In this contribution we group the operator basis for $d^2$ dimensional Hilbert space in a way that enables us to relate bases of entangled states with single particle mutually unbiased state bases (MUB), each in dimensionality $d$. We utilize these sets of operators to show that an arbitrary density matrix for this $d^2$ dimensional Hilbert space is analyzed by via $d^2 + d + 1$ measurements, $d^2 - d$ of which involve those entangled states that we associate with MUB of the $d$-dimensional single particle constituents. The number $d^2 + d + 1$ lies in the middle of the number of measurements needed for bipartite state reconstruction with two-particle MUB ($d^2 + 1$) and those needed by single-particle MUB $[(d^2 + 1)^2]$.\\

**I. INTRODUCTION**

Two orthonormal vector bases, $B_1$, $B_2$, are said to be mutually unbiased bases (MUB) iff

\[
\forall |u_1\rangle, |u_2\rangle \in B_1, B_2 \text{ resp. } \langle u_1|u_2\rangle = \text{constant},
\]

i.e., the absolute value of the scalar product of vectors from different bases is constant and independent of the vectorial labels within either basis. For a finite dimension, $d$, Hilbert spaces the constant is $\frac{1}{\sqrt{d}}$. Schwinger [1] first was to emphasize that there are more than two such bases “that exhibit maximum degree of incompatibility” i.e. more than just the pair of conjugate bases such as $|x\rangle$, (spatial coordinates) and $|k\rangle$ (momentum representation basis). The information theoretic oriented term “mutually unbiased bases” (MUB) is due to Wootters [2].

For the infinite dimensional Hilbert space case Wootters and coworkers [2, 3, 4] related the MUB’s to lines in phase space. The transcription of these notions ("lines in phase space") to the finite, $d$, dimensional cases may be accomplished (3, 4, 5) via the eigenfunctions of the commuting operators

\[X^mZ^l, X^{m'}Z^{l'}, m = sm', l = sl',\]

with $m, m', l, l', s$ integers. Here $X$ and $Z$ are the Schwinger operators (SO) [1]. These are discussed in the following section, and abide by the relation $ZX = \omega XZ$, with $\omega = e^{i\frac{\pi}{2d}}$.

The maximal number of MUB was shown [6] to be $d+1$. Ivanovic, [6] demonstrated that for $d = p$ (a prime) the set of $d + 1$ MUB allows what is probably the most efficient means for determining the density matrix of an arbitrary state. The MUB analysis attracted great deal of research and cogent reviews are given, e.g., [3, 5, 6]. These studies now involve abstract algebra and projective geometry: [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Of particular interest to us are the articles by Planat and coworkers [15, 16] who studied entangled states in conjunction with MUB sets similar to those studied in the present paper. Similar ideas may be found also in [11, 12, 13, 14].

In this work we aim to relate $d$-dimensional Hilbert space MUB (with $d$ a prime) with bases of entangled states in $d^2$-dimensional Hilbert space. To form this relation, we classify the basis for all operators in $d^2$-dimensional Hilbert space via SO for $d$ dimensional Hilbert space. Thus, we give a physical interpretation for the $d^2$-dimensional Hilbert space, as two particles each in $d$-dimensional Hilbert space. The basis for all operators in $d^2$-dimensional Hilbert space contains $d^4$ orthogonal operators. Following the rational of [3], we cluster these operators into subsets of $d^2 - 1$ commuting operators, and construct their common eigenvalues. We show that these states form a basis in the $d^2$-dimensional Hilbert space. Each basis (correspond to different subset of operators) has a simple and transparent relation to one-particle MUB. Some of the bases constitute maximally entangled states for the two-particle system. The relation between these entangled states and one-particle MUB is obtained via a projection of the one particle state onto the two particle entangled state. Finally, we apply our classification for the basis for all operators in $d^2$-dimensional Hilbert space for state reconstruction (tomography). We show that one needs less measurements than are needed for bipartite state tomography analyzed only by single-particle MUB.

In section II, utilizing formalism that stresses the (algebraic) field aspect Bayesian [3] applied to the approach of [3], we obtain $d + 1$ MUB for the $d$ dimensional one particle system, that form subsets each of $d - 1$ commuting orthogonal operators. These operators are selected from the $d^2$ orthogonal complete set of operators that span the operator space. In this section cases with dimensionality, $d$, an odd prime are discussed. Extension to $d$ powers of odd prime is discussed in appendix A. The analysis in section II aims to setup the stage for section III. In section III, we analyze the operator space of two-particle system, each belong to a $d$-dimensional Hilbert space. In this section, we present the main results of this paper. First, we associate the single-particle $d + 1$ sets of operators (each set contains $d$ orthogonal commuting operators) with two-particle bases made of entangled states.
Then we show that the $d^2$-dimensional Hilbert space of two-particle density matrix is analyzed by $d^2 + d + 1$ measurements, $d(d−1)$ of which involve those entangled states that may be associated with single-particle MUB. We show that particularly symmetric density matrix may be accounted for by $d^2 + 1$ measurements, in close analogy with the corresponding single particle MUB case. The numerology of the operator counting is given in Appendix B. The last section includes conclusions and some remarks.

