Multivariable Bergman shifts and Wold decompositions

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Let $H_m(\mathbb{B})$ be the analytic functional Hilbert space on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ with reproducing kernel $K_m(z, w) = (1 - \langle z, w \rangle)^{-m}$. Using algebraic operator identities we characterize those commuting row contractions $T \in L(H)^n$ on a Hilbert space $H$ that decompose into the direct sum of a spherical coisometry and copies of the multiplication tuple $M_z \in L(H_m(\mathbb{B}))^n$. For $m = 1$, this leads to a Wold decomposition for partially isometric commuting row contractions that are regular at $z = 0$. For $m = 1 = n$, the results reduce to the classical Wold decomposition of isometries. We thus extend corresponding one-variable results of Giselsson and Olofsson [5] to the case of the unit ball.

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§1 Introduction

By the classical Wold decomposition theorem each isometry $T \in L(H)$ on a Hilbert space $H$ is a direct sum $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)$ of a unitary operator $T_0 \in L(H_0)$ and an operator $T_1 \in L(H_1)$ which is unitarily equivalent to a Hardy space shift $M_z \in L(H^2(\mathbb{D}, D))$. Isometries are characterized by the the operator identity $T^*T = 1_H$, and the Hardy space is the analytic functional Hilbert space on the unit disc with reproducing kernel $K(z, w) = (1 - z\overline{w})^{-1}$. Our aim is to prove corresponding decomposition theorems for commuting tuples $T \in L(H)^n$ of Hilbert space operators which satisfy higher order operator identities related to the reproducing kernel $K_m(z, w) = (1 - \langle z, w \rangle)^{-m}$ on the unit ball.

An operator $T \in L(H)$ on a Hilbert space $H$ is unitarily equivalent to a Hardy space shift $M_z \in L(H^2(\mathbb{D}, D))$ if and only if it is an isometry which is pure in the sense that $\bigcap_{k=0}^{\infty} T^kH = \{0\}$. We replace the Hardy space $H^2(\mathbb{D})$ on the unit disc by the analytic functional Hilbert spaces $H_m(\mathbb{B})$ on the open unit ball $\mathbb{B} \subset \mathbb{C}^n$ defined by the reproducing kernels $K_m(z, w) = (1 - \langle z, w \rangle)^{-m}$, where $m \geq 1$ is a positive integer. It is well known that the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H_m(\mathbb{B}))^n$ is a row contraction such that its Koszul complex

$$K(M_z, H_m(\mathbb{B})) \xrightarrow{\epsilon_\lambda} \mathbb{C} \rightarrow 0$$

augmented by the point evaluations $\epsilon_\lambda : H_m(\mathbb{B}) \rightarrow \mathbb{C}, f \mapsto f(\lambda)$, at arbitrary points $\lambda \in \mathbb{B}$, is exact [7] (Proposition 2.6). In particular, the row operators

$$H_m(\mathbb{B})^n \rightarrow H(\mathbb{B}), (h_i)_{i=1}^n \mapsto (\lambda - M_z)(h_i)_{i=1}^n = \sum_{i=1}^n (\lambda_i - M_{z_i})h_i$$

have closed range and the operator-valued map $\mathbb{B} \rightarrow L(H_m(\mathbb{B})^n, H(\mathbb{B})), \lambda \mapsto \lambda - M_z$, is regular in the sense of [9] (Theorem II.11.4).

The reciprocal of the kernel $K_m$ is given by the binomial sum

$$K_m(z, w)^{-1} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \langle z, w \rangle^j.$$
Since the row operator $M_z : H_m(\mathbb{B})^n \to H_m(\mathbb{B})$ has closed range, the operator $M_z^*M_z : \text{Im}M_z^* \to \text{Im}M_z^*$ is invertible. In [2] it was shown that its inverse satisfies the identity

$$(M_z^*M_z)^{-1} = \left( \bigoplus_{j=0}^{m-1} (-1)^j \sigma_{M_z}(1_H) \right) \text{Im}M_z^*, $$

where $\sigma_{M_z}(X) = \sum_{i=1}^n M_{z^i}X M_{z^i}$. We show that the commuting row contractions $T \in L(H)^n$ for which the operator-valued function $\mathbb{B} \to L(H^n, H)$, $\lambda \mapsto \lambda - T$, is regular at $z = 0$ and which satisfy the operator identity

$$(T^*T)^{-1} = \left( \bigoplus_{j=0}^{m-1} (-1)^j \right) \sigma_{T}(1_H) \text{Im}T^*$$

are precisely the commuting tuples which decompose into an orthogonal direct sum

$$T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n$$

of a spherical coisometry $T_0 \in L(H)^n$ and a tuple $T_1 \in L(H_1)^n$ which is unitarily equivalent to the $m$-shift $M_z \in L(H_m(\mathbb{B}, D))$ for some Hilbert space $D$. We show that the coisometric part $T_0$ is absent if and only if

$$\bigcap_{k=0}^\infty \sum_{|\alpha| = k} T_0 \cap \{0\}. $$

We thus extend corresponding one-variable results proved by Giselsson and Olafsson [5] for the standard weighted Bergman spaces on the unit disc to the case of the analytic Besov spaces $H_m(\mathbb{B})$ on the unit ball.

For $m = 1$, the space $H_1(\mathbb{B})$ is the Drury-Arveson space and the validity of the above operator identity means precisely that the row operator $T : H^n \to H$ is a partial isometry. Thus up to unitary equivalence, the commuting tuples $T \in L(H)^n$ that are regular at $z = 0$ and for which the row operator $T : H^n \to H$ is a partial isometry, are precisely the direct sums $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n$ of a spherical coisometry $T_0$ and a Drury-Arveson shift $T_1$. Specializing further to the case $n = 1$ one obtains Wold-type decompositions for partial isometries that contain the classical Wold decomposition theorem and are closely related to corresponding results of Halmos and Wallen [8] for power partial isometries.

