Abstract The approach to study properties of charge-exchange excitations in hot nuclei is presented. The approach is based on the extension of the finite rank separable approximation for Skyrme interactions to finite temperatures employing the TFD formalism. We present the formulae to obtain charge-exchange strength distributions within the Thermal Quasiparticle Random Phase Approximation (TQRPA).

1 Introduction

The properties of nuclei at finite temperatures appear to be interesting for many reasons. Currently, one of the most popular fields to apply the nuclear theory at finite temperatures is the astrophysical studies. In the astrophysical context the theory is used to calculate thermal modifications of spin-isospin transition distributions over nuclear spectra, as they play an important role in weak-interaction mediated reactions in stellar environment.

At first, temperature effects were introduced into the calculations via quite straightforward way taking into account a possibility of thermal feeding of nuclear excited states which energies were calculated via independent-particle shell model or taken from experimental compilations [1]. Later this approach was developed and refined employing the large scale shell-model calculations of low-lying excited states [2]. Within this approach a coupling with high-lying resonant states like, e.g., the Gamow-Teller resonance, was considered in the framework of the Brink hypothesis.

The other approach to calculate the rates of weak-interaction mediated processes at finite temperatures employs the thermal quasiparticle random phase approximation (TQRPA). For the first time it was used in [3] to study the electron captures on neutron-rich nuclei. Recently, there appeared several papers [4, 5, 6] where the authors applied TQRPA (or TRPA) with different nuclear Hamiltonians.

In Ref. [5], the TQRPA was applied in a framework of a general context of thermo-field dynamics (TFD) [7, 8] and with the Hamiltonian consisting of schematic separable effective interactions. It is well known that schematic separable interactions allow one to avoid many of computational difficulties and limitations which are inherent for “realistic” residual interactions. However, the cost is a lack of selfconsistency and limited predictive power of the calculations for nuclei far from the valley of stability. Thus, it seems reasonable to use in calculations for hot nuclei a finite rank separable approximation elaborated for the residual forces based on an effective interaction of Skyrme type [10, 11, 12].

*On leave from Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, 141980 Russia
enables to combine the advantages of consistency (the mean field and the residual interaction of RPA are determined from the same effective interaction) with a computational simplicity (the size of the RPA problem does not increase with increasing configuration space).

Here we present the formulae which extend the approach of Refs. [10] to finite temperatures. The TFD formalism presented in [7, 8, 9] and adapted to nuclear structure problems in [13] is used. Having in mind further application of the formalism to charge-exchange excitations we consider only the isovector part of the residual particle-hole interaction.

2 Separabilization of residual interactions

The exact $p-h$ residual interaction $V_{res}$ corresponding to the Skyrme force can be obtained as the second derivative of the energy density functional with respect to the particle density. In Ref. [10], $V_{res}$ was approximated by its Landau-Migdal form. For the Skyrme interactions all the Landau parameters with $l > 1$ are zero. Moreover, the Landau parameters with the $l = 1$ terms are neglected. Therefore, the residual interaction in the isovector channel has the following form:

$$V_{res}(r_1, r_2) = N_0^{-1} [F'_0(r_1) + G'_0(r_1)\sigma^{(1)}\sigma^{(2)}] \tau^{(1)}\tau^{(2)} \delta(r_1 - r_2).$$  (1)

where $\sigma^{(i)}$ and $\tau^{(i)}$ are the spin and isospin operators, and $N_0 = 2k_F m^*/\pi^2\hbar^2$ with $k_F$ and $m^*$ standing for the Fermi momentum and nucleon effective mass in nuclear matter. $F'_0$, $G'_0$ are functions of the coordinate $r$. Their expressions in terms of the Skyrme parameters can be found in Ref. [14].

