Abstract

The relative Dolbeault cohomology which naturally comes up in the theory of Čech-Dolbeault cohomology turns out to be canonically isomorphic with the local (relative) cohomology of A. Grothendieck and M. Sato so that it provides a handy way of representing the latter. In this paper we use this cohomology to give simple explicit expressions of Sato hyperfunctions, some fundamental operations on them and related local duality theorems. This approach also yields a new insight into the theory of hyperfunctions and leads to a number of further results and applications. As one of such, we give an explicit embedding morphism of Schwartz distributions into the space of hyperfunctions.

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1 Introduction

The theory of hyperfunctions was initiated by M. Sato in [28] and then the theory together with its philosophy and methodology has been vastly developed to form a branch of mathematics called algebraic analysis (see M. Sato, T. Kawai and M. Kashiwara [29], M. Kashiwara and P. Schapira [19], P. Schapira [30] and references therein). The space $\mathcal{B}(M)$ of hyperfunctions on a real analytic manifold $M$ is considered to be large enough in the sense that all the solutions to a linear differential equation (with irregular singularities) are exhausted in $\mathcal{B}(M)$, while that of, for example, Schwartz distributions is too small as a solution space as shown in H. Komatsu [21]. Thus the hyperfunctions may be thought of as natural generalizations of functions and play important roles in several areas of mathematics, in particular in the study of linear differential equations.

A hyperfunction on a one-dimensional space is represented by a holomorphic function on the complement of the real axis in the complex plane and this representative is zero as a hyperfunction if and only if it extends across the real axis as a holomorphic function. Contrary to its relative simplicity and concreteness in the one-dimensional case, the theory in higher dimensions is rather abstract, as hyperfunctions are defined in terms of local...
cohomology, with support in $M$ and coefficients in the sheaf $\mathcal{O}$ of holomorphic functions on the complexification $X$ of $M$, and the theory is described in the language of derived functors. Also the representation of local cohomology is done via relative Čech cohomology with coefficients in $\mathcal{O}$. In order to understand hyperfunctions without heavy machineries, A. Kaneko [17] and M. Morimoto [27] introduced the so-called “intuitive representation” of a hyperfunction, where it is represented by a formal sum of several holomorphic functions on infinitesimal wedges with the edge along $M$. As a recent development in this direction, D. Komori and K. Umetsu [24] generalize the intuitive representation to the case of local cohomology with coefficients in an arbitrary sheaf under suitable conditions.

On the other hand, the third named author of this paper observed that the relative Dolbeault cohomology, which naturally appears in the theory of Čech-Dolbeault cohomology in [35] (see also [1], [36] and [37]), is canonically isomorphic with the local cohomology with coefficients in the sheaf of holomorphic forms and that, if we use this cohomology, a hyperfunction $u \in \mathcal{B}(M)$ in an arbitrary dimension has a similar representation as that on the one-dimensional space. That is, $u$ has a representative $\tau = (\tau_1, \tau_{01})$, where $\tau_1$ is a $C^\infty$ form of type $(0, n)$ defined on a neighborhood of $M$ in $X$ and $\tau_{01}$ is a $C^\infty$ form of type $(0, n-1)$ defined on the complement of $M$, with a natural cocycle condition. Such a representation gives a new approach to the hyperfunction theory and makes its treatment more manurable, for instance:

1. We can employ tools available in the $C^\infty$ category such as partitions of unity. In particular, we may take a representative with a compact support if $u$ has a compact support, which is not possible in the framework of intuitive representation, as representatives are holomorphic functions.

2. The conventional representation of a hyperfunction is obtained via Čech cohomology with coefficients in $\mathcal{O}$ and hence the existence of a Stein open covering is indispensable, which sometimes makes the theory rather complicated. In our framework, however, arguments can proceed without Stein coverings.

3. The integration of a hyperfunction is easily performed as it is represented by a pair of $C^\infty$ forms. Furthermore, we may concretely describe the morphism associated with the cohomological residue map, since all the related morphisms are constructed using fine resolutions of sheaves.

The purpose of this paper is to reestablish the theory of hyperfunctions in the framework of relative Dolbeault cohomology and to further pursue the theory along this line. It should be noted, however, that although our approach simplifies various expressions substantially, it is not certain if it also leads to the simplification of the proofs of such fundamental facts as the pure codimensionality of $M$ in $X$ with respect to $\mathcal{O}$ and the flabbiness of the sheaf $\mathcal{B}$ of hyperfunctions. Aside from this, the advantages as listed above provide a new insight and enable us to perform novel treatments of the theory. Furthermore, this point of view leads to several results that are hardly be obtained by the traditional way. One of such is in Theorem 8.1 below, which gives an explicit representative of the embedded image of a Schwartz distribution in the space of hyperfunctions.

In the course of our study, we introduce several fundamental methods and ideas, which become important ingredients for further works. In fact this kind of cohomology
theory serves to establish the foundation of various topics in algebraic analysis such as the theory of Laplace hyperfunctions ([22], [15] and [16]) and the symbol theory of analytic pseudodifferential operators ([2], see also [3]). Recently, K. Umeta is working on the former in [41] and D. Komori is doing for the latter in [23], where our fundamental methods introduced in this paper play essential roles in their arguments.

The paper is organized as follows. In Section 2, we recall orientation sheaves and the Thom class, which are necessary for the orientation free expression of hyperfunctions and the description of the topological aspect of the theory. The definition and fundamental properties of Sato hyperfunctions are also briefly recalled. We review, in Sections 3, 4 and 5, the theory of relative de Rham and Dolbeault cohomologies. In this paper we need the relative de Rham cohomology in two ways. One is for the integration theory and the other for the expression of sections of the complexified relative orientation sheaf. Some canonical morphisms between the relative de Rham and the relative Dolbeault cohomologies and local duality morphism are also discussed.

In Section 6, we first express hyperfunctions via relative Dolbeault cohomology and describe some basic operations such as differentiation and multiplication by real analytic functions. In fact we work more generally on hyperforms, which have been traditionally referred to as forms with hyperfunction coefficients. Then we study the integration of hyperforms and establish the duality theorem of Martineau in our settings. We also give some examples related to Dirac’s delta function. We note that the theory of integration on Čech-de Rham cohomology directly descends to the integration theory on relative Dolbeault cohomology and in turn to the integration theory of hyperforms. The duality pairing that appears in the theorem of Martineau is explicitly expressed in this context.

Several important morphisms in the hyperfunction theory are studied in Section 7. We first discuss the embedding of real analytic functions into the space of hyperfunctions. Particularly noteworthy here is that the canonical morphism from the relative de Rham cohomology to the relative Dolbeault cohomology, which is induced from the projection of a $q$-form to its $(0, q)$-component, is effectively used in the construction of the embedding morphism. Also the Thom class in the relative de Rham cohomology plays an essential role in this scene of interaction of topology and analysis, in particular, combined with the above projection, it is used to give an explicit expression of the embedding morphism (Corollary 7.4). We then generalize this to the construction of the boundary value morphism, by which we may regard a holomorphic function on an open edge along $M$ as a hyperfunction. The essential idea here is that we may “confine” the Thom class in the edge under a suitable condition. We also give an interpretation of the microlocal analyticity in our language (Proposition 7.22). This is a direct consequence of our expression of the spectral map, which is simply a morphism of relative Dolbeault cohomologies induced from the restriction of differential forms. Such operations as the external product of hyperforms, the restriction of hyperfunctions and the fiber integration of hyperforms are also treated in our framework. In Section 8, we give an explicit embedding morphism of distributions into the space of hyperfunctions in our context (Theorem 8.1), as mentioned above.

In Appendix A, we show the compatibility of the boundary value morphisms between our construction and the original functorial one in [29] and [19].

Some parts of this paper are summarized in [14], [36], [38] and [39].
We would like to thank the referee for valuable suggestions that improved the presentation of the paper.

2 Sato hyperfunctions

2.1 Relative cohomology

In the sequel, by a sheaf we mean a sheaf with at least the structure of Abelian groups. As general references for the sheaf cohomology theory, we list [6] and [7]. For a sheaf \( \mathcal{S} \) on a topological space \( X \) and an open set \( V \) in \( X \), we denote by \( \mathcal{S}(V) \) the group of sections on \( V \). For an open subset \( V' \subset V \) we denote by \( \mathcal{S}(V,V') \) the group of sections on \( V \) that vanish on \( V' \).

As reference cohomology theory we adopt the one via flabby resolution. Thus for an open set \( X' \subset X \), \( H^q(X,X';\mathcal{S}) \) denotes the \( q \)-th cohomology of the complex \( \mathcal{F}^\bullet(X,X') \) with 0 \( \rightarrow \mathcal{S} \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow \cdots \). It is uniquely determined modulo canonical isomorphisms, independently of the chosen flabby resolution. Considering the closed set \( S = X \setminus X' \), it will also be denoted by \( H^q_S(X;\mathcal{S}) \). This cohomology in the first expression is referred to as the relative cohomology of \( S \) on \( (X,X') \) (cf. [28]) and in the second expression the local cohomology of \( S \) on \( X \) with support in \( S \) (cf. [11]).

We recall some of the fundamental properties of the cohomology:

**Proposition 2.1**

1. For a triple \( (X,X',X'') \) of open sets with \( X'' \subset X' \subset X \), there is an exact sequence

\[
\cdots \rightarrow H^{q-1}(X',X'';\mathcal{F}) \xrightarrow{\delta} H^q(X,X';\mathcal{S}) \xrightarrow{J} H^q(X,X'';\mathcal{S}) \xrightarrow{i^*} H^q(X',X'';\mathcal{S}) \rightarrow \cdots.
\]

2. (Excision) For any open set \( V \) in \( X \) containing \( S \), there is a canonical isomorphism

\[
H^q(X,X \setminus S;\mathcal{F}) \simeq H^q(V,V \setminus S;\mathcal{F}) \quad \text{or equivalently} \quad H^q_S(X;\mathcal{F}) \simeq H^q_S(V;\mathcal{F}).
\]

2.2 Derived sheaves

Let \( X \) be a topological space, \( S \) a closed set in \( X \) and \( \mathcal{S} \) a sheaf on \( X \).

**Definition 2.2** The \( q \)-th derived sheaf \( \mathcal{H}_S^q(\mathcal{S}) \) of \( \mathcal{S} \) with support in \( S \) is the sheaf defined by the presheaf \( V \mapsto H^q_{S\cap V}(V;\mathcal{S}) \).

By definition, for a point \( x \) in \( X \),

\[
\mathcal{H}_S^q(\mathcal{S})_x = \lim_{\overset{\longrightarrow}{V \ni x}} H^q_{S\cap V}(V;\mathcal{S}). \tag{2.3}
\]

Thus \( \mathcal{H}_S^q(\mathcal{S}) \) is supported on \( S \) and we may think of it as a sheaf on \( S \). As such \( \mathcal{H}_S^q(\mathcal{S}) \) is defined by the presheaf \( U \mapsto H^q_U(V;\mathcal{S}) \), where \( U \) is an open set in \( S \) and \( V \) an open set in \( X \) containing \( U \) as a closed set. Note that by excision \( H^q_U(V;\mathcal{S}) \) does not depend on the choice of such a \( V \). We quote (cf. [18]):
Proposition 2.4 If $H_S^i(S) = 0$ for $i < q$, then the above presheaf is a sheaf, i.e., for any open set $U$ in $S$, there is a canonical isomorphism

$$H_S^q(U) \cong H_U^q(V; S),$$

where $V$ is an open set in $X$ as above.

If the above is the case, we identify $H^q_S(S)(U)$ and $H^q_U(V; S)$ hereafter.

Definition 2.5 We say that $S$ is purely $q$-codimensional in $X$ with respect to $S$ if

$$H_i^q(S) = 0 \quad \text{for} \quad i \neq q.$$

Thus if this is the case, the statement of Proposition 2.4 holds.

2.3 Orientation sheaves and the Thom class

We list [6] and [10] as references for this subsection. As to the Alexander duality, we refer to [5] and [34]. Throughout the paper, the manifolds are assumed to have a countable basis, thus they are paracompact and have countably many connected components. We discuss orientation sheaves and orientations in detail below. For the moment recall that orientations of vector spaces, manifolds and simplices are defined as in [32].

Let $X$ be a $C^\infty$ manifold of dimension $m$ and $\mathbb{Z}_X$ the constant sheaf on $X$ with stalk $\mathbb{Z}$. Then $H^q(X; \mathbb{Z}_X)$ is canonically isomorphic with the simplicial cohomology $H^q(X; \mathbb{Z})$ of $X$ with $\mathbb{Z}$-coefficients on finite chains and, for a closed set $S$, $H_S^q(X; \mathbb{Z}_X)$ is canonically isomorphic with the relative cohomology $H^q(X, X \setminus S; \mathbb{Z})$. We denote by $H_q(X; \mathbb{Z})$ the Borel-Moore homology of $X$, which in our case is canonically isomorphic with the simplicial homology of $X$ of locally finite chains.

Suppose $X$ is orientable and is specified with an orientation, i.e., oriented. For a triangulation of $X$, we orient simplices and dual cells so that the orientation of the cell dual to a simplex followed by that of the simplex gives the orientation of $X$. For a subcomplex $S$ of $X$ with respect to some triangulation, we have the Alexander duality

$$A : H^q(X, X \setminus S; \mathbb{Z}) \xrightarrow{\sim} H_{m-q}(S; \mathbb{Z}),$$

which is given by the left cap product with the fundamental class $[X]$, the class of the sum of all the $m$-simplices in $X$.

First note that we have

$$H_q^0(\mathbb{R}^l, \mathbb{Z}_{\mathbb{R}^l}) \cong \begin{cases} \mathbb{Z} & q = l, \\ 0 & q \neq l, \end{cases}$$

which can be seen from the long exact sequence. We may as well interpret it as the Alexander duality

$$A : H^l(\mathbb{R}^l, \mathbb{R}^l \setminus \{0\}; \mathbb{Z}) \xrightarrow{\sim} H_{l-q}(\{0\}; \mathbb{Z}).$$

In particular, since the isomorphism $H^l(\mathbb{R}^l, \mathbb{R}^l \setminus \{0\}; \mathbb{Z}) \cong H^0(\{0\}; \mathbb{Z}) = \mathbb{Z}$ is given by the left cap product with $[\mathbb{R}^l]$, specifying a generator of $H^l_0(\mathbb{R}^l; \mathbb{Z}_{\mathbb{R}^l})$ is equivalent to specifying an orientation of $\mathbb{R}^l$.  

Let $\pi : E \to M$ be a $C^\infty$ real vector bundle of rank $l$ on a $C^\infty$ manifold $M$. We identify $M$ with the image of the embedding $i : M \hookrightarrow E$ by zero section. By excision, the sheaf $\mathcal{H}^q_M(\mathbb{Z}_E)$ may be identified with the sheaf $\mathcal{H}^q_M(\pi; \mathbb{Z}_E)$ defined by the presheaf $U \mapsto H^0_U(\pi^{-1}(U); \mathbb{Z}_E)$. Setting $E_x = \pi^{-1}(x)$ for $x \in M$, we have an isomorphism

$$r^*_x : \mathcal{H}^q_M(\pi; \mathbb{Z}_E) \xrightarrow{\sim} H^0_{\{0\}}(E_x; \mathbb{Z}_{E_x}) \cong \begin{cases} \mathbb{Z} & q = l, \\ 0 & q \neq l. \end{cases}$$

Thus $M$ is purely $l$-codimensional in $E$ with respect to $\mathbb{Z}_E$.

**Definition 2.7** We set $or_{M/E} = \mathcal{H}^1_M(\mathbb{Z}_E)$ and call it the orientation sheaf of the bundle $E \to M$.

**Remark 2.8** The sheaf $or_{M/E}$ is a sheaf on $M$. It is what is referred to as the relative orientation sheaf of the embedding $i : M \hookrightarrow E$ and describes the orientations of the fibers of $\pi : E \to M$.

The sheaf $or_{M/E}$ is locally constant, i.e., locally isomorphic with a product sheaf with stalk $\mathbb{Z}$ and, by Proposition 2.4, we have

$$or_{M/E}(U) = H^1_U(\pi^{-1}(U); \mathbb{Z}_E).$$

In particular $or_{M/E}(M) = H^1_M(E; \mathbb{Z}_E)$. Note that there is a canonical isomorphism

$$or_{M/E} \otimes_{\mathbb{Z}_M} or_{M/E} \cong \mathbb{Z}_M, \quad \psi \otimes \psi \leftrightarrow 1,$$

where $\psi$ is any generator of $or_{M/E,x}$, $x \in M$.

The bundle $E$ is said to be orientable if $or_{M/E}$ is a constant sheaf. This is equivalent to saying that $or_{M/E}$ admits a global section $\psi \in or_{M/E}(M) = H^1_M(E; \mathbb{Z}_E)$ such that $r^*_x(\psi(x))$ generates $H^0_{\{x\}}(E_x; \mathbb{Z}_{E_x})$ over $\mathbb{Z}$ for every $x \in M$, i.e., we may specify an orientation of each fiber in a "coherent manner". A section $\psi$ as above is called an orientation of $E$. Once we specify such a $\psi$ we say that $E$ is oriented. From the definition, if the bundle $E$ is orientable, we have $or_{M/E}(M) \cong \mathbb{Z}_M(M)$. The isomorphism is not uniquely determined, however once we specify an orientation $\psi$, it is determined so that $\psi$ is sent to the constant function 1. Note that if $M$ is connected, there are exactly two orientations.

Let $\pi : E \to M$ be an oriented real vector bundle of rank $l$. A frame $(e_1, \ldots, e_l)$ of $E$ on an open set $U$ in $M$ is said to be positive if $(e_1(x), \ldots, e_l(x))$ determines the prescribed orientation of $E_x$ for every $x \in U$.

**Definition 2.9** Let $E$ be an oriented real vector bundle of rank $l$. The Thom class $\Psi_E$ of $E$ is the global section of $or_{M/E}$ that defines the orientation. It may be thought of as being in $H^1_M(E; \mathbb{Z}_E) \cong H^l(E, E \setminus M; \mathbb{Z})$.

Let $M$ be a $C^\infty$ manifold of dimension $n$. The orientation sheaf $or_M$ of $M$ is defined to be the orientation sheaf of the tangent bundle $TM$. By excision and the exponential map, we have, for any $x \in M$ and an open neighborhood $U$ of $x$, a canonical isomorphism

$$or_{M,x} \cong H^n_{\{x\}}(U; \mathbb{Z}),$$
which relates the orientation of $TM$ and that of $M$. The manifold $M$ is orientable if and only if $TM$ is and, if this is the case, we orient them so that if $(x_1, \ldots, x_n)$ is a positive coordinate system on $M$, $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ is a positive frame of $TM$.

Letting $\mathcal{E}_M^{(n)}$ be the sheaf of $C^\infty$ $n$-forms on $M$, the sheaf $\mathcal{Y}_M = \mathcal{E}_M^{(n)} \otimes \mathcal{O}_M$ is called the sheaf of densities on $M$. Denoting by $\Gamma_c(M; \mathcal{Y}_M)$ the space of sections with compact support, we have the integration

$$\Gamma_c(M; \mathcal{Y}_M) \xrightarrow{\int_M} \mathbb{R}.$$ 

If $M$ is orientable and if an orientation is specified, there is an isomorphism $\mathcal{Y}_M \simeq \mathcal{E}_M^{(n)}$.

Let $E \to M$ be a real vector bundle and $i : M \to E$ the embedding by zero section. Noting that the normal bundle $T_M E$ of $M$ in $E$ is the bundle $E$ itself, we have the exact sequence of vector bundles on $M$:

$$0 \to TM \to TE|_M \to E \to 0.$$ 

From this we have isomorphisms

$$i^{-1} o_{E} \simeq o_{M/E} \otimes o_M, \quad o_{M/E} \simeq i^{-1} o_{E} \otimes o_M, \quad (2.10)$$

where $o_{E}$ denotes the orientation sheaf of the total space $E$ and the tensor products are over $\mathbb{Z}_M$. Note that there are several choices for the above isomorphisms. From (2.10), we see that if $M$ is orientable and $E$ is orientable as a bundle, the total space $E$ is also orientable. Also if $M$ and the total space $E$ are orientable, $E$ is orientable as a bundle.

Let $X$ be a $C^\infty$ manifold of dimension $m$ and $M \subset X$ a closed submanifold of dimension $n$. Then we have the exact sequence

$$0 \to TM \to TX|_M \xrightarrow{\pi} T_M X \to 0 \quad (2.11)$$

with $T_M X$ the normal bundle of $M$ in $X$. The relative orientation sheaf $o_{M/X}$ is defined to be the orientation sheaf of the normal bundle $\pi : T_M X \to M$. By the tubular neighborhood theorem and excision, we have, for an open set $U \subset M$,

$$o_{M/X}(U) = H^l_U(\pi^{-1}(U); \mathbb{Z}_{T_M X}) \simeq H^l_U(V; \mathbb{Z}_X), \quad (2.12)$$

where $l = m - n$ and $V$ is an open set in $X$ containing $U$ as a closed set. Thus we have a canonical isomorphism

$$o_{M/X} \simeq \mathcal{H}_M^l(\mathbb{Z}_X).$$

Denoting by $i : M \hookrightarrow X$ the embedding, we have isomorphisms (cf. (2.10))

$$i^{-1} o_{X} \simeq o_{M/X} \otimes o_M, \quad o_{M/X} \simeq i^{-1} o_{X} \otimes o_M. \quad (2.13)$$

Thus if $M$ and $X$ are orientable, so is $T_M X$.

From (2.12), we have

$$o_{M/X}(M) = H^l_M(T_M X; \mathbb{Z}_{T_M X}) \simeq H^l_M(X; \mathbb{Z}_X).$$

Suppose the bundle $T_M X$ is oriented. The Thom class $\Psi_M$ of $M$ in $X$ is then defined to be the class in $H^l_M(X; \mathbb{Z}_X) \simeq H^l(X, X \setminus M; \mathbb{Z})$ that corresponds to the Thom class of $T_M X$ by the isomorphism above.
2.4 Hyperfunctions and hyperforms

Let $M$ be a real analytic manifold of dimension $n$ and $X$ its complexification. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on $X$. We quote (cf. [18], [19], [29]):

**Theorem 2.14 1.** $M$ is purely $n$-codimensional in $X$ with respect to $\mathcal{O}_X$, i.e., $\mathcal{H}^i_M(\mathcal{O}_X) = 0$ for $i \neq n$.

2. The sheaf $\mathcal{H}^n_M(\mathcal{O}_X)$ is flabby.

We recall that the sheaf of Sato hyperfunctions on $M$ is defined by

$$\mathcal{B}_M = \mathcal{H}^n_M(\mathcal{O}_X) \otimes_{\mathbb{Z}^n M} \text{or}_M.$$

Denoting by $\mathcal{O}^{(p)}_X$ the sheaf of holomorphic $p$-forms on $X$, we introduce the following

**Definition 2.15** The sheaf of $p$-hyperforms is defined by

$$\mathcal{B}^{(p)}_M = \mathcal{H}^n_M(\mathcal{O}^{(p)}_X) \otimes_{\mathbb{Z}^n M} \text{or}_M.$$

It is what is referred to as the sheaf of $p$-forms with coefficients in hyperfunctions. Note that $M$ is purely $n$-codimensional with respect to $\mathcal{O}^{(p)}_X$ and that $\mathcal{B}^{(p)}_M$ is flabby.

Since $X$ is a complex manifold, it is always orientable. Thus, by (2.13), the bundle $T_MX$ is orientable if and only if $M$ is. If this is the case, for any open set $U \subset M$, we have (cf. (2.12))

$$\mathcal{B}^{(p)}_M(U) = H^n_U(V; \mathcal{O}^{(p)}_X) \otimes_{\mathbb{Z}^n M(U)} H^n_U(V; \mathbb{Z}^n X),$$

where $V$ is an open set in $X$ containing $U$ as a close set. We refer to such $V$ a complex neighborhood of $U$ in $X$. Moreover, if we specify an orientation of $T_MX$, we have a canonical isomorphism $\text{or}_M \simeq \mathbb{Z}^M$ so that we have a canonical isomorphism

$$\mathcal{B}^{(p)}_M \simeq \mathcal{H}^n_M(\mathcal{O}^{(p)}_X)$$

and, for any open set $U \subset M$, $\mathcal{B}^{(p)}_M(U) \simeq H^n_U(V; \mathcal{O}^{(p)}_X)$ with $V$ a complex neighborhood of $U$.

In the sequel, at some point the cohomology $H^n_U(V; \mathbb{Z}^n X)$ is embedded in $H^n_U(V; \mathbb{C}^n X)$, which is represented by the relative de Rham cohomology, while $H^n_U(V; \mathcal{O}^{(p)}_X)$ will be represented by the relative Dolbeault cohomology.

3 Relative de Rham cohomology

Let $X$ be a $C^\infty$ manifold of dimension $m$. We denote by $\mathcal{E}^{(q)}_X$ the sheaf of $C^\infty q$-forms on $X$. We omit the suffix $X$ if there is no fear of confusion.

3.1 Čech-de Rham cohomology

We refer to [4] and [33] for details on the Čech-de Rham cohomology. For relative de Rham cohomology and the Thom class in this context, see [33].
**de Rham cohomology:** The $q$-th de Rham cohomology $H^q_d(X)$ of $X$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(\bullet)}(X), d)$, $d : \mathcal{E}^{(q)}(X) \rightarrow \mathcal{E}^{(q+1)}(X)$. The de Rham theorem says that there is an isomorphism
\[ H^q_d(X) \cong H^q(X; \mathbb{C}). \]

Among the isomorphisms, there is a canonical one, i.e., the one that regards a $q$-form $\omega$ as a cochain that assigns to each oriented $q$-simplex the integration of $\omega$ on the simplex.

**Čech-de Rham cohomology:** The Čech-de Rham cohomology may be defined for an arbitrary covering of $X$. Here we recall the case of coverings consisting of two open sets.

Let $\mathcal{V} = \{ V_0, V_1 \}$ be an open covering of $X$ and set $V_{01} = V_0 \cap V_1$. We set
\[ \mathcal{E}^{(q)}(\mathcal{V}) = \mathcal{E}^{(q)}(V_0) \oplus \mathcal{E}^{(q)}(V_1) \oplus \mathcal{E}^{(q-1)}(V_{01}). \]

Thus an element in $\mathcal{E}^{(q)}(\mathcal{V})$ is expressed by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$. We define the differential
\[ D : \mathcal{E}^{(q)}(\mathcal{V}) \rightarrow \mathcal{E}^{(q+1)}(\mathcal{V}) \]
by $D(\sigma_0, \sigma_1, \sigma_{01}) = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01})$.

Then we see that $D \circ D = 0$.

**Definition 3.1** The $q$-th Čech-de Rham cohomology $H^q_D(\mathcal{V})$ of $\mathcal{V}$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(\bullet)}(\mathcal{V}), D)$.

**Theorem 3.2** The inclusion $\mathcal{E}^{(q)}(X) \hookrightarrow \mathcal{E}^{(q)}(\mathcal{V})$ given by $\omega \mapsto (\omega|_{V_0}, \omega|_{V_1}, 0)$ induces an isomorphism
\[ H^q_d(X) \xrightarrow{\sim} H^q_D(\mathcal{V}). \]

Note that the inverse is given by assigning to the class of $(\sigma_0, \sigma_1, \sigma_{01})$ the class of $\rho_0\sigma_0 + \rho_1\sigma_1 - d\rho_0 \wedge \sigma_{01}$, where $\{ \rho_0, \rho_1 \}$ is a $C^\infty$ partition of unity subordinate to $\mathcal{V}$.

### 3.2 Relative de Rham cohomology

Let $S$ be a closed set in $X$. Letting $V_0 = X \setminus S$ and $V_1$ a neighborhood of $S$ in $X$, we consider the covering $\mathcal{V} = \{ V_0, V_1 \}$ of $X$. We also set $\mathcal{V}' = \{ V_0 \}$ and
\[ \mathcal{E}^{(q)}(\mathcal{V}, \mathcal{V}') = \{ \sigma \in \mathcal{E}^{(q)}(\mathcal{V}) \mid \sigma_0 = 0 \} = \mathcal{E}^{(q)}(V_1) \oplus \mathcal{E}^{(q-1)}(V_{01}). \]

Then we see that $(\mathcal{E}^{(\bullet)}(\mathcal{V}, \mathcal{V}'), D)$ is a subcomplex of $(\mathcal{E}^{(\bullet)}(\mathcal{V}), D)$.

