Critical Graphs for Minimum Vertex Cover

Andreas Jakoby, Naveen Kumar Goswami, Eik List, Stefan Lucks
Faculty of Media, Bauhaus-Universität Weimar, Bauhauserstr. 11, D-99423, Weimar, Germany
<firstname.lastname@uni-weimar.de
July 13, 2017

Abstract

An $\alpha$-critical graph is an instance where the deletion of any element would decrease some graph’s measure $\alpha$. Such instances have shown to be interesting objects of study for deepen the understanding of optimization problems.

This work explores critical graphs in the context of the Minimum-Vertex-Cover problem. We demonstrate their potential for the generation of larger graphs with hidden a priori known solutions. Firstly, we propose a parametrized graph-generation process which preserves the knowledge of the minimum cover. Secondly, we conduct a systematic search for small critical graphs. Thirdly, we illustrate the applicability for benchmarking purposes by reporting on a series of experiments using the state-of-the-art heuristic solver NuMVC.

Keywords: critical graphs, minimum vertex cover, graph generation, benchmark generator

1 Introduction

The Minimum Vertex Cover Problem. A vertex cover $C$ for a given graph $G = (V, E)$ defines a subset of vertices $C \subseteq V$ such that every edge in $E$ is incident to at least one vertex in $C$. A minimum vertex cover (MVC) is a vertex cover with the smallest possible size. The task of finding a minimum vertex cover in a given graph is a classical $NP$-hard optimization problem [5], and its decision version one the 21 original $NP$-complete problems listed by Karp [8]. While the construction of a maximal matching yields a trivial approximation-2 algorithm, it is $NP$-hard to approximate MVC within any factor smaller than 1.3606 [3] unless $P = NP$, according to the Unique Game Conjecture; though, one can achieve an approximation ratio of $2 - o(1)$ [7].

The MVC problem is strongly related to at least three further $NP$-hard problems. Finding a minimum vertex cover is equivalent to finding a Maximum Independent Set (MIS), i.e. a subset of vertices wherein no pair of vertices shares an edge. An MIS problem instance can again be transformed into an instance of the Maximum-size Clique (MC) problem; moreover, there is a straight-forward reduction of a (binary) Constraint Satisfaction Problem (CSP) to an MIS problem. In practice, the MVC problem plays an important role in network security, industrial machine assignment, or facility location [9, 13, 23]. Furthermore, MVC algorithms can be used for solving MIS problems e.g., in the analysis of social networks, pattern recognition, and alignment of protein sequences in bioinformatics [13, 15].

$\alpha$-Critical Graphs. Graphs are called critical if they are minimal with regards to a certain measure. More precisely, an edge of a given graph $G$ is called a critical element iff its deletion would decrease the measure. $G$ is then called edge-critical (or simply critical) iff every edge is a critical element. The concept of critical graphs is mostly used in works on the chromatic number. Though, it can also be of significant interest for the MVC problem, where we call a graph critical iff the deletion of any edge would decrease the size of the minimal cover. This concept has been introduced by Erdös and Gallai [4] in 1961 using the term $\alpha$-critical where $\alpha$ denotes the certain measure on a graph. E.g., let $\alpha$ denote a function determining the size of a maximum independent set, then we call a graph $G = (V, E)$ $\alpha$-critical if for any edge $e \in E$ it holds that $\alpha(G) < \alpha(G')$ where $G' = (V, E \setminus \{e\})$. Since the size of a minimum vertex cover of a graph $G = (V, E)$ is given by $|V| - \alpha(G)$, this definition and the correlated results apply directly to the MVC problem. The insights of studying such $\alpha$-critical graphs could help to deepen our understanding on the
complexity of the Vertex Cover problem or to find more efficient solvers. Useful summaries on α-critical graphs can be found in [10] [6].

Randomized Graph Generation. Critical graphs can further serve as the base for constructing larger graphs. Since small critical instances possess an easily determined cover size, a parametrized graph-generation process that preserves the criticality could create large instances while maintaining the knowledge about the solution. A potential application for such graphs could be, e.g., the dedicated generation of particularly hard instances for benchmarking purposes. Following a series of previous graph-generation models [14] [20], the idea for generating such graphs for the minimum-vertex cover problem had been introduced by Xu and Li [21] [22] and revisited in [19]. During the past decade, Xu’s BHOSLIB suite [17] has established as a valuable benchmark suite for the evaluation of MVC, MIS, MC, and CSP solvers.

Contribution. This work studies critical graphs for the Minimum Vertex Cover problem. First, we propose a (not necessarily efficiently implementable) graph-generation process which can create all possible graphs while preserving the knowledge of the minimum cover size. To implement this process efficiently, we restrict it to a certain set of extensions that enlarge a critical graph while maintaining the criticality. Second, we systematically search for small critical instances. As a useful observation, we show that, if all critical graphs for the MVC problem were known, our restricted process could efficiently generate all possible graphs. Third, we illustrate the applicability of instances generated by our process for benchmarking. We report on a series of experiments with a state-of-the-art heuristic solver NuMVC [1] on exemplary instances that were generated by our randomized process.

Outline. In the following, Section 2 defines α-critical graphs for the Vertex-Cover problem. Section 3 describes our approach for generating hard random graphs from α-critical graphs. Section 5 presents our used extensions. Section 6 details the results of our experiments and Section 7 concludes.

2 α-Critical Graphs for the Minimum-Vertex-Cover Problem

Definition 2.1 A connected graph \( G = (V, E) \) is edge-uncritical (uncritical hereafter) according to an optimization problem \( P \) on graphs iff there exists an edge \( e \in E \) such that every solution for \( P \) at \( G' = (V, E \setminus \{e\}) \) is a solution for \( P \) at \( G \). A connected graph is edge-α-critical (α-critical hereafter) according to an optimization problem \( P \) iff it is not uncritical.

For the vertex cover problem, this definition implies:

Observation 2.2 A connected graph \( G = (V, E) \) is α-critical according to the vertex cover problem (in short: α-critical) iff deleting any edge reduces the minimum cover size.

There are several simple α-critical graphs.

Fact 2.3 Cliques and cycles of odd length are α-critical.

Proof. Note that the size of a minimum vertex cover of a clique \( C_k = (V, E) \) of size \( k \) is \( k - 1 \). If we delete an edge \( e = \{u, v\} \in E \) from \( C_k \), then \( V \setminus \{u, v\} \) is a vertex cover of size \( k - 2 \) for \( (V, E \setminus \{e\}) \). Thus, deleting any edge from \( C_k \) gives a graph with reduced vertex cover size. Cliques are α-critical.

Note that the size of a minimum vertex cover of a cycle \( G_k = (V, E) \) of odd length \( 2k + 1 \) is \( k + 1 \). If we delete an edge \( e \in E \) from \( G_k \) the resulting graph is a simple chain of \( 2k + 1 \) vertices. W.l.o.g., let \( V = \{v_1, \ldots, v_{2k+1}\} \) and \( E \setminus \{e\} = \{\{v_i, v_{i+1}\} | 1 \leq i \leq 2k\} \). Then \( \{v_{2i} | 1 \leq i \leq k\} \) gives a vertex cover of size \( k \) for \( (V, E \setminus \{e\}) \). Thus, deleting any edge from \( G_k \) gives a graph with reduced vertex cover size. Cycles of odd length are α-critical.

For an overview on α-critical graphs see for example [11] [5].

Recall that a perfect matching of a graph is defined as follows:

Definition 2.4 Let \( G = (V, E) \) be an undirected graph and \( E' \subseteq E \). Then \( E' \) is called a matching if for every pair \( e_1, e_2 \in E' \) of edges either \( e_1 = e_2 \) or \( e_1 \cap e_2 = \emptyset \), i.e. if two edges in \( E' \) are different then the adjacent pairs of nodes are disjoint.

A matching \( E' \) is a perfect matching if each node of \( G \) is incident to an edge in \( E' \).
Analyzing graphs with a perfect matching, one can show that cycles of even length are uncritical. Now, we can easily prove the following observations:

**Observation 2.5** Let \( G = (V,E) \) be a connected undirected graph that has a perfect matching \( E' \). The minimal size of a vertex cover is at least \(|V|/2\). Moreover, if the minimal size of a vertex cover is exactly \(|V|/2\), then either \( E = E' \), i.e., \( G \) is its own perfect matching, or \(|E| > |E'|\), and then \( G \) is uncritical.

**Proof.** Observe that \(|V|/2 = |E'|\), and \(|E'| \) vertices are needed to cover all edges in \( E' \). This proves the first claim. For the second claim, assume that there exists a minimum vertex cover of size \(|V|/2\) of \( G \) and \(|E| > |E'|\). This implies at least one edge \( \{u,v\} \in E \), with \( \{u,v\} \notin E' \). Deleting \( \{u,v\} \) from \( G \) gives a smaller graph \( G' \). \( E' \) is a perfect matching, not only of \( G \), but also of \( G' \). From the first claim, we know the minimal size of a vertex cover of \( G' \) is at least \(|V|/2\). Since removing \( \{u,v\} \) from \( G \) does not change the size of a minimum vertex cover, \( G \) is uncritical. \( \square \)

Since there exists a perfect matching for cycles of even length, Observation 2.5 implies:

**Corollary 2.6** Cycles of even length \( 2k \) with \( k \geq 2 \) are uncritical.

**Observation 2.7** Let \( G = (V,E) \) be a connected undirected graph and let \( U \subseteq V \) be a minimum vertex cover for \( G \). Assume that there exists an edge \( e \in E \) such that both endpoints of \( e \) are in the cover \( U \). Then, either \( G \) is uncritical or there exists a minimum vertex cover \( U' \) for \( G \) such that only one of the endpoints of \( e \) is in the cover \( U' \).

**Proof.** Let \( G = (V,E) \) be a connected undirected graph and let \( U \subseteq V \) be a minimum vertex cover for \( G \). Let \( e = \{v_1,v_2\} \in E \) such that both endpoints of \( e \) are in \( U \). Assume that \( G \) is \( \alpha \)-critical. Then \( G' = (V,E \setminus \{e\}) \) has a minimum vertex cover \( U'' \) such that non of the two endpoints of the deleted edge \( e \) is in the cover; otherwise, \( U'' \) has already been a cover for \( G \) – contradicting the assumption that \( U \) is a minimum vertex cover.

Since \( U'' \) covers \( G' \) both sets \( U'' \cup \{v_1\} \) and \( U'' \cup \{v_2\} \) cover \( G \) and include only one of the endpoints of \( e \). \( \square \)

Given a graph \( G = (V,E) \) and a minimum cover \( U \subseteq V \), such that no \( \{u,v\} \in E \) exists with \( u,v \in U \), then the graph is bipartite. In that case, the minimum cover is one side of the bipartite decomposition of the vertex set.

Furthermore, one can show that \( \alpha \)-critical graphs have to be 2-connected.

**Theorem 2.8** Let \( G = (V,E) \) be a graph with an articulation vertex \( u \), then \( G \) is uncritical.

To prove Theorem 2.8 we have to start with showing some useful lemmas. Recall that a vertex is called an articulation vertex of a connected graph if its removal will disconnect the graph.

![Figure 1: Two graphs with a minimum vertex cover size of 3. Since the second graph is a subgraph of the first one, the first is uncritical.](image)

**Lemma 2.9** Let \( G = (V,E) \) be a graph with an articulation vertex \( u \) that subdivides \( G \) into two subgraphs \( G_1 \) and \( G_2 \) (both include \( u \)). For a vertex cover \( C \) of \( G \), let \( C_1 \) denote the subset of \( C \) that denotes a cover for \( G_1 \) and let \( C_2 \) denote the subset of \( C \) that denotes a cover for \( G_2 \). If \( C \) is a minimum vertex cover for \( G \) then either \( C_1 \) is a minimum vertex cover for \( G_1 \), or \( C_2 \) is a minimum vertex cover for \( G_2 \), or both sets are minimum vertex covers for the two corresponding subgraphs.

**Proof.** We prove this observation by a contradiction. Let \( C \) be an optimal vertex cover of \( G \), \( C_1 \) be the subset of \( C \) that denotes a cover for \( G_1 \), and \( C_2 \) be the subset of \( C \) that denotes a cover for \( G_2 \).
Assume that neither $C_1$ nor $C_2$ is an optimal vertex cover for $G_1$ or $G_2$, respectively. Let $C'_1$ be an optimal vertex cover for $G_i$, then $C'' = C'_1 \cup C'_2$ is a vertex cover for $G$ of size $|C''| \leq |C'_1| + |C'_2|$. According to our assumption $C''$ is not an optimal vertex cover; thus, we have

$$|C'_1| + |C'_2| - |C| \geq |C'_1| + |C'_2| - 1 > |C'_1| + |C'_2| + 1$$

– a contradiction.

**Lemma 2.10** Let $G = (V, E)$ be a graph with an articulation vertex $u$ that subdivides $G$ into two subgraphs $G_1$ and $G_2$ (both include $u$). If there exists a minimum vertex cover $C_1$ for $G_1$ with $u \in C_1$, then $G$ is uncritical.

