The periodic rotary motions of a rigid body in a new domain of angular velocity

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Introduction

Some asymptotic perturbed techniques [1–3] are widely used by many authors for solving the ordinary linear and nonlinear systems for differential equations in different problems of engineering, mathematical physics, and astronomy. As an extension of this type of problem, we use some perturbation and numerical techniques in the movement of coherent bodies around a fixed point in the presence of new conditions to obtain periodic solutions of different scales to those obtained before.

As in [4], let the fixed z-axis of the ellipsoid of inertia of the body makes an angle \( \theta_0 \approx \pi/2 \) with the fixed Z-axis in space. We assume the new value of \( r_0 \), which is sufficiently small. Under this assumption, we must achieve a large parameter and search for another technique for solving this problem. This technique is named; a large parameter technique instead of the small one well known previously. We see the advantage of the new technique which appears in saving high energy used to begin the motion and give the solution of the problem in another domain. The obtained solutions by the new technique depend on \( r_0 \). We consider that the center of mass of this body does not necessarily coincide with the fixed point O. We reduce the six nonlinear differential equations of the body and their three first integrals to a quasilinear autonomous system of two degrees of freedom and one first integral. We solve the rational case when the frequencies of the generating system are rational except \( (\omega = 1, 2, 1/2, 3, 1/3, \ldots) \) under the condition \( \gamma''_0 = \cos \theta_0 \approx 0 \). We use the fourth-order Runge–Kutta method to find the periodic solutions in the closed interval of the time t and to compare the analytical method with the numerical one.

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sufficiently small instead of sufficiently large value in [4]. We define a large parameter \( \mu \) proportional to \( 1/r_o \) instead of the small one in [4]. Consider A, B, C represent the moments of inertia of the body, \( p, q, r \) are the components of the angular velocity vector and \( \gamma, \gamma', \gamma'' \) are the direction cosines of the unit vector in direction of the Z-axis. So, the equations of motion and their three first integrals are derived in the form:

\[
\begin{align*}
\dot{p}_1 + A_1 q_1 r_1 &= \mu^{-1} a^{-1}(\gamma'' y_0' - \gamma' z_0' + k a A_1 \gamma''), \\
\dot{q}_1 + B_1 q_1 r_1 &= \mu^{-1} b^{-1}(\gamma z_0' - \gamma'' x_0' + k b B_1 \gamma''), \\
\dot{r}_1 &= \mu^{-2}(-C_1 q_1 + \gamma' x_0' - \gamma y_0' + k C_1 \gamma''), \\
\dot{\gamma} &= r_1 \gamma' - \mu^{-1} q_1 \gamma'' - r_1 \gamma, \quad \dot{\gamma}' = \mu^{-1} p_1 \gamma'' - r_1 \gamma, \quad \dot{\gamma}'' = \mu^{-1}(q_1 \gamma' - p_1 \gamma').
\end{align*}
\]  

(1)

\[
\begin{align*}
r_1^2 &= 1 + \mu^{-2} s_1, \quad r_1 \gamma'' &= \gamma_0'' + \mu^{-1} s_2, \quad \gamma^2 + \gamma'^2 + \gamma''^2 = 1.
\end{align*}
\]  

(2)

Such that:

\[
\begin{align*}
s_1 &= s_3 - 2z_0'(\gamma_0'' - \gamma'') + k(\gamma_0''^2 - \gamma''^2), \\
s_2 &= a(p_1 y_0 - p_1 \gamma) + b(q_1 y_0 - q_1 \gamma') + k[a(\gamma_0'' - \gamma'') + b(\gamma_0''^2 - \gamma''^2)].
\end{align*}
\]  

(3)

\[
p_1 = p/c, \quad (pqr), \quad \gamma_0 > 0, \quad \mu = c/r_0, \quad l^2 = x_0^2 + y_0^2 + z_0^2,
\]  

(4)

\[
k = N/c^2, \quad A_1 = (C - B)/A, \quad (ABC).
\]  

(5)

Reduction of the equations of motion to a quasilinear autonomous system

Solving the first and the second equations of (2) for \( r_1 \) and \( \gamma'' \), we get:

\[
r_1 = 1 + \frac{1}{2}\mu^{-2} s_3 + \ldots, \quad \gamma'' = \gamma_0'' + \mu^{-1} s_2 - \frac{1}{2}\mu^{-2} s_3 \gamma_0'' + \ldots.
\]  

(6)

