Parallel and Orthogonal Cylindrically Symmetric Self-Similar Solutions

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Abstract

In this paper, we evaluate kinematic self-similar perfect fluid and dust solutions for the most general cylindrically symmetric spacetime. We explore kinematic self-similar solutions of the first, second, zeroth and infinite kinds for parallel and orthogonal cases. It is found that the parallel case gives solutions for both perfect fluid and dust cases in all kinds except the zeroth kind of the dust case where there exists no solution. The orthogonal perfect fluid case gives stiff fluid solution only in the first kind and vacuum solution for the dust case. We obtain a total of thirteen solutions out of which eleven are independent. The correspondence of these solutions with those already available in the literature is also given.

Keywords: Cylindrical symmetry, Self-similar variable.
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1 Introduction

Self-similarity simplifies the mathematical complexity of partial differential equations (PDEs). For an appropriate matter field, a set of field equations remains unchanged under a scale transformation. The solutions which are

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invariant under scale transformation are known as self-similar solutions. The special feature of these solutions is that, by a suitable coordinate transformations, the number of independent variables can be reduced by one and hence reduces the PDEs into ordinary differential equations (ODEs). This variable, a dimensionless combination of the independent variables, is called self-similar variable.

Einstein’s field equations (EFEs) are highly non-linear, second order PDEs. In order to solve these equations, several kinds of symmetry restrictions have been imposed by different people. Self-similarity is one of these restrictions used to obtain the exact solutions of the EFEs. In Newtonian gravity, self-similar solutions have been investigated by many authors to obtain the realistic solutions of gravitational collapse leading to star formation [1]. However, in General Relativity, self-similar solutions were first studied by Cahill and Taub [2] which correspond to Newtonian self-similarity of the homothetic class (also called first kind). Carter and Henriksen [3,4] defined the self-similarity of second, zeroth and infinite kinds.

McIntosh [5] found that a stiff fluid $k = 1$ is the only compatible perfect fluid with the homothety in the orthogonal case. Benoit and Coley [6] studied analytic spherically symmetric solutions of the EFEs coupled with a perfect fluid and admitting a kinematic self-similar (KSS) vector of the first, second and zeroth kinds. Carr et al. [7] investigated solution space of self-similar spherically symmetric perfect fluid models and physical aspects of these solutions. They combined the state space description of the homothetic approach with the use of the physically interesting quantities arising in the co-moving approach. Coley and Goliath [8] discussed self-similar spherically symmetric cosmological models with a perfect fluid and a scalar field with an exponential potential. Sintes et al. [9] considered spacetime which admits a KSS vector of the infinite kind without using the equation of state (EOS).

Maeda et al. [10-12] investigated the KSS vector of the second, zeroth and infinite kinds in the tilted, orthogonal and parallel perfect fluid cases for spherically symmetric spacetime. Chan et al. [13] studied the static vacuum plane symmetric spacetimes with two non-trivial solutions: the Taub solution and the Rindler solution. The same authors [14] studied the spherical gravitational collapse of a compact packet consisting of perfect fluid. They concluded that when the collapse has continuous self-similarity, the formation of black holes always starts with zero mass and when collapse has no self-similarity, the formation of black holes always starts with a finite non-zero mass.
In recent papers [15-17], Sharif and Aziz have explored the KSS solutions of a special cylindrically symmetric, special plane symmetric and the most general plane symmetric spacetimes. This analysis has been extensively given for the perfect fluid and dust cases with tilted, parallel and orthogonal vectors. The same authors [18-20] have also studied the physical properties of such solutions for spherically, cylindrically and plane symmetric spacetimes respectively. In this paper, we investigate the KSS solutions of the most general cylindrically symmetric spacetimes for both perfect fluid and dust cases. We explore the KSS solutions of the first, second, zeroth and infinite kinds for the parallel and orthogonal cases.

The paper has been organized as follows. In section 2, we discuss kinematic self-similarity for cylindrically symmetric spacetimes. Sections 3 and 4 are devoted to the parallel perfect fluid and dust solutions respectively. The orthogonal perfect fluid and dust solutions are investigated in section 5. Finally, we summarize and discuss the results in section 6.