**Definition 3.3** The $q$-th relative de Rham cohomology $H^q_D(\mathcal{V}, \mathcal{V}')$ of $(\mathcal{V}, \mathcal{V}')$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(\bullet)}(\mathcal{V}, \mathcal{V}'), D)$.

From the exact sequence of complexes
\[ 0 \rightarrow \mathcal{E}^{(\bullet)}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} \mathcal{E}^{(\bullet)}(\mathcal{V}) \xrightarrow{i^*} \mathcal{E}^{(\bullet)}(V_0) \rightarrow 0, \]
where $j^*(\sigma_1, \sigma_{01}) = (0, \sigma_1, \sigma_{01})$ and $i^*(\sigma_0, \sigma_1, \sigma_{01}) = \sigma_0$, we have the exact sequence
\[ \cdots \rightarrow H^{q-1}_d(V_0) \xrightarrow{\delta} H^q_D(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} H^q_D(\mathcal{V}) \xrightarrow{i^*} H^q_d(V_0) \rightarrow \cdots, \] (3.4)
where $\delta$ assigns to the class of $\theta$ the class of $(0, -\theta)$. From the above sequence and Theorem 3.2, we have:
Proposition 3.5 The cohomology $H_D^q(V, V')$ is determined uniquely, modulo canonical isomorphisms, independently of the choice of $V_1$.

In view of the above we denote $H_D^q(V, V')$ also by $H_D^q(X, X\setminus S)$ and call it the relative de Rham cohomology of the pair $(X, X\setminus S)$. It is not difficult to see the following:

**Proposition 3.6 (Excision)** For any open set $V$ containing $S$, there is a canonical isomorphism

$$H_D^q(X, X\setminus S) \simeq H_D^q(V, V\setminus S).$$

The relative cohomology share other fundamental properties with the relative cohomology of $X$ with coefficients in $\mathbb{C}$. In fact we have (cf. [34], [37]):

**Theorem 3.7 (Relative de Rham theorem)** There is a canonical isomorphism

$$H_D^q(X, X\setminus S) \simeq H^q(X, X\setminus S; \mathbb{C}).$$

The excision in Proposition 3.6 is compatible with the excision in Proposition 2.1.2 for $\mathcal{F} = \mathbb{C}$ via the above isomorphism.

**Complexification of the relative orientation sheaf:** Let $X$ be a $C^\infty$ manifold of dimension $m$ and $M \subset X$ a closed submanifold of dimension $n$. Set $l = m - n$. We define the complexification of the relative orientation sheaf by $or_{M/X}^c = or_{M/X} \otimes_{\mathbb{Z}} \mathbb{C}_M$. Then by (2.12) and Theorem 3.7, we have, for any open set $U$ in $M$,

$$or_{M/X}^c(U) \simeq H_D^l(V, V\setminus U),$$

where $V$ is an open set in $X$ containing $U$ as a closed set.

### 3.3 Integration

The integration on the Čech-de Rham cohomology in general is defined by considering honeycomb systems. Here we recall the relevant case.

Let $X$ be an oriented $C^\infty$ manifold of dimension $m$. First we assume that $X$ is compact. Then the integration of $m$-forms induces the integration

$$H_D^m(X) \xrightarrow{\int_X} \mathbb{C}. \quad (3.9)$$

Now let $K$ be a compact set in $X$ ($X$ may not be compact). Letting $V_0 = X\setminus K$ and $V_1$ a neighborhood of $K$, we consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{V}'_K = \{V_0\}$ of $X$ and $X\setminus K$. Let $R_1$ be an $m$-dimensional compact manifold with $C^\infty$ boundary in $X$ containing $K$ in its interior. We set $R_{01} = -\partial R_1$ and define

$$\mathcal{C}^{(m)}(\mathcal{V}_K, \mathcal{V}'_K) \xrightarrow{\int_X} \mathbb{C} \quad \text{by} \quad \int_X \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}.$$  

Then it induces the integration

$$H_D^m(X, X\setminus K) \xrightarrow{\int_X} \mathbb{C}. \quad (3.10)$$
3.4 Thom class in differential forms

Let $\pi : E \to M$ be a real oriented vector bundle of rank $l$ on a $C^\infty$ manifold $M$. We have the Thom class $\Psi_E$ in $H^l(E, E \setminus M; \mathbb{Z}) \simeq \mathcal{H}_M^l(Z_E)(M)$ (cf. Definition 2.9). Recall that $\Psi_E$ is characterized as the class whose restriction $r_x^*(\Psi_E(x)) \in H^l(E_x, E_x \setminus \{0\}; \mathbb{Z}) \simeq \mathbb{Z}$ to each fiber $E_x$ is the prescribed generator of $H^l(E_x, E_x \setminus \{0\}; \mathbb{Z})$, i.e., the Thom class of the bundle $E_x \to \{x\}$. We denote by $\overline{\Psi_E}$ the image of $\Psi_E$ by the canonical morphism $H^l(E, E \setminus M; \mathbb{Z}) \to H^l(E, E \setminus M; \mathbb{C}) \simeq H_D^l(E, E \setminus M)$.

In view of Theorem 3.7, we may describe the Alexander duality (2.6) with $\mathbb{C}$-coefficients in terms of relative de Rham cohomology. We recall this in the case of duality for the pair $(\mathbb{R}^l, 0)$. Thus let $V_0 = \mathbb{R}^l \setminus \{0\}$ and $V_1$ a neighborhood of 0 in $\mathbb{R}^l$ and consider the coverings $V = \{V_0, V_1\}$ and $V' = \{V_0\}$ of $\mathbb{R}^l$ and $\mathbb{R}^l \setminus \{0\}$. Let $R_1$ be a closed $n$-ball around 0 in $V_1$ and set $R_{01} = -\partial R_1$. Then we have (cf. [33, Ch.II, Example 3.13]):

**Proposition 3.11** Suppose $\mathbb{R}^l$ is oriented some way. Then the Alexander isomorphism

$$H_D^l(V, V') \simeq H^l(\mathbb{R}^l, \mathbb{R}^l \setminus \{0\}; \mathbb{C}) \xrightarrow{\sim} H_0(\{0\}; \mathbb{C}) = \mathbb{C}.$$ 

is induced from the morphism

$$\delta^{(l)}(V, V') : \mathbb{C} \to \mathbb{C} \quad \text{given by} \quad (\sigma_1, \sigma_{01}) \mapsto \int_{R_l} \sigma_1 + \int_{R_{01}} \sigma_{01}.$$ 

Note that, since $H^l(\mathbb{R}^l; \mathbb{C}) = 0$, $\delta$ in (3.4) is surjective and we may always take a cocycle of the form $(0, -\theta)$ with $\theta$ a closed $(l - 1)$-form on $\mathbb{R}^l \setminus \{0\}$.

**Corollary 3.12** The Thom class $\Psi_{\mathbb{R}^l}$ of the bundle $\mathbb{R}^l \to \{\text{pt.}\}$ is characterized as a class in $H_D^l(\mathbb{R}^l, \mathbb{R}^l \setminus \{0\})$ that is represented by a cocycle $(0, -\theta)$ with $\int_{\partial R_l} \theta = 1$.

As a particular choice for $\theta$ as above, we have the angular form $\psi_l$ on $\mathbb{R}^l = \{(t_1, \ldots, t_l)\}$. It is given by

$$\psi_l = C_l \sum_{i=1}^{l} \frac{\Phi_i(t)}{||t||^l}, \quad \Phi_i(t) = (-1)^{i-1} t_i dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_l,$$

where $\widehat{\cdot}$ means the form under it is to be omitted. The constant $C_l$ above is given by

$$\frac{(k-1)!}{2\pi^k l} \quad \text{if} \quad l = 2k \quad \text{and} \quad \frac{(2k)!}{2\pi^k k!} \quad \text{if} \quad l = 2k + 1.$$ 

In particular,

$$\psi_l = \frac{1}{2} \frac{t}{||t||^l}.$$ 

The important fact is that the form is defined and closed in $\mathbb{R}^l \setminus \{0\}$ and, if we orient $\mathbb{R}^l$ so that $(t_1, \ldots, t_l)$ is a positive coordinate system, $\int_{S^{l-1}} \psi_l = 1$ for an $(l - 1)$-sphere $S^{l-1}$ around 0 in $\mathbb{R}^l$, oriented as the boundary of an $l$-ball.

We may think of Corollary 3.12 also as the characterization of the Thom class $\Psi_E^l$ of the product bundle $E = M \times \mathbb{R}^l$. Letting $W_0 = E \setminus M$ and $W_1 = E$, we consider the coverings $\mathcal{W} = \{W_0, W_1\}$ and $\mathcal{W}' = \{W_0\}$ of $E$ and $E \setminus M$. Then we have:

**Proposition 3.14** For the product bundle $E = M \times \mathbb{R}^l$, whose fiber $\mathbb{R}^l$ is oriented as above, the cocycle $(0, -\psi_l) \in \delta^{(l)}(\mathcal{W}, \mathcal{W}')$ represents the Thom class $\Psi_E^l$. 

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4 Relative Dolbeault cohomology

Čech-Dolbeault cohomology and relative Dolbeault cohomology are defined the same way as in the de Rham case, replacing the de Rham complex with the Dolbeault complex. In this section we recall the relevant part of the theory and refer to [35], [36], [37] for details.

Let $X$ be a complex manifold of dimension $n$. We denote by $\mathcal{E}^{(p,q)}_X$ and $\mathcal{G}^{(p)}_X$ the sheaves of $C^\infty$ $(p,q)$-forms and holomorphic $p$-forms on $X$, respectively.

4.1 Čech-Dolbeault cohomology

**Dolbeault cohomology**: The Dolbeault cohomology $H^{p,q}_\bar{\partial}(X)$ of $X$ of type $(p,q)$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(p,\bullet)}(X), \bar{\partial}), \bar{\partial} : \mathcal{E}^{(p,q)}(X) \rightarrow \mathcal{E}^{(p,q+1)}(X)$. The Dolbeault theorem says that there is an isomorphism

$$H^{p,q}_\bar{\partial}(X) \simeq H^q(X, \mathcal{G}^{(p)}). \quad (4.1)$$

Note that among the isomorphisms, there is a canonical one.

**Čech-Dolbeault cohomology**: The Čech-Dolbeault cohomology may be defined for an arbitrary covering of a complex manifold. Here we recall the case of coverings consisting of two open sets.

Let $\mathcal{V} = \{V_0, V_1\}$ be an open covering of $X$ and set $V_{01} = V_0 \cap V_1$. We set

$$\mathcal{E}^{(p,q)}(\mathcal{V}) = \mathcal{E}^{(p,q)}(V_0) \oplus \mathcal{E}^{(p,q)}(V_1) \oplus \mathcal{E}^{(p,q-1)}(V_{01}).$$

Thus an element in $\mathcal{E}^{(p,q)}(\mathcal{V})$ is expressed by a triple $\xi = (\xi_0, \xi_1, \xi_{01})$. We define the differential

$$\bar{\partial} : \mathcal{E}^{(p,q)}(\mathcal{V}) \longrightarrow \mathcal{E}^{(p,q+1)}(\mathcal{V}) \quad \text{by} \quad \bar{\partial}(\xi_0, \xi_1, \xi_{01}) = (\bar{\partial}\xi_0, \bar{\partial}\xi_1, \xi_1 - \xi_0 - \bar{\partial}\xi_{01}).$$

Then we see that $\bar{\partial} \circ \bar{\partial} = 0$.

**Definition 4.2** The Čech-Dolbeault cohomology $H^{p,q}_\bar{\partial}(\mathcal{V})$ of $\mathcal{V}$ of type $(p,q)$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(p,\bullet)}(\mathcal{V}), \bar{\partial})$.

**Theorem 4.3** The inclusion $\mathcal{E}^{(p,q)}(X) \hookrightarrow \mathcal{E}^{(p,q)}(\mathcal{V})$ given by $\omega \mapsto (\omega|_{V_0}, \omega|_{V_1}, 0)$ induces an isomorphism

$$H^{p,q}_\bar{\partial}(X) \sim \to H^{p,q}_\bar{\partial}(\mathcal{V}).$$

The inverse is induced from the assignment $(\xi_0, \xi_1, \xi_{01}) \mapsto \rho_0\xi_0 + \rho_1\xi_1 - \bar{\partial}\rho_0 \land \xi_{01}$.

4.2 Relative Dolbeault cohomology

Let $S$ be a closed set in $X$. Letting $V_0 = X \setminus S$ and $V_1$ a neighborhood of $S$ in $X$, we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of $X$ and $X \setminus S$. We set

$$\mathcal{E}^{(p,q)}(\mathcal{V}, \mathcal{V}') = \{ \xi \in \mathcal{E}^{(p,q)}(\mathcal{V}) \mid \xi_0 = 0 \} = \mathcal{E}^{(p,q)}(V_1) \oplus \mathcal{E}^{(p,q-1)}(V_{01}).$$

Then we see that $(\mathcal{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\partial})$ is a subcomplex of $(\mathcal{E}^{(p,\bullet)}(\mathcal{V}), \bar{\partial})$. 

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**Definition 4.4** The relative Dolbeault cohomology $H^{p,q}_\partial(V, V')$ of $(V, V')$ of type $(p, q)$ is the $q$-th cohomology of the complex $(\mathcal{E}^{(p, \bullet)}(V, V'), \partial)$.

The following exact sequence is obtained as (3.4):

$$
\cdots \longrightarrow H^{p,q}_\partial(V_0) \overset{\delta}{\longrightarrow} H^{p,q}_\partial(V, V') \overset{j^*}{\longrightarrow} H^{p,q}_\partial(V) \overset{i^*}{\longrightarrow} H^{p,q}_\partial(V_0) \longrightarrow \cdots ,
$$

(4.5)

where $\delta$ assigns to the class of $\theta$ the class of $(0, -\theta)$. As in the de Rham case, we have:

**Proposition 4.6** The cohomology $H^{p,q}_\partial(V, V')$ is determined uniquely, modulo canonical isomorphisms, independently of the choice of $V_1$.

In view of the above we denote $H^{p,q}_\partial(V, V')$ also by $H^{p,q}_\partial(X, X \setminus S)$.

**Proposition 4.7** For a triple $(X, X', X'')$, there is a long exact sequence

$$
\cdots \longrightarrow H^{p,q-1}_\partial(X', X'') \overset{\delta}{\longrightarrow} H^{p,q}_\partial(X, X') \overset{j^*}{\longrightarrow} H^{p,q}_\partial(X, X'') \overset{i^*}{\longrightarrow} H^{p,q}_\partial(X', X'') \longrightarrow \cdots .
$$

**Proposition 4.8 (Excision)** For any open set $V$ containing $S$, there is a canonical isomorphism

$$
H^{p,q}_\partial(X, X \setminus S) \simeq H^{p,q}_\partial(V, V \setminus S).
$$

The relative cohomology share other fundamental properties with the local cohomology (relative cohomology) of $X$ with coefficients in $\mathcal{O}^{(p)}$. In fact we have (cf. [36], [37]):

**Theorem 4.9 (Relative Dolbeault theorem)** There is a canonical isomorphism

$$
H^{p,q}_\partial(X, X \setminus S) \simeq H^q_S(X; \mathcal{O}^{(p)}).
$$

The excision in Proposition 4.8 is compatible with the excision in Proposition 2.1.2 for $\mathcal{I} = \mathcal{O}^{(p)}$ via the above isomorphism.

**Differential:** First, the map $\partial : \mathcal{E}^{(p, q)}(X) \rightarrow \mathcal{E}^{(p+1, q)}(X)$ given by $\omega \mapsto (-1)^q \partial \omega$ induces $\partial : H^{p,q}_\partial(X) \rightarrow H^{p+1,q}_\partial(X)$ and it is compatible with $d : H^q(X; \mathcal{O}^{(p)}) \rightarrow H^q(X; \mathcal{O}^{(p+1)})$ via the canonical isomorphism (4.1).

In the case $V = \{V_0, V_1\}$, $\partial : H^{p,q}_\partial(V) \rightarrow H^{p+1,q}_\partial(V)$ is induced by

$$(\xi_0, \xi_1, \xi_{01}) \mapsto (-1)^q (\partial \xi_0, \partial \xi_1, -\partial \xi_{01}).$$

In the case $V_0 = X'$ we have the differential

$$
\partial : H^{p,q}_\partial(X, X') \rightarrow H^{p+1,q}_\partial(X, X') \quad \text{induced by} \quad (\xi_1, \xi_{01}) \mapsto (-1)^q (\partial \xi_1, -\partial \xi_{01}).
$$

(4.10)

We have the following commutative diagram:

$$
\begin{array}{ccc}
H^{p,q}_\partial(X, X') & \overset{\partial}{\longrightarrow} & H^{p+1,q}_\partial(X, X') \\
\downarrow & & \downarrow \\
H^q(X, X'; \mathcal{O}^{(p)}) & \overset{d}{\longrightarrow} & H^q(X, X'; \mathcal{O}^{(p+1)}),
\end{array}
$$

where the vertical isomorphisms are the ones in Theorem 4.9.

**Remark 4.11** A relative cohomology such as the relative de Rham or relative Dolbeault cohomology defined above may also be interpreted as the cohomology of a complex dual to the mapping cone of a morphism of complexes in the theory of derived categories and a theorem as Theorems 3.7 or 4.9 may be proved from this viewpoint as well. This way we also see that this kind of relative cohomology goes well with derived functors (cf. [37]).
4.3 Relative de Rham and relative Dolbeault cohomologies

Let $X$ be a complex manifold of dimension $n$. We consider the following two cases where there is a natural relation between the two cohomology theories (cf. [36]).

(I) Noting that, for any $(n, q)$-form $\omega$, $\bar{\partial}\omega = d\omega$, there exist natural morphisms

$$H^{n,q}_\partial(X) \to H^{n+q}_d(X) \quad \text{and} \quad H^{n,q}_\partial(X, X \setminus S) \to H^{n+q}_D(X, X \setminus S). \quad (4.12)$$

In particular, this is used to define the integration on the relative Dolbeault cohomology in the subsequent section.

(II) We define $\rho^q : \mathcal{E}^{(q)} \to \mathcal{E}^{(0,q)}$ by assigning to a $q$-form $\omega$ its $(0, q)$-component $\omega^{(0,q)}$. Then $\rho^{q+1}(d\omega) = \bar{\partial}(\rho^q\omega)$ and we have a natural morphism of complexes

$$
\begin{array}{c}
0 \to \mathbb{C} \xrightarrow{\iota} \mathcal{E}^{(0)} \xrightarrow{d} \mathcal{E}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{(q)} \xrightarrow{d} \cdots \\
0 \to \mathcal{O} \xrightarrow{\rho^0} \mathcal{E}^{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,q)} \xrightarrow{\bar{\partial}} \cdots
\end{array}
$$

Thus there is a natural morphism $\rho^q : H^q_D(X, X') \to H^0\partial(X, X')$, which makes the following diagram commutative:

$$
\begin{array}{c}
H^q_D(X, X') \xrightarrow{\rho^q} H^0\partial(X, X') \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
H^q(X, X'; \mathbb{C}) \xrightarrow{\iota} H^q(X, X'; \mathcal{O}).
\end{array}
$$

Note that, if we take coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ with $V_0 = X'$ and $V_1$ a neighborhood of $X \setminus X'$, then $\rho^q : H^q_D(X, X') = H^q_D(\mathcal{V}, \mathcal{V}') \to H^0\partial(X, X') = H^0\partial(\mathcal{V}, \mathcal{V}')$ assigns to the class of $(\omega_0, \omega_0)$ the class of $(\omega_0^{(0,q)}, \omega_0^{(0,q-1)})$.

Recalling that we have the analytic de Rham complex

$$
0 \to \mathbb{C} \xrightarrow{\iota} \mathcal{O} \xrightarrow{d} \mathcal{O}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{O}^{(n)} \to 0,
$$

the above diagram is extended to an isomorphism of complexes

$$
\begin{array}{c}
0 \to H^q_D(X, X') \xrightarrow{\rho^q} H^0\partial(X, X') \xrightarrow{\bar{\partial}} H^1\partial(X, X') \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} H^n\partial(X, X') \xrightarrow{\bar{\partial}} 0 \\
0 \to H^q(X, X'; \mathbb{C}) \xrightarrow{\iota} H^q(X, X'; \mathcal{O}) \xrightarrow{d} H^q(X, X'; \mathcal{O}^{(1)}) \xrightarrow{d} \cdots \xrightarrow{d} H^q(X, X'; \mathcal{O}^{(n)}) \xrightarrow{d} 0.
\end{array}
$$

In the above situation we have:

**Theorem 4.14** If $H^q(X, X'; \mathcal{O}) = 0$ and $H^q(X, X'; \mathcal{O}^{(p)}) = 0$ for $p \geq 0$ and $q \neq q_0$, then the following sequence is exact:

$$
0 \to H^{q_0}(X, X'; \mathcal{O}) \xrightarrow{\iota} H^{q_0}(X, X'; \mathcal{O}) \xrightarrow{d} H^{q_0}(X, X'; \mathcal{O}^{(1)}) \xrightarrow{d} \cdots \xrightarrow{d} H^{q_0}(X, X'; \mathcal{O}^{(n)}) \to 0.
$$
5 Local duality morphism

We recall the cup product and integration theory on Čech-Dolbeault cohomology in the relevant case. Then we recall the local duality morphism.

Let $X$ be a complex manifold of dimension $n$ and $\mathcal{V} = \{V_0, V_1\}$ an open covering of $X$.

**Cup product:** We define the cup product

$$E^{(p,q)}(\mathcal{V}) \times E^{(p',q')}(\mathcal{V}) \to E^{(p+p',q+q')}(\mathcal{V}), \quad (\xi, \eta) \mapsto \xi \cup \eta$$

by

$$(\xi \cup \eta)_0 = \xi_0 \wedge \eta_0, \quad (\xi \cup \eta)_1 = \xi_1 \wedge \eta_1 \quad \text{and} \quad (\xi \cup \eta)_{01} = (-1)^{p+q} \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1.$$ 

Then $\xi \cup \eta$ is linear in $\xi$ and $\eta$ and we have

$$\bar{\partial}(\xi \cup \eta) = \bar{\partial} \xi \cup \eta + (-1)^{p+q} \xi \cup \bar{\partial} \eta.$$ 

Thus it induces the cup product

$$H^{p,q}_\bar{\partial}(\mathcal{V}) \times H^{p',q'}_\bar{\partial}(\mathcal{V}) \to H^{p+p',q+q'}_\bar{\partial}(\mathcal{V})$$

compatible, via the isomorphism of Theorem 4.3, with the product in the Dolbeault cohomology induced from the exterior product of forms.

Let $S$ be a closed set in $X$. Letting $V_0 = X \setminus S$ and $V_1$ a neighborhood of $S$, we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$. Then (5.1) induces a pairing

$$E^{(p,q)}(\mathcal{V}, \mathcal{V}') \times E^{(p',q')}(V_1) \to E^{(p+p',q+q')}(\mathcal{V}, \mathcal{V}')$$

assigning to $\xi = (\xi_1, \xi_{01})$ and $\eta_1$ the cochain $(\xi_1 \wedge \eta_1, \xi_{01} \wedge \eta_1)$. It induces a pairing

$$H^p_\bar{\partial}(X, X \setminus S) \times H^{p',q'}_\bar{\partial}(V_1) \to H^{p+p',q+q'}_\bar{\partial}(X, X \setminus S).$$

**Integration:** First we make the following

**Remark 5.3** Since $X$ is a complex manifold, it is always orientable. However the orientation we consider is not necessarily the “usual one”. Here we say an orientation of $X$ is usual if $(x_1, y_1, \ldots, x_n, y_n)$ is a positive coordinate system when $(z_1, \ldots, z_n)$, $z_i = x_i + \sqrt{-1}y_i$, is a complex coordinate system on $X$.

First we assume that $X$ is compact. If $X$ is oriented, from (4.12) and (3.9), we have the integration

$$H^n_\bar{\partial}(X) \xrightarrow{\int_X} \mathbb{C}.$$ 

In the case we do not specify the orientation, we define

$$H^n_\bar{\partial}(X) \otimes_{\mathbb{C}} \text{or}_X(X) \xrightarrow{\int_X} \mathbb{C}.$$
as follows. For simplicity we assume that $X$ is connected. It suffices to define it for a decomposable element $[\omega] \otimes a$. Once we fix an orientation, we have a canonical isomorphism $or_*X \simeq \mathbb{Z}$ so that $a$ determines an integer $n(a)$. On the other hand we have a well-defined integral $\int_X [\omega]$. We set

$$\int_X [\omega] \otimes a = n(a) \int_X [\omega].$$

If we take the opposite orientation for $X$, the above remains the same.

Suppose $K$ is a compact set in $X$ ($X$ may not be compact). Letting $V_0 = X \setminus K$ and $V_1$ a neighborhood of $K$, we consider the covering $\mathcal{V}_K = \{V_0, V_1\}$. Let $R_1$ and $R_{01}$ be as in Subsection 3.3. If $X$ is oriented, we have the integration (cf. (4.12) and (3.10))

$$H^n_{\bar{\partial}}(X, X \setminus K) \xrightarrow{\int_X} \mathbb{C} \quad \text{given by} \quad \int_X [\xi] = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}. \quad (5.4)$$

In the case we do not specify the orientation, we may define, as before, the integration

$$H^n_{\bar{\partial}}(X, X \setminus K) \otimes or_*X \xrightarrow{\int_X} \mathbb{C}. \quad (5.5)$$

**Duality morphisms:** We assume that $X$ is oriented for simplicity. If $X$ is compact, then the bilinear pairing

$$H^p_{\bar{\partial}}(X) \times H^{n-p-n-q}_{\bar{\partial}}(X) \xrightarrow{\wedge} H^n_{\bar{\partial}}(X) \xrightarrow{\int_X} \mathbb{C}$$

induces the Kodaira-Serre duality

$$KS_X: H^p_{\bar{\partial}}(X) \xrightarrow{\sim} H^{n-p-n-q}_{\bar{\partial}}(X)^*.$$ 

Let $K$ be a compact set in $X$ ($X$ may not be compact). The cup product (5.2) followed by the integration (5.4) gives a bilinear pairing

$$H^p_{\bar{\partial}}(X, X \setminus K) \times H^{n-p-n-q}_{\bar{\partial}}(V_1) \xrightarrow{\sim} H^p_{\bar{\partial}}(X, X \setminus K) \xrightarrow{\int_X} \mathbb{C}.$$ 

This induces a morphism

$$\bar{A}_{X,K}: H^p_{\bar{\partial}}(X, X \setminus K) \rightarrow H^{n-p-n-q}_{\bar{\partial}}[K]^* = \lim_{\nu \uparrow \supset K} H^{n-p-n-q}_{\bar{\partial}}(V_1)^*,$$ 

which we call the $\bar{\partial}$-**Alexander morphism.** Here we consider algebraic duals, however in order to have the duality, we need to take topological duals (cf. Theorem 6.10 below).

If $X$ is compact, we have the following commutative diagram:

$$
\begin{array}{ccc}
H^p_{\bar{\partial}}(X, X \setminus K) & \xrightarrow{j^*} & H^p_{\bar{\partial}}(X) \\
\downarrow \bar{A}_{X,K} & & \downarrow KS_X \\
H^{n-p-n-q}_{\bar{\partial}}[K]^* & \xrightarrow{j_*} & H^{n-p-n-q}_{\bar{\partial}}(X)^*.
\end{array}
$$
6 Hyperforms via relative Dolbeault cohomology

In this section we let $M$ denote a real analytic manifold of dimension $n$ and $X$ its complexification. We assume $M$ to be orientable so that $or_M$ is trivial, i.e., a constant sheaf. Thus $or_{M/X}$ is also trivial and, for any open set $U$ in $M$, the space of $p$-hyperforms is given by (cf. (2.16))

$$B^{(p)}_M(U) = H^n_U(V; \mathcal{O}_X^{(p)}) \otimes \mathcal{O}_M(U) or_{M/X}(U),$$

where $V$ is a complex neighborhood of $U$ in $X$.

