Symplectic entropy

Sergio De Nicola,1 Renato Fedele,2 Margarita A. Man’ko3 and Vladimir I. Man’ko3

1 Istituto di Cibernetica “Eduardo Caianiello” del CNR Comprensorio “A. Olivetti” Fabbr. 70, Via Campi Flegrei, 34, I-80078 Pozzuoli (NA), Italy
E-mail: s.denicola@cib.na.cnr.it
2 Dipartimento di Scienze Fisiche, Università Federico II and INFN Sezione di Napoli, Complesso Universitario di M.S. Angelo, via Cintia, I-80126 Napoli, Italy
E-mail: renato.fedele@na.infn.it
3 P. N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991 Russia
E-mail: mmanko@sci.lebedev.ru manko@sci.lebedev.ru

Abstract. The tomographic-probability description of quantum states is reviewed. The symplectic tomography of quantum states with continuous variables is studied. The symplectic entropy of the states with continuous variables is discussed and its relation to Shannon entropy and information is elucidated. The known entropic uncertainty relations of the probability distribution in position and momentum of a particle are extended and new uncertainty relations for symplectic entropy are obtained. The partial case of symplectic entropy, which is optical entropy of quantum states, is considered. The entropy associated to optical tomogram is shown to satisfy the new entropic uncertainty relation. The example of Gaussian states of harmonic oscillator is studied and the entropic uncertainty relations for optical tomograms of the Gaussian state are shown to minimize the uncertainty relation.

1. Introduction

There exist several equivalent formulations of quantum mechanics (see, for example, [1]). The important formulation is based on the phase-space representation of quantum observables and quantum states described by the Wigner function (see [2, 3]. Recent formulation of quantum mechanics, namely, the probability representation of quantum mechanics was suggested in [4, 5] where the probability-distribution density is used as an alternative to the wave function or density matrix. Recent review of this formulation is done in [6]. The operator symbol framework for quantum information based on the probability representation is presented in [7]. This formulation of quantum mechanics uses the invertible map of density matrix onto the probability distribution found for the density matrix in the form of its Wigner function [8] for one degree of freedom in [9] in operator form and for two degrees of freedom in [10].

The mathematical nature of the invertible map of the density matrix (or wave function) onto the probability distribution is based on the properties of known integral Radon transform [11]. In view of this, the approach with symplectic tomography can be extended to other areas where mathematical structures are similar to formalism of quantum mechanics.

For example, in signal analysis the analytic signal [12] is a complex function of time $f(t)$. This function is analogous to the wave function $\psi(x)$ of the quantum state (see, e.g., [13, 14].
Similar ansatz can be applied to solitons [15–20]. There are several standard integral transforms of analytic signal used in information processing. If the signal is considered as a complex function of several variables $f(t_1, t_2, \ldots, t_N) = f(\vec{t})$, the analogous function $\psi(\vec{x})$ in quantum mechanics corresponds to multimode quantum state.

The analogy of analytic signal and quantum wave function was used in [21] to connect integral transforms with the Green function of the Schrödinger evolution equation considered as a kernel of the map (integral transform) of the initial wave function onto the wave function of the quantum system at the final moment of time $\psi(x, 0) \rightarrow \psi(x, t)$.

Such analogy can be extended also to the consideration of the functions of several variables. For this aim, one has to apply the analogy of integral transforms with the Green function of the Schrödinger evolution equation for multimode quantum systems (for review of the approach, see [22]).

Our aim is to study the tomographic map of functions of one or several variables onto probability distributions. Applying the known Shannon construction of entropy associated to the probability distribution [23], we introduce tomographic entropy of wave functions, analytic signals, solitons and obtain some inequality for the entropy.

The general setting for our construction is as follows [24]. Functions are considered to be vectors $| f \rangle$. This is true for both one-variable $f(t)$ and multi-variable $f(\vec{t})$ functions. Vector variable $\vec{t}$ can have both time and space components.

We have a family of self-adjoint operators $B(\alpha)$, where $\alpha$ is a collective index for the set of parameters. Since $B(\alpha)$ has a real valued spectrum, the transform may be defined by means of Dirac delta-function

$$M_f^{(B)}(X) = \langle f | \delta(B(\alpha) - X) | f \rangle. \quad (1)$$

Equation (1) defines what we call the tomographic transform of function $f$ (vector $| f \rangle$) or tomogram. The transform $M_f^{(B)}(X)$ is positive and it can be correctly interpreted as a probability distribution [22].

