MODULAR THEORY BY EXAMPLE

FERNANDO LEDÓ

CONTENTS

1. Introduction 1
2. Modular Theory: definitions, results and first examples 2
2.1. Comments and elementary consequences of the Tomita-Takesaki theorem 5
2.2. Examples 6
3. Modular objects for a crossed product 8
4. Modular objects for the CAR-algebra 12
4.1. Two subspaces in generic position 12
4.2. Modular objects for \((M(q), \Omega)\): 15
4.3. Modular objects for double cones in Fermi models 18
5. Some classical applications of Modular Theory 19
5.1. The commutant of tensor products 19
5.2. Structure of type III factors 20
5.3. The KMS condition 21
6. Appendix: Crossed products and the CAR-algebra 21
6.1. Crossed products 21
6.2. The self-dual CAR-algebra 22
References 24

Abstract. The present article contains a short introduction to Modular Theory for von Neumann algebras with a cyclic and separating vector. It includes the formulation of the central result in this area, the Tomita-Takesaki theorem, and several of its consequences. We illustrate this theory through several elementary examples. We also present more elaborate examples and compute modular objects for a discrete crossed product and for the algebra of canonical anticommutation relations (CAR-algebra) in a Fock representation.

1. Introduction

Modular Theory has been one of the most exciting subjects for operator algebras and for its applications to mathematical physics. We will give here a short introduction to this theory and state some of its main results. There are excellent textbooks and review articles which cover this subject, e.g. [32, 38, 34, 40, 15, 11, Section 2.5.2] or [21, Chapter 9]. For an overview and

Date: January 8, 2009.
further applications to quantum field theory see also [9, 10, 33] and references cited therein. A beautiful alternative approach to Modular Theory in terms of bounded operators is given in [31]. This approach is close in spirit to the example presented in the context of the CAR-algebra in Section 4. The origin of the terminology is explained in Example 2.9.

This article is not intended as a systematic study of Modular Theory for von Neumann algebras. Rather, the emphasis lies on the examples. The hope is that the reader will recognize through the examples some of the power, beauty and variety of applications of Modular Theory. We have also included a few exercises to motivate further thoughts on this topic. Additional aspects and applications of this theory will also appear in [16].

In the present article we will present Modular Theory in the special case when the von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ have a common cyclic vector $\Omega$. This approach avoids introducing too much notation and is enough for almost all examples and applications presented in this school. (An exception to this is Example 2.9.) The reader interested in the more general context described in terms of Hilbert algebras is referred to [38, Chapter VI] or [35, 32].

2. Modular Theory: definitions, results and first examples

In Modular Theory one studies systematically the relation of a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ in the case where both algebras have a common cyclic vector $\Omega$. We begin introducing some standard terminology and stating some elementary results:

**Definition 2.1.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{h}$. A vector $\Omega \in \mathfrak{h}$ is called cyclic for $\mathcal{M}$ if the set $\{M\Omega \mid M \in \mathcal{M}\}$ is dense in $\mathfrak{h}$. We say that $\Omega \in \mathfrak{h}$ is separating for $\mathcal{M}$ if for any $M \in \mathcal{M}$, $M \Omega = 0$ implies $M = 0$.

**Proposition 2.2.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{h}$ and $\Omega \in \mathfrak{h}$. Then $\Omega$ is cyclic for $\mathcal{M}$ iff $\Omega$ is separating for $\mathcal{M}'$.

**Proof.** Assume that $\Omega$ is cyclic for $\mathcal{M}$ and take $M' \in \mathcal{M}'$ such that $M'\Omega = 0$. Then $M'M\Omega = MM'\Omega = 0$ for all $M \in \mathcal{M}$. Since $\{M\Omega \mid M \in \mathcal{M}\}$ is dense in $\mathfrak{h}$ it follows that $M' = 0$.

Assume that $\Omega$ is separating for $\mathcal{M}$ and denote by $P'$ the orthogonal projection onto the closed subspace generated by $\{M\Omega \mid M \in \mathcal{M}\}$. Then $P' \in \mathcal{M}'$ and $(1 - P')\Omega = \Omega - \Omega = 0$. Since $\Omega$ is separating for $\mathcal{M}'$ we have that $P' = 1$, hence $\Omega$ is also cyclic for $\mathcal{M}$.

If $\Omega \in \mathfrak{h}$ is cyclic for the von Neumann algebras $\mathcal{M}$ and its commutant $\mathcal{M}'$ (hence also separating for both algebras by the preceding proposition) one can naturally introduce the following two antilinear operators $S_0$ and
$F_0$ on $\mathfrak{h}$:

\[
\begin{align*}
S_0(M\Omega) &:= M^*\Omega, \quad M \in \mathcal{M} \\
F_0(M\Omega) &:= (M')^*\Omega, \quad M' \in \mathcal{M}'.
\end{align*}
\]

Both operators are well defined on the dense domains $\text{dom} S_0 = M\Omega$ and $\text{dom} F_0 = M'\Omega$, respectively, and have dense images. It can be shown that the operators $S_0$ and $F_0$ are closable and that $S = F_0^*$ as well as $F = S_0^*$, where $S$ and $F$ denote the closures of $S_0$ and $F_0$, respectively. The closed, antilinear operator $S$ is called the Tomita operator for the pair $(\mathcal{M}, \Omega)$, where $\Omega$ is cyclic and separating for $\mathcal{M}$. The operators $S$ and $F$ are involutions in the sense that if $\xi \in \text{dom} S$, then $S\xi \in \text{dom} S$ and $S^2\xi = \xi$ (similarly for $F$).

Let $\Delta$ be the unique positive, selfadjoint operator and $J$ the unique antiunitary operator occurring in the polar decomposition of $S$, i.e.

\[ S = J\Delta^{\frac{1}{2}}. \]

We call $\Delta$ the modular operator and $J$ the modular conjugation associated with the pair $(\mathcal{M}, \Omega)$.

We mention next standard relations between the previously defined modular objects $S, F, \Delta$ and $J$. For a complete proof see Proposition 2.5.11 in [1].

**Proposition 2.3.** The following relations hold

\[
\begin{align*}
\Delta &= FS, \quad \Delta^{-1} = SF, \quad F = J\Delta^{-\frac{1}{2}} \\
J &= J^*, \quad J^2 = 1, \quad \Delta^{-\frac{1}{2}} = J\Delta^{\frac{1}{2}}J.
\end{align*}
\]

We conclude stating the main result of modular theory, the so-called Tomita-Takesaki theorem. We will give a proof only for the case where all modular objects are bounded. This situation covers Examples 2.8 and 2.10 as well as the results in Section 3 below. To state the theorem we need to introduce the following notation: given the modular operator $\Delta$, we construct the strongly continuous unitary group

\[ \Delta^it = \exp \left( it (\ln \Delta) \right), \quad t \in \mathbb{R}, \]

via the functional calculus. It is called the modular group and

\[ \sigma_t(M) := \Delta^it M \Delta^{-it}, \quad M \in \mathcal{M}, \quad t \in \mathbb{R} \]

gives a one parameter automorphism group on $\mathcal{M}$, the so-called modular automorphism group.

**Theorem 2.4.** (Tomita-Takesaki) Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector $\Omega$ in the Hilbert space $\mathfrak{h}$. The operators $\Delta$ and $J$ are the corresponding modular operator and modular conjugation, respectively, and denote by $\sigma_t, \ t \in \mathbb{R}$, the modular automorphism group. Then we have

\[ J\mathcal{M}J = \mathcal{M}' \quad \text{and} \quad \sigma_t(\mathcal{M}) = \mathcal{M}, \quad t \in \mathbb{R}. \]
Proof. We will prove this result assuming that the Tomita $S$ operator is bounded. This implies that $S^*$, $\Delta = S^* S$ and $\Delta^{-1} = SS^*$ are also bounded.

i) We show first $S M S = {\mathcal M}'$. For any $M,M_0 \in {\mathcal M}$ and using $S\Omega = \Omega$ we have

$$(1) \quad SM_0 S M\Omega = SM_0 M^* \Omega = M M_0^* \Omega = M S(M_0 \Omega) = M S M_0 S \Omega$$

or, equivalently, $(SM_0 S M\Omega - M S M_0 S)\Omega = 0$. Putting $M = M_1 M_2$ and using again Eq. (1) we get

$$(SM_0 S M_1 M_2 - M_1 M_2 SM_0 S) \Omega = 0 \quad \text{for all } M_2 \in {\mathcal M},$$

$$(SM_0 S M_1 - M_1 SM_0 S) M_2 \Omega = 0 \quad \text{for all } M_2 \in {\mathcal M}.$$  

Since $\{M\Omega \mid M \in {\mathcal M}\}$ is dense in $\mathfrak{h}$ it follows that $S M S \subseteq {\mathcal M}'$. Similarly it can be shown that $S^* {\mathcal M}' S^* \subseteq {\mathcal M}'' = {\mathcal M}$. Taking in the last inclusion adjoints and multiplying both sides with $S$ we get the reverse inclusion ${\mathcal M}' \subseteq S M S$, hence

$$S M S = {\mathcal M}' \quad \text{and} \quad S^* {\mathcal M}' S^* = {\mathcal M}.$$  

ii) Next we prove the following statement: $\Delta^z M \Delta^{-z} = {\mathcal M}, \ z \in \mathbb{C}$. It implies the required equation taking $z = it$. By the preceding item and using $\Delta = S^* S$, $\Delta^{-1} = SS^*$ we get

$$(2) \quad \Delta M \Delta^{-1} = S^* S M S S^* = {\mathcal M}, \quad \text{hence} \quad \Delta^n M \Delta^{-n} = {\mathcal M}, \ n \in \mathbb{N}.$$  

