Abstract. By using Bernstein-type inequality we define analogs of spaces of entire functions of exponential type in \( L_p(X) \), \( 1 \leq p \leq \infty \), where \( X \) is a symmetric space of non-compact. We give estimates of \( L_p \)-norms, \( 1 \leq p \leq \infty \), of such functions (the Nikolskii-type inequalities) and also prove the \( L_p \)-Plancherel-Polya inequalities which imply that our functions of exponential type are uniquely determined by their inner products with certain countable sets of measures with compact supports and can be reconstructed from such sets of “measurements” in a stable way.

1. Introduction and Main Results

Consider the subspace \( E_\omega^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), \( \omega \geq 0 \), of \( L_p(\mathbb{R}^d) \) which consists of all functions which have extension to \( \mathbb{C}^d \) as entire functions of exponential type \( \leq \omega \). The latest means that for any \( \varepsilon > 0 \) there exists a \( C_\varepsilon > 0 \) such that

\[
|f(z_1, z_2, \ldots, z_d)| \leq C_\varepsilon e^{\sum_{j=1}^d (\omega + \varepsilon)|z_j|},
\]

where \( f \in E_\omega^p(\mathbb{R}^d) \), \( (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d \).

A function \( f \) belongs to the space \( E_\omega^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), if and only if it satisfies the Bernstein inequality

\[
\left\| \frac{\partial^k f}{\partial x_{j_1} \ldots \partial x_{j_k}} \right\|_{L_p(\mathbb{R}^d)} \leq \omega^k \|f\|_{L_p(\mathbb{R}^d)},
\]

for any sequence \( 1 \leq j_1, \ldots, j_k \leq d \). The Paley-Wiener theorem says that the distributional Fourier transform

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx
\]

of a function from a space \( E_\omega^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), has support in the cube

\[
Q_\omega = \{ |\xi_j| \leq \omega, j = 1, 2, \ldots, d \}.
\]

The following inequality plays an important role in approximation theory and in the theory of function spaces \cite{9}, \cite{18}, \cite{19}, and is known as the Nikolskii inequality.

\[
\|f\|_{L_q(\mathbb{R}^d)} \leq 2^d \omega^{\frac{d}{p} - \frac{d}{q}} \|f\|_{L_p(\mathbb{R}^d)}, 1 \leq p \leq q \leq \infty,
\]

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where \( f \in E_p^\omega(\mathbb{R}^d) \). Note that this inequality is exact in the sense that for the function
\[
f(x_1, x_2, \ldots, x_d) = \prod_{i=1}^d x_i^{-2} \sin^2 \frac{\omega x_i}{2}
\]
one has the equality
\[
\|f\|_{L^q(\mathbb{R}^d)} = C(p, q)\omega^{\frac{d}{2} - \frac{d}{q}}\|f\|_{L^p(\mathbb{R}^d)},
\]
for any \( 1 \leq p \leq q \leq \infty \). The detailed proofs of all these results can be found in \([2, 4, 9, 10]\).

It is also known that there exist two positive constants \( C_1, C_2 \) such that for any sufficiently dense discrete set of points \( \{x_j\} \in \mathbb{R}^d \) and any \( f \in E_p^\omega(\mathbb{R}^d) \) the following Plancherel-Polya-type inequalities hold true \([17, 19]\),
\[
C_1 \left( \sum_{j \in \mathbb{N}} |f(x_j)|^p \right)^{1/p} \leq \|f\|_{L_p(\mathbb{R}^d)} \leq C_2 \left( \sum_{j \in \mathbb{N}} |f(x_j)|^p \right)^{1/p}.
\]

The Plancherel-Polya inequalities imply that every function from \( E_p^\omega(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), \( \omega > 0 \), is uniquely determined by its values on sufficiently dense (depending on \( \omega \)) discrete sets of points \( \{x_j\} \in \mathbb{R}^d \) and can be reconstructed from these values in a stable way.

The goal of this paper is to develop similar theory in \( L_p(X) \), \( 1 \leq p \leq \infty \), where \( X \) is a non-compact symmetric manifold. In the case \( p = 2 \) it was partially done in our previous papers \([13-16]\).

In the sections 2-4 we consider a symmetric space of non-compact type \( X = G/H \) where \( G \) is a semi-simple Lie group with finite center and \( H \) its maximal compact subgroup \([7, 8]\). The elements of the corresponding Lie algebra \( \mathfrak{g} \) of \( G \) will be identified with left-invariant vector fields on \( G \). The action of the group \( G \) on functions defined on the space \( X \) is given by the formula
\[
T_g f(x) = f(g \cdot x), \quad g \in G, \quad x \in X = G/H.
\]

The corresponding representation of the group \( G \) in any space \( L_p(X) \), \( 1 \leq p \leq \infty \), is known as quasi-regular representation.

One can consider the so-called differential associated with action \((1.2)\). The differential is a map from the Lie algebra \( \mathfrak{g} \) into algebra of differential operators on the space \( X \). We will use the same notation for elements of \( \mathfrak{g} \) and their images under the differential.

Using a set of vector fields \( \mathcal{V} \) we define the sets of functions \( E_p^\omega(\mathcal{V}) \), \( 1 \leq p \leq \infty \), \( \omega > 0 \), as the sets of all \( f \in L_p(X) \), \( 1 \leq p \leq \infty \), for which the following Bernstein inequality holds true
\[
\|V_{i_1}V_{i_2} \ldots V_{i_k} f\|_{L_p(X)} \leq \omega^k \|f\|_{L_p(X)}, \quad k \in \mathbb{N}.
\]

It is not clear from this definition if the set \( E_p^\omega(\mathcal{V}) \) is linear. It becomes obvious after we prove that this set coincide with the set of all functions \( f \in L_p(X), \leq p \leq \infty \), such that for any choice of indices \( 1 \leq i_1, \ldots, i_k \leq d \), any \( 1 \leq j \leq d \) and any functional \( h \) on \( L_p(X) \) the function
\[
\langle h, e^{tV_{i_1}} \ldots V_{i_k} f \rangle : \mathbb{R} \to \mathbb{C},
\]
of the real variable \( t \) is entire function of the exponential type \( \omega \) which is bounded on the real line.
Although different bases of vector fields will produce different scales of spaces in the sense that for a particular \( \omega \)

\[
E_p^\omega(\mathcal{V}) \neq E_p^\omega(\mathcal{U}),
\]

their unions \( \bigcup_{\omega > 0} E_p^\omega(\mathcal{V}) \) and \( \bigcup_{\omega > 0} E_p^\omega(\mathcal{U}) \) will be the same.

The spectral resolution of the Laplace-Beltrami operator \( \Delta \) in the space \( L_2(X) \) is given by the Helgason-Fourier transform on \( X \). The existence of such transform allows to introduce Paley-Wiener spaces \( PW_\omega(X) \) as sets of all functions from \( L_2(X) \) for which Helgason-Fourier transform has compact support bounded by \( \omega \) in the non-compact direction (see below). As a consequence of our general result (Theorem 3.2) about Paley-Wiener vectors for self-adjoint operators we obtain that a function \( f \) belongs to the Paley-Wiener space \( PW_\omega(X) \) if and only if the following Bernstein inequality holds true

\[
\|\Delta^{s/2} f\|_{L^2(X)} \leq (\omega^2 + \|\rho\|^2)^{s/2} \|f\|_{L^2(X)},
\]

where \( \rho \) is the half-sum of all positive restricted roots and its norm is calculated with respect to the Killing form.

