A NOTE ON VANISHING OF EQUIVARIANT DIFFERENTIABLE COHOMOLOGY OF PROPER ACTIONS AND APPLICATION TO CR-AUTOMORPHISM AND CONFORMAL GROUPS

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Abstract. We establish that for any proper action of a Lie group on a manifold the associated equivariant differentiable cohomology groups with coefficients in modules of $C^\infty$-functions vanish in all degrees except than zero. Furthermore let $G$ be a Lie group of CR-automorphisms of a strictly pseudo-convex CR-manifold $M$. We associate to $G$ a canonical class in the first differential cohomology of $G$ with coefficients in the $C^\infty$-functions on $M$. This class is non-zero if and only if $G$ is essential in the sense that there does not exist a CR-compatible strictly pseudo-convex pseudo-Hermitian structure on $M$ which is preserved by $G$. We prove that a closed Lie subgroup $G$ of CR-automorphisms acts properly on $M$ if and only if its canonical class vanishes. As a consequence of Schoen’s theorem, it follows that for any strictly pseudo-convex CR-manifold $M$, there exists a compatible strictly pseudo-convex pseudo-Hermitian structure such that the CR-automorphism group for $M$ and the group of pseudo-Hermitian transformations coincide, except for two kinds of spherical CR-manifolds. Similar results hold for conformal Riemannian and Kähler manifolds.

1. Introduction

The original motivation for this article arises from the observation that the dynamics of certain groups of automorphisms of geometric structures, for example in the cases of CR- and conformal manifolds, are to a large extent controlled by a one-cocycle for the differentiable cohomology of those groups with coefficients in the $C^\infty$-functions. In fact, as we will show, automorphism groups of such geometric structures act properly if and only if their canonical cohomology class, which is defined by this one-cocycle, vanishes.

Behind the vanishing of the canonical class for proper actions in the above special cases there is a general vanishing principle for the equivariant cohomology groups of function spaces related to proper actions of Lie groups. This vanishing principle not only holds in degree one, but for arbitrary...
degree different from zero. The considerably more simple special case concerning only the first differentiable cohomology suffices for the original applications for $CR$ and conformal manifolds though.

In this article, we will work out the details of the theory of equivariant continuous and differentiable cohomology groups associated to actions of Lie groups on manifolds and we will prove the vanishing theorem for proper actions. This material is harking back and based on original work of Van Est, Mostow, Mostow-Hochschild [40, 24, 20]. Our exposition, as presented in Sections 2 and 3, will employ the language and techniques of sheaf cohomology to develop the necessary steps of the proofs in a structured and concise way.

This introduction is organized as follows. Before diving into the presentation of the theory of differentiable cohomology, we start by discussing the aforementioned applications in the context of automorphism groups of $CR$- and conformal manifolds in Section 1.1. The structure of proofs and the relation with differentiable cohomology will be discussed in Section 1.2. Finally Section 1.3.1 gives a general account on equivariant differentiable cohomology groups associated to proper actions.

1.1. Automorphisms of $CR$ and conformal manifolds. Let $\omega$ be a contact form on a connected smooth manifold $M$. Assume further that there exists a complex structure $J$ on the contact bundle $\ker \omega$ which is compatible with $\omega$ in the sense that the Levi form $d\omega \circ J$ is a positive definite Hermitian form. Then

$$(M, \{\omega, J\})$$

is called a pseudo-Hermitian manifold. Putting $D = \ker \omega$, the pseudo-Hermitian structure $\{\omega, J\}$ induces a $CR$-structure $\{D, J\}$ on $M$, which is then a strictly pseudo-convex $CR$-structure.

These structures naturally associate to $M$ two important groups. Namely the group of pseudo-Hermitian transformations

$$\text{Psh}(M, \{\omega, J\})$$

and the group of $CR$-automorphisms

$$\text{Aut}_{CR}(M, \{D, J\}) .$$

The isometry group of a Riemannian manifold $M$ and also any closed subgroup of the isometry group itself are Lie groups, by the theorem of Myers and Steenrod [33]. Moreover, the isometry group acts properly on $M$. The pseudo-Hermitian group $\text{Psh}(M, \{\omega, J\})$ is a Lie group, since it preserves the associated contact Riemannian metric

$$g = \omega \cdot \omega + d\omega \circ J .$$

The automorphism group of a strictly pseudo-convex $CR$-manifold is also a Lie group which follows by [37, 11].
Strictly pseudo-convex CR-manifolds. For any strictly pseudo-convex CR-manifold $(M, \{D, J\})$, there is, in general, no canonical choice in the conformal class of one-forms

$$\omega = \{f \cdot \omega \mid f \in C^\infty(M, \mathbb{R}^\geq 0)\}$$

which are representing $D$. Remark that the pseudo-Hermitian group

$$\text{Psh} (M, \{f \cdot \omega, J\})$$

is always contained in the group $\text{Aut}_{CR}(M, \{D\})$, but it may vary considerably with the choice of $f$. Moreover, the Lie groups $\text{Psh} (M, \{f \cdot \omega, J\})$ act properly on $M$, whereas in special cases of certain spherical CR-manifolds $\text{Aut}_{CR}(M, \{D\})$ is too large and cannot act properly on $M$ (compare Theorem 3 below).

Now let $M$ be a strictly pseudo-convex CR-manifold. Assuming that $\text{Aut}_{CR}(M)$ is acting properly on $M$, we shall prove that there exists a pseudo-Hermitian structure on $M$ compatible with its CR-structure such that $\text{Psh}(M)$ and $\text{Aut}_{CR}(M)$ coincide. This fact is already mentioned in the literature (see for example the remark following Conjecture 1.4 in [30]) but it is hard to locate a concise proof.

In the light of the celebrated result of R. Schoen [35] on the properness of CR-automorphism groups the following thus holds:

**Theorem 1.** Let $(M, \{D, J\})$ be a strictly pseudo-convex CR-manifold. Then either one of the following holds:

(i) There exists a pseudo-Hermitian structure $\{\eta, J\}$, with $D = \ker \eta$, such that

$$\text{Aut}_{CR}(M, \{D, J\}) = \text{Psh} (M, \{\eta, J\}) .$$

(ii) $M$ has a spherical CR-structure isomorphic to either the standard sphere $S^{2n+1}$ or the Heisenberg Lie group $N = N_{2n+1}$.

In case (ii) the following holds:

$$(\text{Psh}(M), \text{Aut}_{CR}(M)) = \begin{cases} (U(n + 1), PU(n + 1, 1)) & (M = S^{2n+1}), \\ (N \rtimes U(n), N \rtimes (U(n) \times \mathbb{R}^\geq 0)) & (M = N). \end{cases}$$

For a compact strictly pseudo-convex CR-manifold $M$, this result is originally due to Webster [41], see also [21], and [5, Proposition 4.4] for a related result in the context of compact Sasaki manifolds, compare also [2]. For further background on spherical CR-manifolds see [10, 23].
**Conformal Riemannian manifolds.** When we replace $\omega$ by a Riemannian metric $g$ on $M$, there is a conformal analogue of Theorem 1. For this recall that a diffeomorphism $\alpha : M \to M$, which satisfies

$$\alpha^* g = f \cdot g,$$

for some positive function $f \in C^\infty(M, \mathbb{R}^>)$, is called a conformal automorphism. Let $\text{Iso}(M, g)$ denote the group of isometries and $\text{Conf}(M, g)$ the group of conformal automorphisms for the metric $g$. The conformal automorphism group is a Lie group, since the $G$-structure underlying conformal geometry has its second prolongation vanishing [25], but in general it doesn’t act properly on $M$.

Similarly as in the CR-case, we will prove that if $\text{Conf}(M, g)$ acts properly on $M$ then there exists a Riemannian metric $h$ conformal to $g$ such that $\text{Conf}(M, g) = \text{Iso}(M, h)$. In fact, this was first observed long time ago by D.V. Alekseevsky, see [1, proof of Theorem 1].

Together with the classification of conformal Riemannian manifolds with non-proper automorphisms groups completed by the proofs of J. Ferrand [15] and R. Schoen [35], this gives rise to the following (see also [1]):

**Theorem 2.** Let $(M, g)$ be a Riemannian manifold. Then either one of the following holds:

(i) There exists a Riemannian metric $h$ conformal to $g$ such that

$$\text{Conf}(M, g) = \text{Iso}(M, h).$$

(ii) $M$ is conformal to either the standard sphere $S^n$ or the Euclidean flat space $\mathbb{R}^n$.

For (ii), it occurs

$$(\text{Iso}(M), \text{Conf}(M)) = \begin{cases} (O(n + 1), PO(n + 1, 1)) & (M = S^n), \\ (\mathbb{R}^n \rtimes O(n), \mathbb{R}^n \rtimes (O(n) \times \mathbb{R}^>)) & (M = \mathbb{R}^n). \end{cases}$$

**Conformal Kähler manifolds.** We shall also be concerned with the holomorphic conformal deformation of Kähler manifolds. Let $(X, \{g, J\})$ be a Kähler manifold, where $g$ is a Kähler metric and $J$ denotes the complex structure on $X$. We let $\text{Iso}(X, \{g, J\})$ denote the associated group of holomorphic isometries and $\text{Conf}(X, \{g, J\})$ the holomorphic conformal group. In this context, we obtain:

**Theorem 3.** Let $(X, \{g, J\})$ be a Kähler manifold, $\dim_{\mathbb{R}} X = 2n \geq 4$. Then either one of the following holds:

(i) There exists a Hermitian manifold $(X, \{h, J\})$ with the Hermitian metric $h$ conformal to the Kähler metric $g$ and

$$\text{Conf}(X, \{g, J\}) = \text{Iso}(X, \{h, J\}).$$

(ii) $X$ is holomorphically isometric to $\mathbb{C}^n$. In this case,

$$(\text{Iso}(X, \{g, J\}), \text{Conf}(X, \{g, J\})) = (\mathbb{C}^n \rtimes U(n), \mathbb{C}^n \rtimes (U(n) \times \mathbb{R}^>)).$$
Note that Theorem 3 gives a strong metric rigidity property while in Theorem 1 and Theorem 2 we have rigidity only up to a conformal map.

The Hermitian metric \( h \) in Theorem 3 is globally conformal to a Kähler metric. There is an important class of non-Kähler complex manifolds which carry Hermitian metrics locally conformal to Kähler metrics, see \([14, 34, 39]\), for example Hopf manifolds \( S^3 \times S^1 \) fall into this class. We apply Theorem 3 to prove the existence of locally conformal Kähler metrics whose holomorphic isometry group is maximal among all Hermitian metrics in a given conformal class, see the discussion in Section 4.5 for further details.

1.2. Structure of proofs. Theorem 1 respectively Theorem 2 are based mainly on two ingredients. One corner stone are the following celebrated rigidity results which were obtained in their final form by R. Schoen (see also J. Lee \([30]\)) and J. Ferrand \([15]\):

1.2.1. Schoen’s rigidity theorem and related results.

**Theorem 4.** Let \((M, \{D, J\})\) be a strictly pseudo-convex CR-manifold of dimension \(2n + 1\). Then \(\text{Aut}_{\text{CR}}(M, \{D, J\})\) acts properly on \(M\) unless \(M\) has a spherical CR-structure isomorphic to either the standard sphere \(S^{2n+1}\) or the Heisenberg Lie group \(N_{2n+1}\).

