PERVERSE SHEAVES AND THE REDUCTIVE BORREL-SERRE COMPACTIFICATION

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Dedicated to Steve Zucker, with great respect and admiration

1. Introduction

This paper is a report on work in progress to better understand the category of perverse sheaves on the Baily-Borel compactification $X^*$ of a locally symmetric variety by using perverse sheaves on the reductive Borel-Serre compactification $\hat{X}$.

1.1. Definition. Perverse sheaves were introduced in [5] by Beilinson, Bernstein, and Deligne following Goresky and MacPherson’s introduction of intersection homology [19]. (For a full account of the exciting discoveries and interactions around that time, see Kleiman’s excellent history [26]). We begin by briefly recalling the definition (and to avoid confusion, we point out, as in [5], that a perverse sheaf is not actually a sheaf nor is it perverse).

Let $Y$ be a stratified pseudomanifold and fix an integer-valued function on the set of strata of $Y$ called the perversity $p$; $1$ if all the strata of $Y$ have even dimension, it is most useful to take for $p$ the middle perversity, $p(S) = -(1/2)\dim S$. Let $D^b_c(Y)$ be the constructible bounded derived category of $Y$; objects may be represented by complexes of sheaves on $Y$ (up to quasi-isomorphism) whose cohomology lives in a bounded range of degrees and is locally constant along each stratum. By definition, a perverse sheaf $P$ on $Y$ (constructible with respect to the given stratification) is an object $S$ whose local cohomology $H(i^*_S S)$ along each stratum $i_S: S \hookrightarrow Y$ lives in degrees $\leq p(S)$ and whose local cohomology supported on $S$, $H(i^!_S S)$, lives in degrees $\geq p(S)$. The perverse sheaves on $Y$ form an abelian category $\mathcal{P}(Y)$ which is Artinian and Noetherian. For every connected stratum $S$ and irreducible local system $E$ on $S$ there is a simple perverse sheaf $P_S(E) = \mathcal{P}_{p,S}(E)$ and all simple objects are obtained in this fashion. In fact $\mathcal{P}_S(E)$ is the intersection cohomology sheaf of $S$ with coefficients in $E$ shifted by $-p(S)$.

When $Y$ is a complex algebraic variety, we will always consider the middle perversity. In addition, one usually does not fix the (algebraic) stratification in the definitions of $D^b_c(Y)$ and $\mathcal{P}(Y)$.

1.2. Two original applications. Perverse sheaves have played a critical role in the topology of algebraic varieties and representation theory from their beginning; we mention just two examples from the early days. For a fuller survey of applications, see [14].

The first example is the Kazdhan-Lusztig conjecture. Let $Y = G/B$ be the flag variety associated to a simply connected semisimple complex algebraic group $G$. It is stratified by

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The origin of the term “perversity” goes back to the creation of intersection homology by Goresky and MacPherson [19] where chains were constrained so that the dimension of their intersection with a singular stratum $S$ was at most $p(S) - p(S_0) - 1$ more than that allowed by transversality ($S_0$ being the open dense stratum).
its $B$-orbits $S_w$ which are indexed by $w$ in the Weyl group $W$ of $G$; each closure $\overline{S_w}$ is a Schubert variety. The correspondence $w \leftrightarrow S_w$ is order preserving in the sense that $y \leq w$ if and only if $S_y \subset S_w$. For $y \leq w \in W$, Kazdhan and Lusztig \cite{24} gave a conjectural formula for the multiplicity of the irreducible $\mathfrak{g}$-module $L_y$ with highest weight $y\rho - \rho$ in the composition series of the Verma module $M_w$ with highest weight $w\rho - \rho$. The formula involved a combinatorially defined polynomial $P_{y,w}(q)$ which appeared to be related to the failure of Poincaré duality on $\overline{S_w}$; later they showed \cite{25} that the coefficients in $P_{y,w}(q)$ were $\dim H^k(i^*_y\mathcal{P}_{S_w}(C))$, the local Betti numbers along $S_y$ of the simple perverse sheaf associated to $S_w$.

The conjecture was resolved independently by Beilinson and Bernstein \cite{6} and by Brylinski and Kashiwara \cite{11} by transferring the problem from representations to perverse sheaves. Specifically one shows that a certain category of $\mathfrak{g}$-modules (including all $L_y$ and $M_w$) is equivalent to the category of regular holonomic $\mathscr{D}_Y$-modules and that by the Riemann-Hilbert correspondence this category is equivalent to $\mathcal{P}(Y)$.

The second original example of the importance of perverse sheaves is the decomposition theorem. This deep result about the topology of proper algebraic maps was first proved by Beilinson, Bernstein, Deligne, and Gabber \cite[Théorème 6.2.5]{5} and later extended by Saito \cite{38, 39}. It says that if $X \to Y$ is a proper morphism of algebraic varieties, then the direct image of a simple perverse sheaf of Hodge type on $X$ decomposes into the direct sum of shifted simple perverse sheaves, likewise of Hodge type, supported on subvarieties of $Y$. Here by a simple perverse sheaf of Hodge type we mean one that corresponds to a polarizable Hodge module in the sense of Saito; by \cite[Lemmes 5.1.10, 5.2.12, 39, Theorem 0.2]{38}, such simple perverse sheaves are precisely those of the form $\mathcal{P}_S(E)$ where $E$ underlies a real polarizable variation of Hodge structure. As a special case, if $\widetilde{Y} \to Y$ is a resolution of singularities, the theorem implies that the ordinary cohomology $H^\cdot(\widetilde{Y}; \mathbb{C})$ is the direct sum of shifted middle perversity intersection cohomology groups of subvarieties of $Y$ with various local systems as coefficients. Furthermore one summand will be $IH^\cdot(Y; \mathbb{C})$, the intersection cohomology of $Y$ itself.

The proof in \cite{5} is for a simple perverse sheaf of geometric origin and proceeds by first dealing with the theorem over the algebraic closure of a finite field and then lifting the result to $\mathbb{C}$. The version of the theorem stated above was proved by Saito \cite{38, 39} using purely characteristic 0 methods. It uses mixed Hodge modules, which are regular holonomic $\mathscr{D}$-modules equipped with various filtrations and satisfying certain conditions; they correspond to the mixed perverse sheaves over finite fields considered in \cite{5}. A pure Hodge module corresponds under the Riemann-Hilbert correspondence to a simple perverse sheaf of Hodge type.

More recently, other proofs and generalizations of the decomposition theorem have been given. A proof of the constant coefficient case using classical Hodge theory was given by de Cataldo and Migliorini \cite{13}; for more details and a discussion of the different approaches to the decomposition theorem see also their survey \cite{14}. Kashiwara has conjectured \cite{23} that the decomposition theorem should hold for any simple perverse sheaf. This conjecture has been proven analytically using polarizable pure twistor $\mathscr{D}$-modules by work of Sabbah \cite{37} and Mochizuki \cite{32, 33}. At about the same time, an arithmetic proof was given independently by Böckle and Khare \cite{7} and by Gaitsgory \cite{17}; they proved de Jong’s conjecture \cite{15} which by Drinfeld \cite{16} implies Kashiwara’s conjecture. Note that Kashiwara actually conjectures the theorem should hold for a simple holonomic $\mathscr{D}_X$-module with possibly irregular singularities; this has been settled by Mochizuki \cite{34}.
1.3. Locally symmetric varieties. Our main interest here is the category of perverse sheaves on a locally symmetric variety. Let \( X = \Gamma \backslash D = \Gamma \backslash G(\mathbb{R})/A_G(\mathbb{R})K \) be an arithmetic quotient of a symmetric space of noncompact type; here \( G \) is a reductive algebraic group defined over \( \mathbb{Q} \), \( K \) a maximal compact subgroup of \( G(\mathbb{R}) \), \( A_G \) is the maximal \( \mathbb{Q} \)-split torus in the center of \( G \), and \( \Gamma \) is an arithmetic subgroup. Our main focus will be when \( D \) is Hermitian symmetric and unless otherwise noted we shall assume that in this introduction, however some non-Hermitian symmetric spaces will arise as well.

A natural choice of compactification in the Hermitian case is the Baily-Borel compactification \( X^* \) [3] (topologically it is one of Satake’s compactifications). The Baily-Borel compactification is a projective algebraic variety defined over a number field; it is commonly called a locally symmetric variety. Locally symmetric varieties and Shimura varieties, their adelic variants, play an important role in number theory, automorphic forms, and the Langlands’s program.

The Baily-Borel compactification \( X^* \) has a natural stratification \( \bigsqcup_Q F_Q \), where \( Q \) ranges over all \( \Gamma \)-conjugacy classes of saturated parabolic \( \mathbb{Q} \)-subgroups of \( G \); when \( G \) is almost \( \mathbb{Q} \)-simple the saturated condition simply means \( Q \) is maximal parabolic \( \mathbb{Q} \)-subgroup. Each \( F_Q \) is again a Hermitian locally symmetric space associated to a reductive group \( L_{Q,h} \), a quotient of \( Q \); its closure \( \overline{F_Q} \) in \( X^* \) is a subvariety and is the Baily-Borel compactification \( F_Q^* \) of \( F_Q \). We fix this stratification when considering perverse sheaves on \( X^* \).

On the locally symmetric space \( X \) one usually considers local systems \( E \) that arise from a representation \( E \) of \( G \) as opposed to merely a representation of \( \Gamma \). We say that the simple perverse sheaf \( \mathcal{P}_X(E) \) on \( X^* \) is reductively constructible if the coefficient system \( E \) arises from a representation \( E \) of \( G \) as above; similarly \( \mathcal{P}_{F_Q}(E) \) is reductively constructible if \( E \) arises from a representation of \( L_{Q,h} \).

