Abstract

This paper is about the feasibility and means of root-n consistently estimating linear, mean-square continuous functionals of a high dimensional, approximately sparse regression. Such objects include a wide variety of interesting parameters such as regression coefficients, average derivatives, and the average treatment effect. We give lower bounds on the convergence rate of estimators of a regression slope and an average derivative and find that these bounds are substantially larger than in a low dimensional, semiparametric setting. We also give debiased machine learners that are root-n consistent under either a minimal approximate sparsity condition or rate double robustness. These estimators improve on existing estimators in being root-n consistent under more general conditions than previously known.

1 Introduction

A regression slope is of interest in many settings. The coefficient of a binary treatment variable is a weighted average treatment effect in treatment models under the conditional independence assumptions of Rosenbaum and Rubin (1983). The coefficient of a price variable is important for economic demand models. There are also other important objects that depend on a regression, including average derivatives and average treatment effects.

We give necessary conditions for root-n consistent estimation of a regression slope or average derivative of an approximately sparse, high dimensional regression. We show that these conditions are substantially stronger than corresponding conditions for low dimensional regression. We also give estimators that are root-n consistent under minimal approximate sparsity of the regression or under the rate double robustness condition of the debiased machine learning literature. These estimators improve on existing estimators in being root-n consistent under more general conditions than previous estimators.

Approximate sparsity is the key condition here that determines whether it is possible to root-n consistently estimate a regression slope and other objects. Approximate sparsity refers to the minimum root mean square error in approximating a high dimensional function by a linear combination of $s$ regressors that are unknown elements of a known, large set of $p$ regressors, as in Belloni et al. (2014). We consider sparse approximation rates of the form $s^{-\xi}$ for $\xi > 0$. Our results depend on two such rates. The first rate $\xi_1$ is the sparse approximation rate for the regression and the second rate $\xi_2$ is the sparse rate for another function we refer to as the Riesz representer. The key condition for root-n consistency is

$$\max\{\xi_1, \xi_2\} > 1/2.$$  \hfill (1.1)

We show this condition is necessary for existence of a root-n consistent estimator of a regression slope or average derivative.

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We also show that, along with a few other regularity conditions, the minimal condition $\xi_1 > 1/2$ is sufficient for root-n consistent estimation of a regression slope, average derivative, average treatment effect, and any other linear functional that is mean-square continuous. The estimators are based on Lasso regression, Lasso estimation of a bias correction term, and special cross-fitting as explained in Section 3. These estimators are also root-n consistent under the rate double robustness condition $\xi_1 \xi_2 > 1/4$, which is equivalent to the product of Lasso convergence rates for the regression and Riesz representer going to zero faster than $n^{-1/2}$, i.e. is equivalent to $\xi_1/(2\xi_1 + 1) + \xi_2/(2\xi_2 + 1) > 1/2$. These approximate sparsity conditions are more general than previously given in allowing for either $\xi_1 > 1/2$ or $\xi_1 \xi_2 > 1/4$. In addition we give a regression slope estimator that attains root-n consistency under $\xi_2 > 1/2$, where the approximate sparsity of the regression is not restricted, i.e. the regression can be “dense.” This is a debiased machine learning estimator similar to Chernozhukov et al. (2018a) with another kind of special cross-fitting. We are not aware of any existing estimator that only requires nice Riesz representer ($\xi_2 > 1/2$) and achieves the parametric rate for a regression slope under weak conditions allowing for non-Gaussian regressors and heteroskedastic errors.

An important feature of sparse approximations is the unknown identity of the $s$ regressors that give the sparse approximation rate $s^{-4}$, which regressors we will refer to as the “best regressors.” This feature distinguishes sparse approximations from series or sieve approximations for low dimensional regressions where the identity of the best regressors is known. We show that with known identity of the best regressors a necessary condition for root-n consistent estimation of a regression slope is

$$\xi_1 + \xi_2 \geq 1/2.$$  \hspace{1cm} (1.2)

Equation (1.1) is more restrictive than Equation (1.2), showing increased requirements for attaining root-n consistency when the identity of the best regressors is unknown.

Figure 1 summarizes these conditions and their relationship to rate double robustness. Values of $(\xi_1, \xi_2)$ outside the box are those where the necessary condition of Equation (1.1) is satisfied, for root-n consistency under approximate sparsity. Values outside the lower triangle are those where the necessary condition of Equation (1.2) is satisfied, for root-n consistent estimation when the identity of the best regressors is known. The wedge below the box but above the triangle is the set of $(\xi_1, \xi_2)$ where the necessary condition is satisfied for known identity of the best regressors but not for unknown identity. The existence of this wedge shows that it is more difficult to attain root-n consistency when the identity of the best regressors is unknown.

Points above or to the right of the hyperbola in Figure 1 are those where the rate double robustness condition $\xi_1 \xi_2 > 1/4$ is satisfied, e.g. as in Farrell (2015) and Chernozhukov et al. (2018a). We give estimators that attain root-n consistency on the union of points above the hyperbola with points to the right of the box where $\xi_1 > 1/2$, a larger set of $(\xi_1, \xi_2)$ values than previously obtained. The estimator of a regression slope that attains root-n consistency for a dense regression does so for points above the box where $\xi_2 > 1/2$.

We also obtain a lower bound for convergence rates under approximate sparsity when $\hat{\xi} := \max\{\xi_1, \xi_2\} \leq 1/2$ that is $(\ln(p)/n)^{2\xi/(2\xi+1)}$, which is illustrated in Figure 2. This lower bound is also determined by the maximum of the sparse approximation rates. We can see here that approximate sparsity leads to convergence rate bounds that are slower by a power of $n$ than bounds where the best regressors are known. For example when $\xi_1 = \xi_2 = 1/4$ Equation (1.2) is satisfied so that the lower bound is $n^{-1/2}$ when the best regressors are known but under approximate sparsity the lower bound is $(\ln(p)/n)^{1/3}$, that goes to zero slower than $n^{-1/3}$.

The first estimator of a regression coefficient we give is different than Belloni et al. (2014) and is more like the debiased Lasso estimators of Javanmard and Montanari (2014, 2018), Van de Geer et al. (2014), Zhang and Zhang (2014), and Cai and Guo (2017). To explain the relationship of the conditions here to the conditions in these papers it is helpful to relate the strict sparsity conditions of these papers to the size of $\xi_1$ and $\xi_2$. The regression strict sparsity condition in these papers is $s_0 \ln(p)/\sqrt{n} \rightarrow 0$, where $s_0$ is the number of nonzero coefficients (and the second moment matrix of the regressors is unknown). This condition is equivalent to the root mean square convergence rate for Lasso regression being faster than $n^{-1/4}$. With approximate sparsity Lasso will converge faster than $n^{-1/4}$ when $\xi_1 > 1/2$, so that $\xi_1 > 1/2$ corresponds to the strict sparsity conditions in the previous papers. Except for Cai and Guo (2017) and Javanmard and Montanari (2014), these papers also impose the...
The above figure displays $2\xi/(2\xi + 1)$, where $\xi = \max\{\xi_1, \xi_2\}$. 
corresponding approximate or strict sparsity condition on the residual from regressing the variable of interest on the other regressors. Because the Riesz representer here is proportional to this residual that sparsity condition corresponds to $\xi_2 > 1/2$. We do not require this condition. In addition we obtain root-n consistency under the doubly robust rate condition $\xi_1, \xi_2 > 1/4$ that allows for $\xi_1 < 1/2$ if $\xi_2$ is large enough. The conditions here also differ from the previous literature, except for Belloni et al. (2014), in providing standard errors that are robust to heteroskedasticity and in allowing for non Gaussian disturbances and regressors; see Corollary 6 in Section 5. The results here are also for approximately sparse specifications that may be more credible in some applications than strictly sparse specifications.

The regression coefficient estimator that attains root-n consistency when $\xi_2 > 1/2$ and the regression is dense is a debiased machine learning estimator like Chernozhukov et al. (2018a). This estimator builds on Zhu and Bradic (2018) and Bradic et al. (2022). It differs from the previous literature in standard errors being robust to heteroskedasticity and in not requiring Gaussian regression or regressors. As far as we are aware the conditions given here are the weakest yet given for root-n consistent estimation of a slope when the regression is dense.

The estimators of an average derivative, average treatment effect, and a general linear functional are automatic debiased machine learners similar to Chernozhukov et al. (2018b) and Chernozhukov et al. (2018d) with special cross-fitting. These attain root-n consistency under either $\xi_1 > 1/2$ or $\xi_1, \xi_2 > 1/4$. Athey et al. (2018) gave an estimator of the average treatment effect on the treated that is root-n consistency for a homoskedastic regression with sparsity corresponding to $\xi_1 > 1/2$. Also, Hirshberg and Wager (2021) recently showed in independent work that their minimax estimator of a regression functional is root-n consistent for a high dimensional model under conditions similar to $\xi_1 > 1/2$. In comparison the automatic debiased machine learner given here is relatively simple to compute, has simple heteroskedasticity robust standard errors, is doubly robust, and is root-n consistent under $\xi_1, \xi_2 > 1/4$, as discussed in Section 4.

The minimal approximate sparsity condition of Equation (1.1) is analogous to the minimal strict sparsity conditions of (Javanmard and Montanari, 2018, Theorem 3.13 and Proposition 4.2). A crucial difference is that our results do not depend on a quantity that is closely related to the $\ell_1$-norm of the rows of the precision matrix, i.e., the quantity $\rho$ in Javanmard and Montanari (2018). Moreover, in contrast to Remark 4.3 therein, the lower and upper bounds here match, thereby establishing the minimax rate. Our work differs from Cai and Guo (2017) in involving the approximate sparsity of the Riesz representer, a feature that is not present for the regression functionals considered there.

Targeted maximum likelihood (van der Laan Mark and Daniel, 2006) based on machine learners has been considered by Van der Laan et al. (2011) and large sample theory given by Luedtke and van der Laan (2016), Toth and van der Laan (2016), and Zheng et al. (2018). Mackey et al. (2018) showed that weak sparsity conditions would suffice for $1/\sqrt{n}$ consistency of a certain estimator of a partially linear conditional mean when certain variables are independent and non Gaussian. The estimator given there will not be consistent for the objects and model we consider.

Our model specification is like Van de Geer et al. (2014) in being an explicit semiparametric model, or more precisely a nonparametric model where the object of interest is a functional of a regression. This modeling approach facilitates comparison with conditions for root-n consistent estimation in the semiparametric literature where the best approximating functions are known, especially conditions of Robins et al. (2009). Robins et al. (2009) showed that Equation (1.2) is a necessary conditions for root-n consistent estimation when unknown functions are in Holder classes for a regression coefficient and the average treatment effect. In a Holder class each of $\xi_1$ and $\xi_2$ will equal the ratio of the number of derivatives of the regression and Riesz representer, respectively, to the dimension of the regressors when the regressors are splines. We consider instead necessary conditions in terms of sparse approximation rates, which is more general but perhaps less intuitive. In this setting the series estimator of Donald and Newey (1994) attains root-n consistency under the minimal condition of Equation (1.2).

It is interesting to note that in the approximately sparse setting debiased machine learning based on regression that converges at the optimal nonparametric rate is root-n consistent under minimal conditions. In contrast, in the semiparametric, low dimensional setting a root-n consistent estimator can be obtained under the minimal condition
\(\xi_1 = 1/2\) without debiasing but seems to require undersmoothing where the convergence rate of the regression is not optimal, e.g. see Newey and Robins (2018). Thus we see that the debiasing of regression learners that converge at optimal nonparametric rates seems particularly well suited to approximately sparse models.

In Section 2 we describe the parameters of interest. Section 3 explains the approximately sparse setting of the paper and gives minimax lower bounds. Section 4 describes the estimators we give. Section 5 derives their properties.

## 2 A Regression Slope and Other Linear Functionals of a Regression

To describe a regression slope let \(W\) denote a data observation, \(Y\) an outcome of interest, \(D\) a variable of interest, and \((Z_1,Z_2,...)\) an infinite sequence of potential covariates. A regression slope will be the coefficient \(\theta_0\) of \(D\) in a linear regression of \(Y\) on \(D\) and \((Z_1,Z_2,...)\), that is

\[
E[Y|D,Z_1,Z_2,...] = \theta_0D + \sum_{j=1}^{\infty} \gamma_0 Z_j, \quad E[D^2] < \infty, \quad E[Z_j^2] < \infty, \quad (j = 1,2,...). \tag{2.1}
\]

This model is a familiar and important multiple regression with \(\theta_0\) being the coefficient of the regressor \(D\) and \(Z_j\), \((j = 1,...,\bar{\rho})\), being covariates, where we allow for an infinite number of covariates. When all but a finite number of \(\gamma_0\) are zero this is a sparse specification. In general we regard the sum as a mean square limit of sparse specifications but keep the sum notation for expositional purposes. We do not require sparsity but do require that \(\theta_0D + \sum_{j=1}^{\infty} \gamma_0 Z_j\) can be approximated in mean square by a linear combination of \(s\) covariates at rate \(s^{-\bar{\xi}_1}\). This is the approximate sparsity condition highlighted in the Introduction and further discussed in the rest of this paper.

As noted in the Introduction, the regression coefficient \(\theta_0\) is of wide interest. To motivate the estimators and other results it is helpful to interpret \(\theta_0\) as a functional of a regression. Let \(X = (X_1,X_2,...)\) denote the regressor sequence \(X_1 = D\) and \(X_j = Z_{j-1}\), \((j \geq 2)\). Then Equation (2.1) can be written as

\[
E[Y|X] = \rho_0(X), \quad \rho_0(X) = \sum_{j=1}^{\infty} \beta_j X_j, \quad E[X_j^2] < \infty, \quad (j = 1,2,...), \tag{2.2}
\]

where \(\beta_{10} = \theta_0\) and \(\beta_j = \gamma_j - \theta_0\), \((j \geq 2)\). Define \(\mathcal{B}\) to be the set of mean square limits of sparse specifications, i.e. limits of linear combinations of a finite number of elements of the sequence \(X\). Equation (2.2) (and Equation (2.1)) impose the condition that

\[
E[Y|X] \in \mathcal{B},
\]

In general this condition restricts the conditional mean to be linear in elements of \(X = (X_1,X_2,...)\). This condition is not restrictive when linear combinations of \(X\) can approximate in mean square any function of \(X\), but we do not limit ourselves to this case. We will maintain \(E[Y|X] \in \mathcal{B}\) throughout much of this paper.

The regression coefficient \(\theta_0\) can be interpreted as a functional (scalar valued function) of the regression \(\rho_0(X)\) given by

\[
\theta_0 = \frac{\rho_0(\bar{d},Z) - \rho_0(d,Z)}{d - \bar{d}}, \tag{2.3}
\]

where \(d\) and \(\bar{d}\) are distinct values in the support of \(D\). We see from this equation that \(\theta_0\) is a linear functional of the regression function \(\rho_0(x)\). We will focus on specifications where the functional \(\theta_0\) is also mean square continuous in possible \(\rho \in \mathcal{B}\), an important property that has a key role in our results. Let \(\tilde{D}\) denote the linear projection of \(D\) on the mean square closure of linear combinations of a finite number of the covariates \((Z_1,Z_2,...)\) and let \(U = D - \tilde{D}\). We will assume throughout that \(U \neq 0\), i.e. that \(E[U^2] > 0\). Let \(\tilde{\alpha}(X) = (E[U^2])^{-1} U\). Then for any \(\rho(X) = D\theta + \sum_{j=1}^{\infty} \gamma_j Z_j \in \mathcal{B}\),

\[
E[\tilde{\alpha}(X)\rho(X)] = (E[U^2])^{-1} \{\theta E[UD] + E[U(\sum_{j=1}^{\infty} \gamma_j Z_j)]\} = (E[U^2])^{-1} \{\theta E[U^2] + 0\} = \theta.
\]
Noting that \( E[\alpha(X)^2] = 1/E[U^2] < \infty \) it follows that \( |E[m(W, \rho)]| \leq (E[\rho(X)^2]/E[U^2])^{1/2} < \infty \), so that \( E[m(W, \rho)] \) is mean square continuous in \( \rho \). This interpretation of \( \theta_0 \) will be useful in the necessary conditions for root-n consistency and in constructing an estimator of \( \theta_0 \).

There are many other interesting examples of objects of interest that can be formulated as a linear, mean square continuous functional of an infinite dimensional linear regression \( \rho_0(X) \) similar to Equation (2.3). A regression slope as just described constitutes our first example, that we will refer to as Example 1. Two other examples we consider are an average derivative and the average treatment effect. In each example there will be a functional \( m(W, \rho) \) that is linear in \( \rho \) such that the object of interest \( \theta_0 \) is given by

\[
\theta_0 = E[m(W, \rho_0)].
\]

Thus the parameter of interest is an expectation of some known formula \( m(W, \rho) \) of a data observation \( W \) and the regression function \( \rho \). We focus on objects where there is \( \tilde{\alpha}(X) \) to the semiparametric variance bound for \( \theta_0 \) consistency and in constructing an estimator of \( \rho \) continuous functional of an infinite dimensional linear regression with \( \rho \). By the Riesz representation theorem, existence of such a \( \bar{\alpha}(X) \) is equivalent to \( E[m(W, \rho)] \) being a mean-square continuous functional of \( \rho \), i.e. there is \( C > 0 \) such that \( |E[m(W, \rho)]| \leq C\|\rho\|_2 \) for all \( \rho \in \mathcal{B} \), where \( \|\rho\|_2 = \sqrt{E[\rho(X)^2]} \). We will refer to this \( \bar{\alpha}(X) \) as the Riesz representer. Existence of the Riesz representer is equivalent to the semiparametric variance bound for \( \theta_0 \) being finite, as mentioned in Newey (1994) and shown in Hirshberg and Wager (2021) and Chernozhukov et al. (2018c).

We turn now to the average derivative and average treatment effect examples.

**Example 2: (Average Derivative).** Here \( X_j = X_j (D, Z) \) \( (j \geq 1) \) for a continuously distributed variable \( D \) and covariates \( Z = (Z_1, Z_2, \ldots) \), \( \rho_0(x) = \rho_0(d, z) \), \( \omega(d) \) is a pdf, and

\[
\theta_0 = E \left[ \omega(u) \frac{\partial \rho_0(u, Z)}{\partial d} du \right] = E[m(W, \rho_0)], \ m(W, \rho) = S(U) \rho(U, Z),
\]

where \( S(u) = -\omega(u)^{-1} \partial \omega(u)/\partial u \) is the negative of the location score for the pdf \( \omega(u) \). \( U \) is a random variable that is independent of \( (Y, X) \) and has pdf \( \omega(u) \), and the second equality follows by integration by parts. This \( \theta_0 \) can be interpreted as an average treatment effect on \( Y \) of a continuous treatment \( D \); see Chernozhukov et al. (2018d). Multiplying and dividing by the conditional pdf \( f(d|z) \) of \( D = d \) given \( Z = z \) we find that for any \( \rho(X) \) with \( E[\rho(X)^2] < \infty \),

\[
E[m(W, \rho)] = E \left[ \int S(u) \rho(u, Z) \omega(u) du \right] = E[f(D|Z)^{-1}S(D)\omega(D)\rho(X)] = E[\alpha_0(X)\rho(X)],
\]

\[
\alpha_0(X) = f(D|Z)^{-1}S(D)\omega(D).
\]

Letting \( \text{proj}(a(W)|\mathcal{B}) \) denote the linear projection of any \( a(W) \) with finite second moment on \( \mathcal{B} \), it follows that Equation (2.4) is satisfied for

\[
\tilde{\alpha}(X) = \text{proj}(f(D|Z)^{-1}S(D)\omega(D)|\mathcal{B}).
\]

**Example 3: (Average Treatment Effect).** Here we can take \( X = (Z_1, Z_2, \ldots, DZ_1, DZ_2, \ldots) \) and \( \rho_0(x) = \rho_0(d, z) \), where \( D \in \{0, 1\} \) is a treatment indicator and \( Z \) are covariates. The object of interest is

\[
\theta_0 = E[\rho_0(1, Z) - \rho_0(0, Z)] = E[m(W, \rho_0)], \ m(W, \rho) = \rho(1, Z) - \rho(0, Z).
\]

If potential outcomes are mean independent of treatment \( D \) conditional on covariates \( Z \), then \( \theta_0 \) is the average treatment effect (Rosenbaum and Rubin, 1983). Let \( \pi_0(Z) = \Pr(D = 1|Z) \) be the propensity score. Note that \( E[\rho(1, Z)] = E[\pi_0(Z)^{-1}D\rho(1, Z)] = E[\pi_0(Z)^{-1}D\rho(X)] \) and similarly \( E[\rho(0, Z)] = E[(1 - \pi_0(Z))^{-1}(1-D)\rho(X)] \). Then for any \( \rho \in \mathcal{B} \)

\[
E[m(W, \rho)] = E[\rho(1, Z) - \rho(0, Z)] = E[\alpha_0(X)\rho(X)], \ \alpha_0(X) = \frac{D}{\pi_0(Z)} - \frac{1-D}{1-\pi_0(Z)}.
\]
Similarly to Example 2, Equation (2.4) is satisfied for 

\[ \bar{\alpha}(X) = \text{proj} \left( \frac{D}{\pi_0(Z)} - \frac{1-D}{1-\pi_0(Z)} | B \right) . \]

An important feature of the set up here is that the Riesz representer \( \bar{\alpha}(X) \) is an element of \( \mathcal{B} \) and so can be approximated by a sparse linear combination of the regressors \( X = (X_1, X_2, \ldots) \). Consequently a sparse approximation for \( \bar{\alpha}(X) \) exists by the definition of \( \mathcal{B} \) and does not require approximation of nonlinear functions by a linear regression. For instance in Example 3 the Riesz representer \( \bar{\alpha}(X) \) is the linear projection of the inverse propensity score term \( \alpha_0(X) \) on the set of linear combinations of \( X = (Z_1, Z_2, \ldots, DZ_1, DZ_2, \ldots) \). Thus for the average treatment effect and in general a sparse approximation of \( \bar{\alpha}(X) \) exists by construction and does not require that the inverse propensity score be well approximated by linear regressors.

3 Lower Bounds on Convergence Rates

In this Section we give lower bounds for the convergence rates of estimators of \( \theta_0 \). We clarify the distinction between approximately sparse models and those where identity of the important regressors is known and explain the key conditions on which our results are based. We give lower bounds under approximate sparsity and when the identity of the important regressors is known.

3.1 Approximate Sparsity

For the sequence of regressors \( (X_1, X_2, \ldots) \) specified in Section 2 let \( b(x) = (x_1, \ldots, x_p) \) denote the \( p \times 1 \) vector of the first \( p \) regressors. For a scalar random variable \( a(X) \) let \( \|a\|_2 = \sqrt{E[a(X)]^2} \) denote the mean square norm. For a \( p \times 1 \) constant vector \( c \) let \( \|c\|_0 \) denote the number of nonzero elements of \( c \). For any constants \( C, \xi > 0 \), and positive integer \( t \in \mathbb{N} \) we define

\[ \mathcal{M}_{C, \xi} := \left\{ v \in \mathbb{R}^p : \min_{\|a\|_0 \leq t} \|b(\cdot)'(v - a)\|_2 \leq Ct^{-\xi} \forall t \in \mathbb{N} \right\} . \]

In this definition \( t \) is the number of nonzero elements of \( a \). This \( \mathcal{M}_{C, \xi} \) is the set of \( p \times 1 \) coefficients \( v \) such that \( b(X)'v \) can be approximated in mean square by \( b(X)'a \), at a rate \( t^{-\xi} \), where \( t \) is the number of nonzero components of \( a \). The idea is that \( b(X)'v \) is the true regression and \( a \) are the coefficients of a sparse approximation to \( b(X)'v \) with approximation rate \( t^{-\xi} \). Here \( \mathcal{M}_{C, \xi} \) is a set of approximately sparse specifications.

Approximately sparse specifications are different than more familiar nonparametric specifications in ways that are useful in high dimensional settings. Approximate sparsity allows for very many potential regressors (possibly many more than sample size) when relatively few important regressors give a good approximation but the identity of those few is not known. In contrast, series approximations are based on relatively few regressors, often many fewer than the sample size. Approximately sparse and series approximations are similar in that they both depend on a few regressors giving a good approximation. They differ in that series regression requires that the identity of the important regressors is known, while approximate sparsity allows their identity to be unknown. This difference is useful in high dimensional settings, where there are potentially very many regressors needed to approximate a function of many variables. Typically, there is little guidance about which of many regressors is important, such as interaction terms in a multivariate approximation. With approximate sparsity, such information is not needed, since very many terms can be included among the potential regressors.

We can be precise about this key difference between approximate sparsity and series approximations by comparing \( \mathcal{M}_{C, \xi} \) with a class of functions corresponding to series approximations. For \( \mathcal{M}_{C, \xi} \) the nonzero components of \( a \) are allowed to be coefficients of any of the \( p \) dictionary functions, where \( p \) can be large, even larger than sample size. In the definition of \( \mathcal{M}_{C, \xi} \) it is unknown which dictionary functions are used in the sparse approximate at
rate $Ct^{-\xi}$. A series approximation from the semiparametric literature would require that the unknown function be well approximated by a linear combination of the first $t$ functions. The set of unknown $v$ allowed here would be

$$\mathcal{F}_{C, \xi} = \left\{ v \in \mathbb{R}^p : \min_{a = (a_1, \ldots, a_0, 0, \ldots, 0)} \| b(a) (v - a) \|_2 \leq Ct^{-\xi} \forall t \in \mathbb{N} \right\}$$

For example, suppose that $X$ is continuously distributed with compact support and that dictionary functions are products of all nonnegative powers of $X$ that are weakly increasing in order with $j$. If $X$ has dimension $d$ and $v$ is such that $b(x)^j$ has bounded derivatives of order $s$ then it is well known that there is $C$ and an ordering of $b(x)$ such that the inequality in the definition of $\mathcal{F}_{C, \xi}$ is satisfied with $\xi = s/d$. This $\xi$ is the well known rate for approximation of functions in a Holder class. Comparing $\mathcal{F}_{C, \xi}$ with $\mathcal{H}_{C, \xi}$ we see that the approximately sparse class $\mathcal{H}_{C, \xi}$ extends the notion of a series approximation to allow the best approximating functions to be unknown. Approximate sparsity means there is an $t^{-\xi}$ approximation rate without specifying the order/direction/location of the elements of $b(x)$ that give this rate. Notice that if $\xi \geq \tilde{\xi}$, then $\mathcal{H}_{C, \xi} \subseteq \mathcal{H}_{C, \tilde{\xi}}$, similarly to $\mathcal{F}_{C, \xi}$ shrinking with $\xi$.

