Cosmological Evolution of Interacting Dark Energy in Lorentz Violation

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Abstract. The cosmological evolution of an interacting scalar field model in which the scalar field interacts with dark matter, radiation, and baryon via Lorentz violation is investigated. We propose a model of interaction through the effective coupling $\bar{\beta}$. Using dynamical system analysis, we study the linear dynamics of an interacting model and show that the dynamics of critical points are completely controlled by two parameters. Some results can be mentioned as follows. Firstly, the sequence of radiation, the dark matter, and the scalar field dark energy exist and baryons are sub dominant. Secondly, the model also allows the possibility of having a universe in the phantom phase with constant potential. Thirdly, the effective gravitational constant varies with respect to time through $\bar{\beta}$. In particular, we consider a simple case where $\bar{\beta}$ has a quadratic form and has a good agreement with the modified $\Lambda$CDM and quintessence models. Finally, we also calculate the first post–Newtonian parameters for our model.

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1 Introduction

There has been a growing appreciation of the importance of the violations of Lorentz invariance in recent years. The intriguing possibility of the Lorentz violation is that an unknown physics at high-energy scales could lead to a spontaneous breaking of Lorentz invariance by giving an expectation value to certain non Standard Model fields that carry Lorentz indices, such as vectors, tensors, and gradients of scalar fields [1]. Recently, it has been proposed a relativistic theory of gravity where gravity is mediated by a tensor, a vector, and a scalar field, thus called TeVeS gravitational theory [2]. It provides modified Newtonian dynamics (MOND) and Newtonian limits in the weak field nonrelativistic limit, and is devoid of a causal propagation of perturbations. TeVeS could also explain the large-scale structure formation of the Universe without recurring to cold dark matter [3], which is composed of very massive slowly moving and weakly interacting particles. On the other hand, the Einstein–Aether theory [4] is a vector-tensor theory, and TeVeS can be written as a vector-tensor theory which is the extension of the Einstein–Aether theory [5]. In the case of generalized Einstein–Aether theory [6], the effect of a general class of such theories on the solar system has been considered in Ref. [7]. Moreover, as has been shown by authors in Ref. [8], the Einstein–Aether theory may lead to significant modifications of the power spectrum of tensor perturbation. The strong gravitational cases including black holes of such theories have been studied in Refs. [9].

The existence of vector fields in a scalar-vector-tensor theory of gravity also leads to its applications in modern cosmology and it might explain inflationary scenarios [10][11] and accelerated expansion of the universe [6][12]. The accelerated expansion and crossing of the phantom divided line has been studied recently by authors in Ref. [13]. Based on a dynamical vector field coupled to the gravitational and scalar fields, we have studied to some extent the cosmological implications of a scalar-vector-tensor theory of gravity [14].

Since the discovery of accelerated expansion of our Universe [15], identifying the contents of dark energy and dark matter is one of the most important subjects in modern cosmology. The dark energy is described by an equation of state parameter $\omega = p/\rho$, the ratio of the spa-
tially homogeneous dark energy pressure $p$ to its energy density $\rho$. A value of $\omega < -1/3$ is required for accelerated expansion. The classification of dark energy might be due to: quintessence field [16], tachyon models [17], Chaplygin gas [18] if $\omega > -1$, cosmological constant if $\omega = -1$ [19, 20, 21, 22], or phantom field if $\omega < -1$ [23]. A recent comprehensive review on dark energy is available in [24]. Of course, as it has been discussed in [25] the vector field is also a viable dark energy candidate and effects on the cosmic microwave background radiation and the large scale structure [27].

In the previous work [28], the attractor solutions in Lorentz violating scalar-vector-tensor theory of gravity without interaction with background matter was studied. In this framework, both the effective coupling and potential functions determine the stabilities of the fixed points. In the model, we considered the constants of slope of the effective coupling and potential functions which lead to the quadratic effective coupling with the (inverse) power-law potential. Differing from the previous work, in this work, we investigate the cosmological evolution of the scalar field dark energy and background perfect fluid by means of dynamical system. We study the cases of scalar field dark energy interacting with background perfect fluid. The interaction terms are taken to be two different forms which are mediated by the slope of the effective coupling. For more realistic model we assume that the background matter fields might be dark matter, radiation, and baryons.

Furthermore, to test the model in the solar system we present the post–Newtonian parameters (PN). In the PN approximation we restrict ourselves to the first post–Newtonian. The parameterized post–Newtonian (PPN) parameters are determined by expanding the modified field equations in the metric perturbation. Then, we compare the solution to the PPN formalism in first PN approximation proposed by Will and Nordtvedt [29, 30] and read off the coefficients (the PPN parameters) of post Newtonian potentials of the theory.

This paper is organized as follows. In Section 2 we set down the general formalism of the scalar field interacting with background perfect fluid in the scalar-vector-tensor theory where the Lorentz symmetry is spontaneously broken due to the unit–norm vector field. We derive the governing equations of motion for the canonical Lagrangian of the scalar field. In Section 3 we study the interaction models and the attractor solutions. The critical points of the system and their stability are presented. The cosmological implication is discussed in Section 4. In Section 5 we present the parameterized post-Newtonian parameters of the model. The final Section is devoted to the conclusions.

In what follows, the conventions that we use throughout this work are the following: Greek letters represent spacetime indices, while Latin letters stand for spatial indices and repeated indices mean Einstein’s summation. The symbol $\mathcal{O}(N)$ stands for terms of order $N$. Finally, we use the metric signature $(-, +, +, +)$.

## 2 The action and field equations

In the present section, we develop the general reconstruction scheme for the scalar-vector-tensor gravitational theory. We will consider the properties of general four-dimensional universe, i.e. the universe where the four-dimensional space-time is allowed to contain any non-gravitational degree of freedom in the framework of Lorentz violating scalar-tensor-vector theory of gravity. Let us assume that the Lorentz symmetry is spontaneously broken by imposing the expectation values of a vector field $u^\mu$ as $< 0|u^\mu u_\mu|0> = -1$. The action can be written as the sum of four distinct parts:

$$S = S_g + S_u + S_\phi + S_m, \quad (1)$$

where the actions for the tensor field $S_g$, the vector field $S_u$, the scalar field $S_\phi$, and the ordinary matter $S_m$, respectively, are given by

$$S_g = \int d^4x \sqrt{-g} \frac{1}{16\pi G} R, \quad (2)$$

$$S_u = \int d^4x \sqrt{-g} \left[ -\beta_1 \nabla^\mu u^\nu \nabla_\mu u_\nu + \beta_2 (\nabla_\mu u^\mu)^2 \right. \left. -\beta_3 \nabla^\mu u^\nu \nabla_\nu u_\mu + \lambda (u^\mu u_\mu + 1) \right], \quad (3)$$

$$S_\phi = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\nabla \phi)^2 - V(\phi) \right], \quad (4)$$

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m(\phi, \Psi, \rho_m, \rho_g). \quad (5)$$

In the above $\beta_1(\phi)$ ($i = 1, 2, 3$) are the functions of $\phi$, and $\lambda$ is a Lagrange multiplier. In Eq. (5), we allow for an arbitrary coupling between the matter fields $\Psi$ and the scalar field $\phi$.

Varying the action (1) with respect to $g^{\mu \nu}$, we have field equations

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi G T_{\mu \nu}, \quad (6)$$

where $R_{\mu \nu}$ is the Ricci tensor, $R$ is the scalar curvature, $g_{\mu \nu}$ is the metric tensor, and $T_{\mu \nu}$ is the energy-momentum tensor for all the fields present, $T_{\mu \nu}^{(u)} + T_{\mu \nu}^{(\phi)} + T_{\mu \nu}^{(m)}$. $T_{\mu \nu}^{(u)}$, $T_{\mu \nu}^{(\phi)}$ and $T_{\mu \nu}^{(m)}$ are the energy-momentum tensors of vector fields, scalar fields, and ordinary matter, respectively, given by

$$T_{\mu \nu}^{(u)} = 2\beta_1 (\nabla_\mu u^\tau \nabla_\tau u_\nu - \nabla_\tau u_\mu \nabla_\nu u_\tau) - 2\nabla_\tau (u_{(\mu} J^{\tau \nu)}) \left. -2\nabla_\tau (u^\tau J_{\mu \nu}) + 2\nabla_\tau (u_{(\mu} J^{\tau \nu}) \right. \left. -2u_\mu \nabla_\tau J^{\tau \nu} u_\nu + g_{\mu \nu} L_\mu \right), \quad (7)$$

$$T_{\mu \nu}^{(\phi)} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} (|\nabla \phi|^2 + 2V(\phi)), \quad (8)$$

$$T_{\mu \nu}^{(m)} = (\rho_m + p_m) u_\mu u_\nu + p_m g_{\mu \nu}, \quad (9)$$

where $v^\mu$ is the four velocity and the current tensor $J_{\mu \nu}$ in Eq. (7) is given by

$$J_{\mu \nu} = -\beta_1 \nabla_\mu u_\nu - \beta_2 \delta_\mu^\nu \nabla_\tau u_\tau - \beta_3 \nabla_\nu u_\mu. \quad (10)$$
The energy-momentum tensor $T_{\mu\nu}$ is conserved

$$\nabla^\nu \left( T^{(u)}_{\nu\mu} + T^{(\phi)}_{\nu\mu} + T^{(m)}_{\nu\mu} \right) = 0 .$$  \hspace{1cm} (11)

