LÊ'S POLYHEDRON FOR LINE SINGULARITIES

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ABSTRACT. We study the topology of a line singularity, which is a complex hypersurface with non-isolated singularity given by a complex line. We describe the degeneration of its Milnor fibre to the singular hypersurface by means of a pair of polyhedra, one in the Milnor fibre and other in the singular fibre, which are deformation retracts of the corresponding fibres; and a continuous map taking the Milnor fibre to the singular fibre and the first polyhedron to the second one, which restrict to a homeomorphism outside the polyhedra. In the same sense, we also study the topology of a complex isolated singularity hypersurface under a non-local viewpoint.

INTRODUCTION

The idea of studying the critical level of a complex function by looking at the non-critical level is classical, used by many authors like Milnor, Hirzebruch, Brieskorn, Pham and others. This lead to the classic Fibration Theorem of Milnor and to the study of the vanishing homology of a singularity.

In the case of an isolated singularity, Lê Dung Trang refined in [10] the idea of vanishing homology and proved that there exists a vanishing polyhedron in the Milnor fibre such that the Milnor fibre deformation retracts to it, and that there is a continuous map from the Milnor fibre to the singular one which restricts to a homeomorphism outside the polyhedron and takes the polyhedron to the singular point.

It is unlikely that there is a natural extension of these results to holomorphic functions with arbitrary singular locus. In [13] J. Seade and the author proved that there is a vanishing polyhedron in the boundary of the Milnor fibre of any complex hypersurface with one-dimensional singular set. This describes how the link of the singularity is obtained from this boundary, whose topology has been studied by many authors (see [19], [14], [17] and [5], for instance).

The main goal of this paper is to show that there is a vanishing polyhedron in the sense of [10] for an important class of singularities called line singularities, defined and first studied by D. Siersma in [18]. These are nothing but complex hypersurface singularity germs with singular locus a complex smooth curve. This is Theorem 3.2.

To prove this theorem, we need to consider the non-local situation of an isolated singularity, that is, the restriction of a holomorphic function $f : \mathbb{C}^n \rightarrow \mathbb{C}$, with an isolated critical point at $0 \in \mathbb{C}^n$, to a closed ball around 0 with radius $\epsilon$, denoted by $B_\epsilon$, where $\epsilon$ is not necessarily a Milnor radius such that the restriction of $f$ to

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the ball $B_{\varepsilon}$ has only one critical point and such that $f^{-1}(0)$ intersects the sphere $S_{\varepsilon}$ transversally, in the stratified sense.

Then in section 1 we generalize Lê's idea of vanishing polyhedron to such non-local situation of an isolated singularity. We show that there exists a pair of polyhedra, one in the smooth fibre $f^{-1}(t) \cap B_{\varepsilon}$, for $t \neq 0$ small, and other in the singular fibre $f^{-1}(0) \cap B_{\varepsilon}$, which are deformation retracts of the corresponding fibres; and a continuous map taking the smooth fibre to the singular one and the first polyhedron to the second one, which restricts to a homeomorphism outside the polyhedra. This is Theorem 1.3. In section 2 we prove this theorem.

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1. Lê’s Polyhedron for isolated singularities

In this section, we study isolated singularity hypersurfaces. In the first subsection we recall the main theorem of [10], which describes the degeneration of the Milnor fibre of a holomorphic germ of function $f : (X, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, defined on a germ of complex analytic set $X$ (with arbitrary singularity). Then in the second subsection we extend that theorem to the non-local case, when $X$ is a complex affine space.

1.1. The local case. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and suppose that $f(0) = 0$, in order to simplify notation. It is well known (see [10] for instance) that there exists a positive real number $\varepsilon > 0$ sufficiently small such that for any $\varepsilon'$ with $0 < \varepsilon' \leq \varepsilon$ one has that $f^{-1}(0)$ intersects transversally the sphere $S_{\varepsilon'}$ around $0 \in \mathbb{C}^n$ with radius $\varepsilon'$. This property gives the so-called (local) conical structure of a complex analytic variety. A real number $\varepsilon > 0$ as above is called a Milnor radius for $f$ and the ball $B_{\varepsilon}$ around 0 with radius $\varepsilon$ is said to be a Milnor ball for $f$.

Milnor showed in [10] that for any Milnor radius $\varepsilon > 0$ there exists a positive real number $\eta$, with $0 < \eta << \varepsilon$, such that the restriction:

$$f| : f^{-1}(D^*_\eta) \cap B_{\varepsilon} \to D^*_\eta$$

is a locally trivial fibration, where $D_\eta$ is the closed ball around 0 in $\mathbb{C}$ with radius $\eta$ and $D^*_\eta := D_\eta \setminus \{0\}$. This is the so-called Milnor fibration theorem.

If $\varepsilon$ is a Milnor radius for $f$ and $\eta$ is sufficiently small as above, then for any $t \in D^*_\eta$ the set $F_t := f^{-1}(t) \cap B_{\varepsilon}$ is called the Milnor fibre of $f$ and the set $F_0 := f^{-1}(0) \cap B_{\varepsilon}$ is called the special fibre of $f$ (since the topological type of $F_t$ does not depend on the Milnor radius $\varepsilon$).

In [9] Lê Dũng Tráng extended the Milnor fibration theorem for any complex analytic germ of function $f$ defined on a reduced complex analytic space $X$ with any singularity, with the only difference that the locally trivial fibration induced by the restriction of the function is a topological fibration (instead of a differentiable fibration). That is, given a Milnor radius $\varepsilon > 0$ there exists a real number $\eta$ with $0 < \eta << \varepsilon$ such that the restriction:

$$f| : f^{-1}(D^*_\eta) \cap B_{\varepsilon} \cap X \to D^*_\eta$$
is a (topological) locally trivial fibration. For any \( t \in D^*_η \), we say that \( X_t := f^{-1}(t) \cap B_η \) is the Milnor fibre of \( f \), with boundary \( \partial X_t := X_t \cap S_η \).

Let \( S = (X_α)_{α \in A} \) be a Whitney stratification of a reduced equidimensional complex analytic space \( X \). We say that a complex analytic function \( f : X \to \mathbb{C} \) has an isolated singularity at \( x \in X \) if the restriction of \( f \) to each stratum \( X_α \) that does not contain \( x \) but whose closure contains \( x \) is a submersion and if the restriction of \( f \) to the stratum \( X_α(x) \) that contains \( x \) has an isolated singularity at \( x \).

If \( f : (X, 0) \to (\mathbb{C}, 0) \) has an isolated singularity at \( 0 \in X \) and if \( ϵ \) and \( η \) are sufficiently small as above, the first author proved in [10] the following theorem:

**Theorem 1.1.** For each \( t \in D^*_η \) there exist:

(i) a polyhedron \( P_t \) in \( X_t \), compatible with the stratification \( S_t \), and a continuous simplicial map \( ξ_t : \partial X_t \to P_t \), compatible with \( S_t \), such that \( X_t \) is homeomorphic to the mapping cylinder of \( ξ_t \);

(ii) a continuous map \( Ψ_t : X_t \to X_0 \) that sends \( P_t \) to \( \{0\} \) and that restricts to a homeomorphism \( X_t \setminus P_t \to X_0 \setminus \{0\} \).

