Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations.

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Abstract

Within the framework of the theory of the column and row determinants, we obtain explicit representation formulas (analogs of Cramer's rule) for the minimum norm least squares solutions of quaternion matrix equations $AX = B$, $XA = B$ and $AXB = D$.

Keywords: Matrix equation, Least squares solution, Moore-Penrose generalized inverse, Quaternion matrix, Cramer rule, Column determinant, Row determinant.

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1 Introduction

In recent years quaternion matrix equations have been investigated by many authors (see, e.g., [1]-[16]). For example, Jiang, Liu, and Wei [1] studied the solutions of the general quaternion matrix equation $AXB - CYD = E$, and Liu [3] studied the least squares Hermitian solution of the quaternion matrix equation $(A^HXA, B^HXB) = (CD)$, Wang, Chang, and Ning [10] derived the common solution to six quaternion matrix equations, etc.

However the understanding of the problem for determinantal representing the least squares solutions of the quaternion matrix equations, in particular

$$AX = B,$$
\[ \begin{align*}
XA &= B, \\
AXB &= D,
\end{align*} \tag{2} \tag{3} \]

has not yet reached a satisfactory level. The reason was the lack of appropriate noncommutative determinant. Many authors had tried to give the definitions of the determinants of a quaternion matrix, (see, e.g. [17]-[23]). Unfortunately, by their definitions it is impossible for us to give a determinant representation of an inverse matrix.

But in [24], we defined the row and column determinants and the double determinant of a square matrix over the quaternion skew field. As applications we obtained the determinantal representations of an inverse matrix by an analogue of the adjoint matrix in [24] and the Moore-Penrose inverse over quaternion skew field \( \mathbb{H} \) in [25]. In [16] we also gave Cramer’s rule for the solution of nonsingular quaternion matrix equations (1), (2) and (3) within the framework of the theory of the column and row determinants. In [13][14], the authors obtained the generalized Cramer rules for the unique solution of the matrix equation (3) in some restricted conditions within the framework of the theory of the column and row determinants as well.

In this paper we aim to obtain explicit representation formulas (analogs of Cramer’s rule) for the minimum norm least squares solutions of quaternion matrix equations (1), (2) and (3) without any restriction. The paper is organized as follows. In Section 2, we start with some basic concepts and results from the theory of the column and row determinants which are necessary for the following. The theory of the column and row determinants of a quaternionic matrix is considered completely in [24]. In Section 3, we give the theorem about determinantal representations of the Moore-Penrose inverse over the quaternion skew field derived in [25]. In Section 4, we obtain explicit representation formulas for the minimum norm least squares solutions of quaternion matrix equations (1), (2) and (3). In Section 5, we show a numerical example to illustrate the main result.

2 Elements of the theory of the column and row determinants.

Throughout the paper, we denote the real number field by \( \mathbb{R} \), the set of all \( m \times n \) matrices over the quaternion algebra

\[ \mathbb{H} = \{ a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, \ a_0, a_1, a_2, a_3 \in \mathbb{R} \} \]
by $\mathbb{H}^{m \times n}$ and its subset of matrices of rank $r$ by $\mathbb{H}^{r \times n}$. Let $M(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices.

The conjugate of a quaternion $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ is defined by $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$. The Hermitian adjoint matrix of $A = (a_{ij}) \in \mathbb{H}^{n \times m}$ is called the matrix $A^* = (a^*_{ij})_{m \times n}$ if $a^*_{ij} = \overline{a_{ji}}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The matrix $A = (a_{ij}) \in \mathbb{H}^{n \times m}$ is Hermitian if $A^* = A$.

Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

**Definition 2.1** The $i$th row determinant of $A = (a_{ij}) \in M(n, \mathbb{H})$ is defined by

$$\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i_1 k_1} a_{i_1 k_1+1} \cdots a_{i_{k_1+t_1}} \cdots a_{i_{k_r} i_{k_r+1}} \cdots a_{i_{k_r+l_r} i_{k_r}}$$

for all $i = 1, \ldots, n$. The elements of the permutation $\sigma$ are indices of each monomial. The left-ordered cycle notation of the permutation $\sigma$ is written as follows,

$$\sigma = (i_{k_1} i_{k_1+1} \cdots i_{k_1+t_1}) (i_{k_2} i_{k_2+1} \cdots i_{k_2+t_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r}).$$

The index $i$ opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_2} < i_{k_3} < \ldots < i_{k_t}$ and $i_{k_t} < i_{k_{t+s}}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

**Definition 2.2** The $j$th column determinant of $A = (a_{ij}) \in M(n, \mathbb{H})$ is defined by

$$\text{cdet}_j A = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+1}} \cdots a_{j_{k_r+t_r}} \cdots a_{j i_{k_1+t_1}} \cdots a_{j i_{k_1+l_1} i_{k_1}}$$

for all $j = 1, \ldots, n$. The right-ordered cycle notation of the permutation $\tau \in S_n$ is written as follows,

$$\tau = (j_{k_r+t_r} \cdots j_{k_r+1}) \cdots (j_{k_2+t_2} \cdots j_{k_2+1}) j_{k_1+t_1} \cdots j_{k_1+1} i_{k_1}.$$

The index $j$ opens the first cycle from the right and other cycles satisfy the following conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_t}$ and $j_{k_t} < j_{k_{t+s}}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

The following lemmas enable us to expand $\text{rdet}_i A$ by cofactors along the $i$th row and $\text{cdet}_j A$ along the $j$th column respectively for all $i, j = 1, \ldots, n$. 

3
Lemma 2.1 \[24\] Let $R_{ij}$ be the right $ij$-th cofactor of $A \in M(n, \mathbb{H})$, that is, $r\text{det}_i A = \sum_{j=1}^{n} a_{ij} \cdot R_{ij}$ for all $i = 1, \ldots, n$. Then

$$R_{ij} = \begin{cases} -r\text{det}_j A_{ij}^j(a_{i.}), & i \neq j, \\ r\text{det}_k A_{ii}^i, & i = j, \end{cases}$$

where $A_{ij}^j(a_{i.})$ is obtained from $A$ by replacing the $j$th column with the $i$th column, and then by deleting both the $i$th row and column, $k = \min \{I_n \setminus \{i\}\}$.

Lemma 2.2 \[24\] Let $L_{ij}$ be the left $ij$-th cofactor of $A \in M(n, \mathbb{H})$, that is, $c\text{det}_j A = \sum_{i=1}^{n} L_{ij} \cdot a_{ij}$ for all $j = 1, \ldots, n$. Then

$$L_{ij} = \begin{cases} -c\text{det}_i A_{ij}^{ij}(a_{j.}), & i \neq j, \\ c\text{det}_k A_{jj}^j, & i = j, \end{cases}$$

where $A_{ij}^{ij}(a_{j.})$ is obtained from $A$ by replacing the $i$th row with the $j$th row, and then by deleting both the $j$th row and column, $k = \min \{J_n \setminus \{j\}\}$.

