The ordinary Levy motion is a random process whose stationary independent increments are statistically self-affine and distributed with a stable probability law characterized by the Levy index $\alpha$, $0 < \alpha < 2$. The divergence of statistical moments of the order $q > \alpha$ leads to an important role of the finite sample effects. The objective of this paper is to study the influence of these effects on the self-affine properties of the ordinary Levy motion, namely, on the "$1/\alpha$ laws", that is, time dependence of the $q$-th order structure function and of the range. Analytical estimates and simulations of the finite sample effects clearly demonstrates three phenomena: spurious multi-affinity of the Levy motion, strong dependence of the structure function on the sample size at $q > \alpha$, and pseudo-Gaussian behavior of the second-order structure function and of the normalized range. We discuss these phenomena in detail and propose the modified Hurst method for empirical rescaled range analysis.

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I. INTRODUCTION.

By Levy motions, one designates a class of random functions, which are a natural generalization of the Brownian motion, and whose increments are stationary, statistically self-affine and stably distributed in the sense of P. Levy [1]. Two important subclasses are (i) $\alpha$-stable processes, or the ordinary Levy motion (oLM), which generalizes the ordinary Brownian motion, or the Wiener process, and whose increments are independent, and (ii) the fractional Levy motion, which generalizes the fractional Brownian motion and has an infinite span of interdependence.

The theory of processes with independent increments was developed beginning from the Bachelier’s paper [2] concerning Brownian motion. However, the rigorous construction of this process and studies of properties of its trajectories were undertaken by Wiener [3]. The modern presentation of the general theory of processes with independent increments is contained in [4]. The theory of the processes with independent increments possessing stable distributions has begun its history from the already cited work [1] and, later on, was developed by other prominent mathematicians. In particular, the properties of extremes of $\alpha$-stable symmetric processes were studied in [5]. The geometric properties of their trajectories were considered in [6]. The monograph [7] contains a modern presentation of the theory of $\alpha$-stable processes.

The Levy random processes play an important role in different areas of application for at least two reasons.

The first one is that the Levy motion can be considered as a generalization of the Brownian motion. Indeed, the mathematical foundation of the generalization are remarkable properties of stable probability laws. From the limit theorems point of view, the stable distributions are a generalization of widely used Gaussian distribution. Namely, stable distributions are the limit ones for the distributions of (properly normalized) sums of independent identically distributed (i.i.d.) random variables [8]. Therefore, these distributions (like the Gaussian one) occur, when the evolution of a physical system or the result of an experiment are determined by the sum of a large number of identical independent random factors. An important distinction of stable probability densities is the power law tails decreasing as $|x|^{-1-\alpha}$, $\alpha$ is the Levy index, $0 < \alpha < 2$. Hence, the distribution moments of the order $q \geq \alpha$ diverge. In particular, stably distributed variables possess a non-finite variance.

