RECIPIROCY SHEAVES
AND MOTIVES WITH MODULUS

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ABSTRACT. We exhibit an intimate relationship between “motives with modulus” from [10] and “reciprocity sheaves” from [9]. Assuming resolution of singularities, we prove that the triangulated category of effective motivic complexes $\text{MDM}^{\text{eff}}$ admits a $t$-structure whose heart is equivalent to the category of reciprocity Nisnevich sheaves. We also give an isomorphism between some Hom groups in $\text{MDM}^{\text{eff}}$ and the hypercohomology of some Suslin complexes with modulus. As an application, we prove some Mayer-Vietoris sequences for Suslin homology with modulus.

Contents

Introduction 1
1. Review of basic definitions and results 6
2. Complements on interval structures 8
3. $\square$-invariance and SC-reciprocity 18
4. Relation with [9] 30
References 39

Introduction

This paper is a synthesis of [9] and [10], using the results of [15]. In particular, the latter imply [9, Conj. 1] under suitable hypotheses on the base field $k$.

In [9], we introduced reciprocity (pre)sheaves as a generalization of Voevodsky’s homotopy invariant (pre)sheaves with transfers, which are the main building block for constructing his triangulated categories of

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motives in [19]. Let $\text{Sm}$ be the category of separated smooth schemes over $k$. There is an additive category $\text{Cor}$ which has the same objects as $\text{Sm}$ and whose morphisms are finite correspondences; the category $\text{PST}$ of presheaves with transfers is defined as the additive dual of $\text{Cor}$ [12, Lect. 1 and 2]. A presheaf with transfers $F$ is $A^1$-invariant if the projection $X \times A^1 \to X$ induces an isomorphism $F(X) \cong F(X \times A^1)$ for all $X \in \text{Sm}$. Let $\text{HI} \subset \text{PST}$ be the full subcategory of $A^1$-invariant presheaves with transfers. The reciprocity presheaves defined in [9] form a full subcategory $\text{Rec} \subset \text{PST}$, which contains $\text{HI}$.

In this paper, we introduce a new full subcategory $\text{RSC} \subset \text{PST}$ which is fairly close to $\text{Rec}$ and fits better to the new framework of modulus presheaves of transfers. The latter were introduced in [10] to construct a new triangulated category $\text{MDM}^{\text{eff}}$ of motivic nature which enlarges Voevodsky’s triangulated category of motives $\text{DM}^{\text{eff}}$ [12, Lect. 14], [4]. Here we prove by aid of [15] some basic properties of $\text{RSC}$ and use them to relate $\text{Rec}$ to $\text{MDM}^{\text{eff}}$, assuming (unfortunately) resolution of singularities.

To give an idea of how we define $\text{Rec}$ and $\text{RSC}$, we need to reformulate the definition of $A^1$-invariance. Recall [12, Lem. 2.16] that the inclusion $\text{HI} \to \text{PST}$ has a left adjoint $h_{A^1}^0 : \text{PST} \to \text{HI}$. Thus $F \in \text{PST}$ is in $\text{HI}$ if and only if for any $X \in \text{Sm}$ and $a \in F(X)$, the map $Z_{\text{tr}}(X) \to F$ in $\text{PST}$ associated to $a$ by Yoneda’s lemma factors through $h_{A^1}^0(X) := h_{A^1}^0(Z_{\text{tr}}(X))$, where $Z_{\text{tr}}(X)$ is the presheaf with transfers represented by $X$. To define reciprocity presheaves, we introduced bigger quotients $h(M)$ of $Z_{\text{tr}}(X)$ associated to a modulus pair $M = (X, X^\infty)$, consisting of a proper scheme $\overline{X}$ over $k$ and an effective Cartier divisor $X^\infty$ on it, such that $X = \overline{X} \setminus |X^\infty|$. Then a presheaf with transfers $F \in \text{PST}$ belongs to $\text{Rec}$ [9, Definition 2.1.3] if

(*) For any quasi-affine $X \in \text{Sm}$ and any $a \in F(X)$, the associated map $a : Z_{\text{tr}}(X) \to F$ factors through $h(M)$ for some $M$ as above.

The definition of the quotients $h(M)$ is very technical; it is inspired by the theorem of Rosenlicht-Serre on reciprocity for morphisms from curves to commutative algebraic groups [16, Ch. III].

Let us now recall the story of [10]. We defined a category $\text{MCor}$: its objects are modulus pairs as above, and its morphisms are finite correspondences satisfying an admissibility condition with respect to $X^\infty$ (see Definition 1.1.1). Let $\text{MPST}$ be the additive dual of $\text{MCor}$. There is a pair of adjoint functors

$$\text{MPST} \xrightarrow{\omega^!} \text{PST}. $$
Here $\omega^*$ is induced by the functor

$$\omega : \text{MCor} \to \text{Cor} : (\overline{X}, X^\infty) \mapsto \overline{X} \setminus |X^\infty|,$$

and $\omega_!$ is the left Kan extension of $\omega$. Let now $\square = (\mathbb{P}^1, \infty) \in \text{MCor}$: we say that $F \in \text{MPST}$ is $\square$-invariant if the “projection” $M \otimes \square \to M$ induces an isomorphism $F(M) \cong F(M \otimes \square)$ for all $M \in \text{MCor}$ (see §1.2 for the monoidal structure $\otimes$ on $\text{MCor}$). We let $\text{CI} \subset \text{MPST}$ denote the full subcategory of $\square$-invariant objects.

We show in Proposition 3.2.6 that the inclusion $\text{CI} \to \text{MPST}$ has a left adjoint $h_0^\square : \text{MPST} \to \text{CI}$. Define $h_0^M (M) \in \text{PST}$ to be $\omega_! h_0^\square \mathbb{Z}_{\text{tr}} (M)$, where $\mathbb{Z}_{\text{tr}} (M) \in \text{MPST}$ is the presheaf represented by $M$. Then $\text{RSC}$ is the full subcategory of $\text{PST}$ consisting of those presheaves verifying Condition (*) above, modified by dropping the quasi-affine condition on $X$ and replacing $h(M)$ by $h_0(M)$.

We have $\text{HI} \subset \text{RSC}$ (Corollary 3.4.3). The following results do not depend on [15].

**Theorem 1.**

1. (Th. 3.4.2 and Prop. 3.4.6) $\omega_!(\text{CI}) = \text{RSC}$.
2. (Th. 4.2.1) Let $M = (\overline{X}, Y) \in \text{MCor}$ be such that $X := \overline{X} \setminus |Y|$ is quasi-affine. Then $h_0(M) = h(M)$.

Consequently, we have $\text{RSC} \subset \text{Rec}$.

In [19], Voevodsky used $\text{Cor}$ and $\text{PST}$ as his main building blocks to construct triangulated categories of motives $\text{DM}_{\text{gm}}^{\text{eff}} \subset \text{DM}^{\text{eff}}$, where $\text{DM}^{\text{eff}}$ sits as the full subcategory of $\mathbb{A}^1$-local objects in $D(\text{NST})$, the derived category of Nisnevich sheaves with transfers [4]. In exact parallel, we used $\text{MCor}$ and $\text{MPST}$ in [10] as the main building blocks to construct triangulated categories of motives with modulus $\text{MDM}_{\text{gm}}^{\text{eff}} \subset \text{MDM}^{\text{eff}}$; $\text{MDM}^{\text{eff}}$ sits as the full triangulated subcategory of $\square$-local objects in the derived category $D(\text{MNST})$, where $\text{MNST} \subset \text{MPST}$ is the category of Nisnevich modulus sheaves with transfers [10, Th. 6.9.4].

The reader should be warned that, in contrast to [19], the definition of the subcategory $\text{MNST} \subset \text{MPST}$ is highly sophisticated and involves a larger category $\text{MCor}$ of not necessarily proper modulus pairs: see the introduction of [10] and Subsection 3.10 below.

For the sequel, let us consider the following condition, which holds if $k$ is of characteristic zero.

---

1. More accurately, a bounded above version of $\text{DM}^{\text{eff}}$; the latter is studied in [4].
For any \( M = (\bar{X}, X_\infty) \in \text{MCor} \), there exists a proper birational morphism \( p : \bar{X}' \to \bar{X} \) such that \( \bar{X}' \in \text{Sm} \), \( p \) is an isomorphism over \( \bar{X} \setminus |X_\infty| \), and the reduced part of \( p^{-1}(X_\infty) \) is a simple normal crossing divisor.

Using \([15, \text{Th. 0.8}]\), we prove the following parallel to the main results of \([19]\):

**Theorem 2.** Assume (RS).

1. (Th. 3.7.1). The standard \( t \)-structure of \( D(\text{MNST}) \) induces a \( t \)-structure on \( \text{MDM} \). Its heart is \( \text{CI}_{\text{Nis}} := \text{CI} \cap \text{MNST} \), which is a Serre subcategory of \( \text{MNST} \), and we have

\[
\text{MDM} = \{ D \in D(\text{MNST}) \mid H^i(D) \in \text{CI}_{\text{Nis}} \text{ for all } i \in \mathbb{Z} \}.
\]

2. (Th. 3.10.2). For \( \mathcal{X}, \mathcal{Y} \in \text{MCor} \) with \( \mathcal{X} = (\bar{X}, X_\infty) \), we have a canonical isomorphism

\[
\text{Hom}_{\text{MDM}_{\text{gm}}} (M(\mathcal{X}), M(\mathcal{Y})[i]) \simeq H^{-i}(\bar{X}_{\text{Nis}}, C_{\bullet}(\mathcal{Y})_{\mathcal{X}}) \quad (i \in \mathbb{Z}).
\]

Here \( M : \text{MCor} \to \text{MDM}_{\text{gm}} \) is the motive functor, \( C_{\bullet}(\mathcal{Y}) \in C(\text{MPST}) \) is the Suslin complex attached to \( \mathcal{Y} \) (see Definition 3.2.4) and \( C_{\bullet}(\mathcal{Y})_{\mathcal{X}} \) is a complex of Nisnevich sheaves on \( \bar{X} \) canonically attached to \( C_{\bullet}(\mathcal{Y}) \) \([10, \text{Not. 3.6.1}]\).

We note that \( C_{\bullet}(\mathcal{Y})_{\mathcal{X}} \) depends not only on \( \bar{X} \) but also on \( X_\infty \). (2) refines \([10, \text{Theorem 3}]\), which used a “derived Suslin complex” \( RC_{\bullet}(\mathcal{Y}) \) instead of \( C_{\bullet}(\mathcal{Y}) \). The right hand side of (0.1) is directly related to algebraic cycles. For instance, taking \( \mathcal{X} = (\text{Spec } k, \emptyset) \), it is

\[
H_i^S(\mathcal{Y}) := H_i(C_{\bullet}(\mathcal{Y})(\text{Spec } k, \emptyset))
\]

which is the Suslin complex of \( \mathcal{Y} \) studied in \([14]\). Thus (0.1) can be considered as a first step toward a description of \( \text{Hom}_{\text{MDM}_{\text{gm}}} (M(\mathcal{X}), M(\mathcal{Y})[i]) \) in terms of algebraic cycles.

As an application of Theorem 2 and \([10, \text{Th. 7.5.2}]\), we get the following Nisnevich descent results for Suslin homology with modulus:

**Theorem 3** (Th. 3.9.1 and 3.10.3). Assume (RS).

1. Let \( X \) be proper and let \( D, D_1, D_2, D' \) be effective Cartier divisors on \( X \) such that

\[
X - D \text{ is smooth} \quad D \leq D_1 \leq D' \quad |D_1 - D| \cap |D_2 - D| = \emptyset \quad D' - D_2 = D_1 - D.
\]
Then we have a long exact sequence of abelian groups

\[
\cdots \rightarrow H_n^S(X, D') \rightarrow H_n^S(X, D_1) \oplus H_n^S(X, D_2) \rightarrow H_n^S(X, D) \\
\rightarrow H_{n-1}^S(X, D') \rightarrow \cdots
\]

(2) Let \( \mathcal{X} = (\overline{X}, X^\infty) \in \textbf{MCor} \) and

\[
\begin{array}{ccc}
U \times_{\overline{X}} V & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & \overline{X},
\end{array}
\]

be an elementary Nisnevich square; put \( U = (U, U \times_{\overline{X}} X^\infty), \ V = (V, V \times_{\overline{X}} X^\infty), \ U \times_X V = (U \times_{\overline{X}} V, U \times_{\overline{X}} V \times_{\overline{X}} X^\infty). \)

Then we have a long exact sequence

\[
\cdots \rightarrow H_i^S(U \times_X V) \rightarrow H_i^S(U) \oplus H_i^S(V) \rightarrow H_i^S(\mathcal{X}) \rightarrow H_{i-1}^S(U \times_X V) \rightarrow \cdots
\]

In (2), we use the extension of Suslin homology with modulus to non-proper modulus pairs introduced in Subsection 3.10. Note that Nisnevich (or even Zariski) descent for higher Chow groups with modulus is false in general (see \cite[Remark above Th. 3]{13}).

We now use the results of \cite{15} to compare the Nisnevich versions of \textbf{CI}, \textbf{Rec} and \textbf{RSC}. Theorem 1 (1) induces a functor

\[
(0.2) \quad \text{CI}_{\text{Nis}} \rightarrow \text{RSC}_{\text{Nis}} := \text{RSC} \cap \text{NST}.
\]

The following relies on \cite[Theorem 2.2]{15}.

**Theorem 4** (Th. 3.6.5). Under (RS), (0.2) is an equivalence of categories.

Together with \cite[Theorems 2 and 4]{14} allow us to compute the motive of a 1-dimensional modulus pair (see Theorem 3.8.1).

The next result refines Theorem 1 (2). It relies on \cite[Theorem 0.1]{15} and does not require (RS).

**Theorem 5** (Theorem 4.2.1 and Corollary 4.2.3). We have \( \text{RSC}_{\text{Nis}} = \text{Rec}_{\text{Nis}} := \text{Rec} \cap \text{NST}. \)

Putting Theorems 2, 4 and 5 together, we finally get:

**Corollary 6.** Assume (RS). Then the heart of the \( t \)-structure on \( \text{MDM}^{\text{eff}} \) from Theorem 2 (1) is equivalent to \( \text{Rec}_{\text{Nis}}. \)

Thus, remarkably, the reciprocity sheaves introduced in \cite{9} are intimately related with the motives with modulus introduced in \cite{10}, a fact that had eluded us when we started our research on \cite{10}. 
Remark 7. In [9, Conj. 1], we formulated two conjectures for $F \in \text{Rec}$ extending Voevodsky’s results on objects of $\text{HI}$ over a perfect field to objects of $\text{Rec}$: a version of Gersten’s conjecture and a reciprocity version of “$\mathbb{A}^1$-invariance implies strict $\mathbb{A}^1$-invariance”. These conjectures are proven in [15, Cor. 0.3 and Th. 0.4] for $F \in \text{RSC}$ (under (RS) for the second one). Thanks to Theorem 5, we can say that the same holds for $F \in \text{Rec}$.

In [10, Cor 7.3.2], we constructed a full embedding
\begin{equation}
\omega_{\text{gm}}^{\text{eff}} : \text{DM}_{\text{gm}}^{\text{eff}}[1/p] \to \text{MDM}_{\text{gm}}^{\text{eff}}[1/p]
\end{equation}
where $p$ is the exponential characteristic of $k$. Combining Theorem 5 with the work of Binda et al [6], we get the following comparison result:

**Theorem 8 (Cor. 4.2.6).** Assume (RS), and suppose $p > 1$ (sic). Then (0.3) is an equivalence of categories.

See also Theorem 4.2.8 for an integral refinement. Thus the “novel” features of $\text{MDM}_{\text{gm}}^{\text{eff}}$ in positive characteristic $p$ consist fully of $p$-primary phenomena. Of course, the situation is expected to be completely different in characteristic 0. See also Remark 4.2.7 b) for a comparison result with coefficients $\mathbb{Z}/n$, $n$ invertible in $k$.