### 3.2 Lower Bounds Under Approximate Sparsity

We work with data $\{W_i\}_{i=1}^n$ that are i.i.d. where the distribution of $W_i$ can change with $n$. This setup is common for results on lower bounds for convergence rates where the lower bound is uniform across a set of data generating processes.

**ASSUMPTION 1:** $(Y_i, D_i, Z_i')$ is jointly Gaussian with mean zero, $E[Z_iZ_i'] = I_p,$

$$Y_i = D_i\theta + Z_i'\gamma + \epsilon_i, \ D_i = Z_i'\pi + u_i,$$

where $E[Z_i\epsilon_i] = E[Z_iu_i] = E[D_i\epsilon_i] = 0$.\n
The lower bound derived under this condition is a minimax lower bound in any model where Assumption 1 is satisfied as a special case. A lower bound on a convergence rate obtained for a particular model is a lower bound over any class of models that include the particular model as a special case. Let $\sigma_n^2 = E\epsilon_i^2$ and $\sigma_\epsilon^2 = E\epsilon_2^2$. The distribution of the data is now parameterized by $\lambda = (\theta, \gamma, \pi, \sigma_n^2, \sigma_\epsilon^2)$. Let $C_1, C_2, \xi_1, \xi_2 > 0$. We define the following parameter space

$$\Lambda_{\xi_1, \xi_2} = \left\{ \lambda = (\theta, \gamma, \pi, \sigma_n^2, \sigma_\epsilon^2) : \theta \in \mathbb{R}, \gamma \in \mathcal{M}_{C_1, \xi_1}, \pi \in \mathcal{M}_{C_2, \xi_2}, \{\sigma_n^2, \sigma_\epsilon^2\} \subset [M^{-1}, M] \right\},$$

where $M \geq 2$ is a constant.

Let $\mathcal{C}(\Lambda)$ be the set of $1 - \alpha$ confidence intervals for $\theta$ that are valid uniformly over $\lambda \in \Lambda$. We are interested in the following

$$\mathcal{L}(\Lambda, \tilde{\Lambda}) = \inf_{C \in \mathcal{C}(\Lambda)} \sup_{\lambda \in \Lambda} E(\lambda) / |C|$$

for $\Lambda \subseteq \tilde{\Lambda}$, where $|C|$ denotes the width (the Lebesgue measure) of the confidence interval $C$. If $\mathcal{L}(\Lambda, \tilde{\Lambda})$ depends on $\tilde{\Lambda}$ instead of $\Lambda$, then there is no adaptivity between $\Lambda$ and $\tilde{\Lambda}$. If $\Lambda = \tilde{\Lambda}$, then $\mathcal{L}(\Lambda, \Lambda)$ is the minimax rate over $\Lambda$. The primary goal is to study this object with $\Lambda = \Lambda_{\xi_1, \xi_2}$ and $\tilde{\Lambda} = \Lambda_{\tilde{\xi}_1, \tilde{\xi}_2}$, where $\xi_1 \geq \tilde{\xi}_1$ and $\xi_2 \geq \tilde{\xi}_2$. This means $\Lambda_{\xi_1, \xi_2} \subseteq \Lambda_{\tilde{\xi}_1, \tilde{\xi}_2}$. We will assume that we are in a high-dimensional setting where $p > n$ for large enough $n$ by imposing the condition that there exists a constants $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 \ln p \leq n \leq \kappa_2 p \ln p$ for large enough $n$.

**THEOREM 1:** If Assumption 1 is satisfied and there exists a constant $\kappa > 0$ such that $\kappa_1 \ln p \leq n \leq \kappa_2 p \ln p$ for large enough $n$, then for any $\xi_1 \geq \tilde{\xi}_1 \geq 0$, $\xi_2 \geq \tilde{\xi}_2 \geq 0$, and $\xi = \max\{\xi_1, \xi_2\} \leq 1/2$,

$$\mathcal{L}(\Lambda_{\xi_1, \xi_2}, \Lambda_{\tilde{\xi}_1, \tilde{\xi}_2}) \geq C(n^{-1} \ln p)^{\xi/(2\xi + 1)}.$$
where $C > 0$ is a constant depending only on $\xi_1, \xi_2, \kappa, \alpha, C_0$.

Theorem 1 has two important implications. First, when $\xi = \max\{\xi_1, \xi_2\}$ is much smaller than $1/2$, the rate $\mathcal{L}(A_{\xi_1, \xi_2}, A_{\xi_1, \xi_2})$ can be much slower than the parametric rate $n^{-1/2}$. Second, Theorem 1 implies that adaptivity to the size of $\xi_1$ and $\xi_2$ is not possible. Notice that in Theorem 1, the lower bound for $\mathcal{L}(A_{\xi_1, \xi_2}, A_{\xi_1, \xi_2})$ only depends on max $\{\xi_1, \xi_2\}$ and has nothing to do with $(\xi_1, \xi_2)$. This means that any confidence interval that is valid over $A_{\xi_1, \xi_2}$ with max $\{\xi_1, \xi_2\} \leq 1/2$ cannot have expected width $n^{-1/2}$ even at points in a smaller parameter space $A_{\xi_1, \xi_2}$, no matter how small $A_{\xi_1, \xi_2}$ is. Hence, there does not exist a confidence interval that satisfies both of the following properties: (1) being valid over $A_{\xi_1, \xi_2}$ with max $\{\xi_1, \xi_2\} < 1/2$ and (2) having expected width $O(n^{-1/2})$ on a smaller (potentially much smaller) space $A_{\xi_1, \xi_2}$. As a result, it is not possible to distinguish between max $\{\xi_1, \xi_2\} < 1/2$ and max $\{\xi_1, \xi_2\} > 1/2$ from the data. In particular, no test for this condition can allow us to obtain a confidence interval of length $1/\sqrt{n}$ when max $\{\xi_1, \xi_2\} < 1/2$.

We would also like to point out that although $\mathcal{L}(A_{\xi_1, \xi_2}, A_{\xi_1, \xi_2})$ measures the expected length of confidence intervals, the rates are not due to the possibility of $|CI|$ taking extreme values with a small probability because the parameter set is bounded.

The average derivative is a harder problem than partial linear models and hence the lower bound in Theorem 1 applies to the problem of average derivative. To see this, consider a function of $(Z_i, X_i)$. A special case is when the partial derivative with respect to $Z_i$ is constant. In this special case, the average derivative problem becomes learning a coefficient in a partial linear model. By Theorem 1, even in this special problem, max $\{\xi_1, \xi_2\}$ is a necessary condition for attaining root-$n$ consistency. Therefore, in general, one needs to impose max $\{\xi_1, \xi_2\} > 1/2$ to obtain the $1/\sqrt{n}$ rate for the average derivative.

### 3.3 Lower Bound Under Series Approximation

We observe $\{(Y_i, Z_i, X_i)\}_{i=1}^n$ from

$$Y_i = Z_i \beta + X_i' \phi + v_i,$$

where $\gamma, \pi \in \mathbb{R}^p, X_i \sim N(0, I_p)$ is independent of $V_i = (e_i, u_i)' \sim N(0, \Sigma)$ with $\Sigma = \text{diag}(\sigma_2^2, \sigma_3^2)$. The distribution of a data observation $W$ is determined by the parameter $\theta = (\beta, \phi, \pi, \sigma_2^2, \sigma_3^2)$. We are interested in $\beta$. We consider the ordered space

$$\mathcal{S}_2 = \left\{ x = (x_1, \ldots, x_p)' \in \mathbb{R}^p : \sqrt{\sum_{j=r+1}^{t} x_j^2} \leq M_1 \tau^{-\xi} \quad \forall \tau \in \{1, \ldots, p\} \right\}.$$  

We consider

$$\Theta(\xi_1, \xi_2) = \left\{ \theta = (\beta, \phi, \pi, \sigma_2^2, \sigma_3^2) : \gamma \in \mathcal{S}_{M_1, \xi_1}, \pi \in \mathcal{S}_{M_2, \xi_2}, \frac{1}{M_2} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq M_2, \sigma_2^2, \sigma_3^2 \in [M_3^{-1}, M_3] \right\},$$

where $M_1, M_2, M_3 > 0$ are constants. The minimax rate for estimating $\beta$ is

$$\mathcal{R}(\xi_1, \xi_2) = \inf_{f_0} \sup_{\theta = (\beta, \phi, \pi, \sigma_2^2, \sigma_3^2) \in \Theta(\xi_1, \xi_2)} E_{\theta} \left| \hat{\beta}(W) - \beta \right|,$$

where $\inf_{f_0}$ is taken over all measurable functions $\hat{\beta}$ of the data. The goal is to derive a lower bound for $\mathcal{R}(\xi_1, \xi_2)$ when $\xi_1 + \xi_2 < 1/2$.

**Theorem 2:** Assume that $p \geq n^2$. If $\xi_1 + \xi_2 < 1/2$, then $R(\xi_1, \xi_2) \gg n^{-1/2}$.

We know that this lower bound is sharp because Donald and Newey (1994) showed that a series estimator is root-$n$ consistent under $\xi_1 + \xi_2 \geq 1/2$. 


We note that although Theorem 1 is about the optimal length of confidence intervals, it implies that the parametric rate is impossible for any estimator in the unordered case if \( \max\{\xi_1, \xi_2\} \leq 1/2 \). Suppose that there were a root-n consistent estimator under \( \max\{\xi_1, \xi_2\} \leq 1/2 \). Then one can construct a confidence interval centered at this estimator with radius that shrinks at the root-n rate. This would be a confidence interval with length being \( O(n^{-1/2}) \). However, by Theorem 1 (take \( \bar{\xi}_1 = \xi_1 \) and \( \bar{\xi}_2 = \xi_2 \)) such confidence intervals do not exist under \( \max\{\xi_1, \xi_2\} \leq 1/2 \). Hence, if \( \max\{\xi_1, \xi_2\} \leq 1/2 \), there does not exist any estimator that is consistent at the parametric rate.

By comparing Theorems 1 and 2, we see the role of ordering. When the dictionary is not ordered (i.e., the most important terms might not be the first a few basis in the dictionary), the parametric rate is only possible when \( \max\{\xi_1, \xi_2\} > 1/2 \). On the other hand, for ordered models (the most important terms show up first in the dictionary), once can achieve the parametric rate once \( \xi_1 + \xi_2 \geq 1/2 \). As illustrated in Figure 1, the requirement of \( \xi_1 + \xi_2 \geq 1/2 \) is much weaker than \( \max\{\xi_1, \xi_2\} > 1/2 \): triangle versus square. This is price to pay for being unable to order the basis functions by their approximation power.

### 4 Estimation and Inference

We consider estimators based on Lasso regression and bias correction terms with special cross-fitting. For the cross-fitting we partition the observation indices \( \{1, \ldots, n\} \) into two sets of about equal size, \( I_\ell \ (\ell = 1, 2) \), and let \( n_\ell \) denote the number of elements of \( I_\ell \). Also let \( |c|_1 := \sum_{j=1}^p |c_j| \), for a \( p \times 1 \) vector \( c \), \( r > 0 \) be a Lasso penalty to be specified in Section 5, and \( b(X) = (X_1, \ldots, X_p)' \) continue to be the dictionary of regressors. The Lasso regression estimator for each split \( \ell \) is

\[
\hat{\theta}_\ell(x) := b(x)'\hat{\gamma}_\ell, \quad \hat{\gamma}_\ell := \arg\min_\gamma \left\{ \gamma'\hat{\Sigma}_\ell \gamma - 2\hat{\mu}_\ell'\gamma + 2r \|\gamma\|_1 \right\}, (\ell = 1, 2), \tag{4.1}
\]

The first estimator we give is a debiased Lasso estimator of a regression slope. For this estimator \( b(X) = (D, Z_1, \ldots, Z_{p-1})' \) with first element being the regressor of interest. The estimator \( \hat{\theta} \) of the coefficient \( \theta_0 \) of that regressor is

\[
\hat{\theta} = \frac{n_1}{n} \hat{\theta}_1 + \frac{n_2}{n} \hat{\theta}_2, \tag{4.2}
\]

\[
\hat{\theta}_1 := \hat{\gamma}_1 + \frac{1}{n_1} \sum_{i \in I_1} \hat{\alpha}_1(X_i) \gamma - \hat{\rho}_2(X_i) \gamma, \quad \hat{\theta}_2 := \hat{\gamma}_1 + \frac{1}{n_2} \sum_{i \in I_2} \hat{\alpha}_2(X_i) \gamma - \hat{\rho}_1(X_i) \gamma.
\]

Here each \( \hat{\alpha}_\ell(x) \) is a bias correction term given by

\[
\hat{\alpha}_\ell(x) = b(x)'\hat{\pi}_\ell, \quad \hat{\pi}_\ell := \arg\min_{\pi} \left\{ \pi'\hat{\Sigma}\pi - 2\hat{\pi}_1 + 2r \|\pi\|_1 \right\}, (\ell = 1, 2). \tag{4.3}
\]

The estimator \( \hat{\theta} \) is a bias-corrected Lasso similar to those of Javanmard and Montanari (2014, 2018), Van de Geer et al. (2014), Zhang and Zhang (2014), and Cai and Guo (2017). It differs in the form of the bias correction terms \( \hat{\alpha}_\ell(x) \) and in the use of cross-fitting. In the supplementary materials, we provide an estimator without any cross-fitting.

The cross-fitting in \( \hat{\theta} \) sums over the same observations used to construct \( \hat{\alpha}_\ell(x) \) but different observations than used to construct \( \hat{\theta}_\ell(x) \). Using the same observations to construct \( \hat{\alpha}_\ell(x) \) as in the sums enables the first order conditions for \( \hat{\alpha}_\ell(x) \) to help bound a key quadratic remainder under weak conditions on Riesz representer \( \hat{\alpha}(x) \). As a result this estimator will attain root-n consistency under \( \xi_1 > 1/2 \) and some regularity conditions. Using different observations for \( \hat{\theta}_\ell(x) \) than in the sums leads to root-n consistency under the rate double robustness condition \( \xi_1 \xi_2 > 1/4 \). In this way the special cross-fitting in Equation (4.2) gives an estimator that is root-n consistent if either \( \xi_1 > 1/2 \) or \( \xi_1 \xi_2 > 1/4 \).
To construct an estimator of the asymptotic variance of \( \sqrt{n}(\hat{\theta} - \theta_0) \) that allows for very weak conditions on \( \bar{\alpha}(X) \) it is theoretically convenient to trim the estimator \( \hat{\alpha}_\ell(x) \). Let \( \tau_n > 0 \) be a constant that will be assumed to go to infinity with \( n \) and for a scalar \( a \) let

\[
\tau_n(a) := \begin{cases} 
\bar{\tau}_n, & a \geq \bar{\tau}_n, \\
\alpha_n, & \alpha_n < a < \tau_n, \\
-\bar{\tau}_n, & a \leq -\bar{\tau}_n.
\end{cases}
\]

Also let \( \hat{\alpha}_\ell(x) = \tau_n(\hat{\alpha}_\ell(x)) \), \( (\ell = 1, 2) \). The asymptotic variance estimator is

\[
\hat{\psi} = \frac{1}{n} \sum_{i=1}^n \psi_i^2, \quad \psi_i = \begin{cases} 
\hat{\gamma}_1 + \hat{\alpha}_1(X_i)[Y_i - \hat{\rho}_1(X_i)] - \hat{\theta}, & i \in I_1, \\
\hat{\gamma}_2 + \hat{\alpha}_2(X_i)[Y_i - \hat{\rho}_2(X_i)] - \hat{\theta}, & i \in I_2.
\end{cases}
\]

This will be a heteroskedasticity consistent estimator of the asymptotic variance of \( \sqrt{n}(\hat{\theta} - \theta_0) \) under the conditions given in Section 5.

The estimator \( \hat{\theta} \) of Equation (4.2) is root-n consistent for those values of \( (\xi_1, \xi_2) \) to the right of the box in Figure 1, where \( \xi_1 > 1/2 \). We now give another estimator that is root-n consistent for the rest of \( (\xi_1, \xi_2) \) values outside the box where \( \xi_2 > 1/2 \). Here the regression is allowed to be dense with no restriction placed on its rate of sparse approximation \( \xi_1 \). As before we partition the observations into \( I_1 \) and \( I_2 \). Let \( b(z) = (z_1, \ldots, z_{p-1})' \) denote the \((p-1) \times 1 \) vector of all dictionary functions except \( d \) and

\[
\hat{\xi}(z) := b(z)'\hat{\gamma}, \quad \hat{\gamma} = \arg\min_{\gamma} \{ \gamma'\hat{\Sigma}_I \gamma - 2\hat{\mu}'_I \gamma + 2r ||\gamma||_1 \}, \\
\hat{\eta}(z) := b(z)'\hat{\pi}_I, \quad \hat{\pi}_I = \arg\min_{\pi} \{ \pi'\hat{\Sigma}_I \pi - 2\hat{\mu}'_I \pi + 2r ||\pi||_1 \}, \\
\hat{\Sigma}_I := \frac{1}{n} \sum_{i \notin I_1} b(Z_i) b(Z_i)' \hat{\mu}_I := \frac{1}{n} \sum_{i \notin I_1} b(Z_i) Y_i, \quad \hat{M}_I := \frac{1}{n} \sum_{i \notin I_1} b(Z_i) D_i.
\]

Here \( \hat{\xi}(z) \) and \( \hat{\eta}(z) \) are Lasso like estimators where different data observations are used for the estimator \( \hat{\Sigma}_I \) of the second moment matrix of \( b(z) \) than for the cross moment estimators \( \hat{\mu}_I \) and \( \hat{M}_I \). An estimator of the regression slope

\[
\hat{\theta} = \hat{H}^{-1} \hat{\Sigma} \hat{H}^{-1}, \quad \hat{\Sigma} = \sum_{i \in I_1} (D_i - \hat{\eta}(Z_i)) (Y_i - \hat{\xi}(Z_i)) / n, \quad \hat{H} = \sum_{i \in I_1} (D_i - \hat{\eta}(Z_i))^2 / n.
\]

Here \( \hat{\xi}(z) \) and \( \hat{\eta}(z) \) are not based on different observations than in \( I_1 \). Instead \( \hat{\gamma} \) and \( \hat{\pi}_I \) use \( \hat{\Sigma}_I \) that is constructed from observations in \( I_1 \). Using the same observations to construct \( \hat{\Sigma}_I \) as in the sums enables the first order conditions for \( \hat{\gamma} \) and \( \hat{\pi}_I \) to help bound a key quadratic remainder under weak conditions. Using a different samples for \( \hat{\gamma} \) and \( \hat{\pi}_I \) than for \( \hat{\Sigma}_I \) also helps bound key remainders. We will show that \( \hat{\theta} \) is root-n consistent and asymptotically normal under the minimal approximate sparsity condition \( \xi_2 > 1/2 \).

An estimator of the asymptotic variance of \( \sqrt{n}(\hat{\theta} - \theta_0) \) can be constructed as

\[
\hat{\psi} = \hat{H}^{-1} \hat{\Sigma} \hat{H}^{-1}, \quad \hat{\Sigma} = \sum_{i \in I_1} (D_i - \hat{\eta}_I(Z_i))^2 / n
\]

\[
\hat{M}_I = \frac{1}{n} \sum_{i \in I_1} m(W, b), \quad m(W, b) = (m(W, b_1), \ldots, m(W, b_p))'.
\]
The estimator of $\theta_0$ is given by
\[ \hat{\theta} = \frac{n_1}{n} \hat{\theta}_1 + \frac{n_2}{n} \hat{\theta}_2, \quad \hat{\theta}_1 = \frac{1}{n_1} \sum_{i \in I_1} \{ m(W_i, \hat{\rho}_2) + \hat{\alpha}_1(X_i) | Y_i - \hat{\rho}_2(X_i) \}, \tag{4.7} \]
\[ \hat{\theta}_2 = \frac{1}{n_2} \sum_{i \in I_2} \{ m(W_i, \hat{\rho}_1) + \hat{\alpha}_2(X_i) | Y_i - \hat{\rho}_1(X_i) \}. \]

Here the cross-fitting is like that of the first regression slope estimator for analogous reasons. The estimators in Equations (4.5) and (4.7) differ from those in Chernozhukov et al. (2018a,d) by having a different type of cross-fitting.

A consistent asymptotic variance estimator can be constructed similar to the debiased Lasso variance estimator given above. For $\hat{\alpha}_\ell(x) = \tau_n(\hat{\alpha}_\ell(x)), (\ell = 1, 2)$, we let
\[ \hat{\psi} = \frac{1}{n} \sum_{i=1}^n \hat{\psi}_i = \begin{cases} m(W_i, \hat{\rho}_2) + \hat{\alpha}_1(X_i) | Y_i - \hat{\rho}_2(X_i) | - \hat{\theta}, & i \in I_1, \\ m(W_i, \hat{\rho}_1) + \hat{\alpha}_2(X_i) | Y_i - \hat{\rho}_1(X_i) | - \hat{\theta}, & i \in I_2. \end{cases} \tag{4.8} \]

This estimator is heteroskedasticity robust for any $m(W, \rho)$.

**EXAMPLE 1:** (Regression Coefficient) Recall that $b(X) = (D, Z_1, ..., Z_{p-1})$ for the regression slope, i.e. the first component of $b(X)$ is the regressor of interest $D$. Let $\hat{\theta}_{\ell}$ denote the Lasso regression coefficient of $d$ for the Lasso regressions in Equation (4.1) with $\ell = 1, 2$. Note that by Equation (2.3) we have $m(W, \hat{\rho}_{\ell}) = \hat{\theta}_{\ell}$. Also note that $m(W, b_1) = 1$ and $m(W, b_j) = 0$ for all $j \geq 2$, so that $m(W, b) = e_1 = (1, 0, ..., 0)'$ is the first unit vector. Then the previously specified estimators of $\hat{\alpha}(X) = U, \theta_0$, and $V$ given in Equations (4.3), (4.2), and (4.4), respectively, are the same as those of Equations (4.6), (4.7), and (4.8).

**EXAMPLE 2:** (Weighted Average Derivative) Here $X_j = X_j(D, Z) (j \geq 1)$ where $D$ is continuously distributed and
\[ m(W_i, \hat{\rho}_j) = S(U_i) \hat{\rho}_j(U_i, Z_i), \]
where $U_i$ has pdf $\omega(u)$ and is independent of $(Y_i, X_i)$. One can think of $U_i$ as a single simulation draw from the distribution with pdf $\omega(u)$. Also, we have
\[ m(W_i, b_j) = S(U_i) b_j(U_i, Z_i), \quad (j = 1, ..., p). \]

A high dimensional weighted average derivative estimator can then be obtained from Equations (4.6) and (4.7), with asymptotic variance estimator from Equation (4.8).

**EXAMPLE 3:** (Average Treatment Effect) Here the dictionary can be chosen in a variety of ways. One way is to choose $b(x)$ to consist of products of a $p/2$ dimensional vector of covariates $x' = (z_1, ..., z_{p/2})$ with the treatment and nontreatment dummies $d$ and $1 - d$, that is
\[ b(x) = (d' (1 - d) z'). \]

Here $m(W_i, \hat{\rho}_j) = \hat{\rho}_j(1, Z_i) - \hat{\rho}_j(0, Z_i), m(W, b) = (Z'_j - Z'_i)'$. A high dimensional average treatment effect estimator can then be obtained from Equations (4.6) and (4.7), with asymptotic variance estimator from Equation (4.8). As discussed in Chernozhukov et al. (2018d) the resulting $\hat{\alpha}_\ell(x)$ can be interpreted as balancing weights for the covariates $Z_i$.

## 5 Large Sample Properties of the Estimators

We consider large sample inference when $W_1, ..., W_n$ are i.i.d., with the distribution of an observation not changing with $n$. We specify approximate sparsity and other conditions sufficient for root-$n$ consistency of the estimators we are considering. We also obtain consistency of the estimator $\hat{\theta}$ for the asymptotic variance of $\sqrt{n}(\hat{\theta} - \theta_0)$.  

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In what follows we work with the population least squares coefficients $\gamma$ from the regression of $Y$ on $b(X)$ and $\pi$ from the regression of $\tilde{\alpha}(X)$ on $b(X)$. These are given by solutions to

$$
\mu = \Sigma \gamma, \quad M = \Sigma \pi,
$$

where $\mu = E[b(X)Y]$, $\Sigma = E[b(X)b(X)^T]$, and $M = E[m(W,b)] = E[b(X)\tilde{\alpha}(X)]$. Such $\gamma$ and $\pi$ exist by standard projection argument and $\mu$ and $M$ being expected products of $b(X)$ with a random variable.

To specify conditions for $\tilde{\theta}$ we begin with an approximate sparsity condition for the regression of $Y$ on $b(X)$. For a $p \times c$ vector $c$ let $\|c\|_2 = \sqrt{\sum_{j=1}^p c_j^2}$. For any nonincreasing function $f : \mathbb{N} \mapsto [0, \infty)$, we define

$$
\mathcal{M}_f := \left\{ v \in \mathbb{R}^p : \min_{\|a\|_0 \leq t} \|v - a\|_2 \leq f(t) \forall t \in \mathbb{N} \right\}.
$$

We impose the following approximate sparsity condition on $\gamma$.