**Proof.** Assume that there exists a minimum vertex cover $C_1$ for $G_1$ with $u \in C_1$. Furthermore, assume that $G$ is $\alpha$-critical. Let $G'$ be the subgraph of $G$ that does not contain any edge $\{u, v\}$ that connects $u$ with another vertex of $G_2$. Since $G$ is $\alpha$-critical, there exists vertex cover $C'$ for $G'$ that is smaller than the minimum vertex cover size of $G$. If we now replace the part of $C'$ that denotes a vertex cover for $G_1$ by $C_1$, we get a vertex cover $C$ for $G'$ of the same size as $C'$ that contains $u$. Thus, $C$ is also a vertex cover for $G$ – contradicting our assumption that the minimum vertex cover size of $G$ is larger than $|C'| = |C|$ and thus $G$ is $\alpha$-critical.

**Conclusion 2.11** Let $G = (V, E)$ be a $\alpha$-critical graph with a articulation vertex $u$. Then every minimum vertex cover of $G$ does not include $u$.

**Conclusion 2.12** Let $G = (V, E)$ be a $\alpha$-critical graph with a articulation vertex $u$. Then every neighbor of $u$ in $G$ has to be an element of every minimum vertex cover of $G$.

Now we can prove Theorem 2.8.

**Proof.** of Theorem 2.8 For the contrary, assume that $G$ is $\alpha$-critical. Let $v$ be a neighbor of $u$ in $G$ and let $G' = (V, E \setminus \{u, v\})$ be the subgraph of $G$ after removing the edge $\{u, v\}$, then we can distinguish between the following two cases:

- The minimum vertex cover $C'$ of $G'$ is smaller than the minimum vertex cover for $G$. But this implies that $C' \cup \{u\}$ gives a vertex cover for $G$ of size $|C|$ – contradicting conclusion 2.11.
- The minimum vertex cover $C'$ of $G'$ is of the same size as the minimum vertex cover for $G$. But this implies that $G$ is uncritical.

Another useful observation is the following:

**Observation 2.13** Let $G = (V, E)$ be a $\alpha$-critical graph. Then, for every vertex $u \in V$, there exists a minimum vertex cover $C$ of $G$ with $u \in C$.

**Proof.** Let $v$ be a neighbor of $u$ in $G$ and let $G' = (V, E \setminus \{u, v\})$ be the subgraph of $G$ after removing the edge $\{u, v\}$, then (because $G$ is $\alpha$-critical) the minimum vertex cover $C'$ of $G'$ is smaller than the minimum vertex cover for $G$. Thus, $C' \cup \{u\}$ is a minimum vertex cover of $G$.  

\[\square\]
3 Generating Larger Graphs from $\alpha$-Critical Instances

One relevant application of critical graphs is the construction of larger graphs. For this purpose, we need a (randomized) generation process which (1) allows to construct all possible graphs, and (2) preserves the knowledge of a hidden solution, i.e. about the minimum vertex cover. This section presents such a generation process.

Definition 3.1 Let $B = \{B_1, B_2, \ldots\}$ be a set of graphs where each graph is given by a triple $B_i = (U, V, E)$ with two disjoint sets of vertices $U$ and $V$ such that $U$ gives a minimum vertex cover for the graph $(U \cup V, E)$. Define the following random processes

- Given $B$ and $\ell, m, n \in \mathbb{N}$ for the vertex cover size $\ell$, an upper bound $m$ for the number of edges, and an upper bound $n$ for the number of vertices, then $G_{B, \ell, m, n}^3$ is a random variable that uniformly at random gives a collection $S_1, \ldots, S_k$ of elements of $B$ (some elements of $B$ may repeat) with $S_i = (U_i, V_i, E_i)$ s.t.

\[ \left| \bigcup_{i=1}^k U_i \right| = \ell \text{ and } \left| \bigcup_{i=1}^k V_i \right| + \ell \leq n \text{ and } \left| \bigcup_{i=1}^k E_i \right| \leq m. \]

Let $C(S_1, \ldots, S_k) = (\bigcup_{i=1}^k U_i, \bigcup_{i=1}^k V_i, \bigcup_{i=1}^k E_i)$.

- Given a triple $B = (U, V, E)$ with $|V| + |U| \leq n$, then $G_{n}^2(B)$ will be the triple $(U, V \cup V', E)$ where $V'$ denotes a set of $n - |U| - |V|$ new vertices $(V' \cap (U \cup V) = \emptyset)$.

- Given a triple $B = (U, V, E)$ with $|E| \leq m$, let $G_{m}^3(B)$ be a random variable that uniformly at random adds $m - |E|$ new edges $E' \subset U \times (U \cup V)$ to $B$.

Since none of the defined processes reduces the cover size, we can conclude:

Theorem 3.2 Let $B = \{B_1, B_2, \ldots\}$ be a set of graphs, where each graph is given by a triple $B_i = (U, V, E)$ with two disjoint sets of vertices $U$ and $V$ such that $U$ gives a minimum vertex cover for $(U \cup V, E)$. Then, for every random graph $(U', V', E')$ in the range of $G_{m}^3(G_{n}^2(C(G_{B, \ell, m, n}^3))))$, $U'$ is a minimum vertex cover for $(U' \cup V', E')$ of size $\ell$, and the graph $(U' \cup V', E')$ has $n$ vertices and $m$ edges.

Proof. The second claim follows directly from the definition of the two processes $G_{m}^3$ and $G_{n}^2$.

Furthermore, the process $G_{m}^3$, $G_{n}^2$, or $G_{B, \ell, m, n}^3$ adds an edge to the resulting graph that is not covered by the cover sets that are given by the subgraphs $B_i = (U, V, E) \in B$ that are chosen by the first process $G_{B, \ell, m, n}^3$. Thus, the covers are maintained.

Thus, it suffices to prove that the defined processes do not reduce the cover size.

Let $G = (V, E)$ be an arbitrary graph and let $G' = (V', E')$ be a subgraph of $G$, i.e. $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. Furthermore, let $C$ be a vertex cover of $G$. Then $C \cap V'$ is a vertex cover of $G'$.

Now assume that there exists a graph $(U', V', E')$ in the range of $G_{m}^3(G_{n}^2(C(G_{B, \ell, m, n}^3))))$ such that $(U' \cup V', E')$ has a cover $C$ of size $< \ell$. Since the subgraphs $S_i$ chosen by the first process are vertex disjoint there has to be such a chosen subgraph $S_i = (U_i, V_i, E_i)$ such that $C_i = C \cap (U_i \cup V_i)$ is strictly smaller than $|U_i|$ and $C_i$ is a vertex cover for $(U_i \cup V_i, E_i)$. This contradicts the assumption that $U_i$ is a minimum vertex cover for $(U_i \cup V_i, E_i)$.

Thus, the three processes do not affect our a-priori knowledge of the minimum cover size. It remains to show that any graph can be constructed by the three processes. This observation follows by analyzing the reverse processes:

Theorem 3.3 Let $B = \{B_1, B_2, \ldots\}$ with $B_i = (U, V, E)$ be a set of all $\alpha$-critical graphs $G_i = (U \cup V, E)$, where $U$ and $V$ are disjoint and $U$ gives a minimum vertex cover for the graph $G_i$. Then the range of $G_{m}^3(G_{n}^2(C(G_{B, \ell, m, n}^3))))$ determines the set of all graphs of $n$ vertices, $m$ edges, and minimum vertex cover size $\ell$. 

5
Proof. For the contrary, assume that there exists a graph \( G = (V, E) \) of \( n \) vertices, \( m \) edges, and minimum vertex cover size \( \ell \) that is not in the range of \( G_3^\ell_n(C(G_{B,\ell,m,n}^1)) \). Let \( U \subset V \) be a vertex cover of \( G \) of size \( \ell \).

- Then either \( G \) is connected and \( G \not\in \mathcal{B} \) and therefore \( G \) is uncritical or
- there exists a connected component \( G' \) of \( G \) with \( n' \) vertices, \( m' \) edges, and minimum vertex cover size \( \ell' \) where \( G' \) is not in the range of
  \[
  G_3^{\ell'}_{m'}(C(G_{B,\ell',m',n'}^1))
  \]
  and therefore \( G' \not\in \mathcal{B} \) and thus \( G' \) is uncritical.

In the following we will restrict ourselves to the first case. The second case follows analogously by focusing on the components \( G' \) that are not in the range of \( G_3^{\ell'}_{m'}(C(G_{B,\ell',m',n'}^1)) \).

Since \( G \) is uncritical, there exists an edge \( e \) that can be deleted from the corresponding graph without reducing the minimum size of a vertex cover. Since at least one the endpoints of \( e \) is in \( U \) this edge can be generated by the process \( G_3^\ell_n \). Hence, if \( G \) is not in the range of
\[
G_3^\ell_n(C(G_{B,\ell,m,n}^1))
\]
then \( G'' = (V, E \setminus \{e\}) \) is not in the range of
\[
G_3^{\ell-1}_{m-1}(C(G_{B,\ell-1,m-1,n}^1)) .
\]
This reduction step reduces the graph until we have deleted all edges from the graph without changing the initial vertex cover \( C \). Thus, \( C \) has to be empty and therefore also the initial set of edges has to be empty. Hence, \( G \) has to be a set of isolated vertices and \( \ell = 0 \). But this graph is in the range of
\[
G_3^0_n(C(G_{B,0,0,n}^1))
\]
and will be generated by the subprocess \( G_{B,0,0,n}^2 \) – a contradiction. \( \blacksquare \)

4 Circulant Graphs

We will now discuss graph classes that can be seen as a generalization of cycles and as extensions of cliques.

**Definition 4.1** A circulant graph \( \text{CirculantGraph}(n, L) \) with \( n \in \mathbb{N} \) and \( L \subseteq \{1, \ldots, \lceil n/2 \rceil \} \) is an undirected graph with \( n \) vertices \( v_0, \ldots, v_{n-1} \) where each vertex \( v_i \) is adjacent to both vertices \( v_{(i+j) \mod n} \) and \( v_{(i-j) \mod n} \) for all \( j \in L \).

To determine critical graphs, we analyzed circulant graphs of degree 4, i.e. \( \text{CirculantGraph}(n, L) \) with \( n \in [2, 80], L = \{1, j\}, \) and \( j \in [2, 20] \). We identified 121 critical graphs with degree 4 in this range which are visualized in Figure 3. Furthermore, we implemented a search for all critical circulant graphs of degree 6, i.e. for \( \text{CirculantGraph}(n, L) \) with \( |L| = 3 \). For \( n \in [4, 60] \), \( L = \{1, i, j\}, \) and \( i, j \in [2, 20] \). We determined in total 427 critical graphs within this range. These are listed in Table 3 in the Appendix.

To conclude this section we will present a general rule for a subset of circulant graphs to determine whether they are critical or not. One can see that Fact 2.3 is a conclusion of the following result.

**Definition 4.2** For \( n, d_h \in \mathbb{N} \), define the undirected graph \( C_{n,d_h} = (V_{n,d_h}, E_{n,d_h}) \) by \( V_{n,d_h} = \{u_0, \ldots, u_{n-1}\} \) and \( E_{n,d_h} = \bigcup_{j=0}^{n-1} \{\{u_j, v\} | v \in N_{j,n,d_h}^+ \cup N_{j,n,d_h}^-\} \), where for all \( i \in \{0, \ldots, n-1\} \), it holds
\[
N_{i,n,d_h}^+ = \{u_{(i+1) \mod n}, \ldots, u_{(i+d_h) \mod n}\} \quad \text{and} \quad N_{i,n,d_h}^- = \{u_{(i-1) \mod n}, \ldots, u_{(i-d_h) \mod n}\} .
\]

For \( n, d_h, \delta, i \in \mathbb{N} \) and the graph \( C_{n,d_h} = (V_{n,d_h}, E_{n,d_h}) \) define the sets of vertices
\[
V_{i,n,d_h}^\delta = \{u_{(i-(d_h+1)+\delta) \mod n}, \ldots, u_{((i+1)-(d_h+1)+\delta-1) \mod n}\} .
\]
Thus, $C_{n,dh} = \text{CirculantGraph}(n, L)$ where $L = [1, dh]$.

**Lemma 4.3** The minimum vertex cover of each graph $C_{n,dh}$ is $n - \left\lceil \frac{n-dh}{dh+1} \right\rceil$.

**Proof.** One can easily verify that each subset $V_{i,n,dh}^\delta$ forms a complete graph. Thus, every vertex cover of a graph $C_{n,dh}$ has to include all vertices of every subset $V_{i,n,dh}^\delta$ except at most one vertex. W.l.o.g., we can assume that $u_0$ is not an element of the cover. Thus, there exists at most one vertex that is not an element of the cover for each subgraph $V_{i,n,dh}^\delta$ for $i \in \{0, \ldots, \left\lceil \frac{(n-dh)}{(dh+1)} \right\rceil \}$. Since also all vertices of $N_{0,n,dh}$ have to be in this cover.

A cover of size $n - \left\lceil \frac{(n-dh)}{(dh+1)} \right\rceil$ is given by $C = V \setminus \{ u_{i(dh+1)} \mid i \in \{0, \ldots, \left\lceil \frac{(n-dh)}{(dh+1)} \right\rceil \} \}$. $\blacksquare$

**Theorem 4.4** $C_{n,dh}$ is a connected $\alpha$-critical graph iff either $n \leq 2dh + 1$ or $n - dh$ is a multiple of $dh + 1$.

