Parisian & cumulative Parisian ruin probability for two-dimensional Brownian risk model

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ABSTRACT
Parisian ruin probability in the classical Brownian risk model, unlike the standard ruin probability can not be explicitly calculated even in one-dimensional setup. Resorting on asymptotic theory, we derive in this contribution the asymptotic approximations of both Parisian and cumulative Parisian ruin probabilities and simultaneous ruin time for the two-dimensional Brownian risk model when the initial capital increases to infinity.

1. Introduction
Calculation of Parisian ruin for Brownian risk model has been initially considered in [6]. For general Gaussian risk models Parisian ruin cannot be calculated explicitly. As shown in [1,2] methods from the theory of extremes of Gaussian random fields can be successfully applied to approximate the Parisian ruin for general Gaussian risk models. In this paper, we shall focus on the classical bivariate Brownian motion risk model, which in view of recent findings in [5], appears naturally as the limiting model of some general bivariate insurance risk model. Consider therefore two insurance risk portfolios with corresponding risk models

\[ R_1(t) = u + c_1 t - W_1(t), \quad R_2(t) = au + c_2 t - W_2(t), \quad t \geq 0, \]

where \( W_1, W_2 \) are two standard Brownian motions and the initial capital for the first portfolio is \( u > 0 \), whereas for the second it is equal \( au \) for some real constant \( a \). Further \( c_1 \) and \( c_2 \) are some constants which denote the premium rates of the first and the second portfolio, respectively. In this contribution we shall consider the benchmark model where \( (W_1(t), W_2(t)), t \geq 0 \) are assumed to be jointly Gaussian with the same law as

\[ \left( B_1(t), \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t) \right), \quad t \geq 0, \quad \rho \in (-1, 1), \quad (1) \]

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where \( B_1, B_2 \) are two independent standard Brownian motions. As mentioned above, this model is supported by the findings of [5].

For given \( A > 0 \) and \( H \geq 0 \) define the simultaneous Parisian ruin probability on finite time horizon \([0, A]\)

\[
P_{H,A}(u, c_1, au, c_2) = \mathbb{P}\left\{ \exists t \in [0, A], \forall s \in [t, t + H] : R_1(s) < 0, R_2(s) < 0 \right\}, \quad u > 0
\]

When \( H = 0 \), the simultaneous Parisian ruin reduces to the simultaneous classical ruin. Such model has been recently studied in [3].

It follows that for any \( A, H, u \) positive

\[
P_{H,A}(u, c_1, au, c_2) \leq P_{0,A}(u, c_1, au, c_2) = \mathbb{P}\left\{ \exists t \in [0, A] : R_1(t) < 0, R_2(t) < 0 \right\}
\]

since \( P_{H,A}(u, c_1, au, c_2) \) is monotone in \( H \). In [3] it is shown that the simultaneous ruin probability can be bounded as follows

\[
\mathbb{P}\left\{ W_t^+(A) > u, W_{2t}^+(A) > au \right\} \leq \mathbb{P}\left\{ W_t^+(A) > u, W_{2t}^+(A) > au \right\} \leq \frac{\mathbb{P}\left\{ W_t^+(A) > u, W_{2t}^+(A) > au \right\}}{\mathbb{P}\left\{ W_t^+(A) > (c_1A)^+, W_{2t}^+(A) > (c_2A)^+ \right\}},
\]

(3)

where we set \( W_t^+(A) = W_t(t) - c_it, i = 1, 2, \) and \( x^+ = \max(x, 0) \).

A simple lower bound for \( P_{H,A} \) is valid for any \( u > 0 \)

\[
\mathbb{P}\left\{ \forall t \in [A, A + H] : R_1(t) < 0, R_2(t) < 0 \right\} \leq P_{H,A}(u, c_1, au, c_2).
\]

(4)

The above lower bound is very difficult to evaluate even asymptotically when \( u \) tends to infinity. The most simple case is when \( a < \rho, \rho > 0 \). We have (see Appendix) that for all large \( u \) and some \( C \in (0, 1) \)

\[
C\mathbb{P}\left\{ \forall t \in [A, A + H] : W_1^+(t) > u \right\} \leq P_{H,A}(u, c_1, au, c_2) \leq \mathbb{P}\left\{ \sup\limits_{t \in [0, A]} W_1^+(t) > u \right\}.
\]

(5)

Since \( \mathbb{P}\left\{ \sup_{t \in [0, A]} W_1^+(t) > u \right\} \) can be evaluated explicitly, it follows easily that as \( u \to \infty \) it is asymptotically equal to \( 2\mathbb{P}\left\{ W_1^+(A) > u \right\} \) and by [2, Thm 2.1] the lower bound is proportional to \( \mathbb{P}\left\{ W_1^+(A) > u \right\} /u \) as \( u \to \infty \). Therefore, even for this simple case, the bounds derived above do not capture the exact decrease of the Parisian ruin probability as \( u \to \infty \). The reason for this is that the interval \([A, A + H]\) is quite large. In the sequel, under the restriction that \( H = S/u^2 \) for any \( S \geq 0 \) we show that it is possible to derive the exact approximations of the Parisian ruin probability.

Motivated by [4] in this paper we shall also investigate the so-called cumulative Parisian ruin probability on the finite time interval \([0, A]\), i.e.

\[
\Psi_{L,A}(u, au) = \mathbb{P}\left\{ \int_0^A \mathbb{I}(R_1(t) < 0, R_2(t) < 0) \, dt > L/f(u) \right\},
\]

where \( L > 0 \) is a given constant and \( f(u) \) is some positive function that depends on \( u \). It is clear that the above is bounded by \( P_{0,T}(u, c_1, au, c_2) \) and the calculation of the cumulative
Parisian ruin probability is not possible for any fixed \( u \) and \( x \) positive. A natural question here is (see [4] for the infinite time-horizon case) if we can approximate the cumulative Parisian ruin probability as \( u \to \infty \). This in particular requires to determine explicitly the function \( f(u) \). In the case of one-dimensional risk model it is shown in [4] that the cumulative Parisian ruin probability (or in the language of that paper the tail of the sojourn time/occupation time) can be approximated exactly when \( u \) becomes large. In that aforementioned paper \( f(u) \) equals \( u^2 \). We shall show that this is the right choice also for our setup.

Section 2 presents the exact asymptotics of both Parisian and cumulative Parisian ruin. Additionally, we discuss therein the approximation of the cumulative Parisian ruin time

\[
\tau_L(u) = \inf_{A > 0} \int_0^A \mathbb{I}(R_1(t) < 0, R_2(t) < 0) \, dt > L/f(u). \tag{6}
\]

Section 3 is dedicated to the proofs. We conclude this contribution with an Appendix composed of two auxiliary lemmas and a short discussion of general parisan ruin.

