Efficient, deterministic voting rules that approximate Dodgson and Young scores

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Abstract

We provide deterministic, polynomial-time computable voting rules that approximate Dodgson’s and (the “minimization version” of) Young’s scoring rules to within a logarithmic factor. Our approximation of Dodgson’s rule is tight up to a constant factor, as Dodgson’s rule is \( NP \)-hard to approximate to within some logarithmic factor. The “maximization version” of Young’s rule is known to be \( NP \)-hard to approximate by any constant factor. Both approximations are simple, and natural as rules in their own right: Given a candidate we wish to score, we can regard either its Dodgson or Young score as the edit distance between a given set of voter preferences and one in which the candidate to be scored is the Condorcet winner. (The difference between the two scoring rules is the type of edits allowed.) We regard the marginal cost of a sequence of edits to be the number of edits divided by the number of reductions (in the candidate’s deficit against any of its opponents in the pairwise race against that opponent) that the edits yield. Over a series of rounds, our scoring rules greedily choose a sequence of edits that modify exactly one voter’s preferences and whose marginal cost is no greater than any other such single-vote-modifying sequence.

1 Introduction

A voting rule takes a collection of voter preferences (over some fixed set of candidates, or alternatives) and aggregates them into a single ranking, ideally in a way that is as “fair” as possible to the voters. Arrow’s famous impossibility theorem \cite{Arr50} states that no such rule over three or more candidates meets all reasonable fairness criteria. So when considering such rules it may be important to know which criteria they do and do not meet.

One such criterion, which actually predates those studied explicitly by Arrow, is credited to the Marquis de Condorcet \cite{Con85}.\footnote{In fact, centuries earlier Lull considered essentially the same criterion \cite{HP01}.} A candidate that, against any opposing candidate, is preferred by a majority of voters is called the Condorcet winner. Note
that different majorities may prefer the Condorcet winner to different opponents. Note also that such a winner may not exist, but if it does then it is unique. The Condorcet criterion states that, whenever a Condorcet winner does exist, it must be declared the winner. See [You77] for a nice discussion of its virtues.

Unfortunately, many widely used rules, such as plurality, instant-runoff, and Borda count do not have this very natural property. Many that do bring with them undesirable features, for instance Copeland elections [Cop51] tend to frequently result in ties. Others, such as those due to Dodgson [Dod76], Kemeny [Kem59], and Young [You77] are \( NP \)-hard to compute [BTT89, RSV03]. In fact, they are complete with respect to parallel access to \( NP \) [HHR97, HSV05, RSV03], which means that even if the problem of determining the winner according to one these rules is “merely” in \( NP \), the polynomial hierarchy would collapse (to \( NP \)).

We can view Dodgson’s and Young’s rules as variations on a theme: Given a list, or profile, of the voters’ preferences (here as is standard in the theory of voting we take each voter’s preferences to be a total ranking over all the candidates) and a candidate we wish to score, either rule takes as the candidate’s score the edit distance [CLRS01] between the given preference profile and one that makes the candidate a Condorcet winner. In other words, it is the number of edits (exactly what an edit is depends on the particular scoring rule) needed to reduce to zero the vote deficit between the given candidate and each of its rivals. Candidates are then ranked in increasing order by their scores. In Young’s rule, an edit simply deletes one voter from the list. For Dodgson, an edit takes one voter’s ranking and replaces it with one just like it, except that in the new one the positions of one pair of candidates ranked adjacently in the original list are swapped. Clearly, both rules satisfy the Condorcet criterion, as any Condorcet winner has a score of zero.