Differentiating the first and the fourth equations of (1) and using (6) for reduction in the four remaining equations into two differential equations of the second order, we get:

\[
\begin{align*}
\ddot{p}_1 + \omega^2 p_1 &= \mu^{-1}[z_0'(a^{-1} - A_1 b^{-1}) \gamma + A_1 b^{-1} x_0'' y_0'' - k(A_1 - \omega^2) y_0''] \\
&+ \mu^{-2}[A_1 x_0'(b^{-1} s_2 - q_1 \gamma') - \omega^2 p_1 s_3 + p_1(A_1 C_1 q_1^2 - a^{-1} z_0'' y_0'] \\
&+ y_0'[(2A_1 + a^{-1} b) q_1 \gamma' - a^{-1} p_1 \gamma'] + k A_1[p_1(\gamma_0''^2 - \gamma''^2) \\
&+ q_1(1 - C_1) \gamma' \gamma' - s_2(1 + B_1) \gamma] + \mu^{-3}[0.5(a^{-1} - A_1 b^{-1}) z_0'' s_3 \gamma \\
&- s_2 p_1[z_0'(2\omega^2 - a^{-1}) - 2k(A_1 + \omega^2) y_0''] + \ldots, \\
\ddot{\gamma} + \gamma &= \mu^{-1} y_0'' p_1(1 + B_1) + \mu^{-2}[p_1(1 + B_1) s_2 + (1 - C_1) q_1 \gamma'] \\
&- \gamma[s_3 + y_0' \gamma' + z_0'(a^{-1} y_0'' + q_1^2 + k(C_1 \gamma^2 - B_1 y_0'^2) \\
&+ x_0'(b^{-1} y_0'' + \gamma'')) + \mu^{-3}[2b^{-1} x_0'' y_0' s_2 \\
&+ 2k(1 - B_1) y_0'' - z_0'(2 + b^{-1}) s_2 y_0] + \ldots.
\end{align*}
\]  

(7)

Here:
\[ \omega^2 = -A_1 B_1 = (A - C)(B - C)/AB = (a - 1)(b - 1)/ab. \]

Solving the first and the fourth equations of the system (1) and using (6), we get:

\[ \begin{align*}
q_1 &= -p_1 A_1^{-1} + \mu^{-1} a^{-1} A_1^{-1}(y_0' y''_0 - z_0' y' + a k A_1 y' y''_0) + \ldots, \\
\gamma' &= \dot{\gamma} - \mu^{-1} A_1^{-1} y_0'' p_1 + \ldots. \tag{8}
\end{align*} \]

Introducing the new variables \( p_2 \) and \( \gamma_2 \) such that:

\[ \begin{align*}
p_1 &= p_2 + \mu^{-1} x_1 \gamma_2 + \mu^{-1} \frac{x_0'}{1 - a} y''_0, \\
\gamma &= \gamma_2 + \mu^{-1} a y''_0 p_2, \tag{9}
\end{align*} \]

where:

\[ \chi_1 = (1 - \omega^2)^{-1}[-z_0'(a^{-1} - A_1 b^{-1}) + ky''_0(A_1 - \omega^2)]. \]

Substituting from (9) into (8), we get:

\[ \begin{align*}
q_1 &= -A_1^{-1} \dot{p}_2 + \mu^{-1} A_1^{-1}(y_0' y''_0 - \chi_2 \gamma_2) + \ldots, \\
\gamma' &= \dot{\gamma}_2 - \mu^{-1} \gamma_0'' A_1^{-1} b p_2 + \ldots, \quad \chi_2 = \chi_1 + z_0' a^{-1} - k A_1 y''_0. \tag{10}
\end{align*} \]

Substituting (10) and (9) into (3), we get:

\[ s_i = s_{i1} + 2^{2-i} \mu^{-1} s_{i2} + \ldots, \quad i = 1, 2, \tag{11} \]

where:

\[ \begin{align*}
s_{11} &= a(p_{20}^2 - p_2^2) + b(p_{20}^2 - p_2^2)/A_1^2 - 2[x_0'(y_{20} - \gamma_2) \\
+ y_0'(y_{20} - \gamma_2)] + k[a(y_{20}^2 - \gamma_2^2) + b(\dot{y}_{20}^2 - \gamma_2^2)], \\
s_{12} &= a \left[ \frac{x_0'}{1 - a} y''_0 (p_{20} - p_2) + x_1 (p_{20} y_{20} - p_2 \gamma_2) \right] \\
&- b A_1^{-2} y''_0 (\dot{p}_{20} - \dot{p}_2) - \chi_2 (\dot{p}_{20} y_{20} - \dot{p}_2 \gamma_2) \\
&- ax_0' y''_0 (p_{20} - p_2) + y_0' b A_1^{-1} y''_0 (p_{20} - \dot{p}_2) + (z_0' - k y''_0) s_{21} \\
&+ k y''_0 [a^2 (p_{20} y_{20} - p_2 \gamma_2) - b^2 A_1^{-1} (\dot{p}_{20} y_{20} - \dot{p}_2 \gamma_2)], \\
s_{21} &= a(p_{20} y_{20} - p_2 \gamma_2) - b A_1^{-1} (\dot{p}_{20} y_{20} - \dot{p}_2 \gamma_2), \\
s_{22} &= a \left[ a y''_0 (p_{20}^2 - p_2^2) + \frac{x_0'}{1 - a} y''_0 (y_{20} - \gamma_2) + x_1 (y_{20}^2 - \gamma_2^2) \right] \\
+ b A_1^{-1} [b A_1^{-1} y''_0 (p_{20}^2 - p_2^2) + y_0' a^{-1} y''_0 (y_{20} - \gamma_2) - \chi_2 (y_{20}^2 - \gamma_2^2)], \\
s_3 &= s_{11} + 2 \mu^{-1} (s_{12} - z_0' s_{21} + k s_{21} y''_0) + \ldots.
\end{align*} \]

