Moderate deviations for non-linear functionals
and empirical spectral density of moving average
processes

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Abstract

A moderate deviation principle for functionals, with at most quadratic growth,
of moving average processes is established. The main assumptions on the moving
average process are a Logarithmic Sobolev inequality for the driving random variables
and the continuity, or weaker, of the spectral density of the moving average process.
We also obtain the moderate deviations for the empirical spectral density, exhibiting
an interesting new form of the rate function, i.e. with a correction term compared to
the Gaussian rate functionnal.

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Key Words: moderate deviations; moving average processes; logarithmic Sobolev in-
equalities, toeplitz matrices.

1 Introduction

Consider the moving average process

\[ X_n := \sum_{j=-\infty}^{+\infty} a_{j-n} \xi_j = \sum_{j=-\infty}^{+\infty} a_j \xi_{n+j}, \forall n \in \mathbb{Z}. \] (1.1)

where the innovations \((\xi_n)_{n \in \mathbb{Z}}\) is a sequence of \(\mathbb{R}^d\)-valued centered square integrable
i.i.d.r.v., with common law \(\mathcal{L}(\xi_0) = \mu\), and \((a_n)_{n \in \mathbb{Z}}\) be a sequence of real numbers such
that

\[ \sum_{n \in \mathbb{Z}} |a_n|^2 < +\infty. \] (1.2)

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This last condition (1.2) is necessary and sufficient for the a.s. convergence or convergence in law of the serie (1.1). The sequence \((X_k)\) is strictly stationary having spectral density 
\[
f(\theta) := \text{Var}(\xi_0 | g(\theta))^2
\]
where 
\[
g(\theta) := \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}.
\]
(1.3)

The moving average processes are of special importance in time series analysis and they arise in a wide variety of contexts. Applications to economics, engineering and physical sciences are very broad and a vast amount of literature is devoted to the study of the limit theorems for moving average processes under various conditions (e.g. Brockwell and Davis [4] and references therein). For example, the minimal condition for the central limit theorem for \((X_n)\) is (see [16, Corollary 5.2, p.135]) that \(g\) is continuous at \(\theta = 0\). The large deviations theorems have attracted much attention and many work, see Burton and Dehling [7], Jiang, Rao and Wang [17],[18], Djellout and Guillin [11] and recently by Wu [21] on the linear case, under different assumptions on the law \(\xi_0\), and the spectral density function of \(X\), see Wu [21], for relevant reference and more details.

The main purpose of this paper consists to investigate the Moderate Deviation Principle (in short MDP) for the so-called empirical periodogram of order \(n\) of the process \((X_k)\) defined by 
\[
I_n(\theta) := \frac{1}{n} \left| \sum_{k=1}^{n} X_k e^{i k \theta} \right|^2
\]
(1.4)
which are random elements in the space \(L^p(\mathbb{T}, d\theta)\) of \(p\)-integrable function on the torus \(\mathbb{T}\) identified with \([-\pi, \pi]\) equipped with the weak convergence topology. We present a simple proof under some conditions such as the \(L^q(\mathbb{T}, d\theta)\)-boundedness of the spectral density of \((X_k)\) and a Logarithmic Sobolev Inequality (in short LSI) for \(\mu\).

We also establish the MDP for additive non-linear functionals of the moving average processes:
\[
\frac{1}{n} \sum_{k=1}^{n} F(X_k, \ldots, X_{k+l})
\]
(1.5)
where \(F\) takes its value in \(\mathbb{R}^m\), under some regularity for the derivatives of \(F\). This regularity enables us in particular to obtain the MDP for 
\[
F(X_k, \ldots, X_{k+l}) = (X_k X_k^*, X_k X_{k+1}^*, \ldots, X_k X_{k+l}^*)
\]
which is of particular interest in statistics.

To our knowledge, it is the first time a MDP for functionals of moving average is established, for a general class of measurable functions \(F\) (and not only in the Gaussian case).
Bryc and Dembo [6] have considered quadratic functional of Gaussian processes both at the level of large and moderate deviations. We extend their results for the MDP as our r.v. are not necessarily Gaussian (under the same hypothesis on the density), and we consider the autocorrelation vector (in a non i.i.d. setting). Moreover, and compare with Bercu and al [2], we also establish the MDP for the empirical spectral density, not only for marginals of the empirical spectral measures. We exhibit an interesting new form of the rate function, i.e. with a correction term compared to the Gaussian rate functional.

Recall that any real stationary Gaussian process $(X_n)$ with a square integrable spectral density function $f$ can be represented as (1.1), so that one may see our results as the moderate deviations alternative to the seminal work of Donsker and Varadhan [13] on large deviations of Gaussian processes.

This paper is structured as follows. The MDP for the empirical spectral density is stated in next section. The MDP for non-linear functionals is given in section 3. We establish the key a priori estimation in section 4. The last section is devoted to the proofs of the main results.

2 MDP for the empirical spectral density

In this section we only consider, without loss of generality, and to simplify notations, the real case. Let $(\xi_n)_{n \in \mathbb{Z}}$ is a sequence of $\mathbb{R}$-valued centered i.i.d.r.v., with common law $\mathcal{L}(\xi_0) = \mu$, and let $a := (a_n)_{n \in \mathbb{Z}}$ be a sequence of real, and define $(X_n)$ by (1.1). We will always assume that $\mu$ satisfies a LSI, i.e. there exists $C > 0$ such that

$$\text{Ent}_\mu(h^2) \leq 2C\mathbb{E}_\mu(|\nabla h|^2)$$

for every smooth $h$ such that $\mathbb{E}_\mu(h^2 \log^+ h^2) < \infty$, where

$$\text{Ent}_\mu(h^2) = \mathbb{E}_\mu(h^2 \log h^2) - \mathbb{E}_\mu(h^2) \log \mathbb{E}_\mu(h^2).$$

See Ledoux [19] for further details on LSI. Note that it implies in particular that there exists some positive $\delta$ such that

$$\mathbb{E}_\mu(e^{\delta|x|^2}) < \infty. \tag{2.2}$$

Let $(b_n)$ a sequence of real number such that

$$1 \ll b_n \ll \sqrt{n}. \tag{2.3}$$

For any measure $\lambda$ on the torus $\mathbb{T}$ (identified with $[-\pi, \pi]$, in the usual way), let

$$L^p(\mathbb{T}, d\lambda) := \left\{ h \text{ measurable} : ||h||_p = \left( \int_{\mathbb{T}} |h(\lambda)|^p d\lambda \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty,$$

and

$$L^\infty(\mathbb{T}, d\lambda) := \left\{ h \text{ measurable} : ||h||_\infty = \text{esssup}_{\lambda \in \mathbb{T}} |h(\lambda)| < \infty \right\}.$$
We are interested in the MDP of the empirical spectral density of \( (X_n) \) defined by

\[
I_n(\theta) := \frac{1}{n} \left| \sum_{k=1}^{n} X_k e^{ik\theta} \right|^2
\]

which are random elements in the space \( L^p(T, d\theta) \) equipped with the weak convergence topology.

We first present here the MDP for the empirical autocorrelation vector which will be our main tool for the MDP of the empirical spectral density, and has its own interest for statistics. Let \( \kappa_4 = \frac{\mathbb{E}(\xi^4) - 3\mathbb{E}(\xi^2)^2}{\mathbb{E}(\xi^2)^2} \).

**Theorem 2.1.** Suppose that \( \mu \) satisfies the LSI (2.1), that \( (a_n)_{n \in \mathbb{Z}} \) satisfies (1.2). Suppose moreover that the spectral density function \( f \) is in \( L^q(T, d\theta) \), where \( 2 < q \leq +\infty \) and \( b_n \sqrt{n^{1/q}} \to 0 \), then

\[
\left( \frac{1}{b_n \sqrt{n}} \sum_{k=1}^{n} (X_k X_{k+\ell} - \mathbb{E}X_k X_{k+\ell}) \right)_{0 \leq \ell \leq m}
\]

satisfies the MDP on \( \mathbb{R}^{m+1} \) with speed \( b_n^2 \) and with the rate function given by

\[
I(z) = \sup_{\lambda \in \mathbb{R}^{m+1}} \left\{ \langle \lambda, z \rangle - \frac{1}{2} \lambda^* \Sigma^2 \lambda \right\};
\]

where \( \Sigma^2 = (\Sigma^2_{k,\ell})_{0 \leq k,\ell \leq m} \) and

\[
\Sigma^2_{k,\ell} = \frac{1}{2\pi} \int_T \left( e^{i(k-\ell)\theta} + e^{i(k+\ell)\theta} \right) f^2(\theta) d\theta + \kappa_4 \left( \frac{1}{2\pi} \int_T f(\theta) e^{ik\theta} d\theta \right) \left( \frac{1}{2\pi} \int_T f(\theta) e^{i\ell\theta} d\theta \right).
\]

**Remark 2.1.** The additional assumption on the normalizer \( b_n \) is exactly the one supposed in Bryc-Dembo [6, Th. 2.3], but they only consider the case \( l = m = 0 \) in the Gaussian setting. Their large deviations result (namely Prop. 2.5 in [6]) for the empirical autocorrelation is moreover restricted to the i.i.d. case.

