FRACTIONAL DIFFUSION IN GAUSSIAN NOISY ENVIRONMENT

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Abstract. We study the fractional diffusion in a Gaussian noisy environment as described by the fractional order stochastic partial equations of the following form: 

\[ D_t^\alpha u(t,x) = Bu + u \cdot W^H, \]

where \( D_t^\alpha \) is the fractional derivative of order \( \alpha \) with respect to the time variable \( t \), \( B \) is a second order elliptic operator with respect to the space variable \( x \in \mathbb{R}^d \), and \( W^H \) a fractional Gaussian noise of Hurst parameter \( H = (H_1, \cdots, H_d) \). We obtain conditions satisfied by \( \alpha \) and \( H \) so that the square integrable solution \( u \) exists uniquely.

1. Introduction

In recent years there have been a great amount of works on anomalous diffusions in the study of biophysics and so on (see for example [1], [4], [10], [11] to mention just a few). In mathematics, some of these anomalous diffusions (such as sub-diffusions) can be described by the so-called fractional order diffusion processes. As for the term “fractional order diffusion”, one has to distinguish two completely different types. One is the equation of the form

\[ \partial_t u(t,x) = (-\Delta)^\alpha u(t,x), \]

where \( t \geq 0, x \in \mathbb{R}^d, \alpha \in (0,2) \) is a positive number, \( \partial_t = \frac{\partial}{\partial t} \) and \( \Delta = \sum_{i=1}^d \partial^2_{x_i} \) is the Laplacian. This equation is not associated with the anomalous diffusion. Instead, it is associated with the so-called stable process (or in general Lévy process), which has jumps. Another equation is of the form

\[ D_t^{(\alpha)} u(t,x) = \Delta u(t,x), \]

where \( D_t^{(\alpha)} \) is the Caputo fractional derivative with respect to \( t \) (see [9] for the study of various fractional derivatives). This equation is relevant to the anomalous diffusion which we mentioned and has been studied by a number of researchers, see for example [2] and the references therein. If one considers the anomalous diffusion in a random environment, then it is naturally led to the study of a fractional order stochastic partial differential equation of the form

\[ D_t^{(\alpha)} u(t,x) = Bu(t,x) + u(t,x) \dot{W}(t,x), \]

where \( B \) is a second order differential operator, including the Laplacian as a special example, and \( \dot{W} \) is a noise. In this paper, we shall study this fractional order stochastic partial differential equation when \( \dot{W}(t,x) = \dot{W}^H(x) \) is a time homogeneous fractional Gaussian noise of Hurst parameter \( H = (H_1, \cdots, H_d) \). Mainly, we shall find a relation between \( \alpha \) and \( H \) such that the solution to the above equation has a unique square integrable solution.

If \( \alpha \) is formally set to 1, then the above stochastic partial differential equation has been studied in [5]. So, our work can be considered as an extension of the work [5] to the case of fractional diffusion (in Gaussian noisy environment). Let us also mention that when we formally set \( \alpha = 1 \), we recover one of the main results in [5]. Thus, our condition \((2.10)\) given below is also optimal.

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Here is the organization of the paper, Section 2 will describe the operator $B$ and the noise $W^H$ and state the main result of the paper. In our proof we need to use the properties of the two fundamental solutions (Green’s functions) $Z(t,x)$ and $Y(t,x,\xi)$ associated with the equation $D^{(\alpha)}_t u(t,x) = Bu(t,x)$, which is represented by the Fox’s $H$-function. We shall recall some most relevant results on the $H$-function and the Green’s function $Z(t,x,\xi)$ and $Y(t,x,\xi)$ in Section 3. A number of preparatory lemmas are needed to prove main results and they are presented in Section 4. Finally, the last section is devoted to the proof of our main theorem.

2. Main result

Let

$$B = \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

be a uniformly elliptic second-order differential operator with bounded continuous real-valued coefficients. Let $u_0$ be a given bounded continuous function (locally Hölder continuous if $d > 1$). Let $\{W^H(x), x \in \mathbb{R}^d\}$ be a time homogeneous (time-independent) fractional Brownian field on some probability space $(\Omega, \mathcal{F}, P)$ (Like elsewhere in probability theory, we omit the dependence of $W^H(x) = W^H(x, \omega)$ on $\omega \in \Omega$). Namely, the stochastic process $\{W^H(x), x \in \mathbb{R}^d\}$ is a (multi-parameter) Gaussian process with mean 0 and its covariance is given by

$$\mathbb{E} \left( W^H(x) W^H(y) \right) = \prod_{i=1}^{d} R_{H_i}(x_i, y_i),$$

where $H_1, \cdots, H_d$ are some real numbers in the interval $(0,1)$. Due to some technical difficulty, we assume that $H_i > 1/2$ for all $i = 1, 2, \cdots, d$; the symbol $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, P)$ and

$$R_{H_i}(x_i, y_i) = \frac{1}{2} \left( |x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i} \right), \quad \forall x_i, y_i \in \mathbb{R}$$

is the covariance function of a fractional Brownian motion of Hurst parameter $H_i$.

Throughout this paper we fix an arbitrary parameter $\alpha \in (0,1)$ and a finite time horizon $T \in (0, \infty)$. We study the following stochastic partial differential equation of fractional order:

$$\begin{align*}
\begin{cases}
D^{(\alpha)}_t u(t,x) = Bu(t,x) + u(t,x) \cdot \dot{W}^H(x), & t \in (0,T], \quad x \in \mathbb{R}^d; \\
u(0,x) = u_0(x),
\end{cases}
\end{align*}$$

where

$$D^{(\alpha)}_t u(t,x) = \frac{1}{\gamma(1-\alpha)} \left[ \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(\tau,x) d\tau - t^{-\alpha} u(0,x) \right]$$

is the Caputo fractional derivative (see e.g. [9]) and $\dot{W}^H(x) = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W^H(x)$ is the distributional derivative (generalized derivative) of $W^H$, called fractional Brownian noise.

Our objective is to obtain condition on $\alpha$ and $H$ such that the above equation has a unique solution. But since $W^H$ is not differentiable or since $\dot{W}^H(x)$ does not exist as an ordinary function, we have to describe under what sense a random field $\{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a solution to the above equation (2.2).
To motivate our definition of the solution, let us consider the following (deterministic) partial differential equation of fractional order with the term $u(t, x) \cdot \dot{W}^H(x)$ in (2.2) replaced by $f(t, x)$:

\begin{equation}
\begin{cases}
D_t^{(\alpha)} \tilde{u}(t, x) = B\tilde{u}(t, x) + f(t, x), & t \in (0, T], \ x \in \mathbb{R}^d; \\
\tilde{u}(0, x) = u_0(x),
\end{cases}
\end{equation}

where the function $f$ is bounded and jointly continuous in $(t, x)$ and locally H"older continuous in $x$. In [2], it is proved that there are two Green’s functions
\[
\{Z(t, x, \xi), Y(t, x, \xi), 0 < t \leq T, x, \xi \in \mathbb{R}^d\}
\]
such that the solution to the Cauchy problem (2.3) is given by

\begin{equation}
\tilde{u}(t, x) = \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi + \int_0^t ds \int_{\mathbb{R}^d} Y(t - s, x, y)f(s, y)dy.
\end{equation}

In general, there is no explicit form for the two Green’s functions \( \{Z(t, x, \xi), Y(t, x, \xi)\} \). However, their constructions and properties are known (see [2], [6], [7], and the references therein). We shall recall some needed results in the next section.

From the classical solution expression (2.4), we expect that the solution $u(t, x)$ to (2.2) satisfies formally

\[ u(t, x) = \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi + \int_0^t ds \int_{\mathbb{R}^d} Y(t - s, x, y)u(s, y)\dot{W}^H(y)dy. \]

The above formal integral $\int_0^t ds \int_{\mathbb{R}^d} Y(t - s, x, y)u(s, y)\dot{W}^H(y)dy$ can be defined by Itô-Skorohod stochastic integral $\int_{\mathbb{R}^d} \left[ \int_0^t Y(t - s, x, y)u(s, y)ds \right] \dot{W}^H(dy)$ as given in [5].

Now, we can give the following definition.

