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Abstract. This manuscript discusses the approximation of a global maximizer of the Kantorovich mass transfer problem through the approach of $p$-Laplacian equation. Using an approximation mechanism, the primal maximization problem can be transformed into a sequence of minimization problems. By applying the canonical duality theory, one is able to derive a sequence of analytic solutions for the minimization problems. In the final analysis, the convergence of the sequence to a global maximizer of the primal Kantorovich problem will be demonstrated.

1. Introduction

Complementary variational method has been applied in the study of finite deformation by Hellinger since the beginning of the 20th century. During the last few years, considerable effort has been taken to find minimizers for non-convex strain energy functionals with a double-well potential. In this respect, Ericksen bar is a typical model for the research of elastic phase transitions. In [10], R. W. Ogden et al. treated the Ericksen bar as a 1-D smooth compact manifold and discussed two classical loading devices, namely, hard device and soft device, by introducing a distributed axial body force. By applying the canonical duality method, the authors characterized the local energy extrema and the global energy minimizer for both hard device and soft device. This method proved to be very efficient in solving lots of open problems in the mechanical fields such as non-convex optimal design and control, nonlinear stability analysis of finite deformation, nonlinear elastic theory with residual strain, existence results for Nash equilibrium points of non-cooperative games etc. Interested readers can refer to [9, 10, 16] for more details.

This paper mainly addresses the Kantorovich problem in higher dimensions. Let $\Omega = \mathbb{B}(O_1, R_1)$ and $\Omega^* = \mathbb{B}(O_2, R_2)$ denote two open balls with centers $O_1$ and $O_2$.
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radii $R_1$ and $R_2$ in the Euclidean space $\mathbb{R}^n$, respectively, and we denote $U := \Omega \cup \Omega^*$. Here we focus on the following two representative cases:

- $\Omega = B(O_1, R_1)$, $\Omega^* = B(O_2, R_2)$, $\Omega \cap \Omega^* = \emptyset$;
- $\Omega = B(O_1, R_1)$, $\Omega^* = B(O_1, R_2)$, $R_1 \neq R_2$.

Let $f^+$ and $f^-$ be two nonnegative density functions in $\Omega$ and $\Omega^*$, respectively, and satisfy the normalized balance condition

$$\int_{\Omega} f^+ dx = \int_{\Omega^*} f^- dx = 1.$$ 

For convenience’s sake, let $f := f^+ - f^-$. First, let the admissible set $\mathscr{A}$ be defined as

$$\mathscr{A} := \left\{ \phi \in W^{1,\infty}_0(U) \cap C(U) \mid \|\nabla \phi\|_{L^\infty} \leq 1, \phi \text{ radially symmetric}, \phi = 0 \text{ on } \Omega \cap \Omega^* \right\},$$

where $W^{1,\infty}_0(U)$ is a Sobolev spaces. The aim is to find an analytic global maximizer (so-called Kantorovich potential) $u \in \mathscr{A}$ for the Kantorovich problem in the following form,

$$(P) : K[u] = \max_{w \in \mathscr{A}} \left\{ K[w] := \int_U w f dx \right\}.$$ 

In this paper, we consider the Kantorovich problem through a $p$-Laplacian approach by introducing an approximation of the primal $(P)$ [6],

$$(P^{(p)}) : \min_{w_p \in \mathscr{A}} \left\{ I^{(p)}[w_p] := \int_U \left( H^{(p)}(\nabla w_p) - w_p f \right) dx \right\},$$

where $p > 2$ and $H^{(p)} : \mathbb{R}^n \to \mathbb{R}^+$ is defined as

$$H^{(p)}(\gamma) := \|\gamma\|_{p}^{p}/p,$$

and $I^{(p)}$ is called the potential energy functional. It’s evident that

$$- \lim_{p \to +\infty} \min_{w_p \in \mathscr{A}} \{ I^{(p)}[w_p] \} = \max_{w \in \mathscr{A}} \{ K[w] \}.$$ 

Consequently, once a function $\bar{u}_p$ satisfying $I^{(p)}[\bar{u}_p] = \min_{w_p \in \mathscr{A}} \{ I^{(p)}[w_p] \}$ is obtained, then it will help find out an analytic Kantorovich potential $u = \lim_{p \to +\infty} \bar{u}_p$ in the $L^\infty$ sense, which maximizes the primal problem $(P)$.

