G-Lévy processes under sublinear expectations#  

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Abstract  We introduce G-Lévy processes which develop the theory of processes with independent and stationary increments under the framework of sublinear expectations. We then obtain the Lévy–Khintchine formula and the existence for G-Lévy processes. We also introduce G-Poisson processes.  

Keywords  Sublinear expectation, G-normal distribution, G-Brownian motion, G-expectation, Lévy process, G-Lévy process, G-Poisson process, Lévy-Khintchine formula, Lévy-Itô decomposition  

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1. Introduction  

Distribution and independence are two important notions in the theory of probability and statistics. These two notions were introduced in [15,14] under the framework of sublinear expectations. Recently, a new central limit theorem (CLT) under sublinear expectations was obtained in [19] based on a new i.i.d. assumption. The corresponding limit distribution of the CLT is a G-normal distribution. This new type of sublinear distribution was first introduced in [16] (see also [17–20]) for a new type of G-Brownian motion and the related calculus of Itô's type.  

G-Brownian motion has a very interesting structure which non-trivially generalizes the classical one. Briefly speaking, a G-Brownian motion is a continuous process with independent and stationary increments under a given sublinear expectation. An interesting phenomenon of G-Brownian motion is that its quadratic process is also a continuous process with independent and stationary increments and thus can still be regarded as a G-Brownian motion. A natural problem is how to develop the theory of Lévy processes, i.e., processes with independent and stationary increments but not necessarily continuous,
under sublinear expectations. In particular, how to define Poisson processes under sublinear expectations.

The purpose of this paper is to study the distribution property, i.e., Lévy–Khintchine formula, of a Lévy process under sublinear expectations. The corresponding Lévy–Itô decomposition will be discussed in our forthcoming work. We consider types of Lévy processes under sublinear expectations, called \( G \)-Lévy processes, and obtain that the corresponding distributions satisfy a new type of nonlinear parabolic integro-partial differential equation. Conversely, we can directly construct \( G \)-Lévy processes from these types of equations. A specific case is \( G \)-Poisson processes. By comparison with classical methods, our methods are more simple and direct. Books on Lévy processes, e.g., \([5,13,21]\), are recommended for understanding the present results.

This paper is organized as follows: in section 2, we recall some important notions and results of sublinear expectations and \( G \)-Brownian motions. In section 3, we introduce \( G \)-Lévy processes. We discuss the characterization of \( G \)-Lévy processes in section 4. In section 5, we obtain the Lévy–Khintchine formula for \( G \)-Lévy processes. The existence of \( G \)-Lévy processes is given in section 6. For convenience, we present some basic results of this new type of nonlinear parabolic integro-partial differential equation in the appendix.

2. Basic settings

We present some preliminaries in the theory of sublinear expectations and the related \( G \)-Brownian motions. More detail can be found in \([14–20]\).

2.1. Sublinear expectation

Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a linear space of real functions defined on \( \Omega \) such that if \( X_1, \ldots, X_n \in \mathcal{H} \), then \( \varphi(X_1, \ldots, X_n) \in \mathcal{H} \) for each \( \varphi \in \mathcal{C}_{\text{Lip}}(\mathbb{R}^n) \), where \( \mathcal{C}_{\text{Lip}}(\mathbb{R}^n) \) denotes the space of Lipschitz functions.

**Remark 1** In particular, all constants belong to \( \mathcal{H} \) and \( |X|, X^+, X^- \in \mathcal{H} \) if \( X \in \mathcal{H} \).

Here, we use \( \mathcal{C}_{\text{Lip}}(\mathbb{R}^n) \) in our framework only for convenience. In fact, our essential requirement is that \( \mathcal{H} \) contains all constants and, moreover, \( X \in \mathcal{H} \) implies \( |X| \in \mathcal{H} \). In general, \( \mathcal{C}_{\text{Lip}}(\mathbb{R}^n) \) can be replaced with other spaces for specific problems. We list two other spaces used in this paper.

- \( \mathcal{C}_{b,\text{Lip}}(\mathbb{R}^n) \): the space of bounded and Lipschitz functions;
- \( \mathcal{C}^k_b(\mathbb{R}^n) \): the space of bounded and \( k \)-time continuously differentiable functions with bounded derivatives of all orders less than or equal to \( k \).

**Definition 2** A sublinear expectation \( \hat{\mathbb{E}} \) on \( \mathcal{H} \) is a functional \( \hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), we have

(a) Monotonicity: \( \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y] \) if \( X \geq Y \).
(b) Constant preserving: \( \hat{\mathbb{E}}[c] = c \) for \( c \in \mathbb{R} \).
(c) Sub-additivity: \( \hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y] \).
(d) Positive homogeneity: \( \hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X] \) for \( \lambda \geq 0 \).
The triplet \((\Omega, \mathcal{H}, \mathbb{E})\) is called a sublinear expectation space (compare with a probability space \((\Omega, \mathcal{F}, P)\)).

**Remark 3** If \((c)\) results in equality, then \(\mathbb{E}\) is a linear expectation on \(\mathcal{H}\). We recall that the notion of the above sublinear expectations was systematically introduced by Artzner, Delbaen, Eber, and Heath [2,3] in the case where \(\Omega\) is a finite set, and by Delbaen [8] for the general situation with the notation of risk measure: \(\rho(X) := \mathbb{E}[-X]\). See also Huber [10] for even earlier study of this notion \(\mathbb{E}\) (called the upper expectation \(\mathbb{E}^*\) in Chapter 10 of [10]).

Let \(X = (X_1, \ldots, X_n), \ X_i \in \mathcal{H},\) denoted by \(X \in \mathcal{H}^n\), be a given \(n\)-dimensional random vector on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). We define a functional on \(CLip(\mathbb{R}^n)\) by

\[
\hat{\mathbb{E}}_X[\varphi] := \mathbb{E}[\varphi(X)] \text{ for all } \varphi \in CLip(\mathbb{R}^n).
\]

The triplet \((\mathbb{R}^n, CLip(\mathbb{R}^n), \hat{\mathbb{E}}_X[-])\) forms a sublinear expectation space. \(\hat{\mathbb{E}}_X\) is called the distribution of \(X\).

**Remark 4** If the distribution \(\hat{\mathbb{E}}_X\) of \(X \in \mathcal{H}\) is not a linear expectation, then \(X\) is said to have distributional uncertainty. The distribution of \(X\) has the following four typical parameters:

\[
\hat{\mu} := \mathbb{E}[X], \ \mu := -\hat{\mathbb{E}}[-X], \ \hat{\sigma}^2 := \mathbb{E}[X^2], \ \sigma^2 := -\hat{\mathbb{E}}[-X^2].
\]

The intervals \([\hat{\mu}, \mu]\) and \([\hat{\sigma}^2, \sigma^2]\) characterize the mean-uncertainty and the variance-uncertainty of \(X\).

The following simple properties are very useful in sublinear analysis.

**Proposition 5** Let \(X, \ Y \in \mathcal{H}\) be such that \(\hat{\mathbb{E}}[Y] = -\hat{\mathbb{E}}[-Y]\), i.e., \(Y\) has no mean uncertainty. Then we have

\[
\hat{\mathbb{E}}[X + Y] = \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y].
\]

In particular, if \(\hat{\mathbb{E}}[Y] = \hat{\mathbb{E}}[-Y] = 0\), then \(\hat{\mathbb{E}}[X + Y] = \hat{\mathbb{E}}[X]\).

**Proof** It is simply because we have \(\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]\) and

\[
\hat{\mathbb{E}}[X + Y] \geq \hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[-Y] = \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y].
\]

Noting that \(\hat{\mathbb{E}}[c] = -\hat{\mathbb{E}}[-c] = c\) for all \(c \in \mathbb{R}\), we immediately have the following corollary.

**Corollary 6** For each \(X \in \mathcal{H}\), we have \(\hat{\mathbb{E}}[X + c] = \hat{\mathbb{E}}[X] + c\) for all \(c \in \mathbb{R}\).

**Proposition 7** For each \(X, \ Y \in \mathcal{H}\), we have

\[
|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[X - Y] \vee \hat{\mathbb{E}}[Y - X].
\]

In particular, \(|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[|X - Y|]\).
Proof By sub-additivity and monotonicity of \(\hat{E}[-]\), it is easy to prove the inequalities. □

We recall some important notions under sublinear expectations.

Definition 8 Let \(X_1\) and \(X_2\) be two \(n\)-dimensional random vectors defined, respectively, on sublinear expectation spaces \((\Omega_1, \mathcal{H}_1, \hat{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \hat{E}_2)\). They are called identically distributed, denoted by \(X_1 \overset{d}{=} X_2\), if

\[
\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)] \text{ for all } \varphi \in C_{\text{Lip}}(\mathbb{R}^n).
\]

It is clear that \(X_1 \overset{d}{=} X_2\) if and only if their distributions coincide.

Definition 9 Let \((\Omega, \mathcal{H}, \hat{E})\) be a sublinear expectation space. A random vector \(Y = (Y_1, \ldots, Y_n) \in \mathcal{H}^n\) is said to be independent from another random vector \(X = (X_1, \ldots, X_m) \in \mathcal{H}^m\) under \(\hat{E}[-]\) if for each test function \(\varphi \in C_{\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)\) we have

\[
\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x = X}].
\]

\(\hat{X} = (\hat{X}_1, \ldots, \hat{X}_m) \in \mathcal{H}^m\) is said to be an independent copy of \(X\) if \(\hat{X} \overset{d}{=} X\) and \(\hat{X}\) is independent from \(X\).

Remark 10 Under a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\). \(Y\) is independent from \(X\) means that the distributional uncertainty of \(Y\) does not change after the realization of \(X = x\). Or, in other words, the “conditional sublinear expectation” of \(Y\) knowing \(X\) is \(\hat{E}[\varphi(x, Y)]_{x = X}\). In the case of linear expectation, this notion of independence is just the classical one.