In fact, for a normalized vector $| f \rangle$,

$$\langle f | f \rangle = 1,$$

the tomogram is a non-negative and normalized function

$$\int M_f^{(B)}(X) \, dX = 1$$

and, therefore, it may be interpreted as a probability distribution for the random variable $X$ corresponding to the observable defined by the operator $B(\alpha)$.

2. **Green function of quantum harmonic oscillator**

For constructing the new entropic uncertainty relations, in the next sections, we need to remind the properties of the Green function of harmonic oscillator.

The Hamiltonian of quantum harmonic oscillator of mass $m$ has the form

$$H = \frac{\hat{p}^2}{2m} + \frac{m \omega^2 \hat{q}^2}{2}, \quad (2)$$

where in the coordinate representation the momentum operator reads $\hat{p} = -i\hbar \partial / \partial x$, with $\hbar$ being the Planck’s constant, and the position operator is $\hat{q} = x$. 


The Green function $G_\omega(x, y, t)$ of the Schrödinger equation determines the wave function $\psi(x, t)$ in terms of the initial wave function $\psi(y, 0)$ by the relation

$$\psi(x, t) = \int G_\omega(x, y, t) \psi(y, 0) \, dy. \tag{3}$$

The Green function $G_\omega(x, y, t)$, where $t$ is time variable, is the kernel of the unitary evolution operator $\hat{U}(t)$ of the quantum harmonic oscillator, i.e.,

$$\psi(x, t) = \hat{U}(t) \psi(x, 0). \tag{4}$$

From Eq. (4) follows the equality

$$\psi(x, 0) = \hat{U}^{-1}(t) \psi(x, t). \tag{5}$$

In view of the physical meaning of the evolution operator,

$$\hat{U}^{-1}(t) = \hat{U}(-t),$$

i.e., we can find the initial value of the oscillator’s wave function, if we know the wave function at time $t$, using the relation

$$\psi(x, 0) = \int \psi(y, t) G_\omega(x, y, -t) \, dy. \tag{6}$$

From the mathematical point of view, relations (3) and (6) can be interpreted as the integral transform and its inverse, respectively.

The Green function of quantum harmonic oscillator $G_\omega(x, y, t)$ is written as follows:

$$G_\omega(x, y, t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \exp \left\{ \frac{i m\omega}{2\hbar} \left[ (x^2 + y^2) \cot \omega t - \frac{2xy}{\sin \omega t} \right] \right\}. \tag{7}$$

This means that relation (3) takes the explicit form of integral transform of the initial wave function $\psi(y, 0)$

$$\psi(x, t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \int dy \psi(y, 0) \exp \left\{ \frac{i m\omega}{2\hbar} \left[ (x^2 + y^2) \cot \omega t - \frac{2xy}{\sin \omega t} \right] \right\}. \tag{8}$$

The inverse relation (6) for arbitrary time $t$ has the form

$$\psi(x, 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin (-\omega t)}} \int dy \psi(y, t) \exp \left\{ \frac{i m\omega}{2\hbar} \left[ -(x^2 + y^2) \cot \omega t + \frac{2xy}{\sin \omega t} \right] \right\}. \tag{9}$$

For arbitrary time $t$, relation (8) is the integral transform of the initial wave function with the Gaussian kernel, periodic in time. Its inverse is given by (9). The transform (8) is known as fractional Fourier transform [25].
3. Entropy and entropic uncertainty relations

Below we consider the notion of entropy for functions $f(t)$ (vectors $|f\rangle$) following [26].

In the information-theory context, entropy is related to an arbitrary probability-distribution function [23]. For example, given the probability distribution $P(n)$, where $n$ is a discrete random variable, i.e.,

$$P(n) \geq 0,$$

and the normalization condition holds

$$\sum_n P(n) = 1,$$

one has, by definition, the entropy

$$S = -\sum_n P(n) \ln P(n) = -\langle \ln P(n) \rangle.$$  \hspace{1cm} (12)

In quantum mechanics, for continuous variables wave function $\psi(x)$ provides the probability-distribution density of position $P(x) = |\psi(x)|^2$. The corresponding entropy $S_x$ related to the position-probability density $|\tilde{\psi}(x)|^2$ reads (see, for example, [27])

$$S_x = -\int |\psi(x)|^2 \ln |\psi(x)|^2 \, dx.$$  \hspace{1cm} (14)

In the momentum representation, the wave function reads

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(x)e^{-ipx} \, dx \hspace{1cm} (\hbar = 1).$$

The corresponding entropy $S_p$ related to the momentum-probability density $|\tilde{\psi}(p)|^2$ is

$$S_p = -\int |\tilde{\psi}(p)|^2 \ln |\tilde{\psi}(p)|^2 \, dp.$$  \hspace{1cm} (16)

It is worth noting that one can construct entropies $S_x$ and $S_p$ not only in quantum mechanics. If the function $\psi(x)$ is replaced by a signal function $f(t)$ depending on time $t$, the function $\tilde{\psi}(p)$ is replaced by the function $\tilde{f}(\omega)$ describing the signal spectrum.

