Near-optimal frequency-weighted interpolatory model reduction

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Abstract
This paper develops an interpolatory framework for weighted-$H_2$ model reduction of MIMO dynamical systems. A new representation of the weighted-$H_2$ inner products in MIMO settings is introduced and used to derive associated first-order necessary conditions satisfied by optimal weighted-$H_2$ reduced-order models. Equivalence of these new interpolatory conditions with earlier Riccati-based conditions given by Halevi is also shown. An examination of realizations for equivalent weighted-$H_2$ systems leads then to an algorithm that remains tractable for large state-space dimension. Several numerical examples illustrate the effectiveness of this approach and its competitiveness with Frequency Weighted Balanced Truncation and an earlier interpolatory approach, the Weighted Iterative Rational Krylov Algorithm.

Keywords: frequency-weighting, interpolation, controller reduction, $H_2$ model reduction.

1. Introduction

Consider a multiple input/multiple output (MIMO) linear dynamical system having a state-space realization (which will be presumed minimal) given by
\begin{align}
    \dot{x}(t) &= Ax(t) + Bu(t) \\
    y(t) &= Cx(t) + Du(t)
\end{align}
where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are constant matrices. $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are, respectively, the state, the input, and the output of the system. The transfer function of this system is $G(s) = C(sI - A)^{-1}B + D$. Following common usage, the underlying system will also be denoted by $G$. The circumstances of interest for us presume very large state-space dimensions relative to the input/output dimensions, $n \gg m, p$. This leads to fundamental difficulties for any task that involves optimization or control of this system. This in turn motivates model reduction: finding a reduced order model (ROM),
\begin{align}
    \dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \\
    y_r(t) &= C_r x_r(t) + D_r u(t)
\end{align}
with an associated transfer function $G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r$ where $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times m}$, $C_r \in \mathbb{R}^{p \times n_r}$, and $D_r \in \mathbb{R}^{p \times m}$. The goal is to produce a greatly reduced state-space dimension, $n_r \ll n$, yet still assure that $y_r(t) \approx y(t)$ over a large class of inputs $u(t)$. This is accomplished by requiring $G_r(s)$ to approximate $G(s)$ very well, in an appropriate sense, which we interpret as making $G_r(s) - G(s)$ small with respect to an appropriate system norm.

For example, one may consider approximations...
that attempt to minimize either the $\mathcal{H}_2$-error:
\[
\|G - G_r\|_{\mathcal{H}_2} \overset{\text{def}}{=} \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|G(i\omega) - G_r(i\omega)\|^2_F d\omega \right)^{1/2},
\]
or the $\mathcal{H}_\infty$-error:
\[
\|G - G_r\|_{\mathcal{H}_\infty} \overset{\text{def}}{=} \sup_{\omega \in \mathbb{R}} \|G(i\omega) - G_r(i\omega)\|_2.
\]
Here $\|M\|_F = \sum_{i,j} |m_{ij}|^2$ denotes the Frobenius norm and $\|M\|_2$ denotes the spectral norm of the matrix $M$. Notice that to ensure that the first error measure is even finite, it is necessary that $D_r = D$.

“Typical” inputs, $u(t)$, often will have their power concentrated in known frequency ranges, and so, some frequency ranges will naturally be more important than others with regard to ROM fidelity. This leads in a natural way to consideration of weighted system errors designed in such a way as to enhance accuracy in certain frequency ranges while permitting larger errors at other frequencies, and towards that end we consider, weighted measures of system error such as
\[
\|G_r - G\|_{\mathcal{H}_2(W)} \overset{\text{def}}{=} \| (G_r(s) - G(s)) \cdot W(s) \|_{\mathcal{H}_2}
\]
and
\[
\|G_r - G\|_{\mathcal{H}_\infty(W)} \overset{\text{def}}{=} \| (G_r(s) - G(s)) \cdot W(s) \|_{\mathcal{H}_\infty}
\]
where $W(s)$ is a given input weighting (a “shaping filter”). One may specify an output weighting as well, however in the interest of clarity and brevity, we do not do this here. We focus on weighted-$\mathcal{H}_2$ measures of error so that for a given system, $G \in \mathcal{H}_2$, one seeks a reduced system $G_r \in \mathcal{H}_2$ solving:
\[
G_r = \arg \min_{G \in \mathcal{H}_2} \|G - \tilde{G}\|_{\mathcal{H}_2(W)}
\] (3)

A variety of shaping filters can be considered. For example, if $W(s)$ were to be chosen to be a transfer function associated with a band-pass filter then approximation errors at frequencies within the passband would be penalized, while approximation error at frequencies lying outside the passband would be discounted.

Another choice of shaping filter arises from controller reduction: Consider a linear dynamical system, $P$ (the plant), with order $n_P$ together with an associated stabilizing controller, $G$, having order $n$, that is connected to $P$ in a feedback loop. Many control design methodologies, such as LQG and $\mathcal{H}_\infty$ methods, lead ultimately to controllers whose order is generically as high as the order of the plant, $n \approx n_P$, see \cite{30,34} and references therein. Thus, high-order plants will generally lead to high-order controllers. However, high-order controllers are usually undesirable in real-time applications because this typically translates into unduly complex and costly hardware implementation that may suffer degraded performance both in terms of speed and accuracy. Thus, one may prefer to replace $G$ with a reduced order controller, $G_r$, having order $n_r \ll n$.

It is often not enough to simply require $G_r$ to be a good approximation to $G$. In order to accurately recover closed-loop performance, plant dynamics need to be taken into account during the reduction process. This may be achieved through frequency weighting: Given a stabilizing controller $G$, if a reduced model, $G_r$, has the same number of unstable poles as $G$ and
\[
\|G - G_r\|_F \cdot \|P[I + PG]^{-1}\|_{\mathcal{H}_\infty} < 1,
\]
then, if $G_r$ is used to replace $G$, $G_r$ will also be a stabilizing controller \cite{11,34}. Seeking $G_r$ to minimize a weighted measure of $\mathcal{H}_2$ error as in (3) is an effective proxy, using the weight $W(s) = P(s)[I + P(s)G(s)]^{-1}$. This approach has been considered in \cite{30,11,21,13,9,22,18,31,29} and references therein, leading then to variants of frequency-weighted balanced truncation. Related methods in \cite{10,22,28} are tailored instead towards minimizing a similarly weighted $\mathcal{H}_2$ error, as we do here.

The main contributions of this paper are threefold. First, we develop a new analysis framework through the introduction of a linear mapping from $\mathcal{H}_2(W)$ to $\mathcal{H}_2$ that gives a new representation of the weighted-$\mathcal{H}_2$ inner product for MIMO systems. This representation allows us to rewrite the weighted-$\mathcal{H}_2$ inner product as a regular (unweighted) $\mathcal{H}_2$ inner product and leads to interpolatory first-order necessary conditions for optimal weighted-$\mathcal{H}_2$ approximation. This analysis framework allows us to extend the interpolatory conditions of \cite{2} for the SISO weighted-$\mathcal{H}_2$ problem to the MIMO case, and more generally
allows us greater flexibility in treating more general settings that involve non-trivial feedthrough terms, which play a crucial role in the weighted-$H_2$ problem. Second, we show that this new interpolation framework is equivalent to the Riccati-based formulation of Halevi [16], thus assuring the accuracy of the Riccati-based optimality formulation at a much lower cost. Finally, via a detailed examination and a new state-space realization for equivalent weighted-$H_2$ systems, we propose a numerical algorithm for weighted-$H_2$ approximation that remains tractable for large state-space dimension. Unlike the heuristic algorithm introduced in [2], which is inspired by optimality conditions but does not attempt to satisfy them, the algorithm proposed here is “near optimal” in the sense that it directly approximates the weighted optimality conditions and approaches true optimality as reduction order grows.

The rest of the paper is organized as follows: In Section 2, we introduce the new formulation for the weighted-$H_2$ inner product for MIMO systems based on a bounded linear transformation from $H_2(W)$ to $H_2$ with which we derive interpolatory optimality conditions. The equivalence of these conditions to those of Halevi [16] is proved in Section 3 followed by a detailed examination and a summary and conclusions are offered in Section 6.

2. Optimal approximations in a weighted-$H_2$ norm.

$H_\infty$ denotes here the set of $m \times m_w$ matrix-valued functions, $W(s)$, having entries, $w_{ij}(s)$, that are analytic for $s$ in the open right half plane and uniformly bounded along the imaginary axis: $\sup_{\omega \in \mathbb{R}} |w_{ij}(i\omega)|$ is finite for all $i, j$. A norm may be defined on $H_\infty$ as $\|W\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \|W(i\omega)\|_2$, where $\|M\|_2$ here represents the induced matrix 2-norm. We assume throughout that the weighting functions, $W(s)$, are drawn from $H_\infty$.