Substituting (11), and (12) into (6) yields:

\[ r_1 = 1 + \frac{1}{2} \mu^{-2} s_{11} + \ldots, \quad \gamma'' = y''_0 + \mu^{-1} s_{21} + \mu^{-2} s_{22} - \frac{1}{2} \mu^{-2} s_{11} y''_0 + \ldots. \tag{13} \]

Substituting (9), (10), (11), and (12) into (7), we obtain a quasilinear autonomous system [5]:

\[ \ldots \]
\[
\ddot{p}_2 + \omega^2 p_2 = \mu^{-2}F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \epsilon), \quad \dot{\gamma}_2 + \gamma_2 = \mu^{-2}\Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \epsilon), \tag{14}
\]

where:

\[
F = F_2 + \mu^{-1}F_3 + \ldots, \quad \Phi = \Phi_2 + \mu^{-1}\Phi_3 + \ldots,
\]

\[
F_2 = f_2 - a\gamma''_0 \chi_1(1 - \omega^2)p_2, \quad \Phi_2 = \varphi_2 + a\gamma''_0(1 - \omega^2)(\chi_1\gamma_2 + \gamma''_0\chi_0(1 - a)^{-1}),
\]

\[
F_3 = f_3 - \chi_1[\varphi_2 + a\gamma''_0(1 - \omega^2)(\chi_1\gamma_2 + \gamma''_0\chi_0(1 - a)^{-1})],
\]

\[
\Phi_3 = \varphi_3 + a\gamma''_0[a\gamma''_0 \chi_1(1 - \omega^2)p_2 - f_2],
\]

\[
f_2 = x'_0(A_1\gamma_1^{-1}s_1 + \dot{p}_2\gamma_2) + A_1^{-1}\dot{p}_2[C_1\ddot{p}_2 - \gamma_0'(2A_1 + ba^{-1})\gamma_2] - p_2(\omega^2s_1 + \\
+ \gamma''_0\gamma_2 + \gamma''_0\gamma_2') - k[\gamma_2((1 - C_1)\ddot{p}_2\gamma_2 + A_1 + B_1)s_21] - A_1\ddot{p}_2(\gamma''_2 - \gamma''_2^2),
\]

\[
\varphi_2 = (1 + B_1)p_2s_21 + x'_0(\gamma''_2 + b^{-1}\gamma''_0) - A_1^{-1}\dot{p}_2[A_1\gamma_2\dot{p}_2 + (1 - C_1)p_2\gamma_2] - \\
- \gamma_2(s_1 + \gamma''_0\gamma_0' - \gamma_0'\gamma_2 - k(C_1\gamma_2 - B_1\gamma''_2)),
\]

\[
f_3 = -\omega^2\left[\chi_1\gamma_2 + \frac{x'_0}{1 - a} \gamma''_0\right] + 2p_2s_21 + C_1A_1^{-1}\ddot{p}_2
\]

\[
\times \left[\ddot{p}_2\left(\chi_1\gamma_2 + \frac{x'_0}{1 - a} \gamma''_0\right) + 2p_2\ddot{p}_2(\chi\gamma_2 - \gamma''_0\gamma_2)\right]
\]

\[
+ x'_0[A_1\gamma_1^{-1}s_22 - baA_1^{-1}\gamma''_2] - \gamma''_0\gamma_2(\gamma''_0a^{-1} - \chi_2\gamma_2)
\]

\[
- \gamma''_0\gamma_0'\gamma_2\left[\chi_1\gamma_2 + \frac{x'_0}{1 - a} \gamma''_0\right] - bA_1^{-1}\ddot{p}_2
\]

\[
+ \gamma''_0(2 - b)(1 - \omega^2)[\gamma_2(\gamma''_0a^{-1} - \chi_2\gamma_2) - a\gamma''_0\gamma_2\gamma_2] + \\
+ k\left[\gamma_2((1 - C_1)(\gamma''_2a^{-1} - \chi_2\gamma_2)\gamma_2 - a\gamma''_0\gamma_2\gamma_2) + bA_1^{-1}\gamma''_2\gamma_2\right]
\]

\[
+ p_2A_1(2b\gamma_2 - aA_1(1 + B_1)s_21 + 2A_1s_21)
\]

\[
+ \frac{1}{2}2s_21 + a\gamma''_0p_2s_21 + (1 + B_1)\left[p_2s_22 + \left(\frac{x'_0}{1 - a} \gamma''_0 + \chi_1\gamma_2\right)\right],
\]

\[
\psi_3 = -2s_21\gamma_2 - a\gamma''_0p_2s_21 + (1 + B_1)\left[p_2s_22 + \left(\frac{x'_0}{1 - a} \gamma''_0 + \chi_1\gamma_2\right)\right]
\]

\[
+ 2x''_0(\gamma''_0a^{-1}s_21 - A_1^{-1}\ddot{p}_2\gamma_2) - x''_0(\gamma''_0a^{-1}s_21 + A_1^{-1}(1 - C_1))
\]

