Bimatrix variate generalised beta distributions

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Abstract
In this paper, we extend the study of bivariate generalised beta type I and II distributions to the matrix variate case.

1 Introduction
Matrix variate beta type I and II distributions have been studied by different authors utilising diverse approaches, see Olkin and Rubin (1964), Khatri (1970), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2007, 2006, 2008), among many others. These distributions play a very important role in various approaches to proving hypotheses in the context of multivariate analysis, including canonical correlation analysis, the general linear hypothesis in MANOVA and multiple matrix variate correlation analysis, see Muirhead (1982) and Srivastava and Khatri (1973). All these techniques are based on the hypothesis that some matrices $A$ and $B$ are independent with Wishart distributions. In the present paper, these results are generalised, assuming $A$ and $B$ to have a matrix variate gamma distributions.

$A : m \times m$ is said to have a matrix variate gamma distribution with parameters $a$ and $m \times m$ positive definite matrix $\Theta$, this fact being denoted as $A \sim G_m(a, \Theta)$, if its density function is

$$
\frac{1}{\Gamma_m[a]}|\Theta|^{-a}|A|^{-\frac{m+1}{2}}\text{etr}(-\Theta^{-1}A)(dA), \quad A > 0, \quad (1)
$$

where

$$(dA) = \bigwedge_{i \leq j} da_{ij}, \quad (2)
$$

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see Muirhead (1982, pp. 57 and 61) and Gupta and Nagar (2000); where \( \Gamma_m[a] \) denotes the multivariate gamma function and is defined as

\[
\Gamma_m[a] = \int_{V>0} \text{etr}(-V)|V|^{-(m+1)/2}(dV),
\]

\( \text{Re}(a) > (m - 1)/2 \) and \( \text{etr}(-) \equiv \exp(\text{tr}(-)) \).

As well as the classification of the beta distribution, as beta type I and type II (see Gupta and Nagar (2000) and Srivastava and Khatri (1979)), two definitions have been proposed for each of these distributions. Let us focus initially on the beta type I distribution; if \( A \) and \( B \) have a matrix variate gamma distribution, i.e. \( A \sim G_m(a, I_m) \) and \( B \sim G_m(b, I_m) \) independently, then the beta matrix \( U \) can be defined as

\[
U = \begin{cases} 
(A + B)^{-1/2}A((A + B)^{-1/2})', & \text{Definition 1 or,} \\
A^{1/2}(A + B)^{-1/2}(A^{1/2})', & \text{Definition 2},
\end{cases}
\]

where \( C^{1/2}(C^{1/2})' = C \) is a reasonable nonsingular factorization of \( C \), see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982). It is readily apparent that under definitions 1 and 2 its density function is denoted as \( BI_m(U; a, b) \) and given by

\[
\frac{1}{\beta_m[a, b]} |U|^{a-(m+1)/2}|I_m - U|^{b-(m+1)/2}(dU), \quad 0 < U < I_m,
\]

this being denoted as \( U \sim BI_m(a, b) \), with \( \text{Re}(a) > (m - 1)/2 \) and \( \text{Re}(b) > (m - 1)/2 \); where \( \beta_m[a, b] \) denotes the multivariate beta function defined by

\[
\beta_m[b, a] = \int_{S \leq I_m} |S|^{a-(m+1)/2}|I_m - S|^{b-(m+1)/2}(dS) = \int_{R > 0} |R|^{a-(m+1)/2}|I_m + R|^{-(b+b)}(dR) = \frac{\Gamma_m[a]\Gamma_m[b]}{\Gamma_m[a+b]}.
\]

A similar situation arises with the beta type II distribution, with which we have the following three definitions:

\[
F = \begin{cases} 
B^{-1/2}A(B^{-1/2})', & \text{Definition 1,} \\
A^{1/2}B^{-1}(A^{1/2})', & \text{Definition 2},
\end{cases}
\]

with the distribution being denoted as \( F \sim BII_m(a, b) \). In this case under definition 1 and 2, the density function of \( F \) is denoted \( BII_m(F; a, b) \) by and defined as

\[
\frac{1}{\beta_m[a, b]} |F|^{a-(m+1)/2}|I_m + F|^{-(a+b)}(dF), \quad F > 0.
\]

\( (dF) \) is given in analogous form to (2).