**ASSUMPTION 3:** There is $C > 0$ and $\xi > 0$ such that for all $p$ we have $\gamma \in \mathcal{M}_f$ for $f(t) = Ct^{-\xi}$.

When the maximum eigenvalue of $\Sigma = E[b(X)b(X)^T]$ is bounded, which we will assume, this condition strengthens the sparse approximation rate condition of Section 3 to apply to the vector of population least squares coefficients $\gamma$ rather than the mean-square projection $b(X)^T \gamma$. Assumption 3 is equivalent to the sparse approximation rate of Section 3 when the smallest eigenvalue of $\Sigma$ is bounded away from zero. Since $\Sigma = I$ for the model from which the lower bound is constructed, the lower bound continues to hold under this Assumption 3. The next condition specifies $\tilde{\alpha}(X)$ to be the Riesz representer.

**ASSUMPTION 4:** There is $\tilde{\alpha}(X) \in \mathcal{R}$, $C > 0$ such that $E[m(W,p)] = E[\tilde{\alpha}(X)p(X)]$ for all $p \in \mathcal{R}$, $m(W,b_j) = \tilde{m}_0(W)\tilde{m}_j(W)$ where $E[\tilde{m}_0(W)^2] < \infty$, $\max_{j \leq p} |\tilde{m}_j(W)| \leq C$, and $m(W,\gamma)$ depends only on $W$ such that $E[Y|\tilde{W}] = \rho_0(X)$.

We also impose a condition on the Lasso penalty. Let $e_n = \sqrt{\ln(p)/n}$.

**ASSUMPTION 5:** $e_n = o(r)$ and $r = o(n^c e_n)$ for every $c > 0$.

Here we allow $r$ to go to zero slightly slower than than $e_n$ which simplifies the conditions without affecting their sharpness. This assumption could be avoided by specifying that $r \approx C e_n$ for a large enough $C$ and that certain conditions hold with large probability. We impose Assumption 5 for simplicity.

The next condition imposes that the elements of the dictionary $b(X)$ are uniformly bounded. For a $p \times 1$ vector $d$ let $\|d\|_\infty = \max_{j \leq p} |d_j|$.

**ASSUMPTION 6:** There is $C > 0$ such that for all $n$, $\|b(X)\|_\infty \leq C$ and the largest eigenvalue of $\Sigma$ is bounded uniformly in $p$.

This condition simplifies the analysis considerably. It could be changed to allow for sub Gaussian regressors as in the lower bounds of Section 3.

We also make use of a sparse eigenvalue condition as in much of the Lasso literature. Let $J$ denote a subvector of $(1, ..., p)$, $|J|$ denote the number of elements of $J$, $\gamma_J$ be the vector consisting of $\gamma_{f_j} = \gamma_j$ for $j \in J$ and $\gamma_{f_j} = 0$ otherwise, and $\gamma_J$ be the corresponding vector for $J'$ (so that $\gamma = \gamma_J + \gamma_{J'}$).

**ASSUMPTION 7:** There are $c, C > 0$ such that with probability approaching one for all $s = O(n/\ln(p))$,

$$
\min_{\|\gamma\|_1 \leq 3\|\gamma\|_1} \min_{|J| \leq s} \frac{\gamma^T \Sigma \gamma}{\gamma_{J}^T \gamma_{J}} \geq c.
$$

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The next condition is about $E[Y|X]$ and $Var(Y|X)$.

**ASSUMPTION 8:** $E[Y|X] \in \mathcal{B}, E[Y^2] < \infty$, and there is $C > 0$ with $Var(Y|X) \leq C$.

The next two Assumptions will specify conditions for the two cases $\xi_1 > 1/2$ and $\xi_1 \xi_2 > 1/4$ respectively. With $\xi_1 > 1/2$, for simplicity we will constrain $r$ to not grow too slowly in a way determined by a sparse approximation to $\tilde{\alpha}(X)$. Existence of a sparse approximation to $\tilde{\alpha}(X)$ follows from the definition of $\mathcal{B}$ as shown by the following result.

**LEMMA 3:** If $\tilde{\alpha}(X) \in \mathcal{B}$ there is $C > 0$, $\delta_n \rightarrow 0$, and $\pi$ with $\|\pi\| = O(\delta_n^2 n / \ln(p))$ such that $(\pi - \tilde{\pi})^\alpha \Sigma(\pi - \tilde{\pi}) = O(\delta_n^2)$.

The $\delta_n$ is allowed to shrink to zero at any rate. For $\xi_1 > 1/2$ we will impose weak conditions on $\tilde{\alpha}(X)$ that do not require a sparse approximation rate for $\tilde{\alpha}(X)$ and allow for unbounded $\tilde{\alpha}(X)$. Unbounded $\tilde{\alpha}(X)$ is attractive in treatment effect settings, such as Examples 2 and 3, where conditional probabilities or densities will be allowed that require that the approximation errors from the least squares regression of $\rho_0(X)$ and $\tilde{\alpha}(X)$ respectively on the $p \times 1$ dictionary $b(X)$ go to zero faster certain rates. These approximation errors are different than those from using $s$ elements of $b$ with $s < n$. Consequently the condition here concerns the error from using all $p$ of the potential regressors $b(X)$ and $p$ is allowed to be much bigger than $n$. The sparse approximation rate is about the best rate of approximation just using $s$ elements of $b$ with $s < n$. Consequently the condition here is much weaker than requiring that the sparse approximation rate is faster than root-$n$. For example suppose that $\|\rho_0 - \rho_n\|_2 = o(n^{-c})$ for some $c > 0$. Then by choosing $p \geq n^{1/2c}$ we would have $\sqrt{n} ||\rho_0 - \rho_n||_2 = o(n^{-1/2}) \rightarrow 0$.

The next condition gives regularity condition under the rate double robustness condition $\xi_1 \xi_2 > 1/4$.

**ASSUMPTION 10:** i) $\xi_1 > 1/2$; ii) $E[Y|X] \in \mathcal{B}$ and $\xi_2$ replacing $\xi_1$; iii) $\tilde{\alpha}(X)$ is bounded, iv) there is $C > 0$ with $E|\rho(W)|^2 < \infty$ for all $\rho \in \mathcal{B}$; and v) $\sqrt{n} \rho_0 - \rho_n \rightarrow 0$.

Condition v) is the rate double robustness condition with $(r/\epsilon_n)\epsilon_n^{2\xi_1/(2\xi_1+1)} + \|\rho_0 - \rho_n\|_2 \rightarrow 0$ being the rate of mean square convergence of Lasso regression to $\rho_0(X) = E[Y|X]$.

Let

$$\psi_0(w) = \rho(w) - \theta_0 + \tilde{\alpha}(x)[y - \rho_0(x)].$$

**THEOREM 4:** If Assumptions 3-8 are satisfied and either Assumption 9 or Assumption 10 is satisfied then for $V = E[\psi_0(W)]^2$,

$$\sqrt{n} (\tilde{\alpha} - \tilde{\theta}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_0(W_i) + o_P(1) \rightarrow N(0, V).$$

If in addition $\epsilon_n \rightarrow \infty$ and $(r/\epsilon_n)\epsilon_n^{2\xi_1/(2\xi_1+1)} + \|\rho_0 - \rho_n\|_2 \rightarrow 0$ then $\hat{V} \rightarrow V$. 

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This result shows that the automatic debiased machine learning estimator given in Equation (4.2) is asymptotically Gaussian and the asymptotic variance estimator is consistent under either $\xi_1 > 1/2$ or $\xi_1 \xi_2 > 1/4$ and the other conditions we have imposed. Recall from Section 3 that the parametric rate is impossible whenever $\max\{\xi_1, \xi_2\} \leq 1/2$ for a regression slope. Hence, the only hope for the parametric rate is either $\xi_1 > 1/2$ or $\xi_2 > 1/2$; in other words, $\max\{\xi_1, \xi_2\} > 1/2$. By Theorem 4, the proposed estimator attains the parametric rate when $\xi_1 > 1/2$. In Theorem 8, we show that the parametric rate can be achieved for $\xi_2 > 1/2$ using another estimator. Therefore, the requirement of $\max\{\xi_1, \xi_2\} > 1/2$ is indeed sufficient and necessary for the parametric rate for a regression slope. Another important property of the above estimators is that they are doubly robust and have specification robust standard errors under rate double robustness $\xi_1 \xi_2 > 1/4$.

Theorem 4 does assume that $\rho_0(X) \in \mathcal{B}$, i.e., that the regression function is a linear combination of the dictionary $(b_1(X), b_2(X), \ldots)$. This condition is not restrictive when $(b_1(X), b_2(X), \ldots)$ can approximate any function of $X$ in mean square but otherwise is restrictive. We intend to explore in future research whether it is possible to attain root-n consistency when $\xi_1 > 1/2$ but $\rho_0(X)$ is not the conditional expectation.

The estimator $\hat{\theta}$ is doubly robust, meaning that the estimator of $\theta_0$ is consistent if either $\rho$ or $\alpha$ converge to a correct limit; see Equations (2.5) and (2.6) of Chernozhukov et al. (2022). For conciseness we omit here the demonstration of double robustness.

Theorem 4 applies to many interesting objects of interest including Examples 1-3. First, consider the estimator of a regression slope in Example 1.

**Corollary 5 (Example 1):** If Assumptions 3 and 5–8 are satisfied, the residual $U$ from the linear projection of $D$ on the mean square limits of finite linear combinations of $(Z_1, Z_2, \ldots)$ is nonzero and bounded, and either a) Assumption 9 i) and ii) are satisfied and $\sqrt{n}\|\rho_0 - \rho_n\|_2 \overset{d}{\rightarrow} 0$; or Assumption 10 i), ii), and v) are satisfied then $\sqrt{n}(\hat{\theta} - \theta_0) \overset{d}{\rightarrow} N(0,0)$. If in addition $\tilde{\tau}_n \rightarrow \infty$ and $\tilde{\tau}_n(r/e_n)^{2\xi_1/(2\xi_1+1)} + \|\rho_0 - \rho_n\|_2 \overset{d}{\rightarrow} 0$ then $\hat{\V} \overset{p}{\rightarrow} V$.

This result differs from previous debiased Lasso results of Javanmard and Montanari (2014, 2018), Van de Geer et al. (2014), Zhang and Zhang (2014), and Cai and Guo (2017) in several ways. The estimator of the bias correction is obtained from a Lasso like objective function, approximate sparsity is imposed here rather than strict sparsity, the regressors are not Gaussian, and standard errors are robust to heteroskedasticity. Also, the estimators here use cross-fitting that enables them to be root-n consistent under the double robust rate condition $\xi_1 > 1/2$ as well as under the minimal regression approximate sparsity condition $\xi_1 > 1/2$. Also, under $\xi_1 > 1/2$ no additional sparsity conditions are imposed, unlike Javanmard and Montanari (2014, 2018), Van de Geer et al. (2014), Zhang and Zhang (2014), where additional sparsity conditions are imposed on the projection of the variable of interest on the covariates or/and the inverse second moment matrix of the regression dictionary. A crucial difference is that our results do not depend on a quantity that is closely related to the $\ell_1$-norm of the rows of the precision matrix, i.e., the quantity $\rho$ in Javanmard and Montanari (2018). Moreover, in contrast to Remark 4.3 therein, the lower and upper bounds here match, thereby establishing the minimax rate.

Here is an asymptotic efficiency result for the weighted average derivative of Example 2:

**Corollary 6 (Example 2):** If Assumptions 3 and 5–8 are satisfied,

$$E[\{1 + S(D)^4\} \omega(D)^2 f(D)Z^{-1}] \leq \infty, \int S(u)^2 \omega'(u)du < \infty, S(D)^2 \omega(D)f(D)Z^{-1} \leq C,$$

and either a) Assumption 9 i) and ii) are satisfied, $\sqrt{n}\|\rho_0 - \rho_n\|_2 \overset{d}{\rightarrow} 0$, or b) Assumption 10 i), ii), iii), and v) are satisfied and $\hat{\alpha}(X)$ is bounded then $\sqrt{n}(\hat{\theta} - \theta_0) \overset{d}{\rightarrow} N(0,0)$. If in addition $\tilde{\tau}_n \rightarrow \infty$ and $\tilde{\tau}_n(r/e_n)^{2\xi_1/(2\xi_1+1)} + \|\rho_0 - \rho_n\|_2 \overset{d}{\rightarrow} 0$ then $\hat{\V} \overset{d}{\rightarrow} V$.

This result allows weaker conditions on approximate sparsity than in Chernozhukov et al. (2022) by allowing for $\xi_1 > 1/2$ as in Assumption 9 rather than requiring rate double robustness.

Here is an asymptotic efficiency result for the average treatment effect of Example 3.
COROLLARY 7 (Example 3): If Assumptions 3 and 5–8 are satisfied, $E|\pi_0(Z)^{-1}\{1 - \pi_0(Z)\}^{-1}| < \infty$, and either a) Assumption 9 i) and ii) are satisfied, $\sqrt{n}\|\theta_0 - \bar{\theta}_n\|_2 \to 0$, or $\rho_0(X)$ is bounded; or b) Assumption 10 i), ii), iii), and v) are satisfied, $\pi_0(Z)$ is bounded away from zero and one, and $\bar{\alpha}(X)$ is bounded, then $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0,V)$. If in addition $\bar{\tau}_n \to \infty$ and $\tau_n[(r/\tilde{e}_n)^{2\tilde{e}_n/4}] + \|\rho_0 - \bar{\rho}_n\|_2 \to 0$ then $\hat{\theta} \xrightarrow{p} \theta_0$.

This result differs from other results for the ATE in the estimator being asymptotically normal under either the rate double robustness condition $\xi_1 > 1/2$ or the minimal regression approximate sparsity condition $\xi_2 > 1/4$. We are not aware of any other ATE estimator that is root-n consistent under either of these conditions. Athey et al. (2018) root-n consistency result is for sparsity corresponding to $\xi_1 > 1/2$. Corollary 7 for $\xi_1 > 1/2$ was developed independently of a related result by Hirshberg and Wager (2021). Also, the condition $E|\pi_0(Z)^{-1}\{1 - \pi_0(Z)\}^{-1}| < \infty$ is an overlap condition that is required for root-n consistency. Imposing this condition rather than a stronger overlap condition may be of interest in applications where keeping overlap conditions as weak as possible may be warranted.

We also give root-n consistency and asymptotic normality for the estimator $\hat{\theta}$ of a regression coefficient from Equation (4.5) under the minimal condition $\xi_2 > 1/2$ for the Riesz representer that allows a dense regression. Let $\bar{D}$ denote the set of mean square limits of finite linear combinations of $\bar{Z} = (Z_1, Z_2, \ldots)$, $\eta_0(\bar{Z}) = \text{proj}([D]\bar{Z})$, and $\eta(\bar{Z})$ be the least squares projection of $D$ on $b(\bar{Z})$ with coefficients $\pi$.

THEOREM 8: If $\xi_2 > 1/2$, Assumption 3 is satisfied with $\xi_2$ and $\pi$ replacing $\xi_1$ and $\gamma$ respectively, Assumptions 5-7 are satisfied, $r = o(e_n\tilde{\alpha}_n^{1-1})$, $E[D|\bar{Z}] \in \bar{D}$, there is $C > 0$ with $E[D^2|\bar{Z}] + E[|Y^2||\bar{Z}] + \|\text{proj}(Y|\bar{Z})\| \leq C$, and $\sqrt{n}|\eta - \eta_0|_2 \to 0$ then $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0,V)$. If in addition $\bar{\tau}_n \to \infty$ and $\tau_n[(r/\tilde{e}_n)^{2\tilde{e}_n/(2\tilde{e}_n+1)} + \|\rho_0 - \bar{\rho}_n\|_2] \to 0$ then $\hat{\theta} \xrightarrow{p} \theta_0$.

This result differs from previous results on estimation with a dense regression by Zhu and Bradic (2018), Janvannard and Montanari (2018) and Bradic et al. (2022) in several ways. The estimator has a relatively simple form of a double/debiased Lasso machine learner as in Chernozhukov et al. (2018a), approximate sparsity is imposed here rather than strict sparsity, the regressors are not Gaussian, and the inference here is robust to heteroskedasticity. Also, the estimator here uses cross-fitting that enables it to be root-n consistent under the double robust rate condition $\xi_1 \xi_2 > 1/4$ as well as under the minimal approximate sparsity condition $\xi_2 > 1/2$ for the projection $D$ of $D$ on the covariates.

This result combined with that of Corollary 5 shows that it is possible to construct root-n consistent estimators of a coefficient of a high dimensional regression under either of the minimal approximate sparsity conditions $\xi_1 > 1/2$ or $\xi_2 > 1/2$. This is accomplished with a different estimator for the two cases. One estimator that attained root-n consistency under max$\{\xi_1, \xi_2\} > 1/2$ would be good. We leave the search for such an estimator to future work.

It would also be of interest to have estimators for linear functionals other than a regression coefficient that attain root-n consistency when Riesz representer is approximately sparse with $\xi_2 > 1/2$. Such is given for the ATE by Wang and Shah (2020) for a sparse logit propensity score. We leave to future work the search for such for other functionals.

A Proofs for Section 3

Lemma A.1. Let $k \in \mathbb{N}$ and define $\mathcal{D}_k = \{v \in \{0,1\}^p : \|v\|_0 = k\}$. Let $v$ and $u$ be two independent vectors that have a uniform distribution on $\mathcal{D}_k$. Then for any $D \geq 0$,

$$E\exp(Du'v) < \exp\left(\exp(D + \ln(k^2/p))\right).$$

Proof. Let $N = |\mathcal{D}_k|$. We list elements in $\mathcal{D}_k$, i.e., $\mathcal{D}_k = \{x_1, \ldots, x_N\}$. Then

$$E\exp(Du'v) = N^{-2} \sum \sum \exp(Dx'_i,x_j) \overset{(\Omega)}{=} N^{-1} \sum \exp(Dx'_i,x_j) = E\exp(Dx'_i,v),$$
(i) follows by the observation that \( \sum_{i=1}^{n} \exp(Dx_i^*v) \) does not depend on \( j_2 \). Without loss of generality, we take \( x_1 = (1, \ldots, 1, 0, \ldots, 0)^T \), i.e., the vector whose first \( k \) entries are nonzero.

Let \( \mathcal{C}_{n,k} \) be the population that consists of \( n \) elements with \( n - k \) elements being 0 and the remaining \( k \) being 1. Let \( \{\xi_i\}_{i=1}^{k} \) be a random sample without replacement from the population of \( \mathcal{C}_{n,k} \). We observe that \( x_i^*v \) has the same distribution as \( \sum_{i=1}^{k} \xi_i \). Then

\[
E \exp(Dx_i^*v) = E \exp \left( D \sum_{i=1}^{k} \xi_i \right).
\]

Let \( \{\xi_i\}_{i=1}^{k} \) be a random sample with replacement from \( \mathcal{C}_{n,k} \). In other words, \( \{\xi_i\}_{i=1}^{k} \) is i.i.d Bernoulli with \( E(\xi_i) = k/p \). Since \( x \mapsto \exp(Dx) \) is a convex function, we can use Theorem 4 of Hoeffding (1963) and obtain that

\[
E \exp \left( D \sum_{i=1}^{k} \xi_i \right) \leq E \exp \left( D \sum_{i=1}^{k} \xi_i \right) = \exp \left( \frac{k^2}{p} \right) \exp \left( D + \ln \left( \frac{k^2}{p} \right) \right)
\]

where (i) follows by the moment generating function of Bernoulli distributions, (ii) follows by the elementary inequality \( 1 + x \leq \exp(x) \) for \( x \geq 0 \). The proof is complete.

\[\square\]

**Lemma A.2.** Let \( k \) be a positive integer and define \( \mathcal{D}_k = \{v \in \{0, 1\}^p : \|v\|_0 = k\} \). Let \( N = |\mathcal{D}_k| \). We list elements in \( \mathcal{D}_k \), i.e., \( \mathcal{D}_k = \{\delta_1, \ldots, \delta_N\} \). Let

\[
\Sigma_j = \begin{pmatrix}
1 & 0 & q_1 \delta_j^1 \\
0 & 1 & q_2 \delta_j^2 \\
q_1 \delta_j & q_2 \delta_j & I_p
\end{pmatrix}
\]

and \( \Sigma_* = I_{p+2} \). Let \( P_j \) denote the distribution of \( N(0, \Sigma_j) \) and \( P_* \) denote the distribution of \( N(0, I_{p+2}) \). If \( n^{-1} k \ln(p/k^2) \leq 3 \) and \( 6n(q_1^2 + q_2^2) \leq \ln(p/k^2) \), then

\[
\text{TV} \left( N^{-1} \sum_{j=1}^{N} P_j, P_* \right) \leq \sqrt{\exp \left( \sqrt{k^2/p} \right)} - 1.
\]

**Proof.** Let \( E_* \) denotes the expectation under \( P_* \). By Lemma 3 in Cai and Guo (2017), we have that

\[
E_* \frac{dP_{j1}}{dP_*} \frac{dP_{j2}}{dP_*} = \left[ \det \left( I_{p+2} - (\Sigma_*^{-1} \Sigma_j - I_{p+2})(\Sigma_*^{-1} \Sigma_j - I_{p+2}) \right) \right]^{-n/2}.
\]

By the definitions of \( \Sigma_j \) and \( \Sigma_* \), we have that

\[
\Sigma_*^{-1} \Sigma_j - I_{p+2} = \begin{pmatrix}
0 & 0 & q_1 \delta_j^1 \\
0 & 0 & q_2 \delta_j^2 \\
q_1 \delta_j & q_2 \delta_j & 0
\end{pmatrix}.
\]

Then we have

\[
I_{p+2} - (\Sigma_*^{-1} \Sigma_j - I_{p+2})(\Sigma_*^{-1} \Sigma_j - I_{p+2})
\]

\[
= I_{p+2} - \begin{pmatrix}
0 & 0 & q_1 \delta_j^1 \\
0 & 0 & q_2 \delta_j^2 \\
q_1 \delta_j & q_2 \delta_j & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & q_1 \delta_j^1 \\
0 & 0 & q_2 \delta_j^2 \\
q_1 \delta_j & q_2 \delta_j & 0
\end{pmatrix}
\]

\[
= I_{p+2} - \begin{pmatrix}
q_1^2 \delta_j^1 \delta_j & q_1 q_2 \delta_j^1 \delta_j & 0 \\
q_1 q_2 \delta_j^1 \delta_j & q_2^2 \delta_j^2 \delta_j & 0 \\
0 & 0 & (q_1^2 + q_2^2) \delta_j \delta_j
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 - q_1^2 \delta_j^1 \delta_j & -q_1 q_2 \delta_j^1 \delta_j & 0 \\
-q_1 q_2 \delta_j^1 \delta_j & 1 - q_2^2 \delta_j^2 \delta_j & 0 \\
0 & 0 & I_p - (q_1^2 + q_2^2) \delta_j \delta_j
\end{pmatrix}.
\]
Therefore,

\[
\det \left( I_{p+2} - (\Sigma_{j=1}^{-1} \Sigma_j - I_{p+2}) (\Sigma_{j=1}^{-1} \Sigma_j - I_{p+2}) \right) = \det \left( I_p - (q_1^2 + q_2^2) \delta_j \delta_j' \right) \times \det \begin{pmatrix} 1 - q_1^2 \delta_j' \delta_j & -q_1 q_2 \delta_j' \delta_j \\ -q_1 q_2 \delta_j' \delta_j & 1 - q_2^2 \delta_j' \delta_j \end{pmatrix}^n \leq \exp \left( 3n(q_1^2 + q_2^2) \delta_j' \delta_j \right),
\]

where (i) follows by Sylvester’s determinant identity. By (A.1), we have that

\[
E_s \frac{dP_{j_1}}{dP_s} \frac{dP_{j_2}}{dP_s} = (1 - (q_1^2 + q_2^2) \delta_j' \delta_j)^{-n} \leq \exp \left( 3n(q_1^2 + q_2^2) \delta_j' \delta_j \right),
\]

where (i) follows by the definition of \((q_1^2 + q_2^2) \delta_j' \delta_j \leq (q_1^2 + q_2^2) \delta_j \delta_j \leq 1/2\) and the fact that \((1 - x)^{-n} < \exp(3nx)\) for any \(x \in [0, 1/2]\). (To see this, define \(f(x) = -3x \ln(1 - x)\). Notice that \(f(\cdot)\) is convex on \([0, 1/2]\) by checking \(f''(\cdot)\). Also notice that \(f(0) < 0\) and \(f(1/2) < 0\). Hence, \(f(x) < 0\) on \([0, 1/2]\). This means \(-\ln(1 - x) \leq 3x\). Multiplying both sides by \(n\) and taking exponential, we obtain \((1 - x)^{-n} \leq \exp(3nx)\).)

By (A.2), we have

\[
E_s \left( N^{-1} \sum_{j=1}^{N} \frac{dP_{j_1}}{dP_s} \frac{dP_{j_2}}{dP_s} - 1 \right)^2 \leq N^{-2} \sum_{j=1}^{N} \sum_{j' = 1}^{N} \exp \left( 3n(q_1^2 + q_2^2) \delta_j' \delta_j \right) - 1
\]

\[
\leq \exp \left( \exp \left( 3n(q_1^2 + q_2^2) \ln(k^2/p) \right) \right) - 1
\]

\[
\leq \exp \left( \left( 1/2 \right) \ln \left( k^2/p \right) \right) - 1 = \exp \left( \sqrt{k^2/p} \right) - 1,
\]

where (i) follows by Lemma A.1 and (ii) follows by \(q_1^2 + q_2^2 \leq (6n)^{-1} \ln(p/k^2)\). The desired result follows by Equation (2.27) of Tsybakov (2009).

