Non-local Equations and Optimal Sobolev Inequalities on Compact Manifolds

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Abstract
This paper deals with the theory of fractional Sobolev spaces on a compact Riemannian manifold \((M, g)\). Our first main result shows that the fractional Sobolev spaces \(W^{s,p}(M)\) introduced by Guo et al. (Electron J Differ Equ 2018(156): 1–17, 2018) coincide with the classical Triebel–Lizorkin spaces (which in turn coincide with the Besov spaces). As an application, we study a non-local elliptic equation of the form

\[
\mathcal{L}_K u + h |u|^{p-2} u = f |u|^{q-2} u,
\]

where the operator \(\mathcal{L}_K u\) is an integro-differential operator a little more general than the fractional Laplacian, defined on \(W^{s,p}(M)\). We use the Mountain Pass Theorem to show an existence result under a coercivity condition when we have a sub-critical non-linearity on the right-hand side of the Eq. (1). Our second main result is a Sobolev inequality in the critical range with an optimal constant for the fractional Sobolev spaces \(W^{s,2}(M)\). This inequality gives us a sufficient existence condition for (1) with \(p = 2\) and \(q = 2^* = \frac{2n}{n-2s}\) the fractional critical Sobolev exponent.

Keywords Fractional Laplacian · Sobolev inequality · Riemannian manifold · Critical equation

Mathematics Subject Classification 35R01 · 35R11 · 35A15

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1 Introduction

Equations involving non-local operators on an open subset of $\mathbb{R}^n$ are the subject of an intense research activity which focuses on the classical questions of existence, uniqueness, regularity, and qualitative properties of the solutions, for both linear and non-linear operators, like the fractional and the p-fractional Laplacian (see [9, 28, 35, 38]). This acute effort to understand this type of equation is due to its multiple applications in several contexts: continuum mechanics, phase transition, population dynamics, optimal control, game theory and image processing, as is explained in [9, 23] and references therein.

The main purpose of this paper is first to provide a unified functional framework to study non-local equations on a Riemannian manifold $(M, g)$, and then start extending classical existence results available for non-local equation on $\mathbb{R}^n$ or local equations on $M$.

We are particularly interested in equations on Riemannian manifolds involving a non-linearity whose growth is critical from the point of view of the Sobolev embedding. The first results in this direction started in the ’60s in the context of the Yamabe problem, a classical problem in differential geometry formulated as the question of finding a non-trivial solution to a particular critical equation with the Laplace–Beltrami operator. The problem was solved completely three decades ago, and since then, many authors have devoted their work to extend the techniques used for its resolution in different directions (e.g. [7, 18, 27]). Recently, an analogous theory has been developed for the so-called fractional Yamabe problem, which relies on finding a solution to a critical equation with a specific fractional operator on manifolds (see e.g. [14, 30]).

To the best of our knowledge, there is still no work dealing with critical equations on manifolds for general linear and non-linear integro-differential operators. The present work aims at providing a functional framework suitable to extend to the p-fractional Laplacian on a compact Riemannian manifold $(M, g)$, many results obtained for the standard Laplacian on manifolds (see e.g. [32]). We then prove an optimal Sobolev inequality when $p = 2$ from which we can deduce an existence result for critical equations involving a singular non-local operator.

In the Euclidean setting, the fractional Sobolev spaces are defined for $s \in (0, 1)$, $sp < n$, as

$$W^{s, p}(\mathbb{R}^n) := \{ u \in L^p(\mathbb{R}^n), [u]_{s, p}^p < \infty \},$$

endowed with the norm $\|u\|_{s, p} = \|u\|_p + [u]_{s, p}$, where

$$[u]_{s, p}^p := \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy,$$

known as the Gagliardo-seminorm. Basic properties of $W^{s, p}(\mathbb{R}^n)$ can be found in [3, 17]. These spaces also belong to the larger families of Besov and Triebel–Lizorkin (T–L) spaces on $\mathbb{R}^n$, $F^s_{p, p}(\mathbb{R}^n)$ and $B^s_{p, p}(\mathbb{R}^n)$ (see Eqs. (21) and (22)) below.)
since $F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. We refer the reader to Triebel’s classical monograph for a complete presentation of these spaces.

Recently, Guo et al. [24] extended straightforwardly (2) and (3) to a compact Riemannian manifold $(M, g)$ defining, for $s \in (0, 1)$, $sp < n$,

$$\tilde{W}^{s,p}(M) := \{ u \in L^p(\mathbb{R}^n), \|u\|^p_{s,p} := \|u\|^p_p + \|u\|^p \}_{s,p} < \infty, \quad (4)$$

where

$$[u]_{s,p}^p := \int\int_{M \times M} \frac{|u(x) - u(y)|^p}{d_g(x,y)^{n+sp}} dv_g(x)dv_g(y). \quad (5)$$

It was then proved in [29] that there exists a Brezis–Bourgain–Mironescu [4, 5] type result for these spaces in the sense that $\lim_{s \to 1} (1 - s)[u]_{s,p}^p = C\|\nabla u\|_p^p$.

On the other hand, H. Triebel introduced in [40, 41] the whole scales of Besov $B^s_{p,q}(M)$ and Triebel–Lizorkin (T–L) spaces $F^s_{p,q}(M)$ on manifolds. The T–L spaces on manifolds are modelled on the classical T–L spaces on $\mathbb{R}^n$ through exponential charts and partition of unity (see Eq. (23) below). On the other hand, the Besov spaces on manifolds are defined by interpolation of the T–L spaces, as in $\mathbb{R}^n$. Both spaces have attracted considerable attention in the last decades. For example in [8], the authors study the theory of Besov and T–L spaces on general non-compact Lie groups endowed with a sub-Riemannian structure. H. Triebel extended to these spaces on $M$ part of the whole theory available in $\mathbb{R}^n$. He proved in particular that $F^s_{p,p}(M) = B^s_{p,p}(M)$, the spaces we are interested in, and call $W^{s,p}(M)$.

Both spaces $W^{s,p}(M)$ and $\tilde{W}^{s,p}(M)$ satisfy the usual properties of Sobolev spaces (such as Banach, reflexivity, the density of smooth functions and embedding Theorems) and thus seem equally natural and appealing. Their definitions is however quite different. It is therefore logical to investigate if they define different spaces or not. Our first main result shows that they indeed coincide with equivalence of norms:

**Theorem 1.1** Let $(M, g)$ be a compact Riemannian manifold without boundary of dimension $n$, $s \in (0, 1)$ and $sp < n$. Then the fractional Sobolev spaces $W^{s,p}(M)$ and $\tilde{W}^{s,p}(M)$ introduced by H. Triebel and L. Guo, B. Zhang, and Y. Zhang coincide with norm equivalence.

From now on, we fix a boundaryless compact Riemannian manifold of dimension $n$ and denote $W^{s,p}(M) = \tilde{W}^{s,p}(M)$. As a consequence of H. Triebel and L. Guo, B. Zhang, and Y. Zhang’s work, $W^{s,p}(M)$ enjoys all the usual properties: it is a Banach space with the norm $\| \cdot \|_{s,p}$, reflexive if $p > 1$, and the space $C^\infty(M)$ of smooth functions is dense in $W^{s,p}(M)$. Moreover, the embedding from $W^{s,p}(M)$ into $L^q(M)$ is continuous for $q \leq p^*$ and compact for $q < p^*$, where $p^*$ is the fractional critical Sobolev exponent given by

$$p^* = p^*(n, s) = \frac{np}{n - sp}. \quad (6)$$
We then consider non-local equations like

\[ L_K u + h|u|^{p-2}u = f|u|^{q-2}u. \quad (7) \]

for some smooth functions \( f, h \), and where the r.h.s. has at most critical growth \( q \leq p^* \). The operator \( L_K \) is defined weakly on \( W^{s,p}(M) \) by

\[
(L_K u, v) = \iint_{M \times M} |u(x) - u(y)|^{p-2}(u(y) - u(y))(v(x) - v(y))K(x, y; g) \, dv_g(x)dv_g(y),
\]

for some symmetric kernel \( K(x, y; g) \) as singular as \( d_g(x, y)^{-(n+sp)} \). Notice that \( L_K \) appears naturally when looking for critical points of the non-local energy

\[ \frac{1}{p} \iint_{M \times M} |u(x) - u(y)|^p K(x, y; g) \, dv_g(x)dv_g(y), \]

which coincides in particular with the Gagliardo-seminorm \( [u]_{L^p}^p \) defined in (5) when \( K(x, y; g) = d_g(x, y)^{(n+sp)} \).

The motivation for considering such operator comes directly from the fractional \( p \)-Laplacian on \( \mathbb{R}^n \), the operator associated with the energy \( \frac{1}{p}[u]_{L^p}^p \) for \( u \in W^{s,p}(\mathbb{R}^n) \). In the linear case \( p = 2 \), this operator is the fractional Laplacian \( (-\Delta)^s \) which has received considerable attention since the publication of [10]. Notice \( (-\Delta)^s \) in \( \mathbb{R}^n \) can be defined in several equivalent ways, for example, as a Fourier multiplier [17], as fractional powers of the Laplacian operator [22], as a generator of a Levy process [10], using the heat semi-group, or as a singular integral, which appears naturally when looking for critical points of the seminorm \( [u]_{L^p}^p \), our point of view here. It is important to note that, while these critical points do not precisely yield the fractional Laplacian as seen in subsets of \( \mathbb{R}^n \), they give the most singular part.

On a compact manifold, the authors of [1] proved a pointwise representation formula for the fractional Laplacian \( (-\Delta)^s \), which shows in particular that the associated kernel is as singular as \( d_g(x, y)^{-(n+2s)} \). A more precise result can be obtained in the case of the standard sphere \( S^n \) of dimension \( n \). Indeed using expansion in spherical harmonics and clever connections with some functions appearing in number theory, the authors in [16] obtained exact explicit formulas for the fractional Laplacian on \( S^n \). It is thus essential to consider operators \( L_K \) associated to general symmetric kernel \( K \) as singular as \( d_g(x, y)^{-(n+2s)} \).

To state our assumptions on \( K \), we denote \( D = \{(x, x) : x \in M\} \), and let \( g_\varepsilon \) be the metric on \( \mathbb{R}^n \) obtained blowing-up \( g \) at \( x_0 \), namely \( g_\varepsilon(x) = (\exp_{x_0}^\varepsilon)(\varepsilon x) \). We then consider kernel \( K(\cdot, \cdot; g) : (M \times M) \setminus D \to (0, +\infty) \) satisfying the following assumptions:

(K1) \( mK \in L^1(M \times M) \), where \( m = m(x, y) = \min\{d_g(x, y)^p, 1\} \);

(K2) \( K(x, y; g) = K(y, x; g) \) for any \( (x, y) \in (M \times M) \setminus D \).

(K3) There is a constant \( \Lambda > 1 \) such that

\[ \Lambda^{-1} < K(x, y; g)d_g(x, y)^{n+ps} < \Lambda \text{ for all } x, y \in (M \times M) \setminus D. \]
K4 Let \( x_0 \in M \) and \( G : T_{x_0}M \to M \) be a smooth function. If we denote
\[
\tilde{K}(X, Y, G^*g) := K(G(X), G(Y); g)
\]
for all \( X \neq Y \in T_{x_0}M \),
then it holds
\[
\varepsilon^{n+sp} \tilde{K}(X, Y; \varepsilon^2 g_\varepsilon) \to |X - Y|^{-(n+ps)}
\]
as \( \varepsilon \to 0 \) uniformly on compacts.

