Within the framework of metric-affine gravity (MAG, metric and an independent linear connection constitute spacetime), we find, for a specific gravitational Lagrangian and by using prolongation techniques, a stationary axially symmetric exact solution of the vacuum field equations. This black hole solution embodies a Kerr-deSitter metric and the post-Riemannian structures of torsion and nonmetricity. The solution is characterized by mass, angular momentum, and shear charge, the latter of which is a measure for violating Lorentz invariance.

Keywords: Metric-affine gravity, prolongation, exact solutions, Kerr-deSitter metric, torsion, nonmetricity

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1. Introduction

In the spirit of a gauge-theoretical approach to gravity, a metric-affine gauge field theory of gravitation (“metric-affine gravity” MAG) has been proposed based on the metric $g$ and the affine group $A(4, R)$, i.e., the semi-direct product of the four...
dimensional translational group $R^4$ and the general linear group $GL(4, R)$. Besides the usual “weak” Newton-Einstein type gravity, additional “strong gravity” pieces will arise that are supposed to be mediated by additional geometrical degrees of freedom related to the independent linear connection 1-form $\Gamma^\alpha_\beta = \Gamma^i_\alpha_\beta dx^i$. Here $\alpha, \beta, \ldots = 0, 1, 2, 3$ denote frame (or anholonomic) indices and $i, j, \ldots = 0, 1, 2, 3$ coordinate (or holonomic) indices. Alternatively, the strong gravity pieces can also be expressed in terms of the nonmetricity 1-form $Q^\alpha_\beta = Q^i_\alpha_\beta dx^i$ and the torsion 2-form $T^\alpha = \frac{1}{2} T_{ij}^\alpha dx^i \wedge dx^j$. The propagating modes related to the new degrees of freedom manifest themselves in post-Riemannian pieces of the curvature $R^\alpha_\beta$.

2. Geometrical structures of a metric-affine spacetime

We briefly summarize the basic notions of metric-affine geometry. Let us start from a n-dimensional differentiable manifold $M_n$. At each point $P \in M_n$ we can construct the n-dimensional tangent vector space $T_P(M_n)$ with vector basis (or frame) $e_\alpha$. In the space $T^*_P(M_n)$, dual to $T_P(M_n)$, we introduce a local one-form basis (or coframe) $\vartheta^\alpha$ such that

$$e_\alpha | \vartheta^\beta = \delta^\beta_\alpha,$$

where $|$ symbolizes the interior product. Generally, the coframe is not integrable, i.e., we have

$$C^\alpha := d\vartheta^\alpha = \frac{1}{2} C^\mu_\nu_\alpha \vartheta^\mu \wedge \vartheta^\nu \neq 0,$$

where the 1-form $C^\alpha$ is a measure of the anholonomity.

We assume that the manifold is endowed with a metric $g$. We decompose it with respect to the coframe $\vartheta^\alpha$ and find

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta.$$

Furthermore, we assume that the manifold $M_n$ carries additionally a metric-independent linear connection $\Gamma^\alpha_\beta$. Accordingly, nonmetricity and torsion emerge as geometrical field strengths, to be defined as

$$Q_{\alpha\beta} := -D g_{\alpha\beta}$$

and

$$T^\alpha := d\vartheta^\alpha + \Gamma^\alpha_\mu \wedge \vartheta^\mu = D \vartheta^\alpha,$$

respectively, together with the curvature 2-form

$$R^\alpha_\beta := d \Gamma^\alpha_\beta - \Gamma^\alpha_\mu \wedge \Gamma^\beta_\mu.$$

Here $D$ is the exterior covariant derivative with respect to the connection $\Gamma^\alpha_\beta$. 
The geometrical field strengths give rise to integrability conditions, namely to \textit{Bianchi} and \textit{Ricci} identities. We have, with $b$

\begin{align*}
DDg_{\alpha\beta} &= -DQ_{\alpha\beta} = -2R(\alpha^\mu g_\beta)_\mu = -2Z_{\alpha\beta}, \quad (7) \\
DD\vartheta^\alpha &= DT^\alpha = R^\alpha_\mu \wedge \vartheta^\mu, \quad (8) \\
DR_{\alpha\beta} &= 0; \quad (9) \\
DDT^\alpha &= (DR^\alpha_\mu) \wedge \vartheta^\mu + R^\alpha_\mu \wedge T^\mu = R^\alpha_\mu \wedge T^\mu, \quad (10) \\
DDQ_{\alpha\beta} &= -2R(\alpha^\mu \wedge Q_\beta)_\mu =: S_{\alpha\beta}, \quad (11) \\
DS_{\alpha\beta} &= DDDDQ_{\alpha\beta} = -4R(\alpha^\mu \wedge Z_\beta)_\mu. \quad (12)
\end{align*}

We can, in terms of the metric field of MAG, always construct a Riemannian (or Levi-Civita) connection. Therefore, for the purpose of a comparison with general relativity, e.g., it is useful to decompose the connection $\Gamma_{\alpha\beta} := \Gamma_{\alpha\mu}^\rho g^\rho_\beta$ into a Riemannian piece $\tilde{\Gamma}_{\alpha\beta}$ and a tensorial post-Riemannian piece $N_{\alpha\beta}$,

$$\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} + N_{\alpha\beta}. \quad (13)$$

The distortion 1-form $N_{\alpha\beta}$ allows us to recover nonmetricity and torsion according to

$$Q_{\alpha\beta} = 2N_{(\alpha\beta)}, \quad T^\alpha = N^\alpha_\beta \wedge \vartheta^\beta. \quad (14)$$

Explicitly, we have

$$N_{\alpha\beta} = -e_{[\alpha\beta]} T_\beta + \frac{1}{2}(e_\alpha e_\beta T_\gamma) \vartheta^\gamma + (e_{[\alpha} Q_{\beta]\gamma) \vartheta^\gamma + \frac{1}{2}Q_{\alpha\beta}, \quad (15)$$

see Ref.\textsuperscript{25}, Eq.(3.10.7).

In a metric-affine spacetime we can separate the curvature 2-form $R_{\alpha\beta}$ into a tracefree symmetric part (“shear”) $Z_{\alpha\beta}$, a trace part (“dilation”) $Z$, and an antisymmetric part (“rotation”) $W_{\alpha\beta}$ according to

$$R_{\alpha\beta} = R_{(\alpha\beta)} + R_{[\alpha\beta]} = Z_{\alpha\beta} + \frac{1}{4}Zg_{\alpha\beta} + W_{\alpha\beta}, \quad (16)$$

with the definitions

$$Z_{\alpha\beta} := R_{(\alpha\beta)}, \quad Z_{\alpha\beta} := Z_{\alpha\beta} - \frac{1}{4}Zg_{\alpha\beta}, \quad Z := Z^\alpha_\alpha, \quad W_{\alpha\beta} := R_{[\alpha\beta]}.$$

The symmetric part $Z_{\alpha\beta}$ represents the post-Riemann-Cartan part of the curvature, that is, it vanishes together with $Q_{\alpha\beta}$, whereas $W_{\alpha\beta}$ includes the Riemannian contributions, inter alia.

We now specialize to 4-dimensional spacetime with Lorentz signature $(-+++)$. Quite generally, nonmetricity $Q_{\alpha\beta}$, torsion $T^\alpha$, and curvature $R_{\alpha\beta}$ can then be split into smaller pieces, they can be decomposed irreducibly under the Lorentz group, $b$

Parentheses surrounding indices $(\alpha\beta) := (\alpha\beta + \beta\alpha)/2$ denote symmetrization and brackets $[\alpha\beta] := (\alpha\beta - \beta\alpha)/2$ antisymmetrization.
see Appendix A. In the following table, we will give, for \( n = 4 \), an overview of the number of independent components of these quantities:

| \( Q_{\alpha\beta}^{(1)} \) | \( Q_{\alpha\beta}^{(2)} \) | \( Q_{\alpha\beta}^{(3)} \) | \( Q_{\alpha\beta}^{(4)} \) | - | - | \( Q_{\alpha\beta} \) |
|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 16                     | 16                     | 4                      | 4                      | -                      | -                      | \( \Sigma = 40 \)      |
| \( T^\alpha \)         | \( T^\alpha \)         | \( T^\alpha \)         | \( T^\alpha \)         | -                      | -                      | \( \Sigma = 24 \)      |
| 16                     | 4                      | 4                      | -                      | -                      | -                      | \( \Sigma = 36 \)      |
| \( R_{[\alpha\beta]}^{(1)} \) | \( W_{\alpha\beta}^{(2)} \) | \( W_{\alpha\beta}^{(3)} \) | \( W_{\alpha\beta}^{(4)} \) | \( W_{\alpha\beta}^{(5)} \) | \( W_{\alpha\beta}^{(6)} \) | \( W_{\alpha\beta} \) |
| 10                     | 9                      | 1                      | 9                      | 6                      | 1                      | \( \Sigma = 60 \)      |
| \( R_{(\alpha\beta)}^{(1)} \) | \( Z_{\alpha\beta}^{(2)} \) | \( Z_{\alpha\beta}^{(3)} \) | \( Z_{\alpha\beta}^{(4)} \) | \( Z_{\alpha\beta}^{(5)} \) | -                      | \( Z_{\alpha\beta} \) |
| 30                     | 9                      | 6                      | 6                      | 9                      | -                      | \( \Sigma = 60 \)      |

The exterior products of the coframe \( \vartheta^\alpha \) are denoted by \( \vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta \), etc.. Since a metric is prescribed, we can define a Hodge star operator \( \ast \) which maps, in 4 dimensions, \( p \)-forms into \( (4-p) \)-forms. Then, we can introduce the eta-basis \( \eta := \ast 1 \), \( \eta^\alpha := \ast \vartheta^\alpha \), \( \eta^{\alpha\beta} := \ast \vartheta^{\alpha\beta} \), etc..

### 3. Lagrangian and field equations

We consider in the first order Lagrangian formalism the geometrical variables \( \{ g_{\alpha\beta}, \vartheta^\alpha, \Gamma^{\alpha\beta}_{\gamma} \} \) to be minimally coupled to matter fields, collectively called \( \Psi \), such that the total Lagrangian, i.e., the geometrical part plus the matter part, reads

\[
L_{\text{tot}} = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_{\alpha\beta}) + L_{\text{matter}}(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi) .
\] (18)

By using the excitations as place holders,

\[
M^{\alpha\beta} = -2 \frac{\partial V}{\partial Q_{\alpha\beta}}, \quad H_\alpha = - \frac{\partial V}{\partial T^\alpha}, \quad H^\alpha_\beta = - \frac{\partial V}{\partial R^\alpha_\beta},
\] (19)

the field equations of metric-affine gravity can be given in a very concise form:

\[
DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta},
\] (20)

\[
DH_\alpha - E_\alpha = \Sigma_\alpha,
\] (21)

\[
DH^\alpha_\beta - E^\alpha_\beta = \Delta^\alpha_\beta,
\] (22)

\[
\frac{\delta L}{\delta \Psi} = 0 \quad \text{(matter)}.
\] (23)

On the right-hand-sides of each of the three gauge field equations \( 20 \) to \( 22 \), there act the material currents as sources, on the left-hand-side there are typical Yang-Mills like terms governing the gauge fields, their first derivatives, and the...
Rotating black holes in metric-affine gravity

corresponding gauge field currents. The gauge currents turn out to be the metrical (Hilbert) energy-momentum of the gauge fields

\[ m^{\alpha \beta} := 2 \frac{\partial V}{\partial \eta_{\alpha \beta}} = g^{(\alpha \wedge E^{\beta})} + Q^{(\beta \wedge M^{\alpha \gamma} - T^{(\alpha \wedge H^{\beta})} - R^{\gamma (\alpha \wedge H^{\gamma \beta})} + R^{[\beta \gamma \wedge H^{\alpha \gamma}}}, \]  

(24)

the canonical (Noether) energy-momentum of the gauge fields

\[ E_\alpha := \frac{\partial V}{\partial \eta^{\alpha}} = e_\alpha [ V + (e_\alpha \lambda^3) \wedge H_\beta + (e_\alpha \lambda^2) \wedge H^2_{\gamma} + \frac{1}{2} (e_\alpha \lambda^1) Q_{\beta \gamma}] M^{\beta \gamma}, \]  

(25)

and the hypermomentum of the gauge fields

\[ E^{\alpha \beta} := \frac{\partial V}{\partial \Gamma^{\alpha \beta}} = - \partial^\alpha \wedge H_\beta - g_{\beta \gamma} M^{\alpha \gamma}, \]  

(26)

respectively.