2 Cylindrically Symmetric Spacetimes and Kinematic Self-Similarity

The most general cylindrically symmetric spacetime is given in the form [21]

\[ ds^2 = e^{2\nu(t,r)}dt^2 - e^{2\phi(t,r)}dr^2 - e^{2\mu(t,r)}d\theta^2 - e^{2\lambda(t,r)}dz^2, \]  

(1)

where \( \nu, \phi, \mu \) and \( \lambda \) are arbitrary functions of \( t \) and \( r \). The energy-momentum tensor for a perfect fluid is given by

\[ T_{ab} = [\rho(t,r) + p(t,r)]u_a u_b - p(t,r)g_{ab}, \quad (a, b = 0, 1, 2, 3), \]  

(2)

where \( \rho \) is the mass density, \( p \) is the pressure and \( u_a \) is the four-velocity of the fluid element in the co-moving coordinate system given as \( u_a = (e^\nu, 0, 0, 0) \). The EFEs for the line element (1) take the following form

\[
8\pi G\rho = e^{-2\nu}(\mu_t \phi_t + \lambda_t \phi_t + \lambda_t \mu_t) + e^{-2\phi}(-\mu_{rr} + \mu_r \phi_r - \mu_r \phi_r)
- \mu_r^2 - \lambda_{rr} + \lambda_r \phi_r - \lambda_r^2 - \mu_r \lambda_r),
\]

(3)

\[
8\pi G\rho = e^{-2\nu}(-\mu_{tt} - \lambda_{tt} + \mu_t \nu_t + \lambda_t \nu_t - \lambda_t \mu_t - \mu_t^2 - \lambda_t^2)
+ e^{-2\phi}(\mu_r \nu_r + \lambda_r \nu_r + \lambda_r \mu_r),
\]

(4)

\[
8\pi G\rho = e^{-2\nu}(-\phi_{tt} - \lambda_{tt} + \nu_t \phi_t + \lambda_t \nu_t - \lambda_t \phi_t - \phi_t^2 - \lambda_t^2)
\]
The conservation of energy-momentum tensor, $T^{ab}_{;b} = 0$, gives

$$
\begin{align*}
\phi_t &= -\frac{\rho_t}{\rho + p} - \mu_t - \lambda_t, \\
\nu_r &= -\frac{p_r}{\rho + p}.
\end{align*}
$$

(8) (9)

For a cylindrically symmetric spacetime, the vector field $\xi$ can be written as

$$
\xi^a \frac{\partial}{\partial x^a} = h_1(t, r) \frac{\partial}{\partial t} + h_2(t, r) \frac{\partial}{\partial r},
$$

where $h_1$ and $h_2$ are arbitrary functions. For $h_2 = 0$, we obtain parallel case while for the orthogonal case $h_1 = 0$. When both $h_1$ and $h_2$ are non-zero, we have tilted case which is the most general. In this paper, we restrict ourselves to investigate the parallel and orthogonal cases. The tilted case has been explored separately [22].

A KSS vector $\xi$ satisfies the following conditions

$$
\begin{align*}
\mathcal{L}_\xi h_{ab} &= 2\delta h_{ab}, \\
\mathcal{L}_\xi u_a &= \alpha u_a,
\end{align*}
$$

(11) (12)

where $\alpha$ and $\delta$ are dimensionless constants and $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor. The similarity transformation is characterized by the scale independent ratio, $\alpha/\delta$, known as similarity index which gives rise to the following possibilities according to $\delta \neq 0$ or $\delta = 0$.

- $\delta \neq 0$.

For this possibility, the KSS vector takes the form

$$
\xi^a \frac{\partial}{\partial x^a} = (\alpha t + \beta) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}.
$$

(13)

If $\alpha = 1$ ($\beta$ can be set to zero) then this case is referred to as self-similarity of the first kind which corresponds to a homothety.
When $\alpha = 0$ ($\beta$ can be re-scaled to unity), this corresponds to self-similarity of the zeroth kind.

When $\alpha \neq 0, 1$ ($\beta$ can be re-scaled to zero), this is known as self-similarity of the second kind.

• $\delta = 0$.

In this case $\alpha \neq 0$ ($\alpha = 1$ is possible, $\beta$ can be re-scaled to zero), the self-similarity is referred to the infinite kind.