The second reason for ubiquity of the Levy motions is their remarkable property of scale - invariance. From this point of view the Levy motions (like the Brownian ones) belong to the so - called fractal random processes. Indeed, the objects in nature rarely exhibit exact self - similarity ( like the Von Koch curve), or self - affinity. On the contrary, these properties have to be understood in a probabilistic sense [9,10]. The random fractals are believed to be widely spread in nature. A coastline is a simple example of statistically self - similar object [9], as well as the spot of the Chernobyl contamination in the nearest zone [11]. On the contrary, the trace of the Brownian motion is statistically self - affine. Several numerical algorithms were developed in order to simulate fractional Brownian motion [10,12 - 15]. They allow one to model many highly irregular natural objects, which can be viewed as random fractals [10]. The traces of the Levy motions are also statistically self - affine, therefore, one may expect that they are also suited for modeling and studies of natural random fractals.
The stable distributions and the Levy, or Levy-like, random processes are widely used in different areas, where the phenomena possessing scale invariance (in a probabilistic sense) are viewed or, at least, can be suspected, e.g., in economy [16-19], biology and physiology [20], turbulence [21] and chaotic dynamics [22], solid state physics [23], plasma physics [24], geophysics [25] etc. In this respect, the problems connected with experimental data processing are of great importance. It is also necessary to develop different numerical algorithms which allow one to simulate Levy motion with the given statistical properties. This, in turn, allows one to improve the methods aimed at analysis and interpretation of experimental data. Recently, three models of the oLm were proposed [26]. They can be interpreted as "difference schemes" to approximate the evolution equation for the distribution density of the ordinary Levy motion. In our paper we employ the different approximation, which is based on using Gnedenko limit theorem along with the inversion method for generating random variables. With the help of this approximation we study the consequences of self-affinity of the motion, namely, the time dependence of the structure functions and of the range. We show that the finiteness of the sample size plays an essential role when the consequences of self-affinity are considered. We demonstrate both analytically and numerically, that the finiteness of the sample size violates self-affinity, thus giving rise to spurious multi-affinity of the oLm. Further, the second order structure function and the normalized range (just these quantities are widely used in processing a huge number of various experimental data [12,27]) possess spurious, "pseudo-Gaussian" time-dependence. This circumstance allows one to suggest that at estimating the second order structure function and normalized range from experimental data the "Levy nature" of them can be easily masked. In order to avoid pseudo-Gaussianity when studying the range, we propose a modified Hurst method for rescaled range analysis.

The paper is organized as follows. In Sec. 2 we discuss the property of self-affinity and its consequences. In Sec. 3 we propose a model for numerical simulation of the oLm. In Sec. 4 the effects of a finite sample size are studied. In Sec. 5 we present numerical results. Finally, the conclusions are exposed in Sec. 6.

II. SELF-AFFINITY OF THE ORDINARY LEVY MOTION

Let us proceed with the self-affine properties of the ordinary Levy motion denoted below as $L_\alpha(t)$. In this paper we restrict ourselves by symmetric stable distributions. The characteristic function of the oLm increments is [4,7]

$$\hat{p}_{\alpha,D}(k,\tau) \equiv \langle \exp[ik(L_\alpha(t+\tau) - L_\alpha(t))] \rangle = \exp(-D|k|^\alpha \tau)$$  (1)

Here $\alpha$ is the Levy index, $0 < \alpha \leq 2$, and $D > 0$ is a scale parameter. The probability density

$$p_{\alpha,D}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx)\hat{p}_{\alpha,D}(k,t)$$  (2)

is expressed in terms of elementary functions in two cases:

(i) ordinary Brownian motion, $L_\alpha(t) \equiv B(t)$, which has $\alpha = 2$ and the probability density
\[ p_{2,D}(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp \left[ -\frac{x^2}{4Dt} \right] \]

and

(ii) ordinary Cauchy motion, which has \( \alpha = 1 \) and the probability density

\[ p_{1,D}(x,t) = \frac{Dt/\pi}{D^2t^2 + x^2} \]

At \( |x| \to \infty \) the probability densities of the oLm have power law tails,

\[ p_{\alpha,D}(x,t) \propto \frac{Dt}{|x|^{1+\alpha}} \]

The increments of the oLm are stationary in a narrow sense,

\[ L_\alpha(t_1 + \tau) - L_\alpha(t_2 + \tau) \overset{d}{=} L_\alpha(t_1) - L_\alpha(t_2), \] (3)

and self-affine with the parameter \( H = 1/\alpha \), that is, for an arbitrary \( \kappa > 0 \)

\[ L_\alpha(t + \tau) - L_\alpha(t) \overset{d}{=} \left\{ \kappa^{-1/\alpha} [L_\alpha(t + \kappa \tau) - L_\alpha(t)] \right\}, \] (4)

where \( \overset{d}{=} \) implies that the two random functions have the same distribution functions.

The exponent \( 1/\alpha \), by analogy with the theory of fractional Brownian motion [28], is named

the Hurst index of the ordinary Levy motion.

We consider two corollaries of Eqs. (1) - (4), which, again by analogy with the definitions
of Ref. [28], may be called "1/\alpha laws" for the structure function and for the range.