The fact that the Laplace-Beltrami operator (1.4) commutes with the fields \( V_1, V_2, ..., V_d \) allows to obtain the following continuous embeddings

\[
E_2^{\Omega/\sqrt{d}}(\mathcal{V}) \subset PW_\omega(X) \subset E_2^\Omega(\mathcal{V}), d = \dim X,
\]

where \( \Omega = \sqrt{\omega^2 + \|\rho\|^2} \). These embeddings imply that the spaces \( E_2^\Omega(\mathcal{V}) \) are not trivial at least if \( \omega \geq \|\rho\| \) and their union \( \bigcup_{\omega > 0} E_2^\omega(\mathcal{V}) \) is dense in \( L_2(X) \).

In the Theorem 4.2 we prove an inequality which is in the case of \( \mathbb{R}^d \) known as the Nikolskii inequality. Namely, we show that for any \( m > d/p \) and any sufficiently discrete set \( \{g_i\} \subset G \) there exist constants \( C(X), C(X,m) \) such that

\[
\|f\|_{L^q(X)} \leq C(X)r^{d/p} \sup_{g \in G} \left( \sum_{i} (|f(g \cdot o)|)^p \right)^{1/p} \leq C(X,m)r^{d/q-d/p} (1 + (r\omega)^m) \|f\|_{L^q(X)},
\]

(1.4)

for all functions from \( E_p^\omega(\mathcal{V}) \) and \( 1 \leq p \leq q \leq \infty \). Here \( o \) is the "origin" of \( X \) and \( r \) is essentially the "distance" between points: \( \sup_j \inf_i \text{dist}(g_j, g_i) \). Using these inequalities we prove the continuous embedding

\[
E_p^\omega(\mathcal{V}) \subset E_p^\omega(\mathcal{V}), 1 \leq p \leq q \leq \infty.
\]

(1.5)

The embedding (1.7) has an important consequence that the spaces \( E_q^\omega(\mathcal{V}) \) are not trivial at least if \( q \geq 2 \) and \( \omega \geq \|\rho\| \). This result is complementary to a result of the classical paper [3] of L. Ehrenpreis and F. Mautner which says that in \( L_1(X) \) there are not non-trivial functions whose Helgason-Fourier transform has compact support. As another consequence of the inequalities (1.6) we obtain a generalization of the Nikolskii inequality (1.1) for functions from \( E_p^\omega(\mathcal{V}) \)

\[
\|f\|_{L^q(X)} \leq C(X)\omega^{\frac{d}{2} - \frac{d}{4}} \|f\|_{L^p(X)}, d = \dim X, 1 \leq p \leq q \leq \infty.
\]

We also prove a generalization of the Plancherel-Polya inequalities for functions from \( E_p^\omega(\mathcal{V}) \), \( 1 \leq p \leq \infty \). We show that there exist constants \( C(X), c(X) \) such that for every \( \omega > 0 \), every "sufficiently dense" discrete set of measures \( \{\Phi_\nu\} \) with compact supports the following inequalities hold true.
A have Lie algebras $f$ that every $\{distributions\}$ $X$ this inner product it is possible to construct $G$ as a boundary. A centralizer of system $(r,\theta)$ and can be reconstructed in a stable way. The Iwasawa decomposition $G$ is a connected non-compact semi-simple group Lie whose Lie algebra $h$ as a finite center and $K$ is a connected non-compact semi-simple group Lie with respect to the invariant measure $dx$. We also show that our spaces are not trivial at least in the case $p > 2$, $\omega \geq ||\rho||$.

(1.6) $c(X) \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p} \leq r^{-d/p} \|f\|_p \leq C(X) \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p}.

$f \in E^\omega_p(\mathcal{V}), 1 \leq p \leq \infty$ and $r$ is comparable to the "distance" between supports of distributions $\{\Phi_\nu\}$. The Plancherel-Polya-type inequalities (1.8) obviously imply that every $f \in E^\omega_p(\mathcal{V}), 1 \leq p \leq \infty$, is uniquely determined by the values $\{\Phi_\nu(f)\}$ and can be reconstructed in a stable way.

Note that an approach to Paley-Wiener functions and the Bernstein inequality in a Hilbert space in which a strongly continuous representation of a Lie group is given were developed by author in [11]-[15].

2. Bernstein-type inequality in $L_p(X), 1 \leq p \leq \infty$.

A non-compact Riemannian symmetric space $X$ is defined as $G/K$, where $G$ is a connected non-compact semi-simple group Lie whose Lie algebra has a finite center and $K$ its maximal compact subgroup. Their Lie algebras will be denoted respectively as $g$ and $k$. The group $G$ acts on $X$ by left translations and it has the "origin" $o = eK$, where $e$ is the identity in $G$. Every such $G$ admits so called Iwasawa decomposition $G = NAK$, where nilpotent Lie group $N$ and abelian group $A$ have Lie algebras $n$ and $a$ respectively. Letter $M$ is usually used to denote the centralizer of $A$ in $K$ and letter $B$ is used for the factor $B = K/M$ which is known as a boundary.

The Killing form on $G$ induces an inner product on tangent spaces of $X$. Using this inner product it is possible to construct $G$-invariant Riemannian structure on $X$. The Laplace-Beltrami operator of this Riemannian structure is denoted as $\Delta$.

In particular, if $X$ has rank one (dim$A = 1$) then in a polar geodesic coordinate system $(r, \theta_1, ..., \theta_{d-1})$ on $X$ at every point $x \in X$ the operator $\Delta$ has the form [8]

$$\Delta = \partial_r^2 + \frac{1}{S(r)} \frac{dS(r)}{dr} \partial_r + \Delta_S,$$

where $\Delta_S$ is the Laplace-Beltrami operator on the sphere $S(x,r)$ of the induced Riemannian structure on $S(x,r)$ and $S(r)$ is the surface area of a sphere of radius $r$ which depends just on $r$ and is given by the formula

$$S(r) = \Omega_d 2^{-d} e^{-a-b} sh^a(cr) sh^b(2cr),$$

where $d = \text{dim}X = a + b + 1, c = (2a + 8b)^{-1/2}, a$ and $b$ depend on $X$ and $\Omega_d = 2\pi^{d/2}(\Gamma(d/2))^{-1}$ is the surface area of the unit sphere in $d$-dimensional Euclidean space.

In this section we will use the notation $L_p(X), 1 \leq p \leq \infty$, with understanding that in the case $1 \leq p < \infty$ the space $L_p(X)$ represents the usual $L_p(X)$ with respect to the invariant measure $dx$ on $X$ and in the case $p = \infty$ we have the space of uniformly continuous bounded functions on $X$.

The goal of the section is to introduce a scale of closed linear subspaces in $L_p(X), 1 \leq p \leq \infty$, for which an analog of the Bernstein inequality holds true. In the case $p = 2$ our spaces consist of functions whose Helgason-Fourier transform has compact support (see Section 3). We also show that our spaces are not trivial at least in the case $p \geq 2, \omega \geq ||\rho||$. 

(1.6) $c(X) \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p} \leq r^{-d/p} \|f\|_p \leq C(X) \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p}.$
Every vector \( V \) in the Lie algebra \( \mathfrak{g} \) can be identified with a left-invariant vector field on \( G \), which will be denoted by the same letter \( V \).

There exists a basis \( V_1, ..., V_d, V_n \in \mathfrak{g}, n = \dim G, d = \dim X \), in \( \mathfrak{g} \) such that \( V_{d+1}, ..., V_n \) form a basis of the algebra \( \text{Lie} \) of the compact group \( K \) and such that
\[
(V_i, V_j) = \delta_{ij}
\]
if \( 1 \leq i, j \leq d \), and
\[
(V_i, V_j) = -\delta_{ij}
\]
if \( d + 1 \leq i, j \leq n \), where \( \langle \cdot, \cdot \rangle \) is the Killing form and \( \delta_{ij} \) is the Kronecker symbol.