For any such weight, $W \in H_\infty$, denote by $H_2(W)$ the set of $p \times m$ matrix-valued functions, $G(s)$, that have components analytic for $s$ in the open right half plane, and such that for each fixed $Re(s) = s > 0$, $G(x + iy)$ is square integrable with respect to $W$ as a function of $y \in (-\infty, \infty)$ in the sense that

$$\sup_{x > 0} \int_{-\infty}^{\infty} \|G(x + iy)W(x + iy)\|^2_F \, dy < \infty.$$ 

If $G, H \in H_2(W)$ are transfer functions representing real dynamical systems then an inner product may be defined as

$$\langle G, H \rangle_{H_2(W)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left( G(i\omega)W(i\omega)G^T(i\omega) \right) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left( G(-i\omega)W(-i\omega)G^T(i\omega) \right) \, d\omega.$$ 

The associated norm on $H_2(W)$ is

$$\|G\|_{H_2(W)} = \left( \langle G, G \rangle_{H_2(W)} \right)^{1/2}.$$ 

$H_2$ will denote precisely the set $H_2(W)$ with the particular choice $W(s) = I$ (so that $m = m_w$). Note that $H_2 \subset H_2(W)$ and for $G, H \in H_2$,

$$\|G \otimes H\|_{H_2(W)} \leq \|W\|_{H_\infty}^2 \|G\|_{H_2} \|H\|_{H_2}. \tag{4}$$

In all that follows, we suppose the weight $W \in H_\infty$ is a rational function with simple poles at $\{\gamma_1, \ldots, \gamma_{n_w}\}$ and that it has alternative representations given by

$$W(s) = C_w (sI - A_w)^{-1} B_w + D_w \tag{5}$$

and

$$W(s) = \sum_{k=1}^{n_w} \frac{e_k f_k^T}{s - \gamma_k} + D_w. \tag{6}$$

with $A_w \in \mathbb{R}^{n_w \times n_w}$, $B_w \in \mathbb{R}^{n_w \times m_w}$, $C_w \in \mathbb{R}^{m \times n_w}$, and $D_w \in \mathbb{R}^{m \times m_w}$. Echoing the setting of [16], our analysis does not require $m = m_w$, though this may be a natural choice. The (matrix-valued) residue of a meromorphic matrix-valued function, $M(s)$, at a point $\zeta \in \mathbb{C}$ will be denoted as $\text{res}[M(s), \zeta]$, so for example, with $W$ as in (5), $\text{res}[W, \gamma_k] = e_k f_k^T$.

Notice that the transfer function, $G$, associated with the system (1) will be in $H_2(W)$ if and only if
A is stable and $DD_w = 0$. For $G \in \mathcal{H}_2(W)$, define

$$\hat{G}(G)(s) = G(s)W(s)W(-s)^T + \sum_{k=1}^{n_w} G(-\gamma_k)W(-\gamma_k)\frac{f_k e_k^T}{s + \gamma_k}$$

(7)

**Lemma 1.** For $\hat{G}$ as defined in (7)

a. $\hat{G}$ is a bounded linear transformation from $\mathcal{H}_2(W)$ to $\mathcal{H}_2$.

b. For any $G, H \in \mathcal{H}_2$, $\langle G, H \rangle_{\mathcal{H}_2(W)} = \langle \hat{G}(G), H \rangle_{\mathcal{H}_2}$. Hence, $\hat{G}$ is a positive-definite, selfadjoint linear operator on $\mathcal{H}_2$.

The proof of this lemma and subsequent arguments employ an elementary result that we list here. It is an immediate corollary to Lemma 1:

**Proposition 2.** Let $G_1, G_2 \in \mathcal{H}_2$, $G_3(s) = \frac{cb^T}{s-\mu} \in \mathcal{H}_2$, and $G_3(s) = \frac{cb^T}{s-\mu} \in \mathcal{H}_2$. Then,

$$\langle G_1, G_2 \rangle_{\mathcal{H}_2} = c^T G_1(-\mu), \|G_2\|_{\mathcal{H}_2} = \frac{\|c\||b|}{\sqrt{2|\text{Re}\mu|}}$$

and $\langle G_1, G_3 \rangle_{\mathcal{H}_2} = -c^T G_1(-\mu)b$.

**Proof of Lemma 1.** Clearly, $\hat{G}(G)$ is linear in $G$. Let $G \in \mathcal{H}_2(W)$. $G(s)W(s)W(-s)^T$ has simple poles in the right half plane at $-\gamma_1, -\gamma_2, \ldots, -\gamma_{n_w}$, and

$$\text{res}(G(s)W(s)W(-s)^T, -\gamma_k) = \lim_{s \to -\gamma_k} (s + \gamma_k)G(s)W(s)W(-s)^T$$

$$= G(-\gamma_k)W(-\gamma_k)\lim_{s \to -\gamma_k} (s + \gamma_k)W(-s)^T$$

$$= -G(-\gamma_k)W(-\gamma_k)\lim_{s \to -\gamma_k} (s - \gamma_k)W(s)^T$$

$$= -G(-\gamma_k)W(-\gamma_k)\text{res}(W(s)^T, \gamma_k)$$

$$= -G(-\gamma_k)W(-\gamma_k)f_k e_k^T .$$

Thus $\hat{G}(G)(s)$ is analytic in the right-half plane. To show that $\hat{G} : \mathcal{H}_2 \to \mathcal{H}_2$, observe first that $G \cdot W \in \mathcal{H}_2$ so that for each $k = 1, \ldots, n_w$:

$$\|G(-\gamma_k)W(-\gamma_k)\|_2 = \max_{u,v} \frac{u^* [G(-\gamma_k)W(-\gamma_k)] v}{\|u\|_2 \|v\|_2}$$

$$= \max_{u,v} \frac{1}{\|u\|_2 \|v\|_2} \left\langle \frac{u^*}{s-\gamma_k} \right\rangle_{\mathcal{H}_2}$$

$$\leq \|GW\|_{\mathcal{H}_2} \cdot \max_{u,v} \frac{\|u^*\|}{\|u\| \|v\|} = \frac{\|G\|_{\mathcal{H}_2(W)}}{\sqrt{2|\text{Re}\gamma_k|}} ,$$

where the inequality follows from the Cauchy-Schwarz inequality in $\mathcal{H}_2$ and the final equality follows from Proposition 2. Notice that this amounts to the observation that point evaluation in the right half-plane is a continuous map from $\mathcal{H}_2(W)$ to $\mathbb{C}^{m \times p}$. We now use this to calculate

$$\|\hat{G}\|_{\mathcal{H}_2} \leq \|W\|_{\mathcal{H}_\infty} \|G(s)W(s)\|_{\mathcal{H}_2}$$

$$+ \sum_{k=1}^{n_w} \|G(-\gamma_k)W(-\gamma_k)\frac{f_k e_k^T}{s + \gamma_k}\|_{\mathcal{H}_2}$$

$$\leq \left( \|W\|_{\mathcal{H}_\infty} + \sum_{k=1}^{n_w} \frac{\|f_k\| \|e_k\|}{\sqrt{2|\text{Re}\gamma_k|}} \right) \|G\|_{\mathcal{H}_2(W)},$$

where we have used the triangle inequality in $\mathcal{H}_2$ and the observation that $\|MN\|_{\mathcal{F}} \leq \|M\|_{\mathcal{F}}\|N\|_{\mathcal{F}}$ for conforming matrices $M$ and $N$. Thus, $\hat{G}$ is a bounded linear transformation from $\mathcal{H}_2(W)$ to $\mathcal{H}_2$.

For assertion 1b, suppose first that $H$ has simple poles $\{\mu_1, \ldots, \mu_\ell\}$. Note that since $\hat{G}(G)(-s)$ is analytic in the left half plane, $\hat{G}(G)(-s)H(s)^T$ will have poles in the left halfplane exactly at $\{\mu_1, \ldots, \mu_\ell\}$.

