On the infimum of the absolute value of successive derivatives of a real function defined on a bounded interval

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Abstract

A study of the greatest possible ratio of the smallest absolute value of a higher derivative of some function, defined on a bounded interval, to the $L^p$-norm of the function.

Keywords

Chebyshev polynomials, Legendre polynomials, extremal problems, inequalities for derivatives

MSC classification: 26D10, 41A10

To the memory of Eduard Wirsing,
master of analysis,
and of its applications to number theory.

1 Introduction

Let $n$ be a positive integer, $I = [a, b]$ a bounded segment of the real line, of length $L = b - a$. Define $\mathcal{D}^n(I)$ as the set of real functions $f$ defined on $I$, with successive derivatives $f^{(k)}$ defined and continuous on $I$ for $0 \leq k \leq n - 1$, and $f^{(n)}$ defined on $\dot{I} = ]a, b[$. We will use the notation

$$m_n(f) = \inf_{a < t < b} |f^{(n)}(t)|.$$

Let $p$ be a positive real number, or $\infty$.

The problem addressed in this article is that of determining the best constant $C^* = C^*(n, p, I)$ in the inequality

$$m_n(f) \leq C^* \|f\|_p \quad (f \in \mathcal{D}^n(I)),$$
where
\[ \|f\|_p = \left( \int_a^b |f(t)|^p \, dt \right)^{1/p}, \]
with the usual convention when \( p = \infty \), here:
\[ \|f\|_\infty = \max |f|. \]

This problem has been posed by Kwong and Zettl in their 1992 Lecture Notes [11] (see Lemma 1.1, p. 6). They give upper bounds for \( C^*(n, p, I) \), but their reasoning and results are erroneous. In her 1993 PhD Thesis [5], Huang has pointed out that this problem is equivalent to a classical problem in the theory of polynomial approximation: that of determining the minimal \( L^p \)-norm of a monic polynomial of given degree on a given bounded interval. Our purpose in this text is to give a new proof of the equivalence, and to list the consequences of the known results about this extremal problem for the evaluation of \( C^*(n, p, I) \).

2 First observations

2.1 Homogeneity

Defining \( g(u) = f(a + uL) \) for \( f \in D^n(I) \) and \( 0 \leq u \leq 1 \), one has
\[
g \in D^n([0, 1]) ; \quad g^{(n)}(u) = L^n f^{(n)}(a + uL) \quad (0 < u < 1) ; \quad \|g\|_p = L^{-1/p} \|f\|_p.
\]

Hence,
\[ C^*(n, p, I) = C^*(n, p, [0, 1]) L^{-n-1/p}, \]
and one is left with determining \( C^*(n, p, [0, 1]) = C(n, p) \), or in fact \( C^*(n, p, I) \) for any fixed, chosen segment \( I \). We will see that \( I = [-1, 1] \) is particularly convenient.

2.2 An extremal problem

One has
\[
C^*(n, p, I) = \sup\{m_n(f)/\|f\|_p : f \in D^n(I), m_n(f) \neq 0\}
= \sup\{m_n(f)/\|f\|_p : f \in D^n(I), m_n(f) = \lambda\} \quad \text{(for every } \lambda > 0)\]
\[ = \lambda/D^*(n, p, \lambda, I), \]
where
\[
D^*(n, p, \lambda, I) = \inf\{\|f\|_p : f \in D^n(I), m_n(f) = \lambda\}
= \inf\{\|f\|_p : f \in D^n(I), m_n(f) \geq \lambda\},
\]
the last equality being true since \( D^*(n, p, \mu, I) = \frac{\mu}{\lambda} D^*(n, p, \lambda, I) \geq D^*(n, p, \lambda, I) \) if \( \mu \geq \lambda \).
Also, since a derivative has the intermediate value property (cf. [3], pp. 109-110), the inequality \( m_n(f) \geq \lambda > 0 \) implies that \( f^{(n)} \) has constant sign on \( I \), so that

\[
D^*(n,p,\lambda,I) = \inf \{ \| f \|_p, \; f \in D^n(I), \; f^{(n)}(t) \geq \lambda \text{ for } a < t < b \}.
\]

Thus, determining \( C^*(n,p,I) \) is equivalent to minimizing \( \| f \|_p \) for \( f \in D^n(I) \) with the constraint \( f^{(n)}(t) \geq \lambda > 0 \) for \( a < t < b \). We will denote this extremal problem by \( \mathcal{E}^*(n,p,\lambda,I) \).

