A CONNECTION BEHIND THE TERWILLIGER ALGEBRAS OF $H(D, 2)$ AND $\frac{1}{2}H(D, 2)$

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Abstract. The universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is a unital associative algebra over $\mathbb{C}$ generated by $E, F, H$ subject to the relations

$$\{H, E\} = 2E, \quad \{H, F\} = -2F, \quad \{E, F\} = H.$$  

The distinguished central element

$$\Lambda = EF + FE + \frac{H^2}{2}$$

is called the Casimir element of $U(\mathfrak{sl}_2)$. The universal Hahn algebra $\mathcal{H}$ is a unital associative algebra over $\mathbb{C}$ with generators $A, B, C$ and the relations assert that $[A, B] = C$ and each of

$$\alpha = [C, A] + 2A^2 + B, \quad \beta = [B, C] + 4BA + 2C$$

is central in $\mathcal{H}$. The distinguished central element

$$\Omega = 4ABA + B^2 - C^2 - 2\beta A + 2(1 - \alpha)B$$

is called the Casimir element of $\mathcal{H}$. By investigating the relationship between the Terwilliger algebras of the hypercube and its halved graph, we discover the algebra homomorphism $\natural: \mathcal{H} \rightarrow U(\mathfrak{sl}_2)$ that sends

$$A \mapsto \frac{H}{4},$$

$$B \mapsto \frac{E^2 + F^2 + \Lambda - 1}{4} - \frac{H^2}{8},$$

$$C \mapsto \frac{E^2 - F^2}{4}.$$  

We determine the image of $\natural$ and show that the kernel of $\natural$ is the two-sided ideal of $\mathcal{H}$ generated by $\beta$ and $16\Omega - 24\alpha + 3$. By pulling back via $\natural$ each $U(\mathfrak{sl}_2)$-module can be regarded as an $\mathcal{H}$-module. For each integer $n \geq 0$ there exists a unique $(n + 1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module $L_n$ up to isomorphism. We show that the $\mathcal{H}$-module $L_n$ ($n \geq 1$) is a direct sum of two non-isomorphic irreducible $\mathcal{H}$-modules.

Keywords: Askey–Wilson relations, halved cubes, hypercubes, Lie algebras, Terwilliger algebras

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1. Introduction

Throughout this paper we adopt the following conventions: Let $\mathbb{Z}$ denote the ring of integers. Let $\mathbb{N}$ denote the set of all nonnegative integers. Let $\mathbb{C}$ denote the complex number field. An algebra is meant to be a unital associative algebra. A subalgebra has the same unit as the parent algebra. An algebra homomorphism is meant to be a unital algebra homomorphism. In an algebra the notation $[x, y]$ stands for the commutator $xy - yx$. Given any nonempty set $X$ let $\mathbb{C}^X$ denote the free vector space over $\mathbb{C}$ generated by $X$; in other words $\mathbb{C}^X$ can be regarded as a vector space over $\mathbb{C}$ that has the basis $X$.  

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Definition 1.1. The universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ generated by $E, F, H$ subject to the relations

(1) $[H, E] = 2E,$
(2) $[H, F] = -2F,$
(3) $[E, F] = H.$

The element

(4) $\Lambda = EF + FE + \frac{H^2}{2}$

is central in $U(\mathfrak{sl}_2)$ and it is called the Casimir element of $U(\mathfrak{sl}_2)$.

Definition 1.2. The universal Hahn algebra $\mathcal{H}$ is an algebra over $\mathbb{C}$ generated by $A, B, C$ and the relations assert that

(5) $[A, B] = C$

and each of

(6) $[C, A] + 2A^2 + B,$
(7) $[B, C] + 4BA + 2C$

is central in $\mathcal{H}$.

Let $\alpha$ and $\beta$ denote the elements (6) and (7) of $\mathcal{H}$ respectively. By [6] Equation 2.2] the element

(8) $\Omega = 4ABA + B^2 - C^2 - 2\beta A + 2(1 - \alpha)B$

is central in $\mathcal{H}$ and it is called the Casimir element of $\mathcal{H}$.

The main result of [14] implies that the underlying space of the Leonard pair of Hahn type supports a finite-dimensional irreducible $\mathcal{H}$-module. In [8] Section 3] a classification of finite-dimensional irreducible $\mathcal{H}$-modules is provided. In [3,8] the algebra $\mathcal{H}$ is connected to the Terwilliger algebra of the Johnson graph. Additionally, there are some applications of $\mathcal{H}$ to physics. Please see [4, 5] for instance.

By investigating the relationship between the Terwilliger algebras of the hypercube and its halved graph, we find the following connection between $U(\mathfrak{sl}_2)$ and $\mathcal{H}$:

Theorem 1.3. There exists a unique algebra homomorphism $\natural : \mathcal{H} \rightarrow U(\mathfrak{sl}_2)$ that sends

(9) $A \mapsto \frac{H}{4},$
(10) $B \mapsto \frac{E^2 + F^2 + \Lambda - 1}{4} - \frac{H^2}{8},$
(11) $C \mapsto \frac{E^2 - F^2}{4},$
(12) $\alpha \mapsto \frac{\Lambda - 1}{4},$
(13) $\beta \mapsto 0.$
To be a $\mathbb{Z}$-graded algebra we can grade a free algebra by assigning arbitrary degrees to the generators. We set the degrees of $E, F, H$ to be $1, -1, 0$ respectively. Since the elements corresponding to the relations (1)–(3) are homogeneous of degrees $1, -1, 0$, the two-sided ideal generated by these elements is homogeneous. Thus the factor algebra $U(\mathfrak{sl}_2)$ inherits the $\mathbb{Z}$-grading. For each $n \in \mathbb{Z}$ let $U_n$ denote the $n^{\text{th}}$ homogeneous subspace of $U(\mathfrak{sl}_2)$. The subspaces $\{U_n\}_{n \in \mathbb{Z}}$ of $U(\mathfrak{sl}_2)$ satisfy the following properties:

(G1): $U(\mathfrak{sl}_2) = \bigoplus_{n \in \mathbb{Z}} U_n$.

(G2): $U_m \cdot U_n \subseteq U_{m+n}$ for all $m, n \in \mathbb{Z}$.

Definition 1.4. Define $U(\mathfrak{sl}_2)_e = \bigoplus_{n \in \mathbb{Z}} U_{2n}$.

Since $1 \in U_0$ and by (G2) the space $U(\mathfrak{sl}_2)_e$ is a subalgebra of $U(\mathfrak{sl}_2)$.

We determine the image of $\natural$ and characterize the kernel of $\natural$ as follows:

Theorem 1.5. (i) $\text{Im} \natural = U(\mathfrak{sl}_2)_e$.

(ii) $\text{Ker} \natural$ is the two-sided ideal of $\mathcal{H}$ generated by $\beta$ and $16\Omega - 24\alpha + 3$.

By pulling back via $\natural$ each $U(\mathfrak{sl}_2)$-module can be regarded as an $\mathcal{H}$-module. We now recall the finite-dimensional irreducible $U(\mathfrak{sl}_2)$-modules.

Lemma 1.6. For any $n \in \mathbb{N}$ there exists an $(n+1)$-dimensional $U(\mathfrak{sl}_2)$-module $L_n$ satisfying the following conditions:

(i) There exists a basis $\{v_i\}_{i=0}^n$ for $L_n$ such that

\[
E v_i = (n-i+1)v_{i-1} \quad (1 \leq i \leq n), \quad E v_0 = 0, \\
F v_i = (i+1)v_{i+1} \quad (0 \leq i \leq n-1), \quad F v_n = 0, \\
H v_i = (n-2i)v_i \quad (0 \leq i \leq n).
\]

(ii) The element $\Lambda$ acts on $L_n$ as scalar multiplication by $\frac{n(n+2)}{2}$.

Observe that the $U(\mathfrak{sl}_2)$-module $L_n$ $(n \in \mathbb{N})$ is irreducible. For each $n \in \mathbb{N}$ it is well-known that every $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module is isomorphic to $L_n$.

Theorem 1.7. (i) The $\mathcal{H}$-module $L_0$ is irreducible.

(ii) For any integer $n \geq 1$ the $\mathcal{H}$-module $L_n$ is a direct sum of two non-isomorphic irreducible $\mathcal{H}$-modules.