In this basis the differential operator
\[
(V_1^2 + \ldots + V_d^2 - V_{d+1}^2 - \ldots - V_n^2)
\]
on \( G \) belongs to the center of the algebra of all left-invariant differential operators on \( G \) and is known as the Casimir operator.

We are going to use the same notations for the vectors \( V_1, ..., V_d \in \mathfrak{g} \) and for their images under the differential of the quasi-regular representation of \( G \) in \( L_p(X), 1 \leq p \leq \infty \).

The image of every \( V_1, ..., V_d \) under the differential of the quasi-regular representation of \( G \) in the space \( L_p(X), 1 \leq p \leq \infty \), is a generator of strictly continuous isometric one-parameter group in \( L_p(X), 1 \leq p \leq \infty \), which is given by the formula
\[
e^{tV_j} f(x) = f(\exp tV_j \cdot x), x \in X, f \in L_p(X), 1 \leq p \leq \infty,
\]
where \( \exp tV_j \) is the flow generated by the vector field \( V_j \). In the case \( p = 2 \) these generators are skew-symmetric operators.

Note that the Laplace-Beltrami operator \( \Delta \) on \( X \) commutes with the operators \( V_1, ..., V_d \).

In what follows the notation \( \|f\|_p, 1 \leq p \leq \infty \), will always mean the norm \( \|f\|_{L_p(X)}, 1 \leq p \leq \infty \), of a function \( f \).

**Definition 1.** A function \( f \in L_p(X), 1 \leq p \leq \infty \), belongs to the set \( E^\omega_p(X) \) if and only if for every \( 1 \leq i_1, ..., i_k \leq d \) the following Bernstein inequality holds true
\[
\|V_{i_1} \ldots V_{i_k} f\|_p \leq \omega^k \|f\|_p.
\]

**Definition 2.** The linear space \( E^\omega_p(X), \omega > 0 \), is the set of all functions \( f \in L_p(X), 1 \leq p \leq \infty \), such that for any \( 1 \leq i_1, ..., i_k \leq d \), any \( 1 \leq j \leq d \) and any functional \( h \in L_p(X)^* \) the function
\[
\langle h, e^{tV_{i_1} \ldots V_{i_k} f} \rangle : \mathbb{R} \to \mathbb{C},
\]
for the real variable \( t \) is entire function of the exponential type \( \omega \).

We are going to show that these definitions are equivalent. All the necessary information about one-parameter groups of operators can be found in \([5]\) and \([13]\).

**Theorem 2.1.** The sets \( E^\omega_p(X) \) and \( E^\omega_p(X), \omega > 0 \), coincide.

**Proof.** Suppose that \( f \in E^\omega_p(X) \), then for any function \( g = V_{i_1} \ldots V_{i_k} f, 1 \leq i_1, ..., i_k \leq d \), and any \( 1 \leq j \leq n \) because we have the estimate (2.2) the series
\[
e^{tV_j} g = \sum \frac{(tV_j)^r}{r!} g
\]
is convergent in $L_p(X)$ and represents an abstract entire function. Since $\|V_r^* g\|_p \leq \omega^{k+r}\|f\|_p$ we have the estimate

$$\|e^{zV_r^* g}\|_p = \left\|\sum_{r=0}^{\infty} \frac{z^r V_r^* g}{r!}\right\|_p \leq \omega^k\|f\|_p \sum_{r=0}^{\infty} \frac{|z|^r \omega^r}{r!} = \omega^k |z|^{k+|\omega|}\|f\|_p,$$

which shows that the function (2.3) has exponential type $\omega$. Since $e^{zV_j}$ is a group of isometries, the abstract function $e^{zV_j} g$ is bounded by $\omega^k\|f\|_p$. It implies that for any functional $h$ on $L_p(X), 1 \leq p \leq \infty$, the scalar function

$$F(z) = \langle h, e^{zV_j} g \rangle$$

is entire because it is defined by the series

$$F(z) = \langle h, e^{zV_j} g \rangle = \sum_{r=0}^{\infty} \frac{z^r \langle h, V_j^r g \rangle}{r!}$$

and because $|\langle h, V_j^r g \rangle| \leq \omega^{k+r}\|h\|\|f\|_p$ we have

$$|F(z)| \leq e^{\omega|z|}\omega^k\|h\|\|f\|_p.$$

For real $t$ we also have $|F(t)| \leq \omega^k\|h\|\|f\|_p$. Thus, we proved the inclusion $E^\omega_p (V) \subset E^\omega_p (\mathbb{V})$.

Now we prove the inverse inclusion by induction. The fact that $f \in E^\omega_p (\mathbb{V})$ means in particular that for any $1 \leq j \leq d$ and any functional $h$ on $L_p(X), 1 \leq p \leq \infty$, the function $F(z) = \langle h, e^{zV_j} f \rangle$ is an entire function of exponential type $\omega$ which is bounded on the real axis $\mathbb{R}^1$. Since $e^{zV_j}$ is a group of isometries in $L_p(X)$, an application of the Bernstein inequality for functions of one variable gives

$$\|\langle h, e^{zV_j} V_j^m f \rangle\|_{C(\mathbb{R}^1)} = \left\|\frac{d^m}{dt^m} \langle h, e^{V_j t} f \rangle\right\|_{C(\mathbb{R}^1)} \leq \omega^m\|h\|\|f\|_p, m \in \mathbb{N}.$$

The last one gives for $t = 0$

$$|\langle h, V_j^m f \rangle| \leq \omega^m\|h\|\|f\|_p.$$

Choosing $h$ such that $\|h\| = 1$ and

$$\langle h, V_j^m f \rangle = \|V_j^m f\|_p$$

we obtain the inequality

$$\|V_j^m f\|_p \leq \omega^m\|f\|_p, m \in \mathbb{N}.$$

It was the first step of induction. Now assume that we already proved that the fact that $f$ belongs to the space $E^\omega_p (\mathbb{V})$ implies the inequality

$$\|V_{i_1}...V_{i_k} f\|_p \leq \omega^k\|f\|_p$$

for any choice of indices $1 \leq i_1, i_2, ..., i_k \leq d$. Then we can apply our first step of induction to the function $g = V_{i_1}...V_{i_k}$. It proves the inclusion $E^\omega_p (\mathbb{V}) \subset E^\omega_p (\mathbb{V})$.

**Theorem 2.2.** The set $E^\omega_p (\mathbb{V})$ has the following properties:

1) it is invariant under every $V_j$,
2) it is a linear subspace of $L_p(X), 1 \leq p \leq \infty$,
3) it is a closed subspace of $L_p(X), 1 \leq p \leq \infty$.
Proof: We have to show that if \( f \in E_p^\omega(\mathbb{V}) \) then for any \( 1 \leq i_1, i_2, ..., i_k, \nu \leq d \) the inequality
\[
\|V_{i_1}...V_{i_k}g\|_p \leq \omega^k\|g\|_p, g = V_\nu f,
\]
holds true. If \( f \in E_p^\omega(\mathbb{V}) \), then for any \( V_\nu, V_j \) and \( g = V_\nu f \) the inequality
\[
\|V_j f\|_p \leq \omega^k\|f\|_p = \omega^k (\omega\|f\|_p), k \in \mathbb{N},
\]
takes place. But then for any \( z \in \mathbb{C} \) we have
\[
\|e^{zV_j}g\|_p = \left\| \sum_{r=0}^{\infty} \frac{(zV_j^r g)}{r!} \right\|_p \leq \omega\|f\|_p \sum_{r=0}^{\infty} \frac{|z|^r \omega^r}{r!} = \omega e^{\omega |z|}\|f\|_p.
\]
As in the proof of the Theorem 2.1 it implies that for any functional \( h \) on \( L_p(X), 1 \leq p \leq \infty \), the scalar function
\[
F(z) = \langle h, e^{zV_j} \rangle
\]
is an entire function of exponential type \( \sigma \) which is bounded on the real axis \( \mathbb{R}^1 \) by the constant \( \|h\|\|g\|_p \). An application of the Bernstein inequality gives the inequality
\[
\|\langle h, e^{tV_j}V_j^k g \rangle\|_{C(\mathbb{R}^1)} = \left\| \frac{d}{dt} \right\|^k \langle h, e^{tV_j} \rangle \leq \omega^k \|h\|\|g\|_p
\]
which leads (see the proof of the Theorem 2.1) to the inequality
\[
\|V_j f\|_p \leq \omega\|f\|_p.
\]
It is clear that by repeating these arguments we can prove the inequality (2.8). The first part of the Theorem is proved. The second follows from the fact that the set \( E_p^\omega(\mathbb{V}), \omega > 0 \), is obviously linear.