For any $R > 0$, define a semicircular contour in the left halfplane: $\mathcal{C}_R = \{z | z = \omega \text{ with } \omega \in [-R, R] \} \cup \{z | z = Re^{i\theta} \text{ with } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\}$. For $R$ large enough, the region bounded by $\mathcal{C}_R$ contains $\{\mu_1, \ldots, \mu_\ell\}$. Using the Residue Theorem and linearity of the trace,
we find
\[
\langle \tilde{\mathcal{I}}[G], H \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr} \left( \tilde{\mathcal{I}}[G](-\omega) H(\omega)^T \right) d\omega
\]
\[
= \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R} \text{tr} \left( \tilde{\mathcal{I}}[G](-s) H(s)^T \right) ds
\]
\[
= \sum_{k=1}^{\ell} \text{tr} \left(\text{res}[\tilde{\mathcal{I}}[G]](-s)H(s)^T, \mu_k \right)
\]
\[
= \sum_{k=1}^{\ell} \text{tr} \left(\tilde{\mathcal{I}}[G](-\mu_k)\text{res}[H, \mu_k]^T \right)
\]
\[
= \sum_{k=1}^{\ell} \text{tr} \left( G(-\mu_k)W(-\mu_k)W(\mu_k)^T\text{res}[H, \mu_k]^T \right)
\]
\[
+ \sum_{k=1}^{\ell} \sum_{i=1}^{n_w} \text{tr} \left( G(-\gamma_i)W(-\gamma_i) \frac{f_i e_i^T}{-\mu_k + \gamma_i} \text{res}[H, \mu_k]^T \right)
\]
\[
= \sum_{k=1}^{\ell} \text{tr} \left( G(-\mu_k)W(-\mu_k)W(\mu_k)^T\text{res}[H, \mu_k]^T \right)
\]
\[
+ \sum_{k=1}^{\ell} \sum_{i=1}^{n_w} \text{tr} \left( G(-\gamma_i)W(-\gamma_i) f_i e_i^T \sum_{k=1}^{\ell} \frac{\text{res}[H, \mu_k]^T}{\gamma_i - \mu_k} \right)
\]
Since $H$ has simple poles and is in $\mathcal{H}_2$, $\sum_{k=1}^{\ell} \frac{\text{res}[H, \mu_k]^T}{s-\mu_k} = H(s)^T$. Note that $\{\mu_1, \ldots, \mu_\ell\} \cup \{\gamma_1, \ldots, \gamma_{n_w}\}$ is precisely the set of poles in the left half plane for the meromorphic function $G(-s)W(-s)W(s)^TH(s)^T$.

So, we continue:

\[
\langle \tilde{\mathcal{I}}[G], H \rangle_{\mathcal{H}_2} = \sum_{k=1}^{\ell} \text{tr} \left( G(-\mu_k)W(-\mu_k)W(\mu_k)^T\text{res}[H, \mu_k]^T \right)
\]
\[
+ \sum_{i=1}^{n_w} \text{tr} \left( G(-\gamma_i)W(-\gamma_i) f_i e_i^T \sum_{k=1}^{\ell} \frac{\text{res}[H, \mu_k]^T}{\gamma_i - \mu_k} \right)
\]
\[
= \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R} \text{tr} \left( G(-s)W(-s)W(s)^TH(s)^T \right) ds
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr} \left( G(-\omega)W(-\omega)W(\omega)^TH(\omega)^T \right) d\omega
\]
\[
= \langle G, H \rangle_{\mathcal{H}_2(W)}
\]

This remains true independent of whether $H$ has simple poles or not: Take a sequence, $H_k$, converging to $H$ in $\mathcal{H}_2$ with each $H_k$ having simple poles. Then, appeal to the continuity of the expressions $\langle G, H_k \rangle_{\mathcal{H}_2(W)} = \langle \tilde{\mathcal{I}}[G], H_k \rangle_{\mathcal{H}_2}$ with respect to the $\mathcal{H}_2$ norm.

$\tilde{\mathcal{I}}$ is positive-definite and selfadjoint on $\mathcal{H}_2$ because, for $G, H \in \mathcal{H}_2$,

\[
\langle \tilde{\mathcal{I}}[G], H \rangle_{\mathcal{H}_2} = \langle G, H \rangle_{\mathcal{H}_2(W)} = \overline{\langle H, G \rangle_{\mathcal{H}_2(W)}} = \langle \tilde{\mathcal{I}}[H], G \rangle_{\mathcal{H}_2}
\]

and $\langle \tilde{\mathcal{I}}[G], G \rangle_{\mathcal{H}_2} = \langle G, G \rangle_{\mathcal{H}_2(W)} > 0$ if $G \neq 0$.

Given state-space realizations for $W \in \mathcal{H}_\infty$ and $G \in \mathcal{H}_2(W)$, one may obtain an explicit state-space realization for $\tilde{\mathcal{I}}[G](s)$.

**Lemma 3.** Suppose $W \in \mathcal{H}_\infty$ has simple poles at $\{\gamma_1, \ldots, \gamma_p\}$ and $G \in \mathcal{H}_2(W)$. Suppose further that $W(s)$ has a realization as given in (7) and $G(s) = C(sI - A)^{-1}B + D$ from (7).

Then $\tilde{\mathcal{I}}[G](s)$ as defined in (7) has a realization given by

\[
\tilde{\mathcal{I}}[G](s) = e^s(sI - A\bar{\gamma})^{-1}B\bar{g}
\]

where $P_w$ and $Z$ solve, respectively,

\[
A_w P_w + P_w A_w^T + B_w B_w^T = 0 \quad \text{and} \quad AZ + ZA_w^T + B(w)P_w + D_w B_w^T = 0.
\]

**Proof** We evaluate (8) in two parts. Note first that since $G \in \mathcal{H}_2(W)$, $DD_w = 0$. We may directly compute a realization of $G(s) \cdot W(s)$:

\[
\begin{bmatrix} C & DC_w \end{bmatrix} \begin{bmatrix} sI - A & -BC_w \\ 0 & sI - A_w \end{bmatrix}^{-1} \begin{bmatrix} BD_w \\ P_w C_w^T + B_w D_w^T \end{bmatrix}.
\]

$A_w$ has distinct eigenvalues by hypothesis; let its eigenvalue decomposition be given as $A_w = U \Gamma U^{-1}$, with $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_{n_w})$. Postmultiply (9) with $U^{-T}$:

\[
A_w \hat{P}_w + \hat{P}_w \Gamma + B_w \hat{F} = 0,
\]
where $P_w U^{-T} = \tilde{P}_w = [\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n_w}]$ and $B_w U^{-T} = \tilde{F} = [\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{n_w}]$. Since $\Gamma$ is a diagonal matrix, we may solve for each column of $\tilde{P}_w$ independently: $\tilde{p}_k = (-\gamma_k I - A_w)^{-1} B_w \tilde{f}_k$. Then defining $E = C_w U = [\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{n_w}]$, we have

$$P_w C_w^T = P_w U^{-T} U^T C_w^T = \tilde{P}_w \tilde{E}^T$$

$$= \sum_{k=1}^{n_w} (-\gamma_k I - A_w)^{-1} B_w \tilde{f}_k \tilde{e}_k^T.$$

We follow the same development for (10); postmultiplication with $U^{-T}$ gives

$$A \tilde{Z} + \tilde{Z} \Gamma + B (C_w \tilde{P}_w + D_w \tilde{F}) = 0,$$

where $\tilde{Z} = Z U^{-T} = [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{n_w}]$. Note that

$$C_w \tilde{P}_w + D_w \tilde{F} = W(-\gamma_k) \tilde{f}_k$$

so that $\tilde{z}_k = (-\gamma_k I - A)^{-1} BW(-\gamma_k) \tilde{f}_k$. Drawing all together, we obtain

$$ZC_w^T = ZU^{-T}U^TC_w^T = \tilde{Z}E^T$$

$$= \sum_{k=1}^{n_w} (-\gamma_k I - A)^{-1} BW(-\gamma_k) \tilde{f}_k \tilde{e}_k^T.$$

With these expressions, the remaining contribution to (8) becomes

$$[C \quad DC_w] [sI - A \quad -BC_w]^{-1} [ZC_w^T + BD_w D_w^T] \quad [P_w C_w^T]$$

$$= C(sI - A)^{-1} ZC_w^T + G(s) C_w (sI - A_w)^{-1} P_w C_w^T$$

$$= \sum_{k=1}^{n_w} C(sI - A)^{-1} (-\gamma_k I - A)^{-1} BW(-\gamma_k) \tilde{f}_k \tilde{e}_k^T$$

$$+ \sum_{k=1}^{n_w} G(s) C_w (sI - A_w)^{-1} (-\gamma_k I - A_w)^{-1} B_w \tilde{f}_k \tilde{e}_k^T.$$

The following easily verified resolvent identity allows further simplification:

$$(sI - A)^{-1} (-\gamma_k I - A)^{-1} = \frac{1}{s + \gamma_k} (-\gamma_k I - A)^{-1} - \frac{1}{s + \gamma_k} (sI - A)^{-1}.$$

(12)

Which then yields,

$$\ldots = \sum_{k=1}^{n_w} \frac{1}{s + \gamma_k} (G(-\gamma_k) - G(s)) W(-\gamma_k) \tilde{f}_k \tilde{e}_k^T$$

$$+ \sum_{k=1}^{n_w} \frac{1}{s + \gamma_k} G(s) (W(-\gamma_k) - W(s)) \tilde{f}_k \tilde{e}_k^T$$

$$= \sum_{k=1}^{n_w} G(-\gamma_k) W(-\gamma_k) \tilde{f}_k \tilde{e}_k^T - G(s) W(s) \sum_{k=1}^{n_w} \tilde{f}_k \tilde{e}_k^T.$$

