Long-range order for spin-1 Heisenberg model with a small antiferromagnetic interaction

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Abstract
We look at the general SU(2) invariant spin-1 Heisenberg model. This family includes the well known Heisenberg ferromagnet and antiferromagnet as well as the interesting nematic (biquadratic) and the largely mysterious staggered-nematic interaction. Long range order is proved using the method of reflection positivity and infrared bounds on a purely nematic interaction. This is achieved through the use of a type of matrix representation of the interaction making clear several identities that would not otherwise be noticed. Using the reflection positivity of the antiferromagnetic interaction one can then show that the result is maintained if we also include an antiferromagnetic interaction that is sufficiently small.

1 Introduction
Showing the existence of phase transitions at low temperatures for Heisenberg models is a well known difficult problem. There have been several positive results in this area over the years in both the classical and quantum cases. The first rigorous proof of a phase transition in a Heisenberg model was the result of Fröhlich, Simon and Spencer [10] for the classical Heisenberg ferromagnet (and hence for the antiferromagnet also as it is equivalent to the ferromagnet in the classical case). The result was later extended to the quantum system by Dyson, Lieb and Simon [6]. The case of spin-1/2 in dimension three was not covered, the result was extended to this case by Kennedy, Lieb and Shastry [13]. The result also shows long-range order for dimension two at zero temperature. In the nematic case (also called the biquadratic interaction) there is known to be a phase transition. In the classical system there is nematic order (also called quadrupolar long-range order) at low temperatures, as was shown by Angelescu and Zagrebnov [3]. By contrast for the quantum case there is known to be Néel order, as was recently proved in the work of Ueltschi [21]. The paper extended and combined the works of Tóth [18] and Aizenmann and Nachtergaele [2] who introduced probabilistic representations of some quantum Heisenberg models. This work also showed the existence of nematic order in a region with an extra ferromagnetic interaction. All of these results apply in dimension at least three for positive temperature. In dimensions one and two there is the famous result of Mermin and Wagner [14] that rules out a phase transition at positive temperature, this does not contradict the result for dimension two in [13]. For the ground state there are some rigorous results, the work of Tanaka, Tanaka and

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Idogaki shows long range order for an antiferromagnetic interaction accompanied by a small enough nematic (biquadratic) interaction in dimensions two and three. In dimension three they also show long-range order in part of the nematic region investigated in [21], these results were obtained independently. The aim of this article is to show that there is also a phase transition in a region with a nematic interaction accompanied by a small antiferromagnetic interaction, this result was already expected, although an explicit proof has not been presented before. Curiously the result only shows the existence of nematic order, weaker than the expected antiferromagnetic order, this implies that there is further work to be done to strengthen the result to the full antiferromagnetic order.

The positive results concerning long-range order above use the method of reflection positivity in order to obtain an infra-red bound, that is, a bound on the Fourier transform of the correlation in question. One can then easily show that the correlation function doesn’t decay (for example that \( \langle S^3_x S^3_y \rangle \geq c > 0 \) uniformly) if the infra-red bound is sufficiently strong. This article will follow that approach, starting with the nematic model, obtaining a lower bound that involves some other correlation functions. This bound can be shown to be positive for low temperatures by relating these correlations to probabilities in the random loop model introduced in [2]. It is then clear (due to reflection positivity of the antiferromagnet interaction) that adding an antiferromagnetic interaction will maintain the positivity of the lower bound, providing the interaction is small enough.

2 Spin Systems: The General Setting

Let \( S \in \frac{1}{2} \mathbb{N} \). For a spin-\( S \) model we have local Hilbert spaces \( \mathcal{H}_x = \mathbb{C}^{2S+1} \). Observables are then Hermitian matrices built from linear combinations of tensor products of operators on \( \otimes_{x \in \Lambda} \mathcal{H}_x \). Physically important observables can often be expressed in terms of spin matrices \( S^1, S^2, S^3 \), operators on \( \mathbb{C}^{2S+1} \) that are the generators of a \( (2S+1) \)-dimensional irreducible unitary representation of SU(2) such that

\[
[S^\alpha, S^\beta] = i \sum_y \epsilon_{\alpha\beta\gamma} S^\gamma
\]

where \( \alpha, \beta, \gamma \in \{1, 2, 3\} \) and \( \epsilon_{\alpha\beta\gamma} \) is the Levi-Civita symbol. Denote \( S = (S^1, S^2, S^3) \), its magnitude is then \( S \cdot S = S(S+1) \mathbb{I} \). The case \( S = \frac{1}{2} \) gives the Pauli spin matrices. For \( S = 1 \) there are several choices for spin matrices, to make things concrete we will use the following matrices for \( S = 1 \):

\[
S^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Consider a pair \( (\Lambda, E) \) of a lattice \( \Lambda \subset \mathbb{Z}^d \) and a set of edges \( E \) between points in \( \Lambda \). Here we will take

\[
\Lambda = \left\{ -\frac{L_1}{2} + 1, \ldots, \frac{L_1}{2} \right\} \times \ldots \times \left\{ -\frac{L_d}{2} + 1, \ldots, \frac{L_d}{2} \right\},
\]