Zucker’s conjecture [15] gives an analytic realization of a simple reductively constructible perverse sheaf \( \mathcal{P}_X(E) \). The choice of an admissible inner product [31] on \( E \) induces a metric on \( E \) which we assume fixed. The conjecture then states that there is a natural isomorphism \( \mathcal{P}_X(E) \cong \mathcal{L}_2(X^*;E)[-p(X)] \), where \( \mathcal{L}_2(X^*;E) \) is the sheafification of the presheaf that associates to \( U \subset X^* \) the complex of \( E \)-valued differential forms on \( U \cap X \) which, together with their exterior derivative, are \( L^2 \) with respect to the natural locally symmetric metric on \( X \) and the above metric on \( E \). The conjecture was independently settled at about the same time by Looijenga [29] and Saper and Stern [43] using quite different methods.

The conjecture thus shows that the reductively constructible simple perverse sheaves \( \mathcal{P}_{F_Q}(E) \) are related to representations \( L^2(\Gamma_{L_{Q,h}} \backslash L_{Q,h}(\mathbb{R})) \) (see [10]) and hence to automorphic forms. The full category of (not necessarily simple) perverse sheaves on \( X^* \) is thus clearly worth further study.

1.4. The reductive Borel-Serre compactification. In general \( X^* \) is very singular so it may be profitable to study its perverse sheaves by using a resolution.

A natural choice for an algebraic resolution, though non-unique, is one of the smooth toroidal compactifications \( \tilde{\pi} : \tilde{X}_\Sigma \to X^* \) [1]. Let \( \mathcal{P}_X(E) \) be a reductively constructible simple perverse sheaf on \( X^* \). Let \( U = \tilde{\pi}^{-1}(X) \subset \tilde{X}_\Sigma \); via \( \tilde{\pi} \) we can view \( E \) also as a local system on
U. The decomposition theorem applies\footnote{We only need the version proved by Saito here by the following argument. In the Hermitian case, Zucker \cite{47} has shown E underlies a complex polarizable variation of Hodge structure; it only a real polarizable variation of Hodge structure if the representation E is real. However E \otimes R C does underlie a real polarizable variation of Hodge structure so Saito’s decomposition theorem applies to it. Now E \otimes R C \cong E \oplus \overline{E} and one can prove that the decomposition theorem holds for a direct sum if and only if it holds for each summand.} to show that \( \tilde{\pi}_* P_U(E) \) is the direct sum of (shifted) simple perverse sheaves on \( X^* \), one of which is \( P_X(E) \).

However this argument does not easily extend to treat simple perverse sheaves supported on smaller closed strata of \( X^* \). For example \( \tilde{\pi}^{-1}(F_Q) \) is rarely smooth nor irreducible.

To get around this we look instead at the reductive Borel-Serre compactification \( \hat{X} \). This was introduced by Zucker in the same paper \cite{45} where he made his conjecture about the \( L^2 \)-cohomology of \( X \). (It is actually defined for any arithmetic locally symmetric space, not necessarily Hermitian.) It is non-algebraic (in fact it may have odd dimensional strata) but it is canonically associated to \( X \) and its singularities are easy to describe. Zucker \cite{40} showed that there is a continuous quotient map \( \pi: \hat{X} \to X^* \) extending the identity on \( X \) so that \( \hat{X} \) may be viewed as a partial resolution of singularities. Goresky and Tai \cite{22} show that up to homotopy any \( \tilde{\pi} \) factors through \( \hat{X} \). So in this sense it is a minimal partial resolution.

The reductive Borel-Serre compactification \( \hat{X} \) has a natural stratification \( \coprod_P X_P \), where \( P \) ranges over all \( \Gamma \)-conjugacy classes of parabolic \( \mathbb{Q} \)-subgroups of \( G \). Each \( X_P \) is again an arithmetic locally symmetric space associated to a reductive group, namely \( L_P \), the Levi quotient of \( P \); its closure \( \overline{X}_P \) in \( \hat{X} \) is the reductive Borel-Serre compactification \( \hat{X}_P \) of \( X_P \). We fix this stratification when considering perverse sheaves on \( \hat{X} \). Since \( \hat{X} \) may have odd dimensional strata, there are two middle perversities, \( p_- \) and \( p_+ \) (see (5.1)), that we will consider, both depending only on the dimension of the stratum. A simple perverse sheaf \( P_{X_P}(E) \) on \( \hat{X} \) is reductively constructible if \( E \) is induced from a representation \( E \) of \( L_P \).

Despite \( \hat{X} \) not being algebraic, it “wants” to be algebraic. For example, Zucker showed \cite{48} that its cohomology carries a mixed Hodge structure such that

\[
H^*(X^*) \xrightarrow{\pi^*} H^*(\hat{X}) \xrightarrow{\tilde{i}_*} H^*(X)
\]

are morphisms of mixed Hodge structures. (Here the outer two cohomology groups carry Deligne’s canonical mixed Hodge structure.) Furthermore, Ayoub and Zucker \cite{2} construct a motive corresponding to \( \hat{X} \).

As another example of the algebraic-like nature of \( \hat{X} \) is the conjecture of Rapoport \cite{36} and of Goresky and MacPherson \cite{21}. This conjecture was proved by Saper and Stern \cite{36, Appendix} when the \( \mathbb{Q} \)-rank of \( G \) is 1, and by Saper \cite{40} in general. It states that for a local system \( E \) associated to a representation of \( G \),

\[
(1.1) \quad \pi_* P_{X_G}(E) = P_{F_G}(E);
\]

Here the perversity on the left (for \( \hat{X} \)) can be either of the two middle perversities—one obtains the same pushforward.

Unlike in the situation for \( \tilde{\pi}: \hat{X}_\Sigma \to X^* \), the methods of \cite{40} can be used to generalize (1.1) to all reductively constructible simple perverse sheaves \( P_{X_P}(E) \) on \( \hat{X} \). Specifically (see \cite[\S21]{40}) for every parabolic \( \mathbb{Q} \)-subgroup \( P \), there is a saturated parabolic \( \mathbb{Q} \)-subgroup \( P^\dagger \) so that \( \pi(X_P) = F_{P^\dagger} \). In fact, \( \pi|_{X_P}: X_P \to F_{P^\dagger} \) is a flat bundle which becomes trivial over a finite cover of \( F_{P^\dagger} \); the fiber is \( X_{P,\ell} \), a locally symmetric space not usually of Hermitian
type. We have the following extension of the decomposition theorem despite \( \widehat{X} \) not being algebraic:

**Theorem** (Decomposition theorem for \( \pi: \widehat{X} \to X^* \)). Let \( E \) be an irreducible local system on a stratum \( X_P \) of \( \widehat{X} \) which is induced from an algebraic representation of \( L_P \). Let \( p \) be a middle perversity. Then

\[
\pi_* P_{p,X_P}(E) = \bigoplus_i P_{F^i,F^i}(H^i(\widehat{X}_{P,t}; P_{p,X_P}(E)))[-i].
\]

Note that while the left-hand side a priori depends on the choice of middle perversity, the right-hand side does not since \( X^* \) has only even-dimensional strata. Also the theorem makes it clear that \( \pi_* \) is not injective on objects since the map \( P \mapsto P^\dagger \) is not injective.

Here is a sketch of the proof. It suffices to pass to a finite cover so one can arrange that \( X_P = X_{P,t} \times F^i \) whence \( \widehat{X}_P = \widehat{X}_{P,t} \times \widehat{F}^i \), with \( \pi|_{\widehat{X}_P} \) being projection on the second factor. Now apply the Künneth formula of Cohen, Goresky, and Ji [12] (which one checks is applicable for middle perversities) and apply Rapoport’s conjecture to the second factor.

1.5. **Summary of this paper.** The theorem above shows that it is reasonable to study the category of reductively constructible perverse sheaves on the Baily-Borel compactification \( X^* \) by studying the category of perverse sheaves on the reductive Borel-Serre compactification \( \widehat{X} \). By the above version of the decomposition theorem, we understand \( \pi_* \) on simple objects; the goal of this very modest paper is to begin to understand extensions better in this context. After recalling the basics of \( t \)-structures in §2 we carefully define middle perversity perverse sheaves on a stratified pseudomanifold in §3 playing special attention to the issues that arise due to odd dimensional strata. We also note certain non-trivial extensions that arise due to odd codimension strata. In §4 we indicate how one may calculate extensions between two simple perverse sheaves. After discussing needed background on the reductive Borel-Serre compactification in §5 and the link cohomology of simple perverse sheaves in §6 we conclude in §7 by doing the exercise of calculating all extensions between simple perverse sheaves for \( \widehat{\mathcal{A}}_2 \), the reductive Borel-Serre compactification of the moduli of principally polarized abelian surfaces.

1.6. **A potential application: perverse cohomology.** One reason to better understand extensions is to be able to calculate the *perverse cohomology* of an object \( S \) in \( D_{\text{rc}}(\widehat{X}) \) (or \( \pi_* S \) in \( D_{\text{rc}}^b(X^*) \)), the reductively constructible bounded derived category of sheaves. Recall that the definition of a perverse sheaf started with sheaves as the basic objects, proceeded to the derived category, and within it found the abelian category \( \mathbf{P}(Y) \) of perverse sheaves. Thus a perverse sheaf is represented as a complex of ordinary sheaves. However it has become clear that it is also useful to view perverse sheaves themselves as the basic objects and in some cases the roles of the two types of objects can be reversed. Beilinson [4] shows that if \( Y \) is an algebraic variety and we consider all algebraic stratifications (not a fixed one), then the bounded derived category of \( \mathbf{P}(Y) \) is equivalent to \( D^b_c(Y) \) within which one finds the category of ordinary constructible sheaves. Thus in this setting, a constructible sheaf, or in fact any object \( S \) of \( D^b_c(Y) \), can be represented as a complex of perverse sheaves \( \ldots \to P^{i-1} \to P^i \to P^{i+1} \to \ldots \). The cohomology of this complex \( H^i(P) \) is a perverse sheaf called the perverse cohomology \( p^i \) of \( S \).
Even when Beilinson’s theorem does not apply, as it does not for \( \hat{X} \) since we have fixed a stratification, the perverse cohomology can be defined as \( pH^i(S) = \tau_{<0}\tau_{>0}S[i] \). Furthermore, just like for the ordinary cohomology sheaves of a complex of sheaves, there is a spectral sequence with \( E_2^{ij} = H^j(\hat{Y}; pH^i(S)) \) that abuts to \( H^{i+j}(\hat{Y}; S) \). Many important invariants of \( X \) are realized as the cohomology of complexes of sheaves on \( \hat{X} \), for example, the cohomology of the arithmetic group is \( H(\Gamma; E) = H(\hat{X}; i_X, E) \). Thus calculating \( pH^i(S) \), or at least having bounds on the degrees in which \( H(\hat{X}; pH^i(S)) \) can be nonzero, is important.