§2 Analytic models

Let $T \in L(H)^n$ be a commuting tuple of bounded operators on a complex Hilbert space $H$ such that $\sum_{1 \leq i \leq n} T_iH \subset H$ is a closed subspace. As usual we call the space $W(T) = H \oplus \sum_{1 \leq i \leq n} T_iH_i$ the wandering subspace of $T$. If the context is clear, we denote by $T$ also the induced row operator $T : H^n \to H$, $(h_i)_{i=1}^n \mapsto \sum_{i=1}^n T_i h_i$, and we write $T^* : H \to H^n, h \mapsto (T^* h)_{i=1}^n$, for its adjoint. Since $T : H^n \to H$ has closed range, the operator $T^*T : \text{Im}T^* \to \text{Im}T^*$ is invertible. We denote its inverse by $(T^*T)^{-1}$.

Consider the column operator $L = (T^*T)^{-1}T^* \in L(H, H^n)$. Then $LT = P_{\text{im}T^*}$ and

$$L(1_H - ZL)^{-1}(T - Z) = LT - L \sum_{k=0}^\infty (ZL)^k Z(1_{H^n} - LT)$$
Using this algebraic decomposition one obtains the identity

\[ T \text{ is regular at } z = 0 \text{ if and only if the operator-} \]

\[ \text{cohomology groups of the Koszul complex } K^{-}(T, H) \text{ of } T \text{ (see Section 2.2 in [3]). It is well known and elementary to prove that the following conditions suffice to guarantee the regularity of } T \text{ at } z = 0. \]

**1 Lemma.** Under either of the following three conditions:

1. \( TH^n = H \),
2. \( TH^n \subset H \) is closed and \( H^{n-1}(T, H) = \{0\} \),
3. there are an integer \( N \geq 1 \) and a real number \( \delta > 0 \) such that

\[ \dim H/(T - Z)H^n = N \text{ for all } z \in B_\delta(0), \]

the tuple \( T \) is regular at \( z = 0 \).

**Proof.** We sketch the well known proofs. If \( TH^n = H \), then \( (T - Z)H^n = H \) for \( z \) in a suitable neighbourhood of \( z = 0 \) and hence \( T \) is regular at \( z = 0 \). Condition (ii) means precisely that the sequence

\[ \Lambda^{n-2}(\sigma, H) \xrightarrow{(\delta^n_0-2)} \Lambda^{n-1}(\sigma, H) \xrightarrow{(\delta^{n-1}_0-\cdot \cdot \cdot)} W(T) \xrightarrow{P} \Lambda^n(\sigma, H) \rightarrow 0, \]

for \( z \in \mathbb{C}^n \) with \( \|z\| < 1/\|L\| \). Here \( Z : H^n \to H,(h_1)^{\mathbb{C}} \mapsto \sum_{i=1}^n z_i h_i, \) denotes the row operator induced by the complex \( n \)-tuple \( z \) and \( \|z\| = (\sum_{1 \leq i \leq n} |z_i|^2)^{1/2} \) is the Euclidean norm of \( z \). Since \( \Im L \subset \Im T^* = (\Ker T)^\perp \), it follows that \( P(z) \in (T - Z)H^n \subset H \) is a projection with \( \Im P(z) \subset \Im(T - Z) \) for \( z \in \mathbb{C}^n \) as above. The identity \( TL = T(T^*T)^{-1}T^* = P_{\Im T} \) yields that

\[ 1_H - P(z) = 1_H - (T - Z)L(1_H - ZL)^{-1} \]

\[ = (1_H - TL)(1_H - ZL)^{-1} = P_W(T)(1_H - ZL)^{-1} \]

and hence that \( (1_H - P(z))H = W(T) \) for \( z \in \mathbb{C}^n \) with \( \|z\| < 1/\|L\| \).

We call \( T \) regular at \( z = 0 \) if there is a positive real number \( \epsilon > 0 \) such that, for \( \|z\| < \epsilon \), the subspace \( (T - Z)H^n \subset H \) is closed and \( H \) decomposes into the algebraic direct sum

\[ H = (T - Z)H^n \oplus W(T). \]

Using this algebraic decomposition one obtains the identity

\[ (1_H - P(z))(T - Z) = 0 \text{ for } \|z\| < \min(\epsilon, 1/\|L\|). \]

But then the identity theorem implies that \( \Im(T - Z) \subset \Im P(z) \) for \( \|z\| < 1/\|L\| \). Thus we find that

\[ (1_H - P(z))H = W(T) \quad \text{and} \quad P(z)H = (T - Z)H^n \]

for \( \|z\| < 1/\|L\| \).
where \( i : W(T) \to H \) denotes the inclusion map and the operators \( \delta^j_{z-T} \) are the boundary maps of the Koszul complex of \(-T\) (Section 2.2 in \( \mathbb{H} \)), is exact at \( z = 0 \).
By Lemma 2.1.3 in \( \mathbb{H} \) there is a positive real number \( \epsilon < 1/\|L\| \) such that this sequence remains exact for every \( z \in \mathbb{C}^n \) with \( \|z\| < \epsilon \). But then

\[
(T - Z)H^n \oplus W(T) = H
\]

and \( \text{Im}(T - Z) = P(z)H \subset H \) is closed for \( \|z\| < \epsilon \).