Notice that a kernel like \( K(x, y; g) = d_g(x, y)^{-(n+sp)} + d_g(x, y)^{-\alpha} \) with \( \alpha < n + sp \) satisfies these assumptions (see Example (1) below). Assumptions (K1)-(K3) are classical (see e.g. [36]), whereas assumption (K4) is needed to perform the test-functions computations in the proof of the optimal Sobolev inequality (14) below.

Given a kernel \( K \) satisfying the above assumptions and the associated operator \( L_K \), we thus consider an equation of the form (7), namely
\[
L_K u + h|u|^{p-2}u = f|u|^{q-2}u.
\]
(9)

Notice the l.h.s is naturally associated to the functional \( J_K : W^{s,p}(M) \to \mathbb{R} \), defined by
\[
J_K(u) = \frac{1}{p} \int_M \int_M |u(x) - u(y)|^p K(x, y; g) dv_g(x) dv_g(y) + \frac{1}{p} \int_M h|u|^p dv_g.
\]
(10)

Such equations have been the subject of an intense research activity since Caffarelli–Silvestre’s seminal work [10] and it is almost impossible to give an exhaustive list of publications concerning existence, regularity and qualitative properties of the solutions to such equations. As general surveys, we refer e.g. to [3, 33]. Concerning the regularity, we mention [11–13, 34] in the linear case, and [6, 26] in the non-linear case. We also mention the very recent result [20] which deals with G-fractional Laplacian (an operator associated to an Orlicz–Sobolev fractional seminorm). Regularity issues for (7) will be considered in future works.

Local equations like (7) but with the \( p \)-Laplacian on \( \mathbb{R}^n \) or \( M \) (or more recently the \( p(x) \)-Laplacian or the G-Laplacian in \( \mathbb{R}^n \)) are a very classical subject. It is useful to distinguish between the sub-critical case \( q < p^* \) where standard variational methods apply, and the critical case \( q = p^* \) where loss of compactness due to concentration may occur.

In the sub-critical case \( q < p^* \), a standard application of the Mountain Pass Theorem as in [36] gives the following result.

**Theorem 1.2** Let \( s \in (0, 1) \), \( sp < n \), and \( f \geq 0 \) and \( h \) be smooth functions on \( M \). Assume \( q < p^* \), \( q \neq p \), and the coercivity condition
\[
J_K(u) \geq C\|u\|_{s,p}^p \quad \text{for any } u \in W^{s,p}(M).
\]
(11)
Then the Eq. (7) has a non-trivial solution.

In the critical case, we restrict our attention to the linear case $p = 2$ considering the problem

$$\mathcal{L}_K u + hu = f |u|^{2^* - 2} u$$

with $\mathcal{L}_K$ given by (8) with $p = 2$. It is classical since Aubin’s seminal work [2] on Yamabe’s problem (see also [7]), that the existence of a solution to a critical equation is related to best Sobolev inequalities. Let us denote by $K(n, s, 2)$ the best constant in the classical embedding $W^{s,2}(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n)$, namely

$$K(n, s, 2)^{-1} = \inf_{u \in W^{s,2}(\mathbb{R}^n)} \frac{|u|_{s,2}^2}{\|u\|_{2^*}^2}. \quad (13)$$

The second main result of this paper deals with the following optimal fractional Sobolev inequality:

**Theorem 1.3** For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left( \int_M |u|^{2^*} \, dv_g \right)^{2^*/2} \leq (K(n, s, 2) + \varepsilon) \int_{M \times M} |u(x) - u(y)|^2 K(x, y; g) \, dv_g(x) \, dv_g(y)$$

$$+ C_\varepsilon \int_M u^2 \, dv_g \quad (14)$$

for any $u \in W^{s,2}(M)$. Moreover, $K(n, s, 2)$ is the least possible constant.

Unfortunately, in the non-local and non-linear case $p \neq 2$, there is no information about the asymptotic behaviour at infinity of optimizers of the Sobolev inequality in $\mathbb{R}^n$, and we could not prove this Theorem for $p \neq 2$. Recall that, for $p = 2$, the extremals are of the explicit form $c U(\frac{|x-x_0|}{\varepsilon})$ with

$$U(x) = (1 + |x|^{2})^{-\frac{n-2s}{2}}, \quad (15)$$

see [15]. Although it has been conjectured that this extremal has a similar explicit form for the general case, it is still an open problem.

The solutions to (12), with $p = 2$, can be found as the critical points of the functional $J_K$ restricted to

$$H = \{ u \in W^{s,2}(M) : \int_M f |u|^{2^*} \, dv_g = 1 \}. \quad (16)$$

As a corollary of the optimal Sobolev inequality (14), we have the following existence result, which is the non-local counterpart of a result well known in the local setting.
Theorem 1.4 Let $f \geq 0$ and $h$ be smooth functions on $M$. Assume the coercivity condition

$$J_K(u) \geq C \|u\|_{s,2}^2$$

for any $u \in W^{s,2}(M)$ (17)

for some positive constant $C$. If

$$\inf_H J_K < \left(2 \left(\max f\right)^{2/2^*} K(n, s, 2)\right)^{-1},$$

(18)

then the infimum in the l.h.s. of (18) is attained at some nonzero $u_0 \in H$. In particular, $u_0$ is a non-trivial solution to (12).

The rest of this paper is organized as follows: In Sect. 2, we set down some notation that we will use throughout the paper and we prove Theorem 1.1, showing the equivalences between $\tilde{W}^{s,p}(M)$ and the usual fractional spaces $B^s_{p,p}(M)$ and $F^s_{p,p}(M)$. In Sect. 3, we prove Theorem 1.2, which establishes the existence of a solution for the sub-critical problem. In Sect. 4, we find the optimal Sobolev embedding given by Theorem 1.3. In Sect. 5, we apply the results of Sect. 4 to prove Theorem 1.4, which states the existence of a non-trivial solution to the problem (12). In Sect. 6, we give some technical computations related to the function $U$ defined in (15).

2 Fractional Sobolev Spaces on Manifolds

This section is devoted to defining the fractional Sobolev spaces on Riemannian manifolds and proving some results related to the Triebel–Lizorkin and Besov spaces. For further details on the fractional Sobolev spaces in $\mathbb{R}^n$, we refer to [17] and the references therein.

2.1 Preliminaries and Notation

Here, we collect some elementary results, which will be helpful in the main estimates of the paper.

Given $(M, g)$ a smooth Riemannian manifold and $\gamma: [a, b] \to M$, a curve of class $C^1$, the length of $\gamma$ is

$$L(\gamma) = \int_a^b \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$ 

For $x, y \in M$ let $\mathcal{C}$ be the space of piecewise $C^1$ curves $\gamma: [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then $d_g(x, y) = \inf_{\mathcal{C}} L(\gamma)$ is the distance associated with $g$. We denote by $dv_g(x) = \sqrt{\det(g_{ij})} dx$ the Riemannian volume element on $(M, g)$, where the $g_{ij}$ are the components of the Riemannian metric $g$ in the chart and $dx$ is the Lebesgue volume element of $\mathbb{R}^n$. 

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For any \( x \in M \), consider the exponential map \( \exp_x : T_x M \to M \). Assuming \((M, g)\) has positive injectivity radius, we can fix \( r > 0 \) such that \( \exp_x \mid_{B_r} : B_r \to B_r(x) \) is a diffeomorphism for any \( x \in M \). Throughout the paper, we will denote by \( B_R \) the ball in \( \mathbb{R}^n \) centred at 0 with radius \( R \), and by \( B_R(x) \) the ball in \( M \) centred at \( x \) with radius \( R \) for the distance \( d_g \). Also, we denote by \((\exp_x^* g)\) the metric in \( \mathbb{R}^n \) defined as the pullback of \( g \) via the exponential map.

We shall need the following elementary result:

**Lemma 2.1** Suppose \( M \) compact. Given \( \varepsilon > 0 \), there exist \( \delta > 0 \) smaller than the injectivity radius of \((M, g)\) and a covering of \( M \) by balls \( \{B_\delta(x_i), i = 1, \ldots, N\} \), such that for any \( i = 1, \ldots, N \), the following properties hold in the exponential chart \((B_\delta(x_i), \exp_{x_i}^{-1})\):

\[
(1 - \varepsilon) d_{\xi} \leq d_{\exp_{x_i}^* g} \leq (1 + \varepsilon) d_{\xi},
\]

and

\[
(1 - \varepsilon) d_g(x, y) \leq d_g(\exp_{x_i}(x), \exp_{x_i}(y)) \leq (1 + \varepsilon) d_g(x, y),
\]

where \( \xi \) is the Euclidean metric.

In the following, for any \( \alpha > 0 \), we will say that \( I_\varepsilon = O(\varepsilon^\alpha) \) as \( \varepsilon \to 0 \) if there exists a \( C > 0 \) such that \( |I_\varepsilon| \leq C\varepsilon^\alpha \) as \( \varepsilon \to 0 \). Additionally, if \((a_i)\) and \((b_i)\) are two real sequences then \( a_i = o(b_i) \) means that for any \( \varepsilon > 0 \) and \( i \) big enough one has \( |a_i| \leq \varepsilon|b_i| \).

### 2.2 Triebel–Lizorkin and Besov Spaces on Riemannian Manifolds

On the Euclidean \( n \)-space \( \mathbb{R}^n \), consider the Triebel–Lizorkin and Besov spaces \( F^{s,p}_p(\mathbb{R}^n) \) and \( B^{s,q}_{p,q}(\mathbb{R}^n) \) with \( -\infty < s < \infty \), \( 0 < p \leq \infty \) (\( p < \infty \) in the case of the F-spaces), \( 0 < q \leq \infty \). These two families of spaces contain many usual spaces, among which the fractional Sobolev spaces we are interested in. Here, we just recall their definitions and some of their properties referring to the monograph [39] for a complete study.

One possible definition of these spaces is based on the Fourier transform \( F \). Let \( \varphi_j \in \mathcal{S}(\mathbb{R}^n), j = 0, 1, 2, \ldots \), where \( \mathcal{S}(\mathbb{R}^n) \) denotes the Schwarz space, with the following properties:

\[
\varphi_j(x) = \varphi(2^{-j}x) \quad \text{if} \ j \geq 1,
\]

\[
\text{supp } \varphi_0 \subset \{ |x| \leq 2 \},
\]

\[
\text{supp } \varphi \subset \left\{ \frac{1}{2} \leq |x| \leq 2 \right\},
\]

\[
\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for every} \ x \in \mathbb{R}^n.
\]
Following [40], a tempered distribution \( f \in \mathcal{S}((\mathbb{R}^n)') \) belongs to \( F_{p,q}^s(\mathbb{R}^n) \) if

\[
\| f \|_{F_{p,q}^s(\mathbb{R}^n)} := \left\| \left( \sum_{j=0}^{\infty} 2^{sj}q \left| (F^{-1} [\varphi_j Ff]) (\cdot) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{21}
\]

Similarly, \( f \) belongs to \( B_{p,q}^s(\mathbb{R}^n) \) if

\[
\| f \|_{B_{p,q}^s(\mathbb{R}^n)} := \left( \sum_{j=0}^{\infty} 2^{sj}q \left\| F^{-1} [\varphi_j Ff] \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty. \tag{22}
\]

Note that these spaces are all quasi-Banach spaces (Banach spaces if \( p, q \geq 1 \)) and that different choices of \( \{ \varphi_j \} \) yield equivalent quasi-norms. It also hold that \( B_{p,p}^s(\mathbb{R}^n) = F_{p,p}^s(\mathbb{R}^n) = W_{s,p}^s(\mathbb{R}^n) \) is the usual fractional Sobolev space (with equivalence of norms).

