GENERALISED HILBERT NUMERATORS II

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Abstract. Let $K[[X]]$ denote the large polynomial ring (in the sense of Halter-Koch) on the set $X = \{x_1, x_2, x_3, \ldots\}$ of indeterminates. For each integer $n$, there is a truncation homomorphism $\rho_n: K[[X]] \to K[x_1, \ldots, x_n]$. If $I$ is a homogeneous ideal of $K[[X]]$, then the $\mathbb{N}$-graded Hilbert series of $\frac{K[[x_1, \ldots, x_n]]}{\rho_n(I)}$ can be written as $\frac{g_n(t)}{(1-t)^n}$; it was shown in [13] that if in addition $I$ is what we call locally finitely generated, then $g_n(t) \to g(t) \in \mathbb{Z}[[t]]$, the so-called Hilbert numerator of $I$.

In this article, we generalise this result to $\mathbb{N}'$-graded locally finitely generated ideals. For monomial ideals in $K[[X]]$, we define the $[X]$-graded Hilbert numerator as the Hilbert numerator of the contracted monomial ideal in $K[X]$, for which the standard combinatorial and homological methods for calculating multi-graded Hilbert series of monomial ideals in finitely many variables apply. Finally, we show that all polynomial $\mathbb{N}$-graded Hilbert numerators can be obtained from ideals generated in finitely many variables, and that the the closure in $\mathbb{Z}[[t]]$ of this set is the set of all $\mathbb{N}$-graded Hilbert numerators.

Our main tools are the approximation theorem of [11], relating the initial ideal with the initial ideal of the truncated ideal, and the identification of $K[[X]]$ with the ring of all number-theoretic functions [5] which allows the passing from the characteristic function of the complement of a monomial ideal to its Hilbert numerator to be seen as an example of Möbius inversion.

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1. Introduction

The ring $K[[X]]$ of formal polynomials was used by Halter-Koch [3] to study polynomial functions on modules. The author used it to study initial ideals of generic ideals of the same type, for instance generated by a quadratic and a cubic generic form, but in ever more variables [10, 11, 12]. In brief, it is the largest $\mathbb{N}$-graded subring of the power series ring $K[[X]]$ on a countable set $X$ of indeterminates. There are truncation maps $\rho_n : K[[X]] \to K[x_1, \ldots, x_n]$, and inclusion maps going the other way, which relate ideals in $K[[X]]$ with sequences $(I_n)_{n=1}^\infty$ of ideals, where $I_n$ is an ideal in $K[x_1, \ldots, x_n]$, and where $I_{n+1}$ maps to $I_n$ under the map

$$K[x_1, \ldots, x_{n+1}] \to \frac{K[x_1, \ldots, x_{n+1}]}{(x_{n+1})} \simeq K[x_1, \ldots, x_n].$$

As an example, choose positive integers $d_1, \ldots, d_r$, and let for each positive integer $n$ $f_1^{(n)}, \ldots, f_r^{(n)}$ be forms in $n$ variables of degree $d_1$ to $d_r$. Suppose furthermore that $f_1^{(n+1)} - f_1^{(n)}$ is divisible by $x_{n+1}$, and that the coefficients of the forms are chosen randomly. Let $I_n = (f_1^{(n)}, \ldots, f_r^{(n)})$, and let $r > t$ the lexicographic term order on the various polynomial rings. Then the initial ideals in $(I_n)$ will converge to a monomial ideal in infinitely many indeterminates, as $n \to \infty$, and this monomial ideal is the lex-initial ideal of $I = (f_1, \ldots, f_r) \subset K[[X]]$, where $f_\ell = \lim f_\ell^{(n)}$. In this case, we have that for $n \geq r$, the Hilbert series of $\frac{K[x_1, \ldots, x_n]}{I_n}$ is

$$(1 - t)^n \prod_{j=1}^r (1 - t^{d_j}).$$

For arbitrary homogeneous, finitely generated ideals $I \subset K[[X]]$, we conjecture that

$$(1 - t)^n H \left( \frac{K[x_1, \ldots, x_n]}{\rho_n(I)}; t \right)$$

is eventually constant, where $H \left( \frac{K[x_1, \ldots, x_n]}{\rho_n(I)}; t \right)$ denotes the Hilbert series of $\frac{K[x_1, \ldots, x_n]}{\rho_n(I)}$. What we have in fact proved [13] is that for all homogeneous ideal which are countably generated and which have but finitely many minimal generators of each total degree (we call such an ideal locally finitely generated), the polynomials

$$(1 - t)^n H \left( \frac{K[x_1, \ldots, x_n]}{\rho_n(I)}; t \right) \to p(t) \in \mathbb{Z}[t],$$

we call the power series $p(t)$ the generalised Hilbert numerator of $I$. The outstanding question is thus whether for finitely generated ideals, $p(t)$ is always a polynomial.

In this article, we first show that for monomial ideals in $K[X]$, the polynomial ring on countably many indeterminates, the usual methods of calculating multigraded Hilbert series can be used, and that this Hilbert series can always be written $\frac{p(X)}{\prod_{i=1}^r (1-x_i)}$. We call $p(X)$ the $[X]$-multigraded Hilbert numerator of the ideal. For a locally finitely generated ideal in $K[[X]]$, we form its initial ideal, contract it to a monomial ideal in $K[X]$, calculate the $[X]$-multigraded Hilbert numerator of that, then collapse the grading to a $\mathbb{N}$-grading to get the $\mathbb{N}$-graded Hilbert numerator. Apart from providing a more attractive proof of the existence of Hilbert numerators, this methodology yields immediately $\mathbb{N}^r$-graded Hilbert numerators for $\mathbb{N}^r$-graded locally finitely generated ideals of $K[[X]]$.

We give exact (although opaque) descriptions of the set of all $[X]$-graded Hilbert numerators of monomial ideals, and the set of all $\mathbb{N}$-graded Hilbert numerators of locally finitely generated ideals of $K[[X]]$. The latter result can be described briefly as follows: the set of all polynomial
\(N\)-graded Hilbert numerators is the set of sufficiently high iterated differences of admissible \(H\)-vectors (in the sense of Macaulay’s characterisation of admissible Hilbert functions of finitely generated algebras), and the set of all \(N\)-graded Hilbert numerators is the closure in \(\mathbb{Z}[[t]]\) of the previous set.

2. Notation

Let \(\mathbb{Z}, \mathbb{N}, \mathbb{N}^+\) denote the set of integers, non-negative integers, and positive integers, respectively. For any set \(A\) and any positive integer \(k\), we denote by \(\binom{A}{k}\) the set of \(k\)-subsets of \(A\), by \([A]\) the free abelian monoid on \(A\), and by \([A]_k\) the subset of monomials of total degree \(k\).

If \(m \in [A]_k\) we write \(|m| = k\).

If \(B\) is another set, then \(B^A\) denotes the set of all functions \(A \to B\). If \(A\) is pointed, that is, has an extinguished zero element (for instance, \([A]\) is pointed), then \(B^{([A])}\) denotes the set of all finitely supported maps \(A \to B\).

If \(K\) is a commutative ring, then \(K[[A]]\) becomes a commutative \(K\)-algebra under component-wise addition and multiplication of scalars, and with multiplication given by the convolution product

\[ f \times g(m) = \sum_{t|m} f(t)g(m/t). \]  

We denote this ring by \(K[[A]]\). The set \(B^{([A])}\) is a subring, which we denote by \(K[A]\).

3. Rings of formal power series and formal polynomials in countably many indeterminates

Let \(K\) be a field of containing the rational numbers, and let \(X = \{x_1, x_2, x_3, \ldots\}\) be a set of indeterminates. Form the large power series ring \(K[[X]]\) and the polynomial ring \(K[X]\) as above. For \(K[[X]] \ni f = \sum_{m \in [X]} c_m m\) we define

\[ \text{Supp}(f) = \{ m | c_m \neq 0 \} \]  

If \(t \in [X]\), we define

\[ [t] f = c_t. \]  

The ring \(K[X]\) is \([X]\)-graded, and in particular \(\mathbb{N}\)-graded, whereas \(K[[X]]\) is not. The largest \([X]\)-graded subring of \(K[[X]]\) is \(K[X]\), whereas the largest \(\mathbb{N}\)-graded subring is the ring \(K\langle X \rangle\) generated by all bounded elements: an element \(f \in K[[X]]\) is bounded if

\[ |f| := \sup \{ |m|m \in \text{Supp}(f) \} < \infty. \]

Another way of putting this is the following.