**Definition 2.1.** A random field $\{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}^d\}$ is called a mild solution to the equation (2.2) if

1. $u(t, x)$ is jointly measurable in $t \in [0, T]$ and $x \in \mathbb{R}^d$;
2. $\forall (t, x) \in [0, T] \times \mathbb{R}^d$, $\int_0^t \int_{\mathbb{R}^d} Y(t - s, x, y)u(s, y)dsW^H(dy)$ is well defined in $L^2$;
3. The following holds in $L^2$

\begin{equation}
\begin{aligned}
&u(t, x) = \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x, y)u(s, y)W^H(dy)ds.
\end{aligned}
\end{equation}

Let us return to the discussion of the two Green’s functions $\{Z(t, x, \xi), Y(t, x, \xi)\}$. If $\alpha = 1$, namely, if $D_t^{(\alpha)}$ in (2.3) is replaced by $\partial_t$ and $B = \Delta := \sum_{i=1}^d \partial_{x_i}^2$, then

\begin{equation}
Z(t, x, \xi) = Y(t, x, \xi) = (4\pi t)^{-d/2}\exp \left\{ -\frac{|x - \xi|^2}{4t} \right\}.
\end{equation}

In this case the stochastic partial differential equation of the form

\begin{equation}
\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + u \cdot W^H(x), \quad x \in \mathbb{R}^d,
\end{equation}
was studied in [5]. The mild solution to the above equation (2.7) is proved to exist uniquely under conditions

\[(2.8) \quad H_i > 1/2, \quad i = 1, \cdots, d \quad \text{and} \quad \sum_{i=1}^{d} H_i > d - 1.\]

The main result of this paper is to extend the above result in [5] to our equation (2.2).

**Theorem 2.2.** Let the coefficients \(a_{ij}(x), b_i(x), i, j = 1, \cdots, d,\) be bounded and continuous and let them be Hölder continuous with exponent \(\gamma.\) Let \(a_{ij}(x)\) be uniformly elliptic. Namely, there is a constant \(a_0 \in (0, \infty)\) such that

\[
\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \forall \quad \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d.
\]

Let \(u_0\) be a bounded continuous (and locally Hölder continuous if \(d > 1\)). Assume

\[(2.9) \quad H_i > \begin{cases} \frac{1}{2} & \text{if } d = 1, 2, 3, 4 \\ 1 - \frac{\gamma}{2d} & \text{if } d \geq 5 \end{cases}
\]

and

\[(2.10) \quad \sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha}.\]

Then, the mild solution to (2.2) exists uniquely in \(L^2(\Omega, \mathcal{F}, P).\)

**Remark 2.3.** (i) When \(\alpha\) is formally set to 1, the condition (2.10) is the same as the condition (2.8) given in [5]. So, in some sense our condition (2.10) is optimal.

(ii) Since \(H_i < 1\) for all \(i = 1, 2, \cdots, d\) the condition is possible only when \(\alpha > 1/2.\)

### 3. Green’s functions \(Z\) and \(Y\)

**3.1. Fox’s \(H\)-function.** We shall use \(H\)-function to express the Green’s functions \(Z\) and \(Y\) in Definition 2.1. In this subsection we recall some results about the \(H\)-functions and the two Green’s functions. We shall follow the presentation in [8] (see also [2] and references therein).

**Definition 3.1.** Let \(m, n, p, q\) be integers such that \(0 \leq m \leq q, 0 \leq n \leq p.\) Let \(a_i, b_i \in \mathbb{C}\) be complex numbers and let \(\alpha_j, \beta_j\) be positive numbers, \(i = 1, 2, \cdots, p, j = 1, 2, \cdots, q.\) Let the set of poles of the gamma functions \(\Gamma(b_j + \beta_j s)\) doesn’t intersect with that of the gamma functions \(\Gamma(1 - a_i - \alpha_i s),\) namely,

\[
\left\{b_{jl} = \frac{-b_j - l}{\beta_j}, l = 0, 1, \cdots \right\} \bigcap \left\{a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset
\]

for all \(i = 1, 2, \cdots, p\) and \(j = 1, 2, \cdots, q.\) The \(H\)-function

\[
H_{pq}^{mn}(z) \equiv H_{pq}^{mn} \left[ z \bigg| \frac{(a_1, \alpha_1)}{b_1, \beta_1} \cdots \frac{(a_p, \alpha_p)}{b_q, \beta_q} \right]
\]

is defined by the following integral

\[
H_{pq}^{mn}(z) = \frac{1}{2\pi i} \int_L \prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s) z^{-s} ds, \quad z \in \mathbb{C},
\]
where an empty product in (3.1) means 1 and \( L \) in (3.1) is the infinite contour which separates all the points \( b_{ij} \) to the left and all the points \( a_{ik} \) to the right of \( L \). Moreover, \( L \) has one of the following forms:

1. \( L = L_{-\infty} \) is a left loop situated in a horizontal strip starting at point \(-\infty + i\phi_1\) and terminating at point \(-\infty + i\phi_2\) for some \(-\infty < \phi_1 < \phi_2 < \infty\)
2. \( L = L_{+\infty} \) is a right loop situated in a horizontal strip starting at point \( +\infty + i\phi_1\) and terminating at point \( +\infty + i\phi_2\) for some \(-\infty < \phi_1 < \phi_2 < \infty\)
3. \( L = L_{i\gamma\infty} \) is a contour starting at point \( \gamma - i\infty \) and terminating at point \( \gamma + i\infty \) for some \( \gamma \in (-\infty, \infty) \)

To illustrate \( L \) we give the following graphs.

The integral (3.1) exists when \( \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0 \) (see [8], Theorem 1.1).

Example 3.2. To compare with the classical case \( \alpha = 1 \), we consider the case \( m = 2, n = 0, p = 1, q = 2, a_1 = \alpha_1 = b_2 = \beta_1 = \beta_2 = 1 \) and \( b_1 = \frac{d}{2} \). Let \( L = L_{-\infty} \). Then, we have

\[
H_{12}^{20} \left[ \begin{array}{l} (1,1) \\ (\frac{d}{2},1), (1,1) \end{array} \right] = \frac{1}{2\pi i} \int_{L} \frac{\Gamma(d/2 + s)\Gamma(1 + s)}{\Gamma(1 + s)} z^{-s} ds \\
= \frac{1}{2\pi i} \int_{L} \Gamma(d/2 + s) z^{-s} ds \\
= \sum_{v=0}^{\infty} \lim_{s \to -(d/2 + v)} (s + \frac{d}{2} + v) \Gamma(d/2 + s) z^{-s} \\
= \sum_{v=0}^{\infty} \lim_{s \to -(d/2 + v)} \Gamma(v + \frac{d}{2} + s + 1) (s + \frac{d}{2} + v - 1) \cdots (s + \frac{d}{2}) z^{-s} \\
= \sum_{v=0}^{\infty} z^{d/2} (-1)^v \frac{1}{v!} z^v \\
= z^{d/2} \exp(-z).
\]

3.2. Green’s functions \( Z \) and \( Y \) when \( B \) has constant coefficients. In this subsection let us consider \( Z \) and \( Y \) when the operator \( B \) in (2.2) has the following form

\[
B = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},
\]
where the matrix $A = (a_{ij})$ is positive definite. In this case, $Z$ and $Y$ (we call them $Z_0$ and $Y_0$ to distinguish with the general coefficient case) are given as follows.

$$Z_0(t, x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}} \left[ \sum_{i,j=1}^{d} A^{(ij)} x_i x_j \right]^{-d/2}$$

$$\times H_{12}^{20} \left[ \frac{1}{4} t^{-\alpha} \sum_{i,j=1}^{d} A^{(ij)} x_i x_j \left\{ \frac{1}{2}, 1, (1, 1) \right\} \right]$$

where $(A^{(ij)}) = A^{-1}$ and

$$Y_0(t, x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}} \left[ \sum_{i,j=1}^{d} A^{(ij)} x_i x_j \right]^{-d/2} t^{\alpha-1}$$

$$\times H_{12}^{20} \left[ \frac{1}{4} t^{-\alpha} \sum_{i,j=1}^{d} A^{(ij)} x_i x_j \left\{ \frac{\alpha}{2}, 1, (1, 1) \right\} \right].$$

It is easy to see that for the constant coefficient case, the both of the Green’s functions are homogeneous in time and space. Namely,

$$Z_0(t, x, \xi) = Z_0(t, x - \xi), \quad Y_0(t, x, \xi) = Y_0(t, x - \xi).$$

In particular, when $\alpha = 1$, it is easy to see from the above expression and the explicit form (3.2) of $H_{12}^{20}(z)$ that

$$Z_0(t, x, \xi) = Y_0(t, x, \xi) = (4\pi)^{-d/2} \det(A)^{-1/2} \exp \left\{ - \frac{\sum_{i,j=1}^{d} A^{(ij)} (x_i - \xi_i)(x_j - \xi_j)}{4t} \right\}.$$

which reduces to (2.6) when $A = I$ is the identity matrix.

