A functional-analytic construction of the stochastic parallel transport in Hermitian bundles over Riemannian manifolds

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Abstract

This article presents a purely functional-analytic construction of the concept of stochastic parallel transport in Hermitian bundles over Riemannian manifolds. As a byproduct, we also obtain a form of the Feynman-Kac formula in vector bundles that is, to our best knowledge, the most general found so far.

Keywords: stochastic calculus, parallel transport, Hermitian bundle, Wiener measure, Riemannian manifold

1. Motivation and outline of this article

The concept of "stochastic parallel transport" in a vector bundle $E$ over a Riemannian manifold $M$ is usually presented as a byproduct of the concept of "stochastic differential equation"; this is the approach taken in most texts, for instance in [IW89] and in [Meyer82]. Nevertheless, K. Itô had originally conceived it differently ([Itô63], [Itô75a], [Itô75b]): for every continuous curve $c : [0, t] \to M$, consider the unique geodesic segment joining the consecutive "dyadic" points $c(\frac{j}{2^k})$ and $c(\frac{(j+1)}{2^k})$, join these $2^k$ geodesic segments into a single zig-zag piecewise-geodesic line, and parallel-transport the vector $v \in E_{c(0)}$ along this line to $E_{c(t)}$; for Wiener-almost all continuous curves $c$, the limit when $k \to \infty$ will exist and will be called "the stochastic parallel transport of $v$ along $c$". Both approaches are equivalent, as shown in [Meyer82] and [Emery90], and both are constructed within the framework of probability theory, therefore being accessible mostly to probabilists. The aim of this article is to reconstruct the concept of "stochastic parallel transport" using only functional-analytic tools and concepts, thus opening it up to a much larger class of mathematicians.

Since the constructions in this text will be fairly technical, let us sketch the intuition underpinning them. Let $D_t = \{ \frac{j}{2^k} \mid k \in \mathbb{N}, j \in \mathbb{N} \cap [0, 2^k] \}$ - the "dyadic" numbers between 0 and $t$. Following Itô's idea, the parallel transport of $v \in E_{c(0)}$ along the zig-zag line determined by the points $\{ c(0), c(\frac{1}{2^k}), \ldots, c(\frac{2^k-1}{2^k}), c(t) \}$ is the parallel transport $T_{k,0}$ from $c(0)$ to $c(\frac{1}{2^k})$, followed by the parallel transport $T_{k,1}$ from $c(\frac{1}{2^k})$ to $c(\frac{2}{2^k})$ and so on, ending with the parallel transport $T_{k,2^k-1}$ from $c(\frac{2^k-1}{2^k})$ to $c(t)$; symbolically, it is $T_{k,2^k-1} \ldots T_{k,0}v$. Now comes the remark that is

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the backbone of the present work: \( T_{k,2^k-1} \ldots T_{k,0}v \) can be viewed as the "contraction" of all the tensor products in

\[
T_{k,2^k-1} \otimes \cdots \otimes T_{k,0} \otimes v \in \left( E_{c(t)} \otimes E^{*\perp}_{c(t)} \right) \otimes \cdots \otimes \left( E_{c(0)} \otimes E^{*\perp}_{c(0)} \right) \otimes E_{c(0)} \approx \\
\approx E_{c(t)} \otimes \left( E^{*\perp}_{c(t)} \otimes E^{*\perp}_{c(t)} \right) \otimes \cdots \otimes \left( E_{c(0)} \otimes E_{c(0)} \right) \approx \\
\approx E_{c(t)} \otimes \text{End} E^{*\perp}_{c(t)} \otimes \cdots \otimes \text{End} E^{*\perp}_{c(0)}.
\]

Let us see now what "contraction" means. If \( U_1, \ldots, U_N \) are finite-dimensional vector spaces, if \( u \in U_N \) and \( \omega \in U_1^t \), and \( A_j : U_{j+1} \to U_j \) is a linear operator for all \( 1 \leq j \leq N-1 \), then

\[
\omega \otimes A_1 \otimes \cdots \otimes A_{N-1} \otimes u \in U_1^t \otimes (U_1 \otimes U_2^t) \otimes \cdots \otimes (U_{N-1} \otimes U_N^t) \otimes U_N \approx \text{End} U_1^t \otimes \cdots \otimes \text{End} U_N^t;
\]

if \( \text{Id}_{U_j} \) is the identity operator on \( U_j \), then \( \text{Id}_{U_1} \otimes \cdots \otimes \text{Id}_{U_N} \in \text{End} U_1 \otimes \cdots \otimes \text{End} U_N \), therefore it makes sense to apply \( \omega \otimes A_1 \otimes \cdots \otimes A_N \otimes u \) on \( \text{Id}_{U_1} \otimes \cdots \otimes \text{Id}_{U_N} \), the result being \( \omega(A_1, \ldots, A_Nu) \).

We see that in order to perform this contraction in the product of parallel transports considered above we need to add in a supplementary factor \( E^{*\perp}_{c(t)} \) with which to pair the factor \( E_{c(t)} \) in order to obtain \( E^{*\perp}_{c(t)} \) and be able to perform the contraction described above. This means that if \( \eta_{c(t)} \in E_{c(t)} \), then

\[
\eta_{c(t)} \otimes T_{k,2^k-1} \otimes \cdots \otimes T_{k,0} \otimes v \in \text{End} E^{*\perp}_{c(t)} \otimes \cdots \otimes \text{End} E^{*\perp}_{c(0)}
\]

and

\[
\eta_{c(t)}(T_{k,2^k-1} \ldots T_{k,0}v) = (\eta_{c(t)} \otimes T_{k,2^k-1} \otimes \cdots \otimes T_{k,0} \otimes v)(\text{Id}_{E_{c(t)}} \otimes \cdots \otimes \text{Id}_{E_{c(0)}}).
\]

Following now in the footsteps of Itô, we let \( k \to \infty \); what we get, then, will be a contraction between tensor products with infinitely many factors; the rigorous construction of these tensor products will be our first task, but we can say that these tensor product spaces will be \( \mathcal{E}_c = \otimes_{c \in C_t} \text{End} E_{c(t)} \) and its dual. If we denote the space of continuous curves by \( C_t \), the fact that \( \mathcal{E}_c \) depends on \( c \in C_t \) suggests that the disjoint union \( \bigsqcup_{c \in C_t} \mathcal{E}_c \) will be a (topological) vector bundle of infinite rank over \( C_t \). Since \( \eta_{c(t)} \otimes T_{k,2^k-1} \otimes \cdots \otimes T_{k,0} \otimes v \) takes values in the fiber \( \mathcal{E}_c \) for all \( k \in \mathbb{N} \) and all \( c \in C_t \), we deduce that these tensor products will all be some kind of sections in \( \mathcal{E}_c \), whence it is reasonable to assume that their limit for \( k \to \infty \) (the stochastic parallel transport, once we get rid of \( \eta \)) will be a section of the same kind. Indeed, this will turn out to be the case, and in order to obtain this we shall resort to Chernoff’s theorem about the approximation of contraction semigroups.

An unexpected byproduct of the construction in this article is a new version of the Feynman-Kac formula in vector bundles: not only will its proof be completely new, but its hypotheses seem to be the most general considered so far in the literature, to the author’s best knowledge; more precisely, the potential will be taken to be only locally-integrable and lower-bounded, while no restrictions will be imposed upon the manifold.

The plan of the article is the following, the notations going to be explained as soon as they become necessary:

- we shall construct a Hermitian vector bundle \( \mathcal{E} \) over \( C_t \), the fibers of which will be infinite-dimensional Hilbert spaces.
• we shall consider spaces of square-integrable sections in \( E \) and \( E^* \) and, in particular, we shall obtain by an abstract argument a specific essentially bounded section \( \rho_{t,\omega,\eta} \), which will be the limit of a sequence of sections \((P_{t,\omega,\eta,k})_{k \in \mathbb{N}}\) given by explicit formulae;

• we shall emphasize a conjugate-linear continuous map \( P_{t,v}^2 : \Gamma^2(E) \to \Gamma^2(p_t^*E) \), which we shall see to enclose a lot of information about both the geometry of the bundle \( E \to M \) and the Wiener measure \( w_t \) on \( C_t \);

• using the map \( P_{t,v}^2 \) we shall be able to give meaning to the concept of stochastic parallel transport from a functional-analytic point of view;

• finally, using the same map \( P_{t,v}^2 \), we shall study an extension of the Feynman-Kac formula in the bundle \( E \).

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2. A Hermitian vector bundle of infinite rank

In the following, \( M \) will be a separable connected Riemannian manifold of dimension \( n \), and \( x_0 \in M \) some fixed arbitrary point. We shall denote by \( d : M \times M \to [0, \infty) \) the distance induced on \( M \) by the Riemannian structure.

If \( t > 0 \), we shall repeatedly make use of the space

\[
C_t = \{ c : [0,t] \to M \mid c \text{ is continuous, with } c(0) = x_0 \},
\]

that we shall endow with the topology given by the distance \( D(c_1, c_2) = \max_{s \in [0,t]} d(c_1(s), c_2(s)) \) and with the natural Wiener measure \( w_t \) (a non-probabilistic, functional-analytic and geometric construction of the latter may be found in \([BP1]\)). It is known that \( C_t \) endowed with this topology is separable (see \([Michael61]\)).

Since we shall be working with various Banach or Hilbert spaces, the norm and the Hermitian product on each of them will be displayed as a lower index: if \( v, w \in X \), then \( \|v\|_X \) will be the norm of \( v \) and \( \langle v, w \rangle_X \) will be the Hermitian product of \( v \) and \( w \). For bounded linear operators between normed spaces, \( \| \cdot \|_{op} \) will denote the operator norm, without us specifying the spaces when they are clear from the context.

Let \( E \to M \) be a Hermitian vector bundle of complex rank \( r \in \mathbb{N} \), endowed with a Hermitian connection \( \nabla \). The fiber of \( E \) over \( x \in M \) will be denoted by \( E_x \), and the Hermitian product on it will be \( \langle \cdot, \cdot \rangle_{E_x} \) (all the Hermitian products used in this text will be linear in the first argument).

Let \( D_t = \{ \frac{1}{2^k} \mid k \in \mathbb{N}, \ j \in \mathbb{N} \cap [0, 2^k] \} \) - the "dyadic" numbers between 0 and \( t \). Our purpose in this section is to give meaning to the Hermitian vector bundle described intuitively by \( E = \bigotimes_{s \in D_t} \text{End } E \to C_t \). If \( c \in C_t \) we let the fiber \( E_c \) of \( E \) over \( c \) be \( \bigotimes_{s \in D_t} \text{End } E_{s(c)} \). This is a tensor product of countably many factors, the definition of which is not trivial and deserves some clarifications. As such, we endow the space \( \text{End } E_x \) with the Hermitian product given by \( \langle A, B \rangle_{\text{End } E_x} = \frac{1}{r} \text{Trace}(AB^*) \) for \( A, B \in \text{End } E_x \). For every \( x \in M \) Notice that \( \langle \cdot, - \rangle_{\text{End } E_x} = \frac{1}{r} \langle \cdot, - \rangle_{E_x \otimes E_x^*} \), the
Let $c$.

Lemma 2.1. For every continuous curve $c \in \mathcal{C}_t$ and every $\varepsilon > 0$ there exists a piecewise-smooth curve $c' : [0, t] \to M$ such that $D(c, c') < \varepsilon$.

Proof. Let $c \in \mathcal{C}_t$. The idea of the proof is the following: if $k \in \mathbb{N}$ is large enough, then the points $c(0), c\left(\frac{t}{2^k}\right), \ldots, c\left(\frac{(2^k-1)t}{2^k}\right), c(t)$ will be close enough to each other so that any two consecutive of them may be joined by a unique minimizing geodesic; by joining these geodesic segments together, we shall obtain a piecewise-smooth curve $c'$ (a geodesic interpolation of the $2^k + 1$ points above) which, for sufficiently large $k$, will be at distance at most $\varepsilon$ from $c$. The rest of the proof formalizes this idea rigorously.

Being defined on a compact interval, $c$ will be uniformly continuous; let $\delta$ be an increasing modulus of continuity for it. Since

$$d\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) \leq \delta\left(\frac{t}{2^k}\right) \to 0,$$

for every $0 \leq j < 2^k - 1$, we deduce that for large enough $k \in \mathbb{N}$ the points $c\left(\frac{jt}{2^k}\right)$ and $c\left(\frac{(j+1)t}{2^k}\right)$ may be joined by a unique minimizing geodesic $\gamma_{k,j} : [0, 1] \to M$ with $\gamma_{k,j}(0) = c\left(\frac{jt}{2^k}\right)$ and $\gamma_{k,j}(1) = c\left(\frac{(j+1)t}{2^k}\right)$, for all $0 \leq j < 2^k - 1$. Let us consider the piecewise-geodesic curve $c' : [0, t] \to M$ obtained by gluing these geodesic segments together: on every interval $\left[\frac{jt}{2^k}, \frac{(j+1)t}{2^k}\right]$ it will be given by $c'(s) = \gamma_{k,j}\left(\frac{t}{2^k}s - j\right)$, for all $0 \leq j < 2^k - 1$. Using the triangle inequality in the triangle of vertices $c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)$ and $c'(s)$ for $s \in \left[\frac{jt}{2^k}, \frac{(j+1)t}{2^k}\right]$, let us notice that

$$D(c, c') = \max_{0 \leq j < 2^k - 1} \max_{\frac{jt}{2^k} \leq s \leq \frac{(j+1)t}{2^k}} d\left(c(s), \gamma_{k,j}\left(\frac{2^k}{t}s - j\right)\right) \leq \max_{0 \leq j < 2^k - 1} \max_{\frac{jt}{2^k} \leq s \leq \frac{(j+1)t}{2^k}} d\left(c(s), c\left(\frac{jt}{2^k}\right)\right) + d\left(c\left(\frac{jt}{2^k}\right), \gamma_{k,j}\left(\frac{2^k}{t}s - j\right)\right) \leq \max_{0 \leq j < 2^k - 1} \max_{\frac{jt}{2^k} \leq s \leq \frac{(j+1)t}{2^k}} d\left(c(s), c\left(\frac{jt}{2^k}\right)\right) + d\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) \leq \frac{\varepsilon}{2^k}.$$
\[ \leq \max_{0 \leq j \leq 2^k-1} \max_{\frac{j}{2^k} \leq s \leq \frac{j+1}{2^k}} \delta \left( s - \frac{j1}{2^k} \right) + \delta \left( \frac{j1}{2^k} \right) \leq 2 \delta \left( \frac{j1}{2^k} \right) \]

whence it follows that \( D(c, c') < \varepsilon \) for large enough \( k \).

We shall construct the topology on \( E \) first locally, on the restrictions of \( E \) to open balls \( B(c, r) \) centered at each curve \( c \in C_t \), and then we shall show that all these local topologies are compatible with each other, which will allow us to glue them together into a global topology on \( E \).

For every \( r \in (0, \min_{s \in [0, t]} \text{injrad}(c(s))) \) consider the open metric ball \( B(c, r) = \{ \gamma \in C_t \mid D(c, \gamma) < r \} \) and a piecewise-smooth curve \( c' \in B(c, r) \), the existence of which being guaranteed by lemma \( 2.1 \). If \( \gamma \in B(c, r) \) and \( s \in [0, t] \) then

\[ d(\gamma(s), c(s)) < D(\gamma, c) < \min_{s \in [0, t]} \text{injrad}(c(s)) < \text{injrad}(c(s)), \]

so there exists a unique minimizing geodesic defined on \([0, 1]\) from \( \gamma(s) \) to \( c(s) \). Next, using the same argument, there exists a unique minimizing geodesic defined on \([0, 1]\) from \( c(s) \) to \( c'(s) \). We may then parallel-transport the vector \( e \in E_{\gamma(s)} \) to \( c(s) \), and then to \( c'(s) \), each time along the geodesics found above; we finally parallel-transport the vector obtained so far from \( c'(s) \) to \( c'(0) = x_0 \) along \( c' \), which is piecewise-smooth, thus obtaining a vector in \( E_{x_0} \). The procedure just described gives a linear isometry from \( E_{\gamma(s)} \) to \( E_{x_0} \); it is clear that it may be inverted (by traversing the same curves in the opposite direction and in the inverse order), so this procedure is an isometric isomorphism. We may extend it in the natural way to tensor monomials of the form \( e_{\gamma(s_1)} \otimes \cdots \otimes e_{\gamma(s_N)} \in E_\gamma \) with \( N \in \mathbb{N}\setminus\{0\} \) and \( s_1, \ldots, s_N \in D_t \), thus obtaining tensor monomials in \((\text{End } E_{x_0})^\otimes (s_1, \ldots, s_N) \subset (\text{End } E_{x_0})^\otimes D_t \). This extension will still be a surjective isometry between monomials.

Let us now introduce two helpful auxiliary notations: if \( x, y \in M \) and if \( c \) is a piecewise-smooth curve from \( x \) to \( y \), and if \( e \in E_x \), then we shall denote by \( PT_{x \rightarrow y,c}(e) \in E_y \) the parallel transport of \( e \) from \( x \) to \( y \) along \( c \). The unique minimizing geodesic defined on \([0, 1]\) from \( x \) to \( y \) will be denoted by \( \gamma_{x,y} \), whenever it exists.