Postmultiplying (11) with $D_w^T$ and combining with this last expression gives

$$[C \quad DC_w] \left[ \begin{array}{cc} sI - A & -BC_w \\ 0 & sI - A_w \end{array} \right]^{-1} [ZC_w^T + BD_w D_w^T] \quad [P_w C_w^T + B_w D_w^T]$$

$$= G(s) W(s) \left( \sum_{k=1}^{n_w} \tilde{f}_k \tilde{e}_k^T - s - \gamma_k + D_w^T \right)$$

$$+ \sum_{k=1}^{n_w} G(-\gamma_k) W(-\gamma_k) \tilde{f}_k \tilde{e}_k^T$$

$$= \delta[G](s).$$

Lemma 4. Suppose $M_1$ and $M_2$ are stable matrices. The unique solution, $X$, to the Sylvester equation

$$M_1 X + XM_2 + N = 0,$$

is given by

$$X = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-i\omega I - M_1)^{-1} N(i\omega I - M_2)^{-1} d\omega.$$

Lemma 5. For $\delta$ as defined in (7) and any $G, H \in \mathcal{H}_2(W)$, let $H = C_w (sI - A_w)^{-1} B_{\omega} + D_{\omega}$. Then,

a. $\langle \delta[G], \tilde{D}_H \rangle_{\mathcal{H}_2(W)} = \frac{1}{2} \langle G, \tilde{D}_H \rangle_{\mathcal{H}_2(W)}$

b. $\langle \delta[G], H \rangle_{\mathcal{H}_2(W)} = \langle G, H \rangle_{\mathcal{H}_2(W)} - \frac{1}{2} \langle G, \dot{H} \rangle_{\mathcal{H}_2(W)}$

Proof. We may decompose $H$ as $H(s) = H_0(s) + D_H$ with $H_0 = 0 \in \mathcal{H}_2(W)$. Since $G, H \in \mathcal{H}_2(W), D_H \cdot D_w = 0$ and $D \cdot D_w = 0$. Using the realization of $GW$ in (11), we calculate

$$\langle G, \dot{D}_H \rangle_{\mathcal{H}_2(W)} = \langle GW, \dot{D}_H W \rangle_{\mathcal{H}_2(W)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr} \left( G(-i\omega) W(-i\omega) W(i\omega)^T \dot{D}_H^T \right) d\omega$$

$$= \text{tr} \left( [C \quad DC_w] \times C_w^T \dot{D}_H^T \right)$$

6
where
\[ X = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\omega I - A_\delta)^{-1} \begin{bmatrix} BD_w \\ B_w^T \end{bmatrix} \, d\omega. \]

From Lemma 3, this \( X \) is the unique solution to the Sylvester equation
\[ A_\delta X + X A_w^T + \begin{bmatrix} BD_w \\ B_w^T \end{bmatrix} = 0. \]

Recalling (9) and (10), \( X \) evidently may be expressed as \( X = \begin{bmatrix} Z \\ P_w \end{bmatrix} \). Thus, the \( \langle G, \mathcal{D}_H \rangle_{\mathcal{H}_2(W)} \) can be written as
\[ \langle \tilde{\delta}\langle G, \mathcal{D}_H \rangle \rangle_{\mathcal{H}_2(W)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} \tilde{Z} \\ \tilde{P}_w \end{bmatrix} \begin{bmatrix} \tilde{Z} \\ \tilde{P}_w \end{bmatrix}^T \, d\omega = \begin{bmatrix} \tilde{Z} \tilde{Z}^T \\ \tilde{P}_w \tilde{P}_w^T \end{bmatrix} \left( \begin{array}{c} \tilde{Z}^T \\ \tilde{P}_w \end{array} \right), \]

where the integral limit is to be interpreted as a principal value. Because the matrix \( A_\delta \) is stable, the integral reduces to \( \pi \mathbf{I} \), so we have:
\[ \langle \tilde{\delta}\langle G, \mathcal{D}_H \rangle \rangle_{\mathcal{H}_2(W)} = \frac{1}{2} \text{tr} \begin{bmatrix} \tilde{Z} \tilde{Z}^T + \text{DC}_w \mathcal{C}_w^T \mathcal{D}_H^T \end{bmatrix} = \frac{1}{2} \langle G, \mathcal{D}_H \rangle_{\mathcal{H}_2(W)}. \]

Part (b) is shown similarly. We omit details. \( \square \)

2.1. Interpolatory weighted-\( \mathcal{H}_2 \) optimality conditions

The feasible set for (3) consists of all stable transfer functions in \( \mathcal{H}_2(W) \) having order \( n_r \) or less. This is a nonconvex set, hence as a practical matter, finding a global minimizer is extremely difficult. Instead, one typically seeks efficient local minimizers. Methods proposed in [16] and [28] may be used to find local minimizers to (3). However, these methods require solving a sequence of large-scale Lyapunov or Riccati equations and so, rapidly become computationally intractable as system order, \( n_r \), and shaping filter order, \( n_w \), increase.

We approach (3) instead within an interpolatory framework similar to that developed in [2] for SISO systems. Computational complexity for interpolatory methods grows more slowly with increasing \( n \) and \( n_w \), hence much larger problems are feasible. In contrast to the (SISO) results of [2], we are able to treat general MIMO settings including non-zero feedthrough terms, which proves essential for weighted-\( \mathcal{H}_2 \) approximation. The algorithm derived in [2] is heuristic, to the extent that it is inspired by necessary (SISO) optimality conditions but does not seek directly to satisfy them. Our new algorithm proposed in Section 4, on the other hand, directly originates from newly derived MIMO necessary conditions and uses significantly different model reduction spaces, ultimately producing near-optimal reduced models that will approach true optimality as reduction order \( n_r \) grows.

We first derive interpolatory conditions that necessarily must hold for any reduced system, \( G_r \), that solves (3).

Theorem 6. Suppose that \( G_r \in \mathcal{H}_2(W) \) is a solution to (3). Suppose further that \( G_r \) has only simple poles, \( \{\lambda_1, \ldots, \lambda_{nr}\} \), and is represented as:
\[ G_r(s) = C_r (sI - A_r)^{-1} B_r + D_r = \sum_{k=1}^{nr} c_k b_k^T, \quad D_r = \sum_{k=1}^{nr} c_k b_k^T + D_r \]

where \( A_r \in \mathbb{R}^{n_r \times n_r} \) and \( B_r \in \mathbb{R}^{n_r \times m} \), and \( C_r \in \mathbb{R}^{p \times n_r} \). Then \( G_r \) must satisfy for each \( k = 1, \ldots, n_r \),
\[ \tilde{\delta}[G_r(-\lambda_k)b_k] = \tilde{\delta}[G_{r}(-\lambda_k)b_k] \quad \text{(14a)} \]
\[ c_k^T \tilde{\delta}[G_r(-\lambda_k)b_k] = c_k^T \tilde{\delta}[G_{r}(-\lambda_k)b_k], \quad \text{and} \quad \text{(14b)} \]
\[ c_k^T \tilde{\delta}[G_r(-\lambda_k)b_k] = c_k^T \tilde{\delta}[G_{r}(-\lambda_k)b_k], \quad \text{where} \] \( \tilde{\delta} \) is defined in (7) and \( \tilde{\delta} [-\cdot(s)] = d \frac{d}{ds} \tilde{\delta}[\cdot(s)]. \) (Theorem 7 provides one additional condition.)

**Proof** Pick an arbitrary vector \( g \in \mathbb{C}^p \) with \( \|g\| = 1 \) and an index \( k \) with \( 1 \leq k \leq n_r \). Suppose that
\[ \langle G - G_r, \frac{gb_k^T}{s - \lambda_k} \rangle_{\mathcal{H}_2(W)} = \alpha_k = 0. \]

Define \( \theta_0 = \arg(\alpha_0) \) and for arbitrary \( \varepsilon > 0 \), define a perturbation to \( G_r \) as
\[ \tilde{G}_r(s) = \frac{c_k + \varepsilon e^{-\theta_0}b_k^T}{s - \lambda_k} = \frac{\sum_{i \neq k} c_i b_i^T}{s - \lambda_i}. \]
Then, using (4) and Proposition 2 we obtain
\[
\|G_r - \bar{G}_r^{(c)}\|_{\mathcal{H}_2(W)} = \frac{-\varepsilon e^{-\lambda_0 \theta}}{s - \lambda_k} gb^T \|_{\mathcal{H}_2(W)} \\
\leq \|W\|_{\mathcal{H}_2} \frac{\|b_k\|_{\mathcal{H}_2}}{\sqrt{2|\text{Re}(\lambda_k)|}}
\]
Thus, \(\|G_r(s) - \bar{G}_r^{(c)}(s)\|_{\mathcal{H}_2(W)} = O(\varepsilon)\) as \(\varepsilon \to 0\). Since \(G_r\) solves (5).

