Strong Insertion of a Contra-continuous Function between Two Comparable Contra-precontinuous (Contra-semi-continuous) of Real-valued Functions

Majid Mirmiran¹ and Binesh Naderi²

¹Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
e-mail: mirmir@sci.ui.ac.ir

²Department of General Courses, School of Management and Medical Information Sciences, Isfahan University of Medical Sciences, Isfahan, Iran
e-mail: naderi@mng.mui.ac.ir

Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int} (\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [21], while the concept of a, locally dense, set was introduced by Corson and Michael [4].
The concept of a semi-open set in a topological space was introduced by Levine in 1963 [18]. A subset \( A \) of a topological space \( (X, \tau) \) is called semi-open [10] if \( A \subseteq Cl(\text{Int}(A)) \). A set \( A \) is called semi-closed if its complement is semi-open or equivalently if \( \text{Int}(Cl(A)) \subseteq A \).

A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them \( V \)-sets. Complements of \( V \)-sets, i.e., sets that are intersection of open sets are called \( \Lambda \)-sets [20].

Recall that a real-valued function \( f \) defined on a topological space \( X \) is called \( A \)-continuous [28] if the preimage of every open subset of \( \mathbb{R} \) belongs to \( A \), where \( A \) is a collection of subsets of \( X \). Most of the definitions of function used throughout this paper are consequences of the definition of \( A \)-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 26].

Hence, a real-valued function \( f \) defined on a topological space \( X \) is called contra-continuous (resp. contra-semi-continuous, contra-precontinuous) if the preimage of every open subset of \( \mathbb{R} \) is closed (resp. semi-closed, pre-closed) in \( X \) [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that \( \Lambda \)-sets or kernel of sets are open [20].

If \( g \) and \( f \) are real-valued functions defined on a space \( X \), we write \( g \leq f \) in case \( g(x) \leq f(x) \) for all \( x \) in \( X \).

The following definitions are modifications of conditions considered in [16].

A property \( P \) defined relative to a real-valued function on a topological space is a \( cc \)-property provided that any constant function has property \( P \) and provided that the sum of a function with property \( P \) and any contra-continuous function also has property \( P \). If \( \mathbb{R} \)
and $P_2$ are cc-properties, the following terminology is used: (i) A space $X$ has the weak cc-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-continuous function $h$ such that $g \leq h \leq f$. (ii) A space $X$ has the strong cc-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-continuous function $h$ such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose $\Lambda$-sets or kernel of sets are open, is given a sufficient condition for the weak cc-insertion property. Also for a space with the weak cc-insertion property, we give necessary and sufficient conditions for the space to have the strong cc-insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion and strong insertion of a contra-$\alpha$-continuous function and insertion of a contra-continuous function between two comparable real-valued functions has also recently considered by the authors in [22, 23, 24].

2. The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

The abbreviations $cc$, $cpc$ and $csc$ are used for contra-continuous, contra-precontinuous and contra-semi-continuous, respectively.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^\Lambda$ and $A^V$ as follows:

$$A^\Lambda = \bigcap \{O : O \supseteq A, O \in (X, \tau)\} \text{ and } A^V = \bigcup \{F : F \subseteq A, F^c \in (X, \tau)\}.$$ 

In [7, 19, 25], $A^\Lambda$ is called the kernel of $A$.

The family of all preopen, preclosed, semi-open and semi-closed will be denoted by $pO(X, \tau)$, $pC(X, \tau)$, $sO(X, \tau)$, and $sC(X, \tau)$, respectively.

We define the subsets $p(A^\Lambda)$, $p(A^V)$, $s(A^\Lambda)$ and $s(A^V)$ as follows:

$$p(A^\Lambda) = \bigcap \{O : O \supseteq A, O \in pO(X, \tau)\}.$$
\[ p(A^V) = \bigcup \{ F : F \subseteq A, F \in pC(X, \tau) \}. \]

\[ s(A^\Lambda) = \bigcap \{ O : O \supseteq A, O \in sO(X, \tau) \}. \]

and

\[ s(A^V) = \bigcup \{ F : F \subseteq A, F \in sC(X, \tau) \}. \]

\( p(A^\Lambda) \) (resp. \( s(A^\Lambda) \)) is called the prekernel (resp. semi-kernel) of \( A \).