Now, consider a smooth connected complete Riemannian manifold \((M, g)\) with positive injectivity radius \( r_0 \). Assume moreover that there exist \( c > 0 \) and, for any multi-index \( \alpha \), constants \( c_\alpha > 0 \) such that

\[
\det(g_{ij}) \geq c, \quad |D^\alpha g_{ij}| \leq c_\alpha
\]

in the normal chart at any point in \( M \).

Triebel defines in [40, Definition 3] the Triebel–Lizorkin spaces \( F_{p,q}^s(M) \) on \( M \) using a uniformly locally finite covering of \( M \) by exponential charts at points \( x_j \) and a subordinate partition of unity \( (\psi_j)_j \) as the space of distribution \( f \in (C^\infty_c(M))' \) such that

\[
\| f \|_{F_{p,q}^s(M)} := \sum_j \| \psi_j f \circ \exp_{x_j} \|_{F_{p,q}^s(\mathbb{R}^n)}^p < \infty. \tag{23}
\]

The Besov spaces \( B_{p,q}^s(M) \) are then defined by interpolation:

\[
B_{p,q}^s(M) = \left( F_{p,p}^{s_0}(M), F_{p,p}^{s_1}(M) \right)_{\theta,q} \quad s = (1 - \theta)s_0 + \theta s_1.
\]

Triebel proved in [40] that the definition of \( F_{p,q}^s(M) \) is independent of the choice of the points \( x_j \) and the partition of unity, and that the definition of \( B_{p,q}^s(M) \) is independent of the choice of \( s_0 \) and \( s_1 \). Moreover, they are Banach spaces when \( p, q \geq 1 \). Several equivalent norms of these spaces are then provided to mimic the existing theory in \( \mathbb{R}^n \). It is proved in particular in [40, Thm 2] that \( F_{p,p}^s(M) = B_{p,p}^s(M) \) and

\[
\| f \|_{F_{p,q}^s(M)}^p \simeq \sum_j \| \psi_j f \|_{F_{p,q}^s(M)}^p \quad \text{if} \ p < \infty \tag{24}
\]
with equivalence of norm (and usual modification when \( p = \infty \)). Further character-
izations of these spaces are proved in [41], using e.g. finite differences in the spirit of \( W^{s,p} (\mathbb{R}^n) \) (see [41, Thm 2]). Unfortunately, as the author mentions himself, with
unnatural restrictions on the \( s, p, q \)—in particular, the case \( s \in (0, 1) \) is excluded (see
[41, Remark 14]).

2.3 Proof of Theorem 1.1

As a preliminary step towards the proof of Theorem 1.1, we verify that the spaces
\( \tilde{W}^{s,p} (M) \) defined (4), namely

\[
\tilde{W}^{s,p} (M) := \{ u \in L^p (\mathbb{R}^n), \| u \|_{s,p}^p := \| u \|_p^p + [u]_s^p < \infty \}
\]

where

\[
[u]_s^p := \int \int_{M \times M} \frac{|u(x) - u(y)|^p}{d_g (x, y)^{n+sp}} \, dv_g (x) dv_g (y),
\]

satisfy a localization property analogous to the one satisfied by the Triebel–Lizorkin
spaces \( F^{s}_{p,q} (M) \) described in (24).

**Proposition 2.1** Let \((M, g)\) be a compact Riemannian manifold that we cover with
a finite number of exponential charts at points \( \{ x_i : i = 1, \ldots, N \} \). Let \( \{ \eta_i : i = 1, \ldots, N \} \) be a partition of unity associated to this covering. Then there exist constants
\( C, C' > 0 \) depending only on \( N \) and \( p \) such that for any \( u \in \tilde{W}^{s,p} (M) \),

\[
C' \| u \|_{s,p}^p \leq \sum_{i=1}^{N} \| \eta_i u \|_{s,p}^p \leq C \| u \|_{s,p}^p.
\]  

**(Proof)** First, we observe that

\[
|\eta_i u(x) - \eta_j u(y)|^p \leq 2^{p-1} (|\eta_i (x)|^p |u(x) - u(y)|^p + |u(y)|^p |\eta_i (x) - \eta_j (y)|^p) \leq 2^{p-1} (|u(x) - u(y)|^p + |u(y)|^p |\eta_i (x) - \eta_j (y)|^p).
\]

Then we get

\[
[\eta_i u]_{s,p}^p \leq 2^{p-1} \left( [u]_{s,p}^p + \int_M |u(y)|^p \int_M \frac{|\eta_i (x) - \eta_j (y)|^p}{d_g (x, y)^{n+sp}} \, dv_g (x) dv_g (y) \right).
\]

Moreover,

\[
\int_M \frac{|\eta_i (x) - \eta_j (y)|^p}{d_g (x, y)^{n+sp}} \, dv_g (x) \leq C
\]  

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where the constant is independent of $y \in M$ and $i = 1, ..., N$. Indeed since
\[
\int_M \frac{|\eta_i(x) - \eta_i(y)|^p}{d_g(x, y)^{n+sp}} \, dv_g(x) < \infty \quad \text{and} \quad \int_M |\eta_i u(x)|^p \, dv_g(x) \leq \int_M |u(x)|^p \, dv_g(x),
\]
we have
\[
\sum_{i=1}^N \|\eta_i u\|^p_{s,p} \leq C\|u\|^p_{s,p},
\]
with $C > 0$ depending on $N$, $p$, $|M|$ and $\|\eta_i\|\infty$, for all $i = 1, \ldots, N$.

On the other hand, we note that $\|u\|^p_p = \|\sum_i \eta_i u\|_p \leq \sum_i \|\eta_i u\|_p$ so that
\[
\|u\|^p_p \leq N^{p-1} \sum_{i=1}^N \|\eta_i u\|^p_p. \tag{27}
\]
Furthermore, we can write
\[
|u(x) - u(y)|^p = \left| \sum_{i=1}^N (\eta_i u(x) - \eta_i u(y)) \right|^p \leq N^{p-1} \sum_{i=1}^N |\eta_i u(x) - \eta_i u(y)|^p
\]
and then
\[
[u]_{s,p}^p \leq N^{p-1} \sum_{i=1}^N [\eta_i u]_{s,p}^p. \tag{28}
\]
Thus, from (27) and (28), it follows
\[
\|u\|_{s,p}^p = [u]_{s,p}^p + \|u\|_p^p \leq N^{p-1} \sum_{i=1}^N \left( [\eta_i u]_{s,p}^p + \|\eta_i u\|_p^p \right) = N^{p-1} \sum_{i=1}^N \|\eta_i u\|_{s,p}^p.
\]

We are now in position to prove the Theorem 1.1.

**Proof of Theorem 1.1** For $\varepsilon > 0$, let $\{\eta_i, i = 1, \ldots, N\}$ be a partition of unity adapted to the covering $\{B_\delta(x_i), i = 1, \ldots, N\}$ given by Lemma (2.1). In order to prove Theorem 1.1, we first verify that
\[
\|\eta_i u\|_{s,p} \simeq \|\eta_i u \circ \exp_{x_i}\|_{s,p}. \tag{29}
\]
Let $v_i = (\eta_i u) \circ \exp_{x_i}$. Combining
\[
\int_M |\eta_i u|^p \, dv_g = \int_{\mathbb{R}^n} |v_i|^p \, dv_{\exp_{x_i}^* g}
\]
with (19) yields
\[ (1 - \varepsilon) \|v_i\|_p^p \leq \|\eta_i u\|_p^p \leq (1 + \varepsilon) \|v_i\|_p^p. \] \tag{30}

Now we shall estimate $[\eta_i u]_{s,p}$. Denote $U_i := \exp_{x_i}^{-1}(B_\delta(x_i))$, $i = 1, \ldots, N$. Since $\text{supp}(v_i) \subset U_i$, we can write
\[ [v_i]_{s,p}^p = 2 \iint_{x \in U_i, y \notin U_i} \frac{|v_i(x)|^p}{|x - y|^{n+sp}} \, dx \, dy + \iint_{x \in U_i, y \in U_i} \frac{|v_i(x) - v_i(y)|^p}{|x - y|^{n+sp}} \, dx \, dy. \] \tag{31}

Recall that $\text{supp} \, \eta_i$ is a compact subset of $B_\delta(x_i)$. Then $K_i := \exp_{x_i}^{-1}(\text{supp}(\eta_i))$ is compact and we can take some $\alpha > 0$ such that
\[ |x - y| \geq \alpha > 0 \quad \text{for any } x \in K_i, y \in \mathbb{R}^n \setminus U_i, i = 1, \ldots, N. \]

The first integral in the r.h.s of (31) is then
\[ \iint_{x \in K_i, y \notin U_i} \frac{|v_i(x)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq C_{a,\varepsilon} \int_M |\eta_i u|^p \, dv_g, \]

To bound the second integral in the r.h.s of (31), we write
\[ \iint_{x \in U_i, y \notin U_i} \frac{|v_i(x) - v_i(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \leq C_\varepsilon \iint_{x \in U_i, y \notin U_i} \frac{|v_i(x) - v_i(y)|^p}{|x - y|^{n+sp}} \, dv_{\exp_{x_i} g}(x) \, dv_{\exp_{x_i} g}(y) \]
\[ = C_\varepsilon \int_{M \times M} \frac{|(\eta_i u)(x) - (\eta_i u)(y)|^p}{d_g(x, y)^{n+sp}} \, dv_g(x) \, dv_g(y). \]

Then we have $[v_i]_{s,p}^p \leq C_{\varepsilon} \|\eta_i u\|_{s,p}^p$. Since $dx \geq C'_\varepsilon \, dv_{\exp_{x_i} g}(x)$, we also have
\[ \iint_{x \in U_i, y \notin U_i} \frac{|v_i(x) - v_i(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \geq C'_\varepsilon [\eta_i u]_{s,p}^p, \]

which, combined with (31), gives $[v_i]_{s,p}^p \geq C'_\varepsilon [\eta_i u]_{s,p}^p$. This proves that (29).

