CHARACTERS OF THE SYLOW p-SUBGROUPS OF THE CHEVALLEY GROUPS $D_4(p^n)$

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Abstract. Let $U(q)$ be a Sylow $p$-subgroup of the Chevalley groups $D_4(q)$ where $q$ is a power of a prime $p$. We describe a construction of all complex irreducible characters of $U(q)$ and obtain a classification of these irreducible characters via the root subgroups which are contained in the center of these characters. Furthermore, we show that the multiplicities of the degrees of these irreducible characters are given by polynomials in $q-1$ with nonnegative coefficients.

1. Introduction

Let $q$ be a power of a prime $p$ and $\mathbb{F}_q$ a field with $q$ elements. The group $U_n(q)$ of all upper triangular $n \times n$-matrices over $\mathbb{F}_q$ with all diagonal entries equal to 1 is a Sylow $p$-subgroup of $\text{GL}_n(\mathbb{F}_q)$. It was conjectured by G. Higman [6] that the number of conjugacy classes of $U_n(q)$ is given by a polynomial in $q$ with integer coefficients.

Higman’s conjecture was refined using the (complex) character theory of $U_n(q)$. I.M. Isaacs [8] showed that the degrees of the irreducible characters of $U_n(q)$ are of the form $q^e$, $0 \leq e \leq \mu(n)$ where the upper bound $\mu(n)$ is known explicitly. G. Lehrer [9] conjectured that the numbers $N_{n,e}(q)$ of irreducible characters of $U_n(q)$ of degree $q^e$ are given by a polynomial in $q$ with integer coefficients. I.M. Isaacs suggested a strengthened form of Lehrer’s conjecture stating that $N_{n,e}(q)$ is given by a polynomial in $q-1$ with nonnegative integer coefficients. So, Isaac’s conjecture implies Higman’s and Lehrer’s conjectures.

It is natural to consider these questions in the context of finite groups of Lie type. Let $G(q)$ be a Chevalley group defined over $\mathbb{F}_q$ and let $U(q)$ be a maximal unipotent subgroup of $G(q)$ such that $U(q)$ is a Sylow $p$-subgroup of $G(q)$. S. Goodwin and G. Röhrle [5] developed an algorithm for parametrizing the conjugacy classes of $U(q)$ which is valid when $p$ is good for the underlying root system. Using an implementation of this algorithm in the computer algebra system GAP [3], they determined the number of conjugacy classes of $U(q)$ for all Chevalley groups $G(q)$ of rank at most 6. As a consequence, they were able to show that this number is given by a polynomial in $q$.

A promising approach to an understanding of the complex irreducible characters of $U(q)$ is the general concept of supercharacters which was developed by P. Diaconis and I.M. Isaacs [2]. C. André and A.M. Neto defined and studied supercharacters of $U(q)$ if $G(q)$ is a classical group of type $B_n$, $C_n$ or $D_n$ and $p \geq 3$. These supercharacters are defined as products of so-called elementary characters. Under the
additional assumption \( p \geq 2n \) André and Neto were able to show that every irreducible character of \( U(q) \) is a constituent of a unique supercharacter. In particular, the supercharacters induce a partition on the set of irreducible characters of \( U(q) \).

In this article we study the irreducible characters of a maximal unipotent subgroup \( U(q) \), that is a Sylow \( p \)-subgroup, of the Chevalley groups \( G(q) \) of type \( D_4 \). We describe a construction of all irreducible characters of \( U(q) \) which also works for bad characteristic \( p = 2 \). Our main result Theorem 4.1 is a classification of the irreducible characters of \( U(q) \) via the root subgroups which are contained in the center of the irreducible characters. In this way we get a natural partition of the set of irreducible characters of \( U(q) \) into families. With each positive root we associate \( q - 1 \) distinct irreducible characters of \( U(q) \) which we call midafis. They are a fundamental tool in the proof of our classification result and in some sense they play a role similar to André’s and Neto’s elementary characters. We also obtain the degrees and for each degree the number of irreducible characters of \( U(q) \) of this degree. From this we can conclude that an analogue of Isaac’s conjecture is true for \( U(q) \), even in bad characteristic \( p = 2 \).

There are several differences between André’s and Neto’s methods and ours:

- We do not make use of the natural matrix representation of the classical group of type \( D_4 \). Instead we use Lie theory and the underlying root system whenever it is possible.
- Our midafis are always irreducible characters while André’s and Neto’s elementary characters are not necessarily irreducible.
- We can describe a construction of all irreducible characters of \( U(q) \).
- Our methods also work in bad characteristic \( p = 2 \).

There are two reasons why we consider the Chevalley groups of type \( D_4 \). First, they are not too far away from the classical \( A_n \)-case. And second, the maximal unipotent subgroups of the Chevalley groups of type \( D_4 \) are isomorphic to factor groups of the maximal unipotent subgroups of the exceptional groups of type \( E_6 \), \( E_7 \) and \( E_8 \). Therefore, we hope that this paper is a step towards the classification of the irreducible characters of the maximal unipotent subgroups of the “large” exceptional groups.

This paper is organized as follows: In Section 2, we introduce the general setup and fix notation. In Section 3, we develop several tools which are essential for our construction of the irreducible characters of \( U(q) \): hook subgroups and midafis. Finally, in Section 4, we apply these tools to obtain a construction and classification of the irreducible characters of \( U(q) \). Also in this section we determine the degrees and numbers of the irreducible characters of \( U(q) \) and prove an analogue of Isaac’s conjecture for \( U(q) \).

### 2. Notation and Setup

In this section, we introduce the setup and notation which will be used throughout this paper.

#### 2.1. Root system of type \( D_4 \)

Let \( \Phi \) be a root system of type \( D_4 \) in some Euclidean space, with basis \( \Delta = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) of simple roots such that \( \alpha_1, \alpha_2, \alpha_4 \) are orthogonal to each other. The Dynkin diagram of \( \Phi \) is
The positive roots are those roots which can be written as linear combinations of the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with nonnegative coefficients and we write $\Phi_+$ for the set of positive roots. We use the notation $\frac{1}{1 \ 2 \ 1}$ for the root $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ and we use a similar notation for the remaining positive roots. The 12 positive roots of $\Phi$ are given in Table 1.

### Table 1. Positive roots of the root system $\Phi$ of type $D_4$.

| Height | Roots |
|--------|-------|
| 5      | $\alpha_{12} := \frac{1}{1 \ 2 \ 1}$ |
| 4      | $\alpha_{11} := \frac{1}{1 \ 1 \ 1}$ |
| 3      | $\alpha_8 := \frac{1}{1 \ 1 \ 0}$, $\alpha_9 := \frac{0}{1 \ 1 \ 1}$, $\alpha_{10} := \frac{1}{0 \ 1 \ 1}$ |
| 2      | $\alpha_5 := \frac{0}{1 \ 1 \ 0}$, $\alpha_6 := \frac{1}{0 \ 1 \ 0}$, $\alpha_7 := \frac{0}{0 \ 1 \ 1}$ |
| 1      | $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ |

The numbering $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ of these roots is in accordance with the output of the CHEVIE [4] command

\[ \text{CoxeterGroup("D", 4);} \]

We say that a nonempty subset $S \subseteq \Phi$ is **closed** if for all $\alpha, \beta \in S$ we have $\alpha + \beta \in S$ or $\alpha + \beta \notin \Phi$. We write $\text{ht}(\alpha)$ for the height of a positive root $\alpha$.

