Fair Ride Allocation on a Line

Yuki Amano¹, Ayumi Igarashi², Yasushi Kawase³, Kazuhisa Makino¹, and Hirotaka Ono⁴

¹ Kyoto University, Kyoto, Japan. {ukiamano,makino}@kurims.kyoto-u.ac.jp
² National Institute of Informatics, Tokyo, Japan. ayumi.igarashi@nii.ac.jp
³ Tokyo Institute of Technology, Tokyo, Japan. kawase.y.ab@m.titech.ac.jp
⁴ Nagoya University, Aichi, Japan. ono@nagoya-u.jp

Abstract. With the advent of the ride-sharing platform, the carpooling has become increasingly more popular and widespread. In this paper, we propose a new model of the fair ride-sharing problem, where agents with different destinations share rides and divide the total cost among the members of each group according to the Shapley value. We consider several fairness and stability notions, such as envy-freeness and Nash stability, and obtain a number of existence and computational complexity results. Specifically, we show that when the agents’ destinations are aligned on a line graph, a Nash stable allocation that minimizes the social welfare of agents exists and can be computed efficiently. For this simple spatial structure, we also obtain parameterized complexity results for finding an envy-free allocation with respect to various parameters.

Keywords: Ride-sharing · Coalition formation game · Shapley value · Envy-free allocation

1 Introduction

Imagine a group of university students sharing a ride on a taxi. Each student has a different destination. For example, Alice may want to directly go back home while Bob prefers to go to the downtown to meet with his friends. Students may split into multiple groups and benefit from sharing the cost. It is then very natural to ask how they should form coalitions and divide the cost fairly? This question serves as a fundamental problem that has been studied over decades in the theory of cooperative games with transferable utilities.

The seminal paper by Shapley [25] offers an elegant solution to the latter problem. The Shapley’s rule calculates the expectation of each agent’s marginal contribution. It is known to be the unique efficient solution that satisfies the basic desideratum of ‘equal treatment of equals’ together with several other desirable properties. Intuitively, if two agents have the exactly same contribution, they will pay the same amount of money. The axiomatic studies of the Shapley value have resulted in a variety of applications, including voting [26], network analysis [16], facility location [7], and machine learning [13], to name a few.

While the division by the Shapley value ensures fairness within the same group sharing the cost, it does not guarantee fairness across different groups. In the preceding example of the ride-sharing, Alice may not envy Bob who happens to be in the same group; however, she may envy another student who is in a different group, if she could reduce her cost by swapping among them. Nevertheless, the literature on cooperative games typically ignores fairness across different coalitions, rather focusing on the fair treatment of individuals in the same group. Such restrictions on coalition structures have been extensively investigated in the study of hedonic games, initiated by [5,9]. However, the main concern is the existence of stable outcomes, i.e., coalition structures that are immune to agents’ deviations, in terms of both group and individual deviations.

In this paper, we initiate the study of the fair ride sharing problem. We consider a simple setting where agents, represented by a point of an interval, would like to go to their own destinations by multiple taxis with sharing the cost. Although agents start riding at the same point, the destinations of the agents can vary. The total cost charged to each taxi is determined by the distance between the riding point and the dropping point where the last agent in the taxi dropped off. The cost of each taxi is assumed to be divided according to the Shapley value, which is in fact used in a popular fair division website of Spliddit [15]: in our model, the Shapley value coincides with the payment rule where each agent who still rides a taxi evenly pays the cost for each interval. We formulate the notions of stability and fairness, including envy-freeness,
Nash stability, and Pareto optimality, and study the existence and complexity of allocations satisfying such properties. Our central technical contributions are several algorithms for finding fair and stable solutions.

We start by showing that an allocation that simultaneously minimizes the total cost and satisfies several stability requirements always exists and can be computed efficiently by a simple backward greedy strategy. This stands in sharp contrast to the standard results of hedonic games in two respects. First, while a stable outcome does not necessarily exist in the general setting [3], we show that an instance of our problem admits an outcome that satisfies some stability requirements, such as Nash stability and swap stability. In addition, we are able to ensure both efficiency and stability of an allocation, which are in general incomparable with each other except for some restricted classes of games [6,9].

For envy-freeness, it turns out that such allocation does not necessarily exist. We thus study the problem of finding an envy-free outcome from a parameterized complexity viewpoint. Specifically, we show that one can decide the existence of an envy-free allocation in polynomial time when the number of destination types is small. To this end, we will exploit the structural properties of an envy-free outcome. In essence, the property of envy-freeness ensures that agents with the same destination can be split into different coalitions only if they are the first passengers to drop off. Thus, by enumerating all possible ‘shapes’ of an envy-free outcome, and computing whether there is a size vector of coalitions that makes a given shape envy-free, one can efficiently decide the existence of an envy-free outcome.

Further, we will show that the problem of computing an envy-free allocation can be solved in polynomial time when the number of taxis is a constant. The design of the algorithm crucially relies on the fact that each agent must be assigned to the taxi that charges the minimum cost. This allows us to design a greedy algorithm to assign an agent to a taxi in an ascending manner. Using a similar technique, we show that one can compute an envy-free allocation when the maximum capacity of each taxi is at most four. Finally, we focus on an envy-free allocation that is consecutive with respect to agents’ destinations. This restriction is important in practical implementation. We show that the consecutive constraint enforces an envy-free allocation to have a specific structure, which allows one to check envy-freeness only by looking at the envy between the boundary agents. This enables us to decide the existence of such allocations efficiently.

1.1 Related work

The problem of fairly dividing the cost among multiple agents has been long studied in the context of cooperative games with transferable utilities; we refer the reader to the book of Chalkiadakis et al. [11] for an overview. Following the seminal work of Shapley [25], a number of researchers have investigated the axiomatic property of the Shapley value as well as its applications to real-life problems. Littlechild and Owen [18] analyzed the property of the Shapley value, when the cost of each subset of agents is given by the maximum cost associated with the agents in that subset. The work of Chun et al. [12] further studied the strategic process in which agents divide the cost of the resource, showing that the division by the Shapley value is indeed a unique subgame perfect Nash equilibrium under a natural three-stage protocol.

There exists a rich body of literature studying fairness and stability in the context of hedonic coalition formation games and fair division problems [3,6,8,10]. In hedonic games, agents have preferences over coalitions to which they belong, and the goal is to find a partition of agents into disjoint coalitions. Barrot and Yokoo [6] recently studied the compatibility between fairness and stability requirements, showing that top responsive games always admit an envy-free, individually stable, and Pareto optimal partition. Further, our work is similar in spirit to the complexity study of congestion games [19,22]. In fact, without capacity constraints, it is not difficult to see that the fair ride-sharing problem can be formulated as a congestion game. The fairness notions, including envy-freeness in particular, have been well-explored in the fair division literature. Although much of the focus is on the resource allocation among individuals, several recent papers study the fair division problem among groups [17,24]. Our work is different from theirs in that agents’ utilities depend not only on allocated resources, but also on the group structure.

Finally, our work is related to the growing literature on ride-sharing problem [1,2,4,12,15,21,23,27]. Santi et al. [23] empirically showed a large portion of taxi trips in New York City can be shared while keeping passengers’ prolonged travel time low. Motivated by an application to the ride-sharing platform, Ashlagi et
al. [2] considered the problem of matching passengers for sharing rides in an online fashion. They did not, however, study the fairness perspective of the resulting matching.

2 Model

For a natural number $s \in \mathbb{N}$, we write $[s] = \{1, 2, \ldots, s\}$. Our setting includes a finite set of agents, denoted by $A = [n]$, and a finite set of $k$ taxis. The non-empty subsets of agents are referred to as coalitions. Each agent $a \in A$ is endowed with a destination $x_a \in \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ is the set of positive reals. We call each $x_a$ the destination type (or shortly type) of agent $a \in A$.

We assume that the agents ride a taxi at the same initial location and they are sorted in non-decreasing order of their destinations, i.e., $x_1 \leq x_2 \leq \cdots \leq x_n$. Each taxi $i \in [k]$ has a quota $q_i$ representing the capacity of the taxi where $q_1 \geq q_2 \geq \cdots \geq q_k$ ($> 0$). An allocation $\mathcal{T} = (T_1, \ldots, T_k)$ is an ordered $k$-partition of $A$, i.e., (i) $\bigcup_{i \in [k]} T_i = A$ and (ii) $T_i \cap T_j = \emptyset$ for any distinct $i, j \in [k]$. By abuse of notation, we write $T \in \mathcal{T}$ to denote $T \in \{T_1, \ldots, T_k\}$. An allocation $\mathcal{T}$ is called feasible if $|T_i| \leq q_i$ for all $i \in [k]$. For each subset $T$ of agents and $s \in \mathbb{R}_{>0}$, we denote by $n_T(s)$ the number of agents in $T$ whose destinations $x_a$ appear at or after $s$, i.e.,

$$n_T(s) := |\{a \in T \mid x_a \geq s\}|.$$

In addition, we will use notations $T_{<s}, T_{=s},$ and $T_{>s}$ to denote the set of agents with type smaller than $s$ (i.e., $\{a \in T \mid x_a < s\}$), equal to $s$ (i.e., $\{a \in T \mid x_a = s\}$), and larger than $s$ (i.e., $\{a \in T \mid x_a > s\}$), respectively. Also, for a set of types $C$, we will write $T_{\in C}$ to denote the set of agents of a type in $C$ (i.e., $\{a \in T \mid x_a \in C\}$).

In model, we assume that the cost charged to each taxi $T_i$ is simply the distance of the furthest destination $\max_{a \in T_i} x_a$, which has to be divided among the agents in $T_i$. We remark that our results still hold in a more general scenario where the cost is given by a monotone non-decreasing function of the furthest destination, i.e., $f(\max_{a \in T} x_a)$ with a monotone function $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$. We can handle the case in our model by reconfiguring the destination of $a$ as $f(x_a)$ because $\max_{a \in T} f(x_a) = f(\max_{a \in T} x_a)$.

