Macroscopic Quantum Criticality in a Circuit QED

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Cavity quantum electrodynamic (QED) is studied for two strongly-coupled charge qubits interacting with a single-mode quantized field, which is provided by a on-chip transmission line resonator. We analyze the dressed state structure of this superconducting circuit QED system and the selection rules of electromagnetic-induced transitions between any two of these dressed states. Its macroscopic quantum criticality, in the form of ground state level crossing, is also analyzed, resulting from competition between the Ising-type inter-qubit coupling and the controllable on-site potentials.

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Introduction.— In cavity quantum electrodynamics (QED) [1], the coupling effects of two atoms [2] inside a cavity have been theoretically studied to explore exotic quantum coherent phenomena [3], e.g., coherent population trapping and dark states. However, the weak dipole-dipole interaction between atoms make it difficult to experimentally demonstrate these phenomena.

Recent experiments using on-chip superconducting qubits [4, 5, 6] show the strong-coupling effects between: i) two superconducting qubits [5, 6]; ii) a charge qubit and a superconducting transmission line resonator (TLR) [7, 8]; and iii) a flux qubit and an LC oscillator [9, 10]. The later two ones have been referred to as circuit QED. Moreover, recent experiments [8] demonstrated Rabi splitting and AC Stark shift in these QED circuits. Combining these approaches [4, 5, 6, 8], we propose how to observe the above mentioned quantum coherence effects using two coupled artificial atoms, inside a cavity, instead of two natural atoms [2].

Here, we present a circuit QED architecture for two capacitively-coupled charge qubits, interacting with a single-mode quantized field. We not only study the influence of strong inter-qubit coupling on quantum coherent effects, but also explore its macroscopic quantum criticality, via level-crossing, for quantum phase transition (QPT) [11]. Here, the non-analyticity of the ground state is depicted by the fact that the eigenstates are independent of the inter-qubit coupling, while the corresponding eigenvalues depend on the inter-qubit coupling. Of course, a rigorous QPT can only be realized in the thermodynamic limit, but some of the basic features of QPT can still be demonstrated in the form of level crossings for a system of few qubits [11, 12]. A toy-system for QPT, with two qubits, has been experimentally studied using NMR [13].

In contrast with a previous investigation for QPT [11], the two Ising-type-coupled charge qubits in our macroscopic quantum system are also coupled to a TLR. Thus, for the first time, we incorporate the dressed-state structure in the QPTs. Here, we consider a minimal circuit QED model to simulate the quantum critical behavior of the ground state.

Circuit QED model with two qubits.— Our circuit QED system is schematically shown in Fig. 1. Two identical SQUID-based charge qubits are capacitively coupled to each other with the coupling strength \( J = e^2 C_m / (2(C_2^2 - C_m^2)) \) through a capacitance \( C_m \), where \( C_2 \) is the sum of the capacitances connected to a single Cooper pair box. The level-spacing of each qubit is \( \omega_q = 2 E_c (C_g V_g - 1/2) \), with capacitance \( C_g \), bias voltage \( V_g \), and \( E_c = 2 e^2 C_2 / (C_2^2 - C_m^2) \). The two qubits are coupled to a single-mode quantized field which is realized as a 1D TLR with resonant frequency \( \omega_r = n_0 \pi / (L \sqrt{lc}) \). Here \( n_0 \) and \( L \) are the mode number of the resonant mode and the length of the TLR; \( l \) and \( c \) are the inductance and capacitance per unit length of the TLR. When the two dc
SQUIDs are placed at \(x_n = nL/n_0\), the coupling between the qubit and TLR is induced by the quantized magnetic field threading the dc SQUID. The model Hamiltonian \(H = H_Q + H_C\) includes the Ising part

\[
H_Q = \frac{1}{2} \omega_a \left( \sigma_z^{(1)} + \sigma_z^{(2)} \right) + J \sigma_z^{(1)} \sigma_z^{(2)}
\]

and the Jaynes-Cummings terms

\[
H_C = \omega a^\dagger a + \frac{g}{\sqrt{2}} \left( \sigma_+^{(1)} + \sigma_-^{(2)} \right) a + \text{h.c.}
\]