\[
\times \left[bA_1^{-1}\gamma''_0p_2\ddot{p}_2^2 - \left(\frac{x'_0}{1 - a} \gamma''_0 + \chi_1\gamma_2\right)\right]p_2\gamma_2 + b^{-1}\gamma''_0\chi_0 + p_2\gamma_2(\gamma''_0a^{-1} - \chi_2\gamma_2)
\]

\[
- \gamma''_0(a_2\gamma_2 - bA_1^{-1}\ddot{p}_2\gamma_2) + A_1^{-2}[2p_2(\gamma''_0a^{-1} - \chi_2\gamma_2)\gamma_2 - a\gamma''_0p_2\gamma_2]
\]

\[
+ k\gamma_0''(ap_2\gamma_2 - bA_1^{-1}\ddot{p}_2\gamma_2 - B_1\gamma''_2) - 2\gamma_2(bA_1^{-1}C_1\ddot{p}_2\gamma_2 + B_1s_21)]
\]

The last equation of (2) gives the first integral of the system (14) as follows [6]:

\[
\gamma_2^2 + \dot{\gamma}_2^2 + 2\mu^{-1}\gamma''_0(ap_2\gamma_2 - bA_1^{-1}\ddot{p}_2\gamma_2 + s_21) + \ldots = 1 - \gamma''_0.
\tag{15}
\]

In the next, we will look for the periodic solutions of the system (14) under the conditions \(A > B > C\) or \(A < B < C\) (\(\omega^2\) is positive). The first condition [7] gives the slow rotation of the body about the major axis of the ellipsoid of inertia and the second gives
a slow rotation of the body about the minor axis of the ellipsoid of inertia. We apply the large parameter method [8] to solve the autonomous system (14).

**The formal construction of the periodic solutions for a rational value of the natural frequency \(\omega\)**

We achieve the periodic solutions

\[
p_2(0, 0) = \dot{p}_2(0, 0) = \dot{\gamma}_2(0, \mu^{-1}) = 0.
\]

The generating system of (14) is obtained when \(\mu \to \infty\) in the form:

\[
\ddot{p}_2^{(0)} + \omega^2 p_2^{(0)} = 0, \quad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0.
\]

So the periodic solutions of system (17), when the period \(T_0 = 2\pi n\), become:

\[
p_2^{(0)} = M_1 \cos \omega \tau + M_2 \sin \omega \tau, \quad \gamma_2^{(0)} = M_3 \cos \tau,
\]

where \(M_i, \ i = 1, 2, 3\) are constants to be determined.

Assuming the following solutions of system (14) [9]:

\[
p_2(\tau, \mu) = \tilde{M}_1 \cos \omega \tau + \tilde{M}_2 \sin \omega \tau + \sum_{k=2}^{\infty} \mu^{-k} G_k(\tau),
\]

\[
\gamma_2(\tau, \mu) = \tilde{M}_3 \cos \tau + \sum_{k=2}^{\infty} \mu^{-k} H_k(\tau),
\]

with a period \(T(\mu^{-1}) = T_0 + \alpha(\mu^{-1})\) which reduces to (18) at \(\mu \to \infty\). Let us define the quantities \(\tilde{M}_i, \ i = 1, 2, 3\) as follow:

\[
\tilde{M}_i = M_i + \beta_i(\mu^{-1}), \quad i = 1, 2, 3,
\]

where \(\beta_i\) are functions of \(\mu^{-1}\) which represent the deviations of the initial values of \(p_2, \dot{p}_2, \gamma_2\) for system (14) from their initial values of generating system (17) such that \(\beta_i(0) = 0\).

Let us express the initial conditions (16) by the relations:

\[
p_2(0, \mu^{-1}) = \tilde{M}_1, \quad \dot{p}_2(0, \mu^{-1}) = \omega \tilde{M}_2, \quad \gamma_2(0, \mu^{-1}) = \tilde{M}_3, \quad \dot{\gamma}_2(0, \mu^{-1}) = 0.
\]

We rewrite the periodic solutions (18) in the form:

\[
p_2^{(0)} = E \cos (\omega \tau - \epsilon), \quad \gamma_2^{(0)} = M_3 \cos \tau,
\]

where \(E = \sqrt{M_1^2 + M_2^2}\) and \(\epsilon = \tan^{-1} M_2 / M_1\). Using (22) and (12), we get:
$$J_{11}^{(0)} = E^2[ (a \cos^2 \varepsilon - 0.5) + b o^2 A_1^{-2} (\sin^2 \varepsilon - 0.5) + 0.5(b o^2 A_1^{-2} - a) \cos 2(\omega \tau - \varepsilon)]$$
$$- 2 M_3 x_0'(1 - \cos \tau) + y_0' \sin \tau] - 0.5 k M_3^2 C_1 (1 - \cos 2 \tau),$$