Some of these generalisations from a univariate beta distribution to the matrix variate case are inappropriate because, in some applications, the researcher is interested in a vector variate, not in a symmetric matrix, see Libby and Novick (1982). In other words, the researcher is interested in a vector, say, \( X = (x_1, \ldots, x_m)' \), such that \( x_i \) has a marginal beta type I or II distribution for all \( i = 1, \ldots, m \). In this respect, Libby and Novick (1982) and Chen and Novick (1984) proposed a multivariate version of the beta type I and II distributions. Let us consider the following bivariate version, see Olkin and Liu (2003) and Nagar et al. (2008).
Let $X_0, X_1, X_2$ be distributed as independent gamma random variates with parameters $a = a_0, a_1, a_2$, respectively and $\Theta = 1$ in the three cases, and define

$$U_1 = \frac{X_1}{X_1 + X_0}, \quad U_2 = \frac{X_2}{X_2 + X_0}. \quad \text{Clearly, } U_1 \text{ and } U_2 \text{ each have a beta type I distribution, } U_1 \sim BI_1(a_1, a_0) \text{ and } U_2 \sim BI_2(a_2, a_0), \text{ over } 0 \leq u_1, u_2 \leq 1. \text{ However, they are correlated such that } (U_1, U_2)' \text{ has a bivariate generalised beta type I distribution over } 0 \leq u_1, u_2 \leq 1. \text{ The joint density function of } U_1 \text{ and } U_2 \text{ is}

$$

$$u_1^{a_1-1}u_2^{a_2-1}(1-u_1)^{a_2+a_0-1}(1-u_2)^{a_1+a_0-1}$$

$$\frac{\beta_n^* [a_1, a_2, a_0] (1-u_1u_2)^{a_1+a_2+a_0}}{(1-u_1)^{a_2+a_0-1}(1-u_2)^{a_1+a_0-1}}, \quad 0 \leq u_1, u_2 \leq 1$$

where

$$\beta_n^*[a, b, c] = \frac{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]}{\Gamma_m[a+b+c]}. \quad \text{A similar result is obtained in the case of beta type II. Here define}

$$F_1 = X_1 \frac{X_2}{X_0}, \quad F_2 = \frac{X_2}{X_0}. \quad \text{Once again it is evident that } F_1 \text{ and } F_2 \text{ each have a beta type II distribution, } F_1 \sim BI_1(a_1, a_0) \text{ and } F_2 \sim BI_2(a_2, a_0), \text{ over } f_1, f_2 \geq 0. \text{ As in the beta type I case, they are correlated such that } (F_1, F_2)' \text{ has a bivariate generalised beta type II distribution over } f_1, f_2 \geq 0. \text{ The joint density function of } F_1 \text{ and } F_2 \text{ is}

$$

$$f_1^{a_1-1}f_2^{a_2-1}(1+f_1+f_2)^{a_1+a_2+a_0}, \quad f_1, f_2 \geq 0$$

Some applications to utility modelling and Bayesian analysis are presented in [Libby and Novick (1982) and Chen and Novick (1984)], respectively. Properties such as the moments $u_1^nu_2^m$, conditional distribution, the distributions of the product $u_1u_2$, and the $u_1/u_2$ and $u_1/(u_1 + u_2)$ quotients are studied in [Libby and Novick (1982), Chen and Novick (1984), Olkin and Liu (2003) and Nagar et al. (2008)].

In the present paper, we extend the bivariate generalised beta type I and II distributions to the matrix variate case, see Section 3 and 4. These distributions are termed as bimatrix variate generalised beta type I and II distributions. In Section 5 some properties of these distributions are studied.