By variational calculus, one derives a corresponding Euler-Lagrange equation for $(P^{(p)})$,

$$\text{div}(|\nabla u_p|^{-2} \nabla u_p) + f = 0, \quad \text{in } U \setminus \{\Omega \cap \Omega^*\},$$

equipped with the Dirichlet boundary condition. For the integer case, $p = 1$, by variational calculus, one obtains the mean curvature operator; $p = 2$, one has the Laplace operator (see [3]). For $p = n$, one derives the $n$-harmonic equation which is invariant under Möbius transformation. While for the fractional case, such as $p = 3/2$, $p-$Laplacian describes the flow through porous media. And glaciologist usually study the case $p \in (1, 4/3]$. For more background materials, please refer to [15].

Clearly, (3) is a nonlinear $p$-Laplacian problem which is difficult to solve by the direct approach [3, 8, 15]. However, by the canonical duality theory, one is able to
demonstrate the existence and uniqueness of the solution for the nonlinear differential equation, which establishes the equivalence between the global minimizer of \( P^{(p)} \) and the solution of Euler-Lagrange equation (3).

At the moment, we would like to introduce the main theorems.

**Theorem 1.1.** For any positive density functions \( f^+ \in C(\overline{\Omega}) \) and \( f^- \in C(\overline{\Omega^*}) \) satisfying the normalized balance condition, there exists a unique solution \( \bar{u}_p \in \mathcal{A} \) for the Euler-Lagrange equation (3), which is at the same time a global minimizer for the approximation problem \( P^{(p)} \). In particular, let

\[
E_p(x) := x^{(2p-2)/(p-2)}, \quad x \in [0,1],
\]

and \( E_p^{-1} \) stands for the inverse of \( E_p \), then one has

- \( \Omega = \mathbb{B}(O_1,R_1), \Omega^* = \mathbb{B}(O_2,R_2), \Omega \cap \Omega^* = \emptyset \). \( \bar{u}_p \) can be represented explicitly as

\[
\bar{u}_p(r) = \begin{cases} 
\int_{R_1}^r F(\rho)/E_p^{-1}(F^2(\rho)\rho^2)d\rho, & r \in [0,R_1], \\
\int_{R_2}^r G(\rho)/E_p^{-1}(G^2(\rho)\rho^2)d\rho, & r \in [0,R_2],
\end{cases}
\]

where \( F \) and \( G \) are defined as

\[
\begin{align*}
F(r) &= -\Gamma(n/2)/(2\pi^{n/2}r^n) + \int_{r}^{R_1} f^+(\rho)\rho^{n-1}/r^n d\rho, \quad r \in [0,R_1], \\
G(r) &= \Gamma(n/2)/(2\pi^{n/2}r^n) - \int_{r}^{R_2} f^-(\rho)\rho^{n-1}/r^n d\rho, \quad r \in [0,R_2].
\end{align*}
\]

- \( \Omega = \mathbb{B}(O_1,R_1), \Omega^* = \mathbb{B}(O_2,R_2), R_1 > R_2 > 0 \). \( \bar{u}_p \) can be represented explicitly as

\[
\bar{u}_p(r) = \int_{R_2}^r F_p(\rho)/E_p^{-1}(F_p^2(\rho)\rho^2)d\rho, \quad r \in [R_2,R_1],
\]

where

\[
F_p(r) := C_p R_2^n/r^n - \int_{R_2}^r f^+(\rho)\rho^{n-1}/r^n d\rho,
\]

and \( C_p \in (0,\Gamma(n/2)/(2\pi^{n/2}R_2^n)) \).