Since $\Delta$ is bounded, selfadjoint and invertible we may apply the spectral theorem and functional calculus to obtain the following representations

$$\Delta = \int_\varepsilon^\infty \lambda dE(\lambda) \quad \text{and} \quad \Delta^z = \int_\varepsilon^\infty \lambda^z dE(\lambda), \quad z \in \mathbb{C},$$

for some positive $\varepsilon$. It can be shown that for any $M \in {\mathcal M}$, $M' \in {\mathcal M}'$ and $\xi, \eta \in \mathfrak{h}$ the function defined by

$$f(z) = \|\Delta^z\|^{-2z} \langle [\Delta^z M \Delta^{-z}, M'] \xi, \eta \rangle$$

is entire. Moreover, it is also bounded since

$$|f(z)| \leq \|\Delta\|^{-2 \Re(z)} 2 \|\Delta^z\| \|\Delta^{-z}\| \|M'\| \|\xi\| \|\eta\| \leq 2 \|M\| \|M'\| \|\xi\| \|\eta\|,$$

where the last inequality holds if $\Re(z) \geq 0$ and we have used $\|\Delta^z\| = \|\Delta^{-z}\|$. Altogether, we have constructed an entire function $f$ which is bounded if $\Re(z) \geq 0$ and by Eq. (2) satisfies $f(n) = 0$, $n \in \mathbb{N}$. Therefore $f(z) = 0$ for all $z \in \mathbb{C}$ and

$$(3) \quad \Delta^z M \Delta^{-z} \subseteq {\mathcal M}'', \quad z \in \mathbb{C}.$$  

Multiplying the last inclusion from the left by $\Delta^{-z}$ and from the right by $\Delta^z$ and using (3) changing $z$ by $-z$ we obtain $\Delta^z M \Delta^{-z} = {\mathcal M}$.

iii) Finally we show the relation involving the modular conjugation. Using the preceding step (ii) we get

$$J M J = J \Delta^z M \Delta^{-z} J = S M S^* = {\mathcal M}'$$  

\footnote{This proof is close to the one given in [33, p. 48-49]. See also [33].}
2.1. Comments and elementary consequences of the Tomita-Takesaki theorem. One can recognize in the first part of the preceding theorem the interplay between algebraic and analytic structures (cf. \[25\]). In fact, the commutant of a von Neumann algebra is obtained by conjugation with an analytic object like $J$, which is obtained in terms of the polar decomposition of an antilinear closed operator.

Remark 2.5. (i) There are various approaches to a complete proof of Theorem 2.4: one of them stresses more the analytic aspects and techniques from the theory of unbounded operators; another one emphasizes more the algebraic structure (cf. \[21\] Chapter 9). In \[31\] Rieffel and van Daele present a different proof based on projection techniques and bounded operators. This approach is justified by the fact that the main ingredients of the theorem, namely the modular conjugation $J$ and the modular group $\Delta^it$ can be characterized in terms of real subspaces which have suitable relative positions within the underlying complex Hilbert space.

(ii) An immediate application of the preceding theorem is that the modular conjugation $J: \mathfrak{h} \to \mathfrak{h}$ is a $*$-anti-isomorphism between $\mathcal{M}$ and its commutant $\mathcal{M}'$.

(iii) Assume that the cyclic and separating vector $\Omega$ for the von Neumann algebra has norm 1. Then the Tomita operator $S$ measures to what extent the corresponding vector state $\omega$ on $\mathcal{M}$ defined by

\begin{equation}
\omega(M) := \langle \Omega, M\Omega \rangle, \quad M \in \mathcal{M}
\end{equation}

is tracial. In fact, note that $S$ is an isometry iff $\omega$ is a trace, since

$$
\|M\Omega\|^2 = \omega(M^*M) = \omega(MM^*) = \|M^*\Omega\|^2 = \|S(M\Omega)\|^2.
$$

The vector state $\omega$ associated to $\Omega$ is a faithful normal state. Conversely, to any faithful normal state of $\mathcal{M}$ one can associate, via the GNS construction, a cyclic and separating vector in the GNS Hilbert space. Modular Theory may be extended to the situation of von Neumann algebras with faithful, normal and semifinite weights (see e.g. \[33\]).

The following proposition, which can be shown directly (recall the exercise proposed in \[25\] Subsection 2.2]), is an easy to prove if we use Theorem 2.4.

Proposition 2.6. Let $\mathcal{A} \subset \mathcal{L}(\mathfrak{h})$ be an Abelian von Neumann algebra with a cyclic vector $\Omega \in \mathfrak{h}$. Then $\mathcal{A}$ is maximal Abelian, i.e. $\mathcal{A} = \mathcal{A}'$.

Proof. Since the algebra $\mathcal{A}$ is Abelian we have $\mathcal{A} \subseteq \mathcal{A}'$ and any cyclic vector $\Omega$ for $\mathcal{A}$ will also be cyclic for $\mathcal{A}'$. Therefore we can apply Theorem 2.4 and the following chain of inclusions

$$
\mathcal{A} \subseteq \mathcal{A}' = J\mathcal{A}J \subseteq J\mathcal{A}'J \subseteq J(J\mathcal{A}J)J = \mathcal{A}.
$$

conclude the proof. \qed
Remark 2.7. One of the origins of Modular Theory can be traced back to the original work by Murray and von Neumann. A vector \( u \in \mathfrak{h} \) is called a trace vector for a von Neumann algebra \( \mathcal{M} \subset L(\mathfrak{h}) \) if
\[
\langle u, MNu \rangle = \langle u, NMu \rangle , \quad M, N \in \mathcal{M} .
\]
If \( \mathcal{M} \) has a cyclic trace vector \( u \), then for any \( M \in \mathcal{M} \) there is a unique \( M' \in \mathcal{M}' \) satisfying
\[
Mu = M'u .
\]
In this case we say that \( M \) and \( M' \) are reflections of one another about \( u \).

2.2. Examples. The construction of the modular objects given in the beginning of this section is rather involved. The modular operator \( \Delta \) and modular conjugation \( J \) appear as the components of the polar decomposition associated to the closure of the operator \( S_0 \) defined above. To gain some intuition on the modular objects it is useful to compute \( J \) and \( \Delta \) in concrete cases. We begin with some natural examples related to the representation theory of groups.

Example 2.8. We will see in this example that the corresponding modular objects are bounded. Let \( G \) be a discrete group and consider the Hilbert space \( \mathcal{H} := \ell^2(G) \) with orthonormal basis given by delta functions on the group \( \{ \delta_g(\cdot) \mid g \in G \} \). Define the left- resp. right regular (unitary) representations on \( \mathcal{H} \) as
\[
L(g_0)\delta_g := \delta_{g_0g} \text{ resp. } R(g_0)\delta_g := \delta_{gg_0^{-1}} , \quad g \in G .
\]
We introduce finally the von Neumann algebra generated by the left regular representation
\[
\mathcal{M} := \{ L(g) \mid g \in G \}'' \subset L(\mathcal{H}) .
\]
Using Eq. (5) it is immediate to verify that \( \{ R(g) \mid g \in G \} \subset \mathcal{M}' \) (hence \( \{ R(g) \mid g \in G \}'' \subseteq \mathcal{M}' \) and that \( \Omega := \delta_e \) is a cyclic vector for \( \mathcal{M} \) and \( \mathcal{M}' \).

To determine the Tomita operator it is enough to specify \( S \) on the canonical basis of \( \ell^2 \) and extend this action anti-linearly on the whole Hilbert space:
\[
S(\delta_g) = S \left( L(g)\delta_e \right) = L(g)S^*\delta_e = L(g^{-1})\delta_e = \delta_{g^{-1}} .
\]
This implies
\[
S^* = S , \quad \Delta = 1 \quad \text{and} \quad J = S .
\]
Moreover, it is also straightforward to check the relation between the left- and right regular representations in terms of \( J \):
\[
JL(g)J = R(g) , \quad g \in G .
\]
Finally, we can improve the inclusion \( \{R(g) \mid g \in \mathcal{G}\}'' \subseteq \mathcal{M}' \) mentioned above applying the first part of Theorem \ref{thm:main_theorem}. In fact, the commutant of \( \mathcal{M} \) is generated, precisely, by the right regular representation:

\[
\mathcal{M}' = \mathcal{J}\mathcal{M}\mathcal{J} = J\{L(g) \mid g \in \mathcal{G}\}'' J = \{JL(g)J \mid g \in \mathcal{G}\}'' = \{R(g) \mid g \in \mathcal{G}\}''.
\]

**Example 2.9.** Apparently one of the original motivations of Tomita for developing Modular Theory was the harmonic analysis of nonunimodular locally compact groups. For the following example the notion of Hilbert algebras is needed (see \cite{34} Sections 2.3 and 2.4 for details). Let \( \mathcal{G} \) be a locally compact group with left invariant Haar measure \( dg \) and modular function 

\[
\tilde{\Delta} : \mathcal{G} \rightarrow \mathbb{R}_+.
\]

(Recall that the modular function is a continuous group homomorphism that relates the left and right Haar integrals, cf. \cite{19} §15). As in the preceding example, the modular objects associated to the left regular representation on the Hilbert space \( L^2(\mathcal{G}, dg) \) are given in this case by

\[
(S\varphi)(g) = \tilde{\Delta}(g)^{-1} \varphi(g^{-1}), \quad \varphi \in L^2(\mathcal{G}, dg)
\]

\[
(J\varphi)(g) = \tilde{\Delta}(g)^{-\frac{1}{2}} \varphi(g^{-1}), \quad \varphi \in L^2(\mathcal{G}, dg)
\]

\[
(\Delta\varphi)(g) = \tilde{\Delta}(g) \varphi(g), \quad \varphi \in L^2(\mathcal{G}, dg).
\]

This example shows the origin of the name Modular Theory, since the modular operator is just multiplication by the modular function of the group. Moreover, the preceding expressions of the modular objects are in accordance with the preceding example. Recall that if \( \mathcal{G} \) is discrete, then it is also unimodular, i.e. \( \Delta(g) = 1, g \in \mathcal{G} \).