2. Main results

Using the self-similarity of Brownian motion we have that

\[
P_{H,A}(u, c_1, au, c_2)
= \mathbb{P} \{ \exists t \in [0,1], \forall s \in [A, At + H]: R_1(s) < 0, R_2(s) < 0 \}
= \mathbb{P} \{ \exists t \in [0,1], \forall s \in [t, t + (H/A)]: R_1(As) < 0, R_2(As) < 0 \}
= \mathbb{P} \{ \exists t \in [0,1], \forall s \in [t, t + (H/A)]: u + Ac_1s < W_1(As), au + Ac_1s < W_1(As) \}
= \mathbb{P} \left\{ \exists t \in [0,1], \forall s \in [t, t + (H/A)]: \frac{u}{\sqrt{A}} + \sqrt{A}c_1s < W_1(s), \frac{au}{\sqrt{A}} + \sqrt{A}c_1s < W_1(s) \right\}
= P_{H/A,1}(u/\sqrt{A}, \sqrt{A}c_1, au/\sqrt{A}, \sqrt{A}c_1).
\]

In addition, we have

\[
P_{H,A}(u, c_1, au, c_2) = P_{H,A}^{rev}(u, c_2, u/a, c_1),
\]

where

\[
P_{H,A}^{rev}(u, c_1, au, c_2) = \mathbb{P} \{ \exists t \in [0,A], \forall s \in [t, t + H]: R_1^{rev}(s) < 0, R_2^{rev}(s) < 0 \}
\]

and

\[
R_1^{rev}(t) = u + c_1t - W_2(t), \quad R_2^{rev}(t) = au + c_2t - W_1(t), \quad t \geq 0.
\]

Hence, we can consider only the case \( a \leq 1 \) and \( A = 1 \). Let in the following

\[
\lambda_1 = \frac{1 - a \rho}{1 - \rho^2}, \quad \lambda_2 = \frac{a - \rho}{1 - \rho^2}, \tag{7}
\]

which are both positive if \( a \in (\rho, 1] \). For the particular choice of \( H = S/u^2 \) we shall denote \( P_{H,A}(u, au) \) simply as \( \psi_S(u, au) \). We consider first the approximation of the Parisian ruin, recall \( W_i^\tau(t) = W_i(t) - c_it \).
**Theorem 2.1:** Let $c_1, c_2$ be two given real constants and let $S \geq 0$ be given.

(i) If $a \in (\rho, 1]$, then as $u \to \infty$

$$\psi_S(u, au) \sim C_{a, \rho}(S)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where

$$C_{a, \rho}(S) = \lambda_1 \lambda_2 \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists t \geq 0, \quad \forall s \in [t - S, t]: \quad W_1(s) - s > x \right\} \times e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty).$$

(ii) If $a \leq \rho$, then as $u \to \infty$

$$\psi_S(u, au) \sim C(S)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where $C(S) = \mathbb{E}\left\{e^{\sup_{t \geq 0} \inf_{s \in [t-S, t]} (W_1(s) - s)}\right\} \in (0, \infty)$.

The approximation of the cumulative Parisian ruin requires some different arguments since the sojourn functional is different from the supremum functional. In the following we shall choose the scaling function $f(u)$ to be equal to $u^2$. Since we consider $A = 1$, we can omit it and write simply $\Psi_L(u, au)$ instead of $\Psi_{L, A}(u, au)$.

**Theorem 2.2:** Under the setup and the notation of Theorem 2.1 for any $L > 0$ we have:

(i) If $a \in (\rho, 1]$, then as $u \to \infty$

$$\Psi_L(u, au) \sim K_{a, \rho}(L)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where

$$K_{a, \rho}(L) = \lambda_1 \lambda_2 \int_{\mathbb{R}^2} \mathbb{P}\left\{\int_0^\infty \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y) \, dt > L\right\} \times e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty).$$

(ii) If $a \leq \rho$, then as $u \to \infty$

$$\Psi_L(u, au) \sim K(L)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where

$$K(L) = \int_{\mathbb{R}} e^x\mathbb{P}\left\{\int_0^\infty \mathbb{I}(W_1(t) - t > x) \, dt > L\right\} \, dx \in (0, \infty).$$

**Remark 2.3:** Theorems 2.1 and 2.2 may be used also if $c_1, c_2, S$ and $L$ depend on $u$, but have finite limits as $u \to \infty$ ($c_1(u) \to c_1^*$, $c_2(u) \to c_2^*$, $S(u) \to S^*$, $L(u) \to L^*$). In this case all constants $S$ and $L$ on the right-hand sides should be replaced by $S^*$ and $L^*$, respectively.

The asymptotic distribution of the ruin time $\tau_L(u)$ defined in (6) may be explicitly calculated from Theorem 2.2 by using the self-similarity of Brownian motion.
Proposition 2.4: (i) If \( a \in (\rho, 1] \), then for any \( 0 \leq L_2 \leq L_1 \leq 1 \) with \( K_{a,\rho} \) defined in (12)
\[
\lim_{u \to \infty} \mathbb{P} \{ u^2(1 - \tau_{L_1}(u)) \geq x | \tau_{L_2}(u) \leq 1 \} = \frac{K_{a,\rho}(L_1)}{K_{a,\rho}(L_2)} e^{-\frac{x^{1-2a+\rho^2}}{2(1-\rho^2)}} , \quad x \in (0, \infty).
\]
(ii) If \( a \leq \rho \), then for any \( 0 \leq L_2 \leq L_1 \leq 1 \) with \( K \) defined in (14)
\[
\lim_{u \to \infty} \mathbb{P} \{ u^2(1 - \tau_{L_1}(u)) \geq x | \tau_{L_2}(u) \leq 1 \} = \frac{K(L_1)}{K(L_2)} e^{-\frac{x^2}{2}}, \quad x \in (0, \infty).
\]

3. Proofs

Proof of Theorem 2.1: Let in the following \( T > 0 \) and set \( \delta(u, T) = 1 - Tu^{-2} \) for \( T, u > 0 \).

For any \( S \) positive and all \( u \) large
\[
m(u, S, T) := \mathbb{P} \{ \exists t \in [0, \delta(u, T)], \forall s \in [t, t + S/u^2] : W^*_1(s) > u, W^*_2(s) > au \}
\leq \mathbb{P} \{ \exists t \in [0, \delta(u, T)] : W^*_1(t) > u, W^*_2(t) > au \}
\leq e^{-T/8} \frac{\mathbb{P} \{ W^*_1(1) \geq u, W^*_2(1) \geq au \}}{\mathbb{P} \{ W_1(1) > \max(c_1, 0), W_2(1) > \max(c_2, 0) \}},
\]
where the upper bound follows from [3][Lemma 4.1].