Moreover, the construction of the polyhedron \( P_t \), the map \( ξ_t \), and the map \( Ψ_t \), can be done simultaneously for all \( t \) in a simple path \( γ \subset D^*_η \) connecting an arbitrary \( t_0 \in D^*_η \) to \( 0 \in D^*_η \). This gives a polyhedron \( P \subset f^{-1}(γ) \cap B_ϵ \) such that \( f^{-1}(γ) \cap B_ϵ \) deformation retracts to \( P \) and such that \( P \cap X_0 = P_0 \), for any \( t \in γ \).

In the theorem above and in the rest of this paper, a polyhedron is a triangulable topological space.

1.2. The non-local case. When a given real number \( ϵ \) is not a Milnor radius for \( f : \mathbb{C}^n \to \mathbb{C} \) but \( f^{-1}(0) \) intersects \( S_t \) transversally, the Milnor fibration theorem is still true in some sense. Precisely, there exists \( η > 0 \), with \( 0 < η << ϵ \), such that the restriction \( f_t : f^{-1}(D^*_η) \cap B_ϵ \to D^*_η \) is a locally trivial fibration (this is a consequence of Ehresmann’s fibration theorem), but the topology of the sets \( F_t \) and \( F_0 \) do depend on the radius \( ϵ \). We want to obtain a result like Theorem 1.1 in this non-local situation.

**Definition 1.2.** Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a holomorphic function and let \( ϵ \) be a positive real number such that the restriction \( f_t : f^{-1}(D^*_η) \cap B_ϵ \to D^*_η \) is a locally trivial fibration, for some \( η > 0 \) sufficiently small. We say that a pair of polyhedra \( (P_t, P_0) \) is a Lé’s polyhedra pair for \( f \) relative to \( ϵ \) if:

(i) \( P_t \) is contained in \( F_t \), for some \( t \in D^*_η \), and \( F_t \) deformation retracts to \( P_t \);

(ii) \( P_0 \) is contained in \( F_0 \) and \( F_0 \) deformation retracts to \( P_0 \);

(iii) There exists a continuous map \( Ψ_t : F_t \to F_0 \) which sends \( P_t \) to \( P_0 \) and such that \( Ψ_t \) restricts to a homeomorphism from \( F_t \setminus P_t \) to \( F_0 \setminus P_0 \).

We say that \( f \) admits a Lé’s polyhedra pair relative to a positive real number \( ϵ \) if there exist \( η \) and \( (P_t, P_0) \) as in Definition 1.2. In this case, the polyhedron \( P_t \) is called a Lé’s polyhedron for \( f \) relative to \( ϵ \) and the polyhedron \( P_0 \) is called a special polyhedron for \( f \) relative to \( ϵ \). The map \( Ψ_t \) is called a collapse map for \( f \) relative to \( ϵ \). A Lé’s polyhedra pair describes the degeneration of \( F_t \) to \( F_0 \) by means of the collapse map \( Ψ_t \).

Theorem 1.1 above says that any isolated singularity holomorphic function-germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) admits a Lé’s polyhedra pair \( (P_t, P_0) \) relative to any Milnor
radius $\epsilon$, with $\dim_k P_t = n - 1$ and $P_0 = \{0\}$ (in fact, that theorem is even more general since $f$ can be defined in an arbitrary complex analytic set).

In this section we generalize Lê’s construction to the (isolated singularity) situation when $\epsilon$ is not necessarily a Milnor radius for $f$, but satisfying the following conditions:

(1) $f^{-1}(0)$ intersects $S_\epsilon$ transversally;
(2) $B_\epsilon$ contains exactly one critical point of $f$ (at $0 \in B_\epsilon$).

Precisely, we will prove the following theorem:

**Theorem 1.3.** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function that takes $0 \in \mathbb{C}^n$ to $0 \in \mathbb{C}$. Then $f$ admits a Lê’s polyhedra pair $(P_1, P_0)$ relative to $\epsilon$, for any $\epsilon > 0$ satisfying (1) and (2) above. Moreover:

(i) $P_1$ has real dimension $n - 1$;
(ii) If $\epsilon$ is not a Milnor radius for $f$, then $P_0$ has real dimension $n - 1$;
(iii) If $\epsilon$ is a Milnor radius for $f$, then $P_0 = \{0\}$.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3. For any linear form $l : \mathbb{C}^n \to \mathbb{C}$ taking $0 \in \mathbb{C}^n$ to $0 \in \mathbb{C}$, the restriction of both $f$ and $l$ to $B_\epsilon$ induces an analytic morphism

$$\phi_1 : B_\epsilon \to \mathbb{C}^2$$

defined by $\phi_l(z) = (l(z), f(z))$, for any $z \in B_\epsilon$. We have the following lemma (see theorem-definition 1.4.1 of [10]):

**Lemma 2.1.** There exists a non-empty Zariski open set $\Omega$ in the space of non-zero linear forms of $\mathbb{C}^n$ to $\mathbb{C}$ that take $0 \in \mathbb{C}^n$ to $0 \in \mathbb{C}$, such that for any $l \in \Omega$, the analytic morphism $\phi_l : B_\epsilon \to \mathbb{C}^2$ satisfies:

(i) If $C$ is the critical locus of $\phi_l$ and $\Gamma_l \subset B_\epsilon$ is the union of the irreducible components of $C$ which are not contained in $f^{-1}(0)$, then $\Gamma_l$ is either empty or a complex curve;
(ii) If $\Gamma_l \cap f^{-1}(0) = \{p_1, \ldots, p_r\}$, then for each $p_i$ there exists a small neighbourhood $V_i$ of $p_i$ in $B_\epsilon$ such that the restriction of $\phi_l$ to $\Gamma_l \cap V_i$ defines a biholomorphism from $\Gamma_l \cap V_i$ to its image $\Delta_i := \phi_l(\Gamma_l \cap V_i)$.

We say that $l \in \Omega$ is a good linear form relative to $f$. From now on, we shall fix a good linear form $l$, and in order to simplify notation, we shall denote $\Gamma := \Gamma_l$ and $\Delta := \Delta_l$.

Consider small enough positive reals $\epsilon, \eta_1, \eta_2$ with $0 < \eta_2 << \eta_1 << \epsilon << 1$, such that $\phi$ induces a complex analytic map

$$\phi_1 : \phi_l^{-1}(D_{\eta_1} \times D_{\eta_2}) \cap B_\epsilon \to D_{\eta_1} \times D_{\eta_2}.$$

It restricts to a fibre bundle over $(D_{\eta_1} \times D_{\eta_2}) \setminus \Delta$, where $\Delta = \phi_l(C)$ (see Theorem 2.1 of [2] for instance).