## 2 Preliminary results

In this section, some results for the hypergeometric function with a matrix argument are shown.

**Definition 2.1.** The hypergeometric functions of a matrix argument are given by

$$pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \mathbf{X}) = \sum_{\tau} \frac{C_{\tau}(\mathbf{X})}{t!}$$

where $\sum_{\tau}$ denotes the summation over all the partitions $\tau = (t_1, \ldots, t_m)$, $t_1 \geq \cdots \geq t_m \geq 0$, of $t$, $C_{\tau}(\mathbf{X})$ is the zonal polynomial of $\mathbf{X}$ corresponding to $\tau$ and the generalised hypergeometric coefficient $(a)_\tau$ is given by

$$(a)_\tau = \prod_{i=1}^{m} (a - (i - 1)/2)_{t_i},$$
where \((a)_t = a(a+1)(a+2)\cdots(a+t-1)\), \((a)_0 = 1\). Here \(X\), the argument of the function, is a complex symmetric \(m \times m\) and the parameters \(a_i, \ b_j\) are arbitrary complex numbers.

Some other characteristics of the parameters \(a_i\) and \(b_j\) and the convergence of (7) appear in [Muirhead (1982, p. 258)].

A special case of (7) is

\[
1F_0(a; X) = \sum_{t=0}^{\infty} \frac{(a)_t}{t!} \frac{C_t(X)}{||X||} (||X|| < 1)
\]

where \(||X||\) denotes the maximum of the absolute values of the eigenvalues of \(X\).

An interesting relation is derived from the following Lemma 2.1, which gives an induction method for constructing hypergeometric functions, i.e. integration involving \(pF_q\) leads to the new hypergeometric function \(p+1F_{q+1}\). A motivation for the general recursion comes from the following well-known expressions, see [Muirhead (1982, Theorem 7.4.2, p. 264)]:

\[
1F_1(a; c; X) = \frac{1}{\beta_m[a, c-a]} \times \int_{0 < Y < I_m} 0F_0(XY)|Y|^{a-(m+1)/2}|I - Y|^{c-a-(m+1)/2}(dY).
\]

and

\[
2F_1(a, a_1; c; X) = \frac{1}{\beta_m[a, c-a]} \times \int_{0 < Y < I_m} 1F_0(a_1; XY)|Y|^{a-(m+1)/2}|I - Y|^{c-a-(m+1)/2}(dY).
\]

Thus we have

**Lemma 2.1.** Let \(X < I\), \(\text{Re}(a) > (m-1)/2\), \(\text{Re}(c) > (m-1)/2\) and \(\text{Re}(c-a) > (m-1)/2\).

Then

\[
p+1F_{q+1}(a, a_1, \ldots, a_p; c, b_1, \ldots, b_q; X) = \frac{1}{\beta_m[a, c-a]} \times \int_{0 < Y < I_m} pF_q(a_1 \cdots a_p; b_1 \cdots b_q; XY)|Y|^{a-(m+1)/2}|I - Y|^{c-a-(m+1)/2}(dY).
\]

**Proof.** First, we apply an expansion in terms of zonal polynomials

\[
pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; XY) = \sum_{\tau} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau} \frac{C_\tau(XY)}{t!}.
\]

Then, after integrating term by term, see [Muirhead (1982, Theorem 7.2.10, p. 254)], we have that

\[
\int_{0 < Y < I_m} pF_q(a_1 \cdots a_p; b_1 \cdots b_q; XY)|Y|^{a-(m+1)/2}|I - Y|^{c-a-(m+1)/2}(dY)
\]

\[
= \sum_{\tau} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau} \times \int_{0 < Y < I_m} |Y|^{a-(m+1)/2}|I - Y|^{c-a-(m+1)/2}C_\tau(X)(dY)
\]

\[
= \beta_m[a, c-a] \sum_{\tau} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(c)_\tau (b_1)_\tau \cdots (b_q)_\tau} \times \frac{C_\tau(X)}{t!}
\]