**Example 2.10.** Let \( \mathcal{H}, \mathcal{H}' \) be finite dimensional Hilbert spaces with \( \dim \mathcal{H} = \dim \mathcal{H}' = n \) and orthonormal basis \( \{e_k\}_{k=1}^n \) and \( \{e'_k\}_{k=1}^n \), respectively. Consider on the tensor product \( \mathcal{H} \otimes \mathcal{H}' \) the von Neumann algebra \( \mathcal{M} = \mathcal{L}(\mathcal{H}) \otimes \mathbb{C}1_{\mathcal{H}'} \). It is easy to verify that the vector

\[
\Omega := \sum_k \lambda_k (e_k \otimes e'_k) \in \mathcal{H} \otimes \mathcal{H}', \quad \text{with } \lambda_k > 0, \ k = 1, \ldots, n, \text{ and } \sum_{k=1}^n \lambda_k^2 = 1,
\]

is a cyclic and separating vector for \( \mathcal{M} \) with norm 1.

A direct computation shows that the modular objects for the pair \( (\mathcal{M}, \Omega) \) are given by

\[
S(e_k \otimes e'_s) = \frac{\lambda_k}{\lambda_s} (e_s \otimes e'_k), \quad k, s \in \{1, \ldots, n\}.
\]

\[
S^*(e_k \otimes e'_s) = \frac{\lambda_s}{\lambda_k} (e_s \otimes e'_k), \quad k, s \in \{1, \ldots, n\}.
\]

\[
\Delta(e_k \otimes e'_s) = \left(\frac{\lambda_k}{\lambda_s}\right)^2 (e_k \otimes e'_s).
\]

\[
J(e_k \otimes e'_s) = (e_s \otimes e'_k).
\]
In this example the modular conjugation \( J \) acts as a flip of indices for the given basis of the tensor product Hilbert space. This action extends anti-linearly to the whole tensor product \( \mathcal{H} \otimes \mathcal{H}' \). Using Theorem 2.4 and the explicit expression for \( J \) we can again improve the inclusion \( \mathbb{C}1 \otimes \mathcal{L}(\mathcal{H}) \subseteq \mathcal{M}' \). In fact, note that for any \( A \in \mathcal{L}(\mathcal{H}) \) we have

\[
J (A \otimes 1) J = (1 \otimes \overline{A}) , \quad \text{where } (\overline{A})_{rs} = (A_{rs}).
\]

Now applying Theorem 2.4 we get

\[
\mathcal{M}' = J \mathcal{M} J = J (\mathcal{L}(\mathcal{H}) \otimes \mathbb{C}1) J = \mathbb{C}1 \otimes \mathcal{L}(\mathcal{H}) .
\]

**Exercise 2.11.** Generalize the preceding example to the case where \( \mathcal{H} \) is an infinite dimensional separable Hilbert space.

### 3. Modular Objects for a Crossed Product

The crossed product construction is a procedure to obtain a new von Neumann algebra out of a given von Neumann algebra which carries a certain group action. This principle, in particular the group measure space construction given below, goes back to the pioneering work of Murray and von Neumann on rings of operators (cf. [28, 29]). Standard references which present the crossed product construction with some variations are [34, Chapter 4], [37, Section V.7] or [21, Section 8.6 and Chapter 13].

Let \((\Omega, \Sigma, \mathbb{P})\) be a separable measure space with probability measure \(\mathbb{P}\) defined on Borel \(\sigma\)-algebra \(\Sigma\). Consider the Hilbert space \(\mathcal{H} := L^2(\Omega, \mathbb{P})\) and identify with \(\mathcal{M} := L^\infty(\Omega, \mathbb{P})\) the abelian von Neumann algebra that acts on \(\mathcal{H}\) by multiplication. Suppose that there is an infinite countable discrete group \(\Gamma\) acting on \((\Omega, \Sigma, \mathbb{P})\) by measure preserving automorphisms, i.e., \(T: \Gamma \to \text{Aut}(\Omega, \Sigma, \mathbb{P})\). This action induces a canonical action \(\alpha\) of \(\Gamma\) on the von Neumann algebra \(\mathcal{M}\):

\[
\alpha: \Gamma \to \text{Aut}\mathcal{M} \quad \text{with} \quad (\alpha_\gamma f)(\omega) := (f \circ T_{\gamma}^{-1})(\omega) , \quad f \in \mathcal{M}, \omega \in \Omega .
\]

**Definition 3.1.** Let \(\alpha: \Gamma \to \text{Aut}\mathcal{M}\) be an action of the discrete group \(\Gamma\) on \(\mathcal{M}\) as above.

(i) The action \(\alpha\) is called free if any \(\alpha_\gamma, \gamma \neq e\), satisfies the following implication: the equation \(fg = \alpha_\gamma(g)f\) for all \(g \in \mathcal{M}\) implies \(f = 0\).

(ii) The action \(\alpha\) is called *ergodic* if the corresponding fixed point algebra is trivial, i.e.

\[
\mathcal{M}^\alpha := \{ f \in \mathcal{M} | \alpha_\gamma(f) = f , \ \gamma \in \Gamma \} = \mathbb{C}1 .
\]

**Remark 3.2.** The preceding properties of a group action on the von Neumann algebra also translate into properties of the corresponding group action \(T: \Gamma \to \text{Aut}(\Omega, \Sigma, \mathbb{P})\) on the probability space.

---

\(^2\)An automorphism \(T\) of the measure space \((\Omega, \Sigma, \mathbb{P})\) is a bijection \(T: \Omega \to \Omega\) such that

(i) for \(S \in \Sigma\) we have \(T(S), T^{-1}(S) \in \Sigma\) and

(ii) if \(S \in \Sigma\), then \(\mu(S) = 0 \iff \mu(T^{-1}) = 0\).

The action \(T\) is measure preserving if \(\mu \circ T_\gamma = \mu\) for all \(\gamma \in \Gamma\).
(i) The action $T$ is called free if the sets $\{ \omega \in \Omega \mid T_{\gamma} \omega = \omega \}$ are of measure zero for all $\gamma \neq e$.
(ii) The action $T$ is called ergodic if for any $S \in \Sigma$ such that
$$\mu ((T_{\gamma}(S) \backslash S) \cup (S \backslash T_{\gamma}(S))) = 0,$$
for all $\gamma \in \Gamma$,
one has either $\mu(S) = 0$ or $\mu(X \backslash S) = 0$.

The crossed product is a new von Neumann algebra $\mathcal{N}$ which can be constructed from the dynamical system $(\mathcal{M}, \alpha, \Gamma)$. It contains a copy of $\mathcal{M}$ and a copy of the discrete group $\Gamma$ by unitary elements in $\mathcal{N}$ and the commutation relations between both ingredients are given by the group action. We will describe this construction within the group measure space context specified above. We begin introducing a new Hilbert space on which the crossed product will act:

$$\mathcal{K} := \ell_2(\Gamma) \otimes \mathcal{H} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma} \cong \int_{\Omega} \ell^2(\Gamma) d\mathbb{P},$$

where $\mathcal{H}_{\gamma} \equiv \mathcal{H} = L^2(\Omega, \mathbb{P})$ for all $\gamma \in \Gamma$. Moreover, we consider the following representations of $\mathcal{M} = L^\infty(\Omega, \mathbb{P})$ and $\Gamma$ on $\mathcal{K}$: for $\xi = (\xi_\gamma)_{\gamma \in \Gamma} \in \mathcal{K}$ with $\xi_\gamma \in \mathcal{H}$ we define

$$\pi(f)\xi_\gamma := \alpha_{\gamma}^{-1}(f)\xi_\gamma = f(T_{\gamma} \cdot)\xi_\gamma, \quad f \in \mathcal{M},$$

$$U(\gamma_0)\xi_\gamma := \xi_{\gamma_0^{-1} \gamma}.$$ 

The discrete crossed product of $\mathcal{M}$ by $\Gamma$ is the von Neumann algebra acting on $\mathcal{K}$ and generated by these operators, i.e.,

$$\mathcal{N} = \mathcal{M} \otimes_\alpha \Gamma := \left( \{ \pi(f) \mid f \in \mathcal{M} \} \cup \{ U(\gamma) \mid \gamma \in \Gamma \} \right)'' \subset \mathcal{L}(\mathcal{K}),$$

where the prime denotes the commutant in $\mathcal{L}(\mathcal{K})$. A characteristic relation for the crossed product is

$$\pi(\alpha_{\gamma}(f)) = U(\gamma)\pi(f)U(\gamma)^{-1}.$$ 

In other words, $\pi$ is a covariant representation of the $W^*$-dynamical system $(\mathcal{M}, \Gamma, \alpha)$.

Remark 3.3. It is useful to characterize explicitly all elements in the crossed product and not just a generating family. For this consider the identification $\mathcal{K} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma}$ with $\mathcal{H}_{\gamma} \equiv \mathcal{H}$. Then every $T \in \mathcal{L}(\mathcal{K})$ can be written as an infinite matrix $(T_{\gamma'})_{\gamma', \gamma \in \Gamma}$ with entries $T_{\gamma' \gamma} \in \mathcal{L}(\mathcal{H})$. Any element $N$ in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ has the form

$$N_{\gamma' \gamma} = \alpha_{\gamma'}^{-1}(A(\gamma' \gamma^{-1})), \quad \gamma', \gamma \in \Gamma,$$

for some function $A: \Gamma \to \mathcal{M} \subset \mathcal{L}(\mathcal{H})$. Since any $N \in \mathcal{N}$ is a bounded operator in $\mathcal{K}$ it follows that

$$\sum_{\gamma \in \Gamma} \|A(\gamma)\|_{\mathcal{H}}^2 < \infty.$$
For example, the matrix expression of the product of generators $N := π(g) · U(γ_0)$, $g ∈ M$, $γ_0 ∈ Γ$, is given by

\[ N_{γ′,γ} = α_γ^{-1}(g) δ_{γ′,γ_0γ} = α_γ^{-1} (A(γ′γ^{-1})) , \]

where $A(γ) := \begin{cases} g & \text{if } ⅃ = γ_0 \\ 0 & \text{otherwise} \end{cases}$.

