On Hausdorff measure and an inequality due to Maz’ya

Hendrik Vogt and Jürgen Voigt

Abstract

We give an “elementary” proof of an inequality due to Maz’ya. As a prerequisite we prove an approximation property for the Hausdorff measure. We also comment on the relations between Maz’ya’s inequality, the isoperimetric inequality and the Sobolev inequality.

MSC 2010: 49Q15, 28A75, 46E35

Keywords: Hausdorff measure, isoperimetric inequality, Sobolev inequality, functions of bounded variation, perimeter of subsets of \( \mathbb{R}^n \)

Introduction

The objective of this note is to present an “elementary” proof of the following inequality, due to Maz’ya (see [9; Theorem 4.6.3]). If \( \Omega \subseteq \mathbb{R}^n \) is a bounded open set, \( \mathcal{H}_{n-1}(\partial \Omega) < \infty \), \( q = \frac{n}{n-1} \), and \( u \in C(\overline{\Omega}) \cap W_1^1(\Omega) \), then

\[
\|u\|_{L^q(\Omega)} \leq c(n) \left( \int_{\Omega} |\nabla u(x)| \, dx + \int_{\partial \Omega} |u(x)| \, d\mathcal{H}_{n-1}(x) \right), \tag{0.1}
\]

where \( c(n) \) is a constant only depending on the dimension \( n \). For the Hausdorff measure \( \mathcal{H}_{n-1} \) we refer to Section 1.

In Remark 2.2(b) we will comment on the relations between (0.1), the isoperimetric inequality

\[
\text{vol}_n(\Omega)^{\frac{n-1}{n}} \leq c(n) \mathcal{H}_{n-1}(\partial \Omega), \tag{0.2}
\]

valid for any bounded Borel set \( \Omega \subseteq \mathbb{R}^n \), and the Sobolev inequality

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq c(n) \int_{\mathbb{R}^n} |\nabla u(x)| \, dx, \tag{0.3}
\]

for all \( u \in W_1^1(\mathbb{R}^n) \). All three inequalities hold in fact with the same optimal constant

\[
c(n) = \frac{1}{n}\omega_n^{1/n} = \frac{\Gamma(n/2 + 1)^{1/n}}{n\sqrt{\pi}},
\]
the isoperimetric constant. In our proof we will suppose the validity of (0.3), with some (non-optimal) constant $c(n)$; we refer to [1; Remark 5.11], [4; Théorème IX.9] for a proof. We will then derive (0.1) with a constant that is larger than the constant in (0.3).

Rondi [10; Theorem 2.2] has shown an inequality more general than (0.1), where in particular the first term on the right-hand side is replaced by the variation of $u$ on $\Omega$.

Inequalities (0.2) and (0.1) are notorious for their challenging technical level. The desire to find an accessible proof of (0.1) arose from an application concerning the Dirichlet-to-Neumann operator for “rough domains”; see [3]. The inequality needed in this application is

$$\|u\|^2_{L^2(\Omega)} \leq c(\Omega) \left( \int_{\Omega} |\nabla u(x)|^2 \, dx + \int_{\partial \Omega} |u(x)|^2 \, dH_{n-1}(x) \right),$$

(0.4)

for all $u \in C(\overline{\Omega}) \cap W^1_2(\Omega)$, with a constant $c(\Omega)$ only depending on the domain, but where the optimality of the constant is inessential. In Remark 2.2(a) we will indicate how this inequality follows from (0.1). An earlier application of inequality (0.4) can be found in [5].

In Section 1 we treat an approximation property in the context of Hausdorff measures that will be used in the proof of (0.1) in Section 2. In Section 3 we recall the notion of functions of bounded variation, and we show that our method also yields an estimate of the total variation of $u$ by the right-hand side of (0.1).

1 On the Hausdorff measure $\mathcal{H}_d$

In this section we present an auxiliary result concerning the approximation of the Hausdorff measure of dimension $d \in [0, \infty)$ on a metric space $(M, \rho)$. We start by introducing some notation and recalling Hausdorff measures.

For a set $C \subseteq M$ we define $\text{rd}(C) := \frac{1}{2} \text{diam}(C)$, and for a countable collection $C$ of subsets of $M$ we define $\text{rd}(C) := \sup_{C \in C} \text{rd}(C)$ and

$$S_d(C) := \omega_d \sum_{C \in C} \text{rd}(C)^d,$$

where

$$\omega_d := \frac{\pi^{d/2}}{\Gamma(d/2 + 1)},$$

which is the volume of the unit ball in $\mathbb{R}^d$ if $d \in \mathbb{N}_0$. (Even though the notation ‘rd’ should be remindful of ‘radius’, the reader should be aware that a set $C$ will not necessarily be contained in a ball with radius $\text{rd}(C)$.)