Note that, in the above situation, there is an isomorphism $or_{M/X} \cong \mathbb{Z}_M$, however there are various choices of the isomorphism. Once we fix an orientation of $T_MX$, the isomorphism is determined uniquely.

6.1 Expressions of hyperforms and some basic operations

In the above situation there is a canonical isomorphism $H^n_U(V, \mathcal{O}_M^{(p)}) \cong H^{p,n}_{\bar{\partial}}(V, V \setminus U)$ (cf. Theorem 4.9) so that there is a canonical isomorphism

$$B^{(p)}(U) \cong H^{p,n}_{\bar{\partial}}(V, V \setminus U) \otimes \mathbb{Z}_M(U) or_{M/X}(U).$$

In the sequel we give explicit expressions of the classes in $H^{p,n}_{\bar{\partial}}(V, V \setminus U)$ and some of the basic operations on them. In fact, in the case the orientation of $T_MX$ is specified, there is a canonical isomorphism

$$B^{(p)}(U) \cong H^{p,n}_{\bar{\partial}}(V, V \setminus U)$$

and these may be thought of as giving descriptions for the hyperforms themselves.

One-dimensional case: Before we proceed further, we review the original expression of hyperfunctions in one-dimensional case by Sato, with some fundamental examples. In fact we will see below that our expression is in a sense a natural generalization of this. In particular, the integration we discuss in Subsection 6.2 may be thought of as a direct generalization of that in one-dimensional case.

Let $U$ be an open set in $\mathbb{R} = \{(x)\}$ and $V$ a complex neighborhood of $U$ in $\mathbb{C} = \{(z)\}$, $z = x + \sqrt{-1}y$. Here we fix the orientation of $\mathbb{C}$ so that $(y, x)$ is a positive coordinate system. The space of hyperfunctions on $U$ is originally defined by

$$B(U) = \mathcal{O}(V \setminus U)/\mathcal{O}(V).$$

Thus a hyperfunction is represented by a holomorphic function $F$ on $V \setminus U$. In our framework, it is represented by the pair $(0, -F)$ (cf. (6.2) below).

For example, the constant function 1, as a hyperfunction, is represented by such functions as

$$\psi = \begin{cases} \frac{1}{2}, & \psi_+ = \begin{cases} 1, & \psi_- = \begin{cases} 0, \quad \text{and so forth}, \end{cases} \end{cases} \end{cases}$$

where the value in the upper column is the value on $V_+ = \{ z \in V \mid y > 0 \}$ and that in the lower column the value on $V_- = \{ z \in V \mid y < 0 \}$. We will see later that $\psi$ is a natural
representation, while \( \psi_\pm \) is a representation that has support in \( V_\pm \). They correspond to the representations \( (0, -\psi) \) and \( (0, -\psi_\pm) \), respectively, in our framework (cf. Example 7.6 and Remark 7.7 below).

Also the \( \delta \)-function is represented by the function

\[
-\frac{1}{2\pi \sqrt{-1}} \frac{1}{z} = -\frac{1}{2\pi \sqrt{-1}} \frac{\psi_+ - \psi_-}{z},
\]

which is expressed as

\[
-\frac{1}{2\pi \sqrt{-1}} \left( \frac{1}{x + \sqrt{-1}0} - \frac{1}{x - \sqrt{-1}0} \right)
\]

to emphasize that it is the difference of the boundary values of holomorphic functions. In our framework, the \( \delta \)-function is represented by the pair (cf. Definition 6.13 and Example 7.17 below)

\[
(0, \frac{1}{2\pi \sqrt{-1}} \frac{1}{z}).
\]

Note that a direct proof of the fact that \( \mathcal{B}(U) \) is independent of the choice of \( V \) requires Runge’s theorem. In our framework, it follows from the excision property of the relative Dolbeault cohomology (cf. Proposition 4.8).

**Expression of hyperforms:** Coming back to the general situation, let \( U \) be an open set in \( M \) and \( V \) a complex neighborhood of \( U \) in \( X \), as above. Letting \( V_0 = V \setminus U \) and \( V_1 \) a neighborhood of \( U \) in \( V \) (it could be \( V \) itself), we consider the coverings \( V = \{ V_0, V_1 \} \) and \( \mathcal{V}' = \{ V_0 \} \) of \( V \) and \( V \setminus U \). Then \( H_{\partial}^{p,n}(V, V \setminus U) = H_{\partial}^{p,n}(V, \mathcal{V}') \) and a class in \( H_{\partial}^{p,n}(V, V \setminus U) \) is represented by a cocycle \( (\tau_1, \tau_{01}) \) with \( \tau_1 \) a \((p, n)\)-form on \( V_1 \), which is automatically \( \bar{\partial} \)-closed, and \( \tau_{01} \) a \((p, n - 1)\)-form on \( V_{01} \) such that \( \tau_1 = \bar{\partial}\tau_{01} \) on \( V_{01} \). We have the exact sequence (cf. (4.5))

\[
H_{\partial}^{p,n-1}(V) \longrightarrow H_{\partial}^{p,n-1}(V \setminus U) \overset{\delta}{\longrightarrow} H_{\partial}^{p,n}(V, V \setminus U) \overset{j^*}{\longrightarrow} H_{\partial}^{p,n}(V),
\]

where \( \delta \) assigns to the class of \( \theta \) the class of \((0, -\theta)\).

Here we quote the following (cf. [8]):

**Theorem 6.1 (Grauert)** Every real analytic manifold admits a fundamental system of neighborhoods consisting of Stein open sets in its complexification.

By the above theorem, we may further simplify the expression. Namely, if we take as \( V \) a Stein neighborhood, we have \( H_{\partial}^{p,n}(V) \simeq H^n(V; \mathcal{O}(\partial)) = 0 \). Thus \( \delta \) is surjective and every element in \( H_{\partial}^{p,n}(V, V \setminus U) \) is represented by a cocycle of the form \((0, -\theta)\) with \( \theta \) a \( \bar{\partial} \)-closed \((p, n - 1)\)-form on \( V \setminus U \).

In the case \( n > 1 \), \( H_{\partial}^{p,n-1}(V \setminus U) \simeq H^{n-1}(V; \mathcal{O}(\partial)) = 0 \) and \( \delta \) is an isomorphism:

\[
H_{\partial}^{p,n-1}(V \setminus U) \simeq H_{\partial}^{p,n}(V, V \setminus U), \quad [\theta] \leftrightarrow [(0, -\theta)].
\]

In the case \( n = 1 \), we have the exact sequence

\[
H_{\partial}^{p,0}(V) \longrightarrow H_{\partial}^{p,0}(V \setminus U) \overset{\delta}{\longrightarrow} H_{\partial}^{p,1}(V, V \setminus U) \longrightarrow 0,
\]
where \( H^p_0(V \setminus U) = H^0(V \setminus U; \mathcal{O}^{(p)}) \) and \( H^p_0(V) = H^0(V; \mathcal{O}^{(p)}) \). In particular, for \( p = 0 \), we have the isomorphism

\[
H^0(V \setminus U; \mathcal{O})/H^0(V; \mathcal{O}) \cong H^0_0(V, V \setminus U), \quad [F] \leftrightarrow [(0, -F)], \quad (6.2)
\]

where \( F \) is a holomorphic function on \( V \setminus U \). The left hand side is the original expression discussed above, while the right hand side is the expression in terms of relative Dolbeault cohomology.

**Remark 6.3** Although a hyperform may be represented by a single differential form in most of the cases, it is important to keep in mind that it is represented by a pair \((\tau_1, \tau_{01})\) of forms in general.

**Multiplication by real analytic functions:** Let \( \mathcal{A}_M \) denote the sheaf of real analytic functions on \( M \), which is given by \( \mathcal{A}_M = i^{-1}\mathcal{O}_X \) with \( i : M \hookrightarrow X \) the inclusion. We define the multiplication

\[
\mathcal{A}(U) \times H^p_0(V, V \setminus U) \longrightarrow H^p_0(V, V \setminus U)
\]

by assigning to \((f, [\tau])\) the class of \((\tilde{f}\tau_1, \tilde{f}\tau_{01})\) with \( \tilde{f} \) a holomorphic extension of \( f \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{A}(U) \times H^p_0(V, V \setminus U) & \longrightarrow & H^p_0(V, V \setminus U) \\
\downarrow & & \downarrow \\
\mathcal{A}(U) \times H^0_n(V; \mathcal{O}^{(p)}) & \longrightarrow & H^0_n(V; \mathcal{O}^{(p)}).
\end{array}
\]

**Partial derivatives:** Suppose that \( U \) is a coordinate neighborhood with coordinates \((x_1, \ldots, x_n)\). We define the partial derivative

\[
\frac{\partial}{\partial x_i} : H^0_n(V, V \setminus U) \longrightarrow H^0_n(V, V \setminus U)
\]

as follows. Let \((\tau_1, \tau_{01})\) represent a hyperfunction on \( U \). We write \( \tau_1 = f \, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \) and \( \tau_{01} = \sum_{j=1}^n g_j \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \). Then \( \frac{\partial}{\partial x_i}[\tau] \) is represented by the cocycle

\[
\left( \frac{\partial f}{\partial \bar{z}_i} \, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n, \sum_{j=1}^n \frac{\partial g_j}{\partial \bar{z}_i} \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \right).
\]

With this the following diagram is commutative:

\[
\begin{array}{ccc}
H^0_n(V, V \setminus U) & \xrightarrow{\frac{\partial}{\partial x_i}} & H^0_n(V, V \setminus U) \\
\downarrow & & \downarrow \\
H^0_n(V; \mathcal{O}) & \xrightarrow{\frac{\partial}{\partial x_i}} & H^0_n(V; \mathcal{O}).
\end{array}
\]

Thus for a differential operator \( P(x, D) : H^0_n(V, V \setminus U) \to H^0_n(V, V \setminus U) \) is well-defined.
Differential: We define the differential (cf. (4.10), here we denote $\partial$ by $d$)

$$d : H^{p,n}_{\partial}(V, V \setminus U) \longrightarrow H^{p+1,n}_{\partial}(V, V \setminus U). \quad (6.4)$$

by assigning to the class of $(\tau_1, \tau_0)$ the class of $(-1)^n(\partial \tau_1, -\partial \tau_0)$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
H^{p,n}_{\partial}(V, V \setminus U) & \xrightarrow{d} & H^{p+1,n}_{\partial}(V, V \setminus U) \\
\downarrow & & \downarrow \\
H^p_U(V; \mathcal{O}^{(p)}) & \xrightarrow{d} & H^p_U(V; \mathcal{O}^{(p+1)}).
\end{array}
$$

The above operations are readily carried over to those for the hyperforms $\mathcal{B}^{(p)}(U) = H^{p,n}_{\partial}(V, V \setminus U) \otimes \text{or}_{M/X}(U)$.

In particular, since we have a canonical isomorphism (cf. (3.8))

$$H^p_{\partial}(V, V \setminus U) \otimes \text{or}_{M/X}(U) \simeq \mathcal{C}_M(U),$$

we have, from Theorem 4.14, an exact sequence of sheaves on $M$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{B} \xrightarrow{d} \mathcal{B}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{B}^{(n)} \longrightarrow 0.$$ 

We come back to this topic in Subsection 7.1 below.

6.2 Integration and related topics

Support of a hyperform: Let $U$ be an open set in $M$ and $K$ a compact set in $U$. We define the space $\mathcal{B}^{(p)}(U)$ of $p$-hyperforms on $U$ with support in $K$ as the kernel of the restriction $\mathcal{B}^{(p)}(U) \to \mathcal{B}^{(p)}(U \setminus K)$.

Proposition 6.5 For any open set $V$ in $X$ containing $K$, the cohomology $H^{p,n}_{\partial}(V, V \setminus K)$ may be thought of as a $\mathbb{Z}_M(U)$-module and there is a canonical isomorphism:

$$\mathcal{B}^{(p)}_K(U) \simeq H^{p,n}_{\partial}(V, V \setminus K) \otimes_{\mathbb{Z}_M(U)} \text{or}_{M/X}(U).$$

Proof: By Proposition 4.7 for the triple $(V, V \setminus K, V \setminus U)$, we have the exact sequence

$$H^{p,n-1}_{\partial}(V \setminus K, V \setminus U) \xrightarrow{\delta} H^{p,n}_{\partial}(V, V \setminus K) \xrightarrow{j^*} H^{p,n}_{\partial}(V, V \setminus U) \xrightarrow{i^*} H^{p,n}_{\partial}(V \setminus K, V \setminus U).$$

By Proposition 4.8, we may assume that $V$ is a complex neighborhood of $U$ and that each connected component of $V$ contains at most one connected component of $U$. This shows that each of the cohomologies in the sequence has a natural $\mathbb{Z}_M(U)$-module structure. Since $\text{or}_{M/X}(U) \simeq \mathbb{Z}_M(U)$, taking the tensor product with $\text{or}_{M/X}(U)$ over $\mathbb{Z}_M(U)$ is an exact functor. By definition, $H^{p,n}_{\partial}(V, V \setminus U) \otimes \text{or}_{M/X}(U) = \mathcal{B}^{(p)}(U)$. Noting that $V \setminus K$ is a complex neighborhood of $U \setminus K$ and that $V \setminus U = (V \setminus K) \setminus (U \setminus K)$, we have $H^{p,n}_{\partial}(V \setminus K, V \setminus U) \otimes \text{or}_{M/X}(U \setminus K) = \mathcal{B}^{(p)}(U \setminus K)$, where the tensor product is over $\mathbb{Z}_M(U \setminus K)$. Since the restriction

$$r^* : H^{p,n}_{\partial}(V \setminus K, V \setminus U) \otimes_{\mathbb{Z}_M(U)} \text{or}_{M/X}(U) \longrightarrow H^{p,n}_{\partial}(V \setminus K, V \setminus U) \otimes_{\mathbb{Z}_M(U \setminus K)} \text{or}_{M/X}(U \setminus K)$$

is injective, $\text{Ker} i^* = \text{Ker}(r^* \circ i^*)$. On the other hand, since $U \setminus K$ is pure $n$-codimensional in $V \setminus K$ with respect to $\mathcal{O}_X^{(p)}$, we have $H^{p,n-1}_{\partial}(V \setminus K, V \setminus U) \simeq H^{n-1}_{U \setminus K}(V \setminus K; \mathcal{O}_X^{(p)}) = 0$. □

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Remark 6.6 By the flabbiness of $\mathcal{B}^{(p)}$, we have the following exact sequence:

$$0 \rightarrow \mathcal{B}^{(p)}_K(U) \rightarrow \mathcal{B}^{(p)}(U) \rightarrow \mathcal{B}^{(p)}(U \setminus K) \rightarrow 0.$$ 

**Orientation convention:** Recall the sequence (2.11) and the isomorphisms (2.13) with $M$ and $X$ as above. Note that, since we assumed that $M$ is orientable, all the orientation sheaves involved are trivial. Let $(z_1, \ldots, z_n)$, $z_i = x_i + \sqrt{-1}y_i$, be a complex coordinate system on $X$ such that $(x_1, \ldots, x_n)$ is a coordinate system on $M$. Thus $(\varphi(\frac{\partial}{\partial y_1}), \ldots, \varphi(\frac{\partial}{\partial y_n}))$ is a frame of $T_M X$. When we orient $X$, $M$ and $T_M X$, we adopt the following:

**Convention 1.** Let $\psi_M$ and $\psi_M/X$ be prescribed orientations of $M$ and $T_M X$, respectively. Then we take the orientation $\psi$ of $X$ so that, if $(x_1, \ldots, x_n)$ is a positive coordinate system on $M$ and if $(\varphi(\frac{\partial}{\partial y_1}), \ldots, \varphi(\frac{\partial}{\partial y_n}))$ is a positive frame of $T_M X$, then $(y_1, \ldots, y_n, x_1, \ldots, x_n)$ is a positive coordinate system on $X$.

The above amounts to saying that we make identification

$$i^{-1}or_X = or_{M/X} \otimes or_M \quad \text{by} \quad i^{-1}\psi_X \leftrightarrow \psi_{M/X} \otimes \psi_M \quad (6.7)$$

with $\psi_{M/X}$, $\psi_M$ and $\psi_X$ as in Convention 1.

In the case $M$ is an open set in $\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$, we may think of $X$ as an open set in $\mathbb{C}^n = \{(z_1, \ldots, z_n)\}$. In this case, $T_M X$ is trivial and we make identification $T_M X = M \times \mathbb{R}^n_y$, $\mathbb{R}^n_y = \{(y_1, \ldots, y_n)\}$, by $\sum_{i=1}^n \eta_i \varphi(\frac{\partial}{\partial y_i}) \leftrightarrow (x; \eta_1(x), \ldots, \eta_n(x)), x \in M$.

Remark 6.8 1. Let $\psi_M$, $\psi_{M/X}$ and $\psi_X$ be as in Convention 1. Then the difference between $\psi_X$ and the usual orientation of $X$ (cf. Remark 5.3) is a sign of $(-1)^{\frac{n(n+1)}{2}}$.

2. The choice of $\psi_X$ as above is natural from the fiber bundle viewpoint in the following sense. Note that $\psi_{M/X}$ above is the Thom class $\Psi_M \in H^n_M(X; \mathbb{Z})$ of $M$ in $X$ (cf. Subsection 2.3). Recall that the Alexander isomorphism

$$A : H^n_M(X; \mathbb{Z}) = H^n(X, X \setminus M) \simeq H_n(M; \mathbb{Z})$$

is given by the left cap product with the fundamental class $[X]$. If we take $\psi_X$ as the orientation of $X$, then $A(\psi_M) = [M]$, the fundamental class of $[M]$ with orientation $\psi_M$.

**Integration:** We consider the sheaf of real analytic densities on $M$:

$$\mathcal{V}^*_M = \mathcal{S}^{(n)}_M \otimes c_M or_M. $$

The sheaf of hyperdensities is defined by

$$\mathcal{W}^*_M = \mathcal{B}_M \otimes c_M^{(n)} \mathcal{V}^*_M = \mathcal{B}^{(n)}_M \otimes c_M or_M. $$

We define

$$\Gamma_c(M; \mathcal{W}^*_M) \xrightarrow{f_M} \mathbb{C}$$

as follows. We assume $M$ (and $X$) to be connected for simplicity.

For any compact set $K$ in $M$, we have, by the identification (6.7),

$$\Gamma_K(M; \mathcal{W}^*_M) = H^{n,n}_{\bar{\partial}}(X, X \setminus K) \otimes or_{M/X}(M) \otimes or_M(M) = H^{n,n}_{\bar{\partial}}(X, X \setminus K) \otimes i^{-1}or_X(M),$$
where the tensor products are over $\mathbb{Z}_M(M) = \mathbb{Z}$. Thus we have the integration

$$\Gamma_K(M; \mathcal{W}_M) \xrightarrow{f_M} \mathbb{C}$$

as defined in (5.5), which we recall for the sake of completeness. Given

$$u \otimes a \in H^{n,n}_\partial(X, X \setminus K) \otimes i^{-1} \text{or}_X(M).$$

Once we fix an orientation of $X$, we have a well-defined integer $n(a)$ as we saw before (cf. Section 5). If we take the opposite orientation of $X$, the sign changes. Letting $V_0 = X \setminus K$ and $V_1 = X$, consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{W}_K' = \{V_0\}$. Then $u \in H^{n,n}_\partial(X, X \setminus K) = H^{n,n}_\partial(\mathcal{V}_K, \mathcal{W}_K')$ is represented by

$$\tau = (\tau_1, \tau_{01}) \in \mathcal{E}^{(n,n)}(\mathcal{V}_K, \mathcal{W}_K') = \mathcal{E}^{(n,n)}(V_1) \oplus \mathcal{E}^{(n,n-1)}(V_{01}).$$

Let $R_1$ be a $2n$-dimensional compact manifold with $C^\infty$ boundary in $X$ containing $K$ in its interior and set $R_{01} = -\partial R_1$. Then, once we fix an orientation of $X$, we may define

$$\int_M u = \int_{R_1} \tau_1 + \int_{R_{01}} \tau_{01}.$$

If we take the opposite orientation of $X$, the sign changes. Thus

$$\int_M u \otimes a = n(a) \int_M u$$

does not depend on the choice of the orientation of $X$.

**Remark 6.9** In the case $n = 1$, the above definition is consistent with the original one under the correspondence (6.2).

**A theorem of Martineau:** The following theorem of A. Martineau [26] (see also [12], [20]) may be interpreted in our framework as one of the cases where the $\partial$-Alexander morphism (cf. (5.6)) is an isomorphism with topological duals so that the duality pairing is given by the cup product followed by integration as described above. See [36] for a little more detailed discussions on this. The essential point of the proof is that the Serre duality holds for $\mathcal{W} \setminus K$, which is a consequence of a result of B. Malgrange [25].

In the below we assume that $\mathbb{C}^n$ is oriented, however the orientation may not be the usual one.

**Theorem 6.10** Let $K$ be a compact set in $\mathbb{C}^n$ such that $H^{p,q}_\partial[K] = 0$ for $q \geq 1$. Then for any open set $V \supset K$, $H^{p,q}_\partial(V, V \setminus K)$ and $H^{n-p,n-q}_\partial[K]$ admits natural structures of FS and DFS spaces, respectively, and we have:

$$\bar{A} : H^{p,q}_\partial(V, V \setminus K) \xrightarrow{\sim} H^{n-p,n-q}_\partial[K]' = \begin{cases} 0 & q \neq n \\ \mathcal{E}^{(n-p)}[K]' & q = n, \end{cases}$$

where $'$ denotes the strong dual.
Theorem 6.10 we have, without specifying the orientation of $C$, so that $C$ is the Bochner-Martinelli form on $V$ where $V$ and $q$ framework the duality (in the case $\nabla$).

Note that the hypothesis $H^\otimes_q[K] = 0$, $q \geq 1$, is satisfied if $K$ is a subset of $\mathbb{R}^n$ by Theorem 6.1.

Suppose $K \subset \mathbb{R}^n$ and denote by $\mathcal{A}^{(p)}$ the sheaf of real analytic $p$-forms on $\mathbb{R}^n$. Then

$$\mathcal{G}^{(p)}[K] = \lim_{V_1 \supset K} \mathcal{G}^{(p)}(V_1) \simeq \lim_{V_1 \supset K} \mathcal{A}^{(p)}(U_1) = \mathcal{A}^{(p)}[K],$$

where $V_1$ runs through neighborhoods of $K$ in $\mathbb{C}^n$ and $U_1 = V_1 \cap \mathbb{R}^n$. In the sequel we set $X = \mathbb{C}^n$ and $M = \mathbb{R}^n$. Then noting that $\mathcal{B}^{(p)}_K(U) = H^\otimes_q(V, V \setminus K) \otimes \text{or}_M(U)$, from Theorem 6.10 we have, without specifying the orientation of $\mathbb{C}^n$ or of $\mathbb{R}^n$,

**Corollary 6.12** For any open set $U \subset \mathbb{R}^n$ containing $K$, there is an isomorphism

$$\mathcal{B}^{(p)}_K(U) \simeq (\mathcal{A}^{(n-p)}[K] \otimes \text{or}_M(U))^\prime.$$

Note that the duality pairing above is given by (cf. (6.11))

$$\mathcal{B}^{(p)}_K(U) \times (\mathcal{A}^{(n-p)}[K] \otimes \text{or}_M(U)) \longrightarrow H^\otimes_q(V, V \setminus K) \otimes i^{-1} \text{or}_X(U) \rightarrow \mathbb{C}.$$

**Delta function:** We consider the case $K = \{0\} \subset \mathbb{R}^n$. We set

$$\Phi(z) = dz_1 \wedge \ldots \wedge dz_n \quad \text{and} \quad \Phi_i(z) = (-1)^{i-1}z_i dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n.$$ 

The 0-Bochner-Martinelli form on $\mathbb{C}^n \setminus \{0\}$ is defined as

$$\beta_0^n = C_n' \sum_{i=1}^{n} \Phi_i(z) \overline{\Phi_i(z)}, \quad C_n' = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{\pi}^n},$$

so that

$$\beta_n = \beta_0^n \wedge \Phi(z)$$

is the Bochner-Martinelli form on $\mathbb{C}^n \setminus \{0\}$. Note that

$$\beta_1^0 = \frac{1}{2\pi \sqrt{-1} z} \quad \text{and} \quad \beta_1 = \frac{1}{2\pi \sqrt{-1}} \frac{dz}{z}.$$ 

We denote by $\psi_{M/X}$ the section of $\text{or}_{M/X}$ that corresponds to $+1$ when we choose $(y_1, \ldots, y_n)$ as a positive coordinate system in the normal direction.
**Definition 6.13** The *delta function* \( \delta(x) \) is the hyperfunction in

\[
\mathcal{B}_{\{0\}}(\mathbb{R}^n) = H^{0,n}_{0}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}) \otimes or_{M/X}(\mathbb{R}^n)
\]
that is represented by

\[
(0, -(-1)^{n(n+1)/2} \beta_n^0) \otimes \psi_{M/X}.
\]

Recall the isomorphism in Corollary 6.12 in this case:

\[
\mathcal{B}_{\{0\}}(\mathbb{R}^n) \simeq (\mathcal{A}_0^{(n)} \otimes or_{M}(\mathbb{R}^n))'.
\]

For a representative \( h(x) \Phi(x) \) of a germ in \( \mathcal{A}_0^{(n)} \), \( h(z) \Phi(z) \) is its complex representative.

Let \( \psi_M \) denote the section of \( or_{M} \) that corresponds to +1 when we choose \( (x_1, \ldots, x_n) \) as a positive coordinate system on \( \mathbb{R}^n \). Thus, by Convention 1, \( \psi_{M/X} \otimes \psi_M \) is identified with \( i^{-1} \psi_X \), where \( \psi_X \) is the section of \( or_X \) that corresponds to +1 when we choose \( (y_1, \ldots, y_n, x_1, \ldots, x_n) \) as a positive coordinate system on \( \mathbb{C}^n \). Let \( R_1 \) be a closed ball around 0 in \( \mathbb{C}^n \) so that \( R_{01} = -\partial R_1 = -S^{2n-1} \). Here \( S^{2n-1} \) is a \((2n-1)\)-sphere, whose orientation may not be the usual one.

**Theorem 6.14** The *delta function* \( \delta(x) \) is the hyperfunction that assigns the value \( h(0) \) to a representative \( \omega \otimes \psi_M \), \( \omega = h(x) \Phi(x) \).

**Proof:** By definition \( \delta(x) \) assigns to \( \omega \otimes \psi_M \) the value

\[
-(-1)^{n(n+1)/2} n(\psi_{M/X} \otimes \psi_M) \int_{R_{01}} h(z) \beta_n.
\]

If we take the usual orientation for \( \mathbb{C}^n \), \( n(\psi_{M/X} \otimes \psi_M) = (-1)^{n(n+1)/2} \) and the above is equal to

\[
- \int_{R_{01}} h(z) \beta_n = \int_{S^{2n-1}} h(z) \beta_n = h(0),
\]

where \( S^{2n-1} \) is the sphere with the usual orientation. \( \square \)

**Delta form:** We again consider the case \( K = \{0\} \subset \mathbb{R}^n \).

**Definition 6.15** The *delta form* \( \delta^{(n)}(x) \) is the \( n \)-hyperform in

\[
\mathcal{B}_{\{0\}}^{(n)}(\mathbb{R}^n) = H^{0,n}_{\bar{0}}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}) \otimes or_{M/X}(\mathbb{R}^n)
\]
that is represented by

\[
(0, -(-1)^{n(n+1)/2} \beta_n) \otimes \psi_{M/X}.
\]

Recall the isomorphism in Corollary 6.12 in this case:

\[
\mathcal{B}_{\{0\}}^{(n)}(\mathbb{R}^n) \simeq (\mathcal{A}_0 \otimes or_{M}(\mathbb{R}^n))'.
\]

For a representative \( h(x) \) of a germ in \( \mathcal{A}_0 \), \( h(z) \) is its complex representative.
Theorem 6.16 The delta form $\delta^{(n)}(x)$ is the $n$-hyperform that assigns the value $h(0)$ to a representative $h(x) \otimes \psi_M$.