#### 2.2. Chevalley groups of type $D_4$. Let $L$ be a simple complex Lie algebra with root system $\Phi$. We choose a Chevalley basis $\{h_r | r \in \Delta \} \cup \{e_r | r \in \Phi \}$ of $L$ such that the structure constants $N_{rst}$ in $[e_r, e_s] = N_{rst} e_t$ are positive for all extraspecial pairs of roots $(r, s) \in \Phi \times \Phi$, see [1, Section 4.2]. Fix a power $q$ of some prime $p$ and let $G = G(q) = D_4(q)$ be the Chevalley group of type $D_4$ over the field $F_q$ constructed from $L$, see [1, Section 4.4]. The group

\[ G = D_4(q) \cong P\Omega^+_8(q) \]

is simple and is generated by the root elements $x_\alpha(t)$ for $\alpha \in \Phi$ and $t \in F_q$. Let $X_\alpha := \langle x_\alpha(t) | t \in F_q \rangle$ be the root subgroup corresponding to $\alpha \in \Phi$. For positive roots, we use the abbreviation $x_i(t) := x_{\alpha_i}(t)$, $i = 1, 2, \ldots, 12$. The commutators $[x_i(t), x_j(u)] = x_i(t)^{-1}x_j(u)^{-1}x_i(t)x_j(u)$ are given in Table 2. All $[x_i(t), x_j(u)]$ not listed in this table are equal to 1.

Let $U = U(q)$ be the subgroup of $G = G(q)$ generated by the elements $x_i(t)$ for $i = 1, 2, \ldots, 12$ and $t \in F_q$. So $U$ is a maximal unipotent subgroup and a Sylow $p$-subgroup of $G$. In this paper, we are interested in the complex irreducible characters of $U$. Note that each element of $u \in U$ can be written uniquely as

\[ u = x_1(d_1)x_2(d_2)\cdots x_{12}(d_{12}) \]
where $d_1, \ldots, d_{12} \in \mathbb{F}_q$. The multiplication of the elements of $U$ is described by the commutator relations. The center $Z(U) = X_{\alpha_{12}}$ is elementary abelian of order $q$.

2.3. Characters, induction and restriction. For any finite group $H$ let $\text{Irr}(H)$ be the set of complex irreducible characters of $H$ and let $(\cdot, \cdot)_H$ or $(\cdot, \cdot)$ be the usual scalar product on the space of class functions of $H$. Let $1_H$ or 1 denote the trivial character of $H$. If $\chi$ is a character of a subgroup $H_1$ of $H$, then we write $\chi^H$ for the induced character, and if $\chi$ is a character of $H$, we write $\chi|_{H_1}$ for the restriction of $\chi$ to the subgroup $H_1$.

**Definition 2.1.** Let $H$ be a finite group. We say that $\chi \in \text{Irr}(H)$ is almost faithful if $Z(H) \not\subseteq \ker(\chi)$.

Note that if $q$ is not prime, then the center $Z(U)$ is not cyclic and in this case $U$ does not have any faithful irreducible characters. The almost faithful irreducible characters of $U$ are in some sense closest to being faithful.

For a field $K$ let $K^\times$ be its multiplicative group. In the whole paper, we fix a nontrivial linear character $\phi$ of the group $(\mathbb{F}_q, +)$. So for $\alpha \in \Phi_+$ and $s \in \mathbb{F}_q$, the map $\varphi_{\alpha, s} : X_{\alpha} \to \mathbb{C}^\times, x_{\alpha}(d) \mapsto \phi(s \cdot d)$ is a linear character of the root subgroup $X_{\alpha}$, and all irreducible characters of $X_{\alpha}$ arise in this way.

3. Hook subgroups and midafis

For each positive root $\alpha$ we construct $q - 1$ distinct irreducible characters of $U$, called *midafis*, which play a fundamental role in the classification and construction of the irreducible characters of $U$ in this paper.

3.1. Hook subgroups. With each positive root $\alpha$, we associate certain sets of positive roots and a certain subgroup of $U$ which will be used to define the midafis.

**Definition 3.1.** Let $\alpha \in \Phi_+$ be a positive root.

(a) The set $h_\alpha := \{ \gamma \in \Phi_+ \mid \text{there is } \gamma' \in \Phi_+ \cup \{ 0 \} \text{ such that } \gamma + \gamma' = \alpha \} \subseteq \Phi_+$ is called the hook corresponding to $\alpha$.

(b) The subgroup $H_\alpha := \prod_{\gamma \in h_\alpha} X_{\alpha} \subseteq U$ is called the hook subgroup of $U$ corresponding to $\alpha$.

(c) We call

$$\text{arm}(h_\alpha) := \begin{cases} (h_\alpha \cap h_{\alpha_{12}}) \setminus \{ \alpha \}, & \text{if } \alpha \neq \alpha_{12}, \\ \{ \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11} \}, & \text{if } \alpha = \alpha_{12} \end{cases}$$

the arm and $\text{leg}(h_\alpha) := h_\alpha \setminus (\text{arm}(h_\alpha) \cup \{ \alpha \})$ the leg of the hook $h_\alpha$. 

| Table 2. Commutator relations for type $D_4$. |
|-----------------------------------------------|
| $[x_1(t), x_3(u)] = x_5(tu)$,                  |
| $[x_1(t), x_3(u)] = x_6(tu)$,                  |
| $[x_2(t), x_3(u)] = x_3(tu)$,                  |
| $[x_2(t), x_3(u)] = x_7(tu)$,                  |
| $[x_3(t), x_3(u)] = x_9(tu)$,                  |
| $[x_3(t), x_3(u)] = x_4(tu)$,                  |
| $[x_4(t), x_3(u)] = x_8(tu)$,                  |
| $[x_4(t), x_3(u)] = x_5(tu)$,                  |
| $[x_6(t), x_3(u)] = x_9(tu)$,                  |
| $[x_6(t), x_3(u)] = x_{12}(tu)$,               |
| $[x_7(t), x_3(u)] = x_{12}(tu)$,               |
| $[x_7(t), x_3(u)] = x_{12}(tu)$,               |
Remark: (a) By the commutator relations, the hook subgroup $H_\alpha$ does not depend on the order of the root subgroups in the product.

(b) Also, the commutator relations imply that the each hook subgroup $H_\alpha$ is a special $p$-group of type $q^{|\gamma|-\log(p)}$.

Example: The hook subgroup

$$H_{\check{\alpha}12} = X_{\alpha_1}X_{\alpha_5}X_{\alpha_6}X_{\alpha_7}X_{\alpha_8}X_{\alpha_9}X_{\alpha_{10}}X_{\alpha_{11}}X_{\alpha_{12}}$$

is the unipotent radical of the maximal parabolic subgroup of $G$ corresponding to the set $\{\alpha_1, \alpha_2, \alpha_4\}$ of simple roots. So, $H_{\check{\alpha}12}$ is a normal subgroup of $U$. Using the notation from Subsection 2.1, we can picture $H_{\check{\alpha}12}$ as follows:

$$\begin{array}{ccccccc}
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{array}$$

3.2. Midařis. For positive roots $\alpha \in \Phi_+$ we set $V_\alpha := \prod_{\gamma \in \Phi_+ \setminus \text{leg}(\alpha)} X_\gamma$. The midařa of $U$ associated with $\alpha$ will be characters which are induced from $V_\alpha$. We are going to use the following lemma:

Lemma 3.2. Let $\alpha \in \Phi_+$ and $s \in \mathbb{F}_q^\times$.