Notice that for any $t \in D_{\eta_2}^*$ the fibre $f^{-1}(t) \cap B_\epsilon$ can be identified with $F_t := B_\epsilon \cap f^{-1}(D_{\eta_1} \times \{t\})$. Set

$$F_{\eta_2} := B_\epsilon \cap f^{-1}(D_{\eta_2}).$$
If we choose $\eta_2 > 0$ sufficiently small, we can suppose that $\Gamma' := \Gamma \cap F_{\eta_2}$ is contained in $\bigcup_{i=1}^{r} V_i$. For just a moment, we denote by $f'$ the restriction of $f$ to $F_{\eta_2}$, and consider $\phi'$ defined on $F_{\eta_2}$ by setting $\phi'(z) := (\ell(z), f'(z))$. Without lost of generality, we can also suppose that $\ell(p_i) \neq \ell(p_j)$, for any $i, j \in \{1, \ldots, r\}$ with $i \neq j$, and then it follows from the previous Lemma that the restriction of $\phi'$ to $\Gamma'$ is a biholomorphism $\Gamma' \to \Delta' := \phi(\Gamma')$. In fact, if $\eta_2$ is sufficiently small, then $\Gamma'$ has exactly $r$-connected components $\Gamma'(p_i)$, that is, $\Gamma'$ is the disjoint union

$$\Gamma' = \bigcup_{i=1}^{r} \Gamma'(p_i),$$

and $\Delta'$ is the disjoint union

$$\Delta' = \bigcup_{i=1}^{r} \phi(\Gamma'(p_i)) := \bigcup_{i=1}^{r} \Delta'(p_i).$$

In order to simplify notation, from now on we shall denote $f := f'$ (defined on $F_{r, \eta}$), $\Gamma := \Gamma'$ and so on. See figure 1.

![Figure 1](image)

As in [10], the proof of Theorem 1.3 is done by induction on the dimension $n$.

2.1. Case $n = 2$: constructing the polyhedra.

Now we consider $n = 2$. For any $t \in D_{\eta_2}^{*}$, set

$$D_t := D_{\eta_1} \times \{t\}.$$

Then $\phi_t : F_{\eta_2} \to D_{\eta_1} \times D_{\eta_2}$ induces a projection

$$\phi_t : F_t \to D_t,$$

which is a finite covering over $D_t \setminus (\Delta \cap D_t)$. Set

$$\Delta \cap D_t := \{y_1(t), \ldots, y_k(t)\}.$$

Note that each $y_j(t)$, for $j = 1, \ldots, k$, is contained in some $\Delta(p_i)$, for some $i = 1, \ldots, r$. Let $\lambda_t$ be the barycenter of $\{y_1(t), \ldots, y_k(t)\}$ in $D_t \setminus \{y_1(t), \ldots, y_k(t)\}$ and
for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path (differentiable and with no double points) starting at $\lambda_t$ and ending at $y_j(t)$, such that two of them intersect only at $\lambda_t$. See figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}

Set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

and

$$P_t := \varphi_t^{-1}(Q_t).$$

Let $v_t$ be a vector field in $D_t$ such that $v_t$ is:

- $C^\infty$;
- null over $Q_t$;
- transversal to $\partial D_t$ and points inwards.

Then the associated flow $q_t: [0, \infty[ \times (D_t \setminus Q_t) \to D_t$ defines a map

$$\xi_t : \partial D_t \to Q_t$$

such that $\xi_t$ is continuous, surjective and differentiable.

Since $\varphi_t$ is a covering over $D_t \setminus Q_t$, which is differentiable in this case of dimension $n = 2$, we can lift $v_t$ to a vector field $E_t$ in $F_t$ such that $E_t$ is:

- continuous over $F_t$;
- differentiable over $F_t \setminus P_t$;
- null over $P_t$;
- integrable;
- transversal to $\partial F_t$ and points inwards.

Then the associated flow $\tilde{q}_t: [0, \infty[ \times (F_t \setminus P_t) \to F_t$ defines a map

$$\tilde{\xi}_t : \partial F_t \to P_t$$

such that $\tilde{\xi}_t$ is continuous, surjective and differentiable. Then one can check that $F_t$ is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$. 
2.2. Case \( n = 2 \): the collapse along a path.

Now, in order to construct the collapse map, we do the construction of the vector field \( E_t \) simultaneously for all \( t \) in a simple path \( \gamma \) in \( D_\eta \) joining 0 and some \( t_0 \in \partial D_\eta \), such that \( \gamma \) is transverse to \( \partial D_\eta \). To simplify, we may assume that \( \gamma \) is the closed segment of line in \( D_\eta \) joining 0 and \( t_0 \).

The natural projection \( \pi : D_{\eta_1} \times D_{\eta_2} \to D_{\eta_2} \) restricted to \( \Delta \) induces a ramified covering

\[
\pi_1 : \Delta \to D_{\eta_2}
\]

whose ramification locus is \( D_\eta \cap \Delta = \{ \phi(p_1), \ldots, \phi(p_r) \} \).

Hence the inverse image of \( \gamma \setminus \{ 0 \} \) by this covering defines \( k \) disjoint simple paths in \( \Delta \), and each one of them is diffeomorphic to \( \gamma \setminus \{ 0 \} \). Each of these paths have \( \phi(p_i) \) in its closure, for some \( i = 1, \ldots, r \), and it contains the points \( y_j(t) \), for some \( j = 1, \ldots, k \) and any \( t \in \gamma \setminus \{ 0 \} \). We shall denote by \( \varsigma_{i,j} \) the respective path that has \( \phi(p_i) \) in its closure and contains \( y_j(t) \). In particular, we have that \( r \leq k \). See figure 3.

![Figure 3.](image)

Moreover, the set \( \Lambda = \bigcup_{t \in \gamma} \lambda_t \) defines a simple path in \( D_{\eta_1} \times D_{\eta_2} \) such that either \( \Lambda \cap \Delta = \phi(p_1) \), if \( r = 1 \), or \( \Lambda \cap \Delta = \emptyset \), if \( r > 1 \).

We can choose the paths \( \delta(y_j(t)) \) in such a way that

\[
T_j := \bigcup_{t \in \gamma} \delta(y_j(t))
\]

forms either a triangle, if \( r = 1 \), or a square, if \( r > 1 \), differentiably immersed in

\[
\bigcup_{t \in \gamma} D_t = D_{\eta_1} \times \gamma
\]

outside \( \delta(y_j(0)) \). For any \( j, j' \in \{ 1, \ldots, k \} \) with \( j \neq j' \), note that either \( T_j \cap T_{j'} = \Lambda \), if both \( \varsigma_{i,j} \) and \( \varsigma_{i',j'} \) are defined for some \( i, i' \in \{ 1, \ldots, r \} \) with \( i \neq i' \); or \( T_j \cap T_{j'} = \Lambda \cup \gamma(y_j(0)) = \Lambda \cup \gamma(y_{j'}(0)) \), if both \( \varsigma_{i,j} \) and \( \varsigma_{i,j'} \) are defined for some \( i \in \{ 1, \ldots, r \} \).

See figure 4.

Set

\[
Q := \bigcup_{j=1}^{k} T_j
\]

and let \( V \) be a vector field in \( D_{\eta_1} \times \gamma \) such that \( V \) is:

- continuous;
- null over \( Q \);
• differentiable over \((D_{\eta_1} \times \gamma) \setminus Q\);
• transversal to \(\partial D_{\eta_1} \times \gamma\); and such that
• the projection of \(V\) on \(\gamma\) is null.

