PARAMODULAR GROUPS AND Theta SERIES

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Abstract. For a paramodular group of any degree and square free level we study the Hecke algebra and the boundary components. We define paramodular theta series and show that for square free level and large enough weight they generate the space of cusp forms (basis problem), using the doubling or pullback of Eisenstein series method. For this we give a new geometric proof of Garrett’s double coset decomposition which works in our more general situation.

1. Introduction

Paramodular groups are rational discrete subgroups of the real symplectic group which have been considered in [10, 41, 13] and treated by Siegel under the name “Stufengruppe der Stufe $T$” in [43]. From an adelic point of view a maximal paramodular group is a group of type $Sp_n(\mathbb{Q}) \cap \prod_{p \text{ prime}} K_p$ where $K_p \subseteq Sp_n(\mathbb{Q}_p)$ is a maximal compact subgroup which is not special for all primes $p$ dividing the determinant of the defining matrix $T \in M_n^{\text{sym}}(\mathbb{Z}) \cap GL_n(\mathbb{Q})$ and special for all other primes $p$. The maximal paramodular groups are also called paramodular groups of square free level since they can also be characterized by the condition that $NT^{-1}$ is integral for some square free integer $N$; notice that the concept of level used here is quite different from the one for congruence subgroups of the integral symplectic group.

The theory of Siegel modular forms for paramodular groups has recently gained increasing interest, in particular in the case of degree 2. Reasons for this are the investigations of Gritsenko and Hulek of moduli spaces of abelian surfaces with $(1, t)$-polarization, see e. g. [22], the theory of newforms for the group $GSp(4)$ (written as $GSp_2$ in our terminology) of Roberts and Schmidt [38], and the modularity conjecture of Brumer and Kramer [8] together with the numerical evidence provided by Poor and Yuen [37] and the partial proof in [3], see also [17] for the connection to a conjecture of Harder about congruences.

In contrast to the theory of Siegel modular forms for paramodular groups of higher degree $n > 2$, very little is known for paramodular groups of higher degree $n > 2$. In particular, the methods of Poor and Yuen for constructing concrete examples of paramodular forms in degree 2 are based on Gritsenko’s lift generalizing the classical lift of Maass from Jacobi forms to Siegel modular forms of degree 2 and can not be adapted to cases of higher degree without first finding a substitute for Gritsenko’s lift in those cases. Theta series of positive definite quadratic forms as used for the modular group and its congruence subgroups of type $\Gamma_0^{(n)}(N)$, however, very little is known for paramodular groups of higher degree $n > 2$. In particular, the methods of Poor and Yuen for constructing concrete examples of paramodular forms in degree 2 are based on Gritsenko’s lift generalizing the classical lift of Maass from Jacobi forms to Siegel modular forms of degree 2 and can not be adapted to cases of higher degree without first finding a substitute for Gritsenko’s lift in those cases. Theta series of positive definite quadratic forms as used for the modular group and its congruence subgroups don’t have a good transformation behaviour under paramodular groups. In some cases they could be used by Poor and Yuen with the help of tracing constructions, but a definition of theta series that is adapted to the paramodular situation has been missing so far. Moreover, the Hecke algebras of paramodular groups, even if they are maximal, can not be treated using the
general results for Hecke algebras of special maximal compact subgroups of $p$-adic reductive groups \cite{11} Sec. 3.5] since some of their $p$-adic components are not special. This is probably the reason why they have so far been studied only for degree 2 by Gallenkämper and Krieg in \cite{19}.

The goal of this article is to study the Hecke algebras of maximal paramodular groups in spite of these obstacles and to show that in the theory of paramodular forms of any degree suitably defined theta series of positive definite quadratic forms can play a similar role as the usual theta series do for modular forms for the full modular group $Sp_n(\mathbb{Z})$ or its congruence subgroups of type $\Gamma_0^{(n)}(N)$. In particular we give a definition of paramodular theta series and study in the case of square free level the so called basis problem, i.e., the question whether these theta series generate the full space of paramodular forms of the type in question and can hence be used for the explicit construction of a basis of this space. As an important tool for this we give explicit representatives of the basic double cosets in Hecke algebras of maximal paramodular groups of arbitrary degree, generalizing the result of \cite{19}. An interesting feature of this Hecke algebra is that, although it is not commutative, it contains a commutative subalgebra which is very similar to the subalgebra generated by the $T(m)$ in the theory for the modular group and can be used in our applications to play the same role as that algebra does for the modular group. Our methods rely very much on the maximality of the groups and we do not attempt to treat more general levels, corresponding to non maximal groups. In fact we have no idea whether our results or analogues thereof are true or false in such more general cases.

In Section 2 we study lattices in a vector space with nondegenerate alternating bilinear form and their totally isotropic primitive submodules. We use the lattice approach to define, following \cite{10}, para-symplectic groups as isometry groups of lattices and obtain the usual integral paramodular groups as matrix groups of these with respect to suitable bases. Using the study of orbits of primitive totally isotropic submodules we determine and count the boundary components of the Siegel upper half space under the action of a paramodular group of square free level and arbitrary degree.

In Section 3 we study Hecke algebras for paramodular groups of square free level and arbitrary degree and more general spaces of double cosets with respect to two possibly different maximal paramodular groups. As in the case of the full integral symplectic group and its congruence subgroups the double coset space factors by strong approximation into a restricted tensor product of local spaces. The key idea for the investigation of these is to view a local double coset as an orbit of a lattice $L$ in $\mathbb{Q}_p^2$ under the action of the isometry group of a standard lattice $\Lambda$. In the classical case of level 1 these orbits are characterized by the elementary divisors of $L$ with respect to $\Lambda$, which leads to the usual diagonal representatives of the double cosets. In our case one has to consider in addition the elementary divisors in the dual $\Lambda^#$ of $\Lambda$ with respect to the alternating form. In this way we obtain explicit sets of representatives of these double cosets, generalizing results of Gallenkämper and Krieg \cite{19} for degree 2, and find a commutative subalgebra of the Hecke algebra that is important for our applications in the sequel. An analogous approach should work for the study of the Hecke algebra of a (not necessarily special) maximal compact local orthogonal group, but we don’t pursue this further here. It may also
be possible to use our methods for the determination of the structure of the Hecke algebra.

Section 4 is devoted to a generalization of Garrett’s decomposition from \[20\] of the group \(Sp_{m+n}(\mathbb{Z})\) into double cosets with respect to the Siegel maximal parabolic subgroup on one side and a diagonally embedded product \(Sp_m(\mathbb{Z}) \times Sp_n(\mathbb{Z})\) on the other side. This decomposition lies at the heart of the doubling method or method of pullbacks of Eisenstein series, which is an often used tool for the study of the analytic properties of \(L\)-functions of Siegel modular forms and for the study of the basis problem.

To generalize it to the paramodular situation we have to replace \(Sp_m(\mathbb{Z}) \times Sp_n(\mathbb{Z})\) by paramodular groups \(\Gamma_1 = Sp(\Lambda_1) \subseteq Sp_m(\mathbb{Q}), \Gamma_2 = Sp(\Lambda_2) \subseteq Sp_n(\mathbb{Q})\) of square free levels attached to suitable lattices \(\Lambda_1 \subseteq \mathbb{Q}^{2m}, \Lambda_2 \subseteq \mathbb{Q}^{2n}\). We found it difficult to generalize Garrett’s “tedious bit of linear algebra” approach of finding representatives by explicit elementary matrix operations and then relate these to Hecke double coset representatives.

Instead, we choose a geometric approach. We identify the double cosets with orbits of maximal totally isotropic submodules \(X\) of \(\Lambda := \Lambda_1 \oplus \Lambda_2\) under the action of \(Sp(\Lambda_1) \times Sp(\Lambda_2)\) and investigate the projections \(X_1, X_2\) to \(\Lambda_1, \Lambda_2\) of such a module (which are in general no longer totally isotropic). This idea originates in \[36\] and has been used by Murase \[35\] and by Müller \[34\], who, in the case of the integral symplectic group, constructs explicit bases of the \(X_i\) in order to find representatives of Garrett’s double cosets. Modifying that approach we find that a triple consisting of the \(Sp(\Lambda_i)\)-orbits of the radicals \(\text{rad}(X_i)\) together with the orbit of a natural isomorphism \(\phi : X_1/\text{rad}(X_1) \rightarrow X_2/\text{rad}(X_2)\) under the action of suitable paramodular subgroups \(\Gamma, \Gamma'\) of the \(Sp(\Lambda_i)\) characterizes the orbit of \(X\). The orbits of the primitive totally isotropic submodules \(\text{rad}(X_i)\) of \(\Lambda_i\) have been classified in Section 2 and the orbit of the natural isomorphism \(\phi\) can be described by a Hecke double coset with respect to paramodular groups of degree \(r = \text{rk}(\text{rad}(X_i))/2\).

In this way we obtain a direct proof that representatives of Garrett’s double cosets can be expressed in terms of the representatives of Hecke double cosets with respect to paramodular groups \(\Gamma, \Gamma'\) and then use our computation of the latter from the previous section in order to return to explicit matrix representatives. In the situation of the full modular group this gives a more conceptual simplification of Garrett’s proof.

In Section 5, we define paramodular theta series associated to lattice chains and show that this gives examples of paramodular forms. In Section 6, we prove a paramodular version of Siegel’s theorem, expressing a weighted linear combination of these theta series as a paramodular Eisenstein series.

Section 7 then puts together the results obtained so far and generalizes the approach from [5] to the basis problem for the modular group to the paramodular situation, again only for square free levels. We treat here only the case of paramodular cusp forms and show that all paramodular cusp forms of square free level can be written as linear combinations of paramodular theta series of the same level. We notice that it is this level aspect that causes most of the technical difficulties in our approach; we don’t see a way to circumvent them by using the adelic doubling method instead of our classical pullback method.
We have not been able to treat the non-cuspidal case, which presents some interesting challenges caused by the more complicated structure of (equivalence classes of) cusps, see Section 2. In particular, we get only one Siegel Eisenstein series, but for more general boundary components we have to consider Klingen Eisenstein series for lower rank paramodular groups with several different polarization matrices. To deal with these seems to be more difficult than one might think at first sight.

2. Paramodular groups and their modular forms

Definition 2.1. Let $R$ be a principal ideal domain with field of fractions $F$ and $\Lambda \subseteq V$ an $R$-lattice on the $2m$-dimensional $F$-vector space $V$ with nondegenerate alternating bilinear form $\langle \cdot, \cdot \rangle$, assume $\langle \Lambda, \Lambda \rangle \subseteq R$.

We denote the group of isometries of the alternating module $\langle \cdot, \cdot \rangle$ by $\text{Sp}(\Lambda)$ and call it its symplectic or para-symplectic group.

We call an $R$-basis $\mathcal{B} = \{e_i, f_i \mid 1 \leq i \leq m\}$ of $\Lambda$ a para-symplectic basis if one has $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, $\langle e_i, f_j \rangle = d_i\delta_{ij}$ with $d_i \neq 0$ for all $i, j$. It is called ordered or of elementary divisor type if one has $\langle e_i, f_i \rangle \mid \langle e_{i+1}, f_{i+1} \rangle$ for all $i < m$.

We call the group of matrices of the elements of $\text{Sp}(\Lambda)$ with respect to such a para-symplectic basis an integral paramodular group of matrix level (Matrixstufe) $T = \text{diag}(d_1, \ldots, d_m)$ and denote this group by $\Gamma^{(m)}(T)$. The subgroup of the symplectic matrix group $\text{Sp}_m(F) \subseteq \text{GL}_{2m}(F)$ consisting of the matrices of the elements of $\text{Sp}(\Lambda)$ with respect to the standard symplectic basis $\{e_i, d_i^{-1}f_i \mid 1 \leq i \leq m\}$ of $V$ is called a symplectic paramodular group or a paramodular subgroup of $\text{Sp}_m(F)$ of matrix level (Matrixstufe) $T$ and will be denoted by $\Gamma^{(m)}(T)$. If all the $d_i$ are square free we say that the matrix level $T$ is square free.

We say that $\Lambda$ has level $N \in R$ if $N$ is the least common multiple of the $d_i$ above, or equivalently if $N$ is the least common multiple of all $N'$ for which $\Lambda^\# \supseteq \Lambda \supseteq N'\Lambda^\#$ holds, where $\Lambda^\#$ is the dual lattice with respect to $\langle \cdot, \cdot \rangle$.

The determinant of $\Lambda$ is the square class $\det(A)(R^\times)^2$ of the determinant of the matrix $A = (a_{ij}) = (\langle v_i, v_j \rangle)$ of the bilinear form $\langle \cdot, \cdot \rangle$ with respect to a basis $\{v_i\}$ of $\Lambda$.

Remark 2.2. a) If $F$ is a nonarchimedean local field with ring of integers $R$ and prime element $\pi$, the groups $\Gamma^{(m)}(\text{diag}(1, \ldots, 1, \pi, \ldots, \pi))$ with $1 \leq r < m$ entries $\pi$ are stabilizers of non-special vertices in the Bruhat-Tits building of $\text{Sp}(m, F)$ and represent the conjugacy classes of maximal compact subgroups of $\text{Sp}_m(F)$ which are not special, see e. g. [21].

b) A para-symplectic basis of elementary divisor type as described in the definition always exists [10].

c) Siegel [43] defined for any nonsingular matrix $T \in M_{2m}(\mathbb{Z})$ the “Modulgruppe der Stufe $T$” to be the set of all matrices $M \in M_{2m}(\mathbb{Z})$ with

$$tM \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}.$$ 

If $T$ is diagonal this is an integral paramodular group as defined above. If $T$ is symmetric it is the matrix group attached to a group $\text{Sp}(\Lambda)$ as above with respect to a more general basis of $\Lambda$, hence conjugate to a group for diagonal $T$ in elementary divisor form. For diagonal $T$ (assumed to be in elementary divisor form) Kappler [25] called such a group “Siegelsche Stufengruppe $\Gamma(m, T)$ der Stufe $T$” and studied generators of these groups. Later work on
such groups and their modular forms also used the symplectic realization and introduced the now common word "paramodular group". We summarize the relations between the various realizations used in the literature: One has

$$\Gamma^{(m)}(T) = Sp_m(\mathbb{Q}) \cap \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} M_{2m}(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix}$$

and

$$\Gamma^{(m)}(T) = \left( \begin{array}{c} 1_m \\ 0_m & T \end{array} \right) \hat{\Gamma}^{(m)}(T) \left( \begin{array}{c} 1_m \\ 0_m & T^{-1} \end{array} \right)$$

$$\Gamma^{(m)}(T) = \left( \begin{array}{c} T^{-1} \\ 0_m & T \end{array} \right) \left( \begin{array}{c} T \\ 0_m & 1_m \end{array} \right) \hat{\Gamma}^{(m)}(T) \left( \begin{array}{c} T^{-1} \\ 0_m & 1_m \end{array} \right) \left( \begin{array}{c} 0_m \\ 0_m & T^{-1} \end{array} \right).$$

Mainly to fix notation, we recall some definitions concerning Siegel modular forms: Let $\mathcal{H}_n$ be Siegel’s upper half space of degree $n$ consisting of complex symmetric matrices of size $n$ with positive definite imaginary part. The real symplectic group $Sp_n(\mathbb{R})$ acts on $\mathcal{H}_n$ by

$$(M, Z) \mapsto MZ := M \cdot Z := (AZ + B)(CZ + D)^{-1}$$

with $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$; this group then also acts on functions $f : \mathcal{H}_n \rightarrow \mathbb{C}$ by

$$(M, f) \mapsto (f | k M)(Z) := \det(CZ + D)^{-k} f(M \cdot Z)$$

**Definition 2.3.** Let $T = \text{diag}(d_1, \ldots, d_n) \in M_n(\mathbb{Z})$ be a diagonal matrix. A modular form for $\Gamma^{(n)}(T)$ of weight $k$ is for $n \geq 2$ a holomorphic function $f : \mathcal{H}_n \rightarrow \mathbb{C}$ satisfying

$$(f | k \gamma)(Z) := f(\gamma Z) \det(CZ + D)^{-k} = f(Z)$$

for all $\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Gamma^{(n)}(T)$, $Z \in \mathcal{H}_n$. It is a cusp form if for all $g \in Sp_n(\mathbb{Q})$ the Fourier coefficients of $(f | k g)$ are zero at all degenerate matrices.