We first consider the structure function of the oLm. A "1/\alpha law" for the structure
function can be stated as follows:

for all \( 0 < \alpha < 2 \)

\[ S_q(\tau, \alpha) = \langle |L_\alpha(t + \tau) - L_\alpha(t)|^q \rangle = \left\{ \begin{array}{ll}
\tau^{q/\alpha} V(q; \alpha), & 0 \leq q < \alpha \\
\infty, & q \geq \alpha
\end{array} \right. , \] (5)

where

\[ V(\mu; \alpha) = \int_{-\infty}^{\infty} dx_2 |x_2|^\mu \int_{-\infty}^{\infty} \frac{dx_1}{2\pi} \exp(-ix_1x_2 - |x_1|^{\alpha}), \] (6)

whereas for \( \alpha = 2 \) (ordinary Brownian motion)

\[ S_q(\tau; 2) = \tau^{q/2} V(q; 2) \] (7)

for an arbitrary \( q \).

Equations (5) - (7) have a direct physical consequence for description of an anomalous
diffusion. Indeed, for the ordinary Brownian motion the characteristic displacement \( \Delta x(\tau) \)
of a particle may be written in terms of the second order structure function as
\[ \Delta x(\tau) = S_2^{1/2}(\tau; 2) = \sqrt{2} \tau^{1/2}; \]  

where the prefactor \( \sqrt{2} \) is simply \( V^{1/2}(2; 2) \). One may note from Eqs.(6), (7) that, for the normal diffusion

\[ S_{\alpha}^{1/q}(q; 2) \propto \tau^{1/2}, \]

at any \( q \) and, thus any order of the structure function may serve as a measure of a normal diffusion rate:

\[ \Delta x(\tau) \approx S_{\alpha}^{1/q}(q; 2) \propto \tau^{1/2}, \]

if one is interested in time-dependence of the characteristic displacement, but not in the value of the prefactor. We remind that usually just the time-dependence, but not the prefactor, serves as an indicator of normal or anomalous diffusion [23]. In analogy with Eqs. (9), (10) it follows from Eqs. (5), (6) that the quantity \( S_{\alpha}^{1/q}(\tau; \alpha) \) at \( 0 < \alpha < 2 \) and any \( q < \alpha \) can serve as a measure of an anomalous diffusion rate:

\[ \Delta x(\tau) \approx S_{\alpha}^{1/q}(q; \alpha) \propto \tau^{1/\alpha}, \quad 0 < q < \alpha < 2. \]

Here we have the case of a fast anomalous diffusion, or hyperdiffusion.

The second corollary of Eqs. (1) - (4) is for the range of the oLm. A ”1/\( \alpha \) law” for the range can be stated as follows:

\[ R(\tau) = \sup_{0 \leq t \leq \tau} L_\alpha(t) - \inf_{0 \leq t \leq \tau} L_\alpha(t) \overset{d}{=} \tau^{1/\alpha} R(1). \]

For the ordinary Brownian motion \( \tau^{1/2} R(\tau) \) has a distribution independent of \( \tau \) [29]. For the oLm, \( 0 < \alpha < 2 \), the statistical mean of the range, \( \langle R(\tau) \rangle \), behaves as \( \tau^{1/\alpha} \) at \( 1 < \alpha < 2 \), and turn to infinity at \( 0 < \alpha \leq 1 \).

Both ”1/\( \alpha \) law” are studied in numerical simulation in Sec. 5. However, at the end of this Section we point once more to an important property of the moments of stable distributions with the Levy index \( \alpha < 2 \): the moments of the order \( q \geq \alpha \) diverge. This property, in turn, manifests itself in divergence of the \( q \) -th order structure function at \( q \geq \alpha \), see Eq.(5), and of the mean of the range at \( \alpha \leq 1 \). However, at experimental data processing both quantities are finite due to the finiteness of a sample size. In fact, one may expect that in the case of the Levy motion the finiteness of a sample size has a stronger influence on the results than in the case of the Brownian motion. Therefore, the estimates of such influence are needed. We study this problem in Sec. 4 before discussing results of numerical simulation.

