Permuting the Roots of Univariate Polynomials Whose Coefficients Depend on Parameters

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We address two interrelated problems concerning permutation of roots of univariate polynomials whose coefficients depend on parameters. First, we compute the Galois group of polynomials \( \varphi(x) \in \mathbb{C}[t_1, \cdots, t_k][x] \) over \( \mathbb{C}(t_1, \cdots, t_k) \). Provided that the corresponding multivariate polynomial \( \varphi(x, t_1, \cdots, t_k) \) is generic with respect to its support \( A \subset \mathbb{Z}^{k+1} \), we determine the latter Galois group for any \( A \). Second, we determine the Galois group of systems of polynomial equations of the form \( p(x, t) = q(t) = 0 \) where \( p \) and \( q \) have prescribed supports \( A_1 \subset \mathbb{Z}^2 \) and \( A_2 \subset \{0\} \times \mathbb{Z} \) respectively. For each problem, we determine the image of an appropriate braid monodromy map in order to compute \( \text{Gal}(\mathcal{U}) \).

1. Introduction

1.1. BKK-enumerative problems. Recall that the support of a \( n \)-variate Laurent polynomial \( f(x_1, \cdots, x_n) \) is the finite subset of \( \mathbb{Z}^n \) consisting of tuples \( (a_1, \cdots, a_n) \) such that \( f \) has a monomial of the form \( cx_1^{a_1} \cdots x_n^{a_n} \) with \( c \in \mathbb{C}^* \). Any collection \( A := (A_1, \cdots, A_n) \) of finite subsets \( A_j \subset \mathbb{Z}^n \) gives rise to the following enumerative problem. For any generic tuple \( f := (f_1, \cdots, f_n) \) of Laurent polynomials \( f_j \) with support \( A_j \), the Bernstein-Kouchnirenko-Khovanskii Theorem guarantees that the system of equations \( f_1 = \cdots = f_n = 0 \) has \( N \) distinct solutions in \( (\mathbb{C}^*)^n \), where \( N \) is the mixed volume \( \text{MV}(A_1, \cdots, A_n) \). Define the space of conditions \( \mathcal{U}_A \subset \prod_j (\mathbb{C}^*)^{A_j} \) as the Zariski-open subset of systems \( f \) with \( N \) distinct solutions. Consider now the incidence variety

\[
\mathcal{U}_A := \{(x, f) \in (\mathbb{C}^*)^n \times \mathcal{U}_A : f_1(x) = \cdots = f_n(x) = 0\}.
\]

Then, the \( A \)-enumerative problem is the degree \( N \) covering

\[
\mathcal{U}_A \to \mathcal{U}_A
\]

induced by the projection \( (\mathbb{C}^*)^n \times \mathcal{U}_A \to \mathcal{U}_A \).

In the context of enumerative geometry, it is of interest to compute the monodromy group of the covering \((\mathcal{U}_A, \mathcal{U}_A)\), see [SY21] for an overview. In particular, the latter monodromy group can be interpreted as a Galois group, according to Harris [Har79]. Therefore, we denote this group \( G_A \) and refer to its computation as the Galois A-problem.

The most classical case is \( n = 1 \) and \( A = \{0, 1, \cdots, N\} \) where \( G_A \) is the Galois group of the univariate polynomial \( f(x) := c_0 + c_1 x + \cdots + c_N x^N \) over the field \( \mathbb{C}(c_1, \cdots, c_N) \). Plainly, the group \( G_A \) is the full symmetric group \( \mathfrak{S}_N \). For arbitrary \( n \), consider the case where each \( A_j \) is the set of integer points of the standard simplex of size \( N_j \). In other words, each \( f_j \) is an honest polynomial of degree \( N_j \). Again, the group \( G_A \) is the full symmetric group \( \mathfrak{S}_N \) where \( N = \prod_j N_j \).

In general, the tuple \( A \) can manifest two geometric defects that prevent \( G_A \) from being the full symmetric group. First, the tuple \( A \) can be non-reduced, that is when each subset \( A_j \) can be translated to a common strict sublattice of \( \mathbb{Z}^n \). For \( n = 1 \), the support \( A \) of the polynomial \( f \) is non-reduced whenever \( f(x) = g(x^d) \) for some other polynomial \( g \) and some integer \( d > 1 \).
The Galois group of \( f \) is necessarily imprimitive, see Section \[1.2\]. Second, the tuple \( A \) can be reducible, that is when \( k \) of the polynomials \( f_j \) depend only on \( k \) variables (in some appropriate coordinate system). The simplest instance of reducibility is \( n = 2 \) with \( A_2 \subset \{0\} \times \mathbb{Z} \). In that case, the roots of the system \( f_1(x_1, x_2) = f_2(x_2) = 0 \) are distributed among the collection of horizontal lines \( f_2(x_2) = 0 \), leading again to an imprimitive Galois group.

**Remark 1.1.** Non-reduced supports are also called lacunary in the literature, while reducible supports are also called triangular. Throughout the paper, we adopt the former terminology which is rather common in Khovanskii’s school. This terminology is motivated by the fact that a support is non-reduced (respectively reducible) if and only if the corresponding discriminant is non-reduced (respectively reducible).

In [Est19], the first author showed that \( G_A \) is the full symmetric group, provided that \( A \) is free from the above defects. In [EL22b], we studied tuples \( A \) that are non-reduced but irreducible, i.e., there is a strict sublattice in the background but no square subsystem. In that case, the Galois group is at most as large as an explicit wreath product, see Section \[1.2\]. In [EL22b], we gave combinatorial criteria on \( A \) that ensure that \( G_A \) is equal to the latter wreath product, but also criteria that ensure that \( G_A \) is strictly smaller. Already for \( n = 2 \), the group \( G_A \) is not known in general.

That the Galois group \( G_A \) is smaller than expected indicates that the corresponding enumerative problem possesses some intrinsic structure that we failed to uncover from the start. For instance, the set of roots of any univariate polynomial \( f(x) := g(x^d) \) is invariant under multiplication by any \( d^{th} \) root of unity, while the solutions to any system \( f_1(x_1, x_2) = f_2(x_2) = 0 \) are equi-distributed on the collection of horizontal lines defined by \( f_2(x_2) = 0 \). In each case, the corresponding Galois group has to preserve this internal structure of the set of solutions. Another wide category of enumerative problems carrying intrinsic structures is provided by Schubert calculus. We refer once again to [SY21], and references therein.

The general goal of this paper is to make progress in the Galois \( A \)-problem and in particular to identify the missing intrinsic structure that would explain the unsatisfied expectations of [EL22b]. To do so, we consider two types of yet unsolved Galois \( A \)-problems, in the hope that the latter missing structure will reveal itself. The first type of problem is a parametric version of the univariate \( A \)-enumerative problems considered in [EL21] and [EL22b] Section 1.1]. The second type concerns reducible systems in two variables, that is systems of the form \( f_1(x_1, x_2) = f_2(x_2) = 0 \). We determine the associated Galois group for both problems (see Section \[3\]) and uncover an additional intrinsic structure for arbitrary non-reduced tuples \( A \) (see Remark \[3.3\]). To achieve this, we take a wider perspective on the Galois \( A \)-problem, both for computational reasons and because there are other interesting problems within reach.

First, the covering \( \mathcal{X} \) possesses an invariant that is finer than \( G_A \), namely its braid monodromy group, that we denote \( B_A \). The group \( B_A \) is the image of the map \( \pi_1(\mathcal{C}_A) \rightarrow \pi_1(C_N((\mathbb{C}^*)^n)) \) induced by the map \( \mathcal{C}_A \rightarrow C_N((\mathbb{C}^*)^n) \) sending a system of equations to its set of solutions. Here, \( C_N(X) \) denotes the configuration space of \( N \) unordered points on the topological space \( X \). For \( n = 1 \), the fundamental group \( \pi_1(C_N(\mathbb{C}^*)) \) is the braid group on \( N \) strands of \( \mathbb{C}^* \) and the group \( B_A \) has been computed in [EL21]. As far as we know, the case \( n > 1 \) remains unexplored although there are several motivations to study this problem. To start with, the computation of \( B_A \) leads to a solution to the Galois \( A \)-problem, since \( G_A \) is the image of \( B_A \) under the natural map \( \pi_1(C_N(\mathbb{C}^*)) \rightarrow \mathfrak{S}_N \). Also, braid monodromy is a
classical tool used to compute fundamental groups of complement to hypersurfaces, see \[Lib21\] and references therein. Eventually, the group $B_A$ has functorial properties in $A$ that $G_A$ does not have, see \[EL21\] Remark 1.1.(i) and Remark 5.3. This functoriality greatly simplifies computations. We refer to the computation of $B_A$ as the \emph{braid $A$-problem}.

As a second variation of the Galois $A$-problem, one can ask for the computation of the monodromy of group of the restriction of the covering $[1]$ over a connected subvariety $Y \subset \mathscr{C}_A$. An instance of such subvarieties arises from tuples $\hat{A} := (\hat{A}_1, \cdots, \hat{A}_n)$, $\hat{A}_j \subset \mathbb{Z}^n \times \mathbb{Z}^k$, whose projection under $\mathbb{Z}^n \times \mathbb{Z}^k \rightarrow \mathbb{Z}^n$ is $A$. Choose a tuple of generic polynomials $f_j(x, t)$, $x \in (\mathbb{C}^*)^n$, $t \in (\mathbb{C}^*)^k$, with respective support $\hat{A}_j$. The under-determined system $f_1 = \cdots = f_n = 0$ in $(x, t)$ and supported at $\hat{A}$ becomes a determined system supported at $A$ once we assign a generic value for $t$. Denote $Y := Y(\hat{A}) \subset \mathscr{C}_A$ the set of systems obtained in this way and denote $G_{\hat{A}} \subset G_A$ the monodromy group of the restriction of $[1]$ over $Y$. We refer to the computation of $G_{\hat{A}}$ as the \emph{Galois $\hat{A}$-problem} or as a \emph{Galois $A$-problem with parameters}. Since the group $G_{\hat{A}}$ is again a Galois group, the latter problem fits into the inverse Galois problem. In particular, the computation of the Galois group of a rational function $p(x)/q(x) \in \mathbb{C}(x)$, whenever $p$ and $q$ are generic with respect to their respective support, amounts to a certain Galois $\hat{A}$-problem (see Remark 3.1.3). This is also the case for the computation of the monodromy of rational curves in weighted projective planes, see \[Lan20\] Theorem 3]. To our knowledge, the only general statements about Galois $A$-problems with parameters are confined to tuples $\hat{A}$ such that the coefficients of the polynomials $x \mapsto f_j(x, t)$ are linear forms in $t$, see for instance \[Coh80\]. Under this linearity assumption, Zariski’s Theorem \[Che75\] allows to reduce Galois $A$-problems with parameters to the underlying Galois $A$-problem. In this paper, we solve all Galois $A$-problems with parameters in the case $n = 1$.

Eventually, one can consider a combination of the two variations of the Galois $A$-problem, that is the computation of the braid monodromy group of the restriction of the covering $[1]$ over a subvariety $Y \subset \mathscr{C}_A$. Again, the computation of this braid monodromy group enjoys computational advantages and relates to other interesting problems (see Remark 3.1.2). This is the approach we will take to solve Galois $A$-problems with parameters in the case $n = 1$.

### 1.2. Imprimitive permutation groups and wreath products

Let $\mathcal{N}$ be a finite set, with corresponding permutation group $\mathfrak{S}_\mathcal{N}$. A subgroup $\hat{G} \subset \mathfrak{S}_\mathcal{N}$ is imprimitive if its action by permutation on $\mathcal{N}$ preserves a partition $\mathcal{N} = \sqcup_{y \in \mathcal{Q}} \mathcal{N}_y$ where $\mathcal{Q}$ is a set with at least 2 elements and each subset $\mathcal{N}_y \subset \mathcal{N}$ is non-empty. In particular, any element of $\hat{G}$ induces a permutation on the set $\mathcal{Q}$ of blocks of the partition. As mentioned above, the Galois groups of the polynomial systems $f(x^d) = 0$ and $f_1(x_1, x_2) = f_2(x_2) = 0$ are imprimitive. Imprimitive groups associated to $\mathcal{A}$-enumerative problems are often isomorphic to wreath products, whose definition we now recall.

Given an action $\mathcal{P} \rightarrow \mathfrak{S}_\mathcal{Q}$ of a group $\mathcal{P}$ on the finite set $\mathcal{Q}$ and another group $\mathcal{U}$, the group $\mathcal{P}$ acts on the group $\mathcal{U}^\mathcal{Q}$ by permutation of the coordinates. By definition, the \emph{wreath product} $H \wr_{\mathcal{Q}} \mathcal{P}$ of $H$ with $\mathcal{P}$ over $\mathcal{Q}$ (and relative to the action $\mathcal{P} \rightarrow \mathfrak{S}_\mathcal{Q}$) is the corresponding semidirect product $H^\mathcal{Q} \rtimes \mathcal{P}$. In particular, this group is equal to $H^{\mathcal{Q}} \times \mathcal{P}$, as a set. Throughout the paper, the group $\mathcal{P}$ will either be a subgroup of $\mathfrak{S}_\mathcal{Q}$ or a subgroup of a certain braid group $B^*_\mathcal{Q}$ with a natural projection $B^*_\mathcal{Q} \rightarrow \mathfrak{S}_\mathcal{Q}$.
Consider the example \( A = \{0, d, 2d, \cdots, dN\} \) for some integers \( d \geq 2 \) and \( N \geq 1 \). Any univariate polynomial \( f(x) \in \mathbb{C}^A \) is of the form \( g(x^d) \) where \( g \in \mathbb{C}^B \) and \( B := \{0, 1, \cdots, N\} \). In particular, the set \( \{ f = 0 \} \) is acted upon by the group \( U_d \) of \( d \)th roots of unity. Any permutation \( \sigma \) in the Galois group \( G_A \) is \( U_d \)-equivariant, in particular it preserves the partition of \( \{ f = 0 \} \) into \( U_d \)-orbits. These orbits are in natural correspondence with the set \( \mathcal{Q} := \{ g = 0 \} \) through the covering \( x \mapsto x^d \). It turns out that the Galois group \( G_A \) is isomorphic to the subgroup of \( U_d \)-equivariant permutations of \( \{ f = 0 \} \), see [EL22b, Proposition 1.5]. The latter group is itself isomorphic to the wreath product \( U_d \wr \mathfrak{S}_Q \). To see this, fix a \( U_d \)-equivariant permutation \( \sigma \) of \( \{ f = 0 \} \). Its second factor in \( U_d \wr \mathfrak{S}_Q \) is the induced permutation on the set \( \mathcal{Q} \) of \( U_d \)-orbits of \( \{ f = 0 \} \). For the first factor \( (U_d)^{\mathcal{Q}} \), some choices are needed. Fix a reference point in each \( U_d \)-orbit of \( \{ f = 0 \} \), and thus identify each orbit with \( U_d \) itself. These identifications allow to encode the restriction of \( \sigma \) from one orbit to another as an element in \( U_d \). It defines a map \( \mathcal{Q} \to U_d \) that is the first factor of \( \sigma \) in the wreath product. The reader can check that the composition of \( U_d \)-equivariant permutations translates to the group law of \( U_d \wr \mathfrak{S}_Q \) via the latter identification.

The above example fits into a more general context. Consider a surjective morphism of finite degree \( \psi : \mathcal{F} \to \mathcal{G} \) between Abelian algebraic groups, with kernel \( K \subset \mathcal{F} \). For a finite subset \( \mathcal{Q} \subset \mathcal{G} \), the finite subset \( \mathcal{N} := \psi^{-1}(\mathcal{Q}) \) is acted upon by \( K \). The subgroup of \( \mathfrak{S}_\mathcal{N} \) consisting of \( K \)-equivariant permutations is isomorphic to \( K \wr \mathfrak{S}_\mathcal{Q} \). Such an isomorphism can be constructed exactly as above, by choosing a reference point in every \( K \)-orbit of \( \mathcal{N} \).

In the context of the BKK-enumerative problem, the relevant example is

\[
\psi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n, (x_1, \cdots, x_n) \mapsto (x_1^{a_{11}} \cdots x_1^{a_{1n}}, \cdots, x_1^{a_{nn}})
\]

with \( a_{ij} \in \mathbb{Z} \). A system \( f = 0 \) supported on a given tuple \( B \) induces a second system \( f \circ \psi = 0 \) whose support we denote \( A \). The Galois group \( G_A \) consists of \( K \)-equivariant permutations and is therefore isomorphic to a subgroup of \( K \wr \mathfrak{S}_Q \).

**Remark 1.2.** Two different choices of reference points lead to different identifications of the \( K \)-equivariant subgroup of \( \mathfrak{S}_\mathcal{N} \) with \( K \wr \mathfrak{S}_\mathcal{Q} \). These two identifications differ from each other by an isomorphism of \( K \wr \mathfrak{S}_\mathcal{Q} \) that we now describe. The second choice of reference points corresponds to a tuple \( (\xi_y)_{y \in \mathcal{Q}} \in K^\mathcal{Q} \) according to the first choice. The corresponding isomorphism of \( K \wr \mathfrak{S}_\mathcal{Q} \) is given by

\[
((\xi_y)_{y \in \mathcal{Q}}, \sigma) \mapsto ((\xi_y \cdot \xi^{-1}_{\sigma(y)})_{y \in \mathcal{Q}}, \sigma).
\]

In particular, this isomorphism restricts on each factor \( K^\mathcal{Q} \times \{ \sigma \} \) of \( K \wr \mathfrak{S}_\mathcal{Q} \) to a translation on \( K^\mathcal{Q} \) by an element in the kernel of the map

\[
\prod : K \wr \mathfrak{S}_\mathcal{Q} \to K, \quad ((\xi_y)_{y \in \mathcal{Q}}, \sigma) \mapsto \prod_{y \in \mathcal{Q}} \xi_y.
\]

As a consequence, for any subgroup \( L \subset K \), the pullback of the subgroup \( \prod^{-1}(L) \subset K \wr \mathfrak{S}_\mathcal{Q} \) to the \( K \)-equivariant subgroup of \( \mathfrak{S}_\mathcal{N} \) does not depend on the choices made above. This observation will be relevant while studying one of the morphisms \( \prod \in \{ \text{ind}_\mathcal{N}, \text{ind}_\mathcal{N}^\sigma, \text{ind}_A, \text{ind}_A^\sigma \} \) that we will meet later on.
2. Main results

2.1. The first problem. Fix a support \( A \subset \mathbb{Z} \) such that \( \{0, N\} \subset A \subset \{0, \ldots, N\} \) for some integer \( N \geq 0 \). Fix another support \( A \subset \mathbb{Z} \times \mathbb{Z}^k \) whose projection to the first factor \( \mathbb{Z} \) is \( A \). We aim at computing the Galois group of a generic polynomial \( \varphi(x, t) \in \mathbb{C}^A \) over the field \( \mathbb{C}(t) \), where \( x \in \mathbb{C}^\star \) and \( t \in (\mathbb{C}^\star)^k \). In other words, we are interested in the monodromy group of the map

\[
(x, t) \in \mathbb{C}^\star \times (\mathbb{C}^\star)^k : \varphi(x, t) = 0 \rightarrow (\mathbb{C}^\star)^k, \quad (x, t) \mapsto t
\]

which is a degree \( N \) covering over the complement to a Zariski-closed subset \( \mathcal{B} \subset (\mathbb{C}^\star)^k \). This subset is classically referred to as the bifurcation set or the discriminant. Fix a base point \( t_0 \in (\mathbb{C}^\star)^k \setminus \mathcal{B} \) and define \( \mathcal{N} := \{ x \in \mathbb{C}^\star : \varphi(x, t_0) = 0 \} \). As in Section 1, we denote by \( G_A \subset \mathcal{G}_N \) the monodromy group of the covering (2). In the above terminology, the computation of \( G_A \) amounts to a Galois \( A \)-problem with parameters.