In this case, the entropy of the signal

$$S_t = -\int |f(t)|^2 \ln |f(t)|^2 \, dt$$

and the entropy of its frequency spectrum

$$S_\omega = -\int |\tilde{f}(\omega)|^2 \ln |\tilde{f}(\omega)|^2 \, d\omega$$

provide some information characteristics of the signal. Analogous approach can be applied also for soliton solutions of nonlinear equations [17].

From mathematical point of view, there exists correlation of entropies $S_x$ and $S_p$ ($S_t$ and $S_\omega$), since function $\psi(x)$ [$f(t)$] determines Fourier component $\tilde{\psi}(p)$ [$\tilde{f}(\omega)$]. This means that entropies
$S_x$ and $S_p$ have to obey some constrains. These constrains are well-known entropic uncertainty relations (some inequalities).

For one-mode system, the inequalities read (see p. 28 in review [27])

$$S_x + S_p \geq \ln(\pi e), \quad S_t + S_\omega \geq \ln(\pi e). \quad (19)$$

For Gaussian wave functions (Gaussian signals) describing the states without correlations of the position and momentum, e.g., the ground state of harmonic oscillator

$$\psi(x) = \pi^{-1/4} e^{-x^2/2}, \quad \tilde{\psi}(p) = \pi^{-1/4} e^{-p^2/2}, \quad (20)$$

one has

$$S_x^{(0)} = S_p^{(0)} = 2^{-1} \ln(\pi e). \quad (21)$$

Consequently,

$$S_x^{(0)} + S_p^{(0)} = \ln(\pi e). \quad (22)$$

Inequalities (19) are closely related to the position–momentum uncertainty relations for the position dispersion

$$\sigma_x = (\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2)$$

and for the momentum dispersion

$$\sigma_p = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2) .$$

For squeezed and correlated states, wave functions have the Gaussian form, i.e.,

$$\psi(x) = \mathcal{N} \exp(-ax^2 + bx), \quad a = a_1 + ia_2, \quad a_1 > 0, \quad b = b_1 + ib_2,$$

with the normalization constant

$$\mathcal{N} = (2a_1)^{-1/4} \pi^{-1/4} e^{-b_1^2/4a_1},$$

and the product of the position and momentum uncertainties reads

$$\sigma_x \sigma_p = \left[4(1 - R^2)\right]^{-1}, \quad (23)$$

where $R$ is the correlation coefficient of the position and momentum, i.e.,

$$R = \frac{\frac{1}{2} \langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle}{\sqrt{\sigma_x \sigma_p}}, \quad |R| < 1. \quad (24)$$

For squeezed but not correlated states, $R = 0$.

The sum of entropies for squeezed and correlated states reads

$$S_x + S_p = \ln(\pi e) + \ln \frac{1}{\sqrt{1 - R^2}} \geq \ln(\pi e). \quad (25)$$

For squeezed states, the entropy $S_x$ differs from $S_p$.

For multimode quantum systems (or multicomponent signals), the entropy uncertainty relation reads

$$S_x + S_p \geq N \ln(\pi e), \quad (26)$$

where $N$ is the number of degrees of freedom of the system and

$$S_x = - \int |\psi(x)|^2 \ln |\psi(x)|^2 \, dx, \quad S_p = - \int |\tilde{\psi}(p)|^2 \ln |\tilde{\psi}(p)|^2 \, dp. \quad (27)$$

The functions $\psi(x)$ and $\tilde{\psi}(p)$ are connected by the Fourier transform

$$\tilde{\psi}(p) = (2\pi)^{-N/2} \int \psi(x) e^{-ipx} \, dx. \quad (28)$$

For the Gaussian wave function corresponding to factorized squeezed state of several modes,

$$S_x + S_p = N \ln(\pi e). \quad (29)$$
4. Symplectic tomography

As we discussed in the Introduction, there exists invertible map, which is tomographic map of function $f$. In terms of the density operator (matrix) $\hat{\rho}$, the symplectic tomogram [4, 5], which is the probability-density of random position $X$, can be presented in the form

$$w(X, \mu, \nu) = \text{Tr} \hat{\rho} \delta(X - \mu \hat{q} - \nu \hat{p}),$$

(30)

where $\hat{\rho}$ is the density operator, $\mu$ and $\nu$ are real parameters, $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively.