In general, however, the Bianchi identity implies that each energy species in the cosmic mixture is not conserved, namely

$$\nabla^\nu T^{(u)}_{\nu\mu} = \sigma^{(u)}_{\mu} , \quad \nabla^\nu T^{(\phi)}_{\nu\mu} = \sigma^{(\phi)}_{\mu} , \quad \nabla^\nu T^{(m)}_{\nu\mu} = \sigma^{(m)}_{\mu}$$  \hspace{1cm} (12)

Here $\sigma^{(k)}_{\mu} (k = u, \phi, m)$ is an arbitrary vector function of the space-time coordinates that determines the rate of transfer of energy, where $\sigma^{(u)}_{\mu} + \sigma^{(\phi)}_{\mu} + \sigma^{(m)}_{\mu} = 0$. This is in accordance with Eq. (11). Equation (12) is the basic feature of interacting models in which there is exchange of energy between the components of the cosmic fluid. Moreover, the projection of the non conservation equation along the velocity of the whole fluid $n^\mu$ is

$$Q^{(u)} = -Q^{(\phi)} - Q^{(m)} ,$$  \hspace{1cm} (13)

where $Q^{(k)} = n^\mu \sigma^{(k)}_{\mu}$ is a scalar.

Using Eq. (13), the Friedmann and Raychaudhuri equations can be written as

$$3H^2 = 8\pi G \left( \rho_u + \rho_{\phi} + \rho_m \right) ,$$  \hspace{1cm} (14)

and

$$2H = -8\pi G \left( \rho_u + \rho_{\phi} + \rho_m + p_u + p_{\phi} + p_m \right) ,$$  \hspace{1cm} (15)

where

$$\rho_u = -3\beta H^2 , \quad p_u = -\rho_u + 2 \left( \beta \dot{H} + \dot{\beta} H \right) ,$$  \hspace{1cm} (16)

$$\rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V , \quad p_{\phi} = -\rho_{\phi} + \dot{\phi}^2 .$$  \hspace{1cm} (17)

Here, we have defined $\beta \equiv \beta_1 + 3\beta_2 + \beta_3$.

Substituting Eqs. (16) and (17) into Eqs. (14) and (15), respectively, we obtain

$$3 \left( \beta + \frac{1}{8\pi G} \right) H^2 = \frac{1}{2} \dot{\phi}^2 + V + \rho_m$$  \hspace{1cm} (18)

and

$$2 \left( \beta + \frac{1}{8\pi G} \right) \dot{H} = -2\beta H - \dot{\phi}^2 - 2(\rho_m + p_m) .$$  \hspace{1cm} (19)

Let us define the effective coupling as follows

$$\bar{\beta} \equiv \beta + \frac{1}{8\pi G} ,$$  \hspace{1cm} (20)

then Eqs. (18) and (19) can be simplified as

$$H^2 = \frac{1}{3\bar{\beta}} \left( \frac{1}{2} \dot{\phi}^2 + V + \rho_m \right) ,$$  \hspace{1cm} (21)

$$\frac{\dot{H}}{H} = -\frac{\bar{\beta}}{3\bar{\beta}} - \frac{1}{2} \frac{\dot{\phi}^2}{H \bar{\beta}} - \gamma_m \frac{\rho_m}{H \bar{\beta}} .$$  \hspace{1cm} (22)

Here, we have defined $p_m = (\gamma_m - 1) \rho_m$, where $\gamma_m$ is the ordinary matter barotropic parameter, which is related to the equation of state parameter $\omega_m$ through the relationship $\gamma_m = 1 + \omega_m$. Similarly, we also defined the scalar field barotropic parameter, $p_{\phi} = (\gamma_{\phi} - 1) \rho_{\phi}$ and $\gamma_{\phi} = 1 + \omega_{\phi}$.

Then the effective equation of state for the total cosmic fluid is

$$\gamma^{(c)} = 1 + \frac{p_u + p_{\phi} + p_m}{\rho_u + \rho_{\phi} + \rho_m} ,$$  \hspace{1cm} (23)

which, again, is related to the equation of state parameter $\gamma^{(c)}$ through $\gamma^{(c)} = 1 + \omega^{(c)}$. The condition for an accelerated universe is $\gamma^{(c)} < 2/3$. When $0 < \gamma^{(c)} < 2/3$, the universe is in quintessence phase while it is in phantom phase when $\gamma^{(c)} < 0$.

From Eq. (16) we obtain

$$\dot{\rho}_u + 3H(\rho_u + p_u) = 3H^2 \frac{2}{\bar{\beta}} .$$  \hspace{1cm} (24)

In order to preserve the conservation of total energy equation $\dot{\rho}_{tot} + 3H(\rho_{tot} + p_{tot}) = 0$, where $\rho_{tot} = \rho_u + \rho_{\phi} + \rho_m$ and $p_{tot} = p_u + p_{\phi} + p_m$ are the total energy density and the pressure, respectively, one can write the conservation of scalar field and matter field:

$$\dot{\rho}_{\phi} + 3H(\rho_{\phi} + p_{\phi}) = -3H^2 \frac{2}{\bar{\beta}} + Q_m ,$$  \hspace{1cm} (25)

$$\dot{\rho}_m + 3H(p_m + \rho_m) = Q_m .$$  \hspace{1cm} (26)

The interaction term can be interpreted as a transfer from one energy component to another energy component of the cosmic fluid. These interactions are completely associated with Lorentz violation. In our case, the scalar field decays into the matter field and the vector field. The conservation of scalar field, Eq. (25), is equivalent to a dynamical equation for the scalar field $\phi$,

$$Q_m = -\dot{\phi} \left( \dot{\phi} + 3H \dot{\phi} + V_{,\phi} + 3H^2 \frac{2}{\bar{\beta}} \dot{\phi} \right) .$$  \hspace{1cm} (27)

The above equation reduces to Refs. (11,28) for $Q_m = 0$. Equations (21), (22), and (27) represent the basic set of equations of the model of interacting components of the cosmic fluid in the framework of Lorentz violating scalar-vector-tensor theory of gravity. In what follows we shall apply a dynamical system to analyze the cosmological dynamics of this set of equations.

### 3 Interacting model

Some models that allow interaction between the scalar field and the matter field have been proposed as a solution to the cosmic coincidence problem. These models are compatible with observational data but so far there has been no evidence on the existence of this interaction. A solution will be achieved if the dynamical system presents scaling solutions which are characterized by a constant dark matter to dark energy ratio. Even more important are those scaling solutions that are also attractors and have the accelerated solution. In this way, the coincidence
problem gets substantially alleviated because, regardless of the initial conditions, the system evolves towards a final state where the ratio of dark matter to dark energy remains constant.

The explicit form of Eq. (13) can be expressed in the form

$$Q_\phi + Q_m = -\frac{\dot{\beta}}{\beta} (\rho_\phi + \rho_m). \quad (28)$$

We assume the interaction term as follows

$$Q_m = \frac{\dot{\beta}}{\beta} \rho_\phi = -\frac{\dot{\beta}}{\beta} \rho_m \dot{\phi}. \quad (29)$$

The interaction term (29) means that the scalar field can exchange energy with the background matter, through the interaction between them. In this case the exchange energy is mediated by the slope of the effective coupling.