We will identify two geometric defects of \( A \) that prevent \( G_A \) from being the full symmetric group \( \mathcal{G}_N \). The first defect is captured by the largest integer \( d := d(A) \) such that \( A \subset d\mathbb{Z} \). Since \( \varphi(x, t) = \widetilde{\varphi}(x^d, t) \) for some other polynomial \( \widetilde{\varphi} \), the set \( \{ x \in \mathbb{C}^\star : \varphi(x, t) = 0 \} \) is acted upon by the group \( \mathcal{U}_d := \{ x \in \mathbb{C}^\star : x^d = 1 \} \) for any given \( t \in \mathbb{C}^\star \setminus \mathcal{B} \). We denote by \( \mathcal{G}_{N,d} \subset \mathcal{G}_N \) the subgroup of \( \mathcal{U}_d \)-equivariant permutations, i.e. permutations \( \sigma \) for which \( \sigma(\xi \cdot x) = \xi \cdot \sigma(x) \) for any \( \xi \in \mathcal{U}_d \) and \( x \in \mathcal{N} \). Plainly, we have \( G_A \subset \mathcal{G}_{N,d} \).
The second defect of $A$ is captured by the integer $\vartheta := \vartheta(A)$ that we now define. We say that $A$ is \textit{sharp} if the sets $A \cap \{(0) \times \mathbb{Z}^k\}$ and $A \cap \{(N) \times \mathbb{Z}^k\}$ consists of a single point. Writing $\varphi(x, t) = c_0(t) + c_1(t)x + \cdots + c_N(t)x^N,$ the support $A$ is sharp if and only if $c_0(t)/c_N(t) = ct_1^{a_1} \cdots t_k^{a_k}$ for some $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $c \in \mathbb{C}^\ast$. Define $\vartheta := \text{GCD}(a_1, \ldots, a_k)$ (with the convention that $\text{GCD}(0, \ldots, 0) = 0$) and set $\vartheta \neq 1$ if $A$ is not sharp.

The way $\vartheta$ affects the Galois group $G_A$ is seen through the map $\text{ind}_N^\vartheta$ that we now define. Call a \textit{representative} any subset $E \subset N$ that intersect each $U_d$-orbit of $N$ exactly once. Denote $x_E := \prod_{x \in E} x$ and define $P := \{x_E \in \mathbb{C}^\ast : E \text{ representative}\}$. Then $P$ is acted upon by $U_d$ and consists of a single orbit. Observe moreover that for any representative $E$ and any $\sigma \in \mathfrak{S}_{N,d}$, the set $\sigma(E)$ is again a representative. Therefore, we have the following surjective map

$$\text{ind}_N^\vartheta : \mathfrak{S}_{N,d} \to \mathfrak{S}_P,$$

$$\sigma \mapsto (x_E \mapsto x_{\sigma(E)})$$

where $\mathfrak{S}_P$ is the $U_d$-equivariant subgroup of $\mathfrak{S}_P$. Since $P$ consists of a single orbit, we have a canonical isomorphism between $\mathfrak{S}_P$ and the Abelian group $U_d$. In particular, the subset $(\mathfrak{S}_P)^M \subset \mathfrak{S}_P$ of $M$-powers of elements in $\mathfrak{S}_P$ is a subgroup. We prove the following.

\textbf{Theorem 2.1.} \textit{For any support} $A \subset \mathbb{Z}^{k+1}$ \textit{not contained in a line, the group} $G_A$ \textit{is the subgroup} $(\text{ind}_N^\vartheta)^{-1}(\mathfrak{S}_P)$ \textit{of} $\mathfrak{S}_{N,d}$. \textit{In particular,} $G_A$ \textit{is the full symmetric group} $\mathfrak{S}_N$ \textit{if and only if} $d = 1$, \textit{and it is the subgroup} $\mathfrak{S}_{N,d}$ \textit{if and only if} $d$ \textit{and} $\vartheta$ \textit{are coprime}.

The support sets $A$ for which $G_A$ is the full symmetric group can also be described via the study of spatial symmetric curves, see \cite[Theorem 1.3]{EL22a}.

Let us briefly comment on Theorem 2.1. Recall from Section 1.2 that the group $\mathfrak{S}_{N,d}$ is non-canonically isomorphic to the wreath product $U_d \wr \mathfrak{S}_{N/d}$. Via this identification, the map $\text{ind}_N^\vartheta$ reads as

$$U_d \wr \mathfrak{S}_{N/d} \to U_d$$

$$(\xi_1, \ldots, \xi_{N/d}, \sigma) \mapsto \prod_j \xi_j.$$

In particular, Theorem 2.1 states that $G_A$ is isomorphic to

$$\left\{(\xi_1, \ldots, \xi_{N/d}, \sigma) \in U_d \wr \mathfrak{S}_{N/d} : \prod_j (\xi_j)^g = 1\right\},$$

where $g := d/\text{GCD}(d, \vartheta)$. Although the isomorphism between $\mathfrak{S}_{N,d}$ and $U_d \wr \mathfrak{S}_{N/d}$ is not canonical, the description \textbf{1} of $G_A$ does not depend on the choice of the latter isomorphism, see Remark 1.2. To see why $G_A = (\text{ind}_N^\vartheta)^{-1}(\mathfrak{S}_P)$, observe that when $t$ travels along a loop in $(\mathbb{C}^*)^k \setminus B$, the product of the roots of $\bar{\varphi}(x, t)$ closes a loop in $\mathbb{C}^\ast$. The map $\text{ind}_N^\vartheta$ records the reduction $\mod d$ of the winding number of the latter loop. That this reduction is a multiple of $\vartheta$ follows from Vieta’s formula, see Section 4.1.

In order to prove the above theorem, we compute the braid monodromy group of the covering \textbf{2}. Recall that $C_N(\mathbb{C}^\ast)$ is the unordered configuration space of $N$ distinct points in $\mathbb{C}^\ast$ and denote $B_{N} := \pi_1(C_N(\mathbb{C}^\ast), N)$. The map $\mathbb{C}^k \setminus B \to C_N(\mathbb{C}^\ast)$ that associates to every tuple $t$ the set of roots of $\varphi(x, t)$ induces at the level of fundamental groups the map that we denote

$$\mu_{N}^{br} : \pi_1(\mathbb{C}^k \setminus B, t_0) \to B_{N}.$$
Since the group $B_N^*$ is known as the braid group of $\mathbb{C}^*$ with $N$ strings, see [FN62], we will refer to $\mu^br_\varphi$ as the braid monodromy map associated to $\varphi$. Since $\varphi \in \mathbb{C}^A$ is generic, the image of $\mu^br_\varphi$ depends only on $A$. We denote it by $B_A$ and refer to it as the braid monodromy group of $A$.

Denote $\pi_N : B_N^* \to \mathcal{G}_N$ the natural projection. The composition of $\mu^br_\varphi$ with $\pi_N$ leads to the monodromy map

$$\mu_\varphi : \pi_1(\mathbb{C}^k \setminus \mathcal{B}, t_0) \to \mathcal{G}_N$$

of the covering [2]. In particular, we have that $\pi_N(B_A) = G_A$.

In order to describe $B_A$, there are 2 types of subgroups of $B_N^*$ of interest. First, recall that $B_N^*$ is acted upon by $U_d$, since $N \subset \mathbb{C}^*$ is. Following [EL21], we denote by $B_{N,d}^* \subset B_N^*$ the corresponding invariant subgroup.

For the second type of subgroup, consider the multiplication map $C_N(\mathbb{C}^*) \to \mathbb{C}^*$ given by $c \mapsto \prod_{x \in c} x$, and denote $x_0$ the image of $N$. We denote by

$$\text{ind}_N : B_N^* \to \pi_1(\mathbb{C}^*, x_0) \simeq \mathbb{Z}$$

the induced map at the level of fundamental groups. Define $R_N^0 := \text{ind}_N^{-1}(\partial \mathbb{Z})$.

**THEOREM 2.2.** For any support $A \subset \mathbb{Z}^{k+1}$ not contained in a line, the group $B_A$ is the subgroup $B_{N,d}^* \cap R_N^0$ of $B_N^*$.

Theorem 2.1 will come as a consequence of Theorem 2.2. In Section 4.5 we will use a commutative diagram

$$\begin{array}{ccc}
B_{N,d}^* & \xrightarrow{\pi_N} & \mathcal{G}_{N,d} \\
\text{ind}_N \downarrow & & \downarrow \text{ind}_\mathcal{G} \\
\pi_1(\mathbb{C}^*, x_0) & \xrightarrow{q} & \mathcal{G}_{P,d}
\end{array}$$



to show that the image under $\pi_N$ of $B_{N,d}^* \cap R_N^0$ is the subgroup $(\text{ind}_\mathcal{G})^{-1}((\mathcal{G}_{P,d})^0)$ of $\mathcal{G}_{N,d}$.

**REMARK 2.3.** If the support set $A \subset \mathbb{Z}^{k+1}$ is contained in a line, then it is sharp and $\varphi(x, t)$ is equal to $\widetilde{\varphi}(x_0 t_{a_1} \cdots t_{a_k})$ for some non-singular univariate polynomial $\widetilde{\varphi}$. The integer $\vartheta$ equals $N \cdot \text{GCD}(a_1, \cdots, a_k)$ and the bifurcation set $\mathcal{B}$ is the union of the coordinate hyperplanes $\{t_j = 0\}$ for which $a_j \neq 0$. The image of $\mu^br_\varphi$ is the cyclic group generated by $\tau^\vartheta$ where $\tau$ is defined in Figure 1. The corresponding Galois group is isomorphic to the cyclic group $\vartheta \cdot (\mathbb{Z}/d\mathbb{Z})$.

### 2.2. The second problem.

The second problem we consider is the Galois $A$-problem for reducible pairs $A := (A_1, A_2)$, $A_j \subset \mathbb{Z}^2$. Using a harmless affine linear change of coordinates in $\mathbb{Z}^2$, we can assume that $A_2 \subset \{0\} \times \mathbb{Z}$ and that the projection $A_1 \subset \mathbb{Z}$ of $A_1$ onto the first factor satisfies $\{0, n\} \subset A_1 \subset \{0, \cdots, n\}$ for some integer $n \geq 1$. Geometrically, we are interested in the monodromy group of the covering

$$\{((x, y), (p, q)) \in (\mathbb{C}^*)^2 \times \mathcal{C}_A : p(x, y) = q(y) = 0\} \to \mathcal{C}_A$$

where $\mathcal{C}_A$ is the space of conditions defined previously. As before, we denote by $G_A \subset \mathcal{G}_N$ the monodromy group of the covering [7].

Similarly to the first problem, we will deduce $G_A$ from a certain braid monodromy group associated to the covering [7]. In order to define the underlying braid monodromy map, we
fix a base point \((p_0, q_0)\) in \(\mathcal{C}_A\), define the set \(\mathcal{N} := \{(x, y) \in ((\mathbb{C}^*)^2 : p_0(x, y) = q_0(y) = 0\}\) and denote its cardinality by \(N\). Then, denote by \(\mathcal{U}_A\) the subset of \(C_N((\mathbb{C}^*)^2)\) consisting of point configurations of the form

\[
\{(x, y) \in (\mathbb{C}^*)^2 : q(y) = 0, \ p_y(x) = 0\}
\]

for some non-singular polynomials \(q \in \mathbb{C}^A\) and \(p_y \in \mathbb{C}^A, \ y \in \{q = 0\}\). Plainly, configurations of the form \(\{p(x, y) = q(y) = 0\}\), with \((p, q) \in \mathcal{C}_A\), are contained in \(\mathcal{U}_A\). Thus, we have a map \(\mathcal{C}_A \to \mathcal{U}_A, \ (p, q) \mapsto \{p = q = 0\}\). The induced map at the level of fundamental groups is our braid monodromy map

\[
\mu^b_A : \pi_1(\mathcal{C}_A, (p_0, q_0)) \to B_N^*
\]

where \(B_N^*: = \pi_1(\mathcal{U}_A, \mathcal{N})\). The image of \(\mu^b_A\), that we denote \(B_A\), is our braid monodromy group. The natural projection \(\pi_N : B_N^* \to \mathfrak{S}_N\) maps \(B_A\) onto the sought Galois group \(G_A\).

The description of \(B_A\) inside \(B_N^*\) relies on a second system of equations constructed from the initial system \(p(x, y) = q(y) = 0\). If we write \(p(x, y) = c_0(y) + c_1(y)x + \cdots + c_n(y)x^n\), then the second system is

\[
c_n(y)x - (-1)^nc_0(y) = 0, \quad q(y) = 0.
\]

The system \((10)\) relates to the initial system as follows. If we denote \(h\) the degree of \(q\), then Vieta’s formula implies that the map

\[
\mathcal{U}_A \to C_h((\mathbb{C}^*)^2)
\]

sends the set \(\{p(x, y) = q(y) = 0\}\) to the set of solutions to \((10)\).

Denote \(\nu(A):= (\nu(A_1), A_2)\) the reducible support of the system \((10)\). We define its braid monodromy group \(B_{\nu(A)}\) exactly as we defined \(B_A\). In particular, the map \(\nu\) takes values in the set \(\mathcal{U}_{\nu_1}((\mathbb{C}^*)^2)\) of configurations in \(C_h((\mathbb{C}^*)^2)\) that are equi-distributed among the \(h\) many connected components of \(\{(x, y) : q(y) = 0\}\) for some non-singular \(q \in \mathbb{C}^A\), and \(B_{\nu(A)}\) is a subgroup of \(B_Q^*: = \pi_1(\mathcal{U}_{\nu(A)}, \mathcal{Q})\), where \(\mathcal{Q}\) the image of \(\mathcal{N}\) under \((11)\). The map \((11)\) induces at the level of fundamental groups the surjective map that we denote

\[
\text{ind}_A : B_N^* \to B_Q^*.
\]

We prove the following.

**Theorem 2.4.** For any pair \(A := (A_1, A_2)\) such that \(A_2\) is contained in a line and \(A_1\) is not, the braid monodromy group \(B_A\) is the subgroup \(\text{ind}_A^{-1}(B_{\nu(A)}))\) of \(B_N^*\).

According to the above theorem, the explicit map \(\text{ind}_A\) reduces the description of \(B_A\) to the description of the smaller group \(B_{\nu(A)}\). We describe explicitly the groups \(B_N^*, B_Q^*\) and \(B_{\nu(A)}\) in Remark 3.2.1.

The Galois group \(G_A\) will admit a description similar to the one of \(B_A\). To see this, recall that the Galois group \(G_{A_2}\) of the univariate polynomial \(q\), as considered in [EL22a, Section 1.1], acts by permutation on the blocks of the partition of \(\mathcal{N}\) among the horizontal lines \(\mathbb{C}^* \times \{y\}\), \(y \in \mathcal{Q}\). Moreover, for \(d := d(A_1)\), the action \(\xi \cdot (x, y) := (\xi x, y)\) of \(U_d\) on \((\mathbb{C}^*)^2\) preserves the set \(\mathcal{N}\). Define \(\mathfrak{S}_{\mathcal{N}, A}\) as the intersection of the \(G_{A_2}\)-invariant subgroup of \(\mathfrak{S}_N\) with the \(U_d\)-equivariant subgroup of \(\mathfrak{S}_N\). The group \(\mathfrak{S}_{\mathcal{N}, A}\) is non canonically isomorphic to the wreath
product $G_{A_1} \wr_{\mathbb{Q}} G_{A_2}$, where $G_{A_1}$ is the Galois group of the univariate polynomials $p_y$ from [8]. Plainly, we have that $G_A \subseteq S_{N, A}$.

The description of $G_A$ inside $S_{N, A}$ relies on another system of equations constructed from the initial system $p(x, y) = q(y) = 0$, namely the system

$$c_n(y)x^d - (-1)^{n/d}c_0(y) = 0, \quad q(y) = 0. \tag{12}$$

The system (12) relates to the initial system as follows. Denote $N_y := N \cap (\mathbb{C}^* \times \{y\})$ for any $y \in \mathbb{Q}$ and call a representative any subset $E \subset N_y$ that intersect each $U_d$-orbit of $N_y$ exactly once. Denote $x_E := \prod_{x \in E} x$ and define

$$\mathcal{P} := \{(x, y) \in \mathbb{C}^* \times \mathbb{Q} : x = x_E \text{ for some representative } E \subset N_y\}.$$

Vieta’s formula implies that the set $\mathcal{P}$, constructed from the set of solutions $N$ to the initial system, is the set of solutions to (12).

Plainly, the subset $\mathcal{P}$ is acted upon by $U_d$. We denote $S_{\mathcal{P}, A} \subset S_{\mathcal{P}}$ the corresponding $U_d$-equivariant subgroup. For any permutation $\sigma \in S_{N, A}$, denote $\sigma_\mathcal{Q} \in S_{\mathcal{Q}}$ the induced permutation on the blocks $N_y \subset N, y \in \mathbb{Q}$. Then, we have a surjective map

$$\text{ind}_A^\mathcal{Q} : S_{N, A} \to S_{\mathcal{P}, A}, \quad \sigma \mapsto \left(\left((x_E, y) \mapsto (x_{\sigma(E)}, \sigma_\mathcal{Q}(y))\right)\right).$$

The map $\text{ind}_A^\mathcal{Q}$ is a well defined group morphism. Indeed, for any representative $E \subset N_y$ and any $\sigma \in S_{N, A}$, the subset $\sigma(E)$ is a representative of $\sigma_\mathcal{Q}(y)$. Moreover, for any two representatives $E, E' \subset N_y$ such that $x_E = x_{E'}$, then $x_{\sigma(E)} = x_{\sigma(E')}$. We denote $\nu_\mathcal{Q}(A) := (\nu_\mathcal{Q}(A_1), A_2)$ the reducible support of the system (12) and define its Galois group $G_{\nu_\mathcal{Q}(A)} \subset S_{\mathcal{P}, A}$ exactly as we defined $G_A \subset S_{N, A}$. We prove the following.