**Proof.** If $n \leq 2dh + 1$ then $C_{n,dh}$ is a clique and the claim follows directly. Thus it remains to show the claim for parameters $n$ and $dh$ where $n > 2dh + 1$.

For $k \in \{1, \ldots, dh\}$ let $C_{n,dh}^k = (V_{n,dh}, E_{n,dh} \setminus \{u_0, u_{n-k}\})$. Because of symmetry $C_{n,dh}$ is $\alpha$-critical iff for any $k \in \{1, \ldots, dh\}$ the size of a minimum vertex cover has to be smaller than $n - \left\lceil \frac{(n-dh)}{(dh+1)} \right\rceil$.

Let us assume that for given parameters $n$ and $dh$ with $n > 2dh + 1$, $C_{n,dh}$ is $\alpha$-critical. In the following, we will show that this implies that $n - dh$ is a multiple of $dh + 1$. 7
Assume that for given parameters $n, \alpha, a$, a technique to merge two for the generation of diverse critical graphs, one needs efficient methods and patterns. In 1970, Wessel [16] introduced intersection of an $H$ Hence, we can assume that $i_0 \leq 2d_h + 1$, otherwise we can remove $u_{i_0}$ from $C$ to get a vertex cover of smaller size for $C_{n,d_h}$ - thus, $C$ is not a minimum vertex cover for $C_{n,d_h}$. Analogously, it holds that $i_{\ell-1} - i_{\ell - 2} \leq 2d_h + 1$ and $n - 1 - i_{\ell - 1} \leq 2d_h + 1$.

Now, assume that $i_0 > 0$ or $i_{\ell - 1} \neq n - k$ then $C$ is also a vertex cover for $C_{n,d_h}$ - contradicting our assumption that the size of the minimum vertex cover of $C_{n,d_h}$ is smaller than the size of the minimum vertex cover of $C_{n,h}$ and $C$ is such a minimum vertex cover of $C_{n,d_h}$. Thus, we assume that $i_0 = 0$ and $i_{\ell - 1} = n - k$ in the following.

Finally, if $(i_{\ell - 1} - i_{\ell - 2}) + k - 1 \geq 2d_h + 1$ then $(C \cup \{u_{i_{\ell - 1}}\}) \setminus \{u_{i_{\ell - 2} + d_h + 1}\}$ to get a vertex cover for $C_{n,d_h}$ and $C_{n,h}$ of size $\ell$ - contradicting our assumption that $C_{n,d_h}$ has a minimum vertex cover of size larger than the size of the minimum vertex cover of $C_{n,d_h}$.

Because the regularity of the graph as well as the regularity of the discussed cover, we can assume that $\ell = 3$ and therefore

$$n = 1 + d_h + 1 + (i_2 - i_1 - 1) + 1 + (k - 1)
= d_h + k + (i_2 - i_1) + 1 < 3d_h + 3$$

(1)

Assume that for given parameters $n$ and $d_h$ the graph $C_{n,d_h}$ is $\alpha$-critical, then the minimum vertex cover size of $C_{n,d_h}^k$ for any value $k \in \{1, \ldots, d_h\}$ has to be smaller than the size of the minimum vertex cover of $C_{n,d_h}$. As shown above, it is sufficient to restrict our analysis to $\ell = 3$ with $i_0 = 0$ and $i_2 = n - k$ and therefore $i_1 \leq n - d_h - 1 - k$ and $i_0 \leq n - 2d_h - 2 - k$. If we consider the case where $k = d_h$, then this implies that $3d_h + 2 \leq n$.

Using Equation (1) we get $n = 3d_h + 2$. With respect to our simplifications above this implies:

If $C_{n,d_h}$ is $\alpha$-critical, then $n - d_h$ is a multiple of $d_h + 1$.

It remains to show that $C_{n,d_h}$ is $\alpha$-critical, if $n - d_h$ is a multiple of $d_h + 1$. Using our simplifications above it is sufficient to show:

The minimum vertex cover size of $C_{3d_h+2,d_h}^k$ for any value $k \in \{1, \ldots, d_h\}$ is $3d_h - 1$.

More precisely, we will show that $\{u_0, u_{d_h+1}, u_{3d_h+2-k}\}$ is an independent set and therefore its complement is a vertex cover of the desired size.

From our construction of $C_{3d_h+2,d_h}$ it follows that $u_{d_h+1}$ is not a neighbor of $u_0$. Furthermore, $C_{3d_h+2,d_h}^k$ is constructed by deleting the edge $\{u_0, u_{3d_h+2-k}\}$. Thus, it remains to show that $u_{d_h+1}$ and $u_{3d_h+2-k}$ are not directly connected. Note that

$$(3d_h + 2 - k) - (d_h + 1) = 2d_h + 1 - k > d_h.$$  

Hence $u_{3d_h+2-k} \not\in N_{d_h+1,3d_h+2,d_h}^+$. Hence, $C_{n,d_h}$ is $\alpha$-critical, if $n - d_h$ is a multiple of $d_h + 1$.

5 Extensions for the Efficient Generation of Graphs

The problem of deciding whether a given graph is $\alpha$-critical is complete for the second level of the Boolean hierarchy $D_2 = BH_2$ (see [12], or [10] for a survey on $\alpha$-critical graphs), where $BH_2$ is the class of languages that form the intersection of an $NP$ language with a co-$NP$ language. Thus, finding $\alpha$-critical graphs is a hard problem. Moreover, for the generation of diverse critical graphs, one needs efficient methods and patterns. In 1970, Wessel [16] introduced a technique to merge two $\alpha$-critical graphs:
Definition 5.1 For a graph $G = (V, E)$ and a vertex $u \in V$, let $N_G(u)$ denote the neighborhood of $u$ in $G$.

Given two connected graphs $G_1$, $G_2$, a distinguished edge $\{u, v\} \in E(G_1)$, and a vertex $w \in V(G_2)$ with degree at least two. Let $G$ be the disjoint union of $G_1$ and $G_2$. Then, for every neighbor $x \in N_{G_2}(w)$, choose one vertex $y$ from $\{u, v\}$ and add the edge $\{x, y\}$ to $G$. Ensure that $u$ and $v$ are each chosen at least once. Finally, remove the vertex $w$ and the edge $\{u, v\}$ from $G$. The resulting graph $G$ is said to be pasted together from $G_1$ and $G_2$.

By investigating the maximum independent sets, Wessel showed the following theorem:

Theorem 5.2 [Wessel] If $G$ is pasted together from two connected $\alpha$-critical graphs $G_1$ and $G_2$ of size at least three, then $G$ is $\alpha$-critical. If $G$ is a connected $\alpha$-critical graph of size at least four and if $G$’s (vertex) connectivity is two, then there exist two connected $\alpha$-critical graphs $G_1$ and $G_2$ each of size at least three, such that $G$ is pasted together from $G_1$ and $G_2$. The size of the maximum independent set of $G$ is given by the sum of the sizes of the maximum independent sets of $G_1$ and $G_2$.

Following the proof of this observation (see e.g. [10, 11]), one can see that this method for extending $\alpha$-critical graphs does not preserve the knowledge of a concrete maximum independent set or a concrete minimum vertex cover in all cases. It preserves only the knowledge of the corresponding cardinalities.

A second technique for enlarging an $\alpha$-critical graph can be found in [11]:

Definition 5.3 Let $G$ be a $\alpha$-critical graph and $u$ be a vertex of $G$ with degree at least two. The operation of splitting $u$ works as follows: Let $F$ denote a strictly non-empty subset of the neighbors of $u$. Add two new vertices $v$ and $w$ to $G$. Connect $v$ with $u$ and $w$. Connect all vertices of $F$ with $v$ and $w$, and delete the edges between $u$ and the vertices of $F$.

Let $\mathcal{G}$ be a family of undirected graphs, then define $\Gamma_{\text{split}}(\mathcal{G})$ to be the set of graphs that can be generated from any graph $G = (V, E) \in \mathcal{G}$ by splitting any vertex of $G$. Let $\Gamma_{\text{split}}^k(\mathcal{G})$ denote the transitive closure of $\mathcal{G}$ according to $\Gamma_{\text{split}}$. If we apply the extension $k$ times, then the set of resulting graphs is denoted by $\Gamma_{\text{split}}^k(\mathcal{G})$. If $\mathcal{G}$ consists of a single graph $G$, then we use the notions $\Gamma_{\text{split}}(G)$ and $\Gamma_{\text{split}}^k(G)$.

For the proof of the following theorem see e.g. [11]:

Theorem 5.4 Let $G = (V, E)$ be a connected $\alpha$-critical graph of size at least three. Then, every vertex has degree at least two, and splitting any vertex $v \in V$ results in a new $\alpha$-critical graph $G'$ where the maximum independent set and the minimum vertex cover is increased by one. Furthermore, if $G'$ be a connected $\alpha$-critical graph of size at least four, and if $v \in V(G')$ has exactly two neighbors $u, w$ in $G'$ where $u$ and $w$ are not adjacent; then identifying $u$ with $w$ and deleting $v$ results in another $\alpha$-critical graph $G$ where the maximum independent set and the minimum vertex cover is decreased by one.

One can see that this method for extending $\alpha$-critical graphs preserves the knowledge of a concrete maximum independent set and of a concrete minimum vertex cover.

Observation 5.5 Let $G = (V, E)$ be a connected $\alpha$-critical graph of size at least three and let $v \in V$ be an arbitrary vertex of $G$. Let $U$ be an maximum independent set and $\overline{U} = V \setminus U$ be the corresponding minimum vertex cover. Let $v$ and $w$ denote the two new vertices, generated by the splitting operation of $v$ (see Definition 5.3) and let $G'$ be the resulting graph. Then

$$U' = \begin{cases} U \cup \{w\} & \text{if } u \in U \\ U \cup \{v\} & \text{if } u \notin U \end{cases}$$

and

$$\overline{U}' = \begin{cases} \overline{U} \cup \{w\} & \text{if } u \in \overline{U} \\ \overline{U} \cup \{v\} & \text{if } u \notin \overline{U} \end{cases}$$

determine an increased independent set and the corresponding vertex cover and are therefore optimal.

This section introduces a new method which can efficiently produce critical graphs from extending cycles of odd length, called the parallel extension in (see Figure 5[11]). Furthermore, we will discuss a restricted version of the splitting operation, called chain extension (see Figure 5[11]) which had been used for our graph generator. Let us consider the parallel extension first. Prior, we need a useful definition of what we call a VC-Overlap.
Definition 5.6 Let \( G = (V, E) \) be an undirected connected graph and let \( U \subseteq V \) be a subset of vertices of \( G \). \( U \) is called a VC-Overlap iff for every minimum vertex cover \( C \) of \( G \), it holds that \( U \not\subseteq C \). If additionally, for any vertex \( u \in U \), there exists a minimum vertex cover \( C_u \) of \( G \) such that \( u \not\in C_u \) and \( U \setminus \{u\} \subseteq C \), then \( U \) is a 1-VC-Overlap. Let \( u \) be a new vertex and \( U \subseteq V \). Then, define the extension of \( G \) according to \( u \) and \( U \) as

\[
\Gamma(G, U, u) = (V \cup \{u\}, E \cup \{\{u, v\} | v \in U\}).
\]

Observation 5.7 Given a graph \( G = (V, E) \), a VC-Overlap \( U \subseteq V \), and a new vertex \( u \), then the size of a minimum vertex cover of \( \Gamma(G, U, u) \) is given by the size of a minimum vertex cover of \( G \) plus one.

Proof. Let \( C \) be a minimum vertex cover of \( G \), then \( C \cup \{u\} \) is a vertex cover of \( \Gamma(G, U, u) \). Consider now a minimum vertex cover \( C' \) of \( \Gamma(G, U, u) \), then either \( u \in C' \) or \( N_{\Gamma(G, U, u)}(u) \subseteq C' \). In the first case \( |C'| \) has to be larger than the minimum vertex cover size of \( G \) since \( C' \setminus \{u\} \) is a vertex cover of \( G \). For the second case one can conclude that \( |C'| \) is larger than the minimum vertex cover size of \( G \) since no minimum vertex cover of \( G \) includes all vertices of \( U = N_{\Gamma(G, U, u)}(u) \).

Observation 5.8 Given an \( \alpha \)-critical graph \( G = (V, E) \), a VC-Overlap \( U \subseteq V \), and a new vertex \( u \). If \( U \) is not a 1-VC-Overlap, then \( \Gamma(G, U, u) \) is uncritical.

Proof. Since \( U \) is a VC-Overlap but not an 1-VC-Overlap there exists a vertex \( v \in U \) such that for every minimum vertex cover \( C \) of \( G \) with \( v \not\in C \) there exists a second vertex \( v' \in U \) that is also not an element of \( C \). Thus, \( U \setminus \{v\} \) is a VC-Overlap and the claim follows directly from Observation 5.7.

Theorem 5.9 Given a graph \( G = (V, E) \), a set \( U \subseteq V \) with a 1-VC-Overlap, and a new vertex \( u \), then if \( G \) is \( \alpha \)-critical, then \( \Gamma(G, U, u) \) is \( \alpha \)-critical.