Since these matrix functionals do not satisfy the axioms of noncommutative determinant, then they are called "determinants" shareware. But by the following theorems we introduce the concepts of a determinant of a Hermitian matrix and a double determinant which both satisfy the axioms of noncommutative determinant and can be expanded by cofactors along an arbitrary row or column using the row-column determinants. This enables us to obtain determinantal representations of the inverse matrix.

Theorem 2.1 \[24\] If $A = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $r\text{det}_1 A = \cdots = r\text{det}_n A = c\text{det}_1 A = \cdots = c\text{det}_n A \in \mathbb{R}$.

Remark 2.1 Since Theorem 2.1 we can define the determinant of a Hermitian matrix $A \in M(n, \mathbb{H})$ putting for all $i = 1, \ldots, n$

$$\text{det} A := r\text{det}_i A = c\text{det}_i A.$$
then there exist a unique right inverse matrix \((RA)^{-1}\) and a unique left inverse matrix \((LA)^{-1}\) of \(A\), where \((RA)^{-1} = (LA)^{-1} = A^{-1}\), and they possess the following determinantal representations

\[
(RA)^{-1} = \frac{1}{\det A} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix},
\]

(4)

\[
(LA)^{-1} = \frac{1}{\det A} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix},
\]

(5)

where \(R_{ij}, L_{ij}\) are right and left \(ij\)th cofactors of \(A\) respectively for all \(i, j = 1, ..., n\).

**Theorem 2.3** [24] If \(A \in M(n, \mathbb{H})\), then \(\det A A^* = \det A^* A\).

According to Theorem 2.3 we introduce the concept of a double determinant.

**Definition 2.3** The determinant of the corresponding Hermitian matrices, \((A^* A\) or \(AA^*)\), of \(A \in M(n, \mathbb{H})\) is called its double determinant, i.e. \(\text{ddet} A := \det (A^* A) = \det (AA^*)\).

Suppose \(A^i_j\) denotes the submatrix of \(A\) obtained by deleting both the \(i\)th row and the \(j\)th column. Let \(a^i_j\) be the \(j\)th column and \(a^i\) be the \(i\)th row of \(A\). Denote by \(a^*_j\) and \(a^*_i\) the \(j\)th column and the \(i\)th row of a Hermitian adjoint matrix \(A^*\) as well. Suppose \(A^i_j(b)\) denotes the matrix obtained from \(A\) by replacing its \(j\)th column with the column \(b\), and \(A^i_i(b)\) denotes the matrix obtained from \(A\) by replacing its \(i\)th row with the row \(b\).

We have the following theorem on the determinantal representation of the inverse matrix over \(\mathbb{H}\).

**Theorem 2.4** [24] The necessary and sufficient condition of invertibility of \(A \in M(n, \mathbb{H})\) is \(\text{ddet} A \neq 0\). Then there exists \(A^{-1} = (LA)^{-1} = (RA)^{-1}\), where

\[
(LA)^{-1} = (A^* A)^{-1} A^* = \frac{1}{\text{ddet} A} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix},
\]

(6)
\[(RA)^{-1} = A^* (AA^*)^{-1} = \frac{1}{\text{ddet} A^*} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix} \quad (7)\]

and
\[L_{ij} = \text{cdet}_j (A^* A)_{j} (a^*_{i,j}), \quad R_{ij} = \text{rdet}_i (A A^*)_{i} (a^*_{i,j}),\]
for all \(i, j = 1, \ldots, n.\)

Remark 2.2 Since by Theorem 2.4
\[\text{ddet} A = \text{cdet}_j (A^* A) = \sum_i L_{ij} \cdot a_{ij}, \quad \text{ddet} A = \text{rdet}_i (A A^*) = \sum_j a_{ij} \cdot R_{ij}\]
for all \(j = 1, \ldots, n,\) then \(L_{ij}\) is called the left double \(ij\)th cofactor and \(R_{ij}\) is called the right double \(ij\)th cofactor for the entry \(a_{ij}\) of \(A \in \mathbb{M}(n, \mathbb{H}).\)

3 Determinantal representation of the Moore-Penrose inverse.

For a quaternion matrix \(A \in \mathbb{H}^{m \times n},\) a generalized inverse of \(A\) is a quaternion matrix \(X\) with following Penrose conditions
\[1) \ (AA^+)^* = AA^+; \]
\[2) \ (A^+A)^* = A^+A; \]
\[3) \ AA^+A = A; \]
\[4) \ A^+AA^+ = A^+. \quad \text{ (8)}\]

Definition 3.1 For \(A \in \mathbb{H}^{m \times n},\) \(X \in \mathbb{H}^{n \times m}\) is said to be a \((i,j,...)\) generalized inverse of \(A\) if \(X\) satisfies Penrose conditions \((i), (j), \ldots\) in \((8)\). We denote the \(X\) by \(A^{(i,j,...)}\) and the set of all \(A^{(i,j,...)}\) by \(A\{i,j,...\}.\)

By [26] we know that for \(A \in \mathbb{H}^{m \times n},\) the \(A\{i,j,...\}\) exists and the \(A^{(1,2,3,4)}\) exists uniquely. The matrix \(A^{(1,2,3,4)}\) is called the Moore-Penrose inverse of \(A\) and denote \(A^+ := A^{(1,2,3,4)}\).

In [25] the determinantal representations of the Moore-Penrose inverse over the quaternion skew field was derived based on the limit representation.
Theorem 3.1 If \( A \in \mathbb{H}^{m \times n} \) and \( A^+ \) is its Moore-Penrose inverse, then
\[
A^+ = \lim_{\alpha \to 0} A^* (AA^* + \alpha I)^{-1} = \lim_{\alpha \to 0} (A^* A + \alpha I)^{-1} A^* , \quad \text{where} \quad \alpha \in \mathbb{R}_+.
\]

Corollary 3.1 If \( A \in \mathbb{H}^{m \times n} \), then the following statements are true.

i) If \( \text{rank} A = n \), then 
\[ A^+ = (A^* A)^{-1} A^* . \]

ii) If \( \text{rank} A = m \), then 
\[ A^+ = A^* (AA^*)^{-1} . \]

iii) If \( \text{rank} A = n = m \), then 
\[ A^+ = A^{-1} . \]

We shall use the following notations. Let \( \alpha := \{ \alpha_1, \ldots, \alpha_k \} \subseteq \{ 1, \ldots, m \} \) and \( \beta := \{ \beta_1, \ldots, \beta_k \} \subseteq \{ 1, \ldots, n \} \) be subsets of the order \( 1 \leq k \leq \min \{ m, n \} \). By \( A^\alpha_\beta \) denote the submatrix of \( A \) determined by the rows indexed by \( \alpha \) and the columns indexed by \( \beta \). Then \( A^\alpha_\alpha \) denotes the principal submatrix determined by the rows and columns indexed by \( \alpha \). If \( A \in M(n, \mathbb{H}) \) is Hermitian, then by \( |A^\alpha_\alpha| \) denote the corresponding principal minor of \( \det A \). For \( 1 \leq k \leq n \), denote by \( L_{k,n} := \{ \alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n \} \) the collection of strictly increasing sequences of \( k \) integers chosen from \( \{ 1, \ldots, n \} \).