\[
= \beta_m[a, c-a] p+1F_{q+1}(a, a_1 \cdots a_p; c, b_1 \cdots b_q; X),
\]
Theorem 3.1. Assume that \( U \sim BGI_{2m \times m}(a, b, c) \). Then its density function is
\[
|U_1|^{a-(m+1)/2} |U_2|^{b-(m+1)/2} |I_m - U_1|^{1/2} |I_m - U_2|^{1/2} \frac{\beta_m[a, b, c]|I_m - U_1 U_2|^{a+b+c}}{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]} \] (dU)
\]
\( 0 < U_1 < I_m, \ 0 < U_2 < I_m, \) where the measure
\[
(dU) = (dU_1) \wedge (dU_2).
\]
and Re(\( a \)) > (\( m - 1 \))/2, Re(\( b \)) > (\( m - 1 \))/2 and Re(\( c \)) > (\( m - 1 \))/2.

Proof. The joint density of \( A, B \) and \( C \) is
\[
|A|^{a-(m+1)/2} |B|^{b-(m+1)/2} |C|^{c-(m+1)/2} \frac{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]}{(a+c-(m+1)/2) |I_m - U_1|^{a-c-(m+1)/2} |I_m - U_2|^{b-c-(m+1)/2} |I_m - U_1 U_2|^{a+b+c}} \] (dA)(dB)(dC)
\]
By effecting the change of variable (9), then
\[
(dA)(dB)(dC) = |C|^{m+1} |I_m - U_1|^{-(m+1)} |I_m - U_2|^{-(m+1)} (dU_1)(dU_2)(dC).
\]
The joint density of \( U_1, U_2 \) and \( C \) is
\[
|U_1|^{a-(m+1)/2} |U_2|^{b-(m+1)/2} \frac{\Gamma_m[a] \Gamma_m[b] \Gamma_m[c]}{(a+c-(m+1)/2) |I_m - U_1|^{a-c-(m+1)/2} |I_m - U_2|^{b-c-(m+1)/2} |I_m - U_1 U_2|^{a+b+c}} \]
\[
\times \frac{\Gamma_m[a+b+c]}{|I_m - U_1 U_2|^{a+b+c}} \times |I_m - U_1 U_2|^{-1} |I_m - U_1|^{-1} |I_m - U_2|^{-1} (dC)(dU_1)(dU_2).
\]
Integrating with respect to \( C \) using
\[
\int_{C>0} |C|^{a+b+c-(m+1)/2} \frac{\Gamma[a+b+c]}{|I_m - U_1 U_2|^{a+b+c}} \times \frac{|I_m - U_1|^{1/2} |I_m - U_2|^{1/2} |I_m - U_1 U_2|^{a+b+c}}{(dC)} \]
\[= \Gamma[a+b+c] |I_m - U_1 U_2|^{a+b+c} \]
\[(\text{from (11)} \) gives the stated marginal density function for \((U_1; U_2)'\).
As in the bivariate case (Olkin and Liu, 2003), the joint density \( (10) \) can be represented as a mixture. Let us first note that

\[
|I_m - U_1U_2|^{-(a+b+c)} = \sum_{t=0}^{\infty} \sum_r (a+b+c)_r \frac{C_r(U_1U_2)}{t!}.
\]

By substituting in \( (10) \) we obtain that the joint density function of \( (U_1;U_2) \) is

\[
\sum_{t=0}^{\infty} \sum_r \frac{(a+b+c)_r}{\beta^*_m[a,b,c]} |U_1|^{a-(m+1)/2} |U_2|^{b-(m+1)/2} \times |I_m - U_1|^{b+c-(m+1)/2} |I_m - U_2|^{a+c-(m+1)/2} \frac{C_r(U_1U_2)}{t!}.
\]

Moreover

\[
\sum_{t=0}^{\infty} \sum_r \frac{(a+b+c)_r}{\Gamma_m[a+b+c] \Gamma_m[a+c]} \frac{1}{\Gamma_m[a+b+c] \Gamma_m(a+c)} B \mathcal{I}_m(U_1; a, b+c) \times B \mathcal{I}_m(U_2; b, a+c) \frac{C_r(U_1U_2)}{t!}.
\]