Condition (iii) means that \( \text{Cowen-Douglas tuple} \) on \( B_\delta(0) \) in the sense of \( \mathbb{H} \). By the proof of Theorem 1.6 in \( \mathbb{H} \) the tuple \( T \) is regular at \( z = 0 \). □

In the following let \( T \in L(H)^n \) be a commuting tuple that is regular at \( z = 0 \). We denote by \( L_i \in L(H) \) \( (1 \leq i \leq n) \) the components of the column operator \( L = (T^*T)^{-1}T^* \in L(H, H^n) \) and we use the notation \( L_i = L_{i_1} \cdots L_{i_k} \) for arbitrary index tuples \( i = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \). To simplify the notation we write \( \Omega_T = B_1/\|L\| \) for the open Euclidean ball with radius \( 1/\|L\| \) at \( z = 0 \). We equip the space \( \mathcal{O}(\Omega_T, W(T)) \) of all analytic \( W(T) \)-valued functions on \( \Omega_T \) with its usual Fréchet space topology of uniform convergence on all compact subsets.

**2 Theorem.** Let \( T \in L(H)^n \) be regular at \( z = 0 \). Then the map

\[
V : H \to \mathcal{O}(\Omega_T, W(T)), \quad (Vx)(z) = (1_H - P(z))x
\]

is continuous linear with \( Vx \equiv x \) for \( x \in W(T) \) and

(i) \( VT_i = M_zV \quad (i = 1, \ldots, n) \),

(ii) \( \text{Ker} V = \cap_{k=0}^\infty \sum_{|\alpha|=k} T^\alpha H = \cap_{z \in \Omega_T} (T - Z)H^n \).

**Proof.** By construction, for \( z \in \Omega_T \) and \( x \in H \), the vector

\[
x(z) = (1_H - P(z))x = P_{W(T)}(1_H - ZL)^{-1}x
\]

is the unique element in \( W(T) \) such that \( x - x(z) \in \text{Im}(T - Z) \). Obviously the vector \( x(z) \) depends analytically on \( z \) and the map \( V \) is continuous linear with \( Vx \equiv x \) for \( x \in W(T) \). Since for \( z \) and \( x \) as above,

\[
T_i x - z_i x(z) = T_i (x - x(z)) + (T_i - z_i) x(z) \in \text{Im}(T - Z),
\]

the map \( V \) intertwines the tuples \( T \) on \( H \) and \( M_z \) on \( \mathcal{O}(\Omega_T, W(T)) \) componentwise. To calculate the kernel of \( V \), note that, for \( x \in H \) and \( z \in \Omega_T \),

\[
Vx(z) = \sum_{k=0}^\infty P_{W(T)}(ZL)^k x = \sum_{k=0}^\infty \sum_{|\alpha|=k} (P_{W(T)} \sum_{i \in I(\alpha)} L_i x) z^\alpha,
\]

where for each \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \) with \( |\alpha| = k \), the set \( I(\alpha) \) consists of all index tuples \( i = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \) such that, for each \( j = 1, \ldots, n \), exactly \( \alpha_j \) of the indices \( i_1, \ldots, i_k \) equal \( j \). The map \( \Sigma_T : L(H) \to L(H), X \mapsto \sum_{i=1}^n T_i X L_i \), is continuous linear with \( P_{W(T)} = 1_H - TL = 1_H - \Sigma_T(1_H) \) and

\[
\sum_{j=0}^{k-1} \Sigma_T^j(P_{W(T)}) = 1_H - \Sigma_T^k(1_H) \quad (k \geq 0).
\]
Hence for \( x \in \ker V \) and \( k \geq 0 \),

\[
0 = \sum_{j=0}^{k-1} \sum_{|\alpha|=j} T^\alpha (P_{W(T)} \sum_{i \in I(\alpha)} L_i x)
\]

\[
= \sum_{j=0}^{k-1} \sum_{|\alpha|=k} T^\alpha \left( \sum_{i \in I(\alpha)} L_i x \right).
\]

Thus \( \ker V \subset \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} T^n H \). Conversely, if a vector \( x \in H \) belongs to the intersection on the right-hand side, then

\[
Vx \in \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} VT^n H \subset \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} M^\alpha \mathcal{O}(\Omega_T, W(T)) = \{0\}.
\]

Thus the first equality in part (ii) has been shown. The second equality is obvious, since \( \ker (1_H - P(z)) = \im P(z) = (T - Z)H^n \) for all \( z \in \Omega_T \).

Elementary, even finite dimensional, examples show that Theorem 2 need not be true if instead of the regularity at \( z = 0 \) one only demands that the space \( TH^n \subset H \) is closed.

Condition (ii) in Theorem 2 implies that \( W(T) \subset (\ker V)^\perp \). An elementary argument shows that \( W(T) \) coincides with the wandering subspace of the compression of \( T \) to \( (\ker V)^\perp \).

In the following we use the notation \( H_\infty = \bigcap_{k=0}^{\infty} \sum_{|\alpha|=k} T^n H \). We call a commuting tuple \( T \in L(H)^n \) analytic if \( H_\infty = \{0\} \). If a commuting tuple \( T \in L(H)^n \) is unitarily equivalent to the multiplication tuple \( M_z \in L(\mathcal{H})^n \) on a functional Hilbert space \( \mathcal{H} \subset \mathcal{O}(\Omega, D) \) on a connected open zero neighbourhood \( \Omega \subset \mathbb{C}^n \), then \( T \) is necessarily analytic. The next result shows that, under the additional hypothesis that \( T \) is regular at \( z = 0 \), also the converse implication holds.

