A general exhaustive generation algorithm for Gray structures

Antonio Bernini  Elisabetta Grazzini  Elisa Pergola  Renzo Pinzani

Dipartimento di Sistemi e Informatica, Università di Firenze. Viale G. B. Morgagni 65, 50134 Firenze, Italy.

Abstract

Starting from a succession rule for Catalan numbers, we define a procedure encoding and listing the objects enumerated by these numbers such that two consecutive codes of the list differ only for one digit. Gray code we obtain can be generalized to all the succession rules with the stability property: each label \(k\) has in its production two labels \(c_1\) and \(c_2\), always in the same position, regardless of \(k\). Because of this link, we define Gray structures the sets of those combinatorial objects whose construction can be encoded by a succession rule with the stability property. This property is a characteristic that can be found among various succession rules, as the finite, factorial or transcendental ones.

We also indicate an algorithm which is a very slight modification of the Walsh’s one, working in a \(O(1)\) worst-case time per word for generating Gray codes.

1 Introduction

The matter of encoding and listing the objects of a particular class is common to several scientific topics, ranging from computer science and hardware or software testing to chemistry, biology and biochemistry. Often, it is very useful to have a procedure for listing or generating the objects in a particular order. A very special kind of list is the so called Gray code, where two successive objects are encoded in such a way that their codes differ as little as possible (see below for more details and [15]). There are many applications of the theory of Gray codes for several combinatorial objects, involving permutations [10], binary strings, Motzkin and Schröder...
words [11], derangements [8], involutions [16]. They are also used in other technological subjects as circuit testing, signal encoding [11], data compression and other (we refer to [11] for an exhaustive bibliography on the general matter).

The generation of a Gray code is often strictly connected with the nature of the objects which we are dealing with. So, it seems to have some importance the definition of a Gray code for the objects of the classes with some common characteristic, as the classes enumerated by the same sequence. From the idea of [11], which we briefly recall in the sequel, in this work we develop a procedure for listing the objects of those structures whose exhaustive generation can be encoded by particular succession rules (see below), say succession rules satisfying the stability property (see Section 5). In order to point out the relation between such structures and the possibility to list their objects in a Gray code, we define them Gray structures.

Our discussion moves from the well known succession rule $\Omega_C$, which is a system defined by an axiom $a \in \mathbb{N}$ and a set of productions. The usual notation for a succession rule is the following:

$$\Omega : \left\{ \begin{array}{l} (a) \\ (k) \leadsto (e_1(k))(e_2(k)) \ldots (e_k(k)) , \quad k \in \mathbb{N} \end{array} \right.$$  

The succession rule $\Omega$ can also be described with a rooted tree where the nodes are the labels of $\Omega$: the axiom $(a)$ is the root of the tree and each node with label $(k)$ generates $k$ sons with labels $(e_1(k)), (e_2(k)), \ldots, (e_k(k))$. The structure we obtain is the so called generating tree of $\Omega$ [3, 9].

Our discussion moves from the well known succession rule $\Omega_C$,

$$\Omega_C : \left\{ \begin{array}{l} (2) \\ (k) \leadsto (2)(3) \ldots (k)(k + 1) , \quad k \geq 2 \end{array} \right.$$  

defining the sequence of Catalan numbers and whose first levels of the related generating tree are shown in Figure 1. Each object $x$ with size $n$ corresponds to a node at level $n - 1$ (being the root of the tree at level 0, corresponding to the object of size 1) and can be described by a word $w = w_1 w_2 \ldots w_n$ encoding the path from the root to the node corresponding to $x$: each $w_i$ is the label of a node of the path and is generated by $\Omega_C$. In [1] the authors give a method to exhaustively generate all the objects (words) of a given size $n$ which substantially coincide with the reading from left to right in the $(n - 1)$-th level of the tree. So, the words at level 3 are generated in the following order (see figure 1):

2222, 2223, 2232, 2233, 2234, 2322, 2323, 2332, 2333, 2334, 2342, 2343, 2344, 2345.

In the above list it is possible that two consecutive words differ more than one digits: for instance, 2223 and 2232 differing in two digits or 2334 and 2332 with three different digits. Our aim is to generate all the words of
length \( n \) (naturally without repetitions) in such a way that two consecutive words differ only for one digit. Such a property is strictly related to the concept of “Gray code”, which definition we relate can be found in [15]. Substantially, it can be summed up in the following: a Gray code is an infinite set of word-lists with unbounded word-length such that the Hamming distance between any two adjacent words is bounded independently of the word-length (the Hamming distance is the number of positions in which two words differ). For a complete discussion on Gray codes we refer the reader to the paper of T. Walsh [15].

In Section 2 an informal description of the used strategy for our purpose is presented, referring to objects whose construction can be described by \( \Omega_C \). Then, in Section 3 a rigorous definition of the list (Definition 3.1) and a proof that it is a Gray code (Theorem 3.1) are given. Section 4 presents the application and the analysis to the particular case of Dyck paths, enumerated by Catalan numbers. Finally, Section 5 generalizes the construction of the Gray code to those objects whose generation can be described by succession rules with the stability property. In that section, we also present some examples of Gray structures.

2 The procedure

The strategy used in [1] for listing the objects of size \( n \) corresponds to a visit of all the nodes at level \( n - 1 \) in the generating tree from left to right. So, after the visit of a subtree \( T_i \) is completed, the path from the root to the leftmost node of the successive subtree \( T_{i+1} \) has at least two different nodes with respect to those ones of the last path of the preceding subtree \( T_i \). This is due to the fact that the labels of the sons of a node are visited in the same order they have in the production of the succession rule \( \Omega_C \), where the list of the successors of a label \( (k) \) is \( <2, 3, \ldots, k, k+1> \).

For our purpose we must check that when a subtree has been completely visited and if \( v \) is the last path generated in such a visit, then the successive path \( w \) has only one different digit with respect to the digits of \( v \). We now illustrate the procedure we are going to use referring to Figure 1 where the words of length 4 are generated.