Hereafter, \(h = 1\), the Pauli matrices \(\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\), and \(\sigma_+ = |\uparrow\rangle\langle\downarrow|\) and \(\sigma_- = |\downarrow\rangle\langle\uparrow|\) are defined by the charge eigenstates \(|\uparrow\rangle\) and \(|\downarrow\rangle\) denoting 1 or 0 excess Cooper pair state, respectively. \(a^\dagger (a)\) is the creation (annihilation) operator of the resonant mode. The coupling strength is \(g = \frac{E_J \sqrt{A/4L}}{\Phi_0 \sqrt{L}}\) with the tunnelling energy \(E_J\), the enclosed area \(S\) of the dc SQUID, the distance \(d\) between the dc SQUID and the transmission line, and the flux quantum \(\Phi_0 = h/2e\).

The symmetry of two qubits enables us to rewrite the Hamiltonian \(H\) as a function of the total spin operators \(S = (\sigma^{(1)} + \sigma^{(2)})/2\) and \(S_\pm = \sigma_+^{(1)} + \sigma_-^{(2)}, \), e.g., \(\sigma_+^{(1)} \sigma_-^{(2)} = 2S_+^2 - 1\). The dynamical symmetry SO(3) of the Hamiltonian \(H\) results in a direct sum decomposition of the two-qubit Hilbert space \(V\), spanned by \(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\), i.e., \(V = V^{(0)} \otimes V^{(1)}\). Here, \(V^{(0)}\) is spanned by the singlet \(|\psi_s\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}\), while \(V^{(1)}\) is spanned by the triplet \(|\uparrow\uparrow\rangle, |\psi^+\rangle = (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)/\sqrt{2}\), and \(|\downarrow\downarrow\rangle\). \(V^{(0)}\) and \(V^{(1)}\) are invariant under \(H\). Thus, the total Hamiltonian \(H\) can be decomposed into a quasi-diagonal matrix with two blocks. Obviously, the symmetric couplings of the two qubits to the quantized field do not induce transitions between the singlet \(|\psi_s\rangle\) in \(V^{(0)}\) and any other states in the space \(V^{(1)}\). This is because the collective spin operator \(S_\pm = \sigma_+^{(1)} + \sigma_-^{(2)}\) can only change the state vectors within an irreducible subspace. Therefore, here, we denote the singlet \(|\psi_s\rangle\) as a “dark state” \([2]\). This consideration, based on group representations, automatically predicts the coherent population trapping in this superconducting macroscopic quantum system, without resorting to any dynamical evolution calculations for the natural atoms \([2]\).

The population on the singlet \(|\psi_s\rangle\) will be trapped to keep its initial value, due to the coherent cancelling of the two transitions from both, \(|\uparrow\downarrow\rangle\) and \(|\downarrow\uparrow\rangle\), to any state.

Dressed Spectrum.—For quantum criticality, the Ising-type Hamiltonian \(H_Q\) has been extensively studied in the thermodynamic limit, i.e., the case with infinite qubits \([11]\). To demonstrate this with a few qubits, we consider the eigenvalues of \(H_Q\): \(E_s = -J, E_{\uparrow\uparrow} = J + \omega_a, E_{\downarrow\downarrow} = -J\) and \(E_{\uparrow\downarrow} = J - \omega_a\). We use the dimensionless parameter \(\xi = \omega_a/J\) to describe the QPT character of the ground state. If \(\xi < -2\) to \(-2 < \xi < 2\), and then to \(\xi > 2\), the ground state of \(H_Q\) changes from the state \(|\uparrow\uparrow\rangle\) to the state \(|\psi^+\rangle\) \((|\psi^-\rangle\) and then to \(|\downarrow\downarrow\rangle\). Correspondingly, the “order” of the magnetic system changes from ferromagnetic to antiferromagnetic through the maximal entangled state \(|\psi^+\rangle\) \(|\psi^-\rangle\) corresponding to the point \(\xi = 0\).