**Theorem 2.5.** For any pair $A := (A_1, A_2)$ such that $A_2$ is contained in a line and $A_1$ is not, the Galois group $G_A$ is the subgroup $(\text{ind}_A^\mathcal{Q})^{-1}(G_{\nu_\mathcal{Q}(A)})$ of $S_{\mathcal{P}, A}$.

According to the above theorem, the explicit map $\text{ind}_A^\mathcal{Q}$ reduces the description of $G_A$ to the description of the smaller group $G_{\nu_\mathcal{Q}(A)}$ that we provide in Remark 3.2.4. Theorem 2.5 will come as a consequence of Theorem 2.4. Indeed, we will see that $S_{N, A}$ is the image of $B_N^*$ under $\pi_N$ (see Remark 3.2.2) and that there exists a commutative diagram

$$\begin{array}{ccc}
B_N^* & \xrightarrow{\pi_N} & S_{N, A} \\
\text{ind}_A \downarrow & & \downarrow \text{ind}_A^\mathcal{Q} \\
B_Q^* & \xrightarrow{q} & S_{\mathcal{P}, A}
\end{array}$$

see Remark 5.7. We treat the case of reducible pairs where both $A_1$ and $A_2$ are contained in a line in Section 5.2.

### 3. Discussions

In the remark below, we comment on the results mentioned in Section 2.1 while we comment on the results of Section 2.2 in the next remark. In the third remark, we speculate on the missing intrinsic structure of non-reduced irreducible tuples as studied in [EL22b] and provide a general intrinsic structure of the solutions to enumerative problems over algebraic groups.
Remark 3.1. 1. Let us describe polynomials \( \varphi \) to which Theorems 2.2 and 2.1 apply. Recall that \( A \) denotes the projection of \( A \) on the first coordinate axis in \( \mathbb{Z}^{k+1} \). Then the bifurcation set \( B \) is the pullback of the principal \( A \)-determinant \( E_A \subset \mathbb{C}^A \) by the coefficient map \( \mathcal{C}_\varphi : (t_1, \ldots, t_k) \mapsto (c_j)_{j \in A_1} \), see [GKZ08, Chapter 10]. Theorems 2.2 and 2.1 apply to any polynomial \( \varphi \) such that the image of \( \mathcal{C}_\varphi \) intersects \( E_A \) transversely. This is a Zariski-open condition in \( \mathbb{C}^A \), see [ESV23, Theorem 6.6.2]). The actual Zariski-open subset of polynomials \( \varphi \) for which the conclusions of Theorems 2.2 and 2.1 hold might be larger. We study this question in more depth in [EL22a, Section 2], in the case of support sets \( A \) with full symmetric Galois group \( G_A \).

2. There are at least 2 reasons to consider the braid monodromy map \( \mu_{br} \) rather than \( \mu \). First, it is not possible to deduce the case \( d > 1 \) from the case \( d = 1 \) in Theorem 2.1 simply by studying \( \mu \). Indeed, a certain solution lattice has to be computed, see [EL22b]. On the contrary, the case \( d > 1 \) follows from the case \( d = 1 \) in Theorem 2.2 as we explain in Proposition 4.3. Actually, the authors were led to prove Theorem 2.2 while trying to compute the aforementioned solution lattice. Second, the braid monodromy group \( B_A \) is a finer approximation of the fundamental group \( \pi_1(\mathbb{C}^k \setminus B) \) than \( G_A \) and is often used to compute presentations of the latter fundamental group, see [Lib21]. Collecting information on the fundamental group \( \pi_1(\mathbb{C}^k \setminus B) \) is of great interest in the context of [DL81], see again [Lib21] for a recent account.

3. Theorem 2.1 allows to determine the Galois group of any rational function \( p(x)/q(x) \in \mathbb{C}(x) \), provided that \( p(x) \) and \( q(x) \) are generic with respect to their support. Indeed, the monodromy group of the latter function is the image of \( \mu_{\varphi} \) for \( \varphi(x, t) = p(x) + t \cdot q(x) \). In this context, the integer \( d \) can be arbitrary but \( \varphi \) can only assume the values 0 and 1.

4. The case of rational functions justifies the genericity assumption in Theorems 2.2 and 2.1. Indeed, Galois groups of such functions come in many more shapes than the ones appearing in Theorem 2.1, see for instance [Neu93].

5. Theorem 2.2 generalises [EL21] (Theorem 1) in which we studied the braid monodromy of the universal univariate polynomial supported on \( B \subset \mathbb{Z} \). Indeed, by Zariski’s Theorem [Che75 Théorème], the braid monodromy of the latter univariate polynomial is the image of \( \mu_{br}^\varphi \) for a generic polynomial \( \varphi(x, t) \) with support \( A := B \times \{0\} \cup B \times \{1\} \subset \mathbb{Z}^2 \), since \( \varphi \) corresponds to a generic line in \( \mathbb{C}^B \). In general, Zariski’s Theorem relates the \( j \)th homotopy group of the complement to (quasi) projective hypersurface with the \( j \)th homotopy group of the complement of its pullback under a linear map. It is natural to investigate what happens when linear maps are replaced with arbitrary polynomial maps. Theorem 2.1 is one step in this direction: we restrict ourselves to the 1st homotopy group with the polynomial map \( \mathcal{C}_\varphi \) and the hypersurface \( E_A \subset \mathbb{C}^A \) from part 1 of this remark.

6. To our knowledge, Galois groups \( G_A \) were unknown except for supports of the form \( A := B \times \{0\} \cup B \times \{1\} \) as in part 5. As explained above, Zariski’s Theorem do not apply in general so that \( G_A \) cannot be deduced from the Galois group \( G_A \) of the one-dimensional support \( A \subset \mathbb{Z} \). In the case \( k = 1 \), the group \( G_A \) is the monodromy group of the branched cover \( \{ \varphi(x, t) = 0 \} \to \mathbb{P}^1 \), \( (x, t) \mapsto t \), for an appropriate compactification of the source. The group \( (4) \) is imprimitive exactly when the latter cover is a composite cover. The Galois groups of composite covers were studied in [CR21] and references therein. In this context, the appearance of groups of the form \( (4) \) seems to be new.
7. Theorem 2.1 shows that there might be a discrepancy between the sectional monodromy group of a given smooth projective curve and the monodromy of linear subspaces of sections. Indeed, according to Theorem 2.1 and [Kad21, Proposition 1.1], it is possible to find a planar group of a given smooth projective curve and the monodromy of linear subspaces of sections. The braid monodromy group of the univariate polynomial $q$ is the group of $B$ the monodromy group of the univariate polynomial $q$.

8. To prove Theorem 2.2, we use considerations involving coamoebas, similar to those of [EL21]. In the case $k = 1$ and $|A| = 3$, we compute explicitly in Section 4.3 the image under $\mu_\varphi^{br}$ of a set of generators of $\pi_1(\mathbb{C} \setminus \mathcal{B}, t_0)$. This allows to compute the kernel of the map as well as a presentation of the braid monodromy group $B_A$. In section 4.6, we discuss how considerations from tropical geometry allow to extend these results to arbitrary supports $A \subset \mathbb{Z}^2$. We also discuss how such results allow to study isomonodromy loci, namely subsets of the form $\{ \varphi \in \mathbb{C}^A : \text{im}(\mu_\varphi^{br}) = B \}$ for a given subgroup $B \subset B_N$.

**Remark 3.2.** 1. The description of the group $B_A$ given in Theorem 2.4 relies on the groups $B_N^\ast$, $B_Q^\ast$ and $B_{\nu(A)}$ that we now describe. First, observe that the group $B_N^\ast$ is isomorphic to the wreath product $B_A \wr B_{A_2}$ over the set $\{g_0(y) = 0\}$. Here, $B_{A_2}$ stands for the braid monodromy group of the univariate polynomial $q$ as considered in [EL21], while $B_{\Delta_1}$ stands for the braid monodromy group of the polynomials $p_y$ of $\mathbb{A}$. The codomain $B_Q^\ast$ of the map $\text{ind}_A$ is isomorphic to the wreath product $\pi_1(\mathbb{C}^\ast) \wr B_{A_2}$ over the set $\{g_0(y) = 0\}$ and the map $\text{ind}_A$ reads factor-wise as the map $B_{\Delta_1} \rightarrow \pi_1(\mathbb{C}^\ast)$ induced by the product map $c \mapsto \prod_{x \in \mathbb{C}} x$ on point configurations. In particular, the map $\text{ind}_A$ is surjective and the inclusion $B_A \subset B_N^\ast$ is strict exactly when the inclusion $B_{\nu(A)} \subset B_Q^\ast$ is. This happens in two mutually exclusive situations.

In the first situation, the reducible support $\nu(A) = (\nu(A_1), A_2)$ is non-reduced and $A_1$ is not sharp. Then, there exist integers $\kappa := \kappa(A)$ and $\vartheta := \vartheta(A_1)$ with $\kappa > 1$ and $0 \leq \vartheta < \kappa$ such that the smallest lattice in $\mathbb{Z}^2$ containing $\nu(A_1)$ and $A_2$ is generated by $\big(\frac{1}{\kappa}\big)$ and $\big(\frac{1}{\vartheta}\big)$. In particular, we can write

$$c_0(y) = \tilde{c}_0(y^\kappa), \quad c_n(y) = y^{\vartheta} \cdot \tilde{c}_n(y^\kappa) \quad \text{and} \quad q(y) = \tilde{q}(y^\kappa).$$

In that case, the set of solutions to (10) is acted upon by $U_\kappa$ via multiplication by $(e^{-\frac{2\pi i \vartheta}{\kappa}}, e^{\frac{2\pi i}{\vartheta}})$. The braid group $B_{\nu(A)}$ is the $U_\kappa$-invariant subgroup of $B_Q^\ast$. To see this, observe that the map $(x, y) \mapsto (xy^\vartheta, y^\kappa)$ induces an isomorphism between $B_{\nu(A)}$ and the braid monodromy group of the reducible and reduced system

$$\tilde{c}_n(y)x - (-1)^nc_0(y) = 0, \quad \tilde{q}(y) = 0.$$

Straightforward computations show that the latter braid monodromy group is as large as possible, namely $\pi_1(\mathbb{C}^\ast) \wr B_{\tilde{A}_2}$ where $\tilde{A}_2$ is the support of $\tilde{q}$.

In the second situation, the support $A_1$ is sharp. In that case, we can write

$$c_0(y) = c_0, \quad c_n(y) = c_n \cdot y^{\vartheta\kappa + \vartheta} \quad \text{and} \quad q(y) = \tilde{q}(y^\kappa).$$

The map $(x, y) \mapsto (xy^\vartheta, y^\kappa)$ induces an isomorphism between $B_{\nu(A)}$ and the braid monodromy group of the reducible and reduced system

$$c_nx - (-1)^nc_0 = 0, \quad \tilde{q}(y) = 0.$$
The latter group is isomorphic to $\pi_1(\mathbb{C}^*) \times B_{\Delta_2}$.

2. The description of the group $G_A$ given in Theorem 2.5 relies on the group $G_{\nu_0}(A)$ that we now describe. First, observe that the image of the group $B_N^* \simeq B_{\Delta_1} \wr B_{A_2}$ under $\pi_N$ is the subgroup $\mathfrak{S}_{\mathcal{N},A} \wr G_{A_2}$ of $\mathfrak{S}_N$. Since the map $\text{ind}_{A_1}^A$ is surjective, the inclusion $G_A \subset \mathfrak{S}_{\mathcal{N},A}$ is strict exactly when the inclusion $G_{\nu_0}(A) \subset \mathfrak{S}_{\mathcal{P},A}$. This happens in the same two situations as above.

When $\nu_0(A)$ is non-reduced and $A_1$ not sharp, then $G_{\nu_0}(A)$ is the $U_\kappa$-equivariant subgroup of $\mathfrak{S}_{\mathcal{P},A}$. Recall in this case that we can rewrite $c_0$, $e_n$ and $q$ as in (13). Then, the $U_\kappa$-equivariant subgroup of $\mathfrak{S}_{\mathcal{P},A}$ is isomorphic to the wreath product $U_d \wr_{(q=0)} G_{A_2}$ while $\mathfrak{S}_{\mathcal{P},A}$ is isomorphic to $U_d \wr_{(q=0)} G_{A_2}$. When $A_1$ is sharp, the Galois group $G_{\nu_0}(A)$ is isomorphic to the direct product $U_d \times G_{A_2}$.

**Remark 3.3.** 1. Theorems 2.2 and 2.4 invite to speculate on the Galois group of irreducible and non-reduced tuples $A := (\tilde{A}_1, \ldots, \tilde{A}_n)$, $\tilde{A}_i \subset \mathbb{Z}^n$, as studied in (12). Each such tuple is obtained from an irreducible and reduced tuple $A := (A_1, \ldots, A_n)$, $A_i \subset \mathbb{Z}^n$, as follows. There is a finite covering $\psi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$, with dual map $L : \mathbb{Z}^n \to \mathbb{Z}^n$ between character lattices such that $A_i = L(A_i)$. By (13) Theorem 1.5, the Galois group $G_A$ of the tuple $A$ is the full permutation group $\mathfrak{S}_N$ on the set $\mathcal{N}$ of solutions to a generic system

$$f_1 = \cdots = f_n = 0, \quad f_i \in \mathbb{C}^{A_i}.$$  

The set $\tilde{\mathcal{N}}$ of solutions to the corresponding system

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0, \quad \tilde{f}_i := f_i \circ \psi \in \mathbb{C}^{\tilde{A}_i}$$

is acted upon by $K := \text{Ker}(\psi)$ and the Galois group $G_{\tilde{A}}$ is a subgroup of the $K$-equivariant subgroup of $\mathfrak{S}_{\mathcal{N}}$. As discussed in Section 1.2, the latter subgroup is isomorphic to the wreath product $K \wr_{A_1} \mathfrak{S}_N$. In (12), we showed that the inclusion $G_{\tilde{A}} \subset K \wr_{A_1} \mathfrak{S}_N$ is proper for infinitely many tuples $\tilde{A}$, for any $n \geq 2$. There is therefore another obstruction, in addition to the action of $K$ on $\tilde{\mathcal{N}}$, that affects the Galois group $G_{\tilde{A}}$. One such obstruction reveals itself in the two problems considered in Section 2. Indeed, the statements of Section 2 indicate that the product of the solutions to the systems of equation under consideration play a central role in the determination of the braid monodromy group and the Galois group of the enumerative problem. As we explain below, this takes place in a more general context.

2. Recall that an enumerative problem $\mathcal{P}$ consists of a pair of smooth algebraic varieties $\mathcal{F}$ and $\mathcal{C}$, with $\mathcal{C}$ connected, together with an algebraic variety $\mathcal{U} \subset \mathcal{F} \times \mathcal{C}$ such that the projection $c : \mathcal{U} \to \mathcal{C}$ has finite degree $N$. The sets $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{U}$ are respectively referred to as the ambient space, the condition space and the solution space (or incidence variety) of the enumerative problem. The Galois/monodromy group $G_\mathcal{P}$ of the enumerative problem $\mathcal{P}$ is the monodromy group of the covering $c : \mathcal{U} \to \mathcal{C}$. We can define the braid monodromy group $B_\mathcal{P}$ of the problem $\mathcal{P}$ as the image of the braid monodromy map $\mu^b_{\mathcal{P}} : \pi_1(\mathcal{C}) \to \pi_1(C_N(\mathcal{F}))$, again with a projection $\pi_\mathcal{P} : B_\mathcal{P} \to G_\mathcal{P}$. Since the ambient space $\mathcal{F}$ is an algebraic group, we have a product map $C_N(\mathcal{F}) \to \mathcal{F}$, $c \mapsto \prod_{z \in \mathcal{C}_c} z$, and an induced map

$$\text{ind}_{\mathcal{P}} : \pi_1(C_N(\mathcal{F})) \to \pi_1(\mathcal{F}).$$

The subgroup $\mathcal{I} := \text{ind}_{\mathcal{P}}(B_\mathcal{P})$ may provide obstructions on the Galois group $G_\mathcal{P}$ of any enumerative problem $\mathcal{P}$ that is a covering of $\mathcal{P}$. Let us explain.
Any finite covering between connected algebraic groups \( \psi : \widetilde{T} \to T \) induces another enumerative problem \( \widetilde{P} \) given by the triple \( \widetilde{e} := e \) and \( \mathcal{U} := (\psi, \text{id})^{-1}(\mathcal{U}) \subseteq \widetilde{T} \times \mathcal{U} \). Let us add a tilde to every piece of notation coming from \( e \). Then, we have that
\[
G_{\widetilde{P}} \subset \text{Ker}(\psi) \wr G_P,
\]
see [EL22b, Observation 2.5]. One the one hand, the map \( C_N(T) \to C_N(\widetilde{T}) \) obtained by pulling back point configurations by \( \psi \) induces an isomorphism \( \psi^* : B_P \to B_{\widetilde{P}} \). On the other hand, we have a short exact sequence
\[
0 \longrightarrow \pi_1(\widetilde{T}) \overset{\psi}{\longrightarrow} \pi_1(T) \overset{m}{\longrightarrow} \text{Ker}(\psi) \longrightarrow 0
\]
where \( m \) is the monodromy map of the covering \( \psi \). Since \( \text{Ker}(\pi_{\widetilde{P}} \circ \psi^*) \subset \text{Ker}(m \circ \text{ind}_P) \), there is a map \( \text{ind}_P^\mathcal{G} : G_{\widetilde{P}} \to \text{Ker}(\psi) \) that fits into the following commutative diagram
\[
\begin{array}{ccc}
B_P & \overset{\pi_{\widetilde{P}} \circ \psi^*}{\longrightarrow} & G_{\widetilde{P}} \\
\downarrow{\text{ind}_P} & & \downarrow{\text{ind}_P^\mathcal{G}} \\
\pi_1(T) & \overset{m}{\longrightarrow} & \text{Ker}(\psi)
\end{array}
\]

The latter diagram provides the inclusion
\[
G_{\widetilde{P}} \subset (\text{ind}_P^\mathcal{G})^{-1}(m(I)).
\]

Therefore, it is natural to ask whether the Galois group \( G_{\widetilde{P}} \) of the enumerative problem \( \widetilde{P} \) is determined by the inclusions (14) and (15). This applies to the enumerative problem associated to the irreducible non-reduced tuples \( A := (\tilde{A}_1, \ldots, \tilde{A}_n) \), \( \tilde{A}_i \subset \mathbb{Z}^n \), studied in [EL22b]. There, the obstruction provided by (15) appeared, in disguise, through the Poisson-type formula [DS15, Theorem 1.1].