A result analogous to Theorem 4 holds for conformal Riemannian manifolds, see \([35, \text{Theorems 3.3, 3.4}]\) and \([15]\):

**Theorem 5.** Let \((M, g)\) be a Riemannian manifold. Then \(\text{Conf}(M, g)\) acts properly on \(M\), unless \(M\) is conformally diffeomorphic to either the standard round sphere \(S^n\) or Euclidean flat space \(\mathbb{R}^n\).

In the case of Kähler manifolds of complex dimension at least two, we strengthen the above conformal rigidity to the following holomorphic metric rigidity:

**Theorem 6.** Let \((X, \{g, J\})\) be a Kähler manifold, with \(\dim X = 2n \geq 4\). Then the holomorphic conformal group \(\text{Conf}(X, \{g, J\})\) acts properly on \(X\), unless \(X\) is holomorphically isometric to the complex space \(\mathbb{C}^n\).

The proof of Theorem 6 can be found in Appendix A.2. It combines Schoen’s conformal rigidity with classical rigidity results on conformally flat Kähler manifolds (see Appendix A.1).

1.2.2. Differentiable cohomology of Lie groups associated to CR and conformal actions. Schoen’s theorems setting the stage, the main additional ingredient used in the proofs of Theorem 1 and Theorem 2 is the following vanishing result for the first differentiable cohomology group of Lie groups:

**Theorem 7.** Let \(G\) be a Lie group that acts smoothly and properly on a manifold \(M\). Then

\[
H^1_d(G, C^\infty(M, \mathbb{R})) = \{0\}.
\]
In Theorem 7 the coefficients for cohomology are taken in the differentiable $G$-module of smooth functions $C^\infty(M, \mathbb{R})$, which has the $C^\infty$-topology of maps and with the natural $G$-action on functions. The equivariant differentiable cohomology group $H^1_d(G, C^\infty(M, \mathbb{R}))$ is then a topological vector space, whose elements may be represented by certain smooth maps from $G$ to the locally convex vector space $C^\infty(M, \mathbb{R})$.

**Canonical cohomology class associated to CR-actions.** To each CR-action of a Lie group $G$ there exists in the differentiable cohomology of $G$ a natural associated class

$$\mu_{CR} = [\lambda_{CR}] \in H^1_d(G, C^\infty(M, \mathbb{R}^>0))$$

which is induced by the CR-structure on $M$. The class $\mu_{CR}$ vanishes if and only if there exists a contact form $\eta$ compatible with the CR-structure, such that $G$ is contained in the group of pseudo-Hermitian transformations $\text{Psh}(M, \{\eta, J\})$. See Proposition 26 in Section 4.2 for further details.

Together with Theorem 7 this reasoning reveals the following strong relationship between the above type of equivariant cohomology group and the properness of actions on CR- and conformal manifolds:

**Theorem 8.** Let $G$ be a closed Lie subgroup of diffeomorphisms that preserves either a strictly pseudo-convex CR-structure or a conformal Riemannian structure on $M$. Then $G$ acts properly on $M$ if and only if the first differentiable cohomology group $H^1_d(G, C^\infty(M, \mathbb{R}))$ vanishes.

Conversely, taking into account that $\text{SU}(n+1, 1) = \text{Aut}_{CR}(S^{2n+1})$ does not act properly on $S^{2n+1}$, and similarly for the other geometric group actions appearing in (ii) of Theorem 1 and Theorem 2, it follows that these give examples of actions of semisimple Lie groups whose associated first equivariant differentiable cohomology groups are non-vanishing:

**Corollary 9.** The following hold:

$$H^1_d(\text{SU}(n+1, 1), C^\infty(S^{2n+1}, \mathbb{R})) \neq \{0\}$$

$$H^1_d(\text{SO}(n+1, 1), C^\infty(S^n, \mathbb{R})) \neq \{0\}.$$

The proof of Theorem 8 is given in Section 4.4. In fact, Theorem 29 which can be found there gives a much stronger characterization of proper actions which is also involving vanishing of higher cohomology.

1.3. **Actions on manifolds and differentiable cohomology of Lie groups.** The methods which apply to prove Theorem 7 are based on foundational ideas on continuous and differentiable cohomology originally developed in the works of Mostow and Mostow-Hochschild [24, 20], and also on somewhat later expositions [6, 12]. In particular, we use the slice theory of proper actions, integration over compact groups, and Shapiro Lemma type results for equivariant differentiable cohomology of groups with coefficients in function spaces. Finally, the language of sheaf cohomology is applied to
pursue the necessary patching of local data to a global vanishing theorem, see Section 2 and Section 3 for details.

1.3.1. Vanishing of differentiable cohomology associated to proper actions. In the most general form presented in Section 3 the vanishing theorem asserts, that, for any smooth and proper action of $G$ on $X$ and any differentiable $G$-module $V$, the differentiable cohomology module

$$H^r_d (G, \mathcal{C}^\infty (X,V))$$

is acyclic, that is, $H^r_d (G, \mathcal{C}^\infty (X,V)) = \{ 0 \}$, $r \geq 1$.

With respect to the trivial $G$-module $V = \mathbb{R}$, this amounts to:

**Theorem 10.** Let $G$ be a Lie group which acts smoothly and properly on a manifold $X$. Then the differentiable cohomology groups of $G$ satisfy

$$H^r_d (G, \mathcal{C}^\infty (X,\mathbb{R})) = \{ 0 \} , \text{ for all } r \geq 1.$$

The proof of Theorem 10 and its further generalizations will be given in Section 3.

Further applications. We would like to point out that vanishing results of the above type are interesting in their own right and potentially bear many applications to the study of proper actions of Lie groups on geometric manifolds. For example, note that we do not assume that $G$ is connected, so that the theorems also apply to properly discontinuous actions and the covering theory of manifolds. If $G$ is discrete, the differentiable cohomology groups of $G$ with coefficients in a differentiable module $V$ reduce to the ordinary discrete cohomology groups of $G$ with coefficients $V$. Note further that, in case $G$ is acting properly discontinuously on $X$, Theorem 10 is known due to seminal work of Conner-Raymond [13], see also [31, Chapter 7.5]. The equivariant cohomology theory associated with properly discontinuous actions of groups and applications of the corresponding vanishing results to the topology of manifolds are discussed and surveyed in the book [31].

2. Equivariant continuous cohomology

In this section we start by reviewing several facts about continuous cohomology of locally compact groups (compare [12], and also [3, Chapter IX]). Based on this we establish vanishing results for certain equivariant continuous cohomology groups related to proper actions on locally compact spaces. In the proof of the vanishing result we take a sheaf theoretic approach to equivariant cohomology.

**Conventions.** All spaces $X$ and $Y$ are assumed to be locally compact (and Hausdorff). We let $\mathcal{C}(X,Y)$ denote the space of continuous maps with the compact open topology.
2.1. Continuous cohomology of locally compact groups. Let $G$ be a locally compact topological group and $V$ a continuous $G$-module. By definition, $V$ is a topological abelian group with a continuous action of $G$ by automorphisms. In addition we shall always assume that $V$ is a Hausdorff locally convex topological real vector space. An isomorphism of continuous $G$-modules is a $G$-equivariant isomorphism (linear homeomorphism) of topological vector spaces.

2.1.1. Definition of continuous cohomology groups. Denote by $C^r(G;V) := C(G^r, V)$ the $G$-module of continuous (inhomogeneous) $r$-cochains of $G$ into $V$, which consists of continuous maps of the $r$-fold product $G^r = G 	imes \cdots 	imes G$ into $V$.

The inhomogeneous coboundary operator

$$\partial^r : C^r(G;V) \to C^{r+1}(G;V),$$

satisfying $\partial^{r+1} \circ \partial^r = 0$, is defined via:

$$\partial^0(v)(\alpha) = \alpha \cdot v - v \quad (\alpha \in G, v \in V),$$

$$\partial^r \lambda(\alpha_1, \ldots, \alpha_{r+1}) = \alpha_1 \cdot \lambda(\alpha_2, \ldots, \alpha_{r+1})$$

$$+ \sum_{i=1}^{r} (-1)^i \lambda(\alpha_1, \ldots, \alpha_i \alpha_{i+1}, \ldots, \alpha_{r+1})$$

$$+ (-1)^{r+1} \lambda(\alpha_1, \ldots, \alpha_r).$$

(2.1)

Define the continuous cohomology groups

$$H^r(G,V) := \ker(\partial^r) / \operatorname{im}(\partial^{r-1}), \ r \geq 1$$

and put

$$H^0(G,V) := \ker(\partial^0) = V^G.$$  

2.1.2. Cohomology of compact groups and integration. Recall that the locally convex vector space $V$ is called quasi-complete if all closed bounded subsets of $V$ are complete. For quasi-complete $V$ integration of compactly supported continuous functions in $C(G, V)$ is defined with respect to a left-invariant measure on $G$. That is, $V$ is a $G$-integrable module in the sense of [20, §3] (see also [24, 2.13]).

Remark. For a survey concerning the existence of vector valued integrals refer to [11].

Proposition 1 ([12, Lemma 7]). Suppose that $G$ is compact and let $V$ be a quasi-complete $G$-module. Then

$$H^r(G,V) = \{0\}, \ r \geq 1.$$

Proof. Let $\lambda \in C^r(G;V)$ satisfy $\partial^r \lambda = 0$. With respect to a normalized finite Haar measure $d\alpha$ on $G$ define

$$\tau(\alpha_1, \ldots, \alpha_{r-1}) = (-1)^r \int_G \lambda(\alpha_1, \ldots, \alpha_{r-1}, \alpha) \, d\alpha,$$
to obtain \( \tau \in C^{r-1}(G;V) \), which satisfies
\[
\partial^{r-1} \tau = \lambda.
\]

2.1.3. Shapiro’s lemma. Let \( H \) be a closed subgroup of \( G \) and let \( W \) be an \( H \)-module. Put
\[
\text{Ind}_{H}^{G} W = \{ f \in C(G,W) \mid f(gh^{-1}) = h \cdot f(g), \ g \in G, h \in H \}.
\]
Then \( \text{Ind}_{H}^{G} W \) turns into a continuous \( G \)-module, by the action
\[
(\alpha \cdot f)(g) = f(\alpha^{-1}g).
\]
Remark that \( G/H \) is paracompact [7, Ch III §6, Proposition 13]. Suppose that the quotient map
\[
G \to G/H
\]
admits continuous local cross sections. Then we have:

**Proposition 2** (Shapiro Lemma, cf. [12, Propositions 3,4]).
\[
H^r(H,W) \cong H^r(G, \text{Ind}_{H}^{G} W), \ r \geq 0.
\]

2.2. Equivariant cohomology groups. Let \( X \) be a \( G \)-space and \( V \) a continuous \( G \)-module. We consider the space of maps
\[
C(X,V)
\]
as a continuous \( G \)-module, where, for all \( f \in C(X,V), \alpha \in G \), we declare the action of \( G \) as
\[
(\alpha \cdot f)(x) = \alpha \cdot (f(\alpha^{-1} \cdot x)).
\]
We are interested in the properties of the continuous cohomology of \( G \) with coefficients in the \( G \)-module \( C(X,V) \).

