NORMING SUBSPACES OF BANACH SPACES

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Dedicated to the memory of Professor Joseph Diestel

Abstract. We show that, if \( X \) is a closed subspace of a Banach space \( E \) and \( Z \) is a closed subspace of \( E^* \) such that \( Z \) is norming for \( X \) and \( X \) is total over \( Z \) (as well as \( X \) is norming for \( Z \) and \( Z \) is total over \( X \)), then \( X \) and the pre-annihilator of \( Z \) are complemented in \( E \) whenever \( Z \) is \( w^* \)-closed or \( X \) is reflexive.

Let \( E \) be a Banach space, let \( X \) be a subspace of \( E \) and \( Z \) be a subspace of \( E^* \) (the dual space to \( E \)). We say that \( Z \) is norming for \( X \) if the formula
\[
\|\|x\|\| = \sup_{f \in B_Z} |f(x)|, \quad x \in X
\]
defines an equivalent norm on \( X \) (where \( B_Z \) denotes the unit ball of \( Z \)). It is clear that if \( Z \) is norming for \( X \), then \( Z \) is total over \( X \) (that is, \( X \cap Z_{\perp} = \{0\} \), where \( Z_{\perp} = \{x \in E : f(x) = 0 \text{ for every } f \in Z\} \)). Analogously, if \( X \) is norming for \( Z \) (namely, if the image of \( X \) through the canonical maping \( \pi : E \to E^{**} \) is norming for \( Z \)), then \( X \) is total over \( Z \) (that is, \( X_{\perp} \cap Z = \{0\} \), where \( X_{\perp} = \{f \in E^* : f(x) = 0 \text{ for every } x \in X\} \)).

A systematic treatment of these properties was carried out in the paper [3], devoted to the study of norming bibasic systems in Banach spaces. In particular, there it was shown that, if \( X \) is a closed subspace of \( E \) and \( Z \) is a closed subspace of \( E^* \), then \( X \) is norming for \( Z \) if and only if the restriction to \( Z \) of the restriction map \( q^* : E^* \to X^* \) is an isomorphic embedding, if and only if \( X \) is total over \( Z \) and the (direct) sum \( X_{\perp} \oplus Z \) is closed in \( E^* \).

In this note, we obtain a generalization of this statement, which provides a characterization of the property that \( Z \) is norming for \( X \) whenever the ball \( B_Z \) is \( w^* \)-dense in \( B_{Z^*} \). This result will be used to show that, in the case \( Z \) is \( w^* \)-closed, the pair of conditions “\( X \) is total over \( Z \)” and “\( Z \) is norming for \( X \)” (as well as “\( Z \) is total over \( X \)” and “\( X \) is norming for \( Z \)”)
entail that \( E = X \oplus Z_{\perp} \). Using a duality argument, we deduce that the same assertion holds true in the case \( X \) is reflexive. As an application of this result, we provide a criterion for the existence of a sequence of extensions \( \{f_i\}_{i=1}^{\infty} \) of the functionals associated to an \( M \)-basis.
from a reflexive subspace of $X$ of a separable Banach space $E$ with the property that $\{f_i\}$ (the closed linear span of $\{f_i\}$) is norming for $E$.

The main result of this note reads as follows.

**Theorem 1.** Let $E$ be a Banach space and $X$ be a closed subspace of $E$. If $Z$ is a $w^*$-closed subspace of $E^*$, then the following assertions are equivalent:

1. $Z$ is norming for $X$ and $X$ is total over $Z$.
2. $X$ is norming for $Z$ and $Z$ is total over $X$.
3. $X$ is norming for $Z$ and $Z$ is norming for $X$.
4. $E = X \oplus Z_\perp$.

In the proof of this result we shall use the following lemma.

**Lemma 2.** Let $E$ be a Banach space. If $X$ is a closed subspace of $E$ and $Z$ is a closed subspace of $E^*$, then the following conditions are equivalent:

1. $Z^{w^*}$ is norming for $X$.
2. The restriction to $X$ of the quotient map $Q : E \to E/Z_\perp$ is an isomorphic embedding.
3. $Z$ is total over $X$ and the (direct) sum $X \oplus Z_\perp$ is closed in $E$.

If in addition, $B_Z$ is $w^*$-dense in $B_{Z^{w^*}}$, then these conditions are equivalent to:

4. $Z$ is norming for $X$.