For the parallel fluid flow, the KSS vector will take the form

$$\xi^a \frac{\partial}{\partial x^a} = f(t) \frac{\partial}{\partial t},$$

where $f(t)$ is an arbitrary function and self-similar variable is $r$ for all kinds. The metric functions for the first, second, zeroth and infinite kinds, respectively, will be

$$\nu = \tilde{\nu}(r), \quad \phi = \ln t + \tilde{\phi}(r), \quad \mu = \ln t + \tilde{\mu}(r), \quad \lambda = \ln t + \tilde{\lambda}(r),$$

$$\nu = (\alpha - 1) \ln t + \tilde{\nu}(r), \quad \phi = \ln t + \tilde{\phi}(r), \quad \mu = \ln t + \tilde{\mu}(r),$$

$$\lambda = \ln t + \tilde{\lambda}(r),$$

$$\nu = -\ln t + \tilde{\nu}(r), \quad \phi = \ln t + \tilde{\phi}(r), \quad \mu = \ln t + \tilde{\mu}(r),$$

$$\lambda = \ln t + \tilde{\lambda}(r),$$

$$\nu = \tilde{\nu}(r), \quad \phi = \tilde{\phi}(r), \quad \mu = \tilde{\mu}(r), \quad \lambda = \tilde{\lambda}(r).$$

(15)

Hereafter we omit the bar for the sake of simplicity.

$$\nu = \nu(r), \quad \phi = \ln t + \phi(r), \quad \mu = \ln t + \mu(r), \quad \lambda = \ln t + \lambda(r),$$

$$\nu = (\alpha - 1) \ln t + \nu(r), \quad \phi = \ln t + \phi(r), \quad \mu = \ln t + \mu(r),$$

$$\lambda = \ln t + \lambda(r),$$

$$\nu = -\ln t + \nu(r), \quad \phi = \ln t + \phi(r), \quad \mu = \ln t + \mu(r),$$

$$\lambda = \ln t + \lambda(r),$$

$$\nu = \nu(r), \quad \phi = \phi(r), \quad \mu = \mu(r), \quad \lambda = \lambda(r).$$

(16)

If the KSS vector is orthogonal to the fluid flow, it becomes

$$\xi^a \frac{\partial}{\partial x^a} = g(r) \frac{\partial}{\partial r},$$

(17)
where \( g(r) \) is an arbitrary function and self-similar variable is \( t \) for all kinds. The corresponding metric functions for the first, second, zeroth and infinite kinds, respectively, will take the following form (after omitting the bar)

\[
\begin{align*}
\nu &= \ln r + \nu(t), \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= \alpha \ln r + \nu(t), \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= \nu(t), \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= \ln r + \nu(t), \quad \phi = -\ln r + \phi(t), \quad \mu = \mu(t), \quad \lambda = \lambda(t).
\end{align*}
\] (18)

Using coordinate transformation \( \bar{r} = \bar{r}(r) \) and \( \bar{t} = \bar{t}(t) \), Eqs.(16) and (18) become

\[
\begin{align*}
\nu &= \nu(r), \quad \phi = \ln t, \quad \mu = \ln t + \mu(r), \quad \lambda = \ln t + \lambda(r), \\
\nu &= (\alpha - 1) \ln t + \nu(r), \quad \phi = \ln t, \quad \mu = \ln t + \mu(r), \\
\lambda &= \ln t + \lambda(r), \\
\nu &= -\ln t + \nu(r), \quad \phi = \ln t, \quad \mu = \ln t + \mu(r), \\
\lambda &= \ln t + \lambda(r), \\
\nu &= \nu(r), \quad \phi = 0, \quad \mu = \mu(r), \quad \lambda = \lambda(r),
\end{align*}
\] (19)

and

\[
\begin{align*}
\nu &= \ln r, \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= \alpha \ln r, \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= 0, \quad \phi = \phi(t), \quad \mu = \ln r + \mu(t), \quad \lambda = \ln r + \lambda(t), \\
\nu &= \ln r, \quad \phi = -\ln r + \phi(t), \quad \mu = \mu(t), \quad \lambda = \lambda(t),
\end{align*}
\] (20)

respectively.