Note that this definition makes sense not only for paramodular groups but more generally for any group $\Gamma$ commensurable with $Sp(n, \mathbb{Z})$. For general properties of such modular forms we refer to the standard literature [2, 16, 27], in particular, all the results on modular forms for congruence subgroups from [24, chap.2] are also valid for paramodular forms.

**Remark 2.4.**

a) If $T = \text{diag}(d_1, \ldots, d_n)$ is in elementary divisor form, one can, as is usual, without loss of generality assume $d_1 = 1$.

b) The action of $\hat{\gamma} \in \hat{\Gamma}^{(n)}(T)$ on the upper half space $\mathcal{H}_n$ described in [33] is the same as the usual action of

$$\left( \begin{array}{cc} T & 0_n \\ 0_n & 1_n \end{array} \right) \hat{\gamma} \left( \begin{array}{cc} T^{-1} & 0_n \\ 0_n & 1_n \end{array} \right).$$

c) It is sometimes useful to consider for a number $N \in \mathbb{N}$ prime to $\det(T)$ slightly more general the groups $\hat{\Gamma}^{(n)}_0(T, N)$ for the set of $\hat{\gamma} \in \hat{\Gamma}^{(n)}(T)$ for which the lower left $n \times n$-block has entries divisible by $N$ and

$$\Gamma^{(n)}_0(T, N) := \left( \begin{array}{cc} 1_n & 0_n \\ 0_n & T \end{array} \right) \hat{\Gamma}^{(n)}_0(T, N) \left( \begin{array}{cc} 1_n & 0_n \\ 0_n & T^{-1} \end{array} \right).$$
Equivalently we have
\[
\Gamma_0^{(n)}(T, N) = \text{Sp}_n(\mathbb{Q}) \cap \begin{pmatrix} 1 & 0 \\
0 & NT \end{pmatrix} M_{2n}(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\
N^{-1}T^{-1} & 1 \end{pmatrix}
\]

In particular, for \( n = 2, T = \begin{pmatrix} 1 & 0 \\
0 & t \end{pmatrix} \) we have
\[
\Gamma_{T,0}^{(2)}(N) := \Gamma_0^{(2)}(T, N)\{ \begin{pmatrix} * & t* & * \\
* & * & */t \\
N* & N_t* & * \\
N_t* & N_t* & t* \\
& & * 
\end{pmatrix} \} \subseteq \text{Sp}_2(\mathbb{Q})
\]
where the asterisks symbolize integral entries.

For \( T = 1_n \) we have \( \Gamma_0^{(n)}(T, N) = \Gamma_0^{(n)}(N) \subseteq \text{Sp}_n(\mathbb{Z}) \).

In order to discuss the boundary components of the quotient of Siegel’s space \( \mathcal{H}_n \) by a paramodular group we return to the setting of a module \( \Lambda \) with alternating bilinear form over a principal ideal domain \( R \) with field of quotients \( F \). It is well known that any such module \( \Lambda \) has an ordered para-symplectic basis. We will later need that for square free level such a basis can be adapted to any given primitive totally isotropic submodule of \( \Lambda \):

**Theorem 2.5.** Let \( R \) be a principal ideal domain with field of fractions \( F \) and \( \Lambda \subseteq V \) be a \( 2n \)-dimensional \( F \)-vector space with non-degenerate alternating bilinear form \( \langle , \rangle \), assume \( \langle \Lambda, \Lambda \rangle \subseteq R \) and that the level of \( \Lambda \) divides the square free element \( N \in R \).

Let \( Z \) be a primitive submodule of rank \( r \leq m \) of \( \Lambda \) (i.e., \( Z \) is a direct summand of \( \Lambda \), equivalently \( FZ \cap \Lambda = Z \)) which is totally isotropic with respect to \( \langle , \rangle \).

Then there are a basis \( (e_1, \ldots, e_r) \) of \( Z \) and vectors \( f_1, \ldots, f_r \in \Lambda \) generating a totally isotropic submodule of \( \Lambda \) and satisfying \( \langle e_i, f_j \rangle = d_i \delta_{ij} \) with \( d_i \mid N \) such that
\[
M := M(Z) := \bigoplus_{i=1}^r R e_i \oplus \bigoplus_{i=1}^r R f_i
\]
can be split off orthogonally in \( \Lambda \), i.e., one has \( \Lambda = M \perp \Lambda' \) for some submodule \( \Lambda' \subseteq \Lambda \).

Moreover, if \( X \supseteq Z \) is a sublattice of \( \Lambda \) with \( Z \subseteq \text{rad}(X) := \{ x \in X \mid \langle x, X \rangle = \{ 0 \} \} \) one has \( X = Z \perp X' \) with \( X' := X \cap \Lambda' \), and \( X' \) is non-degenerate if and only if \( Z = \text{rad}(X) \) holds.

In particular, if \( X = Z \) is maximal totally isotropic there exist a basis \( (e_1, \ldots, e_m) \) of \( X \) and vectors \( f_1, \ldots, f_m \in \Lambda \) generating a totally isotropic submodule of \( \Lambda \) and satisfying \( \langle e_i, f_j \rangle = d_i \delta_{ij} \) with \( d_i \mid N \) such that
\[
\Lambda = \bigoplus_{i=1}^m R e_i \oplus \bigoplus_{i=1}^m R f_i.
\]

**Proof.** This is a modification of [10, Theorem 3.1]: If \( e \in Z \) is a primitive vector we have \( \langle e, \Lambda \rangle \supseteq \langle e, N\Lambda' \rangle = NR \), hence \( \langle e, \Lambda \rangle = dR \) for some \( d \mid N \).

We have then \( \langle e, d\Lambda' \cap \Lambda \rangle \subseteq dR \), and if there were a prime \( p \in R \) with \( \langle e, d\Lambda' \cap \Lambda \rangle \subseteq pdR \) we had \( e \in \Lambda \cap pd(d\Lambda' \cap \Lambda)' = p\Lambda + (pd\Lambda' \cap \Lambda) \). If one had \( p \nmid d \) this would imply \( p \mid \langle e, \Lambda \rangle \), which is a contradiction. On the other hand, \( p \mid d \) implies
isotropic submodules of rank $u$ and $d$ such that $e$ is primitive.

We can therefore find $f \in d\Lambda \cap \Lambda$ with $\langle e, f \rangle = d$. Since we have $(Re + Rf, \Lambda) \subseteq dR$ we can again split off the $d$-modular hyperbolic plane $Re + Rf$ in $\Lambda$ with orthogonal complement $\Lambda_1$. In any case we consider $Z' := Z \cap \Lambda_1$ with $\operatorname{rk}(Z') = \operatorname{rk}(Z) - 1$, $Z'$ primitive in $\Lambda_1$, and see that by induction we can obtain the vectors $e_1, \ldots, e_r, f_1, \ldots, f_r$ as asserted.

Let now $x \in X$. We put

$$x' := x - \sum_{i=1}^{r} \frac{\langle x, f_i \rangle}{\langle e_i, f_i \rangle} e_i$$

and have $\langle x', f_i \rangle = 0$ for $1 \leq i \leq r$. Moreover, since $\langle e_i, f_i \rangle = d | N$ if and only if $\langle \Lambda, f_i \rangle = dR$ holds, we have $x' \in X \cap M^\perp$, with $x - x' \in Z$, which gives $X = Z \perp (X \cap M^\perp)$ as asserted.

The rest of the assertion is obvious. □

Obviously, the $d_iR^\times$ for the elementary divisors $d_i$ associated to an ordered para-symplectic basis determine the isometry class of $(\Lambda, \langle , \rangle)$ and one has $d_1 \ldots d_m R^\times = DR^\times$, where $D^2R^\times = (D(\Lambda))^2$ is the determinant of $\Lambda$. If the level of $\Lambda$ is square free the product $D = d_1 \ldots d_m$ determines already the isometry class:

**Lemma 2.6.** Let $R$ be a principal ideal domain with field of fractions $F$ and $\Lambda$ an $R$-lattice of rank $2m$ on the $2m$-dimensional $F$-vector space $V$, equipped with a nondegenerate alternating bilinear form $\langle , \rangle$.

Assume that $\Lambda$ with this form has square free level $N$, let $(e_1, \ldots, e_m, f_1, \ldots, f_m)$ be a basis of $\Lambda$ for which the alternating form has matrix $\left( \begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix} \right)$. Assume that the determinant of the matrix of $\langle , \rangle$ is $D^2$ and that $d_1, \ldots, d_m$ are divisors of $N$ with $d_1 \ldots d_m = D$.

Then $M := \oplus_i Re_i$ has a basis $\{ \bar{e}_i = \sum_j s_{ij} e_i \}$ and $M' := \oplus_i Rf_i$ has a basis $\{ \bar{f}_j = \sum_i t_{ij} f_i \}$ with $\langle \bar{e}_i, \bar{f}_j \rangle = d_i d_j$. In particular, the isometry class of $(\Lambda, \langle , \rangle)$ is determined by $DR^\times = \operatorname{det}(A)R^\times$.

**Proof.** Use the elementary divisor theorem. □

**Corollary 2.7.** With the notation as in Theorem 2.5, $d(Z) := d_1 \ldots d_r$ is uniquely determined up to units.

Two primitive totally isotropic submodules $Z_1, Z_2$ of $\Lambda$ are in the same $\operatorname{Sp}(\Lambda)$-orbit if and only if they have the same rank and $d(Z_1)R^\times = d(Z_2)R^\times$ holds.

If $D^2$ is the determinant of $\Lambda$ and $d \mid D$, there exists an $\operatorname{Sp}(\Lambda)$-orbit of primitive totally isotropic submodules $Z = Z_d$ of $\Lambda$ of rank $u$ with $d = d(Z)$ if and only if one has $d \mid N^u$ and $d \mid N^{m-u}$.

In particular, if $N \in R^\times$ holds, there is only one $\operatorname{Sp}(\Lambda)$-orbit of primitive totally isotropic submodules of rank $u$ for each $u \leq m = \operatorname{rk}(\Lambda)$.

**Proof.** Obvious. □

**Corollary 2.8.** Let $\Lambda$ be a lattice on $V$ of rank $2m$ and square free level $N$.

a) Let $U$ be a totally isotropic subspace of $V$ of dimension $u$, let $P_U = P_u = \{ g \in \operatorname{Sp}(V) \mid gU = U \}$ be the maximal parabolic subgroup of $\operatorname{Sp}(V)$ fixing $U$.

Let $\{ g_1, \ldots, g_t \}$ denote a set of representatives of the double cosets $\operatorname{Sp}(\Lambda)gP_U$ with $g \in \operatorname{Sp}(V)$.
lemma 2.10. with notations as before let $U_\Lambda$ be a set of representatives of the $Sp(\Lambda)$-orbits of primitive totally isotropic submodules of rank $u$ of $\Lambda$.

b) let $R = \mathbb{Z}$ and $F = \mathbb{Q}$, let the determinant of $\Lambda$ be $D^2 = \prod_{p | N} p^{2e_p}$.

then the number of boundary components of type $\mathcal{H}_{m-u}$ in the satake compactification (or cusps in the sense of [24]) of $\mathcal{H}_m$ for $Sp(\Lambda)$ is equal to

$$\prod_{p | N} (\min(u, m-u, \ell_p, m-\ell_p) + 1).$$

In particular for $u = m$ there is only one zero dimensional cusp.

proof. by the previous corollary the totally isotropic subspaces of $V$ of fixed dimension are in a single $Sp(V)$-orbit, and assertion a) follows. for b), the number of boundary components of type $\mathcal{H}_{m-u}$ of $\mathcal{H}_m$ for $Sp(\Lambda)$ equals the number of double cosets $Sp(\Lambda)gP_u$.

by the strong approximation theorem for $Sp(V)$ over $\mathbb{Q}$ it is enough to prove the assertion for the case that $N = p$ is a prime. if (using the notation of theorem 2.5) the determinant of the nondegenerate module $M(U \cap \Lambda)$ is denoted by $p^{2s}$, the $Sp(\Lambda)$-orbit of $U$ is determined by $s$ by corollary 2.7, so we have to count the possible values of $s$. one must have $0 \leq s \leq u$ and $s \leq \ell_p$, and all values of $s$ with $0 \leq s \leq \min(u, \ell_p)$ occur if $\ell_p \leq m - u$ holds. in this case we have $\min(u, m-u, \ell_p, m-\ell_p) = \min(u, \ell_p)$, so the assertion is true in this case. if $\ell_p > m-u$ holds, we must have $\ell_p - s \leq m-u$, hence $s \geq \ell_p - (m-u)$ and $\ell_p - (m-u) \leq s \leq \min(\ell_p, u)$, and all these value of $s$ occur, so that we obtain $1 + m - \max(\ell_p, u) = 1 + \min(m-\ell_p, m-u) = 1 + \min(m-\ell_p, m-u, \ell_p, u)$ values, which proves the assertion.

remark 2.9. a) alternatively we can use proposition 3.6 of [24] to determine the number of boundary components and count the $\theta$-admissible $\sigma \in S_r$ defined in [24, definition 3.3] to obtain the formula for the number of boundary components; notice, however, that the condition (2) in that definition should also contain that one has $b_{t+1}(\sigma, \theta) = \{i \in X \mid t+1 \leq i \leq t + 1 + \kappa\}$ for some $\kappa$ if $b_{t+1}(\sigma, \theta) \neq \emptyset$.

b) if we identify the elements of $Sp(V)$ with their matrices with respect to the symplectic basis $\{e_1, d^{-1}_1 f_1\}$, the group $P_u$ becomes the parabolic subgroup of $Sp_m(\mathbb{Q})$ with a block $0_{u, 2m-u}$ in the lower left corner and we obtain representatives of the double cosets $\Gamma^{(m)}((d_1 \ldots d_m))gP_u$. in particular, for $u = m-1$ our formula for the number of double cosets coincides with the result of [32].

c) by the corollary, as in the case of the full modular group the fact that there is only one zero dimensional cusp implies that a modular form for a parabolic group $\Gamma^{(n)}(T)$ of square free matrix level $T$ is a cusp form if and only if its fourier expansion has nonzero fourier coefficients only at nondegenerate matrices.

for later use we construct representatives of the orbits in the corollary.

lemma 2.10. with notations as before let $B = \{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ be a parasymplectic basis of $\Lambda$ of elementary divisor type with $\langle e_i, f_j \rangle = d_i \delta_{ij}$ and $d_i \mid d_{i+1}$, let $v_i = d_i^{-1} f_i$. for $0 \leq r < m$ let $U_0 = U_0(r) = \sum_{i=r+1}^{m} Q e_i$ and $M(U_0 \cap \Lambda) = U_0 \cap \Lambda + \sum_{i=r+1}^{m} i f_i$, let $d$ be such that there exists a totally isotropic subspace $U$ of $V$ of dimension $u = m-r$ with $d(U \cap \Lambda) = d$. let $\tilde{d}_1, \ldots, \tilde{d}_m$ be such that
\[ \tilde{d}_i \mid \tilde{d}_{i+1} \text{ for } 1 \leq i \leq r - 1 \text{ and for } r + 1 \leq i \leq m - 1 \text{ with } d = \prod_{i=r+1}^{m} \tilde{d}_i \text{ and } \prod_{i=1}^{m} \tilde{d}_i = \prod_{i=1}^{m} d_i. \]

Then there is \( g = g(B,d,u) \in Sp(V) \) with \( d(gU_0 \cap \Lambda) = d \) such that the matrix \( \gamma \) of \( g \) with respect to the basis \( \{ e_1, \ldots, e_m, v_1, \ldots, v_m \} \) of \( V \) is \( \gamma = \gamma(B,u,d) = \left( \begin{smallmatrix} S & 0 \\ 0 & S_{S^{-1}} \end{smallmatrix} \right) \), where \( S = S(B,u,d) \in SL_m(\mathbb{Z}) \) is congruent to a permutation matrix \( S_p = S_p(B,u,d) \mod p \) modulo each prime \( p \mid N \) (where we admit possible entries \(-1\) in the permutation matrix). Moreover, \( g \) can be chosen such that the \( \tilde{e}_i := g e_i, \tilde{f}_i := \tilde{d}_i g v_i \) for \( 1 \leq i \leq m \) form a basis of \( \Lambda \), with the last \( u \) pairs \( \tilde{e}_i, \tilde{f}_i \) forming a basis of \( M(gU_0 \cap \Lambda) \). The matrices \( \gamma(B,u,d) \) with \( d \mid D, d \mid N^u, D \mid N^{m-u} \) form a set of representatives of the double cosets \( Sp(\Lambda) \tilde{g} P_u \) and hence of the set of boundary components of type \( \mathcal{S}_{m-u} \) of the Siegel upper half space \( \mathcal{H}_m \).