We may now define a local trivialization \( \varphi : pr^{-1}_E(B(c, r)) \rightarrow B(c, r) \times (\text{End } E_{x_0})^\otimes D_t \) as follows:

- if \( \alpha \in B(c, r) \) and \( e_{\alpha(s_1)} \otimes \cdots \otimes e_{\alpha(s_N)} \in E_\alpha \), then

\[
\varphi((\alpha, e_{\alpha(s_1)} \otimes \cdots \otimes e_{\alpha(s_N)})) = (c, TP_{\gamma(s_1) \rightarrow x_0, c'} TP_{c(s_1) \rightarrow c'(s_1), \gamma(c(s_1), c'(s_1))} TP_{\alpha(s_1) \rightarrow c(s_1), \gamma_{\alpha(s_1), c(s_1)}(c(s_1))} e_{\alpha(s_1)} \otimes \cdots) \]

\[
\cdots \otimes TP_{c(s_N) \rightarrow x_0, c'} TP_{c(s_N) \rightarrow c(s_N), \gamma_{c(s_N), c'(s_N)}(c(s_N))} TP_{\alpha(s_N) \rightarrow c(s_N), \gamma_{\alpha(s_N), c(s_N)}(c(s_N))} e_{\alpha(s_N)})
\]

as explained above:

- on linear combinations of such tensor monomials we extend \( \varphi \) by linearity, thus obtaining a linear isometric isomorphism;

- since \( E_\alpha \) is the Hilbert completion of an algebraic inductive limit, we define \( \varphi \) on limits of elements from the algebraic inductive limit by continuity.

The map \( \varphi : E_{\mid_{B(c, r)}} \rightarrow B(c, r) \times (\text{End } E_{x_0})^\otimes D_t \) allows us now to define a topology on \( E_{\mid_{B(c, r)}} \) by transporting the topology from \( B(c, r) \times (\text{End } E_{x_0})^\otimes D_t \) back under \( \varphi^{-1} \). In particular, since \( B(c, r) \)
is a metric space and \((\text{End } E_{x_0}) \otimes D_t\) is a Hilbert space, the topology so constructed on \(\mathcal{E}|_{B(c,r)}\) will be first-countable.

It remains to show that these topologies defined only locally are compatible with each other. More precisely, let us show that if the curves \(c_1, c_2 \in \mathcal{C}_t\) and the numbers \(r_1, r_2 > 0\) are such that \(B(c_1, r_1) \cap B(c_2, r_2) \neq \emptyset\), and if \(\varphi_1, \varphi_2\) are two local trivializations above these two balls constructed as above, then the local topologies induced by \(\varphi_1\) and \(\varphi_2\) coincide on \(\mathcal{E}|_{B(c_1, r_1) \cap B(c_2, r_2)}\).

But this is easy, since the map \(\phi_1 \circ \phi_2^{-1} : B(c_1, r_1) \cap B(c_2, r_2) \times (\text{End } E_{x_0}) \otimes D_t \to B(c_1, r_1) \cap B(c_2, r_2) \times (\text{End } E_{x_0}) \otimes D_t\) is the identity on the first factor, and is a continuous map defined on \(B(c_1, r_1) \cap B(c_2, r_2)\) with values in the group of isometries of \((\text{End } E_{x_0}) \otimes D_t\) on the second factor, whence the conclusion is clear.

Since the local topologies constructed above have turned out to be compatible with each other, they may be glued together into a unique (first-countable) global topology on \(\mathcal{E}\). In this topology, the maps \(\varphi\) constructed above become continuous local trivializations.

**Remark 2.2.** Since \(\mathcal{C}_t\) is separable, it follows from the above considerations that \(\mathcal{C}_t\) may be covered by a countable family of trivialization domains, a fact which will be useful later on.

Let \(\pi_k : \mathcal{C}_t \to M^{2^k+1}\) denote the projection given by

\[
\pi_k(c) = \left( c(0), c\left(\frac{t}{2^k}\right), \ldots, c\left(\frac{(2^k-1)t}{2^k}\right), c(t) \right).
\]

**Proposition 2.3.** The projections \(\pi_k : \mathcal{C}_t \to M^{2^k+1}\) and the projection \(\text{pr}_\mathcal{E} : \mathcal{E} \to \mathcal{C}_t\) are continuous, for all \(k \in \mathbb{N}\).

**Proof.** If we denote by \(d_k\) the distance induced by the Riemannian tensor on \(M^{2^k+1}\) for every \(k \in \mathbb{N}\), then

\[
d_k(\pi_k(c), \pi_k(c')) = d\left( \left( c(0), c\left(\frac{t}{2^k}\right), \ldots, c(t) \right), \left( c'(0), c'\left(\frac{t}{2^k}\right), \ldots, c'(t) \right) \right) = \\
= \sqrt{\sum_{j=0}^{2^k} d\left( c\left(\frac{j t}{2^k}\right), c'\left(\frac{j t}{2^k}\right) \right)^2} \leq 2^k \sup_{0 \leq s \leq 2^k} d\left( c\left(\frac{j t}{2^k}\right), c'\left(\frac{j t}{2^k}\right) \right) \leq 2^k D(c, c'),
\]

so \(\pi_k\) is continuous.

Since the local trivializations constructed above are continuous, and since the restriction of the projection \(\text{pr}_\mathcal{E}\) to such trivializations has the form \((c, \ldots) \mapsto c\), the continuity of this projection is clear. \(\square\)

### 3. Integrable sections in bundles of infinite rank

If \(s\) is a section of \(E\), the notation \(\|s\|\) (without any indices) will denote the function \(M \ni x \mapsto \|s(x)\|_{E_x} \in [0, \infty)\). The space \(\Gamma_0(E)\) will be the space of compactly-supported smooth sections in \(E\), the space \(\Gamma_c(E)\) will be the space of compactly-supported continuous sections in \(E\), and \(\Gamma_{cb}(E)\) will be the space of bounded continuous sections in \(E\) (i.e. those continuous sections \(\eta\) such that
\( \text{sup}_{x \in M} \| \eta_x \|_{E_x} < \infty \). For each \( 1 \leq p \leq \infty \) the space \( \Gamma^p(E) \) will be the space of classes of measurable sections that coincide almost everywhere, with the property that \( |s| \in L^p(M) \). It is known that \( \Gamma_0(E) \) is dense in \( \Gamma^p(E) \) in the norm topology if \( p \neq \infty \), and in the weak-* topology if \( p = \infty \). The corresponding spaces of locally \( p \)-integrable sections will be \( \Gamma^p_{loc}(E) \).

The quadratic form \( Q_{E, \nabla} : \Gamma_0(E) \subset \Gamma^2(E) \to \mathbb{R} \) defined by \( Q_{E, \nabla}(\eta) = \int_M \| (\nabla \eta)_x \|^2_{E_x} \, dx \) gives rise to a self-adjoint, positive, densely-defined operator \( H_{\nabla} \) in \( \Gamma^2(E) \) (the Friedrichs extension of the connection Laplacian); by functional calculus one may define next the contraction semigroup \( (e^{-sH_{\nabla}})_{s \geq 0} \) acting in \( \Gamma^2(E) \), which we shall call "the heat semigroup in \( E \) corresponding to \( \nabla \)" (full details can be found in [Davies80]). It is shown then in chapter XI of [Güneysu17] that this semigroup admits a unique integral kernel ("the heat kernel in \( E \) corresponding to \( \nabla \)), that is a jointly measurable map \((0, \infty) \times M \times M \ni (s, x, y) \mapsto h_{\nabla}(s, x, y) \in E_x \otimes E_y^* \subset E \otimes E^* \) such that \( h_{\nabla}(s, x, \cdot) \in \Gamma^2(E^*) \), \( h_{\nabla}(\cdot, y) \in \Gamma^2(E) \), and \( (e^{-sH_{\nabla}})(x) = \int_M h_{\nabla}(s, x, y) \eta(y) \, dy \) for almost all \( x, y \in M \), all \( s > 0 \) and all \( \eta \in \Gamma^2(E) \). It is proved in the same chapter that \( h_{\nabla}(s, x, y)^* = h_{\nabla}(s, y, x) \) for all \( s > 0 \) and almost all \( x, y \in M \), where the star denotes the adjoint with respect to the Hermitian products on the fibers \( E_x \) and \( E_y \). One then shows that \( h_{\nabla} \) satisfies locally the partial differential equation \( (2\partial_s + H_{\nabla} x + H_{\nabla} y)u = 0 \) in the distributional sense (where \( H_{\nabla} x \) means the operator \( H_{\nabla} \) acting on the argument \( x \)), whence it follows that \( h_{\nabla} \) is smooth using theorem 1 in [Mizohata57].

The same conclusions hold if instead of working on \( M \) we work on some relatively compact open subset of it with smooth boundary. If \( M = \bigcup_{i \in \mathbb{N}} U_i \) is an exhaustion of \( M \) with such subsets, we shall use the notation \( H_{\nabla}^{(i)} \) for the Friedrichs extension of the connection Laplacean acting in \( \Gamma^2(E|_{U_i}) \), and the corresponding heat kernel will be \( h_{\nabla}^{(i)} \). In the special case when the vector bundle is \( M \times \mathbb{C} \) endowed with the usual Hermitian product and with the trivial connection, the Friedrichs extension of the connection Laplacean will be denoted simply by \( H \), and the corresponding heat kernel simply by \( h \); when working on a domain \( U_i \) as above these will be \( H^{(i)} \) and, respectively, \( h^{(i)} \). It is known that \( h_i \to h \) pointwise and monotonically (theorem 4 in chapter VIII of [Chavel84]). It is shown in subchapter VII.3 of [Güneysu17] that \( \| h_{\nabla}(t, x, y) \|_{\text{op}} \leq h(t, x, y) \) for all \( t > 0 \) and almost all \( x, y \in M \); since both these heat kernels have been seen to be smooth, and since co-null subsets are dense in \( M \), it follows that the inequality is in fact true for all \( x, y \in M \). A similar inequality holds on domains \( U_i \) as above. This result is known as the "diamagnetic inequality" and will turn out to be crucial in our construction below.

**Definition 3.1.** We shall say that the section \( \sigma : C_t \to \mathcal{E} \) is a **cylindrical section** if and only if there exists a section \( s \in \Gamma^\infty \left( \text{End} \, E \otimes \mathbb{C}^2+1 \right) \) such that \( \sigma = s \circ \pi_t \).

**Definition 3.2.** We define the Lebesgue space \( \Gamma^2(\mathcal{E}) \) of square-integrable sections as the space of measurable sections \( \sigma : C_t \to \mathcal{E} \) identified under equality almost everywhere, with the property that the function \( C_t \ni c \mapsto \| \sigma(c) \|_{\mathcal{E}_c} \in [0, \infty) \) is in \( L^2(C_t, w_t) \).

**Theorem 3.3.** The space \( \Gamma^2(\mathcal{E}) \) endowed with the scalar product

\[
\langle \sigma_1, \sigma_2 \rangle_{\Gamma^2(\mathcal{E})} = \int_{C_t} \langle \sigma_1(c), \sigma_2(c) \rangle_{\mathcal{E}_c} \, dw_t(c)
\]

is a Hilbert space. Its dual is \( \Gamma^2(\mathcal{E}^*) \), where \( \mathcal{E}^* \) is the dual bundle of \( \mathcal{E} \) in which the fiber \( \mathcal{E}_c^* \) is the dual space of \( \mathcal{E}_c \) for all \( c \in C_t \).
Proof. That $\Gamma^2(\mathcal{E})$ is an inner product space is easy. The proof of its metric completeness follows faithfully the usual one of the completeness of the space $L^2$. The main ingredients are the fact that each fiber is, in turn, complete (being a Hilbert space), and the fact that $\mathcal{C}_t$ may be covered by a countable family of trivialization domains (a consequence if its separability). More specifically, assume that $(\sigma_k)_{k \in \mathbb{N}} \subset \Gamma^2(\mathcal{E})$ is a Cauchy sequence. There exists a subsequence $(\sigma_{k_m})_{m \in \mathbb{N}}$ such that

$$\|\sigma_{k_{m+1}} - \sigma_{k_m}\|_{\Gamma^2(\mathcal{E})} \leq \frac{1}{2^{m+1}}$$

for each $m \in \mathbb{N}$. Define the function

$$f_{m+1}(x) = \sum_{i=0}^{m} \|\sigma_{k_{m+1}}(c) - \sigma_{k_m}(c)\|_{\mathcal{E}_c}$$

for every $m \in \mathbb{N}$ and notice that $\|f_m\|_{L^2(\mathcal{C}_t)} \leq 1$. As a consequence of the monotone convergence theorem, $(f_m)_{m \geq 1}$ has a limit $f \in L^2(\mathcal{C}_t)$, finite almost everywhere.

If $m \geq l \geq 0$, then for almost all $c \in \mathcal{C}_t$ we have

$$\|\sigma_{k_m}(c) - \sigma_{k_l}(c)\|_{\mathcal{E}_c} \leq \|\sigma_{k_m}(c) - \sigma_{k_{m-1}}(c)\|_{\mathcal{E}_c} + \cdots + \|\sigma_{k_{l+1}}(c) - \sigma_{k_l}(c)\|_{\mathcal{E}_c} \leq f(c) - f_{k_l}(c) \to 0,$$

therefore for almost all $c \in \mathcal{C}_t$ the sequence $(\sigma_{k_m}(c))_{m \in \mathbb{N}} \subset \mathcal{E}_c$ is Cauchy. Since the space $\mathcal{E}_c$ is, by construction, a Hilbert space, hence complete, it follows that for almost all $c \in \mathcal{C}_t$ there exists a unique element $\sigma(c) \in \mathcal{E}_c$ such that $\sigma_{k_m}(c) \to \sigma(c)$. We have already noticed that $\mathcal{C}_t$ may be covered by a countable family of trivialization domains (open balls) of $\mathcal{E}$; on each of them, $\sigma_{k_m} \to \sigma$ almost everywhere, therefore the restriction of $\sigma$ to each such trivialization domain is measurable. Since this family of trivialization domains is countable, it follows that $\sigma$ is measurable. Furthermore, passing to the limit in the inequality

$$\|\sigma_{k_m}(c) - \sigma_{k_l}(c)\|_{\mathcal{E}_c} \leq f(c) - f_{k_l}(c) \leq f(c)$$

we obtain $\|\sigma(c) - \sigma_{k_l}(c)\|_{\mathcal{E}_c} \leq f(c)$, whence

$$\|\sigma\|_2 - \|\sigma_{k_l}\|_2 \leq \|\sigma - \sigma_{k_l}\|_{\mathcal{E}_c} \leq f(c),$$

hence $\|\sigma\| \in L^2(\mathcal{C}_t)$, which means that $\sigma \in \Gamma^2(\mathcal{E})$. Finally, applying the dominated convergence theorem to the sequence $c \mapsto \|\sigma(c) - \sigma_{k_l}(c)\|_{\mathcal{E}_c}^2$ (which is dominated by $f^2$), we conclude that $\sigma_{k_m} \to \sigma$ in $L^2(\mathcal{E})$, so $\sigma_k \to \sigma$ in $\Gamma^2(\mathcal{E})$.

That the dual of $\Gamma^2(\mathcal{E})$ is $\Gamma^2(\mathcal{E}^*)$ is now easy, using the same techniques. \hfill $\Box$

More generally, and along the same lines of thought, one may introduce the space $\Gamma^p(\mathcal{E})$ for every $p \in [1, \infty]$, which will be a Banach space. In particular, $\Gamma^p(\mathcal{E}) \subseteq \Gamma^q(\mathcal{E})$ if $p \leq q$, because the Wiener measure is finite. Also, $\Gamma^p(\mathcal{E}^*)$ is the dual of $\Gamma^{\frac{p}{p-1}}(\mathcal{E})$ for every $p \in (1, \infty]$. The proofs are analogous to those for the spaces $L^p$, the latter being found, for instance, in chap.4 of [Brezis11].

**Theorem 3.4.** The space $\text{Cyl}_1(\mathcal{E})$ of continuous and bounded cylindrical sections is dense in $\Gamma^2(\mathcal{E})$.

**Proof.** The inclusion $\text{Cyl}_1(\mathcal{E}) \subset \Gamma^2(\mathcal{E})$ is trivial: if $s : M^{k+1} \to (\text{End } E)^{\otimes (2^k+1)}$ is essentially bounded, then

$$\int_{\mathcal{C}_t} \|(s \circ \pi_k)(c)\|_{\mathcal{E}_c} \, dw_t(c) \leq \sup_{c \in \mathcal{C}_t} \|(s \circ \pi_k)(c)\|_{\mathcal{E}_c} \, w_t(\mathcal{C}_t) < \infty.$$
Let now $\sigma' \in \text{Cyl}_1(\mathcal{E})^\perp$; we shall show that $\sigma' = 0$. If $f \in \text{Cyl}(\mathcal{C}_t)$ is a cylindrical function (the definition of which is given in [Mustățeanu], and $\sigma \in \text{Cyl}_1(\mathcal{E})$ is a cylindrical section, then it is easy to show that $f\sigma \in \text{Cyl}_1(\mathcal{E})$ and, since $\sigma' \in \text{Cyl}_1(\mathcal{E})^\perp$, we shall have in particular that

$$0 = \langle f\sigma, \sigma' \rangle_{\Gamma^2(\mathcal{E})} = \int_{\mathcal{C}_t} f(c) \langle \sigma(c), \sigma'(c) \rangle_{\mathcal{E}} \, dw_t(c).$$

Using theorem 2.1 in [Mustățeanu], the cylindrical functions are dense in $L^2(\mathcal{C}_t)$, so

$$\int_{\mathcal{C}_t} f(c) \langle \sigma(c), \sigma'(c) \rangle_{\mathcal{E}} \, dw_t(c) = 0$$

for all $f \in L^2(\mathcal{C}_t)$, whence we deduce that $\langle \sigma(c), \sigma'(c) \rangle_{\mathcal{E}} = 0$ for all $c$ in some co-null subset $C_\sigma \subseteq \mathcal{C}_t$.

