ON EXTRA COMPONENTS IN THE FUNCTIORAL COMPACTIFICATION OF $A_g$

VALERY ALEXEEV

Recall the following from the theory of toroidal compactifications of moduli of polarized abelian varieties (Mumford et al [AMMT75] over C, Faltings and Chai [FC90] over $\mathbb{Z}$). Denote $X = \mathbb{Z}^g$ and let $C$ be the convex hull in the space $\text{Sym}^2(X_\mathbb{R})$ of semipositive symmetric matrices $q$ with rational null-space. For any admissible $GL(X)$-invariant decomposition $\tau$ of $C$ (i.e. it is a face-fitting decomposition into finitely generated rational cones such that there are only finitely many cones modulo $GL(X)$) there is a compactification $\overline{A}_g^\tau$ of the moduli space $A_g$ of principally polarized abelian varieties. $\overline{A}_g^\tau$ comes with a natural stratification, and strata correspond in a 1-to-1 way to cones in $\tau$ modulo $GL(X)$. There are infinitely many such decompositions $\tau$ and none of the seems to be better than another. True, some decompositions are smooth and projective but still there are infinitely many of these as well.

There is, however, a decomposition $\tau_{\text{Vor}}$ for the 2nd Voronoi decomposition which has a nicer geometric description. Strata of this compactification $\overline{A}_g^{\text{Vor}}$ still are in bijection with cones of $\tau_{\text{Vor}}$, however, these cones now correspond in a 1-to-1 way to special $X$-periodic face fitting decompositions of $X_\mathbb{R}$ with vertices in $X$, called Delaunay. A form $q$ defines a distance function $d_q$ on $X_\mathbb{R}$ and a cell of the Delaunay decomposition Del$_q$ (Delaunay cell) is a convex hull of integral points circumscribed by an “empty sphere”, a sphere that does not contain any integral points in its interior. As an example, for $g = 2$ there are 4 such decompositions: by 4-gons, by triangles, by infinite strips and finally the decomposition consisting of one big cell covering the whole plane. The decompositions appearing are either polytopal or the preimages of such from a lower dimension.

On the other hand, in [Ale99] I have constructed a functorial compactification $\overline{A}_g^P$ of $A_g$ as the moduli of triples $G \acts P \supset \Theta$ whose geometric fibers have the following description: $G$ is semiabelian, $P$ is projective reduced connected and $\Theta$ is a Cartier divisor, all satisfying a few natural conditions:

1. $P$ is seminormal,
2. there are only finitely many orbits,
3. $\Theta$ does not contain any orbit entirely,
4. for any $p \in P$, the stabilizer of $p$ is connected and reduced and lies in the toric part of $G$.

This compactification comes with a stratification as well, with strata corresponding to all $X$-periodic face fitting decompositions of $X_\mathbb{R}$ with vertices in $X$. Just from this rough description we see that $\overline{A}_g^{\text{Vor}}$ and $\overline{A}_g^P$ must be very closely related.
and at the same time be different. The connection, according to [Ale99], is that \( \overline{X}_g^{\text{Vor}} \) coincides with the main irreducible component of \( \overline{A}_g^g \). The most obvious difference is that, unlike \( \overline{X}_g^{\text{Vor}}, \overline{A}_g^g \) may have Extra Types of irreducible components, ETs for short, which will be the focus of this note. We would like to discuss:

1. Where and why ETs appear.
2. How to find ETs and how to study them.
3. Some concrete evidence of ETs.
4. Dimension 4 case in detail.
5. Relationship between ETs and the Jacobian locus.

1. WHERE AND WHY ETs APPEAR

There is actually a very simple reason for their appearance. Say, \((P', \Theta')\) is a pair with an abelian action and let \((P, \Theta)\) be its degeneration. Assume that \((P, \Theta)\) has several components \(P_i\) and set \(\Theta_i = \Theta|_{P_i}\). Think of the deformations of \((P, \Theta)\) that keep the decomposition into the irreducible components. These deformations correspond to deformations of components \((P_i, \Theta_i)\) that are compatible on intersections. In the simplest case, when the intersections are elementary, there are, perhaps, no gluing conditions at all. Now, it is not hard to imagine the situation when the sum of the dimensions of the deformation spaces for pairs \((P_i, \Theta_i)\) is larger than the dimension of the deformation space of the constrained smooth pair \((P', \Theta')\), i.e. \(g(g + 1)/2\). In this case there must be another irreducible component.

So, the answer to the “where” part is that ETs appear at the boundary of the world as we know it.