\[
\|G - G_r\|_{\mathcal{H}_2(W)} \leq \|G - \bar{G}_r^{(c)}\|_{\mathcal{H}_2(W)} \\
\leq \|G - G_r\|_{\mathcal{H}_2(W)} + 2\text{Re} \left\langle G - G_r, G_r - \bar{G}_r^{(c)} \right\rangle_{\mathcal{H}_2(W)} \\
+ \|G_r - \bar{G}_r^{(c)}\|_{\mathcal{H}_2(W)}^2.
\]
Thus,
\[
0 \leq 2 \text{Re} \left\langle G - G_r, G_r - \bar{G}_r^{(c)} \right\rangle_{\mathcal{H}_2(W)} + \|G_r - \bar{G}_r^{(c)}\|_{\mathcal{H}_2(W)}^2.
\]
This implies that \(0 \leq -\varepsilon|\alpha_0| + O(\varepsilon^2)\), which then leads to a contradiction; it must be that \(\alpha_0 = 0\). But then
\[
0 = \left\langle G - G_r, \frac{gb^T}{s - \lambda_k} \right\rangle_{\mathcal{H}_2(W)} = \left\langle \tilde{G}(G - G_r), \frac{gb^T}{s - \lambda_k} \right\rangle_{\mathcal{H}_2(W)} \\
= g^T (\tilde{G}(G - G_r)(-\lambda_k)) b_k,
\]
(using Proposition 2) and since \(g\) was chosen arbitrarily, we must have
\[
0 = \tilde{G}(G - G_r)(-\lambda_k) b_k = \tilde{G}(G)(-\lambda_k) b_k - \tilde{G}(G_r)(-\lambda_k) b_k
\]
which confirms (14a), (14b) is shown similarly, replacing \(\frac{gb^T}{s - \lambda_k}\) in the argument above with \(\frac{c_k g^T}{s - \lambda_k}\) for arbitrary \(g \in \mathbb{C}^n\).

To show (14c), suppose that \(\tilde{G}(G - G_r, \frac{c_k b_k^T}{s - \lambda_k})_{\mathcal{H}_2(W)} = \alpha_1 \neq 0\). and define \(\theta_1 = \text{arg}(\alpha_1)\). For \(\varepsilon > 0\) sufficiently small, define
\[
\tilde{G}_r^{(c)}(s) = \frac{c_k b_k^T}{s - (\lambda_k + \varepsilon e^{-\theta_1})} + \sum_{i \neq k} \frac{c_i b_i^T}{s - \lambda_i}
\]
As \(\varepsilon \to 0\), we have
\[
\|G_r - \tilde{G}_r^{(c)}\|_{\mathcal{H}_2(W)} = O(\varepsilon)
\]
Following a similar argument as before, we find that \(0 \leq -\varepsilon|\alpha_1| + O(\varepsilon^2)\) as \(\varepsilon \to 0\), which leads to a contradiction, forcing \(\alpha_1 = 0\). This, in turn, implies from Proposition 2
\[
0 = \left\langle G - G_r, \frac{c_k b_k^T}{(s - \lambda_k)^2} \right\rangle_{\mathcal{H}_2(W)} = \left\langle \tilde{G}(G - G_r), \frac{c_k b_k^T}{(s - \lambda_k)^2} \right\rangle_{\mathcal{H}_2} \\
= -\frac{d}{ds} \begin{bmatrix} c_k \bar{b}_k \end{bmatrix} (-\lambda_k) b_k^T \bigg|_{s = \lambda_k}.
\]
which gives (14c). □

We have an additional necessary condition for optimality that arises from the presence of the weighting filter. For \(G, G_r \in \mathcal{H}_2(W)\), let \(F(t)\) and \(F_r(t)\) denote the impulse response functions associated respectively with \(\tilde{G}(G)(s)\) and \(\tilde{G}(G_r)(s)\). That is, \(\tilde{G}(G) = \mathcal{L}\{F\}\) and \(\tilde{G}(G_r) = \mathcal{L}\{F_r\}\), where \(\mathcal{L}\{\cdot\}\) is the Laplace transform.

**Theorem 7.** Assume the hypotheses and notation of Theorem 6. Then for all \(n \in \text{Ker}(D_w^T)\),
\[
F(0)n = F_r(0)n.
\quad (14d)
\]

**Proof** Pick \(m \in \mathbb{R}^p\) and \(n \in \text{Ker}(D_w^T)\), arbitrarily. From (0), \(mn^T W(s) = \sum_{k=1}^n \left(n^T e_k\right) m f_k^T = 0\) is evidently an \(\mathcal{H}_2\) function. Hence, \(mn^T \in \mathcal{H}_2(W)\). Suppose that
\[
\left\langle G - G_r, mn^T \right\rangle_{\mathcal{H}_2(W)} = \alpha_0 \neq 0.
\]
Define \(\theta_0 = \text{arg}(\alpha_0)\) and for arbitrary \(\varepsilon > 0\), define a perturbation to \(G_r\) as
\[
\tilde{G}_r^{(c)}(s) = \varepsilon e^{-\theta_0} mn^T + G_r(s)
\]
Arguments identical to those in the proof of Theorem 6 lead to
\[
0 \leq -2 \text{Re} \left\langle G - G_r, \varepsilon mn^T \right\rangle_{\mathcal{H}_2(W)} + \|\varepsilon mn^T\|_{\mathcal{H}_2(W)}^2,
\]
implying that \(0 \leq -\varepsilon|\alpha_0| + O(\varepsilon^2)\), and leading to a contradiction as before; as a consequence, \(\alpha_0 = 0\). But then
\[
0 = \left\langle G - G_r, mn^T \right\rangle_{\mathcal{H}_2(W)} = \left\langle \tilde{G}(G - G_r), mn^T \right\rangle_{\mathcal{H}_2} \\
= m^T \left\{ \int_{-\infty}^{+\infty} \tilde{G}(G - G_r)(i\omega) d\omega \right\} n.
\]
Since \( m \) was chosen arbitrarily, we must have
\[
0 = \left[ \int_{-\infty}^{+\infty} \frac{1}{\pi} (G - G_r)(s) \, ds \right] n = [F(0) - F_r(0)] n.
\]
which confirms \( (14d) \). \( \square \)

3. The Halevi optimality conditions

Following [16, Appendix A], the first-order necessary conditions for a locally optimal reduced model \( G_r \) can be stated in terms of solutions to linear matrix equations. Consider the set of matrix equations defined by \( G, G_r \in H_2(W) \) and \( W \in H_\infty \) as follows:

\[
A_r X + X A_r^T + 2B_r B_r^T = 0, \tag{15a}
\]

\[
A_r P_r + P_r A_r^T + B_r [0 \ C_w] X + \left( X^T \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} + B_r D_r D_r^T \right) B_r^T = 0, \tag{15b}
\]

\[
A_r^T Q_r + Q_r A_r + C_r^T C_r = 0. \tag{15c}
\]

\[
A_r^T Y + YA_r = \left( [(D - D_r) C_w] C_w^T \right) C_r - \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} B_r^T Q_r. \tag{15d}
\]

If \( G_r \) is locally \( H_2(W) \)-optimal, then:

\[
Y^T X + Q_r P_r = 0, \tag{16a}
\]

\[
e_3 X - C_r P_r - D_r [0 \ C_w] X = 0, \tag{16b}
\]

\[
Y^T B_r + Q_r \left( B_r D_r D_r^T + X^T \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} \right) = 0, \tag{16c}
\]

\[
C_r X^T \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} \ N - C Z C_r^T N = (D - D_r) C_w P_w C_w^T N, \tag{16d}
\]

where \( N = [n_1, \ldots, n_d] \) is a basis for \( \text{Ker}(D_r^T) \).

Notice that for \( W(s) = I \), conditions \( (16a)-(16c) \) coincide with the Wilson optimality conditions from [33], while the final condition \( (16d) \) is satisfied vacuously since in this case, \( \text{Ker}(D_r^T) = \{0\} \).

3.1. Equivalence of the optimality conditions

The close connection between Sylvester equations and tangential interpolation in the unweighted case has been established in [11]. The model reduction bases that enforce tangential interpolation can be obtained as solutions to special Sylvester equations. Moreover, in [14], the necessary \( H_2 \) optimality conditions in the form of Sylvester equations from [33] have been shown to be equivalent to the interpolatory conditions from [19]. For the weighted case, there are two frameworks as well: the interpolatory conditions \( (14a)-(14d) \) we developed here and the linear matrix equations based conditions \( (16a)-(16d) \) of Halevi [16]. Since these are only necessary conditions, their equivalence is not obvious. We formally establish this equivalency.

**Theorem 8.** Let \( G, G_r \in H_2(W) \) and \( W \in H_\infty \). Assume that \( G_r \) has simple poles at \( \{\lambda_1, \ldots, \lambda_n\} \). Then optimality conditions \( (14a)-(14d) \) and \( (16a)-(16d) \) are equivalent.