**Proof.** (a) $\Rightarrow$ (b) Let us write $F = Z^{w^*}$, and let $\lambda \in (0, 1]$ be a number satisfying

$$\sup_{f \in B_F} |f(x)| \geq \lambda \|x\| \quad \text{for every} \quad x \in X.$$

Fix $x \in X$ and pick $f \in B_F$ such that $f(x) \geq \lambda \|x\|/2$. As $(Z_\perp)^\perp = F$, it easily follows that $F_\perp = Z_\perp$. Therefore, for each $y \in Z_\perp$ we have

$$\|x - y\| \geq f(x) \geq \lambda \|x\|/2$$

and consequently, $\|Qx\| = \inf \{\|x - y\| : y \in Z_\perp\} \geq \lambda \|x\|/2$. Hence the operator $Q_{\|x\|}$ is an isomorphic embedding.

(b) $\Rightarrow$ (c) It suffices to show that $\inf \{\|x - y\| : x \in S_X, y \in S_{Z_\perp}\} > 0$, where $S_X$ and $S_{Z_\perp}$ denote respectively the unit spheres of $X$ and $Z_\perp$. Assume the contrary. Then there exist sequences $(x_n)_n \subset S_X$ and $(y_n)_n \subset S_{Z_\perp}$ such that $\|x_n - y_n\| \to 0$. Thus, $\|Qx_n - Qy_n\| \to 0$, and hence $\|Qx_n\| \to 0$. But, because of our assumption we have $\|Qx_n\| \geq \lambda \|x_n\| = \lambda$ for some $\lambda \in (0, 1]$ and every $n \in N$. Therefore, the manifold $X \oplus Z_\perp$ is a closed subspace of $E$.

(c) $\Rightarrow$ (a) Set $U = X \oplus Z_\perp$ and let $M$ and $N$ denote the annihilator subspaces of $X$ and $Z_\perp$ relative to $U$, that is, $M = \{f \in U^* : f \upharpoonright X = 0\}$ and $N = \{g \in U^* : g \upharpoonright Z_\perp = 0\}$. Since $U$ is closed, according to [6, Exercise 4.36] it follows that $U^* = M \oplus N$. In particular, there exists $\alpha > 0$ such that

$$\alpha (\|f\| + \|g\|) \leq \|f + g\| \leq \|f\| + \|g\| \quad \text{whenever} \quad f \in M \text{ and } g \in N.$$
of the previous inequality, we have \( \|g\| \leq \alpha^{-1} \). Therefore, the functional \( \psi = \alpha g \) belongs to \( B_N \) and \( \psi(x) \geq \alpha \). Now, let \( \hat{\psi} \in E^* \) be such that \( \hat{\psi} \upharpoonright U = \psi \) and \( \|\hat{\psi}\| = \|\psi\| \). Then, \( \hat{\psi} \in B_{(Z \perp)^*} = B_{Z^{w^*}} \) and \( \hat{\psi}(x) \geq \alpha \). Consequently, the subspace \( Z^{w^*} \) is norming for \( X \).

Finally, it is clear that (d) implies (a) (with no additional assumption). Further, assuming \( B_{Z^{w^*}} = B_{Z^{w^*}} \), we have \( \sup_{f \in B_Z} f(x) = \sup_{f \in B_{Z^{w^*}}} f(x) \) for every \( x \in X \), hence (a) \( \Rightarrow \) (d). \( \square \)

Remarks. (1) In general, without the assumption \( B_{Z^{w^*}} = B_{Z^{w^*}} \), assertion (d) in the previous lemma is not implied by the other ones. Indeed, if \( E \) is any non quasi-reflexive Banach space then, according to the main result in [4] there exists a closed subspace \( Z \subset E^* \) such that \( Z \) is total but not norming for \( E \). Hence \( Z^{w^*} = E^* \), so \( Z^{w^*} \) is norming for \( E \).

(2) If \( E \) is a weakly compactly generated Banach space not isomorphic to a Hilbert space, then for every non-complemented subspace \( X \subset E \) there is a \( w^* \)-closed subspace \( Z \subset E^* \) such that \( Z \) is total but not norming for \( X \). Indeed, thanks to [6] Theorem 13.48 there exists a subspace \( Y \subset E \) such that \( X \cap Y = \{0\} \) and \( X + Y \) is dense in \( E \). Set \( Z = Y^⊥ \). Then \( Z \) is a \( w^* \)-closed subspace of \( E^* \) and \( X \cap Z^⊥ = X \cap Y = \{0\} \). Hence \( Z \) is total over \( X \). Since \( X + Z^⊥ \) is a proper dense manifold in \( E \), it follows that it is not closed. By using Lemma 2 we deduce that \( Z \) is not norming for \( X \).

As a particular case of the former lemma we get the aforementioned result from [3].