It is important to note that under sublinear expectations the condition “\(Y\) is independent from \(X\)” does not automatically imply that “\(X\) is independent from \(Y\)” See the following example:

Example 11 We consider a case where \(\hat{E}\) is a sublinear expectation and \(X, Y \in \mathcal{H}\) are identically distributed with \(\hat{E}[X] = \hat{E}[-X] = 0\) and \(\hat{\sigma}^2 = \hat{E}[X^2] > \sigma^2 = -\hat{E}[-X^2]\). We also assume that \(\hat{E}[|X|] = \hat{E}[X^+ + X^-] > 0\), thus \(\hat{E}[X^+] = \frac{1}{2}\hat{E}[|X| + X] = \frac{1}{2}\hat{E}[|X|] > 0\). In the case where \(Y\) is independent from \(X\), we have

\[
\hat{E}[XY^2] = \hat{E}[X^+ \sigma^2 - X^- \sigma^2] = (\sigma^2 - \hat{\sigma}^2)\hat{E}[X^+] > 0.
\]

But if \(X\) is independent from \(Y\), we have

\[
\hat{E}[XY^2] = 0.
\]

2.2. \(G\)-Brownian motion

For a given positive integer \(n\), we denote by \(\langle x, y \rangle\) the scalar product of \(x, y \in \mathbb{R}^n\) and by \(|x| = \langle x, x \rangle^{1/2}\) the Euclidean norm of \(x\). We also denote by \(S(d)\) the space of all \(d \times d\) symmetric matrices and by \(\mathbb{R}^{n \times d}\) the space of all \(n \times d\) matrices. For \(A, B \in S(d)\), \(A \geq B\) means that \(A - B\) is non-negative.

Definition 12 \((G\)-normal distribution with zero mean\) A \(d\)-dimensional random vector
\( X = (X_1, \cdots, X_d) \) on a sublinear expectation space \((\Omega, \mathcal{H}, \widehat{\mathbb{E}})\) is said to be \(G\)-normally distributed if for each \(a, b \geq 0\) we have
\[
aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,
\]
where \(\bar{X}\) is an independent copy of \(X\). Here, the letter \(G\) denotes the function
\[
G(A) := \frac{1}{2} \widehat{\mathbb{E}}[\langle AX, X \rangle] \text{ for } A \in \mathbb{S}(d).
\]

It is easy to prove that \(\widehat{\mathbb{E}}[X_i] = \widehat{\mathbb{E}}[-X_i] = 0\) for \(i = 1, \ldots, d\) and the function \(G\) is a monotonic and sublinear function.

Let \((\Omega, \mathcal{H}, \widehat{\mathbb{E}})\) be a sublinear expectation space, \((X_t)_{t \geq 0}\) is called a \(d\)-dimensional process if \(X_t \in \mathcal{H}^d\) for each \(t \geq 0\).

**Definition 13** (G-Brownian motion) Let \(G: \mathbb{S}(d) \to \mathbb{R}\) be a given monotonic and sublinear function. A \(d\)-dimensional process \((B_t)_{t \geq 0}\) on a sublinear expectation space \((\Omega, \mathcal{H}, \widehat{\mathbb{E}})\) is called a G-Brownian motion if the following properties are satisfied:

(i) \(B_0 = 0\);

(ii) For each \(t, s \geq 0\), \(B_{t+s} - B_t\) is independent from \((B_{t_1}, B_{t_2}, \ldots, B_{t_n})\) for each \(n \in \mathbb{N}\) and \(0 \leq t_1 \leq \cdots \leq t_n \leq t\);

(iii) \(B_{t+s} - B_t \overset{d}{=} \sqrt{s}X\) for \(s \geq 0\), where \(X\) is \(G\)-normally distributed.

**Remark 14** If \(\widehat{\mathbb{E}}\) is a linear expectation in the above two definitions, then the function \(G\) is a linear function, \(X\) is classically normal, and \((B_t)_{t \geq 0}\) is a classical Brownian motion.

The above two definitions can be non-trivially generalized to the following situations.

**Definition 15** (G-normal distribution with mean uncertainty) A pair of \(d\)-dimensional random vectors \((X, \eta)\) on a sublinear expectation space \((\Omega, \mathcal{H}, \widehat{\mathbb{E}})\) is called \(G\)-distributed if for each \(a, b \geq 0\) we have
\[
(aX + b\bar{X}, a^2 \eta + b^2 \bar{\eta}) \overset{d}{=} (\sqrt{a^2 + b^2}X, (a^2 + b^2)\eta),
\]
where \((\bar{X}, \bar{\eta})\) is an independent copy of \((X, \eta)\). Here, the letter \(G\) denotes the function
\[
G(p, A) := \widehat{\mathbb{E}}[\frac{1}{2}(AX, X) + \langle p, \eta \rangle] \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).
\]

Obviously, \(X\) is \(\bar{G}\)-normally distributed with \(\bar{G}(A) = G(0, A)\). The distribution of \(\eta\) can be seen as the pure uncertainty of mean (see [17–20]). It is easy to prove that \(G\) is a sublinear function monotonic in \(A \in \mathbb{S}(d)\).

**Definition 16** (Generalized G-Brownian motion) Let \(G: \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}\) be a given sublinear function monotonic in \(A \in \mathbb{S}(d)\). A \(d\)-dimensional process \((B_t)_{t \geq 0}\) on a sublinear expectation space \((\Omega, \mathcal{H}, \widehat{\mathbb{E}})\) is called a generalized \(G\)-Brownian motion if the following properties are satisfied:

(i) \(B_0 = 0\);

(ii) For each \(t, s \geq 0\), \(B_{t+s} - B_t\) is independent from \((B_{t_1}, B_{t_2}, \ldots, B_{t_n})\) for each \(n \in \mathbb{N}\) and \(0 \leq t_1 \leq \cdots \leq t_n \leq t\);
(iii) \( B_{t+s} - B_t \overset{d}{=} \sqrt{s}X + s\eta \) for \( t, s \geq 0 \), where \((X, \eta)\) is G-distributed.

The construction of G-Brownian motion was first given in [16,17] and G-distributed random vector was given in [19].

Moreover, we have the characterization of the generalized G-Brownian motion (see [18,20]).

**Theorem 17** Let \((X_t)_{t \geq 0}\) be a d-dimensional process defined on a sublinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\) such that

(i) \( X_0 = 0 \);

(ii) For each \( t, s \geq 0 \), \( X_{t+s} - X_t \) and \( X_s \) are identically distributed and \( X_{t+s} - X_t \) is independent from \((X_{t_1}, X_{t_2}, \ldots, X_{t_n})\) for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \);

(iii) \( \lim_{t \downarrow 0} \tilde{E}[|X_t|^3]t^{-1} = 0 \).

Then \((X_t)_{t \geq 0}\) is a generalized G-Brownian motion, where

\[
G(p, A) = \lim_{t \downarrow 0} \tilde{E}\left[\frac{1}{2} \langle AX_t, X_t \rangle + \langle p, X_t \rangle \right]t^{-1} \text{ for } (p, A) \in \mathbb{R}^d \times S(d).
\]

**Remark 18** In fact, paths of \((X_t)_{t \geq 0}\) in the above theorem are continuous due to condition (iii) (see [9,11]). In the following sections, we consider Lévy processes without condition (iii) which contain jumps.

3. G-Lévy processes

A process \( \{X_t(\omega) : \omega \in \Omega, t \geq 0\} \) defined on a sublinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\) is called càdlàg if for each \( \omega \in \Omega \), \( \lim_{\delta \downarrow 0} X_{t+\delta}(\omega) = X_t(\omega) \) and \( X_{t-}(\omega) := \lim_{\delta \downarrow 0} X_{t-\delta}(\omega) \) exists for all \( t \geq 0 \). We now give the definition of Lévy processes under sublinear expectations.

**Definition 19** A d-dimensional càdlàg process \((X_t)_{t \geq 0}\) defined on a sublinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\) is called a Lévy process if the following properties are satisfied:

(i) \( X_0 = 0 \);

(ii) Independent increments: for each \( t, s > 0 \), the increment \( X_{t+s} - X_t \) is independent from \((X_{t_1}, X_{t_2}, \ldots, X_{t_n})\), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \);

(iii) Stationary increments: the distribution of \( X_{t+s} - X_t \) does not depend on \( t \).

**Remark 20** If \((X_t)_{t \geq 0}\) is a Lévy process, then the finite-dimensional distribution of \((X_t)_{t \geq 0}\) is uniquely determined by the distribution of \( X_t \) for each \( t \geq 0 \).

**Proposition 21** Let \((X_t)_{t \geq 0}\) be a d-dimensional Lévy process defined on a sublinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\). Then, for each \( A \in \mathbb{R}^{n \times d} \), \((AX_t)_{t \geq 0}\) is an n-dimensional Lévy process.

**Proof** By the definition of distribution and independence, it is easy to prove the result. \(\Box\)

Let \((X_t)_{t \geq 0}\) be a d-dimensional Lévy process defined on a sublinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\). In this paper, we suppose that there exists a 2d-dimensional Lévy process...
(\(X^c_t, X^d_t\))_{t \geq 0} defined on a sublinear expectation space \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\) such that the distributions of \((X^c_t + X^d_t)_{t \geq 0}\) and \((X_t)_{t \geq 0}\) are the same. In this paper, we only consider the distribution property of \((X^c_t)_{t \geq 0}\). Hence, we can suppose \(X_t = X^c_t + X^d_t\) on the same sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\).

**Remark 22** In the classical linear expectation case, by the Lévy–Itô decomposition, the above assumption of \((X_t)_{t \geq 0}\) obviously holds, where \((X_t^c)_{t \geq 0}\) is the continuous part and \((X_t^d)_{t \geq 0}\) is the jump part.

Furthermore, we suppose \((X^c_t + X^d_t)_{t \geq 0}\) satisfying the following assumption:

\[
\lim_{t \downarrow 0} \tilde{\mathbb{E}}[|X^c_t|^3]t^{-1} = 0; \tilde{\mathbb{E}}[|X^d_t|] \leq Ct \text{ for } t \geq 0,
\]

where \(C\) is a constant.

**Remark 23** By the assumption on \((X^c_t)_{t \geq 0}\), we know that \((X^c_t)_{t \geq 0}\) is a generalized G-Brownian motion. The assumption on the jump part \((X^d_t)_{t \geq 0}\) implies that it is of finite variation. The more complicated situation will be discussed in our forthcoming work.