**Remark 2.2.** First note that there exists some practical criteria ensuring the fact that a measure \( \mu \) satisfies some LSI. For example, consider a \( C^2 \) function \( W \) on \( \mathbb{R}^d \) such that \( e^{-W} \) is integrable with respect to Lebesgue measure and let

\[
d\mu(x) = Z^{-1} e^{-W(x)} dx \tag{2.4}
\]

and suppose that for some \( c \) in \( \mathbb{R} \), \( W''(x) \geq cd \) for every \( x \) and that for some \( \epsilon > 0 \),

\[
\int \int e^{(c^- + \epsilon)|x-y|^2} d\mu(x) d\mu(y) < \infty \tag{2.5}
\]

where \( c^- = -\min(c, 0) \). Then \( \mu \) satisfies [21] by the criterion of Wang [19]. Obviously Gaussian variables fulfill this criterion. See Bobkov-Götze [3] for a necessary and sufficient condition in the real case, relying on Hardy’s inequalities.

The following corollary follows from Theorem 2.1
Corollary 2.2. Under the assumptions of Theorem 2.1, we have for all \( \ell \geq 0 \),
\[
\left( \frac{1}{\sqrt{n}b_n} \sum_{k=1}^{n} (X_k X_{k+l} - EX_k X_{k+l}) \right)
\]
satisfies the MDP on \( \mathbb{R} \) with speed \( b_n^2 \) and rate function given by
\[
I^\ell(z) = \frac{2}{2\pi} \int_{\mathbb{T}} (1 + \cos(2\ell \theta)) f^2(\theta) d\theta + \kappa_4 \left( \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) \cos(\ell \theta) d\theta \right)^2.
\]

Remark 2.3. Now assume that \( (\xi_n) \) is a sequence of real i.i.d. normal random variables, so \( (X_n) \) is a stationary Gaussian process and inversely any real Gaussian stationary process \( (X_n) \) with a square integrable spectral density function \( f \) can be represented as \( (1.1) \).

In this case, we have \( \mathbb{E}(\xi^4) = 3\mathbb{E}(\xi^2)^2 \) and thus \( \kappa_4 = 0 \), so we obtain
\[
I^\ell(z) = \frac{2}{2\pi} \int_{\mathbb{T}} (1 + \cos(2\ell \theta)) f^2(\theta) d\theta.
\]

Let us present now the main result of this paper. From Theorem 2.1 (and its proof) together with the projective limit method, we yield the functional type’s MDP below, for
\[
\mathcal{L}_n(\theta) = \frac{\sqrt{n}}{b_n} (I_n(\theta) - \mathbb{E} I_n(\theta)).
\]

Theorem 2.3. Suppose that \( \mu \) satisfies the LSI \( (2.1) \), that \( (a_n)_{n \in \mathbb{Z}} \) satisfies \( (1.2) \). Suppose moreover that the spectral density function \( f \in L^3(\mathbb{T}, d\theta) \), where \( 2 < q \leq +\infty \) and \( \frac{b_n}{\sqrt{n}} n^{1/q + 1/p'} \to 0 \). Let \( \frac{1}{p'} + \frac{1}{p'} = 1 \) and \( \frac{1}{p'} + \frac{1}{q} < \frac{1}{2} \), then \( (\mathcal{L}_n)_{n \geq 0} \) satisfies the MDP on \((L^p(\mathbb{T}, \mathbb{R}), \sigma(L^p(\mathbb{T}, d\theta), L^p(\mathbb{T}, d\theta))) \) with speed \( b_n^2 \) with the rate function given for all even \( \eta \in L^p(\mathbb{T}, d\theta) \) by
\[
I(\eta) = \begin{cases} 
\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\eta^2(\theta)}{4f^2(\theta)} d\theta - \frac{\kappa_4}{2 + \kappa_4} \left( \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\eta(\theta)}{2f(\theta)} d\theta \right)^2 \\
\quad \text{if } \eta(\theta) d\theta \text{ is absolutely continuous w.r.t. } f(\theta) d\theta \text{ and } \frac{\eta(\theta)}{f(\theta)} \in L^2(\mathbb{T}, d\theta); \\
+\infty, \quad \text{otherwise.}
\end{cases}
\]

Remark 2.4. Now assume \( (X_n) \) is a stationary Gaussian process, so we obtain that
\( (\mathcal{L}_n)_{n \geq 0} \) satisfies the MDP on \( L^p(\mathbb{T}, d\theta) \) with speed \( b_n^2 \) with the rate function given by
\[
I(\eta) = \begin{cases} 
\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\eta^2(\theta)}{4f^2(\theta)} d\theta \\
\quad \text{if } \eta(\theta) d\theta \text{ is absolutely continuous w.r.t. } f(\theta) d\theta \text{ and } \frac{\eta(\theta)}{f(\theta)} \in L^2(\mathbb{T}, d\theta); \\
+\infty, \quad \text{otherwise.}
\end{cases}
\]

We thus give the MDP for the spectral empirical measure in the setting of Bercu and
al [2], note however that they only consider the marginal LDP, i.e. LDP for \( \mathcal{I}_n(h) \) for
somebounded \( h \) on the torus with an extra assumption on the eigenvalues of the Toeplitz
matrix, where \( \mathcal{I}_n(h) = \frac{1}{2\pi} \int_{\mathbb{T}} \mathcal{I}_n(\theta) h(\theta) d\theta \).
Remark 2.5. Notice that the extra term with respect to the Gaussian case in the evaluation of the rate function was also found by L. Giraitis and D. Surgailis \cite{15} in their investigations of the CLT for $I_n(h)$. The result of \cite{15} can be summarized as below: if
\[
\lim_{n \to \infty} \frac{1}{n} \text{tr} ((T_n(f)T_n(h))^2) = \frac{1}{2\pi} \int_T f^2(\theta)h^2(\theta)d\theta;
\]
where $T_n(h)$ is the Toeplitz matrix of $h$ then $\sqrt{n}(I_n(h) - \mathbb{E}I_n(h))$ converges in law (as $n \to \infty$) to the normal distribution $\mathcal{N}(0,\sigma^2)$ with $\sigma^2 := \frac{2}{2\pi} \int_T (f(\theta)h(\theta))^2 d\theta + \kappa_4\left(\frac{1}{2\pi} \int_T f(\theta)h(\theta)d\theta\right)^2$. In Gaussian case this result was already proved by Avram \cite{1} and Fox and Taqqu \cite{13}.

Remark 2.6. Our main tool in the proof of our Theorem 2.3 is \eqref{2.6}, which is valid under our conditions on $f$ and $h$. It seems that the single condition that the integral on the right hand side of \eqref{2.6} is finite (i.e. $h \in L^2(T,f^2d\theta)$) is not sufficient to obtain \eqref{2.6}. This explains why we cannot obtain the MDP of the empirical spectral density in $L^2(T,f^2d\theta)$.

Remark 2.7. One can not hope that the MDP in Theorem 2.3 holds w.r.t. the strong topology of $L^p(T,d\theta)$, because the rate function $I(\eta)$ is not inf-compact w.r.t. this topology.

As a consequence of Theorem 2.3 we have the following

Corollary 2.4. Under the assumptions of Theorem 2.3, we have that for all $h \in L^p(T,d\theta)$
\[
\limsup_{n \to \infty} \frac{1}{b_n} \mathbb{E} \log \left( e^{b_n \int_T h(\theta)L_n(\theta)d\theta} \right) = \frac{1}{2} \left( \frac{2}{2\pi} \int_T h^2(\theta)f^2(\theta)d\theta + \kappa_4\left(\frac{1}{2\pi} \int_T h(\theta)f(\theta)d\theta\right)^2 \right).
\]
In the next corollary of Theorem 2.3, we replace $\mathbb{E}I_n(\theta)$ by $f(\theta)$, more useful in practice.

Corollary 2.5. Under the assumptions of Theorem 2.3, assume moreover that $f' \in L^2(T,d\theta)$. The same conclusion holds for $\hat{L}_n$ instead of $L_n$ where
\[
\hat{L}_n(\theta) = \sqrt{\frac{n}{b_n}} (I_n(\theta) - f(\theta)).
\]

Remark 2.8. By looking carefully at the proof of this corollary, one may see that the needed convergence of $\mathbb{E}I_n(\theta)$ to $\int f h$ is ensured by our assumption on $f'$ which is surely too strong (as the negligibility of this term is in $\frac{1}{\sqrt{nb_n}}$) but remains practical, solely relying on the spectral density. Other possibilities impose implicit, and thus difficult to check, conditions linking $h$ and $f$.

