind, along a horizontal x-direction \(x\). The equation for the temperature field \(T(x, z, t)\) is

\[
\partial_t T + u \cdot \nabla T = \kappa \nabla^2 T, \tag{1}
\]

where \(u(x, z, t) \equiv u(x, z, t) \hat{x} + v(x, z, t) \hat{z}\) is the velocity field, and the flow is incompressible:

\[
\nabla \cdot u = 0. \tag{2}
\]

Using Reynolds decomposition of the flow fields into sums of time-averages and fluctuations,

\[
u = U + u', \quad v = V + v', \quad T = \Theta + \theta', \tag{3}
\]

in (1) and averaging it in time afterwards, we obtain the following equation for the time-averaged temperature:

\[
U \partial_x \Theta + V \partial_z \Theta + \partial_x \langle u' \theta' \rangle_t + \partial_z \langle v' \theta' \rangle_t = \kappa \partial^2_x \Theta + \kappa \partial^2_z \Theta, \tag{4}
\]

where \(\langle \cdot \rangle_t\) denotes the time-averaging. The continuity equation (2) holds for both the mean and fluctuating velocities:

\[
\partial_x U + \partial_z V = 0, \tag{5}
\]

\[
\partial_x u' + \partial_z v' = 0. \tag{6}
\]

As usual for BLs, we assume that within the BL \(\partial^2_x \Theta \ll |\partial^2_z \Theta|\) and \(\partial_x \langle u' \theta' \rangle_t \ll |\partial_x \langle v' \theta' \rangle_t|\) and obtain

\[
U \partial_x \Theta + V \partial_z \Theta + \partial_x \langle v' \theta' \rangle_t = \kappa \partial^2_x \Theta. \tag{7}
\]

Introducing the eddy thermal diffusivity \(\kappa_t(x, z)\) for the fluctuation term in (7), which is defined by

\[
\langle v' \theta' \rangle_t \equiv -\kappa_t \partial_z \Theta, \tag{8}
\]

we obtain

\[
U \partial_x \Theta + (V - \partial_z \kappa_t) \partial_z \Theta = (\kappa + \kappa_t) \partial^2_x \Theta. \tag{9}
\]
In the BL in turbulent thermal convection the eddy thermal diffusivity is not negligible [37]. To satisfy [37] we introduce the stream function \( \Psi \), such that

\[
U = \partial_x \Psi, \quad V = -\partial_x \Psi.
\]

(10)

We define the similarity variable \( \xi \) and the dimensionless stream function \( \psi(\xi) \) and temperature \( \theta(\xi) \):

\[
\xi \equiv z/\lambda(x), \\
\Psi \equiv U_0 \lambda(x) \psi(\xi), \\
\Theta \equiv T_h - (\Delta/2)\theta(\xi).
\]

(11-13)

and look for a similarity solution of (9) in terms of \( \xi \). Here \( \lambda(x) \) is the local thickness of the thermal BL, \( U_0 \) is the maximal horizontal velocity (wind velocity), \( T_h \) is the temperature of the heated bottom plate, and \( \Delta/2 \) is the temperature difference between the bottom plate and the bulk of the flow. Substituting (11-13) into (9), we obtain the following dimensionless thermal BL equation:

\[
(1 + \kappa_t/\kappa)\theta_{\xi \xi} + [(\kappa_t/\kappa)\xi + B\psi]\theta_\xi = 0
\]

(14)

with

\[
B = U_0 / \lambda(x) / \kappa.
\]

(15)

The subscripts \( \xi \) and \( x \) denote the ordinary derivative with respect to \( \xi \) and \( x \). For a similarity solution to exist, \( \kappa_t/\kappa \) should depend on \( \xi \) only and \( B \) must be a constant, independent of \( x \), therefore \( \lambda(x) \propto \sqrt{x} \).

