BFKL Pomeron calculus: nucleus-nucleus scattering

Carlos Contreras\textsuperscript{a}\textsuperscript{*}, Eugene Levin\textsuperscript{a,b} \textsuperscript{†} and Jeremy S. Miller\textsuperscript{b,c} \textsuperscript{‡}

\textsuperscript{a} Departamento de Física, Universidad Técnica Federico Santa María, Avda. España 1680 and Centro Científico-Tecnológico de Valparaíso, Casilla 110-V, Valparaíso, Chile
\textsuperscript{b} Department of Particle Physics, School of Physics and Astronomy, Tel Aviv University, Tel Aviv, 69978, Israel
\textsuperscript{c} CENTRA, Departamento de Física, Instituto Superior Técnico (IST), Av. Rovisco Pais, 1049-001 Lisboa, Portugal

ABSTRACT: In this paper the action of the BFKL Pomeron calculus is re-written in momentum representation, and the equations of motion for nucleus-nucleus collisions are derived, in this representation. We found the semi-classical solutions to these equations, outside of the saturation domain. Inside this domain these equations reduce to the set of delay differential equations, and their asymptotic solutions are derived.

KEYWORDS: BFKL Pomeron calculus, semi-classical approach, Pomeron action, equations of motion

PACS: 13.85.-t, 13.85.Hd, 11.55.-m, 11.55.Bq

\textsuperscript{*}Email: carlos.contreras@usm.cl
\textsuperscript{†}Email: leving@post.tau.ac.il, eugeny.levin@usm.cl
\textsuperscript{‡}Email: jeremy.miller@ist.utl.pt
1. Introduction

High energy QCD has reached a mature stage of development, in its description of dilute-dense scattering. Deep inelastic scattering with nuclei provides a good example of this. The main physical phenomena that emerges for this type of scattering has been discussed, and the non-linear equations that govern such processes have been derived and discussed in detail [1–8]. On the other hand, the scattering of the dense system of partons with the dense system of partons has been actively studied [9–15], but with limited
success. This is in spite of the fact that this scattering is closely related to nucleus-nucleus scattering, which is the source of most of the experimental information, for the dense parton system.

At the moment there exist two general approaches to high energy QCD: the BFKL Pomeron calculus [1, 2, 9, 16, 17], and the Colour Glass Condensate approach (CGC) [3, 7], which lead to the same non-linear equations [5, 6] for dilute-dense scattering. The interrelation between these two approaches is not clear at the moment. However, the equations that describe nucleus-nucleus collisions have not been derived, in spite of considerable progress made in this direction [10,12,13], while in the BFKL Pomeron calculus, such equations have been proposed in Ref. [9]. These equations have been on the market for some time, but unfortunately, only three attempts to solve them are available in Refs. [18–20].

The main goal of this paper is to find the solution to these equations. In the next section we re-derive the equation of Ref. [9] in momentum representation, which turn out to be the most economical way of finding the solution. In section 3 we will find the semi-classical solutions to the equation, which describe dense-dense scattering outside of the saturation region. Section 4 is devoted to finding the solution inside of the saturation region. In the conclusion section we summarize our results.

2. The BFKL Pomeron Calculus

The goal of this section is to find the equation for nucleus-nucleus scattering in the momentum representation based on the BFKL Pomeron calculus, based on the main idea of Ref. [9] that the equation for nucleus-nucleus collisions can be found from the equation of motion for Pomerons.

2.1 Functional integral formulation of the BFKL Pomeron Calculus \( S_0 \)

The BFKL Pomeron calculus can be written through the functional integral [9]

\[
Z[\Phi, \Phi^+] = \int D\Phi D\Phi^+ e^S \quad \text{with} \quad S = S_0 + S_I + S_E
\]  

(2.1)

where \( S_0 \) describes free Pomerons, \( S_I \) corresponds to their mutual interaction while \( S_E \) relates to the interaction with the external sources (target and projectile). Here the free action is given by:

\[
S_0 = \int dY' \int d^2 x_1 d^2 x_2 \Phi^\dagger (x_1, x_2, Y') \nabla_1^2 \nabla_2^2 \left[ \frac{\partial}{\partial Y} \right] \Phi (x_1, x_2, Y')
\]  

(2.2)

Define the following Fourier transform,

\[
\Phi^\dagger (x_1, x_2, Y') = \Phi^\dagger (x_{12}, b, Y') = x_{12}^2 \int d^2 k_1 e^{-i k_1 \cdot x_{12}} \Phi^\dagger (k_1, b, Y')
\]  

(2.3)

\[
\Phi (x_1, x_2, Y') = \Phi (x_{12}, b, Y') = x_{12}^2 \int d^2 k_2 e^{i k_2 \cdot x_{12}} \Phi (k_2, b, Y')
\]  

(2.4)

Consider the free action as the sum of two terms:
\[ S_0 = S'_0 + S''_0 \]  \hspace{1cm} (2.5)

where:

\[ S'_0 = \int dY' \int d^2x_1 d^2x_2 \Phi^\dagger (x_1, x_2, Y') \nabla_1^2 \nabla_2^2 \frac{\partial}{\partial Y} \Phi (x_1, x_2, Y') \]  \hspace{1cm} (2.6)

This can be re-written as:

\[ S'_0 = 4 \int dY' \int d^2b \, d^2x_{12} \int d^2k_1 d^2k_2 \, e^{ik_1 \cdot x_{12}} \Phi^\dagger (k_1, b, Y') \, x_{12}^2 \nabla_1^2 \nabla_2^2 \left( x_{12}^2 \, e^{-ik_2 \cdot x_{12}} \frac{\partial}{\partial Y} \Phi (k_2, b, Y') \right) \]  \hspace{1cm} (2.7)

where in the last step the replacement \( d^2x_1 \, d^2x_2 = 4 \, d^2b \, d^2x_{12} \), where,

\[ b = \frac{x_1 + x_2}{2} \hspace{1cm} x_{12} = x_1 - x_2 \]  \hspace{1cm} (2.8)

where \( b \) is the impact parameter. In two-dimensional polar coordinates:

\[ \nabla_{k_2}^2 = \frac{\partial^2}{\partial k_2^2} + \frac{1}{k_2} \frac{\partial}{\partial k_2} + \frac{1}{k_2^2} \frac{\partial}{\partial \theta} \]  \hspace{1cm} (2.9)

Let us introduce the next change of variable \( l = \ln k_2^2 \), such that the two-dimensional Laplacian \( \nabla_{k_2}^2 \) with respect to \( k_2 \), simplifies to the following differential operator:

\[ \nabla_{k_2}^2 = 4e^{-l_{k_2}} \frac{\partial^2}{\partial l^2} + e^{-l_{k_2}} \frac{\partial}{\partial \theta} \]  \hspace{1cm} (2.10)

According to the definition of Eq. (2.10), the next term \( x_{12}^2 \nabla_{x_1}^2 \nabla_{x_2}^2 (x_{12}^2 \, e^{-ik_1 \cdot x_{12}}) \) can be recast as,

\[
x_{12}^2 \nabla_{x_1}^2 \nabla_{x_2}^2 \left( x_{12}^2 \, e^{-ik_2 \cdot x_{12}} \right) = -x_{12}^2 \nabla_{k_2}^2 \nabla_{x_{12}}^4 \left( x_{12}^2 \, e^{-ik_2 \cdot x_{12}} \right)
\]

\[
= -x_{12}^2 \nabla_{k_2}^2 \left( k_2^4 \, e^{-ik_2 \cdot x_{12}} \right)
\]

\[
= \nabla_{k_2}^2 \left( k_2^4 \, \nabla_{k_2}^2 \left( e^{-ik_2 \cdot x_{12}} \right) \right)
\]

\[
= 16 \left( \left( \frac{\partial}{\partial l} + 1 \right)^2 + \frac{\partial}{\partial \theta} \right) \left( \frac{\partial^2}{\partial l^2} + \frac{\partial}{\partial \theta} \right) e^{-ik_2 \cdot x_{12}}
\]  \hspace{1cm} (2.11)
where $\nabla^2_{k_2}$ is the two-dimensional Laplacian derivative with respect to $k_2$. Inserting Eq. (2.11) back into Eq. (2.7), leads to the expression:

$$S'_0 = 4 \int dY' \int d^2 b \int d^2 x_{12} \int d^2 k_1 d^2 k_2$$

$$\times e^{ik_1 \cdot x_{12}} \Phi^\dagger (k_1, b, Y') \left\{ \nabla^2_{k_2} \left( k_2^4 \nabla^2_{k_2} e^{-ik_2 \cdot x_{12}} \right) \right\} \frac{\partial}{\partial Y} \Phi (k_2, b, Y')$$