The first object of the list is the word 2222, corresponding to the path from the root to the leftmost node at level 3 in the generating tree. Then, in order to complete the visit of the current subtree, the second word is 2223. At this point, the next path in the list will have a different digit with respect to the digits of 2223, which is not the last one: in order to respect the above definition of Gray code, the third word in the list could be 2323 or 2233. The choice is determined by the leading idea that a successive path \( w = aw_2 \ldots w_n \) must have as much as possible the same edges of the preceding path \( v = av_2 \ldots v_n \) in the list and if \( v_j \) and \( w_j \) are the first nodes
Figure 1: first levels of the generating tree for Catalan numbers (upper figure); generation of the words of length 4 (lower figure).
necessarily different in \( v \) and \( w \), then all the nodes \( v_r \) and \( w_r \) must have the same labels for \( r = j + 1, \ldots, n - 1, n \), in order to respect the Gray code definition. So, the third word is 2233. The fourth and the fifth one are 2234 and 2232, respectively. From the generation of these last two words we can deduce that only the last digit is changed when a same subtree is visited and that the order for changing the last digit is *shifted* with respect to the classical one in a cyclic way in order to complete the set of the sons of the second-last digit: for the sake of clearness in this case the shifted list of the successors of the second-last digit 3 is \( < 3, 4, 2 > \), while the classical one would be \( < 2, 3, 4 > \). This fact can be generalized. Let \( e \) be the first path of a new subtree and let \( i \) and \( k \) be the the last and the second-last digit of \( e \), respectively \((i \neq 2, \text{ see below})\). Then the right order for changing the last digit is \( < i, i + 1, \ldots, k, k + 1, 2, 3, \ldots, i - 1 > \).

The sixth path which is now generated is \( f = 2332 \), according to the above leading idea. Note that the second digit is changed with respect of the second digit of the fifth word and that the third and the fourth digits in \( f \) are the same you find in 2232. The word \( f \) is the first path of a new subtree and then only the last digit has to be changed, till the whole set of the sons of the second-last digit 3 is completed. Since the last digit of \( f \) is 2, one could think that in this case the *shifted production* of the digit 3 coincides with the classical production \( < 2, 3, 4 > \), obtaining that the 6-th, 7-th and 8-th words are 2332, 2333, 2334, respectively. But so doing the procedure fails when it is used to list the words of length 6, as the reader can easily check when he arrives at the generation of the word 234565. The reason of the failure will be clear in the next section, where the rigorous formalization of our procedure is presented. The right way for changing the last digit of \( f \) is to consider the list \( < 2, 3, 4 > \) of the sons of the digit 3, then obtaining the 6-th, 7-th and 8-th words as follows: 2332, 2334 and 2333, respectively. This fact suggest us that if \( f \) is the first word of a new subtree, if its last digit is 2 and if \( k \) is its second-last digit, then the right order for changing the last digit is \( < 2, k + 1, k - 1, k - 2, \ldots, 4, 3 > \). The remaining objects can be now easily obtained, as in Figure 1.

We now summarize the definition of the *shifted production* which is used to change the last digit in the words. Let \( v = v_1v_2 \ldots v_n \) be the first path of a new subtree. Let \( k \) and \( i \) be the second-last and the last digit of \( v \), respectively, then the list \( s(k, i) \) of the sons of \( k \) such that the first son is \( i \), is:

\[
\{ \begin{array}{l}
  s(k, 2) = < 2, k + 1, k, k - 1, \ldots, 4, 3 > \\
  s(k, i) = < i, i + 1, \ldots, k - 1, k, k + 1, 2, 3, \ldots, i - 1 >
\end{array}
\]

### 3 A Gray code for Catalan structures

First we define the lists for the objects whose generating tree can be described by the succession rule for the Catalan numbers we presented in
the previous section, then we will prove (Theorem 3.1) that these lists form a
Gray code, in the sense of the definition in Section 1. The following notation
is used:

- \( \mathcal{L}_k \) = list of the codes of the objects of length \( k \);
- \( l^k_i \) = \( i \)-th element of \( \mathcal{L}_k \);
- \( |\mathcal{L}_k| \) = cardinality of \( \mathcal{L}_k \);
- if \( x \) is a sequence of digits, then \( \overrightarrow{x} \) is the rightmost digit of \( x \);
- \( \Theta \) is the concatenation of lists;
- if \( L \) is a list, then:
  - first(\( L \)) denotes the first element of the list \( L \);
  - last(\( L \)) denotes the last element of the list \( L \);
  - \( x \circ L \) is the list obtained by pasting \( x \) with each element of \( L \).

Our definition is a recursive definition and it is based on a generation of
sublists with increasing length:

**Definition 3.1** The list \( \mathcal{L}_n \) of all the elements of length \( n \) is

\[
\begin{align*}
\mathcal{L}_1 &= < 2 > \\
\mathcal{L}_n &= \Theta^M_{i=1} L^i_n \quad \text{if } n > 1
\end{align*}
\]

where \( M = |\mathcal{L}_{n-1}| \) and \( L^i_n \) is defined by

\[
\begin{align*}
L^1_n &= l^{n-1}_1 \circ s(2, 2) \\
L^i_n &= l^{n-1}_i \circ s(\overrightarrow{l^{n-1}_i}, \overrightarrow{\text{last}(L^{i-1}_n)}) \quad \text{if } i > 1
\end{align*}
\]

The list \( L^1_n \) is obtained by linking together the first element of the list of the
objects of size \( n-1 \) (i.e. \( l^{n-1}_1 \)) with the elements of the list \( s(2, 2) = < 2, 3 > \);
then \( L^1_n \) has always two elements: \( l^{n-1}_1 2 \) and \( l^{n-1}_1 3 \). The next lists \( L^i_n \) with
\( i > 1 \) are obtained as follows:

- consider the \( i \)-th element of \( \mathcal{L}_{n-1} \) (i.e. \( l^{n-1}_i \));
- consider the list of the successors of the rightmost digit of \( l^{n-1}_i \) shifted
  starting from the rightmost digit of the rightmost element of \( L^{i-1}_n \) (i.e. \( s(\overrightarrow{l^{n-1}_i}, \overrightarrow{\text{last}(L^{i-1}_n)}) \));
- paste \( l^{n-1}_i \) with each element of the list \( s(\overrightarrow{l^{n-1}_i}, \overrightarrow{\text{last}(L^{i-1}_n)}) \).
Let us construct for instance the list $L$:  

$\mathcal{L}_1 = < 2 >$;  
$L_2 = 2 \circ s(2, 2) = 2 < 2, 3 > = < 22, 23 >$, then  
$L_3 = 22 \circ s(2, 2) = 22 < 2, 3 > = < 222, 223 >$;  
$L_4 = 23 \circ s(3, 3) = 23 < 3, 4, 2 > = < 233, 234, 232 >$, then  
$L_5 = 232 \circ s(2, 2) = 232 < 2, 3 > = < 2322, 2323 >$, then  

$$L_4 = < 2222, 2223, 2233, 2234, 2232, 2332, 2334, 2333, 2343, 2344 >$$  

We now prove the following:

**Theorem 3.1** Two consecutive elements of the list $L_n$ differ only for one digit.

**Proof.** We can proceed by induction on $n$:

**base:** if $n = 1$, then the theorem is trivially true since $\mathcal{L}_1 = < 2 >$;

**inductive hypothesis:** let us suppose that $l_i^{n-1}$ and $l_{i+1}^{n-1}$, with $1 \leq i \leq |\mathcal{L}_{n-1}| - 1$, differ only for one digit;

**inductive step:** the list $\mathcal{L}_n$ is obtained by linking together the lists $L_i^n$ for $i = 1, \ldots, |\mathcal{L}_{n-1}|$. Since the elements of each list $L_i^n$ differ only for one digit by construction, we must prove the statement only for $\text{last}(L_i^n)$ and $\text{first}(L_i^{i+1})$, with $1 \leq i \leq |\mathcal{L}_{n-1}| - 1$.

Let $J$ be the last element of $s(l_i^{n-1}, \text{last}(L_i^{i-1}))$. Then we have:

$$\text{last}(L_i^n) = l_i^{n-1} J.$$
We also have:
\[ L_{n+1}^{i+1} = l_{n+1}^{i+1} \circ s(l_{n+1}^{i+1}, \text{last}(L_i^n)) = l_{n+1}^{n-1} \circ s(l_{n+1}^{n-1}, J). \]

From the definition of the shifted list of the successors we deduce that the first element of a list \( s(i, k) \) is always \( k \), then:
\[ \text{first}(L_{n+1}^{i+1}) = l_{n+1}^{n-1} J. \]

Since \( l_{n}^{i} \) and \( l_{n+1}^{i} \) differ only for one digit by the inductive step, this statement holds also for \( \text{last}(L_i^n) \) and \( \text{first}(L_{n+1}^{i+1}) \). So, the theorem is proved.

At this point it is easily seen that \( \text{first}(L_{n+1}^{i+1}) \) is a son of the second-last digit of \( \text{first}(L_i^n) \) and that \( \text{last}(L_i^n) \) does not belong to the set of sons of the second-last digit of \( \text{first}(L_{n+1}^{i+1}) \), since from the definition of the shifted production, the construction we described above and the axiom of \( \Omega_C \) (which is 2), we deduce that \( \text{last}(L_i^n) \in \{2, 3\} \), which are present in the production of each possible label.

3.1 The algorithm to generate \( L_n \)

The aim is defining an algorithm which is not recursive for generating all the words of length \( n \) encoding the objects of size \( n \). We base our procedure on the general idea that if a word \( c_j \) has been generated, then a single digit must be changed to generate the next word \( c_{j+1} \), as the authors made in [1].

The first word of the list is \( w = 222\ldots2 \), where \( w_i = 2 \), for \( i = 1, 2, \ldots n \). The digit \( w_i \) to be modified at each step is determined using the algorithm of Walsh [15], i.e. using a \((n+1)\)-dimensional array \( e \), which is updated in such a way that, at each step, \( e_{n+1} \) points to \( w_i \). Once \( w_i \) is determined, it can not be modified by simply increasing it by one [1], but the definition of the shifted production must be taken in account. So, we use another array \( d \) \((n+1)\)-dimensional), which is defined as follows: \( d_i = 0 \) if \( w_i \) is modified according to the shifted production \( s(k, 2) \); \( d_i = 1 \) if \( w_i \) is modified according to \( s(k, 3) \).

It is easy to prove that the introduction of the array \( d \) does not exchange the complexity of the recalled procedure of Walsh for generating Gray codes in \( O(1) \) worst-case time per word [15]: his clever algorithm remains the starting point for the implementation of our method.

We note that \( d \) can also be used to establish when \( w_i \) is no more modifiable: from the definition of \( s(k, j) \) it happens if \( (d_i = 0 \land w_i = 3) \) or if \( (d_i = 1 \land w_i = 2) \).

The generating procedure stops when the digit to be modified is \( w_1 \).
4 The case of Dyck paths

We consider now the specific class of Dyck paths. Each of them can be associated with a binary string according to the substitution, for example, of the up steps with the 1 bit and the down steps with 0. Let us consider a word of length $n$ of the Gray code defined in Section 3. It has a correspondent Dyck path which, in turn, is associated with a binary string, both of length $2n$ (in Section 4.2 we present an algorithm to directly translate a word in the associated binary string). We want to prove that, if we consider two consecutive binary strings corresponding to two consecutive words in the Gray code, they differ only for two bits (note that the Hamming distance between two binary strings encoding two Dyck paths is at least 2). For this aim we base on the ECO construction of Dyck paths [5]. We recall briefly its main features: if $p$ is a Dyck paths of length $2n$ with the last descent of $k$ steps, then it has $k + 1$ active sites; we obtain each of its $k + 1$ sons by inserting a peak in each active sites; the insertion of a peak in an active sites at height $h$ generates a Dyck path with $h + 2$ active sites. Now we state the next proposition:

**Proposition 4.1** Two words of the Gray code differing for one digit correspond to binary strings which differ only for two bits.