### 3 The relevance of monic polynomials

Let \( P_n \) be the set of monic polynomials of degree \( n \), with real coefficients, identified with the set of the corresponding polynomial functions on \( I \), which is a subset of \( D^n(I) \). Since \( m_n(f) = n! \) for \( f \in P_n \), one has

\[
D^*(n,p,n!,I) \leq D^{**}(n,p,I),
\]

where

\[
D^{**}(n,p,I) = \inf \{ \| Q \|_p, \; Q \in P_n \}.
\]

A basic fact in the study of the extremal problem \( \mathcal{E}^*(n,p,\lambda,I) \) is that (2) is in fact an equality.

**Proposition 1** For all \( n,p,I \), one has \( D^*(n,p,n!,I) = D^{**}(n,p,I) \).

It follows from this proposition that \( C^*(n,p,I) = n!/D^{**}(n,p,I) \) and, by (1),

\[
C(n,p) = L_{n+1/p,n!}/D^{**}(n,p,I).
\]

Let us review the history of Proposition 1.

For \( p = \infty \), it is a corollary to a theorem of S. N. Bernstein from 1937. Denoting by \( E_k(f) \) the distance (for the uniform norm on \( I \)) between \( f \) and the set of polynomials of degree at most \( k \), he proved in particular that

\[
E_{n-1}(f_0) > E_{n-1}(f_1) \quad (f_0, f_1 \in D^n(I)),
\]

provided that the inequality \( f_0^{(n)}(\xi) > |f_1^{(n)}(\xi)| \) is valid for every \( \xi \in \tilde{I} \) (cf. [2], p. 48, inequalities (47bis)-(48bis)). Proposition 1 follows by taking \( f_1(x) = x^n \) and \( f_0(x) = \lambda f(x) \), where \( f \) is a generic element of \( D^n(I) \) such that \( f^{(n)}(t) \geq n! \) for \( a < t < b \), and \( \lambda > 1 \), then letting \( \lambda \to 1 \).

This theorem of Bernstein was generalized by Tsenov in 1951 to the case of the \( L^p \)-norm on \( I \), where \( p \geq 1 \) (cf. [15], Theorem 4, p. 477), thus providing a proof of Proposition 1 for \( p \geq 1 \). The case \( 0 < p < 1 \) was left open by Tsenov.

The study of the extremal problem \( \mathcal{E}^*(n,p,\lambda,I) \) was one of the themes of the 1993 PhD thesis of Xiaoming Huang [5]. In Lemma 2.0.7, pp. 9-10, she gave another proof (due to Saff)
of Proposition 1 in the case \( p = \infty \). For \( 1 \leq p < \infty \), she gave a proof of Proposition 1 which is unfortunately incomplete (cf. [5], pp. 28-30). Again, the case \( 0 < p < 1 \) was left open.

We present now a self-contained proof of Proposition 1, valid for \( 0 < p \leq \infty \). As it proceeds by induction on \( n \), we will need the following classical-looking division lemma, for which we could not locate a reference (compare with [16] or [13]).

**Proposition 2** Let \( n \geq 2 \) and \( f \in D^n(I) \). Let \( c \in [a,b] \). Put

\[
g(x) = \begin{cases} 
  \frac{f(x) - f(c)}{x - c} & (x \in I, x \neq c) \\
  f'(c) & (x = c).
\end{cases}
\]

Then \( g \in D^{n-1}(I) \). For every \( x \in ]a,b[ \), one has

\[
g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},
\]

where \( \xi \in ]a,b[ \).

**Proof**

Since \( f' \) is continuous, one has

\[
g(x) = \int_0^1 f'(c + t(x - c)) \, dt \quad (x \in I).
\]

Using the rule of differentiation under the integration sign, one sees that \( g \) is \( n - 2 \) times differentiable on \( I \), with

\[
g^{(n-2)}(x) = \int_0^1 t^{n-2} f^{(n-1)}(c + t(x - c)) \, dt \quad (x \in I).
\]

As \( f^{(n-1)} \) is continuous on \( I \), this formula yields the continuity of \( g^{(n-2)} \) on \( I \).

The function \( g \) is \( n \) times differentiable on \( I \setminus \{c\} \) (this set is just \( \tilde{I} \) if \( c = a \) or \( c = b \)), being a quotient of \( n \) times differentiable functions, with non-vanishing denominator. In the case \( a < c < b \), we have now to check that \( g \) is \( n - 1 \) times differentiable at the point \( c \).