For fixed \( i \in \alpha \) and \( j \in \beta \), let \( I_{r,m} \{ i \} := \{ \alpha : \alpha \in L_{r,m}, i \in \alpha \} \), \( J_{r,n} \{ j \} := \{ \beta : \beta \in L_{r,n}, j \in \beta \} \).

The following theorem and remarks introduce the determinantal representations of the Moore-Penrose inverse which we shall use below.

Theorem 3.2 \[ \text{[25]} \] If \( A \in \mathbb{H}^{m \times n} \), then the Moore-Penrose inverse \( A^+ = (a^+_{ij}) \in \mathbb{H}^{n \times m} \) possess the following determinantal representations:

\[
a^+_{ij} = \frac{\sum_{\beta \in J_{r,n} \{ i \}} \text{cdet}_i ( (A^* A), (a^*_j) )^\beta_{\beta} } {\sum_{\beta \in J_{r,n}} |(A^* A)|^\beta_{\beta} } , \quad (9)
\]

or

\[
a^+_{ij} = \frac{\sum_{\alpha \in I_{r,m} \{ j \}} \text{rdet}_j ( (AA^*), (a^*_i) )^\alpha_{\alpha} } {\sum_{\alpha \in I_{r,m}} |(AA^*)|^\alpha_{\alpha} } . \quad (10)
\]

for all \( i = 1, \ldots, n, j = 1, \ldots, m \).
Remark 3.1 If \( \text{rank } A = n \), then by Corollary 3.1 \( A^+ = (A^*A)^{-1} A^* \). Considering \( (A^*A)^{-1} \) as a left inverse, we get the following representation of \( A^+ \):

\[
A^+ = \frac{1}{\text{d} \text{det } A} \begin{pmatrix}
\text{cdet}_1(A^*A)_1(a^*_1) & \ldots & \text{cdet}_1(A^*A)_1(a^*_m) \\
\vdots & \ddots & \vdots \\
\text{cdet}_n(A^*A)_n(a^*_1) & \ldots & \text{cdet}_n(A^*A)_n(a^*_m)
\end{pmatrix}.
\] (11)

If \( m > n \), then by Theorem 3.2 for \( A^+ \) we have (9) as well.

Remark 3.2 If \( \text{rank } A = m \), then by Corollary 3.1 \( A^+ = A^* (AA^*)^{-1} \). Considering \( (AA^*)^{-1} \) as a right inverse, we get the following representation of \( A^+ \):

\[
A^+ = \frac{1}{\text{d} \text{det } A} \begin{pmatrix}
\text{rdet}_1(AA^*)_1(a^*_1) & \ldots & \text{rdet}_m(AA^*)_m(a^*_1) \\
\vdots & \ddots & \vdots \\
\text{rdet}_1(AA^*)_1(a^*_n) & \ldots & \text{rdet}_m(AA^*)_m(a^*_n)
\end{pmatrix}.
\] (12)

If \( m < n \), then by Theorem 3.2 for \( A^+ \) we also have (10).

4 Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations

Denote by \( \|A\| \) the Frobenius norm of the quaternion matrix \( A \in \mathbb{H}^{m \times n} \).

Definition 4.1 Consider a matrix equation

\[
AX = B,
\] (13)

where \( A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times s} \) are given, \( X \in \mathbb{H}^{n \times s} \) is unknown. Suppose

\[
H_R = \{ X | X \in \mathbb{H}^{n \times s}, \|AX - B\| = \text{min} \},
\]

Then matrices \( X \in \mathbb{H}^{n \times s} \) such that \( X \in H_R \) are called least squares solutions of the matrix equation (13). If \( X_{LS} = \min_{X \in H_R} \| X \| \), then \( X_{LS} \) is called the minimum norm least squares solution of (13).

If the equation (13) has no precision solutions, then \( X_{LS} \) is its optimal approximation.

The following important theorem is well-known.
Theorem 4.1 (2) The least squares solutions of (13) are

\[ X = A^+ B + (I_n - A^+ A)C, \]

where \( C \in \mathbb{H}^{n \times s} \) is an arbitrary quaternion matrix and the minimum norm least squares solution is \( X_{LS} = A^+ B \).

We denote \( A^* B =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{H}^{n \times s} \).

Theorem 4.2 (i) If \( \text{rank } A = r \leq m < n \), then for the minimum norm least square least squares solution \( X_{LS} = (x_{ij}) \in \mathbb{H}^{n \times s} \) of (13) for all \( i = 1, \ldots, n, \ j = 1, \ldots, s \), we have

\[
x_{ij} = \frac{\sum_{\beta \in J_{r, n}(i)} \text{cdet}_i \left( (A^* A)_{i} \left( \hat{b}_{j} \right) \right) \beta}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|}. \tag{14}
\]

(ii) If \( \text{rank } A = n \), then for \( X_{LS} = (x_{ij}) \in \mathbb{H}^{n \times s} \) of (13) for all \( i = 1, \ldots, n, \ j = 1, \ldots, s \), we have

\[
x_{ij} = \frac{\text{cdet}_i (A^* A)_{i} \left( \hat{b}_{j} \right)}{d\text{det} A} \tag{15}
\]

where \( \hat{b}_{j} \) is the \( j \)th column of \( \hat{B} \) for all \( j = 1, \ldots, s \).

Proof. (i) If \( \text{rank } A = r \leq m < n \), then by Theorem 3.2 we can represent the matrix \( A^+ \) by (9). Therefore, we obtain for all \( i = 1, \ldots, n, \ j = 1, \ldots, s \)

\[
x_{ij} = \sum_{k=1}^{m} a_{ik}^+ b_{kj} = \sum_{k=1}^{m} \frac{\text{cdet}_i \left( (A^* A)_{i} (a^*_k) \right) \beta}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|} \cdot b_{kj} = \sum_{\beta \in J_{r, n}(i)} \frac{\text{cdet}_i \left( (A^* A)_{i} (a^*_k) \right) \beta}{\sum_{\beta \in J_{r, n}} \left| (A^* A)_{\beta} \right|} \cdot b_{kj}
\]
Since $\sum_k a^*_k b_{kj} = \begin{bmatrix} \sum_k a^*_k b_{kj} \\ \sum_k a^*_k b_{kj} \\ \vdots \\ \sum_k a^*_k b_{kj} \end{bmatrix} = \hat{b}_j$ and denoting the $j$th column of $\hat{B}$ by $\hat{b}_j$ as well, then it follows (14).

ii) If $\text{rank } A = n$, then by Corollary 3.1 $A^+ = (A^* A)^{-1} A^*$. Representing $(A^* A)^{-1}$ by $(\hat{B})$, we obtain for all $i = 1, \ldots, n, j = 1, \ldots, s$

$$x_{ij} = \frac{1}{\text{det} A} \sum_{k=1}^n L_{ki} \hat{b}_{kj},$$

where $L_{ij}$ is a left $ij$th cofactor of $(A^* A)$ for all $i, j = 1, \ldots, n$. From this by Lemma 2.2 and denoting the $j$th column of $\hat{B}$ by $\hat{b}_j$, it follows (15).