A natural question arises as to how the cost of each taxi should be divided. Among several payment rules of cooperative games, we consider a scenario where agents divide the cost using the well-known Shapley value [25], which, in our setting, coincides with the following specific function: for each coalition $T \subseteq A$ and positive real $x \in \mathbb{R}_{>0}$, let

$$\varphi(T, x) = \int_0^x \frac{1}{n_T(r)} \, dr.$$

For an allocation $\mathcal{T}$ and taxi $T_i \in \mathcal{T}$, the cost of agent $a \in T_i$ is defined as $\Phi_\mathcal{T}(a) := \varphi_i(T_i, x_a)$ where

$$\varphi_i(T_i, x_a) = \begin{cases} \varphi(T_i, x) & \text{if } |T_i| \leq q_i, \\ \infty & \text{if } |T_i| > q_i. \end{cases}$$

It is not difficult to verify that the payment rule $\varphi_i$ satisfies efficiency: the sum of the payments in $T_i$ equals the total cost if $|T_i| \leq q_i$, i.e., $\max_{a \in T_i} x_a = \sum_{a' \in T_i} \varphi_i(T_i, x_{a'})$ for any $T_i \in \mathcal{T}$ with $|T_i| \leq q_i$. The following proposition formally states that the payment rule for each taxi coincides with the Shapley value. We note that while Littlechild and Owen [18] presented a similar formulation of the Shapley value for airport games, our model is slightly different from theirs with the presence of capacity constraints.

**Proposition 1.** The payment rule $\varphi_i$ is the Shapley value.

**Proof.** We write $c : 2^A \rightarrow \mathbb{R}$ as a cost function where $c(T) = \max_{a \in T} x_a$ for each $T \subseteq A$ with $T \neq \emptyset$, and $c(T) = 0$ for $T = \emptyset$. Take any $T \subseteq A$ and $a \in T$. Let $T = \{a_1, \ldots, a_t\}$ such that $x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_t}$, and let $a = a_t$. We denote by $\Pi$ the set of permutations $\pi : T \rightarrow [t]$; we write $S_\pi(a) = \{a' \in T \mid \pi(a') \leq \pi(a)\}$
for each $\pi \in \Pi$. Recall that according to the Shapley value, the amount agent $a$ has to pay in the game $(T, c)$ is given by:

$$\frac{1}{t!} \sum_{\pi \in \Pi} (c(S_\pi(a)) - c(S_\pi(a) - a)).$$

Letting $x_{ao} = 0$, we have

$$\sum_{\pi \in \Pi} (c(S_\pi(a)) - c(S_\pi(a) - a) = \sum_{\pi \in \Pi} \sum_{j=1}^{i} (x_{a_j} - x_{a_{j-1}}) \mathbf{1}_{S_\pi(a)\cap\{a_j,\ldots,a_t\} = \{a\}}$$

$$= \sum_{j=1}^{i} (x_{a_j} - x_{a_{j-1}}) \sum_{\pi \in \Pi} \mathbf{1}_{S_\pi(a)\cap\{a_j,\ldots,a_t\} = \{a\}}$$

$$= \sum_{j=1}^{i} (x_{a_j} - x_{a_{j-1}}) \frac{t!}{t - j + 1},$$

where the third equality holds because $\mathbf{1}_{S_\pi(a)\cap\{a_j,\ldots,a_t\} = \{a\}}$ takes value 1 if and only if agent $a_i$ appears first at $\pi$ among agents $a_j, a_{j+1}, \ldots, a_t$. Thus,

$$\frac{1}{t!} \sum_{\pi \in \Pi} (c(S_\pi(a)) - c(S_\pi(a) - a)) = \sum_{j=1}^{i} \frac{x_{a_j} - x_{a_{j-1}}}{t - j + 1} = \int_{0}^{x_a} \frac{1}{n_T(r)} \bar{r} = \varphi(T, a). \quad \square$$

To illustrate our payment rule, consider the following example. We will use a succinct notation to specify examples. An instance will be denoted as a single arrow where the black circles on each arrow will denote the set of agents who drop off at the same destination. The labels above and below the circles represent the agents and the destinations, respectively. An allocation will be written as a sequence of arrows.

Example 1. Suppose that agents $a$, $b$, $c$, and $d$ ride on one taxi together from a starting point and drop off respectively at points of 12, 24, 36, and 40 on a line. The total cost is 40, which corresponds to the drop-off point of $d$. According to the Shapley value, each of agents $a$, $b$, $c$, and $d$ pays 3, 7, 13, and 17, respectively. In fact, from the starting point to $a$’s drop-off point, there are four agents riding the taxi, so they equally divide the cost of 12, which means that $a$ should pay 3. Then, from $a$’s dropping point to $b$’s dropping point, there are three agents riding the taxi, so they equally divide the cost of 24 − 12 = 12, which results in the cost of 4 for each of the three agents. Thus agent $b$ pays $3 + 4 = 7$. By repeating similar arguments, $c$ pays $7 + (36 - 24)/2 = 13$, and $d$ pays $13 + (40 - 36) = 17$.

Example 2. Suppose that there are 8 agents, say $a_i, b_i, c_i$, and $d_i, i = 1, 2$, and two taxis $T_1$ and $T_2$ with quota $q_i = 4$ for each $i = 1, 2$. Agents $a_i, b_i, c_i$, and $d_i (i = 1, 2)$ drop off at the corresponding points of $a, b, c$ and $d$, respectively. Consider an allocation $T$ that $a_1, b_1, c_1$, and $d_1$ ride on $T_1$ for $i = 1, 2$. See Fig. 1. The cost of each taxi is 40, which should be divided by the Shapley value as explained above. However, this allocation is not reasonable from several points of view. First, the total cost of the agents is $40 + 40 = 80$, but another allocation $T'$ of $T_1' = \{a_1, a_2, b_1, b_2\}$ and $T_2' = \{c_1, c_2, d_1, d_2\}$ costs $24 + 40 = 64$ only; see Fig. 2.

![Fig. 1. The allocation $T$ in Example 1](image1)

![Fig. 2. The allocation $T'$ in Example 2](image2)
Furthermore, there is an envy between agents. For example, \(d_1\) envies \(c_2\), because if \(d_1\) were to replace with \(c_2\), i.e., \(T_2 - c_2 + d_1 = \{a_2, b_2, d_1, d_2\}\), agent \(d_1\) needs to pay \(3 + 4 + (40 - 24)/2 = 15\) only, which is cheaper than the current cost of 17.

3 Solution concepts

Agents split into coalitions and use the Shapley value to divide the final cost. Our goal is to find a partitioning of agents that satisfies several natural desiderata. Below, we introduce fairness, stability, and efficiency criteria that are inspired from coalition formation games and resource allocation problems [3,9,10,14].

**Fairness:** *Envy-freeness* is one of the most studied fairness concepts in resource allocation, which requires that no agent prefers another agent's coalition over his own. Formally, for an allocation \(T\), agent \(a_i \in T_i\) envies \(a_j \in T_j\) if \(a_i\) can be made strictly better off by replacing \(a_j\) with himself, i.e., \(\varphi_i(T_j - a_j + a_i, x_{a_i}) < \varphi_i(T_i, x_{a_i})\). A feasible allocation \(T\) is *envy-free (EF)* if no agent envies the others. Without capacity constraints, i.e., \(q_1 \geq n\), envy-freeness can be trivially achieved, for example, by partitioning the agents into the grand coalition. Also, when the number of taxis is at least the number of agents, i.e., \(k \geq n\), an outcome that partitions the agents into the singletons is envy-free.

**Stability:** We adapt the following four definitions of stability concepts of hedonic games [3,8,9] to our setting. The first stability concepts we introduce are those that are immune to individual deviations. For an allocation \(T\) and taxis \(i, j \in [k]\), agent \(a_i \in T_i\) has a *Nash-deviation* to \(T_j\) if \(\varphi_j(T_j + a_i, x_{a_i}) < \varphi_i(T_i, x_{a_i})\). Note that by the definition of the function \(\varphi_j\), agent \(a_i\) does not have a Nash-deviation to \(T_j\) if adding \(a_i\) to \(T_j\) violates the capacity constraint, i.e., \(|T_j| \geq q_j\). We say that a feasible allocation \(T\) is

- *Nash stable (NS)* if no agent has a Nash deviation;
- *contractually individually stable (CIS)* if for each \(T_i \in T\) and \(a_i \in T_i\), agent \(a_i\) does not have a Nash deviation or the cost of some agent \(a' \in T_i\) increases if \(a_i\) leaves \(T_i\), i.e., \(\varphi_i(T_i - a_i, x_{a'}) > \varphi_i(T_i, x_{a'})\).

In our model, the definition of CIS only requires that no agent \(a_i\) in a singleton coalition has a Nash deviation since a deviation of an agent necessarily increases the costs of the other agents in his coalition.

We will also consider stability notions that capture resistance to swap deviations. For an allocation \(T\) and \(i, j \in [k]\), agent \(a_i \in T_i\) *can replace \(a_j \in T_j\) if* \(\varphi_j(T_j - a_j + a_i, x_{a_i}) \leq \varphi_i(T_i, x_{a_i})\) [6,20]. A feasible allocation \(T\) is

- *weakly swap-stable (WSS)* if there is no pair of agents \(a_i \in T_i\) and \(a_j \in T_j\) \((i \neq j)\) such that \(a_i\) and \(a_j\) envy each other;
- *strongly swap-stable (SSS)* if there is no pair of agents \(a_i \in T_i\) and \(a_j \in T_j\) \((i \neq j)\) such that \(a_i\) envies \(a_j\), and \(a_j\) can replace \(a_i\).

**Efficiency:** Besides fairness and stability, another important property of allocation is *efficiency*. For two feasible allocations \(T' \) and \(T\), we say that \(T' \) *strictly Pareto dominates* \(T\) if \(\Phi_T(a) > \Phi_{T'}(a)\) for all \(a \in A\); \(T' \) *weakly Pareto dominates* \(T\) if \(\Phi_T(a) \geq \Phi_{T'}(a)\) for all \(a \in A\) with \(\Phi_T(a) > \Phi_{T'}(a)\) for at least one \(a \in A\).

A feasible allocation \(T\) is *weakly Pareto optimal (WPO)* if there is no feasible allocation \(T'\) that strictly Pareto dominates \(T\); it is *strongly Pareto optimal (SPO)* if there is no feasible allocation \(T'\) that weakly Pareto dominates \(T\). The total cost of an allocation \(T\) is defined as \(\sum_{T \in T} \sum_{a \in T} \varphi(T, x_a)\). Note that the total cost of a feasible allocation \(T\) is equal to \(\sum_{T \in T : T \neq \emptyset} \max_{a \in T} x_a\). A feasible allocation \(T\) is *social optimum (SO)* if it minimizes the total cost over all feasible allocations.

Fig. 3 describes the implication relationships among the solution concepts defined in Section 3. We will show below that any two concepts that do not have containment relationships in Fig. 3 are incomparable. Namely, there are examples that are (i) SO and NS but not WSS (Example 3), (ii) EF but not CIS (Example 4), (iii) NS and EF but not WPO (Example 5), and (iv) SO and EF but not NS (Example 6), respectively. In addition, we will show that the relationship of any two concepts that are directly connect in 3 is proper inclusion by providing the following examples: (a) SSS but not EF (Example 5), (b) WSS but
not SSS (Example 7), (c) SPO but not SO (Example 8), (d) WPO but not SPO (Example 7), (e) CIS but not WPO (Example 5), (f) CIS but not NS (Example 6).