One potential approach to calculating, or at least approximating, the perverse cohomology of \( S \) is through its micro-support. In [10] §7, we defined the micro-support \( \text{SS}(S) \) and proved a vanishing theorem [10] §10 for \( H(\hat{X}; S) \) based on \( \text{SS}(S) \). As an application, we were able to prove [12], for example, that in the Hermitian case, \( H^i(\Gamma; E) = 0 \) for \( i < (1/2)\dim X \) provided \( E \) had regular highest weight; this result was independently proved by Li and Schwermer [28] by different methods.

We also determined [10] §17 the micro-support of a simple perverse sheaf \( \mathcal{P}_{X, p}(E) \) provided the \( \mathbb{Q} \)-root system of \( L_P \) did not have a factor of type \( D_n, F_4 \), or \( E_n \) (a restriction that should be removable). The result shows that if a perverse sheaf is semi-simple and satisfies a certain conjugate self-contragredient assumption on the coefficients, the micro-support determines the perverse sheaf. In fact we will show elsewhere that under certain conditions, even for a not necessarily semi-simple perverse sheaf, the micro-support determines the simple constituents and thus potentially the micro-support can be used as a tool to calculate the perverse cohomology.

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It was a great honor to be invited to speak at the birthday conference for Steve Zucker and to contribute to this volume. Steve has always been a source of inspiration and encouragement to me. When I was a graduate student and he was a visitor at the Institute, I remember him sitting down with me in the common room and carefully showing me the derivation of the Poincaré punctured disk metric. In later encounters he always took a deep interest in my work which I appreciated. I also enjoyed long evenings with him spent listening to classical music as well as many delicious meals (particularly Maryland crabs whacked open with a wooden mallet!).

Notation. Morphisms in a category \( \mathbf{C} \) will be denoted \( \text{Mor}_\mathbf{C}(A, B) \); when \( \mathbf{C} \) is a category of representations of a group \( G \), we simply write \( \text{Mor}_G(A, B) \).

If \( A \) is an object of some derived category of sheaves on a space \( Y \) and \( V \subset Y \) is an open subset, we will sometimes abuse notation and also denote by \( A \) the inverse image of \( A \) to the derived category of sheaves on \( V \), that is \( A|_V \).

If \( S \subset Y \), we let \( i_S: S \hookrightarrow Y \) denote the inclusion; we use the same notation to denote the inclusion into any subset of \( Y \) containing \( S \). If \( V \) is an open subset of \( Y \) containing \( S \) for which \( S \) is closed in \( V \), we let \( j_S: U = V \setminus S \hookrightarrow V \). We thus have inclusions of complementary open and closed subsets

\[
U \subset j_S V \subseteq i_S S.
\]
Often \( S \) is a stratum of a stratification of \( Y \) and \( V \) is an open union of strata for which \( S \) is a minimal stratum.

If \( S, S' \subset Y \), we write \( S \preceq S' \) if \( S \subset \overline{S'} \) and \( S \prec S' \) if \( S \subsetneq \overline{S'} \).

2. \( t \)-STRUCTURES

We briefly recall the theory of \( t \)-structures on a triangulated category \( D \) following \([5, \S 1.3]\).

A \( t \)-structure on \( D \) consists of two full subcategories \( D^{\leq 0} \) and \( D^{\geq 0} \), closed under isomorphism, satisfying the following conditions. First set \( D^{\leq k} = D^{\leq 0}[−k] \) and \( D^{\geq k} = D^{\geq 0}[−k] \). We require

\[ (2.1) \quad \text{Mor}_D(A, B) = 0 \text{ for } A \in D^{\leq 0}, B \in D^{\geq 1}. \]

Secondly we require that

\[ (2.2) \quad D^{\leq 0} \subset D^{\leq 1} \quad \text{and} \quad D^{\geq 0} \supset D^{\geq 1}. \]

And finally we assume that for any object \( X \in D \), there exists a distinguished triangle

\[ (2.3) \quad A \to X \to B \xrightarrow{[1]} (A \in D^{\leq 0}, B \in D^{\geq 1}). \]

In fact the distinguished triangle above is unique up to unique isomorphism and one can use it to define truncation functors \( \tau_{\leq 0} X = A \) and \( \tau_{\geq 1} X = B \). By shifting we obtain truncation functors \( \tau_{\leq k} : D \to D^{\leq k} \) and \( \tau_{\geq k} : D \to D^{\geq k} \) for all \( k \in \mathbb{Z} \). There are natural morphisms \( \tau_{\leq k} \to \text{id} \) and \( \text{id} \to \tau_{\geq k} \), which induce adjoint relations

\[ (2.4) \quad \text{Mor}_D(A, B) \cong \text{Mor}_{D^{\leq k}}(A, \tau_{\leq k} B) \quad (A \in D^{\leq k}, B \in D) \]
\[ (2.5) \quad \text{Mor}_D(A, B) \cong \text{Mor}_{D^{\geq k}}(\tau_{\geq k} A, B) \quad (A \in D, B \in D^{\geq k}). \]

The heart of a \( t \)-structure is \( C = D^{\leq 0} \cap D^{\geq 0} \). It is a full abelian subcategory of \( D \). Furthermore, a short exact sequence \( 0 \to C' \to C \to C'' \to 0 \) in \( C \) corresponds to a distinguished triangle \( C' \to C \to C'' \xrightarrow{[1]} \) in \( D \) with all objects belonging to \( C \) and vice-versa. Thus \( \text{Ext}^1_C(C'', C') = \text{Mor}_D(C'', C'[1]). \)

The functor \( H^0 : D \to C \) given by \( H^0(A) = \tau_{\geq 0} \tau_{\leq 0} A = \tau_{\leq 0} \tau_{\geq 0} A \) is a cohomological functor. We set \( H^k(A) = H^0(A[k]) \)

**Lemma 2.1.** If \( A \in D^{\leq k} \) and \( B \in D^{\geq k} \), then \( \text{Mor}_D(A, B) \cong \text{Mor}_C(H^k(A), H^k(B)). \)

**Proof.**

\[
\text{Mor}_D(A, B) = \text{Mor}_D(\tau_{\leq k} A, B) = \text{Mor}_D(\tau_{\geq k} \tau_{\leq k} A, B) = \text{Mor}_D(H^k(A)[−k], B).
\]

Similarly one may replace \( B \) here by \( H^k(B)[−k] \) whence the lemma since \( C \) is full. \( \square \)

In this paper, \( D \) will be the constructible bounded derived category of sheaves on a stratified pseudomanifold \( Y \). The standard \( t \)-structure has \( D^{\leq 0} \) consisting of objects \( S \) satisfying

\[ (2.6) \quad H^k(S) = 0 \text{ for } k > 0, \]

and \( D^{\leq 0} \) consisting of objects \( S \) satisfying

\[ (2.7) \quad H^k(S) = 0 \text{ for } k < 0. \]
In this case, the truncation functors are the usual truncations of a complex, and objects in the heart may be represented by constructible sheaves viewed as complexes with only one nonzero term in degree 0.

In §3 we will define the perverse $t$-structure whose heart is the category of perverse sheaves. The subcategories, truncation functors, and cohomology functors for this $t$-structure will always be distinguished by a left superscript $p$, as in $^p\mathbf{D}^\leq 0$.

3. Perverse sheaves

3.1. Definition. Let $Y$ be a stratified topological pseudomanifold of dimension $n$ with stratification $\mathcal{X} = \{S\}$. We will assume that all strata of $Y$ are connected which implies that the frontier condition holds: the closure of any stratum is a union of strata. We also assume that there are finitely many strata. The strata of $Y$ are partially ordered by $S \leq T$ if and only if $S \subsetneq T$.

We now briefly recall the definition of the category of perverse sheaves on $Y$ following [5] §2.1.

Fix a $\mathbb{Z}$-valued function $p$ on the set of strata, the perversity (in the sense of [5]); we assume the perversity satisfies $p(S) \geq p(T)$ when $S \not\leq T$. Our main interest is when $p$ is one of the two middle perversities:

$$p_-(S) = -\left\lfloor \dim S + 1 \right\rfloor, \quad p_+(S) = -\left\lceil \dim S \right\rceil.$$  

If $Y$ has only even dimensional strata, both $p_-$ and $p_+$ are equal to the self-dual perversity $p_{1/2}(S) = -(1/2)\dim S$. (A non-middle perversity is introduced in [6.4] for technical reasons.)

Let $\mathbf{D}^b_c(Y)$ denote the bounded derived category of $\mathbf{C}$-sheaves on $Y$, constructible with respect to $\mathcal{X}$. (For an algebraic variety one usually does not fix a stratification, but since our main focus here is $Y = \hat{X}$ it seems appropriate.) An object in $\mathbf{D}^b_c(Y)$ may be represented by a complex of sheaves $\mathbf{S}$ on $Y$ whose local cohomology sheaves $\check{H}^i(i_S^*\mathbf{S})$ along a stratum $S$ are locally constant and finitely generated; such a locally constant sheaf may be equivalently viewed as a $\pi_1(S)$-module, once we have picked as base point in $S$ (omitted in the notation). Since $Y$ is a stratified pseudomanifold, it can be shown that $H(i_S^*\mathbf{S})$ is likewise locally constant and finitely generated [9, V,§3].