A simple example illustrates how scoring works. Let \( a, b, c, d, \) and \( e \) be five candidates and let

\[
\begin{align*}
    a &>_1 b >_1 c >_1 d >_1 e \\
    a &>_2 b >_2 c >_2 d >_2 e \\
    d &>_3 a >_3 c >_3 b \\
    d &>_4 a >_4 c >_4 b \\
    c &>_5 e >_5 b >_5 d >_5 a
\end{align*}
\]

be a preference profile having five voters. In this example, no candidate is a Condorcet winner. Note that \( c \) is preferred over \( b, d, \) and \( e \) by majorities of voters and is losing to \( a \) by four votes. To make \( c \) the Condorcet winner, we could swap \( c \) with \( b \) and then with \( a \) in voter one and two’s rankings. It turns out there is no shorter sequence of swaps that makes \( c \) the Condorcet winner, so the Dodgson score of \( c \) is four. Note that in this case, the swaps between \( c \) and \( b \) do not actually reduce \( c \)'s vote deficit, since \( c \) is already beating \( b \).

\[\text{Kemeny’s voting rule is sometimes called the Kemeny-Young rule, as Young studied it and made some important breakthroughs. [YL78, You88], e.g., he showed that it satisfies the Condorcet criterion. The Young-only rule to which we refer is distinct from the Kemeny-Young rule, which to avoid confusion we will call simply “Kemeny.”}\]
Candidate $d$ is losing to $c$ and $b$ by one vote each. To make $d$ the Condorcet winner, we could remove voters one and two. Thus $d$’s Young score is two. In this case, both removals yield two deficit reductions, but in general the number of deficit reductions that each removal yields will vary.

As Procaccia et al. observe \cite{PFR07}, McCabe-Dansted effectively proves that it is hard to $\Omega(\log m)$-approximate Dodgson elections, where $m$ is the number of candidates \cite{MD06}. In the same paper, Procaccia et al. show that it is hard to approximate the “maximization version” of Young’s score—i.e., where the Young score is taken to be the largest subset of voters that makes a given candidate the Condorcet winner—by any constant factor \cite{PFR07}.

In this paper we present a framework for efficient, edit-based scoring rules. From this framework, we obtain $O(\log m)$ approximations of the scoring rules due to Dodgson and Young. The basic idea is very simple: Given a profile of voter preferences and a candidate we wish to score, let the marginal cost of a sequence of edits be the number of edits divided by the number of times that, as the edits are applied, the vote deficit against the candidate we wish to score is reduced. Now, proceed over a series of rounds to edit the profile until the chosen candidate becomes the Condorcet winner. In each round, greedily choose a voter and a sequence of edits on that voter’s preferences that, over all such voters and sequences, has the minimum marginal cost.

It turns out that, when we restrict the edits the algorithm makes to those allowed by Dodgson’s (respectively, Young’s) scoring rule, the result is a polynomial-time $O(\log m)$-approximation, where $m$ is the number of candidates. Thus, in the case of Dodgson elections, the approximation is tight up to a constant factor.

Why care about approximations to voting rules in the first place? One reason is that they are themselves voting rules, ones that in some way relate to the rules they approximate. We feel that our framework supplies approximations that are simple and natural enough to function as voting rules in their own right.

For instance, suppose a group of voters agrees to only accept a Condorcet winner. If their stated individual preferences fail to yield one, then the election controller holds an auction, to entice some of the voters to change their minds.

Taking one candidate at a time, the controller offers to pay each voter for each reduction in the candidate’s vote deficit it can deliver by changing its stated preferences. The cost to the voter is the number of edits it needs to make. If the price offered is less than the cost to the voter, the voter will not accept. If not enough voters accept, the controller increases the amount offered and the process repeats until the candidate becomes the Condorcet winner. The score of the candidate is then the total amount of money offered to the voters and the candidate having the lowest score is the winner. (No payoffs occur until after all candidates are scored, and only those deals made during the winning candidate’s scoring round are actually honored, so in effect the voters “choose” a Condorcet winner.)

The auction thus encourages voters to reveal the true value of their edits, as those

\footnote{This is the actual definition due to Young \cite{You77}. Our formulation in terms of deletions is used elsewhere (see, e.g., \cite{RSV03, BGN07, Fis77}), and is in many respects equivalent to the original definition (though certainly not with respect to optimization and approximation results, at least not directly). Moreover, the deletion-based version we use allows us to more naturally build Dodgson’s and Young’s rules into a single framework.}
who are willing to take the least amount of money per deficit reduction delivered are
rewarded first, while those holding out for more may get nothing. Assuming that all
voters uniformly value their edits at some common unit price, the score the auction
provides (and the order in which it selects the swaps to make) coincides with our rules.