**Proposition 2.** The concepts of allocations satisfies the inclusion relationships shown in Fig. 3.

![Inclusion relationships among the concepts. Any two concepts without any path between them are incomparable.]

**Example 3.** Consider an instance where $n = 9$, $k = 2$, $q_1 = 5$, $q_2 = 4$, and $x_1 = 1, x_2 = x_3 = 2, x_4 = \cdots = x_9 = 4$. An allocation $\mathcal{T} = (\{2, 3, 7, 8, 9\}, \{1, 4, 5, 6\})$ (see Fig. 4) is socially optimal and Nash stable. However, agents 1 and 9 envy each other, which implies that $\mathcal{T}$ is not WSS.

**Example 4.** Consider an instance where $n = 2$, $k = 2$, $q_1 = q_2 = 2$, and $x_1 = x_2 = 1$. Then, an allocation $\mathcal{T} = (\{1\}, \{2\})$ is envy-free but not contractually individually stable.

**Example 5.** Consider an instance where $n = 4$, $k = 3$, $q_1 = q_2 = 2, q_3 = 4$, and $x_1 = x_2 = x_3 = x_4 = 1$. Then, an allocation $\mathcal{T} = (\{1, 2\}, \{3, 4\}, \emptyset)$ is Nash stable and envy-free. However, another feasible allocation $\mathcal{T}' = (\{1, 2, 3, 4\}, \emptyset, \emptyset)$ strictly Pareto dominates $\mathcal{T}$, and hence $\mathcal{T}$ is not weakly Pareto optimal.

**Example 6.** Consider an instance where $n = 5$, $k = 2$, $q_1 = q_2 = 3, x_1 = 1$, and $x_2 = x_3 = x_4 = x_5 = 2$. Then, an allocation $\mathcal{T} = (\{1, 2, 3\}, \{4, 5\})$ (see Fig. 5) is socially optimum. However, agents 2 and 3 have a Nash deviation to $T_2$, and thus $\mathcal{T}$ is not Nash stable.

**Example 7.** Consider an instance where $n = 4$, $k = 2$, $q_1 = q_2 = 2, x_1 = x_2 = 1$, and $x_3 = x_4 = 2$. Then, an allocation $\mathcal{T} = (\{1, 3\}, \{2, 4\})$ (see Fig. 6) is weakly swap-stable but not strongly swap-stable. Moreover, $\mathcal{T}$ is weakly Pareto optimal but not strongly Pareto optimal.

**Example 8.** Consider an instance where $n = 3$, $k = 2$, $q_1 = q_2 = 1, x_1 = 1$, and $x_2 = x_3 = 2$. Then, an allocation $\mathcal{T} = (\{1, 2\}, \{3\})$ (see Fig. 7) is strongly Pareto optimal but not socially optimum. In addition, $\mathcal{T}$ is strongly swap-stable but not envy-free (since agent 3 envies 1).

## 4 Stable and socially optimal allocations

In this section, we will show that an allocation that greedily groups agents with furthest destinations together satisfies Nash stability, strongly swap-stability, and social optimality. Specifically, we will design the following ‘backward greedy’ allocation: We greedily add an agent with the furthest destination to the current coalition $T_i$ as long as $T_i$ has size strictly less than the maximum size $q_i$ of available taxis. If $T_i$ becomes saturated, then we proceed to create a new coalition of size $q_{i+1}$. The formal description is shown in Algorithm 1, which runs in $O(n + k)$ time. Here, recall that the agents are sorted in non-decreasing order of their destinations and the taxis are sorted in non-increasing order of their capacities.
Theorem 1. If there is a feasible outcome, the backward greedy (Algorithm 1) returns a feasible allocation that is socially optimal, Nash stable, and strongly swap stable in polynomial time.

Proof. It is not difficult to see that Algorithm 1 returns a feasible allocation if there exists such an allocation. In what follows, suppose that the algorithm returns a feasible allocation $\mathcal{T} = (T_1, T_2, \ldots, T_k)$.

First, we show that the allocation $\mathcal{T}$ satisfies the Nash stability. Let $k'$ be the index at the end of Algorithm 1, i.e., $T_{k'}$ is the last coalition that was made. Note that agents can have a deviation to the coalition $T_{k'}$ only, as the other groups $T_1, T_2, \ldots, T_{k'-1}$ do not have an empty seat and a deviation to an empty taxi $j$ (> $k'$) is not profitable. Further, if agent $a \in T_i$ can deviate to $T_{k'}$, agent $a$ would become the last passenger to drop off but $|T_{k'}| + 1 \leq q_{k'} \leq q = |T_i|$, which implies

$$\varphi(T_{k'} + a, x_a) - \varphi(T_i, x_a) \geq \varphi(T_{k'} + a, x_b) + (x_a - x_b) - \varphi(T_i, x_b) - (x_a - x_b),$$

$$= \varphi(T_{k'} + a, x_b) - \varphi(T_i, x_b) \geq \frac{x_b}{|T_{k'}| + 1} - \frac{x_b}{|T_i|} > 0,$$

where $x_b = \max_{a \in T_{k'}, x_{k'}}$, yielding a contradiction. Thus $\mathcal{T}$ is Nash stable.

Next, we show that $\mathcal{T}$ is strongly swap-stable. Since swapping a pair of agents in the same taxi has no effect on the cost, it suffices to show that there is no beneficial swap of agents who belong to different taxis. That is, we will show that for any pair of agents $a_i$ and $a_j$ with $a_i \in T_i$ and $a_j \in T_j$ ($i < j$), if agent $a_i$ can replace $a_j$, i.e., $\varphi(T_j - a_j + a_i, x_{a_j}) \leq \varphi(T_i, x_{a_i})$, then swapping $a_i$ and $a_j$ does not change the cost for each of the agents, i.e.,

- $\varphi(T_j - a_j + a_i, x_{a_j}) = \varphi(T_i, x_{a_i})$; and
- $\varphi(T_i - a_i + a_j, x_{a_i}) = \varphi(T_j, x_{a_j})$.

Algorithm 1: Backward greedy

1. Initialize $T_i \leftarrow \emptyset$ for each $i \in [k]$ and let $k' \leftarrow 1$;
2. for $a \leftarrow n$ to 1 do
3.   if $|T_{k'}| = q_{k'}$ then
4.     $k' \leftarrow k' + 1$;
5.   if $k' > k$ then return “No feasible allocation”;
6.   Set $T_{k'} \leftarrow T_{k'} + a$;
7. return $(T_1, T_2, \ldots, T_k)$;
To see this, take any pair of agents \(a_i \in T_i\) and \(a_j \in T_j\) \((i < j)\) and suppose that \(\varphi(T_j - a_j + a_i, x_{a_i}) \leq \varphi(T_i, x_{a_i}).\) Observe first that by construction of \(\mathcal{T},\) \(|T_i| \geq |T_j|\) and \(x_{a_i} \geq x_a\) for all \(a \in T_j\). Thus, for all \(s \in \mathbb{R}_{\geq 0},\)

\[
    n_{T_j - a_j + a_i}(s) \leq n_{T_i}(s)
\]

meaning the number of agents who ride the \(i\)th taxi at \(s\) is greater or equal to the number of agents who ride the \(j\)th taxi at \(s\) after \(a_i\) swaps with \(a_j.\) On the other hand, our supposition implies:

\[
    \int_0^{x_{a_i}} \frac{r}{n_{T_j - a_j + a_i}(r)} = \varphi(T_j - a_j + a_i, x_{a_i}) \leq \varphi(T_i, x_{a_i}) = \int_0^{x_{a_i}} \frac{r}{n_{T_i}(r)}.
\]

Hence, we have \(n_{T_j - a_j + a_i}(s) = n_{T_i}(s)\) for all \(s \in [0, x_{a_i}],\) which implies \(|T_i| = |T_j|\) and \(x_a = x_{a_i}\) for all \(a \in T_j.\) Thus, we have

\[
    \varphi(T_j - a_j + a_i, x_{a_i}) = \int_0^{x_{a_i}} \frac{r}{n_{T_j - a_j + a_i}(r)} = \int_0^{x_{a_1}} \frac{r}{n_{T_i}(r)} = \varphi(T_i, x_{a_i})
\]

and

\[
    \varphi(T_i - a_i + a_j, x_{a_j}) = \int_0^{x_{a_1}} \frac{r}{n_{T_i - a_i + a_j}(r)} = \int_0^{x_{a_1}} \frac{r}{n_{T_j}(r)} = \varphi(T_j, x_{a_j}),
\]

which proves the claim.

It remains to show that \(\mathcal{T}\) is socially optimal, i.e., \(\mathcal{T}\) minimizes the total cost. Recall that the total cost of a feasible allocation is simply the sum of the furthest destinations in each taxi \(\sum_{i=1}^{k'} y_i,\) where \(y_i = \max_{a \in T_i} x_a\) for each \(i \in [k']\). We will show that the sequence of the last drop-off locations in the socially optimal outcomes must be unique and identical to that of the backward greedy allocation, which implies that \(\mathcal{T}\) is socially optimal as well. Consider any feasible allocation \(\mathcal{T}'\) that is socially optimal. Let \(y'_1, y'_2, \ldots, y'_{\ell}\) denote the furthest destinations among agents of each coalition \(T' \in \mathcal{T}'\) with \(T' \neq \emptyset\) aligned in non-increasing order, i.e., \(y'_1 \geq y'_2 \geq \cdots \geq y'_{\ell}.\) Clearly, \(y'_1 = y_1.\) Also, by construction of \(\mathcal{T},\) it can be easily verified that \(\ell \geq k'.\)

We will show that \(y'_j \geq y_j\) for \(j = 1, \ldots, k'.\) Suppose otherwise and take minimum such \(j > 1\) violating the inequality: \(y'_j < y_j.\) This means that all agents \(a\) whose destinations \(x_a\) appear after \(y'_j\) (i.e., \(x_a > y'_j\)) are put into \(j - 1\) coalitions under \(\mathcal{T}'\). We obtain

\[
    \sum_{i=1}^{j-1} q_i = \sum_{i=1}^{j-1} |T_i| < \left| \{ a \in A \mid x_a > y'_j \} \right| \leq \sum_{i=1}^{j-1} q_i,
\]

where the second inequality follows from the fact that \(y_j > y'_j,\) but \(j\)th furthest drop-off location is \(y_j\) under \(\mathcal{T}\). This yields a desired contradiction. \(\square\)

An obvious corollary of the above theorem is that an outcome that satisfies all the fairness, stability, and efficiency notions defined in Section 3, except for envy-freeness, exists whenever a feasible allocation exists.