Let $M = \bigcup_{i \in \mathbb{N}} V_i'$ be a cover of $M$ with open trivialization domains for $E$. Let $V_0 = V_0'$ and $V_i = V_i' \setminus (V_0 \cup \cdots \cup V_{i-1})$ for $i \geq 1$; these subsets will be measurable, pairwise disjoint, trivialization domains. Let $\{\eta_l^1, \ldots, \eta_l^2\}$ be a measurable orthonormal frame in $\text{End} E|_{V_i}$ in which $\eta_l^1(x) = \text{Id}_{E_x}$ for all $x \in V_i$. Defining $\eta_l^i$ by $\eta_l^i|_{V_i} = \eta_l^j$ for all $1 \leq l \leq r^2$ and $i \in \mathbb{N}$, we obtain a global measurable orthonormal frame $\{\eta^1, \ldots, \eta^{r^2}\}$ in $\text{End} E$ made of sections from $\Gamma^x(\text{End} E)$, in which $\eta^1(x) = \text{Id}_{E_x}$ for all $x \in M$.

For every $k \in \mathbb{N}$ and $1 \leq j_0, \ldots, j_{2k} \leq r^2$ define

$$\sigma_{j_0\ldots j_{2k}} (c) = \eta_{j_0} (c(0)) \otimes \eta_{j_1} (c(\frac{t}{2^n})) \otimes \cdots \otimes \eta_{j_{2k}} (c(t))$$

and notice that $\sigma_{j_0\ldots j_{2k}} \in \text{Cyl}_1(\mathcal{E})$ and that the subset $\{\sigma_{j_0\ldots j_{2k-1}} (c) \mid k \in \mathbb{N}, \ 1 \leq j_0, \ldots, j_{2k} \leq r^2\}$ is a countable orthonormal basis in the fiber $\mathcal{E}_c$ for all $c \in \mathcal{C}_t$. We then deduce that there exists a co-null subset $C_{j_0\ldots j_{2k}} \subseteq \mathcal{C}_t$ such that

$$\langle \sigma_{j_0\ldots j_{2k-1}} (c), \sigma'(c) \rangle_{\mathcal{E}_c} = 0$$

for all $c \in C_{j_0\ldots j_{2k}}$, all $k \in \mathbb{N}$ and $1 \leq j_0, \ldots, j_{2k} \leq r^2$. If

$$C = \bigcap_{k \in \mathbb{N}} \bigcap_{1 \leq j_1, \ldots, j_{2k} \leq r^2} C_{j_0\ldots j_{2k}}$$

then $C$ will be co-null and

$$\langle \sigma_{j_0\ldots j_{2k-1}} (c), \sigma'(c) \rangle_{\mathcal{E}_c} = 0$$

for all $c \in C$, for all $k \in \mathbb{N}$ and $1 \leq j_0, \ldots, j_{2k} \leq r^2$, whence $\langle u, \sigma'(c) \rangle_{\mathcal{E}_c} = 0$ for all $u \in \mathcal{E}_c$, hence $\sigma'(c) = 0$ for all $c \in C$, so $\sigma' = 0$ in $\Gamma^2(\mathcal{E})$, so $\text{Cyl}_1(\mathcal{E})^\perp = 0$, meaning that $\text{Cyl}_1(\mathcal{E})$ is dense in $\Gamma^2(\mathcal{E})$. \hfill \Box

In what follows, the main technical result (theorem 3.6) will be based upon the use of Chernoff’s approximation theorem for 1-parameter semigroups. This, in turn, will require us to work on compact subsets of $M$ in order to be able to guarantee the boundedness of certain complicated continuous functions. For this reason we shall consider an exhaustion $M = \bigcup_{i \in \mathbb{N}} U_i$ of $M$ with relatively compact connected domains with smooth boundary, such that $x_0 \in U_0$. In particular, these domains will be Riemannian manifolds, therefore all the above considerations will apply to them, too. All the mathematical objects on $U_i$ obtained as restrictions of some extrinsic objects will
be represented visually by the restriction symbol (such as in, for instance, the bundle $E|_{U_i}$), and all the objects intrinsically associated to $U_i$ will carry the index $(i)$ (for instance: the heat kernel associated to the connection $\nabla$ in $E|_{U_i}$ will be $h^{(i)}$, the Laplacean understood as the generator of the heat semigroup acting in $C(U_i^\gamma)$ will be $L^{(i)}$ etc.).

For each $i \in \mathbb{N}$ we shall consider the space

$$C_t(U_i) = \{ c \in C_t \mid c([0,t]) \subseteq U_i \}$$

endowed with the restriction of the distance $D$ introduced on $C_t$. The natural measure on $C_t(U_i)$ will not be the restriction of the Wiener measure $w_t$, but rather the intrinsic Wiener measure $w^{(i)}_t$ obtained from the intrinsic heat kernel $h^{(i)}$ on $U_i$. It is elementary that $C_t(U_i)$ is closed (and therefore Borel) in $C_t$: the evaluation map $\text{ev} : [0,t] \times C_t \to M$ defined by $\text{ev}(s, \gamma) = \gamma(s)$ is obviously continuous, whence

$$C_t(U_i) = \{ \gamma \in C_t \mid \gamma(s) \in U_i \ \forall s \in [0,t] \} = \bigcup_{s \in [0,t]} \text{ev}(s, \cdot)^{-1}(U_i)$$

is obviously closed. One shows similarly that, if $i \leq j$, then $C_t(U_i)$ is closed in $C_t(U_j)$. It is also known that $w^{(i)}_t \leq w_t|_{C_t(U_i)}$. For details about the Wiener measure, the article [BP11] contains all the necessary constructions and explanations; note that the constructions therein are not probabilistic, but functional-analytic, therefore our project of a purely functional-analytic construction of the stochastic parallel transport is not compromised.

In the following, we shall define a continuous linear functional on $\Gamma^2(\mathcal{E}|_{C_t(U_i)})$ to which, by Riesz's representation theorem, there will correspond a section from $\Gamma^2(\mathcal{E}^*|_{C_t(U_i)})$ which will be seen to be intimately linked to the stochastic parallel transport. Fix $\omega \in E^*_{x_0}$ and $\eta \in \Gamma_{cb}(E)$, and define the functional $W^{(i)}_{t,\omega,\eta}$ on continuous and bounded cylindrical sections as follows: if $s : U_i^{2^k+1} \to (\text{End } E)^{\otimes(2^k+1)}|_{U_i^{2^k+1}}$ is a continuous and bounded section, define

$$W^{(i)}_{t,\omega,\eta}(s \circ \pi_k) = \int_{U_i} \int_{U_i^{2^k}} \int_{U_i^{2^k}} \cdots \int_{U_i^{2^k}} \omega \otimes h^{(i)} \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)} \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \cdot s(x_0, x_1, \ldots, x_{2^k}).$$

The dot inside the integral denotes not a scalar product but a tensor contraction which, in order to be understood, requires a brief discussion. The term $\omega \otimes h^{(i)} \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)} \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k})$ belongs to the space $E^*_{x_0} \otimes (E_{x_0} \otimes E^*_{x_1}) \otimes \cdots \otimes (E_{x_{2^k-1}} \otimes E^*_{x_{2^k}}) \otimes E_{x_{2^k}}$, which is naturally isomorphic to $(E^*_{x_0} \otimes E_{x_0}) \otimes \cdots \otimes (E^*_{x_{2^k}} \otimes E_{x_{2^k}})$, which in turn is isomorphic to $(\text{End } E^*)_{x_0} \otimes \cdots \otimes (\text{End } E^*)_{x_{2^k}}$ (notice that the latter isomorphism is not the natural one, but the natural one multiplied by a normalization factor, because the scalar product of two endomorphisms has been defined such that the identity should have norm 1). In turn, $s(x_0, \ldots, x_{2^k})$ belongs to the space $(\text{End } E)_{x_0} \otimes \cdots \otimes (\text{End } E)_{x_{2^k}}$, therefore the term on the left of the dot may be naturally applied to the one on the right of the dot, this being the meaning of the tensor contraction inside the integral.

Let us show that, indeed, the functional is well defined. First, if $l > k$ then there exists a projection $\pi_{kl} : M^{2^{l+1}} \to M^{2^k+1}$ given by $\pi_{kl}(x_0, \ldots, x_{2^l}) = (x_{2^l-j})_{0 \leq j \leq 2^k}$, so that $s \circ \pi_k = (s \circ \pi_{kl}) \circ \pi_l$. This shows that a cylindrical section may have multiple writings of the form $s \circ \pi_k$. 

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This fact is fortunately compensated inside the integral by the convolution property of the kernel \( h^{(i)}_c \), which insures that the formula of definition of \( W_{t,\omega,\eta}^{(i)} \) does not depend on the writing of the cylindrical sections.

In order to show that the integral in the definition of \( W_{t,\omega,\eta}^{(i)} \) exists, let us notice that

\[
\left\| \omega \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \right\| \leq \left\| \omega \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \right\| \leq \left\| \omega \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \right\| \, ,
\]

each of the two norms being considered in the appropriate space. We shall leave the second as it is, but we shall work on the first one. Let us consider the orthonormal basis \( \{ e_1, \ldots, e_r \} \) in each fiber \( E_{x_0} \), and the dual basis \( \{ f_1, \ldots, f_r \} \) in each fiber \( E^*_{x_0} \). In these bases we have (using Einstein’s summation convention) \( \omega = \omega_i^j \, f_0^i \, \Omega_c \left( \frac{t}{2^k}, x_{j-1}, x_j \right) = h^{(i), j-1}_j \, e_i \otimes f_j \) (for \( 1 \leq j \leq 2^k \)) and \( \eta(x_{2^k}) = \eta^{j-2^k} j^{j^2} \), hence

\[
\left\| \omega \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \otimes \cdots \otimes h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \right\| \leq \frac{1}{r} \left\| \omega \right\|_{E^*_{x_0}}^2 \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \right\|_{E_{x_0} \otimes E^*_1}^2 \ldots \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \right\|_{E_{x_{2^k-1}} \otimes E^*_{x_{2^k}}}^2 \left\| \eta(x_{2^k}) \right\|_{E_{x_{2^k}}}^2 \leq \frac{1}{r} \left\| \omega \right\|_{E^*_0}^2 \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \right\|_{E_{x_0} \otimes E^*_1}^2 \ldots \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \right\|_{E_{x_{2^k-1}} \otimes E^*_{x_{2^k}}}^2 \left\| \eta(x_{2^k}) \right\|_{E_{x_{2^k}}}^2 \leq \frac{1}{r} \left\| \omega \right\|_{E^*_0}^2 \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_0, x_1 \right) \right\|_{E_{x_0} \otimes E^*_1}^2 \ldots \left\| h^{(i)}_c \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \right\|_{E_{x_{2^k-1}} \otimes E^*_{x_{2^k}}}^2 \left\| \eta(x_{2^k}) \right\|_{E_{x_{2^k}}}^2 \, ,
\]
where we have used that \( \|A\|_{\mathcal{V} \otimes U^*} \leq \sqrt{r} \|A\|_{op} \) for any linear map \( A : U \rightarrow V \) between vector spaces of dimension \( r \). We have also used the diamagnetic inequality \( \|h^{(i)}_{\nabla}(s, x, y)\|_{op} \leq h^{(i)}(s, x, y) \).

We conclude that
\[
\left| \frac{1}{\sqrt{r}} \|\omega \otimes h^{(i)}_{\nabla} \left( \frac{t}{2k}, x_0, x_1 \right) \cdots \otimes h^{(i)}_{\nabla} \left( \frac{t}{2k}, x_{2k-1}, x_{2k} \right) \otimes \eta(x_{2k}) \right| \cdot s(x_0, x_1, \ldots, x_{2k}) \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{\rho}} \int_{C_t(\Omega_i)} |(\eta(c(t)) \| (s \circ \pi_k)(c)) \| \, dw_t^{(i)}(c) \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{\rho}} \left( e^{-t} L^{(i)} \| \eta \|_{E^*_{\rho}}(x_0) \right)^{\frac{1}{2}} \| (s \circ \pi_k)(c) \| \Gamma^2(\mathcal{E}^*|\mathcal{C}_t(\Omega_i)) ,
\]

hence that \( W_{t,\omega,\eta}^{(i)} \) is well defined, trivially linear, and that
\[
|W_{t,\omega,\eta}^{(i)}(s \circ \pi_k)| \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{\rho}} \int_{C_t(\Omega_i)} |(\eta(c(t)) \| (s \circ \pi_k)(c)) \| \, dw_t^{(i)}(c) \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{\rho}} \left( e^{-t} L^{(i)} \| \eta \|_{E^*_{\rho}}(x_0) \right)^{\frac{1}{2}} \| (s \circ \pi_k)(c) \| \Gamma^2(\mathcal{E}^*|\mathcal{C}_t(\Omega_i)) ,
\]

where \( \|\eta\| \) denotes the function \( M \ni x \mapsto |\eta(x)|_{E_x} \in [0, \infty) \).

Since the continuous and bounded cylindrical sections are dense in \( \Gamma^2(\mathcal{E}|\mathcal{C}_t(\Omega_i)) \), it follows that \( W_{t,\omega,\eta}^{(i)} \) extends uniquely to a continuous linear functional on this space, therefore there exists a unique \( \rho_{t,\omega,\eta}^{(i)} \in \Gamma^2(\mathcal{E}^*|\mathcal{C}_t(\Omega_i)) \) such that
\[
W_{t,\omega,\eta}^{(i)}(\sigma) = \int_{\mathcal{C}_t(\Omega_i)} \rho_{t,\omega,\eta}^{(i)}(c)(\sigma(c)) \, dw_t^{(i)}(c)
\]
for every \( \sigma \in \Gamma^2(\mathcal{E}^*|\mathcal{C}_t(\Omega_i)) \). Furthermore, \( \|\rho_{t,\omega,\eta}^{(i)}\|_{\Gamma^2(\mathcal{E}^*|\mathcal{C}_t(\Omega_i))} \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{\rho}} \left( e^{-t} L^{(i)} \| \eta \|_{E^*_{\rho}}(x_0) \right)^{\frac{1}{2}} \| (s \circ \pi_k)(c) \| \Gamma^2(\mathcal{E}|\mathcal{C}_t(\Omega_i)) \).

In the following we shall try to uncover some of the geometrical properties of \( \rho_{t,\omega,\eta}^{(i)} \); more precisely, we shall investigate its connection with the parallel transport in \( E \). To this end, let us define the "cut-off parallel transport" \( P(x, y) : E_y \rightarrow E_x \) for every \( (x, y) \in M \times M \) by:

- \( P(x, y) = \) the parallel transport in \( E \) from \( y \) to \( x \), whenever there exists a unique minimizing geodesic in \( M \) defined on \([0, 1]\) between \( x \) and \( y \),

- \( P(x, y) = 0 \) otherwise.

Let us notice that \( P \) so defined is a section in the external tensor product bundle \( E \boxtimes E^* \rightarrow M \times M \). Since the subset
\[
\{(x, y) \in M \times M \mid \text{there exists a unique minimizing geodesic between } x \text{ and } y \text{ defined on } [0, 1]\}
\]
is open in \( M \times M \), it will be Borel measurable. Since the section \( (x, y) \mapsto P(x, y) \) in \( E \boxtimes E^* \) is continuous on this subset, \( P \) will be a measurable section in this bundle.

With \( P \) so defined, define
\[
P_{t,\omega,\eta,k}(c) = \omega \otimes P \left( c(0), c \left( \frac{t}{2k} \right) \right) \otimes \cdots \otimes P \left( c \left( \frac{(2k-1)t}{2k} \right), c(t) \right) \otimes \eta(c(t))
\]
for every curve \( c \in \mathcal{C}_t \) and every \( k \in \mathbb{N} \). Since \( P \) is measurable, with operator norm bounded by 1 at every point of \( M \times M \), we conclude that \( P_{t,\omega,\eta,k} \) is a measurable and bounded cylindrical section.
in the bundle $\mathcal{E}^*$. We shall show that $\rho_{t,\omega,\eta}$ is the limit of the sequence $(P_{t,\omega,\eta,k})_{k \in \mathbb{N}}$ in the norm topology of $\mathcal{F}^2(\mathcal{E}^*)$.

