ALMOST PERIODIC DISCRETE SETS

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Abstract. Using a special metric in the space of sequences, we give a geometric description of almost periodic sets in the $k$-dimensional Euclidean space. We prove the completeness of the space of almost periodic sets and some analogue of the Bochner criterion of almost periodicity. Also, we show the connection between these sets and almost periodic measures.

Introduction. Almost periodic (with respect to the real shifts) sets were first regarded by M.G.Krein and B.Ya.Levin [1] (see also the monograph [2]) when studying the zero distribution of entire almost periodic functions.

Definition. A discrete set $\{a_n\} \subset \{z : |Imz| < M < \infty\}$ is called almost periodic if for each $\varepsilon > 0$ there exists $L < \infty$ such that every real interval of length $L$ contains at least one number $\tau$ such that for some bijection $\rho : \mathbb{Z} \to \mathbb{Z}$

$$|(a_n + \tau) - a_{\rho(n)}| < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$  

M.G.Krein and B.Ya.Levin have shown that the zero set of an almost periodic function from some special class (the class $[\Delta]$, see [1]) is an almost periodic set. Moreover, any almost periodic set in a strip can be completely described as the zero set of some analytic function with almost periodic modulus (see [3]).

In the present paper we study sets in $\mathbb{R}^k$ which are almost periodic with respect to arbitrary shifts. Such sets arise in the models describing quasicristallic structures (see, for example, [4]).

Using a special metric in the space of sequences, we give a geometric description of almost periodic sets. We prove the completeness of the space of almost periodic sets and some analogue of the Bochner criterion of almost periodicity. Following [5], we introduce a notion of almost periodic measure in $\mathbb{R}^k$. We show that a set is almost periodic measure.

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if and only if the discrete measure with unit masses at the points of this set is almost periodic.

Throughout the work we denote $i$-coordinate of a point $x \in \mathbb{R}^k$ by $x^i$, an open $k$-dimensional ball of radius $R$ with center $x \in \mathbb{R}^k$ by $B(x, R)$, $k$-dimensional cube \( \{y \in \mathbb{R}^k \mid x^i - L/2 \leq y^i < x^i + L/2, i = 1, k\} \) by $Q(x, L)$. For any set $A \subset \mathbb{R}^k$ for any $\rho > 0$ we write $A_\rho = \bigcup_{x \in A} B(x, \rho)$ and $A_{-\rho} = \{x \in \mathbb{R}^k \mid B(x, \rho) \subset A\}$. We denote the interior of $A$ by $IntA$, $k$-dimensional Lebesgue measure of $A$ by $m(A)$, the inner product of elements $x, y \in \mathbb{R}^k$ by $\langle x, y \rangle$, usual Euclidean norm by $|x|$. For a mapping $g(x) : \mathbb{R}^k \rightarrow \mathbb{R}^n$ and $\tau \in \mathbb{R}^k$ we put $g^\tau(x) = g(x - \tau)$, for a measure $\mu$ on $\mathbb{R}^k$ we put $\mu^\tau(E) = \mu(E + \tau)$ for any Borel set $E \subset \mathbb{R}^k$.

1. Almost periodic mappings. Call to mind some definitions concerning almost periodicity.

Let $g$ be a continuous mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$. A vector $\tau \in \mathbb{R}^k$ is called an $\varepsilon$-almost period of $g$ if

$$|g^\tau(x) - g(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}^k.$$ 

A set $E \subset \mathbb{R}^k$ is called relatively dense if there exists $L < \infty$ such that every $k$-dimensional ball of radius $L$ has a nonempty intersection with $E$. It is obvious that we can replace a ball with a cube in this definition.

A continuous mapping $g$ from $\mathbb{R}^k$ to $\mathbb{R}^n$ is called almost periodic if for every $\varepsilon > 0$ the set of $\varepsilon$-almost periods of $g$ is relatively dense in $\mathbb{R}^k$.

Point out some properties of almost periodic mappings. First of all, the almost periodicity of a mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$ is equivalent to the almost periodicity of its coordinates, therefore, it is sufficient to consider almost periodic functions on $\mathbb{R}^k$.

