INITIAL COEFFICIENTS OF BI-UNIVALENT FUNCTIONS

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Abstract. An analytic function $f$ defined on the open unit disk $D = \{z : |z| < 1\}$ is bi-univalent if the function $f$ and its inverse $f^{-1}$ are univalent in $D$. Estimates for the initial coefficients of bi-univalent functions $f$ are investigated when $f$ and $f^{-1}$ respectively belong to some subclasses of univalent functions. Some earlier results are shown to be special cases of our results.

1. Introduction

Let $S$ be the class of all univalent analytic functions $f$ in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. For $f \in S$, it is well known that the $n$th coefficient is bounded by $n$. The bounds for the coefficients give information about the geometric properties of these functions. Indeed, the bound for the second coefficient of functions in the class $S$ gives rise to growth, distortion, covering theorems for univalent functions. In view of the influence of the second coefficient in the geometric properties of univalent functions, it is important to know the bounds for the (initial) coefficients of functions belonging to various subclasses of univalent functions. In this paper, we investigate this coefficient problem for certain subclasses of bi-univalent functions.

Recall that the Koebe one-quarter theorem [8] ensures that the image of $D$ under every univalent function $f \in S$ contains a disk of radius $1/4$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$, ($z \in D$) and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4).$$

A function $f \in S$ is bi-univalent in $D$ if both $f$ and $f^{-1}$ are univalent in $D$. Let $\sigma$ denote the class of bi-univalent functions defined in the unit disk $D$. Lewin [12] investigated this class $\sigma$ and obtained the bound for the second coefficient of the bi-univalent functions.

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Several authors subsequently studied similar problems in this direction (see [7, 13]). A function \( f \in \sigma \) is bi-starlike or strongly bi-starlike or bi-convex of order \( \alpha \) if \( f \) and \( f^{-1} \) are both starlike, strongly starlike or convex of order \( \alpha \), respectively. Brannan and Taha \[6\] obtained estimates for the initial coefficients of bi-starlike, strongly bi-starlike and bi-convex functions. Bounds for the initial coefficients of several classes of functions were also investigated in \[2–5, 9, 11, 14–16–20\].

An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), if there is an analytic function \( w : \mathbb{D} \to \mathbb{D} \) with \( w(0) = 0 \) satisfying \( f(z) = g(w(z)) \). Ma and Minda \[13\] unified various subclasses of starlike (\( S^* \)) and convex functions (\( C \)) by requiring that either of the quantity \( zf'(z)/f(z) \) or \( 1+zf''(z)/f'(z) \) is subordinate to a more general superordinate function \( \varphi \) with positive real part in the unit disk \( \mathbb{D} \), \( \varphi(0) = 1, \varphi'(0) > 0 \), \( \varphi \) maps \( \mathbb{D} \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class \( S^*(\varphi) \) of Ma-Minda starlike functions with respect to \( \varphi \) consists of functions \( f \in S \) satisfying the subordination \( zf'(z)/f(z) \prec \varphi(z) \). Similarly, the class \( C(\varphi) \) of Ma-Minda convex functions consists of functions \( f \in S \) satisfying the subordination \( 1+zf''(z)/f'(z) \prec \varphi(z) \). Ma and Minda investigated growth and distortion properties of functions in \( S^*(\varphi) \) and \( C(\varphi) \) as well as Fekete-Szegö inequalities for \( S^*(\varphi) \) and \( C(\varphi) \). Their proof of Fekete-Szegö inequalities requires the univalence of \( \varphi \). Ali et al. \[3\] have investigated Fekete-Szegö problems for various other classes and their proof does not require the univalence or starlikeness of \( \varphi \). In particular, their results are valid even if one just assume the function \( \varphi \) to have a series expansion of the form \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, B_1 > 0 \). So in this paper, we assume that \( \varphi \) has series expansion \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, B_1, B_2 \) are real and \( B_1 > 0 \). A function \( f \) is Ma-Minda bi-starlike or Ma-Minda bi-convex if both \( f \) and \( f^{-1} \) are respectively Ma-Minda starlike or convex. Motivated by the Fekete-Szegö problem for the classes of Ma-Minda starlike and Ma-Minda convex functions \[13\], Ali et al. \[1\] recently obtained estimates of the initial coefficients for bi-univalent Ma-Minda starlike and Ma-Minda convex functions.