The paper is organized as follows: In §2 we prove Theorem 1.3. In §3 we give a presentation for $U(\mathfrak{sl}_2)_e$. In §4 we give a proof for Theorem 1.5. In §5 we prove Theorem 1.7 and classify the finite-dimensional irreducible $U(\mathfrak{sl}_2)_e$-modules. In §6 we explain the connection of our results with the Terwilliger algebras of the hypercube and its halved graph.

2. Proof for Theorem 1.3

Lemma 2.1. There exists a unique algebra homomorphism $\rho : U(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)$ that sends

$$(E, F, H) \mapsto (F, E, -H).$$

Moreover $\rho$ satisfies the following properties:

(i) $\rho^2 = 1$. 

(ii) \( \rho(\Lambda) = \Lambda \).

(iii) \( \rho(U_n) = U_{-n} \) for each \( n \in \mathbb{Z} \).

Proof. It is routine to verify the existence of \( \rho \) by using Definition 1.1. Since the algebra \( U(\mathfrak{sl}_2) \) is generated by \( E, F, H \) the uniqueness of \( \rho \) follows.

(i): The algebra homomorphism \( \rho^2 \) fixes \( E, F, H \).

(ii): Apply \( \rho \) to either side of (4).

(iii): Observe that \( \rho(U_n) \subseteq U_{-n} \) for all \( n \in \mathbb{Z} \).

(14) Applying \( \rho \) to either side of (14) it follows from Lemma 2.1(i) that

(15) \( U_n \subseteq \rho(U_{-n}) \) for all \( n \in \mathbb{Z} \).

Lemma 2.1(iii) follows from (14) and (15).

Observe that

(16) \( [xy, z] = x[y, z] + [x, z]y \)

for any elements \( x, y, z \) in an algebra.

Lemma 2.2. For each \( n \in \mathbb{N} \) the following equations hold in \( U(\mathfrak{sl}_2) \):

(i) \( [H, E^n] = 2nE^n \).

(ii) \( [H, F^n] = -2nF^n \).

Proof. (i): We proceed by induction on \( n \). There is nothing to prove when \( n = 0 \). Suppose that \( n \geq 1 \). Applying (16) with \( (x, y, z) = (E, E^{n-1}, H) \) yields that

\[ [E^n, H] = E[E^{n-1}, H] + [E, H]E^{n-1}. \]

Lemma 2.2(i) follows by applying the induction hypothesis and (11) to the above equation.

(ii): Lemma 2.2(ii) follows by applying \( \rho \) to Lemma 2.2(i) and evaluating the resulting equation by Lemma 2.1.

Lemma 2.3. For each \( n \in \mathbb{N} \) the following equations hold in \( U(\mathfrak{sl}_2) \):

(i) \( [H^2, E^n] = 4n(H - n)E^n \).

(ii) \( [H^2, F^n] = -4n(H + n)F^n \).

Proof. (i): Lemma 2.3(i) follows by applying (16) with \( (x, y, z) = (H, H, E^n) \) and simplifying the resulting equation by using Lemma 2.2(i).

(ii): Lemma 2.3(ii) follows by applying \( \rho \) to Lemma 2.3(i) and evaluating the resulting equation by Lemma 2.1.

Lemma 2.4. For each integer \( n \geq 1 \) the following equations hold in \( U(\mathfrak{sl}_2) \):

(i) \( E^n F^n = \prod_{i=1}^{n} \frac{2\Lambda - (H - 2i + 2)(H - 2i)}{4} \).

(ii) \( F^n E^n = \prod_{i=1}^{n} \frac{2\Lambda - (H + 2i - 2)(H + 2i)}{4} \).
Proof. (i): We proceed by induction on \( n \). By (3) we have \( FE = EF - H \). Substituting this into (4) yields that Lemma 2.4(i) holds for \( n = 1 \). Suppose that \( n \geq 2 \). Observe that

\[
E^n F = E^{n-1}(EF) = E^{n-1}\left(\frac{\Lambda + H}{2} - \frac{H^2}{4}\right).
\]

Applying Lemmas 2.2(i) and 2.3(i) to (17) yields that

\[
E^n F = 2\Lambda - (H - 2n + 2)(H - 2n)E^{n-1}.
\]

Lemma 2.4(i) follows by right multiplying the above equation by \( F^{n-1} \) and applying the induction hypothesis to the resulting equation.

(ii): Lemma 2.4(ii) follows by applying \( \rho \) to Lemma 2.4(i) and evaluating the resulting equation by Lemma 2.1. \( \square \)

Proof of Theorem 1.3. Let \( A^\sharp, B^\sharp, C^\sharp, \alpha^\sharp, \beta^\sharp \) denote the right-hand sides of (9)–(13) respectively. Observe that

\[
[A^\sharp, B^\sharp] = \frac{[H, E^2 + F^2]}{16},
\]

\[
[C^\sharp, A^\sharp] = \frac{[H, F^2 - E^2]}{16},
\]

\[
[B^\sharp, C^\sharp] = \frac{[F^2, E^2]}{8} + \frac{[H^2, F^2 - E^2]}{32}.
\]

Applying Lemma 2.2 to (18) and (19) yields that

\[
[A^\sharp, B^\sharp] = \frac{E^2 - F^2}{4} = C^\sharp,
\]

\[
[C^\sharp, A^\sharp] = -\frac{E^2 + F^2}{4} = -2A^\sharp^2 - B^\sharp + \alpha^\sharp.
\]

Applying Lemmas 2.2, 2.4 to (20) yields that

\[
[B^\sharp, C^\sharp] = \frac{(1 - E^2 - F^2 - \Lambda)H}{4} + \frac{H^3}{8} + \frac{F^2 - E^2}{2} = -4B^\sharp^2A^\sharp - 2C^\sharp + \beta^\sharp.
\]

By Definition 1.2 the existence of \( \sharp \) follows. Since the algebra \( H \) is generated by \( A, B, C \) the uniqueness of \( \sharp \) follows. \( \square \)

3. A Presentation for \( \mathbf{U}(\mathfrak{sl}_2)_e \)

Lemma 3.1. For each \( n \in \mathbb{Z} \) the elements

\[
E^i F^j H^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } i - j = n
\]

are a basis for \( U_n \).

Proof. Immediate from Poincaré–Birkhoff–Witt theorem. \( \square \)

Lemma 3.2. For all \( n \in \mathbb{N} \) the following hold:

(i) The elements

\[
E^n \Lambda^i H^k \quad \text{for all } i, k \in \mathbb{N}
\]

are a basis for \( U_n \).
(ii) The elements

\[ F^n \Lambda^i H^k \quad \text{for all } i, k \in \mathbb{N} \]

are a basis for \( U_{-n} \).

**Proof.** (i): Let \( n \in \mathbb{N} \) be given. For each \( i \in \mathbb{N} \) let \( U_n^{(i)} \) denote the subspace of \( U_n \) spanned by \( E^{n+j} F^j H^k \) for all \( j, k \in \mathbb{N} \) with \( j \leq i \). To see Lemma 3.2(i) it suffices to show that \( U_n^{(i)} \) (\( i \in \mathbb{N} \)) has the basis

\[ E^n \Lambda^i H^k \quad \text{for all } j, k \in \mathbb{N} \text{ with } j \leq i. \]  

To see this we proceed by induction on \( i \). There is nothing to prove when \( i = 0 \). Suppose that \( i \geq 1 \). By construction the quotient space \( U_n^{(i)} / U_n^{(i-1)} \) has the basis \( E^{n+i} F^i H^k + U_n^{(i-1)} \) for all \( k \in \mathbb{N} \). It follows from Lemma 3.2(i) and the induction hypothesis that

\[ E^n \Lambda^i H^k + U_n^{(i-1)} = 2^i E^{n+i} F^i H^k + U_n^{(i-1)} \quad \text{for all } k \in \mathbb{N} \]

are a basis for \( U_n^{(i)} / U_n^{(i-1)} \). Combined with the induction hypothesis the elements \( 21 \) form a basis for \( U_n^{(i)} \).

