A Discrete Variation of Littlewood–Offord Problem

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Abstract

Littlewood–Offord Problem concerns the number of subsums of a set of vectors that fall in a given convex set. We present a discrete variation of this problem where we estimate the number of subsums that are $(0,1)$-vectors. We then utilize this to find the maximum order of graphs with given rank or corank. The rank of a graph $G$ is the rank of its adjacency matrix $A(G)$ and the corank of $G$ is the rank of $A(G)+I$.

Keywords: Littlewood–Offord Problem; Rank of graph; Corank of graph.

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1 Introduction

1.1 Littlewood–Offord Problem and its variants

Littlewood and Offord\textsuperscript{18} dealt with the following problem in studying the number of real zeros of random polynomials: given $\ell$ complex numbers of modulus at least 1, from all

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2^\ell subsums, at most how many can differ from each other by less than 1? They obtained the bound $O\left(\frac{\log \ell}{\sqrt{\ell}}\right)$, which was good enough for their purpose. Erdős [4] noticed that for real numbers, Sperner’s theorem (stating that any family of subsets of an $\ell$-set no two of which being comparable by inclusion has size at most $\left(\begin{array}{c} \ell \\ \frac{\ell}{2} \end{array}\right)$) implies a best possible bound. Suppose $x_1, \ldots, x_\ell$ are real numbers of modulus at least 1. For $S \subset \{1, \ldots, \ell\}$, set $x_S = \sum_{i \in S} x_i$. Then $|x_S - x_{S'}| < 1$ implies that $S$ and $S'$ are not comparable by inclusion.

So Sperner’s theorem implies the following:

**Theorem 1** (Erdős [4]). Let $x_1, \ldots, x_\ell$ be real numbers with $|x_i| \geq 1$ for all $i$. Let $\Lambda$ be an open interval of length 1. Then the total number of $\ell$-tuples $(\epsilon_1, \ldots, \epsilon_\ell) \in \{0, 1\}^\ell$ with $\epsilon_1 x_1 + \cdots + \epsilon_\ell x_\ell \in \Lambda$ is at most $\left(\begin{array}{c} \ell \\ \left\lfloor \frac{\ell}{2} \right\rfloor \end{array}\right)$.

This bound is clearly best possible: if $x_1 = \cdots = x_\ell = 1$, then $\left(\begin{array}{c} \ell \\ \left\lfloor \frac{\ell}{2} \right\rfloor \end{array}\right)$ of the subsums are equal to $\left\lfloor \frac{\ell}{2} \right\rfloor$. Kleitman [14] and Katona [13] proved that the bound $\left(\begin{array}{c} \ell \\ \left\lfloor \frac{\ell}{2} \right\rfloor \end{array}\right)$ holds for sums of complex numbers as well. Later, Kleitman (settling a conjecture of Erdős [4]) proved that instead of complex numbers, vectors in a Hilbert space can be taken.

**Theorem 2** (Kleitman [15]). Let $x_1, \ldots, x_\ell$ be vectors in a Hilbert space, each with length at least 1. Let $\Lambda$ be an open ball of diameter 1. Then the total number of $\ell$-tuples $(\epsilon_1, \ldots, \epsilon_\ell) \in \{0, 1\}^\ell$ with $\epsilon_1 x_1 + \cdots + \epsilon_\ell x_\ell \in \Lambda$ is at most $\left(\begin{array}{c} \ell \\ \left\lfloor \frac{\ell}{2} \right\rfloor \end{array}\right)$.

These results attracted the attention of many researchers and numerous variants of the Littlewood–Offord problem have been proposed and investigated. Tao and Vu [22] initiated a line of work known as inverse Littlewood–Offord theorems. This theory and its variants played a key role in estimating the singularity probability of random matrices (see, for instance, [6, 21, 22, 23]). The Littlewood–Offord type theorems has also arisen in other contexts. In [10, 24] a modular version of the Littlewood–Offord problem is considered with application to database security.

In the present paper, we address the following question:

**Discrete Variation of Littlewood–Offord Problem.** Given $x_1, \ldots, x_\ell \in \mathbb{R}^k$, consider all $2^\ell$ subsums $x_S = \sum_{i \in S} x_i$ for $S \subseteq \{1, \ldots, \ell\}$. How many of these are $(0, 1)$-vectors? In other words, among the $2^\ell$ linear combinations of the columns of the matrix $\begin{bmatrix} x_1 & x_2 & \cdots & x_\ell \end{bmatrix}$ with 0, 1 coefficients, how many result in a $(0, 1)$-vector?

We observe that (see Remark 5 below) it is enough to consider reduced matrices, that is, matrices with all distinct rows, each having at least two non-zero components. Throughout, we denote the set of all $(0, 1)$-vectors of length $\ell$ by $\{0, 1\}^\ell$. 

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Theorem 3. Let \( x_1, \ldots, x_\ell \in \mathbb{R}^k \) such that the matrix \[
abla\begin{array}{c|c|c|c}
abla x_1 & x_2 & \cdots & x_\ell
\end{array}\] is reduced. Let \( \Lambda = \{0, 1\}^k \). Then the total number of \( \ell \)-tuples \((\epsilon_1, \ldots, \epsilon_\ell) \in \{0, 1\}^\ell \) with \( \epsilon_1 x_1 + \cdots + \epsilon_\ell x_\ell \in \Lambda \) is at most \( \frac{2^{k+1}}{2^k+1} \cdot 2^\ell \) if \( k \leq \ell - 1 \), and \( 2^\ell - 1 \) if \( k \geq \ell \).

The proof of Theorem 3 as well as the characterization of the equality cases for \( 1 \leq k \leq \ell - 1 \) will be given in Section 2. Our subsequent results in Section 3 delve into applications of Theorem 3 within the subject of rank-order problems in algebraic graph theory, a domain that is described in the next subsection.

1.2 Rank-order problems for graphs

Let \( G \) be a simple graph with vertex set \( \{v_1, \ldots, v_n\} \). The adjacency matrix of \( G \) is an \( n \times n \) matrix \( A(G) \) whose \((i, j)\)-entry is 1 if \( v_i \) is adjacent to \( v_j \) and 0 otherwise. The order of \( G \) is the number of vertices of \( G \). We denote the set of neighbors of a vertex \( v \) of \( G \) by \( N(v) \). By eigenvalues and rank of \( G \), we mean the eigenvalues and the rank of \( A(G) \) over the reals. We denote the latter by \( \text{rank}(G) \).