4 Bimatrix variate generalised beta type II distribution

Let \( A, B \) and \( C \) be independent, where \( A \sim G_m(a, I_m) \), \( B \sim G_m(b, I_m) \) and \( C \sim G_m(c, I_m) \) with \( Re(a) > (m-1)/2 \), \( Re(b) > (m-1)/2 \) and \( Re(c) > (m-1)/2 \) and let us define

\[
F_1 = C^{-1/2} AC^{-1/2} \quad \text{and} \quad F_2 = C^{-1/2} BC^{-1/2}
\]

Clearly, \( F_1 \sim B \mathcal{I}_m(a, c) \) and \( F_2 \sim B \mathcal{I}_m(b, c) \). But they are correlated and so the distribution of \( F = (F_1; F_2)' \in \mathbb{R}^{2m \times m} \) can be termed a bimatrix variate generalised beta type II distribution, which is denoted as \( F \sim B \mathcal{G} \mathcal{B} \mathcal{I}_{2m \times m}(a, b, c) \).

Theorem 4.1. Assume that \( F \sim B \mathcal{G} \mathcal{B} \mathcal{I}_{2m \times m}(a, b, c) \). Then its density function is

\[
\frac{|F_1|^{a-(m+1)/2} |F_2|^{b-(m+1)/2}}{\beta^*_m[a,b,c] |I_m + F_1 + F_2|^{a+b+c}} (dF)
\]

\( F_1 > 0, F_2 > 0 \), where the measure

\[
(dF) = (dF_1) \wedge (dF_2).
\]

and \( Re(a) > (m-1)/2, Re(b) > (m-1)/2 \) and \( Re(c) > (m-1)/2 \).

Proof. As an alternative to proceeding as in Theorem 3.1. Let us recall that if \( U \sim \mathcal{B} \mathcal{I}_m(a, b) \), then \( (I_m - U)^{-1} - I_m \sim \mathcal{B} \mathcal{I}_m(a, b) \), see Srivastava and Khatri (1979) and Díaz-García and Gutiérrez-Jáimez (2007). Then

\[
F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} (I_m - U_1)^{-1} - I_m \\ (I_m - U_2)^{-1} - I_m \end{pmatrix}
\]

with the Jacobian given by

\[
(dU_1)(dU_2) = |I_m + F_1|^{-(m+1)} |I_m + F_2|^{-(m+1)} (dF_1)(dF_2).
\]

Also, note that

\[
I_m - (I_m + F_1)^{-1} = (I_m + F_1)^{-1}((I_m + F_1) - I_m) = (I_m + F_1)^{-1}F_1
\]

\[
I_m - (I_m + F_2)^{-1} = (I_m + F_2)^{-1}F_2.
\]
Then the joint density of \((F_1, F_2)'\) is
\[
\frac{|F_1|^{(a+c)/2} |F_2|^{-(m+1)/2} |I_m + F_1|^{-(a+b+c)} |I_m + F_2|^{-(a+b+c)}}{\beta_m[a, b, c] |I_m - (I_m + F_1)^{-1} F_1 F_2 (I_m + F_2)^{-1}|^{a+b+c}}(dF).
\]

The desired results is follow noting that
\[
\frac{|I_m + F_1|^{-1} |I_m + F_2|^{-1}}{|I_m - (I_m + F_1)^{-1} F_1 F_2 (I_m + F_2)^{-1}|} = |I_m + F_1 + F_2|^{-1}.
\]

Other properties of the distribution \(BGBI_{2m \times m}(a, b, c)\) can be found in a similar way.

## 5 Properties

In this section we calculate the moments \(E(|U_1|'^r|U_2|^s)\) and the distributions of the product \(Z = U_2^{1/2} U_1 U_2^{1/2}\) and the inverse \((U_1^{-1}: U_2^{-1})\).