(2.12)

$$= 32 \int dY' \int d^2 b \int d^2 x_{12} \int d^2 k_1 d\theta dl e^l$$

$$\times e^{ik_1 \cdot x_{12}} \Phi^\dagger (k_1, b, Y') \left\{ \left( \frac{\partial}{\partial l} + 1 \right)^2 \left( \frac{\partial^2}{\partial l^2} + \frac{\partial}{\partial \theta} \right) e^{-ik_2 \cdot x_{12}} \right\} \frac{\partial}{\partial Y} \Phi (k_2, b, Y')$$

(2.13)

where in the last step, the new variable $l$ was introduced. We use the convention that derivatives only act on terms inside of the curly brackets $\{}$. Hence in Eq. (2.13) derivatives only act on $e^{-ik_2 \cdot x_{12}}$. Assuming that the $\theta$-dependent part of the integrand is a purely periodic function of $\theta$, then the derivatives can be re-ordered through integration by parts, to yield:

$$S'_0 = 32 \int dY' \int d^2 b \int d^2 x_{12} \int d^2 k_1 d\theta dl e^l$$

$$\times e^{ik_1 \cdot x_{12}} \Phi^\dagger (k_1, b, Y') \left\{ \left( \frac{\partial}{\partial l} + 1 \right)^2 \left( \frac{\partial^2}{\partial l^2} + \frac{\partial}{\partial \theta} \right) e^{-ik_2 \cdot x_{12}} \right\} \frac{\partial}{\partial Y} \Phi (k_2, b, Y')$$

(2.14)

$$= 32 \int dY' \int d^2 b \int d^2 x_{12} \int d^2 k_1 d\theta dl e^l e^{ik_1 \cdot x_{12}} e^{-ik_2 \cdot x_{12}}$$

$$\times \Phi^\dagger (k_1, b, Y') \left\{ \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \frac{\partial}{\partial Y} \Phi (k_2, b, Y') \right\}$$

(2.15)

where in the last step it was assumed that $\Phi (k_2, b, Y')$ is purely a function of $k_2^2 = e^l$ and not a function of the angular coordinate $\theta$, such that $\theta$-derivatives vanish. Now returning to the integration variables $\int d\theta dl e^l = 2 \int d^2 k_2$ where $l = \ln k_2^2$, and integrating over $x_{12}$ leads to the delta function $(2\pi)^2 \delta^2 (k_1 - k_2)$. The delta function is absorbed by the integral over $k_2$-space resulting in the expression:

$$S'_0 = 64 (2\pi)^2 \int dY' \int d^2 b \int d^2 k_1 \Phi^\dagger (k_1, b, Y') \left\{ \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \frac{\partial}{\partial Y} \Phi (k_1, b, Y') \right\}$$

(2.16)

where $l = \ln k_1^2$. This part of the action gives the following contribution to the equation of motion, which stems from the condition $\delta S_0 / \delta \Phi^\dagger (k, b, Y) = 0$: 

$$\frac{\partial}{\partial l} + 1 = 0$$

$$\frac{\partial^2}{\partial l^2} \frac{\partial}{\partial Y} \Phi (k_1, b, Y') = 0$$

(2.17)
\[ \frac{\delta S_0'}{\delta \Phi^\dagger (k, b, Y)} = 64(2\pi)^2 \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \frac{\partial}{\partial Y} \Phi (k, b, Y') \]  

(2.17)

Recall that in the above calculation, for the sake of simplicity we considered \( S_0 = S_0' + S_0'' \), and we calculated the Fourier transform of the \( S_0' \) part. The full variation \( \frac{\delta S_0'}{\delta \Phi^\dagger (k, b, Y)} \) is given by the expression:

\[ \frac{\delta S_0'}{\delta \Phi^\dagger (k, b, Y)} = 64(2\pi)^2 \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \left( \frac{\partial}{\partial Y} - H \right) \Phi (k, b, Y') \]  

(2.18)

The general properties of the Hamiltonian \( H \) have been discussed in Refs. [9, 17] in coordinate representation, which read as follows in momentum space representation:

\[ H f(k, Y) = \frac{\alpha_s}{2\pi} \int d^2l K(k, l) \{ f(k, l, Y) - f(k, Y) \} \]  

(2.19)

\[ K(k, l) = \frac{k^2}{l^2} \frac{1}{(k - \vec{l})^2} \]  

(2.20)

### 2.2 Interaction term of the action \( S_I \) and the Field equation

The mutual interaction of the Pomerons is described by \( S_I \). The interaction which is related to the external sources (target and projectile) are not considered in this paper. In this approach \( S_I \) is given by:

\[ S_I = \frac{2\pi \alpha_s^2}{N_c} \int dY' \int \frac{d^2x_1 d^2x_2 d^2x_3}{x_1^2 x_2^3 x_3^2} \left\{ (L_{12} \Phi (x_1, x_2, Y')) \Phi^\dagger (x_2, x_3, Y') \Phi^\dagger (x_3, x_1, Y') \right\} \]  

\[ + \left( L_{12} \Phi^\dagger (x_1, x_2, Y') \right) \Phi (x_2, x_3, Y') \Phi (x_3, x_1, Y') \right\} \]  

(2.21)

\[ = \frac{2\pi \alpha_s^2}{N_c} \int dY' \int \frac{4d^2b d^2x_1 d^2x_2}{x_1^2 x_2^3 x_3^2} \left\{ (L_{12} \Phi (x_1, x_2, Y')) \Phi^\dagger (x_2, x_3, Y') \Phi^\dagger (x_3, x_1, Y') \right\} \]  

\[ + \left( L_{12} \Phi^\dagger (x_1, x_2, Y') \right) \Phi (x_2, x_3, Y') \Phi (x_3, x_1, Y') \right\} \]  

(2.22)

where in the last step the replacement \( d^2x_1 d^2x_2 = 4d^2bd^2x_{12} \), (see Eq. (2.8)), and where the following differential operator was introduced:

\[ L_{12} = x_1^4 \nabla_{x_1}^2 \nabla_{x_2}^2 \]  

(2.23)
Now the above defined Fourier transform of Eqs. (2.3) and (2.4) are inserted into Eq. (2.22), where it is assumed that $b \gg x_{12}$, where $b = (x_1 + x_2) / 2$, i.e. the impact parameter is much bigger than the size of the dipole. Inserting the Fourier transforms of Eqs. (2.3) and (2.4) into Eq. (2.22) gives,

$$S_I = \frac{2\pi \bar{\alpha}^2}{N_c} \int dY' \int 4 d^2 b d^2 x_{12} d^2 x_3 \int d^2 k_1 d^2 k_2 d^2 k_3$$

$$\left\{ \exp (i k_2 \cdot x_{23} + i k_3 \cdot x_{31}) \left( \frac{L_{12} \left( x_{12}^2 e^{-ik_1 \cdot x_{12}} \right)}{x_{12}^2} \right) \Phi \left( k_1, b, Y' \right) \Phi^\dagger \left( k_2, b, Y' \right) \Phi^\dagger \left( k_3, b, Y' \right) \right\}$$

The integration over $x_3$ leads to the Dirac delta function $(2\pi)^2 \delta^2 (k_2 - k_3)$, where $\delta^2 (k_2 - k_3)$ labels the delta function in two dimensions. The delta function is absorbed by the integration over $k_3$ which leads to:

$$S_I = \frac{2\pi \bar{\alpha}^2}{N_c} (2\pi)^2 \int dY' \int 4 d^2 b d^2 x_{12} \int d^2 k_1 d^2 k_2$$

$$\left\{ \exp (-i k_2 \cdot x_{12}) \left( \frac{L_{12} \left( x_{12}^2 e^{-ik_1 \cdot x_{12}} \right)}{x_{12}^2} \right) \Phi \left( k_1, b, Y' \right) \Phi^\dagger \left( k_2, b, Y' \right) \Phi^\dagger \left( k_3, b, Y' \right) \right\}$$