Equations (28) and (29), respectively, become

$$\dot{\rho}_\phi + 3H (\rho_\phi + p_\phi) = -\frac{\dot{\beta}}{\beta} \rho_\phi \dot{\phi}, \quad (30)$$

$$\dot{\rho}_m + 3H (\rho_m + p_m) = -\frac{\dot{\beta}}{\beta} \rho_m \dot{\phi}. \quad (31)$$

For a more realistic model we assume that the matter fields might be a combination of dark matter, $\rho_c$, radiation, $\rho_r$, and baryons, $\rho_b$: $\rho_m = \rho_c + \rho_r + \rho_b$. We also assume that the barotropic equation of state for the radiation field $p_r = \rho_r/3$ and that the baryons are non-relativistic particles so that $p_b = 0$ holds. Hence, the equations for the energy densities of radiation and baryons are

$$\dot{\rho}_r + 4H \rho_r = 0, \quad \dot{\rho}_b + 3H \rho_b = 0, \quad (32)$$

respectively, and we find the well-known relationships: $\rho_r \propto a^{-4}$ and $\rho_b \propto a^{-3}$, $a$ is a scale factor. For the scalar field and the dark matter we have

$$\dot{\rho}_\phi + 3H \gamma^{(e)}_\phi \rho_\phi = 0, \quad \dot{\rho}_c + 3H \gamma^{(e)}_c \rho_c = 0, \quad (33)$$

where $\gamma^{(e)}_\phi$ and $\gamma^{(e)}_c$ are the effective barotropic equation of state for scalar field and dark matter, respectively,

$$\gamma^{(e)}_\phi = \gamma_\phi + \frac{\dot{\beta}}{3H \beta}, \quad \gamma^{(e)}_c = 1 + \frac{\dot{\beta}}{3H \beta} \left(1 + \frac{\rho_r + \rho_b}{\rho_c}\right) \quad (34)$$

Notice that for $\dot{\beta}/\beta < 0$ we have $\gamma^{(e)}_\phi < \gamma_\phi$, $\gamma^{(e)}_c < \gamma_c$ and both $\rho_\phi$ and $\rho_c$ with Lorentz violation will dilute slower than that without Lorentz violation or $\beta = $ const. Thus $\dot{\beta}/\beta$ will determine both the effective equations of state $\gamma^{(e)}_\phi$ and $\gamma^{(e)}_c$.

### 3.1 Dynamical analysis

In order to study the dynamics of the model, we shall introduce the following dimensionless variables $[14, 28]$:

$$x^2 \equiv \frac{\phi^2}{6\beta H^2}, \quad y^2 \equiv \frac{V}{3H^2 \beta}, \quad (35)$$

$$\lambda_1 = -\frac{\dot{\beta}}{\beta} \frac{\dot{\phi}}{\phi}, \quad \lambda_2 = -\sqrt{\frac{\beta}{V}}, \quad (36)$$

$$\Gamma_1 \equiv \frac{\beta \dot{\phi}}{\beta \phi}, \quad \Gamma_2 \equiv \frac{V \dot{\phi}}{V} + 1 + \frac{3\beta}{2 \dot{\phi}/\phi}, \quad (37)$$

and, accordingly, the governing equations of motion could be reexpressed as the following system of equations:

$$H' = -\frac{3}{2} H \left(1 + x^2 - y^2 + \frac{1}{3} z^2 - \sqrt{6} \lambda_1 x\right), \quad (38)$$

$$x' = -x \left(3 + \frac{H'}{H}\right) + \sqrt{\frac{3}{2}} (\lambda_1 + \lambda_2) y^2 + 2 \sqrt{\frac{3}{2}} \lambda_1 x^2, \quad (39)$$

$$y' = -y \left(\frac{H'}{H} - \sqrt{\frac{3}{2}} (\lambda_1 - \lambda_2) x\right), \quad (40)$$

$$z' = -z \left(2 + \frac{H'}{H} - \sqrt{\frac{3}{2}} \lambda_1 x\right), \quad (41)$$

$$u' = -u \left(\frac{3}{2} + \frac{H'}{H} - \sqrt{\frac{3}{2}} \lambda_1 x\right), \quad (42)$$

where

$$z = \sqrt{\frac{\rho_r}{3\beta H^2}}, \quad u = \sqrt{\frac{\rho_b}{3\beta H^2}}. \quad (43)$$

A prime denotes a derivative with respect to the natural logarithm of the scale factor, $d/d \ln a = H^{-1} d/dt$. Equation (21) gives the following constraint equation:

$$\Omega_\phi = \frac{\rho_\phi}{3\beta H^2} = 1 - x^2 - y^2 - z^2 - u^2, \quad (44)$$

where $\Omega_\phi = \rho_\phi/3\beta H^2 = x^2 + y^2$, $\Omega_r = \rho_r/3\beta H^2 = z^2$, and $\Omega_b = \rho_b/3\beta H^2 = u^2$. Notice that $\Omega_i$, $(i = \phi, c, r, b)$ are the effective cosmological density parameters which are associated with the Lorentz violation. In general, the parameters $\lambda_1$, $\lambda_2$, $\Gamma_1$ and $\Gamma_2$ are variables dependent on $\phi$ and completely associated with the Lorentz violation.

In order to construct viable Lorentz violation model, we require that the effective coupling $\beta$ and the potential function $V$ should satisfy the conditions $\Gamma_1 > 1/2$ and $\Gamma_2 > 1 - \lambda_1/2 \lambda_2$, respectively. In this paper, we want to discuss the phase space, then we need certain constraints on the effective coupling and potential function. Note that for $\beta_i = \text{const.}$, $\lambda_1 \rightarrow 0$, the scalar field dynamics in the Lorentz violating scalar-vector-tensor theories is then reduced to the scalar field dynamics in the conventional
we have $\lambda$ in which for the scalar field \([31]\). In this paper we consider the case of points. But, the effective gravitational constant is rescaled about the critical points into Eqs. (39)–(46). At the transition era from matter to radiation, dark matter and scalar field epochs because the fields are almost frozen into Eqs. (47) one finds $\gamma^{(e)} = \frac{1}{3}(\lambda_1 + \lambda_2)^2$, $\gamma^{(e)} = \frac{1}{3}(\lambda_1^2 - \lambda_2^2)$.

3.2 Attractor solutions

The critical points $(x_c, y_c, z_c, u_c)$ are obtained by imposing the conditions $x' = y' = z' = u' = 0$. Substituting linear perturbation $x \to x_c + \delta x$, $y \to y_c + \delta y$, $z \to z_c + \delta z$ and $u \to u_c + \delta u$ about the critical points into Eqs. (39)–(42), we obtain, up to first-order in the perturbation, the equation of motion

$$\frac{d}{d\alpha} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta u \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta u \end{pmatrix}. \quad (46)$$

Notice from (39)–(42) that the dynamical equations are invariant under the change of sign $(y, z, u) \to (-y, -z, -u)$, and in consequence we dont have to include the points with $(y, z, u) < 0$ in our analyzes. The properties of the critical points are summarized in Table 1. There are eight critical points at all and two of them lead to attractor solutions, depending on the values of the parameters $\lambda_1$ and $\lambda_2$. The scalar field dominated solution, point $B$ in Table 2 are characterized by $\Omega = 1$, and the effective equations of state are given by

$$\gamma^{(e)} = \frac{1}{3}(\lambda_1 + \lambda_2)^2, \quad \gamma^{(e)} = \frac{1}{3}(\lambda_1^2 - \lambda_2^2). \quad (47)$$

The solution of this point exists for $(\lambda_1 + \lambda_2)^2 < 6$ and the universe is accelerated for $\lambda_1^2 < \lambda_2^2 + 2$. From eq. (17) one can see that the de Sitter epoch corresponds to $\lambda_2 = \lambda_1$. The scalar field is dark energy when $\lambda_1^2 < 1/2$. In this case the effective coupling $\beta$ and the potential function are quadratic in $\phi$, $\beta(\phi) \sim V(\phi) \sim \phi^2$. The inflationary solution of this model has been studied in Ref. 11. Figure 1 shows that the sequence of radiation, dark matter and scalar field dark energy. The baryon is sub-dominant in this case. The parameters correspond to $\lambda_2 = \lambda_1$ and $\lambda_1 = -1/\sqrt{3}$. The scalar field equation of state parameter $\omega_\phi = \gamma^{(e)} - 1$ is nearly a constant, during the radiation and matter epochs because the fields are almost frozen for which $\omega_\phi = \omega_\phi^{(e)}$. At the transition era from matter domination to the scalar field dark energy domination, $\omega_\phi$ and $\omega_\phi^{(e)}$ begin to grow because the kinetic energies of the fields become important. However, the universe enters the de Sitter phase during which the field $\phi$ rolls up the potential. More interesting of this attractor solution is of the constant potential, $\lambda_2 = 0$. The universe is in phantom phase in this case because $\omega^{(e)}$ is crossing $-1$ and it is accelerated for $\lambda_1^2 > -2$. 