Following the method presented in [10, 11] the residual interaction is reduced to a finite rank separable form

$$V_{res} = V_M + V_{SM},$$  (2)

where

$$V_M = -2 \sum_{JM} \sum_{n=1}^{N} \kappa_F^{(n)} : \hat{M}^{(n)}_{JM} \hat{M}^{(n)}_{JM} :,$$

$$V_{SM} = -2 \sum_{JM} \sum_{L=J, J\pm 1} \sum_{n=1}^{N} \kappa_G^{(n)} : \hat{S}^{(n)}_{LJM} \hat{S}^{(n)}_{LJM} :,$$  (3)

and

$$\begin{pmatrix} \kappa_F^{(n)} \\ \kappa_G^{(n)} \end{pmatrix} = -N_0^{-1} R \omega_k \begin{pmatrix} F'_0(r_k) \\ G'_0(r_k) \end{pmatrix}. $$  (4)

Here $R$ is a large enough cutoff radius for a $N$-point integration Gauss formula with abscessas $r_k$ and weights $\omega_k$ [10].

The operators entering the normal product in Eq. (3) are defined as follows:

$$\hat{M}^{(n)}_{JM} = (-1)^{JM-J} \hat{j}^{-1} \sum_{j_a m_j m_a} \langle j_a m_j m_a | JM \rangle f_{j_a j_p}^{(Jk)} (-1)^{j_p-m_p} a^\dagger_{j_a m_j} a_{j_p m_p},$$

$$\hat{S}^{(n)}_{JM} = (-1)^{JM-J} \hat{j}^{-1} \sum_{j_a m_j m_a} \langle j_a m_j m_a | JM \rangle g_{j_a j_p}^{(Jk)} (-1)^{j_p-m_p} a^\dagger_{j_a m_j} a_{j_p m_p}. $$  (5)

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where \( f_{jnjp}^{(Jk)} \) is the single-particle matrix elements of the multipole operators,

\[
f_{jnjp}^{(Jk)} = u_{jn}(r_k)u_{jp}(r_k)\langle j_n||i^J Y_j||j_p \rangle,
\]

and where \( g_{jnjp}^{(Lk)} \) is the single-particle matrix elements of the spin-multipole operators,

\[
g_{jnjp}^{(Lk)} = u_{jn}(r_k)u_{jp}(r_k)\langle j_n||i^J T_j||j_p \rangle.
\]

In the above equations, \( \langle j_n||i^J Y_j||j_p \rangle \) is the reduced matrix element of the spherical harmonic \( Y_{JM} \), \( \hat{J} = \sqrt{2J + 1} \), \( T_{LJM} = [Y_J \times \sigma]_{LM} \). The radial wave functions \( u_j(r) \) are related to the HF single-particle wave functions

\[
\phi_{i,m}(1) = \frac{u_i(r_1)}{r_1}Y_{l,m}^{-} \sqrt{y_{l,j}}(\hat{r}_1, \sigma_1).
\]

An interaction in the particle-particle channel has the surface peaked density-dependent zero-range shape

\[
V_{\text{pair}}(r_1, r_2) = V_0 \left( 1 - \eta \left( \frac{\rho(r_1)}{\rho_c} \right)^\alpha \right) \delta(r_1 - r_2).
\]

Here \( \rho(r) \) is the particle density in coordinate space, \( \rho_0 \) is equal to the nuclear saturation density, \( V_0, \eta \) and \( \alpha \) are the parameters fixed to reproduce the odd-even mass difference of nuclei in the study region. The interaction (9) is responsible for the pairing correlations as well.

### 3 TBCS and TQRPA equations at \( T \neq 0 \)

#### 3.1 Fundamentals of the thermo-field dynamics

Thermo-field dynamics [7, 8, 9] is a real-time formalism for treating thermal effects in quantum field theory and many-body theories. In TFD, the thermal average of a given operator \( A \) is calculated as the expectation value in a specially constructed, temperature-dependent state \( |0(T)\rangle \) which is termed the thermal vacuum. This expectation value is equal to the usual grand canonical average of \( A \). In this sense, the thermal vacuum describes the thermal equilibrium of the system.