Related work

The study of the approximability of voting rules is rather new. Ailon et al. [ACN05],
Coppersmith et al. [CFR06], and Kenyon-Mathieu and Schudy [KMS07] study approx-
imation algorithms on Kemeny elections.

As noted above, McCabe-Dansted [MD06] (respectively, Procaccia [PFR07]) pro-
vides lower (respectively, upper) bounds on approximating Dodgson (respectively, Young)
scores. Additionally, Procaccia et al. provide a polynomial-time, randomized algo-
rithm that with probability at least 1/2 \(O(\log m)\)-approximates the Dodgson score [PFR07].
They use a linear program whose optimal solution may assign fractional values to
counts of the swaps made. They then use randomness to help assign integer values
to the swap counts, in a way that yields a feasible, integer-valued solution. Our re-
sults improve on this approach in that our algorithm is completely deterministic and,
we feel, more straightforward and natural. Additionally, we provide a polynomial-time
approximation of Young scores.

Several researchers provide algorithms that run in polynomial time on key subsets
of the problem domain. Bartholdi et al., in the same seminal paper that established
\(NP\)-hardness results for Dodgson and Kemeny elections [BTT89], show that Dodgson
elections can be scored in polynomial time when either the number of candidates or the
number of voters is fixed. Our algorithm runs in polynomial time on all inputs, however
it is does not guarantee to provide a correct answer. Rather, it guarantees upper bounds
on the degree of error.

Homan and Hemaspaandra [HH07] and McCabe-Dansted et al. [MPS07] use a
common insight to provide polynomial-time, deterministic heuristics that, in cases
where the voters greatly outnumber the candidates, compute with high probability the
exact Dodgson score on a candidate and preference profile chosen uniformly at random
from all profiles of some fixed size. Our Dodgson-score-approximating algorithm is a
generalization of sorts of their approach. Though we do not analyze the probability of
exactness our algorithm has, we note here that whenever the Homan and Hemaspaan-
dra approach correctly computes the Dodgson score, so does ours. However, when
their algorithm is not exact, it returns a score that is less than the true edit distance. Our
algorithm never returns a score that is less than the edit distance. Moreover, our algo-
rithm always builds as a side effect an actual sequence of edits leading to a Condorcet
winner.

Finally, Rothe et al. (in the same paper where they establish optimal bounds on the
complexity of Young elections) give a polynomial-time algorithm for computing the
“homogeneous” versions (see [Fis77]) of Dodgson’s and Young’s voting rules [RSV03].
(A voting rule is homogenous if cloning each voter’s preferences some fixed number of
times does not affect the score). They do not discuss the degree to which these scores
approximate Dodgson and Young rules.
2 Definitions

2.1 Elections

Let \( V = \{1, \ldots, n\} \) be a set of voters and \( C \) be a set \( \{1, \ldots, m\} \) of candidates. A ranking of the candidates is a total ordering over \( C \), i.e., \( \langle c_m > c_{m-1} > \cdots > c_1 \rangle \), where \( \{c_1, \ldots, c_m\} = C \). We denote the set of all such rankings \( \mathcal{L}(C) \). The voters’ preference profile is an \( n \)-tuple in \( \mathcal{L}(C)^n \). For a given preference profile \( \langle >_1, \ldots, >_n \rangle \in \mathcal{L}(C)^n, i \in V \), and \( c \in C \), let \( e_{>_i} \) denote \( \| \{d \in C \mid c >_i d\} \| \).

For every pair of distinct candidates \( c, d \in C \) and every preference profile \( P = \langle >_1, \ldots, >_n \rangle \), \( c \)'s vote deficit in \( P \) with \( d \) is \( Deficit(P; c, d) = \min\{0, \| \{i \in V \mid d >_i c\} \| - \| \{i \in V \mid c >_i d\} \| \} \). The total deficit of \( c \) is

\[
Deficit(P; c) = \sum_{d \in C - \{c\}} Deficit(P; c, d).
\]

Thus \( c \) is a Condorcet winner if and only if \( Deficit(P; c) = 0 \). \( Deficit(P; c) \) is sometimes known as the Tideman score \(^{[Tid87]}\), which forms the basis of the Tideman (a.k.a., ranked pairs) voting rule.