3 MDP for non-linear functionals

Let us present now the following slightly more general model: $(\xi_n)_{n \in \mathbb{Z}}$ is a sequence of $\mathbb{R}^d$-valued centered i.i.d.r.v., with common law $\mathcal{L}(\xi_0) = \mu$, and let $a := (a_n)_{n \in \mathbb{Z}}$ be a sequence of real $p \times d$-matrix. We now present the MDP for a functional $F : (\mathbb{R}^p)^{l+1} \to \mathbb{R}^m$, i.e. the MDP of
\[
S_n(F) = \frac{1}{\sqrt{nb_n}} \sum_{k=1}^n (F(X_k,\ldots,X_{k+l}) - \mathbb{E}(F(X_k,\ldots,X_{k+l}))).
\]
and we use the notation $F(x_0, \ldots, x_l)$, so that $\partial_{x_i} F$ should be understood as usual. Let $f(\theta) = g(\theta) \Gamma(\xi_0) g^*(\theta)$, $\Gamma(\xi_0) := (\text{cov}(\xi_i^0, \xi_j^0))_{i,j=1}^{d}$.

**Theorem 3.1.** Suppose that $\mu$ satisfies the LSI (2.1), that $(a_n)_{n \in \mathbb{Z}}$ satisfies (1.2) and $g$ is continuous on $\mathbb{T}$. Suppose moreover that $\partial_{x_i} F$ is Lipschitz for $i = 0, \ldots, l$, then $S_n(F)$ satisfies the MDP with speed $b_n^2$ and good rate function $I_F$ given by

$$I_F(z) = \sup_{\lambda \in \mathbb{R}^m} \left\{ \langle \lambda, z \rangle - \frac{1}{2} \lambda^* \Sigma_F^{-2} \lambda \right\} = \frac{1}{2} \Sigma_F^{-2} z.$$  

where $\Sigma_F^{-2}$ is the generalized inverse of the covariance matrix $\Sigma_F^2$ given by

$$\Sigma_F^2 := \lim_{n \to +\infty} \frac{1}{n} \Gamma \left( \sum_{k=1}^{n} F(X_k, \ldots, X_{k+l}) \right)$$  

which exists.

**Remark 3.1.** Note also that under our assumption on $F$ it enables us to obtain the MDP for

$$F(X_k, \ldots, X_{k+l}) = (X_k X_k^*, X_k X_{k+1}^*, \ldots, X_k X_{k+l}^*)$$

as the derivatives in each coordinate is Lipschitz, without further assumption on the normalizer $b_n$ but with a bounded spectral density.

Note also the following corollary in the linear case $F(x_0, \ldots, x_l) = x_0$ which weakens the assumptions on $g$.

**Corollary 3.2.** Suppose that $\mu$ satisfies the integrability condition (2.2), that $(a_n)_{n \in \mathbb{Z}}$ satisfies (1.2) and $g$ is continuous on a neighborhood of 0, then $S_n(F)$ satisfies a MDP with speed $b_n^2$ and rate $I(z) = \sup_{\lambda \in \mathbb{R}^m} \{ \langle z, \lambda \rangle - \frac{1}{2} \lambda^* f(0) \lambda \}$.

It generalizes Th. 3.1 of Djellout and Guillin [11] to the case of unbounded r.v. Under assumption (2.2), the crucial inequality (5.3), as a consequence of the LSI, may not be used. However, we may encompass this difficulty by noting that integrability (2.2) is, by Djellout and al. [12, Th. 2.3], equivalent to a Transport inequality in $L_1$-Wasserstein distance which is itself equivalent to the inequality (5.3) with the Lipschitz norm instead of the gradient in the right hand side, but for this particular linear case, the gradient and Lipschitz norm are equal so that the same proof works. The release of the assumptions of the continuity of $g$ comes from the fact that in this case, Lemma 4.3 is not used.

### 4 A priori estimation

We recall the following well known elementary result

**Lemma 4.1.** Suppose $Y = [Y_1, \ldots, Y_n]^*$ is a real valued centered Gaussian vector with covariance matrix $R$ and let $A$ be a symmetric real valued $n \times n$-matrix. Then with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix $AR$

$$\log \mathbb{E} \exp(z(Y, AY)) = \begin{cases} 
- \frac{1}{2} \sum_{j=1}^{n} \log(1 - 2z\lambda_j) & \text{if} \quad z \max_{1 \leq j \leq n} \lambda_j < 1/2 \\
+ \infty & \text{otherwise}
\end{cases}$$  

(4.1)
We give a crucial lemma which was first stated in Wu [21], and reproduced here for completeness.

**Lemma 4.2.** If the centered r.v. $\xi_0$ satisfies (2.2), then there is some constant $K > 0$ such that

$$L(y) := \mathbb{E}\exp(\langle \xi_0, y \rangle) \leq \exp\left(\frac{K^2}{2} |y|^2 \right), \forall y \in \mathbb{R}^d.$$  

**Proof:** By Chebychev’s inequality,

$$\mathbb{P}(|\xi_0| > t) \leq \exp(-t^2 \delta) \mathbb{E}\exp(\delta |\xi_0|^2) := C(\delta) \exp(-t^2 \delta), \forall t > 0,$$

consequently

$$L(y) \leq \mathbb{E}\exp(|\xi_0||y|) = 1 + \int_0^\infty |y|e^{t|y|} \mathbb{P}(|\xi_0| > t)dt$$

$$\leq 1 + C(\delta)|y| \int_0^\infty \exp(t|y| - t^2 \delta)dt$$

$$\leq 1 + C(\delta)|y| \int_{-\infty}^\infty \exp(t|y| - t^2 \delta)dt$$

$$= 1 + C(\delta)\sqrt{\frac{\pi}{\delta}}|y|\exp\left(\frac{|y|^2}{4\delta}\right).$$

Thus there is $C_1 > 0$ such that (2.1) holds for all $|y| > 1$.

For $|y| \leq 1$, notice that $\log L(y) \in C^\infty(\mathbb{R}^d)$, and $\log L(0) = 0$, $\nabla \log L(y)|_{y=0} = \mathbb{E}\xi_0 = 0$.

By Taylor’s formula of order 2, we have for all $y$ with $|y| \leq 1$,

$$\log L(y) \leq \frac{1}{2} C_2^2 |y|^2,$$

where $C_2 := \sup_{|y| \leq 1} \left( \sum_{k,l=1}^{d} [\partial_{y_k}\partial_{y_l} \log L(y)]^2 \right)^{1/4}$. Thus (2.1) follows with $K := C_1 \vee C_2$.  

We extend (4.1) from Gaussian distribution to general law $\mu$ satisfying (2.2), which is a slight generalization of the preceding lemma.

**Lemma 4.3.** Let $X = [X_1, \cdots, X_n]' \in (\mathbb{R}^p)^n$ with covariance matrix $A = (A_{k,l})_{1 \leq k,l \leq n}$ where $A_{k,l}$ is a $p \times p$ matrix given by

$$A_{k,l} := \mathbb{E}(X_k X_l^*) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(k-l)|\theta|} g(\theta) g(\theta)^* d\theta.$$  

Let $B$ be a symmetric real valued $pn \times pn$-matrix. Assume (2.2). Let $K > 0$ given in lemma 4.2. Then with $\mu_1^{pn}, \cdots, \mu_{pn}^{pn}$ the eigenvalues of the matrix $\sqrt{B}A\sqrt{B}$

$$\log \mathbb{E}\exp(\lambda X, BX) \leq \begin{cases} 
- \frac{1}{2} \sum_{j=1}^{pn} \log(1 - 2K^2 \lambda \mu_j^{pn}) & \text{if } \lambda \max_{1 \leq j \leq pn} \mu_j^{pn} < \frac{1}{2K^2} \\
+ \infty, & \text{otherwise.}
\end{cases}$$
Proof: The main difficulty resides in the nonlinear property of $< x, Bx >$. The trick consists to reduce it to an estimation of linear type in the following way:

$$
\mathbb{E}\left\{\exp\left[\frac{1}{2} t^2 < X, BX >\right]\right\} = \mathbb{E}\left\{\exp\left[\frac{1}{2} t^2 \| \sqrt{B}X \| ^2\right]\right\} = \int_{(\mathbb{R}^p)^n} \mathbb{E}\left\{\exp\left[t\langle \sqrt{B}X, Y \rangle\right]\right\} \gamma(dY)
$$

where $\gamma$ is the standard Gaussian law $N(0, I)$ on $(\mathbb{R}^p)^n$.

