When is $F(p)$ the Laplace transform of a bounded $f(t)$?

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Abstract

Sufficient conditions are given for a function $F(p)$, analytic in $Re p > 0$, to be a Laplace transform of a function $f(t)$, such that $\max_{t \geq 0} |f(t)| < \infty$, $f(0) = 0$.

1 Introduction

There is a large literature on the Laplace transform, e.g., [2], [3]. Tables of the Laplace transform of distributions are published, see, e.g., [1]. The Laplace transform of a function $f(t)$, $f(t) = 0$ for $t < 0$, $|f(t)| \leq Ce^{at}$ for some positive constants $C$ and $a$, is defined by the formula

$$L(f) := F(p) := \int_0^\infty e^{-pt} f(t) dt, \quad p = s + i\eta, \quad s > a.$$ (1)

Under this assumption $F(p)$ is an analytic function of $p$ in the half-plane $s > a$ and $|F(p)| \leq \frac{C}{s-a}$, $s = Re p$, as follows from the estimate

$$|F(p)| = \left| \int_0^\infty e^{-pt} f(t) dt \right| \leq C \int_0^\infty e^{-(s-a)t} dt = \frac{C}{s-a}, \quad s > a.$$ (2)

We assume that

$$\lim_{|p| \to \infty, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}} F(p) = 0,$$ (3)

where $\phi = arg p$ is the argument of $p$.

If $f \in L^2(0, \infty)$, then $a = 0$. In Section 3 we discuss possible generalizations.

We are interested in the sufficient conditions for $F(p)$ to be the Laplace transform of a function $f(t)$, such that

$$\sup_{t \geq 0} |f(t)| < \infty, \quad f(0) = 0.$$ (4)

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These conditions are (2), (3) and
\[ |F(i\eta)| = O(|i\eta|^{-b}), \quad b > 1 \quad \text{for} \quad |\eta| \gg 1; \quad F(p) \text{is analytic for} \quad s > 0. \] (5)

The condition \( F(i\eta) \in L^2(-\infty, \infty) \) follows from the assumption \( f(t) \in L^2(0, \infty) \) by the Parce-val’s identity. This condition is not necessary for \( F(p) \) to be the Laplace transform of \( f(t) \) satisfying conditions (4).

Our result is formulated in the following theorem.

**Theorem 1.** If conditions (2), (3) and (5) hold, then \( F(p) = L(f) \) and \( f(t) \) satisfies conditions (4).

This result is new. It differs from the known results: usually some assumptions are made on \( f(t) \) and necessary conditions are derived for \( F(p) \) to be the Laplace transform. In Theorem 1 assumptions are made on \( F(p) \) and sufficient conditions are given for \( F(p) \) to be the Laplace transform of a function \( f(t) \) satisfying conditions (4) and vanishing for \( t < 0 \).

## 2 Proof

Let
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta t} F(i\eta) d\eta. \] (6)

The integral (6) converges since \( b > 1 \). Therefore, \( L(f) = F(p) \), \( f(t) \) is a continuous function and \( \sup_{t \geq 0} |f(t)| < \infty \). This follows from formula (6) and the known inversion formula for the Laplace transform, (3), namely:
\[ f(t) = \frac{1}{2\pi i} \int_{C_\sigma} e^{pt} F(p) dp, \] (7)

where \( C_\sigma, \sigma > a \), is the straight line \( \sigma - i\eta, \sigma + i\eta, -\infty < \eta < \infty \). In our case \( \sigma = 0 \), \( dp = id\eta \) and \( C_\sigma \) is the straight line \( -i\eta, i\eta, -\infty < \eta < \infty \).