With the group measure space construction one can produce examples of von Neumann algebras of any type. In this section we concentrate on the case of finite factors. (For a general statement see Theorem 6.2.)

**Theorem 3.4.** If the action of the discrete group $Γ$ on the probability space $(Ω, Σ, P)$ is measure preserving, free and ergodic, then the crossed product $N$ constructed before is a finite factor, i.e. a factor of type $I_1$ or of type $II_1$.

Next we specify a cyclic and separating vector for the crossed product $N$ defined previously.

**Lemma 3.5.** The vector $Ω ∈ K$ defined by $(Ω)_{γ} = δ_{eγ, 1}$, where $1 ∈ H$ is the identity function, is cyclic and separating for the finite factor $N$.

**Proof.** 1. We show first that $Ω$ is separating for $N$. Let $NΩ = 0$ for some $N ∈ N$. Then by the matrix form of the elements of $N$ mentioned in Remark 3.3 we have

\[ 0 = (NΩ)_{γ} = A(γ)1 , \quad \text{for all } γ ∈ Γ. \]

This implies $A = 0$, hence $N = 0$.

2. To show that $Ω$ is cyclic note that it is enough to verify that the set

\[ D := \{ g = (g_γ)_{γ} \mid g_γ ∈ M \text{ and finitely many } g_γ ≠ 0 \} \]

is dense in $K$. In fact, using Eq. (12) we have

\[ D = \text{span} \{ π(g) · U(γ) Ω \mid g ∈ M , γ ∈ Γ \} ⊂ \{ NΩ \mid N ∈ N \}. \]

Now, for any $φ = (φ_γ)_{γ} ∈ K$ and any $ε > 0$ choose a subset $Γ_0 ⊂ Γ$ with finite cardinality, i.e. $|Γ_0| < ∞$, such that

\[ \sum_{γ ∈ (Γ_0)^c} \|φ_γ\|^2_H < \frac{ε^2}{2}. \]

Since $M = L^∞(Ω, P)$ is dense in $H = L^2(Ω, P)$ choose an element $g = (g_γ)_{γ} ∈ D$ such that $g_γ = 0$ for $γ ∈ (Γ_0)^c$ and

\[ \|φ_γ − g_γ\|_H ≤ \frac{ε^2}{2|Γ_0|} \quad \text{for } γ ∈ Γ_0. \]

Then we have

\[ \|φ − g\|_K^2 = \sum_{γ ∈ Γ_0} \|φ_γ − g_γ\|_H^2 + \sum_{γ ∈ (Γ_0)^c} \|φ_γ\|_H^2 ≤ |Γ_0| \cdot \frac{ε^2}{2|Γ_0|} + \frac{ε^2}{2} = ε^2. \]

This shows that $D$ is dense in $K$ and the proof is concluded. \(\square\)
Remark 3.6. In the proof of the cyclicity of $\Omega$ in the preceding lemma it is crucial that $L^\infty(\Omega, \mathbb{P})$ is dense in $L^2(\Omega, \mathbb{P})$ or, equivalently, $1 \in L^2(\Omega, \mathbb{P})$ is cyclic with respect to $L^\infty(\Omega, \mathbb{P})$. This is true because we are working with a finite measure space.

We finish this section specifying the Modular objects for the pair $(N, \Omega)$.

**Theorem 3.7.** The Tomita operator associated to the pair $(N, \Omega)$ given in Theorem 3.4 and Lemma 3.5 is an isometry, hence $\Delta = 1$ and $S = J$. An explicit expression for the Tomita operator is given on the dense set

\[
\{N\Omega \mid N \in \mathcal{N}\} = \{g = (A(\gamma))_\gamma \mid \text{for some mapping } A: \Gamma \to \mathcal{M}\} \subset K
\]

by

\[
S_{\gamma'\gamma} = \delta_{\gamma'\gamma}^{-1} \alpha_\gamma(C(\cdot)),
\]

where $C$ means complex conjugation in $\mathcal{M} = L^\infty(\Omega, \mathbb{P})$. The vector state associated to $\Omega$ and defined by

\[
\omega(M) := \langle \Omega, M\Omega \rangle, \quad M \in \mathcal{M}.
\]

is a trace on the finite factor $N$.

**Proof.** Note first that Eq. (13) is an immediate consequence of the characterization of the matrix elements of the crossed product given in Eq. (10),

\[
N_{\gamma'\gamma} = \alpha_\gamma^{-1} (A(\gamma'\gamma^{-1})),
\]

for some function $A: \Gamma \to \mathcal{M}$. Next we verify that the matrix elements $S_{\gamma'\gamma}$ given above correspond to the Tomita operator on the subset $\{N\Omega \mid N \in \mathcal{N}\}$ (which is dense in $K$ by Lemma 3.5):

\[
(S(N\Omega))_\gamma = \sum_{\gamma'} S_{\gamma'\gamma} A(\gamma') = \sum_{\gamma'} \delta_{\gamma'\gamma}^{-1} \alpha_{\gamma'} \left(\overline{A(\gamma')}\right) = \alpha_\gamma^{-1} \left(\overline{A(\gamma^{-1})}\right)
\]

\[= N_{\gamma'\gamma}^* \delta_{\gamma'\gamma} 1 = \sum_{\gamma} (N_{\gamma'\gamma})_{\gamma'\gamma} \Omega_{\gamma'} = (N^*\Omega)_\gamma.
\]

Moreover, for any $\xi = N\Omega = (A(\gamma))_\gamma$ we have

\[
\|S\xi\|_K^2 = \sum_{\gamma} \|(S\xi)_\gamma\|_H^2 = \sum_{\gamma} \|\alpha_{\gamma}^{-1} \left(\overline{A(\gamma^{-1})}\right)\|_H^2 = \sum_{\gamma} \|(A(\gamma))\|_H^2 = \|\xi\|_K^2.
\]

This shows that the Tomita operator $S$ is an isometry on the dense subspace $\{N\Omega \mid N \in \mathcal{N}\}$, hence it extends uniquely to an isometry on the whole Hilbert space $K$. Therefore $\Delta = S^*S = 1$ and $S = J$. By Remark 2.5 (iii) it follows that the vector state associated to the cyclic and separating vector $\Omega$ is a trace.

**Remark 3.8.** For an expression of the modular operator on general crossed products which are not necessarily finite factors see [34, §4.2].
4. Modular objects for the CAR-algebra

In this section we construct the modular objects for the algebra of the canonical anticommutation relations (CAR-algebra). This is a more elaborate example that uses standard results on the CAR-algebra and its irreducible representations (Fock representations). We give a short review of these results in Subsection 6.2. Let \((\mathfrak{h}, \Gamma)\) be a reference space as in Theorem 6.3 and denote by \(q\) a closed \(\Gamma\)-invariant subspace of \(\mathfrak{h}\). The orthogonal projection associated with \(q\) is denoted by \(Q\). We can naturally associate with the subspace \(q\) a von Neumann algebra that acts on the antisymmetric Fock space \(\mathfrak{F}\) characterized by the basis projection \(P\) (cf. (20)):

\[
\mathcal{M}(q) := \left( \{ a(q) \mid q \in q \} \right)^{\prime\prime} \subset \mathcal{L}(\mathfrak{F}).
\]

In the present subsection we will analyze the modular objects corresponding to the pair \( (\mathcal{M}(q), \Omega) \), where \(\Omega\) is the so-called Fock vacuum in \(\mathfrak{F}\). More details and applications of the Modular Theory in the context of the CAR-algebra can be found in [7] and [24, Part I].

4.1. Two subspaces in generic position. We will address next the question when the Fock vacuum \(\Omega\) is cyclic and separating for the von Neumann algebra \(\mathcal{M}(q)\). The answer to this question has to do with the relative position that the subspace \(q\) has with the one particle space \(p := P\mathfrak{h}\).

**Proposition 4.1.** Let \(q\) be a closed \(\Gamma\)-invariant subspace of \(\mathfrak{h}\) as before and denote the one-particle Hilbert space associated to the basis projection \(P\) by \(p\). Then we have:

(i) The vacuum vector \(\Omega\) is cyclic for \(\mathcal{M}(q)\) iff \(p \cap q^\perp = \{0\}\).

(ii) The vacuum vector \(\Omega\) is separating for \(\mathcal{M}(q)\) iff \(p \cap q = \{0\}\).

**Proof.** We will only proof part (i). Similar arguments can be used to show (ii).

Assume that \(\Omega\) is cyclic for \(\mathcal{M}(q)\) and let \(p \in p\) be a vector satisfying \(p \perp Pq\). From Proposition 6.7 and from the structure of the Fock space \(\mathfrak{F}\) (recall Eq. (20)) we have

\[
p \perp \text{span} \{ a(q_1) \cdot \ldots \cdot a(q_n) \Omega \mid q_1, \ldots, q_n \in q, n \in \mathbb{N} \}, \quad \text{thus}
\]

\[
p \perp \{ A\Omega \mid A \in \mathcal{M}(q) \}.
\]

Now since \(\Omega\) is cyclic for \(\mathcal{M}(q)\) we must have \(p = 0\). This shows that \(Pq\) is dense in \(p\) which is equivalent to \(p \cap q^\perp = \{0\}\).