4. **Galois group of a univariate polynomial whose coefficients depend on parameters**

In this section, we address the problem described in Section 2.1 and prove Theorems 2.1 and 2.2. We refer to the aforementioned section for notations.

4.1. **Obstructions.** In this section, we show that the braid monodromy group of \( \varphi \) is constrained as claimed in Theorem 2.2.

**Lemma 4.1.** For any support \( A \subset \mathbb{Z}^{k+1} \) not contained in a line, the group \( B_A \) is a subgroup of \( B_{N,d}^* \cap R_N^d \).

**Proof.** The inclusion \( B_A \subset B_{N,d}^* \) follows from the fact that \( \varphi(x, t) = \widetilde{\varphi}(x^d, t) \) for some Laurent polynomial \( \widetilde{\varphi} \) supported at \( \tilde{A} \subset \mathbb{Z}^{k+1} \). Let us add a tilde to every piece of notation coming from \( \widetilde{\varphi} \). Thus, the map \( (x, t) \mapsto (x^d, t) \) induces an isomorphism from \( \mathbb{C}^{\tilde{A}} \) to \( \mathbb{C}^A \) mapping the
bifurcation set $\mathcal{B} \subset \mathbb{C}^A$ to the bifurcation set $\mathcal{B} \subset \mathbb{C}^A$. In turn, we have the commutative diagram

\[
\begin{array}{ccc}
\pi_1(\mathbb{C}^k \setminus \mathcal{B}) & \overset{\sim}{\longrightarrow} & \pi_1(\mathbb{C}^k \setminus \mathcal{B}) \\
\downarrow \mu_{\varphi}^b & & \downarrow \mu_{\varphi}^b \\
B_N^* & \overset{f_{N,d}}{\longrightarrow} & B_N^* \\
\end{array}
\]

where $f_{N,d} : B_N^* \to B_N^* \cdot d$ is the isomorphism induced by the map $C_{N/d}(\mathbb{C}^*) \to C_N(\mathbb{C}^*)$ taking a configuration of points to its preimage under $x \mapsto x^d$. The inclusion $B_A \subset B_N \cdot d$ follows from the commutativity of the above diagram.

Since $R_N^d = B_N^*$ when $d = 1$, we can restrict to the case $d > 1$. Thus, the coefficients $c_q$ and $c_{q+N}$ of $\varphi$ are monomials and $c_q/c_{q+N} = ct_1^{\varphi_1} \cdots t_k^{\varphi_k}$ with $\vartheta := \text{GCD}(a_1, \ldots, a_k)$. For a given tuple $t := (t_1, \ldots, t_k)$, the product of the $N$ roots of $\varphi(x, t)$ is $ct_1^{\varphi_1} \cdots t_k^{\varphi_k}$, by Vieta’s formula. Therefore, for any loop $\gamma \in \pi_1(\mathbb{C}^{k+1} \setminus \mathcal{B}, t_0)$, we have that $\text{ind}_N(\mu_{\varphi}^b(\gamma))$ is the rotational index of the composition $(t_1^{\varphi_1} \cdots t_k^{\varphi_k}) \circ \gamma$ around $0 \in \mathbb{C}$. The latter index is necessarily in $\vartheta \mathbb{Z}$, which implies the inclusion $B_A \subset R_N^\vartheta$. □

4.2. Reductions. In this section, we argue that we can restrict to simpler cases while proving Theorem 2.2. First, we show that we can restrict to a single parameter to determine $B_A$.

**Proposition 4.2.** Theorem 2.2 holds only if it holds for $k = 1$.

**Proof.** Let us fix a support $A \subset \mathbb{Z}^{k+1}$ as in Theorem 2.2 and a generic polynomial $\varphi(x, t) \in \mathbb{C}^A$. Thus, it suffices to find a polynomial map $L : \mathbb{C}^* \to (\mathbb{C}^*)^k$, $s \mapsto L(s)$, such that $\varphi_L(x, s) := \varphi(x, L(s))$ is generic with respect to its support $A_L$, with the same invariants $N, d$ and $\vartheta$ as $\varphi$. Moreover, the support $A_L$ should not be contained in a line. To see that this implies the result, observe that $\text{im}(\mu_{\varphi}^b) \subset \text{im}(\mu_{\varphi_L}^b)$, that $\text{im}(\mu_{\varphi_L}^b) = R_N^d \cap B_N^* \cdot d$ by Theorem 2.2, and that $\text{im}(\mu_{\varphi_L}^b) \subset R_N^\vartheta \cap B_N^* \cdot d$ by Lemma 4.1. Eventually, this implies the sought equality $\text{im}(\mu_{\varphi_L}^b) = R_N^\vartheta \cap B_N^* \cdot d$.

We take $L$ to be a monomial map, that is $L(s) = (s^{n_1}, \ldots, s^{n_k})$. The dual map $\mathbb{Z}^k \to \mathbb{Z}$ between character lattices is a linear projection and the map $\mathbb{C}^A \to \mathbb{C}^A_L$, $\varphi \mapsto \varphi_L$, between the corresponding spaces of polynomials is surjective. Consequently, the polynomial $\varphi_L$ can be made generic with respect to its support $A_L$ by choosing the coefficients of $\varphi$ accordingly. Moreover, the polynomials $\varphi$ and $\varphi_L$ share the same integers $d(A) = d(A_L)$ and $N(A) = N(A_L)$ since the latter integers only depend on the projection of the support $A$ (respectively $A_L$) on the first coordinate axis of $\mathbb{Z}^{k+1}$ (respectively $\mathbb{Z}^{1+1}$).

Let us show that we can choose the exponents $n_j$ involved in $L$ so that $\vartheta(A) = \vartheta(A_L)$. If $A$ is sharp, that is $c_q/c_{q+N} = ct_1^{\varphi_1} \cdots t_k^{\varphi_k}$ for some $c \in \mathbb{C}^*$ and some tuple $(a_1, \ldots, a_k) \in \mathbb{Z}^k$, then $c_q/c_{q+N} \circ L = ct_1^{an_1+\cdots+an_k}$. If we take $n_1, \ldots, n_k$ so that $a_1n_1 + \cdots + an_k = \vartheta$, which is possible since $\vartheta = \text{GCD}(a_1, \ldots, a_k)$, then we have $\vartheta(A) = \vartheta(A_L)$. If $A$ is not sharp, we can always take $(n_1, \ldots, n_k)$ such that the map $A \to A_L$ induced by $L$ is injective (take $(n_1, \ldots, n_k)$ not perpendicular to the difference between any pair of points in $A$). In particular, $(c_q/c_{q+N}) \circ L$ is not a monomial and $\vartheta(A) = \vartheta(A_L) = 1$. 
Eventually, it remains to show that $L$ can be chosen so that $A_L$ is not contained in a line. If $A$ is not sharp, at least one of $c_q \circ L$ or $c_{q+N} \circ L$ has at least 2 monomials (recall that we took $L$ to be injective in that case). Thus the support $A_L$ is not contained in a line. Assume that $A$ is sharp. Then the set of vectors $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ such that $A_L$ is contained in a line is itself contained in an affine hyperplane $V \subset \mathbb{R}^k$ not parallel to $H := \{(n_1, \ldots, n_k) \in \mathbb{Z}^k : n_1a_1 + \cdots + n_ka_k = \vartheta\}$. This is because $A$ is not contained in a line. In particular, the set $H \setminus V$ is not empty and any vector $(n_1, \ldots, n_k) \in H \setminus V$ will do. The result follows.

We conclude this section with another reduction.

**Proposition 4.3.** Theorem 2.2 holds only if it holds for supports $A$ with $d(A) = 1$.

**Proof.** Let us fix a support $A \subset \mathbb{Z}^{k+1}$ as in Theorem 2.2 and a generic polynomial $\varphi(x, t) \in \mathbb{C}^A$. As in the proof of Lemma 4.1, we can write

$$\varphi(x, t) = \bar{\varphi}(x^d, t)$$

for some Laurent polynomial $\bar{\varphi}$ supported at $\bar{A} \subset \mathbb{Z}^{k+1}$. Again, we add a tilde to every piece of notation coming from $\bar{\varphi}$. In view of the commutative diagram (16), it suffices to prove that $f_{N,d}$ maps $R^d_{N,x}$ to $B^*_N \cap R^d_{N,x}$. To see this, observe first that $\bar{\vartheta} = \vartheta$. Then, consider the commutative diagram

$$
\begin{array}{ccc}
C_{N/d}(\mathbb{C}^*) & \longrightarrow & C_N(\mathbb{C}^*) \\
\downarrow \text{c} \longmapsto \Pi_{x \in c} x & & \downarrow \text{c} \longmapsto \Pi_{x \in c} x \\
\mathbb{C}^* & \longrightarrow & \mathbb{C}^*
\end{array}
$$

where $C_{N/d}(\mathbb{C}^*) \to C_N(\mathbb{C}^*)$ is the pullback by $x \mapsto x^d$ and $\mathbb{C}^* \to \mathbb{C}^*$ is the multiplication by $(\prod_{0 \leq j < d} e^{2i\pi j / d})^{N/d} = (-1)^{N(d-1)/d}$. At the level of fundamental groups, we obtain the following commutative diagram

$$
\begin{array}{ccc}
B^*_N & \longrightarrow & B^*_N \\
\downarrow \text{ind}_{\bar{N}} & & \downarrow \text{ind}_N \\
\pi_1(\mathbb{C}^*, \bar{x}_0) \sim \longrightarrow & \pi_1(\mathbb{C}^*, x_0)
\end{array}
$$

The result follows.

### 4.3. Trinomials.

According to the previous section, it suffice to prove Theorem 2.2 for polynomials $\varphi$ whose coefficients depend on a single parameter $t \in \mathbb{C}^*$ and whose support $A$ satisfies $d := d(A) = 1$. In this section, we restrict our attention even further, namely to trinomials $\varphi(x, t) = \alpha t^{n_1} x^{n_1} + \beta t^{n_2} x^{n_2} + \gamma t^{n_3} x^{n_3}$ with $\alpha, \beta, \gamma \in \mathbb{C}^*$ and integers $n_1 \leq n_2 \leq n_3$. The aim of this section is to prove the following.

**Theorem 4.4.** Let $\{0\} \subset A \subset \mathbb{Z}^2$ be any support set not contained in a line, with associated integers $d := d(A)$ and $\vartheta := \vartheta(A)$ and satisfying $|A| = 3$. For any generic Laurent polynomial $\varphi \in \mathbb{C}^A$, the image of the braid monodromy map $\mu_{\varphi}^{br}$ is the subgroup $B^*_{N,d} \cap R^d_{N,x}$ of $B^*_N$. 
For the sake of simplicity, we will assume that \( \alpha = \beta = \gamma = 1 \). There is no loss of generality in doing so since one of the coefficients \( \alpha, \beta \) or \( \gamma \) can be brought to 1 by projective equivalence while the remaining two can be compensated by a harmless change of coordinates \( (x, t) \to (ux, v t) \), \( u, v \in \mathbb{C}^* \).

4.3.1. Braids and their diagrams. Braids in \( B_N^* \) can be represented by braid diagrams similarly to the elements of the Artin braid group \( B_N := \pi_1(\mathbb{C}_N, N) \). Braids live in the 3-dimensional space \( \mathbb{C} \times \text{timeline} \) and their diagram are obtained by a real linear projection on \( \mathbb{R} \times \text{timeline} \), see for instance [FM11] Chapter 9.1.1. Here, we associate to any braid in \( B_N^* \) a braid diagram using the argument map \( \arg : \mathbb{C}^* \to S^1 \) as our projection, as done in [EL21] Section 2.1.

Precisely, a braid \( \beta \in B_N^* \) is an isotopy class of continuous map \( \beta : [0, 1] \to \mathbb{C}_N(\mathbb{C}^*) \) into the configuration space \( \mathbb{C}_N(\mathbb{C}^*) \), such that \( \beta(0) = \beta(1) = N \). The argument map \( \arg : \mathbb{C}^* \to S^1 \) defines a map from the open dense subset \( U \subset \mathbb{C}_N(\mathbb{C}^*) \) of configurations with pairwise distinct arguments to \( \mathbb{C}_N(S^1) \). If \( C \subset \mathbb{C}_N(\mathbb{C}^*) \) is the complement of \( U \), we can ensure, using an isotopy if necessary, that exactly 2 points of \( \beta(s) \) have common argument whenever \( \beta(s) \in C \). At such meeting point of the projection under \( \arg \) of 2 strands of \( \beta \), we can make sense of which of them has smaller modulus in \( \mathbb{C}^* \) than the other. We choose to represent the strand with smaller modulus as crossing over the other strand. This way, we can represent the element \( \beta \) by the corresponding diagram in \( S^1 \times [0, 1] \) picturing simultaneously the trajectories of the \( N \) points \( \beta(s) \), keeping track of which strand is passing over which one at a crossing. In order to be able to draw braid diagrams in \( S^1 \times [0, 1] \), we will choose a fundamental domain \([\theta, \theta + 2\pi] \times [0, 1] \) instead. Thus, strings in a diagram are allowed to hit boundary points \((\theta, s)\), disappear and reappear at \((2\pi + \theta, s)\).

In Figure 1, we represent the braid diagrams of some useful elements of \( B_N^* \). From the braids \( b_1 \) and \( \tau \), we define \( b_j := \tau^{j-1} b_1 \tau^{-j+1} \), \( j \in \{1, \cdots, N\} \). By a slight abuse of language, we will identify braids with their diagrams and therefore view braid diagrams as elements of the relevant braid group.

\[ b_1 \quad \cdots \quad b_N \quad \tau \]

**Figure 1.** The braid diagrams of \( b_1 \), \( b_N \) and \( \tau \) in \( B_N^* \).

**Lemma 4.5.** The elements \( b_1, \cdots, b_N \in B_N^* \) generates \( \text{Ker} (\text{ind}_N) \).

**Proof.** Plainly, the elements \( b_j, j \in \{1, \cdots, N\} \), belong to \( \text{Ker} (\text{ind}_N) \). As shown in [EL21] Lemma 2.2, the elements \( b_j \) generate \( B_N^* \) together with \( \tau \). Moreover, any element of \( B_N^* \) can be written as \( b \tau^k \) for some integer \( k \) where \( b \) is a word in the \( b_j \)'s. Indeed, the commutator \( \tau b_j \tau^{-1} b_j^{-1} \) equals the product \( b_{j+1} b_{j}^{-1} \), or equivalently \( \tau b_j = b_{j+1} \tau \). Eventually, since \( \text{ind}_N(b \tau^k) = k \), the element \( b \tau^k \) is in \( \text{Ker}(\text{ind}_N) \) if and only if \( k = 0 \). Thus, we have the inclusion \( \text{Ker}(\text{ind}_N) \subset \langle b_1, \cdots, b_N \rangle \) and the result follows. \( \square \)
4.3.2. Further reductions. In this section, we state some elementary lemmas that allow us to restrict our attention to trinomials \( \varphi(x, t) \) with specific support sets.

**Lemma 4.6.** Let \( \{0\} \subset A \subset \mathbb{Z}^2 \) be any support set not contained in a line and \( \varphi \in \mathbb{C}^d \) any generic Laurent polynomial. Then, the conclusion of Theorem 2.2 holds for \( \varphi \) if and only if it holds for \( \varphi(x, t) := \varphi(x^e, t^\nu) \) where \( |e| = |
u| = 1 \).

**Proof.** The statement follows from the fact that the change of variable \( (x, t) \rightarrow (x^e, t^\nu) \) induces an isomorphism between the two fibrations \( \{(x, t) \in \mathbb{C}^* \times (\mathbb{C} \setminus \mathcal{B}) : \varphi(x, t) = 0\} \rightarrow \mathbb{C} \setminus \mathcal{B} \) and \( \{(x, t) \in \mathbb{C}^* \times (\mathbb{C} \setminus \mathcal{B}) : \varphi(x, t) = 0\} \rightarrow \mathbb{C} \setminus \mathcal{B} \). We have the desired commutative diagrams, similar to (16) and (17). The details are left to the reader.

**Corollary 4.7.** Theorem 4.4 holds only if it holds for at least one trinomial in each orbits of the following group action

\[ (p, q, d, \nu) \in \mathbb{Z}^3 \times \{-1, 1\}, \quad (p, q, d, \nu) \cdot \varphi(x, t) := x^{pq} \varphi(x^d, t^\nu). \]

In particular, it suffices to prove Theorem 4.4 for trinomials \( \varphi \) whose support set \( A \) is of either of the following types:

1. \( A = \{(0, 0), (m, a), (n, b)\} \) with \( 0 < m < n \), \( \text{GCD}(m, n) = 1 \) with \( na - mb > 0 \),
2. \( A = \{(0, a), (0, b), (1, 0)\} \) with \( a < b \).

**Proof.** The first part of the statement follows from Proposition 4.3, Lemma 4.6 and the fact that the fibration \( \{(x, t) \in \mathbb{C}^* \times (\mathbb{C} \setminus \mathcal{B}) : \varphi(x, t) = 0\} \rightarrow \mathbb{C} \setminus \mathcal{B} \) associated to \( \varphi \) is the same as the fibration associated to \( x^{pq} \varphi \).

For the second part, observe that the involutions \( x \mapsto x^{-1} \) and \( t \mapsto t^{-1} \) induce the symmetry along the first and second coordinate axis in the lattice of monomials \( \mathbb{Z}^2 \). The multiplication by \( x^{pq} \) induces the translation in the lattice of monomials.

If every vertical line intersects \( A \) in at most one point, then we can translate \( A \) so that \( \{(0, 0)\} \subset A \subset \mathbb{N} \times \mathbb{Z} \). Using the involution \( t \mapsto t^{-1} \), we can achieve \( na - mb > 0 \). Eventually, using the action of \( d \), we can achieve \( \text{GCD}(m, n) = 1 \). Thus, we can always reduce to (1).

If there is a vertical line intersecting \( A \) in 2 points, we can translate \( A \) so that this line becomes the second coordinate axis and so that the third point of \( A \) lies on the first coordinate axis. Using \( x \mapsto x^{-1} \) and \( t \mapsto t^{-1} \), we can achieve \( A = \{(0, a), (0, b), (d, 0)\} \) with \( a < b \) and \( d > 0 \). It remains to rescale \( A \) by \( d \) along the first coordinate axis to achieve (2).