**Definition 3.1.** We define the total-degree filtration on \(K[[X]]\) and its various subrings by

\[ \mathcal{T}^d K[[X]] = \{ f \in K[[X]] | |f| \leq d \} \]  

For \(K[[X]] \ni f = \sum_{m \in [X]} c_m m\), we put

\[ \mathcal{T}^d f = \sum_{m \in [X]} \left[ \sum_{|m| \leq d} c_m m \right]. \]
Then $K[[X]] = \cup_{d \geq 0} T^d K[[X]]$.

It is shown in [10] that $K[[X]]$ is also the maximal subring of $K[[X]]$ with the following property: given any multiplicative total order $>$ on $[X]$ whose restriction to $[X]_1$ is order-isomorphic to $-\omega$ (such a $>$ will be called a term order on $[X]$), the support of any non-constant element $f$ contains a maximal element in $> (f)$. Putting $in_>(1) = 0$, $in_>(0) = -\infty$, we can regard

$$in_> : K[[X]] \rightarrow [X] \cup \{-\infty\}$$

as a $[X]$-valuation, which induces a $[X]$-filtration of $K[[X]]$ by

$$\mathfrak{F}^< m K[[X]] = \{ f \in K[[X]] | in_>(f) < m \}$$

$$\mathfrak{F}^\leq m K[[X]] = \{ f \in K[[X]] | in_>(f) \leq m \}$$

We then have a canonical map

$$K[[X]] \rightarrow \text{gr}(K[[X]]) = \bigoplus_{m \in [X]} \mathfrak{F}^\leq m K[[X]] \cong K[X]$$

$$f \mapsto in_>(f)$$

This map sends an ideal $I \subset K[[X]]$ to its initial ideal

$$in_>(I) = K[[X]] \{ in_>(f) \mid f \in I \}$$

which is a monomial ideal, that is, generated by monomials. We note that

1. Every monomial ideal is its own initial ideal,

2. Extension and contraction of ideals gives a bijection between monomial ideals in $K[X]$, $K[[X]]$, and $K[[X]]$,

3. Monomial ideals in $K[X]$, $K[[X]]$, or $K[[X]]$ correspond bijectively to monoid ideals in $[X]$.

Because of this identification, we shall say that a monoid ideal has a certain property whenever the corresponding monomial ideal has.

**Theorem 3.2** (Snellman [10]). For a $\mathbb{N}$-graded ideal $I \subset K[[X]]$, the following are equivalent:

1. $I$ is generated by a locally finite set, that is a set

   $$F = \bigcup_{d=1}^\infty F_i, \quad \forall i : F_i \in K[[X]]_i, \quad \forall i : \# F_i < \infty$$

2. For each positive integer $d$,

   $$\dim_K \left( \frac{I_d}{\sum_{i=1}^{d-1} K[[X]]_i I_{d-i}} \right) < \infty$$

We call such ideals locally finitely generated (lfg). By our previous remark, we can talk about lfg monoid ideals, as well.

**Theorem 3.3** (Snellman [10]). Let $>$ be a term-order on $[X]$, and $I \subset K[[X]]$ a homogeneous ideal. Then $I$ is lfg if and only if $in_>(I)$ is.
3.1. **Inverse limits.** We shall need to relate elements in $K[\langle X \rangle]$ with their *truncations* in $K[X_n]$. The necessary machinery is as follows.

For any positive integer $n$, we put $X_n = \{x_1, \ldots, x_n\}$, and let $[X_n]$ be the free abelian monoid on $X_n$. We define the polynomial ring $K[X_n]$ and the power series ring $K[[X_n]]$ as above. For $i < j$ there is a commutative diagram of $K$-multilinear maps

\[
\begin{array}{ccc}
K[X] & \longrightarrow & K[[X]] \\
\downarrow & & \downarrow \\
K[X_j] & \longrightarrow & K[[X_j]] \\
\downarrow & & \downarrow \\
K[X_i] & \longrightarrow & K[[X_i]]
\end{array}
\]

with the horizontal arrows given by inclusions, and the remaining ones given by (restrictions of) the truncation maps

\[
\rho_n : [X] \rightarrow [X_n] \cup \{0\}
\]

\[
m \mapsto \begin{cases} 
m & m \in [X_n] \\
0 & m \notin [X_n]
\end{cases}
\]

\[
\rho_n : K[[X]] \rightarrow K[[X_n]] \\
\sum_{m \in [X]} c_m m \mapsto \sum_{m \in [X]} c_m \rho_n(m),
\]

With respect to these inverse systems, we have that $\varprojlim K[X_n] \simeq K[[X]]$, whereas $K[X] \subsetneq K[\langle X \rangle] \subsetneq \varprojlim K[X_n] \subsetneq K[[X]]$.

In fact,

\[
\varprojlim K[X_n] = \{ f \in K[[X]] \mid \forall n : \rho_n (f) \in K[X_n]\}
\]

\[
K[\langle X \rangle] = \{ f \in \varprojlim K[X_n] \mid f \text{ is bounded}\}.
\]

Furthermore, the ring $\varprojlim K[X_n]$ is endowed with a natural topology, the inverse limit topology (where all $K[X_n]$ are discrete), and the ring $K[\langle X \rangle]$ is a dense subring. The topology on $K[\langle X \rangle]$ can be characterised by giving the closure of an arbitrary subset $A \subset K[\langle X \rangle]$:

\[
\bar{A} = \{ f \in K[\langle X \rangle] \mid \forall n : \rho_n (f) \in \rho_n (A)\}.
\]

It was proved in [14] that with respect to this topology, lfg ideals in $K[\langle X \rangle]$ are closed. It was also proved that the closed monomial ideals are precisely the lfg monomial ideals.

3.2. **Topologies on the set of ideals of $K[\langle X \rangle]$, and a “continuity” result.**

**Definition 3.4.** Let $\mathcal{J}, \mathfrak{cJ}, \mathfrak{hJ}, \mathfrak{I}, \mathfrak{mJ}$ denote the following sets of ideals in $K[\langle X \rangle]$: all ideals, closed ideals, homogeneous ideals, lfg ideals, monomial ideals. We will also use combinations of letters to denote intersections, for instance

\[
\mathfrak{ImJ} = \mathfrak{I} \cap \mathfrak{mJ}
\]

denotes the lfg monomial ideals.
Proposition 3.5. (i) The function
\[ d(I, J) = 2^{-n}, \quad n = \max \{ \nu | \rho_n(I) = \rho_n(J) \} \] (15)
gives a metric on \( \mathcal{J} \).

(ii) The function
\[ d(I, J) = 2^{-d}, \quad d = \max \{ \rho(I^d) = \rho(J^d) \} \] (16)
gives a metric on \( \mathcal{H} \).

(iii) Define a convergence structure on \( \mathcal{M} \) by dictating that \( I_n \xrightarrow{\ast} I \in \mathcal{M} \) if and only if,
\[ \forall m \in [X] : \exists N(m) \in \mathbb{N}^+ : \forall n > N(m) : m \in I \iff m \in I_n \] (17)
Then the corresponding topology is weaker than both the previous topologies.

Proof. (i) It is clear the \( d(I, J) = d(I, J) \geq 0 \). Since \( I, J \) are closed, \( d(I, J) = 0 \) if and only if \( I = J \). If \( A, B, C \) are closed ideals, and if \( d(A, B) \leq 2^{-n}, d(B, C) \leq 2^{-n}, \) then \( \rho_n(A) = \rho_n(B) = \rho_n(C) \), hence \( d(A, C) \leq 2^{-n} \). Thus the triangle inequality holds.

(ii) Obvious.

