Phaseless inverse problems with interference waves

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\textbf{Abstract.} We consider two phaseless inverse problems for elliptic equation. The statements of these problems differ from have considered. Namely, instead of given information about modulus of scattering waves, we consider the information related to modulus of full fields, which consist of sums of incident and scattering fields. These full fields are the interference fields generated by point sources. We introduce a set of auxiliary point sources for solving the inverse problems and demonstrate that the corresponding data allow us to solve the inverse problems in a way similar to the case of measurements of scattering waves.

\textbf{Key words:} inverse problem, phaseless information, elliptic equation, uniqueness, tomography problem, inverse kinematic problem

\textbf{AMS subject classification:} 35R30.

\section{Introduction}

The phaseless inverse problem was stated first in the book \cite{2} by Chadan and Sabatier "Inverse Problems in Quantum Scattering Theory". The authors noted that a phase of a solution of the Schrödinger equation cannot be measured for the large frequencies (energies). Therefore they suggested to study inverse problems when only moduli of the fields are measured. The first results for the phaseless inverse problem related to recovering a potential in the Schrödinger equation were obtained in the papers \cite{3}-\cite{5}, \cite{7}-\cite{9} by Klibanov and Romanov and \cite{10}-\cite{18} by Novikov. Then in the papers \cite{6, 10, 11} the phaseless problems were considered for a generalized Helmholtz operator $\Delta + k^2 n^2(x)$ with incident point sources or incident plane waves. In the paper \cite{12}, a procedure for extracting Riemannian distances $\tau(x, y)$ from given data of the phaseless inverse problem was developed for the Helmholtz operator. Recently the inverse phaseless problem of recovering the permittivity coefficient in the Maxwell equations was considered in \cite{21} for a point incident source and in \cite{22} for an incident plane wave.

In the papers \cite{7} and \cite{11} by Klibanov and Romanov, phaseless inverse problem for elliptic equations were studied. The main features of the formulation of these problems consist in the following: it is assumed that a coefficient of an elliptic equation with a frequency parameter $k$ is known outside of some compact domain $\Omega_0 \subset \mathbb{R}^3$ with smooth boundary $S_0 = \partial \Omega_0$ and it should be recovered in $\Omega_0$ from given modulus on $S_0$ of scattering fields generated by point sources $y \in S_0$ which run over the whole $S_0$. The given scattering field is then a function of variable $x, y \in S_0$ and the frequency $k$. Then the asymptotic expansion of the solutions to direct problems with a point source is studied as $k \to \infty$. As a result, the given information

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allows us to reduce the inverse problems under consideration to well-known problems, that is, the tomography problem for recovering a potential \( q(x) \) in the Schrödinger equation (e.g., [7]) or the inverse kinematic problem for recovering a refractive index \( n(x) \) in a generalized Helmholtz equation (III). Related to this, we pose the question whether or not the modulus not of the scattering field but a full field which is the sum of an incident field in the homogeneous medium and the scattering field in non-homogeneities. As it is seen from the above mentioned papers, given scattering field simplifies study of the phaseless inverse problem, but the inverse problem with data of the full field is much more complicated. Related results for an inverse problem with the full field were obtained in a linearized approximation only (III).