With

\[
\lambda(x) \propto \sqrt{x}/U_0,
\]

(16)

from (15) we obtain that \( \lambda \propto \text{Pr} \). It follows from (16) that the viscous BL thickness scales as \( \text{Re}^{-1/2} \), where \( \text{Re} \equiv U_0 x / \nu \), if the ratio of thicknesses of the viscous and thermal BLs depends only on \( \text{Pr} \).

To solve the thermal BL equation (14) with the boundary conditions

\[
\theta(0) = 0, \quad \theta_\xi(0) = 1, \quad \theta(\infty) = 1,
\]

(17)

and obtain the dimensionless temperature profiles \( \theta(\xi) \), we need to know \( \kappa_t(\xi)/\kappa \) and \( \psi(\xi) \). In the next two sections we will establish an approximation of \( \kappa_t(\xi) \) and propose an approximate relation between \( \kappa_t(\xi) \) and \( \psi(\xi) \).

III. EDDY THERMAL DIFFUSIVITY

Very close to the plate, the eddy thermal diffusivity can be approximated as a cubic function of \( \xi \) [37]. In this regard, the eddy thermal diffusivity and the eddy viscosity exhibit similar behavior near the plate [38]. From the continuity equation (9) of the fluctuating velocity it follows that \( \partial_z v' = 0 \) at the plate (\( z = 0 \)). From this result and the fact that all fluctuations \( u' \), \( v' \) and \( \theta' \) vanish at \( z = 0 \), we obtain consequentially

\[
\langle v' \theta' \rangle|_{z=0} = 0, \quad \partial_z \langle v' \theta' \rangle|_{z=0} = 0, \quad \partial^2_z \langle v' \theta' \rangle|_{z=0} = 0.
\]

(18)

Using the definition of \( \kappa_t(\xi) \) and the linear dependency of \( \xi \) on \( z \) [see (11)], these results imply

\[
\kappa_t|_{\xi=0} = (\kappa_t)_{\xi}|_{\xi=0} = (\kappa_t)_{\xi\xi}|_{\xi=0} = 0
\]

(19)

and, hence, close to the plate, \( \kappa_t/\kappa \) can be approximated as a cubic function of \( \xi \),

\[
\kappa_t/\kappa \approx a^3 \xi^3,
\]

(20)

with a certain constant \( a \), which measures the size of fluctuations.

Relatively far away from the plate, the mean temperature \( \Theta \) behaves as a logarithmic function of the distance \( z \) from the plate [22, 39, 40]. In this logarithmic or inner region, the fluctuations are so strong that the term \( \partial_z \langle v' \theta' \rangle \) dominates the other terms on the left-hand side of (17),

\[
\partial_z \langle v' \theta' \rangle \approx \kappa \partial^2_z \Theta,
\]

(21)

which implies

\[
\langle v' \theta' \rangle|_{z=0} \approx \kappa [\partial_z \Theta - \partial_z \Theta|_{z=0}] \approx -\kappa \partial_z \Theta|_{z=0}
\]

(22)

as in the inner region the mean temperature \( \Theta \) changes very slowly with \( z \) so that \( |\partial_z \Theta|_{z=0}| \ll |\partial_z \Theta|_{z=0}| \). Using (22) for small \( \xi \) and (23) for large \( \xi \), we have

\[
\kappa_t/\kappa \approx \partial_z \Theta|_{z=0}
\]

(23)

The logarithmic dependence on \( z \) of \( \Theta \) thus implies that \( \kappa_t/\kappa \) behaves as a linear function of \( z \) or \( \xi \) in this region:

\[
\kappa_t/\kappa \sim \xi.
\]

(24)

These two different behaviors of \( \kappa_t/\kappa \) on \( \xi \), (20) for small \( \xi \) and (24) for large \( \xi \), have both been demonstrated in [37].