(iii) Let \( m \in [X] \), let \( I_1, I_2, I_3, \ldots \) be monomial ideals, and suppose that either \( d(I_n, I) \to 0 \) or \( d(I_n, I) \to 0 \). In the first case, there is an \( N(v) \) such that \( n \geq N(v) \): since \( m \in I \iff m \in \rho_v(I) \), and similarly for \( I_n \), it follows that \( m \in I \) if and only if \( m \in I_n \). Hence the first result follows.

\[ \forall n \geq N(v) : m \in I \iff m \in I_n \] (16)
In the second case, there is an \( \hat{N}(d) \) such that \( \hat{T}^d I_n = \hat{T}^d I \) whenever \( n \geq \hat{N}(d) \): since \( m \in I \iff m \in \hat{T}^d I \), and similarly for \( I_n \), it follows that \( m \in I \) if and only if \( m \in I_n \).

\[ \forall n \geq \hat{N}(d) : m \in I \iff m \in I_n \] (17)

Theorem 3.6. Let > be a term-order on \([X]\). Then the map
\( \text{in} > : \mathcal{J} \to \text{Im} \mathcal{M} \)
\[ I \mapsto \text{in}(I) \] (18)
is continuous with respect to the \( \hat{d} \)-metric. If > is the degree-reverse lexicographic term order, then
\[ \forall n : \rho_n(\text{in}(I)) = \text{in}(\rho_n(I)) \] (19)
from which it follows that \( \{1\} \) is continuous with respect to the \( d \)-metric.

Proof. Using the results of [10], it is straight-forward to show that if \( I, J \) are lfg ideals such that \( \hat{T}^d I = \hat{T}^d J \), then for any term-order \( > \), \( \hat{T}^d \text{in}(I) = \hat{T}^d \text{in}(J) \). Hence the first result follows.

It is immediate that the identity \( \{1\} \) implies continuity of \( \{1\} \). For all term orders, the LHS of \( \{1\} \) is included in the RHS, so we need to prove that the reverse inclusion holds for the degree-reverse lexicographic term order. Let \( f \in I \) be homogeneous of degree \( d \); then the monomials in \( \text{Supp}(f) \) are ordered as follows: first the ones in \([X_1]^d \cap \text{Supp}(f)\), if any, then the ones in \((X_2)^d \setminus [X_1]^d \cap \text{Supp}(f)\), and so on. Let \( m = \text{in}(f) \), then \( \text{in}(\rho_n(I)) = m \) for \( n \) sufficiently large, and 0 otherwise. In the same way, \( \rho_n(m) = m \) for sufficiently large \( n \), and 0 otherwise. So RHS \( \supset \text{in}(\rho_n(I)) = \text{in}(\rho_n(f)) \in \text{LHS} \).

The following result is a key one: it is what will allow us to define Hilbert numerators of lfg ideals by passing to their initial ideals.
**Theorem 3.7** (Snellman [1]). If $I$ is a lfg ideal, and $>$ is a term-order on $[X]$, then

$$d(\text{in}_>(\rho_n(I))^c, \text{in}_>(I)) \to 0 \quad \text{as} \quad n \to \infty. \quad (20)$$

4. **Monoid ideals and arithmetic on $\mathbb{Z}[[X]]$**

4.1. **Topologies on $\mathbb{Z}[[X]]$**. Unless otherwise stated, we henceforth assume that $\mathbb{Z}[[X]] = \mathbb{Z}[x]$ is given the product topology. With this topology, $f_n \to f$ if for all $m \in [X]$, there is an $N(m)$ so that for $n \geq N(m)$, $[m]f = m[f_n]$. An infinite sum $\sum_n f_n$ is convergent if and only if each monomial $m \in [X]$ occurs in but finitely many of the sets $\text{Supp}(f_n)$.

We can also topologise $\mathbb{Z}[[X]]$ by means of the total degree filtration: a sequence $(f_n)_{n=1}^{\infty}$, $f_n \to f$ if and only if

$$\forall d \in \mathbb{N} : \exists N(d) \in \mathbb{N}^+ : \forall v > N(d) : \forall m \in [X]_d : [m]f_v = [m]f.$$ 

This is a stronger topology than the previous one. We shall use it in particular for the study of the subring $\mathcal{S}$, to be defined later. For later use, we note the following:

**Lemma 4.1.** The total degree filtration topology on $\mathbb{Z}[[X]]$ gives a linear topology, and hence additive translation with arbitrary elements, and multiplicative translation with invertible elements, are closed mappings.

**Proof.** We put

$$J_d = \{0\} \cup \{ f \in \mathbb{Z}[[X]] \mid \text{Supp}(f) \subseteq \bigcup_{v=d}[X]_v \}.$$ 

Then the $J_d$’s are clopen ideals which form a fundamental system of neighbourhoods of zero.

It follows [3] that for any subset $A \subset \mathbb{Z}[[X]]$, the closure is given by

$$\bar{A} = \bigcap_{d=1}^{\infty} (A + J_d).$$

Hence if $h \in \mathbb{Z}[[X]]$ and $A \subset \mathbb{Z}[[X]]$, then

$$h + A = \bigcap_{d=1}^{\infty} (h + A + J_d) = h + \bigcap_{d=1}^{\infty} (A + J_d) = h + \bar{A},$$

where the crucial inclusion $\bigcap_{d=1}^{\infty} (h + A + J_d) \subset h + \bigcap_{d=1}^{\infty} (A + J_d)$ is proved as follows. If $f \in h + A + J_d$ for all $d$, then $f - h \in A + J_d$ for all $d$, hence $f - h \in \bigcap_{d=1}^{\infty} (A + J_d)$, hence $f \in h + \bigcap_{d=1}^{\infty} (A + J_d)$.

If $h$ has a multiplicative inverse $h^{-1}$, then

$$h^{-1}A = \bigcap_{d=1}^{\infty} (hA + J_d) = h \bigcap_{d=1}^{\infty} (A + J_d) = h\bar{A};$$

the inclusion $\bigcap_{d=1}^{\infty} (hA + J_d) \subset h \bigcap_{d=1}^{\infty} (A + J_d)$ is proved as follows. Suppose that $f \in hA + J_d$ for all $d$, then $h^{-1}f \in A + h^{-1}J_d \subset A + J_d$ for all $d$, hence $h^{-1}f \in \bigcap_{d=1}^{\infty} (A + J_d)$, hence $f \in h \bigcap_{d=1}^{\infty} (A + J_d)$. \qed

4.2. **The ring of number-theoretic functions**. Define $\Gamma$ to be the set of all maps $\mathbb{N}^+ \to \mathbb{Z}$. With component-wise addition and multiplication by scalars, and with the *Dirichlet convolution*

$$f * g(n) = \sum_{k|n} f(k)g(n/k), \quad (21)$$

$\Gamma$ becomes a commutative ring, often referred to as the ring of number-theoretic functions [3]. The well-known isomorphism, given by unique factorisation of integers, between the
multiplicative monoid \((\mathbb{N}^+, \cdot)\) of the positive integers and a denumerable sum of copies of 
\((\mathbb{N}, +)\), induces an isomorphism
\[
\Gamma \rightarrow \mathbb{Z}[[X]]
\]
\[
f \mapsto \sum_{m=x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in [X]} f(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) m,
\]
(22)
Define the elements \(\nu, \mu \in \mathbb{Z}[[X]]\) by
\[
\nu = \sum_{m \in [X]} m
\]
\[
\mu = \prod_{i=1}^{\infty} (1 - x_i) = 1 - \sum_{i=1}^{\infty} x_i + \sum_{i<j} x_i x_j - \sum_{i<j<k} x_i x_j x_k + \cdots .
\]
(23)
Then the image of \(\mu\) in \(\Gamma\) is the well-known Möbius function, and Möbius inversion can be 
expressed by the formula
\[
\nu \mu = 1.
\]
(24)
We note that we can write
\[
\nu = 1 + \sum_{i=1}^{\infty} \nu_i, \quad \mu = 1 + \sum_{i=1}^{\infty} \mu_i
\]
\[
\nu_i = \sum_{m \in [X]_i} m, \quad \mu_i = (-1)^i \sum_{\sigma \in \binom{[X]}{i}} \sigma
\]
(25)
where \(\nu_i\) is the \(i\)th complete symmetric function \([8]\) and \((-1)^i \mu_i\) is the \(i\)th elementary symmetric function.