We shall need to use $P$ (which is not smooth) in contexts requiring differential calculus methods; in order to do this, we shall now introduce some smooth cut-off functions. Let $\kappa : [0, \infty) \to [0, 1]$ be a smooth function such that $\kappa|_{[0, \frac{1}{4}]} = 1$ and $\kappa|_{[\frac{1}{4}, \infty)} = 0$. Let $\text{injrad}_U : U \to (0, \infty)$ be the injectivity radius function on $U$; we emphasize that this is not the restriction of $\text{injrad}_M$ to $U$, but rather is is computed intrinsically, using the restriction of the Riemannian structure on $U$ (for basic details about the injectivity radius, see p.118 of [Chavel06]). Being continuous and strictly positive, we may find a smooth function $\text{rad} : U \to [0, \infty)$ such that $\text{rad} \leq d_U(x, \partial U)$ for every $x \in U$.

We shall denote this semigroup by $\hat{C}_t(U) \to [0, 1]$ by

$$\chi_k(c) = \kappa \left( c(0), c \left( \frac{t}{2^k} \right), \ldots, c \left( \frac{2^k - 1}{2^k} \right), c(t) \right)$$

for every $k \geq 0$.

If $h^{(i)}$ is the intrinsic heat kernel of $\bar{U}_i$, the operators defined by

$$C(\bar{U}_i) \ni f \mapsto \int_{U_i} h^{(i)}(t, \cdot, \cdot) f(y) \, dy \in C(\bar{U}_i)$$

together with the identity operator form a strongly continuous one-parameter semigroup in $C(\bar{U}_i)$. This will have a generator (closed operator) that we shall denote by $L^{(i)}$, densely defined, with the domain given by (see [Davies80], chap. 1)

$$\text{Dom}(L^{(i)}) = \left\{ f \in C(\bar{U}_i) : \lim_{t \to 0} \frac{1}{t} \left( \int_{U_i} h^{(i)}(t, \cdot, \cdot) f(y) \, dy - f \right) \in C(\bar{U}_i) \right\}.$$ 

We shall denote this semigroup by $(e^{-sL^{(i)}})_{s \geq 0}$. Integrating twice by parts, it is obvious that $C^{\infty}_0(U_i) \subset \text{Dom}(L^{(i)})$. An essential domain for $L^{(i)}$ is

$$\mathcal{E} = \bigcup_{s \geq 0} e^{-sL^{(i)}}(C(\bar{U}_i)).$$

Since the heat semigroup is smoothing (again, one may use [Mizohata57], or one’s favourite Sobolev spaces techniques, to see this), the functions in $\mathcal{E}$ will be smooth. Since $h^{(i)}$ vanishes on the boundary $\partial U_i$, the functions in $\mathcal{E}$ will also vanish on $\partial U_i$.

With exactly the same arguments, but using now the integral kernel $h^{(i)}_\chi$ instead of $h^{(i)}$, we shall obtain a semigroup acting on $\Gamma_c(\bar{U}_i)$, the generator of which will be denoted by $L^{(i)}_\chi$, and the domain of which will contain $\Gamma_c^\infty(E|U_i)$. This semigroup will be denoted $(e^{-sL^{(i)}_\chi})_{s \geq 0}$.

The crucial tool to be used in the following will be Chernoff’s theorem (lemma 3.28 in [Davies80]). For the reader’s convenience, we shall give its statement here.

**Theorem 3.5** (Chernoff). Assume that $(R_t)_{t \geq 0}$ is a family of contractions in a Banach space $X$, with $R_0 = \text{Id}_X$. Let $\mathcal{E} \subseteq X$ be an essential domain for the generator $L$ of a strongly continuous
one-parameter semigroup \((e^{-tL})_{t \geq 0}\) on \(X\). If \(\lim_{t \to 0} \frac{1}{t}(R_t f - f) = -Lf\) for every \(f \in \mathcal{E}\), then \(e^{-tL} = \lim_{k \to \infty} (R_k^\frac{t}{k})^k\) strongly for every \(t \geq 0\). Furthermore, the convergence is uniform with respect to \(t\) on bounded subsets of \([0, \infty)\).

With all these preparations, we are ready now for the main technical result of this work, from which all the developments announced in the introduction will unravel.

**Theorem 3.6.** If \(R^{(i)}_t : C(U_i) \to C(U_i)\), with \(t \geq 0\), is the family of operators given by \(R^{(i)}_0 f = f\) and

\[
(R^{(i)}_t f)(x) = \frac{1}{r} \int_{U_i} \langle h^{(i)}(t, x, y), \chi(x, y)P(x, y) \rangle_{E_x \otimes E_y^*} f(y) \, dy
\]

for \(f \in C(U_i)\), then \(\lim_{k \to \infty} (R^{(i)}_k)^k f = e^{-tL^{(i)}} f\) for every \(f \in C(U_i)\), uniformly with respect to \(t\) from compact subsets of \([0, \infty)\).

**Proof.** The proof reduces to the verification of the hypotheses in Chernoff’s theorem. To begin with, notice that for \(t > 0\)

\[
|(R^{(i)}_t f)(x)| \leq \int_{U_i} \chi(x, y) \frac{1}{\sqrt{t}} \|h^{(i)}(t, x, y)\|_{E_x \otimes E_y^*} \frac{1}{\sqrt{t}} \|P(x, y)\|_{E_x \otimes E_y^*} |f(y)| \, dy \leq \\
\leq \int_{U_i} \chi(x, y) \|h^{(i)}(t, x, y)\|_{op} \|P(x, y)\|_{op} |f(y)| \, dy \leq \\
\leq \int_{U_i} h^{(i)}(t, x, y) f(y) \, dy \leq \|f\|_{C(U_i)},
\]

so \(R^{(i)}_t\) is a contraction for every \(t \geq 0\) (we have used again the fact that \(\|A\|_{V \otimes U^*} \leq \sqrt{r} \|A\|_{op}\), the diamagnetic inequality and the obvious inequalities \(\chi \leq 1\) if \(\|P(x, y)\|_{op} \leq 1\). It remains to show that \(\lim_{t \to 0} \frac{1}{t} (R^{(i)}_t f) - f + L^{(i)} f\|_{C(U_i)} = 0\) for every \(f \in \mathcal{E}\); to this end, let us show first that \((R^{(i)}_t f)(x)\) is smooth with respect to \(t\) for every \(x \in U\). If Trace denotes the trace in \(\text{End} E_x\), notice that

\[
(R^{(i)}_t f)(x) = \frac{1}{r} \int_{U_i} \langle h^{(i)}(t, x, y), \chi(x, y)P(x, y) \rangle_{E_x \otimes E_y^*} f(y) \, dy = \\
= \frac{1}{r} \int_{U_i} \text{Trace}[h^{(i)}(t, x, y)\chi(x, y)P(x, y)^*] f(y) \, dy = \\
= \frac{1}{r} \int_{U_i} \text{Trace}[h^{(i)}(t, x, y)\chi(x, y)P(y, x)] f(y) \, dy = \\
= \frac{1}{r} \text{Trace}[e^{-tL^{(i)}}[(\chi(x, \cdot) P(\cdot, x) f)](x)].
\]

Examining the construction of \(\chi\), it is clear that \(\chi(x, \cdot) P(\cdot, x)\) is a smooth section in \(E_{\Gamma x}\) with compact support, the possible singularities of \(P(\cdot, x)\) being away from the support of \(\chi(x, \cdot)\); since \(f\) is smooth, being from \(\mathcal{E}\), their product is a smooth section in \(E_{\Gamma x}\) with compact support, therefore in the domain of every power of \(L^{(i)}\). Under these conditions, we know from the general theory of 1-parameter \(C_0\)-semigroups in Banach spaces that the map

\[
[0, \infty) \ni t \mapsto e^{-tL^{(i)}}[(\chi(x, \cdot) P(\cdot, x) f)](x) \in \Gamma_c(E_{\Gamma x})
\]
is smooth. If \( \{e_1, \ldots, e_r\} \) is an orthonormal basis in \( E_x \), and if \( \delta_x \) is the Dirac measure concentrated at \( x \), then \( \delta_x \otimes e_i \) is easily seen to be a continuous linear functional on \( \Gamma_c(E_\Sigma) \) for each \( 1 \leq i \leq r \); since

\[
\{e^{-tL_{\Sigma}^{(i)}} [\chi(x, \cdot) P(x, \cdot) f]\}(x) = \sum_{i=1}^{r} (\delta_x \otimes e_i) \left( e^{-tL_{\Sigma}^{(i)}} [\chi(x, \cdot) P(x, \cdot) f] \right) e_i,
\]

the smoothness of the map \( [0, \infty) \ni t \mapsto \{e^{-tL_{\Sigma}^{(i)}} [\chi(x, \cdot) P(x, \cdot) f]\}(x) \in E_x \) is clear, whence the smoothness of the function \( [0, \infty) \ni t \mapsto (R_{t}^{(i)} f)(x) \in \mathbb{C} \) follows immediately.

Expanding with respect to \( t \) we have, for every \( x \in U_i \),

\[
(R_{t}^{(i)} f)(x) = f(x) + \partial_{t}|_{t=0}(R_{t}^{(i)} f)(x) t + \int_{0}^{t} (t - s) \partial_{s}^{2}(R_{s}^{(i)} f)(x) \, ds.
\]

For the calculation of the first derivative of \( (R_{t}^{(i)} f)(x) \) at \( t = 0 \) we have

\[
\partial_{t}|_{t=0}(R_{t}^{(i)} f)(x) = \partial_{t}|_{t=0} \int_{U_i} \frac{1}{r} \text{Trace}[h_{\Sigma}^{(i)}(t, x, y) \chi(x, y) P(x, y, y)] f(y) \, dy =
\]

\[
= \frac{1}{r} \lim_{t \to 0} \int_{U_i} \text{Trace}\left[\left[-L_{\Sigma, (y)}^{(i)} h_{\Sigma}^{(i)}(t, x, y)\right] \chi(x, y) P(y, x) f(y)\right] \, dy =
\]

\[
= \frac{1}{r} \text{Trace}\left\{ \lim_{t \to 0} \int_{U_i} h_{\Sigma}^{(i)}(t, x, y)\left[-L_{\Sigma, (y)}^{(i)} \chi(x, y) P(y, x) f(y)\right] \, dy \right\} =
\]

\[
= \frac{1}{r} \text{Trace}\left\{ \left[-L_{\Sigma, (y)}^{(i)} \chi(x, y) P(y, x) f(y)\right]_{y=x} \right\}.
\]

Some clarifications about the above calculations are in order. First, the notation \( L_{\Sigma, (y)}^{(i)} \) means that the Laplacian acts with respect to \( y \in U_i \). Second, we have been able to move the Laplacian from acting on \( h_{\Sigma}^{(i)}(t, x, \cdot) \) over to acting on the product \( \chi(x, \cdot) P(\cdot, x) f \) because \( \chi(x, \cdot) \) is smooth with compact support, and the other two factors are also smooth inside this support; this is, in fact, the only reason for which the introduction of \( \chi \) in our reasoning was necessary.

Since \( L_{\Sigma}^{(i)} \) is a local operator, since \( \chi(x, \cdot) = 1 \) near \( x \), and since \( \chi(x, \cdot) P(\cdot, x) f \) is smooth near \( x \), we may replace \( L_{\Sigma}^{(i)} \) with \( \nabla^* \nabla \) and we may also drop \( \chi \) in order to obtain the simpler formula

\[
\partial_{t}|_{t=0}(R_{t}^{(i)} f)(x) = \frac{1}{r} \text{Trace}[\nabla^* \nabla [P(\cdot, x) f]](x).
\]

Choosing an orthonormal basis \( \{e_1, \ldots, e_r\} \) in \( E_x \), the above formula becomes

\[
\partial_{t}|_{t=0}(R_{t}^{(i)} f)(x) = \frac{1}{r} \sum_{k=1}^{r} \langle \nabla^* \nabla [P(\cdot, x) e_k f] (x), e_k \rangle_{E_x}.
\]

Since \( P \) is the parallel transport with respect to \( \nabla \), its covariant derivative will be 0, so

\[
\partial_{t}|_{t=0}(R_{t}^{(i)} f)(x) = \frac{1}{r} \sum_{k=1}^{r} \langle \nabla^* [P(\cdot, x) e_k \otimes df] (x), e_k \rangle_{E_x}.
\]
It is a known result in Riemannian geometry that $\nabla^*(\eta \otimes \alpha) = -\nabla^{\alpha} \eta - (d^* \alpha) \eta$ for every real smooth 1-form $\alpha$ and every smooth section $\eta$ in $E$, where $\alpha^*$ is the tangent field dual to the 1-form $\alpha$ under the usual "musical" isomorphisms. In particular, if $f$ is real
\[ \nabla^*(\eta \otimes df) = -\nabla_{\text{grad}} f \eta + (\Delta f) \eta, \]
whence it follows that
\[ \partial_t |_{t=0}(R_t^{(i)} f)(x) = \frac{1}{r} \sum_{k=1}^r \langle \nabla_{\text{grad}} f P(\cdot, x) e_k \rangle(x) + (\Delta f)(x) [P(\cdot, x) e_k](x), e_k \rangle_{E_x} = \frac{1}{r} \sum_{k=1}^r \langle (\Delta f)(x) [P(x, x) e_k](x), e_k \rangle_{E_x} = (\Delta f)(x) = -(L^{(i)} f)(x), \]
where we have used again that $\nabla_{\text{grad}} f P(\cdot, x) e_k = 0$ for the very same geometrical reasons as above.

The result, obtained for real $f$, extends now trivially to complex $f$.

Returning to formula (I), we have
\[ \| (R_t^{(i)} f - f) + t L^{(i)} f \|_{C(\mathcal{U}_t)} \leq \sup_{x \in \mathcal{U}_t} \left| \int_0^t (t - s)^2 (R_s^{(i)} f)(x) \, ds \right| \leq \frac{t^2}{2} \sup_{x \in \mathcal{U}_t} \sup_{s \in [0, t]} \| (R_s^{(i)} f)(x) \| \leq \frac{1}{2} \left| \int_{\mathcal{U}_t} (\Delta (x, y) P(x, y) \chi(x, y) f(y) \rangle_{E_x \otimes E_y} f(y) \, dy \right| \leq \frac{1}{r} \left| \int_{\mathcal{U}_t} (\Delta (x, y) P(x, y) \chi(x, y) f(y) \rangle_{E_x \otimes E_y} f(y) \, dy \right| \leq \frac{1}{r} \left| \int_{\mathcal{U}_t} (\Delta (x, y) P(x, y) \chi(x, y) f(y) \rangle_{E_x \otimes E_y} f(y) \, dy \right|, \]
where in the last inequality we have used the diamagnetic inequality and the sub-markovianity of $h^{(i)}$. Since the function
\[ \mathcal{U}_t \times \mathcal{U}_t \ni (x, y) \mapsto |(L^{(i)} f(x, y) P(x, y) \chi(x, y) f(y))| \in [0, \infty) \]
is smooth, the double supremum obtained in the last inequality will have a finite value $C \in [0, \infty)$, so
\[ \left\| \frac{1}{t} (R_t^{(i)} f - f) + L^{(i)} f \right\|_{C(\mathcal{U}_t)} \leq \frac{C}{2rt} \to 0, \]
which checks the last hypothesis in Chernoff’s theorem, which we may now apply in order to obtain the conclusion of our theorem.

The following corollary is essentially the above theorem in the trivial bundle $U_t \times \mathcal{C}$ endowed with the trivial connection given by differentiation (a situation in which the cut-off parallel transport $P$ may be replaced by the constant function 1). The proof is essentially the same, but in an even simpler context, so we shall omit it.

**Corollary 3.7.** If $S_t^{(i)} : C(\mathcal{U}_t) \to C(\mathcal{U}_t)$, with $t \geq 0$, is the family of operators given by $S_0^{(i)} f = f$ and
\[ (S_t^{(i)} f)(x) = \int_{\mathcal{U}_t} h^{(i)}(t, x, y) \chi(x, y) f(y) \, dy \]
for $f \in C(U_i)$, then $\lim_{k \to \infty} \left( S^{(i)} \right)^k f = e^{-tL(i)} f$ for every $f \in C(U_i)$, uniformly with respect to $t$ from compact subsets of $[0, \infty)$.

**Remark 3.8.** Before going any further, let us pause for a moment and examine where in the above proof we have used the compactness of $U_i$, and whether this compactness assumption is essential or not. It turns out that the only step in the proof where this assumption was used was in the bounding of the function

$$U_i \times U_i \ni (x, y) \mapsto |(L(i)^2 | \chi(x, y) P(x, y) f(y)| | \in [0, \infty) .$$

If instead of working on $U_i$ we had worked on $M$, we would have needed to choose $f$ with the properties that:

- $f$ should be in the domain of $L^2$, where $L$ is the Friedrichs extension of the Laplace-Beltrami operator $-\Delta$ of $M$;
- the product of $f$ with any compactly-supported smooth function should again be in the domain of $L^2$;
- $f$ should have compact essential support, in order to guarantee the desired boundedness.

If $M$ had been metrically complete, then an essential domain for $L$ made of such functions would have been the space of compactly-supported smooth functions (see theorem 11.5 in [Grigor'yan09]). For arbitrary Riemannian manifolds, though, no essential domain satisfying the above three conditions is known to the author, hence the need to treat the problem on relatively compact domains - a technical restriction that we shall see later on how to get rid of.

