On the Synchrotron Spectrum of GRB Prompt Emission

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Abstract

The prompt emission spectrum of gamma-ray bursts is characterized by a smoothly joint broken power-law spectrum known as the Band function. The typical low-energy photon index is $\sim -1$, which poses a challenge to standard synchrotron radiation models. We investigate the electron energy spectrum as a result of the interplay among adiabatic stochastic acceleration (ASA), particle injection, and synchrotron cooling. In the ASA-dominated low-energy range, ASA enables an efficient hardening of the injected energy spectrum to approach a spectral index $-1$. In the synchrotron cooling-dominated high-energy range, the injected high-energy electrons undergo fast synchrotron cooling and have a softer photon spectrum. With the energy range of the injected electrons broadly covering both the ASA- and synchrotron cooling-dominated ranges, the resulting photon number spectrum has low- and high-energy indices of $\alpha_s \sim -1$ and $\beta_s \sim -p/2 - 1$, respectively. The break energy is of the order of $\sim 100$ keV, depending on the turbulence properties.

\textit{Key words:} acceleration of particles -- gamma-ray burst: general -- radiation mechanisms: non-thermal -- turbulence

1. Introduction

After decades of observations, the radiation mechanism of gamma-ray bursts (GRBs) is still subject to debate (e.g., Zhang 2014; Kumar & Zhang 2015; Pe’er 2015). Even though it is agreed that some GRBs have a narrow thermal-like spectrum (e.g., GRB 090902B) of Comptonized photospheric origin (e.g., Ryde et al. 2010; Pe’er et al. 2012), the origin of the so-called Band function component (Band et al. 1993; Preece et al. 2000) commonly observed in GRBs is still unclear. This is because the typical value of the low-energy photon index, $\alpha \sim -1$, is not straightforwardly expected from either the thermal model (Beloborodov 2010; Deng & Zhang 2014) or the synchrotron model (Rees & Meszaros 1992; Katz 1994; Tavani 1996). This motivates the exploration of alternative solutions to the problem (e.g., Brainerd 1994; Liang et al. 1997; Preece et al. 1998; Ghisellini et al. 2000; Medvedev 2000; Mészáros & Rees 2000; Pe’er & Zhang 2006; Daigne et al. 2011; Beniamini & Piran 2014; Uhm & Zhang 2014; Asano & Terasawa 2015; Asano & Mészáros 2016). 

Electron acceleration is the fundamental physical ingredient for understanding the photon spectrum of the prompt emission. Turbulence is naturally expected in the magnetized GRB outflow via the magnetic reconnection. Observational evidence for the turbulent GRB outflow can be found in, e.g., Beloborodov et al. (1998). The reconnection of regular magnetic fields can generate turbulence, which in turn facilitates rapid reconnection on macroscopic scales and determines the reconnection rate (Lazarian & Vishniac 1999; Kowal et al. 2017). Collision-induced turbulent reconnection in moderately high-$\sigma$ regimes provides an efficient dissipation mechanism for the magnetic energy in the GRB outflow (Zhang & Yan 2011; Deng et al. 2015). Advances in particle acceleration theory in magnetohydrodynamic (MHD) turbulence provide us with new insights into the GRB emission.

As a new stochastic acceleration mechanism identified by Brunetti & Lazarian (2016), electrons entrained on turbulent magnetic field lines, i.e., with Larmor radii smaller than the characteristic length scale of magnetic fields, are randomly accelerated and decelerated. This is different from gyroresonant interaction with MHD waves because (1) it originates from the second adiabatic invariant; (2) it accelerates non-resonant particles; and (3) the particles undergo the first-order Fermi process and thus a large fractional energy change occurs within individual turbulent eddies. Xu & Zhang (2017; hereafter Paper I) first applied this adiabatic stochastic acceleration (ASA) to the GRB context, showing that it can efficiently harden any initial energy distribution of electrons at low energies, where acceleration dominates over synchrotron cooling.

The observed GRB spectra during the prompt phase are characterized by a smoothly connected broken power law (Band et al. 1993; Preece et al. 2000). While the hard, low-energy spectrum can be accounted for by the ASA, as shown in Paper I, the soft spectrum at high energies, where the ASA is insignificant, should result from different physical processes. One important physical process that was not included in Xu & Zhang (2017) was fast synchrotron cooling of instantaneously accelerated electrons, which is believed to play a significant role in GRB physics (e.g., Sari et al. 1998; Uhm & Zhang 2014). Based on this knowledge and the findings in Paper I, in this paper we introduce the particle injection, which not only serves as the particle source for the ASA at low energies, but also contributes to developing the high-energy spectrum, together with synchrotron cooling. The injected particles can result from a separate instantaneous acceleration process. Therefore, apart from the continuous ASA and synchrotron cooling, we also incorporate the instantaneous acceleration through invoking particle injection. Our purpose is to model the interplay among different processes and investigate the physical origin of the synchrotron spectrum of GRB prompt emission.