4.3.3. Trinomials of type (2). As we have seen in Corollary 4.7, we can restrict to two types of trinomials while proving Theorem 4.4, namely trinomials of type (1) or (2). In this subsection, we prove the theorem for type (2) trinomials. Let \( \varphi(x, t) = (t^a + t^b) + x \) with integers \( a < b \). In this case, \( N = 1 \) and \( B^*_1 = \pi_1(\mathbb{C}^*) \). Additionally, we have \( d = \vartheta = 1 \).

The bifurcation set \( \mathcal{B} \) is defined by the equation \( t^a + t^b = 0 \Leftrightarrow t^a(t^{b-a} + 1) = 0 \) that has \( b-a \) simple roots in \( \mathbb{C}^* \). When \( t \) travels along a small circle around one of the aforementioned roots, then the root \( x = -(t^a + t^b) \) of \( \varphi(x) \) travels a simple loop around 0. The corresponding element in \( \text{im}(\mu^b_0) \) is thus a generator of \( B^*_1 \). Since \( d = \vartheta = 1 \) and \( R^1_1 = B^*_1 \), this proves Theorem 4.4 for trinomials of type (2).

4.3.4. The bifurcation set. From now until the end of Section 4.3.6, we restrict to trinomials of type (1) according to the dichotomy given in Corollary 4.7, that is \( \varphi(x, t) = 1 + t^a x^m + t^b x^n \) with \( 0 < m < n \), \( \text{GCD}(m, n) = 1 \) and \( na - mb > 0 \).
Let us determine the bifurcation set $B \subset \mathbb{C}$ of $\phi$. To that aim, consider for a moment the polynomial $f(x) = 1 + ux^m + vx^n$. Besides the obvious case $v = 0$, the polynomial $f(x)$ has strictly less than $n$ roots if the triple $(u, v, x)$ satisfies

$$\begin{align*}
1 + ux^m + vx^n &= 0 \\ mu^m + nvx^n &= 0 \
\iff \quad v &= \frac{m}{n} x^{-n} \\ u &= \frac{m}{n} x^{-m}.
\end{align*}$$

Therefore, the polynomial $f$ is singular if and only if $v = m/n x^{-n} = m/n x^{-m}$. It follows that the bifurcation set is defined by the equation $t^b \cdot (n - m)^m = t^a \cdot (m - n)^n$. In particular, the points of $B \setminus \{0\}$ are equi-distributed on the circle $\{t \in \mathbb{C} : |t| = \rho\}$ for some $\rho := \rho(a, b, m, n) > 0$.

### 4.3.5. The coamoeba of $\phi$.

Denote by $\text{Arg}_\phi$ the (closed) coamoeba of $\phi$, that is the closure in $(S^1)^2$ of the subset

$$\{((\theta, \nu) \in (S^1)^2 : \theta = \arg(x), \nu = \arg(t), \phi(x, t) = 0\}.$$

The set $\text{Arg}_\phi$ can easily be described in terms of $\text{Arg}_\ell$ where $\ell(x, t) = 1 + x + t$. Indeed, the polynomial $\phi$ is projectively equivalent to the composition of the polynomial $\ell$ with $\psi(x, t) = (t^a x^m, t^b x^n)$. Therefore, $\text{Arg}_\phi$ is the pullback of $\text{Arg}_\ell$ under the covering map $(\theta, \nu) \mapsto (m\theta + a\nu, n\theta + b\nu)$ from $(S^1)^2$ to itself. Straightforward computations show that the set $\text{Arg}_\ell$ is the union of the 2 closed triangles in $(S^1)^2$ delimited by the 3 geodesics

$$\{\theta = \pi\}, \quad \{\nu = \pi\} \quad \text{and} \quad \{\theta + \nu = \pi\}.$$ 

These two triangles happen to be copies of the Newton polygon of $\ell$ rotated by an angle of $\pi/2$ and $-\pi/2$ respectively. As a consequence, the closed coamoeba $\text{Arg}_\phi$ consists of $2\delta$ many triangles. Each such triangle is bounded by 3 geodesics which are respectively the pullback under the covering $\psi$ of each of the 3 geodesics in (19). 

We refer to the boundary geodesics of $\text{Arg}_\phi$ with slope $(b, -n)$, $(-a, m)$ and $(a - b, n - m)$ respectively as the D-, L- and the R-geodesics, see Figures 2 and 3. The latter slopes are the outer normal vectors to the edges of the convex hull $\text{conv}(A)$. The letters D, L and R stand respectively for down, left and right in relation to the respective positions of the corresponding edges of $\text{conv}(A)$.

![Figure 2](https://via.placeholder.com/150)

**Figure 2.** The closed coamoeba $\text{Arg}_\ell$ of the line $\ell$. 
The braid monodromy map $\mu^{br}_\varphi$ is about tracking the configuration of points given by horizontal sections of the curve $\{(x, t) \in (\mathbb{C}^*)^2 : \varphi(x, t) = 0\}$. Below, we argue that we can study the horizontal sections of $\text{Arg}_\varphi$ in order to determine the image of $\mu^{br}_\varphi$.

While generic horizontal sections of $\{(x, t) \in (\mathbb{C}^*)^2 : \varphi(x, t) = 0\}$ consist of $n$ distinct points, generic horizontal sections of $\text{Arg}_\varphi$ consist of $n$ distinct line segments. Indeed, observe that each such segment has a single endpoint lying on a $D$-geodesic and that $D$-geodesics intersect any horizontal section $n$ times all together. This is true unless the horizontal section passes through the intersection point of a $L$-geodesic with a $R$-geodesic as the corresponding segment is thus bounded by two points on $D$-geodesics (possibly distinct). For practical sake, we refer to the intersection points between $L$- and $R$-geodesics as the singular vertices of $\text{Arg}_\varphi$. We denote by $\mathcal{S} \subset (S^1)^2$ the set of all of them.

The coamoeba of $\varphi$ with $A = \{(0, 0), (2, 4), (5, 2)\}$.

**Figure 3.**

**Lemma 4.8.** The $\delta$ many singular vertices of $\text{Arg}_\varphi$ have pairwise distinct $\nu$-coordinates.

**Proof.** The set $\mathcal{S} \subset \text{Arg}_\varphi$ is the preimage under the covering $(\theta, \nu) \to (m\theta + a\nu, n\theta + b\nu)$ from $S^1$ to itself of the vertex $(0, \pi)$ of $\text{Arg}_\ell$. This is because the latter covering maps $L$- and $R$-geodesics respectively to $\{\nu = \pi\}$ and $\{\theta + \nu = \pi\}$ which intersect at $(0, \pi)$. Therefore, $\mathcal{S}$ consists of $\delta$ many points and it is equal, up to a translation, to the projection onto $(\mathbb{R}/2\pi\mathbb{Z})^2$ of the lattice $\frac{2\pi}{\delta} \left( \frac{b}{m} - \frac{a}{n} \right) \cdot (2\pi\mathbb{Z})^2$. Two points in that lattice have the same $\nu$-coordinate in $(\mathbb{R}/2\pi\mathbb{Z})^2$ if their difference, which is of the form $\frac{2\pi}{\delta} (\lambda \left( \begin{bmatrix} b \\ m \end{bmatrix} \right) + \mu \left( \begin{bmatrix} -a \\ m \end{bmatrix} \right))$, has a second coordinate in $2\pi\mathbb{Z}$, that is $\mu m - \lambda n = \kappa \delta$ for some $\kappa \in \mathbb{Z}$. The general solution to the latter equation is the sum of a particular solution, for instance $(\lambda, \mu) = -\kappa \cdot (a, b)$, with a solution to $\mu m - \lambda n = 0$, which is of the form $(\lambda, \mu) = \ell \cdot (m, n)$ since $\text{GCD}(n,m) = 1$. Thus $(\lambda, \mu) = (\ell m - \kappa a, \ell n - \kappa b)$ and the first coordinate of $\frac{2\pi}{\delta} (\lambda \left( \begin{bmatrix} b \\ m \end{bmatrix} \right) + \mu \left( \begin{bmatrix} -a \\ m \end{bmatrix} \right))$ is therefore $\frac{2\pi}{\delta} (b(\ell m - \kappa a) - a(\ell n - \kappa b)) = \frac{2\pi}{\delta} \ell (bm - an) = -2\pi \ell \in 2\pi\mathbb{Z}$. In conclusion, the two points have the same image in $(\mathbb{R}/2\pi\mathbb{Z})^2$. The result follows. □
Lemma 4.9.

1. For any $t \in \mathbb{C}^*$, the intersection of $\{(\theta, \nu) \in (S^1)^2 : \nu = \arg(t)\}$ with $\text{Arg}_{\varphi}$ consists of $n$ connected components if $\{\nu = \arg(t)\}$ does not pass through $\mathcal{S}$. Otherwise, the intersection consists of $n - 1$ components.

2. The map $\text{Arg} : (\mathbb{C}^*)^2 \to (S^1)^2$ induces a bijection between $\mathcal{S}$ and the set of singular points $(x, t) \in (\mathbb{C}^*)^2$ of $\varphi$. In particular, the projection of $\mathcal{S}$ on $\{1\} \times S^1$ is equal to $\arg(\mathcal{B} \setminus \{0\})$.

3. For any $t \in \mathbb{C} \setminus \mathcal{B}$, each connected component $\mathcal{C}$ of

\[
\{(\nu = \arg(t)) \cap \text{Arg}_{\varphi}\} \setminus \mathcal{S}
\]

contains the projection under Arg of a single point $p$ of $\{(x, t) \in (\mathbb{C}^*)^2 : \varphi(x, t) = 0\}$. If $|t|$ is arbitrarily small, then $\text{Arg}(p)$ is arbitrarily close to the $D$-geodesic bounding $\mathcal{C}$. If $|t|$ is arbitrarily small, then $\text{Arg}(p)$ is arbitrarily close to the other geodesic bounding $\mathcal{C}$.

Proof. 1. This part of the statement follows from the discussion prior to Lemma 4.8 and the lemma itself.

2. From [18], we deduce that if $(x, t)$ is a singular point of $\varphi$, then the pair $(\theta, \nu) := (\arg(x), \arg(t))$ satisfies the equations $n\theta + bv = 0$ and $n\theta + a\nu = \pi$. This is the description of $\mathcal{S}$ used in the proof of Lemma 4.8. Conversely, any point of $\mathcal{S}$ is clearly of the form $(\arg(x), \arg(t))$. The rest of the statement follows.

3. Since $\varphi$ is of type (1) (see Corollary 4.7), the outer normal to the edge $\text{conv}\{0, 0\}, (n, b)$ of $\text{conv}(A)$ points downward while the outer normals to the two remaining edges of $\text{conv}(A)$ point upwards. Therefore, it follows from the Newton-Puiseux Theorem that the $n$ solutions to $\varphi(x, t) = 0$ are asymptotically equivalent to $(x, t) = ((-1/t^b)^{1/n}, t)$ for any of the $n$ determinations of $(-1/t^b)^{1/n}$ when $|t|$ tends to 0. Similarly, the $n$ solutions to $\varphi(x, t) = 0$ split into two groups when $|t|$ tends to $\infty$, those equivalent to $((-1/t^1)^{1/m}, t)$ and those equivalent to $((-1/a-b)^{1/(n-m)}, t)$. The image under Arg of the parametrization $((-1/t^b)^{1/n}, t)$, $((-1/t^a)^{1/m}, t)$ and those equivalent to $((-1/a-b)^{1/(n-m)}, t)$ map surjectively under Arg to the $D$, $L$- and $R$-geodesics respectively. This implies the second part of the statement.

The first part of the statement follows from the second. Indeed, if $t$ is such that $\{\nu = \arg(t)\}$ is disjoint from $\mathcal{S}$, then the above asymptotic implies that, for $|t|$ arbitrarily small, there is exactly one point of $\text{Arg}\{\varphi(x, t) = 0\}$ in each of the $n$ components of $\{\nu = \arg(t)\} \cap \text{Arg}_{\varphi}$. Fixing $\arg(t)$, the latter property remains true when $|t|$ vary from 0 to $\infty$ since the zero-set $\{\varphi(x, t) = 0\}$ depends continuously on $t$, with constant cardinality $n$. If $\{\nu = \arg(t)\}$ intersects $\mathcal{S}$ at $s$, we can apply the same reasoning on continuity, except that the half ray in $\mathbb{C}^*$ given by fixing $\arg(t)$ meets $\mathcal{B}$ exactly once. So for $|t|$ small, the above argument works. For $|t|$ large, we can use the two other asymptotics to argue that the two points in question in $\{\nu = \arg(t)\} \cap \text{Arg}_{\varphi} \setminus \mathcal{S}$ are close to $s$. It remains to argue that they are on different sides of $s$ which follows from continuity, by perturbing $\arg(t)$.

4.3.6. Proof of Theorem 4.4. Here, we compute explicitly the map $\mu_{br}$ of $\varphi$. To do so, we fix a generating set of $\pi_1(\mathbb{C} \setminus \mathcal{B}, t_0)$ and compute its image under $\mu_{br}$, relying essentially on Lemma 4.9.

Let us fix a graph $\{t_0\} \subset \Gamma \subset \mathbb{C} \setminus \mathcal{B}$ that is a deformation retraction of its ambient space, so that $\pi_1(\mathbb{C} \setminus \mathcal{B}, t_0) = \pi_1(\Gamma, t_0)$. We take $\Gamma$ to be the union of two circles

$$C_\varepsilon := \{z \in \mathbb{C}^* : |z| = \varepsilon\} \quad \text{and} \quad C_M := \{z \in \mathbb{C}^* : |z| = M\}$$
for $\varepsilon, M > 0$ arbitrarily small and large respectively, together with $\delta$ many segments joining $C_\varepsilon$ to $C_M$. Each segment is contained in a half ray $\{ z \in \mathbb{C}^* : \arg(z) = \theta \}$, the segments have equidistributed arguments $\theta$ on $S^1$ and share the same distance to $B \setminus \{0\}$. This is possible since $B \setminus \{0\}$ is equidistributed on some circle $C_\rho$, see Section 4.3.4. All together, the graph $\Gamma$ looks like a circular railway track as pictured in Figure 4. We fix the base point $t_0 \in \Gamma$ as a midpoint on an edge of $\Gamma$ contained in $C_\varepsilon$. We also fix the set of generators $\ell_0, \ell_1, \cdots, \ell_\delta$ of $\pi_1(\Gamma, t_0)$ as pictured in Figure 4. We have the following.

**Lemma 4.10.** For any $j \in \{1, \cdots, \delta\}$, we have $\mu^b_\varphi(\ell_j) = b_k$ for some $k \in \{1, \cdots, n\}$. Conversely, we have the inclusion $\{b_1, \cdots, b_n\} \subset \text{im}(\mu^b_\varphi)$. Moreover, we have $\mu^b_\varphi(\ell_0) = \tau b$.

Before proving the Lemma, let us fix the labelling of the roots of $\varphi(x, t_0)$ as follows: we choose arbitrarily one root to be labelled 1 and label the remaining roots by increasing argument. Therefore, their projection under $\text{Arg}$ appears ordered on the horizontal section $\{\nu = \text{arg}(t_0)\}$ of $\text{Arg}_{\varphi}$, see Figure 5. The horizontal section $\{\nu = \text{arg}(t_0)\}$ splits the union of the D-geodesics into $n$ open segments that we refer to as the D-segments. Recall that, by the construction of $\Gamma$, the choice of $t_0$ and Lemma 4.9, the upper endpoint of each D-segment is arbitrarily close to a labelled point of $\text{Arg}(\{\varphi(x, t_0) = 0\})$. We label the D-segments accordingly. The union of the D-geodesics and the horizontal geodesic $\{\nu = \text{arg}(t_0)\}$ split $(S^1)^2$ into $n$ components that we refer to as D-parallelograms and that we label according to the D-segment on their left, see again Figure 5. Eventually, the $\text{GCD}(n, b)$ many D-geodesics split $(S^1)^2$ into $\text{GCD}(n, b)$ many components that we refer to as the D-stripes.

**Proof.** Fix a parametrisation $s \mapsto t(s)$ of $\ell_0$. According to Lemma 4.9.3, the projection under $\text{Arg}$ of $\{(x, t(s)) \in (\mathbb{C}^*)^2 : \varphi(x, t(s)) = 0\}$ is arbitrarily close to the intersection of the horizontal section $\{\nu = \text{arg}(t(s))\}$ with the union of the D-geodesics. Thus, the projection under

![Figure 4. The graph $\Gamma$ and the generators $\ell_0, \ell_1, \cdots, \ell_\delta$ of $\pi_1(\Gamma, t_0)$.](image-url)
Arg of each point of \( \{(x,t(s)) \in (\mathbb{C}^*)^2 : \varphi(x,t(s)) = 0\} \) follows a trajectory as close as desired to a straight line of slope \((-b,n)\). Thus, we have that \( \mu_{\varphi}^{br}(\ell_0) = \tau_b \).

For any \( j \in \{1, \cdots, \delta\} \), we can find a locally injective parametrisation \([0,1] \rightarrow \mathbb{C}^*\), \( s \mapsto t(s) \) of \( \ell_j \) such that there exist \( 0 \leq s_1 < s_2 \leq 1 \) satisfying \( t([0,s_1]) = t([s_2,1]) \) and such that the restriction of \( s \mapsto t(s) \) to \([s_1,s_2]\) is injective. Observe that \( (s_1,s_2) = (0,1) \) exactly when \( j = 1 \).

Let us restrict to the case \( j > 1 \) as the result for \( j = 1 \) is similar.