The function \( f^{(n-1)} \) being continuous on \( I \) and differentiable at the point \( c \), there exists a function \( \varepsilon(h) \), defined and continuous on the segment \( [a - c, b - c] \) (the interior of which contains 0), vanishing for \( h = 0 \), such that

\[
f^{(n-1)}(c + h) = f^{(n-1)}(c) + h f^{(n)}(c) + h \varepsilon(h) \quad (a \leq c + h \leq b).
\]
Hence,

\[ g^{(n-2)}(x) = \int_0^1 t^{n-2} f^{(n-1)}(c + t(x - c)) \, dt \]

\[ = \int_0^1 t^{n-2} \left( f^{(n-1)}(c) + (x - c) f^{(n)}(c) + t(x - c) \varepsilon(t(x - c)) \right) \, dt \]

\[ = \frac{f^{(n-1)}(c)}{n - 1} + \frac{f^{(n)}(c)}{n} (x - c) + (x - c) \int_0^1 t^{n-1} \varepsilon(t(x - c)) \, dt \]

When \( x \) tends to \( c \), the last integral tends to 0, so that the function \( g^{(n-2)} \) is differentiable at the point \( c \), with

\[ g^{(n-1)}(c) = \frac{f^{(n)}(c)}{n}. \]

If \( x \in I \setminus \{c\} \), one may use the general Leibniz rule and Taylor’s theorem with the Lagrange form of the remainder in order to compute \( g^{(n-1)}(x) \):

\[ g^{(n-1)}(x) = \frac{d^{n-1}}{dx^{n-1}} \left( (f(x) - f(c)) \cdot \frac{1}{x - c} \right) \]

\[ = (f(x) - f(c)) \cdot \frac{(-1)^{n-1}(n-1)!}{(x - c)^n} + \sum_{k=1}^{n-1} \binom{n-1}{k} f^{(k)}(x) \cdot \frac{(-1)^{n-1-k}(n-1-k)!}{(x - c)^{n-k}} \]

\[ = \frac{(n-1)!}{(c-x)^n} \left( f(c) - f(x) - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} (c-x)^k \right) \]

\[ = \frac{(n-1)!}{(c-x)^n} \cdot \frac{f^{(n)}(\xi)}{n!} (c-x)^n \quad \text{(where } \xi \text{ belongs to the open interval bounded by } c \text{ and } x) \]

\[ = \frac{f^{(n)}(\xi)}{n}. \]

In the next proposition, we stress the main element of our proof of Proposition 1, namely the fact that the condition \( f^{(n)} \geq n! \), for some \( f \in \mathcal{D}^n(I) \), implies that the absolute value of \( f \) dominates the absolute value of some monic polynomial of degree \( n \).

**Proposition 3** Let \( n \geq 1 \) and \( f \in \mathcal{D}^n(I) \) such that \( f^{(n)}(x) \geq n! \) for every \( x \in [a,b] \).

Then there exists a monic polynomial \( P \) of degree \( n \), with all its zeros in \( I \), such that the inequality \( |f(x)| \geq |P(x)| \) is valid for every \( x \in I \).

Moreover, if \( |f(x)| = |Q(x)| \) for every \( x \in I \), where \( Q \) is a monic polynomial of degree \( n \) with real coefficients, then \( f(x) = Q(x) \) for every \( x \in I \).
Proof

The assertion about the zeros may be obtained a posteriori, by replacing the zeros of $P$ by their projections on $I$. The following proof leads directly to a polynomial $P$ with all zeros in $I$.

We use induction on $n$.

For $n = 1$, the function $f$ is continuous on $[a,b]$, differentiable on $]a,b[$, with $f'(x) \geq 1$ for $a < x < b$.

If $f(a) \geq 0$, one has, for $a < x \leq b$, $f(x) = f(a) + (x - a)f'(\xi)$ (where $a < \xi < x$), thus $f(x) \geq x - a$. Hence, one has $|f(x)| \geq |x - a|$ for every $x \in I$.

If $f(b) \leq 0$, one proves similarly that $|f(x)| \geq |x - b|$ for every $x \in I$.

If $f(a) < 0 < f(b)$, there exists $c \in ]a,b[$ such that $f(c) = 0$. One has then, for every $x \in I$,

$$f(x) = f(x) - f(c) = (x - c)f'(\xi) \quad \text{(where } a < \xi < b).$$

Hence $|f(x)| \geq |x - c|$ for every $x \in I$, and the result is proven for $n = 1$.

Let now $n \geq 2$, and suppose that the result is valid with $n - 1$ instead of $n$. Let $f \in D^n(I)$ such that $f^{(n)}(x) \geq n!$ for every $x \in ]a,b[$.

If $f$ vanishes at some point $c \in I$, it follows from Proposition 2 that the function $g$ defined on $I$ by

$$g(x) = \begin{cases} \frac{f(x)}{x-c} & (x \in I, x \neq c) \\ f'(c) & (x = c), \end{cases}$$

belongs to $D^{n-1}(I)$ and that, for every $x \in ]a,b[$, one has

$$g^{(n-1)}(x) = \frac{f^{(n)}(\xi)}{n},$$

where $\xi \in ]a,b[$, thus $g^{(n-1)}(x) \geq (n - 1)!$. By the induction hypothesis, there exists a monic polynomial $Q$ of degree $n - 1$, with all its roots in $I$, such that $|g(x)| \geq |Q(x)|$ for every $x \in I$.