Proof. For the contrary, assume that \( \Gamma(G, U, u) \) is uncritical, then we can either delete an edge connected to \( u \) or an edge of \( E \) without changing the minimum vertex cover size of \( \Gamma(G, U, u) \):

1. Assume that we can delete the edge \( \{u, v\} \) with \( v \in U \) from \( \Gamma(G, U, u) \) without reducing the minimum vertex cover size of the resulting graph \( G' \). Since \( U \) is a 1-VC-Overlap, there exists a minimum vertex cover for \( G \) that contains all nodes of \( U \setminus \{v\} \). This vertex cover is also a vertex cover of \( G' \). Hence, deleting \( \{u, v\} \) from \( \Gamma(G, U, u) \) reduces the minimum vertex cover size of the resulting graph.

2. Assume that we can delete an edge \( e \) of \( G \) without reducing the minimum vertex cover size of the resulting subgraph \( G' \) of \( \Gamma(G, U, u) \). Since \( G \) is \( \alpha \)-critical, there exists a vertex cover \( C \) of \( G \) after deleting \( e \) where \( |C| \) is smaller than minimum vertex cover size of \( G \). Since \( C \cup \{u\} \) is a vertex cover of \( G' \), deleting the edge \( e \) from \( \Gamma(G, U, u) \) reduces minimum vertex cover size of the resulting graph.

Thus, \( \Gamma(G, U, u) \) is \( \alpha \)-critical.

Note that, if \( U \) is given by a vertex of \( G \) plus its neighborhood, then \( U \) is a VC-Overlap. However, Theorem 5.9 does not provide us with an efficient method for constructing \( \alpha \)-critical graphs. For this purpose, the following definition simplifies the extension \( \Gamma(G, U, u) \).

Figure 5: Two \( \alpha \)-critical graphs which are generated from a cycle of five vertices.
Definition 5.10 (Parallel Extension) Let \( G = (V, E) \) be an undirected connected graph and let \( v \in V \) be a vertex of \( G \). Then, define \( G \setminus v = (V \setminus \{v\}, E \setminus \{\{v, u\} | u \in V\}) \) to be the graph \( G \) without \( v \). For an undirected graph \( G = (V, E) \) and a vertex \( v \in V \), let \( N_G(v) = \{u \in V | \{v, u\} \in E\} \) denote the neighborhood of \( v \) in \( G \).

Let \( e = \{u, v\} \in E \) be an edge of \( G \). Then, we call \( u \) and \( v \) neighbor-equivalent iff \( N_G(v) \setminus \{v\} = N_G(u) \setminus \{v\} \).

Moreover, let \( \mathcal{G} \) be a family of undirected graphs, then define the set of parallel extensions \( \Gamma_{par}(\mathcal{G}) \) of \( \mathcal{G} \) to be the set of graphs that can be generated from any graph \( G = (V, E) \in \mathcal{G} \) by adding a new node \( u \notin V \) and new edges \( E' \) such that for at least one vertex \( v \in V \), it holds that \( E' = \{\{u, v'\} | v' \in N_G(v) \cup \{v\}\} \). We call \( u \) the parallel extension of \( v \).

Define \( \Gamma_{par}^0(\mathcal{G}) = \mathcal{G} \) and \( \Gamma_{par}^k(\mathcal{G}) = \Gamma_{par}^{k-1}(\Gamma_{par}(\mathcal{G})) \). We write

\[
\Gamma_{par}^*(\mathcal{G}) = \bigcup_{k \in \{0, \ldots, \infty\}} \Gamma_{par}^k(\mathcal{G})
\]

for the transitive closure of \( \mathcal{G} \). If \( \mathcal{G} = \{G\} \) consists of a single \( G \), we usually write \( \Gamma_{par}(G) \) and \( \Gamma_{par}^*(G) \) instead of \( \Gamma_{par}(\mathcal{G}) \) and \( \Gamma_{par}^*(\mathcal{G}) \).

A minimum vertex cover for any graph \( G \in \Gamma_{par}(G') \) can easily be determined from a minimum vertex cover of \( G' \).

Theorem 5.11 Let \( G = (V, E) \) be an undirected connected graph and let \( e = \{u, v\} \in E \) such that \( u \) and \( v \) are neighbor-equivalent. Let \( U \) be a minimum vertex cover for \( G \setminus u \). Then, \( U \cup \{u\} \) is a minimum vertex cover for \( G \).

Proof. Let \( G = (V, E) \), \( u, v \in V \), and \( U \) fulfill the preconditions of the theorem. If \( U \) is a cover of size \( m \) for \( G \setminus u \) then \( U \cup \{u\} \) will be a cover for \( G \) of size \( m + 1 \). Thus, it remains to show that if \( U \) is a minimum cover for \( G \setminus u \), then \( U \cup \{u\} \) will be a minimum cover for \( G \).

Assume that \( U \cup \{u\} \) is not a minimum vertex cover for \( G \). Then there exists a vertex cover \( U' \) for \( G \) with \( |U'| \leq m \). Note that for every vertex cover for \( G \) either \( u \) or \( v \) has to be in the cover.

Assume that \( u \notin U' \), then also the set \( \{u\} \cup U' \setminus \{v\} \) has to be a vertex cover for \( G \) of the same size.

But if \( u \) is an element of a vertex cover \( U' \) for \( G \), then \( U' \setminus \{u\} \) will be a vertex cover for \( G \setminus u \) of size \( m - 1 \) – contradicting the condition that \( U \) is a minimum vertex cover for \( G \setminus u \) and \( U \cup \{u\} \) is a minimum vertex cover for \( G \).

Corollary 5.12 Let \( G \) be an \( \alpha \)-critical graph with minimum vertex cover size \( m \). Then, the minimum vertex cover size of each graph of \( \Gamma_{par}(G) \) is \( m + 1 \). More general, the minimum vertex cover size of each graph in \( \Gamma_{par}^k(G) \) is \( m + k \).

We show that a parallel extension of any \( \alpha \)-critical graph gives a new \( \alpha \)-critical graph.

Theorem 5.13 If \( G \) is a connected \( \alpha \)-critical graph, then all graphs in \( \Gamma_{par}(G) \) are connected and \( \alpha \)-critical.

Proof. The claim that any graph in \( \Gamma_{par}(G) \) is connected if \( G \) is connected is obvious. Thus, we will focus on the claim that any graph in \( \Gamma_{par}(G) \) is \( \alpha \)-critical if \( G \) is \( \alpha \)-critical.

Let \( G = (V_G, E_G) \) be a connected \( \alpha \)-critical graph with a minimum vertex cover of size \( m \). Assume that there exists a uncritical graph \( H = (V_H, E_H) \in \Gamma_{par}(G) \). Furthermore, let \( v \in V_G \) and \( u \in V_H \) such that \( v, u \) defines the extension of \( G \), i.e.

- \( v \) and \( u \) are neighbor-equivalent for \( H \),
- \( u \notin V_G \) and \( G = H \setminus u \).

According to Corollary 5.12 the minimum vertex cover size of \( H \) is \( m + 1 \). Since \( H \) is uncritical, some edge \( e \in E_H \) exists, such that the minimum vertex cover size of the smaller graph \( H' = (V_H, E_H \setminus \{e\}) \) is \( m + 1 \), too. Let \( U' \) be a minimum vertex cover for \( H' \).

We distinguish three cases:
1. The edge $e$ is neither incident with $u$, nor with $v$. In this case, $U'$ must contain $u$ or $v$ (or both). If $u \not\in U'$, then $v \in U'$, and $\{u\} \cup U' \setminus \{v\}$ would be another cover of $H'$ of the same size. Thus, assume $u \in U'$.

Since $u$ and $v$ are neighbor-equivalent for $H'$, Theorem 5.11 implies that $U' \setminus \{u\}$ determines a minimum vertex cover for $G' = (V_G, E_G \setminus \{e\})$ of size $m$. This is a contradiction to $G$ being $\alpha$-critical.

2. The edge $e$ is incident with either $u$ or $v$ (but not with both). By symmetry we can assume that $e$ is incident with $v$.

Since $G$ is irreducible, $G'$ has minimum vertex cover $U'$ of size $m - 1$. On the other hand, all the edges added to $G$ by the parallel extension are incident with $v$; thus, $U' \setminus \{u\}$ determines a minimum vertex cover for $H'$ of size $m$ – contradicting the assumption, that have the same minimum vertex cover size.

3. The edge $e$ is incident with both $u$ and $v$, i.e., $e = \{u, v\}$.

Consider a minimum vertex cover $U$ for $G$ with $|U| = m$. If $v \not\in U$, then $U$ is a vertex cover of $H'$, which would contradict the size $m + 1$ for a minimum vertex cover of $H'$. Thus, assume $v \in U$.

Now, assume that a neighbor $w$ of $v$ is not in the cover $U$. By Observation 2.7, such a cover $U$ exists, except when $v$ is isolated. But then each $G$ is not connected or $G$ consist of a single vertex.

Now remove $\{w, v\}$ from $G$ to get a smaller graph $G''$. Since $G$ is $\alpha$-critical, $G''$ has a vertex cover $U''$ of size $m - 1$. Note that $U''$ covers all edges in in $H'$, except (possibly) $\{v, w\}$ and the connections between $u$ and the neighbors of $v$. If $v \in U''$, then $U \cup \{u\}$ covers all edges in $H'$ (and even in $H$). If $v \not\in U''$, all neighbors of $v$ (except for $w$) would be in $U''$; thus, $U'' \cup \{w\}$ would cover all edges in $H'$.

Since $|U'' \cup \{w\}| = m = |U'' \cup \{u\}|$, the size of our vertex cover for $H'$ contradicts the assumption of $H$ being uncritical.

\begin{flushright}
\textbox{\textbullet}
\end{flushright}

**Corollary 5.14** Let $G$ be a $\alpha$-critical graph. Then, all graphs of $\Gamma_{par}(G)$ are $\alpha$-critical.

We will now show an inverse observation to Corollary 5.14. Analogously to Theorem 5.4, our goal is to show that if an $\alpha$-critical graph $G$ is the parallel extension $G'$, then also $G'$ is $\alpha$-critical.

**Theorem 5.15** Let $G = (V, E)$ be an $\alpha$-critical connected graph and $e = \{u, v\} \in E$ an edge of $G$ such that $u$ and $v$ are neighbor-equivalent. Then $H = G \setminus u$ is $\alpha$-critical. The size of a minimum vertex cover of $H$ is reduced by 1 according to the minimum vertex cover of $G$.

**Proof.** The second claim, that the size of a minimum vertex cover of $H = (V_H, E_H)$ is reduced by 1 according to the minimum vertex cover of $G$ follows directly from Theorem 5.11 if $H$ is irreducible.

To show that $H$ is $\alpha$-critical if $G$ is $\alpha$-critical, we have transform the minimum vertex cover of $G$ after deleting an edge $e' \in E \cap E_H$, which occurs in both graphs, into a minimum vertex cover of $H$.

Assume that the size of a minimum vertex cover of $G$ is $m$. Since $G$ is $\alpha$-critical, the size of the minimum vertex cover of $G' = (V, E \setminus \{e'\})$ is $m - 1$. Since $e' \in E \cap E_H$, the edge $e'$ is not incident with $u$. Thus, each vertex cover of $G'$ must contain either $u$ or $v$ or both. Since $N_{G'}(v) \subseteq N_{G'}(u)$, we can easily transform each vertex cover for $G'$ into a vertex cover for $G'$ that contains $u$ and has the same size. Hence, such a minimum vertex cover for $G'$ gives us a minimum vertex cover for $G' \setminus u$ of size $m - 2$. Since $G' \setminus u$ is isomorphic to the graph $H$ after deleting $e'$, $H$ is $\alpha$-critical.

\begin{flushright}
\textbox{\textbullet}
\end{flushright}

Note that the observations on the size of a minimum vertex cover of the shrunked graphs presented in Theorem 5.15 also work for uncritical graphs. Thus, we get:

**Observation 5.16** Let $G = (V, E)$ be an undirected connected graph with minimum vertex cover size $m$. Let $e = \{u, v\} \in E$ be an edge of $G$ such that $u$ and $v$ are neighbor-equivalent. Then size of a minimum vertex cover of $H = G \setminus u$ is $m - 1$.  

12
Let \( \mathcal{C} \) denote the family of all cliques and let \( \mathcal{OC} \) denote the family of all cycles of odd length. For easier notion, we assume that a graph that consists of only a single node is in \( \mathcal{C} \) and in \( \mathcal{OC} \). Furthermore, we assume that a graph which consists of only two connected nodes is in \( \mathcal{C} \). Then, we have:

**Observation 5.17** \( \mathcal{C} \subseteq \Gamma^*_{\text{par}}(\mathcal{OC}) \) and \( \mathcal{C} = \Gamma^*_{\text{par}}(G_1) \) where \( G_1 = (\{v\}, \emptyset) \). Moreover, if \( G_2 = (\{u, v\}, \\{\{u, v\}\}) \) and if \( G_3 \) denotes the cycle of length three, then we have \( \mathcal{C} \setminus \{G_1\} = \Gamma^*_{\text{par}}(G_2) \) and \( \mathcal{C} \setminus \{G_1, G_2\} = \Gamma^*_{\text{par}}(G_3) \).