With the above expression for $Z_0$ and $Y_0$ and the properties of the $H$-function, one can obtain the following estimates.

**Proposition 3.3.** Denote

$$p(t, x) = \exp \left( - \sigma t^{-2\alpha} |x|^{2\alpha} \right), \quad t > 0, \quad x \in \mathbb{R}^d,$$

where $\sigma \in (0, \infty)$ is a constant whose exact value is irrelevant in the paper. Then, we have the following estimates:

$$|Z_0(t, x)| \leq \begin{cases} 
C t^{\frac{\alpha}{2}} p(t, x) & \text{when } d = 1 \\
C t^{-\alpha} \left\| \log \frac{|x|^2}{t^\alpha} \right\| + 1 |p(t, x) & \text{when } d = 2 \\
C t^{-\alpha} |x|^{2-d} p(t, x) & \text{when } d \geq 3,
\end{cases}$$

where for instance, $|Z_0(t, x)| \leq C t^{-\frac{\alpha}{2}} p(t, x)$ means that there are positive constant $C$ and positive constant $\sigma$ such that the above inequality holds. In what follows the positive constants $C$ and $\sigma$ are generic, which may be different in different appearances.

**Proof.** Denote $R = x^2/t^\alpha$. From [2], Proposition 1, it follows that when $R \leq 1$, we have

$$|Z_0(t, x)| \leq \begin{cases} 
C t^{\frac{\alpha}{2}} & \text{when } d = 1 \\
C t^{-\alpha} \left\| \log \frac{|x|^2}{t^\alpha} \right\| + 1 & \text{when } d = 2 \\
C t^{-\alpha} |x|^{2-d} & \text{when } d \geq 3,
\end{cases}$$
Since when $R \leq 1$, $p(t, x)$ is bounded from below. This proves the inequality (3.4) when $R \leq 1$.

When $R > 1$, then by [2], Proposition 1 we have $|Z_0(t, x)| \leq Ct^{-\sigma d}p(t, x)$. It is clear that this implies the inequality (3.4) when $d = 1$ and $d = 2$. Now, we assume that $d \geq 3$. We have

$$
|Z_0(t, x)| \leq Ct^{-\sigma d}p(t, x) \leq Ct^{-\sigma}x^{2-d}\left(\frac{x^2}{t^\alpha}\right)^{\frac{d}{2}-1}p(t, x)
$$

where we used the fact that $\left(\frac{x^2}{t^\alpha}\right)^{\frac{d}{2}-1}p(t, x) \leq p(t, x)$ for a different $\sigma$ in the later $p(t, x)$. □

Similarly, we can use [2], Proposition 2 (for $d = 1$ case) and [2], Section 4.2 (for $d \geq 2$ case) to obtain the following estimates for $Y_0(t, x)$.

**Proposition 3.4.** We follow the same notation $p(t, x)$ as defined by (3.3). We have

1. When $d = 1$, we have the following estimates:

$$
|Y_0(t, x)| \leq \begin{cases} 
Ct^{\frac{d}{2}-1}p(t, x) & \text{when } t^{-\alpha}|x|^2 \geq 1 \\
Ct^{\frac{d}{2}} & \text{when } t^{-\alpha}|x|^2 \leq 1.
\end{cases}
$$

2. When $d \geq 2$, we have the following estimates:

$$
|Y_0(t, x)| \leq \begin{cases} 
Ct^{-1}p(t, x) & \text{when } d = 2 \\
Ct^{-\frac{d}{2}-1}p(t, x) & \text{when } d = 3 \\
Ct^{-\alpha-1}[\log|\frac{|x|^2}{t^\alpha}|+1]p(t, x) & \text{when } d = 4 \\
Ct^{-\alpha-1}|x|^{1-d}p(t, x) & \text{when } d \geq 5,
\end{cases}
$$

where for instance, $|Y_0(t, x)| \leq Ct^{-1}p(t, x)$ means that there are positive constant $C$ and positive constant $\sigma$ such that the above inequality holds. In what follows the positive constants $C$ and $\sigma$ are generic, which may be different in different appearances.

### 3.3. Green’s functions $Z$ and $Y$ in general coefficient case.

If the coefficients of $B$ are not constant, then the Green’s functions $Z$ and $Y$ are more complicated and may be obtained by a method similar to the Levi parametrix for the parabolic equations.

Denote

$$
M(t, x, \xi) = \sum_{i,j=1}^{d} [a_{ij}(x) - a_{ij}(\xi)]\frac{\partial^2}{\partial x_i \partial x_j}Z_0(t, x - \xi)
+ \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}Z_0(t, x - \xi) + c(x)Z_0(t, x - \xi)
$$

$$
K(t, x, \xi) = \sum_{i,j=1}^{d} [a_{ij}(x) - a_{ij}(\xi)]\frac{\partial^2}{\partial x_i \partial x_j}Y_0(t, x - \xi)
+ \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}Y_0(t, x - \xi) + c(x)Y_0(t, x - \xi).
$$
Let $Q(s, y, \xi)$ and $\Phi(s, y, \xi)$ be defined by
\[Q(t, x, \xi) = M(t, x, \xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t - s, x, y)Q(s, y, \xi)dy,\]
\[\Phi(t, x, \xi) = K(t, x, \xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t - s, x, y)\Phi(s, y, \xi)dy\]

**Proposition 3.5.** Let the coefficients $a_{ij}(x)$ and $b_i(x)$ satisfy the conditions in Theorem 2.2. Recall that $\gamma$ is the Hölder exponent of the coefficients with respect to the spatial variable $x$. Then, the Green’s functions $\{Z(t, x, \xi), Y(t, x, \xi)\}$ have the following form:

\[Z(t, x, \xi) = Z_0(t, x - \xi) + V_Z(t, x, \xi);\]

\[Y(t, x, \xi) = Y_0(t, x - \xi) + V_Y(t, x, \xi),\]

where

\[V_Z(t, x, \xi) = \int_0^t ds \int_{\mathbb{R}^d} Y_0(t - s, x, y)Q(s, y, \xi)dy;\]

\[V_Y(t, x, \xi) = \int_0^t ds \int_{\mathbb{R}^d} Y_0(t - s, x, y)\Phi(s, y, \xi)dy.\]

Moreover, the function $V_Z(t, x, \xi), V_Y(t, x, \xi)$ satisfy the following estimates.

\[|V_Z(t, x, \xi)| \leq \begin{cases} Ct(\gamma-1)^{\frac{\alpha}{2}}p(t, x - \xi), & \text{when } d = 1; \\
Ct^{\frac{\alpha}{\alpha - 1}}(t - x)^{\gamma - \gamma_0}p(t, x - \xi), & \text{when } d = 2; \\
Ct^{\frac{\alpha}{\alpha - 1}}|x - \xi|^{2-d+\gamma-\gamma_0}p(t, x - \xi), & \text{when } d = 3 \text{ or } d \geq 5; \\
Ct^{(\gamma-\gamma_0)^{\frac{\alpha}{2}}-1}|x - \xi|^{-2\gamma+2\gamma_0}p(t, x - \xi), & \text{when } d = 4.\end{cases}\]

and

\[|V_Y(t, x, \xi)| \leq \begin{cases} Ct^{\alpha-1+(\gamma-1)^{\frac{\alpha}{2}}}p(t, x - \xi), & \text{when } d = 1; \\
Ct^{\alpha-1}(t - x)^{\gamma - \gamma_0}p(t, x - \xi), & \text{when } d = 2; \\
Ct^{(\gamma_0+\gamma)^{\frac{\alpha}{2}}-1}|x - \xi|^{2-d+(\gamma - \gamma_0)/2}p(t, x - \xi), & \text{when } d = 3 \text{ or } d \geq 5; \\
Ct^{(\gamma-\gamma_0)^{\frac{\alpha}{2}}-1}|x - \xi|^{-2+\gamma-\gamma_0}p(t, x - \xi), & \text{when } d = 4.\end{cases}\]

Here $\gamma_0$ is any number such that $0 < \gamma_0 < \gamma$ and in the case $d \geq 3$, the constant $C$ depends on $\gamma_0$.

4. **Auxiliary lemmas**

To prove our main theorem, we need to dominate certain multiple integral involving $Y(t, x, \xi)$ and $Z(t, x, \xi)$. Since both $Y(t, x, \xi)$ and $Z(t, x, \xi)$ are complicated, we shall first bounded them by $p(t, x - \xi)$ from the estimations of $|Y_0(t, x)|$ and $|V_Y(t, x, \xi)|$. More precisely, we have the following bounds for $Y(t, x, \xi)$.

