A NOTE ON COHERENT ORIENTATIONS FOR EXACT LAGRANGIAN COBDIRSMs

CECILIA KARLSSON

ABSTRACT. Let $L \subset \mathbb{R} \times J^1(M)$ be a spin, exact Lagrangian cobordism in the symplectization of the 1-jet space of a smooth manifold $M$. Assume that $L$ has cylindrical Legendrian ends $\Lambda_\pm \subset J^1(M)$. It is well known that the Legendrian contact homology of $\Lambda_\pm$ can be defined with integer coefficients, via a signed count of pseudo-holomorphic disks in the cotangent bundle of $M$. We prove that this count can be lifted to a signed count of pseudo-holomorphic disks in $\mathbb{R} \times J^1(M)$, and then we use this to prove that $L$ induces a morphism between the $\mathbb{Z}$-valued DGA:s of the ends, in a functorial way. These results have been indicated in several papers before, our aim is to give rigorous proofs of these facts.

The proofs are built on the technique of orienting the moduli spaces of pseudo-holomorphic disks using capping operators at the Reeb chords. We give an expression for how the DGA:s change if we change the capping operators.

1. Introduction

1.1. Background. Let $M$ be an $n$-dimensional manifold and consider the 1-jet space $J^1(M) = T^*M \times \mathbb{R}$ of $M$. This space can be given the structure of a contact manifold, with contact form $\alpha = dz - \sum_j y_j dx_j$. Here $(x, y)$ are coordinates on $T^*M$ and $z$ is the coordinate in the $\mathbb{R}$-direction. An $n$-dimensional submanifold $\Lambda \subset J^1(M)$ is called Legendrian if it is everywhere tangent to the contact distribution $\xi = \text{Ker} \alpha$, and a Legendrian isotopy is a smooth 1-parameter family of Legendrian submanifolds. A major problem in contact geometry is to determine whether two given Legendrian submanifolds are Legendrian isotopic, i.e. if there is a Legendrian isotopy connecting them. To that end, a number of Legendrian invariants have been introduced. These are objects associated to Legendrian submanifolds, invariant under Legendrian isotopies.

One such invariant is Legendrian contact homology, which is the homology of a differential graded algebra (DGA) associated to the Legendrian $\Lambda$. This algebra is called the Chekanov-Eliashberg algebra of $\Lambda$, and we denote it by $A(\Lambda)$. It is a free, unital algebra generated by the Reeb chords of $\Lambda$, which are flow segments of the Reeb vector field $\partial_z$ having its start and end points on $\Lambda$. We assume that $\Lambda$ is chord generic.

Legendrian contact homology fits into the machinery of Symplectic field theory, introduced by Eliashberg, Givental and Hofer in [EGH00]. In particular, let $L$ be an exact Lagrangian cobordism in the symplectization $(\mathbb{R} \times J^1(M), d(e^t \alpha))$ of $J^1(M)$. Assume that $L$ is asymptotic to cylinders $\mathbb{R} \times \Lambda_\pm$ at $\pm \infty$, where $\Lambda_\pm \subset J^1(M)$ are Legendrians. Then $L$ induces a DGA-morphism $\Phi_L : A(\Lambda_+) \to A(\Lambda_-)$ in a functorial way. This

The author was funded by the Knut and Alice Wallenberg foundation and the ERC grant Geodycon.
is proven for $\mathbb{Z}_2$-coefficients in [Ekh08] and used in [EHK16] to derive results about isotopy classes of exact Lagrangians with prescribed boundary. More precisely, these results were derived from explicit descriptions of $\Phi_L$ in the case $L$ is induced by the trace of an elementary Legendrian isotopy.

The purpose of the present paper is to prove that the constructions used in [EHK16] also can be done with $\mathbb{Z}$-coefficients. This result has been indicated in a number of papers, see e.g. [CDRGG15], [CDRGG], [Ekh16]. Our intention is to give a rigorous proof of this, and also to give an explicit description of the orientation scheme for the moduli space of pseudo-holomorphic disks, which are used to define everything over $\mathbb{Z}$. This description seems to be needed to be able to perform explicit calculations similar to those in [EHK16], but now with integer coefficients.

That Legendrian contact homology can be defined over $\mathbb{Z}$, provided $\Lambda$ is spin, is proven in [EES05b]. In that paper the differential of $A(\Lambda)$ is defined by a count of rigid pseudo-holomorphic disks in the Lagrangian projection $\Pi_C : J^1(M) \to T^*M$, with the disks having boundary of $\Pi_C(\Lambda)$. However, there is another way to define the differential, which is more convenient if one wants to consider the functorial properties in Symplectic field theory. That method is to count rigid pseudo-holomorphic disks in the symplectization of $J^1(M)$, with the disks having boundary on $\mathbb{R} \times \Lambda$.

In [DR16] it is proved that these two different counts give the same DGA, given that we work with $\mathbb{Z}_2$-coefficients. We will prove that this also holds with $\mathbb{Z}$-coefficients, provided that $\Lambda$ is spin. More precisely, we will prove that the coherent orientation scheme given in [EES05b] can be lifted to give a coherent orientation scheme for moduli spaces of pseudo-holomorphic disks in $\mathbb{R} \times J^1(M)$ with boundary on $\mathbb{R} \times \Lambda$. Then we prove that this lifted orientation scheme allows us to extend the definition of $\Phi_L$ to $\mathbb{Z}$-coefficients, provided that $L$ is spin and that $\Lambda_\pm$ are given the induced spin structure as boundary of $L$.

The orientation scheme for the moduli spaces of pseudo-holomorphic disks will be defined by using something called capping operators, which are $\bar{\partial}$-operators defined on the 1-punctured unit disk in $\mathbb{C}$ with trivialized Lagrangian boundary conditions. Using the DGA-morphism induced by the trivial cobordism $\mathbb{R} \times \Lambda$, we will derive an expression of how the DGA changes if we change capping operators. In this way we can relate the orientation scheme of pseudo-holomorphic disks in $T^*M$ given in [EES05b] with the one given in [Kar].

The orientation scheme defined in [Kar] is adapted to the situation when the differential of Legendrian contact homology is defined by counting rigid Morse flow trees instead of pseudo-holomorphic disks. We refer to [Ekh07] for the definition of these trees, and for the proof that the trees can replace the pseudo-holomorphic disks in the definition of the differential if we work with $\mathbb{Z}_2$-coefficients. In [Kar], this result is extended to also hold for $\mathbb{Z}$-coefficients.

The advantage of using Morse flow trees instead of pseudo-holomorphic disks is that the former ones can be found using finite-dimensional flow techniques, while the latter ones give rise to non-linear PDE:s, which in general are hard to solve. In [EHK16] it is shown that one can use Morse flow trees to compute the DGA-morphism induced by an exact Lagrangian cobordism, in the case when the coefficients are given by $\mathbb{Z}_2$. 
This is one of the reasons why the DGA-morphisms induced by traces of elementary Legendrian isotopies can be described explicitly. In Section 5 we show that Morse flow trees can be used to compute DGA-morphism also with integer coefficients, given our orientation scheme of moduli spaces.

1.2. Organization of the paper. In Section 2 we give a proper definition of the DGA associated to a Legendrian \( \Lambda \subset J^1(M) \), and the DGA-morphism induced by an exact Lagrangian cobordism. We also state the main theorems. In Section 3 we recall the definition of punctured pseudo-holomorphic disks, and give a more detailed definition of the relevant moduli spaces. In Section 4 we fix orientation conventions, and prove that these conventions make it possible to define Legendrian contact homology with integer coefficients in the symplectization setting. In Section 5 we prove that this also gives the desired results for the DGA-morphisms induced by exact Lagrangian cobordisms. In Section 6 we discuss how the orientation scheme can be used also to orient the moduli space of Morse flow trees associated to exact Lagrangian cobordisms.

Acknowledgments. The author would like to thank Tobias Ekholm and Paolo Ghiggini for useful discussions.

2. Main results

Here we formulate the main results. To be able to do this, we first need to introduce some more notation.

2.1. Legendrian contact homology. As outlined in the Introduction, there are two different ways of defining the differential \( \partial \) of \( \mathcal{A}(\Lambda) \). One method is to compute punctured, rigid, pseudo-holomorphic disks in \( T^*M \) with boundary on \( \Pi_C(\Lambda) \). I.e., the differential is defined on generators \( a \) by

\[
\partial_l(a) = \sum_{\dim \mathcal{M}_{l,\Lambda}(a,b) = 0} |\mathcal{M}_{l,\Lambda}(a,b)| b,
\]

and extended by the Leibniz rule to the rest of the algebra. Here \( b = b_1 \cdots b_m \) is a word of Reeb chords, \( \mathcal{M}_{l,\Lambda}(a,b) \) is the moduli space of rigid pseudo-holomorphic punctured disks with a positive puncture at \( a \), negative punctures at \( b_1, \ldots, b_m \), and with boundary on \( \Pi_C(\Lambda) \), and \( |\mathcal{M}_{l,\Lambda}(a,b)| \) denotes the algebraic count of disks in the moduli space. We refer to Section 3 for more details.

We denote the DGA defined in this way by \( (\mathcal{A}(\Lambda), \partial_l; R) \), where \( R \) indicates the coefficient ring. In \[EES07\] it is proven that for a generic choice of compatible almost complex structure on \( T^*M \), this differential satisfies \( \partial_l^2 = 0 \), and the homology of this complex gives a well-defined Legendrian isotopy invariant if we choose the coefficient ring to be \( \mathbb{Z}_2 \). In \[EES05b\] these results were extended to hold for \( \mathbb{Z} \)-coefficients in the case when \( \Lambda \) is spin. In the special case \( n = 1 \) and \( M = \mathbb{R} \), these results were first established in \[Che02\] for the case of \( \mathbb{Z}_2 \)-coefficients, and in \[ENS02\] for \( \mathbb{Z} \)-coefficients.

The other method of computing the differential, which was discussed in \[EGH00\], and where the details were worked out in \[Eli98\] for \( n = 1 \) and in \[Ekh08\] for higher
dimensions, is to count rigid pseudo-holomorphic disks in the symplectization of $J^1(M)$. That is, in this case the differential is defined by
\[
\partial_s(a) = \sum_{\dim \hat{M}_{s,\Lambda}(a,b) = 1} |\hat{M}_{s,\Lambda}(a,b)|/|b|
\]
on generators, and again extended by the Leibniz rule to the whole algebra. Here $\hat{M}_{s,\Lambda}(a,b)$ is the moduli space of punctured pseudo-holomorphic disks with boundary on $\mathbb{R} \times \Lambda$, having a positive puncture asymptotic to a strip over the Reeb chord $a$ at $t = +\infty$, and having negative punctures asymptotic to strips over the Reeb chords $b_1, \ldots, b_m$ at $t = -\infty$. Moreover, we assume that the given almost complex structure is cylindrical, so that we get an induced an $\mathbb{R}$-action on $\hat{M}_{s,\Lambda}$, given by translation in the $t$-direction. See Section 3.

We let $M_{s,\Lambda} = \hat{M}_{s,\Lambda}(a,b)/\mathbb{R}$ be the space where we have divided out this $\mathbb{R}$-action. In [Ekh08] it is proven that for a generic choice of cylindrical almost complex structure we have $\partial^2_s = 0$, given that we are using $\mathbb{Z}_2$-coefficients, and that the homology of $A(\Lambda)$ is invariant under Legendrian isotopies.

In [DR16] it is shown that under certain, not too restrictive, choices of almost complex structures of $T^*M$ and $\mathbb{R} \times J^1(M)$ we have that
\[(A(\Lambda), \partial_s; \mathbb{Z}_2) \simeq (A(\Lambda), \partial_l; \mathbb{Z}_2)\]
where the isomorphism is induced by the projection
\[\pi_P : \mathbb{R} \times (T^*M \times \mathbb{R}) \to T^*M.\]
In particular, it is proven that the induced map
\[\pi_P : M_{s,\Lambda}(a,b) \to M_{l,\Lambda}(a,b)\]
\[u \mapsto \pi_P(u)\]
is a diffeomorphism. In the present paper we extend this result to $\mathbb{Z}$-coefficients, by proving that there is a choice of orientation conventions so that the coherent orientation scheme given for $M_{l,\Lambda}$ in [EES05b] can be lifted under $\pi_P$ to give a coherent orientation scheme for $M_{s,\Lambda}$. Compare [DR16, Remark 2.4].

