VECTOR-VALUED HEAT EQUATIONS AND NETWORKS
WITH COUPLED DYNAMIC BOUNDARY CONDITIONS

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Abstract. Motivated by diffusion processes on metric graphs and ramified spaces, we consider an abstract setting for interface problems with coupled dynamic boundary conditions belonging to a quite general class. Beside well-posedness, we discuss positivity, $L^\infty$-contractivity and further invariance properties. We show that the parabolic problem with dynamic boundary conditions enjoy these properties if and only if so does its counterpart with time-independent boundary conditions. Furthermore, we prove continuous dependence of the solution to the parabolic problem on the boundary conditions in the considered class.

1. Introduction

Elliptic systems with coupled boundary conditions have been attracting broad attention at least since [1]. A classical approach is based on interpreting interface conditions of an elliptic system as boundary conditions of a vector-valued elliptic equation. This leads to introducing differential operators acting on spaces of vector-valued functions. A parabolic theory for this kind of operators has been recently developed, see e.g. [4, 24].

A particularly interesting application of the theory of elliptic systems is given by so-called networks and quantum graphs, see e.g. [5, 39] and references therein. Their generalisation to $n$-dimensional problems has appeared already in [13], where the related notion of ramified space has been proposed. Having in mind applications to quantum graphs, Kuchment has proposed in [38] a class of coupled, time-independent boundary conditions for 1-dimensional elliptic systems. Kuchment’s formalism allows for a very efficient variational approach, but the tradeoff is that his boundary conditions are only a proper subset of those considered in [1] – or, in the specific context of quantum graphs, in [37]. However, it is remarkable that Kuchment’s conditions give rise exactly to all self-adjoint realisations of the Schrödinger operator on a metric graph, under a mild locality assumption.

In the companion paper [17], Cardanobile and the author have generalized Kuchment’s formalism to the case of $n$-dimensional vector-valued diffusion and characterized several properties of the parabolic problem in dependence on the chosen boundary conditions. The aim of this paper is to provide the extension of the theory in [17] to the case of dynamic boundary conditions of Wentzell–Robin-type.

Although we are soon going to consider the general case, let us start by briefly focusing on the 1-dimensional setting of networks (or quantum graphs).

Example 1.1. Let $N \in \mathbb{N}$ and consider the prototypical case of a diffusion problem

\[
\begin{aligned}
\dot{u}_j(t,x) &= u''_j(t,x), & t \geq 0, \ x \in (0, \infty), \ j = 1, \ldots, N, \\
\dot{u}_j(t,0) &= u_j(t,0) = : \psi(t), & t \geq 0, \ j, \ell = 1, \ldots, N, \\
\dot{\psi}(t) &= \sum_{j=1}^N u_j(t,0) & t \geq 0,
\end{aligned}
\]

(TDPS)

on a metric graph – more precisely, on a semi-infinite star with $N$ edges $e_1, \ldots, e_N$ on whose center a dynamic Kirchhoff-type boundary condition is imposed along with a standard continuity assumption. Each edge is parametrized as a $(0, \infty)$-interval, where $0$ is identified as the center of the star. Therefore, the function $u_j$ describing the diffusion on the edge $e_j$ maps $[0, \infty) \times [0, \infty)$ to $\mathbb{C}$, while $\psi: [0, \infty) \rightarrow \mathbb{C}$ describes the time evolution

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of the common boundary value in the center. It is known that the associated initial value problem is well-posed, as discussed, e.g. in [3, 10, 55].

Laplace operators with dynamic boundary conditions appear as limiting cases of approximation schemes considered in [40, 26]. The cable model of a dendritical tree proposed by Rall in [62] also leads to analogous network diffusion problems, cf. [14, 57]: a thorough biomathematical investigation of them has been performed in a series of four papers beginning with [44].

\[
\begin{array}{c}
\text{A semi-infinite star with 6 edges.}
\end{array}
\]

We can rephrase (TDPS) by considering the orthogonal projection \( P_Y \) of \( \mathbb{C}^N \) onto the subspace \( Y := \langle 1 \rangle \) spanned in \( \mathbb{C}^N \) by the vector

\[ 1 := (1, \ldots, 1). \]

Observe that the unknown can be thought of as a function \( u : (0, \infty) \to \mathbb{C}^N \), so that the network diffusion problem simply becomes

\[ \dot{u}(t, x) = u''(t, x), \quad t \geq 0, \quad x \in (0, \infty), \]

with suitable boundary conditions in \( 0 \). More precisely, the continuity condition in the star’s center – given by the second equation in (TDPS) – amounts to require that \( u(t, 0) \in \langle 1 \rangle \) for all \( t \geq 0 \), i.e.,

\[ P_Y(u(t, 0)) = u(t, 0), \quad t \geq 0, \]

while the dynamic boundary condition equivalently reads

\[ \dot{u}(t, 0) = P_Y(\dot{u}(t, 0)) = N P_Y(u'(t, 0)) = - P_Y \left( \frac{\partial u}{\partial \nu}(t, 0) \right), \quad t \geq 0. \]

Hence, the dynamic boundary condition is an equation living in the \((1\text{-dimensional})\) boundary space \( Y = \langle 1 \rangle \).

This kind of boundary conditions also arises in the mathematical modelling of string networks with masses at the nodes. They play an important role in the control theory of wave and beam equations: investigations in this direction go back at least to [11 §2.7] and [31].

The goal of the present article is to generalize the setting discussed in the above example. Let \( \Omega \) be a smooth open domain in \( \mathbb{R}^n \) with boundary \( \Gamma := \partial \Omega \). Let \( H \) be a separable complex Hilbert space. In particular, Bochner spaces \( L^2(\Omega; H) \) and \( L^2(\Gamma; H) \) become separable complex Hilbert spaces when endowed with the canonical scalar products

\[ (f|g)_{L^2(\Omega; H)} := \int_{\Omega} (f(x)|g(x))_H dx, \quad f, g \in L^2(\Omega; H), \]

and

\[ (f|g)_{L^2(\Gamma; H)} := \int_{\Gamma} (f(z)|g(z))_H d\sigma(z), \quad f, g \in L^2(\Gamma; H). \]
Let $\mathcal{Y}$ be a closed subspace of $L^2(\Gamma; H)$ and hence a Hilbert space in its own right with respect to the scalar product induced by $L^2(\Gamma; H)$. Vector-valued Sobolev spaces can be introduced recursively just like in the scalar-valued case. I.e., one first lets $H^0(\Omega; H) := L^2(\Omega; H)$, hence defines

\begin{equation}
H^k(\Omega; H) := \left\{ f \in H^{k-1}(\Omega; H) : \exists \nabla f := g \in L^2(\Omega; H^2) \text{ s.t.} \int_{\Omega} f(x) \nabla h(x) \, dx = - \int_{\Omega} g(x) h(x) \, dx \text{ for all } h \in C_c^\infty(\Omega; \mathbb{C}) \right\}, \quad k = 1, 2, \ldots,
\end{equation}

and finally introduces spaces of fractional order by standard complex interpolation. (Here we denote by $H^n$ the Hilbert space defined as the Cartesian product of $n$ copies of $H$.) In particular, $H^1(\Omega; H)$ is a Hilbert space with respect to the scalar product

\begin{equation}
(f|g)_{H^1(\Omega; H)} := \int_{\Omega} \langle \nabla f(x), \nabla g(x) \rangle_{H^2} \, dx + \int_{\Omega} (f(x)|g(x))_{H^0} \, dx, \quad f, g \in H^1(\Omega; H).
\end{equation}

We emphasize that vector-valued Sobolev spaces are introduced using scalar-valued test functions, hence those appearing in (1.2) are scalar-valued (i.e., a Bochner integral) whereas those appearing in (1.1) is vector-valued (i.e., an $L^2$ integral).

For Example 1.4.

Let again $\mathcal{Y}_N$ denote by $L^2(\Gamma; H)$ the orthogonal projection of $Y := \{0\}$ onto the closed subspace $\mathcal{Y}$. In the 1-dimensional case of finite quantum graphs, the investigation of such a problem has been sketched in [34, §4].

Example 1.2. Let $\Omega = (0, \infty)$ and $H = \mathbb{C}^N$, so that $L^2(\Gamma) = L^2(\{0\}) = \mathbb{C}^N$. Take

$\mathcal{Y} := \langle (1) \rangle = \{ c \in \mathbb{C}^N : c_1 = \ldots = c_N \}$.

Then

$$
P_{\mathcal{Y}} = \frac{1}{N} \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}
$$

and one sees that (AS) is just a reformulation of (TDPS) considered in Example 1.1.

Example 1.3. Let again $\Omega = (0, \infty)$ and $H = \mathbb{C}^N$. If $N = 1$ and $\mathcal{Y} = L^2(\Gamma; H) = \mathbb{C}$, then the first boundary condition in (AS) is void and (AS) is the reformulation of a scalar-valued heat equation with Wentzell–Robin boundary conditions, see e.g. the recent contributions in [27, §11], [54, §11, 65]. If instead $\mathcal{Y} = \{0\}$, then (AS) reduces to a heat equation with Dirichlet boundary conditions. For $N = 1$, these are the only possible choices for $\mathcal{Y}$, but for $N \geq 2$ we have infinitely many new boundary conditions that in some sense interpolate between Dirichlet and Wentzell–Robin ones. This is crucial when setting up a Courant–Fischer min-max formula, cf. [11].

Example 1.4. For $H = \mathbb{C}^N$ the elliptic problem with dynamic interface conditions – a vector-valued version of Wentzell–Robin boundary conditions – has been considered in [63, §III.4.5]. In [58], even more general elliptic interface problems have been considered under the very general assumption that the given system can even consist of several metric spaces with different Hausdorff dimensions, see also [12], see also [3].

As already mentioned, the general case of a diffusion equation equipped with coupled (either dynamic or time-independent) boundary conditions is mostly motivated by the theories of quantum graphs and parabolic network
equations, but it also appears in higher dimensional applications, in particular in biomathematical models – see e.g. [35] and references therein.

In this article we restrict to the case of dynamic boundary conditions only. However, the general case of mixed dynamic/time-independent boundary conditions (typically appearing in models from the applied sciences, see e.g. [44]) can be easily treated combining the results presented here and those from [17].

In Section 2 we introduce our abstract framework and deduce a well-posedness result. The above examples suggest that the vector-valued setting – although equivalent to the that based on a network (or ramified space) formalism – is more efficient. In fact, its flexibility allows to simply introduce whole families of spaces \( \mathcal{Y} \). Consequently, completely new questions arise. For example, one may wonder how the solution to the heat equation with boundary conditions as in (AS) depends on \( \gamma \): it will be shown in Theorem 2.6 that this dependence is continuous in norm under very natural assumptions. This result is interesting in that it does not have a scalar-valued pendant. We also extend to the vector-valued case a result on continuous dependence on parameters obtained in the scalar-valued case in [21].

We consider invariance of order intervals and subspaces in Section 3, showing in particular a tight relation between the properties of the heat semigroup governing the problem with time-independent (i.e., Robin-type vector-valued) boundary conditions and its dynamic counterpart. We will observe some unexpected phenomena: e.g., the semigroups governing these diffusion problems are in general not submarkovian – not even positivity preserving.

To discuss these behaviours in detail, in Section 4 we focus on the setting of Example 1.1. It turns out that even in the simple context of diffusion on a semi-infinite star with finitely many edges, unexpected dynamics arises after choosing appropriate boundary conditions.

Finally, in Section 5 we briefly discuss the general properties of a similar but different kind of dynamic boundary condition, where the normal derivative – rather than the trace – undergoes a time evolution.

2. Preliminary results

To begin with, we make our standing assumptions precise.

As in the previous section, let \( H \) be a separable complex Hilbert space, \( \Omega \) be an open domain in \( \mathbb{R}^n \) with \( C^1 \) boundary \( \Gamma := \partial \Omega \) and \( \mathcal{Y} \) be a closed subspace of \( L^2(\Gamma; H) \). In the rest of the paper we are going to investigate the general abstract initial-boundary value problem

\[
\begin{cases}
\frac{\partial}{\partial t} u(t) = \Delta u(t), & t \geq 0, \\
u(t) \in \mathcal{Y}, & t \geq 0, \\
\frac{\partial}{\partial \nu} u(t) + (\gamma \Delta u - S) u(t) = 0, & t \geq 0, \\
u(0) = u_0, & \\
u(0) = v_0.
\end{cases}
\tag{AV}
\]

Here \( \gamma \in \mathbb{R}_+ \),

\[ S \in \mathcal{L}(H^{1/2}(\Gamma; H); L^2(\Gamma; H)) \]

and \( \Delta_{\Gamma} \) denotes the (dissipative) Laplace–Beltrami operator on the \((n-1)\)-dimensional (differentiable, orientable) manifold \( \Gamma \), with the convention that \( \gamma = 0 \) if \( n = 1 \), and hence if \( \Gamma \) only consists of isolated points. The vector-valued Sobolev space \( H^1(\Gamma; H) \) can be defined in the usual way as the vector-valued version of the scalar-valued space \( H^1(\Gamma) \) as introduced, i.e., in [42] §I.7.3. The Laplace operator appearing in (AV) is defined weakly.