**Lemma A.3.** Suppose that \(q \neq 0\) and \(\delta \in \{0, 1\}^p\) with \(\|\delta\|_0 = k\). Let \(\xi > 0\). If \(|q|k^{\xi + 1/2} \leq \sqrt{C}\), then \((k - t)q^2 \leq Ct^{-2\xi} \) for \(1 \leq t \leq k\).

**Proof.** We shall show that for any \(0 \leq t \leq k\), we have

\[
kt^{2\xi} - t^{2\xi + 1} \leq Cq^{-2}.
\]

The maximum of this mapping on \([0,k]\) is either at the end points or at an interior point. Taking the first-order condition of the mapping \(t \mapsto kt^{2\xi} - t^{2\xi + 1}\) and setting it zero, we have that \(t = \alpha k\) with \(\alpha = 2\xi / (2\xi + 1)\), which corresponds to a function value of \(k^{2\xi} + (\alpha^{2\xi} - \alpha^{2\xi + 1})\). Notice that the requirement of \(k^{2\xi + 1} (\alpha^{2\xi} - \alpha^{2\xi + 1}) \leq Cq^{-2}\) is satisfied by \(k^{2\xi + 1} \leq Cq^{-2}\) since \(\alpha \in (0, 1)\). Clearly, the value of the mapping \(t \mapsto kt^{2\xi} - t^{2\xi + 1}\) at the end points \((t = 0\) and \(t = k)\) is less than \(Cq^{-2}\). The desired result follows.

**Proof of Theorem 1.** Let

\[
k = \left\lfloor c_0 (n/\ln p)^{1/(2\xi + 1)} \right\rfloor,
\]

where \(c_0 > 0\) is a constant to be chosen. Let \(q_n = c_1 \sqrt{n^{-1} \ln p}\) and \(c_1 > 0\) is a constant to be chosen.

Define \(\mathcal{D}_k = \{v \in \{0, 1\}^p : \|v\|_0 = k\}\). Let \(N = |\mathcal{D}_k|\). Clearly, \(N = \binom{n}{k}\). We list elements in \(\mathcal{D}_k\), i.e., \(\mathcal{D}_k = \{\delta_1, \ldots, \delta_N\}\). For \(1 \leq j \leq N\), define \(\gamma_j = -2q_\delta \delta_j\) and \(\pi_j = q_\delta \delta_j\). Clearly, we can choose \(c_0, c_1\) such that

\[
2q_\delta k^{\xi + 1/2} \leq 2c_1 c_0^{\xi + 1/2} \leq C_0 \quad \text{and} \quad \exp(c_0 k\xi) - 1 \leq c_2^2.
\]
By Lemma A.3, this implies that $-2q_n \delta_j, q_n \delta_j \in \mathcal{M}_{c_1 \xi}$. We define $\lambda_n = (0, 0, 0, 1, 1)$ and for $1 \leq j \leq N$, $\lambda_j = (2q_n^2 k, -2q_n \delta_j, q_n \delta_j, 1 - q_n^2 k, \sigma_{E,1}^2)$ with $\sigma_{E,1}^2 = 1 - 4q_n^2 k - 4q_n^4 k^2 (1 - q_n^2 k)$. Notice that
\[
q_n^2 k \leq c_1^2 c_0 (n/\ln p)^{-1+1/(2\xi+1)} = c_1^2 c_0 (n/\ln p)^{-2\xi/(2\xi+1)} \leq c_1^2 c_0 \kappa_1^{-2\xi/(2\xi+1)}.
\]

Then we can choose $c_1, c_0$ small enough such that $1 - q_n^2 k, \sigma_{E,1}^2 \in [1/2, 2] \subseteq [M^{-1}, M]$ (due to $M \geq 2$). Hence, $\lambda_n, \lambda_j \in \Lambda_{\xi, \xi_2}$.

Then under $P_{\lambda_n}$, the distribution of $(Y_i, Z_i, X_i)$ is $N(0, I_{p+2})$. Under $P_{\lambda_j}$, $Y_i = Z_i \cdot (2q_n^2 k - 2q_n \delta_j + \epsilon_i$ and $Z_i = q_n X_i \delta_j + u_i$, where $EE_i^2 = \sigma_{E,1}^2$ and $E\epsilon_i^2 = 1 - q_n^2 k$. After simple calculations, we have that under $P_{\lambda_j}$, the distribution of $(Y_i, Z_i, X_i)$ is $N(0, \Sigma_j)$ with
\[
\Sigma_j = \begin{pmatrix}
1 & 0 & -q_n \delta_j' \\
0 & 1 & q_n \delta_j' \\
-q_n \delta_j & q_n \delta_j & I_p
\end{pmatrix}.
\]

Clearly, we can choose $c_0, c_1$ small enough such that
\[
n^{-1} k \ln (p/k^2) \leq n^{-1} k \ln p \leq c_0 n^{-1} (n/\ln p)^{1/(2\xi+1)} \ln p = c_0 (n/\ln p)^{-2\xi/(2\xi+1)} \leq c_0 \kappa_1^{-2\xi/(2\xi+1)} \leq 3.
\]

and
\[
\frac{6n(q_n^2 + q_n^4)}{\ln (p/k^2)} \leq \frac{6n(q_n^2 + q_n^4)}{\ln (p)} = 12c_1^2 \leq 1.
\]

Hence, the assumptions of Lemma A.2 hold. Therefore,
\[
TV\left( N^{-1} \sum_{j=1}^N P_{\lambda_j}, P_{\lambda_n} \right) \leq \sqrt{\exp \left( \sqrt{k^2/p} \right)} - 1. \tag{A.4}
\]

Let $CL_b = [l_n, u_n]$ be an arbitrary confidence interval for $\beta = \phi(\lambda)$ with nominal coverage probability $1 - \alpha$ on $\Lambda_{\xi, \xi_2}$. In other words,
\[
\inf_{\lambda \in \Lambda_{\xi, \xi_2}} P_{\lambda} (l_n \leq \phi(\lambda) \leq u_n) = \inf_{\lambda \in \Lambda_{\xi, \xi_2}} P_{\lambda} (\phi(\lambda) \in CL_b) \geq 1 - \alpha. \tag{A.5}
\]

We now define the random variable
\[
\psi_n = 1 \{ 2q_n^2 k \in CL_b \}.
\]

Thus, by (A.4), $n \leq \kappa_2 p \ln p + \exp(c_1^2 \kappa_2) - 1 \leq \alpha^2$, we have
\[
\left| N^{-1} \sum_{j=1}^N E_{\lambda_j} \psi_n - E_{\lambda_n} \psi_n \right| \leq TV\left( N^{-1} \sum_{j=1}^N P_{\lambda_j}, P_{\lambda_n} \right) \leq \sqrt{\exp \left( \sqrt{k^2/p} \right)} - 1
\leq \sqrt{\exp \left( \frac{c_1^2 n}{p \ln p} \right)} - 1 \leq \sqrt{\exp(c_1^2 \kappa_2)} - 1 \leq \alpha.
\]

Therefore,
\[
P_{\lambda_n} (2q_n^2 k \in CL_b) = E_{\lambda_n} \psi_n \geq N^{-1} \sum_{j=1}^N E_{\lambda_j} \psi_n - \alpha = N^{-1} \sum_{j=1}^N P_{\lambda_j} (2q_n^2 k \in CL_b) - \alpha
\]
\[
\overset{(i)}{=} N^{-1} \sum_{j=1}^N P_{\lambda_j} (\phi(\lambda_j) \in CL_b) - \alpha \overset{(ii)}{=} N^{-1} \sum_{j=1}^N P_{\lambda_j} (\phi(\lambda_j) \in CL_b) - \alpha \geq 1 - 2\alpha,
\]

where (i) follows by $\phi(\lambda_j) = 2q_n^2 k$ and (ii) follows by (A.5). On the other hand, by $\phi(\lambda_n) = 0$ and (A.5), we have
\[
P_{\lambda_n} (0 \in CL_b) \geq 1 - \alpha.
\]
The above two displays imply that $P_{\alpha} \left( \{ 0, 2q_n^2 k \} \subset CI_n \right) \geq 1 - 3\alpha$. Since $CI_n$ is an interval, the event of $\{ 0, 2q_n^2 k \} \subset CI_n$ is the same as the event $\{ 0, 2q_n^2 k \} \subset CI_n$. Hence, we have $P_{\alpha} \left( \{ 0, 2q_n^2 k \} \subset CI_n \right) \geq 1 - 3\alpha$, which means

$$\frac{E_{\alpha} | CI_n|}{2q_n^2 k} \geq P_{\alpha} \left( | CI_n| \geq 2q_n^2 k \right) \geq 1 - 3\alpha.$$  

Since $2q_n^2 k \approx (n/\ln)^{-2/((2\xi_2 + 1))}$ (by the definitions of $q_n$ and $k$), the desired result follows.

We next give several Lemmas that are useful in the proof of Theorem 2.

**Lemma A.4.** Let $C_0, C_1, C_2, \xi_1, \xi_2 > 0$ be constants. Define $k = \left[ C_0 n^{1/(\xi_1 + \xi_2 + 1/2)} \right]$, $c_1 = C_1 k^{-\xi_1-1/2}$ and $c_2 = C_2 k^{-\xi_2-1/2}$, where $\lfloor \cdot \rfloor$ denotes the integer part. Let $\lambda \sim N(0, I_k)$. There exist a constant $D > 0$ depending only on $(\xi_1, \xi_2, M_1)$ such that for any $C_1, C_2 \in (0, D)$, we have $P(c_1 \lambda \in \mathcal{N}_{M_1, \xi_1}) = 1 - o(1)$ and $P(c_2 \lambda \in \mathcal{N}_{M_1, \xi_2}) = 1 - o(1)$.

**Proof.** We now show $P(c_1 \lambda \in \mathcal{N}_{M_1, \xi_1}) = 1 - o(1)$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)' \in \mathbb{R}^k$ follow $N(0, I_k)$. The goal is to show that the following event occurs with probability $1 - o(1)$,

$$\bigcap_{t=1}^{k-1} \left\{ c_1 \sum_{j=t+1}^{k} \lambda_j^2 \leq M_1 t^{-\xi_1} \right\}.$$  

We can rewrite this event as

$$\bigcap_{t=1}^{k-1} \left\{ \sum_{j=t+1}^{k} (\lambda_j^2 - 1) \leq \left( c_1^{-2} M_1^2 t^{-2\xi_1} - (k-t) \right) \right\}.$$  

Notice that $E(\lambda_j^2 - 1)^2 = 3$. By Kolmogorov’s maximal inequality, we have that for $x > 0$,

$$P \left( \max_{1 \leq t \leq k-1} \left| \sum_{j=t+1}^{k} (\lambda_j^2 - 1) \right| > x \right) \leq \frac{3k}{x^2}.$$  

Hence,

$$P \left( \max_{1 \leq t \leq k-1} \left| \sum_{j=t+1}^{k} (\lambda_j^2 - 1) \right| > \sqrt{k \ln k} \right) = o(1).$$  

Therefore, we only need to show

$$\min_{1 \leq j \leq k} c_1^{-2} M_1^2 t^{-2\xi_1} - (k-t) \geq \sqrt{k \ln k}.$$  

Recall $c_1 = C_1 k^{-\xi_1-1/2}$ for a constant $C_1 > 0$. We need to show

$$\min_{1 \leq j \leq k} M_1^2 C_1^{-2} k^{2\xi_1+1} t^{-2\xi_1} + t \geq k + \sqrt{k \ln k}.$$  

Notice that the left-hand side is the minimum of a convex function of $t$. Therefore, the minimum occurs at $t = t_*$ with $t_*$ being the solution of the first-order condition

$$-M_1^2 C_1^{-2} k^{2\xi_1+1} t_*^{-2\xi_1-1} / (2\xi_1) + 1 = 0.$$  

This means $t_* = (2\xi_1 M_1^{-2} C_1^{-2})^{-1/(2\xi_1+1)}$, which means

$$\min_{t \in \mathbb{R}} M_1^2 C_1^{-2} k^{2\xi_1+1} t^{-2\xi_1} + t = \left[ (2\xi_1 M_1^{-2})^{-1/(2\xi_1+1)} + M_1^2 (2\xi_1 C_1^{-2})^{2\xi_1/(2\xi_1+1)} \right] k \times C_1^{-2/(2\xi_1+1)}.$$  

Hence, the right-hand side is larger than $k + \sqrt{k \ln k}$ for small enough $C_1$. Therefore, we have proved that for $c_1 = C_1 k^{-\xi_1-1/2}$ with a small constant $C_1 > 0$, $P(c_1 \lambda \in \mathcal{N}_{M_1, \xi_1}) = 1 - o(1)$. The result for $P(c_2 \lambda \in \mathcal{N}_{M_1, \xi_2}) = 1 - o(1)$ is analogous.  

\[ \square \]
Lemma A.5. Let $W \sim N(0, I_k)$. Suppose that $A \in \mathbb{R}^{k \times k}$ is positive semi-definite and $b \in \mathbb{R}^k$. Then
\[
E \exp \left( -\frac{1}{2} W' AW + b' W \right) = (\det(A + I_k))^{-1/2} \exp \left( b'(A + I_k)^{-1} b \right).
\]

Proof. Let $A = U B U'$, where $B$ is diagonal with $B_{jj} \geq 0$ and $U' U = I_k$. This is possible because $A$ is positive semi-definite. Define $T = U' W$ and $c = U' b$. Clearly, $T$ is Gaussian with mean zero and variance $U' I_k U = I_k$.

Then we need to compute
\[
E \exp \left( -\frac{1}{2} T' B T + c' T \right) = E \exp \left( \sum_{j=1}^k \left( -T_j^2 B_{jj} / 2 + c_j T_j \right) \right).
\]

Since $T_j \sim N(0, 1)$ is independent across $j$, it follows that
\[
E \exp \left( -\frac{1}{2} T' B T + c' T \right) = E \exp \left( \sum_{j=1}^k \left( -T_j^2 B_{jj} / 2 + c_j T_j \right) \right)
= \prod_{j=1}^k E \exp \left( -T_j^2 B_{jj} / 2 + c_j T_j \right)
= \prod_{j=1}^k \left[ \int_{-\infty}^{\infty} \exp \left( -x^2 B_{jj} / 2 + c_j x \right) \cdot \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx \right]
= \prod_{j=1}^k \left[ \frac{1}{\sqrt{B_{jj} + 1}} \exp \left( \frac{c_j^2}{2(B_{jj} + 1)} \right) \right]
= \left[ \prod_{j=1}^k (B_{jj} + 1) \right]^{-1/2} \exp \left( \frac{1}{2} \sum_{j=1}^k \frac{c_j^2}{B_{jj} + 1} \right).
\]

Since $U' U = I_k$, we have
\[
\prod_{j=1}^k (B_{jj} + 1) = \det(B + I_k) = \det(U B U' + I_k) = \det(A + I_k).
\]

Moreover, since $B$ is diagonal, we have
\[
\sum_{j=1}^k \frac{c_j^2}{B_{jj} + 1} = c' (B + I_k)^{-1} c = (U' b)' \left( (U' A U + I_k)^{-1} (U' b) \right) = b'(A + I_k)^{-1} b.
\]

The above three displays imply
\[
E \exp \left( -\frac{1}{2} T' B T + c' T \right) = (\det(A + I_k))^{-1/2} \exp \left( b'(A + I_k)^{-1} b / 2 \right).
\]

The proof is complete.

Lemma A.6. Let $C_0, C_1, C_2, \xi_1, \xi_2 > 0$ be constants. Define $k = \left\lfloor C_0 \eta^{1/(\xi_1 + \xi_2 + 1/2)} \right\rfloor$, $c_1 = C_1 k^{\xi_1 - 1/2}$ and $c_2 = C_2 k^{\xi_2 - 1/2}$, where $\lfloor \cdot \rfloor$ denotes the integer part. For any $\lambda \in \mathbb{R}^k$, let $\Phi_{1, \lambda}$ denote the distribution of i.i.d. $\{(Y_i, Z_i, X_i)\}_{i=1}^n$ given by $X_i \sim N(0, I_k)$,
\[
\begin{pmatrix} Y_i \\ Z_i \\ X_i \end{pmatrix} \sim N \left( \begin{pmatrix} c_1 X_i' \lambda \\ 0 \\ 0 \end{pmatrix}, \Sigma \right) \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & c_1 c_2 k \\ c_1 c_2 k & 1 \end{pmatrix}.
\]

Let $\Phi_{2, \lambda}$ denote the distribution of i.i.d. $\{(Y_i, Z_i, X_i)\}_{i=1}^n$ given by $X_i \sim N(0, I_k)$,
\[
\begin{pmatrix} Y_i \\ Z_i \end{pmatrix} \sim N \left( \begin{pmatrix} c_1 X_i' \lambda \\ c_2 X_i' \lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).
\]

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Assume that $\xi_2 \geq 1/4 > \xi_1$ and $\xi_1 + \xi_2 < 1/2$. Let $X_i$ be i.i.d from $N(0, I_k)$. Then

$$TV(\tilde{\phi}_1, \tilde{\phi}_2) \leq 2C_1C_2c_0^{-2(\xi_1 + \xi_2 + 1/2)} + o(1),$$

where $TV$ denotes the total variation distance, $\tilde{\phi}_1 = f_{\tilde{\phi}_1, \lambda} \mu(\lambda) d\lambda$, $\tilde{\phi}_2 = f_{\tilde{\phi}_2, \lambda} \mu(\lambda) d\lambda$ and $\mu$ is the density function of $N(0, I_k)$, i.e., $\mu(\lambda) = (2\pi)^{-k/2} \exp(-\parallel \lambda \parallel^2_2/2)$.

**Proof.** For simplicity, we shall use $\parallel \cdot \parallel$ to denote the determinant of a square matrix. Since $\xi_1 + \xi_2 < 1/2$, we have $k \gg n$.

For $j \in \{1, 2\}$, let $f_{j, \lambda}$ denote the conditional distribution of $(Y, Z)$ given $X$ under $\phi_{j, \lambda}$ and define $\tilde{f}_j = \int_{\mathbb{R}^k} f_{j, \lambda} \mu(\lambda) d\lambda$. Let KL denote the Kullback-Leibler divergence. We will proceed in four steps. In the first three steps, we bound $\text{KL}(\tilde{f}_1, \tilde{f}_2)$; in the last step, we prove the final result.

**Step 1:** Deriving the mixture $\tilde{f}_1$

Notice that

$$R = \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} c_1X \\ 0 \end{pmatrix} \lambda + \begin{pmatrix} \varepsilon \\ u \end{pmatrix} = G \lambda + \begin{pmatrix} \varepsilon \\ u \end{pmatrix},$$

where $G = \begin{pmatrix} c_1X \\ 0 \end{pmatrix} = g \otimes X$ with $g = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ and

$$E\begin{pmatrix} \varepsilon \\ u \end{pmatrix} E'\begin{pmatrix} \varepsilon \\ u \end{pmatrix}' = \begin{pmatrix} \Sigma I_n & \Sigma_3 I_n \\ \Sigma_3 I_n & \Sigma_2 I_n \end{pmatrix} = \Sigma \otimes I_n$$

with

$$\Sigma = \begin{pmatrix} 1 & c_1 c_2 k \\ c_1 c_2 k & 1 \end{pmatrix}.$$ 

Notice that

$$\Sigma^{-1} = D = \begin{pmatrix} D_1 & D_1 \\ D_3 & D_2 \end{pmatrix} = \begin{pmatrix} 1 & -c_1 c_2 k \\ -c_1 c_2 k & 1 \end{pmatrix} \frac{1}{1 - c_1^2 c_2^2 k^2}$$

and $|\Sigma| = 1 - c_1^2 c_2^2 k^2$.

Thus, under $f_{1, \lambda}$, $R$ is Gaussian mean $G \lambda$ and variance $\Sigma \otimes I_n$. Hence, the density is

$$f_{1, \lambda} = f_{1, \lambda}(R) = (2\pi)^{-n} |\Sigma \otimes I_n|^{-1/2} \exp\left(-\frac{1}{2} (R - G \lambda)'(\Sigma \otimes I_n)^{-1}(R - G \lambda)\right)$$

$$= (2\pi)^{-n} |D \otimes I_n|^{1/2} \exp\left(-\frac{1}{2} (R - G \lambda)'(D \otimes I_n)(R - G \lambda)\right)$$

$$= (2\pi)^{-n} |D \otimes I_n|^{1/2} \exp\left(-\frac{1}{2} R'(D \otimes I_n)R - \frac{1}{2} \lambda' G'(D \otimes I_n)G \lambda + R'(D \otimes I_n)G \lambda\right)$$

Hence, we can apply Lemma A.5 and obtain

$$\tilde{f}_1 = \int f_{1, \lambda} \mu(\lambda) d\lambda$$

$$= E_{\lambda \sim N(0, I_k)} \tilde{f}_{1, \lambda}$$

$$= (2\pi)^{-n} |D \otimes I_n|^{1/2} \exp\left(-\frac{1}{2} R'(D \otimes I_n)R\right) \times E_{\lambda \sim N(0, I_k)} \exp\left(-\frac{1}{2} \lambda' G'(D \otimes I_n)G \lambda + R'(D \otimes I_n)G \lambda\right)$$

$$= (2\pi)^{-n} |D \otimes I_n|^{1/2} \exp\left(-\frac{1}{2} R'(D \otimes I_n)R\right) \times |I + G'(D \otimes I_n)G|^{-1/2} \times$$

$$\exp\left(\frac{1}{2} R'(D \otimes I_n)G(I + G'(D \otimes I_n)G)^{-1} G'(D \otimes I_n)R\right)$$

$$= (2\pi)^{-n} |D \otimes I_n|^{1/2} \cdot |I + G'(D \otimes I_n)G|^{-1/2} \exp\left(-\frac{1}{2} R'MR\right),$$
where

\[ M = D \otimes I_n - (D \otimes I_n)G (I + G'(D \otimes I_n))^{-1} G'(D \otimes I_n) \]
\[ = D \otimes I_n - (Dg \otimes X) (I + g'Dg'X')^{-1} (g'D \otimes X'). \]

We notice that

\[
|M| = \left| D \otimes I_n - (D \otimes I_n)G (I + G'(D \otimes I_n))^{-1} G'(D \otimes I_n) \right|
\]
\[ \overset{(i)}{=} \left| D \otimes I_n \right| \left| I - G (I + G'(D \otimes I_n))^{-1} G'(D \otimes I_n) \right| \]
\[ \overset{(ii)}{=} \left| D \otimes I_n \right| \left| I - (I + G'(D \otimes I_n))^{-1} G'(D \otimes I_n) \right| \]
\[ = \left| D \otimes I_n \right| \left| I - G'(D \otimes I_n) \right| \]
\[ = \lambda_1 \pi_1 \]
\[ \lambda_2 \pi_2 \]
\[
\Rightarrow \quad \exp \left( \frac{1}{2} (R - \tilde{G} \bar{\lambda})' \left( R - \tilde{G} \bar{\lambda} \right) \right) \]
\[ = \exp \left( -\frac{1}{2} R'R - \frac{1}{2} \lambda' \tilde{G} \bar{\lambda} + R' \tilde{G} \bar{\lambda} \right). \]

By a similar calculation as in step 1, we have

\[
\tilde{f}_2 = \int f_{2,\lambda} \mu(\lambda) d\lambda
\]
\[ = E_{\tilde{G}_{\lambda} \sim N(0, \tilde{I}_n), \tilde{M}_{\lambda} \sim N(0, \tilde{I}_n)} \exp \left( -\frac{1}{2} \lambda' \tilde{G} \bar{\lambda} + R' \tilde{G} \bar{\lambda} \right)\]
\[ = \exp \left( -\frac{1}{2} \lambda' \tilde{G} \bar{\lambda} \right) \exp \left( -\frac{1}{2} \lambda' \tilde{G} \bar{\lambda} \right) \exp \left( \frac{1}{2} R'R \right) \]
\[ = \exp \left( -\frac{1}{2} \lambda' \tilde{G} \bar{\lambda} \right) \exp \left( \frac{1}{2} R'R \right) \]
\[ \lambda_{\tilde{M}} = \tilde{I}_n - G (I + G'G)^{-1} G' = I - (\tilde{g} \otimes X) (I + \|\tilde{g}\| X'X)^{-1} (\tilde{g} \otimes X'). \]

**Step 3**: Compute KL divergence between the two mixtures

Hence, \( \tilde{f}_1 \) and \( \tilde{f}_2 \) represents \( N(0, \Sigma_1) \) and \( N(0, \Sigma_2) \), respectively, where \( \Sigma_1 = M^{-1} \) and \( \Sigma_2 = \tilde{M}^{-1} \). Now we consider the KL divergence

\[
\text{KL}(\tilde{f}_1, \tilde{f}_2) = \frac{1}{2} \left( \ln \frac{\Sigma_2}{\Sigma_1} - 2n + \text{trace}(\Sigma_2^{-1} \Sigma_1) \right) = \frac{1}{2} \left( \ln \frac{|M|}{|\tilde{M}|} - 2n + \text{trace}(M^{-1}\tilde{M}) \right) .
\]
Using the Woodbury’s identity, we have

\[ M^{-1} = \Sigma \otimes I_n + (gg') \otimes (XX'). \]

Therefore,

\[
\text{trace}(M^{-1}\tilde{M}) = \text{trace} \left[ (\Sigma \otimes I_n + (gg') \otimes (XX')) \left( I - (\tilde{g} \otimes X) (I + ||\tilde{\Sigma}||_2^2 X'X)^{-1} (\tilde{g}' \otimes X') \right) \right].
\]