3 Parallel Perfect Fluid Case

3.1 Self-similarity of the First Kind

For this kind, the EFEs imply that the quantities \( \rho \) and \( p \) can be written as

\[
\begin{align*}
8\pi G\rho &= t^{-2}\rho(r), \\
8\pi Gp &= t^{-2}p(r).
\end{align*}
\] (21) (22)
If the EFEs and the equations of motion for the matter field are satisfied for $O[(t)^{-2}]$, we obtain a set of ODEs and thus Eqs.(3)-(9) reduce to

\begin{align*}
\nu' &= 0, \quad (23) \\
\rho &= 3e^{-2\nu} + (-\mu'' - \mu'^2 - \lambda'' - \lambda'^2 - \mu' \lambda'), \quad (24) \\
p &= -e^{-2\nu} + (\lambda' \mu'), \quad (25) \\
p &= -e^{-2\nu} + (\lambda'' + \lambda'^2), \quad (26) \\
p &= -e^{-2\nu} + (\mu'' + \mu'^2), \quad (27) \\
0 &= \rho + 3p, \quad (28) \\
0 &= -p', \quad (29) \\
\end{align*}

where prime represents derivative with respect to $r$. Equation (28) is an equation of state (EOS) which gives along with Eq.(29) $p = -\rho/3 = -1$ and consequently, Eqs(24)-(27) lead to $\lambda'' + \lambda'^2 = 0 = \mu'' + \mu'^2$. Integration of these equations yield the following solution

\begin{align*}
\nu &= c_1, \quad \phi = 0, \quad \mu = \ln(\xi - c_3) + c_2, \quad \lambda = \ln(\xi - c_5) + c_4 \\
p &= -\rho/3 = -1. \quad (30)
\end{align*}

Equation (30) implies that $e^\mu = rc_6 + c_7$ and $e^\lambda = rc_8 + c_9$. Using Eq.(30) with Eq.(25), we obtain $c_6c_8 = 0$ which yields either $c_6 = 0$, or $c_8 = 0$, or $c_6 = 0 = c_8$.

In the first case, we set $c_7 = 1 = c_8$. Also, $c_9 = 0$ by re-defining the origin of $r$, i.e., $\bar{r} = r + constant$. Thus the resulting spacetime becomes

\begin{equation}
\begin{aligned}
ds^2 &= dt^2 - t^2(dr^2 + d\theta^2 + r^2dz^2). \quad (31)
\end{aligned}
\end{equation}

For the case, when $c_8 = 0$, we follow the same procedure as above and obtain the same spacetime by interchanging $z$ and $\theta$. In the last case, the resulting spacetime takes the form

\begin{equation}
\begin{aligned}
ds^2 &= dt^2 - t^2(dr^2 + d\theta^2 + dz^2). \quad (32)
\end{aligned}
\end{equation}

In all these cases, the spacetime corresponds to FRW metric.

### 3.2 Self-similarity of the Second Kind

Here the EFEs imply that the quantities $\rho$ and $p$ are of the form

\begin{align*}
8\pi G\rho &= t^{-2}\rho_1(r) + t^{-2\alpha}\rho_2(r), \quad (33) \\
8\pi Gp &= t^{-2}p_1(r) + t^{-2\alpha}p_2(r). \quad (34)
\end{align*}
The corresponding set of ODEs become

\[ \nu' = 0, \]
\[ \rho_1 = -\mu'' - \mu'^2 - \lambda'' - \lambda'^2 - \mu'\lambda', \]
\[ e^{2\nu}\rho_2 = 3, \]
\[ p_1 = \lambda'\mu', \]
\[ e^{2\nu}p_2 = (2\alpha - 3), \]
\[ p_1 = \lambda'' + \lambda'^2, \]
\[ p_1 = \mu'' + \mu'^2, \]
\[ 0 = \rho_1 + 3p_1, \]
\[ 0 = (2\alpha - 3)p_2 - 3p_2, \]
\[ -p_1' = 0, \]
\[ -p_2' = 0. \]

Using Eq.(35), we obtain from Eq.(37) \( c_1^2 = \rho_2/3. \) Equations (42) and (43) are two EOS where \( p_1 = -\rho_1/3 = \text{constant}, \) and \( p_2 = (2\alpha - 3)c_1^2 \) respectively. Solving Eqs.(36), (38), (40) and (41), we obtain

\[ \mu'' + \mu'^2 = p_1, \]
\[ \lambda'' + \lambda'^2 = p_1. \]