The following assertions are well known for almost periodic functions on the real axis (see, for example, [6], [7]). Their proofs don’t change much in the case of higher dimension (see, for example, [5]).

**Proposition 1.** a) An almost periodic function on $\mathbb{R}^k$ is bounded and uniformly continuous,

b) if for an almost periodic function $g$ the sequence $(g^{\tau_n})$ converges uniformly on $\mathbb{R}^k$ then its limit $\tilde{g}$ is an almost periodic function as well. In addition, $g \neq const$ if and only if $\tilde{g} \neq const$.

**Proposition 2.** For a continuous mapping $g(x)$ on $\mathbb{R}^k$ the following conditions are equivalent:
a) $g(x)$ is almost periodic,  

b) for each sequence $(h_n) \subset \mathbb{R}^k$ there exists a subsequence $(h_{n'})$ such that $|g(x + h_{n'}) - g(x + h_{m'})| \to 0$ uniformly on $\mathbb{R}^k$,  

c) there is a sequence of finite exponential sums  

$$S_n(x) = \sum_{i=1}^{N(n)} c_{i,n} e^{\langle \lambda_{i,n}, x \rangle}$$  

that converges to $g(x)$ uniformly on $\mathbb{R}^k$.

The assertion b) is called the Bohner criterion for almost periodic mappings.

2. Almost periodic discrete multiple sets.

As it was introduced in [8], we call a value set of a sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}^k$ a discrete multiple set (we write $\{a_n\}_{n \in \mathbb{N}}$) if this sequence doesn’t possess any finite limit points. In other words, a discrete multiple set is a discrete set where each point has a finite multiplicity.

For any two discrete multiple sets $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ we define a distance between them be the formula

$$dist(\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}) = \inf_{\sigma: \mathbb{N} \to \mathbb{N}} \sup_{n \in \mathbb{N}} |a_n - b_{\sigma(n)}|,$$

where infimum is taken over all bijections $\sigma: \mathbb{N} \to \mathbb{N}$. As was shown in [8], this function satisfies all the axioms of metric except the finiteness.

**Definition 1.** A vector $\tau \in \mathbb{R}^k$ is called an $\varepsilon$-almost period of a discrete multiple set $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^k$, if

$$dist(\{a_n\}_{n \in \mathbb{N}}, \{a_n + \tau\}_{n \in \mathbb{N}}) < \varepsilon.$$

**Definition 2.** A discrete multiple set $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^k$ is called almost periodic, if for each $\varepsilon > 0$ the set of its $\varepsilon$-almost periods is relatively dense in $\mathbb{R}^k$.

Note that the sum and the difference of any two $\varepsilon$-almost periods are $2\varepsilon$-almost periods.

**Theorem 1.** The limit of almost periodic multiple sets is almost periodic as well.

**Proof.** Let $(\{a_n^{(p)}\}_{n \in \mathbb{N}})$ be a sequence of almost periodic sets. Let $\{b_n\}_{n \in \mathbb{N}}$ be such a set that $dist\left(\{a_n^{(p)}\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}\right) \to 0$. Let us take any $\varepsilon > 0$. Let $p_0$ be such a number that

$$dist\left(\{a_n^{(p_0)}\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}\right) < \frac{\varepsilon}{3}.$$
Let $E_\frac{\varepsilon}{3}$ be a relatively dense set of $\frac{\varepsilon}{3}$-almost periods of $\{a_n^{(p_0)}\}_{n \in \mathbb{N}}$. For $\tau \in E_\frac{\varepsilon}{3}$ we have

$$\text{dist} \left( \{a_n^{(p_0)}\}_{n \in \mathbb{N}}, \{a_n^{(p_0)} + \tau\}_{n \in \mathbb{N}} \right) < \frac{\varepsilon}{3}.$$ 

Therefore,

$$\text{dist} \left( \{b_n\}_{n \in \mathbb{N}}, \{b_n + \tau\}_{n \in \mathbb{N}} \right) \leq \text{dist} \left( \{b_n\}_{n \in \mathbb{N}}, \{a_n^{(p_0)}\}_{n \in \mathbb{N}} \right) + \text{dist} \left( \{a_n^{(p_0)}\}_{n \in \mathbb{N}}, \{a_n^{(p_0)} + \tau\}_{n \in \mathbb{N}} \right) + \text{dist} \left( \{a_n^{(p_0)} + \tau\}_{n \in \mathbb{N}}, \{b_n + \tau\}_{n \in \mathbb{N}} \right) < \varepsilon.$$ 

Hence, $\tau$ is an $\varepsilon$-almost period of $\{b_n\}_{n \in \mathbb{N}}$, and the set of almost periods is relatively dense.