In this paper, we present a new approach for studying phaseless inverse problems when the information in these problems is given for a full field. In this approach, we suppose that a potential or a refractive index is unknown only in the ball \( \Omega_0 = \{ x \in \mathbb{R}^3 | |x| < R_0 \} \) with \( R_0 > 0 \). Let \( \Omega = \{ \mathbb{R}^3 \ | |x| \leq R \} \) be the ball with the radius \( R > R_0 \). We set \( S = \{ x \in \mathbb{R}^3 \ | |x| = R \} \). We consider an interference of two waves: a wave produced by a point source located at \( y \in S \) and a wave located at the auxiliary point \( z = z(y) \in \mathbb{R}^3 \setminus \Omega \). The modulus of the field obtained by the interference of these waves is measured at a receivers \( x \in S_+(z) := \{ x \in S \ | x \cdot (z - x) > 0 \} \), and we regard the modulus as a function in \( x, y \in S \) and sufficiently large \( k \). Here we can consider \( S_+(z) \) as the illuminated part of \( S \) by the light source placed at the point \( z \). Moreover, in our approach, for every \( y \in S \) we need 3 different sources \( z^{(j)} = z^{(j)}(y), j = 1, 2, 3 \). It is assumed that these sources are on the tangent plane of \( S \) at the point \(-y \in S\), lie enough far from the point \(-y\), and form a right triangle on this tangent plane. Hence the observation data are a function of \( y \in S, x \in S_+(z^{(j)}) \) and \( k \geq k_0 > 0 \), where \( k_0 \) is a positive number. More detailed description is given in the next section.

The structure of this paper is as follows. In section 2, we formulate our observation system which we use for consideration of two phaseless inverse problems. In section 3, we consider the Schrödinger equation and the problem of recovering a potential from phaseless data related to the full field. Here we demonstrate that by given data, we can reduce the inverse problem under consideration to a tomography problem. In section 4, we consider the problem of finding a refractive index \( n(x) \) in a generalized Helmholtz equation from given modulus for 3 interference fields and show that this problem is reduced to an inverse kinematic problem. Thus it turns out that the phaseless data in our paper allow us to reduce the inverse problems exactly to the same problems as in [7 III].

## 2 Formulation of the inverse problems: auxiliary point sources for refractive waves

Here for the formulation of our inverse problems, we define an observation system. We shall consider differential equations with a source like \( \delta(x - y) + \delta(x - z) \), which is a superposition of interference waves produced by two point sources \( y, z \in \mathbb{R} \).

Let \( \Omega_0 := \{ x \in \mathbb{R}^3 \ | |x| < R_0 \} \) and \( \Omega = \{ x \in \mathbb{R}^3 \ | |x| < R \} \) with \( R > R_0 \), and \( S = \partial \Omega \).

Assume that \( y \) is an arbitrary point of \( S \) and for \( y \), we choose a point \( z = z(y) \in \mathbb{R}^3 \setminus \Omega \) in the following way. That is, consider the point \(-y\). This point obviously belongs to \( S \). By \( \Sigma(-y) \) we denote the tangent plane on \( S \) at \(-y\). Let \( z(y) \in \Sigma(-y) \) be a point satisfying \( |z(y) - (-y)| = \ell \).
with \( \ell \geq \ell^* \), where \( \ell^* \) will be estimate below. We set \( S_+(z) := \{ x \in S | x \cdot (z - x) > 0 \} \), which can be regarded as the part of \( S \) illuminated by the light source placed at \( z \). Denote by \( S^*(y) = \{ x \in S | -y \cdot (x - y) > |x - y| \sqrt{R^2 - R_0^2} \} \). It means that \( S^*(y) \) is the shadow part of \( S \) for the light source placed at \( y \) when the light meets the non-transparent domain \( \Omega_0 \) on its way. Later we need to construct for every \( y \in S \) some refractive waves with point sources at \( z^{(j)}(y), j = 1, 2, \ldots, m \), such that the union of \( S_+(z^{(j)}), j = 1, 2, \ldots, m \), contains \( S^*(y) \). A simple analysis shows that it is impossible to do for arbitrary \( R \) and \( R_0 \) if \( m = 1 \) or \( m = 2 \), but it is possible if \( m = 3 \) (see however Remark 1 below). There are many possibilities in choosing such \( z^{(j)}(y), j = 1, 2, 3 \). Among them, we choose \( z^{(j)}(y), j = 1, 2, 3 \) such that \( z^{(j)} \in \Sigma(-y) \), \( |z^{(1)} + y| = |z^{(2)} + y| = |z^{(3)} + y| = \ell \) and the points \( z^{(j)}(y), j = 1, 2, 3 \), are the vertices of a right triangle. Then

**Lemma 1.** Let \( z^{(j)} \in \Sigma(-y), j = 1, 2, 3 \), \( |z^{(1)} + y| = |z^{(2)} + y| = |z^{(3)} + y| = \ell \), and the points \( z^{(j)}(y), j = 1, 2, 3 \) be the vertices of a right triangle and \( \ell \geq \ell^* \), where

\[
\ell^* = \frac{2R_0 R}{\sqrt{R^2 - R_0^2}} \tag{2.1}
\]

Then \( S^*(y) \subset \left( S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)}) \right) \).