2.2.1. Associated bundle over \( G/H \). Let \( H \) be a closed subgroup of \( G \). For any \( H \)-space \( Y \) the diagonal action of \( H \) on \( G \times Y \) is
\[
h \cdot (g,y) = (gh^{-1}, h \cdot y), \ \text{for} \ h \in H.
\]
We then have the associated bundle with fiber \( Y \)
\[
G \times_{H} Y \to G/H.
\]
The space \( G \times_{H} Y \) is defined by taking the quotient of \( G \times Y \) by the action \[[2.3]]\). Observe that \( G \) acts on \( G \times_{H} Y \) by left-multiplication on the first factor, which turns \( G \times_{H} Y \) into a \( G \)-space and \[[2.4]]\) into a \( G \)-map.
Local cross sections. In order that $\underline{[2.4]}$ is a fiber bundle we need to require that $G \to G/H$ is a fiber bundle:

**Lemma 3.** Suppose that $G/H$ admits local cross sections. Then:

1. The map $G \to G/H$ is a fiber bundle.
2. The map $G \times_H Y \to G/H$ is a fiber bundle.
3. The quotient map $G \times Y \to G \times_H Y$ admits local cross sections.

**Proof.** Let $s : U \to G$ be a local section for $\pi : G \to G/H$. Then the map $g \mapsto (gH, s(gH)^{-1}y)$, $\pi^{-1}(U) \to U \times H$ is an $H$-equivariant homeomorphism. Hence, $\pi$ is an $H$-principal bundle over $G/H$.

In the view of (1), (2) is implied by $[2.4]$ II Theorem 2.4. In fact, for any section $s$ of $\pi$, the map $U \times Y \to G \times_H Y$, $(gH, y) \mapsto [s(gH), y]$ defines a local bundle chart for $G \times_H Y$, showing (3). \qed

Put $\mathcal{C}(G \times Y,V)^H = \{ f : G \times Y \to V \mid f(gh^{-1}, h \cdot y) = f(g, y) \}$ for the subspace of $H$-invariant functions in $\mathcal{C}(G \times Y,V)$.

**Lemma 4.** Suppose that $G/H$ admits local cross sections. Then the natural map

$$\iota : \mathcal{C}(G \times Y,V)^H \to \mathcal{C}(G \times_H Y,V)$$

is a homeomorphism.

**Proof.** Given $\tilde{f} \in \mathcal{C}(G \times Y,V)^H$, $\iota(\tilde{f}) = f$ is the unique map satisfying the relation $\tilde{f} = f \circ p$, where $p : G \times Y \to G \times_H Y$ is the quotient map. Now, for any $C \subseteq G \times_H Y$ compact and $U \subseteq V$ open with $f(C) \subseteq U$, consider

$$\mathcal{N}(C,U) = \{ \ell : G \times_H Y \to V \mid \ell(C) \subseteq U \}$$

which is an open neighborhood of $f$ in $\mathcal{C}(G \times_H Y,V)$. By (3) of Lemma $\underline{2.4}$ p : $G \times Y \to G \times_H Y$ admits local cross sections. Since $G \times_H Y$ is locally compact, we may cover $C$ with finitely many compact neighborhoods $C_i$ such that $p$ admits a section $s_i$ over $C_i$. Then put $\tilde{C} = \bigcup_i s_i(C_i \cap C)$. Hence, $\tilde{f} \in \mathcal{N}(\tilde{C},U)$ and, for any $\tilde{\ell} \in \mathcal{N}(\tilde{C},U) \cap \mathcal{C}(G \times Y,V)^H$, we have that $\iota(\tilde{\ell}) \in \mathcal{N}(C,U)$. Hence, $\iota$ is continuous.

Since $G \times_H Y$ is locally compact Hausdorff, the inverse map

$$\iota^{-1} : \mathcal{C}(G \times_H Y,V) \to \mathcal{C}(G \times Y,V)^H, \ f \mapsto \tilde{f} = f \circ \pi$$

is clearly continuous (with respect to the compact open topology). \qed

2.2.2. **Adjointness properties of Ind$_H^G$.** Let $V$ be a continuous $G$-module. Since $H$ is a subgroup of $G$, the $G$-module $V$ is an $H$-module by restriction. The latter will be denoted by $\text{res}_H^G V$. Similarly, if $X$ is a $G$-space, we let $\text{res}_H^GX$ denote the restricted $H$-space.
Lemma 5. Let $V$ be a continuous $G$-module. There is a natural isomorphism of continuous $G$-modules
\[
\mu : \text{Ind}_H^G \mathcal{C}(Y, \text{res}_H^G V) \to \mathcal{C}(G \times_H Y, V).
\]
Similarly, if $X$ is a $G$-space and $W$ an $H$-module, then there is a natural isomorphism of continuous $G$-modules
\[
\nu : \mathcal{C}(X, \text{Ind}_H^G W) \to \text{Ind}_H^G \mathcal{C}(\text{res}_H^G X, W).
\]

Proof. Given $f \in \text{Ind}_H^G \mathcal{C}(Y, \text{res}_H^G V) \subseteq \mathcal{C}(G, \mathcal{C}(Y, V))$, we define
\[
\bar{\mu}(f) \in \mathcal{C}(G, \times_H Y, V), \text{ via } \bar{\mu}(f)(g, y) = g \cdot (f(g)(y))
\]
Since $f$ is $H$-equivariant (by definition of $\text{Ind}_H^G$), it follows
\[
\bar{\mu}(f)(h \cdot (g, y)) = \bar{\mu}(f)(g, y).
\]
That is, $\bar{\mu}(f)$ is $H$-invariant for the diagonal action \((2.3)\) and descends to
\[
\mu(f) \in \mathcal{C}(G \times_H Y, V).
\]
Conversely, every element of $\mathcal{C}(G \times_H Y, V)$ arises in this way. One can verify easily that $\mu$ is $G$-equivariant, and also that it is a linear map. It remains to show that $\mu$ is a homeomorphism.

Note first, since $G$ is Hausdorff and $Y$ is locally compact, the natural map
\[
\mathcal{C}(G, \mathcal{C}(Y, V)) \to \mathcal{C}(G \times Y, V)
\]
is a homeomorphism of function spaces with respect to the compact open topologies \([7\text{ Ch X §3, Corollaire 2}]\). It follows that
\[
\bar{\mu} : \text{Ind}_H^G \mathcal{C}(Y, V) \to \mathcal{C}(G \times_H Y, V)^H
\]
gives a homeomorphism. Finally, the push forward map $\mathcal{C}(G \times Y, V)^H \to \mathcal{C}(G \times_H Y, V)$ is a homeomorphism, by Lemma \([4]\). Therefore, $\mu$ is a homeomorphism. This proves \((2.5)\).

The proof of \((2.6)\) follows similarly, using the adjointness \((2.8)\). □

2.2.3. Slices, tubes and induced representations. Let $X$ be a $G$-space and $p \in X$. Put $G_p = \{ g \in G \mid g \cdot p = p \}$. Let $S_p$ be a $G_p$-invariant locally closed subset of $X$ with $p \in S_p$. Then $S_p$ is called a slice for the $G$-action on $X$ if
\[
U_p = G \cdot S_p
\]
is an open subset of $X$ and the map
\[
G \times_{G_p} S_p \to U_p, \ (g, s) \mapsto g \cdot s
\]
is a homeomorphism. If a slice exists then $U_p$ is called a tube around the orbit $G \cdot p$.

For any $G$-invariant open subset $U \subseteq U_p$, $p \in U$, we define
\[
S_U = U \cap S_p.
\]
Then $S_U$ is a $G_p$-space. For any $G$-module $V$, since $U$ is a $G$-space, $\mathcal{C}(U, V)$ is a $G$-module.
Lemma 6 \((G\text{-tubes})\). There is a natural isomorphism of continuous \(G\)-modules

\[
\text{Ind}^G_{G_p} \mathcal{C}(S_U, \text{res}^G_{G_p} V) \to \mathcal{C}(U, V).
\]

Proof. Since \(S_p\) is a slice at \(p\), the natural map \((2.9)\) is a \(G\)-equivariant homeomorphism, and so are the restricted maps

\[
G \times_{G_p} S_U \to U.
\]

In particular, \(U\) is a \(G\)-tube around \(G \cdot p\). In view of the homeomorphism \((2.11)\), the claim follows from Lemma 5 by taking \(H = G_p\). \qed

We conclude:

Lemma 7 \((\text{Local vanishing})\). Suppose there exists a slice \(S_p\) for \(p \in X\), such that \(G_p\) is compact and \(G \to G/G_p\) has local cross sections. Let \(V\) be a quasi-complete continuous \(G\)-module. Then, for all sufficiently small \(G\)-invariant neighbourhoods \(U\) of \(p\),

\[
H^r(G, \mathcal{C}(U, V)) = \{0\}, \ r \geq 1.
\]

Proof. By Lemma 6 \(\mathcal{C}(U, V) = \text{Ind}^G_{G_p} \mathcal{C}(S_U, \text{res}^G_{G_p} V)\). Shapiro’s lemma \(\text{(Proposition 2)}\) states that

\[
H^r\left(G, \text{Ind}^G_{G_p} \mathcal{C}(S_U, \text{res}^G_{G_p} V)\right) = H^r\left(G_p, \mathcal{C}(S_U, \text{res}^G_{G_p} V)\right).
\]

As \(V\) is quasi-complete also \(\mathcal{C}(S_U, V)\) is quasi-complete. The module \(\mathcal{C}(S_U, V)\) is therefore \(G_p\)-integrable \(\text{(see in particular [20 Proposition 3.1])}\). As \(G_p\) is compact it follows from Proposition 1 that

\[
H^r\left(G_p, \mathcal{C}(S_U, \text{res}^G_{G_p} V)\right) = \{0\}, \ r \geq 1.
\]

\qed

2.3. Sheaf theoretic interpretation of equivariant cohomology. Let \(X\) be a \(G\)-space and let \(V\) be a continuous \(G\)-module. We let

\[
\pi : X \to X/G
\]

denote the quotient map for the \(G\)-action. Furthermore, let

\[
\mathcal{C}_{X, V} = \mathcal{C}(\cdot, V)
\]

denote the sheaf of continuous functions on \(X\) with values in \(V\), as well as,

\[
\mathcal{C}_X \text{ and } \mathcal{C}_{X/G}
\]

the structure sheaves of continuous real valued functions on \(X\) and \(X/G\), respectively. Since \(G\) acts on \(X\) and \(V\), \(\mathcal{C}_{X, V}\) is a \(G\)-sheaf. That is, the sheaf \(\mathcal{C}_{X, V}\) has an action of \(G\) by co-morphisms which is defined by \((2.2)\). Remark further that \(\mathcal{C}_{X, V}\) is also a sheaf of \(\mathcal{C}_X\)-modules. \(\text{(For general background on sheaf theory, see eg. [42 Chapter II] or [8]. For the notion of \(G\)-sheaves, see [17 Chapitre V].)}\)
**Direct image sheaves.** Let \( \mathcal{A} \) denote any sheaf on \( X \). The direct image \( \pi_* \mathcal{A} \) of \( \mathcal{A} \) is the sheaf on \( X/G \), where, for any open subset \( U \) of \( X/G \),
\[
\pi_* \mathcal{A} (U) = \mathcal{A} (\pi^{-1}(U)).
\]
We may view \( \pi_* \) as a left-exact functor taking \( G \)-sheaves on \( X \) to \( G \)-sheaves on \( X/G \) (where \( X/G \) has the trivial \( G \)-action). Of particular interest is the direct image sheaf of rings \( \pi_* \mathcal{C}_X \). Its subsheaf
\[
\pi^G_* \mathcal{C}_X
\]
of \( G \)-invariant functions is called the *equivariant direct image* of \( \mathcal{C}_X \). Note that the sheaf \( \pi^G_* \mathcal{C}_X \) is canonically isomorphic to the sheaf \( \mathcal{C}_{X/G} \) of continuous functions on \( X/G \).