The present work is motivated by the results of Kędzierawski \[10\] who considered functions \( f \) belonging to certain subclasses of univalent functions while its inverse \( f^{-1} \) belongs to some other subclasses of univalent functions. Among other results, he obtained the following coefficient estimates.
Theorem 1.1. \[10\] Let \( f \in \sigma \) with Taylor series \( f(z) = z + a_2z^2 + \cdots \) and \( g = f^{-1} \). Then

\[
|a_2| \leq \begin{cases} 
1.5894 & \text{if } f \in S, \ g \in S, \\
2 & \text{if } f \in S^*, \ g \in S^*, \\
1.507 & \text{if } f \in S^*, \ g \in S, \\
1.224 & \text{if } f \in C, \ g \in S.
\end{cases}
\]

We need the following classes investigated in [1–3].

Definition 1.1. Let \( \varphi : \mathbb{D} \to \mathbb{C} \) be analytic and \( \varphi(z) = 1 + B_1z + B_2z^2 + \cdots \) with \( B_1 > 0 \) and \( B_2 \in \mathbb{R} \). For \( \alpha \geq 0 \), let

\[
\mathcal{M}(\alpha, \varphi) := \left\{ f \in S : (1 - \alpha)zf'(z) + \alpha \left( z + \frac{zf''(z)}{f'(z)} \right) \prec \varphi(z) \right\},
\]

\[
\mathcal{L}(\alpha, \varphi) := \left\{ f \in S : \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \prec \varphi(z) \right\},
\]

\[
\mathcal{P}(\alpha, \varphi) := \left\{ f \in S : \frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f(z)} \prec \varphi(z) \right\}.
\]

In this paper, we obtain the estimates for the second and third coefficients of functions \( f \) when

(1) \( f \in \mathcal{P}(\alpha, \varphi) \) and \( g := f^{-1} \in \mathcal{P}(\beta, \psi) \), or \( g \in \mathcal{M}(\beta, \psi) \), or \( g \in \mathcal{L}(\beta, \psi) \),

(2) \( f \in \mathcal{M}(\alpha, \varphi) \) and \( g \in \mathcal{M}(\beta, \psi) \), or \( g \in \mathcal{L}(\beta, \psi) \),

(3) \( f \in \mathcal{L}(\alpha, \varphi) \) and \( g \in \mathcal{L}(\beta, \psi) \).

2. Coefficient Estimates

In the sequel, it is assumed that \( \varphi \) and \( \psi \) are analytic functions of the form

\[
\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, \quad (B_1 > 0)
\]

and

\[
\psi(z) = 1 + D_1z + D_2z^2 + D_3z^3 + \cdots, \quad (D_1 > 0).
\]

Theorem 2.1. Let \( f \in \sigma \) and \( g = f^{-1} \). If \( f \in \mathcal{P}(\alpha, \varphi) \), \( g \in \mathcal{P}(\beta, \psi) \) and \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
then
\begin{equation}
|a_2| \leq \frac{B_1D_1\sqrt{B_1(1+3\beta) + D_1(1+3\alpha)}}{\sqrt{|\sigma B_1^2D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}}
\end{equation}
and
\begin{equation}
2\sigma|a_3| \leq B_1(3+10\alpha) + D_1(1+2\alpha) + (3+10\beta)|B_2 - B_1| + \frac{(1+2\beta)^2B_1^2|D_2 - D_1|}{D_1^2(1+2\alpha)}
\end{equation}
where \(\sigma := 2 + 7\alpha + 7\beta + 24\alpha\beta\).