Corollary 4.1 (Theorem 3.1 in [10]) Suppose

$$AX = B$$

is a right matrix equation, where $\{A, B\} \in M(n, H)$ are given, $X \in M(n, H)$ is unknown. If $\text{det} A \neq 0$, then (16) has a unique solution, and the solution is

$$x_{ij} = \frac{\text{cdet}_i (A^* A)_{ij} (\hat{b}_j)}{\text{det} A}$$

where $\hat{b}_j$ is the $j$th column of $\hat{B}$ for all $i, j = 1, \ldots, n$.

Definition 4.2 Consider a matrix equation

$$XA = B,$$

where $A \in H_{m \times n}, B \in H_{s \times n}$ are given, $X \in H_{s \times m}$ is unknown. Suppose

$$H_L = \{X | X \in H_{s \times m}, ||XA - B|| = \text{min} \}.$$

Then matrices $X \in H_{s \times m}$ such that $X \in H_L$ are called least squares solutions of the matrix equation (18). If $X_{LS} = \min_{X \in H_L} ||X||$, then $X_{LS}$ is called the minimum norm least squares solution of (18).

The following theorem can be obtained by analogy to Theorem 4.1.
Theorem 4.3 The least squares solutions of (18) are

\[ \mathbf{X} = \mathbf{B} \mathbf{A}^+ + (\mathbf{I}_m - \mathbf{A} \mathbf{A}^+) \mathbf{C}, \]

where \( \mathbf{C} \in \mathbb{H}^{n \times s} \) is an arbitrary quaternion matrix and the minimum norm least squares solution is \( \mathbf{X}_{LS} = \mathbf{B} \mathbf{A}^+ \).

We denote \( \mathbf{B} \mathbf{A}^* =: \tilde{\mathbf{B}} = (\tilde{b}_{ij}) \in \mathbb{H}^{s \times m} \).

Theorem 4.4 (i) If \( \text{rank} \mathbf{A} = r \leq n < m \), then for the minimum norm least squares solution \( \mathbf{X}_{LS} = (x_{ij}) \in \mathbb{H}^{s \times m} \) of (18) for all \( i = 1, \ldots, s \), \( j = 1, \ldots, m \), we have

\[ x_{ij} = \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*_{\alpha})_{j} \cdot (\tilde{b}_{i \alpha}))}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*_{\alpha})_{\alpha}|}. \tag{19} \]

(ii) If \( \text{rank} \mathbf{A} = m \), then for \( \mathbf{X}_{LS} = (x_{ij}) \in \mathbb{H}^{s \times m} \) of (18) for all \( i = 1, \ldots, s \), \( j = 1, \ldots, m \), we have

\[ x_{ij} = \frac{\text{rdet}_{j}(\mathbf{A}\mathbf{A}^*_{j}) (\tilde{b}_{i \cdot})}{\text{ddet}_{\mathbf{A}}}. \tag{20} \]

where \( \tilde{b}_{i \cdot} \) is the \( i \)th row of \( \tilde{\mathbf{B}} \) for all \( i = 1, \ldots, s \).

Proof. (i) If \( \text{rank} \mathbf{A} = r \leq n < m \), then by Theorem 3.2 we can represent the matrix \( \mathbf{A}^+ \) by (10). Therefore, for all \( i = 1, \ldots, s \), \( j = 1, \ldots, m \), we get

\[ x_{ij} = \sum_{k=1}^{m} b_{ik} a^+_{kj} = \sum_{k=1}^{m} b_{ik} \cdot \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*_{\alpha})_{j} \cdot (a^*_{k \alpha}))_{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*_{\alpha})_{\alpha}|} = \]

\[ \frac{\sum_{k=1}^{m} b_{ik} \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j((\mathbf{A}\mathbf{A}^*_{\alpha})_{j} \cdot (a^*_{k \alpha}))_{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*_{\alpha})_{\alpha}|} \]

Since \( \sum_{k} b_{ik} a^*_{k} = (\sum_{k} b_{ik} a^*_{k1} \sum_{k} b_{ik} a^*_{k2} \cdots \sum_{k} b_{ik} a^*_{kn}) = \tilde{b}_{i \cdot} \) and denoting the \( i \)th row of \( \tilde{\mathbf{B}} \) by \( \tilde{b}_{i \cdot} \) as well, then it follows (19).
(ii) If rank $A = m$, then by Corollary 3.1 $A^+ = A^*(AA^*)^{-1}$. Representing $(AA^*)^{-1}$ by (4), we obtain for all $i = 1, ..., s, j = 1, ..., m$,

$$x_{ij} = \frac{1}{d\text{det}\, A} \sum_{k=1}^{n} \tilde{b}_{ik} R_{jk},$$

where $R_{ij}$ is a left $ij$th cofactor of $(AA^*)$ for all $i, j = 1, ..., m$. From this by Lemma 2.1 and denoting the $i$th row of $\tilde{B}$ by $\tilde{b}_i$, it follows (20).

**Corollary 4.2** *(Theorem 3.2 in [10])* Suppose

$$XA = B$$

is a left matrix equation, where $\{A, B\} \in \text{M}(n, \mathbb{H})$ are given, $X \in \text{M}(n, \mathbb{H})$ is unknown. If $d\text{det}\, A \neq 0$, then (21) has a unique solution, and the solution is

$$x_{ij} = \frac{r\text{det}_j(AA^*)_j, (\tilde{b}_i)}{d\text{det}\, A}$$

where $\tilde{b}_i$ is the $i$th column of $\tilde{B}$ for all $i, j = 1, ..., n$.

**Definition 4.3** Consider a matrix equation

$$AXB = D,$$

where $A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{p \times q}, D \in \mathbb{H}^{m \times q}$ are given, $X \in \mathbb{H}^{n \times p}$ is unknown. Suppose

$$H_D = \{X \mid X \in \mathbb{H}^{s \times m}, \|AXB - D\| = \text{min}\}.$$ 

Then matrices $X \in \mathbb{H}^{s \times m}$ such that $X \in H_D$ are called least squares solutions of the matrix equation (23). If $X_{LS} = \text{min}_{X \in H_D} \|X\|$, then $X_{LS}$ is called the minimum norm least squares solution of (23).

The following important theorem is well-known.