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This paper is organized as follows. In Section 2, we analyze the time evolution of the electron energy spectrum under the effects of the ASA, injection, and radiation losses. In Section 3, we provide the synchrotron spectrum in both the slow-cooling and fast-cooling regimes, as well as the characteristic spectral parameters. A discussion and a summary of our results are provided in Sections 4 and 5, respectively.

2. Adiabatic Stochastic Acceleration and Steady Injection

To study the time evolution of an electron energy spectrum $N(E, t)$, we consider the kinetic equation of electrons

$$\frac{dN}{dt} = a_2 \frac{\partial}{\partial E} \left( E \frac{\partial N}{\partial E} \right) + \frac{\beta \frac{\partial (E^2 N)}{\partial E}}{E} + Q(E).$$

(1)

The first term on the right side accounts for the ASA introduced in Paper I. It arises in the trans-Alfvénic turbulence (Goldreich & Sridhar 1995) with balanced magnetic and kinetic energies. The acceleration is stochastic as the particles encounter the statistically equally distributed reconnecting and dynamo regions in MHD turbulence. Resulting from the second adiabatic invariant, they are accelerated when the magnetic field lines shrink during the turbulent reconnection, and decelerated when the field lines are stretched via the turbulent dynamo. On the other hand, the acceleration/deceleration within individual turbulent eddies is the first-order Fermi process, analogous with the particles being trapped between approaching/receding mirrors.

The acceleration rate $a_2$ is expressed as (Paper I)

$$a_2 = \xi \frac{u_{\text{tur}}}{l_{\text{tur}}},$$

(2)

where $l_{\text{tur}}$ and $u_{\text{tur}}$ are the injection scale and velocity of the trans-Alfvénic turbulence, and $\xi = \Delta E/E$ is the fractional energy change within each turbulent eddy-turnover time $\tau_{\text{tur}} = l_{\text{tur}}/u_{\text{tur}}$. In non-relativistic turbulence, $\tau_{\text{tur}}$ is sufficiently long for particles with Larmor radii $r_L$ smaller than $l_{\text{tur}}$ to have many crossings within individual eddies. The significant energy change during every interaction time $\tau_{\text{tur}}$ leads to a large energy diffusion rate

$$D_{\text{EE}} = \frac{\xi E^2}{\tau_{\text{tur}}},$$

(3)

and thus efficient acceleration, which is in contrast to the pitch-angle scattering of particles by MHD waves with a small energy change per collision. In the case of highly relativistic turbulence, one crossing in a turbulent eddy is efficient enough to drastically change the particle energy with $\xi$ of the order $\gamma_{\text{tur}}^2$. It comes from two Lorentz transformations, where $\gamma_{\text{tur}}$ is the turbulence Lorentz factor.

The second term on the right side of Equation (1) describes the synchrotron and synchrotron-self-Compton (SSC) losses. The parameter $\beta$ is defined as

$$\beta = \frac{P_{\text{syn}} + P_{\text{ssc}}}{E^2} = \frac{P_{\text{syn}}(1 + Y)}{E^2} = \frac{4}{3} \sigma_T e U_B \gamma_{\text{tur}}^2 (1 + Y) = \sigma_T e B^2 (1 + Y) \left( \frac{m_e c^2}{\gamma_{\text{tur}}^2} \right)^2 = \sigma_T e B^2 (1 + Y) \left( \frac{m_e c^2}{\gamma_{\text{tur}}^2} \right)^2.$$

(4)

where $P_{\text{syn}}$ and $P_{\text{ssc}}$ are the powers of the synchrotron and SSC radiation, respectively, with $Y$ as the ratio between $P_{\text{ssc}}$ and $P_{\text{syn}}$. Other parameters are the magnetic field strength $B$, the magnetic energy density $U_B = B^2/(8\pi)$, the Thomson cross section $\sigma_T$, the electron Lorentz factor $\gamma_e$, the electron rest mass $m_e$, and the speed of light $c$.

The third term in Equation (1) represents the steady injection of a constant electron spectrum $Q(E) = CE^{-p}$ with a power-law index $p$. At some fixed reference energy, the constant parameter $C$ corresponds to the number of injected electrons per unit energy per unit time. The steady injection of new particles was not included in Paper I. By taking it into account, we involve other possible instantaneous acceleration mechanisms that produce power-law electron spectra. The instantaneous acceleration processes, taking place in a separate region, provide the particle source in the magnetized and turbulent medium for further ASA, as described by the first term in Equation (1). The remaining terms in the general kinetic equation (see, e.g., Ginzburg 1957; Kardashev 1962; Chevalier et al. 1978) are excluded, as they are of marginal importance to our analysis.

In the Appendix, we present a more general analysis of the evolving electron energy distribution under the effects of both stochastic and systematic acceleration. It recovers the situation considered here when the stochastic acceleration is the dominant acceleration process in the turbulence region.