**Proof.** The last digit of a word denotes the number of active sites of the corresponding Dyck path, so if it is $k$, then the path has $k − 1$ down steps in the last descent, according to the above mentioned ECO construction.

A Let us consider the case when the two words differs in the last digit. Let their codes be:

\[ w_1 w_2 \ldots w_i w_{i+1} \]

and

\[ w_1 w_2 \ldots w_i z_{i+1}. \]

We indicate a generic bit with the star $*$, so $w_1 \ldots w_i$ corresponds to

\[
\begin{array}{c}
1 * * * * 1 0 0 0 0 \ldots 0 \\
2i−w_{i+1} \quad w_i-1
\end{array}
\]

The adding of $w_{i+1}$ corresponds to the insertion of a peak at height $w_{i+1} − 2$ in the last descent of the Dyck path associated to $w_1 w_2 \ldots w_i$. So, the corresponding binary string is

\[
\begin{array}{c}
1 * * * * 1 0 0 0 \ldots 0 \\
2i−w_{i+1} \quad (w_i−1)−(w_{i+1}−2) \\
\end{array}
= \begin{array}{c}
1 * * * * 1 0 0 \ldots 0 \\
2i−w_{i+1} \quad w_{i+1}−1 \quad w_{i+1}−1
\end{array}
\]

(note that after the adding of $w_{i+1}$, the total number of bits is properly $2i + 2$). In particular we have:
• in the case $w_{i+1} = w_i + 1$, when the peak is inserted in the active site with maximal height, the binary string becomes

\[
\begin{array}{c}
\underline{1 * * \ldots * * 1} 00 \ldots 0 \\
\text{2} - w_i + 1 \\
\end{array}
\]

in other words, the last ascent is longer than one step with respect to the Dyck path codified by the word $w_1 w_2 \ldots w_i$;

• in the case $w_{i+1} = 2$, when the peak is added at height 0 at the end of the Dyck path corresponding to $w_1 w_2 \ldots w_i$, the binary string is

\[
\begin{array}{c}
\underline{1 * * \ldots * * 1} 00 \ldots 0 10 \\
\text{2} - w_i + 1 \\
\w_i - 1 \\
\end{array}
\]

In a similar manner, the addition of $z_{i+1}$ after $w_i$ transforms the corresponding binary string in

\[
\begin{array}{c}
\underline{1 * * \ldots * * 1} 00 \ldots 0 100 \ldots 0 \\
\text{2} - w_i + 1 \\
\w_i - z_{i+1} + 1 \\
\w_{i+1} - 1 + j \\
\end{array}
\]

Let us suppose that $z_{i+1} = w_{i+1} + j$, where $j$ can also assume negative values. If $j > 0$, then $j \in \{1, w_i + 1 - w_{i+1}\}$; if $j < 0$, then $j \in \{-1, 2 - w_{i+1}\}$. The word $w_1 w_2 \ldots w_{i+1}$ corresponds to the binary string

\[
\begin{array}{c}
\underline{1 * * \ldots * * 1} 00 \ldots 0 1 00 \ldots 0 \\
\text{2} - w_i + 1 \\
\w_i - w_{i+1} + 1 - j \\
\w_{i+1} - 1 + j \\
\end{array}
\] (2)

The difference between the words (1) and (2) is the location of the rightmost 1 bit, which in (2) is shifted of $|j|$ positions towards left ($j > 0$) or right ($j < 0$) with respect to (1). It easily seen that the two strings differ only for the two bits in position $w_{i+1}$ and $w_{i+1} + j$ from the right of the word.

**B** Let us consider now the case when the two words differ for two digits which are not the last ones:

\[
w_1 w_2 \ldots w_i w_{i+1} w_{i+2} \ldots w_n
\] (3)

and

\[
w_1 w_2 \ldots w_i z_{i+1} w_{i+2} \ldots w_n.
\] (4)

The associated binary strings after the insertion of $w_{i+2}$ (i.e. the binary strings coding $w_1 \ldots w_i w_{i+1} w_{i+2}$ and $w_1 \ldots w_i z_{i+1} w_{i+2}$) are

\[
\begin{array}{c}
\underline{1 * * \ldots * * 1} 00 \ldots 0 1 00 \ldots 0 100 \ldots 0 \\
\text{2} - w_i + 1 \\
\w_i - w_{i+1} + 1 \\
\w_{i+1} - w_{i+2} + 1 \\
\w_{i+2} - 1 \\
\end{array}
\]
and
\[
\begin{array}{cccccccc}
\overset{1}{2i-w_i+1} & \overset{00,\ldots,0}{\vdots} & 1 & \overset{00,\ldots,0}{\vdots} & \overset{100,\ldots,0}{\vdots} \\
\end{array}
\]
where, as in the preceding case, \( z_{i+1} = w_{i+1} + j \). The insertions of the next digits \( w_k \) with \( k = i+3, \ldots, n \), which are equal in the two words, modify in the same way the last descent in the associated Dyck paths. Then, the difference between the two binary strings corresponding to them is not due to these insertions. So, also in this case, the binary strings related to (3) and (4) differ only for two bits.

\[\Box\]

4.1 From a binary string to the next one

The structure of the above proof can be used to derive an algorithm to generate a binary string \( p_{h+1} \) from the preceding one \( p_h \), taking into account the generation order of the corresponding words in the Gray code. If \( u_h \) and \( u_{h+1} \) are two consecutive words in the Gray codes and \( p_h \) is the binary string corresponding to \( u_h \), then:

- if \( u_h \) and \( u_{h+1} \) differ in the last digit and \( j = u_{h+1} - u_h \) is the difference between these ones, then \( p_{h+1} \) is obtained from \( p_h \) by the shifting of \(|j|\) positions of the rightmost 1 bit towards left if \( j > 0 \) or right if \( j < 0 \);

- if \( u_h \) and \( u_{h+1} \) differ in the \( i \)-th digit and \( j \) is the difference between the \( i \)-th digit of \( u_{h+1} \) and the \( i \)-th digit of \( u_h \), then \( p_{h+1} \) is obtained from \( p_h \) by the shifting of \(|j|\) positions of the second rightmost 1 bit towards left if \( j > 0 \) or right if \( j < 0 \).