While weak defining the Laplace operator on open domains is standard, a more detailed introduction of the (weakly defined) Laplace–Beltrami operator is in order. A definition of the Laplace–Beltrami operator by means of

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1 Observe that any differentiable function \( g : \Gamma \to H \) is a mapping between the differentiable manifold \( \Gamma \) and the (trivial) Hilbert manifold \( H \), whose tangent bundles are \( T\Gamma \cong \Gamma \times \mathbb{R}^{n-1} \) and \( T\mathbb{H} \cong H \times H \), respectively. Accordingly, at any point \( x \in \Gamma \) the derivative \( \nabla g(x) : T_x \Gamma \to T_{g(x)} H \) is a bounded linear operator from \( \mathbb{R}^{n-1} \) to \( H \) – hence it can actually be seen as a vector in \( H^{n-1} \).
of Hilbert space techniques has been performed in the recent preprint [8]. In fact,
\[(\nabla_\Gamma f(\cdot) | \nabla_\Gamma g(\cdot))_{H^{n-1}} : \Gamma \to \mathbb{C}, \quad f, g \in H^1(\Gamma; H),\]
can be defined as the Lebesgue-integrable mapping such that its restriction to any chart \((V, \xi)\) on \(\Gamma\) satisfies
\[\langle \nabla_\Gamma f(\cdot) | \nabla_\Gamma g(\cdot) \rangle_{H^{n-1}} |_V = \sum_{i,j=1}^{n-1} (g^{ij} D_i(f \circ \xi^{-1}) \circ \xi | D_j(g \circ \xi^{-1}) \circ \xi)_{H},\]
where \(g\) is the canonical Riemannian metric of the surface \(\Gamma\). This expression defines in turn a sesquilinear form, and the linear operator associated with this sesquilinear form is the (weakly defined) Laplace–Beltrami operator \(\Delta_\Gamma\). We refer to [8 §1] for details.

**Remark 2.1.** Clearly, both \(\Delta\) and \(\Delta_\Gamma\) may be replaced by general elliptic operators with real-valued coefficients in pretty much the same way [54] generalizes [9]. Similarly, lower order terms may be added.

It is known that the right setting for the study of systems of this kind is either the space of continuous functions on \(\overline{\Omega}\) or else an \(L^p\)-product space. We are going to follow the latter approach throughout this note.

**Lemma 2.2.** The space
\[(2.1) \quad V_\gamma : = \left\{ f := \left( f_{|\Gamma} \right) \in H^1(\Omega; H) \times (H^s(\Gamma; H) \cap \mathcal{Y}) \right\}\]
is dense in \(L^2(\Omega; H) \times \mathcal{Y}\) for all \(s \geq 0\).

In no confusion is possible, in the following we will write \(L^2\) instead of \(L^2(\gamma)\).

**Proof.** This is a slight modification of [54] Lemma 5.6. More precisely, the assumptions in [54] Lemma 5.6 can be weakened by merely assuming that \(H^1(\Gamma; H) \cap \mathcal{Y}\) is dense in the range of the trace operator, instead of coinciding with it. This density condition is satisfied by assumption, hence the claim follows. \(\square\)

In the following we set either \(s = 1\) if \(\gamma > 0\), or \(s = \frac{1}{2}\) if \(\gamma = 0\). Accordingly,
\[V_\gamma : = \left\{ f := \left( f_{|\Gamma} \right) \in H^1(\Omega; H) \times (H^1(\Gamma; H) \cap \mathcal{Y}) \right\} \quad \text{if } \gamma > 0\]
or
\[V_\gamma : = \left\{ f := \left( f_{|\Gamma} \right) \in H^1(\Omega; H) \times \left( H^{\frac{1}{2}}(\Gamma; H) \cap \mathcal{Y} \right) \right\} \quad \text{if } \gamma = 0.\]
We consider a form \((a_\gamma, V_\gamma)\) defined by
\[a_\gamma(f, g) := \int_\Omega \langle \nabla f(x) | \nabla g(x) \rangle_{H^n} \, dx + \gamma \int_\Gamma \langle \nabla_\Gamma f(z) | \nabla_\Gamma g(z) \rangle_{H^{n-1}} \, d\sigma(z) + (S f_{|\Gamma} | g_{|\Gamma})_{\mathcal{Y}}, \quad f, g \in V_\gamma,\]
where the second addend on the right hand side corresponds to the Laplace–Beltrami operator on the Riemannian manifold \(\Gamma\) (recall that by convention \(\gamma = 0\) whenever \(n = 1\)). We remark that
\[(S f_{|\Gamma} | g_{|\Gamma})_{\mathcal{Y}} = (S f_{|\Gamma} | P_\gamma g_{|\Gamma})_{\mathcal{Y}} = (P_\gamma S f_{|\Gamma} | g_{|\Gamma})_{\mathcal{Y}} \quad \text{for all } f, g \in V_\gamma,\]
so that the third addend in the definition of \((a_\gamma, V_\gamma)\) is well-defined.

By a principle presented in [17] Appendix and based on [30] Thm. 4.5.1, the classical Maz’ya inequality (cf. [40] §4.11.2) can be extended to the vector-valued case. Accordingly, in either case \(V_\gamma\) is a Hilbert space with respect to the norm defined by
\[(f | g)_{V_\gamma} := \int_\Omega \langle \nabla f(x) | \nabla g(x) \rangle_{H^n} \, dx + \int_\Gamma \langle \nabla_\Gamma f(z) | \nabla_\Gamma g(z) \rangle_{H^{n-1}} \, d\sigma(z) + (f_{|\Gamma} | g_{|\Gamma})_{\mathcal{Y}} \quad \text{if } \gamma > 0\]
or
\[(f | g)_{V_\gamma} := \int_\Omega \langle \nabla f(x) | \nabla g(x) \rangle_{H^n} \, dx + (f_{|\Gamma} | g_{|\Gamma})_{\mathcal{Y}} \quad \text{if } \gamma = 0.\]
Theorem 2.3. The operator $\Delta_{Y,S}$ associated with $(a_S, V_Y)$ generates an analytic semigroup $(e^{t\Delta_{Y,S}})_{t \geq 0}$ with angle $\frac{\pi}{2}$ on $L^2$.

The operator $\Delta_{Y,S}$ is dissipative if the operator $S$ is accretive and in this case the semigroup $(e^{t\Delta_{Y,S}})_{t \geq 0}$ is contractive. The operator $\Delta_{Y,S}$ is self-adjoint if and only if the operator $S$ is self-adjoint and in this case the semigroup $(e^{t\Delta_{Y,S}})_{t \geq 0}$ is self-adjoint. The operator $\Delta_{Y,S}$ has compact resolvent if and only if $\Omega, \Gamma$ have finite measure, provided that $H$ is finite dimensional; in this case the semigroup $(e^{t\Delta_{Y,S}})_{t \geq 0}$ is compact.

The proof is based on the approach presented, e.g., in [23, Chapt. VI]. We borrow our terminology from [6].

Proof. We are going to show that $(a_S, V_Y)$ generates a cosine family with phase space $V_Y \times L^2$ in the sense of [7, §3.14]. To this aim, we show that for all $\gamma \in \mathbb{R}_+$ the densely defined sesquilinear form $(a_S, V_Y)$ is continuous and elliptic (with respect to $L^2$), i.e.,

$$\text{Reas}(f, f) + \omega \|f\|_{L^2}^2 \geq \alpha \|f\|_{V_Y}^2 \quad \text{for all } f \in V_Y$$

for some $\alpha > 0$ and a suitable $\omega \in \mathbb{R}$.

Continuity follows from the Cauchy–Schwarz inequality. Ellipticity (with respect to $L^2$) follows from ellipticity (with respect to $L^2(\Omega; H)$ and $L^2(\Gamma; H)$) of the forms associated with the Laplace and Laplace–Beltrami operators, corresponding to the first two addends of $(a_S, V_Y)$. The third addend in the definition of $a_S$ is sesquilinear and defined on $H^\frac{1}{2}(\Gamma; H) \times H^\frac{1}{2}(\Gamma; H)$, hence it can be neglected by a perturbation argument (see [51, Lemma 2.1]). Finally, because

$$|\text{Im}a_S(f, f)| = |\text{Im}(Sf, f)| \leq |S\| \|f\|_{H^\frac{1}{2}(\Gamma; H)} \|f\|_{L^2(\Gamma; H)} \leq M |S\| \|f\|_{H^\frac{1}{2}(\Gamma; H)} \|f\|_{L^2(\Gamma; H)}$$

for some $M > 0$ and all $f \in V_Y$, due to boundedness of the trace operator from $H^1(\Omega; H)$ to $H^\frac{1}{2}(\Gamma; H)$, the announced generation of a cosine family follows by [22, Thm. 4]. It is known that generators of cosine operator functions also generate analytic semigroups with angle $\frac{\pi}{2}$, see [7, Thm. 3.14.17].

Because the forms associated with the Laplace and Laplace–Beltrami operators are accretive, accretivity of $(a_S, V_Y)$ is clear provided $S$ is accretive. A direct computation shows that $(a_S, V_Y)$ is symmetric if and only if $S$ is self-adjoint. The assertion on compactness follows from the Aubin–Lions Lemma, see [61, Prop. III.1.3].

The proof of the following is based on [9, Rem. 2.2].

Proposition 2.4. Assume $\Omega$ to have $C^2$-boundary. For all $\gamma \in \mathbb{R}_+$ and $S \in \mathcal{L}(L^2(\Gamma; H))$ the operator $\Delta_{Y,S}$ associated with $(a_S, V_Y)$ is given by

$$D(\Delta_{Y,S}) = \left\{ f := \left( \frac{f}{f|_\Gamma} \right) \in V_Y : \Delta f \in L^2(\Omega; H), \Delta f|_\Gamma \in L^2(\Gamma; H), \text{ and } \frac{\partial f}{\partial \nu} \in L^2(\Gamma; H) \right\},$$

$$\Delta_{Y,S} f := \left( \begin{array}{c} \Delta f \\ \gamma \Delta f|_\Gamma + Sf \end{array} \right) = \left( \begin{array}{c} 0 \\ -P_y \frac{\partial}{\partial \nu} P_y (\gamma \Delta f|_\Gamma - Sf) \end{array} \right),$$

hence $(e^{t\Delta_{Y,S}})_{t \geq 0}$ yields the solution to (AV). If in particular $f \in D(\Delta_{Y,S})$, then $f \in H^\frac{1}{2}(\Omega; H) \cap H^2_{0,\text{loc}}(\Omega; H)$.

Observe that in general $A_Y$ would not operate on $L^2$ if we would drop the term $P_y$.

Proof. By definition, the operator associated with $(a_S, V_Y)$ is given by

$$D(B_{Y,S}) := \left\{ f \in V_Y : \exists g \in L^2 \text{ s.t. } \alpha(f, h) = (g|h)_{L^2}, \forall h \in V_Y \right\},$$

$$B_{Y,S} f := -g.$$
In order to prove that \( \Delta_{Y,S} \subset B_{Y,S} \) take \( f, h \in V_Y \). By the Gauß–Green formulae and the (weak) definition of the Laplace–Beltrami operator we obtain

\[
a_{S}(f, h) = \int_{\Omega} (\nabla f(x)|\nabla h(x))_{H^s} \, dx + \gamma \int_{\Gamma} (\nabla \Gamma f|\Gamma(z)|\nabla \Gamma h|\Gamma(z))_{H^{n-1}} \, d\sigma(z)
\]

\[
= -\int_{\Omega} (\Delta f(x)|h(x))_{H^s} \, dx
\]

\[
+ \int_{\Gamma} \left( \frac{\partial f(z)}{\partial n} | h|_{\Gamma(z)} \right)_{H} d\sigma(z) - \gamma \int_{\Gamma} (\Delta \Gamma f|\Gamma(z)|g|\Gamma(z))_{H} d\sigma(z)
\]

\[
= -\int_{\Omega} (\Delta f(x)|h(x))_{H^s} \, dx + \left( P_{\gamma} \left( \frac{\partial f(z)}{\partial n} - \gamma \Delta \Gamma f|\Gamma(z) \right) \right)_{\ell^2} =: (g|h)_{\ell^2},
\]

and the operator \( \Delta_{Y,S} \) has the claimed form.