Since

$$
\langle \sqrt{B}X, Y \rangle = \langle X, \sqrt{B}Y \rangle = \sum_{k=1}^{n} \langle X_k, (\sqrt{B}Y)_k \rangle = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{n} \langle \xi_j, a_{j-k}^*(\sqrt{B}Y)_k \rangle.
$$

where $(a_{j,k})$ is the hermitian transposition of the matrix $(a_{j,k})$, we get by Lemma 4.2. and the i.i.d. property of $(\xi_j)$,

$$
\mathbb{E}\left\{\exp\left[t\langle \sqrt{B}X, Y \rangle\right]\right\} \leq \exp\left[K^2 t^2 \sum_{j \in \mathbb{Z}} \sum_{k=1}^{n} a_{j-k}^*(\sqrt{B}Y)_k\right]^2.
$$

Now observe that

$$
\sum_{j \in \mathbb{Z}} \sum_{k=1}^{n} \left| a_{j-k}^*(\sqrt{B}Y)_k \right|^2 = \sum_{k,l=1}^{n} \sum_{j \in \mathbb{Z}} \langle a_{j-k}^*(\sqrt{B}Y)_k, a_{j-l}^*(\sqrt{B}Y)_l \rangle
$$

$$
= \sum_{k,l=1}^{n} \langle (\sqrt{B}Y)_k, \sum_{j \in \mathbb{Z}} a_{j-k} a_{j-l}^*(\sqrt{B}Y)_l \rangle
$$

$$
= \sum_{k,l=1}^{n} \langle (\sqrt{B}Y)_k, A_{k,l}(\sqrt{B}Y)_l \rangle
$$

Then letting $\mu_1^{pm}, \cdots, \mu_n^{pm}$ be the eigenvalues of the matrix $\sqrt{B}A\sqrt{B}$ (which are also the eigenvalues of $AB$), we get for all $\lambda$ such that $t^2 K^2 \max_{1 \leq j \leq n} \mu_j^{pm} < 1$

$$
\mathbb{E}\left\{\exp\left[\frac{1}{2} t^2 < X, BX >\right]\right\} \leq \int_{(\mathbb{R}^p)^n} \left\{\exp\left[\frac{1}{2} K^2 t^2 \langle Y, \sqrt{BA}\sqrt{B}Y \rangle\right]\right\} \gamma(dY)
$$

$$
= -\frac{1}{2} \sum_{j=1}^{nm} \log(1 - K^2 t^2 \mu_j^{pm})
$$

and it follows with $\lambda = t^2 / 2$.

$\diamond$

Remark 4.1. If we assume $\|g\|_{\infty} = \|g(\theta)\|_{L_\infty(\mathbb{R}, d\theta)}$, and $B = I$ we obtain exactly the result in Wu [21]. In fact in this case, we have for any $\lambda > 0$ such that $2\lambda K^2 \|g\|_{\infty}^2 < 1$,

$$
\log \mathbb{E}e^{\lambda < X, X >} \leq -\frac{1}{2} \log \left(1 - 2\lambda K^2 \|g\|_{\infty}^2\right)^{np}.
$$

Remark 4.2. Instead of lemma 4.2., we can use the consequence of the LSI (5.3 ) to prove Lemma 4.3., but (5.3 ) is more stronger than (2.2) (see below).
5 Proofs

Introduce first the following coefficients for each \( N \in \mathbb{N}^* \) : 
\[
a_j^N = a_j \left( 1 - \frac{|j|}{N} \right) \quad \text{if} \ |j| \leq N \quad \text{and} \quad 0 \quad \text{otherwise},
\]
and define the Fejer approximation of \( X_k \) and \( g \)
\[
X_k^N = \sum_{j \in \mathbb{Z}} a_j^N \xi_{k+j}, \quad g^N(\theta) = \sum_{j \in \mathbb{Z}} a_j^N e^{ij\theta} \quad \forall \theta \in \mathbb{R},
\]
that will enable us to first consider the finite case and then extend it to the infinite case by approximation. Remark that if \( f \in L^2(\mathbb{T}, d\theta), \ q > 2, \) then \( \int_{\mathbb{T}} (g - g^N)^4 d\theta \to 0 \) as \( N \to \infty. \)

For any real and symmetric function \( h \in L^1(\mathbb{T}, d\theta) \), let \( T_n(h) \) be the Toeplitz matrix associated with \( h \) i.e. \( T_n(h) = (\hat{r}_{k-l}(h))_{1 \leq k,l \leq n} \) where \( \hat{r}_k(h) \) is the \( k \)th Fourier coefficient of \( h \)
\[
\hat{r}_k(h) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{ik\theta} h(\theta) d\theta, \quad \forall k \in \mathbb{Z}.
\]
The matrix \( T_n(h) \) is obviously real and symmetric, is positive definite whenever \( h \geq 0. \)

For an \( n \times n \) matrix \( A \), we consider the usual operator norm 
\[
||A|| = \sup_{x \in \mathbb{R}^n} \frac{|Ax|}{|x|}.
\]

We shall need the two following lemmas. The first gives an estimate for the maximal eigenvalue of the covariance matrices \( T_n(f) \) which is Lemma 4.7 of Bryc-Dembo [6]. The second one concerning the asymptotic behavior of the trace of the products of Toeplitz matrices see ([15]).

**Lemma 5.1.** If \( 1 \leq q \leq \infty \) then for all \( n > 1 \) we have 
\[
||T_n(f)|| \leq n^{1/q} ||f||_q.
\]

**Lemma 5.2.** Let \( f_k \in L^1(\mathbb{T}, d\theta) \cap L^{q_k}(\mathbb{T}, d\theta) \) with \( 0 \leq q_k \leq \infty \) for \( k = 1, \cdots s \) and 
\[
\sum_{k=1}^{s} \frac{1}{q_k} \leq 1.
\]
The following assertion hold
\[
\lim_{n \to \infty} \frac{1}{n} \text{tr} \left( \prod_{k=1}^{s} T_n(f_k) \right) = \hat{r}_0 \left( \prod_{k=1}^{s} f_k \right).
\]

5.1 Proof of Theorem 2.1

We shall prove it only in the real valued case. The proof is divided into three steps. In the first one, we prove that the MDP holds for some suitable approximation of our process, then we will show this approximation is a good one in the sense of the moderate deviations and we will finally establish the convergence of the rate function and the subsequent existence of the limiting variance.

**Step 1.** Let 
\[
Q_n^N = \frac{1}{\sqrt{n} b_n} \sum_{k=1}^{n} \left( X_k^N X_{k+l}^N - \mathbb{E} X_k^N X_{k+l}^N \right).
\]
The crucial remark is that the sequence \( X_k^N X_{k+l}^N \) is a \( 2N \)-dependent identically distributed sequence. Using [22], we get for all \( N \) and for some positive \( \eta \) that 
\[
\mathbb{E} \left( e^{\eta |X_k^N X_{k+l}^N|} \right) < \infty.
\]
We may then apply results of Chen [8] on Banach valued MDP of $m-$dependent sequence, enabling us to get that for each $N$ fixed, for all $\lambda$

$$\lim_{n \to \infty} \frac{1}{b^2_n} \log \mathbb{E} \left( e^{\lambda b^2_n Q^N_n} \right) = \frac{\lambda^2}{2} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left( \sum_{k=1}^{n} X^N_k X^N_{k+l} - \mathbb{E}X^N_k \mathbb{E}X^N_{k+l} \right)^2$$

$$:= \frac{\lambda^2}{2} \Sigma^2_N \in \mathbb{R},$$

$$= \frac{\lambda^2}{2} \sum_{k=-N}^{N} \text{Cov} \left( X^N_0 X^N_k, X^N_k X^N_{k+l} \right)$$

and that $Q^N_n$ satisfies the MDP with the good rate function $I^N(x) = \sup_{\lambda \in \mathbb{R}} \left\{ \lambda x - \frac{\lambda^2}{2} \sum_{k=-N}^{N} \right\}$.

**Step 2.** The purpose of this step will be to prove the asymptotic negligibility as $N \to \infty$ of $Q_n - Q^N_n$ with respect to the MDP, i.e. we will establish that for all $\lambda \in \mathbb{R}$

$$\lim_{N \to \infty} \limsup_{n \to \infty} \frac{1}{b^2_n} \log \mathbb{E} \left( e^{\lambda b^2_n (Q_n - Q^N_n)} \right) = 0. \quad (5.2)$$

Remark that, by Jensen inequality and as our functionals are centered, we only have to establish the upper inequality in (5.2).