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \), then \( \overline{\rho} \) is defined as follows: \( x \overline{\rho} y \) if and only if \( y \rho v \) implies \( x \rho v \) and \( u \rho x \) implies \( u \rho y \) for \( u \) and \( v \) in \( S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a strong binary relation in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

1. If \( A_i \rho B_j \) for any \( i \in \{ 1, ..., m \} \) and for any \( j \in \{ 1, ..., n \} \), then there exists a set \( C \) in \( P(X) \) such that \( A_i \rho C \) and \( C \rho B_j \) for any \( i \in \{ 1, ..., m \} \) and any \( j \in \{ 1, ..., n \} \).
2. If \( A \subseteq B \), then \( A \overline{\rho} B \).
3. If \( A \rho B \), then \( A^\Lambda \subseteq B \) and \( A \subseteq B^V \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f, \ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f, \ell) \) is called a lower indefinite cut set in the domain of \( f \) at the level \( \ell \).

We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), in which kernel sets are open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the
power set of X and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \leq A(g, t_2) \), then there exists a contra-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on the \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \leq A(g, t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( \mathbb{Q} \) into the power set of \( X \) by \( F(t) = A(f, t) \) and \( G(t) = A(g, t) \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then \( F(t_1) \leq F(t_2) \), \( G(t_1) \leq G(t_2) \), and \( F(t_1) \leq G(t_2) \). By Lemmas 1 and 2 of [15] it follows that there exists a function \( H \) mapping \( \mathbb{Q} \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \leq H(t_2) \), \( H(t_1) \leq H(t_2) \) and \( H(t_1) \leq G(t_2) \).

For any \( x \) in \( X \), let \( h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \): If \( x \) is in \( H(t) \), then \( x \) is in \( G(r') \) for any \( r' > t \); since \( x \) is in \( G(r') = A(g, r') \) implies that \( g(x) \leq r' \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(r') \) for any \( r' < t \); since \( x \) is not in \( F(r') = A(f, r') \) implies that \( f(x) > r' \), it follows that \( f(x) \geq t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1, t_2) = H(t_2)^\Lambda \setminus H(t_1)^\Lambda \). Hence \( h^{-1}(t_1, t_2) \) is closed in \( X \), i.e., \( h \) is a contra-continuous function on \( X \).

The above proof used the technique of Theorem 1 in [14].

If a space has the strong \( cc \)-insertion property for \((P_1, P_2)\), then it has the weak \( cc \)-insertion property for \((P_1, P_2)\). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak \( cc \)-insertion property to satisfy the strong \( cc \)-insertion property.

---

*Earthline J. Math. Sci. Vol. 3 No. 2 (2020), 279-295*
Theorem 2.2. Let \( P_1 \) and \( P_2 \) be cc-property and \( X \) be a space that satisfies the weak cc-insertion property for \( (P_1, P_2) \). Also assume that \( g \) and \( f \) are functions on \( X \) such that 
\[ g \leq f, \quad g \text{ has property } P_1 \text{ and } f \text{ has property } P_2. \]
The space \( X \) has the strong cc-insertion property for \( (P_1, P_2) \) if and only if there exist lower cut sets \( A(f - g, 2^{-n}) \) and there exists a sequence \( \{F_n\} \) of subsets of \( X \) such that (i) for each \( n \), \( F_n \) and \( A(f - g, 2^{-n}) \) are completely separated by contra-continuous functions, and (ii) 
\[ \{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n. \]

