Cauchy problem and multi-soliton solutions for a two-component short pulse system

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Abstract: In this paper, we study the Cauchy problem and multi-soliton solutions for a two-component short pulse system. For the Cauchy problem, we first prove the existence and uniqueness of solution with an estimate of the analytic lifespan, and then investigate the continuity of the data-to-solution map in the space of analytic function. For the multi-soliton solutions, we first derive an \(N\)-fold Darboux transformation from the Lax pair of the two-component short pulse system, which is expressed in terms of the quasideterminant. Then by virtue of the \(N\)-fold Darboux transformation we obtain multi-loop and breather soliton solutions. In particular, one-, two-, three-loop soliton, and breather soliton solutions are discussed in details with interesting dynamical interactions and shown through figures.

Keywords: loop soliton, breather soliton, Darboux transformation, two-component short pulse equation, analytical solution, Cauchy problem, existence and uniqueness.

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1 Introduction

The short pulse equation

\[
-u_{xt} = u + \frac{1}{6}(u^3)_{xx}
\]  \hspace{1cm} (1)

where \(u = u(x, t)\) is a real-valued function, representing the magnitude of the electric field, and the subscripts stand for partial derivatives with respect to \(x\) and \(t\), has attracted much attention in the past decades. This equation was derived as a nonlinear model to describe the propagation of ultra-short pulses in isotropic optical fibers from an approximation to the solutions of the Maxwell’s
equations [1]. It could also be used to construct integrable differential equations associated with pseudospherical surfaces [2]. A numerical analysis reveals that as the pulse length is shortens, equation (1) serves as a better approximation to solving the Maxwell’s equation in comparison with the prediction of the nonlinear Schrödinger (NLS) equation [3]. The short pulse equation (1) is integrable with the Lax pair [4], bi-Hamiltonian structure and infinitely many conservation laws [5]. The Lax pair of the short pulse equation (1) is kind of the WKI-type [6]. Such a kind of Lax pairs, recursion operator, C Neumann and Bargamann constraints to finite dimensional integrable systems, symmetries and transformation to the sine-Gordon equation were studied in [7, 8, 9]. Moreover, a suitable hodograph transformation was found to send the short pulse equation (1) to the well-known Sine-Gordon (SG) equation to get multi-loop solitary wave solutions [10]. Therefore, various solutions to the short pulse equation (1) have been obtained, for instance, its loop and pulse solutions in [10], periodic and solitary wave solutions in [13], two-loop soliton solutions in [14], and bilinear forms, multi-loop solutions, multi-breather and periodic solutions in [11, 12]. The loop soliton solutions to the short pulse equation (1) could also be derived from a Darboux transformation [15].

Similar to the case of the NLS equation [16], there are several different versions to generalize the short pulse equation (1), for instance, the higher-order nonlinearity corrections in [17], and the vector short pulse equations in [18, 19]. Recently, Matsuno proposed a novel multi-component short pulse model [20], in particular, the following two-component short pulse (2SP) system

\[
\begin{align*}
u_{xt} &= u + \frac{1}{2} (uvu_x)_x, \\
v_{xt} &= v + \frac{1}{2} (uvv_x)_x
\end{align*}
\]  

(2)

was addressed from its multi-component model. Actually, the 2SP system (2) was able to be produced from the negative order Wadati-Konno-Ichikawa (WKI) hierarchy in [21, 22], where the Lax pair for the whole WKI hierarchy and algebaric structure with \( r \)-matrix were discussed. The regular Wadati-Konno-Ichikawa (WKI) hierarchy and its Lax representations were discussed in [23] and [6], respectively. Apparently, putting \( u = v \) sends (2) to the short pulse equation (1). The 2SP system (2) is integrable with the following Lax pair [21, 22, 20]

\[
\begin{align*}
\Psi_x &= P \Psi, \\
\Psi_t &= Q \Psi,
\end{align*}
\]  

(3)

where

\[
P = \lambda \begin{pmatrix}
1 & u_x \\
v_x & -1
\end{pmatrix}, \\
Q = \frac{1}{2} \begin{pmatrix}
0 & -u \\
v & 0
\end{pmatrix} + \frac{1}{4\lambda} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix}
uv & uvu_x \\
vuv_x & -uv
\end{pmatrix}.
\]  

(4)

Let

\[
u = q + ir, \\
v = q - ir,
\]  

(5)

then the 2SP system (2) is cast into the following integrable system

\[
q_{xt} = q + \frac{1}{2}, [\nu \nu + r^2)q_t]_x, \\
r_{xt} = r + \frac{1}{2}, [(q^2 + r^2)r_x]_x.
\]  

(6)
One- and two-soliton solutions and breather solutions of equations (2) and (6) have been given in terms of pfaffians by virtue of Hirota’s bilinear method in [20]. If \( r = 0 \) (or \( q = 0 \)), then the system (6) is reduced to the the short pulse equation (1).

In general, once a Lax pair is given one may use it to derive many integrable properties of a nonlinear wave equation such as conservation laws, bi-Hamiltonian structure, Darboux transformation [24, 25, 26, 27, 28, 29] etc. On the other hand, the study of analyticity for nonlinear wave equations is another important field in the theory of partial differential equations. For instance, the hydrodynamics of Euler equations was initiated by Ovsyannikov [38, 39] and later developed with a further study in [34, 36, 37, 40] and in [30, 31] where the approach is based on a contraction type argument in a suitable scale of Banach spaces. The analyticity of the Cauchy problem for the two-component Camassa-Holm shallow water and the two-component Hunter-Saxton systems were studied in [43, 44]. Recently, Barostichi, Himonas and Petronilho [33] established the well-posedness for a class of nonlocal evolution equations in spaces of analytic functions. Furthermore, they proved a Cauchy-Kovalevsky theorem for a generalized Camassa-Holm equation \((g − kbCH)\) in [32]. Very Recently, Luo and Yin investigated the Gevrey regularity and analyticity for a class of Camassa-Holm type systems [35]. Thus, an amazing topic is to study the analytic solutions for the 2SP system (2). It is worthy to point out that our approach is strongly motivated from the Cauchy-Kovalevsky type results in [31, 33].

The present paper is two fold: studying Darboux transformation and Cauchy problem of the 2SP system (2). The whole paper is organized as follows. In section 2, we adopt a hodograph transformation to transform the 2SP equation (2) to another nonlinear partial differential equation, which is a Lax integrable system belonging to the Heisenberg ferromagnet (HF) hierarchy [20, 45]. In section 3, we define a Darboux transformation in terms of Darboux matrix operator, and then provide a detailed proof for the Darboux transformation and its quasideterminant representation of the \(N\)-fold case. In section 4, based on scalar solutions of the Lax pair, we construct a \(N\)-fold Darboux transformation to derive the explicit multi-loop soliton and breather soliton solutions with their dynamical interactions. Three special examples are discussed details and shown through their graphs. Starting from section 5, we study the Cauchy problem for the 2SP system (2). To do so, some preliminary results are first provided including the abstract Ovsyannikov type theorem and the basic properties of the analytic space \(C^{\delta,s}\). In section 5.1, we prove the existence and uniqueness of analytic solution with an estimate about the analytic lifespan, and in section 5.2, we investigate the continuity of the data-to-solution map in spaces of analytic functions. Last section concludes the paper with a brief summary and a future outlook.
2 Hodograph transformation

In this section, we show how the Lax pair (3) is related by a hodograph transformation to a negative order flow in the HF hierarchy. As per [20], let us introduce a dependent variable $w$ satisfying

$$w^2 = 1 + v_x u_x,$$

which is able to send the 2SP system (2) to the following form of conservation law

$$w_t = (\frac{1}{2}uvw)_x.$$  

(7)

(8)

We then define a hodograph transformation $HT: (x,t) \rightarrow (y,\tau)$ by means of

$$dy = wdx + \frac{1}{2}uvwdt, \quad d\tau = dt,$$

or equivalently

$$\frac{\partial}{\partial x} = w \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{2}uvw \frac{\partial}{\partial y}.$$

(9)

(10)

The 2SP system (2) can now be expressed in terms of new variables in the following form

$$x_{y\tau} = -\frac{1}{2}(uv)y, \quad u_{y\tau} = x_yu, \quad v_{y\tau} = x_yv.$$  

(11)

System (11) arises as a zero curvature equation $U_{\tau} - V_y + [U,V] = 0$, which is exactly the compatibility condition for the following Lax pair

$$\Psi_y = U(y,\tau;\lambda)\Psi, \quad \Psi_{\tau} = V(y,\tau;\lambda)\Psi,$$

where

$$U(y,\tau;\lambda) = \lambda \partial_y R, \quad V(y,\tau;\lambda) = S + \frac{1}{\lambda}W,$$

and

$$R = \begin{pmatrix} x & u \\ v & -x \end{pmatrix}, \quad S = \frac{1}{2} \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix}, \quad W = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(12)

(13)

(14)

Thus, system (11) is actually a negative order flow in the HF hierarchy.

3 Darboux transformation

Based on the Lax pair (12) of the integrable equation (11), let us consider the following Darboux transformation:

$$\Psi[1] = D\Psi,$$  

(15)

where $D$ is a Darboux matrix and $\Psi[1]$ recovers the form of the Lax pair (12)

$$\Psi[1]_y = U[1]\Psi[1] = \lambda(\partial_y R[1])\Psi[1], \quad U[1] = (D_y + DU)D^{-1},$$

(16)
\[ \Psi[1]_\tau = V[1] \Psi[1] = (S[1] + \frac{1}{\lambda} W[1]) \Psi[1], \quad V[1] = (D_\tau + D V) D^{-1}, \]

with
\[ R[1] = \begin{pmatrix} x[1] & u[1] \\ v[1] & -x[1] \end{pmatrix}, \quad S[1] = \frac{1}{2} \begin{pmatrix} 0 & -u[1] \\ v[1] & 0 \end{pmatrix}, \quad W[1] = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

and \( x[1], \ u[1] \) and \( v[1] \) being new solutions to the equation (11), that is,
\[ x[1]_\tau = -\frac{1}{2} (u[1]v[1])_\tau, \quad u[1]_\tau = x[1]_\tau u[1], \quad v[1]_\tau = x[1]_\tau v[1]. \]

In order to make the covariance of the Lax pair (12) under the Darboux transformation (15), the key step is to find an appropriate Darboux matrix \( D \) such that \( R[1], S[1] \) and \( W[1] \) in Eq. (18) have the same form as \( R, S \) and \( W \) in Eq. (14). Meanwhile, the old potentials \( x, u \) and \( v \) in \( R, S \) are mapped into the new potential \( x[1], u[1] \) and \( v[1] \) in \( R[1], S[1] \).