Let \( \mu \) be a graph eigenvalue. An extremal problem in algebraic graph theory asks for finding the maximum order \( n \) of a graph \( G \) where \( \text{rank}(A(G) - \mu I) \) is a given integer \( r \). Rowlinson \[19\] showed that if \( \mu \notin \{0, -1\} \), then \( n < r + 2^r \). This was improved in \[20\] to \( n \leq \frac{1}{2} r(r+5) - 2 \). Bell and Rowlinson \[2\] finally proved that if \( \mu \notin \{0, -1\} \), then either (i) \( n \leq \frac{1}{2} r(r+1) \) or (ii) \( \mu = 1 \) and \( G = K_2 \) or \( 2K_2 \).

As the above result suggests, \( \mu = 0, -1 \) are somewhat exceptional. We first discuss the case of \( \mu = 0 \). In general, the order of graphs \( G \) with a fixed \( r = \text{rank}(G) \) can be unbounded. In fact, the order of \( G \) can be increased without changing its rank by adding a new vertex \( v \) twin with a vertex \( u \) (i.e. with \( N(u) = N(v) \)) to \( G \) or adding isolated vertices. For this reason, only reduced graphs, that is, graphs with no isolated vertices and no twins are taken into account. For the reduced graphs with rank \( r \), it is easily seen that the order is bounded above by \( 2^r - 1 \). This bound is far from being sharp. Kotlov and Lovász \[16\] solved the problem asymptotically. They proved that any reduced graph of rank \( r \) has order \( O(2^{r/2}) \) and, for every \( r \geq 2 \), they constructed a reduced graph of rank \( r \) and order

\[
m(r) = \begin{cases} 
2^{r+1} - 2 & r \text{ even,} \\
5 \cdot 2^{r-3} - 2 & r \text{ odd.}
\end{cases}
\]

This is conjectured to be the precise value of the maximum order:

**Conjecture 4** (Akbari, Cameron and Khosrovshahi \[1\]). The maximum order of a reduced graph with rank \( r \geq 2 \) is equal to \( m(r) \).
Haemers and Peeters [11] proved Conjecture 4 for graphs containing an induced matching of size $r/2$ for even $r$ or an induced subgraph consisting of a matching of size $(r-3)/2$ and a cycle of length 3 for odd $r$. Ghorbani, Mohammadian and Tayfeh-Rezaie [9] proved that if Conjecture 4 is valid for all reduced graphs of rank at most 46, then it is true in general. Further, they showed that the order of every reduced graph of rank $r$ is at most $8m(r) + 14$. This problem has been also investigated within specific families of graphs. In [7], it is proved that the maximum order of every reduced tree and bipartite graph of rank $r$ is $3r/2 - 1$ and $2^{r/2} + r/2 - 1$, respectively. This value is shown to be $3 \cdot 2^{\lfloor r/2 \rfloor} - 2 + \lfloor r/2 \rfloor$ for non-bipartite triangle-free graphs in [8].

For the other exceptional eigenvalue, namely $\mu = -1$, one should consider the rank of $A(G) + I$ which we call it the corank of $G$ denoted by corank($G$). Similar to the case of rank, the order of graphs with a fixed corank can be unbounded. In fact, in any graph $G$, adding a new vertex $v$ cotwin with a vertex $u$ (i.e. with $N(u) \cup \{u\} = N(v) \cup \{v\}$) to $G$, increases the order of $G$ without changing its corank. Therefore, one should consider coreduced graphs, i.e. graphs with no cotwins. Similar to the case of rank, in [5], we showed that the order of coreduced graphs with corank $r$ is $O(2^{r/2})$. It was also shown that the order of any tree and bipartite graph of corank $r$ is at most $2r - 3$ and $2r - 2$, respectively, and the order of any coreduced cotree (i.e. the complement of a tree) of corank $r$ is at most $\lfloor 3r/2 - 2 \rfloor$.

As applications for our discrete variation of Littlewood–Offord Problem, we (i) determine the maximum order of a coreduced graph with a bipartite complement of given corank, and (ii) give a new proof for the result of [7] on the maximum order of a reduced bipartite graph of given rank. In both cases, we characterize the graphs achieving the maximum order. These results will be presented in Section 3.

\section{Discrete Variation of Littlewood–Offord Problem}

Our objective in this section is to prove Theorem 3. Some notation is in order. In the remainder of the paper all vectors are treated as “row vectors.” Let $v$ be a real vector. The weight of $v$, denoted by $\text{wt}(v)$, is the number of non-zero components of $v$. Let $A$ be a $k \times \ell$ matrix. We set

$$\Omega(A) := \{b \in \{0,1\}^\ell : bA^\top \in \{0,1\}^k\}.$$ 

In other words, $\Omega(A)$ is the set of $(0,1)$-vectors $b$ of length $\ell$ such that the linear combination of the columns of $A$ with the coefficients from $b$ gives a $(0,1)$-vector. As a discrete variation of Littlewood–Offord Problem, in this section we deal with estimating the size
of \( \Omega(A) \). We call a real matrix reduced if all its rows are distinct and have weight at least 2. Our main result is that if \( A \) is reduced, then \( \Omega(A) \) has size at most \( 2^\ell - 1 \) for \( k \geq \ell \), and \( \frac{2^{k+1}}{2^k + 1} \cdot 2^k \) for \( k \leq \ell - 1 \).

**Remark 5.** Here we justify the restriction to the reduced matrices. If \( v \) is vector of length \( \ell \), then \( \Omega(v) \) is the set of all \( b \in \{0, 1\}^\ell \) such that the inner product \( v \cdot b \) is 0 or 1. Note that if \( v_1, \ldots, v_k \) are all the rows of \( A \), then

\[
\Omega(A) = \Omega(v_1) \cap \cdots \cap \Omega(v_k).
\]

So deleting repeated rows does not alter \( \Omega(A) \). If some \( v_i \) has weight one and its non-zero component is not 1, then \( |\Omega(v_i)| = 2^{\ell - 1} \), and thus by (1), \( |\Omega(A)| \leq 2^{\ell - 1} \), so we are done. Otherwise, assume that any weight-one row \( v_i \) is a \((0, 1) \)-vector. In that case, \( \Omega(v_i) = \{0, 1\}^\ell \). It follows that \( \Omega(A) = \Omega(A') \) where \( A' \) is obtained from \( A \) by removing repeated rows as well as any row of weight at most 1.

As we shall see, our main problem on bounding \( |\Omega(A)| \) for real matrices \( A \), can be reduced to \((0, \pm 1)\)-matrices. So in the next few lemmas, we deal with matrices/vectors with 0, \pm 1 entries.