- \( \Omega = \mathbb{B}(O_1,R_1), \Omega^* = \mathbb{B}(O_1,R_2), R_2 > R_1 > 0 \). \( \bar{u}_p \) can be represented explicitly as

\[
\bar{u}_p(r) = \int_{R_1}^r G_p(\rho)/E_p^{-1}(G_p^2(\rho)\rho^2)d\rho, \quad r \in [R_1,R_2],
\]

where

\[
G_p(r) := -D_p R_1^n/r^n + \int_{R_1}^r f^-(\rho)\rho^{n-1}/r^n d\rho,
\]

and \( D_p \in (0,\Gamma(n/2)/(2\pi^{n/2}R_1^n)) \).
Theorem 1.2. For any positive density functions $f^+ \in C(\Omega)$ and $f^- \in C(\Omega^*)$ satisfying the normalized balance condition, there exists a global maximizer for the Kantorovich problem $(\mathcal{P})$.

The rest of the paper is organized as follows. In Section 2, first we introduce some useful notations which will simplify our proof considerably. Then we apply the canonical dual transformation to deduce a perfect dual problem $(\mathcal{P}^D_p)$ corresponding to $(\mathcal{P}(p))$ and a pure complementary energy principle. Next we apply the canonical duality theory to prove Theorem 1.1 and Theorem 1.2.

2. Proof of the main results

2.1. Some useful notations.
- $\theta^p$ is given by
  $$\theta^p(x) = (\theta_{p,1}(x), \cdots, \theta_{p,n}(x)) = |\nabla w_p|^{p-2}\nabla w_p.$$
- $\Phi(p): \mathcal{A} \rightarrow L^\infty(U)$ is a nonlinear geometric mapping defined as
  $$\Phi(p)(w_p) := |\nabla w_p|^2.$$
  For convenience’s sake, denote $\xi_p := \Phi(p)(w_p)$. It is evident that $\xi_p$ belongs to the function space $\mathcal{U}$ given by
  $$\mathcal{U} := \left\{ \phi \in L^\infty(U) \mid 0 \leq \phi \leq 1 \right\}.$$
- $\Psi(p): \mathcal{U} \rightarrow L^\infty(U)$ is a canonical energy defined as
  $$\Psi(p)(\xi_p) := \xi_p^{p/2}/p,$$
  which is a convex function with respect to $\xi_p$. For simplicity, denote $\zeta_p := \xi_p^{(p-2)/2}/2$, which is the Gâteaux derivative of $\Psi(p)$ with respect to $\xi_p$. Moreover, $\zeta_p$ is invertible with respect to $\xi_p$ and belongs to the function space $\mathcal{U}$,
  $$\mathcal{U} := \left\{ \phi \in L^\infty(U) \mid 0 \leq \phi \leq 1/2 \right\}.$$
- $\Psi^*_p: \mathcal{U} \rightarrow L^\infty(U)$ is defined as
  $$\Psi^*_p(\xi_p) := \xi_p - \Psi(p)(\xi_p) = (1 - 2/p)2^{2/(p-2)}\xi_p^{p/(p-2)}.$$
- $\lambda_p$ is defined as $\lambda_p := 2\zeta_p$, and belongs to the function space $\mathcal{V}$,
  $$\mathcal{V} := \left\{ \phi \in L^\infty(U) \mid 0 \leq \phi \leq 1 \right\}.$$

2.2. Canonical duality techniques.

Definition 2.1. By Legendre transformation, one defines a Gao-Strang total complementary energy functional $\Xi(p)$,
$$\Xi(p)(u_p, \xi_p) := \int_U \left\{ \Phi(p)(u_p)\zeta_p - \Psi^*_p(\xi_p) - fu_p \right\} dx.$$ 

Next we introduce an important criticality criterium for the Gao-Strang total complementary energy functional.
Definition 2.2. $(\bar{u}_p, \bar{\zeta}_p) \in \mathcal{A} \times \mathcal{W}$ is called a critical pair of $\Xi^{(p)}$ if and only if

\begin{align*}
(4) & \quad D_{u_p} \Xi^{(p)}(\bar{u}_p, \bar{\zeta}_p) = 0, \\
(5) & \quad D_{\zeta_p} \Xi^{(p)}(\bar{u}_p, \bar{\zeta}_p) = 0,
\end{align*}

where $D_{u_p}, D_{\zeta_p}$ denote the partial Gâteaux derivatives of $\Xi^{(p)}$, respectively.