Moreover, the totally isotropic submodules \( g(B,u,d) U_0(r) \cap \Lambda \) of \( \Lambda \) form a set of representatives of the \( Sp(\Lambda) \)-orbits of primitive totally isotropic submodules of \( \Lambda \).

**Proof.** For \( p \mid N \) there is a permutation \( \sigma_p \in S_m \) such that \( d((\sum_{i=r+1}^{m} Q e_{\sigma_p(i)}) \cap \Lambda) \in d \mathbb{Z}_p^r \), and we may choose the permutation so that for \( 1 \leq i \leq r - 1 \) and \( r + 1 \leq i \leq m - 1 \) one has \( (e_{\sigma_p(i)}, f_{\sigma_p(i)}) \mathbb{Z}_p \supseteq (e_{\sigma_p(i+1)}, f_{\sigma_p(i+1)}) \mathbb{Z}_p \).

We let \( S_p \in SL_m(\mathbb{Z}) \) be the associated matrix (with an entry \(-1\) if \( \text{sgn}(\sigma_p) = -1 \)).

If \( N = p \) is a prime we set \( S = S_p \), and with \( \gamma \) and \( g \) as in the assertion we have \( gU_0 = \sum_{i=r+1}^{m} Q e_{\sigma_p(i)} \) and are done. For composite \( N \) we have different permutations for its prime factors and have to combine them into a single \( S \).

Indeed, by the strong approximation theorem for \( SL_m \), we find \( S \in SL_m(\mathbb{Z}) \) congruent to \( S_p \mod p \) for each \( p \mid N \). Then \( g \in Sp(V) \) with matrix \( \left( \begin{smallmatrix} S & 0 \\ 0 & S_{S^{-1}} \end{smallmatrix} \right) \) with respect to the basis \( \{ e_1, \ldots, e_m, v_1, \ldots, v_m \} \) of \( V \) is as desired. \( \square \)

**Remark 2.11.** For \( r = 0 \) with \( d = D \) we may set \( S = 1_m \), in the trivial case \( r = m \), where \( U_0 = \{ \theta \} \), we set \( d = 1 \) and \( S = 1_m \).

3. **Hecke algebras for paramodular groups**

We want to study Hecke algebras associated to paramodular groups and slightly more general spaces of double cosets \( \Gamma \Gamma' \) where \( \Gamma, \Gamma' \) are possibly different paramodular groups of the same degree.

We notice first that by [9, 4.7] for any group \( \Gamma \subseteq Sp_n(\mathbb{Q}) \) which can be obtained as \( \Gamma = Sp_n(\mathbb{Q}) \cap \prod_p K_p \), where the \( K_p \) are compact open subgroups of \( Sp_n(\mathbb{Q}_p) \) the Hecke algebra \( \mathcal{H}(Sp_n(\mathbb{Q}), \Gamma) \) factors into a restricted tensor product of the local Hecke algebras \( \mathcal{H}_p(Sp_n(\mathbb{Q}_p), K_p) \). This is proved using the strong approximation theorem for the group \( Sp_n \), and the argument carries over to the situation of spaces of double cosets \( \Gamma \Gamma' \), where \( \Gamma, \Gamma' \) are groups of this type. In particular, the result holds for the case of a pair of paramodular groups.

We can hence restrict attention to the study of the local double coset spaces. We therefore let now \( R \) be a complete discrete valuation ring with prime element \( p \) and field of fractions \( F \) and \( V \) a vector space of dimension \( 2n \) over \( F \) equipped with a nondegenerate alternating bilinear form \( \langle \cdot, \cdot \rangle \). A lattice \( \Lambda \) on \( V \) of level dividing \( p \) (also called a \( p \)-elementary lattice in the sequel) then has for some uniquely determined integers \( a, b \) with \( a + b = n \) a para-symplectic basis of elementary divisor type \( \{ e_1, \ldots, e_n, f_1, \ldots, f_n \} \) with \( \langle e_i, f_i \rangle = 1 \) for \( 1 \leq i \leq a \), \( \langle e_i, f_i \rangle = p \) for \( a + 1 \leq i \leq n \). We say then that the canonical decomposition of \( \Lambda \) has a unimodular hyperbolic planes \( Re_i + Rf_i \) and \( p \)-modular hyperbolic planes.
Proposition 3.1. Let $\Lambda$ be a lattice of level dividing $p$ on $V$ and $e_1, \ldots, e_n, f_1, \ldots, f_n$ a para-symplectic basis of $\Lambda$ with $\langle e_i, f_j \rangle = \delta_{ij}$ for $1 \leq i \leq a$, $\langle e_i, f_j \rangle = p\delta_{ij}$ for $a + 1 \leq i \leq n = a + b$. For $0 \leq a' \leq n$ and $b' = n - a'$ let

$$L(r_+, r_-, \mu_1, \ldots, \mu_n) :=$$

$$Z_p p^{\mu_{a+1}+1} e_{a+1} + Z_p p^{-\mu_{a+1}-1} f_{a+1} + \cdots + Z_p p^{\mu_{a+r_+}+1} e_{a+r_+} + Z_p p^{-\mu_{a+r_+}-1} f_{a+r_+}$$

$$+ Z_p p^{\mu_{r_+}+1} e_{r_+} + Z_p p^{-\mu_{r_+}-1} f_{r_+} + \cdots + Z_p p^{\mu_a+1} e_a + Z_p p^{-\mu_a-1} f_a$$

$$+ Z_p p^{\mu_1} e_1 + Z_p p^{-\mu_1-1} f_1 + \cdots + Z_p p^{\mu_{n_1}} e_{n_1} + Z_p p^{-\mu_{n_1}-1} f_{n_1}$$

$$= Z_p e_1 + Z_p f_1 + \cdots + Z_p e_n + Z_p f_n,$$

with $0 \leq r_- \leq \min(a, b'), 0 \leq r_+ \leq \min(a', b')$, $b' = b - r_+ + r_-, a' = a - r_+ + r_+$,

$1 \leq \mu_1 \leq \cdots \leq \mu_{r_+}, 0 \leq \mu_{r_+} \leq \cdots \leq \mu_a$,

$0 \leq \mu_{a+1} \leq \cdots \leq \mu_{a+r_+}, 0 \leq \mu_{a+r_+} \leq \cdots \leq \mu_n$.

a) Let $L, L'$ be $p$-elementary lattices on $V$ which are isometric.

Then $L'$ and $L$ are in the same $Sp(\Lambda)$-orbit if and only if they have the same elementary divisors with respect to both $\Lambda$ and the dual lattice $\Lambda^\#$. $\Lambda$.

b) The orbits in a) of $L$ with $0 \leq a' \leq n$ unimodular hyperbolic planes in the canonical decomposition are represented by the lattices $L(r_+, r_-, \mu_1, \ldots, \mu_n)$ with $r_+, r_-, \mu_1, \ldots, \mu_n$ as above.

Proof. For a) we denote by $a', b'$ the (common) numbers of unimodular respectively $p$-modular hyperbolic planes in the canonical decompositions of $L$ and $L'$. It is well known (see [21] [18] [11]) that there exist a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $\Lambda$ with $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, $\langle e_i, f_j \rangle = \delta_{ij}$ for $1 \leq i \leq a$, $\langle e_i, f_j \rangle = p\delta_{ij}$ for $a + 1 \leq i \leq n = a + b$ and non negative integers $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_{r_+}$ such that the $p^{\mu_i} e_i, p^{-\mu_i} f_i$ form a basis of $L$. Since $\Lambda$ and $L$ are both assumed to be $p$-elementary, we must obviously have $\nu_i = \min(\mu_i, \mu_i - 1)$ for $1 \leq i \leq a$ and $\nu_i = \min(\mu_i, \mu_i + 1)$ for $a < i \leq n$; moreover, if $r_+$ is the number of of indices $i$ with $\nu_i = \mu_i + 1$ and $r_-$ the number of indices $i$ with $\nu_i = \mu_i - 1$, we have $b' = b - r_+ + r_-$ and $a' = a - r_+ + r_+$.

In other words: There are precisely $r_-$ among the subspaces $Q_i e_i + Q_i f_i$ for which the intersection with $\Lambda$ is unimodular and the intersection with $L$ is $p$-modular and for $r_+$ such subspaces the roles of $\Lambda, L$ are reversed. We say that the $e_i, f_i$ are a para-symplectic elementary divisor basis for the pair $\Lambda, L$ with exponent pairs $(\mu_i, \nu_i)$.

It is clear that $L, L'$ in the same $Sp(\Lambda)$-orbit have the same elementary divisors with respect to $\Lambda$ and with respect to $\Lambda^\#$. On the other hand, assume that there are para-symplectic elementary divisor bases $\{e_i, f_i\}$ with exponent pairs $(\mu_i, \nu_i)$ as above for $\Lambda, L$ and $\{e'_i, f'_i\}$ with the same exponent pairs $(\mu'_i, \nu'_i) = (\nu_i)$ for $\Lambda, L'$.

One has then a $\phi \in Sp(\Lambda)$ with $\phi(e_i) = e'_i, \phi(f_i) = f'_i$ for $1 \leq i \leq n$, and this $\phi$ maps $L$ onto $L'$, i.e, $L$ and $L'$ are in the same orbit.

We are therefore left with the task of proving that the elementary divisor condition of the assertion implies that if one has para-symplectic elementary divisor bases $\{e_i, f_i\}$ with exponent pairs $(\mu_i, \nu_i)$ for $\Lambda, L$ and $\{e'_i, f'_i\}$ with exponent pairs $(\mu'_i, \nu'_i)$
for \( \Lambda, L' \), those pairs must be the same \((\mu'_1 = \mu, \nu'_1 = \nu)\). Notice that this will imply in particular that two elementary divisor bases for a fixed \( L \) must have the same exponent pairs \((\mu, \nu)\).

Let \( \mu \) be maximal such that the number of pairs \((\mu_1, \nu_1)\) with \( \mu_1 = \mu, \nu_1 = \nu \) is not equal to the number of pairs \((\mu'_1, \nu'_1)\) with \( \mu'_1 = \mu, \nu'_1 = \nu \) for one of the possible values \( \nu = \mu - 1, \nu = \mu, \nu = \mu + 1 \) of \( \nu, \nu'_1 \). The pairs with \( \nu_1 = \mu - 1 \) contribute elementary divisors \( p^\mu, p^{-\mu + 1} \) of \( L \) in \( \Lambda^\# \), the pairs with \( \nu_1 = \mu \) contribute elementary divisors \( p^\mu, p^{-\mu} \) or \( p^{\mu+1}, p^{-\mu+1} \) of \( L \) in \( \Lambda^\# \), the pairs with \( \nu_1 = \mu + 1 \) contribute elementary divisors \( p^{\mu+1}, p^{-\mu} \) of \( L \) in \( \Lambda^\# \), and analogously for the \( \nu'_1 \) and \( L' \).

Since here the only contributions to elementary divisors \( p^{-\mu-1} \) in \( \Lambda \) come from pairs \((\mu, -\mu - 1)\) the number of such pairs \((\mu_1, \nu_1)\) and \((\mu'_1, \nu'_1)\) must be equal. These pairs contribute elementary divisors \( p^{\mu+1} \) in \( \Lambda^\# \), with the only other contributions coming from pairs \((\mu, -\mu)\) and pairs with higher first entry. The latter ones occur in equal numbers for \( L, L' \) by assumption, so the number of pairs \((\mu_1 = \mu, \nu_1 = -\mu)\) and \((\mu'_1 = \mu, \nu'_1 = -\mu)\) must also be equal.

We are then left with the possibility that the number \( s \) of pairs \((\mu, -\mu)\) with elementary divisors \( p^\mu, p^{-\mu} \) in \( \Lambda^\# \) are different for \( L \) and \( L' \), say there are more such pairs for \( L \) than for \( L' \). Consequently we have at this level fewer elementary divisors \( p^{-\mu} \) in \( \Lambda^\# \) for \( L' \) than for \( L \). But by our considerations above levels \( \mu - 1 \) and lower can not contribute such elementary divisors, so we have a contradiction to the assumption that \( L \) and \( L' \) have the same elementary divisors in \( \Lambda \) and in \( \Lambda^\# \), and we have finished the proof of \( a \).

Moreover, the proof of \( a \) shows that the lattices \( L(r_+, r_-, \mu_1, \ldots, \mu_n) \) in \( b \) are indeed a set of representatives of the orbits. \( \square \)

**Theorem 3.2.** With notations as in the proposition denote by \( \Gamma_{a,b} \) the paramodular group \( \Gamma\left( \begin{smallmatrix} a & \frac{1}{p} & b \end{smallmatrix} \right) \subseteq Sp_n(\mathbb{Q}_p) \) over the \( p \) -adic integers \( \mathbb{Z}_p \), where \( a \) is the number of unimodular hyperbolic planes in \( \Lambda \) and \( b \) the number of \( p \)-modular hyperbolic planes. Then the double cosets \( \Gamma_{a,b} \alpha \Gamma_{a',b'} \) with \( \alpha \in Sp_n(\mathbb{Q}_p) \) have as a set of representatives the block diagonal matrices

\[
D(r_+, r_-, \mu_1, \ldots, \mu_n) = \begin{pmatrix} B(r_+, r_-, \mu_1, \ldots, \mu_n) & 0_n \\ 0_n & (B(r_+, r_-, \mu_1, \ldots, \mu_n))^{-1} \end{pmatrix}
\]

with \( r_-, r_+ \) and the \( \mu_i \) as in the above proposition, where \( B(r_+, r_-, \mu_1, \ldots, \mu_n) \)

\[
\begin{pmatrix}
p^{\mu_1} & \cdots & p^{\mu_{r-}} \\
p^{\mu_{r-+}} & \cdots & p^{\mu_{r-+}+} \\
p^{\mu_{a+}} & \cdots & p^{\mu_{a+r+}} \\
p^{\mu_{a+r+}} & \cdots & p^{\mu_{a+r+}+} \\
p^{\mu_1} & \cdots & p^{\mu_{r-}}
\end{pmatrix}
\]
with the rest of the entries being zero.

Proof. We let $\Lambda, L$ and the basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of $\Lambda$ be as in the proof of the proposition and obtain a standard symplectic basis \( \{ x_1 = e_1, \ldots, x_n = e_n, y_1 = p^{-1}f_1, \ldots, y_n = p^{-1}f_n \} \) of $V$. Then with respect to this standard basis $Sp(\Lambda)$ has $\Gamma_{a,b}$ and $Sp(L)$ has $\Gamma_{a',b'}$ as its group of matrices. The double cosets $\Gamma_{a,b} \alpha \Gamma_{a',b'}$ correspond then bijectively to the $Sp(\Lambda)$-orbits of lattices isometric to $L$ and are represented by the matrices $D(r_+, r_-, \mu_1, \ldots, \mu_n)$ which transform $L$ into the standard lattices given in part b) of the proposition. \( \Box \)

**Lemma 3.3.** With notation as above and $T := \begin{pmatrix} 1_\alpha & p_1 b' \\ -1 p_1 b' & 1 \end{pmatrix}$, $T' := \begin{pmatrix} 1_\alpha & p_1 b \\ -1 p_1 b & 1 \end{pmatrix}$ the matrix

$$T^{-1} B(r_+, r, \mu_1, \ldots, \mu_n) T'$$

has integral entries.