Define the full subcategory $^p\mathbf{D}^{\leq 0}_c(Y)$ of $\mathbf{D}^b_c(Y)$ to consist of objects $\mathbf{S}$ which satisfy

$$H^k(i_S^*\mathbf{S}) = 0 \text{ for } k > p(S), \text{ for all strata } S;$$

Likewise define $^p\mathbf{D}^{\geq 0}_c(Y)$ to consist of objects $\mathbf{S}$ which satisfy

$$H^k(i_S^*\mathbf{S}) = 0 \text{ for } k < p(S), \text{ for all strata } S.$$

We refer to (3.2) and (3.3) as the perverse vanishing condition and the perverse covanishing condition respectively. As usual, we let $^p\mathbf{D}^{\leq k}_c(Y) = ^p\mathbf{D}^{\leq 0}_c(Y)[−k]$ and $^p\mathbf{D}^{\geq k}_c(Y) = ^p\mathbf{D}^{\geq 0}_c(Y)[−k]$.

The pair $(^p\mathbf{D}^{\leq 0}_c(Y), ^p\mathbf{D}^{\geq 0}_c(Y))$ forms a $t$-structure; its heart, $^p\mathbf{D}^{\leq 0}_c(Y) \cap ^p\mathbf{D}^{\geq 0}_c(Y)$, is the category of $p$-perverse sheaves on $Y$, denoted $\mathbf{P}(Y) = \mathbf{P}_p(Y)$. For $p = p_-$ (resp. $p_+$) we simply write $\mathbf{P}_-(Y)$ (resp. $\mathbf{P}_+(Y)$).

For $\mathbf{S} \in \mathbf{D}^b_c(Y)$, $D_S(i_S^*\mathbf{S}) = i_S^*D_Y(\mathbf{S})$ where $D_Y$ and $D_S$ denote the respective Verdier duality involutions. Set

$$p^*(S) = -p(S) − \dim S,$$
the dual perversity. Then $D_Y$ sends $p$-perverse sheaves to $p^*$-perverse sheaves. From (3.1) we see that $p_-(S) + p_+(S) = -\dim S$ and thus $p^*_\pm = p_\pm$ and $p^*_\pm = p_\pm$. In particular, $D_Y$ interchanges $P_-$($Y$) and $P_+$($Y$).

Warning. In §5.2 after introducing the reductive Borel-Serre compactification, we will impose a condition we call reductively constructible on our objects which is more appropriate to use on $\tilde{X}$. All of the material of the current section as well as §4 hold without change under the assumption of reductively constructible (with a minor exception noted later).

3.2. Simple perverse sheaves. For every stratum $T$ of $Y$ and every irreducible local system $E$ on $T$ (corresponding to an irreducible representation $E$ of $\pi_1(T)$) there is a simple object

$$\mathcal{P}_T(E) = \mathcal{P}_{p,T}(E) = \tau_{\leq p(S_1)} \cdots \tau_{\leq p(S_N)}(i_{T*}E[-p(T)])$$

of $P(Y)$ and these are all the simple objects. Here $S_1, \ldots, S_N$ is an enumeration of the strata $S \prec T$ such that if $S_i \prec S_j$ then $i < j$.

$\mathcal{P}_T(E)$ is supported on $\tilde{T}$ and satisfies the usual perverse vanishing and covanishing condition on $T$:

$$H^k(i^*_T\mathcal{P}_T(E)) = 0 \text{ for } k > p(T) \text{ and } H^k(i^*_T\mathcal{P}_T(E)) = 0 \text{ for } k < p(T).$$

In fact, $i^*_T\mathcal{P}_T(E) = i^*_T\mathcal{P}_T(E) = E[-p(T)]$. However $\mathcal{P}_T(E)$ satisfies stronger conditions on strata smaller than $T$:

$$H^k(i^*_S\mathcal{P}_T(E)) = 0 \text{ for } k \geq p(S), \text{ for all strata } S \prec T$$

and

$$H^k(i^*_S\mathcal{P}_T(E)) = 0 \text{ for } k \leq p(S), \text{ for all strata } S \prec T.$$  

We call (3.5) and (3.6) the intersection cohomology vanishing condition and the intersection cohomology covanishing condition respectively.

Let $S$ be a stratum of $Y$, $V_S$ an open union of strata containing $S$ as a minimal stratum, and $j_S$: $U_S = V_S \setminus S \hookrightarrow V_S$. Define the link cohomology functor $\kappa_S$: $D^b_c(U_S) \rightarrow D^b_c(S)$ by

$$\kappa_SA = i^*_Sj_{S*}A.$$ 

The local cohomology of $\kappa_SA$ is the cohomology of $A$ pulled back to the topological link of the stratum $S$. If $A$ is defined on a subset larger than $U_S$, for example $Y$, we write $\kappa SA$ for $\kappa_S(A|_{U_S})$. Thus it fits into a distinguished triangle for $A \in D^b_c(Y)$:

$$i^*_SA \rightarrow i^*_SA \rightarrow \kappa_SA \xrightarrow{[1]}.$$ 

Now the values of the cohomology groups appearing in (3.5) and (3.6) in the other degrees can be easily calculated:

$$H^k(i^*_S\mathcal{P}_T(E)) \cong H^k(\kappa_S\mathcal{P}_T(E)) \text{ for } k < p(S), \text{ for all strata } S \prec T$$

and

$$H^k(i^*_S\mathcal{P}_T(E)) = H^{k-1}(\kappa_S\mathcal{P}_T(E)) \text{ for } k > p(S), \text{ for all strata } S \prec T.$$ 

All degrees of the $\mathcal{P}_T(E)$-cohomology of the link of $S$ appear, split between $i^*_S\mathcal{P}_T(E)$ and (with a shift) $i^*_S\mathcal{P}_T(E)$. 


3.3. Comparison with intersection cohomology. Assume $p$ is one of the middle perversities $p_-$ or $p_+$ Define
\[(3.9)\]
\[
\bar{p}_T(S) = p(S) - p(T) - 1, \quad (S \preceq T).
\]
Then by pulling out the shift in (3.4) we see that
\[
\mathcal{P}_{p,T}(E) = \mathcal{I}_{\bar{p}_T}(T, E)[-p(T)]
\]
where for $\bar{p}: X(T) \to \mathbb{Z}$ we set
\[(3.10)\]
\[
\mathcal{I}_{\bar{p}}(T, E) = \tau_{\leq \bar{p}(S_1)}j_{S_1}\ldots \tau_{\leq \bar{p}(S_N)}j_{S_N}^*i_{T*}E.
\]
When $\bar{p}(S)$ depends only on $k = \text{codim}_T S$, as is the case for $\bar{p}_T(S)$ from (3.9), the object $\mathcal{I}_{\bar{p}}(T, E)$ is essentially Deligne’s sheaf for intersection cohomology as in [9, V].

In fact $\bar{p}_T(S)$ agrees with one of the two “classical” middle perversities of Goresky and MacPherson [20]:
\[
m(k) = \left\lfloor \frac{k - 2}{2} \right\rfloor, \quad n(k) = \left\lfloor \frac{k - 1}{2} \right\rfloor.
\]
Specifically for $p = p_-,$
\[
\bar{p}_T(S) = \begin{cases} 
   n(\text{codim}_T S) & \text{if dim } T \text{ is odd}, \\
   m(\text{codim}_T S) & \text{if dim } T \text{ is even},
\end{cases}
\]
and for $p = p_+,$
\[
\bar{p}_T(S) = \begin{cases} 
   m(\text{codim}_T S) & \text{if dim } T \text{ is odd}, \\
   n(\text{codim}_T S) & \text{if dim } T \text{ is even},
\end{cases}
\]
Thus $\mathcal{P}_{p,T}(E) = \mathcal{I}_{\bar{p}_T}(T, E)[-p(T)]$ is either $\mathcal{I}_m(\tau_{\leq p(S)}^*j_{S}^*E)$ or $\mathcal{I}_n(\tau_{\leq p(S)}^*j_{S}^*E)$ depending on the parity of dim $T$.

3.4. A non-trivial extension of perverse sheaves. Let $p$ be a middle perversity and let $E$ be an irreducible local system on a stratum $T$ of $Y$. While $\mathcal{P}_{p,T}(E) = \mathcal{I}_{\bar{p}_T}(T, E)[-p(T)]$ is a simple perverse sheaf in $P_p(Y)$, in fact the object
\[
\bar{P}_{p,T}(E) \equiv \mathcal{P}_{p,-T}(E)[p^*(T) - p(T)] = \mathcal{I}_{\bar{p}_T}(T, E)[-p(T)]
\]
is also a perverse sheaf in $P_p(Y)$, though not necessarily simple.

To see this, note that the $p$-perversity condition on the stratum $T$ is satisfied due to the shift. To deal with the other strata, note first that for a middle perversity
\[
p^*(S) - p(S) \in \begin{cases} 
   \{0, 1\} & \text{if } p = p_-, \\
   \{0, -1\} & \text{if } p = p_+,
\end{cases}
\]
for any stratum $S$; here the value 0 occurs if and only if dim $S$ is even. Consequently
\[(3.11)\]
\[
-1 \leq (p^*(S) - p(S)) - (p^*(T) - p(T)) \leq 1
\]
for all strata $S$ and $T$. Now for $S \prec T$, the intersection cohomology vanishing condition (3.3) with respect to $p^*$ implies that
\[
H^k(i_S^*\mathcal{P}_{p^*,T}(E)[p^*(T) - p(T)]) = 0 \text{ for } k \geq p^*(S) - (p^*(T) - p(T)).
\]
This implies the $p$-perversity vanishing condition (3.2) for $k > p(S)$ since
\[
k \geq p(S) + 1 \geq p(S) + (p^*(S) - p(S)) - (p^*(T) - p(T)) = p^*(S) - (p^*(T) - p(T))
\]
by (3.11). The $p$-perverse covanishing condition (3.3) follows similarly.