**Lemma 4.1.** Let $x \in \mathbb{R}^d, t \in (0, T]$. Then

\[|Y(t, x, \xi)| \leq \begin{cases} Ct^{1-\frac{\alpha}{2}}p(t, x - \xi), & \text{when } d = 1; \\
Ct^{1-1}p(t, x - \xi), & \text{when } d = 2; \\
Ct^{-(\gamma-\gamma_0)^{\frac{\alpha}{2}}-1}|x - \xi|^{-2+\gamma-\gamma_0}p(t, x - \xi), & \text{when } d = 4; \\
Ct^{-(\gamma-\gamma_0)^{\frac{\alpha}{2}}-1}|x - \xi|^{2-d+(\gamma - \gamma_0)/2}p(t, x - \xi), & \text{when } d = 3 \text{ or } d \geq 5.\end{cases}\]
Proof. We shall prove the lemma case by case. First, when \( d = 1 \), by Proposition 3.4, we have

\[
|Y_0(t, x - \xi)| \leq \begin{cases} 
C t^{\frac{\alpha}{2} - 1} p(t, x - \xi), & t^{-\alpha}|x - \xi|^2 \geq 1; \\
C t^{\frac{\alpha}{2} - 1}, & t^{-\alpha}|x - \xi|^2 \leq 1.
\end{cases}
\]

If \( t^{-\alpha}|x - \xi|^2 \leq 1 \), then

\[
|Y_0(t, x - \xi)| \leq C t^{-\frac{\alpha}{2} + 2} \cdot \frac{p(x, t)}{e^{-\sigma}} \leq C t^{-\frac{\alpha}{2}} p(t, x - \xi).
\]

Therefore

\[
|Y(t, x, \xi)| \leq |Y_0(t, x - \xi)| + |V_Y(t, x, \xi)| 
\leq C t^\alpha (\gamma - 1)^\frac{\alpha}{2} p(t, x - \xi) + C t^{-\frac{\alpha}{2}} p(t, x - \xi) 
\leq C t^\frac{\alpha}{2} p(t, x - \xi).
\]

Now we consider the case \( d = 2 \). From the following inequalities:

\[
|V_Y(t, x, \xi)| \leq C t^\frac{\alpha}{2} p(t, x - \xi); 
|Y_0(t, x - \xi)| \leq C t^{-\alpha} p(t, x - \xi)
\]

we have easily

\[
|Y(t, x, \xi)| \leq |Y_0(t, x - \xi)| + |V_Y(t, x, \xi)| \leq C t^{-1} p(t, x - \xi).
\]

We are going to prove the lemma when \( d = 3 \). From Proposition 3.4 we have

\[
|Y_0(t, x - \xi)| \leq C t^{-\frac{\alpha}{2} - 1} p(t, x - \xi)
\]

\[
= C t^{-(\gamma - \eta - 1)\frac{\alpha}{2} - 1} |x - \xi|^{-(\gamma - \eta - 1)/2} \left|\frac{x - \xi}{t^{\frac{\alpha}{2}}}\right|^{1-(\gamma - \eta - 1)/2} p(t, x - \xi) 
\leq C t^{-(\gamma - \eta - 1)\frac{\alpha}{2} - 1} |x - \xi|^{-1+(\gamma - \eta - 1)/2} p(t, x - \xi).
\]

Combining this inequality with Proposition 3.5 we obtain

\[
|Y(t, x, \xi)| \leq C t^{-(\gamma - \eta - 1)\frac{\alpha}{2} - 1} |x - \xi|^{-1+(\gamma - \eta - 1)/2} p(t, x - \xi).
\]

We turn to consider the case \( d = 4 \). Proposition 3.4 yields that for any \( \theta > 0 \) the following holds true:

\[
|Y_0(t, x - \xi)| \leq C t^{-\alpha - 1} \left[\left(\frac{|x - \xi|^2}{t^{\alpha}}\right)^{\theta} + \left(\frac{t^{\alpha}}{|x - \xi|^2}\right)^{\theta}\right] p(t, x - \xi); 
\]

\[
= C t^{-\alpha - 1}\left[\frac{t^{\alpha}}{|x - \xi|^2}\right]^{\theta} \left[\left(\frac{|x - \xi|^2}{t^{\alpha}}\right)^{2\theta} + 1\right] p(t, x - \xi).
\]

If \( \frac{|x - \xi|^2}{t^{\alpha}} > 1 \), then

\[
\left[\left(\frac{|x - \xi|^2}{t^{\alpha}}\right)^{2\theta} + 1\right] p(t, x - \xi) \leq 2 \left(\frac{|x - \xi|^2}{t^{\alpha}}\right)^{2\theta} p(t, x - \xi) \leq C p(t, x - \xi).
\]

As a consequence, we have

\[
|Y_0(t, x - \xi)| \leq C t^{-\alpha - 1}\left(\frac{t^{\alpha}}{|x - \xi|^2}\right)^{\theta} p(t, x - \xi).
\]
If $\frac{|x-\xi|^2}{t^2} \leq 1$, then the above inequality is obviously true. Now, we can choose $\theta > 0$, such that $-2\theta \geq (-2 + \gamma - 2\gamma_0)$. Thus, we have

$$|Y_0(t, x - \xi)| = Ct^{-(\frac{\alpha-1}{\gamma}+(-2\theta-(-2+\gamma-2\gamma_0))\frac{\alpha}{2})} |x - \xi|^{-2+\gamma-2\gamma_0} \left(\frac{|x - \xi|}{t^{\frac{\alpha}{2}}}\right)^{-2\theta-(-2+\gamma-2\gamma_0)} p(t, x - \xi) \leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{-2+\gamma-2\gamma_0} p(t, x - \xi).$$

Combining the above inequality with Proposition 3.5 we have

$$|Y(t, x, \xi)| \leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{-2+\gamma-2\gamma_0} p(t, x - \xi) + Ct^{(\gamma_0+\gamma)\frac{\alpha}{2}-1} |x - \xi|^{-2+\gamma-2\gamma_0} p(t, x - \xi) \leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{-2+\gamma-2\gamma_0} p(t, x - \xi)$$

since $-(\gamma - 2\gamma_0)\frac{\alpha}{2} - 1 \leq (\gamma_0 + \gamma)\frac{\alpha}{2} - 1$.

Finally we consider the case $d \geq 5$. From the estimates $|Y_0(t, x - \xi)| \leq Ct^{-\alpha-1} |x - \xi|^{4-d} p(t, x - \xi)$ we obtain

$$|Y_0(t, x - \xi)| \leq Ct^{-(\gamma+\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{2-d+(\gamma-\gamma_0)/2} \left|\frac{x - \xi}{t^{\frac{\alpha}{2}}}\right|^{2-(\gamma-\gamma_0)/2} p(t, x - \xi) \leq t^{-(\gamma-\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{2-d+(\gamma-\gamma_0)/2} p(t, x - \xi).$$

Therefore, we have

$$|Y(t, x, \xi)| \leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{2}-1} |x - \xi|^{2-d+(\gamma-\gamma_0)/2} p(t, x - \xi).$$

The proposition is then proved. □

The bound (4.1) will greatly help to simplify our estimation of the multiple integrals that we are going to encounter. However, when the dimension $d$ is greater than or equal to 2, the multiple integrals are still complicated to estimate and our main technique is to reduce the computation to one dimensional. This means we shall further bound the right hand side of the inequality (4.1) by product of functions of one variable. Before doing so, we denote the exponents of $t$ and $|x - \xi|$ in (4.1) by $\zeta_d$ and $\kappa_d$. Namely, we denote

$$\zeta_d = \begin{cases} 
-1 + \frac{\alpha}{2}, & d = 1; \\
-1, & d = 2; \\
-(\gamma - 2\gamma_0)\frac{\alpha}{2} - 1, & d = 4; \\
-(\gamma - \gamma_0)\frac{\alpha}{4} - 1, & d = 3 \text{ or } d \geq 5.
\end{cases}$$

and

$$\kappa_d = \begin{cases} 
0, & d = 1, 2; \\
-2 + \gamma - 2\gamma_0, & d = 4; \\
2 - d + (\gamma - \gamma_0)/2, & d = 3 \text{ or } d \geq 5.
\end{cases}$$

From now on we shall exclusively use $p(t, x) = \exp\left(-\sigma t^{\frac{\alpha}{2}} |x|^\frac{\alpha}{2}\right)$ to denote a function of one variable. However, the constant $\sigma$ may be different in different appearances of $p(t, x)$ (for notational simplicity, we omit the explicit dependence on $\sigma$ of $p(t, x)$).