We give below the exact asymptotics of
\[
M(u, S, T) := \mathbb{P} \{ \exists t \in [\delta(u, T), 1], \forall s \in [t, t + S/u^2] : W^*_1(s) > u, W^*_2(s) > au \}
\]
as \( u \) tends to infinity.

Lemma 3.1: (i) For any \( a \in (\rho, 1] \) and any positive \( S \) and \( T \) as \( u \to \infty \)
\[
M(u, S, T) \sim u^{-2} \varphi_\rho(u + c_1, au + c_2) I(S, T),
\]
where
\[
I(S, T) := \int_{\mathbb{R}^2} \mathbb{P} \left\{ t \in [0, T], \forall s \in [t - S, t] \begin{array}{l} W_1(s) - s > x \\
W_2(s) - as > y \end{array} \right\} 
\times e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty).
\]
(ii) For any \( a \leq \rho \) and any positive \( S \) and \( T \) as \( u \to \infty \)
\[
M(u, S, T) \sim u^{-1} \varphi_\rho(u + c_1, \rho u + c_2) I(S, T),
\]
where
\[
I(S, T) := \int_{\mathbb{R}^2} \mathbb{P} \{ \exists t \in [0, T], \forall s \in [t - S, t] : W_1(s) - s > x \}
\times \left[ \mathbb{I}(a < \rho) + \mathbb{I}(y < 0, a = \rho) \right] e^{-\frac{y^2}{2(1-\rho^2)}} \, dx \, dy.
\]
The proof of Lemma 3.1 postponed to the Appendix.
In view of Lemma 3.1, (15) and asymptotics of \( P \{ W_1^+ (1) \geq u, W_2^+ (1) \geq au \} \) (see Appendix, Lemma A.1) we immediately obtain that

\[
\lim_{T \to \infty} \lim_{u \to \infty} \frac{m(u, S, T)}{M(u, S, T)} = 0.
\]

Hence, using that

\[ M(u, S, T) \leq \psi_S(u, au) \leq m(u, S, T) + M(u, S, T) \]

we obtain

\[
\lim_{T \to \infty} \lim_{u \to \infty} \frac{M(u, S, T)}{\psi_S(u, au)} = 1.
\]

Consequently, it suffices to prove that

\[
\lim_{T \to \infty} I(S, T) \in (0, \infty).
\]

Since \( I(S, T) \leq I(0, T) \), \( I(S, T) \) is growing and the finiteness of \( \lim_{T \to \infty} I(0, T) \) follows from [3], the claim follows according to the asymptotics of \( P \{ W_1^+ (1) \geq u, W_2^+ (1) \geq au \} \).

\[ \Box \]

**Proof of Theorem 2.2:** First recall \( \delta(u, T) = 1 - Tu^{-2} \).

For given \( L > 0 \) if \( \int_0^1 \mathbb{I}(R_1(t) < 0, R_2(t) < 0) \, dt > L/f(u) \), then either the same integral but from \( 1 - \delta \) to \( 1 \) is larger than \( L/f(u) \), or for some point \( t_1 \in [0, 1 - \delta(u, T)] \) both \( R_1(t_1) \) and \( R_2(t_1) \) are smaller than zero. In terms of probabilities it means that for any \( T > 0 \)

\[
M(u, T) \leq \Psi_L(u, au) \leq M(u, T) + m(u, T),
\]

where we set for \( u > 0 \)

\[
M(u, T) = \mathbb{P}\left\{ \int_{1 - \delta(u, T)}^1 \mathbb{I}(W_1^+(t) > u, W_2^+(t) > au) \, dt > L/f(u) \right\},
\]

\[
m(u, T) = \mathbb{P}\left\{ \exists t \in [0, 1 - \delta(u, T)] : W_1^+(t) > u, W_2^+(t) > au \right\}.
\]

In view of [3][Lemma 4.1] for all large enough \( u \)

\[
m(u, T) \leq e^{-T/8} \frac{\mathbb{P}\{ W_1^+(1) \geq u, W_2^+(1) \geq au \}}{\mathbb{P}\{ W_1(1) > \max(c_1, 0), W_2(1) > \max(c_2, 0) \}}.
\]

The following Lemma establishes the exact asymptotics of \( M(u, T) \).

**Lemma 3.2:** (i) For any \( a \in (\rho, 1] \) and any \( T > 0 \) as \( u \to \infty \)

\[
M(u, T) \sim u^{-2} \varphi_{\rho}(u + c_1, au + c_2)I(T),
\]

where

\[
I(T) := \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y) \, dt > L \right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy \in (0, \infty).
\]
(ii) For any $a \leq \rho$ and any $T > 0$ as $u \to \infty$

$$M(u, T) \sim u^{-1} \varphi_{\rho}(u + c_1, \rho u + c_2) I(T),$$

where

$$I(T) := \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I}(W_1(t) - t > x) \, dt > L \right\} \left[ \mathbb{I}[a < \rho] + \mathbb{I}[a = \rho, y < 0] \right]$$

\times \exp \left( -\frac{y^2 - 2c_2(c_2 - c_1)}{2(1-\rho)} \right) \, dx \, dy \in (0, \infty).

The proof of Lemma 3.2 postponed to the Appendix.

In view of Lemma 3.2, (18) and asymptotics of $\mathbb{P} \left\{ W_1^+(1) \geq u, W_2^+(1) \geq au \right\}$ we immediately obtain that

$$\lim_{T \to \infty} \lim_{u \to \infty} \frac{m(u,T)}{M(u,T)} = 0.$$

Hence, using (17) we have

$$\lim_{T \to \infty} \lim_{u \to \infty} \frac{M(u,T)}{\Psi_L(u,au)} = 1.$$

Consequently, it suffices to show that

$$\lim_{T \to \infty} I(T) \in (0, \infty),$$

where $I(T)$ is defined in Lemma 3.2. Since $I(T) \leq I(L,T)$ defined in Lemma 3.1, $I(T)$ is growing and $\lim_{T \to \infty} I(L,T) < \infty$, the claim follows. ■