Let \( H/\ker V \cong (\ker V)^\perp \) be the quotient space of \( H \) modulo the kernel of \( V \). We denote the elements of \( H/\ker V \) by \( x + \ker V \).

**3 Corollary.** Let \( T \in L(H)^n \) be regular at \( z = 0 \) and let

\[
V : H \to \mathcal{O}(\Omega_T, W(T))
\]

be the map from Theorem 2. Then \( \mathcal{H} = \im V \subset \mathcal{O}(\Omega_T, W(T)) \) equipped with the norm \( \|x\| = \|x + \ker V\| \) is a functional Hilbert space such that

(i) \( P_{(\ker V)^\perp} T((\ker V)^\perp) \) is unitarily equivalent to \( M_z \in L(\mathcal{H})^n \) via the unitary operator \( V : (\ker V)^\perp \to \mathcal{H} \),

(ii) the reproducing kernel \( K_T : \Omega_T \times \Omega_T \to L(W(T)) \) of \( \mathcal{H} \) is given by

\[
K_T(z, w) = P_{W(T)}(1_H - ZL)^{-1}(1_H - L^*W^*)^{-1}|W(T)|.
\]

**Proof.** For \( f \in \mathcal{H} \), there is a unique vector \( x(f) \in (\ker V)^\perp \) with \( f = Vx(f) \). Since \( \lim_{k \to \infty} f_k = f \) in \( \mathcal{H} \) if and only if \( \lim_{k \to \infty} x(f_k) = x(f) \) in \( H \), all point evaluations on \( \mathcal{H} \) are continuous. Thus \( \mathcal{H} \) is a functional Hilbert space. For \( y \in W(T) \) and \( z \in \ker V \),

\[
\langle (1_H - L^*W^*)^{-1}y, z \rangle = \langle y, (Vz)(w) \rangle = 0
\]
for every \( w \in \Omega_T \). Let \( f \in \mathcal{H} \), \( y \in W(T) \) and \( w \in \Omega_T \) be given. Define \( x = x(f) \). Then

\[
(f(w), y)_{W(T)} = (P_{W(T)}(1_H - W L)^{-1} x, y)_{\mathcal{H}} = \langle x, (1_H - L^* W^*)^{-1} y \rangle_{(\text{Ker } V)^\perp}
\]

and hence \( K_T \) is the reproducing kernel of the analytic functional Hilbert space \( \mathcal{H} \). By construction the compression of \( T \) to \( (\text{Ker } V)^\perp \) and \( M_z \in L(\mathcal{H})^n \) are unitarily equivalent via the unitary operator induced by \( V \). □

In particular we obtain that each analytic tuple \( T \in L(H)^n \) which is regular at \( z = 0 \) is unitarily equivalent to a multiplication tuple \( M_z \in L(\mathcal{H})^n \) on a suitable analytic functional Hilbert space \( \mathcal{H} \) defined on a ball with center \( 0 \in \mathbb{C}^n \). For single left invertible analytic operators, Corollary 3 is due to Shimorin [11].

In the setting of Corollary 3 the functional Hilbert space \( \mathcal{H} \subset \mathcal{O}(\Omega_T, W(T)) \) contains all polynomials \( p(z) = \sum_{|\alpha| \leq m} x_{\alpha} z^\alpha \) with coefficients in \( W(T) \). The polynomials with coefficients in \( W(T) \) are dense in \( \mathcal{H} \) if and only if the space \( (\text{Ker } V)^\perp \) is generated (as an invariant subspace) by the wandering subspace of \( P_{(\text{Ker } V)^\perp} T |(\text{Ker } V)^\perp \).

### §3 Characterizations of Bergman shifts

Let \( T \in L(H)^n \) be a commuting tuple that is regular at \( z = 0 \). As before we denote by \( \sigma_T : L(H) \to L(H) \) the positive linear map acting as \( \sigma_T(X) = \sum_{1 \leq i \leq n} T_i X T_i^* \). We suppose in addition that \( T \) satisfies the identity

\[
(T^* T)^{-1} = (\oplus \Delta_T) \text{Im } T^*,
\]

where \( \Delta_T \in L(H) \) is the operator defined by

\[
\Delta_T = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_j^* (1_H).
\]

Let us define \( \delta_T \in L(H) \) by

\[
\delta_T = (\text{Im } T^* \to \text{Im } T^*)^{-1} (T^* T)^{-1} T^*.
\]

Then \( \text{Im } \delta_T = \text{Im } T \) and

\[
T_i^* \delta_T = \Delta_T T_i^* \quad (i = 1, \ldots, n).
\]

Using these intertwining relations, we find that \((\Delta_T T_i^*)(\Delta_T T_j^*) = \Delta_T T_i^* T_j^* \delta_T = (\Delta_T T_i^*) (\Delta_T T_j^*)\) for \( i, j = 1, \ldots, n \). In the following we use the same notation for the column operator \( L : H \to H^n, x \mapsto (T^* T)^{-1} T^* x = (\Delta_T T_i^* x)_{i=1}^n \), and the commuting tuple \( L = (\Delta_T T_i^*)_{i=1}^n \in L(H)^n \). Since \( L \) is commuting, the representation of the map \( V \) obtained in the proof of Theorem 2 simplifies to

\[
(V x)(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha \quad (x \in H, \ z \in \Omega_T),
\]

where \( \gamma_\alpha = |\alpha|! / \alpha! \) for \( \alpha \in \mathbb{N}^n \).
4 Lemma. For $\alpha, \beta \in \mathbb{N}^n$, we have
\[ \gamma_\alpha P_{W(T)} L^\alpha T^\beta = \gamma_{\alpha-\beta} P_{W(T)} L^{\alpha-\beta}, \]
where the right-hand side has to be read as zero whenever $\alpha - \beta$ has negative components.