(ii): Let \( n \in \mathbb{N} \) be given. By Lemma 2.1 the algebra homomorphism \( \rho \) maps \( E^n \Lambda^i H^k \) to \( (-1)^k E^n \Lambda^i H^k \) for all \( i, k \in \mathbb{N} \). From Lemma 2.1(i), (iii) we see that the map \( \rho|_{U_n} : U_n \to U_{-n} \) is a linear isomorphism. Lemma 3.2(ii) follows by Lemma 3.2(i) and the above comments. \( \square \)

Recall the subalgebra \( U(\mathfrak{sl}_2)_e \) of \( U(\mathfrak{sl}_2) \) from Definition 1.4

**Lemma 3.3.** The elements

\[ E^{2n} \Lambda^i H^k \quad \text{for all integers } n \geq 1 \text{ and all } i, k \in \mathbb{N}; \]

\[ \Lambda^i H^k \quad \text{for all } i, k \in \mathbb{N}; \]

\[ F^{2n} \Lambda^i H^k \quad \text{for all integers } n \geq 1 \text{ and all } i, k \in \mathbb{N} \]

form a basis for \( U(\mathfrak{sl}_2)_e \).

**Proof.** Immediate from Lemma 3.2 \( \square \)

**Theorem 3.4.** The algebra \( U(\mathfrak{sl}_2)_e \) has a presentation given by generators \( E^2, F^2, \Lambda, H \) and relations

\[ [H, E^2] = 4E^2, \]

\[ [H, F^2] = -4F^2, \]

\[ 16E^2 F^2 = (H^2 - 2H - 2\Lambda)(H^2 - 6H - 2\Lambda + 8), \]

\[ 16F^2 E^2 = (H^2 + 2H - 2\Lambda)(H^2 + 6H - 2\Lambda + 8), \]

\[ \Lambda E^2 = E^2 \Lambda, \quad \Lambda F^2 = F^2 \Lambda, \quad \Lambda H = H \Lambda. \]

**Proof.** Let \( U(\mathfrak{sl}_2)'_e \) denote the algebra generated by the symbols \( E^2, F^2, \Lambda, H \) subject to the relations \( 25 \)–\( 29 \). By Lemmas 2.2, 2.4 and since \( \Lambda \) is central in \( U(\mathfrak{sl}_2) \), the algebra \( U(\mathfrak{sl}_2)_e \) satisfies the relations \( 25 \)–\( 29 \). Thus there exists a unique algebra homomorphism \( \Phi : U(\mathfrak{sl}_2)'_e \to U(\mathfrak{sl}_2)_e \) that sends

\[ (E^2, F^2, \Lambda, H) \quad \to \quad (E^2, F^2, \Lambda, H). \]

By Lemma 3.3 the elements \( 22 \)–\( 24 \) of \( U(\mathfrak{sl}_2)'_e \) are linearly independent. To prove that \( \Phi \) is an isomorphism, it remains to show the elements \( 22 \)–\( 24 \) of \( U(\mathfrak{sl}_2)'_e \) span \( U(\mathfrak{sl}_2)_e \).
Let $U(\mathfrak{sl}_2)'_{e,+}$ denote the subspace of $U(\mathfrak{sl}_2)'_e$ spanned by $\{22\}$. Let $U(\mathfrak{sl}_2)'_{e,0}$ denote the subspace of $U(\mathfrak{sl}_2)'_e$ spanned by $\{23\}$. Let $U(\mathfrak{sl}_2)'_{e,-}$ denote the subspace of $U(\mathfrak{sl}_2)'_e$ spanned by $\{24\}$. Observe that

\begin{align}
E^2U(\mathfrak{sl}_2)'_{e,+} &\subseteq U(\mathfrak{sl}_2)'_{e,+}, \\
F^2U(\mathfrak{sl}_2)'_{e,-} &\subseteq U(\mathfrak{sl}_2)'_{e,-}.
\end{align}

It follows from (29) that

\begin{align}
E^2U(\mathfrak{sl}_2)'_{e,0} &\subseteq U(\mathfrak{sl}_2)'_{e,0}, \\
F^2U(\mathfrak{sl}_2)'_{e,0} &\subseteq U(\mathfrak{sl}_2)'_{e,0}.
\end{align}

We claim that

\begin{align}
H \Lambda U(\mathfrak{sl}_2)'_{e,+} &\subseteq U(\mathfrak{sl}_2)'_{e,+}, \\
H \Lambda U(\mathfrak{sl}_2)'_{e,-} &\subseteq U(\mathfrak{sl}_2)'_{e,-}, \\
E^2U(\mathfrak{sl}_2)'_{e,0} &\subseteq U(\mathfrak{sl}_2)'_{e,0} + U(\mathfrak{sl}_2)'_{e,-}, \\
F^2U(\mathfrak{sl}_2)'_{e,+} &\subseteq U(\mathfrak{sl}_2)'_{e,0} + U(\mathfrak{sl}_2)'_{e,+}.
\end{align}

To see (34) it suffices to show that

\begin{align}
HE^{2n}\Lambda^iH^k &\in U(\mathfrak{sl}_2)'_{e,+}
\end{align}

for all integers $n \geq 1$ and all $i, k \in \mathbb{N}$. We proceed by induction on $n$. Using (25) yields that the left-hand side of (38) is equal to

\begin{align}
E^2H E^{2n-2}\Lambda^iH^k + 4E^{2n}\Lambda^iH^k.
\end{align}

Clearly the second summand is in $U(\mathfrak{sl}_2)'_{e,+}$. If $n = 1$ then the first summand $E^2H \Lambda^iH^k \in U(\mathfrak{sl}_2)'_{e,+}$ by (29). Hence (38) holds for $n = 1$. If $n \geq 2$ then the first summand is in $U(\mathfrak{sl}_2)'_{e,+}$ by (30) and the induction hypothesis. Therefore (34) follows. By a similar argument (35) follows. We now show (36). Let $n \geq 1$ be an integer and $i, k \in \mathbb{N}$. Using (27) yields that $E^2F^{2n}\Lambda^iH^k$ is equal to

\begin{align}
\frac{(H^2 - 2H - 2\Lambda)(H^2 - 6H - 2\Lambda + 8)}{16} F^{2n-2}\Lambda^iH^k.
\end{align}

If $n = 1$ then (39) is in $U(\mathfrak{sl}_2)'_{e,0}$ by (29). If $n \geq 2$ then (39) is in $U(\mathfrak{sl}_2)'_{e,-}$ by (38) and (35). Therefore (36) holds. By a similar argument (37) follows. By (30)–(37) the space $U(\mathfrak{sl}_2)'_{e,+} + U(\mathfrak{sl}_2)'_{e,0} + U(\mathfrak{sl}_2)'_{e,-}$ is a left ideal of $U(\mathfrak{sl}_2)'_e$. Since $1 \in U(\mathfrak{sl}_2)'_{e,0}$ it follows that $U(\mathfrak{sl}_2)'_{e,+} + U(\mathfrak{sl}_2)'_{e,0} + U(\mathfrak{sl}_2)'_{e,-} = U(\mathfrak{sl}_2)'_e$. In other words $U(\mathfrak{sl}_2)'_e$ is spanned by $\{22\}$–(24). The result follows.

\section{Proof for Theorem 1.5}

Define

\begin{align}
\hat{E}^2 &= 4A^2 + 2B + 2C - 2\alpha, \\
\hat{F}^2 &= 4A^2 + 2B - 2C - 2\alpha, \\
\hat{\Lambda} &= 1 + 4\alpha, \\
\hat{H} &= 4A.
\end{align}
Lemma 4.1. The images of $\hat{E}^2, \hat{F}^2, \Lambda, \hat{H}$ under $\sharp$ are equal to $E^2, F^2, \Lambda, H$ respectively.

Proof. It is routine to verify the lemma by using Theorem 1.3. □

For convenience we use the notation $x\sharp$ to denote the image of $x$ under $\sharp$ for all $x \in H$.

Lemma 4.2. (i) $A^\sharp \in U_0$.
(ii) $B^\sharp \in U_2 \oplus U_0 \oplus U_{-2}$.
(iii) $C^\sharp \in U_2 \oplus U_{-2}$.
(iv) $\alpha^\sharp \in U_0$.

Proof. (i), (iii): Immediate from (9) and (11).
(ii), (iv): Immediate from (10), (12) and the fact that $\Lambda \in U_0$ by (4). □

The proof of Theorem 1.5(i) is as follows:

Proof of Theorem 1.5(i). By Definition 1.2 the algebra $H$ is generated by $A, B, C$. Combined with Lemma 4.2(i)–(iii) this implies that $\text{Im } \sharp \subseteq U(sl_2)e$. By Theorem 3.4 the algebra $U(sl_2)e$ is generated by $E^2, F^2, \Lambda, H$. Combined with Lemma 4.1 this implies that $U(sl_2)e \subseteq \text{Im } \sharp$. Therefore Theorem 1.5(i) follows. □

Lemma 4.3. There exists a unique algebra homomorphism $\tilde{\rho} : H \rightarrow H$ that sends $(A, B, C, \alpha, \beta) \mapsto (-A, B, -C, \alpha, -\beta)$.