For the Lax pair (12), let us define the following Darboux matrix
\[ \Psi[1] = D \Psi \equiv (\lambda^{-1} I - M) \Psi, \]

where \( I \) is the 2 \( \times \) 2 identity matrix and
\[ M = H \Lambda^{-1} H^{-1}, \]

with
\[ \Lambda^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}, \quad H = (\Psi(\lambda_1)|e_1), \quad (\Psi(\lambda_2)|e_2)). \]

In equation (22), \( |e_1 \rangle \) and \( |e_2 \rangle \) are two constant vectors, and \( \Psi(\lambda_1) \) and \( \Psi(\lambda_2) \) are two fundamental-matrix solutions of the Lax pair (12) corresponding to the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. Thus, the Lax pair (12) can be rewritten in the following matrix form
\[ H_\tau = R_\tau H \Lambda, \]
\[ H_\tau = S H + W H \Lambda^{-1}, \]

where \( H \) is the matrix solution of the Lax form (12) corresponding to the eigen-matrix \( \Lambda \) consisting of distinct eigenvalues.

Using the above facts, we may have the following propositions.

**Proposition 1.** Under the Darboux transformation (20), the matrix \( R[1] \) given by the first equation of (18) has the same form as \( R \) in the first equation of (14) with the following condition
\[ R[1] = R - M, \]

where the matrix \( M \) satisfies
\[ M_\tau M = [R_\tau, M]. \]
Proof 1. Let \( R - R[1] = M \). Let us show that under the Darboux transformation \((20)\), \( M = H\Lambda^{-1}H^{-1} \) is a solution of equation \((26)\). Taking derivative with respect to \( y \) on both sides of \( M = H\Lambda^{-1}H^{-1} \) yields:
\[
M_y = H_y\Lambda^{-1}H^{-1} + H\Lambda^{-1}H_y^{-1} \\
= R_y - H\Lambda^{-1}H^{-1}H_yH^{-1} \\
= R_y - H\Lambda^{-1}H^{-1}R_yH\Lambda^{-1}H^{-1} \\
= R_y - MR_yM^{-1},
\]
which is equivalent to the condition \((26)\). The proof is thus completed.

Proposition 2. Under the Darboux transformation \((20)\), the matrices \( S[1] \) and \( W[1] \) given by the second and third equations of \((18)\) have the same form as \( S \) and \( W \) in the second and third equations of \((14)\) with the following conditions
\[
S[1] = S + [W,M], \quad W[1] = W,
\]
where the matrix \( M \) satisfies
\[
M_\tau = [S,M] + [W,M]M. \quad (29)
\]

Proof 2. Let \( S[1] - S = [W,M] \), where \( M \) is to be determined. Apparently, the third equation of \((18)\) and \((14)\) imply \( W[1] = W \). Let us now prove that under the Darboux transformation \((20)\), \( M = H\Lambda^{-1}H^{-1} \) is a solution of equation \((29)\). Taking derivative with respect to \( \tau \) on both sides of \( M = H\Lambda^{-1}H^{-1} \) leads to:
\[
M_\tau = H_\tau\Lambda^{-1}H^{-1} + H\Lambda^{-1}H_\tau^{-1} \\
= S\Lambda^{-1}H^{-1} + WH\Lambda^{-2}H^{-1} - H\Lambda^{-1}H^{-1}S - H\Lambda^{-1}H^{-1}WH\Lambda^{-1}H^{-1} \\
= SM + WM^2 - MS - MWM \\
= [S, M] + [W, M]M,
\]
which is exactly Eq. \((29)\). This completes the proof of the proposition 2.

The interesting thing is to iterate the above Darboux transformation \((20)\) \( N \)-times to generate a quasideterminant representation of the so-called \( N \)-fold Darboux transformation. To do so, we will adopt the notion of the following quasideterminant about the \( n \times n \) matrix \( D \) introduced by Gelfand and Retakh [46]:
\[
\begin{vmatrix}
A & B \\
C & D
\end{vmatrix} = D - CA^{-1}B, \quad (31)
\]
where \( A, B \) and \( C \) are \( n \times n \) matrices and \( A \) is invertible. From Eqs. \((20), (25)\) and \((28)\), the one-fold Darboux transformation \((20)\) and the matrices \( R, S \) and \( W \) can be expressed in terms of
quasideterminants as follows

\[
\Psi[1] = (\lambda^{-1}I - H\Lambda^{-1}H^{-1})\Psi = \begin{vmatrix} H & \Psi \\ H\Lambda^{-1} & \lambda^{-1}\Psi \end{vmatrix},
\]

(32)

\[
R[1] = R - H\Lambda^{-1}H^{-1} = R + \begin{vmatrix} H & I \\ H\Lambda^{-1} & 0 \end{vmatrix},
\]

(33)

\[
S[1] = S - [W, -H\Lambda^{-1}H^{-1}] = S - \begin{bmatrix} H & I \\ H\Lambda^{-1} & 0 \end{bmatrix},
\]

(34)

\[
W[1] = W,
\]

(35)

where 0 is an null matrix. The \(N\) times iteration of the Darboux transformation gives the quasideterminant matrix solution to the 2SP system (2). Let \(H_k (k = 1, 2, \ldots, N)\) be the invertible matrix solution to the Lax system (12) corresponding to the eigenvalue \(\Lambda = \Lambda_k (k = 1, 2, \ldots, N)\). Then the \(N\)-fold Darboux transformation for the Lax pair (12) can be written in the form of

\[
\Psi[N] = \begin{vmatrix} H_1 & H_2 & \cdots & H_N & \Psi \\ H_1\Lambda_1^{-1} & H_2\Lambda_2^{-1} & \cdots & H_N\Lambda_N^{-1} & \lambda^{-1}\Psi \\ H_1\Lambda_1^{-2} & H_2\Lambda_2^{-2} & \cdots & H_N\Lambda_N^{-2} & \lambda^{-2}\Psi \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_1\Lambda_1^{-N} & H_2\Lambda_2^{-N} & \cdots & H_N\Lambda_N^{-N} & \lambda^{-N}\Psi \end{vmatrix},
\]

(36)

The \(N\)-fold Darboux transformation sends the matrices \(R, S\) and \(W\) to the following forms

\[
R[N] = R + \begin{vmatrix} H_1 & H_2 & \cdots & H_N & 0 \\ H_1\Lambda_1^{-1} & H_2\Lambda_2^{-1} & \cdots & H_N\Lambda_N^{-1} & 0 \\ H_1\Lambda_1^{-2} & H_2\Lambda_2^{-2} & \cdots & H_N\Lambda_N^{-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_1\Lambda_1^{-(N-1)} & H_2\Lambda_2^{-(N-1)} & \cdots & H_N\Lambda_N^{-(N-1)} & I \\ H_1\Lambda_1^{-N} & H_2\Lambda_2^{-N} & \cdots & H_N\Lambda_N^{-N} & 0 \end{vmatrix},
\]

(37)

\[
S[N] = S - \begin{bmatrix} W, & \vdots & \vdots & \vdots & \vdots \\ H_1 & H_2 & \cdots & H_N & 0 \\ H_1\Lambda_1^{-1} & H_2\Lambda_2^{-1} & \cdots & H_N\Lambda_N^{-1} & 0 \\ H_1\Lambda_1^{-2} & H_2\Lambda_2^{-2} & \cdots & H_N\Lambda_N^{-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_1\Lambda_1^{-(N-1)} & H_2\Lambda_2^{-(N-1)} & \cdots & H_N\Lambda_N^{-(N-1)} & I \\ H_1\Lambda_1^{-N} & H_2\Lambda_2^{-N} & \cdots & H_N\Lambda_N^{-N} & 0 \end{bmatrix},
\]

(38)

\[
W[N] = W.
\]

(39)

4 Explicit loop soliton and multi-loop soliton solutions

According to Propositions 1 and 2, we will use the \(N\)-fold Darboux transformation to construct scalar solutions of the Lax system (12), and further give multi-loop and breather soliton solutions with their interactional dynamics for the 2SP system (2).
Let \((\psi_{1,k}, \phi_{1,k})^T \) and \((\psi_{2,k}, \phi_{2,k})^T \) be two linearly independent solutions to the Lax system \([12]\) associated with the eigenvalue \(\lambda_k \) \((k = 1, 2, \ldots, N)\), then its general solution has the following form
\[
\begin{pmatrix}
\psi_k \\
\phi_k
\end{pmatrix} = \begin{pmatrix}
\psi_{1,k} & \psi_{2,k} \\
\phi_{1,k} & \phi_{2,k}
\end{pmatrix} \begin{pmatrix}
|e_k|
\end{pmatrix},
\]
where \(|e_k|\) is a two-dimensional constant column vector. Without loss of generality, we choose \(|e_k| = (\mu_k, 1)^T\), where \(\mu_k \in \mathbb{R} \) \((k = 1, 2, \ldots, N)\).

In order to construct the \(N\)-fold Darboux transformation of the Lax system \([12]\) in the form of
\[
\Psi[N] = D\Psi = \begin{pmatrix}
A & B \\
C & E
\end{pmatrix} \Psi,
\]
let us define entries of the Darboux matrix \(D\) as follows
\[
A = \lambda^{-N} + \sum_{k=0}^{N-1} A_k \lambda^{-k}, \quad B = \sum_{k=0}^{N-1} B_k \lambda^{-k}, \quad C = \sum_{k=0}^{N-1} C_k \lambda^{-k}, \quad E = \lambda^{-N} + \sum_{k=0}^{N-1} E_k \lambda^{-k},
\]
where \(A_k, B_k, C_k, \) and \(E_k \) \((k = 0, 1, 2, \ldots, N - 1)\) are functions of \(y\) and \(\tau\). Then, we have the following \(N\)-fold Darboux transformation on the scalar solutions to the Lax system \([12]\).