**Example 24** Suppose \((X^d_t)_{t \geq 0}\) is a 1-dimensional positive Lévy process, i.e., jumps are positive. Note that

\[
\tilde{\mathbb{E}}[X^d_{t+s}] = \tilde{\mathbb{E}}[X^d_t] + \tilde{\mathbb{E}}[X^d_{t+s} - X^d_t] = \tilde{\mathbb{E}}[X^d_t] + \tilde{\mathbb{E}}[X^d_s]
\]

and \(\tilde{\mathbb{E}}[X^d_t]\) is increasing in \(t\), then we obtain \(\tilde{\mathbb{E}}[X^d_t] = \tilde{\mathbb{E}}[X^d_1]t\). Obviously, it satisfies (1).

**Definition 25** A \(d\)-dimensional Lévy process \((X_t)_{t \geq 0}\) is called a G-Lévy process if there exists a decomposition \(X_t = X^c_t + X^d_t\) for each \(t \geq 0\), where \((X^c_t, X^d_t)_{t \geq 0}\) is a 2\(d\)-dimensional Lévy process satisfying (1).

By Proposition 21, we immediately have:

**Proposition 26** Let \((X_t)_{t \geq 0}\) be a \(d\)-dimensional G-Lévy process defined on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). Then, for each \(A \in \mathbb{R}^{n \times d}\), \((AX_t)_{t \geq 0}\) is an \(n\)-dimensional G-Lévy process.

### 4. Characterization of G-Lévy processes

Let \((X_t)_{t \geq 0}\) be a \(d\)-dimensional G-Lévy process with the decomposition \(X_t = X^c_t + X^d_t\). In this section, we show that for each given \(\varphi \in C_{b,Lip}(\mathbb{R}^d)\), \(u(t, x) := \tilde{\mathbb{E}}[\varphi(x + X_t)]\) is a viscosity solution of the following equation:

\[
\partial_t u(t, x) - G_X[u(t, x + \cdot) - u(t, x)] = 0, \quad u(0, x) = \varphi(x), \quad (2)
\]

where \(G_X[f(\cdot)]\) is a nonlocal operator defined by

\[
G_X[f(\cdot)] := \lim_{\delta \downarrow 0} \frac{\tilde{\mathbb{E}}[f(X_\delta)]}{\delta^{-1}} \text{ for } f \in C^1_b(\mathbb{R}^d) \text{ with } f(0) = 0. \quad (3)
\]

We first show that the definition of \(G_X[f(\cdot)]\) is meaningful. For this, we need the
following lemmas.

**Lemma 27** For each $\delta \leq 1$, we have
\[ \mathbb{E}[|X_0^\delta|^p] \leq C_p \delta^{p/2} \text{ for each } p > 0, \]
where $C_p$ is a constant depending only on $p$.

**Proof** This is a direct consequence of Theorem 17. \[ \square \]

**Lemma 28** For each given $f \in C_b^2(\mathbb{R}^d)$ with $f(0) = 0$, we have
\[ \mathbb{E}[f(X_0^\delta)] = \mathbb{E}[f(X_0^\delta) + \langle Df(0), X_0^\delta \rangle + \frac{1}{2} \langle D^2f(0)X_0^\delta, X_0^\delta \rangle] + o(\delta). \]

**Proof** It is easy to check that
\[ I_1^\delta := f(X_0^\delta) - f(X_0^\delta) - f(X_0^d) \]
\[ = \int_0^1 \langle X_0^\delta, Df(X_0^d + \alpha X_0^\delta) - Df(\alpha X_0^\delta) \rangle d\alpha \]
and
\[ I_2^\delta := f(X_0^\delta) - \langle Df(0), X_0^\delta \rangle - \frac{1}{2} \langle D^2f(0)X_0^\delta, X_0^\delta \rangle \]
\[ = \int_0^1 \int_0^1 \langle (D^2f(\alpha X_0^\delta) - D^2f(0))X_0^\delta, X_0^\delta \rangle d\alpha d\beta d\alpha. \]

Note that $Df$ is bounded. Then we get
\[ \mathbb{E}[|I_1^\delta|] \leq (\mathbb{E}[|X_0^\delta|^3])^{1/2} (\mathbb{E}[\int_0^1 |Df(X_0^d + \alpha X_0^\delta) - Df(\alpha X_0^\delta)|^3 d\alpha])^{1/2} \]
\[ \leq C(\mathbb{E}[|X_0^\delta|^3])^{1/2} (\mathbb{E}[\int_0^1 |D^2f(\alpha X_0^\delta)|^3 d\alpha])^{1/2} \]
\[ \leq C_1(\mathbb{E}[|X_0^\delta|^3])^{1/2} (\mathbb{E}[|X_0^d|^3])^{1/2} \]
\[ \leq C_2 \delta^{7/6} = o(\delta). \]

It is easy to obtain $\mathbb{E}[|I_2^\delta|] \leq C\mathbb{E}[|X_0^\delta|^3] = o(\delta)$. Noting that
\[ |\mathbb{E}[f(X_0^\delta)] - \mathbb{E}[f(X_0^d) + \langle Df(0), X_0^\delta \rangle + \frac{1}{2} \langle D^2f(0)X_0^\delta, X_0^\delta \rangle]| \leq \mathbb{E}[|I_1^\delta + I_2^\delta|], \]
we conclude the result. \[ \square \]

**Lemma 29** Let $(p, A) \in \mathbb{R}^d \times S(d)$ and $f \in C_b^2(\mathbb{R}^d)$ with $f(0) = 0$ be given. Then
\[ \lim_{\delta \downarrow 0} \mathbb{E}[f(X_0^\delta) + \langle p, X_0^\delta \rangle + \frac{1}{2} \langle AX_0^\delta, X_0^\delta \rangle] \delta^{-1} \text{ exists.} \]

**Proof** We define
\[ g(t) = \mathbb{E}[f(X_t^d) + \langle p, X_t^\delta \rangle + \frac{1}{2} \langle AX_t^\delta, X_t^\delta \rangle]. \]

Obviously, $g(0) = 0$. For each $t, s \in [0, 1], \]
\[ |g(t + s) - g(t)| \leq Cs + \mathbb{E}[Y] \vee \mathbb{E}[\neg Y] \leq C_1 s, \]
where \( Y = \langle p + AX, X, X + sX \rangle + \frac{1}{2} \langle A(X + sX), X + sX \rangle \). Thus, \( g(\cdot) \) is differentiable almost everywhere on \([0, 1]\). For each fixed \( t_0 < 1 \) such that \( g'(t_0) \) exists, we have
\[
\frac{g(\delta)}{\delta} = \frac{g(t_0 + \delta) - g(t_0)}{\delta} - \Lambda_\delta,
\]
where
\[
\Lambda_\delta = \delta^{-1}(g(t_0 + \delta) - \mathbb{E}[f(X^d_{t_0}) + f(X^d_{t_0+\delta}) - \langle p, X^c_{t_0+\delta} \rangle + \frac{1}{2} \langle A(X^c_{t_0+\delta} - X^c_{t_0}); X^c_{t_0+\delta} - X^c_{t_0} \rangle].
\]
Similar to the above estimate, it is not difficult to prove that \(|\Lambda_\delta| \leq C\sqrt{t_0}\), where \( C \) is a constant independent of \( \delta \) and \( t_0 \). Thus,
\[
|\limsup_{\delta \downarrow 0} g(\delta)\delta^{-1} - \liminf_{\delta \downarrow 0} g(\delta)\delta^{-1}| \leq 2C\sqrt{t_0}.
\]
Letting \( t_0 \downarrow 0 \), we get the result.

By the above two lemmas, we know that the definition of \( G_X[f(\cdot)] \) is meaningful. It is easy to check that \( G_X[f(\cdot)] \) satisfies the following properties: for each \( f \), \( g \in C^3_b(\mathbb{R}^d) \) with \( f(0) = 0 \), \( g(0) = 0 \),
\begin{enumerate}
\item Monotonicity: \( G_X[f(\cdot)] \geq G_X[g(\cdot)] \) if \( f \geq g \).
\item Sub-additivity: \( G_X[f(\cdot) + g(\cdot)] \leq G_X[f(\cdot)] + G_X[g(\cdot)] \).
\item Positive homogeneity: \( G_X[\lambda f(\cdot)] = \lambda G_X[f(\cdot)] \) for all \( \lambda \geq 0 \).
\end{enumerate}
Now, we give the definition of viscosity solution for equation (2).

**Definition 30** A bounded upper semicontinuous (lower semicontinuous) function \( u \) is called a viscosity subsolution (viscosity supersolution) of the equation (2) if \( u(0, x) \leq \varphi(x) \geq \varphi(x) \) for each \( (t, x) \in (0, \infty) \times \mathbb{R}^d \) and for each \( \psi \in C^{2,3}_b \) such that \( \psi \geq u \) and \( \psi(t, x) = u(t, x) \), we have
\[
\partial_t \psi(t, x) - G_X[\psi(t, x + \cdot) - \psi(t, x)] \leq 0 \quad (\geq 0).
\]
A bounded continuous function \( u \) is called a viscosity solution of equation (2) if it is both a viscosity subsolution and a viscosity supersolution.

We now give the characterization of \( G\)-Lévy processes.

**Theorem 31** Let \( (X_t)_{t \geq 0} \) be a d-dimensional G-Lévy process. For each \( \varphi \in C_{b, Lip}(\mathbb{R}^d) \), define \( u(t, x) = \mathbb{E}[\varphi(x + X_t)] \). Then \( u \) is a viscosity solution of equation (2).