Let us compare the above description with the traditional way of expressing the delta function. The difference becomes apparent in the case $n > 1$ and we consider this case. We also choose, for simplicity, the orientation of $\mathbb{C}^n$ so that the usual system is positive. Set $W_i = \{z_i \neq 0\}, i = 1, \ldots, n$, and consider the coverings $\mathcal{W} = \{\mathbb{C}^n, W_i\}$ and $\mathcal{W}' = \{W_i\}$ of $\mathbb{C}^n$ and $\mathbb{C}^n \setminus \{0\}$, which are Stein coverings. Then there is a canonical isomorphism $H^0_{(0)}(\mathbb{C}^n; \mathcal{O}) \simeq H^0(\mathcal{W}, \mathcal{W}'; \mathcal{O})$, the relative Čech cohomology. In our case, the long exact sequence for Čech cohomology yields the isomorphism $H^n(\mathcal{W}, \mathcal{W}'; \mathcal{O}) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O})$.

On the other hand, we have the canonical Dolbeault isomorphism $H^{0,n-1}(\mathbb{C}^n \setminus \{0\}) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O})$, under which the class of $\beta_0^n$ corresponds to the class of $\omega_n^0 = (-1)^{\frac{n(n-1)}{2}} \omega_n^n = \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \frac{1}{z_1 \cdots z_n}$ (cf. [36], see also [13] where this correspondence is first studied in general, with a different sign convention). The class corresponding to $[\omega_0^n]$ in $H^n(\mathcal{W}, \mathcal{W}'; \mathcal{O})$ is the traditional delta function. We set $\omega_n = \omega_0^n \Phi(z)$.

Let $\Gamma$ be the $n$-cycle $\bigcap_{i=1}^n \{|z_i| = \varepsilon\}$ oriented so that $\arg(z_1) \wedge \cdots \wedge \arg(z_n)$ is positive. Let $S^{2n-1}$ denote the sphere of radius $\sqrt{n}\varepsilon$ with usual orientation. Then, for a holomorphic function $h$ on a neighborhood of $0$, we have

$$\int_{S^{2n-1}} h(z) \beta_n = \int_{\Gamma} h(z) \omega_n$$

and either way we have the value $h(0)$. Note that the right hand side is a special case of Grothendieck residues. See, e.g., [40] for applications of the above residue pairing from the computational aspect.

7 Further operations

In this section, we let $M$ be a real analytic manifold of dimension $n > 0$ and $X$ its complexification. We assume, for simplicity, $M$ to be orientable.

7.1 Embedding of real analytic functions

The embedding of the sheaf $\mathcal{A}_M$ of real analytic functions on $M$ into the sheaf $\mathcal{B}_M$ of hyperfunctions or more generally the embedding of the sheaf of real analytic forms $\mathcal{A}_M^{(p)}$ into the sheaf of hyperforms $\mathcal{B}_M^{(p)}$ is determined by the canonical identification of the constant function $1$ as a hyperfunction.

From the canonical identification $\mathbb{Z}_M = \text{or}_{M/X} \otimes \text{or}_{M/X}$ and the canonical morphism $\text{or}_{M/X} = \mathcal{H}_M(\mathcal{O}_X) \to \mathcal{H}_M(\mathcal{O}_X)$, we have the canonical morphism:

$$\mathbb{Z}_M = \text{or}_{M/X} \otimes \text{or}_{M/X} \to \mathcal{B}_M = \mathcal{H}_M(\mathcal{O}_X) \otimes \text{or}_{M/X}.$$
The image of 1 by this morphism is the corresponding hyperfunction. We try to find it explicitly in our framework. For this we consider the complexification \( \partial_{M/X} = H^m_M(\mathbb{C}^X) \) of \( \partial_{M/X} \). Then the above morphism is extended as

\[
\mathbb{C}_M = \partial_{M/X} \otimes \partial_{M/X} \longrightarrow \mathcal{B}_M. \tag{7.1}
\]

Note that this is injective (cf. Theorems 2.14.1 and 4.14). We express the above morphism in terms of relative de Rham and Dolbeault cohomologies.

As \( M \) is assumed to be orientable, the sheaf \( \partial_{M/X} \) admits a global section which generates each stalk and for any of such sections \( \psi \in \partial_{M/X}(M) = H^m_M(X; \mathbb{Z}) \), we have

\[
\psi \otimes \psi = 1. \tag{7.2}
\]

We fix such a section and denote it by \( 1 \) hereafter. Note that it is what is referred to as the Thom class \( \Psi_M \) of \( M \) in \( X \) in Subsection 2.3. The image of \( 1 \) by the canonical morphism \( \partial_{M/X}(M) = H^m_M(X; \mathbb{Z}) \rightarrow \partial_{M/X}(M) = H^m_M(X; \mathbb{C}) \), which is injective, is denoted by \( 1^c \).

Let \( U \) be an open set in \( M \) and \( V \) a complex neighborhood of \( U \) in \( X \). Then from (7.1), we have the following commutative diagram (cf. (3.8) and (4.13)):

\[
\begin{array}{ccc}
\mathbb{C}_M(U) = H^m_U(V; \mathbb{C}) \otimes \partial_{M/X}(U) & \xrightarrow{i} & H^m_U(V; \mathcal{O}) \otimes \partial_{M/X}(U) = \mathcal{B}(U) \\
| & & | \\
H^n_D(V,V \setminus U) \otimes \partial_{M/X}(U) & \xrightarrow{\rho^c \otimes 1} & H^n_D(\bar{V},V \setminus U) \otimes \partial_{M/X}(U).
\end{array}
\]

Setting \( V_0 = V \setminus U \) and \( V_1 = V \), consider the coverings \( \mathcal{V} = \{V_0, V_1\} \) and \( \mathcal{V}' = \{V_0\} \). From the above considerations we have:

**Theorem 7.3** If \( (\nu_1, \nu_{01}) \) is a representative of \( 1^c \) in \( H^m_D(V,V \setminus U) = H^m_D(V, V') \), then the constant function \( 1 \) is identified with the hyperfunction represented by \( (\nu_1^{(0,n)}, \nu_{01}^{(0,n-1)}) \otimes 1 \) in \( \mathcal{B}(U) \cong H^{(0,n)}_D(V, V') \otimes \partial_{M/X}(U) \).

Note that the above identification of \( 1 \) does not depend on the choice of \( (\nu_1, \nu_{01}) \), as \( \rho^c : H^m_D(V,V') \rightarrow H^{(0,n)}_D(V, V') \) is a well-defined morphism. It does not depend on the choice of \( 1 \) either by (7.2).

Let \( \nu = (\nu_1, \nu_{01}) \) be as in the above theorem. We define a morphism

\[
\mathcal{G}^{(p)}(U) \longrightarrow \mathcal{B}^{(p)}(U) = H^{p,n}_D(V, V \setminus U) \otimes \partial_{M/X}(U)
\]

by assigning to \( \omega(x) \in \mathcal{G}^{(p)}(U) \) the class \( [\nu_1^{(0,n)} \wedge \omega(z), \nu_{01}^{(0,n-1)} \wedge \omega(z)] \otimes 1 \), where \( \omega(z) \) denotes the complexification of \( \omega(x) \). Note that \( (\nu_1^{(0,n)} \wedge \omega, \nu_{01}^{(0,n-1)} \wedge \omega) \) is a cocycle, as \( \omega \) is holomorphic. Then it can be readily shown that it does not depend on the choice of the generator \( 1 \) or its representative \( \nu \). Thus it induces a sheaf morphism \( \iota^{(p)} : \mathcal{G}^{(p)} \rightarrow \mathcal{B}^{(p)} \), which is injective. In particular in the case \( p = 0 \), we have:

**Corollary 7.4** The embedding \( \mathcal{G} \hookrightarrow \mathcal{B} \) is locally given by assigning to a real analytic function \( f \) the hyperfunction \( [(\tilde{f} \nu_1^{(0,n)}, \tilde{f} \nu_{01}^{(0,n-1)})] \otimes 1 \), where \( \tilde{f} \) is a complexification of \( f \).

The above embeddings are compatible with differentials:
Proposition 7.5 The following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{A}(p) & \xrightarrow{i(p)} & \mathcal{B}(p) \\
\downarrow d & & \downarrow d \\
\mathcal{A}(p+1) & \xrightarrow{i(p+1)} & \mathcal{B}(p+1).
\end{array}
\]

**Proof:** Recall that \(d: H^p_\partial(V, V \setminus U) \to H^{p+1}_\partial(V, V \setminus U)\) assigns to the class of \((\nu_1^{0,n} \wedge \omega, \nu_0^{1,(n-1)} \wedge \omega)\) the class of \((-1)^n(\partial(\nu_1^{0,n} \wedge \omega), -\partial(\nu_0^{0,(n-1)} \wedge \omega))\) (cf. (6.4)). From \(D(\nu_1, \nu_0) = 0\), we have \(\partial\nu_1^{0,n} + \bar{\partial}\nu_1^{1,(n-1)} = 0\) and \(\partial\nu_0^{0,(n-1)} + \bar{\partial}\nu_0^{1,(n-2)} = \nu_1^{1,n-1}\). Then, using \(\bar{\partial}\omega = 0\), we compute

\[
(-1)^n(\partial(\nu_1^{0,n} \wedge \omega), -\partial(\nu_0^{0,(n-1)} \wedge \omega)) = (\nu_1^{0,n} \wedge \partial\omega, \nu_0^{0,(n-1)} \wedge \partial\omega) + (-1)^{n-1} \bar{\partial}(\nu_1^{1,n-1} \wedge \omega, \nu_0^{1,(n-2)} \wedge \omega).
\]

Since \(\partial\omega(z)\) is the complexification of \(d\omega(x)\), we have the proposition. \(\square\)

We finish this subsection by giving particular representatives of \(\mathbf{1}^c\) and \(\rho^n(\mathbf{1}^c)\).

**Example 7.6** Let \(U\) and \(V\) coordinate neighborhoods with coordinates \((x_1, \ldots, x_n)\) and \((z_1, \ldots, z_n)\), \(z_i = x_i + \sqrt{-1}y_i\). We set \(V_0 = V \setminus U\) and \(V_1 = V\). We orient \(T_MX\) so that \((y_1, \ldots, y_n)\) is a positive fiber coordinate system and this specifies the generator \(\mathbf{1}\) of \(\text{or}_{M/X}(U)\).

Let

\[
\psi_n = C_n \sum_{i=1}^n (-1)^{i-1} y_i dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n / \|y\|^n
\]

be the angular form on \(\mathbb{R}^n_\mathbb{C}\) (cf. (3.13)). Then by Proposition 3.14,

\[
\nu = (0, -\psi_n) \in \mathcal{E}^{(n)}(V_1) \oplus \mathcal{E}^{(n-1)}(V_0) = \mathcal{E}^{(n)}(\mathcal{V}, \mathcal{V}')
\]

represents \(\mathbf{1}^c\). In this case, \(\rho^n(\nu) \in \mathcal{E}^{(n,0)}(\mathcal{V}, \mathcal{V}')\) is given by \((0, -\psi_n^{(0,n-1)})\), where \(\psi_n^{(0,n-1)}\) is the \((0, n-1)\)-component of \(\psi_n\). We may compute it using \(y_i = \frac{1}{\sqrt{-1}}(z_i - \bar{z}_i)\):

\[
\psi_n^{(0,n-1)} = (\sqrt{-1})^n C_n \sum_{i=1}^n (-1)^i (z_i - \bar{z}_i) d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n / \|z - \bar{z}\|^n
\]

In particular, in the case \(n = 1\),

\[
\psi_1^{(0,0)} = \psi_1 = \frac{1}{2} \frac{y}{|y|}.
\]

**Remark 7.7** In the case \(n = 1\), \((0, -\frac{1}{2} \frac{y}{|y|})\) represents both \(\mathbf{1}^c \in \text{or}^c_{M/X}(U)\) and \(\rho^1(\mathbf{1}^c) \in H^0_{\partial}(V, V \setminus U)\). They are also represented by \((0, -\psi_\pm), \psi_\pm = \frac{1}{2} \frac{y}{|y|} \pm \frac{1}{2}\). Note that the support of \(\psi_\pm\) in \(V \setminus U\) is \(V_\pm = \{\pm y > 0\}\) (cf. Lemma 7.10 and Example 7.17 below, \(\psi_\pm = \pm \psi^{0,1}_\Omega\) by the notation there). Any of those cocycles may be thought of as representing the generator \(\mathbf{1}\) of \(\text{or}_{M/X}(U)\), as \(\text{or}_{M/X} \to \text{or}^c_{M/X}\) is injective.

The contents of this subsection are generalized in the next subsection.
7.2 Boundary value morphism

The boundary value morphism is one of the most important tools in the theory of hyperfunctions, by which we can regard a holomorphic function on an open wedge along $M$ as a hyperfunction. In this subsection, we will define the boundary value morphism in the framework of relative de Rham and Dolbeault cohomologies.

We consider a pair $(V, \Omega)$ of an open neighborhood $V$ of $M$ in $X$ and an open set $\Omega$ in $X$ satisfying the following two conditions:

(B1) $\overline{\Omega} \supset M$.

(B2) The inclusion $(V \setminus \Omega) \setminus M \hookrightarrow V \setminus \Omega$ is a homotopy equivalence.

We give some examples of such pairs $(V, \Omega)$.

Example 7.8 If we take $\Omega$ to be $V$, the pair $(V, \Omega)$ satisfies the above the conditions, in particular, the condition (B2) is automatically satisfied as the both subsets are empty. This is the situation we considered in the previous subsection, where a real analytic function is regarded as a hyperfunction. This is a special case of boundary value morphism.

Example 7.9 Let $M$ be an open subset of $\mathbb{R}^n$ and $X = M \times \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$, $z_i = x_i + \sqrt{-1}y_i$. Let $V$ be an open neighborhood of $M$ such that $V \cap \{(x) \times \sqrt{-1}\mathbb{R}^n\}$ is convex for any $x \in M$, and let $\Gamma$ be a non-empty open cone in $\mathbb{R}^n\ y$ for which $\mathbb{R}^n\ \setminus (\Gamma \cup \{0\})$ is contractible. Then the pair of $V$ and $\Omega = (M \times \sqrt{-1}\Gamma) \cap V$ satisfies the conditions (B1) and (B2).

Let us now define the boundary value morphism:

$$b_\Omega : \mathcal{O}(\Omega) \rightarrow \mathcal{B}(M) = H^0_M(X; \mathcal{O}) \otimes_{\mathcal{O}_M(M)} \mathcal{O}_{M/X}(M) \cong H^0_{\bar{\partial}}(\mathcal{V}, \mathcal{V}') \otimes_{\mathcal{O}_M(M)} \mathcal{O}_{M/X}(M),$$

where $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ with $V_0 = V \setminus M$ and $V_1 = V$. To give a concrete representative of $b_\Omega(f)$ ($f \in \mathcal{O}(\Omega)$) in the last cohomology of the above diagram, we need some preparations.

Let $\mathcal{I} \in \mathcal{O}_{M/X}(M) = H^0_M(X; \mathcal{O}_X)$ and $\mathcal{I}^c \in \mathcal{O}_M^c(M) = H^0_M(X; \mathcal{O}_X) \cong H^0_D(\mathcal{V}, \mathcal{V}')$ be as in the previous subsection. The following lemma is crucial to our construction of $b_\Omega$:

Lemma 7.10 The class $\mathcal{I}^c \in H^0_D(\mathcal{V}, \mathcal{V}')$ has a representative

$$(\nu_1, \nu_0) \in \mathcal{O}(V_1) \otimes \mathcal{O}(V_0) = \mathcal{O}(\mathcal{V}, \mathcal{V}')$$

which satisfies $\text{Supp}_{\mathcal{V}_1}(\nu_1) \subset \Omega$ and $\text{Supp}_{\mathcal{V}_0}(\nu_0) \subset \Omega$.

Proof: Replacing $\Omega$ with $\Omega \cap V$ if necessary, we may assume $\Omega \subset V$. Let $j : \Omega \hookrightarrow V$ (resp. $i : V \setminus \Omega \hookrightarrow V$) be an open (resp. a closed) embedding. Then we have the exact sequence of sheaves on $V$:

$$0 \rightarrow j_* j^{-1}\mathcal{C}_V \rightarrow \mathcal{C}_V \rightarrow i_* i^{-1}\mathcal{C}_V \rightarrow 0,$$
where \( j_t \) denotes the direct image with proper supports. This yields the exact sequence
\[
\cdots \rightarrow H^q_M(V; i_* i^{-1} \mathbb{C}) \rightarrow H^q_M(V; j_* j^{-1} \mathbb{C}) \xrightarrow{\iota} H^q_M(V; \mathbb{C}) \rightarrow H^q_M(V; i_* i^{-1} \mathbb{C}) \rightarrow \cdots.
\]

We claim that
\[
H^q_M(V; i_* i^{-1} \mathbb{C}) = 0 \quad \text{for } q \geq 0 \quad (7.11)
\]
so that \( \iota \) is an isomorphism. For this we set \( \mathcal{F} = i_* i^{-1} \mathbb{C} \) and consider the exact sequence
\[
H^{q-1}(V; \mathcal{F}) \rightarrow H^{q-1}(V \setminus M; \mathcal{F}) \rightarrow H^q_M(V; \mathcal{F}) \rightarrow H^q(V; \mathcal{F}) \rightarrow H^q(V \setminus M; \mathcal{F}).
\]

In the above, \( H^q(V; \mathcal{F}) = H^q(V \setminus \Omega; \mathbb{C}) \) and \( H^q(V \setminus M; \mathcal{F}) = H^q((V \setminus \Omega) \setminus M; \mathbb{C}) \). Thus by the condition (B2) above, we have (7.11) and \( \iota \) is an isomorphism.

Let \( \mathcal{E}_V^{(*)} \) denote the de Rham complex on \( V \), which gives a fine resolution of \( \mathbb{C}_V \). Since any of the sections of \( j_* j^{-1} \mathcal{E}_V^{(q)} \) may be thought of as a \( q \)-form with support in the intersection of its domain of definition and \( \Omega \), the sheaf \( j_* j^{-1} \mathcal{E}_V^{(q)} \) admits a natural action of the sheaf \( \mathcal{E}_V \) of \( C^\infty \) functions on \( V \) and thus it is fine. Therefore the complex \( j_* j^{-1} \mathcal{E}_V^{(*)} \) gives a fine resolution of \( j_* j^{-1} \mathbb{C}_V \). We denote by \( d' \) its differential \( j_* j^{-1} d \), which is in fact the usual exterior derivative \( d \) on forms with support in \( \Omega \). We set \( D' = \delta + (-1)^* d' \) and consider the cohomology \( H^n_{D'}(\mathcal{V}, \mathcal{V}') \) of the complex \( (j_* j^{-1} \mathcal{E}_V^{(*)}(\mathcal{V}, \mathcal{V}'), D') \), which is defined similarly as the relative de Rham cohomology, replacing \( \mathcal{E}_V^{(*)} \) and \( D \) by \( j_* j^{-1} \mathcal{E}_V^{(*)} \) and \( D' \) in Definition 3.3. Then there is a canonical morphism \( H^n_{D'}(\mathcal{V}, \mathcal{V}') \rightarrow H^n_M(\mathcal{V}, \mathcal{V}') \). Moreover \( H^n_{D'}(\mathcal{V}, \mathcal{V}') \) is canonically isomorphic with \( H^n_{M}(\mathcal{V}; j_* j^{-1} \mathbb{C}) \) and we have the following commutative diagram (cf. [37]):

\[
\begin{array}{ccc}
H^q_M(\mathcal{V}; j_* j^{-1} \mathbb{C}) & \xrightarrow{\sim} & H^q_M(V; \mathbb{C}) \\
\downarrow & & \downarrow \\
H^n_{D'}(\mathcal{V}, \mathcal{V}') & \longrightarrow & H^n_{D}(\mathcal{V}, \mathcal{V}').
\end{array}
\]

In particular we have
\[
H^n_{D'}(\mathcal{V}, \mathcal{V}') \simeq H^n_{D}(\mathcal{V}, \mathcal{V}'), \quad (7.12)
\]
which assures the existence of a desired representative. \( \square \)

**Remark 7.13** In the above lemma, since \( \nu_{01} \) is a section defined only on \( V_{01} = V \setminus M \), its support is a closed set in \( V \setminus M \), however, it is not necessarily closed in \( V \).

Now we give some examples of representatives of \( \mathbf{1}^c \) described in Lemma 7.10. In the situation of Example 7.8, we already gave a particular example in Example 7.6.

**Example 7.14** Let us consider the situation described in Example 7.9. Here we may assume \( M = \mathbb{R}^n \), \( X = V = \mathbb{C}^n \) and \( \Omega = M \times \sqrt{-1} \Gamma \), as the other cases are obtained by restriction of this case. We set \( V_0 = X \setminus M \) and \( V_1 = X \) as usual.

We first take \( n \) linearly independent unit vectors \( \eta_1, \ldots, \eta_n \) in \( \mathbb{R}_y^n \) so that
\[
\bigcap_{1 \leq k \leq n} H_k \subset \Gamma
\]
holds, where we set \( H_k = \{ y \in \mathbb{R}^n_y \mid \langle y, \eta_k \rangle > 0 \} \). We also set \( \eta_{n+1} = - (\eta_1 + \cdots + \eta_n) \). Then, let \( \varphi_k, k = 1, \ldots, n + 1 \), be \( C^\infty \) functions on \( X \setminus M \) which satisfy
(1) $\text{Supp}_{X\setminus M}(\varphi_k) \subset M \times \sqrt{-1}H_k$ for any $k = 1, \ldots, n + 1$.

(2) $\sum_{k=1}^{n+1} \varphi_k = 1$ on $X \setminus M$.

Set $\nu_{01} = (-1)^n(n-1)! \chi_{H_{n+1}} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1}$, where $\chi_{H_{n+1}}$ is the anti-characteristic function of the set $H_{n+1}$, that is,

$$\chi_{H_{n+1}}(z) = \begin{cases} 0 & z \in H_{n+1}, \\ 1 & \text{otherwise}. \end{cases}$$

Then we can easily confirm that $\nu_{01} \in \mathcal{E}^{(n-1)}(X \setminus M)$ and

$$\text{Supp}_{X\setminus M}(\nu_{01}) \subset M \times \sqrt{-1} \bigcap_{1 \leq k \leq n} H_k \subset \Omega.$$ 

Furthermore, as will be shown in Lemma A.3 in Appendix,

$$\nu = (0, \nu_{01}) \in \mathcal{E}^{(n)}(V_1) \oplus \mathcal{E}^{(n-1)}(V_{01}) = \mathcal{E}^{(n)}(\mathcal{V}, \mathcal{V}').$$

gives the image of a positively oriented generator of $or_{M/X}(M)$ with the orientations on $M$ and $X$ as in Example 7.6. By definition of $\rho^n : \mathcal{E}^{(n)}(\mathcal{V}, \mathcal{V}') \to \mathcal{E}^{(0,n)}(\mathcal{V}, \mathcal{V}')$, we have

$$\rho^n(\nu) = (0, \nu_{01,(0,n-1)}) = (0, (-1)^n(n-1)! \chi_{H_{n+1}} \bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}) \in \mathcal{E}^{(0,n)}(\mathcal{V}, \mathcal{V}'),$$

which satisfies $\bar{\partial}\rho^n(\nu) = \rho^n(D(\nu)) = 0$.

Now we are ready to define the boundary value morphism

$$b_{\Omega} : \mathcal{E}(\Omega) \to H^{0,n}_{\partial} = \mathcal{E}^{0,n}(\mathcal{V}, \mathcal{V}') \otimes_{\mathbb{Z}_M(M)} or_{M/X}(M) = \mathcal{B}(M).$$

Let $\nu = (\nu_1, \nu_{01}) \in \mathcal{E}^{(n)}(\mathcal{V}, \mathcal{V}')$ be a representative of $1^c$ with $\text{Supp}_{V_1}(\nu_1) \subset \Omega$ and $\text{Supp}_{V_{01}}(\nu_{01}) \subset \Omega$ (cf. Lemma 7.10). Take $f \in \mathcal{E}(\Omega)$. Since $\text{Supp}_{V_1}(\rho^n(\nu_1)) \subset \Omega$ and $\text{Supp}_{V_{01}}(\rho^{n-1}(\nu_{01})) \subset \Omega$, we may regard $f\rho^n(\nu_1)$ as a $(0,n)$-form on $V_1$ and $f\rho^{n-1}(\nu_{01})$ as a $(0,n-1)$-form on $V_{01}$. Hence, we have

$$f\rho(\nu) = (f\rho^n(\nu_1), f\rho^{n-1}(\nu_{01})) \in \mathcal{E}^{(0,n)}(V_1) \oplus \mathcal{E}^{(0,n-1)}(V_{01}) = \mathcal{E}^{(0,n)}(\mathcal{V}, \mathcal{V}').$$

Moreover it is a $\bar{\partial}$-cocycle, as $f$ is holomorphic, and defines a class $[f\rho(\nu)] \in H^{0,n}_{\partial}(\mathcal{V}, \mathcal{V}')$. Then we define the boundary value morphism by

$$b_{\Omega}(f) = [f\rho(\nu)] \otimes 1 \in H^{0,n}_{\partial} \otimes_{\mathbb{Z}_M(M)} or_{M/X}(M).$$

**Lemma 7.15** The above $b_{\Omega}$ is well-defined.

**Proof:** Let $\nu' = (\nu_1', \nu'_{01}) \in \mathcal{E}^{(n)}(\mathcal{V}, \mathcal{V}')$ be another representative of $1^c$ with $\text{Supp}_{V_1}(\nu_1') \subset \Omega$ and $\text{Supp}_{V_{01}}(\nu'_{01}) \subset \Omega$. By (7.12), we can find $\omega = (\omega_1, \omega_{01}) \in \mathcal{E}^{(n-1)}(\mathcal{V}, \mathcal{V}')$ with $\text{Supp}_{V_1}(\omega_1) \subset \Omega$ and $\text{Supp}_{V_{01}}(\omega_{01}) \subset \Omega$ satisfying $\nu - \nu' = D\omega$. Then we have $f\rho(\omega) \in \mathcal{E}^{(0,n-1)}(\mathcal{V}, \mathcal{V}')$ and $f\rho(\nu) - f\rho(\nu') = \bar{\partial}(f\rho(\omega))$, which shows

$$[f\rho(\nu)] \otimes 1 = [f\rho(\nu')] \otimes 1 \text{ in } H^{0,n}_{\partial}(\mathcal{V}, \mathcal{V}').$$

Hence $b_{\Omega}$ does not depend on the choice of the representative of $1^c$. It does not depend on the choice of $1$ either by (7.2).

As a corollary, we have the following:
Corollary 7.16 Let $V' \subset V$ be an open neighborhood of $M$, and let $\Omega' \subset \Omega$ be an open subset in $X$. Assume that the pair $(V', \Omega')$ also satisfies the conditions $(B_1)$ and $(B_2)$. Then we have

$$b_{\Omega'}(f|_{\Omega'}) = b_{\Omega}(f), \quad f \in \mathcal{O}(\Omega).$$

Example 7.17 Let $M = \mathbb{R}$ and $X = \mathbb{C}$ with coordinate $z = x + \sqrt{-1}y$ and set $V_0 = \mathbb{C} \setminus \mathbb{R}$ and $V_1 = \mathbb{C}$. Note that $V_0$ is a disjoint union of $\Omega_{\pm} = \{ \pm y > 0 \}$. Define coverings $\mathcal{V} = \{ V_0, V_1 \}$ and $\mathcal{V}' = \{ V_0 \}$ as usual.