for integers \( L_i, i = 1, \ldots, d \). For the set of edges \( E \) we take nearest-neighbour with periodic boundary conditions. Then we take the operator \( S^i_x \) for \( i = 1, 2, 3 \) to be shorthand for the operator \( S^i_x \otimes \text{Id}_{\mathcal{H}_x} \).
Figure 1: The phase diagram for the general SU(2) invariant spin-1 model. Some regions have rigorous proofs that the expected order is indeed correct. The line $J_1 < 0$, $J_2 = 0$ is the Heisenberg antiferromagnet where antiferromagnetic order has been proven [6], this region extends slightly into the dark yellow region. The dark green region has nematic order at low temperatures [21], with Néel order on the line $J_2 > 0$, $J_1 = 0$, the adjacent dark yellow region also has long range order, however only the nematic correlation function has been shown not to decay, antiferromagnet order is expected here but is not yet proved.

3 The Spin-1 SU(2)-invariant model

The Hamiltonian of interest is general the Spin-1 SU(2)-invariant Hamiltonian with a two-body interaction, it is known that this can be written as

$$
H_{\Lambda, \theta}^{J_1, J_2} = -2 \sum_{\{x, y\} \in E} \left( J_1 \left( S_x \cdot S_y \right) + J_2 \left( S_x \cdot S_y \right)^2 \right).
$$

(4)

The phase diagram for this model is only partially understood. If $J_2 = 0$ and $J_1 < 0$ we have the Heisenberg antiferromagnet that is known to undergo a phase transition at low temperatures [6]. As the interaction when $J_2 > 0$ is reflection positive it is also possible to extend this result to $J_2 > 0$ when the ratio $J_1 / J_2$ is sufficiently small. The line $J_1 = 0$ has been shown to exhibit Néel order for low temperatures when $J_2 > 0$ [21], for $J_2 < 0$ there are no rigorous results, it would be a challenging task to obtain results. The line $J_2 = J_1 / 3 < 0$ is the AKLT model [1].

The main result of this paper is to show that there is a phase transition in this model for $J_2 > 0$ and $J_1 < 0$ with $|J_1|$ sufficiently small compared to $|J_2|$, the statement will be made precise below.
First we define the partition function and Gibb’s states of our model as
\[
Z_{\beta, \Lambda, 0}^{J_1, J_2} = \text{Tr} e^{-\beta H_{\Lambda, 0}^{J_1, J_2}},
\]
\[
\langle \cdot \rangle_{\beta, \Lambda, 0}^{J_1, J_2} = \frac{1}{Z_{\beta, \Lambda, 0}^{J_1, J_2}} \text{Tr} \cdot e^{-\beta H_{\Lambda, 0}^{J_1, J_2}}.
\]
We will soon drop the parameters in the notation for readability. The quantity of interest is then the correlation
\[
\rho(x) = \left\langle \left( (S_3^0)^2 - \frac{2}{3} \right) \left( (S_x^3)^2 - \frac{2}{3} \right) \right\rangle_{\beta, \Lambda, 0}^{J_1, J_2}.
\]
This correlation is specifically of interest for spin-1, in general spin-$S$ will be replaced with $\frac{1}{2}S(S + 1)$. The result is then given by the following theorem.

**Theorem 2.1 (Long-range order).** Let $S = 1$, $J_2 > 0$ and $L_1, ..., L_d$ be even. Then there exists $J_1^0 < 0$, $\beta_0$ and $C = C(\beta, J_1) > 0$ such that if $J_1 < J_1^0 < 0$ and $\beta > \beta_0$ then
\[
\frac{1}{|A|} \sum_{x \in A} \rho(x) \geq C.
\]
The proof of the result will be in two steps, first the result will be proved for $J_1 = 0$, this will be the content of the next section. Second it will be shown how the result for $J_1 = 0$ extends to sufficiently small $J_1 < 0$, this should come as no surprise as the interaction is reflection positive for $J_1 < 0$ hence adding a small interaction in this direction should not alter the result too much.

4 The model $J_2 > 0$, $J_1 = 0$

We will now consider the so-called quantum nematic model $J_2 > 0$, $J_1 = 0$, the aim is to prove long-range order for this model using a similar approach to the proofs in [6, 8, 9, 10]. To do this we will use a representation that is an analogue of the matrix representation used in [3]. Care must be taken as now we are working with matrices rather than vectors and so commutativity becomes an issue. We introduce an external field, $h$, to the Hamiltonian
\[
H_{\Lambda, h}^{J_1, J_2} = -2 \sum_{\{x,y\} \in E} (S_x \cdot S_y)^2 - \sum_{x \in A} h_x \left( (S_x^3)^2 - \frac{1}{3} S(S + 1) I \right).
\]
Here $I$ is the identity matrix. Equilibrium states are given by
\[
\langle A \rangle_{\beta, \Lambda, h}^{J_1, J_2} = \frac{1}{Z_{\beta, \Lambda, h}^{J_1, J_2}} \text{Tr} A e^{-\beta H_{\Lambda, h}^{J_1, J_2}}.
\]
Note that the $J_2$ has been absorbed into the parameter $\beta$. Using the direct analogue of [3] will not work here, the reason is that reflection positivity will fail as $S^2 = -S^2$. All other attempts to directly obtain a matrix representation of the interaction $(S_x, S_y)^2$ have also failed, however, there is a solution. We will instead use a matrix representation of a Hamiltonian that is unitarily equivalent to [3].
Similarly to before, equilibrium states are given by (neglecting the parameters for sake of readability)

\[ \langle A \rangle = \frac{1}{Z(\beta, \Lambda, h)} Tr A e^{-\beta H_{\Lambda, h}}. \] (12)