By substituting $EN = \exp(-\epsilon E)u(x, \tau)$, $\epsilon = \beta/a_2$, $x = ln E$, and $\tau = a_2 t$ into Equation (1), we have

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} - \frac{E}{E_{\text{cf}}} \frac{\partial u}{\partial x} + \frac{C}{a_2} \exp(\epsilon E)E^{-p+1},$$

(5)

where we define the cutoff energy for the ASA as

$$E_{\text{cf}} = \frac{a_2}{\beta} = \frac{1}{\epsilon},$$

(6)

corresponding to the balance between the ASA and the energy losses due to the radiation. Under the condition $E \ll E_{\text{cf}}$, Equation (5) approximately becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + C' \exp((-p + 1)x),$$

(7)

where $C' = C/a_2$. Equation (7) has the same form as an inhomogeneous heat equation. With the initial condition $u(x, 0) = 0$, its solution is given by (see Cannon 1984),

$$u(x, \tau) = C' \int_0^\tau \frac{1}{2\sqrt{\pi(\tau - s)}} \times \int_{y_s}^{y_e} \exp \left( -\frac{(x - y)^2}{4(\tau - s)} \right) \exp((-p + 1)y) dy ds,$$

(8)

where $y_i = \ln E_i$ and $y_e = \ln E_e$, with $E_i$ and $E_e$ as the lower and upper limits of the injected energy spectrum. We note that when $s = 0$, the above solution becomes

$$u^0(x, \tau) = u(x, \tau)|_{s=0} = \frac{C'}{2\sqrt{\pi}\tau} \times \int_{y_s}^{y_e} \exp \left( -\frac{(x - y)^2}{4\tau} \right) \exp((-p + 1)y) dy.$$

(9)
The corresponding electron energy spectrum is

\[ N^s(E, \tau) = E^{-1} \frac{C'}{2\sqrt{\pi} \tau} \exp \left( -\frac{E}{E_{cf}} \right) \times \int_{1}^{\infty} \exp \left( -\frac{(\ln E - y)^2}{4\tau} \right) \exp((-p + 1)y)dy. \]  

(10)

As discussed in Paper I, this is the case of impulsive injection with the initial condition

\[ N^s(E, 0) = C' E^{-p} \exp \left( -\frac{E}{E_{cf}} \right), \quad E_i < E < E_a. \]  

(11)

For the steady injection considered here, Equation (8) can be treated as the superposition of a series of \( u^s(x, \kappa) \) (Equation (9)), with \( \kappa \) ranging from 0 to \( \tau \), i.e.,

\[ u(x, \tau) = \int_{0}^{\tau} u^s(x, \kappa) d\kappa. \]  

(12)

When \( \tau \) is small, by integrating the simplified form of \( u^s(x, \kappa) \) at the limit of small \( \kappa \),

\[ u^s(x, \kappa)_{small \kappa} = \frac{C'}{2\sqrt{\pi} \kappa} \exp((-p + 1)x), \]  

(13)

Equation (12) yields

\[ u(x, \tau) = \frac{C' \sqrt{\tau}}{\sqrt{\pi}} \exp((-p + 1)x). \]  

(14)

Then we obtain the short-time asymptotic energy spectrum

\[ N(E, \tau) = \frac{C' \sqrt{\tau}}{\sqrt{\pi}} E^{-p} \exp \left( -\frac{E}{E_{cf}} \right) \]  

(15)

The spectral form is governed by the injected particle distribution. Notice that different from the situation with an impulsive source \( N^s(E, \tau) \) in Equation (10), the above expression shows \( N(\tau) \propto \sqrt{\tau} \). It means that the steady injection leads to an increase of the electron number per unit energy with time.

When \( \tau \) is sufficiently large, we integrate the reduced form of \( u^s(x, \kappa) \) (Equation (9)) at the limit of a large \( \kappa \), which is

\[ u^s(x, \kappa)_{large \kappa} = \frac{C'}{2(-p + 1) \sqrt{\pi} \kappa} \times \left[ \exp((-p + 1)y_u) - \exp((-p + 1)y_l) \right], \]  

(16)

and find (Equation (12)),

\[ u(x, \tau) = \frac{C' \sqrt{\tau}}{(-p + 1) \sqrt{\pi}} \times \left[ \exp((-p + 1)y_u) - \exp((-p + 1)y_l) \right]. \]  

(17)

So at a long time, the energy spectrum asymptotically approaches

\[ N(E, \tau) = \frac{C'(E_u^{p+1} - E_l^{p+1}) \sqrt{\tau} E^{-1} \exp \left( -\frac{E}{E_{cf}} \right)}{(-p + 1) \sqrt{\pi}}. \]  

(18)

The ASA plays a dominant role in regulating the spectral form and the resulting spectral index converges to a universal value of \(-1\) at \( E < E_{cf} \), irrespective of the injected power-law index \( p \). This is valid for both impulsive and steady injection, but obviously in the latter situation it requires relatively longer time for the overall spectrum to comply with the universal power law.