**Theorem 5.1.** Assume that \((U_1: U_2) \sim BGBI_{2m \times m}(a, b, c)\) then
\[
E(|U_1|'^r|U_2|^s) = \frac{\beta_m[a + r, b + c] \beta_m[b + s, a + c] \times 3F_2(a + r, b + s, a + b + c; a + b + c + r, a + b + c + s; I_m)}{\beta_m[a, b, c]}
\]

with \(\text{Re}(b + r) > (m - 1)/2\), and \(\text{Re}(a + c) > (m - 1)/2\).

**Proof.**
\[
E(|U_1|'^r|U_2|^s) = \frac{1}{\beta_m[a, b, c]} \int_{0 < U_1 < I_m} |U_1|^a |U_2|^{b+c-(m+1)/2} |I_m - U_1|^{b+c-(m+1)/2}
\]
\[
\times \int_{0 < U_2 < I_m} |U_2|^b |U_2|^{a+c-(m+1)/2} |I_m - U_2|^{a+c-(m+1)/2}(dU_2)(dU_1).
\]

From (8) we have
\[
E(|U_1|'^r|U_2|^s) = \frac{\beta_m[a + c, b + s]}{\beta_m[a, b, c]} \int_{0 < U_1 < I_m} |U_1|^a |I_m - U_1|^{b+c-(m+1)/2}
\]
\[
\times 2F_1(b + s, a + b + c; a + b + c + s; U_1)(dU_1).
\]

Then, integrating using Lemma 2.1 the desired result is obtained.

**Theorem 5.2.** Assume that \((U_1: U_2) \sim BGBI_{2m \times m}(a, b, c)\). Then the density function of \(Z = U_2^{1/2} U_1 U_2^{1/2}\) is
\[
\frac{\beta_m[a + c, b + c] |Z|^{-(m+1)/2} |I_m - Z|^{-(m+1)/2}}{\beta_m[a, b, c]} 2F_1(a + c, a + c; a + b + 2c; I_m - Z)(dZ)
\]

and
\[
E(|Z|^r) = \frac{\beta_m[a + c, b + c] \beta_m[a + r, c]}{\beta_m[a, b, c]} 3F_2(c, a + c, a + c; a + c + r, a + b + 2c; I_m)
\]

with \(0 < \text{Re}(Z) < I_m\), \(\text{Re}(a + b) > (m - 1)/2\) and \(\text{Re}(b + c) > (m - 1)/2\).
Proof. Consider the transformation \( Z = U_2^{1/2} U_1 U_2^{1/2}, \) with
\[
(dU_1)(dU_2) = |U_2|^{-(m+1)/2} (dZ)(dU_2).
\]
Then the joint density of \( Z \) and \( U_2 \) is
\[
\frac{|Z|^{a-(m+1)/2} |U_2|^{-(a+c)} |I_m - U_2|^{a+c-(m+1)/2} |U_2 - Z|^{b+c-(m+1)/2}}{\beta_m[a,b,c] |I_m - Z|^{a+b+c}} (dZ)(dU_2),
\]
\( 0 < U_2 < Z < I_m. \)

Now, by making the transformation \( W = (I_m - Z)^{-1/2} (I_m - U_2)(I_m - Z)^{-1/2} \) and \( Z = Z. \) Noting that
\[
(dZ)(dU_2) = |I_m - Z|^{(m+1)/2} (dZ)(dW),
\]
Then the joint density of \( Z \) and \( W \) is
\[
\frac{|Z|^{a-(m+1)/2} |I_m - Z|^{c-(m+1)/2} |W|^{a+c-(m+1)/2} |I_m - W|^{b+c-(m+1)/2}}{\beta_m[a,b,c] |I_m - (I_m - Z)W|^{a+c}} (dW)(dZ),
\]
\( 0 < Z < I_m, \) \( 0 < W < I_m. \)