With these notation Lemma 4.1 yields
Lemma 4.2. The following bound holds true for the Green’s function $Y$:

$$
|Y(t, x, \xi)| \leq C \prod_{i=1}^{d} \xi_i^{\kappa_i/d} |x_i - \xi_i|^{\kappa_i/d} p(t, x_i - \xi_i).
$$

Proof. It is easy to see that

$$
|x| = \left( \sum_{i=1}^{d} x_i^2 \right)^{1/2} \geq \max_{1 \leq i \leq d} |x_i| \geq \prod_{i=1}^{d} |x_i|^{\frac{1}{d}}.
$$

Thus for any positive number $\alpha > 0$, $|x|^{-\alpha} \leq \prod_{i=1}^{d} |x_i|^{-\frac{\alpha}{d}}$.

On the other hand,

$$
|x|^{2-\alpha} = \left( \sum_{i=1}^{d} |x_i|^2 \right)^{\frac{1}{2-\alpha}} \geq \left[ \max_{1 \leq i \leq d} |x_i|^2 \right]^{\frac{1}{2-\alpha}} = \max_{1 \leq i \leq d} |x_i|^2 \frac{2}{2-\alpha} \geq \frac{1}{d} \sum_{i=1}^{d} |x_i|^\frac{2}{2-\alpha}.
$$

Combining the above with (4.1) yields (4.4) since the exponents in $|x - \xi|$ in (4.1) are negative.

□

Lemma 4.3. Let $-1 < \beta \leq 0, x \in \mathbb{R}$. Then, there is a constant $C$, dependent on $\sigma$, $\alpha$ and $\beta$, but independent of $\xi$ and $s$ such that

$$
\sup_{\xi \in \mathbb{R}} \int_{\mathbb{R}} |x|^\beta p(s, x - \xi) dx \leq Cs^{\alpha \beta + \frac{\alpha}{2}}.
$$

Proof. Making the substitution $x = ys^{\frac{1}{2}}$ we obtain

$$
\int_{\mathbb{R}} |x|^\beta p(s, x - \xi) dx = s^{\alpha \beta + \frac{\alpha}{2}} \int_{\mathbb{R}} |y|^\beta \cdot \exp \left( -\sigma \left| y - \frac{\xi}{s^{\frac{1}{2}}} \right|^{2-\alpha} \right) dy
$$

$$
\leq s^{\alpha \beta + \frac{\alpha}{2}} \left( \int_{|y| \leq 1} |y|^\beta dy + \int_{\mathbb{R}} \exp \left( -\sigma \left| y - \frac{\xi}{s^{\frac{1}{2}}} \right|^{2-\alpha} \right) dy \right)
$$

$$
\leq Cs^{\alpha \beta + \frac{\alpha}{2}}
$$

since the two integrals inside the parenthesis are finite (and independent of $s$ and $\xi$). □

The following is a slight extension of the above lemma.

Lemma 4.4. There is a constant $C$, dependent on $\sigma$, $\alpha$ and $\beta$, but independent of $\xi$ and $s$ such that

$$
\sup_{\xi \in \mathbb{R}} \int_{\mathbb{R}} |x|^\beta |\log |x|| p(s, x - \xi) dx \leq Cs^{\alpha \beta + \frac{\alpha}{2}} [1 + |\log s|].
$$
Proof. We shall follow the same idea as in the proof of Lemma 4.3. Making the substitution \( x = ys^{\frac{a}{2}} \) we obtain

\[
\int_{\mathbb{R}} |x|^\beta |\log |x|| p(s, x - \xi)dx \\
\leq C s^{\alpha \beta + \frac{a}{2}} \int_{\mathbb{R}} |y|^\beta \left( |\log |y|| + |\log s| \right) \cdot \exp \left( - \sigma |y - \frac{\xi}{s^{\frac{a}{2}}}| \right) dy \\
\leq C s^{\alpha \beta + \frac{a}{2}} (1 + |\log s|) \left( \int_{|y| \leq e} |y|^\beta |\log |y||dy + \int_{\mathbb{R}} \exp \left( - \sigma |y - \frac{\xi}{s^{\frac{a}{2}}}| \right) dy \right) \\
\leq C s^{\alpha \beta + \frac{a}{2}} (1 + |\log s|).
\]

This proves the lemma. \qed

Lemma 4.5. Let \( \theta_1 \) and \( \theta_2 \) satisfy \(-1 < \theta_1 < 0, -1 < \theta_2 \leq 0\). Then for any \( \rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2 \),

1. If \( \theta_1 + \theta_2 = -1 \), then

\[
\int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \leq C |\log (\rho_2 - \tau_1)|.
\]

2. If \( \theta_1 + \theta_2 < -1 \), then

\[
\int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \leq C |\rho_2 - \tau_1|^{1+\theta_1+\theta_2}.
\]

Proof. Without loss of generality, suppose \( \tau_1 \leq \rho_2 \). We divide the integral domain into four intervals.

\[
\int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\
= \int_{\infty}^{3\tau_1 - \rho_1} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\
+ \int_{3\tau_1 - \rho_2}^{\frac{3\tau_1 + \rho_1}{2}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\
+ \int_{\frac{3\rho_2 - \tau_1}{2}}^{\infty} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\
=: I_1 + I_2 + I_3 + I_4.
\]
Let us consider $I_2$ first. When $\rho_1 \in \left[\frac{3\tau_1 - \rho_2}{2}, \frac{\tau_1 + \rho_2}{2}\right]$, we have $|\rho_2 - \rho_1| \geq \frac{\rho_2 - \tau_1}{2}$. Noticing $p(s_2 - s_1, \rho_2 - \rho_1) \leq 1$, we have the following estimate for $I_2$:

\[
I_2 \leq \left(\frac{\rho_2 - \tau_1}{2}\right)^{\theta_2} \int_{3\tau_1 - \rho_2}^{\tau_1 + \rho_2} |\rho_1 - \tau_1|^{\theta_1} d\rho_1
\leq \left(\frac{\rho_2 - \tau_1}{2}\right)^{\theta_2} \left[\int_{\tau_1}^{\tau_1 + \rho_2} (\rho_1 - \tau_1)^{\theta_1} d\rho_1 + \int_{\frac{3\tau_1 - \rho_2}{2}}^{\tau_1} (\tau_1 - \rho_1)^{\theta_1} d\rho_1\right]
= C\left(\rho_2 - \tau_1\right)^{1+\theta_1+\theta_2}.
\]

With the same argument, we have

\[
I_3 \leq C\left(\rho_2 - \tau_1\right)^{1+\theta_1+\theta_2}.
\]

Now, we study $I_1$. The term $I_4$ can be analyzed in a similar way. Since $\rho_1 < \frac{3\tau_1 - \rho_2}{2} < \tau_1 < \rho_2$, we have

\[
I_1 \leq \int_{-\infty}^{3\tau_1 - \rho_2} (\tau_1 - \rho_1)^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1.
\]

To estimate the above integral, we divide our estimation into three cases.

Case i): $\theta_1 + \theta_2 < -1$.
In this case, we bound $p(s_2 - s_1, \rho_2 - \rho_1)$ by 1. Thus, we have

\[
I_1 \leq \int_{-\infty}^{3\tau_1 - \rho_2} (\tau_1 - \rho_1)^{\theta_1+\theta_2} d\rho_1 = \frac{1}{1+\theta_1+\theta_2} \left(\frac{\rho_2 - \tau_1}{2}\right)^{1+\theta_1+\theta_2}.
\]

Case ii): $\theta_1 + \theta_2 = -1, \frac{\rho_2 - \tau_1}{2} \geq 1$.
In this case, we have $\frac{3\tau_1 - \rho_2}{2} \leq \tau_1 - 1$. Thus, we have

\[
I_1 \leq \int_{-\infty}^{\tau_1 - 1} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1
\leq \int_{-\infty}^{\tau_1 - 1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1
\leq \int_{-\infty}^{\infty} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1
\]

which is bounded when $s_1$ and $s_2$ are in a bounded domain.

Case iii): $\theta_1 + \theta_2 = -1, \frac{\rho_2 - \tau_1}{2} < 1$. 