According to the definition of Eq. (2.23), the term $L_{12} \left( x_{12}^2 e^{-ik_1 \cdot x_{12}} \right) / x_{12}^2$ can be recast as,

$$L_{12} \left( x_{12}^2 e^{-ik_1 \cdot x_{12}} \right) / x_{12}^2 = x_{12}^2 \nabla_{x_1}^2 \nabla_{x_2}^2 \left( x_{12}^2 e^{-ik_1 \cdot x_{12}} \right)$$

$$= \nabla_{k_1}^2 \left( k_1^4 \nabla_{k_1}^2 e^{-ik_1 \cdot x_{12}} \right)$$

$$= 16 \left( \left( \frac{\partial}{\partial l} + 1 \right)^2 + \frac{\partial}{\partial \theta} \right) \left( \frac{\partial^2}{\partial l^2} + \frac{\partial}{\partial \theta} \right) e^{-ik_1 \cdot x_{12}}$$

where $l = \ln k_1^2$ and where the last expression was discussed and introduced in the first section (see Eq. (2.11)). Inserting this result into the interaction action of Eq. (2.25), after some algebra one arrives at the expression:
\[ S_I = 16 \frac{2\pi\alpha_s^2}{N_c} (2\pi)^2 \int dY' \int 4 d^2 b \int d^2 k_1 d^2 k_2 \]

\[
\left\{ \exp (-i (k_1 + k_2) \cdot x_{12}) \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi (k_1, b, Y') \Phi^\dagger (k_2, b, Y') \Phi^\dagger (k_2, b, Y') \\
+ \exp (i (k_1 + k_2) \cdot x_{12}) \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi^\dagger (k_1, b, Y') \Phi (k_2, b, Y') \Phi (k_2, b, Y') \right\}
\]

where it was assumed that the functions \( \Phi (k_1, b, Y') \) and \( \Phi^\dagger (k_1, b, Y') \) are purely functions of the radial \( k_1^2 \) coordinate and do not depend on the angular coordinate \( \theta \), such that \( \theta \) derivatives that appeared in Eq. (2.26) have been dropped. The integration over \( x_{12} \) leads to the Dirac delta function \((2\pi)^2 \delta^2 (k_1 + k_2)\). The delta function is absorbed by the integration over \( k_2 \) which leads to:

\[ S_I = 16 \frac{2\pi\alpha_s^2}{N_c} (2\pi)^4 \int dY' \int 4 d^2 b \int d^2 k_1 \]

\[
\left\{ \Phi^\dagger (-k_1, b, Y') \Phi^\dagger (-k_1, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi (k_1, b, Y') \\
\Phi (-k_1, b, Y') \Phi (-k_1, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi^\dagger (k_1, b, Y') \right\}
\]

where \( l = \ln k_1^2 \). Integrating the last term in Eq. (2.29) by parts, and taking into account that \( d^2 k_1 = \frac{1}{2} d\theta \, dl \, e^l \) where \( \theta \) is the azimuthal angle, Eq. (2.29) simplifies to the following expression:

\[ S_I = 16 \frac{2\pi\alpha_s^2}{N_c} (2\pi)^4 \int dY' \int 4 d^2 b \int d^2 k_1 \]

\[
\left\{ \Phi^\dagger (-k_1, b, Y') \Phi^\dagger (-k_1, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi (k_1, b, Y') \\
\Phi^\dagger (k_1, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi (-k_1, b, Y') \right\}
\]

The last equation Eq. (2.30) allows us to study the classical equation of the action, which can be obtained from the conditions \([9, 20]\):

\[ \frac{\delta S}{\delta \Phi (k, b, Y)} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \Phi^\dagger (k, b, Y)} = 0 \]

2.3 Classical equations

The functional derivative of the action for the effective pomeron field theory, with respect to \( \Phi^\dagger (k, b, Y) \) can be found from the results of Eqs. (2.18) and (2.30), which give:
\[ \delta (S_0 + S_I) / \delta \Phi^\dagger (k, b, Y) = 64(2\pi)^2 \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \left( \frac{\partial}{\partial Y} - H \right) \Phi (k, b, Y) \]

\[ + 16 \left( \frac{2\alpha_S}{N_c} \right) (2\pi)^4 \left\{ 2\Phi^\dagger (-k, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi (k, b, Y') + \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \Phi^2 (-k, b, Y') \right\} \] (2.31)

where \( l = \ln k^2 \). The equation for nucleus-nucleus scattering can be derived from the following equation of motion [9]:

\[ \delta (S_0 + S_I) / \delta \Phi^\dagger (k, b, Y) = 0 \quad \text{and} \quad \delta (S_0 + S_I) / \delta \Phi (k, b, Y) = 0 \] (2.32)

Now the following approach is used to average these equations:

\[ \langle O(k, Y; b) \rangle = \frac{\int D\Phi D\Phi^\dagger O (k, Y; b) e^S(\Phi, \Phi^\dagger)}{\int D\Phi D\Phi^\dagger e^S(\Phi, \Phi^\dagger)} \] (2.33)

Introducing the following new functions:

\[ N (k, Y; b) = 2\pi^2 \alpha_S \langle \Phi (k, Y; b) \rangle \quad \quad N^\dagger (k, Y; b) = 2\pi^2 \alpha_S \langle \Phi^\dagger (k, Y; b) \rangle \] (2.34)

then the equation of motion of Eq. (2.31) reduces to:

\[ 0 = \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \left( \frac{\partial}{\partial Y'} - H \right) N (k_l, b, Y') \]

\[ + \alpha_S \left\{ 2 N^\dagger (-k_l, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} N (k_l, b, Y') + \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} N^2 (-k_l, b, Y') \right\} \] (2.35)

Using the following property of the BFKL Pomeron, namely,

\[ p_1^2 p_2^2 \left( \frac{\partial}{\partial Y} - H \right) = \left( \frac{\partial}{\partial Y} - H^\dagger \right) p_1^2 p_2^2, \] (2.36)

we can derive the second equation of motion, by taking the functional derivative with respect to \( \Phi \). This generates the following equation of motion:
\[0 = \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \left( \frac{\partial}{\partial Y} - H \right) \mathcal{N}^\dagger \left( k_l, b, Y \right) \]

\[+ \tilde{\alpha}_S \left\{ 2N \left( -k_l, b, Y' \right) \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \mathcal{N}^\dagger \left( k_l, b, Y' \right) + \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial^2}{\partial l^2} \left( \mathcal{N}^\dagger \right)^2 \left( -k_l, b, Y' \right) \right\} \]

In Eq. (2.35) and Eq. (2.37) the following identities were implemented:

\[\langle \Phi^2 (k, Y; b) \rangle = \left( \langle \Phi (k, Y; b) \rangle \right)^2 \]

\[\langle \left( \Phi^\dagger \right)^2 (k, Y; b) \rangle = \left( \langle \Phi^\dagger (k, Y; b) \rangle \right)^2 \]

\[\langle \Phi (k, Y; b) \Phi^\dagger (k, Y; b) \rangle = \langle \Phi (k, Y; b) \rangle \times \langle \Phi^\dagger (k, Y; b) \rangle \]

For a discussion on why these equations are correct, within an accuracy of about \(1/A^{1/3}\) in the case of nucleus-nucleus scattering, see Refs. [9, 20].