We present the above analytically derived approximate results in Figure 1, in good agreement with the numerical solutions to Equation (5) at \( E < E_{cf} \). For the numerical illustration, we adopt a steep spectral slope \( p = 2.2 \) of the injected electrons and \( C' = 10 \) in arbitrary units. The value of \( C' \) does not affect the spectral form, but only the amplitude of the spectrum. \( E \) is normalized by \( E_{cf} \), and \( N(E, \tau) \) is in arbitrary units. The numerical results indeed confirm that at an early time, the form of \( N(E, \tau) \) is dictated by the injected spectrum. At a later time, i.e., several tens of 1/\( \alpha_{d}\), a hard spectrum with an index approaching \(-1\) is developed below \( E_{cf} \) under the influence of the ASA.

3. Distribution of High-energy Electrons and the Synchrotron Spectrum

3.1. Slow-cooling and Fast-cooling Regimes

The electrons injected into the trans-Alfvénic turbulence undergo the ASA, and consequently their original power-law distribution evolves into a hard spectrum. The hard spectrum can only extend over the energy range below \( E_{cf} \), since above \( E_{cf} \), the ASA becomes ineffective due to the more important radiation losses of electrons. Therefore, in the following analysis of the high-energy portion of the electron spectrum, we assume that it is only shaped by the particle injection and synchrotron losses (\( \gamma = 0 \)). The narrow relativistic Maxwellian distribution that can be piled up above \( E_{cf} \) (Schlickeiser 1984, 1985) is neglected.

The high-energy electron spectrum falls in the cooling regimes, as discussed in Sari et al. (1998). To evaluate the synchrotron cooling effect, we define a critical energy \( E_c \),

\[ E_c = P_{syn} t. \]  

(19)
Combining it with Equation (4), there is
\[ E_c = \frac{1}{\beta t}, \] (20)
which decreases with time \( t \). The electrons with \( E > E_c > E_{ct} \) cool down to \( E_c \) within the time \( t \).

Depending on the relation between \( E_c \) and \( E_t \), which is the lower limit of the energies of the injected electrons, we next separately discuss the slow-cooling regime with \( E_c > E_t \) and the fast-cooling regime with \( E_c < E_t \).

(1) Slow-cooling: the energy spectrum of the electrons with \( E > E_{ct} \) follows the kinetic equation
\[ \frac{\partial N}{\partial t} = \beta (E^2 N) + CE^{-\nu}, \] (21)
where the acceleration term is absent. With the initial condition \( N(E, 0) = 0 \), its solution was provided by Kardashev (1962) as
\[ N(E, t) = \frac{CE^{-(p+1)}}{\beta (p-1)} [1 - (1 - \beta t)^{p-1}], \] (22)
which is asymptotically a broken power law,
\[ N(E, t) \approx \frac{CE}{\beta \nu} E^{-\nu}, \quad E_{ct} < E \ll E_c, \] (23a)
\[ N(E, t) \approx \frac{CE^{-(p+1)}}{\beta (p-1)} E \ll E_c. \] (23b)
This spectrum can only arise at high energies above \( E_{ct} \) with negligible acceleration effects.

Depending on the time \( t \), the relation between \( E_{ct} \) and \( E_c \) varies. Correspondingly, there exist two scenarios for the electron energy spectrum in the slow-cooling regime, as shown in Figures 2(a) and (c), respectively.

At an early time with \( E_c > E_{ct} \) (Case (i), see Figure 2), i.e., \( t < 1/a_2 \), the time is so insufficient that only the very high-energy electrons undergo significant synchrotron cooling and have their distributions comply with Equation 23(b). With the decrease of \( E_c \) with time, the spectral steepening extends to lower energies. But as soon as \( t > 1/a_2 \), i.e., \( E_c < E_{ct} \), is reached (Case (ii), see Figure 2(c)), the ASA dominates over the synchrotron cooling, and no further steepening is expected below \( E_{ct} \). Additionally, as a result of the energy diffusion facilitated by the ASA, the minimum energy \( E_m \) is
\[ E_m = E_t \exp(-2\sqrt{\tau}), \] (24)
which is smaller than the lower bound \( E_t \) to the energy range of the injected electrons. Therefore, the injected electron spectrum is extended over a broader energy range.

Given the electron energy spectrum
\[ N(E) \propto E^{-\nu}, \] (25)
the synchrotron photon number spectrum in terms of frequency \( \nu \) is (Rybicki & Lightman 1979)
\[ N(\nu) \propto \nu^\alpha, \quad \alpha = -\frac{p+1}{2}. \] (26)
The flux \( F_\nu \propto \nu N(\nu) \) corresponding to the above two cases are Case (i) \( E_t < E_{ct} < E_c \),
\[ F_\nu = F_{\nu, max} \left( \frac{\nu}{\nu_m} \right)^{-\frac{1}{2}}, \quad \nu < \nu_m, \] (27a)
time dependences (Equations (20) and (24)), we see that after reaching below $E_{cf}$, $E_m$ can soon become smaller than $E_c$.