**Theorem 2.1.** Let $J_P$ and $\tilde{J}_P$ be almost complex structures on $T^*M$ and $\mathbb{R} \times J^1(M)$, respectively, satisfying the assumptions in [[DR16], Theorem 2.1]. Further assume that $\Lambda \subset J^1(M)$ is a spin Legendrian submanifold. Then there are choices of coherent orientation schemes of the moduli spaces $M_{s,\Lambda}(a,b)$ and $M_{l,\Lambda}(a,b)$ so that
\[\pi_P : M_{s,\Lambda}(a,b) \to M_{l,\Lambda}(a,b)\]
\[u \mapsto \pi_P(u)\]
is orientation preserving. Moreover, for $i = s, l$ we have that
\[(2.1) \quad \partial_i(a) = \sum_{\dim \mathcal{M}_{i,\Lambda}(a,b) = 0} |\mathcal{M}_{i,\Lambda}(a,b)|/|b|\]
satisfies $\partial^2 = 0$. Here $|\mathcal{M}_{i,\Lambda}(a, b)|$ denotes the algebraic number of disks in the moduli space, where the signs of the disks come from the coherent orientation scheme.

**Remark 2.2.** We also get that the stable tame isomorphism class of the DGA:s are invariant under Legendrian isotopies. Compare [EES05b, Section 4.3].

**Remark 2.3.** We will use slightly different orientation conventions than in [EES05b], to simplify the expression of the differential. In that paper it is instead given by

$$\partial_L(a) = \sum_{\dim \mathcal{M}_{i,\Lambda}(a, b)=0} (-1)^{(n-1)|a|+1} |\mathcal{M}_{i,\Lambda}(a, b)||b|.$$

### 2.2. Exact Lagrangian cobordisms.

Here we describe how an exact Lagrangian submanifold $L \subset \mathbb{R} \times J^1(M)$ with cylindrical Legendrian ends induces a morphism between the DGA:s of the ends.

**Definition 2.4.** Let $\Lambda_+, \Lambda_- \subset J^1(M)$ be Legendrian submanifolds. An exact Lagrangian cobordism from $\Lambda_+$ to $\Lambda_-$ is an exact Lagrangian submanifold $L$ of the symplectization of $J^1(M)$, satisfying

\[
\begin{align*}
\mathcal{E}_+(L) &:= L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+, \\
\mathcal{E}_-(L) &:= L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-,
\end{align*}
\]

for some $T > 0$, and so that

1. each function $f$ that satisfies $df = e^t \alpha|_L$, also satisfies that $f|_{\mathcal{E}_\pm(L)}$ is constant,
2. $L \setminus (\mathcal{E}_+(L) \cup \mathcal{E}_-(L))$ is compact with boundary $\Lambda_+ - \Lambda_-$. 

An exact Lagrangian cobordism induces a DGA-morphism

$$\Phi_L : (\mathcal{A}(\Lambda_+), \partial_s; \mathbb{Z}_2) \to (\mathcal{A}(\Lambda_-), \partial_s; \mathbb{Z}_2).$$

Indeed, in [Ekh08] and [EHK16] it is proven that we can define $\Phi_L$ by

$$\Phi_L(a) = \sum_{\dim \mathcal{M}_{L}(a, b)=0} |\mathcal{M}_{L}(a, b)||b|, \quad b = b_1 \cdots b_m,$$

if $a$ is a generator, and extend it to the rest of the algebra by

$$\Phi(a + b) = \Phi(a) + \Phi(b) \quad \text{(2.3)}$$

$$\Phi(ab) = \Phi(a)\Phi(b) \quad \text{(2.4)}$$

Here $\mathcal{M}_{L}(a, b)$ denotes the moduli space of punctured pseudo-holomorphic disks with boundary on $L$, positive puncture mapped asymptotically to a strip over the Reeb chord $a$ at $t = +\infty$, negative punctures mapping asymptotically to strips over the Reeb chords $b_1, \ldots, b_m$ at $t = -\infty$, and $|\mathcal{M}_{L}(a, b)|$ is the modulo 2 count of elements. See Section 3. We prove that we can replace the modulo 2 count by a signed count, so that $\Phi_L$ gives a DGA-morphism also with $\mathbb{Z}$-coefficients.

**Theorem 2.5.** Let $L \subset \mathbb{R} \times J^1(M)$ be a spin, exact Lagrangian cobordism from $\Lambda_+$ to $\Lambda_-$. Then there is an orientation scheme for $\mathcal{M}_{L}(a, b), \mathcal{M}_{a_+}(a, b)$ and $\mathcal{M}_{\Lambda_+}(a, b)$ so that

$$\Phi_L : (\mathcal{A}(\Lambda_+), \partial_s; \mathbb{Z}) \to (\mathcal{A}(\Lambda_-), \partial_s; \mathbb{Z}),$$
defined by (2.2)–(2.4) is a DGA-morphism. Now \(|M_L(a, b)|\) represents the algebraic count of disks in the moduli space.

Moreover, we will prove that the functorial properties of \(\Phi\), as described in [EHK16], continue to hold with integer coefficients.

**Theorem 2.6.** Assume that \(L_1, L_2 \subset \mathbb{R} \times J^1(M)\) are two spin, exact Lagrangian cobordisms with fixed spin structures. Assume that they can be concatenated along a common end, giving rise to a spin exact Lagrangian cobordism \(L_1 \# L_2\), where the spin structure of \(L_1 \# L_2\) restricted to \(L_i\) agrees with the fixed spin structure on \(L_i\) for \(i = 1, 2\).

Then

\[
\Phi_{L_2} \circ \Phi_{L_1} = \Phi_{L_1 \# L_2}.
\]

Moreover, if \(\Lambda \subset J^1(M)\) is a spin Legendrian then

\[
\Phi_{\mathbb{R} \times \Lambda} = \text{id}.
\]

We will prove that the orientation scheme from Theorem 2.1 can be used to derive these results. As indicated in the Introduction, this orientation scheme is defined using capping operators. Briefly, this works as follows.

Let \(u \in \mathcal{M}_{t, \Lambda}(a, b)\). Along the boundary components of \(u\) the spin structure of \(\Lambda\) induces a trivialized Lagrangian boundary condition. This boundary condition is then “closed up” by gluing capping disks to the punctures of \(u\). That is, for each Reeb chord \(c\) of \(\Lambda\) we define two different capping operators \(\bar{\partial}_c^+\) and \(\bar{\partial}_c^-\). These are \(\bar{\partial}\)-operators defined on the unit disk in \(\mathbb{C}\) with one positive puncture. For a disk \(u \in \mathcal{M}_{\Lambda}(a, b)\) we then glue \(\bar{\partial}_a^+\) to the positive puncture of \(u\), and \(\bar{\partial}_b^-\) to the negative puncture corresponding to the chord \(b_i, i = 1, \ldots, m\). We require the trivialized boundary conditions for the capping operators to be defined in such a way so that these gluings induce a trivialized Lagrangian boundary condition for the non-punctured unit disk in \(\mathbb{C}\). Then we use the fact that there is a canonical orientation of the determinant line bundle for the \(\bar{\partial}\)-operator over the space of trivialized Lagrangian boundary conditions for the unit disk in \(\mathbb{C}\). Then this canonical orientation is given via evaluation at the boundary, see [FOOO09]. The canonical orientation for the \(\bar{\partial}\)-operator associated to the capped boundary condition can then be used to give an orientation of \(\text{det} \, \bar{\partial}_u\). This is explained in more detail in Section 3.

Notice that the differential of the DGA of \(\Lambda\) depends on the choice of capping operators. We will prove that for certain systems of capping operators, the associated DGAs are isomorphic.

**Theorem 2.7.** Let \(\Lambda\) be a spin Legendrian submanifold of \(J^1(M)\) with a fixed spin structure. Let \(\mathcal{S}\) denote a system of capping operators for \(\Lambda\) satisfying (c1)–(c3) in Section 4.7. Let \(\partial_{t, \mathcal{S}}\) denote the induced differential as defined in (2.1), where the orientation of the moduli space is induced by the system \(\mathcal{S}\). Then \((\mathcal{A}(\Lambda), \partial_{t, \mathcal{S}} : \mathbb{Z})\) is a DGA whose homology is invariant under Legendrian isotopies.

Moreover, if \(\mathcal{S}'\) is another system of capping operators for \(\Lambda\) satisfying (c1)–(c3), then there is a DGA-isomorphism

\[
\Phi_{\mathcal{S}, \mathcal{S}'} : (\mathcal{A}(\Lambda), \partial_{t, \mathcal{S}} : \mathbb{Z}) \rightarrow (\mathcal{A}(\Lambda), \partial_{t, \mathcal{S}'} : \mathbb{Z}).
\]
We refer to Section 5 for an explicit description of the map (2.7).

**Remark 2.8.** From the proofs of Theorem 2.1, Theorem 2.5 and Theorem 2.6 it follows that any system of capping operators satisfying (c1) – (c3) give coherent orientation schemes so that the statements of the theorems hold.

**Remark 2.9.** All orientation schemes above depend on choices of orientations of $\mathbb{R}^n$ and of $\mathbb{C}$, which we from now on assume to be fixed.

### 3. Punctured pseudo-holomorphic disks

In this section we give a definition of punctured pseudo-holomorphic disks. We also define the moduli spaces that will be relevant for us.

#### 3.1. Pseudo-holomorphic disks

An almost complex structure $J$ on a symplectic manifold $(X, \omega)$ is an endomorphism $J : TX \to TX$ satisfying $J^2 = -\text{id}$. We say that $J$ is compatible with $\omega$ if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on $X$. If $(X, \omega) = (\mathbb{R} \times J^1(M), d(e^t\alpha))$, then $J$ is cylindrical if it is compatible with $\omega$, is invariant under $t$-translation, and satisfies $J(\xi) = \xi$, $J(\partial_t) = R_{\alpha}$. Here $R_{\alpha}$ denotes the Reeb vector field of $\alpha$.

Let $D$ be the compact unit disk in $\mathbb{C}$ and let $D_{m+1}$ denote the punctured disk with $m + 1$ punctures $p_0, \ldots, p_m \in \partial D$, cyclically ordered along the boundary in the counter-clockwise direction. Let $\tilde{D}_{m+1}$ denote the corresponding disk with the punctures removed. We will assume that $p_0 = 1 \in \mathbb{C}$, and call it the *positive puncture*. We say that $p_1, \ldots, p_m$ are the *negative punctures*.

A map $u : D_{m+1} \to X$ (or $u : \tilde{D}_{m+1} \to X$) is J-holomorphic if it satisfies

$$\bar{\partial}_J(u) := du + J \circ du \circ i = 0.$$

If we want to neglect the choice of $J$ we say that $u$ is pseudo-holomorphic.

#### 3.2. Gradings

Each Reeb chord $a$ of $\Lambda$ comes equipped with a grading $|a|$, given by

$$|a| = CZ(a) - 1$$

where $CZ(a)$ is the Conley-Zehnder index of $a$. For a proper definition we refer to \cite{EES05a}.

#### 3.3. Moduli spaces

In this section we give definitions of the relevant moduli spaces of pseudo-holomorphic disks.