Remark. As it was shown in [8], theorem 2, a space $(X, \text{dist})$ of all discrete multiple sets is complete. Hence, almost periodic multiple sets form a complete closed subspace in $(X, \text{dist})$.

The following theorem is an analogue of the Bochner criterion for almost periodic multiple sets.

**Theorem 2.** A discrete multiple set $\{a_n\}_{n \in \mathbb{N}}$ is almost periodic if and only if for every sequence $(h_p)_{p=1}^\infty \subset \mathbb{R}^k$ there is a subsequence $(h'_p)_{p=1}^\infty$ such that $(\{a_n + h'_p\}_{n \in \mathbb{N}})_{p=1}^\infty$ has a limit.

The proof of this theorem is based on the following lemma:

**Lemma 1.** Let $D = \{a_n\}_{n \in \mathbb{N}}$ be an almost periodic multiple set. Then every sequence $(h_p)_{p=1}^\infty \subset \mathbb{R}^k$ has a subsequence $(h'_p)_{p=1}^\infty$ such that for each $\varepsilon > 0$ and arbitrary numbers $l, m > N = N(\varepsilon)$ one can find an $\varepsilon$-almost period $\tau$ of $D$ satisfying the inequality

$$|h'_l - h'_m - \tau| < \varepsilon.$$ 

**Proof.** First put $\varepsilon = 1$. Let $E_\frac{1}{2}$ be the relatively dense set of $\frac{1}{2}$-almost periods of $D$. Consider the sets

$$A_p = \bigcup_{\tau \in E_\frac{1}{2}} B(h_p + \tau, 1/2), p \in \mathbb{N}.$$ 

There exists $L < \infty$ such that for all $p = 1, 2, \ldots$ the ball $B(-h_p, L)$ contains a $\frac{1}{2}$-almost period $\tau_p$ of $D$. Since $\tau_p + h_p \in B(0, L)$, we have $A_p \bigcap B(0, L) \neq 0$ for all $p = 1, 2, \ldots$.

Let us show that there exists a point $x \in B(0, L)$ belonging to an infinite sequence of the sets $A_p$. Cover the ball $B(0, L)$ by a finite number of mutually disjoint $k$-dimensional cubes with edges of length $\frac{\varepsilon}{2k}$. The Dirichlet principle implies that there is a cube containing

\footnote{Precisely, a metric space of all discrete multiple sets lying at a finite distance from some fixed discrete multiple set is complete.}
an infinite sequence of points $h_p + \tau$. The diagonal of such a cube is less than $\frac{1}{2\sqrt{k}}$, hence this cube is contained in the balls $B(h_p + \tau, \frac{1}{2})$, therefore, in an infinite sequence of the sets $A_p$. Hence there exists a subsequence $(A_p^{(1)})_{p=1}^{\infty} \subset (A_p)_{p=1}^{\infty}$ such that

$$A_p^{(1)} = \bigcup_{\tau \in E_{\frac{1}{4}}} B(h_p^{(1)} + \tau, 1/2), \quad p \in \mathbb{N}$$

and

$$\bigcap_{p=1}^{\infty} A_p^{(1)} \neq 0.$$

Take a number $h \in A_i^{(1)} \cap A_j^{(1)}$ with some $i, j$. There exists $\tau' \in E_{\frac{1}{4}}$ such that $|h - (h_i^{(1)} + \tau')| < \frac{1}{2}$ and $\tau'' \in E_{\frac{1}{4}}$ such that $|h - (h_j^{(1)} + \tau'')| < \frac{1}{2}$. Therefore we have

$$|h_i^{(1)} - h_j^{(1)} - (\tau'' - \tau')| \leq |h - (h_i^{(1)} + \tau'')| + |h - (h_j^{(1)} + \tau')| < 1.$$

Thus, for each $i, j$ there exists 1-almost period $\tau = \tau'' - \tau'$ of $D$ such that the inequality

$$|h_i^{(1)} - h_j^{(1)} - \tau| < 1$$

holds.

