Primes in arithmetic progressions to large moduli

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How many primes are less than $x$ and congruent to $a \pmod{q}$?

Theorem (Siegel-Walfisz)
If $q \leq (\log x)^A$ and $\gcd(a, q) = 1$ then
$$\pi(x; q, a) = (1 + o(1)) \pi(x) \phi(q).$$

Theorem (GRH Bound)
Assume GRH. If $q \leq x^{\frac{1}{2} - \epsilon}$ and $\gcd(a, q) = 1$ then
$$\pi(x; q, a) = (1 + o(1)) \pi(x) \phi(q).$$

Conjecture (Montgomery)
If $q \leq x^{1 - \epsilon}$ and $\gcd(a, q) = 1$ then
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Introduction

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**Theorem (Siegel-Walfisz)**

If $q \leq (\log x)^A$ and $\gcd(a, q) = 1$ then

$$\pi(x; q, a) = (1 + o(1)) \frac{\pi(x)}{\phi(q)}.$$

**Theorem (GRH Bound)**

Assume GRH. If $q \leq x^{1/2-\epsilon}$ and $\gcd(a, q) = 1$ then

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**Conjecture (Montgomery)**

If $q \leq x^{1-\epsilon}$ and $\gcd(a, q) = 1$ then

$$\pi(x; q, a) = (1 + o(1)) \frac{\pi(x)}{\phi(q)}.$$
Often we don’t need such a statement to be true for every $q$, just for most $q$. 

Theorem (Bombieri-Vinogradov) 

Let $Q < x^{1/2 - \epsilon}$. Then for any $A$

$$\sum_{q \sim Q} \underset{(a, q)}{\text{sup}} \left| \pi(x; q, a) - \frac{x}{\phi(q)} \right| \ll A x (\log x)^A$$

Corollary 

For most $q \leq x^{1/2 - \epsilon}$, we have

$$\pi(x; q, a) = \left(1 + o(1)\right) \frac{x}{\phi(q)}$$

for every $a$ with $\gcd(a, q) = 1$. 

From the point of view of e.g. sieve methods, this is essentially as good as the Riemann Hypothesis!
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Let $Q < x^{1/2-\epsilon}$. Then for any $A$

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\sum_{q \sim Q} \sup_{(a, q) = 1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}
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**Corollary**

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Beyond GRH

Pioneering work by Bombieri, Fouvry, Friedlander, Iwaniec went beyond the $x^{1/2}$ barrier in special circumstances.

Theorem (BFI1)

Fix $a$. Then we have (uniformly in $\theta$)

$$\sum_{q \sim x^\theta}(q,a) = 1 \left| \pi(x;q,a) - \pi(x) \phi(q) \right| \ll a(\theta - 1/2)^2 x \log \log x O(1) \log x + x \log \frac{3}{x}.$$  

This is non-trivial when $\theta$ is very close to 1/2.

Theorem (BFI2)

Fix $a$. Let $\lambda(q)$ be 'well-factorable'. Then we have

$$\sum_{q \sim x^{4/7 - \epsilon}}(q,a) = 1 \lambda(q) \left( \pi(x;q,a) - \pi(x) \phi(q) \right) \ll a, A x \log A x.$$  

This is often an adequate substitute for BV with exponent $4/7$.
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**Theorem (BFI1)**

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$$\sum_{q \sim x^\theta, (q,a)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_a (\theta - 1/2)^2 \frac{x(\log \log x)^{O(1)}}{\log x} + \frac{x}{\log^3 x}.$$ 

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**Theorem (BFI2)**

Fix $a$. Let $\lambda(q)$ be ‘well-factorable’. Then we have

$$\sum_{q \sim x^{4/7-\epsilon}, (q,a)=1} \lambda(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll_{a,A} \frac{x}{\log^A x}.$$ 

This is often an adequate substitute for BV with exponent $4/7$!
More recently, Zhang went beyond $x^{1/2}$ for smooth/friable moduli.