Moreover $\tilde{\rho}$ satisfies the following properties:

(i) $\tilde{\rho}^2 = 1$.
(ii) $\tilde{\rho}(\Omega) = \Omega$.
(iii) $\sharp \circ \tilde{\rho} = \rho \circ \sharp$.

Proof. It is routine to verify the existence of $\tilde{\rho}$ by using Definition 1.2. Since the algebra $H$ is generated by $A, B, C$ the uniqueness of $\tilde{\rho}$ follows.

(i): The homomorphism $\tilde{\rho}^2$ fixes $A, B, C$.
(ii): Apply $\tilde{\rho}$ to either side of (8).
(iii): By Theorem 1.3 and Lemma 2.1 the algebra homomorphisms $\sharp \circ \tilde{\rho}$ and $\rho \circ \sharp$ agree at $A, B, C$. □

Lemma 4.4. The following equations hold in $H$:

(i) $\Omega - B^2 + C^2 + \beta A = 2BA^2 + 2CA + (1 - \alpha)B$.
(ii) $\Omega - B^2 - C^2 = 2^2B + BA^2 - 2A^2 - \alpha B + \alpha$.

Proof. (i): By (5) we have $ABA = (C + BA)A$. Lemma 4.4(i) follows by substituting this into (8).

(ii): The subtraction of the right-hand side of Lemma 4.4(i) from the right-hand side of Lemma 4.4(i) is equal to

\[
[B, A^2] + 2A^2 + 2CA + B - \alpha.
\]
Applying (16) with $(x, y, z) = (A, A, B)$ yields that $[A^2, B] = AC + CA$. Hence (44) is equal to $[C, A] + 2A^2 + B - \alpha = 0$ by the setting (16) of $\alpha$. Therefore Lemma 4.4(ii) follows. □

Lemma 4.5. (i) $\Omega^\sharp \in U_4 \oplus U_2 \oplus U_0 \oplus U_{-2} \oplus U_{-4}$.
(ii) $\Omega^\sharp = \frac{3}{16}(2\Lambda - 3)$. 


Proof. (i): Recall from (13) that \( \beta^4 = 0 \). Combined with Lemma 4.2 this yields Lemma 4.5(i).

(ii): Let \( x \in U(\mathfrak{sl}_2) \) be given. By (G1) there are unique \( x_n \in U_n \) for all \( n \in \mathbb{Z} \) such that
\[
x = \sum_{n \in \mathbb{Z}} x_n.
\]

The element \( x_n \) is called the \( n \)th homogeneous component of \( x \) for each \( n \in \mathbb{Z} \). Using Lemma 4.4(i) yields that

\[
\begin{align*}
\Omega^4_0 &= \beta^4 = 0, \\
\Omega^2_0 &= 2(1 - \alpha^0)B^0_2, \\
\Omega^0_0 &= 4(1 - \alpha^0)B^0_0.
\end{align*}
\]

In the table below we list the nonzero homogeneous components of \( A^4, B^4, C^4, \alpha^4 \):

| \( x \) | \( A \) | \( B \) | \( C \) | \( \alpha \) |
|-------|-------|-------|-------|-------|
| \( x^4_2 \) | \( 0 \) | \( E^2 \) | \( E^2 \) | \( 0 \) |
| \( x^2_0 \) | \( H \) | \( \Lambda - \frac{1}{4} - \frac{H^2}{8} \) | \( 0 \) | \( \frac{\Lambda - 1}{4} \) |
| \( x^{-2}_2 \) | \( 0 \) | \( -\frac{E^2}{4} \) | \( -\frac{E^2}{4} \) | \( 0 \) |

A direct calculation shows that \( \Omega^4_4 = 0 \) and
\[
\begin{align*}
\Omega^2_2 &= \frac{E^2}{2} + \frac{E^2}{4}H - \frac{[H^2, E^2]}{32}, \\
\Omega^0_0 &= \frac{E^2F^2 + F^2E^2}{8} - \frac{(H^2 - 2\Lambda + 2)(H^2 - 2\Lambda + 18)}{64}.
\end{align*}
\]

Applying Lemmas 2.2(i) and 2.3(i) yields that (15) is equal to 0. Applying Lemma 2.4 yields that (46) is equal to \( \frac{3}{16}(2\Lambda - 3) \). Applying \( \hat{\rho} \) to either side of Lemma 4.3(ii), it follows from Lemma 4.3(iii) that \( \rho(\Omega^4_4) = \Omega^4_4 \). Combined with Lemma 2.1(iii) this yields that
\[
\rho(\Omega^4_n) = \Omega^4_{-n} \quad \text{for all} \quad n \in \mathbb{Z}.
\]

Hence \( \Omega^4_{-4} = 0 \) and \( \Omega^4_{-2} = 0 \). Lemma 4.5(ii) follows from the above comments and Lemma 4.5(i).

Recall the elements \( \hat{E}^2, \hat{F}^2, \hat{\Lambda}, \hat{H} \) of \( \mathcal{H} \) from (40)–(43).

Lemma 4.6. The algebra homomorphism \( \hat{\rho} : \mathcal{H} \rightarrow \mathcal{H} \) sends
\[
(\hat{E}^2, \hat{F}^2, \hat{\Lambda}, \hat{H}) \mapsto (\hat{F}^2, \hat{E}^2, \hat{\Lambda}, -\hat{H}).
\]

Proof. It is routine to verify the lemma by using Lemma 4.3.

Lemma 4.7. The following equations hold in \( \mathcal{H} \):
\[
\begin{align*}
(i) \ [\hat{H}, \hat{E}^2] &= 4\hat{E}^2, \\
(ii) \ [\hat{H}, \hat{F}^2] &= -4\hat{F}^2.
\end{align*}
\]

Proof. □
Proof. (i): Observe that \( \hat{H}, \hat{E}^2 = 8[A, B + C] \). By \( (5) \) and \( (6) \) it is equal to \( 4\hat{E}^2 \).

(ii): Lemma 4.7(ii) follows by applying \( \tilde{\rho} \) to Lemma 4.7(i) and evaluating the resulting equation by Lemma 4.6.

Lemma 4.8. The following equations hold in \( \mathcal{H} \):

(i) \( [A, C] = 2A^2 + B - \alpha \).

(ii) \( [A^2, C] = 4A^3 + 2AB - 2\alpha A - C \).

(iii) \( [[A, C], C] = 8A^3 - 4\alpha A + \beta \).

Proof. (i): Immediate from the setting \( (3) \) of \( \alpha \).

(ii): Applying \( (16) \) with \( (x, y, z) = (A, A, C) \) yields that \( [A^2, C] = A[A, C] + [A, C]A \).

Substituting Lemma 4.8(i) into the right-hand side of the above equation yields that \( [A^2, C] = 4A^3 + AB + BA - 2\alpha A \).

By \( (5) \) we have \( BA = AB - C \). Substituting this into the above equation yields Lemma 4.8(ii).

(iii): By Lemma 4.8(i) the commutator \( [[A, C], C] = 2[A^2, C] + [B, C] \).

Lemma 4.8(iii) follows by evaluating the right-hand side of the above equation by using \( (5) \), the setting \( (7) \) of \( \beta \) and Lemma 4.8(ii).

Lemma 4.9. The following equations hold in \( \mathcal{H} \):

(i) \( 16\hat{E}^2\hat{F}^2 - (\hat{H}^2 - 2\hat{H} - 2\hat{\Lambda})(\hat{H}^2 - 6\hat{H} - 2\hat{\Lambda} + 8) = 4(16\Omega - 24\alpha + 3) + 64\beta(2A - 1) \).

(ii) \( 16\hat{F}^2\hat{E}^2 - (\hat{H}^2 + 2\hat{H} - 2\hat{\Lambda})(\hat{H}^2 + 6\hat{H} - 2\hat{\Lambda} + 8) = 4(16\Omega - 24\alpha + 3) + 64\beta(2A + 1) \).