**Proof** Assume \( G_r \) satisfies \( (16a)-(16d) \) and that \( A_r = RAR^{-1} \) is an eigenvalue decomposition of \( A_r \). Multiplying \( (15a) \) with \( R^{-T} \) from right gives

\[
A_r X + X A_r^T + 2B_r B_r^T = 0,
\]

where \( X = XR^{-T} \) and \( B = B_r^T R^{-T} \). This implies

\[
\tilde{X} s_k = \tilde{X} k = (-\lambda_k I - A_{r_{1}})^{-1} 2B_r b_k, \tag{17}
\]

where \( s_k \) is the \( k \)th unit vector. Similarly, multiplying \( (15b) \) from right with \( R^{-T} \) yields

\[
A_r \tilde{P} + \tilde{P} A_r + B_r [0 \ C_w] \tilde{X} = 0,
\]

where \( \tilde{P} = P_r R^{-T} \). Since for \( X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \) we can conclude that \( X_2 = Z_r^T \), where \( Z_r \) satisfies

\[
A_r Z_r + Z_r A_r^T + B_r(C_r P_r + D_r B_r^T) = 0. \tag{18}
\]

It also follows

\[
\tilde{P} s_k = \tilde{P} k = (-\lambda_k I - A_{r_{1}})^{-1}(Z_r C_r^T + B_r D_r D_r^T)b_k + (-\lambda_k I - A_{r_{1}})^{-2} B_r C_r (-\lambda_k I - A_{r_{1}})^{-1} \times (P_r C_r^T + B_r D_r^T)b_k. \tag{19}
\]
Right multiplication of (16b) with $R^{-T}$, gives
\[ cT_\delta \tilde{X} - C_r \tilde{P} - D_r [0 \quad C_w] \tilde{X} = 0. \]
Hence, due to Lemma 3, each column is equivalent to (14a). Now postmultiply (15c) with $R$ to obtain
\[ A_r^T \tilde{Q} + QA + C_r^T C = 0, \]
where $\tilde{Q} = Q, R$ and $C = C_r, R$. Hence, it follows
\[ \tilde{Q} s_k \tilde{Q}_k = (-\lambda_k I - A_r^{T})^{-1} C_r^T c_k. \tag{20} \]
Also, postmultiplication of (15d) with $R$ leads to
\[ A_r^T \tilde{Y} + YA = \left[ (D - D_r) C_w^T \right]^T [0 \quad C_w^T] B_r \tilde{Q}, \]
where $\tilde{Y} = YR$. In particular, we get
\[ \tilde{Y} s_k = Y_k = (-\lambda_k I - A_\delta)^{-T} \]
\[ \times \left( \begin{bmatrix} 0 \\ C_w \end{bmatrix} B_r (-\lambda_k I - A_r)^{-1} C_r^T + D_r - cT_\delta \right) c_k, \tag{21} \]
We further have $\tilde{Y}^T B_\delta + \tilde{Q} (B_r D_w D_r^T + Z_r C_w^T) = 0$ due to (16c). Together with (20) and (21), for each row it thus holds
\[
0 = -cT_\delta C_\delta (-\lambda_k I - A_\delta)^{-1} B_\delta + cT_\delta (C_r (-\lambda_k I - A_r)^{-1} B_r + D_r)
\times C_w (-\lambda_k I - A_w)^{-1} (B_w D_w^T + P_w C_w^T)
+ cT_\delta C_r (-\lambda_k I - A_r)^{-1} (B_r D_w D_r^T + Z_r C_r^T).
\]
Again, using Lemma 3 this leads to (14b). Finally, pre- and postmultiplication of (16a) with $R^T$ and $R^{-T}$ yields
\[ \tilde{Y}^T \tilde{X} + \tilde{Q} \tilde{P} = 0. \tag{22} \]
Using (17) - (21) for the diagonal of (22), we find
\[
0 = -cT_\delta C_\delta (-\lambda_k I - A_\delta)^{-2} B_\delta b_k
+ cT_\delta (C_r (-\lambda_k I - A_r)^{-1} B_r + D_r)
\times C_w (-\lambda_k I - A_w)^{-2} (B_w D_w^T + P_w C_w^T) b_k
+ cT_\delta C_r (-\lambda_k I - A_r)^{-2} (Z_r C_r^T + B_r D_w D_w^T)
+ cT_\delta C_r (-\lambda_k I - A_r)^{-2} B_r
\times C_w (-\lambda_k I - A_w)^{-1} (B_w D_w^T + P_w C_w^T) b_k.
\]
Then, due to Lemma 3 this implies (14c). Finally, due to (16d) we note that
\[ [C_r \quad D_r C_w^T] Z_r C_w^T N = [C \quad D_w C_w^T] N. \]
From (11), $\int_{-\infty}^{\infty} (i\omega I - M)^{-1} d\omega = \pi I$, for any stable matrix $M$, and we conclude that
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} [C_r \quad D_r C_w^T] \left[ \begin{bmatrix} i\omega I - A_r & -B_r C_w \\ i\omega I - A_w & 0 \end{bmatrix} \right]^{-1} Z_r C_w^T P_w C_w^T N \, d\omega \]
\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} [C \quad D_w C_w^T] \left[ \begin{bmatrix} i\omega I - A_r & -B_r C_w \\ i\omega I - A_w & 0 \end{bmatrix} \right]^{-1} [Z_r C_r^T P_r C_r^T] N \, d\omega \]
Hence, for all $n \in \ker (D_r^T)$,
\[ \int_{-\infty}^{\infty} \tilde{G}(s) \omega N \, d\omega \]
\[ = \int_{-\infty}^{\infty} \tilde{G}(s) \omega N \, d\omega, \]
which is equivalent to (14d). Reversing the arguments and using (12) for the offdiagonal entries of (16a) shows that (14a)-(14d) also imply (16a)-(16d).

4. Frequency-weighted rational interpolation

We henceforth assume that the feedthrough term of the original system, $G_r$, is zero: $D = 0$. This is without loss of generality since the general case may be recovered by reassigning $D_r \leftarrow D_r - D$. From the previous discussion, we have seen that frequency-weighted $H_2$-optimal approximants are mapped to Hermitian interpolants via the mapping $\tilde{\gamma}$ introduced in (7). This presents a practical problem of how to construct reduced order systems, $G_r$, such that $\tilde{\gamma}(G_r)(s)$ interpolates $\tilde{\gamma}(G)(s)$ at selected points in $\mathbb{C}$, say at $\{\sigma_1, \sigma_2, \ldots, \sigma_{n_r}\}$, in selected tangent directions $\{b_1, \ldots, b_{n_r}\}$ and $\{c_1, \ldots, c_{n_r}\}$. Using the realization developed in Lemma 3 and standard interpolation results, we construct reduction subspaces that will force interpolation:
\[ \operatorname{Ran} \begin{bmatrix} \tilde{\gamma}(s) \\ \tilde{\gamma}^{(s)} \end{bmatrix} = \operatorname{span} \{ (\sigma_i I - A_\delta)^{-1} 2 \delta b_i \}. \tag{23} \]
The reduced feedthrough term is computed from
\[ \mathcal{W} = \{\{\sigma_i I - A_{\beta}^T\}^{-1}c_i^T e_i\}. \tag{24} \]
Define \( V_r, W_r \in \mathbb{C}^{n \times n_r} \) so that \( W_r^T V_r = I \) and
\[
\begin{align*}
\text{Ran}(V_r) & \supset \text{Ran}\{\mathcal{W}(a)\} \\
\text{Ran}(W_r) & \supset \text{Ran}\{\mathcal{W}(b)\}.
\end{align*}
\tag{25}
\]

The reduced feedthrough term is computed from \cite{16}:
\[
D_r = C(Z - V_r, Z_r) C_w^T N(N^T C_w P_w C_w^T N)^{-1} N^T, \tag{26}
\]
where \( N \) is a basis for \( \text{Ker}(D_w^T) \).

**Theorem 9.** Let \( A_r = W_r^T A V_r \), \( B_r = W_r^T B \), \( C_r = C_r V_r \), with \( V_r \) and \( W_r \) constructed as in \cite{23, 24}, and \cite{25}. Suppose \( D_r \) is determined by (20). Then pick any interpolation point \( \sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \), with associated tangent directions: \( b \) and \( c \). Provided \( \sigma \not\in \{\Lambda(A), \Lambda(A_r)\} \), we have
\[
\hat{\mathcal{G}}[s] = \hat{\mathcal{G}}[\sigma] - \hat{\mathcal{G}}[\sigma_r](s) = \mathcal{H}_1^T(s) (Z - V_r, Z_r) C_w^T b - C(Z - V_r, Z_r) H_2(\sigma)b
\]
and
\[
\mathbf{F}(0)n = \mathbf{F}_r(0)n,
\]
where \( \mathbf{F}(t) \) and \( \mathbf{F}_r(t) \) are the impulse responses of \( \hat{\mathcal{G}}[\sigma] \) and \( \hat{\mathcal{G}}[\sigma_r] \), respectively, \( n \in \text{Ker}(D_w^T) \) is arbitrary,

\[
\mathbf{H}_1(s) = C_r(sI - A_r)^{-1} W_r^T, \quad \text{and}
\]
\[
\mathbf{H}_2(s) = C_w^T N(N^T C_w P_w C_w^T N)^{-1} N^T \times C_r (sI - A_r)^{-1} (P_w C_w^T + B_w D_w^T).
\]

**Proof:** We follow a pattern of proof given in \cite{31}.

Define \( \mathcal{V} = \begin{bmatrix} V_r & 0 \\ 0 & I \end{bmatrix} \), \( \mathcal{W} = \begin{bmatrix} W_r & 0 \\ 0 & I \end{bmatrix} \), and \( A_{\beta} = \begin{bmatrix} A_r & B_r C_w \\ 0 & A_w \end{bmatrix} \).