Proof. By the invariance of \( z^{(j)}, j = 1, 2, 3 \), it suffices to consider the case where \( y = (0, 0, -R) \). Then \( -y = (0, 0, R) \). On the plane \( \Sigma(-y) \) we consider the points \( z^{(1)} = (\ell/2, \ell \sqrt{3}/2, R) \), \( z^{(2)} = (-\ell/2, \ell \sqrt{3}/2, R) \), \( z^{(3)} = (0, -\ell, R) \) and demonstrate that the relation \( S^*(y) \subset \left( S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)}) \right) \) holds if \( \ell \geq \ell^* \), where \( \ell^* \) is defined by (2.1). Let \( z = (\ell \cos \varphi, \ell \sin \varphi, R) \) with \( \varphi \in (0, \pi) \), and let \( C(z) = \{ \xi \in S | |\xi - z| = \ell \} \).

Then \( \xi \in C(z), \xi = (\xi_1, \xi_2, \xi_3) \) if and only if \( \xi = z + \ell \beta \) where \( \beta = (\beta_1, \beta_2, \beta_3) \) is a unit vector satisfying the conditions \( z \cdot \beta = -\ell \) and \( \beta_3 \in (-1, 0) \). The first of these conditions means that \( |\beta|^2 = R^2 \), and the second one means that the component \( \beta_3 \) should be non positive. Since \( \varphi \in (0, \pi) \), the circumference \( C(z) \) has two points of intersections with the plane \( \xi_1 = 0 \). One of these points is \( -y = (0, 0, R) \). Let us find the other intersection point. Note that for both intersection points the equality \( 0 = \ell \cos \varphi + \ell \beta_3 \) holds. Hence, \( \beta_3 = -\cos \varphi \). Then \( \beta_3 = -\sqrt{\sin^2 \varphi - \beta_3^2} \). One can calculate \( \beta_2 \) using the relation \( z \cdot \beta = -\ell \). Taking into account that \( z = (\ell \cos \varphi, \ell \sin \varphi, R), \beta_1 = -\cos \varphi \) and \( \beta_3 = -\sqrt{\sin^2 \varphi - \beta_3^2} \), we obtain the equality

\[
-\ell \cos^2 \varphi + \beta_2 \ell \sin \varphi - R \sqrt{\sin^2 \varphi - \beta_3^2} = -\ell.
\]

From this, we have the quadratic equation in \( \beta_2 \):

\[
\beta_2^2 (R^2 + \ell^2 \sin^2 \varphi) + \frac{2 \beta_2 \ell^2 \sin^3 \varphi + \ell^2 \sin^4 \varphi - R^2 \sin^2 \varphi}{2} = 0.
\]

Solving this equation, we obtain

\[
(\beta_2)_* = \sin \varphi \frac{-\ell^2 \sin^2 \varphi \pm R^2}{R^2 + \ell^2 \sin^2 \varphi}.
\]

Here \( (\beta_2)_- = -\sin \varphi \) corresponds to the intersection point \( -y \), while

\[
(\beta_2)_+ = \sin \varphi \frac{R^2 - \ell^2 \sin^2 \varphi}{R^2 + \ell^2 \sin^2 \varphi}.
\]

Here (2.1) holds.
corresponds to the second intersection point. This point has the coordinates

\[
\xi_1 = 0, \quad \xi_2 = \ell \sin \varphi + \beta_2, \quad \xi_3 = R - \ell \sqrt{\sin^2 \varphi - (\beta_2)^2}.
\]

(2.2)

Hence,

\[
\xi_1 = 0, \quad \xi_2 = \frac{2\ell R^2 \sin \varphi}{R^2 + \ell^2 \sin^2 \varphi}, \quad \xi_3 = R - \frac{2\ell R^2 \sin^2 \varphi}{R^2 + \ell^2 \sin^2 \varphi}.
\]

(2.3)

Obviously that \( \xi_3 \in (-R, R) \) for \( \varphi \in (0, \pi) \).