Proof. (i): Using the setting \( (1) \) of \( \alpha \) yields that \( \hat{E}^2 \) and \( \hat{F}^2 \) are equal to \( 2([A, C] + C) \) and \( 2([A, C] - C) \) respectively. Hence \( \hat{E}^2\hat{F}^2 \) is equal to 4 times

\[
[A, C]^2 - [[A, C], C] - C^2.
\]

By Lemma 4.8(i), (iii) the element \( (47) \) is equal to

\[
4A^4 - 8A^3 + B^2 - C^2 + 2A^2B + 2BA^2 - 4\alpha A^2 + 4\alpha A - 2\alpha B + \alpha^2 - \beta.
\]

By \( (42) \) and \( (43) \) the product \( (\hat{H}^2 - 2\hat{H} - 2\hat{\Lambda})(\hat{H}^2 - 6\hat{H} - 2\hat{\Lambda} + 8) \) is equal to 4 times

\[
64A^4 - 128A^3 + 64A^2 - 64\alpha A^2 + 64\alpha A + 16\alpha^2 - 8\alpha - 3.
\]

Using the above results yields that the left-hand side of Lemma 4.9(i) is equal to 4 times

\[
16B^2 - 16C^2 + 32A^2B + 32BA^2 - 64A^2 - 32\alpha B + 8\alpha - 16\beta + 3.
\]

By Lemma 4.4(ii) the element \( (48) \) is equal to \( 16\Omega - 24\alpha + 3 + 16\beta(2A - 1) \). Therefore Lemma 4.9(i) follows.

(ii): Lemma 4.9(ii) follows by applying \( \tilde{\rho} \) to Lemma 4.9(i) and evaluating the resulting equation by Lemmas 4.3 and 4.6. \( \square \)
Proof of Theorem 1.5(ii). Let $\mathcal{K}$ denote the two-sided ideal of $\mathcal{H}$ generated by $\beta$ and $16\Omega - 24\alpha + 3$. By (13) the element $\beta \in \text{Ker } \natural$. Using (12) and Lemma 4.5(ii) yields that $16\Omega - 24\alpha + 3 \in \text{Ker } \natural$. Hence $\mathcal{K} \subseteq \text{Ker } \natural$. There exists a unique algebra homomorphism $\natural : \mathcal{H}/\mathcal{K} \to U(\mathfrak{sl}_2)_e$ given by

$$\natural(x + \mathcal{K}) = x^2 \quad \text{for all } x \in \mathcal{H}.$$ 

To see that $\text{Ker } \natural \subseteq \mathcal{K}$ it suffices to show that $\natural$ is injective. Since $\alpha$ is central in $\mathcal{H}$ the element $\hat{\Lambda}$ commutes with $\hat{E}^2, \hat{F}^2, \hat{H}$. Combined with Lemmas 4.7, 4.9 the cosets $\hat{E}^2 + \mathcal{K}, \hat{F}^2 + \mathcal{K}, \hat{\Lambda} + \mathcal{K}, \hat{H} + \mathcal{K}$ satisfy the relations (25)–(29). By Theorem 3.4 there exists a unique algebra homomorphism $\sharp : U(\mathfrak{sl}_2)_e \to \mathcal{H}/\mathcal{K}$ that sends $(\hat{E}^2, \hat{F}^2, \hat{\Lambda}, \hat{H}) \mapsto (\hat{E}^2 + \mathcal{K}, \hat{F}^2 + \mathcal{K}, \hat{\Lambda} + \mathcal{K}, \hat{H} + \mathcal{K})$.

Using (40)–(43) yields that $A = \frac{\hat{H}}{4}$, $B = \frac{\hat{E}^2 + \hat{F}^2}{4} - \frac{\hat{H}^2}{8} + \frac{\hat{\Lambda} - 1}{4}$, $C = \frac{\hat{E}^2 - \hat{F}^2}{4}$.

Since the algebra $\mathcal{H}$ is generated by $A, B, C$ it follows that the algebra $\mathcal{H}/\mathcal{K}$ is generated by $\hat{E}^2 + \mathcal{K}, \hat{F}^2 + \mathcal{K}, \hat{\Lambda} + \mathcal{K}, \hat{H} + \mathcal{K}$. By Lemma 4.1 the algebra homomorphism $\sharp \circ \natural$ fixes each of $\hat{E}^2 + \mathcal{K}, \hat{F}^2 + \mathcal{K}, \hat{\Lambda} + \mathcal{K}, \hat{H} + \mathcal{K}$. It follows that $\sharp \circ \natural = 1$ and thus $\natural$ is injective. Theorem 1.5(ii) follows.

5. Proof for Theorem 1.7 and finite-dimensional irreducible $U(\mathfrak{sl}_2)_e$-modules

For any $U(\mathfrak{sl}_2)$-module $V$ and any $\theta \in \mathbb{C}$ let $V(\theta) = \{v \in V \mid Hv = \theta v\}$.

Proposition 5.1. Let $V$ denote a $U(\mathfrak{sl}_2)$-module. Then

$$\bigoplus_{n \in \mathbb{Z}} V(\theta + 4n)$$

is a $U(\mathfrak{sl}_2)_e$-submodule of $V$ for any $\theta \in \mathbb{C}$.

Proof. Let $\theta \in \mathbb{C}$ be given. Clearly $HV(\theta) \subseteq V(\theta)$. Using (11) and (2) yields that $EV(\theta) \subseteq V(\theta + 2)$ and $FV(\theta) \subseteq V(\theta - 2)$ respectively. It follows that

$$U_n V(\theta) \subseteq V(\theta + 2n) \quad \text{for all } n \in \mathbb{Z}.$$ 

By Definition 1.4 the proposition follows.

Recall the $U(\mathfrak{sl}_2)$-module $L_n$ ($n \in \mathbb{N}$) from Lemma 1.6.

Definition 5.2. (i) For each $n \in \mathbb{N}$ let

$$L_n^{(0)} = \bigoplus_{i \in \mathbb{Z}} L_n(n - 4i).$$

(ii) For each integer $n \geq 1$ let

$$L_n^{(1)} = \bigoplus_{i \in \mathbb{Z}} L_n(n - 4i - 2).$$
Lemma 5.3. For any \( n \in \mathbb{N} \) the space \( L_n \) is equal to
\[
\begin{cases} 
L_n^{(0)} & \text{if } n = 0, \\
L_n^{(0)} \oplus L_n^{(1)} & \text{if } n \geq 1.
\end{cases}
\]

Proof. Immediate from Lemma 1.6(i). \( \square \)

It follows from Proposition 5.1 that \( L_n^{(0)} \) and \( L_n^{(1)} \) are two \( U(\mathfrak{sl}_2)_e \)-submodules of \( L_n \).

Lemma 5.4. For each \( n \in \mathbb{N} \) the \( U(\mathfrak{sl}_2)_e \)-module \( L_n^{(0)} \) satisfies the following properties:

(i) There exists a basis \( \{u_i^{(0)}\}_{i=0}^n \) for \( L_n^{(0)} \) such that
\[
E^2 u_i^{(0)} = (n - 2i + 1)(n - 2i + 2)u_{i-1}^{(0)} \quad (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor), \quad E^2 u_0^{(0)} = 0,
\]
\[
F^2 u_i^{(0)} = (2i + 1)(2i + 2)u_{i+1}^{(0)} \quad (0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1), \quad F^2 u_1^{(0)} = 0,
\]
\[
H u_i^{(0)} = (n - 4i)u_i^{(0)} \quad (0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor).
\]

(ii) The element \( \Lambda \) acts on \( L_n^{(0)} \) as scalar multiplication by \( n(n+2) \).

Proof. (i): Recall the basis \( \{v_i\}_{i=0}^n \) for \( L_n \) from Lemma 1.6(i). Observe that
\[
L_n(n-4i) = \begin{cases} 
\text{span}\{v_{2i}\} & \text{if } i = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \\
\{0\} & \text{else}
\end{cases}
\]
for all \( i \in \mathbb{Z} \). Let \( u_i^{(0)} = v_{2i} \) for \( i = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \). By Definition 5.2(i) the \( U(\mathfrak{sl}_2)_e \)-module \( L_n^{(0)} \)
has the basis \( \{u_i^{(0)}\}_{i=0}^n \). It is routine to verify the actions of \( E^2, F^2, H \) by using Lemma 1.6(i).

(ii): Immediate from Lemma 1.6(ii). \( \square \)

Lemma 5.5. For any \( n \in \mathbb{N} \) the \( U(\mathfrak{sl}_2)_e \)-module \( L_n^{(0)} \) is irreducible.