By Sylvester’s theorem, we have

\[ \text{trace}(\Sigma) + ||g||^2 \text{trace}(X'X) = \text{trace}(g \Sigma g' X'X) (I + ||\tilde{\Sigma}||_2^2 X'X)^{-1}. \]

Let \( \sigma_1, \ldots, \sigma_n \) be the eigenvalues of \( X'X \). By Corollary 5.35 of Vershynin (2012) (with \( t = \sqrt{n} \)), we have that

\[ P \left( \max_{1 \leq i \leq n} |\sqrt{\sigma_i} - \sqrt{k}| > 2\sqrt{n} \right) \leq 2 \exp(-n/2). \]

Thus, with probability at least \( 1 - 2 \exp(-n/2) \), \( \sqrt{\sigma_i} + \sqrt{k} \leq 2\sqrt{k} + 2\sqrt{n} \) for all \( i \). By \( k \gg n \), it follows that \( P(\omega') \geq 1 - o(1) \), where

\[ \omega' = \left\{ \max_{1 \leq i \leq n} |\sigma_i - k| \leq \min\{k, 4\sqrt{n}k\} \right\}. \]

Now we notice

\[
\ln \frac{|M|}{|\tilde{M}|} = n \ln |D| + n \ln \left| \frac{I + (c_1^2 + c_2^2)X'X}{I + c_1^2 D_1 X'X} \right|
\]

\[
= n \ln |D| + \sum_{i=1}^n \ln \left( 1 + \frac{(c_1^2 + c_2^2)\sigma_i}{1 + c_1^2 D_1 \sigma_i} \right)
\]

\[
\leq n \ln |D| + \sum_{i=1}^n \frac{(c_1^2(1-D_1) + c_2^2)\sigma_i}{1 + c_1^2 D_1 \sigma_i}
\]

\[
(i) = n \ln \left( \frac{1}{1 - c_1^2 c_2^2 k} \right) + \sum_{i=1}^n \frac{(c_1^2(1-D_1) + c_2^2)\sigma_i}{1 + c_1^2 D_1 \sigma_i}
\]

\[
(ii) \leq n c_1^2 c_2^2 k^2 + n c_1^4 c_2^4 k^4 + \sum_{i=1}^n \frac{(c_1^2(1-D_1) + c_2^2)\sigma_i}{1 + c_1^2 D_1 \sigma_i}.
\]
where (i) follows by \( \ln(1 + x) \leq x \) for \( x \geq -1 \) and (ii) follows by \( |D| = (1 - c_1^2 c_2^2 k^2)^{-1} \) and (iii) follows by the elementary inequality \( \ln(\frac{1}{1-x}) \leq x + x^2 \) for \( x \in [0, 1/2] \) and \( c_1^2 c_2^2 k^2 \approx k^{-2q_1}2^{-q_2} = O(1) \). Observe that on the event \( \mathcal{A} \),

\[ 2 \cdot \text{KL}(\mathbf{f}_1, \mathbf{f}_2) \]

\[
= \ln \left( \frac{M}{\bar{M}} \right) - 2n + \text{trace}(M^{-1} \bar{M})
\]

\[
\leq n c_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right) - 2n + n \text{trace}(\Sigma)
\]

\[
+ \|g\|^2 \text{trace}(X^\top X) - \text{trace} \left[ \left( g' \Sigma gX'X + c_1^4 XX'X \right) (I + \|g\|^2 X'X)^{-1} \right]
\]

\[
= nc_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right)
\]

\[+ c_1^2 \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \left( \frac{c_1^2 + c_2^2 + 2c_1^2 c_2^2 \sigma_i + c_1^4 \sigma_i^2}{1 + (c_1^2 + c_2^2) \sigma_i} \right)
\]

\[
= nc_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right)
\]

\[+ c_1^2 \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \left( \frac{c_1^2 + c_2^2 + 2c_1^2 c_2^2 \sigma_i + c_1^4 \sigma_i^2}{1 + (c_1^2 + c_2^2) \sigma_i} \right)
\]

\[
= nc_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right)
\]

\[
+ c_1^2 \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \left( \frac{c_1^2 + c_2^2 + 2c_1^2 c_2^2 \sigma_i + c_1^4 \sigma_i^2}{1 + (c_1^2 + c_2^2) \sigma_i} \right)
\]

\[
= nc_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right)
\]

\[
+ c_1^2 \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \left( \frac{c_1^2 + c_2^2 + 2c_1^2 c_2^2 \sigma_i + c_1^4 \sigma_i^2}{1 + (c_1^2 + c_2^2) \sigma_i} \right)
\]

\[
= nc_1^2 c_2^2 k^2 + nc_1^4 c_2^4 k^4 + \sum_{i=1}^{n} \left( \frac{(c_1^2(1 - D_i) + c_2^2)}{1 + c_1^2 D_i \sigma_i} \right)
\]

\[+ c_1^2 \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} \left( \frac{c_1^2 + c_2^2 + 2c_1^2 c_2^2 \sigma_i + c_1^4 \sigma_i^2}{1 + (c_1^2 + c_2^2) \sigma_i} \right)
\]
Thus, we have proved that on the event $A$

Lemma A.7.

By first Pinsker’s inequality (Lemma 2.5 of Tsybakov (2009)), we have that on the event $\mathcal{A}$,

Step 4: derive the final result.

By first Pinsker’s inequality (Lemma 2.5 of Tsybakov (2009)), we have that on the event $\mathcal{A}$,

$$TV(\bar{f}_1, \bar{f}_2) \leq \sqrt{KL(\bar{f}_1, \bar{f}_2)/2} \leq 2C_1C_2C_0^{-(\xi_1+\xi_2+1/2)} + o(1).$$

(A.6)

By Schæffer’s theorem (Lemma 2.1 of Tsybakov (2009)),

$$TV(\bar{f}_1, \bar{f}_2) = \frac{1}{2} \int |\bar{f}_1(r,X) - \bar{f}_2(r,X)|dr.$$  

(A.7)

Since $X_i$ is i.i.d $N(0, I_k)$, we have that for $j \in \{1, 2\}$, $\phi_{j,k}(r,x) = f_{j,k}(r,x) \mu_n(x)$ with $r = (y,z)$, where $\mu_n(x) = \prod_{i=1}^n \mu(x_i)$ with $x = (x_1, \ldots, x_n)$. This means that

$$\phi_j(r,x) = \int_{R_k} \phi_{j,k}(r,x) \mu(\lambda) d\lambda = \int_{R_k} f_{j,k}(r,x) \mu_n(x) \mu(\lambda) d\lambda = \bar{f}_j(r,x) \mu_n(x).$$

In the following argument, $E$ denotes expectation with respect to the randomness in $X$ given by $X_i \sim i.i.d. N(0, I_k)$.

Then

$$TV(\bar{\phi}_1, \bar{\phi}_2) = \frac{1}{2} \int \mu_n(x) |\bar{f}_1(r,x) - \bar{f}_2(r,x)|drdx$$

$$= \frac{1}{2} E \int |\bar{f}_1(r,X) - \bar{f}_2(r,X)|dr$$

$$= \frac{1}{2} E \int |\bar{f}_1(r,X) - \bar{f}_2(r,X)|dr \cdot 1\{X \in \mathcal{A}\} + \frac{1}{2} E \int |\bar{f}_1(r,X) - \bar{f}_2(r,X)|dr \cdot 1\{X \notin \mathcal{A}\}$$

(i)

$$\leq 2C_1C_2C_0^{-(\xi_1+\xi_2+1/2)} + o(1) + \frac{1}{2} E \int |\bar{f}_1(r,X) - \bar{f}_2(r,X)|dr \cdot 1\{X \notin \mathcal{A}\}$$

(ii)

$$\leq 2C_1C_2C_0^{-(\xi_1+\xi_2+1/2)} + o(1) + P(X \notin \mathcal{A}),$$

where (i) follows by (A.6) and (A.7) and (ii) follows by the fact that $\int \bar{f}_j(r,X)dr = 1$ almost surely. Since $P(\mathcal{A}) = 1 - o(1)$, we have $TV(\bar{\phi}_1, \bar{\phi}_2) \leq 2C_1C_2C_0^{-(\xi_1+\xi_2+1/2)} + o(1)$.

Lemma A.7. Assume that $p \geq n^2$. If $\xi_1 + \xi_2 < 1/2$ and $\xi_2 \geq 1/4 > \xi_1$, then $\mathcal{P}(\xi_1, \xi_2) \gtrsim n^{-(\xi_1+\xi_2)/(\xi_1+\xi_2+1/2)}$. 


Proof. Define $k = \left\lfloor c_0 n^{1/(\xi_1 + \xi_2 + 1/2)} \right\rfloor$, $c_1 = c_1 k^{-\xi_1 - 1/2}$ and $c_2 = c_2 k^{-\xi_2 - 1/2}$, where $C_0, C_1, C_2 > 0$ are constants to be determined. Since $\xi_1 + \xi_2 < 1/2$, we have $k \gg n$. Since $p \gtrsim n^2$ and $k \asymp n^{1/(\xi_1 + \xi_2 + 1/2)} \ll n^2$, we have $p \gg k$.

In the following analysis, we set $\gamma_j = \pi_j = 0$ for $j > k$. This effectively reduces the dimension from $p$ to $k$ in our analysis.

For any $\lambda \in \mathbb{R}^k$, let $\theta_{\lambda} = (c_1 c_2 k, c_1 \lambda, 0, 1 - c_1^2 c_2^2 k^2, 1)$ and $\tilde{\theta}_{\lambda} = (0, c_1 \lambda, c_2 \lambda, 1, 1)$. Notice that $1 - c_1^2 c_2^2 k^2$ is well defined as the value of a variance (i.e., $1 - c_1^2 c_2^2 k^2 > 0$) if we choose $C_0, C_1, C_2 > 0$ small enough; to see this notice that $c_1 c_2 k \gg k^{-\xi_1 - \xi_2} = o(1)$.

We consider two probability measures: $P_1 = \int P_{\theta_{\lambda}} \mu(\lambda) d\lambda$ and $P_2 = \int P_{\tilde{\theta}_{\lambda}} \mu(\lambda) d\lambda$, where $\mu$ is the density of $N(0, I_k)$, i.e., $\mu(\lambda) = (2\pi)^{-k/2} \exp(-\|\lambda\|^2/2)$.

Under $P_{\theta_{\lambda}}$, $X_i \sim N(0, I_k)$ and

$$
\begin{pmatrix}
Y_i \\
Z_i
\end{pmatrix} \quad \mid X \sim N \left( \begin{pmatrix} c_1 X_i \lambda + c_1 k u_i + v_i \\ 0 \\
\end{pmatrix} \right), \quad \Sigma = \begin{pmatrix} 1 & c_1 c_2 k \\ c_1 c_2 k & 1 \end{pmatrix}.
$$

with $E u_i^2 = 1$ and $E v_i^2 = 1 - c_1^2 c_2^2 k^2$. Therefore, under $P_{\theta_{\lambda}}$,

$$
\begin{pmatrix}
Y_i \\
Z_i
\end{pmatrix} \mid X \sim N \left( \begin{pmatrix} c_1 X_i \lambda + v_i \\ c_2 X_i \lambda + u_i \\
\end{pmatrix} \right).
$$

Similarly, we observe that under $P_{\tilde{\theta}_{\lambda}}$, $X_i \sim N(0, I_k)$ and

$$
\begin{pmatrix}
Y_i \\
Z_i
\end{pmatrix} \quad \mid X \sim N \left( \begin{pmatrix} c_1 X_i \lambda + v_i \\ c_2 X_i \lambda + u_i \\
\end{pmatrix} \right).
$$

By Lemma A.6, we have $TV(P_1, P_2) \leq 2C_1 C_2 c_0^{-\xi_1 - \xi_2 + 1/2} + o(1)$. We can choose constants $C_1, C_2, C_0 > 0$ small enough such that $C_1 C_2 C_0^{-\xi_1 - \xi_2 + 1/2} \leq \alpha/4$. Thus,

$$
TV(P_1, P_2) \leq \alpha/2 + o(1). \tag{A.8}
$$

Let $\psi$ be a test for

$$
H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1,
$$

where $\Theta_1 = \{ \theta = (\beta, \phi, \pi, \alpha_1^2, \alpha_2^2) \in \Theta(\xi_1, \xi_2) : |\beta| \geq C_3 n^{-(\xi_1 + \xi_2)/(1/2 + \xi_1 + \xi_2)} \}$ and $\Theta_0 = \{ \theta = (\beta, \phi, \pi, \alpha_1^2, \alpha_2^2) \in \Theta(\xi_1, \xi_2) : \beta = 0 \}$ with $C_3 = C_1 C_2 C_0^{-\xi_1 - \xi_2 + 1/2}/2$. Let $\eta(\theta) = E_{P_\theta}(\psi)$ be the power function. By definition of $\psi$, we have $\sup_{\theta \in \Theta_0} \eta(\theta) \leq \alpha$.

By Lemma A.4, we can choose small enough constants $C_1, C_2 > 0$ such that

$$
\int_{\mathbb{R}^k} 1\{\theta_{\lambda} \in \Theta_0\} \mu(\lambda) d\lambda = 1 - o(1). \tag{A.9}
$$

and

$$
\int_{\mathbb{R}^k} 1\{\tilde{\theta}_{\lambda} \in \Theta_1\} \mu(\lambda) d\lambda = 1 - o(1). \tag{A.10}
$$

Then

$$
\left| \int_{\mathbb{R}^k} \eta(\theta_{\lambda}) \mu(\lambda) d\lambda - \int_{\mathbb{R}^k} \eta(\tilde{\theta}_{\lambda}) \mu(\lambda) d\lambda \right| = \left| \int \left( \int \psi dP_{\theta_{\lambda}} \right) \mu(\lambda) d\lambda - \int \left( \int \psi dP_{\tilde{\theta}_{\lambda}} \right) \mu(\lambda) d\lambda \right|
\leq \int |\psi| dP_1 - \int |\psi| dP_2 \leq 2 \cdot TV(P_1, P_2) \leq \alpha + o_P(1),
$$

where (i) follows by $|\psi| \leq 1$ and Scheffe’s theorem (Lemma 2.1 of Tsybakov (2009)) and (ii) follows by (A.8).
Since the power function is bounded by 1, (A.9) and (A.10) imply
\[
\left| \int_{R^k} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda - \int_{R^k} 1\{\hat{\lambda}_k \in \Theta_0\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda \right| = \int_{R^k} 1\{\hat{\lambda}_k \notin \Theta_0\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda \leq \int_{R^k} 1\{\hat{\lambda}_k \notin \Theta_0\} \mu(\lambda) d\lambda = o(1)
\]
and similarly
\[
\left| \int_{R^k} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda - \int_{R^k} 1\{\hat{\lambda}_k \in \Theta_1\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda \right| = o(1).
\]
By the above three displays, we have
\[
\int_{R^k} 1\{\hat{\lambda}_k \in \Theta_1\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda \leq \alpha + \int_{R^k} 1\{\hat{\lambda}_k \in \Theta_0\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda + o(1) \leq 2\alpha + o(1),
\]
where (i) follows by sup_{\theta \in \Theta_0} \eta(\theta) \leq \alpha. On the other hand, by (A.10),
\[
\int_{R^k} 1\{\hat{\lambda}_k \in \Theta_1\} \eta(\hat{\lambda}_k) \mu(\lambda) d\lambda \geq (\inf_{\theta \in \Theta_1} \eta(\theta)) \times \int_{R^k} 1\{\hat{\lambda}_k \in \Theta_1\} \mu(\lambda) d\lambda = (\inf_{\theta \in \Theta_1} \eta(\theta)) \times (1 - o(1)).
\]
Combining the above two displays, we have that
\[
\inf_{\theta \in \Theta_1} \eta(\theta) \leq \frac{2\alpha + o(1)}{1 - o(1)} = 2\alpha + o(1).
\]
Therefore, for testing \(H_0: \phi(\theta) = 0\) versus \(H_1: |\phi(\theta)| \geq C_3 n^{-\xi_1 + \xi_2}/(\xi_1 + \xi_2 + 1/2)\), any test of size \(\alpha\) can have the worst-case power bounded by \(2\alpha + o(1)\). The desired result follows.

Lemma A.8. Let \(C_0, C_1, C_2, \xi_1, \xi_2 > 0\) be constants. Define \(k = \left\lfloor C_0 n^{1/(\xi_1 + \xi_2 + 1/2)} \right\rfloor\), \(c_1 = C_1 k^{-\xi_1 - 1/2}\) and \(c_2 = C_2 k^{-\xi_2 - 1/2}\), where \(\left\lfloor \cdot \right\rfloor\) denotes the integer part. For any \(\lambda \in \mathbb{R}^3\), let \(\phi_{1,\lambda}\) denote the distribution of \(i.i.d\) \((Y_i, Z_i, X_i)\)_{i=1}^n given by \(X_i \sim N(0, I_k)\)
\[
\begin{pmatrix} Z_i \\ Y_i \end{pmatrix} \mid X \sim N \left( \begin{pmatrix} c_2X_i'\lambda \\ 0 \end{pmatrix}, \Sigma \right) \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & c_1c_2k \\ c_1c_2k & 1 \end{pmatrix}.
\]
Let \(\phi_{2,\lambda}\) denote the distribution of \(i.i.d\) \((Y_i, Z_i, X_i)\)_{i=1}^n given by \(X_i \sim N(0, I_k)\).
\[
\begin{pmatrix} Z_i \\ Y_i \end{pmatrix} \mid X \sim N \left( \begin{pmatrix} c_2X_i'\lambda \\ c_1X_i'\lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).
\]
Assume that \(\xi_1 \geq 1/4 > \xi_2\) and \(\xi_1 + \xi_2 < 1/2\). Let \(X_i\) be \(i.i.d\) from \(N(0, I_k)\). Then
\[
TV(\hat{\phi}_1, \hat{\phi}_2) \leq 2C_1C_2C_0^{-\xi_1 + \xi_2 + 1/2} + o(1),
\]
where \(TV\) denotes the total variation distance. \(\hat{\phi}_1 = \int_{R^3} \phi_{1,\lambda} \mu(\lambda) d\lambda, \hat{\phi}_2 = \int_{R^3} \phi_{2,\lambda} \mu(\lambda) d\lambda\) and \(\mu\) is the density function of \(N(0, I_k)\), i.e., \(\mu(\lambda) = (2\pi)^{-k/2} \exp(-||\lambda||^2_2/2)\).

Proof. The proof is analogous to that of Lemma A.6. We simply swapped the role of \(c_1\) and \(c_2\) as well as the role of \(Y_i\) and \(Z_i\).

Lemma A.9. Assume that \(p \leq n^2\). If \(\xi_1 + \xi_2 < 1/2\) and \(\xi_1 \geq 1/4 > \xi_2\), then \(\mathcal{D}(\xi_1, \xi_2) \geq n^{-\xi_1 + \xi_2}/(\xi_1 + \xi_2 + 1/2)\).
Proof. The argument is analogous to that of Lemma A.7 with \( \xi_1 \) and \( \xi_2 \) swapped. Define \( k = C_0 n^{(1/\xi_1 + \xi_2 + 1/2)} \), \( c_1 = C_1 k^{-\xi_1-1/2} \) and \( c_2 = C_2 k^{-\xi_2-1/2} \), where \( C_0, C_1, C_2 > 0 \) are constants to be determined. Since \( \xi_1 + \xi_2 < 1/2 \), we have \( k \gg n \). Since \( p \geq n^2 \) and \( k \gg n^{1/(\xi_1 + \xi_2 + 1/2)} \ll n^2 \), we have \( p \gg k \).

In the following analysis, we set \( \gamma_j = \pi_j = 0 \) for \( j > k \). This effectively reduces the dimension from \( p \) to \( k \) in our analysis.

For any \( \lambda \in \mathbb{R}^k \), let \( \theta_\lambda = (c_1 c_2 k, -c_1 c_2 k \lambda, c_2 \lambda, 1 - c_1 c_2 k^2, 1) \) and \( \tilde{\theta}_\lambda = (0, c_1 \lambda, c_2 \lambda, 1, 1) \). Notice that \( 1 - c_1 c_2 k^2 \) is well defined as the value of a variance (i.e., \( 1 - c_1 c_2 k^2 > 0 \)) if we choose \( C_0, C_1, C_2 > 0 \) small enough; to see this notice that \( c_1 c_2 k \gg k^{-\xi_1 + \xi_2} = o(1) \).

We consider two probability measures: \( P_1 = \int P_{\theta_0} \mu(\lambda) d\lambda \) and \( P_2 = \int P_{\tilde{\theta}_0} \mu(\lambda) d\lambda \), where \( \mu \) is the density of \( N(0, I_k) \), i.e., \( \mu(\lambda) = (2\pi)^{-k/2} \exp(-||\lambda||^2/2) \).

Under \( P_{\theta_0} \), \( X_i \sim N(0, I_k) \) and
\[
\begin{pmatrix}
Z_i \\
Y_i
\end{pmatrix}
= \begin{pmatrix}
Z_i \\
Z_i c_1 c_2 k - c_1 c_2 k X_i' \lambda + v_i
\end{pmatrix}
= \begin{pmatrix}
c_1 X_i' \lambda + u_i \\
c_1 c_2 k u_i + v_i
\end{pmatrix}
\]
with \( E Z_i^2 = 1 \) and \( E Y_i^2 = 1 - c_1^2 c_2^2 k^2 \). Therefore, under \( P_{\theta_0} \),
\[
\begin{pmatrix}
Z_i \\
Y_i
\end{pmatrix} \mid X \sim N \left( \begin{pmatrix} c_1 X_i' \lambda \\ 0 \end{pmatrix}, \Sigma \right) \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & c_1 c_2 k \\ c_1 c_2 k & 1 \end{pmatrix}.
\]

Similarly, we observe that under \( P_{\tilde{\theta}_0} \), \( X_i \sim N(0, I_k) \) and
\[
\begin{pmatrix}
Z_i \\
Y_i
\end{pmatrix}
= \begin{pmatrix}
(c_1 X_i' \lambda + u_i) \\
(c_1 X_i' \lambda + v_i)
\end{pmatrix},
\]
which means that
\[
\begin{pmatrix}
Z_i \\
Y_i
\end{pmatrix} \mid X \sim N \left( \begin{pmatrix} c_1 X_i' \lambda \\ c_1 X_i' \lambda \end{pmatrix}, I_2 \right).
\]

By Lemma A.8, we have \( \text{TV}(P_1, P_2) \leq 2C_1 C_2 C_0^{-(\xi_1 + \xi_2 + 1/2)} + o(1) \). We can choose constants \( C_1, C_2, C_0 > 0 \) small enough such that \( C_1 C_2 C_0^{-(\xi_1 + \xi_2 + 1/2)} \leq \alpha/4 \). Thus,
\[
\text{TV}(P_1, P_2) \leq \alpha/2 + o(1). \tag{11}
\]

Let \( \psi \) be a test for
\[
H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1,
\]
where \( \Theta_1 = \{ \theta = (\beta, \phi, \pi, \sigma_1^2, \sigma_2^2) \in \Theta(\xi_1, \xi_2) : |\beta| \geq C_3 n^{(1/\xi_1 + \xi_2)/\xi_1)/(\xi_1 + \xi_2 + 1/2) \} \) and \( \Theta_0 = \{ \theta = (\beta, \phi, \pi, \sigma_1^2, \sigma_2^2) \in \Theta(\xi_1, \xi_2) : \beta = 0 \} \) with \( C_3 = C_1 C_2 C_0^{-(\xi_1 + \xi_2)} \). Let \( \eta(\theta) = E_\theta(\psi) \) be the power function. By definition of \( \psi \), we have \( \sup_{\theta \in \Theta_0} \eta(\theta) \leq \alpha \).

By Lemma A.4, we can choose small enough constants \( C_1, C_2 > 0 \) such that
\[
\int_{\mathbb{R}^k} \mathbf{1}\{\theta \in \Theta_0\} \mu(\lambda) d\lambda = 1 - o(1). \tag{12}
\]

and
\[
\int_{\mathbb{R}^k} \mathbf{1}\{\tilde{\theta}_0 \in \Theta_1\} \mu(\lambda) d\lambda = 1 - o(1). \tag{13}
\]

Then
\[
\left| \int_{\mathbb{R}^k} \eta(\tilde{\theta}_0) \mu(\lambda) d\lambda - \int_{\mathbb{R}^k} \eta(\theta_0) \mu(\lambda) d\lambda \right| = \left| \int \left( \int \psi dP_{\theta_0} \right) \mu(\lambda) d\lambda - \int \left( \int \psi dP_{\tilde{\theta}_0} \right) \mu(\lambda) d\lambda \right| \leq 2 \cdot \text{TV}(P_1, P_2) \leq \alpha + o_p(1),
\]
where (i) follows by $|\psi| \leq 1$ and Scheffe’s theorem (Lemma 2.1 of Tsybakov (2009)) and (ii) follows by (A.11).