2.2 Edit-based scoring rules

The building blocks of this paper are edits and deficit reductions. It will be useful to view them as objects we can label. We now show how to do this.

An edit is a mapping \( e : \bigcup_{i=0}^{\infty} \mathcal{L}(C)^i \to \bigcup_{i=0}^{\infty} \mathcal{L}(C)^i \). Let \( P \circ e \) denote the application of \( e \) to some preference profile \( P \). A sequence of edits \( \langle e_1, \ldots, e_p \rangle \) is called a Condorcet sequence if \( Deficit(P \circ e_1 \circ \cdots \circ e_p, c) = 0 \).

A swap is an edit, designated by an ordered pair \( (i, j) \in \mathbb{N}^2 \), that takes a preference profile \( P = \langle >_1, \ldots, >_n \rangle \) and outputs \( \langle >'_1, \ldots, >'_n \rangle \), which is just like \( P \) except that, if \( 1 \leq i \leq n \) and \( 0 < j < m \), then for \( c, d \in C \) satisfying \( d(>_i) = j = c(>_i) + 1 \) it holds that \( c(>_i') = j = d(>_i') + 1 \), i.e., \( c, d \) are adjacent in both rankings, \( d >_i c \), and \( c >'_i d \). Candidates \( c \) and \( d \) are said to be involved in the swap.

A deletion is an edit, designated by some \( i \in \mathbb{N} \), that takes a preference profile \( P = \langle >_1, \ldots, >_n \rangle \in \mathbb{N}^n \) and outputs \( \langle >_1, \ldots, >_{i-1}, >_{i+1}, \ldots, >_n \rangle \).

A deficit reduction is a 4-tuple \( (P, c, e, d) \) where \( P \) is a preference profile, \( c \) and \( d \) are candidates, and \( e \) is an edit such that \( Deficit(P, c, d) > Deficit(P \circ e, c, d) \). The full sequence of deficit reductions with respect to candidate \( c \) over a sequence of edits \( \langle e_1, \ldots, e_p \rangle \) on a preference profile \( P \), denoted \( D(P, c, \langle e_1, \ldots, e_p \rangle) \), is the nonrepeating sequence of deficit reductions \( \langle (P_1, e_{i_1}, c, d_1), \ldots, (P_{i_k}, e_{i_k}, c, d_k) \rangle \) of maximum length such that, for all \( k \in \{1, \ldots, q\} \), \( P_k = P \circ e_{i_1} \circ \cdots \circ e_{i_{k-1}}, Deficit(P_k, c, d_k) > Deficit(P_k \circ e_{i_k}, c, d_k) \), and for all \( j \in \{1, \ldots, k - 1\}, i_j \leq i_k \).

We now define, using the terms given above, Dodgson and Young’s scoring rules. Let \( S \) be the collection of all sequences of swaps. The Dodgson score of candidate \( c \) in profile \( P \) is the smallest \( p \in \mathbb{N} \) such that

\[ (\exists \langle e_1, \ldots, e_p \rangle \in S) [Deficit(P \circ e_1 \circ \cdots \circ e_p, c) = 0]. \]
Let $D$ be the collection of all sequences of deletions. The Young score of candidate $c$ in profile $P$ is the smallest $p \in \mathbb{N}$ such that

$$\exists (e_1, \ldots, e_p) \in D \left[ \text{Deficit}(P \circ e_1 \circ \cdots \circ e_p, c) = 0 \right].$$

### 2.3 The generic framework

Below is a generic algorithm for the voting rules we study and approximate. Here, $\mathcal{E}$ is a collection of “legal” sequences of edits, whose exact makeup depends on the particular scoring rule in question. The variable $\mathcal{E}$ is implemented as a priority queue, where priority is given to sequences of edits $S'$ that, when applied to the preference profile $P$, have the fewest edits per deficit reduction, i.e., that minimize $|S'|/|D(P, c, S')|$. We call this quantity the marginal cost of $S'$. We define $|S'|/|D(P, c, S')| = \infty$ whenever $|D(P, c, S')| = 0$.