To show the reverse implication assume that \(p \cap q^\perp = \{0\}\). Then \(Pq\) is dense in \(p\) and, consequently, the algebraic direct sum \(\bigoplus_{n=0}^{\infty} \left( \wedge^n Pq \right)\) is also dense in the antisymmetric Fock space \(\mathfrak{F}\). From Proposition 6.7 we obtain the inclusions

\[
\bigoplus_{n=0}^{\infty} \left( \wedge^n Pq \right) \subset \mathcal{M}(q) \Omega \subset \mathfrak{F},
\]

which imply that \(\Omega\) is cyclic for \(\mathcal{M}(q)\). \(\square\)
Let $P$ and $Q$ be the orthoprojections corresponding to the subspaces $p$ and $q$ and satisfying the usual relations w.r.t. the antiunitary involution $\Gamma$:

$$\Gamma P \Gamma = 1 - P = P^\perp$$

and

$$\Gamma Q = \Gamma Q.$$ 

Proposition 4.1 above says that a necessary and sufficient condition for doing Modular Theory with the pair $(M(q), \Omega)$ is that

$$(15) \quad p \cap q = \{0\} = p \cap q^\perp.$$ 

Using the fact that $\Gamma p = p^\perp$ we obtain, in addition,

$$(16) \quad p^\perp \cap q = \{0\} = p^\perp \cap q^\perp,$$

where $p^\perp = \perp h$.

According to Halmos terminology (cf. [18]) if (15) and (16) hold, then the subspaces $p$ and $q$ are said to be in generic position. In other words the maximal subspace where $P$ and $Q$ commute is $\{0\}$. This is, in fact, a very rich mathematical situation. For example, the following useful density statements are immediate consequences of the assumption that $p$ and $q$ are in generic position. If $r \subseteq q$ (or $r \subseteq q^\perp$) is a dense linear submanifold in $q$ (respectively in $q^\perp$), then $Pr$ is dense in $p$ and $P^\perp r$ is dense in $p^\perp$. The same holds if $Q$ and $P$ are interchanged. In particular, we have that $Qp^\perp$ is dense in $q$, $PQp$ is dense in $p$ etc. Moreover, in the generic position situation the mapping

$$(17) \quad P: q \rightarrow p$$

is a bounded injective linear mapping with dense image. (Similar results hold if we interchange $q$ with $q^\perp$, $P$ with $Q$ or $Q^\perp$ etc.)

Exercise 4.2. Let $q \subset h$ be a closed $\Gamma$-invariant subspace. Show that the following conditions are equivalent:

(i) $q \in q$ and $Pq = 0$ implies $q = 0$.

(ii) $P(q^\perp)$ is a dense submanifold of $p$.

(iii) $q \cap p = \{0\}$.

Exercise 4.3. Show that for any pair $P, Q$ of orthogonal projections in $L(\mathcal{H})$ one has $\|P - Q\| \leq 1$. [Hint: Consider the operators $A := P - Q$, $B := 1 - P - Q$ and the equation $A^2 + B^2 = 1$.]

Next we give a criterion for the bicontinuity of the mapping (17). First, note that because $P$ is a basis projection we already have

$$\|PQ\| = \|QP\| = \|(1 - P)Q\| = \|Q(1 - P)\| = : \delta$$

and $0 < \delta \leq 1$. So we can distinguish between the two cases: $\delta < 1$ and $\delta = 1$.

Proposition 4.4. Let $P, Q$ and $\delta$ given as before.

(i) If $\delta < 1$, then the mapping (17) is bicontinuous. In particular, the image coincides with $p$. Moreover, the relations

$$\|P - Q\| = \|(1 - Q)P\| = \|(1 - Q)(1 - P)\| = \delta$$

hold.
(ii) If $\delta = 1$, then the inverse mapping of (17) is unbounded and densely defined, i.e. the image of (17) is nontrivial proper dense set in $p$.

Proof. (i) This result is a special case of Theorem 6.34 in \[22, p. 56\]. Note that the second alternative stated in Kato’s result cannot appear in the present situation, as a consequence of the fact that $p$ and $q$ are in generic position.

(ii) We will only show the assertion for the mapping (17), since one can easily adapt the following arguments to the other cases. Put $A := QP^\perp Q | q \in L(q)$, so that $A = A^*$ and $A \geq 0$. From

$$\text{spr } A = \|A\| = \|QP^\perp P^\perp Q\| = \|P^\perp Q\|^2 = \delta^2 = 1$$

we obtain $1 \in \text{sp } A$. However, 1 is not an eigenvalue of $A$, because $Aq = q$, $q \in q$, implies $s - \lim_{n\to\infty} (QP)^n q = q$ and this means $q \in q \cap p^\perp = \{0\}$. Thus $\ker (1_q - A) = \{0\}$ or $(1_q - A)^{-1}$ exists and is unbounded since $1 \notin \text{res } A$. Therefore $\vartheta := \text{dom } (1_q - A)^{-1}$ is a proper dense subset in $q$ and this means $\text{ima } (1_q - A) = \vartheta = \text{ima } (Q - QP^\perp Q) = \text{ima } (QPQ)$. Finally, from the polar decomposition of $PQ$,

$$PQ = \text{sgn } (PQ) \cdot (QPQ)^{\frac{1}{2}},$$

we have that the partial isometry $\text{sgn } (PQ)$ maps $\text{ima } (QPQ)^{\frac{1}{2}}$ isometrically onto $\text{ima } (PQ) = Pq$. Thus $Pq$ is a proper dense set in $p$, i.e. $P: q \to p$ is unbounded invertible. $\square$

Remark 4.5. The situation in Proposition 4.4 (i) corresponds to the case where the index of $P$ and $Q$ is 0 (cf. [5, Theorem 3.3]).

Example 4.6. As we have seen in Propositions 4.4 there are two characteristic situations when the subspaces $q$ and $p$ are in generic position. First, when $\|PQ\| < 1$. This case may be realized when $h$ has finite dimension. Second, when $\|PQ\| = 1$. This condition implies that the reference space is infinite dimensional. We will give here two simple examples for both situations.

(i) The case $\|PQ\| < 1$:

Put $h := \mathbb{C}^2$ and $\Gamma(\alpha, \beta) := (\overline{\beta}, \overline{\alpha})$, $(\alpha, \beta) \in \mathbb{C}^2$. The generators of $\text{CAR}(\mathbb{C}^2, \Gamma)$ are simply given by

$$\mathbb{C}^2 \ni (\alpha, \beta) \mapsto a(\alpha, \beta) := \begin{pmatrix} 0 & \overline{\beta} \\ \overline{\alpha} & 0 \end{pmatrix}.$$

As a basis projection we take

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{hence } \quad p = \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which satisfies $\Gamma P \Gamma = P^\perp$. As invariant projection we choose

$$Q := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{hence } \quad q = \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the $\Gamma$-invariance condition $\Gamma Q \Gamma = Q$ is trivially satisfied.
In this example $p$ and $q$ are in generic position and it is straightforward to compute

$$\|PQ\| = \frac{1}{\sqrt{2}}.$$ 

(ii) The case $\|PQ\| = 1$:
Put $h := L^2(\mathbb{R})$ and $\Gamma f := f$, $f \in L^2(\mathbb{R})$. As invariant projection define

$$(Qf)(x) := \chi_+(x)f(x),$$

where $\chi_+$ is the characteristic function of the nonnegative real numbers $\mathbb{R}_+ = [0, \infty)$. The corresponding $\Gamma$-invariant projection space is $q = L^2(\mathbb{R}_+)$. To specify $P$ we consider first the following projection in momentum space

$$(\hat{P} \hat{f})(k) := \chi_+(k)\hat{f}(k),$$

where the Fourier transformation $F$ is defined as usual by

$$F(f)(k) = \hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) dx, \quad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Finally, the basis projection $P$ is given by

$$P := F\hat{P}F^{-1}.$$ 

The corresponding projection space is the Hardy space, i.e. $p = H^2(\mathbb{R})$, and $P$ satisfies $\Gamma P \Gamma = P\perp$. (For a brief introduction to Hardy spaces see [6]). Since by the theorem of Paley and Wiener the subspace $H^2_-(\mathbb{R})$ may be characterized in terms of holomorphic functions on the upper half plane, it is clear that $p$ and $q$ are in generic position. Using now the invariance of $H^2_-(\mathbb{R})$ under the regular representation $(U(a)f)(x) := f(x-a)$, $a \in \mathbb{R}$, it can be shown that

$$\|PQ\| = 1.$$ 

4.2. Modular objects for $(\mathcal{M}(q), \Omega)$: Let $(\mathcal{M}(q), \Omega)$ be as in the preceding subsection and assume that the subspace $q$ and $p$ are in generic position. We will compute in the present subsection the corresponding modular objects (recall Section 2). For this purpose it is enough to restrict the analysis to the one-particle Hilbert space $p$ of the Fock space $\mathcal{F}$.

Motivated by the following direct computation for the Tomita operator

$$S(Pq) = S(a(\Gamma q) \Omega) = a(\Gamma q)^* \Omega = a(q) \Omega = P\Gamma q,$$
we introduce the following antilinear mappings defined by the following graphs:

$${\text{gra}} \beta := \{(Pq, P\Gamma q) \in p \times p \mid q \in q\}$$

$${\text{gra}} \alpha := \{(Pq^\perp, -P\Gamma q^\perp) \in p \times p \mid q^\perp \in q^\perp\}.$$
(Note that the r.h.s. of the preceding equations define indeed graphs of antilinear mappings, because the assignments $q \to Pq$ and $q^\perp \to Pq^\perp$ are injective.)

The following result can be verified directly:

**Lemma 4.7.** The mappings $\alpha, \beta$ defined by the preceding graphs are antilinear, injective and closed with dense domains and images $\text{dom} \alpha = \text{ima} \alpha = P(q^\perp)$, $\text{dom} \beta = \text{ima} \beta = Pq$. Further, we have $\alpha^2 = \text{id}$, $\beta^2 = \text{id}$ on $P(q^\perp)$ resp. $Pq$ and $\alpha = \beta^*$. 