The above results allow us to finally approach the statement that we were after, namely to prove that the sequence $(P_{t, \omega, \eta, k})_{k \in \mathbb{N}}$ approximates $\rho^{(i)}_{t, \omega, \eta}$.

**Theorem 3.9.** The sequence $(P_{t, \omega, \eta, k} U_i)_{k \in \mathbb{N}}$ converges to $\rho^{(i)}_{t, \omega, \eta} \in \Gamma^2(\mathcal{E}^* | c_i(U_i))$, uniformly with respect to $t$ from bounded subsets of $[0, \infty)$.

**Proof.** In order to simplify the notations, we shall no longer indicate visually the restriction of functions or sections to $c_i(U_i)$ where this is obvious. In the equality

$$\|\rho^{(i)}_{t, \omega, \eta} - P_{t, \omega, \eta, k}\|_{L^2(\mathcal{E}^* | c_i(U_i))}^2 = \|\rho^{(i)}_{t, \omega, \eta}\|_{L^2(\mathcal{E}^* | c_i(U_i))}^2 - \langle \rho^{(i)}_{t, \omega, \eta}, P_{t, \omega, \eta, k}\rangle_{L^2(\mathcal{E}^* | c_i(U_i))}^2 - \langle P_{t, \omega, \eta, k}, \rho^{(i)}_{t, \omega, \eta}\rangle_{L^2(\mathcal{E}^* | c_i(U_i))}^2 + \|P_{t, \omega, \eta, k}\|_{L^2(\mathcal{E}^* | c_i(U_i))}^2$$

the first term is less or equal than $\frac{1}{p} \|\omega\|_{L^p(U_i)}^2 \|e^{-tL(i)} | \eta\|^2(x_0)$. Performing majorizations similar to the ones made when we showed that $W^{(i)}_{t, \omega, \eta}$ is well defined, in which we only replace $h^{(i)}_{\gamma} (\gamma, x_{j-1}, x_j)$ with $P(x_{j-1}, x_j)$ (the operator norm of which is less or equal than 1), we obtain that

$$\|P_{t, \omega, \eta, k}(c)\|_{L^2(\mathcal{E}^* | c_i(U_i))} \leq \frac{1}{p} \|\omega\|_{L^p(U_i)}^2 \|\eta(c(t))\|_{L^2(\Gamma, c)}^2,$$

whence we may bound the last term in the above right-hand side by

$$\|P_{t, \omega, \eta, k}\|_{L^2(\mathcal{E}^* | c_i(U_i))}^2 = \int_{\Gamma, c} \omega \otimes P(c(0), c\left(\frac{t}{2k}\right)) \otimes \ldots$$
\[ \cdots \otimes P \left( c \left( \frac{(2^k - 1)t}{2^k} \right), c(t) \right) \otimes \eta(c(t)) \right)^2_{E^*_t} \ dw_t^{(i)}(c) \leq \frac{1}{r} \|\omega\|^2_{E^*_0} (e^{-tL^{(i)}} \|\eta\|^2)(x_0). \]

We shall now show that \( \lim_{k \to \infty} \langle \rho_t^{(i)}(P_{t,\omega,\eta,k} ) \rangle_{\Gamma^2(E^*_t|_{C_t(\overline{U})})} = \frac{1}{r} \|\omega\|^2_{E^*_0} (e^{-tL^{(i)}} \|\eta\|^2)(x_0) \). If \( P_{t,\omega,\eta,k}^* \in \Gamma^2(E^*_t) \) denotes the element dual to \( P_{t,\omega,\eta,k} \) with respect to the Hermitian product on \( \Gamma^2(E^*_t) \), we have that

\[ \langle \rho_t^{(i)}(P_{t,\omega,\eta,k} ) \rangle_{\Gamma^2(E^*_t|_{C_t(\overline{U})})} = \int_{C_t(U_t)} \rho_t^{(i)}(P_{t,\omega,\eta,k}^*)(c) \|\eta\|^2_{E^*_t} \ dw_t^{(i)}(c) = W_t^{(i)}(P_{t,\omega,\eta,k}^*) = W_t^{(i)}(\chi_k P_{t,\omega,\eta,k}^*) + W_t^{(i)}((1 - \chi_k) P_{t,\omega,\eta,k}^*). \]

The first term in the right-hand side is

\[ \int_{U_t} dx_1 \cdots \int_{U_t} dx_{2^k+1} \omega \otimes \chi(x_0, x_1) P(x_0, x_1) \otimes \cdots \otimes \chi(x_{2^k-1}, x_{2^k}) P(x_{2^k-1}, x_{2^k}) \otimes \eta(x_{2^k}) = \]

\[ = \left( \frac{1}{r} \right)^{2^k+1} \|\omega\|^2_{E^*_0} \int_{U_t} dx_1 \cdots \int_{U_t} dx_{2^k} \left( \frac{t}{2^k}, x_{2^k}, \chi(x_0, x_1) P(x_0, x_1) \right)_{E^*_t} \cdots \]

\[ \cdots \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \chi(x_{2^k-1}, x_{2^k}) P(x_{2^k-1}, x_{2^k}) \otimes \eta(x_{2^k}) \right)_{E^*_t} = \]

\[ = \frac{1}{r} \|\omega\|^2_{E^*_0} \left( \omega \left( R^{(i)} \right)^{2^k} \right)(x_0), \]

which converges to \( \frac{1}{r} \|\omega\|^2_{E^*_0} (e^{-tL^{(i)}} \|\eta\|^2)(x_0) \) uniformly with respect to \( t \) from bounded subsets of \( (0, \infty) \) according to theorem 3.6.

Using first the diamagnetic inequality, then the Cauchy-Schwarz inequality in the fiber \( E^*_t \), the second term in the above right-hand side may be bounded as such:

\[ |W_t^{(i)}((1 - \chi_k) P_{t,\omega,\eta,k})| = \int_{C_t(U_t)} \rho_t^{(i)}(P_{t,\omega,\eta,k}) \left( (1 - \chi_k)(c) \right) \|\eta\|^2_{E^*_t} \ dw_t^{(i)}(c) = \]

\[ = \left| \int_{U_t} dx_1 \cdots \int_{U_t} dx_{2^k} \left[ 1 - \chi(x_0, x_1) \cdots \chi(x_{2^k-1}, x_{2^k}) \right) \chi(x_0, x_1) P(x_0, x_1) \otimes \cdots \otimes P(x_{2^k-1}, x_{2^k}) \otimes \eta(x_{2^k}) \right| \leq \]

\[ \leq \frac{1}{r} \left| \int_{U_t} dx_1 h_t^{(i)} \left( \frac{t}{2^k}, x_0, x_1 \right) \cdots \int_{U_t} dx_{2^k} h_t^{(i)} \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \right| \]

\[ \left| 1 - \chi(x_0, x_1) \cdots \chi(x_{2^k-1}, x_{2^k}) \right| \|\omega\|^2_{E^*_0} \|\eta(x_{2^k})\|^2_{E^*_t} = \]

\[ = \frac{1}{r} \|\omega\|^2_{E^*_0} \int_{C_t(U_t)} (1 - \chi_k(c)) \|\eta(c(t))\|^2_{E^*_t} \ dw_t^{(i)}(c) = \frac{1}{r} \|\omega\|^2_{E^*_0} (e^{-tL^{(i)}} \|\eta\|^2)(x_0). \]
\[
-\frac{1}{r}\|\omega\|_{E_{x_0}^\#}^2 \int_{U_i} dx_1 h^{(i)} \left( \frac{t}{2^k}, x_0, x_1 \right) \chi(x_0, x_1) \ldots \\
\ldots \int_{U_i} dx_{2^k} h^{(i)} \left( \frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \chi(x_{2^k-1}, x_{2^k}) \|\eta(x_{2^k})\|_{E_{x_{2^k}}^\#}^2 \frac{\eta}{2^k} = \\
= \frac{1}{r}\|\omega\|_{E_{x_0}^\#}^2 (e^{-tL^{(i)}} \|\eta\|^2)(x_0) - \frac{1}{r}\|\omega\|_{E_{x_0}^\#}^2 \left[ \left( S^{(i)}_{(i)} \frac{2^k}{r} \right) \|\eta\|^2 \right](x_0),
\]
which converges to 0 uniformly with respect to \( t \) from bounded subsets of \((0, \infty)\) according to corollary [3.7].

Passing to the limit, we obtain that

\[
\lim_{k \to \infty} \langle \rho^{(i)}_{t, \omega, \eta}, P_{t, \omega, \eta, k} \rangle_{\Gamma^2(E^\# | \mathcal{C}_1(\Gamma_1))} = \frac{1}{r} \|\omega\|_{E_{x_0}^\#}^2 \lim_{k \to \infty} \left( \delta_{x_0}, \left( R^{(i)}_{t, \omega, \eta} \right) \frac{2^k}{r} \right) \|\eta\|^2 = \frac{1}{r} \|\omega\|_{E_{x_0}^\#}^2 (e^{-tL^{(i)}} \|\eta\|^2)(x_0)
\]
uniformly with respect to \( t \) from bounded subsets of \((0, \infty)\), whence it follows that

\[
\lim_{k \to \infty} \|\rho^{(i)}_{t, \omega, \eta} - P_{t, \omega, \eta, k} \|_{\Gamma^2(E^\# | \mathcal{C}_1(\Gamma_1))} \leq 0
\]
uniformly with respect to \( t \) from bounded subsets of \((0, \infty)\), the conclusion being now immediate. \( \square \)

We already knew that \( \rho^{(i)}_{t, \omega, \eta} \in \Gamma^2(E^\# | \mathcal{C}_1(\Gamma_1)) \), but the convergence that we have just proved allows us to obtain an even stronger conclusion, which will be useful later on, in particular in proving the Feynman-Kac formula in fiber bundles.

**Corollary 3.10.** \( \rho^{(i)}_{t, \omega, \eta} \in \Gamma^\infty(E^\# | \mathcal{C}_1(\Gamma_1)) \) s.t. \( \|\rho^{(i)}_{t, \omega, \eta}(c)\|_{E^\#} = \frac{1}{\sqrt{r}} \|\omega\|_{E_{x_0}^\#} \|\eta(c(t))\|_{E_{c(t)}} \) for almost every \( c \in \mathcal{C}_1(\Gamma) \) (with respect to the measure \( u^{(i)}_{t} \)).

**Proof.** We know that \( P_{t, \omega, \eta, k} \in \Gamma^2(E^\# | \mathcal{C}_1(\Gamma_1)) \). After choosing a measurable representative of \( \rho^{(i)}_{t, \omega, \eta} \), which we shall denote \( \rho^{(i)}_{t, \omega, \eta} \) again, for simplicity, there exists a subsequence \((k_j)_{j \in \mathbb{N}} \subseteq \mathbb{N} \) and a co-null subset \( C \subseteq \mathcal{C}_1(\Gamma_1) \) such that \( P_{t, \omega, \eta, k_j}(c) \to \rho^{(i)}_{t, \omega, \eta}(c) \) for every \( c \in C \) (the proof of this fact is almost identical to the proof of the completeness of \( \Gamma^2(E) \)). Reusing the argument in lemma [2.1] for each curve \( c \in C \) there exists \( k_c \in \mathbb{N} \) such that \( P(c(\frac{j}{k_c}), c(\frac{(j+1)}{k_c})) = 0 \) (therefore it is precisely the parallel transport between the two points on \( c \)) for all \( k \geq k_c \) and \( 0 \leq j \leq 2^k - 1 \). It follows that

\[
\|P_{t, \omega, \eta, k}(c)\|_{E^\#} = \frac{1}{\sqrt{r}} \|\omega\|_{E_{x_0}^\#} \|\eta(c(t))\|_{E_{c(t)}}
\]
for \( c \in C \) and \( k \geq k_c \), whence it follows that

\[
\|\rho^{(i)}_{t, \omega, \eta}(c)\|_{E^\#} = \lim_{k \to \infty} \|P_{t, \omega, \eta, k}(c)\|_{E^\#} = \frac{1}{\sqrt{r}} \|\omega\|_{E_{x_0}^\#} \|\eta(c(t))\|_{E_{c(t)}} \leq \frac{1}{\sqrt{r}} \|\omega\|_{E_{x_0}^\#} \|\eta\|_{\Gamma_{c(t)}(E)}. \]

\( \square \)
So far we have worked on the spaces of curves $C_t(U_j)$ associated to the relatively compact domains $U_j$ that exhaust $M$. We have done this purely for technical reasons, the compactness having to do with the need to bound certain continuous functions in the proof of theorem 3.6 that were difficult to control otherwise. It is the appropriate moment now to get rid of this exhaustion and obtain global geometrical objects and global pieces of relationship among these objects.

**Theorem 3.11.** If $i \leq j$ then $\rho_{t,\omega,\eta}(j)|_{C_t(U_j)} = \rho_{t,\omega,\eta}(i)$.

**Proof.** For all $k \in \mathbb{N}$ we have

$$
\|\rho_{t,\omega,\eta}(j)|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))} \leq \|\rho_{t,\omega,\eta}(j)|_{C_t(U_j)} - P_{t,\omega,\eta,k}|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))} + \|P_{t,\omega,\eta,k}|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))} =\\
= \sqrt{\int_{C_t(U_j)} |\rho_{t,\omega,\eta}(j)(c) - P_{t,\omega,\eta,k}(c)|^2 \mathcal{E}^* \mathcal{C}_t(U_j) + \|P_{t,\omega,\eta,k}|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))}} \leq\\
\leq \sqrt{\int_{C_t(U_j)} |\rho_{t,\omega,\eta}(j)(c) - P_{t,\omega,\eta,k}(c)|^2 \mathcal{E}^* \mathcal{C}_t(U_j) + \|P_{t,\omega,\eta,k}|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))}} \leq\\
\leq \sqrt{\int_{C_t(U_j)} |\rho_{t,\omega,\eta}(j)(c) - P_{t,\omega,\eta,k}(c)|^2 \mathcal{E}^* \mathcal{C}_t(U_j) + \|P_{t,\omega,\eta,k}|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))}} =\\
= \|\rho_{t,\omega,\eta}(j) - P_{t,\omega,\eta,k}|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))} + \|P_{t,\omega,\eta,k}|_{C_t(U_j)} - \rho_{t,\omega,\eta}(i)|_{C_t(U_j)}\|_{L^2(\mathcal{E}^*:\mathcal{C}_t(U_j))},
$$

whence the conclusion is clear with the aid of theorem 3.9.

This compatibility relationship among the sections $(\rho_{t,\omega,\eta}(j))_{j \in \mathbb{N}}$ insures that the global section defined by $\rho_{t,\omega,\eta} = \lim_{j \to \infty} \rho_{t,\omega,\eta}(j)$ is well defined, and that $\rho_{t,\omega,\eta}|_{C_t(U_j)} = \rho_{t,\omega,\eta}(j)$. In defining $\rho_{t,\omega,\eta}$ as we have done it is understood that we work with measurable representatives of the equivalence classes $\rho_{t,\omega,\eta} \in \Gamma(\mathcal{E}^*:\mathcal{C}_t(U_j)) \subseteq \Gamma(\mathcal{E}^*)$, and that changing these representatives in turn changes the limit only on some null subset, therefore its equivalence class stays the same.

**Theorem 3.12.** The section $\rho_{t,\omega,\eta}$ so defined is measurable and essentially bounded. Furthermore, $\|\rho_{t,\omega,\eta}(c)\|_{\mathcal{E}^*} = \frac{1}{\sqrt{\mathcal{E}^*}} \|\omega\|_{\mathcal{E}^*} \|E_t(c)\|_{L^2(\mathcal{E}^*)}$ for almost all $c \in \mathcal{C}_t$.

**Proof.** Since $M = \bigcup_{j \in \mathbb{N}} U_j$, it follows that $\mathcal{C}_t = \bigcup_{j \in \mathbb{N}} \mathcal{C}_t(U_j)$. Since $\rho_{t,\omega,\eta}|_{\mathcal{C}_t(U_j)} = \rho_{t,\omega,\eta}(j)$, it follows that if $S \subseteq \mathcal{E}^*$ is a measurable subset then

$$
\rho_{t,\omega,\eta}(j)^{-1}(S) = \bigcup_{j \in \mathbb{N}} \rho_{t,\omega,\eta}(j)^{-1}(S) \cap \mathcal{C}_t(U_j) = \bigcup_{j \in \mathbb{N}} \rho_{t,\omega,\eta}(j)^{-1}(S)
$$

is measurable because each $\rho_{t,\omega,\eta}(j)$ is measurable.

The value of the pointwise norm of $\rho_{t,\omega,\eta}$ is a consequence of corollary 3.10.

We have seen in theorem 3.9 that $P_{t,\omega,\eta,k}|_{\mathcal{C}_t(U_j)} \to \rho_{t,\omega,\eta}|_{\mathcal{C}_t(U_j)}$ in $\Gamma(\mathcal{E}^*:\mathcal{C}_t(U_j))$, for all $j \in \mathbb{N}$. We shall now prove that it is possible to remove the restriction to $\mathcal{C}_t(U_j)$ and obtain the convergence globally, on the whole $\mathcal{C}_t$. 

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Theorem 3.13. The sequence \((P_{t,\omega,k})_{k\in\mathbb{N}}\) converges to \(\rho_{t,\omega}\) in \(\Gamma^2(\mathcal{E}^*)\), uniformly with respect to \(t \in (0,T]\) for all \(T > 0\).