Indeed, by variational calculus, we have the following observation from (4) and (5).

Lemma 2.3. On the one hand, for any fixed $\zeta_p \in \mathcal{W}$, (3.4) is equivalent to the equilibrium equation

$$\text{div}(\lambda_p \nabla \bar{u}_p) + f = 0 \quad \text{in} \ U \setminus \{\Omega \cap \Omega^*\}.$$ 

On the other hand, for any fixed $u_p \in \mathcal{A}$, (5) is consistent with the constructive law

$$\Phi^{(p)}(u_p) = D_{\zeta_p} \Psi^{(p)}(\bar{\zeta}_p).$$

Lemma 3.2.3 indicates that $\bar{u}_p$ from the critical pair $(\bar{u}_p, \bar{\zeta}_p)$ solves the Euler-Lagrange equation (3).

Definition 2.4. From Definition 3.2.1, one defines the Gao-Strang pure complementary energy $I^{(p)}_d$ in the form

$$I^{(p)}_d[\zeta_p] := \Xi^{(p)}(\bar{u}_p, \zeta_p),$$

where $\bar{u}_p$ solves the Euler-Lagrange equation (3).

To simplify the discussion, we use another representation of the pure energy $I^{(p)}_d$ given by the following lemma.

Lemma 2.5. The pure complementary energy functional $I^{(p)}_d$ can be rewritten as

$$I^{(p)}_d[\zeta_p] = -\int_U \left\{ \frac{|\theta_p|^2}{4\zeta_p} + (1 - 2/p)2^{2/(p-2)}\zeta_p^{p/(p-2)} \right\} dx,$$

where $\theta_p$ satisfies

\begin{align*}
(6) & \quad \text{div} \theta_p + f = 0 \quad \text{in} \ U,
\end{align*}

equipped with a hidden boundary condition.

Proof. Through integrating by parts, one has

\begin{align*}
I^{(p)}_d[\zeta_p] &= -\int_U \left\{ \text{div}(2\zeta_p \nabla \bar{u}_p) + f \right\} \bar{u}_p dx \\
& \quad \tag{I} - \int_U \left\{ \zeta_p |\nabla \bar{u}_p|^2 + (1 - 2/p)2^{2/(p-2)}\zeta_p^{p/(p-2)} \right\} dx. \tag{II}
\end{align*}

Since $\bar{u}_p$ solves the Euler-Lagrange equation (3), then the first part (I) disappears. Keeping in mind the definition of $\theta_p$ and $\zeta_p$, one reaches the conclusion. \hfill \Box
With the above discussion, next we establish a variational problem to the approximation problem \((P^{(p)})\).

\[
(P_{d}^{(p)}): \max_{\zeta_{p} \in \mathbb{W}} \left\{ I_{d}^{(p)}[\zeta_{p}] = -\int_{U} \left\{ \left| \nabla \theta_{p} \right|^{2}/(4 \zeta_{p}) + (1 - 2/p)^{2/(p-2)} \zeta_{p}^{p/(p-2)} \right\} \right\}.
\]

Indeed, by calculating the Gâteaux derivative of \(I_{d}^{(p)}\) with respect to \(\zeta_{p}\), one has

**Lemma 2.6.** The variation of \(I_{d}^{(p)}\) with respect to \(\zeta_{p}\) leads to the dual algebraic equation (DAE), namely,

\[
\left| \nabla \theta_{p} \right|^{2} = (2 \tilde{\zeta}_{p})^{(2p-2)/(p-2)},
\]

where \(\tilde{\zeta}_{p}\) is from the critical pair \((\bar{u}_{p}, \tilde{\zeta}_{p})\).