**Remark 4.8.** The Tomita operator $S$ associated to $\langle \mathcal{M}(q), \Omega \rangle$ satisfies $S \upharpoonright p \supseteq \beta$ and $S^* \upharpoonright p \supseteq \alpha$. Moreover, the mappings $\alpha, \beta$ are bicontinuous iff $\|PQ\| < 1$ (recall Proposition 4.4).

We introduce next the notation

$$\Delta_p := \beta^* \beta,$$

since it will later turn out that $\Delta_p$ is actually the modular operator restricted to the one-particle Hilbert space $p = P\mathfrak{h}$.

**Theorem 4.9.** The mapping $\Delta_p : p \to p$ is a densely defined linear positive self-adjoint operator on $p$ with graph

$$\text{gra} \Delta_p = \{(PQp, PQ^\perp p) \mid p \in p\}.$$

Moreover, $\Delta_p^{-1} = \beta^* = \alpha^* \alpha$. An expression for the modular conjugation is given by

$$J(Pq) = \Delta_p^{1/2} (P\Gamma q).$$

**Proof.** We compute first the domain of $\beta^* \beta$. Recalling that $\beta^* = \alpha$ we have

$$\text{dom} \ (\Delta_p) = \left\{Pq \mid q \in q \text{ and } P\Gamma q \in \text{dom} \alpha = P(q^\perp) \right\}$$

$$= \left\{Pq \mid q \in q \text{ and } P\Gamma q = Pq^\perp \text{ for some } q^\perp \in q^\perp \right\}$$

$$= \left\{Pq \mid q \in q \text{ and } q \in \Gamma Q(p^\perp) \right\}$$

$$= \left\{Pq \mid q \in q \text{ and } q \in \Gamma Q(p^\perp) = Q(\Gamma p^\perp) = Qp \right\}$$

$$= PQp = PQP\mathfrak{h}.$$

For the third equation note that $\Gamma q - q^\perp \in p^\perp$, hence $Q(\Gamma q) = \Gamma q \in Q(p^\perp)$. Since $P\Gamma Qp = -PQ^\perp \Gamma p$, $p \in p$ (recall $P\Gamma p = 0$, $p \in p$), we have

$$\Delta_p(PQp) = \alpha \left(P\Gamma Qp\right) = -\alpha \left(PQ^\perp \Gamma p\right) = P\Gamma Q^\perp \Gamma p = PQ^\perp p, \ p \in p.$$ 

Since $p$ and $q$ are in generic position the domain and image of $\Delta_p$ are dense in $p$. 


The last equations concerning the inverse of \( \Delta \) follow from the preceding computation and from the fact that \( \alpha^2 = \text{id} \) and \( \beta^2 = \text{id} \) on the corresponding domains (recall Lemma 4.7). Finally, for the expression of the modular conjugation use \( J = \Delta^{\frac{1}{4}} S \).

\[ \square \]

**Remark 4.10.**

(i) Since \( S = J \Delta^{1/2} \) we have from the preceding theorem the following inclusion of domains:

\[ \text{dom } \Delta^{1/2} = \text{dom } S = Pq \supset Pqp = \text{dom } \Delta. \]

In this example we can characterize precisely how the domain of the square root increases.

(ii) The present model is also useful to test many expressions that appear in general computation done in Modular Theory. For example, for certain calculations one needs to work with the dense set \( D := \text{dom } \Delta^{1/2} \cap \text{dom } \Delta^{-1/2} \). In the present example involving the CAR-algebra it is straightforward to verify that \( D = PQp^\perp \), which is in fact dense in \( p \) since the corresponding two subspaces are in generic position.

**Exercise 4.11.** Show that, in general, for a positive self-adjoint operator \( T \) in a Hilbert space one has the inclusion:

\[ \text{dom } T^{\frac{1}{2}} \supseteq \text{dom } T. \]

We conclude mentioning the behavior of the modular objects with respect to the direct sums that appear in the Fock space \( \mathcal{F} \) (cf. Eq. (FockSpace)). For a complete proof see \[7\]. Let \( (\mathcal{H}, \Gamma) \), \( P \) and \( Q \) be as before and denote by \( S = J \Delta^{\frac{1}{4}} \) the polar decomposition of the Tomita operator for the pair \( (\mathcal{M}(q), \Omega) \).

Note that the different modular objects leave the \( n \)-particle submanifolds \( \wedge \) invariant. (This fact is well known in the context of CCR-algebras \[23\], where one can use the so-called exponential vectors which are specially well-adapted to the Weyl operators.)

**Proposition 4.12.** Let \( q_1, \ldots, q_n \in q \) and \( q_1^\perp, \ldots, q_n^\perp \in q^\perp \). Then the following equations hold

\[
S(Pq_1 \wedge \ldots \wedge Pq_n) = P\Gamma q_n \wedge \ldots \wedge P\Gamma q_1 = S(Pq_n) \wedge \ldots \wedge S(Pq_1)
\]

\[
S^*(Pq_1^\perp \wedge \ldots \wedge Pq_n^\perp) = S^*(Pq_n^\perp) \wedge \ldots \wedge S^*(Pq_1^\perp)
\]

Moreover,

\[
\text{span} \left\{ a(q_1) \cdots a(q_n)\Omega \mid q_1, \ldots, q_n \in q, \ n \in \mathbb{N} \cup \{0\} \right\}
\]

is a core for the Tomita operator \( S \).

The following result together with Theorem \[12\] gives a complete picture of the modular objects in the context of CAR-algebras.
Theorem 4.13. Let \( (\mathcal{M}(\mathfrak{q}), \Omega) \) be as in the preceding subsection and assume that \( \mathfrak{q} \) and \( \mathfrak{p} \) are in generic position. Let \( S = J\Delta \bar{\Delta} \) be the polar decomposition of the Tomita operator. The modular operator \( \Delta = S^*S \) and the modular conjugation \( J \) can be restricted to the respective \( n \)-particle subspaces. In particular we have:

(i) **Modular operator:** We have \( \Delta \upharpoonright \mathfrak{p} = \Delta_{\mathfrak{p}} \), where \( \Delta_{\mathfrak{p}} = \beta^*\beta \), and

\[
\text{dom} \Delta \upharpoonright P_n \mathfrak{F} = \bigwedge \text{dom} \Delta_{\mathfrak{p}}.
\]
Moreover, the action on the \( n \)-particle vector is given by

\[
\Delta(p_1 \wedge \ldots \wedge p_n) = (\Delta_{\mathfrak{p}} p_1) \wedge \ldots \wedge (\Delta_{\mathfrak{p}} p_n), \quad p_1, \ldots, p_n \in \text{dom} \Delta_{\mathfrak{p}} = \mathfrak{P} Q \mathfrak{p}.
\]

(ii) **Modular conjugation:** Its action on the \( n \)-particle vector is given by

\[
J(p_1 \wedge \ldots \wedge p_n) = (Jp_n) \wedge \ldots \wedge (Jp_1), \quad p_1, \ldots, p_n \in \mathfrak{p}.
\]

Remark 4.14. The CAR-algebra is typically used to model Fermi systems in quantum physics, while bosonic systems are described in terms of the CCR-algebra. A formula for the modular operator was given for the (bosonic) free scalar field in [14]. In this paper the reference space is specified in terms of the Cauchy data of the Klein-Gordon operator and the formula for the modular operator on the one-particle Hilbert space reads

\[
\delta = \frac{B + 1}{B - 1},
\]

where the operator \( B \) is defined in terms of two other densely defined closed operators \( A_{\pm 1} \) and these are again defined using suitable idempotents \( P_{\pm 1} \) (see [14, p. 425] for details).

The simplicity of the formulas obtained in the context of the self-dual CAR-algebra (see e.g. Theorem 4.9) suggest that also for the bosonic models the self-dual approach to the CCR-algebra may be better adapted to problems concerning Modular Theory. In fact, in this case one can also characterize the Fock representations in terms of basis projections (cf. [4, 2, 30]). Therefore, it seems likely that similar simple formulas as the ones presented in this chapter also hold in the context of the CCR-algebra.

4.3. **Modular objects for double cones in Fermi models.** We mention finally that the formulas established previously also apply to the localized algebras that appear in the context of Fermi free nets (see e.g. [26, 27] and references therein). For more details on local quantum theories see Section 2 in [16]. Let \( \mathcal{O} \subset \mathbb{R}^4 \) be a double cone in Minkowski space and denote by \( \overline{\mathfrak{q}(\mathcal{O})} \) the closure of the subspaces \( \mathfrak{q}(\mathcal{O}) \) of the reference Hilbert space \( (\mathfrak{h}, \Gamma) \). The subspaces \( \mathfrak{q}(\mathcal{O}) \) are defined in terms of the embeddings that characterize the free nets (essentially Fourier transformation of \( C^\infty \) functions with compact support restricted to the positive mass shell/light cone). It is easily shown
that $\Gamma q(O) = q(O)$, hence $\Gamma q(O) = q(O)$. Moreover the localized C*-algebras are again CAR-algebras:

$$A(O) := C^\ast\{a(\varphi) \mid \varphi \in q(O)\} = \text{CAR}(q(O), \Gamma \upharpoonright q(O))$$

$$= \text{CAR}(q(O), \Gamma \upharpoonright q(O)) \subset \text{CAR}(\mathfrak{h}, \Gamma),$$

where for the last equation we have used Proposition 6.8. For the canonical basis projection $P$ given in the context of Fermi free nets (see e.g. [28, p. 1157]) and for double cones $O$ one has

$$\mathfrak{p} \cap \overline{q(O)} = \mathfrak{p} \cap \overline{q(O)}^\perp = \{0\}, \quad \text{where} \mathfrak{p} = P\mathfrak{h},$$

(see also Section 2 in [16] for similar relations in the case of Bose fields). Therefore $\mathfrak{p}$ and $q(O)$ are in generic position and we can apply the results and formulas of the present section to the von Neumann algebras in a Fock representation specified by $P$ and localized in double cones $O$:

$$\mathcal{M}(O) := \left\{a(\varphi) \mid \varphi \in q(O)\right\}''.$$
5.2. **Structure of type III factors.** The technically more tractable cases mentioned previously in Examples 2.8 and 2.10, as well as in Section 3, have in common that the corresponding von Neumann algebra $\mathcal{M}$ is finite or, equivalently, that the vector state associated with the cyclic and separating vector is a trace. In order to treat infinite algebras one has to consider more general states or even weights\(^3\).