Consider now two points \( z^{(1)} = (\ell \cos(\pi/6), \ell \sin(\pi/6), R) \), \( z^{(2)} = (\ell \cos(5\pi/6), \ell \sin(5\pi/6), R) \). These points lie symmetrically with respect to the plane \( \xi_1 = 0 \). Therefore they have the same intersection points with the plane \( \xi_1 = 0 \), and the coordinates of \( \xi^* \neq -y \) are determined by the formulae (2.3) with \( \varphi = \pi/6 \), that is,

\[
\xi_1^* = 0, \quad \xi_2^* = \frac{2\ell R^2}{4R^2 + \ell^2}, \quad \xi_3^* = R - \frac{2\ell R^2}{4R^2 + \ell^2}.
\]

(2.4)

Denote by \( S_{12}(\ell) \) the piece of \( S \) bounded by the meridional semi-planes \( \varphi = \pi/6, \varphi = 5\pi/6 \) and by the plane \( \xi_3 = \xi_3^* \). Obviously \( S_{12}(\ell) \subset (S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)}) \cup S_+(z^{(4)})). \) Now we consider 3 points: \( z^{(1)} = (\ell \cos(\pi/6), \ell \sin(\pi/6), R) \), \( z^{(2)} = (\ell \cos(5\pi/6), \ell \sin(5\pi/6), R) \) and \( z^{(3)} = (\ell \cos(3\pi/2), \ell \sin(3\pi/2), R) \). All these points are in a symmetrical position with respect to one to other. Therefore if we denote by \( S_{13}(\ell) \) the part of \( S \) bounded by the meridional semi-planes \( \varphi = \pi/6, \varphi = 3\pi/2 \) and by the plane \( \xi_3 = \xi_3^* \), then we find that \( S_{13}(\ell) \subset (S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)})) \).

Similarly, if \( S_{23}(\ell) \) denotes the part of \( S \) bounded by the meridional semi-planes \( \varphi = 5\pi/6, \varphi = 3\pi/2 \) and by the plane \( \xi_3 = \xi_3^* \), then \( S_{23}(\ell) \subset (S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)})). \) Hence \( S(\ell) := (S_{12}(\ell) \cup S_{13}(\ell) \cup S_{23}(\ell)) \subset (S_+(z^{(1)}) \cup S_+(z^{(2)}) \cup S_+(z^{(3)})) \). On the other hand, \( S(\ell) = \{ \xi \in \mathbb{R}^3 \mid |\xi| = R, \xi_3 \geq \xi_3^* \} \). Note that \( \xi_3^* \to -R \) as \( \ell \to \infty \). Choose \( \ell \) such that \( S^*(y) \subset S(\ell) \). For this we first need find \( h = \inf \xi_3 \) for all \( \xi \in S^*(y) \). By \( \gamma \) we denote the angle between the \( \xi_3 \)-axis and a straight line which passes \( y \) and is tangent on \( \Omega_0 \). Then \( \sin \gamma = R/\ell \). The length of the piece of this straight line included in \( \Omega \) is \( 2R \cos \gamma \) and \( h = 2R \cos^2 \gamma - R \). Making simple calculations we find

\[
h = 2R \left( 1 - \frac{R^2}{R^2} \right) - R.
\]

The inclusion \( S^*(y) \subset S(\ell) \) holds if \( h \geq \xi_3^* \), that is,

\[
2R \left( 1 - \frac{R^2}{R^2} \right) - R \geq R - \frac{2R\ell^2}{4R^2 + \ell^2}.
\]