Proof. Suppose that there is a sequence \( (A(f - g, 2^{-n})) \) of lower cut sets for \( f - g \) and suppose that there is a sequence \( \{F_n\} \) of subsets of \( X \) such that 
\[ \{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n \]
and such that for each \( n \), there exists a contra-continuous function \( k_n \) on \( X \) into \([0, 2^{-n}]\) with \( k_n = 2^{-n} \) on \( F_n \) and \( k_n = 0 \) on \( A(f - g, 2^{-n}) \). The function \( k \) from \( X \) into \([0, 1/4]\) which is defined by
\[ k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \]
is a contra-continuous function by the Cauchy condition and the properties of contra-continuous functions, (1) \( k^{-1}(0) = \{x \in X : (f - g)(x) = 0\} \) and (2) if \( (f - g)(x) > 0 \), then \( k(x) < (f - g)(x) \): In order to verify (1), observe that if \( (f - g)(x) = 0 \), then \( x \in A(f - g, 2^{-n}) \) for each \( n \) and hence \( k_n(x) = 0 \) for each \( n \). Thus \( k(x) = 0 \).
Conversely, if \( (f - g)(x) > 0 \), then there exists an \( n \) such that \( x \in F_n \) and hence \( k_n(x) = 2^{-n} \). Thus \( k(x) \neq 0 \) and this verifies (1). Next, in order to establish (2), note that
\[ \{x \in X : (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n}) \]
and that $(A(f - g, 2^{-n}))$ is a decreasing sequence. Thus if $(f - g)(x) > 0$, then either $x \not\in A(f - g, 1/2)$ or there exists a smallest $n$ such that $x \not\in A(f - g, 2^{-n})$ and $x \in A(f - g, 2^{-j})$ for $j = 1, ..., n - 1$.

In the former case,

$$k(x) = \frac{1}{4} \sum_{n=1}^{\infty} k_n(x) \leq \frac{1}{4} \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f - g)(x),$$

and in the latter,

$$k(x) = \frac{1}{4} \sum_{j=n}^{\infty} k_j(x) \leq \frac{1}{4} \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x).$$

Thus $0 \leq k \leq f - g$ and if $(f - g)(x) > 0$, then $(f - g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$, then $g(x) < g_1(x) < f_1(x) < f(x)$.

Since $P_1$ and $P_2$ are cc-properties, then $g_1$ has property $P_1$ and $f_1$ has property $P_2$. Since by hypothesis $X$ has the weak cc-insertion property for $(P_1, P_2)$, then there exists a contra-continuous function $h$ such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$, then $g(x) < h(x) < f(x)$. Therefore $X$ has the strong cc-insertion property for $(P_1, P_2)$. (The technique of this proof is by Lane [16].)

Conversely, assume that $X$ satisfies the strong cc-insertion for $(P_1, P_2)$. Let $g$ and $f$ be functions on $X$ satisfying $P_1$ and $P_2$ respectively such that $g \leq f$. Thus there exists a contra-continuous function $h$ such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$. We follow an idea contained in Powderly [27]. Now consider the functions $0$ and $f - h$. $0$ satisfies property $P_1$ and $f - h$ satisfies property $P_2$. Thus there exists a contra-continuous function $h_1$ such that $0 \leq h_1 \leq f - h$ and if $0 < (f - h)(x)$ for any $x$ in $X$, then $0 < h_1(x) < (f - h)(x)$. We next show that

$$\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$
If \( x \) is such that \((f - g)(x) > 0\), then \( g(x) < f(x) \). Therefore \( g(x) < h(x) < f(x) \). Thus \( f(x) - h(x) > 0 \) or \((f - h)(x) > 0\). Hence \( h_1(x) > 0 \). On the other hand, if \( h_1(x) > 0 \), then since \((f - h) \geq h_1\) and \( f - g \geq f - h \), therefore \((f - g)(x) > 0\). For each \( n \), let

\[
A(f - g, 2^{-n}) = \{ x \in X : (f - g)(x) \leq 2^{-n} \},
\]

\[
F_n = \{ x \in X : h_1(x) \geq 2^{-n+1} \}
\]

and

\[
k_n = \sup \{ \inf \{ h_1, 2^{-n+1} \}, 2^{-n} \} - 2^{-n}.
\]

Since \( \{ x \in X : (f - g)(x) > 0 \} = \{ x \in X : h_1(x) > 0 \} \), it follows that

\[
\{ x \in X : (f - g)(x) > 0 \} = \bigcup_{n=1}^{\infty} F_n.
\]