Proof. Let \( V \) denote a nonzero \( U(\mathfrak{sl}_2)_e \)-submodule of \( L_n^{(0)} \). Note that each eigenspace of \( H \) in \( L_n^{(0)} \) has dimension one. Since \( V \) is a nonzero \( H \)-invariant subspace of \( L_n^{(0)} \) there exists the smallest integer \( i \) with \( 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \) such that \( u_i^{(0)} \in V \).

Suppose that \( i \geq 1 \). Then \( E^2 u_i^{(0)} = (n - 2i + 1)(n - 2i + 2)u_{i-1}^{(0)} \in V \). Since \( 2i \not\in \{n+1, n+2\} \) it follows that \( u_i^{(0)} \in V \), a contradiction. Hence \( i = 0 \). Since \( F^2 u_0^{(0)} = (2i)u_0^{(0)} \in V \) for all \( i = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) it follows that \( V = L_n^{(0)} \). The lemma follows. \( \square \)

Lemma 5.6. The \( U(\mathfrak{sl}_2)_e \)-modules \( L_n^{(0)} \) for all \( n \in \mathbb{N} \) are mutually non-isomorphic.

Proof. Suppose that there are \( m, n \in \mathbb{N} \) such that the \( U(\mathfrak{sl}_2)_e \)-modules \( L_n^{(0)} \) and \( L_n^{(0)} \) are isomorphic. By Lemma 5.3(i) the dimension of \( L_m^{(0)} \) is \( \left\lfloor \frac{m}{2} \right\rfloor + 1 \) and the dimension of \( L_n^{(0)} \) is \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \). Hence \( m \in \{n-1, n, n+1\} \). It follows from Lemma 5.3(ii) that \( m(m+2) = n(n+2) \). Using the above results yields that \( m = n \). The lemma follows. \( \square \)

Lemma 5.7. For each integer \( n \geq 1 \) the \( U(\mathfrak{sl}_2)_e \)-module \( L_n^{(1)} \) satisfies the following properties:

(i) There exists a basis \( \{u_i^{(1)}\}_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \) for \( L_n^{(1)} \) such that
\[
E^2 u_i^{(1)} = (n - 2i)(n - 2i + 1)u_{i-1}^{(1)} \quad (1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor), \quad E^2 u_0^{(1)} = 0,
\]
\[
F^2 u_i^{(1)} = (2i + 1)(2i + 2)u_{i+1}^{(1)} \quad (0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1), \quad F^2 u_1^{(1)} = 0,
\]
\[
H u_i^{(1)} = (n - 4i)u_i^{(1)} \quad (0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor).
\]
\[ F^2 u_i^{(1)} = (2i + 2)(2i + 3) u_{i+1}^{(1)} \quad (0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1), \quad F^2 u_i^{(1)} = 0, \]
\[ H u_i^{(1)} = (n - 4i - 2) u_i^{(1)} \quad (0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor). \]

(ii) The element \( \Lambda \) acts on \( L_n^{(1)} \) as scalar multiplication by \( \frac{n(n+2)}{2} \).

**Proof.** (i): Recall the basis \( \{v_i\}_{i=0}^n \) for \( L_n \) from Lemma 1.6(i). Observe that
\[ L_n (n - 4i - 2) = \begin{cases} \text{span} \{v_{2i+1}\} & \text{if } i = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor, \\ \{0\} & \text{else} \end{cases} \]
for all \( i \in \mathbb{Z} \). Let \( u_i^{(1)} = v_{2i+1} \) for \( i = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \). By Definition 5.2(ii) the \( U(\mathfrak{sl}_2) \)-module \( L_n^{(1)} \) has the basis \( \{u_i^{(1)}\}_{i=0}^n \). It is routine to verify the actions of \( E^2, F^2, H \) by using Lemma 1.6(i).

(ii): Immediate from Lemma 1.6(ii).

\[ \square \]

**Lemma 5.8.** For any integer \( n \geq 1 \) the \( U(\mathfrak{sl}_2) \)-module \( L_n^{(1)} \) is irreducible.

**Proof.** Let \( V \) denote a nonzero \( U(\mathfrak{sl}_2) \)-submodule of \( L_n^{(1)} \). Note that each eigenspace of \( H \) on \( L_n^{(1)} \) has dimension one. Since \( V \) is a nonzero \( H \)-invariant subspace of \( L_n^{(1)} \) there exists the smallest integer \( i \) with \( 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) such that \( u_i^{(1)} \in V \).

Suppose that \( i \geq 1 \). Then \( E^2 u_i^{(1)} = (n - 2i)(n - 2i + 1) u_{i-1}^{(1)} \in V \). Since \( 2i \not\in \{n, n+1\} \) it follows that \( u_{i-1}^{(1)} \in V \), a contradiction. Hence \( i = 0 \). Since \( F^2 u_0^{(1)} = (2i + 1)! u_i^{(1)} \in V \) for all \( i = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \) it follows that \( V = L_n^{(1)} \). The lemma follows.

\[ \square \]

**Lemma 5.9.** The \( U(\mathfrak{sl}_2) \)-modules \( L_n^{(1)} \) for all integers \( n \geq 1 \) are mutually non-isomorphic.

**Proof.** Suppose that there are two integers \( m, n \geq 1 \) such that the \( U(\mathfrak{sl}_2) \)-modules \( L_m^{(1)} \) and \( L_n^{(1)} \) are isomorphic. By Lemma 5.7(i) the dimension of \( L_m^{(1)} \) is \( \left\lfloor \frac{m-1}{2} \right\rfloor + 1 \) and the dimension of \( L_n^{(1)} \) is \( \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \). Hence \( m \in \{n - 2, n - 1, n\} \). It follows from Lemma 5.7(ii) that \( m(m + 2) = n(n + 2) \). Using the above results yields that \( m = n \). The lemma follows.

\[ \square \]

**Theorem 5.10.** The \( U(\mathfrak{sl}_2) \)-modules \( L_n^{(0)} \) for all \( n \in \mathbb{N} \) and the \( U(\mathfrak{sl}_2) \)-modules \( L_n^{(1)} \) for all integers \( n \geq 1 \) are mutually non-isomorphic.

**Proof.** Pick any two integers \( m, n \) with \( m \in \mathbb{N} \) and \( n \geq 1 \). Suppose that the \( U(\mathfrak{sl}_2) \)-modules \( L_m^{(0)} \) and \( L_n^{(1)} \) are isomorphic. By Lemma 5.4(i) the dimension of \( L_m^{(0)} \) is \( \left\lfloor \frac{m}{2} \right\rfloor + 1 \). By Lemma 5.7(i) the dimension of \( L_n^{(1)} \) is \( \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \). Hence \( m \in \{n - 2, n - 1, n\} \). It follows from Lemmas 5.4(ii) and 5.7(ii) that \( m(m + 2) = n(n + 2) \). Using the above results yields that \( m = n \). By Lemma 5.4(i) the element \( H \) has the eigenvalue \( m \) in \( L_m^{(0)} \). However \( m \) is not an eigenvalue of \( H \) in \( L_n^{(1)} \) by Lemma 5.7(i), a contradiction. Hence the \( U(\mathfrak{sl}_2) \)-modules \( L_m^{(0)} \) and \( L_n^{(1)} \) are non-isomorphic. Combined with Lemmas 5.6 and 5.9 this implies Theorem 5.10.

\[ \square \]

The proof of Theorem 1.7 is given below.

**Proof of Theorem 1.7.** Recall from Theorem 1.6(i) that \( \text{Im} \mathcal{H} = U(\mathfrak{sl}_2) \). Hence the \( \mathcal{H} \)-module \( L_n^{(0)} \) \((n \in \mathbb{N})\) is irreducible by Lemma 5.5 and the \( \mathcal{H} \)-module \( L_n^{(1)} \) \((n \geq 1)\) is irreducible by Lemma 5.8. By Theorem 5.10 the \( \mathcal{H} \)-modules \( L_n^{(0)} \) for all \( n \in \mathbb{N} \) and the \( \mathcal{H} \)-modules \( L_n^{(1)} \) for
all integers \( n \geq 1 \) are mutually non-isomorphic. Theorem 1.7 follows by Lemma 5.3 and the above comments. \( \square \)

We finish this section with a classification of the finite-dimensional irreducible \( \mathfrak{sl}_2 \)-modules.

**Theorem 5.11.** For any \( d \in \mathbb{N} \) the \( \mathfrak{sl}_2 \)-modules \( L^{(0)}_{2d} \), \( L^{(0)}_{2d+1} \), \( L^{(1)}_{2d+1} \), \( L^{(1)}_{2d+2} \) are all \((d + 1)\)-dimensional irreducible \( \mathfrak{sl}_2 \)-modules up to isomorphism.