The latter inequality holds if \( \ell \geq \ell^* \) where \( \ell^* \) is given by (2.1).

\[ \square \]

**Remark 1.** If \( R > \sqrt{2}R_0 \) then instead of 3 points \( z^j, j = 1, 2, 3 \), one can use only one point \( z = -\lambda y \) with \( \lambda \geq R^2/(R^2 - \sqrt{2}R_0^2) \). In this case \( S^*(y) \subset S_+(z) \). Thus in this case one auxiliary point \( z = -\lambda y \) and the data on the full interference field corresponding point sources placed at \( y \) and \( z \) play the same role as the points \( z^j, j = 1, 2, 3 \), and data on 3 full interference fields related to them.
3 The phaseless inverse problem of determining a potential for the Schrödinger equation

Throughout this paper, let $i = \sqrt{-1}$ and $\delta$ be the Dirac delta function.

We study here the phaseless inverse problem for the equation

$$-(\Delta + k^2 - q(x))u = \delta(x - y) + \delta(x - z), \ x \in \mathbb{R}^3. \quad (3.1)$$

Let a function $u = u(x, y, z, k)$ satisfy the equation (3.1) and the radiation conditions

$$u = O(r^{-1}), \ \frac{\partial u}{\partial r} - iku = o(r^{-1}) \ \text{as} \ r = |x| \to \infty. \quad (3.2)$$

Here conditions (3.2) is assumed to be valid uniformly for all the direct ions $x/r$.

We assume that a potential $q(x)$ satisfies

$$q \in C^4(\mathbb{R}^3), \ q(x) \geq 0, \ x \in \mathbb{R}, \ \text{supp} \ q \subset \Omega_0. \quad (3.3)$$

The solution of the problem (3.1) and (3.2) describes an interference of waves produced by the sources at points $y$ and $z$. We suppose that the modulus of the function $u(x, y, z, k)$ can be measured on the sets $S_+^{(j)}(z_j)$, $j = 1, 2, 3$, introduced above.

**Problem 1.** Let $y \in S$ and $z_j = z_j(y)$, $j = 1, 2, 3$ be defined as above. Then find $q(x)$ in $\Omega_0$ from

$$|u(x, y, z_j, k)|^2 = f_j(x, y, k), \ y \in S, \ x \in S_+^{(j)}(z_j), \ k \geq k_0, \ j = 1, 2, 3, \quad (3.4)$$

where $u(x, y, z_j, k)$ is the solution to the problem (3.1)-(3.2) with $z = z_j$ and $k_0 > 0$ is a fixed positive number.

Our first main result is the uniqueness in the inverse problem.

**Theorem 1.** Let a potential $q(x)$ satisfy (3.3) and the additional condition $|z_j(y) + y| = \ell > R\sqrt{8}, \ j = 1, 2, 3$. Then $q$ is uniquely determined by the information (3.4).

Proof. Represent the solution $u(x, y, z, k)$ of the problem (3.1)-(3.2) in the form

$$u(x, y, z, k) = v(x, y, k) + v(x, z, k), \quad (3.5)$$

where $v(x, y, k)$ satisfies the equation

$$-(\Delta + k^2 - q(x))v = \delta(x - y), \ x \in \mathbb{R}^3 \quad (3.6)$$

and the radiation conditions. Then $v(x, y, k)$ can be represented in the form

$$v(x, y, k) = u_0(x, y, k) + v_{sc}(x, y, k), \quad (3.7)$$

where $u_0(x, y, k)$ is defined by the formula

$$v_0(x, y, k) = A_0(x, y)e^{ik|x-z|}, \ A_0(x, y) = \frac{1}{4\pi|x-y|}, \quad (3.8)$$
and is the fundamental solution of the Helmholtz operator $-\Delta - k^2$ with the conditions (3.2) and $v_{sc}(x, y, k)$ is the scattering field from the point source located at $y$.