Thus, simple edges and cycles of length three are interesting candidates to start with a generation process for \( \alpha \)-critical graphs. Based on the following observation, one can show that the relation of neighbor-equivalence can be used to define equivalence classes of vertices.

**Observation 5.18** Given an undirected connected graph \( G = (V, E) \). Assume that \( u \) is neighbor-equivalent to \( v \) and to \( w \), then \( v \) is neighbor-equivalent to \( w \).

**Proof.** Since \( u \) is equivalent to \( v \) and to \( w \) the two nodes \( v \) and \( w \) have to be adjacent. Furthermore, by definition we have \( N_G(w) \setminus \{u, v\} = N_G(u) \setminus \{v, w\} = N_G(v) \setminus \{w, u\} \). Hence, \( N_G(w) \setminus \{v\} = N_G(v) \setminus \{w\} \) and therefore \( v \) is equivalent to \( w \).

**Definition 5.19** Let \( G = (V, E) \) be an undirected connected graph and \( u \in V \). By \([u]_{\text{par}}\) we denote the set of all vertices that are neighbor-equivalent to \( u \) plus \( u \) itself. We call the different sets \([u]_{\text{par}}\) of \( G \) the node equivalence classes of \( G \).

**Observation 5.20** Every cliques has only one node equivalence class. For \( i \geq 4 \) every cycle of length \( i \) has exactly \( i \) node equivalence classes.

From Observation 5.18 we conclude:

**Corollary 5.21** Let \( G = (V, E) \) be an undirected connected graph and \( u \in V \). Then for all \( v \in [u]_{\text{par}} \) we have \([u]_{\text{par}} = [v]_{\text{par}}\).

**Theorem 5.22** Let \( G \) be an undirected connected graph of \( k \) node equivalence classes. Then every graph in \( \Gamma^*_{\text{par}}(G) \) has exactly \( k \) node equivalence classes.

**Proof.** Since every graph \( G' \in \Gamma^*_{\text{par}}(G) \) can be generated by sequence of steps where in each step a new node is added (together with some edges connecting this node to the rest of the graph) to the graph, it sufficient to prove that such a step does not change the number equivalence classes. Thus, we have to analyze such a step:

Let \( G' = (V', E') \) be an arbitrary graph, \( u \) be a new node and \( V'' \subseteq V' \) be a subset of nodes such that for at least one vertex \( v \in V'' \) it holds that \( N_G(v) = V'' \cup \{v\} \). Then the new graph is given by \( G''' = (V' \cup \{u\}, E' \cup \{\{u, v\} | v \in V''\}) \).

From the construction it follows that \( u \) and \( v \) are equivalent in \( G''' \); thus, it remains to show that for any pair of nodes \( x, y \) of \( G' \) it holds that

1. if \( x \) and \( y \) are not equivalent in \( G' \) then they are not equivalent in \( G''' \), and
2. if \( x \) and \( y \) are equivalent in \( G' \) then they are equivalent in \( G''' \).

Since our step to extend \( G' \) only adds new edges to the graph that are incident with the new node, this step does not extend the neighborhood of an old node according to the node in \( V' \). Thus, if two nodes are not equivalent in \( G' \) then they are not equivalent in \( G''' \).

Now assume that \( x \) and \( y \) are equivalent in \( G' \). If the new node is a parallel extension of \( x \) then this extension adds the new node also to the neighborhood of \( y \). Thus, \( x \) and \( y \) remain equivalent in \( G''' \).

From Observation 5.20 and Theorem 5.22 we can directly conclude that:
Corollary 5.23 Let $G_i$ be a cycle of length $2i+1$ and let $k, m \geq 1$ with $k \neq m$ the $\Gamma_{par}(G_k) \cap \Gamma_{par}(G_m) = \emptyset$.

The analysis of the equivalence classes leads to the observation that the sets of graphs which can be generated by parallel extensions from two different cycles of different odd length $i, j \geq 3$ are always distinct. Thus, we have to use for a second type of extension that helps us to generate cycles of odd length. Consider the $\alpha$-critical graph of Figure 5.b) which cannot be generated by parallel extensions of a cycle. This graph yields a second type of extension, the splitting extension (see Definition 5.3).

Investigating the vertex covers of cycles of odd length, one can see that every minimum cover contains two adjacent vertices. Investigating bipartite graphs, one can see that this is not a general behaviour of all graphs. In the following we will investigate whether this behaviour will be preserved by parallel and splitting extensions:

Definition 5.24 An edge $e = \{u, v\}$ of a graph $G$ fulfills the double-cover condition if there exists a minimum vertex cover $U$ of $G$ with $u, v \in U$. A graph $G$ fulfills the double-cover condition if any edge of $G$ fulfills it.

Lemma 5.25 Let $G = (V, E)$ be an $\alpha$-critical graph such that for every edge $e = \{u, v\}$ there exists a minimum vertex cover $U$, such that both endpoints of $e$ are in the cover $U$. Then for every graph $G' = (V', E')$ $\in \Gamma_{split}(G) \cup \Gamma_{par}(G)$ and for every edge $e' = \{u', v'\}$ there exists a minimum vertex cover $U'$, such that both endpoints of $e'$ are in the cover $U'$.

Proof. Assume that $G' \in \Gamma_{split}(G)$ and assume that $G' = (V', E')$ can be generated by splitting $u'' \in V$, i.e. there exists a subset $F \subset N_G(u'')$ and $x, y \in V' \setminus V$ such that

$$
V' = V \cup \{x, y\} \quad \text{and} \quad E' = \{(u'', x), \{x, y\} \cup \{y, v''\}|v'' \in F\} \cup \{\{u'', v''\}|v'' \in F\}.
$$

For every edge $e$ let $U_e$ denote a minimum vertex cover of $G$ that contains both endpoints of $e$.

Now assume that $e \notin \{(u'', v'')|v'' \in F\}$. Since either $u''$ or all vertices in $F$ have to be in $U_e$, we can extend the cover $U_e$ to a vertex cover for $G'$ by adding either $x$ or $y$ to $U_e$. The resulting set is a minimum vertex cover for $G'$ that covers both endpoints of $e$.

Next assume that $e \in \{(u'', v'')|v'' \in F\}$. Since both nodes $u''$ and $v''$ are in $U_e$, we can extend the cover $U_e$ to a vertex cover for $G'$ by adding either $x$ or $y$ to $U_e$. Depending on the added node we get a minimum vertex cover for $G'$ that covers both endpoints of $\{u'', x\}$ or of $\{y, v''\}$ for all $v'' \in F$.

To get a minimum vertex cover that contains $x$ and $y$, let us focus on a minimum vertex cover $U$ for $G$ after deleting an edge $\{u'', v''\}$ for $v'' \in F$. Since $G$ is $\alpha$-critical, the maximal independent set of the resulting graph will be increased and includes both vertices $u''$ and $v''$. This implies that the minimum vertex cover $U$ will be reduced by 1 (compared to the minimum vertex cover of $G$) and neither contains $u''$ nor $v''$. Thus, adding $x$ and $y$ to $U$ gives us a minimum vertex cover for $G'$ that covers both endpoints of $\{x, y\}$.

Let us now focus on graphs $G' \in \Gamma_{par}(G)$. Assume that $G' = (V', E')$ and let $v \in V$ and $u \in V'$ such that $u$ is the parallel extension of $v$, i.e.

- $v$ and $u$ are neighbor-equivalent for $G'$,
- $u \notin V$ and $G = H \setminus u$.

Since every minimum vertex cover $U$ for $G$ can be transformed into a minimum vertex cover for $G'$ by adding $u$ to $U$ (see Theorem 5.11), the claim holds for all edges of $G'$ that are already in $G$. Since $G$ fulfills the double cover condition, there exists a minimum vertex cover $U$ for each vertex $v'$ that contains $v'$. Since $U \cup \{u\}$ is a minimum vertex cover for $G'$, the new edge $\{u, v'\}$ fulfills the double cover property. ▶

This lemma implies:

Theorem 5.26 Let $\mathcal{G}$ be a family of $\alpha$-critical undirected graphs. If $\mathcal{G}$ neither contains the graph with a single node nor the graph with a single edge and if each graph of $\mathcal{G}$ fulfills the double-cover condition, then all graphs of $\Gamma_{split}(\mathcal{G})$ and $\Gamma_{par}(\mathcal{G})$ are $\alpha$-critical and fulfill the double-cover condition.
**Corollary 5.27** Let \( G \) be a family of \( \alpha \)-critical undirected graphs without the single-node graph and without the single-edge graph, then any graph of \( \Gamma_{\text{split}}^*(G) \) is \( \alpha \)-critical. If in addition all graphs of \( G \) fulfil the double-cover condition then all graphs of \( \Gamma_{\text{split}}^*(G) \) fulfil the double-cover condition.

To combine splitting and parallel extensions we define:

**Definition 5.28** Let \( G \) be a family of undirected graphs, then define

\[
\Gamma_{\text{split, par}}(G) = \Gamma_{\text{split}}(G) \cup \Gamma_{\text{par}}(G).
\]

Let \( \Gamma_{\text{split, par}}(G) \) denote the transitive closure of \( G \) according to \( \Gamma_{\text{split, par}} \). If we apply the extension \( k \) times, then the set of resulting graphs is denoted by \( \Gamma_{\text{split, par}}^k(G) \). If \( G \) consists of a single graph \( G \), we use the notions \( \Gamma_{\text{split, par}}(G) \) and \( \Gamma_{\text{split, par}}^k(G) \).

There are connected \( \alpha \)-critical graphs that cannot be generated by splitting and parallel extensions from a cycle of length three. An example is illustrated in Figure 6.

![Figure 6: A \( \alpha \)-critical graph that cannot be generated by splitting and parallel extensions from a cycle of length three.](image)

For our experimental benchmark test we have restricted ourselves to a restricted version of our the splitting operations. More general experimental benchmark tests will appear within future work.

**Definition 5.29 (Chain Extension)** Let \( G = (V, E) \) be an undirected connected graph and let \( e = \{u, v\} \in E \) be an edge of \( G \). Let \( x, y \not\in V \) denote two new nodes. Then the graph \( G' = (V \cup \{x, y\}, \{\{u, x\}, \{x, y\}, \{y, v\}\} \cup E \setminus \{e\}) \) is called a chain extension of \( G \).

Let \( G \) be a family of undirected graphs, then define \( \Gamma_{\text{chain}}(G) \) to be the set of graphs that can be generated from any graph \( G = (V, E) \in G \) by one chain extension. Let \( \Gamma_{\text{chain}}^k(G) \) denote the transitive closure of \( G \) according to \( \Gamma_{\text{chain}} \). If we apply the extension \( k \) times, then the set of resulting graphs is denoted by \( \Gamma_{\text{chain}}^k(G) \). If \( G \) consists of a single graph \( G \), then we use the notions \( \Gamma_{\text{chain}}(G) \) and \( \Gamma_{\text{chain}}^k(G) \). Analogously we define \( \Gamma_{\text{chain, par}}(G), \Gamma_{\text{chain, par}}^k(G), \Gamma_{\text{chain, par}}^k(G), \Gamma_{\text{chain, par}}(G), \Gamma_{\text{chain, par}}^k(G), \Gamma_{\text{chain, par}}^k(G) \).

Since the splitting operation is a generalization of the chain extension, we can conclude that the chain extension also preserves the \( \alpha \)-criticality of a graph and the knowledge of known minimum vertex covers and maximum independent sets. Furthermore, we can conclude that these properties are also preserved for the inverse chain extension, i.e. for the corresponding shrinkage operation. For the shrinkage of uncritical we can show a similar result. This result follows directly from the following proof the properties of shrinkage of \( \alpha \)-critical graphs:

**Theorem 5.30** Let \( G = (V, E) \) be an \( \alpha \)-critical connected graph and let \( x, y \) be two adjacent nodes of \( G \) of degree 2. Furthermore, let \( u, v \) denote the two remaining neighbors of \( x \) and \( y \) then the graph \( H = (V \setminus \{x, y\}, \{\{u, v\}\} \cup E \setminus (\{x, y\} \times V)) \) is \( \alpha \)-critical. The size of a minimum vertex cover of \( H \) is reduced by 1 according to the minimum vertex cover of \( G \).

**Proof.** Let us start to show the second claim. Let \( m \) denote the size of a minimum vertex cover of \( G \). By contradiction assume that there exists minimum vertex cover \( U \) for \( H = (V_H, E_H) \) of size \( m - 2 \). Since \( u \) and \( v \) are adjacent, we can transform this vertex cover of \( H \) into a vertex cover for \( G \) by either adding \( x \) or \( y \) to \( U \). The resulting vertex cover has size \( m - 1 \) – contradicting our assumption the exists minimum vertex cover size of \( G \) is \( m \). Thus, the minimum vertex cover size is at least \( m - 1 \). On the other hand every vertex cover of \( G \) can easily be transformed into a vertex cover for \( H \), where the size of the cover is reduced by one.

To show that \( H \) is \( \alpha \)-critical if \( G \) is \( \alpha \)-critical, we have to consider two cases:
1. the deleted edge occurs in both graphs, i.e. \( e' \in E \cap E_H \),
2. the deleted edge \( e' \in E \) only occurs in \( E \).