To construct the state \( |0(T)\rangle \), a formal doubling of the system degrees of freedom is introduced. In TFD, a tilde conjugate operator \( \tilde{A} \) – acting in the independent Hilbert space – is associated with \( A \), in accordance with properly formulated tilde conjugation rules [7, 8, 9]

\[
(A_1 A_2) = \tilde{A}_1, \tilde{A}_2, \quad (c_1 A_1 + c_2 A_2) = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2, \\
(A_1^\dagger) = \tilde{A}_1^\dagger, \quad (\tilde{A}) = A,
\]

where \( A_1 \) and \( A_2 \) stand for any operators and \( c_1 \) and \( c_2 \) are \( c \)-numbers. The asterisk denotes the complex conjugate. It is assumed that any bosonic (fermionic) operator with tilde commutes (anticommutes) with all bosonic (fermionic) operators without tilde.
Let $H$ be the Hamiltonian of the system. In the doubled Hilbert space the thermal vacuum is defined as the zero-energy eigenstate of the so-called thermal Hamiltonian $\mathcal{H} = H - \tilde{H}$ and it satisfies the thermal state condition \[ A|0(T)\rangle = \sigma e^{i\mathcal{H}/2T}\tilde{A}^\dagger|0(T)\rangle, \] where $\sigma = 1$ for bosonic $A$ and $\sigma = i$ for fermionic $A$. The thermal state condition is one of the most fundamental relations in TFD. The celebrated Kubo-Martín-Schwinger condition, which is one of the basic axioms in the $c^*$-algebra formalism for statistical mechanics, is a result of the thermal state condition in TFD. Furthermore, from (11) we can see immediately that, in TFD, there always exists a certain combination of $A$ and $\tilde{A}$ which annihilates the thermal vacuum. That mixing is promoted by a specific canonical transformation called the thermal Bogoliubov transformation \[ A \] and \[ \tilde{A}^\dagger. \] The temperature dependence comes from the transformation parameters.

The important point is that in the doubled Hilbert space the time-translation operator is the thermal Hamiltonian $\mathcal{H} = H - \tilde{H}$. This means that the excitations of the thermal system are obtained by the diagonalization of $\mathcal{H}$. The existence of the thermal vacuum annihilation operators allows for straightforward extensions of different zero-temperature approximations to diagonalize the thermal Hamiltonian. As follows from the definition of $\mathcal{H}$ each of its eigenstates with positive energy has the counterpart – the tilde-conjugate eigenstate – with negative but the same absolute energy value. Transitions from the thermal vacuum to positive (non-tilde) energy states correspond to excitation of the system, while transitions to negative (tilde) energy states - to deexcitation.

### 3.2 Equations for pairing correlations at $T \neq 0$

Applying the TFD formalism we suppose that the nuclear proton and neutron mean fields are already produced according the Hartree-Fock procedure and our Hamiltonian $H$ consists of the mean fields and separableized residual interactions presented in the previous Section 2. In particular, it means that we ignore the influence of temperature on the nuclear mean field.

At first, following the TFD prescription, we double the original nuclear degrees of freedom introducing the so-called tilde creation and annihilation operators $\tilde{a}_{jm}^\dagger, a_{jm}$ and construct the thermal Hamiltonian

$$\mathcal{H} = H - \tilde{H}$$

Then we transform $\mathcal{H}$ to the thermal quasiparticle representation by means of the two canonical transformations. The first one is the standard Bogoliubov transformation to quasiparticle operators

$$a_{jm}^\dagger = u_j a_{jm}^\dagger + v_j a_{jm}^\dagger,$$

$$\tilde{a}_{jm}^\dagger = u_j \tilde{a}_{jm}^\dagger + v_j \tilde{a}_{jm}^\dagger, \quad (u_j^2 + v_j^2 = 1),$$