The most general parity conserving quadratic Lagrangian, which is expressed in terms of the 4 + 3 + 6 + 5 irreducible pieces of \( Q_{\alpha \beta} \), \( T^\alpha \), \( W_\alpha \beta \), and \( Z_\alpha \beta \), respectively, reads

\[ V_{\text{MAG}} = \frac{1}{2 \kappa} \left[ -a_0 \alpha^{\alpha \beta} \wedge \eta_{\alpha \beta} - 2 \lambda_0 \eta \right. \]

\[ + \left. T^\alpha \wedge \star \left( \sum_{i=1}^{3} a_1 \alpha_i \right) + Q_{\alpha \beta} \wedge \star \left( \sum_{i=1}^{4} b_1 \alpha_i \right) \right] \]

\[ + 2 \left( \sum_{i=2}^{4} c_1 \alpha_i \right) \wedge \partial^\alpha \wedge \star T^\beta + b_5 \left( \sum_{i=3}^{5} Q_{\alpha \gamma} \wedge \partial^\alpha \right) \wedge \star \left( \sum_{i=4}^{5} Q_{\beta \gamma} \wedge \partial^\beta \right) \]

\[ - \frac{1}{2 \rho} R_{\alpha \beta} \wedge \star \left( \sum_{l=1}^{6} u_1 \beta l \right) W_{\alpha \beta} + \sum_{l=1}^{5} z_1 \beta l \right) \]

\[ + w_7 \partial_\alpha \wedge (e_\gamma \wedge (e_\alpha \wedge Z_{\gamma \beta}) + \sum_{l=7}^{9} z_1 \partial_\alpha \wedge (e_\gamma \wedge (e_\alpha \wedge Z_{\gamma \beta})) \right]. \]

(27)

see Ref. [13][14][27] and the literature quoted there. Here \( \kappa \) is the dimensionful “weak” gravitational constant, \( \lambda_0 \) the “bare” cosmological constant, and \( \rho \) the dimensionless “strong” gravity coupling constant. The constants \( a_0, \ldots, a_3, b_1, \ldots, b_5, c_2, c_3, c_4, w_1, \ldots, w_7, z_1, \ldots, z_9 \) are dimensionless and are expected to be of order unity. The constant \( a_0 \) can only have the values 1 or 0 depending on whether a Hilbert-Einstein term is present or not.

We ordered the Lagrangian (27) in the following way: In the first line, we have the linear pieces, a Hilbert-Einstein type term and the cosmological term. Some algebra yields \( R_{\alpha \beta} \wedge \eta_{\alpha \beta} = W_{\alpha \beta} \), that is, only the curvature scalar is left over, as expected. In the second line, we have the pure Yang-Mills type terms for torsion and nonmetricity. If we expand them, we find \( a_1 \alpha \beta \beta \wedge \star \left( \alpha \beta \beta \right) + \ldots \). For a Yang-Mills field strength \( F \) we have always the Lagrangian \( \sim \beta F \wedge \star F \). In our case, for \( T^\alpha \) and \( Q_{\alpha \beta} \), the field strength are reducible and we can put open weighting factors in front of each square piece. Nevertheless,
the second line is the obvious analog of a Yang-Mills Lagrangian for \( T^\alpha \) and \( Q^{\alpha\beta} \).

In the third line, we have “interactions” between \( Q^{\alpha\beta} \) and \( T^\alpha \) and between different irreducible pieces of \( Q^{\alpha\beta} \). In the fourth line, we have the pure Yang-Mills terms for the rotational and the strain curvature \( \sim w_1(1) W^{\alpha\beta} \land *^{(1)} W_{\alpha\beta} + \ldots + z_1(1) Z^{\alpha\beta} \land *^{(1)} Z_{\alpha\beta} + \ldots \), and, eventually, in the last line, “exotic” interactions between different irreducible pieces of the curvature enter that we will drop subsequently. In other words, we restrict ourselves to

\[
w_7 = z_6 = z_8 = z_9 = 0. \tag{28}
\]

Taking into consideration (28), the various excitations \( \{ M^{\alpha\beta}, H_\alpha, H^{\alpha\beta} \} \) are found to be

\[
M^{\alpha\beta} = -\frac{2}{\kappa} \left( \sum_{I=1}^{4} b_I(I) Q^{\alpha\beta} \right)
- \frac{2}{\kappa} \left[ c_2 \theta^{\alpha} \land *^{(1)} T^\beta + c_3 \theta^{(\alpha} \land *^{(2)} T^\beta + \frac{1}{4} (c_3 - c_4) *^8 g^{\alpha\beta} \right]
- \frac{b_0}{\kappa} \left[ \theta^{(\alpha} \land *(Q \land \theta^\beta) - \frac{1}{4} g^{\alpha\beta} *^4 (3Q + \Lambda) \right], \tag{29}
\]

\[
H_\alpha = -\frac{1}{\kappa} \left( \sum_{I=1}^{3} a_I(I) T_\alpha + \sum_{K=2}^{4} c_K(K) Q_{\alpha\beta} \land \theta^\beta \right), \tag{30}
\]

\[
H^{\alpha\beta} = \frac{a_0}{2\kappa} \eta_{\alpha\beta} + \sum_{I=1}^{6} w_I(I) W^{\alpha\beta} + \sum_{K=1}^{5} z_K(K) Z^{\alpha\beta}. \tag{31}
\]

The last equation can be slightly rewritten as

\[
H^{\alpha\beta} = \left( \frac{a_0}{2\kappa} - \frac{w_6}{12\rho} \right) \eta_{\alpha\beta} + \frac{1}{\rho} \sum_{n=1}^{5} \left( w_n *^{(n)} W^{\alpha\beta} + z_n *^{(n)} Z^{\alpha\beta} \right), \tag{32}
\]

where \( ^{(6)} W^{\alpha\beta} = -W \theta^{\alpha\beta}/12 \) corresponds to the curvature scalar \( W \).

4. Master equation for solving the field equations algebraically

Generally, it is a very delicate task to solve the nonlinear partial differential equations (20) to (22) for the set of variables \( \{ g_{\alpha\beta}, \theta^\alpha, T^\alpha, Q_{\alpha\beta} \} \). Even for high symmetries, there will be very few chances to find exact solutions. Therefore, we developed an algebraic method for solving the field equations.

The main observation is that we can construct an algebraic relation between torsion and nonmetricity. This is a result of Prolongation Theory that has been applied very successfully in the context of Einstein’s field equation by M.Gürses \cite{22} and Bilge et al. \cite{11}, amongst others. Application of this method to the Poincaré gauge field theory, i.e., to MAG with vanishing nonmetricity, \( Q_{\alpha\beta} = 0 \), leads to the construction of stationary axisymmetric solutions with dynamic torsion, see
Rotating black holes in metric-affine gravity

Baekler et al. [10]. This method has been developed further systematically and, in ref. [10], we used a quite general ansatz for solving also the field equations of MAG.

It has been shown [10] that the following linear relationship between nonmetricity and torsion can be exploited for solving the field equations of MAG straightforwardly:

$$ T^\alpha = \sum_{A=2}^{4} \tilde{\xi}_A (A) Q^\alpha_\mu \wedge \vartheta^\mu + (3) T^\alpha. $$ (33)

The parameters $\tilde{\xi}_A$ have to be determined by the field equations.

To demonstrate the consequences of such an ansatz, we will consider a simplified version of (33) in the form of

$$ T^\alpha = \xi_0 Q^\alpha_\mu \wedge \vartheta^\mu + \xi_1 Q^\wedge \vartheta^\alpha + (3) T^\alpha. $$ (34)

We name this equation master equation. The constants $\xi_0$ and $\xi_1$ will be picked later in the context of solving the field equations.

Alternatively, we can write it as

$$ T^\alpha = \xi_0 P^\alpha_\beta \wedge \vartheta^\beta + (\xi_0 + \xi_1) Q \wedge \vartheta^\alpha + (3) T^\alpha. $$ (35)

The Weyl covector $Q$ and the traceless nonmetricity $\tilde{Q}^\alpha_\beta$ are defined by

$$ Q := \frac{1}{4} Q^\alpha_\alpha, \quad \tilde{Q}^\alpha_\beta := Q^\alpha_\beta - Q^\delta_\alpha \delta^\beta. $$ (36)

If we make use of the 2-form $P^\alpha$ of (A.2) and the 1-form $\Lambda$ of (A.1), eq. (35) translates into

$$ T^\alpha = \xi_0 P^\alpha_\beta \wedge \vartheta^\beta + (\xi_0 + \xi_1) Q \wedge \vartheta^\alpha + (3) T^\alpha. $$ (37)

We compute the trace of this equation by contracting it with the frame $e_\alpha$. Since $e_\alpha | P^\alpha = 0$ and $e_\alpha | (3) T^\alpha = 0$, we find

$$ T = \xi_0 \Lambda - 3(\xi_0 + \xi_1) Q, $$ (38)

with the 1-form $T := e_\alpha | T^\alpha$.

Empirically, a special case of relation (38) has been used in MAG for constructing exact solutions in the form of the triplet ansatz [12, 62, 41, 19]

$$ Q/k_0 = \Lambda/k_1 = T/k_2, $$ (39)

with some constants $k_0, k_1, k_2$. We refer here to the triplet of 1-forms $Q, \Lambda, T$. Spherically symmetric solutions [12] were found as well as stationary axially symmetric ones [62, 41, 19]. A deeper understanding of why this ansatz [48] could work successfully in those approaches has been elaborated systematically by Baekler et

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*b*We could slightly generalize this expression for the torsion to

$$ T^\alpha = \tilde{\xi}_0 P^\alpha_\beta + \tilde{\xi}_1 \Lambda \wedge \vartheta^\alpha + \tilde{\xi}_2 Q \wedge \vartheta^\alpha + (3) T^\alpha, $$

with suitable constants $\tilde{\xi}_0, \tilde{\xi}_1, \text{and} \tilde{\xi}_2$. 
al.\textsuperscript{10}, see also Heinicke et al.\textsuperscript{27}, demonstrating that the key is to look for further integrability conditions. Especially the first Bianchi identity \textsuperscript{33} turns out to be helpful in answering this question.