Proof. For $\beta \in \mathbb{N}^n$, $x \in H$ and $z \in \Omega_T$, 
\[ \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha P_{W(T)}(L^\alpha x) z^{\alpha+\beta} = z^\beta (Vx)(z) = (VT^\beta x)(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha P_{W(T)}(L^\alpha T^\beta x) z^\alpha. \]
The proof follows by comparing the coefficients of these convergent power series. $\square$

Let us apply the above constructions to the particular case of the multiplication tuple $T = M_z \in L(H_m(\mathbb{B}))^n$. By definition $H_m(\mathbb{B})$ is the analytic functional Hilbert space with reproducing kernel $K_m : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{C}$, $K_m(z, w) = (1 - \langle z, w \rangle)^{-m}$. We consider only the case where the exponent $m \geq 1$ is a positive integer. The commuting tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H_m(\mathbb{B}))^n$, consisting of the multiplication operators $M_{z_i} : H_m(\mathbb{B}) \rightarrow H_m(\mathbb{B})$, $f \mapsto z_i f$, with the coordinate functions, is regular at $z = 0$. Indeed, for each point $\lambda \in \mathbb{B}$, the Koszul complex $K(\lambda - M_z, H_m(\mathbb{B}))$ is exact in degree $p = 0, \ldots, n - 1$ and $\dim H^n(K(\lambda - M_z, H_m(\mathbb{B}))) = 1$ (see e.g. Proposition 2.6 in [7]).

By Lemma 1 and Lemma 3 in [2] (see also the proof of Lemma 3 in [2]) we know that 
\[ (M_z^* M_z)^{-1} = (\oplus \Delta_{M_z}) \operatorname{Im} M_z^*, \]
where $\Delta_{M_z} = \sum_{j=0}^{m-1} (-1)^j m! \sigma^j_{M_z} 1_{H_m(\mathbb{B})}$ acts as the diagonal operator 
\[ \Delta_{M_z} \sum_{k=0}^{\infty} \left( \sum_{|\alpha| = k} f_\alpha z^\alpha \right) = \sum_{k=0}^{\infty} \frac{m+k}{1+k} \left( \sum_{|\alpha| = k} f_\alpha z^\alpha \right). \]
In this particular example, $W(M_z) = \mathbb{C}$ and $P_{W(M_z)} \in L(H_m(\mathbb{B}))$ is the orthogonal projection onto the closed subspace $\mathbb{C} \subset H_m(\mathbb{B})$ consisting of all constant functions. Furthermore, the intertwining relation $M_z^* \delta = (\oplus \Delta_{M_z}) M_z^*$ holds with the diagonal operator $\delta = \delta_{M_z} : H_m(\mathbb{B}) \rightarrow H_m(\mathbb{B})$, 
\[ \delta \sum_{k=0}^{\infty} \left( \sum_{|\alpha| = k} f_\alpha z^\alpha \right) = f_0 + \sum_{k=1}^{m+k-1} \frac{m+k-1}{k} \left( \sum_{|\alpha| = k} f_\alpha z^\alpha \right), \]
(see the proof of Lemma 3 in [2]).

5 Lemma. The commuting tuple $L_{M_z} = (\Delta_{M_z} M_z^*, \ldots, \Delta_{M_z} M_{z_n}^*) \in L(H_m(\mathbb{B}))^n$ satisfies the identities 
\[ P_{W(M_z)} L_{M_z}^\alpha = \binom{m + |\alpha| - 1}{|\alpha|} P_{W(M_z)} M_{z_\alpha} \quad (\alpha \in \mathbb{N}^n). \]

Proof. For $\alpha = 0$, the identity obviously holds. Suppose that the result has been shown for each multiindex $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ and fix an $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ as
Hence we may conclude that

\[ P_{W(M_z)} L_{M_z}^{α+ε_i} = \binom{m+k-1}{k} P_{W(M_z)} M_z^{α} \Delta M_z M_z^{ε_i} \]

\[ = \binom{m+k-1}{k} P_{W(M_z)} M_z^{(α+ε_i)i} \]

\[ = \binom{m+k-1}{k} \frac{m+k}{k+1} P_{W(M_z)} M_z^{(α+ε_i)i} \]

\[ = \binom{m+k-1}{k+1} P_{W(M_z)} M_z^{(α+ε_i)i}. \]

Thus the assertion follows by induction on \(|α|\).

We use the result proved in Lemma 4 for \(M_z \in L(H_m(\mathbb{B}))^n\) to prove the corresponding result for the commuting tuple \(T\) fixed at the beginning of Section 3.

**6 Lemma.** For \(α \in \mathbb{N}^n\), the identity

\[ P_{W(T)} L^α = \binom{m + |α| - 1}{|α|} P_{W(T)} T^{α} \]

holds.