Consider the asymptotic behavior of $v_{sc}(x, y, k)$ as $k \to \infty$. Theorem 1 in [7] yields

**Lemma 2.** Suppose that a potential $q(x)$ satisfies conditions (6.3). Then for each pair of points $x, y$ with $x \neq y$ the asymptotic behavior of the function $v_{sc}(x, x^0, k)$ is

$$v_{sc}(x, y, k) = \frac{i \exp(ik|x-y|)}{8\pi|x-y|k} \left[ \int_{L(x,y)} q(\xi)d\sigma + O\left(\frac{1}{k}\right) \right], \quad k \to \infty,$$

where $L(x, y)$ is the segment of the straight line connecting $x$ and $y$, and $d\sigma$ is the arc length.

By this lemma, we can obtain the asymptotic behavior of the data (3.4):

$$f_j(x, y, k) = \left| A_0(x, y)e^{ik|x-y|} + \frac{i e^{ik|x-y|}}{8\pi|x-y|k} \int_{L(x,y)} q(\xi)d\sigma + A_0(x, z^{(j)}(y))e^{ik|x-z^{(j)}(y)|} \right|^2 + O\left(\frac{1}{k^2}\right)$$

$$= A_0^2(x, y) + A_0^2(x, z^{(j)}(y)) + 2A_0(x, y)A_0(x, z^{(j)}(y)) \cos(k|x-z^{(j)}(y)| - k|x-y|)$$

$$+ \frac{A_0(x, z^{(j)}(y))}{4\pi|x-y|k} \int_{L(x,y)} q(\xi)d\sigma \sin(k|x-z^{(j)}(y)| - k|x-y|) + O\left(\frac{1}{k^2}\right),$$

$$y \in S, \quad x \in S_+(z^{(j)}), \quad k \geq k_0, \quad j = 1, 2, 3.$$

For $x \in S_+(z^{(j)})$, we have

$$\min_{x \in S_+(z^{(j)})} |x - z^{(j)}(y)| \geq \sqrt{\ell^2 + R^2} - R, \quad \max_{x \in S_+(z^{(j)})} |x - y| \leq 2R.$$

Therefore under the condition $\sqrt{\ell^2 + R^2} - R > 2R$, that is, $\ell > R\sqrt{8}$, the inequality $|x - z^{(j)}(y)| - |x - y| > 0$ holds for all $x \in S_+(z^{(j)})$. Fix $y$ and $x \in S_+(z^{(j)})$ and take $k = k_m(x, y) = \pi/2 + 2\pi/(|x - z^{(j)}(y)| - |x - y|)$. Then we obtain

$$f_j(x, y, k_m(x, y)) = A_0^2(x, y) + A_0^2(x, z^{(j)}(y))$$

$$+ \frac{A_0(x, z^{(j)}(y))}{4\pi|x-y|k_m(x, y)} \int_{L(x,y)} q(\xi)d\sigma + O\left(\frac{1}{k_m^2(x, y)}\right),$$

$$y \in S, \quad x \in S_+(z^{(j)}), \quad k_m(x, y) \geq k_0, \quad j = 1, 2, 3.$$

Hence,

$$\int_{L(x,y)} q(\xi)d\sigma = g_j(x, y), \quad y \in S, \quad x \in S_+(z^{(j)}), \quad j = 1, 2, 3,$$

where

$$g_j(x, y) = 4\pi|x-y| \lim_{k_m(x,y) \to \infty} \frac{[f_j(x, y, k_m(x, y)) - A_0^2(x, y) - A_0^2(x, z^{(j)}(y))]k_m(x, y)}{A_0(x, z^{(j)}(y))},$$

$$y \in S, \quad x \in S_+(z^{(j)}), \quad j = 1, 2, 3.$$
Because $S^*(y) \subset (S_+^{(1)} \cup S_+^{(2)} \cup S_+^{(3)})$, the equality yields that integrals over $q(x)$ along $L(x, y)$ are given for all $y \in S$ and $x \in S^*(y)$. Hence we know the integrals for all straight lines that cross out the ball $\Omega_0$. Hence we reach a usual tomography problem for recovering $q(x)$ inside $\Omega_0$. It proves Theorem 1.