where $a_{jm} = (-1)^{j-m}\alpha_{j-m}$. The second transformation is the thermal Bogoliubov transformation \[ A \] and \[ \tilde{A}^\dagger. \] It mixes the quasiparticle and tilde quasiparticle operators, thus producing thermal quasiparticle operators and their tilde partners $\beta_{jm}^\dagger, \beta_{jm}, \beta_{jm}^\dagger, \beta_{jm}$

$$\beta_{jm}^\dagger = x_j a_{jm}^\dagger - iy_j \tilde{a}_{jm}^\dagger,$$

$$\tilde{\beta}_{jm}^\dagger = x_j \tilde{a}_{jm}^\dagger + iy_j a_{jm}, \quad (x_j^2 + y_j^2 = 1).$$

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The coefficients of both the transformations are determined by the diagonalization of the sum of single-particle and pairing parts of $\mathcal{H}$ and additional demand for the BCS thermal vacuum to obey the thermal state condition \cite{[13]}. At the end, we get the following equations for the coefficient $u_j, v_j$ and $x_j, y_j$:

$$v_j = \frac{1}{\sqrt{2}} \left( 1 - \frac{E_j - \lambda}{\varepsilon_j} \right)^{1/2}, \quad u_j = (1 - v_j^2)^{1/2},$$

$$y_j = \left[ 1 + \exp \left( \frac{\varepsilon_j}{T} \right) \right]^{-1/2}, \quad x_j = (1 - y_j^2)^{1/2},$$

where $\varepsilon_j = \sqrt{(E_j - \lambda)^2 + \Delta_j^2}$ is a quasiparticle energy. The pairing gaps $\Delta_j$ and the chemical potential $\lambda$ are the solutions to the finite-temperature BCS equations

$$\Delta_j(T) = - \sum_{j'} (1)^{j_{j'} + l_{j'}} \sqrt{\frac{2j_{j'} + 1}{2j_j + 1}} G_0(j_{j}j'_{j'}) (1 - 2y_j^2) u_{j'} v_{j'},$$

$$N = \sum_j (2j + 1)(v_j^2 x_j^2 + u_j^2 y_j^2),$$

where $N$ is the number of neutrons or protons in a nucleus, and two-body matrix element $G_0(j_{j}j'_{j'})$ is given by the $J = 0$ particle-particle matrix element of the interaction $V_{\text{pair}}$ \cite{[9]}

$$G_0(j_{j}j'_{j'}) = \langle j_{j} | V_{\text{pair}} | j'_{j'} \rangle_{00}.$$  \hspace{1cm} (17)

Now, the sum of the single-particle and the pairing parts of the Hamiltonian $\mathcal{H}$ becomes diagonal

$$\mathcal{H}_{\text{BCS}} = \sum_{\tau=p,n} \sum_{jm} \varepsilon_j \left( \beta_{jm}^\dagger \beta_{jm} - \beta_{jm}^\dagger \beta_{jm} \right).$$

Here $\sum^\tau$ implies a summation over proton or neutron single-particle states only. The Hamiltonian $\mathcal{H}_{\text{BCS}}$ describes a system of noninteracting non-tilde and tilde thermal quasiparticles with energies $\varepsilon_j$ and $-\varepsilon_j$, respectively. The vacuum of thermal quasiparticles, $|0(T); \text{qp}\rangle$, is the thermal vacuum in BCS approximation.