**Proof of Proposition 2.4:** Using the formula of conditional probability and the self-similarity of Brownian motion for $L_1, L_2, u$ positive

$$\mathbb{P} \left\{ u^2 (1 - \tau_{L_1}(u)) \geq x | \tau_{L_2}(u) \leq 1 \right\}$$

\[= \frac{\mathbb{P} \left\{ \tau_{L_1}(u) \leq 1 - x/u^2 \right\}}{\mathbb{P} \left\{ \tau_{L_2}(u) \leq 1 \right\}} \]

\[= \frac{\mathbb{P} \left\{ \int_0^{1-x/u^2} \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_1/u^2 \right\}}{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_2/u^2 \right\}} \]

\[= \frac{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(1-x/u^2)t}{W_2^+(1-x/u^2)t} > u \right) \, dt > L_1/u^2 \right\}}{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_2/u^2 \right\}} \]

\[= \frac{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_2/u^2 \right\}}{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u + c_1 \sqrt{1 - x/u^2} + c_2 \sqrt{1 - x/u^2} \right) \, dt > L_2/u^2 \right\}} \]

\[= \frac{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_2/u^2 \right\}}{\mathbb{P} \left\{ \int_0^1 \mathbb{I} \left( \frac{W_1^+(t)}{W_2^+(t)} > u \right) \, dt > L_2/u^2 \right\}} \cdot \]
Applying Theorem 2.2 yields

\[
\Pr \left\{ \int_0^1 \left( W_1(t) > \frac{u}{\sqrt{1-x/u^2}} + c_1 \sqrt{1-x/u^2} \right) \, dt > \frac{L(1-x/u^2)^2}{(u/\sqrt{1-x/u^2})^2} \right\} \\
= \Pr \left\{ \int_0^1 \left( \frac{W_1^*(t) > u}{W_2^*(t) > au} \right) \, dt > \frac{L}{u^2} \right\} \\
\sim \Gamma(L_1, L_2) \Pr \left\{ W_1(1) > \frac{u}{\sqrt{1-x/u^2}} + c_1 \sqrt{1-x/u^2}, W_2(1) > \frac{u}{\sqrt{1-x/u^2}} + c_2 \sqrt{1-x/u^2} \right\}
\]

where

\[
\Gamma(L_1, L_2) = \begin{cases} 
K_{a,\rho}(L_1), & a \in (\rho, 1], \\
K_{a,\rho}(L_2), & a \leq \rho.
\end{cases}
\]

Notice that (write \( \varphi(x, y) \) for the pdf of vector \((W_1(1), W_2(1))\))

\[
\varphi \left( \frac{u}{\sqrt{1-x/u^2}} + c_1 \sqrt{1-x/u^2}, \frac{u}{\sqrt{1-x/u^2}} a + c_2 \sqrt{1-x/u^2} \right) \\
= \varphi(u+c_1, au+c_2) \psi^*_u(a, c_1, c_2),
\]

where

\[
\lim_{u \to \infty} \log \psi^*_u(a, c_1, c_2) = -x \frac{1 - 2a\rho + a^2}{2 - 2\rho^2},
\]

hence by Lemma A.1 the claim follows if \( a > \rho \). For the case \( a \leq \rho \) notice that

\[
\varphi \left( \frac{u}{\sqrt{1-x/u^2}} + c_1 \sqrt{1-x/u^2}, \frac{u}{\sqrt{1-x/u^2}} \rho + c_2 \sqrt{1-x/u^2} \right) \\
= \varphi(u+c_1, \rho u+c_2) \psi^*_u(\rho, c_1, c_2),
\]

where

\[
\lim_{u \to \infty} \log \psi^*_u(\rho, c_1, c_2) = -x/2.
\]

This finishes the proof in the case \( a \leq \rho \) again using Lemma A.1

\[\blacksquare\]

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**References**

[1] K. Dębicki, E. Hashorva, and L. Ji, *Parisian ruin of self-similar Gaussian risk processes*, J. Appl. Probab. 52(3) (2015), pp. 688–702.
Theorem 2.2.

The following Lemma shows the exact asymptotics of the right-hand sides in Theorem 2.1 and Auxiliary lemmas.

Lemma A.1: Let $X_1$ and $X_2$ be Gaussian random variables with correlation coefficient $\rho \in (-1, 1)$. Let also $c_1$ and $c_2$ be two given real constants and $a \leq 1$ be given. Write further $\varphi_\rho(x, y)$ for the joint density function of vector $(X_1, X_2)$.
(i) If \( a \in (\rho, 1] \), then as \( u \to \infty \)
\[
\mathbb{P} \{ X_1 > u + c_1, X_2 > au + c_2 \} \sim \frac{u^{-2}}{\lambda_1 \lambda_2} \varphi_\rho(u + c_1, au + c_2),
\]
where
\[
\lambda_1 = \frac{1 - a\rho}{1 - \rho^2}, \quad \lambda_2 = \frac{a - \rho}{1 - \rho^2}.
\]

(ii) If \( a \leq \rho \), then we have as \( u \to \infty \)
\[
\mathbb{P} \{ X_1 > u + c_1, X_2 > au + c_2 \}
\sim \sqrt{2\pi(1 - \rho^2)} \Phi^*(c_1\rho - c_2) e^{\frac{(c_1 - \rho\rho)^2}{2(1 - \rho^2)}} u^{-1} \varphi_\rho(u + c_1, \rho u + c_2),
\]
where \( \Phi^*(c_1\rho - c_2) = 1 \) if \( a < \rho \) and \( \Phi^* \) is the df of \( \sqrt{1 - \rho^2}X_1 \) when \( a = \rho \).

**Proof of Lemma A.1:** (i) Using the dominated convergence theorem as \( u \to \infty \)
\[
\mathbb{P} \{ X_1 > u + c_1, X_2 > au + c_2 \}
\sim \varphi_\rho(u + c_1, au + c_2) \int_{x,y > 0} e^{-\lambda_1 x - \lambda_2 y} \varphi_\rho(c_1 + x/u, c_2 + y/u) \varphi_\rho(c_1, c_2) dx dy,
\]
\[
\sim \varphi_\rho(u + c_1, au + c_2) \int_{x,y > 0} e^{-\lambda_1 x - \lambda_2 y} dx dy.
\]

(ii) Again using the dominated convergence theorem as \( u \to \infty \) (denote \( C = 0 \) if \( a = \rho \) and \( C = -\infty \) otherwise)
\[
\mathbb{P} \{ X_1 > u + c_1, X_2 > au + c_2 \}
\sim \frac{\varphi_\rho(u + c_1, \rho u + c_2)}{u} \int_{x > 0 \atop y > C} e^{-x} \varphi_\rho(c_1, c_2 + y) dx dy
\]
\[
= \frac{\varphi_\rho(u + c_1, \rho u + c_2)}{u} \frac{(c_2 - \rho\rho)^2}{1 - \rho^2} \sqrt{2\pi(1 - \rho^2)} \int_{y < \rho c_1 - c_2 - C} e^{-\frac{y^2}{2(1 - \rho^2)}} dy
\]
\[
= \frac{\varphi_\rho(u + c_1, \rho u + c_2)}{u} \frac{(c_2 - \rho\rho)^2}{1 - \rho^2} \sqrt{2\pi(1 - \rho^2)} \Phi^*(\rho c_1 - c_2).
\]

Then next lemma helps to go from ‘almost all \( L \)’ to ‘all \( L \)’ in Lemma 3.2