Hence, one has the inequality $|f(x)| \geq |P(x)|$ for every $x \in I$, where $P(x) = (x - c)Q(x)$ is a monic polynomial of degree $n$, with all its roots in $I$.

If $f > 0$, it reaches a minimum at some point $c \in I$. Again, it follows from Proposition 2 that the function $g$ defined on $I$ by (4) satisfies the required hypothesis for degree $n - 1$. Thus there exists a monic polynomial $Q$ of degree $n - 1$, with all its roots in $I$, such that $|g(x)| \geq |Q(x)|$ for every $x \in I$. Hence, one has the inequality

$$f(x) - f(c) = |f(x) - f(c)| \geq |P(x)| \quad (x \in I),$$

where $P(x) = (x - c)Q(x)$. It follows that

$$|f(x)| = f(x) \geq f(c) + |P(x)| > |P(x)| \quad (x \in I)$$
If \( f < 0 \), the reasoning is similar by considering a point \( c \in I \) where \( f \) reaches a maximum.

Let us prove the last assertion. The hypothesis \(|f| = |P|\) is equivalent to the equality \( f^2 = P^2 \), that is \((f - P)(f + P) = 0\). The set \( E = \{x \in I, f(x) + P(x) = 0\} \) has empty interior, since \( f^{(n)}(x) + P^{(n)}(x) = 0 \) on every open subinterval of \( E \), whereas \( f^{(n)}(x) + P^{(n)}(x) \geq 2n! \) on \( I \). The set \( I \setminus E \) is therefore dense in \( I \); its elements \( x \) all verify \( f(x) = P(x) \), hence \( f = P \) on \( I \) by continuity.

Proposition 1 is an immediate corollary of Proposition 3: by taking \( f \) and \( P \) as stated there, one has \(|f(x)| \geq |P(x)|\) for every \( x \in I \), so that

\[
\int_a^b |f(x)|^p \, dx \geq \int_a^b |P(x)|^p \, dx,
\]

for every \( p > 0 \) (for \( p = \infty \): \( \max |f| \geq \max |P| \)).

Moreover, if \( p < \infty \), equality in (5) implies that \(|f| = |P|\) on \( I \), hence \( f = P \).

In other words, if \( 0 < p < \infty \), the extremal problem \( \mathcal{E}^*(n, p, n! I) \) has exactly the same solutions (value of the infimum and extremal functions) as the problem \( \mathcal{E}^{**}(n, p, I) \) obtained by considering only monic polynomials of degree \( n \), which one may even take with all their roots in \( I \).

For \( p = \infty \), our reasoning does not prove that an extremal function for \( \mathcal{E}^*(n, p, n! I) \) (if it exists) must be a polynomial. This is true anyway, as proved by Huang in [5], pp. 10-13.

## 4 Extremal polynomials

One may now use the results of the well developed theory of the extremal problem \( \mathcal{E}^{**}(n, p, I) \) for polynomials. Thus, since the integral

\[
\int_a^b |(x - x_1) \cdots (x - x_n)|^p \, dx \quad (x_1, \ldots, x_n \in I)
\]

(or the value \( \max_{x \in I} |(x - x_1) \cdots (x - x_n)| \)) is a continuous function of \( (x_1, \ldots, x_n) \), the compactness of \( I^n \) yields the existence of an extremal (polynomial) function for \( \mathcal{E}^{**}(n, p, I) \), hence for \( \mathcal{E}^*(n, p, n!, I) \).

It is a known fact that the polynomial extremal problem \( \mathcal{E}^{**}(n, p, I) \) has a unique solution for all \( p \in [0, \infty] \), but there is no proof valid uniformly for all values of \( p \).

- For \( p = \infty \), uniqueness was proved by Young in 1907 (cf. [18], Theorem 5, p. 340)) and follows from the general theory of uniform approximation (cf. [12], Theorem 1.8, p. 28).
- For \( 1 < p < \infty \), as proved by Jackson in 1921 (cf. [7], §6, pp. 121-122), this is a consequence of the strict convexity of the space \( L^p(I) \).
- For \( p = 1 \), this is also due to Jackson in 1921 (cf. [6], §4, pp. 323-326).
• For $0 < p < 1$, the uniqueness of the extremal polynomial was proved in 1988 by Kroó and Saff (cf. [10], Theorem 2, p. 184). Their proof uses the uniqueness property for $p = 1$ and the implicit function theorem.