**Corollary 3.** ([3] Theorem 2) Let \( E \) be a Banach space. If \( X \) is a closed subspace of \( E \) and \( Z \) is a closed subspace of \( E^* \), then the following conditions are equivalent:

1. \( X \) is norming for \( Z \).
2. The restriction to \( Z \) of the restriction map \( q^* : E^* \to X^* \) is an isomorphic embedding.
3. \( X \) is total over \( Z \) and the (direct) sum \( X^⊥ \oplus Z \) is closed in \( E^* \).

**Proof.** Put \( \bar{E} = E^* \), \( \bar{X} = Z \) and \( \bar{Z} = \pi(X) \). Then, \( \bar{Z}^⊥ = X^⊥ \). Hence, assertion (1) is satisfied if and only if \( \bar{Z} \) is norming for \( \bar{X} \), and condition (3) is equivalent to the properties that \( \bar{Z} \) is total over \( \bar{X} \) and \( \bar{Z}^⊥ \oplus \bar{X} \) is closed in \( \bar{E} \). Further, since \( E^*/X^⊥ \cong X^* \), the quotient map \( Q : \bar{X} \to \bar{E}/\bar{Z} \) can be identified with the restriction operator \( q^*_Z : Z \to X^* \), thus condition (2) is equivalent to the fact that \( Q : \bar{X} \to \bar{E}/\bar{Z} \) is an isomorphic embedding. Finally, thanks to Goldstine’s theorem we have \( B_{\bar{Z}^{w^*}} = B_{\bar{Z}^{w^*}} \) and Lemma 2 applies. \( \square \)

**Proof of Theorem 7.** It is clear that (4) entails (1) and (2). Thus, it is enough to prove the implications (1) \( \Rightarrow \) (4), (2) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (3).

(1) \( \Rightarrow \) (4) Because of the hypothesis we have \( X \cap Z^⊥ = \{0\} \). We claim that the direct sum \( X \oplus Z^⊥ \) is dense in \( E \). Indeed, since \( Z \) is \( w^* \)-closed, the adjoint operator of the map \( Q_{X \oplus Z^⊥} : X \oplus E/Z^⊥ \to X^* \) can be identified with the restriction map \( q^*_Z : Z \to X^* \). It is clear that \( \ker q^*_Z = X^⊥ \cap Z \). Bearing in mind that \( X \) is total over \( Z \), it follows that \( q^*_Z \) is one-to-one. Hence, the operator \( Q_{X \oplus Z^⊥} \) has dense range, and using the Hahn-Banach theorem we deduce that the manifold \( X \oplus Z^⊥ \) is dense in \( E \). On the other hand, as \( Z \) is norming for \( X \), Lemma 2 guarantees that \( X \oplus Z^⊥ \) is closed. Consequently, \( E = X \oplus Z^⊥ \).
(2) ⇒ (4) As $Z$ is $w^*$-closed we have $Z = (Z_\perp)$. Therefore, according to [6, Exercise 4.16], it is enough to show that $E^* = X_\perp \oplus Z$, and this is clearly equivalent to the fact that the operator $q^*_Z$ is an isomorphism from $Z$ onto $X^*$. Since $X$ is norming for $Z$, Corollary 3 yields the existence of a number $\lambda > 0$ such that
\[
\|q^*(z)\| \geq \lambda\|z\| \quad \text{for every} \quad z \in Z.
\]

Therefore, by the open mapping theorem, it is enough to check that $q^*_Z$ is onto. We claim that $M = q^*(Z)$ is a $w^*$-closed subspace of $X^*$. According to the Banach-Dieudonné Theorem (see e.g. [6, Theorem 3.92]), it is enough to prove that the set $B_M$ is $w^*$-closed. Let $\{x^*_\alpha\}_{\alpha \in \Lambda}$ be a net in $B_M$ such that $x^*_\alpha \xrightarrow{w^*} x^*$ for some $x^* \in X^*$. For each $\alpha \in \Lambda$ there is (a unique) $z_\alpha \in Z$ with $q^*(z_\alpha) = x^*_\alpha$. By the previous inequality we have $\|z_\alpha\| \leq \lambda^{-1}$ for each $\alpha \in \Lambda$. Thus the net $\{z_\alpha\}_\alpha$ has a $w^*$-cluster point, say $z \in E^*$. As $Z$ is $w^*$-closed, we get $z \in Z$. Moreover, since the map $q^*$ is $w^*$-$w^*$ continuous, we have $q^*(z_\alpha) \xrightarrow{w^*} q^*(z)$, that is $x^*_\alpha \xrightarrow{w^*} q^*(z)$. Consequently, $x^* = q^*(z)$, so $x^* \in B_M$. Therefore, $B_M$ (hence also $M$) is $w^*$-closed. Since $M$ is also total over $X$ we have $M = X^*$.