For convenience of the reader, let us give a version of a proof of the inversion formula. Under our assumptions \( f(t) \), defined in (6) is a continuous function on \([0, \infty)\), uniformly bounded because the function \( F(i\eta) \) is absolutely integrable on the whole axis \( -\infty < \eta < \infty \). Consider
\[ \int_0^\infty f(t) e^{-qt} dt = \frac{1}{2\pi} \int_{-\infty}^\infty d\eta F(i\eta) \int_0^\infty e^{-(q-i\eta)t} dt = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} F(p) \frac{1}{p-q} dp, \quad p = i\eta. \] (8)

Let \( C_n \) be the closed contour consisting of \([in, -in] \cup L_n\), where \( L_n \) is the semi-circle \( p = ne^{i\phi}, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \). Since \( b > 1 \), it follows that
\[ \lim_{n \to \infty} \int_{L_n} F(p) \frac{1}{p-q} dp = 0. \] (9)

Therefore, the integral on the right side of (8) can be considered as the integral over the closed contour \( C_n = \) with \( n \to \infty \). The minus sign in (8) is used to get the closed contour passed counterclockwise. Consequently, the integral in the right side of (8) is equal to \( F(q) \) by the Cauchy formula and the analyticity of \( F(p) \) in the half-plane \( \text{Re}p > 0 \). Thus,
\[ \int_0^\infty f(t)e^{-qt} dt = F(q). \] (10)
Therefore, the inversion formula for the Laplace transform is proved.

Let us prove that \( f(0) = 0 \). It follows from (6) that

\[
f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\eta) d\eta. \tag{11}
\]

If \( N \) is sufficiently large for the estimate

\[
|F(p)| \leq c(1 + |p|)^{-b}, \quad |p| > N, \quad b > 1,
\]

to hold, then

\[
f(0) = \frac{1}{2\pi} \int_{-N}^{N} F(i\eta) d\eta + o(1) \quad \text{as} \quad N \to \infty. \tag{13}
\]

The function \( f(t) \) is continuous and uniformly bounded on \([0, \infty)\). Indeed, by the inversion formula (7) with \( \sigma = 0 \), where the integral over the straight line \( \sigma = 0 \) absolutely converges if \( b > 1 \), one has

\[
f(t) = \frac{1}{2\pi i} \int_{-iN}^{iN} e^{pt} F(p) dp + \frac{1}{2\pi} \int_{-\infty}^{-iN} e^{pt} F(p) dp + \frac{1}{2\pi} \int_{iN}^{i\infty} e^{pt} F(p) dp. \tag{14}
\]

The first integral is a continuous function of \( t \) because it is taken over a compact set and \( F(p) \in L^1(-iN, iN) \), the second and third integrals are continuous functions of \( t \) because \( b > 1 \). At \( t = 0 \) formula (14) reduces to (13).

If we prove that

\[
I_N := \frac{1}{2\pi} \int_{-N}^{N} F(i\eta) d\eta \to 0 \quad \text{as} \quad N \to \infty, \tag{15}
\]

then relations (13)-(15) imply that \( f(0) = 0 \) and Theorem 1 is proved.

To prove (15), consider a closed contour \( C_N \), consisting of \([-iN, iN]\) and the semi-circle \( L_N := Ne^{i\phi} \), where \(-\pi/2 \leq \phi \leq \pi/2\). Since \( F(p) \) is analytic in the half-plane \( s > 0 \), one has:

\[
\int_{C_N} F(p) dp = 0. \tag{16}
\]

Since \( \lim_{N \to \infty} \frac{1}{2\pi} \int_{L_N} F(p) dp = 0 \) because \( b > 1 \), it follows from (16) that equation (15) is valid.

Finally, the condition \( f(t) = 0 \) for \( t < 0 \) follows from the following argument. If \( t < 0 \) then the function \( e^{-pt} F(p) \), \( t > 0 \) is analytic in \( \text{Rep} > 0 \) and

\[
f(-t) = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_N} e^{-pt} F(p) dp = 0, \quad t > 0, \tag{17}
\]

because by the Cauchy theorem the integral over the closed contour \( C_N \), inside which the function \( e^{-pt} F(p) \) is analytic, is equal to zero.

Theorem 1 is proved. \( \square \)

The Jordan lemma in the following form, compare [3], pp. 412, 469, is useful:

**Lemma.** If a function \( h(p) \) tends to zero uniformly with respect to the argument \( \phi \) of \( p \), \(-\pi/2 \leq \phi \leq \pi/2\) on the contour \( L_n \) as \( n \to \infty \) then for \( t > 0 \) one has \( \lim_{n \to \infty} \int_{L_n} h(p)e^{pt} dp = 0 \).
3 Discussion

In this Section we discuss possible generalization and applications of Theorem 1. We may replace the assumption $F(i\eta) \in L^2(-\infty, \infty)$ by the assumption

$$F(i\eta) \in L^p(-iN, iN), \quad p \geq 1,$$

(18)

The proof of Theorem 1 remains unchanged because the assumption $F(i\eta) \in L^p(-iN, iN), \quad p \geq 1$ implies that the first integral in (14) is a continuous function of $t$ and assumption (12) guarantees that the second and third integrals in (14) are continuous functions of $t$, which tend to zero as $N \to \infty$.