Taking into account the notation of \(\lambda_{p}\), the identity (8) can be rewritten as

\[
\left| \nabla \theta_{p} \right|^{2} = E_{p}(\lambda_{p}) = \lambda_{p}^{(2p-2)/(p-2)}.
\]

It is evident \(E_{p}\) is monotonously increasing with respect to \(\lambda \in [0, 1]\).

**2.3. Proof of Theorem 1.1.** From the above discussion, one deduces that, once \(\theta_{p}\) is given, then the analytic solution of the Euler-Lagrange equation (3) can be represented as

\[
\bar{u}_{p}(x) = \int_{x_{0}}^{x} \eta_{p}(t)dt,
\]

where \(x \in \overline{U}, x_{0} \in \partial U, \eta_{p} := \theta_{p}/\lambda_{p}\). Together with (9), one sees that \(\lim_{p \to +\infty} |\nabla \bar{u}_{p}| = 1\), which is consistent with the a-priori estimate in [5]. Next we verify that \(\bar{u}_{p}\) is exactly a global minimizer for \((P_{d}^{(p)})\) and \(\tilde{\zeta}_{p}\) is a unique global maximizer for \((P_{d}^{(p)})\).

**Lemma 2.7.** (Canonical duality theory) For any positive density functions \(f^{+} \in C(\overline{\Omega})\) and \(f^{-} \in C(\overline{\Omega^{*}})\) satisfying the normalized balance condition, there exists a unique radially symmetric solution \(\bar{u}_{p} \in \mathcal{A}\) for the Euler-Lagrange equations (3) with Dirichlet boundary in the form of (10), which is a unique global minimizer over \(\mathcal{A}\) for the approximation problem \((P^{(p)})\). And the corresponding \(\tilde{\zeta}_{p}\) is a unique global maximizer over \(\mathcal{W}\) for the dual problem \((P_{d}^{(p)})\). Moreover, the following duality identity holds,

\[
I^{(p)}(\bar{u}_{p}) = \min_{u_{p} \in \mathcal{A}} I^{(p)}(u_{p}) = \Xi^{(p)}(\bar{u}_{p}, \tilde{\zeta}_{p}) = \max_{\zeta_{p} \in \mathcal{W}} I_{d}^{(p)}(\zeta_{p}) = I_{d}^{(p)}(\tilde{\zeta}_{p}).
\]

Lemma 3.2.7 shows that the maximization of the pure complementary energy functional \(I_{d}^{(p)}\) is perfectly dual to the minimization of the potential energy functional \(I^{(p)}\). Indeed, identity (11) indicates there is no duality gap between them.

**Proof.** We divide our proof into three parts. In the first and second parts, we discuss the uniqueness of \(\theta_{p}\) for both cases. Global extremum will be studied in the third part. It is worth noticing that the first and second parts are similar to the proof of Theorem 1.2.

**First Part:** \(\Omega = \mathbb{B}(O_{1}, R_{1}), \Omega^{*} = \mathbb{B}(O_{2}, R_{2}), \Omega \cap \Omega^{*} = \emptyset\)
(1) Discussion in $\Omega$

Let $O_1 = (a_1, a_2, \cdots, a_n)$. Actually, a radially symmetric solution for the Euler-Lagrange equation (3) is of the form

$$\vec{\theta}_p = F_p(r) ((x_1 - a_1, \cdots, x_n - a_n)) = F_p \left( \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \right) ((x_1 - a_1, \cdots, x_n - a_n)),$$

where

$$F_p(r) = C_p R_1^n / r^n + \int_{r}^{R_1} f^+ (\rho) \rho^{n-1} / r^n d\rho$$

is the unique solution of the differential equation

$$F_p'(r) + n F_p(r) / r = - f^+(r) / r, \quad r \in (0, R_1].$$

Recall that $\bar{u}_p(R_1) = 0$, as a result,

$$\bar{u}_p(r) = \int_{r}^{R_1} \left( R_1^n C_p + \int_{\rho}^{R_1} f^+(r) r^{n-1} dr \right) / \left( \rho^{n-1} \lambda_p(\rho) \right) d\rho, \quad r \in (0, R_1].$$