(4) ⇒ (3) Taking into account that $Z$ is $w^*$-closed and the manifold $X \oplus Z_\perp$ is closed in $E$, from Lemma 2 we deduce that $Z$ is norming for $X$. Moreover, a new appeal to [6, Exercise 4.16] yields $E^* = X_\perp \oplus Z$, which implies that $X$ is norming for $Z$.

The next result constitutes an analogue of Theorem 1 in case $X$ is reflexive.

**Corollary 4.** Let $E$ be a Banach space, let $X$ be a closed subspace of $E$ and $Z$ be a closed subspace of $E^*$. If $X$ is reflexive then the following conditions are equivalent:

1. $X$ is norming for $Z$ and $Z$ is total over $X$.
2. $Z$ is norming for $X$ and $X$ is total over $Z$.
3. $X$ is norming for $Z$ and $Z$ is norming for $X$.
4. $Z$ is $w^*$-closed and $E = X \oplus Z_\perp$.

**Proof.** Notice that the facts that $X$ is reflexive and norming for $Z$ entail that $Z$ is $w^*$-closed. Indeed, in such a case, by Corollary 3 the map $q^*_Z : Z \to X^*$ is an isomorphic embedding. Therefore, $q^*(Z)$ is a closed subspace of $X^*$ that is isomorphic to $Z$. Since $X^*$ is reflexive, $q^*(Z)$ is reflexive as well, and using [6, Lemma 4.62] we deduce that $Z$ is $w^*$-closed. Thus Theorem 1 yields that (1) implies (4). The reverse implication is obvious. Hence, it remains to show that assertions (1), (2) and (3) are equivalent. This follows by duality applying Theorem 1 to space $\widetilde{E} = E^*$ and to subspaces $\widetilde{X} = Z \subset \widetilde{E}$ and $\widetilde{Z} = \pi(X) \subset \widetilde{E}^*$. Observe that, since $X$ is reflexive, $\widetilde{Z}$ is a $w^*$-closed subspace of $\widetilde{E}^*$.

**Remarks.** (1) A Banach space $E$ is called *indecomposable* if $E$ cannot be written as the direct sum of two infinite dimensional closed subspaces. The first example of an indecomposable Banach space was constructed by Gowers and Maurey (c.f. [7]): it enjoys the much stronger property of being *hereditarily indecomposable* (i.e., every infinite dimensional closed subspace of that space is indecomposable). As an immediate consequence of Lemma 2 the fact (pointed out by V. D. Milman, c.f. [1, Lemma 1.1]) follows that a Banach space $E$ is hereditarily indecomposable if, and only if, for any closed subspace $X \subset E$ with $\dim(X) = \infty$ and
each $w^*$-closed subspace $Z \subset E^*$ such that $Z$ is norming for $X$, we have $\text{codim}(Z) < \infty$. Analogously, an easy consequence of Theorem 1 yields the following characterization of indecomposable spaces: A Banach space $E$ is indecomposable if, and only if, for every closed subspace $X \subset E$ with $\dim(X) = \infty$ and every $w^*$-closed subspace $Z \subset E^*$ such that $Z$ is norming for $X$ and $X \perp Z = \{0\}$ (as well as $X$ is norming for $Z$ and $X \cap Z_{\perp} = \{0\}$) we have $\text{codim}(Z) < \infty$.

(2) A Banach space $E$ is said to have the Dunford-Pettis property (in short, DPP), if for any two weakly null sequences $\{x_n\}_n \subset E$ and $\{x^*_n\}_n \subset E^*$, we have $x^*_n(x_n) \to 0$ (for some equivalent formulations of this property see e.g. [5, Theorem 1]). Typical examples of spaces with this property are $L^1(\Omega, \mu)$, where $(\Omega, \mu)$ is any $\sigma$-finite measure space, and $C(K)$ for any compact Hausdorff space $K$. If $E$ is a Banach space with the DPP and $X$ is a reflexive infinite dimensional subspace of $E$, then:

(a) for every closed subspace $Z \subset E^*$ such that $Z$ is norming for $X$, we have that $X$ is not total over $Z$, and

(b) for every closed subspace $Z \subset E^*$ such that $X$ is norming for $Z$, we have that $Z$ is not total over $X$.

Indeed, being reflexive and infinite-dimensional, $X$ fails to have the DPP (c.f. [6, p. 597]). According to a classical result by Grothendieck (c.f. [6, Lemma 13.44]) it follows that $X$ is not complemented in $E$. Corollary 4 applies.