**Proof.** Let \( d \in \mathbb{N} \) be given. Suppose that \( V \) is a \((d + 1)\)-dimensional irreducible \( \mathfrak{sl}_2 \)-module. Since \( \mathbb{C} \) is algebraically closed and the \( \mathfrak{sl}_2 \)-module \( V \) is finite-dimensional, it follows from Schur’s lemma that the central element \( \Lambda \) acts on \( V \) as scalar multiplication by some scalar \( \lambda \in \mathbb{C} \). In addition, there exists a scalar \( \theta \in \mathbb{C} \) such that \( \theta \) is an eigenvalue of \( H \) but \( \theta + 4 \) is not an eigenvalue of \( H \) in \( V \).

Let \( w \) denote a \( \theta \)-eigenvector of \( H \) in \( V \). Set \( w_i = F^{2i}w \) for all \( i \in \mathbb{N} \).

(49) Applying \( w_0 \) to either side of Lemma 2.2(ii) with even \( n \) yields that \( Hw_i = (\theta - 4i)w_i \) for all \( i \in \mathbb{N} \).

(50) Applying \( w_0 \) to either side of (25) yields that \( HE^2w_0 = (\theta + 4)E^2w_0 \). By the choice of \( \theta \) it follows that \( E^2w_0 = 0 \).

(51) We apply \( w_{i-1} \) (\( i \geq 1 \)) to either side of (27) and use the equation (50) to evaluate the resulting equation. It follows that \( E^2w_i \in \text{span}\{w_{i-1}\} \) for all integers \( i \geq 1 \).

By (50) the nonzero vectors among \( \{w_i\}_{i \in \mathbb{N}} \) are linearly independent. Since \( V \) is finite-dimensional there are only finitely many indices \( i \) such that \( w_i \neq 0 \). Since \( w_0 \neq 0 \) there is an integer \( d' \geq 0 \) such that \( w_i \neq 0 \) for all \( i = 0, 1, \ldots, d' \) and \( w_{d'+1} = 0 \). Let \( W \) denote the subspace of \( V \) spanned by \( \{w_i\}_{i=0}^{d'} \). By (49) the space \( W \) is \( F^2 \)-invariant. By (51) and (52) the space \( W \) is \( E^2 \)-invariant. By (50) the space \( W \) is \( H \)-invariant. Hence \( W \) is a nonzero \( \mathfrak{sl}_2 \)-submodule of \( V \). By the irreducibility of \( V \) it follows that \( d' = d \). In other words the vectors \( \{w_i\}_{i=0}^{d} \) form a basis for \( V \). Note that

(53) \[ F^2w_d = 0. \]

We apply \( w_0 \) to either side of (28) and evaluate the resulting equation by using (50) and (51). It follows that

(54) \[ \lambda = \frac{\theta(\theta + 2)}{2} \]
or

(55) \[ \lambda = 4 + \frac{\theta(\theta + 6)}{2}. \]

We apply \( w_d \) to either side of (27) and evaluate the resulting equation by using (50) and (53). It follows that

(56) \[ \lambda = \frac{(\theta - 4d)(\theta - 4d - 2)}{2}. \]
or
\[
\lambda = 4 + \frac{(\theta - 4d)(\theta - 4d - 6)}{2}.
\]

Hence there are only the following four possibilities for \( V \):

Case 1: The equations (54) and (56) hold. Solving (54) and (56) for \( \theta \) yields that
\[
\theta = 2d.
\]

We apply \( w_{i-1} \) (\( 1 \leq i \leq d \)) to either side of (27) and evaluate the resulting equation by using (49), (50), (54) and (58). It follows that
\[
E^2w_i = 4i(2i - 1)(d - i + 1)(2d - 2i + 1)w_{i-1} \quad (1 \leq i \leq d).
\]

By Lemma 5.4 there exists a unique \( U(\mathfrak{sl}_2) \)-module isomorphism \( V \to L^{(0)}_{2d} \) that maps \( w_i \) to \((2i)!u_i^{(0)}\) for \( i = 0, 1, \ldots, d \).

Case 2: The equations (54) and (57) hold. Solving (54) and (57) for \( \theta \) yields that
\[
\theta = 2d + 1.
\]

We apply \( w_{i-1} \) (\( 1 \leq i \leq d \)) to either side of (27) and evaluate the resulting equation by using (49), (50), (54) and (59). It follows that
\[
E^2w_i = 4i(2i - 1)(d - i + 1)(2d - 2i + 3)w_{i-1} \quad (1 \leq i \leq d).
\]

By Lemma 5.4 there exists a unique \( U(\mathfrak{sl}_2) \)-module isomorphism \( V \to L^{(0)}_{2d+1} \) that maps \( w_i \) to \((2i)!u_i^{(0)}\) for \( i = 0, 1, \ldots, d \).

Case 3: The equations (55) and (56) hold. Solving (55) and (56) for \( \theta \) yields that
\[
\theta = 2d - 1.
\]

We apply \( w_{i-1} \) (\( 1 \leq i \leq d \)) to either side of (27) and evaluate the resulting equation by using (49), (50), (55) and (60). It follows that
\[
E^2w_i = 4i(2i + 1)(d - i + 1)(2d - 2i + 1)w_{i-1} \quad (1 \leq i \leq d).
\]

By Lemma 5.7 there exists a unique \( U(\mathfrak{sl}_2) \)-module isomorphism \( V \to L^{(1)}_{2d+1} \) that maps \( w_i \) to \((2i + 1)!u_i^{(1)}\) for \( i = 0, 1, \ldots, d \).

Case 4: The equations (55) and (57) hold. Solving (55) and (57) for \( \theta \) yields that
\[
\theta = 2d.
\]

We apply \( w_{i-1} \) (\( 1 \leq i \leq d \)) to either side of (27) and evaluate the resulting equation by using (49), (50), (55) and (61). It follows that
\[
E^2w_i = 4i(2i + 1)(d - i + 1)(2d - 2i + 3)w_{i-1} \quad (1 \leq i \leq d).
\]

By Lemma 5.7 there exists a unique \( U(\mathfrak{sl}_2) \)-module isomorphism \( V \to L^{(1)}_{2d+2} \) that maps \( w_i \) to \((2i + 1)!u_i^{(1)}\) for \( i = 0, 1, \ldots, d \). \( \square \)
6. The connection to the Terwilliger algebras of \( H(D, 2) \) and \( \frac{1}{2} H(D, 2) \)

Fix an integer \( D \geq 2 \). Let \( X = X(D) \) denote the set
\[
\{ (x_1, x_2, \ldots, x_D) \mid x_1, x_2, \ldots, x_D \in \{0, 1\} \}.
\]
For any \( x \in X \) let \( x_i \) \((1 \leq i \leq D)\) denote the \( i \)th coordinate of \( x \). Recall that the \( D \)-dimensional hypercube \( H(D, 2) \) is a finite simple connected graph with vertex set \( X \) and \( x, y \in X \) are adjacent if and only if there exists exactly one index \( i \) \((1 \leq i \leq D)\) such that \( x_i \neq y_i \). For any \( x, y \in X \) let \( \partial(x, y) \) denote the distance from \( x \) to \( y \) in \( H(D, 2) \). Note that \( \partial(x, y) = |\{ i \mid 1 \leq i \leq D, x_i \neq y_i \}| \).

The adjacency operator \( A \) of \( H(D, 2) \) is a linear endomorphism of \( \mathbb{C}^X \) given by
\[
Ax = \sum_{y \in X} y \quad \text{for all } x \in X.
\]
Let \( x \in X \) be given. By \([1, 11–13]\) the dual adjacency operator \( A^*(x) \) of \( H(D, 2) \) with respect to \( x \) is a linear endomorphism of \( \mathbb{C}^X \) given by
\[
A^*(x)y = (D - 2\partial(x, y))y \quad \text{for all } y \in X.
\]
The Terwilliger algebra \( T(x) \) of \( H(D, 2) \) with respect to \( x \) \([2, 7, 9, 13, 15]\) is the subalgebra of \( \text{End}(\mathbb{C}^X) \) generated by \( A \) and \( A^*(x) \).