As a matter of fact, if $\bar{u}_p \in C[0, R_1]$, we have

$$\lim_{\rho \to 0^+} \left\{ R_1^n C_p + \int_{\rho}^{R_1} f^+(r) r^{n-1} dr \right\} = 0,$$

which indicates

$$C_p = - \Gamma(n/2) / (2 \pi^{n/2} R_1^n),$$

from the normalized balance condition

$$\int_{\Omega} f^+(x) dx = 2 \pi^{n/2} / \Gamma(n/2) \int_{0}^{R_1} f^+(r) r^{n-1} dr = 1.$$

(2) Discussion in $\Omega^*$

Let $O_2 = (b_1, b_2, \cdots, b_n)$. In fact, a radially symmetric solution for the Euler-Lagrange equation (3) is of the form

$$\vec{\theta}_p = G_p(r) ((x_1 - b_1, \cdots, x_n - b_n)) = G_p \left( \sqrt{\sum_{i=1}^{n} (x_i - b_i)^2} \right) ((x_1 - b_1, \cdots, x_n - b_n)),$$

where

$$G_p(r) = D_p R_2^n / r^n - \int_{r}^{R_2} f^- (\rho) \rho^{n-1} / r^n d\rho$$

is the unique solution of the differential equation

$$G_p'(r) + n G_p(r) / r = f^-(r) / r, \quad r \in (0, R_2].$$

Recall that $\bar{u}_p(R_2) = 0$, as a result,

$$\bar{u}_p(r) = \int_{R_2}^{r} \left( R_2^n D_p - \int_{\rho}^{R_2} f^-(r) r^{n-1} dr \right) / \left( \rho^{n-1} \lambda_p(\rho) \right) d\rho, \quad r \in (0, R_2].$$
Indeed, if $\bar{u}_p \in C[0, R_2]$, then by applying the similar contradiction method as above, one has

$$\lim_{\rho \to 0^+} \left\{ R_2^n D_p - \int_\rho^{R_2} f^-(r)r^{n-1}dr \right\} = 0,$$

which indicates

$$D_p = \Gamma(n/2)/(2\pi^{n/2} R_2^n)$$

from the normalized balance condition

$$\int_{\Omega^*} f^-(x)dx = 2\pi^{n/2}/\Gamma(n/2) \int_0^{R_2} f^-(r)r^{n-1}dr = 1.$$

Second Part: $\Omega = \mathbb{B}(O_1, R_1)$, $\Omega^* = \mathbb{B}(O_1, R_2)$, $R_1 \neq R_2$

(1) $R_1 > R_2 > 0$

Let $O_1 = (a_1, a_2, \cdots, a_n)$. Actually, a radially symmetric solution for the Euler-Lagrange equation (3) is of the form

$$\overrightarrow{\theta}_p = F_p(r)(x_1 - a_1, \cdots, x_n - a_n) = F_p\left( \sum_{i=1}^{n} (x_i - a_i)^2 \right)$$

where

$$F_p(r) = C_p R_2^n/r^n - \int_0^{r} f^+(\rho)\rho^{n-1}/r^n d\rho$$

is the unique solution of the differential equation

$$F'_p(r) + nF_p(r)/r = -f^+(r)/r, \quad r \in [R_2, R_1].$$

Recall that $\bar{u}_p(R_2) = 0$, consequently,

$$\bar{u}_p(r) = \int_{R_2}^{r} \left( R_2^n C_p - \int_{R_2}^{\rho} f^+(r)r^{n-1}dr \right)/\rho^{n-1}\lambda_p(\rho) d\rho, \quad r \in [R_2, R_1].$$