\textbf{Proof.} Since \(f \in \mathcal{P}(\alpha, \varphi)\) and \(g \in \mathcal{P}(\beta, \psi)\), \(g = f^{-1}\). There exist analytic functions \(u, v : \mathbb{D} \to \mathbb{D}\), with \(u(0) = v(0) = 0\), satisfying
\begin{equation}
\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = \psi(v(w)).
\end{equation}
Define the functions \(p_1\) and \(p_2\) by
\begin{align*}
p_1(z) := \frac{1+u(z)}{1-u(z)} &= 1 + c_1z + c_2z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1+v(z)}{1-v(z)} = 1 + b_1z + b_2z^2 + \cdots,
\end{align*}
or, equivalently,
\begin{align*}
u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} &= \frac{1}{2} \left( c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right) \\
v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} &= \frac{1}{2} \left( b_1z + \left( b_2 - \frac{b_1^2}{2} \right) z^2 + \cdots \right).
\end{align*}
Then \(p_1\) and \(p_2\) are analytic in \(\mathbb{D}\) with \(p_1(0) = 1 = p_2(0)\). Since \(u, v : \mathbb{D} \to \mathbb{D}\), the functions \(p_1\) and \(p_2\) have positive real part in \(\mathbb{D}\), and \(|b_i| \leq 2\) and \(|c_i| \leq 2\). In view of (2.6), (2.7) and (2.8), clearly
\begin{align*}
zf'(z) + \alpha z^2 f''(z) &= \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) \quad \text{and} \quad wg'(w) + \beta w^2 g''(w) = \psi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right).
\end{align*}
Using (2.7) and (2.8) together with (2.1) and (2.2), it is evident that
\begin{align*}
\varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) &= 1 + \frac{1}{2}B_1c_1z + \left( \frac{1}{2}B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2 \right) z^2 + \cdots \\
\psi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right) &= 1 + \frac{1}{2}D_1b_1w + \left( \frac{1}{2}D_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4}D_2b_1^2 \right) w^2 + \cdots.
\end{align*}
Since $f$ has the Maclaurin series given by (2.3), a computation shows that its inverse $g = f^{-1}$ has the expansion
\[ g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \]

Since
\[ \frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1 + 2\alpha)z + (2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2)z^2 + \cdots \]
and
\[ \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = 1 - (1 + 2\beta)a_2 w + ((3 + 10\beta)a_2^2 - 2(1 + 3\beta)a_3)w^2 + \cdots, \]
it follows from (2.9), (2.10) and (2.11) that
\[
\begin{align*}
(2.12) & \quad a_2(1 + 2\alpha) = \frac{1}{2}B_1c_1, \\
(2.13) & \quad 2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \\
(2.14) & \quad -(1 + 2\beta)a_2 = \frac{1}{2}D_1b_1 \\
(2.15) & \quad (3 + 10\beta)a_2^2 - 2(1 + 3\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2.
\end{align*}
\]
It follows from (2.12) and (2.14) that
\[
(2.16) \quad b_1 = -\frac{B_1(1 + 2\beta)}{D_1(1 + 2\alpha)}c_1.
\]

Equations (2.12), (2.13), (2.15) and (2.16) lead to
\[
a_2^2 = \frac{B_1^2D_1^2[B_1(1 + 3\beta)c_2 + D_1(1 + 3\alpha)b_2]}{2[\sigma B_1^2D_1^2 - (1 + 2\alpha)^2(1 + 3\beta)(B_2 - B_1)D_1^2 - (1 + 2\beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2]},
\]
which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.4).

By using (2.13), (2.15) and (2.16) lead to
\[
2\sigma a_3 = \frac{1}{2} \left[ B_1(3 + 10\beta)c_2 + D_1(1 + 2\alpha)b_2 \right] + \frac{c_1^2}{4} \left[ (3 + 10\beta)(B_2 - B_1) + \frac{(1 + 2\beta)^2B_1^2(D_2 - D_1)}{D_1^2(1 + 2\alpha)} \right],
\]
and this yields the estimate given in (2.5). \qed
Remark 2.1. When $\alpha = \beta = 0$ and $B_1 = D_1 = 2$, then (2.4) reduces to Theorem 1.1. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.1 reduces to [1, Theorem 2.2].

Theorem 2.2. Let $f \in \sigma$ and $g = f^{-1}$. If $f \in P(\alpha, \varphi)$ and $g \in M(\beta, \psi)$, then

\[
|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1 + 2\beta) + D_1(1 + 3\alpha)}}{\sqrt{|\sigma B_1^2 D_1^2 - (1 + 2\alpha)^2(1 + 2\beta)(B_2 - B_1)D_1^2 - (1 + \beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2|}}
\]

and

\[
2\sigma |a_3| \leq B_1(3 + 5\beta) + D_1(1 + 2\alpha) + (3 + 5\beta)|B_2 - B_1| + \frac{(1 + \beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1 + 2\alpha)}
\]

where $\sigma := 2 + 7\alpha + 3\beta + 11\alpha\beta$.

