A modified of FR method to solve unconstrained optimization

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Abstract. There are many methods derived from the conjugate gradient method, the most famous of which is the FR method (Fletcher–Reeves). Most of the methods are found to solve large unconstrained optimization problems. In this paper, we made a modified to the FR method, so that it achieves better numerical results as well as the conditions of global convergence. The numerical experiment showed the efficiency and robustness of the new method.

Keywords: Conjugate gradient method, Fletcher–Reeves method, global convergence.

1. Introduction

Conjugate gradient (CG) method is a typical method to solve large scale unconstrained optimization problems in the form

$$\min f(x)_{x \in \mathbb{R}^n},$$

(1.1)

where $f: \mathbb{R} \rightarrow \mathbb{R}^n$ is differentiable and continuously. Using the iterative form

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \ldots$$

(1.2)

where $x_k$ is the $k_{th}$ iterative point and $d_k$ is the search direction, $\alpha_k > 0$ represented the step length and $d_k$ is calculated by:

$$d_k = \begin{cases} -F_k & \text{if } k = 0 \\ -F_k + \beta_k d_{k-1} & \text{if } k \geq 1 \end{cases}$$

(1.3)

$\beta_k \in \mathbb{R}$ Defined as a coefficient for the CG method, Note that they are of various formats and have undergone many and various modifications. Here we mention the most famous of them:

- Fletcher-Reeves (FR), Polak-Ribière-Polyak (PRP), Hestenes-Stiefel (HS), and Dai-Yuan (DY).

$$\beta_k^{FR} = \frac{||F_k||^2}{||F_{k-1}||^2}, \quad \beta_k^{PRP} = \frac{F_k^T y_{k-1}}{||F_{k-1}||^2}, \quad \beta_k^{HS} = \frac{F_k^T y_k}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{DY} = \frac{||F_k||^2}{d_{k-1}^T y_{k-1}} [1].$$

Where $F_k$ is the derivative of $f(x)$ at $x_k$.

The global convergence of the FR method has been proven. Noting that it was used many different line search as exact line search and SWP line search ... etc. [2]. In our method, we used the strong Wolfe-
Powell line search \([3, 4]\). The presented numerical experiments showed the good performance and competitiveness of the new method.

The rest of this work is arranged as follows: in section 2 we will exhibit the new formula and the algorithm based on the modifications of the FR method. In Section 3 we prove the global convergence of suggestion method, while numerical results and conclusion are given in sections 4, 5 respectively.

2. The new algorithm

In this section we suggest a new algorithm by modified (FR) method as follows:

Consider
\[
\beta_k^{mFR} = \frac{\gamma \|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|w_k\|^2 F_k^T(F_k - F_{k-1})}{\|F_{k-1}\|^2},
\]
(2.1)

where \(\gamma > 0\), and \(w_k = x_{k+1} - x_k\) we will use the following search direction
\[
d_k = \begin{cases} -F_k & \text{if } k = 0 \\ -F_k + \beta_k^{mFR} w_k & \text{if } k \geq 1. \end{cases}
\]
(2.2)

As we previously mentioned we use in our proposed algorithm the following line search
\[
\beta_k^m \min_{\alpha_k} f(x_k + \alpha_k d_k) \leq \alpha_k \|d_k\|^2, \]
(2.3)

Where \(\sigma > 0\). The proposed algorithm will be as the following:

2.1 Algorithm

Step 1. Select a primary point \(x_0 \in \mathbb{R}^n\), \(\varepsilon \in (0,1)\), \(\gamma > 0\), \(\rho > 0\), \(d_0 = -F_0 = -\nabla f(x_0)\), \(k = 0\).

Step 2. if \(\|F_{k-1}\| \leq \varepsilon\), then stop, otherwise, go to the next step.

Step 3. Compute \(\alpha_k\) from (2.3).

Step 4. \(x_{k+1} = x_k + \alpha_k d_k\), if \(\|F_k\| \leq \varepsilon\), then stop.

Step 5. Compute the search direction \(d_k\) by (2.2), where \(\beta_k^{mFR}\) calculated by (2.1).

Step 6. Set \(k := k + 1\), go to step 3.