**Proposition 1’.** Under the Darboux transformation \([41]\), the matrix \(R[1]\) given by the first equation of \([18]\) has the same form as \(R\) in the first equation of \([14]\), while the transformation sends the old potentials \(u, v\) to the following new ones
\[
x[N] = x[0] + \frac{\Delta_{AX}}{\Delta_{N-1}}, \quad u[N] = u[0] + \frac{\Delta_{BN}}{\Delta_{N-1}}, \quad v[N] = v[0] + \frac{\Delta_{CN}}{\Delta_{N-1}},
\]
where
\[
\Delta_{N-1} = \begin{vmatrix}
\psi_1 & \psi_1^{(N-2)} & \cdots & \psi_1^{(N-1)} & \phi_1 & \phi_1^{(N-2)} & \cdots & \phi_1^{(N-1)} \\
\psi_2 & \psi_2^{(N-2)} & \cdots & \psi_2^{(N-1)} & \phi_2 & \phi_2^{(N-2)} & \cdots & \phi_2^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\psi_{2N} & \psi_{2N}^{(N-2)} & \cdots & \psi_{2N}^{(N-1)} & \phi_{2N} & \phi_{2N}^{(N-2)} & \cdots & \phi_{2N}^{(N-1)}
\end{vmatrix},
\]
\[
\Delta_{AX} = \begin{vmatrix}
\psi_1 & \psi_1^{(N-2)} & \cdots & \psi_1^{(N)} & -\psi_1^{(N)} & \phi_1 & \phi_1^{(N-2)} & \cdots & \phi_1^{(N-1)} \\
\psi_2 & \psi_2^{(N-2)} & \cdots & \psi_2^{(N)} & -\psi_2^{(N)} & \phi_2 & \phi_2^{(N-2)} & \cdots & \phi_2^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\psi_{2N} & \psi_{2N}^{(N-2)} & \cdots & \psi_{2N}^{(N)} & -\psi_{2N}^{(N)} & \phi_{2N} & \phi_{2N}^{(N-2)} & \cdots & \phi_{2N}^{(N-1)}
\end{vmatrix},
\]
\[
\Delta_{BN} = \begin{vmatrix}
\psi_1 & \psi_1^{(N-2)} & \psi_1^{(N-1)} & \phi_1 & \phi_1^{(N-2)} & \phi_1^{(N-1)} & \psi_1^{(N)} \\
\psi_2 & \psi_2^{(N-2)} & \psi_2^{(N-1)} & \phi_2 & \phi_2^{(N-2)} & \phi_2^{(N-1)} & \psi_2^{(N)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\psi_{2N} & \psi_{2N}^{(N-2)} & \psi_{2N}^{(N-1)} & \phi_{2N} & \phi_{2N}^{(N-2)} & \phi_{2N}^{(N-1)} & \psi_{2N}^{(N)}
\end{vmatrix},
\]
\[
\Delta_{CN} = \begin{vmatrix}
\psi_1 & \psi_1^{(N-2)} & \phi_1^{(N-2)} & \psi_1^{(N)} \\
\psi_2 & \psi_2^{(N-2)} & \phi_2^{(N-2)} & \psi_2^{(N)} \\
\vdots & \vdots & \vdots & \vdots \\
\psi_{2N} & \psi_{2N}^{(N-2)} & \phi_{2N}^{(N-2)} & \psi_{2N}^{(N)}
\end{vmatrix},
\]
\[ \Delta_{C_{N-1}} = \begin{vmatrix} \psi_1 & \psi_1^{(N-2)} & \ldots & \psi_1^{(N)} & -\phi_1^{(N)} & \phi_1 & \phi_1^{(1)} & \ldots & \phi_1^{(N-2)} & \phi_1^{(N-1)} \\ \psi_2 & \psi_2^{(N-2)} & \ldots & \psi_2^{(N)} & -\phi_2^{(N)} & \phi_2 & \phi_2^{(1)} & \ldots & \phi_2^{(N-2)} & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \psi_{2N} & \psi_{2N}^{(N-2)} & \ldots & \psi_{2N}^{(N)} & -\phi_{2N}^{(N)} & \phi_{2N} & \phi_{2N}^{(1)} & \ldots & \phi_{2N}^{(N-2)} & \phi_{2N}^{(N-1)} \end{vmatrix}. \] (47)

In Eqs. (43)-(47), we have used the notations \( x[0] = x, u[0] = u, v[0] = v, \) \( \psi^{(N)}_k = \lambda^{-j} \psi_k \) and \( \phi^{(j)}_k = \lambda^{-j} \phi_k \) \( (k = 1, 2, \ldots, 2N; \ j = 1, 2, \ldots, N) \).

**Proof 1’.** Let \( D^{-1} = D^*/\det D \) and

\[ (D_y + DU)D^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \] (48)

It is not hard for us to see that \( f_{11}(\lambda), f_{12}(\lambda), f_{21}(\lambda), \) and \( f_{22}(\lambda) \) are \((-2N+1)\)th degree polynomials in \( \lambda \). Substituting (40) into the first equation of (12) yields

\[ \psi_{ky} = \lambda_k(x_y \psi_k + u_y \phi_k), \quad \phi_{ky} = \lambda_k(v_y \psi_k - x_y \phi_k), \quad (0 \leq k \leq 2N). \] (49)

A direct calculation reveals that all \( \lambda_k(1 \leq k \leq 2N) \) are the roots of \( f_{mn}(m, n = 1, 2) \). Therefore from (48), we have

\[ (D_y + DU)D^* = (\det D)\lambda P, \] (50)

where

\[ \lambda P = \lambda \begin{pmatrix} p^{(1)}_{11} & p^{(1)}_{12} \\ p^{(1)}_{21} & p^{(1)}_{22} \end{pmatrix}. \] (51)

and \( p^{(i)}_{mn} (m, n = 1, 2; i = 1) \) are independent of \( \lambda \). Apparently, Eq. (50) can be rewritten as

\[ D_y + DU = \lambda PD. \] (52)

Comparing the coefficients of \( \lambda^{-N+1} \) and \( \lambda^{-N+m} \) on both sides of (52) and noticing (43), we have

\[ p^{(1)}_{11} = x_y + A_{N-1,y} = x[1]_y, \quad p^{(1)}_{12} = u_y + B_{N-1,y} = u[1]_y \]
\[ p^{(1)}_{21} = v_y + C_{N-1,y} = v[1]_y, \quad p^{(1)}_{22} = -x_y + E_{N-1,y} = -x[1]_y, \]
\[ A_{N-m,y} = -A_{N-m+1}(x_y - x[1]_y) + C_{N-m+1}u[1]_y - B_{N-m+1}v_y, \]
\[ B_{N-m,y} = B_{N-m+1}(x_y + x[1]_y) - A_{N-m+1}u_y + E_{N-m+1}v[1]_y, \]
\[ C_{N-m,y} = C_{N-m+1}(x_y + x[1]_y) + A_{N-m+1}v[1]_y - E_{N-m+1}v_y, \]
\[ E_{N-m,y} = E_{N-m+1}(x_y - x[1]_y) + B_{N-m+1}v[1]_y - C_{N-m+1}u_y, \quad (2 \leq m \leq N). \] (54)

From Eqs. (18) and (51), one can readily see that \( P = R[1] \). Thus, the proof is completed.

**Proposition 2’**. Under the Darboux transformation (41), the matrices \( S[1] \) and \( W[1] \) defined by the second and third equations of (18) have the same form as \( S \) and \( W \) in the second and third equation of (14).
**Proof 2’.** Let \( D^{-1} = D^* / \det D \) and

\[
(D_\tau + DV)D^* = \begin{pmatrix}
g_{11}(\lambda) & g_{12}(\lambda) 
g_{21}(\lambda) & g_{22}(\lambda)
\end{pmatrix}.
\]

One may easily see that \( g_{11}(\lambda), g_{12}(\lambda), g_{21}(\lambda) \), and \( g_{22}(\lambda) \) are the \((-2N-1)\)th degree polynomials in \( \lambda \). Substituting \((40)\) into the second equation of \((12)\) leads to

\[
\psi_{k\tau} = \frac{1}{4\lambda_k} \psi_k - \frac{1}{2} u \phi_k, \quad \phi_{k\tau} = \frac{1}{2} v \psi_k - \frac{1}{4\lambda_k} \phi_k, \quad (0 \leq k \leq 2N).
\]

A direct calculation shows that all \( \lambda_k (1 \leq k \leq 2N) \) are the \((−2N−1)\)th degree polynomials \( g_{mn}(m, n = 1, 2) \). Therefore, we arrive at

\[
(D_\tau + DV)D^* = (\det D)Q(\lambda),
\]

where

\[
Q(\lambda) = \frac{\lambda}{2} \begin{pmatrix}
q_{11}^{(0)} & q_{12}^{(0)} 
q_{21}^{(0)} & q_{22}^{(0)}
\end{pmatrix} + \frac{1}{4\lambda} \begin{pmatrix}
q_{11}^{(1)} & q_{12}^{(1)} 
q_{21}^{(1)} & q_{22}^{(1)}
\end{pmatrix}
\]

and \( q_{mn}^{(i)}(m, n = 1, 2; i = 1, 2) \) are independent of \( \lambda \). Hence, Eq. \((57)\) can be rewritten as

\[
D_\tau + DV = Q(\lambda)D.
\]

Comparing the coefficients of \( \lambda^{-N-1} \) and \( \lambda^{-N} \) on both sides of Eq. \((59)\), we obtain

\[
q_{11}^{(0)} = -q_{22}^{(0)} = 1, \quad q_{12}^{(0)} = q_{21}^{(0)} = q_{12}^{(1)} = q_{21}^{(1)} = 0,
\]

and

\[
q_{21}^{(0)} = v + CN_{-1} = v[1], \quad q_{12}^{(0)} = -u - BN_{-1} = -u[1], \quad q_{11}^{(0)} = q_{22}^{(0)} = 0.
\]

From Eqs. \((18)\) and \((58)\), it is not hard for us to see that \( Q(\lambda) = V \), which completes the proof.

**Remark 1:**

(i) If \((x(y, \tau), u(y, \tau), v(y, \tau))\) is a parametric representation of the solution to the 2SP system \([2]\), then \((x(-y, \tau), u(-y, \tau), v(-y, \tau))\) is a solution, too; and for (an arbitrary complex constant \( C, (x(y, \tau) + C, u(y, \tau), v(y, \tau)) \) is also a solution.

(ii) The 2SP system \([2]\) has a trivial solution \((x, u, v) = (\alpha y + \beta \tau + x_0, 0, 0)\), where \( \alpha, \beta, x_0 \in \mathbb{R} \).

Substituting the trivial solution \( x = \alpha y + \beta \tau + x_0, u = v = 0 \) into the Lax system \((12)\), we obtain

\[
\begin{pmatrix}
\psi^{(1,k)} \\
\phi^{(1,k)}
\end{pmatrix} = \begin{pmatrix} e^{\xi_k} \\ 0
\end{pmatrix}, \quad \begin{pmatrix}
\psi^{(2,k)} \\
\phi^{(2,k)}
\end{pmatrix} = \begin{pmatrix} 0 \\ e^{-\xi_k}
\end{pmatrix},
\]

where \( \xi_k = \alpha \lambda_k y + \frac{1}{\alpha x_k} \tau, \quad (1 \leq k \leq 2N) \). Thus as per \((40)\), we accordingly have

\[
\begin{pmatrix}
\psi_k \\
\phi_k
\end{pmatrix} = \begin{pmatrix} \mu_k e^{\xi_k} \\ e^{-\xi_k}
\end{pmatrix}.
\]
Under the condition $\lambda_{2k-1} \lambda_{2k} < 0$, $\mu_{2k-1} \mu_{2k} < 0$ ($k = 1, 2, \cdots, N$), substituting (63) into (43) generates the $N$-soliton solution of the 2SP system (2).

**Remarks 2:** If $\lambda_{2k} = -\lambda_{2k-1}$ and $\mu_{2k} \mu_{2k-1} = -1$ ($k = 1, 2, \cdots, N$), (43) and (63) can be reduced to the $N$-soliton solution of the short pulse equation (1).

Let us take three special cases $N = 1$, $N = 2$ and $N = 3$ as examples.