### Table 1. Properties of the critical points.

| Point | $(x, y, z, u)$ | $\Omega$ | $\gamma$ | $\gamma_{\text{eff}}$ |
|-------|---------------|---------|---------|------------------|
| $A_+$  | $(+1, 0, 0, 0)$ | 1       | 2       | $2 - 2\sqrt{\frac{2}{3}}\lambda_1$ |
| $A_-$  | $(-1, 0, 0, 0)$ | 1       | 2       | $2 + 2\sqrt{\frac{2}{3}}\lambda_1$ |
| $B$    | $(\frac{\lambda_1 + \lambda_2}{\sqrt{6}}, 0, 0, 0)$ | $1$     | $\frac{(-\lambda_1 + \lambda_2)^2}{3}$ | $-\frac{(\lambda_1^2 - \lambda_2^2)}{3}$ |
| $C_r$  | $\left(\frac{2}{3}, 0, 0, 0\right)$ | $\frac{2}{3}$ | $2$ | $\frac{2}{3}$ |
| $D_r$  | $(\frac{2}{3\sqrt{3}}, 0, 0, 0)$ | $\frac{3}{\lambda_1 + \lambda_2}$ | $1$ | $1 - \frac{2\lambda_1}{\lambda_1 + \lambda_2}$ |
| $E_r$  | $(0, 0, 0, 0)$ | $0$     | $-\frac{4}{3}$ | $1$ |

### Table 2. Stabilities and acceleration conditions of the critical points.

| Point | Existence | Stability | Acceleration |
|-------|-----------|-----------|--------------|
| $A_+$  | $\forall \lambda_1, \lambda_2$ | unstable | $\lambda_1 > \sqrt{\frac{2}{3}}$ |
| $A_-$  | $\forall \lambda_1, \lambda_2$ | unstable | $\lambda_1 < -\sqrt{\frac{2}{3}}$ |
| $B$    | $(\lambda_1 + \lambda_2)^2 < 6$ | stable | $\lambda_2 < \lambda_1^2 + 2$ |
| $C_r$  | $\lambda_2 > \frac{2}{3}$ | unstable | never |
| $D_r$  | $(\lambda_1 + \lambda_2)^2 > 3$ | stable | $\lambda_2 < 5\lambda_1$ |
| $E_r$  | $\lambda_2(\lambda_1 + \lambda_2) > 4$ | unstable | never |
| $E_b$  | $\forall \lambda_1, \lambda_2$ | unstable | never |

The scalar field equation of state parameter is given by

$$V(\phi) = V_0(\beta(\phi))^s, \quad (45)$$

where $s = \lambda_2/\lambda_1$ is a constant parameter. In general, one can write the potential as a function of effective coupling, $V(\phi) \equiv f(\beta(\phi))$. 

$$\gamma^{(e)} = \frac{1}{3}(\lambda_1 + \lambda_2)^2, \quad \gamma^{(e)} = -\frac{1}{3}(\lambda_1^2 - \lambda_2^2). \quad (47)$$
Fig. 1. Evolution of the density parameters and the equation of state parameters as a function of \( \ln a \). Top panel corresponds to the case of \( \lambda_2 = \lambda_1 = -1/\sqrt{3} \) while the bottom panel corresponds to the cases of constant potential and \( \lambda_1 = -1 \).

Fig. 2. Evolution of the density parameters and the equation of state parameters as a function of \( \ln a \). Top panel corresponds to the case of \( \lambda_2 = \lambda_1 = -3/\sqrt{2} \) while the bottom panel corresponds to the cases of constant potential and \( \lambda_1 = 3/2\sqrt{2} \).

The second attractor solution is the scalar field scaling solution, point \( D \) in Table 2. The solution of this point exists for \((\lambda_1 + \lambda_2)^2 > 3\), corresponding to energy density parameter \( \Omega_\phi = 3/(\lambda_1 + \lambda_2)^2 \). The effective equations of state are given by

\[
\gamma_\phi = \gamma_m = 1, \quad \gamma_\phi^{(e)} = \gamma_m^{(e)} = \frac{\lambda_2}{(\lambda_1 + \lambda_2)^2}, \quad (48)
\]

\[
\gamma^{(e)} = 1 - \frac{2\lambda_1}{\lambda_1 + \lambda_2}. \quad (49)
\]

The universe is accelerated for \( \lambda_2 < 5\lambda_1 \). In the case of the effective coupling \( \bar{\beta} \) and the potential function are quadratic in \( \phi \), i.e. \( \lambda_2 = \lambda_1 \), the universe is always accelerated. For the constant potential, \( \lambda_2 = 0 \), the scalar field behaves as a cosmological constant while the universe is in phantom phase. Figure 2 shows the sequence of radiation, dark matter and scalar field dark energy. The baryon is sub dominant in this case. The parameters correspond to \( \lambda_2 = \lambda_1 = -3/\sqrt{2} \) (top panel), and \( \lambda_1 = 3/2\sqrt{2} \) (bottom panel).


4 A comparison of the model using supernova data

From the above analysis, we may investigate the cosmological consequences of a Lorentz violating scalar-vector-tensor theory which incorporates time variations in the gravitational constant. It was raised by Dirac who introduced the large number hypothesis \(^{32}\), and has recently become a subject of intensive experimental and theoretical studies \(^{33}\). The effective gravitational constant, \(G^{(e)}\), is obtained from the Friedmann equation,

\[
G^{(e)} = \frac{1}{8\pi\beta} = \frac{G}{1 + 8\pi G\beta},
\]

(50)

where \(G\) is the parameter in the action \(^{1}\). Therefore the time variation of \(G^{(e)}\) can be written as

\[
\dot{G}^{(e)} - \frac{\dot{\beta}}{\beta} = 3\lambda_1 (\lambda_1 + \lambda_2) H,
\]

(51)

and the effective gravitational constant is determined by the effective coupling \(\beta\). For the quadratic effective coupling, \(\beta \propto \phi^2\), the effective gravitational constant is inversely proportional to \(\phi^2\), \(G^{(e)} \propto [\phi(t)]^{-2}\). Recently using the data provided by the pulsating white dwarf star G117-B15A the asteroseismological bound on \(\dot{G}/G\) is found \(^{34}\) to be \(-2.5 \times 10^{-10} \text{ yr}^{-1} < \dot{G}/G < 4.0 \times 10^{-10} \text{ yr}^{-1}\). In the present model the time variation in the gravitational constant is given by

\[
\dot{G}^{(e)} = \frac{3\lambda_1}{\lambda_1 + \lambda_2} H,
\]

(52)

in the scaling solution and

\[
\dot{G}^{(e)} = \lambda_1 (\lambda_1 + \lambda_2) H,
\]

(53)

in the scalar field dominated solution, where the evolution of the Hubble parameter is given by Eq. \(^{58}\). For instance, in the case of power law expansion of the universe \(a(t) \propto t^p\) with \(p > 0\), the time variation of \(G^{(e)}\) leads to

\[
\dot{G}^{(e)} \propto \frac{3\lambda_1}{\lambda_1 + \lambda_2} t^{-1},
\]

(54)

in the scaling solution. Assuming the present age of the Universe as 14 Gyr, it is straightforward to derive from Eq. \(^{54}\) the estimate \(G^{(e)}/G^{(c)} \sim 2.14 \times 10^{-10} \text{ yr}^{-1}\) for the case of constant potential. Our model also allows the negative value of \(G^{(e)}/G^{(c)}\). Let us focus on the scaling solution. If \(\Omega_\phi = 2/3\) we find

\[
\dot{G}^{(e)} = \pm\sqrt{2}\lambda_1 H.
\]

(55)

A negative \(G^{(e)}/G^{(c)}\) implies a time-decreasing \(G^{(c)}\), while a positive \(G^{(e)}/G^{(c)}\) means \(G^{(c)}\) is growing with time. From Eq. \(^{55}\), it is clear that the Lorentz violation leads to time variation of the gravitational constant.

In the following, we study the expansion history of the universe using the 194 SnIa data \(^{35,36}\). We simplify our model by considering an interaction between dark matter and the scalar field dark energy given by Eqs. \(^{30}\) and \(^{31}\). The evolution of the dark matter and scalar field dark energy are given by

\[
\rho_i(z) = \rho_0 e^{3\int_0^z \frac{1 + \omega_i(z)}{1 + \omega_\phi(z)} dz'}, \quad (i = m, \phi),
\]

(56)

where \(z = 1/a - 1\) is the redshift. Using the above relation, the Hubble parameter as a function of the redshift can be written as

\[
H^2(z) = \left(\frac{H_0 \beta_0}{\beta(z)}\right)^2 \left[\Omega_m(1 + z)^3 + (1 - \Omega_m)(1 + z)^{3(1 + \omega_\phi(z))}\right],
\]

(57)

where \(\zeta\) is a constant.

Let us first consider the modified \(Λ\) Cold Dark Matter (ΛCDM) model. We have

\[
H^2(z; \zeta, \Omega_m) = \left(\frac{H_0}{1 + \zeta z^2}\right)^2 \left[\Omega_m(1 + z)^3 + (1 - \Omega_m)\right],
\]

(59)

Equation \(^{59}\) has two free parameters \(\zeta\) and \(\Omega_m\) which are determined by minimizing

\[
\chi^2 = \sum_i \left[\frac{\mu_{\text{obs}}(z_i) - \mu(z_i)}{\sigma_i}\right]^2,
\]

(60)

where \(\mu\) is the extinction-corrected distance modulus,

\[
\mu(z) = 5 \log_{10} \left(\frac{d_L(z)}{1 Mpc}\right) + 25,
\]

(61)

and \(\sigma_i\) is the total uncertainty in the SnIa data. The luminosity distance is given by

\[
d_L(z) = \frac{c(1 + z)}{H_0} \int_0^z \frac{dz'}{H(z')},
\]

(62)

Fitting the model to 194 SnIa data, we get \(\chi^2_{\text{min}} = 195.68\), \(\zeta = -0.33\), and \(\Omega_m = 0.24\). For comparison, we also fit the cosmological constant model to the 194 SnIa data and find \(\chi^2 = 198.74\), and \(\Omega_m = 0.34\).