(a) The subset $\Phi_+ \setminus \text{leg}(\alpha) \subseteq \Phi$ is closed.

(b) The set $V_\alpha$ is a subgroup of $U$.

(c) We have $X_\alpha \cap [V_\alpha, V_\alpha] = \{1\}$.

(d) For each $s \in \mathbb{F}_q^\times$ there is a linear $\lambda_{\alpha,s} \in \text{Irr}(V_\alpha)$ such that $\lambda_{\alpha,s}|_{X_\alpha} = \varphi_{\alpha,s}$ and $X_\gamma \subseteq \ker(\lambda_{\alpha,s})$ for all $\gamma \in \Phi_+$ with $\text{ht}(\gamma) > \text{ht}(\alpha)$.

Proof. (a) Suppose not. So there is $\beta \in \text{leg}(\alpha)$ and $\gamma, \gamma' \in \Phi_+ \setminus \text{leg}(\alpha)$ such that $\beta = \gamma + \gamma'$. But then $\gamma \in \text{leg}(\alpha)$ or $\gamma' \in \text{leg}(\alpha)$, a contradiction.

(b) follows from (a).

(c) Suppose $X_\alpha \cap [V_\alpha, V_\alpha] \neq \{1\}$. The commutator relations imply that there are $\gamma, \gamma' \in \Phi_+ \setminus \text{leg}(\alpha)$ such that $\gamma + \gamma' = \alpha$. But then $\gamma \in \text{leg}(\alpha)$ or $\gamma' \in \text{leg}(\alpha)$, which is a contradiction.

(d) follows from (c). □

Definition 3.3. Let $\alpha$ be a positive root. We call the characters $\mu_{\alpha,s} := \lambda_{\alpha,s}^U$ for $s \in \mathbb{F}_q^\times$ the midařa of $U$ associated with $\alpha$.

Proposition 3.4. Let $\alpha \in \Phi_+$ be a positive root. The midafis $\mu_{\alpha,s}$ for $s \in \mathbb{F}_q^\times$ are $q - 1$ distinct irreducible characters of $U$ and $\mu_{\alpha,s}|_{X_\alpha} = \mu_{\alpha,s}(1) \cdot \varphi_{\alpha,s}$.

Proof. Note that for all $\gamma \in \Phi_+$ with $\text{ht}(\gamma) > \text{ht}(\alpha)$, we have $X_\gamma \subseteq \ker(\lambda_{\alpha,s})$. So the statement about the restriction $\mu_{\alpha,s}|_{X_\alpha}$ is clear by the commutator relations and the definition of induced characters, and from this we immediately get $\mu_{\alpha,s} \neq \mu_{\alpha,s'}$ for $s \neq s' \in \mathbb{F}_q^\times$. So we only have to show that $\mu_{\alpha,s}$ is irreducible.
Let \( \text{leg}(\alpha) = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \) with \( \text{ht}(\alpha_1) \geq \text{ht}(\alpha_2) \geq \cdots \geq \text{ht}(\alpha_m) \). From the commutator relations we get

\[
V_\alpha \leq X_{\alpha_1}V_\alpha \leq X_{\alpha_2}X_{\alpha_1}V_\alpha \leq \cdots \leq \left( \prod_{j=1}^{m} X_{\alpha_j} \right) V_\alpha = U.
\]

Considering character values on \( X_{\alpha-\alpha_1} \), we see that the inertia subgroup of \( \lambda_{\alpha,s} \) in \( X_{\alpha_1}V_\alpha \) is equal to \( V_\alpha \). Hence by Clifford theory [7, Theorem (6.11)], the induced character \( \lambda_{\alpha,s}^{X_{\alpha_1}V_\alpha} \) is irreducible. Similarly, considering character values on \( X_{\alpha-\alpha_2} \) we see that the inertia subgroup of \( \lambda_{\alpha,s}^{X_{\alpha_2}V_\alpha} \) in \( X_{\alpha_1}X_{\alpha_2}V_\alpha \) is equal to \( X_{\alpha_1}V_\alpha \). Again, [7, Theorem (6.11)] implies the irreducibility of the induced character \( \lambda_{\alpha,s}^{X_{\alpha_1}X_{\alpha_2}V_\alpha} \). Continuing in this way, we get that \( \mu_{\alpha,s} = \lambda_{\alpha,s}^{U} \) is irreducible. □

3.3. Hook subgroups for the group of upper triangular matrices. The notation of hooks, arms, legs and midafis is motivated by root systems of type \( A \), that is, by the structure of the Sylow \( p \)-subgroups of \( \text{GL}_n(\mathbb{F}_q) \). The group \( U_n(q) \) of all upper unitriangular matrices over \( \mathbb{F}_q \) is a Sylow \( p \)-subgroup of \( \text{GL}_n(\mathbb{F}_q) \). The root system of \( \text{GL}_n(\mathbb{F}_q) \) with respect to the maximal torus of diagonal matrices has simple roots \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) such that the nodes corresponding to \( \alpha_i \) and \( \alpha_{i+1} \) are joined in the Dynkin diagram for \( i = 1, 2, \ldots, n-2 \). The positive roots are the roots \( \alpha_{ij} := \alpha_i + \cdots + \alpha_j \) for all \( 1 \leq i \leq j \leq n-1 \). The root subgroup \( X_{\alpha_{ij}} \) consists of the matrices \( I_n + t \cdot e_{i,j+1} \) for \( t \in \mathbb{F}_q \), where \( I_n \) is the \( n \times n \)-identity matrix and \( e_{i,j+1} \) is the \( n \times n \)-matrix with zero entries except a single entry \( 1 \) in position \((i, j+1)\). Hooks and hook subgroups can be defined for \( U_n(q) \) in the same way as in Definition 3.1 (a), (b). For \( n = 8 \), the hooks subgroup corresponding to \( \alpha_{2,6} \) can be pictured as follows:

\[
\begin{pmatrix}
1 & & & & & & & \\
1 & * & * & & & & & \\
. & 1 & * & & & & & \\
. & . & 1 & * & & & & \\
. & . & . & 1 & * & & & \\
. & . & . & . & 1 & & & \\
. & . & . & . & . & 1 & & \\
. & . & . & . & . & . & 1 & \\
\end{pmatrix}
\]

Associated with each positive root \( \alpha_{ij} \) are \( q-1 \) irreducible characters of \( U_n(q) \) which can be constructed in a way analogous to our midafis for type \( D_4 \). The word \emph{midafi} is an abbreviation for \emph{minimal degree almost faithful irreducible} which comes from the fact that the midafis of \( U_n(q) \) can be interpreted as almost faithful irreducible characters of minimal degree of suitable factor groups of \( U_n(q) \). For details, see [11, Section 2.2].

4. Irreducible characters of \( U \)

In this section, we describe a construction of all irreducible characters of \( U \) for all prime powers \( q \). In particular, we obtain the number and the degrees of all irreducible characters of \( U \). The main result is the following theorem.