Proof. If \(\omega = 0\) the result is trivially true; we shall assume then that \(\omega \neq 0\). Let \(\varepsilon > 0\). Using the fact that \(\rho_{t,\omega,n}|_{\mathcal{C}(U_j)} = \rho_{t,\omega,n}^{(j)}\) for all \(j \in \mathbb{N}\), we may write that

\[
\|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 = \|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 + \|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 \leq \|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 + \frac{4}{r}\|\omega\|_{\text{mod}}^2 \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c)
\]

and the integral on the right-hand side is

\[
\int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) = \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) - \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) 
\]

On the other hand,

\[
\|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 = \int_{\mathcal{C}(U_j)} \|P_{t,\omega,k}(c) - \rho_{t,\omega,\omega}(c)\|_{\mathcal{E}^*}^2 d(w_t(c)) + \int_{\mathcal{C}(U_j)} \|P_{t,\omega,k}(c) - \rho_{t,\omega,\omega}(c)\|_{\mathcal{E}^*}^2 d(w_t(c)) \leq \|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 + \frac{4}{r}\|\omega\|_{\text{mod}}^2 \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) - \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c). 
\]

We conclude that

\[
\|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 \leq \|P_{t,\omega,k} - \rho_{t,\omega,\omega}\|_{\Gamma^2(\mathcal{E}^*)}^2 + \frac{8}{r}\|\omega\|_{\text{mod}}^2 \left(\int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) - \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t^{(j)}(c)\right).
\]

We shall choose \(j\) by a careful examination of the difference between these two latter integrals:

\[
\int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t(c) - \int_{\mathcal{C}(U_j)} \|\eta(c(t))\|_{\mathcal{E}(t)}^2 dw_t^{(j)}(c) = \int_{\mathcal{C}(U_j)} h(t,x_0,x) \|\eta(x)\|_{\mathcal{E}_x}^2 dx - \int_{\mathcal{C}(U_j)} h^{(j)}(t,x_0,x) \|\eta(x)\|_{\mathcal{E}_x}^2 dx,
\]

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and since $h^{(j)} \to h$ pointwise and monotonically, for each $t > 0$ there exists an $j_{e,t} \in \mathbb{N}$ such that

$$\left| \int_M h(t, x_0, x) \eta(x) \| \eta(x) \|_{E_x}^2 \, dx - \int_{U_{j_{e,t}}} h^{(j)}(t, x_0, x) \| \eta(x) \|_{E_x}^2 \, dx \right| = \left| \langle \delta_{x_0}, e^{-tL} \| \eta \|^2 \rangle - \langle \delta_{x_0}, e^{-tL} \| \eta \|_{U_{j_{e,t}}}^2 \rangle \right| \leq \frac{r \varepsilon}{16 \| \omega \|_{E_{x_0}}^2}$$

for every $j \geq j_{e,t}$, where $\langle \cdot, - \rangle$ denotes the duality pairing between the space of bounded continuous functions and its dual (to which $\delta_{x_0}$ belongs). Since the two heat semigroups seen above are strongly continuous, the above expression that contains them is continuous with respect to $t \in [0, T]$, therefore every $t$ from $[0, T]$ admits an open neighbourhood $V_{e,t} \subseteq [0, T]$ such that

$$\left| \langle \delta_{x_0}, e^{-sL} \| \eta \|^2 \rangle - \langle \delta_{x_0}, e^{-sL} \| \eta \|_{U_{j_{e,t}}}^2 \rangle \right| \leq \frac{r \varepsilon}{16 \| \omega \|_{E_{x_0}}^2}$$

for every $j \geq j_{e,t}$ uniformly with respect to $s \in V_{e,t}$. Since $[0, T]$ is compact, we may cover it with a finite number of such neighbourhoods, $[0, T] = \bigcup_{i=1}^N V_{e,t_i}$. Choosing $j_{e} = \max\{j_{e,t_1}, \ldots, j_{e,t_N}\}$ we have

$$\left| \langle \delta_{x_0}, e^{-tL} \| \eta \|^2 \rangle - \langle \delta_{x_0}, e^{-tL} \| \eta \|_{U_{j_{e,t}}}^2 \rangle \right| \leq \frac{r \varepsilon}{16 \| \omega \|_{E_{x_0}}^2}$$

for all $t \in (0, T)$.

Using now theorem 3.9 on $U_{j_e}$, we may find a $k_{e} \in \mathbb{N}$ such that

$$\| P_{t, \omega, \eta, k} - \rho_{t, \omega, \eta} \|_{E^*}^2 \leq \frac{\varepsilon}{2}$$

for all $k \geq k_{e}$, uniformly with respect to $t \in (0, T)$.

Combining all these upper bounds we obtain that

$$\| P_{t, \omega, \eta, k} - \rho_{t, \omega, \eta} \|_{E^*}^2 \leq \varepsilon$$

for all $k \geq k_{e}$, uniformly with respect to $t \in (0, T)$, whence the conclusion is clear.

\[\square\]

**Remark 3.14.** The section $\rho_{t, \omega, \eta}$ does not depend on the exhaustion with regular domains used to construct it, because it is the limit of the sequence of sections $(P_{t, \omega, \eta, k})_{k \in \mathbb{N}}$ that does not depend on any exhaustion.

**4. Getting rid of the auxiliary section**

Let us remember now that one the aims of this article is to give a new construction of the stochastic parallel transport. Whatever this object may be, it is clear that the stochastic parallel transport of a vector $v \in E_{x_0}$ should not depend on any section $\eta \in \Gamma_{x_0}(E)$. Indeed, if one looks back at the proof of theorem 3.9, one sees that $\eta$ was needed only for technical reasons, in order for us to be able to use theorem 3.0 and corollary 3.7 which in turn were based upon Chernoff’s theorem. Since $\eta$ is seen to play an exclusively auxiliary role, in the following we shall concentrate our efforts on eliminating it from our results. In order to achieve this, we shall need a number of useful auxiliary results.
Let us begin by showing that \( \rho_{t, \omega, \eta} \), which so far has been constructed under the hypothesis that \( \eta \in \Gamma_{cb}(E) \), can be extended to a significantly larger class of sections \( \eta \). More precisely, let

\[
\Gamma'_t = \left\{ \eta : M \to E \text{ measurable section } \mid \int_M h(t, x_0, x) \| \eta_x \|_{E_x}^2 \, dx < \infty \right\}
\]

and let \( \Gamma_t \) be the quotient of \( \Gamma'_t \) under equality almost everywhere. It is easy to show that \( \Gamma_t \) is a Hilbert space, the Hermitian product being

\[
\langle \eta, \eta' \rangle_{\Gamma_t} = \int_M h(t, x_0, x) \langle \eta_x, \eta'_x \rangle_{E_x} \, dx = \int_{\mathcal{C}_t} \langle \eta_{c(t)}, \eta'_{c(t)} \rangle_{E_{c(t)}} \, d\omega_t(c).
\]

It is also clear that \( \Gamma^x(E) \subseteq \Gamma_t \).

**Theorem 4.1.** The space \( \Gamma_{cb}(E) \) is dense in \( \Gamma_t \).

**Proof.** It is obvious that \( \Gamma_{cb}(E) \subseteq \Gamma_t \). Let \( \eta \in \Gamma_{cb}(E)^1 \); we shall show that \( \eta = 0 \). If \( f \in C_b(M) \) and \( \eta' \in \Gamma_{cb}(E) \), then \( f \eta' \in \Gamma_{cb}(E) \subseteq \Gamma_t \) and

\[
0 = \langle f \eta', \eta \rangle_{\Gamma_t} = \int_M f(x) \langle \eta_x, \eta'_x \rangle_{E_x} \, dx.
\]

Since \( f \) is arbitrary, we conclude that there exists a co-null subset \( C_{\eta'} \subseteq M \) such that \( \langle \eta_x', \eta_x \rangle_{E_x} = 0 \) for every \( x \in C_{\eta'} \) (it is understood that we work with some measurable representative of \( \eta \)).

Since \( M \) is separable, we may cover it with a countable family of trivialization open domains \( (V_i)_{i \in \mathbb{N}} \); by possibly shrinking them we may assume that each \( V_i \) is a (closed) domain of trivialization. Choose an orthonormal frame in \( E|_{V_i} \) and use Tietze's theorem to extend it continuously to the whole \( M \); let \( \{ \eta_{i1}, \ldots, \eta_{i2} \} \) be the resulting continuous global frame in \( E \); the sections making it up will belong to \( \Gamma_{cb}(E) \). Fix \( i \in \mathbb{N} \). There exists a co-null \( C_{i,j} \subseteq M \) such that \( \langle \eta_{ij}, \eta_x \rangle_{E_x} = 0 \) for all \( x \in C_{i,j} \). Letting \( C_t = \bigcap_{j=1}^{\infty} C_{i,j} \cap V_i \), we immediately obtain that \( \langle u, \eta_x \rangle_{E_x} = 0 \) for all \( x \in C_t \) and \( u \in E_x \), whence \( \eta|_{V_t} = 0 \) almost everywhere and therefore \( \eta = 0 \) in \( \Gamma_t \).

If we integrate the result of theorem 3.12 with respect to \( c \in \mathcal{C}_t \), we obtain that

\[
\| \rho_{t, \omega, \eta} \|_{\Gamma^2(\mathcal{E}^*)} \leq \frac{1}{\sqrt{r}} \| \omega \|_{E_{\omega_0}} \| \eta \|_{\Gamma_t},
\]

when \( \eta \in \Gamma_t \); since we have just shown that \( \Gamma_{cb}(E) \) is dense in \( \Gamma_t \), the map \( \Gamma_{cb}(E) \ni \eta \mapsto \rho_{t, \omega, \eta} \in \Gamma^2(\mathcal{E}^*) \) extends to a continuous linear map \( \Gamma_t \ni \eta \mapsto \rho_{t, \omega, \eta} \in \Gamma^2(\mathcal{E}^*) \).

**Lemma 4.2.** If \( \eta \in \Gamma_t \) then \( \rho_{t, \omega, \eta}(c) \|_{\mathcal{E}^*} = \frac{1}{\sqrt{r}} \| \omega \|_{E_{\omega_0}} \| \eta_{c(t)} \|_{E_{c(t)}} \) for almost all \( c \in \mathcal{C}_t \).

**Proof.** If \( \eta \in \Gamma_t \), let \( (\eta_k)_{k \in \mathbb{N}} \subseteq \Gamma_{cb}(E) \) be a sequence that converges to \( \eta \) in \( \Gamma_t \); it follows that \( \rho_{t, \omega, \eta_k} \to \rho_{t, \omega, \eta} \) in \( \Gamma^2(\mathcal{E}^*) \), so there exists a subsequence \( (k_l)_{l \in \mathbb{N}} \subseteq \mathbb{N} \) such that \( \eta_{k_l} \to \eta \) almost everywhere on \( M \) and \( \rho_{t, \omega, \eta_{k_l}} \to \rho_{t, \omega, \eta} \) almost everywhere on \( \mathcal{C}_t \), whence

\[
\| \rho_{t, \omega, \eta}(c) \|_{\mathcal{E}^*} = \frac{1}{\sqrt{r}} \| \omega \|_{E_{\omega_0}} \| \eta_{c(t)} \|_{E_{c(t)}}
\]

foe almost all \( c \in \mathcal{C}_t \) (again, we have tacitly worked with some arbitrary measurable representative of \( \eta \); if we choose another one, this will coincide with the former on a co-null subset of \( M \), which does not change the conclusion of the lemma).
Let \( p_t : C_t \rightarrow M \) be the projection given by \( p_t(c) = c(t) \). For every \( v \in E_{x_0} \) we shall denote by \( \nu^v \in E^*_{x_0} \) the linear form given by \( \nu^v = \sqrt{\langle , \rangle}_{E_{x_0}} \in E^*_{x_0} \). The notation \( p^*_t E \) will denote the fiber bundle above \( C_t \) obtained as the pull-back of \( E \rightarrow M \) under \( p_t \). Its fiber \( (p^*_t E)_c \) over the curve \( c \in C_t \) will be, by definition, \( E_{c(t)} \), and we shall use the latter notation for its simplicity. In the following we shall construct, for every \( p \in (1, \infty) \), a continuous conjugate-linear map \( \Gamma^p(\mathcal{E}) \ni \xi \mapsto \mathcal{P}^p_{t,v}(\xi) \in \Gamma^p(p^*_t E) \) such that

\[
[p_{t,v}, \eta(c)](\xi(c)) = \langle \mathcal{P}^p_{t,v}(\xi)(c), \eta(c(t)) \rangle_{E_{c(t)}}
\]

for every \( \eta \in \Gamma^\infty(E) \).

Let \( M = \bigcup_{i \in \mathbb{N}} V'_i \) be a cover of \( M \) with open trivialization domains for \( E \). Let \( V_0 = V'_0 \) and \( V_i = V'_i \setminus (V_0 \cup \cdots \cup V_{i-1}) \) for \( i \geq 1 \); these subsets will be measurable, pairwise disjoint, trivialization domains. Let \( \{ \eta_1, \ldots, \eta_r \} \) be a measurable orthonormal frame in \( E|_{V_i} \). Defining \( \eta_i \) by \( \eta_i|_{V_i} = \eta_i \) for all \( 1 \leq i \leq r \) and \( i \in \mathbb{N} \), we obtain a global measurable orthonormal frame \( \{ \eta_1, \ldots, \eta_r \} \) in \( E \) made of sections from \( \Gamma^\infty(E) \subseteq \Gamma_t \), that is sections \( \eta_i \) for which \( \rho_{t,v}^{\eta_i} \) is a well-defined object as we have seen above. Let \( \{ \eta_1, \ldots, \eta_r \} \) be the dual frame in \( E^* \) defined by \( \eta_k(\eta^l) = \delta_k^l \) (Kronecker’s symbol).

If \( \sigma \in \Gamma^\infty(p^*_t E^*) \), then

\[
\sigma(c) = \sum_{l=1}^r [\sigma(c)] [\eta_i'(c(t))] \eta_i(c(t)) \in E^*_{c(t)}
\]

for every \( c \in C_t \). We define then the functional \( \mathcal{F}^p_{t,v,\xi} : \Gamma^\infty(p^*_t E^*) \rightarrow \mathbb{C} \) by

\[
\mathcal{F}^p_{t,v,\xi}(\sigma) = \sum_{l=1}^r \int_{C_t} \| [\sigma(c)] [\eta_l'(c(t))] \|_{E^*_{c(t)}} \| [\eta_l'(c(t))] \|_{E_{c(t)}} \| [\xi(c)] \|_{E^*_{c(t)}} \| [\rho_{t,v}^{\eta_l'}(c)] \|_{E_{c(t)}} \| [\xi(c)] \|_{E^*_{c(t)}} \| [\rho_{t,v}^{\eta_l'}(c)] \|_{E_{c(t)}} \text{d}w_k(c)
\]

and it is obvious that it is linear. We have seen above that

\[
\| \rho_{t,v}^{\eta_l'}(c) \|_{E^*_{c(t)}} = \frac{1}{\sqrt{r}} \| \nu^v \|_{E_{c(t)}} \| [\eta_l')(c(t))] \|_{E_{c(t)}} = \| v \|_{E_{x_0}},
\]

whence we obtain that

\[
|\mathcal{F}^p_{t,v,\xi}(\sigma)| \leq \sum_{l=1}^r \int_{C_t} \| [\sigma(c)] \|_{E^*_{c(t)}} \| [\eta_l'(c(t))] \|_{E_{c(t)}} \| [\xi(c)] \|_{E^*_{c(t)}} \| [\rho_{t,v}^{\eta_l'}(c)] \|_{E_{c(t)}} \| [\xi(c)] \|_{E^*_{c(t)}} \| [\rho_{t,v}^{\eta_l'}(c)] \|_{E_{c(t)}} \text{d}w_k(c) \leq
\]

\[
\leq r \| v \|_{E_{x_0}} \int_{C_t} \| [\sigma(c)] \|_{E^*_{c(t)}} \| [\xi(c)] \|_{E^*_{c(t)}} \text{d}w_k(c) \leq
\]

\[
\leq r \| v \|_{E_{x_0}} \| [\xi] \|_{\Gamma^p(E)} \| [\sigma] \|_{\Gamma^\infty(p^*_t E^*)}.
\]

We conclude that there exists a unique section \( \mathcal{P}^p_{t,v}(\xi) \in \Gamma^p(p^*_t E) \) such that

\[
\mathcal{F}^p_{t,v,\xi}(\sigma) = \int_{C_t} [\sigma(c)] [\mathcal{P}^p_{t,v}(\xi)(c)] \text{d}w_k(c)
\]

for all \( \sigma \in \Gamma^\infty(p^*_t E^*) \), and that

\[
\| \mathcal{P}^p_{t,v}(\xi) \|_{\Gamma^p(p^*_t E)} \leq r \| v \|_{E_{x_0}} \| [\xi] \|_{\Gamma^p(E)}.
\]

The continuity and conjugate-linearity of \( \xi \mapsto \mathcal{P}^p_{t,v}(\xi) \) are obvious.
Corollary 4.3. With the notations above, $\langle \eta_{c(t)}, \mathcal{P}^p_t(\xi)(c) \rangle_{E_{c(t)}} = [\rho_{t,v,\eta}(c)] [\xi(c)]$ for all $\eta \in \Gamma^\infty(E)$ and almost all $c \in \mathcal{C}$.