### III. A SIMPLE WAY TO APPROXIMATE ORDINARY LEVY MOTION.

The process of constructing approximation to the oLm can be divided into two steps.

**Step 1.** At the first step we generate random sequence of i.i.d. random variables possessing stable probability law. These variables play the role of increments of the oLm having the Levy index \( \alpha \), \( 0 < \alpha < 2 \). The value \( \alpha = 2 \) corresponds to the ordinary Brownian motion, hence in this case the sequence of independent increments is generated with the use of a
standard Gaussian generator. Since in Ref. [12] the sequence generated by the Gaussian generator is called "approximate discrete-time white Gaussian noise", we call the sequence generated at $0 < \alpha < 2$ "approximate discrete-time white Levy noise".

We generate approximate white Levy noise possessing characteristic function

$$\hat{p}_{\alpha,D}(k) = \langle \exp(ikx) \rangle = \exp(-D|k|^\alpha).$$

At $0 < \alpha < 2$ they have power law asymptotic tails [8],

$$p_{\alpha,D}(x) \propto D \frac{\Gamma(1 + \alpha) \sin(\pi \alpha/2)}{\pi |x|^{1+\alpha}}, \quad x \to \pm \infty.$$ (14)

In the literature there exist different algorithms for generating random variables distributed with stable probability law. We only mention two recently proposed schemes, which use the combinations of random number generators [30] and the family of chaotic dynamical systems with broad probability distributions [31], respectively. However, we believe that the ways of generating stably distributed variables and, then the Levy motion, are not exhausted, and various simulation models are needed, each of them may appear to be useful when studying some particular problem. In this paper we propose a simple approximation based on the Gnedenko limit theorem along with the method of inversion. Indeed, among the methods of generating random sequence with the given probability law $F(x)$ the method of inversion seems most simple and effective [32]. However, it is well-known fact that its effectiveness is limited by the laws possessing analytic expressions for $F^{-1}$, hence the direct application of the method of inversion to the stable law is not expedient. In this connection, it is natural to exploit an important property of stable distributions. Namely, such distributions are limiting for those of properly normalized sums of i.i.d. random variables [8]. To be more concrete, we generate the needed random sequence in two steps. At the first one we generate an "auxiliary" sequence of i.i.d. random variables $\{\xi_j\}$, whose distribution density $F'(x)$ possesses asymptotics having the same power law dependence as the stable density with the Levy index has, see Eq.(14). However, contrary to the stable law, the function $F(x)$ is chosen as simple as possible in order to get analytic form of $F^{-1}$. For example,

$$F(x) = \frac{1}{2(1 + |x|^\alpha)}, \quad x < 0,$$ (15)

$$F(x) = 1 - \frac{1}{2(1 + x^\alpha)}, \quad x > 0.$$  

Then, the normalized sum is estimated,

$$X = \frac{1}{am^{1/\alpha}} \sum_{j=1}^{m} \xi_j,$$ (16)

where

$$a = \left( \frac{\pi}{2\Gamma(\alpha) \sin(\pi \alpha/2)} \right)^{1/\alpha}.$$
According to the Gnedenko theorem on the normal attraction basin of the stable law \[8\], the distribution of the sum (16) is then converges to the stable law with the characteristic function (13) and \( D = 1 \). It is reasonable to generate random variables having stable distribution with the unit \( D \), with a consequent rescaling, if necessary. Repeating \( N \) times the above procedure, we get a sequence of i.i.d. random variables \( \{X(t)\}, \quad t = 1, 2, ..., N \). This is an approximate discrete-time white Levy noise.