Then the associated flow \(w : [0, \infty[ \times ((D_{\eta_1} \times \gamma) \setminus Q) \rightarrow D_{\eta_1} \times \gamma\) defines a map
\[
\xi : \partial D_{\eta_1} \times \gamma \rightarrow Q \quad z \mapsto \lim_{\tau \to \infty} w(\tau, z)
\]
such that \(\xi\) is continuous, surjective and differentiable. For any real \(A > 0\), set
\[
V_A(Q) := (D_{\eta_1} \times \gamma) \setminus w([0, A[\times \partial D_{\eta_1} \times \gamma]),
\]
a closed neighbourhood of \(Q\) in \(D_{\eta_1} \times \gamma\). Note that \(\partial V_A(Q)\) is a differentiable manifold that fibres over \(\gamma\) with fibre a circle, by the restriction of the projection \(\pi\). Moreover, \(D_{\eta_1} \times \gamma\) is clearly the mapping cylinder of \(\xi\). Set
\[
F_\gamma := \phi^{-1}(D_{\eta_1} \times \gamma) \cap B_c.
\]
Since
\[
\phi_1 : F_\gamma \setminus \phi^{-1}(Q) \rightarrow (D_{\eta_1} \times \gamma) \setminus Q
\]
is a fibre bundle, it follows that \(\phi^{-1}(\partial V_A(Q))\) is a differentiable submanifold of \(F_\gamma\) which is a fibre bundle over \(\gamma\).

Now set
\[
P_\gamma := \phi^{-1}(Q),
\]
which we call the *collapse polyhedron of \(f\) along \(\gamma\). It is a polyhedron in \(F_\gamma\) of real dimension 2. Let \(\theta\) be a vector field in \(\gamma\) that goes from \(t_0\) to 0 in time \(a > 0\). Set
\[
Z := F_\gamma \setminus P_\gamma.
\]
Since
\[
Z = \phi^{-1}((D_{\eta_1} \times \gamma) \setminus Q) \xrightarrow{\phi} (D_{\eta_1} \times \gamma) \setminus Q \xrightarrow{\pi} \gamma
\]
and
\[
\phi^{-1}(\partial V_A(Q)) \xrightarrow{\phi} \partial V_A(Q) \xrightarrow{\pi} \gamma
\]
are (differentiable) fibre bundles, we can lift \(\theta\) to obtain a vector field \(E\) such that:
induces a topological fibre bundle

Case 2.4. we have that $F \in E \varphi$.

Since the dimension $n$ starting at $\lambda$, we collapse polyhedron of $\mathcal{L} \ast \mathcal{P}$, and consider a vector field $P \ast \mathcal{L} \ast \mathcal{P}$, as defined before, and that $\phi(p_i)$, if $r = 1$, or empty, if $r > 1$. Fix $j \in \{1, \ldots, k\}$. The union of paths $\delta(y_j(t))$ for all $t \in \mathcal{L}$ gives a 3-dimensional polyhedron $T_j$ in $\mathcal{L} \times \mathcal{P}$. Note that either $T_j \cap T_j' = \Lambda$ or $T_j \cap T_j' = \Lambda \cup \gamma(y_j(0)) = \Lambda \cup \gamma(y_j'(0))$.

Then we define

$$Q := \bigcup_{j=1}^{k} T_j$$

and consider a vector field $V$ in $\mathcal{L} \times \mathcal{P}$ that retracts $\mathcal{L} \times \mathcal{P}$ onto $Q$. Now set

$$P := \phi^{-1}(Q),$$

which we call the collapse polyhedron of $f$ along a disk. It is a polyhedron of real dimension $n + 1$ contained in $F_{n_2}$.

Also set

$$Z := F_{n_2} \setminus P.$$ 

Since $\phi$ is a submersion over $(\mathcal{L} \times \mathcal{P}) \setminus Q$, it follows that $V$ lifts to a vector field $E$ in $Z$ with the desired properties, which gives the collapse map $\Psi_t$. In particular, we have that $F_{n_2}$ deformation retracts to $P$.

2.3. Case $n = 2$: the collapse along a disk.

We can go further and describe the collapse simultaneously along all the disk $\mathcal{L} \ast \mathcal{P}$ in the following way:

Consider the intersection of $\Delta$ with $\mathcal{L} \ast \mathcal{P} \times \mathcal{P}$. Then we obtain $k$ punctured disks $T_j$. The barycenter points of these punctured disks also give a punctured disk $\Lambda$ such that the intersection of the closure of all this punctured disks is either $\phi(p_i)$, if $r = 1$, or empty, if $r > 1$. Fix $j \in \{1, \ldots, k\}$. The union of paths $\delta(y_j(t))$ for all $t \in \mathcal{L}$ gives a 3-dimensional polyhedron $T_j$ in $\mathcal{L} \times \mathcal{P}$. Note that either $T_j \cap T_j' = \Lambda$ or $T_j \cap T_j' = \Lambda \cup \gamma(y_j(0)) = \Lambda \cup \gamma(y_j'(0))$.

Then we define

$$Q := \bigcup_{j=1}^{k} T_j$$

and consider a vector field $V$ in $\mathcal{L} \times \mathcal{P}$ that retracts $\mathcal{L} \times \mathcal{P}$ onto $Q$. Now set

$$P := \phi^{-1}(Q),$$

which we call the collapse polyhedron of $f$ along a disk. It is a polyhedron of real dimension $n + 1$ contained in $F_{n_2}$.

Also set

$$Z := F_{n_2} \setminus P.$$ 

Since $\phi$ is a submersion over $(\mathcal{L} \times \mathcal{P}) \setminus Q$, it follows that $V$ lifts to a vector field $E$ in $Z$ with the desired properties, which gives the collapse map $\Psi_t$. In particular, we have that $F_{n_2}$ deformation retracts to $P$.

2.4. Case $n \geq 3$: the construction of the polyhedra.

We know that for each $t \in \mathcal{L}$ the projection

$$\varphi_t : F_t \to D_t$$

induces a topological fibre bundle

$$\varphi| : F_t \setminus \varphi^{-1} \{\{y_1(t), \ldots, y_k(t)\}\} \to D_t \setminus \{y_1(t), \ldots, y_k(t)\},$$

where $\{y_1(t), \ldots, y_k(t)\} = \Delta \cap D_t$. Note that each $y_j(t)$, for $j \in \{1, \ldots, k\}$, is contained in some $\Delta(p_i)$, $i \in \{1, \ldots, r\}$, as defined before, and that $k \geq r$ if $t \neq 0$, and $k = r$ if $t = 0$.

As before, for each $t \in \mathcal{P}$ and $j = 1, \ldots, k$, let $\lambda_t := (u', t)$ be the barycenter point of $\{y_1(t), \ldots, y_k(t)\}$ in $D_t = \mathcal{L} \times \mathcal{P}$ and let $\delta(y_j(t))$ be a simple path starting at $\lambda_t$ and ending at $y_j(t)$, such that two of them intersect only at $\lambda_t$.