**Proof.** Again we use induction on \(|α|\). Suppose that the result holds for \(|α| \leq k\). Let \(α \in \mathbb{N}^n\) be a multiindex with \(|α| = k\) and let \(i \in \{1, \ldots, n\}\) be arbitrary. Using Lemma 4 and the induction hypothesis we obtain

\[ P_{W(T)} L^{α+ε_i} = P_{W(T)} L^α \Delta T_i T^{-α}_i \]

\[ = P_{W(T)} L^α \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sum_{|β|=j, α ≥ β} \gamma_β T^{-β} T^{α} \]

\[ = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sum_{|β|=j, α ≥ β} \frac{γ_β γ_α - γ_β}{γ_α} P_{W(T)} L^{α-β} T^{α+ε_i} \]

\[ = \left( \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sum_{|β|=j, α ≥ β} \frac{γ_β γ_α - γ_β}{γ_α} \left( \frac{m + |α-β| - 1}{|α-β|} \right) \right) P_{W(T)} T^{α+ε_i}. \]

Here by definition \(α ≥ β\) means that \(α_i ≥ β_i\) for \(i = 1, \ldots, n\). Next observe that the preceding chain of equalities remains true if \(T, Δ T\) and \(L\) are replaced by \(M_z, Δ M_z\) and \(L_{M_z}\). But in this case we know from Lemma 4 that

\[ P_{W(M_z)} L_{M_z}^{α+ε_i} = \binom{m + |α|}{|α| + 1} P_{W(M_z)} M_z^{(α+ε_i)i}. \]

Hence we may conclude that

\[ \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sum_{|β|=j, α ≥ β} \frac{γ_β γ_α - γ_β}{γ_α} \left( \frac{m + |α-β| - 1}{|α-β|} \right) = \binom{m + |α|}{|α| + 1}. \]

This observation completes the inductive proof. \(□\)
Since \( V : H \to \mathcal{O}(\Omega_T, W(T)) \), \( V x(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha \), is a continuous linear map that intertwines the tuples \( T \) on \( H \) and \( M_z \) on \( \mathcal{O}(\Omega_T, W(T)) \) componentwise, the kernel of \( V \) is a closed invariant subspace for \( T \). Much more than this is true.

7 Lemma. The kernel of \( V \) is reducing for \( T \) with

(a) \( \ker V = H_\infty = \{ x \in H; P_{W(T)} T^{*\alpha} x = 0 \text{ for all } \alpha \in \mathbb{N}^n \} \),

(b) \( \langle \ker V \rangle^\perp = \bigvee_{\alpha \in \mathbb{N}^n} T^\alpha W(T) \).

Proof. The first equality in part (a) holds by Theorem 2. Since \( x \) for part (b) follows from (a). Both parts together imply that \( \ker V \) is a reducing subspace for \( T \).

In the following we write \([M] \subset H\) for the smallest closed linear subspace of \( H \) which contains a given subset \( M \subset H \). For a complex Hilbert space \( \mathcal{E} \), we denote by \( H_m(\mathbb{B}, \mathcal{E}) \) the \( \mathcal{E} \)-valued analytic functional Hilbert space with reproducing kernel

\[
K_m^\mathcal{E} : \mathbb{B} \times \mathbb{B} \to L(\mathcal{E}), K_m^\mathcal{E}(z, w) = \frac{1_{\mathcal{E}}}{(1 - \langle z, w \rangle)_m}
\]
on \( \mathbb{B} \). A well known alternative description of the space \( H_m(\mathbb{B}, \mathcal{E}) \) is given by

\[
H_m(\mathbb{B}, \mathcal{E}) = \{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}, \mathcal{E}); \| f \|^2 = \sum_{\alpha \in \mathbb{N}^n} \| f_\alpha \|^2 \rho_m(\alpha) < \infty \},
\]
where \( \rho_m(\alpha) = \frac{\lfloor m + |\alpha| \rfloor !}{\alpha ! (m - 1 !_m !)} \).

8 Theorem. Let \( T \in L(H)^n \) be a commuting tuple that is regular at \( z = 0 \) and satisfies the identity \( (T^* T)^{-1} = (\oplus \Delta_T) |\Im T^*| \). Then the map

\[
U : [W(T)] \to H_m(\mathbb{B}, W(T)), U x(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha
\]
is a unitary operator which componentwise intertwines the tuples \( T |[W(T)] \) and \( M_z \in L(H_m(\mathbb{B}, W(T)))^n \).

Proof. For \( N \in \mathbb{N} \) and \( x_\alpha \in W(T) (|\alpha| \leq N) \), we have

\[
\| \sum_{|\alpha| \leq N} T^\alpha x_\alpha \|^2 = \sum_{|\alpha|, |\beta| \leq N} \langle P_{W(T)} T^{*\beta} T^\alpha x_\alpha, x_\beta \rangle = \sum_{|\alpha|, |\beta| \leq N} \langle x_\alpha, P_{W(T)} T^{*\alpha} T^\beta x_\beta \rangle.
\]
Using first Lemma 6 and then twice Lemma 4 we find that

\[ \| \sum_{|\alpha| \leq N} T^\alpha x_\alpha \|^2 = \sum_{|\alpha| \leq N} \left( m + |\alpha| - 1 \right)^{-1} \langle P_{W(T)} L^\alpha T^\alpha x_\alpha, x_\alpha \rangle \]

\[ = \sum_{|\alpha| \leq N} \left( m + |\alpha| - 1 \right) \gamma_{\alpha}^{-1} \| x_\alpha \|^2 \]

\[ = \sum_{|\alpha| \leq N} \| x_\alpha \|^2 \rho_m(\alpha) = \sum_{|\alpha| \leq N} x_\alpha z_\alpha^\alpha \| T^m(z_\alpha)^2 \|_{H_m(\mathbb{B}, W(T))}. \]