**Lemma A.2:** Let \( X_1(t), X_2(t) \) for \( t \geq 0 \) satisfy the representation (1). Let also \( \lambda_1, \lambda_2, a, T \) be positive constants and \( c_1, c_2 \) be real constants. Then the functions
\[
I_1(L) = \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I}(X_1(t) - t > x, X_2(t) - at > y) dt > L \right\} e^{\lambda_1 x + \lambda_2 y} dx dy,
\]
\[
I_2(L) = \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I}(X_1(t) - t > x) dt > L \right\} \left[ \mathbb{I}(a < \rho) + \mathbb{I}(a = \rho, y < 0) \right] e^{\frac{-y^2 - 2\rho c_2 - c_1\rho}{2(1 - \rho^2)}} dx dy.
\]
are continuous for \( L \in (0, \infty) \).
**Proof of Lemma A.2:** Consider the function $I_1(L)$. The proof for $I_2(L)$ will be the same. To show the continuity of $I_1(L)$ it is sufficient to verify that

$$I_1^*(L) := \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y) \, dt = L \right\} e^{\lambda_1 x + \lambda_2 y} \, dx \, dy = 0$$

for all positive $L$. Fix some $L > 0$ and let

$$A_{xy} = \left\{ f_1, f_2 \in C[0, T] : \int_0^T \mathbb{I}(f_1(t) - t > x, f_2(t) - at > y) \, dt = L \right\}.$$

For any fixed $y_0 \in \mathbb{R}$ the sets $A_{x_1,y_0}$ and $A_{x_2,y_0}$ are non-overlapping for $x_1 \neq x_2$. Define

$$\mathcal{X} = \{ x \in \mathbb{R} : \mathbb{P} \{ A_{x,y_0} \} > 0 \}, \quad \mathcal{X}_n = \{ x \in \mathbb{R} : \mathbb{P} \{ A_{x,y_0} \} > 1/n \}.$$

Since $A_{x,y_0}$ are non-overlapping for different $x \in \mathbb{R}$, $|\mathcal{X}_n| < n$. In addition, $\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n$. Thus, the set $\mathcal{X}$ is countable, establishing the proof.

**Proofs of lemmas**

This part contains proofs of all the lemmas presented above in the contribution.

**Proof of Lemma 3.1:** (i) For any $x, y \in \mathbb{R}$ put

$$u_x = u + c_1 - x/u, \quad u_y = au + c_2 - y/u.$$

Writing $\varphi(x, y)$ for the joint pdf of $(W_1(1), W_2(1))^T$ we have

$$\varphi_p(u_x, u_y) =: \varphi_p(u + c_1, au + c_2) \psi_u(x, y),$$

where (write $\Sigma$ for the covariance matrix of $(W_1(1), W_2(1))^T$)

$$\log \psi_u(x, y) = \frac{1}{u^2} (u + c_1, au + c_2) \Sigma^{-1}(x, y)^T - \frac{1}{2u^2} (x, y) \Sigma^{-1}(x, y)^T$$

$$\quad \rightarrow (1, a) \Sigma^{-1}(x, y)^T, \quad u \to \infty$$

$$\quad = \frac{1 - a \rho}{1 - \rho^2} x + \frac{a - \rho}{1 - \rho^2} y = \lambda_1 x + \lambda_2 y. \quad (A2)$$

Denote further

$$u_{x,y} = u_y - \rho u_x = (a - \rho)u - (y - \rho x)/u + c_2 - \rho c_1.$$

Let $B_1, B_2$ be two independent Brownian motions. The representation of $(W_1(t), W_2(t))$ in terms of $B_1$ and $B_2$ is given in (1). Define the following transform

$$\bar{s}_u = 1 - s/u^2,$$

and set $F(u) = u^{-2} \varphi_p(u + c_1, au + c_2)$.

For the function $M(u, S, T)$ we have using $\psi_u$ defined in (A1)

$$M(u, S, T)$$

$$= F(u) \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{array}{l} B_1(\bar{s}_u) - c_1 \bar{s}_u > u \\ \rho B_1(\bar{s}_u) + \rho^2 B_2(\bar{s}_u) - c_2 \bar{s}_u > au \end{array} \right\} \psi_u(x, y) \, dx \, dy$$

$$=: F(u) \int_{\mathbb{R}^2} h_u(T, S, x, y) \psi_u(x, y) \, dx \, dy.$$
Define two auxiliary processes for $s \in [-S, T]$ as follows

\[ B_{u,1}(s) := \{B_1(s) \mid B_1(1) = u_x\} - \tilde{s}_u u_x, \quad B_{u,2}(s) := \{B_2(s) \mid \rho^* B_2(1) = u_{xy}\} - \tilde{s}_u u_{xy}/\rho^* \]  

(A3)

Represent the function $h_u(T, S, x, y)$ in terms of these processes as

\[ h_u(T, S, x, y) = \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{align*}
  u(B_{u,1}(s) + \tilde{s}_u u_x - c_1 \tilde{s}_u - u) &> 0 \\
  u\rho (B_{u,1}(s) + \tilde{s}_u u_x - c_1 \tilde{s}_u - u) + u\rho^* B_{u,2}(s) + u[\tilde{s}_u u_{xy}] &- (c_2 - \rho c_1) \tilde{s}_u - u(a - \rho) > 0 
\end{align*} \right\}. \]

We have the following weak convergence in the space $C([-S, T])$ as $u \to \infty$

\[ uB_{u,1}(t) \to B_1(t), \quad uB_{u,2}(t) \to B_2(t), \quad t \in [-S, T], \]  

(A4)

and further

\[ u(\tilde{s}_u u_x - c_1 \tilde{s}_u - u) = u \left[ \left(1 - \frac{s}{u^2}\right) \left(u + c_1 - \frac{x}{u}\right) - c_1 \left(1 - \frac{s}{u^2}\right) - u \right] \to -s - x, \]

\[ u[\tilde{s}_u u_{xy}] - (c_2 - \rho c_1) \tilde{s}_u - u(a - \rho) \to -(a - \rho)s - (y - \rho x). \]

Consequently, as $u$ tends to infinity

\[ h_u(T, S, x, y) \to h(T, S, x, y), \]

where in view of (1)

\[ h(T, S, x, y) = \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{align*}
  B_1(s) - s - x &> 0 \\
  \rho(B_1(s) - s - x) + \rho^* B_2(s) &> 0 \\
  -(a - \rho)s - (y - \rho x) &> 0 
\end{align*} \right\}. \]