Define two (skew) projectors via
\[
\mathcal{P}_r(s) = \mathcal{V}(sI - A_{\beta})^{-1} \mathcal{W}^T (sI - A_{\beta})
\]
\[
\mathcal{Q}_r(s) = (sI - A_{\beta}) \mathcal{P}_r(s) (sI - A_{\beta})^{-1}
\]
\[
= (sI - A_{\beta}) \mathcal{V}(sI - A_{\beta})^{-1} \mathcal{W}^T.
\]

For all \( s \) in a neighborhood of \( \sigma \), we have
\( \mathcal{V} = \text{Ran}(\mathcal{P}_r(s)) = \text{Ker}(I - \mathcal{P}_r(s)) \) and \( \mathcal{W}^\perp = \text{Ker}(\mathcal{Q}_r(s)) = \text{Ran}(I - \mathcal{Q}_r(s)) \). Now observe that
\[
\hat{\mathcal{G}}[\sigma_r](s) = [C_r^T 0] [sI - A_r - B_r C_w]^{-1} \begin{bmatrix} W^T Z C_w^T + B_w D_w^T \end{bmatrix} - [C_r^T 0] [sI - A_r - B_r C_w]^{-1} \begin{bmatrix} W^T Z C_w^T \end{bmatrix} - D_w C_w(sI - A_w)^{-1}(P_w C_w^T + B_w D_w^T).
\]

Hence, we can write
\[
\hat{\mathcal{G}}[\mathbf{G}_r](s) - \hat{\mathcal{G}}[\mathbf{G}_r](s) = H_1(s)(Z - V_r, Z_r) C_w^T b - C(Z - V_r, Z_r) H_2(\sigma)b
\]
\[
+ C_r(sI - A_{\beta})^{-1} (I - \mathcal{P}_r(s))(sI - A_{\beta})
\]
\[
\times (I - \mathcal{P}_r(s))(sI - A_{\beta})^{-1} B_{\beta}.
\]

Evaluating this expression at \( s = \sigma \) and postmultiplying by \( b \) yields the first assertion; premultiplying by \( c^T \) yields the second. We find that
\[
((\sigma + \varepsilon) I - A_{\beta})^{-1} = (sI - A_{\beta})^{-1} - \varepsilon(sI - A_{\beta})^{-2} + \mathcal{O}(\varepsilon^2).
\]
Evaluating (28) at \( s = \sigma + \varepsilon \), premultiplying by \( c^T \), and postmultiplying by \( b \) together with \( \varepsilon \rightarrow 0 \) yields the third statement. The last statement results from the proof of Theorem 8 and the fact that \( N \) is a basis of \( \text{Ker}(D_w^T) \). Note also that we have \( D_r D_w = 0 \). \( \square \)

Conditions for exact interpolation are now evident:

**Corollary 10.** Let \( \mathbf{G}_r \) denote the reduced order model of Theorem 9. If \( \mathbf{G}_r \) is stable and \( \text{Ran}(Z) \subseteq \text{Ran}(V_r) \) then \( \hat{\mathcal{G}}[\mathbf{G}_r] \) is an exact bitangential Hermite interpolant to \( \hat{\mathcal{G}}[\mathbf{G}] \) at each interpolation point, \( \{s_1, s_2, \ldots, s_n\} \), in corresponding tangent directions, \( \{b_1, \ldots, b_n\} \) and \( \{c_1, \ldots, c_n\} \).
\[ A_r W_r^T (Z - V_r Z_r) + W_r^T (Z - V_r Z_r) A_w^T = 0. \]

Since \( A_r \) and \( A_w \) are both stable,

\[ W_r^T (Z - V_r Z_r) = W_r^T Z - Z_r = 0 \]

and so, \( Z = V_r Z_r \).

The deviation from exact interpolation is quantified in Theorem 9 and depends on the deviation of \( V_r, Z_r \) from \( Z \). For shaping filters of modest order with \( n_w \ll n \), exact interpolation can be induced since one may include \( \text{Ran}(Z) \) in the projection space, \( \text{Ran}(V_r) \).

More generally, \( V_r, Z_r \) may be viewed as a Petrov-Galerkin approximation to the solution \( Z \) of the Sylvester equation \( 10 \) in the following sense: \( Z_r \) that solves \( 18 \) is a solution to the problem of finding \( Z_r \in \mathbb{R}^{n \times n_w} \) such that with respect to the usual (Euclidean) inner product in \( \mathbb{R}^n \),

\[ \text{Ran} \left( A (V_r Z_r) + (V_r Z_r) A_w^T + B (C_u P_w + D_w B_w^T) \right) \perp \text{Ran}(W_r). \]

Since \( m, m_w \ll n \), the singular values of the original solution, \( Z \), to \( 10 \) will typically decay rapidly \( 12, 21, 26, 27 \); there will be good low rank approximations to \( Z \) and among them will be approximations of the form \( V_r Z_r \). In our approach, the subspace \( V_r \) is closely related to a \( H_2 \) optimal approximation. And in the unweighted case, projection subspaces associated with \( H_2 \)-optimal reduced models are known to yield very accurate approximations. This has been underlined in \( 8, 10 \) by the fact that the approximations are equivalent to those obtained from the alternating directions implicit (ADI) iteration. Moreover, \( 5 \) showed that for symmetric state space systems, low rank approximations from an \( H_2 \)-optimal reduced model in fact locally minimize the energy norm naturally induced by the corresponding Lyapunov operator. Overall, this leads to the expectation that as \( n_r \) increases, \( V_r Z_r \approx Z \). If furthermore, the interpolation points that determine a reduced model coincide with the reflected poles of the model, then Theorem 9 asserts that the optimality conditions \( 14a, 14d \) will very nearly be satisfied; the reduced model draws closer to \( H_2(W) \)-optimality as \( n_r \) increases.

The practical difficulty in constructing such near optimal reduced models is that one doesn’t know a priori how to choose interpolation data determining a reduced model so as to coincide with the reflected poles of the model. The parallel circumstance for (unweighted) optimal \( H_2 \) model reduction has been largely resolved with an iterative correction process \( 14 \); we propose an analogous approach here:

\begin{algorithm}
\textbf{Algorithm NOWI:} Nearly Optimal Weighted Interpolation

\textbf{Input:} Interpolation points: \( \{\sigma_1, \ldots, \sigma_{n_r}\} \);
\hspace{1em} Tangent directions: \( \tilde{\mathbf{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_{n_r}] \)
\hspace{1em} and \( \tilde{\mathbf{C}} = [\mathbf{c}_1, \ldots, \mathbf{c}_{n_r}] \).

\textbf{Output:} \( A_r, B_r, C_r, D_r \)

\begin{enumerate}
\item \textbf{while} relative change in \( \{\sigma_i\} > \text{tol} \) \textbf{do}
\item Compute \( V_r \) and \( W_r \) from \( 23, 24, \) and \( 25 \).
\item Update ROM: \( A_r = W_r^T A V_r, B_r = W_r^T B, \)
\hspace{1em} \( C_r = C V_r, \) and \( D_r \) as in \( 26 \).
\item \( \sigma_i = -\lambda_i(\mathbf{A}), A_r = \mathbf{R} A \mathbf{R}^{-1}, \mathbf{B} = \mathbf{B} \mathbf{R}^{-T}, \)
\hspace{1em} and \( \tilde{\mathbf{C}} = \mathbf{C} \mathbf{R}^{-1} \).
\item \textbf{end while}
\end{enumerate}
\end{algorithm}