Since the power function is bounded by 1, (A.12) and (A.13) imply

$$
\left| \int_{\ellk} \eta(\theta_\lambda) \mu(\lambda) d\lambda - \int_{\ellk} 1\{\theta_\lambda \in \Theta_0\} \eta(\theta_\lambda) \mu(\lambda) d\lambda \right| = \int_{\ellk} 1\{\theta_\lambda \notin \Theta_0\} \eta(\theta_\lambda) \mu(\lambda) d\lambda \leq \int_{\ellk} 1\{\theta_\lambda \notin \Theta_0\} \mu(\lambda) d\lambda = o(1)
$$

and similarly

$$
\left| \int_{\ellk} \eta(\theta_\lambda) \mu(\lambda) d\lambda - \int_{\ellk} 1\{\tilde{\theta}_\lambda \in \Theta_1\} \eta(\tilde{\theta}_\lambda) \mu(\lambda) d\lambda \right| = o(1).
$$

By the above three displays, we have

$$
\int_{\ellk} 1\{\tilde{\theta}_\lambda \in \Theta_1\} \eta(\tilde{\theta}_\lambda) \mu(\lambda) d\lambda \leq \alpha + \int_{\ellk} 1\{\theta_\lambda \in \Theta_0\} \eta(\theta_\lambda) \mu(\lambda) d\lambda + o(1) \tag{i}
$$

where (i) follows by $\sup_{\theta \in \Theta_0} \eta(\theta) \leq \alpha$. On the other hand, by (A.13),

$$
\int_{\ellk} 1\{\tilde{\theta}_\lambda \in \Theta_1\} \eta(\tilde{\theta}_\lambda) \mu(\lambda) d\lambda \geq (\inf_{\theta \in \Theta_1} \eta(\theta)) \times \int_{\ellk} 1\{\tilde{\theta}_\lambda \in \Theta_1\} \mu(\lambda) d\lambda = (\inf_{\theta \in \Theta_1} \eta(\theta)) \times (1 - o(1)).
$$

Combining the above two displays, we have that

$$
\inf_{\theta \in \Theta_1} \eta(\theta) \leq \frac{2\alpha + o(1)}{1 - o(1)} = 2\alpha + o(1).
$$

Therefore, for testing $H_0: \phi(\theta) = 0$ versus $H_1: |\phi(\theta)| \geq C_3 n^{-\left(\xi_1 + \xi_2\right)/(\xi_1 + \xi_2 + 1/2)}$, any test of size $\alpha$ can has the worst-case power bounded by $2\alpha + o(1)$. The desired result follows. \hfill \Box

**Proof of Theorem 2.** We consider two cases: (1) max$\{\xi_1, \xi_2\} \geq 1/4$ and (2) max$\{\xi_1, \xi_2\} < 1/4$. We first consider the case of max$\{\xi_1, \xi_2\} \geq 1/4$. By Lemmas A.7 and A.9, we have proved that for any $\xi_1, \xi_2 > 0$ satisfying $\xi_1 + \xi_2 < 1/2$ and max$\{\xi_1, \xi_2\} \geq 1/4$, we have $\mathcal{R}(\xi_1, \xi_2) \geq n^{-\left(\xi_1 + \xi_2\right)/(\xi_1 + \xi_2 + 1/2)}$. Thus the result follows for the case of max$\{\xi_1, \xi_2\} \geq 1/4$.

Now we consider the case of max$\{\xi_1, \xi_2\} < 1/4$. Without loss of generality, we assume $\xi_1 \geq \xi_2$. Then we have $1/4 > \xi_1 \geq \xi_2$. Now we set $\tilde{\xi}_1 = 1/4$ and $\tilde{\xi}_2 = \xi_2$. Thus, we have $\tilde{\xi}_1 + \tilde{\xi}_2 = 1/4 + \xi_2 < 1/4 + 1/4 = 1/2$ and max$\{\tilde{\xi}_1, \tilde{\xi}_2\} = 1/4$. The results for the case of max$\{\xi_1, \xi_2\} < 1/4$ implies $\mathcal{R}(\tilde{\xi}_1, \tilde{\xi}_2) \geq n^{-\left(\tilde{\xi}_1 + \tilde{\xi}_2\right)/(\tilde{\xi}_1 + \tilde{\xi}_2 + 1/2)} \gg n^{-1/2}$. By $\tilde{\xi}_1 > \xi_1$, $\Theta(\tilde{\xi}_1, \tilde{\xi}_2) \subset \Theta(\xi_1, \xi_2)$, which means that $\mathcal{R}(\xi_1, \xi_2) \gg \mathcal{R}(\tilde{\xi}_1, \tilde{\xi}_2)$. Hence, $\mathcal{R}(\xi_1, \xi_2) \gg n^{-1/2}$. \hfill \Box

## B Proofs for Section 5

For the proof of results in Section 5 let $\epsilon_n = \sqrt{\ln(p)/n}$, $s_0 \geq C_6 n^{-2/(2\xi_1 + 1)}$, and $\pi$ be coefficients of the least squares projection of $\alpha_0(X)$ on $b(X)$, satisfying

$$
M - \Sigma \pi = E[b(X) \{\alpha_0(X) - b(X)\pi\}] = 0.
$$

By Assumption 3 we can define $J_0 \subset \{1, ..., p\}$ as indices of a sparse approximation with $|J_0| = s_0$, where $|A|$ denotes the number of elements of a matrix, and coefficients $\pi_j$ for $j \in J_0$ such that for $\pi = (\pi_1, ..., \pi_J)'$, with $\pi_j = 0$ for $j \notin J_0$,

$$
\|\pi - \tilde{\pi}\|_2 \leq C s_0^{-\xi_2} \leq C \epsilon_n^{-2\xi_2/(2\xi_1 + 1)}
$$

Also define $\pi_n$ as

$$
\pi_n \in \arg \min_v (\pi - v)\Sigma(\pi - v) + 2\epsilon_n \sum_{j \in J_0} |v_j|. \tag{B.1}
$$

30
Lemma B.1. $\|\Sigma(\pi_s - \pi)\|_\infty \leq C_{\epsilon_n}$.

Proof. Let $e_j \in \mathbb{R}^p$ denote the $j$-th column of $I_p$. The first-order condition for $\pi^*$ imply that for $j \in J_0$, we have $e_j' \Sigma(\pi_s - \pi) = 0$; for $j \in J_0^c$, we have $e_j' \Sigma(\pi_s - \pi) + \epsilon_n z_j = 0$, where $z_j = \text{sign}(\pi_s, j)$ if $\pi_s, j \neq 0$ and $z_j \in [-1, 1]$ if $\pi_s, j = 0$. Therefore, for any $j$, we have that $|e_j' \Sigma(\pi_s - \pi)| \leq \epsilon_n$. Hence, $\|\Sigma(\pi_s - \pi)\|_\infty \leq \epsilon_n$. \hfill \Box

Lemma B.2. $(\pi - \pi_s)' \Sigma(\pi - \pi_s) \leq C_{\epsilon_n}^{4/2^{2-1}}$.

Proof. By the definition of $\pi_s$, we have that by the largest eigenvalue of $\Sigma$ bounded,

$$\begin{align*}
(\pi - \pi_s)' \Sigma(\pi - \pi_s) + \epsilon_n \sum_{j \in J_0^c} |\pi_s, j| & \leq (\pi - \pi_s)' \Sigma(\pi - \pi_s) + \epsilon_n \sum_{j \in J_0^c} |\pi_s| = (\pi - \pi_s)' \Sigma(\pi - \pi_s) \\
& \leq C \|\pi - \pi_s\|^2 \leq C_{\epsilon_n}^{-2/2^{2-1}} + 1.
\end{align*}$$

Let $J$ be the vector of indices of nonzero elements of $\pi_s$.

Lemma B.3. $|J| \leq C_{\epsilon_n}^{-2/2^{2-1}}$.

Proof. For all $j \in J \setminus J_0$ the first order conditions to Equation (B.1) imply $|e_j' \Sigma(\pi_s - \pi)| = \epsilon_n$. Therefore, It follows that

$$\begin{align*}
\sum_{j \in J \setminus J_0} (e_j' \Sigma(\pi_s - \pi))^2 & = \frac{1}{4} \epsilon_n^2 |J \setminus J_0|.
\end{align*}$$

In addition,

$$\begin{align*}
\sum_{j \in J \setminus J_0} (e_j' \Sigma(\pi_s - \pi))^2 & \leq \sum_{j=1}^p (e_j' \Sigma(\pi_s - \pi))^2 = (\pi_s - \pi)' \Sigma \left( \sum_{j=1}^p e_j e_j' \right) \Sigma (\pi_s - \pi) \\
& = (\pi_s - \pi)' \Sigma^2 (\pi_s - \pi) \leq \lambda_{\max}(\Sigma) \{(\pi_s - \pi)' \Sigma (\pi_s - \pi)\} \leq C_{\epsilon_n}^{4/2^{2-1}},
\end{align*}$$

where the last inequality follows by Lemma B.2 and $\lambda_{\max}(\Sigma) \leq C$. Combining the above two displays, we obtain

$$\frac{1}{4} \epsilon_n^2 |J \setminus J_0| \leq C_{\epsilon_n}^{4/2^{2-1}}.$$

Dividing through by $\epsilon_n^2$ gives $|J \setminus J_0| \leq C_{\epsilon_n}^{-2/2^{2-1}}$. Thus by $s_0 \leq C_{\epsilon_n}^{-2/2^{2-1}}$,

$$|J| = |J_0| + |J \setminus J_0| = s_0 + |J \setminus J_0| \leq s_0 + C_{\epsilon_n}^{-2/2^{2-1}} \leq C_{\epsilon_n}^{-2/2^{2-1}}.$$

\hfill \Box

Lemma B.4. If $\xi_2 > 1/2$ then $\|\pi_s - \pi\|_1 \leq C_{\epsilon_n}^{(2^{2-1})/(2^{2-1})}$.

Proof. By Lemma C.1 and $\xi_2 > 1/2$, we have that

$$\|\pi_{J_0^c}\|_1 \leq \frac{1}{2} - \frac{\xi_2}{2} \leq C_{\epsilon_n}^{(2^{2-1})/(2^{2-1})}.$$

Let $J_1 = J \cup J_0$ note that $J \subset J_1$ and $J_0 \subset J_1$ imply $J_1' \subset J'$ and $J_1^c \subset J_0^c$, so that

$$\|\pi_s|_{J_1'} - \pi_{J_1'}\|_1 \leq \|\pi_s - \pi\|_1 \leq \|\pi_{J_0}^c\|_1.$$

Also, by Lemma B.3,

$$|J_1| \leq |J| + |J_0| \leq C_{\epsilon_n}^{-2/2^{2-1}} + s_0 \leq C_{\epsilon_n}^{-2/2^{2-1}}.$$

Therefore we have

$$\begin{align*}
\|\pi_s - \pi\|_1 & = \|\pi_s|_{J_1'} - \pi_{J_1'}\|_1 + \|\pi_s|_{J_1^c} - \pi_{J_1^c}\|_1 \leq \|\pi_s|_{J_1'} - \pi_{J_1'}\|_1 + \|\pi_{J_0}^c\|_1 \\
& \leq \sqrt{|J_1|} \|\pi_s|_{J_1'} - \pi_{J_1'}\|_2 + C_{\epsilon_n}^{(2^{2-1})/(2^{2-1})} \\
& \leq C_{\epsilon_n}^{1/(2^{2-1})} - \|\pi_s - \pi\|_2^2 + C_{\epsilon_n}^{(2^{2-1})/(2^{2-1})} \leq C_{\epsilon_n}^{(2^{2-1})/(2^{2-1})}.
\end{align*}$$

\hfill \Box
Lemma B.5. \( \| \hat{\Sigma}_\pi - \Sigma \pi \|_\infty = O_p(\varepsilon_n) \), \( \| \hat{\Sigma}_\pi - \Sigma \pi \|_1 = O_p(\varepsilon_n) \).

Proof. By \( (\pi - \pi')\Sigma(\pi - \pi) \rightarrow 0 \) and \( \pi' \Sigma \pi \leq E[\varepsilon_0(X)^2] \) it follows that \( E[(b(X')\pi)^2] = \pi' \Sigma \pi \leq C \). The first conclusion then follows by uniform boundedness of the elements of \( b(X) \) and Lemma C.2 with \( X_{i,j} = b_j(X_i) \) and \( X_0 = b(X')\pi_n \). The second conclusion follows similarly.

Next let
\[
\hat{\pi} = \arg \min_{\pi} \{-2\hat{M}'\pi + \pi' \hat{\Sigma} \pi + 2r\|\pi\|_1\},
\]
for \( \hat{M} \) to be specified later in this appendix.

Lemma B.6. If \( \| \hat{M} - M \|_\infty = O_p(\varepsilon_n) \) and \( \varepsilon_n = o(r) \) then for \( \Delta = \hat{\pi} - \pi' \) and any \( \hat{J} \) such that \( (\pi')_\hat{J} = 0 \), with probability approaching one,
\[
\Delta' \hat{\Sigma} \Delta \leq 3r\|\Delta\|_1, \quad \|\Delta\|_1 \leq 3\|\Delta\|_1.
\]

Proof. By the definition of the estimator, we have
\[
\hat{\pi}' \hat{\Sigma} \hat{\pi} - 2M'\hat{\pi} + 2r\|\hat{\pi}\|_1 \leq \pi' \Sigma \pi - 2M'\pi_n + 4r\|\pi\|_1.
\]
Plugging \( \hat{\pi} = \pi + \Delta \) into the above equation and rearranging the terms gives
\[
\Delta' \hat{\Sigma} \Delta + 2r\|\pi + \Delta\|_1 \leq 2r\|\pi_n\|_1 + 2(\hat{M} - \Sigma \pi)'\Delta. \tag{B.2}
\]
Note that \( \| \hat{M} - M \|_\infty = O_p(\varepsilon_n) \) and by Lemma B.5 \( \| \hat{\Sigma}_\pi - \Sigma \pi \|_\infty = O_p(\varepsilon_n) \). Then by Lemma B.1, \( M = \Sigma \pi \), and the triangle inequality,
\[
\| \hat{M} - \Sigma \pi \|_\infty \leq \| \hat{\Sigma}_\pi - \Sigma \pi \|_\infty + \| M - M \|_\infty + \| M - \Sigma \pi \|_\infty \leq O_p(\varepsilon_n) + \| M - \Sigma \pi \|_\infty + \| \Sigma (\pi - \pi') \|_\infty = O_p(\varepsilon_n).
\]
Therefore, by the Holder inequality we have \( \| (\hat{M} - \Sigma \pi)'\Delta \| \leq \| \hat{M} - \Sigma \pi \|_\infty \| \Delta\|_1 \), so that
\[
\Delta' \hat{\Sigma} \Delta + 2r\|\pi + \Delta\|_1 \leq 2r\|\pi_n\|_1 + 2\varepsilon_n \| \Delta\|_1.
\]
By \( \varepsilon_n = o(r) \) it follows that with probability approaching one \( 2\varepsilon_n \leq r \) and
\[
\Delta' \hat{\Sigma} \Delta + 2r\|\pi + \Delta\|_1 \leq 2r\|\pi_n\|_1 + r\| \Delta\|_1.
\]
The triangle inequality implies \( \| \pi\|_1 = \| \pi + \Delta - \Delta\|_1 \leq \| \pi + \Delta\|_1 + \| \Delta\|_1 \) so subtracting \( 2r\|\pi + \Delta\|_1 \) from both sides gives the first conclusion.

Next, since \( \Delta' \hat{\Sigma} \Delta \geq 0 \) it also follows from Equation (B.2) that \( 2r\|\pi + \Delta\|_1 \leq 2r\|\pi\|_1 + r\| \Delta\|_1 \) with probability approaching one, so dividing through by \( r \) gives
\[
2\|\pi\|_1 + \| \Delta\|_1 \leq 2\|\pi\|_1 + \| \Delta\|_1.
\]
It follows by \( (\pi')_\hat{J} = 0 \) that \( \| \pi + \Delta\|_1 = \| (\pi)_J + \Delta J\|_1 + \| \Delta\|_1 \) and \( \| \pi\|_1 = \| (\pi)_J\|_1 \). Substituting in the previous display then gives
\[
2\| (\pi)_J + \Delta J\|_1 + 2\| \Delta\|_1 \leq 2\| (\pi)_J + \Delta J\|_1 + \| \Delta\|_1 = 2\| (\pi)_J\|_1 + 2\| \Delta J\|_1 + \| \Delta\|_1 \leq 2\|\pi\|_1 + 2\| \Delta J\|_1 + \| \Delta\|_1 = 2\|\pi\|_1 + 3\| \Delta J\|_1 + \| \Delta\|_1.
\]
Subtracting \( 2\| (\pi)_J + \Delta J\|_1 + \| \Delta\|_1 \) from both sides gives the second conclusion.

Lemma B.7. If \( \| \hat{M} - M \|_\infty = O_p(\varepsilon_n) \) and \( \varepsilon_n = o(r) \) then \( \Delta' \hat{\Sigma} \Delta = O_p((r/\varepsilon_n)^2 \varepsilon_n^{2\xi_2/(2\xi_2 + 1)}) \), \( \| \Delta\|_1 = O_p((r/\varepsilon_n) \varepsilon_n^{2\xi_2/(2\xi_2 + 1)}) \), \( \| \Delta\|_2 = O_p((r/\varepsilon_n) \varepsilon_n^{2\xi_2/(2\xi_2 + 1)}) \).
Proof. For $J = J$ it follows from the sparse eigenvalue condition and Lemma B.6 that with high probability
\[
\|\Delta_j\|^2 \leq C\bar{\delta}\Delta \leq Cr\|\Delta\|_1 = Cr(\|\Delta_j\|_1 + \|\Delta_j\|_1) \leq Cr\|\Delta_j\|_1
\]
\[
\leq Cr\sqrt{|J|}\|\Delta_j\|_2 \leq C\epsilon_n^{-1/(2\delta_j^2 + 1)}\|\Delta_j\|_2 = C(r/\epsilon_n)\epsilon_n^{2\delta_j^2/(2\delta_j^2 + 1)}\|\Delta_j\|_2.
\]
Dividing through by $\|\Delta_j\|_2$ then gives
\[
\|\Delta_j\|_2 \leq C(r/\epsilon_n)\epsilon_n^{2\delta_j^2/(2\delta_j^2 + 1)}.
\]
Plugging this back in the final expression in the previous inequality gives the first conclusion.

For the second conclusion note that by Lemma B.6,
\[
\|\Delta_j\|_2 \leq (|J|)^{-1/2}\|\Delta_j\|_1 \leq (|J|)^{-1/2}3\|\Delta_j\|_1 \leq 3(|J|)^{-1/2}\sqrt{|J|}\|\Delta_j\|_2 \leq C(r/\epsilon_n)\epsilon_n^{2\delta_j^2/(2\delta_j^2 + 1)}.
\]
Therefore, by the triangle inequality,
\[
\|\Delta\|_2 \leq \|\Delta_j\|_2 + \|\Delta_j\|_2 \leq C(r/\epsilon_n)\epsilon_n^{2\delta_j^2/(2\delta_j^2 + 1)},
\]
giving the third conclusion. □

Lemma B.8. If $\|\bar{X} - M\|_\infty = O_p(\epsilon_n)$ and $\epsilon_n = o(r)$ then $\|\bar{\Sigma}(\hat{\pi} - \pi)\|_\infty = O_p(r)$.

Proof. The Lasso first order conditions imply $\|\bar{\Sigma}(\hat{\pi} - \pi)\|_\infty = O(r)$. By Lemma B.5 $\|\bar{\Sigma}(\hat{\pi} - \pi)\|_\infty = O_p(\epsilon_n)$. Then by the triangle inequality,
\[
\|\bar{\Sigma}(\hat{\pi} - \pi)\|_\infty \leq \|\bar{\Sigma}\bar{\pi} - \bar{M}\|_\infty + \|\bar{M} - M\|_\infty + \|M - \Sigma\pi\|_\infty + \|\Sigma - \hat{\Sigma}\|_\infty \|\pi\|_\infty
\]
\[
= O_p(r) + O_p(\epsilon_n) + 0 + O_p(\epsilon_n) = O_p(r).
\]
□

Lemma B.9. If Assumptions 4-7 are satisfied then $(\hat{\pi}_1 - \pi)^T\bar{\Sigma}_1(\hat{\pi}_1 - \pi) = o_p(1)$.

Proof. Consider $\hat{\pi}$ from Lemma 3 and let $\pi_*$ be as defined in Equation (B.1) for $J_0 = \{j : \hat{\pi}_j \neq 0\}$. By the definition of $\pi_*$, we have
\[
(\pi - \pi_*)^T\Sigma(\pi - \pi_*) + \epsilon_n \sum_{j \in J_0^c} \pi_j \leq (\pi - \hat{\pi})^T\Sigma(\pi - \hat{\pi}) + \epsilon_n \sum_{j \in J_0^c} \hat{\pi}_j \leq (\pi - \hat{\pi})^T\Sigma(\pi - \hat{\pi})
\]
\[
= O(\delta_0^2) = o(1).
\]
Let $J$ be the vector of indices of nonzero elements of $\pi_*$. For all $j \in J \setminus J_0$ the first order conditions to Equation (B.1) imply $|v_j^*\Sigma(\pi_* - \pi)| = \epsilon_n$. Therefore, it follows that
\[
\sum_{j \in J \setminus J_0} (v_j^*\Sigma(\pi_* - \pi))^2 = \epsilon_n^2 |J \setminus J_0|.
\]
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In addition
\[ \sum_{j \in J \setminus J_0} (e_j, \Sigma (\pi - \pi))^2 \leq \sum_{j=1}^p (e_j, \Sigma (\pi - \pi))^2 = (\pi - \pi) \Sigma (\sum_{j=1}^p e_j e_j^T) \Sigma (\pi - \pi) \]
\[= (\pi - \pi) \Sigma^2 (\pi - \pi) \leq \lambda_{\text{max}}(\Sigma) \{(\pi - \pi) \Sigma (\pi - \pi)\} \leq C \delta_h^2. \]

Then from the previous equation \(e_n^T J \setminus J_0 \leq C \delta_h^2\) implying \(|J \setminus J_0| \leq C \delta_h^2 e_n^{-2}\). By Lemma 3 we also have \(|J_0| \leq C \delta_h^2 e_n^{-2}\), so by summing
\[ |J| \leq C \delta_h^2 e_n^{-2}. \]

For convenience we drop the 1 subscript on \(\hat{\pi}_1\) and \(\hat{\Sigma}_1\) for the rest of the proof. Then by Lemma B.6, we have
\[ \Delta' \hat{\Sigma} \Delta \leq 3r||A||_1 = 3r(||A_F||_1 + ||A_J||_1) \leq 12r||A_J||_1 \leq C \sqrt{|J||A_J||_2} \leq C \epsilon_n^{-1} \delta_n ||A_J||_2, \]

By Assumption 7, \( ||A_J||_2 \leq C \Delta' \hat{\Sigma} \Delta \) with probability approaching one. Then dividing through by \( ||A_J||_2 \) gives \( ||A_J||_2 \leq C \epsilon_n^{-1} \delta_n \) with probability approaching one. Plugging this inequality in the previous Equation gives
\[ \Delta' \hat{\Sigma} \Delta \leq C (r \epsilon_n^{-1} \delta_n)^2 \]
with probability approaching one. Therefore \( \Delta' \hat{\Sigma} \Delta = O_p((r \epsilon_n^{-1} \delta_n)^2) \). Also, by Assumption 5 \( \delta_n \leq C \epsilon_n^{-1} \delta_n \) for large enough \( n \), so by the Markov inequality and Equation (B.3),
\[ (\pi - \pi) \Sigma (\pi - \pi) = O_p(E[(\pi - \pi) \Sigma (\pi - \pi)]) = O_p((\pi - \pi) \Sigma (\pi - \pi)) \]
\[= O_p(\delta_n^2) = O_p((r \epsilon_n^{-1} \delta_n)^2). \]

Then by the Cauchy-Schwartz and triangle inequalities,
\[ (\hat{\pi} - \pi) \Sigma (\hat{\pi} - \pi) \leq C \Delta' \hat{\Sigma} \Delta + C (\pi - \pi) \Sigma (\pi - \pi) \leq O_p((r \epsilon_n^{-1} \delta_n)^2). \]

\[\square\]

Proof of Lemma 3: By \( \alpha(X) \in \mathcal{B} \) there exists a sequence \( \alpha_k(x) \) of finite dimensional linear combinations of \( (b_1(x), b_2(x), \ldots) \) such that \( ||\alpha - \alpha_k||_2 \to 0 \). Since each \( \alpha_k(x) \) is a finite dimensional linear combination there exists \( p_k \) such that for \( b^k(x) = (b_1(x), \ldots, b_{p_k}(x)) \) we have \( \bar{\alpha}_k(x) = b^k(x)' \gamma_k \) for some \( \gamma_k \). Let \( \alpha_k(X) \) be the least square projection of \( \alpha(X) \) on \( b^k(X) \). Then by \( ||\alpha - \alpha_k||_2 \leq ||\alpha - \bar{\alpha}_k||_2 \) it follows that
\[ ||\bar{\alpha} - \alpha_k||_2 \to 0. \]

Let \( \delta_k = ||\bar{\alpha} - \alpha_k||_2 \). By \( n/\ln(p) \to \infty \) and \( p \to \infty \) we can choose \( k_n \) so that \( p_{k_n} \leq p \) and
\[ p_{k_n} \leq \delta_{k_n}^2 \frac{n}{\ln(p)}. \]

Let \( \delta_k = \delta_{k_n} \) and \( \bar{\pi} = (\pi_{k_n}', 0)' \) where \( \pi_{k_n}' \) are the coefficients for \( \alpha_{k_n} \). Then \( \|\bar{\pi}\|_0 \leq p_{k_n} \leq \delta_{k_n}^2 n/\ln(p) \) and \( ||\alpha_{k_n} - \bar{\pi}||_2 \leq ||\alpha_{k_n} - \bar{\alpha}||_2 \) so that
\[ (\pi - \bar{\pi}) \Sigma (\pi - \bar{\pi}) = ||\alpha_{k_n} - \bar{\pi}||_2^2 \leq 2 ||\alpha_{k_n} - \bar{\alpha}||_2^2 + 2 ||\alpha_{k_n} - \bar{\pi}||_2^2 \leq 4 \delta_{k_n}^2 = 4 \delta_n^2. \]

\[\square\]
**Lemma B.10.** If Assumptions 3 and 5-7 are satisfied then
\[
\|\hat{\rho} - \rho_n\|_2 = O_p\left(\frac{r}{\varepsilon^n}\right)
\]
and in addition if \(\xi_1 > 1/2\) and \(|\gamma| \leq C\) for all \(p\) then \(\sup_{x \in X} |\hat{\rho}(x)| = O_p(1)\).