**3.3.A. Moduli spaces in the Lagrangian projection.** Fix an almost complex structure $J$ on $T^*M$, compatible with $\omega$. We let $\mathcal{M}_{t, \Lambda}(a, b)$, $b = b_1 \cdots b_m$, denote the moduli space of pseudo-holomorphic maps $u : (\tilde{D}_{m+1}, \partial \tilde{D}_{m+1}) \to (T^*M, \Pi_{\mathbb{C}}(\Lambda))$ satisfying the following:

1. $u|_{\partial \tilde{D}_{m+1}}$ has a continuous lift $\tilde{u}$ to $\Lambda$;
2. $u(p_0) = \Pi_{\mathbb{C}}(a)$, where $a$ is a Reeb chord of $\Lambda$, and the $z$-coordinate of $\tilde{u}$ makes a positive jump when passing through $p_0$ in the counterclockwise direction;
3. (3) $u(p_i) = \Pi_C(b_i)$, $i = 1, \ldots, m$, where $b_i$ is a Reeb chord of $\Lambda$, and the $z$-coordinate of $\tilde{u}$ makes a negative jump when passing through $p_i$ in the counterclockwise direction.

Moreover, we consider two maps $u_1, u_2$ satisfying the above to be equal if they differ by a biholomorphism of $D_{m+1}$.

In [EES05a] it is proven that for generic $J$ the moduli spaces are transversely cut out manifolds of of dimension

$$\dim \mathcal{M}_{t,\Lambda}(a, b) = |a| - \sum_{i=1}^{m} |b_i| - 1.$$  

3.3.B. Moduli spaces in the symplectization. Fix a cylindrical almost complex structure $J$ on $\mathbb{R} \times J^1(M)$. We let $\mathcal{M}_{s,\Lambda}(a, b)$ denote the moduli space of pseudo-holomorphic maps $u : (\hat{D}_{m+1}, \partial \hat{D}_{m+1}) \to (\mathbb{R} \times J^1(M), \mathbb{R} \times \Lambda)$ satisfying the following:

(s1) in a neighborhood of the positive puncture $p_0$ the map $u$ is asymptotic to the Reeb chord strip $[0, \infty) \times c$;

(s2) in a neighborhood of the negative puncture $p_i$ the map $u$ is asymptotic to the Reeb chord strip $(-\infty, 0) \times b_i$ $i = 1, \ldots, m$.

Again, we consider two maps $u_1, u_2$ satisfying the above to be equal if they differ by a biholomorphism of $D_{m+1}$.

We let $\mathcal{M}_{s,\Lambda}(a, b) = \mathcal{M}_{s,\Lambda}(a, b)/\mathbb{R}$ where the $\mathbb{R}$-action is given by translation in the $t$-direction.

In [Ekh08] it is proven that for generic $J$ the moduli spaces are transversely cut out manifolds of dimension

$$\dim \mathcal{M}_{s,\Lambda}(a, b) = |a| - \sum_{i=1}^{m} |b_i| - 1.$$  

3.3.C. Moduli spaces associated to an exact cobordism. Fix a compatible almost complex structure $J$ on $\mathbb{R} \times J^1(M)$, and assume that it is cylindrical for $|t| > N$ for some $N$. We let $\mathcal{M}_{L}(a, b)$ denote the moduli space of pseudo-holomorphic maps $u : (\hat{D}_{m+1}, \partial \hat{D}_{m+1}) \to (\mathbb{R} \times J^1(M), L)$ satisfying [s1] and [s2] and again consider two maps $u_1, u_2$ to be equal if they differ by a biholomorphism of $D_{m+1}$.

From [EHK16] it follows that for generic $J$ the moduli spaces are transversely cut out manifolds of dimension

$$\dim \mathcal{M}_{L}(a, b) = |a| - \sum_{i=1}^{m} |b_i|.$$  

From now on we assume that the almost complex structures are chosen so that the relevant moduli spaces are transversely cut out manifolds of the expected dimension. We call a disk $u \in \mathcal{M}_{i,\Lambda}(a, b)$, $i = l, s$, a pseudo-holomorphic disk of $\Lambda$ with positive puncture $a$ and negative punctures $b_1, \ldots, b_m$. If moreover $\dim \mathcal{M}_{i,\Lambda}(a, b) = 0$ we say that $u$ is rigid. We use similar language for disks $u \in \mathcal{M}_{L}(a, b)$. 

3.4. The linearized $\bar{\partial}$-operator. If $u$ is a pseudo-holomorphic disk of $\Lambda$ or of $L$, then the linearized $\bar{\partial}$-operator $\bar{\partial}_u$ at $u$ gives a Fredholm operator (i.e., for certain choices of Sobolev spaces, the operator $\bar{\partial}_u$ has closed range, and finite-dimensional kernel and cokernel.) An orientation of $u$ is given by an orientation of the determinant line
\[
\det \bar{\partial}_u = \bigwedge \text{max} \, \text{Ker} \bar{\partial}_u \otimes \bigwedge \text{max} \, (\text{Coker} \bar{\partial}_u)^*.
\]
Since the moduli spaces used to define the DGA-differentials and the DGA-morphisms are assumed to be 0-dimensional, they are always orientable, but since we require that $\partial^2 = 0$ and $\partial \circ \Phi_L = \Phi_L \circ \partial$, we need to choose the orientations in a coherent way. This is done in Section 4, but to that end we need to discuss boundary conditions associated to the map $\bar{\partial}_u$.

If $u : (D_{m+1}, \partial D_{m+1}) \to (T^*M, \Pi_\mathbb{C}(\Lambda))$ is a pseudo-holomorphic disk, then $u^*T\Pi_\mathbb{C}(\Lambda)$ induces a Lagrangian boundary condition on $D_{m+1}$. Using that $\Lambda$ is spin, we can pick a well-defined trivialization of this Lagrangian boundary condition, following [EES05b, Section 3.4.2]. This gives a collection of maps
\[
A = (A_0, \ldots, A_{m+1}) : \partial D_{m+1} \to U(n),
\]
where
\[
A_i : [p_i, p_{i+1}] \to U(n), \quad i = 0, \ldots, m+1, \quad m+2 = 0.
\]
From this we get an associated linearized $\bar{\partial}$-operator
\[
\bar{\partial}_A = \bar{\partial}_{t,A} : \mathcal{W}[A](u^*T^*M) \to \mathcal{V}[0](u^*T^*M \otimes \Omega^{0,1}(D_{m+1})),
\]
where $\mathcal{W}[A], \mathcal{V}[0]$ are suitable Sobolev spaces making the operator Fredholm. We refer to [EES05a] and [EES07] for a deeper discussion on this subject.

The boundary condition $A$ lifts to a boundary condition $\text{id} \oplus A$ under $\pi_P$, and give rise to a similar operator
\[
\bar{\partial}_{s,A} : \mathcal{W}[\text{id} \oplus A](\tilde{u}^*T(\mathbb{R} \times J^1(M))) \to \mathcal{V}[0](\tilde{u}^*T(\mathbb{R} \times J^1(M)) \otimes \Omega^{0,1}(D_{m+1}))
\]
where $\tilde{u}$ is the lift of $u$ under $\pi_P$. Again, there is a choice of Sobolev spaces $\mathcal{W}$ and $\mathcal{V}$ so that $\bar{\partial}_{s,A}$ is Fredholm.

Similar constructions are done for the linearized $\bar{\partial}$-operator at a holomorphic disk $u \in \mathcal{M}_L(a, b)$.

4. Orientation conventions

The signs in the algebraic count of elements in the DGA-morphisms, and also in the DGA-differentials, come from orientations of the moduli spaces of $J$-holomorphic disks. These orientations depend on several choices, which we fix in this section.

We mainly use the approach of [EES05b] where the moduli spaces $\mathcal{M}_{L,\Lambda}$ are oriented, but we will make some slight modifications of these conventions to make them fit into the symplectization setting.

We close this section by proving that the chosen conventions implies the statements in Theorem 2.1.
4.1. **Short exact sequences.** First of all, it is a standard fact that an exact sequence

\begin{equation}
0 \to V_1 \xrightarrow{\alpha} W_1 \xrightarrow{\beta} W_2 \xrightarrow{\gamma} V_2 \to 0
\end{equation}

of finite-dimensional vector spaces induces an isomorphism

\begin{equation}
\phi : \bigwedge^{\text{max}} V_1 \otimes \bigwedge^{\text{max}} V_2^* \xrightarrow{\cong} \bigwedge^{\text{max}} W_1 \otimes \bigwedge^{\text{max}} W_2^*,
\end{equation}

where \( \bigwedge^{\text{max}} V \) is the top exterior power of the vector space \( V \). See e.g. [FH93]. This isomorphism is not canonical, but depends on choices. We will use the following convention, described in terms of oriented bases:

First we identify \( \bigwedge^{\text{max}} V^* \) with \( \bigwedge^{\text{max}} V \) via \( v_1 \wedge \cdots \wedge v_k \mapsto v_1^* \wedge \cdots \wedge v_k^* \) where \((v_1, \ldots, v_k)\) is any basis for \( V \), and \( v_i^* \) is the vector dual to \( v_i \). Now pick a basis \((v_1, \ldots, v_k)\) for \( V_1 \), and vectors \((u_1, \ldots, u_l)\) \( \in W_2 \) so that \((\gamma(u_1), \ldots, \gamma(u_l))\) gives a basis for \( V_2 \). Then pick vectors \((w_1, \ldots, w_m)\) \( \in W_1 \) so that \((\alpha(v_1), \ldots, \alpha(v_k), w_1, \ldots, w_m)\) gives a basis for \( W_1 \). From the exactness of the sequence (4.1) it then follows that \((\beta(w_1), \ldots, \beta(w_m), u_1, \ldots, u_l)\) gives a basis for \( W_2 \). We fix the isomorphism (4.2) to be given by

\begin{equation}
v_1 \wedge \cdots \wedge v_k \otimes \gamma(u_1) \wedge \cdots \wedge \gamma(u_l) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_m \otimes u_1 \wedge \cdots \wedge u_l \wedge \beta(w_1) \wedge \cdots \wedge \beta(w_m),
\end{equation}

and extend by linearity. It is straightforward to check that this definition does not depend on the choice of bases.

4.2. **Exact gluing sequences, and order of gluing.** We will repeatedly make use of exact gluing sequences of pseudo-holomorphic disks. For a detailed description we refer to [EES05b]. Here we give an outline of the construction.

Let \( D_{m_1+1} \) be a disk with punctures \((q_0, q_1, \ldots, q_{m_1})\) and with an associated Lagrangian boundary condition \( A : \partial D_{m_1+1} \to U(n) \). Similarly, let \( D_{m_2+1} \) be a disk with punctures \((p_0, p_1, \ldots, p_{m_2})\) and with an associated Lagrangian boundary condition \( B : \partial D_{m_2+1} \to U(n) \). If \( A \) and \( B \) are asymptotically equal to the same constant map at the punctures \( q_0 \) and \( p_j \), say, then we can glue \( D_{m_1+1} \) to \( D_{m_2+1} \) at \( q_0 = p_j \), and get a trivialized Lagrangian boundary condition \( A \# B \) on the glued disk \( D_{m_1+m_2+1} = D_{m_1+1} \# D_{m_2+1} \).

This gluing induces an exact sequence for the kernels and cokernels of the associated operators \( \partial_A, \partial_B \) and \( \partial_{A \# B} \), given by

\begin{equation}
0 \to \text{Ker} \partial_{A \# B} \to \begin{bmatrix} \text{Ker} \partial_B \\ \text{Ker} \partial_A \end{bmatrix} \to \begin{bmatrix} \text{Coker} \partial_B \\ \text{Coker} \partial_A \end{bmatrix} \to \text{Coker} \partial_{A \# B} \to 0.
\end{equation}

Here we use the notation

\[
\begin{bmatrix} V \\ W \end{bmatrix} = V \oplus W.
\]

For an explicit description of the maps in this exact sequence we refer to [EES05b, Remark 3.3].
Using the isomorphism (4.2) we see that orientations of $\det \bar{\partial} A$ and $\det \bar{\partial} B$ induces an orientation of $\det \bar{\partial} A \# B$. Note that this induced orientation depends on the pairwise order of the vector spaces in the second and third column of the gluing sequence, and that we have chosen the opposite order compared to [EES05b]. The reason for this change is that the order in (4.4) seems more feasible when working with an extra $\mathbb{R}$-direction, which shows up when we consider pseudo-holomorphic disks in the symplectization instead of in the Lagrangian projection.