Conversely, let \( f \in D(B_{Y,S}) \). The above computation also shows that \( \Delta f \) and \( \Delta \Gamma f|\Gamma \) are well defined elements of \( L^2(\Omega; H) \) and \( L^2(\Gamma; H) \), respectively, and that \( f \) has a weak normal derivative in \( L^2(\Gamma; H) \). We deduce that \( f \in H^s(\Omega; H) \) by [32, Thm. 2.7.4] – suitably extended to the vector-valued case by virtue of [30, Thm. 4.5.1].

\[\text{Remark 2.5.} \] The vectors in \( D(A_{Y}^{2}) \) also satisfy the additional boundary condition

\[
(\Delta u)|_{\Gamma} \in Y \quad \text{and} \quad (\Delta u)|_{\Gamma} + P_{\gamma} \frac{\partial u}{\partial \nu} + P_{\gamma} (S u|_{\Gamma} - \gamma \Delta \Gamma u|_{\Gamma}) = 0
\]

for all \( z \in \Gamma \). Conditions 2.2 can be interpreted as a formulation of Wentzell–Robin boundary conditions which is stronger than the dynamic one that is usual in the context of \( L^p \)-spaces. Due to the regularising effect of the analytic semigroup \( (e^{t\Delta_{Y,S}})_{t \geq 0} \), these additional conditions are satisfied by the solution \( (AV) \) for any time \( t > 0 \).

Consider a sequence \( (Y_n)_{n \in \mathbb{N}} \) of closed subspaces of \( L^2(\Gamma; H) \) such that the associated sequence of orthogonal projections \( (P_{Y_n})_{n \in \mathbb{N}} \) converges in operator norm. Then its limit is also necessarily a projection and a contraction, i.e., an orthogonal projection – say, onto a subspace \( Y \). Consider moreover a sequence \( (S_n)_{n \in \mathbb{N}} \) in \( \mathcal{L}(H^{\frac{1}{2}}(\Gamma; H); L^2(\Gamma; H)) \) that converges in operator norm to some \( S \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma; H); L^2(\Gamma; H)) \). Now, it is quite natural to conjecture that \( \Delta_{Y_n,S_n} \) converges to \( \Delta_{Y,S} \) in a suitable sense.

Observe that no kind of convergence from above or below of the form family \( (a_{S_n}, V_{Y_n})_{n \in \mathbb{N}} \) holds – in our case one typically has \( V_{Y_n} \cap V_{Y_m} = V_{Y_0} \) for some lower-dimensional \( Y_0 \), whenever \( n \neq m \) – so that in general \( V_{Y_n} \) is not dense in any \( V_{Y_m} \). Furthermore, the operators \( \Delta_{Y_n,S_n} \) and \( \Delta_{Y,S} \) act on \( L^2(\Omega; H) \times Y_n \) and \( L^2(\Omega; H) \times Y \), respectively, i.e., they generally act on different spaces. All in all, it seems that well-known results for convergence of operators associated with forms (e.g., those due to Kato and Simon) cannot be applied to our setting. Some results on approximation of operators acting on different spaces have been recently obtained by Ito and Kappel (see e.g., [32, Chapt. 4]), but it seems that they fall short of fitting our framework, too.

The different approach proposed by Post in [30] and further developed in [33] seems to be more appropriate. In order to apply Post’s results, we need to impose a structural assumption on \( Y \) that will prove a significant simplification in our framework.

\[\text{Theorem 2.6.} \] Let \( (Y_n)_{n \in \mathbb{N}} \) be a sequence of closed subspaces of \( H \). Consider a further closed subspace \( Y \) of \( H \) and a family \( (J_{Y_n})_{n \in \mathbb{N}} \) of unitary operators on \( H \) such that \( J_{Y_n} Y_n = Y \) for all \( n \in \mathbb{N} \). Assume furthermore that \( \lim_{n \to \infty} J_{Y_n} = \text{Id} \) in operator norm and consider the spaces

\[
Y := \{ f \in L^2(\Gamma; H) : f(z) \in Y \text{ for a.e. } z \in \Gamma \}, \quad \text{and}
\]

\[
Y_n := \{ f \in L^2(\Gamma; H) : f(z) \in Y_n \text{ for a.e. } z \in \Gamma \}, \quad n \in \mathbb{N}.
\]
Let \((S_n)_{n \in \mathbb{N}}\) be a sequence of accretive bounded linear operators on \(H\) that converges in operator norm to some \(S \in \mathcal{L}(H)\) and define linear operators \(S_n, S \in \mathcal{L} \left( H^{\frac{1}{2}} (\Gamma; H) \right) \) by
\[
S_n g := S_n \circ g, \quad n \in \mathbb{N}, \quad \text{and} \quad S g := S \circ g, \quad g \in H^{\frac{1}{2}} (\Gamma; H).
\]
Then both families \((R(\lambda, \Delta_{Y_n, S_n}))_{n \in \mathbb{N}}\) and \((e^{t(s\lambda, S_n)})_{n \in \mathbb{N}}\) of bounded linear operators on \(L^2_{Y_n}\) converge in operator norm to the bounded linear operators \(R(\lambda, \Delta_{Y, S})\) and to \(e^{t \Delta_{Y, S}}\) on \(L^2_Y\), for all \(\Re \lambda > 0\) and for all \(t > 0\) respectively. Moreover, if \(H\) is finite dimensional and \(\Omega, \Gamma\) have finite measure, then the (discrete) spectrum of \(\Delta_{Y_n, S_n}\) converges to the (discrete) spectrum of \(\Delta_{Y, S}\).

**Remark 2.7.** Observe that the phenomenon observed in Theorem 2.6 is intrinsically related to the vector-valued case. If in fact \(\dim H = 1\), then each sequence \((Y_n)_{n \in \mathbb{N}}\) of subspaces of \(H = \mathbb{C}\) such that \((P_{Y_n})_{n \in \mathbb{N}}\) converges is eventually constant – with value either \(\{0\}\) or \(H\) – so that the assertion becomes trivial.

The proof is based on an abstract convergence scheme discussed in [60] Appendix, which we briefly recall for the sake of self-containedness. The following collects results from [60] Thms. A.5 and A.10.

**Proposition 2.8.** Let \(\mathcal{H}, \mathcal{H}_1, \tilde{\mathcal{H}}, \hat{\mathcal{H}}_1\) be Hilbert spaces such that \(\mathcal{H}_1 \hookrightarrow \mathcal{H}\) and \(\hat{\mathcal{H}}_1 \hookrightarrow \hat{\mathcal{H}}\) with dense embeddings.

Let \(\mathfrak{h} : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{C}\) and \(\tilde{\mathfrak{h}} : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{C}\) be continuous, accretive and elliptic (with respect to \(\mathcal{H}\) and \(\mathcal{H}_1\), respectively) with associated operators \(\mathfrak{A}\) and \(\tilde{\mathfrak{A}}\). Consider operators \(J \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})_1 \in \mathcal{L}(\hat{\mathcal{H}_1}, \mathcal{H}_1)\). Let moreover the above spaces and operators satisfy the following conditions:

\[
\begin{align*}
\| \mathfrak{J} f - J^1 f \|_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1}, \\
\| \mathfrak{J} u - \tilde{J}^1 u \|_{\tilde{\mathcal{H}}} & \leq \delta \| u \|_{\mathcal{H}_1}, \\
\| (\mathfrak{J} f | u)_{\tilde{\mathcal{H}}} - (f | \tilde{J}^1 u)_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}} \| u \|_{\tilde{\mathcal{H}}}, \\
\| \mathfrak{h}(\mathfrak{J} f | u)_{\tilde{\mathcal{H}}} - \tilde{\mathfrak{h}}(f | \tilde{J}^1 u)_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1} \| u \|_{\tilde{\mathcal{H}}}, \\
\| f - \tilde{J} \mathfrak{J} f \|_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1}, \\
\| u - \tilde{J} \mathfrak{J} u \|_{\tilde{\mathcal{H}}} & \leq \delta \| u \|_{\tilde{\mathcal{H}}}, \\
\| \mathfrak{J} f \|_{\tilde{\mathcal{H}}} & \leq 2 \| f \|_{\mathcal{H}_1}, \\
\| \tilde{J} \mathfrak{J} u \|_{\tilde{\mathcal{H}}} & \leq 2 \| u \|_{\tilde{\mathcal{H}}},
\end{align*}
\]

for some \(\delta > 0\). Then
\[
\| R(\lambda, \tilde{\mathfrak{A}}) - \mathfrak{J} R(\lambda, \mathfrak{A}) \tilde{J} \| \leq M \delta
\]
for some \(M > 0\).

We emphasize that the convergence assertion is rather poor at a numerical level, but fairly strong at a functional analytical level: it states convergence in operator norm, rather than just strong convergence as done e.g. by the various Trotter–Kato-type theorems. We are now in the position to prove Theorem 2.6.

**Proof of Theorem 2.6.** Fix \(n \in \mathbb{N}\). We apply Proposition 2.8 setting
\[
\mathcal{H} := L^2_{Y_n}, \quad \mathcal{H}_1 := V_{Y_n}, \quad \tilde{\mathcal{H}} := V_{Y},
\]
along with
\[
\mathfrak{h} := (a_{S_n}, V_{Y_n}) \quad \text{and} \quad \tilde{\mathfrak{h}} := (a_S, V_Y).
\]
Observe that accretivity of \(\mathfrak{h}, \tilde{\mathfrak{h}}\) follows from accretivity of the operators \(S_n, S\). Define moreover \(J \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})\) by
\[
J f := \begin{pmatrix} J^{1n} \circ f_1 \\ J^{1n} \circ f_2 \end{pmatrix}, \quad f := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H},
\]
and
\[
J u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \tilde{\mathcal{H}},
\]
and
\[
\begin{align*}
\| J f - J^1 f \|_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1}, \\
\| J u - \tilde{J}^1 u \|_{\tilde{\mathcal{H}}} & \leq \delta \| u \|_{\mathcal{H}_1}, \\
\| (J f | u)_{\tilde{\mathcal{H}}} - (f | \tilde{J}^1 u)_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}} \| u \|_{\tilde{\mathcal{H}}}, \\
\| \mathfrak{h}(J f | u)_{\tilde{\mathcal{H}}} - \tilde{\mathfrak{h}}(f | \tilde{J}^1 u)_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1} \| u \|_{\tilde{\mathcal{H}}}, \\
\| f - \tilde{J} J f \|_{\tilde{\mathcal{H}}} & \leq \delta \| f \|_{\mathcal{H}_1}, \\
\| u - \tilde{J} J u \|_{\tilde{\mathcal{H}}} & \leq \delta \| u \|_{\tilde{\mathcal{H}}}, \\
\| J f \|_{\tilde{\mathcal{H}}} & \leq 2 \| f \|_{\mathcal{H}_1}, \\
\| J u \|_{\tilde{\mathcal{H}}} & \leq 2 \| u \|_{\tilde{\mathcal{H}}},
\end{align*}
\]
for some \(\delta > 0\). Then
\[
\| R(\lambda, \tilde{\mathfrak{A}}) - J R(\lambda, \mathfrak{A}) \tilde{J} \| \leq M \delta
\]
for some \(M > 0\).
and moreover \( \mathcal{J}_1 := \mathcal{J} \) and \( \tilde{\mathcal{J}}_1 := \tilde{\mathcal{J}} \).

It is apparent that \((2.9), (2.10)\) and \((2.11)\) are trivially satisfied for \(\delta\) large enough (and getting smaller and smaller as \(n\) increases). Moreover, \((2.7), (2.9)\) and \((2.10)\) hold because \(\mathcal{J}, \tilde{\mathcal{J}}\) are unitary with \(\mathcal{J}^* = \mathcal{J}\).