The natural morphism $\mathcal{I}_m \mathcal{C}(\mathcal{T}, \mathcal{E})[-p(T)] \to \mathcal{I}_n \mathcal{C}(\mathcal{T}, \mathcal{E})[-p(T)]$ thus induces short exact sequences

$$0 \to \mathcal{P}_{p,T}(\mathcal{E}) \to \tilde{\mathcal{P}}_{p,T}(\mathcal{E}) \to \mathcal{P} \to 0 \quad (\text{if } \tilde{p}_T = m)$$

or

$$0 \to \mathcal{P} \to \tilde{\mathcal{P}}_{p,T}(\mathcal{E}) \to \mathcal{P}_{p,T}(\mathcal{E}) \to 0 \quad (\text{if } \tilde{p}_T = n)$$
in $\mathcal{P}_p(Y)$ and hence extensions in $\mathcal{P}_p(Y)$.

If $\mathcal{P}_{p,T}(\mathcal{E}) \neq \tilde{\mathcal{P}}_{p,T}(\mathcal{E})$, the resulting extensions are not trivial. For if $\tilde{\mathcal{P}}_{p,T}(\mathcal{E}) = \mathcal{P} \oplus \mathcal{P}_{p,T}(\mathcal{E})$ in $\mathcal{P}_p(Y)$, then applying $i_S^\sharp$ or $i_S^!$ and shifting by $p(T) - p^*(T)$, we see that $\mathcal{P}[p(T) - p^*(T)]$ and $\tilde{\mathcal{P}}_{p,T}(\mathcal{E}) = \mathcal{P}_{p,T}(\mathcal{E})[p(T) - p^*(T)]$ satisfy the $p^*$-perverse conditions (since $\mathcal{P}_{p^*,T}(\mathcal{E}) = \tilde{\mathcal{P}}_{p,T}(\mathcal{E})[p(T) - p^*(T)]$ does). This implies that $\mathcal{P}_{p^*,T}(\mathcal{E}) = \mathcal{P}[p(T) - p^*(T)] \oplus \tilde{\mathcal{P}}_{p^*,T}(\mathcal{E})$ in $\mathcal{P}_p(Y)$, contradicting the fact that $\mathcal{P}_{p^*,T}(\mathcal{E})$ is simple in $\mathcal{P}_p(Y)$.

Remarks. (i) The inequality $\mathcal{P}_{p,T}(\mathcal{E}) \neq \tilde{\mathcal{P}}_{p,T}(\mathcal{E})$ will occur precisely when there is a odd codimension stratum $S \prec T$ with nonvanishing middle degree link cohomology

$$H^k(\kappa_S \mathcal{P}_{p,T}(\mathcal{E})) \neq 0 \quad (k = (\text{codim}_T S - 1)/2 + p(T)).$$

(ii) If $Y = \tilde{X}$, the reductive Borel-Serre compactification of a locally symmetric variety, and $\mathcal{E}$ is induced from an algebraic representation of $\pi$, then decomposition theorem for $\pi: \tilde{X} \to X^*$ (see [114]) implies that this extension always becomes trivial when pushed down to the Baily-Borel compactification. In fact $\pi_* \mathcal{P}_{p,X}(\mathcal{E}) = \pi_* \tilde{\mathcal{P}}_{p,X}(\mathcal{E})$ by the solution to Rapoport’s conjecture [40] since it applies to both middle perversities and they agree on $X^*$; more generally the theorem shows that $\pi_* \mathcal{P}_{p,X}(\mathcal{E})$ and $\pi_* \tilde{\mathcal{P}}_{p,X}(\mathcal{E})$ differ only in the coefficient system, but in the reductively constructible setting, the category of coefficient systems on a stratum is semisimple.

(iii) When $\mathcal{P} = \bar{\pi}_*$ (with $X$ not necessarily Hermitian), the inequality $\mathcal{P}_{p,T}(\mathcal{E}) \neq \tilde{\mathcal{P}}_{p,T}(\mathcal{E})$ has an intimate relation with the presence of infinite dimensional local $L^2$-cohomology on $\tilde{X}$.

4. Extensions

Fix a perversity $p$ (not necessarily middle). In this section we indicate how one may compute the extensions $\text{Ext}^1_{P(Y)}(\mathcal{P}_T(\mathcal{E}), \mathcal{P}_{T'}(\mathcal{E}'))$ between two simple perverse sheaves corresponding to strata $T, T' \in \mathcal{X}(Y)$. Note that by the theory of $t$-structures,

$$\text{Ext}^1_{P(Y)}(\mathcal{P}_T(\mathcal{E}), \mathcal{P}_{T'}(\mathcal{E}')) = \text{Mor}_{\mathcal{D}_c(Y)}(\mathcal{P}_T(\mathcal{E}), \mathcal{P}_{T'}(\mathcal{E}'))[1].$$

Lemma 4.1. Let $V$ be an open union of strata with $S$ a minimal stratum of $V$; set $U = V \setminus S$. For $A, B \in \mathcal{D}_c(U)$ and $k \in \mathbb{Z}$, assume $\text{Mor}_{\mathcal{D}_c(U)}(j_S^* A, j_S^* B[k - 1]) = \text{Mor}_{\mathcal{D}_c(U)}(j_S^* A, j_S^* B[k]) = 0$. Then $\text{Mor}_{\mathcal{D}_c(V)}(A, B[k]) \cong \text{Mor}_{\mathcal{D}_c(S)}(i_S^! A, i_S^! B[k])$.

Proof. The distinguished triangle $i_S^! j_S^* B \to B \to j_S^* j_S^* B \Rightarrow$ yields a long exact sequence

$$\text{Mor}_{\mathcal{D}_c(V)}(A, j_S^* j_S^* B[k - 1]) \to \text{Mor}_{\mathcal{D}_c(V)}(A, i_S^! j_S^* B[k]) \to \text{Mor}_{\mathcal{D}_c(V)}(A, B[k]) \to \text{Mor}_{\mathcal{D}_c(V)}(A, j_S^* j_S^* B[k]).$$
The outer two groups are zero by hypothesis, thus
\[ \text{Mor}_{D^b(V)}(A, B[k]) \cong \text{Mor}_{D^b(S)}(A, {i}_S^* {i}_S B[k]) \cong \text{Mor}_{D^b(S)}(i_S^* A, i_S^* B[k]). \]

**Lemma 4.2.** If all maximal strata of \( \overline{T} \cap \overline{T'} \) are strictly smaller than \( T \) and \( T' \), then \( \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[k]) = 0 \) for \( k \leq 1 \).

**Proof.** Let \( V \) be an open union of strata containing \( Y \setminus \overline{T} \cap \overline{T'} \). We will prove that
\[ \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[k]) = 0, \quad k \leq 1, \]
by induction on the number of strata in \( V \). When \( V = Y \) this proves the lemma.

If \( V = Y \setminus \overline{T} \cap \overline{T'} \), the supports of \( \mathcal{P}_T(E) \) and \( \mathcal{P}_{T'}(E') \) are disjoint and
\[ \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[k]) = 0, \quad \text{for all } k. \]

For larger \( V \), let \( S \) be a minimal stratum in \( V \cap (\overline{T} \cap \overline{T'}) \) and set \( U = V \setminus S \). By Lemma 4.1 (which applies by the inductive hypothesis)
\[ \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[k]) \cong \text{Mor}_{D^b(S)}(i_S^* \mathcal{P}_T(E), i_S^* \mathcal{P}_{T'}(E')[k]), \]
for \( k \leq 1 \). Since \( S \) is dominated by a maximal stratum in \( \overline{T} \cap \overline{T'} \) which is thus strictly smaller than \( T \) and \( T' \), the intersection cohomology vanishing and covanishing conditions apply to yield \( i_S^* \mathcal{P}_T(E) \in D^{-p(S)+1}_c(S) \) and \( i_S^* \mathcal{P}_{T'}(E')[k] \in D^{-p(S)+1-k}_c(S) \). Now \( p(S) - 1 < p(S) + 1 - k \) for \( k \leq 1 \), so \( \text{Mor}_{D^b(S)}(i_S^* \mathcal{P}_T(E), i_S^* \mathcal{P}_{T'}(E')[k]) = 0 \) by (2.1).

We conclude that a non-trivial extension can only exist when \( T = T' \), \( T < T' \), or \( T > T' \).

**Lemma 4.3.** Let \( V_T = (Y \setminus \overline{T}) \cup T \) and similarly for \( T' \).

(i) If \( T = T' \), \( \text{Mor}_{D^b(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[1]) \cong \text{Mor}_{D^b(T)}(E, E'[1]) \).

(ii) If \( T < T' \), \( \text{Mor}_{D^b(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[1]) \cong \text{Mor}_{\pi_1(T)}(E, H^p(T)(\kappa_T \mathcal{P}_{T'}(E'))). \)

(iii) If \( T > T' \), \( \text{Mor}_{D^b(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[1]) \cong \text{Mor}_{\pi_1(T)}(H^p(T)^{-1}(\kappa_T \mathcal{P}_{T'}(E)), E'). \)

**Proof.** The case \( T = T' \) follows from Lemma 4.1. For \( T < T' \) we calculate
\[ \text{Mor}_{D^b(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E')[1]) \cong \text{Mor}_{D^b(T)}(E - p(T), i_T^* \mathcal{P}_{T'}(E')[1]) \text{ (by Lemma 4.1)} \]
\[ \cong \text{Mor}_{\pi_1(T)}(E, H^p(T) + 1(i_T^* \mathcal{P}_{T'}(E'))) \text{ (by 3.6 and Lemma 2.1)} \]
\[ \cong \text{Mor}_{\pi_1(T)}(E, H^p(T)(\kappa_T \mathcal{P}_{T'}(E'))) \text{ by 3.8}. \]

The case where \( T > T' \) is similar.