It is well-known that Dirac’s delta function is the difference of boundary values on $\Omega_+$ and $\Omega_-$ of \( \frac{-1}{2\pi \sqrt{-1}z} \), that is,

$$\delta(x) = b_{\Omega_+} \left( \frac{-1}{2\pi \sqrt{-1}z} \right) - b_{\Omega_-} \left( \frac{-1}{2\pi \sqrt{-1}z} \right).$$

Define the functions

$$\varphi_{\Omega_+}(z) = \begin{cases} 1 & z \in \Omega_+, \\ 0 & z \in \Omega_- \end{cases}, \quad \varphi_{\Omega_-}(z) = \begin{cases} 0 & z \in \Omega_+, \\ 1 & z \in \Omega_- \end{cases},$$

and set

$$\nu_{\Omega_\pm} = (0, -\varphi_{\Omega_\pm}) \in \mathcal{E}^{(1)}(V_1) \oplus \mathcal{E}^{(0)}(V_0) = \mathcal{E}^{(1)}(\mathcal{V}, \mathcal{V}').$$

Note that $\nu_{\Omega_\pm}$ may be thought of as representing both a generator $1$ of $or_{M/X}(M)$ and $\rho^!(1^e) \in H^{0,1}_\psi(V, \mathcal{V})$ (cf. Remark 7.7). Thus from the definition of $b_{\Omega_\pm} : \mathcal{O}(\Omega_\pm) \to \mathcal{B}(M)$, we have, for $F \in \mathcal{O}(\Omega_\pm)$,

$$b_{\Omega_\pm}(F) = \left[ (0, -F(z)\varphi_{\Omega_\pm}) \right] \otimes [\nu_{\Omega_\pm}] \in H^{0,1}_\psi(\mathcal{V}, \mathcal{V}') \otimes_{\mathcal{Z}_M(M)} or_{M/X}(M),$$

which is often written $F(x \pm \sqrt{-1}0)$ in Sato’s context. Hence we have

$$\delta(x) = \left[ \left( 0, \frac{1}{2\pi \sqrt{-1}z} \varphi_{\Omega_+} \right) \right] \otimes [\nu_{\Omega_+}] - \left[ \left( 0, \frac{1}{2\pi \sqrt{-1}z} \varphi_{\Omega_-} \right) \right] \otimes [\nu_{\Omega_-}]$$

$$= \frac{-1}{2\pi \sqrt{-1}} \left( \frac{1}{x + \sqrt{-1}0} - \frac{1}{x - \sqrt{-1}0} \right).$$

We may also express it as

$$\delta(x) = \left[ \left( 0, \frac{1}{2\pi \sqrt{-1}z} (\varphi_{\Omega_+} + \varphi_{\Omega_-}) \right) \right] \otimes [\nu_{\Omega_+}] = \left[ \left( 0, \frac{1}{2\pi \sqrt{-1}z} \right) \right] \otimes [\nu_{\Omega_+}]$$

in $H^{0,1}_\psi(\mathcal{V}, \mathcal{V}') \otimes_{\mathcal{Z}} or_{M/X}(M)$. Recall that (cf. Remark 7.7) if we orient the normal direction so that $y$ is a positive coordinate, $[\nu_{\Omega_+}]$ is the canonical generator of $or_{M/X}(M)$ and thus the above delta function coincides with the one in Definition 6.13 for $n = 1$.

If we fix the orientation as above, we have a canonical isomorphism

$$\mathbb{Z} = \mathbb{Z}_M(M) \simeq or_{M/X}(M) = H^1_M(X; \mathbb{Z}_X)$$

which sends $1 \in \mathbb{Z}$ to $[\nu_{\Omega_+}] \in or_{M/X}(M)$. If we identify $\mathcal{B}(M)$ with $H^1_M(X; \mathcal{O}) = H^{0,1}_\psi(\mathcal{V}, \mathcal{V}')$ via the above isomorphism, we have the expression

$$\delta(x) = \left[ \left( 0, \frac{1}{2\pi \sqrt{-1}z} \right) \right] \in H^{0,1}_\psi(\mathcal{V}, \mathcal{V}). \quad (7.18)$$
Remark 7.19 Since we always assume $M$ to be orientable, as we see in the above example, we may omit the relative orientation $or_{M/X}(M)$ in the definition of hyperfunctions via the isomorphism $Z_M(M) \simeq or_{M/X}(M)$. Here we fix the isomorphism so that $1 \in Z_M(M)$ is sent to the positively oriented generator of $or_{M/X}(M)$. This omission does not have much impact on usual treatment of hyperfunctions, however, some particular operations such as coordinate transformations or integration of hyperfunctions require special attention: For example, let us consider the coordinate transformation $x \mapsto -x$ in the above example. It follows from the definition that $\delta(x)$ remains unchanged by this. The element defined by (7.18), however, changes its sign under this transformation. Thus, if we omit the orientation sheaf, we are required to change the sign of a hyperfunction manually under a coordinate transformation reversing the orientation. For the integration of hyperfunction densities, a similar consideration is needed, see Subsection 7.6 for details.

### 7.3 Microlocal analyticity

We first recall the notion of microlocal analyticity of a hyperfunction ([29], [19]). Then we will give its interpretation in our framework. In this subsection, we assume that $X$ and $M$ are oriented so that we omit the orientation sheaves.

Let $T^*_M X$ denote the conormal bundle of $M$ in $X$, which is isomorphic to $\sqrt{-1}T^*M$, and $\pi : T^*_M X \to M$ the projection. We describe the spectral map $sp : \pi^{-1}\mathcal{B} \to \mathcal{C}$ in our frame work, where $\mathcal{C}$ denotes the sheaf of microfunctions on $T^*_M X$. Let $p_0 = (x_0; \sqrt{-1}\xi_0)$ be a point in $T^*_M X = \sqrt{-1}T^*M$. Then it is known that the stalk $\mathcal{C}_{p_0}$ at $p_0$ is given by the following formula (see Theorem 4.3.2 and Definition 11.5.1 in [19]): Under a $(C^1$-class) local trivialization near $x_0$, i.e., $(M, X) \simeq (\mathbb{R}^n_x, \mathbb{C}^n = \mathbb{R}^n_x \times \sqrt{-1}\mathbb{R}^n_y)$ near $x_0$, we have

$$\mathcal{C}_{p_0} = \lim_{V \in G} H^0_{\mathbb{R}^n_y \times \sqrt{-1}G}(V; \mathcal{O}),$$

where $V$ runs through open neighborhoods of $x_0$ and $G$ ranges through closed cones in $\mathbb{R}^n_y$ satisfying

$$G \setminus \{0\} \subset \{ y \in \mathbb{R}^n \mid \text{Re}(\sqrt{-1}y, \sqrt{-1}\xi_0) > 0 \} = \{ y \in \mathbb{R}^n \mid \langle y, \xi_0 \rangle < 0 \}. \quad (7.20)$$

Therefore, setting $W_{V,G} = \{ V \setminus (\mathbb{R}^n_x \times \sqrt{-1}G), V \}$ and $W'_{V,G} = \{ V \setminus (\mathbb{R}^n_x \times \sqrt{-1}G) \}$, we get the expression in our framework:

$$\mathcal{C}_{p_0} \simeq \lim_{V \in G} H^0_{\mathbb{R}^n_y}(W_{V,G}, W'_{V,G}).$$

We also set $W_V = \{ V_0 = V \setminus M, V_1 = V \}$ and $W'_V = \{ V_0 \}$. Then the spectral map at $p_0$

$$\mathcal{B}_{x_0} \simeq \lim_{V} H^0_{\mathbb{R}^n_y}(W_V, W'_V) \xrightarrow{sp} \mathcal{C}_{p_0} \simeq \lim_{V \in G} H^0_{\mathbb{R}^n_y}(W_{V,G}, W'_{V,G})$$

is simply the one canonically induced from the restriction map $V \setminus M \hookrightarrow V \setminus (\mathbb{R}^n_x \times \sqrt{-1}G)$.

**Definition 7.21** For a hyperfunction $u$ at $x_0$, we say that $u$ is *microlocally analytic* at $p_0$ if $sp(u)$ becomes zero at $p_0$ as a microfunction.
We have the following equivalent characterization:

**Proposition 7.22** Let \( u \) be a hyperfunction at \( x_0 \). Then \( u \) is microlocally analytic at \( p_0 \) if and only if there exist a closed cone \( G \) satisfying (7.20), an open neighborhood \( V \) of \( x_0 \) and a representative

\[
(\tau_0, \tau_1) \in \mathcal{E}^{(0,n)}(V_1) \oplus \mathcal{E}^{(0,n-1)}(V_0) = \mathcal{E}^{(0,n)}(W_V, W'_V)
\]
of \( u \) near \( x_0 \) which satisfies \( \tau_1 = 0 \) and \( \text{Supp}_{V \cap M}(\tau_0) \subset \mathbb{R}^n_x \times \sqrt{-1}G \).

**Proof:** Let \( u \) be represented by \((\xi_1, \xi_0) \in \mathcal{E}^{(0,n)}(W'_V, W_V)\). If \( u \) is microlocally analytic at \( p_0 \), there exist \( G' \) and a cochain \((\eta_1, \eta_0) \in \mathcal{E}^{(0,n-1)}(W'_V, W''_V)\) such that

\[
(\xi_1, \xi_0) = \bar{\partial}(\eta_1, \eta_0) = (\bar{\partial}\eta_1, \eta_1 - \bar{\partial}\eta_0)
\]
in \( \mathcal{E}^{(0,n)}(W'_V, W''_V) \),

where \( \xi_0 \) is to be restricted to \( V \setminus (\mathbb{R}^n_x \times \sqrt{-1}G') \). Let \( G \) be a closed cone with (7.20) containing \( G' \setminus \{0\} \) in its interior. Let \( \psi \) be a \( C^\infty \) function on \( V \setminus M \) such that \( \psi \equiv 1 \) on the complement of \( \mathbb{R}^n_x \times \sqrt{-1}\text{Int } G \) in \( V \setminus M \) and \( \psi \equiv 0 \) on \( \mathbb{C}^n_x \times \sqrt{-1}(G' \setminus \{0\}) \).

Note that the both sets are closed in \( V \setminus M \) and that such a \( \psi \) may be constructed making it “radially constant”. Then \( \psi\eta_0 \) is a \((0, n - 2)\)-form on \( V \setminus M \). Set \( \tau_1 = 0 \) and \( \tau_0 = \xi_0 - \eta_1 + \bar{\partial}(\psi\eta_0) \). Then \( \text{Supp}_{V \cap M}(\tau_0) \subset \mathbb{R}^n_x \times \sqrt{-1}G \) and

\[
(\xi_1, \xi_0) - (\tau_0, \tau_1) = \bar{\partial}(\eta_1, \psi\eta_0)
\]
in \( \mathcal{E}^{(0,n)}(W'_V, W''_V) \),

so that \( u \) is represented by \((\tau_1, \tau_0)\). \( \square \)

We denote by \( \text{SS}(u) \) the set of points in \( T^*_M X \) at which \( u \) is not microlocally analytic. By the construction of the boundary value morphism in the previous subsection and the definition of microlocal analyticity, we have:

**Proposition 7.23** Let \( M \) be an open subset of \( \mathbb{R}^n_x \) and \( X = M \times \sqrt{-1}\mathbb{R}^n_y \). Also let \( V \) be an open neighborhood of \( M \) in \( X \) and \( \Omega \) an open set in \( X \). Assume that \( \Omega \cap (\{x_0\} \times \sqrt{-1}\mathbb{R}^n_y) \) (resp. \( V \cap (\{x_0\} \times \sqrt{-1}\mathbb{R}^n_y) \)) is a non-empty convex cone (resp. a convex set) for any \( x_0 \in M \). Then we have

\[
\text{SS}(b_{\Omega \cap V}(f)) \subset \Omega^c, \quad f \in \mathcal{E}(\Omega \cap V),
\]

where \( \Omega^c \) is the polar set of \( \Omega \) defined by

\[
\bigcup_{x \in M} \{ \sqrt{-1}\xi \in (T^*_M X)_x \mid \langle \xi, y \rangle \geq 0 \text{ for any } y \text{ with } x + \sqrt{-1}y \in \Omega \} \subset T^*_M X.
\]

### 7.4 External product of hyperforms

For each \( k = 1, 2, \ldots, \ell \), let \( M_k \) be a real analytic manifold of dimension \( n_k \), \( X_k \) its complexification and \( p_k \) a non-negative integer. All the manifolds are assumed to be oriented, and thus, we omit the relative orientations \( \text{or}_{M_k/X_k}(M_k) \) in the definition of hyperforms \( \mathcal{B}^{(p_k)}(M_k) \) throughout this subsection. Set

\[
M = M_1 \times \cdots \times M_\ell, \quad X = X_1 \times \cdots \times X_\ell
\]
and denote by \( \pi_k : X \to X_k \) the canonical projection. We also set \( n = n_1 + \cdots + n_k \) and \( p = p_1 + \cdots + p_r \). For each \( k \), we consider the coverings \( \mathcal{V}_k = \{ V_{k,0}, V_{k,1} \} \) and \( \mathcal{V}'_k = \{ V_{k,0} \} \) of \( X_k \) and \( X_k \setminus M_k \) given by \( V_{k,0} = X_k \setminus M_k \) and \( V_{k,1} = X_k \). We set \( V_{k,01} = V_{k,0} \cap V_{k,1} \). We also consider coverings \( \mathcal{V} = \{ V_0, V_1 \} \) and \( \mathcal{V}' = \{ V_0 \} \) of \( X \) and \( X \setminus M \) with \( V_0 = X \setminus M \), \( V_1 = X \) and \( V_{01} = V_0 \cap V_1 \).

Given \( \ell \) hyperforms \( u_k \in \mathcal{B}^{(p_k)}(M_k) \), \( k = 1, \ldots, \ell \), and their representatives

\[
\tau_k = (\tau_{k,1}, \tau_{k,01}) \in \mathcal{E}^{(p_k,n_k)}(V_{k,1}) \oplus \mathcal{E}^{(p_k,n_k-1)}(V_{k,01}) = \mathcal{E}^{(p_k,n_k)}(V_k, \mathcal{V}'_k),
\]

we compute a concrete representative of the external product

\[
u = u_1 \times u_2 \times \cdots \times u_\ell \in \mathcal{B}^{(p)}(M)
\]
from the representatives \( \tau_k \). Let \( \varphi_k, k = 1, \ldots, \ell \), be \( C^\infty \)-functions on \( V_0 \) which satisfy

1. \( \text{Supp}_{V_0}(\varphi_k) \subset \pi_k^{-1}(V_{k,0}), k = 1, \ldots, \ell \),
2. \( \varphi_1 + \cdots + \varphi_\ell = 1 \) on \( V_0 \).

First we introduce two families of forms \((\bar{\partial}\varphi)_{j}\)'s and \( \tau_\alpha \)'s. Set \( \Lambda = \{ 1, 2, \ldots, \ell \} \). For \( \beta = (\beta_1, \ldots, \beta_k) \in \Lambda^k \), we define

\[
(\bar{\partial}\varphi)_{j} = \bar{\partial}\varphi_{\beta_1} \wedge \cdots \wedge \bar{\partial}\varphi_{\beta_k}.
\]

Note that \((\bar{\partial}\varphi)_{j}\) is a \( (0, k) \)-form on \( V_0 \) whose support is in \( \pi_{\beta_1}^{-1}(V_{01}) \cap \cdots \cap \pi_{\beta_k}^{-1}(V_{01}) \). Furthermore, for \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \Lambda^k \) with \( \alpha_1 < \alpha_2 < \cdots < \alpha_k \), we define \( \tau_\alpha \) by

\[
\tau_\alpha = (-1)^{\sigma(\alpha)} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_\ell,
\]
where, for \( j = 1, 2, \ldots, \ell \),

\[
\omega_j = \begin{cases} 
\tau_{j,01} & \text{if } \alpha \text{ contains the index } j, \\
\tau_{j,1} & \text{otherwise},
\end{cases}
\]

and

\[
\sigma(\alpha) = \frac{k(k-1)}{2} + \sum_{j=1}^{k} \sum_{i=1}^{\alpha_i-1} (n_i + p_i).
\]

We extend \( \tau_\alpha \) to general \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \Lambda^k \) in the usual way, that is, \( \tau_\alpha = \text{sgn}(\mu) \tau_{\mu(\alpha)} \) for any permutation \( \mu \) on \( \alpha \). It is easy to see that \( \tau_\alpha \) is a \( (p, n-k) \)-form defined on

\[
\pi_{\alpha_1}^{-1}(V_{01}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(V_{01}).
\]

Then, for \( i = 1, 2, \ldots, \ell \), we set

\[
\kappa_i = \sum_{\beta \in \Lambda^{k-1}} (\bar{\partial}\varphi)_{j} \wedge \tau_{\beta i} + \sum_{0 \leq k \leq \ell, \beta \in \Lambda^k} (\bar{\partial}\varphi)_{j} \wedge \left( \sum_{\lambda \in \Lambda, 1 \leq j \leq k+1} (-1)^{j+1} \varphi_\lambda \tau_{j} \lambda \beta i \right),
\]

where \( \lambda \beta i \) is the sequence \( \lambda \beta i \) with the \( j \)-th component removed. Here the first \( \lambda \) in \( \lambda \beta i \) is considered to be the 0-th component and the last \( i \) the \( (k+1) \)-st component.
Lemma 7.24 Each $\kappa_i$ is a $(p,n-1)$-form defined on $\pi_i^{-1}(V_i,0)$. Furthermore, for $i$ and $j$ in $\Lambda$, we have $\kappa_i = \kappa_j$ on $\pi_i^{-1}(V_i,0) \cap \pi_j^{-1}(V_j,0)$.

It follows from the above lemma and the fact $V_0 = \bigcup_{i \leq \ell} \pi_i^{-1}(V_i,0)$ that the family $\{\kappa_j\}_{j=1}^{\ell}$ determines a $(p,n-1)$-form on $V_0$, which is denoted by $\kappa_0$. We also define a $(p,n)$-form $\kappa_1$ on $V_1$ by

$$\kappa_1 = \tau_{1,1} \wedge \tau_{2,1} \wedge \cdots \wedge \tau_{\ell,1}.$$ 

Proposition 7.25 Thus constructed

$$\kappa = (\kappa_1, \kappa_0) \in \mathcal{E}^{(p,n)}(V_1) \oplus \mathcal{E}^{(p,n-1)}(V_0) = \mathcal{E}^{(p,n)}(V, V')$$

is a representative of the external product $u = u_1 \times u_2 \times \cdots \times u_\ell \in \mathcal{B}^{(p)}(M)$.

PROOF: This formula is obtained by the cup product formula, and then, by repeated applications of the remark after Lemma A.5 in Appendix. Note that Lemma 7.24 is an immediate consequence of this procedure.

The above expression appears to be rather complicated, however, it becomes much simpler in some particular but important cases:

Example 7.26 Assume all the $X_k$’s are Stein. Then we may take, for each $k$, a representative $(\tau_{k,1}, \tau_{k,01})$ of $u_k$ so that $\tau_{k,1} = 0$. A representative of $u$ is then given by

$$(0, (-1)^e (\ell - 1)! \bar{\partial} \varphi_1 \wedge \cdots \wedge \bar{\partial} \varphi_{\ell-1} \wedge \tau_{1,01} \wedge \cdots \wedge \tau_{01}) \in \mathcal{E}^{(p,n)}(V, V').$$

Here the constant $e$ is $\frac{\ell(\ell-1)}{2} + \sum_{k=1}^{\ell-1}(\ell - k)(n_k + p_k)$.

For example, the $n$-dimensional Dirac’s delta function $\delta(x)$ is just the external product $\delta(x_1) \times \delta(x_2) \times \cdots \times \delta(x_n)$ of the ones on $\mathbb{R}$. Hence, its representative is given by the above formula using a representative of the one-dimensional Dirac’s delta function, namely,

$$\left(0, \frac{1}{(2\pi \sqrt{-1})^n} \frac{\varphi_1 \wedge \cdots \wedge \varphi_{n-1}}{z_1 \cdots z_n} \right) \in \mathcal{E}^{(0,n)}(V, V').$$

Example 7.27 In the case $\ell = 2$, a representative of $u = u_1 \times u_2 \in \mathcal{B}^{(p)}(M)$ is given by

$$\left(\tau_{1,1} \wedge \tau_{2,1}, \varphi_1 \tau_{1,01} \wedge \tau_{2,1} + (-1)^{n_1+p_1} \varphi_2 \tau_{1,1} \wedge \tau_{2,01}
- (-1)^{n_1+p_1} \bar{\partial} \varphi_1 \wedge \tau_{1,01} \wedge \tau_{2,01}\right) \in \mathcal{E}^{(p,n)}(V, V').$$

We can easily show the following two propositions:

Proposition 7.28 We have, for $u_k \in \mathcal{B}(M_k)$, $k = 1, \ldots, \ell$,

$$\text{SS}(u_1 \times \cdots \times u_\ell) = \text{SS}(u_1) \times \cdots \times \text{SS}(u_\ell).$$

Proposition 7.29 For each $k = 1, \ldots, \ell$, let $\Omega_k$ be an open subset of $X_k$ satisfying the conditions (B_1) and (B_2) in $X_k$, and let $f_k \in \mathcal{O}(\Omega_k)$. Then we have

$$b_{\Omega_1 \times \cdots \times \Omega_\ell}(f_1 f_2 \cdots f_\ell) = b_{\Omega_1}(f_1) \times b_{\Omega_2}(f_2) \times \cdots \times b_{\Omega_\ell}(f_\ell).$$
7.5 Restriction of hyperfunctions

Let $N$ be a closed real analytic hypersurface in $M$ and $Y$ its complexification in $X$. It is known that the restriction to $N$ of a hyperfunction $u$ on $M$ cannot be defined in general. However, if $SS(u)$ is non-characteristic to $N$, i.e., $SS(u) \cap \sqrt{-1} T_N^* M \subset T_X^* X$ holds, then we can consider its restriction to $N$. In this subsection, we will define the restriction of a hyperfunction from the viewpoint of relative Dolbeault representation. We assume that $M$ and $N$ are oriented. Then we can take a non-vanishing continuous section

$$s : N \to T^*_N M \setminus T^*_X X.$$ 

Note that, when $N$ is connected, there are essentially two choices of $s$, i.e., either $s$ or $-s$. For such a choice, by noticing the morphisms of vector bundles

$$0 \to T^*_N M \to N \times_M T^* M \to T^* N \to 0,$$

we determine it so that, for any $x_0 \in N$, the vector $s(x_0)$ and a positively oriented frame of $(T^* N)_{x_0}$ form that of $(T^* M)_{x_0}$, where the frame of $(T^* N)_{x_0}$ follows $s(x_0)$.

Let $t : N \to T^*_N M$ be a continuous section on $N$ and $G$ a closed set in $X$.

**Definition 7.30** We say that $G$ is conically contained in the half space spanned by $\sqrt{-1} t$ if, for any $x_0 \in N$, there exist an open neighborhood $W$ of $x_0$ with a $(C^1$-class) local trivialization $\iota : (M \cap W, W) \simeq (\mathbb{R}^n_\times, \mathbb{C}^n = \mathbb{R}^n_\times \times \sqrt{-1} \mathbb{R}^n_y)$ and a closed cone $\Gamma \subset \mathbb{R}^n_y$ with

$$\Gamma \setminus \{0\} \subset \{ y \in \mathbb{R}^n_y \mid \text{Re} \langle \sqrt{-1} y, \sqrt{-1} t(x_0) \rangle > 0 \} = \{ y \in \mathbb{R}^n_y \mid \langle y, t(x_0) \rangle < 0 \}$$

for which the following holds:

$$\iota(G \cap W) \subset \mathbb{R}^n_\times \times \sqrt{-1} \Gamma.$$ 

Recall that, for a closed set $G$, we set $W_{V,G} = \{ V \setminus (\mathbb{R}^n_\times \times \sqrt{-1} \Gamma), V \}$ and $W'_{V,G} = \{ V \setminus (\mathbb{R}^n_\times \times \sqrt{-1} \Gamma) \}$. We also set $W_V = \{ V_0 = V \setminus M, V_1 = V \}$ and $W'_V = \{ V_0 \}$. Then we have a global version of Proposition 7.22:

**Lemma 7.31** Let $u$ be a hyperfunction on $M$. Assume $SS(u) \cap \sqrt{-1} s = \emptyset$. Then there exist an open neighborhood $V$ of $N$ in $X$, a closed set $G$ which is conically contained in the half space spanned by $\sqrt{-1} s$ and an element

$$\tau = (0, \tau_{01}) \in \mathcal{E}^{(0,n)}(V_1) \oplus \mathcal{E}^{(0,n-1)}(V_{01}) = \mathcal{E}^{(0,n)}(W_V, W'_V)$$

for which $\tau$ is a representative of $u$ near $N$ and $\text{Supp}_{V,M}(\tau_{01}) \subset G$ holds.

**Proof:** Set $\sqrt{-1} T_N^* M^+ = \sqrt{-1} \mathbb{R}^+ s$. Then it follows from [19, Theorem 4.3.2] that

$$\mathcal{E}(\sqrt{-1} T_N^* M^+) = \lim_{V, \mathcal{O}} \mathcal{H}_G^n(V; \mathcal{O}),$$

where $V$ runs through open neighborhoods of $N$ and $G$ ranges through closed sets conically contained in the half space spanned by $\sqrt{-1} s$. Therefore the argument goes the same way as that in Subsection 7.3. 

\[\square\]
We first give the cohomological definition of restriction to \(N\) of a hyperfunction on \(M\). Set \(T_N^*M^\pm = \pm \mathbb{R}^+s \subset T_N^*M\). Let us consider the map

\[
\mathcal{B}(M) \xrightarrow{\text{sp}|_{T_N^*M^+} \oplus \text{sp}|_{T_N^*M^-}} \mathcal{C}(\sqrt{-1}T_N^*M^+) \oplus \mathcal{C}(\sqrt{-1}T_N^*M^-).
\]

By [19, Theorem 4.3.2], the above diagram is equivalent to

\[
H^n_M(X; \theta) \xrightarrow{\lim_{v, \mathcal{G}^+}} H^n_{G^+}(V; \theta) \oplus \lim_{v, \mathcal{G}^-} H^n_{G^-}(V; \theta),
\]

where \(V\) runs through open neighborhoods of \(N\) in \(X\) and \(G^+\) (resp. \(G^-\)) ranges through conic sets contained in the half space spanned by \(\sqrt{-1}s\) (resp. \(-\sqrt{-1}s\)), and the morphism is just the canonical restriction. Furthermore, we may assume that \((G^+ \cap G^-) \cap V = M \cap V\) holds. Then we have (see the proof of the Lemma 7.33 below)

\[
\lim_{v, \mathcal{G}^\pm} H^k_G(V; \theta) = 0 \quad \text{for} \quad k < n.
\]

From the Mayer-Vietoris sequence for the pair \((G^+, G^-)\), we have the exact sequence

\[
0 \xrightarrow{\lim_{v, \mathcal{G}^\pm}} H^{n-1}_{G^+ \cup G^-}(V; \theta) \rightarrow H^n_M(V; \theta) \rightarrow \lim_{v, \mathcal{G}^+} H^n_{G^+}(V; \theta) \oplus \lim_{v, \mathcal{G}^-} H^n_{G^-}(V; \theta).
\]

Note that the morphisms of the above Mayer-Vietoris sequence

\[
H^k_{G^+ \cap G^-}(V; \theta) \rightarrow H^k_{G^+}(V; \theta) \oplus H^k_{G^-}(V; \theta) \rightarrow H^k_{G^+ \cup G^-}(V; \theta)
\]

are defined by sending \(u\) to \(u \oplus u\) and \(u \oplus v\) to \(u - v\), respectively, for which the choices of sign, i.e., either \(u - v\) or \(v - u\), is determined by taking our choice of the orientation of fibers of \(T_N^*M\) into account. Let \(i : Y \hookrightarrow X\) be the closed embedding. Then we have the canonical sheaf morphism \(\theta_X \rightarrow i_* \theta_Y\), which induces the morphism

\[
H^{n-1}_{G^+ \cup G^-}(V; \theta_X) \rightarrow H^{n-1}_{G^+ \cup G^-}(V; i_* \theta_Y) = H^{n-1}_{G}(V \cap Y; \theta_Y) = \mathcal{B}(N)
\]

because of \(i^{-1}(G^+ \cup G^-) = N\). Summing up, we have the diagram with the exact row

\[
0 \xrightarrow{\lim_{v, \mathcal{G}^\pm}} H^{n-1}_{G^+ \cup G^-}(V; \theta) \rightarrow \mathcal{B}(M) \xrightarrow{\text{sp} \oplus \text{sp}|_{T_N^*M^+} \oplus \text{sp}|_{T_N^*M^-}} \mathcal{C}(\sqrt{-1}T_N^*M^+) \oplus \mathcal{C}(\sqrt{-1}T_N^*M^-)
\]

\[
\mathcal{B}(N).
\]

Therefore, if a hyperfunction \(u \in \mathcal{B}(M)\) satisfies \(\text{SS}(u) \cap \sqrt{-1}T_N^*M \subset T_X^*X\), which is equivalent to saying that the image of \(u\) is zero by the morphism \(\text{sp}|_{T_N^*M^+} \oplus \text{sp}|_{T_N^*M^-}\), then we have the unique hyperfunction \(u|_N \in \mathcal{B}(N)\) by tracing the above diagram.