3. Semiclassical solution

3.1 The system of equations: general approach

In this section we find the semiclassical solutions to Eq. (2.35) and Eq. (2.37). In the semi-classical approximation, we are searching for solutions of the following form [21]:

\[N \left( l \equiv \ln(k^2), b, Y' \right) = \exp \left( S \left( l, b, Y' \right) \right) \quad \text{where} \quad S = \omega Y' - (1 - \gamma) l + \beta(b) \]

\[\mathcal{N}^\dagger \left( l \ln(k^2), b, Y' \right) = \exp \left( S^\dagger \left( l, b, Y' \right) \right) \quad \text{where} \quad S^\dagger = \omega^\dagger Y' - (1 - \gamma^\dagger) l + \beta^\dagger(b) \]

where \(\omega(l, b, Y')\) and \(\gamma(l, b, Y')\) are smooth functions of \(Y'\) and \(l\), and the following conditions are assumed:

\[\frac{d\omega(l, b, Y')}{dY'} \ll \omega^2(l, b, Y'); \quad \frac{d\omega(l, b, Y')}{dl} \ll \omega(l, b, Y') \left( 1 - \gamma(l, b, Y') \right); \]

\[\frac{d\gamma(l, b, Y')}{dl} \ll \left( 1 - \gamma(l, b, Y') \right)^2; \quad \frac{d\gamma(l, b, Y')}{dY'} \ll \omega(l, b, Y') \left( 1 - \gamma(l, b, Y') \right); \]

with analogous conditions for the functions \(\omega^\dagger(l, b, Y')\) and \(\gamma^\dagger(l, b, Y')\). Assuming that for Eq. (3.1), the method of characteristics can be applied (see, for example, Ref. [22]) to solve the non-linear equation. Notice that

\[He^S = \tilde{\alpha}_S \int K (k, k') e^{S(k', b, Y)} = \tilde{\alpha}_S \chi(\gamma) e^{S(k, b, Y)} \]

where \(\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)\)
where $\psi(z) = d \ln \Gamma(z)/dz$ is the Di-Gamma function. In the semiclassical approach (for $\gamma$ a smooth function), inserting the definition of Eq. (3.1) into the equations of motion of Eq. (2.35) and Eq. (2.37), and using the conditions of Eqs. (3.2) - (3.3), leads to the following formulae in the semi-classical approach:

\[
\begin{align*}
    \omega - \tilde{\alpha}_S \chi(\gamma) &+ \kappa(\gamma^\dagger) \tilde{\alpha}_S e^{\tilde{S}_l} + \tilde{\alpha}_S e^{\tilde{S}} &= 0 \\
    -\omega^\dagger - \tilde{\alpha}_S \chi(\gamma^\dagger) &+ \kappa(\gamma) \tilde{\alpha}_S e^{\tilde{S}^\dagger} + \tilde{\alpha}_S e^\tilde{S} &= 0
\end{align*}
\]

(3.6)

where the following functions were introduced with definitions:

\[
\kappa(\gamma) = \frac{2 \gamma^2}{4(2\gamma - 1)^2} \quad \tilde{S} = S - \ln \kappa(\gamma) \quad \tilde{S}^\dagger = S^\dagger - \ln \kappa(\gamma^\dagger)
\]

For the equation in the form

\[
F(Y', l, \tilde{S}, \gamma, \omega) = 0
\]

(3.8)

where $S$ is given by Eq. (3.1), we can introduce the set of characteristic lines on which $l(t), Y'(t), S(t), \omega(t),$ and $\gamma(t)$ are functions of the variable $t$ (which we call artificial time), that satisfy the following equations:

\[
\begin{align*}
    (1.) \quad \frac{dl}{dt} &= F_{\gamma} = -\tilde{\alpha}_S \frac{d\chi(\gamma)}{d\gamma} \\
    (2.) \quad \frac{dY'}{dt} &= F_{\omega} = 1 \\
    (3.) \quad \frac{d\tilde{S}}{dt} &= \gamma F_{\gamma} + \omega F_{\omega} = \tilde{\alpha}_S (1 - \gamma) \frac{d\chi(\gamma)}{d\gamma} + \omega \\
    (4.) \quad \frac{d\gamma}{dt} &= -(F_l + \gamma F_S) = \tilde{\alpha}_S \kappa(\gamma^\dagger) (1 - \gamma^\dagger) e^{\tilde{S}^\dagger} + \tilde{\alpha}_S (1 - \gamma) e^\tilde{S} \\
    (5.) \quad \frac{d\omega}{dt} &= -(F_{Y'} + \omega F_S) = -\tilde{\alpha}_S \omega \kappa(\gamma^\dagger) e^{\tilde{S}^\dagger} - \tilde{\alpha}_S \omega e^\tilde{S}
\end{align*}
\]

(3.9)

where $F_l = \frac{\partial F(Y', l, \tilde{S}, \gamma, \omega)}{\partial l}$, and the terms in the first of Eqns. (3.6) that depend on $\tilde{S}^\dagger, \omega^\dagger, \gamma^\dagger$, are treated as if they depend explicitly on $Y'$ and $l$. The same five equations we can write for the second of Eqns. (3.6) which have the following form:

\[
\begin{align*}
    (1.) \quad \frac{dl^\dagger}{dt} &= F_{\gamma^\dagger} = -\tilde{\alpha}_S \frac{d\chi(\gamma^\dagger)}{d\gamma^\dagger} \\
    (2.) \quad \frac{dY'^\dagger}{dt} &= F_{\omega^\dagger} = -1, \\
    (3.) \quad \frac{d\tilde{S}^\dagger}{dt} &= \gamma^\dagger F_{\gamma^\dagger} + \omega^\dagger F_{\omega^\dagger} = \tilde{\alpha}_S (1 - \gamma^\dagger) \frac{d\chi(\gamma^\dagger)}{d\gamma^\dagger} - \omega^\dagger \\
    (4.) \quad \frac{d\gamma^\dagger}{dt} &= -(F_l + \gamma^\dagger F_S) = \tilde{\alpha}_S \kappa(\gamma) (1 - \gamma) e^{\tilde{S}} + \tilde{\alpha}_S (1 - \gamma^\dagger) e^{\tilde{S}^\dagger} \\
    (5.) \quad \frac{d\omega^\dagger}{dt} &= -(F_{Y'} + \omega^\dagger F_S) = -\tilde{\alpha}_S \omega \kappa(\gamma) e^{\tilde{S}} - \tilde{\alpha}_S \omega^\dagger e^{\tilde{S}^\dagger}
\end{align*}
\]

(3.10)
3.2 The system of equations: linear approximation

We start with the solution of Eq. (2.35) and Eq. (2.37) in the kinematic region where the non-linear corrections are small. In this region, we can reduce this system of equations to the solution of the BFKL equation, both for $N(l, b, Y')$ and $N^\dagger(l, b, Y')$. Actually in order to find the solution, it isn’t necessary to solve the equation, since the $t$-channel unitarity constraints for the BFKL Pomeron can be used instead (see Refs. [1, 16, 23]). This is given by:

$$N_{BFKL}(L, b, Y) = \int d^2b' \int dl \ N^\dagger_{BFKL} \left( L - l, \vec{b} - \vec{b}', Y - Y' \right) \ N_{BFKL} \left( l, b', Y' \right)$$

(3.11)

Using the explicit form for the BFKL solution [16,17], then Eq. (3.11) reduces to the following expression:

$$N_{BFKL}(L, b, Y) = \int \frac{d\gamma}{2\pi i} \ n_{BFKL}(\gamma, b) \ e^{\bar{\alpha_S}\chi(\gamma)} Y - (1-\gamma) L$$

(3.12)

$$= \int d^2b' \int dl \ \int \frac{d\gamma}{2\pi i} \ N^\dagger_{BFKL}(\gamma, \vec{b} - \vec{b}') \ e^{\bar{\alpha_S}\chi(\gamma)} (Y - Y') - (1-\gamma)(L - l) \ n_{BFKL}(\gamma', b') \ e^{\bar{\alpha_S}\chi(\gamma')} Y' - (1-\gamma') l$$

where $L = \ln \left( k_p^2 / k_t^2 \right)$ and $l = \ln \left( k^2 / k_2^2 \right)$. $k_p$ and $k_t$ are the momenta of the dipoles in the projectile and the target, respectively. The function $n_{BFKL}(\gamma, b)$ is determined by the initial condition at $Y = 0$.