For pure state $\hat{\rho} = |\psi\rangle \langle \psi|$, transform (30) yields [28]

$$w(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \psi(y) \exp \left( \frac{i\mu}{2\nu} y^2 - \frac{iX}{\nu} y \right) dy \right|^2. \quad (31)$$

The function $w(X, \mu, \nu)$ (the state tomogram) is the position-probability density of the position $X$, i.e., $w(X, \mu, \nu) \geq 0$ and

$$\int w(X, \mu, \nu) dX = 1. \quad (32)$$

The tomogram has homogeneity property which reads [29]

$$w(\lambda X, \lambda \mu, \lambda \nu) = |\lambda|^{-1} w(X, \mu, \nu). \quad (33)$$

Also for the pure state, one has

$$w(X, 1, 0) = |\psi(X)|^2, \quad w(X, 0, 1) = |\tilde{\psi}(X)|^2, \quad \text{where } \psi(X) [\tilde{\psi}(X)] \text{ is the wave function in the position (momentum) representation.}$$

Tomogram (31) can be rewritten in the form [30, 31]

$$w(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \psi(y) \exp \left[ \frac{i}{2} \left( \frac{\mu y^2}{\nu} - \frac{2X}{\nu} y + \frac{\mu X^2}{\nu^2} \right) \right] dy \right|^2. \quad (35)$$

For $\mu = \cos t$ and $\nu = \sin t$, one has the optical tomogram

$$w(X, t) = \left| \int \psi(y) \exp \left[ \frac{i}{2} \left( \cot t \left( y^2 + X^2 \right) - \frac{2X}{\sin^2 t} y \right) \right] \frac{dy}{\sqrt{2\pi i \sin t}} \right|^2. \quad (36)$$

The optical tomogram of the quantum state given in terms of the Wigner function was introduced in [32, 33].

If we denote

$$\psi(X, t) = \frac{1}{\sqrt{2\pi i \sin t}} \int \exp \left[ \frac{i}{2} \left( \cot t \left( y^2 + X^2 \right) - \frac{2X}{\sin^2 t} y \right) \right] \psi(y) dy, \quad (37)$$

then (36) means that

$$w(X, t) = |\psi(X, t)|^2. \quad (38)$$

Comparing (37) with (8) one can see that the wave function $\psi(X, t)$ corresponds to the wave function of a harmonic oscillator with $\hbar = m = \omega = 1$ taken at the time moment $t$ provided the wave function $\psi(y, 0)$ at the initial time equals to $\psi(y)$.

In optical tomography, one denotes $\mu = \cos \theta$ and $\nu = \sin \theta$. Thus the parameter of time $t$ for oscillator with unit frequency is identical to the angle parameter $\theta$. Below we denote in optical tomograms the angle as $t$.

Thus, we pointed out that symplectic tomogram of a pure quantum state can be interpreted as modulus squared of the harmonic-oscillator’s wave function for the pure state. This observation provides the possibility to apply entropic uncertainty relations known for wave functions and reviewed in previous sections to symplectic tomograms.
5. Tomographic entropies

Since the symplectic tomogram is the standard probability distribution, one can introduce entropy associated with tomogram of quantum state [34] or with tomogram of analytic signal [35]. Thus one has entropy as the function of two real variables

\[ S(\mu, \nu) = -\int w(X, \mu, \nu) \ln w(X, \mu, \nu) dX. \]

(39)

We call this entropy symplectic entropy.

In view of the homogeneity [29] and normalization conditions for tomogram, one has the additivity property

\[ S(\lambda \mu, \lambda \nu) = S(\mu, \nu) + \ln |\lambda|. \]

(40)

For pure state \( |\psi\rangle \), in view of

\[ w(X, 1, 0) = |\psi(X)|^2 \quad \text{and} \quad w(X, 0, 1) = |\tilde{\psi}(X)|^2, \]

one obtains the entropies \( S_x \) and \( S_p \) putting in (39) \( \mu = 1 \) and \( \nu = 0 \)

\[ S(1, 0) = S_x, \]

(41)

or \( \mu = 0 \) and \( \nu = 1 \)

\[ S(0, 1) = S_p. \]

(42)

In view of inequality (19), one has the inequality for tomographic entropies

\[ S(1, 0) + S(0, 1) \geq \ln(\pi e). \]