Now we compute a concrete representative of \(u|_N\) in our framework. Assume that \(\text{SS}(u) \cap \sqrt{-1}T_N^*M \subset T_X^*X\). Let \(\tau = (\tau^n_0, \tau^{n-1}_0) \in \theta^{(0,n)}(W, W')\) be a representative of \(u\) near \(N\). Then, by the assumption and the above lemma, there exist an open neighborhood
V of N, a closed set $G^+$ (resp. $G^-$) conically contained in the half space spanned by $\sqrt{-1}s$ (resp. $-\sqrt{-1}s$) and representatives

$$\tau^{n-1,\pm} = (\tau_1^{n-1,\pm}, \tau_{01}^{n-2,\pm}) \in \mathcal{E}^{(0,n-1)}(\mathcal{W}_{V,G^\pm}, \mathcal{W}_{V,G^\pm}')$$

such that $\tau = \bar{\partial}\tau^{n-1,\pm}$ in $\mathcal{E}^{(0,n)}(\mathcal{W}_{V,G^\pm}, \mathcal{W}_{V,G^\pm}')$. Define

$$\tau_Y = (\tau^{n-1,-} - \tau^{n-1,+})|_{\mathcal{Y}} = ((\tau_1^{n-1,-} - \tau_1^{n-1,+})|_{\mathcal{Y}}, (\tau_{01}^{n-2,-} - \tau_{01}^{n-2,+})|_{\mathcal{Y}}).$$

Here $\bullet|_{\mathcal{Y}}$ denotes the restriction of a differential form to $\mathcal{Y}$. Note that the choices of sign, i.e., either $(\tau^{n-1,-} - \tau^{n-1,+})|_{\mathcal{Y}}$ or $(\tau^{n-1,+} - \tau^{n-1,-})|_{\mathcal{Y}}$, is a consequence of that in the Mayer-Vietoris sequence. If $V$ is a sufficiently small neighborhood of $N$, then we have

$$(V \setminus G^+ \cap Y) = (V \setminus N) \cap Y \quad \text{and} \quad (V \setminus G^- \cap Y) = (V \setminus N) \cap Y.$$ 

Hence $\tau_Y$ belongs to $\mathcal{E}^{(0,n-1)}(\mathcal{W}_{V \cap Y}, \mathcal{W}_{V \cap Y}')$ with coverings

$$\mathcal{W}_{V \cap Y} = \{(V \cap Y) \setminus N, V \cap Y\}, \quad \mathcal{W}_{V \cap Y}' = \{(V \cap Y) \setminus N\}.$$ 

Furthermore we have

$$\bar{\partial}\tau_Y = \bar{\partial}((\tau^{n-1,-} - \tau^{n-1,+})|_{\mathcal{Y}}) = (\bar{\partial}\tau^{n-1,-} - \bar{\partial}\tau^{n-1,+})|_{\mathcal{Y}} = (\tau - \tau)|_{\mathcal{Y}} = 0,$$

which implies that the representative $\tau_Y$ defines a hyperfunction on $N$.

**Definition 7.32** The hyperfunction on $N$ defined by $\tau_Y \in \mathcal{E}^{(0,n-1)}(\mathcal{W}_{V \cap Y}, \mathcal{W}_{V \cap Y}')$ is denoted by $u|_N$ and is called the restriction of $u$ to $N$.

**Lemma 7.33** The restriction $u|_N$ is well-defined, that is, $u|_N$ does not depend on the choice of $\tau$, $\tau^{n-1,+}$ or $\tau^{n-1,-}$ in the above construction.

**Proof:** Recall that we set $\sqrt{-1}T_N^*M^+ = \sqrt{-1}\mathbb{R}^+s$. Clearly, by construction, $u|_N$ is independent of the choice of $\tau$. By the same argument as that in the proof for Lemma 7.15, the independency of the choice of $\tau^{n-1,+}$ comes from the fact

$$\lim_{V \setminus G^+} H^0_{\bar{\partial}}(\mathcal{W}_{V,G^+}, \mathcal{W}_{V,G^+}') = 0,$$

which can be shown in the following way: Thanks to the edge of the wedge theorem for $\mathcal{O}$ and [19, Theorem 4.3.2], we have the formula, for any $k \in \mathbb{Z}$,

$$H^k(\sqrt{-1}T_N^*M^+; \mathcal{O}) = \lim_{V \setminus G^+} H^{n+k}_{G^+}(V; \mathcal{O}),$$

where $V$ runs through open neighborhoods of $N$ in $X$ and $G^+$ ranges through closed sets conically contained in the half space spanned by $\sqrt{-1}s$. From the above we have

$$0 = H^{k-n}(\sqrt{-1}T_N^*M^+; \mathcal{O}) = \lim_{V \setminus G^+} H^k_{G^+}(V; \mathcal{O}) = \lim_{V \setminus G^+} H^0_{\bar{\partial}}(\mathcal{W}_{V,G^+}, \mathcal{W}_{V,G^+}') \quad \text{for} \quad k < n.$$ 

Hence $u|_N$ is independent of the choice of $t^{n-1,+}$. 

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The independency of the choice of $t^{n-1,-}$ can be proved the same way. □

The following theorem assures that our construction coincides with the original one in [29]. Let $V$ be an open neighborhood of $N$ in $X$ and $\Omega$ an open subset of $X$. Set

$$\Omega_Y = \Omega \cap Y, \quad V_Y = V \cap Y.$$  

Before stating the theorem, we introduce two conditions (B$^\dagger_1$) and (B$^\dagger_2$): The condition (B$^\dagger_1$) is the one (B$^\dagger_1$) given in Appendix with a non-characteristic condition of $\Omega$ along $N$.

(B$^\dagger_1$) For any $x_0 \in N$, there exist an open neighborhood $W$ of $x_0$ with a ($C^1$-class) local trivialization $\iota : (N \cap W, M \cap W, W) \cong (\mathbb{R}^{n-1}_x, \mathbb{R}^n_x, \mathbb{C}^n)$ and a non-empty open convex cone $\Gamma \subset \mathbb{R}^n$ such that

$$\mathbb{R}^n_x \times \sqrt{-1}\Gamma \subset \iota(W \cap \Omega) \quad \text{and} \quad \mathbb{R}^{n-1}_x \cap \Gamma \neq \emptyset.$$

The condition (B$^\dagger_2$) is a local version of (B$^\dagger_2$) introduced in Subsection 7.2.

(B$^\dagger_2$) For any $x_0 \in M$, there exist a fundamental system $\{V_\lambda\}_{\lambda \in \Lambda}$ of neighborhoods of $x_0$ in $V$ that satisfies the same condition as (B$^\dagger_2$) with $V$ replaced by $V_\lambda$, that is, the canonical inclusion $(V_\lambda \setminus \Omega) \setminus M \hookrightarrow V_\lambda \setminus \Omega$ is a homotopy equivalence for any $\lambda \in \Lambda$.

Theorem 7.34 Asssume that the pair $V$ and $\Omega$ satisfies the conditions (B$^\dagger_1$), (B$^\dagger_2$) and (B$^\dagger_2$). Assume also that the pair $V_Y$ and $\Omega_Y$ satisfies the conditions (B$^\dagger_2$) and (B$^\dagger_2$) in $Y$. Let $f \in \mathcal{O}(\Omega)$. Then we have

$$SS(b_\Omega(f)) \cap \sqrt{-1}T^*_N M \subset T^*_X X$$

and

$$b_\Omega(f)|_N = b_{\Omega_Y}(f|_Y).$$

Proof: By (B$^\dagger_2$), it suffices to show the claim locally. We may assume $M$ is a convex open neighborhood of $0 \in \mathbb{R}^n_x$ with coordinates $(x_1, \ldots, x_n) = (x_1, x')$, $X = M \times \sqrt{-1}\mathbb{R}^n_y$ with coordinates $(z_1, \ldots, z_n) = (z_1, z')$, $z_k = x_k + \sqrt{-1}y_k$, $k = 1, \ldots, n$, $N = M \cap \{x_1 = 0\}$ and $Y = X \cap \{z_1 = 0\}$. We write $(y_1, \ldots, y_n) = (y_1, y')$. Further, we assume

$$V = \{ z \in X \mid |x_1| < \epsilon, |y| < \epsilon \} \quad \text{and} \quad \Omega = (M \times \sqrt{-1}\Gamma) \cap V$$

for some $\epsilon > 0$ and an open proper convex cone $\Gamma \subset \mathbb{R}^n_y$ with $\Gamma \cap \{y_1 = 0\} \neq \emptyset$.

Let $e = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Then we may assume that our coordinate systems of $N$ and $M$ are positively oriented and the section $s$ is given by $s(x) = e \in (T^*_N M)_x$ ($x \in N$). Now we determine, in the similar way as those in Example 7.14, convex subsets $H_k$’s in $\mathbb{R}^n_y$ and $C^\infty$ functions $\varphi_k$’s on $X \setminus M$ where the index $k$ is either $k = \pm$ or $k = 1, 2, \ldots, n$. We first take linearly independent vectors $\tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1}$ in $\mathbb{R}^{n-1}_y$ such that

$$\bigcap_{1 \leq j \leq n-1} \{ y' \in \mathbb{R}^{n-1}_y \mid \langle y', \tilde{\eta}_j \rangle > 0 \} \subset \Gamma \cap \{y_1 = 0\}.$$  

Set $\tilde{\eta}_n = -(\tilde{\eta}_1 + \cdots + \tilde{\eta}_{n-1}) \in \mathbb{R}^{n-1}_y$. Let $c > 0$ and define convex subsets $H_j, j = 1, 2, \ldots, n$, in $\mathbb{R}^n_y$ by

$$H_j = \{ y \in \mathbb{R}^n_y \mid -c\langle y', \tilde{\eta}_j \rangle < y_1 < c\langle y', \tilde{\eta}_j \rangle \}.$$  

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We also define convex subsets $H_+$ and $H_-$ in $\mathbb{R}^n_y$ by

$$H_+ = \{ y \in \mathbb{R}^n_y \mid \pm y_1 > c^2|y'| \}.$$ 

Note that, for a sufficiently small $c > 0$, we have

$$H_1 \cup \cdots \cup H_n \cup H_+ \cup H_- = \mathbb{R}^n_y \setminus \{0\} \quad \text{and} \quad H_1 \cap \cdots \cap H_{n-1} \subset \Gamma.$$ 

We fix such a $c > 0$ in what follows. Note also that the intersection of $(n+1)$-choices in $(n+2)$-subsets $H_1, \ldots, H_n, H_+$ and $H_-$ is always empty.

Now let $\varphi_k, k = \pm, 1, \ldots, n$, be $C^\infty$ functions on $V \setminus M$ such that

1. $\text{Supp}_{V,M}(\varphi_k) \subset M \times \sqrt{-1}H_k$ for $k = 1, \ldots, n, \pm$,

2. $\varphi_1 + \cdots + \varphi_n + \varphi_+ + \varphi_- = 1$.

In particular, it follows from the definition of $H_k$ that $\varphi_\pm |_Y = 0$ and

$$\varphi_1 |_Y + \cdots + \varphi_n |_Y = 1 \quad (7.35)$$

holds. Set

$$\kappa^n = (0, \kappa^n_{01}^{-1}) = (0, (-1)^n(n-1)! \check{\chi}_{H_n \cup H_-} \check{\partial} \varphi_+ \wedge \check{\partial} \varphi_1 \wedge \cdots \wedge \check{\partial} \varphi_{n-2}),$$

where $\check{\chi}_{H_n \cup H_-}$ is the anti-characteristic function of the set $H_n \cup H_-$. Then, by the same arguments as those in Example 7.14 and Lemma A.3, we see that

$$\text{Supp}_{V,M}(\kappa^n_{01}^{-1}) \subset M \times \sqrt{-1} \Gamma$$

and that $[\kappa^n] \in H^n_M(X; \mathcal{O})$ corresponds to the image of the positively oriented generator in $\text{or}_{M/X}(M)$ under the standard orientation of $\mathbb{R}^n_y$. Hence, by the definition of the boundary value map, $\tau = f \kappa^n = (0, f \kappa^n_{01}^{-1})$ is a representative of $b_{\Omega}(f)$. Then, as

$$\text{Supp}_{X,M}(\kappa^n_{01}^{-1}) \subset H^+$$

holds, we can take $\tau^{n-1,-} = 0$ in the construction of $\tau^{n-1}_{\Omega}$. Let us compute $\tau^{n-1,+}$. Set

$$\kappa^{n-2}_{01} = -(n-2)! \check{\chi}_{H_n \cup H_-} \sum_{j=1}^{n-1} (-1)^j \varphi_j \check{\partial} \varphi_1 \wedge \cdots \wedge \check{\partial} \varphi_j \wedge \cdots \wedge \check{\partial} \varphi_{n-1}.$$ 

Then there exists a closed set $G^+$ conically contained in the half space spanned by $\sqrt{-1}s$ such that $\kappa^{n-2}_{01}$ is a $C^\infty (0, n-2)$-form on $V \setminus G^+$. Define

$$\tau^{n-1,+} = (0, f \kappa^{n-2}_{01}).$$

Then, since $\kappa^{n-1}_{01} = -\check{\partial} \kappa^{n-2}_{01}$ holds, we see

$$\tau = \bar{\partial} \tau^{n-1,+} \in \mathcal{E}^{(0,n-1)}(\mathcal{W}_{V,G^+}, \mathcal{W}_{V,G^+}).$$

Furthermore, it follows from (7.35) that we have

$$\kappa^{n-2}_{01} |_Y = (-1)^n(n-2)! \check{\chi}_{H_n \cap Y} \check{\partial} (\varphi_1 |_Y) \wedge \cdots \wedge \check{\partial} (\varphi_{n-2} |_Y),$$

for which, by Example 7.14, the element $[(0, -\kappa^{n-2}_{01} |_Y)] \in H^{n-1}_N(Y; \mathcal{O})$ corresponds to the image of the positively oriented generator in $\text{or}_{N/Y}(N)$ under the standard orientation of $\mathbb{R}^{n-1}_y$. Therefore we have obtained

$$\tau_Y = (\tau^{n-1,-} - \tau^{n-1,+}) |_Y$$

$$= (0, f |_Y (-1)^{n-1}(n-2)! \check{\chi}_{H_n \cap Y} \check{\partial} (\varphi_1 |_Y) \wedge \cdots \wedge \check{\partial} (\varphi_{n-2} |_Y)).$$

This implies that the representative $\tau_Y$ gives the hyperfunction $b_{\Omega Y}(f |_Y)$.
7.6 Fiber integration of hyperfunction densities

Let $M$ and $N$ be real analytic manifolds of dimensions $m$ and $n$, respectively, $m \geq n$. We denote by $X$ and $Y$ their complexifications. Let $f : M \rightarrow N$ be a submersion whose complexification is denoted by $\tilde{f} : X \rightarrow Y$. We assume that $M$ and $N$ are orientable.

In this subsection, we introduce an integration morphism $\int_f$ of hyperfunction densities along fibers of $f$:

$$
\int_f (\mathcal{B}_M \otimes_{\mathcal{A}_M} \mathcal{V}_M)(N) \xrightarrow{\int_f} (\mathcal{B}_N \otimes_{\mathcal{A}_N} \mathcal{V}_N)(N),
$$

where $\mathcal{V}_M$ is the sheaf of real analytic densities on $M$, i.e., $\mathcal{V}_M = \mathcal{A}_M^{(m)} \otimes_{\mathbb{C}} \mathcal{O}_M$ with $\mathcal{A}_M^{(m)}$ the sheaf of real analytic $m$-forms on $M$ and $\mathcal{O}_M$ the orientation sheaf of $M$. The sheaf $\mathcal{V}_N$ is defined similarly for $N$. Note that $\mathcal{B}_M \otimes_{\mathcal{A}_M} \mathcal{V}_M = \mathcal{B}_M^{(m)} \otimes_{\mathbb{C}} \mathcal{O}_M$. Note also that, as the integration we define is performed on densities, the morphism $\int_f$ is independent of the orientation of fibers of $f$.

We may assume each fiber of $f$ to be connected. Since otherwise, $M$ is a disjoint union of real analytic manifolds $M_\lambda$ such that each fiber of $f|_{M_\lambda}$ is connected. Then the integrations along $f|_{M_\lambda}$ can be summed up, as the support of the integrand is proper along $f$.

Set $d = m - n$ and let $p$ be a non-negative integer. We will construct slightly extended version of the integration morphism:

$$
\int_f (\mathcal{B}_M^{(p)} \otimes_{\mathbb{C}} \mathcal{O}_M)(N) \xrightarrow{\int_f} (\mathcal{B}_N \otimes_{\mathbb{C}} \mathcal{O}_N)(N),
$$

where $\mathcal{B}_M^{(p)}$ is the sheaf of $p$-hyperforms on $M$. Note that $\mathcal{B}_M^{(p)} = \mathcal{B}_M \otimes_{\mathcal{A}_M} \mathcal{A}_M^{(p)}$.

Since $M$ is assumed to be orientable, for any closed set $K$ in $M$,

$$
\Gamma_K(M; \mathcal{B}_M^{(p)}) = H^m_K(X; \mathcal{O}_X^{(p)} \otimes_{\mathcal{A}_M^{(m)}} \mathcal{O}_M). 
$$

Hence it suffices to construct the following integration morphism for any closed set $K \subset M$ with $f|_K : K \rightarrow N$ being proper:

$$
H^m_K(X; \mathcal{O}_X^{(p)} \otimes_{\mathcal{A}_M^{(m)}} \mathcal{O}_M)(M) \xrightarrow{\int_f} H^n_N(Y; \mathcal{O}_Y^{(p-d)} \otimes_{\mathcal{A}_N^{(n)}} \mathcal{O}_N). 
$$

Denoting by $i : M \hookrightarrow X$ and $j : N \hookrightarrow Y$ the inclusions, we have the identifications by Convention 1:

$$
or_{M/X} \otimes_{\mathbb{C}} \mathcal{O}_M = i^{-1} or_X, \quad or_{N/Y} \otimes_{\mathbb{C}} \mathcal{O}_N = j^{-1} or_Y. \quad (7.36)
$$

With these, the above integration morphism is expressed as

$$
H^m_K(X; \mathcal{O}_X^{(p)} \otimes_{\mathcal{A}_M^{(m)}} i^{-1} or_X)(M) \xrightarrow{\int_f} H^n_N(Y; \mathcal{O}_Y^{(p-d)} \otimes_{\mathcal{A}_N^{(n)}} j^{-1} or_Y(N). \quad (7.37)
$$

Before we further proceed, we recall the usual fiber integration of $C^\infty$ forms with some more orientation convention.

In general, let $g : P \rightarrow Q$ be a submersion of $C^\infty$ manifolds with fiber dimension $r$. We denote by $Tg$ the bundle of tangent vectors on $P$ that are tangent to the fibers of $g$. 

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We define the orientation sheaf of $g$, denoted by $\text{or}_g$, to be the orientation sheaf of the bundle $Tg$ (cf. Subsection 2.3). It is a sheaf on $P$ and describes the orientation of fibers of $g$. From the exact sequence
\[ 0 \rightarrow Tg \rightarrow TP \rightarrow g^*TQ \rightarrow 0, \]
we have an isomorphism
\[ \text{or}_P \simeq \text{or}_g \otimes g^{-1} \text{or}_Q. \]
Note that there are several possible choices for the above isomorphisms.

Now we assume that $P$ and $Q$ are orientable. Thus $Tg$ is also orientable.

**Convention 2.** Let $\psi_P$ and $\psi_Q$ be prescribed orientations of $P$ and $Q$. Then we take the orientation $\psi_g$ of $Tg$ so that a positive fiber coordinate system followed by (the pull-back by $g$ of) a positive coordinate system on $Q$ gives a positive coordinate system on $P$.

Thus we make identification
\[ \text{or}_P = \text{or}_g \otimes g^{-1} \text{or}_Q \quad \text{by} \quad \psi_P \longleftrightarrow \psi_g \otimes g^{-1} \psi_Q \quad (7.38) \]
with $\psi_P$, $\psi_Q$ and $\psi_g$ as in Convention 2.

Once we fix various orientations according to Convention 2, we have the fiber integration
\[ (g_! \mathcal{E}_P^{(p)})_Q \xrightarrow{f_g} \mathcal{E}_Q^{(p-r)}(Q), \]
which has the property $\int f_g \circ d = (-1)^r d \circ \int g$. Thus it induces a morphism of complexes $g_! \mathcal{E}_P^{(\bullet)}[\dim P] \rightarrow \mathcal{E}_Q^{(\bullet)}[\dim Q]$.

If $P$ and $Q$ are complex manifolds and if $g$ is a holomorphic submersion with complex fiber dimension $s$, the above integration induces
\[ (g_! \mathcal{E}_P^{(p,q)})_Q \xrightarrow{f_g} \mathcal{E}_Q^{(p-s,q-s)}(Q), \]
which commutes with both $\partial$ and $\bar{\partial}$.

Coming back to our situation, we have two submersions, namely, $f : M \rightarrow N$ and its complexification $\tilde{f} : X \rightarrow Y$. We make the following identifications according to Convention 2:
\[ \text{or}_M = \text{or}_f \otimes \mathbb{Z}_M f^{-1} \text{or}_N, \quad \text{or}_X = \text{or}_{\tilde{f}} \otimes \mathbb{Z}_X \tilde{f}^{-1} \text{or}_Y. \quad (7.39) \]
We may make the identifications (7.36) and (7.39) compatible, by determining various coordinate systems as follows:

1. Let $\psi_M$ and $\psi_N$ be prescribed orientations of $M$ and $N$. We orient the fibers of $f$ so that if $x' = (x_1, \ldots, x_d)$ is a positive fiber coordinate system, $d = m - n$, and if $x'' = (x_{d+1}, \ldots, x_m)$ is a positive coordinate system on $N$, then $x = (x', x'')$ is a positive coordinate system on $M$. 

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(2) The orientations \( \psi_X \) and \( \psi_Y \) of \( X \) and \( Y \) are determined by Convention 1 (cf. (7.36)). Let \((z_1, \ldots, z_d)\) be a complex fiber coordinate system of \( \tilde{f} \), \( z_i = x_i + \sqrt{-1}y_i \). We orient the fibers of \( \tilde{f} \) so that if \((-1)^{dn}(y', x')\), \( y' = (y_1, \ldots, y_d)\), is a positive fiber coordinate system and if \((y'', x'')\), \( y'' = (y_{d+1}, \ldots, y_m)\), is a positive coordinate system on \( Y \), then \((y, x)\) is a positive coordinate system on \( X \), where \( y = (y', y'') \) and \( x \) is as above.

Now we define the integration morphism (7.37). We do it separately for \( H_{K}^{m}(X; \mathcal{E}_{X}^{(p)}) \) and \( i^{-1}or_{X}(M) \). First we do this for the latter. Note that we have the identifications
\[
i^{-1}or_{X} = i^{-1}(or_{f} \otimes_{x_{X}} f^{-1}or_{Y}) = i^{-1}or_{f} \otimes_{z_{M}} f^{-1}j^{-1}or_{Y}.
\]
Given \( a_{X} \in i^{-1}or_{X}(M) \), we take \( a_{f} \in i^{-1}or_{f}(M) \) and \( a_{Y} \in j^{-1}or_{Y}(N) \) so that
\[
a_{X} = a_{f} \otimes f^{-1}a_{Y}.
\]
For any point \( x \in M \), we have the identification
\[
or_{f,x} = H_{2d}^{2d}(\tilde{f}^{-1}f(x); \mathbb{Z}f^{-1}j^{-1}(f(x))).
\]
Hence, once we fix an orientation of the fiber, which is a complex manifold of dimension \( d \), we can determine an integer \( n(a_{f})(x) \) as the cap product
\[
[\tilde{f}^{-1}f(x)] \cap a_{f}(x) \in \mathbb{Z}
\]
with the fundamental class \([\tilde{f}^{-1}f(x)] \in H_{2d}(\tilde{f}^{-1}f(x); \mathbb{Z})\) of the fiber \( \tilde{f}^{-1}f(x) \) (cf. Subsection 2.3). Note that \( n(a_{f})(x) \) is a locally constant function on \( M \). Furthermore, as each fiber of \( f \) is assumed to be connected, we can regard \( n(a_{f}) \) as a locally constant function on \( N \), i.e., \( n(a_{f}) \in \mathbb{Z}_{N}(N) \). Then we define the integration of \( a_{X} \) by
\[
\int_{f} a_{X} = n(a_{f}) a_{Y} \in j^{-1}or_{Y}(N).
\]
Clearly, the integration thus defined is independent of the choice of \( a_{f} \) or \( a_{Y} \).

**Remark 7.40** The point here is that we choose an orientation of the fiber of \( \tilde{f} \) and fix it throughout the integration procedure. The final outcome, after multiplication by the integration of an element \( u \) in \( H_{K}^{m}(X; \mathcal{E}_{X}^{(p)}) \), does not depend on such a choice.

Now we define the integration morphism for \( u \in H_{K}^{m}(X; \mathcal{E}_{X}^{(p)}) \approx H_{\bar{\delta}}^{p,m}(\mathcal{V}_{K}, \mathcal{V}'_{K}) \), where \( \mathcal{V}_{K} = \{V_0 = X \setminus K, V_1 = X\} \) and \( \mathcal{V}'_{K} = \{V_0\} \). We also set \( \mathcal{W} = \{W_0 = Y \setminus N, W_1 = Y\} \) and \( \mathcal{W}' = \{W_0\} \). Let
\[
\tau = (\tau_1, \tau_{01}) \in \mathcal{E}^{(p,m)}(V_1) \oplus \mathcal{E}^{(p,m-1)}(V_{01}) = \mathcal{E}^{(p,m)}(\mathcal{V}_{K}, \mathcal{V}'_{K})
\]
be a representative of \( u \in H_{\bar{\delta}}^{p,m}(\mathcal{V}_{K}, \mathcal{V}'_{K}) \). Let \( \varphi \) be a \( C^\infty \)-function on \( X \) satisfying
\[
(1) \quad \tilde{f}|_{\text{Supp}(\varphi)} : \text{Supp}(\varphi) \to Y \text{ is proper},
\]
\[
(2) \quad \varphi \text{ is identically } 1 \text{ on an open neighborhood of } K.
\]
Set
\[
\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_0) = (\varphi \tau_1 + \bar{\partial} \varphi \wedge \tau_0, \varphi \tau_0) \in \mathcal{E}^{(p,m)}(V_1) \oplus \mathcal{E}^{(p,m-1)}(V_0).
\]
Then, by noticing \(\hat{\partial} \tau_0 = \tau_1\) on \(V_1\), we have
\[
\tau - \hat{\tau} = \hat{\vartheta}((1 - \varphi)\tau_0, 0),
\]
and hence, \(\hat{\tau}\) is also a representative of \(u\). Furthermore, \(\hat{f}\) is proper on \(\text{Supp}(\hat{\tau}_1)\) and \(\text{Supp}(\hat{\tau}_0)\). Hence we may apply the usual integration of differential forms along fibers of \(\hat{f}\) to \(\hat{\tau}_1\) and \(\hat{\tau}_0\), and we see that
\[
\left(\int_{\hat{f}} \hat{\tau}_1, \int_{\hat{f}} \hat{\tau}_0\right) \in \mathcal{E}^{(p-d,n)}(W_1) \oplus \mathcal{E}^{(p-d,n-1)}(W_0)
\]
gives an element of \(H_{\partial}^{p-d,n}(\mathcal{W}, \mathcal{W}')\). Here the fiber integration \(\int_{\hat{f}}\) is to be performed with the orientation of fibers chosen for the integration on \(i^{-1} \text{or}_X(M)\) (cf. Remark 7.40).

Summing up, the integration along fibers of \(f\) for \([(\tau_1, \tau_0)] \otimes a_X\) is given by
\[
\left[\left(\int_f \varphi \tau_1 + \bar{\partial} \varphi \wedge \tau_0, \int_f \varphi \tau_0\right)\right] \otimes \int_f a_X \in H_{\partial}^{p-d,n}(\mathcal{W}, \mathcal{W}') \otimes_{z_N(N)} j^{-1} \text{or}_Y(N).
\]
It can easily be confirmed that the above definition does not depend on the choice of the representative of the hyperform or of \(\varphi\). It does not depend on the choices of various orientations involved either.