$S$ is a list of edits made.

In order to emphasize the key components of this algorithm, we have omitted important but mundane steps. For instance, the algorithm needs to compute $\text{Deficit}(P, c)$. We will discuss such details when we discuss the actual Dodgson—and Young—approximation rules.

**Input:** A preference profile $P$ and a candidate $c$.

1. let $S = \langle \rangle$
2. while $\text{Deficit}(P, c) > 0$
3. \quad let $S' = \text{argmin}_{S'' \in \mathcal{E}} |S''|/|D(P, c, S'')|$
4. \quad let $\langle e_1, \ldots, e_p \rangle = S'$
5. \quad let $P = P \circ e_1 \circ \cdots \circ e_p$
6. \quad concatenate($S, \langle e_1, \ldots, e_p \rangle$)
7. output $|S|

### 3 Approximating Dodgson’s scoring rule

For any candidate $c$, we say that a sequence of swaps $s_1, \ldots, s_p$ is $c$-normal on $P$ if, for each $k \in \{1, \ldots, p\}$, $c$ is involved in swap $s_k = (i, j)$ on $P \circ s_1 \circ \cdots \circ s_{k-1}$ and $c(<i) = j - 1$.

Let $P$ be a preference profile and let $\mathcal{E}'$ be the collection of all $c$-normal swap sequences where, for each sequence, there is a single voter’s preference list to which all swaps in the sequence apply. Note then that every such sequence has a distinct last element, so we can represent each sequence in $\mathcal{E}'$ by storing its last element only. Let...
us call the voting rule based on the generic algorithm with \( E = E' \) “Marginal-Cost-Greedy-Dodgson.”

**Theorem 1.** The running time of Marginal-Cost-Greedy-Dodgson, when \( E = E' \), is \( O(N^2 \log N) \), where \( N \) is the length of the input.

**Proof.** Let \((P, c)\) be the input to the algorithm, where \( E = E' \) and \( P \) has \( m \) candidates and \( n \) voters. We first need to initialize the data structures used. It takes linear time to calculate \( \text{Deficit}(P, c, d) \) on all \( d \in C - \{d\} \) (note that we can compute \( \text{Deficit}(P, c) \) at the same time). Next we need to initialize \( E' \). There are at most \( n(m - 1)/2 \) distinct values for \( |S'|/|D(S')| \) that any such sequence can take. So (regarding \( E \) as a priority queue) it takes \( O(\log m) \) comparisons to add any such sequence (which we recall is represented by the last element of the sequence) to \( E \). Note that we can calculate \( |S'|/|D(S')| \) for every sequence \( S' \) in \( E \) in a single pass through \( P \). The worst case is when \( n \) is as small as possible, so the worst case running time for initialization is \( O(N \log N) \).

After initialization, the algorithm performs swaps on \( P \) until \( c \) is the Condorcet winner. Note that any given swap is performed at most once. For each swap applied, the algorithm must remove the corresponding swap from the queue (since whenever a swap is applied it follows that the swap sequence ending with that swap has also been applied), and it must update the marginal cost of each swap sequence remaining in \( E \) that applies to the current voter’s preferences. Thus, every swap may require \( O(m) \) updates to \( E \). Assuming that all swaps in \( E \) sharing a common voter are connected via a linked list, each update can happen in constant time. As during initialization, the worst case for these procedures occurs when \( n \) is as small as possible, so the running time for this part of the algorithm is \( O(N^2) \).

Finally, every time a swap causes the deficit against some opponent to go from positive to zero the entire queue needs to be reprioritized, which means we must pass through all swap sequences and recalculate. This can happen at most \((m - 1)\) times. Again, the worst-case running time is when \( n \) is as small as possible, so it is \( O(N^2 \log N) \).

We turn now to the approximation bound. Our proof assumes there is a \( c \)-normal Condorcet sequence of swaps witnessing the Dodgson score of \( c \). The following proposition shows that our assumption is valid.