2. HOW TO FIND ETs AND HOW TO STUDY THEM

There are two basic methods:

1. Find a periodic non-Delaunay decomposition. The first such decomposition appears in dimension 4, and we will describe it in more detail below.
2. For a Delaunay decomposition \(\Delta\), compute the dimension of Stratum \(\Delta\) in \(\overline{A}_g^g\). If it is higher than dimension of the corresponding stratum in \(\overline{X}_g^{\text{Vor}}\), then there must be an ET nearby. There is a remarkably simple formula for the first of these dimensions which we are now going to describe.

Let \(\Delta\) be an \(X\)-periodic face-fitting decomposition of \(X_\mathbb{R}\) with vertices in \(X\). We will identify \(\Delta\) with its quotient \(\Delta\) which is a decomposition of the real torus \(X_\mathbb{R}/X \cong \mathbb{R}^g/\mathbb{Z}^g\). If we work over \(\mathbb{C}\), this decomposition is directly related to the properties of each pair \((P, \Theta)\) in the \(\Delta\)-stratum of \(\overline{A}_g^g\), and this connection is described by using the moment map. (Even if the base field is not \(\mathbb{C}\), this decomposition describes the main properties of \((P, \Theta)\) very faithfully.)

Say, \(G \curvearrowright P \supset \Theta\) is a triple as before, and start with the case when \(G = T = (\mathbb{C}^*)^g\) is a torus with the character group \(X\). The moment map sends \(P(\mathbb{C})\) to its quotient by the action of the compact torus \(CT = (S^1)^g = U(1)^g \subset T\). If \(L = \mathcal{O}(\Theta)\) is \(T\)-linearized then we can describe the moment map more directly. Let \(\theta \in H^0(P, L)\) be an equation of \(\Theta\). \(H^0(P, L)\) splits into the direct sum of \(T\)-eigenspaces and we can write \(\theta\) as a finite sum \(\sum_{x \in \mathcal{X}} \xi_x\). Then

\[
\text{Mom} : P(\mathbb{C}) \ni p \mapsto \frac{\sum_{x \in \mathcal{X}} |\xi_x(p)|^2 \cdot x}{\sum |\xi_x(p)|^2} \in X_\mathbb{R}
\]
It is well defined if at each \( p \in P \) at least one of \( \xi_x \)'s is not zero, and that is one of the conditions on \((G, P, \Theta)\) that is satisfied by our definition of a triple.

If \( L \) is not linearized then \((P, L)\) is in a canonical way the quotient by \( X \) of a pair \((\tilde{P}, \tilde{L})\) with linearized \( \tilde{L} \) (the scheme \( \tilde{P} \) is only locally of finite type but the action of \( X \) is properly discontinuous in Zariski topology). We take \( \tilde{\theta} \in H^0(\tilde{P}, \tilde{L}) \) to be the pullback of \( \theta \). It is a fact that for any \( \tilde{p} \in \tilde{P} \) the sum \( \sum_{x \in X} \xi_x \) is finite as almost all of \( \xi_x \) vanish at \( \tilde{p} \). The moment map \( \text{Mom}_{\tilde{P}, \tilde{\theta}} \) commutes with the translation action of \( X \), and we define \( \text{Mom}_{P, \theta} \) from the following diagram

\[
\begin{array}{ccc}
\tilde{P}(\mathbb{C}) & \xrightarrow{\text{Mom}_{P, \theta}} & X_R \\
/ \sim & / \sim & / \sim \\
\tilde{P}(\mathbb{C}) & \xrightarrow{\text{Mom}_{\tilde{P}, \tilde{\theta}}} & X_R / X
\end{array}
\]

Finally, if \( G \) is an arbitrary semiabelian variety then we can (there is a choice involved) write \( G \) as the quotient \((\mathbb{C}^*)^{g'} / X', X \supset X' = \mathbb{Z}^{g'} \) and \( g' \) as the dimension of the abelian part of \( G \), making \( P \) into a “toric” variety. The \( T \)-action here is not algebraic, of course, but we can still repeat the above definition. The sum \( \sum |\xi_x(p)|^2 \cdot x \) in this case is truly infinite, however, it is convergent because theta functions have exponential decline for large \( x \).