If \( \Lambda \) has a bipartite structure, \( \Lambda = \Lambda_A \cup \Lambda_B \), then if we define \( U = \prod_{x \in \Lambda_B} e^{i \pi S_i^0} \) we have

\[ U^{-1} H_{\Lambda, h} U = H_{\Lambda, h}. \] (13)

Before the theorem we introduce an integral, it is also introduced in [13],

\[ I_d = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sqrt{2(k + \pi)} e^{ik} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right) dk, \] (14)

where

\[ \epsilon(k) = 2 \sum_{i=1}^d (1 - \cos k_i). \] (15)

Then we have the following result:

**Theorem 3.1 (Long-range order for the quantum nematic model).** Let \( S = 1 \). Assume \( h = 0 \) and \( L_1, \ldots, L_d \) are even. Then we have the bound

\[ \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{1}{|A|} \sum_{x \in \Lambda} \rho(x) \geq \rho(e_1) - I_d \sqrt{\langle S_0^1 S_0^3 S_1^3 e_{1} \rangle}. \]

If this lower bound is strictly positive it implies the existence of a phase transition at low temperatures. Using the loop model introduced in [23] and extended in [21] we can relate the expectations in the lower bound to the probability of the event \( E_{0, e_i} \), that two nearest neighbours are in the same loops as

\[ \rho(e_1) = \frac{2}{9} \mathbb{P}[E_{0, e_i}], \quad \langle S_0^1 S_0^3 S_1^3 e_{1} \rangle = \frac{1}{3} \mathbb{P}[E_{0, e_i}]. \] (16)

So we can write the lower bound as

\[ \sqrt{\mathbb{P}[E_{0, e_i}]} \left( \frac{2}{9} \mathbb{P}[E_{0, e_i}] - \frac{1}{3} \right). \]

Because in the nematic model all the diagonal terms \( \left( S_i^1 \right)^2 = \frac{1}{2} \) have the same expectation, as do all the cross terms \( S_i^0 S_j^3 S_k^1 e_{i} S_{k+\epsilon_i}^i \), for \( i \neq j \), we can relate the expectation \( \langle H_{\Lambda, 0}^{0 Ja} \rangle_{\beta, \Lambda, 0} \) in the ground state to \( \mathbb{P}[E_{0, e_i}] \) using (16) and the translation invariance of the model

\[ \langle H_{\Lambda, 0}^{0 Ja} \rangle_{\beta, \Lambda, 0} = -8d|A| \left( \frac{2}{3} \mathbb{P}[E_{0, e_i}] + 1 \right). \] (17)
Then if we define the Néel state $\psi = \otimes_{\alpha \in \Lambda} (-1)^{\alpha}$ we have the bound $\langle H_{J_x J_y}^{0, J_z} | \psi, H_{J_x J_y}^{0, J_z} \rangle \leq \langle \psi, H_{J_x J_y}^{0, J_z} \psi \rangle$, giving $E_{0, \alpha} \geq \frac{1}{2}$. From this we can show that the lower bound is positive in $d \geq 6$ for large enough $\beta$.

The rest of the section will be dedicated to the proof of this theorem. We will proceed with calculations for general spin until it becomes necessary to restrict to the case $S = 1$. Fortunately for this Hamiltonian we can find a matrix representation. Define $Q_x$ as

$$Q_x = \begin{pmatrix}
(S^1_x)^2 - \frac{1}{2} S(S + 1) & S^1_x i S^2_x & S^1_x S^3_x \\
S^1_x i S^2_x & (S^2_x)^2 - \frac{1}{2} S(S + 1) & i S^2_x S^3_x \\
S^1_x S^3_x & i S^2_x S^3_x & (S^3_x)^2 - \frac{1}{2} S(S + 1)
\end{pmatrix}. \tag{18}
$$

We introduce the operation $\mathcal{T} R$, which is the sum of diagonal entries of matrices of the form of $Q_x$, however this ‘trace’ will return an operator, not a number, so we distinguish what we mean by multiplication. The representation (18) is not at all essential to the proof, the advantage of using it is that once (19) has been verified other relations can be stated much more concisely and clearly and easily checked, these relations are not at all obvious or easy to come up with without using (19).

By the product $Q_x Q_y$, we follow the ‘normal’ matrix multiplication with the added stipulation that for the $i^{th}$ diagonal entry of $Q_x Q_y$, the operator $S^i$ will appear first. For example in entry [1, 1] of $Q_x Q_y$, there is the term $S^1_x i S^2_x S^1_x i S^2_x$, in the entry [2, 2] this term will become $i S^2_x S^1_x i S^2_x S^1_x$. this ensures that we have each of the cross terms in the right-hand side of (19). For off-diagonal entries we are not concerned as we are always taking a ‘trace’.