Equations (38) and (46) lead to the following solution

\[ \nu = c_1, \quad \phi = 0, \quad \mu = c_3 + \ln[\cosh[\sqrt{p_1}(\xi + c_2)]]], \]
\[ \lambda = \ln[\sinh[\sqrt{p_1}(\xi + c_2)]]]. \]

The resulting spacetime is

\[ ds^2 = dt^2 - t^2(dr^2 + \cosh^2[\sqrt{p_1}(r + c_2)]d\theta^2 + \sinh^2[\sqrt{p_1}(r + c_2)]dz^2). \]

Equation (47) leads to the same spacetime by interchanging \( \theta \) and \( z. \)

### 3.3 Self-similarity of the Zeroth Kind

For this kind, we set \( \alpha = 0 \) in Eqs.(33)-(45). Equations (42) and (43) give \( p_1 = -\rho_1/3 = \text{constant}, \) \( p_2 = -\rho_2 = -3 \) which are equations of state. Solving Eqs.(35)-(45), we obtain the same solution as given for the second kind.
3.4 Self-similarity of the Infinite Kind

For this kind, the metric functions of (1) are given by Eq.(19) and a set of ODEs turn out to be

\[ \rho = -\mu'' - \mu'^2 - \lambda'' - \lambda'^2 - \mu'\lambda', \]  
\[ p = \mu'\nu' + \lambda'\nu' + \lambda'\mu', \]  
\[ p = \nu'' + \nu'^2 + \lambda'\nu' + \lambda'' + \lambda'^2, \]  
\[ p = \nu'' + \nu'^2 + \mu'\nu' + \mu'' + \mu'^2, \]  
\[ -p' = \nu'(\rho + p). \]  

We take the following possibilities to solve the above set of equations

(1) \( \nu' = \mu' \), (2) \( \nu' = \lambda' \),
(3) \( \lambda' = \mu' \), (4) \( \mu' = \nu' = \lambda' \).

The possibilities (1) and (2) yield contradiction. The possibility (3) gives the following solution

\[ \phi = 0, \ \nu = c_1, \ \mu = \lambda = c_2\xi + c_3, \ p = -\rho/3 = \text{constant}, \]  
and the corresponding metric is

\[ ds^2 = dt^2 - dr^2 - e^{2c_1r}(d\theta^2 + dz^2). \]

The last possibility gives the following solution

\[ \phi = 0, \ \nu = \mu = \lambda = c_1\xi + c_2, \ p = -\rho = \text{constant}, \]  
and the resulting spacetime is

\[ ds^2 = e^{2c_1r}(dt^2 - d\theta^2 - dz^2) - dr^2. \]

4 Parallel Dust Case

4.1 Self-similarity of the First Kind

For the dust case, we take \( p = 0 \) in Eqs.(23)-(29). Solving Eqs.(24)-(27), we obtain \( \mu'' + \mu'^2 - \mu'\lambda' = 0 \) and finally, we have the following solution

\[ \nu = c_1, \ \phi = 0, \ \mu = \ln(e^{2\xi} - c_2) + c_3 - \xi, \]
\[ \lambda = \ln(e^{2\xi} + c_2) - \xi, \ \rho = 0 = p. \]
The resulting spacetime is
\[ ds^2 = dt^2 - t^2[dr^2 + e^{-2r}((e^{2r} - c_2)^2d\theta^2 + (e^{2r} + c_2)^2dz^2)]. \]  

4.2 Self-similarity of the Second Kind

Setting \( p_1 = 0 = p_2 \) in Eqs.(35)-(45), we obtain the same solution as given by Eqs.(31) and (32) with \( p_1 = 0, \ p_2 = 3c_1^2 \) and \( \alpha = 3/2 \), where \( c_1 \) is the same as given in perfect fluid case.

4.3 Self-similarity of the Zeroth Kind

For \( p_1 = 0 = p_2 \), we have contradiction from basic equations for perfect fluid case.

4.4 Self-similarity of the Infinite Kind

When we replace \( p = 0 \) in Eqs.(50)-(54), there arise three cases from Eq.(54)
\[ (a) \ \nu' = 0, \ \rho \neq 0, \ \ (b) \ \rho = 0, \ \nu' \neq 0, \ \ (c) \ \nu' = 0 = \rho. \]  

The first case (a) yields contradiction and hence there is no solution.