In the next model we replace the cosmological constant energy density by a scalar field dark energy with constant equation of state parameter \((\omega_\phi(z) = \text{constant})\). We set
here \( \Omega_{m0} = 0.3 \). We evaluate \( \chi^2(\zeta, \omega_\phi) \) and minimize with respect to \( \zeta \) and \( \omega_\phi \). We find
\[
\chi^2_{\text{min}} = \chi^2(\zeta = -0.29, \omega_\phi = -1.13) = 195.71 . \tag{63}
\]

Figure 3 shows a comparison of the observed 194 SNIa Hubble free luminosity distances along the predicted curves in the context of Lorentz violating scalar-vector-tensor theory. We see that the effect of Lorentz violation appears at \( z > 0.75 \). We define the reduced form of Hubble parameter compared to the standard case as
\[
H^2_{\text{red}} = \frac{H^2_{LV} - H^2_{std}}{H^2_{std}} , \tag{64}
\]
where
\[
H^2_{std}(z) = H^2_0 \left[ \Omega_{m0}(1+z)^3 + (1 - \Omega_{m0})(1+z)^{3(1+\omega_\phi(z))} \right] . \tag{65}
\]
Thus the reduced form of Hubble parameter, due to the effect of Lorentz violation, is
\[
H^2_{\text{red}}(z) = \left( \frac{\beta}{\beta(z)} \right)^2 - 1 . \tag{66}
\]

5 Parameterized Post-Newtonian

In order to confront the predictions of a given gravity theory with experiment in the solar system, it is necessary to compute its PPN parameters. The post-Newtonian approximation is based on the assumptions of weak gravitational fields and slow motions. It provides a way to estimate general relativistic effects in the fully nonlinear evolution stage of the large scale cosmic structures. The procedure of parameterizing our model is following to that of Ref [37], in which the authors has derived the PPN parameters in the frame of Einstein aether theory.

In the weak field approximation, we choose a system of coordinates in which the metric can be perturbatively expanded around Minkowski spacetime. The decomposition is as follows
\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \tag{67}
\]
where \( \eta_{\mu\nu} \) is the Minkowski metric and \( h_{\mu\nu} \) is the metric perturbations and we take \( |h_{\mu\nu}| << 1 \).

The equations governing the perturbation \( h_{\mu\nu} \) in the model Eq. 11 are found by computing the Einstein field equations in the perturbative limit. The full field equations are given by
\[
R_{\mu\nu} = 8\pi G \left( T^{(m)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(u)}_{\mu\nu} \right) \left( \frac{\delta^{\alpha}_\beta \delta^{\beta}_\nu}{2} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \right) \tag{68}
\]

We allow for an arbitrary coupling between the matter fields and the scalar field \( \phi \). We assume that the scalar field \( \phi \) is coupled to a barotropic perfect fluid with a coupling function given by
\[
q(\phi) = -\frac{1}{\rho_m \sqrt{-g}} \frac{\delta S_m}{\delta \phi} , \tag{69}
\]
Therefore, the equation of motion for the scalar field is
\[
\Box \phi - \frac{dV}{d\phi} - \sum_{i=1}^{3} \frac{d\beta_i}{d\phi} K_i = q(\phi) \rho_m , \tag{70}
\]
where
\[
K_1 = \nabla^\mu u^\nu \nabla_\mu u_\nu , \quad K_2 = (\nabla^\mu u^\mu)^2 , \quad K_3 = \nabla^\mu u^\nu \nabla_\mu u_\nu \tag{71}
\]
In the previous discussion we have considered the coupling between the scalar field and the matter field is given by the effective coupling, \( q(\phi) = d\ln \beta / d\phi \), where \( \beta \) is defined by Eq. 20.

We also have the vector field equation,
\[
\nabla_\mu J^\mu_\nu = \lambda u_\nu \tag{72}
\]
where \( J^\mu_\nu \) is given by Eq. 10. The constraint for the vector field is
\[
g_{\mu\nu} u^\mu u^\nu + 1 = 0 . \tag{73}
\]

The vector field is purely timelike at the zeroth order and the fluid variables are assigned orders of \( \rho \sim \Pi \sim p / \rho \sim (v^i)^2 \sim \mathcal{O}(1) \). The scalar field is expanded as \( \phi = \phi_0 + \mathcal{O}(1) \), where \( \phi_0 \) is determined by the cosmological solution.
Then, the metric perturbations $h_{\mu\nu}$ will be of orders $h_{00} \sim O(1) + O(2)$, $h_{ij} \sim O(1)$, and $h_{0i} \sim O(1.5)$.

The general form of the first post Newtonian metric is given by [30]

$$ g_{00} = -1 + 2U - 2\beta_{PPN} U^2 - 2\xi \Phi_W - (\zeta_1 - 2\xi) A $$
$$ + (2\gamma_{PPN} + 2 + \alpha_3 + \zeta_1 - 2\xi) \Phi_1 $$
$$ + 2(3\gamma_{PPN} - 2\beta_{PPN} + 1 + \zeta_2 + \xi) \Phi_2 $$
$$ + 2(1 + \zeta_3) \Phi_3 + 2(3\gamma_{PPN} + 3\zeta_4 - 2\xi) \Phi_4 $$

$$ g_{ij} = (1 + 2\gamma_{PPN} U) \delta_{ij} $$
$$ g_{0i} = -\frac{1}{2} (4\gamma_{PPN} + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi) V_i $$
$$ - \frac{1}{2} (1 + \alpha_2 - \zeta_1 - 2\xi) W_i $$

(74)

The PPN potentials $(U, \Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, A, V_i, W_i)$ are defined by

$$ F(x) = G_N \int d^3y \frac{\rho(y) f}{|x-y|} $$

(75)

where the correspondences $F : f$ are given by

$$ U : 1, \quad \Phi_1 : v_i v_i, \quad \Phi_2 : U, \quad \Phi_3 : \Pi, \quad \Phi_4 : \frac{P}{\rho_m} $$
$$ \Phi_W : \int d^3z \rho_m(z) \left( \frac{x-y)_i}{|x-y|^2} \right) \left( \frac{y-z)_j}{|y-z|^2} \right), \quad V_i : v_i $$
$$ A : \left( \frac{v_i(x-y)_i}{|x-y|^2} \right), \quad W_i : \left( \frac{v_j(x-y)_j}{|x-y|^2} \right) $$

(76)

These potentials satisfy the following relations

$$ F_{,ii} = -4\pi G_N \rho_m f $$

(77)

for $U, \Phi_{1,2,3,4},$ and $V_i$. The superpotential $\chi$ is defined by

$$ \chi = -G_N \int d^3 y \rho_m |x-y| $$

(78)

which satisfies

$$ \chi_{,ii} = -2U $$

(79)

We also note the identity

$$ \chi_{,0i} = V_i - W_i $$

(80)

The PPN metric Eq. (74) contains ten parameters $(\gamma_{PPN}, \beta_{PPN}, \xi, \alpha_{1,2,3}, \zeta_{1,2,3,4})$. The parameter $\gamma_{PPN}$ measures how much space-curvature is produced by a unit rest mass, while the parameter $\beta_{PPN}$ determines how much non-linearity is there in the superposition law of gravity. On the other hand, the parameter $\xi$ determines whether there are preferred-location effects, while $\alpha_{1,2,3}$ represent preferred-frame effects. Finally, the parameters $\zeta_{1,2,3,4}$ measure the amount of violation of conservation of total momentum. In terms of conservation laws, one can interpret these parameters as a measure whether a theory is fully conservative, i.e. the linear and angular momenta are conserved $(\zeta_{1,2,3,4}$ and $\alpha_{1,2,3}$ vanish), semi-conservative, i.e. the linear momentum is conserved $(\zeta_{1,2,3,4}$ and $\alpha_{1,2,3}$ vanish), or non-conservative, where only the energy is conserved through lowest Newtonian order. One can verify that in general relativity $\gamma_{PPN} = \beta_{PPN} = 1$ and all other parameters vanish, which implies that there are no preferred-location or frame effects and that the theory is fully conservative.

The PPN parameters are determined as follows: expand the modified field equations in the metric perturbation and in the PN approximation; iteratively solve for the metric perturbation to $O(2)$ in $h_{00}$, to $O(1.5)$ in $h_{0i}$ and to $O(1)$ in $h_{ij}$; compare the solution to the PPN metric of Eq. (74) and read off the PPN parameters of the theory.