**Proof** We first show that \( u \) is a continuous function. Obviously, \(|u(t, x) - u(t, y)| \leq C|x - y|\). Note that
\[
u(t + s, x) = \mathbb{E}[\varphi(x + X_s + X_{t+s} - X_s)] = \mathbb{E}[u(t, x + X_s)],
\]
then for \( s \leq 1 \), \( \nu(t + s, x) - \nu(t, x) \leq C\mathbb{E}[|X_s^c + X_{s}^d|] \leq C_1 \sqrt{s} \). Thus, \( u \) is continuous. For each fixed \( (t, x) \in (0, \infty) \times \mathbb{R}^d \) and \( \psi \in C^{2,3}_b \) such that \( \psi \geq u \) and \( \psi(t, x) = u(t, x) \), we have
\[
\psi(t, x) = u(t, x) = \mathbb{E}[u(t - \delta, x + X_\delta)] \leq \mathbb{E}[\psi(t - \delta, x + X_\delta)].
\]
Therefore,
\[
0 \leq \mathbb{E}[\psi(t-\delta, x+X_\delta) - \psi(t, x)] = -\partial_t \psi(t, x)\delta + \mathbb{E}[\psi(t, x + X_\delta) - \psi(t, x) + I_\delta] \leq -\partial_t \psi(t, x)\delta + \mathbb{E}[\psi(t, x + X_\delta) - \psi(t, x)] + \mathbb{E}|I_\delta|,
\]
where \( I_\delta = \delta \int_0^1 [\partial_t \psi(t, x) - \partial_t \psi(t - \beta\delta, x + X_\delta)]d\beta \). It is easy to show that
\[
\mathbb{E}|I_\delta| \leq C\mathbb{E}[\delta + |X_\delta|] = o(\delta).
\]
By the definition of \( G_X \), we get
\[
\partial_t \psi(t, x) - G_X[\psi(t, x + \cdot) - \psi(t, x)] \leq 0.
\]
Hence, \( u \) is a viscosity subsolution of (2). Similarly, we prove that \( u \) is a viscosity supersolution of (2). Thus, \( u \) is a viscosity solution of (2).

\[\square\]

**Remark 32** We do not know the uniqueness of viscosity solution for (2). For this, we need the representation of \( G_X \) presented in the following section.

### 5. Lévy–Khintchine representation of \( G_X \)

In this section, we give a representation of the infinitesimal generator \( G_X \), which can be seen as the Lévy–Khintchine formula for \( G \)-Lévy processes. We first give some lemmas.

**Lemma 33** Let \((p, A) \in \mathbb{R}^d \times \mathbb{S}(d)\) and \( f \in C_{b, Lip}(\mathbb{R}^d) \) with \( f(0) = 0 \) and \( f(x) = o(|x|) \) be given. Then \( \lim_{\delta \to 0} \mathbb{E}[f(X^d_\delta) + \langle p, X^c_\delta \rangle + \frac{1}{2}\langle AX^c_\delta, X^c_\delta \rangle] \delta^{-1} \) exists.

**Proof** Since \( f(x) = o(|x|) \), there exists a sequence \( \{\delta_n : n \geq 1\} \) such that \( \delta_n \downarrow 0 \) and \( |f(x)| \leq \frac{1}{n}|x| \) on \( |x| \leq \delta_n \). For each fixed \( \delta_n \), we choose \( f^n_\epsilon \in C^2_b(\mathbb{R}^d) \) with \( f^n_\epsilon(0) = 0 \) such that
\[
|f(x) - f^n_\epsilon(x)| \leq \frac{4L\varepsilon}{\delta_n}|x| + \frac{1}{n}|x|,
\]
where \( L \) is the Lipschitz constant of \( f \). Thus,
\[
\frac{\mathbb{E}|f(X^d_\delta) - f^n_\epsilon(X^d_\delta)|}{\delta} \leq \left( \frac{4L\varepsilon}{\delta_n} + \frac{1}{n} \right) \frac{\mathbb{E}|X^d_\delta|}{\delta} \leq C\left( \frac{4L\varepsilon}{\delta_n} + \frac{1}{n} \right).
\]
By Lemma 29 and the above estimate, we obtain
\[
\limsup_{\delta \downarrow 0} \mathbb{E}[f(X^d_\delta) + \langle p, X^c_\delta \rangle + \frac{1}{2}\langle AX^c_\delta, X^c_\delta \rangle] \delta^{-1} \leq \liminf_{\delta \downarrow 0} \mathbb{E}[f(X^d_\delta) + \langle p, X^c_\delta \rangle + \frac{1}{2}\langle AX^c_\delta, X^c_\delta \rangle] \delta^{-1} + 2C\left( \frac{4L\varepsilon}{\delta_n} + \frac{1}{n} \right),
\]
which implies the result by first letting \( \varepsilon \downarrow 0 \) and then \( n \to \infty \).

\[\square\]

We denote by
\[
\mathcal{L}_0 = \{ f \in C_{b, Lip}(\mathbb{R}^d) : f(0) = 0 \text{ and } f(x) = o(|x|) \}
\]
and
\[
\mathcal{L} = \{ (f, p, q, A) : f \in \mathcal{L}_0 \text{ and } (p, q, A) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}(d) \}.
\]
It is clear that \( \mathcal{L}_0 \) and \( \mathcal{L} \) are both linear spaces. Now, we define a functional \( \tilde{F}[\cdot] \) on \( \mathcal{L} \) by
\[
\tilde{F}[(f, p, q, A)] := \lim_{\delta \to 0} \tilde{E}[f(X^d_\delta) + \frac{\langle p, X^d_\delta \rangle}{1 + |X^d_\delta|^2} + \langle q, X^c_\delta \rangle + \frac{1}{2} \langle AX^c_\delta, X^c_\delta \rangle]^{\delta - 1}.
\]
Similar to the proof of the above lemma, we know that the definition of \( \tilde{F}[\cdot] \) is meaningful.

**Remark 34** For \( g \in C^2_0(\mathbb{R}^d) \) with \( g(0) = 0 \), define
\[
f(x) = g(x) - \frac{\langle Dg(0), x \rangle}{1 + |x|^2} \text{ for } x \in \mathbb{R}^d.
\]
One can verify that \( f \in C_{b, L^p}(\mathbb{R}^d) \) with \( f(0) = 0 \) and \( f(x) = o(|x|) \). So we add the term \( \langle p, X^d_\delta \rangle(1 + |X^d_\delta|^2)^{-1} \) in the definition of \( \tilde{F}[\cdot] \).

**Lemma 35** The functional \( \tilde{F} : \mathcal{L} \to \mathbb{R} \) satisfies the following properties:
1. \( \tilde{F}[(f_1, p, q, A_1)] \geq \tilde{F}[(f_2, p, q, A_2)] \) if \( f_1 \geq f_2 \) and \( A_1 \geq A_2 \).
2. \( \tilde{F}[(f_1 + f_2, p_1 + p_2, q_1 + q_2, A_1 + A_2)] \leq \tilde{F}[(f_1, p_1, q_1, A_1)] + \tilde{F}[(f_2, p_2, q_2, A_2)] \).
3. \( \tilde{F}[\lambda(f, p, q, A)] = \lambda \tilde{F}[(f, p, q, A)] \) for all \( \lambda \geq 0 \).
4. If \( f_n \in \mathcal{L}_0 \) satisfies \( f_n \downarrow 0 \), then \( \tilde{F}[(f_n, 0, 0, 0)] \downarrow 0 \).

**Proof** It is easy to prove (1), (2), and (3). We now prove (4). For each fixed \( 0 < \eta_1 < \eta_2 < \infty \), it is easy to check
\[
f_n(x) \leq \left( \sup_{0 < |y| \leq \eta_1} \frac{f_1(y)}{|y|} \right)|x| + \left( \sup_{\eta_1 \leq |y| \leq \eta_2} f_n(y) \right) \frac{|x|}{\eta_1} + \left( \sup_{|y| \geq \eta_2} f_1(y) \right) \frac{|x|}{\eta_2}.
\]
Thus,
\[
\tilde{F}[(f_n, 0, 0, 0)] \leq C \left( \sup_{0 < |y| \leq \eta_1} \frac{f_1(y)}{|y|} + \frac{\sup_{\eta_1 \leq |y| \leq \eta_2} f_n(y)}{\eta_1} + \frac{\sup_{|y| \geq \eta_2} f_1(y)}{\eta_2} \right).
\]
Noting that \( \sup_{\eta_1 \leq |y| \leq \eta_2} f_n(y) \downarrow 0 \), we then have
\[
\lim_{n \to \infty} \tilde{F}[(f_n, 0, 0, 0)] \leq C \left( \sup_{0 < |y| \leq \eta_1} \frac{f_1(y)}{|y|} + \sup_{|y| \geq \eta_2} \frac{f_1(y)}{\eta_2} \right).
\]
First letting \( \eta_1 \to 0 \) and then \( \eta_2 \to \infty \), we obtain (4).

By (2) and (3) of the above lemma, we immediately obtain that there exists a family of linear functionals \( \{F_u : u \in \mathcal{U}_0\} \) defined on \( \mathcal{L} \) such that
\[
\tilde{F}[(f, p, q, A)] = \sup_{u \in \mathcal{U}_0} F_u[(f, p, q, A)].
\]
The proof is found in [19]. Note that (1) and (4) of the above lemma, then for each \( F_u \), by the Daniell–Stone theorem, there exist \( (p', q', Q) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \) and a unique measure \( v \) on \( (\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})) \) such that
\[
F_u[(f, p, q, A)] = \int_{\mathbb{R}^d \setminus \{0\}} f(z) v(dz) + \langle p, p' \rangle + \langle q, q' \rangle + \frac{1}{2} \text{tr}[AQQT].
\]
Thus,
\[
\hat{F}(f, p, q, A) = \sup_{(v, p', q', Q) \in U_0} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} f(z)v(dz) + \langle p, p' \rangle + \langle q, q' \rangle + \frac{1}{2} \text{tr}[AQQT] \right\}.
\]  
(4)

In particular,
\[
\lim_{\delta \downarrow 0} \hat{E}[f(X^\delta_0)]/\delta = \sup_{v \in V} \int_{\mathbb{R}^d} f(z)v(dz),
\]  
where \( V = \{ v : \exists (p', q', Q) \text{ such that } (v, p', q', Q) \in U_0 \} \).  
(6)

Taking a specific \( f \), we easily prove that
1. For each \( \varepsilon > 0 \), \( \sup_{v \in V} v(\{ z : |z| \geq \varepsilon \}) < \infty \).
2. For each \( \varepsilon > 0 \), the restriction of \( V \) on the set \( \{ z : |z| \geq \varepsilon \} \) is tight.
3. \( \sup_{v \in V} \int_{\mathbb{R}^d} |z|v(dz) < \infty \).