**Case 1 ($N = 1$).** From (43), we have the following one-soliton solution of the 2SP system (2)

\[
x[1] = \alpha y + \beta \tau + x_0 - \frac{\lambda_2 \mu_1 \mu_2 e^{\zeta_1} - \lambda_1 \mu_2 e^{\zeta_2}}{\lambda_1 \lambda_2 (\mu_1 e^{\zeta_1} - \mu_2 e^{\zeta_2})},
\]

(64)

\[
u[1] = v[0] + \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2 (\mu_1 e^{\zeta_1} - \mu_2 e^{\zeta_2})},
\]

(65)

where $\zeta_k = 2(\alpha \lambda_k y + \frac{1}{4 \lambda_k} \tau)$ ($k = 1, 2$), $u[0] = u = 0$, $v[0] = v = 0$ and $x_0 = 0$, and parameters satisfy $\lambda_1 \lambda_2 < 0$, $\mu_1 \mu_2 < 0$.

According to Remark 2, if we substitute $\lambda_2 = -\lambda_1$, $\mu_2 = -\mu_1^{-1}$ into (64)-(65), then Eqs. (65) and (66) read one-loop soliton solution of the short pulse equation (1) as follows

\[
x[1] = \alpha y + \beta \tau - \frac{1}{\lambda_1} \tanh(\zeta_1 + \ln |\mu_1|),
\]

(67)

\[
u[1] = v[1] = -\frac{1}{\lambda_1} \sech(\zeta_1 + \ln |\mu_1|),
\]

(68)

where $\zeta_1 = 2(\alpha \lambda_1 y + \frac{1}{4 \lambda_1} \tau)$. 

![Diagram](image-url)
Figure 1. The time evolution of one-loop soliton solution (64)-(65) with $\lambda_1 = -3$, $\lambda_2 = \frac{10}{3}$, $\mu_1 = 2$, $\mu_2 = -\frac{1}{2}$, $\alpha = 1$, and $\beta = 0$.

Choosing appropriately different parameters may send (64)-(65) to different types of soliton solutions for the 2SP system (2). For instance, the time evolutions of one-loop soliton and one-anti-loop soliton solutions are captured and illustrated in Figure 1 and Figure 2, respectively.

Case 2 ($N = 2$). From (43), we may obtain a two-soliton solution of the 2SP system (2) in the following form

$$x[2] = \alpha y + \beta \tau + x_0 + \frac{\Delta A_1}{\Delta_1}, \quad u[2] = u[0] + \frac{\Delta B_1}{\Delta_1}, \quad v[2] = v[0] + \frac{\Delta C_1}{\Delta_1},$$

where

$$\Delta_1 = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & \phi_1 & \phi_1^{(1)} \\ \psi_2 & \psi_2^{(1)} & \phi_2 & \phi_2^{(1)} \\ \psi_3 & \psi_3^{(1)} & \phi_3 & \phi_3^{(1)} \\ \psi_4 & \psi_4^{(1)} & \phi_4 & \phi_4^{(1)} \end{vmatrix}, \quad \Delta A_1 = \begin{vmatrix} \psi_1 & -\psi_1^{(2)} & \phi_1 & \phi_1^{(1)} \\ \psi_2 & -\psi_2^{(2)} & \phi_2 & \phi_2^{(1)} \\ \psi_3 & -\psi_3^{(2)} & \phi_3 & \phi_3^{(1)} \\ \psi_4 & -\psi_4^{(2)} & \phi_4 & \phi_4^{(1)} \end{vmatrix},$$

$$\Delta B_1 = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & \phi_1 & -\psi_1^{(2)} \\ \psi_2 & \psi_2^{(1)} & \phi_2 & -\psi_2^{(2)} \\ \psi_3 & \psi_3^{(1)} & \phi_3 & -\psi_3^{(2)} \\ \psi_4 & \psi_4^{(1)} & \phi_4 & -\psi_4^{(2)} \end{vmatrix}, \quad \Delta C_1 = \begin{vmatrix} \psi_1 & -\phi_1^{(2)} & \phi_1 & \phi_1^{(1)} \\ \psi_2 & -\phi_2^{(2)} & \phi_2 & \phi_2^{(1)} \\ \psi_3 & -\phi_3^{(2)} & \phi_3 & \phi_3^{(1)} \\ \psi_4 & -\phi_4^{(2)} & \phi_4 & \phi_4^{(1)} \end{vmatrix},$$

$u[0] = v[0] = x_0 = 0$, $\psi_k^{(j)} = \lambda_k^{-j}\psi_k$, $\phi_k^{(j)} = \lambda_k^{-j}\phi_k$ ($k = 1, 2, 3, 4; j = 1, 2$), and $(\psi_k, \phi_k)^T$ are given through (63). The two-soliton solution (69) has interactional dynamics, for instance, two-loop
soliton with $\lambda_1 = -\lambda_2 = 3$, $\lambda_3 = -\lambda_4 = -2$, $\mu_1 = 2$, $\mu_2 = -\frac{1}{2}$, $\mu_3 = 3$, $\mu_4 = -1$, $\alpha = 1$, $\beta = 0$ (see figure 3), two-anti-loop soliton with $\lambda_1 = -\lambda_2 = -3$, $\lambda_3 = -\lambda_4 = 2$, $\mu_1 = 2$, $\mu_2 = -\frac{1}{2}$, $\mu_3 = 3$, $\mu_4 = -1$, $\alpha = 1$, $\beta = 0$ (see figure 4), and loop-anti-loop soliton with $\lambda_1 = -\lambda_2 = -3$, $\lambda_3 = -\lambda_4 = -5$, $\mu_1 = 2$, $\mu_2 = -\frac{1}{2}$, $\mu_3 = 3$, $\mu_4 = -1$, $\alpha = 1$, $\beta = 0$ (see figure 5).

The following Figures 3-5 show the dynamical interaction of two-soliton solutions ($u[2], v[2]$) at three different times: (a) the short dashed line stands for the wave elevation at $t = -30$, (b) the long dashed line represents for the wave elevation at $t = 0$, and (c) the solid line refers to the wave elevation at $t = 30$. Initially, two-solitons are well separated at $t = -30$ and the soliton with faster velocity is located at the right. At $t = 0$, the soliton with faster velocity catches up with the other soliton with slower velocity and they have a collision. Afterward, they gradually separate as time increases and eventually at $t = 30$, the two solitons recover their original shapes apart from some phase shifts. Apparently, such a collision is elastic, and there is no change in shape and amplitude of solitons except a phase shift. This interaction is very similar to that of two solitons of the Korteweg-de Vries (KdV) equation. However, an interesting phenomenon is that a soliton (see $v[2]$ in Figures 3-4) with a smaller amplitude could travel faster than the one with larger amplitude when they interact.

Figure 3. The dynamical interaction of two-loop soliton solution \[\text{soliton with } \lambda_1 = -\lambda_2 = 3, \lambda_3 = -\lambda_4 = -2, \mu_1 = 2, \mu_2 = -\frac{1}{2}, \mu_3 = 3, \mu_4 = -1, \alpha = 1, \beta = 0.\]
Figure 4. The dynamical interaction of two-anti-loop soliton solution \((69)\) with \(\lambda_1 = -\lambda_2 = -3, \ \lambda_3 = -\lambda_4 = 2, \ \mu_1 = 2, \ \mu_2 = -\frac{1}{2}, \ \mu_3 = 3, \ \mu_4 = -1, \ \alpha = 1,\) and \(\beta = 0.\)

Figure 5. The dynamical interaction of loop-anti-loop soliton solution \((69)\) with \(\lambda_1 = -\lambda_2 = -3, \ \lambda_3 = -\lambda_4 = -5, \ \mu_1 = 2, \ \mu_2 = -\frac{1}{2}, \ \mu_3 = 3, \ \mu_4 = -1, \ \alpha = 1,\) and \(\beta = 0.\)

Casting \(\lambda_1 = \lambda_{1R} + i\lambda_{1I}, \ \lambda_2 = -(\lambda_{1R} - i\lambda_{1I}), \ \lambda_3 = \lambda_1^*, \ \lambda_4 = \lambda_2^*\) and \(\mu_2 = -\mu_3 = \mu_4 = -\mu_1\) in
Eq. (69) produces the following breather soliton solution of the 2SP system (2)

\[
x[2] = \alpha y + \beta \tau + \frac{2\lambda_{1R}\lambda_{1I}}{|\lambda_1|^2} \frac{\lambda_{1I} \sinh \Omega_1 + \lambda_{1R} \sin \theta_1}{2\lambda_{1I}^2 \cosh^2 \Omega_1 + \lambda_{1R}^2(1 - \cos \theta_1)},
\]

(72)

\[
u[2] = -\frac{4\lambda_{1R}\lambda_{1I}\mu_1}{|\lambda_1|^2} \frac{\lambda_{1I} \cos \frac{1}{2} \theta_1 \cosh \frac{1}{2} \Omega_1 - \lambda_{1R} \sin \frac{1}{2} \theta_1 \sinh \frac{1}{2} \Omega_1}{2\lambda_{1I}^2 \cosh^2 \frac{1}{2} \Omega_1 + \lambda_{1R}^2(1 - \cos \theta_1)},
\]

(73)

\[
v[2] = -\frac{4\lambda_{1R}\lambda_{1I}}{\mu_1 |\lambda_1|^2} \frac{\lambda_{1I} \cos \frac{1}{2} \theta_1 \cosh \frac{1}{2} \Omega_1 + \lambda_{1R} \sin \frac{1}{2} \theta_1 \sinh \frac{1}{2} \Omega_1}{2\lambda_{1I}^2 \cosh^2 \frac{1}{2} \Omega_1 + \lambda_{1R}^2(1 - \cos \theta_1)},
\]

(74)

where \(\Omega_1 = 4\alpha \lambda_{1R} \tau + \frac{\lambda_{1R}}{|\lambda_1|^2} \tau, \quad \theta_1 = -4\alpha \lambda_{1I} \tau + \frac{\lambda_{1I}}{|\lambda_1|^2} \tau\). The following figure (Figure 6) shows the time evolution of the breather soliton solutions (72)-(74) with the parameters \(\lambda_{1R} = \lambda_{1I} = 1, \mu_1 = \mu_3 = -\mu_2 = -\mu_4 = \frac{1}{2}, \alpha = 1, \beta = 0\).

Figure 6. The dynamical interaction of breather-soliton solutions (72)-(75) with \(\lambda_1 = \lambda_2^* = -\lambda_3 = -\lambda_4^* = 1 + i, \mu_1 = 2, \mu_2 = -\frac{1}{2}, \mu_3 = 4, \mu_4 = -1, \alpha = 1, \beta = 0\).