Now put $\varepsilon = \frac{1}{4}$. Similarly, construct the sets

$$A_p^{(2)} = \bigcup_{\tau \in E_{\frac{1}{4}}} B(h_p^{(2)} + \tau, 1/4), \quad p \in \mathbb{N}$$

and $(h_p^{(2)})_{p=1}^{\infty} \subset (h_p^{(1)})_{p=1}^{\infty}$ such that for all $i, j$ the inequality

$$|h_i^{(2)} - h_j^{(2)} - \tau| < \frac{1}{2}$$

holds for some $\frac{1}{2}$-almost period $\tau$ of $D$.

Repeating this construction for $\varepsilon = \frac{1}{4}, \varepsilon = \frac{1}{4}, \ldots$ and choosing a diagonal subsequence $(h_p')_{p=1}^{\infty}$, we obtain the assertion of the lemma.

\[\square\]

**Proof of the theorem** Suppose that for every sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$ there is a subsequence $(h_p')_{p=1}^{\infty}$ such that a sequence $(\{a_n + h_p'\}_{p=1}^{\infty})_{n=\mathbb{N}}$ has a limit. If $(a_n)_{n=\mathbb{N}}$ is not almost periodic, then for some $\varepsilon_0 > 0$ there exists a sequence of $k$-dimensional balls $(B_p)_{p=1}^{\infty}$ with infinitely increasing diameters $l_p$, such that no ball contains any $\varepsilon_0$-almost period of $(a_n)_{n=\mathbb{N}}$. 5
Let us take an arbitrary \( h_1 \in \mathbb{R}^k \) and a number \( \nu_1 \) such that \( l_{\nu_1} > 1 \). For some \( h_2 \in \mathbb{R}^k \) the difference \( h_2 - h_1 \) belongs to \( B_{\nu_1} \). Let \( \nu_2 \) be the first number such that \( l_{\nu_2} > \max\{2, |h_2 - h_1|\} \). Take \( h_3 \in \mathbb{R}^k \) such that the differences \( h_3 - h_1, h_3 - h_2 \) belong to \( B_{\nu_2} \) (it is possible, since the latter condition is equivalent to \( |h_2 - h_1| < l_{\nu_2} \)). Generally, take a number \( \nu_n \) such that \( l_{\nu_n} > \max\{n, |h_2 - h_1|, |h_3 - h_2|, \ldots, |h_n - h_{n-1}|\} \) and \( h_{n+1} \in \mathbb{R}^k \) such that the differences \( h_{n+1} - h_1, h_{n+1} - h_2, \ldots, h_{n+1} - h_n \) belong to \( B_{\nu_n} \).

Take arbitrary \( p, m (p > m) \). By construction \( h_p - h_m \in B_{\nu_p-1} \), and \( B_{\nu_p-1} \) doesn’t contain any \( \varepsilon_0 \)-almost period of \( \{a_n\}_{n \in \mathbb{N}} \). We have

\[
\text{dist}(\{a_n + h_p\}_{n \in \mathbb{N}}, \{a_n + h_m\}_{n \in \mathbb{N}}) = \text{dist}(\{a_n + (h_p - h_m)\}_{n \in \mathbb{N}}, \{a_n\}_{n \in \mathbb{N}}) \geq \varepsilon_0.
\]

Thus, there are no convergent subsequences in the sequence \( \{a_n + h_p\}_{n \in \mathbb{N}} \).