Integrating with respect to \( W \) using (8) with \( \Re(Z) < I_m, \) \( \Re(a+c) > (m-1)/2, \) and \( \Re(b+c) > (m-1)/2, \) we then obtain the stated marginal density function for \( Z. \)

Now in order to find the expression for \( E(|Z|^r) \), let us first observe that by \( p = 2 \) and \( q = 1 \) in Lemma 2.1 we obtain
\[
3F_2(a, a_1, a_2; c, b_1; X) = \frac{1}{\beta_m[a,c-a]} \times \int_{0 < Y < I_m} 2F_1(a_1 a_2; b_1; XY)|Y|^{a-(m+1)/2} |I_m - Y|^{c-a-(m+1)/2} (dY).
\]
Now, by making the transformation \( W = I_m - Y, \) with \( (dY) = (dW). \) Then
\[
3F_2(a, a_1, a_2; c, b_1; X) = \frac{1}{\beta_m[a,c-a]} \times \int_{0 < W < I_m} 2F_1(a_1 a_2; b_1; X(I_m - W))|I_m - W|^{a-(m+1)/2} |W|^{c-a-(m+1)/2} (dW). \tag{13}
\]
Then the expression for \( E(|Z|^r) \) follows immediately from the density function of \( Z \) and (13).

Theorem 5.3. Assume that \( (U_1; U_2)' \sim \mathcal{BGBI}_{2m \times m}(a,b,c). \) Then the density function of \( V = (V_1; V_2)' = (U_1^{-1}; U_2^{-1})' \) is
\[
\frac{|V_1|^{-a-(m+1)/2} |V_2|^{-b-(m+1)/2} |I_m - V_1^{-1}|^{b+c-(m+1)/2} |I_m - V_2^{-1}|^{a+c-(m+1)/2}}{\beta_m[a,b,c] |I_m - (V_1 V_2)^{-1}|^{a+b+c}} (dV)
\]
\( 0 < V_1 < I_m, \) \( 0 < V_2 < I_m, \) where the measure
\[
(dV) = (dV_1) \wedge (dV_2).
\]
and \( \Re(a) > (m-1)/2, \) \( \Re(b) > (m-1)/2 \) and \( \Re(c) > (m-1)/2. \)

Proof. The desired result follows immediately by making the change of variable \( (V_1; V_2)' = (U_1^{-1}; U_2^{-1})' \) in the joint density of \( (U_1; U_2)', \) taking into account that \( (dU_1)(dU_2) = |V_1|^{-m-1} |V_2|^{-m-1} (dV_1)(dV_2). \) \( \square \)
6 Conclusions

With the algorithms proposed by Koev and Edelman (2006) for the calculation of Jack polynomials and hypergeometric functions with matrix arguments, together with the Mat- Lab implementation by Koev (2004), it is now is feasible to evaluate in a very efficient way expressions such as density functions and moments, as shown in the preceding sections, as well as highly complex expressions. The theory developed in this paper has not yet been applied, with the exception of the bivariate case. However, its potential role is apparent, for example, in multidimensional scaling (MDS) in the following context. Bimatrix variate generalised beta distributions can be used as distributions of the matrices of similarities (or dissimilarities) for an individual when these matrices of similarities have been obtained at two different times. Given the genesis of bimatrix variate generalised beta distributions, such distributions may allow us, in some sense, to model the learning problem. Statistical approaches to MDS have been studied assuming independence between times (without learning) in the univariate case by Ramsay (1982) and by Vera et al. (2008).

Another potential use appears in the context of shape theory, specifically in the approach known as affine shape or configuration densities. This approach is currently being studied, see Caro-Lopera et al. (2008). It was first proposed by Goodall and Mardia (1993), and at present only the Cauchy configuration densities, have been explored. The use of beta configuration densities, only was proposed by Goodall and Mardia (1993), but no additional information was provided. Perhaps bimatrix variate generalised beta distributions can be obtained as configuration densities if we assume that two figures (images) of a single individual or a single object, obtained at two different times, are not independent. Currently, both applications are under study.

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