## 5 Envy-free allocations

We have seen that Nash stability as well as Pareto optimality are possible to achieve simultaneously. In contrast, the set of envy-free feasible allocations may be empty even when a feasible outcome exists, as shown in the following example.

**Example 9.** Consider an instance where \(n = 4, k = 2, q_1 = q_2 = 2, x_1 = 2,\) and \(x_2 = x_3 = x_4 = 4.\) We will show that no feasible allocation is envy-free. To see this, let \(\mathcal{T} = (T_1, T_2)\) be an envy-free feasible allocation. By feasibility, the capacity of each taxi must be full, i.e., \(|T_1| = |T_2| = 2.\) Suppose without loss of generality that \(T_1 = \{1, 2\}.\) Then, \(T_2 = \{3, 4\}\) (see Fig. 5). Thus, agent 2 envies the agents of the same type. Indeed, he needs to pay the cost of 3 at the current coalition while he would only pay 2 if he were to replace 3 (or 4) with himself. Hence, no feasible allocation is envy-free.
Given that an envy-free outcome may not exist, we turn our attention to the problem of deciding the existence of an envy-free feasible allocation and finding one if it exists. We first present the parameterized complexity results with respect to various parameters. We then study the complexity of finding consecutive envy-free allocations where each group of agents is consecutive with respect to their destinations.

5.1 Parameterized setting

A natural approach is to explore the complexity of our problem from the perspective of fixed parameter tractability. A problem is said to be fixed parameter tractable (FPT) with respect to a parameter $p$ if each instance $I$ of this problem can be solved in time $f(p) \cdot \text{poly}(|I|)$, and to be slice-wise polynomial (XP) with respect to $p$ if each instance $I$ of this problem can be solved in time $f(p) \cdot |I|^g(p)$; here $f(\cdot)$ and $g(\cdot)$ are computable functions that depend on $p$ only, and poly(\cdot) is an arbitrary polynomial.

In our setting, a particularly relevant parameter is the number of destinations: generally, we expect the number of destinations to be small in many practical applications. For instance, a workshop organizer may offer a few excursion opportunities to the participants of the workshop. The number of students using a school bus of the same route may be limited. The other parameters we consider include the number of taxis and the maximum capacity of a taxi. These restrictions are also relevant in many real-life scenarios: for example, a taxi company may have a limited resource, both in terms of quantity and capacity.

In what follows, we will design efficient algorithms to compute an envy-free feasible allocation when each of the above parameters is small. We first present an FPT algorithm to compute an envy-free allocation with respect to the number of types. Further, we show that obtaining an envy-free allocation is polynomial time solvable when the number of taxis is constant, or the maximum capacity of a taxi is at most four.

Before we proceed, we present a few auxiliary lemmas that will play a key role in designing such algorithms. One important property of an envy-free allocation is the monotonicity of the size of coalitions in terms of the first drop-off point, which is formalized in the following lemma.

**Lemma 1 (Monotonicity lemma).** For any envy-free allocation $\mathcal{T}$ and non-empty coalitions $T, T' \in \mathcal{T}$, we have the following:

\[
\min_{a \in T} x_a \leq \min_{a' \in T'} x_{a'} \implies |T| \geq |T'|, \\
\min_{a \in T} x_a = \min_{a' \in T'} x_{a'} \implies |T| = |T'|.
\]

**Proof.** To prove (1) by contradiction, suppose that $\min_{a \in T} x_a \leq \min_{a' \in T'} x_{a'}$ and $|T| < |T'|$. Let $a^* \in \arg \min_{a \in T} x_a$. Then, $a^*$ envies any $a' \in T'$ because

\[
\varphi(T, x_{a^*}) = \frac{x_{a^*}}{|T|} > \frac{x_{a'}}{|T'|} = \varphi(T' - a' + a^*, x_{a^*}).
\]

In addition, (2) holds by (1). \qed

Agents of type $x$ are called split in an allocation $\mathcal{T}$ if there exist distinct $T, T' \in \mathcal{T}$ such that $T_{=x} \neq \emptyset$ and $T'_{=x} \neq \emptyset$. The next lemma states that, the agents of type $x$ can be split in an envy-free allocation only if they are the first passengers to drop off in their coalitions, and such coalitions are of the same size; further, if two taxis have an equal number of agents of split type, then no agent of the other type ride these taxis.
Lemma 2 (Split lemma). For an envy-free allocation \( T \) and a type \( x \), suppose that there exist distinct \( T, T' \in \mathcal{T} \) such that \( T_{=x} \neq \emptyset \) and \( T'_{=x} \neq \emptyset \). Then, the following hold:

(i) The agents of type \( x \) are the first passengers to drop off in both \( T \) and \( T' \), i.e., \( T_{<x} = T'_{<x} = \emptyset \).

(ii) Both \( T \) and \( T' \) are of the same size, i.e., \( |T| = |T'| \), and \( |T_{=x}| = |T'_{=x}| \).

(iii) If \( |T_{=x}| = |T'_{=x}| \), then \( T = T_{=x} \) and \( T' = T'_{=x} \).

Proof. Let \( a \in T_{=x} \) and \( b \in T'_{=x} \). As \( a \) and \( b \) do not envy each other, we have

\[
\varphi(T, x) = \varphi(T', x, a) \leq \varphi(T' - b + a, x) = \varphi(T', x),
\]

\[
\varphi(T', x) = \varphi(T', x, b) \leq \varphi(T - a + b, x) = \varphi(T, x).
\]

Hence, \( \varphi(T, x) = \varphi(T', x) \).

To show (i), suppose to the contrary that \( T_{<x} \) is non-empty and let \( \hat{a} \) be its element. Then, \( b \) envies \( \hat{a} \) because

\[
\varphi(T', x, b) = \varphi(T, x) > \varphi(T - \hat{a} + b, x).
\]

Hence, \( T_{<x} = \emptyset \). By symmetry, we also have \( T'_{<x} = \emptyset \), proving (i). This implies that \( \varphi(T, x) = x/|T| \) and \( \varphi(T', x) = x/|T'| \). Since \( \varphi(T, x) = \varphi(T', x) \), we have \( |T| = |T'| \), which proves (ii).

To see (iii), suppose towards a contradiction that \( |T_{=x}| = |T'_{=x}| \) but there is an agent in \( T \) or \( T' \) whose destination appears strictly after \( x \), i.e., \( (T \cup T') \setminus A_{=x} \neq \emptyset \). Let \( a^* \) be the agent with closest destination among such agents, i.e., \( a^* \in \arg \min_{a^* \in (T \cup T')} A_{=x} x_{a^*} \). Assume without loss of generality \( a^* \in T \). Then, \( a^* \) envies \( b \) because

\[
\varphi(T' - b + a^*, x_{a^*}) = \frac{x}{|T'|} + \frac{(x_{a^*} - x)}{|T'| - |T'_{=x}|} + 1 < \frac{x}{|T|} + \frac{(x_{a^*} - x)}{|T| - |T_{=x}|} = \varphi(T, x_{a^*}),
\]

yielding a contradiction. Hence, \( |T_{=x}| = |T'_{=x}| \) implies \( T = T_{=x} \) and \( T' = T'_{=x} \).

Small number of types The goal of this subsection is to show that we can find an envy-free feasible allocation in FPT time with respect to the number of destination types. Due to the split lemma, once we know the first drop-off points of each coalition, no agent of the other type will be split in an envy-free allocation. Thus, the ‘shapes’ of an envy-free allocation representing the agent types of each taxi have a particular structure where the first drop-off points can be considered as roots, followed by the agents of the other type. The main idea of the following algorithm is to enumerate all such structures and decide whether each agent can be assigned to each taxi in an envy-free manner that is consistent with the given shape.

Throughout, we assume that there are \( p \) types of agents, i.e., \( p = |\{ x_a \mid a \in A \}| \). Let \( \{ y_1, y_2, \ldots, y_p \} \) be the set of destination types, namely, \( \{ y_1, y_2, \ldots, y_p \} = \{ x_a \mid a \in A \} \) where \( y_1 < y_2 < \cdots < y_p \). We call a directed graph over the set of types star-forest if it is a directed out-forest in which every non-root node has out-degree at most one and every edge is oriented from a smaller type to a larger type. We write \( T \simeq T' \) to denote that \( T \) and \( T' \) contains the same number of agents for any type, i.e., \( |T_{=y}| = |T'_{=y}| \) for all \( y \in V \).

Enumerating allocation graphs. We will first define the notion of an allocation graph. For a feasible allocation \( T = (T_1, \ldots, T_k) \), the allocation graph of \( T \) is a directed graph \( G_T = (V, E) \) over the set of types where there is an edge from \( y_i \) to \( y_j \) if and only if an agent of type \( y_j \) drops off just after an agent of type \( y_i \) in some coalition \( T \in \mathcal{T} \), i.e.,

\[
V = \{ y_1, y_2, \ldots, y_p \},
\]

\[
E = \bigcup_{T \in \mathcal{T}} \left\{(y_i, y_j) \in V^2 \mid y_i, y_j \in \{ x_a \mid a \in T \}, y_i < y_j, y_i < x_a < y_j (\forall a \in T) \right\}.
\]

Note that the allocation graph is always acyclic because every edge is oriented from a smaller type to a larger type. Now let us consider the allocation graph of an envy-free feasible allocation \( T \). Recall that by the split
lemma (Lemma 2), the agents who are not the first passengers to drop off are not split in $T$, implying that all such agents are assigned to the same taxi. Hence, the in-degree of such type must be one and the out-degree is at most one in $G^{T}$. This readily implies that $G^{T}$ is a star-forest with roots being the first passengers to drop off in each taxi.

**Lemma 3.** If $T$ is an envy-free feasible allocation, then $G^{T}$ is star-forest.

An example of the allocation graph for an envy-free feasible allocation is depicted in Fig. 9. Here, a non-empty path from a root to a leaf corresponds to the corresponding taxi. We note that agents of root type may also form their own coalitions (e.g., $T_{3}$ in the example of Fig. 9). In addition, any singleton corresponds to a set of taxis that carry the agents of one type only.