The corresponding flux in the two cases takes the following forms (see Figures 3(b) and (d)).

Case (i) $E_c < E_{cf} < E_l$,

\begin{align}
F_\nu &= F_{\nu,\text{max}} \left( \frac{\nu}{\nu_m} \right)^{1/p}, \quad \nu < \nu_m, \quad (31a) \\
F_\nu &= F_{\nu,\text{max}}, \quad \nu_m < \nu < \nu_{cf}, \quad (31b) \\
F_\nu &= F_{\nu,\text{max}} \left( \frac{\nu}{\nu_{cf}} \right)^{-q}, \quad \nu_{cf} < \nu < \nu_l, \quad (31c) \\
F_\nu &= F_{\nu,\text{max}} \left( \frac{\nu_l}{\nu_{cf}} \right)^{1/q} \left( \frac{\nu}{\nu_l} \right)^{-q}, \quad \nu_l < \nu; \quad (31d)
\end{align}

and Case (ii) $E_l < E_c < E_{cf}$,

\begin{align}
F_\nu &= F_{\nu,\text{max}} \left( \frac{\nu}{\nu_m} \right)^{1/p}, \quad \nu < \nu_m, \quad (32a) \\
F_\nu &= F_{\nu,\text{max}}, \quad \nu_m < \nu < \nu_{cf}, \quad (32b) \\
F_\nu &= F_{\nu,\text{max}} \left( \frac{\nu}{\nu_{cf}} \right)^{-q}, \quad \nu_{cf} < \nu. \quad (32c)
\end{align}

For the $\nu F_\nu$ spectrum, if $p > 2$, it peaks at $\nu_l$ in Case (i) and at $\nu_{cf}$ in Case (ii), depending on the energy range of the injected electrons.

3.2. Characteristic Spectral Parameters

The time-averaged GRB spectra are typically modeled by the empirical Band spectrum (Band et al. 1993), which is characterized by a low-energy spectral index $\alpha_s$, a break energy $E_b$, and a high-energy spectral index $\beta_b$. As shown by Preece et al. (2000) based on a large burst sample, the indices $\alpha_s$ and $\beta_b$ have their distributions centered on $-1$ and $-2.25$, respectively. The break energy at the transition is on the order of $100 \text{ keV}$, which corresponds to the peak energy $E_p$ in $\nu F_\nu$ spectrum if $\beta_b < -2 < \alpha_s$.

For the prompt phase of a GRB in the fast-cooling regime, by comparing the above analysis with the observed spectral characteristics, we find that Case (ii) in the fast-cooling regime can satisfactorily reproduce the spectral behavior revealed by observations. The hard low-energy spectrum $N(\nu)$ with $\alpha_s \sim -1$ naturally arises over the range $(\nu_m, \nu_{cf})$ out of the ASA. The slope of the injected energy spectrum inferred from $\beta_s = -2.25$ is $p = 2.5$. This is close to the typical spectral index of the accelerated electrons in relativistic shocks ($p = 2.2$), as well as that resulting from the reconnection acceleration ($p = 2.5$) (de Gouveia dal Pino & Lazarian 2005).
The peak energy \( E_p \) (i.e., \( E_0 \)) of the synchrotron spectrum is determined by the cutoff energy \( E_{\text{cf}} \) of the electron energy spectrum. Its evaluation was discussed in Paper I, which is again presented here for the sake of completeness. By combining Equations (2), (4), and (6), \( E_{\text{cf}} \) has the expression

\[
E_{\text{cf}} = \frac{6\pi \xi (m_e c^2)^2}{\sigma_T B^2 I_{\text{tur}}}.
\]  

(33)

where \( u_{\text{tur}} \approx c \) for the relativistic turbulence. Furthermore, we consider that turbulence is mainly driven by the magnetic reconnection, with the turbulent energy converted from the magnetic energy released during the magnetic reconnection. The turbulence becomes trans-Alfvénic when the growing turbulent energy reaches equipartition with the magnetic energy. According to the evolutionary behavior of the resulting turbulence (Kowal et al. 2017) and the expected global decay of magnetic field strength in the emission region due to adiabatic expansion of the jet (Pe'er & Zhang 2006; Uhm & Zhang 2014; Zhao et al. 2014), we assume that they are anticorrelated by

\[
I_{\text{tur}} = l_0 \left( \frac{B}{B_0} \right)^{-\zeta}, \quad \zeta > 0.
\]  

(34)

with normalization parameters \( l_0 \) and \( B_0 \). Using the above relation, we can further write Equation (33) as

\[
E_{\text{cf}} = \frac{6\pi \xi (m_e c^2)^2}{\sigma_T l_0 B_0^2}. \quad \text{(35)}
\]