Conversely, let a discrete multiple set \( \{a_n\}_{n \in \mathbb{N}} \) be almost periodic. Consider an arbitrary sequence \( (h_p)_{p=1}^{\infty} \subset \mathbb{R}^k \). By Lemma \([\text{1}]\) there exists a subsequence \( (h'_p)_{p=1}^{\infty} \) such that for every \( \varepsilon > 0 \) and arbitrary \( l, m > N = N(\varepsilon) \) one can find an \( \varepsilon/2 \)-almost period \( \tau \) of \( \{a_n\}_{n \in \mathbb{N}} \) with the property

\[
|h'_l - h'_m - \tau| < \frac{\varepsilon}{2}.
\]

We get

\[
dist(\{a_n + h'_l\}_{n \in \mathbb{N}}, \{a_n + h'_m\}_{n \in \mathbb{N}}) = \inf_{bij \sigma: \mathbb{N} \to \mathbb{N}} \sup_{n \in \mathbb{N}} |a_n + h'_l - a_{\sigma(n)} - h'_m| \\
\leq \inf_{bij \sigma: \mathbb{N} \to \mathbb{N}} \sup_{n \in \mathbb{N}} |a_n - a_{\sigma(n)} + \tau| + |h'_l - h'_m - \tau| \\
< dist(\{a_n\}_{n \in \mathbb{N}}, \{a_{\sigma(n)} - \tau\}_{n \in \mathbb{N}}) + \varepsilon/2 < \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we see that the sequence \( (\{a_n + h'_p\}_{n \in \mathbb{N}})_{p=1}^{\infty} \) is fundamental. Using the remark after Theorem \([\text{1}]\) we finish the proof.

\[\Box\]

**Definition 3.** \([\text{8}]\) A discrete multiple set \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^k \) possesses \( S \)-property, if there exists \( L < \infty \) such that for any \( \tau \in \mathbb{R}^k \) there is a bijection \( \sigma : \mathbb{N} \to \mathbb{N} \) with the property

\[
\sup_{n \in \mathbb{N}} |(a_n + \tau) - a_{\sigma(n)}| \leq L.
\]

In the other words, a discrete multiple set \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^k \) possesses \( S \)-property, iff there exists \( L < \infty \) such that the inequality

\[
\text{dist}(\{a_n\}_{n \in \mathbb{N}}, \{a_n + \tau\}_{n \in \mathbb{N}}) \leq L
\]

holds for any \( \tau \in \mathbb{R}^k \).

By \([\text{8}]\), any almost periodic discrete multiple set possesses \( S \)-property. Therefore Theorem 4 and Proposition 4 from \([\text{8}]\) imply the following statements:
Theorem 3. Let $D$ be an almost periodic multiple set. Then there exists $M < \infty$ such that

(1) \[ \text{card} \left( D \cap B(c,1) \right) < M \quad \text{for all } c \in \mathbb{R}^k. \]

Theorem 4. For any almost periodic multiple set $D$ there exists $C < \infty$ such that for any convex bounded set $E \subset \mathbb{R}^k$ and $t \in \mathbb{R}^k$ the inequality

\[ |\text{card} \left( D \cap E \right) - \text{card} \left( D \cap (E + t) \right)| < C((\text{diam } E)^{k-1} + 1) \]

is fulfilled.

Following [8], the density of a discrete multiple set is the value

\[ \Delta = \lim_{T \to \infty} \frac{\text{card} \left( D \cap Q(0,T) \right)}{T^k}. \]

From Theorem 5 of [8] we get

Theorem 5. Any almost periodic multiple set possesses finite nonzero shift invariant density.

3. The connection between almost periodic measures and almost periodic discrete multiple sets.

Definition 4. ([5]) A locally finite complex-valued Radon measure $\mu$ on $\mathbb{R}^k$ is called almost periodic if for each compactly supported continuous function $\varphi$ on $\mathbb{R}^k$ the convolution

\[ \varphi * \mu(z) = \int_{\mathbb{R}^k} \varphi(y - z)d\mu(y) \]

is almost periodic on $\mathbb{R}^k$.

In other words, a locally finite complex-valued Radon measure $\mu$ is almost periodic if for each compactly supported continuous function $\varphi$ on $\mathbb{R}^k$ and any $\varepsilon > 0$ there exists a relatively dense set $E_{\varepsilon, \varphi}$ (the set of $(\varepsilon, \varphi)$-almost periods of the measure $\mu$) with the property

\[ |\varphi * \mu(z) - \varphi * \mu(z - \tau)| \leq \varepsilon \quad \forall \tau \in E_{\varepsilon, \varphi}, \forall z \in \mathbb{R}^k. \]

Note that the sum and the difference of any two $(\varepsilon, \varphi)$-almost periods are $(2\varepsilon, \varphi)$-almost periods.