In the case $x \neq y$ less care is needed as components of $S_x$ and $S_y$ commute (in fact $\mathcal{T} R Q_x = \mathcal{T} R Q_y$, hence we must only take care that the product order of components of spin at the same site is maintained).

We also have that $\mathcal{T} R Q_x^2 = C^S_x - \frac{1}{2} S^2(S + 1)^2$ acting on $\mathcal{H}_x$. In $S = 1$

$$C^1_x = \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
2 & 0 & 2
\end{pmatrix}_x. \tag{20}$$

Using this we can represent our interaction as

$$(S^1_x)^2 - S^2_x S^3_x + S^3_x S^1_x)^2 = \frac{1}{2} \left( C^S_x + C^S_y - \mathcal{T} R (Q_x - Q_y)^2 \right). \tag{21}$$

We introduce the field $v$ with value $v_x \in \mathbb{R}$ at the site $x \in \Lambda$, we denote by $v$ the field of $3 \times 3$ matrices such that each $v_x$ has one non-zero entry, the entry [3, 3] being $v_x \in \mathbb{R}$. We define

$$H(v) = \sum_{(x, y) \in E} \left( \mathcal{T} R [(Q_x - Q_y)^2] - C^S_x - C^S_y \right) - \sum_{x \in \Lambda} (\Delta v)_x \left( (S^3_x)^2 - \frac{1}{3} S(S + 1) \right), \tag{22}$$

$$Z(v) = T re^{-\beta H(v)}. \tag{23}$$

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Note that from \[21\] \( H(v) = H_{\Delta v} \). Here we have used the lattice Laplacian and below we use the inner product \((f, g) = \sum_{x \in \Lambda} f_x g_x\) with the identity \((f_x - \Delta g_x) = \sum_{x \in \Lambda} (f_x - f_x)(g_x - g_x)\). Then we can calculate as follows:

\[
H(v) = \sum_{x} T \mathcal{R} \left[ (Q_x + \frac{v_x}{2} - Q_y - \frac{v_y}{2}) \right] - T \mathcal{R} \left[ (Q_x - Q_y)(v_x - v_y) \right] - C_x^1 - C_y^1 + (v_x - v_y) \left( (S^1_x)^2 - (S^1_y)^2 \right) - \frac{1}{4} (v_x - v_y)^2 \tag{24}
\]

We must check carefully when dealing with the cross terms \((Q_x - Q_y)(v_x - v_y)\) and \((v_x - v_y)(Q_x - Q_y)\), they are not equal but \(T \mathcal{R}(Q_x - Q_y)(v_x - v_y) = T \mathcal{R}(v_x - v_y)(Q_x - Q_y)\), so the calculation is correct. From this it makes sense to define the following Hamiltonian and partition function:

\[
H'(v) = H(v) + \frac{1}{4} (v, -\Delta v), \tag{25}
\]

\[
Z'(v) = T e^{-\beta H'(v)}. \tag{26}
\]

Now the property of Gaussian Domination is

\[
Z(v) \leq Z(0)e^{\beta(\Delta \mu)} \iff Z'(v) \leq Z'(0), \tag{27}
\]

as in the classical case it follows from reflection positivity.

**Lemma 3.2 (Reflection positivity).** Let \( \mathcal{H} = h \otimes h, \dim h < \infty \), fix a basis. Let \( A, B, C, D \) for \( i = 1, \ldots, k \) be matrices in \( h \), then

\[
\left| \text{Tr}_{\mathcal{H}} \exp \left\{ A \otimes \mathbb{1} + \mathbb{1} \otimes B - \sum_{i=1}^{k} (C_i \otimes \mathbb{1} - \mathbb{1} \otimes D_i) \right\} \right|^2 \leq \text{Tr}_{\mathcal{H}} \exp \left\{ A \otimes \mathbb{1} + \mathbb{1} \otimes \bar{A} - \sum_{i=1}^{k} (C_i \otimes \mathbb{1} - \mathbb{1} \otimes \bar{C}_i) \right\} \times \text{Tr}_{\mathcal{H}} \exp \left\{ \bar{B} \otimes \mathbb{1} + \mathbb{1} \otimes B - \sum_{i=1}^{k} (D_i \otimes \mathbb{1} - \mathbb{1} \otimes \bar{D}_i) \right\} \tag{28}
\]

where \( \bar{A} \) is the complex conjugate of \( A \).

The proof uses Trotter’s formula. As in the classical case, reflection positivity is a very powerful tool, for more information see \([4, 6, 8, 9, 10, 19, 20, 21]\).