**Lemma 4.4.** Assume \( S < T \) and \( S < T' \). Let \( V \) be an open union of strata with \( S \subset V \) a minimal stratum and set \( U = V \setminus S \). A morphism \( \phi_U \in \text{Mor}_{D^b(U)}(j_S^* \mathcal{P}_T(E), j_S^* \mathcal{P}_{T'}(E')[1]) \) prolongs to a morphism \( \phi_V \) over \( V \) if and only if
\[ H^{p(S) - 1}(\kappa_S(\phi_U)) = 0 \text{ in } \text{Mor}_{\pi_1(S)}(H^{p(S) - 1}(\kappa_S \mathcal{P}_T(E)), H^{p(S)}(\kappa_S \mathcal{P}_{T'}(E'))). \]

If this prolongation exists it is unique.

**Proof.** We apply \( \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \cdot) \) to the distinguished triangle \( i_S^* j_S^* \mathcal{P}_T(E') \to \mathcal{P}_{T'}(E') \to i_S^* j_S^* \mathcal{P}_T(E') \xrightarrow{[1]} \) which yields, after using adjointness, the long exact sequence
\[ \text{Mor}_{D^b(S)}(i_S^* \mathcal{P}_T(E), i_S^* \mathcal{P}_{T'}(E') [1]) \to \text{Mor}_{D^b(V)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E') [1]) \to \text{Mor}_{D^b(S)}(i_S^* \mathcal{P}_T(E), i_S^* \mathcal{P}_{T'}(E') [2]). \]
The condition (4.1) needed to prolong an extension can be nontrivial even in $\mathbb{U}$ for all mainly on its structure rather than its construction. We recall it following [18], focusing to zero in the last term, which by a similar calculation and using Lemma 2.1 is

$$\text{Mor}_{\pi_1(S)}(H^{p(S)-1}(i_S^* \mathcal{P}_T(E)), H^{p(S)-1}(i_S^* \mathcal{P}_{T'}(E'))[2])).$$

By (5.7) and (5.8) this is the group of morphisms in (4.1).

Note that the map on link cohomology at $S$ which needs to be zero in order to prolong an extension is the map from the highest degree of link cohomology occurring in $i_S^* \mathcal{P}_T(E)$ to the lowest degree of link cohomology that does not occur in $i_S^* \mathcal{P}_{T'}(E').$

This section is summarized in the following

**Proposition 4.5.** $\text{Ext}^1_{\mathcal{P}(Y)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))$ is zero unless $T = T'$, $T \prec T'$ or $T \succ T'$.

(i) For $T = T'$, $\text{Ext}^1_{\mathcal{P}(Y)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))$ is isomorphic to a subgroup of

$$\text{Mor}_{\mathcal{D}_c(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))[1] \cong \text{Mor}_{\mathcal{D}_c(T)}(E, E'[1]).$$

(ii) For $T \prec T'$, $\text{Ext}^1_{\mathcal{P}(Y)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))$ is isomorphic to a subgroup of

$$\text{Mor}_{\mathcal{D}_c(V_T)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))[1] \cong \text{Mor}_{\pi_1(T)}(E, H^{p(T)}(\kappa_T \mathcal{P}_T(E'))).$$

(iii) For $T \succ T'$, $\text{Ext}^1_{\mathcal{P}(Y)}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))$ is isomorphic to a subgroup of

$$\text{Mor}_{\mathcal{D}_c(V_{T'})}(\mathcal{P}_T(E), \mathcal{P}_{T'}(E'))[1] \cong \text{Mor}_{\pi_1(T')}(H^{p(T')-1}(\kappa_T \mathcal{P}_T(E)), E').$$

In either case, the group of extensions consists of those morphisms $\phi$ that recursively satisfy

$$H^{p(S)-1}(\kappa_S(\phi|_{U_S})) = 0 \quad \text{in} \quad \text{Mor}_{\pi_1(S)}(H^{p(S)-1}(\kappa_S \mathcal{P}_T(E)), H^{p(S)}(\kappa_S \mathcal{P}_{T'}(E'))).$$

for all $S$ satisfying $S \prec T$ and $S \prec T'$. Here $V_S$ is an open union of strata containing $S$ as a minimal stratum and $U_S = V_S \setminus S$; the condition above allows one to uniquely extend $\phi$ from $U_S$ to $V_S$.

**Remarks.** (i) The condition (4.1) needed to prolong an extension can be nontrivial even in the case $T = T'$; one example of an extension of coefficient systems failing this condition (and hence not prolonging to an extension of perverse sheaves) is [14, Examples 2.2.5 and 2.7.1]. The point is that the intermediate extension function $E \mapsto \mathcal{P}_T(E)$ (3.4) is not exact even though it preserves injective and surjective maps [14, §2.7].

(ii) In §5.2 we will define a more refined notion of constructibility in which local systems on a stratum $T$ are associated to algebraic representations of a reductive group $L_T$. All the results of this section hold in this context except one should replace $\text{Mor}_{\pi_1(T)}$ by $\text{Mor}_{L_T}$ as needed. In addition, since algebraic representations of a reductive group are semisimple, there are no extensions between simple objects when $T = T'$.

## 5. Reductive Borel-Serre compactification

The reductive Borel-Serre compactification $\tilde{X}$ first appeared in work of Zucker [45] and has grown in importance far beyond its original use. We recall it following [18], focusing mainly on its structure rather than its construction.
5.1. Stratification. For $G$ a reductive algebraic group defined over $Q$ and $\Gamma$ an arithmetic subgroup, let $X = X_G = \Gamma \backslash G(R)/A_G(R)K$ be the corresponding locally symmetric space. Here $A_G$ is the maximal $Q$-split torus in the center of $G$, and $K$ is a maximal compact subgroup of $G(R)$.

For a parabolic $Q$-subgroup $S$ of $G$, let $L_S = S/N_S$ denote its reductive Levi quotient, where $N_S$ denotes the unipotent radical of $S$; it is also defined over $Q$. We have an almost direct product $L_S = M_S A_S$ where $M_S = 0L_S = \bigcap_\chi \ker \chi^2$, where $\chi$ runs over characters of $L_S$ defined over $Q$. Let $\Gamma_{N_S} = \Gamma \cap N_S$ and $\Gamma_{L_S} = (\Gamma \cap S)/(\Gamma \cap N_S)$ be the induced arithmetic subgroups. Starting from $L_S$ and $\Gamma_{L_S}$ we again obtain a locally symmetric space $X_S$. If $S'$ is $\Gamma$-conjugate to $S$, then $X_{S'}$ and $X_S$ may be canonically identified.

The reductive Borel-Serre compactification of $X$ is the stratified pseudomanifold $\hat{X} = \bigsqcup_S X_S$ (where $S$ ranges over the $\Gamma$-conjugacy classes of parabolic $Q$-subgroups of $G$) endowed with an appropriate topology. The closure of a stratum $X_S$ is denoted $\hat{X}_S$; it is indeed the reductive Borel-Serre compactification of $X_S$ and its strata $X_T$ correspond to $\Gamma$-conjugacy classes of parabolic $Q$-subgroups having a representative $T \subseteq S$.

From now on we will abuse notation by using the same letter $S$ to denote a stratum $X_S$, the corresponding $\Gamma$-conjugacy class of parabolic $Q$-subgroups, and a representative of that class. Thus the partial order $S \leq T$ on strata corresponds on parabolic subgroups to $S \subseteq \gamma T$ for some $\gamma \in \Gamma$.

The depth of a stratum $S$ of $\hat{X}$, depth $S$, is the maximal length $d$ of a chain $S = S_0 \prec S_1 \prec \cdots \prec S_d = X$. If $S \leq T$, the relative depth of $S$ viewed as a stratum of $\hat{T}$, depth$_T S$, is the maximal length of a chain $S = S_0 \prec S_1 \prec \cdots \prec S_d = T$. We have $\text{depth}_T S = \dim A_S - \dim A_T$.

An algebraic representation $E$ of $L_S$ induces a representation of $\Gamma_{L_S}$ and hence a local system $E$ on $S$.

5.2. Reductive constructibility. A constructible complex of sheaves $S$ on a pseudomanifold has the property that for all strata $S$, the cohomology sheaves of $i_S^* S$, $i_S^! S$, and $\kappa_S S$ are all finitely generated locally constant and hence are associated to finite-dimensional representations of $\Gamma_{L_S}$. Furthermore the maps on cohomology induced by the distinguished triangle $i_S^! S \to i_S^* S \to \kappa_S S \overset{[1]}{\to}$ are morphisms of $\Gamma_{L_S}$-representations. A reductively constructible complex of sheaves $S$ on $\hat{X}$ is as above but has been enriched with extra structure so that all of these locally constant sheaves arise from algebraic representations of $L_S$ and that the morphisms above are morphisms of $L_S$-modules. Morphisms between such sheaves must also induce morphisms of $L_S$-modules on the cohomology sheaves over a stratum.

Rather than construct the derived category of reductively constructible sheaves $D_{\text{rc}}^b(\hat{X})$ directly as suggested above, we note instead that if one starts with the category of $\mathcal{L}$-modules constructed in \[40\] and pass to the homotopy category, one obtains the desired category of reductively constructible sheaves. (In the homotopy category of $\mathcal{L}$-modules, every quasi-isomorphism is already an isomorphism and thus one does not need to localize further.) We will discuss the details elsewhere.

Exactly as in \[33\] one can define a perverse $t$-structure on $D_{\text{rc}}^b(\hat{X})$ and obtain a category of perverse sheaves which we again denote $\mathcal{P}(\hat{X})$. The description of simple objects is the same except we start with $E$ on $T$ coming from an algebraic representation of $L_T$. The fact that the pushforward functors in the definition of the simple perverse sheaves do preserve reductive constructibility is a consequence of the theorems of Nomizu, van Est, and Kostant.
which we recall in [3]. The results in [4] all remain true with identical proofs however one must replace \( \pi_1(S) \)-morphisms with \( L_S \)-morphisms in Lemmas [1.3] and [4.4].