Since the polynomials with coefficients in \( W(T) \) are dense in \( H_m(\mathbb{B}, W(T)) \), there is a unique unitary operator \( U : [W(T)] \to H_m(\mathbb{B}, W(T)) \) with \( U(\sum_{|\alpha| \leq N} T^\alpha x_\alpha) = \sum_{|\alpha| \leq N} x_\alpha z_\alpha^\alpha \) for all finite families \((x_\alpha)_{|\alpha| \leq N} \) in \( W(T) \). In particular it follows that, for \( h \in \text{span}\{T^\alpha x; \alpha \in \mathbb{N}^n \text{ and } x \in W(T)\} \), the analytic functions \( U h \in \mathcal{O}(\mathbb{B}, W(T)) \) and \( V h \in \mathcal{O}(\Omega_T, W(T)) \) have the same Taylor coefficients at \( z = 0 \). The continuity of the maps \( U : [W(T)] \to \mathcal{O}(\mathbb{B}, W(T)) \) and \( V : [W(T)] \to \mathcal{O}(\Omega_T, W(T)) \) implies that \( Ux \) and \( Vx \) have the same Taylor coefficients at \( z = 0 \) for every \( x \in [W(T)] \). But then

\[ Ux(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z_\alpha \]

for \( x \in [W(T)] \) and \( z \in \mathbb{B} \). Since \( V \) intertwines \( T \in L(H)^n \) and \( M_z \) on \( \mathcal{O}(\Omega_T, W(T)) \), the identity theorem implies that \( U \) satisfies the same intertwining relation.

For a commuting tuple \( S \in L(H)^n \) its \( k \)th order defect operators are defined by

\[ \Delta_S^{(k)} = (I - \sigma_S)^k(1_H) = \sum_{j=0}^k (-1)^j \binom{k}{j} \sigma_S^j(1_H) \quad (k \in \mathbb{N}). \]

The tuple \( S \) is called an \( m \)-hypercontraction if \( \Delta_S^{(1)} \geq 0 \) and \( \Delta_S^{(m)} \geq 0 \). A commuting tuple \( S \in L(H)^n \) is said to be of type \( C_0 \) if \( \text{SOT-}\lim_{k \to \infty} \sigma_S^k(1_H) = 0 \).

9 Corollary. Let \( T \in L(H)^n \) be as in Theorem 8. The following conditions on \( T \) are equivalent:

(i) \( T \) is analytic,

(ii) \( \|x\|^2 = \| \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z_\alpha^\alpha \|^2 \|_{H_m(\mathbb{B}, W(T))} \) for all \( x \in H \),

(iii) \( T \) is of type \( C_0 \).

(iv) \( T \) is unitarily equivalent to \( M_z \in L(H_m(\mathbb{B}, D))^n \) for some Hilbert space \( D \).

Proof. The equivalence of (i) and (ii) follows from Lemma 7 and Theorem 8.

The implication (i) to (iv) follows from Theorem 8. It is well known that \( M_z \in L(H_m(\mathbb{B}, D))^n \) satisfies the \( C_0 \)-condition

\[ \text{SOT-} \lim_{k \to \infty} \sigma_{M_z}^k(1_{H_m(\mathbb{B}, D)}) = 0. \]

Since this condition is preserved under unitary equivalence, the implication (iv) to (iii) holds.
Let us suppose that $T$ satisfies condition (iii). To complete the proof note first that

$$P_{W(T)} = 1_H - T(T^* T)^{-1} T^* = 1_H - T(\oplus \Delta_T) T^*$$

$$= 1_H - \sigma_T \left( \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma_T^k (1_H) \right)$$

$$= 1_H + \sum_{k=1}^m (-1)^k \binom{m}{k} \sigma_T^k (1_H)$$

$$= (I - \sigma_T)^m (1_H).$$

It is well known that a commuting tuple $T \in L(H)^n$ of type $C_0$ for which the $m$th order defect operator $\Delta_T^{(m)} = (I - \sigma_T)^m (1_H)$ is positive is an $m$-hypercontraction (see [2] [10]). Thus it follows from the dilation theory for $m$-hypercontractions (see e.g. [2]) that the map

$$j : H \to H_m(\mathbb{B}, H) \quad jx = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) ((\Delta_T^{(m)})^{1/2} T^{* \alpha} x) z^\alpha$$

defines an isometric intertwiner between the tuples $T^*$ on $H$ and $M_z^*$ on $H_m(\mathbb{B}, H)$. Using Lemma [3] we find that

$$(Vx)(z) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (P_{W(T)} L^\alpha x) z^\alpha = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) (P_{W(T)} T^{* \alpha} x) z^\alpha = (jx)(z)$$

for $x \in H$ and $z \in \Omega_T$. Hence $\text{Ker} V = \text{Ker} j = \{0\}$ and the proof is complete. □

Let $S \in L(H)^n$ be an $m$-hypercontraction. Since

$$0 \leq \sigma_S^{k+1} (1_H) \leq \sigma_S^k (1_H) \leq 1_H \quad (k \geq 0),$$

the strong limit $S_\infty = \text{SOT} \lim_{k \to \infty} \sigma_S^k (1_H)$ exists. It is well known that the map

$$j : H \to H_m(\mathbb{B}, H), \quad jx = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha) ((\Delta_S^{(m)})^{1/2} S^{* \alpha} x) z^\alpha$$

defines a contraction that intertwines the tuples $S^* \in L(H)^n$ and $M_z^* \in L(H_m(\mathbb{B}, H))^n$. More precisely one can show that

$$\|jx\|^2 + \|S_\infty x\|^2 = \|x\|^2$$

for every vector $x \in H$.

Let us recall that a commuting tuple $S \in L(H)^n$ is called a spherical isometry if

$$\sum_{1 \leq i \leq n} S_i^* S_i = 1_H$$

or, equivalently, if $\sum_{1 \leq i \leq n} \|S_i x\|^2 = \|x\|^2$ for each vector $x \in H$. By a result of Athavale [11] each spherical isometry $S \in L(H)^n$ is subnormal and its minimal normal extension is a spherical unitary, that is, a commuting tuple $N \in L(K)^n$ of normal operators such that $\sum_{1 \leq i \leq n} N_i^* N_i = 1_K$. A spherical coisometry is a commuting tuple $S \in L(H)^n$ such that its adjoint $S^* \in L(H)^n$ is a spherical isometry.