Remark 7.41 By definition of \(d : \mathcal{B}^{(p)} \to \mathcal{B}^{(p+1)}\) (cf. (6.4)), we see that the above integration \(\int_f\) induces a morphism of complexes \(f_!(\mathcal{D}^\bullet_M \otimes_{z_M \text{or}_M}[m] \to (\mathcal{D}^\bullet_N \otimes_{z_N \text{or}_N}[n])\). Moreover, it is compatible with the definition in terms of derived functors as given in [29].

Example 7.42 Let \(M = \mathbb{R}^m = \{(x_1, \ldots, x_m)\}\) and \(N = \mathbb{R}^n = \{(y_1, \ldots, y_m)\}\), \(d = m - n\). Also let \(X = \mathbb{C}^m = \{(z_1, \ldots, z_m)\}\) and \(Y = \mathbb{C}^n = \{(z_{d+1}, \ldots, z_m)\}\), \(z_i = x_i + \sqrt{-1} y_i\). For \(\hat{f} : X \to Y\), we take the projection \(\hat{f}(z_1, \ldots, z_m) = (z_{d+1}, \ldots, z_m)\). We take, for \(X, Y\) and the fibers of \(\hat{f}\), the orientations for which \((y_1, \ldots, y_{d+1}, x_{d+1}, \ldots, x_m)\) and \((-1)^{dn}(y_1, \ldots, y_d, x_1, \ldots, x_d)\) are positive systems, according to Conventions 1 and 2.

In this case, we usually omit the orientation sheaves appearing in the above construction via the identifications \(i^{-1} \text{or}_X(M) = \mathbb{Z}, j^{-1} \text{or}_Y(N) = \mathbb{Z}\) and \(i^{-1} \text{or}_f(M) = \mathbb{Z}\), where each orientation specified as above corresponds to \(1 \in \mathbb{Z}\). If we denote by \(a_X, a_Y\) and \(a_f\) such orientations, respectively, we have
\[
a_X = a_f \otimes f^{-1} a_Y \quad \text{so that} \quad \int_f a_X = 1.
\]
Thus the integration along fibers of \(f\) for the hyperform \(u = [(\tau_1, \tau_0)] \in H_{\partial}^{p,m}(\mathcal{V}_K, \mathcal{V}_K')\) with the orientations as above is given by
\[
\left[\left(\int_f \varphi \tau_1 + \bar{\partial} \varphi \wedge \tau_0, \int_f \varphi \tau_0\right)\right] \in H_{\partial}^{p-d,n}(\mathcal{W}, \mathcal{W}').
\]
An auxiliary function \( \varphi \) is involved in the definition of integration along fibers, however we can eliminate it in some special cases as we will see. Let \( N \) and \( N_1 \) be real analytic manifolds with \( \dim N = n \) and \( \dim N_1 = d \), and let \( Y \) and \( Y_1 \) be their complexifications. Set \( X = Y_1 \times Y \), \( M = N_1 \times N \) and \( m = d + n \). We take \( \bar{f} \) to be the canonical projection \( X = Y_1 \times Y \to Y \) and orient the various space involved according to Conventions 1 and 2, regarding \( Y_1 \) as the fiber. Let \( K_1 \) be a compact set in \( Y_1 \) and set \( K = K_1 \times Y \). We may choose a compact set \( D_1 \) in \( Y_1 \) with \( C^\infty \) boundary \( \partial D_1 \) containing \( K_1 \) in its interior. Further, we may assume that \( \varphi \) depends only on the variables in \( Y_1 \) and \( Y \)

\[
K = K_1 \times Y \subset \text{Supp}(\varphi) \subset \text{Int}(D_1) \times Y.
\]

For any differential form \( \kappa \) on \( X \), we denote by \( \int_{D_1} \kappa \) the partial integration of \( \kappa \) with respect to the variables in \( Y_1 \) on the domain \( D_1 \).

Now we have

\[
\left( \int_{\bar{f}} \varphi \tau_1 + \bar{\partial}_X \varphi \wedge \tau_0, \int_{\bar{f}} \varphi \tau_0 \right) = \left( \int_{D_1} \varphi \tau_1 + \bar{\partial}_X \varphi \wedge \tau_0, \int_{D_1} \varphi \tau_0 \right)
\]

\[
= \left( \int_{D_1} \tau_1 - \bar{\partial}_X((1 - \varphi)\tau_0), \int_{D_1} \varphi \tau_0 \right).
\]

By applying Stokes’ formula and noticing \( \bar{\partial}_X = \bar{\partial}_{Y_1} + \bar{\partial}_Y \), the above is equal to

\[
\left( \int_{D_1} \tau_1 - \int_{\partial D_1} \tau_0 |_{\partial D_1 \times Y} - \bar{\partial}_Y \int_{D_1} (1 - \varphi)\tau_0, \int_{D_1} \varphi \tau_0 \right).
\]

Since \( \int_{D_1} (1 - \varphi)\tau_0 \) is a \( C^\infty \) form on the total space \( Y \), by adding the coboundary \( \bar{\partial}_Y \left( \int_{D_1} (1 - \varphi)\tau_0, 0 \right) \) to the above expression, we have the following proposition:

**Proposition 7.43** In the situation above, the integration of \( [(\tau_1, \tau_0)] \otimes \alpha_X \) along fibers of \( f \) is given by

\[
\left[ \left( \int_{D_1} \tau_1 - \int_{\partial D_1} \tau_0 |_{\partial D_1 \times Y}, \int_{D_1} \tau_0 \right) \right] \otimes \int_{\bar{f}} \alpha_X.
\]

We finish this subsection by clarifying the relation between the external product and the integration. Let \( M_1 \) and \( M_2 \) be real analytic manifolds.

**Proposition 7.44** Let \( u_k \otimes v_k \in (\mathcal{B}_{M_k} \otimes \mathcal{E}_{k} \mathcal{V}_{M_k})(M_k), k = 1, 2 \), with \( u_k \) having a compact support in \( M_k \). Then we have

\[
\int_{M_1 \times M_2} (u_1 \times u_2) \otimes (v_1 \wedge v_2) = \left( \int_{M_1} u_1 \otimes v_1 \right) \left( \int_{M_2} u_2 \otimes v_2 \right).
\]

8 Embedding of distributions to the space of hyperfunctions

In this section, we study embedding of distributions to the space of hyperfunctions. Let \( M = \mathbb{R}^n \) and \( X = \mathbb{C}^n \), and let \( u \) be a distribution on an open set \( U \) in \( M \). Then, in our theory, a representative of \( u \) as a hyperfunction is given by the following theorem:
Theorem 8.1 The pair

\[
\left( \int_U (\theta \tau_1 + \bar{\partial}_2 \theta \wedge \tau_0) (z - t, t) \, u(t) \, dt, \int_U (\theta \tau_0) (z - t, t) \, u(t) \, dt \right) \in \mathcal{E}^{(0,n)}(V_1) \oplus \mathcal{E}^{(0,n-1)}(V_0)
\]

is well-defined and represents the hyperfunction image of \( u \). Here \( \theta(z,t) \) is a \( C^\infty \) function on \( X \times U \) such that the conditions (1) and (2) before Lemma 8.7 hold for \( K = \{0\} \) and an open set \( T \) satisfying (8.8), and \( (\tau_1(z), \tau_0(z)) \) is a representative of Dirac's delta density, for example, we may choose the one of Bochner-Martinelli type (cf. Definition 6.15).

Note that the precise definition of the above integrations in the theorem is given in (8.10).

Traditionally, embedding of a distribution \( u \) was realized by taking a convolution of \( u \) with the Cauchy kernel \( K(z) \). To make the integration converge, we first decompose \( u \) as a sum \( \sum \lambda \, u_\lambda \) of distributions \( u_\lambda \)'s with compact support. Unfortunately, since the support of each \( \langle u_\lambda(t), K(z-t) \rangle \) is not compact, the sum \( \sum \lambda \langle u_\lambda(t), K(z-t) \rangle \) itself does not converge in general. Therefore, to obtain an embedding image of \( u \), we should sum up these terms as an element of quotient classes, i.e., we consider the locally finite sum \( \sum \lambda \langle u_\lambda(t), K(z-t) \rangle \) in the space of hyperfunctions. Hence, in the traditional approach, we cannot directly obtain a representative of an embedding image of \( u \). As we have already seen, our formula (8.2) gives globally a representative of the embedding image of a distribution, which is one of advantages in our framework.

In the subsequent arguments, we establish the above theorem and several properties of the embedding morphism, in particular, it is canonical.

Let \( M = \mathbb{R}^n \) and \( X = \mathbb{C}^n \). We denote by \( \mathcal{O}_\mathcal{E}^{(p,r)} \) the sheaf on \( X \times M \) of \( C^\infty \) forms on \( X \times M \) that are holomorphic \( p \)-forms in the complex variables \( z \) of \( X \) and \( r \)-forms in the variables \( t \) of \( M \), i.e., they have the form

\[
\sum_{|I|=p, |J|=r} f^{I,J}(z,t) \, dz_I dt_J,
\]

where each \( f^{I,J}(z,t) \) is a \( C^\infty \) function that is holomorphic in \( z \). Let \( U \) be an open set in \( M \) and \( K \) a compact set in \( X \). Let also \( \pi : X \times U \to X \) be the canonical projection. Then the canonical sheaf morphism \( \pi^{-1} \mathcal{O}^{(n)} \to \mathcal{O}_\mathcal{E}^{(n;0)} \) induces the canonical one

\[
\iota : H^n_K(X; \mathcal{O}^{(n)}) \longrightarrow H^n_{K \times U}(X \times U; \mathcal{O}_\mathcal{E}^{(n;0)}).
\]

Let \( \mathcal{E}_{X \times M}^{(p,q;r)} \) denote the sheaf of \( C^\infty \) forms on \( X \times M \) that are \( (p,q) \)-forms in the variables \( z \) of \( X \) and \( r \)-forms in the variables \( t \) of \( M \). Note that \( \mathcal{E}_{X \times M}^{(p,q;r)} \) is fine.

Theorem 8.3 Let \( \Omega \) be a Stein open set in \( X \) and \( U \) an open set in \( M \). Then the following complex is exact:

\[
0 \to \mathcal{O}_\mathcal{E}^{(p,r)}(\Omega \times U) \to \mathcal{E}_{X \times M}^{(p,0;r)}(\Omega \times U) \xrightarrow{\partial_z} \mathcal{E}_{X \times M}^{(p,1;r)}(\Omega \times U) \xrightarrow{\partial_z} \cdots \xrightarrow{\partial_z} \mathcal{E}_{X \times M}^{(p,n;r)}(\Omega \times U) \to 0.
\]

In particular, the \( \bar{\partial}_z \)-complex \( \mathcal{E}_{X \times M}^{(p,q;\bullet)} \) is a fine resolution of \( \mathcal{O}_\mathcal{E}^{(p,r)} \).
we have the isomorphism
for any open set $W$.

For any sheaf $H$.

The above morphism $\tau$ is just regarding a representative $(G)$ and the differential is defined as before. For a complex $(K)$ satisfying $\tau$.

Let us consider the coverings $S_K = \{S_0 = X \setminus K, S_1 = X\}$ and $S'_K = \{S_0\}$ of the pair $(X, X \setminus K)$. Consider also the coverings $S_K \times U = \{S_0 \times U, S_1 \times U\}$ and $S'_K \times U = \{S_0 \times U\}$ of $(X \times U, (X \setminus K) \times U)$. Hereafter, we sometime denote $(S_K \times U, S'_K \times U)$ by $(S_K, S'_K) \times U$.

For a complex $\mathcal{F}^\bullet$ of fine sheaves on $X$, we define a complex $C(S_K, S'_K)(\mathcal{F}^\bullet)$ as in the case of Dolbeault complex. Namely its $q$-th degree term is given by

$$C^q(S_K, S'_K)(\mathcal{F}^\bullet) = \mathcal{F}^q(S_1) \oplus \mathcal{F}^{q-1}(S_{01})$$

and the differential is defined as before. For a complex $\mathcal{G}^\bullet$ of fine sheaves on $X \times U$, the complex $C((S_K, S'_K) \times U)(\mathcal{G}^\bullet)$ is similarly defined. Then, by the above theorem and the same reasoning as that for the relative Dolbeault cohomology, we have the isomorphisms

$$H^n_K(X; \mathcal{O}^{(n)}) \simeq H^n(C(S_K, S'_K)(\mathcal{O}^{(n, \bullet)})),$$

$$H^n_{K \times U}(X \times U; \mathcal{O}^{(n, 0)}) \simeq H^n(C(S_K, S'_K) \times U)((\mathcal{O}^{(n, \bullet)}))_X \times M).$$

Hence an element in $H^n_{K \times U}(X \times U; \mathcal{O}^{(n, 0)})$ is represented by

$$(\tau_1(z, t), \tau_{01}(z, t)) \in \mathcal{O}^{(n, n-1, 0)}(S_1 \times U) \oplus \mathcal{O}^{(n, n-1, 0)}_X(S_0 \times U)$$

satisfying $\tau_1 = \partial_z \tau_{01}$ on $(X \setminus K) \times U$. Under these identifications, the canonical morphism

$$\iota : H^n_K(X; \mathcal{O}^{(n)}) \rightarrow H^n_{K \times U}(X \times U; \mathcal{O}^{(n, 0)})$$

is just regarding a representative $(\tau_1(z), \tau_{01}(z))$ of an element in $H^n_K(X; \mathcal{O}^{(n)})$ as that in $H^n_{K \times U}(X \times U; \mathcal{O}^{(n, 0)})$. Furthermore, by fixing $t$ to some point $t_0 \in U$, we can easily see:

**Lemma 8.4** The above morphism $\iota$ is injective.

Let $T$ be an open set in $X \times U$ containing $K \times U$ with $j_T : T \hookrightarrow X \times U$ the open embedding. For any sheaf $\mathcal{F}$, let us consider the subsheaf $j_T j_T^{-1} \mathcal{F} \subset \mathcal{F}$. By definition,

$$(j_T j_T^{-1} \mathcal{F})(W) = \{ s \in \mathcal{F}(W) \mid \text{Supp}_W(s) \subset T \}$$

for any open set $W \subset X \times U$, in particular, we have $(j_T j_T^{-1} \mathcal{F})|_T = \mathcal{F}|_T$.

Since $j_T j_T^{-1} \mathcal{O}^{(p, q, r)}_X$ is a $C^\infty$-module, it is a fine sheaf. Furthermore, since $j_T j_T^{-1}$ is an exact functor, the complex $j_T j_T^{-1} \mathcal{O}^{(n, \bullet)}$ is a fine resolution of $j_T j_T^{-1} \mathcal{O}^{(n, 0)}$. Hence, we have the isomorphism

$$H^n_{K \times U}(X \times U; j_T j_T^{-1} \mathcal{O}^{(n, 0)}) \simeq H^n(C((S_K, S'_K) \times U)(j_T j_T^{-1} \mathcal{O}^{(n, \bullet)}))_X \times M).$$
This implies that an element in $H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)})$ is represented by

$$((\tau_1(z,t), \tau_0(z,t)) \in \mathcal{E}^{(n,0)}(S_1 \times U) \oplus \mathcal{E}^{(n-1,0)}(S_0 \times U)$$

which satisfies $\tau_1 = \partial_2 \tau_0$ on $(X \setminus K) \times U$ and the support condition

$$\text{Supp}_{X \times U}(\tau_1) \subset T, \quad \text{Supp}_{(X \setminus K) \times U}(\tau_0) \subset T.$$ (8.5)

Further, the canonical morphism (induced from the one $j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)} \to \mathcal{O} \mathcal{E}^{(n,0)}$)

$$H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)}) \xrightarrow{\psi} H^n_{K \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n,0)})$$

is isomorphic. In fact, we have

$$H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)}) \simeq H^n_{K \times U}(T; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)})$$

$$\simeq H^n_{K \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n,0)}).$$

Here the first and third isomorphisms are due to excision and the second morphism is isomorphic thanks to $(j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)})|_T = \mathcal{O} \mathcal{E}^{(n,0)}|_T$.

Summing up, we have the commutative diagram whose arrows are all isomorphisms.

$$H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)}) \xrightarrow{\sim} H^n_{K \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n,0)})$$

This also implies:

**Lemma 8.6** Let $T$ be an open set in $X \times U$ containing $K \times U$. Then, any element in $H^n_{K \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n,0)})$ has a representative

$$(\tau_1(z,t), \tau_0(z,t)) \in C^n((S_K, S'_K) \times U)(\mathcal{E}^{(n,0)}\mathcal{M})$$

which satisfies the support condition (8.5).

Now we give a concrete method to obtain such a representative appearing in the above lemma. Let $\theta(z,t)$ be a $C^\infty$-function on $X \times U$ satisfying that

1. $\theta(z,t)$ is identically 1 in a neighborhood of $K \times U$.
2. $\text{Supp}(\theta) \subset T$.

**Lemma 8.7** The inverse of the above isomorphism $H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)}) \to H^n_{K \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n,0)})$ is given by

$$[(\tau_1, \tau_0)] \mapsto [((\theta \tau_1 + \partial_2 \theta \wedge \tau_0, \theta \tau_0)] \in H^n_{K \times U}(X \times U; j_T \gamma_T^{-1} \mathcal{O} \mathcal{E}^{(n,0)}).$$
Let we formally define $M \in \mathcal{E}$.
Then the following lemma easily follows from (8.8):

\[ \text{It follows from Lemma 8.9 that} \]

**Proof:** This comes from the fact

\[ (\tau_1, \tau_{01}) - (\theta \tau_1 + \bar{\partial}_2 \theta \wedge \tau_{01}, \theta \tau_{01}) = \bar{\partial}((1 - \theta)\tau_{01}, 0). \]

\[ \square \]

In what follows, we assume $K = \{0\}$. We first take an open set $T$ in $X \times U$ satisfying

\[ \{0\} \times U \subset T \subset \{ (z, t) \in X \times U \mid |z| < 3^{-1} \min \{1, \text{dist}(t, M \setminus U)\} \}. \tag{8.8} \]

Here we set $\text{dist}(t, \emptyset) = +\infty$. Let $i : M \hookrightarrow X$ be the canonical embedding. Set

\[ V_U = U \times \sqrt{-1} \mathbb{R}^n \subset X. \]

Then the following lemma easily follows from (8.8):

**Lemma 8.9** For any compact set $L$ in $V_U \setminus U \subset X \setminus M$, the following set is compact in $U$:

\[ \bigcup_{z \in L} \{ t \in U \mid (z - t, t) \in T \}. \]

Let $u \in \mathcal{D}(U)$, that is, $u$ is an element of the topological dual of the space

\[ \Gamma_c(U; \mathcal{E}_{\mathcal{M}}^{(n)}) \otimes_{\mathcal{Z}(U)} \mathcal{O}_M(U) = \Gamma_c(U; \mathcal{E}_M) \otimes_{\mathcal{Z}(U)} \mathcal{V}_M(U), \]

where $\mathcal{V}_M$ is the sheaf of density on $M$. Recall that $i$ denotes the canonical inclusion $M \hookrightarrow X$. In what follows, the canonical inclusion $M \times M \hookrightarrow X \times M$ is also denoted by $i$. Let $\pi : X \times M \rightarrow X$ be the canonical projection. For any $\varphi \in \mathcal{E}_{X \times M}^{(n,q,0)} \otimes_{\mathcal{Z}_{X \times M}} i^*_q i^{-1} \pi^{-1} \mathcal{O}_X$, we formally define $u \ast \varphi = \langle u, \varphi(z - t, t) \rangle$ in the following way: Fix the coordinates $x$ and $z = x + \sqrt{-1}y$ of $M$ and $X$, respectively, and assume $\varphi$ has a form

\[ \varphi = \sum_{|J| = q} \varphi^J(z, t) d\bar{z}_J \wedge dz \otimes a_X \in \mathcal{E}_{X \times M}^{(n,q,0)} \otimes_{\mathcal{Z}_{X \times M}} i^*_q i^{-1} \pi^{-1} \mathcal{O}_X, \]

where $dz = dz_1 \wedge \cdots \wedge dz_n$. Then we define

\[ u \ast \varphi = \langle u(t), \varphi(z - t, t) \rangle = \sum_{|J| = q} \langle u(t), \varphi^J(z - t, t)dt \otimes e_M \rangle d\bar{z}_J \otimes (a_X \otimes e^{-1}_M). \tag{8.10} \]

Here $e_M$ is a section in $\mathcal{O}_M(U)$ which generates $\mathcal{O}_{M,x}$ over $\mathcal{Z}$ at each $x \in U$ and, through the identification $i^{-1} \mathcal{O}_X \simeq \mathcal{O}_{M/X} \otimes_{\mathcal{Z}_M} \mathcal{O}_M$, we regard $a_X \otimes e^{-1}_M \in (i^{-1} \mathcal{O}_X \otimes \mathcal{O}_M(U))$ as a section in $\mathcal{O}_{M/X}(U)$. Set the coverings $\mathcal{V}_U = \{ V_0 = V_U \setminus U, V_1 = V_U \}$, $\mathcal{V}_U^\prime = \{ V_0 \}$, where $V_U = U \times \sqrt{-1} \mathbb{R}^n \subset X$. Then, for

\[ \mu \otimes a_X = (\mu_1 \otimes a_X, \mu_{01} \otimes a_X) \in C^q((\mathcal{S}_{(0)}^t, \mathcal{S}_{(0)}^t) \times U)(j_T, t^{-1} \mathcal{E}_{X \times M}^{(n,q,0)} \otimes_{\mathcal{Z}(X \times U)} i^{-1} \mathcal{O}_X(M), \]

it follows from Lemma 8.9 that $u \ast (\mu_1 \otimes a_X)$ and $u \ast (\mu_{01} \otimes a_X)$ are well-defined, and they belong to $\mathcal{E}^{(0,q)}(V_1) \otimes \mathcal{O}_{M/X}(U)$ and $\mathcal{E}^{(0,q-1)}(V_{01}) \otimes \mathcal{O}_{M/X}(U)$, respectively. That is,

\[ u \ast (\mu \otimes a_X) = (u \ast (\mu_1 \otimes a_X), u \ast (\mu_{01} a_X)) \in \mathcal{E}^{(0,q)}(V_U, \mathcal{V}_U^\prime) \otimes_{\mathcal{Z}(U)} \mathcal{O}_{M/X}(U). \]

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Let $h \otimes a_X \in H^n_{[0,1]}(X \times U; \mathcal{O}\mathcal{E}^{(n,0)}(X \times U) \otimes \mathcal{O}\mathcal{E}^{(n,0)}(X \times U))$. It follows from Lemma 8.6 that we can find a representative

$$\tau = (\tau_1(z,t), \tau_0(z,t)) \in \mathcal{E}^{(n,n;0)}_{X \times M}(X \times U) \oplus \mathcal{E}^{(n,n-1;0)}_{X \times M}((X \setminus \{0\}) \times U)$$

of $h$ which satisfies

$$\text{Supp}_{X \times U}(\tau_1) \subset T, \quad \text{Supp}_{(X \setminus \{0\}) \times U}(\tau_0) \subset T.$$  \hspace{1cm} (8.11)

**Lemma 8.12** The morphism

$$u \ast (\bullet) : C^\bullet((S_{\{0\}}, S'_{\{0\}}) \times U)(j_{T!}j_T^{-1}\mathcal{E}^{(n,\bullet;0)}_{X \times M}) \otimes i^{-1} \mathcal{O}\mathcal{E}^{(n,0)}_{X \times M} \to \mathcal{E}^{(0,\bullet)}(\mathcal{V}_U, \mathcal{V}'_U) \otimes i^{-1} \mathcal{O}\mathcal{E}^{(0,0)}_{M/X}(U)$$

is that of complexes. In particular, $[u \ast (\tau \otimes a_X)]$ defines a hyperfunction on $U$ and it does not depend on the choices of a representative $\tau = (\tau_1(z,t), \tau_0(z,t))$ of $h$.

**Proof:** For fixed coordinates, we set

$$\varphi = \sum_{|J|=q} \varphi(z,t)^J d\bar{z}_J \wedge dz \in j_{T!}j_T^{-1}\mathcal{E}^{(n,q;0)}_{X \times M}.$$  

Then it is easy to check

$$u \ast \partial_z (\varphi \otimes a_X) = u \ast \left( \sum_{|J|=q} \partial_z \varphi^J \wedge d\bar{z}_J \wedge dz \otimes a_X \right)$$

$$= \sum_{|J|=q} \sum_{j=1}^n \left( \sum_{j=1}^n \partial_z \varphi^J(z - t, t) dt \otimes e_M \right) d\bar{z}_j \wedge d\bar{z}_j \otimes (a_X \otimes e_M^{-1})$$

$$= \sum_{|J|=q} \left( \sum_{j=1}^n \partial_z \varphi^J(z - t, t) dt \otimes e_M \right) \wedge d\bar{z}_j \otimes (a_X \otimes e_M^{-1})$$

$$= \partial (u \ast (\varphi \otimes a_X)).$$

Now if we set

$$\tau \otimes a_X = (\tau_1 \otimes a_X, \tau_0 \otimes a_X) \in C^q((S_{\{0\}}, S'_{\{0\}}) \times U)(j_{T!}j_T^{-1}\mathcal{E}^{(n,\bullet;0)}_{X \times M}) \otimes i^{-1} \mathcal{O}\mathcal{E}^{(n,0)}_{X \times M},$$

the lemma follows from the following commutativity:

$$u \ast \partial_z (\tau \otimes a_X) = \partial \left( u \ast (\tau_1 \otimes a_X), (\tau_1 - \partial_z \tau_0) \otimes a_X \right)$$

$$= (\partial (u \ast (\tau_1 \otimes a_X)), u \ast (\tau_1 \otimes a_X) - \partial (u \ast (\tau_0 \otimes a_X)))$$

$$= \partial (u \ast (\tau \otimes a_X)).$$

\[\square\]

Let $h \otimes a_X \in H^n_{[0,1]}(X \times U; \mathcal{O}\mathcal{E}^{(n,0)}(X \times U) \otimes \mathcal{O}\mathcal{E}^{(n,0)}(X \times U))$ and $\tau$ a representable of $h$. Then thanks to the lemma, hereafter, we write $[u \ast (h \otimes a_X)]$ instead of $[u \ast (\tau \otimes a_X)]$.

**Lemma 8.13** We have $\text{Supp}([u \ast (h \otimes a_X)]) \subset \text{Supp}(u)$.  

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Then we can easily see that

\[ \text{Proof:} \]

We forget the orientation for simplicity. For \( \epsilon > 0 \), take an open set \( T_\epsilon \) with

\[ \{0\} \times U \subset T_\epsilon \subset \{ (z,t) \in X \times U \mid |z| < 3^{-1} \min \{\epsilon, \text{dist}(t, M \setminus U)\} \}. \]

Choose a representative \( \tau_\epsilon = (\tau_{1,\epsilon}, \tau_{01,\epsilon}) \) of \( h \) with the condition

\[ \operatorname{Supp}_{X \times U}(\tau_{1,\epsilon}) \subset T_\epsilon, \quad \operatorname{Supp}_{(X, \{0\}) \times U}(\tau_{01,\epsilon}) \subset T_\epsilon. \]

Then we can easily see that

\[ \operatorname{Supp}(u \ast \tau_{1,\epsilon}) \cup \operatorname{Supp}(u \ast \tau_{01,\epsilon}) \subset \{ z = x + \sqrt{-1} y \in \mathbb{C}^n \mid \text{dist}(x, \operatorname{Supp}(u)) \leq \epsilon \}, \]

from which we have

\[ \operatorname{Supp}([u \ast h]) = \operatorname{Supp}([u \ast \tau]) \subset \{ x \in U \mid \text{dist}(x, \operatorname{Supp}(u)) \leq \epsilon \}. \]

Since \( \epsilon > 0 \) is arbitrary, this completes the proof. \( \square \)

Now we take \( h \otimes a_X \) in the above argument to be the image of Dirac’s density \( \delta(x)dx \otimes a_X \in H^n_{\{0\}}(X; \mathcal{O}^{(n)} \otimes _{\mathbb{Z}(x)} i^{-1} or_X(M)) \) by \( \iota \). Note that \( \delta(x)dx \otimes a_X \) is defined by the image of \( 1 \in \mathbb{C} \) by the canonical morphism

\[ \mathbb{C} \rightarrow H^n_{\{0\}}(X; \mathcal{O}^{(n)} \otimes _{\mathbb{Z}(x)} i^{-1} or_X(M)), \]

which is the topological dual of the continuous morphism

\[ \mathcal{A}(\{0\}) = \lim_{\{0\} \subset V} \mathcal{O}(V) \rightarrow \mathbb{C}, \quad \varphi \mathcal{A}(\{0\}) = \varphi(0). \tag{8.14} \]

We denote by the same symbol \( \delta(x)dx \otimes a_X \) its image through the morphism \( \iota \otimes \text{id} : H^n_{\{0\}}(X; \mathcal{O}^{(n)} \otimes _{\mathbb{Z}(x)} i^{-1} or_X(M)) \rightarrow H^n_{\{0\} \times U}(X \times U; \mathcal{O} \mathcal{E}^{(n;0)} \otimes _{\mathbb{Z}(x \times U)} i^{-1} or_X(M)). \]

**Definition 8.15** We define the map \( \iota_{\mathcal{D}(U)} : \mathcal{D}(U) \rightarrow \mathcal{B}(U) \) by

\[ \iota_{\mathcal{D}(U)}(u) = [u \ast (\delta(x)dx \otimes a_X)]. \tag{8.16} \]

**Remark 8.17** The definition of \( u \ast \varphi \) given in (8.10) depends on the choices of coordinates. However, \( \iota_{\mathcal{D}(U)} \) is independent of such choices, which will be shown later.