As we have seen above, the \( n \) points of \( \operatorname{Arg} \{ (x,t(s)) \in \mathbb{C}^* : \varphi(x,t(s)) = 0 \} \) follow tightly the D-geodesics downwards in \( S^1 \times S^1 \) when \( s \) travels along \([0,s_1]\) and will follow the same trajectory upwards between \( s_2 \) and 1. We now focus on the segment \([s_1,s_2]\). Let \( \nu_1, \nu_2 \in S^1 \) be such that \( \operatorname{arg} \{ t([s_1,s_2]) \} \subset S^1 \) is the arc between \( \nu_1 \) and \( \nu_2 \) with \( \operatorname{arg} \{ t(s_1) \} = \operatorname{arg} \{ t(s_2) \} = \nu_1 \). We denote this arc, somehow abusively, by \([\nu_1,\nu_2]\). The intersection of \( \operatorname{Arg}_{\varphi} \) with \( S^1 \times [\nu_1,\nu_2] \) consists of exactly \( n-1 \) connected components. Indeed, the intersection of \( S^1 \times [\nu_1,\nu_2] \) with the union of the D-geodesics consists of \( n \) segments and each component of \( \operatorname{Arg}_{\varphi} \cap S^1 \times [\nu_1,\nu_2] \) contains exactly one such segment, except the one component that contains a singular vertex of \( \operatorname{Arg}_{\varphi} \) (by Lemma 4.9.2, there is exactly one such vertex since \( \ell_j \) encloses exactly one point of \( \mathcal{B} \)). According to Lemma 4.9.3, when \( s \) goes from \( s_1 \) to \( s_2 \) then each of the \( n \) points of the subset \( \operatorname{Arg} \{ (x,t(s)) \in \mathbb{C}^* : \varphi(x,t(s)) = 0 \} \) travels arbitrarily close the boundary of one of the connected components of \( \operatorname{Arg}_{\varphi} \cap S^1 \times [\nu_1,\nu_2] \), see Figure 6. More precisely, each of the \( n \) points of \( \operatorname{Arg} \{ (x,t(s)) \in \mathbb{C}^* : \varphi(x,t(s)) = 0 \} \) travels (almost) horizontally when \( t(s) \) travels along part of the circles \( C_{\epsilon} \) and \( C_{\nu} \) and follows tightly part of a boundary geodesic of \( \operatorname{Arg}_{\varphi} \) otherwise. In particular, exactly two branches of \( s \mapsto \operatorname{Arg} \{ (x,t(s)) \in \mathbb{C}^* : \varphi(x,t(s)) = 0 \} \) cross each other at the unique point of \( \mathcal{S} \cap S^1 \times [\nu_1,\nu_2] \). The image of \( \ell_j \) under \( \mu_{\varphi}^{br} \) is therefore of the form \( b_k^{1+} \) for some \( 1 \leq k \leq n \). Plainly, the integer \( k \) is the label of the D-parallelogram containing the singular vertex involved. Although it is not of crucial importance, it can be shown that, according to our conventions, we have \( \mu_{\varphi}^{br}(\ell_j) = b_k \), see Figure 6.

**Figure 5.** The labelling of the roots of \( \varphi(x,t_0) \), of the D-segments and the first D-parallelogram \( P_1 \).
In order to conclude, we need to prove that all \( b_k, 1 \leq k \leq n \), are in the image of \( \mu^{br}_\varphi \). It is not necessarily true that every \( b_k \) appears as \( \mu^{br}_\varphi (\ell_j^{\pm 1}) \) since it is not granted that every D-parallelogram contains a singular vertex. However, we claim that any \( b_k \) appears as \( \mu^{br}_\varphi (\ell_j^{\pm 1} \circ \ell_0^p) \) for an appropriate integer \( p \). According to the above discussion, this is equivalent to saying that each D-stripe contains a singular vertex. But this is clear since at least one such stripe contains a singular vertex and since, for two different stripes \( S_1 \) and \( S_2 \), we have that Arg\(_{\varphi} \cap S_1 \) is a translation of Arg\(_{\varphi} \cap S_2 \) in the argument torus \( S^1 \times S^1 \). Therefore, any such stripe contains a singular vertex. This implies the inclusion \( \{ b_1, \ldots, b_n \} \subset \text{im}(\mu^{br}_\varphi) \) and concludes the proof. \( \square \)

![Figure 6. The image of \( \ell_j \), \( j > 1 \), under \( \mu^{br}_\varphi \).](image)

**Proof of Theorem 4.4.** By Corollary 4.7, we could reduce the proof of the Theorem to the case of trinomials of type (2), in which case we need to show that \( \text{im}(\mu^{br}_\varphi) = R^\vartheta_n \) since \( d = \text{GCD}(n, m) = 1 \). Here, \( \vartheta = b \). Since the elements \( b_1, \ldots, b_n \) generate Ker(ind\(_N\)), by Lemma 4.5 and \( \text{ind}_{\mathcal{N}}(r^b) = b \), they generate \( R^\vartheta_N \) all together. The result now follows from Lemma 4.10. \( \square \)

4.4. **Proof of Theorem 2.2.** According to Proposition 4.2, we can restrict our attention to the case \( k = 1 \) while proving Theorem 2.2. By Proposition 4.3, we can even restrict to supports \( A \) so that \( d(A) = 1 \). Therefore, we fix a support \( \{(0, 0)\} \subset A \subset \mathbb{Z}^2 \) with \( d(A) = 1 \) and horizontal width \( N := N(A) \) and fix a generic polynomial \( \varphi \in \mathbb{C}^A \).

Let us shortly describe the overall strategy of the proof. Let \( \rho : [0, 1] \rightarrow \mathbb{C}^A, s \mapsto \varphi_s \), be a continuous path from \( \varphi \) to some other polynomial \( \varphi \in \mathbb{C}^A \) and denote \( \mathcal{B}_s \subset \mathbb{C} \) the bifurcation set of \( \varphi_s \). Fix a continuous path \( t : s \mapsto t_{0,s} \in \mathbb{C} \setminus \mathcal{B}_s \) of base points. The pair \( (\rho, t) \) is said to be \( A \)-suitable if
- the support \( A_s \) of \( \varphi_s \) has horizontal width \( N(A_s) = N \) for any \( 0 \leq s \leq 1 \),
- the family of fundamental groups \( \pi_1(\mathbb{C} \setminus \mathcal{B}_s, t_{0,s}) \) is locally trivial for \( 0 \leq s < 1 \).

Plainly, the pair \( (\rho, t) \) induces an injective morphism \( \text{im}(\mu^{br}_{\varphi_s}) \rightarrow \text{im}(\mu^{br}_{\varphi}) \). In order to prove Theorem 2.2, we will take \( \varphi \) to be a trinomial and determine the image of \( \text{im}(\mu^{br}_{\varphi_s}) \rightarrow \text{im}(\mu^{br}_{\varphi}) \).
with the help of Theorem 4.4. Using the Euclidean algorithm [EL21 Proposition 4.1], we will then argue that that the image of the latter morphisms generate the expected braid monodromy group, once sufficiently many trinomials \( \bar{\varphi} \) are taken. It will be crucial to observe that different \( A \)-suitable pairs \((\rho, t)\) lead to different morphisms \( \text{im}(\mu^b_\varphi) \to \text{im}(\mu^b_{\bar{\varphi}}) \). We will choose these pairs carefully. In that regard, it will be helpful to have the following in mind.

**Remark 4.11.** For a general support \( \{(0,0)\} \subset A \subset \mathbb{Z}^2 \), the base point \( N \) of the braid group \( B^*_N := \pi_1(C_N(C^*), N) \) is globally invariant under the multiplication \( x \mapsto e^{2\pi i/d}x \), where \( d := d(A) \). In turn, the latter multiplication induces an automorphism \( \iota \) on \( B^*_N \) and \( B^*_{N,d} \) can be alternatively defined as the invariant subgroup of \( \varphi \). To make sense of the subgroup \( \varepsilon > 0 \) and \( N^d \), choose these pairs carefully. In that regard, it will be helpful to have the following in mind.

Recall that the Hausdorff distance on compact subsets of \( C^* \) induces a metric on the configuration space \( C_N(C^*) \). Fix a binomial \( \beta \in C^{A_0} \) where \( A_0 \subset A \) has horizontal width \( N \). Fix \( \varepsilon > 0 \) arbitrarily small and fix an open ball \( V \subset C^A \) containing \( \beta \) and such that
- for any \( \varphi \in \mathcal{V} \), the corresponding bifurcation set \( \mathcal{R}_\varphi \subset C \) does not contain \( 1 \),
- for any \( \varphi \in \mathcal{V} \), the configuration \( N^\varphi := \{ x \in C^* : \varphi(x, 1) = 0 \} \) belongs to the \( \varepsilon \)-neighbourhood of \( N^\beta \) in \( C_N(C^*) \).

The existence of \( \varphi \) is obvious since \( 1 \notin \mathcal{R}_\beta = \emptyset \) and since \( \mathcal{R}_\beta \) and \( N^\varphi \) depend continuously on \( \varphi \).

As discussed in Remark 4.11, any path \( \rho \subset \mathcal{V} \) from \( \varphi \) to \( \beta \) allows to define the subgroup \( B^*_{N,\beta} \subset B^*_{N,\beta} \) for any divisor \( d \) of \( N \). This is because the set \( N^\beta \) is invariant under multiplication by \( e^{2\pi i/N} \). Since \( \mathcal{V} \) is simply connected, the latter subgroup is independent of the path \( \rho \). Moreover, any such path defines an isomorphism between \( B^*_{N,\beta} \) and \( B^*_{N,\beta} \) and the isomorphism is independent from the path, for the same reason as above. In particular, the braid groups \( B^*_{N,\beta} \) are mutually isomorphic to each other. (\( * \))

Since we are allowed to choose \( \varphi \) in some Zariski-open subset of \( C^A \), we choose \( \varphi \in \mathcal{V} \). In particular, we can make sense of the subgroup \( B^*_{N,\beta} \subset B^*_{N} \) for any divisor \( d \) of \( N \). We have the following.

**Lemma 4.12.** For any \( A_0 \subset \bar{A} \subset A \) with \( |\bar{A}| = 3 \) and any generic \( \bar{\varphi} \in C^{\bar{A}} \cap \mathcal{V} \), there exists a path \( \rho \subset \mathcal{V} \) from \( \varphi \) to \( \bar{\varphi} \) such that the pair \((\rho, t)\) is \( A \)-suitable, where \( t \) is constant and equal to 1. For any such path, the image of the corresponding morphism \( \text{im}(\mu^b_\varphi) \to \text{im}(\mu^b_{\bar{\varphi}}) \) is

\[ B^*_{N,\beta} \subset B^*_{N,\beta}. \]

**Proof.** It is an elementary fact that the cardinality of \( \mathcal{R}_\beta \) is constant and maximal on a Zariski-open subset of \( C^A \). In particular, the fundamental group of \( \pi_1(C \setminus \mathcal{R}_\beta, 0) \) is locally constant on an open dense subset of \( \mathcal{V} \). This proves the existence of the \( A \)-suitable pair \((\rho, t)\).

By Theorem 4.4, we have that \( \text{im}(\mu^b_\varphi) = B^*_{N,\beta} \subset B^*_{N,\beta} \). Therefore, it suffices to show that the isomorphism \((*)\) between \( B^*_{N,\beta} \) and \( B^*_{N,\beta} \) maps \( B^*_{N,\beta} \) to \( B^*_{N,\beta} \) and \( R^\varphi_{N,\beta} \) to
Let \( \{ \tilde{A}_i \}_{i \in I} \) be the set of all supports \( \tilde{A} \) as in the above lemma. Denote \( \tilde{d}_i := d(\tilde{A}_i) \) and \( \tilde{\varphi}_i := \varphi(\tilde{A}_i) \). Then, we have \( \text{GCD}(\{ \tilde{d}_i \}_{i \in I}) = d(A) = 1 \). By the above lemma, we have that \( B^*_{N, \tilde{d}_i} \cap R_N^0 \subset \text{im}(\mu_{\varphi}^{br}) \) for all \( i \in I \). It follows from the Euclidean algorithm \([EL21, \text{Proposition 4.1}]\) and Lemma \( 4.5 \) that the subgroups \( B^*_{N, \tilde{d}_i} \cap R_N^0 \) generate \( R_N^0 = \text{Ker}(\text{ind}_N) \) all together and therefore that \( \text{Ker}(\text{ind}_N) \subset \text{im}(\mu_{\varphi}^{br}) \).

To conclude, it remains to construct an element \( \sigma \in \text{im}(\mu_{\varphi}^{br}) \) such that \( \text{ind}_N(\sigma) = \varphi(A) \). By the above lemma, there exists \( \sigma_i \in \text{im}(\mu_{\varphi}^{br}) \) such that \( \text{ind}_N(\sigma) = \tilde{\varphi}_i \). Thus, there is \( \sigma \in \text{im}(\mu_{\varphi}^{br}) \) such that \( \text{ind}_N(\sigma) = \text{GCD}(\{ \tilde{\varphi}_i \}_{i \in I}) = \varphi(A) \). The result follows. \( \square \)

4.5. **Proof of Theorem 2.1** In order to prove Theorem 2.1, it suffices to show that the image of \( B_A = R_N^0 \cap B^*_{N, d} \) under the map \( \pi_N : B^*_N \rightarrow \mathcal{G}_N \) is \( (\text{ind}_N)^{-1}(1) \). To see this, we claim that there is a commutative diagram

\[
\begin{array}{ccc}
B^*_N, d & \xrightarrow{\pi_N} & \mathcal{G}_N, d \\
\text{ind}_N & & \downarrow \text{ind}_N^0 \\
\pi_1(\mathbb{C}^*, x_0) & \xrightarrow{q} & \mathcal{G}_P, d
\end{array}
\]

such that \( \pi_N(\text{Ker}(\text{ind}_N^0)) = \text{Ker}(\text{ind}_N^0) \). If so, the commutativity of the diagram and the surjectivity of the horizontal arrows implies that \( \text{ind}_N^0(G_A) = q(R_N^0) = (U_d)^\varphi \). To prove that \( G_A = (\text{ind}_N^0)^{-1}(1) \), it remains to show that \( \text{Ker}(\text{ind}_N^0) \subset G_A \). Since \( \text{Ker}(\text{ind}_N) \subset B_A \) and \( G_A = \pi_N(B_A) \), the inclusion \( \text{Ker}(\text{ind}_N) \subset G_A \) follows from \( \pi_N(\text{Ker}(\text{ind}_N)) = \text{Ker}(\text{ind}_N^0) \).

Let us show that we have the above commutative diagram. Plainly, the image of \( B^*_N, d \) under \( \pi_N \) is the \( U_d \)-equivariant subgroup \( \mathcal{G}_N, d \subset \mathcal{G}_N \). As we mentioned before, the group \( \mathcal{G}_P, d \) is canonically isomorphic to \( U_d \). Choosing the generator \( t \mapsto x_0 \cdot e^{2\pi i t} \), \( 0 \leq t \leq 1 \), induces an isomorphism from \( \pi_1(\mathbb{C}^*, x_0) \) to \( U_d \). Any braid \( \beta \in B^*_N, d \) can be represented by a collection of path \( t \mapsto x(t) \), \( 0 \leq t \leq 1 \), indexed by \( x \in \mathcal{N} \). From this representative of \( \beta \), we construct the element \( \gamma : t \mapsto \prod_{x \in \mathcal{N}} x(t) \) of \( \pi_1(\mathbb{C}^*, x_0) \). With the latter identifications, the sought diagram is the following

\[
\begin{array}{ccc}
B^*_N, d & \xrightarrow{\beta \mapsto (x \mapsto x(1))_{x \in \mathcal{N}}} & \mathcal{G}_N, d \\
\beta \mapsto \frac{1}{2\pi} \int_{\gamma} \frac{dz}{z} & & \sigma \mapsto \prod_{x \in E} \frac{\sigma(x)}{x} \\
\theta \mapsto e^{2\pi i \theta / d} & & \sigma \mapsto U_d
\end{array}
\]

where the formula \( \sigma \mapsto \prod_{x \in E} \frac{\sigma(x)}{x} \) for \( \text{ind}_N^0 \) does not depend of the choice of a representative \( E \subset \mathcal{N} \). The commutativity of the diagram is now an easy exercise in complex analysis, showing that the composition of two arrows in the diagram equals the map \( \beta \mapsto \exp \left( \frac{1}{2\pi} \int_{\gamma} \frac{dz}{z} \right) \).
where $\gamma_E : t \mapsto \prod_{x \in E} x(t)$. To see that $\pi_N'(\text{Ker}(\text{ind}_N)) = \text{Ker}(\text{ind}_N^0)$, recall that $\mathcal{G}_{N,d}$ is isomorphic to $U_d \wr \mathcal{T}_{N/d} \mathcal{S}_{N/d}$ and that $\text{ind}_N^0$ reads as $(\xi_1, \cdots, \xi_{N/q}, \sigma) \mapsto \prod_j \xi_j$. The kernel of $\text{ind}_N^0$ is generated by elements of the form $(1, \cdots, 1, \sigma)$ and $(\xi_1, \cdots, \xi_{N/q}, \text{id})$ such that $\prod_j \xi_j = 0$. Plainly, such elements are contained in $\pi_N'(\text{Ker}(\text{ind}_N))$. \[\square\]

**Remark 4.13.** There is another point of view on the commutative diagram 20. Observe that the set $\mathcal{P}$ is the preimage of the point $x_0$ under the covering $x \mapsto x^d$. The latter covering induces an isomorphism from the fundamental group $\pi_1(\mathcal{C}^*, x_0)$ to the subgroup $B^*_{\mathcal{P},d} \subset B^*_{\mathcal{P}}$ of $U_d$-invariant braids, where $B^*_{\mathcal{P}} := \pi_1(C_d(\mathbb{C}^*, \mathcal{P}))$. We have again a natural projection $\pi_Q : B^*_{\mathcal{P}} \rightarrow \mathcal{G}$ mapping $B^*_{\mathcal{P},d}$ to $\mathcal{G}_{\mathcal{P},d}$. Via the isomorphism $\pi_1(\mathcal{C}^*, x_0) \simeq B^*_{\mathcal{P},d}$, the commutative diagram 20 translates to

\[
\begin{array}{ccc}
B^*_{\mathcal{P},d} & \xrightarrow{\pi_N} & \mathcal{G}_{\mathcal{P},d} \\
\downarrow \text{ind}_N & & \downarrow \text{ind}_N^0 \\
B^*_{\mathcal{P},d} & \xrightarrow{\pi_Q} & \mathcal{G}_{\mathcal{P},d}
\end{array}
\]

### 4.6. Discussions: kernel, presentation and isomonodromy.

In this section, we discuss how the explicit methods of Section 4.3.5 allow to determine the kernel of $\mu^{br}_\varphi$, obtain a presentation of the braid monodromy group $B_A$ and study isomonodromy loci.

As discussed in the course of the proof of Lemma 4.10, we can determine explicitly the image under $\mu^{br}_\varphi$ of the generators $\ell_j$, $0 \leq j \leq \delta$, of $\pi_1(\mathcal{C}, t_0)$ when $\varphi$ is a trinomial. Recall that $\mu^{br}_\varphi(\ell_0) = \tau^b$ and that $\mu^{br}_\varphi(\ell_j) = b_k$ for $j > 1$ and for some $1 \leq k \leq n$. The integer $k$ is determined as follows: $\ell_j$ encompasses exactly one point of $\mathcal{B}$ in $\mathbb{C}^*$, this point maps to a singular vertex under $\text{Arg}$ and this singular vertex belongs to the interior of one of the $D$-parallelograms covering $(S^1)^2$. The latter parallelogram is indexed by some integer $k$ so that $\mu^{br}_\varphi(\ell_j) = b_k$.

In order to determine the kernel of $\mu^{br}_\varphi$ as well as a presentation of the corresponding braid monodromy group, it suffices to
- know the relations between the elements $\tau^b, b_1, \cdots, b_n$ of $B^*_N$, see e.g. [Lam00],
- determine the image and cardinality of the fibers of the map $j \mapsto k$.