**Lemma 6.** Let \( v \) be a \pm 1-vector of length \( \ell \). If the number of 1’s in \( v \) is \( k \), then, \( |\Omega(v)| = \binom{\ell + 1}{k} \leq \binom{\ell + 1}{\left\lfloor \frac{\ell + 1}{2} \right\rfloor} \).

**Proof.** With no loss of generality, we may assume that \( v = (1, \ldots, 1, -1, \ldots, -1) \), where the number of 1’s is \( k \). Let \( b = (b_1, \ldots, b_\ell) \in \Omega(v) \) and \( b' = (1-b_1, \ldots, 1-b_k, b_{k+1}, \ldots, b_\ell) \). Assume that \( wt((b_1, \ldots, b_k)) = s \) and \( wt((b_{k+1}, \ldots, b_\ell)) = t \). Hence \( wt(b') = k - s + t \). We have \( s - t = b \cdot v \in \{0, 1\} \) and hence \( wt(b') \in \{k, k - 1\} \). So the number of different \( b' \) (and so the number of different \( b \in \Omega(v) \)) is equal to \( \binom{\ell}{k} + \binom{\ell}{k - 1} = \binom{\ell + 1}{k} \). We know that \( \binom{\ell + 1}{k} \leq \binom{\ell + 1}{\left\lfloor \frac{\ell + 1}{2} \right\rfloor} \), so the proof is complete.

Given a matrix \( A \), we denote its submatrix consisting of all the non-zero columns by \( A^* \). If \( A^* \) is obtained by removing \( j \) zero columns, then it is clear that

\[
|\Omega(A)| = 2^j \cdot |\Omega(A^*)|.
\]

We say that the matrix \( A' \) is equivalent with \( A \) and write \( A' \simeq A \), if \( A \) can be transformed into \( A' \) by row and/or column permutations. It is observed that

\[
|\Omega(A')| = |\Omega(A)|.
\]

From (1), it is also clear that if the matrix \( B \) is obtained by removing some of the rows of \( A \), then

\[
|\Omega(A)| \leq |\Omega(B)|.
\]

We denote the all 1’s and all 0’s vectors by \( \mathbf{1} \) and \( \mathbf{0} \), respectively.
Lemma 7. Let $A$ be a $k \times (k+2)$ matrix of the form

$$
\begin{bmatrix}
\pm 1 & \pm 1 & \pm 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\pm 1 & \pm 1 & 0 & \ldots & \pm 1 & 0 \\
\pm 1 & \pm 1 & 0 & \ldots & 0 & a \\
\end{bmatrix},
$$

(3)

where $a \in \{0, \pm 1\}$. Then $|\Omega(A)| \leq 2^{k+1} + 2$ and the equality holds if and only if $A$ is of the form

$$
A_1 = \begin{bmatrix}
1 & 1 & \ldots & -I_{k-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 0 & \ldots & b \\
\end{bmatrix}, \\
A_2 = \begin{bmatrix}
a_1 & -a_1 & \ldots & I_{k-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{k-1} & -a_{k-1} & \ldots & 0 & c \\
1 & -1 & \ldots & 0 & c \\
\end{bmatrix},
$$

(4)

where $a_i \in \{1, -1\}$, $b \in \{0, -1\}$ and $c \in \{0, 1\}$.

Proof. If in some row of $A$ with weight 3 there are not two 1’s, then by Lemma 6 and (2), $|\Omega(A)| \leq \binom{4}{2} \cdot 2^{k-1} = 2^{k+1}$ and we are done. So assume that in any row of $A$ with weight 3, there are exactly two 1’s. First, suppose that in the right block of $A$ there exist two entries with different signs. Then $A$ contains a $2 \times (k+2)$ submatrix $B$ with

$$
B^* = \begin{bmatrix}
1 & -1 & 1 & 0 \\
1 & 1 & 0 & -1 \\
\end{bmatrix}.
$$

We see that

$$
\Omega(B^*) = \{0000, 0010, 0110, 0111, 1000, 1001, 1101, 1111\}.
$$

Thus $|\Omega(A)| \leq |\Omega(B)| = |\Omega(B^*)| \cdot 2^k = 2^{k+1}$, and so we are done. Hence, we assume that in the right block of $A$ all the non-zero entries have the same sign. It follows that $A$ is of the form either $A_1$ or $A_2$. We have

$$
\Omega(A_1) = \begin{cases} 
\{0, 01\} \cup (\{01, 10\} \times \{0, 1\}^k) & \text{if } b = 0, \\
\{0, 1\} \cup (\{01, 10\} \times \{0, 1\}^k) & \text{if } b = -1.
\end{cases}
$$

For $A_2$, consider the $(0, 1)$-vectors $b = \frac{1}{2}(1 - a_1, \ldots, 1 - a_{k-1})$ and $b' = \frac{1}{2}(1 + a_1, \ldots, 1 + a_{k-1})$. Then

$$
\Omega(A_2) = \begin{cases} 
\{10b0, 10b1\} \cup (\{00, 11\} \times \{0, 1\}^k) & \text{if } c = 0, \\
\{10b0, 01b1\} \cup (\{00, 11\} \times \{0, 1\}^k) & \text{if } c = 1.
\end{cases}
$$

Therefore, $|\Omega(A_1)| = |\Omega(A_2)| = 2^{k+1} + 2$. \qed
Similar to Lemma 7, the following can be obtained.

**Lemma 8.** Let $A$ be a $k \times (k + 1)$ matrix of the form

\[
\begin{bmatrix}
\pm 1 & \pm 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\pm 1 & 0 & \ldots & \pm 1
\end{bmatrix}.
\]

Then $|\Omega(A)| \leq 2^k + 1$. The equality holds if and only if $A$ is one of the following matrices:

\[
A_3 = \begin{bmatrix}
1 \\
\vdots \\
1 \\
\end{bmatrix} - I_k, \quad A_4 = \begin{bmatrix}
\pm 1 \\
\vdots \\
\pm 1
\end{bmatrix} I_k.
\]

We also need the following lemma on $(0, \pm 1)$-matrices with two or three rows.

**Lemma 9.** Let $A$ be a $k \times s$ reduced $(0, \pm 1)$-matrix and $t$ be the maximum weight of the rows of $A$.

(i) If $k = 2$, $t = 6, 7$ and $s \leq 14$, then $|\Omega(A)| \leq 2^{s-1}$.

(ii) If $k = 2$, $t = 4, 5$ and $s \leq 10$, then $|\Omega(A)| < \frac{5}{8} \cdot 2^s$.