Proof. Let $f \in P(\alpha, \varphi)$ and $g \in M(\beta, \psi)$, $g = f^{-1}$. There exist analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = v(0) = 0$, such that

\[
\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad (1 - \beta) \frac{wg'(w)}{g(w)} + \beta \left(1 + \frac{wg''(w)}{g'(w)}\right) = \psi(v(w)),
\]

Since

\[
\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1 + 2\alpha)z + (2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2)z^2 + \cdots
\]

and

\[
(1 - \beta) \frac{wg'(w)}{g(w)} + \beta \left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 - (1 + \beta)a_2w + ((3 + 5\beta)a_2^2 - 2(1 + 2\beta)a_3)w^2 + \cdots,
\]

then (2.10), (2.11) and (2.19) yield

\[
a_2(1 + 2\alpha) = \frac{1}{2}B_1 c_1,
\]

\[
2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2 = \frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2,
\]

\[
-(1 + \beta)a_2 = \frac{1}{2}D_1 b_1
\]

and

\[
(3 + 5\beta)a_2^2 - 2(1 + 2\beta)a_3 = \frac{1}{2}D_1 \left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2.
\]

It follows from (2.20) and (2.22) that

\[
b_1 = -\frac{B_1(1 + \beta)}{D_1(1 + 2\alpha)c_1}.
\]
Equations (2.20), (2.21), (2.23) and (2.24) lead to
\[ a_2^2 = \frac{B_1^2 D_1^2 [B_1(1 + 2\beta)c_2 + D_1(1 + 3\alpha)b_2]}{2[\sigma B_1^2 D_1^2 - (1 + 2\alpha)^2(1 + 2\beta)(B_2 - B_1)D_1^2 - (1 + 2\beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2]}, \]
which gives us the desired estimate on \(|a_2|\) as asserted in (2.17) when \(|b_2| \leq 2\) and \(|c_2| \leq 2\).

Since (2.21), (2.23) and (2.24) lead to
\[
2\sigma a_3 = \frac{1}{2} [B_1(3 + 5\beta)c_2 + D_1(1 + 2\alpha)b_2] + c_2^2 \left[ (3 + 5\beta)(B_2 - B_1) + \frac{(1 + \beta)^2 B_1^2(D_2 - D_1)}{D_1^2(1 + 2\alpha)} \right],
\]
and this yields the desired estimate given in (2.18). \(\square\)

**Theorem 2.3.** Let \(f \in \sigma\) and \(g = f^{-1}\). If \(f \in \mathcal{P}(\alpha, \varphi)\) and \(g \in \mathcal{L}(\beta, \psi)\), then
\[
|a_2| \leq \frac{B_1 D_1 \sqrt{2B_1(3 - 2\beta) + D_1(1 + 3\alpha)}}{\sqrt{\sigma B_1^2 D_1^2 - 2(1 + 2\alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2}}
\]
and
\[
|\sigma a_3| \leq \frac{1}{2} B_1(\beta^2 - 11\beta + 16) + D_1(1 + 2\alpha) + \frac{1}{2}(\beta^2 - 11\beta + 16)|B_2 - B_1| + \frac{(2 - \beta)^2 B_1^2|D_2 - D_1|}{D_1^2(1 + 2\alpha)}
\]
where \(\sigma := 10 + 36\alpha - 7\beta - 25\alpha\beta + \beta^2 + 3\alpha\beta^2\).

**Proof.** Let \(f \in \mathcal{P}(\alpha, \varphi)\) and \(g \in \mathcal{L}(\beta, \psi)\), \(g = f^{-1}\). Then there are analytic functions \(u, v : \mathbb{D} \to \mathbb{D}\), with \(u(0) = v(0) = 0\), satisfying
\[
zf'(z) + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \left( \frac{wg'(w)}{g(w)} \right)^\beta \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\beta} = \psi(v(w))
\]
Since
\[
zf'(z) + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1 + 2\alpha)z + (2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2)z^2 + \cdots
\]
and
\[
\left( \frac{wg'(w)}{g(w)} \right)^\beta \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\beta}
= 1 - (2 - \beta)a_2 w + \left( (8(1 - \beta) + \frac{1}{2}\beta(\beta + 5))a_2^2 - 2(3 - 2\beta)a_3 \right) w^2 + \cdots,
\]
then (2.10), (2.11) and (2.27) yield

\begin{equation}
    a_2(1 + 2\alpha) = \frac{1}{2}B_1c_1,
\end{equation}

(2.28)

\begin{equation}
    2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2 = \frac{1}{2}B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2,
\end{equation}