4.3. Orientations of the space of conformal variations. Let $u : D_{m+1} \to X$, $X = T^*M$ or $X = \mathbb{R} \times J^1(M)$, be a rigid $J$-holomorphic disk of $\Lambda$ or $L$. If $m > 1$, then the linearized $\bar{\partial}$-operator at $u$, restricted to the Sobolev space of candidate maps, will have cokernel isomorphic to the tangent space of the space of conformal structures of $D_{m+1}$. We call this tangent space the space of conformal variations, and the orientation (i.e. the sign) of $u$ will depend on which orientation we choose on this space. See [EES05b], Lemma 3.17. We fix this orientation as follows.

Let $C_m$ denote the space of conformal structures on $D_{m+1}$. If we fix the positions of three of the punctures of $D_{m+1}$, then the position of the other punctures parametrize $C_m$. To describe the orientation of the tangent space $T_\kappa C_m$ at a conformal structure $\kappa$, let $\partial_{p_j}$ denote the vector tangent to $\partial D_{m+1}$ at $p_j$, pointing in the counterclockwise direction. Then if we choose $m - 2$ of the vectors $\partial_{p_1}, \ldots, \partial_{p_m}$ we get a basis for $T_\kappa C_m$. We define the positive orientation of $T_\kappa C_m$ to be given by

$$(\partial_{p_m}, \ldots, \partial_{p_3}).$$

This somewhat unnatural orientation is forced by the convention (4.4).

Remark 4.1. This gives the same orientation as the oriented basis

$$(\partial_{p_m}, \ldots, \partial_{p_k+1}, -\partial_{p_{k-1}}, \ldots, -\partial_{p_{j+1}}, \partial_{p_{j-1}}, \ldots, \partial_{p_1}).$$

Remark 4.2. If $m < 1$ then we can add marked points to the boundary of $D_{m+1}$ to get the setting above. See [EES05b], Section 4.2.3.

Now recall the gluing of $\bar{\partial} A$ and $\bar{\partial} B$ described in Section 4.1.2. The direct sum of the conformal structures of the disks $D_{m_1+1}$ and $D_{m_2+1}$ (which were joined at $q_0 = p_j$, with $q_0$ denoting the positive puncture of $D_{m_1+1}$) can be seen as an element of the boundary of the space $C_m$, $m = m_1 + m_2$. In addition, the outward normal at this conformal structure can be given by $\partial_{q_1} = -\partial_{p_{j-1}}$, or alternatively $\partial_{p_{j+1}} = -\partial_{q_{m_1}}$. We orient the boundary by outward normal last.

Lemma 4.3. We have

$$TC_{m_2} \oplus TC_{m_1} \oplus \mathbb{R} = (-1)^{(m_1-1)j+1}TC_m$$

as oriented vector spaces, where $\mathbb{R}$ is given the orientation from the outward normal.

Proof. This is similar to the proof of Lemma 4.7 in [EES05b]. $\square$
4.4. Canonical orientation of the closed disk, trivializations, and spin structures. The determinant line bundle of the $\bar{\partial}$-operator over the space of trivialized Lagrangian boundary conditions on the closed unit disk in $\mathbb{C}$ is orientable. Moreover, if we fix an orientation of $\mathbb{R}^n$ and of $\mathbb{C}$, then this induces an orientation, via evaluation at the boundary. See [FOOO09]. We denote this induced orientation the canonical orientation (recall that we assume that we have fixed orientations of $\mathbb{C}$ and $\mathbb{R}^n$ already, see Remark 2.9).

In [EES05b] it is described how a choice of spin structure of $\Lambda$ induces a well-defined Lagrangian trivialization of the linearized boundary condition of the $\bar{\partial}$-operator associated to a pseudo-holomorphic disk with boundary on $\Lambda$. Also compare with the similar discussion of relative spin structure in [FOOO09], from which the results in [EES05b] are derived.

In this paper we use the following conventions. If $\Lambda$ is a spin Legendrian and $L$ is the Lagrangian cylinder $\mathbb{R} \times \Lambda$, then we give $L$ the spin structure induced from the spin structure of $\Lambda$ and the trivial spin structure on $\mathbb{R}$. If $L$ is a spin, exact Lagrangian cobordism with cylindrical ends $\Lambda_\pm$, then we require that $\Lambda_\pm$ are given the boundary spin structures induced by $L$. We refer to [EES05b], Section 4.4] for a discussion on how other choices of spin structure affect the orientations of the moduli space of pseudo-holomorphic disks.

4.5. Capping operators. Let $u$ be a holomorphic disk of $\Lambda$ or of $L$. As pointed out above, to give a sign to $u$ is the same thing as give an orientation to the determinant line of $\bar{\partial}_u$. All this must be done in a coherent way, so that we get $\partial^2 = 0$ and $\Phi_L \circ \partial = \partial \circ \Phi_L$ in the very end.

The idea from [EES05b] is to use the trivialized Lagrangian boundary condition of $u$, induced by the spin structure of $\Lambda$ or $L$, together with the canonical orientation of $\det \bar{\partial}$ over the space of trivialized Lagrangian boundary conditions on the non-punctured disk. To make this work, we need to choose a way to close up the trivialized boundary conditions of $u$ at the punctures. In [EES05b] this is done by using something called capping operators, and this is the method that we will use. We give an outline of the constructions, and also explain the modifications needed to carry it over to the symplectization.

4.5.A. Capping trivializations. The capping operators are $\bar{\partial}$-operators defined on the 1-punctured unit disk in $\mathbb{C}$, and we have two operators, $\bar{\partial}_{p,+}$ and $\bar{\partial}_{p,-}$, associated to each Reeb chord $p$ of $\Lambda$. The reason for this is that we need one capping operator for $p$ in the case when $p$ occurs as a positive puncture of a disk, and another capping operator for $p$ when $p$ occurs as a negative puncture.

To each capping operator $\bar{\partial}_{p,\pm}$ we have an associated trivialized Lagrangian boundary condition $R_{p,\pm}$, which is chosen in a way so that we get a trivialized boundary condition on the non-punctured disk after having glued all the capping operators corresponding to the punctures of $u$ to $\bar{\partial}_u$. We call the boundary conditions $R_{p,\pm}$ the capping trivializations.

There are different possibilities to define $R_{p,\pm}$. See for example [EES05b], Section 3.3], [EES05b], Section 4.5], and [Kar]. We will not fix a specific system of capping
trivializations in the present paper, instead we consider any system that satisfies certain conditions, listed below. To that end, we first have to discuss a stabilization of the tangent bundle of $\Lambda$ and $L$, made by adding a trivial bundle.

4.5.B. Auxiliary directions. In [EES05b], something called auxiliary directions are introduced. These are artificial extra directions that are added to the capping trivializations and to the trivializations induced by the pseudo-holomorphic disks, to get the invariance proof of Legendrian contact homology over $\mathbb{Z}$ to work out well. These extra directions also simplify the work of assuring that we get a trivializable boundary condition on the non-punctured disk when we glue the capping operators to $\bar{\partial}_u$.

In the case when we are considering Legendrian knots $\Lambda \subset \mathbb{R}^3$ (i.e. when $n = 1$) we add one auxiliary direction, denoted by $\text{Aux}_1 \simeq \mathbb{R}$. In the more general setting when $n \geq 2$ we add two auxiliary directions $\text{Aux}_1 \oplus \text{Aux}_2 \simeq \mathbb{R}^2$. We will in what follows use $d_A$ for the dimension of the auxiliary space added. That is, if $n = 1$ then $d_A = 1$, and if $n > 1$ then $d_A = 2$.

If $u$ is a pseudo-holomorphic disk of $\Lambda$ or of $L$, then the linearized $\bar{\partial}_u$-problem is extended to the auxiliary directions so that it gives an isomorphism here. See [EES05b], Section 3.3.3]. Thus we get a canonical isomorphism between the determinant line of the original $\bar{\partial}_u$-problem and the extended one. With abuse of notation, we let $\bar{\partial}_u$ denote the extended problem from now on.

4.5.C. System of capping operators for disks in $T^*M$. The capping operators $\bar{\partial}_{p,\pm}$ are also extended to the auxiliary directions, but will in general not give isomorphisms in these directions. To describe the properties that we require the capping operators to have, recall that we assume to have fixed a trivialization of the Lagrangian boundary conditions of $\bar{\partial}_u$ (now also extended to the auxiliary directions, using the trivial spin structure here). If now $p$ is a puncture of $u$, then notice that we have two Lagrangian subspaces associated to $u$ at $p$, given by the two stabilized tangent spaces of $\Pi_C \Lambda$ at $p$. From the fixed trivialization we then get oriented frames for these two spaces. Let $p_+, p_-$ denote the endpoints of the Reeb chord of $\Lambda$ corresponding to $p$, where $p_+$ corresponds to the end with largest $z$-coordinate. Let $X_{\pm}$ denote the oriented frame of the stabilized tangent space of $\Pi_C \Lambda$ at $p$ that lifts to $T_{p\pm} \Lambda$.

We define a system of capping operators for $\Lambda$ to be a set $\mathcal{S}$ consisting of $\bar{\partial}$-operators defined on the one-punctured disk in $\mathcal{S}$, such that for each Reeb chord $p$ of $\Lambda$ we have a pair of $\bar{\partial}$-operators $\bar{\partial}_{p,+}, \bar{\partial}_{p,-} \in \mathcal{S}$ with associated trivialized boundary conditions $R_{p,\pm} : \partial D_1 \to U(n + d_A)$. Moreover, as a part of the data of the system we choose an orientation of $\det \bar{\partial}_{p,-}$ for each Reeb chord $p$.

We say that the system is admissible if the operators satisfy the following:

(c1) $R_{p,-}$ takes the oriented frame $X_1$ to the oriented frame $X_2$;
(c2) $R_{p,+}$ takes the oriented frame $X_2$ to the oriented frame $X_1$;
(c3) 
\[ \dim \ker \bar{\partial}_{p,+} \equiv 0, \quad \dim \text{coker } \bar{\partial}_{p,+} \equiv |p| + n + d_A + 1, \]
\[ \dim \ker \bar{\partial}_{p,-} \equiv 1, \quad \dim \text{coker } \bar{\partial}_{p,-} \equiv |p|, \]
everything modulo 2.

**Remark 4.4.** The author has not been able to prove Theorem 2.1–2.7 for capping operators not satisfying (c1)–(c3), but believe it should be possible.

4.5.D. **Capping trivialization in the symplectization-direction.** We extend the boundary conditions $R_{r,\pm}$ to the symplectization, by defining them to be given by the identity in the $\mathbb{R}_t$-direction. We denote the induced capping operators by $\partial_{s,r,\pm}$, and we use the notation $\partial_{l,r,\pm}$ for the capping operators for disks in $T^*M$ defined above (that is, $\partial_{l,r,\pm}$ is the restriction of $\partial_{s,r,\pm}$ to $T^*M$).

At the ends of a cobordism $L$, we may assume that we have an asymptotic behavior $L = \mathbb{R} \times \Lambda_{\pm}$. If $u$ is a holomorphic disk of $L$ (or of $\mathbb{R} \times \Lambda$), then this implies that we need to weight the Sobolev spaces in the domain and target of $\partial_n$ in the $\mathbb{R}_t$-direction, to get a Fredholm problem. We do this by imposing a small negative exponential weight $e^{-\delta|\tau|\partial_t}$ at all punctures of $D_{m+1}$, where $t$ is the coordinate in the symplectization-direction and where we have chosen coordinates $(\tau, s) \in [0, \infty) \times [0,1]$ in a neighborhood of each puncture. For a more detailed discussion on this subject, see e.g. [Don02] and [EES05a].