With these definitions in mind, let \((G, P, \Theta)\) be a triple corresponding to a point in \( \text{Stratum}[\Delta] \subset A^g_q \). Then the moment map sends \( P \) to \( X_R / X \) and for any \( z \in \delta^0 \), interior of \( \delta \), one has \( \text{Mom}^{-1}(z) = (S^1)^{\dim \delta} \). Moreover, \( P \) has as many irreducible components as there are cells in \( \Delta \), and these intersect exactly in the way the corresponding cells intersect. Moreover, if \( \delta \) is a polytope then the irreducible component \( P_{\delta} \) is the projective toric variety corresponding to \( \delta \), glued in a way that the decomposition suggests (in particular, all of its 0-dimensional orbits are glued together).

As triples degenerate, so do the moment maps. If the fiber over a particular point was 0-dimensional, it is not going to get bigger. On the other hand, a big fiber may get “squashed” to a smaller one in the limit, so the degeneration of triples of type \( \Delta \) must correspond to a subdivision of \( \Delta \). A familiar example is that of an elliptic curve degenerating to a nodal curve. An elliptic curve has a moment map to a circle whose every fiber is \( S^1 \), and in the limit a fiber over one point becomes 0-dimensional. An elliptic curve corresponds to a decomposition \( \Delta \) with one big cell, and the nodal curve – to a decomposition of \( \mathbb{R} \) into intervals. In higher dimensions exactly the same happens but the decompositions \( \Delta \) that appear are more sophisticated.

**Definition 2.1.** For a “simple” cell \( \delta \) we define \( \mathbb{L}_{\Delta, \mathbb{R}} \) to be the space of all \( \mathbb{R} \)-valued functions on \( \delta \cap X \) modulo the subspace of linear nonhomogeneous functions. \( C^0(\Delta, \mathbb{L}) \) is defined as

\[
\bigoplus_{\dim \delta = g} \mathbb{L}_{\Delta, \mathbb{R}}
\]

and \( H^0(\Delta, \mathbb{L}) \) as the subspace of \( C^0 \) consisting of functions that coincide on “intersections”.
The reader will notice that we are computing here the space of global sections of a certain constructible sheaf $\hat{\mathcal{L}}$ on $X_R/X$ which is constant on locally closed strata defined by $\Delta$.

**Theorem 2.2** ([Ale99]). \( \dim \text{Stratum}[\Delta] = h^0(\Delta, \hat{\mathcal{L}}) \)

The meaning of “simple” and “intersections” should be clear from the following instructive examples.

**Example 2.3.** Say, $\tilde{\Delta}$ consists of just one big cell, $\mathbb{R}^g$ – this is the stratum corresponding to triples with abelian $G$. Then $C^0(\hat{\mathcal{L}})$ is the space of all real-valued functions on $X = \mathbb{Z}^g$ modulo the $(g + 1)$-dimensional subspace of linear nonhomogeneous functions. $H^0(\hat{\mathcal{L}})$ is the subspace of functions that are invariant under the translations by $X$. If $[f]$ is the equivalence class function of such a function then for any $y \in X$ the function $g(x) = f(y + x) - f(x)$ must be linear, so $f(x)$ must be quadratic. Therefore, $H^0(\hat{\mathcal{L}})$ is the space of quadratic modulo linear functions and its dimension is $g(g + 1)/2$, i.e. the dimension of $A_g$, as expected.

**Example 2.4.** Let $\tilde{\Delta}$ be the decomposition of $\mathbb{R}^2$ into squares. There is only 1 maximal-dimensional cell modulo $X$, a square. It is easy to see that for any polytopal cell appearing in an $X$-periodic decomposition one has

$$l_\delta = \dim \hat{\mathcal{L}}_{\delta, \mathbb{R}} = \#(\text{vertices of } \delta) - \dim \delta - 1$$

For the square we have $l = 1$ and for the intervals on the boundary: $l = 0$, so there are no gluing conditions. Hence, the dimension of the corresponding stratum is $h^0(\hat{\mathcal{L}}) = 1$.

**Example 2.5.** Let us generalize the previous example slightly. Assume that all cells of $\tilde{\Delta}$ are polytopal and that all maximal cells are simplicial, i.e. all of its proper faces are simplices. Then again $C^1(\hat{\mathcal{L}}) = \oplus_{\dim \delta = g-1} \hat{\mathcal{L}}_{\delta, \mathbb{R}} = 0$ and one obtains

$$h^0(\hat{\mathcal{L}}) = \sum_{\dim \delta = g} l_\delta.$$  

This very simple formula has many applications as we will see.

[Ale99] contains a more detailed information about $\text{Stratum}[\Delta]$. In particular, if all the cells are polytopal then the normalization of the closure of this stratum is the quotient by the finite group $\text{Sym} \Delta$ of the projective toric variety corresponding to the so called generalized secondary polytope $\Sigma(\Delta)$.