The corresponding photon energy in the observer frame is

\[
E_{\nu,\text{cf,obs}} = (h\nu_{\text{cf}})_{\text{obs}} = \frac{h\nu_{\text{cf}}}{m_e c^2} \gamma_{e,\text{cf}}^2 \Gamma(1 + z)^{-1}
\]

\[
= \frac{2^{-\frac{1}{2}}}{\sqrt{m_e c} e^{-\frac{1}{2}}} \left( \frac{6\pi \xi}{\sigma_T l_0 B_0^2} \right)^2 \times \Gamma^{-2\zeta+4}(1 + z)^{-1} L_{L^2}^{-\frac{1}{2}} r^{-2\zeta+3}, \quad \text{(36)}
\]

with \( \gamma_{e,\text{cf}} = E_{\text{cf}}/(m_e c^2) \), the Planck constant \( h \), the electron charge \( e \), the redshift \( z \), and the substitution (Zhang & Mészáros 2002),

\[
B = \frac{2}{\sqrt{c}} L_{L^2}^{\frac{1}{2}} r^{-1} \Gamma^{-1}. \quad \text{(37)}
\]

By adopting \( \Gamma = 100 \) for the bulk Lorentz factor, \( L = 10^{32} \text{ erg s}^{-1} L_{32} \) for the total isotropic luminosity, \( r = 10^{15} \text{ cm} r_{15} \) for the radius of the emission region, \( \zeta \approx 2.1 \) for
based on the observational constraints (Liang et al. 2015), as well as $\xi = 10^4$, $B_0 = 10^5$ G, $l_0 = 2 \times 10^9$ cm, we can then evaluate $E_{\rm{s,cf,obs}}$.

$$E_{\rm{s,cf,obs}} \simeq 385 \, \text{keV} \left( \frac{1 + \zeta}{2} \right)^{-1} \Gamma_2^{-0.2} L_5^{0.6} r_\zeta^{-1.2},$$  

which is compatible with the observationally revealed peak energy.

One necessary condition for Case (ii) in the fast-cooling regime is $E_c < E_{\rm{cf}}$, which is equivalent to (Equations 6, 20)

$$t > \frac{1}{\alpha_2} = \frac{l_{\rm{dur}}}{\zeta l_{\rm{dur}}},$$  

(39)

Using Equations (34) and (37), we find

$$t > \frac{l_0 B_0}{\zeta c} \left( \frac{g}{c} \right)^{\frac{3}{2}} L_2^{\varphi_T} \Gamma_2^{-\zeta}$$

$$= 1.3 \times 10^{-3} s L_5^{0.5} r_\zeta^{0.2} \Gamma_2^{1.1},$$  

(40)

t = \text{as the time required for } E_c \text{ to decrease below } E_{\rm{cf}}. \text{ In the observer frame, it becomes}

$$t_{\rm{obs}} > 1.3 \times 10^{-5} s L_5^{0.5} r_\zeta^{0.2} \Gamma_2^{1.1},$$  

(41)

which is so short that we can safely assume that the relation $E_c < E_{\rm{cf}}$ always holds.

The other condition $E_l < E_{\rm{cf}}$ implies that the energy range of the injected electrons should be broad enough to reach down to $E_{\rm{cf}}$. Given the estimate of $E_{\rm{cf}}$, there is (Equations (35), (37))

$$E_l < E_{\rm{cf}} = \frac{6 \pi \xi (m_e c^2)^2}{\sigma T l_0 B_0^2} \left( \frac{g}{c} \right)^{\frac{3}{2}} L_2^{\varphi_T} \Gamma_2^{-\zeta}$$

$$= 4.6 \, \text{GeV} L_5^{0.05} r_\zeta^{-0.1} \Gamma_2^{-0.1},$$  

(42)

Starting from $E_l$, the minimum energy of electrons further decreases with time driven by the ASA. Following Equation (24), we can write

$$E_m = E_l \exp(-2\sqrt{\Omega_T t_l})$$

$$= E_l \left[ -\left( \frac{\xi c}{l_0 B_0} \right)^{\frac{3}{2}} L_2^{\varphi_T} \Gamma_2^{-\zeta} \right] t_l^{\frac{1}{2}}$$

$$= E_l \exp(-5.58 L_5^{0.525} r_\zeta^{-0.5} \Gamma_2^{-0.55} t_{\rm{obs},0}),$$  

(43)

where $t_{\rm{obs},0} = 1$ s, and the expression of $\alpha_2$ can be extracted from Equations (39) and (40). We see that $E_m$ rapidly decreases to a negligibly small value compared with $E_l$, forming an extended hard-energy spectrum of electrons. It leads to the photon energy

$$E_{\gamma,\ell,\text{obs}} \simeq h \frac{e B}{m_e c} \gamma_{\ell,\text{obs}}^{-2} (1 + z)^{-1}$$

$$= E_{\ell,\text{obs}} \exp(-1.1 \times 10^3 L_5^{0.525} r_\zeta^{-1.05} \Gamma_2^{-0.55} t_{\text{obs},0}),$$  