4.3. Characteristic/generating functions of monoid ideals.

4.3.1. Definitions.

**Definition 4.2.** If \(I\) is a monoid ideal in \([X]\) then
\[
W(I) = I \setminus ([X] \setminus \{1\}) I
\]
(26)
denote the canonical set of minimal generators of \(I\). We define
\[
\text{char}(I) = \sum_{m \in I} m
\]
(27)
\[
w(I) = \sum_{m \in W(I)} m
\]
(28)
\[
q(I) = \nu - \text{char}(I)
\]
(29)
\[
p(I) = \mu q(I)
\]
(30)
We call \(\text{char}(I)\) the characteristic function of \(I\), \(q(I)\) the \([X]\)-graded Hilbert series of \(I\), and 
\(p(I)\) the \([X]\)-graded Hilbert numerator of \(I\).
For a monomial ideal $J$ in $K[X]$ or $K[[X]]$, we put
\[
\text{char}(J) = \text{char}(J \cap [X])
\]
\[
w(J) = w(J \cap [X])
\]
\[
q(J) = q(J \cap [X])
\]
\[
p(J) = p(J \cap [X])
\]
(31)

Similarly, if $n$ is a positive integer, and if $I$ is a monoid ideal in $[X^n]$, then we put
\[
\text{char}^n(I) = \sum_{m \in I} m
\]
\[
q^n(I) = \sum_{m \in [X^n] \setminus I} m
\]
\[
p^n(I) = \rho_n(\mu)q^n(I)
\]
(32)

Remark 4.3. char($I$) and $q(I)$ are the $[X]$-graded Hilbert series of $I$, regarded as a monomial ideal in $K[X]$, and $K[[X]]$, respectively. However, the ring $K[[X]]$ is not $[X]$-graded, so in order to attach a meaning to $q(I)$ for a monomial ideal we regard it as a limit of the Hilbert series of $\rho_n(I)$, that is, as a limit of $q^n(I)$.

We note that char($I$), $w(I)$, $q(I)$, and $p(I)$ all lie in $\mathbb{Z}[[X]]$.

4.3.2. Distributiveness properties.

Proposition 4.4. Suppose that $I, I_1, I_2, I_3, \ldots$ are monomial ideals, and suppose that
\[
\sum_{i=1}^{\infty} I_n = I,
\]
and that the sum is convergent with respect to the $\to$ topology. Then
\[
\text{char}(I) = \sum_i \text{char}(I_i) - \sum_{i<j} \text{char}(I_i \cap I_j) + \sum_{i<j<k} \text{char}(I_i \cap I_j \cap I_k) - \cdots,
\]
(33)
and the sum is convergent (with respect to the product topology on $\mathbb{Z}[[X]]$).

Putting $\hat{p}(I) = p(I) - 1$, we also have that
\[
\hat{p}(I) = \sum_i \hat{p}(I_i) - \sum_{i<j} \hat{p}(I_i \cap I_j) + \sum_{i<j<k} \hat{p}(I_i \cap I_j \cap I_k) - \cdots,
\]
(34)
and this is a convergent sum.

Proof. If we identify monomial ideals with their characteristic functions, and write $\wedge$ for intersections of ideals, and $\vee$ for sum of ideals, then $\wedge$ and $\vee$ correspond to component-wise minimum and maximum, and (33) to the identity
\[
\bigvee_{i=1}^{\infty} f_i = \sum_i f_i - \sum_{i<j} f_i \wedge f_j + \cdots,
\]
(35)
where the sum is component-wise. The LHS of (35) is always defined; for the RHS to be defined, it is necessary and sufficient that
\[
\forall m \in [X] \colon \exists N(M) : \forall n > N(M) : f_i(m) = 0.
\]
If this holds, then denoting by $S$ the cardinality of the finite subset $\{ j \in \mathbb{N}^+ | f_j(m) \neq 0 \}$, the formula (33) becomes
\[
S - \binom{S}{2} + \binom{S}{3} - \cdots = \begin{cases} 
0 & S = \emptyset \\
1 & S \neq \emptyset 
\end{cases}
\]
a well-known binomial identity.

To prove (34), note that $\hat{p}(I_i) = -\mu \text{char}(I_i)$, hence from (35) we get that
\[
\hat{p}(I) = \mu \text{char}(I) 
\]
\[
= -\mu \left( \sum_i \text{char}(I_i) - \sum_{i < j} \text{char}(I_i \cap I_j) + \sum_{i < j < k} \text{char}(I_i \cap I_j \cap I_k) - \cdots \right) 
\]
\[
= \sum_i (-\mu \text{char}(I_i)) - \sum_{i < j} (-\mu \text{char}(I_i \cap I_j)) + \cdots 
\]
\[
= \sum_i \hat{p}(I_i) - \sum_{i < j} \hat{p}(I_i \cap I_j) + \sum_{i < j < k} \hat{p}(I_i \cap I_j \cap I_k) - \cdots 
\]

\[\square\]

4.3.3. Inclusion-exclusion for Hilbert numerators.

**Theorem 4.5.** Let $I \subset [X]$ be a monoid ideal. If $\sigma \subset W(I)$ is finite, let $\text{lcm}(\sigma)$ be the least common multiple of the elements in $\sigma$, and let $\#\sigma$ be the cardinality of $\sigma$. Then
\[
p(I) = \sum_{\sigma} (-1)^{(#\sigma)} \text{lcm}(\sigma), \tag{36}
\]
where the sum is over all finite subsets of $W(I)$. Alternatively,
\[
p(I) = 1 - \sum_{m \in W(I)} m + \sum_{\sigma \in \binom{W(I)}{2}} \text{lcm}(\sigma) - \sum_{\sigma \in \binom{W(I)}{3}} \text{lcm}(\sigma) + \cdots \tag{37}
\]

**Proof.** We have that $I = \sum_{m \in W(I)} (m)$, and that $(m_i) \cap (m_j) = (\text{lcm}(m_i, m_j))$. Hence the result follows from (34), once we have proved that that $p((m)) = 1 - m$ for all $m \in [X]$. But $\text{char}((m)) = \sum_{m \mid t} t$, hence by Möbius inversion
\[
p((m)) = 1 - \mu \text{char}((m)) = 1 - \mu \left( \sum_{m \mid t} t \right) = 1 - \sum_{m \mid t} \mu t = 1 - m.
\]

\[\square\]

4.3.4. Homology methods.

**Lemma 4.6.** Let $I \subset [X]$ be a monoid ideal. Then
\[
\forall n \in \mathbb{N}^+ : \quad \rho_n(p(I)) = p^n(\rho_n(I)) \tag{38}
\]
Proof.
\[
p^n(\rho_n(I)) = \rho_n(\mu)q^n(\rho_n(I)) \\
= \rho_n(\mu) \sum_{m \in [X_n] \setminus \rho_n(I)} m \\
= \rho_n(\mu)\rho_n \left( \sum_{m \in [X] \setminus I} m \right) \\
= \rho_n(\mu)\rho_n(q(I)) = \rho_n(\mu q(I)) = \rho_n(p(I)).
\]
\[
\square
\]

Using this lemma, we can immediately extend the various homological methods for getting the multigraded Hilbert series of monoid ideals in \([X_n]\) (see [1, 9]) to work for monoid ideals in \([X]\).

We get

**Theorem 4.7.** Let \(I \subset [X]\) be a monoid ideal, and let \(m \in [X]\). Let \(\Delta_m = \Delta_m(I) \subset 2^{\mathbb{N}^+}\) be the following simplicial complex:

\[
\sigma = \{\sigma_1, \ldots, \sigma_r\} \in \Delta_m \iff m \prod_{i=1}^r x_{\sigma_i} \in I.
\]

Then \(\Delta_m\) is finite, and

\[
[m]p(I) = \tilde{\chi}(\Delta_m(I)) = \sum_{F \in \Delta_m(I)} (-1)^{|F|} = \sum_{i=-1}^{\infty} (-1)^i \dim H_i(\Delta_m, K),
\]

where \(\tilde{\chi}\) denotes the the reduced Euler characteristic of an abstract simplicial complex, counting the empty set as a \(-1\)-face.