Let

$$\tilde{F}(r) := 1/R_2^n \int_{R_2}^{r} f^+(\rho)\rho^{n-1} d\rho, \quad r \in [R_2, R_1].$$

Since $f^+ > 0$, then $\tilde{F} \in C[R_2, R_1]$ is a strictly increasing function with respect to $r \in [R_2, R_1]$ and consequently is invertible. Let $\tilde{F}^{-1}$ be its inverse function, which is also a strictly increasing function. From (9), we see that there exists a unique piecewise continuous function $\lambda_p(x) \geq 0$. Since

$$\lim_{r \to \tilde{F}^{-1}(C_p)} (-\tilde{F}(r) + C_p R_2^n/(r^{n-1}\lambda_p(r))) = 0,$$

thus $\bar{u}_p$ is continuous at the point $r = \tilde{F}^{-1}(C_p)$. As a result, $\bar{u}_p \in C[R_2, R_1]$. Notice that $\bar{u}_p(R_1) = 0$ and we can determine the constant $C_p$ uniquely. Indeed, let

$$\mu_p(\rho, t) := \left( R_2^n t - \int_{R_2}^{\rho} f^+(r)r^{n-1}dr \right)/\left( \rho^{n-1}\lambda_p(\rho, t) \right)$$
where $\lambda_p(\rho, t)$ is from (9). It is evident that $\lambda_p$ depends on $C_p$. As a matter of fact, it is easy to check $M$ is strictly increasing with respect to $t$, which leads to

$$C_p = M_p^{-1}(0).$$

Furthermore, by a similar discussion as in [16], we have

$$\lim_{k \to \infty} C_p = \tilde{F}((R_1 + R_2)/2).$$

(2) $0 < R_1 < R_2$

In fact, a radially symmetric solution for the Euler-Lagrange equation (3) is of the form

$$\tilde{\theta}_p = G_p(r)((x_1 - a_1, \ldots, x_n - a_n)) = G_p\left(\sum_{i=1}^{n} (x_i - a_i)^2\right)((x_1 - a_1, \ldots, x_n - a_n),$$

where

$$G_p(r) = -D_p R_1^n/r^n + \int_{R_1}^{r} f^-(\rho)\rho^{n-1}/r^n d\rho$$

is the unique solution of the differential equation

$$G'(r) + nG_p(r)/r = f^-(r)/r, \quad r \in [R_1, R_2].$$

Recall that $\tilde{u}_p(R_1) = 0$, as a result,

$$\tilde{u}_p(r) = \int_{R_1}^{r} \left(- R_1^n D_p + \int_{R_1}^r f^-(\rho)\rho^{n-1} d\rho\right)/\left(\rho^{n-1}\lambda_p(\rho)\right) d\rho, \quad r \in [R_1, R_2].$$

Let

$$\tilde{G}(r) := 1/R_1^n \int_{R_1}^{r} f^-(\rho)\rho^{n-1} d\rho, \quad r \in [R_1, R_2].$$

Since $f^- > 0$, then $\tilde{G} \in C[R_1, R_2]$ is a strictly increasing function with respect to $r \in [R_1, R_2]$ and consequently is invertible. Let $\tilde{G}^{-1}$ be its inverse function, which is also a strictly increasing function. From (9), we see that there exists a unique piecewise continuous function $\lambda_p(x) \geq 0$. Since

$$\lim_{r \to \tilde{G}^{-1}(D_p)} (G(r) - D_p) R_1^n/(r^{n-1}\lambda_p(r)) = 0,$$

thus $\tilde{u}_p$ is continuous at the point $r = G^{-1}(D_p)$. As a result, $\tilde{u}_p \in C[R_1, R_2]$. Notice that $\tilde{u}_p(R_2) = 0$ and we can determine the constant $D_p$ uniquely. Indeed, let