Case 3 \((N = 3)\). AS per Eq. (43), putting \(N = 3\) generates three-soliton solutions of the 2SP system (2)

\[
x[3] = \alpha y + \beta \tau + x_0 + \frac{\Delta A_2}{\Delta_2}, \quad u[3] = u[0] + \frac{\Delta B_2}{\Delta_2}, \quad v[3] = v[0] + \frac{\Delta C_2}{\Delta_2},
\]

(75)

where

\[
\Delta_2 = \begin{vmatrix}
\psi_1 & \psi_1^{(1)} & \psi_1^{(2)} & \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} \\
\psi_2 & \psi_2^{(1)} & \psi_2^{(2)} & \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} \\
\psi_3 & \psi_3^{(1)} & \psi_3^{(2)} & \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} \\
\psi_4 & \psi_4^{(1)} & \psi_4^{(2)} & \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} \\
\psi_5 & \psi_5^{(1)} & \psi_5^{(2)} & \phi_5 & \phi_5^{(1)} & \phi_5^{(2)} \\
\psi_6 & \psi_6^{(1)} & \psi_6^{(2)} & \phi_6 & \phi_6^{(1)} & \phi_6^{(2)} \\
\end{vmatrix}.
\]

(76)
\[ \Delta_{A_2} = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & -\psi_1^{(3)} & \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} \\ \psi_2 & \psi_2^{(1)} & -\psi_2^{(3)} & \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} \\ \psi_3 & \psi_3^{(1)} & -\psi_3^{(3)} & \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} \\ \psi_4 & \psi_4^{(1)} & -\psi_4^{(3)} & \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} \\ \psi_5 & \psi_5^{(1)} & -\psi_5^{(3)} & \phi_5 & \phi_5^{(1)} & \phi_5^{(2)} \\ \psi_6 & \psi_6^{(1)} & -\psi_6^{(3)} & \phi_6 & \phi_6^{(1)} & \phi_6^{(2)} \end{vmatrix}, \quad (77) \]

\[ \Delta_{B_2} = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & \phi_1 & \phi_1^{(1)} & -\psi_1^{(3)} \\ \psi_2 & \psi_2^{(1)} & \phi_2 & \phi_2^{(1)} & -\psi_2^{(3)} \\ \psi_3 & \psi_3^{(1)} & \phi_3 & \phi_3^{(1)} & -\psi_3^{(3)} \\ \psi_4 & \psi_4^{(1)} & \phi_4 & \phi_4^{(1)} & -\psi_4^{(3)} \\ \psi_5 & \psi_5^{(1)} & \phi_5 & \phi_5^{(1)} & -\psi_5^{(3)} \\ \psi_6 & \psi_6^{(1)} & \phi_6 & \phi_6^{(1)} & -\psi_6^{(3)} \end{vmatrix}, \quad (78) \]

\[ \Delta_{C_2} = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & -\phi_1^{(3)} & \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} \\ \psi_2 & \psi_2^{(1)} & -\phi_2^{(3)} & \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} \\ \psi_3 & \psi_3^{(1)} & -\phi_3^{(3)} & \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} \\ \psi_4 & \psi_4^{(1)} & -\phi_4^{(3)} & \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} \\ \psi_5 & \psi_5^{(1)} & -\phi_5^{(3)} & \phi_5 & \phi_5^{(1)} & \phi_5^{(2)} \\ \psi_6 & \psi_6^{(1)} & -\phi_6^{(3)} & \phi_6 & \phi_6^{(1)} & \phi_6^{(2)} \end{vmatrix}, \quad (79) \]

where \( u[0] = v[0] = x_0 = 0 \), \( \psi_k(j) = \lambda_k^{-j}\psi_k \), \( \phi_k(j) = \lambda_k^{-j}\phi_k \) \( (k = 1, 2, 3, 4, 5, 6; \ j = 1, 2, 3) \) and \( (\psi_k, \phi_k)^T \) are given by (63).

The following graphs (Figures 7-9) describe the dynamical interactions of the three-loop-soliton solutions (75) with those designated parameters. One can easily see that the parameters have little influence on the profiles and characters of the three-soliton solutions of \((u[3],v[3])\). All the interactional dynamics is very similar to the case of \( N = 2 \).
Figure 7. The interactional dynamics of three-soliton solution (75) with \( \lambda_1 = -\lambda_2 = -3, \lambda_3 = -\lambda_4 = 2, \lambda_5 = -\lambda_6 = -1 \)
\( \mu_1 = 2, \mu_2 = -\frac{1}{2}, \mu_3 = 3, \mu_4 = -1, \mu_5 = 4, \mu_6 = -2, \alpha = 1, \) and \( \beta = 0. \)

Figure 8. The interactional dynamics of three-soliton solution (75) with \( \lambda_1 = -\lambda_2 = 3, \lambda_3 = -\lambda_4 = -2, \lambda_5 = -\lambda_6 = 1 \)
\( \mu_1 = 2, \mu_2 = -\frac{1}{2}, \mu_3 = 3, \mu_4 = -1, \mu_5 = 4, \mu_6 = -2, \alpha = 1, \) and \( \beta = 0. \)
Figure 9. The interactional dynamics of three-soliton solution (75) with $\lambda_1 = -\lambda_2 = 3, \lambda_3 = -\lambda_4 = -2, \lambda_5 = -\lambda_6 = -1$
\[
\mu_1 = 2, \mu_2 = -\frac{1}{2}, \mu_3 = 3, \mu_4 = -1, \mu_5 = 4, \mu_6 = -2, \alpha = 1, \text{and } \beta = x_0 = 0.
\]

5 Cauchy problem for the 2SP system (2)

Since the 2SP system (2) is integrable, from the view point of soliton solutions, we already derived its multi-loop soliton solutions through the Darboux transformation described in the above sections. In this section, from the view point of analysis, we want to study the Cauchy problem of the 2SP system (2). To do so, let us present some preliminary works. First, let us recall the definition of a scale of Banach spaces \{X_\delta\}_{0 < \delta \leq 1}.

**Definition 1** A scale of complex Banach spaces is a one-parameter family of complex Banach spaces \{X_\delta\}_{0 < \delta \leq 1} such that:

(Scale): If for any $0 < \delta' < \delta \leq 1$ we have
\[
X_\delta \subset X_{\delta'}, \quad \| \cdot \|_{\delta'} \leq \| \cdot \|_{\delta}.
\]

Then, we present the framework in an analytic space. In the following contexts, we denote the Fourier transform of $f$ in $\mathbb{R}$ or $\mathbb{T}$ by $\hat{f}$. Assume that the initial data belong to the decreasing Banach space in the following scale. For $\delta > 0$ and $s \geq 0$, in the periodic case the Banach space is defined by
\[
G^{\delta,s}(\mathbb{T}) = \{ f \in L^2(\mathbb{T}) : \| f \|_{G^{\delta,s}(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s e^{2\delta |k|} |\hat{f}(k)|^2 < \infty \}.
\]
While in the non-periodic case the Banach space is defined by

\[
G_{\delta,s}^\delta(R) = \{ f \in L^2(R) : \| f \|^2_{G_{\delta,s}^\delta(R)} := \int_R (1 + |\xi|^2)^s e^{2i\xi}\hat{f}(\xi)|^2 d\xi < \infty \}. \tag{82}
\]

**Remark 2** For \( f \in G_{\delta,s}^\delta \), the following properties are obvious by the definition of \( G_{\delta,s}^\delta \):

(i) \( 0 < \delta' < \delta \) and \( s \geq 0 \), then \( \| \cdot \|_{\delta,s} \leq \| \cdot \|_{\delta',s} \), i.e. \( G_{\delta,s}^\delta \hookrightarrow G_{\delta',s}^\delta \).

(ii) \( 0 < s' < s \) and \( \delta > 0 \), then \( \| \cdot \|_{\delta,s} \leq \| \cdot \|_{\delta,s'} \), i.e. \( G_{\delta,s}^\delta \hookrightarrow G_{\delta,s'}^\delta \).

**Remark 3** By Remark 2 (i), it is not hard for us to see that the spaces \( \{ G_{\delta,s}^\delta \}_{0 < \delta < 1} \) with norm \( \| \cdot \|_{\delta,s} \) form a scale of decreasing Banach spaces.

Throughout the paper, \( G_{\delta,s}^\delta \) represents the space for both the periodic and non-periodic cases if a result holds for both cases, and \( \| \cdot \|_{\delta,s} \) stands for the norm in the space \( G_{\delta,s}^\delta \).

Next, let us provide some basic properties of the \( G_{\delta,s}^\delta \) spaces, which can also be seen in [33].

**Lemma 4** Let \( s \geq 0 \) for any \( x \in T \), and \( s > \frac{1}{2} \) if \( x \in \mathbb{R} \). Assume that \( f \in G_{\delta,s}^\delta \), with \( 0 < \delta \leq 1 \), then \( f \) is a holomorphic function in a symmetric strip \( D := \{ z \in \mathbb{C} : |y| < \delta \} \), where \( z = x + iy \). \( \delta \) is called the analytic radius.

**Lemma 5** If \( 0 < \delta' < \delta \leq 1 \), \( s \geq 0 \) and \( f \in G_{\delta,s}^\delta \), then

\[
\| f_x \|_{\delta',s} \leq \frac{e^{-1}}{\delta - \delta'} \| f \|_{\delta,s} \tag{83}
\]

**Lemma 6** (Algebraic property)

If \( 0 < \delta < 1 \), \( s > \frac{1}{2} \) and \( f, g \in G_{\delta,s}^\delta \), then we have

\[
\| fg \|_{\delta,s} \leq C_s \| f \|_{\delta,s} \| g \|_{\delta,s}. \tag{84}
\]

with \( C_s = \sqrt{2(1 + 2^2s)} \sum_{k=0}^{\infty} \frac{1}{1 + |k|^2} \) in the periodic case and \( C_s = \sqrt{\frac{2(1 + 2^2s)}{2s - 1}} \) in the non-periodic case.

**Lemma 7** If \( u_0 \in C^w(T) \), there exists \( \delta_0 > 0 \) such that \( u_0 \in G_{\delta_0,s}^\delta(T) \) for any \( s > 0 \).
Furthermore, we present a brief description of the autonomous Ovsyannikov theorem [40, 41, 42] that are used in the following sections. Given a decreasing scale of Banach spaces \( \{X_\delta\}_{0 < \delta \leq 1} \) and initial data \( u_0 \in X_1 \), we consider the Cauchy problem

\[
\frac{du}{dt} = F(u), \quad u(0) = u_0
\]

(85)

where \( F : X_\delta \to X_{\delta'} \) satisfies the following conditions:

1. \( F : X_\delta \to X_{\delta'} \) is a function, and for any given \( u_0 \in X_1 \) and \( R > 0 \) there exist two positive constants \( L \) and \( M \), depending on \( u_0 \) and \( R \), such that for all \( 0 < \delta' < \delta < 1 \) and \( u_1, u_2 \in X_\delta \) with \( \|u_1 - u_0\|_\delta < R \) and \( \|u_2 - u_0\|_\delta < R \) we have the following Lipschitz type condition

\[
\|F(u_1) - F(u_2)\|_{\delta'} \leq \frac{L}{\delta - \delta'} \|u_1 - u_2\|_\delta,
\]

(86)

and the following bound in the \( X_\delta \) norm of \( F(u_0) \)

\[
\|F(u_0)\|_\delta \leq \frac{M}{1 - \delta}, \quad 0 < \delta < 1.
\]

(87)

2. For \( 0 < \delta' < \delta < 1 \) and \( a > 0 \), if the function \( t \mapsto u(t) \) is holomorphic on \( \{t \in \mathbb{C} : |t| < a(1 - \delta)\} \) with values in \( X_\delta \) and \( \sup_{|t| < a(1 - \delta)} \|u(t) - u_0\|_\delta < R \), then the function \( t \mapsto F(u(t)) \) is holomorphic on \( \{t \in \mathbb{C} : |t| < a(1 - \delta)\} \) with values in \( X_{\delta'} \).