The integration over $l$ leads to the delta function $(2\pi)^2 \delta(\gamma - \gamma')$, and hence the $\gamma'$ integral is immediately solvable by setting $\gamma = \gamma'$ everywhere in the integrand. Integrating over the impact parameter $b'$ yields:

$$\int d^2b' \ n^\dagger_{BFKL}(\gamma, \vec{b} - \vec{b}') n_{BFKL}(\gamma, b') = n_{BFKL}(\gamma, b)$$

(3.13)

In light of this Eq. (3.12) simplifies to

$$N_{BFKL}(L, b, Y) = \int \frac{d\gamma}{2\pi i} \ e^{\bar{\alpha_S}\chi(\gamma)} Y - (1-\gamma) L \ n_{BFKL}(\gamma, b)$$

(3.14)

Incidentally Eq. (3.14) satisfies Eq. (3.11). Therefore, we have reproduced Eq. (3.11), where $N^\dagger$ and $N$ are given explicitly by:

$$N^\dagger_{BFKL} \left( L - l, \vec{b} - \vec{b}', Y - Y' \right) = \int \frac{d\gamma}{2\pi i} \ n^\dagger_{BFKL}(\gamma, \vec{b} - \vec{b}') \ e^{\bar{\alpha_S}\chi(\gamma)} (Y - Y') - (1-\gamma)(L - l)$$

(3.15)

$$N_{BFKL}(l, b', Y') = \int \frac{d\gamma'}{2\pi i} \ n_{BFKL}(\gamma', b') \ e^{\bar{\alpha_S}\chi(\gamma')} Y' + (1-\gamma') l$$

(3.16)

where the functions $n^\dagger_{BFKL}(\gamma, \vec{b} - \vec{b}')$ and $n_{BFKL}(\gamma', b')$ can be found from the initial conditions for $N^\dagger_{BFKL}$ at $Y' = Y$ and $N_{BFKL}$ at $Y' = 0$. 

- 11 -
The integral expressions of Eqs. (3.15) and (3.16) can be used to derive the explicit expressions for $N_{BFKL}^\dagger \left(L-l, \vec{b} - \vec{b}', Y - Y'\right)$ and $N_{BFKL} (l, b', Y')$ in the semi-classical approach, by integrating using the method of steepest decent. This is assuming that $n_{BFKL}^\dagger (\gamma, \vec{b} - \vec{b}')$ and $n_{BFKL} (\gamma', b')$ are continuous, and differentiable functions of $\gamma$ and $\gamma'$.

The saddle point equations, for the functions in the exponent in Eqs. (3.15) and (3.16) are:

$$\bar{\alpha}_S d\chi(\gamma'_{SP}) (Y - Y') + L - l = 0 \quad \text{and} \quad \bar{\alpha}_S \frac{d\chi(\gamma'_SP)}{d\gamma'_SP} Y' + l = 0 \quad (3.17)$$

The integrals of Eqns. (3.15) and (3.16) can be solved using the method of steepest descents to yield:

$$N_{BFKL}^\dagger \left(L-l, \vec{b} - \vec{b}', Y - Y'\right) = \sqrt{\frac{2\pi}{\bar{\alpha}_S \chi''(\gamma_{SP}) (Y - Y')}} n_{BFKL}^\dagger (\gamma_{SP}, \vec{b} - \vec{b}') \exp (\bar{\alpha}_S \chi(\gamma_{SP}) (Y - Y') - (L - l) (1 - \gamma_{SP}))$$

$$= \bar{n}_{BFKL}(\gamma_{SP}, \vec{b} - \vec{b}') e^{S^\dagger} \quad (3.18)$$

$$N_{BFKL} (l, b', Y') = \sqrt{\frac{2\pi}{\bar{\alpha}_S \chi''(\gamma'_SP) Y'}} n_{BFKL} (\gamma'_SP, \vec{b} - \vec{b}') \exp (\bar{\alpha}_S \chi(\gamma'_SP) Y' p - l (1 - \gamma'_SP))$$

$$= \bar{n}_{BFKL} (\gamma'_SP, b') e^S \quad (3.19)$$

$$S = \bar{\alpha}_S \left(\chi(\gamma') - (1 - \gamma_{SP}') \frac{d\chi(\gamma'_{SP})}{d\gamma'_{SP}}\right) Y'$$

$$S^\dagger = \bar{\alpha}_S \left(\chi(\gamma_{SP}) - (1 - \gamma_{SP}) \frac{d\chi(\gamma_{SP})}{d\gamma_{SP}}\right) (Y - Y')$$

where all slowly changing terms, have been absorbed by the functions $\bar{n}^\dagger$ and $\bar{n}$. Eq. (3.3) together with Eq. (2.35) have the following form in the linear approximation:

$$\begin{align*}
(1.) \frac{d l}{d Y'} &= -\bar{\alpha}_S \frac{d\chi(\gamma)}{d\gamma}; \\
(2.) \omega &= \bar{\alpha}_S \chi(\gamma); \\
(3.) \frac{d \tilde{S}}{d Y'} &= \bar{\alpha}_S \left(1 - \gamma \right) \frac{d\chi(\gamma)}{d\gamma} + \omega; \\
(4.) \frac{d \gamma}{d Y'} &= 0; \quad (3.20)
\end{align*}$$

It is easy to see that Eq. (3.20) leads to the same $S$ as Eq. (3.18) and Eq. (3.19). One can see that for $\gamma = \gamma_{cr}$ for which

$$\chi(\gamma_{cr}') + (1 - \gamma_{cr}') \frac{d\chi(\gamma_{cr}')}{d\gamma_{cr}'} = 0 \quad (3.21)$$
Figure 2: The trajectories of the linear equations for $N^\dagger$ and $N$: $L = \ln \left( k_i^2/k_i^2 \right)$, $l = \ln \left( k^2/k_i^2 \right)$ and $\alpha = \bar{\alpha}S(\gamma_{cr})/\gamma_{cr}$. Red lines denote the critical trajectories. Only the trajectories to the left (for $N^\dagger$) and to the right (for $N$) are shown.

Then $S = 0$. The equation for this line takes the form:

$$l = \bar{\alpha}S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} Y'$$

(3.22)

Repeating all the steps of the calculations above for $S^\dagger$, we find that $S^\dagger = 0$ on the line

$$L - l = \bar{\alpha}S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} (Y - Y')$$

(3.23)

using the same $\gamma_{cr}$ from Eq. (3.21). The general pattern of trajectories for the linear equation is shown in Fig. 2. It should be stressed that for Eq. (3.9) and Eq. (3.10), we have different trajectories but they are parallel and shifted by $\Delta l = \bar{\alpha}S(\chi(\gamma_{cr})/\gamma_{cr})Y$.

We need to know the initial conditions for $S(S^\dagger)$ and $\gamma (\gamma^\dagger)$ at $Y' = 0$ ($Y' = Y$). We choose the McLerran-Venugopalan formula [3] which is written for the dipole-target amplitude and is given in terms of $N$ as:

$$N(r, b, Y' = 0) = 1 - \exp(-r^2k_i^2(b)/4)$$

(3.24)

where the initial characteristic momentum $k_i^2(b) \propto T_A(b)$, and $T_A(b)$ is used to denote the number of nucleons inside the nucleus with fixed impact parameter $b$. In momentum representation $N$ is equal to (see Fig. 3):

$$N_0(k, b, Y' = 0) = \int rdr J_0(kr) N(r, b, Y' = 0) /r^2 = \frac{1}{2} \Gamma(0, \tau = \frac{k_i^2}{k^2})$$

or $S = \ln \left( \frac{1}{2} \Gamma_0(\tau) \right)$

(3.25)

$$\gamma_0 - 1 = \frac{\partial \ln \left( N(k, b, Y' = 0) \right)}{\partial \ln \left( k^2/k_i^2 \right)} = -e^{-\tau}/\Gamma_0(\tau)$$

(3.26)

$\Gamma_0(\tau)$ that appears in Eq. (3.25) is the Euler incomplete gamma function (see formulae 8.50 in Ref. [24]).
Caution should be taken here, since we cannot trust the McLerran-Venugopalan formula at small dipole sizes. Indeed, we know that in the limit of perturbative QCD, \( \gamma_0 \) is \( \gamma \to 0 \) contradicts this formula. The problem is that the simplified version of Eq. (3.24) does not reproduce the perturbative QCD limit at \( r \to 0 \). The relation in Eq. (3.24) leads to \( N \propto r^2 \) at \( r \to 0 \), while the correct behaviour should be \( N \propto r^2 \ln r^2 \). Nevertheless we will use Eq. (3.24), because our main interest lies in the region in the vicinity of the saturation scale, where Eq. (3.24) reproduces the amplitude for \( N \) quite well.