**Resolution of structure sheaves on \( X/G \).** We declare a differential sheaf
\[
C^0_{X,V} \xrightarrow{\partial^0} C^1_{X,V} \xrightarrow{\partial^1} C^2_{X,V} \xrightarrow{\partial^2} \ldots
\]
of \( \pi_* \mathcal{C}_X \)-modules on \( X/G \). The module of sections of \( C^r_{X,V} \) over an open subset \( U \subseteq X/G \) is defined as
\[
C^r_{X,V} (U) = C^r (G; C(\pi^{-1}(U), V)).
\]
Here \( C(\pi^{-1}(U), V) \) is a \( G \)-module and, by definition, \( C^r_{X,V} (U) \) consists of the inhomogeneous \( r \)-cochains with coefficients \( C(\pi^{-1}(U), V) \). The differential
\[
C^r_{X,V} (U) \xrightarrow{\partial^r} C^{r+1}_{X,V} (U)
\]
is obtained by the usual formula \([2.1]\) for inhomogeneous cochains. It is trivial to verify that these local maps patch together to define a homomorphism of sheaves
\[
\partial^r : C^r_{X,V} \longrightarrow C^{r+1}_{X,V}.
\]

**Equivariant direct image of \( \mathcal{C}_{X,V} \).** We remark that \( C^0_{X,V} = \pi_* \mathcal{C}_{X,V} \) is a sheaf of \( G \)-modules, since \( C^0_{X,V} (U) = \mathcal{C}_{X,V} (\pi^{-1}(U)) \). The subsheaf
\[
\pi^G_* \mathcal{C}_{X,V}
\]
of \( G \)-invariant functions is generated by the presheaf
\[
U \mapsto C(\pi^{-1}(U), V)^G.
\]
Thus, we note that
\[
\pi^G_* \mathcal{C}_{X,V} = \ker \partial^0.
\]

**Remark.** Observe that \( \pi^G_* \mathcal{C}_{X,V} \) is not necessarily a sheaf of functions on \( X/G \), unless the action of \( G \) on \( V \) is trivial or \( G \) acts freely on \( X \). (In case \( V \) is a trivial \( G \)-module, we may identify \( \pi^G_* \mathcal{C}_{X,V} \) with the structure sheaf \( \mathcal{C}_{X/G,V} \) of continuous functions on \( X/G \) taking values in \( V \).)
Local vanishing of continuous cohomology. Suppose that, for all sufficiently small open neighbourhoods $U$ on $X/G$, we have

$$H^r (G, C(\pi^{-1}(U), V)) = \{0\}, \; r \geq 1.$$  

Then we shall say that the continuous cohomology groups of $G$ with coefficients $\pi_* \mathcal{C}_{X,V}$ vanish locally. This is clearly equivalent to the condition that the sequence (2.13) is an exact sequence of sheaves:

**Lemma 8.** If the continuous cohomology groups of $G$ with coefficients $\pi_* \mathcal{C}_{X,V}$ vanish locally then

$$\{0\} \longrightarrow \pi_*^G \mathcal{C}_{X,V} \longrightarrow C^0_{X,V} \xrightarrow{\partial^0} C^1_{X,V} \xrightarrow{\partial^1} C^2_{X,V} \xrightarrow{\partial^2} \ldots$$

is an exact sequence of sheaves on $X/G$.

**Cohomology of the equivariant direct image sheaf $\pi_*^G \mathcal{C}_{X,V}$.** By a standard argument on double complexes (compare [16, Théorème 2.4.1]), we can compute the sheaf cohomology groups $H^*_X/G(\pi_*^G \mathcal{C}_{X,V})$ as follows:

**Proposition 9.** Suppose that the continuous cohomology groups of $G$ with coefficients $\pi_* \mathcal{C}_{X,V}$ vanish locally. Then there exists a spectral sequence converging to the sheaf cohomology $H^*_X/G(\pi_*^G \mathcal{C}_{X,V})$ with

$$E_{2}^{p,q} = H^p \left( H^q_{X/G}(C^*_X,V) \right).$$

Note that the induced complex of global sections for the resolution (2.15) takes the form

$$0 \longrightarrow \mathcal{C}(X,V)^G \longrightarrow \mathcal{C}(X,V) \xrightarrow{\partial^0} C^1(G; \mathcal{C}(X,V)) \xrightarrow{\partial^1} \ldots.$$  

By construction this is the inhomogeneous bar complex for the continuous cohomology of $G$ with coefficients in the $G$-module $\mathcal{C}(X,V)$. Therefore

$$E_{2}^{0,0} = H^0 \left( H^0_{X/G}(C^*_X,V) \right) = H^0 \left( C^*_G(G; \mathcal{C}(X,V)) \right) = H^0 \left( G, \mathcal{C}(X,V) \right).$$

**Paracompact quotient space $X/G$.** Suppose that $X/G$ is a paracompact Hausdorff space. Then (2.15) gives a resolution of $\pi_*^G \mathcal{C}_{X,V}$ by fine sheaves:

**Lemma 10.** If $X/G$ is a paracompact Hausdorff space, then the sheaves

1. $C^r_{X,V}, \; r \geq 0,$ and
2. $\pi_*^G \mathcal{C}_{X,V}$

are fine sheaves.

**Proof.** Since $X/G$ is a paracompact Hausdorff space, any locally finite covering by open sets admits a subordinate partition of unity. This implies that the structure sheaf $\mathcal{C}_{X/G}$ is a fine sheaf of rings. In particular, any sheaf of $\mathcal{C}_{X/G}$-modules is a fine sheaf (see [8, Theorem 9.16]). Now $C^r_{X,V}$ is a sheaf of $\mathcal{C}_{X/G}$-modules, where, given $\epsilon \in \mathcal{C}_{X/G}$, an open subset $U \subseteq X/G$ and $c \in C^r_{X,V}(U)$, we declare

$$\epsilon \cdot c \left( g_1, \ldots, g_r \right) = \left( \epsilon \circ \pi \right) \cdot c \left( g_1, \ldots, g_r \right).$$
It follows that $C^r_{X,V}$ is a fine sheaf, hence (1).

Since $\pi_*^G C_{X,V}$ is a sheaf of $\pi_*^G C_X = C_{X/G}$ -modules on $X/G$, it is a fine sheaf as well. Thus (2) holds. □

In this situation the equivariant continuous cohomology groups of $G$ may be expressed in terms of sheaf cohomology on $X/G$:

**Corollary 11.** Suppose that $X/G$ is a paracompact Hausdorff space and that (2.15) is exact. Then, for all $r \geq 0$, there is a natural isomorphism

$$H^r (G, C(X,V)) \cong H^r (\pi_*^G C_{X,V}) .$$

**Proof.** The homology of the complex of global sections for any resolution of $\pi_*^G C_{X,V}$ by fine sheaves is isomorphic to the sheaf cohomology $H^* (\pi_*^G C_{X,V})$ (see, for example [42, Section II.3]). By Lemma 10 part (1), the resolution (2.15) of $\pi_*^G C_{X,V}$ is fine. Thus the cohomology of the sheaf $\pi_*^G C_{X,V}$ is isomorphic to the homology of the complex (2.16). (In particular, the spectral sequence in Proposition 9 collapses at $E_2$. That is $E_2^{p,q} = \{0\}$, $q > 0$.) □

### 2.4. Vanishing of equivariant continuous cohomology.

If $X/G$ is a paracompact Hausdorff space, then $\pi_*^G C_{X,V}$ is a fine sheaf (by Lemma 11 part (2)). Therefore its sheaf cohomology must be acyclic. In the view of Corollary 11 this proves:

**Theorem 12** (Vanishing theorem, continuous case). Suppose that the continuous cohomology groups of $G$ with coefficients $\pi_* C_{X,V}$ vanish locally and that $X/G$ is a paracompact Hausdorff space. Then

$$H^r (G, C(X,V)) = \{0\} , \text{ for all } r \geq 1 .$$

**Proper actions of Lie groups.** A typical case for application arises in smooth actions on manifolds. Let $G$ be a Lie group and let $V$ be a quasi-complete continuous $G$-module. Then we have:

**Corollary 13** (Vanishing theorem, smooth manifolds). Let $G$ be a Lie group and $X$ a $G$-manifold on which $G$ acts smoothly and properly. Then

$$H^r (G, C(X,V)) = 0 , \text{ for all } r \geq 1 .$$

**Proof.** Since $X$ is paracompact and $G$ acts properly, $X/G$ is a paracompact Hausdorff space. By the differentiable slice theorem (see [27, §4, Lemma 4], for example), every point $p \in X$ admits a $G$-tube $U_p$. Moreover, since $G$ is a Lie group, for any closed subgroup $H$ of $G$, $G/H$ is a manifold and admits local (smooth) sections. Since $V$ is assumed quasi-complete, it follows by Lemma 7 that the continuous cohomology of $G$ with coefficients $C(X,V)$ vanishes locally. Therefore, Theorem 12 applies. □
3. Smooth coefficients and vanishing of equivariant differentiable cohomology

Let $G$ be a Lie group and $X$ a smooth manifold on which $G$ acts smoothly. Such $X$ will be called a differentiable $G$-space. For any continuous $G$-module $V$, where $V$ is a locally convex topological vector space (as in Section 2), let $C^\infty(X,V)$ denote the vector space of smooth functions on $X$ with values in $V$. Endowed with the $C^\infty$-topology of maps, $C^\infty(X,V)$ is a locally convex vector space and (quasi-) complete if $V$ is (quasi-) complete [18, Chapter 1, §10], see also [11, §3] and [36]. Moreover, since $G$ acts smoothly on $X$, with respect to (2.2), $C^\infty(X,V)$ becomes a continuous $G$-module in an obvious way.

Van Est and Mostow-Hochschild [20] introduced the notion of differentiable $G$-modules and differentiable cohomology groups $H^r_d(G, \cdot)$ based on smooth cochains (see Section 3.1 below for definitions). In particular, if $X$ is a differentiable $G$-space and $V$ a differentiable $G$-module then $C^\infty(X,V)$ is a differentiable $G$-module.

The main result for this section will be:

**Theorem 14 (Vanishing theorem, smooth case).** Let $X$ be a differentiable $G$-space on which $G$ acts properly, and let $V$ be a differentiable $G$-module. Then

$$H^r_d(G, C^\infty(X,V)) = H^r(G, C^\infty(X,V)) = \{0\}, \ r \geq 1.$$  

This result implies Theorem 10 in the introduction.

Note that Theorem 14 is a differentiable version of Corollary 13. Here we are dealing with smooth functions as coefficients instead of continuous functions. Likewise, the differentiable cohomology groups $H^*_d(G, \cdot)$ are using smooth cochains in their definition, and it is required that the coefficient modules for the differentiable cohomology functor $H^*_d(G, \cdot)$ are differentiable $G$-modules. We shall explain these notions right away in the following Section 3.1.

3.1. Differentiable cohomology groups. Since $G$ is a Lie group we can introduce a smooth analogue of the continuous cohomology theory, which was first systematically studied by Van Est [40]. Its foundations were further developed by Mostow and Hochschild, see [24, 20]. Another good reference is [10]. The differentiable cohomology of $G$ is a functor defined on the category of differentiable $G$-modules.