3. Global Convergence

To prove the global convergence of the suggest conjugate gradient method, we will assume the following

3.1. Assumption

(I) the set \(\Omega = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}\) is bounded if \(x\) is the initial point.

(II) For some neighborhood \(N\) of the set \(\Omega\), assume \(F\) be Lipchitz continuous on \(\Omega\), i.e., \(\exists\) a positive number \(L > 0\), such that
\[
\|F[x] - F[y]\| \leq L\|x - y\|, \forall x, y \in \Omega
\]

Lemma 3.1. Suppose Assumption holds. Form (1.2), and (1.3), where \(\alpha_k\) satisfies (2.3) then the following condition holds.
\[
\sum_{k=0}^{\infty} \frac{(F_k^T d_k)^2}{\|d_k\|^2} < +\infty \quad (3.1)
\]

It equivalent to
\[
\sum_{k=0}^{\infty} \frac{\|F_k\|^2}{\|d_k\|^2} < \infty \quad (3.2)
\]

Proof: Zoutendijk, G. gave the proof \([5]\). □

Theorem 3.2 Suppose Assumption 3.1 holds then
\[
\lim_{k \to 0} \|F_k\| = 0, \forall k \geq 0
\] (3.3)

Proof: suppose there exists a positive number \(\epsilon > 0\) such that
\[
\|F_k\| > \epsilon, \forall k \geq 0
\] (3.4)

Now squaring both sides of (2.2) we get
\[
\|d_k\|^2 = \frac{1}{\gamma} (\|F_k\|^2 - 2 \beta_k m_{FR} w_k F_k^T + (\beta_k m_{FR})^2 (w_k)^2)
\] (3.5)

Substituting (2.1) in equation (3.5), we get
\[
\beta_k m_{FR} = \frac{\gamma \|F_k\|^2}{\|F_k\|^2 \|F_{k-1}\|^2} + \frac{\|w_k\|^2 F_k^T (F_k - F_{k-1})}{\|F_{k-1}\|^2} \leq \frac{\|F_k\|^2}{\|F_{k-1}\|^2}
\] (3.6)

That is mean
\[
\|d_k\|^2 \leq \frac{\gamma \|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|w_k\|^2 \|F_{k-1}\|^2}{\|F_{k-1}\|^2}
\] (3.7)

Dividing both sides by \(\|F_k\|^4\) then
\[
\frac{\|d_k\|^2}{\|F_k\|^4} \leq \frac{\gamma}{\|F_{k-1}\|^2} + \frac{\|w_k\|^2}{\|F_{k-1}\|^4}
\] (3.8)

And repeatedly to the (3.8), we will get
\[
\frac{\|d_k\|^2}{\|F_k\|^4} \leq \frac{1}{\|F_{k-1}\|^2} \sum_{l=1}^{k} \frac{1}{\|F_l\|^2}
\] (3.9)

From (3.3) and (3.8) we get
\[
\frac{\|d_k\|^2}{\|F_k\|^4} \geq \frac{\epsilon^2}{k}
\] (3.10)

This means
\[
\sum_{k=0}^{\infty} \frac{\|F_k\|^4}{\|d_k\|^2} = \infty
\] (3.11)

We have a contradiction with (3.2), then (3.3) hold, the proof is completed.

4. Numerical Results

In this section we will compare our method with some famous algorithms, they are as follows:

Q. Li et.al. [6], Wasi, H. A. and Shiker, M. A. K. [7] and M. Rivaie et.al.[8], which was indicated in the tables of numerical results as QD, AP and MM respectively, we have also indicated our proposed algorithm by MFR. The numerical results showed a clear differentiation from other algorithms concerning functions evaluations, a number of iterations, and processing time, measured in seconds. Many researchers had adopted the principle of modification as well as hybridization to obtain more accurate results [see 8- 24].

The parameters selected as follows: \(\rho = 0.7, \sigma = 0.3, \epsilon = 10^{-8}, \theta = 0.2\), and the stop condition is \(\|F_{k-1}\| \leq 10^{-8}\).