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**Notation and conventions.** Throughout this paper we work over a base field $k$. Denote by $\text{Sch}$ the category of separated schemes of finite type over $k$, and by $\text{Sm}$ the full subcategory of $\text{Sch}$ consisting of all smooth $k$-schemes.

1. **Review of basic definitions and results**

1.1. **Modulus pairs.** Let $Z \subset T$ be an integral closed subscheme on $T \in \text{Sch}$. For two effective Cartier divisors $D, E$ on $T$, we write $D|_Z \prec E|_Z$ when $\nu^*D \leq \nu^*E$, where $\nu : Z^N \to T$ is the composition of the normalization $Z^N \to Z$ and the inclusion $Z \hookrightarrow T$. The following definition is taken from [10, Definitions 1.3.1, 1.4.1].

**Definition 1.1.1.**
(1) A pair $M = (\overline{X}, X_\infty)$ of $\overline{X} \in \text{Sch}$ and an effective divisor $X_\infty$ on $\overline{X}$ is called a modulus pair if $\overline{X} \setminus |X_\infty| \in \text{Sm}$. It is called \textit{proper} if $\overline{X}$ is proper over $k$.

(2) Let $M = (\overline{X}, X_\infty), N = (\overline{Y}, Y_\infty)$ be two proper modulus pairs and put $X = \overline{X} \setminus |X_\infty|, Y = \overline{Y} \setminus |Y_\infty|$. We define $\text{MCor}(M,N)$ to be the subgroup of $\text{Cor}(X,Y)$ generated by all elementary correspondences $V \in \text{Cor}(X,Y)$ such that the closure $\overline{V}$ of $V$ in $\overline{X} \times \overline{Y}$ satisfies $(\overline{X} \times Y_\infty)|_{\overline{V}} \subset (X_\infty \times \overline{Y})|_{\overline{V}}$. This defines a category $\text{MCor}$ of proper modulus pairs.

(3) We define a symmetric monoidal structure on $\text{MCor}$ by

$$(\overline{X}, X_\infty) \otimes (\overline{Y}, Y_\infty) = (\overline{X} \times \overline{Y}, X_\infty \times Y + \overline{X} \times Y_\infty).$$

1.2. \textbf{Modulus presheaves with transfers}. Here is the definition of our main object of study (see [10, Definition 2.1.1]).

\textbf{Definition 1.2.1.}

(1) We denote by $\text{MPST}$ the abelian category of all additive functors $\text{MCor}^{\text{op}} \to \text{Ab}$.

(2) For $M \in \text{MCor}$, we denote by $\mathbb{Z}_{\text{tr}}(M) \in \text{MPST}$ the object represented by $M$.

By [10, Prop. 2.1.2] $\text{MPST}$ has a symmetric monoidal structure induced by that of $\text{MCor}$. It admits an internal Hom such that

\begin{equation}
\text{Hom}_{\text{MPST}}(\mathbb{Z}_{\text{tr}}(M), F)(N) = F(M \otimes N)
\end{equation}

for $M, N \in \text{MCor}$ and $F \in \text{MPST}$.

1.3. \textbf{Relation with PST}. There is a functor $\omega : \text{MCor} \to \text{Cor}$ defined by $\omega(\overline{X}, X_\infty) = \overline{X} \setminus |X_\infty|$ It induces a functor $\omega^* : \text{PST} \to \text{MPST}$, $\omega^*(F) = F \circ \omega$.

\textbf{Proposition 1.3.1.}

(1) The functor $\omega^*$ is fully faithful and exact.

(2) There is a left adjoint $\omega_l : \text{MPST} \to \text{PST}$ of $\omega^*$, which is monoidal and exact. We have

$$\omega_l F(X) \cong \lim_{\longrightarrow} F(M) \quad (F \in \text{MPST}, \ X \in \text{Sm}).$$

In (2), $\text{MSm}(X)$ is the inverse system $\{M = (\overline{X}, X_\infty) \in \text{MCor} | \ X = \overline{X} \setminus |X_\infty|\}$, where transition maps are given by the diagonal $X \subset X \times X$ whenever it defines a morphism in $\text{MCor}$.

\textit{Proof.} See [10, Prop. 2.2.1 and (2.1)]. \qed
Proposition 1.3.2. Let $M = (\mathcal{X}, X_\infty), N = (\mathcal{Y}, Y_\infty) \in \text{MCor}$ and let $U \in \text{Sm}$. Put $X = \mathcal{X} \backslash |X_\infty|$ and $Y = \mathcal{Y} \backslash |Y_\infty|$. Then
\[
\omega!(\text{Hom}_{\text{MPST}}(Z_{\text{tr}}(N), Z_{\text{tr}}(M))(U))
\]
is the subgroup of $\text{Cor}(Y \times U, X)$ generated by all elementary correspondences $Z \in \text{Cor}(Y \times U, X)$ such that
\[
(X_\infty \times \mathcal{Y} \times U)|_Z \prec (X \times Y_\infty \times U)|_Z.
\]

Proof. See [10, Prop. 2.2.2].

1.4. Modulus sheaves with transfers. In [10, Def. 3.7.1], we define a full subcategory $\text{MNST} := \text{MPST}_{\text{Nis}} \subset \text{MPST}$ of “modulus Nisnevich sheaves with transfers”. (See §3.10 for more details.) In this paper we need the following:

Proposition 1.4.1.  

(1) The full embedding $i_{\text{Nis}} : \text{MNST} \hookrightarrow \text{MPST}$ has an exact left adjoint $a_{\text{Nis}}$ (“sheafification”).

(2) The functors $\omega_!$ and $\omega^*$ of Proposition 1.3.1 preserve $\text{MNST}$ and $\text{NST}$; they induce an adjunction $(\omega_{\text{Nis}}, \omega_{\text{Nis}}^*)$ between these two categories, and $\omega_{\text{Nis}}, \omega_{\text{Nis}}^*$ are both exact. Moreover, the pairs $(\omega_!, \omega_{\text{Nis}})$ and $(\omega^*, \omega_{\text{Nis}}^*)$ both commute with the sheafification functors $a_{\text{Nis}}$ and $a_{V_{\text{Nis}}} : \text{PST} \to \text{NST}$ [19, Th. 3.1.4].

(3) The category $\text{MNST}$ has a symmetric monoidal structure such that $a_{\text{Nis}}$ is monoidal. It admits an internal Hom such that
\[
\text{Hom}_{\text{MNST}}(Z_{\text{tr}}(M), F)(N) = F(M \otimes N)
\]
for $M, N \in \text{MCor}$ and $F \in \text{MNST}.$

Proof. See [10, Prop. 3.7.3] for (1), [10, Prop. 3.8.1] for (2) and [10, Prop. 3.10.1] for (3).
Let \((\mathcal{A}, I)\) be as in the definition, and suppose that \(\mathcal{A}\) is pseudo-abelian. We recall a construction from [10, §5.1–5.2]. Put \(I^n := I^{\otimes n} \in \mathcal{A}\) for each \(n \geq 0\). For \(n > 0\) and \(i \in \{1, \ldots, n\}\), define \(p^n_i : I^n \to I^{n-1}\) to be the projection omitting the \(i\)-th component. For \(n \geq 0\), \(i \in \{1, \ldots, n+1\}\) and \(\varepsilon \in \{0,1\}\), define \(\delta^n_{i,\varepsilon} : I^n \to I^{n+1}\) to be the map inserting \(\varepsilon\) at the \(i\)-th component. Defining coboundary maps by

\[
d^n := \sum_{i=1}^{n+1} (-1)^{i-1}(\delta^n_{i,0} - \delta^n_{i,1}) : I^n \to I^{n+1},
\]

we obtain a cochain complex \((I^\bullet, d^\bullet)\). It turns out that there is a decomposition of complexes \(I^\bullet = I^\bullet_\nu \oplus I^\bullet_{\text{deg}}\), where

\[
I^n_\nu := \ker \left( \bigoplus_{i=1}^n p^n_i : I^n \to \bigoplus_{i=1}^n I^{n-1} \right),
\]

\[
I^n_{\text{deg}} := \text{im} \left( \bigoplus_{i=1}^n \delta^n_{i,0} : \bigoplus_{i=1}^n I^{n-1} \to I^n \right).
\]

By definition and [10, Rem. 5.1.5], we have \(I^n_\nu = I_\nu \otimes \cdots \otimes I_\nu\) with \(I_\nu = \ker(I \xrightarrow{p} 1)\).

**Proposition 2.1.2.** The morphism \(1 \otimes p : I^\bullet_\nu \otimes I \to I^\bullet_\nu\) is a homotopy equivalence.

**Proof.** See [10, Prop. 5.3.1]. \(\square\)

### 2.2. Suslin complex and \(I\)-invariant objects

Suppose that \(\mathcal{A}\) is a closed tensor abelian category equipped with an interval structure \(I\).

**Definition 2.2.1.** Let \(F \in \mathcal{A}\). Its Suslin complex is defined by

\[
C_\bullet(F) := \text{Hom}_\mathcal{A}(I^\bullet_\nu, F).
\]

This is a chain complex in \(\mathcal{A}\). We define for each \(n \in \mathbb{Z}\)

\[
h^n_I(F) := H_n(C_\bullet(F)) \in \mathcal{A}.
\]

For \(n = 0\) the definition simplifies to

\[
h^0_I(F) = \text{coker}(i^*_0 - i^*_1 : \text{Hom}_\mathcal{A}(I, F) \to F).
\]

**Definition 2.2.2.** We say \(F \in \mathcal{A}\) is \(I\)-invariant if the map \(p^* : F \to \text{Hom}_\mathcal{A}(I, F)\) is an isomorphism. Denote by \(\mathcal{H}I(\mathcal{A}, I) \subset \mathcal{A}\) the full subcategory of the \(I\)-invariant objects.

**Lemma 2.2.3.** The following conditions are equivalent for \(F \in \mathcal{A}\):

1. \(F\) is \(I\)-invariant.
2. The maps \(i^*_0, i^*_1 : \text{Hom}_\mathcal{A}(I, F) \to F\) are equal.
(3) The canonical map $F \to h^I_0(F)$ is an isomorphism.

Proof. The equivalence of (2) and (3) follows from (2.1). The rest is proven in [10, Lemma 5.2.5]. □

**Proposition 2.2.4.** For any $F \in \mathcal{A}$ and $n \in \mathbb{Z}$, we have $h^I_n(F) \in HI(\mathcal{A}, I)$.

Proof. This is a consequence of Proposition 2.1.2. □

**Proposition 2.2.5.** Let us abbreviate $HI(\mathcal{A}, I)$ into $HI$. Suppose that the tensor product of $\mathcal{A}$ is right exact.

1. The inclusion $ι : HI \hookrightarrow \mathcal{A}$ has a left adjoint $h^I_0$ given by (2.1).
2. $HI$ is closed in $\mathcal{A}$ under taking subobjects and extensions.
3. Any morphism $f$ in $HI$ has a kernel and a cokernel, respectively given by
   \[ \text{Ker}_{HI} f = \text{Ker}_{\mathcal{A}} f, \quad \text{Coker}_{HI} f = h^I_0(\text{Coker}_{\mathcal{A}} f). \]
4. The following conditions are equivalent:
   \begin{enumerate}
   
   
   
   \item [(i)] $HI$ is closed in $\mathcal{A}$ under taking quotients.
   \item [(ii)] $HI$ is closed in $\mathcal{A}$ under taking cokernels.
   \item [(iii)] $HI$ is abelian.
   \item [(iv)] $HI$ is a Serre subcategory of $\mathcal{A}$.
   
   
   \end{enumerate}
   They are fulfilled if $\text{Hom}_A(I, -)$ is exact.
5. If the conditions of (4) hold, $ι$ has a right adjoint $h^0_I$ characterised by the identity
   \[ \mathcal{A}(G, h^0_I(F)) = \mathcal{A}(ι h^0_I(G), F)). \]

Proof. (1) Note that the composite
   \[ \text{Hom}(I, F) \xrightarrow{i^*_I - i^*_I} F \xrightarrow{p^*} \text{Hom}(I, F) \]
   is the zero map, where $p : I \to 1$ is from Definition 2.1.1. Hence any morphism from $F$ to an $I$-invariant object factors uniquely through $\text{Coker}(\text{Hom}(I, F)) = h^I_0(F)$, which itself is $I$-invariant by the previous proposition. This implies (1). (2) follows from the 5 lemma since $\text{Hom}(I, -)$ is left exact. (3) is obvious.

In (4), (iv) $⇒$ (iii) is obvious. For $f$ as in (3), a diagram chase shows that $\text{Coim}^\mathcal{A} f \xrightarrow{≅} \text{Coim}^HI f$ and that $\text{Coim}^HI f \to \text{Im}^HI f$ is an isomorphism if and only if $\text{Coker}^\mathcal{A} f \in HI$. Hence (iii) $⇒$ (ii). Moreover, (ii) $⇒$ (i) thanks to (2), which also shows that (i) $⇒$ (iv). The last statement is obvious.

Under these conditions, define for $F \in \mathcal{A}$
   \[ h^I_0(F) = \bigcup_{F'} \text{Im}(F' \to F) \]
where $F'$ runs through the objects of $\mathbf{HI}$. By (i), $h_0^I(F) \in \mathbf{HI}$, hence it yields the desired right adjoint. The identity then follows by a double adjunction.

**Proposition 2.2.6.** Via $h_0^I$, the tensor structure of $\mathcal{A}$ induces a symmetric monoidal structure in $\mathbf{HI}(\mathcal{A}, I)$, given explicitly by

$$F \otimes_{\mathbf{HI}(\mathcal{A}, I)} F' = h_0^I(\iota F \otimes_A \iota F')$$

for $F, F' \in \mathbf{HI}(\mathcal{A}, I)$.

**Proof.** Since $h_0^I$ is a localisation, we have to show that if $f \in \mathcal{A}(G_1, G_2)$ is such that $h_0^I(f)$ is an isomorphism, then $g = h_0^I(f \otimes_A 1_{G'})$ is an isomorphism for any $G' \in \mathcal{A}$. By (co)Yoneda, it suffices to show that

$$g^* : \mathbf{HI}(\mathcal{A}, I)(h_0^I(G_2 \otimes_A G'), H) \xrightarrow{\sim} \mathbf{HI}(\mathcal{A}, I)(h_0^I(G_1 \otimes_A G'), H)$$

for any $H \in \mathbf{HI}(\mathcal{A}, I)$. By adjunction, it suffices to show that

$$f^* : \mathcal{A}(G_2, \text{Hom}_A(G', \iota H)) \xrightarrow{\sim} \mathcal{A}(G_1, \text{Hom}_A(G', \iota H)),$$

and, for this, it suffices to show that $\text{Hom}_A(G', \iota H) \in \iota \mathbf{HI}(\mathcal{A}, I)$. This follows from the isomorphisms (easily checked by means of Yoneda’s lemma):

$$\text{Hom}_A(I, \text{Hom}_A(G', \iota H)) \cong \text{Hom}_A(I \otimes_A G', \iota H)$$

$$\cong \text{Hom}_A(G', \text{Hom}_A(I, \iota H)) \xleftarrow{\sim} \text{Hom}_A(G', \iota H)$$

where the last isomorphism is induced by $p : I \rightarrow 1$ by the assumption $H \in \mathbf{HI}(\mathcal{A}, I)$. \hfill \qed

2.3. Derived Suslin complex. From now on, we assume that $\mathcal{A}$ is a closed tensor Grothendieck category equipped with an interval $(I, p, i_0, i_1, \mu)$. We assume further that $\mathcal{A}$ has a set of compact generators which is preserved by tensor product with $I$, and that the tensor structure of $\mathcal{A}$ is right exact.