**Differentiable $G$-modules.** Let $V$ be a differentiable $G$-module. By this we mean a continuous $G$-module $V$ (with all the assumptions of Section 2 in place, in particular $V$ is a quasi-complete Hausdorff locally convex topological real vector space) that satisfies (see [10, 20]) that, for all $v \in V$,

1. the orbit map $G \to V, \ o_v : g \mapsto g \cdot v$ is smooth.
2. the map $V \to C^\infty(G,V), \ v \mapsto o_v$ is smooth.
An isomorphism of differentiable $G$-modules is a $G$-equivariant isomorphism of topological vector spaces.

**Smooth functions on $G$-spaces.** A differentiable $G$-space is a smooth manifold $X$ on which $G$ acts smoothly. In this situation,

$$C^\infty(X, V)$$

with the usual action (defined by (2.2)) is a differentiable $G$-module. (Compare [6, 8°) Proposition.]

**Smooth cochains.** The differentiable cohomology groups of $G$ with coefficients in the differentiable module $V$ are defined by using differentiable cochains instead of continuous cochains. For this, we consider in the complex of continuous inhomogeneous cochains

$$(C^r(G; V), \partial)$$

(see Section 2.1.1) the subcomplex of differentiable inhomogeneous cochains

$$(C^r_d(G; V), \partial)$$, where $C^r_d(G; V) := C^\infty(G^r, V)$.

We put $B^r_d(G; V) = \partial(C^{r-1}_d(G; V))$ and $Z^r_d(G; V) = \ker \partial \cap C^r_d(G; V)$. This defines the differentiable cohomology groups

$$H^r_d(G; V) = Z^r_d(G; V)/B^r_d(G; V).$$

We mention the following comparison result with continuous cohomology.

**Theorem 15** (Hochschild, Mostow [20, Theorem 5.1]). Let $G$ be a real Lie group and $V$ a differentiable $G$-module. Then the natural map

$$H^r_d(G; V) \to H^r(G, V)$$

is an isomorphism of (topological) vector spaces.

3.1.1. **Compact Lie groups.** Let $V$ be a differentiable $G$-module. By Proposition 1, vanishing of continuous cohomology with coefficients in $V$ follows for all compact Lie groups $G$. In fact, the same proof (or application of Theorem 14) and Theorem 15) shows:

**Lemma 16.** Let $G$ be a compact Lie group and let $V$ be a differentiable $G$-module. Then $H^r_d(G, V) = \{0\}$, for all $r \geq 1$.

3.1.2. **Smooth Shapiro lemma.** Let $H$ be a closed subgroup of $G$ and $W$ a differentiable $H$-module. Put

$$\text{Ind}^\infty_G^H W = \{ f \in C^\infty(G, W) \mid f(gh^{-1}) = h \cdot f(g), \ g \in G, \ h \in H \}.$$ 

Then the space $\text{Ind}^\infty_G^H W$ turns into a differentiable $G$-module, by declaring

$$(\alpha \cdot f)(g) = f(\alpha^{-1} g).$$

As it turns out the usual proof of Shapiro’s lemma (Proposition 2) works for differentiable cochains with respect to the functor $\text{Ind}^\infty_G^H$, compare [6]. It relies on a differentiable equivariant version of Frobenius reciprocity related to Lemma 20 below.

Let $W$ be a differentiable $H$-module.
Proposition 17 (Smooth Shapiro lemma, see [6, Théorème 11]). There is a natural isomorphism
\[ H^r_d(G, \text{Ind}^\infty_H^G W) \cong H^r_d(H, W), \quad r \geq 0. \]

Remark. Note that, in the view of Theorem 15, Proposition 17 implies that the continuous cohomology groups \( H^r(H, W) \) and \( H^r(G, \text{Ind}^\infty_H^G W) \) are isomorphic.

Associated bundles over \( G/H \). Let \( Y \) be a differentiable \( H \)-space. Then the associated bundle \( G \times_H Y \) is a smooth \( G \)-manifold, and a locally trivial smooth fiber bundle over \( G/H \) with fiber \( Y \). The following is the differentiable analogue of Lemma 5.

Lemma 18. Let \( V \) be a differentiable \( G \)-module. There is a natural isomorphism of differentiable \( G \)-modules
\[ \mu : \text{Ind}^\infty_H^G C^\infty(Y, \text{res}^G_H V) \to C^\infty(G \times_H Y, V). \]

Proof. First note that for any two smooth manifolds \( A, B \), we have (cf. [18, Ch. 1, §10 and Ch. 3, §8]) the adjoint formula
\[ C^\infty(A, C^\infty(B, V)) = C^\infty(A \times B, V). \]

Also, since \( H \) acts properly and freely on \( G \times Y \), the differentiable slice theorem (compare (3.4)), shows that the quotient map \( G \times Y \to G \times_H Y \) is a smooth \( H \)-principal bundle map which is locally trivial.

Now, for any trivial \( H \)-principal bundle \( A \times H, C^\infty(A \times H, V)^H = C^\infty(A, V) \). This allows to show the homeomorphism
\[ C^\infty(G \times Y, V)^H \to C^\infty(G \times_H Y, V) \]
(compare Lemma 4). The proof of Lemma 18 carries now through analogously to the one of Lemma 5 (1), using (3.2) and (3.3). \( \Box \)

Corollary 19 (Smooth Shapiro lemma for actions on functions). There is a natural isomorphism
\[ H^r_d(H, C^\infty(Y, \text{res}^G_H V)) \to H^r_d(G, C^\infty(G \times_H Y, V)). \]

Equivariant reciprocity lemma. Let \( X \) be a differentiable \( G \)-space. Then \( X \) is also a differentiable \( H \)-space by restriction. We denote this space by \( \text{res}^G_H X \).

We mention the analogue of Lemma 5 (2):

Lemma 20. There is a natural isomorphism of differentiable \( G \)-modules
\[ \nu : C^\infty(X, \text{Ind}^\infty_H^G W) \to \text{Ind}^\infty_H^G C^\infty(\text{res}^G_H X, W). \]

3.1.3. Local vanishing for smooth coefficients.
Differentiable slices and smooth $G$-tubes. Let $S_p$ be a differentiable slice at $p \in X$, and $U_p = G \cdot S_p$ the corresponding smooth $G$-tube. By definition a differentiable slice $S_p$ is a submanifold such that the natural map
\begin{equation}
G \times G_p S_p \to U_p
\end{equation}
is a $G$-equivariant diffeomorphism (see [27, §2 Lemma 4]). As before, for any, $G$-invariant open subset $U \subseteq U_p$ of $X$, $p \in U$, define $S_U = S_p \cap U$, which is an open submanifold of the manifold $S_p$. Moreover, then $S_U$ is a differentiable slice with tube $U$. In fact, since (3.4) is a $G_p$-equivariant diffeomorphism, so are the restricted maps $G \times G_p S_U \to U$. In particular, if a tube $U_p$, exists we may always find a differentiable slice $S_U$ near $p$ which has compact closure. That is, by taking $U = G \cdot S$, where $S$ is a $G_p$-invariant neighborhood of $p$ in $S_p$ with compact closure in $S_p$.

Lemma 21 (Smooth $G$-tubes). There is a natural isomorphism of differentiable $G$-modules
\begin{align}
\text{Ind}^G_{G_p} C^\infty(S_U, \text{res}_{G_p}^G V) & \to C^\infty(U, V) \quad \text{and} \\
H^r_d \left( G, C^\infty(U, V) \right) & = \{0\}, \quad \text{for all } r \geq 1.
\end{align}

Proof. By Lemma 15, $\text{Ind}^G_{G_p} C^\infty(S_U, \text{res}_{G_p}^G V) = C^\infty(G \times G_p S_U, V)$. Using the differentiable Shapiro’s lemma (Proposition 17) we conclude
\begin{equation}
H^r_d \left( G, \text{Ind}^G_{G_p} C^\infty(S_U, V) \right) = H^r_d \left( G_p, C^\infty(S_U, V) \right).
\end{equation}
Since $G_p$ is compact, we have $H^r \left( G_p, C^\infty(S_U, V) \right) = \{0\}$, $r \geq 1$. \qed

3.2. Proof of Theorem 14.
Let $X$ be a differentiable $G$-space and $V$ a differentiable $G$-module. We are also assuming that $X$ is a proper $G$-space. As before we let

$$
\pi : X \to X/G
$$
denote the quotient map for the $G$-action. Then $X/G$ is a locally compact, paracompact Hausdorff space, since $G$ acts properly. Moreover, by the differentiable slice theorem [27, §2 Lemma 4] differentiable slices do exist for every $p \in X$. Therefore, the local vanishing property for the differentiable cohomology of $G$ with coefficients in the function space $C^\infty(X, V)$ is satisfied by application of Lemma 21.

Differentiable structure sheaves. Following Section 2.3 we are now considering the properties of various sheaves of functions on $X$ and $X/G$ which are associated to the action of $G$. First let

$$
C^\infty_{X,V} = C^\infty(\cdot, V)
$$
denote the sheaf of smooth functions on $X$ with values in $V$. Since $G$ acts on $X$ and $V$, the sheaf $C^\infty_{X,V}$ has an action of $G$ by co-morphisms which is defined by (2.2). Furthermore, let

$$
C^\infty_X \text{ and } C^\infty_{X/G}
$$
denote the structure sheaves of smooth real valued functions on $X$ and $X/G$, respectively. Note that $X/G$ is in general not a smooth manifold, but we can define

$$\mathcal{C}^\infty_{X/G} = \pi^G_* \mathcal{C}^\infty_X.$$  

Similarly, for any differentiable $G$-module $V$, we have the equivariant direct image sheaf $\pi^G_* \mathcal{C}^\infty_{X,V}$ (compare Section 2.3).

**Lemma 22.** The sheaves $\pi^G_* \mathcal{C}^\infty_{X,V}$ are fine sheaves.

**Proof.** Since $\pi^G_* \mathcal{C}^\infty_{X,V}$ is a sheaf of $\mathcal{C}^\infty_{X/G} = \pi^G_* \mathcal{C}^\infty_X$-modules, it is sufficient to show that the latter is a fine sheaf of rings (see for example [8, §9]). This amounts to constructing, for any covering of the form $\{\pi^{-1}(U_j)\}$ of $X$, where $\{U_j\}$ is a locally finite covering of $X/G$, an associated subordinate partition of unity by $G$-invariant smooth functions $\{\tilde{\eta}_j \in \mathcal{C}^\infty(X)^G\}$. To this end, it is sufficient to construct a partition of unity subordinate to any locally finite covering of $X$ by $G$-invariant open subsets that refines the covering $\{\pi^{-1}(U_j)\}$. Therefore, since $X$ is covered by $G$-tubes, we may assume that $\pi^{-1}(U_j) = G \times_{G_j} S_j$ is a differentiable $G$-tube and that $S_j$ has compact closure. In particular, $U_j$ has compact closure in $X/G$.

Since $X/G$ is paracompact, the covering $\{U_j\}$ has a subordinate continuous partition of unity $\epsilon_j$. Therefore the functions $\epsilon_j \circ \pi \in \mathcal{C}(X)^G$ give a continuous $G$-invariant partition of unity subordinate to $\{\pi^{-1}(U_j)\}$.