(2.29)

\begin{equation}
    -(2 - \beta)a_2 = \frac{1}{2}D_1b_1
\end{equation}

and

\begin{equation}
    [8(1 - \beta) + \frac{\beta}{2} (\beta + 5)]a_2^2 - 2(3 - 2\beta)a_3 = \frac{1}{2}D_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4}D_2b_1^2.
\end{equation}

(2.31)

It follows from (2.28) and (2.30) that

\begin{equation}
    b_1 = -\frac{B_1(2 - \beta)}{D_1(1 + 2\alpha)}c_1.
\end{equation}

(2.32)

Equations (2.28), (2.29), (2.31) and (2.32) lead to

\begin{equation}
    a_2^2 = \frac{B_1^2D_1^2[B_1(3 - 2\beta)c_2 + D_1(1 + 3\alpha)b_2]}{\sigma B_1^2D_1^2 - 2(1 + 2\alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2},
\end{equation}

which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.25).

By using (2.29), (2.31) and (2.32) lead to

\begin{equation}
    \sigma a_3 = \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1 + 2\alpha)b_2
\end{equation}

\begin{equation}
    + \frac{c_1^2}{4} \left[ \frac{1}{2}(\beta^2 - 11\beta + 16)(B_2 - B_1) + \frac{(2 - \beta)^2B_1^2(D_2 - D_1)}{D_1^2(1 + 2\alpha)} \right]
\end{equation}

and this yields the estimate given in (2.26).

\[ \square \]

Theorem 2.4. Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{M}(\alpha, \varphi)$, $g \in \mathcal{M}(\beta, \psi)$, then

\begin{equation}
    |a_2| \leq \frac{B_1D_1\sqrt{B_1(1 + 2\beta) + D_1(1 + 2\alpha)}}{\sqrt{\sigma B_1^2D_1^2 - (1 + \alpha)^2(1 + 2\beta)(B_2 - B_1)D_1^2 - (1 + \beta)^2(1 + 2\alpha)(D_2 - D_1)B_1^2}}
\end{equation}

(2.33)

and

\begin{equation}
    2\sigma |a_3| \leq B_1(3 + 5\beta) + D_1(1 + 3\alpha) + (3 + 5\beta)|B_2 - B_1| + \frac{(1 + \beta)^2(1 + 3\alpha)B_1^2|D_2 - D_1|}{D_1^2(1 + \alpha)^2}
\end{equation}

(2.34)

where $\sigma := 2 + 3\alpha + 3\beta + 4\alpha\beta$. 

























Proof. Let \( f \in \mathcal{M}(\alpha, \varphi) \) and \( g \in \mathcal{M}(\beta, \psi) \), \( g = f^{-1} \). Then there are analytic functions \( u, v : \mathbb{D} \to \mathbb{D} \), with \( u(0) = v(0) = 0 \), satisfying

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \varphi(u(z)), \quad (1 - \beta) \frac{wg'(w)}{g(w)} + \beta \left( 1 + \frac{wg''(w)}{g'(w)} \right) = \psi(v(w)),
\]

Since

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + (1 + \alpha)a_2z + (2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2)z^2 + \cdots
\]

and

\[
(1 - \beta) \frac{wg'(w)}{g(w)} + \beta \left( 1 + \frac{wg''(w)}{g'(w)} \right) = 1 - (1 + \beta)a_2w + ((3 + 5\beta)a_2^2 - 2(1 + 2\beta)a_3)w^2 + \cdots,
\]

then (2.10), (2.11) and (2.35) yield

\[
a_2(1 + \alpha) = \frac{1}{2} B_1 c_1,
\]

(2.36)

\[
2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2,
\]

(2.37)

\[- (1 + \beta)a_2 = \frac{1}{2} D_1 b_1
\]

and

\[
(3 + 5\beta)a_2^2 - 2(1 + 2\beta)a_3 = \frac{1}{2} D_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} D_2 b_1^2.
\]

(2.38)

It follows from (2.36) and (2.38) that

\[
b_1 = - \frac{B_1 (1 + \beta)}{D_1 (1 + \alpha)} c_1.
\]

(2.40)

Equations (2.36), (2.37), (2.39) and (2.40) lead to

\[
a_2^2 = \frac{B_1^2 D_1^2 [B_1 (1 + 2\beta)c_2 + D_1 (1 + 2\alpha)b_2]}{2\sigma B_1^2 D_1^2 - 2(1 + \alpha)^2(1 + 2\beta)(B_2 - B_1)D_1^2 - 2(1 + \beta)^2(1 + 3\alpha)(D_2 - D_1)B_1^2},
\]

which, in view of the well-known inequalities \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, gives us the desired estimate on \(|a_2|\) as asserted in (2.33).