(iii) If $k = 2$, $t = 3$, $s \leq 6$, and $A^*$ is not equivalent with

\[
B_0 = \begin{bmatrix}
\pm 1 & \pm 1 & \pm 1 & 0 \\
\pm 1 & \pm 1 & 0 & a
\end{bmatrix},
\]

where $a \in \{0, \pm 1\}$, then $|\Omega(A)| \leq \frac{9}{16} \cdot 2^s$.

(iv) If $k = 3$, $t = 4, 5$ and $s \leq 15$, then $|\Omega(A)| \leq 2^{s-1}$.

(v) If $k = 3$, $t = 3$, $s \leq 9$, and $A^*$ is not equivalent with the matrix given in (3), then $|\Omega(A)| \leq 2^{s-1}$.

We verified Lemma 9 by performing an exhaustive computer search. As it may not be clear from the statement, we discuss here why such a search is feasible. As an instance, we give an enumeration on the total number of inner products required to verify the part (i) of the lemma with $t = 7$. Let $v$ be the first row of $A$ of weight 7 and $d$ be the number of 1’s in $v$. If $d \neq 4$, then by Lemma 9 $|\Omega(v^*)| \leq \binom{8}{3} < 2^6$ implying that

\[1\text{The Python code of the program is available at: https://wp.kntu.ac.ir/ghorbani/ComputFiles/P}
\text{ythonCode.txt}
\[ |\Omega(v)| \leq |\Omega(v^*)| \cdot 2^{s-7} < 2^{s-1} \], and we are done. So let \( d = 4 \). Then \( A \) is equivalent with a matrix of the form

\[
\begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & c_1 & c_2 & c_3 & c_4
\end{bmatrix},
\]

where \( a_1 \leq a_2 \leq a_3, b_1 \leq \cdots \leq b_7 \) and \( c_1 \leq \cdots \leq c_4 \). Let \( a = (a_1, a_2, a_3), b = (b_1, \ldots, b_7) \) and \( c = (c_1, \ldots, c_4) \). We must have \( 2 \leq \text{wt}(a) + \text{wt}(b) + \text{wt}(c) \leq 7 \). If \( \text{wt}(b) = 7 \), then \( \text{wt}(a) = \text{wt}(c) = 0 \) and thus \( |\Omega(A)| = |\Omega(v^*)| \cdot |\Omega(b)| \leq (8)^2 < 2^{13} \), and we are done. So \( \text{wt}(b) \leq 6 \). Suppose that \( \text{wt}(a) = i, \text{wt}(b) = j \) and \( \text{wt}(c) = r \). Given that the components of these vectors are increasing, the numbers of choices for \( a, b, \) and \( c \) are \( i+1, j+1, \) and \( r+1 \), respectively. We have \( 0 \leq i \leq 3, 0 \leq j \leq 6, \) and \( 0 \leq k \leq 4 \). Furthermore, since \( i + j + k \leq 7 \), we must have \( j \leq 7 - i \) and \( k \leq 7 - i - j \). Taking into account these conditions on \( i, j, k \), it follows that the number of different choices for the second row of \( A \) is at most

\[
\sum_{i=0}^{3} (i+1) \sum_{j=0}^{\min(6,7-i)} (j+1) \sum_{r=0}^{\min(4,7-i-j)} (r+1) = 1267.
\]

Now, for any choice of \( A \) we should compute \( \mathbf{x}A^\top \) for any \( \mathbf{x} \in \{0,1\}^{14} \). Since \( A^* \) has \( j+7 \) columns, it suffices to compute \( \mathbf{x}A^{*\top} \) for any \( \mathbf{x} \in \{0,1\}^{j+7} \). It turns out that the total number of required inner products to verify the assertion is at most

\[
2 \sum_{i=0}^{3} (i+1) \sum_{j=0}^{\min(6,7-i)} 2^{j+7}(j+1) \sum_{r=0}^{\min(4,7-i-j)} (r+1) = 3035648,
\]

which shows the feasibility of the exhaustive search.

We are now prepared to prove the main result of the paper. For convenience, we repeat Theorem 3 here, including the equality cases.

**Theorem 10.** If \( A \) is a \( k \times \ell \) reduced matrix, then

\[
|\Omega(A)| \leq \begin{cases} 2^k \cdot 2^\ell & \text{if } k \leq \ell - 1, \\
2^\ell - 1 & \text{if } k \geq \ell. \end{cases}
\]

For \( 1 \leq k \leq \ell - 1 \), the equality holds if and only if \( A^* \) is equivalent with one of the matrices \( A_1, A_2, A_3, A_4 \) given in (4) and (6).

**Proof.** We first show that if \( A \) has an entry other than 0,\( \pm 1 \), then we are done. To see this, with no loss of generality, assume that \( \mathbf{v} = (v_1, v_2, \ldots, v_\ell) \), with \( v_1 \notin \{0, \pm 1\} \), is
some row of $A$. Let $a = (1, a_2, \ldots, a_{\ell}) \in \{0, 1\}^\ell$ and $a' = (0, a_2, \ldots, a_{\ell})$. We claim that at most one of $a$ and $a'$ belong to $\Omega(v)$, since otherwise
\[
|v_1| = |a \cdot v - a' \cdot v| \in \{0, 1\},
\]
which is a contradiction. Thus, at most one of $a$ or $a'$ belong to $\Omega(v)$. This implies that $|\Omega(A)| \leq |\Omega(v)| \leq 2^{\ell-1}$. So we may assume that all the entries of $A$ are $0, \pm 1$.

Assume that the row $v$ with $\text{wt}(v) = t$ has the largest weight among the rows of $A$. By Lemma 6 and (2), we have $|\Omega(v)| \leq 2^{\ell-t}$ for $t \geq 8$, by induction, we have $2^{\ell-t} \leq 2t$. Hence if $t \geq 8$, then $|\Omega(A)| \leq |\Omega(v)| < 2^{\ell-1}$, and we are done. Therefore, we suppose that $t \leq 7$. We consider the following four cases.

**Case 1.** $k = 1$

Since $k = 1$, and $A$ is a reduced matrix, the weight of each row of $A$ is at least two. Thus, $k \leq \ell - 1$, which means that we only need to show that $|\Omega(A)| \leq \frac{2}{9} \cdot 2^\ell$.

As $t \geq 2$, we have $\binom{\frac{t+1}{2}}{2} \leq \frac{3}{4} \cdot 2^t$ with equality for $t = 2, 3$. Now, from Lemma 6 it follows that $|\Omega(A)| \leq |\Omega(v)| \leq \binom{\frac{t+1}{2}}{2} \cdot 2^{\ell-t} \leq \frac{3}{4} \cdot 2^\ell$. The equality holds if and only if $t = 2, 3$ which agrees with the equality cases of the theorem.