### 3.3 Thermal QRPA equations

The transformations \cite{[12]} and \cite{[13]} are applied to the whole thermal nuclear Hamiltonian $\mathcal{H}$. At the next step we approximately diagonalize $\mathcal{H}$ within the Thermal Quasiparticle Random Phase Approximation

$$\mathcal{H} \approx \mathcal{H}_{\text{TQRPA}} = \sum_{JM_i} \omega_{JM_i} (Q_{JM_i}^\dagger Q_{JM_i} - \bar{Q}_{JM_i}^\dagger \bar{Q}_{JM_i}).$$

Here the thermal (charge-exchange) phonon creation operator is defined as a linear superposition of the proton-neutron thermal quasiparticle pair creation and annihilation operators

$$Q_{JM_i}^\dagger = \sum_{Jnlp} \left( \psi_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \beta_{Jnlp}^\dagger + \bar{\psi}_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger + i\eta_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger + i\bar{\eta}_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \right)_{JM_i}$$

$$+ \phi_{Jnlp}^\dagger \left( \bar{\beta}_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \beta_{Jnlp}^\dagger + \bar{\phi}_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \right)_{JM_i} + i\xi_{Jnlp}^\dagger \left( \beta_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \right)_{JM_i} + i\bar{\xi}_{Jnlp}^\dagger \left( \bar{\beta}_{Jnlp}^\dagger \beta_{Jnlp}^\dagger \bar{\beta}_{Jnlp}^\dagger \right)_{JM_i}. $$  \hspace{1cm} (19)
and \( |J\rangle \) denotes the coupling of single-particle angular momenta \( j_n, j_p \) to total angular momentum \( J \). The tilde conjugate thermal phonon operator \( \tilde{Q}_{JM}^{\dagger} \) can be obtained from \( |J\rangle \) by applying tilde conjugation rules \( (10) \). Now the thermal equilibrium state is treated as the vacuum \( |0(T);\phi\rangle \) for thermal phonon annihilation operators and it obeys the thermal state condition \( (11) \). The excited thermal one-phonon states are \( Q_{JM}^{\dagger} |0(T);\phi\rangle \) and \( \tilde{Q}_{JM}^{\dagger} |0(T);\phi\rangle \).

The thermal phonon operators are assumed to commute as bosonic operators, that is, \( [Q_{JM}, \tilde{Q}_{J'M'}^\dagger] = \delta_{JJ'}\delta_{MM'}\delta_{ii'} \). This assumption imposes the normalization condition on the phonon amplitudes

\[
\sum_{Jnjp} \left( \psi_{Jnjp}^{Ji} \psi_{Jnjp}^{J'i'} + \tilde{\psi}_{Jnjp}^{Ji} \tilde{\psi}_{Jnjp}^{J'i'} + \eta_{Jnjp}^{Ji} \eta_{Jnjp}^{J'i'} + \tilde{\eta}_{Jnjp}^{Ji} \tilde{\eta}_{Jnjp}^{J'i'} \right) - \phi_{Jnjp}^{Ji} \phi_{Jnjp}^{J'i'} - \tilde{\phi}_{Jnjp}^{Ji} \tilde{\phi}_{Jnjp}^{J'i'} - \xi_{Jnjp}^{Ji} \xi_{Jnjp}^{J'i'} - \tilde{\xi}_{Jnjp}^{Ji} \tilde{\xi}_{Jnjp}^{J'i'} = \delta_{ii'} \].

Furthermore, additional constraints on the amplitudes come from the thermal state condition. Namely, putting \( A = [\alpha_{JM}^\dagger \alpha_{J'M'}^\dagger] \) in \( (11) \) we get \( (\omega_{Ji} > 0) \)

\[
(x_n, x_p, \phi_{Jnjp}^{Ji} + J_n x_p \phi_{Jnjp}^{J'i'}) = \exp \left( \frac{\omega_{Ji}}{2T} \right) (x_n, x_p, \phi_{Jnjp}^{J'i'} + J_n x_p \phi_{Jnjp}^{Ji}),
\]

\[
(x_n, x_p, \phi_{Jnjp}^{Ji} + J_n y_p \phi_{Jnjp}^{J'i'}) = \exp \left( -\frac{\omega_{Ji}}{2T} \right) (x_n, x_p, \phi_{Jnjp}^{J'i'} + J_n y_p \phi_{Jnjp}^{Ji}),
\]