Let $B \subseteq M$. For $\delta > 0$ we put

$$\mathcal{H}_{d, \delta}(B) := \inf\{S_d(C); C \text{ countable covering of } B, \text{ rd}(C) \leq \delta\}.$$
Then
\[ H^*_d(B) := \lim_{\delta \to 0} H_{d,\delta}(B) = \sup_{\delta > 0} H_{d,\delta}(B) \]
is the outer \(d\)-dimensional Hausdorff measure of \(B\). Carathéodory's construction of measurable sets yields a measure \(H_d\), the \(d\)-dimensional Hausdorff measure, and it turns out that all Borel sets are measurable. If \(d \in \mathbb{N}\), and \(E = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^n\), then \(H_d\) is the Lebesgue measure on \(\mathbb{R}^d \cong E\). For all of these properties we refer to [6; Chap. 2] and [7; 2.10.2].

Observe that, in the definition of \(H_{d,\delta}(B)\), one can also take the infimum over all countable open coverings of \(B\) and still obtain the same resulting value for \(H^*_d(B)\). Indeed, for \(\varepsilon > 0\) there exists a countable covering \(C\) with \(rd(C) \leq \frac{\delta}{2}\) and \(S_d(C) \leq H_d^*(B) + \varepsilon/2\). Choose \((\varepsilon_C)_{C \in C} \in (0, \infty)^C\) such that \(\sum_{C \in C} \varepsilon_C \leq \varepsilon/2\). Then for all \(C \in C\) there exists an open set \(U_C \supset C\) such that
\[ rd(U_C) \leq \delta \quad \text{and} \quad \omega_d rd(U_C)^d \leq \omega_d rd(C)^d + \varepsilon_C. \]

Then \(U := \{U_C; C \in C\}\) is a countable open covering of \(B\) with \(rd(U) \leq \delta\) and \(S_d(U) \leq H_d^*(B) + \varepsilon\).

1.1 Proposition. Let \(M\) be a metric space, \(d \in [0, \infty)\), and assume that \(H_d(M) < \infty\). Let \(\varepsilon > 0\).

Then for all \(\delta > 0\) there exists a countable partition \(A\) of \(M\) with \(rd(A) \leq \delta\), consisting of Borel subsets of \(M\) and such that
\[ \sum_{C \in A} |H_d(C) - \omega_d rd(C)^d| \leq \varepsilon. \] (1.1)

If \(M\) is compact, then the partition \(A\) can be chosen finite.

Proof. (i) The definition of \(H_d\) implies that there exists \(\delta_\varepsilon > 0\) such that for all countable coverings \(C\) of \(M\) with \(rd(C) \leq \delta_\varepsilon\) one has
\[ H_d(M) - \varepsilon \leq S_d(C). \] (1.2)

(ii) Next we show: for all \(\delta > 0\) there exists a countable partition \(A\) of \(M\) with \(rd(A) \leq \delta\), consisting of Borel subsets of \(M\) and such that
\[ S_d(A) \leq H_d(M) + \varepsilon; \]
if \(M\) is compact, then the partition \(A\) can be chosen finite.

As pointed out above, there exists a countable open covering \(U\) of \(M\) with \(rd(U) \leq \delta\) and \(S_d(U) \leq H_d(M) + \varepsilon\). If \(M\) is compact, then there exists a finite subcovering of \(U\). A standard procedure to produce a disjoint covering yields the desired partition \(A\).
H. Vogt, J. Voigt

(iii) Let $0 < \delta \leq \delta_\varepsilon$ (from step (i)), and let $\mathcal{A}$ be a partition of $M$ as in (ii). Let $\mathcal{A}_1 \subseteq \mathcal{A}$; then $\mathcal{A}_1$ is a partition of $M_1 := \bigcup \mathcal{A}_1$. We show that

$$\mathcal{H}_d(M_1) - \varepsilon \leq S_d(\mathcal{A}_1) \leq \mathcal{H}_d(M_1) + 2\varepsilon. \quad (1.3)$$

Let $\mathcal{C}_2$ be a countable covering of $M_2 := M \setminus M_1$ with $\text{rd}(\mathcal{C}_2) \leq \delta_\varepsilon$. Then the covering $\mathcal{A}_1 \cup \mathcal{C}_2$ of $M$ satisfies $\text{rd}(\mathcal{A}_1 \cup \mathcal{C}_2) \leq \delta_\varepsilon$, so from (1.2) we obtain

$$\mathcal{H}_d(M_1) + \mathcal{H}_d(M_2) - \varepsilon = \mathcal{H}_d(M) - \varepsilon \leq S_d(\mathcal{A}_1 \cup \mathcal{C}_2) \leq S_d(\mathcal{A}_1) + S_d(\mathcal{C}_2).$$

Since one can approximate $\mathcal{H}_d(M_2)$ arbitrarily well by $S_d(\mathcal{C}_2)$, choosing $\mathcal{C}_2$ suitably, this inequality implies

$$\mathcal{H}_d(M_1) - \varepsilon \leq S_d(\mathcal{A}_1),$$

the left-hand inequality of (1.3).