**Case 2.** $k = 2$

In this case, we need to show that for $\ell = 2$, $|\Omega(A)| \leq 2$, and for $\ell \geq 3$, $|\Omega(A)| \leq \frac{5}{8} \cdot 2^\ell$.

(The only possibility for $A$ in the case $\ell = 2$ is that $A$ is equivalent to the matrix $B_1$ below.)

First, assume that $t = 2$. Then, $A^*$ is equivalent with one of
\[
B_1 = \begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 \end{bmatrix}.
\]
It is easy to check that at most two vectors from $\{0, 1\}^2$ can belong to $\Omega(B_1)$, that is, $|\Omega(B_1)| \leq 2$. So if $A^* \simeq B_1$, then $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-2} \leq 2^{\ell-1}$, implying the result. We have $|\Omega(B_2)| = |\Omega((\pm 1, \pm 1))|^2 \leq 9$. Thus, if $A^* \simeq B_2$, then $|\Omega(A)| = |\Omega(B_2)| \cdot 2^{\ell-4} = \frac{9}{16} \cdot 2^\ell < \frac{5}{8} \cdot 2^\ell$, and we are done. Finally, let $A^* \simeq B_3$. By Lemma 8, $|\Omega(B_3)| \leq 5$. It follows that $|\Omega(A)| \leq \frac{5}{8} \cdot 2^\ell$ and the equality holds if and only if $A^*$ is equivalent with $A_3$ or $A_4$ of (6).

If $t = 3$, then $A^*$ has $s \leq 6$ columns because the weight of the second row of $A$ is at most $t$. If $A^*$ is not equivalent with $B_0$ of (7), then Lemma 9(iii) implies that $|\Omega(A^*)| \leq \frac{9}{16} \cdot 2^s$ and thus $|\Omega(A)| \leq \frac{9}{16} \cdot 2^t < \frac{5}{8} \cdot 2^t$. If $A^* \simeq B_0$, then $s = 4$ and by Lemma 7 $|\Omega(A^*)| \leq 10$. It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-4} \leq \frac{5}{8} \cdot 2^\ell$ and the equality holds if and only if $A^*$ is equivalent with either $A_1$ or $A_2$ of (4).
If $t = 4, 5$, then $A^*$ has $s \leq 10$ columns. By Lemma 3(ii), $|\Omega(A^*)| < \frac{5}{8} \cdot 2^s$. It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{s-t} < \frac{5}{8} \cdot 2^t$.

If $t = 6, 7$, then $\left(\frac{t+1}{2^{t+1}}\right) = \frac{35}{64} \cdot 2^t < \frac{5}{8} \cdot 2^t$. Then by Lemma 3, $|\Omega(A)| \leq |\Omega(v)| \leq \left(\frac{t+1}{2^{t+1}}\right)2^{t-t} < \frac{5}{8} \cdot 2^t$.

**Case 3. $k = 3$**

In this case, we need to show that for $\ell = 2, 3$, $|\Omega(A)| \leq 2^{\ell-1}$, and for $\ell \geq 4$, $|\Omega(A)| \leq \frac{9}{16} \cdot 2^\ell$.

First, let $t = 2$. Comparing the $2 \times \ell$ submatrices of $A$ with $B_1, B_2, B_3$ of Case 2, we see that $A$ satisfies in one of the following three cases.

(i) For some $2 \times \ell$ submatrix $B$ of $A$, we have $B^* \simeq B_1$. Thus $|\Omega(A)| \leq |\Omega(B)| \leq 2^{\ell-1}$.

(ii) For all $2 \times \ell$ submatrices $B$ of $A$, we have $B^* \simeq B_2$. Then $A^*$ is equivalent either with the matrix given in (5), or with

$$\begin{bmatrix}
\pm1 & \pm1 & 0 \\
\pm1 & 0 & \pm1 \\
0 & \pm1 & \pm1
\end{bmatrix}.
\tag{8}
$$

If the former occurs, then by Lemma 8, $|\Omega(A)| \leq \frac{2^{k+1}}{2^{k+1}} \cdot 2^\ell = \frac{9}{16} \cdot 2^\ell$ and the equality holds if and only if $A^*$ is equivalent with $A_3$ or $A_4$ of (5). So assume that $A^*$ is equivalent with (8). If some $2 \times 3$ submatrix $B$ of $A^*$ is equivalent to neither of $A_3, A_4$ of (5), then by Lemma 8, $|\Omega(A^*)| \leq |\Omega(B)| \leq 4$. It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-3} \leq 2^{\ell-1}$, as desired. Otherwise, $A^*$ is equivalent with either of

$$\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{bmatrix}.
$$

Then it can be easily checked that $|\Omega(A^*)| = 4$ and thus $|\Omega(A)| \leq 4 \cdot 2^{\ell-3} = 2^{\ell-1}$, and we are done.

(iii) $A$ has two $2 \times \ell$ submatrices that are either both equivalent with $B_2$, or one is equivalent with $B_2$ and the other one with $B_3$. It turns out that $A^*$ is equivalent with either of

$$\begin{bmatrix}
\pm1 & \pm1 & 0 & 0 & 0 & 0 \\
0 & 0 & \pm1 & \pm1 & 0 & 0 \\
0 & 0 & 0 & 0 & \pm1 & \pm1
\end{bmatrix},
\begin{bmatrix}
\pm1 & \pm1 & 0 & 0 & 0 \\
0 & 0 & \pm1 & \pm1 & 0 & 0 \\
0 & 0 & 0 & 0 & \pm1 & \pm1
\end{bmatrix},
\begin{bmatrix}
\pm1 & \pm1 & 0 & 0 \\
0 & 0 & \pm1 & \pm1 \\
0 & 0 & 0 & 0 & \pm1 & \pm1
\end{bmatrix}.$$
For the first one, we have $|\Omega(A^*)| \leq |\Omega((\pm 1, \pm 1))|^3 \leq 27$, and thus $|\Omega(A)| \leq |\Omega(A^*)| \cdot 2^\ell - 6 < 2^\ell - 1$. For the second one, $|\Omega(A^*)| \leq |\Omega((\pm 1, \pm 1))| \cdot |\Omega(B_2)| \leq 27$, and thus $|\Omega(A)| \leq |\Omega(A^*)| \cdot 2^{\ell - 5} < 2^{\ell - 1}$. For the third one, if we have $|\Omega(A^*)| \leq 8$, then it will follow that $|\Omega(A)| \leq 2^{\ell - 5}$. Otherwise, $|\Omega(A^*)| \geq 9$. On the other hand, $\Omega(A^*) \subseteq \Omega(B_2)$. Since $|\Omega(B_2)| \leq 9$, it follows that $\Omega(A^*) = \Omega(B_2)$. This in turn implies that $\Omega(B_2) \subseteq \Omega(x)$ where $x = (0, \pm 1, \pm 1, 0)$. At least one of $0100$ or $1100$ and at least one of $0010$ or $0011$ belong to $\Omega(B_2)$. This implies that $x = (0, 1, 1, 0)$. Also $\Omega(B_2)$ contains a vector of the form $\ast 11 \ast$. Such a vector cannot belong to $\Omega(x)$, a contradiction.