(43)

For multimode system, the symplectic entropy reads

\[ S(\vec{\mu}, \vec{\nu}) = -\int w(\vec{X}, \vec{\mu}, \vec{\nu}) \ln w(\vec{X}, \vec{\mu}, \vec{\nu}) d\vec{X}. \]

(44)

Since the symplectic entropy is related to entropies \( S_x \) and \( S_p \) of multimode-system state, one can use inequality (26) to obtain the entropic uncertainty relation in the form of inequality for symplectic entropies

\[ S(\vec{1}, \vec{0}) + S(\vec{0}, \vec{1}) \geq N \ln(\pi e), \]

(45)

where \( \vec{\mu} = \vec{1} = (1, 1, \ldots, 1) \) and \( \vec{\nu} = \vec{1} = (1, 1, \ldots, 1) \).

For the optical tomogram \( w(X, t) \) given by (36), entropy is defined by the formula

\[ S(t) = -\int w(X, t) \ln w(X, t) dX. \]

(46)

For pure state,

\[ S(0) = S_x \]

(47)

and

\[ S(\pi/2) = S_p. \]

(48)

In view of (37) and (38), one has the entropic uncertainty relation in the form

\[ S(t) + S(t + \pi/2) \geq \ln \pi e. \]

(49)

Thus we extended the entropic uncertainty relation to arbitrary value of parameter \( t \) using the interpretation of symplectic tomogram as modulus squared of the wave function of an ‘artificial harmonic oscillator’.
Since symplectic and optical tomograms are connected
\[ w(X, \mu = \cos t, \nu = \sin t) = w(X, t), \]
the corresponding entropies are also connected
\[ S(t) = S(\mu = \cos t, \nu = \sin t). \]
For given symplectic entropy of any pure state \( S(\mu, \nu) \), inequality (49) reads
\[ S(\cos t, \sin t) + S(- \sin t, \cos t) \geq \ln \pi e. \]
In view of the homogeneity property [29], optical \( w(X, t) \) and symplectic \( w(X, \mu, \nu) \) tomograms are also related as follows:
\[ w(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} w\left(\frac{X}{\sqrt{\mu^2 + \nu^2}}, t\right). \]
This means that for given optical tomogram \( w(X, t) \) one can reconstruct symplectic tomogram \( w(X, \mu, \nu) \).
Inserting (53) into the basic equation defining the entropy (39), in view of the additivity property of the tomographic entropy (40), we obtain
\[ S(t) = S\left(\sqrt{\mu^2 + \nu^2} \cos t, \sqrt{\mu^2 + \nu^2} \sin t\right) - \frac{1}{2} \ln(\mu^2 + \nu^2). \]
And for symplectic entropy (39), the entropic uncertainty relation yields
\[ S\left(\sqrt{\mu^2 + \nu^2} \cos t, \sqrt{\mu^2 + \nu^2} \sin t\right) + S\left(-\sqrt{\mu^2 + \nu^2} \sin t, \sqrt{\mu^2 + \nu^2} \cos t\right) - \ln(\mu^2 + \nu^2) \geq \ln \pi e. \]
This inequality is the new result obtained for tomographic entropy.
One can extend this inequality for multimode system
\[
S\left(\sqrt{\mu_1^2 + \nu_1^2} \cos t_1, \sqrt{\mu_1^2 + \nu_1^2} \sin t_1, \sqrt{\mu_2^2 + \nu_2^2} \cos t_2, \ldots, \sqrt{\mu_N^2 + \nu_N^2} \cos t_N, \sqrt{\mu_1^2 + \nu_1^2} \sin t_1, \sqrt{\mu_2^2 + \nu_2^2} \sin t_2, \ldots, \sqrt{\mu_N^2 + \nu_N^2} \sin t_N\right)
+ S\left(-\sqrt{\mu_1^2 + \nu_1^2} \sin t_1, -\sqrt{\mu_2^2 + \nu_2^2} \sin t_2, \ldots, -\sqrt{\mu_N^2 + \nu_N^2} \sin t_N, \sqrt{\mu_1^2 + \nu_1^2} \cos t_1, \sqrt{\mu_2^2 + \nu_2^2} \cos t_2, \ldots, \sqrt{\mu_N^2 + \nu_N^2} \cos t_N\right)
- \sum_{k=1}^{N} \ln \left(\mu_k^2 + \nu_k^2\right) \geq N \ln(\pi e),
\]
where entropy \( S(\vec{\mu}, \vec{\nu}) \) is given by (44).
5.1. Some Examples