Next, assume that a sequence \( f_n \in E_p^\omega(\mathbb{V}) \) converges in \( L_p(X) \) to a function \( f \). Because of the Bernstein inequality for any \( 1 \leq j \leq d \) the sequence \( V_j f_n \) will be fundamental in \( L_p(X) \). Since the operator \( V_j \) is closed the limit of the sequence \( V_j f_n \) will be the function \( V_j f \), which implies the inequality
\[
\|V_j f\|_p \leq \omega\|f\|_p.
\]
By repeating these arguments we can show that if a sequence \( f_n \in E_p^\omega(\mathbb{V}) \) converges in \( L_p(X) \) to a function \( f \) then the Bernstein inequality (2.8) for \( f \) holds true. The Theorem is proved.

\( \Box \)

3. Paley-Wiener Spaces of Functions \( PW_\omega(X) \) in \( L_2(X) \)

Let \( a^* \) be the real dual of \( a \) and \( W \) be the Weyl’s group. The \( \Sigma \) will be the set of all bounded roots, and \( \Sigma^+ \) will be the set of all positive bounded roots. The notation \( a^+ \) has the meaning \( a^+ = \{ h \in a | \alpha(h) > 0, \alpha \in \Sigma^+ \} \) and is known as positive Weyl’s chamber. Let \( \rho \in a^{*} \) is defined in a way that \( 2\rho \) is the sum of all positive bounded roots. The Killing form \( \langle , \rangle \) on \( g \) defines a metric on \( a \). By duality it defines a scalar product on \( a^* \). The \( a^*_+ \) is the set of \( \lambda \in a^* \), whose dual belongs to \( a^+ \).
According to Iwasawa decomposition for every $g \in G$ there exists a unique $A(g) \in \mathfrak{a}$ such that $g = n \exp A(g)k$, $k \in K$, $n \in N$, where $\exp : \mathfrak{a} \to A$ is the exponential map of the Lie algebra $\mathfrak{a}$ to Lie group $A$. On the direct product $X \times \mathcal{B}$ we introduce function with values in $\mathfrak{a}$ using the formula $A(x, b) = A(u^{-1}g)$ where $x = gK$, $g \in G$, $b = uM$, $u \in K$.

For every $f \in C_0^\infty(X)$ the Helgason-Fourier transform is defined by the formula
\[
\hat{f}(\lambda, b) = \int_X f(x)e^{-i\lambda \rho A(x, b)}dx,
\]
where $\lambda \in \mathfrak{a}^*$, $b \in \mathcal{B} = K/M$, and $dx$ is a $G$-invariant measure on $X$. This integral can also be expressed as an integral over group $G$. Namely, if $b = uM$, $u \in K$, then
\[
\hat{f}(\lambda, b) = \int_G f(x)e^{-i\lambda \rho A(\omega^{-1}g)}dg.
\]
The following inversion formula holds true
\[
f(x) = w^{-1} \int_{\mathfrak{a}^* \times \mathcal{B}} \hat{f}(\lambda, b)e^{-i\lambda \rho A(x, b)}|c(\lambda)|^{-2}d\lambda db,
\]
where $w$ is the order of the Weyl’s group and $c(\lambda)$ is the Harish-Chandra’s function, $d\lambda$ is the Euclidean measure on $\mathfrak{a}^*$ and $db$ is the normalized $K$-invariant measure on $\mathcal{B}$. This transform can be extended to an isomorphism between spaces $L_2(\mathcal{X}, dx)$ and $L_2(\mathfrak{a}_+^* \times \mathcal{B}, |c(\lambda)|^{-2}d\lambda db)$ and the Plancherel formula holds true
\[
\|f\| = \left(\int_{\mathfrak{a}_+^* \times \mathcal{B}} |\hat{f}(\lambda, b)|^2|c(\lambda)|^{-2}d\lambda db\right)^{1/2}.
\]

An analog of the Paley-Wiener Theorem holds true that says in particular that a Helgason-Fourier transform of a compactly supported distribution is a function which is analytic in $\lambda$.

It is known, that
\[
\hat{\Delta} \hat{f}(\lambda, b) = -(|\lambda|^2 + |\rho|^2) \hat{f}(\lambda, b), f \in C_0^\infty(X),
\]
where $|\lambda|^2 = <\lambda, \lambda>$, $|\rho|^2 = <\rho, \rho>$, $<,>$ is the Killing form on $\mathfrak{a}^*$.

In the case $p = 2$ we can introduce the Paley-Wiener spaces $PW_\omega(X)$ which depend just on the symmetric space $X$.

**Definition 3.** In what follows by the Paley-Wiener space $PW_\omega(X)$ we understand the space of all functions $f \in L_2(X)$ whose Helgason-Fourier transform has support in the set $(\mathfrak{a}_+^*)_\omega \times \mathcal{B}$, where
\[
(\mathfrak{a}_+^*)_\omega = \left\{\lambda \in \mathfrak{a}_+^* : <\lambda, \lambda>^{1/2} = |\lambda| \leq \omega\right\}, \omega \geq 0,
\]
and $<,>$ is the Killing form on $\mathfrak{a}^*$.

The next theorem is evident.

**Theorem 3.1.** The following statements hold true:

1) the set $\bigcup_{\omega > 0} PW_\omega(X)$ is dense in $L_2(X)$;

2) the $PW_\omega(X)$ is a linear closed subspace in $L_2(X)$.

We have the following Theorem in which we use notation $\rho \in \mathfrak{a}^*$ for the half-sum of the positive bounded roots.
Theorem 3.2. A function $f$ belongs to $PW_\omega(X)$ if and only if
\begin{equation}
\|D^f\|_2 \leq (\omega^2 + \|\rho\|^2)^{1/2} \|f\|_2.
\end{equation}
where $D$ is the positive square root from the Laplace-Beltrami operator $\Delta$, $D = \Delta^{1/2}$.

Proof. By using the Plancherel formula and (2.8) we obtain that for every $\omega$- band limited function
\[
\|\Delta^f\|^2 = \int_{a^+} (\|\lambda\|^2 + \|\rho\|^2)^{1/2} |\hat{f}(\lambda, b)|^2 |c(\lambda)|^2 \rho \lambda d\lambda \leq
\]
\[
(\omega^2 + \|\rho\|^2)^{1/2} \int_{a^+} |\hat{f}(\lambda, b)|^2 |c(\lambda)|^2 \rho \lambda d\lambda = (\omega^2 + \|\rho\|^2)^{1/2} \|f\|^2.
\]
Conversely, if $f$ satisfies (3.1), then for any $\varepsilon > 0$ we have
\[
\int_{a^+} (\|\lambda\|^2 + \|\rho\|^2)^{1/2} |\hat{f}(\lambda, b)|^2 |c(\lambda)|^2 \rho \lambda d\lambda \leq
\]
\[
(\omega^2 + \|\rho\|^2)^{1/2} \|f\|^2.
\]
It means, that for any $\varepsilon > 0$ the function $\hat{f}(\lambda, b)$ is zero on $\{a^+ \setminus (a^+)_\omega\} \times B$. The statement is proved.