![Fig. 9. An example of the allocation graph for an envy-free feasible allocation $T = (T_{1}, T_{2}, \ldots, T_{9})$ where $T_{1} = \{r_{1}, a_{1}^{1}, a_{2}^{1}, a_{3}^{1}, a_{4}^{1}\}$, $T_{2} = \{r_{2}^{1}, r_{1}^{1}, b_{1}^{2}, b_{2}^{2}, b_{3}^{2}\}$, $T_{3} = \{r_{1}^{3}, r_{3}^{3}, r_{1}^{1}, c_{1}, c_{2}\}$, $T_{4} = \{r_{2}^{3}, r_{2}^{2}, r_{2}^{5}, r_{2}^{2}\}$, $T_{5} = \{r_{5}^{2}, r_{2}^{2}, d_{1}, d_{2}, d_{3}\}$, $T_{6} = \{r_{1}^{3}, r_{6}^{3}, r_{5}^{3}, r_{1}^{6}\}$, $T_{7} = \{r_{9}^{2}, r_{3}^{3}, r_{3}^{3}, r_{3}^{10}\}$, $T_{8} = \{r_{4}^{1}, e_{1}^{1}, e_{2}^{1}\}$, $T_{9} = \{r_{4}^{1}, r_{1}^{3}, f_{1}, f_{2}\}$. There are seven agents of type $r_{1}$ ($r_{1}^{1}, r_{1}^{5}$), seven agents of type $r_{2}$ ($r_{2}^{2}, r_{2}^{3}, r_{2}^{5}$), ten agents of type $r_{3}$ ($r_{3}^{1}, r_{3}^{2}, r_{3}^{10}$), and three agents of type $r_{4}$ ($r_{4}^{1}, r_{4}^{2}, r_{4}^{3}$).

The number of star-forest graphs is at most $p^{n}$ because the in-degree of every node is at most one. Hence, we can enumerate all the possible allocation graphs that could be induced by an envy-free feasible allocation in FPT time.

**Computing size vectors.** Next, we consider whether given a star-forest graph, there is an envy-free allocation that is consistent with the graph. Formally, an allocation $T$ is consistent with a star-forest graph $G$ if $G = G^{T}$. In order to find such allocation that is envy-free, one naive approach is to treat each path of the trees starting with a root independently, and guess the possible size of the respective coalitions. However, due to the split lemma, we know that all the taxis that involves the same split type have the same size in an envy-free allocation. Thus, it suffices to assume that the number of agents assigned to each path is the same within each of the connected components, and guess the size $\lambda_{j}$ of a coalition with the first drop-off point being the root of each rooted tree $C_{j}$. We may then try the brute-force approach to enumerate all possible such vectors, which gives rise to an $O(n^{p})$ algorithm. It turns out that there is a more sophisticated way to reduce this upper bound to $O(n^{4})$ by iteratively updating the component-wise maximum size vector; the underlying lattice structure of size vectors will play a key role in designing the algorithm.

Now, fix a star-forest graph $G$ and let us consider an envy-free feasible allocation $T$ that is consistent with $G$. We will closely look at the properties of $\lambda_{j}$ in an envy-free feasible allocation. Specifically, let $C = \{C_{1}, \ldots, C_{t}\}$ be the set of the vertex sets of the trees of $G$, and let $r_{j} = \min C_{j}$ be the root for each $C_{j} \in C$. We assume that the roots are arranged in ascending order, i.e., $r_{1} < \cdots < r_{t}$. For each $j \in [t]$, we let $\lambda_{j}$ be the size of $T \in \mathcal{T}$ that consists of agents of a type in $C_{j}$ (i.e., $T \subseteq A_{\in C_{j}}$). Such a size can be uniquely determined by the split lemma (Lemma 2) because the first drop-off point of $T$ must be $r_{j}$ and all such coalitions have the same size. For instance, $\lambda_{j}^{T} = 6$, $\lambda_{j}^{T} = 5$, and $\lambda_{j}^{T} = 4$ in the example of Fig. 9. We call $\lambda_{T}^{T} = (\lambda_{j}^{T})_{j \in [t]}$ a size vector induced by $T$. In the graph $G$, the types in $C_{j} - r_{j}$ are partitioned into paths by the structure of the star-forest graph. Let $C_{j}^{1}, \ldots, C_{j}^{k}$ be the vertex sets of the paths. The size vector induced by an envy-free allocation satisfies the following conditions that are efficiently verifiable.
Lemma 4. The size vector $\lambda^T$ induced by an envy-free feasible allocation $T$ satisfies the following conditions:

\[
\begin{align*}
\lambda^T_j & \text{ is a divisor of } |A_{\in C_j}| \quad (\forall j \in [t]), \\
|A_{\in C_j}| & \geq \lambda^T_j \cdot h_j \quad (\forall j \in [t]), \\
\lambda^T_j & > \max_{t \in [n]} |A_{\in C_j^t}| \quad (\forall j \in [t]), \\
\sum_{j \in [t]} |A_{\in C_j}|/\lambda^T_j & \leq k, \text{ and } \\
\lambda^T_1 & \geq \lambda^T_2 \geq \cdots \geq \lambda^T_t.
\end{align*}
\]

Proof. As the agents of a type in $C_j$ form at least $h_j$ coalitions, each of which consists of $\lambda^T_j$ agents, and $|$ and $\lambda^T$ are satisfied. Also, as every coalition corresponding to $C_j$ contains at least one agent of type $r_j$, $|$ is satisfied. In addition, $|$ holds because the agents of a type in $C_j$ form $|A_{\in C_j}|/\lambda_j$ coalitions. Finally, $|$ holds by Lemma 1. \hfill \Box

Observe, on the other hand, that fixing an allocation graph and a size vector gives rise to an allocation. Formally, we call a vector $\lambda \in [n]^t$ a valid size vector with respect to $G$ if it satisfies the conditions (3)–(7) in Lemma 4 and an invalid size vector with respect to $G$ otherwise. For a valid size vector $\lambda$, we define $T^{\lambda}$ to be an allocation that induces $\lambda$. That is, $T^{\lambda}$ is an allocation such that the cardinality of $T^\lambda_i$ is $\lambda_j$, and

1. $T^\lambda_i$ consists of all the agents of a type in $C_j^\eta$ and some agents of type $r_j$ (i.e., $A_{\in C_j} \subseteq T^\lambda_i \subseteq A_{\in C_j^\eta} \cup A_{=r_j}$ and $|T^\lambda_i \cap A_{=r_j}| = \lambda_j - |A_{\in C_j}^\eta|$) if $\eta \in [h_j];$
2. $T^\lambda_i$ consists of some agents of type $r_j$ (i.e., $T^\lambda_i \subseteq A_{=r_j}$ and $|T^\lambda_i| = |T^\lambda_i \cap A_{=r_j}| = \lambda_j$) if $\eta \notin [h_j],$

for $i = \sum_{j=1}^{t-1} |A_{\in C_j}|/\lambda_j + j$ with $j \in [t]$ and $\eta \in \left[|A_{\in C_j}|/\lambda_j\right]$. We note that $T^\lambda$ is unique up to isomorphism. More precisely, if there are two allocations $T$ and $T'$ that induces $\lambda$, then there exists a permutation $\sigma: [k] \to [k]$ such that $T^\lambda_i \simeq T^\lambda_{\sigma(i)}$ for all $i \in [k].$

Now let $\Lambda^*$ be the set of vectors $\lambda \in [n]^t$ that are induced by envy-free feasible allocations $T$ consistent with $G$. Note that any $\lambda \in \Lambda^*$ is a valid size vector by Lemma 4. The next lemma states that we can restore an envy-free feasible allocation from any size vector in $\Lambda^*$.

Lemma 5. For any $\lambda \in \Lambda^*$, the allocation $T^\lambda$ is an envy-free feasible allocation.

Proof. Let $T$ be an envy-free feasible allocation that induces $\lambda$. Then, by the definition of $T^\lambda$, there exists a one-to-one correspondence $\sigma: [k] \to [k]$ such that $T^\lambda_i \simeq T_{\sigma(i)}$ for all $i \in [k]$. Recall that the taxis are sorted in non-increasing order of their capacities. As $|T^\lambda_i| \geq |T^\lambda_k| \geq \cdots \geq |T^\lambda_1|$ by the construction of $T^\lambda$ and (7), the allocation $T^\lambda$ is feasible. Hence, $T^\lambda$ is an envy-free feasible allocation. \hfill \Box

Our problem reduces to deciding whether $\Lambda^*$ is empty or not. Let $A_j$ be a set of candidate sizes $\lambda_j$ for each $j \in [t]$ and suppose that $\Lambda^* \subseteq \prod_{j \in [t]} A_j$. Here, by (3), $A_j = \left[|A_{\in C_j}|\right]$ can be used as an initial set of candidate sizes.

Lemma 6. If $A_j = \left[|A_{\in C_j}|\right]$ for all $j \in [t]$, then $\Lambda^* \subseteq \prod_{j \in [t]} A_j$.

Then, we show that (max $A_j$)$_{j \in [t]} \in \Lambda^*$ or efficiently remove a candidate vector. The following lemma is a slightly stronger version of this statement.

Lemma 7. Suppose that $A_j \neq \emptyset \quad (\forall j \in [t])$ and let $\lambda = (\max A_j)_{j \in [t]}$. If $\lambda \notin \Lambda^*$, then there exists an index $j^* \in [t]$ such that

\[
\begin{align*}
\Lambda^* \cap \prod_{j \in [t]} A_j' = \Lambda^* \cap \prod_{j \in [t]} A_j \\
\text{where } A_j' &= \begin{cases} 
A_j - \lambda_j & \text{if } j = j^*, \\
A_j & \text{if } j \neq j^*.
\end{cases}
\end{align*}
\]

In addition, such a $j^*$ can be computed in polynomial time.
Proof. First, suppose that λ is invalid. If it violates (3) or (4) for an index j’ ∈ [t], then λ′_j ≠ λ_j for any valid size vector λ∗ ∈ \prod_{j∈[t]} A_j. If it violates (5) or (6), then A∗ = ∅ because any size vector in \prod_{j∈[t]} A_j is invalid. If it violates (7), let j’ be an index such that λ_j’-1 < λ_j. Then, λ′_j ≠ λ_j for any valid size vector λ∗ ∈ \prod_{j∈[t]} A_j since λ_j’ ≤ λ∗_j’-1 ≤ λ_j-1. Hence, in either case, we can efficiently find j* that satisfies (8) if λ is invalid. From now on, suppose that λ is valid. Note that, as λ ∉ A∗, the allocation T^λ is not an envy-free feasible allocation.