In this subsection, we further suppose that $D(\mathcal{A})$ is provided with a tensor structure $\otimes_{D(\mathcal{A})}$ with the following properties, taken from [10, §5.6]:

**Hypothesis 2.3.1.**

(i) $\otimes_{D(\mathcal{A})}$ is right $t$-exact and strongly biadditive (see [10, Definition 3.3.2]).

(ii) Let $\otimes_{K(\mathcal{A})}$ be the canonical extension of $\otimes_A$ to $K(\mathcal{A})$. Then the localisation functor $\lambda : K(\mathcal{A}) \rightarrow D(\mathcal{A})$ is lax monoidal, i.e., there is a collection of morphisms

$$\lambda C \otimes_{D(\mathcal{A})} \lambda D \rightarrow \lambda(C \otimes_{K(\mathcal{A})} D)$$
binatural in \((C, D) \in K(A) \times K(A)\) and commuting with the associativity and commutativity constraints.

(iii) The object \(\lambda 1_A[0]\) is a unit of \(\otimes_{D(A)}\).

(iv) The map \((\lambda I[0])_{\otimes D(A)^n} \to \lambda(I_{\otimes A^n}[0])\) induced by (ii) is an isomorphism for all \(n \geq 0\).

It follows that there exists a right adjoint \(\text{Hom}_{D(A)}\) to \(\otimes_{D(A)}\) (see [10, beginning of §5.4]). We equip \(D(A)\) with the interval \(\lambda I[0]\); we abbreviate \(\lambda I[0]\) to \(I\). An object \(D \in D(A)\) is said to be \(I\)-local if the map \(D \xrightarrow{p*} \text{Hom}_{D(A)}(I, D)\) induced by \(p : I \to 1\) is an isomorphism. Let \(\mathcal{R}_I\) be the the localising subcategory of \(D(A)\) generated by objects of the form \(\text{Cone}(X \otimes I \xrightarrow{1\otimes p} X)\) for \(X \in D(A)\), and write \(D(A)_I\) for the Verdier quotient \(D(A)/\mathcal{R}_I\). The following result is taken from [10, Prop. 5.4.2, Th. 5.6.3 and 5.6.4].

**Theorem 2.3.2.**

1. The projection functor \(LC : D(A) \to D(A)_I\) has a right adjoint \(j\) which identifies \(D(A)_I\) with the full subcategory of \(I\)-local objects, and \(j\) has itself a right adjoint;
2. There is a natural isomorphism in \(D \in D(A)\):

   \[
   jLC(D) \cong RC_*(D) := \text{Hom}_{D(A)}(I^*_{\nu}, D).
   \]

We call \(RC_*(D)\) the derived cubical Suslin complex.

### 2.4. The homotopy category.

In this subsection we work under the basic setting introduced in the beginning of §2.3 (and forget about Hypothesis 2.3.1). We equip \(K(A)\) with the closed monoidal structure induced by that of \(A\). Thus, for \(C, D \in K(A)\):

\[
C \otimes_{K(A)} D = \text{Tot}^\oplus(C^i \otimes D^j)
\]

\[
\text{Hom}_{K(A)}(C, D) = \text{Tot}^\Pi(\text{Hom}_A(C^i, D^j))
\]

where \(\text{Tot}^\oplus\) (resp. \(\text{Tot}^\Pi\)) denotes the total complex involving direct sums (resp. products). (Recall that \(A\) is complete: [10, Theorem A.12.1 a]).) We further equip \(K(A)\) with the interval \(I[0]\). As above, an object \(D \in K(A)\) is said to be \(I\)-local if the map \(D \xrightarrow{p*} \text{Hom}_{K(A)}(I, D)\) induced by \(p : I \to 1\) is an isomorphism. Let \(\mathcal{R}^K_I\) be the the localising subcategory of \(K(A)\) generated by objects of the form \(\text{Cone}(X \otimes I \xrightarrow{1\otimes p} X)\) for \(X \in K(A)\), and write \(K(A)_I\) for the Verdier quotient \(K(A)/\mathcal{R}^K_I\). By the assumptions on \(A\) and \(I\), \(K(A)\) is compactly generated and \(\mathcal{R}^K_I\) is generated by compact objects of \(K(A)\). The following result is a variant of Theorem 2.3.2 (precisely the same proofs work.)
Theorem 2.4.1.

1. The projection functor $LC^K : K(A) \to K(A)_I$ has a right adjoint $j^K$ which identifies $K(A)_I$ with the full subcategory of $I$-local objects, and $j^K$ has itself a right adjoint;
2. There is a natural isomorphism in $D \in K(A)$:

\begin{equation}
    j^K LC^K(D) \simeq C_*(D) := \text{Hom}_{K(A)}(I^*_\nu, D).
\end{equation}

We call $C_*(D)$ the naive cubical Suslin complex: if $D = A[0]$ for $A \in A$, it is the Suslin complex from Definition 2.2.1.

We will compare $C_*(D)$ and $RC_*(D)$; see Theorem 2.6.4 (4).

Recall from Definition 2.2.2 that we also have the full subcategory of the $I$-invariant objects

\begin{equation}
    HI(A, I) = A \cap j^K K(A)_I = \{ A \in A \mid A \xrightarrow{\sim} \text{Hom}_A(I, A) \}.
\end{equation}

Lemma 2.4.2. Let $C \in j^K K(A)_I$. Then for any $n \in \mathbb{Z}$:

\begin{itemize}
    \item $\tau_{\leq n}C \in j^K K(A)_I$.
    \item $\tau_{\geq n}C \in j^K K(A)_I$.
    \item $H^n(C) \in HI(A, I)$.
\end{itemize}

Proof. For notational simplicity, let us drop the mention of $j^K$. Recall that $I_\nu := \text{Ker}(I \xrightarrow{p} 1)$. Then $C \in K(A)_I \iff \text{Hom}_{\mathcal{A}}(I_\nu, C)$ is contractible \iff there are morphisms $s^n : \text{Hom}_{\mathcal{A}}(I_\nu, C^n) \to \text{Hom}_{\mathcal{A}}(I_\nu, C^{n-1})$ such that $s^{n+1}d^n + d^{n-1}s^n = 1_{\text{Hom}_{\mathcal{A}}(I_\nu, C^n)}$, where $d^n$ is the differential of $\text{Hom}(I_\nu, C^*)$ induced by the differential $d^*$ of $C$.

For $A \in \mathcal{A}$, let $T(A) = \text{Hom}(I_\nu, A) \otimes I_\nu$ and let $ev_A : T(A) \to A$ be the evaluation morphism (counit of the adjunction $- \otimes I_\nu \dashv \text{Hom}(I_\nu, -)$). Then $s^n$ corresponds to $\bar{s}^n : T(C^n) \to C^{n-1}$ such that $ev_{C^n} = \bar{s}^{n+1}T(d^n) + d^{n-1}\bar{s}^n$.

By definition,

\begin{equation}
    \tau_{\leq n}C^q = \begin{cases} 
        C^q & \text{for } q < n \\
        \text{Ker } d^n & \text{for } q = n \\
        0 & \text{for } q > n
    \end{cases}
\end{equation}

with the induced differentials. By restriction $\bar{s}^q$ then gives rise to

\begin{equation}
    \tau_{\leq n}\bar{s}^q : T((\tau_{\leq n}C)^q) \to (\tau_{\leq n}C)^{q-1}
\end{equation}

which yields a contracting homotopy

\begin{equation}
    \tau_{\leq n}\bar{s}^q : \text{Hom}_{\mathcal{A}}(I_\nu, (\tau_{\leq n}C)^q) \to \text{Hom}_{\mathcal{A}}(I_\nu, (\tau_{\leq n}C)^{q-1}).
\end{equation}

Dual reasoning for $\tau_{\geq n}C$; the case of $H_n(C) = \tau_{\leq n}\tau_{\geq n}C$ follows. \qed
2.5. **Induced t-structures.** Let $\mathcal{T}$ be a triangulated category provided with a $t$-structure (= $t$-category, [3, 1.3]), with heart $\mathcal{A}$ (ibid., Th. 1.3.6), and let $\mathcal{S} \subseteq \mathcal{T}$ be a thick subcategory. Then $\mathcal{A}' = \mathcal{A} \cap \mathcal{S}$ is an admissible abelian subcategory of $\mathcal{S}$ in the sense of [3, Def. 1.2.5], and a thick subcategory of $\mathcal{A}$ by [3, Prop. 1.2.4 (i)]. It follows that

\[(2.5) \quad S' = \{ X \in \mathcal{T} \mid \, tH^n(X) \in \mathcal{A}' \text{ for all } n \in \mathbb{Z} \}\]

is triangulated and that the $t$-structure of $\mathcal{T}$ induces a $t$-structure on $S'$. Write as usual $S'^+ = \{ X \in S' \mid tH^n(X) = 0 \text{ for } n \ll 0 \}$.

**Proposition 2.5.1.** Suppose that small direct sums are representable in $\mathcal{T}$ and that $\mathcal{S}$ is localising. Then we have $S'^+ \subseteq S'$. If $\mathcal{T}$ is left complete [10, Definition A.13.10] and $\mathcal{S}$ is closed under products, we have $S' \subseteq S'$, with equality if and only if the $t$-structure of $\mathcal{T}$ induces a $t$-structure on $S$ with heart $\mathcal{A}'$.

**Proof.** The inclusions are proven in the same way as [10, Theorem 3.7.15 b)] (the hypothesis on $\mathcal{S}$ implies that holim $\tau_{\geq -n} C \in \mathcal{S}$ if $C \in \mathcal{S}$). “If” is obvious; if the $t$-structure of $\mathcal{T}$ induces a $t$-structure on $S$ with heart $\mathcal{A}'$, then clearly $S \subseteq S'$, hence “only if”. □

2.6. **The I-homotopy t-structure.** We go back to the situation of §2.3, and assume Hypothesis 2.3.1. We apply the results of §2.5 to $\mathcal{T} = D(\mathcal{A})$ (provided with its canonical $t$-structure) and $\mathcal{S} = jD(\mathcal{A})_I$, the full subcategory of $I$-local objects. Then we have

\[
\mathcal{A}' = \{ A \in \mathcal{A} \mid A \xrightarrow{\sim} \text{Hom}_{D(\mathcal{A})}(I, A) \} \\
= \{ A \in \mathcal{A} \mid A \xrightarrow{\sim} \text{Hom}_\mathcal{A}(I, A) \text{ and } \text{Ext}_A^i(I, A) = 0 \text{ for } i > 0 \}
\]

where $\text{Ext}_A^i(I, A) := H^i(\text{Hom}_{D(\mathcal{A})}(I, A))$. In particular, $\mathcal{A}'$ is a thick subcategory of $\mathcal{A}$.

**Definition 2.6.1.** We write $\mathcal{A}' =: \text{SHI}(\mathcal{A}, I)$ and call its objects **strictly I-invariant**.

Clearly, $\text{SHI}(\mathcal{A}, I) \subseteq \text{HI}(\mathcal{A}, I)$. We write $D_{\text{SHI}}(\mathcal{A})$ for the category of (2.5), namely:

\[
D_{\text{SHI}}(\mathcal{A}) = \{ D \in D(\mathcal{A}) \mid H^i(D) \in \text{SHI}(\mathcal{A}, I) \text{ for all } i \in \mathbb{Z} \}.
\]

We now assume $D(\mathcal{A})$ left complete: note that $jD(\mathcal{A})_I$ is closed under products thanks to the left adjoint $LC$. Then $D_{\text{SHI}}(\mathcal{A}) \subseteq jD(\mathcal{A})_I$ by Proposition 2.5.1. Consider the functor $r : D(\mathcal{A}) \leftrightarrow K(\mathcal{A})$, right adjoint to the localisation functor $\lambda : K(\mathcal{A}) \rightarrow D(\mathcal{A})$ [10, Theorem A.13.1 b)]. By [10, Lemma A.7.1], the natural transformation...
$\lambda C \otimes_{D(A)} \lambda D \to \lambda (C \otimes_{K(A)} D)$ of Hypothesis 2.3.1 (ii) yields morphisms for any $D \in D(A)$:

\[
\text{Hom}_{K(A)}(I, rD) \to r \text{Hom}_{D(A)}(I, D)
\]
\[
C_*(rD) \to rRC_*(D).
\]
(2.6)

By Hypothesis 2.3.1 (iii), the diagram

\[
\begin{array}{ccc}
\text{Hom}_{K(A)}(I, rD) & \to & r \text{Hom}_{D(A)}(I, D) \\
\downarrow i^t & & \downarrow i^t \\
D & \to & \text{Hom}_{D}(A)(I, D)
\end{array}
\]

is commutative for $t = 0, 1$. This is sufficient to imply:

**Proposition 2.6.2.** Let $D \in D(A)$. If $D$ is $I$-local, then $rD$ is $I$-local.

*Proof.* Apply the criterion (1) $\iff$ (2) of Lemma 2.2.3. (By [10, Lemma 5.2.5], it holds for any closed $\otimes$-category with interval.) □

We thus have an inclusion of full subcategories of $K(A)$:

\[
rD \overset{\text{SHI}}{\subseteq} \overset{\text{j}}{\subseteq} \overset{\text{K}}{\subseteq}
\]

where the second inclusion comes from Proposition 2.6.2.

**Corollary 2.6.3.** The localisation functor $\lambda : K(A) \to D(A)$ induces a functor $\overline{\lambda} : K(A)_I \to D(A)_I$, which is left adjoint to the full embedding $\overline{r} : D(A)_I \hookrightarrow K(A)_I$ given by Proposition 2.6.2; one has a tautological isomorphism of functors

\[
\text{LC}^K \simeq LC\lambda
\]
(2.8)

and a base change morphism

\[
\text{LC}^K r \Rightarrow r\text{LC}.
\]
(2.9)

Moreover, the morphism (2.6) is part of a commutative diagram of natural transformations

\[
\begin{array}{ccc}
C_*r & \to & rRC_* \\
\downarrow & & \downarrow \\
\text{j}^K \text{LC}^K r & \to & r\text{j} \text{LC}
\end{array}
\]
(2.10)

where the vertical ones are those from (2.2) and (2.3), and the bottom one is obtained by applying $\text{j}^K$ to (2.9), using the tautological isomorphism of functors $r\text{j} \simeq \text{j}^K \overline{r}$.

The natural transformation (2.6) is an isomorphism if and only if (2.9) is. When this happens, we have $RC_*(\lambda C) \simeq \lambda C_*(C)$ for any $C \in K(A)$. 


Proof. To lighten notation, we drop the mention of $j$ and $j^K$. From (2.7) we deduce the reverse inclusion
\[
\langle \text{Cone}(C \otimes_{K(A)} I \to C) \rangle = \perp K(A)_I \\
\subseteq \perp rD(A)_I = \{ C \in K(A) \mid \lambda C \in \langle \text{Cone}(D \otimes_{D(A)} I \to D) \rangle \}
\]
(see [4, 1.2]). This shows that $\bar{\lambda}$ exists and is left adjoint to $\bar{r}$; (2.9) is derived from (2.8) as usual. One checks that (2.10) commutes; the equivalence between the invertibility of (2.6) and (2.9) then follows from the full faithfulness of $j^K$. When this happens, applying $\lambda$ to (2.6) with $D = \lambda C$ for $C \in K(A)$, we get an isomorphism in $D(A)$
\[
\lambda C_*(r\lambda C) \sim - \rightarrow \lambda rRC_*(\lambda C) \sim - \rightarrow RC_*(\lambda C).
\]
But $\lambda C_*(C) \sim \lambda C_*(r\lambda C)$ since $C \to r\lambda C$ is a quasi-isomorphism and $C_*$ preserves quasi-isomorphisms (it is defined out of $\text{Tot}^{\Pi}$).

\[\Box\]

Theorem 2.6.4. The following are equivalent:

(i) $\text{SHI}(A, I) = HI(A, I)$.
(ii) All inclusions in (2.7) are equalities.