We will denote by $T_{n,p,I}$ the unique solution of the extremal problem $E^{**}(n,p,I)$. Uniqueness gives immediately the relation

$$T_{n,p,I}(a + b - x) = (-1)^n T_{n,p,I}(x) \quad (x \in \mathbb{R}).$$

Another property of these polynomials is the fact that all their roots are simple. For $p = 1$, this fact was proved by Korkine and Zolotareff in 1873 (cf. [8], pp. 339-340), before their explicit determination of the extremal polynomial (see §5.4 below), and their proof extends, *mutatis mutandis*, to the case $1 < p < \infty$. For $p = \infty$, this is a property of the Chebyshev polynomials of the first kind (see §5.2 below). Lastly, for $0 < p < 1$, this was proved by Kroó and Saff in [10], p. 187.

Define $T_{n,p} = T_{n,p,[−1,1]}$, and write $n = 2k + \varepsilon$, where $k \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$. It follows from the mentioned results that

$$T_{n,p}(x) = x^\varepsilon (x^2 - x_{n,1}(p)^2) \cdots (x^2 - x_{n,k}(p)^2) \quad (x \in \mathbb{R}),$$

where

$$0 < x_{n,1}(p) < \cdots < x_{n,k}(p) \leq 1.$$

Kroó, Peherstorfer and Saff have conjectured that all the $x_{n,k}$ are increasing functions of $p$ (cf. [9], p. 656, and [10], p. 192).

5 Results on $C(n,p)$

5.1 The case $n = 1$

The value $n = 1$ is the only one for which $C(n,p)$ is explicitly known for all $p$.

**Proposition 4** One has $C(1,p) = 2(p + 1)^{1/p}$ for $0 < p < \infty$, and $C(1,\infty) = 2$.

**Proof**

By (6), one has $T_{1,p}(x) = x$, so that, for $0 < p < \infty$,

$$D^{**}(1,p,[−1,1]) = \left( \int_{−1}^{1} |t|^p \, dt \right)^{1/p} = \left( \frac{2}{(p + 1)} \right)^{1/p},$$

and, by (3),

$$C(1,p) = 2^{1+1/p}/D^{**}(1,p,[−1,1]) = 2(p + 1)^{1/p}. \quad \Box$$

Note that the Lemma 1.1, p. 6 of [11], asserts that $C(1,p) \leq 2 \cdot 3^{1/p}$ for $p \geq 2$, and that bound is $< 2(p + 1)^{1/p}$ for $p > 2$. 

8
5.2 The case \( p = \infty \)

This is the classical case, solved by Chebyshev in 1853 by introducing the polynomials \( T_n \) defined by the relation

\[
T_n(\cos t) = \cos nt
\]

(now called Chebyshev polynomial of the first kind): the unique solution of the extremal problem \( E^{**}(n, \infty, [-1, 1]) = 2^{1-n}T_n \). Let us record a short proof of this fact.

Take \( I = [-1, 1] \) and suppose that \( P \) is a monic polynomial of degree \( n \) satisfying the inequality \( \|P\|_\infty \leq \|2^{1-n}T_n\|_\infty = 2^{1-n} \). Then, for \( \lambda > 1 \) the polynomial

\[
Q_\lambda = \lambda 2^{1-n}T_n - P
\]

is of degree \( n \), with leading coefficient \( \lambda - 1 \). Moreover, it satisfies

\[
(-1)^k Q_\lambda(\cos k\pi/n) = \lambda 2^{1-n} - (-1)^k P(\cos k\pi/n) > 0 \quad (k = 0, \ldots, n)
\]

By the intermediate value property, \( Q_\lambda \) has at least \( n \) distinct roots, hence exactly \( n \), and these roots, say \( x_1, \ldots, x_n \), have absolute value not larger than 1. Hence,

\[
|Q_\lambda(x)| = (\lambda - 1) |(x - x_1) \cdots (x - x_n)| \leq (\lambda - 1)(1 + |x|)^n \quad (x \in \mathbb{R}).
\]

When \( \lambda \to 1 \), \( Q_\lambda(x) \) tends to 0 for every real \( x \), which means that \( P = 2^{1-n}T_n \).

One deduces from this theorem the value of \( C(n, \infty) \). One has

\[
D^{**}(n, \infty, [-1, 1]) = \max_{|x|\leq 1} |2^{1-n}T_n(x)| = 2^{1-n},
\]

hence

\[
C(n, \infty) = 2^n \cdot n! / D^{**}(n, \infty, [-1, 1]) = 2^{2n-1}n!
\]

(compare with the upper bound \( C(n, \infty) \leq 2^{n(n+1)/2}n^n \) of [4], 3 (a), p. 185). This result is essentially due to Bernstein (1912, cf. [1], p. 65).

Qualitatively, the result expressed by (7) was nicely described by Soula in [14], p. 86, as follows.

Bernstein’s principle: the minimum of the absolute value of the \( n \)-th derivative of an \( n \) times differentiable function and the maximum of the absolute value of the \( n \)-th derivative of an analytic function have similar orders of magnitude.