Suppose that Λ^λ is infeasible. Let i’ be the minimum index that satisfies |T^λ_i| > q∗ and let j’ ∈ [t] be the index such that T^λ_i ⊆ A_in C_j. Since λ is the size vector that selects the possible largest size for each j ∈ [t] and the number of taxis that are used by the agents of a type C_j is inversely proportional to λ_j, some agents of a type in C_j must be allocated to a taxi with index at least i’. Hence, λ^λ is also infeasible for any T ∈ \prod_{j∈[t]} A_j with λ′_j = λ_j. Thus, j* = j’ satisfies (8).

Now, we assume that Λ^λ is feasible but not envy-free. Suppose that agent a ∈ T_i^λ envies another agent a’ ∈ T_i^λ, and let j and j’ denote, respectively, the indices such that T^λ_i ⊆ A_in C_j. If j = j’, the allocation T^λ is not envy-free for any T’j ∈ \prod_{j∈[t]} A_j with λ_j = λ_j because the agents of a type in C_j are partitioned into coalitions in the same way in Λ^λ and T^λ (i.e., there exists η such that T^λ_i ∼ T^λ_i+η for any i with T^λ_i ⊆ A_in C_j). Thus, suppose j ≠ j’. Consider the allocation T^λ’ for a size vector T’ ∈ \prod_{j∈[t]} A_j, with λ_j = λ_j. We will show that j* = j’ satisfies (8). As λ_j ≤ λ_j, the cost of a for T^λ’ is at least that for T^λ (i.e., Φ_a(T^λ’a(a)) ≥ Φ_a(T^λ’a(a))). Moreover, as λ_j = λ_j, there exist an agent a’ and a coalition T^∗_i such that a’ ∈ T^∗_i, x_a = x_a’, and T^∗_i ∼ T^∗_i. Then the cost of a after swapping a with a’ in T^λ is equal to the cost of a after swapping a with a’ in T^λ’ (i.e., Φ_a(T^λ’a) - a + a, x_a) = Φ_a(T^∗_i’a) + a, x_a). Hence, we have

Φ_a(T^λ’a) ≥ Φ_a(T^λ’a) = Φ_a(T^∗_i’a, x_a) + Φ_a(T^∗_i’a) + a, x_a) = Φ_a(T^∗_i’a) - a + a, x_a),

which means that a envies a’ in T^λ’. Thus, j* = j’ satisfies (8).

Remark 1. Lemma 7 implies that any non-empty A∗ ⊆ A* has a least upper bound. Hence, A* is an upper semilattice with respect to the componentwise max operation.

By Lemma 7, we can obtain an envy-free feasible allocation by iteratively eliminating a candidate vector. Formally, we can check the existence of an envy-free feasible allocation by Algorithm 2. The running time of the algorithm is O(p^p · n^4), which is FPT. Hence, we obtain the following theorem.

Theorem 2. We can check the existence of an envy-free feasible allocation, and find one if it exists in FPT with respect to the number p of types of agents.

Proof. We will prove that Algorithm 2 correctly decides the existence of an envy-free feasible allocation. If Algorithm 2 returns an allocation T^λ for some star-forest G, then it is not difficult to verify that the allocation is indeed an envy-free feasible allocation. Conversely, suppose that there exists an envy-free feasible allocation Λ. Obviously, Λ is an allocation consistent with G^τ, which is a star-forest by Lemma 3. Consider the size vector λ := λ^τ. The vector λ is valid by Lemma 4. Further, T^λ is an envy-free feasible allocation by Lemma 5. Finally, Lemma 6 and Lemma 7 ensure that λ remains as a candidate vector during the execution of the algorithm, i.e., λ ∈ \prod_{j∈[t]} A_j, which proves the claim.

Constant number of taxis We now move on to the next question of computing an envy-free allocation when the number of taxis is a constant. The next lemma states that, in an envy-free allocation, every agent must be allocated to a taxi that charges the minimum cost.

Lemma 8. For any envy-free feasible allocation Λ, taxi i ∈ [k], and agent a ∈ T_i, we have

ϕ(T_i, x_a) ≤ ϕ(T_j, x_a)

for all T_j ∈ Λ with i ≠ j. Further, the strict inequality holds if a is not the first passenger to drop off in T_j, i.e., x_a > min_{a’ ∈ T_j} x_a’.

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Algorithm 2: FPT w.r.t. the number of types

1. foreach star-forest graph $G$ do
2. Let $A_j = \{A_{C_j} \mid j \in [t]\}$;
3. while $A_j \neq \emptyset$ (forall $j \in [t]$) do
4. Let $\lambda = (\max A_j)_{j \in [t]}$;
5. if $\lambda$ is not valid then
6. if $\lambda$ violates (8) or (11) then
7. Let $j' \in [t]$ be an index for which $\lambda$ violates (8) or (11);
8. Set $A_{j'} \leftarrow A_{j'} - \lambda_{j'}$;
9. continue;
10. else if $\lambda$ violates (5) or (8) then break;
11. else $\lambda$ violates (7)
12. Let $j' \in [t]$ be an index for which $\lambda_{j'-1} < \lambda_{j'}$;
13. Set $A_{j'} \leftarrow A_{j'} - \lambda_{j'}$;
14. continue;
15. if $T^\lambda$ is infeasible then
16. Let $i' \in [k]$ be the minimum index such that $|T^\lambda_{i'}| > q_{i'}$;
17. Let $j' \in [t]$ be the index such that $T^\lambda_{i'} \subseteq A_{C_{j'}}$;
18. Set $A_{j'} \leftarrow A_{j'} - \lambda_{j'}$;
19. continue;
20. if $T^\lambda$ is not envy-free then
21. Let $i', i'' \in [k]$ be indices such that an agent in $T_{i'}$ envies an agent in $T_{i''}$;
22. Let $j'' \in [t]$ be the index such that $T_{i''} \subseteq A_{C_{j''}}$;
23. Set $A_{j''} \leftarrow A_{j''} - \lambda_{j''}$;
24. continue;
25. return $T^\lambda$;
26. return “No envy-free feasible allocation”;
Proof. To show the first statement, suppose towards a contradiction that there exists \( T_j \in \mathcal{T} \) such that \( \varphi(T_i, x_a) > \varphi(T_j, x_a) \). Let \( a' \in T_j \) where \( x_{a'} \geq x_a \) (such agent \( a' \) exists since otherwise \( \varphi(T_j, x_a) = \int_0^{x_a} \frac{I}{n_{T_j}(r)} = \infty \)). Then we have
\[
\varphi(T_i, x_a) > \varphi(T_j, x_a) = \varphi(T_j - a' + a, x_a),
\]
which contradicts the fact that \( \mathcal{T} \) is envy-free.

To show the second statement, assume towards a contradiction that there exists \( T_j \in \mathcal{T} \) such that \( x_a > \min_{a' \in T_j} x_{a'} \) and \( \varphi_i(T_i, x_a) = \varphi_j(T_j, x_a) \). By our supposition, there is an agent \( a' \in T_j \) such that \( x_{a'} < x_a \); choose such \( a' \in T_j \) with maximum \( x_{a'} \). Then,
\[
\varphi(T_j, x_a) = \int_0^{x_a} \frac{I}{n_{T_j}(r)} > \int_0^{x_{a'}} \frac{I}{n_{T_j}(r)} + \frac{(x_a - x_{a'})}{n_{T_j}(x_a)} + 1 = \varphi(T_j - a' + a, x_a),
\]
which implies that \( \varphi(T_i, x_a) > \varphi(T_j - a' + a, x_a) \), a contradiction. \( \square \)

Recall that the cost of an agent \( a \in A \) in an allocation is determined by the agents of type smaller than \( x_a \) and the number of agents allocated to the same taxi as agent \( a \). Formally, for coalition \( T \subseteq A \), positive real \( x \in \mathbb{R}_{>0} \), and positive integer \( \mu \in [n] \), we define
\[
\psi(T, x, \mu) := \int_0^x \frac{I}{n_T(r) + \mu - |T|}.
\]
For a feasible allocation \( T \), it is not difficult to verify \( \varphi(T, x) = \psi(T, x, |T|) \) for each \( T \in \mathcal{T} \) and \( x \in \mathbb{R}_{>0} \). Lemma \([5]\) thus implies that we can uniquely determine the coalition of each agent in a greedy manner from an agent with nearest destination, when fixing the size of each taxi and the agents who are the first passengers in their coalitions. Building up on this observation, we can design the following greedy algorithm to compute an envy-free allocation when the number of taxis is a constant.

**Theorem 3.** We can decide the existence of an envy-free allocation and find one if it exists in polynomial time when the number \( k \) of taxis is a constant.

**Proof.** We say that allocation \( \mathcal{T} \) is consistent with \( (\mu_i, s_i, r_i)_{i \in [k]} \) if each taxi \( i \in [k] \) satisfies the following:

- the number of allocated agents equals to \( \mu_i \), i.e., \( |T_i| = \mu_i \);
- the first drop-off point equals to \( s_i \), i.e., \( \min_{a \in T_i} x_a = s_i \); and
- the number of agents who get off at the first drop-off point equals to \( r_i \), i.e., \( |T_i \cap A_{s_i}| = r_i \).

We call \( (\mu_i, s_i, r_i)_{i \in [k]} \) a configuration.

Fix a configuration \( (\mu_i, s_i, r_i)_{i \in [k]} \). Our aim is to decide the existence of an envy-free allocation that is consistent with the configuration. We assume that \( \mu_i, r_i \in \{0, 1, \ldots, n\} \), \( s_i \in \{x_a \mid a \in A\} \), \( r_i \leq \mu_i \leq q_i \) for each \( i \in [k] \), and \( \sum_{i \in [k]} \mu_i = n \), since otherwise no allocation is consistent with the configuration. In an envy-free feasible allocation, if agent \( a \) is the first passenger to drop off in his coalition then all the agents of type \( x_a \) are the first passengers in their coalitions by Lemma \([2]\). Hence, we also assume \( |A_{s_i}| = \sum_{j \in [k]\setminus \{s_i\}} r_j \) for each \( i \in [k] \). Note that, the number of such possible configurations is at most \( O(n^{3k}) \), which is polynomial when \( k \) is a constant.

By Lemma \([8]\) we can efficiently compute whether there is an envy-free feasible allocation that is consistent with \( (\mu_i, s_i, r_i)_{i \in [k]} \), by allocating each agent in ascending order of destination to a taxi with the cheapest cost and checking whether the obtained allocation is both envy-free and feasible. The formal description is given in Algorithm \([3]\).

In what follows, we will formally argue that Algorithm \([3]\) correctly decides whether there is an envy-free feasible allocation \( \mathcal{T} \) that is consistent with \( (\mu_i, s_i, r_i)_{i \in [k]} \).