### 3. Some concrete evidence of ETs

**Theorem 3.1.** There are no ETs in dimension $g \leq 3$.

**Theorem 3.2.** For $g = 4$ there is exactly 1 ET and it is isomorphic to $\mathbb{P}^2$.

**Theorem 3.3.** For $g \geq 5$, $2^g - g - 3 \leq \dim \mathcal{N}_g < g!$

Hence, the maximal dimension of ETs grows at least exponentially. The dimension of the “main” component, of course, grows as a particular polynomial of degree 2, namely $g(g + 1)/2$. One might say that in higher dimensions ETs are a dominating life form.
Theorems 3.1 and 3.2 follow from the concrete computations of [AE] where it is proved that all periodic decompositions in dimension \( \leq 3 \) are Delaunay (easy) and that in dimension 4 there are exactly 2 non-Delaunay decompositions (hard). Both of them are sub decompositions of the unique maximal dicing with 9 hyperplanes, which I will denote by \( \Delta_{RT} \). The dimension of the stratum for \( \Delta_{RT} \) in \( \overline{A} \) is 1. On the other hand, \( h^0(\Delta, \hat{L}) = 2 \) and so there must be a second irreducible component. The secondary polytope \( \Sigma(\Delta_{RT}) \) is a square which corresponds to \( \mathbb{P}^1 \times \mathbb{P}^1 \), which gives \( \mathbb{P}^2 \) after dividing by \( \text{Sym} \Delta_{RT} = \mathbb{Z}_2 \times \mathbb{Z}_2 \).

The lower bound in theorem 3.3 follows by computing a particular example, the Delaunay decomposition of the classical lattice \( D_n \). One knows (see f.e. [CS93]) that there are 3 maximal cells in this decomposition, 2 copies of a hemicube \( hgamma_n \) (with \( 2g-1 \) vertices) and a crosspolytope \( beta_n \) (with \( 2g \) vertices), and that these polytopes are simplicial. Applying formula 1 gives the bound. The upper bound follows from a simple observation that for a polytopal decomposition the dimension of a secondary polytope \( \Sigma(\Delta) \) is always less than the volume of \( |\Delta| \) (in the lattice units of volume). The volume of \( X_g/X \) is \( g! \). If the decomposition is not polytopal but is the pullback of a polytopal decomposition in \( \mathbb{R}^g-a \), one also has to add the term \( a(a+1)/2 + a(g-a) \) to \( (g-a)! \) to account for the abelian part. However, it is easy to see that for \( g \geq 3 \) the bound \( g! \) is greater.

I note another nice application of formula 1: lattice \( E_8 \) whose cells are 135 copies of a crosspolytope \( beta_8 \) and 1920 copies of a simplex \( alpha_8 \). The computation
gives \( \dim \text{Stratum}[E_8] = h^0(E_8, \hat{L}) = 945 \). This gives the largest ET in dimension 8 that I am aware of. The normalization of the closure of this stratum is the quotient of \( (\mathbb{P}^7)^{135} \) by a finite group.

4. Dimension 4 case in detail

The following is a classical description, due to Voronoi, of the 2nd Voronoi decomposition in dimension 4. The picture below gives a schematic view of a cross-section of this decomposition. First of all, in dimension 4 \( \tau_{\text{Vor}} \) is a subdecomposition of the perfect decomposition. The perfect decomposition has cones of two types: 1st domain, which are simplicial (with 10 sides) and cones with 64 sides. In \( \tau_{\text{Vor}} \) the cones of the second type are subdivided into 64 simplicial cones, 48 of which belong to the so called 2nd domain, and 16 – to the 3rd domain. As we have indicated, the 2nd domain cones have 1st domains as “across-the-border” neighbors, and 3rd domains – 3rd domains again.

It is exactly the face between two 3rd type domains where an ET attaches itself. The way it happens is easier to see on a dual picture. A dual picture is the picture for a polarization function on \( \tau_{\text{Vor}} \) (\( \overline{A} \) is projective, [Ale99]). Two maximal cones of 3rd type on the dual picture correspond to points, and the face between them – to an interval. In the moduli space this gives a \( \mathbb{P}^1 \) (which we have to divide by automorphisms which gives a \( \mathbb{P}^1 \) again). If we denote by \( \Delta_{RT} \) the corresponding decomposition then \( \dim \text{Stratum}[\Delta_{RT}] \) in \( \overline{A} \) is one. One the other hand, explicitly this decomposition consists of

1. a copy of a cyclic polytope \( C_6 \) with 6 vertices,
2. the inverse of it under the involution \( x \mapsto -x \),
3. 18 simplexes.
Hence, according to the formula of example 2.5 the dimension of the corresponding stratum in $\overline{\mathcal{M}}_g$ is $2 \cdot l_{C_6} = 2$. Hence, we must have an ET. This ET on the dual picture is represented by a square and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. After dividing by the automorphism group $\mathbb{Z}^2 \times \mathbb{Z}^2$ this gives a $\mathbb{P}^2$ in the coarse moduli space. As the picture suggests, the intersection of the main component in $\overline{\mathcal{M}}_g$ and the ET is a diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.