If $f'$ is the restriction of $f$ to the hyperplane section $\{l = u'\}$, of dimension $n - 1$, we can apply the conclusion of the Theorem, by induction hypothesis, to obtain a Lé's polyhedron $P'_t$ and a vector field $E'_t$ as in the previous subsection.
Recall that $\phi$ defines a fibre bundle over $\delta(y_j(t)) \setminus \{y_j(t)\}$. Now, for each $j = 1, \ldots, k$, define the point

$$x_j(t) := \phi^{-1}(y_j(t)) \cap \Gamma,$$

which is the isolated singularity of $\phi^{-1}(y_j(t))$. If we look at the local situation at $x_j(t)$, we can use the result of [10], that is, we can consider $B_j$ a small ball in $\mathbb{C}^n$ centered at $x_j(t)$ and $D_s$ a small disk in $D_t$ centered at $y_j(t)$ such that the restriction

$$\varphi_{t_j} : B_j \cap \varphi_t^{-1}(D_s) \to D_s$$

satisfies the hypothesis (2.3.2) and (2.3.3) of [10]. See figure 5.

Then if $D^+_s$ denotes a semi-disk in $D_t$ containing $D_s \cap \delta(y_j(t))$, we apply the main theorem of [10] to obtain a collapse cone $P_j$ (which is the collapse polyhedron in the isolated singularity case) and a vector field $E_j$ in $\varphi_t^{-1}(D^+_s) \cap B_j$ that gives the degeneration of the map $\varphi_{t_j}$.

Set

$$a_j := \partial D_s \cap \delta(y_j(t)).$$

Then

$$P_j(a_j) := \varphi_t^{-1}(a_j) \cap P_j$$

is a Lé's polyhedron for the germ $\varphi_t : (F_t, x_j(t)) \to (D_t, y_j(t))$.

Next we construct some useful vector fields on $A_j := \varphi_t^{-1}(\delta(y_j(t)) \setminus \{y_j(t)\})$:

- **Vector Field $\Xi$:** let $\xi$ be a $C^\infty$ vector field non-zero on $\delta(y_j(t)) \setminus \{y_j(t)\}$ that goes from $y_j(t)$ to $\lambda_t = (u', t)$. Since $\varphi_t$ is a fibre bundle over $\delta(y_j(t)) \setminus \{y_j(t)\}$, we can lift $\xi$ to a vector field $\Xi$ on $A_j$ which is integrable (see [21] and (2.3.2.2) of [10]) and tangent to $A_j$ and $\partial A_j$;

- **Vector Field $\mathcal{V}$:** Then we can transport the vector field $E^*_t$ of $\varphi_t^{-1}(\lambda_t)$ to all the fibres $\varphi_t^{-1}(u_t)$, for any $u_t \in \delta(y_j(t)) \setminus \{y_j(t)\}$. Then we obtain a vector field $\mathcal{V}$ on $A_j$ whose restriction to $\varphi_t^{-1}(\lambda_t)$ is $E^*_t$ and the restrictions to $\varphi_t^{-1}(u_t)$ are the vector fields of $\varphi_t^{-1}(u_t)$.
• **Vector Field** $V_1$: Let $\theta$ be a differentiable function on $\delta(y_j(t))$ such that $\theta(\lambda_t) = 0$ and such that $\theta$ is non-zero and positive on $\delta(y_j(t)) \setminus \{\lambda_t, y_j(t)\}$.

It induces a function $\tilde{\theta}$ defined on $A_j$. Define

$$V_1 := V + \tilde{\theta} \Xi,$$

which is integrable, tangent on the interior of $A_j$, transversal and pointing inwards on the boundary $\partial A_j$.

Since $V$ and $\Xi$ are transversal, the vector field $V_1$ is zero only on the Lê’s polyhedron $P_t'$ of $\varphi_t^{-1}(\lambda_t)$. Then if $z$ is a point in $A_j \setminus \varphi_t^{-1}(\lambda_t)$, the orbit of $V_1$ that passes through $z$ has its limit point $z_1'$ in $P_t'$.

Moreover, since the orbit of $V$ that passes through $z$ has its limit point $z'$ in the transportation of $P_t'$ to $\varphi_t^{-1}(\varphi_t(z))$ by $\Xi$, it follows that $z_1'$ is the point corresponding to $z'$ by $\Xi$. This comes from the fact that $V$ and $\Xi$ commute by construction.

Now, since $P_j(a_j)$ is obviously contained in $A_j$, it follows that $V_1$ takes $P_j(a_j)$ to $P_t'$. In fact, it takes all the fibre $\varphi_t^{-1}(a_i) \cap B_r$ to $P_t'$. Now, since the action of the flow given by $V$ is simplicial, we can assume that the action of the flow given by $V_1$ is simplicial. Then the image of $P_j(a_j)$ by the action of $V_1$ is a sub-polyhedron $P_j'$ of $P_t'$. Moreover, the orbits of the points in $P_j(a_j)$ give a polyhedron $R_j$.

Set

$$\tilde{P}_j := P_j \cap \varphi_t^{-1}(\delta(y_j(t)))$$

and

$$S_j := \tilde{P}_j \cup R_j \cup P_j'.$$

See figure 6.

![Figure 6](image_url)

**Lemma 2.2.** For each $t \in D_{\eta_2}$, the polyhedron $P_t$ is the union of a $(n - 1)$-dimensional polyhedron $P_t'$ and $k$-copies of a $n$-dimensional cone $S$, all glued together along their bases, which are a sub-polyhedron of $P_t'$.

**Proof.** Let $r > 0$ be small enough such that the ball $B_r(x_j(t))$ of radius $r$ centered at $x_j(t)$ is contained in $B_j$, for any $j = 1, \ldots, k$, and then set $B_j(t) := B_r(x_j(t))$.

Then, for any $j_1, j_2 = 1, \ldots, k$, we have:
(i) $P'_{j_1} = P'_{j_2}$;

(ii) $P'_{j_1}(a_{j_1}) \text{diff} = P_{j_2}(a_{j_2})$;

(iii) $S_{j_1} \text{diff} = S_{j_2}$.

In fact, the natural extension of $\Xi$ to $\delta(y_{j_1}(t))\{y_{j_1}(t)\} \cup \delta(y_{j_2}(t))\{y_{j_2}(t)\}$ gives (i) and (ii). Clearly, $R_j \text{diff} = P'_j \times I$, for any $j = 1, \ldots, k$; where $I$ denotes a real interval, and then (iii) follows. Then we set $S = S_1$ and the lemma is proved for $t \neq 0$.