Moreover, we may arrange things that there exist open $G$-invariant subsets $W^1 \subset W^2 \subset S_j$, which in some coordinate system are diffeomorphic to Euclidean balls of radius 1, respectively 2, such that

$$K_j = \text{supp} (\epsilon_j \circ \pi)|_{S_j} \subset W^1.$$  

Using the same technique as employed in the proof of [9, Ch. VI, §4 4.2 Theorem] (for reference on approximation of continuous functions by smooth functions, see [32] and [19, Theorem 2.2]), we may approximate $\epsilon_j'$ by a smooth function $\eta_j' : S_j \to \mathbb{R}$ such that

1. $\eta_j'$ is a smooth (positive) function on $K_j$,
2. $\eta_j' = 0$ on $S_j - \overline{W^1}$ (where $\overline{W^1}$ denotes the closure of $W^1$).

Next let $\eta_j'' \in \mathcal{C}^\infty(S_j, \mathbb{R})$ be defined by

$$\eta_j''(x) = \int_{G_j} \eta_j'(gx) dg.$$  

Then $\eta_j''$ is a $G_j$-invariant nonnegative smooth function. Since $W^1$ is $G_j$-invariant, $\eta_j''$ also satisfies (1), (2). In particular, note that

$$\text{supp} \epsilon_j' \subseteq \text{supp} \eta_j''.$$
As the restriction map $C^\infty(\pi^{-1}(U_j), \mathbb{R})^G \rightarrow C^\infty(S_j, \mathbb{R})^G_j$ is bijective (as follows from (3.5), for example), we obtain a $G$-invariant smooth function $\eta_j$ on $\pi^{-1}(U_j)$, which restricts to $\eta_j''$ on $S_j$.

With the above provisions in place, and taking into account that $\pi : S_j \rightarrow U_j$ is a quotient map (see [9, Ch. VI, Proposition 3.3]), it follows that $G \cdot (S_j - W_1)$ is an open subset of $\pi^{-1}(U_j)$ and a neighborhood of the boundary of the open subset $G \cdot W_2$. Therefore, extension by 0 outside $\pi^{-1}(U_j)$ shows that $\eta_j$ arises as the restriction of a smooth $G$-invariant function $\eta_j$ defined on $X$ with support in $\pi^{-1}(U_j)$.

By construction, $a = \sum_j \eta_j \in C^\infty(X, \mathbb{R})$, is everywhere positive on $X$ and $G$-invariant. Thus \[ \tilde{\eta}_j = \frac{1}{a} \eta_j \]
defines a partition of unity by $G$-invariant smooth functions subordinate to the covering by $G$-neighborhoods $\{\pi^{-1}(U_j)\}$.

We conclude the proof of Theorem 14 using the reasoning developed in Section 2.3 as follows:

First of all, in the view of (3.6) we have a resolution of sheaves

\[
\{0\} \rightarrow \pi_*^G C_{X,V}^\infty \rightarrow (C_{X,V}^\infty)^0 \overset{\partial^0}{\rightarrow} (C_{X,V}^\infty)^1 \overset{\partial^1}{\rightarrow} (C_{X,V}^\infty)^2 \overset{\partial^2}{\rightarrow} \cdots,
\]

where $(C_{X,V}^\infty)^r$ are sheaves on $X/G$ whose sections over $U$ are differentiable $r$-cochains of $G$ with coefficients in smooth functions $C^\infty(\pi^{-1}(U), V)$. Since the sheaves $(C_{X,V}^\infty)^r$ are sheaves of $C^\infty_{X/G}$ modules (which is a fine sheaf of rings by Lemma 22) these are fine sheaves. It follows

\[
H^r(G, C^\infty(X, V)) \cong H^r(\pi_*^G C_{X,V}^\infty).
\]

Finally, by Lemma 22 $\pi_*^G C_{X,V}^\infty$ is a fine sheaf. It follows that the right hand cohomology in (3.8) is acyclic. Therefore,

\[
H^r_J(G, C^\infty(X, V)) = \{0\}, \ r \geq 1.
\]

This finishes the proof of Theorem 14.

4. PROOF OF THEOREMS 1 AND THEOREM 2

Given a pseudo-Hermitian structure $\{\omega, J\}$ on $M$, the distribution $D = \ker\omega$ defines a strictly pseudo-convex $CR$-structure $\{D, J\}$ on $M$. We have the following naturally associated transformation groups, namely the group

\[
Psh(M, \{\omega, J\}) = \{ \alpha \in \text{Diff}(M) \mid \alpha^*\omega = \omega, \ \alpha_s \circ J = J \circ \alpha_s |D \}\]

of pseudo-Hermitian transformations and the group of $CR$-automorphisms

\[
\text{Aut}_{CR}(M, \{D, J\}) = \{ \alpha \in \text{Diff}(M) \mid \alpha_s D = D, \ \alpha_s \circ J = J \circ \alpha_s |D \}.
\]
In general, the inclusion
\[ \text{Psh} (M, \{\omega, J\}) \leq \text{Aut}_{CR} (M, \{D, J\}) \]
is strict, since the contact form \( \omega \) is determined by \( D \) only up to con-
formal equivalence. Moreover, the Lie group \( \text{Psh} (M, \{\omega, J\}) \) always acts
properly on \( M \), whereas in some cases (as detailed in Theorem 4) the \( CR \)-
automorphism group \( \text{Aut}_{CR} (M, \{D, J\}) \) is too large and doesn’t act properly
on \( M \). About the possible relations of the group \( \text{Psh} (M, \{\omega, J\}) \) and the
\( CR \)-automorphism group \( \text{Aut}_{CR} (M, \{D, J\}) \), we shall prove:

**Theorem 23.** Let \((M, \{D, J\})\) be a strictly pseudo-convex \( CR \)-manifold and
let \( G \leq \text{Aut}_{CR} (M, \{D, J\}) \) be a subgroup of \( CR \)-automorphisms that acts
properly on \( M \). Then there exists on \( M \) a pseudo-Hermitian structure \( \{\eta, J\} \)
compatible with \( \{D, J\} \), such that \( G \leq \text{Psh} (M, \{\eta, J\}) \).

Together with Schoen’s theorem (Theorem 4), this result obviously im-
pies Theorem \( \text{I} \) in the introduction.

This section is organized as follows: In subsection 4.1 we prepare the
proof of Theorem 23 with a brief discussion of equivariant cr-
sewed homomorphisms which are associated to group actions on manifolds. Following
that we construct the canonical cohomology class associated to the action
of \( CR \)-automorphisms. The proof of Theorem 23 is based on the vanis-
hing of this class, see section 4.2. Analogous results for the conformal case and
also for locally conformal Kähler manifolds will be discussed in subsections
4.3 and 4.5.

4.1. **Equivariant crossed homomorphisms.** Let \( G \) be a Lie group which
acts smoothly on the manifold \( X \). We let \( \mathbb{R}^{>0} \) denote the multiplicative
group of positive real numbers. Next consider

\[ C^{\infty} (X, \mathbb{R}^{>0}) \]

the group of all smooth maps from \( X \) into \( \mathbb{R}^{>0} \) endowed with its natural
\( G \)-module structure, where, for \( \alpha \in G \), \( f \in C^{\infty} (X, \mathbb{R}^{>0}) \),

\[ (\alpha \cdot f) (x) = f(\alpha^{-1} x) . \]

In fact, taking the \( C^{\infty} \)-topology of maps, \( C^{\infty} (X, \mathbb{R}^{>0}) \) is a differentiable \( G \)-
module in the sense of Section 3.1 and it is isomorphic to the \( G \)-module
\( C^{\infty} (X, \mathbb{R}) \). Concerning the associated differentiable cohomology groups of
\( G \), we note:

**Theorem 24.** Suppose that \( G \) acts properly on \( X \). Then, for all \( r \geq 1 \), we have
\( H^{r}_{d} (G, C^{\infty} (X, \mathbb{R}^{>0})) = \{0\} \).

**Proof.** The map \( \exp : C^{\infty} (X, \mathbb{R}) \to C^{\infty} (X, \mathbb{R}^{>0}) \) defined by \( u = \exp f \), that is,

\[ u(x) = e^{f(x)} , \ x \in X \]
is easily seen to be an isomorphism of differentiable $G$-modules. In fact, the correspondence is $G$-equivariant, since, for all $\alpha \in G$,\[
\exp (\alpha \cdot f)(x) = e^{f(\alpha^{-1}x)} = u(\alpha^{-1}x) = (\alpha \cdot u)(x).
\]
Hence, for all $r$, there is an isomorphism of differentiable cohomology groups\[
H^r_d (G, C^\infty (X, \mathbb{R})) \cong H^r_d (G, C^\infty (X, \mathbb{R}^>)).
\]
Since $G$ acts properly, Theorem 24 implies\[
H^r_d (G, C^\infty (X, \mathbb{R}^>)) = \{0\}, \ r \geq 1.
\]

4.1.1. **Differentiable crossed homomorphisms.** Smooth one-cocycles \(\lambda \in C^\infty (G, C^\infty (X, \mathbb{R}^>))\), \(\partial^1 \lambda = 0\)
are representing the elements of the first differentiable cohomology group\[
H^1_d (G, C^\infty (X, \mathbb{R}^>)).
\]
The condition \(\partial^1 \lambda = 0\) amounts to the requirement that, for all \(\alpha, \beta \in G\),\[
(4.2) \quad \lambda(\alpha \beta)(x) = \lambda(\beta)(\alpha^{-1}x) \cdot \lambda(\alpha)(x).
\]
Denoting by \(\alpha^*\) the covariant map on forms which is induced by \(\alpha\), this relation can be written in the concise form\[
(4.3) \quad \lambda(\alpha \beta) = \alpha^* \lambda(\beta) \cdot \lambda(\alpha).
\]
Such \(\lambda\) are called **differentiable crossed homomorphisms**.

**Exact crossed homomorphisms.** For any crossed homomorphism \(\lambda\), the cohomology class\[
[\lambda] \in H^1_d (G, C^\infty (X, \mathbb{R}^>))
\]
vanishes if there exists \(v \in C^\infty (X, \mathbb{R}^>)\), with \(\partial^0 v = \lambda\). That is, if\[
(4.4) \quad (\alpha^* v) \cdot v^{-1} = \lambda(\alpha), \ \text{for all } \alpha \in G,
\]
or, equivalently, \(\lambda(\alpha)(x) = v(\alpha^{-1}x) \cdot v^{-1}(x)\), for all \(x \in X\). Moreover, two crossed homomorphisms \(\lambda\) and \(\lambda'\) represent the same class in the group \(H^1_d (G, C^\infty (X, \mathbb{R}^>))\) if and only if \(\lambda' = \lambda \cdot \partial^0 v\), for some \(v \in C^\infty (X, \mathbb{R}^>)\).

4.2. **CR-automorphisms and canonical class.** By definition, for any pseudo-Hermitian manifold \((M, \{\omega, J\})\), the Levi form of the underlying CR-structure\[
B = d\omega (J \cdot, \cdot)
\]
is **positive definite** on \(D = \ker \omega\). For any CR-automorphism \(\alpha \in \text{Aut}_{\text{CR}}(M, \{D, J\})\), the equation \(\alpha^*(D) = D\) implies that there exists \(u_\alpha \in C^\infty (M, \mathbb{R}^>)\) with\[
(4.5) \quad \alpha^* \omega = u_\alpha \cdot \omega.
\]
(In fact, to show \(u_\alpha > 0\), note that, for any \(0 \neq v \in D = \ker \omega\), we have \(B(\alpha^* v, \alpha^* v) = d\omega (J\alpha^* v, \alpha^* v) = d\alpha^* \omega (Jv, v) = u_\alpha B(v, v) > 0\). )
4.2.1. Associated crossed homomorphism and canonical cohomology class.