5.3 The case \( p = 2 \)

In this case, the extremal problem \( E^{**}(n, 2, [-1, 1]) \) is an instance of the general problem of computing the orthogonal projection of an element of a Hilbert space onto a finite dimensional subspace. Here, the Hilbert space is \( L^2(-1, 1) \), the element is the monomial function \( x^n \), and
the subspace is the set of polynomial functions of degree less than \( n \). The solution follows from the theory of orthogonal polynomials: the extremal polynomial for \( E^{**}(n, 2, [-1, 1]) \) is

\[
\frac{2^n(n!)^2}{(2n)!} P_n(x) \quad (|x| \leq 1),
\]

where \( P_n \) is the \( n \)-th Legendre polynomial, defined by

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

Hence,

\[
D^{**}(n, 2, [-1, 1]) = \frac{2^n(n!)^2}{(2n)!} \|P_n\|_2 = \frac{2^n(n!)^2}{(2n)!} \sqrt{\frac{2}{2n + 1}},
\]

(see [17], §15-14, p. 305) and

\[
C(n, 2) = 2^{n+\frac{1}{2}} \cdot n! / D^{**}(n, 2, [-1, 1]) = \frac{(2n)!}{n!} \sqrt{2n + 1},
\]

(8)

a result given by Soula in 1932 (cf. [14], pp. 87-88).

5.4 The case \( p = 1 \)

The problem \( E^{**}(n, 1, [-1, 1]) \) was solved by Korkine and Zolotareff in [8]: the extremal polynomial is \( 2^{-n} U_n(x) \), where \( U_n \) is the \( n \)-th Chebyshev polynomial of the second kind, defined by the relation \( U_n(\cos t) = \sin(n + 1)t / \sin t \).

Therefore, one has

\[
D^{**}(n, 1, [-1, 1]) = 2^{-n} \int_{-1}^{1} |U_n(x)| \, dx = 2^{-n} \int_{0}^{\pi} |U_n(\cos t)| \sin t \, dt
= 2^{-n} \int_{0}^{\pi} |\sin(n + 1)t| \, dt = 2^{-n} \int_{0}^{\pi} \sin u \, du
= 2^{1-n},
\]

and

\[
C(n, 1) = 2^{n+1} \cdot n! / D^{**}(n, 1, [-1, 1]) = 2^{2n} n!.
\]

(9)

5.5 Bounds for \( C(n, p) \)

We begin with a simple monotony result.

**Proposition 5** For every positive integer \( n \), the function \( p \mapsto C(n, p) \) is decreasing on the interval \( 0 < p \leq \infty \).
Proof

Let $I = [0, 1]$. Equivalently, we will see that the function $p \mapsto D^{**}(n, p, I)$ is increasing. This is due to the fact that, for a fixed $f \in L^\infty(I)$ such that $|f|$ is not equal almost everywhere to a constant, the function $p \mapsto \|f\|_p$ is increasing (a consequence of Hölder’s inequality). Thus, for every $Q \in \mathcal{P}_n$ and $0 < p < p' \leq \infty$,

$$\|Q\|_{p'} > \|Q\|_p \geq D^{**}(n, p, I),$$

which implies that $D^{**}(n, p', I) > D^{**}(n, p, I)$. □

In particular, (7) and (9) yield the inequalities

$$2^{2n-1}n! < C(n, p) < 2^{2n}n! \quad (1 < p < \infty).$$

The next proposition implies that the limit of $C(n, p)$ when $p$ tends to 0 is $(2e)^n n!$.

**Proposition 6** For every positive integer $n$ and every positive real number $p$, one has

$$2^n (1 + np)^{1/p} n! \leq C(n, p) \leq (2e)^n n!$$

**Proof**

Equivalently, we will prove that

$$(2e)^{-n} \leq D^{**}(n, p, I) \leq 2^{-n} (1 + np)^{-1/p}, \quad (10)$$

where $I = [0, 1]$.

Let $Q(t) = (t - x_1) \cdots (t - x_n)$, where $0 \leq x_1, \ldots, x_n \leq 1$. One has

$$\ln \|Q\|_p = \frac{1}{p} \ln \int_0^1 |Q(t)|^p \, dt$$

$$\geq \frac{1}{p} \int_0^1 \ln \left(|Q(t)|^p\right) \, dt \quad \text{(by Jensen’s inequality)}$$

$$= \int_0^1 \ln |Q(t)| \, dt$$

$$= \sum_{k=1}^n \int_0^1 \ln |t - x_k| \, dt.$$ 

Now,

$$\int_0^1 \ln |t - x| \, dt = (1 - x) \ln(1 - x) + x \ln x - 1 \quad (0 \leq x \leq 1),$$

attains its minimal value, namely $-1 - \ln 2$, when $x = 1/2$. This implies the first inequality of (10).