In a similar way one can prove the following Corollary.

Corollary 3.1. The following statements hold true:
1) the norm of the operator $D = \Delta^{1/2}$ in the space $PW_\omega(X)$ is exactly $\sqrt{\omega^2 + \|\rho\|^2}$;
2) the following limit takes place
\[
\lim_{k \to \infty} \|D^k f\|_2^{1/k} = \sqrt{\omega^2 + \|\rho\|^2}, \quad 0 < \omega < \infty,
\]
if and only if $\omega$ is the smallest number for which $(a^+)_\omega \times B$ contains the support of a function $F f, f \in L_2(X)$.

In particular, we have the following property.

Corollary 3.2. If a function $f$ belongs to the space $PW_\omega(X)$ then for any vector fields $V_{i_1}, \ldots, V_{i_k}, V_j, 1 \leq i_1, \ldots, i_k, j \leq d$ and any $h \in L_2(X)$ the function
\begin{equation}
\int_X V_{i_1} \ldots V_{i_k} f(\exp z V_j \cdot x) h(x) dx : \mathbb{C} \to \mathbb{C}
\end{equation}
is entire function of the exponential type $\leq \Omega = \sqrt{\omega^2 + \|\rho\|^2}$ which is bounded on the real line. Conversely, if the function (3.10) is an entire function of the exponential type
\[
\frac{\Omega}{\sqrt{d}} = \frac{\sqrt{\omega^2 + \|\rho\|^2}}{\sqrt{d}}
\]
for any $V_{i_1}, \ldots, V_{i_k}, V_j, 1 \leq i_1, \ldots, i_k, j \leq d$ and any $h \in L_2(X)$, then $f \in PW_\omega(X)$. 
The next Lemma describes relations between spaces \(PW_\omega(X)\) and \(E_\omega(D)\). In what follows we assume that \(\sqrt{\omega^2 + \|\rho\|^2} > 1\).

**Lemma 3.3.** The following statements hold

1. there exists a constant \(a = a(X)\) such that
   \[
   PW_\omega(X) \subset E_{a\Omega}(D), \quad \Omega = \sqrt{\omega^2 + \|\rho\|^2};
   \]
2. there exists a constant \(b = b(X)\) such that such that
   \[
   E_{\omega/\sqrt{\pi}(D)} \subset PW_{\omega}(X).
   \]

**Proof.** We prove (3.9). Let \(A = A(X)\) be a constant such that for all \(f \in H^1(X)\)

\[
\|D_j f\| \leq A \left( \|f\| + \|\Delta^{1/2} f\| \right), \quad 1 \leq j \leq d.
\]

Since every \(D_j\) is a generator of an isometry of \(X\) the Laplace-Beltrami operator \(\Delta\) commutes with every \(D_j\). Using (3.11) we obtain the following inequality for \(f \in H^\infty(X)\)

\[
\|D_{j_1} D_{j_2} \cdots D_{j_m} f\| \leq A^m \sum_{0 \leq l \leq m} C_m^l \|\Delta^{l/2} f\| \leq (2A)^m \sum_{0 \leq l \leq m} \|\Delta^{l/2} f\|,
\]

where \(C_m^l\) is the number of combinations from \(m\) elements taken \(l\) at a time. Thus, if \(f \in PW_\omega(X)\) then \(\|\Delta^s f\| \leq (\omega^2 + \|\rho\|^2)^s \|f\|\) and we obtain the inequality

\[
\|D_{j_1} \cdots D_{j_m} f\| \leq (2A)^m \sum_{0 \leq l \leq m} (\omega^2 + \|\rho\|^2)^l \|f\| \leq \left( a \sqrt{\omega^2 + \|\rho\|^2} \right)^m \|f\|,
\]

where \(a = 4A\). The inclusion (3.9) is proved.

Now we prove (3.10). Let \(B = B(X)\) be a constant such that for all \(f \in H^2(X)\)

\[
\|\Delta f\| \leq B(\|f\| + \|Lf\|).
\]

Since the Laplace-Beltrami operator \(\Delta\) commutes with every \(D_j\) it commutes with \(L\) and we have for every \(f \in H^{2k}(X)\)

\[
\|\Delta^k f\| \leq B(\|\Delta^{k-1} f\| + \|\Delta^{k-1} Lf\|) \leq \ldots \leq (2B)^k \sum_{0 \leq l \leq k} \|L^l f\|.
\]

Using definition of the operator \(L\) one can easily verify that for any natural \(l\) the function \(L^l(f)\) is a sum of \(d^l\) terms of the following form:

\[
D_{j_1}^2 \cdots D_{j_l}^2 (f), \quad 1 \leq j_1, \ldots, j_l \leq d.
\]

Thus, for \(f \in E_{\omega/\sqrt{\pi}(D)}\) one has

\[
\|D_{i_1} \cdots D_{i_k} f\| \leq \left( \frac{\omega}{\sqrt{d}} \right)^k \|f\|, \quad \text{and then}
\]

\[
\|\Delta^k f\| \leq (2B)^k \sum_{0 \leq l \leq k} \|L^l f\| \leq (2B)^k \sum_{0 \leq l \leq k} \sum_{1 \leq j_1, \ldots, j_l \leq d} \|D^2_{j_1} \cdots D^2_{j_l} f\| \leq (b(\omega^2 + \|\rho\|^2))^k \|f\|,
\]

where \(b = b(X)\).
where $b = 4B$. Lemma is proved.

\[ \square \]

4. Embedding Theorems

Denote by $T_x(X)$ the tangent space of $X$ at a point $x \in X$ and let $\exp_x : T_x(X) \to X$ be the exponential geodesic map i.e. $\exp_x(u) = \gamma(1)$, $u \in T_x(X)$ where $\gamma(t)$ is the geodesic starting at $x$ with the initial vector $u : \gamma(0) = x, \frac{d\gamma(t)}{dt} = u$. We will always assume that all our local coordinates are defined by $\exp$.

We consider a uniformly bounded partition of unity $\{\varphi_\nu\}$ subordinate to a cover of $X$ of finite multiplicity

$$X = \bigcup_\nu B(x_\nu, r),$$

where $B(x_\nu, r)$ is a metric ball at $x_\nu \in X$ of radius $r$ and introduce the Sobolev space $W^k_p(X), k \in \mathbb{N}, 1 \leq p < \infty$, as the completion of $C^\infty_0(X)$ with respect to the norm

$$\|f\|_{W^k_p(X)} = \left(\sum_\nu \|\varphi_\nu f\|_{W^k_p(B(x_\nu, r))}^p\right)^{1/p}. \quad (4.1)$$

The regularity theorem for $\Delta$ means in particular, that the norm of the Sobolev space $W^k_p(X), k \in \mathbb{N}, 1 \leq p < \infty$, is equivalent to the graph norm $\|f\|_p + \|\Delta^k f\|_p$.