When this happens,

(1) The standard $t$-structure of $D(A)$ induces a $t$-structure on $D(A)_I$ and we have
\[
j D(A)_I = \{ D \in D(A) \mid H^i(D) \in HI(A, I) \text{ for all } i \in \mathbb{Z} \}.
\]
(2) In Corollary 2.6.3, the functors $\bar{\lambda}$ and $\bar{r}$ are quasi-inverse equivalences of categories.
(3) (2.6) is an isomorphism for any $D \in D(A)$.
(4) The objects $RC_*(\lambda C)$ and $C_*(C)$ are isomorphic in $D(A)$ for any $C \in K(A)$.
(5) $HI(A, I)$ is a Serre subcategory of $A$.

Proof. If (ii) holds, let $F \in HI(A, I)$. Then $F[0] \in rD_{\text{SHI}}(A)$, hence $\lambda F[0] = F[0] \in D_{\text{SHI}}(A)$ and $F = H^0(F[0]) \in \text{SHI}(A, I)$. Conversely, (i) $\Rightarrow$ (ii) follows from Lemma 2.4.2. (1) and (2) follow immediately (for (1), use the full faithfulness of $r$). (3) is then formal, but let us spell its proof out for clarity: (2.6) is obtained as the composition
\[
LC^K r \overset{(a)}{\Rightarrow} \bar{r} \lambda LC^K r \overset{(2.8)}{\cong} \bar{r} LC \lambda r \overset{(b)}{\Rightarrow} \bar{r} LC
\]
where (a) (resp. (b)) is induced by the unit of the adjunction $(\bar{\lambda}, \bar{r})$ (resp. by the counit of the adjunction $(\lambda, r)$). Here, (b) is invertible by the full faithfulness of $r$, and (a) is invertible by the full faithfulness of (the equivalence) $\bar{\lambda}$. (4) now follows from (3) and Corollary 2.6.3,
and (5) follows from Proposition 2.2.5 (4) since $\text{SHI}(\mathcal{A}, I)$ is a thick subcategory of $\mathcal{A}$. □

2.7. Comparison of intervals. Let $(\mathcal{A}, I), (\mathcal{A}', I')$ be as in §2.3. Let $T : \mathcal{A} \to \mathcal{A}'$ be a right exact cocontinuous $\otimes$-functor sending $I$ to $I'$ and respecting the constants of structure of $I$ and $I'$. Then $T$ has a right adjoint $S$ [10, Theorem A.12.1 b)]. We assume that $T$ has a total left derived functor $LT : D(\mathcal{A}) \to D(\mathcal{A}')$, which is strongly additive, a $\otimes$-functor and sends $I[0]$ to $I'[0]$ (this is automatic if $T$ is exact). In our application we shall take $\mathcal{A} = \text{MPST}, \mathcal{A}' = \text{MNST}$, and $T = a_{\text{Nis}}$ and $S = i_{\text{Nis}}$ (see Proposition 1.4.1). By Brown representability [10, Lemma A.11.2, Theorem A.13.1 a]), $LT$ has a right adjoint $RS$, which is the total right derived functor of $S$. Then $LT$ induces a triangulated $\otimes$-functor $LT : D(\mathcal{A})_I \to D(\mathcal{A}')_{I'}$. We have a base change morphism (2.11)

\[ LT \circ j \Rightarrow j' \circ LT, \]

where $j$ and $j'$ are the right adjoints of the localisation functors $D(\mathcal{A}) \to D(\mathcal{A})_I$ and $D(\mathcal{A}') \to D(\mathcal{A}')_{I'}$ respectively (see Theorem 2.3.2).

Proposition 2.7.1.

(1) The functor $S$ sends $HI(\mathcal{A}', I')$ into $HI(\mathcal{A}, I)$ and $\text{SHI}(\mathcal{A}', I')$ into $\text{SHI}(\mathcal{A}, I)$.

(2) Suppose that $D(\mathcal{A})$ and $D(\mathcal{A}')$ are left complete, that $T$ is a localisation and that $T \text{HI}(\mathcal{A}, I) \subseteq \text{SHI}(\mathcal{A}', I')$. Then, if $(\mathcal{A}, I)$ verifies Condition (i) of Theorem 2.6.4, so does $(\mathcal{A}', I')$. If moreover $T$ is exact, the base change morphism (2.11) is an isomorphism.

Proof. (1) is formal in view of [10, Lemma A.7.1]. In (2), we have by hypothesis

\[ T \text{HI}(\mathcal{A}, I) \subseteq \text{SHI}(\mathcal{A}', I') \subseteq \text{HI}(\mathcal{A}', I') \]

but the left and right categories coincide by (1), since $TS \simeq \text{Id}_{\mathcal{A}}$. To deduce the last assertion, it suffices by [10, Lemma 5.7.4] to prove that $LT j D(\mathcal{A}) I \subseteq j' D(\mathcal{A}') {r'}$. By condition (ii) of Theorem 2.6.4 and the full faithfulness of $r$ and $r'$, this is equivalent to $LT D_{\text{SHI}}(\mathcal{A}) \subseteq D_{\text{SHI}}(\mathcal{A}')$, which follows from $T \text{SHI}(\mathcal{A}, I) = T \text{HI}(\mathcal{A}, I) \subseteq \text{SHI}(\mathcal{A}', I')$ and the exactness of $T$. □

2.8. Categories of modules. Consider the special case where $\mathcal{A} = \text{Mod} - \mathcal{B}$ for an essentially small additive $\otimes$-category $\mathcal{B}$, and $I \in \mathcal{B}$ is an interval: we work with the interval $y(I) \in \mathcal{A}$, where $y : \mathcal{B} \to \mathcal{A}$ is the Yoneda functor. We may apply all results from previous subsections thanks to [10, Theorem A.13.2]. Moreover:
Theorem 2.8.1. In the above situation, we have $\text{Ext}^i_A(y(I), A) = 0$ for any $A \in \mathcal{A}$ and any $i > 0$; in particular, $\text{SHI}(A, y(I)) = \text{HI}(A, y(I))$. (Hence all consequences from Theorem 2.6.4 follow.)

Proof. It suffices to show that $\mathcal{A}(y(B), \text{Ext}^i_A(I, A)) = 0$ for any $B \in \mathcal{B}$ and any $i > 0$. Since $y(B)$ is projective [1, Prop. 1.3.6], the latter group is isomorphic to

$$D(A)(y(B)[0], \text{Hom}_{D(A)}(y(I)[0], A[i]) \simeq D(A)(y(B \otimes B I)[0], A[i]) = 0$$

as $y(B \otimes B I)$ is also projective. (We used the isomorphism $y(B)[0] \otimes_{D(A)} y(I)[0] \simeq y(B \otimes B I)[0]$, which follows from the definition of $\otimes_{D(A)}$ in [10, Theorem A.13.2 (d)].) □

We will apply Theorem 2.8.1 in Proposition 3.2.6 and Theorem 3.7.1.

3. $□$-invariance and SC-reciprocity

3.1. Interval theory in $\text{MCor}$ and $\text{MPST}$. Let $□ = (\mathbb{P}^1, \infty) \in \text{MCor}$. By [10, Lemma 6.1.1], we have an interval structure on $□$ induced by the interval structure on $A^1 \in \text{Cor}$ from [17]. Note that $\mu : I \otimes I \to I$ is induced by the map $A^1 \times A^1 \to A^1$, $(a, b) \mapsto ab$. It induces an interval structure on $Z_{tr}(□) \in \text{MPST}$.

3.2. $□$-invariance and Suslin complexes. We now apply Definition 2.2.2 to the above interval structure. Concretely, it can be rewritten as follows:

Definition 3.2.1. We say $F \in \text{MPST}$ is $□$-invariant if the projection map $p : M \otimes □ \to M$ induces an isomorphism $p^* : F(M) \simto F(M \otimes □)$ for any $M \in \text{MCor}$.

Recall:

Lemma 3.2.2 ([10, Lemma 6.1.3]). Let $H \in \text{PST}$. Then $H$ is $A^1$-invariant $\iff \omega^* H \in \text{MPST}$ is $□$-invariant. □

Definition 3.2.3. We define $\text{CI} := \text{HI}(\text{MPST}, Z_{tr}(□))$ to be the full subcategory of $\text{MPST}$ consisting of all objects having $□$-invariance.

Definition 3.2.4. For any $F \in \text{MPST}$, let $C_\square^n(F)$ be the complex in $\text{MPST}$ obtained by Definition 2.2.1 applied to $\mathcal{A} = \text{MPST}$. By definition, for $n \in \mathbb{Z}_{\geq 0}$, we have

$$C_\square^n(F)(N) = \text{Hom}_{\text{MPST}}(Z_{tr}(N) \otimes Z_{tr}(□)^{\otimes n}, F) \quad (N \in \text{MCor}).$$
where $Z_{tr}(\mathcal{U}_p) = \text{Ker}(Z_{tr}(\mathcal{U})) \to \mathbb{Z})$. We write $h_{\mathcal{U}}^n(F) = H_n(C^\mathcal{U}_\bullet(F)) \in \text{MPST}$. For $M \in \text{MCor}$, we put $C^\mathcal{U}_\bullet(M) := C^\mathcal{U}_\bullet(Z_{tr}(M)), h^\mathcal{U}_n(M) = h^\mathcal{U}_n(Z_{tr}(M))$.

By Proposition 2.2.4, we have the following:

**Proposition 3.2.5.** Let $F \in \text{MPST}$. Then $h_{\mathcal{U}}^n(F) \in \text{CI}$ for any $n \in \mathbb{Z}$.

**Proposition 3.2.6.**

(1) The category $\text{CI}$ is a Serre subcategory of $\text{MPST}$; in particular, it is abelian. The inclusion $i^\mathcal{U} : \text{CI} \hookrightarrow \text{MPST}$ has

(i) a left adjoint given by $F \mapsto h_0^\mathcal{U}(F)$;

(ii) a right adjoint given by $F \mapsto h_0^\mathcal{U}(F)$, where

$$h_0^\mathcal{U}F(M) = \text{Hom}(h_0^\mathcal{U}(M), F) \quad (M \in \text{MCor}).$$

(2) Via $h_0^\mathcal{U}$, the symmetric monoidal structure on $\text{MPST}$ induces a symmetric monoidal structure on $\text{CI}$ by

$$F \otimes_{\text{CI}} G = h_0^\mathcal{U}(i^\mathcal{U}F \otimes_{\text{MPST}} i^\mathcal{U}G) \quad (F, G \in \text{CI}).$$

**Proof.** (1) follows from Theorems 2.8.1 and 2.6.4, while (2) follows from Proposition 2.2.6. \qed

### 3.3. SC-reciprocity.

**Definition 3.3.1.** Let $F \in \text{MPST}$. For each $n \in \mathbb{Z}$, we define

$$h_n(F) := H_n(\omega_!C^\mathcal{U}_\bullet(F)) = \omega_!h_n^\mathcal{U}(F) \in \text{PST}.$$  

For $M \in \text{MCor}$, we put $h_n(M) := h_n(Z_{tr}(M))$. In particular we have

$$h_0(M) = \text{Coker}(i_0^* - i_1^* : \omega_!(\text{Hom}_{\text{MPST}}(Z_{tr}(\mathcal{U}), Z_{tr}(M)) \to Z_{tr}(M^o))$$  

where $M^o = \omega M$.

**Remark 3.3.2.** Let $M = (\mathcal{X}, \mathcal{X}_\infty) \in \text{MCor}$ and put $X = \mathcal{X} \setminus |X_\infty|$. Define $h_0(M) := h_0(Z_{tr}(M))$.

(1) By definition, $h_n(M)$ is the homology of the complex

$$\omega_!C^\mathcal{U}_\bullet(M) = \omega_!(\text{Hom}_{\text{MPST}}(Z_{tr}(\mathcal{U}_p), Z_{tr}(M))).$$

An explicit description of each term of this complex is given in Proposition 1.3.2. It implies that the group $h_n(M)(\text{Spec } k)$ agrees with the Suslin homology $H^n_S(\mathcal{X}, \mathcal{X}_\infty)$ considered in [14, Definition 3.1].
(2) Suppose that $\overline{X}$ is equidimensional and let

$$CH^r(M, n) = CH^r(\overline{X}|X_\infty, n)$$

be the higher Chow group with modulus defined in [5]. As is explained in [14, Remark 3.4 (2)], there is a canonical morphism

$$h_n(M)(\text{Spec } k) \to CH^{\dim \overline{X}}(M, n)$$

which is bijective for $n = 0$.

**Definition 3.3.3.**

(1) Let $F \in \text{PST}$, $X \in \text{Sm}$ and $a \in F(X) = \text{Hom}_{\text{PST}}(\mathbb{Z}_u(X), F)$. We say $M = (\overline{X}, X_\infty) \in \text{MCor}$ is an SC-modulus for $a$ if $X = \overline{X} \setminus X_\infty$ and $a : \mathbb{Z}_u(X) \to F$ factors through $\mathbb{Z}_u(X) \to h_0(M)$. (SC stands for “Suslin complex”.)

(2) We say $F \in \text{PST}$ has SC-reciprocity if, for any $X \in \text{Sm}$ and any $a \in F(X)$, there is an SC-modulus $M \in \text{MCor}$ for $a$.

(3) We define $\text{RSC}$ to be the full subcategory of $\text{PST}$ consisting of all objects having SC-reciprocity.

**Remark 3.3.4.** The category $\text{RSC}$ is closed under subobjects and quotient objects in $\text{PST}$. This is obvious from the definition. In particular, $\text{RSC}$ is abelian and the inclusion functor $i^\natural$ is exact.

**Proposition 3.3.5.** The functor $\rho := \omega^! h_0^! \omega^* : \text{PST} \to \text{RSC}$, and is right adjoint to the inclusion $i^\natural : \text{RSC} \hookrightarrow \text{PST}$.

**Proof.** For $F \in \text{PST}$ and $X \in \text{Sm}$, we have by successive adjunctions and by Proposition 1.3.1 (2):

$$(3.1) \quad \rho F(X) = \lim_{\longrightarrow} i^\natural h_0^! \omega^* F(M) = \lim_{\longrightarrow} \text{MPST}(h_0^!(M), \omega^* F) = \lim_{\longrightarrow} \text{PST}(h_0(M), F)$$

which realises $\rho F$ as the largest subobject of $F$ which is in $\text{RSC}$. □

**Corollary 3.3.6.** Let $F \in \text{PST}$. The counit map

$$(3.2) \quad i^\natural \rho F \to F$$

of the adjunction in Proposition 3.3.5 agrees with the counit map of the adjunction $(\omega^!, h_0^!, \omega^*)$. Moreover, $F \in \text{RSC}$ if and only if (3.2) is an isomorphism.

**Proof.** The first statement simply restates the computation in (3.1). The second one follows from Proposition 3.3.5 and the definition of SC-reciprocity. □
3.4. Relations between CI, HI and RSC.

Proposition 3.4.1. The composite $\omega \colon \text{MPST} \xrightarrow{\omega_i} \text{PST} \xrightarrow{h^0_{\bullet}} \text{HI}$ factors through $\text{MPST} \xrightarrow{h^0_{\bullet}} \text{CI}$, inducing a functor $\omega_h : \text{CI} \to \text{HI}$. This functor is right exact and monoidal for the $\otimes$-structures given on CI by Proposition 3.2.6 (2), and analogously on HI. It has an exact right adjoint $\omega^h$, given by the restriction of $\omega^*$ to HI.