11
To prove the second inequality of (10), we just compute $\|Q\|_p^p$ when $Q(t) = (t - 1/2)^n$:

$$\int_0^1 |t - 1/2|^{np} dt = 2^{(1/2)^{np+1}}(np+1).$$

For $0 < p < 1$, we can also prove the following result.

**Proposition 7** Let $n$ be a positive integer, and $p$ such that $0 < p < 1$. One has

$$1 \leq C(n,p) \leq \frac{1}{2}(8/\pi)^{1/p}.$$

**Proof**

The first inequality is just $C(n,1) \leq C(n,p)$.

To prove the second inequality, let $r$ and $s$ such that $1 < s < 2$ and $r^{-1} + s^{-1} = 1$. Define

$$I_1(s) = \int_{-1}^1 \frac{dt}{(1 - t^2)^{s/2}},$$

$$I_2(s) = \int_{-1}^1 \frac{|t|^{(s-1)/s} dt}{\sqrt{1 - t^2}}.$$

The integrals $I_1(s)$ and $I_2(s)$ may be computed, using the eulerian identity

$$\int_0^1 t^{x-1}(1 - t)^{y-1} dt = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x > 0, y > 0).$$

The results are

$$I_1(s) = 2^{1-s}\frac{\Gamma(1 - \frac{s}{2})^2}{\Gamma(2 - s)},$$

$$I_2(s) = \frac{\Gamma(1 - \frac{1}{2s})\Gamma(\frac{s}{2})}{\Gamma(\frac{3}{2} - \frac{1}{2s})}.$$

Now, let $Q \in P_n$ and put $p' = p/r$. By Hölder's inequality, one has

$$\int_{-1}^1 |Q(t)|^{p'} \frac{dt}{\sqrt{1 - t^2}} \leq \left( \int_{-1}^1 |Q(t)|^{p'r} dt \right)^{1/r} \left( \int_{-1}^1 \frac{dt}{(1 - t^2)^{s/2}} \right)^{1/s} = \|Q\|_p^{p'} I_1(s)^{1/s}.$$
It was proved by Kroó and Saff (cf. [10], pp. 182-183) that

\[
2^{(n-1)p'} \int_{-1}^{1} |Q(t)|^{p'} \frac{dt}{\sqrt{1-t^2}} \geq \int_{-1}^{1} |T_n(t)|^{p'} \frac{dt}{\sqrt{1-t^2}}
\]

\[
= \int_{0}^{\pi} |\cos nu|^{p'} \, du = \int_{0}^{\pi} |\cos u|^{p'} \, du
\]

\[
= \int_{-1}^{1} |t|^{p'} \frac{dt}{\sqrt{1-t^2}} \geq \int_{-1}^{1} |t|^{1/r} \frac{dt}{\sqrt{1-t^2}} \quad \text{(one has } p' = p/r < 1/r)\]

\[
= I_2(s).
\]

Therefore, with \( I = [-1, 1] \),

\[
\|Q\|_p \geq 2^{1-n} I_2(s)^{1/p'} I_1(s)^{-1/p'} = 2^{1-n} A(s)^{1/p} \quad (1 < s < 2),
\]

where

\[
A(s) = I_2(s)^{s/(s-1)} I_1(s)^{-1/(s-1)}.
\]

Hence

\[
A(s) = 2^{(s/(s-1))s} \left( \frac{\Gamma \left( 1 - \frac{s}{2} \right) \Gamma \left( 1 - \frac{s}{2} \right) \Gamma(2-s) \Gamma \left( \frac{3}{2} - \frac{s}{2} \right)}{\Gamma \left( 1 - \frac{s}{2} \right) \Gamma \left( \frac{3}{2} - \frac{s}{2} \right) \Gamma(2-s)} \right)^{1/(s-1)} \quad (1 < s < 2).
\]

Putting \( f(s) = \ln \Gamma(s) \), one has

\[
\ln A(s) = \ln 2 + \frac{sf(1-1/2s) + sf(1/2) + f(2-s) - 2f(1-s/2) - sf(3/2-1/2s)}{s-1}.
\]

When \( s \) tends to 1, the last fraction tends to

\[
\ln \pi + \frac{3}{2} \psi(1/2) - \frac{3}{2} \psi(1) = \ln \pi - 3 \ln 2,
\]

with the usual notation \( f' = \Gamma'/\Gamma = \psi \). It follows that

\[
A(s) \to \frac{\pi}{4} \quad (s \to 1).
\]

Together with (11), this gives the inequality

\[
D^{**}(n, p, [-1, 1]) \geq 2^{1-n} (\pi/4)^{1/p}
\]

and (3) now implies

\[
C(n, p) \leq 2^{2n-1} n!(8/\pi)^{1/p}.
\]