**Theorem 4.5** *(15)* The least squares solutions of (23) are

$$X = A^+DB^+ + (I_n - A^+A)V + W(I_p - BB^+),$$

where $\{V, W\} \subset \mathbb{H}^{n \times p}$ are arbitrary quaternion matrices and the least squares solution with minimum norm is $X_{LS} = A^+DB^+$. 

12
We denote $\tilde{D} = A^*DB^*$. 

**Theorem 4.6**  
(i) If $\text{rank } A = r_1 < m$ and $\text{rank } B = r_2 < p$, then for the minimum norm least square solution $X_{LS} = (x_{ij}) \in H^{n \times p}$ of (23) we have

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n} \{i\}} \cdet_i ((A^*A)_i \tilde{(d^B)}_j)^\beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_\beta \right| \sum_{\alpha \in I_{r_2,p}} \left| (BB^*)_\alpha \right|},$$

or

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,p}} \left| (A^*A)_\beta \right| \sum_{\alpha \in I_{r_2,q}} \left| (BB^*)_\alpha \right|}{\sum_{\beta \in J_{r_1,n} \{i\}} \left| (A^*A)_\beta \right| \sum_{\alpha \in I_{r_2,q}} \left| (BB^*)_\alpha \right|} \sum_{\beta \in J_{r_1,n} \{i\}} \cdet_i ((A^*A)_i \tilde{(d^B)}_j)^\beta,$$

where

$$d^B_j = \left( \sum_{\alpha \in I_{r_2,p} \{j\}} \text{rdet}_j ((BB^*)_j \tilde{(d^A)}_i)^\alpha \right) \in H^{n \times 1}, \quad k = 1, \ldots, n$$

and

$$d^A_i = \left( \sum_{\beta \in J_{r_1,n} \{i\}} \cdet_i ((A^*A)_i \tilde{(d^B)}_j)^\beta \right) \in H^{1 \times p}, \quad l = 1, \ldots, p$$

are the column vector and the row vector, respectively. $\tilde{d}_i$ and $\tilde{d}_j$ are the $i$th row and the $j$th column of $\tilde{D}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, p$.

(ii) If $\text{rank } A = n$ and $\text{rank } B = p$, then for $X_{LS} = (x_{ij}) \in H^{n \times p}$ of (23) we have

$$x_{ij} = \frac{\cdet_i(A^*A)_i (d^B)}{d^A} \frac{d^B}{d^A},$$

or

$$x_{ij} = \frac{\text{rdet}_j (BB^*)_j (d^A)}{d^A} \frac{d^B}{d^A},$$

where

$$d^B_j := \left( \text{rdet}_j (BB^*)_j (\tilde{d}_1), \ldots, \text{rdet}_j (BB^*)_j (\tilde{d}_n) \right)^T$$

and

$$d^A_i := \left( \cdet_i(A^*A)_i (\tilde{d}_1), \ldots, \cdet_i(A^*A)_i (\tilde{d}_n) \right)$$
are respectively the column-vector and the row-vector. $\tilde{d}_i$ is the $i$th row of $\tilde{D}$ for all $i = 1, \ldots, n$, and $\tilde{d}_{j\cdot}$ is the $j$th column of $\tilde{D}$ for all $j = 1, \ldots, p$.

(iii) If $\text{rank} \ A = n$ and $\text{rank} \ B = r_2 < p$, then for $X_{LS} = (x_{ij}) \in \mathbb{H}^{n \times p}$ of (23) we have

$$x_{ij} = \frac{\text{cdet}_i \left( (A^*A)_{i\cdot} \left( \tilde{d}^B_{\cdot j} \right) \right)}{\text{ddet} A \sum_{\alpha \in I_{r_2,p}} \left| (B^*B)_{\alpha\alpha} \right|},$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2,p}} \text{rdet}_j \left( (B^*B)_{j\cdot} \left( d^A_{\cdot i} \right) \right)_\alpha}{\text{ddet} A \sum_{\alpha \in I_{r_2,p}} \left| (B^*B)_{\alpha\alpha} \right|},$$

where $d^B_{\cdot j}$ is (26) and $d^A_{\cdot i}$ is (27).

(iii) If $\text{rank} \ A = r_1 < n$ and $\text{rank} \ B = p$, then for $X_{LS} = (x_{ij}) \in \mathbb{H}^{n \times p}$ of (23) we have

$$x_{ij} = \frac{\text{rdet}_j \left( (B^*B)_{j\cdot} \left( d^A_{\cdot i} \right) \right)}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{\beta\beta} \right| \cdot \text{ddet} B},$$

or

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n}} \text{cdet}_i \left( (A^*A)_{i\cdot} \left( d^B_{\cdot j} \right) \right)_\beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{\beta\beta} \right| \cdot \text{ddet} B},$$

where $d^B_{\cdot j}$ is (30) and $d^A_{\cdot i}$ is (27).

Proof. (i) If $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{p \times q}$ and $r_1 < m$, $r_2 < p$, then by Theorem 3.2 the Moore-Penrose inverses $A^+ = (a^+_{ij}) \in \mathbb{H}^{n \times m}$ and $B^+ = (a^+_{ij}) \in \mathbb{H}^{q \times p}$ posses the following determinantal representations respectively,

$$a^+_{ij} = \frac{\sum_{\beta \in J_{r_1,n}} \text{cdet}_i \left( (A^*A)_{i\cdot} (a^*_{j\cdot}) \right)_{\beta\beta}}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{\beta\beta} \right|},$$
\[ b_{ij}^+ = \frac{\sum_{\alpha \in I_{2,p}(j)} \text{rdet}_j ((BB^*)_j.(b_i^*)) \alpha}{\sum_{\alpha \in I_{2,p}} |(BB^*)_\alpha^|}. \]  \tag{36}

By Theorem 4.5, \( X_{LS} = A^+DB^+ \) and entries of \( X_{LS} = (x_{ij}) \) are

\[ x_{ij} = \sum_{s=1}^q \left( \sum_{k=1}^m a_{ik}^+ d_{ks} \right) b_{sj}^+. \]  \tag{37}

for all \( i = 1, \ldots, n, j = 1, \ldots, p \).