Suppose that Algorithm \([3]\) returns an allocation \( \mathcal{T} \) for \( (\mu_i, s_i, r_i)_{i \in [k]} \). It is not difficult to verify that \( \mathcal{T} \) is an envy-free feasible allocation, which is consistent with \( (\mu_i, s_i, r_i) \). Conversely, suppose that there is an
enjoy-free feasible allocation $\mathcal{T}$ that is consistent with $(\mu_i, s_i, r_i)_{i \in [k]}$. For each agent $a \in A$ with $x_a = s_i$ for some $i \in [k]$, Algorithm 3 allocates $a$ to such $i$ because $|A_{=s_i}| = \sum_{j \in [k]} s_j = s_i, r_j$. For each agent $a \in A$ with $x_a \neq s_i$ for any $i \in [k]$, Lemma 8 implies that if $a \in T_i$, 

$$\varphi(T_i, x_a) = \psi(T_i \cap A_{<x_a}, x_a, \mu_i) < \psi(T_j \cap A_{<x_a}, x_a, \mu_j) = \varphi(T_j, x_a)$$

for all $j \neq i$ with $s_j < x_a$ and $|T_j| < \mu_j$; thus, $i$ must be uniquely chosen as $i^*$ in Line 6. Hence, Algorithm 3 can return an allocation for $(\mu, s_i, r_i)$.  

In general, we have two difficulties to extend the greedy strategy. The first difficulty is that the number of possible combinations of taxis’ size may be exponential. The second difficulty is that we may have exponentially many possibilities to split each destination type.

**Constant capacity** Finally, we consider the case when the capacity of each taxi is at most four. We will design a greedy algorithm in a similar spirit to the one for a constant number of taxis. In outline, the algorithm guesses the number of passengers in each taxi; the number of such candidate vectors is bounded by polynomial since the maximum capacity of a taxi is bounded by a constant. The algorithm then decides how to split the agents of the first drop-off points. The key property is that when the capacity of each taxi is at most four, such splitting can be uniquely determined.

The next lemma further ensures that, if a coalition $T$ consists of $|T| - 1$ agents of type $x$, then all the other agents of type $x$ form coalitions without any agent of the other type, and the first passengers of the other coalitions of size $|T|$ drop off at destinations smaller than $x$.

**Lemma 9.** For an envy-free feasible allocation $\mathcal{T}$ and $T \in \mathcal{T}$, suppose that all agents in $T$ except for one drop off at the first destination, i.e., $|T_{=x}| = |T| - 1$ for $x = \min_{a \in T} x_a$. Then, for any $T' \in \mathcal{T}$ with $T' \neq T$ and $|T'| = |T|$, the first drop-off point $x'$ of $T'$ ($x' = \min_{a \in T'} x_a$) appears strictly before $x$ (i.e., $x' < x$), or all agents in $T'$ drop off at $x$ (i.e., $T' \subseteq A_{=x}$).

**Proof.** To prove the claim, suppose that $|T_{=x}| = |T| - 1$ for $T \in \mathcal{T}$ with $x = \min_{a \in T} x_a$. Let $a^*$ be the unique agent in $T_{>x}$. Take any $T' \in \mathcal{T} - T$ with $|T'| = |T|$ and let $x' = \min_{a \in T'} x_a$. Assume towards a contradiction that $x' \geq x$; and some agent in $T'$ does not drop off at $x$, i.e., $\max_{a \in T'} x_a > x$. Let $x'' = \min\{x_a, \max_{a \in T'} x_a\}$. Note that $x < x' \leq x_a$. Hence, $a^*$ envies every agent $a$ in $T'_{=x'}$, because 

$$\varphi(T, x_a) = \frac{x}{|T|} + (x_a - x) > \frac{x}{|T|} + \frac{x'' - x}{2} + (x_a - x'') \geq \varphi(T' - a + a^*, x_a),$$

a contradiction.  

Equipped with this, we can show that an envy-free feasible allocation can be computed in polynomial time, when the maximum capacity is at most four.

\[\text{Algorithm 3: XP algorithm w.r.t. the number of taxis}\]

1. **foreach** configuration $(\mu_i, s_i, r_i)_{i \in [k]}$ **do** 
2. Initialize $T_i \leftarrow \emptyset$ for each $i \in [k]$; 
3. for $a \leftarrow 1, 2, \ldots, n$ **do** 
   4. if $x_a = s_i$ for some $i \in [k]$ **then** 
      5. Let $i^* \in [k]$ be an index such that $s_{i^*} = x_a$ and $|T_{i^*}| < r_{i^*}$; 
   6. **else** Let $i^* \leftarrow \arg\min_{i \in [k]} x_a < x_a \land |T_i| < \mu_i$; 
   7. Set $T_{i^*} \leftarrow T_{i^*} + a$; 
8. **if** $(T_1, \ldots, T_k)$ is envy-free **then** return $(T_1, \ldots, T_k)$; 
9. return “No envy-free feasible allocation”;
**Theorem 4.** If \( q_i \leq 4 \) for all \( i \in [k] \), then we can check the existence of an envy-free feasible allocation and find one if it exists in polynomial time.

**Proof.** Suppose that there exists an envy-free feasible allocation and let \( \mathcal{T}^* = (T_1^*, \ldots, T_k^*) \) be such an allocation. Without loss of generality, we may assume that \( \min_{a \in T_i^*} x_a \leq \cdots \leq \min_{a \in T_i^*} x_a \) and \( |T_i^*| \geq \cdots \geq |T_k^*| \) by the monotonicity lemma (Lemma 1). In addition, for every split type \( x \), we may assume that the taxis are arranged in monotone non-increasing order in terms of the number of agents of type \( x \), i.e.,

\[
|T_i^* \cap A_{=x}| \geq |T_{i+1}^* \cap A_{=x}| \geq \cdots \geq |T_k^* \cap A_{=x}|
\]

where \( \{u, u+1, \ldots, v\} = \{i \in [k] \mid T_i^* \cap A_{=x} \neq \emptyset\} \).

We show that an allocation \( \mathcal{T} = (T_1, \ldots, T_k) \) with \( |T_i \cap A_{=x}| = |T_i^\ast \cap A_{=x}| \) (\( \forall i \in [k], x \in \mathbb{R}_{>0} \)) can be found by the following procedure:

(i) guess \( |T_i^*| (i \in [k]) \);
(ii) greedily allocate the agent with nearest destination;
(iii) check whether the resulting allocation is envy-free.

Let \( \mathcal{M} \) be the set of possible \( k \)-tuples \((|T_1^*|, \ldots, |T_k^*|)\), i.e., the set of \( k \)-tuples of integers \((\mu_1, \ldots, \mu_k)\) such that \( \mu_1 \geq \cdots \geq \mu_k \geq 0, \sum_{i \in [k]} \mu_i = n \), and \( \mu_i \leq q_i (i \in [k]) \). As \( q_i \leq 4 \) for all \( i \in [k] \), and \( \mu_1 \geq \cdots \geq \mu_k \), we have \( |\mathcal{M}| = O(n^4) \). Hence, we can enumerate all the possibilities in a polynomial time.

Algorithm description: Fix \((\mu_1, \ldots, \mu_k) \in \mathcal{M}\), and we greedily add agents \( A_{=x} \) with the smallest type \( x \) to a taxi which charges the minimum cost \( \min_i \phi(T_i, x) \). If there are multiple agents of type \( x \) and they are about to be added to taxis which are allocated no agents of type \( x \) smaller than \( x \), we need to consider how to split the agents of type \( x \). For an allocation \( \mathcal{T} \), the multiset of the sizes \( |T_i \cap A_{=x}| \) is referred to as split outcome in \( \mathcal{T} \). We will only denote nonzero sizes to represent a split outcome.

Suppose that the agents of type \( x \) are allocated to taxis which will carry \( \mu \equiv (\in \{1, 2, 3, 4\}) \) agents (such \( \mu \) exists by Lemma 2). Then, by Lemmas 2 and 3 we can uniquely determine the split outcomes as shown in Table 1. For example, when \( \mu = 4 \) and \( |A_{=x}| = 4 \), there are five possibilities of split outcomes:

\[
\{1, 1, 1, 1\}, \{1, 1, 2\}, \{2, 2\}, \{1, 3\}, \{4\}.
\]

However, the first three possibilities (i.e., \( \{1, 1, 1, 1\}, \{1, 1, 2\}, \{2, 2\} \)) violates Lemma 2(iii) (i.e., if two taxis have an equal number of agents of split type, then no agent of the other type ride these taxis). The fourth possibility violates Lemma 3 (i.e., if a coalition \( T \) consists of \(|T| - 1 \) agents of type \( x \), then all the other agents of type \( x \) form coalitions without any agent of the other type).

When \( \mu = 4 \) and \( |A_{=x}| = 4d + 3 \) for an integer \( d \), we have two possible split outcomes that satisfies necessary conditions of an envy-free allocation in Lemmas 2 and 3 \( \{4, \ldots, 4, 2, 1\} \) and \( \{4, \ldots, 4, 3\} \). Note that if \( |T_i^*| = 4 \) and \( |T_i^* \cap A_{=x}| = 3 \), then \( |T_i^* \cap A_{=x}| \leq 3 \) by Lemma 3. Hence, the latter split outcome is possible only when there are exactly \( d + 1 \) empty taxis carrying 4 agents. In contrast, the former split outcome is possible only when there are at least \( d + 2 \) empty taxis which will carry 4 agents. Thus, we can uniquely determine which split outcome can be used. The formal description of the algorithm is given by Algorithm 3.