Our main tool is the following consequence of the LSI (2.1), see Ledoux [19, Th. 2.7] applied to our context (after having extended (2.1) by tensorization to the infinite product measure of $\mu$): for exponentially integrable $G$,

$$\mathbb{E} \left( e^{\lambda b^2_n (G - \mathbb{E}G)} \right) \leq e^{\lambda^2 b^2_n C |\nabla G|^2},$$

with $C$ given in (2.1). Let apply it to

$$G((\xi_i)_{i \in \mathbb{Z}}) = \sum_{k=1}^{n} (X_k X_{k+l} - X^N_k X^N_{k+l}),$$

so that our main estimations are now transferred to the gradient of $G$. Clearly

$$\partial \xi_i G = \sum_{k=1}^{n} (a_{i-k} X_{k+l} + a_{i-k-1} X_k - a^N_{i-k} X^N_{k+l} - a^N_{i-k-1} X^N_k);$$

so

$$|\nabla G|^2 \leq 4 \sum_{i \in \mathbb{Z}} \left( \left( \sum_{k=1}^{n} (a_{i-k} - a^N_{i-k}) X_{k+l} \right)^2 + \left( \sum_{k=1}^{n} (a_{i-k} - a^N_{i-k}) X_k \right)^2 \right)$$

$$+ \left( \sum_{k=1}^{n} a^N_{i-k} (X_{k+l} - X^N_{k+l}) \right)^2 + \left( \sum_{k=1}^{n} a^N_{i-k} (X_k - X^N_k) \right)^2$$

$$= (I) + (II) + (III) + (IV).$$
By Hölder inequality,
\[
\log \mathbb{E} \left( e^{\frac{\lambda b_n}{n} (G - EG)} \right) \leq \log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 \|\nabla G\|^2} \right)
\leq \frac{1}{4} \log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (I)} \right) + \frac{1}{4} \log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (II)} \right) + \frac{1}{4} \log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (III)} \right) + \frac{1}{4} \log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (IV)} \right).
\] (5.4)

Let us deal with the first term of this inequality. We rewrite the expression of (I) as
\[
(I) = 4 \sum_{i \in \mathbb{Z}} \sum_{k, k' = 1}^n (a_{i-k} - a_{i-k}) (a_{i-k'} - a_{i-k'}) X_{k+t} X_{k'+t}
= 4 \sum_{k, k'} \hat{r}_{k' - k} ((g - g^N)^2) X_{k+t} X_{k'+t}
= 4 < X_{+t}, T_n ((g - g^N)^2) X_{+t} >
\]

Let \( \mu_1^{n, N}, \ldots, \mu_n^{n, N} \) be the eigenvalues of the matrix
\[
\sqrt{T_n ((g - g^N)^2) T_n (f) / T_n ((g - g^N)^2)}.
\]
Its operator norm is bounded from above by (using Lemma 5.1)
\[
||T_n (f)|| \cdot ||T_n ((g - g^N)^2)|| \leq n^{1/2} ||f||_q n^{1/2} \| (g - g^N)^2 \|_q.
\]

Since \( \frac{b_n}{n^{1/4}} \rightarrow 0 \) and \( f \in L^q (\mathbb{T}, d\theta) \), we choose \( n \) sufficiently large such that \( 32C^2 \lambda^2 \frac{b_n^2}{n} \max_{1 \leq j \leq n} \mu_j^{n, N} < 1 \). Applying Lemma 4.3, we get
\[
\log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (I)} \right) \leq - \frac{1}{2} \sum_{j=1}^n \log (1 - 32CR^2 \lambda^2 \frac{b_n^2}{n} \mu_j^{n, N}).
\] (5.5)

Similarly, we have
\[
\log \mathbb{E} \left( e^{4C\lambda^2 b_n^2 (II)} \right) = \log \mathbb{E} \left( e^{16C^2 \lambda^2 \frac{b_n^2}{n} X_{+t} ((g - g^N)^2)} X > \right)
\leq - \frac{1}{2} \sum_{j=1}^n \log (1 - 32CR^2 \lambda^2 \frac{b_n^2}{n} \mu_j^{n, N}).
\] (5.6)

Let us deal with the third term. We rewrite the expression of (III) as
\[
(III) = 4 \sum_{i \in \mathbb{Z}} \sum_{k, k' = 1}^n a_{i-k}^{N} a_{i-k'}^{N} (X_{k+t} - X_{k+t}^N) (X_{k'+t} - X_{k'+t}^N)
= 4 \sum_{k, k'} \hat{r}_{k' - k} ((g^N)^2) (X_{k+t} - X_{k+t}^N) (X_{k'+t} - X_{k'+t}^N)
= 4 < X_{+t} - X_{+t}^N, T_n ((g^N)^2) (X_{+t} - X_{+t}^N) > .
\]
Let \( \nu_1^{n,N}, \ldots, \nu_n^{n,N} \) the eigenvalues of the matrix
\[
\sqrt{T_n((g^N)^2)T_n((g - g^N)^2)\sqrt{T_n((g^N)^2)}}.
\]
Its operator norm is bounded from above by (using Lemma 5.1)
\[
||T_n((g^N)^2)|| \cdot ||T_n((g - g^N)^2)|| \leq n^{1/q}||T_n((g^N)^2)||qn^{1/q}||T_n((g^N)^2)||q.
\]
By our assumptions on \( b_n \) and \( f \), once again we take \( n \) sufficiently large such that
\[
32CK^2\lambda^2b_n^2\max_{1 \leq j \leq n} \nu_j^{n,N} < 1.
\]
Applying lemma 4.3., we get
\[
\log \mathbb{E}(e^{4C^2\lambda^2b_n^2 (III)}) \leq -\frac{1}{2} \sum_{j=1}^{n} \log(1 - 32CK^2\lambda^2b_n^2\nu_j^{n,N}).
\]  
(5.7)
Similarly
\[
\log \mathbb{E}(e^{4C^2\lambda^2b_n^2 (IV)}) \leq -\frac{1}{2} \sum_{j=1}^{n} \log(1 - 32CK^2\lambda^2b_n^2\nu_j^{n,N}).
\]  
(5.8)
By (5.4) and the previous estimations (5.5) (5.6) (5.7) (5.8), we have
\[
\frac{1}{b_n^2} \log \mathbb{E}\left(e^{\mu_n^2(Q_n - Q_n^N)}\right) \leq -\frac{1}{4} \frac{1}{n} \sum_{j=1}^{n} \left( \log(1 - 32CK^2\lambda^2b_n^2\nu_j^{n,N}) + \log(1 - 32CK^2\lambda^2b_n^2\nu_j^{n,N}) \right).
\]  
(5.9)
Notice that by the Taylor’s expansion of order 1, we have for \(|z| < 1\)
\[
\log(1 - z) = -z(1 - tz)^{-1}
\]
where \( t = t(z) \in [0,1] \). This applied here to \( z_j^{n,N} = 32CK^2\lambda^2b_n^2\nu_j^{n,N} \), \( \lambda_j^{n,N} = \nu_j^{n,N} \) or \( \lambda_j^{n,N} = \mu_j^{n,N} \) which satisfies \( \sup_{1 \leq j \leq n} |z_j^{n,N}| \to 0 \) as \( n \to \infty \), and hence \( |1 - t(z_j^{n,N})z_j^{n,N}| \to 1 \) uniformly in \( 1 \leq j \leq n \). Thus
\[
\lim_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E}\left(e^{\mu_n^2(Q_n - Q_n^N)}\right) \leq 16C^2\lambda^2 \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \mu_j^{n,N} + \nu_j^{n,N} \right).
\]
Thanks to the elementary formula \( \text{tr} (AC) = \text{tr} (CA) \) and using Lemma 5.2, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mu_j^{n,N} = \lim_{n \to \infty} \frac{1}{n} \text{tr} \left(T_n(f)T_n((g - g^N)^2)) \right) = \tilde{r}_0 \left((g - g^N)^2f \right).
\]
Similarly
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \nu_j^{n,N} = \lim_{n \to \infty} \frac{1}{n} \text{tr} \left(T_n((g^N)^2)T_n((g - g^N)^2)) \right) = \tilde{r}_0 \left((g^N)^2(g - g^N)^4 \right).
\]
So we have
\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E}\left(e^{\mu_n^2(Q_n - Q_n^N)}\right) \leq 32C^2\lambda^2\tilde{r}_0(f^2)\tilde{r}_0((g - g^N)^4).
\]
Letting $N$ to infinity, we get the desired negligibility (5.2). We then obtain that $Q_n$ satisfies the MDP of speed $b_n^2$ and good rate function $\tilde{I}$ by the approximation lemma [21, Th. 2.1], with $\tilde{I}$ given by

$$\tilde{I}(x) = \sup_{\delta > 0} \liminf_{N \to \infty} \inf_{B(x, \delta)} I^N = \sup_{\delta > 0} \limsup_{N \to \infty} \inf_{B(x, \delta)} I^N. \quad (5.10)$$

**Step 3.** We have now to prove the identification of the rate function. First, we show that

$$\Sigma^2 := \lim_{n \to \infty} \frac{1}{b_n^2} \mathbb{E} \left( \sum_{k=1}^{n} (X_k X_{k+l} - \mathbb{E} X_k X_{k+l}) \right)^2$$

exists and $\Sigma^2 = \lim_{N \to +\infty} \Sigma^2_N \in \mathbb{R}. \quad (5.11)$

By the previous estimations, we have that for all $|\lambda|$ small enough

$$\mathbb{E} \left( e^{|G - \mathbb{E}G|} \right) \leq 1 + 16 C^2 \lambda^2 n \hat{r}_0 (f^2)^2 ((g - g^N)^4) + o \left( \frac{\lambda^2}{2} \right)$$

Since, for all $|\lambda|$ small enough $\mathbb{E} \left( e^{|G - \mathbb{E}G|} \right) = 1 + \frac{\lambda^2}{2} \mathbb{E} (G - \mathbb{E}G)^2 + o \left( \frac{\lambda^2}{2} \right)$, we deduce that $\mathbb{E} (G - \mathbb{E}G)^2 \leq 16 C^2 n \hat{r}_0 (f^2)^2 \hat{r}_0 ((g - g^N)^4) + o \left( \frac{\lambda^2}{2} \right)$.