The application of the result obtained so far to the partition $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$ of $M_2$ yields $\mathcal{H}_d(M_2) - \varepsilon \leq S_d(\mathcal{A}_2)$. Putting this inequality together with the inequality stated in step (ii) we obtain

$$S_d(\mathcal{A}_1) = S_d(\mathcal{A}) - S_d(\mathcal{A}_2) \leq \mathcal{H}_d(M) + \varepsilon - (\mathcal{H}_d(M_2) - \varepsilon) = \mathcal{H}_d(M_1) + 2\varepsilon,$$

the right-hand inequality of (1.3).

Now we choose $\mathcal{A}_1 := \{ \mathcal{C} \in \mathcal{A}; \omega_d \text{rd}(\mathcal{C})^d \leq \mathcal{H}_d(\mathcal{C}) \}$ and apply (1.3) with $\mathcal{A}_1$, $M_1$ and with $\mathcal{A}_2$, $M_2$ (as defined above):

$$\sum_{\mathcal{C} \in \mathcal{A}} |\mathcal{H}_d(\mathcal{C}) - \omega_d \text{rd}(\mathcal{C})^d|$$

$$= \sum_{\mathcal{C} \in \mathcal{A}_1} (\mathcal{H}_d(\mathcal{C}) - \omega_d \text{rd}(\mathcal{C})^d) + \sum_{\mathcal{C} \in \mathcal{A}_2} (\omega_d \text{rd}(\mathcal{C})^d - \mathcal{H}_d(\mathcal{C}))$$

$$= (\mathcal{H}_d(M_1) - S_d(\mathcal{A}_1)) + (S_d(\mathcal{A}_2) - \mathcal{H}_d(M_2)) \leq 3\varepsilon. \quad \square$$

1.2 Remark. The crucial point of the inequality in Proposition 1.1 is that not only is the sum $S_d(\mathcal{A})$ close to $\mathcal{H}_d(M)$, but the individual terms $\omega_d \text{rd}(\mathcal{C})^d$ of the sum also approximate the corresponding terms $\mathcal{H}_d(\mathcal{C})$, with a small total error.

2 Proof of Maz’ya’s inequality

Before entering the proof we recall some properties of distributional derivatives.

2.1 Remarks. Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Let $u, v \in L_{1,\text{loc}}(\Omega; \mathbb{R})$, $\nabla u, \nabla v \in L_{1,\text{loc}}(\Omega; \mathbb{R}^n)$. Then $\nabla(u \wedge v) \in L_{1,\text{loc}}(\Omega; \mathbb{R}^n)$.

$$|\nabla(u \wedge v)| \leq |\nabla u| + |\nabla v|. $$
Indeed, from the well-known equality \( \partial_j u^+ = 1_{[u>0]} \partial_j u \) one easily deduces

\[
\partial_j |u| = \partial_j u^+ - \partial_j u^- = (1_{[u>0]} - 1_{[u<0]}) \partial_j u,
\]

for \( 1 \leq j \leq n \). Applying this equality to \( u \wedge v = \frac{1}{2}(u + v - |u - v|) \) one obtains

\[
\nabla (u \wedge v) = 1_{[u\leq v]} \nabla u + 1_{[u>v]} \nabla v,
\]

and this equality implies the asserted inequality.

(b) Let \( u \in L_{1,\text{loc}}(\Omega) \), \( 1 \leq j \leq n \), \( \partial_j u \in L_{1,\text{loc}}(\Omega) \). Then

\[
\partial_j |u| = \Re(\text{sgn} \, u \partial_j u) \in L_{1,\text{loc}}(\Omega).
\]

Indeed, given \( \delta > 0 \), the equality

\[
\partial_j (|u|^2 + \delta)^{1/2} = \partial_j (\overline{u}u + \delta)^{1/2} = (|u|^2 + \delta)^{-1/2} \Re(\overline{u} \partial_j u)
\]

is straightforward if \( u \) is continuously differentiable. Standard approximation arguments show that this equality also holds under the present hypotheses. Letting \( \delta \to 0 \) one concludes that

\[
\partial_j |u| = \Re\left(\frac{\overline{u}}{|u|} 1_{[|u|\neq 0]} \partial_j u\right) = \Re(\text{sgn} \, u \partial_j u).
\]

(c) Let additionally \( \Omega \subseteq \mathbb{R}^n \) be bounded, and let \( u \in C_0(\Omega) \cap W^1_{1,\text{loc}}(\Omega) \). We show that then \( u \in W^1_{1,\text{loc}}(\Omega) \). It is sufficient to treat the case when \( u \) is real-valued. Splitting \( u = u^+-u^- \) we reduce the problem to the case \( u \geq 0 \).