The correctness of the above procedure can be easily checked and the algorithm is based on the proof of the preceding proposition.

4.2 From the word to the binary string

The proof of Proposition 4.1 suggests also the idea for an inductive algorithm which allows to derive the binary string corresponding to a given word in the Gray code. Let us suppose we have already encoded a word \( w_1 \ldots w_{n-1} \) in the binary string \( u \). The adding of a new digit \( w_n \) modifies only the final part of \( u \), as we can deduce from the first part of the proof of Proposition 4.1. In particular, the \( w_{n-1} - 1 \) rightmost 0 bits of \( u \) corresponding to the last descent of the related Dyck path, are replaced by \( w_{n-1} + 1 \) bits as in the following:

\[
\begin{array}{cccccccc}
\overset{000,\ldots,0}{\vdots} & \overset{000,\ldots,0}{\vdots} & \overset{1000,\ldots,0}{\vdots} \\
\overset{w_{n-1}-1}{\vdots} & \overset{w_{n-1}-w_n+1}{\vdots} & \overset{w_n-1}{\vdots} & \overset{w_{n-1}+1}{\vdots} \\
\end{array}
\]
It correspond to the adding of a peak in some site of the last descent of the Dyck path related to \( u \).

Then, starting from the binary string 10 encoding the minimal Dyck path whose relating word in the Gray code is 2, it is possible to get the binary string corresponding to \( w_1 \ldots w_n \) from the knowledge of that one related to \( w_1 \ldots w_{n-1} \) by means of the following inductive procedure:

**base:** the binary string corresponding to the word 2 is 10;

**inductive hypothesis:** assume that \( u \) is the binary string codifying \( w_1 \ldots w_{n-1} \);

**inductive step:** then the binary string corresponding to \( w_1 \ldots w_n \) is obtained replacing the \( w_{n-1} + 1 \) rightmost 0 bits of \( u \) with the \( w_n + 1 \) bits 000...01000...0.

In the following example the encoding of the word 2334 is shown:

\[
10 \rightarrow 1100 \rightarrow 110100 \rightarrow 11011000
\]

(2) (23) (233) (2334)

**Note.** The algorithm of Section 4.1 allows to find a binary string \( p_{h+1} \) starting from the preceding one \( p_j \) and the words \( u_h \) and \( u_{h+1} \) of the Gray code, corresponding to \( p_h \) and \( p_{h+1} \), respectively. The algorithm of this section, whereas, generates the binary string from the corresponding word by means of an inductive procedure which can turn out too heavy for large values of \( n \) (the length of the word).

Hence, the preceding algorithm, having a low complexity, can be used to generate \( p_{h+1} \) in the case the string \( p_h \) and the words \( u_h \) and \( u_{h+1} \) are known.

5 **Generalization to stable succession rules**

The crucial point in the construction of the lists \( L_n \) is that each label \( k \) in the succession rule \( \Omega_C \) has in its production the two labels 2 and 3, as we pointed out at the end of Section 3. This property, together with the definition of the *shifted production* of \( k \), allows \( \text{last} (L_n^i) \) and \( \text{first} (L_n^{i+1}) \) to be different only for one digit (which is not the last one). Starting from this remark, we generalize the procedure to define the Gray code to all those succession rules having a particularity similar to \( \Omega_C \) which we would like to call *stability property*, meaning with this name that in each production of \( k \) we always find two labels, say \( c_1 \) and \( c_2 \), regardless of \( k \).
Definition 5.1 (stability property) We say that a succession rule \( \Omega \) is stable if for each \( k \) there exist two indexes \( i, j \) (\( i < j \)) such that \( e_i(k) = c_1 \) and \( e_j(k) = c_2 \) (\( c_1 \leq c_2 \)).

We need also to extend the definition of shifted production for the labels of succession rules with the stability property, in order to obtain that each list of successors of any \( k \) ends with \( c_1 \) or \( c_2 \). We have the following generalized shifted productions of \( k \), being \( e_i(k) = c_1 \) and \( e_j(k) = c_2 \):

\[
\begin{align*}
\Omega : \{ & (a) \\
& (k) \mapsto (e_1(k)) (e_2(k)) \ldots (e_k(k)) \}, \quad k \in \mathbb{N}
\end{align*}
\]

is stable if for each \( k \) there exist two indexes \( i, j \) (\( i < j \)) such that \( e_i(k) = c_1 \) and \( e_j(k) = c_2 \) (\( c_1 \leq c_2 \)).

In Figure 2 we used two walks, very similar to the factorial walks on the integer half-line [3], to visualize the generalized shifted production of \( k \), the above one starting from \( c_1 \) and ending in \( c_2 \) (corresponding to \( s(k, c_1) \)) and the below one starting from \( c_2 \) and ending in \( c_1 \) (corresponding to \( s(k, c_2) \)).