In the top of Fig.1 the probability densities \( p(x) \) for the members of the sequence \( \{X(t)\} \) (\( m = 30 \) in Eq.(16)) are depicted by black points for (a) \( \alpha = 1.0 \), and (b) \( \alpha = 1.5 \). The functions \( p_{\alpha,1}(x) \) obtained with the inverse Fourier transform, see Eq.(13), are shown by solid lines. In the bottom of Fig.1 the black points depict asymptotics of the same probability densities in log-log scale. The solid lines show the asymptotics given by Eq.(14). The examples presented demonstrate a good agreement between the probability densities for the sequences \( \{X(t)\} \) obtained with the use of the numerical algorithm proposed and the densities of the stable laws.

We would like to stress that a certain merit of the proposed model is its simplicity. It is entirely based on classical formulation of one of the limit theorems and can be easily generalized for the case of asymmetric stable distributions. It is also allows one, after some modifications, to speed up the convergence to the stable law. These problems, as well as the comparison with the other algorithms seems to be the subject of a separate paper.

Step 2. With using approximate discrete-time white Levy noise \( X(t) \) the approximation to the oLm is defined by

\[
L_\alpha(t) = \sum_{\tau=1}^{t} X(\tau) \quad (17)
\]

In Fig.2 the approximate white Levy noises obtained with the numerical algorithm proposed are depicted by thin lines at 4 different Levy indexes. The thick lines depict the sample paths, or the trajectories, of the approximation to the oLm. It is clearly seen that with the Levy index decreasing, the amplitude of the noise increases. The large outliers lead to large “jumps” (often named as "Levy flights") on the trajectory.

IV. EFFECTS OF LIMITED SAMPLE SIZE.

In this Section we discuss what is the time dependence of the \( q \)-th order structure function and the range, which are estimated from experimental data.

We first give an estimate for the mode of the maximum value.

Suppose we have a sequence \( \{X(t)\}, \quad t = 1, 2, ..., N \), of i.i.d. random variables possessing stable probability density \( p_{\alpha,1}(x) \) and cumulative probability \( P_{\alpha,1}(X \leq x) = \int_{-\infty}^{x} du p_{\alpha,1}(u) \). Then, \( P_{\alpha,1}^N \), is the probability that all \( N \) terms of the sequence are less than \( x \). This, in turn, implies that \( P_{\alpha,1}^N \) is the probability that the maximum value of \( N \) terms is less than \( x \). Therefore,

\[
\varphi_N(x) = \left( P_{\alpha,1}^N(x) \right)' = NP_{\alpha,1}^{N-1}(x) p_{\alpha,1}(x) \quad (18)
\]
is the probability density of the maximum value in the sample consisting of \( N \) terms. Let \( X_{\text{max}}(N) \) be the mode of maximum value, that is, the most probable maximum value. It obeys an equation

\[
\varphi'(x)\Big|_{x = X_{\text{max}}} = 0
\]

Since \( p_{\alpha,1}(x) \propto x^{-1-\alpha} \) for large \( x \), we may give the following estimate for the mode:

\[
X_{\text{max}}(N) \propto N^{1/\alpha}, \quad 0 < \alpha < 2.
\] (19)

With the help of Eq.(19) we are able to roughly estimate the diverging statistical moment of the sequence \( \{X(t)\} \), \( t = 1, 2, ..., N \), as

\[
\int_0^{X_{\text{max}}(N)} dX X^q p_{\alpha,1}(X) \propto X_{\text{max}}^{q-\alpha} \propto N^{q/\alpha - 1}, \quad q > \alpha.
\] (20)

Now we turn to the structure function. It can be written as

\[
S_q(\tau) = \langle |\Delta L_\alpha(\tau)|^q \rangle = \int_{-\infty}^{\infty} d\Delta L_\alpha |\Delta L_\alpha|^q p_{\alpha,1}(\Delta L_\alpha, \tau),
\] (21)

where \( \Delta L_\alpha(t) \equiv L_\alpha(t + \tau) - L_\alpha(t) \), and the probability density is given by Eqs. (1), (2).

We may introduce a stochastic variable \( \xi \), such that \( \Delta L_\alpha(\tau) = \xi \tau^{1/\alpha} \), and rewrite \( S_q \),

\[
S_q = \tau^{q/\alpha} \int_{-\infty}^{\infty} d\xi \xi^q p_{\alpha,1}(\xi),
\] (22)

where the probability density for a new variable is given by Eqs. (13), (14).