Finally, observe that by \((2.9), (2.10), (2.11)\) the operators \(\mathcal{J}_1, \tilde{\mathcal{J}}_1\) do not depend on space, hence they commute with the local operators associated with the forms \(\mathfrak{h}, \tilde{\mathfrak{h}}\). Furthermore, for all \(f \in \mathcal{H}_1\) and all \(u \in \mathcal{H}_1\)

\[
(S \mathcal{J}^n f|_\Omega| u|_\Omega)_Y - (S_n f|_\Omega|(\mathcal{J}^n)^{-1} u|_\Omega)_Y = ((S \mathcal{J}^n - \mathcal{J}^n S_n) f|_\Omega| u|_\Omega)_Y,
\]

which converges to 0 because \(\mathcal{J}^n \rightarrow 0\) because \(J\) and elliptic with respect to \(\mathfrak{h}\) for all \(u \in \mathcal{H}\).

Remark 2.9. Let us consider the case of a more general diffusion equation of the form

\[
\left\{\begin{array}{ll}
\frac{\partial}{\partial t} u(t) & \frac{\nabla \cdot (D \nabla u(t))}{\mu} \in \mathcal{Y}, \\
\frac{\partial}{\partial t} u(t)|_\Gamma & = -P_y \frac{\partial \varphi(t)}{\partial \nu} + (\gamma \Delta_\Gamma - \mathcal{S}) u(t)|_\Gamma, \\
u(0) & = u_0, \\
u(0)|_\Gamma & = v_0,
\end{array}\right.
\]

(\(AV_D\))

where \(D \in C^1(\Omega; L(H^n))\) satisfies for some \(\mu > 0\) the following ellipticity condition:

\[
\text{Re}(D(x)|\xi|^2) \geq \mu \|\xi\|_{H^n}^2 \quad \text{for all } \xi \in H^n \text{ and all } x \in \Omega.
\]

The subspace \(\mathcal{Y}\) as well as the operator \(\mathcal{S}\) are now fixed. Then, a variational approach can still be pursued, after introducing suitable weighted Bochner spaces \(L^2_{\mathcal{Y}, D}\) as it has been done in \([34]\). Due to uniform ellipticity, the coefficients do not degenerate on the boundary, yielding that \(\|\cdot\|_{L^2_{\mathcal{Y}, D}}\) and \(\|\cdot\|_{L^2_{\mathcal{Y}}}\) are equivalent norms on \(L^2_{\mathcal{Y}, D}\).

Now, consider a uniformly elliptic family \((D_k)_{k \in \mathbb{N}} \subset C^1(\Omega; L(H^n))\) of coefficients such that \(D_k(x)\) is self-adjoint for all \(x \in \Omega\) and all \(k \in \mathbb{N}\). Consider the sesquilinear form \(a_k\) arising from the problem \((AV_{D_k})\), \(k \in \mathbb{N}\), whose domains all coincide with \(V_\mathcal{Y}\). Denote by \(\Delta_k\) the associated operator. These operators are uniformly sectorial — actually, all their numerical ranges are contained in the negative halfline. If the sequence \((D_k)_{k \in \mathbb{N}}\) converges strongly, then \((a_k(f, f))_{k \in \mathbb{N}}\) is a Cauchy sequence for all \(f \in V_\mathcal{Y}\). Therefore, by a known result due to Kato (see \([32\), §VIII.3]), \((R(\lambda, \Delta_{V_\mathcal{Y}}))_{k \in \mathbb{N}}\) converges strongly for all \(\text{Re}\lambda > 0\). By simple functional calculus arguments this also implies strong convergence of \((e^{t\Delta_k})_{k \in \mathbb{N}}\) for all \(z\) in the open right halfplane. This is comparable with \([21\) Thm. 3.1]. A similar assertion concerning convergence in operator norm can also be obtained applying Proposition 2.8.

3. Lattice-based invariance properties

This section is devoted to the characterization of qualitative properties of \((e^{t\Delta_{\mathcal{Y}, s}})_{t \geq 0}\). These can often be discussed in terms of invariance of relevant subsets of the state space \(L^2\) — most notably, order intervals\(^2\). By Ouhabaz’s well-known invariance criterion, such invariance properties can be characterized by simple, almost linear algebraic properties of a quadratic form. In a more general form presented in \([34\) Thm. 2.1], Ouhabaz’s criterion can be stated as follows.

Lemma 3.1. Let \(\mathcal{H}\) be a separable Hilbert space and \(a\) a sesquilinear form with dense domain \(\mathcal{V}\) that is continuous and elliptic with respect to \(\mathcal{H}\). A closed convex set \(\mathcal{C} \subset \mathcal{H}\) is invariant under the semigroup associated with \(a\) if and only if \(\mathcal{V}\) is invariant under the orthogonal projection \(\mathcal{P}\) of \(\mathcal{H}\) onto \(\mathcal{C}\) and moreover \(\text{Re} a(\mathcal{P}u, u - \mathcal{P}u) \geq 0\) for all \(u \in \mathcal{V}\).

\(^2\)It has been observed in \([19\) §5] that also invariances of some subspaces of the state space often reveal important properties of the evolution equation. In fact, all results in this section also apply when order intervals are replaced by subspace — of course, even dropping all lattice assumptions.
In the remainder of this section assume for simplicity that \( \gamma > 0 \), i.e.,
\[
V_\gamma = \{ f \in H^1(\Omega; H) \times H^1(\Gamma; H) : f|_\Gamma \in \mathcal{Y} \}.
\]
(Still, all assertions hold true in the case \( \gamma = 0 \) with obvious, minor modifications in the proofs).

To warm up, we start by characterising reality of \((e^{t\Delta_{\gamma}})_{t \geq 0}\). A function in \(L^2\) is called \(H_\mathbb{R}\)-valued if it takes values in the real Hilbert space \(H_\mathbb{R}\) underlying \(H\) for a.e. \(x \in \Omega \oplus \Gamma\). As a direct consequence of locality the forms associated with the Laplace and Laplace–Beltrami operators we obtain the following.

**Proposition 3.2.** The semigroup \((e^{t\Delta_{\gamma}})_{t \geq 0}\) leaves invariant the real part of \(L^2\), i.e., the set of all \(H_\mathbb{R}\)-valued functions in \(L^2\), if and only if \((\mathcal{S}\text{Reg}\mathcal{I}m)\_f \in \mathbb{R}\) for all \(f \in \mathcal{Y}\), hence if and only if
\[
\mathcal{S}\{ f \in H^2(\Gamma; H) : f(x) \in H_\mathbb{R} \text{ for a.e. } x \in \Gamma \} \subset \{ f \in L^2(\Gamma; H) : f(x) \in H_\mathbb{R} \text{ for a.e. } x \in \Gamma \}.
\]

In typical applications the space \(H\) is a Hilbert lattice\(^1\) – hence we will assume henceforth that
\[
H \cong L^2(\Xi; C)
\]
for a suitable finite measure space \(\Xi\), cf. [47, Cor. 2.7.5]. In particular, the scalar products of \(L^2(\Omega; \Gamma; H)\) and \(L^2(\Gamma; H)\) read now
\[
(f|g)_{L^2(\Omega; \Gamma; H)} := \int_\Omega \int_\Xi f(x, \xi)g(x, \xi)d\xi dx \quad \text{and} \quad (f|g)_{L^2(\Gamma; H)} := \int_\Gamma \int_\Xi f(x, \xi)g(x, \xi)d\xi dx,
\]
respectively. We can define the positive and negative parts and the absolute value of functions in \(\text{Hilbert lattices. Whenever we refer to an operator on a Hilbert lattice as “positive”, we always mean “positivity preserving”.}

\(^{1}\)For the necessary notions from the theory of Banach lattices we refer to [55, 57]. Consequently, also \(L^2(\Omega; H), \mathcal{Y}\) and \(L^2\) are Hilbert lattices. Whenever we refer to an operator on a Hilbert lattice as “positive”, we always mean “positivity preserving”.  

To illustrate the generalised \(\mathcal{S}\)-regulator expression in \(L^2\), recall that (3.1), (3.2), (3.3), (3.4) represent equalities of functions in \(L^2(\Xi; \mathbb{C})\) and \(L^2(\Gamma; C)\), respectively. In particular, each “slice” \(u(\cdot, \xi)\) defines a scalar-valued function on \(\Omega\): it is the differential of this slice-function that is denoted by \(\nabla u(\cdot, \xi)\). The same is valid for \(v, \tilde{u}, \tilde{v}\).  

Let \(a, b \in L^2(\Omega; H)\) and consider the unbounded order intervals
\[
[a, +\infty)_{L^2(\Omega; H)} := \left\{ f \in L^2(\Omega; H) : a(x) \leq f(x) \text{ for a.e. } x \in \Omega \right\} \cong \left\{ f \in L^2(\Omega \times \Xi; C) : a(x, \xi) \leq f(x, \xi) \text{ for a.e. } (x, \xi) \in \Omega \times \Xi \right\},
\]
\[
(-\infty, b]_{L^2(\Omega; H)} := \left\{ f \in L^2(\Omega; H) : f(x) \leq b(x) \text{ for a.e. } x \in \Omega \right\} \cong \left\{ f \in L^2(\Omega \times \Xi; C) : f(x, \xi) \leq b(x, \xi) \text{ for a.e. } (x, \xi) \in \Omega \times \Xi \right\}.
\]
These subsets of \(L^2(\Omega; H)\) are closed and convex. Similarly, for \(c, d \in L^2(\Gamma; H)\) one considers the unbounded order intervals \([c, +\infty)_\mathcal{Y}, (-\infty, d]_\mathcal{Y}\).

**Lemma 3.3.** Let \(u, v \in H^1(\Omega; H)\) and \(\tilde{u}, \tilde{v} \in H^1(\Gamma; H)\). Then also \(\max\{u, v\} \in H^1(\Omega; H)\) as well as \(\max\{\tilde{u}, \tilde{v}\} \in H^1(\Gamma; H)\). Furthermore,
\[
(3.1) \quad \nabla \max\{u, v\}(\cdot, \xi) = 1\{u(\cdot, \xi) \geq v(\cdot, \xi)\} \nabla u(\cdot, \xi) + 1\{u(\cdot, \xi) < v(\cdot, \xi)\} \nabla v(\cdot, \xi),
\]
\[
(3.2) \quad \nabla(u - v)^-(\cdot, \xi) = 1\{u(\cdot, \xi) < v(\cdot, \xi)\} \nabla v(\cdot, \xi),
\]
\[
(3.3) \quad \nabla \max\{\tilde{u}, \tilde{v}\}(\cdot, \xi) = 1\{\tilde{u}(\cdot, \xi) \geq \tilde{v}(\cdot, \xi)\} \nabla \tilde{u}(\cdot, \xi) + 1\{\tilde{u}(\cdot, \xi) < \tilde{v}(\cdot, \xi)\} \nabla \tilde{v}(\cdot, \xi),
\]
\[
(3.4) \quad \nabla(\tilde{u} - \tilde{v})^-\(\cdot, \xi)) = 1\{\tilde{u}(\cdot, \xi) < \tilde{v}(\cdot, \xi)\} \nabla \tilde{v}(\cdot, \xi),
\]
for a.e. \(\xi \in \Xi\).
Proof. The proof goes in several steps. We will repeatedly use the fact that
\[ u \in L^2(\Omega; H) \cong L^2(\Omega; C) \otimes H \cong L^2(\Omega; C) \otimes L^2(\Xi; C) \cong L^2(\Omega \times \Xi; C) \]
in order to reduce a vector-valued relation to a collection of scalar-valued ones: this follows from the elementary theory of Hilbert tensor products.

1) First of all, we recall the following vector-valued extension of [13, Prop. IX.3], observed in [17, Appendix A]: Let \( G : H \to H \) be a Lipschitz continuous mapping and \( f \in H^1(\Omega; H) \). If \( G(0) = 0 \), then \( G \circ f \in H^1(\Omega; H) \). The proof is an easy modification of [13, Prop. IX.3]. In particular, this result applies to the case where \( G \) is an orthogonal projection onto an order interval
\[
- [\alpha, +\infty)_{L^2(\Omega; H)} := \{ f \in L^2(\Omega; H) : \alpha \leq f(x) \text{ for a.e. } x \in \Omega \} \]
\[
(\alpha, \beta)_{L^2(\Omega; H)} := \{ f \in L^2(\Omega; H) : f(x) \leq \beta \text{ for a.e. } x \in \Omega \},
\]
for \( \alpha, \beta \in H \) with \( -\alpha, \beta \in H_+ \), so that these order intervals actually contain 0. In particular, if \( f \in H^1(\Omega; H) \), then \( f^+, f^- \in H^1(\Omega; H) \).