**Theorem 4.8.** Let \(I \subset [X]\) be a monoid ideal, let \(W = W(I)\) be its minimal set of generators, and let \(L_I\) be the lattice of all finite lcm’s of elements in \(W\), ordered by divisibility. Let \(\hat{0}\) denote the minimal element in \(L_I\), and let, for \(m \in L_I\), \(\mu(\hat{0}, m)\) denote the value of the Möbius function of the poset \(L_I\), evaluated on the interval \([\hat{0}, m]\). Let \(\Delta(\hat{0}, m)\) denote the abstract simplicial complex of all chains in \((\hat{0}, m)\). Then we have:

\[
\forall m \in [X] : \quad [m]p(I) = \begin{cases} 
0 & m \notin L_I \\
\tilde{\chi}(\Delta(\hat{0}, m)) = \mu(\hat{0}, m) & m \in L_I
\end{cases}
\]

**Proof.** It follows from [3] that \(c_m = 0\) for \(m \notin L_I\), and that \(c_m = \tilde{\chi}(\Delta(\hat{0}, m))\) for \(m \in L_I\). By [13], \(\tilde{\chi}(\Delta(\hat{0}, m)) = \mu(\hat{0}, m)\) whenever \(m \in L_I\).

\[
\square
\]

4.3.5. **Classifications.**

**Proposition 4.9.** Let \(f = \sum_{m \in [X]} c_m m \in \mathbb{Z}[[X]]\). Then \(f \in p(\mathfrak{m}\mathfrak{N})\) if and only if the following conditions hold:

1. \(\forall m \in [X] : \quad \sum_{s|m} c_s \in \{0, 1\}\),
2. If \(\sum_{s|m} c_s = 1\) and \(t|m\) then \(\sum_{s|t} c_s = 1\).
Proof. Suppose that \( I \) is a monoid ideal in \([X]\), then \( q(I) = \nu - \text{char}(I) \) is the characteristic function of \( I^c = [X] \setminus I \). This is an order ideal, that is, if \( m \in I^c \) and \( t \mid m \), then \( t \in I^c \). It follows that the set of \( q(I) \)'s is the set of \( g = \sum_{m \in [X]} d_m m \in \mathbb{Z}[[X]] \) such that

1. \( \forall m \in [X] : d_m \in \{0, 1\} \),
2. If \( d_m = 1 \) and \( t \mid m \) then \( d_t = 1 \).

Since \( p(I) = \mu q(I) \), the result follows by Möbius inversion. \( \square \)

**Proposition 4.10.** Let \( f = \sum_{m \in [X]} c_m m \in \mathbb{Z}[[X]] \) be the \([X]\)-graded Hilbert numerator of a monoid ideal. Let \( m = x_1^{a_1} \cdots x_n^{a_n} \). Then

\[
\abs(c_m) \leq \binom{n-1}{\frac{n-1}{2}} \tag{42}
\]

**Proof.** From Theorem 4.1 we have that \( c_m \) is the reduced Euler characteristic of some simplicial complex on \( n \) vertices. Björner and Kalai [2] showed that the absolute value of the reduced Euler characteristic of a simplicial complex on \( n \) vertices is \( \leq \binom{n-1}{\frac{n-1}{2}} \). \( \square \)

**Corollary 4.11.** Let \( f = \sum_{m \in [X]} c_m m \in \mathbb{Z}[[X]] \) be the \([X]\)-graded Hilbert numerator of a monoid ideal. Let \( m = x_1^{a_1} \cdots x_n^{a_n} \), and let \( r \) be the number of \( 1 \leq i \leq n \) such that \( \alpha_i > 0 \). Then

\[
\abs(c_m) \leq \binom{r-1}{\frac{r-1}{2}} \tag{43}
\]

**Proof.** Let \( \sigma \) be a permutation of \( X \). Define \( \sigma(x_1^{a_1} \cdots x_r^{a_r}) = \prod_{i=1}^r x_{\sigma(i)}^{a_i} \), and \( \sigma(\sum_{m \in [X]} c_m m) = \sum_{m \in [X]} c_m \sigma(m) \). We let \( \sigma \) act on monoid ideals in \([X]\) in the obvious way. Then \( \mu \) and \( \nu \) are fix-points for the action of \( \sigma \) on \( \mathbb{Z}[[X]] \), and \( \sigma(\text{char}(I)) = \text{char}(\sigma(I)) \) for all monoid ideals \( I \). Hence

\[
p(\sigma(I)) = \mu(\nu - \text{char}(\sigma(I))) = \mu(\sigma\nu - \sigma(\text{char}(I))) = \sigma(\mu(\nu - \text{char}(I))) = \sigma(p(I)).
\]

Let \( i_1, \ldots, i_r \) be the support of \( m \), that is, \( \alpha_{i_1} > 0, \ldots, \alpha_{i_r} > 0 \), and let \( \sigma \) be a permutation which takes \( i_1 \) to 1, \( i_2 \) to 2, and so on. Then \( \sigma(m) = x_1^{\alpha_1^{-1}(i_1)} \cdots x_r^{\alpha_1^{-1}(i_r)} \), and

\[
c_m = [m]f = [\sigma(m)]\sigma(f),
\]

hence the result follows by applying Proposition 4.10. \( \square \)

5. The subring \( \mathcal{S} \), locally finitely generated ideals, and their generalised Hilbert numerators

For this section, we fix a positive integer \( r \) and set-partition \( Y \) of the set of variables: \( X = \cup_{\ell=1}^r Y_\ell \). There is an associated map \( y : \mathbb{N}^+ \to \{1, \ldots, r\} \) such that \( x_n \in Y_{y(n)} \). We
denote by \( \deg \) the associated \( r \)-multi-grading, that is, the monoid homomorphism
\[
\deg : [X] \to \mathbb{N}^r
\]
\[
x_1 \mapsto e_y(i)
\]
\[
x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \alpha_1 \deg(x_1) + \cdots + \alpha_n \deg(x_n)
\]
where \( e_1, \ldots, e_r \) are the natural basis elements of \( \mathbb{N}^r \). In particular, if \( r = 1 \), then \( \deg(m) = |m| \). Note that, since \( r \) is finite, \( K[[X]] \) is indeed \( \mathbb{N}^r \)-graded by means of \( \deg \), even if it is not \([X]\)-graded. We say that an ideal is \( r \)-homogeneous if it is homogeneous with respect to this grading. Clearly, \( r \)-homogeneous ideals are homogeneous, and all monomial ideals are \( r \)-homogeneous. Furthermore we have:

**Proposition 5.1.** Let \( I \) be an ideal of \( K[[X]] \). Then the following are equivalent:

(i) \( I \) is \( r \)-homogeneous and \( \text{lfg} \),

(ii) \( I \) can be generated by \( F = \bigcup_{\alpha \in \mathbb{N}^r} F_\alpha \), where each \( F_\alpha \) is a finite set of \( r \)-homogeneous elements of multi-degree \( \alpha \).

(iii) For each \( \alpha \in \mathbb{N}^r \),
\[
\dim_K \left( \frac{I_\alpha}{\sum_{\beta + \gamma = \alpha} K[\langle X \rangle^\beta I_\gamma]} \right) < \infty
\]

**Proof.** For each total degree \( d \), there are only finitely many multi-degrees in \( \mathbb{N}^r \) of total degree \( d \). Thus (i) and (i) are equivalent. The equivalence of (ii) and (iii) is parallel to Theorem 3.2 and is proved in the same way (see 11).

**Definition 5.2.** Denote by \( S \subset \mathbb{Z}[[X]] \) the subring consisting of all \( f \in \mathbb{Z}[[X]] \) fulfilling the equivalent conditions below:

1. \( f = f_0 + f_1 + f_2 + f_3 + \cdots \) with \( f_i \in \mathbb{Z}[X]_i \),
2. \( f(t, t, t, \ldots) \), the substitution of each \( x_i \) with the new formal indeterminate \( t \), is defined,
3. \( f = \sum_{\alpha \in \mathbb{N}^r} f_\alpha \) with \( f_\alpha \in \mathbb{Z}[X]_\alpha \),
4. \( f(t_{y(1)}, t_{y(2)}, t_{y(3)}, \ldots) \), the substitution of each \( x_i \) with the new formal indeterminate \( t_{y(i)} \), is defined,

Denote the map \( S \ni f \mapsto f(t_{y(1)}, t_{y(2)}, t_{y(3)}, \ldots) \in \mathbb{Z}[[t_1, \ldots, t_r]] \) by \( E = E^y \).

**Theorem 5.3.** Let \( I \subset [X] \) be a monoid ideal. Then the following are equivalent:

1. \( p(I) \in S \),
2. \( I \) is \( \text{lfg} \),
3. \( w(I) \in S \).