We next show that \( k_n \) is a contra-continuous function which completely separates \( F_n \) and \( A(f - g, 2^{-n}) \). From its definition and by the properties of contra-continuous functions, it is clear that \( k_n \) is a contra-continuous function. Let \( x \in F_n \). Then, from the definition of \( k_n \), \( k_n(x) = 2^{-n} \). If \( x \in A(f - g, 2^{-n}) \), then since \( h_1 \leq f - h \leq f - g \), \( h_1(x) \leq 2^{-n} \). Thus \( k_n(x) = 0 \), according to the definition of \( k_n \). Hence \( k_n \) completely separates \( F_n \) and \( A(f - g, 2^{-n}) \). \( \square \)

**Theorem 2.3.** Let \( P_1 \) and \( P_2 \) be cc-properties and assume that the space \( X \) satisfied the weak cc-insertion property for \((P_1, P_2)\). The space \( X \) satisfies the strong cc-insertion property for \((P_1, P_2)\) if and only if \( X \) satisfies the strong cc-insertion property for \((P_1, cc)\) and for \((cc, P_2)\).

**Proof.** Assume that \( X \) satisfies the strong cc-insertion property for \((P_1, cc)\) and for \((cc, P_2)\). If \( g \) and \( f \) are functions on \( X \) such that \( g \leq f \), \( g \) satisfies property \( P_1 \), and \( f \) satisfies property \( P_2 \), then since \( X \) satisfies the weak cc-insertion property for \((P_1, P_2)\) there is a contra-continuous function \( k \) such that \( g \leq k \leq f \). Also, by hypothesis there exist contra-continuous functions \( h_1 \) and \( h_2 \) such that \( g \leq h_1 \leq k \) and if \( g(x) < k(x) \),

http://www.earthlinepublishers.com
then \( g(x) < h_1(x) < k(x) \) and such that \( k \leq h_2 \leq f \) and if \( k(x) < f(x) \), then \( k(x) < h_2(x) < f(x) \). If a function \( h \) is defined by \( h(x) = (h_2(x) + h_1(x))/2 \), then \( h \) is a contra-continuous function, \( g \leq h \leq f \). and if \( g(x) < f(x) \), then \( g(x) < h(x) < f(x) \). Hence \( X \) satisfies the strong cc-insertion property for \( (P_1, P_2) \).

The converse is obvious since any contra-continuous function must satisfy both properties \( P_1 \) and \( P_2 \). (The technique of this proof is by Lane [17].)

\[ \square \]

3. Applications

Before stating the consequences of Theorems 2.1, 2.2, and 2.3 we suppose that \( X \) is a topological space whose kernel sets are open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then \( X \) has the weak cc-insertion property for \( (cpc, cpc) \) (resp. \( (csc, csc) \)).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on \( X \), such that \( f \) and \( g \) are \( cpc \) (resp. \( csc \)), and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( p(A^\Lambda) \subseteq p(B^V) \) (resp. \( s(A^\Lambda) \subseteq s(B^V) \)), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( Q \) with \( t_1 < t_2 \), then

\[ A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2); \]

since \( \{ x \in X : f(x) \leq t_1 \} \) is a preopen (resp. semi-open) set and since \( \{ x \in X : g(x) < t_2 \} \) is a preclosed (resp. semi-closed) set, it follows that \( p(A(f, t_1)^\Lambda) \subseteq p(A(g, t_2)^V) \) (resp. \( s(A(f, t_1)^\Lambda) \subseteq s(A(g, t_2)^V) \). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 2.1. \[ \square \]

**Corollary 3.2.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \), there exist closed sets \( F_1 \) and \( F_2 \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then every contra-precontinuous (resp. contra-semi-continuous) function is contra-continuous.
Proof. Let $f$ be a real-valued contra-precontinuous (resp. contra-semi-continuous) function defined on $X$. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function $h$ such that $g = h = f$. □

Corollary 3.3. If for each pair of disjoint preopen (resp. semi-open) sets $G_1$, $G_2$ of $X$, there exist closed sets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ has the cc-insertion property for (cpc, cpc) (resp. (csc, csc)).

Proof. Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are cpc (resp. csc), and $g < f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since $g$ and $f$ are contra-continuous functions hence $h$ is a contra-continuous function. □

Corollary 3.4. If for each pair of disjoint subsets $G_1$, $G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist closed subsets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ has the weak cc-insertion property for (cpc, csc) and (csc, cpc).