Proof. Let $\eta \in \Gamma^\infty(E)$ and $f \in L^2_{p,E}(\mathcal{C})$. Denote by $\eta^p \in \Gamma^\infty(E^*)$ the dual section, defined pointwise by $\eta^p(x) = \langle u, \eta(x) \rangle_{E_x}$ for almost all $x \in M$ and all $u \in E_x$. With this notation, $f \eta^p \in \Gamma^\infty(E^*)$.

Using the definitions of $\mathcal{P}^p_t$ and of $\mathcal{F}^p_t$ given above,

$$\int_{\mathcal{C}_t} f(c) \langle \mathcal{P}^p_t(\xi)(c), \eta_{c(t)} \rangle_{E_{c(t)}} \, dw_t(c) = \int_{\mathcal{C}_t} \left[ \langle f \rho^p_t \eta^p(c) \rangle \right] \xi(c) \, dw_t(c) = \mathcal{F}^p_t(f \rho^p_t \eta^p) =$$

$$= \sum_{i=1}^r \int_{\mathcal{C}_t} \left[ \langle f \rho^p_t \eta^p(c) \rangle \right] \xi(c) \, dw_t(c) =$$

$$= \int_{\mathcal{C}_t} f(c) \sum_{i=1}^r \langle \eta_{c(t)}^i, \eta_{c(t)} \rangle_{E_{c(t)}} \left[ \rho_{t,v,\eta}(c) \right] \xi(c) \, dw_t(c) =$$

$$= \int_{\mathcal{C}_t} f(c) \left[ \rho_{t,v,\eta}(c) \right] \xi(c) \, dw_t(c),$$

where for the last equality we have used the linearity of the map $\eta \mapsto \rho_{t,v,\eta}$ and the fact that

$$\eta_{c(t)} = \sum_{i=1}^r \langle \eta_{c(t)}, \eta^i(c) \rangle_{E_{c(t)}} \eta^i(c)$$

for all $c \in \mathcal{C}_t$.

Since $f$ is arbitrary, we conclude that

$$\langle \eta_{c(t)}, \mathcal{P}^p_t(\xi)(c) \rangle_{E_{c(t)}} = \left[ \rho_{t,v,\eta}(c) \right] [\xi(c)]$$

for almost all $c \in \mathcal{C}_t$. \hfill $\Box$

The linearity of the map $\mathcal{P}^p_t(\xi)$ with respect to $v \in E_{x_0}$ allows us to define a section $\mathcal{P}^p_t(\xi) \in \Gamma^p(E_{x_0}) \otimes E^*_{x_0}$ by requiring that $\mathcal{P}^p_t(\xi)(v) = \mathcal{P}^p_t(\xi)(c)$ for all $v \in E_{x_0}$ and almost all $c \in \mathcal{C}_t$.

Furthermore,

$$\|\mathcal{P}^p_t(\xi)(c)\|_{E_{c(t)} \otimes E^*_{x_0}} \leq \sup_{\|v\|_{E_{x_0}} = 1} \|\mathcal{P}^p_t(\xi)(c)\|_{E_{c(t)}} \leq r \|v\|_{E_{x_0}} \|\xi\|_{\Gamma^p(E)} = r \|\xi\|_{\Gamma^p(E)}.$$

The map $\mathcal{P}^p_t$ encloses a great deal of information regarding the differential geometry and the stochastic calculus associated to the bundle $E$. In the rest of this article we shall see just two of its uses, hopefully enough to convince the reader of its usefulness: the stochastic parallel transport and the Feynman-Kac formula.

5. The stochastic parallel transport

Let us begin by defining the sections $\mathcal{P}_{t,v,k}$ by the explicit formula

$$\mathcal{P}_{t,v,k}(c) = P \left( c(t), c \left( \frac{(2^k - 1)t}{2^k} \right) \right) \ldots P \left( c \left( \frac{t}{2^k} \right), c(0) \right) v$$

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where $v \in E_{x_0}$ is arbitrary, $c \in \mathcal{C}_t$ and $k \in \mathbb{N}$. Notice that $\mathcal{P}_{t,v,k}(c)$ belongs to the fiber $E_{c(t)} = (p_t^*E)_c$. Since $P$ has been shown to be a measurable map, $\mathcal{P}_{t,v,k}$ will be a measurable section in $p_t^*E$. Furthermore, since $\|\mathcal{P}_{t,v,k}\|_{L^2(\mathcal{E}_c)} \leq \|v\|_{L^2(\mathcal{E}_c)}$, we deduce that $\mathcal{P}_{t,v,k} \in \Gamma^2(p_t^*E) \subseteq \Gamma^2(p_t^*E)$.

Let us define the section $\text{Id} : \mathcal{C}_t \to \mathcal{E}$ by $\text{Id}(c) = \otimes_{l \in D_t} \text{Id}_{E_{c(l)}} \in \mathcal{E}_c$; more precisely, $\text{Id}(c)$ is the equivalence class (in the sense of the construction of the algebraic inductive limit as a space of equivalence classes), for instance, of the element $\text{Id}_{E_{c(l)}}$, and the map $\mathcal{C}_t \ni c \mapsto \text{Id}_{E_{c(l)}} \in \mathcal{E}_c$ is obviously continuous. Furthermore, it is obvious that $\|\text{Id}(c)\|_{\mathcal{E}_c} = \|\text{Id}(E_{x_0})\|_{\text{End}E_{x_0}} = 1$, so $\text{Id} \in \Gamma^2(p_t^*E) \subseteq \Gamma^2(p_t^*E)$. We notice then that

$$[P_{t,v^*,\eta,k}(c)] = [P_{t,v^*,\eta,k}(c)] \otimes \ldots \otimes [\eta(c(t))], \mathcal{P}_{t,v,k}(c) \rangle \mathcal{E}_{c(t)}.$$  

**Theorem 5.1.** $\mathcal{P}_{t,v,k} \to \mathcal{P}_{t,v}^2(\text{Id}) \in \Gamma^2(p_t^*E)$ for all $v \in E_{x_0}$, uniformly with respect to $t \in (0,T]$ for all $T > 0$.

**Proof.** Using again the global measurable orthonormal frame $\{\eta^1, \ldots, \eta^r\}$ in $E$,

$$\sup_{t \in (0,T]} \|\mathcal{P}_{t,v}^2(\text{Id}) - \mathcal{P}_{t,v,k}(c)\|_{L^2(\mathcal{E}_c)}^2 =$$

$$= \sup_{t \in (0,T]} \int_{\mathcal{C}_t} \|P_{t,v}(\text{Id})(c) - \mathcal{P}_{t,v,k}(c)\|_{E_{c(t)}}^2 \, dw_t(c) =$$

$$= \sup_{t \in (0,T]} \sum_{l=1}^r \int_{\mathcal{C}_t} \|\eta^l(c(t)), P_{t,v}(\text{Id})(c) - \mathcal{P}_{t,v,k}(c)\|_{E_{c(t)}}^2 \, dw_t(c) =$$

$$= \sup_{t \in (0,T]} \sum_{l=1}^r \|\rho_{t,v^*,\eta^l}(c) - P_{t,v^*,\eta^l,k}(c)\|_{E_{c(t)}}^2 \, dw_t(c) \leq$$

$$\leq \sum_{l=1}^r \sup_{t \in (0,T]} \int_{\mathcal{C}_t} \|\rho_{t,v^*,\eta^l}(c) - P_{t,v^*,\eta^l,k}(c)\|_{E_{c(t)}}^2 \, dw_t(c) \leq$$

$$\leq \sum_{l=1}^r \sup_{t \in (0,T]} \|\rho_{t,v^*,\eta^l} - P_{t,v^*,\eta^l,k}\|_{L^2(\mathcal{E}_c)}^2 \to 0,$$

which together with theorem 3.13 shows the desired convergence. □

Comparing this result with the one obtained by probabilistic techniques ([Ito62, Ito75a, Ito75b]), we conclude that $\mathcal{P}_{t,v}^2(\text{Id})$ is the stochastic parallel transport in $E$ of the vector $v \in E_{x_0}$.

In particular, $\mathcal{P}_{t,v}^2(\text{Id})$ does not depend on the choices made in its construction (the domains of trivialization, the orthonormal frames above them etc.), being the limit of a sequence of sections that do not depend on these choices.

**Corollary 5.2.** $\|\mathcal{P}_{t,v}^2(\text{Id})(c)\|_{E_{c(t)}} = \|v\|_{E_{x_0}}$ for almost every curve $c \in \mathcal{C}_t$.

**Proof.** Since $\mathcal{P}_{t,v,k} \to \mathcal{P}_{t,v}^2(\text{Id})$ in $\Gamma^2(p_t^*E)$, there exists a subsequence $(k_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\mathcal{P}_{t,v,k_j}(c) \to \mathcal{P}_{t,v}^2(\text{Id})(c)$ in $E_{c(t)}$ for almost all $c \in \mathcal{C}_t$. Let $c$ be such a curve; using again the argument in lemma 3.12 there exists a $l_c \in \mathbb{N}$ such that $\mathcal{P}_{t,v,k}(c)$ is the parallel transport of $v$ along a zig-zag line made of $2^{l_c} - 1$ minimizing geodesic segments, for all $k \geq l_c$, so

$$\|\mathcal{P}_{t,v}^2(\text{Id})(c)\|_{E_{c(t)}} = \lim_{j \to \infty} \|\mathcal{P}_{t,v,k_j}(c)\|_{E_{c(t)}} = \|v\|_{E_{x_0}}.$$  

□

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Since $\|P_t^2(\operatorname{Id})(v)(c)\|_{E_x(t)} = \|v\|_{E_{x_0}}$ for almost every curve $c \in C_t$, it makes sense to talk about $P_t^2(\operatorname{Id})^{-1}$. One sees immediately that this object will be a section in $p_t^*E$ with values in $E_{x_0}$, or formally $P_t^2(\operatorname{Id})^{-1} \in \Gamma^2(p_t^*E) \otimes E_{x_0}$.

6. The Feynman-Kac formula in vector bundles

In the following we shall state and prove an extension in Hermitian bundles of the Feynman-Kac formula. Consider a "potential" $V \in \Gamma^1_{loc} (\operatorname{End} E)$ with the property $\inf_{x \in M} \min \operatorname{spec} V(x) = \beta > -\infty$ (for short: $V \geq \beta$), and with $V(x)$ self-adjoint for almost all $x \in M$. The quadratic form $\Gamma_0(E) \ni \eta \mapsto \int_M \langle V(x)\eta_x, \eta_x \rangle_{E_x} \, dx \in \mathbb{R}$ will give rise to a densely-defined self-adjoint operator in $\Gamma^2(E)$, that we shall denote again by $V$, for simplicity. Indeed, the quadratic form is well-defined because

$$\left| \int_M \langle V(x)\eta_x, \eta_x \rangle_{E_x} \, dx \right| \leq \sup_{x \in M} \|\eta_x\|^2 \int_{\operatorname{supp} \eta} \|V(x)\|_{\operatorname{op}} \, dx < \infty.$$ 

It is also lower-bounded by $\beta$ because, if $\{e_{1,x}, \ldots, e_{r,x}\}$ is an orthonormal basis in $E_x$ made of eigenvectors of $V(x)$ with corresponding eigenvalues $\lambda_{1,x} \leq \cdots \leq \lambda_{r,x} \in [\beta, \infty)$ for every $x \in M$, and if $\eta_x = \sum_{i=1}^r \alpha_{i,x} e_{i,x}$, we have that

$$\langle V(x)\eta_x, \eta_x \rangle_{E_x} = \left\langle \sum_{i=1}^r \alpha_{i,x} \lambda_{i,x} e_{i,x}, \sum_{j=1}^r \alpha_{j,x} e_{j,x} \right\rangle_{E_x} = \sum_{i=1}^r \lambda_{i,x} |\alpha_{i,x}|^2 = \sum_{i=1}^r \lambda_{i,x} |\eta_x|^2_{E_x} \geq \beta \|\eta_x\|^2_{E_x}.$$

One may construct the self-adjoint, densely-defined operator corresponding to the sum of $H_V$ and $V$ in the same way, using quadratic forms. Of course, the same construction may be performed not only on $M$, but also on any relatively compact open subset with smooth boundary.

When the starting point of the continuous curves is no longer the fixed point $x_0 \in M$, as until now, but some variable $x \in M$, all the objects that depend on it will gain it as a supplementary lower index; this means that the space of continuous curves starting at $x$ will be $C_{t,x}$, on which we shall have the Wiener measure $w_{t,x}$, and all the objects constructed in this article so far will also gain a supplementary lower index $x$, meaning that we shall have the sections $\rho_{t,\omega,\eta,x}$, $\mathcal{P}_{t,v,x}$ and $\mathcal{P}_{t,x}$ etc.

For each $k \in \mathbb{N}$ denote by $V_{t,x,k} \in \Gamma^X(E)$ the section given by

$$V_{t,x,k}(c) = e^{-\frac{t}{\beta}V(c(\frac{k}{t}))} \cdots e^{-\frac{t}{\beta}V(c(t))}.$$ 

Since $V \geq \beta$ if $t \geq 0$, it is immediate that $\|V_{t,x,k}(c)\|_{E_x} \leq e^{-t\beta}$ for almost all $c \in C_{t,x}$, whence we conclude with the Banach-Alaoglu theorem that there exists a subsequence $(k_l)_{l \in \mathbb{N}} \subseteq \mathbb{N}$ such that the subsequence $(V_{t,x,k_l})_{l \in \mathbb{N}}$ has a weak limit denoted $V_{t,x} \in \Gamma^2(E)$. In particular, we conclude that the section $\mathcal{P}_{t,x}^2(V_{t,x})$ exists in $\Gamma^2(p_t^*E) \otimes E_{x_0}$.

**Theorem 6.1** (The Feynman-Kac formula). If $\eta \in \Gamma^2(E)$ then

$$(e^{-tH_V} - tV)\eta(x) = \int_{C_{t,x}} [\mathcal{P}_{t,x}^2(V_{t,x})(c)]^* \eta_{c(t)} \, dw_{t,x}(c)$$

for every $t > 0$ and almost all $x \in M$.
Proof. Let us consider an exhaustion \( M = \bigcup_{j \geq 0} U_j \) with relatively compact connected open subsets with smooth boundary, as we have already done in this article, the notations being the ones already encountered. For every \( x \in M \) there exists a \( j_x \in \mathbb{N} \) such that \( x \in U_j \) for all \( j \geq j_x \). This means that for every \( x \in M \) it makes sense to consider the spaces \( C_{t,x}(U_j) \) for large enough \( j \), and this is enough because in the following we shall let \( j \to \infty \).

From theorem 4.2 in [Simon78] applied to the exponential function we have that
\[
e^{-tH_{\psi} - tV} = \lim_{j \to \infty} e^{-tH_{\psi} - tV}
\]
strongly in \( \Gamma^2(E) \), while from the Trotter-Kato formula (see [Kato74]) we have that
\[
e^{-tH_{\psi} - tV} = \lim_{l \to \infty} \left( e^{-\frac{1}{2^{l+m}} H_{\psi} - \frac{1}{2^{l+m}} V} \right)^{2^{l+m}}
\]
strongly in \( \Gamma^2(E|U_j) \), where \((k_l)_{l \in \mathbb{N}} \subseteq \mathbb{N}\) is the subsequence found right above this theorem. It follows that there exists a sub-subsequence \((k_{l,m})_{m \in \mathbb{N}} \subseteq \mathbb{N}\) such that
\[
e^{-tH_{\psi} - tV} \eta \bigl( x \bigr) = \lim_{m \to \infty} \left( e^{-\frac{1}{2^{l+m}} H_{\psi} - \frac{1}{2^{l+m}} V} \right)^{2^{l+m}} \eta \bigl( x \bigr)
\]
for all \( \eta \in \Gamma^2(E) \) and almost all \( x \in U_j \).