By using (2.37), (2.39) and (2.40) lead to

\[
2\sigma a_3 = \frac{B_1}{2} (3 + 5\beta)c_2 + \frac{D_1}{2} (1 + 3\alpha)b_2 + \frac{c_1^2}{4} \left[ (3 + 5\beta)(B_2 - B_1)
\right.
\]

\[
+ \left. \frac{(1 + \beta)^2(1 + 3\alpha)B_1^2 (D_2 - D_1)}{D_1^2 (1 + \alpha)^2} \right]
\]

which we will discuss further.
and this yields the estimate given in (2.34).

\[ \square \]

**Remark 2.2.** When \( \beta = \alpha \) and \( \psi = \varphi \), Theorem 2.4 reduces to [1, Theorem 2.3].

**Theorem 2.5.** Let \( f \in \sigma \) and \( g = f^{-1} \). If \( f \in M(\alpha, \varphi), g \in L(\beta, \psi), \) then

\[
|a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3 - 2\beta) + D_1(1 + 2\alpha)]}}{\sqrt{\sigma B_1^2 D_1^2 - 2(1 + \alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(1 + 2\alpha)(D_2 - D_1)B_1^2}}
\]

and

\[
|\sigma a_3| \leq \frac{B_1 \left(\frac{\beta^2 - 11\beta + 16}{2}\right) + D_1(1 + 3\alpha) + \frac{1}{2}(\beta^2 - 11\beta + 16)|B_2 - B_1|}{D_1^2(1 + \alpha)^2} + \frac{(2 - \beta)^2(1 + 3\alpha)B_1^2|D_2 - D_1|}{D_1^2(1 + \alpha)^2}
\]

where \( \sigma := 10 + 14\alpha - 7\beta + \beta^2 + 2\alpha\beta^2 - 10\alpha\beta \).

**Proof.** Let \( f \in M(\alpha, \varphi) \) and \( g \in L(\beta, \psi), g = f^{-1} \). Then there are analytic functions \( u, v : \mathbb{D} \to \mathbb{D}, \) with \( u(0) = v(0) = 0, \) satisfying

\[
(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)), \quad \left(\frac{wg'(w)}{g(w)}\right)^\beta \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)),
\]

Since

\[
(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1 + \alpha)a_2 z + (2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2)z^2 + \cdots
\]

and

\[
\left(\frac{wg'(w)}{g(w)}\right)^\beta \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = 1 - (2 - \beta)a_2 w + \left(8(1 - \beta) + \frac{1}{2}\beta(\beta + 5)\right)a_2^2 - 2(3 - 2\beta)a_3 w^2 + \cdots,
\]

then (2.10), (2.11) and (2.43) yield

\[
a_2(1 + \alpha) = \frac{1}{2}B_1 c_1,
\]

\[
2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2,
\]

\[
-(2 - \beta)a_2 = \frac{1}{2}D_1 b_1
\]
and

\[ (2.47) \quad [8(1 - \beta) + \frac{\beta}{2}(\beta + 5)]a_2^2 - 2(3 - 2\beta)a_3 = \frac{1}{2}D_1 (b_2 - \frac{b_1^2}{2}) + \frac{1}{4}D_2 b_1^2. \]

It follows from (2.44) and (2.46) that

\[ (2.48) \quad b_1 = -\frac{B_1(2 - \beta)}{D_1(1 + \alpha)}c_1. \]

Equations (2.44), (2.45), (2.47) and (2.48) lead to

\[ a_2^2 = \frac{B_1^2 D_2^2 [B_1(3 - 2\beta)c_2 + D_1(1 + 2\alpha)b_2]}{\sigma B_1^2 D_1^2 - 2(1 + \alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(1 + 2\alpha)(D_2 - D_1)B_1^2}, \]

which, in view of the well-known inequalities \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, gives us the desired estimate on \(|a_2|\) as asserted in (2.41).