Let 

\[ G \leq {\text{Aut}}_{CR}(M, \{D, J\}) \]

be a group of CR-automorphisms. Next choose a compatible pseudo-Hermitian structure \{\omega, J\} for the CR-structure on \( M \). Define a map

\[ \lambda_{CR} : G \to C^\infty(M, \mathbb{R}^{>0}) \]

by declaring

\[ (4.6) \quad \lambda_{CR}(\alpha) = \alpha_* u_\alpha, \; \alpha \in {\text{Aut}}_{CR}(M), \]

where \( u_\alpha \) is as in \([4.5]\) defined relative to \( \omega \). (More explicitly, \([4.6]\) amounts to \( \lambda_{CR}(\alpha)(x) = u_\alpha(\alpha^{-1}x) \), for all \( x \in M \).

We claim that \( \lambda_{CR} \) is a crossed homomorphism for \( G \).

**Definition 25.** The map \( \lambda_{CR} \) is called crossed homomorphism associated to \( G \) and the pseudo-Hermitian structure \{\omega, J\}. Its cohomology class \( \mu_{CR} \) is defined by the underlying CR-structure only and it is called the canonical class associated to the CR-action of \( G \).

The geometric meaning of the canonical cohomology class is given by:

**Proposition 26** (Cohomology class of CR-transformation group). Suppose that \( M \) is a strictly pseudo-convex CR-manifold and let \( G \) be a Lie subgroup of the group of CR-automorphisms of \( M \). Then:

1. There exists in the differentiable cohomology of \( G \) a natural associated class

\[ \mu_{CR} = [\lambda_{CR}] \in H^1_G (G, C^\infty(M, \mathbb{R}^{>0})) \]

which is induced by the CR-structure on \( M \).

2. Moreover, the class \( \mu_{CR} \) vanishes if and only if there exists a contact form \( \eta \) compatible with the CR-structure, such that \( G \) is contained in the group of pseudo-Hermitian transformations \( \text{Psh}(M, \{\eta, J\}) \).

**Proof.** To show that \( \lambda_{CR} \) is a crossed homomorphism, we calculate

\[ (\alpha \cdot \beta)^* \omega = \beta^* (\alpha^* \omega) = \beta^* (u_\alpha \omega) = (\beta^* u_\alpha) \cdot u_\beta \omega. \]

As, according to \([4.5]\), \((\alpha \cdot \beta)^* \omega = u_{\alpha \beta} \omega \), for some \( u_{\alpha \beta} \in C(X, \mathbb{R}^{>0}) \), we have \( u_{\alpha \beta} = \beta^* u_\alpha \cdot u_\beta \). In particular,

\[ \lambda_{CR}(\alpha \beta) = (\alpha \cdot \beta)_* u_{\alpha \beta} = \alpha_* u_\alpha \cdot (\alpha \cdot \beta)_* u_\beta = \lambda_{CR}(\alpha) \cdot \alpha_* \lambda_{CR}(\beta). \]

So \( \lambda_{CR} \) is a crossed homomorphism as in \([4.3]\).

Suppose next that \( \omega = v \omega' \), for some \( v \in C^\infty(M, \mathbb{R}^{>0}) \). From

\[ u_\alpha \omega = \alpha^* \omega = \alpha^* (v \omega') = \alpha^* v \cdot u'_\alpha \omega' = \alpha^* v \cdot u'_\alpha \cdot v^{-1} \omega \]

we deduce \( u_\alpha = \alpha^* v \cdot u'_\alpha \cdot v^{-1} \). Thus \( \lambda_{CR}(\alpha) = \alpha_* u_\alpha = v \cdot \alpha_* v^{-1} \cdot \alpha_* u'_\alpha = (\partial^0 v^{-1})(\alpha) \cdot \lambda'_{CR}(\alpha) \), showing that the associated crossed homomorphisms \( \lambda_{CR} \) and \( \lambda'_{CR} \) are representing the same cohomology class \( \mu_{CR} \).
Next let $\{\omega, J\}$ be a pseudo-Hermitian structure on $M$ compatible with the $CR$-structure and $\lambda_{CR}$ be the associated crossed homomorphism defined by $\omega$. Suppose that $[\lambda_{CR}] = 0$. Therefore, as in (4.4), there exists $v \in \mathcal{C}^\infty(M, \mathbb{R}^{>0})$ such that $\alpha^* v = \lambda(\alpha) \cdot v$, for all $\alpha \in G$, which amounts to

\begin{equation}
(4.7) \quad v = \alpha^* (\lambda(\alpha) \cdot v) = u_\alpha \cdot \alpha^* v.
\end{equation}

Put a 1-form $\eta = v \cdot \omega$. Then

\begin{equation}
\alpha^* \eta = \alpha^* v \cdot \alpha^* \omega = \alpha^* v \cdot u_\alpha \omega = v \cdot \omega = \eta.
\end{equation}

Hence, $\alpha \in \text{Psh}(M, \{\eta, J\})$ is equivalent to (4.7). This shows (2). □

4.2.2. Proof of Theorem 23. Since $G$ acts properly, Theorem 24 shows that the canonical crossed homomorphism $\lambda_{CR}$ for the group of $CR$-automorphisms $G$, as defined in (4.6), is exact. In the view of (2) of Proposition 26, this proves Theorem 23. □

4.3. Conformal case. Replacing the role of $\omega$ in $CR$-geometry by a Riemannian metric $g$ on $M$, the conformal class of $g$ is said to establish a conformal structure on $M$. Every diffeomorphism $\alpha : M \to M$ that satisfies $\alpha^* g = u_\alpha \cdot g$ for some positive smooth function $u_\alpha \in \mathcal{C}^\infty(M, \mathbb{R}^{>0})$ is correspondingly called a conformal automorphism of $(M, g)$.

As in the $CR$ case, Proposition 26 to any Lie group $G$ of conformal automorphisms there is a natural associated cohomology class $\mu_{Conf}$, which is an obstruction for $G$ being a group of isometries:

Proposition 27 (Cohomology class of conformal transformation group). Let $(M, g)$ be a Riemannian manifold and let $G$ be a Lie subgroup of the group of conformal automorphisms of $M$. Then:

1. There exists in the differentiable cohomology of $G$ a natural associated class

\begin{equation}
\mu_{Conf} = [\lambda_{Conf}] \in H^1_d(G, \mathcal{C}^\infty(M, \mathbb{R}^{>0}))
\end{equation}

which is induced by the conformal structure on $M$.

2. Moreover, the class $\mu_{Conf}$ vanishes if and only if there exists a Riemannian metric $h$ conformal to $g$, such that $G$ is contained in the group of isometries $\text{Iso}(M, h)$.

Proof. We define the cocycle $\lambda_{conf} : G \to \mathcal{C}^\infty(M, \mathbb{R}^{>0})$ by declaring

\begin{equation}
(4.8) \quad \lambda_{conf} (\alpha) = \alpha^* u_\alpha, \quad \alpha \in \text{Aut}_{CR}(M),
\end{equation}

where $u_\alpha \in \mathcal{C}^\infty(M, \mathbb{R}^{>0})$ is defined as above by the relation $\alpha^* g = u_\alpha \cdot g$.

As in the proof of Proposition 26 it can be verified that $\lambda_{conf}$ is a crossed homomorphism for $G$, and its class $\mu_{Conf}$ in $H^1_d(G, \mathcal{C}^\infty(M, \mathbb{R}^{>0}))$ depends only on the conformal class of $g$. This shows (1).

The condition that $[\lambda_{Conf}] \in H^1_d(G, \mathcal{C}^\infty(M, \mathbb{R}^{>0}))$ vanishes means that there exists $v \in \mathcal{C}^\infty(M, \mathbb{R}^{>0})$ with $\partial^0 v = \lambda_{Conf}$, that is, by (4.4),

\begin{equation}
(4.9) \quad (\alpha^* v) \cdot v^{-1} = \lambda_{Conf}(\alpha) = \alpha^* u_\alpha, \quad \text{for all } \alpha \in G.
\end{equation}
On the other hand, putting \( h = v \cdot g \), we have \( \alpha^* h = \alpha^* v \cdot \alpha^* g = \alpha^* v \cdot u_\alpha g \). Therefore \( \alpha^* h = h \) is equivalent to (4.9), which is equivalent \( (\partial^0 v)(\alpha) = \lambda_{\text{Conf}}(\alpha) \). This implies (2).

We also obtain the following analogue of Theorem 23:

**Theorem 28.** Let \((M, g)\) be a Riemannian manifold. Suppose that \( G \leq \text{Conf}(M, g) \) is a subgroup of conformal automorphisms that acts properly on \( M \). Then there exists on \( M \) a Riemannian metric \( h \) conformal to \( g \), such that \( G \leq \text{Iso}(M, h) \).

**Proof.** Since \( G \) acts properly, Theorem 24 implies that the associated differentiable cohomology class \( \mu_{\text{Conf}} \) vanishes. Hence, there exists on \( M \) a Riemannian metric \( h = v \cdot g \), such that \( G \leq \text{Iso}(M, h) \).

### 4.4. Cohomological characterization of proper actions

The following theorem shows that the properness of \( CR \)- and conformal actions is basically a vanishing property for differentiable cohomology. This also implies Corollary 9 in the introduction:

**Theorem 29.** Let \( G \) be a Lie group of diffeomorphisms of the smooth manifold \( M \) that preserves either a strictly pseudo-convex \( CR \)-structure or a conformal Riemannian structure on \( M \). If \( G \) is closed in the group of all diffeomorphisms of \( M \), then the following are equivalent:

1. \( G \) acts properly on \( M \).
2. \( H^1_d(G, \mathcal{C}^\infty(M, \mathbb{R})) = \{0\} \).
3. \( H^r_d(G, \mathcal{C}^\infty(M, V)) = \{0\} \), for all \( r > 0 \), and any differentiable \( G \)-module \( V \).

**Proof.** Suppose that \( G \) acts properly, then by Theorem 14 (3) is satisfied. Now (3) clearly implies (2). Finally, if (2) is satisfied, the canonical class \( \mu \in H^1_d(G, \mathcal{C}^\infty(M, \mathbb{R}^0)) \) associated to either the \( CR \)- or conformal structure on \( M \) vanishes. In the first case, as is implied by Proposition 26, \( G \) preserves an associated contact Riemannian metric on \( M \), respectively in the case of Proposition 27, the group \( G \) preserves a Riemannian metric in the given conformal class. Since the group of isometries of a Riemannian manifold acts properly by the theorem of Myers and Steenrod [33], and \( G \) is a closed group of isometries, \( G \) acts properly on \( M \).