We need a similar construction to get the capping operators $\partial_{s,r,\pm}$ to be Fredholm, but now we put a small positive exponential weight at the puncture in the $\partial_t$-direction. This implies that in the $\mathbb{R}_t$-direction the capping operators are isomorphisms. This can be summarized as follows.

**Proposition 4.5.** For every Reeb chord $p$ of $\Lambda$ the projection $\pi_p$ extends to canonical isomorphisms

$$
\pi_p : \text{Ker } \partial_{s,r,\pm} \to \text{Ker } \partial_{l,r,\pm}
$$

$$
\pi_p : \text{Coker } \partial_{s,r,\pm} \to \text{Coker } \partial_{l,r,\pm}.
$$

4.5.E. **Orientation of capping operators.** Next we define the orientation of the capping operators $\partial_{s,r,\pm}$, following the constructions from [EES05a].

Recall that for each Reeb chord $p$ we are assumed to fix an orientation of $\text{det } \partial_{l,r,-}$ when we specify the system of capping operators that we choose. This will be the capping orientation of $\partial_{l,r,-}$. Notice that this canonically induces an orientation of $\partial_{s,r,-}$ via the isomorphism in Proposition 4.5.

To define the corresponding orientation of $\text{det } \partial_{s,r,+}$, let $\partial_{s,r}$ denote the $\partial$-problem on the closed disk, obtained by gluing the $\partial_{s,r,+}$-problem to the $\partial_{s,r,-}$-problem, and consider the induced exact gluing sequence

$$
0 \to \text{Ker } \partial_{s,r} \to \left[ \begin{array}{c} \text{Ker } \partial_{s,r,+} \\ \mathbb{R}_t \\ \text{Ker } \partial_{s,r,-} \end{array} \right] \to \left[ \begin{array}{c} \text{Coker } \partial_{s,r,+} \\ \text{Coker } \partial_{s,r,-} \end{array} \right] \to \text{Coker } \partial_{s,r} \to 0.
$$

Here the $\mathbb{R}_t$-summand corresponds to a gluing kernel which is born when gluing positive weighted Sobolev spaces, compare [Kar]. The chosen capping orientation of $\partial_{s,r,-}$ together with the canonical orientation of $\text{det } \partial_{s,r}$, induce an orientation $\mathcal{O}(p_+)$ of $\text{det } \partial_{s,r,+}$, via the sequence (4.5) and the isomorphism (4.2).
Definition 4.6. We define the capping orientation of \( \tilde{\partial}_{s,p} \) to be \((-1)^{|p|+n+d_A+1}O(p+)\).

Remark 4.7. We will refer to the \( \tilde{\partial}_{s,p} \)-problem as the glued capping disk at \( p \).

We give the operator \( \tilde{\partial}_{l,p} \) the capping orientation induced by the capping orientation of \( \tilde{\partial}_{s,p} \) under the isomorphism \( \pi_p \) from Proposition [EES05b]. We let \( \tilde{\partial}_{l,p} \) denote the \( \tilde{\partial} \)-problem on the non-punctured disk obtained from gluing \( \tilde{\partial}_{l,p,+} \) to \( \tilde{\partial}_{l,p,-} \), and we notice the following.

Lemma 4.8. Assume that we have chosen an admissible system of capping operators for \( \Lambda \). Then the capping orientations of \( \tilde{\partial}_{l,p,+} \) and \( \tilde{\partial}_{l,p,-} \) glue to the canonical orientation of \( \tilde{\partial}_{l,p} \), times \((-1)^{|p|+n+d_A+1} \), under the exact gluing sequence

\[
\text{(4.6)} \quad 0 \to \text{Ker} \tilde{\partial}_{l,p} \to \begin{bmatrix} \text{Ker} \tilde{\partial}_{l,p,+} \\ \text{Ker} \tilde{\partial}_{l,p,-} \end{bmatrix} \to \begin{bmatrix} \text{Coker} \tilde{\partial}_{l,p,+} \\ \text{Coker} \tilde{\partial}_{l,p,-} \end{bmatrix} \to \text{Coker} \tilde{\partial}_{l,p} \to 0.
\]

Remark 4.9. Since we use the gluing convention [EES05b], this gives the opposite convention of [EES05b]. Also notice that the sign \((-1)^{|p|+n+d_A+1}\) in the definition of the orientation of the capping operators is not used in that paper.

Proof of Lemma 4.8. By construction we have that \( \text{Ker} \tilde{\partial}_{s,p} \simeq \mathbb{R}_t \oplus \text{Ker} \tilde{\partial}_{l,p} \), and from [EES05b], Remark 3.3] we see that the first nontrivial map in (4.5) restricted to the \( \mathbb{R}_t \)-factors is given by projection \( \nu \mapsto \nu \). Since \( \dim \text{Ker} \tilde{\partial}_{s,p} \equiv 0 \mod 2 \) we can remove \( \mathbb{R}_t \) from both the first and second nontrivial column without affecting orientations on the remaining spaces. But after removing \( \mathbb{R}_t \) we get the gluing sequences for the capping operators in the Lagrangian projection, and since the canonical orientation is given via evaluation the result follows.

Remark 4.10. From now on, we use the notation \( \tilde{\partial}_{p\pm} \) to denote the capping operators both in the symplectization-setting and in the setting of the Lagrangian projection.

4.6. Capping orientation of disks. Now we give a definition of the capping orientation of a punctured pseudo-holomorphic disk. Below \( X \) denotes either \( T^*M \) or \( \mathbb{R} \times J^1(M) \), with almost complex structure \( J \) as described in Section 3.

If \( L \) is an exact Lagrangian cobordism with cylindrical Legendrian ends \( \Lambda_\pm \), then assume that \( S_{\pm} \) gives a system of capping operators for \( \Lambda_\pm \). This gives rise to an induced system of capping operators of \( L \), where the positive capping operators \( \tilde{\partial}_{p,+} \) are taken from the system \( S_+ \), and the negative capping operators \( \tilde{\partial}_{p,-} \) are taken from \( S_- \). This system is admissible if both \( S_{\pm} \) are admissible.

Let \( u : D_m \to X \) be a pseudo-holomorphic disk of \( \Lambda \) or \( L \), with positive puncture \( a \) and negative punctures \( b_1, \ldots, b_m \). Assume that we have fixed a system of capping operators, and consider the exact gluing sequence

\[
\text{(4.7)} \quad \text{Ker} \tilde{\partial}_u \to \begin{bmatrix} \text{Ker} \tilde{\partial}_u \\ \text{Ker} \tilde{\partial}_{a,+} \\ \text{Ker} \tilde{\partial}_{b_{m,+}} \\ \vdots \\ \text{Ker} \tilde{\partial}_{b_1,+} \end{bmatrix} \to \begin{bmatrix} \text{Coker} \tilde{\partial}_u \\ \text{Coker} \tilde{\partial}_{a,+} \\ \text{Coker} \tilde{\partial}_{b_{m,+}} \\ \vdots \\ \text{Coker} \tilde{\partial}_{b_1,+} \end{bmatrix} \to \text{Coker} \tilde{\partial}_u.
\]
Here \( \bar{\partial}_u \) denotes the \( \bar{\partial} \)-problem on the non-punctured disk with trivialized boundary condition \( \hat{u} \), which is obtained by gluing the trivialized boundary condition induced by \( u \) to the positive capping trivialization of \( a \) at the positive puncture of \( D_m \), and then to the negative capping trivializations of \( b m, . . . , b_1 \) at the corresponding negative punctures. We refer to the \( \bar{\partial}_u \)-problem as the fully capped problem corresponding to \( u \), and to the sequence (4.7) as the capping sequence for \( u \).

**Definition 4.11.** We define the capping orientation of \( u \in M_{l,\Lambda}(a, b) \) and \( M_{s,\Lambda}(a, b) \) to be the orientation \( O(\bar{\partial}_u) \) on \( \det \bar{\partial}_u \) induced by the gluing sequence (4.7), where \( \det \bar{\partial}_u \) is given the canonical orientation, and the capping operators are given their capping orientations. For \( u \in M_{s,\Lambda}(a, b) \) we define the capping orientation to be given by \( \det \bar{\partial}_\pi(u) \wedge \partial_t = (-1)^{\text{index}(\pi_u)} \partial_t \wedge \det \bar{\partial}_\pi(u) \), where \( \det \bar{\partial}_\pi(u) \) is given its capping orientation and \( \partial_t \) gives the positive orientation in the symplectization direction.

**Remark 4.12.** We will use the notation \( \partial_S \) to indicate the dependence of the chosen capping system \( S \) in the definition for the DGA-differential.

In the case when \( u \) is a rigid disk (and where we assume that we have divided out the \( \mathbb{R}_t \)-action if \( u \) is a disk of \( \mathbb{R} \times \Lambda \)), the capping orientation of \( u \) can be understood as an orientation of the kernel or the cokernel of \( \bar{\partial}_u \). Moreover, by [EES05b, Section 4.2.3], we may assume that we are in the case when \( \text{Ker} \bar{\partial}_u \) is trivial, so that an orientation of \( \det \bar{\partial}_u \) is nothing but an orientation of \( \text{Coker} \bar{\partial}_u \). Let \( \kappa \) denote the conformal structure of \( u \), and recall that \( \text{Coker} \bar{\partial}_u \) is isomorphic to the space of conformal variations at \( \kappa \). Since this latter space was given a fixed orientation in Section 4.3, we can compare the capping orientation of \( u \) with this orientation and get a sign \( \sigma(u) \in \{ -1, 1 \} \). This sign is the capping sign of \( u \), and is the one that we use in the algebraic count of the elements in the moduli spaces when defining the DGA-morphisms and the DGA-differentials. See [EES05b, Section 3.4.3]

4.7. **Proof of Theorem 2.1.** To prove Theorem 2.1 it only remains to establish the following.

**Lemma 4.13.** Let \( i = s \) or \( i = l \). Then the map defined by

\[
\partial_i a = \sum_{\dim M(a, b) = 0} |M_i(a, b)| b
\]

on generators and extended by the signed Leibniz rule to rest of the algebra, satisfies \( \partial_i^2 = 0 \). Here \( |M_i(a, b)| \) is the algebraic count of disks in \( M_i(a, b) \), where each disk is counted with its capping sign.

**Proof.** We follow the proof of Theorem 4.1 in [EES05b], and in particular the notations therein. Briefly, the argument goes as follows. Let \( A \) and \( B \) denote trivialized Lagrangian boundary conditions associated to the punctured disks \( D_m \) and \( D_{r+1} \), respectively, and assume that we glue them together as described in Section 4.2. That is, we glue the positive puncture of \( \bar{\partial}_A \) to the \( j \)-th negative puncture of \( \bar{\partial}_B \). Let \( O(\bar{\partial}_iA) \) and \( O(\bar{\partial}_iB) \) denote the corresponding capping orientations, where \( i = l, s \). Also, let...
\( \mathcal{O}(\bar{\partial}_{i,A \# B}) \) denote the induced orientation of the \( \bar{\partial} \)-problem on the non-punctured disk, obtained by gluing the fully capped problems \( \bar{\partial}_{i,A} \) and \( \bar{\partial}_{i,B} \) together.

Note that \( \text{Ker} \bar{\partial}_{s,A} \cong \mathbb{R}_t \oplus \text{Ker} \bar{\partial}_{l,A} \), \( \text{Coker} \bar{\partial}_{s,A} \cong \text{Coker} \bar{\partial}_{l,A} \), and similarly for \( \hat{A} \), \( B \) and \( \hat{B} \). We write \( \bar{\partial}_C = \bar{\partial}_{l,C} \) to simplify notation, where \( C \) is any trivialized Lagrangian boundary condition on the (possibly punctured) disk.

We assume that these problems are associated to rigid disks of \( \Lambda \) such that we get one of the boundary components of a compactified 1-dimensional moduli space when we glue the disks. We want to calculate the induced orientation of this boundary component from the capping orientation of \( \bar{\partial}_A \) and \( \bar{\partial}_B \). Compare [EES05b], Lemma 4.9.