In fact, it is easy to deduce that (3) implies (1) and (2). Similarly, it is also easy to show that all \( (p', q', Q) \in U_0 \) are bounded. Now, we give the representation of \( G_X \). For each \( f \in C_b^3(\mathbb{R}^d) \) with \( f(0) = 0 \), by Lemma 28 and the above analysis, we have
\[
G_X[f(\cdot)] = \sup_{(v, p', q', Q) \in U_0} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (f(z) - \frac{\langle Df(0), z \rangle}{1 + |z|^2})v(dz) + \langle Df(0), p' + q' \rangle + \frac{1}{2} \text{tr}[D^2f(0)QQ^T] \right\}.
\]  
(7)

Noting that \( \sup_{v \in V} \int_{\mathbb{R}^d} |z|v(dz) < \infty \), we then have the following Lévy–Khintchine representation of \( G_X \):
\[
G_X[f(\cdot)] = \sup_{(v, q, Q) \in U} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} f(z)v(dz) + \langle Df(0), q \rangle + \frac{1}{2} \text{tr}[D^2f(0)QQ^T] \right\},
\]  
(8)

where
\[
U = \{ (v, p' + q' - \int_{\mathbb{R}^d \setminus \{0\}} \frac{z}{1 + |z|^2}v(dz), Q) : \forall (v, p', q', Q) \in U_0 \}.
\]

We summarize the above discussions as a theorem.

**Theorem 36** Let \( (X_t)_{t \geq 0} \) be a d-dimensional G-Lévy process. Then \( G_X[f(\cdot)] \), \( f \in C_b^3(\mathbb{R}^d) \), has the Lévy–Khintchine representation (7), where \( (q, Q) \in \mathbb{R}^d \times \mathbb{R}^{d \times d} \) and \( v \) is a measure on \( (\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})) \) satisfying
\[
\sup_{(v, q, Q) \in U} \left\{ \int_{\mathbb{R}^d} |z|v(dz) + |q| + \text{tr}[QQ^T] \right\} < \infty.
\]  
(8)

**Remark 37** In the above theorem, the set \( U \) represents the uncertainty of the Lévy measure, the drift, and the covariance matrix of \( (X_t)_{t \geq 0} \).

We then have the following theorem.

**Theorem 38** Let \( (X_t)_{t \geq 0} \) be a d-dimensional G-Lévy process. For each \( \varphi \in C_b, \text{Lip}(\mathbb{R}^d) \), define \( u(t, x) = \hat{E}[\varphi(x + X_t)] \). Then \( u \) is the unique viscosity solution of the following integro-partial differential equation:
\[
\partial_t u(t, x) - \sup_{(v, q, Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (u(t, x + z) - u(t, x)) v(dz) + \langle Du(t, x), q \rangle + \frac{1}{2} \text{tr}[D^2 u(t, x) Q Q^T] \right\} = 0, 
\]

where \( \mathcal{U} \) represents \( G_X \).

**Proof** By Theorems 5.4 and 4.5, \( u \) is a viscosity solution of (9). For the uniqueness, see appendix.

**Remark 39** The definition of the viscosity solution for (9) is the same as Definition 30.

### 6. Existence of \( G \)-Lévy processes

We denote by \( \Omega = \mathcal{D}_0(\mathbb{R}^+, \mathbb{R}^d) \) the space of all \( \mathbb{R}^d \)-valued càdlàg functions \( (\omega_t)_{t \in \mathbb{R}^+} \), with \( \omega_0 = 0 \), equipped with the Skorokhod topology. The corresponding canonical process is \( B_t(\omega) = \omega_t \) for \( \omega \in \Omega \), \( t \geq 0 \). We define

\[
\mathcal{F}_t := \sigma\{ B_s : s \leq t \} \text{ and } \mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t.
\]

Following [15,16,17], for each fixed \( T \in [0, \infty) \), we set

\[
L_{ip}(\mathcal{F}_T) := \{ \varphi(B_{t_1 \wedge T}, \ldots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, \infty), \varphi \in C_{b, Lip}(\mathbb{R}^{d \times n}) \}.
\]

It is clear that \( L_{ip}(\mathcal{F}_t) \subset L_{ip}(\mathcal{F}_T) \) for \( t \leq T \). We also set

\[
L_{ip}(\mathcal{F}) := \bigcup_{n=1}^{\infty} L_{ip}(\mathcal{F}_n).
\]

Let \( \mathcal{U} \) be given and satisfy (8). We consider the corresponding integro-partial differential equation (9). For each given initial condition \( \varphi \in C_{b, Lip}(\mathbb{R}^d) \), the viscosity solution \( u^\varphi \) for (9) exists (see appendix). Furthermore, we have the following theorem.

**Theorem 40** Let \( u^\varphi \) denote the viscosity solution of (9) with the initial condition \( \varphi \in C_{b, Lip}(\mathbb{R}^d) \). Then we have

1) \( u^\varphi \geq u^\psi \) if \( \varphi \geq \psi \).
2) \( u^{\varphi + \psi} \leq u^\varphi + u^\psi \).
3) \( u^{\varphi + c} = u^\varphi + c \) for \( c \in \mathbb{R} \).
4) \( u^{\lambda \varphi} = \lambda u^\varphi \) for all \( \lambda \geq 0 \).
5) \( u^\varphi(t + s, x) = u^{u^\varphi(t, x +)}(s, 0) \).

**Proof** It is easy to check 3)–5). 1) and 2) are proved in the appendix.

We now introduce a sublinear expectation \( \hat{\mathbb{E}} \) on \( L_{ip}(\mathcal{F}) \) by the following two steps:

Step 1. For each \( \xi \in L_{ip}(\mathcal{F}) \) of the form \( \xi = \varphi(B_{t+s} - B_t) \), \( t, s \geq 0 \) and \( \varphi \in C_{b, Lip}(\mathbb{R}^d) \), we define \( \hat{\mathbb{E}}[\xi] = u(s, 0) \), where \( u \) is a viscosity solution of (9) with the initial condition \( u(0, x) = \varphi(x) \).
Step 2. For each $\xi \in L_{tp}(F)$, we can find a $\phi \in C_{b,Lip}(\mathbb{R}^{d \times m})$ such that $\xi = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, $t_1 < t_2 < \cdots < t_m$. Then we define $\hat{E}[\xi] = \phi_m$, where $\phi_m \in \mathbb{R}$ is obtained via the following procedure:

$$
\begin{align*}
\phi_1(x_1, \ldots, x_{m-1}) &= \hat{E}[\phi(x_1, \ldots, x_{m-1}, B_{t_m} - B_{t_{m-1}})]; \\
\phi_2(x_1, \ldots, x_{m-2}) &= \hat{E}[\phi_1(x_1, \ldots, x_{m-2}, B_{t_m} - B_{t_{m-2}})]; \\
& \vdots \\
\phi_{m-1}(x_1) &= \hat{E}[\phi_{m-2}(x_1, B_{t_2} - B_{t_1})]; \\
\phi_m &= \hat{E}[\phi_{m-1}(B_{t_1})].
\end{align*}
$$

The related conditional expectation of $\xi$ under $F_{t_j}$ is defined by

$$
\hat{E}[\xi | F_{t_j}] = \hat{E}[\phi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) | F_{t_j}]
= \phi_m - j(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_{j-1}}).
$$

By the above theorem, it is easy to prove that $\hat{E}[\cdot]$ consistently defines a sublinear expectation on $L_{tp}(F)$. Then $L_{tp}(F)$ can be extended to a Banach space under the norm $||X|| := \hat{E}[||X||]$. We denote this space by $L^1_{CLip}(F)$. Note that $|\hat{E}[X] - \hat{E}[Y]| \leq \hat{E}[|X - Y|]$. Then $\hat{E}[\cdot]$ can be extended as a continuous mapping on $L^1_{CLip}(F)$ which is still a sublinear expectation. Similarly, it is easy to check that the conditional expectation $\hat{E}[\cdot | F_t]$ can be also extended as a continuous mapping $L^1_{CLip}(F) \to L^1_{CLip}(F_t)$. We now prove that the canonical process $(B_t)_{t \geq 0}$ is a $G$-Lévy process. For this, we need the following lemma.

**Lemma 41** Let $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ be two sequences of $d$-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. We assume that $Y_n$ is independent from $X_n$ for $n = 1, 2, \ldots$. If there exist $X, Y \in \mathcal{H}^d$ such that $\hat{E}[|X_n - X|] \to 0$ and $\hat{E}[|Y_n - Y|] \to 0$, then $Y$ is independent from $X$.

**Proof** For each fixed $\varphi \in C_{Lip}(\mathbb{R}^{2d})$, we define

$$
\hat{E}[\varphi(x, Y_n)] \text{ and } \hat{E}[\varphi(x, Y)].
$$

It is clear that $|\hat{E}[\varphi(x) - \varphi_n(x)]| \leq C(|x - \bar{x}| + \hat{E}[|Y_n - Y|])$. Thus, $|\hat{E}[\varphi_n(X_n)] - \hat{E}[\varphi(X)]| \leq C(\hat{E}[|X_n - X|] + \hat{E}[|Y_n - Y|])$. Note that $\hat{E}[\varphi_n(X_n)] = \hat{E}[\varphi(X, Y)]$. Then we obtain $\hat{E}[\varphi(X)] = \hat{E}[\varphi(X, Y)]$, which implies that $Y$ is independent from $X$.