In the case $a = a', b = b'$ we have with $r = r_+ = r_-$

$$T^{-1} B(r, r, \mu_1, \ldots, \mu_n) T = B(r, r, \mu_{a+1} + 1, \ldots, \mu_{a+r} + 1, \mu_{r+1}, \ldots, \mu_n) \cdot \mu_1 - 1, \ldots, \mu_r - 1, \mu_{a+r+1}, \ldots, \mu_n)$$

Proof. This is easily checked, using $1 \leq \mu_1 \leq \cdots \leq \mu_r$. \( \Box \)

**Corollary 3.4.** With notations as above assume $a = a', b = b'$, $r := r_+ = r_-$, and write $\Gamma = \Gamma_{a,b}$, let $H(\Gamma)$ denote the Hecke algebra generated by the $\Gamma$ double cosets as defined in [42]. For $j \in \mathbb{N}$ denote by $T(p^j)$ the sum of the double cosets of the $D(r, r, \mu_1, \ldots, \mu_n)$ with $\mu_1 + \cdots + \mu_n = j$. Then $T(p^j)$ is invariant under the involution of $H(\Gamma)$ given by $\Gamma \to \Gamma^{-1} \Gamma$.

Consequently, the subalgebra generated by the $T(p^j)$ is commutative.

Proof. If we conjugate the inverses $D(r = r_+, r = r_-, \mu_1, \ldots, \mu_n)^{-1}$ of the representatives given above of the double cosets occurring in $T(p^j)$ by

$$\begin{pmatrix} 1_a & \cdot \\ -1_a & -p_1 b \end{pmatrix} \in \Gamma_{a,b}$$

we obtain by the previous lemma a permutation of the representatives of the double cosets in $T(p^j)$.

The asserted commutativity can then be seen from an easy modification of the proof of [42] Prop. 3.8. \( \Box \)

**Remark 3.5.** For later use we notice that $T(p^0)$ consists only of the double coset with $r = \mu_1 = \cdots = \mu_n = 0$.

In the case of level 1, i.e., the usual full integral symplectic group, it is well known that the formal power series $\sum_{j=0}^\infty T(p^j) X^j$ is a rational function in $X$, and that this rational function can be explicitly given (essentially equal to the local Euler-factor of the standard $L$-function), see [3]. We have unfortunately not succeeded in finding a similar result in the paramodular case. For the applications we have in mind we can, however, at least partly substitute this result by an upper estimate for the number of one sided cosets that occur when one expands each of the double cosets occurring in $T(p^j)$ into a formal sum of single cosets. For this we need a few preparations. For $j \geq 0$ we denote by $N(p^j)$ the number of left cosets $\Gamma \gamma$
occurring in the decomposition of the double cosets in $T(p^j)$ into left cosets. In analogy to the case of symmetric bilinear forms, see [28], we say that isomorphic lattices $\Lambda, \Lambda' = \alpha \Lambda$ $\alpha \in Sp(V)$ on $V$ are neighbors of each other if $\Lambda \cap \Lambda'$ has index $p$ in both $\Lambda$ and $\Lambda'$.

**Lemma 3.6.** Let $j \geq 0$.

a) The cosets $\alpha Sp(\Lambda)$ with $\alpha \in Sp(V)$ correspond bijectively to the lattices $\alpha \Lambda$ in $V$.

b) The double coset $Sp(\Lambda)\alpha Sp(\Lambda)$ appears in $T(p^j)$ if and only if $\Lambda \cap \alpha \Lambda$ has index $p^j$ in $\Lambda$ and in $\alpha \Lambda$.

c) $N(p)$ is the number of lattices in $V$ which are neighbors of $\Lambda$.

d) $N(p^j) \leq (N(p))^j$.

*Proof.* a) is obvious, b) follows from the definition of $T(p^j)$. For c) and d) we notice that the number of left cosets in $\Gamma \alpha \Gamma$ equals the number of right cosets.

c) is then immediate from a),b). Finally, as noticed in the proof of Theorem 3.2, isomorphic lattices $\Lambda', \Lambda = \alpha \Lambda$ of level dividing $p$ have adapted parasymplectic bases $(e_1, \ldots, e_n, f_1, \ldots, f_n), (p^{\nu_1}e_1, \ldots, p^{\nu_n}e_n, p^{\nu_1}f_1, \ldots, p^{\nu_n}f_n)$. From this we see that the lattices $\Lambda, \Lambda' = \alpha \Lambda$ with $(\Lambda : \Lambda \cap \alpha \Lambda) = p^j$ can be connected by a chain of neighboring lattices of length $j$, which implies d). Notice, however, that $N(p^j)$ will usually be larger than $N(p^j)$, since chains of neighboring lattices may backtrack and since different chains can arrive at the same goal.

**Lemma 3.7.** Let $n = n_1 + n_2$ and assume that $\Lambda$ is the orthogonal sum of $n_1$ unimodular hyperbolic planes and $n_2$ $p$-modular hyperbolic planes. Then the number $N(p)$ of neighbors of $\Lambda$ equals

$$
\frac{p}{p-1} \left( p^{2n_1} - 1 \right) \left( p^{2n_2} - 1 \right) + \frac{p}{p-1} p^{2n_1} \left( p^{2n_2} - 1 \right) + \frac{p}{p-1} p^{2n_2} \left( p^{2n_1} - 1 \right).
$$

If $p > 3$ or $n_2 = 0$ holds one has $N(p) < p^{2n+1}$, for $p = 3$ we have $N(3) < \frac{5}{4} 3^{2n+1}$, for $p = 2$ one has $N(2) < 2^{2n+2}$.

*Proof.* There are three different types of neighbors $\Lambda'$ of a given $\Lambda$ which we will count separately.

In the first type there is $x \in \Lambda$ with $\bar{x} \in \Lambda'$ and $\langle x, \Lambda \rangle = \langle \bar{x}, \Lambda' \rangle = \mathbb{Z}_p$. We have then $\Lambda' =: \Lambda(x) = \mathbb{Z}_p \bar{x} + \Lambda_x$ with $\Lambda_x := \{ z \in \Lambda \mid \langle z, x \rangle \in p\mathbb{Z}_p \}$. If we have two vectors $x, x'$ as above with $\Lambda(x) = \Lambda(x')$ and $z' \in \Lambda_{x'}$, we can write $pz' = \alpha x + pz$ with $\alpha \in \mathbb{Z}_p, z \in \Lambda_x$ and obtain $\langle p^{\nu}z', x \rangle \in p^{2}\mathbb{Z}_p$, thus $\Lambda_{x'} \subseteq \Lambda_x$ and hence $\Lambda_{x} = \Lambda_{x'}$. On the other hand, if we have $x, x' \in \Lambda$ as above with $\Lambda_x = \Lambda_{x'}$ we have $\Lambda(x) = \Lambda(x')$ if and only if $x, x'$ generate the same line modulo $p\Lambda_x = p\Lambda_{x'} \supseteq p^2 \Lambda$. Obviously, one has $\Lambda(x) = \Lambda(x')$ if $x, x'$ generate the same line modulo $p^2 \Lambda$. The number of eligible lines $\mathbb{Z}_p x + p^2 \Lambda$ is

$$
\frac{(p^{2n} - p^{2n_2})p^{2n}}{p^{2} - p} = \frac{p^{2n+2n_2}(p^{2n_1} - 1)}{p^{2} - p}.
$$

Writing $\Lambda = (\mathbb{Z}_p x + p\mathbb{Z}_p \bar{x}) \perp \Lambda$ with $\langle x, \bar{x} \rangle = 1$ we have $\Lambda_x = (\mathbb{Z}_p x + p\mathbb{Z}_p \bar{x}) \perp \Lambda$ and see that each line modulo $p\Lambda_x$ contains $p^{2n}$ points modulo $p^2 \Lambda$, whereas each line modulo $p^2 \Lambda$ consists of $p^2$ points. Each line modulo $p\Lambda_x$ consists therefore of $p^{2n-2}$ lines modulo $p^2 \Lambda$ yielding the same neighbor of $\Lambda$, and we obtain the last summand in our formula for $N(p)$ as the number of neighbors of the first type.
In the second type there is a primitive vector \( y \in \Lambda \) with \( \frac{y}{p} \in \Lambda' \) and \( \langle y, \Lambda \rangle = \langle \frac{y}{p}, \Lambda' \rangle = p\mathbb{Z}_p \), one has then \( \Lambda' = \Lambda(y) = \mathbb{Z}_p \frac{y}{p} + \Lambda_y \) with \( \Lambda_y = \{ z \in \Lambda \mid \langle z, y \rangle \in p^2\mathbb{Z}_p \} \). In the same way as above we see that \( \Lambda(y) = \Lambda(y') \) implies \( \Lambda_y = \Lambda_{y'} \) and that vectors \( y, y' \) as above with \( \Lambda_y = \Lambda_{y'} \) yield the same neighbor if and only if they generate the same line modulo \( p\Lambda_y = p\Lambda_{y'} \supseteq p^3\Lambda \). Obviously, one has \( \Lambda(y) = \Lambda(y') \) if \( y, y' \) generate the same line modulo \( p^3\Lambda \). The number of eligible lines modulo \( p^3\Lambda \) is

\[
\frac{\left(\frac{p^{4n+2n_1}}{p^3} - 1\right)}{p^3 - p^2}.
\]

Writing \( \Lambda = (\mathbb{Z}_p y + \mathbb{Z}_p \tilde{y}) \perp \tilde{\Lambda} \) with \( \langle y, \tilde{y} \rangle = p \) we have \( \Lambda_y = (\mathbb{Z}_p y + p\mathbb{Z}_p \tilde{y}) \perp \tilde{\Lambda} \) and see that the line \( \mathbb{Z}_p y + p\Lambda_y \) consists of \( p^{4n} \) points modulo \( p^3\Lambda \), whereas each line modulo \( p^3\Lambda \) consists of \( p^3 \) points. Each line \( \mathbb{Z}_p y + \Lambda_y \) consists therefore of \( p^{4n-3} \) lines modulo \( p^3\Lambda \) all yielding the same neighbor of \( \Lambda \), and we obtain the second summand in our formula for \( N(p) \) as the number of neighbors of the second type. In the third type there is a vector \( x \in \Lambda \cap \Lambda', x \notin p\Lambda' \) with \( \langle x, \Lambda \rangle = \mathbb{Z}_p \), \( \langle x, \Lambda' \rangle = p\mathbb{Z}_p \) and a primitive vector \( y \in \Lambda \) with \( \frac{y}{p} \in \Lambda' \), \( \langle y, \Lambda \rangle = p\mathbb{Z}_p \), \( \langle \frac{y}{p}, \Lambda' \rangle = \mathbb{Z}_p \), we have then \( \Lambda' = \Lambda(x, y) = \mathbb{Z}_p \frac{y}{p} + \Lambda_x \) with \( \Lambda_x = \{ z \in \Lambda \mid \langle z, x \rangle \in p\mathbb{Z}_p \} \). As in the previous cases we see that \( \Lambda(x, y) = \Lambda(x', y') \) implies \( \Lambda_x = \Lambda_{x'} \) and that for given \( \Lambda_x \) two vectors \( y, y' \) as above yield the same neighbor if and only if they generate the same line modulo \( p\Lambda_x \supseteq p^2\Lambda \). Obviously one has \( \Lambda(x, y) = \Lambda(x', y') \) if \( \Lambda_x = \Lambda_{x'} \) and \( y, y' \) generate the same line modulo \( p^2\Lambda \). We have \( \Lambda_x = \Lambda_{x'} \) if and only if \( x, x' \) generate the same line modulo \( \Lambda \cap p\Lambda^{\#} \), and one has \( \Lambda : \Lambda \cap p\Lambda^{\#} = p^{2n_1} \), the number of possible \( \Lambda_x \) is hence equal to \( \frac{p^{2n}}{p^{2n_1}} \). For a fixed \( \Lambda_x \) we may write \( \Lambda_x + Z_p \frac{y}{p} = (Z_p x + p\mathbb{Z}_p \tilde{x}) \perp (Z_p \frac{y}{p} + Z_p \tilde{y}) \perp \tilde{\Lambda} \) with \( \langle x, \tilde{x} \rangle = 1 \), \( \langle y, \tilde{y} \rangle = p \) by changing \( x \) modulo \( \Lambda \cap p\Lambda^{\#} \) if necessary. We see that \( Z_p y + p\Lambda_x \) consists of \( p^{2n} \) points modulo \( p^2\Lambda \). Since a line modulo \( p^2\Lambda \) consists of \( p^2 \) points, the line \( Z_p y + p\Lambda_x \) consists of \( p^{2n-2} \) lines modulo \( p^2\Lambda \) all yielding, together with \( \Lambda_x \), the same neighbor. The number of eligible lines modulo \( p^2\Lambda \) is

\[
\frac{p^{2n}(p^{2n_2} - 1)}{p^2 - p},
\]

dividing by \( p^{2n-2} \) and multiplying by the number of \( \Lambda_x \) we obtain the first summand in our formula for \( N(p) \) as the contribution of the neighbors of the third type. Finally, the upper bounds for \( N(p) \) are easily checked using our formula. 

\[ \square \]

For use in later sections we have to make the transition from our local considerations to the global setting more explicit. We therefore switch now to ground field \( \mathbb{Q} \) and a lattice \( \Lambda \) of square free level \( N \) on the \( \mathbb{Q} \)-vector space \( V \), put \( \Gamma = Sp(\Lambda) \) and \( \Gamma_p = Sp(\Lambda_p) \). As mentioned in the beginning of this section, the Hecke algebra \( \mathcal{H}(Sp_n(V), \Gamma) \) is isomorphic to the restricted tensor product of the local Hecke algebras \( \mathcal{H}(Sp_n(V_p), \Gamma_p) \).

**Corollary 3.8.** Let \( \Gamma = \Gamma^{(n)}(T) \), \( \Gamma' = \Gamma^{(n)}(T') \) with \( T = \text{diag}(d_1, \ldots, d_n) \), \( T' = \text{diag}(d'_1, \ldots, d'_n) \), \( d_i \mid d_{i+1}, d'_i \mid d'_{i+1} \) be paramodular subgroups of \( Sp_n(\mathbb{Q}) \) of square free levels \( N, N' \). Then each double coset \( \Gamma' \alpha \Gamma \) with \( \alpha \in Sp_n(\mathbb{Q}) \) has a representative of the form

\[
\begin{pmatrix}
B & 0 \\
0 & tB^{-1}
\end{pmatrix},
\]
with $B \in GL_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$ satisfying $T^{-1}BT' \in M_n(\mathbb{Z})$.

Proof. We write $\Gamma = Sp_n(\mathbb{Q}) \cap \prod \Gamma_p$ with local paramodular groups $\Gamma_p \subseteq Sp_n(\mathbb{Q}_p)$ and $\Gamma_p = Sp_n(\mathbb{Z}_p)$ for all primes $p \nmid N$, and similarly for $\Gamma'$. We choose local representatives $\alpha_p$ of the double cosets $\Gamma_p \alpha \Gamma_p$ as in Theorem 3.2, in particular, each $\alpha_p$ is of block diagonal shape $(A_p \ 0 \ 0 \ 0)$, where in $A_p \in M_n(\mathbb{Z}_p)$ each line has a single nonzero entry. Let $M \in \mathbb{N}$ with $N | M, N' | M$ be such that $\alpha_p \in \Gamma_p = \Gamma_p = Sp_n(\mathbb{Z}_p)$ for all $p \nmid M$. Changing the local representatives $\alpha_p$ if necessary we can assume that there exists $m \in \mathbb{N}$ with $\det(A_p) = m$ for all $p \mid M$. We can hence find a diagonal matrix $D \in GL_n(\mathbb{Q})$ with $DA_p \in SL_n(\mathbb{Z}_p)$ for all primes $p \mid M$ and $D^{-1} \in M_n(\mathbb{Z})$. By the strong approximation theorem for $SL_n$ there exists $A \in SL_n(\mathbb{Z})$ with $A \equiv DA_p \mod MM_n(\mathbb{Z}_p)$ for all primes $p \mid M$, and $B = D^{-1}A$ is as asserted, where for the last part of the assertion we use Lemma 3.3 notice that we have reversed the roles of $T, T'$ here in view of the application of this result in the next section. □

Lemma 3.9. In the Hecke algebra $\mathcal{H}(Sp_n(V), \Gamma)$ denote for $m \in \mathbb{N}$ by $T(m)$ the sum of all double cosets with a representative of the form

$$
\begin{pmatrix}
B & 0 \\
0 & tB^{-1}
\end{pmatrix},
$$

as in Corollary 3.8 with $B \in GL_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$ of determinant $m$. Then $T(m)$ is the product of the $T(p_i^j)$, where $m = \prod p_i^j$ is the factorization of $m$ into prime powers.