This convergence is justified by applying continuous mapping theorem for the continuous functional

\[ H_{T,S}(F_1(t), F_2(t)) = \sup_{t \in [0, T]} \inf_{s \in [t - S, t]} \left( \inf_{s \in [t - S, t]} F_1(t), \inf_{s \in [t - S, t]} F_2(t) \right) \]

and random sequence $(F_{1,x,y,u}, F_{2,x,y,u}) \in C([-S, T]^2$

\[ F_{1,x,y,u}(s) = u(B_{u,1}(s) + \tilde{s}_u u_x - c_1 \tilde{s}_u - u), \]

\[ F_{2,x,y,u}(s) = u\rho (B_{u,1}(s) + \tilde{s}_u u_x - c_1 \tilde{s}_u - u) + u\rho^* B_{u,2}(s) \]

\[ + u[\tilde{s}_u u_{xy}] - (c_2 - \rho c_1) \tilde{s}_u - u(a - \rho). \]

To finish the proof it is enough to show the dominated convergence as $u \to \infty$ for

\[ I_u(S, T) = \int_{\mathbb{R}^2} h_u(T, S, x, y) \psi_u(x, y) \, dx \, dy. \]

For $\psi_u(x, y)$ we can show the following upper bound. Fix some $0 < \varepsilon < \min(\lambda_1, \lambda_2)$ (such constant exists as in our case both $\lambda_1$ and $\lambda_2$ are positive), and define constants $\lambda_{1,\varepsilon} = \lambda_1 + \text{sign}(x)\varepsilon$ and $\lambda_{2,\varepsilon} = \lambda_2 + \text{sign}(y)\varepsilon$. Hence for large enough $u$ and all $x, y \in \mathbb{R}$

\[ \psi_u(x, y) \leq \tilde{\psi} := e^{\lambda_{1,\varepsilon} x + \lambda_{2,\varepsilon} y}. \]  

(A5)

For $h_u(S, T, x, y)$ we use Piterbarg inequality (see [7], Thm 8.1), since for all $t, s$ positive

\[ u^2 \mathbb{E} \left\{ (B_{u,i}(t) - B_{u,i}(s))^2 \right\} < \text{Const} |t - s| \]  

(A6)
for some positive constant and sufficiently large $u$. Thus, for such $u$ we have for some positive constant $\tilde{C}$

$$h_u(T, S, x, y) \leq \mathbb{P} \left\{ \exists s \in [0, T] : u(B_{u,1}(s) + \tilde{s}_u(u_x - c_1) - u) > 0 \quad \text{and} \quad u(\rho u_{B_{u,2}}(s) + uB_{u,2}(s)) + u[\tilde{s}_u(u_x - c_2 + \rho c_1) - u(a - \rho)] > 0 \right\}$$

$$\leq \tilde{h} := \begin{cases} \tilde{C}e^{-c(x^2+y^2)}, & x, y \geq 0, \\ \tilde{C}e^{-cx^2}, & x \geq 0, y < 0, \\ \tilde{C}e^{-cy^2}, & y \geq 0, x < 0, \\ 1, & x, y < 0. \end{cases}$$

Since $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$ are positive

$$I_u(S, T) \leq \int_{\mathbb{R}^2} \tilde{h}(T, S, x, y) \psi(x, y) \, dx \, dy$$

$$= \tilde{C} \int_{x,y>0} e^{-c(x^2+y^2) + \lambda_{1,\varepsilon}x + \lambda_{2,\varepsilon}y} \, dx \, dy + \tilde{C} \int_{x>0,y<0} e^{-cx^2 + \lambda_{1,\varepsilon}x + \lambda_{2,\varepsilon}y} \, dx \, dy$$

$$+ \tilde{C} \int_{x<0,y>0} e^{-cy^2 + \lambda_{1,\varepsilon}x + \lambda_{2,\varepsilon}y} \, dx \, dy + \tilde{C} \int_{x,y<0} e^{\lambda_{1,\varepsilon}x + \lambda_{2,\varepsilon}y} \, dx \, dy < \infty.$$  

Hence the proof follows from the dominated convergence theorem.

(ii) In the case $\alpha \leq \rho$ we define

$$u_x = u + c_1 - x/u, \quad u_y = \rho u + c_2 - y$$

and $u_{x,y} = u_y - \rho u_x = c_2 - y - \rho c_1 + \rho x/u$. In the previous notation

$$\varphi_{\rho}(u_x, u_y) := \varphi_{\rho}(u + c_1, \rho u + c_2) \psi_u(x, y),$$

where as $u$ tends to infinity

$$\log \psi_u(x, y) = (u + c_1, \rho u + c_2) \Sigma^{-1}(x/u, y)^\top$$

$$- \frac{1}{2}(x/u, y) \Sigma^{-1}(x/u, y)^\top \rightarrow x - \frac{y^2 - 2y(c_2 - \rho c_1)}{2 - 2\rho^2}.$$  

(A7)

Setting $F(u) = u^{-1} \varphi_{\rho}(u + c_1, \rho u + c_2)$, we have the following representation for the function $M(u, S, T)$ (write $\tilde{s}_u$ for $1 - s/u^2$ and recall (1))

$$M(u, S, T)$$

$$= u^{-1} \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists t \in \delta(u, T), 1 \forall s \in [t, t + S/u^2] : \begin{cases} W_1^u(s) > u \\ W_2^u(s) > au \end{cases} \right\} W_1^u(1) = u_x \quad \text{and} \quad W_2^u(1) = u_y$$

$$\times \varphi_{\rho}(u_x, u_y) \, dx \, dy$$

$$= F(u) \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{cases} B_1(\tilde{s}_u) - c_1 \tilde{s}_u > u \\ \rho B_1(\tilde{s}_u) + \rho^* B_2(\tilde{s}_u) - c_2 \tilde{s}_u > au \end{cases} \right\} B_1(1) = u_x \quad \text{and} \quad \rho^* B_2(1) = u_{x,y}$$

$$\psi_u(x, y) \, dx \, dy$$

$$= F(u) \int_{\mathbb{R}^2} h_u(T, S, x, y) \psi_u(x, y) \, dx \, dy.$$
Using $B_{u,1}$ and $B_{u,2}$ defined in (A3) we can represent the function $h_u(T, S, x, y)$ as

$$h_u(T, S, x, y) = \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{array}{l}
u(B_{u,1}(s) + \bar{\xi}_u u_x - c_1 \bar{\xi}_u - u) > 0 \\
u(\rho (B_{u,1}(s) + \bar{\xi}_u u_x - c_1 \bar{\xi}_u - u) + u \rho^* B_{u,2}(s) + u[\bar{\xi}_u u_{x,y}] - (c_2 - \rho c_1 \bar{\xi}_u - u(a - \rho)) > 0
\end{array} \right\}.$$

As $u$ tends to infinity we have

$$u(\bar{\xi}_u u_x - c_1 \bar{\xi}_u - u) = u \left[ (1 - \frac{s}{u^2}) \left( u + c_1 - \frac{x}{u} \right) - c_1 \left( 1 - \frac{s}{u^2} \right) - u \right] \rightarrow -s - x, $$
$$u[\bar{\xi}_u u_{x,y} - (c_2 - \rho c_1 \bar{\xi}_u - u(a - \rho))] = -u^2(a - \rho) - uy + \rho x + ys/u + \rho xs/u^2.$$

If $a < \rho$, then the above tends to $\infty$, and if $a = \rho$ then it tends to $\infty$ only if $y < 0$ and to $-\infty$ if $y > 0$. Finally, if $a = \rho$ and $y = 0$, then the above tends to $\rho x$.