For \( N^\dagger \) the initial conditions look the same, but \( \tau^\dagger = k^2/k_f^2 \). In other words, \( N^\dagger (k, b, Y' = Y) = e^{S_0^\dagger(k)} = \frac{1}{2} \Gamma_0 (\tau^\dagger) \). One can see that using Eq. (3.9)(4) and Eq. (3.10)(4), we obtain that \( S^\dagger (l, Y') = S(L, Y) - S(l, Y') + S_0^\dagger \). It means that \( \omega^\dagger = -\omega \) and \( 1 - \gamma^+ = -(1 - \gamma) \). It is easy to see that the system of equations in Eqns. (3.10) degenerate to the system of equations in (3.9).

### 3.3 The final system of equations

Therefore, instead of Eq. (3.9) and Eq. (3.10), we can solve the following system of equations.

\[
\begin{align*}
\frac{d l}{d t} &= -\bar{\alpha}_S \frac{d \chi(\gamma)}{d \gamma}, & \frac{d Y'}{d t} &= 1, \\
\frac{d S}{d t} &= \bar{\alpha}_S (1 - \gamma) \frac{d \chi(\gamma)}{d \gamma} + \omega, \\
\frac{d \gamma}{d t} &= -\bar{\alpha}_S \kappa(\gamma) (1 - \gamma) e^{S(L,Y)} - S(l,Y') + S_0 + \bar{\alpha}_S (1 - \gamma) e^{S(l,Y')}, \\
\frac{d \omega}{d t} &= \bar{\alpha}_S \left( \omega \kappa(\gamma) e^{S(L,Y)} - S(l,Y') + S_0 - \omega e^{S(l,Y')} \right),
\end{align*}
\]  

(3.26)

For the sake of simplicity, the label \( \tilde{\gamma} \) has been omitted everywhere in Eq. (3.26). Eq. (3.26) can be re-written in a different form, namely

\[
\begin{align*}
\frac{d l}{d Y'} &= -\bar{\alpha}_S \frac{d \chi(\gamma)}{d \gamma}, & \frac{d S}{d Y'} &= \bar{\alpha}_S (1 - \gamma) \frac{d \chi(\gamma)}{d \gamma} + \omega, & \frac{d \gamma}{d Y'} &= -\frac{1 - \gamma}{\omega}, \\
\frac{d \gamma}{d S} &= -\kappa(\gamma) \frac{e^{S(L,Y)} - S(l,Y') + S_0}{d \chi(\gamma)/d \gamma + \omega/(1 - \gamma)}
\end{align*}
\]  

(3.27)

Eq. (3.27)-4 has the solution \( \omega = \bar{\alpha}_S Const(1 - \gamma) \) where \( Const \) can be determined from the initial conditions.

\[
Const = \frac{\chi(\gamma_0) - \kappa(\gamma_0) e^{S(L,Y)} - e^{S_0}}{1 - \gamma_0}
\]  

(3.28)

Notice that the value of this constant depends on \( S(L,Y) \), but for large and negative \( S(L,Y) \) which we are dealing with, for \( \gamma_0 < \gamma_{cr} \), we can safely neglect this dependence. Introducing \( \hat{S} = S(l,Y') - (S(L,Y) + S_0)/2 \) we can rewrite Eq. (3.27)-5 in the form

\[
\begin{align*}
\frac{d \gamma}{d S} &= -\kappa(\gamma) \frac{\hat{S}}{d \chi(\gamma)/d \gamma + \omega/(1 - \gamma)}
\end{align*}
\]  

(3.29)
\[
\frac{d\gamma}{dS} = e^{(S(L,Y)+S_0)/2} \left( \frac{-\kappa(\gamma) e^{-\hat{S}(l,Y')} + e^{\hat{S}(l,Y')}}{d\chi(\gamma)/d\gamma + \text{Const}} \right)
\] (3.29)

We follow the following strategy for solving this equation. First we assume that we are looking for the solution in the interval \([\hat{S}_0, \hat{S}_{\text{max}}]\), where we have selected \(\hat{S}_{\text{max}}\). Using the initial conditions, we can calculate \(\hat{S}(l,Y=0)\) through \(S_0\) and \(S_{\text{max}}\), and specify the coefficient in front of Eq. (3.29). Solving Eq. (3.27)-2 we will find \(Y'\) as a function of \(\hat{S}\). In particular \(Y'(S_{\text{max}}) \neq Y\), however trying several times we find a value for \(S_{\text{max}}\) that will give \(Y'(S_{\text{max}})\). Fig. 4 shows the solutions for four values of \(\gamma_0\) that are smaller than \(\gamma_{cr}\). One can see that the trajectories and the values of \(S\) and \(\gamma\) are close to the solution of the linear equation. Indeed, \(dS/dY'<0\) for \(\gamma < \gamma_{cr}\) and therefore, the value of \(S\) decreases due to evolution. Starting with \(S_0 < 0\) one can see that the contribution of the non-linear terms becomes less and less at higher values of \(Y'\) than at \(Y' = 0\). Therefore, only at small values of \(Y'\) we can see a deviation from the solution to the linear equation, as shown in Fig. 4 (see Fig. 4-b for example). Eq. (3.27)-4 has the form

\[
\frac{d\gamma}{dY'} = \bar{\alpha}_S (1 - \gamma) e^{(S(L,Y)+S_0)/2} \left\{ -\kappa(\gamma) e^{-\hat{S}(l,Y')} + e^{\hat{S}(l,Y')} \right\}
\] (3.30)

and due to the smallness of the factor \(\exp((S(L,Y)+S_0)/2)\), then \(d\gamma/dY'\) turns out to be small at large \(Y'\), leading to a constant \(\gamma\), as shown in Fig. 4-b. The value of \(\gamma_{cr}\) is slightly different from the one from Eq. (3.21), due to a contribution from the non-linear terms to Eq. (3.27)-3, and it depends on the initial condition for \(S(Y' = 0) = S_0\). Indeed, for the trajectory on which \(S(Y' = 0) = S_0\) is constant we have the
following equation:
\[
(1 - \gamma_{cr}) \frac{d\chi(\gamma_{cr})}{d\gamma_{cr}} + \chi(\gamma_{cr}) - (\kappa(\gamma_{cr}) + 1) e^{S_0} = 0
\]  
(3.31)

The dependence of \(S_0\) on \(\gamma_{cr}\) from Eq. (3.31) is shown in Fig. 5. One can see that for \(S_0 < 0\) (\(N_0 < 1\)), the value of \(\gamma_{cr}\) is close to the solution to Eq. (3.21). From Fig. 4 one can see that we start the evolution from the value of \(\gamma\) that is close to \(\gamma_{cr}\), \(\gamma\) steeply increases to \(\gamma = \gamma_{cr}\) and freezes at this value leading to constant \(S\) almost in the entire kinematic region of \(\bar{\alpha}_S Y'\). For \(\gamma > \gamma_{cr}\), the solution will lead to \(S(l, Y')\) that increases with \(Y'\), and we need to search for a different method of finding the solution, other than the semi-classical approach.