Before we prove reflection positivity for our partition function we should calculate the trace in \( Z'(v) \), recall how we have defined our multiplication.

\[
T \left( \left( Q_x + \frac{v_x}{2} - Q_y - \frac{v_y}{2} \right) \right) = \left( (S^1_x)^2 - (S^1_y)^2 \right) + \left( (S^2_x)^2 - (S^2_y)^2 \right) + \left( (S^3_x)^2 - (S^3_y)^2 \right) + \left( (S^1_x S^1_y - S^1_y S^1_x) + (S^2_x S^2_y - S^2_y S^2_x) \right) + \left( (S^3_x S^3_y - S^3_y S^3_x) + (S^1_x S^1_y - S^1_y S^1_x) \right) \tag{29}
\]
Now we have enough information to use the Lemma, let $R : \Lambda \rightarrow \Lambda$ be a reflection that swaps $\Lambda_1$ and $\Lambda_2$ where $\Lambda = \Lambda_1 \cup \Lambda_2$, each such reflection defines two sub-lattices of $\Lambda$ in this way, we split the field $v = (v_1, v_2)$ on the sub-lattices $\Lambda_1$ and $\Lambda_2$.

**Lemma 3.3 (Reflection positivity for the quantum nematic model).** For any reflection, $R$, across edges and $v = (v_1, v_2)$

$$Z((v_1, v_2))^2 \leq Z((v_1, Rv_1))Z((Rv_2, v_2)).$$

**Proof.** We cast $Z'(v)$ in RP form. Let

$$A = -\beta \sum_{x,y \in E_1} Tr \left[ (Q_x + \frac{v_x}{2} - Q_y - \frac{v_y}{2})^2 \right] - \beta d \sum_{x \in \Lambda_1} C_x^S,$$

$$B = \text{same in } \Lambda_2,$$

where $E_1$ is the set of edges in $\Lambda_1$ and we note that the term $C_x^S$ occurs $d$ times in the sum over $E$ for each $x \in \Lambda$. Further define

$$C^1_x = \sqrt{N}(S^1_x)^2, \quad D^1_x = \sqrt{N}(S^1_x)^2,$$

$$C^2_x = \sqrt{N}(S^2_x)^2, \quad D^2_x = \sqrt{N}(S^2_x)^2,$$

$$C^3_x = \sqrt{N}(S^3_x)^2 + \frac{v_y}{2}, \quad D^3_x = \sqrt{N}(S^3_x)^2 + \frac{v_y}{2},$$

$$C^4_x = \sqrt{B}S^1_x iS^2_x, \quad D^4_x = \sqrt{B}S^1_x iS^2_x,$$

$$C^5_x = \sqrt{B}S^1_x S^3_x, \quad D^5_x = \sqrt{B}S^1_x S^3_x,$$

$$C^6_x = \sqrt{B}iS^2_x S^3_x, \quad D^6_x = \sqrt{B}iS^2_x S^3_x,$$

$$C^7_x = \sqrt{B}iS^2_x iS^3_x, \quad D^7_x = \sqrt{B}iS^2_x iS^3_x,$$

$$C^8_x = \sqrt{B}S^3_x iS^2_x, \quad D^8_x = \sqrt{B}S^3_x iS^2_x.$$

Where $\{x_i, y_i\}$ are edges crossing the reflection plane with $x_i \in \Lambda_1$ and $y_i \in \Lambda_2$. Because $S^1_x = S^1_y, S^3_x = S^3_y, iS^2_x = iS^2_y$ we see from the previous lemma that $Z'((v_1, v_2))^2 \leq Z'((v_1, Rv_1))Z'((Rv_2, v_2))$, from which the result follows. \(\square\)

The Gaussian domination inequality (27) follows from this just as in the classical case, a proof can be found in [6]. The next step in the classical case was to obtain an infra-red bound for the correlation function $\rho(x)$, we cannot do this directly but we can obtain an infra-red bound for the Duhamel correlation function.

**Definition 3.4 (Duhamel correlation function).** For matrices $A, B$ we define the Duhamel correlation function $(A, B)_\text{Duh}$ as

$$(A, B)_\text{Duh} = \frac{1}{Z(0) \beta} \int_0^\beta ds Tr A^* e^{-sH(0)} Be^{-s\beta H(0)}$$

Note that this is an inner product.

Now to use this correlation function we must first fix our definition of the Fourier transform

$$\mathcal{F}(f)(k) = \hat{f}(k) = \sum_{x \in \Lambda} e^{-ikx} f(x) \quad k \in \Lambda^*,$$

$$f(x) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} e^{ikx} \hat{f}(k) \quad x \in \Lambda.$$
where
\[ \Lambda^* = \frac{2\pi}{L_i} \left( \frac{L_1}{2} + 1, \ldots, \frac{L_i}{2} \right) \times \cdots \times \frac{2\pi}{L_d} \left( \frac{L_d}{2} + 1, \ldots, \frac{L_d}{2} \right). \] (33)

**Lemma 3.5.** We have the following infra-red bound
\[
\mathcal{F} \left((S^1_0)^2 - \frac{1}{3} S(S + 1), (S^1_0)^2 - \frac{1}{3} S(S + 1) \right)_{\text{Duh}} (k) \leq \frac{1}{2\beta\epsilon(k)} \] (34)