We claim that with our orientation conventions we should replace the sign \((-1)^\nu\),

\[
\nu = mk + \sum_{j=1}^{k-1} |b_j| + |b_k|(n-1),
\]

in [EES05b], Lemma 4.9, by the sign \((-1)^\sigma\),

\[
\sigma = mk + \sum_{j=1}^{k-1} |b_j| + r + 1.
\]

To prove this, we just copy the techniques from the proof of [EES05b], Lemma 4.9. We give an outline of the arguments for clarity.

Let \( \bigoplus_{j=1}^r V_j = V_r \oplus \cdots \oplus V_1 \). The first sequence in [EES05b], Lemma 4.9 now looks like, with our orientation conventions and in the case when we consider the symplectization setting,

\[
(4.8) \quad \begin{bmatrix} \mathbb{R}_t \\ \text{Ker} \bar{\partial}_{A \# B} \end{bmatrix} \to \begin{bmatrix} \mathbb{R}_t^B \\ \bigoplus_{j=1}^r \text{Ker} \bar{\partial}_{b_j-} \end{bmatrix} \to \begin{bmatrix} \text{Coker} \bar{\partial}_B \\ \text{Coker} \bar{\partial}_{a+} \\ \bigoplus_{j=1}^r \text{Coker} \bar{\partial}_{b_j-} \\ \mathbb{R}_t^{n+d_A} \\ \text{Coker} \bar{\partial}_A \\ \text{Coker} \bar{\partial}_{b_k+} \\ \bigoplus_{j=1}^{m-1} \text{Coker} \bar{\partial}_{f_j-} \end{bmatrix} \to \text{Coker} \bar{\partial}_{A \# B}
\]

and the second reads
Here we use the notation $\mathbb{R}_t^B$ for the $R_t$-factor in $\ker \partial_B$, and similar for $\mathbb{R}_t^A$.

Now we perform the same moves as in the proof of Lemma 4.9 in [EES05b], and compute the permutation signs. We do this in the case when $n > 1$, the case $n = 1$ is similar and left to the reader. Due to our conventions, the only differences that we get are the following. First, we get a sign $(-1)^{\sigma_0}$,

$$\sigma_0 = r + n + d_A,$$

coming from moving $\mathbb{R}_t^A$ in the first gluing sequence to the position right under $\mathbb{R}_t^B$, in the same time as we move the $R_t$-factor in the third column to the position right under $\text{Coker } \partial_B$. We also get an extra sign $(-1)^{\sigma_1}$, where

$$\sigma_1 \equiv |b_k| \cdot (|b_k| + n + d_A + 1) \equiv |b_k| \cdot (n + d_A) \pmod{2},$$

which comes from moving $\text{Coker } \partial_{b_k^+}$ over $\text{Coker } \partial_{b_k^-}$ at the very end.

Thus we get a total sign $(-1)^{\sigma_2}$, where

$$\sigma_2 = mk + \sum_{j=1}^{k-1} |b_j| + |b_k| + n + d_A + r.$$

Indeed, in the case $n > 2$ this follows from the fact that $d_A = 2$, just by adding $\sigma_2$ to $\nu$.

Now, notice that the second gluing sequence does not induce the canonical orientation of $\det \partial_{A^\#B}$, but $(-1)^{|b_k|+n+d_A+1}$ times the canonical orientation, since in our setting the pair

$$\begin{pmatrix} \ker \partial_{b_k^+} \\ \mathbb{R}_t \\ \ker \partial_{b_k^-} \end{pmatrix}, \begin{pmatrix} \text{Coker } \partial_{b_k^+} \\ \text{Coker } \partial_{b_k^-} \end{pmatrix}$$

is oriented by $(-1)^{|b_k|+n+d_A+1}$ times the canonical orientation. Hence the claim follows.

The next claim is that the sign in [EES05b, Lemma 4.11] now equals

$$\sigma(\mathcal{M}_r) = \mu_1 \mu_2 (-1)^{r+m+1+\sum_{j=1}^{k-1} |b_j|}.$$
The occurrence of \( m \) comes from the fact that from the sequence (4.9) we get that \( TC_r \oplus TC_{m-1} \oplus \mathbb{R} \) should be identified with \( \text{Coker} \bar{\partial}_B \oplus \mathbb{R}_t \oplus \text{Coker} \bar{\partial}_A \). This is because of the fact that in the symplectization setting, we should interpret the gluing cokernel that occurs in the \( \mathbb{R}_t \)-direction as the outward normal of the space \( TC_{m+r-1} \).

Now if we trace the proof of [EES05b, Theorem 4.1] we see that all the occurrences of terms of the form \( 1 + r + m \) in \( \sigma(M_r) \) will cancel pairwise, and hence the result follows.

**Proof of Theorem 2.5** The first statement follows by the results in [DR16] together with Definition 4.11. The second statement follows from Lemma 4.13.

5. DGA-morphisms induced by exact Lagrangian cobordisms

Let \( L \) be an exact Lagrangian cobordism with cylindrical Legendrian ends \( \Lambda_{\pm} \), and assume that we are given admissible systems \( S_{\pm} \) of capping operators for \( \Lambda_{\pm} \). Let \( \partial_\pm = \partial_{S_{\pm}} \) to simplify notation. Moreover, let \( S \) denote the induced system of \( L \), and let \( \Phi_L : (A(\Lambda_+), \partial_+, Z) \to (A(\Lambda_-), \partial_-, Z) \) be defined by (2.2) – (2.4), where now \( |M_L(a, b)| \) denotes the algebraic count induced by the capping orientation of \( M_L \) corresponding to the system \( S \). When we want to emphasize that \( \Phi_L \) is given with respect to the system \( S \) we write \( \Phi_{L,S} \).

In this section we prove that this setup gives the statements in Theorem 2.5, Theorem 2.6, and Theorem 2.7.

5.1. Proof of Theorem 2.5 and Theorem 2.7. To prove Theorem 2.5 we must show that

\[
\Phi_L \circ \partial_+ = \partial_- \circ \Phi_L.
\]

This uses similar techniques as the proof of Theorem 2.1. That is, we will use compactness results from [BEH03] to pair up 2-level buildings that emanates from the left hand side of (5.1) with 2-level buildings that emanates from the right hand side. We perform this pairing in a way so that each pair can be interpreted as the boundary of a 1-dimensional moduli space. Then we compute the difference in orientations induced by gluings of such buildings, and argue that all the signs cancel.

As a corollary, we get the statement in Theorem 2.7.

**Proof of Theorem 2.5** Assume that \( L \subset \mathbb{R} \times J^1(M) \) is an exact Lagrangian cobordism with cylindrical Legendrian ends \( \Lambda_{\pm} \). Let \( a \) be a Reeb chord of \( \Lambda_+ \), and let \( u_0 \in M_{s_0, \Lambda_+}(a, b_1 \cdots b_m) \), \( v_0 \in M_L(a, c_1 \cdots c_l) \) be rigid disks. Also pick rigid disks \( v_i \in M_L(b_{i_1} b_{i_2} \cdots b_{i_m}) \) for \( i = 1, \ldots, m \), and \( u_j \in M_{s, \Lambda_-}(c_j f_1 \cdots f_k) \) for some \( j \in \{1, \ldots, l\} \), such that we can interpret \( u_0 \# \sum_{i=1}^m v_i \) and \( v_0 \# u_j \) as broken boundary components of a 1-dimensional moduli space \( M_L \). In particular, we assume that

\[
b_{i_1} \cdots b_{i_m}^1 b_{j_1}^2 \cdots b_{j_m}^2 \cdots b_{i_m}^m \cdots b_{j_m}^m = c_1 \cdots c_{j-1} f_1 \cdots f_k \cdot c_{j+1} \cdots c_l.
\]

Let \( \bar{\partial}_L \) denote the \( \bar{\partial} \)-problem we get when we glue the fully capped \( \bar{\partial}_{a_0} \)-problem to the fully capped \( \bar{\partial}_{v_i} \)-problems, \( i = 1, \ldots, m \). This gives the same problem as if we first
glue $u_0$ to $v_1, \ldots, v_m$, then glue the capping operators to the punctures of this new problem, and then glue this to the glued capping disks at $b_1, \ldots, b_m$.

We also have to analyze the situation at the other boundary component of $\mathcal{M}_L$. To that end, let $\bar{\partial}_T$ denote the dbar-problem that we get if we glue the fully capped $\bar{\partial}_{u_0}$-problem to the fully capped $\bar{\partial}_{u_j}$-problem. This is the same dbar-problem as we get if we first glue $v_0$ to $u_j$, then glue the capping operators to the punctures of this new problem, and then glue this fully capped problem to the glued capping disk at $c_j$.

Now, following the arguments in [EES05b, Lemma 4.9], we first calculate the difference in orientation of $\det \bar{\partial}_T$, induced by the 2 different gluings described above, and then we do the same thing for $\det \bar{\partial}_T$. If these differences cancel modulo 2 when we take the orientation of the space of conformal variations associated to $\mathcal{M}_L$ into account, then we get that $\Phi_L \circ \partial_+ = \partial_- \circ \Phi_L$.

From the gluings for $\bar{\partial}_T$ we get the following two exact sequences. To simplify notation, we assume that $u_i, v_j, i = 0, \ldots, l, j = 0, \ldots, m$, all have at least 2 negative punctures.

The first exact sequence has the form

\[
\begin{align*}
\text{Ker} \bar{\partial}_T &\rightarrow \\
\begin{pmatrix} 
\text{Ker} \bar{\partial}_{u_0} \\
\text{Ker} \bar{\partial}_{a,+} \\
\text{Ker} \bar{\partial}_{b_1,-} \\
\vdots \\
\text{Ker} \bar{\partial}_{b_m,-} \\
\text{Ker} \bar{\partial}_{v_m} \\
\text{Ker} \bar{\partial}_{b_m,+} \\
\text{Ker} \text{cap}(v_m, -) \\
\vdots \\
\text{Ker} \bar{\partial}_{v_1} \\
\text{Ker} \bar{\partial}_{b_1,+} \\
\text{Ker} \text{cap}(v_1, -) 
\end{pmatrix} &\rightarrow \\
\begin{pmatrix} 
\text{Coker} \bar{\partial}_{u_0} \\
\text{Coker} \bar{\partial}_{a,+} \\
\text{Coker} \bar{\partial}_{b_m,-} \\
\vdots \\
\text{Coker} \bar{\partial}_{b_1,-} \\
\mathbb{R}_{t,m} \\
\mathbb{R}^{n+i} \\
\text{Coker} \bar{\partial}_{v_m} \\
\text{Coker} \bar{\partial}_{b_m,+} \\
\text{Coker} \text{cap}(v_m, -) \\
\vdots \\
\text{Coker} \bar{\partial}_{v_1} \\
\text{Coker} \bar{\partial}_{b_1,+} \\
\text{Coker} \text{cap}(v_1, -) 
\end{pmatrix} &\rightarrow \text{Coker} \bar{\partial}_T,
\end{align*}
\]

where $\text{Ker} \text{cap}(v_i, -) = \bigoplus_{j=1}^{m_i} \text{Ker} \bar{\partial}_{b_i,-}$, and similar for the cokernel, and where we use the notation $\mathbb{R}_{t,i}$ to indicate the cokernel that is born when we glue the $\bar{\partial}_1$-disk to the bigger problem.
Similarly, the second gluing sequence has the form

\begin{align}
\text{Ker} \partial_T & \rightarrow \begin{pmatrix}
\text{Ker} \partial_{u_0} \\
\text{Ker} \partial_{v_m} \\
\vdots \\
\text{Ker} \partial_{v_1} \\
\text{Ker} \partial_{a,+} \\
\text{Ker} \text{cap}(v_m, -) \\
\text{Ker} \partial_{b_m,+} \\
\mathbb{R}_t \\
\text{Ker} \partial_{b_m,-} \\
\vdots \\
\text{Ker} \partial_{b_1,+} \\
\mathbb{R}_t \\
\text{Ker} \partial_{b_1,-} \\
\end{pmatrix} \\
& \rightarrow \begin{pmatrix}
\text{Coker} \partial_{u_0} \\
\mathbb{R}_{t,m} \\
\text{Coker} \partial_{v_m} \\
\vdots \\
\mathbb{R}_{t,1} \\
\text{Coker} \partial_{v_1} \\
\text{Coker} \partial_{a,+} \\
\text{Coker} \text{cap}(v_m, -) \\
\vdots \\
\text{Coker} \partial_{b_m,+} \\
\mathbb{R}_t \\
\text{Coker} \partial_{b_m,-} \\
\vdots \\
\text{Coker} \partial_{b_1,+} \\
\mathbb{R}_t \\
\text{Coker} \partial_{b_1,-} \\
\end{pmatrix} \\
& \rightarrow \text{Coker} \partial_T.
\end{align}

(5.3)

To compare the two different orientations induced on det \( \partial_T \), notice that by construction of the gluing map, each \( \mathbb{R}_t \)-summand in the second column in (5.3) is mapped by the identity to the corresponding \( \mathbb{R}_t \)-summand in the third column. That is, the \( \mathbb{R}_t \)-summand in the kernel of the glued capping disk at \( b_i \) is mapped by the identity to the \( \mathbb{R}_t \)-summand that corresponds to the cokernel that is born when we glue the glued capping disk at \( b_i \) to the larger problem. Thus, using that we require the capping operators to satisfy (c3) we can remove all these occurrences of \( \mathbb{R}_t \) from the sequence (5.3) without changing any induced orientation. This is done by moving all these space to the bottom of column 2 and 3, respectively, and recalling our orientation convention (4.3).