Now, since $\phi : F_{T_0} \phi^{-1}(\Delta) \rightarrow (D_{T_1} \times D_{T_0}) \Delta$ is a fibre bundle, it follows that, for any $t \in D_{T_1}$ fixed, one has that $\varphi^{-1}(u_t) \text{diff} = \varphi^{-1}(u_0)$, for any $u_t \in D_t \Delta$ and $u_0 \in D_0 \Delta$. In particular, we have that

$$P'_t \text{diff} = P'_0.$$

Moreover, we have that $\varphi^{-1}(a_{j_1}) \cap B_j(t)$ is homeomorphic to $\varphi^{-1}(a_{j_1}) \cap B_j(0)$, and this implies that the Lé's polyhedron $P_j(a_{j_1})$ of $\varphi_{t_1}$ is homeomorphic to the Lé's polyhedron $P_j(a_{j_1})$ of $\varphi_{0}$. This means that the cone $S$ is the same for the $t$-level and the $0$-level and this completes the proof.

Hence the polyhedron in $F_t$ that we are looking for is given by

$$P_t := P'_t \bigcup_{j=1}^k S_j$$

and the special polyhedron in $M_0$ is given by

$$P_0 := P'_0 \bigcup_{j=1}^r S_j.$$

Figure 7 bellow illustrates the homotopy type of both $P_t$ and $P_0$, as well as the degeneration of the first to the second one, which we will construct next.

Once the polyhedra $P_t$ and $P_0$ are defined, one can construct a vector field $E_t$ that retracts $F_t$ to $P_t$ following the same arguments of sections (4.2.1), (5.1) and (5.2) of [10].
2.5. Case \( n \geq 3 \): constructing the collapse polyhedron along a disk.

In order to describe the collapse of \( f \) along the disk \( D_{\eta_0} \), we first construct the collapse polyhedron (along the disk) \( P \), and latter we will construct the collapse map. Fix \( u' \in D_{\eta_1} \) as before and consider the disk
\[
\Lambda := \{u'\} \times D_{\eta_2}.
\]
Then we define the 3-dimensional polyhedra \( T_j \) in \( D_{\eta_1} \times D_{\eta_2} \), for \( j = 1, \ldots, k \), as before. That is,
\[
T_j := \bigcup_{t \in D_{\eta_2}} \delta(y_j(t)).
\]
For each \( x_j(t) \) over \( y_j(t) \), with \( t \in D_{\eta_2}^* \), choose a small radius \( r(t) \) such that
\[
B_j := \bigcup_{t \in D_{\eta_2}^*} B_{r(t)}(x_j(t))
\]
is a keen neighbourhood of \( \cup_{t \in D_{\eta_2}^*} \{x_j(t)\} \), where \( r(t) \) is a real analytic function of \( t \in D_{\eta_2} \), with \( r(0) = 0 \).
To each \( B_j \) one can associate a neighbourhood
\[
A_j := \bigcup_{t \in D_{\eta_2}^*} D_{s(t)}(y_j(t)),
\]
where \( s(t) \) is an analytic function of \( t \in D_{\eta_2}^* \) with \( 0 < s(t) << r(t) \).

Let \( U \) be a keen neighbourhood of \( \Lambda \setminus \{0\} \) that meets all the \( A_j \)'s, but not containing any \( y_j(t) \), and set
\[
V := \phi^{-1}(U) \cap F_{\eta_2}.
\]
For some \( t_0 \in D_{\eta_2} \), consider the polyhedron \( P_{t_0} \) constructed as before. It is given by the union
\[
P_{t_0} = P_{t_0}' \bigcup_{j=1}^{k} S_j,
\]
glued together along a sub-polyhedron \((P_j)'_{t_0}\) of \( P_{t_0}' \), where \( P_{t_0}' \) is the Lé’s polyhedron of the restriction \( f' \) of \( f \) to \( F_{\eta_2} \cap \{l = u'\} \), which is given by the induction hypothesis. It also provides us a collapse polyhedron \( P' \) in \( F_{\eta_2} \cap \{l = u'\} \) and a continuous vector field \( G' \) (integrable outside \( P' \)) over \( F_{\eta_2} \cap \{l = u'\} \) that gives the degeneration of \( f' \).

Recall that \( P' \cap F_t = P'_t \), for any \( t \in D_{\eta_2} \). Also, if \((P_j)'_{t_0}\) is a sub-polyhedron of \( P_{t_0}' \), the vector field \( G' \) gives a sub-polyhedron \((P_j)'_t\) of \( P' \) such that \((P_j)'_t \cap F_t = (P_j)'_t, \) for any \( t \in D_{\eta_2} \).

From this initial polyhedron \( P_{t_0} \), we are going to construct \( P \) (which latter, in the next sections, will be shown to be a collapse polyhedron for \( f \)) as follows:

Let \( G_U \) be the continuous vector field over \( V \) given by the trivialization of \( G' \) over \( V \). Note that if \( P_U \) is the polyhedron on \( V \) given by the “parallel” transportation of \( P_{t_0}' \), then \( G_U \) sends \( P_U \) to \( P_{t_0} \).

One can also construct an integrable vector field \( G_j \) over \( B_j \) that trivializes it over \( D_{\eta_2} \). Then, using a partition of unity, we glue all the vector fields \( G_j \)'s and \( G_U \) to obtain a trivializing vector field over \( V \cup \bigcup_{j=1}^{k} B_j \) which projects on a radial vector field over \( D_{\eta_2} \) convergent to 0. This allows us to construct the vanishing cone \( P \) from a Lé’s polyhedron \( P_{t_0} \) previously constructed. See figure 8.
2.6. **Case** $n \geq 3$: the collapse.

Following the same arguments of \[10\], one can construct a continuous vector field $E$ in $F_{\eta_2}$ which is integrable and non-null outside $P$ and such that, for any $t \in D_{\eta_2}$, the restriction of $E$ to $F_t$ is a vector field $E_t$ that gives the retraction of $F_t$ to $P_t$. Since this construction is quite technical and analogous to Lé’s construction, we do not do it here.

We have a polyhedron $P_t$ and a vector field $E_t$ simultaneously for any $t \in D_{\eta_2}$, which gives us a vector field $E$ over $F_{\eta_2}$ and a collapse polyhedron $P = \bigcup_{t \in D_{\eta_2}} P_t$.

Let $\tilde{q} : \big[0, \infty\big) \times (F_{\eta_2} \setminus P) \to F_{\eta_2}$ be the flow associated to $E$, which induces a continuous, surjective and differentiable map

$$\xi : \partial F_{\eta_2} \to P \quad z \mapsto \lim_{\tau \to \infty} \tilde{q}(\tau, z).$$

For any positive real $A > 0$, set

$$\tilde{V}_A(P) := F_{\eta_2} \setminus \tilde{q}\big(\big[0, A]\times \partial F_{\eta_2}\big),$$

which is a closed neighbourhood of $P$ in $F_{\eta_2}$. Notice that $\partial \tilde{V}_A(P)$ is a differentiable manifold that fibres over $\gamma^*$ with fibre $\partial F_t$. Moreover, $F_{\eta_2}$ is clearly the mapping cylinder of $\xi$.