**Proof.** Write
\[
w(I) = 0 + w_1 + w_2 + w_3 + \cdots, \quad |w_i| = i
\]
\[
p(I) = 1 + p_1 + p_2 + p_3 + \cdots, \quad |p_i| = i.
\]
It is immediate that \( I \) is \( \text{lfg} \) if and only if \( w(I) \in S \), which occurs precisely when every \( w_i \) is a polynomial. We note that if \( \sigma \subset W(I) \) has cardinality \( u \), and the minimal and maximal total degree of elements in \( \sigma \) is \( c \) and \( d \), respectively, then \( c + 1 \leq |\text{lcm}(\sigma)| \leq ud \). Clearly, the terms of \( w(I) \) contributing to \( p_i \) in (38) have total degree \( \leq i \).
Hence, if \( I \) is \( \text{lfg} \), so that each \( w_i \) is a polynomial, then only the various \( \text{lcm} \)’s of elements in the support of \( w_1, \ldots, w_d \) may contribute to \( p_d \). The number of elements in the support of \( w_d \) is thus \( \leq 2^{(\# w_1 + \cdots + \# w_d)} < \infty \).

Conversely, if \( I \) is not \( \text{lfg} \), suppose that \( w_1, \ldots, w_d \) are polynomials, but that \( w_{d+1} \) is not. Using (20) we see that \( p_{d+1} \) receives contribution from a finite number of terms stemming from \( \text{lcm} \)’s of elements in the support of \( w_1, \ldots, w_d \), and from the non-polynomial \( w_{d+1} \). Thus \( p_{d+1} \) is not a polynomial.

**Corollary 5.4.** Let \( f = \sum_{m \in [X]} c_m m \in \mathbb{Z}[[X]] \). Then \( f \in p(\text{lm}\mathcal{I}) \) if and only if the following conditions hold:

1. \( \forall m \in [X] : \sum_{s|m} c_s \in \{0,1\} \),
2. If \( \sum_{s|m} c_s = 1 \) and \( t|m \) then \( \sum_{s|t} c_s = 1 \),
3. \( f \in \mathcal{S} \).

**Proof.** This follows from Proposition 4.9 and Theorem 5.3.

We henceforth regard \( \mathcal{S} \) as a topological ring having the topology given by the total degree filtration. We have that this topology is the same as the one given by any \( r \)-multi degree filtration in the sense that if \( f_n \to f \) if for each multi-degree \( \alpha \), there is an \( N(\alpha) \) so that \( f_n \) and \( f \) agrees in multi-degree \( \leq \alpha \) whenever \( n \geq N(\alpha) \): here \( \leq \alpha \) is with respect to some term-order on \( \mathbb{N}^r \) which refines the total-degree partial order. Similarly, the \( d \)-metric on homogeneous ideals gives the same topology as an analogous \( r \)-multigraded metric.

**Lemma 5.5.** \( \mathcal{S} \) is a closed subset of \( \mathbb{Z}[[X]] \).

**Proof.** Suppose that \( f_i \to f \), where \( f_i \in \mathcal{S} \). Fix a total degree \( d \). By the definition of the total degree filtration topology, there exists an \( N \) such that \( T^d f_i = T^d f \) for all \( i > N \). Since for all \( f_i \), \( T^d f_i \) is a polynomial, this is true for \( T^d \), as well.

**Theorem 5.6.** \( \mathcal{E} : \mathcal{S} \to \mathbb{Z}[[t_1, \ldots, t_r]] \) is continuous and clopen, when \( \mathbb{Z}[[t_1, \ldots, t_r]] \) is given the \( (t_1, \ldots, t_r) \)-adic topology.

**Proof.** We assume for simplicity that \( r = 1 \). Suppose that \( f_i \to f \) in \( \mathcal{S} \). Fix an integer \( d \), and choose an \( N(d) \) such that \( f_i - f \in \mathfrak{m}^d \) for \( i > N(d) \). Thus for \( i > N(d) \) we have that the \( t^d \) coefficient of \( \mathcal{E}(f_i) \) and \( \mathcal{E}(f) \) coincides. This shows that \( \mathcal{E}(f_i) \to \mathcal{E}(f) \).

To show that this map is clopen, we pick a basic clopen subset \( O_{f,d} = \{ g \in \mathcal{S} | T^d f = T^d g \} \), where \( d \) is a positive integer, and \( f \in \mathcal{S} \). Then \( \mathcal{E}(O_{f,d}) = \{ h \in \mathbb{Z}[[t]] | T^d h = T^d \mathcal{E}(f) \} \), and this is a basic clopen set of \( \mathbb{Z}[[t]] \).

**Lemma 5.7.** The characteristic function is a continuous mapping from the set of \( \text{lfg} \) monomial ideals, with the \( d \) metric, to \( \mathcal{S} \). In fact, it is a homeomorphism onto its image.

**Proof.** Obvious.

**Lemma 5.8.** The set of characteristic functions of \( \text{lfg} \) monomial ideals is a closed subset of \( \mathcal{S} \) (and of \( \mathbb{Z}[[X]] \)).

**Proof.** This follows from the previous Lemma and from Lemma 5.3.

**Lemma 5.9.** The set \( p(\text{lm}\mathcal{I}) \subset \mathcal{S} \) is closed.
Proof. By the previous Lemma, the set of characteristic functions of lfg monoid ideals is a closed subset of $\mathbb{Z}[[X]]$. By Lemma 4.1, the mapping $f \mapsto \mu(\nu - f)$ is a closed mapping, hence $p(\text{im J})$ is a closed subset of $\mathbb{Z}[[X]]$. From Theorem 5.3 we have that $p(\text{im J}) \subset S$, hence it is closed in there.

We henceforth assume that $\text{im J}$ have the $\hat{d}$-topology.

Proposition 5.10. Let $I, I_1, I_2, I_3, \ldots$ be lfg monomial ideals in $K[[X]]$. The following are equivalent:
1. $\hat{d}(I_n, I) \to 0$,
2. $\text{char}(I_n) \to \text{char}(I)$,
3. $q(I_n) \to q(I)$,
4. $p(I_n) \to p(I)$,
5. $w(I_n) \to w(I)$.

Furthermore, if the conditions above are satisfied, then $E(p(I_n)) \to E(p(I))$.

Proof. By the previous lemma, $I_n \to I$ if and only if $\text{char}(I_n) \to \text{char}(I)$. Since the endomorphism given by multiplication with a fixed element in a topological ring is continuous, $\text{char}(I_n) \to \text{char}(I) \iff q(I_n) \to q(I) \iff p(I_n) \to p(I)$.

If $w(I_n) \to w(I)$, then fixing a total degree $d$, we get that there exists an $N(d)$ such that $w(I_n) - w(I) \in \mathfrak{m}^d$ for $n \geq N(d)$. It then follows that $\text{char}(I_n) - \text{char}(I) \in \mathfrak{m}^d$ for $n \geq N(d)$. The converse also holds.

The last assertion follows immediately from the fact that $E$ is continuous.

We now recall a theorem by Macaulay, which says that if $I \subset K[X_n]$ is a homogeneous ideal, and $>$ is a term-order on $[X_n]$, then $K[X_n]$ and $K[X_n]/\rho_n(I)$ have the same $\mathbb{N}$-graded Hilbert series (see for instance [1]). It is also true that if $I$ is $r$-multigraded, when $K[X_n]$ is $\mathbb{N}^r$-graded using the partition $Y \cap X_n$, then the above algebras have in fact the same $\mathbb{N}^r$-graded Hilbert series. Using this, and our previous results, we get:

Theorem 5.11. Suppose that $>$ is a term-order on $[X]$. Let $I \subset K[[X]]$ be a r-homogeneous lfg ideal, and define $g_n \in \mathbb{Z}[t_1, \ldots, t_r]$ by requiring that
\[
g_n = \prod_{i=1}^{n}(1 - t_{y(i)})
\]
is the $\mathbb{N}^r$-graded Hilbert series of $K[X_n]/\rho_n(I)$.

Then, $g_n \to E(p(I))$ as $n \to \infty$, and $E(p(I)) \in \mathbb{Z}[[t_1, \ldots, t_r]]$.