To conclude the proof of Theorem 1.1, we first write using the definition (23) of $F_{p,p}^s(M)$ that for any $i$,
\[ \|\eta_i u\|_{F_{p,p}^s(M)}^p = \sum_j \|\eta_j \eta_i u \circ \exp_{x_j}\|_{F_{p,p}^s(\mathbb{R}^n)}^p \]
\[ \simeq \sum_j \|\eta_j \eta_i u \circ \exp_{x_j}\|_{s,p}^p. \]

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since \( F_{p,p}^s(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n) \). In view of (29) and then (24), we obtain
\[
\|\eta_i u\|_{F_{p,p}^s(M)}^p \simeq \sum_j \|\eta_j \eta_i u\|_{L^p_{s,p}}^p \simeq \|\eta_i u\|_{L^p_{s,p}}^p .
\]

Summing over \( i = 1, \ldots, N \), and using the localization properties (24) and (25) for \( F_{p,p}^s(M) \) and \( \tilde{W}^{s,p}(M) \) yield
\[
\|u\|_{F_{p,p}^s(M)}^p \simeq \sum_i \|\eta_i u\|_{F_{p,p}^s(M)}^p \simeq \sum_i \|\eta_i u\|_{L^p_{s,p}}^p \simeq \|u\|_{L^p_{s,p}}^p .
\]

This concludes the proof of Theorem 1.1.

\[\square\]

### 3 Proof of Theorem 1.2

In this section, we shall prove the existence of a solution to the problem
\[
\mathcal{L}_K u + h|u|^{p-2}u = f|u|^{q-2}u,
\]
where \( 1 < q < p^* \), the functions \( h, f \) satisfy the coercivity condition (11), and \( \mathcal{L}_K \) is the non-local operator defined in (8).

Recall that \( \mathcal{K}(\cdot, \cdot; g) : (M \times M) \setminus \mathcal{D} \to (0, +\infty) \) satisfies assumptions (K1)-(K4). The first three conditions for \( \mathcal{K} \) are standard (see, e.g. [3, 28, 31, 36]), while condition (K4) is necessary to prove our present result. In [21, Assumption 2.1], the authors consider a family of measurable functions \( k_\varepsilon \), for \( \varepsilon > 0 \), on \( \mathbb{R}^n \times \mathbb{R}^n \) satisfying similar properties to \( \varepsilon^{n+sp}\tilde{K}(X,Y,\varepsilon^{-2}g_\varepsilon) \).

In what follows, we give a typical example for the kernel \( \mathcal{K} \).

**Example 1** Consider the fractional kernel given by
\[
\mathcal{K}_0(x, y; g) = d_g(x, y)^{-(n+ps)} + d_g(x, y)^{-\alpha}
\]
with \( \alpha \in (0, n + ps) \). For a particular \( \alpha \), the operator \( \mathcal{L}_{\mathcal{K}_0} \) is the fractional Laplacian operator \( (-\Delta_{p,g})^s \). The operator \( \mathcal{L}_K \) has been studied for particular values of \( \alpha \) in [1, 16]. Then, \( \mathcal{K}_0 \) trivially satisfies conditions (K2) and (K3). Let us now check condition (K4). Indeed, recalling that \( \varepsilon^2 g_\varepsilon(x) = \varepsilon^2 \exp_{x_0}^* g(\varepsilon x) = (\exp_{x_0} o T)^*(x) \) with \( T(x) := \varepsilon x, \) we have
\[
\tilde{\mathcal{K}}_0(X, Y; \varepsilon^2 g_\varepsilon) = \mathcal{K}_0(\exp_{x_0} (\varepsilon X), \exp_{x_0} (\varepsilon Y); g)
= d_g(\exp_{x_0} (\varepsilon X), \exp_{x_0} (\varepsilon Y))^{-(n+ps)} + d_g(\exp_{x_0} (\varepsilon X), \exp_{x_0} (\varepsilon Y))^{-\alpha}
= d_{\exp_{x_0}^* g}(\varepsilon X, \varepsilon Y)^{-(n+ps)} + d_{\exp_{x_0}^* g}(\varepsilon X, \varepsilon Y)^{-\alpha}
\]
Since $\frac{1}{\varepsilon} d_{\exp_{y_0}^\varepsilon}^g(x, y) \tends, as \varepsilon \rightarrow 0$, to the Euclidean distance $|X - Y|_\xi$ between $X$ and $Y$, uniformly for $X, Y$ in a compact set, we obtain that

$$\varepsilon^{(n + p_s)} K_0^*(X, Y; \varepsilon^2 g_\varepsilon^\varepsilon) = |X - Y|_\xi + O(\varepsilon^{n + p_s - \alpha})$$

and (K4) follows.

In the following, we shall use the Mountain Pass Theorem to prove Theorem 1.2, where $p < q$. By definition, a sequence $(u_i)$ of functions in $W^{s, p}(M)$ is said to be a Palais–Smale sequence for $I$ if

(PS1) $I(u_i)$ is bounded, and
(PS2) $I'(u_i) \rightarrow 0$ in $W^{s, p}(M)'$ as $i \rightarrow +\infty$, where $W^{s, p}(M)'$ denotes the dual space of $W^{s, p}(M)$.

We say that $I$ satisfies (PS) condition in $W^{s, p}(M)$, if for any Palais–Smale sequence $\{u_i\} \subset W^{s, p}(M)$, there exists a convergent subsequence of $\{u_i\}$.

**Mountain Pass Theorem (Ambrosetti-Rabinowitz)** Let $I$ be a $C^1$ function on a Banach space $E$. Suppose that $I$ satisfies the Palais–Smale condition. Suppose also

(MP1) $I(0) = 0$,
(MP2) there exist constants $\rho, r$ such that $I(u) \geq \rho$ for all $u \in \partial B_0(r) \subset E$ and
(MP3) there exists an element $u_0 \in E$ with $I(u_0) < \rho$.

Let

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} \Phi(u)$$

where $\Gamma$ stands for the class of continuous paths joining $0$ to $u_0$. Then $c$ is a critical value of $I$.

Let $J_K : W^{s, p}(M) \rightarrow \mathbb{R}$ defined by

$$J_K(u) = \frac{1}{p} \int_M \int_M |u(x) - u(y)|^p K_c(x, y; g)dv_g(x)dv_g(y) + \frac{1}{p} \int_M h|u(x)|^p dv_g(x),$$

and let $I : W^{s, p}(M) \rightarrow \mathbb{R}$ be the energy functional associated with the problem:

$$I(u) = J_K(u) - \frac{1}{q} \int_M f|u|^q \, dv_g.$$

**Lemma 3.1** $I \in C^1(W^{s, p}(M), \mathbb{R})$ and

$$\langle I'(u), v \rangle = \langle L_K u, v \rangle + \int_M h|u|^{p - 2} uv \, dv_g - \int_M f|u|^{p - 2} uv \, dv_g$$

We shall prove the Lemma in two steps.
Step 1 The functional \( J_K \in C^1 (W^{s,p}(M), \mathbb{R}) \) and
\[
\{ J'_K(u), v \} = \langle L_K u, v \rangle + \int_M h|u|^{p-2}uv \, dv_g
\]
for all \( u, v \in W^{s,p}(M) \). Moreover, if \( u \in W^{s,p}(M) \), then \( J'_K(u) \in W^{s,p}(M)' \).

**Proof** Firstly, it is easy to see that for all \( u, v \in W^{s,p}(M) \), it holds
\[
\langle J'_K(u), v \rangle = \langle L_K u, v \rangle + \int_M h|u|^{p-2}uv \, dv_g.
\]
Then, it follows that \( J'_K(u) \in W^{s,p}(M)' \) for each \( u \in W^{s,p}(M) \). Next, we prove that \( J_K \in C^1 (W^{s,p}(M), \mathbb{R}) \). Let \( \{ u_j \} \subset W^{s,p}(M) \) a sequence such that \( u_j \rightarrow u \) for some \( u \in W^{s,p}(M) \), strongly in \( W^{s,p}(M) \) as \( n \rightarrow \infty \).

Now, using Hölder’s inequality, we have
\[
\langle L_K u_j, v \rangle \leq \int_{M \times M} |u_j(x) - u_j(y)|^{p-1} \mathcal{K}(x, y; g)^{p-1/p} (v(x) - v(y)) \mathcal{K}(x, y; g)^{1/p} \, dv_g(x) \, dv_g(y)
\]
\[
\leq \left( \int_{M \times M} |u_j(x) - u_j(y)|^{p} \mathcal{K}(x, y; g) \, dv_g(x) \, dv_g(y) \right)^{1/p} \times \left( \int_{M \times M} |v(x) - v(y)|^{p} \mathcal{K}(x, y; g) \, dv_g(x) \, dv_g(y) \right)^{-1/p}
\]
From there, we get
\[
\langle L_K u_j, v \rangle \leq C[u_j]_{s,p}^p [v]_{s,p}^p \tag{32}
\]
for \( v \in W^{s,p}(M) \). Additionally, the fact that \( u_j \rightarrow u \) strongly in \( W^{s,p}(M) \) implies
\[
\lim_{n \rightarrow \infty} \int_M (h|u_j(x)|^p - h|u(x)|^p) \, dv_g(x) = 0. \tag{33}
\]
Combining (32) and (33), we have
\[
\| J'_K(u_j) - J'_K(u) \| = \sup_{v \in W^{s,p}_0(M), \| v \|_{s,p} \leq 1} |\langle J'_K(u_j) - J'_K(u), v \rangle| \rightarrow 0
\]
as \( n \rightarrow \infty \). \( \square \)

Using the same strategy as in the previous Step, we have

Step 2 If we define
\[
H(u) = \frac{1}{q} \int_M f|u|^q \, dv_g,
\]
then $H \in C^1(W^{s,p}(M), \mathbb{R})$ and

$$\{H'(u), v\} = \int_M f|u|^{q-2}uv \, dv_g$$

for all $u, v \in W^{s,p}(M)$. Due to Steps 1 and 2, critical points of $I : W^{s,p}(M) \to \mathbb{R}$ are weak solutions to the problem (7). We shall study the cases $p < q$ and $q < p$ separately. We intend to apply the Mountain Pass Theorem to $I$ for the first case with $E = W^{s,p}(M)$.

**Proof of Theorem 1.2** for $p < q$

**Step 1** $I$ satisfies the Palais–Smale condition.

**Proof** Let $(u_i)_i \subset W^{s,p}(M)$ be a Palais–Smale sequence for $I$. Then

$$o(\|u_i\|_{s,p})\|u_i\|_{s,p} + O(1) = \left\{I'(u_i), u_i\right\} - qI(u_i) = (1 - q/p)J_k(u_i).$$

Because of the coercivity assumption (11), we have $J_k(u_i) \geq C\|u_i\|_s^p$. Since $p \neq q$, we obtain $\|u_i\|_{s,p} = o(\|u_i\|_{s,p})\|u_i\|_{s,p} + O(1)$ from which we deduce that $(u_i)$ is bounded in $W^{s,p}(M)$. We can thus extract from $(u_i)_i$ a subsequence converging to some $u \in W^{s,p}(M)$ weakly in $W^{s,p}(M)$ and strongly in $L^p(M)$ and $L^q(M)$ (because $q < p^*$). Using this strong convergence and denoting $v_i := u_i - u$, it is easy to see that

$$o(1) = \left\{I'(u_i), v_i\right\} = \iint_{M \times M} |u_i(x) - u_i(y)|^{p-2}(u_i(x) - u_i(y))K(x, y, g)dv_g(x)dv_g(y) = o(1).$$

Moreover, the weak convergence $v_i \to 0$ in $W^{s,p}(M)$ gives that

$$\iint_{M \times M} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v_i(x) - v_i(y))K(x, y, g)dv_g(x)dv_g(y) = o(1).$$

Thus,

$$\iint_{M \times M} \left(|u_i(x) - u_i(y)|^{p-2}(u_i(x) - u_i(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))\right) \times (v_i(x) - v_i(y))K(x, y, g)dv_g(x)dv_g(y) = o(1).$$

Recall ([19, Lemma 1.11]) that for all $a, b \in \mathbb{R}$, we have

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a - b) \leq \frac{1}{2} (a - b)^2.$$
Proof Since \( I \) satisfies conditions (MP1) to (MP3).