We will say that measures $\mu_n$ converge weakly uniformly to some measure $\mu$ if

\[ \varphi * \mu_n(x) \to_{\mu_n \to \mu} \varphi * \mu(x) \]

uniformly on $\mathbb{R}^k$ for every continuous compactly supported function $\varphi$.

Note some properties of almost periodic measures.
Theorem 6. If almost periodic measures converge weakly uniformly to some measure \( \mu \), then \( \mu \) is almost periodic as well.

The proof follows immediately from the definition of an almost periodic measure and Proposition 1 of the present article.

Theorem 7. Let \( \mu \) be an almost periodic measure. Then there exists \( M < \infty \) such that the condition

\[
|\mu|(B(c, 1)) \leq M \quad \text{for all } c \in \mathbb{R}^k
\]

is fulfilled; here \( |\mu| \) is the variation of measure \( \mu \).

This fact follows from Theorem 2.1. in [5] with \( S = \{0\}, \ p = 0 \).

The following theorem is an analogue of the Bochner criterion for almost periodic mappings.

Theorem 8. A Radon measure \( \mu \) is almost periodic if and only if for every sequence \( (h_n) \subset \mathbb{R}^k \) there is a subsequence \( (h_n') \) such that measures \( \mu^{h_n'} \) converge weakly uniformly to some measure \( \mu' \).

This theorem is a corollary of Theorem 2.2. in [5] with \( S = \{0\}, \ p = 0 \).

Definition 5. For any Radon measure \( \mu \) the value

\[
\Delta = \lim_{T \to \infty} \frac{\mu(Q(0, T))}{T^k}
\]

is called the density of \( \mu \).

Theorem 9. Any almost periodic Radon measure possesses finite shift invariant density, i.e.,

\[
\Delta = \lim_{T \to \infty} \frac{\mu(Q(\alpha, T))}{T^k}
\]

uniformly over \( \alpha \in \mathbb{R}^k \).

(See [5], Theorem 2.7. with \( S = \{0\} \).

We can associate with a discrete multiple set \( D = \{a_n\}_{n \in \mathbb{N}} \) the measure

\[
\mu_D = \sum_{a_n \in D} \delta(x - a_n).
\]

Note that for every continuous function \( \varphi \)

\[
\varphi \ast \mu_D(x) = \sum_{n \in \mathbb{N}} \varphi(x - a_n), \ x \in \mathbb{R}^k.
\]
Theorem 10. Let \((D_p) = \{\{a_n^{(p)}\}_{n \in \mathbb{N}}\}_{n \in \mathbb{N}}\) be a sequence of discrete multiple sets. Let \(D = \{a_n\}_{n \in \mathbb{N}}\) be a discrete multiple set satisfying (1). Then the following conditions are equivalent:

a) discrete multiple sets \(D_p\) converge to \(D\),

b) measures \(\mu_{D_p}\) converge weakly uniformly to \(\mu_D\).

We need the following lemma:

Lemma 2. Let \((D_p) = \{\{a_n^{(p)}\}_{n \in \mathbb{N}}\}_{n \in \mathbb{N}}\) be a sequence of discrete multiple sets, which converges to a discrete multiple set \(D = \{a_n\}_{n \in \mathbb{N}}\) satisfying (1). Then for sufficiently large \(p\) any discrete multiple set \(D_p\) satisfies (1) (with a constant \(4^k M\) instead of \(M\)).

Proof. For sufficiently large \(p\) we have \(\inf_{\sigma} \sup_{n \in \mathbb{N}} |a_n - a_n^{(p)}| < 1\). Hence, \(\operatorname{card} (D_p \cap B(c, 1)) \leq \operatorname{card} (D \cap B(c, 2)) \leq 4^k M\) for all \(c \in \mathbb{R}^k\).