We turn now to the connection and thus to the distortion 1-form $N_{\alpha\beta}$. We eliminate the torsion $T_\alpha$ from \textsuperscript{15} by means of our master equation \textsuperscript{35}. After some algebra we find

$$N_{\alpha\beta} = \frac{1}{2} Q_{\alpha\beta} - 2 \left( \xi_0 - \frac{1}{2} \right) Q_{[\alpha\beta] \gamma} \theta^\gamma - 2 \left( \xi_0 + \xi_1 - \frac{1}{2} \right) Q_{\alpha \beta} - \frac{1}{2} e_{[\alpha]} (3) T_{\beta]}, \quad (40)$$

Note that $e_{\alpha} Q_{\beta\gamma} = Q_{\alpha\beta\gamma}$ and $e_{\alpha} Q = Q_{\alpha}$. Moreover, by means of \textsuperscript{35}, we can also express the first two irreducible pieces of the torsion \textsuperscript{A.10} and \textsuperscript{A.9} in terms of the nonmetricity,

$$(1) T^\alpha = \xi_0 (\mathcal{Q}_{\alpha\mu} \wedge \vartheta^\mu + \frac{1}{3} \Lambda \wedge \vartheta^\alpha) = \xi_0 P^\alpha, \quad (41)$$

$$ (2) T^\alpha = -\frac{1}{3} \left[ \xi_0 \Lambda - 3(\xi_0 + \xi_1) Q \right] \wedge \vartheta^\alpha, \quad (42)$$

with the 2-form $P^\alpha$ of \textsuperscript{A.2}. Note that both, \textsuperscript{(1)}$T^\alpha$ and $P^\alpha$, have 16 independent components. Eq.\textsuperscript{42} is equivalent to \textsuperscript{35}.

Further insight into the structure of the metric-affine field equations can be gained if we take care of the master equation \textsuperscript{34} in the excitations \textsuperscript{29} and \textsuperscript{30}. Let us first turn to the simpler expression \textsuperscript{40}. With our master equation, we derived \textsuperscript{(1)}$T^\alpha$ and \textsuperscript{(2)}$T^\alpha$ in \textsuperscript{41} and \textsuperscript{42}, respectively. We substitute these two pieces, together with the irreducible decompositions \textsuperscript{A.4}, \textsuperscript{A.5}, and \textsuperscript{A.6}, into \textsuperscript{30}. We find

$$-\kappa H_\alpha = * \left( a_1 T_\alpha + a_2 T_\alpha + a_3 T_\alpha + c_2 Q_{\alpha \beta} \wedge \vartheta^\beta + c_3 Q_{\alpha \beta} \wedge \vartheta^\beta 
+ c_4 Q_{\alpha \beta} \wedge \vartheta^\beta \right)$$

$$= * \left\{ a_1 \xi_0 P_\alpha - \frac{a_2}{3} [\xi_0 \Lambda - 3(\xi_0 + \xi_1) Q] \wedge \vartheta_\alpha + a_3 T_\alpha - \frac{2c_2}{3} (e_{(\alpha}) P_{\beta]} \Lambda \wedge \vartheta^\beta 
+ \frac{4c_3}{9} \left( \vartheta_{(\alpha} e_{\beta)} \right) \Lambda - \frac{1}{4} g_{\alpha \beta} \Lambda \right] \wedge \vartheta^\beta + c_4 Q \wedge \vartheta_\alpha \right\}. \quad (43)$$

Now we order the right-hand side in terms of $P_\alpha$, $\Lambda$, and $Q$. After some algebra, we have

$$-\kappa H_\alpha = * \left\{ (a_1 \xi_0 + c_2) P_\alpha - \frac{1}{3} (a_2 \xi_0 + c_3) \Lambda \wedge \vartheta_\alpha + [a_2 (\xi_0 + \xi_1) + c_4] Q \wedge \vartheta_\alpha 
+ a_3 T^\alpha \right\}. \quad (44)$$

The evaluation of the excitation \textsuperscript{29} is a bit more complicated. In expanded form, eq.\textsuperscript{29} reads

$$-\frac{\kappa}{2} M^{\alpha\beta} = * \left( b_1 Q^{\alpha\beta} + b_2 Q^{\alpha\beta} + b_3 Q^{\alpha\beta} + b_4 Q^{\alpha\beta} 
+ c_2 \vartheta^{(\alpha} \wedge *^{(1)} T^{\beta)} + c_3 \vartheta^{(\alpha} \wedge *^{(2)} T^{\beta)} + \frac{1}{4} (c_3 - c_4) T^g^{\alpha\beta} \right)$$

where $Q^{\alpha\beta}$ is
Now we substitute into this equation the irreducible pieces \((^2)Q_{\alpha\beta}, (^3)Q_{\alpha\beta}, (^4)Q_{\alpha\beta}\) from \((\Lambda, \xi), (\Lambda, \tilde{\xi}), (\Lambda, \delta)\), respectively, \((^1)T^{\alpha}, (^2)T^{\alpha}\) from \((\roman{11}), (\roman{12})\), and \(T\) from \((\roman{38})\):

\[
-\frac{\kappa}{2} M^{\alpha\beta} = b_1 \star (*) Q^{\alpha\beta} - \frac{2b_2}{3} \left( e^{(\alpha)} P^{\beta} \right) + \frac{4b_3}{9} \left[ \star \left( \vartheta^{(\alpha)} e^{\beta} \right) \right] - \frac{1}{4} g^{\alpha\beta} \Lambda \left( 3Q + \Lambda \right) .
\]  

(45)

This completes our simplifications of \(H\) in terms of \(d^\star\) into the Lagrangian \((27)\). Again, like with the excitations \(\vartheta\) and \(\omega\) in \((9)\), the connection reduces to

\[
\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} + \frac{1}{2} Q_{\alpha\beta} , \quad \text{with} \quad \vartheta_0 = \frac{1}{2}, \vartheta_1 = 0 \quad \text{or} \quad Q = 0 , \quad (^1)T^{\alpha} = 0 .
\]  

(48)

Metric-affine spacetimes with such a simple connection have already been studied before.\(^4\) We will come back to such a connection later in the discussion of our new exact solution in Sec.\(^4\)

Eventually, we can also substitute the master equation \((\roman{44})\) and the choice \((\roman{28})\) into the Lagrangian \((\roman{27})\). Again, like with the excitations \(H\) and \(M^{\alpha\beta}\), we express the Lagrangian in terms of \(P^{\alpha}\), \(\Lambda\), and \(Q\). We find

\[
V = \frac{1}{2\kappa} \left\{ -a_0 P^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0 \eta + b_1 \star (*) Q^{\alpha\beta} \wedge (^1)Q_{\alpha\beta} \right\}.
\]

\(^4\)Note that for any 1-form \(\Phi = \Phi^\alpha \eta^\alpha\) we have \(\star \left[ \vartheta^{(\alpha)} e^{\beta} \right] = \Phi^\beta \wedge \vartheta^\alpha = \Phi^\beta \eta^\alpha\). A bit more complicated is the computation of \(\vartheta^{\alpha} \wedge (*) (\Phi \wedge e^\beta)\). If \(\Omega\) is another 1-form, we have the general rule \(\star \Phi \wedge \Omega = \star \Omega \wedge \Phi\). Moreover, we have the rules for the Hodge star for any form \(\star (\Phi \wedge e^\alpha) = e^\alpha \star \Phi\) and (in four dimensions) \(\star (e^\alpha) \Phi = -\star \Phi \wedge e^\alpha\). Consequently,

\[
\vartheta^{\alpha} \wedge \left( \Phi \wedge e^\beta \right) = \delta^\alpha \wedge \left( e^\beta \wedge \star \Phi \right) = -\delta^\alpha \left( \vartheta^\alpha \wedge \Phi \right) + g^{\alpha\beta} \Phi = e^\beta \left( \star \vartheta^\alpha \wedge \phi + g^{\alpha\beta} \Phi \right).
\]

\(^5\)These spacetimes emerge in the following context: We define the Palatini 3-form \(P^{\alpha\beta} = -\delta \eta_{\alpha\beta} \wedge \tilde{R}^{\alpha\beta} / 2\delta \Gamma_{\alpha\beta}\) and find \(P_{\alpha\beta} = -P_{\beta\alpha} = D \eta_{\alpha\beta} / 2 = -Q \wedge \eta_{\alpha\beta} + T^{\alpha} \wedge \eta_{\alpha\beta} / 2\). If we require \(\epsilon_{[\alpha \beta \gamma]} \tilde{P}^{\gamma} / 2 = 0\), the \((\roman{23})\) \(\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta} + Q_{\alpha\beta} / 2\).
\[ + \left[ \left( a_1 \xi_0 + c_2 \right) \xi_0 + \frac{1}{3} \left( 2b_2 + 3c_2 \xi_0 \right) \right] P^\alpha \wedge * P_\alpha \\
+ \frac{1}{3} \left[ \left( a_2 \xi_0 + c_3 \right) \xi_0 + c_3 \xi_0 + \frac{4}{3} b_3 \right] \Lambda^\mu \Lambda_\mu \eta \\
+ \frac{1}{3} \left[ \left( a_2 \xi_0 + c_4 \right) \xi_0 + 3c_4 \xi_0 + 4b_4 \right] Q^\mu Q_\mu \eta \\
- \left[ \frac{1}{2} \left( a_2 \xi_0 + \xi_1 \right) + c_4 \right] \xi_0 + 2 \left( a_2 \xi_0 + c_3 \right) \left( \xi_0 + \xi_1 \right) - \frac{4}{3} a_2 \xi_0 \left( \xi_0 + \xi_1 \right) + b_5 \right] \\
\times Q^\mu \Lambda_\mu \eta \right\} + V_{R^2} . \] (49)

We put here \( a_3 = 0 \).

5. Finding solutions by nullifying the excitations

We can find exact solutions of the field equations of MAG straightforwardly in a very simple manner. We will ask for non-trivial field configurations with the property of vanishing field excitations, i.e., we will require

\[ H_\alpha = 0 , \quad M^{\alpha \beta} = 0 , \quad H^{\alpha \beta} = 0 . \] (50)

And indeed, because of the inhomogeneity of the excitations in terms of the field strengths, it will be possible to generate solutions with non trivial curvature.

Table 2. The case \( Q = 0 \).

| Excitation | constraints |
|------------|-------------|
| \( H_\alpha = 0 \) | \( a_1 \xi_0 + c_2 = 0 \)  \\
| | \( a_2 \xi_0 + c_3 = 0 \)  \\
| | \( a_3 = 0 \) or (3) \( T^\alpha = 0 \) |
| \( M^{\alpha \beta} = 0 \) | \( b_1 = 0 \)  \\
| | \( 2b_2 + 3c_2 \xi_0 = 0 \)  \\
| | \( 4b_3 + 3c_3 \xi_0 = 0 \)  \\
| | \( b_5 + 2c_4 \xi_0 = 0 \) |
| \( H^{\alpha \beta} = 0 \) | \( w_1 = w_2 = w_3 = w_4 = w_5 = 0 \)  \\
| | \( z_1 = z_2 = z_3 = z_4 = z_5 = 0 \)  \\
| | \( 6 \rho a_0 - \kappa w_3 W = 0 \) |

If we substitute this into the sourcefree field equations \( 21 \) and \( 22 \), only the following truncated equation is left over:

\[ E_\alpha = e_\alpha | V = 0 . \] (51)

Since \( \vartheta_\alpha \wedge E_\beta = 0 \), this equation has only 10 independent components. The field equation \( 20 \) is redundant because \( 21 \) and \( 22 \) are fulfilled. Accordingly, we have just to solve the algebraic relation \( 51 \).
Let us now have a look at the excitations given in equations (44), (47), and (32), respectively, and ask under which conditions these excitations will vanish without being dynamically trivial. Naturally, we can distinguish between the different cases $Q = 0$ and $Q \neq 0$. In the case of vanishing Weyl covector $Q$, the conditions are collected in Table 2, for nonvanishing $Q$ in Table 3.

Having the conditions at our disposal that are listed in the Tables 2 and 3, the construction of exact solutions of MAG is appreciably simplified. The only equation that has to be fulfilled is eq. (51).