Proof. If $\Gamma \alpha \Gamma$ is a double coset occurring in $T(m)$ as a summand and $p = p_i$ for some $i$, the coset $\Gamma_p \alpha \Gamma_p$ occurs as a summand in $T_p(p_i^j)$, where we write $T_p(\ldots)$ for the elements of the local Hecke algebra at $p$. Moreover, the isomorphism between $\mathcal{H}(Sp_n(V), \Gamma)$ and the restricted tensor product of the $\mathcal{H}_p(Sp_n(V_p), \Gamma_p)$ guarantees, together with the previous corollary, that each collection of local double cosets $\Gamma_p \alpha_p \Gamma_p$ with $\alpha_p \in \Gamma_p$ for almost all $p$ arises in this way from a global double coset occurring in some $T(m)$. Consequently, writing each $T(p_i^j)$ as a sum of double cosets and distributively multiplying out their product we obtain the sum of all double cosets occurring in $T(m)$. □

4. Garrett’s double coset decomposition for paramodular groups

Our goal in this section is to generalize Garrett’s double coset decomposition from [20] from the integral symplectic group to paramodular groups. For this we want to investigate orbits of maximal totally isotropic submodules of lattices with non-degenerate alternating bilinear form.

An obvious consequence of Theorem 2.6 and Corollary 2.7 is:

Corollary 4.1. With $\Lambda$ as in Theorem 2.7 the group $Sp(\Lambda)$ acts transitively on the set of maximal totally isotropic submodules of $\Lambda$.

More generally, let $X_1, X_2 \subseteq \Lambda$ be sublattices and $Z_1, Z_2 \subseteq X_1, X_2$ primitive totally isotropic submodules of $\Lambda$ of rank $r$ with $Z_j \subseteq \text{rad}(X_j)$ for $j = 1, 2$. For $j = 1, 2$ let $M_j = M(Z_j)$ as in Theorem 2.6 with $\Lambda = M(Z_j) \perp \Lambda'(Z_j) = M_j \perp \Lambda_j'$, $X_j = (X_j \cap \Lambda_j') \perp Z_j$, put $X_j' = X_j \cap \Lambda_j'$.

Then there exists $\phi \in Sp(\Lambda)$ with $\phi(X_1) = X_2, \phi(Z_1) = Z_2$ if and only if $d(Z_2) \in d(Z_1)R^\times$ holds and there exists an isometry $\psi : \Lambda_1' \to \Lambda_2'$ with $\psi(X_1') = X_2'$. 

A set of representatives of the orbits of pairs \( X \supseteq Z \) as above with \( Z = \text{rad}(X) \) is then obtained by picking a \( Z = Z_d \) for each \( dR^X \) satisfying the conditions of Corollary 2.7 and setting \( X = X' \perp Z_d \), where \( X' \) runs through a set of representatives of the \( \text{Sp}(\Lambda'(Z_d)) \)-orbits of nondegenerate sublattices of \( \Lambda'(Z) \).

If \( X \subseteq \Lambda \) is a fixed maximal totally isotropic submodule and \( P = P_X \subseteq \text{Sp}(\Lambda) \) its stabilizer, we may therefore identify the set of all maximal totally isotropic submodules with the coset space \( \{gP \mid g \in \text{Sp}(\Lambda)\} \), and for an orthogonal splitting \( \Lambda = \Lambda_1 \perp \Lambda_2 \) the orbits of maximal totally isotropic submodules of \( \Lambda \) under the action of \( \text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2) \subseteq \text{Sp}(\Lambda) \) correspond bijectively to the double cosets \( (\text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2))gP \) with \( g \in \text{Sp}(\Lambda) \). Garrett gave explicit representatives for these double cosets for \( \Lambda \) of level 1 in [20]. We will generalize his result to arbitrary square free level by finding representatives of the orbits of maximal totally isotropic submodules of \( \Lambda \) under the action of \( \text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2) \subseteq \text{Sp}(\Lambda) \).

**Theorem 4.2.** Let \( \Lambda \) be a free module over the commutative ring \( R \) with an alternating bilinear form \( \langle \cdot, \cdot \rangle \).

Let \( \Lambda_1, \Lambda_2 \) be mutually orthogonal (with respect to \( \langle \cdot, \cdot \rangle \)) submodules of \( \Lambda \) with \( \Lambda = \Lambda_1 \perp \Lambda_2 \) and denote by \( \pi_1, \pi_2 \) the orthogonal projections from \( \Lambda \) onto \( \Lambda_1, \Lambda_2 \).

Let \( X \) be a totally isotropic submodule of \( \Lambda \) and write \( X_i = \pi_i(X) \) for \( i = 1, 2 \). Then there exists an isometry \( \phi \) from the bilinear module \( (X_1/\text{rad}(X_1), \langle \cdot, \cdot \rangle) \) to \( (X_2/\text{rad}(X_2), -\langle \cdot, \cdot \rangle) \) such that

\[
X = \{x_1 + x_2 \mid x_1 \in X_1, x_2 + \text{rad}(X_2) = \phi(x_1 + \text{rad}(X_1))\}.
\]

If \( X \) is maximal totally isotropic the triple \( (X_1, X_2, \phi) \) is maximal with respect to this property.

Conversely, for \( i = 1, 2 \) let \( Y_i \subseteq \Lambda_i \) be submodules with an isometry \( \psi : (Y_1, \langle \cdot, \cdot \rangle)/\text{rad}(Y_1) \to (Y_2, -\langle \cdot, \cdot \rangle)/\text{rad}(Y_2) \).

Then

\[
Y := \{(y_1, y_2) \mid y_1 \in Y_1, y_2 \in Y_2, \psi(y_1 + \text{rad}(Y_1)) = y_2 + \text{rad}(Y_2)\}
\]

is a totally isotropic submodule of \( \Lambda \) with \( \pi_1(Y) = Y_1, \pi_2(Y) = Y_2 \). If \( Y_1, Y_2, \psi \) are maximal, \( Y \) is a maximal totally isotropic submodule of \( \Lambda \).

**Proof.** Let \( v_1 = \pi_1(x) \in X_1 \) with \( x = v_1 + v_2 \in X, v_2 \in X_2 \). If \( v_1 = \pi_1(v_1 + v'_2) \) for some \( v'_2 \in X_2 \) and \( w_2 = \pi_2(y) \in X_2 \) is arbitrary, we have

\[
\langle v_2 - v'_2, w_2 \rangle = \langle v_2 - v'_2, y \rangle = 0,
\]

and hence \( v_2 + \text{rad}(X_2) = v'_2 + \text{rad}(X_2) \) since \( v_2 - v'_2 = (v_1 + v_2) - (v_1 + v'_2) \in X \) and \( X \) is totally isotropic. Moreover, if \( v_2 \in \text{rad}(X_2) \) holds and \( w_1 = \pi_1(y) \in X_1 \) is arbitrary with \( y = w_1 + w_2, w_2 \in X_2 \) we have

\[
\langle w_1, v_1 \rangle = \langle w_1, v_1 + v_2 \rangle = \langle w_1 + w_2, v_1 + v_2 \rangle = \langle y, x \rangle = 0,
\]

since \( v_2 \in \text{rad}(X_2) \) gives \( \langle w_2, v_2 \rangle = 0 \) and since \( X \) is totally isotropic. Conversely, it is easy to see that a vector \( v_1 \in \text{rad}(X_1) \) gives a vector \( v_2 \in \text{rad}(X_2) \).

By the homomorphism theorem we obtain hence a map \( \phi : X_1/\text{rad}(X_1) \to X_2/\text{rad}(X_2) \) with \( \phi(v_1) + \text{rad}(X_1) = v_2 + \text{rad}(X_2) \) for all \( v_1 + v_2 \in X \) with \( v_1 \in \Lambda_i \) for \( i = 1, 2 \).
For \( v_1 = \pi_1(v_1 + v_2), v'_1 = \pi_1(v'_1 + v'_2) \) we have
\[
\langle v_1, v'_1 \rangle + \langle v_2, v'_2 \rangle = \langle v_1 + v_2, v'_1 + v'_2 \rangle = 0
\]
since \( X \) is totally isotropic, so \( \phi \) is indeed an isometry.

If on the other hand \( Y_1, Y_2, \psi, Y \) are as in the assertion it is clear that \( Y \) is a totally isotropic submodule of \( \Lambda \) with \( \pi_1(Y) = Y_1, \pi_2(Y) = Y_2 \). If \( Y_i \supseteq X_i \) for \( i = 1, 2 \) and \( \psi \) extends \( \phi \) one has \( Y \supseteq X \), and one sees that the maximality of \( X \) implies the maximality of \( X_1, X_2, \phi \). If \( w \in \Lambda \) is such that \( Y + Rw \) is totally isotropic, \( Y_1 := Y_1 + R\pi_1(w), Y_2 := Y_2 + R\pi_2(w) \) have the same properties as \( Y_1, Y_2 \), so by the assumed maximality we have \( \pi_1(w) \in Y_1, \pi_2(w) \in Y_2 \), which implies \( w \in Y \). Hence the maximality of \( Y_1, Y_2, \psi \) implies that \( Y \) is indeed a maximal totally isotropic submodule of \( \Lambda \).

\[\Box\]

**Remark 4.3.** In the situation of the theorem with \( X \) maximal let \( R \) be an integral domain with field of fractions \( F \) and let \( \Lambda, \Lambda_1, \Lambda_2 \) be lattices of full rank on the \( F \)-vector spaces \( V, V_1, V_2 \). Applying the theorem to \( FX \subseteq V, V_1, V_2 \) we obtain subspaces \( U_i = \pi_i(FX) \subseteq V_i \) and an isometry \( \phi : U_1/\text{rad}(U_1) \rightarrow U_2/\text{rad}(U_2) \).

We have then \( X_1 = (U_1 \cap \Lambda_1) \cap \phi^{-1}(U_2 \cap \Lambda_2) \) and \( X_2 = (U_2 \cap \Lambda_2) \cap \phi(U_1 \cap \Lambda_1) \) and may view the map \( X_1/\text{rad}(X_1) \rightarrow X_2/\text{rad}(X_2) \) as the restriction of the map \( U_1/\text{rad}(U_1) \rightarrow U_2/\text{rad}(U_2) \) (with the natural embedding of \( X_i/\text{rad}(X_i) \) into \( U_i/\text{rad}(U_i) \)).

Conversely, given subspaces \( U_i \subseteq V_i \) and an isometric map \( \phi \) as above we can set \( X_1 = (U_1 \cap \Lambda_1) \cap \phi^{-1}(U_2 \cap \Lambda_2) \) and \( X_2 = (U_2 \cap \Lambda_2) \cap \phi(U_1 \cap \Lambda_1) \) and retrieve the associated maximal totally isotropic submodule \( X \) of \( \Lambda \) as described in the theorem.

**Lemma 4.4.** With notations as before let now \( R \) be a principal ideal domain and \( \Lambda, \Lambda_1, \Lambda_2 \) finitely generated free \( R \)-modules, let \( 2m := \text{rk}(\Lambda_1), 2n := \text{rk}(\Lambda_2) \) and assume that \( \Lambda, \Lambda_1, \Lambda_2 \) with the alternating form \( \langle \cdot, \cdot \rangle \) are nondegenerate alternating modules.

Let \( X \) be a maximal totally isotropic submodule of \( \Lambda \), put \( 2r := \text{rk}(X_1/\text{rad}(X_1)) = \text{rk}(X_2/\text{rad}(X_2)), m_1 = \text{rk}(\text{rad}(X_1)), n_1 = \text{rk}(\text{rad}(X_2)) \).

Then \( m_1 = m - r, n_1 = n - r \).

**Proof.** From \( X = \{x_1 + x_2 \mid x_1 \in X_1, x_2 \in X_2, \phi(x_1 + \text{rad}(X_1)) = x_2 + \text{rad}(X_2)\} \) with \( \phi \) as in the proof of the theorem one sees that \( m + n = \text{rk}(X) = m_1 + 2r + n_1 \) holds, and since \( m \) resp. \( n \) is the dimension of a maximal totally isotropic submodule of \( \Lambda_1 \) resp. \( \Lambda_2 \) we have \( m_1 + r \leq m, n_1 + r \leq n \). Taken together we obtain \( m_1 = m - r, n_1 = n - r \).

\[\Box\]

**Lemma 4.5.** Let \( \Lambda_1 \subseteq V_1, \Lambda_2 \subseteq V_2 \) be integral \( \mathbb{Z} \)-lattices of square free level \( N \) in the regular symplectic spaces \( V_1, V_2 \) over \( \mathbb{Q} \), we set \( V = V_1 \perp V_2, \Lambda = \Lambda_1 \perp \Lambda_2 \). Let \( \pi_1, \pi_2 \) denote the orthogonal projections from \( \Lambda \) onto \( \Lambda_1, \Lambda_2 \).

Let \( X \) be a maximal totally isotropic submodule of \( \Lambda \) and put \( X_i = \pi_i(X) \) for \( i = 1, 2 \), let \( Z_1 = X \cap \Lambda_1 = \text{rad}(X_1) \). Then there are primitive sublattices \( M_i \subseteq \Lambda_i, M_i \supseteq Z_i \) and \( Z_1, Z_2 \) maximal totally isotropic in \( M_i \) with \( \Lambda_i = M_i \perp \Lambda'_i \).

With \( M = M_1 \perp M_2, Z = Z_1 \perp Z_2, \Lambda' = \Lambda'_1 \perp \Lambda'_2, X' = X \cap \Lambda' \) one has \( X = X \perp X' \) and \( X_1 = \pi_1(X') \perp Z_1, X_2 = \pi_2(X') \perp Z_2 \).

In particular, the \( \pi_i(X') \) are nondegenerate isometric to \( X_i/\text{rad}(X_i) \), and the isomorphism \( \phi : X_1/\text{rad}(X_1) \rightarrow X_2/\text{rad}(X_2) \) of Theorem 4.2 induces an isomorphism \( \phi' : \pi_1(X') \rightarrow \pi_2(X') \).
Proof. The lattices $M_i$ are obtained from Theorem 2.5, which also implies that one has $X = Z \perp X'$ and $X_1 = \pi_1(X') \perp Z_1$, $X_2 = \pi_2(X') \perp Z_2$ as asserted. That $\phi$ induces $\phi'$ as asserted is clear. \hfill \Box

Lemma 4.6. In the situation of the previous lemma let (using the notation of Corollary 2.7) $d := d(Z_1), d' = d(Z_2)$. Then the orbit of the submodule $X$ under the action of $\text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2) \subseteq \text{Sp}(\Lambda)$ determines $d, d'$ and the orbit of $X'$ under the action of $\text{Sp}(\Lambda_1') \times \text{Sp}(\Lambda_2') \subseteq \text{Sp}(\Lambda')$ and is conversely determined by these data.

Proof. This follows from the previous lemma and Corollary 2.7. \hfill \Box

By these lemmata we can now restrict our attention to the case that $\Lambda_1, \Lambda_2$ have equal rank and that the projections $\pi_1(X), \pi_2(X)$ are nondegenerate alternating modules.

Proposition 4.7. Let $\Lambda_1, \Lambda_2$ be $\mathbb{Z}$-lattices of ranks $2m$ on the vector spaces $V_1, V_2$ over $\mathbb{Q}$ with alternating bilinear forms of square free levels $N_1, N_2$ and determinants $D_1, D_2$, let $N = \text{lcm}(N_1, N_2)$ and $D = D_1D_2$, let $\Lambda = \Lambda_1 \perp \Lambda_2$ and let $X, \tilde{X}$ be maximal totally isotropic submodules of $\Lambda$ for which the projections $\pi_i(X) = X_i, \pi_i(\tilde{X}) = \tilde{X}_i$ to $\Lambda_i$ are nondegenerate alternating modules for $i = 1, 2$.