For \( s > 0 \) one has \( u-s \in W^1_{1,\text{loc}}(\Omega) \), and similarly as in part (a) above one obtains

\[
\nabla (u-s)^+ = 1_{[u>s]} \nabla (u-s) = 1_{[u>s]} \nabla u.
\]

Note that \( u \leq s \) on a neighbourhood of \( \partial \Omega \) because \( u \in C_0(\Omega) \), and therefore \( \text{spt}(u-s)^+ \) is compact, which implies \( (u-s)^+ \in W^1_{1,c}(\Omega) \). For \( s \to 0 \) one has \( (u-s)^+ \to u \), \( \nabla (u-s)^+ \to 1_{[u>0]} \nabla u \) pointwise and dominated, therefore in \( L_1(\Omega), L_1(\Omega; \mathbb{R}^n) \), respectively. Because \( 1_{[u>0]} \nabla u = \nabla u^+ = \nabla u \), this means that \( (u-s)^+ \to u \) in \( W^1_{1}(\Omega) \), hence \( u \in W^1_{1,\text{loc}}(\Omega) \).

As mentioned in the introduction, we will use Sobolev’s inequality (0.3) (where \( c(n) \) need not be the optimal constant) in our proof of (0.1).

**Proof of Maz’ya’s inequality (0.1).** Let \( u \in C(\overline{\Omega}) \cap W^1_1(\Omega) \). It follows from Remark 2.1(b) that \( |u| \in C(\overline{\Omega}) \cap W^1_1(\Omega) \) and that \( |\nabla |u|| \leq |\nabla u| \). This shows that it is sufficient to treat the case \( u \geq 0 \).

Let \( u \in C(\overline{\Omega}) \cap W^1_1(\Omega) \), \( u \geq 0 \), and let \( \varepsilon > 0 \). Then, by the (uniform) continuity of \( u \), there exists \( \delta \in (0,\varepsilon] \) such that \( |u(x) - u(y)| < \varepsilon \) whenever \( x,y \in \Omega, |x-y| < \delta \). By Proposition 1.1 there exists a finite partition \( \mathcal{A} \) of \( \partial \Omega \) with \( \text{rd}(\mathcal{A}) \leq \delta/4 \), consisting of Borel sets and such that (1.1) holds. We choose a family \( (x_C)_{C \in \mathcal{A}} \) with \( x_C \in C \) for all \( C \in \mathcal{A} \). (Recall that, by definition, the sets in a partition are supposed to be non-empty.)
Clearly \( \{ B_{\mathbb{R}^n}[x_C, \text{diam}(C)]; C \in \mathcal{A} \} \) is a covering of \( \partial \Omega \), where we use the notation \( B[x, r] \) for the closed ball with centre \( x \) and radius \( r \). For each \( C \in \mathcal{A} \), \( s \in (0, \delta/2) \) we define a function

\[
\psi_{C,s}(x) := \begin{cases} \frac{1}{2} \text{dist}(x, B(x_C, \text{diam}(C))) & \text{if } x \in B[x_C, \text{diam}(C) + s], \\ \infty & \text{otherwise}. \end{cases}
\]

Note that \( \psi_{C,s} \in W^1_1(B(x_C, \text{diam}(C) + s)) \). Note also that \( \text{diam}(C) + s \leq \delta \), and hence \( \psi_{C,s}(x) = u(x_C) + \varepsilon > u(x) \) for all \( x \in \partial B(x_C, \text{diam}(C) + s) \cap \Omega, \ C \in \mathcal{A} \). These properties and Remark 2.1(a) – applied repeatedly – imply that

\[
u_{\varepsilon,s} := u \wedge \inf_{C \in \mathcal{A}} \psi_{C,s} \quad \text{on } \bar{\Omega}
\]

belongs to \( C(\bar{\Omega}) \cap W^1_1(\Omega) \). Since \( \delta \leq \varepsilon \), the function \( u_{\varepsilon,s} \) coincides with \( u \) on \( \Omega_\varepsilon := \{ x \in \Omega; B(x, \varepsilon) \subseteq \Omega \} \), and as \( u_{\varepsilon,s} \) vanishes on \( \partial \Omega \), Remark 2.1(c) shows that \( u_{\varepsilon,s} \in W^1_1(\Omega) \).