Now, it is easy to prove that:

Proposition 5.1 If \( \Omega \) is a succession rule with the stability property, then the lists \( L_n \) defined by:

\[
\begin{align*}
L_1 & = < a > \\
L_n & = \Theta^M_{i=1} L^n_i \quad \text{if} \quad n > 1
\end{align*}
\]

where \( M = |L_{n-1}| \) and \( L^n_i \) is defined by

\[
\begin{align*}
L^n_1 & = l^{n-1}_1 \circ s(l^{n-1}_1, c_1) \\
L^n_i & = l^{n-1}_i \circ s(l^{n-1}_i, \text{last}(L^n_{i-1})) \quad \text{if} \quad i > 1
\end{align*}
\]

Figure 2: Generalized shifted production.
form a Gray code in the sense of the definition of Section 4, where two consecutive words of length \( n \) differ for one digit (Hamming distance equals to one).

The proof is completely similar to that one of Theorem 3.1 and it is omitted.

Note that in the special case \( i = 1, j = 2 \) the generalized shifted production is:

\[
\begin{align*}
\Omega_{F_o} : \quad & (2) \Rightarrow (2)(3) \\
& (3) \Rightarrow (2)(3)(3)
\end{align*}
\]

We now analyze some particular cases of succession rules with the stability property.

**Example 1.** Let us consider the following rule \( \Omega_{F_o} \),

\[
\Omega_{F_o} : \quad \begin{cases} 
(2) \Rightarrow (2)(3) \\
(3) \Rightarrow (2)(3)(3)
\end{cases}
\]

defining the odd Fibonacci numbers. It is easily seen that it satisfies the stability property, but the rule \( \Omega_{F} \),

\[
\Omega_{F} : \quad \begin{cases} 
(2) \Rightarrow (1)(2) \\
(1) \Rightarrow (2)
\end{cases}
\]

defining Fibonacci numbers, does not satisfy the stability property. This is to say that such a property is not common to all the succession rules of a certain family (finite succession rules, in this case).

In the following examples it is shown that a similar behavior can be found also in factorial or transcendental rules.

**Example 2.** The factorial rule:

\[
\Omega_{M} : \quad \begin{cases} 
(1) \\
(k) \Rightarrow (1)(2) \ldots (k - 1)(k + 1)
\end{cases}
\]
defining the sequence of Motzkin numbers, does not satisfy the stability property, since only for \( k \geq 3 \) each label has \( c_1 = 1 \) and \( c_2 = 2 \) in its production. But the rules \( \Omega_A \) of kind

\[
\Omega_A : \left\{ \begin{array}{l}
(a) \\
(k) \leadsto (a)(a+1)\ldots(k)(k+1)(k+d_1)\ldots(k+d_m)
\end{array} \right.
\]

(with \( a \geq 2, m = a - 2, d_i \geq 0 \) and \( d_i \leq d_{i+1} \)) are factorial and stable rules, with \( i = 1, j = 2, c_1 = a \) and \( c_2 = a + 1 \). The following well-known succession rule \( \Omega_t \), related to the Gray structure of the \( t \)-ary trees \[4\], is a particular case:

\[
\Omega_t : \left\{ \begin{array}{l}
(t) \\
(k) \leadsto (t)(t+1)\ldots(k-1)(k+1)\ldots(k+t-2)(k+t-1)
\end{array} \right.
\]

and the generalized shifted production is:

\[
\left\{ \begin{array}{l}
s(k,t) =< t,k+1,t-1,k+t-2,\ldots,k+1,k,k-1,\ldots,t+2,t+1 > \\
s(k,t+1) =< t+1,t+2,\ldots,k-1,k+1,\ldots,k+t-2,k+t-1,t >.
\end{array} \right.
\]

In the following, we present the construction of the list \( L_3 \) in the case \( t = 3 \) in the above succession rule \( \Omega_t \).

\( L_1 =< 3 >; \)

\( L_2 = 3 \circ s(3,3) = 3 \circ < 3, 5, 4 > = < 33, 35, 34 >, \) then

\( L_3 = 33 \circ s(3,3) = 33 \circ < 3, 5, 4 > = < 333, 335, 334 >; \)

\( L_4 = 35 \circ s(5,4) = 35 \circ < 4, 5, 6, 7, 3 > = < 354, 355, 356, 357, 353 >; \)

\( L_5 = 34 \circ s(4,3) = 34 \circ < 3, 6, 5, 4 > = < 343, 346, 345, 344 >, \) then

\( L_3 = < 333, 335, 334, 354, 355, 356, 357, 353, 343, 346, 345, 344 > . \)

If \( t = 2 \), then we find the succession rule \( \Omega_C \) for Catalan numbers, enumerating, among other things, the binary trees. In \[13\] the author proposes a constant time algorithm for generating binary trees Gray codes. We note that our procedure, combined with the results of Section \[4\] is an alternative approach for this aim.
Example 3. Another particular case of $\Omega_A$ is the following family:

$$\Omega_r : \begin{cases} (r) \\ (k) \rightsquigarrow (r)(r+1)\ldots(k)(k+1)^{r-1} , \end{cases}$$

with $r \geq 2$. They satisfy the stability property, too, with $i = 1$, $j = 2$, $c_1 = r$ and $c_2 = r + 1$. If $r = 3$, then $\Omega_r$ is the well-known succession rule defining the sequence of Schröder numbers. The following rule $\Omega_s$ also codes the construction of Schröder paths, 2-colored parallelogram polyominoes, $(4231, 4132)$-pattern avoiding permutations, $(3142, 2413)$-pattern avoiding permutations [6, 17, 18] (these latter patterns are also considered in [2] for pattern matching decision problem for permutations).

$$\Omega_s : \begin{cases} (2) \\ (k) \rightsquigarrow (3)(4)\ldots(k)(k+1)^2 \end{cases}$$

In this case it is $c_1 = 3$, $c_2 = 4$ and the associated shifted production is:

$$\begin{cases} s(k, 3) =< 3, (k + 1)_2, (k + 1)_1, k, k - 1, \ldots, 5, 4 > \\ s(k, 4) =< 4, 5, \ldots, k, (k + 1)_1, (k + 1)_2, 3 > , \end{cases}$$

where the indexes differentiate labels with the same value. Note that $s(k_2, *) = s(k_1, *)$ ($* = 3$ or $4$). The construction of the list $\mathcal{L}_3$ is:

$\mathcal{L}_1 =< 3 > ;$

$L_1^1 = 3 \circ s(3, 3) =< 33, 34_2, 34_1 >,$ then

$L_2 =< 33, 34_2, 34_1 > ;$

$L_3^1 = 33 \circ s(3, 3) =< 333, 334_2, 334_1 > ;$

$L_3^2 = 34_2 \circ s(4_2, 4_1) =< 34_24_1, 34_25_1, 34_25_2, 34_23 > ;$

$L_3^3 = 34 \circ s(4, 3) =< 343, 345_2, 345_1, 344 > , then$

$\mathcal{L}_3 =< 333, 334_2, 334_1, 34_24_1, 34_25_1, 34_25_2, 34_23, 343, 345_2, 345_1, 344 > .$

Example 4. Succession rules of kind:
\[ \Omega_B : \begin{cases} 
(r) & \rightarrow (b)^l(a)(a+1) \ldots (k)(k+d_1) \ldots (k+d_m) \\
(k) & \rightarrow (b)^k \\
(k) & \rightarrow (b)^{(k-1)}(a) 
\end{cases} \]

if \( k < a \) \&\& \( k \leq l \)

if \( k < a \),

with \( l \geq 2 \), \( b < a \), \( m = a - l - 1 \), satisfy the stability property with \( i = 1 \), \( j = 2 \) and, denoting \( b^l = b_1b_2 \ldots b_l \), \( c_1 = b_1 \), \( c_2 = b_2 \). A well-known particular case is

\[ \Omega_{GD} : \begin{cases} 
(2) & \rightarrow (3)(3) \\
(3) & \rightarrow (3)(3)(4) \\
k & \rightarrow (3)^2(4) \ldots (k)(k + 1) 
\end{cases} \]

which encodes a construction for Gran Dyck paths [12]. The generalized shifted production associated is

\[ s(k, 3_1) = < 3_1, k + 1, k_1, \ldots, 4, 3_2 > \\
s(k, 3_2) = < 3_2, 4, \ldots, k, k + 1, 3_1 > . \]

The list \( L_3 \) is obtained as follows:

\( L_1 = < 2 >; \)

\( L_2^1 = 2 \circ s(2, 3_1) = < 23_1, 23_2 >; \) then

\( L_2 = < 23_1, 23_2 >; \)

\( L_3^1 = 23_1 \circ s(3_1, 3_1) = < 23_13_1, 23_14, 23_13_2 >; \)

\( L_3^2 = 23_2 \circ s(3_2, 3_2) = < 23_23_2, 23_24, 23_23_1 >; \) then

\[ L_3 = < 23_13_1, 23_14, 23_13_2, 23_23_2, 23_24, 23_23_1 > . \]

Example 5. It is possible to find some examples among the transcendental succession rules which are stable or not. The classical rule defining the factorial numbers, which describes the construction of the permutations of length \( n \) by inserting the element \( n \) in any active site of any permutation of length \( n - 1 \), is not stable (its production is: \( (k) \rightarrow (k + 1)^k \)). On the contrary, the following one \( \Omega_p \), defining the same sequence, is stable:

\[ \Omega_p = \begin{cases} 
(2) & \\
(2k) & \rightarrow (2)(4)(6) \ldots (2k)(2k + 2)^k 
\end{cases} \]
Stability property is satisfied since each label \((2k)\) generates in the first two positions labels \((2)\) and \((4)\). The associated generalized shifted production is:

\[
\begin{align*}
 s(2k, 2) &= \langle 2, (2k + 2)_k, (2k + 2)_{k-1}, \ldots, (2k + 2)_1, 2k, 2k - 2, \ldots, 4 \rangle \\
 s(2k, 4) &= \langle 4, 6, \ldots, 2k - 2, 2k, (2k + 2)_1, (2k + 2)_2, \ldots, (2k + 2)_k, 2 \rangle ,
\end{align*}
\]

where the indexes are useful to distinguish different labels but with the same value. In order to illustrate the combinatorial placement of \(\Omega_p\) we propose a probably new ECO construction for the permutations which can be described by this rule. Let \(\pi = \pi_1 \pi_2 \cdots \pi_n\) be a permutation of \(S_n\), we define an operator \(\vartheta : S_n \rightarrow 2^{S_{n+1}}\) (the power set of \(S_{n+1}\)) working as follows \((n \geq 1)\):

- let \(\pi_1 = k\), then \(\vartheta\) generates \(2k\) permutations \(\pi' \in S_{n+1}\) which are indicated by \(\pi'^{(i)}\), with \(i = 1, 2, \ldots, 2k\);
- the entries of \(\pi'^{(i)}\) are:
  1. if \(i = 1, 2, \ldots, k\), then:
     - \(\pi'^{(i)}_1 = i\);
     - the other entries are the same of \(\pi\) where the entry \(i\) is replaced by \(n + 1\).
  2. if \(i = k + 1, k + 2, \ldots, 2k\), then:
     - \(\pi'^{(i)}_1 \pi'^{(i)}_2 = (\pi_1 + 1)j\), where \(j = 1, 2, \ldots, k\);
     - the other entries are obtained as follows:
       - If \(\pi_1 \neq n\), then let \(\rho\) be the sequence, with length \(n - 1\), obtained by \(\pi\) deleting \(\pi_1\) after it has been interchanged with \(\pi_1 + 1\). The remaining entries of \(\pi'^{(i)}\) are the same of \(\rho\) where the entry \(j\) is replaced by \(n + 1\).
       - If \(\pi_1 = n\), then let \(\rho\) be the sequence obtained from \(\pi\) by deleting \(\pi_1\). The remaining entries of \(\pi'^{(i)}\) are the same of \(\rho\) where the entry \(j\) is replaced by \(n\).