### Table 3. The case $Q \neq 0$.

| Excitation | constraints |
|------------|-------------|
| $H_\alpha = 0$ | $a_1 \xi_0 + c_2 = 0$  
| | $a_2 \xi_0 + c_3 = 0$  
| | $a_2 (\xi_0 + \xi_1) + c_4 = 0$  
| | $a_3 = 0$ or $(3)T^\alpha = 0$  
| $M^{\alpha\beta} = 0$ | $b_1 = 0$  
| | $2b_2 + 3c_2 \xi_0 = 0$  
| | $4b_3 + 3c_3 \xi_0 = 0$  
| | $4b_4 + 3c_4 (\xi_0 + \xi_1) = 0$  
| | $b_5 + 2c_3 (\xi_0 + \xi_1) = 0$  
| | $c_4 \xi_0 - c_3 (\xi_0 + \xi_1) = 0$  
| $H^{\alpha\beta} = 0$ | $w_1 = w_2 = w_3 = w_4 = w_5 = 0$  
| | $z_1 = z_2 = z_3 = z_4 = z_5 = 0$  
| | $6\rho a_0 - \kappa w_6 W = 0$  

In this article, we will concentrate on Table 2, that is, on the case of vanishing Weyl covector, $Q = 0$. In particular, we have $b_1 = 0$ and we choose the option $a_3 = 0$. Then the constraints of Table 2 can be used to eliminate the constants $a_1, a_2, a_3$ and $b_2, b_3, b_5$ from the gauge Lagrangian (27) obeying the conditions (28):

$$V = \frac{1}{2\kappa} \left[ -a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0 \eta \right]$$

$$- \frac{1}{\xi_0} T^\alpha \wedge \ast \left( c_2^{(1)} T_\alpha + c_3^{(2)} T_\alpha \right)$$

$$+ 2 \left( c_2^{(2)} Q_{\alpha\beta} + c_3^{(3)} Q_{\alpha\beta} \right) \wedge \theta^\alpha \wedge \ast T^\beta$$

$$- \frac{3\xi_0}{4} Q_{\alpha\beta} \wedge \ast \left( 2c_2^{(2)} Q^{\alpha\beta} + c_3^{(3)} Q^{\alpha\beta} \right)$$

$$- \frac{w_6}{2\rho} R^{\alpha\beta} \wedge \ast (6) W_{\alpha\beta}.$$  \hspace{1cm} (52)

We also dropped $Q$-dependent terms. This is equivalent to $c_4 = 0$ and, according to Table 2, corresponds to $b_5 = 0$. 

We still didn’t apply the last constraint of Table 2. Now we can use it in order to eliminate the constant \( w_6 \). But we first collect the curvature dependent terms in (52). For this purpose we recall the geometric identities

\[
R^\alpha_\beta \wedge \eta_\alpha_\beta = (6) \quad W^\alpha_\beta \wedge \eta_\alpha_\beta = -W_\eta \quad \text{and} \quad R^\alpha_\beta \wedge ^*(6) W_\alpha_\beta = \frac{W^2}{12} \eta. \quad (53)
\]

Then, we have quite generally

\[
-\frac{a_0}{2\kappa} R^\alpha_\beta \wedge \eta_\alpha_\beta - \frac{w_6}{2\rho} R^\alpha_\beta \wedge ^*(6) W_\alpha_\beta = \frac{1}{2\kappa} W \left( a_0 - \frac{\kappa w_6}{12\rho} W \right) \eta, \quad (54)
\]

and the Lagrangian (52) can be rewritten as

\[
V = \frac{1}{2\kappa} \left[ W \left( a_0 - \frac{\kappa w_6}{12\rho} W \right) \eta - 2\lambda_0 \eta 
\right.
- \frac{1}{\xi_0} T^\alpha \wedge ^* \left( c_2(1) T_\alpha + c_3(2) T_\alpha \right) 
+ 2 \left( c_2(2) Q^\alpha_\beta + c_3(3) Q^\alpha_\beta \right) \wedge \vartheta^\alpha \wedge ^* T^\beta 
- \frac{3\xi_0}{4} Q^\alpha_\beta \wedge ^* \left( 2c_2(2) Q^\alpha_\beta + c_3(3) Q^\alpha_\beta \right) \right]. \quad (55)
\]

We recognize that in the curvature dependent pieces we have a weak gravity contribution \( \sim W/\kappa \) and a strong gravity contribution \( \sim W^2/\rho \). This is the final Lagrangian for the field equations of which we find an exact solutions. However, we still have to fulfill the last constraint of Table 2, namely

\[
\kappa w_6 W = 6\rho a_0. \quad (56)
\]

This means that the curvature scalar \( W \) of our solution is required to be a constant. Substituting the constraint into (55), the expression in the first parentheses becomes \( a_0/2 \), i.e., only a linear and constant term in \( W \) is left over, namely \((a_0 W/4\kappa) \eta\).

We could find a relatively trivial solution by putting \( a_0 = 0 \) and \( \lambda_0 = 0 \), but that is not our desire, see, however, Adak and Sert. Therefore, we rather choose \( a_0 = 1 \) in future. Then our Lagrangian (55) depends, besides the (weak) gravitational constant \( \kappa \) and the cosmological constant \( \lambda_0 \), only on the arbitrary parameters \( \xi_0 \) and \( c_2, c_3 \).

6. Seed Solution carrying metric and torsion

The formulation of Einstein’s field equation in terms of differential ideals[22] opens the possibility to create non-trivial solutions of the field equation even by starting from flat Minkowski spacetime. The Kerr-Schild transformation of general relativity provides another example. The method of prolongations suggests, as we have seen, an ansatz in form of the master equation (44). We choose as a seed solution a suitable coframe \( \vartheta^\alpha \) (or metric \( g_{\alpha\beta} \)) and a torsion \( T^\alpha \) and impose further assumptions or constraints that will lead to purely algebraic equations, which will be nonlinear in general. But remember, we could start with any metric and any torsion as a seed.
solution and even in the presence of matter this method works. We don’t need to take recourse to the field equations that the seed solutions have to obey, the field equations of MAG alone will determine the relevant geometrical quantities.

Of course, we will start from seed solutions that are expected to be of physical relevance, such as the Kerr metric given in Ref. [13] and the torsion displayed in Ref. [8]. In the case of vanishing nonmetricity \( Q^{\alpha\beta} \), our solution to be found should go over into a solution of the Poincaré gauge field theory (PGT) and further limits to be taken will lead to Newton-Einstein gravity, provided the coupling constants will be adjusted suitably.

Accordingly, for our purposes, we choose a two-step procedure: We first take a metric \( g \) of an exact solution of general relativity and then, keeping the metric fixed, turn on the torsion \( T^\alpha \) by going over to a known solution of the field equations of the Poincaré gauge theory, in which, as we recall, the nonmetricity vanishes identically.\(^1\)

Taking this as a new starting point, we eventually switch on the nonmetricity \( Q_{\alpha\beta} \) and generate a whole class of solutions of MAG.

Let us start with the Kerr-deSitter metric of general relativity with cosmological constant \( \lambda \) and an exact solution of the Poincaré gauge theory using that metric and providing additionally the torsion by solving the field equations of the Poincaré gauge theory.\(^8\)

### 6.1. Seed metric as solution of Einstein’s field equation

The coframe \( \vartheta^\alpha \), in terms of coordinates \( t, r, \theta, \phi \), reads

\[
\begin{align*}
\vartheta^0 &= \sqrt{\frac{\Delta}{\Sigma}} (dt + a \sin^2 \theta d\phi), \\
\vartheta^1 &= \sqrt{\frac{\Delta}{\Sigma}} dr, \\
\vartheta^2 &= \sqrt{\frac{\Delta}{\Sigma}} d\theta, \\
\vartheta^3 &= \sin \theta \sqrt{\frac{F}{\Sigma}} [adt + (r^2 + a^2) d\phi].
\end{align*}
\]

The structure functions are defined according to

\[
\begin{align*}
\Sigma &= r^2 + a^2 \cos^2 \theta, \\
F &= 1 + \frac{1}{3} \lambda a^2 \cos^2 \theta, \\
\Delta &= r^2 + a^2 - 2Mr - \frac{1}{3} \lambda r^2 (r^2 + a^2).
\end{align*}
\]

\(^1\) Presentations of the Poincaré gauge theory can be found in Nester\(^3\), Blagojević\(^2\), and Gronwald et al.\(^2\); for recent results one should compare Obukhov\(^4\) and, for possible observations, Preuss et al.\(^5\). The Kerr solution and its approximation in a teleparallel spacetime was discussed by Pereira, Vargas, and Zhang\(^6\).\(^7\)
The coframe is orthonormal. Then the metric reads
\[ g = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3 \] (61)
or, in terms of local coordinates,
\[ g = -\frac{\Delta}{\Sigma} (dt + asin^2 \theta d\phi)^2 + \sum \frac{\Sigma}{F} dr^2 + \sum \frac{\Sigma}{F} d\theta^2 \sin^2 \theta + \frac{F}{\Sigma} [adt + (r^2 + a^2)d\phi]^2 . \] (62)
This solution of Einstein’s field equation depends on the set of essential constants \( \{M, a, \lambda\} \), i.e., on mass, angular momentum per mass, and the cosmological constant.

In the limit of vanishing Kerr-parameter \( a \to 0 \), the coframe (57) reduces to the Schwarzschild-deSitter (also known as Kottler) coframe with
\[ \vartheta^0 = \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{1/2} dt, \]
\[ \vartheta^1 = \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{-1/2} dr, \]
\[ \vartheta^2 = rd\theta, \]
\[ \vartheta^3 = r\sin \theta d\phi, \] (63)
whereas in the limit of a vanishing cosmological constant \( \lambda \to 0 \) we recover the well-known Kerr-solution. Further physical and mathematical properties of the solution are compiled in the book of Chandrasekhar.

The Riemannian curvature \( \tilde{R}^{\alpha\beta} \) comprises three irreducible pieces, the Weyl-curvature, the tracefree symmetric Ricci and the curvature scalar:
\[ \tilde{R}^{\alpha\beta} = (1)\tilde{W}^{\alpha\beta} + (4)\tilde{W}^{\alpha\beta} + (6)\tilde{W}^{\alpha\beta}. \] (64)
The numbers (1), (4), (6) refer already to a metric-affine space in which the rotational curvature \( W_{[\alpha\beta]} := R_{[\alpha\beta]} \) has six independent components, see [A.26] to [A.31]. In a Riemannian space only the pieces with the numbers (1), (4), (6) are non-vanishing. Explicitly, these matrices are given by

\[ (1)\tilde{W}^{\alpha\beta} = \frac{Mr}{\Sigma^3} (r^2 - 3a^2 \cos^2 \theta) \begin{pmatrix} 0 & -2\vartheta^{01} & \vartheta^{02} & \vartheta^{03} \\ \diamond & 0 & \vartheta^{12} & \vartheta^{13} \\ \diamond & \diamond & 0 & -2\vartheta^{23} \\ \diamond & \diamond & \diamond & 0 \end{pmatrix}, \] (65)
\[ (4)\tilde{W}^{\alpha\beta} = 0, \] (66)
\[ (6)\tilde{W}^{\alpha\beta} = -\frac{\lambda}{3} \vartheta^{\alpha\beta}. \] (67)
The symbols \( \diamond \) and \( \bullet \) denote matrix elements already known because of the antisymmetry or the symmetry of the matrix involved. The matrices (65) and (67) are
equivalent to the statement that Einstein’s vacuum field equation with cosmological constant $\lambda$ is fulfilled for the metric (61) and the coframe (57).