\textbf{Theorem 4.1.} For every prime power \( q \) the irreducible characters of \( U \) are given by Table 3.
We begin with some comments on Table 3. There are 17 families of irreducible characters of $U$ and each row of Table 3 represents one of these families. The first column gives notation for these families of characters. Note that the family $\mathcal{F}_{8,9,10}^{odd}$ exists only for odd $q$, while $\mathcal{F}_{8,9,10}^{even}$ exists only if $q$ is even. The second column of Table 3 gives notation for the irreducible characters in each family. The first one, two or three indices of this notation describe the positive roots $\alpha_j$ of maximal height such that $X_{\alpha_j}$ is not contained in the kernel of the irreducible characters in the family. If there are two types of characters in the family, there is an additional index which describes the degree of the characters. The remaining indices are parameters which can take values from the parameter set in the third column. The fourth column lists the number of irreducible characters in the family and the last column gives their degrees. Note that we use a slightly different notation for the family of linear characters.

**Example:** Family $\mathcal{F}_3$ consists of 2 types of characters. The characters $\chi_{8,q^2,a_1,a_2}$ where $a_1, a_2$ vary over $\mathbb{F}_q^*$ are $(q - 1)^2$ distinct irreducible characters of degree $q^2$, and the characters $\chi_{8,q^2,a,b_1,b_2}$ where $a$ varies over $\mathbb{F}_q^*$ and $b_1, b_2$ vary over $\mathbb{F}_q$, are $q^2(q - 1)$ distinct irreducible characters of $U$ of degree $q^2$. Furthermore, one has $X_{a_0}X_{a_{10}}X_{a_{11}}X_{a_{12}} \subseteq \ker(\chi_{8,q^2,a_1,a_2}), \ker(\chi_{8,q^2,a,b_1,b_2})$ and $X_{a_8} \not\subset \ker(\chi_{8,q^2,a_1,a_2})$ and $X_{a_8} \not\subset \ker(\chi_{8,q^2,a,b_1,b_2})$.

### 4.1. Proof of Theorem 4.1

We describe the definition and construction of the irreducible characters in each family. Our main tools will be the midafis, Clifford theory and the commutator relations.

**The irreducible characters in family $\mathcal{F}_{12}$**

In this subsection we describe the construction of the irreducible characters in family $\mathcal{F}_{12}$ which is the family of the almost faithful irreducible characters of $U$.

We have already seen in Subsection 3.1 that the hook subgroup $H_{12}$ is normal in $U$. Since $H_{12}$ is a special $p$-group of type $q^{1+8}$, it has $q^8$ linear characters and $q - 1$ irreducible characters of degree $q^2$. We have $X_{a_{12}} \not\subset \ker(\mu_{a_{12}})$ and $\mu_{a_{12}} \not\subset X_{a_{12}}$ for all $s \neq s' \in \mathbb{F}_q^*$. So the restrictions of the midafis $\mu_{a_{12},s}$ are exactly the nonlinear irreducible characters of $H_{12}$. In particular, each nonlinear irreducible character of $H_{12}$ extends to $U$.

By Clifford theory, the almost faithful irreducible characters of $U$ are those lying over the nonlinear irreducible characters of $H_{12}$. The factor group $U/H_{12}$ is isomorphic to $X_{a_1} \times X_{a_2} \times X_{a_4} \cong \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$. So by Gallagher’s theorem [7, Corollary (6.17)], there are $q^2(q - 1)$ almost faithful irreducible characters of $U$.

They all have degree $q^4$ and can be parametrized by $a \in \mathbb{F}_q^*$ (parametrizing the midafi) and $b_1, b_2, b_3 \in \mathbb{F}_q$ (parametrizing the irreducible characters of $U/H_{12}$).

This proves all statements about the family $\mathcal{F}_{12}$ in Table 3.

**The irreducible characters in family $\mathcal{F}_{11}$**

The characters in family $\mathcal{F}_{11}$ are the irreducible characters $\chi$ of $U$ such that $X_{a_{11}} \subseteq \ker(\chi)$ and $X_{a_{11}} \not\subset \ker(\chi)$. We are going to work in the factor group $\overline{U} := U/Z(U) = U/X_{a_{12}}$. By the commutator relations, we have the following subnormal series:

$$
\{1\} \leq \overline{H}_{11} := H_{11}X_{a_{12}}/X_{a_{12}} \leq \overline{N}_{11} := \prod_{i=1}^{12} X_{a_i}/X_{a_{12}} \leq \overline{U},
$$
Table 3. Irreducible characters of $U$.

| Family | Notation | Parameter set | Number | Degree |
|--------|----------|---------------|--------|--------|
| $\mathcal{F}_{12}$ | $\chi_{12.a,b_1,b_2,b_3}$ | $F_q^2 \times F_q \times F_q \times F_q$ | $q^2(q-1)$ | $q^4$ |
| $\mathcal{F}_{11}$ | $\chi_{11.a,b_1,b_2,b_3}$ | $F_q^2 \times F_q \times F_q \times F_q \times F_q$ | $q^4(q-1)$ | $q^4$ |
| $\mathcal{F}_{odd}$ | $\chi_{8,9,10,a_1,a_2,a_3,b}$ | $F_q^8 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{even}$ | $\chi_{8,9,10,q^3.a_1,a_2,a_3}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $(q-1)^3$ | $q^3$ |
| | $\chi_{8,9,10,a_1,a_2,a_3,a_4}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{8}$ | $\chi_{8,q^3,a_1,a_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{9}$ | $\chi_{8,q^3,a_1,a_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{10}$ | $\chi_{10,q^3,a_1,a_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{5,6,7}$ | $\chi_{5,6,7,a_1,a_2,a_3,b_1,b_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q^6(q-1)^3$ | $q^3$ |
| $\mathcal{F}_{5,6}$ | $\chi_{5,6,a_1,a_2,b_1,b_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q^6(q-1)^2$ | $q$ |
| $\mathcal{F}_{7}$ | $\chi_{5,6,a_1,a_2,b_1,b_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q^6(q-1)^2$ | $q$ |
| $\mathcal{F}_{6}$ | $\chi_{5,6,a_1,b_1,b_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q^6(q-1)^2$ | $q$ |
| $\mathcal{F}_{7}$ | $\chi_{5,6,a_1,b_1,b_2}$ | $F_q^9 \times F_q^6 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2 \times F_q^2$ | $q^6(q-1)^2$ | $q$ |