To get deeper into the structure of type III factors it is necessary to consider Modular Theory in the more general context defined by Hilbert Algebras and focus on the crucial information contained in the modular automorphism group. Recall that the action of the modular automorphism group is nontrivial if the von Neumann algebra is infinite. In this more general context one can also associate to any faithful, normal, semifinite weight $\phi$ on a von Neumann algebra $\mathcal{M}$ modular objects $\left(\Delta^\phi, J^\phi\right)$ (for details see [34, 21, 38]). Connes analyzed in [12] (see also [13] for a review or [36, 34]) the dependence of the modular automorphism group $\sigma^\phi_t$ on the weight $\phi$.

In [12] the author established the following fundamental theorem:

**Theorem 5.2.** Let $\phi, \psi$ be faithful, normal, semifinite weights on the von Neumann algebra $\mathcal{M}$. Then there is a $\sigma$-weakly continuous one-parameter family $\left\{U_t\right\}_{t \in \mathbb{R}}$ of unitaries in $\mathcal{M}$ satisfying the cocycle condition

$$U_{t+s} = U_t \sigma^\phi_t(U_s), \quad t, s \in \mathbb{R},$$

and such that

$$\sigma^\psi_t(M) = U_t \sigma^\phi_t(M) U_t^*, \quad M \in \mathcal{M}.$$
factor is hyperfinite, i.e. if it is generated by an increasing sequence of finite-dimensional *-subalgebras.

5.3. The KMS condition. As was seen in the preceding subsection the modular automorphism group \( \{ \sigma_t \}_{t \in \mathbb{R}} \) plays a fundamental role in the classification of type III factors. There is also a characteristic and very useful analytic relation between \( \{ \sigma_t \}_{t \in \mathbb{R}} \) and the corresponding state \( \varphi \). For simplicity, we will formulate it in the case where \( \varphi \) is a faithful normal state.

**Definition 5.3.** A one-parameter automorphism group \( \{ \sigma_t \}_{t \in \mathbb{R}} \) satisfies the modular condition relative to the state \( \varphi \) if invariance holds, i.e. \( \varphi \circ \sigma_t = \varphi \), \( t \in \mathbb{R} \), and if for each \( M, N \in \mathcal{M} \) there is a complex-valued function \( F \) satisfying the following two conditions:

(i) \( F \) bounded and continuous on the horizontal strip \( \{ z \in \mathbb{C} \mid 0 \leq \text{Im}(z) \leq 1 \} \) and analytic on the interior of that strip.

(ii) \( F \) satisfies the following boundary condition:

\[
F(t) = \varphi(\sigma_t(M)N) \quad \text{and} \quad F(t + i) = \varphi(N\sigma_t(M)), \quad t \in \mathbb{R}.
\]

To each state \( \varphi \) as above there corresponds uniquely a one-parameter automorphism group \( \sigma_t \) satisfying the modular condition. In view of the preceding definition, the modular automorphism group gives a measure of the extent to which the state fails to be tracial (see also Remark 2.5 (iii)).

**Remark 5.4.** The modular condition mentioned before is known in quantum statistical mechanics as the KMS (Kubo-Martin-Schwinger) boundary condition (in this case at inverse temperature \( \beta = -1 \)). In this context \( \{ \sigma_t \}_{t \in \mathbb{R}} \) describes the time evolution of the system and KMS condition was proposed as a criterion for equilibrium (see [17] for further details).

6. Appendix: Crossed products and the CAR-algebra

In this appendix we collect some material used in the examples presented in Sections 3 and 4 and that would have interrupted the flow of the article.

6.1. Crossed products. Recall the group measure space construction presented in Section 3. In particular let \( (\Omega, \Sigma, \mu) \) be a \( \sigma \)-finite measure space and denote by \( \mathcal{M} := L^\infty(\Omega, \Sigma, \mu) \) the corresponding maximal abelian von Neumann algebra in \( \mathcal{L}(\mathcal{H}) \), where \( \mathcal{H} := L^2(\Omega, \Sigma, \mu) \). Denote by \( \alpha: \Gamma \to \text{Aut}\mathcal{M} \) the action of the discrete group \( \Gamma \) on \( \mathcal{M} \) and by \( \mathcal{N} = \mathcal{M} \otimes_\alpha \Gamma \) the corresponding crossed product acting on \( \mathcal{K} := \ell_2(\Gamma) \otimes \mathcal{H} \).

**Proposition 6.1.** With the preceding notation we have:

(i) The image \( \pi(\mathcal{M}) \) of \( \mathcal{M} \) in the crossed product \( \mathcal{N} \) is maximal abelian iff the action \( \alpha \) is free (cf. Eq. (7)).

(ii) Assume that the action \( \alpha \) is free. Then \( \alpha \) is ergodic iff \( \mathcal{N} \) is a factor.

The next theorem shows that all types of von Neumann factors mentioned before may be realized explicitly within the group measure space construction described previously. It implies, in particular, Theorem 3.4.
Theorem 6.2. Let \((\Omega, \Sigma, \mu)\) and \(M := L^\infty(\Omega, \Sigma, \mu)\) be as in the preceding proposition. Assume that there is a free and ergodic action \(\alpha: \Gamma \to \text{Aut} M\) of a discrete group \(\Gamma\) on \(M\). For the types of the factor \(N = M \otimes_\alpha \Gamma\) we have the following criteria:

(i) Suppose that there is \(\Gamma\)-invariant \(\sigma\)-finite positive measure \(\nu\) which is equivalent to \(\mu\) (in the sense of mutual absolute continuity). Then

- \(N\) is of type I iff the measure space \((\Omega, \Sigma, \mu)\) contains atoms.
- \(N\) is of type II iff the measure space \((\Omega, \Sigma, \mu)\) contains no atoms.
- \(N\) is finite iff \(\nu\) is a finite measure.

(ii) The factor \(N\) is of type III iff there does not exist a \(\sigma\)-finite positive measure \(\nu\) which is equivalent to \(\mu\) and \(\Gamma\)-invariant.

More concrete and explicit examples (including type III\(_0\), type III\(_\lambda\) and type III\(_1\) factors) can be found in [34, §4.3], [37, §V.7], [39, §XIII.1] or [21, §8.6].

6.2. The self-dual CAR-algebra. In this subsection we recall some standard results on the self-dual CAR-algebra which is needed in the example in Section 4. We will define and state the main properties of the C*-algebra that is associated to the canonical anticommutation relations and its irreducible representations. General references for the present section are [1, 3].

Theorem 6.3. Let \(\mathfrak{h}\) be a complex Hilbert space with scalar product \(\langle \cdot, \cdot \rangle\) and anti-unitary involution \(\Gamma\), i.e. \((\Gamma f, \Gamma h) = \langle h, f \rangle\), for all \(f, h \in \mathfrak{h}\). Then \(\text{CAR}(\mathfrak{h}, \Gamma)\) denotes the algebraically unique C*-algebra generated by \(1\) and \(a(\varphi), \varphi \in \mathfrak{h}\), such that the following relations hold:

(i) The mapping \(\varphi \mapsto a(\varphi)\) is antilinear.

(ii) \(a(\varphi)^* = a(\Gamma \varphi)\), \(\varphi \in \mathfrak{h}\).

(iii) \(a(\varphi_1)a(\varphi_2)^* + a(\varphi_2)^*a(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle 1\), \(\varphi_1, \varphi_2 \in \mathfrak{h}\).

\(\mathfrak{h}\) is called the reference space of \(\text{CAR}(\mathfrak{h}, \Gamma)\). This space is a ‘parameter’ space labeling the generators of the algebra. The name ‘self-dual’ comes from property (ii) above, where the algebra involution \(^*\) is described in terms of the antilinear mapping \(\Gamma\) of the reference space \(\mathfrak{h}\). For a finite-dimensional example see Example 4.6 (i). The preceding uniqueness result implies the following statement concerning the automorphisms of the CAR-algebra.

Theorem 6.4. Let \(U\) be a unitary of the reference space \(\mathfrak{h}\) that satisfies \(U \Gamma = \Gamma U\). Any such \(U\) generates an automorphism \(\alpha_U\) of \(\text{CAR}(\mathfrak{h}, \Gamma)\) (called the Bogoljubov automorphism associated to the Bogoljubov unitarity \(U\)) uniquely determined by the equation

\[\alpha_U(a(\varphi)) := a(U \varphi), \quad \varphi \in \mathfrak{h}.\]

Definition 6.5. An orthoprojection \(P\) on the reference space \(\mathfrak{h}\) is a basis projection if it satisfies the equation \(P + \Gamma P = 1\).
Theorem 6.6. Any basis projection $P$ generates a unique state $\omega_p$ on $\text{CAR}(\mathfrak{h}, \Gamma)$ by means of the relation

$$\omega_p \left(a(\varphi)^*a(\varphi)\right) = 0, \quad \text{if} \quad P\varphi = 0.$$ 

$\omega_p$ is a pure state and is called the Fock state corresponding to the basis projection $P$.