Based on these two behaviors, we make the following approximation of \( (\kappa_t/\kappa)\xi \):

\[
(\kappa_t/\kappa)\xi \approx \frac{3a^3 \xi^2}{1 + b^2 \xi^2},
\]

(25)

where \( b \) is a constant that determines the location \( \xi_{\text{max}} \) of the maximum value of \( (\kappa_t/\kappa)\xi \), namely

\[
\xi_{\text{max}} = (\sqrt{3}b)^{-1}.
\]

(26)

From (25) we obtain

\[
\frac{\kappa_t}{\kappa} \approx \frac{3a^3}{b^2} [b \xi - \arctan(b \xi)].
\]

(27)

which gives the two limiting behaviors discussed above, namely

\[
\frac{\kappa_t}{\kappa} \approx \frac{3a^3 \xi^3}{3} + \frac{b^2 \xi^5}{5} + \frac{b^4 \xi^7}{7} + \mathcal{O}(\xi^9) \approx a^3 \xi^3
\]

(28)

for \( \xi \rightarrow 0 \), and

\[
\frac{\kappa_t}{\kappa} \approx \frac{3a^3 \xi}{b^2} \approx \frac{1}{b^2 \xi} + \mathcal{O}(\xi^{-3}) \approx \frac{a^3}{b^2 \xi}
\]

(29)

for \( \xi \rightarrow \infty \).
IV. PROPOSED RELATION BASED ON MIXING LENGTH MODEL

We first make use of the idea of Prandtl’s mixing length model [41] to relate $\kappa_i$ to the mean velocity gradient. According to Prandtl’s mixing length model, a fluid parcel will retain its velocity for a mixing length $l_v$ before mixing with surrounding fluid in a turbulent environment. Thus, the fluctuation in the velocity can be seen as the difference in velocity between a distance $l_v$. As all fluctuations in the thermal BL are mostly along the vertical $z$-direction, we use this picture to approximate the vertical velocity fluctuation $v'$ by

$$v' \approx l_v \partial_z V.$$  

(30)

Similarly, we approximate the temperature fluctuation $\theta'$ by

$$\theta' \approx -l_\theta \partial_z \Theta,$$  

(31)

where $l_\theta$ is the mixing length for temperature. Using (30) and (31), we have

$$\langle v'\theta' \rangle_t \approx -l_v l_\theta \partial_z V \partial_z \Theta.$$  

(32)

Comparing (32) with (8), we obtain

$$(\kappa_i / \kappa) \approx (l_v / \kappa) \partial_z V,$$  

(33)

which relates the eddy thermal diffusivity to the mean velocity gradient.

Next, we evaluate (33) near the plate to get a direct relation between $(\kappa_i / \kappa)\xi$ and $\psi$. Near the plate, we estimate the mixing lengths to be proportional to $z$:

$$l_v \approx k_v z, \quad l_\theta \approx k_\theta z,$$  

(34)

where $k_v$ and $k_\theta$ are some positive constants. Substituting (34) into (33) and using (10) - (12) and (15), we thus have

$$(\kappa_i / \kappa) \approx B k_v k_\theta \xi^3 \psi \xi.$$  

(35)

Taking the derivative of (35) w.r.t. $\xi$ and keeping only the lowest order term in $\xi$, we obtain

$$(\kappa_i / \kappa)\xi \approx 3 B k_v k_\theta \xi^2 \psi (0)$$  

(36)

for small $\xi$. On the other hand, using $\psi (0) = \psi (0) = 0$ that result from the no-slip boundary condition, we obtain

$$\psi \approx \psi (0) \xi^2 / 2$$  

(37)

for small $\xi$. Hence, (36) and (37) give

$$(\kappa_i / \kappa)\xi \approx 6 B k_v k_\theta \psi$$  

(38)

establishing a similarity between the dimensionless stream function $\psi$ and the derivative of the eddy thermal diffusivity $(\kappa_i / \kappa)\xi$ near the plate.