where $\overline{N}_{11}$ does not depend on the order of the root subgroups in the product. Furthermore, $\overline{H}_{11} \cong H_{11}$ is a special $p$-group of type $q^{11}$, the factor group $\overline{N}_{11}/\overline{H}_{11} \cong X_{\alpha_3} \times X_{\alpha_3} \times X_{\alpha_3}$ is elementary abelian of order $q^3$ and $\overline{G}/\overline{N}_{11} \cong X_{\alpha_3}$ is elementary abelian of order $q$. First, we construct all $\psi \in \text{Irr}(\overline{N}_{11})$ with $X_{\alpha_1}X_{\alpha_2}/X_{\alpha_3} \subset \ker(\psi)$. Since $\overline{H}_{11}$ is special of type $q^{11}$, it has $q^6$ linear characters and $q-1$ almost faithful irreducible characters of degree $q^3$. For each $s \in F_q^3$, one has $X_{\alpha_3} \subseteq \ker(\mu_{\alpha_1,s})$. Hence we can identify the midafi $\mu_{\alpha_1,s}$ with an irreducible character of $\overline{G}$, which we also denote by $\mu_{\alpha_1,s}$. Since $\deg(\mu_{\alpha_1,s}) = q^3$ and $X_{\alpha_1} \subseteq \ker(\mu_{\alpha_1,s})$, the restriction
of $\mu_{\alpha_{11}}$ to $\overline{\Pi}_{11}$ is irreducible. Thus, also $\mu_{\alpha_{11}} | \overline{\Pi}_{11}$ is irreducible and each almost faithful irreducible character of $\overline{\Pi}_{11}$ extends to $\overline{N}_{11}$. Gallagher’s theorem [7, Corollary (6.17)] applied to $\overline{\Pi}_{11} \leq \overline{N}_{11}$ implies that each almost faithful irreducible character of $\overline{\Pi}_{11}$ extends to $\overline{N}_{11}$ in $q^3$ ways. Hence there are $q^3(q - 1)$ irreducible characters of $\overline{N}_{11}$ such that $X_{\alpha_{11}} X_{\alpha_{12}}$ is not contained in their kernel, these characters have degree $q^3$ and can be parametrized by $\psi_{a,b_1,b_2,b_3}$ where $a \in \mathbb{F}_q^*$ and $b_i \in \mathbb{F}_q$.

Next, we show that each $\psi_{a,b_1,b_2,b_3}$ extends to $\overline{U}$. Since $X_{\alpha_3} \cdot \prod_{i \geq 5} X_{\alpha_i}/X_{\alpha_{11}}$ is a normal abelian subgroup of $\overline{U}$ of index $q^3$, we can conclude from Ito’s theorem [7, Theorem 6.15] that the degrees of all irreducible characters of $\overline{U}$ divide $q^3$. So from Clifford theory and Gallagher’s theorem applied to $\overline{N}_{11} \leq \overline{U}$ it follows that each $\psi_{a,b_1,b_2,b_3}$ extends to $\overline{U}$ in $q$ ways. We denote these extensions by $\psi_{a,b_1,b_2,b_3}$. Inflating $\psi_{a,b_1,b_2,b_3}$ to $U$, we get the $q^3(q - 1)$ irreducible characters $X_{a_1,b_1,b_2,b_3,b_4}$ of degree $q^3$ of $U$.

The irreducible characters in family $F_{8,9,10}^{odd}$. We assume that $q$ is odd. The characters in $F_{8,9,10}^{odd}$ are the irreducible characters $\chi$ of $U$ such that $X_{\alpha_i} \subseteq \ker(\chi)$ for $i = 11, 12$ and $X_{\alpha_j} \not\subseteq \ker(\chi)$ for $j = 8, 9, 10$. We are going to work in the factor group $\overline{U} := U/X_{\alpha_{11}} X_{\alpha_{12}}$. Note that $\overline{U}$ is a semidirect product $\overline{U} = \overline{K} \rtimes \overline{A}$ of the elementary abelian groups

$$\overline{K} = X_{\alpha_1} X_{\alpha_2} X_{\alpha_4}$$

and

$$\overline{A} := X_{\alpha_3} \cdot \prod_{i \geq 5} X_{\alpha_i}/X_{\alpha_{11}} X_{\alpha_{12}}.$$

We consider the action of $X_{\alpha_1} X_{\alpha_2} X_{\alpha_4}$ on $\text{Irr}(\overline{A})$ by conjugation. For field elements $x, a, b, c, d, e, f \in \mathbb{F}_q$ we define a linear character $\lambda_{x,a,b,c,d,e,f} \in \text{Irr}(\overline{A})$ by

$$x_3(d_3)x_5(d_5)x_6(d_6) \cdots x_{10}(d_{10})X_{\alpha_{11}} X_{\alpha_{12}} \mapsto \phi(x \cdot d_3 + a \cdot d_5 + b \cdot d_6 + \cdots + f \cdot d_{10}),$$

where $\phi$ is a nontrivial linear character of $(\mathbb{F}_q,+)$ as in Subsection 2.3. We claim that

$$(1) \quad \{ \lambda_{x,0,0,0,d,e,f} \mid x \in \mathbb{F}_q, d, e, f \in \mathbb{F}_q^* \}$$

is a set of representatives for the action of $X_{\alpha_1} X_{\alpha_2} X_{\alpha_4}$ on the set $\lambda \in \text{Irr}(\overline{A})$ such that $\lambda|_{X_{\alpha_i}}$ is nontrivial for $i = 8, 9, 10$. From the commutator relations we obtain

$$(2) \quad x_{1(r)x_2(s)x_4(t)} = x_{3(d_3)x_5(d_5) \cdots x_{10}(d_{10})},$$

where $d_i$ are integers. So, $(\lambda_{x,0,0,0,d,e,f})^{x_{1(r)x_2(s)x_4(t)}} = \lambda_{x',0,0,0,d',e',f'}$ if and only if

$$\phi(x'd_3 + d'd_8 + e'd_9 + f'd_{10}) = \phi((x - drs - c’t - fst)d_3 + (-ds - et)d_5 + (-dr - ft)d_6 + (er + fs)d_7 + dd_8 + ed_9 + fd_{10})$$
for all $d_i \in \mathbb{F}_q$. Thus, if $(\lambda_{x,0,0,0,d,e,f})^{x_1(r)x_2(s)x_4(t)} = \lambda_{x',0,0,0,d',e',f'}$ then the coefficients of $d_5, d_6, \ldots, d_{10}$ imply $d' = d, e' = e, f' = f$ and

$$
\begin{pmatrix}
0 & d & e \\
0 & d & f \\
e & f & 0
\end{pmatrix}
\begin{pmatrix}
r \\
s \\
t
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
$$

The determinant of the coefficient matrix of this system of linear equations is $2def$. Since $q$ is odd, we conclude $r = s = t = 0$ and so $x' = x$. So, (1) is indeed a set of representatives and $\text{Stab}_K(\lambda_{x,0,0,0,d,e,f}) = \{1\}$. By Clifford theory, the induced characters $\psi_{a_1,a_2,a_3} := \chi_{0,0,0,a_1,a_2,a_3}$ are $q(q-1)^3$ distinct irreducible characters of $\overline{U}$. By inflation, we obtain $\chi_{8,9,10,a_1,a_2,a_3} \in \text{Irr}(U)$ of degree $q^3$ of $U$.