For the case (b), we take the four possibilities given in Eq.(55). For the possibility (1), Eq.(51) further gives two more options either \( \mu' = 0 \) or \( \mu' = -2\lambda' \). The first option gives contradiction while the second option yields the following solution
\[ \nu = \mu = -\frac{2}{3}\ln(3\xi + c_1), \ \lambda = c_2 - \frac{1}{3}\ln(3\xi + c_1), \ \phi = 0, \ \rho = p = 0. \]  

The corresponding metric becomes
\[ ds^2 = (3r + c_1)^{4/3}(dt^2 - d\theta^2) - dr^2 - (3r + c_2)^{-2/3}dz^2. \]  

The possibility (2) yields the same solution by interchanging \( z \) and \( \theta \) and the possibility (3) leads to the following solution
\[ \lambda = \mu = c_1, \ \nu = c_3 + \ln(\xi - c_2), \ \phi = 0, \ \rho = p = 0, \]  
\[ \lambda = \mu = -\frac{2}{3}\ln(3\xi + c_1), \ \nu = c_2 - \frac{1}{3}\ln(3\xi + c_1), \ \phi = 0, \ \rho = p = 0. \]
The corresponding metrics are

\[ ds^2 = r^2 dt^2 - dr^2 - d\theta^2 - dz^2, \]  
\[ ds^2 = (3r + c_2)^{-2/3} dt^2 - dr^2 - (3r + c_1)^{4/3}(d\theta^2 + dz^2). \]

For the last possibility (4), we have contradiction and hence there is no solution.

The last case (c) gives the same solution as Eq.(30) with \( \rho = 0 = p \) but the corresponding metrics are

\[ ds^2 = dt^2 - dr^2 - r^2d\theta^2 - r^2dz^2, \]  
\[ ds^2 = dt^2 - dr^2 - r^2d\theta^2 - dz^2, \]

and Minkowski spacetime.

## 5 Orthogonal Perfect Fluid and Dust Cases

Here the self-similar variable is \( \xi = t \) in each kind. The EFEs and the equations of motion for the perfect fluid of the first kind gives the following set of ODEs

\[ \dot{\phi} = 0, \]  
\[ \dot{\mu} + \dot{\lambda} = 0, \]  
\[ (\rho + e^{-2\phi}) = \mu\lambda, \]  
\[ (p - 3e^{-2\phi}) = -\mu - \lambda - \mu\lambda - \mu^2 - \lambda^2, \]  
\[ (p - e^{-2\phi}) = -\lambda^2, \]  
\[ (p - e^{-2\phi}) = -\mu^2, \]  
\[ (\dot{\mu} + \dot{\lambda})(\rho + p) = -\rho, \]  
\[ \rho = p, \]

where dot represents derivative with respect to \( t \). Equation (77) gives an equation of state for this system of equations. Solving Eqs.(71)-(77) by taking \( \dot{\mu} = \dot{\lambda} \) which yields the following solution

\[ \nu = 0, \quad \phi = c_1, \quad \mu = \lambda = -\frac{1}{4}\ln p + \ln c_2, \]  
\[ \dot{\rho} = 8p(3p + 2), \quad p = p(t). \]
The resulting spacetime becomes
\[ ds^2 = dt^2 - dr^2 - \frac{r^2}{\sqrt{p}}(d\theta^2 + dz^2). \] (80)

For the perfect fluid case of the second, zeroth and infinite kinds, we obtain contradictions from the basic equations.

For the dust case (i.e., \( p = 0 \)) of the first kind, we obtain the following solution
\[ \nu = 0, \quad \phi = c_1, \quad \mu = \ln(e^{2\xi} - c_2) + c_3 - \xi, \]
\[ \lambda = \ln(e^{2\xi} + c_2) - \xi, \quad \rho = 0 = p, \] (81)
and the corresponding metric is
\[ ds^2 = dt^2 - dr^2 - r^2 e^{-2t}[(e^{2t} - c_2)^2 d\theta^2 + (e^{2t} + c_2)^2 dz^2]. \] (82)

In self-similarity of the second, zeroth and infinite kinds, we have a contradiction and hence there is no solution for these kinds.