Again using continuous mapping theorem, since (A4) holds, we have the following convergence (except if $y = 0$)

$$h_u(T, S, x, y) \rightarrow h(T, S, x, y), \quad u \rightarrow \infty,$$
where

$$h(T, S, x, y) = \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{array}{l}B_1(s) - s - x > 0, \\
\rho (B_1(s) - s - x) + \rho^* B_2(s) + \infty > 0
\end{array} \right\} \times (\mathbb{I}(a < \rho) + \mathbb{I}(a = \rho, y < 0)) $$
$$= \mathbb{P} \left\{ \exists t \in [0, T] \forall s \in [t - S, t] : W_1(s) - s > x \right\} (\mathbb{I}(a < \rho) + \mathbb{I}(a = \rho, y < 0)).$$

To show the claim we can apply the dominated convergence theorem. Note that for large enough $u$ and all $x, y \in \mathbb{R}$

$$\log \psi_u(x, y) \leq \tilde{\psi}(x, y) = (1 + \text{sgn}(x)/2) x + \frac{c_2 - \rho c_1}{1 - \rho^2} y - \frac{y^2}{2}.$$

By Piterbarg inequality (as (A6) holds here for $i = 1$) we can establish that for some positive constant $\tilde{C}$

$$h_u(T, S, x, y) < \mathbb{P} \left\{ \exists s \in [0, T] : u(B_{u,1}(s) + \bar{\xi}_u(u_x - c_1) - u) > 0 \right\} \leq \tilde{h} := \begin{cases} \tilde{C} e^{\tilde{c} x^2}, & x \geq 0, \\
1, & x < 0.
\end{cases}$$

Since $(1 + \text{sgn}(x)/2) > 0$, then

$$\int_{\mathbb{R}^2} \tilde{h}(x, y) \tilde{\psi}(x, y) \ dx \ dy < \infty$$

and by the dominated convergence theorem the claim follows.

\section*{Proof of Lemma 3.2:}
(i) We use the same notation as in Lemma 3.1 (i). Hence the convergence (A2) holds. For the function $M(u, T)$ we have

$$M(u, T) = F(u) \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I} \left( \begin{array}{l}
B_1(\tilde{\xi}_u) - c_1 \tilde{\xi}_u > u \\
\rho B_1(\tilde{\xi}_u) + \rho^* B_2(\tilde{\xi}_u) - c_2 \tilde{\xi}_u > au
\end{array} \right) \ dt > T \right\} \psi_u(x, y) \ dx \ dy$$
$$= F(u) \int_{\mathbb{R}^2} h_u(L, T, x, y) \psi_u(x, y) \ dx \ dy.$$
Recalling the processes $B_{u,1}$ and $B_{u,2}$ from (A3) we can represent the function $h_u(T, S, x, y)$ as follows

$$h_u(L, T, x, y) = \mathbb{P} \left\{ \int_0^T \mathbb{I} \left( u(B_u(t) + t_iu - c_1\hat{t}_u - u) > 0 \right) \, dt > L \right\}.$$ 

We have the same weak convergence as in (A4) and further as $u$ tends to infinity

$$u(t_iu - c_1\hat{t}_u - u) = u\left[ \left( 1 - \frac{t}{u^2} \right) (u + c_1 - \frac{x}{u}) - c_1 \left( 1 - \frac{t}{u^2} \right) - u \right] \to -t - x,$$

$$u[t_iu - (c_2 - \rho c_1)\hat{t}_u - u(a - \rho)] \to -(a - \rho)t - (y - \rho x).$$

Now we want to apply the continuous mapping theorem to the function

$$H_T(F_1, F_2) = \int_0^T \mathbb{I} (F_1(t) > 0, F_2(t) > 0) \, dt$$

and a random sequence $(F_{1,xy,u}, F_{2,xy,u}) \in C([0, T] \to \mathbb{R}^2)$ defined as

$$F_{1,xy,u} = u(B_{u,1}(t) + t_iu - c_1\hat{t}_u - u),$$

$$F_{2,xy,u} = u\rho(B_{u,1}(t) + t_iu - c_1\hat{t}_u - u) + u\rho^*B_{u,2}(t) + u[t_{xy}u - (c_2 - \rho c_1)\hat{t}_u - u(a - \rho)],$$

with exception set

$$\Lambda = \{ F \in C([0, T]) : \mu(F^{-1}(\partial\{x, y\} \in \mathbb{R}^2|x > 0, y > 0)) > 0 \}.$$

First we need to show that $H_T(F_1, F_2)$ is continuous for $(F_1, F_2) \notin \Lambda$. Define an area

$$\lambda = (F_1, F_2)^{-1}(\partial\{x, y\} \in \mathbb{R}^2|x > 0, y > 0)).$$

For any sequence $(F_{1,n}, F_{2,n})$ converging in $C([0, T] \to \mathbb{R}^2)$ to some function $(F_1, F_2)$ as $n \to \infty$ we can define

$$(F_{1,n}(t), F_{2,n}(t)) = \begin{cases} (F_{1,n}(t), F_{2,n}(t)), & t \notin \lambda, \\ (F_1(t), F_2(t)), & t \in \lambda. \end{cases}$$

In this case for all $t \in [0, T]$ as $n \to \infty$

$$\mathbb{I}(F_{1,n} > 0, F_{2,n} > 0) \to \mathbb{I}(F_1 > 0, F_2 > 0).$$

Since $\mu(\lambda) = 0$, we have $H_T(F_{1,n}, F_{2,n}) \to H_T(F_1, F_2)$. Hence, the dominated convergence theorem establishes the continuity of the function $H_T$ at the point $(F_1, F_2)$.