**The irreducible characters in family $F_{8,9,10}^\text{even}$.** We assume that $q$ is even. The characters in $F_{8,9,10}^\text{even}$ are the irreducible characters of $U$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 11, 12$ and $X_{a_i} \not\subseteq \ker(\chi)$ for $j = 8, 9, 10$. Again, we are going to work in the factor group $\overline{U} := U/X_{a_{11}}X_{a_{12}}$. We define subgroups $\overline{K} = X_{a_1}X_{a_2}X_{a_4}$ and $\overline{A}$ of $\overline{U}$ in the same way as for odd $q$ and obtain the decomposition $\overline{U} = \overline{K} \rtimes \overline{A}$. We consider the conjugation action of $X_{a_1}X_{a_2}X_{a_4}$ on the set of irreducible characters of the elementary abelian normal subgroup $\overline{A}$. As for odd $q$, we define linear characters $\lambda_{x,a,b,c,d,e,f} \in \text{Irr}(\overline{A})$ by

$$x_3(d_3)x_5(d_5)x_6(d_6) \cdots x_{10}(d_{10})X_{a_{11}}X_{a_{12}} \mapsto \phi(x \cdot d_3 + a \cdot d_5 + b \cdot d_6 + \cdots + f \cdot d_{10}),$$

where $\phi$ is a nontrivial linear character of $(\mathbb{F}_q, +)$ and $x, a, b, c, d, e, f \in \mathbb{F}_q$. If $cdef \neq 0$, then $\{defz^2 + cdz \mid z \in \mathbb{F}_q\}$ is a subgroup of $(\mathbb{F}_q, +)$ of index 2. Choose $t_{c,d,e,f} \in \mathbb{F}_q \setminus \{defz^2 + cdz \mid z \in \mathbb{F}_q\}$. We claim that

$$\lambda_{x,a,b,c,d,e,f} \in \text{Irr}(\overline{A})$$

is a set of representatives for the action of $X_{a_1}X_{a_2}X_{a_4}$ on the set $\overline{A} = \text{Irr}(\overline{A})$ such that $\lambda|_{X_{a_i}}$ is nontrivial for $i = 8, 9, 10$. From (2), interpreted in characteristic 2, we see that $(\lambda_{0,0,0,0,d,e,f})^{x_1(r)x_2(s)x_4(t)} = \lambda_{0,0,0,0,d,e,f}$ implies

$$\phi(dd_8 + ed_9 + fd_{10}) = \phi((drs + c+ t + f st)d_3) + (ds + et)d_5 + (dr + ft)d_6 + (er + fs)d_7 + d_8 + ed_9 + fd_{10})$$

for all $d_i \in \mathbb{F}_q$. Thus, if $(\lambda_{0,0,0,0,d,e,f})^{x_1(r)x_2(s)x_4(t)} = \lambda_{0,0,0,0,d,e,f}$ then the coefficients of $d_3, d_5, d_6, \ldots, d_{10}$ imply $r = s = t = 0$ and so $\text{Stab}_K(\lambda_{0,0,0,0,d,e,f}) = \{1\}$.

Now, suppose $(\lambda_{x,a,b,c,d,e,f})^{x_1(r)x_2(s)x_4(t)} = \lambda_{x',0,0,c',d',e',f'}$, where $x \in \mathbb{F}_q$ and $c, c', d, d', e, e', f, f' \in \mathbb{F}_q^\times$. From (2) we see that

$$\lambda_{x,a,b,c,d,e,f} = \lambda_{x',0,0,c',d',e',f'}$$

if and only if

$$\phi(x'd_3 + c'd_7 + d'd_8 + e'd_9 + f'd_{10}) = \phi((x + ct + drs + c + t + f st)d_3) + (ds + et)d_5 + (dr + ft)d_6 + (c + er + fs)d_7 + d_8 + ed_9 + fd_{10})$$

for all $d_i \in \mathbb{F}_q$. So $(\lambda_{x,a,b,c,d,e,f})^{x_1(r)x_2(s)x_4(t)} = \lambda_{x',0,0,c',d',e',f'}$ if and only if $d = d'$, $e = e'$, $f = f'$, $c = c'$, $r = d't$, $s = t$, and $x' = x + ct + \frac{ef}{d}f^2$. It follows that (3) is indeed a set of representatives and $|\text{Stab}_K(\lambda_{x,a,b,c,d,e,f})| = 2$. By Clifford theory,
the induced characters \( \psi_{a_1,a_2,a_3} := \lambda_{10,0,0,0,a_1,a_2,a_3} \) are \((q - 1)^3\) distinct irreducible characters of degree \(q^3\) of \( \overline{U} \). Furthermore, each \( \lambda_{x,0,0,d,e,f} \) extends in two ways to its inertia subgroup and by inducing to \( \overline{U} \) we obtain \(4(q - 1)^4\) irreducible characters \( \psi_{x,a_1,a_2,a_3,a_4} \) of degree \(\frac{q^3}{2}\) of \( \overline{U} \). By inflation, we obtain \( \chi_{8,9,10,9,a_1,a_2,a_3} \in \text{Irr}(U) \) of degree \(q^3\) of \( U \) and \( \chi_{8,9,10,9,a_1,a_2,a_3,a_4} \) of degree \(\frac{q^3}{2}\) of \( U \).

**The irreducible characters in families** \( F_{8,9}, F_{8,10}, F_{9,10} \). The characters in family \( F_{8,9} \) are those \( \chi \in \text{Irr}(U) \) such that \( X_{a_i} \subseteq \ker(\chi) \) for \( i = 9, 11, 12 \) and \( X_{a_i} \nsubseteq \ker(\chi) \) for \( j = 8, 9 \). We are going to work in the factor group \( \overline{U} := U/X_{a_{10}}X_{a_{11}}X_{a_{12}} \). By the commutator relations, the group

\[
\overline{N}_{10} := H_{a_8} \prod_{i \geq 9} X_{a_i}/X_{a_{10}}X_{a_{11}}X_{a_{12}}
\]

is a normal subgroup of \( \overline{U} \) and we have

\[
\overline{N}_{10} \cong H_{a_8} \times X_{a_9} \quad \text{and} \quad \overline{U}/\overline{N}_{10} \cong H_{a_7},
\]

where \( H_{a_i} \) are special \( p \)-groups of type \( q^{i+2} \) and \( q^{i+4} \), respectively. So \( \overline{N}_{10} \) has \((q - 1)^2\) almost faithful irreducible characters and they have degree \( q^2 \). We claim that all of them extend to \( \overline{U} \). Consider the subgroup

\[
\overline{K}_{10} := \prod_{i \neq 1,5} X_{a_i}/X_{a_{10}}X_{a_{11}}X_{a_{12}}
\]

of \( \overline{U} \) of index \( q^2 \) and let \( a_1, a_2 \in \mathbb{F}_q^* \). Since \( \overline{K}_{10} \cong X_{a_2}X_{a_3}X_{a_4}X_{a_5}X_{a_7}X_{a_8}X_{a_9}X_{a_{10}} \), there is a linear character \( \lambda_{a_1,a_2} \) of \( \overline{K}_{10} \) such that

\[
\lambda_{a_1,a_2}|_{X_{a_8}} = \varphi_{a_8,a_1} \quad \text{and} \quad \lambda_{a_1,a_2}|_{X_{a_9}} = \varphi_{a_9,a_2}
\]

(here and in the following we identify the root subgroups with their images in \( \overline{U} \)). The induced characters \( \overline{\chi}^{\overline{U}}_{a_1,a_2} \) have degree \( q^2 \) and are irreducible, because they restrict irreducibly to \( H_{a_7} \). Hence they are extensions of the almost faithful irreducible characters of \( \overline{N}_{10} \). Applying Gallagher’s theorem to \( \overline{N}_{10} \subseteq \overline{U} \), we obtain \((q - 1)^3\) almost faithful irreducible characters \( \psi_{8,9,9,q^2,a_1,a_2,a_3} \) of degree \( q^3 \) and \( q^2(q - 1)^2 \) almost faithful irreducible characters \( \psi_{8,9,9,q^2,a_1,a_2,a_3,a_4} \) of degree \( q^2 \) of \( \overline{U} \), where \( a_i \in \mathbb{F}_q^* \) and \( b_i \in \mathbb{F}_q \). By inflation, we obtain the irreducible characters \( \chi_{8,9,q^2,a_1,a_2,a_3,a_4} \) and \( \chi_{8,9,q^2,a_1,a_2,a_3,a_4,a_5} \) of \( U \). This completes the construction of the irreducible characters in the family \( F_{8,9} \).