A canonical form of the metric for the most general type D solution of the Einstein-Maxwell equations (with cosmological constant) has been given by Plebański and Demianski resulting in a seven parameter solution. This was updated by Debever, Kamran, and McLenaghan, see also Gracía and Macías. This solution could be taken as seed solution as well. However, because of simplicity, we will concentrate on the metric as given in (61), together with the coframe (57) and the structure functions (58) to (60).

### 6.2. Seed torsion of the Poincaré gauge theory

The torsion $T^\alpha$ of a stationery axially symmetric solution of the Poincaré gauge theory reads

\[
T^0 = \sqrt{\frac{\Sigma}{\Delta}} \left[ -v_1 \dot{\vartheta}^{01} + \sqrt{\frac{\Sigma}{\Delta}} \left[ v_2 (\dot{\vartheta}^{02} - \dot{\vartheta}^{12}) + v_3 (\dot{\vartheta}^{03} - \dot{\vartheta}^{13}) \right] - 2v_4 \dot{\vartheta}^{23} \right],
\]

\[
T^1 = T^0,
\]

\[
T^2 = \sqrt{\frac{\Sigma}{\Delta}} \left[ v_5 (\dot{\vartheta}^{02} - \dot{\vartheta}^{12}) + v_4 (\dot{\vartheta}^{03} - \dot{\vartheta}^{13}) \right],
\]

\[
T^3 = -\sqrt{\frac{\Sigma}{\Delta}} \left[ -v_4 (\dot{\vartheta}^{02} - \dot{\vartheta}^{12}) + v_5 (\dot{\vartheta}^{03} - \dot{\vartheta}^{13}) \right],
\]

(68)

together with the functions

\[
v_1 = \frac{M}{\Sigma^2} (r^2 - a^2 \cos^2 \theta),
\]

\[
v_2 = -\frac{Ma^2 r \sin \theta \cos \theta}{\Sigma^2} \sqrt{\frac{F}{\Sigma}},
\]

\[
v_3 = \frac{Mar^2 \sin \theta}{\Sigma^2} \sqrt{\frac{F}{\Sigma}},
\]

\[
v_4 = \frac{Mar^2 \cos \theta}{\Sigma^2},
\]

\[
v_5 = \frac{Mr^2}{\Sigma^2}
\]

(69)

and the coframe (57). Note that also the choice $T^1 = -T^0$ would lead to a viable solution of MAG.

For the torsion trace $T = e_\alpha |T^\alpha|$, we find

\[
T = \sqrt{\frac{\Sigma}{\Delta}} (v_1 - 2v_5)(\dot{\vartheta}^0 - \dot{\vartheta}^1),
\]

(70)

and the axial torsion, see (A.8), turns out to vanish:

\[
(3)T^\alpha = 0.
\]

(71)

This completes our seed solution of the Poincaré gauge theory. We now turn to our ansatz for finding a solution of MAG.
7. Ansatz for the nonmetricity

The nonmetricity 1-form can be decomposed into components according to

\[ Q^\alpha\beta = Q_{\gamma}{}^\alpha{}^\beta{}^\gamma. \]  

(72)

We can represent it as a symmetric 4 \times 4-matrix with 10 independent components, each of which is a sum of suitable 1-forms:

\[
Q^{\alpha\beta} = \begin{pmatrix}
Q_{00} & Q_{01} & Q_{02} & Q_{03} \\
Q_{11} & Q_{12} & Q_{13} & 0 \\
Q_{22} & Q_{23} & 0 & 0 \\
Q_{33} & 0 & 0 & 0
\end{pmatrix}.
\]  

(73)

The bullets denote those matrix elements that, due to symmetry, can be read off from the other matrix elements.

Since we are looking for a stationary axially symmetric solution, it would appear natural to start with the most general axially symmetric form of \( Q_{\alpha\beta} \). So far, this form is unknown. Moreover, we expect that it is so general that it would not help us in our task of solving the field equations of MAG. Therefore, we start from the most general spherically symmetric form of \( Q_{\alpha\beta} \) and generalize it. For vanishing Kerr parameter \( a \), our solution has to reduce to the spherically symmetric form.

For \( SO(3) \)-symmetry, by solving the Killing equations, Tresguerres derived the most general form of the nonmetricity; this has been confirmed by different groups. Tresguerres found 12 independent components only depending on the radial coordinate \( r \), namely

\[
Q^{\alpha\beta}|_{SO(3)} = \begin{pmatrix}
Q_0{}^{00} \vartheta^0 + Q_1{}^{00} \vartheta^1 & Q_0{}^{01} \vartheta^0 + Q_1{}^{01} \vartheta^1 & Q_0{}^{02} \vartheta^0 + Q_1{}^{02} \vartheta^1 & -Q_1{}^{02} \vartheta^0 + Q_2{}^{02} \vartheta^3 \\
Q_0{}^{11} \vartheta^0 + Q_1{}^{11} \vartheta^1 & Q_0{}^{12} \vartheta^0 + Q_1{}^{12} \vartheta^1 & -Q_1{}^{12} \vartheta^0 + Q_2{}^{12} \vartheta^3 & 0 \\
Q_0{}^{22} \vartheta^0 + Q_1{}^{22} \vartheta^1 & 0 & 0 & 0
\end{pmatrix}.
\]  

(74)

For convenience, we would like to abbreviate these 12 functions in a different manner. We translate the holonomic version of Minkevich and Vasilevski\[\text{37}\] into a corresponding anholonomic version. Then, with their \( \Omega_0, \Omega_1, \ldots, \Omega_{11} \), we have

\[
Q^{\alpha\beta}|_{SO(3)} = \begin{pmatrix}
\Omega_0 \vartheta^0 + \Omega_1 \vartheta^1 & \Omega_2 \vartheta^0 + \Omega_3 \vartheta^1 & \Omega_5 \vartheta^2 + \Omega_{10} \vartheta^3 & -\Omega_0 \vartheta^0 + \Omega_6 \vartheta^3 \\
\Omega_4 \vartheta^0 + \Omega_5 \vartheta^1 & \Omega_7 \vartheta^2 + \Omega_{11} \vartheta^3 & -\Omega_{11} \vartheta^0 + \Omega_7 \vartheta^3 & 0 \\
\Omega_8 \vartheta^0 + \Omega_9 \vartheta^1 & 0 & \Omega_8 \vartheta^0 + \Omega_9 \vartheta^1
\end{pmatrix}.
\]  

(75)

In the static \( SO(3) \)-case, the \( \Omega \)'s depend only on the radial coordinate \( r \).

Let us now “blow up” \( Q_{\alpha\beta} \) following considerations as given by Minkevich and Vasilevski\[\text{37}\]. First of all, the \( \Omega \) are assumed to depend on the two variables \( r \) and \( \vartheta \), that is, \( \Omega = \Omega(r, \vartheta) \). Moreover, we need a few more independent components. For the \( SO(3) \)-case we have 12 functions. Since the torsion, if its axial piece vanishes,
Rotating black holes in metric-affine gravity

... carries 20 independent functions, we tentatively introduce 8 more functions for the nonmetricity since, according to our master equation (34), nonmetricity and torsion are algebraically related. The following representation of the nonmetricity is a minimal set that fills our bill,

\[ Q^{\alpha\beta} = \begin{pmatrix}
Q_{00} + Q_{01} \vartheta^0 + Q_{02} \vartheta^1 \\
Q_{03} + Q_{04} \vartheta^0 + Q_{05} \vartheta^1 \\
Q_{06} + Q_{07} \vartheta^0 + Q_{08} \vartheta^1 \\
Q_{09} + Q_{10} \vartheta^0 + Q_{11} \vartheta^1 \\
Q_{12} + Q_{13} \vartheta^0 + Q_{14} \vartheta^1 \\
Q_{15} + Q_{16} \vartheta^0 + Q_{17} \vartheta^1 \\
Q_{18} + Q_{19} \vartheta^0 + Q_{20} \vartheta^1
\end{pmatrix}, \tag{76} \]

with the 1-forms \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are the names in our computer programs.

We introduced here 4+4 new functions \( \mathfrak{Y} \) and \( \mathfrak{Z} \) (or the four 1-forms \( \alpha_1, \cdots, \alpha_4 \)), ending up with 20 independent functions, exactly as planned. The trace of \( Q^{\alpha\beta} \) yields

\[ g_{\alpha\beta} Q^{\alpha\beta} = -q_1 \vartheta^0 - q_0 \vartheta^1 = 4Q, \tag{78} \]

with the abbreviations

\[ q_0 := \frac{1}{4}(\Omega_1 - \Omega_5 - 2\Omega_9), \quad q_1 := \frac{1}{4}(\Omega_0 - \Omega_4 - 2\Omega_8). \tag{79} \]

Hence the expression \( Q \wedge \vartheta^\alpha \), which enters \( \alpha = \mathfrak{M} \), reads

\[ Q \wedge \vartheta^\alpha = q_0 \begin{pmatrix}
\vartheta_0^1 \\
0 \\
-\vartheta_{12} \\
0 \\
-\vartheta_{13} \\
0 \\
\vartheta_0^3
\end{pmatrix}, \tag{80} \]

This parameterization of the nonmetricity supports all four irreducible pieces \( \mathfrak{A} \) to \( \mathfrak{A}_3 \) of \( Q_{\alpha\beta} \). Especially, the two trace-free symmetric second rank tensor pieces \( \{ (1)Q^{\alpha\beta}, (2)Q^{\alpha\beta} \} \) are supported, as well as the two vector pieces proportional to \( \{ Q, \Lambda \} \). Nevertheless, we will put in \( \mathfrak{M} \), in accordance with Table 2, the Weyl-covector \( Q \) to zero.

8. Master equation and solutions

The powerful tool of prolongation theory as applied to the highly nonlinear partial differential equations of metric-affine gravity (MAG) results in a set of linear algebraic equations which interrelates torsion and nonmetricity. This is our master equation \( \alpha = \mathfrak{M} \).
8.1. Generating nonmetricity

To generate nonmetricity, we liberate at first the parameter $M$ occurring in (69), i.e., we let

$$ M \rightarrow L_0 \quad \text{in eq} \,(83). $$

(81)

in order to introduce a new $GL(4, R)$-charge. Everything else in (69) and (68) remains untouched. In particular, the structure functions (68) to (60) keep their old values. This transformation decouples metric-compatible (Riemann-Cartan) quantities from metric-affine quantities. Note that further parameter transformations will lead analogously to viable nonmetricity functions.

Our tool for generating nonmetricity is the master equation (34). We consider its left-hand side specified by the seed torsion (68) and (69). In particular, this implies that the right-hand side encompasses the generated nonmetricity. However, we only allow such a nonmetricity to emerge that is compatible with our ansatz (60) with (61). Only in this way we find equations determining $Q^{\alpha\beta}$ that remain manageable. Altogether, we have then five functions $v$ for $T^\alpha$, three functions $\Sigma, F, \Lambda$ for $\vartheta^\alpha$, and 20 functions $\Omega, \varPsi, \varDelta$ for $Q^{\alpha\beta}$.