An explicit representation $\pi$ of the CAR-algebra associated to a basis projection $P$ is realized on the the antisymmetric Fock space

$$(20) \quad \mathfrak{F} := \bigoplus_{n=0}^{\infty} \left( \wedge^n P\mathfrak{h} \right).$$

At this point we introduce the usual annihilation and creation operators on $\mathfrak{F}$.

$$c(p) \Omega := 0,$$
$$c(p) (p_1 \wedge \ldots \wedge p_n) := \sum_{r=1}^{n} (-1)^{r-1} \langle p, p_r \rangle \mathfrak{h} p_1 \wedge \ldots \wedge \hat{p}_r \wedge \ldots \wedge p_n,$$
$$c(p)^* \Omega := p,$$
$$c(p)^* (p_1 \wedge \ldots \wedge p_n) := p \wedge p_1 \wedge \ldots \wedge p_n,$$

where $\Omega$ is the Fock vacuum in the subspace corresponding to $n = 0$ in the definition $(20)$ and $p, p_1, \ldots, p_n \in P\mathfrak{h}$. The symbol $\hat{p}_r$ means that the vector $p_r$ is omitted in the (antisymmetric) wedge product $\wedge$. Finally, the Fock representation $\pi$ is defined by

$$\pi(a(f)) := c(P\Gamma f)^* + c(Pf), \quad f \in \mathfrak{h}.$$ 

In the rest of this section we assume that a basis projection $P$ is given and when no confusion arises we will also simply write $a(f)$ instead of $\pi(a(f))$. We will later need an explicit expression for $a(f_n) \cdot \ldots \cdot a(f_1) \Omega$. Let $n, k, p$ be natural numbers with $2p + k = n$ and define the following subset of the symmetric group $\mathfrak{S}_n$:

$$\mathfrak{S}_{n,p} := \left\{ \left( \begin{array}{cccccccc} n & n-1 & \cdots & n-2p+2 & n-2p+1 & k & \cdots & 1 \\ \alpha_1 & \beta_1 & \cdots & \alpha_p & \beta_p & j_1 & \cdots & j_k \end{array} \right) \in \mathfrak{S}_n \bigg| \alpha_1 > \ldots > \alpha_p, \alpha_l > \beta_l, l = 1, \ldots, p \quad \text{and} \quad n \geq j_1 > j_2 > \ldots > j_k \geq 1 \right\}.$$ 

Note that $\mathfrak{S}_{n,p}$ contains \( \frac{(2p)!}{p!2^p} \) elements.

Proposition 6.7. For $f_1, \ldots, f_n \in \mathfrak{h}$ the equation

$$(a(f_n) \cdot \ldots \cdot a(f_1)) \Omega = \sum_{\pi \in \mathfrak{S}_{n,p}} (\text{sgn} \, \pi) \prod_{l=1}^{p} \langle Pf_{\alpha_l}, P\Gamma f_{\beta_l} \rangle P\Gamma f_{j_1} \wedge \ldots \wedge P\Gamma f_{j_k}$$

\[0 \leq 2p \leq n\]
holds, where the indices $\alpha_l, \beta_l, j_1, \ldots, j_k$ are given in the definition of $S_{n,p}$ and where for $n = 2p$ in the preceding sum one replaces the wedge product by the vacuum $\Omega$.

Finally, we state the following proposition that shows the stability of the CAR-algebra w.r.t. the operation of taking the closure of the reference space.

**Proposition 6.8.** Let $\mathfrak{h}_0$ be a complex pre-Hilbert space and $\Gamma_0$ an antilinear involution on it. Denote by $(\mathfrak{h}, \Gamma)$ the corresponding closures. Then

$$\text{CAR}(\mathfrak{h}_0, \Gamma_0) := C^*\left( a(\varphi) \mid \varphi \in \mathfrak{h}_0 \right) = \text{CAR}(\mathfrak{h}, \Gamma),$$

where $C^*(\cdot)$ denotes the $C^*$-closure of the argument.

**References**

[1] H. Araki, *On quasifree states of CAR and Bogoliubov automorphisms*, Publ. RIMS, Kyoto Univ. 6 (1970/71), 385–442.

[2] H. Araki, *On quasifree states of the canonical commutation relations (II)*, Publ. RIMS, Kyoto Univ. 7 (1971/72), 121–152.

[3] H. Araki, *Bogoliubov automorphisms and Fock representations of canonical anticommutation relations*, In Operator Algebras and Mathematical Physics, (Proceedings of the summer conference held at the University of Iowa, 1985), P.E.T. Jorgensen and P.S. Muhly (eds.), American Mathematical Society, Providence, 1987.

[4] H. Araki and M. Shiraishi, *On quasifree states of the canonical commutation relations (I)*, Publ. RIMS, Kyoto Univ. 7 (1971/72), 105–120.

[5] J. Avron, R. Seiler, and B. Simon, *The index of a pair of projections*, J. Funct. Anal. 120 (1994), 220–237.

[6] H. Baumgärtel, *Introduction to Hardy Spaces*, Int. J. Theor. Phys. 42 (2003), 2213–2223.

[7] H. Baumgärtel, M. Jurke, and F. Lledó, *Twisted duality of the CAR-Algebra*, J. Math. Phys. 43 (2002), 4158–4179.

[8] H. Baumgärtel and M. Wollenberg, *Causal Nets of Operator Algebras. Mathematical Aspects of Algebraic Quantum Field Theory*, Akademie Verlag, Berlin, 1992.

[9] H.J. Borchers, *On revolutionizing quantum field theory with Tomita’s modular theory*, J. Math. Phys. 41 (2000), 3604–3673.

[10] H.J. Borchers, *Tomita’s modular theory and the development of quantum field theory*, preprint, 2004.

[11] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*, Springer Verlag, Berlin, 1987.

[12] A. Connes, *Une classification des facteurs de type III*, Ann. Sci. Ecole Norm. Sup.(4) 6, 133-252.

[13] A. Connes, *The classification of von Neumann algebras and their automorphisms*, in Symposia Mathematica Vol. XX, Academic Press, London, pp.435-478.

[14] F. Figliolini and D. Guido, *The Tomita operator for the free scalar field*, Ann. Inst. H. Poincaré, Phys. Théor. 51 (1989), 419–435.

[15] P.A. Fillmore, *A User’s guide to Operator Algebras*, John Wiley and Sons, Inc., New York, 1996.

[16] D. Guido, *Modular Theory for the von Neumann algebras of local quantum physics*, in this volume.

[17] R. Haag, *Local Quantum Physics*, Springer Verlag, Berlin, 1992.

[18] P.R. Halmos, *Two subspaces*, Transactions Amer. Math. Soc. 144 (1969), 381–389.
[19] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. I, Springer Verlag, Berlin, 1979.
[20] R.V. Kadison, Reflections relating a von Neumann algebra and its commutant, Mappings of operator algebras (Philadelphia, PA, 1988), Progr. Math., vol. 84, Birkhäuser Boston, Boston, MA, 1991, pp. 295–304.
[21] R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras II, Academic Press, Orlando, 1986.
[22] T. Kato, Perturbation Theory for Linear Operators, Springer Verlag, Berlin, 1995.
[23] P. Leyland, J.E. Roberts, and D. Testard, Duality for quantum free fields, preprint, CNRS Marseille, 1978.
[24] F. Lledó, Operator algebraic methods in mathematical physics: duality of compact groups and gauge quantum field theory, Habilitation Thesis, 244p., RWTH-Aachen University, 2004.
[25] F. Lledó, Operator algebras: an informal overview, in this volume.
[26] F. Lledó, Conformal covariance of massless free nets, Rev. Math. Phys. 13 (2001), 1135–1161.
[27] F. Lledó, Massless relativistic wave equations and quantum field theory, Ann. H. Poincaré. 5 (2004), 607–670.
[28] F.J. Murray and J.v. Neumann, On rings of operators, Ann. Math. 37 (1936), 116–229.
[29] J.v. Neumann, On rings of operators, III., Ann. Math. 41 (1940), 94–161.
[30] D. Petz, An Invitation to the Algebra of Canonical Commutation Relations, Leuven University Press, Leuven, 1990.
[31] M.A. Rieffel and A. van Daele, A bounded operator approach to Tomita–Takesaki theory, Pacific J. Math. 69 (1977), 187–221.
[32] S. Strătilă, Modular Theory in Operator Algebras, Abacus Press, Tunbridge Wells, 1981.
[33] S.J. Summers, Tomita–Takesaki modular theory, In Encyclopedia of Mathematical Physics, J.P. Francoise, G. Naber, and T.S. Tsun (eds.), to be published by the Elsevier publishing house.
[34] V.S. Sunder, An Invitation to von Neumann Algebras, Springer, New York, 1987.
[35] M. Takesaki, Tomita’s Theory of Modular Hilbert Algebras and its Applications (LNM 128), Springer Verlag, Berlin, 1970.
[36] M. Takesaki, Structure of Factors and Automorphism Groups, CBMS Regional Conference Series in Mathematics, Vol. 51, American Mathematical Society, Providence, 1983.
[37] M. Takesaki, Theory of Operator Algebras I, Springer Verlag, Berlin, 2002.
[38] M. Takesaki, Theory of Operator Algebras II, Springer Verlag, Berlin, 2003.
[39] M. Takesaki, Theory of Operator Algebras III, Springer Verlag, Berlin, 2003.
[40] A. van Daele, The Tomita–Takesaki theory for von Neumann algebras with a separating and cyclic vector, in $C^*$-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory (Proc. Internat. School of Physics “Enrico Fermi”, Course LX, Varenna, 1973), North-Holland, Amsterdam, 1976, pp. 19–28.
[41] A. van Daele, Celebration of Tomita’s theorem, in Operator algebras and Applications, Proceedings of Symposia in Pure Mathematics Vol. 38-Part 2, American Mathematical Society, Providence, 1982, pp. 1-4.

Department of Mathematics, University Carlos III Madrid, Avda. de la Universidad 30, E-28911 Leganés (Madrid), Spain and Institute for Pure and Applied Mathematics, RWTH-Aachen University, Templergraben 55, D-52062 Aachen, Germany (on leave)

E-mail address: fllledo@math.uc3m.es and lledo@iram.rwth-aachen.de