Next, let $t = 3$. Since the weight of each row of $A$ is at most $t$, $A^*$ has $s \leq 9$ columns. If $A^*$ is not equivalent with the matrix given in (3), then by Lemma 9(v), $|\Omega(A^*)| \leq 2^{s - 1}$.

It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell - s} \leq 2^{\ell - 1}$, as desired. Otherwise, by Lemma 7, $|\Omega(A)| \leq 9 \cdot 2^\ell$ and the equality holds if and only if $A^*$ is equivalent with $A_1$ or $A_2$ of (1).

If $t = 4, 5$, then $A^*$ has $s \leq 15$ columns. By Lemma 9(iv), $|\Omega(A^*)| \leq 2^{s - 1}$. It follows that $|\Omega(A)| \leq |\Omega(A^*)| \cdot 2^{\ell - s} \leq 2^{\ell - 1}$, and we are done.

If $t = 6, 7$, in a similar manner as above we are done by Lemma 9(i).

Case 4. $k \geq 4$

First let $t = 2$. If $A^*$ is equivalent with the matrix given in (5), then by Lemma 8, $|\Omega(A)| \leq 2^k + 1 \cdot 2^\ell$ and the equality holds if and only if $A^*$ is equivalent with $A_3$ or $A_4$ of (6). Otherwise, as shown in Case 3, for some $3 \times \ell$ submatrix $B$ of $A$ we have $|\Omega(B)| \leq 2^{\ell - 1}$, and so we are done.

If $t = 3$, then we are done similarly as for $t = 2$.

If $4 \leq t \leq 7$, then we are done by Lemma 9 as in Case 3.

3 Applications

In this section, we present two applications for our result on the discrete variation of Littlewood–Offord Problem. We first give a new proof for the result of [7] on the maximum order of a reduced bipartite graph with a given rank. Then we present another application on finding the maximum order of a coreduced cobipartite graph (i.e. the complement of a bipartite graph) with a given corank.

We need further notation. Let $G$ be a bipartite graph. Then its adjacency matrix can
be put in the form:

\[ A(G) = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}. \]

We call \( B = B(G) \) a bipartite adjacency matrix of \( G \). When \( G \) is connected, this is unique up to permutations of rows and columns. We denote the \( \ell \times 2^\ell \) matrix whose columns consist of all \((0, 1)\)-vectors of length \( \ell \) by \( \mathbb{B}_\ell \). The bipartite graph \( G \) with \( B(G) = \mathbb{B}_\ell \) is denoted by \( \mathcal{B}_\ell \). The graph \( \mathcal{B}_\ell \) is in fact the incidence graph of \([\ell] := \{1, \ldots, \ell\}\) versus \( \mathcal{P}([\ell]) \), the power set of \([\ell]\). We also denote the column space and the row space of a matrix \( M \) by \( \text{Col}(M) \) and \( \text{Row}(M) \), respectively.

### 3.1 Bipartite graphs

The graph \( \mathcal{B}_\ell \) has an isolated vertex. We denote the resulting graph by removing this isolated vertex by \( \mathcal{B}'_\ell \). So \( \mathcal{B}'_\ell \) is a reduced bipartite graph of rank \( 2\ell \) and order \( 2\ell + \ell - 1 \).

As the first application of Theorem \([10]\) we give a new proof for the following theorem from \([7]\).

**Theorem 11.** Let \( G \) be a reduced bipartite graph of order \( n \) and rank \( r \). Then \( n \leq 2^{r/2} + r/2 - 1 \) and the equality holds if and only if \( G \) is isomorphic to \( \mathcal{B}'_{r/2} \).

**Proof.** Let \( B = B(G) \) be a \( p \times q \) matrix with rank \( \ell \). We have \( r = 2\ell \). We can assume that \( p \leq q \). First, suppose that \( p = \ell \). Since \( G \) is a reduced graph, \( B \) has no two identical columns nor a zero column. Thus \( q \leq 2^\ell - 1 \) with equality if and only if \( B \) is equal to the matrix \( \mathbb{B}_\ell \) whose zero column is removed. It follows that \( n = p + q \leq 2^\ell + \ell - 1 \) with equality if and only if \( G \) is isomorphic to \( \mathcal{B}'_\ell \).

Now, assume that \( p = \ell + k \) with \( k \geq 1 \). By performing column-elementary operations, we can find a basis for \( \text{Col}(B) \) as follows (a permutation of the rows might be also necessary):

\[ W = \begin{bmatrix} \mathbb{I}_\ell \\ \frac{C_{k\times\ell}}{\mathbb{C}_{k\times\ell}} \end{bmatrix}. \]

Since \( G \) is a reduced graph, \( W \) has no two identical rows and no zero row. This implies that \( C \) is a reduced matrix. Any column of \( B \) is a non-zero \((0, 1)\)-vector, so it is generated by a linear combination of the columns of \( W \) if the corresponding vector of coefficients belong to \( \Omega(W) \setminus \{\mathbf{0}\} \). It turns out that \( q \leq |\Omega(W)| - 1 \). It is also clear that \( \Omega(W) = \Omega(C) \).

If \( k \geq \ell \), by Theorem \([10]\) \(|\Omega(C)| \leq 2^{\ell - 1} \) and then as \( p \leq q \), we have \( n = p + q \leq 2q \leq 2(|\Omega(C)| - 1) < 2^\ell \), so we are done. Hence, assume that \( k \leq \ell - 1 \). By Theorem \([10]\) \(|\Omega(C)| \leq \frac{2^k + 1}{2^{k+1}} \cdot 2^\ell \), and thus \( n \leq \ell + k + \frac{2^k + 1}{2^{k+1}} \cdot 2^\ell - 1 \). If \( \ell = 2 \), then \( k = 1 \), and so...
\[ p = \ell + k = 3 \text{ and } q \leq \frac{2^k + 1}{2^k + 1} \cdot 2^\ell - 1 = 2, \] which is impossible. Hence, \( \ell \geq 3 \). Note that \( k + \frac{2^k + 1}{2^k + 1} \cdot 2^\ell \) is maximized at \( k = 1 \). Thus \( k + \frac{2^k + 1}{2^k + 1} \cdot 2^\ell \leq 1 + \frac{3}{4} \cdot 2^\ell < 2^\ell \) for \( \ell \geq 3 \). Therefore, \( n < 2^\ell + \ell - 1 \), which completes the proof. \[ \square \]

### 3.2 Cobipartite graphs

As the second application of Theorem 10, we determine the maximum order of coreduced cobipartite graphs with a given corank and characterize the graphs achieving the maximum order.