Proof. The first claim and the monoidality of $\omega^h$ follow from that of $\omega^i$, as $\omega^i Z(\square) = Z(\square)$. The existence, characterisation and exactness of $\omega^h$ follows from Lemma 3.2.2, and the right exactness of $\omega^h$ then follows.

Theorem 3.4.2. If $F \in \text{MPST}$ has $\square$-invariance, then $\omega^i F \in \text{PST}$ has SC-reciprocity.

Proof. We have a commutative diagram, for any $F \in \text{MPST}$:

\[
\begin{array}{cccccc}
\omega^i h^0_{\square} F & \xrightarrow{\omega^i h^0_{\square} \eta_F} & \omega^i h^0_{\square} \omega^* \omega^i F \\
\omega^i \varepsilon' & \downarrow & \omega^i \varepsilon' \varepsilon \omega^i F \\
\omega^i F & \xrightarrow{\omega^i \eta_F} & \omega^i \omega^* \omega^i F & \xrightarrow{\varepsilon^i F} & \omega^i F.
\end{array}
\]

Here, $\eta$ and $\varepsilon$ are the unit and counit of the adjunction $(\omega^i, \omega^*)$, while $\varepsilon'$ is the counit of the adjunction $(i^*, h^0_{\square})$. We have $(c) \circ (b) = 1_{\omega^i F}$ by the adjunction identities; since $\omega^i$ is a localisation, $(c)$ is an isomorphism hence so is $(b)$. This shows that $(c)$ factors through $(a)$. On the other hand, $\varepsilon'$ is mono by Proposition 3.2.6 (1), hence so are $(c)$ and $(d)$ since $\omega^i$ is exact. Finally, the diagram boils down to two successive monomorphisms

\[
\omega^i i^* h^0_{\square} F \hookrightarrow \omega^i \varepsilon' F \hookrightarrow \omega^i F
\]

with composition $\omega^i \varepsilon' F$. Therefore, $F \in \text{CI} \Rightarrow \omega^i F \in \text{RSC}$.

Corollary 3.4.3. We have $\text{HI} \subset \text{RSC}$.

Proof. Let $F \in \text{HI}$. By Lemma 3.2.2, $\omega^* F \in \text{CI}$, hence

\[
F \cong \omega^i \omega^* F \in \text{RSC}
\]

by Theorem 3.4.2. (See [15, Lemma 1.22] for a simpler proof.)

Corollary 3.4.4. For any $F \in \text{MPST}$ and $n \in \mathbb{Z}_{\geq 0}$, $h_n(F) \in \text{RSC}$.

Proof. This follows from Proposition 3.2.5 and Proposition 3.4.2.
Corollary 3.4.5. The inclusion functor $i^\natural : \text{RSC} \hookrightarrow \text{PST}$ has a pro-left adjoint $\ell$.

Proof. It suffices to show that $\ell$ is defined on the generators $Z_{tr}(X)$. Since $h_0(M) \in \text{RSC}$ for any $M \in \text{MSm}(X)$ by Corollary 3.4.4, we have $\ell(Z_{tr}(X)) = \lim_{\leftarrow} h_0(M)$.

Proposition 3.4.6. There exist unique functors $\omega_{\text{CI}}$ and $\omega^\text{CI}$ that make the two diagrams

\[
\text{CI} \xrightarrow{i^\natural} \text{MPST} \quad \text{CI} \xleftarrow{h_0^\natural} \text{MPST}
\]

\[
\text{RSC} \xrightarrow{i^\natural} \text{PST} \quad \text{RSC} \xleftarrow{i^\natural} \text{PST}
\]

commutative, where $i^\natural$ is the inclusion. Moreover, $\omega^\text{CI}$ is right adjoint to $\omega_{\text{CI}}$. The counit map $\omega_{\text{CI}} \omega^\text{CI} \Rightarrow \text{Id}_{\text{RSC}}$ is an isomorphism, $\omega_{\text{CI}}$ is a localisation (in particular, is essentially surjective) and $\omega^\text{CI}$ is fully faithfull. Finally, $\omega_{\text{CI}}$ is exact and $\omega^\text{CI}$ is left exact.

Proof. The existence of $\omega_{\text{CI}}$ is the contents of Theorem 3.4.2, and $\omega^\text{CI}$ is defined by the commutativity of the diagram. For the second assertion, let $F \in \text{CI}$ and $G \in \text{RSC}$. Using two successive adjunctions, we compute:

\[
\text{CI}(F, \omega^\text{CI} G) = \text{CI}(F, h_0^\natural \omega_* i^\natural G) \simeq \text{PST}(\omega_{\text{CI}} i^\natural F, i^\natural G) = \text{PST}(i^\natural \omega_{\text{CI}} F, i^\natural G) \simeq \text{RSC}(\omega_{\text{CI}} F, G)
\]

where the last isomorphism uses the (tautological) full faithfulness of $i^\natural$. So the adjunction $(\omega_{\text{CI}}, \omega^\text{CI})$ is obtained by “cancelling” $i^\natural$ from the adjunction $(\omega_{\text{CI}} i^\natural, h_0^\natural \omega_*)$, after applying Theorem 3.4.2. Therefore the third assertion follows from Corollary 3.3.6, and the next two are standard consequences [10, Lemma A.3.1]. The exactness of $\omega_{\text{CI}}$ follows from the exactness of $i^\natural$ and $\omega_1$ (as well as the full faithfulness of $i^\natural$), and $\omega^\text{CI}$ is left exact as a right adjoint.

3.5. Monoidal structure on RSC. Let $M = (\overline{X}, X_\infty), M' = (\overline{X'}, X'_\infty) \in \text{MCor}$. We say $M'$ is a modulus thickening of $M$ if $X := \overline{X} \setminus |X_\infty| = \overline{X} \setminus |X'_\infty|$ and if the diagonal $X \subset X \times X$ belongs to $\text{MCor}(M', M)$. We write $\text{MT}$ for the class of all pairs $(M, M')$ where $M'$ is a modulus thickening of $M$. For any $(M, M') \in \text{MT}$, we define

\[
Z_{tr}(M/M') := \text{Coker}(Z_{tr}(M') \to Z_{tr}(M)),
\]

\[
h_0^\natural(M/M') := \text{Coker}(h_0^\natural(M') \to h_0^\natural(M)).
\]
Proposition 3.5.1.

(1) Suppose $F \in \text{MPST}$ satisfies $\omega F = 0$. Then $F$ is a quotient of a direct sum of objects of the form $\mathbb{Z}_{tr}(M/M')$ with $(M, M') \in \text{MT}$.

(2) Suppose $F \in \text{CI}$ satisfies $\omega_{\text{CI}} F = 0$. Then $F$ is a quotient of a direct sum of objects of the form $h^0_\square(M/M')$ with $(M, M') \in \text{MT}$.

Proof. (1) Recall that any $F \in \text{MPST}$ is a quotient of a direct sum of objects of the form $\mathbb{Z}_{tr}(M)$ with $M \in \text{MCor}$ (cf. [19, §3.2]). Hence it suffices to show that, given $M \in \text{MCor}$ and $F \in \text{MPST}$ with $\omega F = 0$, any morphism $\mathbb{Z}_{tr}(M) \rightarrow F$ factors through $\mathbb{Z}_{tr}(M/M')$ for some $(M, M') \in \text{MT}$. This follows from Proposition 1.3.1 (3) by Yoneda.

(2) Suppose $F \in \text{CI}$ satisfies $\omega_{\text{CI}} F = 0$. We have $\omega_{\square} F = \hat{i} \cdot \omega_{\text{CI}} F = 0$, and hence by (1) we get a surjection $\bigoplus \mathbb{Z}_{tr}(M/M') \rightarrow \hat{i} \cdot F$, where $(M, M')$ ranges through objects in $\text{MT}$. Since $h^0_\square \hat{i} \cdot = \text{Id}_{\text{CI}}$, we arrive at the conclusion by applying the right exact functor $h^0_\square$. □

Corollary 3.5.2. The symmetric monoidal structure of $\text{CI}$ (see Proposition 3.2.6) induces a symmetric monoidal structure on $\text{RSC}$. This tensor structure comes with a binatural epimorphism

$$F \otimes_{\text{PST}} G \longrightarrow F \otimes_{\text{RSC}} G$$

for $F, G \in \text{RSC}$.

Proof. By [10, Lemma A.5.7], we must prove that $\text{Ker} \omega_{\text{CI}}$ is an $\otimes$-ideal. Since $\text{CI}$ is generated by the $h^0_\square(N)$, it suffices by Proposition 3.5.1 (2) to observe that $h^0_\square(M/M') \otimes h^0_\square(N) \simeq h^0_\square(M \otimes N/M' \otimes N)$ by the right exactness of $h^0_\square$ and $\otimes$.

For the last statement, let $F' = \omega_{\text{CI}} F$, $G' = \omega_{\text{CI}} G \in \text{CI}$. There is a natural epimorphism

$$F' \otimes_{\text{MPST}} G' \longrightarrow h^0_\square(F' \otimes_{\text{MPST}} G') = F' \otimes_{\text{CI}} G'$$

whence a natural epimorphism

$$F \otimes_{\text{PST}} G = \omega_{\text{CI}} F' \otimes_{\text{PST}} \omega_{\text{CI}} G' = \omega F' \otimes_{\text{PST}} \omega G' = \omega(F' \otimes_{\text{MPST}} G')$$

$$\longrightarrow \omega(F' \otimes_{\text{CI}} G') = \omega_{\text{CI}} F' \otimes_{\text{RSC}} \omega_{\text{CI}} G' = F \otimes_{\text{RSC}} G$$

where we used the monoidality and right exactness of $\omega_1$ (Proposition 1.3.1 (1)); the last-but-one equality is by definition of $\otimes_{\text{RSC}}$. □

It would be interesting to compare this $\otimes$-structure with the tensor products defined by Ivorra and Rölling in [8]. Meanwhile, let us clarify its relationship with the one of $\text{HI}$ (cf. Corollary 3.4.3): the functor
$h_0^{A_1} : PST \to HI$ restricts to a left adjoint of the inclusion of Corollary 3.4.3, that we write $h_0^{\text{rec}} : RSC \to HI$.

**Proposition 3.5.3.** We have a natural isomorphism $\omega_h \simeq h_0^{\text{rec}} \omega_{CI}$ (see Proposition 3.4.1 for $\omega_h$). The functor $h_0^{\text{rec}}$ is symmetric monoidal.

**Proof.** Applying the natural isomorphism $\omega_h h_0^{\square} G \simeq h_0^{A_1} \omega_G$ to $F \in \text{CI}$, we get a natural isomorphism

$$\omega_h F \simeq \omega_h h_0^{\square} i^{\square} F \simeq h_0^{A_1} \omega_G \simeq h_0^{\text{rec}} \omega_{CI} F$$

as requested. The monoidality of $h_0^{\text{rec}}$ now follows from those of $\omega_h$ and (by definition) $\omega_{CI}$, as well as the isomorphism $\omega_{CI} \omega_{CI} \Rightarrow \text{Id}_{RSC}$ from Proposition 3.4.6. \qed

### 3.6. From presheaves to sheaves

This subsection relies on results of [15], namely Th. 0.1, 0.6 and 2.2. The reader will check that it is not used in [15].

**Definition 3.6.1.** We set:

$$CI_{\text{Nis}} = CI \cap \text{MNST} = HI(\text{MNST}, Z_{tr}(\square))$$

$$RSC_{\text{Nis}} = RSC \cap \text{NST}.$$  

As in Definition 3.2.4, for $F \in \text{MNST}$, we obtain a complex $C_{\square}^{\square, \text{Nis}}(F)$ in $\text{MNST}$ by Definition 2.2.1 applied to $A = \text{MNST}$. We have

$$(3.5) \quad i_{\text{Nis}} C_{\square}^{\square, \text{Nis}}(F) = C_{\square}^{\square}(i_{\text{Nis}} F),$$

where $i_{\text{Nis}} : \text{MNST} \to MPST$ is the inclusion and $C_{\square}(\cdot)$ is from Definition 3.2.4. This follows from

$$i_{\text{Nis}} \text{Hom}_{\text{MNST}}(Z_{tr}(M), F) = \text{Hom}_{\text{MPST}}(Z_{tr}(M), i_{\text{Nis}} F),$$

itself following from [10, Lemma A.7.1]. We write $h_{n}^{\square, \text{Nis}}(F) := h_{n}^{Z_{tr}(\square)}(F)$

$\in \text{MNST}$ for $F \in \text{MNST}$. Applying the exact functor $a_{\text{Nis}}$ to (3.5) and taking homology, we get

$$(3.6) \quad h_{n}^{\square, \text{Nis}}(F) = a_{\text{Nis}} h_{n}^{\square}(i_{\text{Nis}} F).$$

Proposition 2.2.4 implies:

**Proposition 3.6.2.** Let $F \in \text{MNST}$. Then $h_{n}^{\square, \text{Nis}}(F) \in CI_{\text{Nis}}$ for any $n \in \mathbb{Z}$.

**Proposition 3.6.3.** Suppose (RS). The functor $h_0^{\square} : MPST \to CI$ sends $\text{MNST}$ into $CI_{\text{Nis}}$; it yields the right adjoint $h_0^{\square, \text{Nis}}$ to the inclusion $i_{\text{Nis}} : CI_{\text{Nis}} \hookrightarrow \text{MNST}$, which has the left adjoint $F \mapsto h_{0}^{\square, \text{Nis}}(F)$.  

Proof. Let $F \in \text{MNST}$. Considering $F$ as an object of $\text{MPST}$, we may view $i^! h^0_{\square} F$ as the largest subobject of $F$ which belongs to $\text{CI}$. (Note that $i^! h^0_{\square} F \to F$ is a monomorphism by the formula in Proposition 3.2.6 (1) (ii) giving $h^0_{\square}$.) Applying the left exact functor $i_{\text{Nis}}^a$ to this inclusion, we get a sequence

$$i^! h^0_{\square} F \to i_{\text{Nis}}^a i^! h^0_{\square} F \to i_{\text{Nis}}^a F = F$$

where the second map is a monomorphism. But the middle term is in $\text{CI}$ by [15, Th. 0.6], hence the first map must be an isomorphism, which implies the first claim. The last claim follows from [10, Lemma A.9.1 d)] and (3.6).

**Theorem 3.6.4.**

1. The functor $\rho$ of Proposition 3.3.5 sends $\text{NST}$ into $\text{RSC}_{\text{Nis}}$. It yields a right adjoint $\rho_{\text{Nis}}$ to the inclusion $i^a_{\text{Nis}} : \text{RSC}_{\text{Nis}} \hookrightarrow \text{NST}$.

2. The functor $\omega_{\text{CI}}$ of Proposition 3.4.6 sends $\text{CI}_{\text{Nis}}$ to $\text{RSC}_{\text{Nis}}$.

**Proof.** (1) is proven as in Proposition 3.6.3, using [15, Th. 0.1] (which does not requires (RS)) instead of [15, Th. 0.6]. (2) is obvious since $\omega_i$ preserves Nisnevich sheaves by Proposition 1.4.1 (2).

**Theorem 3.6.5.** Assume that $k$ verifies (RS). Then the functor $\omega^\text{Nis}_{\text{CI}} : \text{RSC}_{\text{Nis}} \to \text{CI}_{\text{Nis}}$ defined by Corollary 3.6.4 (2) is an equivalence of categories; its quasi-inverse $\omega^\text{Nis}_{\text{CI}}$ is induced by $\omega_{\text{CI}}$.