So we have

$$\sup_{n} \frac{1}{n} \mathbb{E} (G - \mathbb{E}G)^2 \longrightarrow 0 \quad \text{as} \quad N \to +\infty.$$ 

Whence the limit $\Sigma^2$ in (5.11) exists, and $\Sigma^2_N \longrightarrow \Sigma^2$.

Now we claim that

$$\lim_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E} \exp \left( \lambda b_n^2 Q_n \right) = \frac{\lambda^2}{2} \Sigma^2. \quad (5.12)$$

For fixed $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, by the Hölder inequality we have that

$$\log \mathbb{E} \exp \left( \lambda b_n^2 Q_n \right) \leq \frac{1}{q} \log \mathbb{E} \exp \left( q \lambda b_n^2 (Q_n - Q^N_n) \right) + \frac{1}{p} \log \mathbb{E} \exp \left( p b_n^2 \lambda Q^N_n \right)$$

for all $\lambda$. From (5.1) and previous estimations it follows that for some constant $B > 0$

$$\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E} \left( e^{b_n^2 \lambda Q_n} \right) \leq \frac{p \lambda^2}{2} \Sigma^2_N + q B \lambda^2 \hat{r}_0 ((g - g^N)^4).$$

Letting $N \to \infty$ and using (5.11), we get

$$\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E} \left( e^{b_n^2 \lambda Q_n} \right) \leq \frac{p \lambda^2}{2} \Sigma^2. \quad (5.13)$$

Similarly, by the Hölder inequality, we have

$$\log \mathbb{E} \exp \left( b_n^2 \lambda Q^N_n \right) \leq \frac{1}{q} \log \mathbb{E} \exp \left( \frac{q b_n^2}{p} \lambda (Q^N_n - Q_n) \right) + \frac{1}{p} \log \mathbb{E} \exp \left( b_n^2 \lambda Q_n \right)$$

for every $\lambda$. From (5.1) and previous estimations it follows that
\[
\frac{\lambda^2 \Sigma^2}{2p^2} \leq \liminf_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E} \left( e^{\lambda Q_n^2} \right) + \frac{g \lambda^2}{2p^2} B \tilde{r}_0((g - g^N)\lambda).
\]
Letting \( N \to \infty \) and using (5.11), we obtain
\[
\frac{\lambda^2 \Sigma^2}{2p^2} \leq \liminf_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E} \left( e^{\lambda Q_n^2} \right). \tag{5.14}
\]
Letting \( p \to 1 \) in (5.13) and (5.14) yields (5.12).

So by (5.12) and the Laplace principle \[10, Th. 2.1.10, p.43\], we have
\[
\lim_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left( e^{b_n^2 \lambda Q_n} \right) = \frac{\lambda^2 \Sigma^2}{2}. \tag{5.15}
\]
To conclude, we have now to show that \( \tilde{I}(x) \) defined in (5.10) is convex.
\[
\tilde{I} \left( \frac{1}{2}(x_1 + x_2) \right) = \sup_{\delta > 0} \limsup_{N \to \infty} \inf_{B(\frac{1}{2}(x_1 + x_2), \delta)} I^N
\]
\[
\inf_{B(\frac{1}{2}(x_1 + x_2), \delta)} I^N \leq \frac{1}{2} \inf_{y_1 \in B(x_1, \delta), y_2 \in B(x_2, \delta)} \left( I^N \left( \frac{1}{2}(y_1 + y_2) \right) \right)
\]
\[
\leq \frac{1}{2} \inf_{y_1 \in B(x_1, \delta), y_2 \in B(x_2, \delta)} \left( I^N(y_1) + I^N(y_2) \right)
\]
\[
= \frac{1}{2} \left( \inf_{B(x_1, \delta)} I^N + \inf_{B(x_2, \delta)} I^N \right)
\]
So
\[
\limsup_{N \to \infty} \inf_{B(\frac{1}{2}(x_1 + x_2), \delta)} I^N \leq \frac{1}{2} \left( \limsup_{N \to \infty} \inf_{B(x_1, \delta)} I^N + \limsup_{N \to \infty} \inf_{B(x_2, \delta)} I^N \right)
\]
Letting \( \delta \downarrow 0 \), we get
\[
\tilde{I} \left( \frac{1}{2}(x_1 + x_2) \right) \leq \frac{1}{2} \left( \tilde{I}(x_1) + \tilde{I}(x_2) \right).
\]
Since \( \tilde{I} \) is inf-compact and convex, by Fenchel’s theorem and (5.15), we get for all \( x \in \mathbb{R} \)
\[
\tilde{I}(x) = \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \frac{\lambda^2 \Sigma^2}{2} \},
\]
which is exactly the announced rate function.

5.2 Proof of Theorem 2.3

We begin with the following lemma \[20, Chap.2, Prop. 2.5\] which implies the exponential tightness.

**Lemma 5.3.** Under the hypothesis Theorem 2.3, we have that for all \( h \in L^p(T, d\theta) \)
\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left( e^{b_n^2 \frac{1}{p} f_T h(\theta) L_n(\theta) d\theta} \right) < +\infty
\]
In particular \( \mathbb{P}(L_n \in \cdot) \) is exponentially \(*\)-tight in \((L^p(T, d\theta), \sigma(L^p(T, d\theta), L^p(T, d\theta)))\), where \( \frac{1}{p'} + \frac{1}{p} = 1 \).
Proof : For every function $h \in L^p(\mathbb{T}, d\theta)$, the function $\tilde{h}(\theta) = \frac{1}{2}[h(\theta) + h(-\theta)]$ is even and
\[
\frac{1}{2\pi} \int_{\mathbb{T}} h(\theta) \mathcal{L}_n(\theta) d\theta = \frac{1}{2\pi} \int_{\mathbb{T}} \tilde{h}(\theta) \mathcal{L}_n(\theta) d\theta,
\]
we shall hence restrict ourselves to the case where $h$ is even. Since
\[
\frac{1}{2\pi} \int_{\mathbb{T}} h(\theta) \mathcal{L}_n(\theta) d\theta = \frac{1}{b_n \sqrt{n}} \langle \langle X, T_n(h)X \rangle - \mathbb{E} \langle X, T_n(h)X \rangle \rangle
\]
Let apply Lemma 5.1 to $H((\xi_l)_{l \in \mathbb{Z}}) = \langle X, T_n(h)X \rangle$:
\[
\mathbb{E}(e^{\lambda^2_1 n \frac{h}{\sqrt{n}} f(\theta) \mathcal{L}_n(\theta)}) = \mathbb{E}(e^{\lambda^2_1 n \frac{h}{\sqrt{n}} (H - \mathbb{E}H)}) \leq \mathbb{E}(e^{\frac{\lambda^2_1 n}{2} \frac{1}{p} C|\mathcal{L}_n|}).
\]
Clearly
\[
|\nabla H|^2 = \sum_{i \in \mathbb{Z}} (\partial_{\xi_i} H)^2 = \sum_{i \in \mathbb{Z}} \left( 2 \sum_{l,k=1}^n a_{i-k} X_l T_n(h)_{k,l} \right)^2
\]
\[
= 4 \sum_{l,k,l',k'=1}^n T_n(f)_{k,k'} X_l X_{l'} T_n(h)_{k,l} T_n(h)_{k',l'}
\]
\[
= 4 \langle X, T_n(h) \rangle T_n(f) T_n(h) X.
\]
Let $\alpha_1^n, \cdots, \alpha_n^n$ the eigenvalues of the matrix
\[
\sqrt{T_n(h) T_n(f) T_n(h)} T_n(f) \sqrt{T_n(h) T_n(f) T_n(h)}.
\]
Its operator norm is bounded from above by (using Lemma 5.1)
\[
||T_n(f)|| \cdot ||T_n(h) T_n(f) T_n(h)|| \leq (n^{1/p} ||f||_q)^2 (n^{1/p'} ||h||_p')^2
\]
Since $\frac{b_n}{n} n^{1/q + 1/p'} \to 0$, $f \in L^q(\mathbb{T}, d\theta)$ and $h \in L^p(\mathbb{T}, d\theta)$, we take $n$ large enough such that $8CK^2 \lambda^2 \frac{b_n^2}{n} \max_{1 \leq j \leq n} \alpha_j^n < 1$. Applying Lemma 4.3. we get
\[
\log \mathbb{E}(e^{\lambda^2_1 n \frac{1}{\sqrt{n}} f(\theta) \mathcal{L}_n(\theta)}) \leq -\frac{1}{2} \sum_{j=1}^n \log \left( 1 - 8CK^2 \lambda^2 \frac{b_n^2}{n} \alpha_j^n \right).
\]
Thus
\[
\limsup_{n \to \infty} \frac{1}{b_n^2} \log \left( e^{\frac{b_n^2}{n} f(\theta) \mathcal{L}_n(\theta)} \right) \leq 8C^2 \lambda^2 \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \alpha_j^n.
\]
Since $f \in L^q(\mathbb{T}, d\theta)$ and $h \in L^p(\mathbb{T}, d\theta)$ with $\frac{1}{p'} + \frac{1}{q} < \frac{1}{2}$, applying Lemma 5.2, we obtain
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^n \alpha_j^n = \lim_{n \to +\infty} \frac{1}{n} \text{tr} ((T_n(f) T_n(h))^2) = \hat{r}_0(f^2 h^2) < +\infty.
\]
The proof of the Lemma ends.