The mean horizontal velocity $U$ grows linearly with distance close to the plate. At a certain distance from the plate it attains a maximum value (which gives the wind velocity) and then decays to zero towards the bulk of the flow. From (10) and (12), $U = U_0 \psi \xi$, and $\xi$ is linearly related with the vertical coordinate $z$, therefore, the dimensionless stream function $\psi$ goes as $\xi^2$ near the plate and is almost constant far away from the plate. Thus, the functional dependences of $\psi$ and $(\kappa_i / \kappa)\xi$ on $\xi$ are similar in two limits: both of them $\sim \xi^2$ for $\xi \to 0$ and $\sim const$ for $\xi \to \infty$. This observation together with (38) motivate us to propose the following approximate relation for the whole thermal BL:

$$(\kappa_i / \kappa)\xi \approx K B \psi$$  

(39)

for some constant $K > 0$.

V. THEORETICAL MODEL

Using the proposed relation (39), we obtain

$$(\kappa_i / \kappa)\xi + B \psi \approx (1 + 1 / K) (\kappa_i / \kappa)\xi \equiv c (\kappa_i / \kappa)\xi,$$  

(40)

and (14) becomes

$$(1 + \kappa_i / \kappa) \theta \xi + c (\kappa_i / \kappa) \theta \xi = 0.$$  

(41)

Equation (41) is a thermal BL equation for all values of Pr, including Pr < 1. When the fluctuations are relatively weak so that the flow remains in the transition from laminar to turbulent state, the value of $c$ can be large. When the fluctuations are so strong that the term $(\kappa_i / \kappa)\xi$ dominates $B \psi$ in (10), the constant $c$ is close to 1. The solution of (41) is

$$\theta (\xi) = \int_0^\xi \left[ 1 + \frac{\kappa_i}{\kappa} (\eta) \right]^{-c} d\eta,$$  

(42)

which together with the approximation (27) yields

$$\theta (\xi) = \frac{1}{b} \int_0^{b \xi} \left[ 1 + \frac{3 \alpha^3}{b^3} (\eta - \arctan(\eta)) \right]^{-c} d\eta.$$  

(43)

Note that (43) has two independent parameters only as the following must be fulfilled,

$$b = \int_0^\infty \left[ 1 + \frac{3 \alpha^3}{b^3} (\eta - \arctan(\eta)) \right]^{-c} d\eta.$$  

(44)

due to the boundary conditions far away from the plate, $\theta (\infty) = 1$.

For the particular case of very large Pr, the thermal BL is deeply nested within the viscous BL and the eddy thermal diffusivity can be approximated by (28). Thus, (42) is reduced to the form reported in (37):

$$\theta (\xi) = \int_0^\xi (1 + a^3 \eta^3)^{-c} d\eta.$$  

(45)
VI. VALIDATION OF THE MODEL

In [37, 42] we have shown that (45) describes the temperature profiles obtained in DNS of RBC very well in the large-Pr regime, from Pr = 4 to Pr = 2547.9, as reported in [37]. For intermediate values of Pr between 4.38 and 2547.9, with a fitted value of c equal to 1, which corresponds to the limiting case of large fluctuations:

\[ \theta = \frac{\sqrt{3}}{4\pi} \log \left( \frac{1 + e^3}{1 + (e\xi)^3} \right) + \frac{3}{2\pi} \arctan \frac{2e\xi - 1}{\sqrt{3}} + \frac{1}{4} \]

with \( e = \frac{2\pi}{3\sqrt{3}} \approx 1.2 \) as well as for c = 2:

\[ \theta = \frac{\sqrt{3}}{4\pi} \log \left( \frac{1 + f\xi^3}{1 + (f\xi)^3} \right) + \frac{3}{2\pi} \arctan \frac{2f\xi - 1}{\sqrt{3}} + \frac{1}{4} \]

with \( f = \frac{4\pi}{9\sqrt{3}} \approx 0.8 \).