**Proof.** Since $\omega^*$ and $h^0_{\square}$ preserve Nisnevich sheaves (the first by Proposition 1.4.1 (2), the second by Proposition 3.6.3), we have $\omega_{\text{CI}}(\text{RSC}_{\text{Nis}}) \subseteq \text{CI}_{\text{Nis}}$ and the induced functor $\omega^\text{Nis}_{\text{CI}}$ is right adjoint to $\omega^\text{Nis}_{\text{CI}}$ by restricting the adjunction identities of the adjunction $(\omega_{\text{CI}}, \omega_{\text{CI}})$. This argument also shows that $\omega^\text{Nis}_{\text{CI}}$ is a localisation, and it is exact because $\omega_{\text{Nis}}$ is exact (Proposition 1.4.1 (2) again). To conclude, it suffices by [10, Lemma A.5.7] to show that $\text{Ker} \omega^\text{Nis}_{\text{CI}} = 0$. Let $F \in \text{Ker} \omega^\text{Nis}_{\text{CI}}$. Then $i_{\text{Nis}}^a F \in \text{Ker} \omega_{\text{CI}}$. By Proposition 3.5.1 (2), $i_{\text{Nis}}^a F$ is a quotient of a sum of objects of the form $h^0_{\square}(M/M')$ with $(M,M') \in \text{MT}$. Therefore, $F = a_{\text{Nis}} i_{\text{Nis}}^a F$ is a quotient of a sum of objects of the form $a_{\text{Nis}} h^0_{\square}(M/M')$ with $(M,M') \in \text{MT}$. But under Condition (RS) we have $a_{\text{Nis}} h^0_{\square}(M/M') = 0$ for any $(M,M') \in \text{MT}$, by [15, Th. 2.2] and [10, Cor. 1.10.5]. The theorem is proven.

**Remark 3.6.6.** The Nisnevich analogue of Proposition 3.5.3 is valid.
3.7. **The triangulated picture.** This subsection also uses results of [15]; the same comment as in Subsection 3.6 applies. We assume (RS) throughout.

**Theorem 3.7.1.**

1. The standard $t$-structure of $D(\text{MNST})$ induces a $t$-structure on $\text{MDM}^{\text{eff}}$ via $j$. Its heart is $\text{CI}_{\text{Nis}} := \text{CI} \cap \text{MNST}$, which is a Serre subcategory of $\text{MNST}$, and we have

$$j \, \text{MDM}^{\text{eff}} = \{ D \in D(\text{MNST}) \mid H^i(D) \in \text{CI}_{\text{Nis}} \text{ for all } i \in \mathbb{Z} \}.$$ 

2. Let $\lambda : K(\text{MNST}) \to D(\text{MNST})$ be the localisation functor. For any $K \in K(\text{MNST})$, we have an isomorphism

$$RC^\square_\bullet(\lambda K) \simeq \lambda C^\square_\bullet(K).$$

**Proof.** In Proposition 2.7.1, take $(A, I) = (\text{MPST}, \square)$, $(A', I) = (\text{MNST}, \square)$ and $T = a_{\text{Nis}}$. We check that the hypotheses are verified:

- $A$ and $A'$ verify the conditions of Hypothesis 2.3.1: for $A$ because it is a category of modules [10, Th. A.13.2] and for $A'$ by [10, Prop. 6.9.1 (3)].
- Both categories $D(A)$ and $D(A')$ are left complete: for $A$, see [10, Th. A.3.12] again and for $A'$ see [10, Prop. 3.7.12].
- $T$ is a localisation and is exact: see [10, Prop. 3.7.3].
- $(A, I)$ verifies Condition (i) of Theorem 2.6.4 by Theorem 2.8.1.
- $T \, \text{HI}(A, I) \subseteq \text{SHI}(A', I)$: this follows from [15, Th. 0.8] (and requires Condition (RS)).

Therefore $(\text{MNST}, \square)$ verifies Conditions (i) $\iff$ (ii) of Theorem 2.6.4, hence their consequences (1) – (5). 

**Proposition 3.7.2.** There are canonical isomorphisms for $F \in \text{MPST}$

$$C^\square_{\text{Nis}}(a_{\text{Nis}}F) \simeq a_{\text{Nis}}C^\square_\bullet(F)$$

$$h^\square_{n, \text{Nis}}(a_{\text{Nis}}F) \simeq a_{\text{Nis}}h^\square_n(F).$$

**Proof.** Apply Proposition 2.7.1 (2) (whose hypotheses are verified in the proof of Theorem 3.7.1 above). 

**Remark 3.7.3.** Proposition 3.7.2 cannot be deduced formally from (3.6). It lies much deeper.

**Proposition 3.7.4.** Assume $k$ perfect. Then the functor $\omega^{\text{eff}} : \text{DM}^{\text{eff}} \to \text{MDM}^{\text{eff}}$ from [10, Th. 7.3.1] is $t$-exact with respect to the homotopy $t$-structure on $\text{DM}^{\text{eff}}$ and the $\square$-$t$-structure on $\text{MDM}^{\text{eff}}$ from Theorem 2. The functor $H_{\text{Nis}} \to \text{CI}_{\text{Nis}}$ induced on the hearts coincides with the functor induced by $\omega^h : H_{\text{I}} \to \text{CI}$ (see Proposition 3.4.1).
Proof. We have a naturally commutative diagram of full embeddings

\[
\begin{array}{ccc}
\text{MDM}^{\text{eff}} & \xrightarrow{j} & D(\text{MNST}) \\
\omega^{\text{eff}} & & D(\omega^{\text{Nis}}) \\
\downarrow & & \downarrow \\
\text{DM}^{\text{eff}} & \xrightarrow{j^{\text{N}}} & D(\text{NST})
\end{array}
\]

in which \(j, j^{\text{N}}\) and \(D(\omega^{\text{Nis}})\) are \(t\)-exact; the \(t\)-exactness of \(\omega^{\text{eff}}\) follows. The same reasoning yields the second statement. \(\Box\)

Proposition 3.7.5. The functor \(D(\omega^{\text{Nis}})j : \text{MDM}^{\text{eff}} \to D(\text{NST})\) is conservative; if \(K \in D(\text{NST})\) is in its essential image, then \(H_{i}(K) \in \text{RSC}_{\text{Nis}}\) for any \(i \in \mathbb{Z}\).

Proof. Let \(K \in \text{MDM}^{\text{eff}}\) be such that \(D(\omega^{\text{Nis}})(jK) = 0\). Then we have \(\omega^{\text{Nis}}H_{i}(jK) = 0\) for all \(i \in \mathbb{Z}\) by the exactness of \(\omega^{\text{Nis}}\). But \(H_{i}(jK) \in \text{CI}_{\text{Nis}}\) by Theorem 2 (1), hence \(H_{i}(jK) = 0\) by Theorem 3.6.5 (1), and finally \(K = 0\) by the full faithfulness of \(j\). The last statement follows from Corollary 3.6.4 (2) and the exactness of \(\omega^{\text{Nis}}\). \(\Box\)

Remark 3.7.6. We do not know if \(D(\omega^{\text{Nis}})j\) is fully faithful: this would imply that its essential image in \(D(\text{NST})\) is triangulated. A related question is: is \(\text{RSC}_{\text{Nis}}\) closed under extensions in \(\text{NST}\)?

3.8. The motive of a 1-dimensional modulus pair. We continue to assume (RS). Together with the results of [15], this subsection rests on the main result of [14]

Theorem 3.8.1. Let \(C\) be a smooth projective \(k\)-curve, and let \(D > 0\) be a nonzero effective divisor on \(C\). Then, we have an isomorphism in \(\text{MDM}^{\text{eff}}\):

\[
M(C, D) \simeq \omega^{\text{CI}}_{\text{Nis}} \text{Pic}(C, D)[0]
\]

where \(\text{Pic}(C, D) \in \text{NST}\) is the Nisnevich sheaf with transfers associated to the presheaf \(S \mapsto \text{Pic}(C \times S, D \times S)\).

Proof. By Theorem 3.7.1 (2) and [10, Th. 6.9.4 (2)], we have an isomorphism

\[
jM(C, D) \simeq C^{\text{Nis}}(C, D).
\]

Applying \(D(\omega^{\text{Nis}})\) to the right hand side, we find the complex \(C^{\text{Nis}}(C, D) \in D(\text{NST})\) which is computed in [14, Th. 4.4]: its homology sheaves are 0 in degree \(> 0\) and

\[
H_{0}(C^{\text{Nis}}(C, D)) \simeq \text{Pic}(C, D).
\]
By the exactness of $\omega_{\text{Nis}}$ (Proposition 1.4.1 (2)) and Theorem 3.6.5, we find that the homology sheaves of $C^\bullet_{\text{Nis}}(C, D)$ are 0 in degree $> 0$ and that

$$H_0(C^\bullet_{\text{Nis}}(C, D)) \simeq \omega_{\text{Pic}}(C, D)$$

as claimed.

\[\square\]

Remarks 3.8.2. a) The case where $D = 0$ is handled in [10, Th. 7.3.1]. It is analogous to this one, thanks to [19, Th. 3.4.2].
b) For an object $G \in \text{RSC}_{\text{Nis}}$, $\omega_{\text{Nis}}^\text{CI} G$ is computed on $M \in \text{MCor}$ by the formula

$$\omega_{\text{Nis}}^\text{CI} G(M) = \text{MNST}(\mathbb{Z}_\text{tr}(M), i^0 h^0\omega_{\text{Nis}} G)$$

$$= \text{MNST}(h^0(M), \omega_{\text{Nis}}(G)) = \text{NST}(h^0_{\text{Nis}}(M), G).$$

3.9. Nisnevich descent. We continue to assume (RS). For $X \in \text{MCor}$, we put

$$(3.7) \quad H^S_i(X) := h^\text{CI}_i(X)(\text{Spec } k, \emptyset) = h_i(X)(\text{Spec } k),$$

see Definitions 3.2.4, 3.3.1 and Remark 3.3.2. This is the Suslin homology studied in [14]. We then have:

**Theorem 3.9.1.** Let $X$ be proper and let $D, D_1, D_2, D'$ be effective Cartier divisors on $X$ such that

- $X - D$ is smooth
- $D \leq D_i \leq D'$
- $|D_1 - D| \cap |D_2 - D| = \emptyset$
- $D' - D_2 = D_1 - D$.

Then we have a long exact sequence in $\text{CI}$:

$$\cdots \rightarrow h^S_n(X, D') \rightarrow h^S_n(X, D_1) \oplus h^S_n(X, D_2) \rightarrow h^S_n(X, D) \rightarrow h^S_{n-1}(X, D') \rightarrow \cdots$$

and a long exact sequence of abelian groups

$$\cdots \rightarrow H^S_n(X, D') \rightarrow H^S_n(X, D_1) \oplus H^S_n(X, D_2) \rightarrow H^S_n(X, D) \rightarrow H^S_{n-1}(X, D') \rightarrow \cdots$$

**Proof.** This follows from [10, Th. 7.5.2 (2)] and Theorem 3.7.1 (2). \[\square\]
3.10. **Extension to MCor.** We still assume (RS). In [10], we introduced a larger category $\mathbf{MCor} \supset \mathbf{MCor}$ of not necessarily proper modulus pairs, and an associated category of motivic sheaves which plays a key rôle in the proofs of [15]. Let us review this:

**Definition 3.10.1.** Let $M = (\overline{X}, X_\infty)$, $N = (\overline{Y}, Y_\infty)$ be two (not necessarily proper) modulus pairs and put $X = \overline{X} \setminus |X_\infty|$, $Y = \overline{Y} \setminus |Y_\infty|$ (see Definition 1.1.1). We define $\mathbf{MCor}(M,N)$ to be the subgroup of $\mathbf{Cor}(X,Y)$ generated by all elementary correspondences $V \in \mathbf{Cor}(X,Y)$ such that the closure $\overline{V}$ of $V$ in $X \times Y$ satisfies $(X \times Y_\infty) \prec \overline{V}$ and is proper over $X$.

This still defines a category, and the monoidal structure of Definition 1.1.1 (3) extends to $\mathbf{MCor}$. We denote by $\mathbf{MPST}$ the corresponding category of additive presheaves, as in Definition 1.2.1; it inherits a closed monoidal structure, with the same formula as (1.1) for its internal Hom. The fully faithful $\otimes$-functor $\tau : \mathbf{MCor} \to \mathbf{MCor}$ induces (3.8) $\tau^* : \mathbf{MPST} \to \mathbf{MPST}$ by $\tau^*(F) = F \circ \tau$.

By [10, Prop. 2.4.1] there is a left adjoint $\tau^! : \mathbf{MPST} \to \mathbf{MPST}$ to $\tau^*$, which is fully faithful, monoidal and exact.

There is a full subcategory of “Nisnevich sheaves” $\mathbf{MNST} \subset \mathbf{MPST}$ [10, Def. 3.5.1]. For $F \in \mathbf{MPST}$ to be in $\mathbf{MNST}$ it is necessary and sufficient that $F_M$ is a sheaf on the small Nisnevich site $\mathcal{X}_{\text{Nis}}$ (in the usual sense) for any $M = (\overline{X}, X_\infty) \in \mathbf{MCor}$, where $F_M$ is the presheaf given by $F_M(U) = F(U, p^*X_\infty)$ for $(p : U \to \overline{X}) \in \mathcal{X}_{\text{Nis}}$. (See [10, Lemma 3.1.4].) We have $Z_{\text{tr}}(M) \in \mathbf{MNST}$ for any $M \in \mathbf{MCor}$. We then define $\text{MNST}$ as the full subcategory of $\mathbf{MPST}$ consisting of $F \in \mathbf{MPST}$ such that $\tau^* F \in \mathbf{MNST}$. The functors $\tau^!$ and $\tau^*$ respect these subcategories and the induced functors $\tau_{\text{Nis}}$ and $\tau_{\text{Nis}}^*$ are exact [10, Cor. 3.7.5 and 3.9.6 (3)].

Let $\mathcal{X} \in \mathbf{MCor}$. In [10, Def. 7.5.1], we defined:

$$M(\mathcal{X}) = L^\square(\tau_{\text{Nis}}^* Z_{\text{tr}}(\mathcal{X})[0]) \in \text{MDM}^{\text{eff}},$$

where $L^\square : D(\text{MNST}) \to \text{MDM}^{\text{eff}}$ is the localisation functor. Correspondingly, we extend the notation of Definition 3.2.4 to $\mathcal{X} \in \mathbf{MCor}$ with $C^\square(\mathcal{X}) = C^\square(\tau^* Z_{\text{tr}}(\mathcal{X}))$, $h^\square(\mathcal{X}) := h^\square(\tau^* Z_{\text{tr}}(\mathcal{X}))$ as well as the notation (3.7). Theorem 3.7.1 may then be completed as follows:

**Theorem 3.10.2.** For any $\mathcal{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$ and $\mathcal{Y} \in \mathbf{MCor}$ and any $i \in \mathbb{Z}$, we have a canonical isomorphism

$$\text{MDM}^{\text{eff}}(M(\mathcal{X}), M(\mathcal{Y})[i]) \cong H_{\text{Nis}}^i(\overline{X}, C^\square(\mathcal{Y}) \mathcal{X}[i]),$$

where $\text{MDM}^{\text{eff}}$ denotes the derived category of effective motives.
where and $C^\bullet(\mathcal{V})$ is the complex of Nisnevich sheaves on $\overline{X}$ associated to $\tau_! C^\bullet(\mathcal{V}) \in K(\text{NST})$ as in [10, Not. 3.4.7].