Correctness: Now, we formally show that Algorithm 3 returns an envy-free feasible allocation in polynomial time if there exists such an allocation. Since \( |\mathcal{M}| = O(n^4) \), the algorithm runs in \( O(n^4 \cdot n^2) = O(n^6) \) time. It is not difficult see that Algorithm 4 returns an envy-free feasible allocation if it returns an allocation. Suppose that there exists an envy-free feasible allocation \( \mathcal{T}^\ast \) and let \( \mu_i^* = |T_i^*| \) for all \( i \in [k] \). We prove the claim by showing that the algorithm computes an allocation \( \mathcal{T} \) such that \( \mathcal{T} = (T_1, \ldots, T_k) \) with \( T_i \approx T_i^\ast \) (\( \forall i \in [k] \)) in the for-loop when \( (\mu_1, \ldots, \mu_k) = (\mu_1^*, \ldots, \mu_k^*) \). The proof is by an induction on type \( x \). For each type \( x \) whose agents are not the first passengers to drop off in their coalition of \( \mathcal{T}^\ast \), Lemma 8 implies that if \( a \in T_i \cap A_{=x} \),

\[
\varphi(T_i, x) = \psi(T_i \cap A_{<x}, x, \mu_i^*) = \psi(T_i \cap A_{<x}, x, \mu_i^*) < \psi(T_i \cap A_{<x}, x, \mu_i^*) = \psi(T_j \cap A_{<x}, x, \mu_j^*) = \varphi(T_j, x)
\]

for all \( j \neq i \) with \( |T_j| < \mu_j^* \); thus, \( i \) must be uniquely chosen as \( i^* \) in Line 4. For each type \( x \) of which agents are the first passengers to drop off in their coalition of \( \mathcal{T}^\ast \), the algorithm correctly allocates the agents of type \( x \) because the split outcome of \( \mathcal{T}^\ast \) is determined as shown in Table 1 and the algorithm allocates the agents in the same manner. Hence, we have \( |T_i \cap A_{=x}| = |T_i^* \cap A_{=x}| \) (\( \forall i \in [k], x \in \mathbb{R}_{>0} \)). As \( \mathcal{T} \) must be an envy-free feasible allocation, Algorithm 3 returns an envy-free feasible allocation.
Table 1. Split of type \( x \)

| \( \mu \) | \( A_{=x} \) | split outcome |
|---|---|---|
| 4 | 4d | \{4, 4, \ldots, 4\} |
| 4 | 4d + 1 | \{4, 4, \ldots, 4, 1\} |
| 4 | 4d + 2 | \{4, 4, \ldots, 4, 2\} |
| 4 | 4d + 3 | \{4, 4, \ldots, 4, 2, 1\} (if \( d + 2 \) taxis are available) |
| 3 | 3d | \{3, 3, \ldots, 3\} |
| 3 | 3d + 1 | \{3, 3, \ldots, 3, 1\} |
| 3 | 3d + 2 | \{3, 3, \ldots, 3, 2\} |
| 2 | 2d | \{2, 2, \ldots, 2\} |
| 2 | 2d + 1 | \{2, 2, \ldots, 2, 1\} |
| 1 | d | \{1, 1, \ldots, 1\} |

Algorithm 4: Polynomial-time algorithm for \( q_i \leq 4 \) (\( \forall i \in [k] \))

1. foreach \((\mu_1, \ldots, \mu_k) \in M\) do
2. Let \( T_i \leftarrow \emptyset \) for each \( i \in [k] \);
3. for \( a \leftarrow 1, 2, \ldots, n \) do // from nearest to farthest
4. Let \( i^* \) be the minimum index that minimizes \( \psi(T_i, x_a, \mu) \) while satisfying \( |T_i| < q_i \);
5. if \( |T_{i^*}| = 2 \), \( x_{a-2} = x_{a-1} = x_a < x_{a+1} \), and \( \mu_{i^*} = \mu_{i^*+1} = 4 \) then
6. Set \( T_{i^*+1} \leftarrow T_{i^*+1} + a \);
7. else Set \( T_{i^*} \leftarrow T_{i^*} + a \);
8. if \((T_1, \ldots, T_k)\) is feasible and envy-free then return \((T_1, \ldots, T_k)\);
9. return "No envy-free feasible allocation";

The above result may not be directly extended to the case of \( q_i = 5 \) for some \( i \). This is essentially because the splitting of agents of the same type cannot be uniquely determined: we have two possibilities as to whether a type with 5 agents is allocated into one taxi or two taxis (one for 2 and one for 3).

5.2 Consecutive envy-free allocation

One desirable property that the allocation returned by the backward greedy satisfies is consecutiveness, i.e., agents form consecutive groups according to their destinations. The property is intuitive to the users and hence important in practical implementation. Formally, an allocation \( \mathcal{T} \) is consecutive if \( \max_{a \in T} x_a \leq \min_{a \in T'} x_a \) or \( \min_{a \in T} x_a \geq \max_{a \in T'} x_a \) hold for all distinct \( T, T' \in \mathcal{T} \). However, there is an instance such that any envy-free feasible allocation is not consecutive; see Example 10.

Example 10. Consider an instance where \( n = 10, k = 2, q_1 = 6, q_2 = 4, x_1 = \cdots = x_4 = 1, x_5 = \cdots = x_8 = 10, \) and \( x_9 = x_{10} = 20 \). Then, it can be easily checked that allocation \( \mathcal{T}^* = \{\{1, 2, 3, 4, 9, 10\}, \{5, 6, 7, 8\}\} \) is envy-free but not consecutive (see Fig. 10). Moreover, this allocation is the unique envy-free allocation for the instance. To see this, consider any envy-free feasible allocation \( \mathcal{T} = (T_1, T_2) \). By feasibility, \( |T_1| = 6 \) and \( |T_2| = 4 \). Agents of type \( x = 1 \) must be allocated to \( T_1 \) since the cost they have to pay in \( T_1 \) is \( \frac{1}{6} \) and that for \( T_2 \) is \( \frac{1}{4} \). Hence, \( \{1, 2, 3, 4\} \subseteq T_1 \). Further, all agents of type \( x = 10 \) must be allocated to \( T_2 \); if some agent of type \( x = 10 \) is allocated to \( T_1 \), then he would envy an agent of the same type allocated to \( T_2 \) since the cost he has to pay at \( T_1 \) is \( \frac{1}{6} + \frac{9}{2} \), which is strictly greater than the cost of \( \frac{10}{4} \) at \( T_2 \). Thus, all agents of type \( x = 20 \) must be allocated to \( T_1 \) and hence \( \mathcal{T} = \mathcal{T}^* \). This, in turn, implies that no envy-free feasible allocation is consecutive.

We will show that the existence of a consecutive envy-free allocation can be efficiently checked in polynomial time. By the monotonicity lemma and the assumption that \( q_1 \geq \cdots \geq q_k \), it is sufficient to consider
Case (i): $i < j$

Consider two cases: (i) $i < j$ and $a_i < s_j$ and envies between adjacent boundary agents. Since the “only if” part is clear, we prove the “if” part. Suppose that there is no pair of agents $(a_i, s_j)$ that satisfy

$$\int_0^{x_{a_i}} \frac{r}{n_{T_i}(r)} = \varphi(T_i, x_{a_i}) > \varphi(T_j - a_j + a_i, x_{a_i}) = \frac{x_{a_i}}{|T_j|} \geq \frac{x_{a_i}}{|T_{i+1}|}. \quad (9)$$

Note that $\frac{1}{x_{a_i}} \int_0^{x_{a_i}} \frac{r}{n_{T_i}(r)}$ is monotone non-decreasing in $x$, since $n_{T_i}(r)$ is monotone non-increasing in $r$. Hence, we have

$$\frac{1}{x_{a_i}} \int_0^{x_{a_i}} \frac{r}{n_{T_i}(r)} \geq \frac{1}{x_{a_i}} \int_0^{x_{a_i}} \frac{r}{n_{T_i}(r)} > \frac{1}{|T_{i+1}|}.$$

Thus, we obtain

$$\varphi(T_i, x_{t_i}) = \int_0^{x_{t_i}} \frac{r}{n_{T_i}(r)} > \frac{x_{t_i}}{|T_{i+1}|} = \varphi(T_{i+1} - s_{i+1} + t_i, x_{t_i}),$$

meaning that $t_i$ envies $s_{i+1}$, a contradiction.

Case (ii): $i > j$. As $a_i$ envies $a_j$, we have

$$\varphi(T_i, x_{a_i}) > \varphi(T_j - a_j + a_i, x_{a_i}) = \varphi(T_j - a_j + t_{i-1}, x_{t_{i-1}}) + (x_{a_i} - x_{t_{i-1}})$$

$$\geq \varphi(T_{i-1}, x_{t_{i-1}}) + (x_{a_i} - x_{t_{i-1}})$$

$$= \varphi(T_{i-1} - t_{i-1} + a_i, x_{t_{i-1}}) + (x_{a_i} - x_{t_{i-1}})$$

$$= \varphi(T_{i-1} - t_{i-1} + a_i, x_{t_i}).$$

Fig. 10. An allocation that is envy-free but not consecutive
Moreover, for $k$ the bound on the running time is $O(n^3)$. If this is the case, such a feasible partition can be found using standard dynamic programming techniques.

Proof. Let us consider the subproblem in which $[n']$ is the set of agents and $[k']$ is the set of taxis. Let $z(n', k', \ell) \in \{0, 1\}$ be 1 if and only if there exists a consecutive envy-free allocation $T$ with $|T_{k'}| = \ell$ in the subproblem. When there is only one taxi (i.e., $k' = 1$), it is not difficult to see that

$$z(n', 1, \ell) = \begin{cases} 1 & \text{if } n' = \ell \text{ and } n' \leq q_1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $k' > 1$, we have $z(n', k', \ell) = 1$ if and only if there exists $\ell' \in \min\{\ell, q_{k'-1}\}$ such that

- $z(n' - \ell, k' - 1, \ell') = 1$,
- $\varphi(T, x_{n' - \ell}) \leq \varphi(T' \setminus \{n' - \ell + 1\} \cup \{n' - \ell\}, x_{n' - \ell})$, and
- $\varphi(T', x_{n' - \ell + 1}) \leq \varphi(T' \setminus \{n' - \ell\} \cup \{n' - \ell + 1\}, x_{n' - \ell + 1})$.

where $T = \{n' - \ell, \ldots, n' - \ell\}$ and $T' = \{n' - \ell + 1, \ldots, n'\}$. Lemma[1] and Lemma[10] immediately imply that a consecutive envy-free allocation in the original instance exists if and only if $max_{\ell \in [n]} z(n, k, \ell) = 1$. If this is the case, such a feasible partition can be found using standard dynamic programming techniques. The bound on the running time is $O(n^3k)$, which is polynomial in the input size. \qed

6 Conclusion

In this paper, we introduced a new model of the fair ride allocation problem on a line with an initial point. We proved that the backward greedy allocation satisfies Nash stability, strongly swap-stability, and social optimality. In addition, we showed that envy-free allocations may not exist and designed several algorithms to decide the existence of an envy-free feasible allocation and find one if it exists. The obvious open problem is the complexity of finding an envy-free allocation for the general case. There are several possible extensions of our model. First, while we have assumed that agents ride at the same starting point, it would be very natural to consider a setting where the riding locations may be different. Extending our results to this setting would be a promising research direction. Besides the Shapley value, there are other division rules of cooperative games that can be applied to our model. Examples include the Owen value and the Banzhaf value. It would be interesting to see whether using a different rule of cooperative games results in a different property of an allocation.

Acknowledgement

This work was partially supported by the joint project of Kyoto University and Toyota Motor Corporation, titled “Advanced Mathematical Science for Mobility Society” and by JSPS KAKENHI Grant Numbers 20K19739...
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