The following computations prepare the application of Sobolev’s inequality (0.3) to \( u_{\varepsilon,s} \). Remark 2.1(a) implies

\[
\int_\Omega |\nabla u_{\varepsilon,s}(x)| \, dx \leq \int_\Omega |\nabla u(x)| \, dx + \sum_{C \in \mathcal{A}} \int_{B(x_C, \text{diam}(C) + s)} |\nabla \psi_{C,s}(x)| \, dx. \tag{2.1}
\]

Observing that \( \psi_{C,s} = 0 \) on \( B(x_C, \text{diam}(C)) \) and \( |\nabla \psi_{C,s}| = (u(x_C) + \varepsilon)/s \) on the spherical shell \( B(x_C, \text{diam}(C) + s) \setminus B(x_C, \text{diam}(C)) \), we obtain

\[
\int_{B(x_C, \text{diam}(C) + s)} |\nabla \psi_{C,s}(x)| \, dx = (u(x_C) + \varepsilon) \frac{1}{s} \omega_n ((\text{diam}(C) + s)^n - \text{diam}(C)^n). \tag{2.2}
\]

We note that, for \( s \to 0 \), the latter expression tends to

\[
(u(x_C) + \varepsilon) n \omega_n \text{diam}(C)^{n-1} = 2^{n-1} \frac{n \omega_n}{\omega_{n-1}} (u(x_C) + \varepsilon) \omega_{n-1} \text{rd}(C)^{n-1}.
\]

Recalling that \( u|_{\Omega_\varepsilon} = u_{\varepsilon,s}|_{\Omega_\varepsilon} \), we conclude from (0.3) that

\[
\|u\|_{L^q(\Omega_\varepsilon)} \leq \|u_{\varepsilon,s}\|_{L^q(\Omega)} \leq c(n) \int_\Omega |\nabla u_{\varepsilon,s}(x)| \, dx. \tag{2.3}
\]

Inserting (2.1) and (2.2) into (2.3) and taking \( s \to 0 \) we obtain

\[
\|u\|_{L^q(\Omega_\varepsilon)} \leq c(n) \left( \int_\Omega |\nabla u(x)| \, dx + 2^{n-1} \frac{n \omega_n}{\omega_{n-1}} \sum_{C \in \mathcal{A}} (u(x_C) + \varepsilon) \omega_{n-1} \text{rd}(C)^{n-1} \right). \tag{2.4}
\]
Exploiting (1.1) we can estimate the sum on the right-hand side of (2.4) by
\[
\sum_{C \in \mathcal{A}} (u(x_C) + \varepsilon) \left( |\omega_{n-1} \operatorname{rd}(C)^{n-1} - \mathcal{H}_{n-1}(C)| + \mathcal{H}_{n-1}(C) \right)
\leq (\|u\|_{\infty} + \varepsilon)\varepsilon + \sum_{C \in \mathcal{A}} \int_{C} (u(x_C) + \varepsilon) \, d\mathcal{H}_{n-1}
\leq (\|u\|_{\infty} + \varepsilon)\varepsilon + \int_{\partial \Omega} (u + 2\varepsilon) \, d\mathcal{H}_{n-1}.
\]

Because of this inequality, the estimate (2.4) implies
\[
\|u\|_{L_q(\Omega)} \leq c(n) \left( \int_{\Omega} |\nabla u(x)| \, dx + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \int_{\partial \Omega} u \, d\mathcal{H}_{n-1} + \varepsilon(2\mathcal{H}_{n-1}(\partial \Omega) + \|u\|_{\infty} + \varepsilon) \right).
\]

As this inequality holds for all \(\varepsilon > 0\), we finally obtain
\[
\|u\|_{L_q(\Omega)} \leq c(n) \left( \int_{\Omega} |\nabla u(x)| \, dx + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \int_{\partial \Omega} u \, d\mathcal{H}_{n-1} \right). \tag{2.5}
\]

**2.2 Remarks.** (a) Here we show how (0.4) can be derived from (0.1). Using the continuity of the embedding \(L_q(\Omega) \hookrightarrow L_1(\Omega)\) one obtains
\[
\|u\|_{L_1(\Omega)} \leq c_1(\Omega) \left( \int_{\Omega} |\nabla u(x)| \, dx + \int_{\partial \Omega} |u(x)| \, d\mathcal{H}_{n-1}(x) \right), \tag{2.6}
\]
for all \(u \in C(\overline{\Omega}) \cap W^1_1(\Omega)\). We mention that this inequality can also be obtained directly in our proof given above: in the derivation of (2.3) one can use Poincaré’s inequality instead of Sobolev’s inequality, thereby obtaining an estimate for the \(L_1\)-norm of \(u\).

Let \(u \in C(\overline{\Omega}) \cap W^1_2(\Omega)\). By standard arguments, in particular using Remark 2.1(b), we then conclude that \(\nabla |u|^2 = 2|u| \nabla |u|\), \(|u|^2 \in W^1_1(\Omega)\). Hence, (2.6) implies
\[
\|u\|_{L_2(\Omega)}^2 \leq c_1(\Omega) \left( 2 \int_{\Omega} |u(x)||\nabla |u||^2 \, dx + \int_{\partial \Omega} |u|^2 \, d\mathcal{H}_{n-1} \right). \tag{2.7}
\]
Applying the Cauchy–Schwarz inequality and Young’s inequality we get
\[
\int_{\Omega} |u(x)||\nabla |u||^2 \, dx \leq \|u\|_{L_2(\Omega)} \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{1/2}
\leq \gamma \|u\|_{L_2(\Omega)}^2 + \frac{1}{4\gamma} \int_{\Omega} |\nabla u(x)|^2 \, dx
\]
for all $\gamma > 0$. Inserting this inequality into (2.6), with $\gamma := \frac{1}{4c_1(\Omega)}$, and reshuffling terms we finally obtain