**Proof.** This follows from Theorem 3.7.1 (2) and [10, Th. 3]. □

Moreover,

**Theorem 3.10.3.** Let $\mathcal{X} = (\overline{X}, X^\infty) \in \text{MCor}$ and

$$
\begin{array}{ccc}
U \times \overline{X} & V \\
\downarrow & \downarrow \\
U & \overline{X},
\end{array}
$$

be an elementary Nisnevich square (see Definition 3.5.6) and put $U = (U, U \times \overline{X}X^\infty)$, $V = (V, V \times \overline{X}X^\infty)$, $U \times_{\mathcal{X}} V = (U \times_X V, U \times_X V \times_X X^\infty)$. Then we have a long exact sequence

$$
\cdots \to H^i_\mathcal{S}(U \times_{\mathcal{X}} V) \to H^i_\mathcal{S}(U) \oplus H^i_\mathcal{S}(V) \to H^i_\mathcal{S}(\mathcal{X}) \to H^i_{\mathcal{S}-1}(U \times_{\mathcal{X}} V) \to \cdots
$$

**Proof.** This follows from Theorem 3.7.1 (2) and [10, Th. 7.5.2 (1)]. □

4. Relation with [9]

4.1. **Review of reciprocity presheaves with transfers.** In [9, Definition 2.1.3], we defined a full subcategory $\text{Rec}$ of $\text{PST}$, which we now recall.

Let $\overline{X}, Y) \in \text{MCor}$ and suppose that $X = \overline{X} \setminus |Y|$ is quasi-affine. For $S \in \text{Sm}$, let $\mathcal{C}_{(\overline{X}, Y)}(S)$ be the class of all finite morphisms $\varphi: \overline{C} \to \overline{X} \times S$ satisfying the following conditions:

- $\overline{C} \in \text{Sch}$ is integral and normal.
- There is a generic point $\eta$ of $S$ such that $\dim \overline{C} \times_S \eta = 1$.
- The image of $\gamma_\varphi := \text{pr} \circ \varphi$ is not contained in $|Y|$, where $\text{pr} : \overline{X} \times S \to \overline{X}$ is the projection map.

For an effective Cartier divisor $D$ on $\overline{C}$, we set

$$
G(\overline{C}, D) := \bigcap_{x \in D} \ker (\mathcal{O}^x_{\overline{C}, x} \to \mathcal{O}^x_{D, x}).
$$

We then define

$$
\Phi(\overline{X}, Y)(S) = \bigoplus_{(\varphi: \overline{C} \to \overline{X} \times S) \in \mathcal{C}_{(\overline{X}, Y)}(S)} G(\overline{C}, \gamma_\varphi^* Y)
$$

It is proved in [9, Proposition 2.2.2] that $\Phi(\overline{X}, Y)$ defines a presheaf with transfers. It is also shown there that one has $\varphi_*(\text{div}_{\overline{C}}(f)) \in$
\textbf{Cor}(S, X) for any \((\phi: C \to X \times S) \in \text{Cor}(X, Y)(S)\) and \(f \in G(C, \gamma^*_\varphi Y)\), yielding a map \(\tau: \Phi(X, Y) \to \mathbb{Z}_{\text{tr}}(X)\) in PST. We define
\[
h(M) := \text{Coker}(\tau: \Phi(X, Y) \to \mathbb{Z}_{\text{tr}}(X)) \in \text{PST}.
\]

\textbf{Definition 4.1.1} ([9, Definition 2.1.2, Remark 2.1.6]). We say \(F \in \text{PST}\) has reciprocity if for any quasi-affine \(X \in \text{Sm}\) and \(a \in F(X) = \text{Hom}_{\text{PST}}(\mathbb{Z}_{\text{tr}}(X), F)\), there is an \(M = (X, X_\infty) \in \text{MCor}\) such that \(X = X \setminus |X_\infty|\) and \(a: \mathbb{Z}_{\text{tr}}(X) \to F\) factors through \(\mathbb{Z}_{\text{tr}}(X) \to h(M)\).

We define \(\text{Rec}\) to be the full subcategory of \(\text{PST}\) consisting of all objects having reciprocity.

\textbf{4.2. Statement of the result and consequences.}

\textbf{Theorem 4.2.1.} Let \(M = (X, Y) \in \text{MCor}\) be such that \(X := X \setminus |Y|\) is quasi-affine. Then \(h_0(M) = h(M)\). Hence we have \(\text{RSC} \subset \text{Rec}\).

The proof of Theorem 4.2.1 will occupy §§4.3 and 4.4. We first deduce some consequences.

\textbf{Corollary 4.2.2.} For any \(F \in \text{RSC}\), we have \(F_{\text{Zar}} \cong F_{\text{Nis}}\), where \(F_{\text{Zar}}\) (resp. \(F_{\text{Nis}}\)) is the Zariski (resp. Nisnevich) sheafification of \(F\).

\textbf{Proof.} Combine Theorem 4.2.1 and [9, Theorem 7]. \hfill \Box

The next result depends on [15, Theorem 0.1].

\textbf{Corollary 4.2.3.} We have \(\text{RSC}_{\text{Nis}} = \text{Rec}_{\text{Nis}}\).

\textbf{Proof.} The inclusion follows immediately from Theorem 4.2.1. To prove the equality, let \(F \in \text{Rec}_{\text{Nis}}\). By (3.1) and Theorem 4.2.1, the map \(\iota^*\rho F \to F\) of (3.2) is an isomorphism when evaluated at \(X\) if \(X\) is quasi-affine. By Corollary 3.6.4 (1), this extends to any \(X \in \text{Sm}\) by using a quasi-affine Zariski cover. Thus \(F \in \text{RSC}_{\text{Nis}}\). \hfill \Box

\textbf{Remark 4.2.4.} Here is an example of an object \(F \in \text{Rec} \setminus \text{RSC}\). Define \(F\) as
\[
\text{Coker} \left( \bigoplus_{(X, a)} \mathbb{Z}_{\text{tr}}(X) \to \mathbb{Z}_{\text{tr}}(\mathbb{P}^1) \right)
\]
where \(X\) runs through all smooth quasi-affine \(k\)-schemes and \(a\) runs through all elements of \(\text{Cor}(X, \mathbb{P}^1)\). By construction, \(F(X) = 0\) for any smooth quasi-affine \(X\), hence \(F \in \text{Rec}\). On the other hand, we claim that the image \(\eta \in F(\mathbb{P}^1)\) of the identity map \(1_{\mathbb{P}^1} \in \mathbb{Z}_{\text{tr}}(\mathbb{P}^1)(\mathbb{P}^1)\) does not have an SC modulus. Since \(\mathbb{P}^1\) is proper, this amounts to say that the composition
\[
C_1^{\mathbb{A}^1}(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1)) \xrightarrow{i^*_1 i^*_0} \mathbb{Z}_{\text{tr}}(\mathbb{P}^1) \xrightarrow{\eta} F
\]
is nonzero. The properness of $\mathbb{P}^1$ also yields that for any connected $Y \in \mathbf{Sm}$ the image of $\oplus_{(X,a)} \mathbb{Z}_{\text{tr}}(X)(Y) \to \mathbb{Z}_{\text{tr}}(\mathbb{P}^1)(Y) = \text{Cor}(Y, \mathbb{P}^1)$ is generated by the cycles of the form $Y \times x$ where $x$ ranges over closed points of $\mathbb{P}^1$. In particular, if we take $Y = \mathbb{P}^1$ we find that $F(\mathbb{P}^1)$ is not finitely generated. On the other hand, [11, Th. 2.3.1] shows
\[
\text{Coker}(C^1_1(\mathbb{Z}_{\text{tr}}(\mathbb{P}^1)))(\mathbb{P}^1) \to \mathbb{Z}_{\text{tr}}(\mathbb{P}^1)(\mathbb{P}^1) \cong \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}.
\]
Hence $\eta$ cannot vanish at $\mathbb{P}^1$.

**Corollary 4.2.5.**

1. A presheaf with transfers represented by a smooth commutative algebraic group has SC-reciprocity.
2. The presheaf with transfers $H^0(-, \Omega^i_\cdot)$ has SC-reciprocity for any $i \geq 0$. If $k$ is perfect, the same is true for the presheaf with transfers $H^0(-, \Omega^i_\cdot/k)$.
3. Suppose that $k$ is a perfect field of positive characteristic. Then the presheaf with transfers $H^0(-, W_n \Omega^i_\cdot)$ has SC-reciprocity for any $i \geq 0$ and $n \geq 1$.

**Proof.** Combine Corollary 4.2.3 and [9, Theorems 4, 5].

The next corollary uses the work of Binda et al [6]: we suppose $k$ is of characteristic $p > 0$ and we use the notation $[1/p]$ to designate categories constructed out of sheaves of $\mathbb{Z}[1/p]$-modules: they are full subcategories of those considered in this paper.

**Corollary 4.2.6.** Assume that $\text{char } k = p > 0$ and that $k$ verifies (RS). Then:

1. The isomorphism $\omega_h \simeq h^0_{\text{rec}} \omega_{\text{CI}}$ from Proposition 3.5.3 induces two successive equivalences of categories

   $\text{CI}_{\text{Nis}}[1/p] \xrightarrow{\sim} \text{RSC}_{\text{Nis}}[1/p] \xrightarrow{\sim} \text{HI}_{\text{Nis}}[1/p]$.

   The quasi-inverse of the composition is induced by $\omega_h$ (Proposition 3.4.1).

2. The functors $\omega_{\text{eff}} : \text{MDM}_{\text{eff}} \to \text{DM}_{\text{eff}}$ and $\omega_{\text{eff}} : \text{DM}_{\text{eff}} \to \text{MDM}_{\text{eff}}$ from [10, Prop. 6.10.3] induce quasi-inverse equivalences of categories between the subcategories of $\mathbb{Z}[1/p]$-objects. They induce quasi-inverse equivalences of categories between $\text{MDM}_{\text{gm}}[1/p]$ and $\text{DM}_{\text{gm}}[1/p]$.

**Proof.** (1) The first equivalence follows from Theorem 3.6.5; the second one follows from Corollary 4.2.3 and [6, Th. 3.5 (2)]. The second claim is clear, since $\omega_h$ is right adjoint to $\omega_h$. 

(2) By [10, Prop. 6.10.3 and Cor. 7.3.2], \( \omega_{\text{eff}} \) and \( \omega_{\text{eff}} \) are a pair of adjoint functors and induce a pair of adjoint functors on \( \text{MDM}_{\text{gm}}^{\text{eff}}[1/p] \) and \( \text{DM}_{\text{gm}}^{\text{eff}}[1/p] \); moreover, \( \omega_{\text{eff}} \) is fully faithful, hence we are reduced to showing that the unit \( K \to \omega_{\text{eff}} \omega_{\text{eff}} K \) is an isomorphism for any \( K \in \text{MDM}_{\text{gm}}^{\text{eff}}[1/p] \).

We first show that \( \omega_{\text{eff}} \) is \( t \)-exact. It is right \( t \)-exact by [3, Prop. 1.3.17 (iii)] since \( \omega_{\text{eff}} \) is \( t \)-exact (Proposition 3.7.4), hence it suffices to show its left \( t \)-exactness. Let \( K \in (\text{MDM}_{\text{gm}}^{\text{eff}})^{\geq 0} \); we must show that \( \omega_{\text{eff}} K \in (\text{DM}_{\text{gm}}^{\text{eff}})^{\geq 0} \). Suppose first that \( K = F[0] \) for \( F \in \text{CI}_{\text{Nis}} \). We have

\[
(4.2) \quad \omega_{\text{eff}}(F[0]) = LCV D(\omega_{\text{Nis}}) j(F[0]) = LCV j^V(\omega_{\text{CI}} F[0]) \to \omega_{\text{CI}} F[0]
\]

by (1), hence the claim in this case. From this one gets the case where \( K \in (\text{MDM}_{\text{gm}}^{\text{eff}})^{[0,n]} \) by induction on \( n \), and then the general case in view of the isomorphism

\[
\text{hocolim} \tau_{\leq n} K \to K
\]

since \( \omega_{\text{eff}} \) is strongly additive as a left adjoint.

Since \( \omega_{\text{eff}} \) is also \( t \)-exact, so is the composition \( \omega_{\text{eff}} \omega_{\text{eff}} \). Then the desired isomorphism follows from (4.2), applied a second time. \( \square \)

Remarks 4.2.7. a) There is a finer operation which consists of inverting \( p \) on morphisms rather than on objects, but Corollary 4.2.6 is false for these categories. For example, the sheaf \( \bigoplus_{n \geq 1} W_n \in \text{RSC}_{\text{Nis}} \) is not torsion, but maps to 0 in \( \text{HI}_{\text{Nis}} \). However there is no difference when we restrict to \( \text{MDM}_{\text{gm}}^{\text{eff}} \) and \( \text{DM}_{\text{gm}}^{\text{eff}} \), by the argument in the proof of [2, Prop. 6.1.1].

b) Exactly the same arguments give the same results for motives with coefficients \( \mathbb{Z}/n \) where \( n \) is invertible in \( k \), using [6, Th. 3.5 (1)] instead of [6, Th. 3.5 (2)]. Here the characteristic of \( k \) is arbitrary.

Here is a refinement of Corollary 4.2.6 (2):

Theorem 4.2.8. Assume (RS) and \( \text{char} k = p > 0 \). Then the functor \( \omega_{\text{eff}} : \text{DM}_{\text{gm}}^{\text{eff}} \to \text{MDM}_{\text{gm}}^{\text{eff}} \) sends \( \text{DM}_{\text{gm}}^{\text{eff}} \) into \( \text{MDM}_{\text{gm}}^{\text{eff}} \). Moreover, for any \( M \in \text{MDM}_{\text{gm}}^{\text{eff}} \), the cone of the unit morphism

\[
(4.3) \quad M \to \omega_{\text{eff}} \omega_{\text{eff}} M
\]

is killed by some power of \( p \).
Proof. The first claim is proven as in [10, Cor. 7.3.2] by removing “à la de Jong-Gabber”. Now consider the functor

$$\Psi : \text{MDM}^\text{eff} \to \text{MDM}^\text{eff}[1/p]$$

induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}[1/p]$. It is induced by $\square$-localisation from the same functor $D(\text{MNST}) \to D(\text{MNST}[1/p])$, but it also commutes with the right adjoints because if $K$ is $\square$-local, so is $\Psi(K)$ (which ultimately boils down to the fact that Nisnevich cohomology commutes with $\otimes \mathbb{Z}[1/p]$). Using cohomology sheaves, we see that, for $K \in D(\text{MNST})$,

$$\Psi(K) = 0 \Rightarrow \text{hocolim}_n(K, p) = 0$$

where the hocolim on the right is for the system $K \xrightarrow{p} K \xrightarrow{p} K \ldots$. So, this implication also holds in $\text{MDM}^\text{eff}$.