2) Observe that although \( f^+ \) is formally given by the composition of a Lipschitz continuous mapping on \( H \) and a function in \( H^1(\Omega; H) \), providing a chain rule is not trivial as Rademacher’s theorem fails to hold in infinite dimensional spaces and it is in particular not easy to understand in which sense the orthogonal projection of \( H \) onto \( H_+ \) is “differentiable a.e.”, as one would expect in the finite dimensional case.

To this aim, let \( f \in H^1(\Omega; H) \). By 1), one has in particular and by definition of \( H^1(\Omega; H) \) that
\[
\int_{\Omega} f^+(x) \nabla h(x) dx = \int_{\Omega} \nabla f^+(x) h(x) dx \quad \text{for all } h \in C_0^\infty(\Omega; C)
\]
in the sense of \( H^n \)-valued Bochner integrals. In other words, the above integrals define an element of \( L^2(\Xi, C)^n \). Accordingly,
\[
\left( \int_{\Omega} f^+(x) \nabla h(x) dx \right)(\xi) = - \left( \int_{\Omega} \nabla f^+(x) h(x) dx \right)(\xi) \quad \text{for all } h \in C_0^\infty(\Omega; C) \text{ and a.e. } \xi \in \Xi,
\]
and therefore
\[
\int_{\Omega} f^+(x, \xi) \nabla h(x) dx = - \int_{\Omega} \nabla f^+(x, \xi) h(x) dx \quad \text{for all } h \in C_0^\infty(\Omega; C) \text{ and a.e. } \xi \in \Xi:
\]
this can be checked by first considering step functions and then going to the limit. We deduce that \( f^+(\cdot, \xi) \in H^1(\Omega; C) \) for a.e. \( \xi \in \Xi \). Since this is a scalar-valued function, we can apply the usual differentiation formula and deduce from [29, Lemma 7.6] that
\[
(3.5) \quad \nabla f^+(\cdot, \xi) = 1_{\{f(\cdot, \xi) \geq 0\}} \nabla f(\cdot, \xi) \quad \text{for a.e. } \xi \in \Xi.
\]
Now, because \( f^+ \in H^1(\Omega; H) \), the weak derivative of \( f^+ \) is necessarily given by \( (3.5) \) outside a subset of \( \Omega \times \Xi \) of zero measure.

We emphasize that the characteristic function is defined by means of subsets of \( \Omega \) such that some inequality is satisfied by a scalar-valued function. In fact the two subsets \( \{ f(\cdot, \xi) \geq 0 \}, \{ f(\cdot, \xi) < 0 \} \) define a partition of \( \Omega \) for a.e. \( \xi \in \Xi \).

3) We are now in the position to prove the main assertion. Since \( u - v \in H^1(\Omega; H) \), we deduce from 2) that
\[
\nabla (u - v)^+(\cdot, \xi) = 1_{\{u(\cdot, \xi) \geq v(\cdot, \xi)\}} (\nabla u - \nabla v)(\cdot, \xi), \quad \nabla (u - v)^-(\cdot, \xi) = 1_{\{u(\cdot, \xi) < v(\cdot, \xi)\}} (\nabla u - \nabla v)(\cdot, \xi)
\]
hold for a.e. \( \xi \in \Xi \). Accordingly, both
\[
P_{[\infty, \xi]} u = \min\{u, v\} = u - (u - v)^+ \quad \text{and} \quad P_{[\xi, +\infty)} u = \max\{u, v\} = v + (u - v)^+
\]
belong to \( H^1(\Omega; H) \) and \( (3.1) \) follows.

The remaining assertions are proven likewise.
Theorem 3.4. Let $a \in H^1(\Omega \times \Xi; \mathbb{C})$ be such that $a = (a, a|_{\Gamma}) \in V_Y$ and consider the unbounded order interval

$$[a, +\infty)_{L^2} := [a, \infty)_{L^2(\Omega; H)} \times [a|_{\Gamma}, \infty)_{L^2(\Gamma; H)}$$

$$\cong \left\{ f \in L^2(\Omega \times \Xi; \mathbb{C}) : a(x, \xi) \leq f(x, \xi) \text{ for a.e. } (x, \xi) \in \Omega \times \Xi \right\} \times \left\{ g \in L^2(\Gamma \times \Xi; \mathbb{C}) : a(z, \xi) \leq g(z, \xi) \text{ for a.e. } (z, \xi) \in \Gamma \times \Xi \right\}$$

Then $(e^{t\Delta_{Y,s}})_{t \geq 0}$ leaves invariant $[a, +\infty)_{L^2}$ if and only if

(i) $P_Y[a, +\infty)_{L^2} \subset [a, +\infty)_{L^2}$ and additionally

(ii) the inequality

$$0 \geq \int_{\Xi} \int_{\{a(\cdot, \xi)>f(\cdot, \xi)\}} \nabla a(x, \xi) \overline{(\nabla f - \nabla a)(x, \xi)} \, dx \, d\xi$$

$$+ \gamma \int_{\Xi} \int_{\{a|_{\Gamma}(\cdot, \xi)>f|_{\Gamma}(\cdot, \xi)\}} \nabla a(z, \xi) \overline{(\nabla f(z, \xi) - \nabla a(z, \xi)} \, d\sigma(z) \, d\xi + (S \max\{a|_{\Gamma}, f|_{\Gamma}\} (f|_{\Gamma} - a|_{\Gamma}^-)^-)_{Y}$$

holds for all $f \in H^1(\Omega \times \Xi; \mathbb{R})$ such that $f = (f, f|_{\Gamma}) \in V_Y$.

Proof. By Lemma 3.1 $(e^{t\Delta_{Y,s}})_{t \geq 0}$ leaves invariant the order interval $[a, +\infty)_{L^2}$ if and only if the associated orthogonal projection $P_{[a, +\infty)_{L^2}}$ leaves invariant $V_Y$ and moreover $a(P_{[a, +\infty)_{L^2}} f, f - P_{[a, +\infty)_{L^2}} f) \geq 0$ for all $H^1$-valued $f \in V_Y$. By Lemma 3.3 the first condition is satisfied if and only if $P_{[a, +\infty)_{L^2}} Y \subset Y$. By [45] Lemma 2.3 this is equivalent to $P_{[a, +\infty)_{L^2}} Y \subset [a, +\infty)_{L^2}$.

The second criterion can be deduced applying Lemma 3.3 and observing that for all $f \in V_Y$

$$aS(P_{[a, +\infty)_{L^2}} f, f - P_{[a, +\infty)_{L^2}} f) = -aS(\max\{a, f\}, (f - a)^-)$$

$$= -\int_{\Xi} \left( 1_{\{a>f\}} \nabla a + 1_{\{a\leq f\}} \nabla f \right) (\nabla f - \nabla a) \, d\sigma(x)$$

$$- \gamma \int_{\Gamma} \left( 1_{\{a|_{\Gamma}>f|_{\Gamma}\}} \nabla a|_{\Gamma} + 1_{\{a|_{\Gamma} \leq f|_{\Gamma}\}} \nabla f|_{\Gamma} - \nabla a|_{\Gamma} \right) \, d\sigma(z)$$

$$- \gamma \int_{\Gamma} \left( 1_{\{a|_{\Gamma} \leq f|_{\Gamma}\}} \nabla f|_{\Gamma} - \nabla a|_{\Gamma} \right) \, d\sigma(z)$$

$$- \left( S \max\{a|_{\Gamma}, f|_{\Gamma}\} (f|_{\Gamma} - a|_{\Gamma}^-)^- \right)_{Y}$$

$$= -\int_{\Xi} \int_{\{a(\cdot, \xi)>f(\cdot, \xi)\}} \nabla a(x, \xi) \overline{(\nabla f - \nabla a)(x, \xi)} \, dx \, d\xi$$

$$- \gamma \int_{\Xi} \int_{\{a|_{\Gamma}(\cdot, \xi)>f|_{\Gamma}(\cdot, \xi)\}} \nabla a|_{\Gamma}(z, \xi) \overline{(\nabla f(z, \xi) - \nabla a(z, \xi)} \, d\sigma(z)$$

$$- \gamma \int_{\Xi} \int_{\{a|_{\Gamma}(\cdot, \xi) \leq f|_{\Gamma}(\cdot, \xi)\}} \nabla f(z, \xi) \, d\sigma(z)$$

$$- \left( S \max\{a|_{\Gamma}, f|_{\Gamma}\} (f|_{\Gamma} - a|_{\Gamma}^-)^- \right)_{Y}.$$
Applying Fubini’s theorem we obtain

\[
a_S(P_{[a, +\infty)}\mathcal{L}_2 f, f - P_{[a, +\infty)}\mathcal{L}_2 f) = -\int_{\Xi} \int_{\{a(\cdot, \xi) > f(\cdot, \xi)\}} \nabla a(x, \xi)(\nabla f - \nabla a)(x, \xi)dx \, d\xi
- \gamma \int_{\Xi} \int_{\{a_{\Gamma}(\cdot, \xi) > f_{\Gamma}(\cdot, \xi)\}} \nabla a(z, \xi)(\nabla f(z, \xi) - \nabla a(z, \xi))d\sigma(z) \, d\xi
- \left(S \max\{a_{\Gamma}, f_{\Gamma}\} \right)^{\gamma} f_{\Gamma} - a_{\Gamma})^\gamma.\]

This concludes the proof. \(\square\)

Analogous assertions hold for the order intervals \((-\infty, b]_{2}\).

In general, condition (ii) in Theorem 3.4 will rarely be satisfied. An easy, yet relevant special case is clearly that of constant \(a\), i.e., \(a(x, \xi) \equiv \alpha\) for some \(\alpha \in H\) and a.e. \((x, \xi) \in \Omega \times \Xi\). In this case, condition (ii) reduces to the condition

\[(3.6) \quad (S \max\{a_{\Gamma}, f_{\Gamma}\})^\gamma (f_{\Gamma} - a_{\Gamma})^\gamma Y \leq 0.\]

Observe that if in addition \(S \in \mathcal{L}(L^2(\Gamma; H))\), then by Lemma 3.1 the validity of condition (ii) in Theorem 3.4 is equivalent to the invariance of \([a, +\infty)_{L^2(\Gamma; H)}\) under the semigroup generated by \(S\). E.g., positivity of the semigroup corresponds to invariance of \([0, \infty)_{\mathcal{L}^2}\), while \(\mathcal{L}^\infty\)-contractivity\(^4\) can be formulated in terms of simultaneous invariance of both order intervals \([-1, \infty)_{\mathcal{L}^2}, (-\infty, 1]_{\mathcal{L}^2}\).

For the sake of further reference we introduce the following locality assumptions.

**Assumptions 3.5.** There exist a closed subspace \(Y\) of \(H\), a closed convex subset \(C_H\) of \(H\) and an operator \(S \in \mathcal{L}(H)\) such that

- \(Y = \{f \in L^2(\Gamma; H) : f(z) \in Y\text{ for a.e. } z \in \Gamma\}\),
- \(C_{L^2(\Omega; H)} := \{f \in L^2(\Omega; H) : f(x) \in C_H\text{ for a.e. } x \in \Omega\}\),
- \(C_Y := \{f \in Y : f(z) \in C_H\text{ for a.e. } z \in \Gamma\}\),
- \(C_{\mathcal{L}^2} := C_{L^2(\Omega; H)} \times C_Y\) and
- \(\mathcal{S}_Y = S \circ g\) for all \(g \in H^2(\Gamma; H)\).

Moreover, 0 is in \(C\) or else both \(\Omega\) and \(\Gamma\) have finite measure.

Observe that under Assumptions 3.5 the abstract problem (AV) becomes a parabolic problem with dynamic boundary condition

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= \Delta u(t, x), & t \geq 0, x \in \Omega, \\
u(t, z) &\in Y, & t \geq 0, z \in \Gamma, \\
\Delta u(t, z) &= P_Y \left( -\Delta u(t, z) + (\gamma \Delta u - S) u(t, z) \right), & t \geq 0, z \in \Gamma, \\
u(0, x) &= u_0(x), & x \in \Omega, \\
u(0, z) &= v_0(z), & z \in \Gamma.
\end{align*}
\]

If \(\Omega = (0, \infty) \times \mathbb{R}^{n-1}\), it is common in the literature to refer to this problem as “diffusion on an open book” (with dynamic boundary conditions). If \(n = 1\), this is nothing but the semi-infinite star considered in Example 1.1.

Under the Assumptions 3.5, \(Y\) is a closed subspace of \(L^2(\Gamma; H)\) and \(C_{L^2(\Omega; H)}, C_Y, C_{\mathcal{L}^2}\) are closed and convex subsets of \(L^2(\Omega; H), Y\), and \(\mathcal{L}^2\), respectively. With an abuse of notation we then write \(Y := Y, S := S\) and \(\Delta_{Y,S}\) instead of \(\Delta_{Y,S}\).