We finally verify the remaining hypothesis of the Mountain Pass Theorem.

Step 2 \( I \) satisfies conditions (MP1) to (MP3).

Proof Since \( J_K \) is coercive, and thanks to the Sobolev inequality corresponding to the embedding of \( W^{s,p}(M) \) in \( L^q(M) \), there exists positive constants \( C_1, C_2 > 0 \) such that for any \( u \in W^{s,p}(M) \),

\[
I(u) = J_K(u) - \int_M |u|^q \, dv_g \geq C \left( \|u\|_{s,p}^p - \|u\|_q^q \right) \geq C \left( \|u\|_{s,p}^p - \|u\|_s^q \right)
\]

(35)

Applying (34) with \( a = u_i(x) - u_i(y) \) and \( b = u(x) - u(y) \), and using the Hölder inequality, it is easy to see that

\[
o(1) = \int_M \int_M |v_i(x) - v_i(y)|^p K(x, y, g) dv_g(x) dv_g(y).
\]

Then, \([v_i]_{s,p} \to 0\). Since \( v_i \to 0 \) in \( L^p(M) \), we deduce that \( v_i \to 0 \) strongly in \( W^{s,p}(M) \).

We finally verify the remaining hypothesis of the Mountain Pass Theorem.

Step 2 \( I \) satisfies conditions (MP1) to (MP3).

Proof Since \( J_K \) is coercive, and thanks to the Sobolev inequality corresponding to the embedding of \( W^{s,p}(M) \) in \( L^q(M) \), there exists positive constants \( C_1, C_2 > 0 \) such that for any \( u \in W^{s,p}(M) \),

\[
I(u) = J_K(u) - \int_M |u|^q \, dv_g \geq C \left( \|u\|_{s,p}^p - \|u\|_q^q \right) \geq C \left( \|u\|_{s,p}^p - \|u\|_s^q \right)
\]

(35)

Taking \( r > 0 \) small enough, then it follows that there exists \( \rho > 0 \) such that for any \( u \in \partial B_0(r) \), \( I(u) \geq \rho \). Independently, \( I(0) = 0 \), while for \( v_0 \in W^{s,p}(M), v_0 \neq 0 \),

\[
\lim_{t \to +\infty} I(tv_0) = \lim_{t \to +\infty} \left( t^p J_K(v_0) - \frac{t^q}{q} \int_M |v_0|^q \, dv_g \right) = -\infty
\]

It follows that there exists \( r > 0, \rho > 0 \), and \( u_0 = tv_0 \) such that \( I(u_0) < \rho, u_0 \in W^{s,p}(M) \setminus B_0(r) \). The Mountain Pass Lemma then gives the existence of a critical point \( u_0 \) of \( I \) such that

\[
I(u_0) > 0 = I(0)
\]

so that \( u_0 \neq 0 \). Thus, the assertion of Theorem 1.2 follows.

Proof of Theorem 1.2 for \( q \leq p \) We shall show that the functional \( I \) is weakly lower semi-continuous. Let \( \{u_i\} \subset W^{s,p}(M) \), such that \( u_i \to u \) weakly in \( W^{s,p}(M) \) as \( n \to \infty \). Thus, we get that \( u_i \to u \) strongly in \( L^q(M) \). Due to Lemma 3.1, we have the following inequality

\[
I(u_i) > I(u) + \left[ (I'(u_i), u_i - u) \right].
\]

Then we get that \( I(u) \leq \lim \inf_{r \to \infty} I(u_i) \), i.e. \( I \) is weakly lower semi-continuous in \( W^{s,p}(M) \). On the other hand, we have \( I(tv_0) \to +\infty \) as \( t \to +\infty \) for any \( v_0 \in W^{s,p}(M), v_0 \neq 0 \). Since \( I \) is weakly lower semi-continuous, it has a minimum point in \( W^{s,p}(M) \).
4 Optimal Sobolev Embedding for \( p = 2 \) in the Critical Case

It is well known (see e.g. [17]) that the following fractional Sobolev embedding holds: there exists a constant \( A > 0 \) such that
\[
\left( \int_{\mathbb{R}^n} |v|^p \, dx \right)^{\frac{p}{p^*}} \leq A \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dx \, dy
\]
for any \( v : \mathbb{R}^n \to \mathbb{R} \) measurable and compactly supported. We denote by \( K(n, s, p) \) the best constant in this embedding, namely
\[
K(n, s, p)^{-1} = \inf_{u \in W^{s,p}(\mathbb{R}^n)} \frac{[u]_{s,p}^p}{\|u\|_{p^*}^{p^*}}.
\]

In order to prove our main result, we will consider \( p = 2 \) from now on.

**Proof of Theorem 1.3**

**Step 1** Suppose that there are constants \( C_1, C_2 > 0 \) such that
\[
\left( \int_M |u|^{2^*_s} \, dv_g \right)^{\frac{2}{2^*_s}} \leq C_1 \int_M \int_M |u(x) - u(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x) \, dv_g(y)
+ C_2 \int_M u^2 \, dv_g
\]
for any \( u \in W^{s,p}(M) \). Then \( C_1 \geq K(n, s, 2) \).

**Proof** Let \( \eta : [0, +\infty) \to [0, 1] \) be a smooth test-function with compact support in \([0, 2\delta]\) and such that \( \eta \equiv 1 \) in \([0, \delta]\). We choose \( \delta > 0 \) such that \( 2\delta \) is smaller than the injectivity radius of \((M, g)\). Given a point \( x_0 \in M, \varepsilon > 0 \) and \( U \in W^{s,2}(\mathbb{R}^n) \) given by
\[
U(x) = (1 + |x|^2)^{-\frac{n-2s}{2}}
\]
we consider the test-function
\[
u_\varepsilon(x) = \eta(d_g(x_0, x)) U_\varepsilon(x) \quad \text{where} \quad U_\varepsilon(x) = \varepsilon^{-\frac{n-2s}{2}} U \left( \frac{1}{\varepsilon} \exp_{x_0}^{-1}(x) \right).
\]
Applying (38) to \( u_\varepsilon \), we obtain
\[
\left( \int_M |u_\varepsilon|^{2^*_s} \, dv_g \right)^{\frac{2}{2^*_s}} \leq C_1 \int_M \int_M |u_\varepsilon(x) - u_\varepsilon(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x) \, dv_g(y) + C_2 \int_M u_\varepsilon^2 \, dv_g.
\]
In view of (43), (54) and (55), we can send \( \varepsilon \to 0 \) to obtain
\[
\left( \int_{\mathbb{R}^n} |U|^2 \, dx \right)^{\frac{1}{2s}} \leq C_1 \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx \, dy
\]
Since \( U \) is an extremal for (37), we obtain \( C_1 \geq K(n, s, 2) \).

**Step 2** Inequality (14) holds.

**Proof** Given \( \varepsilon > 0 \), we take \( \delta > 0 \) smaller than the injectivity radius of \( (M, g) \) and a covering of \( M \) by balls \( \{B_\delta(x_i), i = 1, \ldots, N\} \), such that for any \( i = 1, \ldots, N \), the properties (19) and (20) hold. Let \( \{\eta_i, i = 1, \ldots, N\} \) be a partition of unity adapted to the covering \( \{B_\delta(x_i), i = 1, \ldots, N\} \). Then \( \|u\|_{2^*} = \|\sum_i \eta_i u\|_{2^*} \leq \sum_i \|\eta_i u\|_{2^*} \) and, by Jensen’s inequality, we obtain
\[
\|u\|_{2^*}^2 \leq \frac{1}{N} \sum_i \|\eta_i u\|_{2^*}^2.
\]
We now estimate \( \|\eta_i u\|_{2^*} \). Let \( v_i = (\eta_i u) \circ \exp_{x_i} \). Then by (30), we have
\[
\int_M |\eta_i u|^{2^*} \, dv_g \leq (1 + \varepsilon) \int_{\mathbb{R}^n} |v_i|^{2^*} \, dv_\xi.
\]
Using (36), we obtain
\[
\|\eta_i u\|_{2^*}^2 \leq (1 + \varepsilon) A \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \text{ where } A = K(n, s, 2).
\]
Given \( \delta' > 0 \) small to be specified later, we write the integral in the rhs as
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = \int_{|x - y| > \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \int_{|x - y| < \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.
\]
Using that \( (a + b)^2 \leq 2(a^2 + b^2) \), we can bound the first integral in the rhs by
\[
\int_{|x - y| > \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq 2 \int_{\mathbb{R}^n} |v_i(x)|^2 \left( \int_{|x - y| > \delta'} \frac{dy}{|x - y|^{n+2s}} \right) \, dx \\
\leq C_i (\delta')^{-2s} \int_{\mathbb{R}^n} |v_i(x)|^2 \, dx \\
\leq C_{n, \varepsilon} (\delta')^{-2s} \int_M \eta_i |u|^2 \, dv_g.
\]
Thus,

$$\|\eta_i u\|_{2s}^2 \leq C_{n,e}(\delta')^{-2s} \int_M \eta_i |u|^2 dv_g + (1 + \varepsilon) A \int_{|x-y| < \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.$$ 

Denote $$U_i := \exp_{x_i}^{-1}(B_{\delta_i}(x_i))$$, $$i = 1, \ldots, N$$. Noticing that $$v_i(x) = 0$$ if $$x \notin U_i$$, we can write the second integral in the r.h.s. as

$$\int_{|x-y| < \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = \int_{|x-y| < \delta'} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{|x-y| < \delta', \, x \in U_i, \, y \notin U_i} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{|x-y| < \delta', \, x, \, y \in U_i} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \leq 2 \int_{|x-y| < \delta', \, x \in U_i, \, y \notin U_i} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{|x-y| < \delta', \, x \in U_i} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.$$ 

Recall that supp $$\eta_i$$ is a compact subset of $$B_{\delta_i}(x_i)$$. Denote $$K_i := \exp_{x_i}^{-1}(\text{supp}(\eta_i))$$. Since $$K_i$$ is a compact, we can take some $$\alpha > 0$$ such that

$$|x - y| \geq \alpha > 0 \quad \text{for any } x \in K_i, \, y \in \mathbb{R}^n \setminus U_i, \, \text{and any } i = 1, \ldots, N.$$ 

$$\int_{|x-y| < \delta', \, x \in U_i, \, y \notin U_i} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = \int_{|x-y| < \delta', \, x \in K_i, \, y \notin U_i} \frac{|v_i(x)|^2}{|x-y|^{n+2s}} \, dx \, dy \leq \int_{K_i} |v_i(x)|^2 \left( \int_{|x-y| < \delta'} \frac{dy}{\alpha^{n+2s}} \right) \, dx \leq C_{\alpha,n,\delta'} \int_{K_i} |v_i(x)|^2 \, dx \leq C_{\alpha,n,\delta',\varepsilon} \int_M \eta_i |u|^2 dv_g.$$ 