Proof. From Theorem 3.7 we know that
\[
\hat{d}(\text{im } (\rho_n(I))^e, \text{im } (I)) \to 0.
\]
Then Proposition 5.10 gives that
\[
p(\text{im } (\rho_n(I))^e) \to p(\text{im } (I)),
\]
Lemma 5.15. We put

\[ \mathcal{E}(p(\text{in}_{>}(\rho_n(I)^s))) \rightarrow \mathcal{E}(p(\text{in}_{>}(I))). \]

It is clear that

\[ p^n(\text{in}_{>}(\rho_n(I))) = p(\text{in}_{>}(\rho_n(I))^s). \]

As we remarked above, a \( r \)-homogeneous ideal have the same \( \mathbb{N}^r \)-graded Hilbert series as its initial ideal, so

\[ g_n = \mathcal{E}(p^n(\text{in}_{>}(\rho_n(I)))), \]

hence

\[ g_n \rightarrow \mathcal{E}(p(\text{in}_{>}(I))). \]

\[ \square \]

In [13], the result above (for \( r = 1 \)) was proved through a different route, and the power series \( \mathcal{E}(p(I)) \) was called the generalised Hilbert numerator of \( I \).

We note two simple corollaries:

**Corollary 5.12.** If \( I, J \) are \( r \)-homogeneous lfg ideals of \( K[[X]] \), and if \( \rho_n(I) \) and \( \rho_n(J) \) have the same \( \mathbb{N}^r \)-graded Hilbert series for all \( n \), then \( I \) and \( J \) have the same \( r \)-graded Hilbert numerator.

**Corollary 5.13.** If \( I \) is an \( r \)-homogeneous lfg ideal of \( K[[X]] \), and if \( > \) is a term-order on \([X]\), then \( I \) and \( \text{in}_{>}(I) \) have the same \( r \)-graded Hilbert numerator.

In particular, \( \mathcal{E}(p(I)) = \mathcal{E}(p(\text{in}\mathcal{M})) \), that is, all \( r \)-graded Hilbert numerators of lfg ideals can be obtained from lfg monomial ideals.

5.1. **Polynomial \( r \)-graded Hilbert numerators.**

**Definition 5.14.** We put \( \mathcal{T}^Y = \mathcal{E}^{-1}(\mathbb{Z}[t_1, \ldots, t_r]) \). If \( p(I) \in \mathcal{T}^Y \) we say that \( I \) has polynomial \( r \)-graded Hilbert numerator.

**Lemma 5.15.** Suppose that \( Y' \) refines \( Y \). Denote by \( 0 \) the partition \( X = X \). Then \( \mathbb{Z}[X] \subseteq \mathcal{T}^{Y'} \subset \mathcal{T}^Y \subset \mathcal{T}^0 \subset S \).

**Proof.** The inclusions are obvious. To see that the strict ones are indeed strict, consider the following examples: \( \sum_{i=1}^\infty (x_1^i - x_2^i) \in \mathcal{T}^0 \setminus \mathbb{Z}[X] \), \( \sum_{i=1}^\infty x_1^i \in S \setminus \mathcal{T}^Y \). \( \square \)

**Example 5.16.** There are lfg monomial ideals which have Hilbert numerators in \( S \setminus \mathcal{T}^0 \). Let \( I \) be generated by \( a_i = x_1x_2x_3 \cdots x_{i-1}x_i^2 \), for \( i \geq 1 \), and \( b_j = x_1x_2x_3 \cdots x_j-2x_j^0 \), for \( j \geq 2 \). Put \( p_n = p^n(\rho_n(I)) \) for \( n > 0 \), and define \( p_0 = 1 \). We claim that

\[ p_n = p_{n-1} + (-1)^n v_n, \quad v_n = (x_{n-1} + x_n^4)x_1x_2 \cdots x_{n-2}^2 \prod_{i=1}^{n-1} (x_i - 1) \quad (46) \]

To see this, we first note that \( (x_{n-1} + x_n^4)x_1x_2 \cdots x_{n-2}^2 \prod_{i=1}^{n-1} (x_i - 1) \) consists of those monomials which can be formed as a lcm of \( \{a_n\} \cup S \) or of \( \{b_n\} \cup S \) or of \( \{a_n, b_n\} \cup S \), where \( S \subseteq \{a_1, a_2, \ldots, a_{n-1}, b_2, b_3, \ldots, b_{n-1}\} \). Note that monomials \( x_1^\alpha \cdots x_n^\beta \) with \( \alpha_i = \alpha_j = 6 \) for \( i < j \) does not occur. This can be
readily explained: every such monomial can be expressed as a lcm in two different ways, by either including or omitting the superfluous generator \( a_i \).

Hence, it follows that
\[
p_n = 1 + \sum_{i=1}^{\infty} (-1)^i v_i
\]
\[
\lim p_n = p = 1 + \sum_{i=1}^{\infty} (-1)^i v_i
\] (47)

We have that \( p \in S \setminus \mathbb{Z}[X] \). As we shall see, \( p \in T \).

Setting each \( x_i = t \) in (47) we have that
\[
p(t) = 1 - t^2 + \sum_{n=2}^{\infty} (-1)^n(t - 1)^{n-1}(t + t^4)t^n
\]
\[
= 1 - t^2 + \frac{t + t^4}{t - 1} \sum_{n=2}^{\infty} (-1)^n(t - 1)^tn^n
\]
\[
= 1 - t^2 + \frac{t + t^4}{t - 1} \left( \frac{1}{1 + t(t - 1)} - 1 + (t - 1)t \right)
\]
\[
= 1 - t^2 - t^3 + t^5
\] (48)

**Lemma 5.17.** Let \( r = 1 \). Let \( d, a_1, \ldots, a_d \) be integers, with \( d > 0 \). Then the set of all \( |p(I)| \), where \( I \) is finitely generated and generated in degrees \( \leq d \), and has
\[
\mathcal{E}(p(I)) = 1 + a_1 + \ldots + a_d t^d + O(t^{d+1}),
\] (49)
is either empty, or has a maximum, which we denote by \( Q_d(a_1, \ldots, a_d) \).

**Proof.** We claim that there are positive integers \( u_1, \ldots, u_d \) such that if \( I \) is a finitely generated monomial ideal generated in degrees \( \leq d \) satisfying (49), then \( \mathcal{E}(w(I)) = w_1 t + \ldots + w_dt^d \) with \( w_i \leq u_i \) for \( 1 \leq i \leq d \). Assuming the claim, it is clear that the total degree of \( p(I) \) is \( \leq \sum_{i=1}^{d} iu_i \), since this is a bound of the lcm of all the generators.

To establish the claim, we note that \( a_1 = -w_1 \), and assume by induction that we have shown that \( u_1, \ldots, u_i \) exist. We note that the minimal generators which affect \( a_{i+1} \) are those of degree \( i+1 \), which each contribute with \(-1\), and also \( s \)-tuples \( m_1, \ldots, m_s \) with \( m_\ell \in W(I) \), \( |m_\ell| < i + 1 \), and with \( |\text{lcm}(m_1, \ldots, m_s)| = i + 1 \), which each contribute \( (-1)^s \). If we pick \( \lambda_1 \) elements of \( W(I)_1 \), \( \lambda_2 \) elements of \( W(I)_2 \), et cetera, then for the resulting lcm to be of total degree \( i + 1 \) it is necessary that \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq i + 1 \) and that \( \lambda_\ell < i + 1 \) for all \( \ell \); thus only finitely many \( \lambda = (\lambda_1, \ldots, \lambda_i) \) are relevant.

We thus have that
\[
a_{i+1} = -w_{i+1} + \sum_{\lambda} (-1)^c(\lambda) R_\lambda,
\] (50)
where \( \lambda = (\lambda_1, \ldots, \lambda_i) \), \( |\lambda| \geq i + 1 \), \( 0 \leq \lambda_\ell < i + 1 \) for \( 1 \leq \ell \leq j \leq i \), \( c(\lambda) \) is the number of non-zero entries in \( \lambda \). The symbol \( R_\lambda \) denotes a finite interval \([0, L]\) of integers, where \( L \) is the maximal numbers of lcm’s of \( \lambda_1 \) elements of degree 1, drawn from a set of cardinality \( u_1 \), \( \lambda_2 \) elements of degree 2, drawn from a set of cardinality \( u_2 \), and so on, which have total degree \( i + 1 \). For instance, if \( i = 1 \) and \( \lambda = (2) \) then \( L = \binom{u_2}{2} \), if \( i = 2 \) and \( \lambda = (1, 1) \) then \( L = u_1 u_2 \).
These finite intervals are added using interval arithmetic, so that \([a, b] + [c, d] = [a + c, b + d]\).