Next we rearrange vector spaces in the sequence (5.2), and we start with letting Ker \( \partial_{b_i,-} \) switch position with Ker \( \text{cap}(v_i, -) \) for \( i = m, \ldots, 1 \). This costs \((-1)^{\sigma_1}\),

\[
\sigma_1 \equiv (m_m + 1) \cdot (m - 1) + m_m + \ldots + (m_1 + 1) \cdot (m - 1) + m_1
\equiv m \cdot \sum_{i=1}^{m} m_i \pmod{2}.
\]

Next we move \( \mathbb{R}_{t,m} \) and Coker \( \partial_{v_m} \) to the place just below Coker \( \partial_{u_0} \). This costs \((-1)^{\nu}\),

\[
\nu = n + d_A + (m_m + 1) \cdot (n + d_A + n + d_A) = n + d_A.
\]
Perform the similar move with the other such spaces, so that we end up with
\[(5.4) \quad \text{Coker } \tilde{\partial}_{u_0} \oplus \mathbb{R}_{t,m} \oplus \text{Coker } \tilde{\partial}_{v_m} \oplus \ldots \oplus \mathbb{R}_{t,1} \oplus \text{Coker } \tilde{\partial}_{v_1}\]
at the top of the third column in sequence (5.2). These moves have a total cost \((-1)^{\sigma_2},\)
\[\sigma_2 = m \cdot (n + d_A) + m \cdot \sum_{i=1}^{m} m_i + m + \sum_{i=1}^{m} i \cdot (m_i + 1).\]
In the last step we switch position of \(\text{Coker } \tilde{\partial}_{b_i,-} \) and \(\text{Coker } \text{cap}(v_i, -)\) for \(i = m, \ldots, 1.\) This costs \((-1)^{\sigma_3},\)
\[\sigma_3 = \sum_{i=1}^{m} |b_i|.

Thus, in total we get a permutation sign \((-1)^{\sigma},\)
\[\sigma = \sigma_1 + \sigma_2 + \sigma_3 = m \cdot (n + d_A) + m + \sum_{i=1}^{m} i \cdot (m_i + 1) + \sum_{i=1}^{m} |b_i|
= \sum_{i=1}^{m} i \cdot (m_i + 1) + \sum_{i=1}^{m} (|b_i| + n + d_A + 1).
\]

Now we do the similar thing for the \(\tilde{\partial}_{\tilde{T}}\)-problem, which gives a total permutation sign \((-1)^{\bar{\sigma}},\)
\[\bar{\sigma} = (|c_j| + n + d_A + 1) + j \cdot (k + 1) + k + l + \sum_{i=1}^{j-1} |c_i|.
\]
The details are left to the reader.

Now it remains to calculate the contribution from the orientation of the space of conformal variations. This is also similar to the arguments in the proof of Theorem 2.1. That is, from (5.4) we get a sign \((-1)^{\sigma_4},\)
\[\sigma_4 = \sum_{i=1}^{m} m_i + (m_m - 1) \cdot m + 1 + \ldots + (m_1 - 1) \cdot m_1 + 1
= l + k + 1 + m + \sum_{i=1}^{m} i \cdot (m_i + 1).
\]
For the \(\tilde{\partial}_{\tilde{T}}\)-problem we get that the corresponding cokernels in the second sequence equals
\[\text{Coker } \tilde{\partial}_{v_0} \oplus \mathbb{R}_t \oplus \text{Coker } \tilde{\partial}_{u_j},\]
and this gives rise to a sign \((-1)^{\sigma_5},\)
\[\sigma_5 = j \cdot (k + 1) + 1 + k.
\]
Thus, what we get in total after having added \(\sigma_4 + \sigma_5\) to \(\sigma + \bar{\sigma},\) and also having taken into account that the \(\tilde{\partial}_{c}\)-problem corresponding to the capping disk of a Reeb chord \(c\) is oriented by \((-1)^{|c|+n+d_A+1}\) times the canonical orientation, is a sign \((-1)^{k+m+\sum_{i=1}^{j-1} |c_i|,}\)
But \(\sum_{i=1}^{j-1} |c_i|\) comes from the Leibniz rule, and \(m+k\) from the definition of the capping orientation of \(u_0\) and \(u_j.\) The statement in the theorem then follows. \(\square\)
Proof of Theorem 2.7. From [EES05b] we get that there exists an admissible system \( \mathcal{S} \) of capping operators satisfying the first statement in the theorem. Now, if \( \mathcal{S}' \) is another admissible system of capping operators, then lift both systems to the symplectization, as described in Section 4.5.

In this setting, the result about the DGA-isomorphism follows by exactly the same arguments as above, with \( L = \mathbb{R} \times \Lambda \) and where \( \Lambda_+ = \Lambda \) is equipped with the system \( \mathcal{S} \) and \( \Lambda_- = \Lambda \) is equipped with the system \( \mathcal{S}' \). The DGA-isomorphism is then given by

\[
\Phi_{\mathcal{S}, \mathcal{S}'}(a) = \sigma(u_a)a,
\]

where \( a \) is a Reeb chord of \( \Lambda \), \( u_a = \mathbb{R} \times a \), and \( \sigma(u_a) \) is the capping sign of \( u_a \) with respect to the induced system of \( L \).

Thus, any admissible system gives rise to a DGA which is isomorphic to the system in [EES05b], and the result follows. \( \square \)

5.2. Proof of Theorem 2.6. In this section we prove that \( \Phi \) satisfies the functorial properties stated in Theorem 2.6. To that end, if \( L = \mathbb{R} \times \Lambda \) and we wish to prove (2.6), note that we shall assume that the capping system \( \mathcal{S} \) of \( \Lambda_+ = \Lambda \) equals that of \( \Lambda_- = \Lambda \), which in turn equals the induced system of \( L \).

To prove (2.5), we need to have the following setup. Assume that \( L_1 \) is an exact Lagrangian cobordism from \( \Lambda_0 \) to \( \Lambda_1 \), and that \( L_2 \) is an exact Lagrangian cobordism from \( \Lambda_1 \) to \( \Lambda_2 \). Assume that \( L_1 \) and \( L_2 \) are equipped with spin structures such that the induced spin structure on \( \Lambda_1 \) from \( L_1 \), regarded as the negative boundary of \( L_1 \), equals the induced spin structure on \( \Lambda_1 \) induced from \( L_2 \), regarded as the positive boundary of \( L_2 \). Let \( \mathcal{S}_i \) be fixed admissible systems of capping operators for \( \Lambda_i \), \( i = 0, 1, 2 \), let \( \mathcal{S}_{01} \) denote the induced system on \( L_1 \), \( \mathcal{S}_{12} \) the induced system on \( L_2 \), and \( \mathcal{S}_{02} \) the induced system on the concatenation \( L_1 \# L_2 \). We refer to [EHK16] for a discussion on concatenations of exact Lagrangian cobordisms.

Proof of Theorem 2.6. We must prove that

\[
\Phi_{\mathbb{R} \times \Lambda, \mathcal{S}} = \text{id}, \quad \Phi_{L_2, \mathcal{S}_{12}} \circ \Phi_{L_1, \mathcal{S}_{01}} = \Phi_{L_1 \# L_2, \mathcal{S}_{02}}.
\]

We prove the statement for the identity map. The statement for the concatenation is similar to the proof of Theorem 2.5, and details are left to the reader. Indeed, the only thing that has to be checked in that case is the following. Let \( u_0 \in \mathcal{M}_L(a, b_1 \cdots b_m) \), \( v_i \in \mathcal{M}_L(b_i, b_1^i \cdots b_{m_i}^i), i = 1, \ldots, m, u_1 \in \mathcal{M}_{L_1 \# L_2}(a, b_1^1 \cdots b_{m_1}^1 \cdots b_1^m \cdots b_{m_m}^m) \) be rigid disks. Then one must prove that the gluing sequence for \( \bar{\partial} u_0 \# i=1 \bar{\partial} v_i \) induces the same orientation on the total glued problem as the gluing sequence we get when we glue the glued capping disks at \( b_1, \ldots, b_m \), with respect to the \( \mathcal{S}_1 \)-system, to the \( \bar{\partial} \)-problem \( \bar{\partial} u_1 \), capped off with the \( \mathcal{S}_{02} \)-system. But this follows from similar arguments as in the proof of Theorem 2.5.

We turn to the trivial cobordism. Let \( a \) be a Reeb chord of \( \Lambda \), let \( u = \mathbb{R} \times a \), and assume that \( \Phi_{\mathbb{R} \times \Lambda, \mathcal{S}}(a) = (-1)^\sigma a \). Here the sign \( (-1)^\sigma \) satisfies that \( \text{Ker}\bar{\partial}_u \simeq \mathbb{R}_t \) is given the capping orientation \((-1)^\sigma \bar{\partial}_t\) with respect to the system \( \mathcal{S} \). We must prove that \( \sigma \equiv 0 \) (mod 2).
Consider the situation when we concatenate two trivial cobordisms. That is, let $L_1 = L_2 = \mathbb{R} \times \Lambda$, let $u_1$ denote the disk $\mathbb{R} \times a$ of $L_1$, let $u_2$ denote the disk $\mathbb{R} \times a$ of $L_2$ and let $u$ denote the disk $\mathbb{R} \times a$ in $L_1 \# L_2$. Let $\partial_{u,+}$ denote the capping operator associated to $a$, regarded as a Reeb chord of the positive end of $L_1$, and let $\partial_{a,-}$ denote the capping operator associated to $a$, regarded as a Reeb chord of the negative end of $L_1$. Similarly, let $\partial_{u,+}^m, \partial_{a,-}^l$ denote the capping operators of $a$ regarded as Reeb chord of the positive and negative end of $L_2$.

From the concatenation we get the following exact sequence

$$\text{Ker } \tilde{\partial}_u \to \begin{bmatrix} \text{Ker } \tilde{\partial}_{u_2} \\ \text{Ker } \tilde{\partial}_{u_1} \end{bmatrix} \to \mathbb{R}_t \to 0,$$

where all non-trivial spaces are isomorphic to $\mathbb{R}_t$. Notice that this sequence induces a $+\mathbb{R}_t$-orientation of $\text{Ker } \tilde{\partial}_u$, given that $\text{Ker } \tilde{\partial}_{u_i}$ is given the orientation $(-1)^i \mathbb{R}_t$, $i = 1, 2$.