*Proof.* We begin as usual by choosing \( v = \eta \cos(kx) \), then from Taylor’s theorem and using \( h = \Delta v = -\epsilon(k)v \) we see
\[
Z(v) = Z(0) + \frac{1}{2} \left( h, \frac{\partial^2 Z(v)}{\partial h_x \partial h_y} \right)_{h=0} + O(\eta^4). \] (35)

Using the Duhamel formula
\[
e^{\beta(A^* + B)} = e^{\beta A} + \int_0^\beta dx e^{\beta A} B e^{(\beta - x)(A^* + B)} \] (36)
gives
\[
\frac{1}{Z(0)} \frac{\partial^2 Z(v)}{\partial h_x \partial h_y} = \beta^2 \left((S^1_0)^2 - \frac{1}{3} S(S + 1), (S^1_0)^2 - \frac{1}{3} S(S + 1) \right)_{\text{Duh}} \] (37)

Putting this together we have
\[
Z(v) - O(\eta^4) = Z(0) + \frac{1}{2} Z(0) (\eta \epsilon(k) \beta)^2 \sum_{x,y \in \Lambda} \cos(kx) \cos(ky) \left((S^1_0)^2 - \frac{1}{3} S(S + 1), (S^1_0)^2 - \frac{1}{3} S(S + 1) \right)_{\text{Duh}}
\]
\[
= Z(0) + \frac{1}{2} Z(0) \beta^2 \eta^2 \epsilon(k)^2 \mathcal{F} \left((S^1_0)^2 - \frac{1}{3} S(S + 1), (S^1_0)^2 - \frac{1}{3} S(S + 1) \right)_{\text{Duh}} \sum_{x \in \Lambda} \cos^2(kx). \] (38)

Also
\[
e^{-\frac{i}{\beta} \mathcal{F}(v, \Delta v)} = e^{\frac{i}{\beta} \mathcal{F}(v, \Delta v)} \sum_{x} \cos(kx), \] (39)

comparing the order \( \eta^2 \) terms gives the result. \( \square \)

To transfer the infra-red bound to the normal correlation function we would like to use the Falk-Bruch inequality [7]:
\[
\frac{1}{2} (A^* A + AA^*) \leq (A, A)_{\text{Duh}} + \frac{1}{2} \sqrt{(A, A)_{\text{Duh}} ([A^*, [H, A]])} \] (40)

where is the Hamiltonian of the system. If we attempt to use this inequality with \( A = \mathcal{F} \left((S^1_0)^2 - \frac{1}{3} S(S + 1) \right) (k) \) and \( H = \beta H_{N,B} \), we must calculate the double commutator to find \( ([A^*, [H, A]]) \), in general spins this is a huge calculation, instead we specialise to the case \( S = 1 \), in this case we can calculate as below, it uses several special properties.
of the Spin-1 matrices. To make use of this inequality we note that

\[
\mathcal{F} \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) = \sum_{x \in \Lambda} e^{-i k x} \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} e^{-i k x} \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \right. 
\]

(41)

This relation holds for other correlation functions, including the Duhamel correlation function, but for Duhamel

\[
\mathcal{F} \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( P \right) \left( k \right) = \frac{1}{|\Lambda|} \mathcal{F} \left( \left( S_0^3 \right)^2 - \frac{1}{3} S(S + 1) \right) \left( P \right) \left( k \right) \left( k \right) .
\]

(42)

there is no \(-k\) because of the definition of the Duhamel correlation function and the equality \(\mathcal{F} \left( \left( S_0^3 \right)^2 \right) \left( k \right) = \mathcal{F} \left( \left( S_0^3 \right)^2 \right) \left( -k \right)\).

First we prove a preliminary lemma regarding the double commutator

**Lemma 3.6.** For \(S = 1\), \(A = \mathcal{F} \left( \left( S_0^3 \right)^2 - \frac{2}{3} \right) \left( k \right)\) and \(H = \beta \mathcal{H}_{N,0}\) we have

\[
\langle [A^*, [H, A]] \rangle = 8|\Lambda| e(k + \pi) \left( S_0^1 S_0^3 S_1^3 S_1^1 \right)
\]

where \(e_1\) is the first basis vector in \(\mathbb{Z}^d\).

**Proof.** The proof is just a calculation, although it is somewhat complicated, we begin by noting that in the case \(S = 1\) the matrices \((S_i)^2\) and \((S_i)^2\) commute and \((S_i)^3 = S_i\) for \(i, j = 1, 2, 3\).

\[
\left[ H, A \right] = -2\beta \sum_{x \neq \{x', y\} \in \mathcal{E}} e^{-i k x} \left( S_0^1 S_0^3 S_1^3 S_1^1 - S_0^2 S_0^2 S_2^2 S_2^2 + S_0^1 S_0^3 S_1^3 S_1^1 \right)
\]

\[
= -2\beta \sum_{x \neq \{x', y\} \in \mathcal{E}} e^{-i k x} \left( S_0^1 S_0^3 S_1^3 S_1^1 - S_0^1 S_0^3 S_1^3 S_1^1 - S_0^2 S_0^2 S_2^2 S_2^2 + S_0^1 S_0^3 S_1^3 S_1^1 \right)
\]

\[
- S_0^2 S_0^2 S_2^2 S_2^2 + S_0^1 S_0^3 S_1^3 S_1^1 - S_0^2 S_0^2 S_2^2 S_2^2 - S_0^1 S_0^3 S_1^3 S_1^1 
\]