and for \( A = [\alpha_{JM}^\dagger \alpha_{J'M'}^\dagger] \) we have

\[
(x_n, y_p, \eta_{Jnjp}^{Ji} + J_n y_p \eta_{Jnjp}^{J'i'}) = \exp \left( \frac{\omega_{Ji}}{2T} \right) (x_n, y_p, \eta_{Jnjp}^{J'i'} + J_n y_p \eta_{Jnjp}^{Ji}),
\]

\[
(x_n, y_p, \eta_{Jnjp}^{Ji} + J_n x_p \eta_{Jnjp}^{J'i'}) = \exp \left( -\frac{\omega_{Ji}}{2T} \right) (x_n, y_p, \eta_{Jnjp}^{J'i'} + J_n x_p \eta_{Jnjp}^{Ji}).
\]

To find the energy and the structure of thermal phonons we apply the variational principle, i.e. we minimize the expectation value of \( \mathcal{H} \) over the thermal one-phonon state under the constraints \( (20) \). As a result we get the TQRPA eigenvalue equations for the amplitudes and the energy eigenvalues \( \omega_{Ji} \). In contrast to the zero-temperature case, the negative eigenvalues of TQRPA matrix have physical meaning and they can be interpreted as the excitation energies of tilde thermal one-phonon states relative to the thermal vacuum. Besides, each eigenvalue is twice degenerate so that \( \mathcal{H}_{TQRPA} \) is invariant under the thermal Bogoliubov transformation

\[
Q_{JM}^{\dagger} \rightarrow X_{J} Q_{JM}^{\dagger} - Y_{J} Q_{JM}, \quad \tilde{Q}_{JM}^{\dagger} \rightarrow X_{J} \tilde{Q}_{JM}^{\dagger} - Y_{J} Q_{JM}.
\]

with \( X_{J}^2 - Y_{J}^2 = 1 \). To find the structure of thermal phonons unambiguously we demand that the constraints \( (21) \) are valid. Only in this case the vacuum of thermal phonons is the thermal vacuum in the TQRPA approximation.

Using the separable form of the residual interaction one can reduce remarkably the dimensions of the corresponding TQRPA matrixes. To do this we introduce a vector \( \left( D_{JM}^{\dagger} \right) \) by its components:

\[
D_{JM}^{\dagger} = \sum_{Jnjp} \delta_{Jnjp}^{\dagger} \left( u_{Jnjp}^{\dagger} \left[ x_n, x_p (\psi_{Jnjp}^{Ji} \pm \phi_{Jnjp}^{J'i'}) \pm y_n, y_p (\eta_{Jnjp}^{Ji} \pm \xi_{Jnjp}^{J'i'}) \right] + v_{Jnjp}^{\dagger} \left[ x_n, y_p (\eta_{Jnjp}^{Ji} \pm \xi_{Jnjp}^{J'i'}) \pm y_n, x_p (\psi_{Jnjp}^{Ji} \pm \phi_{Jnjp}^{J'i'}) \right] \right),
\]
where \( u_{j_n j_p}^{(\pm)} = u_{j_n} v_{j_p} \pm v_{j_n} u_{j_p} \) and \( v_{j_n j_p}^{(\pm)} = u_{j_n} u_{j_p} \pm v_{j_n} v_{j_p} \). Phonon amplitudes are functions of the vectors