To estimate the integral in Eq.(22), we use Eq.(20) with an important note that here \( N \) is equal to \( T/\tau \), \( T \) is the total length of the sample. Therefore, the integral can be estimated as \( (T/\tau)^{q/\alpha - 1} \), and

\[
S_q \propto \tau T^{q/\alpha - 1}, \quad q > \alpha.
\] (23)

Thus, the effects of a limited sample size manifest itself in Eq.(23), which replaces ”theoretical infinity” in Eq.(5) for \( q > \alpha \). A particular case is the second order structure function, for which we have ”pseudo - Gaussian” relation (see Eq.(8)),

\[
S_{1/2}^{1/2} \propto \tau^{1/2} \] [33]. The linear \( \tau \)- dependence of the \( q \)- th order structure function was found in [34].

Equation (23) points to violation of self - affinity of the oLm. Indeed, self - affinity implies that the - exponent of the structure function depends linearly on \( q \), see Eqs. (4), (5). On the contrary, the rough estimate of the finite sample effects demonstrates that the exponent does not depend on \( q \) at \( q > \alpha \). Thus, in experimental data processing one may expect that the exponent smoothly changes its slope, thus giving rise to a convex curve. Such convexity seems as an indication of multi - affinity, see Ref. [35]. However, in our case this behavior of the exponent is stipulated not by ”intrinsic” reason, but instead by
the influence of a finite size of a sample of the self-affine process. That is why we call this effect "spurious multi-affinity".

Now we consider the finite size effects for the range of the oLm. In the empirical rescaled range analysis, that is, at experimental data processing or in numerical simulation the range of the random process is divided by the standard deviation (that is, the square root of the second moment) for the sequence of increments,

$$\sigma_2 = \left( \frac{1}{\tau} \sum_{t=1}^{\tau} (X(t))^2 \right)^{1/2}$$

after subtraction of a linear trend. This procedure, called the Hurst method, or the method of normalized range, "smoothes" the fluctuations of the range on different segments of time series, and is used in a great variety of applications [27]. H. E. Hurst was the first [36] who has collected large statistical material relating to water levels and other phenomena, which indicates that the observed normalized range do not increase (as it is expected for the ordinary Brownian motion) like the square root of the observational period \( \tau \), but instead like a higher power. However, the Hurst method is not satisfactory for the oLm because of the infinity of the theoretical value of standard deviation. What one should expect when estimating the normalized range of the oLm from the finite sample size? Since \( \langle X^2 \rangle \propto \tau^{2/\alpha-1} \), see Eq. (20), then

$$\sigma_2 \propto \tau^{1/\alpha-1/2}, \quad \text{whereas} \quad R(\tau) \propto \tau^{1/\alpha}, \quad \text{and, thus,}$$

$$\frac{R(\tau)}{\sigma_2} \propto \tau^{1/2}.$$  \hspace{1cm} (24)

Therefore, we conclude, that at the empirical rescaled range analysis the Hurst method gives "spurious", "pseudo-Gaussian" time dependence.

In order to get the correct exponent \(1/\alpha\), and, at the same time, to smooth fluctuations of the range, we propose to modify the Hurst method by exploiting the \(\alpha\)-th root of the \(\alpha\)-th moment instead of standard deviation, that is,

$$\sigma_\alpha = \left( \frac{1}{\tau} \sum_{t=1}^{\tau} |X(t)|^\alpha \right)^{1/\alpha}. \hspace{1cm} (25)$$

Since it has only weak logarithmic divergence with the number of terms in the sum increasing, then one has

$$\left( \frac{R(\tau)}{\sigma_\alpha} \right) \propto \tau^H, \hspace{1cm} (26)$$

where \(H \approx 1/\alpha\) is the Hurst index for the oLm with the Levy index, and the bar denotes averaging over the number of segments (having the length \(\tau\)) of the sample path.