A vertex marked IV corresponds to a non-Delaunay triangulation obtained by triangulating one copy of $C_6$ in one of the two possible ways, and the other – in the non-symmetric way. Clearly, this triangulation cannot be Delaunay, indeed it does not even have the basic symmetry $x \mapsto -x$. The same can be said about the corresponding variety $P$.

5. **Relationship between ETs and the Jacobian locus**

The Torelli map $M_g \to A_g$ extends to a functorial morphism from the Mumford-Deligne compactification $\overline{M}_g$ to $\overline{\mathcal{M}}_g$. On $k$-points, it sends a stable curve $C$ to a triple $\text{Pic}^0 C \cap \text{Jac}^{g-1} C \supset \Theta_{g-1}$, where $\text{Jac}^{g-1} C$ is the moduli space of semistable rank 1 sheaves of degree $g - 1$ on $C$. [Ale96] contains an algorithm for computing this triple in terms of the dual graph of a stable curve $C$.

By checking all genus 4 stable curves we find that precisely 1 of them maps to a point meeting an ET. It is a curve all of whose components are $\mathbb{P}^1$'s and the dual graph is the bipartite graph $K_{3,3}$. By the Kuratowski theorem this is the minimal (in terms of genus) graph that is not planar. The image of this curve in $\overline{\mathcal{M}}_g$ is

**Figure 1.** 2nd Voronoi decomposition in dimension 4

**Figure 2.** Part of the dual picture with an ET attached
precisely the center $1 \in \mathbb{P}^1$ connecting two 3 type domains on the above figure. For any planar graph the corresponding Delaunay decomposition belongs to the 1st domain (or one of its faces) and so is away from an ET. Therefore, we obtain

**Theorem 5.1.** For a stable curve of genus $g \leq 4$ its Torelli image $[C] \in \overline{\mathcal{A}}F_g$ meets an ET if and only if the dual graph $C(\Gamma)$ is not planar.

One can easily make this theorem a little stronger: it is true for a stable curve of arbitrary genus such that $h^1(C(\Gamma)) \leq 4$. This leads me to make the following

**Conjecture 5.2.** Theorem 5.1 holds in arbitrary genus.

It feels that the non-planarity of $C$ somehow opens up new dimensions for deformations for its jacobian. Here is some additional evidence in favor of this conjecture:

1. If $\Gamma(C)$ is planar, the corresponding Delaunay decomposition belongs to the simplest part of the 2nd Voronoi decomposition, the so called 1st domain. If there is any part of $\overline{\mathcal{A}}Vor_g$ where “nothing tricky happens”, that should be it.
2. It is plausible that if $\Gamma(C)$ contains a subgraph $\Gamma(C')$ for a curve $C'$ whose jacobian has extra deformations then the same must be true for the curve $C$ itself. By the Kuratowski theorem a graph $\Gamma$ is non-planar if and only if it contains either a $K_{3,3}$ or a $K_5$. The case of $K_5$, therefore, becomes a crucial test for the validity of the conjecture.

**References**

[AE] V. Alexeev and R. Erdahl, *Periodic tilings in dimension \leq 4*, In preparation.

[Ale96] V. Alexeev, *Compactified jacobians*, Preprint (1996), alg-geom/9608012.

[Ale99] V. Alexeev, *Complete moduli in the presence of semialbelian group action*, Preprint (1999), math.AG/9905103.

[AMMT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactifications of locally symmetric varieties*, Lie groups: history, frontiers and applications, vol. IV, Math Sci Press, 1975.

[CS93] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, 2 ed., A Series of Comprehensive Studies in Mathematics, vol. 290, Springer-Verlag, 1993.

[FC90] G. Faltings and C.-L. Chai, *Degenerations of abelian varieties*, vol. 22, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 3, Springer-Verlag, 1990.

Department of Mathematics, University of Georgia, Athens, GA 30602

E-mail address: valery@math.uga.edu