Let $\phi, \tilde{\phi} : V_1 \to V_2$ with $X_2 = \Lambda_2 \cap \phi(\Lambda_1), X_1 = \Lambda_1 \cap \phi^{-1}(\Lambda_2)$ and $\tilde{X}_2 = \Lambda_2 \cap \tilde{\phi}(\Lambda_1), \tilde{X}_1 = \Lambda_1 \cap \tilde{\phi}^{-1}(\Lambda_2)$ be the isometries associated to $\mathbb{Q}X, \mathbb{Q}\tilde{X}$ by Theorem 4.2 and the remark following it.

Let $W$ be a $2m$-dimensional vector space over $\mathbb{Q}$ with nondegenerate alternating bilinear form and $\Sigma_1 : V_1 \to W, \Sigma_2 : V_2 \to W$ be fixed isometries. Then

a) For $\sigma_1 \in \text{Sp}(V_1), \sigma_2 \in \text{Sp}(V_2)$ one has $\mathbb{Q}\tilde{X} = (\sigma_1, \sigma_2)(\mathbb{Q}X)$ if and only if $\sigma_2 \circ \phi \circ \sigma_1^{-1} = \tilde{\phi}$ holds.

b) $\tilde{X}$ is in the $\text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2)$-orbit of $X$ if and only if one has

$$\Sigma_2 \circ \tilde{\phi} \circ \Sigma_1^{-1} \in \text{Sp}(\Sigma_2 \Lambda_2)(\Sigma_2 \circ \phi \circ \Sigma_1^{-1})\text{Sp}(\Sigma_1 \Lambda_1).$$

Proof. Since one has $\tilde{X} = (\sigma_1, \sigma_2)X$ if and only if $X_1 = \sigma_1 X_1, \tilde{X}_2 = \sigma_2(X_2)$ by Theorem 4.2 a) follows. Assertion b) then follows from a). \hfill \Box

Taken together, Lemma 4.6 and Proposition 4.7 show that the orbits under $\text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2)$ of maximal totally isotropic submodules $X$ of $\Lambda$ are characterized by the rank $\text{rk}(\pi_1(X)/\text{rad}(\pi_1(X))) = \text{rk}(\pi_2(X)/\text{rad}(\pi_2(X)))$, the invariants $d, d'$ of the $\text{rad}(\pi_1(X))$, and a Hecke double coset associated to paramodular groups of rank $\text{rk}(\pi_1(X)/\text{rad}(\pi_1(X)))/2$ derived from $\text{Sp}(\Lambda_1), \text{Sp}(\Lambda_2)$. Translating that into matrix language we obtain the desired generalisation of Garrett’s double coset decomposition. Unfortunately, this requires a somewhat lengthy notation involving the boundary components (associated to $\text{rad}(\pi_1(X)), \text{rad}(\pi_2(X))$ by the results of Section 2) on which the paramodular groups of smaller rank act. Important special cases which look much simpler will be discussed in the remark following the proof.

Theorem 4.8. Let $\Lambda_1, \Lambda_2$ be $\mathbb{Z}$-lattices of ranks $2m, 2n$ on the vector spaces $V_1, V_2$ over $\mathbb{Q}$ with nondegenerate alternating bilinear forms of square free levels $N_1, N_2$ and determinants $D_1, D_2$, let $N = \text{lcm}(N_1, N_2)$ and $D = D_1D_2$, let $V = V_1 \perp V_2, \Lambda = \Lambda_1 \perp \Lambda_2$.

Let $\mathcal{B} = (e_1, \ldots, e_m, f_1, \ldots, f_n)$ be an ordered para-symplectic basis for $\Lambda_1$ with $\langle e_i, f_j \rangle = d_i \delta_{ij}, d_i | d_{i+1}$ and $\mathcal{B}' = (e'_1, \ldots, e'_n, f'_1, \ldots, f'_n)$ an ordered para-symplectic
basis for $\Lambda_2$ with $\langle e'_i, f'_i \rangle = d'_i \delta_{ij}, d'_i \mid d'_i + 1$, let $v_i = d_i^{-1} f_i, v'_i = d_i^{-1} f'_i$. Identify the elements of $\text{Sp}(V)$ with their matrices with respect to the ordered symplectic basis $(e_1, e_m, e'_1, \ldots, e'_n, v_1, \ldots, v_m, v'_1, \ldots, v'_n)$ of $V$ and the elements of $\text{Sp}(V_1)$ with their matrices with respect to the ordered symplectic basis $(e_1, \ldots, e_m, v_1, \ldots, v_m)$ of $V_1,$ the elements of $\text{Sp}(V_2)$ with their matrices with respect to the ordered symplectic basis $(e'_1, \ldots, e'_n, v'_1, \ldots, v'_n)$ of $V_2.$

Let $d \mid D_1, d' \mid D_2, r \leq \min(m, n)$ satisfy $d \mid N_1^{m-r}, d' \mid N_2^{n-r}, \frac{D_1}{d} \mid N_1^{r}, \frac{D_2}{d} \mid N_2^{r},$ set $u_1 = m - r, u_2 = n - r.$ Let $g_1 = g_1(B, u_1, d)$ be as in Lemma 2.10 with associated matrices $\gamma_1 = \gamma_1(B, u_1, d) \in \text{Sp}_m(\mathbb{Q}), S_1 = S_1(B, u_1, d) \in \text{SL}_m(\mathbb{Z})$ and define $g_2 = g_2(B', u_2, d')$ and $\gamma_2, S_2$ analogously for $\Lambda_2,$ for $u_1 = 0$ set $S_1 = S_1(B, 0, 1) = 1_m,$ for $u_2 = 0$ set $S_2 = S_2(B', 0, 1) = 1_n.$ Let $\tilde{v}_i = g_1 v_i, \tilde{f}_i = d_i \tilde{v}_i, \tilde{e}_i = g_1 e_i$ and $\tilde{v}'_i = g_2 v'_i, \tilde{f}'_i = d_i \tilde{v}'_i, \tilde{e}'_i = g_2 e'_i$ as in Lemma 2.10.

Let

$$T = T(d, r) = \begin{pmatrix} \tilde{d}_1 & \cdots & 0 \\ & \ddots & \vdots \\ 0 & \cdots & \tilde{d}_r \end{pmatrix}, T' = T'(d', r) = \begin{pmatrix} \tilde{d}'_1 & \cdots & 0 \\ & \ddots & \vdots \\ 0 & \cdots & \tilde{d}'_r \end{pmatrix}.$$ 

Let $X^{(0)} = \bigoplus_{i=1}^m \mathbb{Z} e_i + \bigoplus_{i=1}^n \mathbb{Z} e'_i$ and let $P = \{ g \in \text{Sp}(\Lambda) \mid g(QX^{(0)}) = QX^{(0)} \}.$ For a maximal totally isotropic submodule $X$ of $\Lambda$ let $\pi_i(X) = X_i$ for $i = 1, 2$ denote the projections to $\Lambda_i.$

Then for each double coset $(\text{Sp}(\Lambda), \text{Sp}(\Lambda)) g P$ with $g \in \text{Sp}(\Lambda)$ the maximal totally isotropic submodule $X = gX^{(0)}$ of $\Lambda$ satisfies $d(\text{rad}(X_i)) = d, d(\text{rad}(X_i)) = d'$ and $\text{rk}(X_i/\text{rad}(X_i)) = 2r$ for $i = 1, 2$ for some triple of values $d, d', r$ as described above, and for each fixed such triple with associated matrices $S_1, S_2,$ a set of representatives of those double cosets $(\text{Sp}(\Lambda_1) \times \text{Sp}(\Lambda_2)) g P$ with $g \in \text{Sp}(\Lambda)$ for which $X = gX^{(0)}$ has $d(\text{rad}(X_i)) = d, d(\text{rad}(X_i)) = d'$ and $X_i/\text{rad}(X_i)$ has rank $2r$ for $i = 1, 2$ is given by the matrices

$$\begin{pmatrix} 1_{m+n} & 0_{m+n} \\ C & 1_{m+n} \end{pmatrix},$$

with

$$C = \begin{pmatrix} tS_1^{-1} & 0 \\ 0 & tS_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & tB(T) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix},$$

where the matrices $\left( \begin{smallmatrix} B & 0 \\ 0 & B^{-1} \end{smallmatrix} \right)$ with $B \in \text{GL}_r(\mathbb{Q}) \cap M_r(\mathbb{Z})$ run through a set of representatives of the double cosets

$$\Gamma^{(r)}(T') h \Gamma^{(r)}(T) \subseteq \text{Sp}_r(\mathbb{Q}),$$

for example the set $\mathcal{R}_{d,d',r}$ given in Corollary 3.8.

Proof. For $B$ as above the automorphism $\rho = \rho(r, d, d', B)$ of $V$ given by the matrix

$$\begin{pmatrix} S^{-1} & 0 \\ 0 & tS \end{pmatrix} \begin{pmatrix} 1_{m+n} & 0_{m+n} \\ C & 1_{m+n} \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & tS^{-1} \end{pmatrix} = \begin{pmatrix} 1_{m+n} & 0_{m+n} \\ tSCS & 1_{m+n} \end{pmatrix}$$
with
\[ S = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0_{r,r} & 0_{r,u_1} & t BT' \\ 0_{u_1,r} & 0_{u_1,u_1} & 0_{u_1,r} \\ T'B & 0_{r,u_1} & 0_{r,r} \\ 0_{u_2,r} & 0_{u_2,u_1} & 0_{u_2,r} \end{pmatrix} \]

with respect to the basis \((e_1, \ldots, e_m, e'_1, \ldots, e'_n, v_1, \ldots, v_m, v'_1, \ldots, v'_n)\) of \(V\) has matrix
\[
\begin{pmatrix} 1_{m+n} & 0_{m+n} \\ C & 1_{m+n} \end{pmatrix}
\]
with respect to the basis \((\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{e}'_1, \ldots, \tilde{e}'_n, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{v}'_1, \ldots, \tilde{v}'_n)\) of \(V\) by definition of \(S_1, S_2\). We have thus
\[
\rho(\tilde{e}_j) = \tilde{e}_j + \sum_{i=1}^r d^*_i b_{ij} \tilde{v}'_i
\]
\[
= \tilde{e}_j + \sum_{i=1}^r b_{ij} \tilde{f}'_i,
\]
\[
\rho(\tilde{e}'_j) = \tilde{e}'_j + \sum_{i=1}^r b_{ji} \tilde{d}'_j \tilde{v}_i
\]
\[
= \tilde{e}'_j + \sum_{i=1}^r \tilde{d}'_i b_{ji} \tilde{d}'_j \tilde{f}_i.
\]

We notice that we have \(\tilde{d}'_i b_{ji} \tilde{d}'_j \in \mathbb{Z}\) by Corollary 5.8 hence \(\rho(\tilde{e}'_j) \in \Lambda_1\) for all \(j\) and therefore \(g \in Sp(\Lambda)\). With \(X = \rho(X^{(0)})\) we have then \(X_1 = \sum_{i=1}^m \mathbb{Z} \tilde{e}_j + \sum_{i=1}^r \mathbb{Z} (\tilde{d}'_j \tilde{v}_i)\) with \(\text{rad}(X_1) = \sum_{i=1}^m \mathbb{Z} \tilde{e}_i, \ d(\text{rad}(X_1)) = d\).

In the same way we have \(X_2 = \sum_{i=1}^m \mathbb{Z} \tilde{e}'_j + \sum_{i=1}^r \mathbb{Z} (\sum_{i=1}^r b_{ij} \tilde{f}'_i)\) with \(\text{rad}(X_2) = \sum_{i=1}^m \mathbb{Z} \tilde{e}'_i, \ d(\text{rad}(X_2)) = d'.\)

We can write \(M(\text{rad}(X_1)) = \text{rad}(X_1) + \sum_{i=1}^m \mathbb{Z} \tilde{f}'_i, \ A_1' = \sum_{i=1}^m (\mathbb{Z} \tilde{e}_i + \mathbb{Z} \tilde{f}'_i)\) and have \(\Lambda_1 = M(\text{rad}(X_1)) \subseteq \Lambda_1'.\) In the same way we have \(M(\text{rad}(X_2)) = \text{rad}(X_2) + \sum_{i=1}^r \mathbb{Z} (\sum_{i=1}^r b_{ij} \tilde{f}'_i)\) and \(\Lambda_2 = M(\text{rad}(X_2)) \subseteq \Lambda_2'.\)

We can identify \(X_1/\text{rad}(X_1)\) with \(\sum_{j=1}^r \mathbb{Z} \tilde{e}_j + \sum_{i=1}^r \mathbb{Z} (\sum_{i=1}^r b_{ij} \tilde{v}_i) \subseteq \Lambda_1'\) and \(X_2/\text{rad}(X_2)\) with \(\sum_{i,j=1}^r \mathbb{Z} \tilde{e}'_j + \sum_{i,j=1}^r \mathbb{Z} (\sum_{i=1}^r b_{ij} \tilde{f}'_i) \subseteq \Lambda_2'\) and obtain an isomorphism \(\phi : X_1/\text{rad}(X_1) \to X_2/\text{rad}(X_2)\) given by \(\phi(\tilde{e}_j) = \sum_{i=1}^r b_{ij} \tilde{f}'_i, \ \phi(\tilde{v}_i) = \sum_{i=1}^r a_{ij} (\tilde{d}'_i)^{-1} \tilde{e}'_j\) with \(A = t B^{-1}\).

We let now \(W\) be a symplectic vector space of dimension \(2r\) over \(Q\) with symplectic basis \((x_1, \ldots, x_r, y_1, \ldots, y_r)\) and define \(\Sigma_1 : Q\Lambda_1' \to W_1, \Sigma_2 : Q\Lambda_2' \to W\) by \(\Sigma_1(\tilde{v}_j) = y_j, \Sigma_1(\tilde{e}_j) = x_j, \Sigma_2(\tilde{e}'_j) = d_j y_j, \Sigma_2(\tilde{f}'_j) = x_j\). The matrix groups attached to \(Sp(\Lambda_1'), Sp(\Lambda_2')\) with respect to the \(x_j, y_j\) are then \(\Gamma^{(r)}(T), \Gamma^{(r)}(T')\) respectively and we have
\[
\Sigma_2 \circ \phi \circ \Sigma_1^{-1}(x_j) = \sum_{i=1}^r b_{ij} x_i,
\]
\[
\Sigma_2 \circ \phi \circ \Sigma_1^{-1}(y_j) = \sum_{i=1}^r a_{ij} y_i.
\]
with $A = tB^{-1}$ as before, i.e., $\Sigma_2 \circ \phi \circ \Sigma_1^{-1}$ has matrix \( \begin{pmatrix} B & 0_r \\ 0_r & tB^{-1} \end{pmatrix} \).

The previous proposition together with the lemmata of this section leading up to it implies then the assertion. \(\square\)

**Remark 4.9.**

**a)** Assume $m = n, N_1 = N_2, D_1 = D_2$ and consider those double cosets $(Sp(\Lambda_1) \times Sp(\Lambda_2))gP$ with $r = m = n$, i.e., the projections $X_1, X_2$ of $X = gX^{(0)}$ have zero radical. In the theorem we have then $T = T' = \text{diag}(d_1, \ldots, d_m)$ and $S_1 = S_2 = 1_m$, and the set of representatives of these double cosets consists in matrix notation of the

\[
\begin{pmatrix}
1_m & 0_m & 0_m & 0_m \\
0_m & 1_m & 0_m & 0_m \\
0_m & 'BT & 1_m & 0_m \\
TB & 0_m & 0_m & 1_m
\end{pmatrix},
\]

where $B$ runs through a set of representatives of the double cosets

\[
\Gamma^{(m)}(T)h^{r(m)}(T) \subseteq Sp_m(\mathbb{Q}).
\]

**b)** In the case that $\Lambda, \Lambda_1, \Lambda_2$ have level 1 we obtain (with $d = d' = D_1 = D_2 = 1$ and the matrices $B$ being diagonal elementary divisor matrices by known results for the Hecke algebra of $Sp_n(\mathbb{Z})$) Garrett’s [20] result with a coordinate free proof. That such a proof should be possible has already been remarked in [20], see also [35]. Whereas in Garrett’s proof the relation between the representatives of the double cosets $Pg(Sp_m(\mathbb{Z}) \times Sp_n(\mathbb{Z}))$ and the representatives of the Hecke double cosets in $Sp_i(\mathbb{Z})$ appears to be a coincidence, the method chosen here shows that it is not.