Equations (47) and (48) are found to be in good agreement with DNS data for Pr = 4.38 and Pr = 2547.9 respectively, as reported in [37]. For intermediate values of Pr between 4.38 and 2547.9, with a fitted value of c is shown to be in good agreement with DNS results [42].

We first check directly the validity of (33) with \( \Theta = (v\Theta - (vT)_z)/\partial_z \), and then averaged over horizontal cross-sections, obtained in the DNS for Pr = 10 and Ra = 10^6 (symbols) together with a fit for \( (l_\theta l_\theta/\kappa) \partial_z v \) averaged over horizontal cross-sections (solid line). Here \( l_\theta \) and \( l_\theta \) are taken according to (33), i.e. \( l_\theta l_\theta \approx z^2 \). Thus, the symbols and the line represent, respectively, the magnitudes of the left- and right-hand sides of the assumption (33).

![Image](image.png)

FIG. 1. Normalized eddy thermal diffusivity \(|\kappa_\ell/\kappa|\), calculated for \( \kappa_\ell = (V \Theta - (vT)_z)/\partial_z \Theta \), and then averaged over horizontal cross-sections, obtained in the DNS for Pr = 10 and Ra = 10^6 (symbols) together with a fit for \( (l_\theta l_\theta/\kappa) \partial_z v \) averaged over horizontal cross-sections (solid line). Here \( l_\theta \) and \( l_\theta \) are taken according to (33), i.e. \( l_\theta l_\theta \approx z^2 \). Thus, the symbols and the line represent, respectively, the magnitudes of the left- and right-hand sides of the assumption (33).
FIG. 2. Temperature profiles, averaged in time and over horizontal cross sections, obtained in the DNS of RBC in a cylindrical container of the aspect ratio 1 for \( Pr = 0.1 \) and different \( Ra \) (symbols). One can see that the profiles converge with increasing \( Ra \). Prandtl–Blasius predictions for \( Pr \to \infty \) (– – –) and \( Pr \to 0 \) (—) bound the gray region of Prandtl–Blasius predictions for all intermediate \( Pr \).

FIG. 3. Temperature profiles, averaged in time and over horizontal cross sections, obtained in the DNS of RBC in a cylindrical container of the aspect ratio 1 for \( Pr = 0.1, Pr = 1 \) and \( Pr = 2547.9 \) and different \( Ra \) (symbols) together with the predictions (43) (lines of the corresponding colours), see Table I. The black dashed line corresponds to the simplification (48) for very large \( Pr \), as reported in [37]. The Prandtl–Blasius region (gray) is as in Fig. 2.

\( b \) and \( c \) obtained for different \( Ra \) and \( Pr \) are presented in Table I. Evidently, (43) perfectly describes the temperature profiles in a wide range of \( Pr \) including \( Pr \ll 1 \).

Finally, we consider transitional RBC flows with very small \( Pr \), specifically \( Pr = 0.01 \) and \( Pr = 0.0232 \) for \( Ra = 10^7 \). Also for these RBC flows, the temperature profiles are in excellent agreement with (43), as illustrated in Fig. 4. The corresponding parameters are given in Table I. The black dashed line corresponds to the limiting case (47) for the simplification (45) for large \( Pr \). The Prandtl–Blasius region (gray) is as in Fig. 2.

TABLE I. Fitted values of the parameters in the temperature profiles approximation (43).