Since

$$(\pi \circ \phi)| : F_{\eta_2} \setminus P \to D_{\eta_2}$$

is a submersion, where $\pi : D_{\eta_1} \times D_{\eta_2} \to D_{\eta_2}$ is the natural projection, it follows that

$$(\pi \circ \phi)| : F_{\eta_2} \setminus \text{int}(\tilde{V}_A(P)) \to D_{\eta_2}$$

and

$$(\pi \circ \phi)| : \partial \tilde{V}_A(P) \to D_{\eta_2}$$

are submersions. Hence it follows from Ehresmann’s fibration lemma that $(\pi \circ \phi)| : F_{\eta_2} \setminus \text{int}(\tilde{V}_A(P)) \to D_{\eta_2}$ is a locally trivial fibration, for any $A > 0$. 
Then we can lift a $C^\infty$ vector field $\theta$ over $D_{n_2}$ that converges to 0 on time $a > 0$ to a $C^\infty$ vector field over $F_{n_2} \setminus \text{int}(\tilde{V}_A(P))$ which is tangent to $\partial \tilde{V}_A(P)$, for any $A > 0$. Hence we can actually lift $\theta$ to a differentiable vector field over $F_{n_2} \setminus P$.

Then the associated flow $g : [0, a] \times F_{n_2} \setminus P \rightarrow F_{n_2} \setminus P$ defines a $C^\infty$-diffeomorphism $\Psi_t$ from $F_t \setminus P_t$ to $M_0 \setminus P_0$ that extends to a continuous map from $F_t$ to $M_0$ and that sends $P_t$ to $P_0$, for any $t \in D_{n_2}^*$. This finishes the proof of Theorem 1.3.

2.7. Remark: a more general situation.

We can actually drop the hypothesis that $f : \mathbb{C}^n \rightarrow \mathbb{C}$ has only one critical point in the (not necessarily Milnor) ball $B_\epsilon$. Instead of that hypothesis, we ask the critical locus of $f$ to be zero-dimensional, and hence it intersects the compact ball $B_\epsilon$ on a finite set of points $\{u_1, \ldots, u_m\}$.

If these points lie in the same special fibre $f^{-1}(0)$, the construction of a Lé’s polyhedra pair is exactly the same. Otherwise, we could take $\eta > 0$ sufficiently small such that $0 \in D_\eta$ is the only critical value of $f$. But we can allow $D_\eta$ to have finitely many critical values $\{t_1, \ldots, t_m\}$, and then we ask as hypothesis that each fibre $f^{-1}(t)$ intersects the sphere $S_\epsilon$ transversally (this will be the situation in the next section).

In this case, the construction of a Lé’s polyhedra pair is still the same, but one should note that the collapse along a path $\gamma$ connecting a regular value $t_0$ to the critical value 0 and passing through one critical value $t_i \neq 0$ actually describes three degenerations:

- the collapse of $F_{t_0}$ to $F_{t_i}$ though the sub-path $\gamma_1 \subset \gamma$ connecting $t_0$ to $t_i$;
- later, the inverse of the collapse of $F_{t'}$ to $F_{t_i}$, for some regular value $t' \in \gamma \setminus t_0$, though the sub-path $\gamma_2 \subset \gamma$ connecting $t_i$ to $t'$;
- finally, the collapse of $F_{t'}$ to $F_0$ though the sub-path $\gamma_3 \subset \gamma$ connecting $t'$ to 0.

3. Line singularities

When one wishes to generalize a property of isolated singularities for non-isolated singularities, the most natural class to be studied is that of line singularities, which were first defined by Siersma in [18] as the class of holomorphic germs of function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with critical locus a smooth germ of curve $(\Sigma, 0)$. The main goal of this paper is to prove that line singularities can be given a Lé’s polyhedra pair (relative to any Milnor radius $\epsilon$). But first we need the following definition:

**Definition 3.1.** Let $f$ be a line singularity as above and let $H_s$ be a family of hyperplane sections of $\mathbb{C}^{n+1}$ transversal to $\Sigma$ at each $s \in \Sigma$. We say that a real number $\epsilon > 0$ is a good Milnor radius for $f$ (and that $B_\epsilon$ is a good Milnor ball for $f$) if $\epsilon$ is a Milnor radius for $f$ and, for any $s \in \Sigma_\epsilon := \Sigma \cap B_\epsilon$, the intersection $B_\epsilon \cap H_s$ is a Milnor ball for the restriction of $f$ to $H_s$. We say that $f$ admits a good Milnor radius if there exists $\epsilon > 0$ which is a good Milnor radius for $f$.

Now we can state the main theorem of this paper:

**Theorem 3.2.** Any line singularity $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ admits a Lé’s polyhedra pair (relative to any Milnor radius $\epsilon$). Moreover:
(i) \( P_t \) has real dimension \( n + 1 \);
(ii) \( P_0 \) has real dimension either \( n+1 \) or 2 (and clearly \( P_0 \) must be contractible);
(iii) If \( f \) admits a good Milnor radius, then \( P_0 = \Sigma \cap B_e \).

In the rest of this section, we prove Theorem 3.2 using two lemmas that we will prove in the next section.

Let \( f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) be a line singularity with critical locus \( \Sigma \). Without lost of generality, we can suppose that \( \Sigma \) is the \( z_{n+1} \)-axis, that is:
\[
\Sigma = \{ z_1 = \cdots = z_n = 0 \},
\]

Consider a generic family of parallel hyperplane sections \( H_s \subset \mathbb{C}^{n+1} \) transversal to \( \Sigma \) at each \( s \in \Sigma \) such that the each \( H_s \) is transversal to the smooth part of the hypersurface defined by \( f \). Then considering the restrictions
\[
f_s := f|_{H_s} : H_s \to \mathbb{C}
\]
we obtain a family, in the parameter \( s \in \Sigma \), of holomorphic functions with isolated singularity.

Let \( \epsilon > 0 \) be small enough such that \( B_e \subset \mathbb{C}^{n+1} \) is a Milnor ball for \( f \) and such that \( B_e \cap H_0 \) is a Milnor ball for \( f_0 \). Also let \( \eta \), with \( 0 < \eta << \epsilon \), be small enough such that the restriction
\[
f_{| : f^{-1}(D^*_\eta) \cap B_e \to D^*_\eta
\]
is a locally trivial fibration.

For any \( t \in D^*_\eta \), our goal is to describe the degeneration of the Milnor fibre \( F_t := f^{-1}(t) \cap B_e \) of \( f \) to the special fibre \( F_0 := f^{-1}(0) \cap B_e \) of \( f \). To simplify notation, from now on we shall denote by \( f_s \) the restriction of \( f \) to \( B_e \cap H_s \), for each \( s \) in \( \Sigma_e := \Sigma \cap B_e \).

Now, for any \( t \in D^*_\eta \) fixed, let
\[
\pi_t : F_t \to \Sigma_e
\]
be the holomorphic function given by the projection of \( \mathbb{C}^{n+1} \) on \( \Sigma \) induced by the hyperplane sections \( H_s \), restricted to \( F_t \). Set:
\[
F_{t,s} := \pi^{-1}_t(s) = f_s^{-1}(t).
\]
If \( t \neq 0 \), note that \( F_{t,s} \) is a Milnor fibre for \( f_s \) if and only if \( B_e \cap H_s \) is a Milnor ball for \( f_s \), which is not always the case.