Since vector fields $V_1, \ldots, V_d$, generate the tangent space at every point of $X$ the norm of the space $W^k_p(X), k \in \mathbb{N}, 1 \leq p < \infty$, is equivalent to the norm

$$\|f\|_p + \sum_{j=1}^k \sum_{1 \leq i_1, \ldots, i_j \leq d} \|V_{i_1} \ldots V_{i_j} f\|_p, 1 \leq p < \infty. \quad (4.2)$$

Using the closed graph Theorem and the fact that every $V_i$ is a closed operator in $L_p(X), 1 \leq p < \infty$, it is easy to show that the norm (4.2) is equivalent to the norm

$$\|f\|_p + \sum_{1 \leq i_1, \ldots, i_k \leq d} \|V_{i_1} \ldots V_{i_k} f\|_p, 1 \leq p < \infty. \quad (4.3)$$

Let $\pi : G \to X = G/K$ be the natural projection and $o \in X$ is the image of identity in $G$. We consider a ball $B(o, r/4)$ in the invariant metric on $X$. Now we choose such elements $g_\nu \in G$ that the family of balls $B(x_\nu, r/4), x_\nu = g_\nu \cdot o$, has the following maximal property: there is no ball in $X$ of radius $r/4$ which would have empty intersection with every ball from this family. Then the balls of double radius $B(x_\nu, r/2)$ would form a cover of $X$. Of course, the balls $B(x_\nu, r)$ will also form a cover of $X$. Let us estimate the multiplicity of this cover.

Note, that the Riemannian volume $B(\rho)$ of a ball of radius $\rho$ in $X$ is independent of its center and is given by the formula

$$B(\rho) = \int_0^\rho S(t)dt,$$

where the surface area $S(t)$ of a sphere of radius $t$.

Every ball from the family $\{B(x_\nu, r)\}$, that has non-empty intersection with a particular ball $B(x_j, r)$ is contained in the ball $B(x_j, 3r)$. Since any two balls from
the family \( \{ B(x_\nu, r/4) \} \) are disjoint, it gives the following estimate for the index of multiplicity \( N \) of the cover \( \{ B(x_\nu, r) \} \):

\[
N \leq \frac{B(3r)}{B(r/4)} \leq \frac{\int_0^{3r} S(t) dt}{\int_{r/4}^r S(t) dt}.
\]

By using some elementary inequalities for the function \( sh \) one can obtain the following rough estimate

\[
N \leq 12^d e^{\sqrt{d-1}}.
\]

So, we proved the following Lemma.

**Lemma 4.1.** For any \( r > 0 \) there exists a set of points \( \{ x_\nu \} \) from \( X \) such that

1) balls \( B(x_\nu, r/4) \) are disjoint,

2) balls \( B(x_\nu, r/2) \) form a cover of \( X \),

3) multiplicity of the cover by balls \( B(x_\nu, r) \) is not greater \( N_d = (12)^d e^{\sqrt{d-1}}. \)

**Definition 4.** We will use notation \( Z(x_\nu, r, N_d) \) for a set of points \( \{ x_\nu \} \in X \) which satisfies the properties 1)- 3) from the last Lemma and we will call such set a \( (r, N_d) \)-lattice in \( X \).

**Definition 5.** We will use notation \( Z_G(g_\nu, r, N_d) \) for a set of elements \( \{ g_\nu \} \) of the group \( G \) such that the points \( \{ x_\nu = g_\nu \cdot o \} \) form a \( (r, N_d) \)-lattice in \( X \) (here \( o \in X \) is the origin of \( X \)). Such set \( Z_G(g_\nu, r, N_d) \) will be called a \( (r, N_d) \)-lattice in \( G \).

**Theorem 4.2.** For any \( (r, N_d) \)-lattice \( Z_G(g_\nu, r, N_d) \subset G \), any \( m > d/p \) there exists constants \( C(X, N_d) \) and \( C(X, N_d, m) \) such that for any \( \omega > 0 \) and any \( 1 \leq p < q \leq \infty \) the following inequalities hold true

\[
\| f \|_q \leq C(X) r^{d/p} \sup_{g \in G} \left( \sum_i (\| f(g_i \cdot o) \|)^{p'} \right)^{1/p},
\]

\[
C(X, m) r^{d/q-p} (1 + (r\omega)^m) \| f \|_p,
\]

for all \( f \in E^*_{\omega}(X) \).

In particular the following embeddings hold true

\[
E^*_{\omega}(V) \subset L_q(X), \; V = \{ V_1, ..., V_d \},
\]

for any \( 1 \leq p \leq q \leq \infty \).

**Proof.** In what follows we fix a \( r > 0 \) and consider a cover of \( X \) of finite multiplicity \( N_d \) by balls \( \{ B(y_i \cdot o, r) \} \) which was constructed in Lemma 4.1. First we are going to use the following inequality

\[
|\psi(y)| \leq C_1(d, m) \sum_{0 \leq j \leq m} r^{d-j/p} \| \psi \|_{W^{d}_p(B(y_i \cdot o, r))}, \; m > d/p,
\]

where \( y \in B(y_i \cdot o, r/2), \psi \in C^\infty(B(y_i \cdot o, r)). \) From the inequality (4.6) we have for \( 1 \leq p < \infty \)

\[
\left( r^{d/p} \| f(g_i \cdot o) \| \right)^p \leq C_2(d, m) \sum_{0 \leq j \leq m} r^{jp} \| f \|_{W^{d}_p(B(y_i \cdot o, r))}^p, \; m > d/p,
\]
Applying this inequality to a translated function 
\begin{equation}
(4.11)
\end{equation}
of the measure 
\begin{equation}
(4.12)
\end{equation}
we obtain for any \(1 \leq r, N \) such that for any \((r, N_d)\)-lattice the following inequality holds true for \(1 \leq p < \infty\)
\[
\left( \sum_i \left( r^{d/p} |f(g_i \cdot o)| \right)^p \right)^{1/p} \leq C(X, N_d, m) \left( \|f\|_p + r^m \sum_{j=1}^{m} \|f\|_{W^p_j(X)} \right).
\]
Since the vector fields \(V_1, ..., V_d\), form a basis of the tangent space at every point of \(X\) the Sobolev norm \(\|f\|_{W^p_j(X)}\) for every \(k \in \mathbb{N}\) is equivalent to the norm
\[
\|f\|_p + \sum_{j=1}^{m} \sum_{0 \leq k_1, ..., k_j \leq d} \|V_{k_1}...V_{k_j}f\|_p, 1 \leq p < \infty.
\]
We obtain
\[
\left( \sum_i \left( r^{d/p} |f(g_i \cdot o)| \right)^p \right)^{1/p} \leq C(X, N_d, m) \left( \|f\|_p + r^m \sum_{j=1}^{m} \|V_{k_1}...V_{k_j}f\|_p \right), m > d/p.
\] (4.8)
Because every \(V_k, k = 1, ..., d\), is a generator of a one-parameter isometric group of bounded operators in \(L_p(X)\), the following interpolation inequality holds true
\[
r^l \|V_k^l f\|_p \leq a^{m-l} r^m \|V_k^m f\|_p + c_m a^{-l} \|f\|_p, 1 \leq p < \infty,
\] (4.9)
for any \(1 \leq l < m, a, r > 0\). The last two inequalities imply the following estimate
\[
\left( \sum_i \left( r^{d/p} |f(g_i \cdot o)| \right)^p \right)^{1/p} \leq C(X, N_d, m) \left( \|f\|_p + r^m \sum_{0 \leq k_1, ..., k_m \leq d} \|V_{k_1}...V_{k_m}f\|_p \right), m > d/p.
\] (4.10)
For \(f \in E^∞_p(\mathcal{V})\) it gives for \(1 \leq p < \infty\)
\[
\left( \sum_i \left( r^{d/p} |f(g_i \cdot o)| \right)^p \right)^{1/p} \leq C(X, N_d, m) (1 + (r\omega)^m) \|f\|_p, m > d/p.
\] (4.11)
Applying this inequality to a translated function \(f(h \cdot x), h \in G\), and using invariance of the measure \(dx\) we obtain for \(f \in E^∞_p(\mathcal{V})\)
\[
\sup_{h \in G} \left( \sum_i \left( r^{d/p} |f(hg_i \cdot o)| \right)^p \right)^{1/p} \leq C(X, m) (1 + (r\omega)^m) \|f\|_p.
\] (4.12)
\[
C(X, m) (1 + (r\omega)^m) \sup_{h \in G} \|f(h \cdot x)\|_p = C(X, m) (1 + (r\omega)^m) \|f\|_p.
\]
We introduce the following neighborhood of the identity in the group $G$

$$Q_r = \{ g \in G : g \cdot o \in B(o, r) \}.$$ 

According to the known formula

$$\int_X f(x)dx = \int_G f(g \cdot o)dg, f \in C_0(X),$$

we have for the characteristic function $\chi_B$ of the ball $B(o, r)$

$$r^d \approx \int_{B(o, r)} dx = \int_X \chi_B(x)dx = \int_G \chi_B(g \cdot o)dg = \int_{Q_r} dg.$$ 