From (A4) and (A8) we can establish that as $u$ tends to infinity

$$F_{1,xy,u}(t) \to B_1(t) - t - x = W_1(t) - t - x,$$

$$F_{2,xy,u}(t) \to \rho(B_1(t) - t - x) + \rho^*B_2(t) - (a - \rho)t - (y - \rho x) = W_2(t) - at - y.$$ 

Since $W_1$ and $W_2$ are standard Brownian motions

$$\mathbb{P} \left\{ \mu((W_1(\cdot) - \cdot)^{-1}(x)) > 0 \right\} = 0, \quad \mathbb{P} \left\{ \mu((W_2(\cdot) - a - \cdot)^{-1}(y)) > 0 \right\} = 0.$$ 

Consequently, $\mathbb{P} \left\{ (W_1(\cdot) - x - \cdot, W_2(\cdot) - a - \cdot - y) \in \Lambda \right\} = 0$, and we can apply continuous mapping theorem, which establish that for almost all $L$ positive

$$h_u(L, T, x, y) \to h(L, T, x, y), \quad u \to \infty,$$

where

$$h(L, T, x, y) = \mathbb{P} \left\{ \int_0^T \mathbb{I} \left( \begin{array}{c} B_1(t) - t - x > 0 \\ \rho(B_1(t) - t - x) + \rho^*B_2(t) - (a - \rho)t - (y - \rho x) > 0 \end{array} \right) \, dt > L \right\} = \mathbb{P} \left\{ \int_0^T \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y) \, dt > L \right\}.$$
To finish the proof it is enough to show the dominated convergence for the integrals

\[ I_u(T) = \int_{\mathbb{R}^2} h_u(L, T, x, y) \psi_u(x, y) \, dx \, dy. \]

In view of (A5) and (A6) for large enough \( u \) we have for some positive constant \( \tilde{C} \) such that for all \( x, y \in \mathbb{R} \)

\[
h_u(L, T, x, y) \leq \mathbb{P} \left\{ \exists t \in [0, T] : \begin{array}{l}
    u(B_{u,1}(t) + \tilde{t}_u(u_x - c_1) - u) > 0 \\
    u\rho(B_{u,1}(t) + \tilde{t}_u(u_x - c_1) - u) + u \rho^* B_{u,2}(t) + \tilde{t}_u[u_x(y - c_2 + \rho c_1) - u(a - \rho)] > 0
  \end{array} \right\}.
\]

\[
\leq \tilde{h}(T, x, y) := \begin{cases}
  \tilde{C} e^{-(x^2+y^2)}, & x, y \geq 0, \\
  \tilde{C} e^{-cx^2}, & x \geq 0, y < 0, \\
  \tilde{C} e^{-cy^2}, & y \geq 0, x < 0, \\
  1, & x, y < 0.
\end{cases}
\]

Since \( \lambda_{1,\varepsilon}, \lambda_{2,\varepsilon} \) are positive,

\[
I_u(T) \leq \int_{\mathbb{R}^2} \tilde{h}(T, x, y) \tilde{\psi}(x, y) \, dx \, dy
\]

\[
= \tilde{C} \int_{x, y > 0} e^{-(x^2+y^2)+\lambda_{1,\varepsilon} x+\lambda_{2,\varepsilon} y} \, dx \, dy + \tilde{C} \int_{x > 0, y < 0} e^{-cx^2+\lambda_{1,\varepsilon} x+\lambda_{2,\varepsilon} y} \, dx \, dy
\]

\[
+ \tilde{C} \int_{x < 0, y > 0} e^{-cy^2+\lambda_{1,\varepsilon} x+\lambda_{2,\varepsilon} y} \, dx \, dy + \tilde{C} \int_{x, y < 0} e^{\lambda_{1,\varepsilon} x+\lambda_{2,\varepsilon} y} \, dx \, dy < \infty.
\]

Thus the dominated convergence theorem may be applied and provides us with the claimed assertion. (The constant \( I(T) \) is continuous with respect to \( L \) (see Appendix, Lemma A.2), so it holds for all \( L \) positive).

(ii) We keep the same notation as in Lemma 3.1 (i).

The following representation for the function \( M(u, T) \) holds (write \( \tilde{t}_u \) for \( 1 - t/u^2 \) and recall (1))

\[
M(u, T) = F(u) \int_{\mathbb{R}^2} \mathbb{P} \left\{ \int_0^T \mathbb{I} \left( \frac{B_1(\tilde{t}_u) - c_1 \tilde{t}_u}{\rho B_1(\tilde{t}_u) + \rho^* B_2(\tilde{t}_u) - c_2 \tilde{t}_u} > u \right) \, dt > L \right\} \psi_u(x, y) \, dx \, dy
\]

\[
=: F(u) \int_{\mathbb{R}^2} h_u(L, T, x, y) \psi_u(x, y) \, dx \, dy.
\]

Using again \( B_{u,1} \) and \( B_{u,2} \) as in (A3) we can represent the function \( h_u(L, T, x, y) \) as

\[
h_u(L, T, x, y) = \mathbb{P} \left\{ \int_0^T \mathbb{I} \left( u(B_{u,1}(t) + \tilde{t}_u u_x - c_1 \tilde{t}_u - u) > 0 \\
    u \rho(B_{u,1}(t) + \tilde{t}_u u_x - c_1 \tilde{t}_u - u) + u \rho^* B_{u,2}(t) + u \tilde{t}_u[u_x y - c_2 - \rho c_1] \tilde{t}_u - u(a - \rho)] > 0 \right) \, dt > L \right\}.
\]

We have the same weak convergence as in (A4). Moreover, in this case we may use the convergence (A7). With the same arguments as in i) we can apply the continuous mapping theorem and establish the following convergence for almost all \( L \) positive and all \( x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\} \)

\[ h_u(L, T, x, y) \to h(L, T, x, y), \quad u \to \infty, \]

where

\[
h(L, T, x, y) = \mathbb{P} \left\{ \int_0^T \mathbb{I}(B_1(t) - t > x) \, dt > L \right\} (\mathbb{I}[a < \rho] + \mathbb{I}[a = \rho, y < 0]).
\]
To show the claim we can apply the dominated convergence theorem. Note that for large enough $u$

$$\log \psi_u(x, y) \leq \hat{\phi}(x, y) = (1 + \text{sign}(x)/2)x + \frac{c_2 - \rho c_1}{1 - \rho^2} y - \frac{y^2}{2}.$$ 

By Piterbarg inequality we can establish that for some positive constant $\tilde{C}$

$$h_u(L, T, x, y) \leq \mathbb{P}\left\{ \exists t \in [0, T] : u(B_{u,1}(t) + \tilde{I}_u(u_x - c_1) - u) > 0 \right\} \leq \tilde{h}(x) := \begin{cases} 
Ce^{-cx^2}, & x \geq 0, \\
1, & x < 0.
\end{cases}$$

Since $(1 + \text{sign}(x)/2) > 0$, then

$$\int_{\mathbb{R}^2} \tilde{h}(x)\hat{\phi}(x, y) \, dx \, dy < \infty,$$

and by the dominated convergence theorem the claim follows for almost all $L \in (0, \infty)$. The function $I(T)$ is continuous with respect to $L$, so the claimed assertion holds for all $L \in (0, \infty)$. ■