The expediency of using modified Hurst method when studying the Levy motion can be explained as follows. In general case, the Hurst exponent in Eq.(4) contains information not only on the Levy index of the increment distribution, but also on long-time correlations between the increments. In case of the ordinary Levy motion the correlations between non-overlapping increments are absent, and the Hurst index is equal \(1/\alpha\). If correlations exist,
the Hurst index for the Levy motion differ from $1/\alpha$, and this circumstance leads to violation of the "$1/\alpha$ laws" for the structure function and for the range. Therefore, when treating experimental data, it seems expedient to estimate with the use of increment distribution (there are different methods for estimating parameters of stable distributions, see, e.g., [37]) and, then by testing "$1/\alpha$ laws" for the structure function and for the range, to get information about the presence (or absence) of long-time correlations.

In the next Section we verify numerically the self-affine properties and finite sample effects with the help of approximation described in Sec. 3.

V. SELF-AFFINE PROPERTIES AND FINITE SAMPLE EFFECTS IN NUMERICAL SIMULATION

We study numerically $\tau$- and $T$- dependence of the $q$-th order structure function,

$$S_q \propto \tau^{\mu(q)} T^{a(q)},$$

where, as before, is the time argument of the structure function, and $T$ is the sample size. According to the analytical estimates

$$\mu(q) = \begin{cases} q/\alpha, & q < \alpha, \\ 1, & q \geq \alpha, \end{cases}$$

whereas

$$a(q) = \frac{q}{\alpha} - 1,$$

see Eqs. (5) and (23).

We first study $\mu$ vs $q$ in the relation at $q$ less than $\alpha$. In Fig.3 a typical example is depicted by crosses at fixed $q = 1/2$. The $q/\alpha$ curve is shown by primes. One can be convinced himself that the $1/\alpha$ law is well confirmed at $q$ smaller than the smallest Levy index in numerical simulation. Also in the figure $\mu$ vs $\alpha$ is depicted by black points at $q = 2$. It is shown that the second order structure function, being estimated from a finite sample, leads to the spurious "pseudo-Gaussian" behavior. At the clarifying inset we show $S_q$ vs $\tau$ in a log-log scale with $q$ and being fixed. The exponent $\mu$ in the main figure is obtained for the fixed $\alpha$ as a slope of a straight line at the inset.

In Fig. 4a, b we plot $\mu$ vs $q$ for the Levy index 1.2 and 1.7, respectively. The analytical estimate, see Eq. (28), is shown by solid line for $q < \alpha$ and by dotted line for $q > \alpha$. The results of simulation are shown by black points. The arrows indicate the value $q = \alpha$, at which the bend of theoretical curves occur. It is shown that the results of simulation are well fitted by the analytical curves.

Figure 5 demonstrates $S_q$ versus sample size $T$ in a log-log scale. The Levy index $\alpha$ is equal 1.2. The dotted lines indicate $S_q$ vs $T$ at $q = \alpha$ (horizontal line), $q = 2.5\alpha$ and $4\alpha$ (sloping lines). The black points indicate simulation results (4 points for each $q$). As in Fig. 4, we see a good agreement between experimental results and analytical estimates. It implies, that Eq. (23), seeming very rough estimate, nevertheless, accounts for the finite sample size effects for the structure function of the oLm.
Figure 6 demonstrates the application of the modified Hurst method to the sample path of the approximation to the oLm with the Levy index $\alpha = 1$. In Fig. 6a fluctuations of the range (thin curve) and those of $\sigma_\alpha$ (thick curve) are shown for the case, when the total length of the sample is divided into 64 segments, each of $\tau = 16$ lengthwise. Below the variations of the ratio $R/\sigma_\alpha$ are depicted. It is shown that fluctuations of the ratio is much smaller than those of the span. This circumstance justify the use of the ratio in the empirical analysis. In Fig. 6b the rescaled span, see Eq. (26) is depicted versus the time interval by black points in a log - log scale. The slope of the solid line is equal to $H = 0.9$.