\[
\begin{align*}
\left( \begin{array}{c} \psi_j \\ \phi_j \end{array} \right)_{J_n j_p}^J &= \frac{\hat{j}^{-2}}{\varepsilon_{j_n j_p}^+ + \omega_{J_i}} (x_{j_n} x_{j_p} X_{\lambda_i} - y_{j_n} y_{j_p} Y_{\lambda_i}) \sum_{n=1}^{2N} \delta_{J_n j_p}^{J(n)} \kappa(n) \left( D_{J_n j_p}^{+} u_{j_n j_p}^{(+)} \pm D_{J_n j_p}^{-} u_{j_n j_p}^{(-)} \right), \\
\left( \begin{array}{c} \eta_j \\ \xi_j \end{array} \right)_{J_n j_p}^J &= \frac{\hat{j}^{-2}}{\varepsilon_{j_n j_p}^- + \omega_{J_i}} (x_{j_n} y_{j_p} X_{\lambda_i} - y_{j_n} x_{j_p} Y_{\lambda_i}) \sum_{n=1}^{2N} \delta_{J_n j_p}^{J(n)} \kappa(n) \left( D_{J_n j_p}^{+} v_{j_n j_p}^{(-)} \pm D_{J_n j_p}^{-} v_{j_n j_p}^{(+)} \right),
\end{align*}
\]

where

\[
Y_{J_i} = \left[ \exp \left( \frac{\omega_{J_i}}{T} \right) - 1 \right]^{-1/2}, \quad X_{J_i} = (1 + Y_{\lambda}^2)^{1/2}.
\]

So, the TQRPA equations are reduced to the set of equations for \( D_{J_n j_p}^{+in} \):

\[
\begin{bmatrix} \mathcal{M}_1 - \frac{1}{2} I & \mathcal{M}_2 \\ \mathcal{M}_2 & \mathcal{M}_3 - \frac{1}{2} I \end{bmatrix} \begin{bmatrix} D_{J_n j_p}^{+} \\ D_{J_n j_p}^{-} \end{bmatrix} = 0.
\]

The matrix elements of the \( 2N \times 2N \) matrices \( \mathcal{M}_\beta \) have the following expressions

\[
\begin{align*}
\mathcal{M}_{1,3}^{mn} &= \frac{\kappa(n)}{J^2} \sum_{J_n j_p} d_{J_n j_p}^{(Jn)} d_{J_n j_p}^{(Jn')} \left\{ \frac{\varepsilon_{j_n j_p}^{(\pm)} (u_{j_n j_p}^{(\pm)})^2}{\varepsilon_{j_n j_p}^{(\pm)} J_j^2 - \omega_{J_i}^2} (1 - y_{J_n}^2 - y_{J_p}^2) - \frac{\varepsilon_{j_n j_p}^{(-)} (v_{j_n j_p}^{(-)})^2}{\varepsilon_{j_n j_p}^{(-)} J_j^2 - \omega_{J_i}^2} (y_{J_n}^2 - y_{J_p}^2) \right\}, \\
\mathcal{M}_{2}^{mn} &= \frac{\kappa(n)}{J^2} \omega_{J_i} \sum_{J_n j_p} d_{J_n j_p}^{(Jn)} d_{J_n j_p}^{(Jn')} \left\{ \frac{u_{j_n j_p}^{(\pm)} v_{j_n j_p}^{(-)}}{\varepsilon_{j_n j_p}^{(\pm)} J_j^2 - \omega_{J_i}^2} (1 - y_{J_n}^2 - y_{J_p}^2) - \frac{v_{j_n j_p}^{(\pm)} v_{j_n j_p}^{(-)}}{\varepsilon_{j_n j_p}^{(-)} J_j^2 - \omega_{J_i}^2} (y_{J_n}^2 - y_{J_p}^2) \right\},
\end{align*}
\]

where \( \varepsilon_{j_n j_p}^{(\pm)} = \varepsilon_{j_n} \pm \varepsilon_{j_p} \). Its solution requires to compute the determinant

\[
\det \begin{bmatrix} \mathcal{M}_1 - \frac{1}{2} I & \mathcal{M}_2 \\ \mathcal{M}_2 & \mathcal{M}_3 - \frac{1}{2} I \end{bmatrix} = 0
\]

and we find the eigenvalues of the TQRPA equations.