In both cases we transform the minimum vertex cover of \( G \) after deleting \( e' \in E \) into a minimum vertex cover of \( H \) after deleting either \( e' \) or, if \( e' \not\in E_H \), after deleting \( \{u, v\} \).

W.l.o.g., assume that \( u \) and \( x \) are adjacent.

We start with the first case: Since \( G \) is \( \alpha \)-critical, the size of the minimum vertex cover of \( G' = (V, E \setminus \{e'\}) \) is \( m - 1 \). Since \( e' \in E \cap E_H \) the edge \( e' \) is not incident with \( x \) or \( y \). Thus, each vertex cover of \( G' \) must contain at least two of the vertices \( u, v, x, y \), and at least one of the vertices \( x, y \). Thus, every the minimum vertex cover \( U \) of \( G' \) can be transformed into a minimum vertex cover of \( H' = (V_H, E_H \setminus \{e'\}) \) such that

- if the cover of \( G' \) contains either \( u \) or \( v \), then a cover of \( H' \) is given by \( U \setminus \{x, y\} \),
- if the cover of \( G' \) contains neither \( u \) nor \( v \), then it has to contain \( x \) and \( y \) and a cover of \( H' \) is given by \( \{u\} \cup U \setminus \{x, y\} \).

In both cases the size of the constructed cover for \( H' \) is \( m - 2 \).

Now, let us consider the case, that the deleted edge \( e' \) only occurs in \( E \), but not in \( E_H \). For this case it suffice, if we restrict our selves to the case where \( e' = \{u, x\} \). As in the previous case the size of the minimum vertex cover of \( G' = (V, E \setminus \{e'\}) \) is \( m - 1 \). Furthermore, each minimum vertex cover of \( G' \) either contains \( x \) or \( y \). Thus, this vertex cover gives us directly a vertex cover for \( H' = (V_H, E_H \setminus \{\{u, v\}\}) \) of size \( m - 2 \).

Summarizing, our two cases implies that the vertex covers of \( G' \) of size \( m - 1 \) can be transformed into a vertex cover of \( H' \) of size \( m - 2 \). Since our analysis takes every edge of \( H \) into account, \( H \) is \( \alpha \)-critical.

Note that the observations on the size of a minimum vertex cover of the shrunk graphs in Theorem 5.30 also work for uncritical graphs. Thus, we get:

**Observation 5.31** Let \( G = (V, E) \) be an undirected connected graph with minimum vertex cover size \( m \). Let \( x, y \) be two adjacent nodes of \( G \) of degree 2 and let \( u, v \) denote the two remaining neighbors of \( x \) and \( y \). Then the size of a minimum vertex cover of the graph \( H' = (V \setminus \{x, y\}, \{\{u, v\}\} \cup E \setminus (\{x, y\} \times V)) \) is \( m - 1 \).

For generating our benchmark graphs, we focus on the set of base graphs \( \mathcal{B} \) that can be generated by parallel and chain extensions from a cycle of length three. To efficiently generate such graphs, we used several bounds on the number of edges according to the number of vertices and the size of a minimum vertex cover. For a given graph \( G = (V, E) \), we define \( n(G) = |V| \) for the number of vertices of \( G \), \( m(G) = |E| \) for the number of edges of \( G \), \( c(G) \) for the size of the minimum vertex cover of \( G \), and \( \pi(G) = n(G) - c(G) \) for the number of vertices not in the minimum cover.

In the following we will focus on some bounds for the number of edges \( m(G) \) of a graph \( G \), if the number of vertices \( n(G) \) and the minimum cover size \( c(G) \) are given. We will first establish some general bounds for \( m(G) \) according to \( n(G) \) and \( c(G) \). Since there are no edges between two vertices that are not in the cover, we can conclude:

**Observation 5.32** For every graph \( G \), it holds that

\[
m(G) \leq \frac{c(G) \cdot (c(G) - 1)}{2} + c(G) \cdot (n(G) - c(G))
= c(G) \cdot \left( n(G) - \frac{c(G) + 1}{2} \right).
\]

Now we would like to establish a lower bound for the number of edges \( m(G) \) for a graph if the number \( n(G) \) of vertices of \( G \) and the size of the minimum vertex cover \( c(G) \) is given.

For a clique \( K_i \) of \( i \) vertices let \( k_i = \frac{i(i - 1)}{2} \) denote the number of edges of \( K_i \).

Given two vectors \( \vec{u} = (u_1, \ldots, u_t) \) and \( \vec{v} = (v_1, \ldots, v_t) \), then we call \( \vec{u} \) lexicographically smaller than \( \vec{v} \), denoted by \( \vec{u} \leq_{\text{lex}} \vec{v} \), if either both vectors are equal or if there exists an index \( i \in \{1, \ldots, t\} \) such that \( u_i < v_i \) and \( u_j = v_j \) for all \( j \in \{1, \ldots, i - 1\} \).
Definition 5.33 Given two values \( n \) and \( c \) with \( n > c \) define \( \text{CoV}(n,c) \) denote the set of vector \( \vec{\alpha} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \) such that
\[
\begin{align*}
    n &= \sum_{i=1}^{n} i \cdot \alpha_i \quad \text{and} \quad c = \sum_{i=1}^{n} (i-1) \cdot \alpha_i .
\end{align*}
\]

Let us analyze the vectors \( \vec{\alpha} \in \text{CoV}(n,c) \). We can easily see that:

- \( n - c = \sum_{i=1}^{n} \alpha_i \); thus, the sum of all entries of the vectors in \( \text{CoV}(n,c) \) is determined by the parameters \( n \) and \( c \).

- We will use the following interpretation of the vectors \( \vec{\alpha} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \in \text{CoV}(n,c) \): To generate a graph of \( n = n(G) \) vertices and of minimum vertex cover size \( c = c(G) \) we can use a collection of cliques, where for \( \alpha_i \) determines the number cliques of size \( i \).

- If \( \vec{\alpha} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \) is a solution for
\[
\begin{align*}
    n &= \sum_{i=1}^{n} i \cdot \alpha_i \quad \text{and} \quad c = \sum_{i=1}^{n} (i-1) \cdot \alpha_i .
\end{align*}
\]
and if \( \alpha_i > 0 \) and \( \alpha_j > 0 \) \( (i \neq j + 1) \) then also the vector \( \vec{\alpha}_{i,j} = (\alpha'_n, \alpha'_{n-1}, \ldots, \alpha'_1) \) with
\[
\alpha'_{k} = \begin{cases} 
    \alpha_{k} & \text{for } k \notin \{i, i-1, j+1, j\} \\
    \alpha_{k} - 1 & \text{for } k \in \{i, j\} \\
    \alpha_{k} + 1 & \text{for } k \in \{i-1, j+1\}
\end{cases}
\]
is a solution for the two equations.

- If the vector \( \vec{\alpha}_{i,j} \) is generated form \( \vec{\alpha} \) as described above and if \( i > j + 1 \), then
\[
\begin{align*}
    \sum_{k=1}^{n} \frac{k \cdot (k-1)}{2} \cdot \alpha'_k \\
    &= - \frac{i \cdot (i-1)}{2} + \frac{(i-1) \cdot (i-2)}{2} + \frac{(j+1) \cdot j}{2} \\
    &\quad - \frac{j \cdot (j-1)}{2} + \sum_{k=1}^{n} \frac{k \cdot (k-1)}{2} \cdot \alpha_k \\
    &= - \frac{(i-i+2) \cdot (i-1)}{2} + \frac{(j+1-j+1) \cdot j}{2} \\
    &\quad + \sum_{k=1}^{n} \frac{k \cdot (k-1)}{2} \cdot \alpha_k \\
    &= -i + 1 + j + \sum_{k=1}^{n} \frac{k \cdot (k-1)}{2} \cdot \alpha_k \\
    &< \sum_{k=1}^{n} \frac{k \cdot (k-1)}{2} \cdot \alpha_k \\
\end{align*}
\]
Thus, the corresponding value of the sum of edges in the collection of cliques is smaller for \( \vec{\alpha}_{i,j} \) than for \( \vec{\alpha} \) if \( i > j + 1 \).

- Hence, the lexicographical minimum vector results in a collection of cliques with the minimum number of edges.

- Moreover the lexicographical minimum vector includes at most two positions \( i, i+1 \) where the entries are greater than 0 (the values on all other positions are 0).
Lemma 5.34 [Lower-Bound Lemma] Let $G = (V, E)$ be a graph generated according to Definition 3.1 where $B$ is given by the graphs in $\Gamma_{\text{chain, par}}^*(K_3) \cup \{K_1, K_2\}$. Let $(\alpha_n(G), \alpha_{n(G)-1}, \ldots, \alpha_1) \in \mathbb{N}^{n(G)}$ denote the lexicographically minimal vector that fulfills the following two equations

$$n(G) = \sum_{i=1}^{n(G)} i \cdot \alpha_i \quad \text{and} \quad c(G) = \sum_{i=1}^{n(G)} (i-1) \cdot \alpha_i .$$

Then, we have

$$m(G) \geq \sum_{i=1}^{n(G)} \frac{i \cdot (i-1)}{2} \cdot \alpha_i .$$

Proof. Let $G = (V, E)$ be a graph generated according to Definition 3.1 where $B$ is given by the graphs in $\Gamma_{\text{chain, par}}^*(K_3) \cup \{K_1, K_2\}$. Furthermore, assume that $G$ is not $\alpha$-critical. Then there exists at least one edge that has been added to $G$ by the process $G_{\alpha}^m(\cdot)$, such that removing this edge from $G$ does not reduce the minimum size of a cover and the number of vertices of the resulting graph $G'$. Thus, the lower bound on the number of edges of this claim for $G$ and $G'$ are equal. In the following we will assume that $G$ is $\alpha$-critical and generated by according to Definition 3.1 where $B$ is given by the graphs in $\Gamma_{\text{chain, par}}^*(K_3) \cup \{K_1, K_2\}$.

Note that if the connected components of $G$ are cliques, then these components determine a vector from CoV$(n, c)$. Thus, the claim follows directly.

Now consider the case that not all the connected components of $G$ are cliques and assume that for the lexicographically minimal vector $\vec{\alpha} \in \text{CoV}(n(G), c(G))$ we have

$$m(G) < \sum_{i=1}^{n(G)} \frac{i \cdot (i-1)}{2} \cdot \alpha_i .$$

By focusing on the connected components $G'$ of $G$, we can conclude that there has to exist such a component $G' \in \Gamma_{\text{chain, par}}^*(K_3)$ which is not a clique and that for the lexicographically minimal vector $\vec{\alpha}' \in \text{CoV}(n(G'), c(G'))$ we have

$$m(G') < \sum_{i=1}^{n(G')} \frac{i \cdot (i-1)}{2} \cdot \alpha_i'.$$

We will now show that such a graph $G'$ does not exist.

For the contrary, assume that such a connected $\alpha$-critical graph exists in $\Gamma_{\text{chain, par}}^*(K_3)$ and that $G'$ denotes such a graph of minimum size.

Since $G'$ is not a clique and since there exists a sequence of graph extensions $\Gamma_{\text{chain}}(\cdot)$, $\Gamma_{\text{par}}(\cdot)$ for generation $G'$ from $K_3$, this sequence contains at least one chain extensions $\Gamma_{\text{chain}}(\cdot)$. Let us fix such a sequence $\Gamma_1, \ldots, \Gamma_k$ such that $G'' = \Gamma_k(\cdots \Gamma_1(K_3) \cdots)$.

One can even show that this sequence contains exactly $n(G') = c(G') - 1$ chain extensions $\Gamma_{\text{chain}}(\cdot)$.

Let $G'_i = \Gamma_i(\cdots \Gamma_1(K_3) \cdots)$ denote the intermediate graphs on the construction of $G'$. And assume that $\Gamma_t$ had been the last chain extensions within the sequence $\Gamma_1, \ldots, \Gamma_k$, i.e. all $\Gamma_h$ with $\ell < h \leq k$ are parallel extensions.

We will now compare $G'$ and $G'' = \Gamma_k(\cdots \Gamma_{t+1}(G'_{t-1} + K_2) \cdots)$ where $G'_{t-1} + K_2$ denotes the graph $G'_{t-1}$ plus an separate component $K_2$ where the two nodes of $K_2$ will have the same name labels than the two new nodes that are added to $G'_{t-1}$ by performing the chain extension $\Gamma_t$. Since $\Gamma_h$ with $\ell < h \leq k$ are parallel extensions we can conclude that

$$n(G') - c(G') = n(G'_t) - c(G'_t)$$

$$= n(G'_{t-1}) - c(G'_{t-1}) - 1$$

$$= n(G'_{t-1} + K_2) - c(G'_{t-1} + K_2)$$

$$= n(G'') - c(G'').$$
Since \( n(G') = n(G'') \) we have \( c(G') = c(G'') \). Furthermore, since \( \Gamma \) with \( \ell < h \leq k \) are parallel extensions we can conclude that \( G'' \) consists of two connected components, where at least the components that has been constructed form \( K_2 \) is a clique.