Proof. Let $g$ and $f$ be real-valued functions defined on $X$, such that $g$ is cpc (resp. csc) and $f$ is csc (resp. cpc), with $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $s(A^\lambda) \subseteq p(B^V)$ (resp. $p(A^\lambda) \subseteq s(B^V)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);$$

since $\{ x \in X : f(x) \leq t_1 \}$ is a semi-open (resp. preopen) set and since $\{ x \in X : g(x) < t_2 \}$ is a preclosed (resp. semi-closed) set, it follows that $s(A(f, t_1)^\lambda) \subseteq p(A(g, t_2)^V)$ (resp. $p(A(f, t_1)^\lambda) \subseteq s(A(g, t_2)^V)$). Hence $t_1 < t_2$, implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. □

Before stating consequences of Theorems 2.2, 2.3 we state and prove the necessary lemmas.
Lemma 3.1. The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $G_1$, $G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist closed subsets $F_1$, $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$.

(ii) If $G$ is a semi-open (resp. preopen) subset of $X$ which is contained in a preclosed (resp. semi-closed) subset $F$ of $X$, then there exists a closed subset $H$ of $X$ such that $G \subseteq H \subseteq H^\Lambda \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where $G$ and $F$ are semi-open (resp. preopen) and preclosed (resp. semi-closed) subsets of $X$, respectively. Hence, $F^c$ is a preopen (resp. semi-open) and $G \cap F^c = \emptyset$.

By (i) there exists two disjoint closed subsets $F_1$, $F_2$ such that $G \subseteq F_1$ and $F_1^c \subseteq F_2$. But

$$F_1^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since $F_2^c$ is an open subset containing $F_1$, we conclude that $F_1^\Lambda \subseteq F_2^\Lambda$, i.e.,

$$G \subseteq F_1 \subseteq F_1^\Lambda \subseteq F.$$

By setting $H = F_1$, condition(ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $G_1$, $G_2$ are two disjoint subsets of $X$, such that $G_1$ is preopen and $G_2$ is semi-open.

This implies that $G_2 \subseteq G_1^c$ and $G_1^c$ is a preclosed subset of $X$. Hence by (ii) there exists a closed set $H$ such that $G_2 \subseteq H \subseteq H^\Lambda \subseteq G_1^c$. 

*Earthline J. Math. Sci. Vol. 3 No. 2 (2020), 279-295*
But

\[ H \subseteq H^\Lambda \Rightarrow H \cap (H^\Lambda)^c = \emptyset \]

and

\[ H^\Lambda \subseteq G_1^c \Rightarrow G_1 \subseteq (H^\Lambda)^c. \]

Furthermore, \((H^\Lambda)^c\) is a closed subset of \(X\). Hence \(G_2 \subseteq H, G_1 \subseteq (H^\Lambda)^c\) and \(H \cap (H^\Lambda)^c = \emptyset\). This means that condition (i) holds. \(\square\)

**Lemma 3.2.** Suppose that \(X\) is a topological space. If each pair of disjoint subsets \(G_1, G_2\) of \(X\), where \(G_1\) is preopen and \(G_2\) is semi-open, can be separated by closed subsets of \(X\), then there exists a contra-continuous function \(h : X \to [0, 1]\) such that \(h(G_2) = \{0\}\) and \(h(G_1) = \{1\}\).

**Proof.** Suppose \(G_1\) and \(G_2\) are two disjoint subsets of \(X\), where \(G_1\) is preopen and \(G_2\) is semi-open. Since \(G_1 \cap G_2 = \emptyset\), hence \(G_2 \subseteq G_1^c\). In particular, since \(G_1^c\) is a preclosed subset of \(X\) containing the semi-open subset \(G_2\) of \(X\), by Lemma 3.1, there exists a closed subset \(H_{1/2}\) such that

\[ G_2 \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq G_1^c. \]

Note that \(H_{1/2}\) is also a preclosed subset of \(X\) and contains \(G_2\), and \(G_1^c\) is a preclosed subset of \(X\) and contains the semi-open subset \(H_{1/2}^\Lambda\) of \(X\). Hence, by Lemma 3.1, there exists closed subsets \(H_{1/4}\) and \(H_{3/4}\) such that

\[ G_2 \subseteq H_{1/4} \subseteq H_{1/4}^\Lambda \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq H_{3/4} \subseteq H_{3/4}^\Lambda \subseteq G_1^c. \]