Enriching our objects \( \mathcal{S} \) so that \( H(i^*_S \mathcal{S}) \) has an action of the central split torus \( A_S \subset L_S \) is new data. However aside from this, passing from the constructible category to the reductively constructible category is not as major a change as it may appear at first glance due to the Borel density theorem [3] and Margulis superrigidity [30].

5.3. Links. We recall the link of a stratum \( S \) of \( \hat{X} \) following [18] (see also [41 §7]).

For any parabolic \( \mathbb{Q} \)-subgroup \( S \), the Levi quotient \( L_S \) acts (via a lift to \( S \)) by conjugation on the Lie algebra \( n_S \) of \( N_S \). Though this action depends on the lift, the weights by which \( A_S \) acts on \( n_S \) are well-defined and have a unique basis denoted \( \Delta_S \). If \( S \subset S' \), then \( N_{S'} \triangleleft N_S \) and group \( A_{S'} \) may be naturally viewed as a subgroup of \( A_S \). We let \( \Delta_S^{S'} \subseteq \Delta_{S'} \) be those weights that restrict trivially to \( A_{S'} \). The correspondence \( S' \mapsto \Delta_S^{S'} \) is an order preserving bijection between parabolic \( \mathbb{Q} \)-subgroups containing \( S \) and subsets of \( \Delta_S \). Restriction to \( A_{S'} \) yields a bijection between \( \Delta_S \setminus \Delta_S^{S'} \) and \( \Delta_{S'} \). Note that \( \Delta_S^S = \Delta_S \).

Let \( |\Delta_S| \) be a topological simplex with vertices indexed by the elements of \( \Delta_S \). Give \( |\Delta_S| \) the stratification by its open faces \( |\Delta_S^{S'}| = \text{int} |\Delta_S^{S'}| \) indexed by \( S' \supset S \). Note that

\[
\dim |\Delta_S| = \text{depth } S.
\]

The topological link of the stratum \( S \subset \hat{X} \) is

\[
\text{Lk}_S = \left( \Gamma_{N_S \setminus N_S(\mathbb{R})} \times |\Delta_S| \right) / \sim
\]

where if \( t \in |\Delta_S^{S'}|, (n, t) \sim (n', n, t) \) for all \( n' \in N_{S'}(\mathbb{R}) \). Thus the intersection of the link of \( S \) with higher strata \( S' \supset S \) is

\[
\text{Lk}_S \cap S' = \Gamma_{N_S^{S'} \setminus N_S^{S'}(\mathbb{R})} \times |\Delta_S^{S'}|,
\]

where \( N_S^{S'} = N_S / N_{S'} \) and \( \Gamma_{N_S^{S'}} = \Gamma_{N_S} / \Gamma_{N_{S'}} \).

More generally, the topological link of \( S \) viewed as a stratum of \( \overline{T} \) (for \( S \ll T \)) is

\[
\text{Lk}_S^T = \left( \Gamma_{N_S^{S} \setminus N_{S}^{S}(\mathbb{R})} \times |\Delta_S^{S}| \right) / \sim
\]

and

\[
\dim |\Delta_S^S| = \text{depth}_S T.
\]

If \( S \ll S' \ll T \), we have again

\[
\text{Lk}_S^T \cap S' = \text{Lk}_S \cap S' \Gamma_{N_S^{S'} \setminus N_S^{S'}(\mathbb{R})} \times |\Delta_S^{S'}|.
\]

6. Link cohomology in \( \hat{X} \)

To actually calculate extensions we need to understand the link cohomology functor \( \kappa_S = i^*_S \iota_* S \) on a stratum \( S \). We calculate the link cohomology of \( \mathcal{P}_T(\mathcal{E}) \) for strata \( S \) of relative depth 1, 2, and 3, following the formula given in [40 §5.5] (see also [41 §18.4]).
6.1. Kostant’s theorem. The calculation of $H(\kappa_S \mathcal{P}_T(E))$ given later will involve the Lie algebra cohomology $H(n_S^T, E)$ where $n_S^T$ is the Lie algebra of $N_S^T$. The adjoint action of $L_S$ on $n_S^T$ (via choice of a lift) induces an defined action of $L_S$ on $H(n_S^T, E)$ which is independent of the lift. We begin by recalling Kostant’s theorem \cite{27} which gives a decomposition of $H(n_S^T, E)$ as an $L_S$-module.

We first consider $T = G$ and let $E$ be an irreducible algebraic representation of $G$. Choose a Cartan subalgebra for the Lie algebra $I_S$ of $L_S$; it lifts to a Cartan subalgebra of the Lie algebra $g$ of $G$. Choose an order on the roots of $g$ so that the roots in $n_S$ are all positive. Let $\rho$ be one-half the sum of the positive roots of $g$. Let $W = W^G$ be the Weyl group of the root system of $g$ with corresponding length function $\ell(w)$. Let $W^S \subseteq W$ be the subgroup generated by the simple reflections in roots of $I_S$. Let $W_S \subseteq W$ be the set of unique minimal length representatives of cosets in $W^S \setminus W$. Thus there is a product decomposition $W = W^SW_S$. For $E$ an irreducible algebraic representation of $G$ with highest weight $\lambda$, Kostant’s theorem says

$$H(n_S, E) = \bigoplus_{w \in W_S} H^{\ell(w)}(n_S, E),[-\ell(w)]$$

where $H^{\ell(w)}(n_S, E)_w$ is an irreducible $L_S$-module and has highest weight $w(\lambda + \rho) - \rho$.

Consider now $H(n_S^T, E)$ where $S < T$ and $E$ is an irreducible algebraic representation of $L_T$ with highest weight $\lambda$. We replace $G$ above by $L_T$. Thus let $\rho^T$ be one-half the sum of the positive roots of $I_T$, let $W^T$ be the Weyl group of the root system of $I_T$. We have a decomposition $W^T = W^S W_S^T$, where $W_S^T$ consists of the minimal length representatives of the cosets in $W^S \setminus W^T$. Kostant’s theorem here says

$$H(n_S^T, E) = \bigoplus_{w \in W_S^T} H^{\ell(w)}(n_S^T, E)_w[-\ell(w)]$$

where $H^{\ell(w)}(n_S^T, E)_w$ has highest weight $w(\lambda + \rho^T) - \rho^T$.

6.2. Relative depth 1. Let $E$ be a local system on $T$ corresponding to an algebraic representation $E$ of $L_T$. By (5.4), since $\text{depth}_T S = 1$, the link $Lk_T S$ is simply the compact nilmanifold $\Gamma_{N_S^T} \setminus N_S^T(R)$. Thus

(6.1) $$H(\kappa_S \mathcal{P}_T(E)) = H(\Gamma_{N_S^T} \setminus N_S^T(R); E)[-p(T)] \cong H(n_S^T, E)[-p(T)] .$$

Here we use the theorem of Nomizu and van Est \cite{35} \cite{44} for the last isomorphism with Lie algebra cohomology. In fact the action of $L_S$ on $n_S^T$ induces an action on $H(n_S^T, E)$ and the isomorphism in (6.1) is an isomorphism of $L_S$-modules.

If we combine this with Kostant’s theorem we obtain

(6.2) $$H(\kappa_S \mathcal{P}_T(E)) \cong \bigoplus_{w \in W_S^T} H^{\ell(w)}(n_S^T, E)_w[-\ell(w) - p(T)] .$$

Note that to go from here to $H(i_S^T \mathcal{P}_T(E))$, according to (3.5) and (3.7), we need to truncate this cohomology, leaving just degrees $< p(S)$. Thus the sum now only include $w$ satisfying $\ell(w) + p(T) \leq p(S) - 1$, or

$$\ell(w) \leq p(S) - p(T) - 1 = \tilde{p}_T(S) .$$
Thus for relative depth 1

\[ H(\gamma_S^* P_T(E)) \cong \bigoplus_{w \in W_S^T, \ell(w) \leq \bar{p}_T(S)} H^{\ell(w)}(n_S^T, E)_w[-\ell(w) - p(T)] \]

(compare \[3.3\]).

6.3. **Relative depth 2.** If \( \text{depth}_T S = 2 \), the simplex \(|\Delta_T^S|\) is a 1-simplex. The endpoints correspond to the two intermediate strata: \( S \prec Q_1, Q_2 \prec T \). The cohomology of the interior of the link is the same as \[6.2\], however each term may be truncated at \( Q_1 \) or at \( Q_2 \) or at both — see \[3.5\] as well as \[6.3\]. The terms that remain are those that are not truncated at either side or (with an additional shift by \(-1\)) those that are truncated at both sides. This corresponds to the fact that the cohomology of a 1-simplex relative to either one of its endpoints vanishes, while the cohomology relative to both endpoints is one dimensional in degree 1.

To precisely express these side truncations, write \( W_T = W^{Q_1}W^{Q_2}Q_1 = W_S^TQ_1 = W_1^TQ_1 \). In fact \( W_1^T = W_S^TQ_1 \) and for \( w \in W_S^T \) we decompose \( w = w^{Q_1}w_Q \), accordingly. Define

\[ \ell_{Q_1}(w) = \ell(w_{Q_1}) \quad (w = w^{Q_1}w_Q) \]

Then we have the formula

\[ H(\kappa_S^* P_T(E)) \cong \bigoplus_{w \in W_S^T, \ell_{Q_1}(w) \leq \bar{p}_T(Q_1), \ell_{Q_2}(w) \leq \bar{p}_T(Q_2)} H^{\ell(w)}(n_S^T, E)_w[-\ell(w) - p(T) - 1] \]

6.4. **Relative depth 3 and higher.** One can similarly write an explicit formula for \( H(\kappa_S^* P_T(E)) \) when \( \text{depth}_T S = 3 \) however there are many more possible configurations of truncations on the 6 intermediate strata \( Q \); these are illustrated in [11], §18.4, Figure 24 and result in additional degree shifts of 0, \(-1\), and \(-2\).