**Theorem (Zhang, Polymath)**

$$
\sum_{q \leq x^{1/2+7/300-\epsilon}} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}
$$

$p|q \Rightarrow p \leq x^{\epsilon^2}$

$(q, a) = 1$

The implied constant is independent of $a$. 
New results

Theorem (M.)

Let \( \delta < 1/42 \) and \( Q_\delta := \{ q \sim x^{1/2+\delta} : \exists d|q \text{ s.t. } x^{2\delta+\epsilon} < d < x^{1/14-\delta} \} \).

\[
\sum_{\substack{q \in Q_\delta \tab (q,a)=1}} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x(\log \log x)^{O(1)}}{\log^5 x}.
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$$

Corollary

Let $\delta < 1/42$. For $(100 - O(\delta))\%$ of $q \sim x^{1/2+\delta}$ we have

$$
\pi(x; q, a) = (1 + o(1)) \frac{\pi(x)}{\phi(q)}
$$

Corollary

$$
\sum_{q_1 \sim x^{1/21}} \sum_{q_2 \sim x^{10/21-\epsilon}} \left| \pi(x; q_1q_2, a) - \frac{\pi(x)}{\phi(q_1q_2)} \right| \ll_a \frac{x (\log \log x)^{O(1)}}{\log^5 x}
$$
Theorem (M.)

Let $\lambda(q)$ be ‘very well factorable’. Then we have

$$
\sum_{\substack{q \leq x^{3/5-\epsilon} \\
(q,a)=1}} \lambda(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll_{a,A} x \frac{x}{(\log x)^A}.
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The $\beta$-sieve weights are ‘very well factorable’ for $\beta \geq 2$. 
Theorem (M.)

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The $\beta$-sieve weights are ‘very well factorable’ for $\beta \geq 2$.

Corollary

Let $\lambda^+(d)$ be sieve weights for the linear sieve. Then

$$\sum_{q \leq x^{7/12-\epsilon}, \gcd(q,a)=1} \lambda^+(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll \frac{x}{(\log x)^A}.$$
| Result | Size of $q$ | Type of $q$ | Proportion of $q$ |
|--------|-------------|-------------|------------------|
| BFI1   | $x^{1/2+o(1)}$ | All         | $(100 - \delta)\%$ |
| BFI2   | $x^{4/7-\epsilon}$ | Factorable  | $\delta\%$     |
| Zhang  | $x^{1/2+7/300-\epsilon}$ | Factorable  | $\delta\%$     |
| M1     | $x^{11/21-\epsilon}$ | Partially Factorable | $(100 - \delta)\%$ |
| M2     | $x^{3/5-\epsilon}$ | Factorable  | $\delta\%$     |

| Result | Coefficients | Residue class | Cancellation |
|--------|--------------|---------------|--------------|
| BFI1   | Absolute values | Fixed         | $o(1)$       |
| BFI2   | Factorable weights | Fixed         | $\log^{A} x$ |
| Zhang  | Absolute values | Uniform       | $\log^{A} x$ |
| M1     | Absolute values | Fixed         | $\log^{5-\epsilon} x$ |
| M2     | Factorable weights | Fixed         | $\log^{A} x$ |

Note that $3/5 > 4/7 > 11/21 > 1/2 + 7/300$. 

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Proof overview

The overall proof follows the same lines as previous approaches:

1. Apply a combinatorial decomposition to $\Lambda(n)$

2. Reduce the problem to estimating exponential sums of convolutions

3. Apply different techniques in different ranges to estimate exponential sums

   - Bounds from the spectral theory of automorphic forms (Kuznetsov Trace Formula)
   - Bounds from Algebraic Geometry (Weil bound/Deligne bounds)

4. Ensure that (essentially) all ranges are covered.

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Let us recall the situation when $q \sim x^{1/2+\delta}$ where $\delta > 0$ is fixed but small. Using BFI proof ideas:

- Heath-Brown Identity/Sieve methods reduces to considering products of few prime factors

$\prod_{i=1}^{5} p_i$ of 5 primes with $p_i = x^{1/5} + O(\delta)$

$\prod_{i=1}^{4} p_i$ of 4 primes with $p_i = x^{1/4} + O(\delta)$

The BFI result follows on noting that these terms are only a $O(\delta)$ proportion of the terms.