The gauge condition we use for the metric is as follows [37]

$$ h_{ij, j} = \frac{1}{2} (h_{jj, i} - h_{00, i}) $$
$$ h_{0i, i} = 3U_{,0} + Bu_i $$

(81)

(82)

where $B$ is a function of $\beta_1$ which will be determined below.

There are two important notes to be considered here since we concern in the post-Newtonian expansion. Firstly, the term $V(\phi_0)$ is the same order as the energy density of the cosmological constant, therefore, these term cannot leads to any observable deviations at Solar system scales. Practically, we can assume $dV(\phi)/d\phi = 0$ as far as the post-Newtonian expansion is concerned. It means that the cosmological solution corresponds to a minimum of the potential and the scalar field will satisfy the extremum condition. Secondly, from the analysis of the tensor perturbations, the velocity of the gravitational waves are different from the velocity of light [11]. In order to have the real velocity, the coupling functions $\beta_1$ and $\beta_3$ must be satisfy $(\beta_1 + \beta_3) < (16\pi G)^{-1}$. Therefore, one can assume that $\beta_1$ and $\beta_3$ are constant. This assumption does not affect our previous calculations because the coupling functions $\beta_1$ and $\beta_3$ have included in the definition of the effective coupling, which is given by Eq. (20). These two assumptions will be used in the following calculations.

### 5.1 Solving $u^\mu$, $\phi$, and $g_{\mu\nu}$ to $O(1)$

We first solve the constraint equation to $O(1)$. We find

$$ u^0 = 1 + \frac{1}{2} h_{00} $$

(83)

From this result, we have

$$ u_0 = -1 + \frac{1}{2} h_{00}, \quad u_i = u^i + h_{0i} $$

(84)

where we used $u_\mu = g_{\mu\nu} u^\nu$.

From Eq. (70), we obtain

$$ \phi_{,ii}^{(1)} - \mu_0 G_m = 0 $$

(85)

Using $U_{,ii} = -4\pi G_N \rho_m$, we have

$$ \phi_{,ii}^{(1)} = -\frac{\mu_0}{4\pi G_N} U_{,ii} $$

(86)
The solution of this equation is
\[ \phi^{(1)} = - \frac{q_0}{4\pi G_N} U = - \frac{\ln' \beta_0}{4\pi G_N} U , \] (87)

where \( q_0 = q(\phi_0) \) and \( \ln' \beta_0 = d \ln \beta(\phi_0) / d\phi \).

The general expression of the covariant derivatives of \( u_{\mu} \) is
\[ \nabla_{\mu} u^\nu = u_{\mu'}^\nu + \frac{\nu^0}{2} g^{\nu\alpha} \left( h_{0\alpha,\mu} + h_{\alpha\mu,0} - h_{0\mu,\alpha} \right) , \] (88)

To \( \mathcal{O}(2) \), we find
\[ \nabla_{\mu} u^0 = 0 , \quad \nabla_0 u^i = u_{i,0} - \frac{1}{2} h_{00,i} + \frac{1}{4} h_{00} h_{00,i} , \] (89)

and
\[ \nabla_i (\nabla_0 u^i) = u_{i,0i} - \frac{1}{2} h_{00,ii} + \frac{1}{4} h_{00} h_{00,ii} - \frac{3}{4} h_{00,i} h_{00,i} \] (90)

We also have
\[ \nabla_i u^j = u_{j,i} + \frac{1}{2} (h_{ji,0} - h_{ij,i} - h_{0i,j}) , \] (91)

to \( \mathcal{O}(1.5) \).

Let us now proceed with the solution to the field equation for the "time-time" component of the metric perturbation. The left hand side of Eq. \( \text{(69)} \) is
\[ R_{00} = - \frac{1}{2} h_{00,ii} + \frac{1}{2} h_{ij} h_{00,ij} + \left( h_{00,ij} - \frac{1}{2} h_{ii,0} \right) , \] (92)

with \( \nabla_i u^i = 0 \) and \( \nabla_0 u^0 = 0 \).

To \( \mathcal{O}(1) \), we have the components of energy-momentum tensor
\[ T_{00}^{(m)} = \rho_m , \quad T_{ij}^{(m)} = 0 , \quad T_{00}^{(\phi)} = 0 = T_{ij}^{(\phi)} , \] (94)

and
\[ T_{00}^{(u)} = -2 \nabla_0 J_0 - 2 \nabla_\mu J_0^\mu = -2 \nabla_i J_0^i , \]
\[ = 2 (\beta_1 \nabla_0 u^0) = -\beta_{10} h_{00,00} , \]
\[ T_{ij}^{(u)} = 0 , \] (95)

here \( \beta_{10} = \beta_1 (\phi_0) \), \( i, j = 1, 2, 3 \).

Then, we obtain
\[ (1 - 8\pi G \beta_{10}) h_{00,00} = -8\pi G \rho_m , \] (96)

which gives \( h_{00} \) to \( \mathcal{O}(1) \),
\[ h_{00} = 2U , \] (97)

with Newton’s constant
\[ G_N = \frac{G}{(1 - 8\pi G \beta_{10})} . \] (98)

To \( \mathcal{O}(1) \) the "space-space" component of left hand side Eq. \( \text{(69)} \) is
\[ R_{ij} = -\frac{1}{2} h_{ij,kk} - \frac{1}{2} h_{kk,ij} + \frac{1}{2} h_{00,ij} , \] (99)

If we use the gauge \( \text{(81)} \), we obtain
\[ R_{ij} = -\frac{1}{2} h_{ij,kk} . \] (100)

This expression similar to the "time-time" component. It means that the spatial metric perturbation to \( \mathcal{O}(1) \) is then simply given by the GR prediction without correction, namely
\[ h_{ij} = h_{00} \delta_{ij} , \] (101)

where \( h_{00} \) is given by Eq. \( \text{(68)} \).

5.2 Solving \( u^i, g_{0i} \) to \( \mathcal{O}(1.5) \)

From the previous results one can see that the Lagrange multiplier is \( \lambda \sim \mathcal{O}(1) \). Therefore, the vector field equation yields \( \nabla_{\mu} J_{\mu i} = 0 \). This means
\[ - J_{0i,0} + J_{ji,j} = 0 , \] (102)

to \( \mathcal{O}(1.5) \), where
\[ J_{0i,0} = - (\beta_1 \nabla_0 u_i) , \] (103)
\[ J_{ji,j} = - (\beta_1 \nabla_j u_i) - (\beta_2 \nabla k u_k)_i - (\beta_3 \nabla j u_i)_j . \] (104)

We notice that \( \nabla_0 u_i = -\frac{1}{2} h_{00,i} \) to \( \mathcal{O}(1) \). To \( \mathcal{O}(1.5) \), we have
\[ J_{0i,0} = \frac{1}{2} \beta_{10} h_{00,00} , \] (105)

and
\[ J_{ji,j} = -\beta_{10} u_{ij} - (\beta_{10} + \beta_{30}) u_{ij} - (\beta_{10} - \beta_{30}) h_{00,0i} , \] (106)

Then, the vector field equation can be written
\[ \beta_{10} u_{ij} + (\beta_{20} + \beta_{30}) u_{ij} + (\beta_{10} - \beta_{30}) h_{00,0i} = 0 . \] (107)

By taking the spatial divergence of this equation, we can solve for \( u^i \),
\[ u^i = C_0 \partial_{0i} . \] (108)
where
\[ C_0 = \frac{(2\beta_{10} + 3\beta_{20} + \beta_{30})}{2(\beta_{10} + \beta_{20} + \beta_{30})}. \] (111)

Using Eq. (111), the gauge [22] can be rewritten
\[ h_{0i,i} = -\frac{1}{2}(3 - 2B_0C_0)\chi_{,0i}. \] (112)

Substituting Eqs. (111) and (112) into Eq. (110), and using the previous results, we can solve Eq. (110) for \( u^i \),
\[
    u^i = -\frac{(\beta_{10} - \beta_{10})}{2\beta_{10}}h_{0i} + \left[ C_0 - \frac{(\beta_{10} - \beta_{10})(3 - 2B_0C_0)}{4\beta_{10}} \right] \chi_{,0i}. \] (113)

Let us now look for solutions to the field equations for the metric perturbation \( g_{0i} \) to \( \mathcal{O}(1.5) \). To \( \mathcal{O}(1.5) \), we have the "time-space" components of left hand side Eq. (69),
\[
    R_{0i} = \frac{1}{2}h_{0i,jj} + \frac{1}{4}(1 + 2BC_0), \] (114)

and the "time-space" components of \( T_{\mu\nu} \),
\[
    T_{0i}^{(m)} = -\rho_m v_i, \quad T_{0i}^{(\phi)} = 0, \] (115)

and
\[
    T_{0i}^{(u)} = -J_{0i,0} - J_{ij,i}, \] (116)

where \( J_{0i,0} \) is given by Eq. (107) and
\[
    J_{ij,i} = -(\beta_{10} + \beta_{20})u_{ij,i} - \beta_{30}u_{ij,i} - (\beta_{10} - \beta_{30})h_{0i,jj,i} - \frac{1}{2}(\beta_{10} + \beta_{30})h_{ij,0j,0i} - \frac{1}{2}\beta_{20}h_{ij,0i}. \] (117)

Using Eqs. (112) and (113), and our previous results, we obtain
\[
    T_{0i}^{(u)} = \frac{(\beta_{10} - \beta_{10})}{2\beta_{10}}h_{0i,jj} - \left[ \beta_{10} - \frac{(\beta_{10} - \beta_{30})(3 - 2B_0C_0)}{4\beta_{10}} \right] \chi_{,0i,jj}. \] (118)

Here, we have assumed that \( B \) is a constant parameter, \( B \rightarrow B_0 \), which will be determined below.