Now, following our standard arguments, consider the exact gluing sequence

$$\text{Ker } \tilde{\partial}_T \to \begin{bmatrix} \text{Ker } \tilde{\partial}_{u_1} \\ \text{Ker } \tilde{\partial}_{u_2} \\ \text{Ker } \tilde{\partial}_{a,+}^m \\ \text{Ker } \tilde{\partial}_{a,-}^l \\ \text{Ker } \tilde{\partial}_{a,-}^m \\ \text{Ker } \tilde{\partial}_{a,+}^l \end{bmatrix} \to \begin{bmatrix} \text{Coker } \tilde{\partial}_{a,+}^h \\ \text{Coker } \tilde{\partial}_{a,-}^m \\ \mathbb{R}_t \\ \mathbb{R}^{n+dA} \\ \text{Coker } \tilde{\partial}_{a,+}^m \\ \text{Coker } \tilde{\partial}_{a,-}^l \end{bmatrix} \to \text{Coker } \tilde{\partial}_T.$$

From [EES05b] we get that this sequence induces the canonical orientation of $\det \tilde{\partial}_T$, where $T$ is the totally glued problem, given that the capping operators are given their capping orientation and $\text{Ker } \tilde{\partial}_{u_i}$ is given the capping orientation $(-1)^i \mathbb{R}_t$. But, similar to our arguments in the proof of Theorem 2.5, the $\tilde{\partial}_T$-problem can also be obtained from the following gluing sequence

$$\text{Ker } \tilde{\partial}_T \to \begin{bmatrix} \text{Ker } \tilde{\partial}_{u_1} \\ \text{Ker } \tilde{\partial}_{u_2} \\ \text{Ker } \tilde{\partial}_{a,+}^m \\ \text{Ker } \tilde{\partial}_{a,-}^l \\ \text{Ker } \tilde{\partial}_{a,-}^m \\ \text{Ker } \tilde{\partial}_{a,+}^l \end{bmatrix} \to \begin{bmatrix} \mathbb{R}_t \\ \text{Coker } \tilde{\partial}_{a,+}^h \\ \text{Coker } \tilde{\partial}_{a,-}^m \\ \mathbb{R}_t \\ \mathbb{R}^{n+dA} \\ \text{Coker } \tilde{\partial}_{a,+}^m \\ \text{Coker } \tilde{\partial}_{a,-}^l \end{bmatrix} \to \text{Coker } \tilde{\partial}_T.$$

We see that the orientation of $\det \tilde{\partial}_T$ induced by this sequence, where we assume that all spaces in column 2 and 3 are oriented as in sequence (5.6), equals the canonical orientation times $(-1)^{|a|+n+dA+1}$. Indeed, this sign we get if we do the usual rearrangements, comparing (5.6) with (5.7): we can remove the bottom-most $\mathbb{R}_t$ in column 2 and 3 in (5.7) without any cost, then move $\text{Ker } \tilde{\partial}_{u_2}$ and the gluing cokernel in (5.6) to their corresponding positions in (5.7) with the cost of $(-1)^{1+n+dA+1}$, and then switch positions of $\text{Ker } \tilde{\partial}_{a,-}^m$ and $\text{Ker } \tilde{\partial}_{a,-}^l$, and similar for the cokernels, with a cost of $(-1)^{1+|a|}$.

Now, if we use associativity of orientations under gluing, see [EES05b], Section 3.2.3, we can use the sequence (5.5) to replace the pair $(\text{Ker } \tilde{\partial}_{u_1} \oplus \text{Ker } \tilde{\partial}_{u_2}, \mathbb{R}_t)$ in (5.7)
by the pair $(\text{Ker} \bar{\partial}_u, 0)$. Recall that the latter pair has the induced orientation $+\mathbb{R}_t$ from
the capping orientation of $\bar{\partial}_u$, and $\bar{\partial}_u$. Also, notice that in the lower part of (5.7) we
have the gluing sequence for the capping disk at $a$, which gives the canonical orientation times $(-1)^{|a|+n+d_A+1}$, given that the capping operators are given their capping orientations. All this implies that the gluing sequence (5.7) induc es the canonical orientation of $\bar{\partial}_T$ times $(-1)^{\sigma+|a|+n+d_A+1}$. Thus, since the sequence (5.6) induces the canonical orientation of $\bar{\partial}_T$ and differs from the sequence (5.7) only by the sign $(-1)^{|a|+n+d_A+1}$, it follows that we must have $\sigma = 0$. This concludes the proof of $\Phi_{\mathbb{R} \times \Lambda, \mathcal{S}} = \text{id}$. □

6. Morse flow trees and abstract perturbations

In [EHK16] the techniques of abstractly perturbed flow trees are used to give explicit descriptions of DGA-morphisms $\Phi_L$ associated to elementary Legendrian isotopies, with coefficients in $\mathbb{Z}_2$. Here we explain how this can be done also with integer coefficients.

Let $L \subset \mathbb{R} \times J^1(M)$ be an exact Lagrangian cobordism. To this cobordism we associate a Morse cobordism $L^{MO}$, which will be exact Lagrangian isotopic to $L$ relative the ends. See [EHK16]. The advantage of considering $L^{MO}$ instead of $L$ is that we can use the Morse flow tree techniques from [Ekh07] to define $\Phi_{L^{MO}}$ to be given by a count of rigid flow trees instead of rigid disks.

For a definition of Morse flow trees we refer to [Ekh07] and [EHK16]. Briefly, these trees are built out of flow lines of local gradient differences associated to $L^{MO}$, and there is a one-to-one correspondence between the rigid Morse flow trees and the rigid pseudo-holomorphic disks of $L^{MO}$. Thus, if we let $\mathcal{M}_{T,L}(a, b)$ denote the moduli space of Morse flow trees of $L^{MO}$ with positive puncture $a$ and negative punctures $b$, it follows that $\Phi_L$ can be given by

$$\Phi_L(a) = \sum_{\dim \mathcal{M}_{T,L}(a, b) = 0} |\mathcal{M}_{T,L}(a, b)|^b.$$

Here $|\mathcal{M}_{T,L}(a, b)|$ denotes the algebraic count of elements in the moduli space, where we have used the oriented identification of $\mathcal{M}_{T,L}(a, b)$ and $\mathcal{M}_L(a, b)$ from [Kar]. In summary, the orientation of a tree $\Gamma \in \mathcal{M}_{T,L}(a, b)$ is defined by considering the cotangent lift the tree, which gives rise to a Lagrangian boundary condition on the punctured disk, and we get an associated linearized $\bar{\partial}$-operator $\bar{\partial}_T$. Moreover, we can glue capping operators associated to the punctures of $\Gamma$ to this operator, to get a corresponding fully capped problem $\hat{\bar{\partial}}_T$ on the non-punctured disk. This gives rise to exactly the same gluing sequences as in the case of true $J$-holomorphic disks, and the capping orientation of $\Gamma$ is defined completely analogous to how it is done for $J$-holomorphic disks. For details we refer to [Kar].

To get the explicit formulas for the DGA-maps in [EHK16], the trees of $L^{MO}$ are perturbed. This is first done by a geometric perturbation, which is a perturbation of $L^{MO}$ together with a perturbation of the Riemannian metric. If we extend the orientation scheme of $\mathcal{M}_{T,L}$ to also be defined for the geometrically perturbed trees, it follows by straightforward arguments that for a generic geometric perturbation, the algebraic count of trees in $\mathcal{M}_{T,L}$ will be equal to the algebraic count of rigid,
geometrically perturbed trees. We let $\Phi_{L,g}$ denote the DGA-map defined by the count of the rigid geometric perturbed trees.

Next we consider the abstract perturbation of the trees of $L$, and we let $\Phi_{L,a}$ denote the corresponding DGA-map. We refer to [EK08] for a detailed description of this perturbation, and instead we explain how the chain homotopy between $\Phi_{L,g}$ and $\Phi_{L,a}$ in [EHK16, Lemma 6.4] can be extended to $\mathbb{Z}$-coefficients.

Indeed, the proof of this lemma makes use of a 1-parameter family of abstract perturbations, starting at the geometric perturbation (which we can interpret as an abstract perturbation) and ending at the desired abstract perturbation. The chain homotopy is then defined by a count of certain Morse flow trees of $L \times D$, where the trees are induced by the 1-parameter family of perturbations.

From [EK08] it follows that the boundary conditions of the abstractly perturbed trees are close to being boundary conditions for true trees. Hence we can extend the orientation scheme of flow trees to also include abstractly perturbed trees, both for the trees occurring in the formula for $\Phi_{L,a}$ and also for the flow trees in the chain homotopy just described. Thus it follows that the algebraic count of abstractly perturbed trees given by $\Phi_{L,a}$ is signed chain homotopic to the algebraic count of geometrically perturbed trees given by $\Phi_{L,g}$. It follows that $\Phi_{L,a}$ is chain homotopic to $\Phi_L$ over $\mathbb{Z}$.

References

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in symplectic field theory. Geom. Topol., 7:799–888, 2003.

[CDRGG] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko. Floer theory for Lagrangian cobordisms. arXiv:1511.09471.

[CDRGG15] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko. Floer homology and Lagrangian concordance. In Proceedings of the Gökova Geometry-Topology Conference 2014, pages 76–113. Gökova Geometry/Topology Conference (GGT), Gökova, 2015.

[Che02] Yuri Chekanov. Differential algebra of Legendrian links. Invent. Math., 150(3):441–483, 2002.

[Don02] S. K. Donaldson. Floer homology groups in Yang-Mills theory, volume 147 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.

[DR16] Georgios Dimitroglou Rizell. Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications. Quantum Topol., 7(1):29–105, 2016.

[EES05a] Tobias Ekholm, John Etnyre, and Michael Sullivan. The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$. J. Differential Geom., 71(2):177–305, 2005.

[EES05b] Tobias Ekholm, John Etnyre, and Michael Sullivan. Orientations in Legendrian contact homology and exact Lagrangian immersions. Internat. J. Math., 16(5):453–532, 2005.

[EES07] Tobias Ekholm, John Etnyre, and Michael Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. Trans. Amer. Math. Soc., 359(7):3301–3335 (electronic), 2007.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. Geom. Funct. Anal., (Special Volume, Part II):560–673, 2000. GAFA 2000 (Tel Aviv, 1999).

[EHK16] Tobias Ekholm, Ko Honda, and Tamás Kálmán. Legendrian knots and exact Lagrangian cobordisms. J. Eur. Math. Soc. (JEMS), 18(11):2627–2689, 2016.

[EK08] Tobias Ekholm and Tamás Kálmán. Isotopies of Legendrian 1-knots and Legendrian 2-tori. J. Symplectic Geom., 6(4):407–460, 2008.
A NOTE ON COHERENT ORIENTATIONS FOR EXACT LAGRANGIAN COBORDISMS

[Tobias Ekholm. Morse flow trees and Legendrian contact homology in 1-jet spaces. Geom. Topol., 11:1083–1224, 2007.]

[Tobias Ekholm. Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms. J. Eur. Math. Soc. (JEMS), 10(3):641–704, 2008.]

[Tobias Ekholm. Non-loose Legendrian spheres with trivial contact homology DGA. J. Topol., 9(3):826–848, 2016.]

[Yakov Eliashberg. Invariants in contact topology. In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), number Extra Vol. II, pages 327–338, 1998.]

[John B. Etnyre, Lenhard L. Ng, and Joshua M. Sabloff. Invariants of Legendrian knots and coherent orientations. J. Symplectic Geom., 1(2):321–367, 2002.]

[A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. Math. Z., 212(1):13–38, 1993.]

[Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian intersection Floer theory: anomaly and obstruction. Part II, volume 46 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI, 2009.]

[Cecilia Karlsson. Orientations of Morse flow trees in Legendrian contact homology, arXiv:1601.07346.

STANFORD UNIVERSITY, MATHEMATICS, 450 SERRA MALL, BUILDING 380 STANFORD, CA 94305, USA

E-mail address: ceka@stanford.edu]