5. **Theta series for the paramodular group**

We consider an $m$-dimensional vector space $V$ over $\mathbb{Q}$ with positive definite quadratic form $Q : V \rightarrow \mathbb{Q}$ and associated symmetric bilinear forms

\[b(x, y) = Q(x + y) - Q(x) - Q(y), \quad B(x, y) = \frac{1}{2}b(x, y).\]

The Gram matrix of an $n$-tuple $(x_1, \ldots, x_n) \in V^n$ of vectors in $V$ with respect to $Q$ is the matrix $Q(x_i, x_j) = (B(x_i, x_j))_{1 \leq i, j \leq n}$.

For a lattice $L = \bigoplus_{j=1}^{m} \mathbb{Z}e_j$ of full rank on $V$ with basis $(e_1, \ldots, e_m)$ the dual lattice is $L^\# = \{ y \in V \mid b(y, L) \subseteq \mathbb{Z} \}$, the level $N(L)$ is the smallest $N \in \mathbb{N}$ with $NQ(L^\#) \subseteq \mathbb{Z}$ and the discriminant $\text{disc}(L)$ is $\det(b(e_i, e_j))$.

The lattice is integral if $Q(L) \subseteq \mathbb{Z}$, it is unimodular if $L = L^\#$ (this corresponds to an even unimodular lattice in the notation of [33, 14]). Lattices $L$ and $K$ on $V$ are in the same class if there is

\[\varphi \in O(V) = \{ \varphi \in \text{GL}(V) \mid Q(\varphi(x)) = Q(x) \text{ for all } x \in V \}\]

with $\varphi(K) = L$, they are in the same genus ($K \in \text{gen}(L)$) if for all primes $p$ there is $\varphi_p \in O(V \otimes \mathbb{Q}_p)$ with $\varphi_p(K \otimes \mathbb{Z}_p) = L \otimes \mathbb{Z}_p$.

Similarly, two $n$-tuples $(K_1, \ldots, K_n), (L_1, \ldots, L_n)$ of lattices on $V$ are in the same class if there exists $\varphi \in O(V)$ with $\varphi(K_i) = L_i$ for $1 \leq i \leq n$, similarly for the genus. We write

\[O(K_1, \ldots, K_n) = \{ \varphi \in O(V) \mid \varphi(K_i) = K_i \text{ for } 1 \leq i \leq n \}\]
and $O_\Lambda(K_1, \ldots, K_n)$ for its adelization, so that $\varphi \in O_\Lambda(K_1, \ldots, K_n)$ is a tuple $(\varphi_p)_{p \in \mathbb{P}}$ with $\varphi_p \in O(V \otimes \mathbb{Q}_p)$ and $\varphi_p(K_i \otimes \mathbb{Z}_p) = K_i \otimes \mathbb{Z}_p$ for all $p$ including $p = \infty$. With this notation the classes in the genus of the $n$-tuple $(K_1, \ldots, K_n)$ correspond to the double cosets $O(V)\varphi O_\Lambda(K_1, \ldots, K_n)$ in the adelic orthogonal group $O_\Lambda(V)$ of the quadratic space $(V, Q)$. Assuming $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq MK_{1}$ for some $M \in \mathbb{N}$ we see that $O(K_1, \ldots, K_n)$ is a congruence subgroup of $O(K_1)$, hence in particular of finite index. Since the number of double cosets $O(V)\varphi O_\Lambda(K_1)$, being the number of classes in the genus of $K_1$, is known to be finite, the number of classes in the genus of $K_1, \ldots, K_n$ is finite too. This assumption can be made without loss of generality since there exist $M_2, \ldots, M_n \in \mathbb{N}$ with $M_{i+1}K_{i+1} \subseteq M_iK_i \subseteq K_i$ for all $i$ and $M$ with $MK_1 \subseteq K_n$.

**Definition 5.1.** Let $L_1, \ldots, L_n$ be lattices on $V$. The theta series $\vartheta^{(n)}(L_1, \ldots, L_n)$ is the function on $\mathcal{S}_n$ given for $Z \in \mathcal{S}_n$ by

$$\vartheta^{(n)}(L_1, \ldots, L_n; Z) = \sum_{x_1 \in L_1, \ldots, x_n \in L_n} \exp(\pi \text{tr}(Q(x_1, \ldots, x_n)Z)).$$

**Remark 5.2.**

a) for $L = L_1 = \cdots = L_n$ we obtain the usual degree (or genus) $n$ theta series of the lattice $L$.

b) We obtain a matrix notation for our theta series by fixing a basis $(e_1, \ldots, e_m)$ of $V$ and matrices $U_1, \ldots, U_n \in \text{GL}_m(\mathbb{Q})$ such that the coordinate vectors with respect to the given basis of vectors in $L_i$ run through $U_i\mathbb{Z}^m$. If $S$ is the Gram matrix of $Q$ with respect to the given basis we obtain

$$\vartheta^{(n)}(L_1, \ldots, L_n; Z) = \sum_G \exp(\pi \text{tr}(S[UG]Z)),$$

where $G = (g_1, \ldots, g_n)$ runs over the integral $(m \times n)$-matrices and where we write $UG = (U_1g_1, \ldots, U_ng_n)$.

**Theorem 5.3.** The theta series satisfy the transformation formula

$$\vartheta^{(n)}(L_1^\#, \ldots, L_n^\#, -Z^{-1}) = \sqrt{\det(Z/i)}^m \prod_{j=1}^n \sqrt{\text{disc}(L_j)} \vartheta^{(n)}(L_1, \ldots, L_n; Z),$$

where $\sqrt{\det(Z/i)}$ is defined as in [16], Hilfssatz 0.10, i.e., it is continuous on $\mathcal{S}_n$ and has positive real values for $Z = iy$ on the imaginary axis.

**Proof.** By using the matrix formulation of our theta series we can proceed as in the proof of [16] Hilfssatz 0.12] by computing the Fourier coefficient $a(H)$ at a matrix $H = (h_1, \ldots, h_n) \in \mathbb{Z}^{mn}$ of the periodic function $f$ at a matrix $H = (h_1, \ldots, h_n) \in \mathbb{Z}^{mn}$ given by

$$f(W) = \sum_G \exp(\pi \text{tr}(S[UG + UW]Z)).$$

We arrive then in the same way as there at

$$a(H) = \exp(-\pi \text{tr}(S^{-1}[t^1U^{-1}H]Z^{-1})) \int \exp(\pi \text{tr}(S[UV]Z))dV,$$

where $V$ runs over $\mathbb{R}^{mn}$ and where we write $t^1U^{-1}H = (t^1U^{-1}h_1, \ldots, t^1U^{-1}h_n)$. Applying the transformation formula for integrals and using that the coordinate vectors with respect to the given basis of the dual lattice $L_i^\#$ run over $U_i^{-1}\mathbb{Z}^m$ we obtain the final result as in [16].
Theorem 5.4. Let $T \in M_n(\mathbb{Z})$ be an elementary divisor matrix with diagonal entries $1 = t_1, t_2, \ldots, t_n$, let $L_j$ for $1 \leq j \leq n$ be positive definite even $t_j$-modular lattices (so $L_j^\# = t_j^{-1}L_j$ and $Q(L_j) = t_j\mathbb{Z}$) of rank $m$ with $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n$ (we will call such a chain of lattices paramodular of level $T$ in the sequel). Then $\vartheta^{(n)}(L_1, \ldots, L_n)$ is a modular form of weight $k = m/2$ for the paramodular group $\Gamma^{(n)}(T)$.

Proof. By Satz 1.12 of [25] the group $\Gamma^{(n)}(T)$ is generated by the matrices

\[
J_T = \begin{pmatrix} 0_n & -T^{-1} \\ T & 0_n \end{pmatrix}, \quad \begin{pmatrix} 1_n & t_i^{-1}E_{ii} \\ 0_n & 1_n \end{pmatrix}, \quad \begin{pmatrix} 1_n & t_i^{-1}(E_{ij} + E_{ji}) \\ 0_n & 1_n \end{pmatrix} (1 \leq i < j \leq n),
\]

where $E_{ij}$ denotes the $n \times n$-matrix with entry 1 in position $(i, j)$ and 0 in all other positions.

We check the transformation behavior under these generating matrices:

By the previous theorem we have

\[
\vartheta^{(n)}(L_1, \ldots, L_n; J_T Z) = \vartheta^{(n)}(L_1, \ldots, L_n; -(T Z T)^{-1})
\]

\[
= \sqrt{\det(Z/i)^m (\det(T))^m} \prod_{j=1}^n \sqrt{\text{disc}(L_j^\#)} \vartheta^{(n)}(L_1^\#, \ldots, L_n^\#; T Z T)
\]

\[
= \sqrt{\det(T Z/i)^m} \vartheta^{(n)}(t_1 L_1^\#, \ldots, t_n L_n^\#; Z)
\]

\[
= \sqrt{\det(T Z/i)^m} \vartheta^{(n)}(L_1, \ldots, L_n; Z)
\]

\[
= \det(T Z)^{m/2} \vartheta^{(n)}(L_1, \ldots, L_n; Z)
\]

since the rank $m$ of a positive definite even unimodular lattice is divisible by 8.

Moreover, the translation matrices $\begin{pmatrix} 1_n & t_i^{-1}E_{ii} \\ 0_n & 1_n \end{pmatrix}$, $\begin{pmatrix} 1_n & t_i^{-1}(E_{ij} + E_{ji}) \\ 0_n & 1_n \end{pmatrix}$ leave $\vartheta^{(n)}(L_1, \ldots, L_n)$ invariant since by assumption we have $t_i \mid Q(x_i)$ and $t_i \mid B(x_i, x_j)$ for $1 \leq i < j \leq n$ and $x_i \in L_i$. \hfill \Box

Remark 5.5. a) As noticed in Remark [24] it is no restriction of generality to assume $t_1 = 1$ for a diagonal $T = \text{diag}(t_1, \ldots, t_n)$, we will hence do so in the sequel.

b) If we permute the lattices $L_i$ by some $\sigma \in S_n$ and apply the same permutation to the diagonal entries of the matrix $T$ we obtain a paramodular form with respect to this permuted matrix, hence with respect to a conjugate paramodular group. We will use this obvious fact later.

c) If we relax the conditions on the $L_i$ by demanding $t_j$-modularity only for the $p$-adic completions of the $L_j$ for all $p$ dividing $t_n$ we obtain modular forms for groups $\Gamma_0^{(n)}(T, N)$ (as mentioned in Section [3]) which are sort of a mix between paramodular groups and groups of type $\Gamma_0^{(n)}(N)$. The details of this will not be worked out here.

6. Siegel’s main theorem

Definition 6.1. Let $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n$ be lattices on $V$. 
a) The weight \( w(\text{gen}(L_1, \ldots, L_n)) \) of the genus of \((L_1, \ldots, L_n)\) is given as

\[
\sum_{(M_1, \ldots, M_n)} \frac{1}{|O(M_1, \ldots, M_n)|},
\]

where the summation is over a set of representatives of the classes in the genus of \((L_1, \ldots, L_n)\).

b) The genus theta series \( \vartheta^{(n)}(\text{gen}(L_1, \ldots, L_n)) \) is given as

\[
\vartheta^{(n)}(\text{gen}(L_1, \ldots, L_n)) = \frac{1}{w(\text{gen}(L_1, \ldots, L_n))} \sum_{(M_1, \ldots, M_n)} \frac{\vartheta^{(n)}(M_1, \ldots, M_n)}{|O(M_1, \ldots, M_n)|},
\]

where the summation is over a set of representatives of the classes in the genus of \((L_1, \ldots, L_n)\).

**Theorem 6.2.** Let \( k \equiv 0 \mod 4 \), and let \( T \) be an elementary divisor matrix of square free level. Then there is exactly one genus of \( n \)-tuples \((L_1, \ldots, L_n)\) of lattices of rank \( m = 2k \) which are paramodular of level \( T \).

**Proof.** Let \( L_1 \) be an even unimodular lattice of rank \( m \). It is well known and can be seen using the corollaries on p. 116 and 119 of [12] that the \( p \)-adic completion \((L_1)_p\) of \( L_1 \) is an orthogonal sum of hyperbolic planes for any prime \( p \), and it is then obvious that it contains a sublattice (of index \( p^k \)) that is \( p \)-modular. We can therefore obtain a chain of sublattices \( L_i \supseteq L_{i+1} \) of \( L_1 \) for which each \( L_i \) is \( t_i \)-modular, which settles the existence claim.

On the other hand, given two such chains of lattices \( L_i, K_i \), we can assume the lattices to be on the same rational quadratic space since there is only one genus of even unimodular lattices of given rank. If \( p \) is a prime dividing \( t_n \), the completion \((L_n)_p\) is a \( p \)-modular sublattice of the maximal lattice \((L_1)_p\), which is an orthogonal sum of hyperbolic planes. By [13] Satz 9.5 there is a hyperbolic basis \( e_1, \ldots, e_k, f_1, \ldots, f_k \) (i.e., the \( e_i, f_i \) are isotropic vectors with \( B(e_i, f_j) = \delta_{ij} \)), such that \( e_1, \ldots, e_k, pf_1, \ldots, pf_k \) is a basis of \((L_n)_p\). In the same way we obtain analogous bases \( e'_1, \ldots, e'_k, f'_1, \ldots, f'_k \) of \( K_1 \) and \( e''_1, \ldots, e''_k, pf'_1, \ldots, pf'_k \) of \( K_n \). Sending \( e_i \) to \( e'_i \) and \( f_i \) to \( f'_i \) we obtain a local isometry at \( p \) mapping \( L_i \) onto \( K_i \) for \( 1 \leq i \leq n \), so the two chains are in the same genus.

**Remark 6.3.** An arithmetic study of the classes in this genus will be quite interesting. For example, for a prime \( p \) and \( T = \begin{pmatrix} 1_a & 0 \\ 0 & p^{1_b} \end{pmatrix} \) the lattice chains are of the type \((L, \ldots, L, K, \ldots, K)\) with a copies of the even unimodular lattice \( L \) and \( b \) copies of the even \( p \)-modular lattice \( K \).

Two such chains \((L, \ldots, L, K, \ldots, K)\) and \((L', \ldots, L', K', \ldots, K')\) belong to the same class if \( L, L' \) are in the same class and (with \( L = L' \)) the sublattice \( K' \) of \( L \) is in the orbit of \( K \) under the action of the group of automorphisms of \( L \), so the number of classes in this genus of chains with first entry in the class of \( L \) is the number of these orbits.

**Lemma 6.4.** For \( T \) of square free level and \( k > n + 1 \) the space of Siegel Eisenstein series of weight \( k \) for \( \Gamma^{(n)}(T) \) has dimension 1.

**Proof.** We recall that Siegel Eisenstein series, i.e. Eisenstein series associated to zero-dimensional cusps, of weight \( k \) define a space of dimension equal to the number...
of equivalence classes of zero-dimensional cusps. The claim follows from Corollary 2.8.

Theorem 6.5 (Siegel’s main theorem for the paramodular group). Let $T$ be an elementary divisor matrix of square free level and let $\gen((L_1, \ldots, L_n))$ be the unique genus of $n$-tuples of positive definite lattices of rank $m = 2k$ (with $8 \mid m$) which are paramodular of level $T$. Assume $k > n + 1$ and define by

$$E_{k}^{(n), T}(Z) := \sum_{g \in P \backslash \Gamma^{(n)}(T)} j(g, Z)^{-k}$$

the unique normalized (Siegel) Eisenstein series which is paramodular of level $T$ (where $j((A B), Z) = \det(CZ + D)$ as usual and where $P$ as in Theorem 4.3 is the subgroup of matrices with upper triangular block decomposition of the paramodular group $\Gamma^{(n)}(T)$).