Thus, since every ball in our cover is a translation of the ball $B(o, r)$ by using $G$-invariance of the measure $dx$, we have for any $f \in L^q(X), 1 \leq q \leq \infty$,

$$\int_X |f(x)|^q dx \leq N_d \sum_i \int_{B(g_i \cdot o, r)} |f(x)|^q dx \leq N_d \sum_i \int_{B(o, r)} |f(g_i \cdot y)|^q dy =$$

$$C(X, N_d) \int_{Q_r} \sum_i |f(g_i h \cdot o)|^q dh \leq C(X, N_d) r^d \sup_{g \in G} \sum_i |f(g_i g \cdot o)|^q,$$

where $d = \text{dim}X$. After all we have

$$\|f\|_q \leq C(X, N_d) r^{d/q} \sup_{g \in G} \left( \sum_i \left( |f(g_i g \cdot o)| \right)^q \right)^{1/q} f \in L^q(X), 1 \leq q \leq \infty.$$ 

Next, using the inequality

$$\left( \sum a_i^q \right)^{1/q} \leq \left( \sum a_i^p \right)^{1/p},$$

which holds true for any $a_i \geq 0, 1 \leq p \leq q \leq \infty$, we obtain the following inequality

$$\|f\|_q \leq C(X, N_d) r^{d/q} \sup_{g \in G} \left( \sum_i \left( |f(g_i g \cdot o)| \right)^q \right)^{1/q} \leq$$

$$C(X, N_d) r^{d/q} \sup_{g \in G} \left( \sum_i \left( |f(g_i g \cdot o)| \right)^p \right)^{1/p} =$$

$$\left( \sum_i \left( r^{d/p} |f(g_i g \cdot o)| \right)^p \right)^{1/p}. \tag{4.13}$$

From the inequalities (4.12) and (4.13) and the observation, that for the element $g = g_i^{-1}hg_i$ the expression

$$\sum_i \left( r^{d/p} |f(g_i g \cdot o)| \right)^p$$

becomes the expression

$$\sum_i \left( r^{d/p} |f(hg_i \cdot o)| \right)^p,$$

we obtain the Theorem 4.2.

As a consequence we have the following Corollary.
Corollary 4.1. For any $1 \leq p \leq q \leq \infty$ the following embeddings hold true

$$E^\omega_p(V) \subset E^\omega_q(V).$$

In particular, the spaces $E^\omega_q(V)$ are not trivial at least if $2 \leq q \leq \infty$ and $\omega \geq \|\rho\|$.

Here the notation $\rho \in a^*$ means the half-sum of all positive restricted roots.

Proof. Since $E^\omega_p(V)$ is invariant under every operator $V_i, 1 \leq i \leq d$, it is enough to show that if $f \in E^\omega_p(V)$, then for any $1 \leq j \leq d, k \in \mathbb{N}$

$$\|V_j^k f\|_q \leq \omega^k \|f\|_q.$$

We are using the same arguments as in the proof of the Theorem 2.2. Namely, since $f \in E^\omega_p(V)$ we have for any $z \in \mathbb{C}$

$$\|e^{zV_j} f\|_q = \sum_{r=0}^{\infty} \left( z^r V_j^r f \right) / r! \leq e^{\|z\|\omega} \|f\|_p.$$

As it was shown in the proof of the Theorem 2.1 it implies that for any functional $h$ on $L^q(X), 1 \leq p \leq \infty$, the scalar function

$$F(z) = \langle h, e^{zV_j} f \rangle,$$

is an entire function of exponential type $\omega$ which is bounded on the real axis $\mathbb{R}^1$ by the constant $\|h\|\|f\|_p$. The classical Bernstein inequality gives

$$\sup_t |\langle h, e^{tV_j} V_j^k f \rangle| = \sup_t \left( \frac{d}{dt} \right)^k \langle h, e^{tV_j} f \rangle \leq \omega^k \|h\|\|f\|_q, m \in \mathbb{N}.$$

When $t = 0$ we obtain

$$|\langle h, V_j^k f \rangle| \leq \omega^k \|h\|\|f\|_q.$$

Choosing $h$ such that $\|h\| = 1$ and

(4.14) $$\langle h, V_j^k f \rangle = \|V_j^k f\|_q$$

we get the inequality

(4.15) $$\|V_j^k f\|_q \leq \omega^k \|f\|_q, k \in \mathbb{N}.$$

The Corollary 4.1 is proved. \qed

Inequalities (4.4) and (4.5) are known as Nikolskii inequalities. Now we are ready to prove a generalization of another inequality which is also attributed to Nikolskii.

Theorem 4.3. There exists a constant $C(X)$ such that for any $1 \leq p \leq q \leq \infty$ the following inequality holds true for all $f \in E^\omega_p(V)$

(4.16) $$\|f\|_q \leq C(X)\omega^{\frac{d}{p} - \frac{d}{q}} \|f\|_p, d = dimX.$$

Proof. The Theorem 4.2 imply that for any $(r, N_d)$-lattice, any $m > d/p$ there exists a constant $C(X, N_d, m) > 0$ such that for any $\omega > 0$ and any $1 \leq p \leq q \leq \infty$ the following inequality holds true

$$\|f\|_q \leq C(X, N_d, m) \omega^{d/q - d/p} (1 + (r\omega)^m) \|f\|_p,$$

for all $f \in E^\omega_p(X)$. We make the substitution $t = r\omega$ into this inequality to obtain
\[ \|f\|_q \leq C(X,N_d,m)\omega^{d/p-d/q} \left( t^{d/q-d/p}(1 + t^m) \right) \|f\|_p = C(X,N_d,m)\eta_{p,q}(t)\omega^{d/p-d/q}\|f\|_p, \quad m > d/p, \]

(4.17)

where

\[ \eta_{p,q}(t) = t^{d/q-d/p}(1 + t^m), \quad t \in (0, \infty). \]

Since \( m \) can be any number greater than \( d/p \) and \( p \geq 1 \), we fix the number \( m = 2d \). At the point

\[ t_{d,p,q} = \frac{\alpha}{2d - \alpha} \in (0, 1), \]

(4.18)

where \( 0 < \alpha = d/p - d/q < 1 \), the function \( \eta_{p,q} \) has its minimum, which is

\[ \eta_{p,q}(t_{d,p,q}) = \frac{1}{(1 - \beta)^{1 - \beta} \beta^\beta} \leq 2, \]

where \( \beta = \alpha/2d \). Thus, if we would substitute this \( t_{d,p,q} \) into (4.17) we would have for any \( \omega > 0 \) and any \( 1 \leq p \leq q \leq \infty \) the desired inequality

\[ \|f\|_q \leq 2C(X,N_d)\omega^{\frac{d}{p} - \frac{d}{q}} \|f\|_p, \quad d = \dim X, \quad 1 \leq p \leq q \leq \infty. \]

We consider the following distribution on \( C^\infty_0(B(x_\nu,r)) \),

\[ (4.20) \quad \Phi_\nu(\varphi) = \int_{K_\nu} \varphi d\mu_\nu, \]

where \( \varphi \in C^\infty_0(B(x_\nu,r)) \). As a compactly supported distribution of order zero it has a unique continuous extension to the space \( C^\infty(B(x_\nu,r)) \).