**Theorem 42** The canonical process $(B_t)_{t \geq 0}$ is a $G$-Lévy process.

**Proof** Consider $D_0(\mathbb{R}^+, \mathbb{R}^{2d})$ and the canonical process $(\tilde{B}_t, \tilde{B}_t)_{t \geq 0}$. Similarly as above, we construct a sublinear expectation, still denoted by $\hat{E}[\cdot]$, on $L_{tp}(F)$ via the following integro-partial differential equation:

$$
\begin{align*}
\partial_t w(t, x, y) &- \sup_{(v, q, Q) \in U} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (w(t, x, y + z) - w(t, x, y))v(dz) + \langle D_x w(t, x, y), q \rangle \\
&+ \frac{1}{2} \text{tr}[D^2_x w(t, x, y)QQ^T] \right\} = 0.
\end{align*}
$$

It is easy to check that the distribution of $(\tilde{B}_t)_{t \geq 0}$ satisfies the following equation:
\[ \partial_t u(t, x) - \sup_{(q; Q) \in \mathcal{U}} \{ \langle Du(t, x), q \rangle + \frac{1}{2} \text{tr}[D^2 u(t, x)QQT] \} = 0. \]

Thus, \((\tilde{B}_t)_{t \geq 0}\) is the generalized G-Brownian motion. The distribution of \((\tilde{B}_t)_{t \geq 0}\) satisfies the following equation:

\[ \partial_t u(t, y) - \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d \setminus \{0\}} (u(t, y + z) - u(t, y))v(dz) = 0. \]

We now show that \(\tilde{B}_t\) belongs to \(L^1_G(\mathcal{F})\) and satisfies (1). Consider the function \(\phi_0(y) = d + \sum_{i=1}^{d} y_i (\arctan y_i + \pi/2)\). Define \(\phi_0^N(y) = \phi_0(y - N)\) for \(N > 0\) and

\[ \phi_{i+1}^N(y) = \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d \setminus \{0\}} (\phi_i^N(y + z) - \phi_i^N(y))v(dz) \text{ for } i = 1, 2, \ldots. \]

Then, it is not difficult to check that \(u^N(t, y) := \sum_{i=0}^{\infty} \frac{t^i}{i!} \phi_i^N(y)\) is the solution of the above equation with the initial condition \(\phi_0^N(y)\). It is easy to check that \(\phi_0(y) \geq \sum_{i=1}^{d} y_i^+\) and \(u^N(t, 0) \to 0\) as \(N \to \infty\). Therefore, we conclude that \(|\tilde{B}_t| \in L^1_G(\mathcal{F})\). Noting that \(u(t, y) := |y| + t\sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} |z|v(dz)\) is a viscosity supersolution of the above equation, then (1) holds. It is also easy to check that the distribution of \((\tilde{B}_t + \tilde{B}_t)_{t \geq 0}\) satisfies (9). By the above lemma, \((\tilde{B}_t)_{t \geq 0}\) is a G-Lévy process.

**Example 43** We consider the following 1-dimensional equation:

\[ \partial_t u(t, x) - G_\lambda(u(t, x + 1) - u(t, x)) = 0, \quad u(0, x) = \varphi(x), \]

where \(G_\lambda(a) = a^+ - \lambda a^-\), \(\lambda \in [0, 1]\). This equation is a special case of the above equation with \(\mathcal{V} = \{\delta_l : l \in [\lambda, 1]\}\). Thus, we can construct the corresponding sublinear expectation \(\hat{E}[\cdot]\). The canonical process \((B_t)_{t \geq 0}\) is called the G-Poisson process under this sublinear expectation \(\hat{E}[\cdot]\). We also have

- If \(\varphi\) is increasing, then \(\hat{E}[\varphi(x + B_t)] = \sum_{i=0}^{\infty} \frac{t^i}{i!} \varphi(x + i)e^{-\lambda t}\).
- If \(\varphi\) is decreasing, then \(\hat{E}[-\varphi(x + B_t)] = \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} \varphi(x + i)e^{\lambda t}\).

In particular, \(\hat{E}[B_t] = t\) and \(-\hat{E}[-B_t] = \lambda t\). Thus, it characterizes a Poisson process with intensity uncertainty in \([\lambda, 1]\).

**Remark 44** We consider the following integro-partial differential equation:

\[ \partial_t u(t, x) - \sup_{(v, q, Q) \in \mathcal{U}} \{ \int_{\mathbb{R}^d \setminus \{0\}} (u(t, x + z) - u(t, x) - \langle Du(t, x), z \rangle) v(dz) \]

\[ + \langle Du(t, x), q \rangle + \frac{1}{2} \text{tr}[D^2 u(t, x)QQ^T] \} = 0, \tag{10} \]

where

\[ \sup_{(v, q, Q) \in \mathcal{U}} \{ \int_{\mathbb{R}^d} (|z|I_{|z| \geq 1} + |z|^2 I_{|z| < 1}) v(dz) + |q| + \text{tr}[QQ^T] \} < \infty \]

and
\[
\limsup_{\kappa \to 0} \int_{|z| \leq \kappa} |z|^2 v(dz) = 0.
\]

For each given initial condition \( \varphi \in C_{b,Lip}(\mathbb{R}^d) \), the viscosity solution \( u^\varphi \) for (10) exists (see appendix). Thus, we can construct the corresponding sublinear expectation. Obviously, the canonical process \((B_t)_{t \geq 0}\) is a Lévy process and has the decomposition \( B_t = B_t^1 + B_t^2 \). But it does not satisfy (1) if \( \sup_{v \in V} \int_{|z| \leq 1} |z| v(dz) = \infty \).

7. Appendix

In the appendix, we mainly consider the domination of viscosity solutions for (9) and (10). We refer to [1,4,6,7,12,19] and the references therein. For simplicity, we consider the following type of integro-partial differential equation:

\[
\partial_t u(t, x) - G(Du(t, x), D^2 u(t, x), u(t, x + \cdot)) = 0, \quad u(0, x) = \varphi \in C_{b,Lip}(\mathbb{R}^d),
\]

where \( G : \mathbb{R}^d \times \mathcal{S}(d) \times C^{1,2}_b(Q_T) \to \mathbb{R} \). We suppose \( G \) satisfies the following conditions:

(A1) If \( p_k \to p \), \( X_k \to X \) and for \( (t_k, x_k) \to (t, x) \), \( \phi_k(t_k, x_k + \cdot) \to \phi(t, x + \cdot) \) locally uniform on \( \mathbb{R}^d \), \( \phi_k \) uniformly bounded and \( D^n \phi_k \to D^n \phi, \ n = 1, 2 \), locally uniform on \( Q_T \), then

\[
G(p_k, X_k, \phi_k(t_k, x_k + \cdot)) \to G(p, X, \phi(t, x + \cdot)).
\]

(A2) If \( X \geq Y \) and \( (\phi - \psi)(t, \cdot) \) has a global minimum at \( x \), then

\[
G(p, X, \phi(t, x + \cdot)) \geq G(p, Y, \psi(t, x + \cdot)).
\]

(A3) For each constant \( c \in \mathbb{R} \), \( G(p, X, \phi(t, x + \cdot) + c) = G(p, X, \phi(t, x + \cdot)) \).

For equation (9), the corresponding \( G \) is

\[
G(p, X, u(t, x + \cdot)) = \sup_{(v, q) \in U} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (u(t, x + z) - u(t, x)) v(dz) + \langle p, q \rangle + \frac{1}{2} \text{tr}[XQQ^T] \right\}.
\]

Obviously, it satisfies all of the above assumptions. The above assumptions also hold for the equation (10).

The definition of viscosity solution for (11) is the same as Definition 30. We also suppose that for each given \( \kappa \in (0, 1) \), there exists \( G^\kappa : \mathbb{R}^d \times \mathcal{S}(d) \times \text{SC}_b(Q_T) \to \mathbb{R} \), where \( \text{SC}_b(Q_T) \) denotes the set of bounded upper or lower semicontinuous functions, satisfying the following assumptions: for \( p \in \mathbb{R}^d \), \( X, Y \in \mathcal{S}(d) \), \( u, -v \in \text{USC}_b(Q_T) \), \( w \in \text{SC}_b(Q_T) \), \( \phi, \psi, \psi_k \in C_b^{1,2}(Q_T) \),

(B1) \( G^\kappa(p, X, \phi(t, x + \cdot), \phi(t, x + \cdot)) = G(p, X, \phi(t, x + \cdot)) \).

(B2) If \( X \geq Y \), \( (v - u)(t, \cdot) \) and \( (\phi - \psi)(t, \cdot) \) have a global minimum at \( x \), then

\[
G^\kappa(p, X, v(t, x + \cdot), \phi(t, x + \cdot)) \geq G^\kappa(p, Y, u(t, x + \cdot), \psi(t, x + \cdot)).
\]

(B3) For \( c_1, c_2 \in \mathbb{R} \),

\[
G^\kappa(p, X, w(t, x + \cdot) + c_1, \phi(t, x + \cdot) + c_2) = G^\kappa(p, X, w(t, x + \cdot), \phi(t, x + \cdot)).
\]
(B4) If \( \psi_k(t, \cdot) \to w(t, \cdot) \) locally uniform on \( \mathbb{R}^d \) and \( \psi_k(t, \cdot) \) uniformly bounded, then \( G^\kappa(p, X, \psi_k(t, x + \cdot), \phi(t, x + \cdot)) \to G^\kappa(p, X, w(t, x + \cdot), \phi(t, x + \cdot)) \).

For equation (9), the corresponding \( G^\kappa \) is

\[
G^\kappa(p, X, u(t, x + \cdot), \phi(t, x + \cdot)) = \sup_{(v, q, Q) \in \mathcal{U}} \left\{ \int_{|z| > \kappa} (u(t, x + z) - u(t, x))v(dz) + \int_{|z| \leq \kappa} (\phi(t, x + z) - \phi(t, x))v(dz) + \langle p, q \rangle + \frac{1}{2} \text{tr}[XQQT] \right\}.
\]

It is easy to check that this \( G^\kappa \) satisfies the above assumptions. The above assumptions also hold for equation (10).

**Remark 45** Our assumptions (A1) and (B4) are different than those in [4,12]. This is because the measures in \( \mathcal{U} \) may be singular, which makes the problem more difficult when taking the limit. See the following example.

**Example 46** Consider \( \mathcal{V} = \{ \delta_x : x \in (1, 2] \} \) and an upper semicontinuous function \( f(x) = I_{[0,1]}(x) \). Let \( f_n \) be a sequence of continuous functions such that \( f_n \to f \) pointwise. Then it is easy to show that \( \sup_{v \in \mathcal{V}} \int f_n(z)v(dz) \) does not tend to \( \sup_{v \in \mathcal{V}} \int f(z)v(dz) \).