We may now turn to the proof of Theorem 2.3.

Proof :
Step 1. Since \( \left( \frac{1}{b_n \sqrt{n}} \sum_{k=n-\ell+1}^{n} (X_k X_{k+\ell} - \mathbb{E}X_k X_{k+\ell}) \right)_{0 \leq \ell \leq m} \) is negligible with respect to the MDP, using Theorem 2.1, we get the finite dimensional MDP on \( \mathbb{R}^{m+1} \) of

\[
\left( \frac{1}{b_n \sqrt{n}} \sum_{k=1}^{n-\ell} (X_k X_{k+\ell} - \mathbb{E}X_k X_{k+\ell}) \right)_{0 \leq \ell \leq m}
\]

with the rate function given by

\[
I(z) = \sup_{\lambda \in \mathbb{R}^{m+1}} \left\{ \langle \lambda, z \rangle - \frac{1}{2} \lambda^{*} \Sigma^{2} \lambda \right\}.
\]

Now notice that

\[
\hat{\Lambda}\left(\frac{\ell}{n}\right) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{i\ell \theta} \mathcal{L}_{n}(d\theta) = \frac{1}{b_n \sqrt{n}} \sum_{k=1}^{n-\ell} (X_k X_{k+\ell} - \mathbb{E}X_k X_{k+\ell}).
\]

Thus \( (\hat{\mathcal{L}}_{n}(\ell))_{0 \leq \ell \leq m} \) satisfies the MDP on \( \mathbb{R}^{m+1} \) with the same rate function. By Lemma 4.3 and the projective limit Theorem [39, Th. 4.6.9], we deduce that \( (\mathcal{L}_{n})_{n \geq 0} \) satisfies the MDP on \( (L^{p}(\mathbb{T}, d\theta), \sigma(L^{p}(\mathbb{T}, d\theta), L^{p}(\mathbb{T}, d\theta))) \) with the rate function given by for even function \( \eta \in L^{p}(\mathbb{T}, d\theta) \)

\[
I(\eta) = \sup_{m \geq 0} \sup_{\lambda_{0}, \ldots, \lambda_{m} \in \mathbb{R}} \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \left( \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right) \eta(\theta)d\theta - \frac{1}{2} \Lambda \left( \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right) \right\}.
\]

where

\[
\Lambda \left( \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right) = \lambda^{*} \Sigma^{2} \lambda = \frac{1}{2\pi} \int_{\mathbb{T}} \left( \left| \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right|^{2} + \left( \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right)^{2} \right) f^{2}(\theta)d\theta
\]

\[
+ \kappa_{4} \left( \frac{1}{2\pi} \int_{\mathbb{T}} \left( \sum_{k=0}^{m} e^{ik\theta} \lambda_{k} \right) f(\theta)d\theta \right)^{2}.
\]

Step 2. Identification of the rate function. Remark as trigonometric polynomials are dense in \( L^{2}(\mathbb{T}, f^{2}d\theta) \), one can find for \( h \in L^{2}(\mathbb{T}, f^{2}d\theta) \), an approximation by some trigonometric polynomials sequence \( h_{n} \), such that

\[
\lim_{n \to \infty} \int_{\mathbb{T}} (h_{n} - h)^{2}(\theta)f^{2}(\theta)d\theta = 0.
\]

So we can extend continuously the definition of \( \Lambda \) to all function \( h \in L^{2}(\mathbb{T}, f^{2}d\theta) \)

\[
\Lambda(h) = \frac{2}{2\pi} \int_{\mathbb{T}} h^{2}(\theta)f^{2}(\theta)d\theta + \kappa_{4} \left( \frac{1}{2\pi} \int_{\mathbb{T}} h(\theta)f(\theta)d\theta \right)^{2}.
\]

(a) Suppose that \( \eta(\theta)d\theta \) is absolutely continuous w.r.t. \( f^{2}(\theta)d\theta \), and \( \frac{\eta}{f} \in L^{2}(\mathbb{T}, d\theta) \). Let \( h_{n} \) the sequence defined below in (5.17), by Cauchy-Schwartz inequality, we get for all even function \( \eta \in L^{p}(\mathbb{T}, d\theta) \)
By Fatou's lemma we get so it follows that

\[
\left( \int_T \left| (h_n - h)(\theta) \eta(\theta) \right| d\theta \right)^2 \leq \int_T |h_n(\theta) - h(\theta)|^2 f^2(\theta) d\theta \int_T \left( \frac{\eta}{f} \right)^2 (\theta) d\theta \xrightarrow{n \to \infty} 0.
\]

So \( I(\eta) \) defined in (5.16) coincides with

\[
I(\eta) = \sup_{h \in L^2(T, f^2 d\theta)} \left\{ \frac{1}{2} \int_T h(\theta) \eta(\theta) d\theta - \frac{1}{2} \Lambda(h) \right\} := \sup_{h \in L^2(T, f^2 d\theta)} D(h).
\]

Let us find explicitly the maximizer \( h_0 \) of \( D(h) \). Let \( k \in L^2(T, f^2 d\theta) \) and \( \epsilon > 0 \),

\[
\lim_{\epsilon \to 0} \frac{D(h + \epsilon k) - D(h)}{\epsilon} = \frac{1}{2\pi} \int_T k(\theta) \eta(\theta) d\theta - \frac{1}{2} \left( \frac{1}{2} \int_T 2f^2(\theta) h(\theta) k(\theta) d\theta \right)
+ \frac{1}{2\pi} \left( \frac{1}{2} \int_T f(\theta) h(\theta) d\theta \right) \left( \frac{1}{2\pi} \int_T f(\theta) k(\theta) d\theta \right).
\]

So

\[
\lim_{\epsilon \to 0} \frac{D(h + \epsilon k) - D(h)}{\epsilon} = 0, \quad \forall k \in L^2(T, f^2 d\theta)
\] (5.18)

implies that

\[
\eta(\theta) = 2f(\theta)^2 h(\theta) + \kappa_4 \left( \frac{1}{2\pi} \int_T f(\theta) h(\theta) d\theta \right) f(\theta).
\] (5.19)

Dividing (5.19) by \( f \) and integrating over \( T \), we obtain

\[
\int_T f(\theta) h(\theta) d\theta = - \frac{\eta(\theta)}{2f(\theta)} - \frac{\kappa_4}{2 + \kappa_4} \left( \frac{1}{2\pi} \int_T \eta(u) 2f(u) du \right).
\]

Replacing this last expression in (5.19), it is then easy to verify that the only functional \( h_0 \in L^2(T, f^2 d\theta) \) realizing (5.18) is given by

\[
h_0(\theta) f(\theta) = \frac{\eta(\theta)}{2f(\theta)} - \frac{\kappa_4}{2 + \kappa_4} \left( \frac{1}{2\pi} \int_T \eta(u) 2f(u) du \right).
\]

Calculating \( D(h_0) \) gives finally the announced rate function.