Note that \textsc{nowi} is not simply a MIMO extension of \textsc{wirka} in \( 2 \), which was developed specifically for SISO settings. \textsc{wirka} is heuristic in nature and does not originate from necessary optimality conditions. On the other hand, \textsc{nowi} directly attempts to satisfy conditions for optimality and will provide progressively better approximations to them as \( n_r \) increases. Even in SISO settings, the difference between \textsc{nowi} and \textsc{wirka} is easily seen by noting that the model reduction bases \( V_r \) and \( W_r \) are completely different. While \textsc{nowi} uses a state-space realization of \( \mathcal{F}[\mathbf{G}](s) \) (as the interpolation conditions require) in order to construct \( V_r \) and \( W_r \), \textsc{wirka} instead uses regular rational Krylov subspaces corresponding to \( \mathbf{G}(s) \) – generally, not even approximately satisfying the necessary optimality conditions. Moreover, in \textsc{wirka}, \( W_r \) is kept constant after initialization unlike in \textsc{nowi} where both \( W_r \) and \( V_r \) are updated iteratively.
**Computational complexity:** Many issues enter in determining the computational resources necessary to produce an effective reduced order model. Estimates of computational complexity serve as a useful proxy for this expense, which may be then further refined according to problem-specific structure and implementation. Notice first that our NOWI Algorithm is an iterative process, requiring in each cycle the construction of left- and right-reduction subspaces. This requires first the solution of two linear matrix equations, (9) and (10) of orders $n_w \times n_w$ and $n \times n_w$, respectively. If $n_w \ll n$, this may be done directly with cost dominated by $n_w$ linear solves of dimension $n$. For larger $n_w$, the numerical rank of $P_w$ and $Z$ is often relatively small allowing for very accurate approximations by low rank methods such as $[20, 15, 23, 6, 17, 25]$. Bases for the left- and right-reduction subspaces then may be computed exploiting the block triangular structure of the $\mathcal{F}$-realization; this leads to $2n_r$ linear solves of dimension $n$ and $n_r$ linear solves of dimension $n_w$. Sparsity in $A$ and $A_w$ may be exploited with either direct or iterative linear solvers. Multiple right-hand sides and small changes among shifts offer further opportunities for efficiency from subspace and preconditioner recycling.

When compared to standard approaches for frequency-weighted balanced truncation (FWBT), we find that as long as the number of iterations of NOWI remains modest (which appears typical), the overhead associated with solving two large Lyapunov equations of dimension $n$, which is necessary for FWBT, has been eliminated. This creates a particularly dramatic advantage for NOWI in the case of a shaping filter where $n_w \ll n$. The computational advantages of NOWI are also significant when compared to Halevi’s approach to weighted-$\mathcal{H}_2$ model reduction $[16]$, which requires solving large-scale Ricatti and Lyapunov equations of order $(n + n_w) \times (n + n_w)$ at every step of the iteration.

5. **Numerical examples**

We study the performance of our NOWI Algorithm for three different examples resulting from controller reduction. We compare the proposed method with frequency weighted balanced truncation (FWBT) of $[9]$, and also with WIRKA of $[2]$ for the SISO example.

**Los Angeles University Hospital.** The plant is a linearized model for the Los Angeles University Hospital with order $n = 48$. An LQG-based controller of the same order as the original system is to be reduced, leading to a weighting $W(s)$ of order $n_w = 96$, see $[2]$. For a given $n_r$, we use the mirror images of the $\nu = 2$ most dominant poles of $W(s)$ and the mirror images $n_r - \nu$ most dominant poles of $G(s)$ as the initial interpolation points for WIRKA, as suggested in $[2]$. We use the same initialization for the NOWI Algorithm. Figure 1 shows the relative $H_2(W)$- and $H_\infty(W)$-errors obtained from NOWI, FWBT, and WIRKA for reduced system orders $n_r = 2, \ldots, 30$. For the $H_2(W)$-case, NOWI outperforms FWBT and WIRKA for all $n_r$ values except for $n_r = 18$, for which WIRKA is slightly better. The superiority of NOWI is especially evident for smaller $n_r$ values. We find similar results for the $H_\infty(W)$-error as well; FWBT yields the smallest $H_\infty(W)$-errors for larger $n_r$, as expected. The fact that NOWI displays better $H_\infty(W)$ performance than FWBT even for a subset of reduction orders suggests the effectiveness of the approach. NOWI
produces reduced models that satisfy the $H_2(W)$-optimality interpolation conditions \ref{eq:optimal_conditions} only approximately (see Theorem \ref{thm:optimality}). Figure 2 shows how the relative interpolation error (deviation from \ref{eq:optimal_conditions}) in final reduced models produced by NOWI evolves with increasing $n_r$. As the figure shows, the relative error in the optimality conditions decreases as $n_r$ increases. This confirms the expectations described in the discussion following Corollary \ref{corollary:optimality}. Figure 3 shows how the relative interpolation error in the optimality conditions \ref{eq:optimal_conditions} evolve (for fixed reduction order, $n_r$) step to step in the NOWI Algorithm. Results for two cases are displayed: $n_r = 16$ and $n_r = 30$. In both cases, we observe that NOWI rapidly reduces interpolation error during the iteration. For example, for $n_r = 16$, relative interpolation errors are in the order of 1 initially; however as the algorithm progresses, relative errors decline to levels of $10^{-3}$, leading to near-optimal interpolation.

**CD player.** The plant is a model for a CD player and belongs to the SLICOT benchmark collection. We consider the original MIMO version with $n = 120$ and $m = p = 2$. As in the previous example, we design an LQG-based controller having the same order as the plant, leading to a weight $W(s)$ with $n_w = 240$. Since WIRKA has been proposed only for SISO systems and a MIMO extension is not immediate, we show comparisons only between FWBT and NOWI, using a random initialization. Figure 4 again compares the quality of reduction in terms of the $H_2(W)$-error and $H_\infty(W)$-error. Both methods perform equally well with slight advantages for NOWI in the case of the $H_2(W)$-error and for FWBT in the case of the $H_\infty(W)$-error. Similar to the previous example, Figure 5 shows how the relative error in the optimal interpolation conditions \ref{eq:optimal_conditions} vary as $n_r$ varies. Once again, the relative residual of the optimality conditions decreases as $n_r$ increases, yielding near-optimal interpolation.

**ISS 1R Module.** The final example is the component 1r of the International Space Station from the SLICOT benchmark collection. The plant is a MIMO system with $n = 270$, and $m = p = 3$. The controller to be reduced is an LQG-based controller as before. We compare NOWI and FWBT for $n_r = 2, 4, \ldots, 40$. For $n_r \leq 30$, we use logarithmically spaced interpolation points for initializing NOWI. For larger values of $n_r$, we aggregate the optimal points from smaller reduced models. The relative $H_2(W)$ errors are shown in Figure 6. The full model is hard to reduce with slowly
decaying Hankel singular values. This is apparent from Figure 6 where FWTB hardly reduces the error for smaller $n_r$ values. The proposed method clearly outperforms FWTB for every reduction order.

1-D Beam Model. The full-order model represents the dynamics of a 1-D beam with order $n = 3000$ with two inputs (point forces applied to the first two states) and one output (the displacement in the middle). The sigma plot, i.e. $\|G(i\omega)\|_2$ vs $\omega \in \mathbb{R}$ is given in Figure 7. For the weighting function $W(s)$, first we construct an order $n_w = 60$, two-inputs/two-outputs band-pass filter with $[10^{-3}, 0.7]$ rad/sec frequency band of interest to focus the emphasis on the first three peaks in the sigma plot. Using both NOWI and FWTB, we reduce the order to $n_r = 16$. NOWI was initiated by a random selection of interpolation points and tangent directions as before. As the Figure 8 depicts, NOWI significantly outperforms FWTB, successfully achieving high accuracy within the frequency interval of interest. We repeat the process using a band-pass filter with $[3 \times 10^{-2}, 0.7]$ rad/sec frequency band of interest. As Figures 9 and 10 depict, NOWI outperforms FWTB in this case as well. In order to achieve this accuracy, NOWI took only 3.34 seconds to run, while FWTB already took more than 277 seconds just to solve for the weighted Gramians.

6. Conclusions

We have extended an interpolatory framework for weighted-$H_2$ model reduction to include MIMO dynamical systems with feed-forward terms. The main tool was a new representation of the weighted-$H_2$ inner product in MIMO settings (the $F$-transformation defined in (7)) which led to associated first-order necessary conditions that must be satisfied by an optimal weighted-$H_2$ reduced-order model. These conditions in turn were found to be equivalent to necessary conditions established earlier by Halevi. An examination of realizations for systems defined by $F \cdot$ then led to an algorithm that remains tractable for large state-space dimension. There are a variety of refinements of the ideas presented here that can exploit the flexibility afforded by the interpolatory model reduction framework. One direction that has been fruitful in the unweighted case is trust-region based descent approaches, as described in [4] and extended to frequency-weighted settings in [7]. We have presented here several numerical examples that illustrate the effectiveness of our basic approach and its competitiveness with weighted balanced truncation.
Reduced system dimension $n_r$

$$\|G - G_r\|_{H^2(W)}$$

$$\|G\|_{H^2(W)}$$

FWBT

NOWI

Figure 6: ISS, $n = 270$, $n_w = 540$.

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Figure 10: Beam, $n = 3000, n_w = 60$. 

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