$$\|u\|_{L^2(\Omega)}^2 \leq 2c_1(\Omega) \left( 2c_1(\Omega) \int_\Omega |\nabla u(x)|^2 \, dx + \int_{\partial\Omega} |u|^2 \, dH_{n-1} \right).$$

(b) Here we show how (0.2) and (0.3), with optimal constant, can be derived from (0.1).

If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, then applying (0.1) with $u := 1_\Omega$ one immediately obtains (0.2). Now let $\Omega \subseteq \mathbb{R}^n$ be a bounded Borel set. We define the bounded open set $\Omega_0 := \overline{\Omega}$ (which may be empty). Then

$$\partial\Omega_0 = \overline{\Omega_0} \setminus \Omega_0 \subseteq \overline{\Omega} \setminus \hat{\Omega} = \partial\Omega,$$

and applying (0.2) for $\Omega_0$ we conclude that

$$\operatorname{vol}_n(\Omega \cap \Omega_0)^{\frac{n-1}{n}} \leq \operatorname{vol}_n(\Omega_0)^{\frac{n-1}{n}} \leq c(n)H_{n-1}(\partial\Omega_0) \leq c(n)H_{n-1}(\partial\Omega).$$

(2.8)

If $\operatorname{vol}_n(\Omega \setminus \Omega_0) = 0$, then (2.8) shows (0.2) for $\Omega$. Note that $\Omega \setminus \Omega_0 \subseteq \overline{\Omega} \setminus \hat{\Omega} = \partial\Omega$. Hence, if $\operatorname{vol}_n(\Omega \setminus \Omega_0) > 0$, then $H_{n-1}(\partial\Omega_0) = \operatorname{vol}_n(\partial\Omega) > 0$ as well, which implies $H_{n-1}(\partial\Omega) = \infty$, and again (0.2) is satisfied for $\Omega$.

Concerning (0.3) we first note that (0.1) clearly implies (0.3) for all $u \in C_c^\infty(\mathbb{R}^n)$. Applying standard cut-off and smoothing procedures one concludes that (0.3) holds for all $u \in W_1^1(\mathbb{R}^n)$.

(c) In this part of the remark we sketch a direct proof of (0.2) for bounded open sets $\Omega$ with $C^2$-boundary, with optimal constant. We first recall the Brunn-Minkowski inequality

$$\lambda^n(A + B)^{1/n} \geq \lambda^n(A)^{1/n} + \lambda^n(B)^{1/n},$$

valid for bounded Borel sets $A, B \subseteq \mathbb{R}^n$, where $\lambda^n$ denotes the Lebesgue measure. We refer to [8; Theorem C.7] for an elementary proof of (2.9). We also recall the Minkowski–Steiner formula

$$H_{n-1}(\partial\Omega) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( \lambda^n(\Omega + \varepsilon B) - \lambda^n(\Omega) \right),$$

where $B$ is the open unit ball in $\mathbb{R}^n$. Under our hypothesis of $C^2$-boundary the formula is not too hard to show; the essential observation for the proof is that, for small $\varepsilon > 0$, the set $(\Omega + \varepsilon B) \setminus \Omega$ can be written as the disjoint union

$$\bigcup_{t \in [0,\varepsilon)} \{ x + t\nu(x); x \in \partial\Omega \},$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial\Omega$. 

Combining these two ingredients one obtains
\[\mathcal{H}_{n-1}(\partial \Omega) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( \lambda^n(\Omega \cap \varepsilon B) - \lambda^n(\Omega) \right) \]
\[\geq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( \left( \lambda^n(\Omega)^{1/n} + \varepsilon \lambda^n(B)^{1/n} \right)^n - \lambda^n(\Omega) \right) \]
\[= n\lambda^n(\Omega)^{(n-1)/n} \lambda^n(B)^{1/n} = n\omega_n^{1/n} \lambda^n(\Omega)^{(n-1)/n},\]
for bounded open sets \(\Omega \subseteq \mathbb{R}^n\) with \(C^2\)-boundary.

On the basis of the above information, a variant of the isoperimetric inequality (see (3.2) below) is proved in [9; Section 9.1.5]. Using [2; Proposition 3.62] one then concludes (0.2) for all bounded open \(\Omega \subseteq \mathbb{R}^n\).