(43)

The square terms have dropped out as they commute with \((S_0^3)^2\), as has the constant term \(S(S + 1)/3\). Now we calculate the commutator for each term in the sum, here we
make use of the fact that $S^i S^i = 0$ for $i \neq j, i, j = 1, 2, 3$ for $S = 1$.

$$[H, A] = -2\beta \sum_{x,y,(l,x) \in \mathcal{E}} e^{-ikx} \left( S^1 S^2 S^1 S^2 + \left( (S^3)^2, S^1 S^1 \right) S^1 S^2 + \left( (S^3)^2, S^1 S^1 \right) S^2 S^1 \right)$$

$$- S^2 S^3 S^2 S^3 - S^2 S^1 S^3 S^1 + S^3 S^2 S^3 S^2 \right)$$

$$= -2\beta \sum_{x,y,(l,x) \in \mathcal{E}} e^{-ikx} \left( -S^2 S^3 S^2 S^3 + S^1 S^2 S^3 S^2 - S^3 S^1 S^3 S^1 + S^1 S^3 S^1 S^1 \right)$$

$$= +2\beta \sum_{x,y,(l,x) \in \mathcal{E}} e^{-ikx} \left( \left[ S^2 S^3 S^2, S^3 S^3 \right] + \left[ S^2 S^3 S^3, S^3 S^3 \right] \right).$$

Now calculating the commutator of these products and using the spin commutation relations we obtain

$$[H, A] = 2\beta i \sum_{x,y,(l,x) \in \mathcal{E}} e^{-ikx} \left( S^2 S^3 S^3 S^2 + S^1 S^2 S^3 + S^3 S^1 S^3 + S^3 S^3 S^3 \right) \tag{44}$$

Now we can use this to calculate the double commutator, firstly we split the commutator into the sum of two similar terms

$$[A^*, [H, A]] = 2\beta i \sum_{x,y,(l,x) \in \mathcal{E}} e^{-ikx} \left[ \left( S^2 S^3 \right)^2, f(S_x, S_y) \right]$$

$$= 2\beta i \sum_{x,y,(l,x) \in \mathcal{E}} \left[ \left( S^2 S^3 \right)^2, f(S_x, S_y) \right] + \cos(k(x-y)) \left( (S^3)^2, f(S_x, S_y) \right). \tag{46}$$

We can calculate each of these commutators separately, the first double commutator can be calculated as follows

$$\left( S^3 \right)^2, f(S_x, S_y) = \left[ (S^3)^2, S^1 S^1 S^3 + S^1 S^1 S^3 + S^1 S^1 S^3 + S^2 S^2 S^3 \right]$$

$$= -S^2 S^1 S^1 + \left( (S^3)^2, S^1 \right) S^1 S^3 + S^1 S^1 S^3 + \left( (S^3)^2, S^1 \right) S^1 S^3$$

$$- S^2 S^1 S^1 + iS^2 S^2 S^1 S^2 + iS^2 S^2 S^1 S^2$$

$$+ S^1 S^1 S^3 - iS^1 S^1 S^3 S^3 - iS^1 S^1 S^3 S^3 \tag{47}$$

now we group the terms above as follows

$$S^2 S^1 S^2 (iS^1 S^1 S^2 - S^1) + iS^2 S^2 S^1 S^3 S^3 + S^1 S^1 S^2 - iS^1 S^1 S^3$$

$$= -iS^1 S^1 S^3 S^3,$$ \tag{48}

and we recognise the commutator relations above to finally give

$$\left( (S^3)^2, f(S_x, S_y) \right) = iS^2 S^2 S^1 S^3 + iS^2 S^2 S^1 S^3 - iS^1 S^1 S^3 S^3 - iS^1 S^1 S^3 S^3. \tag{49}$$

For the other commutator we follow the previous calculation almost exactly and in fact we find the two commutators are equal

$$\left( (S^3)^2, f(S_x, S_y) \right) = \left( (S^3)^2, f(S_x, S_y) \right). \tag{50}$$
To finish the calculation we take expectations

$$
\langle [A^*, [H, A]] \rangle = -4\beta|\Lambda| \sum_{i=1}^{d} (1 + \cos(k_i)) \left( S_{0}^{3}S_{2}^{3}S_{0}^{3}S_{2}^{3} + S_{0}^{3}S_{2}^{3}S_{0}^{3}S_{2}^{3} + S_{0}^{3}S_{2}^{3}S_{0}^{3}S_{2}^{3} - S_{0}^{3}S_{2}^{3}S_{0}^{3}S_{2}^{3}ight)
$$

now use the identities $(S^{3}S^{2})^{T} = -S^{2}S^{3}$ and $(S^{3}S^{1})^{T} = S^{1}S^{3}$ and get

$$
\langle [A^*, [H, A]] \rangle = -8\beta|\Lambda| \sum_{i=1}^{d} (1 + \cos(k_i)) \left( 2S_{0}^{3}S_{2}^{3}S_{0}^{3}S_{2}^{3} \right)_{0,J_{z}}^{(0,2)}
$$

Using this in Falk-Bruch we have the bound

$$
\hat{\rho}(x(k)) \leq \sqrt{\left( \frac{s(k + \pi)}{s(k)} \right) + \frac{1}{2\beta \epsilon(k)}}
$$

The possibility of obtaining a result is not ruled out for other values of $S$, I expect it to be the case for other values of $S$, but computing the double commutator in Falk-Bruch becomes extremely complicated.