For a weak perturbation, not commuting with \(S_z\) or \(H_Q\), the ground state is still determined by the longitudinal field controlled by the gate voltages. Now, we consider the effect of the strong interaction on the ground state when the \(H_Q\) system interacts with a single-mode quantized field. The above discussions show that the quantized field can only mix qubit states, in the invariant subspace \(V^{(0)}\) or \(V^{(1)}\), to form dressed states, but it cannot induce the transitions from \(V^{(0)}\) to \(V^{(1)}\). Therefore, in the qubit subspace \(V^{(0)}\), the eigenstates \(|\varphi_n^{(s)}\rangle\) of the Hamiltonian \(H\) are the product states of the singlet \(|\psi^\pm\rangle\) and photon number states \(|n\rangle\), i.e., \(|\varphi_n^{(s)}\rangle \equiv |n, \psi^\pm\rangle\), which correspond to the eigenvalues \(E_n^{(s)} = n\omega - J\), with \(n = 0, 1, 2, \cdots\).

However, in the qubit subspace \(V^{(1)}\), when we consider the state mixture induced by the quantized field, the Hamiltonian \(H\) possesses the following invariant subspaces

\[
V^{(0)} : \{0, 0\} \equiv \{|0, 4\rangle\}, \quad V^{(1)} : \{0, 0\} \equiv \{|0, 0\}, \{|1, 1\rangle\}, \quad V^{(n+1)} : \{|n-1, 1\} \equiv \{|n-1\} \otimes |\uparrow\uparrow\rangle, \{|n, 0\} \equiv \{|n\} \otimes |\psi^\pm\rangle\}
\]

for \(n = 1, 2, 3, \cdots\). Then, the Hamiltonian \(H\) can be diagonalized in each quasi-diagonal block \(W^{(n+1)}\). The 1-D block \(W^{(0)}\) only contains the eigenstate \(|\varphi_0^{(0)}\rangle = |0, -1\rangle\) with eigenvalue \(E_0^{(0)} = J - \omega_a\). The 2-D block \(W^{(1)}\) is spanned by two eigenstates

\[
\begin{align*}
|\varphi_0^{(0)}\rangle &= \cos(\theta/2)|0, 0\rangle - \sin(\theta/2)|1, -1\rangle, \\
|\varphi_0^{(-)}\rangle &= \sin(\theta/2)|0, 0\rangle + \cos(\theta/2)|1, -1\rangle
\end{align*}
\]

with eigenvalues \(E_0^{(\pm)} = \pm \sqrt{J^2 + g^2}\). The eigenvalues of the 3D blocks \(W^{(n+1)}(n \geq 1)\) can be explicitly solved in the resonant case \(\omega = \omega_a\). In this condition, the eigenvalues are \(E_n^{(0)} = n\omega + J\) and \(E_n^{(\pm)} = n\omega \pm N_n(g)\), which correspond to the eigenstates

\[
\begin{align*}
|\varphi_n^{(0)}\rangle &= \sqrt{\frac{n + 1}{2n + 1}}|n-1, 1\rangle - \sqrt{\frac{n}{2n + 1}}|n+1, -1\rangle, \\
|\varphi_n^{(\pm)}\rangle &= \sqrt{\frac{ng^2}{2N_n(g)}}|n-1, 1\rangle \pm \sqrt{\frac{N_n(g) + J}{2N_n(g)}}|n-1, 1\rangle, \\
&\quad + \sqrt{\frac{(n+1)g^2}{2N_n(g)}}|n+1, -1\rangle
\end{align*}
\]

where \(N_n(g) = \sqrt{J^2 + (2n + 1)g^2}\) and \(\Omega_n(g) = N_n^2(g) + JN_n(g)\).

Quantum criticality due to level crossings.—The eigenvalues \(E_n^{(\alpha)}\) with \(\alpha = 0, \pm, s; \ n = 0, 1, 2, \cdots\) form a complete set for the energy spectrum. Fig. 2 shows a few
The free Hamiltonian $H_\nu$ of the multi-mode bath is $H_\nu = \sum_k \hbar \omega_k a_k^\dagger a_k$. $a_k^\dagger$ (a$_k$) are the creation (annihilation) operators of the $k$th bath mode with the angular frequency $\omega_k$. The bath can be the quasi-normal modes of the TLR or the multi-mode external electromagnetic field.