In this case, we divide the integral into two intervals as follows.

\[ I_1 = \int_{-\infty}^{\frac{3r_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1 + \theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \]
\[ \leq \int_{-\infty}^{\tau_1 - \rho_2} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 + \int_{\tau_1 - \rho_2}^{\frac{3r_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \]
\[ \leq C + \int_{\tau_1 - \rho_2}^{\frac{3r_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{-1} d\rho_1 \]
\[ \leq C + C|\ln(\rho_2 - \tau_1)|. \]

Similar argument works for \( I_4 \). Combining the estimates for \( I_k, k = 1, 2, 3, 4 \) yields the lemma. \( \square \)

**Lemma 4.6.** Let \( \theta_1 \) and \( \theta_2 \) satisfy \(-1 < \theta_1 < 0, -1 < \theta_2 \leq 0 \) and \( \theta_1 + 2\theta_2 > -2 \). Let \( 0 \leq r_1 < r_2 \leq T \) and \( 0 \leq s_1 < s_2 \leq T \). Then for any \( \rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2 \), we have

\[ \int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \]

(4.5) \[ \leq \begin{cases} 
C(s_2 - s_1)^{\alpha (\theta_1 + \theta_2 + 1)} (r_2 - r_1)^{\alpha (\theta_2 + 1)} / 2, & \theta_1 + \theta_2 > -1; \\
C(r_2 - r_1)^{\alpha (\theta_1 + 2\theta_2 + 1)} / 2, & \theta_1 + \theta_2 < -1; \\
C(r_2 - r_1)^{\alpha (\theta_2 + 1)} [1 + |\ln(r_2 - r_1)|], & \theta_1 + \theta_2 = -1.
\end{cases} \]

**Proof.** First, we write

\[ I := \int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \]

(4.6) \[ = \int_{\mathbb{R}} f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 , \]

where

\[ f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) = \int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 . \]

We divide the situation into three cases.

**Case i:** \( \theta_1 + \theta_2 > -1 \).

In this case we apply the Hölder’s inequality to obtain

\[ f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \leq \left\{ \int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1 + \theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \right\}^{\frac{\theta_1}{\theta_1 + \theta_2}} \times \left\{ \int_{\mathbb{R}} |\rho_2 - \rho_1|^{\theta_1 + \theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \right\}^{\frac{\theta_2}{\theta_1 + \theta_2}} \]

(4.7) \[ \leq C(s_2 - s_1)^{\alpha (\theta_1 + \theta_2 + 1)} / 2 . \]
where the last inequality follows from Lemma 4.3. Substituting the above estimate (4.7) into (4.6), we have

\[
I = \int f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2)|\tau_2 - \tau_1|^{\theta_1}p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\
\leq C(s_2 - s_1)^{\alpha(\theta_1 + \theta_2)} + \frac{\alpha}{2} \int |\tau_2 - \tau_1|^\theta p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1.
\]

Using Lemma 4.3 again we have,

\[
I \leq C(s_2 - s_1)^{\alpha(\theta_1 + \theta_2)} + \frac{\alpha}{2} (r_2 - r_1)^{\alpha \theta_2 + \frac{\alpha}{2}}.
\]

Case ii): \(\theta_1 + \theta_2 < -1\).

In this case, from Lemma 4.5, part (ii) it follows

\[
f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \leq C|\rho_2 - \tau_1|^{\theta_1 + \theta_2 + 1}.
\]

Hence, we have

\[
I \leq C \int |\rho_2 - \tau_1|^{\theta_1 + \theta_2 + 1}|\tau_2 - \tau_1|^\theta p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1.
\]

Now, since from the condition of the lemma, \(\theta_1 + 2\theta_2 + 1 > -1\), we can use Hölder’s inequality such as in the inequality (4.7) in the case (i), to obtain

\[
I \leq C(r_2 - r_1)^{\alpha(\theta_1 + 2\theta_2)} + \alpha.
\]

Case iii): \(\theta_1 + 2\theta_2 = -1\).

In this case, we first use Lemma 4.5, part (i) to obtain

\[
f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \leq C [1 + \log |\rho_2 - \tau_1|].
\]

Thus, using Lemma 4.4, we have

\[
I \leq C \int \{1 + \log |\rho_2 - \tau_1|\} |\tau_2 - \tau_1|^\theta p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1
\]

\[
\leq C(r_2 - r_1)^{\alpha(\theta_1 + 1)} [1 + \log |\rho_2 - \tau_1|].
\]

The lemma is then proved. \(\square\)

**Corollary 4.7.** Let \(\theta_1\) and \(\theta_2\) satisfy \(-1 < \theta_1 < 0, -1 < \theta_2 \leq 0\) and \(\theta_1 + 2\theta_2 > -2\). Let \(0 \leq r_1 < r_2 \leq T\) and \(0 \leq s_1 < s_2 \leq T\). Then for any \(\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2\), we have

\[
\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1)(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1
\]

\[
\leq \begin{cases} 
C(s_2 - s_1)^{\alpha(\theta_1 + 2\theta_2)} (r_2 - r_1)^{\alpha(\theta_1 + 2\theta_2)} & \text{if } \theta_1 + \theta_2 \neq -1 \\
C(s_2 - s_1)^{\alpha(\theta_2 + 1)} (r_2 - r_1)^{\alpha(\theta_2 + 1)} [1 + |\log(r_2 - r_1)| + |\log(s_2 - s_1)|] & \text{if } \theta_1 + \theta_2 = -1.
\end{cases}
\]

(4.8)

**Proof.** Consider first the case \(\theta_1 + \theta_2 < -1\). Denote the integral on the left hand side of (4.8) by \(I\). Then the inequality (4.8) implies

\[
I \leq C(r_2 - r_1)^{\alpha(\theta_1 + 2\theta_2)} + \alpha.
\]
In the same way we have
\[ I \leq C(s_2 - s_1) \frac{a^{(\theta_1 + 2\theta_2) + \alpha}}{2}. \]
Now we use the fact that if three numbers satisfying \( a \leq b \) and \( a \leq c \), then \( a = a^{1/2} a^{1/2} \leq b^{1/2} c^{1/2} \).
\[ I \leq C(r_2 - r_1) \frac{a^{(\theta_1 + 2\theta_2) + \alpha/2}}{2}(s_2 - s_1) \frac{a^{(\theta_1 + 2\theta_2) + \alpha/2}}{2}, \]
which simplifies to (4.8). The exactly same argument applied to the case \( \theta_1 + \theta_2 = -1 \) and the case \( \theta_1 + \theta_2 > -1 \). Thus, the inequality (4.7) implies (4.8).

Lemma 4.8. Let \( p_1, \ldots, p_n > 0 \). Then for any \( T > 0 \),
\[ (4.9) \quad \int_{0 \leq s_1 < \cdots < s_n \leq T} (s_1 - s_1 - 1)^p_\nu - 1 \cdots (s_2 - s_1 - 1)^p_2 - 1 s_1^{p_1 - 1} ds = \frac{T^n \prod_{k=1}^n \Gamma(p_k)}{\Gamma(p_1 + \cdots + p_n + 1)}. \]

Proof. This is well-known. For example it is straightforward consequence of formula 4.634 of [3] with some obvious transformations.

Lemma 4.9. Assume that \( u_0 \) is bounded. Then
\[ \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi \leq C. \]

Proof. We use \( Z(t, x, \xi) = Z_0(t, x - \xi) + V_Z(t, x, \xi) \). Since \( u_0 \) is bounded,
\[ \left| \int_{\mathbb{R}^d} Z_0(t, x, \xi)u_0(\xi)d\xi \right| \leq C \int_{\mathbb{R}^d} |Z_0(t, x, \xi)|d\xi \]
which is bounded by the estimates in (3.4) and a substitution \( \xi = x + t^{\gamma/\gamma} y \). In fact, we have, for example, when \( \gamma \geq 3 \),
\[ \int_{\mathbb{R}^d} |Z_0(t, x - \xi)|d\xi \leq C \int_{\mathbb{R}^d} t^{-\gamma/2} t^{\gamma/2} |y|^{2-d} \exp\{-\sigma|y|^{2-\alpha}\}dy \leq Ct^{1-\alpha} \leq C. \]

Similarly, using the estimation for \( V_Z(t, x, \xi) \) given in Proposition 3.5 we can bound \( \int_{\mathbb{R}^d} |V_Z(t, x, \xi)|d\xi \) by a constant. In fact, for example, when \( \gamma = 3 \), we have
\[ \int_{\mathbb{R}^d} |V_Z(t, x, \xi)|d\xi \leq Ct^{\gamma/2-\alpha} \int_{\mathbb{R}^d} t^{\gamma/2} t^{\gamma/2} |y|^{\gamma-\gamma-1} \exp\{-\sigma|y|^{2-\alpha}\}dy \leq Ct^{\gamma/2} \leq C. \]
The other dimension case can be dealt with the same way.