Let \( G''_i = \Gamma_i(\cdots \Gamma_{\ell+1}(G'_{\ell-1} + K_2\cdots) \) denote the intermediate graphs on the construction of \( G'' \) after step \( \ell \) (the addition of \( K_2 \)). By induction one can show that for every parallel extension \( \Gamma_{\ell+j} (1 \leq j \leq k - \ell) \) the degree of the duplicated vertex of \( G''_{\ell+j-1} \) is smaller or equal than the degree of the corresponding duplicated vertex of \( G'_{\ell+j-1} \). Hence, \( m(G'') \leq m(G') \) and therefore

\[
m(G'') < m(G') < \sum_{i=1}^{n(G')} \frac{i \cdot (i - 1)}{2} \cdot \alpha'_i.
\]

Again, by focusing on the connected components \( G'' \) of \( G'' \), we can conclude that there has to exist such a \( G''' = \Gamma^*_n, \text{par}(K_3) \) which is not a clique and that for the lexicographically minimal vector \( \alpha''' \in \text{CoV}(n(G'''), c(G''')) \) we have

\[
m(G''') \leq \sum_{i=1}^{n(G''')} \frac{i \cdot (i - 1)}{2} \cdot \alpha'''.
\]

Since \( G''' \) is a proper subgraph of \( G'' \) we can conclude that the size of \( G''' \) is strictly smaller than size of \( G' \) – a contradiction to our choice of \( G' \).

We conclude this section with the observation that the vectors \( \alpha = (\alpha_n, \ldots, \alpha_1) \) fulfill

\[
n = \sum_{i=1}^{n} i \cdot \alpha_i \quad \text{and} \quad c = \sum_{i=1}^{n} (i - 1) \cdot \alpha_i.
\]

One can show that if for \( \alpha = (\alpha_n, \ldots, \alpha_1) \) there are two indices \( i, j \) with \( \alpha_i, \alpha_j > 0 \) and \( i \notin \{j, j + 1\} \) then also the vector \( \alpha_{i,j} = (\alpha'_n, \ldots, \alpha'_1) \) with

\[
\alpha'_{k} = \begin{cases} \alpha_k & \text{for } k \notin \{i, i - 1, j + 1, j\} \\ \alpha_k - 1 & \text{for } k \in \{i, j\} \\ \alpha_k + 1 & \text{for } k \in \{i - 1, j + 1\} \end{cases}
\]

is a solution for the two equations on \( c \) and \( n \). On the other hand, if \( i > j + 1 \), then

\[
\sum_{k=1}^{n} \frac{k \cdot (k - 1)}{2} \cdot \alpha'_{k} < \sum_{k=1}^{n} \frac{k \cdot (k - 1)}{2} \cdot \alpha_{k}.
\]

Thus, we can conclude that the lexicographical minimal vector \( \alpha \) as at most two non-zero entries. And if it has two non-zero entries, then one can find these entries at two consecutive positions \( h, h + 1 \). Hence, the lexicographical minimal vector can be determined as follows

\[
\text{Let } h = \left\lceil c/(n - c) \right\rceil + 1 \text{ and } g = \left\lceil c/(n - c) \right\rceil + 1. \text{ If } h = g \text{ choose } \alpha_k = (n - c) \text{ and } \alpha_i = 0 \text{ for all } i \neq h. \text{ If } h \neq g \text{ choose } \alpha_k = c \text{ mod } (n - c), \alpha_g = n - c - \alpha_h, \text{ and } \alpha_i = 0 \text{ for all } i \notin \{h, g\}.
\]

Note that a cycle of odd length can be generated by a sequence of chain extensions of an \( K_3 \).

For our graph generator we use a process that generates random graphs from \( \Gamma^*_\text{chain, par}(K_3) \) for \( G_{B, \ell, m, n} \). Since it might be possible that this process generates huge components, such that these components can not be combined with earlier chosen components from \( \Gamma^*_\text{chain, par}(K_3) \) it will be useful to see whether an parallel or an chain extension of an interim graph leads to a component that cannot used within the further construction process.

One idea for such a decision process will be to investigate the vectors \( (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \) for the remaining unused vertices after each chain or parallel extension \( \Gamma_{\text{chain}} \) or \( \Gamma_{\text{par}} \). In the following we will investigate the changes of \( (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \) after such a step. This will allow us to determine the modified bound after each extension very efficiently.
This section shows the results of an exemplary test on a state-of-the-art heuristic and Naive Greedy as reference on version of the randomized graph generator described in Definition 3.1. By generating several graphs, we could verify that choosing two \( \alpha \) gives us the intuition that hard graphs can be found if we choose the cover size so-generated graphs can have the potential use of serving as benchmarks for challenging current and future algorithms.

One application of our \( \alpha \)-critical graphs and generation processes based on them is the construction of graphs that are difficult to solve for algorithms focused on graph problems, e.g. the minimum vertex cover problem. Therefore, \( \alpha \) application example: generating hard instances for benchmarks

6 Application Example: Generating Hard Instances for Benchmarks

Note that a chain extension reduces the total set of unused available vertices by 2 and the set of unused available vertices within the cover by 1. This change results in the following modification of the vector \( \alpha \): Let \( h \) be the maximum value of \( i \) such that \( \alpha_i > 0 \), then one can compute the modified vector \( \alpha' = (\alpha'_1, \alpha'_{n-1}, \ldots, \alpha'_1) \) by

\[
\alpha'_k = \begin{cases} 
\alpha_k & \text{for } k \in \{h, h-1\} \\
\alpha_k - 1 & \text{for } k = h \\
\alpha_k + 1 & \text{for } k = h - 1.
\end{cases}
\]

Note that a chain extension reduces the total set of unused available vertices by 2 and the set of unused available vertices within the cover by 1. This change results in the following modification of the vector \( \alpha \):

- Let \( h \) be the maximum value of \( i \) such that \( \alpha_i > 0 \) and let \( g \) be the minimum value of \( i \) such that \( \alpha_i > 0 \). Note that either \( h = g \) or \( h = g + 1 \).

- To compute the modified vector \( \alpha' = (\alpha'_1, \alpha'_{n-1}, \ldots, \alpha'_1) \) we have to reduce \( \alpha_g \) by 1 and shift \( g - 2 \) ones to a higher position. This can be done by the following algorithm:
  1. Choose the temporary values \( t_i = \alpha_i \) for all \( i \neq g \), and \( t_g = \alpha_g - 1 \).
  2. Set \( s = g - 2 \).
  3. If \( t_g > s \) set \( t_g = t_g - s \), \( t_{g+1} = t_{g+1} + s \), \( s = 0 \).
  4. If \( t_g \leq s \) set \( t_{g+1} = t_{g+1} + t_g \), \( t_g = 0 \), \( s = s - t_g \), \( g = g + 1 \).
  5. If \( g = h \) and \( s \geq g > 0 \) set \( d = \lfloor s/t_g \rfloor \), \( t_{g+d} = t_g \), \( t_g = 0 \), \( s = s - d \cdot t_g \), \( g = g + d \), \( h = h + d \).
  6. If \( g = h \) and \( 0 < s < t_g \) set \( t_{g+1} = s \), \( t_g = t_g - s \), \( h = h + 1 \), \( s = 0 \).
  7. Finally set \( \alpha'_k = t_i \) for all \( i \).

- Since during this computation at most two temporary variables \( t_k \) are greater than 0, we can implement the functionality of this algorithm by using only 6 variables.

Since there are at most 2 entries of \( \vec{\alpha} \) (and \( \vec{\alpha}' \)) that are different from 0, we only need to store these two values and the corresponding coordinates. Thus, four variables suffice to implement the lexicographical minimum vector \( \vec{\alpha} \).

6 Application Example: Generating Hard Instances for Benchmarks

One application of our \( \alpha \)-critical graphs and generation processes based on them is the construction of graphs that are difficult to solve for algorithms focused on graph problems, e.g. the minimum vertex cover problem. Therefore, so-generated graphs can have the potential use of serving as benchmarks for challenging current and future algorithms. This section shows the results of an exemplary test on a state-of-the-art heuristic and Naive Greedy as reference on our graphs.

Generating Hard Instances. To find graphs that are hard to solve within acceptable time, we used a restricted version of the randomized graph generator described in Definition 3.1. By generating several graphs, we could verify that choosing two \( \alpha \)-critical graphs within the first phase by \( G_{B,\ell,m,n}^k \) is sufficient for finding hard graphs. Moreover, our experiments led us to the intuition that hard graphs can be found if we choose the cover size \( \ell \) a little bit larger than \( n/2 \). Our results for the modification of \( G_{B,\ell,m,n}^k \) are given in the following:

- We generate two randomized \( \alpha \)-critical graphs from \( B = \Gamma_{\text{chain, par}}(K_3) \), such that each of their accumulated minimal vertex cover size is approximatively \( n/4 \) (the process generates such a graph with minimum cover size of \( n/4 \) and stops its extension randomly).
- Thereupon, we add the remaining vertices to get a graph of size \( n \).
- Then, we add random edges to obtain a graph with the desired number of edges, ensuring that no two vertices that are not in the optimal cover share an edge.
- Finally, we permute the order of the vertices.
Throughout all steps, we keep track of one optimal vertex cover of the graph.

We analyzed the performance of the updated version of NuMVC (v2015.8), where we used the original code provided by its authors.²

| n(G) | k | Avg. Opt. | #Opt. | Avg. Max. | #Opt. | Avg. Max. |
|------|---|-----------|-------|-----------|-------|-----------|
| 1500 | 1 | 1.1       | 752.7 | 10        | 0     | 24.3      | 28       |
| 1500 | 1.3 | 811.1 | 10 | 0 | 0 | 10.5 | 25 |
| 1500 | 1.5 | 834.9 | 9 | 0 | 0 | 26.9 | 51 |
| 1500 | 1.7 | 793.1 | 9 | 11.3 | 113 | 0 | 154.6 | 188 |
| 1500 | 1.9 | 918.3 | 10 | 0 | 0 | 10 | 0 |
| 2500 | 1.1 | 1270.6 | 10 | 0 | 0 | 41.0 | 52 |
| 2500 | 1.3 | 1372.6 | 10 | 0 | 0 | 28.0 | 54 |
| 2500 | 1.5 | 1363.1 | 9 | 0.1 | 1 | 51.1 | 87 |
| 2500 | 1.7 | 1431.7 | 8 | 29.4 | 175 | 0 | 176.0 | 264 |
| 2500 | 1.9 | 1418.8 | 10 | 0 | 0 | 10 | 0 |
| 3500 | 1.1 | 1797.7 | 10 | 0 | 0 | 59.9 | 91 |
| 3500 | 1.3 | 1884.0 | 2 | 5.4 | 16 | 0 | 38.9 | 56 |
| 3500 | 1.5 | 1975.5 | 10 | 0 | 0 | 62.3 | 96 |
| 3500 | 1.7 | 2009.6 | 4 | 154.0 | 203 | 0 | 189.1 | 336 |
| 3500 | 1.9 | 1955.8 | 10 | 0 | 0 | 10 | 0 |

Table 1: Results for running NuMVC and Naive Greedy for a maximum of 1500 seconds per run on graphs constructed with our approach. \( n(G) \) and \( m(G) = n^k \) denote the number of vertices and edges, respectively. We constructed ten graphs for each combination of \( n \) and \( k \). #Opt. denotes the number of runs where the respective algorithm found the optimal solution. Avg. and Max. distance denote the average and maximal number of vertices above the optimum, respectively.

Results for NuMVC. Tables 1 summarizes our experimental results for NuMVC for varying number of vertices \( n \) and number of edges \( m \), where \( m = n^k \) is determined by a polynomial function on \( n \). The table presents the quality of the found vertex cover, where the column Avg. Opt. provides the average optimum cover size, #Opt. the number of instances for which NuMVC found an optimal solution, and the remaining two columns the average and maximal distance between the vertex cover found by NuMVC and the minimum vertex cover. One can observe that the worst behavior of NuMVC take place for \( k = 1.7 \). For sparse or dense graphs, i.e. \( k = 1.1 \) or \( k = 1.9 \), NuMVC always found an optimal solution. Our results lead to the observation that choosing \( k \) between 1.5 and 1.7 leads to hard instances.

Results for Greedy. We also studied the Naive-Greedy algorithm on our graphs, which always adds a vertex to its resulting cover from those with a maximum degree within the remaining graph. The experiments show that the Naive-Greedy algorithm can determine an optimal solution for dense graphs. As for NuMVC graphs, the choice of \( k = 1.7 \) leads to graphs that seem to be infeasible to solve. So, one can assume that the error of NuMVC results from that of the initial cover generated by its internally used greedy algorithm.

Comparison with Witzel Graphs. The effectiveness of a random benchmark generation process also relies on its frequency of producing hard instances. One could, e. g., preselect some vertices \( C \) and add edges randomly among them and from them to remaining vertices – ensuring that every vertex of \( C \) shares an edge with at least one vertex in \( C \). This process is similar to that discussed above: though, the probability that the generated instance is hard is low and unpredictable. We experimentally constructed 50 instances from this process. None of them appeared hard to solve for NuMVC, i. e. in all cases, it required a single step to find a solution. It is hard to determine if NuMVC always found an optimal cover, but it is at least highly likely; one can see from Table 1 that, whenever NuMVC took a single step to solve an instance, it always returned an optimal solution. We compared our generation process to that of Witzel graphs, which start from \( n \) cliques of \( x \) vertices each and “[…] connect random pairs of cliques by random edges” ², which resembles the BHOSLIB process ¹⁹. For the parameters used to generate BHOSLIB Graphs see ¹⁸.