Proof of Theorem 10. Let the measures \(\mu_{D_p}\) converge weakly uniformly to \(\mu_D\). We will show that \(\operatorname{dist} (\{a_n^{(p)}\}_{n \in \mathbb{N}}, \{a_n\}_{n \in \mathbb{N}}) \to 0\) as \(p \to \infty\), i.e. for any \(\varepsilon > 0\) for each sufficiently large \(p\) there exists a bijection \(\sigma : \mathbb{N} \to \mathbb{N}\) such that

\[
\sup_{n \in \mathbb{N}} |a_n^{(p)} - a_{\sigma(n)}| < \varepsilon.
\]

Theorem 3. Let \((D_p) = \{\{a_n^{(p)}\}_{n \in \mathbb{N}}\}_{n \in \mathbb{N}}\) be a sequence of discrete multiple sets, which converges to a discrete multiple set \(D = \{a_n\}_{n \in \mathbb{N}}\) satisfying (1). Then for sufficiently large \(p\) any discrete multiple set \(D_p\) satisfies (1) (with a constant \(4^k M\) instead of \(M\)).

First note that for any \(\varepsilon > 0\) there is \(\eta > 0\) such that the diameter of every connected component of the union \(\bigcup_n B(a_n, \eta)\) is less than \(\varepsilon\). Let us check it. Put \(\varepsilon \in (0, 1)\) and put \(\eta < \varepsilon/(2M + 1)\), where \(M\) satisfies (1). Let \(A\) be an arbitrary connected component of the union \(\bigcup_n B(a_n, \eta)\) and \(c\) be an arbitrary point of \(A\). By Theorem 3 the ball \(B(c, 1)\) contains at most \(M\) points of \(D\). Choose from them the points \(a_{n_1}, \ldots, a_{n_M}\) \((N \leq M)\) belonging to \(A\) (we suppose that \(B(a_{n_i}, \eta) \cap B(a_{n_{i+1}}, \eta) \neq 0, \ i = 1, N - 1\)). If \(A\) isn’t contained in \(B(c, 1)\), then there exists a point \(a' \in A \setminus B(c, 1)\) with the property \(|a' - a_{n_i}| < 2\eta\) for some \(n_i, \ i = 1, N\). Since \(c \in B(a_{n_p}, \eta)\) for some \(n_p, \ p = 1, N\), we have

\[
|a' - c| \leq |a' - a_{n_i}| + (N - 1) \max_{j=1,N-1} |a_{n_j} - a_{n_{j+1}}| + |c - a_{n_p}|
\]

\[
< 2\eta + (N - 1)2\eta + \eta = \eta(2N + 1) < 1,
\]

that is impossible. Thus \(A \subset B(c, 1)\) and

\[
\operatorname{diam} A < 2\eta M < \varepsilon.
\]
Take a nonnegative continuous function $\varphi$ with the support in $B(0, \eta/2)$ such that $0 \leq \varphi(x) \leq \varphi(0) = 1$. Put $\nu = \int \varphi dm$. We may assume that $\nu < 1$. Since measures $\mu_{D_p}$ converge weakly uniformly to $\mu_D$, for sufficiently large $p$ we have

\begin{equation}
|\varphi \ast \mu_{D_p}(x) - \varphi \ast \mu_D(x)| < \nu/2 \quad \forall x \in \mathbb{R}^k.
\end{equation}

The distance between an arbitrary pair of terms $a_n \in A$ and $a_m \notin A$ is at least $2\eta$, therefore,

\begin{equation}
\int_A \left( \int_{\mathbb{R}^k} \varphi(x - y)d\mu_D(y) \right) dm(x) = \int_A \left( \int_A \varphi(x - y)d\mu_D(y) \right) dm(x)
= \int_A \left( \int_{\mathbb{R}^k} \varphi(x - y)dm(x) \right) d\mu_D(y) = \nu \text{ card } (D \cap A).
\end{equation}