We thus obtain

$$\|\eta_i u\|_{2s}^2 \leq C_{\alpha,n,p,\delta',\varepsilon} \int_M \eta_i |u|^2 dv_g + (1 + \varepsilon) A \int_{|x-y| < \delta', \, x, \, y \in U_i} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.$$
To bound the 2nd integral in the r.h.s., we write

\[ v_i(x) - v_i(y) = \eta_i(\exp_{x_i}(x)) [u(\exp_{x_i}(x)) - u(\exp_{x_i}(y))] + u(\exp_{x_i}(y)) [\eta_i(\exp_{x_i}(x)) - \eta_i(\exp_{x_i}(y))]. \]

Using the inequality \((a + b)^2 \leq (1 + \varepsilon)a^2 + C\varepsilon b^2, a, b \geq 0\), we deduce

\[
\int \int_{|x - y| < \delta', x, y \in U_i} \frac{|v_i(x) - v_i(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
\leq (1 + \varepsilon) \int \int_{|x - y| < \delta', x, y \in U_i} |\eta_i(\exp_{x_i}(x))|^2 \frac{|u(\exp_{x_i}(x)) - u(\exp_{x_i}(y))|^2}{|x - y|^{n+2s}} \, dx \, dy \\
+ C\varepsilon \int \int_{|x - y| < \delta', x, y \in U_i} |\eta_i(\exp_{x_i}(x)) - \eta_i(\exp_{x_i}(y))|^2 |u(\exp_{x_i}(y))|^2 \, dx \, dy \\
=: I + II.
\]

Using that \(\eta_i \circ \exp_{x_i}\) is Lipschitz with a Lipschitz constant that can be chosen to depend on \(\delta\), and thus on \(\varepsilon\), and not on \(i\), we can bound II:

\[
II \leq C\varepsilon, \delta \int_{U_i} |u(\exp_{x_i}(y))|^2 \left( \int_{|x - y| < \delta'} \frac{dx}{|x - y|^{n-(1-s)2}} \right) \, dy \\
\leq \frac{C\varepsilon, \delta}{1 - s} \int_{U_i} |u(\exp_{x_i}(y))|^2 \, dx \\
\leq C\varepsilon, \delta, s \int_{B_\delta(x_i)} |u|^2 \, dv_g.
\]

Concerning I,

\[
I \leq (1 + C\varepsilon) \int \int_{x, y \in B_\delta(x_i), \, |\exp_{x_i}^{-1}(x) - \exp_{x_i}^{-1}(y)| < \delta'} \frac{\eta_i(x) |u(x) - u(y)|^2}{|\exp_{x_i}^{-1}(x) - \exp_{x_i}^{-1}(y)|^{n+2s}} \, dv_g(x) \, dv_g(y) \\
\leq (1 + C\varepsilon) \int \int_{x, y \in B_\delta(x_i)} \eta_i(x) |u(x) - u(y)|^2 \frac{d_g(x, y)^{n+2s}}{d_g(x, y)^{n+2s}} \, dv_g(x) \, dv_g(y) \\
\leq (1 + C\varepsilon) \int \int_{x, y \in B_\delta(x_i)} \eta_i(x) |u(x) - u(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x) \, dv_g(y)
\]

where we used (20) and (3). We thus obtain

\[
\|\eta_i u\|_{L^2}^2 \leq C \int_{M} \eta_i |u|^2 \, dv_g + C \int_{B_\delta(x_i)} |u|^2 \, dv_g \\
+ (1 + C\varepsilon) A \int_{M} \eta_i(x) \left( \int_{M} |u(x) - u(y)|^2 \mathcal{K}(x, y; g) \, dv_g(y) \right) \, dv_g(x).
\]
where \( C = C(n, \alpha, s, \delta, \varepsilon) \). Since the balls \( \{ B_{\delta}(x_i), \ i = 1, \ldots, N \} \) overlap a finite number of times, summing this inequality over \( i = 1, \ldots, N \) gives (14). 

5 Applications to Equations with a Fractional Non-local Operator

In this section, we prove Theorem 1.4. The proof is similar to the classical one in the local case (see, e.g., [25]), but we prefer to give a full demonstration as some technical differences arise.

We consider the critical equation (12) for \( p = 2 \), which is

\[
L_K u + hu = f |u|^{2^*-2}u.
\]

(40)

We are interested in the weak formulation of that equation given by the following problem:

\[
\int_{M \times M} (u(x) - u(y))(v(x) - v(y))K(x, y; g) \, dv_g(x)dv_g(y) + \int_M huv = \int_M f |u|^{2^*} - 2u v \, dv_g(x),
\]

for all \( v \in W^{s, 2}(M) \), and \( u \in W^{s, 2}(M) \). Critical points of \( J_K \) in \( H \) are solutions to the problem (40), where \( J_K \) and \( H \) are defined in (10) and (16).

**Proof of Theorem 1.4** Denote \( \mu_0 := \inf_{u \in H} J_K(u) \). We shall prove that there is a \( u_0 \in H \) which attained the infimum in \( \mu_0 \). We approximate the minimization problem by the sub-critical problem

\[
\mu_q := \inf_{u \in H_q} J_K(u) \quad \text{where} \quad H_q = \{ u \in W^{s, 2}(M) : \int_M f |u|^q = 1 \}, \quad 1 \leq q < 2^*.
\]

This infimum is attained at some \( u_q \in H_q \) due to Theorem 1.2. Using the constant test-function \( 1/(\int_M f \, dv_g)^{1/q} \in H_q \) to estimate \( \mu_q \), we obtain

\[
\mu_q \leq J_K \left( \frac{1}{(\int_M f \, dv_g)^{1/q}} \right) \leq \frac{\|h\|_{\infty}}{(\int_M f \, dv_g)^{2/q}} \leq C.
\]

Since \( J_K \) is coercive, we deduce that \( (u_q) \) is bounded in \( W^{s, 2}(M) \). Then, \( u_q \rightharpoonup u_0 \) as \( q \to 2^* \) weakly in \( W^{s, 2}(M) \) and strongly in \( L^2(M) \). Thus, \( u_0 \) is a weak solution of (40).

An easy claim is that \( \mu_q \to \mu_0 \) as \( q \to 2^* \). Indeed, given \( \varepsilon > 0 \) there is a \( u \in H \) such that \( J_K(u) \leq \mu_0 + \varepsilon \). It follows that

\[
\mu_q \leq J_K \left( \frac{u}{(\int_M f |u|^q)^{1/q}} \right).
\]
It is clear that \((\int_M f|u|^q)^{1/q} \to (\int_M f|u|^{2^*})^{1/2^*} = 1\) as \(q \to 2^*.\) Then,

\[
\lim \sup \mu_q \leq J_{K}(u) \leq \mu_0 + \varepsilon.
\]

Hence, \(\lim \sup \mu_q \leq \mu_0\) as \(q \to 2^*.\) Conversely, it follows from Hölder’s inequality that

\[
1 = \int_M f|u_q|^q \, dv_g = \int_M f^{1-q/2^*} (f^{1/2^*}|u_q|)^q \, dv_g \\
\leq \left( \int_M f \, dv_g \right)^{1-q/2^*} \left( \int_M f|u_q|^{2^*} \, dv_g \right)^{q/2^*}.
\]

Thus, we have \(1 \leq \lim \inf \left( \int_M f|u_q|^{2^*} \right)\) as \(q \to 2^*.\) Noting that

\[
\mu_0 \leq J_{K} \left( \frac{u_q}{(\int_M f|u_q|^{2^*} \, dv_g)^{1/2^*}} \right) = \frac{\mu_q}{(\int_M f|u_q|^{2^*} \, dv_g)^{2/2^*}},
\]

we then get that \(\mu_0 \leq \lim \inf \mu_q\) as \(q \to 2^*.\) It follows that \(\lim_{q \to 2^*} \mu_q = \mu_0,\) and the above claim is proved.

To prove that \(u_0 \neq 0,\) we use Theorem 1.3. We write

\[
1 = \left( \int_M f|u_q|^q \, dv_g \right)^{2/q} \leq \left( \max f \right)^{2/q} \left( \int_M |u_q|^{2^*} \, dv_g \right)^{2/2^*} |M|^{(1-q/2^*)2/q} \\
\leq \left( \max f \right)^{2/q} |M|^{(1-q/2^*)2/q} \left( \left( \mathcal{L}_K u_q, u_q \right)^2 + C_\varepsilon \|u_q\|_2^2 \right) \\
\leq \left( \max f \right)^{2/q} |M|^{(1-q/2^*)2/q} \left( \left( K(n, s, 2) + \varepsilon \right) 2\mu_q + (C_\varepsilon + (K(n, s, 2) + \varepsilon) \|h\|_\infty) \|u_q\|_2^2 \right).
\]

Since \(\lim \sup_{q \to 2^*} \mu_q \leq \mu_0 < (2(\max f)^{2/2^*} K(n, s, 2))^{-1}\) we deduce that \(\|u_0\|_2 \geq C > 0\) so that \(u_0 \neq 0.\)

It remains to prove that \(u_0 \in H\) i.e. that \(\int_M f|u_0|^{2^*} = 1.\) Indeed, from the weak convergence \(u_q \rightharpoonup u_0\) on \(W^{s,2^*}(M),\) and the fundamental property of the weak limit (the norm of a weak limit is less than or equal to the infimum limit of the norms of the sequence), we have

\[
\|u_0\|_{s, 2} \leq \lim \inf \|u_q\|_{s, 2},
\]

from which it follows

\[
(\mathcal{L}_K u_0, u_0)^2 \leq \lim \inf (\mathcal{L}_K u_q, u_q)^2.
\]
As $u_q \to u_0$ strongly on $L^2(M)$, we know that $\|u_q\|_2 \to \|u_0\|_2$ and $\int_M h|u_q|^2 \to \int_M h|u_0|^2$. Then, it follows

$$(\mathcal{L}_K u_0, u_0)^2 + \int_M h|u_0|^2 \leq \lim \inf \left( (\mathcal{L}_K u_q, u_q)^2 + \int_M h|u_q|^2 \right).$$

(41)

Taking $u_0$ as a test-function in the equation where $u_0$ is a weak solution, we have that the l.h.s. of (41) is equal to $\int_M f|u_0|^{2^*} \, dv_g$. Analogously, the r.h.s. is $\lim_{q \to 2^*} \int_M f|u_q|^{q} \, dv_g = 1$. Finally, we obtain

$$\int_M f|u_0|^{2^*} \, dv_g \leq 1.$$  

To prove $\int_M f|u_0|^{2^*} \, dv_g \geq 1$, we note that

$$\mu_0 \leq \frac{J_K(u_0)}{(\int_M f|u_0|^{2^*} \, dv_g)^{2/2^*}}.$$  

Since $J_K(u_0) = \frac{1}{2} \int_M f|u_0|^{2^*} \, dv_g$ and $\mu_q = J_K(u_q)$, we get then

$$\frac{1}{2} \left( \int_M f|u_0|^{2^*} \, dv_g \right)^{1-2^*/2^*} \geq \mu_0 = \lim_{q \to 2^*} \mu_q = \lim_{q \to 2^*} \frac{1}{2} \int_M f|u_q|^{q} \, dv_g = \frac{1}{2},$$

and the Theorem is proved.

Using the constant test-function $v = 1/(\int_M f \, dv_g)^{1/2^*} \in H$, we obtain

**Corollary 5.1** If $f \geq 0$ and $h$ are smooth functions on $M$ such that $J_K$ satisfies the coercivity condition (17) and

$$\left( \frac{\max f \, h} {\left( \int_M f \, dv_g \right)^{2/2^*}} \right) < K(n, s, 2)^{-1},$$

then (18) holds and thus (40) has a non-trivial solution.