(44)

where

$$E_{\gamma,\ell,\text{obs}} \simeq h \frac{e B}{m_e c} \gamma_{\ell,\text{obs}}^{-2} (1 + z)^{-1},$$  

(45)

with $\gamma_{m} = E_m/(m_e c^2)$ and $\gamma_{\ell,\text{obs}} = E_l/(m_e c^2)$. The time evolution of $E_m$ discussed above is solely affected by the ASA. If there is additional systematic acceleration that also comes into play in the emission region, its behavior can be found in the Appendix.

4. Discussion

In the standard model for synchrotron emission from shock-accelerated electrons, the sites for instantaneous acceleration and radiation are separate, under the assumptions of a highly efficient acceleration and thus a negligible cooling effect at relativistic shocks. In a more realistic situation, the instantaneously accelerated electrons, after being injected into the magnetized and turbulent medium, are likely to experience further continuous reacceleration and radiation simultaneously (Lloyd & Petrosian 2000; Asano & Terasawa 2009, 2015). In the model discussed in Paper I and this paper, we introduce the ASA for the electrons injected into the trans-Alfvénic turbulence. This diffusive acceleration acts against the synchrotron cooling and generates a hard spectrum up to the cutoff energy with balanced acceleration and cooling rates (Equation (6)). The stochastic nature of the ASA originates from the equipartition between the turbulent kinetic and magnetic energies in the trans-Alfvénic turbulence, and it does not impose the resonance condition on particles to be accelerated. In contrast, the conventionally adopted stochastic acceleration through the pitch-angle scattering of particles by magnetic fluctuations is disregarded here, because the resonant condition may not be satisfied with a small Larmor radius of particles at a strong magnetization. Moreover, even in the presence of sufficiently small-scale magnetic turbulence, the resonant scattering is very inefficient due to the prominent turbulence anisotropy, especially at small scales (Yan & Lazarian 2002). At high energies away from the cutoff energy, without significant ASA a steep spectrum of the injected electrons arises under the fast-cooling effect.

As discussed above, the underlying physical process of particle injection is an instantaneous acceleration in a separate region from the ASA region. The possible injection mechanisms include the shock acceleration as adopted in the classical GRB model (Rees & Meszaros 1994), which has the acceleration timescale of the particle gyroperiod (Bykov & Meszaros 1996; Bednarz 2000). Other possible injection mechanisms are the reconnection acceleration (e.g., de Gouveia dal Pino & Lazarian 2005), or the hadronic injection via $pp$ and $p\gamma$ reactions (e.g., Murase et al. 2012). The instantaneous acceleration should be able to produce a power-law spectrum of electrons over a relatively broad energy range with lower and upper energy bounds below and above $E_{\text{cf}}$, respectively (see Figure 3(c)). A detailed model for the injection mechanism that connects both the energy dissipation and particle acceleration will be constructed in our future work.

Although the ASA can successfully explain the typical value of the low-energy spectral index $\alpha = -1$, observations suggest that $\alpha$ has a wide distribution around $-1$ (Ghirlanda et al. 2003). Lloyd & Petrosian (2000) pointed out that the dispersion in $\alpha$ can come from the finite bandwidth of the instrument and the spectral fitting effects. Note that $\alpha = -1$ in our analysis is given in the asymptotic limit below the cutoff frequency $f_{\text{cut}}$. Above and near $f_{\text{cut}}$, the spectrum can be piled up due to the interplay between acceleration and cooling.
of death of the synchrotron emission, different physical models based on, e.g., synchrotron self-absorption (Preece et al. 1998), small pitch-angle synchrotron emission (Lloyd & Petrosian 2000; Lloyd-Ronning & Petrosian 2002), or inverse Compton scattering (Liang et al. 1997), have been developed for understanding the spectral hardness.

A caveat in our analysis is that the decay of B was not incorporated in the kinetic equation of electrons, but only added later in our evaluation of the characteristic spectral parameters (Section 3.2). In a more realistic scenario, Uhm & Zhang (2014; see also Geng et al. 2017) showed that a decaying B can affect the fast-cooling process and results in a hard-energy spectrum of the cooled electrons with an index of $\sim -1$ instead of $-2$. Considering this, a hard electron spectrum would arise within the energy range ($E_{\text{cf}}, E$) in Case (i) in the fast-cooling regime, which provides an alternative explanation for the ASA function without invoking the ASA at lower energies. Note that the ASA-generated electron spectrum would also be modified if the temporal dependence of B is taken into account in Equation (1). These effects will be addressed in future work.

5. Summary

In an attempt to fully understand the origin of the GRB Band function within the framework of the synchrotron radiation mechanism, we have analyzed the time evolution of the injected electron spectrum under the effects of both the ASA and synchrotron cooling. Their relative importance depends on the energy range of interest.