Let us now describe the autonomous Ovsyannikov theorem below.

**Theorem 8** [30] (Autonomous Ovsyannikov Theorem) Assume that the scale of Banach spaces \( X_\delta \) and the function \( F(u) \) satisfy the above conditions (1) and (2). Then, for given \( u_0 \in X_1 \) and \( R > 0 \) there exists \( T > 0 \) such that

\[
T = \frac{R}{16LR + 8M},
\]

(88)

and a unique solution \( u(t) \) to the Cauchy problem \( (85) \), which for every \( \delta \in (0, 1) \) is a holomorphic function in the disc \( D(0, T(1 - \delta)) \) valued in \( X_\delta \) satisfying

\[
\sup_{|t| < T(1 - \delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1.
\]

(89)

### 5.1 Existence and uniqueness

Let us consider the following Cauchy problem for the 2SP system [2]

\[
\begin{aligned}
    u_{tx} &= \frac{1}{2}(uvu_x)_x + u, \\
    v_{tx} &= \frac{1}{2}(uvv_x)_x + v, \\
    u(0, x) &= u_0(x), \\
    v(0, x) &= v_0(x).
\end{aligned}
\]

(90)
Casting the integral operator $\partial_x^{-1}$ onto both sides of (90) yields

$$
\begin{aligned}
\left\{ \begin{array}{l}
u_t = \frac{1}{2} uvu_x + \partial_x^{-1} u := F_1(z), \\
v_t = \frac{1}{2} uvv_x + \partial_x^{-1} v := F_2(z),
\end{array} \right.
\end{aligned}
\tag{91}
$$

Putting $z = (u, v)^T, z_0 = (u_0, v_0)^T$ and $F(z) = (F_1(z), F_2(z))^T$ leads (91) to the following concise form

$$
\frac{dz}{dt} = F(z), \quad z(0) = z_0.
\tag{92}
$$

Notice that there is the integral operator $\partial_x^{-1}$ involved in Eq. (91). Let us give a remark about it below.

**Remark 9** $\partial_x^{-1}f$ is defined via the Fourier transform as follows

$$
\hat{\partial_x^{-1}f} = \frac{1}{i\xi} \hat{f}(\xi).
$$

Due to the singularity at $\xi = 0$, one requires that $\hat{f}(0) = 0$, which is clearly equivalent to

$$
\int_{\mathbb{R}} f(x) dx = 0.
$$

In what follows, $\partial_x^{-1}f \in \mathcal{L}^2(\mathbb{R})$ means that there is an $\mathcal{L}^2(\mathbb{R})$ function $g$ such that $g_x = f$, at least in the distributional sense.

To deal with the integral term $\partial_x^{-1}u$ in Eq. (91), we establish the following key lemma.

**Lemma 10** For $0 < \delta \leq 1, s \geq 0$ and $u \in G^{\delta,s}$, the following estimate holds true:

$$
||\partial_x^{-1}u||_{G^{\delta,s}}^2 \leq 2 ||u||_{G^{\delta,s}}^2.
\tag{93}
$$

**Proof:**

$$
||\partial_x^{-1}u||_{G^{\delta,s}}^2 = \int_{\mathbb{R}} (1 + |\xi|^2) e^{\delta|\xi|} |\hat{\partial_x^{-1}u}|^2 d\xi
= \int_{\mathbb{R}} (1 + |\xi|^2) e^{\delta|\xi|} |\hat{u}|^2 d\xi
\geq \int_{\mathbb{R}} (1 + |\xi|^2) e^{\delta(-\Delta)^{1/2}} |u|^2 d\xi
\geq ||\partial_x^{-1}e^{\delta(-\Delta)^{1/2}} u||_{H^s}^2 = ||\partial_x^{-1}f||_{H^s}^2,
\quad (\text{Here } f = e^{\delta(-\Delta)^{1/2}} u)
= ||\partial_x^{-1}f||_{L^2}^2 + ||\partial_x^{-1}f||_{\dot{H}^s}^2
\quad (\because H^s = L^2 \cap \dot{H}^s, \text{ with } s \geq 0)
\leq ||f||_{H^{\delta-1}}^2 + ||f||_{H^{\delta-1}}^2 \leq 2 ||f||_{H^s}^2 = 2 ||e^{\delta(-\Delta)^{1/2}} u||_{H^s}^2
= 2 \int_{\mathbb{R}} (1 + |\xi|^2) e^{2\delta|\xi|} |\hat{u}|^2 d\xi
\leq 2 \int_{\mathbb{R}} (1 + |\xi|^2) e^{2\delta|\xi|} |\hat{u}|^2 d\xi = 2 ||u||_{G^{\delta,s}}^2.
$$
Remark 11 Lemma 10 is also true for the periodic case by using the definition (81).

For the sake of simplicity, we shall assume that our initial data \( u_0 \in G^{1,s} \), and for any \( 0 < \delta < 1 \) and \( s > \frac{1}{2} \), we define \( \|z\|_{\delta,s} = \|u\|_{\delta,s} + \|v\|_{\delta,s} \). Let us now present the existence and uniqueness of Eq. (91).

Theorem 12 Let \( s > \frac{1}{2} \). If \( z_0 = (u_0, v_0)^T \in G^{1,s} \times G^{1,s} \), then there exists a positive time \( T \), which depends on the initial data \( z_0 \) and \( s \), such that for every \( \delta \in (0,1) \), the Cauchy problem (91) has a unique solution \( z(t) = (u(t), v(t))^T \). And \( u(t), v(t) \) are holomorphic functions in the disc \( D(0, T(1-\delta)) = \{ t \in \mathbb{C} : |t| < T(1-\delta) \} \) valued in \( G^{\delta,s} \). Furthermore, the analytic lifespan \( T \) satisfies

\[
T = \frac{1}{68C_2e^{-1}\|z_0\|_{1,s}^2 + 24\sqrt{2}}
\]

where \( \|z_0\|_{1,s} = \|u_0\|_{1,s} + \|v_0\|_{1,s} \) and \( C_s \) comes from (81).

Remark 3 ensures that \( G^{\delta,s} \) satisfies the scale decreasing condition (80) like the space \( X_\delta \) in the autonomous Ovsyannikov theorem. Also, these spaces and \( F_i(z), i = 1, 2 \) satisfy condition (2). Therefore, to prove Theorem 3.1, it suffices to show that the right-hand side \( F_i(z), i = 1, 2 \) of Eq. (91) satisfies conditions (86) and (87). This is included in the following crucial lemma.

Lemma 13 Let \( s > \frac{1}{2} \). Also, let \( R > 0 \) and \( z_0 = (u_0, v_0)^T \in G^{1,s} \times G^{1,s} \) be given. Then, for the Cauchy problem (91) there exist positive constants \( L \) and \( M \), which depend on \( R \) and \( \|z_0\|_{1,s} \) such that for \( z_1, z_2 \in G^{\delta,s} \times G^{\delta,s} \), \( \|z_1 - z_2\|_{\delta,s} < R, \|z_2 - z_0\|_{\delta,s} < R \) and \( 0 < \delta' < \delta < 1 \) we have

\[
\|F_1(z_1) - F_1(z_2)\|_{\delta',s} \leq \frac{L}{\delta - \delta'}\|z_1 - z_2\|_{\delta,s}, \quad i = 1, 2
\]

where \( L = C_2e^{-1}(R + \|z_0\|_{1,s})^2 + \sqrt{2} \), and

\[
\|F_1(z_0)\|_{\delta,s} \leq \frac{M}{1 - \delta'}, \quad i = 1, 2.
\]

with

\[
M = \frac{1}{2}C_2e^{-1}\|z_0\|_{1,s}^3 + \sqrt{2}\|z_0\|_{1,s}.
\]

Moreover, the analytic lifespan \( T \) satisfies the estimate

\[
T = \frac{1}{68C_2e^{-1}\|z_0\|_{1,s}^2 + 24\sqrt{2}}.
\]

Proof: We first prove that \( F_1(z) \) satisfies (86) and (87). For \( s > \frac{1}{2} \), applying the triangle inequality we get

\[
\|F_1(z_1) - F_1(z_2)\|_{\delta',s} \leq \frac{1}{2}\|(u_1v_1u_1,s - u_2v_2u_2,s)\|_{\delta',s} + \|\partial^{-1}_x u_1 - \partial^{-1}_x u_2\|_{\delta',s} := I + II.
\]
and (84) lead to
\[
I \leq \frac{1}{2} C^2 \left[ \|u_1\|_{\delta,s} \|v_1\|_{\delta,s} \|(u_1 - u_2)_x\|_{\delta,s} + \|u_1\|_{\delta,s} \|u_2\|_{\delta,s} \|v_1 - v_2\|_{\delta,s} + \|v_2\|_{\delta,s} \|u_2\|_{\delta,s} \|u_1 - u_2\|_{\delta,s} \right]
\]
\[
\leq \frac{1}{2} C^2 \left[ \|u_1\|_{\delta,s} \|v_1\|_{\delta,s} \|\frac{e^{-1}}{\delta - \delta'} (u_1 - u_2)\|_{\delta,s} + \|u_1\|_{\delta,s} \|\frac{e^{-1}}{\delta - \delta'} (u_2 - u_1)\|_{\delta,s} \right.
\]
\[
+ \left. \frac{1}{2} C^2 \|v_1\|_{\delta,s} \|u_2\|_{\delta,s} \|v_1 - v_2\|_{\delta,s} + \frac{1}{2} C^2 \|v_2\|_{\delta,s} \|u_2\|_{\delta,s} \|u_1 - u_2\|_{\delta,s} \right]
\]
\[
\leq \frac{1}{2} C^2 \left[ \frac{e^{-1}}{\delta - \delta'} (R + \|z_0\|_{1,s})^2 \right] z_1 - z_2 \|\delta,s\|,
\]
where we have used the following estimates
\[
\|u_i\|_{\delta,s} \leq \|u_i - u_0\|_{\delta,s} + \|u_0\|_{\delta,s} \leq R + \|z_0\|_{1,s}, \quad i = 1, 2
\]
and
\[
\|v_i\|_{\delta,s} \leq \|v_i - v_0\|_{\delta,s} + \|v_0\|_{\delta,s} \leq R + \|z_0\|_{1,s}, \quad i = 1, 2.
\]

Next, we estimate the term II. Lemma 10 gives
\[
II = \|\partial_x^{-1}(u_1 - u_2)\|_{\delta',s} \leq \sqrt{2} \|(u_1 - u_2)\|_{\delta',s} \leq \frac{\sqrt{2}}{\delta - \delta'} |z_1 - z_2|_{\delta,s},
\]
where we have adopted the fact that $0 < \delta' < \delta < 1$, which implies $0 < \delta - \delta' < 1$.