We can deduce that
\[
w_{i+1} = -a_{i+1} + \sum_{\lambda} (-1)^{c(\lambda)} R_\lambda = [-C, D]
\]
for some integers \(C, D\). Putting \(u_{i+1} = D\) we have the desired bound. \(\square\)

**Lemma 5.18.** Let \(r = 1\) and suppose that \(f(t) \in \mathcal{E}(p(I))\), in other words, that \(f(t)\) is the \(\mathbb{N}\)-graded Hilbert numerator of some lfg ideal. Suppose that \(f(t) = 1 + a_1 t + \cdots + a_d t^d + t^{d+r+1} g(t)\), with \(r > Q_d(a_1, \ldots, a_d)\). Then \(1 + a_1 t + \cdots + a_d t^d \in \mathcal{E}(p(I))\).

**Proof.** Let \(f = \mathcal{E}(p(I))\), where \(I\) is a lfg monomial ideal. Let \(I_{\leq d}\) denote the ideal generated by everything in \(I\) of total degree \(\leq d\). Since the maximal degree of a lcm of the generators of degree \(\leq d\) is \(Q_d(a_1, \ldots, a_r)\), it follows from (36) and Lemma 5.17 that \(\mathcal{E}(p(I_{\leq d})) = 1 + a_1 t + \cdots + a_d t^d\).

**Theorem 5.19.** Let \(r = 1\). If \(I \subset \langle X \rangle\) is a lfg monoid ideal with \(p(I) \in \mathcal{T}\), then there exists a positive integer \(N\) and a monoid ideal \(J \subset \langle X_N \rangle\) so that \(\mathcal{E}(p(J^r)) = \mathcal{E}(p(I))\).

**Proof.** Let \(f = \mathcal{E}(p(I)) = 1 + a_1 t + a_2 t^2 + \cdots + a_d t^d\), and let \(f_n = \mathcal{E}(p^n(\rho_n(I)))\). Then \(f_n \to f\) in \(\mathbb{Z}[[t]]\), with respect to the \((t)\)-adic topology. Let \(r > Q_d(a_1, \ldots, a_d)\), and choose \(N\) such that for \(n \geq N\), \(f_n - f \in (t^r)\). Then Lemma 5.18 shows that there is a monoid ideal \(J\) in \([X_n]\) with \(f\) as its \(\mathbb{N}\)-graded Hilbert numerator. \(\square\)

**Corollary 5.20.** The set of polynomial \(\mathbb{N}\)-graded Hilbert numerators of lfg ideals in \(K[[X]]\) is equal to the set of \(\mathbb{N}\)-graded Hilbert numerators of homogeneous ideals in finitely many variables. This set is dense in the set of all possible \(\mathbb{N}\)-graded Hilbert numerators of lfg ideals.

**Proof.** From Theorem 5.19 we get that all polynomial \(\mathbb{N}\)-graded Hilbert numerators can be obtained from ideals generated in finitely many variables. To prove the second assertion, we note that if \(I\) is lfg, \(d > 0\), and \(I_{\leq d}\) is the ideal generated by everything in \(I\) of degree \(\leq d\), then \(\mathcal{E}(p(I)) \simeq \mathcal{E}(p(I_{\leq d})) \mod (t^{d+1})\), and since \(I_{\leq d}\) is finitely generated, \(p(I) \in \mathbb{Z}[X]\) hence \(\mathcal{E}(p(I)) \in \mathbb{Z}[t]\). \(\square\)

**Theorem 5.21.** Let, for every pair of integers \(0 < a \leq b\), \(G_{a,b}\) denote the set
\[
\left\{ (1 - t)^b (1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots) \middle| a_1 = a, \quad \forall i : 0 \leq a_{i+1} \leq a_i^{<t>} \right\},
\]
where
\[
u^{<d>} = \binom{k(d) + 1}{d + 1} + \binom{k(d - 1) + 1}{d} + \cdots + \binom{k(1) + 1}{2}
\]
when \(u\) has \(d\)-th Macaulay expansion
\[
u = \binom{k(d)}{d} + \binom{k(d - 1)}{d - 1} + \cdots + \binom{k(1)}{1}
\]
(see [3]). Then the set of polynomial \(\mathbb{N}\)-graded Hilbert numerators is \(\cup_{0 < a \leq b} G_{a,b}\), and the closure of this set in \(\mathbb{Z}[[t]]\) is exactly the set of \(\mathbb{N}\)-graded Hilbert numerators of lfg ideals in \(K[[X]]\).
Proof. It follows from a well-know classification by Macaulay (see [4, Theorem 4.2.10]) that the set of (generating functions of) Hilbert functions of homogeneous quotients of polynomial rings with finitely many indeterminates is

\[
\left\{ 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \mid \forall i : 0 \leq a_{i+1} \leq a_i^{<i>} \right\}.
\]

The function \(1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\) can be realised as the Hilbert function of a quotient of \(K[x_1, \ldots, x_{a_1}]\) with a monomial ideal; we are of course free to use more variables, if we so desire. The first part of the theorem is therefore demonstrated.

To prove the second part, we proceeds as follows. We know by Lemma 5.3 that

\[ p(\text{Im} \mathcal{I}) = \{ \mu(\nu - \text{char}(I)) \mid I \in \text{Im} \mathcal{I} \} \]

is a closed subset of \(S\). We have that \(\mathcal{E} : S \to \mathbb{Z}[[t]]\) is a closed map (Theorem 5.4), hence \(\mathcal{E}(p(\text{Im} \mathcal{I}))\) is closed in \(\mathbb{Z}[[t]]\).

Now, Corollary 5.20 shows that the set of polynomial \(\mathbb{N}\)-graded Hilbert numerators is dense in the set of all \(\mathbb{N}\)-graded Hilbert numerators; since this latter set is closed, the second part of the theorem follows. \(\square\)

References

[1] Dave Bayer. Monomial Ideals and Duality. Preprint, 1996.
[2] Anders Björner and Gil Kalai. An extended Euler-Poincaré theorem. Acta Math., 161(3-4):279–303, 1988.
[3] Nicolas Bourbaki. Commutative Algebra. Hermann, 1972.
[4] Winfried Bruns and Jürgen Herzog. Cohen-Macaulay rings. Cambridge University Press, Cambridge, 1993.
[5] E. D. Cashwell and C. J. Everett. The ring of number-theoretic functions. Pacific Journal of Mathematics, 9:975–985, 1959.
[6] David Eisenbud. Commutative Algebra with a View Toward Algebraic Geometry, volume 150 of Graduate Texts in Mathematics. Springer Verlag, 1995.
[7] Franz Halter-Koch. On the Algebraic and Arithmetical Structure of Generalized Polynomial Algebras. Rendiconti del Seminario Matematico della Università di Padova, 90:121–140, 1993.
[8] I. G. Macdonald. Symmetric functions and Hall polynomials. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
[9] Irena Peeva, Volkmar Welker, and Vesselin Gasharov. The lcm-lattice in monomial resolutions. To appear in Mathematical Research Notes.
[10] Jan Snellman. Gröbner bases and normal forms in a subring of the power series ring on countably many variables. Journal of Symbolic Computation, 25(3):315–328, 1998.
[11] Jan Snellman. Initial ideals of truncated homogeneous ideals. Communications in Algebra, 26(3):813–824, 1998.
[12] Jan Snellman. Reverse lexicographic initial ideals of generic ideals are finitely generated. In Buchberger and Winkler, editors, Gröbner Bases and Applications: Proceedings of the Conference 33 years of Gröbner Bases, volume 251 of London Mathematical Society Lecture Notes Series, 1998.
[13] Jan Snellman. Generalized Hilbert numerators. Communications in Algebra, 27(1):321–333, 1999.
[14] Jan Snellman. Some topological properties of a subring of the power series ring on a countably infinite number of variables over a field. Int. J. Math. Game Theory Algebra, 8(4):231–241, 1999.
[15] Richard P. Stanley. Enumerative combinatorics. Vol. 1. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

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