**Remark:** Permutations \(\pi'^{(i)}\) with \(i = 1, 2, \ldots, k\) start with an ascent, while permutations \(\pi'^{(i)}\) with \(i = k + 1, k + 2, \ldots, 2k\) start with a descent.

It can be easily proved that if \(\pi' \in S_{n+1}\), then there exists a unique \(\pi \in S_n\) such that \(\pi' \in \vartheta(\pi)\) \((n \geq 1)\), then operator \(\vartheta\) satisfies Proposition 2.1 of [5], which ensures that the family of sets \(\{\vartheta(\pi) : \pi \in S_n\}\) is a partition of \(S_{n+1}\), so that \(\vartheta\) provides a recursive construction of the permutations of \(S = \bigcup S_n\).

In Figure 3 the action of \(\vartheta\) on two different permutations of \(S_6\) (the first one starting with an entry different from \(n = 6\)) is illustrated. Permutations
\( \pi'_{(i)}, i = 1, 2, \ldots, 2k \) generated by \( \pi \) by means \( \vartheta \) are listed from the top to the bottom, being \( \pi''(1) \) at the top.

6 Conclusions and further developments

It is possible to find a lot of succession rules satisfying the stability property, but we are interested to the rules having some combinatorial relevance, as the ones presented in the above examples. In this way, with our procedure we are able to give a Gray code for the words (i.e. the paths whose nodes are the labels in the generating tree) encoding combinatorial Gray structures, i.e. those structures whose exhaustive generation can be described by a rule satisfying the stability property, which is not, as we have seen, an infrequent property.

Clearly, it would be better to have a Gray code for the objects instead of their encodes. Nevertheless, as we stated in Section 4 our procedure generates a Gray code which is not related to the nature of a particular class of combinatorial objects. Moreover, in some case it could be possible to translate the word of labels (the path in the generating tree) into the corresponding object. A further effort in this sense could be the research of algorithms for this translation in order to generalize the approach of Section 4 for Dyck paths. For this aim the ECO method can be useful, since by means of it each code is associated to a single object of the structure.

From the above examples it is possible to argue that the stability property of a succession rule does not depend on its "structural properties", which have been discussed by the authors in [3]. In the light of this fact, it is reasonable to ask if a stable succession rule can be considered as the representative, say standard form, of a set of rules which are all equivalent to it (two rules are said equivalent if they define the same number sequence [7]). This is to say that the equivalence problem for succession rules could be amplified with respect to the investigation conducted in [7] where the authors analyze the equivalence problem for some different kinds of rules: is it suitable the research of the set of rules equivalent to a stable succession
rule?

Moreover, it is evident that it is not the sequence defined by the rule that induces it to be stable or not: factorial number sequence can be defined by a stable or not stable rule, as showed in Example 5. Consequently, a problem which naturally arises from this note is the existence of a succession rule with the stability property for any given number sequence. A first concerning question could be the following (to the authors knowledge the answer is open): is there a stable rule defining Motzkin numbers?

References

[1] S. Bacchelli, E. Barcucci, E. Grazzini, E. Pergola, Exhaustive generation of combinatorial objects by ECO, Acta Inform. 40 (8) (2004) 585-602.

[2] P. Bose, J. F. Buss, A. Lubiw, Pattern matching for permutations, Inform. Process. Lett. 65 (1998) 277-283.

[3] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps, Generating functions for generating trees, Discrete Math. 246 (2002) 29-55.

[4] E. Barcucci, A. Del Lungo, E. Pergola, Random generation of trees and other combinatorial objects, Theoret. Comput. Sci 218 (1999) 219-232.

[5] E. Barcucci, A. Del Lungo, E. Pergola, E. Pinzani, ECO: a methodology for the enumeration of combinatorial objects, J. Difference Equ. Appl. 5 (1999) 435-490.

[6] E. Barcucci, A. Del Lungo, E. Pergola, E. Pinzani, Some combinatorial interpretations of q-analogs of Schröder numbers, Ann. Combin. 3 (1999) 171-190.

[7] S. Brlek, E. Duchi, E. Pergola, S. Rinaldi On the equivalence problem for succession rules, Discrete Math. 298 (2005) 142-154.

[8] J. Baril, V. Vajnovszki, Gray code for derangements, Discrete Appl. Math. 140 (2004) 207-221.

[9] F.R.K. Chung, R.L. Graham, V.E. Hoggat, M. Kleiman, The number of Baxter permutations, J. Combin. Theory Ser. A 24 (1978) 382-394.

[10] S.M. Jonson, Generation of permutations by adjacent transposition, Math. Comp. 17 (1963) 282-285.
[11] J.E. Ludman, \textit{Gray code generation for MPSK signals}, IEEE Trans. Commun. COM-29 (1981) 1519-1522.

[12] E. Pegola, R. Pinzani, S. Rinaldi, \textit{Approximating algebraic function by means of rational monos}, Theoret. Comput. Sci. 270 (2002) 643-657.

[13] V. Vajnovszki, \textit{Constant Time Algorithm for Generating Binary Trees Gray Codes}, Studies in Informatics and Control, 5(1) (1996), 15-21.

[14] V. Vajnovszki, \textit{Gray visiting Motzkin}, Acta Inform. 38 (2002) 793-811.

[15] T. Walsh, \textit{Generating Gray Codes in O(1) worst-case time per word}, LNCS 2731 (2003) 73-88.

[16] T. Walsh, \textit{Gray codes for involutions}, J. Combin. Math. Combin. Comput. 36 (2001) 95-118.

[17] J. West, \textit{Generating trees and the Catalan and Schröder numbers}, Discrete Math. 146 (1995) 247-262.

[18] J. West, \textit{Generating trees and forbidden subsequences}, Discrete Math. 157 (1996) 363-374.