We have to understand how \( F_{t,s} \) degenerates to \( F_{t,0} \) as \( s \in \Sigma_e \setminus \{0\} \) goes to \( 0 \in \Sigma_e \), for each \( t \in D^*_\eta \) fixed. We have the following lemmas, which give a Lê’s polyhedra pair for the projection \( \pi_t \), for any \( t \in D^*_\eta \) fixed.

**Lemma 3.3.** For any \( s \in \Sigma_e \), let \((P_{t,s}, P_{0,s})\) be a Lê’s polyhedra pair for \( f_s \) (in the non-local sense of subsection 1.2, considering Remark 2.7). Then the pair of polyhedra \((P_{t,s}, P_{t,0})\) is a Lê’s polyhedra pair for the projection \( \pi_t \). In particular, there exists a continuous map \( \Psi_{t,s} : F_{t,s} \to F_{t,0} \) such that \( \Psi_{t,s} \) restricts to a homeomorphism \( F_{t,s} \setminus P_{t,s} \to F_{t,0} \setminus P_{t,0} \) and takes \( P_{t,s} \) to \( P_{t,0} \).

**Lemma 3.4.** For any \( t \in D^*_\eta \) fixed, the construction of the polyhedron \( P_{t,s} \) and of the collapse \( \Psi_{t,s} : F_{t,s} \to F_{t,0} \) can be done simultaneously for any \( s \in \Sigma_e \). Then we obtain a polyhedron \( P_t \) in \( F_t \) such that \( F_t \) deformation retracts to \( P_t \) and such that, for any \( s \in \Sigma_e \), the fibre \( F_{t,s} \) deformation retracts to the intersection \( P_t \cap F_{t,s} \).
Moreover, \( P_t \) has real dimension \( n + 1 \) if \( t \neq 0 \) and \( P_0 \) has real dimension either 0 or \( n + 1 \).
In the next sections we will prove Lemmas 3.3 and 3.4. We remark that the family \((f_s)\) of isolated singularity hypersurface germs is \(\mu\)-constant if, and only if, the “polar curve of \((f_s)\) with respect to \(\{s = 0\}\) does not split”, that is:

\[
\{(z_1, \ldots, z_n, s) \in \mathbb{C}^n \times \mathbb{C} : \frac{\partial f_s}{\partial z_i} = 0, \ i = 1, \ldots, n\} = \{0\} \times \mathbb{C}
\]

near \((0, 0)\) (see [7] for instance). That is why we need to consider Remark 2.7 in Lemma 3.3.

Now, assuming that the Lemmas above are true, we will show that the pair of polyhedra \((P_t, P_0)\) given by them is a Lê’s polyhedra pair for \(f\). Define the collapse map

\[
\Psi_t : F_t \to F_0
\]

for \(f\) setting

\[
\Psi_t(z) := \Psi_{t, \pi_t(z)}(z),
\]

where \(\Psi_{t,s} : F_{t,s} \to F_{0,s}\) is a collapse map for \(f_s\) (in the non-local sense), which restricts to a homeomorphism from \(F_{t,s}\) to \(F_{0,s}\) and sends \(F_{t,s}\) to \(F_{0,s}\).

Then \(\Psi_t\) clearly restricts to a homeomorphism \(F_t \setminus \partial F_t \to F_0 \setminus \partial F_0\) and \(\Psi_t(P_t) = P_0\), and then we have proved \((ii)\) of Theorem 3.2.

Now suppose that \(f\) admits a good Milnor radius. Then \(B_t \cap H_s\) is a Milnor ball for \(f_s\), for any \(s \in \Sigma_e\), and then \(P_{0,s} = \{s\}\), as in the local situation of [10]. Hence \(P_0 = \Sigma_e\) and we have proved \((iii)\) of Theorem 3.2.

4. Proof of Lemmas 3.3 and 3.4

Since \(F_t\) and \(F_{t,s} = \pi_t^{-1}(s)\) are smooth manifolds for any \(t \in D^*_n\) and \(s \in \Sigma_e\), we can do the same constructions of [10] and of section 1 for the projection \(\pi_t : F_t \to \mathbb{C}\) defined in the previous section. The same happens for the projection \(\pi_0 : F_0 \to \mathbb{C}\), since the critical locus of \(\pi_0\) coincides with the singular set of \(F_0\).

Then we have the following proposition:

**Proposition 4.1.** For each \(t \in D_n\) and \(s \in \Sigma_e\), there exist:

\(i)\) a polyhedron \(P_{t,s}\) in the fibre \(F_{t,s} = \pi_t^{-1}(s) = f^{-1}(t) \cap H_s \cap B_s\), compatible with the Whitney stratification \(S\) of \(F_t\), and a continuous simplicial map \(\hat{\xi}_{t,s} : \partial F_{t,s} \to P_{t,s}\), compatible with \(S\), such that \(F_{t,s}\) is homeomorphic to the mapping cylinder of \(\hat{\xi}_{t,s}\);

\(ii)\) a continuous map \(\Psi_{t,s} : F_{t,s} \to F_{t,0}\) that sends \(P_{t,s}\) to \(P_{t,0}\) and that restricts to a homeomorphism \(F_{t,s} \setminus \partial F_{t,s} \to F_{t,0} \setminus \partial F_{t,0}\).

Moreover, the construction of the polyhedron \(P_{t,s}\), the map \(\hat{\xi}_{t,s}\) and the map \(\Psi_{t,s}\) can be done simultaneously for all \(s\) in the disk \(\Sigma_e\). This gives a polyhedron \(P_t \subset F_t\) such that \(F_t\) deformation retracts to \(P_t\) and such that \(P_t \cap F_{t,s} = P_{t,s}\) for any \(s \in \Sigma_e\).

**Idea of the proof:** For a general linear form \(l : \mathbb{C}^{n+1} \to \mathbb{C}\) taking \(0 \in \mathbb{C}^{n+1}\) to \(0 \in \mathbb{C}\) such that the set \(\mathcal{D} := l(F_t)\) is diffeomorphic to a disk in \(\mathbb{C}\), we define the analytic morphism

\[
\phi : F_t \to \mathbb{C}^2
\]
given by $\phi_t(z) = (l(z), \pi_t(z))$, for any $z \in F_t$. Then we consider the polar curve $\Gamma$ and the polar image of $\pi_t$ relative to $l$. For any $t \in D_\eta$, set

$$D_s := D \times \{s\}.$$ 

Then $\phi : F_t \to D \times D_\eta$ induces a projection

$$\varphi_t : F_{t,s} \to D_s,$$

which is a fibre bundle over $D_s \setminus (\Delta \cap D_s)$. Then the rest of the proof follows combining the techniques of [10] and of the proof of Theorem 1.3.

Then Lemmas 3.3 and 3.4 follow immediately from the proposition above, noting that since the construction of the polyhedra depends only on the linear form $l$ restricted to the Milnor fibre of the respective map; then when one constructs a Lê’s polyhedron for $\pi_t$, one just have to consider the linear form $l$ to be the same of that used in the construction of the Lê’s polyhedron $P_{t,s}$ for $f_s$.

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