**Proof.** Let \(\rho^*(x) = b(x)'\gamma\). By the largest eigenvalue of \(\Sigma\) bounded and Lemma B.7,
\[
\|\hat{\rho} - \rho^*\|_2^2 = \Delta' \Sigma \Delta \leq C \|\Delta\|_2^2 = O_p\left(\frac{r}{\varepsilon^n}\right).
\]
Also, by the Lemma B.2,
\[
\|\rho^* - \rho_n\|_2^2 = (\gamma' - \gamma)' \Sigma (\gamma' - \gamma) = O(\varepsilon^{4\xi_1/(2\xi_1 + 1)}).
\]
The first conclusion then follows by the triangle inequality. Also, by Assumption 3 and the proof of Lemma C.1,
\[
|\gamma|_1 \leq |\gamma| + C \leq C,
\]
so by the triangle inequality, Assumption 6, and Lemmas B.4 and B.7,
\[
\sup_{x \in X} |\hat{\rho}(x)| \leq C |\gamma|_1 \sup_{x \in X: |\gamma|_1 \leq \rho} |b_j(x)| \leq C (|\gamma|_1 + \|\gamma\|_1) = O_p(1).
\]

\[
\hat{\Theta}_n = \hat{\Theta}_n + \frac{1}{n} \sum_{i \in I} \psi_n(W_i) + o_p(n^{-1/2}) = \hat{\Theta}_n + O_p(n^{-1/2}).
\]

**Proof.** By adding and subtracting terms and by linearity of \(m(w, \rho)\) in \(\rho\),
\[
\hat{\Theta}_n - \Theta_n = \frac{1}{n} \sum_{i \in I} \left\{m(W_i, \hat{\rho}_2 + \hat{\Theta}_n) - \hat{\Theta}_n \right\}
\]
\[
= M_1^\prime \hat{\gamma}_2 + \hat{\pi}'(\hat{\mu}_1 - \hat{\Sigma}_1 \hat{\pi}) - \Theta_n = \hat{\pi}'(\hat{\mu}_1 - \hat{\ Theta}_n) + \hat{\pi}'(\hat{\Sigma}_1 - \pi) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \hat{\pi}'(\hat{\mu}_1 - \hat{\Theta}_n + \hat{\Sigma}_1 \gamma)(\hat{\pi}_1 - \pi) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \hat{\pi}'(\hat{\mu}_1 - \hat{\Theta}_n + \hat{\Sigma}_1 \gamma)(\hat{\pi}_1 - \pi) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \hat{\pi}'(\hat{\mu}_1 - \hat{\Theta}_n + \hat{\Sigma}_1 \gamma) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \hat{\pi}'(\hat{\mu}_1 - \hat{\Theta}_n + \hat{\Sigma}_1 \gamma) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \hat{\pi}'(\hat{\mu}_1 - \hat{\Theta}_n + \hat{\Sigma}_1 \gamma) + \hat{\Sigma}_1^\prime \hat{\gamma}_2 - \hat{\pi}' \hat{\Sigma}_1 \hat{\gamma}_2
\]
\[
= \frac{1}{n} \sum_{i \in I} \psi_n(W_i) + T_2 + T_3 + T_1,
\]
\[
T_1 = \langle \hat{\gamma} - \gamma \rangle \hat{\Sigma}_1 (\hat{\pi}_1 - \pi), T_2 = \hat{\Sigma}_1 (\hat{\pi}_1 - \pi), T_3 = \hat{\Sigma}_1 (\hat{\pi}_1 - \pi).
\]

It follows from Assumptions 6 and 8 by Lemma C.2 that \(\|M_1 - M_\|_\infty = O_p(\varepsilon_n)\) and \(\|\hat{\pi}_1 - \mu\|_\infty = O_p(\varepsilon_n)\). By Lemmas B.4 and B.7 applied to \(\hat{\gamma}, \gamma, \gamma\) in place of \(\hat{\pi}, \pi, \pi\) and the triangle inequality, \(\|\hat{\gamma}_2 - \gamma\|_1 = O_p\left((r/\varepsilon_n)^{2\xi_1/(2\xi_1 + 1)}\right)\). Then by Lemma B.8, the Holder inequality, and \(r/\varepsilon_n = o(n^c)\) for any \(c > 0\),
\[
|T_1| \leq \|\hat{\Sigma}_1 (\hat{\pi}_1 - \pi)\|_\infty \|\hat{\gamma}_2 - \gamma\|_1 = O_p(r)O_p\left((r/\varepsilon_n)^{(2\xi_1 - 1)/(2\xi_1 + 1)}\right) = O_p\left((r/\varepsilon_n)^{2\xi_1/(2\xi_1 + 1)}\right) = O_p(n^{-1/2}).
\]

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Also, by Lemma B.5,
\[
\| \hat{R} \|_{\infty} \leq \| \hat{M} - M \|_{\infty} + \| (\Sigma - \hat{\Sigma}) \pi \|_{\infty} + \| M - \Sigma \pi \|_{\infty} = O_p(\epsilon_n) + O_p(\epsilon_n) + 0 = O_p(\epsilon_n). \tag{B.5}
\]

Therefore it follows that
\[
|T_2| \leq \| \hat{R} \|_{\infty} |\hat{\gamma} - \gamma|_1 = O_p(\epsilon_n)O_p((r/\epsilon_n)\epsilon_n^{(2\xi_2-1)/(2\xi_2+1)}) = O_p((r/\epsilon_n)\epsilon_n^{2\xi_2/(2\xi_2+1)}) = o_p(n^{-1/2}).
\]

Next, note that for \( e_i = Y_i - \rho_0(X_i) \) and \( d_i = \rho_0(X_i) - \rho_n(X_i) \),
\[
T_3 = T_{31} + T_{32}, \quad T_{31} = \frac{1}{n_1} \sum_{i \in I_1} e_i \{ b(X_i)'(\hat{\pi}_1 - \pi) \}, \quad T_{32} = \frac{1}{n_1} \sum_{i \in I_1} d_i \{ b(X_i)'(\hat{\pi}_1 - \pi) \},
\]
For \( \hat{X} = \{ X_i | i \in I_1 \} \) we have \( E[\epsilon_i|\hat{X}] = 0 \), so that
\[
E[T_{31}|\hat{X}] = \frac{1}{n_1} \sum_{i \in I_1} E[\epsilon_i|\hat{X}] \{ b(X_i)'(\hat{\pi}_1 - \pi) \} = 0.
\]

Also, by Lemma B.9 and Assumptions 5 and 8,
\[
\text{Var}(T_{31}|\hat{X}) = \frac{1}{n_1^2} \sum_{i \in I_1} \text{Var}(Y_i|X_i) \{ b(X_i)'(\hat{\pi}_1 - \pi) \}^2 \leq \frac{C}{n_1} (\hat{\pi}_1 - \pi)'\hat{\Sigma}_1 (\hat{\pi}_1 - \pi) = o_p(1/n_1),
\]
so that \( T_{31} = o_p(n^{-1/2}) \). Furthermore by the Cauchy-Schwartz and Markov inequalities, Lemma B.9, and Assumption 9,
\[
|T_{32}| \leq \sqrt{\left( \frac{1}{n_1} \sum_{i \in I_1} d_i^2 \right) \\left( \hat{\pi}_1 - \pi \right)' \hat{\Sigma}_1 (\hat{\pi}_1 - \pi) = O_p(\sqrt{E[d_i^2]})O_p(\epsilon_n^{-1}\delta_n)} = O_p(\|\rho_0 - \rho_n\|_2 \epsilon_n^{-1}\delta_n) = o_p(n^{-1/2}).
\]

The conclusion then follows by the triangle inequality. \( \square \)

We also give a result corresponding to Lemma A11 for the case where \( \xi_1 \xi_2 > 1/4 \).

**Lemma B.12.** If Assumptions 3-8 and 10 are satisfied
\[
\hat{\theta}_1 = \theta_n + \frac{1}{n_1} \sum_{i \in I_1} \psi_n(W_i) + o_p(n^{-1/2}) = \theta_n + O_p(n^{-1/2}).
\]

**Proof.** We consider the same remainder decomposition as the proof of Lemma B.11 with \( T_1 = (\gamma - \hat{\gamma})'\hat{\Sigma}_1 (\hat{\pi}_1 - \pi), T_2 = \hat{R}'(\hat{\gamma} - \gamma), T_3 = \hat{U}'(\hat{\pi}_1 - \pi) \). Note that
\[
T_1 = \frac{1}{n_1} \sum_{i \in I_1} [\hat{\beta}_2(X_i) - \rho_n(X_i)][\hat{\alpha}_1(X_i) - \alpha_n(X_i)],
\]
\[
T_2 = \frac{1}{n_1} \sum_{i \in I_1} [m(W_i, \hat{\beta}_2 - \rho_n) + \alpha_n(X_i)]\{ \hat{\beta}_2(X_i) - \rho_n(X_i) \},
\]
\[
T_3 = \frac{1}{n_1} \sum_{i \in I_1} [\hat{\alpha}_2(X_i) - \alpha_n(X_i)]\{ Y_i - \rho_n(X_i) \}.
\]

We consider first \( T_1 \). By Lemmas B.2 and B.7 and the Markov inequality
\[
\frac{1}{n_1} \sum_{i \in I_1} \left[ \hat{\alpha}_1(X_i) - \alpha_n(X_i) \right]^2 \leq (\hat{\pi}_1 - \pi)'\hat{\Sigma}_1 (\hat{\pi}_1 - \pi) \leq C\Delta'\Delta + C(1 - \pi)\sum_{i \in I_1} (\pi - \pi_n)
\]
\[
= O_p((r/\epsilon_n^2)\epsilon_n^{4\xi_2/(2\xi_2+1)}) + O_p(E[\pi - \pi_n]'\hat{\Sigma}_1 (\pi - \pi_n))
\]
\[
= O_p((r/\epsilon_n^2)\epsilon_n^{4\xi_2/(2\xi_2+1)}) + O_p((\pi - \pi_n)'\Sigma (\pi - \pi_n)) = O_p((r/\epsilon_n^2)\epsilon_n^{4\xi_2/(2\xi_2+1)}).
\]

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Also
\[
E\left[ \frac{1}{n_1} \sum_{i \in I_1} [\hat{\gamma}(X_i) - \gamma(X_i)]^2 | \{W_i\}_{i \in I_2} \right] = \| \hat{\gamma} - \gamma \|_2^2 \leq C\Delta' \Sigma\Delta + C(\gamma - \gamma_0)^2(\gamma - \gamma_0) \\
\leq C \| \Delta \|_2^2 + CE_n^{\xi_1/(2k_1+1)} = O_p((r/n)^2 e_n^{\xi_1/(2k_1+1)}). 
\]
It then follows by the conditional Markov inequality that
\[
\frac{1}{n_1} \sum_{i \in I_1} [\hat{\gamma}(X_i) - \gamma(X_i)]^2 = O_p((r/n)^2 e_n^{\xi_1/(2k_1+1)}). 
\]
Then by the Cauchy-Schwarz inequality and Assumption 10,
\[
|T_1| \leq \left( \frac{1}{n_1} \sum_{i \in I_1} [\hat{\gamma}(X_i) - \gamma(X_i)]^2 \right)^{1/2} \left( \frac{1}{n_1} \sum_{i \in I_1} [\hat{a}_i(X_i) - a_n(X_i)]^2 \right)^{1/2} \\
= O_p((r/n)^2 e_n^{\xi_1/(2k_1+1) + 2\xi_1/(2k_1+1)}) = o_p(1/\sqrt{n}). 
\]
Also,
\[
T_2 = T_{21} + T_{22}, 
\]
\[
T_{21} = \frac{1}{n_1} \sum_{i \in I_1} [m(W_i, \hat{\rho}_2 - \rho_n) + \hat{a}(X_i) (\hat{\rho}_2(X_i) - \rho_n(X_i))], 
\]
\[
T_{22} = \frac{1}{n_1} \sum_{i \in I_1} [(a_n(X_i) - \hat{a}(X_i)) (\hat{\rho}_2(X_i) - \rho_n(X_i))]. 
\]
Note that
\[
E[m(W_i, \hat{\rho}_2 - \rho_n) + \hat{a}(X) (\hat{\rho}_2(X) - \rho_n(X)) | \{W_i\}_{i \in I_2}] = 0. 
\]
Also by Assumption 10,
\[
E[m(W_i, \hat{\rho}_2 - \rho_n)^2 | \{W_i\}_{i \in I_2}] \leq C \| \hat{\rho}_2 - \rho_n \|_2^2 = o_p(1), 
\]
\[
E[\hat{a}(X)^2 (\hat{\rho}_2(X) - \rho_n(X))^2 | \{W_i\}_{i \in I_2}] \leq C \| \hat{\rho}_2 - \rho_n \|_2^2 = o_p(1). 
\]
Therefore it follows by independent observations that \( E[T_{21}^2 | \{W_i\}_{i \in I_2}] = o_p(n^{-1}) \), so by the conditional Markov inequality \( T_{21} = o_p(n^{-1/2}) \). Similarly to the result for \( T_1 \), Assumption 10 implies that
\[
|T_{22}| \leq \left( \frac{1}{n_1} \sum_{i \in I_1} [\hat{a}(X_i) - a_n(X_i)]^2 \right)^{1/2} \left( \frac{1}{n_1} \sum_{i \in I_1} [\hat{\gamma}(X_i) - \gamma(X_i)]^2 \right)^{1/2} \\
= O_p(\| \hat{a} - b' \pi \|_2) O_p((r/n)^2 e_n^{\xi_1/(2k_1+1)}) = o_p(n^{-1/2}). 
\]
Then by the triangle inequality \( T_2 = o_p(n^{-1/2}) \). It also follows similarly to the proof of Lemma B.11 that \( T_3 = T_{31} + T_{32} \) for the same \( T_{31} \) and \( T_{32} \) given in Lemma B.11. Then \( T_{31} = o_p(n^{-1/2}) \) follows as in the proof of Lemma B.11 and by the Cauchy-Schwarz and Markov inequalities, Lemma B.9, and Assumption 10,
\[
|T_{32}| \leq \sqrt{\frac{1}{n_1} \sum_{i \in I_1} d_i^2 \sqrt{(\hat{\pi} - \pi) \Sigma (\tilde{\pi} - \pi)}} = O_p(\sqrt{E[|d_i^2|]}) O_p((r/n)^2 e_n^{\xi_1/(2k_1+1)}) \\
= O_p(\| \rho - \rho_0 \|_2) O_p((r/n)^2 e_n^{\xi_1/(2k_1+1)}) = o_p(n^{-1/2}). 
\]
The conclusion then follows by the triangle inequality.

\[\square\]

**Lemma B.13.** If \( E[|Y - \rho_0(X)|^2 | X] \) is bounded and either a) \( \hat{a}(X) \) is bounded, \( E[m(W, \rho)^2] \leq C \| \rho \|_2^2 \), and \( \| a_n - \hat{a} \|_2 \| \rho_n - \rho_0 \|_2 = o(1/\sqrt{n}) \) or b) \( E[|m(W, \rho)|] \leq C \| \rho \|_2 \) and \( \| \rho_n - \rho_0 \|_2 = o(1/\sqrt{n}) \) then
\[
\frac{1}{n} \sum_{i=1}^n (\theta_n + \psi_n(W_i) - \theta_0 - \psi_0(W_i)) = o_p(n^{-1/2}). 
\]
Proof. Note that
\[ \theta_n + \psi_n(w) - \theta_0 - \psi_0(w) = T_{1n}(w) + T_{2n}(w) + T_{3n}(w), \]
where
\[ T_{1n}(w) := m(w, \rho_n - \rho_0) - \bar{\alpha}(x) [\rho_n(x) - \rho_0(x)], \]
\[ T_{2n}(w) := [\alpha_n(x) - \bar{\alpha}(x)][y - \rho_0(x)], \]
\[ T_{3n}(w) := - [\rho_n(x) - \rho_0(x)][\alpha_n(x) - \bar{\alpha}(x)]. \]

Let \( T_j := \sum T_{mj}(W_j) / \sqrt{n}, (j = 1, 2, 3). \) Note that \( E[T_{1n}(W)] = 0. \) If condition a) is satisfied then
\[ E|m(W, \rho_n - \rho_0)^2| \leq C \|\rho_n - \rho_0\|^2 \rightarrow 0, \]
\[ E[\bar{\alpha}(x)^2 \rho_n(x) - \rho_0(x)^2] \leq C \|\rho_n - \rho_0\|^2 \rightarrow 0. \]

By the triangle inequality it then follows that \( E[T_{1n}(W)^2] \rightarrow 0, \) implying \( E[T_j^2] \rightarrow 0, \) so \( T_1 = o_p(1) \) follows by the Markov inequality. If conditional b) is satisfied then by the triangle and Cauchy-Schwartz inequalities
\[ E[|T_{1n}(W)|] \leq E[m(W, \alpha_n - \rho_0)|] + E[\bar{\alpha}(x)\rho_n(x) - \rho_0(x)] \]
\[ \leq C \|\rho_n - \rho_0\|_2 = o(1/\sqrt{n}) \]
so \( E[|T_1|] \leq \sqrt{n}E[|T_{1n}(W)|] = o(1), \) so \( T_1 = o_p(1) \) follows by the Markov inequality.

Next, note that by \( \bar{\alpha}, \alpha_n \in \Gamma \) it follows that \( E[T_{2n}(W)] = 0. \) Also, by \( E[Y - \rho_0(X)]^2 \) bounded,
\[ E[T_{2n}(W)^2] = E[\{\alpha_n(X) - \bar{\alpha}(X)\}^2 \{Y - \rho_0(X)\}^2] \leq C \|\alpha_n - \bar{\alpha}\|^2 \rightarrow 0. \]

Then \( E[T_2^2] \rightarrow 0 \) so \( T_2 = o_p(1) \) follows by the Markov inequality. Also, note that by the Cauchy-Schwartz inequality and either Assumption 9 or Assumption 10,
\[ E[|T_{3n}(W)|] \leq \|\alpha_n - \bar{\alpha}\|_2 \|\rho_n - \rho_0\|_2 = o(1/\sqrt{n}). \]
Then by the triangle inequality \( E[|T_3|] \leq \sqrt{n}E[|T_{3n}(W)|] \rightarrow 0, \) so \( T_2 = o_p(1) \) follows by the Markov inequality. The conclusion then follows from the triangle inequality. \( \square \)

Proof of Theorem 4. By Lemma B.11 or Lemma B.12, the same results for \( T_2, \) and Lemma B.13 the triangle inequality gives
\[ \hat{\theta} = \frac{n_1}{n} \theta_1 + \frac{n_2}{n} \bar{\theta}_2 = \frac{2}{n} \sum_{l=1}^{n} \left\{ \theta_n + \frac{1}{n} \sum \psi_n(W_l) + o_p(n^{-1/2}) \right\} \]
\[ = \theta_0 + \frac{1}{n} \sum \psi_n(W_l) + o_p(n^{-1/2}) = \theta_0 + \frac{1}{n} \sum \psi(W_l) + o_p(n^{-1/2}). \]

Note also that by Assumption 8 \( E[m(W, \rho_0)^2] < \infty \) and
\[ E[\bar{\alpha}(X)^2 \{Y - \rho_0(X)\}^2] = E[\alpha(X)^2 \text{Var}(Y|X)] \leq CE[\alpha(X)^2] < \infty, \]
so that \( E[|\psi(W)|^2] < \infty. \) Then the first conclusion follows by the the Lindebergh-Levy central limit theorem.

Let \( \psi_i = m(W_i, \rho_0) + \alpha(X_i) (y_i - \rho_0(X_i)) - \theta_0, \) by Khintchines law of large numbers \( \sum \psi_i^2 / n = V + o_p(1). \) Let \( \hat{\psi}_i := \hat{\psi}_i - \psi_i. \) By the triangle and Cauchy-Schwartz inequalities,
\[ \left| V - \frac{1}{n} \sum W_i \right| \leq \frac{1}{n} \sum \left| \psi_i^2 - \psi_i^2 \right| \leq \frac{1}{n} \sum (\hat{\psi}_i^2 + 2 \left| \psi_i \right| \left| \hat{\psi}_i \right|) \]
\[ \leq \frac{1}{n} \sum \hat{\psi}_i^2 + 2 \left( \frac{1}{n} \sum \left| \psi_i \right|^2 \right)^{1/2} \left( \frac{1}{n} \sum \hat{\psi}_i^2 \right)^{1/2}. \]
By existence of $V$ and the Markov inequality $\sum |\psi_i|^2 / n = O_p(1)$, so the second conclusion will follow from $\sum 0^2 / n = O_p(1)$. Let $\xi_i = Y_i - \theta_0(X_i)$. Note that for $i \in I_1$,
\[ \delta_i = \delta_1 + \delta_2 + \delta_3 + \theta_0 \delta_1 := m(W_i, \rho_2 - \rho_0), \]
\[ \delta_2 := \alpha_1(X_i)[\rho_0(X_i) - \rho_2(X_i)], \delta_3 := \{ \alpha_1(X_i) - \alpha(X_i) \} \xi_i. \]

If Assumption 9 iii) a) or Assumption 10 is satisfied then
\[ E[\frac{1}{n_1} \sum_{i \in I_1} 0^2 | \{ W_i \}_{i \in I_2}] = \int [m(w, \rho_2 - \rho_0)]^2 F_W(dw) \leq C \| \rho_2 - \rho_0 \|_2^2 + \rho \to 0. \]

If Assumption 9 iii) b) is satisfied then $\sup_{x \in X} |\rho(x)| \leq C$ and by Lemma B.10 $\sup_{x \in X} |\hat{\rho}(x)| = O_p(1)$. Therefore by Lemma B.10, and
\[ E[\frac{1}{n_1} \sum_{i \in I_1} 0^2 | \{ W_i \}_{i \in I_2}] = \int [m(w, \hat{\rho}_2 - \rho_0)]^2 F_W(dw) \leq \sup_x |\hat{\rho}_2(x) - \rho_0(x)| \int [m(w, \hat{\rho}_2 - \rho_0)] a(w) F_W(dw) \leq O_p(1) \| \hat{\rho} - \rho_0 \|_2 \leq O_p(1) \| \hat{\rho} - \rho_0 \|_2 + \| \rho_0 - \rho_0 \|_2 \to 0. \]

In either case the conditional Markov inequality implies \[ \sum_{i \in I_1} 0^2 / n = O_p(1). \] Similarly, by $|\alpha_1(X_i)| \leq 1$, \[ \sum_{i \in I_1} 0^2 / n = O_p(1) \] so that $\sum_{i \in I_1} 0^2 / n = O_p(1)$ follows by the conditional Markov inequality. In addition, since $\bar{\alpha}(X_i)$ depends only on $\{ X_i \}_{i \in I_1}$,
\[ E[\frac{1}{n_1} \sum_{i \in I_1} 0^2 | \{ X_i \}_{i \in I_1}] = \frac{1}{n_1} \sum_{i \in I_1} (\bar{\alpha}_1(X_i) - \bar{\alpha}(X_i))^2 Var(Y_i|X_i) \leq C \frac{1}{n_1} \sum_{i \in I_1} (\bar{\alpha}_1(X_i) - \bar{\alpha}(X_i))^2. \]

By $|\tau_n(\hat{a}) - \tau_n(a)| \leq |\hat{a} - a|$ for any scalars $\hat{a}$ and $a$ and $E[1(|\bar{\alpha}(X_i)| \geq \tau_n) \bar{\alpha}(X_i) ] \to 0$, by $\tau_n \to \infty$, the Cauchy Schwartz and Markov inequalities give
\[ \frac{1}{n_1} \sum_{i \in I_1} (\bar{\alpha}_1(X_i) - \bar{\alpha}(X_i))^2 \leq \frac{C}{n_1} \sum_{i \in I_1} (\bar{\alpha}_1(X_i) - \bar{\alpha}(X_i))^2 + \frac{C}{n_1} \sum_{i \in I_1} 1(|\bar{\alpha}(X_i)| \geq \tau_n) \bar{\alpha}(X_i)^2 \leq C(\bar{\pi}_1 - \pi)|\pi_1(\bar{\pi}_1 - \pi) + \frac{C}{n_1} \sum_{i \in I_1} (\bar{\alpha}_0(X_i) - \bar{\alpha}_0(X_i))^2 + o_p(1) = o_p(1). \]

Then by the conditional Markov inequality we have $\sum_{i \in I_1} 0^2 / n = O_p(1)$. Applying the same arguments for $i \in I_2$ also gives $\sum_{i \in I_2} 0^2 / n_2 = o_p(1)$, $k = 1, 2, 3$. Then combining these results it follows that $\sum_{i=1}^n 0^2 / n = o_p(1)$, $k = 1, 2, 3$, so the second conclusion then follows by $\hat{\theta} = \theta_0 + o_p(1)$ and the triangle and Cauchy-Schwartz inequalities.