By using (2.45), (2.47) and (2.48) lead to

\[ \sigma a_3 = \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1 + 3\alpha)b_2 + \frac{c_1^2}{4}[(\beta^2 - 11\beta + 16)(B_2 - B_1) + (2 - \beta)^2(1 + 3\alpha)B_2^2(D_2 - D_1)] \]

\[ \] \[ \] \[ \]

and this yields the estimate given in (2.42). \(\square\)

**Theorem 2.6.** Let \(f \in \sigma\) and \(g = f^{-1}\). If \(f \in L(\alpha, \varphi), g \in L(\beta, \psi)\), then

\[ (2.49) \quad |a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3 - 2\beta) + D_1(3 - 2\alpha)]}}{\sqrt{\sigma B_1^2 D_1^2 - 2(2 - \alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(3 - 2\alpha)(D_2 - D_1)B_1^2}} \]

and

\[ 2|\sigma a_3| \leq B_1(\beta^2 - 11\beta + 16) + D_1(8 - 5\alpha - \alpha^2) + (\beta^2 - 11\beta + 16)|B_2 - B_1| \]

\[ + \frac{(2 - \beta)^2(\alpha^2 + 5\alpha - 8)B_2^2(D_2 - D_1)}{D_1^2(2 - \alpha^2)} \]

(2.50)

where \(\sigma := 24 + 3\alpha^2 + 3\beta^2 - 17\alpha - 17\beta - 2\beta\alpha^2 - 2\alpha\beta^2 - 12\alpha\beta\).

**Proof.** Let \(f \in L(\alpha, \varphi)\) and \(g \in L(\beta, \psi), g = f^{-1}\). Then there are analytic functions \(u, v : \mathbb{D} \to \mathbb{D}\), with \(u(0) = v(0) = 0\), satisfying

\[ (2.51) \quad \left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = \varphi(u(z)), \quad \left( \frac{wg'(w)}{g(w)} \right)^\beta \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\beta} = \psi(v(w)), \]
Since
\[
\left(\frac{zf''(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1 + (2 - \alpha)a_2 z + \left(2(3 - 2\alpha)a_3 + \frac{(\alpha - 2)^2 - 3(4 - 3\alpha)}{2}a_2^2\right)z^2 + \cdots
\]
and
\[
\left(\frac{wg'(w)}{g(w)}\right)^{\beta} \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = 1 - (2 - \beta)a_2 w + \left(\frac{8(1 - \beta) + \frac{1}{2}\beta(\beta + 5)}{2}a_2^2 - 2(3 - 2\beta)a_3\right)w^2 + \cdots,
\]
then (2.10), (2.11) and (2.51) yield
\[
(2.52) \quad a_2(2 - \alpha) = \frac{1}{2}B_1c_1,
\]
\[
(2.53) \quad 2(3 - 2\alpha)a_3 + \frac{1}{2}[(\alpha - 2)^2 - 3(4 - 3\alpha)]a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_2^2,
\]
\[
(2.54) \quad -(2 - \beta)a_2 = \frac{1}{2}D_1b_1
\]
and
\[
(2.55) \quad [8(1 - \beta) + \frac{\beta}{2}(\beta + 5)]a_2^2 - 2(3 - 2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_2^2.
\]
It follows from (2.52) and (2.54) that
\[
(2.56) \quad b_1 = -\frac{B_1(2 - \beta)}{D_1(2 - \alpha)}c_1.
\]
Equations (2.52), (2.53), (2.55) and (2.56) lead to
\[
a_2^2 = \frac{B_1^2D_1^2[B_1(3 - 2\beta)c_2 + D_1(3 - 2\alpha)b_2]}{\sigma B_1^2D_1^2 - 2(2 - \alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(3 - 2\alpha)(D_2 - D_1)B_1^2},
\]
which, in view of the well-known inequalities \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, gives us the desired estimate on \(|a_2|\) as asserted in (2.49).

By using (2.53), (2.55) and (2.56) lead to
\[
2\sigma a_3 = \frac{B_1}{2}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(8 - 5\alpha - \alpha^2)b_2 + \frac{c_2^2}{4}\left[(\beta^2 - 11\beta + 16)(B_2 - B_1) + (2 - \beta)^2(\alpha^2 + 5\alpha - 8)B_1^2(D_2 - D_1)\right]
\]
and this yields the estimate given in (2.50). \[\square\]
Remark 2.3. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.6 reduces to [11, Theorem 2.4].

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