\(^4\) By this we mean contractivity with respect to the norm of \(L^\infty(\Omega \times \Xi; \mathbb{C}) \times L^\infty(\Gamma \times \Xi; \mathbb{C})\).
It is crucial that whenever Assumptions 3.3 hold the orthogonal projections of \( L^2(\Omega; H) \) onto \( P_{L^2(\Omega; H)} \), of \( \mathcal{Y} \) onto \( P_{\mathcal{Y}} \) and hence of \( \mathcal{L}^2 \) onto \( C_{\mathcal{L}^2} \) satisfy
\[
\begin{align*}
P_{C_{L^2(\Omega; H)}} f &= P_{C_H} \circ f \quad \text{for all } f \in L^2(\Omega; H), \\
P_{\mathcal{Y}} g &= P_{C_H} \circ g \quad \text{for all } g \in \mathcal{Y}, \\
P_{C_{\mathcal{L}^2}} f &= \left( P_{C_H} \circ f \right) \left( P_{C_H} \circ g \right) \quad \text{for all } f := \left( g \right) \in \mathcal{L}^2.
\end{align*}
\]

The fact that the projections onto the above subsets of vector-valued function spaces are the compositions of a Lipschitz continuous mapping (namely, the projection \( P_{C_H} \)) and a function of class \( H^1 \) permits to apply the version of a chain rule obtained in Lemma 3.3. Furthermore, due to the local structure of the sets \( C_{L^2(\Omega; H)} \) and \( C_{\mathcal{Y}} \), one sees that in particular
\[
(P_{C_{L^2(\Omega; H)}} \circ f)|_{\Gamma} = (P_{\mathcal{Y}} \circ f|_{\Gamma}) \quad \text{for all } f \in \mathcal{Y}.
\]

**Theorem 3.6.** Impose the Assumptions 3.3. Then \( C_{\mathcal{L}^2} \) is left invariant under \( (e^{t\Delta_{\mathcal{Y}-s}})_{t \geq 0} \) if and only if
(i) the inclusion \( P_{\mathcal{Y}} C_H \subset C_H \) holds and additionally
(ii) the semigroup \( (e^{-tS})_{t \geq 0} \) leaves \( C_H \) invariant.

Comparable results have been obtained in the context of networks in [36, 34].

**Proof.** First of all, we show that the inclusion \( P_{C_{\mathcal{L}^2}} \mathcal{Y} \subset \mathcal{Y} \) holds if and only if the inclusion \( P_{\mathcal{Y}} C_{\mathcal{L}^2} \subset \mathcal{Y} \). Orthogonal projections onto closed convex subsets of a Hilbert space are Lipschitz continuous mappings, hence as already observed by [17, Lemma 7.3] \( P_{C_{\mathcal{L}^2}} \) maps \( H^1(\Omega; H) \times H^1(\Gamma; H) \) into itself – i.e., the weak differentiability conditions is satisfied independently of the boundary conditions. Consequently, \( P_{C_{\mathcal{L}^2}} \mathcal{Y} \subset \mathcal{Y} \) if and only if \( f_{\mathcal{Y}} \in \mathcal{Y} \) implies \( P_{\mathcal{Y}} f_{\mathcal{Y}} \in \mathcal{Y} \), for all \( f \in H^1(\Omega; H) \). The proof can be completed reasoning as in [17] Prop. 4.2.

By Lemma 3.1, invariance of \( \mathcal{Y} \) under \( (e^{t\Delta_{\mathcal{Y}-s}})_{t \geq 0} \) is now equivalent to \( P_{\mathcal{Y}} C_{\mathcal{L}^2} \subset \mathcal{Y} \) and
\[
\text{Re}(S(P_{C_{\mathcal{L}^2}} f, (I - P_{C_{\mathcal{L}^2}}) f)) \geq 0 \quad \text{for all } f \in \mathcal{Y}.
\]

Due to locality of the forms associated with the Laplacian on \( \Omega \) and the Laplace–Beltrami operator on \( \Gamma \) (and hence of the form \( (a_S, V_{\mathcal{Y}}) \)), a direct computation shows that
\[
\text{Re}(S(P_{C_{\mathcal{L}^2}} f, (I - P_{C_{\mathcal{L}^2}}) f)) = \text{Re}(SP_{\mathcal{Y}} f|_{\Gamma})(I - P_{\mathcal{Y}} f|_{\Gamma})_{\mathcal{Y}}.
\]
By density, the latter term is \( \geq 0 \) for all \( f \in \mathcal{Y} \) if and only if
\[
\text{Re}(SP_{\mathcal{Y}} g)(I - P_{\mathcal{Y}} g)_{\mathcal{Y}} \geq 0 \quad \text{for all } g \in \mathcal{Y}.
\]
By a localisation argument this is equivalent to asking that
\[
\text{Re}(SP_{\mathcal{H}} x)(I - P_{\mathcal{H}} x)_{H} \geq 0 \quad \text{for all } x \in H.
\]
A further application of Lemma 3.1 concludes the proof, since \( (S \cdot | \cdot)_H \) is the form associated with \( -S \).

In the previous theorem, it is not too restrictive to consider sets of the form \( C_{L^2(\Omega; H)} \times C_{L^2(\Gamma; H)} \) – i.e., to restrict ourselves to study invariance of sets of those functions pointwise belonging to the same subset of \( H \), both on \( \Omega \) and on the boundary \( \Gamma \). In fact, the following holds.

**Proposition 3.7.** Let \( C, D \subset H \) be closed convex subsets. If \( C_{L^2(\Omega; H)} \times D_{L^2(\Gamma; H)} \) is invariant under \( (e^{t\Delta_{\mathcal{Y}-s}})_{t \geq 0} \), then \( C = D \).

**Proof.** We only consider the case of \( \Omega, \Gamma \) with bounded measure. The general case will then follow by localisation arguments. Let first \( C \nsubseteq D \), say \( v \in C \setminus D \). Take \( f \in \mathcal{Y} \) such that \( f = 1_{\Omega} \otimes v - \text{i.e., } f \equiv v \) then \( f \in C_{L^2(\Omega; H)} \) and \( f|_{\Gamma} = 1_{\Gamma} \otimes v \notin D_{L^2(\Gamma; H)} \). Then
\[
P_{C_{L^2(\Omega; H)} \times D_{L^2(\Gamma; H)}} f = \begin{pmatrix} 1 \otimes v \\ 1 \otimes P_{D}v \end{pmatrix}.
\]
We mention that domination of semigroups can also be discussed. E.g., the following can be shown mimicking the proof of [59, Cor. 2.22]. This results extends [11, Prop. 2.8] and [55, Prop. 4.2].

**Proposition 3.8.** Impose the Assumptions [55] and let $P_2$ be a positive operator. Let $S_1, S_2$ be $L^\infty(\Gamma; L^\infty(H))$-functions. Define operators $S_1, S_2$ by

$$S_1g = S_1 \circ g \quad \text{and} \quad S_2g = S_2 \circ g \quad \text{for all} \quad g \in H^{1/2}(\Gamma; H).$$

Consider two sesquilinear forms $a_1, a_2$ defined by

$$a_1(f, g) := \int_\Omega (\nabla f \cdot \nabla g) \, dx + \int_\Gamma (\nabla f(z) \cdot \nabla g(z)) \, d\sigma(z) + (S_1 f | r | g |_\Gamma)_Y$$

and

$$a_2(f, g) := \int_\Omega (\nabla f \cdot \nabla g) \, dx + \int_\Gamma (\nabla f(z) \cdot \nabla g(z)) \, d\sigma(z) + (S_2 f | r | g |_\Gamma)_Y,$$

both defined on $V_2$, and the associated operators $\Delta_{Y,S_1}, \Delta_{Y,S_2}$. Then the following assertions hold.

1. The semigroup $(e^{t \Delta_{Y,S_1}})_{t \geq 0}$ is dominated by $(e^{t \Delta_{Y,S_2}})_{t \geq 0}$, i.e.,

$$|e^{t \Delta_{Y,S_1}} f(x, \xi)| \leq e^{t \Delta_{Y,S_2}} |f(x, \xi)|, \quad t \geq 0, \quad f \in L^2(\Omega \times \Xi; \mathbb{C}), \quad x \in \Omega, \quad \xi \in \Xi,$$

if and only if

$$\Re (S_1 f | r | g |_\Gamma)_Y \geq (S_2 f | r | g |_\Gamma)_Y$$

for all $u, v \in V_2$ such that $u \overline{\sigma} \geq 0$.

2. Let $S_1(z), S_2(z)$ be positive operators for a.e. $z \in \Gamma$. Then the semigroup $(e^{t \Delta_{Y,S_1}})_{t \geq 0}$ is dominated by $(e^{t \Delta_{Y,S_2}})_{t \geq 0}$ if and only if $S_1(z) - S_2(z)$ is a positive operator for a.e. $z \in \Gamma$.

**Remark 3.9.** In the usual theory of semigroup domination, both the dominating and the dominated semigroup have to act on the same space, or else one of them has to act on a space of scalar-valued functions, see [15] and references therein. This rules out several interesting cases in our context, due to the fact the boundary conditions also determine the state space – and hence semigroups governing equations with different boundary conditions cannot be compared. E.g., it would be natural to expect that all semigroups $(e^{t \Delta_{Y,s}})$ dominate the semigroup that governs the heat equation with (uncoupled) Dirichlet boundary conditions, provided that condition (3.6) holds.

While it is known that many relevant properties are shared by the heat equation with either non-dynamic or dynamic boundary conditions, to the best of our knowledge a structural relation between these phenomena had not yet been observed. The following is a straightforward consequence of Theorem 3.9 and [17, Prop. 4.3].

**Corollary 3.10.** Impose the Assumptions [55]. Then $C_{L^2}$ is left invariant under $(e^{t \Delta_{Y,s}})_{t \geq 0}$ if and only if $C_{L^2(\Gamma; H)}$ is left invariant under the semigroup governing the parabolic problem

$$(\text{NDBC})$$

$$\begin{align*}
\frac{\partial u(t)}{\partial t} &= \Delta u(t), \quad t \geq 0, \\
\frac{\partial u(t)}{\partial \nu} + S u(t) &\in Y, \quad t \geq 0, \\
u(t) &= u_0,
\end{align*}$$

with time-independent boundary conditions.

Observe that the semigroup governing (NDBC) is generated by the operator associated with $a_S$ (with $\gamma = 0$), but considered as a sesquilinear form acting on the Hilbert space $\{ f \in H^1(\Omega; H) : f|\Gamma \in Y \} \hookrightarrow L^2(\Omega; H)$ rather than $V_2 \hookrightarrow L^2$, cf. [17].
Example 3.11. As shown in [27, 9], remarkable properties of the (scalar-valued) heat equation with Wentzell–Robin (dynamic) boundary conditions include positivity and contractivity with respect to the $\infty$-norm of the semigroup that governs it. In the light of Corollary 3.7, these properties actually follow from the same properties enjoyed by the heat equation with corresponding Robin (time-independent) boundary conditions.

Observe in particular that

$$L^p := L^p(\Omega; H) \times (L^p(\Gamma; H) \cap \mathcal{Y}), \quad p \in [1, \infty],$$

are Bochner spaces with respect to a suitable product measure. Assume both $(e^{t\Delta_{Y,s}})_{t \geq 0}$ and its adjoint to be $L^\infty$-contractive: under the Assumptions 3.5 this can be characterized by means of Theorem 3.6 with $C_H = (-\infty, 1]_H \cap [1, \infty)_H$.

Corollary 3.12. Assume both $(e^{t\Delta_{Y,s}})_{t \geq 0}$ and its adjoint to be $L^\infty$-contractive and let $n \geq 2$. Then $(e^{t\Delta_{Y,s}})_{t \geq 0}$ extrapolates to a consistent family of operator semigroups on $L^p$, $p \in [1, \infty]$. These semigroups are strongly continuous and analytic for $p \in (1, \infty)$.

Moreover, $(e^{t\Delta_{Y,s}})_{t \geq 0}$ is ultracontractive, i.e., it satisfies the estimate

$$\|e^{t\Delta_{Y,s}}f\|_{L^\infty} \leq M_\mu t^{-\frac{n}{2}} \|f\|_{L^2} \quad \text{for all } t \in (0, 1], \ f \in L^2$$

where

$$\mu \in \begin{cases} \lfloor n-1, 1 \rfloor, & \text{if } n \geq 3, \\ (1, \infty), & \text{if } n = 2, \end{cases}$$

and some constant $M_\mu$. The same estimates are satisfied by the dual semigroup.