5. Proof of the main theorem 2.2.

Change \( t \) to \( s \) and \( x \) to \( y \) and the equation (2.5) for mild solution becomes
\[ u(s, y) = \int_{\mathbb{R}^d} Z(s, y, \xi)u_0(\xi)d\xi + \int_0^s \int_{\mathbb{R}^d} Y(s - r, y, z)u(r, z)W^H(\xi)dr. \]
Substituting the above into (2.5), we have
\[ u(t, x) = \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x, y)Z(s, y, \xi)u_0(\xi)d\xi W^H(dy)ds \]
\[ + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x, y)Y(s - r, y, z)u(r, z)W^H(\xi)drW^H(dy)ds. \]
We continue to iterate this procedure to obtain

\begin{equation}
(5.1) \quad u(t, x) = \sum_{n=0}^{\infty} \Psi_n(t, x),
\end{equation}

where \( \Psi_n \) satisfies the following recursive relation:

\begin{align*}
\Psi_0(t, x) &= \int_{\mathbb{R}^d} Z(t, x, \xi) u_0(\xi) d\xi \\
\Psi_{n+1}(t, x) &= \int_0^t \int_{\mathbb{R}^d} Y(t-s, x, y) \Psi_n(s, y) W^H(dy) ds, \quad n = 0, 1, 2, \ldots
\end{align*}

To write down the explicit expression for the expansion (5.1), we denote

\begin{equation}
(5.2) \quad f_n(t, x; x_1, \cdots, x_n) = \int_{T_n} \int_{\mathbb{R}^d} Y(t-s_n, x, x_n) \cdots Y(s_2-s_1, x_2, x_1) Z(s_1, x_1, \xi) u_0(\xi) d\xi ds,
\end{equation}

where

\begin{equation*}
T_n = 0 < s_1 < s_2 < \cdots < s_n \leq t \quad \text{and} \quad ds = ds_1 ds_2 \cdots ds_n.
\end{equation*}

With these notations, we see from the above iteration procedure that

\begin{align*}
\Psi_n(t, x) &= I_n(\tilde{f}_n(t, x)) \\
&= \int_{\mathbb{R}^{nd}} f_n(t, x; x_1, \cdots, x_n) W^H(dx_1) W^H(dx_2) \cdots W^H(dx_n) \\
&= \int_{\mathbb{R}^{nd}} \tilde{f}_n(t, x; x_1, \cdots, x_n) W^H(dx_1) W^H(dx_2) \cdots W^H(dx_n).
\end{align*}

\begin{equation}
(5.3)
\end{equation}

where \( I_n \) denotes the multiple Itô type integral with respect to \( W(x) \) (see [5]) and \( \tilde{f}_n(t, x; x_1, \cdots, x_n) \) is the symmetrization of \( f_n(t, x; x_1, \cdots, x_n) \) with respect to \( x_1, \cdots, x_n \):

\begin{equation*}
\tilde{f}_n(t, x; x_1, \cdots, x_n) = \frac{1}{n!} \sum_{i_1, \cdots, i_n \in \sigma(n)} f_n(t, x; x_{i_1}, \cdots, x_{i_n}),
\end{equation*}

where \( \sigma(n) \) denotes the set of permutations of \( (1, 2, \cdots, n) \).

The expansion (5.1) with the explicit expression (5.3) for \( \Psi_n \) is called the \textit{chaos expansion of the solution}.

If the equation (2.2) has a square integrable solution, then it has a chaos expansion according to a general theorem of Itô. From the above iteration procedure, it is easy to see that this chaos expansion of the solution is given uniquely by (5.1)-(5.3). This is the uniqueness.

If we can show that the series (5.1) is convergent in \( L^2(\Omega, \mathcal{F}, P) \), then it is easy to verify that \( u(t, x) \) defined by (5.1)-(5.3) satisfies the equation (2.5). Thus, the existence of the solution to (2.2) is solved and the explicit form of the solution is also given (by (5.1)-(5.3)). We refer to [5] for more detail.

Thus, our remaining task is to prove that the series defined by (5.1) is convergent in \( L^2(\Omega, \mathcal{F}, P) \). To this end, we need to use the lemmas that we just proved.
Let now \( u(t, x) \) be defined by (5.1)-(5.3). Then we have

\[
E[u(t, x)^2] = \sum_{n=0}^\infty E\left[ I_n(\tilde{f}_n(t, x)) \right]^2 = \sum_{n=0}^\infty n! \langle \tilde{f}_n, \tilde{f}_n \rangle_H
\]

(5.4)

\[
\leq \sum_{n=0}^\infty n! \langle f_n, f_n \rangle_H,
\]

where

\[
\langle f, g \rangle_H = \int_{\mathbb{R}^{2d}} \prod_{i=1}^n \varphi_H(u_i, v_i) f(u_1, \cdots, u_n) g(v_1, \cdots, v_n) du_1 dv_1 du_2 dv_2 \cdots du_n dv_n
\]

and the last inequality follows from Hölder inequality. Here and in the remaining part of the paper, we use the following notations:

\[
u_i = (u_{i1}, \cdots, u_{id}), \quad du_i = du_{i1} \cdots du_{id}, \quad i = 1, 2, \cdots, n;
\]

\[
\varphi_H(u_i, v_i) = \prod_{j=1}^d \varphi_{H_j}(u_{ij}, v_{ij}) = \prod_{j=1}^d H_j(2H_j - 1)|u_{ij} - v_{ij}|^{2H_j - 2}.
\]

We use the idea in [5] to estimate each term \( \Theta_n(t, x) = n! \langle f, f \rangle_H \) in the series (5.4). By the defining formula (5.2) for \( f_n \) we have

\[
\Theta_n(t, x) = n! \int_{T_n^d} \int_{\mathbb{R}^{2nd+2}} \prod_{k=1}^n \varphi_H(\xi - \eta_k) Y(t - s_n, x, \xi_n) \cdots Y(s_2 - s_1, \xi_2, \xi_1)
\]

\[
\cdot \int_{\mathbb{R}^d} Z(s_1, \xi_1, \xi_0) u_0(\xi_0) d\xi_0 \cdot Y(t - r_n, x, \eta_n) \cdots Y(r_2 - r_1, \eta_2, \eta_1)
\]

\[
\cdot \int_{\mathbb{R}^d} Z(r_1, \eta_1, \eta_0) u_0(\eta_0) d\eta_0 d\xi d\eta ds dr.
\]

Application of Lemma 4.9 to the above integral yields

\[
\Theta_n(t, x) \leq Cn! \int_{T_n^d} \int_{\mathbb{R}^{2nd+2}} \prod_{k=1}^n \varphi_H(\xi - \eta_k) Y(t - s_n, x, \xi_n) \cdots Y(s_2 - s_1, \xi_2, \xi_1)
\]

\[
\cdot Y(t - r_n, x, \eta_n) \cdots Y(r_2 - r_1, \eta_2, \eta_1) d\xi d\eta ds dr.
\]

Using Lemma 4.2 to the above integral, we have

\[
(5.6) \quad \Theta_n(t, x) \leq C^m n! \int_{T_n^d} \prod_{i=1}^d \Theta_{i,n}(t, x_i, s_i) ds dr,
\]
where

\[ \Theta_{i,n}(t, x_i, s, r) = \int_{\mathbb{R}^{2n}} \left\{ \prod_{k=1}^{n} \varphi_{H_i}(\rho_k - \tau_k) \right\} \left| t - s_n \right|^\frac{\zeta_d}{d} |x_i - \rho_n|^\frac{\zeta_d}{d} p(t - s_n, x_i - \rho_n) \]

\[ \cdots \cdot |s_2 - s_1|^\frac{\zeta_d}{d} |r_2 - r_1|^\frac{\zeta_d}{d} \]

\[ \cdot |t - r_n|^\frac{\zeta_d}{d} |x_i - \tau_n|^\frac{\zeta_d}{d} p(t - r_n, x_i - \tau_n) \cdots |r_2 - r_1|^\frac{\zeta_d}{d} \]

\[ \cdot |\tau_2 - \tau_1|^\frac{\zeta_d}{d} p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1. \]