We see that the given information related to full fields for reflected waves allows us to reduce the phaseless inverse problem to the same tomography problem as for data related to simple sources at $y$ and the scattering field.

The same idea successively works in many others phaseless inverse problems. Below we consider such an inverse problem for the Helmholtz equation.

### 4 The phaseless inverse problem of determining a refractive index for the Helmholtz equation

Following [11], we consider the phaseless inverse problem for the Helmholtz equation. Let $\Omega_0$, $\Omega$ and $S$ be as in the previous section. Let $n(x), x \in \mathbb{R}^3$ be a real valued function satisfying

$$n \in C^{15}(\mathbb{R}^3), \quad 1 \leq n(x) \leq n_1 < \infty, \quad x \in \mathbb{R}, \quad \text{supp}(n - 1) \subset \Omega_0,$$

where $n_1$ is a given constant. Let $u(x, y, z, k)$ satisfy

$$-(\Delta + k^2 n(x))u = \delta(x - y) + \delta(x - z), \quad x \in \mathbb{R}^3,$$

and the radiation condition (3.2). Consider the following problem

**Problem 2.** Let $y \in S$ and $z^{(j)} = z^{(j)}(y), j = 1, 2, 3$, be defined as above. Find $n(x)$ in $\Omega_0$ from the data

$$|u(x, y, z^{(j)}, k)|^2 = f_j(x, y, k), \quad y \in S, \quad x \in S_+(z^{(j)}), \quad k \geq k_0, \quad j = 1, 2, 3,$$

where $u(x, y, z^{(j)}, k)$ is the solution to problem (4.2) and (3.2) with $z = z^{(j)}$ and $k_0 > 0$ is a fixed positive number.

Introduce the conformal Riemannian metric by the formula

$$d\tau = n(x)|dx|, \quad |dx| = \left(\sum_{i=1}^{3} dx_i^2\right)^{1/2},$$

where $d\tau$ is the element of length and by $\tau(x, y)$ we denote the Riemannian distance between the points $x, y \in \mathbb{R}^3$. As in [11], we shall pose the following assumption.

**Assumption.** The Riemannian metric $d\tau = n(x)|dx|$ is simple, that is, every two points $x, y \in \mathbb{R}^3$ can be connected by a single geodesic line $\Gamma(x, y)$.

**Theorem 2.** Let conditions (4.1) and Assumption be fulfilled. Additionally we assume that $z^{(j)}(y)$ satisfies $|z^{(j)}(y) + y| = \ell > R \sqrt{(1 + 2n_1)^2 - 1}, \quad j = 1, 2, 3$. Then the data (4.3) uniquely determines $n(x)$ inside $\Omega_0$. 

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*Note: The above text is a transcription of the given document. The formatting and some mathematical expressions have been adapted for better readability.*
Proof. We use the following lemma which is a corollary of Theorem 2 and formula (3.18) from [11].

Lemma 3. Let conditions (4.1) and Assumption be fulfilled. If \( v = v(x, y, k) \) satisfies

\[
-(\Delta + k^2 n(x))v = \delta(x - y), \quad x \in \mathbb{R}^3
\]

and the radiation condition

\[
v = O(r^{-1}), \quad \frac{\partial v}{\partial r} - ikv = o(r^{-1}) \quad \text{as} \quad r = |x| \to \infty
\]

for \( x \neq y \), then we have the asymptotic formula

\[
v(x, y, k) = A(x, y)e^{ik\tau(x,y)} + O \left( \frac{1}{k} \right) \quad \text{as} \quad k \to \infty,
\]

where \( A = A(x, y) > 0, \in C^{1,2}((\mathbb{R}^3 \setminus \{y\}) \times S) \).