Additional conditions ensuring strong continuity for $p = 1$ are known, cf. [6, §7.2.1] for the scalar case.

Proof. The assertion on extrapolation follows applying a vector-valued version of Riesz–Thorin’s interpolation theorem, cf. [3] p. 77. The second assertion can be proved as in the scalar-valued case, applying a known characterisation of ultracontractivity (see [1] §12.2) based on standard Sobolev embeddings, cf. [54, Lemma 3.8]. It can be easily seen that all the involved techniques carry over to the vector-valued case. \hfill $\square$

Remarks 3.13. 1) By [52, Lemma 3.3], $(e^{t\Delta_{Y,s}})_{t \geq 0}$ consists of kernel operators for all $t > 0$.

2) It is remarkable that the above mentioned criterion for ultracontractivity based on Sobolev embeddings only applies if $n > 1$. In the scalar case, a common workaround is to deduce ultracontractivity from the Nash inequality. Unfortunately, the Nash inequality seems to extend to the vector-valued case only if the space $H$ is finite dimensional. This is why we are not able to prove the above result in the case of $n = 1$ – which in particular corresponds to the relevant case of networks with infinitely many edges.

A semigroup on an $L^2$-space is said to be irreducible if the only closed ideals of $L^2$ left invariant under the semigroup are the trivial ones. If $Y$ is a closed ideal of $H$, then clearly $(e^{t\Delta_{Y,s}})_{t \geq 0}$ leaves invariant $L^2(\Omega; Y) \times L^2(\Gamma; Y)$, which is a closed ideal of $L^2$. Thus, uncoupled boundary conditions jeopardize irreducibility.

More generally, we observe that if $\mathcal{P}: \Omega \to L(H)$ is a strongly measurable function such that $\mathcal{P}(x)$ is an orthogonal projection onto a closed ideal of $H$ for a.e. $x \in \Omega$, then the subspace

$$I_\mathcal{P} := \{ f \in L^2(\Omega; H) : f(x) \in \text{Range} \mathcal{P}(x) \text{ for a.e. } x \in \Omega \}$$

is a closed ideal of $L^2(\Omega; H)$, too. In fact, all closed ideals of $L^2(\Omega; H)$ are of this form, as it is proven in [18]. Similarly, if the Assumptions 3.5 hold one can see that each closed ideals of $L^2$ is the range of an operator-valued strongly measurable mapping $\mathcal{P}$ defined on the product measure space $\Omega \oplus \Gamma$ and such that

- $\mathcal{P}(x)$ is an orthogonal projection onto a closed ideal of $H$ for a.e. $x \in \Omega$ and
- $\mathcal{P}(z)$ is an orthogonal projection onto a closed ideal of $Y$ for a.e. $z \in \Gamma$.

Proposition 3.14. Impose the Assumptions 3.5. Then $(e^{t\Delta_{Y,s}})_{t \geq 0}$ is irreducible if and only if $\mathcal{P}_Y$ is irreducible and $\Omega$ is connected.
Observe that in the scalar case $H = \mathbb{C}$ the orthogonal projections on both subspaces of $H$ are irreducible.

**Proof.** It is clear that the semigroup is not irreducible if $\Omega$ is unconnected, since it lets invariant the closed ideals consisting of those functions supported in any of the connected components.

Let now $P_Y$ be non-irreducible, i.e., consider a non-trivial closed ideal $J_Y$ of $H$ such that $P_Y J_Y \subset J_Y$. Then by Theorem 3.6, we conclude that $J_{L^2(\Omega; H)} \times J_Y$ is a closed ideal of $L^2$ that is left invariant under the semigroup, i.e., $(e^{t\Delta Y \sigma})_{t \geq 0}$ is not irreducible.

Let conversely $(e^{t\Delta Y \sigma})_{t \geq 0}$ be non-irreducible. Then there exists a non-trivial closed ideal of $L^2$ that is invariant under $(e^{t\Delta Y \sigma})_{t \geq 0}$. By Proposition 3.7, such an ideal is necessarily of the form $C^1_{L^2(\Omega; H)} \times C^1_{L^2(\Omega; H)}$. Now, we can apply Theorem 3.6 and deduce the claim. \hfill \Box

**Remark 3.15.** In the scalar case, it is known that irreducibility is equivalent to a strong parabolic maximum principle, provided that the semigroup is positive, cf. \[39\] [2.2] – but this characterisation fails to hold in the general vector-valued case. E.g., the heat semigroup $(e^{t\Delta})_{t \geq 0}$ on $L^2(\mathbb{R}; \mathbb{R}^2)$ is not irreducible because $L^2(\mathbb{R}; \mathbb{R} \times \{0\})$ is a non-trivial closed ideal left invariant under the semigroup. However, it does map nonzero positive functions $f$ to functions $e^{t\Delta} f$ satisfying $e^{t\Delta} f(x) > f(x)$ for all $t > 0$ and a.e. $x \in \mathbb{R}$.

4. AN EXAMPLE: DIFFUSION ON A STAR-SHAPED NETWORK

Throughout this section we consider the setting presented in Example 3.1. Observe that the Assumptions are satisfied whenever we discuss invariance of a set $C_H$ which is either a subspace or an order interval containing 0. We are going to present some interesting behaviour even in this elementary setting. Actually, same properties hold for more general diffusion on domains, rather than intervals. Also, by Corollary 3.10 all the results in this section hold for the semigroups governing (NDBC) and (DBC) alike. Thus, we explicitly refer to the case of time-independent boundary conditions only.

It has been proved in [17] §5 that the semigroup governing (NDBC) is positive if $Y = \{1\}$ (i.e., under so-called Kirchhoff boundary conditions) and not positive if $Y = \{1\}^\perp$ (i.e., under so-called anti-Kirchhoff boundary conditions) as considered e.g. in [38] [28] [61] [2]), provided that $-S$ generates a positive semigroup on $Y$ (i.e., $-S$ is a real matrix with positive off-diagonal entries).

Similarly, assume that $-S$ generates an $L^\infty$-contractive semigroup on $Y$ and that $H = \mathbb{C}^N$. Then by [49] Lemma 6.1], this can be characterized by the fact that the entries $s_{ij}$ of $S$ satisfy

$$\sum_{j \neq i} |s_{ij}| \leq \text{Res}_{ii} \quad \text{for all } i,$$

cf. also [17] Rem. 3.8.(2)]. Then one can prove that the heat semigroup is $L^\infty$-contractive under Kirchhoff boundary conditions for all $N \in \mathbb{N}$, whereas in the anti-Kirchhoff case it is $L^\infty$-contractive if and only if $N = 2$.

For the sake of simplicity, in the remainder of this section we let $S = 0$.

A semi-infinite star with two edges can be identified with a line. More precisely, up to the canonical isometric isomorphism $U$ defined by

$$(Uf)(x) := \begin{pmatrix} f(x) \\ f(-x) \end{pmatrix}, \quad x \geq 0,$$

functions in $L^2(\mathbb{R}; \mathbb{C})$ and in $L^2((0, +\infty); \mathbb{C}^2)$ may be identified. Accordingly, a function $(f_1, f_2) \in L^2((0, +\infty); \mathbb{C}^2)$ is called even (resp., odd) if $f_1(x) = f_2(x)$ (resp., if $f_1(x) + f_2(x) = 0$) for a.e. $x \in (0, +\infty)$. More generally, we call a function $f \in L^2(\Omega; \mathbb{C}^N)$ even (resp., odd) if $f(x) \in \{1\}$ (resp., if $f(x) \in \{1^\perp\}$) for a.e. $x \in \Omega$.

By Theorem 3.6 both the diffusion semigroups with Kirchhoff (i.e., $Y = \{1\}$) and anti-Kirchhoff (i.e., $Y = \{1^\perp\}$) boundary conditions leave invariant the set of even functions as well as the set of odd ones. If $N = 2$, then it is easy to see that these are in fact the only boundary conditions leading to invariance of any of these both sets.

\footnote{I.e., $e^{t\Delta} f(x)$ is a nonzero, positive vector of $\mathbb{R}^2$.}
Now, consider a semi-infinite star with only two edges, i.e., $H = \mathbb{C}^2$. Neglecting the trivial (uncoupled) boundary conditions defined by $Y = \{0\}$ and $Y = \mathbb{C}^2$ we can consider all 1-dimensional subspaces $Y \equiv Y_\xi$ of $\mathbb{C}^2$ by means of the parametrisation

$$P_{Y_\xi} := \begin{pmatrix} \cos^2 \xi & \sin \xi \cos \xi \\ \sin \xi \cos \xi & \sin^2 \xi \end{pmatrix}, \quad \xi \in [0, \pi),$$

where $Y_\xi$ denotes the range of the orthogonal projection $P_{Y_\xi}$. Observe that $\xi = 0$, $\xi = \frac{\pi}{4}$, $\xi = \frac{\pi}{2}$ and $\xi = \frac{3\pi}{4}$ correspond to uncoupled Dirichlet/Neumann, to Kirchhoff, to uncoupled Neumann/Dirichlet and to anti-Kirchhoff boundary conditions, respectively, as can be checked directly.

We are going to discuss the submarkovian property of the semigroup associated with these subspaces in dependence of $\xi$. A direct computation shows that the semigroup $(e^{t\Delta_{Y_\xi}})_{t \geq 0}$ is positive if and only if $\xi \in [0, \frac{\pi}{4}]$. Furthermore, by Theorem 3.6 the semigroup that governs (NDBC) is $L^\infty(\Omega \times \Xi; \mathbb{C})$-contractive if and only if $P_{Y_\xi}$ is $L^\infty(\Xi; \mathbb{C})$-contractive, i.e., if and only if the inequalities

$$\cos^2 \xi + |\sin \xi \cos \xi| \leq 1 \quad \text{and} \quad |\sin \xi \cos \xi| + \sin^2 \xi \leq 1$$

hold simultaneously. The former (resp., the latter) inequality holds if and only if $\xi \notin (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$ (resp., if and only if $\xi \notin \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$).

![Figure 1](image_url)

Therefore, the $L^\infty$-contractivity of the semigroup associated with Kirchhoff boundary conditions represents a singularity. In particular, a submarkovian semigroup is generated exactly in the following five cases:

- with uncoupled Dirichlet/Dirichlet boundary conditions,
- with uncoupled Neumann/Neumann boundary conditions,
- with uncoupled Dirichlet/Neumann boundary conditions,
- with uncoupled Neumann/Dirichlet boundary conditions and finally
- with Kirchhoff boundary conditions.

Similarly, we can consider general boundary conditions defined by 1-dimensional subspaces of $H$ for a semi-infinite star with 3 edges ($H = \mathbb{C}^3$). They can be investigated by means of spherical boundary conditions, i.e., considering spaces $Y \equiv Y_{\xi, \phi}$ that are ranges of the orthogonal projections

$$P_Y \equiv P_{Y_{\xi, \phi}} = \begin{pmatrix} \sin^2 \xi \cos^2 \phi & \sin^2 \xi \sin \phi \cos \phi & \sin \xi \cos \xi \cos \phi \\ \sin \xi \cos \xi \cos \phi & \sin \xi \cos \xi \sin \phi & \sin^2 \xi \sin \phi \\ \sin \xi \cos \xi \sin \phi & \sin \xi \cos \xi \cos \phi & \cos^2 \xi \end{pmatrix}, \quad \xi, \phi \in [0, 2\pi).$$

Analysing the behaviour of $P_{Y_{\xi, \phi}}$ in dependence of $\xi, \phi$ as done above for $P_{Y_\xi}$ is less elementary. While the matrix is clearly positive if and only if $\xi, \phi \in [0, \frac{\pi}{4}] \cup [\pi, \frac{3\pi}{4}]$, it is not clear how to determine all the values $\xi, \phi$ leading to $L^\infty$-contractivity, i.e., all the values $\xi, \phi$ such that the three functions

$$\sin^2 \xi \cos^2 \phi + |\sin^2 \xi \sin \phi \cos \phi| + |\sin \xi \cos \xi \cos \phi|,$$

$$|\sin^2 \xi \sin \phi \cos \phi| + \sin^2 \xi \sin^2 \phi + |\sin \xi \cos \xi \sin \phi| \quad \text{and} \quad \xi, \phi \in [0, \pi)$$

are simultaneously $\leq 1$, corresponding to the three conditions for $L^\infty$-contractivity associated with the three rows of the matrix $P_{Y_{\xi, \phi}}$ in (4.1).
In Figure 2A we have plotted the level lines of the above functions for the value 1 (in violet, blue and red, respectively). This suggests that the ten parameter choices

- \((\xi, \phi) = (\arctan \sqrt{2}, \frac{\pi}{4})\),
- \((\xi, \phi) = (\pi - \arctan \sqrt{2}, \frac{\pi}{4})\),
- \((\xi, \phi) = (\arctan \sqrt{2}, \frac{3\pi}{4})\),
- \((\xi, \phi) = (\pi - \arctan \sqrt{2}, \frac{3\pi}{4})\),
- \((\xi, \phi) = (\frac{\pi}{4}, \frac{\pi}{2})\),
- \((\xi, \phi) = (\frac{3\pi}{4}, \frac{\pi}{2})\),
- \((\xi, \phi) = (\frac{\pi}{2}, \frac{\pi}{4})\),
- \((\xi, \phi) = (\frac{\pi}{2}, \frac{3\pi}{4})\),

lead to an \(L^\infty\)-contractive semigroup – as in fact can be checked directly.