We executed NuMVC for 1500 seconds over 30 random instances consisting of 4000 vertices each, generated by the Witzel Graph process, the BHOSLIB process, and our benchmark generator. For the construction of Witzel graphs, we chose 100 cliques of 40 vertices each, and added edges randomly such that their total number was located between 550000 and 850000. Analogously, we also chose a number of \( m \) in \([550000, 850000]\) random edges for our benchmark.

²http://lcs.ios.ac.cn/~caisw/MVC.html
generator. For the BHOSLIB process, we chose $n = 100$, $\alpha = 0.8$, $r = 0.8/(\ln(4) - \ln(3)) \approx 2.7808$, and $p = 0.25$, which were used for the hardest instance frb-100-40 according to the BHOSLIB authors. NuMVC found an optimal solution for all 30 Witzel instances and 29 of the BHOSLIB graphs, whereas it returned a minimum cover in only 23 of the 30 cases of our benchmark generator. Detailed results can be seen in Table 2. In a future step, we will compare our generator with the BHOSLIB process.

| Generator         | Avg. Opt. VC | #Opt. | Avg. | Max. | Avg. #Steps | Avg. Time (s) |
|-------------------|--------------|-------|------|------|-------------|---------------|
| Our generator     | 2250.63      | 23    | 16.57| 102  | 8364265.5   | 118.37        |
| BHOSLIB Graphs    | 3900.00      | 29    | 3.00 | 3    | 11465347.9  | 398.22        |
| Witzel Graphs     | 3900.00      | 30    | 0.00 | 0    | 21821546.0  | 441.84        |

Table 2: Comparison of minimum vertex cover size and run time of NuMVC for a maximum of 1500 seconds on 30 graphs each from our graph generator, BHOSLIB Graphs, and Witzel Graphs.

**Generating Instances with Less Structure** Finally, we tackled the question whether the structure that is given by the $\alpha$-critical graphs is necessary for generating hard instances. Therefore, we generated random graphs by the following simplified generator $G_{n,m,n_c}$:

$G_{n,m,n_c}$ outputs a graph of $n$ vertices $V$ and $m$ edges $E$ such that for a subset $V_C \subseteq V$ of size $n_C$ the edges are uniformly chosen from the set of edges connecting a vertex of $V_C$ with an arbitrary other vertex of $V$ (see Figure 7).

Clearly, $n_C$ is an upper bound for the minimum vertex cover of graphs generated by $G_{n,m,n_c}$.

For our experiments, we chose $n = 3500$ and $m = n^{1.7}$ and generated ten random graphs for each value in $n_C \in \{2000, 2050, 2100, 2150, 2200\}$. For all generated graphs, NuMVC found a solution latest after a single improvement step that has been within the upper bound of $n_C$. We conclude from these results that using the structure of $\alpha$-critical graphs increases the probability of finding hard instances. Moreover, knowing the exact size of a cover is highly useful for determining the quality of the solutions determined by NuMVC.

**7 Conclusions**

This paper introduced the concept of $\alpha$-critical graphs for the vertex-cover problem, i.e., graphs which cannot be reduced wrt. their number of vertices without reducing the minimum vertex cover. As starting points, they allow to generate any graph within a random process such that we can determine the optimal cover size of the resulting construction. The random process can be parametrized by the desired number of vertices, cover size, as well as number of edges. We introduced a set of extensions that allow us to generate a sufficiently large subset of $\alpha$-critical graphs as the base for constructing graphs. We showed that our extensions allow to render the generation process very efficiently.

One specific application with apparent potential is the construction of graphs with hidden solutions that are difficult to solve. Our experiments with a recent heuristic NuMVC and Naive Greedy frequently underestimate the minimum vertex cover by more than 100 vertices for graphs generated with our process with 3500 vertices. By analyzing the
behavior of NuMVC and Naive Greedy on random graphs without a substructure like our \( \alpha \)-critical graphs, we could observe that these structures appear necessary for hard instances. We will continue this research and publish the hardest graphs at our website.

In general, \( \alpha \)-critical graphs are interesting objects to study to deepen the understanding of hard problems. Concerning our proposed generation process, if we were aware of all \( \alpha \)-critical instances, one could efficiently create all graphs from our process. Though, it seems that the task of finding all critical instances is far from complete. We have started the systematic search of small \( \alpha \)-critical instances and will continue to report on further findings in the close future.

Furthermore, we started analyzing the quality of \textit{CirculantGraphs} for benchmarking purposes. At an early stage, we found that instances with a small size of internal cliques were still easily solvable for local-search algorithms such as NuMVC, which shows that the verification of partial results is crucial. We plan to continue our research also in the direction of constructing larger \( \alpha \)-critical graphs from \textit{CirculantGraphs} and will also report on those findings.

\textbf{Acknowledgement:} We would like to thank Gwenaël Joret and Ke Xu for their constructive comments on an earlier version of the paper.

\section*{References}

[1] Shaowei Cai, Kaile Su, Chuan Luo, and Abdul Sattar. NuMVC: An Efficient Local Search Algorithm for Minimum Vertex Cover. \textit{J. Artif. Intell. Res. (JAIR)}, 46:687–716, 2013.

[2] Ashay Dharwadker. The Independent Set Algorithm, 2006. \url{http://www.dharwadker.org/independent_set/}, accessed 2015/09/17.

[3] Irit Dinur and Samuel Safra. On the Hardness of Approximating Minimum Vertex Cover. \textit{Annals of Mathematics}, pages 439–485, 2005.

[4] Paul Erdös and Tibor Gallai. On the minimal number of vertices representing the edges of a graph. \textit{Magyar Tudományos Akadémia; Matematikai Kultúrtételek Kőszényei}, A 6:181–203, 1961.

[5] M. R. Garey and David S. Johnson. \textit{Computers and Intractability: A Guide to the Theory of NP-Completeness}. W. H. Freeman, 1979.

[6] Gwenaël Joret. \textit{Entropy and Stability in Graphs}. PhD thesis, Université Libre de Bruxelles, Faculté des Sciences, 2007.

[7] George Karakostas. A better approximation ratio for the vertex cover problem. In \textit{ICALP}, volume 3580 of \textit{LNCS}, pages 1043–1050. Springer, 2005.

[8] Richard M. Karp. \textit{Reducibility Among Combinatorial Problems}. In \textit{Complexity of Computer Computations}, The IBM Research Symposia Series, pages 85–103, 1972.

[9] Vedat Kavalcı, Aybars Ural, and Orhan Dagdeviren. Distributed Vertex Cover Algorithms For Wireless Sensor Networks. \textit{Int. Journ. of Comp. Netw. & Communic.}, 6(1):95, 2014.

[10] László Lovász. \textit{Combinatorial Problems and Exercises (Second Edition)}. North-Holland Publishing Co., Amsterdam, 1993.

[11] László Lovász and Michael David Plummer. \textit{Matching Theory}, volume 121 of \textit{North-Holland Mathematics Studies}. North-Holland, 1986.

[12] Christos H. Papadimitriou and David Wolfe. The Complexity of Facets Resolved. \textit{Journal of Computer and System Science}, 37(1):2–13, 1988.

[13] Shariefuddin Pizizada and Ashay Dharwadker. Applications of Graph Theory. In \textit{KSIAM}, volume 11, pages 19–38, 2007.

[14] Barbara M. Smith and Martin E. Dyer. Locating the Phase Transition in Binary Constraint Satisfaction Problems. \textit{Artificial Intelligence}, 81(1-2):155–181, 1996.

[15] Ulrike Stege. \textit{Resolving Conflicts in Problems from Computational Biology}. PhD thesis, ETH Zurich, Institute of Scientific Computing, 2000.

[16] Walter Wessel. Kanten-kritische graphen mit der zusammenhangszahl 2. \textit{manuscripta mathematica}, 2(4):309–334, 1970.

[17] Ke Xu. BHOSLIB: Benchmarks with Hidden Optimum Solutions for Graph Problems, 2005. \url{http://www.nlsde.buaa.edu.cn/~kexu/benchmarks/graph-benchmarks.htm}.

[18] Ke Xu, Frédéric Boussemart, Fred Hemery, and Christophe Lecoutre. A Simple Model to Generate Hard Satisfiable Instances. In \textit{IJCAI}, pages 337–342, 2005.

[19] Ke Xu, Frédéric Boussemart, Fred Hemery, and Christophe Lecoutre. Random Constraint Satisfaction: Easy Generation of Hard (Satisfiable) Instances. \textit{Artif. Intell.}, 171(8-9):514–534, 2007.

[20] Ke Xu and Wei Li. Exact Phase Transitions in Random Constraint Satisfaction Problems. \textit{Journal of Artificial Intelligence Research (JAIR)}, 12:93–103, 2000.

[21] Ke Xu and Wei Li. Many Hard Examples in Exact Phase Transitions with Application to Generating Hard Satisfiable Instances. \textit{CoRR}, cs.CC/0302001, 2003.

[22] Ke Xu and Wei Li. Many hard examples in exact phase transitions. \textit{Theoretical Computer Science}, 355(3):291–302, 2006.
[23] Yong Zhang, Qi Ge, Rudolf Fleischer, Tao Jiang, and Hong Zhu. Approximating the Minimum Weight Weak Vertex Cover. *Theor. Comput. Sci. (TCS)*, 363(1):99–105, 2006.
## A Critical Graphs of Degree 6

| n  | Tuples \((i, j)\)                                      |
|----|-------------------------------------------------------|
| 4  | \((2, 3)\)                                            |
| 5  | \((2, 3), (2, 4), (3, 4)\)                           |
| 6  | \((2, 3), (3, 4)\)                                   |
| 7  | \((2, 3), (2, 4), (3, 5), (4, 5)\)                   |
| 8  | \((2, 6), (2, 7), (6, 7)\)                           |
| 10 | \((4, 5), (5, 6)\)                                   |
| 11 | \((2, 3), (2, 8), (2, 9), (2, 10), (3, 4), (3, 7), (3, 9), (4, 5), (4, 6), (4, 8), (5, 6), (5, 7), (5, 10), (6, 7), (6, 10), (7, 8), (8, 9), (9, 10)\) |
| 12 | \((3, 4), (3, 9), (4, 10), (5, 8), (5, 12), (8, 12), (9, 10)\) |
| 14 | \((2, 12), (2, 13), (6, 7), (7, 8), (12, 13)\)       |
| 15 | \((2, 3), (2, 12), (3, 13), (6, 7), (6, 8), (7, 9), (8, 9), (12, 13)\) |
| 17 | \((2, 6), (2, 11), (2, 15), (2, 16), (3, 6), (3, 8), (3, 9), (3, 11), (4, 13), (4, 16), (6, 14), (6, 15), (8, 9), (8, 14), (8, 16), (9, 14), (9, 16), (11, 14), (11, 15), (13, 16), (15, 16)\) |
| 18 | \((3, 4), (3, 14), (4, 15), (8, 9), (9, 10), (14, 15)\) |
| 19 | \((2, 3), (2, 16), (3, 17), (6, 7), (6, 12), (7, 8), (7, 11), (7, 13), (8, 9), (8, 10), (8, 12), (9, 11), (10, 11), (11, 12), (12, 13), (16, 17)\) |
| 20 | \((2, 18), (2, 19), (3, 4), (3, 16), (4, 17), (7, 8), (7, 12), (8, 13), (12, 13), (16, 17), (18, 19)\) |
| 30 | \((6, 7), (12, 13), (12, 17), (13, 18), (14, 15), (15, 16), (17, 18)\) |
| 40 | \((5, 12), (6, 7), (10, 14), (10, 18), (12, 18), (14, 15), (15, 16), (17, 18)\) |
| 41 | \((2, 18), (3, 4), (3, 15), (5, 14), (5, 19), (8, 11), (8, 12), (9, 16), (9, 20), (10, 11), (10, 12), (10, 13), (10, 18), (11, 12), (11, 13), (11, 16), (12, 13), (12, 16), (13, 18), (15, 16)\) |
| 42 | \((2, 12), (6, 10), (6, 19), (11, 18)\)               |
| 43 | \((2, 3), (2, 9), (5, 19), (6, 7), (8, 12), (14, 15), (15, 18), (16, 20)\) |
| 44 | \((5, 12), (6, 10), (8, 11), (8, 18), (9, 20)\)      |
| 45 | \((2, 6), (6, 7), (9, 10), (12, 13)\)                |
| 46 | \((3, 4), (5, 12), (6, 7), (9, 16), (10, 11), (13, 14), (14, 15), (16, 17), (18, 19)\) |
| 47 | \((2, 3), (5, 17), (5, 19), (6, 19), (8, 11), (9, 10), (9, 18), (10, 11), (13, 14), (14, 15), (15, 16), (15, 19), (17, 18)\) |
| 48 | \((3, 4)\)                                           |
| 49 | \((18, 19)\)                                         |
| 50 | \((3, 8), (5, 14), (8, 12), (9, 10), (10, 11), (13, 14), (14, 17)\) |

Table 3: Critical circulant graphs of degree 6. Within this table the value 1 is omitted within each entry.