There is a general formula for arbitrary depth that involves the intersection cohomology of \(|\Delta_T^S|\) associated to a special perversity:

\[ \bar{p}_{T,w}(Q) = \bar{p}_T(Q) - \ell_Q(w) = p(Q) - p(T) - 1 - \ell_Q(w) \quad (S \prec Q \prec T) \]

Let

\[ I_{\bar{p}_{T,w}}H(|\Delta_T^S|) = H(I_{\bar{p}_{T,w}}C(|\Delta_T^S|, \mathbb{Z})) \]

be the hypercohomology of the corresponding Deligne sheaf \[3.10\]. Then

\[ H(\kappa_S^* P_T(E)) \cong \bigoplus_{w \in W_S^T} H^{\ell(w)}(n_S^T, E)_w[-\ell(w) - p(T)] \otimes I_{\bar{p}_{T,w}}H(|\Delta_T^S|) \].
7. Example Computations

To illustrate the results of §4 we calculate all extensions between simple perverse sheaves for the reductive Borel-Serre compactification $\hat{X}$ of the locally symmetric spaces associated with $G = \text{Sp}(4, \mathbb{R})$. Since $\text{Ext}_P^1(P_T(E), P_{T'}(E')) = 0$ unless $T < T'$ or $T > T'$, we can assume both $T$ and $T'$ are standard parabolic $Q$-subgroups.

For brevity we only consider $p = p_-$ and we give complete details for $T > T'$; the results for $T < T'$ are summarized in §7.4.

7.1. Preliminaries. Here the root system is type $C_2$ with simple $Q$-roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\}$. The standard parabolic $Q$-subgroups $P_I$ correspond to subsets $I \subseteq \{1, 2\}$ ($\Delta_P = \{\alpha_i\}_{i \in I} \subseteq \Delta$) and so form a lattice

$$
\begin{array}{c}
G \\
\text{P}_1 \\
\text{P}_2 \\
\text{P}_\emptyset
\end{array}
$$

The real dimension of $X$ is 6, while strata corresponding to $P_1$ and $P_2$ have dimension 2, and that corresponding to $P_\emptyset$ has dimension 0. Thus the perversity values $p_-(S)$ are

$$
\begin{array}{c}
-3 \\
-1 \\
-1 \\
0
\end{array}
$$

To calculate link cohomology we need the Weyl group which is generated by the reflections $s_1$ and $s_2$ in the simple roots modulo the relation $(s_1s_2)^3 = e$:

$$
W = W_{P_\emptyset} = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2\}.
$$

For the intermediate strata we have

$$
W_{P_1} = W_{P_1}^{P_1} = \{e, s_1\} \quad \text{and} \quad W_{P_2} = W_{P_2}^{P_2} = \{e, s_2\}
$$

and

$$
W_{P_1} = \{e, s_2, s_2s_1, s_2s_1s_2\} \quad \text{and} \quad W_{P_2} = \{e, s_1, s_1s_2, s_1s_2s_1\}.
$$

7.2. The case $T = G$. We first set $T = G$ and $E$ an irreducible algebraic representation of $G$ with highest weight $\lambda$. The associated classical perversity $\bar{p}_G(S) = p(S) - p(G) - 1$ has values

$$
\begin{array}{c}
-1 \\
1 \\
1 \\
2
\end{array}
$$

We look for nonzero extensions in $\text{Ext}_P^1(P_G(E), P_{T'}(E'))$ for $T' < G$, where $E'$ is an irreducible algebraic representation of $L_{T'}$. To start constructing such an extension, according to Lemma [4.3][iii] we need an $L_{T'}$-module morphism

$$
H^{p(T')-1}(\kappa_{T'} P_G(E)) \longrightarrow E'.
$$
We calculate the above link cohomology. We begin with $T' = P_0$ for which we use (6.4). In degree $p(T') - 1$, a contribution from the first direct sum of (6.4) requires $\ell(w) = \bar{\rho}_G(P_0) = 2$ and no truncation on either side. This means $w = s_1s_2$ or $s_2s_1$. Now $\ell_{P_1}(s_1s_2) = 1 \leq 1 = \bar{\rho}_G(P_1)$ (since $s_1s_2 = (s_1)(s_2) \in W^{F_1}W_{P_1}$) and $\ell_{P_2}(s_1s_2) = 2 \leq 1 = \bar{\rho}_G(P_2)$ (since $s_1s_2 = (e)(s_1s_2) \in W^{F_1}W_{P_1}$). Thus $s_1s_2$ cannot contribute to the link cohomology since the corresponding class is truncated on one side but not the other. Similarly $s_2s_1$ does not contribute since $\ell_{P_1}(s_2s_1) = 2 \leq 1$ and $\ell_{P_2}(s_2s_1) = 1 \leq 1$.

A contribution in degree $p(T') - 1$ from the second direct sum of (6.4) requires $\ell(w) = \bar{\rho}_G(P_0) - 1 = 1$ and truncation on both sides. Thus $w = s_1$ or $w_2$. For $w = s_1$, $\ell_{P_1}(s_1) = 0 \neq \bar{\rho}_G(P_1) = 1$ and $\ell_{P_2}(s_1) = 1 \neq \bar{\rho}_G(P_2) = 1$ and thus does not contribute, and similarly for $w = s_2$.

Thus $H^{p(P_0)-1}(\kappa_{P_0}\mathcal{P}_G(E)) = 0$ and
\[
\text{Ext}^1_{\mathcal{P}_-}(\mathcal{P}_G(E), \mathcal{P}_{P_0}(E')) = 0
\]
for all $E'$.

We now consider the link cohomology at $T' = P_1$ or $P_2$. Since we are interested in degree $p(T') - 1$, a contribution from (6.2) requires $\ell(w) = \bar{\rho}_G(T') = 1$. Thus
\[
H^{p(T')-1}(\kappa_{T'}\mathcal{P}_G(E)) = \begin{cases} 
H^1(n_{P_1}, E)_{s_2} & T' = P_1, \\
H^1(n_{P_2}, E)_{s_1} & T' = P_2.
\end{cases}
\]

This means that $\text{Ext}^1_{\mathcal{P}_-}(\mathcal{P}_G(E), \mathcal{P}_{P_1}(E'))$ can only be nonzero (in which case it is $C$) when $E'$ has highest weight $s_2(\lambda + \rho) - \rho$. By Lemma 4.4, this extension will actually exist provided the map of $L_{P_1}$-modules
\[
\phi: H^{-2}(\kappa_{P_1}\mathcal{P}_G(E)) = H^1(n_{P_1}, E)_{s_2} \longrightarrow E'
\]
(which is an isomorphism), induces the zero map
\[
H^{p(P_0)-1}(\kappa_{P_0}\mathcal{P}_G(E)) \longrightarrow H^{p(P_0)}(\kappa_{P_0}\mathcal{P}_{P_1}(E')).
\]
However we have already seen that $H^{p(P_0)-1}(\kappa_{P_0}\mathcal{P}_G(E)) = 0$, thus this condition is satisfied.

So we find that
\[
\text{Ext}^1_{\mathcal{P}_-}(\mathcal{P}_G(E), \mathcal{P}_{P_1}(E')) = \begin{cases} 
C & \text{if } E' \text{ has highest weight } s_2(\lambda + \rho) - \rho, \\
0 & \text{otherwise}.
\end{cases}
\]

A similar argument shows
\[
\text{Ext}^1_{\mathcal{P}_-}(\mathcal{P}_G(E), \mathcal{P}_{P_2}(E')) = \begin{cases} 
C & \text{if } E' \text{ has highest weight } s_1(\lambda + \rho) - \rho, \\
0 & \text{otherwise}.
\end{cases}
\]

7.3. The case $T = P_1$ and $P_2$. Fix $i = 1$ or $2$, and say $E$ is an irreducible algebraic representation of $L_{P_i}$ with highest weight $\lambda$. By Lemma 4.3(iii)
\[
\text{Ext}^1_{\mathcal{P}_-}(\mathcal{P}_{P_i}(E), \mathcal{P}_{P_0}(E')) \cong \text{Mor}_{L_{P_0}}(H^{p(P_0)-1}(\kappa_{P_0}\mathcal{P}_{P_i}(E)), E')
\]
for any irreducible algebraic representation $E'$ of $L_{P_0}$. Since $L_{P_0}$ is just the maximal torus, $E' = C_\chi$ where $\chi$ is the character by which $L_{P_0}$ acts. We calculate by (6.2) that

$$H^{p(P_0)-1}(\kappa_{P_0} P_{P_i}(E)) = \bigoplus_{w \in W_{P_i}} H^\ell(w)(n_{P_i}^{P_0}, E) = H^0(n_{P_i}^{P_0}, E)$$

since $\bar{p}_{P_i}(P_0) = 0$. Thus we find

$$\text{(7.5)} \quad \text{Ext}^1_{P_{-}}(P_{P_i}(E), P_{P_0}(E')) = \begin{cases} C & \text{if } E' = C_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

### 7.4. Results for $T \prec T'$.

Let $E'$ be an irreducible algebraic representation of $L_{T'}$ with highest weight $\lambda$. Similarly to the above, one may check that for $T \prec T'$, the only nonzero extensions are as follows:

$$\text{(7.6)} \quad \text{Ext}^1_{P_{-}}(P_{P_0}(E), P_{P_1}(E')) = \begin{cases} C & \text{if } E = C_{s_1(\lambda + \rho) - \rho}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(7.7)} \quad \text{Ext}^1_{P_{-}}(P_{P_0}(E), P_{P_2}(E')) = \begin{cases} C & \text{if } E = C_{s_2(\lambda + \rho) - \rho}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(7.8)} \quad \text{Ext}^1_{P_{-}}(P_{P_1}(E), P_G(E')) = \begin{cases} C & \text{if } E \text{ has highest weight } s_2s_1(\lambda + \rho) - \rho, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(7.9)} \quad \text{Ext}^1_{P_{-}}(P_{P_2}(E), P_G(E')) = \begin{cases} C & \text{if } E \text{ has highest weight } s_1s_2(\lambda + \rho) - \rho, \\ 0 & \text{otherwise.} \end{cases}$$

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