(b) Now we have to treat the case where \( \eta(\theta)d\theta \) is absolutely continuous w.r.t. \( f^2(\theta)d\theta \) but \( \frac{\eta}{f} \not\in L^2(T, d\theta) \). So there exists \( g \in L^2(T, d\theta) \) such that \( \int_T g(\theta) \frac{\eta}{f}(\theta) d\theta = +\infty \), and \( g \frac{\eta}{f} \geq 0 \). Let \( h := \frac{g}{f} \), so \( h \in L^2(T, f^2 d\theta) \), we choose \( h_n = (h \vee (-n)) \wedge n \). We get by dominated convergence

\[
\lim_{n \to \infty} \int_T (h_n(\theta) - h(\theta))^2 f(\theta)^2 d\theta = 0,
\]

so it follows that

\[
\lim_{n \to +\infty} \Lambda(h_n) = \Lambda(h).
\]

By Fatou's lemma we get

\[
\liminf_{n \to \infty} \int_T h_n(\theta) \eta(\theta) d\theta \geq \int_T \liminf_{n \to \infty} h_n(\theta) \eta(\theta) d\theta = +\infty.
\]
Since
\[ I(\eta) \geq \frac{1}{2\pi} \int_T h_n(\theta)\eta(\theta)d\theta - \frac{1}{2}\Lambda(h_n), \]
letting \( n \to \infty \), we obtain \( I(\eta) = \infty \).

(c) Now we have to treat the case where \( \eta(\theta)d\theta \) is not absolutely continuous w.r.t. \( f^2(\theta)d\theta \), i.e. there exists a set \( K \subset T \) such that \( \int_K f^2(\theta)d\theta = 0 \) while \( \int_K \eta(\theta)d\theta > 0 \). For any \( t > 0 \), we approximate the function \( t1_K \) by a sequence function \( h_n \in L^2(T, f^2d\theta) \). So \( \forall t \in \mathbb{R} \)
\[ I(\eta) \geq \lim_{n \to +\infty} D(h_n) \geq t \int_K \eta(\theta)d\theta. \]
Letting \( t \) to infinity, we get \( I(\eta) = +\infty \).

5.3 Proof of corollary 2.5

Here we assume \( f' \in L^2(T, d\theta) \), so \( \sum_k |k|^2|\hat{r}_k(f)|^2 < \infty \).

We thus only need to prove that for all \( h \in L^p(T, d\theta) \) ( so \( h \in L^2(T, d\theta) \) since \( p \geq 2 \))
\[ \frac{\sqrt{n}}{b_n} \left( \int_T h(\theta)\mathcal{I}_n(\theta)d\theta - \int_T f(\theta)h(\theta)d\theta \right) \xrightarrow{n \to 0} 0. \]

We have
\[
\left| \int_T h(\theta)\mathcal{I}_n(\theta)d\theta - \int_T f(\theta)h(\theta)d\theta \right| = \left| \sum_{|k| \leq n-1} \left( 1 - \frac{k}{n} \right) \hat{r}_k(f)\hat{r}_k(h) - \sum_k \hat{r}_k(f)\hat{r}_k(h) \right|
- \left| \sum_{|k| \geq n} \frac{|k|}{n} \hat{r}_k(f)\hat{r}_k(h) - \sum_{|k| \geq n} \hat{r}_k(f)\hat{r}_k(h) \right| .
\]

We have \( \sum_{|k| \geq n} |\hat{r}_k(f)||\hat{r}_k(h)| \leq \sum_{|k| \geq n} \frac{|k|}{n} |\hat{r}_k(f)||\hat{r}_k(h)| \). So applying Cauchy-Schwartz inequality we get
\[
\left| \int_T h(\theta)\mathcal{I}_n(\theta)d\theta - \int_T f(\theta)h(\theta)d\theta \right| \leq \frac{1}{n} \sqrt{\sum_k |k|^2|\hat{r}_k(f)|^2} \sqrt{\sum_k |\hat{r}_k(h)|^2}
\leq \frac{C}{n}.
\]
The proof ends.

5.4 Proof of Theorem 3.1

For simplicity, we only consider the problem in \( \mathbb{R} \) and \( F(x_0, \ldots, x_i) = F(x_0) \).
Let us describe briefly how the preceding proof of Theorem 2.1 can be easily extended to
the more general framework of our example.
Since \( F' \) is Lipschitz continuous, we get for some positive \( L \), and for all \( N \)
\[
|F(X_k^N)| \leq L(1 + |X_k^N|^2) \leq 2L(N + 1) \left( 1 + \sum_{j=-N}^{N} a_j^2 \xi_{k+j}^2 \right)
\]
so that, setting \( \delta' = \frac{\delta}{2L(N+1)^2 \sup_i a_i^2} \) where \( \delta \) is given in (4.2), by the assumption on the validity of the LSI, we get
\[
\mathbb{E} \left( e^{\delta' |F(X_k^N)|} \right) \leq e^{\delta' L(N+1)} \mathbb{E} \left( e^{\delta \xi_0^2} \right) < \infty.
\]
Since Chen [8] deals with moderate deviations of \( m \)-dependent Banach space valued random variables, so that the first step is exactly the same in the general case.

To prove the asymptotic negligibility as \( N \to \infty \) of \( S_n(F) - S_n^N(F) \) with respect to the MDP, we need to assume the boundedness of the density. We apply again (5.3) to
\[
G((\xi_t)_{t \in \mathbb{Z}}) = \sum_{k=1}^{n} (F(X_k) - F(X_k^N)),
\]
We have
\[
|\nabla G|^2 = \sum_{i \in \mathbb{Z}} \left( \sum_{k=1}^{n} a_{i-k} F'(X_k) - a_{i-k}^N F'(X_k^N) \right)^2
\leq 2 \sum_{i \in \mathbb{Z}} \left( \sum_{k=1}^{n} (a_{i-k} - a_{i-k}^N) F'(X_k) \right)^2 + 2 \sum_{i \in \mathbb{Z}} \left( \sum_{k=1}^{n} a_{i-k}^N (F'(X_k^N) - F'(X_k)) \right)^2
= 2 \left( T_n((g - g^N)^2) F'(X) \right)^2 + 2 \left( T_n((g^N)^2) F'(X - F'(X^N)) \right)^2.
\]
By the fact that the derivative of \( F \) is Lipschitz and the spectral density is bounded, we have that the last term is bounded by
\[
2L ||g - g^N||^2_\infty (n + \langle X, X \rangle) + 2 ||g^N||^2_\infty (X^N - X, X; X^N - X).
\]
Finally by (4.2), as \( \lambda^2 b_n^2 ||g - g^N||^2_\infty \) can be chosen arbitrary small for large \( n \),
\[
\frac{1}{b_n^2} \log \mathbb{E} \left( e^{\lambda^2 b_n^2 (S_n(F) - S_n^N(F^N))} \right) \leq LC^2 ||g - g^N||^2_\infty
\leq - \frac{n}{4b_n^2} \log \left( 1 - 4CLK^2 \lambda^2 b_n^2 ||g - g^N||^2_\infty ||g||^2_\infty \right)
\leq - \frac{n}{4b_n^2} \log \left( 1 - 4CLK^2 \lambda^2 b_n^2 ||g^N||^2_\infty ||g - g^N||^2_\infty \right)
\]
and the left hand side of this last inequality is easily seen to behave as \( n \to \infty \) as
\[
||g - g^N||^2_\infty (LC^2 + 2CLK^2 \lambda^2 ||g||^2_\infty).
\]
By the famous Fejer Theorem, under the assumption of continuity of \( g \), we get that
\[
\lim_{N \to \infty} ||g - g^N||^2_\infty = 0,
\]
which yields to the desired negligibility.

A careful reading of Step 3 in the proof of Theorem 2.1 shows that the extension to the general case brings no further difficulties. The proof then ends.

**Remark 5.1.** To prove negligibility of Step 2 in general framework, we only have to establish this negligibility for each of the coordinates $F_j$ of $F$ (as there is only a finite number of coordinates), and also that

$$|∇G|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=1}^{m} \left( \partial_{k_i} \sum_{k=1}^{n} F_j(X_k, \ldots, X_{k+l}) - F_j(X^N_k, \ldots, X^N_{k+l}) \right)^2$$

$$= \sum_{i \in \mathbb{Z}} \sum_{j=1}^{m} \left( \sum_{s=0}^{l} \sum_{k=1}^{n} \left( a_{i-k-s} \partial_{x_s} F_j(X_k, \ldots, X_{k+l}) - a^N_{i-k-s} \partial_{x_s} F_j(X^N_k, \ldots, X^N_{k+l}) \right) \right)^2$$

$$\leq (l + 1) \sum_{s=0}^{l} \sum_{j=1}^{m} \sum_{i \in \mathbb{Z}} \left( \sum_{k=1}^{n} \left( a_{i-k-s} \partial_{x_s} F_j(X_k, \ldots, X_{k+l}) - a^N_{i-k-s} \partial_{x_s} F_j(X^N_k, \ldots, X^N_{k+l}) \right) \right)^2$$

which leads to the same estimation as before as $\partial_{x_s} F_j$ is supposed to be Lipschitz for each $j$ and $s$.

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