By continuing this method for every \(t \in D\), where \(D \subseteq [0, 1]\) is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets \(H_t\) with the property that if \(t_1, t_2 \in D\) and \(t_1 < t_2\), then \(H_{t_1} \subseteq H_{t_2}\). We define the function \(h\) on \(X\) by \(h(x) = \inf \{t : x \in H_t\}\) for \(x \notin G_1\) and \(h(x) = 1\) for \(x \in G_1\).
Note that for every \( x \in X, \ 0 \leq h(x) \leq 1 \), i.e., \( h \) maps \( X \) into \([0, 1]\). Also, we note that for any \( t \in D, G_2 \subseteq H_t \); hence \( h(G_2) = \{0\} \). Furthermore, by definition, \( h(G_1) = \{1\} \). It remains only to prove that \( h \) is a contra-continuous function on \( X \). For every \( \alpha \in \mathbb{R} \), we have if \( \alpha \leq 0 \), then \( \{ x \in X : h(x) < \alpha \} = \emptyset \) and if \( 0 < \alpha \), then \( \{ x \in X : h(x) < \alpha \} = \bigcup \{H_t : t < \alpha\} \), hence, they are closed subsets of \( X \). Similarly, if \( \alpha < 0 \), then \( \{ x \in X : h(x) > \alpha \} = X \) and if \( 0 \leq \alpha \), then \( \{ x \in X : h(x) > \alpha \} = \bigcup \{(H_t)^c : t > \alpha\} \), hence, every of them is a closed subset. Consequently \( h \) is a contra-continuous function.

\[ \square \]

**Lemma 3.3.** Suppose that \( X \) is a topological space. If each pair of disjoint subsets \( G_1, G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open, can separate by closed subsets of \( X \), and \( G_1 \) (resp. \( G_2 \)) is a closed subsets of \( X \), then there exists a contra-continuous function \( h : X \to [0, 1] \) such that, \( h^{-1}(0) = G_1 \) (resp. \( h^{-1}(0) = G_2 \)) and \( h(G_2) = \{1\} \) (resp. \( h(G_1) = \{1\} \)).

**Proof.** Suppose that \( G_1 \) (resp. \( G_2 \)) is a closed subset of \( X \). By Lemma 3.2, there exists a contra-continuous function \( h : X \to [0, 1] \) such that, \( h(G_1) = \{0\} \) (resp. \( h(G_2) = \{0\} \)) and \( h(X \setminus G_1) = \{1\} \) (resp. \( h(X \setminus G_2) = \{1\} \)). Hence, \( h^{-1}(0) = G_1 \) (resp. \( h^{-1}(0) = G_2 \)) and since \( G_2 \subseteq X \setminus G_1 \) (resp. \( G_1 \subseteq X \setminus G_2 \)), therefore \( h(G_2) = \{1\} \) (resp. \( h(G_1) = \{1\} \)). \[ \square \]

**Lemma 3.4.** Suppose that \( X \) is a topological space such that every two disjoint semi-open and preopen subsets of \( X \) can be separated by closed subsets of \( X \). The following conditions are equivalent:

(i) For every two disjoint subsets \( G_1 \) and \( G_2 \) of \( X \), where \( G_1 \) is preopen and \( G_2 \) is semi-open, there exists a contra-continuous function \( h : X \to [0, 1] \) such that, \( h^{-1}(0) = G_1 \) (resp. \( h^{-1}(0) = G_2 \)) and \( h^{-1}(1) = G_2 \) (resp. \( h^{-1}(1) = G_1 \)).

(ii) Every preopen (resp. semi-open) subset of \( X \) is a closed subsets of \( X \).

(iii) Every preclosed (resp. semi-closed) subset of \( X \) is an open subsets of \( X \).
Proof. (i) ⇒ (ii) Suppose that $G$ is a preopen (resp. semi-open) subset of $X$. Since $\emptyset$ is a semi-open (resp. preopen) subset of $X$, by (i) there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G$. Set $F_n = \left\{ x \in X : h(x) < \frac{1}{n} \right\}$. Then for every $n \in \mathbb{N}$, $F_n$ is a closed subset of $X$ and $\bigcap_{n=1}^{\infty} F_n = \{ x \in X : h(x) = 0 \} = G$.