Then one has

$$(6.1) \quad \vartheta(\gen(L_1, \ldots, L_n)) = E_k^{(n), T}.$$

Proof. This is a consequence of the general Siegel-Weil theorem as given in [30]. The translation between the adelic setting used there and our present classical setting is provided by [31], see in particular Section IV.2. In the notation used there we take as the test function $\varphi_p : V_p^n \to \mathbb{C}$ at the finite primes $p$ the characteristic function of the $p$-adic completions of the given $n$-tuple of lattices and let $\varphi_{\infty}(x_1, \ldots, x_n) = \exp(-\pi \text{tr}(Q(x_1, \ldots, x_n)))$. A standard argument (see e.g. [45, Section 2]) shows that the integral $I(g; \varphi)$ is the adelic function corresponding to Siegel’s weighted average $\vartheta^{(n)}(\gen(L_1, \ldots, L_n))$.

That the adelic Eisenstein series $E(g, s_0, \Phi)$ corresponds to the classical Eisenstein series above under the usual correspondence between adelic and classical modular forms is checked as in the case of level $N = 1$ in [31], replacing $\Gamma = Sp(n, \mathbb{Z})$ by the paramodular group $\Gamma = \Gamma^{(n)}(T)$. Notice (still using Kudla’s notation) for this that Kudla’s argument for the case $Sp(n, \mathbb{Z})$ goes through unchanged for our situation if we can show that $\Gamma = \Gamma^{(n)}(T)$ satisfies $P(\mathbb{Q})\Gamma = G(\mathbb{Q})$, thus $P(\mathbb{Q}) \backslash G(\mathbb{Q}) = P(\mathbb{Q}) \cap \Gamma \backslash \Gamma$, and that the function $\Phi_f$ associated to the $\varphi_p$ satisfies $\Phi_f(\gamma) = 1$ for all $\gamma \in \Gamma^{(n)}(T)$. Indeed Corollary 2.8 implies $P(\mathbb{Q})\Gamma = G(\mathbb{Q})$. Moreover, using the generators discussed in the proof of Theorem 5.4 and the usual formulas for the action of the local Weil representation $\omega_p$ (see again e.g. [45, Section 2]), it is easily checked that $\omega_p(\gamma)\varphi_p = \varphi_p$ for all $\gamma \in \Gamma^{(n)}(T)$ and all primes $p$ and hence $\Phi_f(\gamma) = 1$ for all $\gamma \in \Gamma^{(n)}(T)$ holds. We notice in passing that the last argument can also be used to give an adelic proof of Theorem 5.4.

Remark 6.6. Using the relation

$$\Gamma^{(n)}(T) = \begin{pmatrix} T^{-1} & 0_n & 0_n \\ 0_n & T & 0_n \\ 0_n & 1_n & T^{-1} \end{pmatrix} \tilde{\Gamma}^{(n)}(T) \begin{pmatrix} T^{-1} & 0_n & 0_n \\ 0_n & T & 0_n \\ 0_n & 1_n & T^{-1} \end{pmatrix}$$

one sees that

$$E_k^{(n), T}(T^{-1}ZT^{-1}) = \tilde{E}_k^{(n), T}(Z),$$

where $\tilde{E}_k^{(n), T}(Z)$ denotes the Eisenstein series of Siegel in [43] defined using the group $\tilde{\Gamma}^{(n)}(T)$, so that Siegel’s Eisenstein series is the genus theta series attached to the lattice $n$-tuple $(L_1^\#, \ldots, L_n^\#)$. 

7. The basis problem for cusp forms of square free level

7.1. Pullback of Eisenstein series. We start from a “polarization matrix” \( \mathcal{T} \) of size \( 2n \) of the special form \( \mathcal{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \). Furthermore we denote the element 
\[
\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \in \mathcal{H}_{2n} \text{ with } z, w \in \mathcal{H}_n \text{ by } \iota(z, w).
\]
For the degree \( 2n \) Eisenstein series \( E^{(2n), \mathcal{T}}_k \) and a degree \( n \) cusp form \( f \) for \( \Gamma(n)(T) \), we consider the function \( \Omega(f) \) on \( H_n \) defined by
\[
w \mapsto \Omega(f)(w) := \int_{\Gamma(n)(T) \setminus H_n} f(z) E^{(2n), \mathcal{T}}_k(\iota(z, w)) \det(y)^k dz_n,
\]
where \( dz_n \) denotes the invariant symplectic volume element on \( \mathcal{H}_n \).

We only give a sketch of the unfolding in this case and refer to [20] and [6], where the case of level 1 was considered in detail. We recall from loc. cit. that one needs a “Garrett double coset decomposition” and a “twisted coset decomposition” describing \( P \setminus \Gamma^{2n}(\mathcal{T}) / (\Gamma(n)(T) \times \Gamma(n)(T)) \).

We can then closely follow the strategy of loc. cit.:

The key new ingredient is Theorem 4.8 applied to the situation \( n = m, T = T' \), which gives the desired double coset decomposition. Noticing that in view of \( \Gamma^{(n)}(T) B \Gamma^{(n)}(T) = \Gamma^{(n)}(T)(-B) \Gamma^{(n)}(T) \), the representatives given there can also be used as representatives for the double cosets \( P g (\Gamma^{(n)}(T) \times \Gamma^{(n)}(T)) \subseteq \Gamma^{(2n)}(\mathcal{T}) \).

Once the former decomposition is available, then the latter decomposition is obtained by a routine calculation.

The natural decomposition is given by collecting all left cosets \( P \cdot g \in \Gamma^{(2n)}(\mathcal{T}) \), which belong to a fixed double coset \( \text{encoded by } C \):
\[
E^{(2n), \mathcal{T}}_k(\iota(z, w)) = \sum_C \omega_C(z, w).
\]

Note that \( \omega_C \) is a modular form in \( z \) and \( w \) for \( \Gamma(n)(T) \). The summands appearing in a fixed \( \omega_C \) are described by the “twisted double coset decomposition”; as in the level one case, this is of type
\[
(7.1) \quad P \cdot \begin{pmatrix} 1_{2n} & 0 \\ C & 1_{2n} \end{pmatrix} (\Gamma^{(n)}(T) \times \Gamma^{(n)}(T)) = \cup_{\gamma, \delta} P \cdot \begin{pmatrix} 1_{2n} & 0 \\ C & 1_{2n} \end{pmatrix} \cdot \gamma \times \delta,
\]
where \( \gamma \) and \( \delta \) run over representatives of left cosets in \( \Gamma^{(n)}(T) \) modulo certain subgroups depending on \( C \), but independently of each other. This follows in the same way as in the level one case.

We can compute the contributions of the \( \omega_C \) to the integral \( \Omega(f) \) individually: The main points are

- By elementary matrix calculation, one has
\[
(7.2) \quad j(\begin{pmatrix} 1_{2n} & 0_{2n} \\ C & 1_{2n} \end{pmatrix}, \iota(z, w)) = \det(1_r - \ell BT^{r-1} w^S_1 T^{r-1} B z^S_1),
\]
where \( w^S \) denotes the submatrix of size \( r \) in the upper left corner of \( S_{2}^{-1} w S_{2}^{-t} \) (and analogously, \( z_{1}^{S} \) = upper left corner of size \( r \) in \( S_{1}^{-1} z S_{1}^{-t} \)).

- None of the \( \omega_C \) with \( r < n \) contributes to the doubling integral:

One may argue in essentially the same way as for level one (see [20, §8] or [26, p.238]). We briefly sketch the proof in our context:

It follows from general principles that \( \omega_C \) as a function of \( z \) is orthogonal to cusp forms: This holds quite generally (under suitable convergence conditions) for any modular form \( F \) on \( \mathcal{H}_n \) for an arithmetic subgroup \( \Gamma \) of \( \text{Sp}(n, \mathbb{Q}) \), constructed by averaging from a function \( \phi \) on \( \mathcal{H}_r \) with \( r < n \) by

\[
F(Z) := \sum_{\gamma} \Phi |_{k} \gamma.
\]

Here \( \Phi(Z) := \phi(z_1) \) with \( z_1 \) as explained above and \( \gamma \) runs over \( \Gamma \) modulo an appropriate subgroup (a subgroup of a Klingen parabolic, containing nontrivial translations). The requested orthogonality follows by a standard unfolding argument, see e.g. [27, §7].

- In the special case \( r = n \) with \( S_1 = S_2 = 1_n \), \( T' = T \) and \( B = 1_n \) the right hand side of (7.2) is just \( \det(1_n - TwTz) \).

The reproducing formula (attributed to Selberg)

\[
\int_{\mathcal{H}_n} f(z) \frac{\det(z - \bar{w})^{-k}}{\det(y)} dz_n = a_{n,k} f(w),
\]

valid for \( k > 2n \) and any holomorphic function \( f \) on \( \mathcal{H}_n \) satisfying a suitable growth condition (where \( a_{n,k} \) is a nonzero constant, for the explicit value see e.g. [27, p.78]) gives then for a cusp form \( f \) - after unfolding - for this special \( C \)

\[
\int_{\Gamma_0(T) \backslash \mathcal{H}_n} f(z) \omega_C(z, -\bar{w}) \frac{\det(y)}{\det(y)} dz_n = a_{n,k} \det(T)^{-k} \left( f |_{k} J_T \right)(w)
\]

Here \( J_T \) is as in [5.4].

- More generally (and again just as in the level one case), for arbitrary \( B \) with \( r = n \), the Hecke operator associated to the double coset \( \Gamma_n(T) h \Gamma_n(T) \) comes in:

\[
\int_{\Gamma_0(T) \backslash \mathcal{H}_n} f(z) \omega_B(z, -\bar{w}) \frac{\det(z)}{\det(h)} dz_n = a_{n,k} \det(T)^{-k} \left( f |_{k} \Gamma^{(n)}(T) h \Gamma^{(n)}(T) \right)(w) \cdot \det(h)^{-k}.
\]

Here we switched notation from \( \omega_C \) to \( \omega_B \).

In the formula above, we used the standard definition of Hecke operators: For any double coset \( \Gamma^{(n)}(T) g \Gamma^{(n)}(T) \) with \( g \in \text{Sp}(n, \mathbb{Q}) \), we get an endomorphism of the
space of modular forms (cusp forms) by
\[
(f \mid \Gamma(n)(T)g\Gamma(n)(T))(z) := \sum_\delta (f \mid \delta)(z),
\]
where \(\Gamma(n)(T)g\Gamma(n)(T) = \cup \Gamma(n)(T)\delta\). Note that the algebra generated by these endomorphisms is in general not commutative. The results from Section 3 show, however, that commutativity holds for the subalgebra generated by the \(T(m)\). Moreover, the \(T(m)\) define selfadjoint operators (w.r.t. the Petersson inner product) on the space of cusp forms; to see this, one can use the same kind of reasoning as for the commutativity in Section 3, using the involution on the Hecke algebra induced by \(\alpha \mapsto -\alpha^{-1}\). In particular, the space of cusp forms has a basis consisting of simultaneous eigenforms of all the \(T(m)\).

7.2. Nonvanishing. Now we assume that \(f\) is an eigenform of all the Hecke operators \(T(m)\) with eigenvalues \(\lambda(m)\). Then \(\Omega(f)\) is proportional to \(f\) with a factor, which equals - up to the factor \(a_{n,k} \det(T)^{-k}\) - the Dirichlet series
\[
L(f, s) := \sum_d \lambda(d)d^{-s}
\]
at \(s = k\).
We will show that this Dirichlet series does not vanish at \(s = k\):
We observe that this series inherits the absolute convergence at \(s = k\) from the corresponding property of the degree 2n Eisenstein series provided that \(k > 2n + 1\).
Also, from Lemma 3.8. we have an Euler product expansion (for \(\Re(s) \gg 0\))
\[
L(f, s) = \prod_p L_p(f, s)
\]
and it suffices to show that all these Euler factors are different from zero at \(s = k\).
We shall do this without explicitly determining the Euler factors (for primes not dividing \(\det(T)\) one can write them as standard Euler factors expressed by Satake parameters, but we do not use this here).

The Euler factors are of the form
\[
L_p(f, s) = \sum_{j=0}^{\infty} \lambda(p^j)p^{-js} = 1 + R_p(f, s).
\]
We show that the subseries \(R_p(f, s)\) of \(L_p(f, s)\) defined by \(R_p(f, s) = \sum_{j=1}^{\infty} \lambda(p^j)p^{-js}\) is of absolute value smaller than 1 at \(s = k\) if \(k\) is large enough:
The consideration in [29] shows that Hecke eigenvalues of cusp forms may quite generally be estimated by the number of left cosets in the double coset defining a Hecke operator.
The results from Lemma 3.6 and Lemma 3.7 allow us then to estimate \(\lambda(p^j)\) by a power of \(p^j\).

We observe that the condition
\[
\sum_{j=1}^{\infty} p^{-js} = \frac{p^{-s}}{1 - p^{-s}} < 1
\]
holds for all real \(s \geq 1\) if \(p\) is odd (and for \(p = 2\) it holds for \(s > 1\)).
Using the explicit estimates from Lemma 3.6, we see that $L_p(s)$ does not vanish at $s = k$ provided that $k \geq 2n + 2$ and $p$ is odd; the case $p = 3$ needs a minor additional consideration.

Furthermore the nonvanishing also holds for $p = 2$ if $k > 2n + 3$.

**Remark 7.1.** The nonvanishing of $L_p(s)$ at $s = k$ for $p$ coprime to $\det(T)$ also follows (under somewhat weaker conditions) from the explicit form of the Euler factor, which is of the same type as in the level one case.

**Proposition 7.2.** Assume that $k$ is even with $k > 2n + 2$ and $f$ is a cusp form of weight $k$ for $\Gamma_0(T)$ and also a Hecke eigenform for all operators $T(d)$. Then $L(f, s)$ is nonzero at $s = k$, in particular, $\Omega(f)$ is nonzero. If $\det(T)$ is odd, then this holds for $k \geq 2n + 2$.

7.3. **Basis problem.** To combine the considerations above about pullbacks of Eisenstein series with Siegel’s theorem, we have to change our setting slightly (this is mainly a matter of notation):

We assume now that $T$ is of elementary divisor form:

$$T = \text{diag}(1, t_2, \ldots, t_n)$$

We let $(L_1, L_2, L_3, \ldots, L_n)$ run over representatives of the classes in the genus of $n$-tuples of lattices of rank $m = 2k$ which are paramodular of level $T$ (see Theorem 6.2). Then $(L_1, L_2, L_3, \ldots, L_n)$ runs over representatives of the classes in the genus of $2n$-tuples of lattices of the same rank with paramodular level $\text{diag}(1, t_2, t_3, \ldots, t_n, t_n)$. By applying a suitable permutation of the entries of $Z \in \mathcal{S}_{2n}$ we may now reformulate Siegel’s theorem: The Eisenstein series $E^{(2n), T}_k$ is a linear combination of theta series $\vartheta^{(2n)}(L_1, L_2, \ldots, L_n, L_1, \ldots, L_n)$, in particular, $E^{(2n), T}_k (\iota(z, w))$ is a linear combination of

$$\vartheta^{(n)}(L_1, \ldots, L_n)(z) \times \vartheta^{(n)}(L_1, \ldots, L_n)(w).$$

We may therefore express $\Omega(f)$ in the usual way as a linear combination of theta series of the desired type. Taking into account that $\Omega$ defines an automorphism of the space of cusp forms (see Proposition 7.2), we obtain

**Theorem 7.3** (Basis problem for paramodular cusp forms). For $k \geq 2n + 4$ and $4 \mid k$ all cusp forms for $\Gamma^{(n)}_0(T)$ are linear combinations of theta series of type $\vartheta^{(n)}(L_1, \ldots, L_n)$ as described in Section 6. If $\det(T)$ is odd, the same holds for $k \geq 2n + 2$.

**Remark 7.4.** Using standard techniques about equivariant holomorphic differential operators [23], one can deduce in the same way statements concerning theta series with harmonic polynomials and one can also give a solution of the basis problem for cuspidal vector-valued modular forms, see e.g. [7].

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