**Proposition 2.** For every preference profile \( P \) and candidate \( c \) there is a \( c \)-normal Condorcet swap sequence of length equal to the Dodgson score of \( c \).

**Proof.** Let \( p \) be the Dodgson score of \( c \) and \( \langle s_1, \ldots, s_p \rangle \) be a Condorcet swap sequence with respect to candidate \( c \) on preference profile \( P = \langle >_1, \ldots, >_n \rangle \). Let \( \langle >'_1, \ldots, >'_n \rangle = P \circ s_1 \circ \cdots \circ s_p \). Choose \( i \in V \) and let \( \langle s'_1, \ldots, s'_q \rangle \) be the subsequence of \( \langle s_1, \ldots, s_p \rangle \) consisting of all swaps on voter \( i \)'s preferences. Let \( d' = \arg \max_{d \in C: >_i} d(>_{i}) - c(>_{i}) \). Since it requires at least \( d'(>_{i}) - c(>_{i}) \) swaps in order for \( c >'_i \) \( d' \) to hold, it must be the case that \( |\langle s'_1, \ldots, s'_q \rangle| \geq d'(>_{i}) - c(>_{i}) \). So, removing from \( S \) each swap in \( \langle s'_1, \ldots, s'_q \rangle \) and appending the sequence \( \langle i, c(>_{i}) \)
Theorem 3. Marginal-Cost-Greedy-Dodgson is an \((\ln m + 1)\)-approximation of Dodgson score, where \(m\) is the number of candidates in the input election.

Proof. Let \(P\) be a preference profile over \(m\) candidates and \(n\) voters and let \(c\) be a candidate in \(\{1, \ldots, n\}\). Let \(x\) be the Dodgson score of \(c\) on \(P\) and let \(S^*\) be a \(c\)-normal Condorcet sequence of \(c\), such that \(S^*\) is the sequence of all swaps the algorithm applies to \(P\), and let \(\langle (P_1, c, s_1^*, d_1^*), \ldots, (P_y, c, s_y^*, d_y^*) \rangle = D(P, c, S^*)\). Let \(S\) be the same as in the algorithm on input \((P, c)\) at the time line 7 is reached (i.e., it is the sequence of all swaps the algorithm applies to \(P\)), and let \(\langle (P_1, c, s_1, d_1), \ldots, (P_y, c, s_y, d_y) \rangle = D(P, c, S)\).

The basic idea behind our proof is that the number of deficit reductions in a sequence witnesses the Dodgson score of \(c\), such as \(S^*\), is equal to the number of deficit reductions in the sequence \(S\) that the algorithm produces. So to compare \(|S|\) to \(|S^*|\) we partition the swaps in \(S\) (respectively, \(S^*\)) among the deficit reductions and then match the deficit reductions in \(S\) to those in \(S^*\). The partitioning is easy: For \(S\) it is just the marginal cost associated with each deficit reduction. For \(S^*\) we fudge the marginal cost in a straightforward way. The matching and the order in which matched elements are compared are the trickiest parts of the proof.

For every \(k \in \{1, \ldots, y\}\), let \(r(s_k)\) denote the marginal cost the algorithm associates with \(s_k\) (i.e., \(|S^*|/|D(P, c, S^*)|\), where \(S^*\) and \(P\) are as in line 3 during the iteration when the algorithm chooses \(s_k\) to be in \(S^*\)). Clearly,

\[|S| = \sum_{k=1}^{y} r(s_k).\]

Let \(\sigma\) denote a permutation over \(\{1, \ldots, y\}\) that satisfies the following constraints.

1. For every \(j \in \{1, \ldots, y\}\), \(d_j^* = d_{\sigma(j)}\).
2. For every \(j, k \in \{1, \ldots, y\}\), if \(s_k^* = s_j\) then \(k = \sigma(j)\).

Clearly, such a mapping exists.