Tomogram of the ground state of multimode isotropic harmonic oscillator with unit masses and frequencies has the form of Gaussian probability distribution

\[ w_0(\vec{X}, \vec{\mu}, \vec{\nu}) = \prod_{k=1}^{N} \frac{1}{\sqrt{\pi (\mu_k^2 + \nu_k^2)}} \exp \left( - \frac{X_k^2}{\mu_k^2 + \nu_k^2} \right). \]  

(57)

Entropy associated with this tomogram reads

\[ S_0(\vec{\mu}, \vec{\nu}) = \frac{N}{2} \ln \pi + \frac{N}{2} + \frac{N}{2} \sum_{k=1}^{N} \ln (\mu_k^2 + \nu_k^2). \]  

(58)

Function (58) transforms relation (56) into the identity.

Entropy of the soliton solution to nonlinear equations was discussed in [17]. In particular, the soliton solution to Gross–Pitaevskii equation [36, 37] was considered in the tomographic-probability representation to study Bose–Einstein condensate (BEC) (see [18–20]).

BEC soliton under consideration is given as the function

\[ \psi(x) = \frac{1}{\sqrt{2\ell_z}} \text{sech} \left( \frac{x}{\ell_z} \right), \]  

(59)

where the parameter \( \ell_z \) describes the soliton width.

Symplectic tomogram of BEC soliton reads

\[ w_S(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \frac{1}{\sqrt{2\ell_z}} \text{sech} \left( \frac{y}{\ell_z} \right) \exp \left( i\mu \frac{y^2}{2\nu} - i\frac{X}{\nu} y \right) dy \right|^2, \]  

(60)

where \( \mu = r \cos t \) and \( \nu = r \sin t \). Since \( \int |\psi(x)|^2 dx = 1 \), tomogram (60) is nonnegative normalized probability distribution of random position \( X \).

The tomographic entropy of BEC soliton equals to

\[ S(r, t) = - \int \frac{1}{2\pi |\nu|} \left| \int \frac{1}{\sqrt{2\ell_z}} \text{sech} \left( \frac{y}{\ell_z} \right) \exp \left( i\mu \frac{y^2}{2\nu} - i\frac{X}{\nu} y \right) dy \right|^2 \times \ln \left\{ \frac{1}{2\pi |\nu|} \left| \int \frac{1}{\sqrt{2\ell_z}} \text{sech} \left( \frac{y}{\ell_z} \right) \exp \left( i\mu \frac{y^2}{2\nu} - i\frac{X}{\nu} y \right) dy \right|^2 \right\} dX. \]  

(61)

We introduce the function

\[ F(r, t) = S(r, t) + S(r, t + \pi/2) - \ln r^2 - \ln(\pi e). \]  

(62)

According to the entropic uncertainty relation (55), this function (we call it entropic uncertainty function) must be nonnegative. Equation (54) and the additivity property (40) mean that the entropic uncertainty function \( F(r, t) \) (62) does not depend on parameter \( r \).

The normalized initial Gaussian profile is given by

\[ \psi_G(y) = \exp \left( -\frac{y^2}{2\sigma^2} \right), \]  

(63)

where \( \sigma \) is the waist of Gaussian profile. The corresponding tomogram calculated with the help of Eq. (31) with \( \mu = r \cos t \) and \( \nu = r \sin t \) is given by

\[ w_G(r, t) = \frac{\sigma}{r \sqrt{\pi (\sin^2 t + \sigma^4 \cos^2 t)}} \exp \left[ -\frac{\sigma^2 X^2}{r^2 (\sin^2 t + \sigma^4 \cos^2 t)} \right]. \]  

(64)
The symplectic Gaussian entropy is given as follows:

\[ S_G(r, t) = \frac{1}{2} - \ln \left( \frac{\sigma}{r \sqrt{\pi (\sin^2 t + \sigma^4 \cos^2 t)}} \right). \] (65)

The corresponding entropic uncertainty function \( F_G(t) \) in this case can be calculated explicitly

\[ F_G(t) = \ln \left( \sqrt{1 + \left( \frac{1 - \sigma^4}{2 \sigma^2} \right)^2 \sin^2 2t} \right). \] (66)

Note that the positive definite function \( F_G(t) \) does not depend on the radial variable \( r \), i.e., it is the same for both the symplectic and optical entropies and it reduces to zero for \( \sigma = 1 \), whereas for \( \sigma \neq 1 \) it is periodic with period \( \pi/2 \).