(d) A proof of (0.2) can also be found in [7; 3.2.43, note also 3.2.44]. A proof of (0.1) is given in [9; Example 5.6.2/1 and Theorem 5.6.3]. Another proof of a more general version of (0.1) on the basis of (0.2) and the coarea formula of geometric measure theory can be found in [10].

3 Functions of bounded variation, the extended Sobolev inequality, and Maz’ya’s inequality

Throughout this section let \(K = \mathbb{R}\). Let \(\Omega \subseteq \mathbb{R}^n\) be an open set. A function \(u \in L^1(\Omega)\) is of bounded variation, \(u \in BV(\Omega)\), if the distributional derivatives \(\partial_j u\), for \(j = 1, \ldots, n\), are finite signed Borel measures. If \(u \in BV(\Omega)\), then the variation of the \(\mathbb{R}^n\)-valued vector measure \(\partial u\) is given by
\[|\partial u|(A) := \sup \left\{ \sum_{B \in \mathcal{B}} |\partial u(B)|; \mathcal{B} \text{ finite partition of } A \text{ consisting of Borel sets} \right\},\]
for Borel subsets \(A\) of \(\Omega\). There exists a measurable function \(\sigma: \Omega \to S_{n-1}\) such that \(\partial u = \sigma|\partial u|\) (where \(S_{n-1}\) denotes the unit sphere in \(\mathbb{R}^n\)); see [6; Chap. 5.1, Theorem 1]. The total variation of \(\partial u\) is given by \(|\partial u|(\Omega)\). For more information on \(BV\) we refer to [2; Chap. 3], [6; Chap. 5], [8; Chap. 13].

With these notions we now sketch a proof of the extended Sobolev inequality
\[\|u\|_{L^q(\mathbb{R}^n)} \leq c(n) |\partial u|(\mathbb{R}^n)\] (3.1)
for all \(u \in BV(\mathbb{R}^n)\), with the optimal constant \(c(n)\) (where \(q = \frac{n}{n-1}\)).

Recall from Remark 2.2(c) the sketch of (0.2) for bounded open \(\Omega\) with \(C^2\)-boundary. Using the coarea formula and Sard’s lemma one can then show (0.3) for \(C^\infty_c\)-functions. As in the last paragraph of Remark 2.2(b) one extends (0.3) to all functions \(u \in W^1_1(\mathbb{R}^n)\). Finally, let \(u \in BV(\mathbb{R}^n)\), and choose a \(\delta\)-sequence \((\rho_k)\) in \(C^\infty_c(\mathbb{R}^n)_+\). Then \(\rho_k \ast u \in W^1_1(\mathbb{R}^n)\) for all \(k \in \mathbb{N}\), \(\rho_k \ast u \to u\) in \(L^1(\mathbb{R}^n)\) as \(k \to \infty\), and – most importantly – \(|\partial(\rho_k \ast u)|(\mathbb{R}^n) \leq |\partial u|(\mathbb{R}^n)\) for all \(k \in \mathbb{N}\). This last inequality follows from
\[|\partial(\rho_k \ast u)| = |\rho_k \ast \partial u| \leq \rho_k \ast |\partial u|\]
and \( \int \rho_k(x) \, dx = 1 \). Now, applying (0.3) to \( \rho_k \ast u \) we conclude that

\[
\| \rho_k \ast u \|_{L^q(\mathbb{R}^n)} \leq c(n) \int_{\mathbb{R}^n} |\nabla \rho_k \ast u(x)| \, dx \leq c(n)|\partial u|_{L^q(\mathbb{R}^n)} \quad (k \in \mathbb{N}).
\]

From these inequalities and the convergence of the sequence \((\rho_k \ast u)\) to \( u \) in \( L_1(\mathbb{R}^n) \) one obtains (3.1).

### 3.1 Remark.
If \( \Omega \subseteq \mathbb{R}^n \) is a Borel set, then one calls \( \Omega \) of finite perimeter if 
\( 1_\Omega \in BV(\mathbb{R}^n) \), and 
\( P(\Omega) := |\partial 1_\Omega|(\mathbb{R}^n) \) is called the perimeter of \( \Omega \). In this case (3.1) reads

\[
\text{vol}_n(\Omega)^{\frac{n-1}{n}} = \|1_\Omega\|_{L^q(\mathbb{R}^n)} \leq c(n)P(\Omega),
\]

an inequality that sometimes is also called “isoperimetric inequality”; see [9; Section 9.1.5].

The final issue of this paper is to show that the method of our proof of (0.1) also yields an estimate of the total variation of \( \partial u \) by the right-hand side of (0.1), with a non-optimal constant.

For this purpose we need the following important lower semi-continuity property of the total variation for \( BV \) functions. Let \((u_k)\) be a sequence in \( BV(\Omega) \), 
\( u_k \to u \) in \( L_1(\Omega) \), \( \sup_{k \in \mathbb{N}} |\partial u_k|(\Omega) < \infty \). Then \( u \in \mathcal{BV}(\Omega) \), and 
\( |\partial u|(\Omega) \leq \liminf_{k \to \infty} |\partial u_k|(\Omega) \). (See [6; Chap. 5.2, Theorem 1].)