In applications Theorem 1 is useful in the definition of hyper-singular integrals, see [4]. To give an idea of this application, consider the integral equation:

$$q(t) + \frac{1}{\Gamma(\lambda)} \int_0^t (t - \tau)^{\lambda-1} q(\tau) d\tau = f(t).$$

(19)

If $\text{Re}\lambda > 0$, then the integral in (19) is defined classically, i.e., from the classical point of view. If $\lambda < 0$, then this integral is a hyper-singular integral, it diverges classically. For $\text{Re}\lambda > 0$ we take the Laplace transform of (19) and get

$$L(q)(1 + p^{-\lambda}) = L(f), \quad L(q) = \frac{L(f)}{1 + p^{-\lambda}},$$

(20)

where the following formula was used: $L(t^{\lambda-1}) = \Gamma(\lambda) p^{-\lambda}$. This formula is valid for all $\lambda \in \mathbb{C}$ except for $\lambda = 0, -1, -2, \ldots$. The $L(q)$ in (20) admits analytic continuation with respect to $\lambda$ from the region $\text{Re}\lambda > 0$ to the region $\text{Re}\lambda < 0$, for example, to the point $\lambda = -\frac{1}{4}$ which is of interest in the Navier-Stokes problem. Theorem 1 is of use to prove that $\frac{L(f)}{1 + p^{-\lambda}}$ for $\lambda = -\frac{1}{4}$ is the Laplace transform of a function $q(t)$ satisfying conditions (14). This can be checked if $f(t)$ smooth and rapidly decaying as $t \to \infty$, so that $|L(f)| \leq c(1 + |p|)^{-1}$. In this case the function $(1 + p^{\frac{1}{2}})^{-1}$ is analytic in the half-plane $\text{Re} p \geq 0$ and is $O\left(\frac{1}{|p|^{\frac{3}{2}}}\right)$ for $|p| \gg 1$, the function $L(f)$ is analytic in the half-plane $\text{Re} p \geq 0$ and is $O\left(\frac{1}{|p|}\right)$ for $|p| \gg 1$ on the imaginary axis of the complex plane $p = s + i\eta$. By Theorem 1, the function $\frac{L(f)}{1 + p^{-\lambda}}$ is the Laplace transform of a function $q(t)$ satisfying (14). In this example $b = \frac{5}{4} > 1$. We have proved the following result.

**Theorem 2.** Assume $\lambda = -\frac{1}{4}$ and $f(t)$ be a smooth rapidly decaying as $t \to \infty$. Then equation (19) has a unique solution $q(t)$ satisfying (14).

The kernel of equation (19) with $\lambda = -\frac{1}{4}$ is hyper-singular. The integral in this equation with $\lambda = -\frac{1}{4}$ diverges classically. Theorem 2 is of prime interest in a study of the Navier-Stokes problem, see [4].

4 Conclusion

Sufficient conditions are given for a function $F(p), p = s + i\eta$, analytic in the half-plane $s > 0$ to be the Laplace transform of a function $f(t), f(t) = 0$ for $t < 0$, $\sup_{t \geq 0} |f(t)| < \infty$ and $f(0) = 0$. This result is useful in a study of the Navier-Stokes problem in $\mathbb{R}^3$ and in a study of integral equations with hyper-singular kernels, see [4].
References

[1] Yu. Brychkov, A. Prudnikov, *Integral tranforms of generalized functions*, Nauka, Moskow, 1977 (in Russian)

[2] J. Schiff, The Laplace transform, Springer, Berlin, 1999.

[3] M.Lavrentiev, B. Shabat, Methods of the theory of functions of complex variable, GIFML, Moscow, 1958. (in Russian)

[4] A.G.Ramm, *The Navier-Stokes problem*, Morgan & Claypool publishers, 2021.