For the latter, observe that the singular vertices are equi-distributed on a collection of $\text{GCD}(n, \beta)$ geodesics parallel to the $D$-geodesics, namely the preimage under $S^1 \rightarrow S^1$, $(\theta, \nu) \mapsto (m\theta + \alpha\nu, n\theta + b\nu)$ of the geodesic $\{\theta = 0\}$, see Section 4.3.5. If we refer to these geodesics as $D'$-geodesics, each $D'$-geodesic decomposes into labelled $D'$-segments, each such segment being the intersection of the geodesic with a labelled $D$-parallelogram. Therefore, the image and cardinality of the fibers of $j \mapsto k$ are determined by the distribution on each $D'$-geodesics of the $d/\text{GCD}(n, \beta)$ equi-distributed singular vertices with respect to the $n/\text{GCD}(n, \beta)$ equi-distributed intersection points with the horizontal section $\{\nu = \text{arg}(t_0)\}$. This can be done for any specific trinomial $\varphi$ and depends solely on the arithmetic of the support $A := \{(0, 0), (m, a), (n, b)\} \subset \mathbb{Z}^2$. 
Next, let us briefly discuss how the above observations on trinomials extend to arbitrary supports \( A \subset \mathbb{Z}^2 \) with the help of tropical geometry. Consider a regular triangulation \( T \) of \( \text{conv}(A) \) whose set of vertices is \( A \), that is there exists a piecewise linear convex function \( f : \text{conv}(A) \to \mathbb{R} \) whose domains of linearity are exactly the triangles of \( T \), see e.g. [HPPS21, Section 2.1] for existence. We assume furthermore that no inner edge of \( T \) is vertical. This can be achieved by taking a smaller subset \( \tilde{A} \subset A \) with the same invariants \( N, d \) and \( \vartheta \) and the triangulation \( T \) supported on \( \tilde{A} \) rather than \( A \). Then, we fix
\[
(21) \quad \varphi(x, t) := \sum_{a := (a_1, a_2) \in A} s f(a) x^{a_1} t^{a_2}
\]
for \( s > 0 \) arbitrarily small. Such polynomial is known as a Viro polynomial, a central object in tropical geometry, see [BIMS15]. The Viro polynomial \( \varphi \) defines a tropical curve \( C \subset \mathbb{R}^2 \) where \( \mathbb{R}^2 \) is the plane with coordinate \( (x, t) := \log(x, t) := (\log|x|, \log|t|) \). The tropical curve \( C \) is a piecewise linear graph dual to the subdivision \( T \) of \( \text{conv}(A) \), see [BIMS15, Proposition 2.5]. Any triangle \( T \subset T \) defines a trinomial
\[
\varphi_T(x, t) := \sum_{a := (a_1, a_2) \in T} s f(a) x^{a_1} t^{a_2}
\]
and is dual to some vertex \( v_T \) of \( C \), see Figure 7(a) and (b). Viro’s Patchworking Theorem states that for any neighbourhood \( V_T \subset \mathbb{R}^2 \) of \( v_T \), the set \( \text{Log}^{-1}(V_T) \cap \{ \varphi(x, t) = 0 \} \) is a small deformation of \( \text{Log}^{-1}(V_T) \cap \{ \varphi_T(x, t) = 0 \} \), see [IMS09, Section 2.3.2]. Moreover, it tells how the pieces \( \text{Log}^{-1}(V_T) \cap \{ \varphi(x, t) = 0 \} \) connect to each other, for all triangles \( T \subset T \). Using the case of trinomials, this allows to compute the image of the generators of \( \pi_1(C \setminus \mathcal{B}, t_0) \) under \( \mu_{\varphi}^{br} \), determine the kernel of \( \mu_{\varphi}^{br} \) as well as a presentation for \( B_A \).

Eventually, let us discuss how the explicit description of the map \( \mu_{\varphi}^{br} \) allows to derive information on isomonodromy loci, that is subsets of the form
\[
\text{Mon}(B) := \{ \varphi \in \mathbb{C}^A : \text{im}(\mu_{\varphi}^{br}) = B \}
\]
for a prescribed subgroup \( B \subset B_N \). The first instance is the set \( \text{Mon}(B_A) \). In Remark 3.1.1, we mentioned that \( \text{Mon}(B_A) \) contains the Zariski-open subset of \( \mathbb{C}^A \) consisting of polynomials \( \varphi \) for which the cardinality of the bifurcation set \( \mathcal{B} \) is maximal. Let us illustrate that the latter containment is strict in general.

Consider for instance a sharp support set \( A \subset \mathbb{Z}^2 \) contained in a vertical strip \( [0, n] \times \mathbb{Z} \), containing \( (0, 0) \) and \( (n, b) \) as vertices and lying in the upper half-plane defined by the line \( \mathbb{R} \cdot (n, b) \). We assume furthermore that \( (n, b) \) is primitive. Eventually, assume that there is point \( (m, a) \in A \) such that \( m \) and \( n \) are coprime. Thus, the trinomial \( \tilde{A} := \{(0, 0), (m, a), (n, b)\} \) satisfies \( d(A) = d(\tilde{A}) = 1 \), \( N(A) = N(\tilde{A}) = n \) and \( \vartheta(A) = \vartheta(\tilde{A}) \). Denote
\[
D := \text{Vol}(A) - \text{Vol}(\tilde{A}) + 1
\]
where \( \text{Vol} \) is twice the Euclidean area of the convex hull of its argument. Also, denote the couple \( \{(0, 0), (n, b)\} \) by \( A_0 \) and its complement by \( A^* := A \setminus A_0 \). Then, we claim that
\[
(22) \quad \{ \tilde{\varphi} \in (\mathbb{C}^*)^{A_0} \times \mathbb{C}^{A^*} : \mathcal{B} \text{ has at least } D \text{ points of multiplicity } 1 \text{ in } \mathbb{C}^* \} \subset \text{Mon}(B_A).
\]
Whenever \( A \) is sharp, the maximal cardinality of \( \mathcal{B} \) for \( \varphi \in \mathbb{C}^A \) is \( \text{Vol}(A) + 1 \). This can easily be seen using the Bernstein-Kouchnirenko-Khovanskii Theorem, taking into account that \( 0 \in \mathbb{C} \).
Figure 7. (a) The triangulated support set $A$ with $\text{conv}(\tilde{A})$ in blue. (b) The corresponding tropical curve $C \subset \mathbb{R}^2$. Coloured halos illustrate the duality with the triangulation of $A$. (c) The bifurcation set $\mathcal{B}$ for $\varphi$ as in (21). We can choose a skeleton $\Gamma$ of $\mathbb{C} \setminus \mathcal{B}$ as the juxtaposition of two railway tracks and compute $\mu^{br}_\varphi$ explicitly on the generators of $\pi_1(\Gamma)$. Again, coloured halos illustrate duality. (d) The deformation of $\mathcal{B}$ along a path from $\varphi$ as in (21) to $\tilde{\varphi}$ as in the left-hand side of (22). In that particular case, we have $D = 5$.

is necessarily in $\mathcal{B}$ in the case of a sharp support. In particular, the left-hand side of (22) strictly contains the set of polynomials $\varphi$ satisfying $|\mathcal{B}| = \text{Vol}(A) + 1$.

Let us outline the proof of (22) in the particular case where $A$ consists of 4 points and admits a triangulation with exactly two triangles, one of them being $\text{conv}(\tilde{A})$, see Figure 7. For $\varphi$ as in (21), the bifurcation set $\mathcal{B}$ is as pictured in Figure 7(c). The generators of $\pi_1(\Gamma)$ encompassing blue points maps to elements of the form $b_k \in B^N$ under $\mu^{br}_\varphi$ and the circle around 0 maps to $\tau^b$. Consider now a path in $\mathbb{C}^A$ from $\varphi$ to $\tilde{\varphi}$ as in the left-hand side of (22). By cardinality, at least one of the blue points of $\mathcal{B}$ remains simple along the deformation of bifurcation sets induced by the latter path. We deduce that $\text{im}(\mu^{br}_\varphi)$ contains $\tau^b$ and $b_k$ for some $k$. As $n$ and $b$ are coprime, any element $b_j$ is obtained as the conjugation of $b_k$ by a suitable power of $\tau_n$. We deduce that $\text{im}(\mu^{br}_\varphi)$ contains $\{b_1, \ldots, b_n\} = \text{Ker}(\text{ind}_N)$. Since $\text{ind}_N(\tau^b) = b = \vartheta(A)$, we conclude that $\text{im}(\mu^{br}_\varphi) \supset R^b_D = B_A$. The latter inclusion, which is necessarily an equality, implies the inclusion (22).

Eventually, let us observe that the integer $D$ can be as small as 2 for some appropriate $A$. Simultaneously, the area $\text{Vol}(A)$ can be chosen as large as desired. It illustrates that for particular choices of $A$, the isomonodromy locus $\text{Mon}(B_A)$ contains polynomials $\tilde{\varphi}$ that are extremely degenerated with respect to the projection $(x, t) \mapsto t$.

5. Galois group of a reducible pair of polynomial equations

In this section, we address the problem described in Section 2.2 and prove Theorems 2.4 and 2.5.

5.1. Reducible pairs with one two-dimensional support. Let $A := (A_1, A_2)$, $A_j \subset \mathbb{Z}^2$, be a reducible pair as in [EL22b, Definition 1.8], that is either $A_1$ or $A_2$ is contained in a line.
We are interested in the determination of the Galois group of the system of equations
\[(23) \quad p(x, t) = q(x, t) = 0\]
where \(p \in \mathbb{C}^{A_1}\) and \(q \in \mathbb{C}^{A_2}\). For now, we will restrict to the case when exactly one of \(A_1\) or \(A_2\), let us say \(A_2\), is one-dimensional.

Recall the bifurcation set \(\mathcal{B} \subset \mathbb{C}^{A_1} \times \mathbb{C}^{A_2}\), that is the hypersurface consisting of pairs \((p, q)\) for which the number of solutions to \((23)\) in \((\mathbb{C}^*)^2\) is not maximal. Loops in the complement of \(\mathcal{B}\) based at a chosen pair \((p_0, q_0)\) induce permutations on the set of solutions \(\mathcal{N}\) to the system \((23)\) corresponding to \((p_0, q_0)\). The latter permutations form the Galois group \(G_A \subset \mathfrak{S}_\mathcal{N}\) of \(A\), see \([\text{CL}22]b\).

As explained in Section 2.2, we will deduce \(G_A\) as the image of \(B_A\) under the natural projection \(\pi_\mathcal{N} : B_N^* \to \mathfrak{S}_\mathcal{N}\). Recall that the braid monodromy group \(B_A\) is the image of the map \(\mu^B_A\) as defined in \([9]\).

For the sake of simplicity, we will choose an appropriate system of coordinates on \(\mathbb{Z}^2\) (and the corresponding coordinates on \((\mathbb{C}^*)^2\) to compute \(B_A\). Since we can afford to translate \(A_1\) and \(A_2\) independently in \(\mathbb{Z}^2\) and apply a common change of coordinate, there is no loss of generality in assuming that
\[(0, 0) \in A_1 \cap A_2, \quad A_1 \subset \mathbb{N} \times \mathbb{Z} \quad \text{and} \quad A_2 \subset \{0\} \times \mathbb{N},\]
\[(24) \quad \text{the projection } A_1 \text{ of } A_1 \text{ on } \mathbb{Z} \times \{0\} \text{ satisfies } \{0, n\} \subset A_1 \subset [0, n] \text{ for some } n \geq 1,\]
\[A_2 \text{ satisfies } \{0\} \times \{0, h\} \subset A_2 \subset \{0\} \times [0, h] \text{ for some } h \geq 1.\]

Consequently, the polynomial \(q(x, t) = q(t)\) is univariate and the system \((23)\) has \(N := n \cdot h\) many solutions for generic \((p, q)\).

5.1.1. Obstructions. In this section, we show that the braid monodromy group of \(B_A\) is constrained as claimed in Theorem 2.4

**Lemma 5.1.** Let \(A\) be as in \((24)\). Then, the braid monodromy group \(B_A\) is contained in the subgroup \(\text{ind}_A^{-1}(B_{\nu(A)})\) of \(B_N^*\). Moreover, we have \(\text{ind}_A(B_A) = B_{\nu(A)}\).

This lemma is straightforward once we unravel all the definitions. Recall that we write
\[p(x, y) = c_0(y) + c_1(y)x + \cdots + c_n(y)x^n\]
and that the map \(\text{ind}_A : B_N^* \to B_Q^*\) comes from the map
\[\{(x, y) : q(y) = p_y(x) = 0\} \mapsto \left\{\prod_{x \in \{p_y=0\}} x, y : q(y) = 0\right\}\]
that takes the product of all the points of a given configuration in \(\mathcal{W}_A\) that lie on the horizontal slice given by some root \(y\) of \(q\). When the configuration in question comes from a system \(\{p(x, y) = q(y) = 0\} \in \mathcal{C}_A\), the product corresponding to a root \(y\) of \(q\) is the solution to the degree 1 equation
\[c_n(y)x - (-1)^nc_0(y) = 0\]
according to Vieta’s formula. It follows that the map \(\text{ind}_A\) takes values in the braid group \(B_{\nu(A)}\) associated to systems of the form
\[c_n(y)x - (-1)^nc_0(y) = 0, \quad q(y) = 0.\]
Moreover, the linear map \(\mathbb{C}^A \to \mathbb{C}^{\nu(A)}\) that sends the system \(\{p(x, y) = q(y) = 0\}\) to the corresponding system supported on \(\nu(A)\) maps the \(A\)-bifurcation set inside the \(\nu(A)\)-bifurcation
set. It follows that any loop in the configuration space \( \mathcal{C}_\nu(A) \) has a lift to \( \mathcal{C}_A \) and therefore that \( \text{ind}_A(B_A) = B_{\nu(A)} \).

5.1.2. Reductions. As mentioned in Remark 3.1.2, one of the advantages of considering braid monodromy maps is that it allows to reduce to simpler cases. Recall that the pair \( A := (A_1, A_2) \) is called irreducible in the terminology of [Est19], if the lattice generated by \( A_1 \) and \( A_2 \) is the full monomial lattice \( \mathbb{Z}^2 \) of \( (\mathbb{C}^*)^2 \). We have the following.

**Lemma 5.2.** Theorem 2.4 holds only if it holds for irreducible pairs \( A \).

**Proof.** There is no loss of generality in restricting ourselves to pairs \( A \) as in (24). Let \( d \) and \( e \) be the largest integer so that any pair \((p, q) \in \mathcal{C}_A\) can be written \((\tilde{p}(x^d, y^e), \tilde{q}(y^e))\) for a pair of polynomials \((\tilde{p}, \tilde{q})\). Denote \( \tilde{A} := (A_1, A_2) \) the irreducible support of the pair \((\tilde{p}, \tilde{q})\), for generic \((p, q)\). Plainly, the map \((x, y) \mapsto (x^d, y^e)\) induces an isomorphism from the configuration space \( \mathcal{U}_{\tilde{A}} \) to the configuration space \( \mathcal{U}_A \) (see Section 2.2 for the definitions). Similarly, the map \((x, y) \mapsto ((-1)^\varepsilon x, y^e)\) induces an isomorphism from \( \mathcal{U}_{\nu(\tilde{A})} \) to \( \mathcal{U}_{\nu(A)} \) (where \( \varepsilon \in \{0, 1\} \) is so that \((-1)^n = (-1)^{n/d} \cdot (-1)^\varepsilon\)). Additionally, consider the map

\[
\{q(y) = 0, \ p_y(x) = 0\} \in \mathcal{U}_A \mapsto \{q(y) = 0, \ c_{0,y} - (-1)^n c_{n,y} x = 0\} \in \mathcal{U}_{\nu(A)}
\]

where \( p_y(x) = c_{0,y} + \cdots + c_{n,y} x^n \), and the similar map \( \mathcal{U}_{\tilde{A}} \to \mathcal{U}_{\nu(\tilde{A})} \). In particular, the above arrows lead to the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{U}_A & \sim & \mathcal{U}_{\tilde{A}} \\
\downarrow & & \downarrow \\
\mathcal{U}_{\nu(A)} & \sim & \mathcal{U}_{\nu(\tilde{A})}
\end{array}
\]

where commutativity follows from the specific choice of \( \varepsilon \). Plainly, the above diagram is compatible with the inclusions \( \mathcal{C}_B \hookrightarrow \mathcal{U}_B \) for \( B \in \{A, \tilde{A}, \nu(A), \nu(\tilde{A})\} \). Precisely, we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_A & \sim & \mathcal{C}_{\tilde{A}} \\
\downarrow & & \downarrow \\
\mathcal{U}_A & \sim & \mathcal{U}_{\tilde{A}} \\
\mathcal{C}_{\nu(A)} & \sim & \mathcal{C}_{\nu(\tilde{A})} \\
\downarrow & & \downarrow \\
\mathcal{U}_{\nu(A)} & \sim & \mathcal{U}_{\nu(\tilde{A})}
\end{array}
\]
inducing in turn the following commutative diagram

\[
\begin{array}{ccc}
B_A & \sim & B_A^* \\
\downarrow & & \downarrow \\
B_N^* & \sim & B_N^* \\
\downarrow & & \downarrow \\
B_{\nu(A)} & \sim & B_{\nu(A)}^* \\
\downarrow & & \downarrow \\
B_Q^* & \sim & B_Q^* \\
\end{array}
\]

at the level of fundamental groups. The vertical arrows are surjective, according to Lemma 5.1. It follows that \( B_A = \text{ind}_{A}^{-1}(B_{\nu(A)}) \) only if \( B_A^* = \text{ind}_{\bar{A}}^{-1}(B_{\nu(\bar{A})}) \). The result follows. \( \square \)

**Remark 5.3.** In the above proof, we saw that the braid monodromy groups \( B_A \) and \( B_{\bar{A}} \) are isomorphic to each other, unlike the Galois groups \( G_A \) and \( G_{\bar{A}} \).

5.1.3. **Proof of Theorems 2.4 and 2.5.**

**Lemma 5.4.** For a reduced pair \( A \) as in (24), the group \( B_A \) contains \( \ker(\text{ind}_{A}) \).

**Proof of Theorem 2.4.** There is no loss of generality in assuming that \( A \) satisfies (24) and moreover that \( A \) is irreducible, thanks to Lemma 5.2. Thus, we have the inclusion \( \ker(\text{ind}_{A}) \subset B_A \), according to Lemma 5.4. Moreover, we have that \( \text{ind}_{A}(B_A) = B_{\nu(A)} \), thanks to Lemma 5.1. We deduce that \( B_A = \text{ind}_{A}^{-1}(B_{\nu(A)}) \). \( \square \)

In order to prove the Lemma 5.4, we need to study the geometry of the bifurcation set \( B \). For \( \check{y} \in \mathbb{C}^* \), define

\[ \mathcal{B}^\check{y} := \{ p \in B : p(x, \check{y}) \text{ is singular, with } n \text{ roots in } \mathbb{C}^* \text{ counted with multiplicities} \}. \]

The set \( \mathcal{B}^\check{y} \) is the pullback of the \( A_1 \)-discriminant in \( \mathbb{C}^{\check{y}} \) under the linear map \( L_{\check{y}} : \mathbb{C}^{A_1} \to \mathbb{C}^{\check{y}} \) defined by \( p(x, y) \to p(x, \check{y}) \). Therefore, \( \mathcal{B}^\check{y} \) is an irreducible non-empty hypersurface of \( \mathbb{C}^{A_1} \) provided that \( n > 1 \).