It follows from Lemma 3 that the solution of the problem (4.2) and (3.2) for \( x \neq y \) posseses the asymptotic behavior of the form

\[
u(x, y, z, k) = A(x, y)e^{ik\tau(x, y)} + A(x, z)e^{ik\tau(x, z)} + O \left( \frac{1}{k} \right) \quad \text{as} \quad k \to \infty.
\]

Hence we conclude that

\[
f_j(x, y, k) = |u(x, y, z^{(j)}(y), k)|^2 = A^2(x, y) + A^2(x, z^{(j)}(y))
\]

\[
+ 2A(x, y)A(x, z^{(j)}(y))\cos(k\tau(x, z^{(j)}(y)) - k\tau(x, y)) + O \left( \frac{1}{k} \right)
\]

\[
as \quad k \to \infty, \quad y \in S, \quad x \in S_+(z^{(j)}(y)).
\]

Note that \( \tau(x, z^{(j)}(y)) = |x - z^{(j)}(y)| \) for \( y \in S \) and \( x \in S_+(z^{(j)}(y)) \). Moreover,

\[
\min \tau(x, z^{(j)}(y)) \geq \sqrt{\ell^2 + R^2} - R,
\]

where \( \ell = |z^{(j)}(y) + y| \). On the other hand, for \( y \in S \) and \( x \in S_+(z^{(j)}(y)) \) we have \( \tau(x, y) \leq 2Rn_1 \), where \( n_1 \) is the bound given in (4.1).

The latter equality holds because for fixed \( x \) and \( y \), the function \( \tau(x, y) \) minimizes integrals of \( n(x) \) over arbitrary smooth curves connecting \( x \) and \( y \). In particular, \( \tau(x, y) \) is less than the integral along the straight line \( L(x, y) \). Therefore \( \tau(x, y) \leq n_1|x - y| \leq 2n_1R \). Since the condition in Theorem 2 implies \( \ell > R\sqrt{(1 + 2n_1)^2 - 1} \), we have

\[
\rho(x, y) = \tau(x, z^{(j)}(y)) - \tau(x, y) \geq \sqrt{\ell^2 + R^2} - R - 2Rn_1 > 0.
\]

Fix \( x \) and \( y \) in (4.8). Then the left-hand side is an almost periodic function of \( k \). Hence, we can extract the period of this function and find the difference \( \rho(x, y) = \tau(x, z^{(j)}(y)) - \tau(x, y) \). The procedure of extracting the period of almost periodic function is given in the paper [12]. Then one can calculate \( \tau(x, y) = \tau(x, z^{(j)}(y)) - \rho(x, y) \) for all \( y \in S \) and \( x \in S_+(z^{(j)}(y)) \). Since \( S^*(y) \subset (S_+(z^{(1)}(y)) \cup S_+(z^{(2)}(y)) \cup S_+(z^{(3)}(y))), \) we find \( \tau(x, y) \) for all \( y \in S \) and \( x \in S^*(y) \).
Note that $\tau(x, y) = |x - y|$ for all $y \in S$ and $x \in S \setminus S^*(y)$. Therefore $\tau(x, y)$ is known for all $(x, y) \in S \times S$. Therefore we reach the well-known inverse kinematic problem: find $n(x)$ in $\Omega_0$ from given $\tau(x, y)$ for all $(x, y) \in S \times S$.

The multidimensional inverse kinematic problem was studied for first time in a linear approximations in [13, 19, 20]. The nonlinear problem was studied under Assumption in the papers [1, 14, 15], where uniqueness and stability theorems were shown. From the results given in these papers, the proof of Theorem 2 is complete.

\begin{acknowledgments}

The work of V. G. Romanov was partially supported by the Russian Foundation for Basic Research grant No. 17-01-00120. The work of M. Yamamoto was supported partly by Grant-in-Aid for Scientific Research (S) 15H05740 of Japan Society for the Promotion of Science and the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02 a03. 21. 0008).

\end{acknowledgments}

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