We substitute (60) and (61) into (34) and find

$$ T^0 = \xi_0 \left[ (\Omega_1 + \Omega_2) \vartheta^{01} + \varPsi_1 \vartheta^{02} + \varPsi_2 \vartheta^{03} + \varDelta_1 \vartheta^{12} + \varDelta_2 \vartheta^{13} - 2 \Omega_{10} \vartheta^{23} \right] + \xi_1 q_0 \vartheta^{01}, $$

$$ T^1 = \xi_0 \left[ (\Omega_3 + \Omega_4) \vartheta^{01} + \varPsi_3 \vartheta^{02} + \varPsi_4 \vartheta^{03} + \varDelta_3 \vartheta^{12} + \varDelta_4 \vartheta^{13} - 2 \Omega_{11} \vartheta^{23} \right] - \xi_1 q_1 \vartheta^{01}, $$

$$ T^2 = \xi_0 \left[ (\varPsi_3 + \varDelta_1) \vartheta^{01} + (\Omega_6 + \Omega_8) \vartheta^{02} + \Omega_{10} \vartheta^{03} + (\Omega_9 - \Omega_7) \vartheta^{12} - \Omega_{11} \vartheta^{13} \right] - \xi_1 (q_1 \vartheta^{01} + q_0 \vartheta^{12}), $$

$$ T^3 = \xi_0 \left[ (\varPsi_4 + \varDelta_2) \vartheta^{01} - \Omega_{10} \vartheta^{02} + (\Omega_6 + \Omega_8) \vartheta^{03} + (\Omega_9 - \Omega_7) \vartheta^{12} + (\Omega_3 + \Omega_4) \vartheta^{13} \right] - \xi_1 (q_1 \vartheta^{03} + q_0 \vartheta^{12}). $$

(82)

If we substitute the torsion (68) into (82), this represents an under determined system of linear algebraic equations for determining the unknown functions $\Omega, \varPsi, \varDelta$. By comparing the coefficients of the 2-forms $\vartheta^{01}, \vartheta^{02}, \vartheta^{03}, \vartheta^{12}, \vartheta^{13}, \vartheta^{23}$, we find for $T^0$ the 6 equations

$$ \xi_0 (\Omega_1 + \Omega_2) = - \sqrt{\frac{\Sigma}{\Delta}} v_1 - \xi_1 q_0, \quad \xi_0 \varPsi_1 = \frac{\Sigma}{\Delta} v_2, \quad \xi_0 \varPsi_2 = \frac{\Sigma}{\Delta} v_3, $$

$$ \xi_0 \varDelta_1 = \frac{\Sigma}{\Delta} v_2, \quad \xi_0 \varDelta_2 = - \frac{\Sigma}{\Delta} v_3, \quad \xi_0 \Omega_{10} = \sqrt{\frac{\Sigma}{\Delta}} v_4. $$

(83)

Similarly, for $T^1$, we have again 6 equations, namely

$$ \xi_0 (\Omega_3 + \Omega_4) = - \sqrt{\frac{\Sigma}{\Delta}} v_1 + \xi_1 q_1, \quad \xi_0 \varPsi_3 = \frac{\Sigma}{\Delta} v_2, \quad \xi_0 \varPsi_4 = \frac{\Sigma}{\Delta} v_3, $$

$$ \xi_0 \varDelta_3 = \frac{\Sigma}{\Delta} v_2, \quad \xi_0 \varDelta_4 = - \frac{\Sigma}{\Delta} v_3, \quad \xi_0 \Omega_{11} = \sqrt{\frac{\Sigma}{\Delta}} v_4. $$

(84)
For $T^2$, one equation vanishes and three are redundant, since contained in (83) or (84). Thus, we find the 2 equations

$$\xi_0(\Omega_6 + \Omega_8) = \sqrt{\frac{\Sigma}{\Delta}} v_5 + \xi_1 q_1, \quad \xi_0(\Omega_9 - \Omega_7) = -\sqrt{\frac{\Sigma}{\Delta}} v_5 + \xi_1 q_0.$$  

Eventually, for $T^3$, one equation vanishes and the rest is redundant.

### 8.2. Solving the master equation

So far, we have $6+6+2=14$ equations in (83), (84), and (85) for the $12+4+4=20$ functions $Q, Y, Z$. In other words, for making the system of equations well determined, we have to pick 6 conditions. Two of them are obvious. We put $q_0 = q_1 = 0$, i.e., see (79),

$$Q_1 - Q_5 - 2Q_9 = 0, \quad Q_0 - Q_4 - 2Q_8 = 0.$$  

The remaining 4 conditions can be selected from those equations in (83), (84), and (85) in which sums of $Q$’s enter. We choose

$$Q_2 = Q_3 = Q_8 = Q_9 = 0.$$  

Then, according to (86),

$$\Omega_2 = \Omega_3 = \Omega_8 = \Omega_9 = 0.$$  

Now, from (83), (84), and (85), it is simple to read off the nonvanishing members of the $\Omega$’s, $Q$’s, and $Z$’s:

$$\Omega_0 = \Omega_1 = \Omega_4 = \Omega_5 = \frac{v_1}{\xi_0} \sqrt{\frac{\Sigma}{\Delta}}, \quad \Omega_6 = \Omega_7 = \frac{v_5}{\xi_0} \sqrt{\frac{\Sigma}{\Delta}}, \quad \Omega_10 = \Omega_11 = \frac{v_4}{\xi_0} \sqrt{\frac{\Sigma}{\Delta}},$$

$$Q_1 = Q_3 = -3_1 = -3_3 = \frac{v_2}{\xi_0} \Sigma, \quad Q_2 = Q_4 = -3_2 = -3_4 = \frac{v_3}{\xi_0} \Sigma.$$  

If we substitute (89) into (76) and (77), the nonmetricity matrix turns out to be

$$Q^{\alpha\beta} = \frac{1}{\xi_0} \Sigma \frac{\Delta}{\Delta} \begin{pmatrix} 0 & 0 & v_2(\vartheta^0 - \vartheta^1) & v_3(\vartheta^0 - \vartheta^1) \\ \bullet & 0 & v_2(\vartheta^0 - \vartheta^1) & v_3(\vartheta^0 - \vartheta^1) \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{pmatrix} + \frac{1}{\xi_0} \sqrt{\frac{\Sigma}{\Delta}} \begin{pmatrix} -v_1(\vartheta^0 + \vartheta^1) & 0 & v_5 \vartheta^2 + v_4 \vartheta^3 & -v_4 \vartheta^2 + v_5 \vartheta^3 \\ \bullet & -v_1(\vartheta^0 + \vartheta^1) & v_5 \vartheta^2 + v_4 \vartheta^3 & -v_4 \vartheta^2 + v_5 \vartheta^3 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}.$$  

(90)
8.3. Irreducible decomposition of the generated nonmetricity

In order to recognize the structure of the nonmetricity, we decompose it irreducibly in accordance with the scheme \( \Lambda \) to \( \Lambda \):

\[
(1) \quad Q^{\alpha\beta} = \frac{2v_2}{3\xi_0} \frac{\Delta}{\Sigma} \begin{pmatrix}
-\vartheta^2 & -\vartheta^2 & 0 \\
0 & -\vartheta^2 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{2v_3}{3\xi_0} \frac{\Delta}{\Sigma} \begin{pmatrix}
-\vartheta^3 & -\vartheta^3 & 0 \\
0 & -\vartheta^3 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{v_1}{g\xi_0} \sqrt{\frac{\Sigma}{\Delta}} \begin{pmatrix}
-6\vartheta^0 - 4\vartheta^1 & 0 & 0 \\
0 & -4\vartheta^0 - 6\vartheta^1 & 0 \\
0 & 0 & -4(\vartheta^0 - \vartheta^1)
\end{pmatrix} + \frac{v_5}{g\xi_0} \sqrt{\frac{\Sigma}{\Delta}} \begin{pmatrix}
-6\vartheta^0 + 2\vartheta^1 & 0 & 0 \\
0 & 2\vartheta^0 - 6\vartheta^1 & 0 \\
0 & 0 & -4(\vartheta^0 - \vartheta^1)
\end{pmatrix},
\]

\[
(2) \quad Q^{\alpha\beta} = \frac{v_2}{3\xi_0} \frac{\Delta}{\Sigma} \begin{pmatrix}
2\vartheta^2 & 0 & 0 \\
0 & 2\vartheta^2 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{v_3}{3\xi_0} \frac{\Delta}{\Sigma} \begin{pmatrix}
2\vartheta^3 & 0 & 0 \\
0 & 2\vartheta^3 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
(3) \quad Q^{\alpha\beta} = -\frac{1}{9\xi_0}(v_1 - 2v_5) \sqrt{\frac{\Sigma}{\Delta}} \begin{pmatrix}
3\vartheta^0 + \vartheta^1 & 2(\vartheta^0 + \vartheta^1) & 0 \\
0 & \vartheta^0 + 3\vartheta^1 & 0 \\
0 & 0 & \vartheta^0 - \vartheta^1
\end{pmatrix},
\]

\[
(4) \quad Q^{\alpha\beta} = 0 .
\]

For the 1-form triplet we find

\[
\Lambda = -\frac{v_5}{\xi_0 r^2} \sqrt{\frac{\Sigma}{\Delta}}(\vartheta^0 - \vartheta^1), \quad Q = 0, \quad T = -\frac{v_5}{r^2} \sqrt{\frac{\Sigma}{\Delta}}(\vartheta^0 - \vartheta^1).
\]

Remember that the function \( v_5 \) carries here an \( L_0 \) instead of the original \( M \), see \( \text{[31]} \). Because of \( (24) \), we have

\[
T = \xi_0 \Lambda .
\]

This is a special case of \( \text{[38]} \), compare also the discussion in Heinicke et al. \( \text{[24]} \), where \( \text{[38]} \) was used in the context of spherically symmetric exact solutions.

Besides \( Q = 0 \), we also have \( (3)T^\alpha = 0 \), see \( \text{[71]} \). Moreover, from \( \text{[93]} \) we read off \( \xi_1 = 0 \). Then a comparison with \( \text{[38]} \) shows that

\[
\xi_0 = 1/2
\]

yields a particular simple connection. Thus, we adopt \( \text{[31]} \).
Eventually we put the cosmological constant in the MAG Lagrangian $\lambda_0$ equal to the corresponding Einsteinian cosmological constant of our seed solution, i.e., $\lambda_0 = \lambda$.

9. Display of the solution

Our new solution is given in terms of the coframe $\vartheta^\alpha$ in (57) [with the structure functions (58) to (60)], of the metric $g$ in (61), of the torsion [see (68)]

$$
T^0 = T^1 = - L_0 (r^2 - a^2 \cos^2 \theta) \vartheta^{01} - \frac{L_0 a^2 r \sin \theta \cos \theta}{\Sigma^2} \frac{F}{\Delta} (\vartheta^{02} - \vartheta^{12}),
$$

$$
+ \frac{L_0 a r^2 \sin \theta}{\Delta \Sigma} \sqrt{\frac{F}{\Sigma}} (\vartheta^{03} - \vartheta^{13}) - \frac{2 L_0 a r \cos \theta}{\Sigma \sqrt{\Delta \Sigma}} \vartheta^{23},
$$

$$
T^2 = \frac{L_0 r^2}{\Sigma \sqrt{\Delta \Sigma}} (\vartheta^{02} - \vartheta^{12}) + \frac{L_0 a r \cos \theta}{\Sigma \sqrt{\Delta \Sigma}} (\vartheta^{03} - \vartheta^{13}),
$$

$$
T^3 = - \frac{L_0 a r \cos \theta}{\Sigma \sqrt{\Delta \Sigma}} (\vartheta^{02} - \vartheta^{12}) + \frac{L_0 r^2}{\Sigma \sqrt{\Delta \Sigma}} (\vartheta^{03} - \vartheta^{13}),
$$

and of the nonmetricity [see (90)]

$$
Q^{\alpha\beta} = \frac{2 L_0 a r \sin \theta}{\Delta \Sigma} \sqrt{\frac{F}{\Sigma}} \left( \begin{array}{ccc}
0 & - \cos \theta (\vartheta^{00} - \vartheta^{01}) & r (\vartheta^{00} - \vartheta^{01}) \\
0 & - \cos \theta (\vartheta^{00} - \vartheta^{01}) & r (\vartheta^{00} - \vartheta^{01}) \\
0 & 0 & 0
\end{array} \right),
$$

$$
- \frac{2 L_0}{\Sigma \sqrt{\Delta \Sigma}} \left( \begin{array}{ccc}
\mathcal{O} & 0 & - (r^2 \vartheta^2 + \arccos \theta \vartheta^3) & \arccos \theta \vartheta^2 - r^2 \vartheta^3 \\
0 & \mathcal{O} & - (r^2 \vartheta^2 + \arccos \theta \vartheta^3) & \arccos \theta \vartheta^2 - r^2 \vartheta^3 \\
0 & 0 & 0
\end{array} \right),
$$

with $\mathcal{O} := (r^2 - a^2 \cos^2 \theta) (\vartheta^{00} + \vartheta^{01})$. This completes our solution belonging to the Lagrangians (52).