(ii) ⇒ (i) Suppose that $G_1$ and $G_2$ are two disjoint subsets of $X$, where $G_1$ is preopen and $G_2$ is semi-open. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0, 1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \left\{ x \in X : f(x) < \frac{1}{2} \right\}$, $F = \left\{ x \in X : f(x) = \frac{1}{2} \right\}$, and $H = \left\{ x \in X : f(x) > \frac{1}{2} \right\}$. Then $G \cup F$ and $H \cup F$ are two open subsets of $X$ and $(G \cup F) \cap G_2 = \emptyset$. By Lemma 3.3, there exists a contra-continuous function $g : X \to \left[\frac{1}{2}, 1\right]$ such that, $g^{-1}(l) = G_2$ and $g(G \cup F) = \left\{\frac{1}{2}\right\}$. Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then $h$ is well-defined and a contra-continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0, 1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) ⇔ (iii) By De Morgan law and noting that the complement of every open subset of $X$ is a closed subset of $X$ and complement of every closed subset of $X$ is an open subset of $X$, the equivalence is hold.

Corollary 3.5. If for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is preopen (resp. semi-open) and $G_2$ is semi-open (resp. preopen), there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, then $X$ has the strong cc-insertion property for $(cpc, csc)$ (resp. $(csc, cpc)$).

Proof. Since for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is preopen (resp. semi-open) and $G_2$ is semi-open (resp. preopen), there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define...
Strong Insertion of a Contra-continuous Function …

\[ F_1 = \left\{ x \in X : h(x) < \frac{1}{2} \right\} \quad \text{and} \quad F_2 = \left\{ x \in X : h(x) > \frac{1}{2} \right\}. \]

Then \( F_1 \) and \( F_2 \) are two disjoint closed subsets of \( X \) that contain \( G_1 \) and \( G_2 \), respectively. Hence by Corollary 3.4, \( X \) has the weak \( cc \)-insertion property for \((cpc, csc)\) and \((esc, cpc)\). Now, assume that \( g \) and \( f \) are functions on \( X \) such that \( g \leq f \), \( g \) is \( cpc \) (resp. \( csc \)) and \( f \) is \( cc \). Since \( f - g \) is \( cpc \) (resp. \( csc \)), therefore the lower cut set \( A(f - g, 2^{-n}) = \{ x \in X : (f - g)(x) \leq 2^{-n} \} \) is a preopen (resp. semi-open) subset of \( X \). Now setting \( H_n = \{ x \in X : (f - g)(x) > 2^{-n} \} \) for every \( n \in \mathbb{N} \), then by Lemma 3.4, \( H_n \) is an open subset of \( X \) and we have \( \{ x \in X : (f - g)(x) > 0 \} = \bigcup_{n=1}^{\infty} H_n \) and for every \( n \in \mathbb{N} \), \( H_n \) and \( A(f - g, 2^{-n}) \) are disjoint subsets of \( X \). By Lemma 3.2, \( H_n \) and \( A(f - g, 2^{-n}) \) can be completely separated by contra-continuous functions. Hence by Theorem 2.2, \( X \) has the strong \( cc \)-insertion property for \((cpc, cc)\) (resp. \((csc, cc)\)).

By an analogous argument, we can prove that \( X \) has the strong \( cc \)-insertion property for \((cc, csc)\) (resp. \((cc, cpc)\)). Hence, by Theorem 2.3, \( X \) has the strong \( cc \)-insertion property for \((cpc, csc)\) (resp. \((cpc, cpc)\)).

\[ \blacksquare \]

Acknowledgement

This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan).

References

[1] A. Al-Omari and M. S. Md. Noorani, Some properties of contra-\( b \)-continuous and almost contra-\( b \)-continuous functions, *European J. Pure. Appl. Math.* 2(2) (2009), 213-230.

[2] F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly* 78 (1971), 1007-1010. https://doi.org/10.1080/00029890.1971.11992929

[3] M. Caldas and S. Jafari, Some properties of contra-\( \beta \)-continuous functions, *Mem. Fac. Sci. Kochi. Univ.* 22 (2001), 19-28.