$$\eta_p(\rho, t) := \left(- R_1^nt + \int_{R_1}^{\rho} f^-(\rho) r^{n-1} dr\right)/\left(\rho^{n-1}\lambda_p(\rho, t)\right)$$

and

$$N_p(t) := \int_{R_1}^{\tilde{G}^{-1}(t)} \eta_p(\rho, t) d\rho + \int_{\tilde{G}^{-1}(t)}^{R_2} \eta_p(\rho, t) d\rho.$$
where \( \lambda_p(\rho, t) \) is from (9). It is evident that \( \lambda_p \) depends on \( C_p \). As a matter of fact, it is easy to check \( N_p \) is strictly increasing with respect to \( t \), which leads to
\[
D_p = N_p^{-1}(0).
\]
Furthermore, by a similar discussion as in [16], we have
\[
\lim_{k \to \infty} D_p = \tilde{G}((R_1 + R_2)/2).
\]

Third Part:

On the one hand, for any test function \( \phi \in \mathcal{A} \) satisfying \( \nabla \phi \neq 0 \) a.e. in \( U \), the second variational form \( \delta_\phi^2 I^{(p)} \) with respect to \( \phi \) is equal to
\[
\int_U \left\{ |\nabla \bar{u}_p|^p |\nabla \phi|^2 + (p - 2)|\nabla \bar{u}_p|^{p-4} (\nabla \bar{u}_p \cdot \nabla \phi)^2 \right\} dx.
\]
On the other hand, for any test function \( \psi \in \mathcal{W} \) satisfying \( \psi \neq 0 \) a.e. in \( U \), the second variational form \( \delta_\psi^2 J^{(p)}_d \) with respect to \( \psi \) is equal to
\[
- \int_U \psi^2 \left\{ |\overrightarrow{\theta}_p|^2/(2\zeta_p^3) + 1/(p - 2)2^{p/(p-2)}c_p/(p-2) \right\} dx.
\]
From (12) and (13), one deduces immediately that
\[
\delta_\phi^2 I^{(p)}(\bar{u}_p) > 0, \quad \delta_\psi^2 J^{(p)}_d(\bar{c}_p) < 0.
\]
Together with the uniqueness of \( \overrightarrow{\theta}_p \) discussed in the first and second parts, the proof is concluded.

Consequently, we reach the conclusion of Theorem 1.1 by summarizing the above discussion.

2.4. Proof of Theorem 1.2: According to Rellich-Kondrachov Compactness Theorem, since
\[
\sup_k |\bar{u}_k| \leq C(R_1, R_2)
\]
and
\[
\sup_k |\nabla \bar{u}_k| \leq 1,
\]
then, there exists a subsequence(without any confusion, we still denote as) \( \{\bar{u}_k\}_k \) and \( u \in W_0^{1, \infty}(U) \cap C(\overline{U}) \) such that
\[
\bar{u}_k \to u \ (k \to \infty) \text{ in } L^\infty(U),
\]
\[
\nabla \bar{u}_k \rightharpoonup \nabla u \ (k \to \infty) \text{ weakly * in } L^\infty(U).
\]
It remains to check that \( u \) satisfies (5). From (19), one has
\[
\|\nabla u\|_{L^\infty(U)} \leq \liminf_{k \to \infty} \|\nabla \bar{u}_k\|_{L^\infty(U)} \leq \sup_{k \to \infty} \|\nabla \bar{u}_k\|_{L^\infty(U)} \leq 1.
\]
Consequently, one reaches the conclusion of Theorem 1.2 by summarizing the above discussion.
Remark 2.8. Frankly speaking, the uniqueness of the global minimizer for the primal problem does not hold when \( U \) is a general Lipschitz domain. As \( p \to \infty \), we have the infinity harmonic equation
\[
\sum_{j,k=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_j \partial x_k} = f.
\]
This equation has often been used in image processing and optimal Lipschitz extensions.

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