The explicit verification that the field equations are fulfilled, indeed, is still a delicate task. We did it by means of Hearn’s computer algebra system REDUCE together with Schrüfer’s EXCALC package, see also Socorro et al. [54], Heinicke et al. [28], and Ref. [26].

In studying the properties of our solution, certainly the computation of the curvature will provide some insight. In Appendix B we collected all the corresponding formulas. It becomes immediately clear that our solution is far from being trivial. It rather displays a fairly complicated structure. In order to get some insight, we will display first a typical component of the rotational curvature, namely a component of the Weyl curvature,

$$
^{(1)}W^{03} = \frac{L_0 a r \sin \theta (r^2 - a^2 \cos^2 \theta)}{2 \Delta \Sigma^3} \frac{F}{\Delta} (r \vartheta^{01} - a \cos \theta \vartheta^{23}),
$$

(97)
and such a component of the strain curvature that is not too complicated,
\begin{equation}
(2) Z_{00}^{(2)} = \frac{M L a r \cos \theta}{\Delta \Sigma_3} (3r^2 - a^2 \cos^2 \theta) \vartheta_{23}.
\end{equation}
(98)
The other pieces are listed in Appendix B.1 and Appendix B.2 respectively.

Alternatively, instead of torsion and nonmetricity, we could display the connection of our solution. It can be read off from (48). Since $Q = 0$, we have
\begin{equation}
\Gamma_{\alpha\beta} = \bar{\Gamma}_{\alpha\beta} + \frac{1}{2} Q_{\alpha\beta} = \bar{\Gamma}_{\alpha\beta} + \frac{1}{2} \bar{Q}_{\alpha\beta}.
\end{equation}
(99)
To the general relativistic Levi-Civita connection $\bar{\Gamma}_{\alpha\beta}$, we have to add half of the nonmetricity $\bar{Q}_{\alpha\beta}$.

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Appendix A. Irreducible decompositions of the geometrical field strengths

At each point of spacetime, we have invariance under the linear group $GL(4, R)$. Therefore we can decompose nonmetricity, torsion, and curvature irreducibly under this group. Moreover, since a metric is defined locally, we can decompose these quantities even finer, namely with respect to the Lorentz group $SO(1,3)$. This is what we will list here, for more details and references to the original literature, see Ref.[25].

Appendix A.1. Decomposition of the nonmetricity

The nonmetricity $Q_{\alpha\beta}$ can be decomposed into four pieces. We have to recapitulate some of these features. In four dimensions, as a symmetric tensor-valued 1-form, the nonmetricity has 40 independent components. Two vector-like pieces can be easily identified. Firstly, the Weyl covector $Q := Q_{\alpha}^\alpha/4$ can be extracted by tracing $Q_{\alpha\beta}$. The remaining tracefree part of the nonmetricity $\bar{Q}_{\alpha\beta}$ contains a second vector-like piece represented by the 1-form $\Lambda$:
\begin{equation}
\Lambda := (e^{\beta} \lceil Q_{\alpha\beta}) \wedge \vartheta^\alpha.
\end{equation}
(A.1)
The 2-form $2^{27}$
\begin{equation}
P_{\alpha} := \bar{Q}_{\alpha\beta} \wedge \vartheta^\beta - \frac{1}{3} \vartheta_\alpha \wedge \Lambda,
\end{equation}
(A.2)
obeyes the 4 + 4 constraints
\begin{equation}
P^\alpha \wedge \vartheta_\alpha = 0, \quad e^\alpha \lceil P^\alpha = 0.
\end{equation}
(A.3)
Accordingly, $P^\alpha$ carries $24 - 4 - 4 = 16$ independent components, and it is related to a further irreducible piece of $Q_{\alpha\beta}$. Explicitly, we have

\[
(4) \quad Q_{\alpha\beta} := Q g_{\alpha\beta},
\]

\[
(3) \quad Q_{\alpha\beta} := \frac{4}{9} \left( \vartheta_{(\alpha} e_{\beta]} - \frac{1}{4} g_{\alpha\beta} \right) \Lambda,
\]

\[
(2) \quad Q_{\alpha\beta} := -\frac{2}{3} e_{(\alpha} | P_{\beta)},
\]

\[
(1) \quad Q_{\alpha\beta} := Q_{\alpha\beta} - (2) Q_{\alpha\beta} - (3) Q_{\alpha\beta} - (4) Q_{\alpha\beta}.
\]

**Appendix A.2. Decomposition of the torsion**

The torsion $T^\alpha$ can be decomposed irreducibly into three independent pieces, the totally antisymmetric axial part $(3)T^\alpha$, its trace $(2)T^\alpha$, and the tracefree symmetric tensor part $(1)T^\alpha$. They read, respectively,

\[
(3)T^\alpha := \frac{1}{3} e_{\alpha} \left( \vartheta^\mu \wedge T_\mu \right),
\]

\[
(2)T^\alpha := \frac{1}{3} \vartheta^\alpha \wedge T \quad \text{with} \quad T := e_\mu | T^\mu,
\]

\[
(1)T^\alpha := T^\alpha - (2)T^\alpha - (3)T^\alpha.
\]

**Appendix A.3. Decomposition of the strain = shear \oplus dilation curvature $Z_{\alpha\beta}$**

From the strain curvature $Z_{\alpha\beta} := R_{(\alpha\beta)}$ we can split off the dilation curvature $Z := R_{\alpha}^\alpha$, see (17). The (tracefree) shear curvature $Z_{\alpha\beta}$ can be cut into different pieces by contraction with $e_\alpha$, transvecting with $\vartheta^\alpha$, and by “hodge”-ing the corresponding expressions:

\[
Z_\alpha := e^\beta \wedge Z_{\alpha\beta}, \quad \hat{\Delta} := \frac{1}{2} \vartheta^\alpha \wedge Z_\alpha, \quad Y_\alpha := \ast (Z_{\alpha\beta} \wedge \vartheta^\beta).
\]

Subsequently we can subtract out traces:

\[
\Xi_\alpha := Z_\alpha - \frac{1}{2} e_\alpha \left( \vartheta^\gamma \wedge Z_\gamma \right), \quad \Upsilon_\alpha := Y_\alpha - \frac{1}{2} e_\alpha \left( \vartheta^\gamma \wedge Y_\gamma \right).
\]

The irreducible pieces may then be written as

\[
(2)Z_{\alpha\beta} := -\frac{1}{2} \ast (\vartheta_{(\alpha} \wedge Y_{\beta)}),
\]

\[
(3)Z_{\alpha\beta} := \frac{1}{3} \left( 2 \vartheta_{(\alpha} \wedge e_{\beta]} - g_{\alpha\beta} \right) \hat{\Delta},
\]

\[
(4)Z_{\alpha\beta} := \frac{1}{4} g_{\alpha\beta} Z,
\]

\[
(5)Z_{\alpha\beta} := \frac{1}{2} \vartheta_{(\alpha} \wedge \Xi_{\beta)},
\]

\[
(1)Z_{\alpha\beta} := Z_{\alpha\beta} - (2)Z_{\alpha\beta} - (3)Z_{\alpha\beta} - (4)Z_{\alpha\beta} - (5)Z_{\alpha\beta}.
\]
Appendix A.4. Decomposition of the rotational curvature

The rotational curvature \( W^{\alpha\beta} := R^{[\alpha\beta]} \) is a sum of six irreducible pieces,

\[
W^{\alpha\beta} = \sum_{k=1}^{6} (k)W^{\alpha\beta} \tag{A.18}
\]

that can be parameterized by using the following four vector-valued 1-forms \( W^\alpha, X^\alpha, \Phi^\alpha, \) and \( \Psi^\alpha \):

\[
W^\alpha := e_\beta \lrcorner W^{\alpha\beta}, \tag{A.19}
\]

\[
X^\alpha := * (W^{\beta\alpha} \wedge \vartheta_\beta), \tag{A.20}
\]

\[
\Phi^\alpha := W^\alpha - \frac{1}{4} W \vartheta_\alpha - \frac{1}{2} e_\alpha \lrcorner (\vartheta^\mu \wedge W_\mu), \tag{A.21}
\]

\[
\Psi^\alpha := X^\alpha - \frac{1}{4} X \vartheta_\alpha - \frac{1}{2} e_\alpha \lrcorner (\vartheta^\mu \wedge X_\mu). \tag{A.22}
\]

The 0-forms \( W \) and \( X \) are the contractions of the corresponding 1-forms, i.e.

\[
W = e_\mu \lrcorner W^\mu, \quad X = e_\mu \lrcorner X^\mu, \tag{A.23}
\]

whereas the contractions of \( \Phi^\alpha \) and \( \Psi^\alpha \) vanish identically, i.e.,

\[
e_\mu \lrcorner \Phi^\mu = 0, \quad e_\mu \lrcorner \Psi^\mu = 0. \tag{A.24}
\]

Furthermore, we find

\[
\Phi^\alpha \land \vartheta^\alpha = 0. \tag{A.25}
\]

For the irreducible pieces of the rotational curvature in terms of these auxiliary 1-forms there results

\[
(2)W^{\alpha\beta} = -* (\vartheta^{[\alpha} \wedge \Psi^{\beta]}), \tag{A.26}
\]

\[
(3)W^{\alpha\beta} = -\frac{1}{12} * (X \wedge \vartheta^{\alpha} \wedge \vartheta^{\beta}), \tag{A.27}
\]

\[
(4)W^{\alpha\beta} = -\frac{1}{2} \vartheta^{[\alpha} \wedge \Phi^{\beta]}, \tag{A.28}
\]

\[
(5)W^{\alpha\beta} = -\frac{1}{2} \vartheta^{[\alpha} \wedge e^{\beta]} \lrcorner (\vartheta^\mu \wedge W_\mu), \tag{A.29}
\]

\[
(6)W^{\alpha\beta} = -\frac{W}{12} \vartheta^{\alpha} \wedge \vartheta^{\beta}, \tag{A.30}
\]

\[
(1)W^{\alpha\beta} = W^{\alpha\beta} - \sum_{n=2}^{6} (n)W^{\alpha\beta}. \tag{A.31}
\]
Appendix B. Decomposing the curvature of our solution

Appendix B.1. Irreducible rotational curvature \( R^{[\alpha\beta]} \)