For each \(i \in \{1, \ldots, n\}\), let \(S_i^*\) (respectively, \(D_i^*\)) be the subsequence of all swaps in \(S^*\) (respectively, \(\langle s_1^*, \ldots, s_y^* \rangle\)) that apply to voter \(i\) only (i.e., all swaps that for some \(j\) are of the form \((i, j)\)). Let \(p = |D_i^*|\) and let \(D_i = \langle s_{k_1}, \ldots, s_{k_p} \rangle\) be the subsequence of all swaps in \(\langle s_1, \ldots, s_y \rangle\) that \(\sigma\) maps to some element in \(D_i^*\). In particular, this subsequence preserves the order in which the algorithm applies the swaps.

We claim, for every \(q \in \{1, \ldots, p\}\), that \(r(s_{k_q}) \leq |S_i^*|/(|D_i^*| + 1 - q)\). This is because, by our construction of \(\sigma\), at the time the algorithm is about to choose \(s_{k_q}\) it has not chosen \(s_{\sigma(k_q)}^*\) nor any of the other swaps in \(S_i^*\) that come after it (in fact, the algorithm may not have chosen a single swap in \(S_i^*\)). Because the subsequence \(\langle s_{k_1}, \ldots, s_{k_p} \rangle\) preserves the order in which the swaps were made, the algorithm still needs at this point to close deficits against the candidates \(d_{k_q}, d_{k_q+1}, \ldots, d_{k_p}\) (as \(d_{\sigma(k_q)}, d_{\sigma(k_q+1)}, \ldots, d_{\sigma(k_p)}\)).

So at the time the algorithm chooses swap \(s_{k_q}\), it could instead take the longest subsequence of \(S_i^*\) that remains unchosen. Obviously, this subsequence is at most \(|S_i^*|\).
swaps long and, as discussed above, it yields at least $|D_i^*| + 1 - q$ deficit reductions. Since $s_{k_q}$ was chosen as part of a sequence $S'$ for which $|S'|/|D(P, S', c)| = r(s_{k_q})$, where $P$ here is taken to be in the same state as when $S'$ was chosen) was as small as possible, our claim holds. But then

$$|S| = \sum_{k=i}^{y} r(s_k)$$

$$\leq \sum_{i=1}^{n} \sum_{q=1}^{m} |S_i^*|/(|D_i^*| + 1 - q)$$

$$\leq \sum_{i=1}^{n} \sum_{q=1}^{m} |S_i^*|/(m + 1 - q)$$

$$\leq |S^*| \ln m + 1$$

4 Approximating Young’s scoring rule

For a given preference profile $P$, let $E''$ be the collection of all single-element sequences of deletions on $P$. Let us call the voting rule based on the generic algorithm with $E = E''$ “Marginal-Cost-Greedy-Young.”

**Theorem 4.** Marginal-Cost-Greedy-Young runs in time $O(N^2 \log N)$.

**Theorem 5.** Marginal-Cost-Greedy-Young is a $O(\log m)$ approximation of the Young score, where $m$ is the number of candidates in a given input preference profile.

The proofs of the above theorems are essentially analogous to those of theorems 1 and 3.

5 Conclusion

We provide scoring rules that approximate Dodgson’s and Young’s rules to within logarithmic factors. Assuming $P \neq NP$, the bound on Dodgson’s scoring rule is within a constant factor of the optimal polynomial-time approximation. Many natural questions arise from this work. What are the actual optimal polynomial-time approximations to Dodgson and Young scores, assuming $P \neq NP$? How frequently do the final candidate rankings according to our scoring rules equal those given by Dodgson’s and Young’s rules on the same input?

Our paper gives a general framework for edit-based scoring rules, in which different types of edits could be combined to produce an endless stream of distinct voting rules. The basic problem of comparing, in such a broadened setting, edit distances against the edit sequences produced by the kind algorithms presented here seems worthy of further research.
Finally, in the introduction we explained our voting rules in terms of an auction-like mechanism, where we assumed that all voters value all edits equally. This suggests an intriguing line of study: What if that is not how voters feel? For instance, it seems natural to us that voters would be less willing to make swaps higher up on their preference lists, and so would require a higher price to make them. And in many settings we would expect the value placed on edits to vary across a population of voters. So how would allowing voters to specify the cost of each edit affect the score our algorithm produces, compared to the corresponding edit-distance-based score?

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