### 6 Test-Function Computations: Case $p = 2$

This section provides some necessary results to prove the Theorem (1.3). We use the assumptions (1)–(4) about $K$ and the behaviour of the function $U$ given by (15) and its derivatives. Unfortunately, we could not extend these results to the case $p \neq 2$ because there is not enough information about the decay of the derivatives of the corresponding minimizer.
Proposition 6.1 Given a function $U : \mathbb{R}^n \to \mathbb{R}$ such that $(\mathcal{L}_K U, U) < \infty$ and a point $x_0 \in M$ consider the function $U_\varepsilon(x) = \varepsilon^{-\frac{n+2s}{2}} U \left( \frac{1}{\varepsilon} \exp_{x_0}^{-1}(x) \right)$ defined on $B_\delta(x_0)$ with $\delta$ smaller than the injectivity radius of $(M, g)$. Then,

$$
\int\int_{B_\delta(x_0) \times B_\delta(x_0)} |U_\varepsilon(x) - U_\varepsilon(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x)dv_g(y) 
= \varepsilon^{n+2s} \int\int_{B_\frac{\delta}{\varepsilon} \times B_\frac{\delta}{\varepsilon}} |U(x) - U(y)|^2 \tilde{\mathcal{K}}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x)dv_{g_\varepsilon}(y). 
$$

(42)

**Proof** Consider $T : B_\frac{\delta}{\varepsilon} \to B_\delta$ given by $T(x) = \varepsilon x$. Then $T^* \exp_{x_0}^* g(x) = \varepsilon^{-2} g_\varepsilon(x)$ so that $d_{T^* \exp_{x_0}^* g} = \varepsilon^n d_{g_\varepsilon}$. Then, we have

$$
I_1 = \int\int_{B_\delta(x_0) \times B_\delta(x_0)} |U_\varepsilon(x) - U_\varepsilon(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x)dv_g(y) 
= \varepsilon^{-n+2s} \int\int_{B_1 \times B_1} |U \left( \frac{x}{\varepsilon} \right) - U \left( \frac{y}{\varepsilon} \right)|^2 \mathcal{K} \left( \exp_{x_0} x, \exp_{x_0} y; \exp_{x_0}^* g \right) \, dv_{\exp_{x_0}^* g}(x)dv_{\exp_{x_0}^* g}(y) 
= \varepsilon^{-n+2s} \int\int_{B_\frac{\delta}{\varepsilon} \times B_\frac{\delta}{\varepsilon}} |U(x) - U(y)|^2 \tilde{\mathcal{K}}(T(x_0), T(y_0); T^* \exp_{x_0}^* g) 
\, dv_{T^* \exp_{x_0}^* g}(x)dv_{T^* \exp_{x_0}^* g}(y) 
= \varepsilon^{n+2s} \int\int_{B_\frac{\delta}{\varepsilon} \times B_\frac{\delta}{\varepsilon}} |U(x) - U(y)|^2 \tilde{\mathcal{K}}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x)dv_{g_\varepsilon}(y) \quad \square
$$

Let $\eta : [0, +\infty) \to [0, 1]$ be a smooth test-function with compact support in $[0, 2\delta]$ and such that $\eta \equiv 1$ in $[0, \delta]$. We choose $\delta > 0$ such that $2\delta$ is smaller than the injectivity radius of $(M, g)$. Given a point $x_0 \in M$, we consider the test-function $u_\varepsilon(x) = \eta(d_g(x_0, x)) U_\varepsilon(x) \in W^{s, 2} (\mathbb{R}^n)$ where

$$
U_\varepsilon(x) = \varepsilon^{-\frac{n-2s}{2}} U \left( \frac{1}{\varepsilon} \exp_{x_0}^{-1}(x) \right) \quad \text{and} \quad U(x) = \left( 1 + |x|^2 \right)^{-\frac{n-2s}{2}}.
$$

Proposition 6.2 There holds

$$
\lim_{\varepsilon \to 0} \sup_{M \times M} |u_\varepsilon(x) - u_\varepsilon(y)|^2 \mathcal{K}(x, y; g) \, dv_g(x)dv_g(y) 
\leq \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx dy. 
$$

(43)

The proof uses ideas of [37, Prop. 21]. We split the proof into two steps.
Step 1 There holds
\[
\int_{M \times M} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) dv_g(y) \leq \int_{x \in B_\delta^2, y \in B_\delta^2} |U(x) - U(y)|^2 \tilde{K}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x) dv_{g_\varepsilon}(y) + O(\varepsilon^{n-2s}) + O(\varepsilon^{2s}).
\] (44)

Proof We write
\[
\int_{M \times M} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) dv_g(y) = I_1 + 2I_2 + I_3
\] (45)
where
\[
I_1 = \int_{B_\delta(x_0) \times B_\delta(x_0)} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) dv_g(y)
\]
\[
I_2 = \int_{x \in B_\delta(x_0), y \not\in B_\delta(x_0)} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) dv_g(y)
\]
\[
I_3 = \int_{x, y \not\in B_\delta(x_0)} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) dv_g(y).
\]

According to Proposition 6.1,
\[
I_1 = \varepsilon^{n+sp} \int_{B_\delta^2 \times B_\delta^2} |U(x) - U(y)|^2 \tilde{K}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x) dv_{g_\varepsilon}(y). \tag{46}
\]

We now prove
\[
I_3 \leq C \varepsilon^{n-2s}.
\] (47)

Indeed
\[
I_3 \leq \Lambda \int_{x, y \not\in B_\delta(x_0)} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{d_g(x, y)^{n+2s}} \, dv_g(x) dv_g(y).
\]

Here, we use again (3). Indeed for \(x \in B_{2\delta}\), let
\[
\tilde{u}_\varepsilon(x) := u_\varepsilon(\exp_{x_0} x) = \eta(|x|) \tilde{U}_\varepsilon(x), \quad \tilde{U}_\varepsilon(x) = \varepsilon^{-\frac{n-2s}{n}} U(x/\varepsilon).
\]

Then,
\[
I_3 \leq \Lambda \int_{x, y \not\in B_\delta} \frac{|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^2}{d_{\exp_{x_0} x, y}^{n+2s}} \, dv_{\exp_{x_0} x} dv_{\exp_{x_0} y}.
\]
There exists $C > 0$ such that as bilinear forms,
\[
C^{-1} \delta_{ij} \leq (\exp_{x_0}^* g)(x) \leq C \delta_{ij} \quad \text{for } x \in B_{2\delta}.
\] (48)

It follows that
\[
I_3 \leq C \iint_{x, y \notin B_{\delta}} \frac{\vert \tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y) \vert^2}{\vert x - y \vert^{n+2s}} \, dx \, dy
\]
\[
\leq C \iint_{x \in B_{2\delta}, y \notin B_{\delta}} \frac{\vert \tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y) \vert^2}{\vert x - y \vert^{n+2s}} \, dx \, dy
\]
where we used that $\tilde{u}_\epsilon$ is supported in $B_{2\delta}$ in the second inequality. Eventually, for $x, y \in \mathbb{R}^n \setminus B_{\delta}$,
\[
\vert \tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y) \vert
\]
\[
\leq 1_{\{ \vert x \vert, \vert y \vert \geq \delta, \vert x - y \vert \leq \frac{\delta}{2} \}} \max_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} (\vert \tilde{U}_\epsilon \vert + \vert \nabla \tilde{U}_\epsilon \vert) \vert x - y \vert + 1_{\{ \vert x \vert, \vert y \vert \geq \delta, \vert x - y \vert \geq \frac{\delta}{2} \}} 2 \max_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} |\tilde{U}_\epsilon|.
\]
\[
\leq C \varepsilon^{n-2s} \left( 1_{\{ \vert x \vert, \vert y \vert \geq \delta, \vert x - y \vert \leq \frac{\delta}{2} \}} \vert x - y \vert + 1_{\{ \vert x \vert, \vert y \vert \geq \delta, \vert x - y \vert \geq \frac{\delta}{2} \}} \right).
\]
Thus,
\[
\varepsilon^{2s-n} I_3 \leq C \iint_{\vert x \vert \leq 2\delta, \vert x - y \vert \leq \frac{\delta}{2}} \frac{dx \, dy}{\vert x - y \vert^{n+2s-2}} + C \iint_{\vert x \vert \leq 2\delta, \vert x - y \vert \geq \frac{\delta}{2}} \frac{dx \, dy}{\vert x - y \vert^{n+2s}}
\]
\[
\leq C \iint_{\vert x \vert \leq 2\delta, \vert t \vert \leq \frac{\delta}{2}} \frac{dx \, dt}{\vert t \vert^{n+2s-2}} + C \iint_{\vert x \vert \leq 2\delta, \vert t \vert \geq \frac{\delta}{2}} \frac{dx \, dt}{\vert t \vert^{n+2s}}
\]
which is finite. We deduce (47).

Concerning $I_2$ we will prove that
\[
I_2 \leq \iint_{x \in B_{\frac{\delta}{2}}, y \in B_{\frac{\delta}{2}} \setminus B_{\delta}} \vert U(x) - U(y) \vert^2 \tilde{K}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x)dv_{g_\varepsilon}(y)
\]
\[
+ O(\varepsilon^{n-2s}) + O(\varepsilon^{2s}).
\] (49)

To prove that we write
\[
I_2 = I_2^1 + I_2^2 + I_2^3
\]
with
\[
I_2^1 = \iint_{x \in B_{\delta}(x_0), y \notin B_{\delta}(x_0), d_{g}(x, y) < \frac{\delta}{4}} \vert u_{\varepsilon}(x) - u_{\varepsilon}(y) \vert^2 K(x, y; g) \, dv_{g}(x)dv_{g}(y)
\]
\[I_2^2 = \int \int_{x \in B_\delta(x_0), \ y \in B_{2\delta}(x_0) \setminus B_\delta(x_0), \ d_\delta(x, y) \geq \frac{\delta}{2}} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) \, dv_g(y)\]

\[I_2^3 = \int \int_{x \in B_\delta(x_0), \ y \notin B_{2\delta}(x_0), \ d_\delta(x, y) \geq \frac{\delta}{2}} |u_\varepsilon(x) - u_\varepsilon(y)|^2 K(x, y; g) \, dv_g(x) \, dv_g(y).\]

We first check, as we did for \(I_3\), that

\[I_2^1 \leq C \varepsilon^{n-2s}. \quad (50)\]

Indeed

\[I_2^1 \leq \Lambda \int \int_{x \in B_\delta(x_0), \ y \notin B_\delta(x_0), \ d_\delta(x, y) < \frac{\delta}{2}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{d_\delta(x, y)^{n+2s}} \, dv_g(x) \, dv_g(y)\]

\[\leq \Lambda \int \int_{\frac{\delta}{2} \leq |x|, \ |y| \leq \frac{3\delta}{4}, \ d_\exp\delta(x, y) < \frac{\delta}{4}} \frac{\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)^2}{d_\exp\delta(x, y)^{n+2s}} \, dv_{\exp\delta}(x) \, dv_{\exp\delta}(y)\]

\[\leq C \int \int_{\frac{\delta}{2} \leq |x|, \ |y| \leq \frac{3\delta}{4}, \ |x - y| < \frac{\delta}{4}} \frac{\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)^2}{|x - y|^{n+2s}} \, dx \, dy\]

where the constant \(C\) can be chosen arbitrarily close to 1 up to taking \(\delta\) small enough. In particular, we assume that \(\rho := \frac{\delta}{2} - C \frac{\delta^2}{4} > 0\). It follows that any segment \([x, y]\) remains far from 0 since for any \(t \in [0, 1]\), \(|x + t(y - x)| \geq |x - t(y - x)| = |x| - t|y - x| \geq \rho\). Then, \(\max_{[x, y]} |\tilde{U}_\varepsilon| + |\nabla \tilde{U}_\varepsilon| \leq C \varepsilon^{n-2s}\). Thus,

\[\varepsilon^{2s-n} I_2^1 \leq C \int \int_{\frac{\delta}{2} \leq |x|, \ |y| \leq \frac{3\delta}{4}, \ |x - y| < C \frac{\delta^2}{4}} \frac{1}{|x - y|^{n+2s-2}} \, dx \, dy \leq C\]

where we used that \(s \in (0, 1)\) to obtain that the inner integral is bounded by a constant. From there, we get (50).