**Proposition 47** Suppose \( u \in \text{USC}_b(Q_T) \) (\( u \in \text{LSC}_b(Q_T) \)) is a viscosity subsolution (viscosity supersolution) of (11). If \( u \) is continuous in \( x \) and for \( \phi \in C^{1,2}(Q_T) \), \( (t, x) \in Q_T \) is a global maximum point (minimum point) of \( u - \phi \), then for each \( \kappa \in (0, 1) \) we have

\[
\partial_t \phi(t, x) - G^\kappa(D\phi(t, x), D^2\phi(t, x), u(t, x + \cdot), \phi(t, x + \cdot)) \leq 0 (\geq 0).
\]

The proof is found in [12] and the references therein.

In the following, we first extend the matrix lemma in [6].

**Theorem 48** Suppose that \( X, Y \in \mathbb{S}(N) \) satisfy \( X \leq Y < \frac{1}{\gamma} I \) for some \( \gamma > 0 \). Define \( X^\gamma = X(I - \gamma X)^{-1} \) and \( Y^\gamma = Y(I - \gamma Y)^{-1} \). Then \( Y^\gamma \geq X^\gamma \geq X \) and \( X^\gamma \geq -\frac{1}{\gamma} I \).

**Proof** For each \( A \in \mathbb{S}(N) \) with \( A < \frac{1}{\gamma} I \), it is easy to check that for each fixed \( y \in \mathbb{R}^N \),

\[
\max_{x \in \mathbb{R}^N} \{ \langle Ax, x \rangle - \frac{1}{\gamma} |x|^2 \} = \langle A(I - \gamma A)^{-1} y, y \rangle. \tag{12}
\]

From the condition \( X \leq Y < \frac{1}{\gamma} I \), we have

\[
\langle Xx, x \rangle - \frac{1}{\gamma} |x|^2 \leq \langle Yx, x \rangle - \frac{1}{\gamma} |x|^2 \text{ for } x, y \in \mathbb{R}^N.
\]

Thus, for each fixed \( y \in \mathbb{R}^N \),

\[
\max_{x \in \mathbb{R}^N} \{ \langle Xx, x \rangle - \frac{1}{\gamma} |x|^2 \} \leq \max_{x \in \mathbb{R}^N} \{ \langle Yx, x \rangle - \frac{1}{\gamma} |x|^2 \}.
\]

By (12), we obtain

\[
\langle X^\gamma y, y \rangle \leq \langle Y^\gamma y, y \rangle \text{ for all } y \in \mathbb{R}^N,
\]
which yields $Y^\gamma \geq X^\gamma$. It is easy to check $X^\gamma \geq X$ and $X^\gamma \geq -\frac{1}{\gamma}I$. The proof is complete.

In particular, we consider

$$J_{nd} := \begin{pmatrix} (n-1)I & -I & \cdots & -I \\ -I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -I \\ -I & \cdots & -I & (n-1)I \end{pmatrix}_{nd \times nd}.$$ 

It is easy to prove that $J_{nd}^2 = nJ_{nd}$ and for each given $\gamma \in (0, \frac{1}{n})$, $J_{nd} < \frac{1}{\gamma}I$ and $J_{nd}^\gamma = \frac{1}{1-n\gamma}J_{nd}$. In the following, we always use $J_{nd}$ for convenience. Then we immediately get the following corollary.

**Corollary 49** Let $X_i \in S(d)$, $i = 1, \ldots, n$, satisfy

$$\begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_n \end{pmatrix} \leq J_{nd}.$$ 

Then, for each given $\gamma \in (0, \frac{1}{n})$, $(I - \gamma X_i)^{-1}$ exists for $i = 1, \ldots, n$, and $X_i^\gamma := X_i(I - \gamma X_i)^{-1}$, $i = 1, \ldots, n$, satisfy $X_i^\gamma \geq X_i$ and

$$-\frac{1}{\gamma}I_{nd} \leq \begin{pmatrix} X_i^\gamma & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_i^\gamma \end{pmatrix} \leq \frac{1}{1-n\gamma}J_{nd}.$$ 

We now give the main lemma (see Lemma 7.8 in [12]).

**Lemma 50** Let $u_i \in USC_b(Q_T)$ be viscosity subsolutions of

$$\partial_t u(t,x) - G_i(Du(t,x), D^2 u(t,x), u(t,x+\cdot)) = 0, i = 1, \ldots, n,$$

on $Q_T$, where $G_i$, $i = 1, \ldots, n$, satisfy (A1) – (A3). Let $\phi \in C_b^{1,2}$ satisfy that $(\bar{t}, \bar{x}_1, \ldots, \bar{x}_n) \in (0, T) \times \mathbb{R}^d$ is a global maximum point of $\sum_{i=1}^n u_i(t, x_i) - \phi(t, x_1, \ldots, x_n)$. Moreover, suppose that there exist continuous functions $g_0 > 0$, $g_1, \ldots, g_n$ such that

$$D^2 \phi \leq g_0(t,x)J_{nd} + \begin{pmatrix} g_1(t, x_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_n(t, x_n) \end{pmatrix}.$$ 

Then, for each $\gamma \in (0, \frac{1}{n})$, there exist $b_i \in \mathbb{R}$ and $X_i \in S(d)$, $i = 1, \ldots, n$, such that

(i) $b_1 + \cdots + b_n = \partial_t \phi(\bar{t}, \bar{x}_1, \ldots, \bar{x}_n)$;

(ii)
where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and

$$K_{nd} = \begin{pmatrix}
g_1(\bar{t}, \bar{x}_1) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & g_n(\bar{t}, \bar{x}_n)
\end{pmatrix};$$

(iii) for each $i = 1, \ldots, n$,

$$b_i - G_i(D_x\phi(\bar{t}, \bar{x}), X_i, \phi(\bar{t}, \bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_i +, \bar{x}_{i+1}, \ldots, \bar{x}_n)) \leq 0.$$

**Remark 51** Applying the above matrix inequalities, the proof in [12] still holds.

**Remark 52** If $u_i$ is continuous in $x$, we further get that for each $\kappa \in (0, 1)$,

$$b_i - G_i^\kappa(D_x\phi(\bar{t}, \bar{x}), X_i, u_i(\bar{t}, \bar{x}_1 +, \phi(\bar{t}, \bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_i +, \bar{x}_{i+1}, \ldots, \bar{x}_n)) \leq 0.$$

We now give the main theorem, which combines the methods in [19] and [4].

**Theorem 53 (Domination theorem)** Let $u_i \in \text{USC}_b([0, T) \times \mathbb{R}^d)$ be viscosity subsolutions of

$$\partial_t u_i(t, x) - G_i(Du_i(t, x), D^2u_i(t, x), u_i(t, x + \cdot)) = 0, i = 1, \ldots, n,$$

on $Q_T$, where $G_i$ and $G_i^\kappa$, $i = 1, \ldots, n$, satisfy (A1)–(A3) and (B1)–(B4). We suppose also that

(i) There exists a constant $C > 0$ such that for each $\kappa \in (0, 1)$, $p, q \in \mathbb{R}^d$, $X, Y \in \mathbb{S}(d)$, $u \in \text{SC}_b(Q_T)$, $\varphi \in C^{1,2}(Q_T)$, $\psi_1 \in C^{1,2}_b(\mathbb{R}^d)$ and $\psi_2 \in C^{2,1}_b(\mathbb{R}^d)$,

$$|G_i^\kappa(p, X, u(t, \cdot) + \varphi(t, \cdot) + \psi_2(\cdot)) - G_i(p, Y, u(t, \cdot), \varphi(t, \cdot))| \leq C(|p - q| + ||X - Y|| + \sup_{x \in \mathbb{R}^d} |D\psi_1(x)| + \sup_{x \in \mathbb{R}^d} (|D\psi_2(x)| + |D^2\psi_2(x)|));$$

(ii) For given constants $\beta_i > 0$, $i = 1, \ldots, n$, the following domination condition holds for $G_i$: for each $(t, x_1, \ldots, x_n) \in (0, T) \times \mathbb{R}^{nd}$, $(p_i, X_i) \in \mathbb{R}^d \times \mathbb{S}(d)$ and $\phi_i \in C^{1,2}(Q_T)$ such that $\sum_{i=1}^n \beta_i p_i = 0$, $\sum_{i=1}^n \beta_i X_i \leq 0$, $\sum_{i=1}^n \beta_i u_i(t, x_i + \cdot) \leq \sum_{i=1}^n \beta_i u_i(t, x_i)$ and $\sum_{i=1}^n \beta_i D\phi_i(t, x_i) = 0$,

$$\sum_{i=1}^n \beta_i G_i^\kappa(p_i, X_i, u_i(t, x_i + \cdot), \phi_i(t, x_i + \cdot)) \leq o_\kappa(1) \text{ as } \kappa \to 0.$$

Then a similar domination also holds for the solutions: if $u_i(0, \cdot) \in C_{b,\text{Lip}}(\mathbb{R}^d)$ and $\sum_{i=1}^n \beta_i u_i(0, \cdot) \leq 0$, then $\sum_{i=1}^n \beta_i u_i(t, \cdot) \leq 0$ for all $t > 0$.