Note that, by Lemma 8.13, \( \iota_{\mathcal{D}(U)} \) is a local operator, that is, we have

\[ \operatorname{Supp}(\iota_{\mathcal{D}(U)}(u)) \subset \operatorname{Supp}(u) \quad \text{for} \quad u \in \mathcal{D}(U). \tag{8.18} \]

Using this local property, we can show

**Lemma 8.19** For any open sets \( U_1 \subset U_2 \) in \( M \), we have the following commutative diagram:

\[ \mathcal{D}(U_2) \xrightarrow{\iota_{\mathcal{D}(U_2)}} \mathcal{B}(U_2) \]

\[ \downarrow \quad \downarrow \]

\[ \mathcal{D}(U_1) \xrightarrow{\iota_{\mathcal{D}(U_1)}} \mathcal{B}(U_1). \]

In particular, \( \iota_{\mathcal{D}} = \{ \iota_{\mathcal{D}(U)} \}_U \) induces the sheaf morphism \( \mathcal{D} \rightarrow \mathcal{B} \).
PROOF: Let $\tau^1$ (resp. $\tau^2$) be representatives of $\delta(x)dx$ on $X \times U_1$ (resp. $X \times U_2$) with the support condition, where the corresponding open set $T$ is denoted by $T_1$ (resp. $T_2$). Let $u \in \mathcal{Db}(U_2)$. Since $\mathcal{D}$ is a sheaf, it is enough to show, for any $W$ being relatively compact in $U_1$,

$$\iota_{\mathcal{Db}(U_2)}(u)|_W = \iota_{\mathcal{Db}(U_1)}(u|_{U_1})|_W.$$  

Let $\theta$ be a $C^\infty$-function on $X$ whose support is contained in $U_1$ and which is identically 1 on a neighborhood of $W$. By the local property of $\iota_{\mathcal{Db}}$, we have

$$\iota_{\mathcal{Db}(U_2)}(u)|_W = \iota_{\mathcal{Db}(U_2)}(\theta u)|_W, \quad \iota_{\mathcal{Db}(U_1)}(u|_{U_1})|_W = \iota_{\mathcal{Db}(U_1)}((\theta u)|_{U_1})|_W.$$  

Hence, from the beginning, we may assume that the support of $u$ is contained in $U_1$. Let $W'$ be a relatively compact open set in $U_1$ which contains the support of $u$ and $W$. Then, we can take $\tau^1$ and $\tau^2$ so that $\tau^1 = \tau^2$ holds on a neighborhood of $X \times W'$ and their supports are contained in the set

$$\{ (z,t) \in X \times U \mid |z| < \epsilon \}$$

for a sufficiently small $\epsilon > 0$. For such $\tau^1$ and $\tau^2$, if $\epsilon > 0$ is sufficiently small, we have, on $W$,

$$\iota_{\mathcal{Db}(U_1)}(u|_{U_1}) = [u * (\tau^1 \otimes a_X)] = [u * (\tau^2 \otimes a_X)] = \iota_{\mathcal{Db}(U_2)}(u).$$

\[\Box\]

**Lemma 8.20** For $u \in \mathcal{Db}(U)$ with a compact support, we have

$$\langle u, \psi \rangle = \langle \iota_{\mathcal{Db}(U)}(u), \psi \rangle$$

for any $\psi \in (\mathcal{A}(n) \otimes \mathcal{O}_M)(U)$, where the left-side is the inner product of a distribution with a compact support and the right-side is that of a hyperfunction.

**PROOF:** Let $h$ be a real analytic function on $U$. We also denote by $h$ its complexification. We set $\psi = h(t)dt \otimes e_M$ on $U$. Then for the delta current $\delta = [(\tau_1, \tau_{01})]$, the lemma follows from

$$\langle u, h(t)dt \otimes e_M \rangle = \langle u, \left( \int_{R_1} h(z)(\tau_1 \otimes a_X) + \int_{R_{01}} h(z)(\tau_{01} \otimes a_X) \right) dt \otimes e_M \rangle$$

$$= \int_{R_1} (u * (\tau_1 \otimes a_X)) \wedge h(z)dz \otimes e_M$$

$$+ \int_{R_{01}} (u * (\tau_{01} \otimes a_X)) \wedge h(z)dz \otimes e_M$$

$$= \int_X \iota_{\mathcal{Db}(U)}(u) \sim (h(z)dz \otimes e_M) = \langle \iota_{\mathcal{Db}(U)}(u), h(t)dt \otimes e_M \rangle.$$  

\[\Box\]

**Corollary 8.21** $\iota_{\mathcal{Db}(U)}$ is independent of the choices of coordinates in (8.10).
Proof: If \( u \) has a compact support, the claim is a consequence of the duality theorem for hyperfunctions with compact support and the above lemma. The general case follows from the local property of \( \iota_{\mathcal{D}(U)} \) by considering \( \theta u \) instead of \( u \) with \( \theta \in \Gamma_c(U; E_M) \). \( \square \)

Proposition 8.22 \( \iota_{\mathcal{D}(U)} \) is injective.

Proof: From the beginning, we may assume \( U \) to be relatively compact. Through the proof, we fix the orientation and forget \( e_M, \alpha_X \), etc. Let \( u \in \mathcal{D}(U) \) with \( \iota_{\mathcal{D}(U)}(u) = 0 \).

Since \( \mathcal{D} \) is a sheaf, it suffices to show \( u|_W = 0 \) for any relatively compact open set \( W \) in \( U \). This is equivalent to saying that

\[
\langle u, \varphi \rangle = 0 \quad \text{for } \varphi \in \Gamma_c(U; E_M^{(n)}) \text{ with Supp(\varphi) } \subset W.
\]

Let \( \varphi \in \Gamma_c(U; E_M^{(n)}) \) with Supp(\( \varphi \)) \( \subset W \), and let \( W' \) and \( W'' \) be relatively compact open sets in \( U \) with \( \overline{W'} \subset W' \subset \overline{W''} \subset W'' \). Let \( \theta \) be a \( C^\infty \) function on \( U \) satisfying \( \theta = 1 \) on a neighborhood of \( \overline{W''} \) and Supp(\( \theta \)) \( \subset W'' \). Then, by the local property of \( \iota_{\mathcal{D}(U)} \), we have

\[
\iota_{\mathcal{D}(U)}(\theta u)|_{W'} = 0.
\]

Set, for \( \epsilon > 0 \),

\[
\varphi_\epsilon(z) = \frac{1}{(\sqrt{\pi \epsilon})^n} \int_{\mathbb{R}^n} \varphi(t) e^{-(z-t)^2/\epsilon} dt.
\]

It is well-known that

1. \( \varphi_\epsilon(z) \in \mathcal{O}^{(n)}(\mathbb{C}^n) \).
2. \( \varphi_\epsilon|_{\mathbb{R}^n} \to \varphi \) uniformly convergent on a compact subset in \( \mathbb{R}^n \).
3. \( \varphi_\epsilon \to 0 \) uniformly convergent on a compact subset in

\[
L = \bigcap_{a \in W} \{ z = x + \sqrt{y} \in \mathbb{C}^n \mid |y| < |x - a| \}.
\]

One should be aware that we can say nothing about convergence of \( \varphi_\epsilon \) outside of the region specified in the above (2) and (3).

Since \( \text{Supp}(\iota_{\mathcal{D}(U)}(\theta u)) \subset W'' \setminus W' \), by the same reasoning as that for Lemma 8.6, we can find a representative \( \tau(z) = (\tau_1(z), \tau_0(z)) \) of \( \iota_{\mathcal{D}(U)}(\theta u) \) which satisfies the condition

\[
\text{Supp}(\tau_1) \subset L, \quad \text{Supp}(\tau_0) \subset L.
\]

Then we can take an open subset \( R_1 \) (with \( C^\infty \) boundary \( R_{01} \)) that contains the support of \( \tau \) and that is contained in \( L \), for which we get

\[
\langle \theta u, \varphi_\epsilon|_U \rangle = \langle \iota_{\mathcal{D}(U)}(\theta u), \varphi_\epsilon \rangle = \int_{R_1} \varphi_\epsilon \tau_1 + \int_{R_{01}} \varphi_\epsilon \tau_{01} = \int_{R_1} \varphi_\epsilon \tau_1.
\]

When \( \epsilon \to 0 \), the rightmost term of the above equation tends to 0 by the property (3) of \( \varphi_\epsilon \). Furthermore, it follows from the property (2) that we have, when \( \epsilon \to 0 \),

\[
\langle \theta u, \varphi_\epsilon|_U \rangle \to \langle \theta u, \varphi \rangle = \langle u, \varphi \rangle.
\]

Therefore we finally obtained \( \langle u, \varphi \rangle = 0 \), which completes the proof. \( \square \)
A Compatibility of boundary value morphisms

The boundary value morphism was first constructed in [29], and then, it was extended to more general cases by P. Schapira from the viewpoint of boundary value problems (see Section 11.5 in [19]). In this appendix, we briefly recall its functorial construction due to P. Schapira, and then, we will show that the boundary value morphism given in Subsection 7.2 coincides with the current one.

Let $V$ be an open neighborhood of $M$ and $\Omega$ an open subset in $X$ which satisfies the conditions (B$_1$) and (B$_2$) given in Subsection 7.2. For simplicity, we also assume the following additional condition in functorial construction:

(B$_3$) $\Omega$ is cohomologically trivial, that is, $\Omega$ satisfies

$$\mathcal{R}\mathcal{H}om_{\mathcal{C}_X}(\mathcal{C}_\Omega, \mathcal{C}_X) = \mathcal{C}_\Omega$$

Note that the above condition is satisfied if the inclusion $j : \Omega \to X$ is locally homeomorphic to the inclusion of an open convex subset into $\mathbb{C}^n$.

From the assumption $\bar{\Omega} \supset M$, we have a canonical morphism $\mathcal{C}_\Omega \to \mathcal{C}_M$. Applying the functor $\mathcal{R}\mathcal{H}om_{\mathcal{C}_X}(\bullet, \mathcal{C}_X)$ to this morphism, thanks to the condition (B$_3$) and the fact $\mathcal{R}\mathcal{H}om_{\mathcal{C}_X}(\mathcal{C}_M, \mathcal{C}_X) \simeq \mathcal{C}_M \otimes_{z_M} H^n_M(X; \mathbb{Z}_X)[-n]$, we have the morphism in $\mathcal{D}^b(\mathbb{Z}_X)$

$$\mathcal{C}_\Omega \xrightarrow{i^*} \mathcal{C}_M \otimes_{z_M} H^n_M(X; \mathbb{Z}_X)[-n], \quad i^* = \mathcal{R}\mathcal{H}om_{\mathcal{C}_X}(i, \mathcal{C}_X). \quad (A.1)$$

Note that, for any complex $\mathcal{F}$, we have the formulas

$$\mathcal{R}\text{Hom}_{\mathcal{C}_X}(\mathcal{C}_\Omega, \mathcal{F}) = \mathcal{R}\Gamma(\Omega; \mathcal{F}) \quad \text{and} \quad \mathcal{R}\text{Hom}_{\mathcal{C}_X}(\mathcal{C}_M, \mathcal{F}) = \mathcal{R}\Gamma_M(X; \mathcal{F}).$$

Then, applying $\mathcal{R}\text{Hom}_{\mathcal{C}_X}(\bullet, \mathcal{O})$ to (A.1) and taking the 0-th cohomology groups, we have obtained the boundary value map $\hat{b}_\Omega$ in a functorial way:

$$\hat{b}_\Omega : \mathcal{O}(\Omega) \xrightarrow{i^{**}} H^n_M(X; \mathcal{O}) \otimes_{z_M(M)} H^n_M(X; \mathbb{Z}_X) = \mathcal{B}(M), \quad i^{**} = \text{Hom}(i^*, \mathcal{O}).$$

Now we give the theorem which guarantees the coincidence of the boundary value morphism in our framework and the functorial one constructed above. First recall the condition (B$_2$) in Subsection 7.2 and its local version (B$_2^{lc}$) in Subsection 7.5. We also introduce the condition (B$_1^l$) which is stronger version of (B$_1$) in Subsection 7.2.

(B$_1^l$) For any $x_0 \in M$, there exist an open neighborhood $W$ of $x_0$ with a ($C^1$-class) local trivialization $\iota : (M \cap W, W) \simeq (\mathbb{R}_x^n, \mathbb{C}^n = \mathbb{R}_x^n \times \sqrt{-1}\mathbb{R}_y^n)$ and a non-empty open cone $\Gamma \subset \mathbb{R}_y^n$ such that $\mathbb{R}_x^n \times \sqrt{-1}\Gamma \subset \iota(W \cap \Omega)$.

**Theorem A.2** Assume the pair $(V, \Omega)$ satisfies the conditions (B$_1^l$), (B$_2$), (B$_2^{lc}$) and (B$_3$). Then the morphisms $b_\Omega$ and $\hat{b}_\Omega$ coincide.
Since \( \{ \mathcal{B}(U) \}_{U \subset M} \) forms a sheaf and since the boundary value morphisms and restriction maps of sheaves commute by Corollary 7.16, by taking the condition \( (\mathcal{B}_Y^\text{loc}) \) into account, it suffices to show the claim locally. Hence we may assume that \( M \) is a convex open subset in \( \mathbb{R}^n \), \( X = M \times \sqrt{1-y} \mathbb{R}^n \) and \( V \) is the open tubular neighborhood

\[
V = \{ (z = x + \sqrt{-1}y) \in \mathbb{C}^n \mid x \in M, |y| < \epsilon \}
\]

with some \( \epsilon > 0 \). Further, \( \Omega \) is assumed to be \( (M \times \sqrt{-1} \Gamma) \cap V \) where \( \Gamma \) is an open proper convex cone in \( \mathbb{R}_n^+ \). Then, we take \( (n+1) \)-vectors \( \eta_1, \ldots, \eta_{n+1} \) in \( \mathbb{R}^n_y \) as was specified in Example 7.14 and define the open half spaces \( H_k, k = 1, \ldots, n+1 \), in \( \mathbb{R}^n \) and \( C^\infty \)-functions \( \varphi_k, k = 1, \ldots, n+1 \), as in the same example. Since the boundary value morphisms and restriction maps of sheaves commute, we may assume

\[
\Gamma = \bigcap_{1 \leq k \leq n} H_k
\]

from the beginning. Now let us define another pair of coverings \( (S, S') \) of \( (V, V \setminus M) \) by

\[
S = \{ S_1, \ldots, S_{n+1}, S_{n+2} \}, \quad S' = \{ S_1, \ldots, S_{n+1} \},
\]

where \( S_{n+2} = V \) and \( S_k = (M \times \sqrt{-1} H_k) \cap V, k = 1, \ldots, n+1 \). Note that \( (S, S') \) is a Leray covering with respect to either of \( \mathcal{C}_X \) and \( \mathcal{O} \). Note also that \( (S, S') \) is a pair of coverings finer than \( (\mathcal{W}, \mathcal{W}') \). We denote by \( C^\bullet(S, S'; \mathcal{I}) \) the complex of relative Čech cochains on \( (S, S') \) with coefficients in a sheaf \( \mathcal{I} \).

Let \( \nu = (0, \nu_{01}) \) be the element of \( \mathcal{E}^{(n)}(\mathcal{W}, \mathcal{W}') \) defined in Example 7.14, and set \( \Lambda = \{ 1, 2, \ldots, n+1, n+2 \} \). We also set, for \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \Lambda^k \),

\[
S_\alpha = S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}.
\]

**Lemma A.3** \( \nu \) is a generator of \( H^n_D(\mathcal{W}, \mathcal{W}') \approx H^n_M(X; \mathbb{C}_X) \).

**Proof:** Let us consider the diagram of complexes:

\[
C^\bullet(S, S'; \mathbb{C}_X) \xrightarrow{h} \mathcal{E}^{\bullet}(S, S') \xleftarrow{r} \mathcal{E}^{\bullet}(\mathcal{W}, \mathcal{W}'),
\]

where \( h \) and \( r \) are canonical morphisms of complexes. Since these complexes are quasi-isomorphic to \( R\Gamma_M(X; \mathbb{C}_X) \), the morphisms \( h \) are \( r \) are quasi-isomorphic.

Let us define \( \sigma = \{ \sigma_\alpha \}_{\alpha \in \Lambda^{n+1}} \in C^n(S, S'; \mathbb{Z}_X) \) by

\[
\sigma_\alpha = \left\{ \begin{array}{ll}
(-1)^n & \text{if } \alpha = (1, 2, \ldots, n, n+2), \\
0 & \text{if } \alpha \text{ contains } n+1.
\end{array} \right. \quad \text{(A.4)}
\]

Then \( [\sigma] \) belongs to \( H^n(S, S'; \mathbb{Z}_X) \) and it is a generator of \( H^n(S, S'; \mathbb{Z}_X) \). Here we take \( (-1)^n \) so that \( [\sigma] \) becomes the positively oriented generator under the standard orientation on \( \mathbb{R}^n_y \). Hence it suffices to show that \( h(\sigma) \) and \( r(\nu) \) are the same in \( H^n_D(S, S') \), and it can be shown by repeated applications of the following easy lemma. \( \square \)
Lemma A.5 Assume \( q_1 > 0 \). Let \( \omega = \{ \omega_\alpha \}_{\alpha \in \Lambda^{q_1+1}} \) be in \( C^{q_1+1}(\mathcal{S}, \mathcal{S}'; \mathcal{E}(q_2)) \) with \( \delta(\omega) = 0 \), and define \( \omega' = \{ \omega'_\beta \}_{\beta \in \Lambda^{q_1}} \in C^{q_1}(\mathcal{S}, \mathcal{S}'; \mathcal{E}(q_2)) \) by

\[
\omega'_\beta = \sum_{\lambda \in \Lambda} \varphi_\lambda \omega_{\lambda \beta} \quad (\beta \in \Lambda^{q_1}).
\]

Then we have \( \omega = \delta(\omega') \).

Remark A.6 The same result holds for \( C^\bullet(\mathcal{S}, \mathcal{S}'; \mathcal{E}(p, \bullet)) \). That is: Let \( q_1 > 0 \) and let \( \omega = \{ \omega_\alpha \}_{\alpha \in \Lambda^{q_1+1}} \) be in \( C^{q_1+1}(\mathcal{S}, \mathcal{S}'; \mathcal{E}(p, q_2)) \) with \( \delta(\omega) = 0 \). Define \( \omega' = \{ \omega'_\beta \}_{\beta \in \Lambda^{q_1}} \in C^{q_1}(\mathcal{S}, \mathcal{S}'; \mathcal{E}(p, q_2)) \) by the same formula as in the above lemma. Then we have \( \omega = \delta(\omega') \).

Let us compute functors \( R\mathcal{H}om_{\mathcal{C}_X}(\bullet, \mathcal{C}_X) \) and \( R\mathcal{H}om_{\mathcal{C}_X}(\bullet, \mathcal{O}) \) concretely. Set \( \Lambda' = \{ 1, 2, \ldots, n + 1 \} \). First define the sheaf \( \wedge_{\alpha \in \Lambda^k} \mathcal{C}_{\alpha}^{-} \). It is a subsheaf of \( \bigoplus_{\alpha \in \Lambda^k} \mathcal{C}_{\alpha}^{-} \) which consists of alternating sections with respect to the index \( \alpha \in \Lambda^k \). We define the differential

\[
\delta_k : \wedge_{\alpha \in \Lambda^k} \mathcal{C}_{\alpha}^{-} \rightarrow \wedge_{\alpha \in \Lambda^{k+1}} \mathcal{C}_{\alpha}^{-}
\]

the same way as for a Čech complex. We have the complex

\[
\mathcal{L} : 0 \rightarrow \wedge_{\alpha \in \Lambda^0} \mathcal{C}_{\alpha}^{-} \rightarrow \wedge_{\alpha \in \Lambda^1} \mathcal{C}_{\alpha}^{-} \rightarrow \cdots \rightarrow \wedge_{\alpha \in \Lambda^n} \mathcal{C}_{\alpha}^{-} \rightarrow 0,
\]

where \( \wedge_{\alpha \in \Lambda^0} \mathcal{C}_{\alpha}^{-} \) is located at degree \( -n \) and \( \wedge_{\alpha \in \Lambda^0} \mathcal{C}_{\alpha}^{-} \) at degree 0. We define the morphism

\[
\iota : \wedge_{\alpha \in \Lambda^0} \mathcal{C}_{\alpha}^{-} \rightarrow \mathcal{C}_M \otimes_{\mathcal{O}(X)} H^n(\mathcal{S}, \mathcal{S}'; \mathcal{Z}_X)
\]

by assigning an alternating section \( c_\alpha \) on \( \mathcal{C}_{\alpha}^{-} \) to \( c_\alpha|_M \otimes 1_\alpha \). Here \( 1_\alpha \) denotes an alternating section with value 1 on \( \mathcal{S}_\alpha \). Then we can extend \( \iota \) to a morphism of complexes from \( \mathcal{L} \) to the single complex \( \mathcal{C}_M \otimes_{\mathcal{O}(X)} H^n(\mathcal{S}, \mathcal{S}'; \mathcal{Z}_X) \). Now we can easily see:

Lemma A.7 The morphism

\[
\mathcal{L} \rightarrow \mathcal{C}_M \otimes_{\mathcal{O}(X)} H^n(\mathcal{S}, \mathcal{S}'; \mathcal{Z}_X)
\]

is quasi-isomorphic. That is, \( \mathcal{L} \) is a resolution complex of \( \mathcal{C}_M \otimes_{\mathcal{O}(X)} H^n(\mathcal{S}, \mathcal{S}'; \mathcal{Z}_X) \).

Proof: For \( k = 1, \ldots, n + 1 \), we have the exact sequence

\[
0 \rightarrow \mathcal{C}_{\mathcal{V} \times_k} \rightarrow \mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{C}_{\mathcal{S}_k} \rightarrow 0.
\]

Define the complex \( \mathcal{L}_k \) by

\[
0 \rightarrow \mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{C}_{\mathcal{S}_k} \rightarrow 0,
\]

where \( \mathcal{C}_{\mathcal{S}_k} \) is located in degree 0 of this complex. Then the above exact sequence implies that \( \mathcal{L}_k \) is quasi-isomorphic to \( \mathcal{C}_{\mathcal{V} \times \mathcal{S}_k}[1] \). Hence the single complex \( \mathcal{L} \) associated with

\[
\mathcal{L}_1 \otimes_{\mathcal{C}_X} \mathcal{L}_2 \otimes_{\mathcal{C}_X} \cdots \otimes_{\mathcal{C}_X} \mathcal{L}_{n+1}
\]

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is quasi-isomorphic to the complex
\[(\mathbb{C} \tau S_1 \otimes_{c_X} \mathbb{C} \tau S_2 \otimes_{c_X} \cdots \otimes_{c_X} \mathbb{C} \tau S_{n+1})[n + 1]\]
which is isomorphic to 0 because of \((\overline{V \setminus S_1}) \cap (\overline{V \setminus S_2}) \cap \cdots \cap (\overline{V \setminus S_{n+1}}) = \emptyset\). Therefore, the complex \(\tilde{\mathcal{L}}\) becomes exact, and the result follows from the fact that the degree 0 term of the complex \(\tilde{\mathcal{L}}\) is
\[\mathbb{C} S_1 \otimes_{c_X} \mathbb{C} S_2 \otimes_{c_X} \cdots \otimes_{c_X} \mathbb{C} S_{n+1} = \mathbb{C} M\]
due to \(\overline{S_1} \cap \overline{S_2} \cap \cdots \cap \overline{S_{n+1}} = M\).

Furthermore, since we have \(S_{12\ldots n} \cap S_{n+2} = (M \times \sqrt{-1}\Gamma) \cap V = \Omega\), we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{C} \times_{\omega_{\mathcal{X}(X)}} H^n(S, S'; \mathbb{Z}) & \xrightarrow{\hat{b}_\Omega} & H^n(S, S'; \mathbb{Z}) \\
\mathbb{C} \times_{\omega_{\mathcal{X}(X)}} H^n(S, S'; \mathbb{Z}) & \xrightarrow{\hat{b}_\Omega} & H^n(S, S'; \mathbb{Z}),
\end{array}
\]
where \(\mathbb{C} \times_{\omega_{\mathcal{X}(X)}} \mathbb{C}\) is the subsheaf of \(\bigwedge_{\alpha \in \Lambda^n} \mathbb{C} X\) which consists of alternating sections only on \(S_{12\ldots n} \cap S_{n+2}\).

Since \((S, S')\) is an acyclic covering with respect to \(\Theta\) and since each open subset \(S_\alpha\) \((\alpha \in \Lambda^k)\) is cohomologically trivial, we can compute \(\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C} M, \Theta)\) by first applying \(\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C}, \mathbb{C})\) to the above resolution \(\mathcal{L}\), and then, applying \(\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C}, \Theta)\) to the resulting complex. As a conclusion, we have
\[\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C} M, \Theta) \simeq C^\bullet(S, S'; \Theta)\).

Further, by applying the functor \(\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C}, \mathbb{C})\) to the above commutative diagram, and then, applying \(\mathcal{R}\text{Hom}_{\mathcal{X}}(\mathbb{C}, \Theta)\) to the resulting diagram, we see that the morphism
\[\hat{b}_\Omega : \Theta(\Omega) \longrightarrow H^n_M(X; \Theta) \otimes_{\omega_M} H^n_M(X; \mathbb{Z}) \simeq H^n(S, S'; \Theta) \otimes_{\omega_{\mathcal{X}(X)}} H^n(S, S'; \mathbb{Z})\]
is given by
\[\hat{b}_\Omega(f) = [f\sigma] \otimes [\sigma] \in H^n(S, S'; \Theta) \otimes_{\omega_{\mathcal{X}(X)}} H^n(S, S'; \mathbb{Z}),\]
where \(\sigma\) is defined in (A.4).

Now we consider the diagram
\[C^\bullet(S, S'; \Theta) \xrightarrow{b'_\Theta} \Theta^{(0, \bullet)}(S, S') \xleftarrow{r'_\Theta} \Theta^{(0, \bullet)}(\mathcal{W}, \mathcal{W}'),\]
where \(b'_\Theta\) and \(r'_\Theta\) are canonical morphisms of complexes and they are quasi-isomorphic. Then, by the remark after Lemma A.5, we see that \(h'(f\sigma)\) and \(r'(f\rho(\nu))\) are the same element in \(H^{0,n}_\Theta(S, S')\). This implies \(b_\Omega(f) = b_\Omega(f)\) and the theorem follows.

Clearly \(b_\Omega\) is a morphism of \(\mathcal{D}\)-modules. Therefore, by the theorem, we have:

**Corollary A.8** The boundary value morphism \(b_\Omega\) is a morphism of \(\mathcal{D}\)-modules.
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