**Lemma 5.5.** For \( A_1 \) as in (24) and distinct \( \check{y}, \check{y}' \in \mathbb{C}^* \), the intersection \( \mathcal{B}^\check{y} \cap \mathcal{B}^\check{y}' \) has codimension 1 in both \( \mathcal{B}^\check{y} \) and \( \mathcal{B}^\check{y}' \) unless \( \check{y}/\check{y}' \) is a \( k \)-th root of unity where \( k \) is the index of the lattice \( \langle A_1 \rangle \) in \( \mathbb{Z}^2 \).

**Proof.** Since \( \mathcal{B}^\check{y} \) and \( \mathcal{B}^\check{y}' \) are irreducible, either they coincide or \( \mathcal{B}^\check{y} \cap \mathcal{B}^\check{y}' \) has codimension 1 in both \( \mathcal{B}^\check{y} \) and \( \mathcal{B}^\check{y}' \). Assume that \( \mathcal{B}^\check{y} \) and \( \mathcal{B}^\check{y}' \) coincide. Take \( B \subset A_1 \) so that the projection \( \mathbb{Z}^2 \to \mathbb{Z} \times \{0\} \) induces a bijection between \( B \) and \( A_1 \). In particular, each of the evaluation maps \( ev_{\check{y}} : (x, y) \to (x, \check{y}) \) and \( ev_{\check{y}'} : (x, y) \to (x, \check{y}') \) realises an isomorphism from \( \mathbb{C}^B \) to \( \mathbb{C}^{\check{y}} \). Since \( \mathcal{B}^\check{y} \cap \mathbb{C}^B = \mathcal{B}^\check{y}' \cap \mathbb{C}^B \), it follows that the automorphism \( ev_{\check{y}} \circ ev_{\check{y}'} : \mathbb{C}^{A_1} \to \mathbb{C}^{A_1} \) preserves the \( A_1 \)-discriminant. This implies that the \( A_1 \)-discriminant is invariant under translation by \( (\check{y}/\check{y}')^{b(a)} \) \( a \in A_1 \), where we denote \( (a, b(a)) \in B \) the point above \( a \in A_1 \). The Horn-Kapranov
uniformisation of the $A_1$-discriminant implies the existence of $u, v \in \mathbb{C}^*$ such that $(\hat{y}/\hat{y})^{b(a)} = u v^a$ for any $a \in A_1$. If we denote $\omega := \hat{y}/\hat{y}$, then either $|\omega| = 1$ or

$$b(a) = \log \frac{|v|}{|\omega|} \cdot a + \log \frac{|u|}{|\omega|}$$

for any $a \in A_1$, in particular $B$ is contained in a line in $\mathbb{Z}^2$. Since $A_1$ is not contained in a line, we can take $B$ not contained in a line and deduce that $|\omega| = 1$. Moreover, for any pair $a_1, a_2 \in A_1$, we have

$$\begin{cases} \omega^{b(0)} = u \\ \omega^{b(a_1)} = u v^{a_1} \Rightarrow \omega^D = 1, \text{ where } D := D(a_1, a_2) = \det \left( \begin{array}{cc} a_1 - 0 & a_2 - 0 \\ b(a_1) - b(0) & b(a_2) - b(0) \end{array} \right) \end{cases}$$

Taking all possible $B \subset A_1$ projecting bijectively onto $A_1$, we deduce from the above equality that $\omega^k = 1$ where $k := \gcd(\{D(a_1, a_2)\}_{a_1, a_2 \in A_1})$. It remains to observe that $k$ is the index of $(A_1)$ in $\mathbb{Z}^2$. 

To prove Lemma 5.4, we also need a stronger version of [EL21, Theorem 3] which was already proven there. Recall that for a support $A_1 \subset \mathbb{N}$ with $\{0, n\} \subset A_1 \subset \{0, \ldots, n\}$, we can consider the braid monodromy map

$$\mu^*_A : \pi_1(\mathcal{C}_{A_1}) \to B_n^*$$

where $\mathcal{C}_{A_1}$ is the subspace of $\mathbb{C}^{\Delta_1}$ consisting of polynomials $p(x) = \Sigma_{a \in A_1} c_a x^a$ with $n$ distinct roots on $\mathbb{C}^*$. We have the following.

**Theorem 5.6.** For a reduced support $A_1 \subset \mathbb{N}$ and any constants $C_0, C_n \in \mathbb{C}^*$, the composition of $\pi_1(\mathcal{C}_{A_1} \cap \{c_0 = C_0, c_n = C_n\}) \to \pi_1(\mathcal{C}_{A_1})$ with $\mu^*_A$ maps surjectively on the subgroup $R_n^0 \subset B_n^*$. In particular, if we denote $K \subset \pi_1(\mathcal{C}_{A_1})$ the kernel of the map induced by the inclusion $\mathcal{C}_{A_1} \hookrightarrow \mathbb{C}^{\Delta_1} \setminus \{c_0c_n = 0\}$, then $\mu^*_A(K) = R_n^0$.

**Proof.** Let us first consider the case $C_0 = C_1 = 1$. Since $R_n^0$ is the subgroup generated by $b_1, \ldots, b_n$, by Lemma 4.5, the proof is identical to the proof of [EL21, Theorem 3], provided that we upgrade [EL21, Corollary 5] accordingly. This upgrade is obviously true, by [EL21, Lemma 3.5].

For the general case, observe that the image of $\pi_1(\mathcal{C}_{A_1} \cap \{c_0 = c_n = 1\})$ under $\mu^*_A$ is not affected by the coordinate change $p(x) \mapsto u \cdot p(vx)$. Choosing $u$ and $v$ appropriately, we can achieve any prescribed value $C_0$ (respectively $C_n$) for the constant (respectively dominant) coefficient of $u \cdot p(vx)$. 

**Proof of Lemma 5.4.** Consider the subset $\mathcal{H}$ of $\mathcal{C}_A$ consisting of pairs $(p, q)$ where $q = q_0$ and the coefficients $c_0(y)$ and $c_n(y)$ of

$$p(x, y) = c_0(y) + c_1(y)x + \cdots + c_n(y)x^n$$

are constant and equal to the corresponding coefficients of $p_0$. In particular, we have $(p_0, q_0) \in \mathcal{H}$ and the “restriction” of $\mu^*_A$ to $\pi_1(\mathcal{H}, (p_0, q_0))$ takes values in $\ker(\ind_A)$, by Vieta’s formula. Let us show that the image of $\pi_1(\mathcal{H}, (p_0, q_0))$ contains $\ker(\ind_A) = \prod_{\hat{y} \in \mathbb{Q}} \ker(\ind_{\hat{y}})$, where $\ind_{\hat{y}}$ is as in (3). Remember that $\prod_{\hat{y} \in \mathbb{Q}} \ker(\ind_{\hat{y}}) = \prod_{\hat{y} \in \mathbb{Q}} R_{\hat{y}}^0$, by definition.
To do so, it suffices, according to Zariski’s Theorem [Che75, Théorème], to compute the image of the “restriction” of \( \mu_A^{br} \) to \( \pi_1(L \setminus \mathcal{B}, \varphi_0) \) for a generic line \( L \) in the affine linear subspace \( \mathcal{K} \subset \mathbb{C}^4 \) that passes through \((p_0, q_0)\). We claim that for any pair \( \hat{y}, \check{y} \in \mathbb{Q} \), the subset \( \mathcal{B}^y \cap \mathcal{B}^\check{y} \) has codimension 1 in both \( \mathcal{B}^y \) and \( \mathcal{B}^\check{y} \). Thus the finite sets \( L \cap \mathcal{B}^y, \hat{y} \in \mathbb{Q}, \) are pairwise disjoint. It follows from Theorem 5.6 and Zariski’s Theorem that the “restriction” of \( \mu_A^{br} \) to \( \pi_1(L \setminus \mathcal{B}, \varphi_0) \) maps surjectively on \( \prod_{\hat{y} \in \mathbb{Q}} \mathcal{B}^\check{y} \), and thus on \( \text{Ker}(\text{ind}_A) \).

It remains to prove the claim. By Lemma 5.5, the claim is true unless \( \hat{y}/\check{y} \) is a \( k^{th} \) root of unity where \( k \) is the index of \( \langle A_1 \rangle \). As \( q_0 \in \mathbb{C}^{A_2} \) is generic, the ratio \( \hat{y}/\check{y} \) can only be an \( l^{th} \) root of unity where \( l \) is the index of \( \langle A_2 \rangle \). Since the pair \( A \) is reduced, the integers \( k \) and \( l \) are necessarily coprime. This proves the claim. The result follows.

\[ \square \]

**Proof of Theorem 2.5** First, we have the obvious analogue of Lemma 5.1, namely we have the inclusion \( G_A \subset \langle \text{ind}_A^e \rangle^{-1}(G_{\nu_\mathcal{B}(A)}) \) and the equality \( \text{ind}_A^e(G_A) = G_{\nu_{\mathcal{B}}(A)} \). This follows again from Vieta’s formula. It remains to show that \( \text{Ker}(\text{ind}_A^e) = \prod_{\hat{y} \in \mathbb{Q}} \text{Ker}(\text{ind}_{\mathcal{B}^\check{y}}) \) is a subset of \( G_A \). We saw in the above proof that \( \text{Ker}(\text{ind}_A^e) = \prod_{\hat{y} \in \mathbb{Q}} \text{Ker}(\text{ind}_{\mathcal{B}^\check{y}}) \) is a subset of \( B_A \). Thus, it suffices to show that the projection \( \pi_{\mathcal{P}}: B_A \to G_A \) maps \( \text{Ker}(\text{ind}_A^e) \) surjectively onto \( \text{Ker}(\text{ind}_{\mathcal{B}^\check{y}}) \). In the proof of Theorem 2.1 we showed that \( \text{Ker}(\text{ind}_{\mathcal{B}^\check{y}}) \) maps surjectively onto \( \text{Ker}(\text{ind}_{\mathcal{B}^\check{y}}) \). The result follows.

\[ \square \]

**Remark 5.7.** The way we deduce Theorem 2.5 from Theorem 2.4 can alternatively be explained with the help of a commutative diagram

\[
\begin{array}{ccc}
B_N^* & \xrightarrow{\pi_N} & \mathcal{G}_{N,A} \\
\downarrow \text{ind}_A & & \downarrow \text{ind}_A^e \\
B_Q^* & \xrightarrow{q} & \mathcal{G}_{Q,A}
\end{array}
\]

as mentioned at the end of Section 2.2. In echo to Remark 4.13, the braid group \( B_N^* \) is isomorphic to the braid group \( B_P^* \) of the system \( (12) \). Indeed, the set \( \mathcal{P} \) is the preimage of \( \mathcal{Q} \) under the covering \((x, y) \to (x^d, y)\). Pulling back braid along the latter covering induces the sought isomorphism \( B_Q^* \to B_P^* \). Via this isomorphism, the above commutative diagram is

\[
\begin{array}{ccc}
B_N^* & \xrightarrow{\pi_N} & \mathcal{G}_{N,A} \\
\downarrow \text{ind}_A & & \downarrow \text{ind}_A^e \\
B_P^* & \xrightarrow{\pi_P} & \mathcal{G}_{P,A}
\end{array}
\]

5.2. **Reducible pairs with two one-dimensional supports.** We now consider the same problem as in the previous section in the special case where both \( A_1 \) and \( A_2 \) are contained in a line. The two lines are assumed to be skew, otherwise the system

\[ p(x, t) = q(x, t) = 0 \]

has no solutions for generic \((p, q)\). We assume with no loss of generality that \( A_1 \) and \( A_2 \) contain \((0, 0)\) as an extremity. In such case, there exist primitive vectors \((a, b), (u, v) \in \mathbb{Z}^2\) such that

\[ p(x, t) = \bar{p}(x^a y^b) \quad \text{and} \quad q(x, y) = \bar{q}(x^u y^v) \]
where \( \tilde{p}(x) \) and \( \tilde{q}(y) \) are univariate polynomials. To the equation system \((25)\), we can associate the following systems

\[
\begin{align*}
(26) & \quad p(x, y) = 0, \quad x^a y^b = 1 \quad \text{with support } A_p, \\
(27) & \quad x^a y^b = 1, \quad q(x, y) = 0 \quad \text{with support } A_q,
\end{align*}
\]

Denote the set of solutions to \((25), (26)\) and \((27)\) respectively by \(N_1, N_1\) and \(N_2\). Plainly, we have a surjective map

\[
N_1 \times N_2 \to N
\]

where \(s_1 \cdot s_2\) is the multiplication in \((\mathbb{C}^*)^2\). The subgroup \(K := \{x^a y^b = x^a y^b = 1\} \subset (\mathbb{C}^*)^2\) acts on \(N, N_1\) and \(N_2\) by multiplication. Define the action \(\xi.(s_1, s_2) := (\xi \cdot s_1, \xi^{-1} \cdot s_2)\) on the product \(N_1 \times N_2\). Then, the product map \((28)\) induces a bijection between the \(K\)-orbits of \(N_1 \times N_2\) (relative to the latter action) and the set \(N\).

The action of \(K \to \mathcal{G}_{N_1} \times \mathcal{G}_{N_2}\) of \(K\) on \(N_1 \times N_2\) induces a morphism \(K \to G_{A_p} \times G_{A_q}\) and the map \((28)\) induces a morphism \(G_{A_p} \times G_{A_q} \to G_A\). These morphisms fit into the following short exact sequence

\[
1 \to K \to G_{A_p} \times G_{A_q} \to G_A \to 1.
\]

We will see that the latter sequence is split and exhibit a section of \(G_{A_p} \times G_{A_q} \to G_A\). First, we will describe the Galois groups \(G_{A_p}\) and \(G_{A_q}\).

To begin with, we will change coordinate. Choose \((m, n) \in \mathbb{Z}^2\) such that \((a, b)\) and \((m, n)\) form a basis of \(\mathbb{Z}^2\) and set \((s, t) := (x^a y^b, x^m y^n)\). The system \((26)\) now reads as

\[
\tilde{p}(s) = 0, \quad s^e t^f = 1
\]

for some \((e, f) \in \mathbb{Z}^2\). Observe that \(K = \{s = 1, s^e t^f = 1\} \simeq U_f\) in these new coordinates. The solutions to \((30)\) is acted upon by \(K\), but also by the group \(U_d\) where \(d\) is the largest integer so that \(\tilde{p}(s) = \tilde{p}(s^d)\) for some other polynomials \(\tilde{p}\). Plainly, the Galois group \(G_{A_p}\) is contained in the \((U_d \times K)\)-invariant subgroup of \(\mathcal{G}_{N_1}\). The latter subgroup is isomorphic to \((U_d \times K)\) wr \(\mathcal{G}_{N_1}\) where \(N_1\) is the set of solutions to

\[
\tilde{p}(s) = 0, \quad s^e t^f = 1
\]

We claim that the containment \(G_{A_p} \subset (U_d \times K)\) wr \(\mathcal{G}_{N_1}\) is an equality.

Indeed, from [EL21, Theorem 1], we know two things. First, the Galois group of \(\tilde{p}\) is the full symmetric group. Second, for any tuple in \(\pi_1(\mathbb{C}^*)\), we can find a loop in the space of non-singular \(\tilde{p}\)-s such that each root of \(N_1\) closes a loop whose class in \(\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}\) is prescribed by the latter tuple. When a root \(\hat{r}\) in \(N_1\) realises the class \(kd + l \in \pi_1(\mathbb{C}^*)\), with \(k\) and \(l\) integers so that \(0 \leq l < d\), the fiber over \(\hat{r}\) in \(N_1\) is translated by \(([l], [k]) \in U_d \times K\). We deduce that the \((U_d \times K)\)-orbits of \(N_1\) can be permuted arbitrarily and that we can realise any tuple of translations on these orbits. The claim follows.

Let us now exhibit a subgroup of \(G_{A_p} \times G_{A_q}\) that is isomorphic to \(G_A\). To that aim, consider the map \((U_d \times K)\) wr \(\mathcal{G}_{N_1} \to K\) that associates to any tuple \(((\xi_\ell, \zeta_\ell)_{\ell \in N_1}, \sigma)\) the product \(\prod_{\ell \in N_1} \xi_\ell \cdot \zeta_\ell\). The kernel of this map defines a subgroup \(\mathbb{P}G_{A_p} \subset G_{A_p}\) that is independent of the identification \(G_{A_p} \simeq (U_d \times K)\) wr \(\mathcal{G}_{N_1}\), see Section \ref{sec:2}. We claim that \(\mathbb{P}G_{A_p} \times G_{A_q}\) is
isomorphic to $G_A$. Indeed, the former group intersect trivially the image of $K$ in $G_{A_p} \times G_{A_q}$ and has index $|K|$ in $G_{A_p} \times G_{A_q}$. Thus, the restriction of $G_{A_p} \times G_{A_q} \to G_A$ to $\mathbb{P}G_{A_p} \times G_{A_q}$ is an isomorphism onto $G_A$.

Observe that we can also define the subgroup $\mathbb{P}G_{A_q} \subset G_{A_q}$ in the same way we define $\mathbb{P}G_{A_p}$. In this section, we proved the following.

**Theorem 5.8.** Let $A := (A_1, A_2)$, $A_i \subset \mathbb{Z}^2$, be any pair such that $A_i$ is contained in a line $L_i$ with $L_1$ and $L_2$ skew. Then, the Galois group $G_A$ is isomorphic to either of the subgroups $\mathbb{P}G_{A_p} \times G_{A_q}$ and $G_{A_p} \times \mathbb{P}G_{A_q}$ of $G_{A_p} \times G_{A_q}$. The isomorphism is provided by the arrow $G_{A_p} \times G_{A_q} \to G_A$ of the short exact sequence (29).

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**Index of Notations**

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|--------|---------|
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| $B_{N_{d}}^{*}$, $\mathcal{B}$ | $6$, $8$ |
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| $\mathcal{C}_{A}$, $\mathcal{C}_{N(X)}$ | $1$, $2$, $6$ |
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