We end this note with an application of Corollary 4 in the setting of $M$-bibasic systems in separable Banach spaces. Recall that a sequence $\{x_i\}_{i=1}^{\infty}$ in a Banach space $E$ is a Markushevich basis (in short, $M$-basis) of $E$ provided that $E = [x_i]$ and there exists a (unique) sequence of functionals $\{x^*_i\}_i \subset E^*$ such that $\{x_i, x^*_i\}_i$ is a biorthogonal system in $E$ and the subspace $[x^*_i] \subset E^*$ is total over $E$ (we refer to [8, Section 1.f] for the fundamental properties of $M$-bases in Banach spaces). We say that a biorthogonal system $\{x_i, z_i\}_{i=1}^{\infty}$ in $E$ is $M$-bibasic whenever $\{x_i\}_i$ is an $M$-basis of $[x_i]$ and $\{z_i\}_i$ is an $M$-basis of $[z_i]$. If $\{x_i\}_i$ and $\{z_i\}_i$ are basic sequences, the system $\{x_i, z_i\}_i$ is called bibasic. It was shown in [8] that every infinite-dimensional Banach space has a bibasic system $\{x_i, z_i\}_{i=1}^{\infty}$ such that $\sup_i \|x_i\|z_i\| < \infty$ ($\{x_i, z_i\}_i$ is said to be bounded) and $[z_i]$ is not norming for $[x_i]$. Actually, the existence of norming bibasic (or $M$-bibasic systems) is a rather strong condition (for instance, from the previous remark it follows that, if $E$ is a Banach space with the DPP and $X$ is a reflexive subspace of $E$, then no $M$-bibasic system $\{x_i, z_i\}_i$ exists in $E$ with $\{x_i\}_i \subset X$ and such that $[z_i]$ is norming for $[x_i]$). In fact, in some cases the presence of an $M$-bibasic system $\{x_i, z_i\}_i$ with this property yields the existence of a biorthogonal sequence of extensions of the functionals $x^*_i$ which is norming for the whole space.

**Corollary 5.** Let $X$ be a reflexive subspace of a separable Banach space $E$ and $\{x_i, z_i\}_{i=1}^{\infty} \subset X \times E^*$ be a bounded $M$-bibasic system. If $[z_i]$ is norming for $[x_i]$ then there exists a bounded sequence $\{f_i\}_i \subset E^*$ such that $\{x_i, f_i\}_i$ is a biorthogonal system and $[f_i]$ is norming for $E$.

**Proof.** We can assume that $X = [x_i]$. The separability of $E$ yields the existence of a normalized sequence $\{u_j\}_j$ in $X_{\perp}$ which is $w^*$-dense in $B_{X_{\perp}}$ and such that, for every $j$, the vector $u_j$ appears infinitely many times in that sequence. Let us write, for each $i \in \mathbb{N}$, $f_i = z_i + u_i$. 

It is clear that \( \{x_i, f_i\}_i \) is a biorthogonal system in \( E \). Put \( Z = [z_i] \) and let \( N \) denote the \( w^* \)-sequential closure of \( \text{span} \{f_i\}_i \). Fix \( j \in \mathbb{N} \) and let \( \{i_k\}_k \) be a strictly increasing sequence of positive integers such that \( u_{i_k} = u_j \) for every \( k \in \mathbb{N} \). As \( E \) is separable and, by Corollary 3, \( Z \) is \( w^* \)-closed, we have that \( B_Z \) is \( w^* \)-sequentially compact. Thus, we can assume that \( z_{i_k} \stackrel{w^*}{\longrightarrow} z \) for some \( z \in Z \). Since \( \{x_i\}_i \) is an \( M \)-basis of \( X \) and \( X \) is total over \( Z \), we easily get \( z = 0 \), hence \( z_{i_k} \stackrel{w^*}{\longrightarrow} 0 \). Consequently, \( f_{i_k} \stackrel{w^*}{\longrightarrow} u_j \). In particular \( u_j \in N \), therefore \( z_j = f_j - u_j \in N \) for every \( j \in \mathbb{N} \). So \( Z \subset N \). Further, bearing in mind that the sequence \( \{u_j\}_j \) is \( w^* \)-dense in \( B_{X^1} \), we have \( X^1 \subset N \). Since, because of Corollary 3, \( E^* = X^1 \oplus (Z^1)^\perp = X^1 \oplus Z \), it follows that \( E^* = N \). Therefore, \( \text{span} \{f_i\}_i \) is \( w^* \)-sequentially dense in \( E^* \). Taking into account that \( E \) is separable, according to a result by Banach (c.f. [2, Annexe, Théorème 2]), we deduce that \( [f_i] \) is norming for \( E \). □

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