In Fig.7 the Hurst index obtained with using Eq.(26) is depicted by crosses, whereas the curve $1/\alpha$ is indicated by primes. By comparing Fig.7 with Fig.3 one can see that the ”$1/\alpha$ law” is better fulfilled for the structure function of the simulated process than for the range. We also note that the same conclusion follows for ”$\tau^H$ laws” for fractional Brownian motion [15]. A more detailed discussion of this problem require the use of the theory of extremes for the processes with stationary increments. This is beyond the scope of our paper. However, we point readers attention on the Hurst index which is obtained with the use of ”traditional” ratio $R/\sigma_2$ and is shown by black points in Fig.7. In accordance with the discussion of Sec.4, it can be clearly seen, that the standard deviation, being used in empirical analysis of the oLm, ”suppresses” the variations of the range, thus giving rise to the spurious value $H \approx 0.5$ for all Levy indexes. On the contrary, only ”smoothes” the variations of the range, thus leading to the (nearly) correct value $H = 1/\alpha$.

VI. CONCLUSIONS.

The ubiquity of the Levy motion rises the problems related to the experimental data processing. In particular, the fat tails of the stable distributions allows one to suggest that the effects of a finite sample size play an important role. To study these effects, we propose an approximation which is based on using Gnedenko limit theorem along with the method of inversion. It allows us to simulate the ordinary Levy motion and study the finite sample effects when estimating structure functions and the range. The results of simulation are in a quantitative agreement with theoretical estimates. In particular, we find, that the second order structure function being estimated from the ordinary Levy motion sample paths as well as the Hurst method both lead to spurious ”pseudo - Gaussian” time dependencies. We also propose to modify the Hurst method of rescaled range analysis in order to avoid ”pseudo - Gaussian” relation.

At the end we note that introduction of correlations into the sequence of i.i.d. stably distributed random variables (with the help of the method used in Ref. [15] for the sequence of i.i.d. Gaussian variables) allows us to study the finite sample effects for fractional Levy motion.

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FIGURE CAPTIONS

Fig. 1. Probability densities (above) and their asymptotics (below) are indicated for the sequences of random variables generated with the use of the proposed numerical algorithm at the Levy indexes (a) $\alpha = 1$, and (b) $\alpha = 1.5$. The probability densities and the asymptotics of the stable laws are indicated by solid lines.

Fig. 2. Stationary sequences (thin lines) and ordinary Levy motion trajectories (thick lines) at the different Levy indexes.

Fig. 3. Plots of the exponent in Eq.(12) versus the Levy index $\alpha$ at $q = 0.5$ (crosses) and $q = 2$ (black points). The $q/\alpha$ curve is depicted by dashed line. At the inset the structure function $S$ versus is shown in a log - log scale at $\alpha = 1, q = 0.5$.

Fig. 4. The exponent $\mu$ (see Eqs. (27), (28)) versus the order $q$ of the structure function for (a) $\alpha = 1.2$ and (b) $\alpha = 1.7$. The analytical estimate, see Eq. (28), is shown by solid line for $q < \alpha$ and by dotted line for $q > \alpha$. The results of simulation are shown by black points. The arrows indicate the value $q = \alpha$, at which bend of theoretical curve occurs.

Fig. 5. Structure function $S$ versus sample size $T$ in a log - log scale for the Levy index $\alpha = 1$ and different values of $q$: $q = \alpha$ (horizontal line), $q = 2.5\alpha$ and $q = 4\alpha$ (sloping lines). The black points indicate simulation results.

Fig. 6. (a) Variations of the range (thin curve), of $\sigma_\alpha$ (thick curve) and of their ratio (below) at the different time intervals for the oLm with $\alpha = 1$. (b) Rescaled range, see Eq. (26), versus time interval in log-log scale (black points). Solid line has a slope $H = 0.9$.

Fig. 7. Plots of the Hurst exponent $H$ vs $\alpha$ estimated with the use of Eq.(16) (crosses) and with the use of the traditional Hurst method (black points). The $1/\alpha$ curve is depicted by dashed line.