$$J_{12}^{(0)} = a^2 E y_0' x_0'(1 - a)^{-1} [\cos \varepsilon - \cos(\omega \tau - \varepsilon)]$$
$$+ E y_0' b y_0'' A_1^{-1} \omega (1 - a^{-1} A_1^{-1}) [\sin \varepsilon + \sin(\omega \tau - \varepsilon)]$$
$$+ a E M_3 (x_1 + y_0''(k) [\cos \varepsilon - 0.5 \cos[(\omega + 1) \tau - \varepsilon] - 0.5 \cos[(\omega - 1) \tau - \varepsilon])$$
$$+ o b A_1^{-1} E M_3 (x_2 A_1^{-1} - b k y_0'') (0.5 \cos[(\omega + 1) \tau - \varepsilon] - 0.5 \cos[(\omega - 1) \tau - \varepsilon])$$
$$+ E M_3 (z_0' - y_0'' k)(a \cos \varepsilon + 0.5(b o A_1^{-1} - a) \cos[(\omega - 1) \tau - \varepsilon]$$
$$- 0.5(b o A_1^{-1} + a) \cos[(\omega + 1) \tau - \varepsilon]],[23]$$

Substituting (22) and (23) into (16), we get:

$$F_2^{(0)} = M_1 L(\omega) \cos \omega \tau + M_2 L(\omega) \sin \omega \tau + \ldots,$$

$$\Phi_2^{(0)} = M_3 N(\omega) \cos \tau + \ldots, [24]$$

$$F_3^{(0)} = M_1 K(\omega) \cos \omega \tau + M_2 K(\omega) \sin \omega \tau + \ldots,$$

where:

$$L(\omega) = \omega^2[-(a M_1^2 + b o^2 A_1^{-2} M_2^2) + 0.25(M_1^2 + M_2^2)(C_1 A_1^{-1} + 3 a + b o^2 A_1^{-2})]$$
$$+ 2 o^2 M_3 x_0' - y_0'' [z_0'' a^{-1} + \alpha x_1 (1 - \omega^2)] + k [A_1 (y_0'') - 0.5 M_3^2]$$
$$+ 0.5 M_3^2 [a(a - 2 \omega^2) + o^2 b)],$$

$$N(\omega) = -(a M_1^2 + b o^2 A_1^{-2} M_2^2) - 0.5(M_1^2 + M_2^2)[a b_1 + \omega^2 A_1^{-2}(1 - b)]$$
$$+ 2 M_3 x_0' - y_0'' [z_0'' b^{-1} - \alpha x_1 (1 - \omega^2)] + k [M_3^2 (b - a) - B_1 y_0'']^2,$$

$$K(\omega) = -2 \omega^2 y_0'' [a x_0'(M_1 (1 - a)^{-1} + o y_0'' b) M_2 (1 - a^{-1} A_1^{-1}) A_1^{-1}$$
$$+ 0.25 a^{-1} A_1^{-1} C_1 Y_0(M_1^2 + M_2^2)] - 2 o^2 a M_1 M_3 [x_1 + A y_0'' k + z_0'' - y_0'' k]$$
$$+ a M_1 M_3 [2 A_k y_0'' - z_0'' a^{-1} - a A_1 k y_0''(1 + B_1) - \chi_1 (1 + B_1)]. [25]$$

Using (24) and (25), the following functions are obtained:

$$g_2(T_0) = -\pi n o^{-1} M_2 L(\omega), \quad g_2(T_0) = \pi n M_3 L(\omega),$$

$$h_2(T_0) = 0,$$  $$h_2(T_0) = \pi n M_3 N(\omega), [26]$$

$$g_3(T_0) = -\pi n o^{-1} M_2 K(\omega), \quad g_3(T_0) = \pi n M_1 K(\omega).$$

Substituting by the initial conditions (21) into the first integration (17) when \(\tau = 0\), we get:

$$M_3^2 + 2 M_3 \beta_3 + \beta_3^2 + 2 \mu^{-1} A y_0'' M_3 (M_1 + \beta_1) = 1 - y_0''^2. [27]$$

Let \(y_0''\) depends on \(\mu^{-1}\), we get:
\( \gamma''' = \mu^{-1} \Gamma, \quad 0 < \Gamma < 1. \) \hfill (28)

Taking into consideration, Eqs. (27) and (28), we get \( M_3, \beta_3 \) as follows:

\[
M_3 = 1, \quad \beta_3 = -aG\mu^{-2}\tilde{M}_1 - \frac{1}{2}\mu^{-2}I^2 + \ldots.
\] \hfill (29)

The independent conditions for periodicity are:

\[
- (L_1(\omega) - \omega^2\tilde{N}_1(\omega))\pi nomega^{-1}\tilde{M}_2 + \mu^{-1}\tilde{G}_3(T_0) + \ldots = 0,
\]

\[
(L_1(\omega) - \omega^2\tilde{N}_1(\omega))\pi n\tilde{M}_1 + \mu^{-1}\tilde{G}_3(T_0) + \ldots = 0,
\]

\[
\mu^{-2}(\tilde{H}_2(T_0) + \mu^{-1}\tilde{H}_3(T_0))\tilde{M}_3^{-1} + \ldots = \alpha(\mu^{-1}),
\] \hfill (30)

where \( L_1(\omega), N_1(\omega) \) are obtained from \( L(\omega), N(\omega) \) replacing \( M_i \) by \( (M_i + \beta_i), \ i = 1, 2, 3 \) to get:

\[
L_1(\omega) - \omega^2\tilde{N}_1(\omega) = W_0(\omega) (\tilde{M}_1^2 + \tilde{M}_2^2) - \gamma''_0 [z_0 W_1(\omega) + k\gamma''_0 W_2(\omega)] - kW_3(\omega)\tilde{M}_2^2,
\]

where:

\[
W_0(\omega) = (a - 1)(a + b - 2)/2b, \quad W_1(\omega) = [3(a + b) - 2(2ab + 1)]/ab,
\]

\[
W_2(\omega) = 2\omega^2[1 - (a + b)], \quad W_3(\omega) = \omega^2b.
\] \hfill (31)