Here we use the notation \( \rho_k = \xi_{ki} \) and \( \tau_k = \eta_{ki}, k = 1, \ldots, n \). The quantity \( \Theta_{i,n} \) can be written as

\[ \Theta_{i,n}(t, x_i, s, r) = \left| t - s_n \right|^\frac{\zeta_d}{d} |t - r_n|^\frac{\zeta_d}{d} \cdots \left| s_2 - s_1 \right|^\frac{\zeta_d}{d} |r_2 - r_1|^\frac{\zeta_d}{d} \]

\[ \cdot \int_{\mathbb{R}^{2n}} \left\{ \prod_{k=1}^{n} \varphi_{H_i}(\rho_k - \tau_k) \right\} \left| x_i - \rho_n \right|^\frac{\zeta_d}{d} p(t - s_n, x_i - \rho_n) \]

\[ \cdot |x_i - \tau_n|^\frac{\zeta_d}{d} p(t - r_n, x_i - \tau_n) \cdots |\rho_2 - \rho_1|^\frac{\zeta_d}{d} p(s_2 - s_1, \rho_2 - \rho_1) \]

\[ \cdots |\tau_2 - \tau_1|^\frac{\zeta_d}{d} p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1. \]

(5.7)

From the definition (4.3) of \( \kappa_d \) we see easily \( \frac{\kappa_d}{d} > -1 \). We assume

\[ 2H_i + \frac{2\kappa_d}{d} > 0. \]

(5.8)

Under the above condition we can apply the Corollary 4.7 with \( \theta_1 = 2H_i - 2 > -1, \theta_2 = \frac{\kappa_d}{d} > -1 \) to the integration \( d\rho_1 d\tau_1 \) in the expression (5.7) (Condition (5.8) implies that \( \theta_1 + 2\theta_2 > -2 \)). Then, when \( \theta_1 + \theta_2 \neq -1 \), we have

\[ \Theta_{i,n}(t, x_i, s, r) \leq C \left| t - s_n \right|^\frac{\zeta_d}{d} |t - r_n|^\frac{\zeta_d}{d} \cdots |s_3 - s_2|^\frac{\zeta_d}{d} |r_3 - r_2|^\frac{\zeta_d}{d} \]

\[ \cdot \left| s_2 - s_1 \right|^\frac{\zeta_d}{d} + \frac{H_i d + \kappa_d}{2d} \alpha \right| r_2 - r_1|^\frac{\zeta_d}{d} \cdots |s_3 - s_2, \rho_3 - \rho_2 \]

\[ \cdot \int_{\mathbb{R}^{2n-2}} \left\{ \prod_{k=2}^{n} \varphi_{H_i}(\rho_k - \tau_k) \right\} \left| x_i - \rho_n \right|^\frac{\zeta_d}{d} p(t - s_n, x_i - \rho_n) \]

\[ \cdot |x_i - \tau_n|^\frac{\zeta_d}{d} p(t - r_n, x_i - \tau_n) \cdots |\rho_3 - \rho_2|^\frac{\zeta_d}{d} p(s_3 - s_2, \rho_3 - \rho_2) \]

\[ \cdots |\tau_3 - \tau_2|^\frac{\zeta_d}{d} p(r_3 - r_2, \tau_3 - \tau_2) d\rho_1 \cdots d\rho_2 d\tau_1 \cdots d\tau_2. \]

Repeatedly applying this argument, we obtain

\[ \Theta_{i,n}(t, x_i, s, r) \leq C^n \prod_{k=1}^{n} |t_{k+1} - t_k|^\ell_i |s_{k+1} - s_k|^\ell_i, \]

(5.9)

where we recall the convention that \( t_{n+1} = t \) and \( s_{n+1} = s \) and where

\[ \ell_i = \frac{\zeta_d}{d} + \frac{H_i d + \kappa_d}{2d} \alpha. \]
Substituting the above estimate of $\Theta_{i,n}$ into the expression for $\Theta_n$, we have

$$
\Theta_n(t, x) \leq C^n \int_{T_n} \prod_{k=1}^{n} (s_{k+1} - s_k)^{\ell} (r_{k+1} - r_k)^{\ell} dsdr
$$

where

$$
\ell = \sum_{i=1}^{d} \ell_i = \zeta_d + \frac{|H|\alpha}{2} + \frac{\kappa_d\alpha}{2} \quad \text{with} \quad |H| = \sum_{i=1}^{d} H_i.
$$

Now, we apply Lemma 4.8 to obtain

$$
\Theta_n(t, x) \leq C^n \left[ \frac{\Gamma(\ell + 1)}{\Gamma(n(\ell + 1))} \right]^2 \leq \frac{C^n}{\Gamma(2n(\ell + 1))}.
$$

This estimate combined with (5.4) proves that if

(5.10) $2(\ell + 1) > 1$,

then $\sum_{n=0}^{\infty} \Theta_n(t, x)$ is bounded which implies that the series (5.1) is convergent in $L^2(\Omega, F, P)$.

From the explicit expressions of $\zeta_d$ and $\kappa_d$ we see by analyzing the condition (5.10) for the cases $d = 1, d = 2, d = 4$ and $d = 3$ or $d \geq 5$ separately. We see that the condition (5.10) is equivalent to

$$
\sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha}.
$$

When $\theta_1 + \theta_2 = -1$, Corollary 4.7 implies that for any $\varepsilon > 0$,

$$
\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^\theta_1 |\rho_2 - \rho_1|^\theta_2 |\tau_2 - \tau_1|^\theta_2 p(s_2 - s_1, \rho_2 - \rho_1)p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \leq C(s_2 - s_1)^{\alpha(\theta_2+1+\varepsilon)} (r_2 - r_1)^{\alpha(\theta_2+1+\varepsilon)}.
$$

Now we can follow the above same argument to obtain that if

(5.12) $2(\ell + 1) > 1$,

where $\ell = \frac{d\varepsilon + \kappa_d + d}{4} \alpha$, then $\Theta_n(t, x)$ is bounded. In the same way as in the case $\theta_1 + \theta_2 \neq -1$, we can show that the condition (5.11) implies (5.12).

Now we consider the condition (5.8). From the definition (4.3) of $\kappa_d$, we see that when $d = 1, 2, 3, 4, H_i > 1/2$ implies (5.8). When $d \geq 5$, then the condition (5.8) is implied by the following

$$
H_i > 1 - \frac{2}{d} - \frac{\gamma}{2d}
$$

by choosing $\gamma_0$ sufficiently small. Theorem 2 is then proved. $\square.$
REFERENCES

1. Bronstein, I.; Israel, Y.; Kepten, E.; Mai, S.; Shavta, Y.; Barkai, E. and Garini, Y. Transient anomalous diffusion of telomeres in the nucleus of mammalian cells. *Physical Review Letters* **2009**, *Volume* 103, 018102, p1-p4.

2. Eidelman, S. D.; Kochubei, A. N. Cauchy problem for fractional diffusion equations. *Journal of Differential Equations* **2004**, *Volume* 199, 211-255.

3. Gradshteyn, I.S. and Ryzhik, I.M. *Table of integrals, Series, and Products*, seventh edition. Academic Press, 2007.

4. Hellmann, M.; Heermann, D. W. and Weiss, M. Enhancing phosphorylation cascades by anomalous diffusion, EPL, **2012**, *Volume* 97, (58004) p1-p5.

5. Hu, Y. Heat equations with fractional white noise potentials. *Applied Mathematics and Optimization* **2001**, *Volume* 43, 221-243.

6. Kochubei, A. N. Fractional-order diffusion. *Differential Equations* **1990**, *Volume* 26, 485-492.

7. Schneider, W. R. Fractional diffusion and wave equations. *J. Math. Phys.* **1989**, *Volume* 30, 134-144.

8. Kilbas, A. A.; Saigo, M. *H*-transforms. *Theory and applications*. Analytical Methods and Special Functions, 9. Chapman & Hall/CRC, Boca Raton, FL, 2004.

9. Samko, S. G.; Kilbas, A. A. and Marichev, O. I. *Fractional integrals and derivatives. Theory and applications*. Gordon and Breach Science Publishers, Yverdon, 1993.

10. Soula, H.; Carre, B.; Beslon, G. and Berry, H. Anomalous versus slowed-Down Brownian Diffusion in the Ligand-Binding Equilibrium, *Biophysical Journal* **2013**, *Volume* 105, 2064-2073.

11. Yuste, S. B.; Abad, E. D and Lindenberg, K. Reaction-subdiffusion model of morphogen gradient formation, *Physical Review E*, **2010**, *Volume* 82, 061123, 1-9.

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