From known relations between ranks of matrix sums (see the item 0.4.5 (d) in [12, p. 13]), we obtain the following:

**Lemma 12.** For a symmetric matrix \( M \), \( \text{rank}(M + J) = \text{rank}(M) + 1 \) if and only if \( 1 \notin \text{Row}(M) \).

The following lemma is crucial for the proof of the main result of this section.

**Lemma 13.** Let \( B \) be a \( p \times q \) \((0, 1)\)-matrix with \( p \leq q \), \( \text{rank}(B) = \ell \) and \( 1 \in \text{Row}(B) \). Also assume that \( B \) has no two identical columns or rows nor a zero row. If \( p + q \geq 2^{\ell - 1} + \ell - 1 \) and \( \ell \geq 6 \), then \( B \) is a submatrix of

\[
\begin{bmatrix}
B_{\ell-1} \\
1 \\
J - B_{\ell-1}
\end{bmatrix}, \tag{9}
\]

with a single exception in the case that \( \ell = 6 \), \( p + q = 2^{\ell - 1} + \ell - 1 \), and the columns of \( B \) are generated by

\[
\begin{bmatrix}
I_6 \\
x \\
1 \\
J_6 - I_6 \\
1 - x
\end{bmatrix}, \tag{10}
\]

for some vector \( x \) of weight 2 or 3.

**Proof.** We first construct a new matrix from \( B \) as follows: if \( 1 \) is not already a row of \( B \), we add it to the rows. Additionally, for any row \( x \neq 1 \) of \( B \), if \( 1 - x \) is not a row, we add that as well. We call the resulting matrix \( B' \). The matrix \( B' \) is of the following form:

\[
B' = \begin{bmatrix}
B_0 \\
1 \\
J - B_0
\end{bmatrix},
\]
where \( B_0 \) consists of the rows of \( B' \) whose first component is zero. As \( B' \) is obtained by adding some rows to \( B \), it follows that \( \text{rank}(B') \geq \text{rank}(B) \). However, each row of \( B' \) can be expressed as a linear combination of \( 1 \) and some row of \( B \). Since \( 1 \in \text{Row}(B) \), this implies \( \text{Row}(B') \subseteq \text{Row}(B) \), leading to \( \text{rank}(B') = \text{rank}(B) = \ell \). Given that \( 1 \notin \text{Row}(B_0) \) and every row of \( B' \) can be formed through a linear combination of the rows of \( B_0 \) and \( 1 \), we conclude that \( \text{rank}(B_0) = \ell - 1 \). Our assumption on \( B \) guarantees that \( B_0 \) has no two identical columns/rows and no zero rows. If \( B_0 \) has \( \ell - 1 \) rows, then \( B_0 \) is a submatrix of \( B_{\ell - 1} \), and we are done. Therefore, assume that \( B_0 \) has \( \ell - 1 + k \) rows for some \( k \geq 1 \). So, \( p \leq 2\ell + 2k - 1 \). By performing column-elementary operations and possibly permuting the rows, we can assume that \( B_0 \) has a basis of the form

\[
\begin{bmatrix}
I_{\ell-1} \\
C_{k \times (\ell-1)}
\end{bmatrix}.
\]

This basis has no identical rows nor a zero row. This implies that \( C \) is a reduced matrix. Every column of \( B \) belongs to \( \{ Ab^T : b \in \Omega(C) \} \). So \( q \leq |\Omega(C)| \). If \( k \geq \ell - 1 \), then by Theorem 10 \( |\Omega(C)| \leq 2^{\ell - 2} \). Thus \( p + q \leq 2q \leq 2|\Omega(C)| \leq 2^{\ell - 1} \), which is a contradiction. Hence, assume that \( 1 \leq k \leq \ell - 2 \). By Theorem 10 we have \( |\Omega(C)| \leq \frac{2k+1}{2k+1} \cdot 2^{\ell - 1} \), and so

\[
p + q \leq f := 2\ell + 2k - 1 + \frac{2k+1}{2k+1} \cdot 2^{\ell - 1}.
\]

If \( \ell = 6 \) and \( 2 \leq k \leq 4 \), by direct computation one can verify that \( f < 2^{\ell - 1} + \ell - 1 \). For \( \ell = 6 \) and \( k = 1 \), we have \( f = 2^{\ell - 1} + \ell - 1 \). This implies that \( q = |\Omega(C)| = \frac{3}{4} \cdot 2^{\ell - 1} \). By the cases of equality in Theorem 10, \( C \) should consists of a vector of weight 2 or 3, and thus \( \text{Col}(B) \) has a basis of the form (10). If \( \ell \geq 7 \), \( 2k + \frac{2k+1}{2k+1} \cdot 2^{\ell - 1} \) is maximized at \( k = 1 \). Therefore,

\[
f \leq 2\ell + 1 + \frac{3}{4} \cdot 2^{\ell - 1} < 2^{\ell - 1} + \ell - 1,
\]

from which the result follows.

We denote the bipartite graph \( G \) with

\[
B(G) = \begin{bmatrix}
B_{\ell} \\
J - B_{\ell}
\end{bmatrix},
\]

by \( D_{\ell} \). In other words, \( D_{\ell} \) is a bipartite graph with parts \( \{1, 1', \ldots, \ell, \ell'\} \) and \( P([\ell]) \), such that each \( S \in P([\ell]) \) has the \( \ell \) neighbors \( \{i : i \in S\} \cup \{j' : j \in [\ell] \setminus S\} \). As an instance, \( D_3 \) is depicted in Figure 1.

Now, we are in a position to prove the main result of this section. Recall that the complement of a graph \( G \) is denoted by \( \overline{G} \).
Theorem 14. If $G$ is a coreduced cobipartite graph with order $n$ and corank $r$, then

$$n \leq \begin{cases} 2^r - 1 + r - 2 & \text{r even,} \\ 2^{r-1} + \frac{r - 1}{2} & \text{r odd.} \end{cases}$$

The equality holds if and only if $G$ is isomorphic to $D_{\frac{r}{2} - 1}$ for even $r$, and to $B_{\frac{r}{2} - 1}$ for odd $r$.