For zeros approximation for power series of \( 1/\mu \), Eq. (30) give:

\[
M_1 = M_2 = 0.
\] \hfill (32)

Since the \( z \)-axis is directed along with the major or the minor axis of the ellipsoid of inertia of the body, we get: \( W_0(\omega) > 0 \) for all \( \omega \) under consideration.

Assume that:

\[
\gamma''_0 [z_0 W_1 + k\gamma''_0 W_2] + kW_3(\omega)\tilde{M}_2^2 \neq 0.
\] \hfill (33)

Using (30), we get \( \beta_1, \beta_2 \) in power series expansions of powers less than \( \mu^{-2} \). Then for the rational values of the natural frequency \( \omega \) does not equal to \( (1, 2, 1/2, 3, 1/3, \ldots) \), we get the required periodic solutions and the correction of the period \( \alpha(\mu^{-1}) \) as:

\[
p_1(\tau, \mu^{-1}) = \mu^{-1}[x_0(1 - a)^{-1}\gamma''_0 + \chi_1 M_3 \cos \tau] + \ldots,
\]

\[
q_1(\tau, \mu^{-1}) = \mu^{-1}[y_0(1 - b)^{-1}\gamma''_0 + A_1^{-1} M_3 \chi_2 \sin \tau] + \ldots,
\]

\[
r_1(\tau, \mu^{-1}) = 1 - 0.25\mu^{-2}M_3[kM_3 C_1 + 4x_0(1 - \cos \tau) + y_0 \sin \tau - kM_3 C_1 \cos 2\tau] + \ldots,
\]

\[
\gamma(\tau, \mu^{-1}) = M_3 \cos \tau - 0.5\mu^{-2}I^2 \cos \tau + \ldots,
\]

\[
\gamma'(\tau, \mu^{-1}) = -M_3 \sin \tau + 0.5\mu^{-2}I^2 \sin \tau + \ldots,
\]

\[
\gamma''(\tau, \mu^{-1}) = \gamma''_0 + \mu^{-2}M_3[x_0(1 - a)^{-1}\gamma''_0 - 0.5M_3 C_1 \left( \frac{z_0}{a + b - 1} + 0.5k\gamma''_0 \right) + x_0(1 - a)^{-1}\gamma''_0 \cos \tau + y_0(1 - b)^{-1}\gamma''_0 \sin \tau] + 0.25M_3 C_1 \left( \frac{z_0}{a + b - 1} + k\gamma''_0 \right) \cos 2\tau] + \ldots.
\] \hfill (35)
\[
\omega(\mu^{-1}) = 2\mu^{-2}\pi n \left\{ M_3x_0' - z_0'y_0'' - 0.5k \left[ y_0'^2(bB_1 - aA_1) + B_1y_0''(1 - y_0'') - 0.125C_1M_2^2 \right] \right\} + \ldots
\] (36)

The obtained solutions (35) and (36) are considered as the generalization of the corresponding problem in gravity field which studied in previous works [10] (when \(k = 0\)), the deviations between them are given by:

\[
\begin{align*}
\Delta p_1 &= \mu^{-1}(1 - \omega^2)^{-1}kM_3y''_0(A_1 - \omega^2)\cos \tau + \ldots, \\
\Delta q_1 &= \mu^{-1}A_1 M_3 \left\{ (1 - \omega^2)^{-1}(k(A_1 - \omega^2)y''_0 - kA_1y''_0) \right\} \sin \tau + \ldots, \\
\Delta r_1 &= -0.25\mu^{-2}M_3kC_1(1 - \cos 2\tau) + \ldots, \\
\Delta \gamma &= -\mu^{-2}[0] + \ldots, \quad \Delta \gamma' = \mu^{-2}[0] + \ldots, \\
\Delta \gamma'' &= -0.25\mu^{-2}M_3^2C_1ky''_0(1 - \cos 2\tau) + \ldots,
\end{align*}
\]

\[
\Delta \alpha(\mu^{-1}) = -\mu^{-2}\pi nk \left[ y''_0(aB_1 - aA_1) - 0.5M_2^2(b - a) + B_1y''_0(1 - y''_0) \right] + \ldots 
\] (37)