It follows that if \( \eta, \eta' \in \Gamma^2(E) \), then
\[
\langle e^{-tH_{\psi} - tV} \eta, \eta' \rangle_{\Gamma^2(E)} = \lim_{j \to \infty} \langle e^{-tH_{\psi} - tV} \eta, \eta' \rangle_{\Gamma^2(E|U_j)} =
\]
\[
= \lim_{j \to \infty} \left( \lim_{m \to \infty} \left( e^{-\frac{1}{2^{l+m}} H_{\psi} - \frac{1}{2^{l+m}} V} \right)^{2^{l+m}} \eta, \eta' \right)_{\Gamma^2(E|U_j)} =
\]
\[
= \lim_{j \to \infty} \int_{U_j} dx \left( \lim_{m \to \infty} \int_{U_j} dx_1 h_{\psi}^{(j)} \left( \frac{t}{2^{k_{l,m}}}, x, x_1 \right) e^{-\frac{1}{2^{l+m}} V(x_1)} \ldots \right.
\]
\[
\left. \ldots \int_{U_j} dx_{2^{k_{l,m}}} h_{\psi}^{(j)} \left( \frac{t}{2^{k_{l,m}}}, x_{2^{k_{l,m}}-1}, x_{2^{k_{l,m}}} \right) e^{-\frac{1}{2^{l+m}} V(x_{2^{k_{l,m}}})} \right) \eta(x_{2^{k_{l,m}}}, \eta'_{x})_{E_x} =
\]
\[
= \lim_{j \to \infty} \int_{U_j} dx \lim_{m \to \infty} W_{t,\eta^*_x,\eta_x}^{(j)} \left( V_{t,x,k_{l,m}} \bigl| C_{t,x}(U_j) \right) =
\]
\[
= \lim_{j \to \infty} \int_{U_j} dx \lim_{m \to \infty} \int_{C_{t,x}(U_j)} \left( \rho_{t,\eta^*_x,\eta_x}^{(j)}\right) \left( V_{t,x,k_{l,m}} \right) \left( c \right) dw_{t,x}^{(j)} (c) =
\]
\[
= \lim_{j \to \infty} \int_{U_j} dx \lim_{m \to \infty} \int_{C_{t,x}(U_j)} \left( \rho_{t,\eta^*_x,\eta_x}^{(j)}\right) \left( V_{t,x,k_{l,m}} \right) \left( c \right) dw_{t,x}(c) +\tag{2}
\]
\[
+ \lim_{j \to \infty} \int_{U_j} dx \lim_{m \to \infty} \int_{C_{t,x}(U_j)} \left( \rho_{t,\eta^*_x,\eta_x}^{(j)}\right) \left( V_{t,x,k_{l,m}} \right) \left( c \right) dw_{t,x}(c) \left( c \right).\tag{3}
\]

Due to the weak convergence of \( V_{t,x,k_{l,m}} \) to \( V_{t,x} \) in \( \Gamma^2(\mathcal{E}) \), and therefore in \( \Gamma^2(\mathcal{E}|C_{t,x}(U_j)) \) (both spaces being considered with respect to the measure \( w_{t,x} \)), the term \( \text{(2)} \) is
\[
\lim_{j \to \infty} \int_{U_j} dx \int_{C_{t,x}(U_j)} \left( \rho_{t,\eta^*_x,\eta_x}^{(j)}\right) \left( V_{t,x} \right) \left( c \right) dw_{t,x}(c) =
\]
\[ = \int_M \text{d}x \int_{C_{t,x}} [\rho_{t,y^*_{\eta},x}(c)] [V_{t,x}(c)] \text{d}w_{t,x}(c) = \]

\[ = \int_M \text{d}x \int_{C_{t,x}} \langle \eta_{\epsilon(t)}, P_{t,x}^2(V_{t,x}(c)) \eta_{\epsilon(t)} \rangle E_{\epsilon(t)} \text{d}w_{t,x}(c) = \]

\[ = \int_M \text{d}x \int_{C_{t,x}} \langle [P_{t,x}^2(V_{t,x}(c))]^* \eta_{\epsilon(t)}, \eta_{\epsilon(t)} \rangle E_{\epsilon(t)} \text{d}w_{t,x}(c) = \]

\[ = \int_M \text{d}x \left( \int_{C_{t,x}} [P_{t,x}^2(V_{t,x}(c))]^* \eta_{\epsilon(t)} \text{d}w_{t,x}(c), \eta_{\epsilon(t)} \right) E_{\epsilon(t)}. \]

In order to obtain the limit when \( j \to \infty \) we have applied the dominated convergence theorem on \( M \) to the limit

\[ \lim_{j \to \infty} 1_{U_j}(x) \int_{C_{t,x} \cap (U_j)} [\rho_{t,y^*_{\eta},x}(c)] [V_{t,x,k_1}(c)] \text{d}w_{t,x}(c) = \int_{C_{t,x}} [\rho_{t,y^*_{\eta},x}(c)] [V_{t,x,k_1}(c)] \text{d}w_{t,x}(c) \]

valid for almost all \( x \in M \), where \( 1_{U_j} \) is the characteristic function of \( U_j \). The domination is insured by the fact that both \( [\rho_{t,y^*_{\eta},x}(c)] \|E_x^* \) and \( [\rho_{t,y^*_{\eta},x}(c)] \|E_x^* \) are bounded by \( \|\eta_{\epsilon(t)}\|E_x \|\eta(c(t))\|E_{\epsilon(t)} \), and \( \|V_{t,x,k_1}(c)\|E_x \) is bounded by \( e^{-t_\beta} \), for almost all \( c \), hence

\[
\left| 1_{U_j}(x) \int_{C_{t,x} \cap (U_j)} [\rho_{t,y^*_{\eta},x}(c)] [V_{t,x,k_1}(c)] \text{d}w_{t,x}(c) \right| \leq e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x \int_{C_{t,x} \cap (U_j)} \|\eta(c(t))\|E_{\epsilon(t)} \text{d}w_{t,x}(c) \leq \]

\[
\leq e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x \int_{C_{t,x}} \|\eta(c(t))\|E_{\epsilon(t)} \text{d}w_{t,x}(c) = e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x \int_M h(t,x,y) \|\eta_{\epsilon(t)}\|E_{\epsilon(t)} \|\eta\|E_y \text{d}y \leq \]

\[
\leq e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x (e^{-tH_{\alpha \eta}}) \|\eta\|E_y (x), \]

and the latter function is finite at every \( x \in M \) and has the integral

\[
\int_M \|\eta_{\epsilon(t)}\|E_x (e^{-tH} \|\eta\|E_y) \text{d}x = \langle \|\eta\|, e^{-tH} \|\eta\| \rangle_{L^2(M)} \leq \|\eta\|_{L^2(E)} \|\eta\|_{L^2(E)} < \infty .
\]

Using the same majorizations as above, and using that \( w_{t,x}^{(j)} \leq w_{t,x} \), the term \( 3 \) is 0 because

\[
\lim_{j \to \infty} 1_{U_j} \int_{C_{t,x} \cap (U_j)} \|P_{t,x}^2(V_{t,x,k_1}(c)) - w_{t,x}^{(j)}(c)\|E_x \text{d}w_{t,x}(c) = \]

\[
\leq \lim_{j \to \infty} \int_{U_j} \text{d}x \int_{C_{t,x} \cap (U_j)} \|P_{t,x}^2(V_{t,x,k_1}(c)) - w_{t,x}^{(j)}(c)\|E_x \|\eta_{\epsilon(t)}\|E_{\epsilon(t)} \text{d}w_{t,x}(c) \leq \]

\[
\leq e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x \int_{C_{t,x} \cap (U_j)} \|\eta_{\epsilon(t)}\|E_{\epsilon(t)} \text{d}w_{t,x}(c) \leq \]

\[
\leq e^{-t_\beta} \|\eta_{\epsilon(t)}\|E_x \int_{C_{t,x} \cap (U_j)} \|\eta_{\epsilon(t)}\|E_{\epsilon(t)} \text{d}w_{t,x}(c) = \]

\[
\leq \|\eta\|_{L^2(E)} \lim_{j \to \infty} \|e^{-tH} \|\eta\| - e^{-tH_{\alpha \eta}} \|\eta\|_{L^2(M)} = 0 .
\]
We conclude that

\[
\langle e^{-tH-V} \eta, \eta' \rangle_{\Gamma^2(E)} = \int_M dx \left\langle \int_{C_{t,x}} \left[ \mathcal{P}_{t,x}^2 (V_{t,x})(c) \right]^* \eta_{c(t)} dW_{t,x}(c), \eta'_x \right\rangle_{E_x},
\]

whence

\[
(e^{-tH-V} \eta)(x) = \int_{C_{t,x}} \left[ \mathcal{P}_{t,x}^2 (V_{t,x})(c) \right]^* \eta_{c(t)} dW_{t,x}(c)
\]

for every \( \eta \in \Gamma^2(E) \) and almost all \( x \in M \). \( \square \)

Notice that if we define the map \( V_{t,x} : C_{t,x} \to \text{End} E_x \) by

\[
V_{t,x}(c) = \left[ \mathcal{P}_{t,x}^2 (V_{t,x})(c) \right]^* \mathcal{P}_{t,x}^2 (\text{Id})(c)
\]

we may trivially rewrite the Feynman-Kac formula under the equivalent form

\[
(e^{-tH-V} \eta)(x) = \int_{C_{t,x}} \left[ V_{t,x}(c) \right] \left[ \mathcal{P}_{t,x}^2 (\text{Id})(c) \right]^{-1} \eta_{c(t)} dW_{t,x}(c);
\]

so rewritten, the Feynman-Kac formula in bundles has been obtained by other authors, too, but in other contexts and under different assumptions:

- the authors of [BP08] use functional-analytic techniques (again based on Chernoff’s theorem) but the potential \( V \) is assumed smooth and \( M \) is a closed manifold;

- the authors of [DT01] use probabilistic techniques to give abstract conditions in proposition 4.5 under which the Feynman-Kac formula is valid, after which proposition 5.1 shows that these conditions are met when \( M \) is closed; the potential (denoted therein by \( R \)) is assumed smooth (p.48);

- in [Güneysu10] the Feynman-Kac formula is proved using functional-analytic techniques, but assuming the existence of the stochastic parallel transport, under the assumption that the manifold is both metrically and stochastically complete, and under very generous hypotheses on the potential (in theorem 3.1 it is assumed essentially bounded, and in theorem 3.3 the result is extended to the more general situation when the potential is locally square-integrable); in remark 1.4 therein the author sketches the modifications to be made to the proof in order for the assumption of metric completeness to be dropped, but does not give further details;

- in [BG20] (arXiv preprint, submitted for publication but still unpublished at the date of writing of this text), the potential \( V \), which may be understood as a differential operator of order 0, is assumed now to be a differential operator of order 0 or 1 acting on the smooth sections in \( E \) (in particular, this means that \( V \) has smooth coefficients), such that the operator \( \nabla^* \nabla + V \) should be sectorial; such a potential gives rise naturally to a stochastic differential equation, the unique solution of which is assumed to be locally square-integrable in a certain uniform way with respect to \( x \in M \) (in our notations); this hypothesis guarantees that the equality in the Feynman-Kac formula will be valid everywhere, not just almost everywhere. No restrictions are placed on the manifold \( M \).

It can be seen, by way of comparison with the cited previous work, that the Feynman-Kac formula presented here seems to be the most general one currently existing in the literature.
Corollary 6.2. If $V : M \to \mathbb{R}$ is continuous and lower-bounded, then the above Feynman-Kac formula reduces to

$$(e^{-iH v^{-1}V} \eta)(x) = \int_{C_{t,x}} e^{-i \int_0^t V(c(s))\,ds} \left[ P_{t,x}^2(\text{Id}) (c) \right]^{-1} \eta(c(t)) \, dw_{t,x}(c).$$

Proof. When $V$ is a continuous scalar function,

$$V_{t,x,k}(c) = e^{-\frac{i}{2}V(c(\frac{t}{k}))} \otimes \cdots \otimes e^{-\frac{i}{2}V(c(t))} = e^{-\frac{i}{2} \sum_{j=1}^k V(c(\frac{t}{j}))} \text{Id}_{E_{c(\frac{t}{k})} \otimes \cdots \otimes E_{c(\frac{t}{2})}} =$$

$$= e^{-\frac{i}{2} \sum_{j=1}^k V(c(\frac{t}{j}))} \text{Id} \to e^{-\int_0^t V(c(s))\,ds} \text{Id},$$

the convergence being valid for all $c \in C_{t,x}$, therefore also weakly in $\Gamma^2(E)$, using the dominated convergence theorem. It follows that

$$P_{t,x}^2(V_{t,x}) = e^{-\int_0^t V(c(s))\,ds} P_{t,x}^2(\text{Id})$$

and the conclusion is immediate. \hfill \square

Remark 6.3. Comparing the results herein with the ones obtained by the author in [Mustătea22], we notice that if $E = M \times \mathbb{C}$ and $\nabla = d + i\alpha$, then

$$P_{t,x}^2(\text{Id}) = e^{-i\text{Strat}_{t,x}(\alpha)}$$

for almost all $x \in M$. This shows once more that the Stratonovich stochastic integral is the "most geometrically-flavoured" of all the stochastic integrals considered therein, since its exponential (including the negative imaginary unit) is the stochastic parallel transport, in perfect analogy with how the parallel transport along some smooth curve $c$ with respect to $\nabla$ is $e^{-i\text{Strat}_{t,x}(\alpha)}$.

When $V = 0$, the Feynman-Kac formula and the disintegration theorem for measures allow us to derive a formula expressing the heat kernel $h_v$ in the bundle $E$ in terms of the heat kernel $h$ acting on functions and the stochastic parallel transport in $E$. In order to state it, we shall need to introduce some notations.

Let us endow the manifold $M$ with the measure $h(t, x, \cdot) \, dx$, where $dx$ is the natural measure on $M$. The map $p_t : C_{t,x} \to M$ satisfies the hypotheses of the disintegration theorem (p.78-III and following of [DM78]), therefore there exists a family $(\nu_{t,x,y})_{y \in M}$ of Borel regular probabilities on $C_{t,x}$, uniquely determined for almost all $y \in M$, such that $\nu_{t,x,y}$ is concentrated on $p_t^{-1}(\{y\}) = \{c \in C_{t,x} \mid c(t) = y\}$ for almost all $y \in M$, and

$$\int_{C_{t,x}} f \, dw_{t,x} = \int_M h(t, x, y) \left( \int_{p_t^{-1}(\{y\})} f \, d\nu_{t,x,y} \right) \, dy$$

for all $f \in L^1(C_{t,x})$.

Corollary 6.4.

$$h_v(t, x, y) = h(t, x, y) \int_{p_t^{-1}(\{y\})} \left[ P_{t,x}^2(\text{Id}) \right]^{-1} \, d\nu_{t,x,y} \in E_x \otimes E^*$$

for all $t > 0$, all $x \in M$ and almost all $y \in M$.  

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Lemma 6.6. \( \textbf{Proof.} \) Choosing \( V = 0 \) in the Feynman-Kac formula in bundles, we obtain

\[
\omega \left[ \int_M h_N(t, x, y) \eta_y \, dy \right] = \omega \left\{ [e^{-tH_N}]\eta(x) \right\} = \omega \left[ \int_{\mathcal{C}_{t,x}} [P_{t,x}^2(Id)(c)]^{-1} \eta_{c(t)} \, dw_{t,x}(c) \right] = \\
= \omega \left[ \int_M dy \, h(t, x, y) \int_{P_{t,x}^{-1}((y))} [P_{t,x}^2(Id)(c)]^{-1} \eta_y \, dv_{t,x,y}(c) \right]
\]

for all \( \eta \in \Gamma^2(E) \) and \( \omega \in E_x^\ast \), whence the conclusion is clear. \( \square \)

Remark 6.5. Since \( h_N(t, x, y) = h_N(t, y, x) \), the above equality may be rewritten, equivalently, as

\[
h_N(t, x, y) = h(t, x, y) \int_{P_{t,x}^{-1}((x))} [P_{t,x}^2(Id)(c)] \, dv_{t,x,y}(c)
\]

for all \( t > 0 \) and \( y \in M \), and for almost all \( x \in M \).

If \( M = \bigcup_{j \in \mathbb{N}} U_j \), we already know that \( h_j \to h \) pointwise; as a final application of all the results obtained above, we shall show that \( h_N^{(j)} \to h_N \) pointwise, too. It is clear that the disintegration theorem may be used, analogously, on each space \( \mathcal{C}_{t,x}(U_j) \) endowed with the measure \( w_{t,x}^{(j)} \), in order to obtain

\[
\int_{\mathcal{C}_{t,x}(U_j)} f \, dw_{t,x}^{(j)} = \int_{U_j} h_N^{(j)}(t, x, y) \left( \int_{P_{t,x}^{-1}((y))} f \, dv_{t,x,y}^{(j)} \right) \, dy
\]

for all \( f \in L^1(\mathcal{C}_{t,x}(U_j), w_{t,x}^{(j)}) \), \( L^1(\mathcal{C}_{t,x}, w_{t,x}) \).

Lemma 6.6.

\[
\lim_{j \to \infty} \int_{P_{t,x}^{-1}((y))} f \, dv_{t,x,y}^{(j)} = \int_{P_{t,x}^{-1}((y))} f \, dv_{t,x,y}
\]

for all \( f \in L^1(\mathcal{C}_{t,x}) \), all \( t > 0 \) and \( x \in M \), and almost all \( y \in M \).

Corollary 6.7. \( h_N(t, x, y) = \lim_{j \to \infty} h_N^{(j)}(t, x, y) \) for all \( t > 0 \) and \( x, y \in M \).

\textbf{Proof.} We know that \( h(t, x, y) = \lim_{j \to \infty} h_N^{(j)}(t, x, y) \) for all \( t > 0 \) and \( x, y \in M \), whence, by corroborating the preceding results, we obtain that there exists a co-null \( C \subseteq M \) such that \( h_N(t, x, y) = \lim_{j \to \infty} h_N^{(j)}(t, x, y) \) for all \( t > 0 \), \( x \in M \) and all \( y \in C \). Since \( h_N \) and \( h_N^{(j)} \) are smooth on \( (0, \infty) \times M \times M \), and since \( C \) is dense in \( M \) by virtue of being co-null, an elementary argument shows that \( C = M \). \( \square \)

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