Observe that \(Y_{\xi, \phi}\) identifies Kirchhoff boundary conditions if and only if \(P_{Y_{\xi, \phi}} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\), i.e., if and only if \(\xi = \arctan \sqrt{2}\) and \(\phi = \frac{\pi}{4}\). A direct computation shows that the remaining nine cases correspond to boundary conditions defined by means of spaces \(Y\) given by

\[
\begin{align*}
\{(c, -c, -c) : c \in \mathbb{C}\}, & \quad \{(c, c, -c) : c \in \mathbb{C}\}, & \quad \{(c, -c, c) : c \in \mathbb{C}\}, \\
\{(c, c, 0) : c \in \mathbb{C}\}, & \quad \{(c, c, 0) : c \in \mathbb{C}\}, & \quad \{(0, c, c) : c \in \mathbb{C}\}, \\
\{(0, 0, c) : c \in \mathbb{C}\}, & \quad \{(c, 0, c) : c \in \mathbb{C}\}, & \quad \{(c, 0, -c) : c \in \mathbb{C}\}.
\end{align*}
\]

While the last six subspaces only describe some decoupling of any of the three edges, we cannot find any physical interpretation for the first three boundary conditions. One can see that analogous boundary conditions give rise to \(L^\infty\)-contractive semigroups also in higher dimensional spaces \(H = \mathbb{C}^N\) for any \(N \in \mathbb{N}\).

It ought to be remarked that not all relevant values become evident through the above plot: one can see that decoupled boundary conditions arise with \(Y_{0, \phi}\) and \(Y_{\pi, \phi}\) for all \(\phi \in [0, \pi]\) as well as with \(Y_{\xi, 0}\) and \(Y_{\xi, \pi}\) for all \(\xi \in [0, \pi]\). Hence, using again the computations performed in the case of \(H = \mathbb{C}^2\), we see that \(Y_{\pi, \phi}\) lead to \(L^\infty\)-contractivity for \(\phi \in \{0, \frac{\pi}{4}, \frac{3\pi}{4}\}\), and so do \(Y_{\xi, 0}\) and \(Y_{\xi, \pi}\) for \(\xi \in \{0, \frac{\pi}{4}, \frac{3\pi}{4}\}\) as well as \(Y_{0, \phi}\) for all \(\phi \in [0, \pi]\). We do not know whether further pairs \((\xi, \phi)\) leading to \(L^\infty\)-contractivity exist.

\footnote{The figure has been obtained using Gnuplot 4.2 with a grid density of 1000 on both axes. For reference we have plotted the level lines for the value 0.5, too. On the \(\xi\)-axis (horizontal) we have highlighted the values \(\frac{\pi}{4}\), \(\arctan \sqrt{2}\), \(\frac{3\pi}{4}\) and \(\pi - \arctan \sqrt{2}\). On the \(\phi\)-axis (vertical) we have highlighted the values \(\frac{\pi}{4}\), \(\frac{\pi}{2}\) and \(\frac{3\pi}{4}\).}
Moreover, a straightforward computation shows that \( P_{Y,\gamma} \) is a positive matrix if and only if \((\xi, \phi) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]\). Again, Kirchhoff boundary conditions are a singularity in a "sea" of non-submarkovian behaviours.

A similar procedure identifies all the 2-dimensional subspaces of \( C^3 \), i.e., all ranges of the matrices \( \text{Id} - P_{Y,\gamma,\phi} \), \( \xi, \phi \in [0, 2\pi] \). However, plotting the level lines of the corresponding three functions (as we have done in Figure 2B in violet, blue and red, respectively), does not suggest any new pairs \((\xi, \phi)\) that lead to \( L^\infty\)-contractivity.

The general case of a semi-infinite star with arbitrarily (finitely) many edges can be treated likewise, using known formulae for hyperspherical coordinates.

As already remarked, the above results carry over to case of dynamic boundary conditions and should be compared with the known properties of the heat equation with Wentzell–Robin boundary conditions in the scalar case, cf. \([10, 55]\) and references therein.

5. Dynamic boundary conditions on the normal derivative

In this section we consider a different setting by discussing a new kind of dynamics on the boundary. While the dynamic boundary conditions introduced in (AV) involve the trace, dynamic boundary conditions on the normal derivative have also been considered in the literature, although less commonly (see \([20, 16]\)). Accordingly, the similar but different abstract initial-boundary value problem

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u(t) &= \Delta u(t), & t \geq 0, \\
\frac{\partial^2}{\partial t \partial \nu} u(t)|_{\Gamma} &= \delta P_{Y} u|_{\Gamma}(t) + P_{Y} (\gamma \Delta - \mathcal{S}) \frac{\partial u}{\partial \nu}(t), & t \geq 0, \\
\frac{\partial u}{\partial \nu}(0) &= u_0, \\
\frac{\partial u}{\partial \nu}(0)|_{\Gamma} &= v_0,
\end{align*}
\]

(AVN)

can be studied for \( \gamma \in \mathbb{R}_+ \) and \( \mathcal{S} \in \mathcal{L}(L^2(\Omega; H)) \). The parameter \( \delta \in \mathbb{C} \) will be shown to influence the behaviour of the solutions to (AVN) in a curious way.

Consider a sesquilinear form \( b_S \) defined by

\[
b_S \left( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right) := \int_{\Omega} (\nabla f_1(x)|\nabla g_1(x))_{H^s} \, dx - \delta \left( f_1|_{\Gamma}|g_2 \right)_Y d\sigma(z) - \left( f_2|g_1|_{\Gamma} \right)_Y d\sigma(z) + \gamma \int_{\Gamma} (\nabla f_2(z)|\nabla g_2(z))_{H^{s-1}} d\sigma(z) + (S f_2|g_2)_Y,
\]

with dense domain

\[
W_Y := H^1(\Omega; H) \times (H'((\Gamma; H) \cap Y)),
\]

where \( s = 0 \) if \( \gamma = 0 \) or \( s = 1 \) if \( \gamma > 0 \). Mimicking the proof of Theorem 5.1, we deduce a corresponding generation result (cf. also the discussion in \([16, \S 4.3]\)).

**Theorem 5.1.** For any \( \gamma \in \mathbb{R}_+ \), \( \delta \in \mathbb{C} \) and \( \mathcal{S} \in \mathcal{L}(L^2(\Omega; H)) \) the sesquilinear form \( b_Y \) is continuous and elliptic (with respect to \( L^2 \)). The operator \( B_{Y,S} \) associated with \( (b_S, W_Y) \) generates an analytic semigroup \( (e^{tB_Y})_{t \geq 0} \) on \( L^2 \) with angle \( \frac{\pi}{2} \). The semigroup is compact if and only if \( \Omega, \Gamma \) have finite measure, provided that \( H \) is finite dimensional. Moreover, \( b_Y \) is accretive if \( \delta = -1 \) and \( \mathcal{S} \) is accretive; it is symmetric if and only if \( \delta = 1 \) and \( \mathcal{S} \) is self-adjoint. In these cases the semigroups are contractive and self-adjoint, respectively.

With a proof similar to that of Proposition 5.2, we can show the following, see also \([15, \S 1.8]\).

**Proposition 5.2.** Assume \( \Omega \) to have \( C^2 \)-boundary. For any \( \gamma \in \mathbb{R}_+ \), \( \delta \in \mathbb{C} \) and \( \mathcal{S} \in \mathcal{L}(L^2(\Omega; H)) \) the operator \( B_{Y,S} \) associated with \( (b_S, W_Y) \) is given by

\[
D(B_{Y,S}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in W_Y : \Delta u \in L^2(\Omega; H), \Delta u|_{\Gamma} \in L^2(\Gamma; H), \text{ and } \frac{\partial f}{\partial \nu} \in L^2(\Gamma; H) \right\},
\]

\[
B_{Y,S} = \begin{pmatrix} \Delta & 0 \\ \delta P_{Y} T & P_{Y} (\gamma \Delta - \mathcal{S}) \end{pmatrix},
\]
where $T$ denotes the trace operator from $H^1(\Omega; H)$ to $H^\frac{1}{2}(\Gamma; H)$, cf. \cite{7} §7.1.

Thus, the semigroup associated with $B_Y$, yields the solution to the abstract initial-boundary value problem
\[
\begin{aligned}
\frac{\partial u}{\partial t}(t) &= \Delta u(t), & t &\geq 0, \\
\frac{\partial u}{\partial n}(t) &\in Y, & t &\geq 0, \\
\frac{\partial u}{\partial n}(t) &= \delta (P_Y u(t)) + ((\gamma \Delta r - S) \frac{\partial u}{\partial n}(t)), & t &\geq 0, \\
u(0) &= u_0, \\
\frac{\partial u}{\partial n}(0) &= u_0.
\end{aligned}
\]

Ouhabaz’s criterion may be promptly applied to this setting, too. We omit the easy proof.

**Proposition 5.3.** Impose Assumptions \textbullet5.5. Let $C_H$ be a closed subspace or a closed ordered interval of $H$. Consider the closed convex subsets $C_{L^2(\Omega; H)}$ and $C_Y$. Then $(e^{tB_Y})$ leaves invariant $C_{L^2(\Omega; H)} \times C_Y$ if and only if the compatibility condition
\[
\delta \text{Re} \left( P_{C_{L^2(\Omega; H)}}, f_1 \right)(I - P_{C_Y}) f_2) \leq \text{Re} \left( P_{C_Y}, f_2 \right)(I - P_{C_Y}) f_2 \leq \text{Re} \left( P_{C_Y}, f_2 \right)(I - P_{C_Y}) f_2
\]
holds for all $f_1 \in H^1(\Omega; H)$ and all $f_2 \in \mathcal{Y}$.

**Example 5.4.** Impose Assumptions \textbullet5.5. Then, by linearity $(e^{tB_Y})$ is positive if and only if $\delta = 1$ and
\[
\text{Re} \left( P_{C_Y}, f_2 \right)(I - P_{C_Y}) f_2 \geq 0,
\]
i.e., if and only if $\delta = 1$ and the semigroup on $H$ generated by $-S$ is positive.

**Remark 5.5.** It is easy to see that by similar methods one can also treat the parabolic problem
\[
\begin{aligned}
\frac{\partial u}{\partial t}(t) &= \Delta u(t), & t &\geq 0, \\
\frac{\partial P_Y u(t)}{\partial \nu}(t) \mid_{\partial \Omega} &= -P_Y \frac{\partial u(t)}{\partial \nu}(t) \mid_{\partial \Omega} + R_1 u(t) \mid_{\partial \Omega}, & t &\geq 0, \\
\frac{\partial P_Y u(t)}{\partial \nu}(t) \mid_{\Gamma} &= -P_Y \frac{\partial u(t)}{\partial \nu}(t) \mid_{\Gamma} + R_2 \frac{\partial u(t)}{\partial \nu}(t) \mid_{\Gamma}, & t &\geq 0, \\
u(0) &= u_0, \\
\frac{\partial u(t)}{\partial \nu}(0) \mid_{\Gamma} &= u_1,
\end{aligned}
\]
for some $R_1 \in \mathcal{L}(H^1(\Omega; H), \mathcal{Y})$ and $R_2 \in \mathcal{L}(H^1(\Omega; H), \mathcal{Y}^\perp)$. In this case the state space is $L^2(\Omega; H) \times \mathcal{Y} \times \mathcal{Y}^\perp$. We omit the details.

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