**Geometric interpretation of the motion**

The geometric interpretation for the motion of the body at any instant of time to Euler’s angles definitions \(\theta, \psi, \phi\) is given by [11]:

\[
\begin{align*}
\theta &= \theta_0 + \mu^{-2}\cosec \theta_0[\theta_2(t + t_0) - \theta_2(t_0)] + \ldots, \\
\psi &= \psi_0 + 0.5Mg\ell C^{-1}r^{-1}_0(x_1 - x_2A^{-1}_1)t + \mu^{-1}\sqrt{Mg\ell/C} [\psi_1(t + t_0) - \psi_1(t_0)] + \ldots, \\
\phi &= \phi_0 + [0]_0^{-1} - 0.5Mg\ell C^{-1}r^{-1}_0 \cos \theta_0(x_1 - x_2A^{-1}_1)t + \ldots,
\end{align*}
\] (38)

where:

\[
\begin{align*}
\theta_2(t) &= M_3 \left[ y''_0(1 - b)^{-1} \sin r^{-1}_0 t - x_0'y''_0(1 - a)^{-1} \cos r^{-1}_0 t ight. \\
&\quad + 0.5C_1 \left( \frac{z'_0}{a + b - 1} - 0.667k_0 y''_0 \right) \cos 2r^{-1}_0 t \right], \\
\psi_1(t) &= 0.25(x_1 + x_2A^{-1}_1) + \sin 2r^{-1}_0 t + y_0'y''_0(1 - b)^{-1} \cos r^{-1}_0 t + x_0'(1 - a)^{-1} y''_0 \sin r^{-1}_0 t.
\end{align*}
\] (39)

**Numerical solutions**

In this section, we use a computer program to determine the obtained solutions (19) and their derivatives for the time in the interval \(t \in [0, 300]\). On the other hand, we use the fourth-order Runge–Kutta method [12] through another program to obtain numerical solutions for the autonomous system (14). In the end, we compare both solutions to check the accuracy of the method of solutions. These results are obtained through Tables 1 and 2. From these Tables, we deduce that the numerical solutions are in agreement with the analytical ones which prove the accuracy of considered methods.
Conclusions

We conclude that the problem of the motion of a rigid body about a fixed point is studied in many works [13–18] in both the uniform and gravity fields. We study our problem in case of a right angle of nutation $\theta_0$ when its center of mass does not necessarily coincide with the fixed point. The equations of motions of the problem are obtained and reduced to a quasilinear autonomous system. The obtained system is solved by assuming a large parameter achieved from an angular velocity component tends to zero. The obtained solutions are treated through computer programs in a bounded interval of time. The autonomous system is treated with the Runge–Kutta method in the same interval of time to obtain the numerical solutions of the motion. Both obtained solutions are in full agreement with others which prove the accuracy of both numerical and analytical techniques used in solving the problem. For the geometric interpretation obtained, we note that:

The precession angle $\psi$

From (38) when $\mu \to \infty$, we deduce that the precession angle $\psi$ is sufficiently large because $r_0$ is sufficiently small, that is, we obtain a case of large precession $\psi = \psi_0 + 0.5Mg\ell C^{-1}r_0^{-1}(\chi_1 - \chi_2 A_1^{-1})t$.

The nutation angle $\theta$

We obtain a case of steady regular permutation: $\theta = \theta_0$.

The pure rotation angle $\phi$

The case of a large pure rotation is obtained which depends on $1/r_0$ in the form:

$$\phi = \phi_0 + [r_0 - 0.5Mg\ell C^{-1}r_0^{-1} \cos \theta_0(\chi_1 - \chi_2 A_1^{-1})]t.$$
Table 1 Represents the values of the obtained analytical solutions using the large parameter method in the interval \( t \in [0, 300] \)

| \( t \)  | \( p_{2a} \)     | \( y_{2a} \)     | \( x_a = dp_{2a}/dt \) | \( y_a = dy_{2a}/dt \) |
|-------|----------------|----------------|------------------------|------------------------|
| 0     | 1.88117E−15    | 1              | 0                      | 0                      |
| 10    | 1.52189E−15    | 0.809017       | −1.10572E−15           | −0.587785              |
| 20    | 5.81312E−16    | 0.309017       | −1.78909E−15           | −0.951057              |
| 30    | −5.81312E−16   | −0.309017      | −1.78909E−15           | −0.951056              |
| 40    | −1.5219E−15    | −0.809017      | −1.10572E−15           | −0.587785              |
| 50    | −1.88117E−15   | −1             | −2.84048E−22           | −1.50996E−07           |
| 60    | −1.52189E−15   | −0.809017      | 1.10572E−15            | 0.587785               |
| 70    | −5.81312E−16   | −0.309017      | 1.78909E−15            | 0.951056               |
| 80    | 5.81312E−16    | 0.309017       | 1.78909E−15            | 0.951056               |
| 90    | 1.5219E−15     | 0.809017       | 1.10572E−15            | 0.587785               |
| 100   | 1.88117E−15    | 1              | 5.68096E−22            | 3.01992E−07            |
| 110   | 1.52189E−15    | 0.809017       | −1.10572E−15           | −0.587785              |
| 120   | 5.81312E−16    | 0.309017       | −1.78909E−15           | −0.951057              |
| 130   | −5.81313E−16   | −0.309017      | −1.78909E−15           | −0.951056              |
| 140   | −1.52189E−15   | −0.809017      | −1.10572E−15           | −0.587785              |
| 150   | −1.88117E−15   | −1             | 4.48654E−23            | 2.38498E−08            |
| 160   | −1.52189E−15   | −0.809017      | 1.10572E−15            | 0.587785               |
| 170   | −5.81311E−16   | −0.309016      | 1.78909E−15            | 0.951057               |
| 180   | 5.81314E−16    | 0.309018       | 1.78909E−15            | 0.951056               |
| 190   | 1.5219E−15     | 0.809017       | 1.10572E−15            | 0.587785               |
| 200   | 1.88117E−15    | 1              | 1.13619E−21            | 6.03983E−07            |
| 210   | 1.5219E−15     | 0.809017       | −1.10572E−15           | −0.587785              |
| 220   | 5.81312E−16    | 0.309017       | −1.78909E−15           | −0.951056              |
| 230   | −5.81312E−16   | −0.309017      | −1.78909E−15           | −0.951056              |
| 240   | −1.5219E−15    | −0.809017      | −1.10572E−15           | −0.587785              |
| 250   | −1.88117E−15   | −1             | 1.27079E−21            | 6.75332E−07            |
| 260   | −1.52189E−15   | −0.809016      | 1.10572E−15            | 0.587786               |
| 270   | −5.81313E−16   | −0.309018      | 1.78909E−15            | 0.951056               |
| 280   | 5.81311E−16    | 0.309017       | 1.78909E−15            | 0.951057               |
| 290   | 1.52189E−15    | 0.809017       | 1.10572E−15            | 0.587785               |
| 300   | 1.88117E−15    | 1              | −8.97307E−23           | −4.76995E−08           |