We say that a family \( \Phi = \{\Phi_\nu\} \) is uniformly bounded, if there exists a positive constant \( C_\Phi \) such that

\[ (4.21) \quad |K_\nu| \leq C_\Phi \]

for all \( \nu \).
where
\( f(4.26) \)
The following inequality holds true
\( C \) constant
\( c(4.22) \)

For any family \( \{\Phi_\nu\} \) of distributions of type (4.20) with supports in \( B \) the following inequalities hold true
\( (4.23) \)
\( (\sum_\nu |\Phi_\nu(f)|^p)^{1/p} \leq C_1 r^{-d/p} \left( \|f\|_p + \sum_{1 \leq k_1, \ldots, k_m \leq d} \|V_{k_1} \cdots V_{k_m} f\|_p \right) \),
\( \|f\|_p \leq \)
\( (4.24) \)
\( C_2 \left\{ \frac{r^{d/p}}{(p)} \left( \sum_\nu |K_\nu|^{-1} |\Phi_\nu(f)|^p \right)^{1/p} + c_\Phi^{-1} r^m \sum_{1 \leq k_1, \ldots, k_m \leq d} \|V_{k_1} \cdots V_{k_m} f\|_p \right\} \).

The following theorem was proved in [14] in the case \( p = 2 \). In what follows we just sketched the proof.

\( \text{Proof.} \) The inequality (4.23) follows from the definitions of the distributions \( \Phi_\nu \) and the inequality
\( (4.25) \)
\( |\psi(y)| \leq C_0(d, k) \sum_{0 \leq j \leq k} r^{j-d/p} \|\psi\|_{W^j_p(B(x_\nu, r))}, k > d/p, \)
where \( y \in B(x_\nu, r/2), \psi \in C^\infty(B(x_\nu, r)) \).

To prove (4.24) we show that for any \( k > d/p \) there exists a constant \( C = C(d, k) > 0 \), such that for any ball \( B(x_\nu, r), x_\nu \in M \), any distribution \( \Phi_\nu \) of type (4.20) the following inequality holds true
\( (4.26) \)
\( \|f - |K_\nu|^{-1} \Phi_\nu(f)\|_{L_p(B(x_\nu, r/2))} \leq C(d, k) \sum_{1 \leq |\alpha| \leq k} r^{|\alpha|} \|\partial^{\alpha} f\|_{L_p(B(x_\nu, r))} \),
where \( f \in W^k_p(M), k > d/p, 1 \leq p \leq \infty, \) and \( \partial^j f \) is a partial derivative of order \( j \).

To prove the inequality (4.26) we make use of the Taylor series. For any \( f \in C^\infty(B(x_\nu, r/2)) \), every \( x, y \in B(x_\nu, r/2) \) we have the following
\( f(x) = f(y) + \sum_{1 \leq |\alpha| \leq k} \frac{1}{|\alpha|!} \partial^{\alpha} f(y)(x - y)^\alpha + \sum_{|\alpha| = k} \frac{1}{k!} \int_0^\eta t^{k-1} \partial^{\alpha} f(y + t\theta) \theta^\alpha dt, \)
where \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d), \alpha = (\alpha_1, \ldots, \alpha_d), (x - y)^\alpha = (x_1 - y_1)^{\alpha_1} \cdots (x_d - y_d)^{\alpha_d}, \eta = \|x - y\|, \theta = (x - y)/\eta. \)
We integrate each term over compact \( K_\omega \subset B(x_\nu, r) \) against \( d\mu_\nu(y) \), where \( d\mu_\nu \) is the measure on \( K_\nu \). After all we obtain

\[
\|f - |K_\nu|^{-1}\Phi_\nu(f)\|_{L_p(B(x_\nu, r/2))} \leq
\]

\[
C(k, d)|K_\nu|^{-1} \sum_{1 \leq |\alpha| \leq k-1} \left( \int_{B(x_\omega, r/2)} \left( \int_{K_\nu} |\partial^{\alpha} f(y)(x - y)^{\alpha}| \, d\mu_\nu(y) \right)^p \, dx \right)^{1/p} +
\]

\[
C(k, d)|K_\nu|^{-1} \sum_{|\alpha| = k} \left( \int_{B(x_\omega, r/2)} \left( \int_{K_\nu} \left| \int_0^1 \partial^{k-1} f(y + t\hat{\alpha}) \, dt \right| \, d\mu_\nu(y) \right)^p \, dx \right)^{1/p}.
\]

(4.27)

By using the Minkowski inequality and the estimate (4.25) we obtain (4.26). Summation over all \( \nu \) gives the inequality (4.24).

The Theorems 4.2, 4.4 and the Bernstein inequality (1.5) imply the following Plancherel-Polya inequalities.

**Theorem 4.5.** For any given \( \omega > 0, C_\Phi > 0, c_\Phi > 0, m = 0, 1, 2, \ldots \), there exist positive constants \( C, c_1, c_2 \), such that for every \( \rho \)-lattice \( Z(x_\nu, r, N_d) \) with \( 0 < r < (C\omega)^{-1} \), every family of distributions \( \{\Phi_\nu\} \) of the form (4.20) with properties (4.21), (4.22) and every \( f \in E_\omega^m(X) \) the following inequalities hold true

\[
(4.28) \quad c_1 \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p} \leq r^{-d/p} \|f\|_p \leq c_2 \left( \sum_\nu |\Phi_\nu(f)|^p \right)^{1/p}.
\]

In the case of Euclidean space when \( \Phi_\nu = \delta_{x_\nu} \) and \( \{x_\nu\} \) is the regular lattice the above inequality represents the classical Plancherel-Polya inequalities.

The notation \( l^p_\nu \) will be used for a linear subspace of all sequences \( \{v_\nu\} \) in \( l_p \) for which there exists a function \( f \) in \( E_\nu^m(X) \) such that

\[
\Phi_\nu(f) = v_\nu, \nu \in \mathbb{N}.
\]

In general \( l^p_\nu \neq l_p \).

**Definition 6.** A linear reconstruction method \( R \) is a linear operator

\[
R : l^\omega_\nu \to E_\nu^\omega(X)
\]

such that

\[
R : \{\Phi_\nu(f)\} \to f.
\]

The reconstruction method is said to be stable, if it is continuous in topologies induced respectively by \( l_p \) and \( L_p(X) \).

The following result is a consequence of the Plancherel-Polya inequalities and the fact that \( E_\nu^m(X), 1 \leq p \leq \infty \), is a linear space.

**Theorem 4.6.** For any given \( \omega > 0, C_\Phi > 0, c_\Phi > 0, m = 0, 1, 2, \ldots \), there exist positive constants \( C, c_1, c_2 \), such that for every \( \rho \)-lattice \( Z(x_\nu, r, N_d) \) with \( 0 < r < (C\omega)^{-1} \), every family of distributions \( \{\Phi_\nu\} \) of the form (4.20) with properties (4.21), (4.22) and every \( f \in E_\nu^m(X) \) the following statements hold true.
1) every function \( f \) from \( E^p_\omega(X), 1 \leq p \leq \infty \), is uniquely determined by the set of samples \( \{ \Phi_\nu(f) \} \);

2) reconstruction method \( R \) from a set of samples \( \{ \Phi_\nu(f) \} \)

\[ R : \{ \Phi_\nu(f) \} \to f \]

is stable.

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