**Proof of Corollary 5.** For Example 1 we let $m(W, \rho) = p(d, Z) - p(d, Z) / (d - d)$, $U = D - \bar{D}$ for $\bar{D}$ the least squares projection of $D$ on the mean square closure of finite linear combinations of the covariates $\{ Z_1, Z_2, \ldots \}$, and $\bar{\alpha}(X) = U / E[U^2]$ as at the beginning of Section 2. It was showed there that $E[m(W, \rho)] = E[\bar{\alpha}(X) p(X)]$ for all $\rho \in \mathcal{P}$. Also $m(W, b_j) = m_0(W)m_j(W)$ for $m_0(W) := 1$ and $m_j(W) := b_j(1, Z) - b_j(0, Z)$, with Assumption 6 implying 
\[ E[m(W)^2] < \infty, \quad |m_j(W)| = |b_j(1, Z) - b_j(0, Z)| \leq C. \]

Therefore Assumption 4 is satisfied.
Let \( \mathcal{R}_Z \) denote the mean square closure of finite linear combinations of \( \{Z_1, Z_2, \ldots \} \). It follows by standard calculations for partially linear projections that any \( \rho(X) \in \mathcal{R} \) has the form \( \theta D + h(Z) \), \( h \in \mathcal{R}_Z \). Therefore by \( E[U(h + D)] = 0 \),
\[
||\rho||^2 = E[(\theta U + h + \bar{D}))^2] = \theta^2 E[U^2] + E[(h + \bar{D})^2] \geq \theta^2 E[U^2].
\]
Therefore
\[
E[m(W, \rho)^2] = \theta^2 \leq C ||\rho||^2, C = 1/E[U^2].
\]
Also \( U \) is bounded by assumption, so that \( \alpha(X) \) is bounded. Therefore the conditions of Assumption 9 a) are Assumption 10 iii) and iv) are satisfied. The conclusions then follow by Theorem 4.

**Proof of Corollary 6.** In Example 2 let \( \alpha_0(X) := S(D)\omega(D) f(D)Z^{-1} \). Note that by the hypotheses,
\[
E[\alpha_0(X)^2] \leq E[\{1 + S(D)^4\} \omega(D)^2 f(D)Z^{-2}] < \infty,
\]
so that \( E[\alpha(X)^2] < \infty \) for \( \alpha(X) = \text{proj}(\alpha_0(X)\mathcal{R}) \). For \( \rho \in \mathcal{R} \) it follows by dividing and multiplying by \( f(D)Z \) that
\[
E[m(W, \rho)] = \int S(u)\rho(u, z)\omega(du)F_Z(dz) = E[\alpha_0(X)\rho(X)] = E[\alpha(X)\rho(X)].
\]
Also \( m(W, b_j) = \tilde{m}_0(W)\tilde{m}_j(W) \) for \( \tilde{m}_0(W) := S(U) \) and \( \tilde{m}_j(W) := b_j(U, Z) \), with Assumption 6 implying
\[
E[\tilde{m}_0(W)^2] = E[S(U)^2] = \int S(u)^2\omega(du) < \infty, \quad |\tilde{m}_j(W)| = |b_j(U, Z)| \leq C.
\]
Therefore Assumption 4 is satisfied. Also, for \( a(W) = S(U) \), \( m(W, \rho) \leq a(W) \sup_{x \in X} |\rho(x)| \), and
\[
E[\{1 + a(W)\} m(W, \rho)] = E[\{1 + S(U)\} S(U) |\rho(U, Z)|] = E[\{1 + S(D)\} S(D) \omega(D)f(D)Z^{-1} |\rho(X)|],
\]
\[
\leq \left\{ E[\{1 + S(D)^2\} S(D)^2 \omega(D)^2 f(D)Z^{-2}] \right\}^{1/2} ||\rho||_2 \leq C ||\rho||_2.
\]
Also, by \( S(D)^2 \omega(D)f(D)Z^{-1} \leq C \),
\[
E[m(W, \rho)^2] = E[S(U)^2 \rho(U, X)^2] = E[S(D)^2 \omega(D)f(D)Z^{-1} \rho(X)^2] \leq C ||\rho||_2^2.
\]
Therefore the inequalities in Assumption 9 iii) a) and b) are satisfied, so the conclusions follow by Theorem 4.

**Proof of Corollary 7.** In Example 3 let \( \alpha_0(X) := \pi_0(Z)^{-1}D - \{1 - \pi_0(Z)^{-1}\}(1 - D) \). Note that by assumption,
\[
E[\alpha_0(X)^2] = E[\pi_0(Z)^{-2}D^2] + E[\{1 - \pi_0(Z)^{-1}\}^2(1 - D)^2]
\]
\[
= E[\pi_0(Z)^{-1} + \{1 - \pi_0(Z)^{-1}\}] = E[\pi_0(Z)^{-1} \{1 - \pi_0(Z)^{-1}\}] < \infty,
\]
so that \( E[\alpha(X)^2] < \infty \) for \( \alpha(X) = \text{proj}(\alpha_0(X)\mathcal{R}) \). For any \( \rho \in \mathcal{R} \) it follows by a standard calculation that
\[
E[m(W, \rho)] = E[\alpha_0(X)\rho(X)] = E[\bar{\alpha}(X)\rho(X)].
\]
Also \( m(W, b_j) = \tilde{m}_0(W)\tilde{m}_j(W) \) for \( \tilde{m}_0(W) := 1 \) and \( \tilde{m}_j(W) := b_j(1, Z) - b_j(0, Z) \), with Assumption 6 implying
\[
E[\tilde{m}_0(W)^2] < \infty, \quad |\tilde{m}_j(W)| = |b_j(1, Z) - b_j(0, Z)| \leq C.
\]
Therefore Assumption 4 is satisfied. Also, for \( a(W) = 2 \), \( m(W, \rho) \leq a(W) \sup_{x \in X} |\rho(x)| \), and
\[
E[\{1 + a(W)\} m(W, \rho)] = 3E[|\rho(1, Z) - \rho(0, Z)|] \leq 3E[|\rho(1, Z)|] + 3E[|\rho(0, Z)|]
\]
\[
= 3E[\left\{ D \frac{\pi_0(Z)}{1 - \pi_0(Z)} \right\}|\rho(X)|] \leq \sqrt{E[\alpha_0(X)^2]} ||\rho||_2 \leq C ||\rho||_2.
\]
Thus all the conditions of Assumption 9 b) are satisfied. Also, under Assumption 10 with \(\pi_0(Z)\) bounded away from zero and one,

\[
E[m(W, \rho)^2] = E[|\rho(1, Z) - \rho(0, Z)|^2] \leq E[\rho(1, Z)^2 + \rho(0, Z)^2] = E[\frac{D}{\pi_0(Z)} + \frac{1 - D}{\pi_0(Z)}] \rho(X)^2 \leq \|\rho\|^2_2,
\]

and all the conditions of Assumption 10 are satisfied. The conclusions then follow by Theorem 4. \(\square\)

**Proof of Theorem 8.** Note that \(\hat{\theta}\) solves

\[
\sum_{t=1}^{2} \sum_{i \in I_t} (D_i - \hat{\eta}_t(Z_i)) [Y_i - \hat{\xi}_t(Z_i) - \theta \{D_i - \hat{\eta}_t(Z_i)\}] = 0.
\]

We will proceed by showing that

\[
\frac{1}{\sqrt{n}} \sum_{i \in I_t} (D_i - \hat{\eta}_t(Z_i)) [Y_i - \hat{\xi}_t(Z_i) - \theta_0 \{D_i - \hat{\eta}_t(Z_i)\}] \overset{d}{\to} N(0, \Omega), \tag{B.6}
\]

\[
\frac{1}{\sqrt{n}} \sum_{i \in I_t} (D_i - \hat{\eta}_t(Z_i))^2 \overset{p}{\to} Q.
\]

The first conclusion then follows by standard arguments (e.g. Newey and McFadden, 1994).

To show the first result in Equation (B.6) note that it suffices to show

\[
\sum_{i \in I_t} (D_i - \hat{\eta}_t(Z_i)) [Y_i - \hat{\xi}_t(Z_i) - \theta_0 \{D_i - \hat{\eta}_t(Z_i)\}] = \frac{1}{\sqrt{n}} \sum_{i \in I_t} (D_i - \eta_0(Z_i)) [Y_i - \zeta_0(Z_i)] + o_p(1),
\]

\[
\frac{1}{\sqrt{n}} \sum_{i \in I_t} (D_i - \eta_0(Z_i))^2 = \frac{1}{\sqrt{n}} \sum_{i \in I_t} (D_i - \eta_0(Z_i))^2 + o_p(1), \quad (\ell = 1, 2),
\]

\[
\eta_0(Z) = E[D/Z], \quad \zeta_0(Z) = \text{proj}(Y|Z, Z_2, \ldots).
\]

Note also that

\[
T := \frac{1}{\sqrt{n}} \sum_{i \in I_t} \left\{ (D_i - \eta_0(Z_i)) [Y_i - \zeta_0(Z_i)] - (D_i - \eta_0(Z_i)) [Y_i - \zeta_0(Z_i)] \right\}
\]

\[
= -\frac{1}{\sqrt{n}} \sum_{i \in I_t} \left\{ Y_i \hat{\eta}_t(Z_i) + \hat{\xi}_t(Z_i) [D_i - \hat{\eta}_t(Z_i)] - Y_i \eta_0(Z_i) + \zeta_0(Z_i) [D_i - \eta_0(Z_i)] \right\}.
\]

Now we follow the proof of Lemma B.11, for \(x = z, \rho(x) = \eta(z), m(w, \rho) = \eta(z), \eta_0(Z) = \text{proj}(D/Z_1, \ldots, Z_n), \) and \(\zeta_0(Z) = \text{proj}(Y|Z_1, \ldots, Z_n), \) with \(\Sigma = E[ZZ'] - \Sigma_Y = 0, E[ZY] - \Sigma \pi = 0.\) In this setting \(T = T_1 + T_2 + T_3\) as in the proof of Lemma B.11, with \(T_1 = o_p(n^{-1/2})\) and \(T_2 = o_p(n^{-1/2})\) as shown there. Also

\[
\mathcal{U} = \frac{1}{n_t} \sum_{i \in I_t} Z_{t_i}[D_i - \eta_0(Z_i)].
\]

Then for \(u_i = D_i - \eta_0(Z_i)\) and \(d_i = \eta_0(Z_i) - \eta_0(Z_i),\)

\[
T_3 = T_{31} + T_{32}, \quad T_{31} = \frac{1}{n_t} \sum_{i \in I_t} u_i \{ \hat{\xi}_i(Z_i) - \zeta_0(Z_i) \}, \quad T_{32} = \frac{1}{n_t} \sum_{i \in I_t} d_i \{ \hat{\xi}_i(Z_i) - \zeta_0(Z_i) \}.
\]

Then for \(\hat{\xi}_i(Z_i)\) depending only on \(Z_i\) in \(I_t\) (that are independent of observations in \(I_t\)) it follows that \(E[u_i|Z, \tilde{W}] = 0.\) Then by \(\hat{\xi}_i(Z_i)\) depending only on \(Z_i\) and \(\tilde{W}\) and \(\text{Var}(D/Z) \leq C,\)

\[
E[T_{31}|Z, \tilde{W}] = \frac{1}{n_t} \sum_{i \in I_t} E[u_i|Z, \tilde{W}] \{ \hat{\xi}_i(Z_i) - \zeta_0(Z_i) \} = 0
\]

\[
\text{Var}(T_{31}|Z, \tilde{W}) = \frac{1}{n_t^2} \sum_{i, j \in I_t} \text{Var}(D_i|Z_{1}, Z_{2}, \ldots) \{ \hat{\xi}_i(Z_i) - \zeta_0(Z_i) \}^2 \leq C \frac{1}{n_t^2} \sum_{i \in I_t} \{ \hat{\xi}_i(Z_i) - \zeta_0(Z_i) \}^2 = C(\tilde{\alpha}_t - \pi)' \tilde{\Sigma}_t(\tilde{\alpha}_t - \pi)/n = o_p(n^{-1}),
\]

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where the last equality follows as in Lemma B.9. It follows by the conditional Markov inequality that $T_{31} = o_p(n^{-1/2})$. Also note that by the triangle and Markov inequalities and Lemma B.9,

$$|T_{32}| \leq \left( \frac{1}{m} \sum_{i \in I} d_i^2 \right)^{1/2} \left( \frac{1}{m} \sum_{i \in I} (\hat{\xi}^i(Z_i) - \zeta^0(Z_i))^2 \right)^{1/2} = O_p(\|\eta_0 - \eta_0\|_2 r^{-1} \delta_n) = o_p(n^{-1/2}).$$

Then $T = o_p(n^{-1/2})$ follows by the triangle inequality. It then follows from Lemma B.13 and $E[Y^2|Z_1, Z_2, ...]$ and proj$(Y|Z_1, Z_2,...)$ bounded that

$$\frac{1}{\sqrt{n}} \sum_{i \in I} \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i) - \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i)] = o_p(1).$$

The triangle inequality then gives

$$\frac{1}{\sqrt{n}} \sum_{i \in I} \{D_i - \hat{\eta}_r(Z_i)\} [Y_i - \hat{\xi}_r(Z_i) - \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i)] = o_p(1).$$

It follows analogously that

$$\frac{1}{\sqrt{n}} \sum_{i \in I} \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i) - \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i)] = o_p(1). \quad (B.7)$$

Also, the central limit theorem gives

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i) - \{D_i - \eta_0(Z_i)\} [Y_i - \zeta_0(Z_i)] \xrightarrow{d} N(0, \Omega)$$

The first result in Equation (B.6) then follows by the triangle inequality and the Slutsky Lemma. The second result follows by Khintchine’s law of large numbers and the triangle inequality.

\[\square\]

## C Some General Lemmas

**Lemma C.1.** For any $a \in \mathbb{R}^p$ such that $\|a - b_s\|_2 \leq Cs^{-r}$ for any $s \geq 0$, where $C, r > 0$ are constants and $b_s = \arg\min_{\|v\|_2 \leq s} \|a - v\|_2$. If $r > 1/2$ and $s \geq 2$ then $\|a - b_s\|_1 \leq Ds^{1/2-r}$, where $D > 0$ is a constant depending only on $C$ and $r$.

**Proof.** Without loss of generality, we assume that $|a_1| \geq |a_2| \geq \cdots \geq |a_p| \geq 0$. Then clearly, $a - b_s = (0, 0, \ldots, 0, a_{k+1}, a_{k+2}, \ldots, a_p)'$ for $k \geq 0$. By assumption, we have that for any $k \geq 0$,

$$\sum_{j=k+1}^p a_j^2 \leq C^2 k^{-2r}. \quad (C.1)$$

Let $g \in \mathbb{N}$ be defined as $2^g < p/s \leq 2^{g+1}$. With a slight abuse of notation, we extend $a$ to be a $2^{g+1} s$-dimensional vector with $a_j = 0$ for $j > p$. Then we have that

$$\sum_{j=1}^p a_j = \sum_{m=0}^g \sum_{j=2^m+1}^{2^{m+1}} a_j \leq \sum_{m=0}^g \frac{2^{m+1}}{2^m s} \sum_{j=2^m+1}^{2^{m+1}} a_j^2 \leq \sum_{m=0}^g \frac{2^m s}{2^m s} \sum_{j=2^m+1}^{2^{m+1}} a_j^2 \leq \sum_{m=0}^g \sqrt{2^m s} C^2 (2^m s)^{-2r} = Cs^{1/2-r} \sum_{m=0}^g \left( \frac{2^{1/2-r}}{2^{1/2-r}} \right)^{m} \leq C \frac{1}{1 - 2^{1/2-r}} s^{1/2-r},$$

where (i) follows by (C.1) applied to $k = 2^m s$ and (ii) follows by the fact that $2^{1/2-r} < 1$ (since $r > 1/2$). \[\square\]

Let $X = (X_1, \ldots, X_p)' \in \mathbb{R}^p$ and $X_{t0}$ be a scalar, with all random variables allowed to depend on $n$. 42
Lemma C.2. If there is C such that $\max_{1 \leq j \leq p} |X_{i,j}| \leq C$ and $E[X_{i,j}^2] \leq C$ then for $D_i = X_iX_{0,i} - E[X_iX_{0,i}]$ and $\tilde{D} = \sum_{i=1}^{n} D_i/n$, 

$$||\tilde{D}||_{\infty} = O_p(\sqrt{\ln(p)/n}).$$

Proof. We prove this result using symmetrization. Note that 

$$E[D_{i,j}^2] \leq E[X_{i,j}^2]X_{0,j}^2 \leq CE[X_{i,j}^2] \leq C.$$ 

Let $e_1, ..., e_n$ be i.i.d Rademacher random variables independent of $X_i$ for for all observations, i.e., $P(e_i = 1) = P(e_i = -1) = 1/2$. Define the symmetrized quantity $W_{n,j} = \sum_{i=1}^{n} D_{i,j}e_i$. Since $e_i$ is sub-Gaussian (due to $e_i \in \{-1, 1\}$), there exists a constant $\kappa > 0$ such that for any $t \in \mathbb{R}$, $E[\exp(t e_i)] \leq \exp(\kappa t^2)$. By Hoeffding’s lemma, we can simply take $\kappa = 1/2$. Since $\{e_i\}_{i=1}^{n}$ is independent of $X$ we have 

$$E[\exp(t W_{n,j})|X] = E\left[ \prod_{i=1}^{n} \exp[t D_{i,j} e_i] | X \right] = \prod_{i=1}^{n} E[\exp[t D_{i,j} e_i] | X] \leq \prod_{i=1}^{n} \exp(t^2 D_{i,j}^2/2) = \exp \left( t^2 \sum_{i=1}^{n} D_{i,j}^2/2 \right).$$

Similarly, apply the same argument to $-W_{n,j}$ to obtain $E[\exp(-t W_{n,j}) | X] \leq \exp \left( t^2 \sum_{i=1}^{n} D_{i,j}^2/2 \right).$ Since $E[\exp(t W_{n,j})] \leq \exp(t W_{n,j}) + \exp(-t W_{n,j})$, we have 

$$E[\exp(t W_{n,j}) | X] \leq 2 \exp \left( t^2 \sum_{i=1}^{n} D_{i,j}^2/2 \right).$$

Next let $z > 0$ be a non-random quantity to be chosen later and $||W_n||_{\infty} = \max_{1 \leq j \leq p} |W_{n,j}|$. By Lemma 2.37 of van der Vaart and Wellner (1996), we have 

$$(1 - \beta_n(z)) P \left( \max_{1 \leq j \leq p} \left| \sum_{i=1}^{n} D_{i,j} \right| > z \right) \leq 2P( ||W_n||_{\infty} > z/4),$$

(C.2) where 

$$\beta_n(z) = 1 - 4z^{-2} n \max_{1 \leq j \leq p} E[D_{i,j}^2] \geq 1 - Cz^{-2}n.$$ 

Note that $|E[D_{i,j}]|^2 \leq E[D_{i,j}^2] \leq C$, so that 

$$D_{i,j}^2 \leq C(X_{i,0}^2 + 1).$$

For any $M_n \rightarrow \infty$ let $\mathcal{A} = \left\{ n^{-1} \sum_{i=0}^{n} X_{i,0}^2 > M_n \right\}$. Since $E[X_{i,0}^2]$ is bounded uniformly in $n$ the Markov inequality, for any $M_n \rightarrow \infty$ we have $P(\mathcal{A}) = o(1)$. On the event $\mathcal{A}^c$ we have $D_{i,j}^2 \leq C(M_n + 1) := \tilde{M}_n$ for all $j$, so that 

$$E[\exp(t ||W_n||_{\infty}) | X] = E\left[ \exp \left( t \max_{1 \leq j \leq p} |W_{n,j}| \right) | X \right] \leq \sum_{j=1}^{p} E(\exp(t |W_{n,j}|) | X) \leq \sum_{j=1}^{p} \exp \left( t^2 \sum_{i=1}^{n} D_{i,j}^2/2 \right) \leq 2\exp(\tilde{M}_n t^2 n),$$

Let $t > 0$ be a non-random quantity to be chosen. By the Markov inequality we have 

$$P( ||W_n||_{\infty} > z/4 | \mathcal{A}^c) \leq P(\exp(t ||W_n||_{\infty}) > \exp(tz/4) | X) \mathbf{1}_{\mathcal{A}^c} \leq \exp(-tz/4)E[\exp(t ||W_n||_{\infty}) | X] \mathbf{1}_{\mathcal{A}^c} \leq \exp(-tz/4) \cdot 2 \exp(\tilde{M}_n t^2 n) = \exp \left( -\frac{1}{4} tz + \ln(2p) + \tilde{M}_n t^2 n \right).$$

Now choose $t = z/|8n\tilde{M}_n|$ to obtain 

$$P( ||W_n||_{\infty} > z/4 | X) \mathbf{1}_{\mathcal{A}^c} \leq \exp \left( -\frac{z^2}{nK_n} + \ln(2p) \right),$$

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Define \( B \) fitting is

It may also be of interest that without cross-fitting the automatic debiased machine learner attains root-n consistency under the minimal approximate sparsity condition \( \xi_1 > 1/2 \) for the regression. The estimator without cross fitting is

\[
\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \{m(W_i, \hat{\beta}) + \hat{\alpha}(X_i)[Y_i - \hat{\beta}(X_i)]\}, \hat{V} = \frac{1}{n} \sum_{i=1}^{n} \hat{\psi}_i^2, \hat{\psi}_i = m(W_i, \hat{\beta}) + \hat{\alpha}(X_i)[Y_i - \hat{\beta}(X_i)] - \hat{\theta},
\]

where \( \hat{\beta}(x) \) and \( \hat{\alpha}(x) \) are obtained as in equations (4.1) and (4.6) respectively, with averages over over all \( n \) observations rather than a subset of observations, and \( \hat{\alpha}(x) = \tau_n(\hat{\alpha}(x)) \). The following result gives the properties of this estimator.

**Corollary 9:** If Assumptions 3-9 then for \( V = E[\gamma_0(W)^2] \),

\[
\sqrt{n}(\hat{\theta} - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \gamma_0(W_i) + o_p(1) \xrightarrow{d} N(0, V).
\]

If in addition \( \xi_n \rightarrow \infty \) and \( \xi_n[(r/\epsilon_n)^{2\xi_i/2}\xi_i+1] + \|\rho_0 - \rho_n\|_2 \rightarrow 0 \) then \( \hat{V} \xrightarrow{p} V \).

This result shows that without cross-fitting the automatic debiased machine learner is root-n consistent and asymptotically normal with consistent asymptotic variance estimator when \( \xi_1 > 1/2 \).

**Proof of Corollary 9.** The proof of the first conclusion follows exactly as for the proof of Theorem 4 for the case where \( \xi_1 =1/2 \), upon noting that the proof of B.11 and B.13 do not use cross fitting. In contrast, the proof of the second conclusion proceeds in a different way.

By Assumption 4 and the Holder inequality,

\[
|m(W_i, \hat{\beta} - \rho_n)| = |m(W_i, b)'(\hat{\gamma} - \gamma)| \leq \|m(W_i, b)\|_\infty \|\hat{\gamma} - \gamma\|_1 \leq C \|\tilde{m}_0(W_i)\|_1 \|\hat{\gamma} - \gamma\|_1.
\]
Also follows similarly to the proof of Theorem 4 that
\[ \| \hat{\gamma} - \gamma \|_1 = o_p \left( \frac{r}{\xi_n} \right) (\xi_n)^{(2\delta_1 - 1)/(2\delta_2 + 1)} = o_p(1). \]

Then by \( E[\hat{m}_0(W)^2] < \infty \) and the Markov inequality,
\[ \frac{1}{n} \sum_i m(W_i, \hat{\rho} - \rho_0)^2 \leq C \left( \frac{1}{n} \sum_i \hat{m}_0(W_i)^2 \right) \| \hat{\gamma} - \gamma \|_1^2 = o_p(1) o_p(1) = o_p(1). \]

If Assumption 9 iii) a) is satisfied then by the Markov inequality
\[ \frac{1}{n} \sum_i m(iW_i, \rho_n - \rho_0)^2 = O_p(E[m(W, \rho_n - \rho_0)^2]) = O_p(\| \rho_n - \rho_0 \|_2) = o_p(1). \]

If Assumption 9 iii) b) is satisfied then by \( \rho_n(x) \) and \( \rho_0(x) \) bounded and the Markov inequality
\[ \frac{1}{n} \sum_i m(W_i, \rho_n - \rho_0)^2 \leq C \frac{1}{n} \sum_i a(W_i) m(W_i, \rho_n - \rho_0) = O_p(\| \rho_n - \rho_0 \|) = o_p(1). \]

Then for \( \hat{\delta}_1 = m(W, \hat{\rho} - \rho_0) \) as in the proof of Theorem 4, the triangle and Cauchy Schwartz inequalities give
\[ \frac{1}{n} \sum_i \hat{\delta}_1^2 \leq C \frac{1}{n} \sum_i \{ m(W_i, \hat{\rho} - \rho_n)^2 + m(W_i, \rho_n - \rho_0)^2 \} = o_p(1). \]

Also, by \( \| b(x) \|_\infty \leq C \) and the Holder inequality we have
\[ [\hat{\rho}(X_i) - \rho_n(X_i)]^2 \leq C \| \hat{\gamma} - \gamma \|_1^2. \]

Then for \( \hat{\delta}_2 = \hat{\alpha}(X_i)[\hat{\rho}(X_i) - \rho_0(X_i)] \) we have
\[ \frac{1}{n} \sum \hat{\delta}_2^2 \leq \hat{\tau}_2 \frac{1}{n} \sum [\hat{\rho}(X_i) - \rho_0(X_i)]^2 \]
\[ \leq \hat{\tau}_2 C \frac{1}{n} \sum \{ [\hat{\rho}(X_i) - \rho_n(X_i)]^2 + [\rho_n(X_i) - \rho_0(X_i)]^2 \} \]
\[ \leq C \frac{\hat{\tau}_2}{\xi_n} (\| \hat{\gamma} - \gamma \|_1^2 + \| \rho_n - \rho_0 \|_2^2) = o_p(1). \]

Furthermore, for \( \hat{\delta}_3 = \{ \hat{\alpha}(X_i) - \alpha(X_i) \} [Y_i - \rho_0(X_i)] \) it follows as in the proof of Theorem 4 that \( \sum \hat{\delta}_3^2/n = o_p(1) \). The conclusion then follows as in proof of Theorem 4.

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