4. Solution inside the saturation domain

4.1 General consideration

As discussed above, the semi-classical approach cannot be used inside of the saturation region. It should be mentioned that at large \(l\), the amplitude behaves as \(\frac{1}{2l}\), which is certainly not the function for which we can use the semi-classical approach. However, it has been noticed in Ref. [21] that introducing new functions: \(\phi(l', Y', b)\) and \(\phi^\dagger(l', Y', b)\) instead of \(N\) and \(N^\dagger\), defined as
\[
N(l, Y', b) = \frac{1}{2} \int dl' \left( 1 - e^{-\phi(l', Y', b)} \right), \quad N^\dagger(l, Y', b) = \frac{1}{2} \int dl' \left( 1 - e^{-\phi^\dagger(l', Y', b)} \right)
\]  
(4.1)

Then we can indeed use the semi-classical approach for these functions. The first observation is that, from the property of Eq. (3.4) and the definition of Eq. (3.16):
\[
H \frac{\partial N(l, Y', b)}{\partial l} = \int \frac{d\gamma}{2\pi i} \bar{\alpha}_S \chi(\gamma) \frac{n((\gamma - 1), Y', b)}{1 - \gamma} \exp \left( \bar{\alpha}_S \chi(\gamma) Y' - (1 - \gamma) l \right)
\]
\[
\equiv \bar{\alpha}_S \int \frac{d\gamma}{2\pi i} \chi(\gamma) \frac{1}{1 - \gamma} \frac{n(\gamma, Y', b)}{1 - \gamma} \exp \left( \bar{\alpha}_S \chi(\gamma) Y' - (1 - \gamma) l \right) - \bar{\alpha}_S N(l, Y', b)
\]
\[
\equiv \bar{\alpha}_S L \left( -\frac{\partial}{\partial l} \right) \frac{\partial N(l, Y', b)}{\partial l} - \bar{\alpha}_S N(l, Y', b)
\]

The second observation is related to the function \(L(\gamma)\), namely that its expansion with respect to \((1 - \gamma)\) starts with \((1 - \gamma)^2\) as the first non-zero term. That is:
\[
L \left( -\frac{\partial}{\partial l} \right) = -\frac{d^2\psi(z)}{dz^2} \bigg|_{z=1} \frac{\partial^2}{\partial l^2} - \frac{1}{12} \frac{d^4\psi(z)}{dz^4} \bigg|_{z=1} \frac{\partial^4}{\partial l^4} - \cdots
\]  
(4.2)
where \( \psi(z) = d \ln \Gamma(z)/dz \) is the Euler \( \psi \)-function (see Ref. [24]). Following the definition of Eq. (4.1) and assuming that \( \phi \) and \( \phi^\dagger \) are smooth functions, we can replace *

\[
\left( \frac{\partial}{\partial l} \right)^n N(l, Y', b) = -\frac{1}{2} \left( \frac{\partial}{\partial l} \right)^{n-1} \left( 1 - e^{-\phi(l, Y', b)} \right) = \frac{1}{2} \left( -\frac{\partial \phi}{\partial l} \right)^{n-1} e^{-\phi(l, Y', b)}
\]

(4.3)

\[
\left( \frac{\partial}{\partial l} \right)^n N^\dagger(l, Y', b) = -\frac{1}{2} \left( \frac{\partial}{\partial l} \right)^{n-1} \left( 1 - e^{-\phi^\dagger(l, Y', b)} \right) = \frac{1}{2} \left( -\frac{\partial \phi^\dagger}{\partial l} \right)^{n-1} e^{-\phi^\dagger(l, Y', b)}
\]

(4.4)

Inserting Eqs. (4.2), (4.3) and (4.4) and into Eq. (2.35) and Eq. (2.37), and introducing the notation \( \partial \phi/\partial Y' = \omega_\phi \) and \( \partial \phi^\dagger/\partial Y' = \omega^\dagger_\phi \) we obtain:

\[
0 = \left( \omega_\phi + \alpha_S \mathcal{L}(-\gamma_\phi) \right) (1 - \gamma_\phi)^2 \gamma_\phi e^{-\phi(l, b, Y')} + 2 \alpha_S \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial \phi}{\partial l} e^{-\phi(l, b, Y')} \]

(4.5)

\[
\alpha_S 2 N^\dagger(l, b, Y') \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial \phi}{\partial Y'} e^{-\phi(l, b, Y')}
\]

\[
0 = \left( -\omega^\dagger_\phi + \alpha_S \mathcal{L}(-\gamma^\dagger_\phi) \right) (1 - \gamma^\dagger_\phi)^2 \gamma^\dagger_\phi e^{-\phi^\dagger(l, b, Y')} + 2 \alpha_S \left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial \phi^\dagger}{\partial l} e^{-\phi^\dagger(l, b, Y')} \}

(4.6)

One can see that

\[
\left( \frac{\partial}{\partial l} + 1 \right)^2 \frac{\partial \phi}{\partial Y'} e^{-\phi(l, b, Y')} N(l, b, Y') \}
\]

\[
= e^{-\phi} \left\{ \frac{1}{2} \left( 1 - 6 \gamma_\phi + 7 \gamma_\phi^2 \right) e^{-\phi} - \frac{1}{2} \left( 1 - 4 \gamma_\phi + 3 \gamma_\phi^2 \right) \right\}
\]

(4.7)

\[
- \gamma_\phi (1 - \gamma_\phi)^2 N(l, b, Y') \}
\]

Dividing both sides of Eq. (4.5) by \( \alpha_S e^{-\phi} \) and introducing the new variable \( \tilde{\omega} = \omega/\alpha_S \), which corresponds to the change \( Y' \) to \( Y' = \alpha_S Y' \) we obtain:

\[
-\tilde{\omega}_\phi - \mathcal{L}(-\gamma_\phi) + 2 N^\dagger(l, b, Y') - \frac{\left( 1 - 6 \gamma_\phi + 7 \gamma_\phi^2 \right) e^{-\phi} \left( 1 - 4 \gamma_\phi + 3 \gamma_\phi^2 \right)}{(1 - \gamma_\phi)^2 \gamma_\phi} + 2 N(l, b, Y') = 0 \quad (4.8)
\]

A similar equation can be written for \( \phi^\dagger \). However, we assume that \( \phi^\dagger(l, b, Y') = \phi(L - l, b, Y - Y') \) based on our experience with the semi-classical solution. Looking for the solution that has a geometric scaling behaviour [25], we expect that \( \phi(l, b, Y') \) is a function of one variable:

\[
z = \ln \left( Q^2_s \right) = \frac{\left( (\gamma_{cr}) / (1 - \gamma_{cr}) \right) Y' - l}{1 - \gamma_{cr}} \quad (4.9)
\]

*For \( \partial \phi/\partial l \) (\( \partial \phi^\dagger/\partial l \)) we use the notations \( \gamma_\phi \) and \( \gamma^\dagger_\phi \). We hope that it will not lead to any confusions with the notation, due to the similarity with \( \gamma \) and \( \gamma^\dagger \), that was heavily used in the previous section.
4.2 Asymptotic solution

Plugging in \( \phi^\dagger (l, b, Y') = \phi (L - l, b, Y - Y') \) and the geometric scaling behavior of Eq. (4.8) that we anticipate, and assuming that \( \phi \gg 1 \) we obtain:

\[
G(\tilde{\gamma}) = z_Y
\]

where

\[
G(\tilde{\gamma}) = \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \tilde{\gamma} + L(\tilde{\gamma}) + \frac{1 + 4\gamma + 3\gamma^2}{(1 + \gamma)^2} \left( 1 + \frac{\gamma}{\chi(\gamma_{cr})} \right) Y - L
\]

It should be noticed that \( \tilde{\gamma} \) is defined as \( \tilde{\gamma} = d\phi/dz \). The function \( G(\gamma_{\phi}) \) is shown in Fig. 6. One can see that we have three types of solutions to Eq. (4.10):

1. At large \( \gamma \), then \( G(\tilde{\gamma}) \to \chi(\gamma_{cr})/(1 - \gamma_{cr}) \tilde{\gamma} \) and we have the solution: \( \tilde{\gamma} = (1 - \gamma_{cr})/\chi(\gamma_{cr}) z_Y \), which translates into \( \phi = \left( (1 - \gamma_{cr})/\chi(\gamma_{cr}) \right) z_Y z \);

2. We have solutions at \( \tilde{\gamma} \to n \) where \( n = 1, 2, \ldots \) which lead to \( \phi = nz \). We only need to take into account \( n = 1 \), since other values of \( n \) give smaller contributions at large \( z \). In vicinity \( \gamma \to 1 \) \( G(\gamma) = -1/(1 - \gamma) \) and the solution to Eq. (4.11) gives \( \phi = (1 + 1/z_Y) z \);

3. Solutions where \( \tilde{\gamma} \to -n \) we do not consider, since they lead to decreasing \( \phi \) at large \( Y \).

It is interesting to notice, that at the case where we restrict ourselves to the leading twist contributions to the BFKL kernel [26], only the first solution survives (see Fig. 6-b).