**Proof** For each given $\bar{\delta} > 0$, it is easy to check that for each $1 \leq i \leq n$, $\bar{u}_i := u_i - \bar{\delta}/(T - t)$ is a viscosity solution of

$$\partial_t \bar{u}_i(t, x) - G_i(D\bar{u}_i(t, x), D^2\bar{u}_i(t, x), \bar{u}_i(t, x + \cdot)) \leq -c, \quad c := \bar{\delta}/T^2. \quad (13)$$

For each $\lambda > 0$, the sup-convolution of $\bar{u}_i$ is defined by
\[ \hat{u}_i^\lambda(t, x) := \sup_{y \in \mathbb{R}^d} \{ u_i(t, y) - \frac{|x - y|^2}{\lambda} \} \leq \sup_{y \in \mathbb{R}^d} \{ u_i(t, y) - \frac{|x - y|^2}{\lambda} \} - \frac{\delta}{T - t}. \]

The function \( \hat{u}_i^\lambda \) is upper semicontinuous in \((t, x)\) and continuous in \(x\). Moreover, \( \hat{u}_i^\lambda \) is still a viscosity solution of (13) (see Lemma 7.3 in [12]). Noting that \( u_i(0, \cdot) \in C_{b, Lip}(\mathbb{R}^d) \), we can then choose a small \( \lambda_0 > 0 \) such that for each \( \lambda \leq \lambda_0 \), \( \sum_{i=1}^n \beta_i \hat{u}_i^\lambda(0, \cdot) \leq 0 \). Since \( \sum_{i=1}^n \beta_i u_i \leq 0 \) follows from \( \sum_{i=1}^n \beta_i \hat{u}_i \leq 0 \) in the limit \( \delta \downarrow 0 \) and \( \sum_{i=1}^n \beta_i \hat{u}_i \downarrow \sum_{i=1}^n \beta_i \hat{u}_i \) as \( \lambda \downarrow 0 \), it suffices to prove the theorem under the additional assumptions: \( \hat{u}_i \) is a viscosity solution of (13), \( \hat{u}_i \) is continuous in \(x\), and \( \lim_{t \to T} \hat{u}_i(t, x) = -\infty \) uniformly in \([0, T) \times \mathbb{R}^d \). To prove the theorem, we assume to the contrary that

\[ \sup_{(t, x) \in [0, T) \times \mathbb{R}^d} \sum_{i=1}^n \beta_i \hat{u}_i(t, x) = m_0 > 0. \]

We define for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{nd} \)

\[ \phi_{\varepsilon, \beta}(x) = \frac{1}{2\varepsilon} \sum_{1 \leq i < j \leq n} |x_i - x_j|^2 + \psi_\beta(x_1), \]

where \( \psi_\beta(x_1) := \psi(\beta x_1) \) and \( \psi \in C^2_b(\mathbb{R}^d) \) such that \( \psi(x_1) = 0 \) for \( |x_1| \leq 1 \) and \( \psi(x_1) = 2 \sum_{i=1}^n \beta_i |u_i|_{\infty} \) for \( |x_1| \geq 2 \). It is easy to check that \( \psi_\beta(x_1) = 2 \sum_{i=1}^n \beta_i |u_i|_{\infty} \) for \( |x_1| \geq 2/\beta \) and \( \sup \{|D\psi_\beta| + |D^2\psi_\beta|\} \to 0 \) as \( \beta \to 0 \). We define

\[ M_{\varepsilon, \beta} = \sup_{(t, x) \in [0, T) \times \mathbb{R}^d} \left\{ \sum_{i=1}^n \beta_i \hat{u}_i(t, x_i) - \phi_{\varepsilon, \beta}(x) \right\}. \]

By the construction of \( \psi_\beta \), for small \( \varepsilon, \beta > 0 \), the maximum of the above function is achieved at some \((\hat{t}, \hat{x}) = (\hat{t}, \hat{x}_1, \ldots, \hat{x}_n)\) with \( |\hat{x}_1| \leq 2/\beta \), \( \sum_{1 \leq i < j \leq n} |\hat{x}_i - \hat{x}_j|^2 \leq (2\varepsilon \sum_{i=1}^n |u_i|_{\infty})^{1/2} \) and \( M_{\varepsilon, \beta} \geq m_0/2 > 0 \). Thus, there exists a constant \( T_0 < T \) independent of \( \varepsilon \) and \( \beta \) such that \( \hat{t} \leq T_0 \). For fixed small \( \beta \), we check that (see Lemma 3.1 in [7])

1) \( \frac{1}{2} \sum_{1 \leq i < j \leq n} |\hat{x}_i - \hat{x}_j|^2 \to 0 \) as \( \varepsilon \to 0 \).

2) \( \lim_{\varepsilon \to 0} M_{\varepsilon, \beta} = \sup_{(t, x) \in [0, T) \times \mathbb{R}^d} \left\{ \sum_{i=1}^n \beta_i \hat{u}_i(t, z) - \psi_\beta(z) \right\} = \sum_{i=1}^n \beta_i \hat{u}_i(\hat{t}, \hat{z}) - \psi_\beta(\hat{z}) \geq m_0/2 \), where \((\hat{t}, \hat{z})\) is any limit point of \((\hat{t}, \hat{x}_1)\).

Since \( \hat{u}_i \in USC((0, T) \times \mathbb{R}^d) \) and \( \sum_{i=1}^n \beta_i \hat{u}_i(0, \cdot) \leq 0 \), it is easy to get \( \hat{t} > 0 \). Thus, \( \hat{t} \) must be strictly positive for small \( \varepsilon \). Applying Lemma 50 at the point \((\hat{t}, \hat{x}) = (\hat{t}, \hat{x}_1, \ldots, \hat{x}_n)\) and taking \( \gamma = 1/(2n) \), we obtain that there exist \( b_i \in \mathbb{R} \) and \( X_i \in \mathbb{S}(d) \) for \( i = 1, \ldots, n \) such that \( \sum_{i=1}^n \beta_i b_i = 0 \),

\[
\begin{pmatrix}
\beta_1 X_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \beta_n X_n
\end{pmatrix} - \frac{2}{\varepsilon} J_{nd} \leq \begin{pmatrix}
D^2 \psi_\beta(\hat{x}_1) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D^2 \psi_\beta(\hat{x}_n)
\end{pmatrix} \leq \begin{pmatrix}
2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 2
\end{pmatrix}
\]

and for each \( \kappa \in (0, 1) \),

\[ b_i - C^\kappa_i(p_i, X_i, \hat{u}_i(\hat{t}, \hat{x}_1 + \cdot), \beta_i^{-1} \phi_{\varepsilon, \beta}(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{x}_i + \cdot, \hat{x}_{i+1}, \ldots, \hat{x}_n)) \leq -c, \]

where \( p_i = \beta_i^{-1} D x_i \phi_{\varepsilon, \beta}(\hat{x}) \). It is easy to check that \( \sum_{i=1}^n \beta_i p_i = D \psi_\beta(\hat{x}_1) \) and
The above theorem still holds for general \( G \), which may contain \((t, x, u, Du, D^2u, u(t, \cdot))\).

We have the following corollaries which are important in this paper.

**Corollary 55** (Comparison theorem) Let the functions \( u \in \text{USC}_b([0, T) \times \mathbb{R}^d) \) and \( v \in \text{LSC}_b([0, T) \times \mathbb{R}^d) \) be, respectively, a viscosity subsolution and a viscosity supersolution of (11). Suppose \( G \) and \( G^\kappa \) satisfy (A1)–(A3) and (B1)–(B4) and the condition (i) of Theorem 53. Furthermore, we suppose that

- \( |G^\kappa(p, X, u(t, x + \cdot), \phi_1(t, x + \cdot)) - G^\kappa(p, X, u(t, x + \cdot), \phi_2(t, x + \cdot))| \to 0 \)
- If \( u(0, \cdot), v(0, \cdot) \in C_{b, Lip}(\mathbb{R}^d) \) and \( u(0, \cdot) \leq v(0, \cdot) \), then \( u(t, \cdot) \leq v(t, \cdot) \) for all \( t > 0 \).

**Corollary 56** (Sub-additivity) Let \( u^\varphi \) denote the viscosity solution of (11) with initial condition \( \varphi \). Suppose \( G \) and \( G^\kappa \) satisfy all conditions of Corollary 55 and the following condition:

- For each \( (p_i, X_i) \in \mathbb{R}^d \times \mathcal{S}(d) \), \( u_i \in \text{SC}_b(Q_T) \), \( \phi_i \in C^{1,2}(Q_T) \), \( i = 1, 2 \),
- \( G^\kappa(p_1 + p_2, X_1 + X_2, (u_1 + u_2)(t, \cdot), (\phi_1 + \phi_2)(t, \cdot)) \leq G^\kappa(p_1, X_1, u_1(t, \cdot), \phi_1(t, \cdot)) + G^\kappa(p_2, X_2, u_2(t, \cdot), \phi_2(t, \cdot)). \)

Then \( u^{\varphi^+} \leq u^\varphi + u^\psi \) for each \( \varphi, \psi \in C_{b, Lip}(\mathbb{R}^d) \).

**Corollary 57** (Convexity) Let \( u^\varphi \) denote the viscosity solution of (11) with initial condition \( \varphi \). Suppose \( G \) and \( G^\kappa \) satisfy all conditions of Corollary 55 and the following condition:

- For each \( \lambda \in (0, 1) \), \( (p_i, X_i) \in \mathbb{R}^d \times \mathcal{S}(d) \), \( u_i \in \text{SC}_b(Q_T) \), \( \phi_i \in C^{1,2}(Q_T) \), \( i = 1, 2 \),
- \( G^\kappa(\lambda p_1, X_1, u_1(t, \cdot), \phi_1(t, \cdot)) + (1 - \lambda)(p_2, X_2, u_2(t, \cdot), \phi_2(t, \cdot)) \leq \lambda G^\kappa(p_1, X_1, u_1(t, \cdot), \phi_1(t, \cdot)) + (1 - \lambda) G^\kappa(p_2, X_2, u_2(t, \cdot), \phi_2(t, \cdot)). \)

Then \( u^{\varphi^+(1-\lambda)} \leq \lambda u^\varphi + (1 - \lambda) u^\psi \) for each \( \varphi, \psi \in C_{b, Lip}(\mathbb{R}^d) \).

For our main equations (9) and (10), it is easy to check that all assumptions (i)–(v)
hold. Perron's method for (11) still holds (the proof is similar to Proposition 1 in [1]).

For each \( \varphi \in C_b^2(\mathbb{R}^d) \), it is easy to find constants \( M_1 \) and \( M_2 \) such that \( u(t, x) := M_1 t + \varphi \) and \( v(t, x) := M_2 t + \varphi \) are, respectively, the viscosity subsolution and supersolution of (9) and (10). By Perron's method and approximation, the solutions of (9) and (10) exist for each \( \varphi \in C_b, \text{Lip}(\mathbb{R}^d) \).

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