The cutoff energy $E_{\text{cf}}$ corresponds to the balance between the ASA and the synchrotron cooling. At energies below $E_{\text{cf}}$, the ASA plays a dominant role and ensures rapid development of a flat energy distribution of electrons, in spite of the steady injection of a steep electron spectrum.

At higher energies, the synchrotron cooling effect manifests. The injected high-energy electrons in the slow-cooling and fast-cooling regimes evolve to various spectral forms, which are all steeper than the low-energy spectral component.

Among different scenarios in the fast-cooling regime, if the lower energy limit of the injected electrons is below $E_{\text{cf}}$ (Equation (42)), the ASA can efficiently harden the injected distribution below $E_{\text{cf}}$ and spread it out to further lower energies. The resulting low-energy and high-energy spectral components can together reproduce the Band spectrum of GRBs, with $E_{\text{cf}}$ corresponding to the break energy.

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Appendix

The Electron Energy Spectrum Affected by Both Stochastic and Systematic Acceleration

The kinetic equation including both the stochastic and systematic acceleration processes is

$$\frac{\partial N}{\partial t} = a_1 \frac{\partial}{\partial E} \left( E^2 \frac{\partial \langle N \rangle}{\partial E} \right) - a_1 \frac{\partial \langle N \rangle}{\partial E},$$

where $a_1$ accounts for the systematic Fermi acceleration. By again substituting the variables $f = EN$, $x = \ln E$, $\tau = a_2 t$, the above equation can be rewritten as

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2} - a_2 \frac{\partial f}{\partial x},$$

where $a_1 = a_1 / a_2$. Following algebra similar to that in Melrose (1969), by introducing

$$X = x - a_2 \tau,$

we find

$$\frac{\partial f}{\partial \tau} = \frac{\partial f(X, \tau)}{\partial \tau} - a_2 \frac{\partial f(X, \tau)}{\partial X},$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial f(X, \tau)}{\partial x}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f(X, \tau)}{\partial X^2}.$$

Then, Equation (47) can be further simplified as

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial X^2}.$$ (51)

The solution to this diffusion equation is (e.g., Evans 1998)

$$f(X, \tau) = \frac{1}{2 \sqrt{\pi \tau}} \int_{y_1}^{y_2} \exp \left[ - \frac{(X - y)^2}{4 \tau} \right] f(y, 0) dy.$$ (52)

The initial condition is given by $f(y, 0)$ within the finite range $(y_1, y_2)$, where $y_1 = \ln E_1$ and $y_2 = \ln E_2$. Therefore, we obtain the energy spectrum as (see e.g., Kardashev 1962; Melrose 1969)

$$N(E, \tau) = E^{-1} \frac{1}{2 \sqrt{\pi \tau}} \int_{y_1}^{y_2} \exp \left[ - \frac{(\ln E - a_2 \tau - y)^2}{4 \tau} \right] f(y, 0) dy.$$ (53)

When $a_1 = 0$, i.e., $a_1 = 0$, it recovers the result with only the stochastic acceleration being considered (see Equation (10)). When $a_2 > 0$, given the time dependence of the peak position of the Gaussian function in Equation (53), we see that electrons initially at $E_0$ spread out in energy space and can reach $E_\tau$ after a time $\tau$,

$$E_\tau = E_0 \exp(\pm 2 \sqrt{\tau} + a_2 \tau).$$ (54)

By taking logarithms of both sides, this can be expressed in a more physically transparent way:

$$\ln E_\tau = \ln E_0 \pm 2 \sqrt{\tau} + a_2 \tau.$$ (55)

Differing from the case of purely stochastic acceleration, which leads to a random walk in the $\ln E$ space, the combination of both the stochastic and systematic acceleration drives an
asymmetric diffusion about $\ln E_0$. With a linear dependence on time, the second term in the exponential function in Equation (54) becomes dominant over the first one at a large $\tau$, irrespective of the value of $a_{12}$ ($a_{12} > 0$). More specifically, when $\tau > 4/a_{12}^2$, there is always $E_\tau > E_0$, and the lower limit of the energy $E_{\infty}$ increases with time.

The asymptotic behavior of $N(E, \tau)$ at a sufficiently large $\tau$ is

$$N(E, \tau) \approx E^{-1} \frac{1}{2\sqrt{\pi \tau}} \exp \left( -\frac{a_{12}^2}{4} \right) \int_0^{\infty} f(y, 0) dy. \quad (56)$$

Compared with the spectrum governed only by stochastic acceleration, under the combined effect of the two acceleration mechanisms the spectrum is still flat, but the number of electrons exponentially decreases as they shift to higher energies.

When the synchrotron cooling is taken into account, around the energy with comparable acceleration and cooling rates, it acts in concert with the systematic acceleration to narrow the energy range and pile up the electron distribution.

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