Adding (99) and (101) gives the desired condition (86)
\[
\|F_1(z_1) - F_1(z_2)\|_{\delta',s} \leq \frac{L}{\delta - \delta'} |z_1 - z_2|_{\delta,s},
\]
with $L = C^2 b^2 (R + \|z_0\|_{1,s})^2 + \sqrt{2}$.

Finally, we estimate $F_1(z_0)$. (83) and Lemma 10 lead to
\[
\|F_1(z_0)\|_{\delta',s} \leq \frac{1}{2} \|u_0 v_0 u_{0,x} s + \|\partial_x^{-1} u_0\|_{\delta',s}
\]
\[
\leq \frac{1}{2} C^2 \|u_0\|_{\delta,s} \|v_0\|_{\delta,s} \|u_{0,x}\|_{\delta,s} + \sqrt{2} \|u_0\|_{\delta,s}
\]
\[
\leq \frac{1}{2} C^2 \|u_0\|_{\delta,s} \|v_0\|_{\delta,s} \|u_{0,x}\|_{\delta,s} \frac{e^{-1}}{\delta - \delta'} \|u_0\|_{\delta,s} + \frac{\sqrt{2}}{\delta - \delta'} \|u_0\|_{\delta,s}
\]
\[
\leq \frac{1}{2} C^2 \frac{e^{-1}}{\delta - \delta'} |z_0|_{\delta,s} + \frac{\sqrt{2}}{\delta - \delta'} |z_0|_{\delta,s}
\]
Replacing $\delta'$ by $\delta$ and $\delta$ by 1, and setting
\[
M = \frac{1}{2} C^2 e^{-1} |z_0|_{1,s}^3 + \sqrt{2} |z_0|_{1,s},
\]
we obtain the desired estimate (87), namely,
\[
\|F_1(z_0)\|_{\delta,s} \leq \frac{M}{1 - \delta}.
\]
The symmetric structure of Eq. (91) immediately reads
\[ \|F_2(z_1) - F_2(z_2)\|_{\delta', s} \leq \frac{L}{\delta - \delta'} \|z_1 - z_2\|_{\delta, s}, \quad (103) \]
and
\[ \|F_2(z_0)\|_{\delta, s} \leq \frac{M}{1 - \delta}, \]
where \( L \) and \( M \) are the same as above. Thus, we obtain
\[ \|F(z_1) - F(z_2)\|_{\delta', s} \leq \frac{L}{\delta - \delta'} \|z_1 - z_2\|_{\delta, s}, \quad (104) \]
and
\[ \|F(z_0)\|_{\delta, s} \leq \frac{M}{1 - \delta}. \quad (105) \]

Therefore, substituting the above \( L \) and \( M \) into (88) yields
\[ T = \frac{R}{16LR + 8M} = \frac{R}{16[C^2e^{1}(R + \|z_0\|_{1, s})^2 + \sqrt{2}]R + 8(\frac{1}{2}C^2e^{1}\|z_0\|_{1, s}^2 + \sqrt{2}\|z_0\|_{1, s})}. \]

Thanks to Theorem 8, there exists a unique solution \( z(t) \) to the Cauchy problem (91), which is a holomorphic vector function for every \( \delta \in (0, 1) \) in \( D(0, T(1 - \delta)) \) \( G^{\delta, s} \times G^{\delta, s} \) and
\[ \sup_{|t| < T(1 - \delta)} \|z(t) - z_0\|_{\delta, s} < R. \]

Let \( R = \|z_0\|_{1, s} \), then we have
\[ T = \frac{1}{68C^2e^{1}\|z_0\|_{1, s}^2 + 24\sqrt{2}}. \]

This completes the proof of Lemma 13 and hence Theorem 12 is true.

5.2 Continuity of the data-to-solution map

We now prove the continuity of the data-to-solution map for initial data and solution in Theorem 3.1.

First, let us recall that the scale of Banach spaces \( X_{\delta} \) and the function \( F(u) \) satisfy the conditions (1) and (2). For \( b > 0 \), we denote by \( H(|t| < b; X_{\delta} \times X_{\delta}) \) the set of holomorphic vector functions \( f(t) \) in \( |t| < b \) valued in \( X_{\delta} \times X_{\delta} \). Also, notice that for \( 0 < \delta \leq 1 \) and \( w \in H(|t| < b; X_{\delta} \times X_{\delta}) \) with \( b > 0 \), the equation
\[ \frac{dz}{dt} = F(z), \quad z(0) = z_0, \quad (106) \]
has a unique solution \( z \in H(|t| < b; X_\delta \times X_\delta) \) given by
\[
z(t) = z_0 + Kw(t) := z_0 + \int_0^t w(\tau)d\tau.
\]
(107)

Therefore, it follows that the existence of \( z \) in Theorem 8 is equivalent to the existence of \( z \in H(|t| < T(1 - \delta); X_\delta \times X_\delta) \) for every \( \delta \in (0, 1) \), satisfying for \( |t| < T(1 - \delta) \)
\[
\| \int_0^t w(\tau)d\tau \|_\delta < R
\]
and
\[
w = F(z_0 + Kw).
\]
(109)

Therefore, our initial value problem is converted to find the fixed point of the equation (109). To see that, let us introduce a new space \( E_a \).

**Definition 14** For \( a > 0 \) we denote by \( E_a = \bigcap_{0 < \delta < 1} H(D(0, a(1 - \delta)); X_\delta) \) the Banach space of all functions \( t \mapsto u(t) \) where for every \( 0 < \delta < 1 \) we have
\[
u : \{t : |t| < a(1 - \delta)\} \rightarrow X_\delta \text{ is holomorphic},
\]
(110)
whose norm is defined by
\[
\||u||_a := \sup_{0 < \delta < 1} \{\|u(t)\|_\delta(1 - \delta)\sqrt{1 - \frac{|t|}{a(1 - \delta)}}\} < \infty.
\]
(111)

**Remark 15** One may easily see that \( E_{T_2} \hookrightarrow E_{T_1} \) for any \( 0 < T_1 < T_2 \).

**Remark 16** It is clear from the proof of Theorem 12 that under the hypotheses (1) and (2) that given \( z_0 \in G^{1,s} \times G^{1,s} \) and \( R > 0 \) there are \( T > 0 \) and a unique solution to the Cauchy problem (97) in the set
\[
E_{T,R} := \{z(t) \in \bigcap_{0 < \delta < 1} H(D(0, T(1 - \delta)); G^{\delta,s} \times G^{\delta,s}) \text{ and } \sup_{|t| < T(1 - \delta)} \|z(t) - z_0\|_{\delta,s} < R, \ 0 < \delta < 1\}.
\]
Notice that if \( z \in E_{T,R} \) then \( z \in E_T \). Thus, this allows us endow \( E_{T,R} \) with the metric \( d(z, w) = \||z - w||_T \).

Using the spaces \( E_a \) and the norm (111) we may readily obtain the following two lemmas. We refer the readers to [33] for the detailed proofs.
Lemma 17 Let \( a > 0, u \in E_\alpha, 0 < \delta < 1 \) and \(|t| < a(1 - \delta)\). Then

\[
\| Ku(t) \|_\delta \leq \int_0^{|t|} \| u(\tau \frac{t}{|t|}) \|_\delta d\tau \leq 2a\| u \|_\alpha.
\]

Lemma 18 For every \( a > 0, u \in E_\alpha, 0 < \delta < 1 \) and \(|t| < a(1 - \delta)\), we have

\[
\int_0^{|t|} \frac{|u(\tau \frac{t}{|t|})| \delta(\tau)}{\delta(\tau) - \delta} d\tau \leq \frac{8a\| u \|_\alpha}{1 - \delta} \sqrt{\frac{a(1 - \delta)}{a(1 - \delta) - |t|}},
\]

where \( \delta(\tau) = \frac{1}{2}(1 + \delta - \frac{|\tau|}{\delta}) \).

Next, let us recall the following definition of the continuity of the data-to-solution map for Eq. 91.

**Definition 19** [23] One says that the data-to-solution map \( z_0 \mapsto z(t) \) is continuous if for a given \( z_\infty(0) \in G^{1,s} \times G^{1,s} \) there exist \( T = T(||z_\infty(0)||_{1,s}) > 0 \) and \( R > 0 \) such that for any sequence of initial data \( z_n(0) \in G^{1,s} \times G^{1,s} \) converging to \( z_\infty(0) \) in \( G^{1,s} \times G^{1,s} \) the corresponding solutions, \( z_n(t), z_\infty(t) \) to the Cauchy problem 91 for all sufficiently large \( n \) satisfy: \( z_n(t), z_\infty(t) \in E_{T,R} \) and

\[
||z_n(t) - z_\infty(t)||_T = ||u_n(t) - u_\infty(t)||_T + ||v_n(t) - v_\infty(t)||_T \to 0,
\]

where

\[
||u||_T := \sup\{|u(t)||\delta(1-\delta)\sqrt{1 - \frac{|t|}{T(1-\delta)}} : 0 < \delta < 1, |t| < T(1-\delta)\} < \infty.
\]

We now give the continuity of the solution map for the Cauchy problem 91.

Theorem 20 Given \( z_0 \in G^{1,s} \times G^{1,s}, s > \frac{1}{2}, \) and \( R > 0 \) there exists \( T = T_{z_0} > 0 \), which is given in [33], such that the Cauchy problem for 91 has a unique solution

\[
z \in E_{T,R} := \{ z(t) \in \bigcap_{0 < \delta < 1} H(D(0,T(1-\delta)); G^{5,s} \times G^{5,s}) \}
\]

and

\[
\sup_{|t| < T(1-\delta)} \| z(t) - z_0 \|_\delta < R, \ 0 < \delta < 1 \}
\]

Moreover the data-to-solution map \( z_0 \mapsto z(t) : G^{1,s} \times G^{1,s} \mapsto E_{T,R} \) is continuous.