The characters in the family \( F_{8,10} \) are those \( \chi \in \text{Irr}(U) \) such that \( X_{a_i} \subseteq \ker(\chi) \) for \( i = 9, 11, 12 \) and \( X_{a_i} \nsubseteq \ker(\chi) \) for \( j = 8, 10 \), and the characters in the family \( F_{9,10} \) are those \( \chi \in \text{Irr}(U) \) such that \( X_{a_i} \subseteq \ker(\chi) \) for \( i = 8, 11, 12 \) and \( X_{a_j} \nsubseteq \ker(\chi) \) for \( j = 9, 10 \). The definition and construction of the irreducible characters in these two families are analogous to the definition and construction of the irreducible characters in \( F_{8,9} \).

**The irreducible characters in families** \( F_8, F_9, F_{10} \). The characters in family \( F_8 \) are those \( \chi \in \text{Irr}(U) \) such that \( X_{a_i} \subseteq \ker(\chi) \) for \( i = 9, 10, 11, 12 \) and \( X_{a_i} \nsubseteq \ker(\chi) \). We are going to work in the factor group \( \overline{U} := U/X_{a_{10}}X_{a_{11}}X_{a_{12}} \). By the commutator relations, the group

\[
\overline{N}_{9,10} := H_{a_8} \prod_{i \geq 9} X_{a_i}/X_{a_{10}}X_{a_{11}}X_{a_{12}}
\]
is a normal subgroup of $\overline{U}$ and $\overline{U}/\overline{N}_{9,10} \cong H_{a_2}$. Since $\overline{N}_{9,10}$ is special of type $q^{1+4}$, it has $q - 1$ almost faithful irreducible characters and they have degree $q^2$ and the midafis $\mu_{a_3,a}, a \in \mathbb{F}_q^\times$, are extensions of these almost faithful characters to $\overline{U}$. Now, Gallager’s theorem applied to $\overline{N}_{9,10} \triangleleft \overline{U}$ gives us $(q - 1)^2$ almost faithful irreducible characters $\psi_{8,q^2,a_1} \alpha$, of degree $q^3$ and $q^2(q - 1)$ almost faithful irreducible characters $\psi_{8,q^2,a_1,b_1,b_2}$ of degree $q^2$ of $\overline{U}$, where $a_1, a_1 \in \mathbb{F}_q^\times$ and $b_1 \in \mathbb{F}_q$. By inflation, we obtain the irreducible characters $\chi_{8,q^2,a_1} \alpha$ and $\chi_{8,q^2,a_1,b_1,b_2}$ of $U$.

This completes the construction of the irreducible characters in the family $F_8$.

The characters in the family $F_9$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 8,10,11,12$ and $X_{a_i} \not\subseteq \ker(\chi)$, and the characters in the family $F_{10}$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 8,9,11,12$ and $X_{a_1} \not\subseteq \ker(\chi)$. The definition and construction of the irreducible characters in these two families are analogous to the definition and construction of the irreducible characters in $F_8$.

**The irreducible characters in family $F_{5,6,7}$**. The characters in $F_{5,6,7}$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 8,9,\ldots,12$ and $X_{a_j} \not\subseteq \ker(\chi)$ for $j = 5,6,7$. We are going to work in the factor group $\overline{U} := U/\prod_{i \geq 8}X_{a_i}$. By the commutator relations, the group

$$\overline{N}_{5,6,7} := \prod_{i \geq 8}X_{a_i}/\prod_{i \geq 8}X_{a_i}$$

is a normal subgroup of $\overline{U}$ and we have

$$\overline{N}_{5,6,7} \cong H_{a_5} \times X_{a_5} \times X_{a_7} \quad \text{and} \quad \overline{U}/\overline{N}_{5,6,7} \cong X_{a_2} \times X_{a_4},$$

where $H_{a_5}$ is a special $p$-group of type $q^{1+2}$. Hence there are $(q - 1)^3$ almost faithful irreducible characters of $\overline{N}_{5,6,7}$ such that $X_{a_5}, X_{a_6}, X_{a_7}$ are not contained in their kernel, these characters have degree $q$ and can be parametrized by $\psi_{a_1,a_2,a_3}$ where $a_1 \in \mathbb{F}_q^\times$.

We show that each $\psi_{a_1,a_2,a_3}$ extends to $\overline{U}$. Since $X_{a_5}, X_{a_6}, \prod_{i \geq 4}X_{a_i}/\prod_{i \geq 8}X_{a_i}$, is a normal abelian subgroup of $\overline{U}$ of index $q$, we can conclude from Ito’s theorem [7, Theorem (6.15)] that the degrees of all irreducible characters of $\overline{U}$ divide $q$. So from Clifford theory and Gallager’s theorem applied to $\overline{N}_{5,6,7} \triangleleft \overline{U}$ it follows that each $\psi_{a_1,a_2,a_3}$ extends to $\overline{U}$ in $q^2$ ways. We denote these extensions by $\psi_{a_1,a_2,a_3,b_1,b_2}$. We inflate $\psi_{a_1,a_2,a_3,b_1,b_2}$ to $U$, we get the $q^2(q - 1)^3$ irreducible characters $\chi_{a_1,a_2,a_3,b_1,b_2}$ of degree $q$ of $U$.

**The irreducible characters in family $F_{5,6}, F_{5,7}, F_{6,7}$**. The characters in $F_{5,6}$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i \geq 7$ and $X_{a_j} \not\subseteq \ker(\chi)$ for $j = 5,6$. Let $\overline{U}$, $\overline{N}_{5,6,7}$ be the groups defined in the construction of the irreducible characters in the family $F_{5,6,7}$. The group $\overline{N}_{5,6,7} \cong H_{a_5} \times X_{a_5} \times X_{a_7}$ has $(q - 1)^2$ irreducible characters such that $X_{a_5}$ is contained in their kernel and $X_{a_5}, X_{a_6}$ are not contained in their kernel, these characters have degree $q$ and can be parametrized by $\psi_{a_1,a_2}$ where $a_1,a_2 \in \mathbb{F}_q^\times$. Using Gallager’s theorem in the same way as for $F_{5,6,7}$ we see that each $\psi_{a_1,a_2}$ extends to $\overline{U}$ in $q^2$ ways leading to the irreducible characters $\chi_{a_1,a_2,b_1,b_2}$ of degree $q$ of $U$.

The characters in the family $F_{5,7}$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 6,8,9,10,11,12$ and $X_{a_j} \not\subseteq \ker(\chi)$ for $j = 5,7$, and the characters in the family $F_{6,7}$ are those $\chi \in \text{Irr}(U)$ such that $X_{a_i} \subseteq \ker(\chi)$ for $i = 5,8,9,10,11,12$.
and $X_{\alpha_j} \not\subseteq \ker(\chi)$ for $j = 6, 7$. The definition and construction of the irreducible characters in these two families are analogous to the definition and construction of the irreducible characters in $\mathcal{F}_{5,6}$.