[4] H. H. Corson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.* 8 (1964), 351-360. https://doi.org/10.1215/ijm/1256059678
[5] J. Dontchev, Characterization of some peculiar topological space via A- and B-sets, *Acta Math. Hungar.* 69(1-2) (1995), 67-71. https://doi.org/10.1007/BF01874608

[6] J. Dontchev, Contra-continuous functions and strongly S-closed space, *Internat. J. Math. Math. Sci.* 19(2) (1996), 303-310. https://doi.org/10.1155/S0161171296000427

[7] J. Dontchev and H. Maki, On sg-closed sets and semi-λ-closed sets, *Questions Answers Gen. Topology* 15(2) (1997), 259-266.

[8] E. Ekici, On contra-continuity, *Annales Univ. Sci. Budapest* 47 (2004), 127-137.

[9] E. Ekici, New forms of contra-continuity, *Carpathian J. Math.* 24(1) (2008), 37-45.

[10] A. I. El-Maghrabi, Some properties of contra-continuous mappings, *Int. J. General Topol.* 3(1-2) (2010), 55-64.

[11] M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar.* 56(3-4) (1990), 299-301. https://doi.org/10.1007/BF01903846

[12] S. Jafari and T. Noiri, Contra-continuous function between topological spaces, *Iranian Int. J. Sci.* 2 (2001), 153-167.

[13] S. Jafari and T. Noiri, On contra-precontinuous functions, *Bull. Malaysian Math. Sc. Soc.* 25 (2002), 115-128.

[14] M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* 38 (1951), 85-91.

[15] M. Katětov, Correction to, “On real-valued functions in topological spaces”, *Fund. Math.* 40 (1953), 203-205. https://doi.org/10.4064/fm-40-1-203-205

[16] E. Lane, Insertion of a continuous function, *Pacific J. Math.* 66 (1976), 181-190. https://doi.org/10.2140/pjm.1976.66.181

[17] E. Lane, PM-normality and the insertion of a continuous function, *Pacific J. Math.* 82 (1979), 155-162. https://doi.org/10.2140/pjm.1979.82.155

[18] N. Levine, Semi-open sets and semi-continuity in topological space, *Amer. Math. Monthly* 70 (1963), 36-41. https://doi.org/10.1080/00029890.1963.11990039

[19] S. N. Maheshwari and R. Prasad, On RO-spaces, *Portugal. Math.* 34 (1975), 213-217.

[20] H. Maki, Generalized A-sets and the associated closure operator, *The Special Issue in Commemoration of Prof. Kazuada IKEDA’s Retirement* (1986), 139-146.

[21] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On pre-continuous and weak pre-continuous mappings, *Proc. Math. Phys. Soc. Egypt* 53 (1982), 47-53.
[22] M. Mirmiran and B. Naderi, Strong insertion of a contra-α-continuous function between two comparable real-valued functions, *Earthline J. Math. Sci.* 2(1) (2019), 223-239. https://doi.org/10.34198/ejms.2119.223239

[23] M. Mirmiran and B. Naderi, Insertion of a contra-α-continuous function, *Earthline J. Math. Sci.* 2(2) (2019), 383-393. https://doi.org/10.34198/ejms.2219.383393

[24] M. Mirmiran and B. Naderi, Insertion of a contra-continuous function between two comparable real-valued functions, *Earthline J. Math. Sci.* 3(1) (2020), 21-35. https://doi.org/10.34198/ejms.3120.2135

[25] M. Mrsevic, On pairwise $R_0$ and pairwise $R_1$ bitopological spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* 30 (1986), 141-145.

[26] A. A. Nasef, Some properties of contra-γ-continuous functions, *Chaos Solitons Fractals* 24 (2005), 471-477. https://doi.org/10.1016/j.chaos.2003.10.033

[27] M. Powderly, On insertion of a continuous function, *Proceedings of the A.M.S.* 81 (1981), 119-120. https://doi.org/10.1090/S0002-9939-1981-0589151-7

[28] M. Przemski, A decomposition of continuity and α-continuity, *Acta Math. Hungar.* 61(1-2) (1993), 93-98. https://doi.org/10.1007/BF01872101