Furthermore, let $a_n^{(p)}$ be a term of $D_p$ such that $a_n^{(p)} \in A$. We have $\varphi \ast \mu_{D_p}(a_n^{(p)}) \geq \varphi(0) = 1$. In view of (3), we get

$$\varphi \ast \mu_{D_p}(a_n^{(p)}) \geq 1 - \nu/2 > 1/2.$$ 

Hence, there exists a term $a_{n'}$ of $D$ such that $|a_{n'} - a_n^{(p)}| < \eta/2$. Then the distance between $a_{n'}$ and $A$ is less than $\eta/2$. Therefore, $a_{n'} \in A$. Similarly, for any $a_j^{(p)} \notin A$ there exists a term $a_j'$ of $D$ such that $|a_j' - a_j^{(p)}| < \eta/2$. Consequently, the distance between $a_j'$ and $\mathbb{R}^k \setminus A$ is less than $\eta/2$, so $a_j' \notin A$. Therefore,

\begin{equation}
\int_A \left( \int_{\mathbb{R}^k} \varphi(x - y)d\mu_{D_p}(y) \right) dm(x) = \int_A \left( \int_A \varphi(x - y)d\mu_{D_p}(y) \right) dm(x)
= \int_A \left( \int_{\mathbb{R}^k} \varphi(x - y)dm(x) \right) d\mu_{D_p}(y) = \nu \text{ card } (D_p \cap A).
\end{equation}

Since

$$\left| \int_A (\varphi \ast \mu_{D_p}(x) - \varphi \ast \mu_D(x))dm(x) \right| \leq m(A) \nu/2 < \nu/2,$$

the values from equalities (4) and (5) coincide. Therefore, $\text{card } (D \cap A) = \text{card } (D_p \cap A)$. The same is valid for all connected components of union $\bigcup_n B(a_n, \eta)$. On the other hand, we have just proved that each term $a_n^{(p)}$ of $D_p$ belongs to $B(a_{n'}, \eta)$ for some term $a_{n'}$ of $D$. Hence, it belongs to a connected component of $\bigcup_n B(a_n, \eta)$. Consequently, there is a bijection $\sigma$ such that (2) is fulfilled.

Conversely, suppose that discrete multiple sets $D_p$ converge to a discrete multiple set $D$. Let $\varphi$ be a nonnegative continuous function
with compact support \( G \) in \( \mathbb{R}^k \). We will show that the convolutions \( \varphi * \mu_{D_p}(x) \) converge to \( \varphi * \mu_D(x) \) as \( p \to \infty \) uniformly over \( x \in \mathbb{R}^k \).

Take \( z \in \mathbb{R}^k \) and \( \varepsilon > 0 \). There is \( \delta > 0 \) such that the inequality \(|x_1 - x_2| < \delta\) implies \(|\varphi(x_1) - \varphi(x_2)| < \varepsilon\). By assumption, for each sufficiently large \( p \) there exists a bijection \( \sigma \) such that \(|a_n - a^{(p)}_{\sigma(n)}| < \delta\) for all \( n \in \mathbb{N} \). By Lemma 2 the set \( G \) contains at most \( M' = M'(G) < \infty \) terms of \( D_p \). Hence,

\[
|\varphi * \mu_{D_p}(x) - \varphi * \mu_{D}(x)| \leq M' \sup_{|x_1 - x_2| < \delta} |\varphi(x_1) - \varphi(x_2)| < M' \varepsilon.
\]

Now we can prove the main result of this section.

**Theorem 11.** A discrete multiple set \( D \) is almost periodic if and only if the corresponding measure \( \mu_{D} \) is almost periodic.

**Proof.** By theorem 2 a discrete multiple set \( D \) is almost periodic if and only if for every sequence \((h_p)_{p=1}^{\infty} \subset \mathbb{R}^k \) there is a subsequence \((h'_p)_{p=1}^{\infty} \) such that the sequence of almost periodic multiple sets \( \{a_n + h'_p\}_{p=1}^{\infty} \) has a limit. By theorem 10 it is true if and only if for every sequence \((h_p)_{p=1}^{\infty} \subset \mathbb{R}^k \) there is a subsequence \((h'_p)_{p=1}^{\infty} \) such that the sequence of almost periodic measures \( (\mu_{D+h'_p})_{p=1}^{\infty} \) converges weakly uniformly to some measure. By theorem 8 it is true if and only if the measure \( \mu \) is almost periodic.

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