By solving the "time-space" components of the field equation, we obtain
\[
    \left[ 1 - \frac{8\pi G(\beta_{10} - \beta_{30})}{\beta_{10}} \right] h_{0i,jj} = 16\pi G\rho v_i
    - \left( 16\pi GE_0 - B_0C_0 - \frac{1}{2} \chi_{,0i,jj} \right), \] (119)

where
\[ E_0 = \beta_{10} - \frac{(\beta_{10} - \beta_{30})(3 - 2B_0C_0)}{4\beta_{10}}. \] (120)

We can thus solve Eq. (119) for \( h_{0i} \),
\[
    h_{0i} = -\left[ 1 - \frac{8\pi G(\beta_{10} - \beta_{30})}{\beta_{10}} \right] \times
        \left\{ \left[ 4(1 - 8\pi G\beta_{10}) + 16\pi GE_0 - B_0C_0 - \frac{1}{2} \right] V_i
        - \left( 16\pi GE_0 - B_0C_0 - \frac{1}{2} \right) W_i \right\}, \] (121)

where we have used that the superpotential \( \chi \) satisfies \( \chi_{,0i} = V_i - W_i \).

5.3 Solving \( g_{00} \) to \( \mathcal{O}(2) \)

A full analysis of the PPN parameters requires that we solve for the "time-time" component of Eq. (69) to \( \mathcal{O}(2) \). To \( \mathcal{O}(2) \) we have
\[
    R_{00} = -\frac{1}{2} \left( h_{00} + 2U + 2U^2 - 8\Phi_2 - 2B_0C_0 \chi_{,00} \right)_{,ii}, \] (122)

where we have defined \( h_{00} = g_{00} + 1 - 2U \). To \( \mathcal{O}(2) \), we also have
\[
    T_{00}^{(m)} = \rho_m(1 + \Pi + v_i v_i - 2U), \quad T_{ii}^{(m)} = \rho_m v_i v_i + 3p_m, \]
\[
    T_{00}^{(\phi)} = \frac{1}{2}(\phi_{,i})^2, \quad T_{ii}^{(\phi)} = -\frac{1}{2}(\phi_{,i})^2, \]
\[
    T_{00}^{(u)} = \beta_1(\nabla_0 u_i)^2 + 2u_0 \nabla_i J_0, \quad T_{ii}^{(u)} = \beta_1(\nabla_0 u_i)^2 - 2\nabla_i J_0. \] (123)

Using the covariant derivative Eqs. (89) and (90), the components of energy-momentum tensor for the vector field become
\[
    T_{00}^{(u)} = -\beta_{10} \left[ h_{00} + 2U + \frac{5}{2} U^2 - 9\Phi_2 \right], \] (124)

and
\[
    T_{ii}^{(u)} = -\beta_{10} \left( \frac{1}{2} U^2 - \Phi_2 \right)_{,ii} \]
\[-\beta_{10}(3 - 2C_0(1 + B_0))\chi_{,00ii}. \] (125)

From Eq. (124), one can evaluate the "time-time" components of the right hand side of Eq. (69),
\[
    T_{00}^{(m)} - \frac{1}{2} g_{00} \rho_m T^{(m)}_{\mu\nu} = \frac{1}{2}(T_{00}^{(m)} + T_{ii}^{(m)})
    = -\frac{1}{2} \frac{8\pi G}{\beta_{10}} (U + 2\Phi_1 - 2\Phi_2 + 3\Phi_4), \] (126)

\[
    T_{00}^{(\phi)} - \frac{1}{2} g_{00} \rho_m T^{(\phi)}_{\mu\nu} = \frac{1}{2}(T_{00}^{(\phi)} + T_{ii}^{(\phi)}) = 0, \] (127)

and
\[
    T_{00}^{(u)} - \frac{1}{2} g_{00} \rho_m T^{(u)}_{\mu\nu} = \frac{1}{2}(T_{00}^{(u)} + T_{ii}^{(u)})
    = -\beta_{10} \left[ h_{00} + 2U + 2U^2 - 8\Phi_2 \right]_{,ii} + \frac{1}{2} 2B_0C_0
    -(2\beta_{10} + 3\beta_2 + \beta_3)(3 - 2C_0)|\chi_{,00ii}|. \] (128)
Using the "time-time" component of Eq. (60) to $O(2)$ and combining Eqs. (122), (126), (127) and (128), we can obtain

\[
\hat{h}_{00} = -2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 + Q_0\gamma_{00},
\]

where

\[
Q_0 = \frac{16\pi G C_0}{1 - 8\pi G \beta_{10}} \left( \beta_{10} + 2\beta_{30} + \frac{1 - 8\pi G \beta_{10}}{8\pi G} B_0 \right)
\]

The last term of Eq. (129) is a new PPN parameter. However, there is no need to introduce any additional PPN parameters when we move into the standard gauge by choosing $B_0$ such that $Q_0 = 0$. We have

\[
B_0 = -\frac{8\pi G (\beta_{10} + 2\beta_{30})}{1 - 8\pi G \beta_{10}}.
\]

We have all the necessary ingredients to read off the PPN parameters. Let us begin by writing the full metric, we have

\[
g_{00} = -1 + 2U - 2U^2 + 4\Phi_2 + 4\Phi_1 + 2\Phi_3 + 6\Phi_4,
\]

\[
g_{ij} = (1 + 2U)\delta_{ij},
\]

\[
g_{0i} = \frac{\beta_{10}}{\beta_{10} - 8\pi G (\beta_{10}^2 - \beta_{30}^2)} \times
\]

\[
\left\{ \left[ 4(1 - 8\pi G \beta_{10}) + 16\pi G E_0 - B_0 C_0 - \frac{1}{2} \right] V_i - \left( 16\pi G E_0 - B_0 C_0 - \frac{1}{2} \right) W_i \right\}.
\]

By comparing Eqs. (74) and (132), the PPN parameters are given by

\[
\gamma_{PPN} = 1, \quad \beta_{PPN} = 1, \quad \chi = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0,
\]

\[
\frac{1}{8\pi G} \alpha_1 = -\frac{8\beta_{30}^2}{\beta_{10}(1 - 8\pi G \beta_{10}) + 8\pi G \beta_{30}^2},
\]

\[
\frac{1}{8\pi G} \alpha_2 = \frac{\beta_{10} + 2\beta_{30}^2}{(1 - 8\pi G \beta_{10})(\beta_{10} + \beta_{30} + \beta_{30})}
\]

\[
- \frac{6\beta_{10}\beta_{30} + 3\beta_{30}^2[1 + 8\pi G(\beta_{10} + 2\beta_{30})]}{(1 - 8\pi G \beta_{10})(1 - 8\pi G \beta_{10} + 8\pi G \beta_{30}^2)}
\]

\[
\frac{2\beta_{30}^2[1 + 2\pi G(\beta_{10} - 6\beta_{30})]}{(1 - 8\pi G \beta_{10})(1 - 8\pi G \beta_{10} + 8\pi G \beta_{30}^2)}.
\]

\[
(133)
\]

Notice that the values of $\gamma_{PPN}$ and $\beta_{PPN}$ are the same as in GR. $\gamma_{PPN} = 1$ implies an identical predicted deflection of light about the Sun as well as identical predictions for radar echo delay. The parameter $\beta_{PPN}$ enters into the expression for anomalous relativistic precession of planetary orbits, but the strongest experimental limit is provided by the lunar laser ranging test. The other non-vanishing ones are $\alpha_1$ and $\alpha_2$ which are the effect of the preferred frame.