## 6 Summary and Discussion

We have evaluated the KSS perfect fluid and dust solutions of the first, second, zeroth and infinite kinds when the KSS vector is parallel and orthogonal to the fluid flow. In the parallel perfect fluid case, the first kind gives only one vacuum solution while the second kind gives one radiation solution. The zeroth kind yields the same solution as for the second kind and the infinite kind yields two independent vacuum solutions. In the orthogonal perfect fluid case, there is only one stiff fluid solution for the first kind.

In the parallel dust case, the first kind yields only one vacuum solution. The second kind yields the same solution as given in the first kind for perfect fluid case while there exists no solution for the zeroth kind. The infinite kind gives four independent vacuum solutions out of which one is Minkowski spacetime. For the orthogonal dust case, the first kind provides only one vacuum solution while in perfect fluid and dust cases there is no solution for the second, zeroth and infinite kinds. Thus we obtain a total of thirteen solutions out of which eleven are independent.
Here we give the correspondence of these solutions to the well-known solutions available in the literature. The metrics given by Eqs.(31) and (32) correspond to
\[ ds^2 = dt^2 - a^2(t)[dr^2 + f(r)d\theta^2 + dz^2]. \] (83)
This spacetime can be matched with FRW metric which has six KVs. The spacetime (64) corresponds to a class of metrics [23]
\[ ds^2 = e^{\nu(r)}(dt^2 - dr^2 - d\theta^2 - e^{\mu(r)}dz^2). \] (84)
The metrics given by Eq.(68) corresponds to a class of metrics [23]
\[ ds^2 = e^{\nu(r)}dt^2 - dr^2 - e^{\mu(r)}(a^2d\theta^2 + dz^2), \] (85)
where \( \mu'' \neq 0 \neq \nu''. \) The spacetimes (84), (85) have four KVs and belong to group \( G_4 = \langle X_0, X_1, X_2, X_3 \rangle. \)

The metric given by Eq.(57) belongs to a class of metrics [24]
\[ ds^2 = dt^2 - dr^2 - e^{r/b}(a^2d\theta^2 + dz^2), \] (86)
where \( b = 1, 2, \ldots, 6 \) and has 7 KVs while the metric (69) corresponds to a class of metrics [24]
\[ ds^2 = dt^2 - dr^2 - a^2d\theta^2 - e^{\lambda(r)}dz^2, \] (87)
where \( a = 1, 2, \ldots, 6. \) The spacetimes given by Eq.(70) correspond to a class of metrics [24]
\[ ds^2 = dt^2 - dr^2 - a^2e^{\mu(r)}d\theta^2 - dz^2. \] (88)
while Eqs.(67) correspond to a class of metrics [24]
\[ ds^2 = e^{\nu(r)}dt^2 - dr^2 - a^2d\theta^2 - dz^2. \] (89)
The metrics given by Eqs.(87)-(89) have 6 KVs. The spacetime (59) can correspond to the metric [24]
\[ ds^2 = (r/b)^2(dt^2 - a^2d\theta^2 - dz^2) - dr^2 \] (90)
which has 6 KVs. The metrics (49), (61), (80) and (82) do not correspond to any solution in the literature.

It is interesting to mention here that the parallel case gives many solutions while there was no solution for this case when dealing with special metric [15]. The orthogonal case yields solutions only in the first kind and contradictory results in all other kinds while for the special metric [15], the infinite kind gives vacuum solution and no solution in any other kind. The summary of the solutions are presented below in tables 1-2.
Table 1. Parallel Perfect Fluid KSS solutions.

| Self-similarity | Solutions                  |
|-----------------|---------------------------|
| First kind      | solutions given by Eqs.(31),(32) |
| Second kind     | solution given by Eq.(49)               |
| Zeroth kind     | solution given by Eq.(49)               |
| Infinite kind   | solutions given by Eqs.(57),(59)         |

Table 2. Parallel Dust KSS solutions.

| Self-similarity | Solutions                  |
|-----------------|---------------------------|
| First kind      | solution given by Eq.(61)               |
| Second kind     | solutions given by Eqs.(31),(32)         |
| Zeroth kind     | None                        |
| Infinite kind   | solutions given by Eqs.(64),(68),(69),(70) |

There is only one solution each for the orthogonal perfect fluid and dust cases of the first kind given by Eqs.(80) and (82) respectively.

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