### 4 Charge-exchange transition probabilities

Charge-exchange transition probabilities (transition strengths) from the thermal vacuum to thermal one-phonon states are given by the squared reduced matrix elements of the corresponding transition operator:

\[
\begin{align*}
\Phi_{J_i}^{(\pm)} &= \left| \left< Q_{JM}^{(\pm)} \| D_{JM}^{(\pm)} \| 0(T); \phi \right> \right|^2, \\
\tilde{\Phi}_{\lambda_i}^{(\pm)} &= \left| \left< Q_{M^{(\pm)}} \| D_{JM}^{(\pm)} \| 0(T); \phi \right> \right|^2.
\end{align*}
\]
Hereinafter the symbol (−) labels the \( n \to p \) transition operators, and the symbol (+) labels the \( p \to n \) transition operators. The explicit expressions for \( \Phi_{\lambda i}^{(+)} \) and \( \Phi_{\lambda i}^{(-)} \) are the following:

\[
\Phi_{\lambda i}^{(+)} = \left( \sum_{j_n,j_p} d_{j_p}^{(+)}(j_n,j_p) \Omega_1(j_n,j_p; J_i) \right)^2,
\]

\[
\Phi_{\lambda i}^{(-)} = \left( \sum_{j_n,j_p} d_{j_p}^{(-)}(j_n,j_p) \Omega_1(j_n,j_p; J_i) \right)^2,
\]

\[
\Phi_{\lambda i}^{(\pm)} = \left( \sum_{j_n,j_p} (-1)^{j_p-j_n} d_{j_p}^{(\pm)}(j_n,j_p) \Omega_2(j_n,j_p; J_i) \right)^2,
\]

\[
\Phi_{\lambda i}^{(\mp)} = \left( \sum_{j_n,j_p} (-1)^{j_p-j_n} d_{j_p}^{(\mp)}(j_n,j_p) \Omega_2(j_n,j_p; J_i) \right)^2,
\]

(30)

where \( d_{\lambda}^{(\pm)}(j_p,j_n) \) is a reduced single-particle matrix element of the transition operator functions \( \Omega_{1,2}(j_n,j_p; J_i) \) are linear combinations of the phonon amplitudes (25):

\[
\Omega_1(j_n,j_p; J_i) = u_{j_n}v_{j_p}(x_{j_n}x_{j_p}\phi_{j_n,j_p}^j + y_{j_n}y_{j_p}\phi_{j_n,j_p}^j)\right)
\]

\[
+u_{j_n}u_{j_p}(x_{j_n}x_{j_p}\eta_{j_n,j_p}^j + y_{j_n}y_{j_p}\eta_{j_n,j_p}^j) - v_{j_n}v_{j_p}(y_{j_n}x_{j_p}\eta_{j_n,j_p}^j + x_{j_n}y_{j_p}\eta_{j_n,j_p}^j)_\right)
\]

\[
\Phi_{\lambda i}^{(\pm)} = \left( \sum_{j_n,j_p} (-1)^{j_p-j_n} d_{j_p}^{(\pm)}(j_n,j_p) \Omega_2(j_n,j_p; J_i) \right)^2,
\]

(32)

and \( \tilde{\Omega}_{1,2}(j_n,j_p; J_i) \) result from \( \Omega_{1,2}(j_n,j_p; J_i) \) by changing non-tilde phonon amplitudes by their tilde partners and vise versa.

The excitation energies with respect to the thermal equilibrium state of the parent nucleus are

\[
E^\pm = \omega_{ji} \mp \Delta \lambda
\]

(33)

for non-tilde phonon states, and

\[
E^\mp = -\omega_{ji} \mp \Delta \lambda
\]

(34)

for tilde phonon states. Here \( \Delta \lambda = \lambda_n - \lambda_p \) is the difference between the neutron and the proton chemical potentials.

Expressions (30) and (33 34) determine charge exchange strength distribution in a hot nucleus within the TQRPA. Note that at finite temperatures some amount of transition strength is always located in the region of negative transition energies.

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