Proof. Suppose that $\overrightarrow{G}$ is a coreduced cobipartite graph with corank $r$ and the maximum possible order $n$. Let $\overrightarrow{A} = A(\overrightarrow{G})$ and $A = A(G)$. Also let $B = B(G)$ be a $p \times q$ matrix. So, $n = p + q$. With no loss of generality, assume that $p \leq q$. Since $\overrightarrow{G}$ is a coreduced graph, $G$ has no twins. So $B$ has no identical rows/columns. Note that $G$ might have an isolated vertex. In which case, we can assume that the isolated vertex lies in the larger part of $G$, that is, $B$ has a zero column rather than a zero row. Recall that $r = \text{rank}(\overrightarrow{A} + I)$. So from $\overrightarrow{A} + I = J - A$, it follows that

$$r - 1 \leq \text{rank}(A) = 2\text{rank}(B) \leq r + 1. \quad (11)$$

We verified the result for $r \leq 10$ by a computer search. This is done by implementing an algorithm from [3] (see also [1]) for constructing coreduced graphs of a fixed corank $r$. For a given $r$, the input of the algorithm is the set of coreduced graphs with both order and corank equal to $r$ (which was generated by using McKay database of small graphs [17]) and the output of the algorithm is the set of all coreduced graphs of corank $r$. So in what follows, we assume that $r \geq 11$.

First suppose that $r = 2\ell$ is even and so $\ell \geq 6$. From (11) it follows that $\text{rank}(A) = r$. Hence, by Lemma [12] $1 \in \text{Row}(A)$. It follows that $1_q \in \text{Row}(B)$ and $1_p^\top \in \text{Col}(B)$. If $n = p + q < 2^{\ell - 1} + 2\ell - 2$, there is nothing to prove. Hence, we assume that $p + q \geq 2^{\ell - 1} + 2\ell - 2$. So $B$ satisfies the conditions of Lemma [13] and thus it is a submatrix of the
matrix $C$ given in (9). However, $1^\top \not\in \operatorname{Col}(C)$ because $\operatorname{Col}(C)$ has the following basis:

$$
\begin{pmatrix}
0^\top & I_{\ell-1} \\
1 & 1_{\ell-1} \\
1^\top & J_{\ell-1} - I_{\ell-1}
\end{pmatrix},
$$

and it is clear that such a basis cannot generate $1^\top$. Therefore, $B$ must have at least one row or one column less than $C$. This shows that $n \leq 2^{\ell-1} + 2\ell - 2$. If we remove the 1 row of $C$, then the resulting matrix is $B(\mathcal{D}_{\ell-1})$. So $G = \mathcal{D}_{\ell-1}$, as desired. To finish the proof, we show that if one deletes any other row or any column from $C$, then $1^\top$ does not belong to the column space of the resulting matrix. If we remove a row other than 1 from $C$ to obtain $C'$, then the restriction of (12) to $C'$ forms a basis for $\operatorname{Col}(C')$. Again such a basis does not generate $1^\top$. A similar argument works in the case that $C'$ is obtained by removing one column from $C$.

Next, suppose that $r = 2\ell - 1$ is odd and so $\ell \geq 6$. Let $n \geq 2^{\ell-1} + \ell - 1$. To establish the theorem, it suffices to show that $G$ is isomorphic to $\mathcal{B}_{\ell-1}$. By (11), we have $\operatorname{rank}(A) = 2\ell - 2$ or $2\ell$. If $\operatorname{rank}(A) = 2\ell - 2$, then we have necessarily $B = \mathcal{B}_{\ell-1}$, that is $G = \mathcal{B}_{\ell-1}$ and we are done. So in what follows, we assume that $\operatorname{rank}(A) = 2\ell$, i.e. $\operatorname{rank}(B) = \ell$. Given that $A = J - (A + I)$, we have $\operatorname{rank}(J - (A + I)) = r + 1 = \operatorname{rank}(-((A + I))) + 1$. By invoking Lemma 12, this implies that $1 \notin \operatorname{Row}(A + I)$. Furthermore, since $\operatorname{rank}(J - A) < \operatorname{rank}(A)$, another application of Lemma 12 establishes that $1 \in \operatorname{Row}(A)$, implying $1 \in \operatorname{Row}(B)$. Given this and the condition $n \geq 2^{\ell-1} + \ell - 1$, the criteria outlined in Lemma 13 are satisfied. Consequently, $\operatorname{Col}(B)$ has a basis of the form (10) or $B$ is a submatrix of (9). If the former occurs, then $1^\top \not\in \operatorname{Col}(B)$, which implies $1 \notin \operatorname{Row}(A)$, leading to a contradiction. Therefore, $B$ is a submatrix of (9). Note that 1 cannot be a row of $B$. Otherwise, similar to the case of even $r$, we observe that $1^\top \not\in \operatorname{Col}(B)$, resulting in $1 \notin \operatorname{Row}(A)$, which is a contradiction. Now, we make use of the fact that $1 \notin \operatorname{Row}(A + I)$. We have

$$
\overline{A} + I = \begin{bmatrix}
J & J - B^\top \\
J - B & J
\end{bmatrix}.
$$

We claim that if some vector $x$ is a row of $B$, then $1 - x$ is not a row of $B$. If this fails, then we can obtain $\begin{bmatrix} 21_p | 1_q \end{bmatrix}$ as sum of two rows of $\begin{bmatrix} J | J - B \end{bmatrix}$. Also, as $B$ has more than $2^{\ell-2}$ columns, it contains some two columns of the forms $y^\top$ and $1^\top - y^\top$. The two corresponding rows in $\begin{bmatrix} J - B^\top | J \end{bmatrix}$ sum up to $\begin{bmatrix} 1_p | 21_q \end{bmatrix}$. It turns out that $1_n = \frac{1}{3} \begin{bmatrix} 21_p | 1_q \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1_p | 21_q \end{bmatrix} \in \operatorname{Row}(\overline{A} + I)$, again a contradiction. This proves the claim. So we have established that $B$ is a submatrix of (9) such that $1_q$ is not a row of $B$ and if $x$ is a row of $B$, then $1 - x$ is not a row of $B$. It follows that $B$ has at most $\ell - 1$ rows. This is a contradiction because $\operatorname{rank}(B) = \ell$. This means that the case
rank($A$) = $2\ell$ is impossible, and the proof is complete. □

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