\section{6 Conclusions}

In this paper, we have investigated the cosmological evolution of an interacting scalar field model in which the scalar field has an interaction with the background matter via Lorentz violation. We propose a model of interaction, specifically $Q_m = -\dot{\beta} \rho_m / \beta$ in which the interaction is mediated by the slope of the effective coupling $\beta$. The equation of state parameter of the scalar field is expressed by Eq. (54) as a candidate of dark energy. The important role of the model is played by the effective coupling in the transition era from the matter dominated to scalar field dominated, which leads to an accelerating universe. The model also predicts a constant fraction of dark energy to dark matter in the future and hence solves the coincidence problem. This is a profitable support to the effective coupling. As a cosmological implication, the dynamic of the effective gravitational constant is determined by the effective coupling and allows one to test the Lorentz violating scalar-vector-tensor theory of gravity using the SNe Ia data. We have studied how a varying $G$ or a effective coupling could modify the evolution of the Hubble parameter which is deviated by the term of $\beta^{-2}$. For a simple polynomial $\beta(z) = \beta_0(1 + \zeta z^2)$ ansatz, the best fit values are $\chi_{min}^2 = 195.68$, $\zeta = -0.33$, and $\Omega_{mn} = 0.24$ for the modified $\Lambda$CDM model and $\chi_{min}^2 = 195.71$, $\zeta = -0.29$, and $\omega_0 = -1.13$ for the modified quintessence model.

We also have presented the 1PPN parameters of the theory. Our result strongly depends on the two assumptions: the scalar field satisfies the extremum condition, and the coupling functions $\beta_1$ and $\beta_2$ are constant. The first assumption stems from the fact that the potential will play the role of an effective cosmological constant if the theory is to account for the late time accelerated expansion of the universe. The second one is to obtain the real velocity of the gravitational waves. Up to 1PPN approximation, important to note that it is impossible to obtain $G_N = G_{\cosmo}$, although $\beta_1$ and $\beta_2$ are constant. Here $G_{\cosmo} = G^{(e)}$ is given by Eq. (50). $G_{\cosmo}$ is still dynamic in this case because of $\beta_2$, while $G_N$ is always constant (99). In the post-Newtonian approximation, the effect of non-constant coupling function $\beta_2$ appears in $O(2.5)$ or higher.

Of course, there are many remaining works to make this scenario more concrete which is beyond the main aim of the present work. For instance, non-linear coupling or more complicate functions are also possible. In the present work we have considered to the case of instantaneous critical points, where $\lambda_1$ and $\lambda_2$ are the constant parameters. However, in order to construct viable model, $\lambda_1$ and $\lambda_2$ should satisfy the condition Eq. (47). We need to solve the dynamical system Eq. (49). Then, $\lambda_1$ and $\lambda_2$ are dynamically changing quantities. For example, the critical point $D$ in Table 1 becomes $x(N) = \sqrt{3/2}/(\lambda_1(N) + \lambda_2(N))$, $y(N) = x(N)$, $z(N) = 0$, and $u(N) = 0$ where $N = \ln a$. As a consequence we obtain the running critical point according to the changing of $\lambda_1(N)$ and $\lambda_2(N)$.

Another important aspect is that the coupling interaction between the scalar field and matter fields affects only
the solution for the scalar field, \( \phi^{(1)} \) represents the local deviation from \( \phi_0 \), which vanishes far from the local system. In conclusion, one can reasonably state that the our gravity model could be a viable candidate theory, even in the PPN approximation. It cannot be a priori excluded at Solar system scales.

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References

1. V. A. Kostelecky and S. Samuel, Phys. Rev. D 39, 683 (1989).
2. J. D. Bekenstein, Phys. Rev. D 70, 083509 (2004).
3. C. Skordis et al., Phys. Rev. Lett. 96, 011301 (2006); C. Skordis, Phys. Rev. D 74, 103513 (2006).
4. T. Jacobson and D. Mattingly, Phys. Rev. D 64, 024028 (2001).
5. T. G. Zlosnik, P. G. Ferreira, and G. D. Starkman, Phys. Rev. D 74, 044037 (2006).
6. T. G. Zlosnik, P. G. Ferreira, and G. D. Starkman, Phys. Rev. D 75, 044017 (2007).
7. C. Bonvin, R. Durrer, P. G. Ferreira, G. Starkman, and T. G. Zlosnik, Phys. Rev. D 77, 024037 (2008).
8. B. Li, D. F. Mota, and J. D. Barrow, Phys. Rev. D 77, 024032 (2008).
9. C. Eling and T. Jacobson, Classical Quantum Gravity 23, 5625 (2006); C. Eling and T. Jacobson, Classical Quantum Gravity 23, 5643 (2006); R. A. Konoplya and A. Zhidenko, Phys. Lett. B 644, 186 (2007); D. Garfinkle, C. Eling, and T. Jacobson, Phys. Rev. D 76, 024003 (2007); C. Eling, T. Jacobson, and M. C. Miller, Phys. Rev. D 76, 024003 (2007); T. Tamaki and U. Miyamoto, Phys. Rev. D 77, 024026 (2008).
10. E. A. Lim, Phys. Rev. D 71, 063504 (2005).
11. S. Kanno and J. Soda, Phys. Rev. D 74, 063505 (2006).
12. A. Tartaglia and M. Capone, arXiv:gr-qc/0601033 A. Tartaglia and N. Radicella, Phys. Rev. D 76, 083501 (2007).
13. K. Nozari and S. D. Sadatian, Eur. Phys. J. C 58, 499 (2008); S. D. Sadatian and K. Nozari, Europhys. Lett. 82, 49001 (2008).
14. Arianto, F. P. Zen, B. E. Gunara, Triyanta, and Supardi, J. High Energy Physics, JHEP 09, 048 (2007).
15. A.G. Riess et al., Astron. J. 116 (1998) 1009; S. Perlmutter et al., Astrophys. J. 517 (1999) 565; P. de Bernardis et al., Nature 404 (2000) 955; A.D. Miller et al. Astrophys. J. Lett. 524 (1999) L1; S. Hanany et al., Astrophys. J. Lett. 545 (2000) L5; N.W. Halverson et al., Astrophys. J. 568 (2002) 38; B.S. Mason et al., Astrophys. J. 591, 540 (2003); D.N. Spergel et al., Astrophys. J. Suppl. 148, 175 (2003) ; L. Page et al., Astrophys. J. Suppl. 148, 233 (2003); R. Scranton et al., astro-ph/0307335 M. Tegmark et al., Phys. Rev. D 69, 103501 (2004); W.L. Freedman, M.S. Turner, Rev. Mod. Phys. 75, 1433 (2003); S.M. Carroll, astro-ph/0310342
16. B. Ratra, P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988); P. J. E. Peebles, B. Ratra, Astrophys. J. 325, L17 (1988); C. Wetterich, Nucl. Phys. B 302, 668 (1988); R. R. Caldwell, R. Dave, P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998); I. Zlatev, L. Wang, P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999).
17. A. Sen, JHEP 0204, 048 (2002); A. Sen, JHEP 0207, 065 (2002); A. Sen, Mod. Phys. Lett. A 17 (2002) 1797; T. Padmanabhan, T. Roy Choudhury, Phys. Rev. D 66, 083501 (2002); J. S. Bagla, H. K. Jassal, T. Padmanabhan, Phys. Rev. D 67, 063504 (2003).
18. A. Yu. Kamenshchik, U. Moschella, V. Pasquier, Phys. Lett. B 511, 265 (2001); N. Bilec, G. B. Tupper, R. D. Viollier, Phys. Lett. B 535, 17 (2002); M. C. Bento, O. Bertolami, A. A. Sen, Phys. Rev. D 66, 043507 (2002).
19. V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000).
20. P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003).
21. S. M. Carroll, Living Rev. Rel. 4, 1 (2001).
22. T. Padmanabhan, Phys. Rept. 388, 235, (2003).
23. R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
24. E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006).
25. C. Armendariz-Picon, J. Cosmol. Astropart. Phys. 07, 007 (2004).
26. V. V. Kiselev, Classical Quantum Gravity 21, 3323 (2004).
27. T. Koivisto and D. F. Mota, Phys. Rev. D 73, 083502 (2006).
28. Arianto, F. P. Zen, Triyanta, and B. E. Gunara, Phys. Rev. D 77, 123517 (2008).
29. C. M. Will and K. J. Nordtvedt, Astrophys. J. 177, 757 (1972).
30. C. M. Will, Theory and experiment in gravitational physics, (Cambridge University Press, Cambridge, 1981).
31. E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D 57, 4686 (1998).
32. P. A. M. Dirac, Nature. 139, 323 (1937).
33. J. Uzan, Rev. Mod. Phys. 75, 403U (2003).
34. O. G. Benvenuto et al., Phys. Rev. D 69, 082002 (2004).
35. J. L. Tonry et al., Astrophys. J. 594, 1 (2003).
36. B. J. Barris et al., Astrophys. J. 602, 571 (2004).
37. B. Z. Foster and T. Jacobson, Phys. Rev. D 73, 064015 (2006).