Table 2. So we confirm that the presented numerical results would correlate to results obtained by other distinguished numerical techniques (except the Runge–Kutta method) will reveal the same results obtained in Table 1.
Table 2 represents the values of the numerical solutions using fourth-order Runge–Kutta method in the interval $t \in [0, 300]$

| $t$  | $P_{2n}$  | $Y_{2n}$  | $x_n = dp_{2n}/dt$ | $y_n = dy_{2n}/dt$ |
|------|-----------|-----------|---------------------|---------------------|
| 0    | 1.8812E−15 | 1         | 0                   | 0                   |
| 10   | 1.5247E−15 | 0.809035  | −1.09756E−15        | −0.587738           |
| 20   | 5.9028E−16 | 0.309102  | −1.77911E−15        | −0.951001           |
| 30   | −5.6779E−16| −0.308864 | −1.78633E−15        | −0.951064           |
| 40   | −1.5106E−15| 0.808859  | −1.11651E−15        | −0.587913           |
| 50   | −1.8809E−15| −0.999934 | −2.35325E−17        | −0.000245929        |
| 60   | −1.5338E−15| 0.809126  | 1.07833E−15         | 0.5875              |
| 70   | −6.1263E−16| −0.309316 | 1.77147E−15         | 0.950862            |
| 80   | 5.4519E−16 | 0.30861   | 1.79318E−15         | 0.951078            |
| 90   | 1.4963E−15 | 0.808661  | 1.13524E−15         | 0.588073            |
| 100  | 1.8803E−15 | 0.999868  | 4.70583E−17         | 0.000491738         |
| 110  | 1.5517E−15 | 0.809217  | −1.05894E−15        | −0.587262           |
| 120  | 6.3487E−16 | 0.309529  | −1.76355E−15        | −0.950724           |
| 130  | −5.225E−16 | −0.308356 | −1.79974E−15        | −0.951091           |
| 140  | −1.4818E−15| −0.808463 | −1.1538E−15         | −0.588234           |
| 150  | −1.8795E−15| −0.999802 | −7.05736E−17        | −0.000737578        |
| 160  | −1.5648E−15| −0.809309 | 1.03937E−15         | 0.587025            |
| 170  | −6.5702E−16| −0.309743 | 1.75336E−15         | 0.950585            |
| 180  | 4.9974E−16 | 0.308102  | 1.80602E−15         | 0.951104            |
| 190  | 1.4671E−15 | 0.808265  | 1.17217E−15         | 0.588394            |
| 200  | 1.8783E−15 | 0.999736  | 9.40748E−17         | 0.000983447         |
| 210  | 1.5777E−15 | 0.809399  | −1.01965E−15        | −0.586787           |
| 220  | 6.7906E−16 | 0.309956  | −1.74689E−15        | −0.950446           |
| 230  | −4.7689E−16| −0.307848 | −1.81201E−15        | −0.951117           |
| 240  | −1.4521E−15| −0.808067 | −1.19036E−15        | −0.588553           |
| 250  | −1.8769E−15| −0.99967  | −1.17558E−16        | −0.00122914         |
| 260  | −1.5903E−15| −0.80949  | 9.99769E−16         | 0.586549            |
| 270  | −7.0098E−16| −0.310169 | 1.73814E−15         | 0.950307            |
| 280  | 4.5398E−16 | 0.307593  | 1.81772E−15         | 0.95113             |
| 290  | 1.4369E−15 | 0.807869  | 1.20835E−15         | 0.588713            |
| 300  | 1.8751E−15 | 0.999603  | 1.41019E−16         | 0.00147495          |

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