**The irreducible characters in families $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_7$.** The characters in $\mathcal{F}_5$ are those $\chi \in \text{Irr}(U)$ such that $X_{\alpha_i} \subseteq \ker(\chi)$ for $i \geq 6$ and $X_{\alpha_5} \not\subseteq \ker(\chi)$. Let $\mathcal{U}$, $\mathcal{N}_{5,6,7}$ be the groups defined in the construction of the irreducible characters in the family $\mathcal{F}_{5,6,7}$. The group $\mathcal{N}_{5,6,7} \cong H_{\alpha_5} \times X_{\alpha_6} \times X_{\alpha_7}$ has $q - 1$ irreducible characters such that $X_{\alpha_5}$ and $X_{\alpha_6}$ are contained in their kernel and $X_{\alpha_7}$ is not contained in their kernel, these characters have degree $q$ and can be parametrized by $\psi_a$ where $a \in \mathbb{F}^\times_q$. Using Gallagher’s theorem in the same way as for $\mathcal{F}_{5,6,7}$ we see that each $\psi_{a_1,a_2}$ extends to $\mathcal{U}$ in $q^2$ ways leading to the irreducible characters $\chi_{a,b_1,b_2}$ of degree $q$ of $U$.

The characters in the family $\mathcal{F}_6$ are those $\chi \in \text{Irr}(U)$ such that $X_{\alpha_i} \subseteq \ker(\chi)$ for $i = 5, 7, 8, 9, 10, 11, 12$ and $X_{\alpha_6} \not\subseteq \ker(\chi)$, and the characters in the family $\mathcal{F}_7$ are those $\chi \in \text{Irr}(U)$ such that $X_{\alpha_i} \subseteq \ker(\chi)$ for $i = 5, 6, 8, 9, 10, 11, 12$ and $X_{\alpha_7} \not\subseteq \ker(\chi)$. The definition and construction of the irreducible characters in these two families are analogous to the definition and construction of the irreducible characters in $\mathcal{F}_5$.

**The irreducible characters in family $\mathcal{F}_{lin}$.** We have

$$U/[U,U] \cong X_{\alpha_1} \times X_{\alpha_2} \times X_{\alpha_3} \times X_{\alpha_4} \cong \mathbb{F}_q^4$$

and so the statements about the linear characters of $U$ are clear. This completes the proof of Theorem 4.1. □

4.2. **Numbers and degrees of irreducible characters.** We see from Table 3 that for odd $q$ the degrees of the irreducible characters of $U$ are powers of $q$. So $U$ is a $q$-power-degree group in the sense of [8]. This observation is a special case of a general theorem of B. Szegedy on the Sylow $p$-subgroups of classical groups defined over finite fields of of good characteristic $p$, see [10, Theorem 2].

The subgroup $U_n(q)$ of $\text{GL}_n(\mathbb{F}_q)$ consisting of all upper unitriangular matrices is a Sylow $p$-subgroup of $\text{GL}_n(\mathbb{F}_q)$. A conjecture of G. Higman [6] states that the number of conjugacy classes of $U_n(q)$ is given by a polynomial in $q$ with integer coefficients. I.M. Isaacs proved that the degrees of the irreducible characters of $U_n(q)$ are of the form $q^\ell$, $0 \leq \ell \leq \mu(n)$ where the upper bound $\mu(n)$ depends on $n$ and is known explicitly. G. Lehrer [9] conjectured that the number $N_{n,e}(q)$ of irreducible characters of $U_n(q)$ of degree $q^e$ are given by a polynomial in $q$ with integer coefficients. I.M. Isaacs suggested a strengthened form of Lehrer’s conjecture stating that $N_{n,e}(q)$ is given by a polynomial in $q - 1$ with nonnegative integer coefficients. Obviously, Isaac’s conjecture implies Higman’s and Lehrer’s conjectures.

From Table 3, we can derive that an analogue of Isaac’s conjecture holds for the Sylow $p$-subgroup $U$ of the Chevalley groups of type $D_4$, even in bad characteristic.

**Corollary 4.2.** The degrees of the irreducible characters of $U$ are given by Table 4. In particular, the number of conjugacy classes of $U$ is

$$\begin{cases} 2q^5 + 5q^4 - 4q^3 - 4q^2 + 2q & , \text{if } q \text{ is odd}, \\ 2q^5 + 8q^4 - 16q^3 + 14q^2 - 10q + 3 & , \text{if } q \text{ is even}. \end{cases}$$

**Proof.** This follows from Theorem 4.1. □
Table 4. Numbers and degrees of the irreducible characters of $U$.
The numbers of the irreducible characters are given as polynomials in $v = q - 1$.

| Degree | Number of irreducible characters | Comments |
|--------|----------------------------------|----------|
| $q^4$  | $v^4 + 3v^3 + 3v^2 + v$          |          |
| $q^3$  | $v^5 + 5v^4 + 10v^3 + 7v^2 + v$  | if $q$ is odd |
|        | $v^5 + 4v^4 + 10v^3 + 7v^2 + v$  | if $q$ is even |
| $q^2/2$ | $4v^4$                          | only if $q$ is even |
| $q^2$  | $3v^4 + 9v^3 + 9v^2 + 3v$        |          |
| $q$    | $v^5 + 5v^4 + 10v^3 + 9v^2 + 3v$ |          |
| $1$    | $v^4 + 4v^3 + 6v^2 + 4v + 1$     |          |

For odd prime powers $q$ the number of conjugacy classes of $U$ was already computed by S.M. Goodwin and G. Röhrle [5, Table 1].

Acknowledgements. Part of this work was done while the authors were participating in the program on Representation Theory of Finite Groups and Related Topics at the Mathematical Sciences Research Institute (MSRI), Berkeley. It is a pleasure to thank the organizers Professors J. L. Alperin, M. Broué, J. F. Carlson, A. S. Kleshchev, J. Rickard, B. Srinivasan for generous hospitality and support. We also thank C. André, P. Diaconis, I.M. Isaacs, N. Thiem and N. Yan for the stimulating seminar on supercharacters at the MSRI.

References

[1] R. W. Carter, Simple Groups of Lie Type, ‘A Wiley-Interscience publication’, London, 1972.
[2] P. Diaconis, I. Isaacs, Supercharacters and superclasses for algebra groups, Trans. Amer. Math. Soc., 360 (2008), 2359–2392.
[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12 (2008), http://www.gap-system.org
[4] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer, CHEVIE - A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, Appl. Algebra Engrg. Comm. Comput., 7 (1996), 175–210.
[5] S. Goodwin, G. Röhrle, Calculating conjugacy classes in sylow $p$-subgroups of finite chevalley groups, J. Algebra, 321 (2009), 3321–3334.
[6] G. Higman, Enumerating $p$-groups. i: Inequalities, Proc. London Math. Soc. (3), 10 (1960), 24–30.
[7] I. Isaacs, Character Theory of Finite Groups, Dover, New York, 1976.
[8] I. Isaacs, Characters of groups associated with finite algebras, J. Algebra, 177 (1995), 708–730.
[9] G. Lehrer, Discrete series and the unipotent subgroup, Compos. Math., 28 (1974), 9–19.
[10] B. Szegedy, Characters of the Borel and Sylow subgroups of classical groups, J. Algebra, 267 (2003), 130–136.
[11] L. Tung, Irreducible characters of the unitriangular groups, PhD thesis, Wayne State University, 2008.
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