Now let $K$ be the cone of (4.3). By Corollary 4.2.6 (2), we have $\Psi(K) = 0$. Hence $\text{hocolim}_n(K, p) = 0$. But $K \in \text{MDM}^\text{eff}_{\text{gm}}$ is compact in $\text{MDM}^\text{eff}$ [10, Def. 6.9.3]. Hence

$$0 = \text{MDM}^\text{eff}(K, \text{hocolim}_n(K, p)) = \lim_n \text{MDM}^\text{eff}(K, K)$$

where the transition maps are again multiplication by $p$. In particular, there exists $n \gg 0$ such that $p^n 1_K = 0$, which is what we claimed. □

4.3. Preliminary lemmas. In the rest of this section, we use a change of coordinates $\square \cong (\mathbb{P}^1, 1)$ given by $\mathbb{A}^1 \to \mathbb{P}^1 \setminus \{1\}, \ t \mapsto t/(t - 1)$. Let $\square := \mathbb{P}^1 \setminus \{1\}$. Take $S \in \text{Sm}$ and a closed integral subscheme $V \subset X \times \square \times S$ that is finite and surjective over $X \times \square$. We have a commutative diagram

$$\begin{array}{ccc}
V & \subset & X \times \square \times S \\
\downarrow & & \downarrow \\
\nabla^N & \subset & X \times \mathbb{P}^1 \times S \\
\downarrow & \quad & \downarrow p \\
\nabla^N & \to & X \times S \\
\downarrow & \quad & \downarrow \gamma \\
W^N & \to & X \\
& & \downarrow \\
& & X
\end{array}$$

where $V$ is the closure of $V$ in $X \times \mathbb{P}^1 \times S$, $W$ is the image of $V$ under the projection $p$, and $\nabla^N \to \nabla^N$ and $\nabla^N \to W$ are the normalizations. Let
\( \varphi_V : \overline{\mathcal{V}}^N \to \overline{X} \times \mathbb{P}^1 \times S \) be the natural map. Let \( \iota_\infty : \overline{X} \times S \to \overline{X} \times \mathbb{P}^1 \times S \) be induce by \( \infty \in \mathbb{P}^1 \). Put

\[
\partial^\infty \mathcal{V} = \iota_\infty^{-1}(\mathcal{V}) = p(\mathcal{V} \cap (\overline{X} \times \{\infty\} \times S)) \subset \overline{X} \times S.
\]

Putting \( W_\infty = W \setminus \partial^\infty \mathcal{V} \) and \( W_{N,o} = \overline{W}^N \times_{\overline{W}} W_\infty \), we have

\[
(4.5) \quad \mathcal{V} \times_{\overline{W}} W_{N,o} \subset W_{N,o} \times (\mathbb{P}^1 - \{\infty\}).
\]

Let \( \mathcal{V}^o \) be the reduced part of an irreducible component of \( \mathcal{V} \times_{\overline{W}} W_{N,o} \) which dominates \( W_{N,o} \). (Thus \( \mathcal{V}^o \to \mathcal{V} \) is birational.)

Lemma 4.3.1. If \( W_\infty = \emptyset \), then \( V = W \times \square \) with \( W = \overline{W} \cap (X \times S) \).

Proof. The assumption implies \( W \subset p(\mathcal{V} \cap (\overline{X} \times \{\infty\} \times S)) \) and hence

\[
\dim W \leq \dim \mathcal{V} \cap (\overline{X} \times \{\infty\} \times S) < \dim \mathcal{V}.
\]

Noting \( \mathcal{V} \hookrightarrow \overline{W} \times \mathbb{P}^1 \), we get \( \mathcal{V} = \overline{W} \times \mathbb{P}^1 \), which implies the desired assertion. \( \square \)

Lemma 4.3.2. If \( W_\infty \neq \emptyset \), \( \mathcal{V}^o \) is finite over \( W_{N,o} \).

Proof. \( \mathcal{V} \) is proper over \( \overline{W} \) so that \( \mathcal{V}^o \) is proper over \( W_{N,o} \). On the other hand \( W_{N,o} \times (\mathbb{P}^1 - \{\infty\}) \) is affine over \( W_{N,o} \) and so is \( \mathcal{V}^o \). This implies the lemma. \( \square \)

Now we consider the modulus condition for \( V \):

\[
(4.6) \quad \varphi_V^{-1}(Y \times \mathbb{P}^1 \times S) \leq \varphi_V^{-1}(\overline{X} \times \{1\} \times S).
\]

Let \( y \) be the standard coordinate on \( \mathbb{P}^1 - \{\infty\} = \text{Spec}(k[y]) \). (Note that the divisor involved in the modulus condition is \( \{1\} \subset \mathbb{P}^1 - \{\infty\} \) defined by the ideal \( (1 - y) \subset k[y] \).) Let \( I \subset \mathcal{O}_{\overline{W}_{N,o}} \) be the ideal sheaf of \( Y \times_{\overline{X}} W_{N,o} \subset W_{N,o} \).

Lemma 4.3.3. Assuming \( \overline{W}^o \neq \emptyset \), (4.6) is equivalent to the conditions:

(i) \( \mathcal{V} \cap (Y \times \square \times S) = \emptyset \).

(ii) Locally on \( W_{N,o} \), \( \mathcal{V}^o \) is defined by an equation

\[
f(y) := (1 - y)^m + \sum_{1 \leq \nu \leq m} a_\nu (1 - y)^{m-\nu} \quad \text{with} \quad a_\nu \in \Gamma(W_{N,o}, I^\nu),
\]

in \( W_{N,o} \times (\mathbb{P}^1 - \{1\}) = \overline{W}_{N,o} \times \text{Spec}(k[y]) \) (see (4.5)).

Proof. By Lemma 4.3.2, the minimal polynomial over \( k(\overline{W}) \) of the image of \( y \) in \( \Gamma(\mathcal{V}^o, \mathcal{O}) \):

\[
f(t) = (1 - t)^m + \sum_{1 \leq \nu \leq m} a_\nu (1 - t)^{m-\nu}
\]
has its coefficients $a_\nu \in A := \Gamma(W^{N,o}, O)$. We claim that $\nabla^o$ coincides with the closed subscheme $T \subset W^{N,o} \times \text{Spec}(k[y])$ defined by the equation $f(y) \in A[y]$. Indeed it is clear that $\nabla^o$ is contained in $T$, hence it suffices to show that $T$ is integral. Note that $T$ is a Cartier divisor in $W^{N,o} \times (\mathbb{P}^1 - \{1\})$ which is finite over $W^{N,o}$. It follows that each irreducible component dominates $W^{N,o}$. Hence the integrality is checked over the generic point, which holds by the irreducibility of $f$. The claim is proved. Thus we are reduced to showing the following.

Claim 4.3.4. The condition (4.6) holds if and only if $V \cap (Y \times \Box \times S) = \emptyset$ and $a_\nu \in \Gamma(W^{N,o}, I^\nu)$ for all $\nu$.

The question is Zariski local and we may assume that the $I$ is generated by $\pi \in \Gamma(W^{N,o}, O)$. Then (4.6) holds if and only if $V \cap (Y \times \Box \times S) = \emptyset$ and

$$
\theta := \frac{1 - \sqrt{f}}{\pi} \in \Gamma(W^{N} \times W^{N,o}, O).
$$

(4.7)

Noting $\pi \in k(W)$, the minimal polynomial of $\theta$ over $k(W)$ is

$$
g(t) = t^m + \sum_{1 \leq \nu \leq m} \frac{a_\nu}{\pi^\nu} t^{m-\nu}.
$$

Since $\nabla^o$ is finite over $W^{N,o}$ as is shown before, $\nabla^o \times \nabla^o$ is finite over $W^{N,o}$. Hence (4.7) is equivalent to the condition that $\theta$ is integral over $\Gamma(W^{N,o}, O)$, which is equivalent to

$$
\frac{a_\nu}{\pi^\nu} \in \Gamma(W^{N,o}, O) \quad \text{for all } \nu.
$$

This proves the claim and the proof of Lemma 4.3.3 is completed. $\square$

4.4. Proof of Theorem 4.2.1. We put

$$
C_1(\overline{X}|Y) := \omega \overline{\text{Hom}}_{\text{MPST}}(Z_{\text{tr}}(\Box), Z_{\text{tr}}(M)) \in \text{PST}
$$

and write by $\partial$ for the boundary map $\delta_{1,0}^0 - \delta_{1,\infty}^0 : C_1(\overline{X}|Y) \to Z_{\text{tr}}(M)$. Fix $S \in \text{Sm}$. By Definitions 3.3.3 and 4.1.1, it suffices to construct a homomorphism:

$$
\xi : C_1(\overline{X}|Y)(S) \to \Phi(\overline{X}, Y)(S)
$$

such that the following diagram commutes:

$$
\begin{array}{c}
C_1(\overline{X}|Y)(S) \\
\downarrow \xi \\
\Phi(\overline{X}, Y)(S) \\
\end{array} \quad \begin{array}{c}
\rightarrow \quad \partial \\
\quad \triangleright \\
\rightarrow \quad \tau \\
\end{array} \quad \begin{array}{c}
Z_{\text{tr}}(X)(S) \\
\end{array}
$$

(4.9)
and such that we have

\[(4.10) \quad \text{Image}(\tau) = \text{Image}(\tau \circ \xi).\]

Take a closed integral subscheme \(V \subset X \times \varnothing \times S\) satisfying \((4.6)\). Consider the commutative diagram \((4.4)\). We first suppose \(\overline{W}^\circ \neq \emptyset\). Then we have (see §4.1 for notations)

\[(\overline{W}^N \to \overline{X} \times S) \in C(\overline{X}, Y)(S)\]

The projection \(V \to \varnothing = \mathbf{P}^1 - \{1\}\) induces a rational function \(g_V \in k(\overline{V})^\times\). By [7, Prop.1.4 and §1.6] we have

\[(4.11) \quad \partial V = \gamma_* \text{div}_{\overline{W}^N}(Ng_V) \in \mathbb{Z}_{tr}(X)(S) = \text{Cor}(S, X),\]

where \(N : k(\overline{V})^\times \to k(\overline{W})^\times\) is the norm map induced by \(V \to W\). By Lemma 4.3.3 we have

\[Ng_V = f(0) = 1 + \sum_{1 \leq \nu \leq m} a_{\nu} \in \Gamma(\overline{W}^{N, o}, I) \subset G(\overline{W}^N, \gamma^* Y) \subset \Phi(\overline{X}, Y)(S).\]

We now define a map

\[(4.12) \quad \xi : C_1(\overline{X}|Y)(S) \to \Phi(\overline{X}|Y)(S)\]

by declaring

\[\xi(V) = \begin{cases} N g_V & \text{if } \overline{W}^\circ \neq \emptyset; \\ 0 & \text{if } \overline{W}^\circ = \emptyset. \end{cases}\]

Note that if \(\overline{W}^\circ = \emptyset\), then we have \(\partial(V) = 0\) by Lemma 4.3.1. It follows that the diagram \((4.9)\) commutes thanks to \((4.11)\).

It remains to show \((4.10)\). It suffices to show the following.

**Lemma 4.4.1.** Let \(\varphi : \overline{W} \hookrightarrow \overline{X} \times S\) be a closed immersion such that \((\overline{W}^N \to \overline{X} \times S) \in C(\overline{X}|Y)(S)\), where \(\overline{W}^N \to \overline{W}\) is the normalization. Then, letting \(\gamma^* Y = \overline{W}^N \times_{\overline{X}} Y\),

\[G(\overline{W}^N, \gamma^* Y) \subset \Phi(\overline{X}|Y)(S),\]

is contained in the image of \(\xi : C_1(\overline{X}|Y)(S) \to \Phi(\overline{X}|Y)(S)\).

**Proof.** Take \(g \in G(\overline{W}^N, \gamma^* Y)\). Let \(\Sigma \subset \overline{W}^N\) be the closure of the union of points \(x \in \overline{W}^N\) of codimension one such that \(v_x(g) < 0\), where \(v_x\) is the valuation associated to \(x\). Since \(\overline{W}^N\) is normal, we have \(g \in \Gamma(\overline{W}^N - \Sigma, \mathcal{O})\) and \(g \in G(\overline{W}^N, \gamma^* Y)\) implies

\[(4.13) \quad g - 1 \in \Gamma(\overline{W}^N - \Sigma, I),\]
where $I \subset O_{\overline{W}}$ is the ideal sheaf of $\gamma^*Y \subset \overline{W}$. Let
\[
\psi_g : \overline{W}^N - \Sigma \to \mathbb{P}^1 - \{\infty\}
\]
be the morphism induced by $g$ and $\Gamma \subset \overline{W}^N \times \mathbb{P}^1$ be the closure of the graph of $\psi_g$. Let
\[
\overline{V} \subset \overline{W} \times \mathbb{P}^1 \subset X \times \mathbb{P}^1 \times S
\]
be the image of $\Gamma$ under $\overline{W}^N \times \mathbb{P}^1 \to \overline{W} \times \mathbb{P}^1$. By (4.13) we have $|\gamma^*Y| \subset \psi_g^{-1}(1)$ and hence
\[
(4.14) \quad \overline{V} \cap (Y \times \square \times S) = \emptyset
\]
so that
\[
V := \overline{V} \cap (X \times \square \times S) \subset X \times \square \times S.
\]
It suffices to show the following.

**Claim 4.4.2.** $V \in C_1(\overline{X} | Y)(S)$ and $\xi(V) = g$.

Once we prove the first assertion, the second follows easily from the construction of $\xi$. To prove the first assertion, noting (4.14), it suffices to check the condition $(ii)$ of Lemma 4.3.3. By definition
\[
(\star) \quad \Gamma \cap ((\overline{W}^N - \Sigma) \times (\mathbb{P}^1 - \{\infty\}))
\]
is the graph of $\psi_g$ and hence is defined by $y - g$ where $y$ is the standard coordinate of $\mathbb{P}^1 - \{\infty\} = \text{Spec}(k[y])$.

We have a diagram of schemes
\[
(4.15)
\]
where $\iota_\infty$ are induced by $\infty \in \mathbb{P}^1$. The natural map $\Gamma \to \Gamma'$ is a closed immersion onto an irreducible component that dominates $\overline{V}$. We claim
\[
(4.16) \quad \Sigma \subset \iota_\infty^{-1}(\Gamma).
\]
The claim implies $\overline{W}^{N,o} := \overline{W}^N \times_{\overline{W}} (\overline{W} \setminus \iota_\infty^{-1}(\overline{V})) \subset \overline{W}^N - \Sigma$. Let $\overline{V}^o$ be the reduced part of the irreducible component of
\[
\overline{V}^o := \overline{V} \times_{\overline{W}} \overline{W}^{N,o} = \Gamma' \times_{\overline{W}} \overline{W}^{N,o} \subset \overline{W}^{N,o} \times (\mathbb{P}^1 - \{\infty\})
\]
which dominates $V$; see the following diagram:

\[
\begin{array}{c}
\xymatrix{
\mathcal{W}^N \ar[r] & \mathcal{W}^N \times (\mathbb{P}^1 - \{\infty\}) \ar[l] & \mathcal{V}' \ar[l] & \mathcal{V} \ar[l] \\
\mathcal{W}^N \ar[r] \ar[d] & \mathcal{W}^N \times \mathbb{P}^1 \ar[d] & \Gamma' \ar[d] \ar[l] & \mathcal{V} \ar[d] \ar[l] \\
\mathcal{W} \ar[r] & \mathcal{W} \times \mathbb{P}^1 \ar[l] & \Gamma \ar[l] \\
\end{array}
\]

By $(\clubsuit)$, $\mathcal{V}'$ is defined in $\mathcal{W}^N \times \text{Spec}(k[y])$ by the equation $y - g$ and thus $V$ satisfies Lemma 4.3.3 $(ii)$.

It remains to show (4.16). From (4.15), it is equivalent to

(4.17) \[\Sigma \subset \text{pr}((\mathcal{W}^N \times \infty) \cap \Gamma).\]

Since $\text{pr}_\Gamma$ is proper birational and $\mathcal{W}^N$ is normal, $\text{pr}_\Gamma$ is an isomorphism in codimension one. For a generic point $x \in \Sigma$, there is a unique codimension one point $y \in \Gamma$ such that $x = \text{pr}_\Gamma(y)$ and we have $v_y(g) = v_x(g) < 0$ for $g \in k(\Gamma) = k(\mathcal{W}^N)$. The projection $\mathcal{W}^N \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a morphism $\Gamma \setminus (\mathcal{W}^N \times \{\infty\}) \to \mathbb{P}^1 - \{\infty\}$, which corresponds to $g$. Hence we must have $y \in (\mathcal{W}^N \times \{\infty\}) \cap \Gamma$ which proves (4.17) by the properness of $\text{pr}_\Gamma$. This completes the proof of Lemma 4.4.1. \[\square\]

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