Denote by \( \hat{d}_s \) the \( s \)th column of \( A^*D =: \hat{D} = (\hat{d}_{ij}) \in \mathbb{H}^{n \times q} \) for all \( s = 1, \ldots, q \). It follows from \( \sum_k a_{ik}^+ d_{ks} = d_{is} \) that

\[
\sum_{k=1}^m a_{ik}^+ d_{ks} = \sum_{k=1}^m \frac{\sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i ((A^*A)_{-i}.(a_{ik}^*)) \beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{-\beta}^\beta \right|} \cdot d_{ks} = \sum_{\beta \in J_{r_1,n}(i)} \frac{\text{cdet}_i ((A^*A)_{-i}.(\hat{d}_s)) \beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{-\beta}^\beta \right|} \cdot d_{ks} = \sum_{\beta \in J_{r_1,n}(i)} \frac{\text{cdet}_i ((A^*A)_{-i}.(\hat{d}_s)) \beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{-\beta}^\beta \right|} \cdot d_{ks} \tag{38}
\]

Suppose \( e_s \) and \( e_s^* \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)th components, which are 1. Substituting (38) and (36) in (37), we obtain

\[
x_{ij} = \sum_{s=1}^q \frac{\sum_{\beta \in J_{r_1,n}(i)} \text{cdet}_i ((A^*A)_{-i}.(\hat{d}_s)) \beta}{\sum_{\beta \in J_{r_1,n}} \left| (A^*A)_{-\beta}^\beta \right|} \cdot \frac{\sum_{\alpha \in I_{2,p}(j)} \text{rdet}_j ((BB^*)_j.(b_i^*)) \alpha}{\sum_{\alpha \in I_{2,p}} |(BB^*)_\alpha^|}.
\]

Since

\[
d_{is} = \sum_{l=1}^n e_s^* \hat{d}_{is}, \quad b^*_s = \sum_{t=1}^p b_{st}^* e_t, \quad \sum_{s=1}^q \hat{d}_{is} b^*_s = \tilde{d}_{it}, \tag{39}
\]

then we have

\[ x_{ij} = \]
Substituting it in (40), we obtain

\[
\sum_{s=1}^{q} \sum_{t=1}^{p} \sum_{n=1}^{n} \text{cdet}_i ((A^*A)_{,i}(e_i)) \beta_d t b^*_s t \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(e_t)) \frac{\alpha}{\alpha} = \sum_{\beta \in J_{r_1,n}} \left( (A^*A) \beta \right) \sum_{\alpha \in I_{2,p}} \left| (BB^*) \frac{\alpha}{\alpha} \right|
\]

\[
\sum_{t=1}^{p} \sum_{n=1}^{n} \sum_{l=1}^{l} \text{cdet}_i ((A^*A)_{,i}(e_i)) \beta_d t \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(e_t)) \frac{\alpha}{\alpha} = \sum_{\beta \in J_{r_1,n}} \left( (A^*A) \beta \right) \sum_{\alpha \in I_{2,p}} \left| (BB^*) \frac{\alpha}{\alpha} \right|
\]

(40)

Denote by

\[
d^A_{it} := \sum_{\beta \in J_{r_1,n}} \text{cdet}_i ((A^*A)_{,i}(\tilde{d}_t i)) \beta_d t = \sum_{t=1}^{n} \sum_{l=1}^{l} \text{cdet}_i ((A^*A)_{,i}(e_i)) \beta_d t \]

the tth component of a row-vector \(d^A_{t} = (d^A_{t1}, ..., d^A_{tp})\) for all \(t = 1, ..., p\). Substituting it in (40), we have

\[
x_{ij} = \frac{\sum_{t=1}^{p} d^A_{it} \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(e_t)) \frac{\alpha}{\alpha}}{\sum_{\beta \in J_{r_1,n}} \left( (A^*A) \beta \right) \sum_{\alpha \in I_{2,p}} \left| (BB^*) \frac{\alpha}{\alpha} \right|}
\]

Since \(\sum_{t=1}^{p} d^A_{it} e_t = d^A_{i} \), then it follows (25).

If we denote by

\[
d^B_{lj} := \sum_{t=1}^{p} \tilde{d}_t \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(\tilde{d}_t)) \frac{\alpha}{\alpha} = \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(\tilde{d}_t)) \frac{\alpha}{\alpha}
\]

(41)

the lth component of a column-vector \(d^B_{l} = (d^B_{l1}, ..., d^B_{ln})^T\) for all \(l = 1, ..., n\) and substitute it in (40), we obtain

\[
x_{ij} = \frac{\sum_{l=1}^{n} \sum_{\beta \in J_{r_1,n}} \text{cdet}_i ((A^*A)_{,i}(e_i)) \beta_d t b^*_s t \sum_{\alpha \in I_{2,p}} \text{rdet}_j ((BB^*)_{j,}(d^B_{lj})) \frac{\alpha}{\alpha}}{\sum_{\beta \in J_{r_1,n}} \left( (A^*A) \beta \right) \sum_{\alpha \in I_{2,p}} \left| (BB^*) \frac{\alpha}{\alpha} \right|}
\]
Since $\sum_{l=1}^{n} e_{lj} d_{lj}^B = d_{lj}^B$, then it follows (24).

(ii) If rank $A = n$ and rank $B = p$, then by Corollary 3.1 $A^+ = (A^*A)^{-1} A^*$ and $B^+ = (BB^*)^{-1}$. If we represent $(A^*A)^{-1}$ as the left inverse by (24) and $(BB^*)^{-1}$ as the right inverse by (24), then we obtain

$$X_{LS} = (A^*A)^{-1} A^* DB^+ (BB^*)^{-1} =$$

$$= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\
x_{21} & x_{22} & \cdots & x_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{np} \\
\end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} L_{A11}^A & L_{A21}^A & \cdots & L_{An1}^A \\
L_{A12}^A & L_{A22}^A & \cdots & L_{An2}^A \\
\vdots & \vdots & \ddots & \vdots \\
L_{An1}^A & L_{An2}^A & \cdots & L_{Anp}^A \\
\end{pmatrix} \times$$

$$\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\
d_{21} & d_{22} & \cdots & d_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1} & d_{n2} & \cdots & d_{nm} \\
\end{pmatrix} \frac{1}{\det B} \begin{pmatrix} R_{B11}^B & R_{B21}^B & \cdots & R_{B1p}^B \\
R_{B12}^B & R_{B22}^B & \cdots & R_{B2p}^B \\
\vdots & \vdots & \ddots & \vdots \\
R_{B1p}^B & R_{B2p}^B & \cdots & R_{Bpp}^B \\
\end{pmatrix},$$

where $L_{ij}^A$ is a left $ij$th cofactor of $(A^*A)$ for all $i, j = 1, \ldots, n$ and $R_{ij}^B$ is a right $ij$th cofactor of $(BB^*)$ for all $i, j = 1, \ldots, p$. This implies

$$x_{ij} = \frac{\sum_{s=1}^{p} \left( \sum_{k=1}^{n} L_{ki}^A \tilde{d}_{ks} \right) R_{js}^B}{\det A \cdot \det B}, \quad (42)$$

for all $i = 1, \ldots, n, \ j = 1, \ldots, p$. We obtain the sum in parentheses by Lemma 2.2 and denote it,

$$\sum_{k=1}^{n} L_{ki}^A \tilde{d}_{ks} = cdet_i (A^*A), \quad (\tilde{d}_s) =: d_i^A,$$

where $\tilde{d}_s$ is the $s$th column-vector of $\tilde{D}$ for all $s = 1, \ldots, p$. Suppose $d_i^A := (d_{i1}^A, \ldots, d_{ip}^A)$ is the row-vector for all $i = 1, \ldots, n$. Reducing the sum $\sum_{s=1}^{n} d_{is}^A R_{js}^B$ by Lemma 2.1 we obtain an analog of Cramer’s rule for the minimum norm least squares solution of (23) by (28).