Now using the Fourier transform in the following way:

$$
\left( (S_{0}^{3})^{2} - \frac{2}{3} \right)(S_{x}^{3})^{2} - \frac{2}{3}
$$

with $y = e_{1}$ we get the lower bound.

$$
\langle (S_{x}^{3})^{2} - \frac{1}{3}S(S + 1) \rangle
$$

5  Extending to $J_{1} < 0$

The proof of long-range order for $J_{1} < 0$ is a straightforward extension of the previous results, like before we will work with a Hamiltonian that is Unitarily equivalent to $H_{A,0}^{J_{1},J_{z}}$, we also introduce an external field $h$ as before. Recall the unitary operator $U = \prod_{x \in \Lambda} e^{\alpha S_{x}^{3}}$, let

$$
\tilde{H}_{A,h} = UH_{A,h}^{J_{1},J_{z}}U^{-1} - \sum_{x \in \Lambda} h_{x} \left( (S_{x}^{3})^{2} - \frac{1}{3}S(S + 1) \right).
$$
The effect of the unitary operator here is to replace $S^1_3$ and $S^3_3$ in $H_{\lambda,0}^{J_1}$ with $-S^1_3$ and $-S^3_3$ respectively. By using the representation (18) we can write $H_{\lambda,0}$ as
\[
H_{\lambda,0} = -\sum_{(x,y)\in E} \left[ J_1 \left( (S^1_3 - S^1_3)^2 - (S^2_3 - S^2_3)^2 + (S^3_3 - S^3_3)^2 \right) \right.
\]
\[
\left. - J_1 \left( \mathcal{T} \mathcal{R} [(Q_x - Q_y)^2] \right) + C_A(J_1, J_2) \right].
\]
(56)

Then similar to before we introduce the field $v$ and associated $3 \times 3$ field of matrices $v$, define
\[
\bar{H}(v) = -\sum_{(x,y)\in E} \left[ J_1 \left( (S^1_3 - S^1_3)^2 - (S^2_3 - S^2_3)^2 + (S^3_3 - S^3_3)^2 \right) \right.
\]
\[
\left. - J_1 \left( \mathcal{T} \mathcal{R} [(Q_x + \frac{v_x}{2} - Q_y - \frac{v_y}{2})^2] \right) + C_A(J_1, J_2) \right] - \frac{1}{4} (v, -\Delta v),
\]
(57)
\[
\bar{Z}(v) = \text{Tr} e^{-i\bar{H}(v)},
\]
(58)
and
\[
\bar{H}'(v) = H(v) + \frac{1}{4} (v, -\Delta v),
\]
(59)
\[
\bar{Z}'(v) = \text{Tr} e^{-i\bar{H}'(v)}.
\]
(60)

From this reflection positivity follow just as in Lemma 3.4, with the obvious changes to $A$ and $B$ and the extra terms
\[
C^{10} = \sqrt{-J_1} S^1_3, \quad D^{10} = \sqrt{-J_1} S^1_3,
\]
\[
C^{11} = \sqrt{-J_1} iS^2_3, \quad D^{11} = \sqrt{-J_1} iS^2_3,
\]
\[
C^{12} = \sqrt{-J_1} S^3_3, \quad D^{12} = \sqrt{-J_1} S^3_3,
\]
(61)
(recall that $J_1 < 0$). From this we obtain the Gaussian domination inequality
\[
\bar{Z}(v) \leq \bar{Z}(0)e^{i\langle v, -\Delta v \rangle} \iff \bar{Z}'(v) \leq \bar{Z}'(0),
\]
(62)
just as before. We also obtain the same infra-red bound as in Lemma 3.7, with an identical proof
\[
\mathcal{F} \left( \left( S^3_0 \right)^2 - \frac{1}{3} S(S + 1), \left( S^1_3 \right)^2 - \frac{1}{3} S(S + 1) \right) \leq \frac{1}{2}\beta e(k).
\]
(63)

Again the results up to here work for general $S \in \frac{1}{2} \mathbb{N}$, at this point we must specialise to $S = 1$ to be able to calculate the quantities in the double commutator of the Falk-Bruch inequality. From this we can see that by using Falk-Bruch inequality with $A = \mathcal{F} \left( (S^3_0)^2 - \frac{2}{3} \right) (k)$ and $H = \beta H_{\lambda,0}$ the linearity of the double commutator ensures that we have a positive lower bound $C = C(\beta, J_1)$ in Theorem 2.1 when $\beta$ and $|J_1|$ are small enough.

**Acknowledgments**

I am pleased to thank my supervisor Daniel Ueltschi for his support and useful discussions.
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