All algorithms perform through MATLAB R2014 and run on PC with 2.5 GHz CPU processor and 12 GB RAM and Windows XP operation system. The results are shown in the following tables.
| problem | Dim   | f eval | Iter |
|---------|-------|--------|------|
| P1      | 500000|        |      |
|         |       | MFR    | QD   | AP  | MM  | MFR | QD  | AP  | MM  |
|         | 58    | 163    | 30   | 53  | 14  | 22  | 3   | 13  |
|         | 58    | 163    | 30   | 352 | 14  | 22  | 3   | 26  |
|         | 52    | 153    | 85   | 47  | 12  | 19  | 8   | 11  |
|         | 52    | 153    | 85   | 300 | 12  | 19  | 8   | 22  |
| P2      | 500000|        |      |
|         |       | MFR    | QD   | AP  | MM  | MFR | QD  | AP  | MM  |
|         | 58    | 163    | 30   | 53  | 14  | 22  | 3   | 13  |
|         | 61    | 12     | 7    | 384 | 15  | 2   | 2   | 28  |
|         | 52    | 153    | 85   | 47  | 12  | 19  | 8   | 11  |
|         | 52    | 270    | 38   | 328 | 12  | 31  | 5   | 24  |
| P3      | 500000|        |      |
|         |       | MFR    | QD   | AP  | MM  | MFR | QD  | AP  | MM  |
|         | 104   | 597    | 357  | 228 | 25  | 78  | 52  | 56  |
|         | 164   | 947    | 470  | 1084| 40  | 114 | 63  | 80  |
|         | 92    | 279    | 308  | 112 | 22  | 44  | 49  | 27  |
|         | 128   | 389    | 253  | 841 | 31  | 65  | 44  | 62  |
| P4      | 500000|        |      |
|         |       | MFR    | QD   | AP  | MM  | MFR | QD  | AP  | MM  |
|         | 276   | 261    | 642  | 261 | 69  | 61  | 80  | 61  |
|         | 295   | 268    | 642  | 268 | 74  | 64  | 79  | 64  |
|         | 371   | 265    | 642  | 265 | 94  | 62  | 80  | 62  |
|         | 283   | 266    | 642  | 266 | 71  | 62  | 79  | 62  |
| P5      | 500000|        |      |
|         |       | MFR    | QD   | AP  | MM  | MFR | QD  | AP  | MM  |
|         | 56    | 10606  | 178  | 56  | 13  | 585 | 39  | 13  |
|         | 60    | 19868  | 54   | 382 | 14  | 1014| 11  | 28  |
|         | 52    | 1978   | 6    | 328 | 12  | 142 | 1   | 24  |
|         | 56    | 4880   | 9    | 355 | 13  | 300 | 2   | 26  |

| problem | Dim   | CPU-Time |
|---------|-------|----------|
| P1      | 500000|          |
|         |       | MFR      | QD     | AP   | MM   |
|         | 0.328125 | 0.71875  | 0.203125 | 0.21875 |
|         | 0.265625 | 0.6875  | 0.1875  | 1.28125 |
|         | 0.203125 | 0.734375 | 0.46875  | 0.1875 |
|         | 0.3125  | 0.65625  | 0.5     | 1.140625 |
| P2      | 500000|          |
|         |       | MFR      | QD     | AP   | MM   |
|         | 0.3125  | 0.65625  | 0.203125 | 0.375 |
|         | 0.328125 | 0.03125  | 0.03125  | 1.421875 |
|         | 0.25    | 0.5625  | 0.484375 | 0.203125 |
|         | 0.265625 | 1.171875 | 0.15625  | 1.171875 |
| P3      | 500000|          |
|         |       | MFR      | QD     | AP   | MM   |
|         | 0.140625 | 0.65625  | 0.5     | 0.3125 |
|         | 0.15625 | 0.90625  | 0.53125 | 0.9375 |
|         | 0.140625 | 0.265625 | 0.375   | 0.125 |
|         | 0.09375  | 0.421875 | 0.296875 | 0.6875 |
|         | 0.171875 | 0.171875 | 0.421875 | 0.1875 |
5. Conclusions

In this paper, the FR (Fletcher–Reeves) algorithm was modified to solve unconstrained optimization problems. This modified proved its effectiveness through numerical results, the global convergence of the proposed method was proven, and the comparison with other algorithms clarify the ability of the proposed method to compete.

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