**Theorem 6.1** (Theorem 13.2, \([7]\)). For each \( x \in X \) there exists a unique algebra homomorphism \( \rho(x) : U(\mathfrak{sl}_2) \to T(x) \) that sends
\[
E \mapsto \frac{A}{2} - \frac{[A, A^*(x)]}{4},
\]
\[
F \mapsto \frac{A}{2} + \frac{[A, A^*(x)]}{4},
\]
\[
H \mapsto A^*(x).
\]

**Lemma 6.2.** For each \( x \in X \) the algebra homomorphism \( \rho(x) \circ \sharp : \mathcal{H} \to T(x) \) maps
\[
A \mapsto \frac{A^*(x)}{4},
\]
\[
B \mapsto \frac{A^2 - 1}{4}.
\]

**Proof.** Let \( x \in X \) be given. Recall \( A^2 \) and \( B^2 \) from \([9]\) and \([10]\). By Theorem 6.1 the image of \( A^2 \) under \( \rho(x) \) is as stated. Using \([11]\) yields that \( B^2 \) is equal to \( \frac{(E + F - 1)(E + F + 1)}{4} \). Hence the image of \( B^2 \) under \( \rho(x) \) is as stated. The result follows. \(\square\)

Let \( x \in X \) be given. By Theorem 6.1 each \( T(x) \)-module is a \( U(\mathfrak{sl}_2) \)-module. We denote by \( \mathbb{C}^X(x) \) the natural \( T(x) \)-module structure on \( \mathbb{C}^X \). Let \( V \) denote a vector space. For any integer \( n \geq 1 \) we write \( n \cdot V \) for \( \underbrace{V \oplus V \oplus \cdots \oplus V}_{n \text{ copies of } V} \).

**Theorem 6.3** (Theorem 10.2, \([7]\)). For each \( x \in X \) the \( U(\mathfrak{sl}_2) \)-module \( \mathbb{C}^X(x) \) is isomorphic to
\[
\bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot L_{D-2k}.
\]
Define
\[ X_e = \left\{ x \in X \mid \sum_{i=1}^{D} x_i \text{ is even} \right\}. \]

The halved graph \(\frac{1}{2}H(D, 2)\) of \(H(D, 2)\) is a finite simple connected graph with vertex set \(X_e\) and \(x, y \in X_e\) are adjacent if and only if \(\partial(x, y) = 2\). Hence the adjacency operator of \(\frac{1}{2}H(D, 2)\) is equal to
\[ \frac{A^2 - D}{2} \mid_{C^{X_e}}. \]

Let \(x \in X_e\) be given. By [11] the dual adjacency operator of \(\frac{1}{2}H(D, 2)\) with respect to \(x\) is equal to
\[ \begin{cases} \frac{1}{2}A^*(x) \mid_{C^{X_e}} & \text{if } D = 2, \\ A^*(x) \mid_{C^{X_e}} & \text{if } D \geq 3. \end{cases} \]

Therefore the Terwilliger algebra \(T_e(x)\) of \(\frac{1}{2}H(D, 2)\) with respect to \(x\) is the subalgebra of \(\text{End}(C^{X_e})\) generated by \(A^2 \mid_{C^{X_e}}\) and \(A^*(x) \mid_{C^{X_e}}\).

**Theorem 6.4.** For each \(x \in X_e\) the following hold:

(i) \(T_e(x) = \{ M \mid_{C^{X_e}} \mid M \in \text{Im}(\rho(x) \circ \varepsilon) \} \).

(ii) \(T_e(x) = \{ M \mid_{C^{X_e}} \mid M \in \text{Im}(\rho(x) |_{\mathfrak{sl}(2)}) \} \).

**Proof.** (i): By Definition [12] the algebra \(H\) is generated by \(A\) and \(B\). Theorem 6.4(i) is now immediate from Lemma 6.2

(ii): Immediate from Theorems 6.4(i) and 6.4(i). \(\square\)

Let \(x \in X_e\) be given. By Theorem 6.4 each \(T_e(x)\)-module is a \(U(\mathfrak{sl}_2)\)-module as well as an \(H\)-module. We denote by \(C^{X_e}(x)\) the natural \(T_e(x)\)-module structure on \(C^{X_e}\).

**Theorem 6.5.** For each \(x \in X_e\) the \(U(\mathfrak{sl}_2)\)-module \(C^{X_e}(x)\) is isomorphic to
\[ \bigoplus_{k=0 \atop k \text{ is even}}^{\lfloor D/2 \rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot L_{D-2k}^{(0)} \oplus \bigoplus_{k=1 \atop k \text{ is odd}}^{\lfloor D/2 \rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot L_{D-2k}^{(1)}. \]

**Proof.** Let \(x \in X_e\) be given. For any \(y \in X\) observe that \(y \in X_e\) if and only if \(A^*(x)y = (D - 4i)y\) for some \(i \in \mathbb{Z}\). It follows from Theorem 6.1 that
\[ C^{X_e} = \bigoplus_{i \in \mathbb{Z}} C^X(D - 4i). \]

Combined with Theorem 6.3 the \(U(\mathfrak{sl}_2)\)-module \(C^{X_e}(x)\) is isomorphic to
\[ \bigoplus_{k=0}^{\lfloor D/2 \rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot \bigoplus_{i \in \mathbb{Z}} L_{D-2k}(D - 4i). \]

By Definition 5.2 the result follows. \(\square\)

By [11] the algebra \(T_e(x)\) is semi-simple. We remark that the first description of all non-isomorphic irreducible \(T_e(x)\)-modules was given in [13].

**Corollary 6.6.** For each \(x \in X_e\) the following hold:
(i) The $\mathbf{T}_e(x)$-modules

\begin{equation}
L_{D-2k}^{(0)} \quad \text{for all even integers } k \text{ with } 0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor;
\end{equation}

\begin{equation}
L_{D-2k}^{(1)} \quad \text{for all odd integers } k \text{ with } 1 \leq k \leq \left\lfloor \frac{D-1}{2} \right\rfloor
\end{equation}

are all irreducible $\mathbf{T}_e(x)$-modules up to isomorphism.

(ii) The algebra $\mathbf{T}_e(x)$ is isomorphic to

\begin{equation}
\bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \text{End} \left( C^X_{\left\lfloor \frac{D}{2} \right\rfloor-k+1} \right) \oplus \bigoplus_{k=1}^{\left\lceil \frac{D}{2} \right\rceil} \text{End} \left( C^X_{\left\lfloor \frac{D-1}{2} \right\rfloor-k+1} \right).\end{equation}

(iii) The dimension of $\mathbf{T}_e(x)$ is equal to

\begin{equation}
\left( \left\lfloor \frac{D}{2} \right\rfloor + 3 \right) + \left( \left\lceil \frac{D}{2} \right\rceil + 1 \right).
\end{equation}

Proof. (i): Since the $\mathbf{T}_e(x)$-module $C^X_{\mathbf{e}(x)}$ is faithful, each irreducible $\mathbf{T}_e(x)$-module is contained in $C^X_{\mathbf{e}(x)}$. Recall from Lemmas 5.5 and 5.8 that the $\mathbf{U}(\mathfrak{sl}_2)_{\mathbf{e}}$-modules (62) and (63) are irreducible. Recall from Theorem 5.10 that the $\mathbf{U}(\mathfrak{sl}_2)_{\mathbf{e}}$-modules (62) and (63) are mutually non-isomorphic. Corollary 6.6(i) now follows by the above comments and Theorem 6.4(ii).

(ii): By Corollary 6.6(i) the algebra $\mathbf{T}_e(x)$ is isomorphic to

\begin{equation}
\bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \text{End}(L_{D-2k}^{(0)}) \oplus \bigoplus_{k=1}^{\left\lceil \frac{D}{2} \right\rceil} \text{End}(L_{D-2k}^{(1)}).
\end{equation}

Combined with Lemmas 5.4(i) and 5.7(i) this implies Corollary 6.6(ii).

(iii): By Corollary 6.6(ii) the dimension of $\mathbf{T}_e(x)$ is equal to

\begin{equation}
\sum_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \left( \left\lfloor \frac{D}{2} \right\rfloor - k + 1 \right)^2 + \sum_{k=1}^{\left\lceil \frac{D-1}{2} \right\rceil} \left( \left\lfloor \frac{D-1}{2} \right\rfloor - k + 1 \right)^2.
\end{equation}

The first summation is equal to $\left\lfloor \frac{D}{2} \right\rfloor + 3$ and the second summation is equal to $\left\lceil \frac{D}{2} \right\rceil + 1$. □

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