We can concentrate on these 'bad products'.
Bad products

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   - Products $p_1 p_2 p_3 p_4 p_5$ of 5 primes with $p_i = x^{1/5+O(\delta)}$
   - Products $p_1 p_2 p_3 p_4$ of 4 primes with $p_i = x^{1/4+O(\delta)}$
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We can concentrate on these ‘bad products’.
Consider terms $p_1 p_2 p_3 p_4 p_5$ with $p_i \in [x^{1/5-\delta}, x^{1/5+\delta}]$

- Zhang-style estimates can handle all terms when the modulus is smooth, but are least efficient for products of 5 primes, so don’t help.
Consider terms $p_1 p_2 p_3 p_4 p_5$ with $p_i \in [x^{1/5 - \delta}, x^{1/5} + \delta]$

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- Instead we refine some of the estimates for exponential sums coming from Kuznetsov/Kloostermaina.
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- Refinement of BFI can handle $p_1 p_2 p_3 p_4 p_5$ with $q < x^{4/7-\epsilon}$ when $p_i \approx x^{1/5}$ except when $p_i \in [x^{1/5} \log^{-A} x, x^{1/5} \log^{A} x]$
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- I still can’t handle these terms, but they now contribute $O((\log \log x)^{O(1)}/\log^4 x)$ proportion for a wide range of $q$. (This is why I only save $4 - \epsilon \log x$ factors.)

**Algebraic Geometry doesn’t help much, but we can refine Kuznetsov-based estimates to handle these terms**
Consider terms $p_1 p_2 p_3 p_4$ with $p_i \in [x^{1/4-\delta}, x^{1/4+\delta}]$

- Kloostermania techniques still can’t handle products of 4 primes
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- Note: In this case there is a factor $p_1 p_4 = x^{1/2+O(\delta)}$ very close to 1/2. This is the situation when Zhang-style arguments are most effective!

- Provided $q$ has a suitable factor close to $x^{1/2}$, we can handle these terms using the Weil bound.

The technical parts which spectral theory estimates can’t handle are precisely parts that the algebraic geometry estimates are best at *when there is a suitable factor*.  

As stated these ideas combine to give a result for $q \sim x^{1/2+\delta}$ for some small $\delta > 0$.

To get good numerics, need to refine estimates for other parts of prime decomposition

- Generalize ideas based on Deligne’s work (Fouvry, Kowalski, Michel) to handle products of 3 primes when the modulus has a convenient small factor.
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- Generalize ideas based on Deligne’s work (Fouvry, Kowalski, Michel) to handle products of 3 primes when the modulus has a convenient small factor.
- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.
As stated these ideas combine to give a result for $q \sim x^{1/2+\delta}$ for some small $\delta > 0$.

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- Generalize ideas based on Deligne’s work (Fouvry, Kowalski, Michel) to handle products of 3 primes when the modulus has a convenient small factor.
- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.
- Auxilliary estimate when there is a very small factor

Together these improve all terms in the decomposition, with a reasonable range of $q$!
Overview

Spectral Theory
- Fouvry-Kowalski-Michel style
- Bombieri-Friedlander-Iwaniec style
- Fouvry style
- Zhang style

Algebraic Geometry
- Fouvry style
- Products of 3 Primes
- Products of 5 Primes
- Product of 7 Primes
- Factor away from $x^{1/2}$
- Factor near $x^{1/2}$

Combinatorial Decomposition

Primes in APs

Figure: Outline of steps to prove primes in arithmetic progressions
Thank you for listening.