Using Eq. (4.11) we can obtain the asymptotic solution for the scattering amplitude, namely

\[
N(z_Y) = \frac{1}{2} \int_0^{z_Y} dz \left( 1 - \exp \left( - \frac{1 - \gamma_{cr}}{\chi(\gamma_{cr})} z_Y z \right) \right) \quad \text{leading twist;} \quad \text{(4.11)}
\]

\[
N(z_Y) = \frac{1}{2} \int_0^{z_Y} dz \left( 1 - \exp (- z) \right) \quad \text{general asymptotic behaviour;} \quad \text{(4.12)}
\]
4.3 General equations

In general, Eq. (4.8) can be reduced to the differential equation by taking derivatives with respect to \( z \) on both sides of the equation. The resulting expression is:

\[
\frac{dG(\tilde{\gamma})}{d\tilde{\gamma}} \frac{d\tilde{\gamma}}{dz} = e^{-\phi(z)} - e^{\phi(z_Y - z)}, \quad \frac{d\phi(z)}{dz} = \tilde{\gamma},
\]

(4.13)

This equation belongs to the class of delay differential equations, and the solution to this equation we hope to publish in an upcoming paper, since it is a separate and rather difficult problem beyond the scope of this paper (see for example Ref. [27]).

5. Conclusions

In conclusion, we summarize our results as follows. First we re-wrote the action of the BFKL Pomeron calculus, and we derived the equations in momentum representation. It turns out that the equations that we obtain have a simpler form, than in coordinate representation in the format that they were originally derived in Ref. [9]. Second, we found the semi-classical solution to these equations, to the right of the critical line. In the saturation domain, we reduced these equations to the class of delay differential equations, and we found their asymptotic solution. This solution shows, that the nucleus-nucleus amplitude in coordinate representation approaches unity as \( 1 - \Delta N \), where \( \Delta N \propto \exp(-z_Y) \), where \( z_Y \equiv \ln \left( \frac{Q_s^2(Y)}{r_i^2} \right) \), where \( r_i \) is the size of the colourless dipole in the nucleus. If we take into account only the leading twist part of the BFKL kernel, the behaviour of \( \Delta N \) is similar to the behaviour of the amplitude of the dilute-dense parton system interaction, given in Ref. [26].

We hope that this paper will show, that the problem of the nucleus-nucleus interaction in the framework of the BFKL Pomeron calculus, can be solved. We hope that this development will motivate further efforts towards understanding the dense-dense scattering system.

6. Acknowledgements

This research was supported by the Fundação para ciência e a tecnologia (FCT), and CENTRA – Instituto Superior Técnico (IST), Lisbon and by the Fondecyt (Chile) grants 1100648, 1095196 and DGIP 11.11.05. One of us (JM) would like to thank Tel Aviv University for their hospitality on this visit, during the time of the writing of this paper.
References

[1] L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rep. 100 (1983) 1.
[2] A. H. Mueller and J. Qiu, Nucl. Phys. B268 (1986) 427.
[3] L. McLerran and R. Venugopalan, Phys. Rev. D49 (1994) 2233, 3352; D50 (1994) 2225; D53 (1996) 458; D59 (1999) 09400.
[4] A. H. Mueller, Nucl. Phys. B 415, 373 (1994); Nucl. Phys. B 437 (1995) 107 [arXiv:hep-ph/9408245].
[5] I. Balitsky, [arXiv:hep-ph/9509348]; Phys. Rev. D60, 014020 (1999) [arXiv:hep-ph/9812311]
[6] Y. V. Kovchegov, Phys. Rev. D60, 034008 (1999), [arXiv:hep-ph/9901281].
[7] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Phys. Rev. D59, 014014 (1999), [arXiv:hep-ph/9706377]; Nucl. Phys. B504, 415 (1997), [arXiv:hep-ph/9701284]; J. Jalilian-Marian, A. Kovner and H. Weigert, Phys. Rev. D59, 014015 (1999), [arXiv:hep-ph/9709432]; A. Kovner, J. G. Milhano and H. Weigert, Phys. Rev. D62, 114005 (2000), [arXiv:hep-ph/0004014]; E. Iancu, A. Leonidov and L. D. McLerran, Phys. Lett. B510, 133 (2001); [arXiv:hep-ph/0102009]; Nucl. Phys. A692, 583 (2001), [arXiv:hep-ph/0011241]; E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. A703, 489 (2002), [arXiv:hep-ph/0109115]; H. Weigert, Nucl. Phys. A703, 823 (2002), [arXiv:hep-ph/0004044].
[8] E. Gotsman, E. Levin, U. Maor and E. Naftali, Nucl. Phys. A750 (2005) 391 [arXiv:hep-ph/0411242].
[9] M. A. Braun, Phys. Lett. B 483 (2000) 115, [arXiv:hep-ph/0003004]; Eur. Phys. J. C 33 (2004) 113 [arXiv:hep-ph/0309293]; Phys. Lett. B 632 (2006) 297, [Eur. Phys. J. C 48 (2006) 511], [arXiv:hep-ph/0512057].
[10] T. Altinoluk, A. Kovner, M. Lublinsky and J. Peressutti, JHEP 0903 (2009) 109 [arXiv:0901.2559 [hep-ph]]; A. Kovner and M. Lublinsky, Phys. Rev. Lett. 94 (2005) 181603 [arXiv:hep-ph/0502119]; Phys. Rev. D 71 (2005) 085004 [arXiv:hep-ph/0501198].
[11] E. Levin and M. Lublinsky, Nucl. Phys. A 763 (2005) 172 [arXiv:hep-ph/0501173].
[12] J. P. Blaizot, E. Iancu, K. Itakura and D. N. Triantafyllopoulos, Phys. Lett. B 615 (2005) 221 [arXiv:hep-ph/0502221].
[13] Y. Hatta, E. Iancu, L. McLerran, A. Stasto and D. N. Triantafyllopoulos, Nucl. Phys. A 764 (2006) 423 [arXiv:hep-ph/0504182].
[14] C. Marquet, A. H. Mueller, A. I. Shoshi and S. M. H. Wong, Nucl. Phys. A 762 (2005) 252 [arXiv:hep-ph/0505229].
[15] E. Levin, J. Miller and A. Prygarin, Nucl. Phys. A 806 (2008) 245 [arXiv:0706.2944 [hep-ph]].
[16] E. A. Kuraev, L. N. Lipatov, and F. S. Fadin, Sov. Phys. JETP 45, 199 (1977); Ya. Ya. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. 28, 22 (1978).
[17] L. N. Lipatov, Phys. Rep. 286 (1997) 131; Sov. Phys. JETP 63 (1986) 904 and references therein. ep-ph]]
[18] S. Bondarenko and L. Motyka, Phys. Rev. D 75 (2007) 114015 [arXiv:hep-ph/0605185].
[19] S. Bondarenko and M. A. Braun, Nucl. Phys. A 799 (2008) 151 [arXiv:0708.3629 [hep-ph]].
[20] A. Kormilitzin, E. Levin, J. S. Miller, Nucl. Phys. A859 (2011) 87-113. [arXiv:1009.1329 [hep-ph]].
[21] S. Bondarenko, M. Kozlov, E. Levin, Nucl. Phys. A727 (2003) 139-178, [hep-ph/0305150] and references therein.

[22] I. N. Sneddon, Elements of partial differential equations, Mc-Graw-Hill, New York, 1957.

[23] A. H. Mueller and A. I. Shoshi, Nucl. Phys. B 692 (2004) 175 [arXiv:hep-ph/0402193].

[24] I. Gradstein and I. Ryzhik, "Tables of Series, Products, and Integrals", Verlag MIR, Moskau, 1981.

[25] J. Bartels, E. Levin, Nucl. Phys. B387 (1992) 617-637; A. M. Stasto, K. J. Golec-Biernat, J. Kwiecinski, Phys. Rev. Lett. 86 (2001) 596-599, [hep-ph/0007192]; L. McLerran, M. Praszalowicz, Acta Phys. Polon. B42 (2011) 99, [arXiv:1011.3403 [hep-ph]]; B41 (2010) 1917-1926, [arXiv:1006.4293 [hep-ph]].

M. Praszalowicz, [arXiv:1104.1777 [hep-ph]], [arXiv:1101.0585 [hep-ph]].

[26] E. Levin and K. Tuchin, Nucl. Phys. B 573 (2000) 833 [arXiv:hep-ph/9908317].

[27] Hal Smith, “An Introduction to Delay Differential Equations with Applications to the Life Sciences,” (Texts in Applied Mathematics), Springer, 2011.