Interchanging the order of summation in (42), we have

$$x_{ij} = \frac{\sum_{k=1}^{n} L_{ki}^A \left( \sum_{s=1}^{p} \tilde{d}_{ks} R_{js}^B \right)}{\det A \cdot \det B}.$$
Further, we obtain the sum in parentheses by Lemma 2.1 and denote it,

$$
\sum_{s=1}^{p} \tilde{d}_{ks} d_{kj}^B = r\det_j(BB^*)_j. \left(\tilde{d}_k\right) := d_{kj}^B,
$$

where $\tilde{d}_k$ is the $k$th row-vector of $\tilde{D}$ for all $k = 1, \ldots, n$. Suppose $d_j^B := (d_{1j}^B, \ldots, d_{nj}^B)^T$ is the column-vector for all $j = 1, \ldots, p$. Using Lemma 2.2 for reducing the sum $\sum_{k=1}^{n} L_{ki}^A d_{kj}^B$, we obtain an analog of Cramer’s rule for (23) by (29).

(iii) If $A \in \mathbb{H}_{r_1 \times n}$, $B \in \mathbb{H}_{p \times q}$ and $r_1 = n$, $r_2 < p$, then by Remark 3.1 and Theorem 3.2 the Moore-Penrose inverses $A^+ = (a^+_i) \in \mathbb{H}^{n \times m}$ and $B^+ = (b^+_ij) \in \mathbb{H}^{q \times p}$ possess the following determinantal representations respectively,

$$
a^+_ij = \frac{\text{cdet}_i(A^*A)_i (a^*_j)}{\text{ddet}A},
$$

$$
b^+_ij = \frac{\sum_{\alpha \in I_{r_2,p}\{j\}} r\det_j((BB^*)_j.(b^*_i))_{\alpha} \sum_{\alpha \in I_{r_2,p}} |(BB^*)_\alpha^+|}{\text{ddet}A}.
$$

(43)

Since by Theorem 4.3 $X_{LS} = A^+DB^+$, then an entry of $X_{LS} = (x_{ij})$ is (37). Denote by $d_s$ the $s$th column of $A^*D =: \tilde{D} = (\hat{d}_{ij}) \in \mathbb{H}^{n \times q}$ for all $s = 1, \ldots, q$.

It follows from $\sum_k a^*_k d_{ks} = \hat{d}_s$ that

$$
\sum_{k=1}^{m} a^+_ik d_{ks} = \sum_{k=1}^{m} \frac{\text{cdet}_i(A^*A)_i (a^*_k)}{\text{ddet}A} \cdot d_{ks} = \frac{\text{cdet}_i(A^*A)_i (\hat{d}_s)}{\text{ddet}A}.
$$

(44)

Substituting (44) and (43) in (37), and using (39) we have

$$
x_{ij} = \sum_{s=1}^{q} \frac{\text{cdet}_i(A^*A)_i (\hat{d}_s)}{\text{ddet}A} \sum_{\alpha \in I_{r_2,p}\{j\}} r\det_j((BB^*)_j.(b^*_i))_{\alpha} \sum_{\alpha \in I_{r_2,p}} |(BB^*)_\alpha^+| =
$$

$$
\sum_{s=1}^{q} \sum_{t=1}^{p} \sum_{l=1}^{n} \text{cdet}_i(A^*A)_i (e_{t}) \hat{d}_{ls} \hat{b}_{st}^* \sum_{\alpha \in I_{r_2,p}\{j\}} r\det_j((BB^*)_j.(e_{t}))_{\alpha} \sum_{\alpha \in I_{r_2,p}} |(BB^*)_\alpha^+| =
$$

$$
\text{ddet}A \sum_{\alpha \in I_{r_2,p}} |(BB^*)_\alpha^+|.
$$
\[
\sum_{l=1}^{p} \sum_{t=1}^{n} \text{cdet}_i(A^*A)_{i.t} \tilde{d}_{lt} \sum_{\alpha \in I_{2,p}(j)} \text{rdet}_j((BB^*)_{j.t}(e_l))_{\alpha} \over \text{ddet}A \sum_{\alpha \in I_{2,p}} |(BB^*)_{\alpha}|. \tag{45}
\]

If we substitute (41) in (45), then we get

\[
x_{ij} = \sum_{l=1}^{n} \text{cdet}_i(A^*A)_{i.t} (e_l) \tilde{d}_{lt} \over \text{ddet}A \sum_{\alpha \in I_{2,p}} |(BB^*)_{\alpha}|.
\]

Since again \(\sum_{l=1}^{n} e_l d^B_{lj} = d^B_{j} \), then it follows (32), where \(d^B_{j} \) is (26).

If we denote by

\[
d^A_{it} := \sum_{l=1}^{n} \text{cdet}_i(A^*A)_{i.t} (\tilde{d}_{lt}) = \sum_{l=1}^{n} \text{cdet}_i(A^*A)_{i.t} (e_l) \tilde{d}_{lt}
\]

the \(t\)th component of a row-vector \(d^A_{i.} = (d^A_{i.1}, ..., d^A_{i.p})\) for all \(t = 1, ..., p\) and substitute it in (45), we obtain

\[
x_{ij} = \sum_{l=1}^{p} d^A_{it} \sum_{\alpha \in I_{2,p}(j)} \text{rdet}_j((BB^*)_{j.t}(e_l))_{\alpha} \over \text{ddet}A \sum_{\alpha \in I_{2,p}} |(BB^*)_{\alpha}|.
\]

Since again \(\sum_{l=1}^{p} d^A_{lt} e_l = d^A_{t.} \), then it follows (33), where \(d^A_{t.} \) is (31).

(iii) The proof is similar to the proof of (iii). \(\blacksquare\)

**Corollary 4.3** *(Theorem 3.3 in [16])* Suppose

\[
AXB = C
\]

is a two-sided matrix equation, where \(\{A, B, C\} \in M(n, \mathbb{H})\) are given, \(X \in M(n, \mathbb{H})\) is unknown. If \(\text{ddet}A \neq 0\) and \(\text{ddet}B \neq 0\), then (46) has a unique solution, and the solution is

\[
x_{i.j} = \text{rdet}_j((BB^*)_{j.t}(e^A_{i.}) \over \text{ddet}A \cdot \text{ddet}B, \tag{47}
\]

19
or
\[ x_{i,j} = \frac{\text{c}_{d,e}^i (A^* A)_i (c^B_j)}{\text{d} \cdot \text{d} \text{d} \text{e}^i A \cdot \text{d} \text{d} \text{e}^i B}, \]  
(48)

where \( c^A_i : = (\text{c}_{d,e}^i (A^* A)_i (\tilde{c}_i)_1, \ldots, \text{c}_{d,e}^i (A^* A)_i (\tilde{c}_i)_n) \) is the row vector and \( c^B_j : = (\text{r} \text{d} \text{e}^i (BB^*)_j (\tilde{c}_j)_1, \ldots, \text{r} \text{d} \text{e}^i (BB^*)_j (\tilde{c}_n)_j)^T \) is the column vector and \( \tilde{c}_i, \tilde{c}_j \) are the \( i \)th row vector and the \( j \)th column vector of \( \tilde{C} \), respectively, for all \( i, j = 1, ..., n \).