To characterize the irreducible pieces of the rotational curvature it is of advantage to introduce the following four curvature structure functions \( \Phi_1, \cdots, \Phi_4 \), all depending on the non-ignorable coordinates \((r, \theta)\) as follows:

\[
\begin{align*}
\Phi_1 &:= \frac{L_0^2 a^2 \sin \theta \cos \theta (r^2 - a^2 \cos^2 \theta)}{2 \Delta \Sigma^3} \sqrt{\frac{F}{\Delta}}, \\
\Phi_2 &:= \frac{L_0^2 a r \sin \theta (r^2 - a^2 \cos^2 \theta)}{2 \Delta \Sigma^3} \sqrt{\frac{F}{\Delta}}, \\
\Phi_3 &:= \frac{L_0^2 \cos \theta (r^2 - a^2 \cos^2 \theta)}{\Delta \Sigma^3}, \\
\Phi_4 &:= \frac{L_0^2 r^2 (r^2 - a^2 \cos^2 \theta)}{\Delta \Sigma^3}.
\end{align*}
\]

(B.1)

In this way, the curvature \( R^{[\alpha\beta]} \) has a relatively simple appearance. Observe that these functions are related algebraically according to

\[
\begin{align*}
\Phi_1 r - \Phi_2 a \cos \theta &= 0, \\
\Phi_3 r - \Phi_4 a \cos \theta &= 0.
\end{align*}
\]

(B.2)

The corresponding relations for the torsion functions \( \{v_6\} \) read

\[
\begin{align*}
v_2 r + v_3 a \cos \theta &= 0, \\
(v_1 - v_5) r + v_4 a \cos \theta &= 0.
\end{align*}
\]

(B.3)

Because of

\[
\frac{\Phi_1}{\Phi_2} = \frac{\Phi_3}{\Phi_4} \quad \text{and} \quad \frac{v_2}{v_1 - v_5} = \frac{v_3}{v_4},
\]

(B.4)

these functions are not functionally independent. It is also remarkable that the \( \Phi \)'s, and thus also the rotational curvature \( W^{\alpha\beta} \), depend on the mass only via the function \( \Delta \). However, the constant \( L_0^2 \) appears in all \( \Phi \)'s as proportionality constant.

For the (generalized) Weyl curvature \( (1) W^{\alpha\beta} \) (WEYL) we find

\[
(1) W^{\alpha\beta} = (1) \tilde{W}^{\alpha\beta} + \Phi_1 \begin{pmatrix}
0 & -(\vartheta_0^0 + \vartheta_1^2) & -\vartheta_0^1 & -\vartheta_0^3 - \vartheta_2^3 \\
\vartheta_0^0 & 0 & -\vartheta_0^1 & \vartheta_2^3 \\
\vartheta_0^3 + \vartheta_1^3 & 0 & 0 & \vartheta_0^3 + \vartheta_1^3 \\
\vartheta_0^3 & 0 & -\vartheta_2^3 & 0
\end{pmatrix}
\]

\[
+ \Phi_2 \begin{pmatrix}
0 & \vartheta_0^0 + \vartheta_1^2 & \vartheta_0^1 & \vartheta_0^3 + \vartheta_1^3 \\
\vartheta_0^0 & 0 & -\vartheta_0^3 - \vartheta_2^3 & 0 \\
\vartheta_0^3 + \vartheta_1^3 & 0 & 0 & \vartheta_0^3 + \vartheta_1^3 \\
\vartheta_0^3 & 0 & \vartheta_2^3 & 0
\end{pmatrix},
\]

(B.5)

where \( (1) \tilde{W}^{\alpha\beta} \) denotes the Riemannian part, cf. \( \{v_6\} \).
The pair-commutator \( W^{\alpha\beta} \) (PAIRCOM) turns out to be
\[
W^{\alpha\beta} = \Phi_1 \begin{pmatrix}
0 & \vartheta^{02} + \vartheta^{12} & -\vartheta^{01} & -\vartheta^{23} \\
\circ & 0 & \vartheta^{01} & -\vartheta^{23} \\
\circ & \circ & 0 & \vartheta^{03} + \vartheta^{13} \\
\circ & \circ & \circ & 0
\end{pmatrix}
+ \Phi_2 \begin{pmatrix}
0 & -\vartheta^{03} + \vartheta^{13} & \vartheta^{23} & \vartheta^{01} \\
\circ & 0 & -\vartheta^{23} & -\vartheta^{01} \\
\circ & \circ & 0 & \vartheta^{02} + \vartheta^{12} \\
\circ & \circ & \circ & 0
\end{pmatrix}
+ \Phi_3 \begin{pmatrix}
0 & 0 & -\vartheta^{03} + \vartheta^{13} & \vartheta^{02} + \vartheta^{12} \\
\circ & \circ & 0 & 0 \\
\circ & \circ & \circ & 0 \
\circ & \circ & \circ & 0
\end{pmatrix}.
\] (B.6)

The pseudoscalar part of the curvature (PSCALAR) vanishes identically, i.e.,
\[
W^{\alpha\beta} = 0.
\] (B.7)

The tracefree symmetric Ricci \( W^{\alpha\beta} \) (RICSYM) turns out to be
\[
W^{\alpha\beta} = \Phi_2 \begin{pmatrix}
0 & \vartheta^{03} + \vartheta^{13} & \vartheta^{23} & \vartheta^{01} \\
\circ & 0 & -\vartheta^{23} & -\vartheta^{01} \\
\circ & \circ & 0 & -\vartheta^{02} - \vartheta^{12} \\
\circ & \circ & \circ & 0
\end{pmatrix}
+ \Phi_1 \begin{pmatrix}
0 & -\vartheta^{02} - \vartheta^{12} & -\vartheta^{01} & -\vartheta^{23} \\
\circ & 0 & \vartheta^{01} & -\vartheta^{23} \\
\circ & \circ & 0 & -\vartheta^{03} - \vartheta^{13} \\
\circ & \circ & \circ & 0
\end{pmatrix}
+ \Phi_4 \begin{pmatrix}
0 & 0 & -\vartheta^{02} - \vartheta^{12} & -\vartheta^{03} - \vartheta^{13} \\
\circ & \circ & 0 & 0 \\
\circ & \circ & \circ & 0 \\
\circ & \circ & \circ & 0
\end{pmatrix},
\] (B.8)

and the antisymmetric Ricci \( W^{\alpha\beta} \) (RICANTI) and the curvature scalar part \( W^{\alpha\beta} \) (SCALAR) read, respectively,
\[
W^{\alpha\beta} = \Phi_1 \begin{pmatrix}
0 & \vartheta^{02} + \vartheta^{12} & -\vartheta^{01} & -\vartheta^{23} \\
\circ & 0 & \vartheta^{01} & -\vartheta^{23} \\
\circ & \circ & 0 & -\vartheta^{03} - \vartheta^{13} \\
\circ & \circ & \circ & 0
\end{pmatrix}
+ \Phi_2 \begin{pmatrix}
0 & -\vartheta^{03} + \vartheta^{13} & -\vartheta^{23} & \vartheta^{01} \\
\circ & 0 & \vartheta^{23} & -\vartheta^{01} \\
\circ & \circ & 0 & -\vartheta^{02} - \vartheta^{12} \\
\circ & \circ & \circ & 0
\end{pmatrix}
and
\[
W^{\alpha\beta} = -\frac{\lambda}{3}\vartheta^{\alpha\beta}.
\] (B.10)
Note that (B.10), because of (A.30), is consistent with (30).

Appendix B.2. Decomposition of the strain curvature $Z^{\alpha\beta}$

In view of the complexity of the irreducible piece $^{(1)}Z^{\alpha\beta}$ we will not display it in terms of matrices but give the result just in terms of components, Appendix B.2.

\[ Z^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ \Delta = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ L_a r^2 \sin^2 \theta \cos \theta \left( F_\Delta - \varrho_0 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) \]

\[ L_0 \Delta \varrho_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ \Delta_{\Sigma^3} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ L_a r^2 \sin^2 \theta \cos \theta \left( F_\Delta - \varrho_0 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) + 2 L_0 a^3 r \sin^2 \theta \cos \theta \left( \frac{F_\Delta}{\Delta} \varrho_0 - \varrho_1 \right) \]

\[ L_0 \Delta \varrho_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ \Delta_{\Sigma^3} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{pmatrix} \]
$$Z^ {13} = \frac{3L_0}{2\Delta \Sigma^3} \left(M(2r^4 - a^2 r^2 \cos^2 \theta + a^4 \cos^4 \theta) - 2r(r^2 - a^2 \cos^2 \theta) \left(\frac{\Delta \Sigma^2}{3} + a^2 \sin^2 \theta \right) F \right) \vartheta^{12}$$

$$L_0 \cos \theta \left(3r^2 - a^2 \cos^2 \theta \right) (\mathcal{M} + \Delta) \vartheta^{13} + \frac{L_0 \sin \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\Sigma - 3r^2) \vartheta^{23},$$

$$Z^ {22} = \frac{2L_0 a^2 r^2 \sin \theta \cos \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\vartheta^{02} - \vartheta^{12}),$$

$$Z^ {23} = -\frac{L_0 \sin \theta (3r^2 - \Sigma)}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\vartheta^{02} - \vartheta^{12}) - \frac{L_0 a^2 r^2 \sin \theta \cos \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\vartheta^{03} - \vartheta^{13}),$$

$$Z^ {33} = -\frac{2L_0 \sin \theta (3r^2 - \Sigma)}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\vartheta^{03} - \vartheta^{13}) + \frac{4L_0 a^2 r^2 \sin \theta \cos \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} (\vartheta^{02} - \vartheta^{12}),$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 \sin \theta \cos \theta}{2\Delta \Sigma^3} \left(\begin{array}{ccc}
2 \vartheta^{01} & 2 \vartheta^{02} & -\vartheta^{13} \\
-2 \vartheta^{01} & 2 \vartheta^{02} & -\vartheta^{13} \\
0 & -\vartheta^{02} & \vartheta^{12}
\end{array}\right),$$

$$Z^ {\alpha \beta} = 0, \quad (B.12)$$

$$Z^ {\alpha \beta} = 0, \quad (B.13)$$

$$Z^ {\alpha \beta} = 0, \quad (B.14)$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 (3r^2 - a^2 \cos^2 \theta)}{2\Delta \Sigma^3} \left(\begin{array}{ccc}
2 \vartheta^{01} & 0 & 0 \\
0 & 0 & 0 \\
0 & 2 \vartheta^{02} & \vartheta^{12}
\end{array}\right),$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 (3r^2 - a^2 \cos^2 \theta)}{2\Delta \Sigma^3} \left(\begin{array}{ccc}
0 & 2 \vartheta^{01} & -\vartheta^{13} \\
0 & 2 \vartheta^{01} & -\vartheta^{13} \\
-\vartheta^{02} & -\vartheta^{12} & 0
\end{array}\right),$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 \sin \theta \cos \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} \left(\begin{array}{ccc}
-2 \vartheta^{02} & -\vartheta^{01} & 2 \vartheta^{23} \\
-2 \vartheta^{02} & -\vartheta^{01} & 2 \vartheta^{23} \\
0 & 0 & 0
\end{array}\right),$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 \sin \theta \cos \theta}{\Sigma^3} \sqrt{\frac{F}{\Delta}} \left(\begin{array}{ccc}
-2 \vartheta^{03} & -\vartheta^{01} & 0 \\
-2 \vartheta^{03} & -\vartheta^{01} & 0 \\
0 & 0 & 0
\end{array}\right),$$

$$Z^{\alpha \beta} = \frac{M L_0 a^2 r^2 \sin^2 \theta F}{\Delta \Sigma^2} \left(\begin{array}{ccc}
0 & -\vartheta^{01} & 0 \\
0 & -\vartheta^{01} & 0 \\
-\vartheta^{02} & -\vartheta^{12} & \vartheta^{03} + \vartheta^{13}
\end{array}\right). \quad (B.15)$$
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