Proof Let \( s > \frac{1}{2}, z_\infty(0) \in G^{1,s} \times G^{1,s} \) be given. And let \( z_n(0) \in G^{1,s} \times G^{1,s} \) be a sequence of initial data converging to \( z_\infty(0) \), that is \( ||z_n(0) - z_\infty(0)||_{1,s} \to 0 \), as \( n \to \infty \). Therefore, there exists a natural integer \( N \in \mathbb{N} \), such that for any \( n \geq N \), we have

\[
||z_n(0)||_{1,s} \leq ||z_\infty(0)||_{1,s} + 1. \quad (112)
\]

Setting

\[
R_\infty = ||z_\infty(0)||_{1,s} + 1, \quad (113)
\]
and for $n > N$

$$R_n = R_\infty + \|z_n(0) - z_\infty(0)\|_{1,s} \leq R_\infty + 1.$$  \hspace{1cm} (114)

For the given initial data $z_\infty(0), z_n(0) \in G^{1,s} \times G^{1,s}$, Theorem 12 ensures the existence and uniqueness of the corresponding solutions $z_\infty(t) \in E_{T_{z_\infty}(0), R_\infty}$ and $z_n(t) \in E_{T_{z_n}(0), R_n}$ with

$$z_\infty(t) = z_\infty(0) + K(F(z_\infty(t))), \text{ for } |t| < T_{z_\infty}(0)(1 - \delta),$$  \hspace{1cm} (115)

$$z_n(t) = z_n(0) + K(F(z_n(t))), \text{ for } |t| < T_{z_n}(0)(1 - \delta),$$  \hspace{1cm} (116)

respectively, where their lifespans are given by

$$T_{z_\infty}(0) = \frac{1}{68C_2^2e^{-1}\|z_\infty(0)\|_{1,s}^2 + 24\sqrt{2}}$$

$$T_{z_n}(0) = \frac{1}{68C_2^2e^{-1}\|z_n(0)\|_{1,s}^2 + 24\sqrt{2}},$$

respectively.

Let us now figure out the same lifespan of $z_\infty(t)$ and $z_n(t)$ by setting $T$ as follows

$$T_{z_\infty}(0), z_n(0) = \frac{1}{68C_2^2e^{-1}R^2 + 24\sqrt{2}},$$

where

$$R = 2R_\infty + 1.$$  \hspace{1cm} (117)

(112), (113) and (117) imply that $T_{z_\infty}(0), z_n(0) < T_{z_\infty}(0)$ and $T_{z_\infty}(0), z_n(0) < T_{z_\infty}(0), z_n(0)$, that is $T_{z_\infty}(0), z_n(0) < \min\{T_{z_\infty}(0), z_n(0)\}$.

For $n \geq N$, noticing $E_{T_{z_\infty}(0), R_\infty} \hookrightarrow E_{T_{z_\infty}(0), z_n(0), R}$ and $E_{T_{z_n}(0), R_n} \hookrightarrow E_{T_{z_\infty}(0), z_n(0), R}$ gives $z_\infty(t), z_n(t) \in E_{T_{z_\infty}(0), z_n(0), R}$.

Next, we need to prove $||z_n - z_\infty||_{T_{z_\infty}(0), z_n(0)} \to 0$ as $n \to \infty$. For $0 < \delta < 1$ and $|t| < T_{z_\infty}(0), z_n(0)(1 - \delta)$ it follows from (115) and (116) that

$$||z_n - z_\infty||_{\delta,s} \leq \|K[F(z_n(t)) - F(z_\infty(t))]|_{\delta,s} + ||z_n(0) - z_\infty(0)||_{\delta,s}. \hspace{1cm} (118)$$

Using Lemma 17 (the complete proof of the lemma can be found in Lemma 6 [33]), we have

$$\int_0^t \|F(z_\infty(y)) - F(z_n(y))\|_{\delta,s} dy \leq \int_0^{|t|} \|F(z_\infty(\tau \frac{t}{|t|})) - F(z_n(\tau \frac{t}{|t|}))\|_{\delta,s} d\tau,$$  \hspace{1cm} (118)
and therefore, we get
\[ \| z_n(t) - z(t) \|_{\delta,s} - \| z_n(0) - z(t) \|_{\delta,s} \]
\[ \leq \int_0^{|t|} \| F(z(\tau_T - t) - F(z(\tau_T)) \|_{\delta,s} d\tau. \] (119)

In order to use (86), we also need to prove that
\[ \| z(\tau_T - t) - z(0) \|_{\delta(\tau),s} < R, \]
and
\[ \| z(\tau_T) - z(0) \|_{\delta(\tau),s} < R. \]

Apparently, if \( 0 < \delta < 1 \) and \( |t| < T_{z_{\infty}(0),z(0)}(1 - \delta), 0 \leq |\tau| = |t|, \delta < \delta(\tau) \leq \frac{1}{2}(1 + \delta - \frac{|\tau|}{T_{z_{\infty}(0),z(0)}}) \) and \( n > N \), then we have
\[ \| z(\tau_T - t) - z(0) \|_{\delta(\tau),s} \]
\[ \leq \| z(\tau_T) - z(0) \|_{\delta(\tau),s} + \| z(0) - z(0) \|_{\delta(\tau),s} \]
\[ < R_\infty + 1 < R. \] (120)

Theorem 8 and (114) ensure that
\[ \| z_n(\tau_T - t) - z_n(0) \|_{\delta(\tau),s} \leq R_n < R. \] (121)

[120], [121], [86] and Lemma 18 lead to
\[ \| z_n(t) - z(\tau_T - t) \|_{\delta,s} - \| z_n(0) - z(0) \|_{\delta,s} \]
\[ \leq \int_0^{|t|} \| F(z(\tau_T - t) - F(z(\tau_T)) \|_{\delta,s} d\tau \]
\[ \leq L_n \int_0^{|t|} \frac{\| z(\tau_T - t) - z(\tau_T) \|_{\delta(\tau),s}}{\delta(\tau) - \delta} d\tau. \] (122)

with \( L_n = C_2 e^{-1}(R_n + \| z_0 \|_{1,s})^2 + \sqrt{2} \) and \( \delta(\tau) = \frac{1}{2}(1 + \delta - \frac{|\tau|}{T_{z_{\infty}(0),z(0)}}) \). Noticing for \( |\tau| < T_{z_{\infty}(0),z(0)}(1 - \delta) \), we have \( 0 < \delta < \delta(\tau) < 1 \).

As per Lemma 18 (with \( a = T_{z_{\infty}(0),z(0)} \), and [122], for \( 0 < \delta < 1 \) and \( |t| < T_{z_{\infty}(0),z(0)}(1 - \delta) \) we obtain
\[ \| z_n(t) - z(\tau_T - t) \|_{\delta,s} - \| z_n(0) - z(0) \|_{\delta,s} \]
\[ \leq 8 T_{z_{\infty}(0),z(0)} L_n \| z_{\infty} - z_n \|_{T_{z_{\infty}(0),z(0)}} \sqrt{T_{z_{\infty}(0),z(0)}(1 - \delta)} \] 
which reveals that
\[ \| z_n(t) - z(\tau_T - t) \|_{T_{z_{\infty}(0),z(0)}} \]
\[ \leq 8 T_{z_{\infty}(0),z(0)} L_n \| z_{\infty} - z_n \|_{T_{z_{\infty}(0),z(0)}} + \| z_n(0) - z(0) \|_{\delta,s} \]
in turns implies that

\[(1 - 8T_{z_\infty(0), z_n(0)} L_n) \|z_n(t) - z_\infty(t)\|_{T_{z_\infty(0), z_n(0)}} \leq \|z_n(0) - z_\infty(0)\|_{\delta,s}\] (123)

For \(n > N\), (113), (114) and (117) give

\[L_n = C_2^s e^{-1} (R_n + \|z_n(0)\|_{1,s})^2 + \sqrt{2}\]
\[\leq C_2^s e^{-1} (R_n + \|z_n(0) - z_\infty(0)\|_{1,s} + \|z_\infty(0)\|_{1,s})^2 + \sqrt{2}\]
\[\leq C_2^s e^{-1} (R_n + 1 + \|z_\infty(0)\|_{1,s})^2 + \sqrt{2}\]
\[= C_2^s e^{-1} (R_n + R_\infty)^2 + \sqrt{2}\]
\[\leq C_2^s e^{-1} R^2 + \sqrt{2}\] (124)

Noticing \(T_{z_\infty(0), z_n(0)} = \frac{1}{68C_2^s e^{-1} R^2 + 24\sqrt{2}}\) and (124), we have

\[8T_{z_\infty(0), z_n(0)} L_n \leq \frac{8C_2^s e^{-1} R^2 + 8\sqrt{2}}{68C_2^s e^{-1} R^2 + 24\sqrt{2}} < \frac{1}{2},\]

which implies that

\[\|z_n(t) - z_\infty(t)\|_{T_{z_\infty(0), z_n(0)}} \leq \frac{1}{1 - 8T_{z_\infty(0), z_n(0)} L_n} \|z_\infty - z_n\|_{1,s}\]
\[\leq 2\|z_n(0) - z_\infty(0)\|_{1,s}.\] (125)

This completes the proof of Theorem 20.

Remark 21 Theorem 20 showed important results since it makes the 2SP system (90) to be well-posed in the spaces \(G^{\delta,s} \times G^{\delta,s}\) in the sense of Hadamard. One may compare these results with the classical Cauchy-Kovalevski theorem, where there is no continuity of the data-to-solution map.

6 Conclusions

In this paper, we have used the Darboux transformation to solve the 2SP system (2) with loop, anti-loop, and multi-loop soliton solutions. Those solutions are explicitly given and graphically depicted through the plotted graphs. The approach utilized in this paper may be developed in the following ways:

(i) In light of the transformation (5) and Darboux transformation (43), we may obtain the
following parametric representation of soliton solutions to equations (6)

\[ x[N] = x[0] + \frac{\Delta_{A_{N-1}}}{\Delta_{N-1}}, \]

\[ q = \frac{1}{2} \left( u[0] + v[0] + \frac{\Delta_{B_{N-1}} + \Delta_{C_{N-1}}}{\Delta_{N-1}} \right), \]

\[ r = -\frac{i}{2} \left( u[0] - v[0] + \frac{\Delta_{B_{N-1}} - \Delta_{C_{N-1}}}{\Delta_{N-1}} \right), \]

where \( \Delta_{N-1}, \Delta_{A_{N-1}}, \Delta_{B_{N-1}}, \) and \( \Delta_{C_{N-1}} \) are given through (44)-(47).

(ii) The \( N \)-soliton solution of the short pulse equation (1) could be regained as a reduction of the Darboux transformation (43) with (63) under the conditions \( \lambda_{2k} = -\lambda_{2k-1} \) and \( \mu_{2k} \mu_{2k-1} = -1 \) \( (k = 1, 2, \ldots, N) \).

(iii) Our 2SP system (2) could be extended to a two-component complex short pulse model, which we shall pursue a deeper study in the near future.

In the analysis aspect of the 2SP system (2), we adopted the abstract Ovsyannikov type theorem and proved the well-posedness of the Cauchy problem for the 2SP system (2) provided that the initial data are analytic. It is very interesting to study the existence and the uniqueness of the global weak solutions, which we will discuss elsewhere. Furthermore, as mentioned above, the analysis work for the complex short pulse model is also deserved to investigate.

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