| \( Pr \) | \( Ra \) | \( a \) | \( b \) | \( c \) |
|-------|-------|-------|-------|-------|
| 0.01  | \( 10^7 \) | 1.59  | 6.19  | 4.99  |
| 0.0232| \( 10^7 \) | 1.56  | 3.59  | 2.64  |
| 0.1   | \( 10^7 \) | 1.52  | 2.27  | 1.84  |
| 0.1   | \( 2 \times 10^7 \) | 1.49  | 2.21  | 1.86  |
| 0.1   | \( 5 \times 10^7 \) | 1.59  | 2.72  | 1.97  |
| 0.1   | \( 10^8 \) | 1.62  | 2.79  | 1.96  |
| 1     | \( 10^7 \) | 1.16  | 0.62  | 1.36  |
| 1     | \( 2 \times 10^7 \) | 1.13  | 0.63  | 1.41  |
| 1     | \( 5 \times 10^7 \) | 1.12  | 0.81  | 1.57  |
| 1     | \( 10^8 \) | 1.15  | 0.64  | 1.39  |
| 4.38  | \( 10^8 \) | 1.00  | 0.61  | 1.68  |
| 4.38  | \( 10^9 \) | 1.02  | 0.62  | 1.64  |
| 2547.9| \( 10^8 \) | 0.77  | 0.51  | 2.61  |
| 2547.9| \( 10^9 \) | 0.75  | 0.52  | 2.77  |

VII. CONCLUSIONS

Utilizing the idea of Prandtl’s mixing length model, we put forward an approximate relation between the eddy thermal diffusivity \( \kappa_t \) and the stream function \( \psi \) within the thermal BL. This proposed relation has allowed us to obtain a thermal BL equation (41) that takes fluctuations for fluids with a general \( Pr \) into account, thus extending our earlier work [67]. Using the present approximation
parameter $t$ is very close to the plate according to (28) while the parameter $c$ is fixed by the boundary condition far away from the plate, which is given by (44). The parameter $a$ measures the intensity of the fluctuations very close to the plate according to (28) while the parameter $c \geq 1$ reflects the relative magnitudes of the stream function and the derivative of the eddy thermal diffusivity. When the BL flow is highly fluctuating so that the term $(\kappa_t/\kappa)$ dominates the term $\bar{B} \psi$ in (40), $c$ is about 1. On the other hand, in a transitional flow with relatively weak fluctuations, the value of $c$ is large. With a proper choice of $a$ and $c$, our theoretical model (43) describes extremely precisely the temperature profiles near the heated or cooled horizontal plates, in transitional and turbulent convective flows, for very large as well as very small Pr. In the present work, we have obtained the fitted value of $a$ by using DNS data of $\kappa_t/\kappa$ near the heated plate. In situations where measurements of $\kappa_t$ are not available, as in most experimental studies, we suggest to fit the measured temperature profiles directly by (43) with the constraint (44) to get the values of the two independent parameters $a$ and $c$. Note that our earlier model (45), (46) for the temperature profiles in large-Prandtl-number RBC, which was proposed in (57), has only one free parameter, while the new model (43), (44) has two free parameters but is applicable to general Pr.

To derive empirical formulas for the parameters $a$ and $c$ in the model (43), (44), further experimental and numerical data are needed, in particular for very high Ra and very small Pr. It should be noted that the DNS of RBC by large Ra and either very large Pr (47) or very small Pr (48, 49) require enormous computational efforts due to the requirement to resolve all relevant spatial and temporal microscales (44). That is, the time stepping must be finer than the time microscale $\tau = (\nu/e_a)^{1/2}$ and the spatial stepping must be smaller than the Kolmogorov microscale $\eta = (\nu^3/e_a)^{1/4}$ if Pr $\leq 1$ or smaller than the Batchelor microscale $\eta_B = (\nu^2/e_a)^{1/4}$ if Pr $> 1$. Here $e_a$ is the mean kinetic dissipation rate, which in RBC equals $e_a = (\nu^3/H^4)(\text{Nu} - 1)\text{Ra}^3\text{Pr}^{-2}$. For large Pr the need to resolve the time microscale is restrictive, while for small Pr the very fine meshes in space are needed to resolve the Kolmogorov spatial microscale. The general dependence of the temperature profiles (43) on Pr, Ra and the geometrical characteristics of the convection cell will therefore be explored in future when more experimental and numerical data, in particular for very small Pr, will be available.

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