**10 Theorem.** Let $T \in L(H)^n$ be a commuting row contraction that is regular at $z = 0$. Then $T$ satisfies the operator identity

$$(T^* T)^{-1} = \left( \oplus \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_T^j (1_H) \right) \text{Im} T^*$$
if and only if \( T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n \) is the direct sum of a spherical coisometry \( T_0 \in L(H_0)^n \) and a tuple \( T_1 \in L(H_1)^n \) which is unitarily equivalent to \( M_z \in L(H_m(\mathbb{B}, D))^n \) for some Hilbert space \( D \).

**Proof.** Suppose that \( T \) satisfies the above operator identity. Then by Lemma \([7]\) the space \( H \) is the orthogonal sum \( H = H_\infty \oplus [W(T)] \) of closed subspaces reducing \( T \). According to Theorem \([5]\) the restriction \( T_1 = T|[W(T)] \) is unitarily equivalent to \( M_z \in L(H_m(\mathbb{B}, W(T)))^n \). The proof of Corollary \([9]\) shows that

\[
\Delta_T^{(m)} = (1 - \sigma_T)^m(1_H) = P_{W(T)} \geq 0.
\]

Since \( T \) is a row contraction, also \( \Delta_T^{(1)} = 1_H - \sum_{1 \leq i \leq n} T_i T_i^* \geq 0 \). Thus the tuple \( T \) is an \( m \)-hypercontraction. Using the remarks following Corollary \([9]\) as well as Lemma \([9]\) we obtain that the map \( j : H \to H_m(\mathbb{B}, H) \),

\[
jx = \sum_{\alpha \in \mathbb{N}_n} \rho_m(\alpha)(P_{W(T)}T^* x)z^\alpha = \sum_{\alpha \in \mathbb{N}_n} \gamma_\alpha(P_{W(T)}L^nx)z^\alpha
\]

is a well defined contraction with \( \|jx\|^2 + \langle T_\infty x, x \rangle = \|x\|^2 \) for all \( x \in H \). Since \( T_\infty \leq \sigma_T(1_H) \leq 1_H \), it follows that

\[
\sum_{i=1}^n \|T_i^* x\|^2 = \langle \sigma_T(1_H)x, x \rangle = \|x\|^2
\]

for \( x \in \text{Ker } j = \text{Ker } V = H_\infty \). Thus \( (T|H_\infty)^* \) is a spherical isometry.

To prove the converse let us first consider a commuting tuple \( T \in L(H)^n \) such that \( T^* \) is a spherical isometry. Since \( \sigma_T(1_H) = TT^* = 1_H \), it follows that \( T^* T = P_{\text{im } T^*} \) and

\[
\sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_T^j(1_H) = \left( \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \right) 1_H = 1_H.
\]

On the other hand, by Lemma 1 and Lemma 3 in \([2]\) it follows that, for any Hilbert space \( D \), the tuple \( M_z \in L(H_m(\mathbb{B}, D))^n \) satisfies the operator identity

\[
(M_z^* M_z)^{-1} = \left( \oplus \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_{M_z}^j(1_{H_m(\mathbb{B}, D)}) \right) \text{Im } M_z^*.
\]

Since the validity of this operator identity is preserved under unitary equivalence and the passage to direct sums, also the reverse implication follows. \( \square \)

In the single-variable case \( n = 1 \) Theorem \([10]\) implies that the left invertible contractions that satisfy the operator identity

\[
(T^* T)^{-1} = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_T^j(1_H)
\]

are precisely the operators \( T \in L(H) \) that decompose into the orthogonal direct sum \( T = T_0 \oplus T_1 \) of a unitary operator \( T_0 \) and an operator \( T_1 \) which is unitarily equivalent to an \( m \)-shift \( M_z \in L(H_m(\mathbb{D}, D)) \). This is a slight variant of the main result of \([5]\).

In the particular case \( m = 1 \) the result stated in Theorem \([10]\) takes the form.
11 Corollary. Let $T \in L(H)^n$ be a commuting tuple that is regular at $z = 0$. Then $T : H^n \to H$ is a partial isometry if and only if $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n$ is the direct sum of a spherical coisometry $T_0 \in L(H_0)^n$ and a tuple $T_1 \in L(H_1)^n$ which is unitarily equivalent to $M_z \in L(H_1(B, D))^n$ for some Hilbert space $D$.

Proof. For $m = 1$, we have $\sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_j^2(1_H) = 1_H$. Thus in this case the operator identity from Theorem 10 means precisely that $T : H^n \to H$ is a partial isometry. Hence the assertion follows immediately from Theorem 10. □

In the case $m = 1 = n$ the preceding results yield a Wold decomposition for partial isometries $T \in L(H)$ that are regular at $z = 0$ which contains the classical Wold decomposition for isometries. Corollary 11 implies that all powers $T^k$ of partial isometries $T \in L(H)$ that are regular at $z = 0$ are partial isometries again. Thus partial isometries that are regular at $z = 0$ are power partial isometries in the sense of Halmos and Wallen [8]. In [8] a Wold-decomposition theorem for general power partial isometries is proved.

Since a non-unitary partial isometry on a finite dimensional Hilbert space cannot admit a decomposition as in Corollary 11 Corollary 11 and Theorem 11 cannot be expected to hold without the hypothesis that the given tuple $T$ is regular at $z = 0$.

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