When $g_k^{(1)} = g_k^{(2)}$, $H_\nu$ cannot induce the transitions between two different diagonal blocks $V^{(0)}$ and $V^{(1)}$ of the Hamiltonian $H$: it can only induce transitions between different states in the same block. However, when $g_k^{(1)} \neq g_k^{(2)}$, the perturbation $H_\nu$ will break the original invariant subspaces $V^{(0)}$ and $V^{(1)}$. Transitions between the different subspaces $V^{(0)}$ and $V^{(1)}$ are possible. This leads to the mixture of the different diagonal blocks of the Hamiltonian $H$ discussed above.

As known in conventional cavity QED, many typical strong-coupling phenomena, e.g., the Rabi splitting, actually refer to transitions between the different invariant subspaces. The corresponding width of the spectral line can be determined by the matrix elements $\langle \varphi^{(\alpha)}_m | H_\nu | \varphi^{(\beta)}_n \rangle$ through the Fermi golden rule. We can analyze this problem in detail in different quantum critical regions. For example, in region (III), $| \varphi^{(0)}_0 \rangle$ is the ground state, and...
qubits are symmetrically coupled to both the bath and "dark state" is not affected by any other states when two ent trapped two-qubit singlet state ("dark state"). This a single-mode quantized field. There exists a coherent coupling effect between the two-qubit system and a single-mode quantized field.

We also study the influence of the strong inter-“atom” coupling and atoms-to-quantized-field couplings on the quantum criticality through level crossing. Specially, an intrinsic singular point is found in the limit of weak-cavity-field coupling. This point is characterized by the discreteness of the spectra in some critical regions. Our study can be easily generalized to the case of the non-resonant interaction between the cavity field and two qubits. We hope that our proposal can further motivate experiments on the circuit QED with two strongly coupled qubits.

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\[ |\varphi_0^{(-)}\rangle \] and \[ |\varphi_1^{(-)}\rangle \] are the first two excited states. Transitions from \[ |\varphi_0^{(-)}\rangle \] and \[ |\varphi_1^{(-)}\rangle \] to \[ |0, -1\rangle \] will lead to a double-peak Rabi split. The distance between the center of the two peaks is determined by the energy difference \( \sqrt{J^2 + g^2} - J \) between two excited states. For instance, if we take \( J \sim 4 \text{ GHz} \) and \( g \sim 2 \text{ GHz} \), this Rabi splitting is about 0.47 GHz; however, for a small cavity-field coupling constant, e.g., \( g \sim 0.2 \text{ GHz} \), this Rabi splitting is about 5 MHz.

We can also evaluate the ratios of the line widths, which are determined by the damping rates. For example, the ratio of the damping rates \( \gamma_1 \) (from \( |\varphi_0^{(-)}\rangle \) to \( |\varphi_0^{(0)}\rangle \)) and \( \gamma_2 \) (from \( |\varphi_1^{(-)}\rangle \) to \( |\varphi_0^{(0)}\rangle \)) is

\[
\frac{\gamma_1(\omega_1)}{\gamma_2(\omega_2)} = \left[ \sin \left( \frac{\theta}{2} \right) \frac{G^+(\omega_1)\rho_1(\omega_1)}{G^-(\omega_2)\rho_2(\omega_2)} \right]^2,
\]

where \( \rho_l(\omega) \) is the given spectral density of the bath, \( \omega_l = \omega - (2 - l)(J + \sqrt{J^2 + g^2}) \) (\( l = 1, 2 \)) and \( G^\pm(\omega_k) = \frac{1}{2} g_k \pm i g_k \). The above quantitative results can also be tested by future experiments.

**Conclusions.**— We analyze the dynamical symmetry of two strongly-coupled charge qubits, interacting with a single-mode quantized field. There exists a coherent trapped two-qubit singlet state (“dark state”). This “dark state” is not affected by any other states when two qubits are symmetrically coupled to both the bath and the quantized field. However, when the symmetry is broken, the “dark state” is no longer “dark”. The transitions from this “dark state” to other states become possible. By analyzing the Rabi splitting, the level shift in the limit of weak-cavity-field coupling, and the dressed state structure of the transmission spectrum, we can probe the coherent coupling effect between the two-qubit system and a single-mode quantized field.