### 4.5. Locally conformal Kähler metrics

In this subsection we let \((X, J)\) denote a connected complex manifold satisfying \( \dim_{\mathbb{R}} X = 2n \geq 4 \). Then a Hermitian metric \( h \) for \( X \) is called a \textit{locally conformal Kähler} metric if it is \textit{locally} conformal to a Kähler metric. (This means that there exists an open covering \( \{U_\ell\} \) of \( X \) and functions \( u_\ell : U_\ell \to \mathbb{R}^0 \) such that \( h = u_\ell \cdot g_\ell \), where \( g_\ell \) is a Kähler metric on \((U_\ell, J)\), compare [39], [14].)
4.5.1. Conformal automorphisms of Kähler metrics. Our first result concerns lcK-metrics which admit a Kähler metric in their conformal class:

**Theorem 30.** Let \((X, \{J, g\})\) be a Kähler manifold which is not holomorphically isometric to \(\mathbb{C}^n\). Then the following hold:

1. There exists an lcK manifold \((X, \{h, J\})\) such that the lcK metric \(h\) is conformal to the Kähler metric \(g\) and satisfying
   \[
   \text{Iso}(X, \{h, J\}) = \text{Conf}(X, \{g, J\}).
   \]
2. Furthermore, the holomorphic isometry group \(\text{Iso}(X, \{h, J\})\) is maximal among all isometry groups of Hermitian metrics conformal to \(g\).

**Proof.** Since \(X\) is not holomorphically isometric to \(\mathbb{C}^n\), we infer from Theorem 6 that the Lie group \(\text{Conf}(X, \{g, J\})\) is acting properly on \(X\). Therefore, according to Theorem 28, there exists a Riemannian metric \(h\) conformal to \(g\), such that \(\text{Conf}(X, \{g, J\}) \leq \text{Iso}(X, h)\). Since \(h\) is conformal to the Kähler metric \(g\), \(h\) is Hermitian for the complex structure \(J\) and \(\text{Conf}(X, \{g, J\}) \leq \text{Iso}(X, \{h, J\})\). From the fact that \(\text{Conf}(X, \{g, J\}) = \text{Conf}(X, \{h, J\})\), we deduce that in fact \(\text{Conf}(X, \{g, J\}) = \text{Iso}(X, \{h, J\})\). This proves (1).

For the proof of (2), note that \(\text{Iso}(X, \{h', J\}) \leq \text{Conf}(X, \{h', J\}) = \text{Conf}(X, \{g, J\}) = \text{Iso}(X, \{h, J\})\), by (1). This shows (2). \(\square\)

**Remark.** For a Kähler manifold \((X, \{g, J\})\) with \(\dim \mathbb{R} X = 2n \geq 4\), the group of holomorphic conformal diffeomorphisms
\[
\text{Conf}(X, \{g, J\})
\]
coincides with the group of holomorphic homothetic transformations
\[
\text{Hoth}(X, \{g, J\}) = \{ \alpha \in \text{Diff}(X) \mid \alpha^* g = c \cdot g, \alpha_s J = J \alpha_s, \text{ for some } c \in \mathbb{R} > 0 \}.
\]
In fact, any biholomorphic conformal map between Kähler manifolds is easily seen to be an isometry up to constant scaling of the metrics, compare [43, Theorem 6.5, p. 66]. Therefore, (1) of Theorem 30 also asserts the equality
\[
(1') \quad \text{Iso}(X, \{h, J\}) = \text{Hoth}(X, \{h, J\}) = \text{Hoth}(X, \{g, J\}).
\]

4.5.2. Conformal automorphisms of lcK-metrics. We are now looking at the conformal class of lcK-metrics and their conformal automorphisms in general. First we recall that lcK-manifolds admit Kähler coverings (compare [39, p. 65]):

**Lemma 31** (Kähler covering). Let \(X\) be a simply connected complex manifold and \(h\) any lcK-metric on \(X\). Then

1. The lcK-metric \(h\) is conformal to a Kähler metric \(g\) (which is unique up to a constant factor).
2. The lcK-metric \(h\) is locally holomorphically conformal to \(\mathbb{C}^n\) if and only if \(g\) is a flat Kähler metric (that is, \(g\) is locally holomorphically isometric to \(\mathbb{C}^n\)).
(3) The lcK-manifold \((X, \{h, J\})\) is holomorphically conformal to \(\mathbb{C}^n\) if and only if the Kähler metric \((X, \{g, J\})\) is holomorphically isometric to \(\mathbb{C}^n\).

Proof. Let \(\Theta = h(J\cdot, \cdot)\) be the fundamental two-form of the lcK-metric \(h\). There exists a closed (global) one-form \(\theta\) on \(X\), called Lee form, such that
\[d\Theta = \theta \wedge \Theta.\]
Indeed, by the definition of lcK metric, \(\theta\) is constructed as \(\theta = d\log u^\ell\) on \(U^\ell\), where \(h = u^\ell \cdot g^\ell\) and \(g^\ell\) is a Kähler metric on \(U^\ell\), for some covering of \(X\). Furthermore, assuming that \(X\) is simply connected, \(\theta = df\) for some function \(f\) on \(X\). Then \(\Omega = e^{-f} \cdot \Theta\) is a Kähler form on \(X\). This proves (1).

The lcK-metric \(h\) being locally holomorphically conformal to \(\mathbb{C}^n\) means that there exists locally a holomorphic conformal map to \(\mathbb{C}^n\). Since the Kähler metric \(g\) is conformal to \(h\), this map is also locally conformal for \(g\), so \(g\) is locally holomorphically conformal to the standard flat Kähler space \(\mathbb{C}^n\). By the above remark following Theorem 30, \(g\) is actually locally homothetic to the standard complex space \(\mathbb{C}^n\), which also implies that \(g\) flat and locally holomorphically isometric to \(\mathbb{C}^n\), proving (2). The remark also implies (3).

The following is now a consequence of Theorem 30:

Corollary 32. Let \((X, \{h, J\})\) be an lcK-manifold whose universal covering manifold is not holomorphically conformal to \(\mathbb{C}^n\). Then there exists an lcK-metric \(h'\) conformal to \(h\) which is satisfying
\[\text{Iso}(X, \{h', J\}) = \text{Conf}(X, \{h, J\}).\]

Proof. Let \(p: (\tilde{X}, \tilde{h}) \to (X, h)\) be the universal covering lcK-manifold. Let \(g\) be the Kähler metric on \(\tilde{X}\) conformal to \(\tilde{h}\). Then the group of deck-transformations \(\Gamma\) for the covering \(p\) is contained in \(\text{Hoth}(\tilde{X}, \{g, J\})\). Since \((X, g)\) is not holomorphically isometric to \(\mathbb{C}^n\), Theorem 30 implies that there exists a Hermitian metric \(\tilde{h}'\) conformal to \(g\) such that \(\text{Iso}(\tilde{X}, \{\tilde{h}', J\}) = \text{Conf}(\tilde{X}, \{g, J\})\). Since \(\Gamma \leq \text{Iso}(\tilde{X}, \{\tilde{h}', J\})\), there exists a unique lcK-metric \(h'\) on \(X\), such that \(p: (\tilde{X}, \tilde{h}') \to (X, h')\) is a holomorphic Riemannian covering, and this metric is conformal to \(h\). Moreover, \(\text{Iso}(X, \{h', J\}) = \text{Conf}(X, \{h, J\})\).

Appendix A. Other related results

A.1. Conformally flat Kähler manifolds. The following results were proved by K. Yano and I. Mogi [44, Theorem 4.1] in the case \(\dim_{\mathbb{R}} X \geq 6\), and by S. Tanno [55] for \(\dim_{\mathbb{R}} X = 4\).

Theorem 33. Any conformally flat Kähler manifold \(X\) of dimension \(n \geq 6\) is locally flat, that is, it is locally holomorphically isometric to \(\mathbb{C}^n\).

Here the complex space \(\mathbb{C}^n\) carries the standard flat Kähler metric.
Theorem 34. Any 4-dimensional conformally flat Kähler manifold $X$ is locally holomorphically isometric to either $\mathbb{C}^2$ or the product of 2-dimensional surfaces $\mathbb{H}_\mathbb{R}^2 \times S^2$ with constant opposite sign.

Recall that the universal covering of any conformally flat $m$-dimensional manifold admits a conformal development map into the sphere $S^m$ and this map is unique up to composition with an element of $\text{Conf}(S^m) = \text{PO}(m + 1, 1)$, see for example [28, Theorem 4]. In the case of the two theorems, the universal covering space $\tilde{X}$ is thus mapped to $S^4 - \{\infty\} = \mathbb{C}^2$ or the sphere complement $S^4 - S^1 = \mathbb{H}_\mathbb{R}^2 \times S^2$ (compare [22]) through this developing map.

A.2. Proof of Theorem 6

(i) When $\dim \mathbb{R} X \geq 6$, by the result of Yano and Mogi Theorem [33], the Kähler manifold $X$ has everywhere holomorphic sectional curvature 0. In particular, it is locally holomorphically isometric to the standard flat complex space $\mathbb{C}^n$ (compare [26], IX, Theorem 7.9). Since $X$ is simply connected, the usual monodromy argument (see [29]) shows that there is a holomorphic map $\varphi : X \to \mathbb{C}^n$, that is also an isometric immersion.

Up to a conformal map we may identify both domains $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ with an open subset $U = S^{2n} - \{p\}$ contained in $S^{2n}$. Thus $\varphi$ and $\psi$ correspond to maps

$$\tilde{\varphi}, \tilde{\psi} : X \to U \subset S^{2n}$$

and both $\tilde{\varphi}, \tilde{\psi}$ are developing maps for the locally flat conformal structure associated with $(X, g)$. The uniformization theorem for locally flat conformal structures [28] implies that the two developing maps $\tilde{\varphi}, \tilde{\psi}$ are equivalent by an element of $\text{Conf}(S^{2n}) = \text{PO}(2n + 1, 1)$, so that $\alpha \circ \tilde{\psi} = \tilde{\varphi}$ on $X$. Clearly, since $\tilde{\psi}$ is a diffeomorphism, this shows that $\tilde{\varphi}$ is an injective embedding of $X$ into $U$. Note that every conformal embedding of Euclidean space $\mathbb{R}^{2n}$ into $S^{2n}$ has an image $S^{2n}$ with a point removed. Since the image of $\alpha \circ \tilde{\psi}$ is contained in $U$, we conclude that $\alpha \circ \tilde{\psi}$ is, in fact, surjective onto $U$. Thus the holomorphic isometric immersion $\varphi : X \to \mathbb{C}^n$ turns out to be an isometry.

(ii) $\dim \mathbb{R} X = 4$. By Tanno’s result, Theorem [34] since $X$ is conformally flat Kähler, as above, the simply connected Kähler manifold $X$ admits a holomorphic immersion $\varphi : X \to \mathbb{C}^2$ or $\varphi : X \to \mathbb{H}_\mathbb{R}^2 \times S^2$, respectively. Recall that there is a developing diffeomorphism $\psi : X \to \mathbb{R}^4$. As in part (i), by the
uniformization theorem for locally flat conformal structures, there exists $\alpha \in \text{PO}(5, 1)$ with $\alpha \circ \bar{\psi} = \bar{\varphi}$. Since the image of $\alpha \circ \bar{\psi}$ is $S^4 - \{p\}$, it cannot occur that $\bar{\varphi}$ takes values in $\mathbb{H}_R^2 \times S^2 = S^4 - S^1$. As in part (i), we conclude that $\varphi : X \to \mathbb{C}^2$ is a holomorphic isometry.

(iii) $\dim \mathbb{R} X = 2$. By Schoen’s theorem [35], $X$ is simply connected and there exists a conformal diffeomorphism to either $S^2$ or $\mathbb{R}^2$. In terms of uniformization of Riemann surfaces we conclude that $X$ is biholomorphic to $S^2$ or $\mathbb{C}$. □

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