6. Inequalities with extra parameters for tomograms

In [38] the new uncertainty relation was obtained for Rényi entropy [39] related to the probability distributions for position and momentum of quantum state with density operator \( \hat{\rho} \). The uncertainty relation reads

\[
\frac{1}{1 - \alpha} \ln \left( \int_{-\infty}^{\infty} dp [\rho(p,p)]^\alpha \right) + \frac{1}{1 - \beta} \ln \left( \int_{-\infty}^{\infty} dx [\rho(x,x)]^\beta \right) \\
\geq - \frac{1}{2(1 - \alpha)} \ln \frac{\alpha}{\pi} - \frac{1}{2(1 - \beta)} \ln \frac{\beta}{\pi},
\]

where positive parameters \( \alpha \) and \( \beta \) satisfy the constrain

\[ \frac{1}{\alpha} + \frac{1}{\beta} = 2. \] (68)

Using the same argument that we employed to obtain inequality (52), one arrived at the condition for optical tomogram [40]

\[
(q - 1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w(X,\theta + \frac{\pi}{2}) \right]^{1/(1-q)} \right\} \\
+ (q + 1) \ln \left\{ \int_{-\infty}^{\infty} dX [w(X,\theta)]^{1/(1+q)} \right\} \\
\geq \frac{1}{2} \left\{ (q - 1) \ln [\pi(1 - q)] + (q + 1) \ln [\pi(1 + q)] \right\},
\]

where the parameter \( q \) is defined by \( \alpha = (1 - q)^{-1} \). We used in Eq. (69) the angle parameter \( \theta \) instead of the notation \( t \) used in the previous sections since the notation \( \theta \) is the standard one for optical tomography.

Below we present a new inequality for symplectic tomogram in the one-mode case. It reads

\[
(q - 1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w(X,-\sqrt{\mu^2 + \nu^2} \sin \theta, \sqrt{\mu^2 + \nu^2} \cos \theta) \right]^{1/(1-q)} \right\} \\
+ (q + 1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w(X,\sqrt{\mu^2 + \nu^2} \cos \theta, \sqrt{\mu^2 + \nu^2} \sin \theta) \right]^{1/(1+q)} \right\} \\
\geq \frac{1}{2} \left\{ (q - 1) \ln [\pi(1 - q)] + (q + 1) \ln [\pi(1 + q)] \right\}.
\]

(70)
This inequality can be interpreted as a generalization of the inequality (69) extended from optical tomogram to symplectic tomogram.

In view of the inequality for Rényi entropy adopted from [38], the above condition for tomogram of quantum state can be generalized for the multimode case as well. Then for symplectic tomogram of quantum state with density operator \( \hat{\rho} \) defined as

\[
\begin{align*}
  w(X_1, X_2, \ldots, X_N, \mu_1, \mu_2, \ldots, \mu_N, \nu_1, \nu_2, \ldots, \nu_N) \\
  = \text{Tr} \left[ \hat{\rho} \left( X_1 - \mu_1 \hat{q}_1 - \nu_1 \hat{p}_1 \right) \delta(X_2 - \mu_2 \hat{q}_2 - \nu_2 \hat{p}_2) \cdots \delta(X_N - \mu_N \hat{q}_N - \nu_N \hat{p}_N) \right],
\end{align*}
\]

(71)

where \( \hat{q}_k \) and \( \hat{p}_k \) \((k = 1, 2, \ldots, N)\) are the position and momentum operators, respectively, we obtain

\[
\begin{align*}
  & (q-1) \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[ w(X_1, X_2, \ldots, X_N, -\sqrt{\mu_1^2 + \nu_1^2} \sin \theta_1, \ldots, \\
  -\sqrt{\mu_N^2 + \nu_N^2} \sin \theta_N, \sqrt{\mu_1^2 + \nu_1^2} \cos \theta_1, \ldots, \sqrt{\mu_N^2 + \nu_N^2} \cos \theta_N \right] \right\}^{1/(1-q)} \\
  & +(q+1) \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[ w(X_1, X_2, \ldots, X_N, \sqrt{\mu_1^2 + \nu_1^2} \cos \theta_1, \ldots, \\
  \sqrt{\mu_N^2 + \nu_N^2} \cos \theta_N, \sqrt{\mu_1^2 + \nu_1^2} \sin \theta_1, \ldots, \sqrt{\mu_N^2 + \nu_N^2} \sin \theta_N \right] \right\}^{1/(1+q)} \\
  \geq & \frac{N}{2} \left\{ (q-1) \ln [\pi(1-q)] + (q+1) \ln [\pi(1+q)] \right\}.
\end{align*}
\]