**Remark 4.1** In Eq. (24), the index \( i \) in \( \text{c}_{d,e}^i (A^* A)_i (d^B)_j^\beta \) designates \( i \)th column of \( (A^* A)_i (d^B)_j^\beta \), but in the submatrix \( (A^* A)_i (d^B)_j^\beta \) the entries of \( d^B_j \) may be placed in a column with a different index. Similarly, we have for \( j \) in (25). In (14) and (19) we have equivalently.

### 5 An example

In this section, we give an example to illustrate our results. Let us consider the matrix equation
\[ AXB = D, \]
(49)
where
\[
A = \begin{pmatrix} 1 & i & j \\ -k & i & 1 \\ k & j & -i \\ j & -1 & i \end{pmatrix}, \quad B = \begin{pmatrix} i & 1 & j \\ j & k & -i \end{pmatrix}, \quad D = \begin{pmatrix} 1 & i & j \\ k & 0 & i \\ 1 & j & 0 \\ 0 & k & i \end{pmatrix}.
\]

Then we have
\[
A^* A = \begin{pmatrix} 4 & 2i + 2j & 2j + 2k \\ -2i - 2j & 4 & -2i - 2k \\ -2j - 2k & 2i + 2k & 4 \end{pmatrix}, \quad BB^* = \begin{pmatrix} 3 & -3k \\ 3k & 3 \end{pmatrix},
\]
\[
\tilde{D} = A^* DB^* = \begin{pmatrix} 2 + 2i + 2j & -i + 2j - 2k \\ 1 - i - j & i - j - k \\ 1 - 2j & 2i - k \end{pmatrix}.
\]

Since \( \text{d} \text{d} \text{e}^i A = \text{d} \text{d} \text{e}^i B = 0 \) and \( \text{d} \text{d} \text{e}^i A = \begin{pmatrix} 4 & 2i + 2j \\ -2i - 2j & 4 \end{pmatrix} = 8 \neq 0 \),
then \( \text{rank} A = 2 \). Similarly \( \text{rank} B = 1 \).
So, we have the case (i) of Theorem 4.6. We shall find the minimum norm least squares solution \( X_{LS} \) of (49) by (24). We obtain

\[
\sum_{\alpha \in I_{1,2}} |(BB^*)^\alpha| = 3 + 3 = 6,
\]

\[
\sum_{\beta \in J_{2,3}} |(A^*A)^\beta| = \det \begin{pmatrix} 4 & 2i + 2j \\ -2i - 2j & 4 \end{pmatrix} + \det \begin{pmatrix} 4 & 2j + 2k \\ -2j - 2k & 4 \end{pmatrix} + \det \begin{pmatrix} 2i + 2k & -2i - 2k \\ 2i + 2k & 4 \end{pmatrix} = 24.
\]

By (26), we can get

\[
d_B^1 = \begin{pmatrix} 2 + 2i + 2j \\ 1 - i - j \\ 1 - 2j \end{pmatrix}, \quad d_B^2 = \begin{pmatrix} -i + 2j + 2k \\ i - j - k \\ 2i - k \end{pmatrix}.
\]

Since

\[
(A^*A)_{11} (d_B^1) = \begin{pmatrix} 2 + 2i + 2j & 2i + 2j & 2j + 2k \\ 1 - i - j & 4 & -2i - 2k \\ 1 - 2j & 2i + 2k & 4 \end{pmatrix},
\]

then finally we obtain

\[
x_{11} = \frac{\sum_{\beta \in J_{2,3}} \text{cdet}((A^*A)_{11} (d_B^1))^\beta}{\sum_{\beta \in J_{2,3}} |(A^*A)^\beta| \sum_{\alpha \in I_{1,2}} |(BB^*)^\alpha|} = \frac{\text{cdet}_1 \begin{pmatrix} 2 + 2i + 2j & 2i + 2j \\ 1 - i - j & 4 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} 2 + 2i + 2j & 2j + 2k \\ 1 - 2j & 4 \end{pmatrix}}{144} = \frac{4 + 5i + 6j - k}{72}.
\]

Similarly,

\[
x_{12} = \frac{\text{cdet}_1 \begin{pmatrix} -i + 2j - 2k & 2i + 2j \\ i - j - k & 4 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} -i + 2j - 2k & 2j + 2k \\ 2i - k & 4 \end{pmatrix}}{144} = \frac{21}{28}.
\]
\[
x_{21} = \frac{\text{cdet}_2 \left( \begin{array}{cc} 4 & 2 + 2i + 2j \\ -2i - 2j & 1 - i - j \end{array} \right) + \text{cdet}_1 \left( \begin{array}{cc} 1 - i - j & -2i - 2k \\ 1 - 2j & 4 \end{array} \right)}{144} = \frac{-1 - 2i + 5j - 4k}{72},
\]

\[
x_{22} = \frac{\text{cdet}_2 \left( \begin{array}{cc} 4 & -i + 2j - 2k \\ -2i - 2j & i - j - k \end{array} \right) + \text{cdet}_1 \left( \begin{array}{cc} i - j - k & -2i - 2k \\ 2i - k & 4 \end{array} \right)}{144} = \frac{i - 2j - k}{72},
\]

\[
x_{31} = \frac{\text{cdet}_2 \left( \begin{array}{cc} 4 & 2 + 2i + 2j \\ -2j - 2k & 1 - 2j \end{array} \right) + \text{cdet}_2 \left( \begin{array}{cc} 4 & 1 - i - j \\ 2i + 2k & 1 - 2j \end{array} \right)}{144} = \frac{1 - 4i - 3j}{72},
\]

\[
x_{32} = \frac{\text{cdet}_2 \left( \begin{array}{cc} 4 & -i + 2j - 2k \\ -2j - 2k & 2i - k \end{array} \right) + \text{cdet}_1 \left( \begin{array}{cc} 4 & i - j - k \\ 2i + 2k & 2i - k \end{array} \right)}{144} = \frac{3i - 3j - 2k}{72}.
\]

Then

\[
X_{LS} = \frac{1}{72} \begin{pmatrix} 4 + 5i + 6j - k & -1 - 2i + 5j - 4k \\ i - 2j - k & -2 + 2i + j - k \\ 1 - 4i - 3j & 3i - 3j - 2k \end{pmatrix}
\]

is the minimum norm least squares solution of (49).
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