We now prove an inequality involving three values of the function \( C \).
Proposition 8 Let $p, q, r$ be positive real numbers such that

\[
\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.
\]

Let $m$ and $n$ be positive integers. Then,

\[
\frac{C(m + n, p)}{(m + n)!} \geq \frac{C(m, q)}{m!} \cdot \frac{C(n, r)}{n!}.
\]

Proof

Equivalently, by (3), one has to prove that

\[
D^{**}(m + n, p, I) \leq D^{**}(m, q, I) \cdot D^{**}(n, r, I),
\]

where $I$ is a segment of the real line.

In fact, if $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$, then $PQ \in \mathcal{P}_{m+n}$ hence

\[
D^{**}(m + n, p, I)^p \leq \int_I |P(t)Q(t)|^p \, dt \leq \left( \int_I |P(t)|^q \, dt \right)^{p/q} \cdot \left( \int_I |Q(t)|^r \, dt \right)^{p/r}
\]

by the definition of $D^{**}(m + n, p, I)$ and Hölder’s inequality. The greatest lower bound of the last term, when $P$ runs over $\mathcal{P}_m$ and $Q$ runs over $\mathcal{P}_n$, is

\[
D^{**}(m, q, I)^p \cdot D^{**}(n, r, I)^p.
\]

The result follows. \square

5.6 An open question

Finally, observing that

\[
C(n, 2) \sim \sqrt{\frac{2}{\pi}} \cdot 2^{2n} n! \quad (n \to \infty),
\]

(an exercise on Stirling’s formula from (8)), we ask the following question.

Is it true that, for every $p > 0$, the quantity $2^{-2n}C(n, p)/n!$ tends to a limit when $n$ tends to infinity?

References

[1] S. Bernstein – “Sur l’ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné.”, \textit{Mém. Cl. Sci. Acad. Roy. Belg.} \textbf{IV} (1912), no. 1-104.
[2] S. N. Bernstein – *Extremal properties of polynomials and the best approximation of continuous functions of a single real variable. part i*, in russian éd., G. R. O . L, Leningrad, Moscow, 1937.

[3] G. Darboux – “Mémoire sur les fonctions discontinues.”, *Ann. de l’Éc. Norm. (2)* 4 (1875), p. 57–112.

[4] J. Dieudonné – *Foundations of modern analysis*, Pure and Applied Mathematics, vol. 10, Academic Press, New York-London, 1969.

[5] X. Huang – “On extremal properties of algebraic polynomials”, Thèse, The Ohio State University, 1993.

[6] D. Jackson – “Note on a class of polynomials of approximation.”, *Trans. Am. Math. Soc.* 22 (1921), p. 320–326.

[7] — , “On functions of closest approximation.”, *Trans. Am. Math. Soc.* 22 (1921), p. 117–128.

[8] A. Korkine et G. Zolotareff – “Sur un certain minimum”, *Nouv. Ann.* 12 (1873), p. 337–356.

[9] A. Kroó et F. Peherstorfer – “On the zeros of polynomials of minimal $L_p$-norm”, *Proc. Am. Math. Soc.* 101 (1987), p. 652–656.

[10] A. Kroó et E. B. Saff – “On polynomials of minimal $L_q$-deviation, $0 < q < 1$”, *J. Lond. Math. Soc., II. Ser.* 37 (1988), p. 182–192.

[11] M. K. Kwong et A. Zettl – *Norm inequalities for derivatives and differences*, Lecture Notes in Mathematics, vol. 1536, Springer, 1992.

[12] T. J. Rivlin – *An introduction to the approximation of functions. Corr. reprint of the 1969 orig*, Dover Publications Inc., Mineola, NY, 1981.

[13] L. Schoenfeld – “On the differentiability of indeterminate quotients”, *Math. Mag.* 41 (1968), p. 152–155.

[14] J. Soula – “Sur une inégalité vérifiée par une fonction et sa dérivée d’ordre $n$”, *Mathematica, Cluj* 6 (1932), p. 86–88.

[15] I. V. Tsenov – “On a question of the approximation of functions by polynomials”, *Mat. Sb., Nov. Ser.* 28 (1951), p. 473–478 (Russian).

[16] H. Whitney – “Differentiability of the remainder term in Taylor’s formula”, *Duke Math. J.* 10 (1943), p. 153–158.
[17] E. T. Whittaker et G. N. Watson – *A course of modern analysis*, 4th éd., Cambridge University Press, 1927.

[18] J. W. Young – “General theory of approximation by functions involving a given number of arbitrary parameters.”, *Trans. Am. Math. Soc.* 8 (1907), p. 331–344.

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