(72)

For optical tomogram \( w(X_1, \ldots, X_N, \theta_1, \ldots, \theta_N) \), the inequality reads

\[
\begin{align*}
  & (q-1) \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[ w(X_1, X_2, \ldots, X_N, \theta_1, \ldots, \theta_N) \right]^{1/(1-q)} \right\} \\
  & +(q+1) \ln \left\{ \int_{-\infty}^{\infty} d\vec{X} \left[ w(X_1, X_2, \ldots, X_N, \theta_1 + \pi/2, \ldots, \theta_N + \pi/2) \right]^{1/(1+q)} \right\} \\
  \geq & \frac{N}{2} \left\{ (q-1) \ln [\pi(1-q)] + (q+1) \ln [\pi(1+q)] \right\}.
\end{align*}
\]

(73)

Inequalities (72) and (73) are saturated for Gaussian tomograms of coherent states. In the limit \( q \to 0 \), they become entropic uncertainty relations found in [40,41] (see, also [42]).

The obtained inequalities are illustrated by plots.

We introduce the function \( R(\theta, q) \) which is equal to the difference of left- and right-hand sides of Eq. (69)

\[
R(\theta, q) = (q-1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w(X, \theta + \frac{\pi}{2}) \right]^{1/(1-q)} \right\} \\
+(q+1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w(X, \theta) \right]^{1/(1+q)} \right\} \\
- \frac{1}{2} \left\{ (q-1) \ln [\pi(1-q)] + (q+1) \ln [\pi(1+q)] \right\},
\]

(74)

This function must be nonnegative for any tomogram describing a quantum state.

The inequality \( R(\theta, q) \geq 0 \) is checked numerically for some tomograms.
Figure 1. Function $R(\theta, q)$ vs angular variable $\theta$ calculated for different values of the parameter $q$ when the initial wave function was assumed a normalized soliton profile with the width parameter $\ell_z = 4$ (left). The corresponding 3D plot of the function $R(\theta, q)$ for the soliton wave function (right).

Plot of the dependence of $R(\theta, q)$ on the angular variable $\theta$ calculated for different values of the parameter $q$ for normalized soliton profile of the wave function given by Eq. (59) with the width parameter $\ell_z = 4$ is shown in Fig. 1 (left). The plot shows that the condition of positivity is satisfied over all the considered $\theta$ range. For $q = 0$, the inequality reduce to the well-known Shannon entropy inequality previously studied. The 3D plot of the same function $R(\theta, q)$ in the $\theta - q$ plane is presented in Fig. 1 (right). One can see that the function is nonnegative and its behavior corresponds to the tomographic entropic inequality.

Figure 2 shows plots similar to those shown in Fig. 1 but this time the initial wave function $\phi(y)$ was assumed a normalized Lorentzian, namely,

$$\psi_L(y) = \frac{1}{2\pi \gamma} \frac{\gamma}{y^2 + (\gamma/2)^2},$$

where $\gamma$ is the full width at half-maximum. The curves are calculated for $\gamma = 12$ and for the same values of $q$ as those of soliton (Fig. 1). The entropy inequality shows a wider distribution of positive values as compared to the distribution shown in Fig. 1. The corresponding 3D perspective displays more clearly the behaviour of inequality $R(\theta, q)$ in the case of the Lorentzian profile of the wave function for different values of $q$. One can see that the behavior of this plot is similar to the case of soliton profile and this statement completely corresponds to the inequality introduced.

7. Conclusions
To conclude, we summarize the main results of our study.

The new uncertainty relations were reviewed within the framework of the probability representation of quantum mechanics. The entropic uncertainty relations have the form of integral condition for tomograms of quantum states which contain the complete information on the states. The new inequality with extra parameter were obtained for some integral expressions containing the quantum-state tomograms on the base of recently found uncertainty relations [38].
Figure 2. Function $R(\theta, q)$ vs angular variable $\theta$ calculated for different values of the parameter $q$ when the initial wave function was assumed a normalized Lorentzian (left). The corresponding 3D plot of the function $R(\theta, q)$ for the Lorentzian wave function (right).

for Rényi entropy of quantum states. The conditions for the one-mode and multimode optical tomograms are of particular interest, since these tomograms are directly measured in quantum-optics experiments [43]. We hope to get analogous new inequalities for tomograms depending on discrete variables.

Acknowledgments
M.A.M. thanks the Organizers of the Third Feynman Festival and University of Maryland at College Park for invitation and kind hospitality.

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