### 3.2 Proposition.
Let \( \Omega \) and \( u \) be as in (0.1), and extend \( u \) to \( \mathbb{R}^n \) by putting 
\( u_{|\mathbb{R}^n \setminus \Omega} := 0 \). Then \( u \in BV(\mathbb{R}^n) \),

\[
|\partial u|(\mathbb{R}^n) \leq \int_{\mathbb{R}^n} |\nabla u(x)| \, dx + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \int_{\partial \Omega} |u| \, dH_{n-1}. \tag{3.3}
\]

**Proof.** First we treat the case \( u \geq 0 \). For \( \varepsilon > 0 \) we start the construction as in the proof of Maz'ya’s inequality above and obtain \( \delta_\varepsilon \in (0, \varepsilon] \) and functions 
\( u_{\varepsilon,s} \in W_1^1(\Omega) \) satisfying (2.1) for all \( s \in (0, \delta_\varepsilon/2) \). We also extend \( u_{\varepsilon,s} \) to \( \mathbb{R}^n \) by zero on the complement of \( \Omega \) and thereby obtain \( u_{\varepsilon,s} \in W_1^1(\mathbb{R}^n) \). Exploiting the considerations following (2.1) – but staying with the left hand side of (2.1) instead of applying (0.3) – we find \( s_\varepsilon \in (0, \delta_\varepsilon/2) \) such that

\[
\int_{\mathbb{R}^n} |\nabla u_{\varepsilon,s_\varepsilon}(x)| \, dx \leq \int_{\Omega} |\nabla u(x)| \, dx
+ 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \left( \int_{\partial \Omega} u \, dH_{n-1} + \varepsilon(2H_{n-1}(\partial \Omega) + \|u\|_\infty + \varepsilon) \right) + \varepsilon.
\]

Clearly \( u_{\varepsilon,s_\varepsilon} \to u \) in \( L_1(\mathbb{R}^n) \) as \( \varepsilon \to 0 \). Applying the lower semi-continuity of the total variation mentioned above we conclude (3.3).

For general \( u \in C(\overline{\Omega}) \cap W_1^1(\Omega) \) one then obtains (3.3) by applying \( |\partial u|(\mathbb{R}^n) \leq |\partial u^+|(\mathbb{R}^n) + |\partial u^-|(\mathbb{R}^n) \).
3.3 Remarks. (a) Combining the inequalities (3.1) and (3.3) one again obtains (2.5).

(b) One might hope that, once one has (3.3) and thereby knows that the function \( u \) defined in Proposition 3.2 belongs to \( BV \), one can continue and estimate the total variation of \( \partial u \) by \( \int_\Omega |\nabla u(x)| \, dx + \int_{\partial\Omega} |u| \, dH_{n-1} \). Combining (3.1) with this estimate one would obtain Maz’ya’s inequality (0.1) with the optimal constant. This kind of procedure is successful for the isoperimetric inequality, where first it is shown that finite Hausdorff measure of the boundary implies finite perimeter, and then fine points of geometric measure theory are applied to show that in fact the perimeter is estimated by the Hausdorff measure of the boundary.

References

[1] R. A. Adams: *Sobolev spaces*. Academic Press, New York, 1975.

[2] L. Ambrosio, N. Fusco and D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*. Clarendon Press, Oxford, 2000.

[3] W. Arendt and A. F. M. ter Elst: *The Dirichlet-to-Neumann operator on rough domains*. J. Differential Equations 251, 2100–2124 (2011).

[4] H. Brezis: *Analyse fonctionnelle – Théorie et applications*. Masson, Paris, 1983.

[5] D. Daners: *Robin boundary value problems on arbitrary domains*. Trans. Amer. Math. Soc. 352, 4207–4236 (2000).

[6] L. C. Evans and R. F. Gariepy: *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, FL, 1992.

[7] H. Federer: *Geometric Measure Theory*. Springer-Verlag, Berlin, 1969.

[8] G. Leoni: *A First Course in Sobolev Spaces*. American Mathematical Society, Providence, R.I., 2009.

[9] V. G. Maz’ya: *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*. 2nd ed., revised and augmented, Springer-Verlag, Berlin, 2011.

[10] L. Rondi: *A Friedrichs–Maz’ya inequality for functions of bounded variation*. Math. Nachr. 290, 1830–1839 (2017).
Hendrik Vogt
Fachbereich Mathematik
Universität Bremen
Postfach 330 440
28359 Bremen, Germany
hendrik.vogt@uni-bremen.de

Jürgen Voigt
Technische Universität Dresden
Fakultät Mathematik
01062 Dresden, Germany
juergen.voigt@tu-dresden.de