Concerning \(I_3\), notice that when \(x \in B_\delta(x_0)\) and \(y \notin B_{2\delta}(x_0)\), we have \(d_\delta(x, y) > \delta\) and \(u_\varepsilon(y) = 0\). From there and (3), we have

\[I_2^3 \leq \Lambda \int \int_{x \in B_\delta(x_0), \ y \notin B_\delta(x_0)} \frac{|u_\varepsilon(x)|^2}{d_\delta(x, y)^{n+2s}} \, dv_g(x) \, dv_g(y)\]

\[\leq \Lambda \, \text{Vol}_g(M) \delta^{n-2s} \int_{x \in B_\delta(x_0)} |u_\varepsilon(x)|^2 \, dv_g(x).\]
In view of (54), we obtain $I_2^3 \leq C \delta e^{-2s}$.

We eventually verify that

$$I_2^2 \leq \int_{x \in B_{\delta}(x_0), \ y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0), \ d_p(x, y) \geq \frac{\delta}{4}} [U_{\epsilon}(x) - U_{\epsilon}(y)]^2 K(x, y; g) \, dv_g(x)dv_g(y)$$

$$+ O(\epsilon^{-n-2s}) + O(\epsilon^{2s}). \tag{51}$$

Indeed for $x \in B_{\delta}(x_0)$, we have $u_{\epsilon}(x) = U_{\epsilon}(x)$ so that

$$|u_{\epsilon}(x) - u_{\epsilon}(y)|^2 = |U_{\epsilon}(x) - U_{\epsilon}(y) + U_{\epsilon}(y) - u_{\epsilon}(y)|^2$$

$$= |U_{\epsilon}(x) - U_{\epsilon}(y)|^2 + |U_{\epsilon}(y) - u_{\epsilon}(y)|^2$$

$$+ 2|U_{\epsilon}(x) - U_{\epsilon}(y)||U_{\epsilon}(y) - u_{\epsilon}(y)|$$

$$\leq |U_{\epsilon}(x) - U_{\epsilon}(y)|^2 + 8|U_{\epsilon}(y)|^2 + 4|U_{\epsilon}(x)||U_{\epsilon}(y)|.$$

As we had for $I_2^3$, we get

$$\int_{x \in B_{\delta}(x_0), \ y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0), \ d_p(x, y) \geq \frac{\delta}{4}} [U_{\epsilon}(x) - U_{\epsilon}(y)]^2 K(x, y; g) \, dv_g(x)dv_g(y)$$

$$\leq \Lambda C_{\delta} \int_{B_{2\delta}(x_0)} |U_{\epsilon}(y)|^2 \, dv_g(y) \leq C \epsilon^{2s}.$$

Moreover, for $x \in B_{\delta}(x_0)$ and $y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0)$, we have $U_{\epsilon}(y) \leq C \epsilon^{(n-2s)/2}$ and

$$U_{\epsilon}(x)U_{\epsilon}(y) \leq CU(\exp_{x_0}^{-1}(x)/\epsilon).$$

Then,

$$\int_{x \in B_{\delta}(x_0), \ y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0), \ d_p(x, y) \geq \frac{\delta}{4}} [U_{\epsilon}(x)||U_{\epsilon}(y)|K(x, y; g) \, dv_g(x)dv_g(y)$$

$$\leq \Lambda \int_{x \in \bar{B}_{\delta}(x_0), \ y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0), \ d_p(x, y) \geq \frac{\delta}{4}} [U_{\epsilon}(x)||U_{\epsilon}(y)|K(x, y; g) \, dv_g(x)dv_g(y)$$

$$\leq C \int_{|x| \leq \delta, \ \delta \leq |y| \leq 2\delta, \ d_{\exp_{x_0}^{-1}}(x, y) \geq \frac{\delta}{4}} \frac{U(x/\epsilon)}{d_{\exp_{x_0}^{-1}}(x, y)^{n+2s}} \, dv_{\exp_{x_0}^{-1}}(x)dv_{\exp_{x_0}^{-1}}(y)$$

$$\leq C \epsilon^{-n-2s} \int_{|x| \leq \delta, \ \delta \leq |y| \leq 2\delta, \ d_{\exp_{x_0}^{-1}}(x, y) \geq \frac{\delta}{4}} \frac{U(x/\epsilon)}{|x-y|^{n+2s}} \, dv_{\exp_{x_0}^{-1}}(x)dv_{\exp_{x_0}^{-1}}(y)$$

$$\leq C \epsilon^{-n-2s} \int_{|x| \leq \delta, \ \delta \leq |y| \leq 2\delta, \ |x-y| \geq \delta/(2/\epsilon)} \frac{U(x)}{|x-y|^{n+2s}} \, dxdy$$

$$\leq C \epsilon^{-n-2s} \int_{|x| \leq \delta/(2/\epsilon)} U(x) \, dx \int_{|\xi| \geq \delta/(2/\epsilon)} |\xi|^{-n-2s} \, d\xi.$$
Since $\int_{|x|\leq \frac{\delta}{R}} U(x) \, dx \leq C e^{-2s}$ and $\int_{|\xi|\geq \delta/(2\varepsilon)} |\xi|^{-n-2s} \, d\xi \leq C \varepsilon^{-2s}$, we obtain

$$\int_{x \in B_{\delta}(x_0), y \in B_{2\delta}(x_0) \setminus B_{\delta}(x_0), \delta(x,y) \geq \frac{\delta}{R}} |U_{\varepsilon}(x)||U_{\varepsilon}(y)|K(x, y; g) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y) \leq C \varepsilon^{-n-2s}. \leqno{(51)}$$

We deduce (51). \hfill {}\Box

**Step 2** (43) holds.

**Proof** In view of the previous Step, it is enough to prove that

$$\limsup_{\varepsilon \to 0} \varepsilon^{n+sp} \int_{B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}} |U(x) - U(y)|^2 \tilde{K}(x, y; g_{\varepsilon}) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \leqno{(52)}$$

Given $R > 0$ let

$$\varepsilon_R = \int_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (B_R \times B_R)} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx \, dy$$

which goes to 0 as $R \to +\infty$ by dominated convergence since $U \in W^{s,2}(\mathbb{R}^n)$. For $\varepsilon > 0$, small enough so that $R < 2\frac{\delta}{\varepsilon}$, we split the integral in the l.h.s of (52) as

$$\int_{B_R \times B_R} |U(x) - U(y)|^2 \tilde{K}(x, y; g_{\varepsilon}) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y)$$

$$+ \int_{(B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}) \setminus (B_R \times B_R)} |U(x) - U(y)|^2 \tilde{K}(x, y; g_{\varepsilon}) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y). \leqno{(53)}$$

Then, for the second integral it holds

$$\int_{(B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}) \setminus (B_R \times B_R)} |U(x) - U(y)|^2 \tilde{K}(x, y; g_{\varepsilon}) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y)$$

$$\leq \Lambda \int_{(B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}) \setminus (B_R \times B_R)} \frac{|U(x) - U(y)|^2}{d_{g_{\varepsilon}}(x, y)^{n+2s}} \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y)$$

$$\leq C \int_{(B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}) \setminus (B_R \times B_R)} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq C \varepsilon_R.$$

Thus,

$$\int_{B_{\frac{\delta}{R}} \times B_{\frac{\delta}{R}}} |U(x) - U(y)|^2 \tilde{K}(x, y; g_{\varepsilon}) \, d\nu_{\varepsilon}(x) \, d\nu_{\varepsilon}(y)$$

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\[
\int_{B_R \times B_R} \left| U(x) - U(y) \right|^2 \tilde{K}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x) dv_{g_\varepsilon}(y) + O(\varepsilon R).
\]

Moreover, for a given \( R > 0 \), we can send \( \varepsilon \to 0 \) in (53). From (4), we obtain
\[
\lim_{\varepsilon \to 0} \varepsilon^{n+s} \int_{B_R \times B_R} \left| U(x) - U(y) \right|^2 \tilde{K}(x, y; g_\varepsilon) \, dv_{g_\varepsilon}(x) dv_{g_\varepsilon}(y)
= \int_{B_R \times B_R} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx dy = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|U(x) - U(y)|^2}{|x - y|^{n+2s}} \, dx dy - \varepsilon R.
\]

\( \square \)

**Proposition 6.3** There hold
\[
\int_M |u_\varepsilon|^2 \, dv_g \leq C \varepsilon^{2s}
\]  
and
\[
\lim_{\varepsilon \to 0} \int_M |u_\varepsilon|^{2^*} \, dv_g = \int_{\mathbb{R}^n} |U|^{2^*} \, dx.
\]

**Proof** We have
\[
\int_M |u_\varepsilon|^2 \, dv_g = \varepsilon^{-n+2s} \int_{B_{2\delta}(x_0)} |U_\varepsilon(x)|^2 \, dv_g \leq \varepsilon^{2s} \int_{B_{2\delta}(x_0)} |U|^2 \, dv_{g_\varepsilon} \leq C \varepsilon^{2s} \int_{\mathbb{R}^n} |U|^2 \, dx.
\]

Moreover, given \( R > 0 \), we have for \( \varepsilon \) small enough so that \( R\varepsilon < \delta \), that
\[
\int_M |u_\varepsilon|^{2^*} \, dv_g = \varepsilon^{-n} \int_{B_{R\varepsilon}(x_0)} |U_\varepsilon|^{2^*} \, dv_g + \varepsilon^{-n} \int_{B_{2\delta}(x_0) \setminus B_{R\varepsilon}(x_0)} |u_\varepsilon|^{2^*} \, dv_g = I + J.
\]

We have that \( I \) tends to \( \int_{\mathbb{R}^n} |U|^{2^*} \, dx \) as \( R \to +\infty \) and \( \varepsilon \to 0 \). Furthermore,
\[
J \leq \int_{B_{2\delta} \setminus B_R} |U|^{2^*} \, dv_{g_\varepsilon} \leq C \int_{B_{2\delta} \setminus B_R} |U|^{2^*} \, dx
\]
and thus \( J \) goes to 0 as \( R \to +\infty \) and \( \varepsilon \to 0 \). \( \square \)

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