II. MUTUALLY UNBIASED BASES - FINITE DIMENSION

Schwinger [1] noted that the physics of finite dimensional, $d$, Hilbert space is expressed by two unitary operators, $X$, and $Z$. Thus if we label the $d$ distinct states, termed the computational basis, by $|n\rangle$, $n = 0, 1, \cdots , d − 1$; $|n + d\rangle = |n\rangle$, these operators are defined by:
\[
Z|n\rangle = \omega^n|n\rangle; \quad X|n\rangle = |n + 1\rangle, \tag{2.1}
\]
with $\omega = e^{2\pi i/d}$. They form a complete set, i.e. only a multiple of the identity commutes with both $X$, $Z$. We shall briefly outline a method to utilize these operators (due mainly to [5]) to construct the $d+1$ MUB for a $d$ dimensional Hilbert space with $d$ being an odd prime. This review will be of help in building our sets of entangled states that we shall associate with these MUB. The computational basis vectors span the Hilbert space. All operators in this space are expressible in terms of the $d^2$ Schwinger-operators $X^m Z^l$ whose number is obviously $d^2$ form an orthogonal basis for all operators in the $d$ dimensional Hilbert space,
\[
\text{Tr} \left[ X^m Z^l \left( X^{m'} Z^{l'} \right)\dagger \right] = d \delta_{m,m'} \delta_{l,l'}. \tag{2.3}
\]
This follows from Eq. (2.1) which implies the commutation formula
\[
X Z = \omega Z X. \tag{2.4}
\]
Now, let us confine ourselves to cases wherein $m,l \in \mathbb{F}_d$ where $\mathbb{F}_d$ is a Galois field with $d$ elements. In this case, we can relate Schwinger operators to MUB. With this aim we group Schwinger operators $\{X^m Z^l\}$ into $d+1$ sets of $d+1$ orthogonal commuting operators (which together with the identity operator form a complete operator basis for the Hilbert space). Each set of (commuting, orthogonal) operators defines a unique vector basis in Hilbert space. All the $d+1$ sets of bases form an MUB set.

Let us first consider the case $m = 0$ in Eq. (2.2). Readily, the operators $Z^l$ with $l = 0, 1, \cdots, d-1$ $(l = 0 \text{ is, trivially, the identity operator})$ form one set of commuting and orthogonal operators. This set is diagonalized in the computational basis (c.f. Eq. (2.1)). Next consider the case $m \neq 0$. In this case a unique inverse $m^{-1}$ is defined on $\mathbb{F}_d$ and thus we can rewrite the operators (2.2) as
\[
X^m Z^l = \omega^\nu (X Z^b)^m, \tag{2.5}
\]
where $b = l/m$, $\nu = -\frac{b}{m} m(m-1)$ and $b = 0, 1, \cdots, d-1$. Thus, we have associated the $d(d-1)$ of the above operators in the following manner
\[
X^m Z^{bm} \sim (X Z^b)^m. \tag{2.6}
\]
That is, these operators differ at most by a unimodular number. Now, for a fixed $b$, we have $(d-1)$ orthogonal and commuting operators. There are $d$ distinct such sets, each labelled by $b$, $b = 0, 1, \cdots, d-1$ which are orthogonal with
\[
\text{Tr} \left[ (X Z^b)^m \left( (X Z^b)^{m'} \right)\dagger \right] = d \delta_{b,b'} \delta_{m,m'}, \quad m, m' \neq 0. \tag{2.7}
\]
When each $b$-labelled set of the $d-1$ orthogonal, commuting operators is supplemented with the identity operator it constitutes a set of $d$ unitary, orthogonal and commuting operators. And as such, defines a vector basis for Hilbert space: The $d$ (orthonormal) vectors diagonalize these operators in the set. (We remark, in passing, that the basis is defined up to a choice of a phase factor which does not affect the following results.) For each ($b$-labelled) set there exist a unique vector basis. Here, we are designating the vectors that form the basis by $|b; c\rangle$, where the index $b$ labels the basis and $c$ that particular vector in the basis $b$, $(b, c = 0, 1, \cdots, d-1)$. The expression for these states in terms of the computational basis is
\[
|b; c\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \omega^{\nu n(n-1)-cn}|n\rangle, \tag{2.8}
\]
The eigenvalues are $\omega^\nu$.

The importance of the classification of the operators in this manner arises when considering the relation between different sets of the vector basis. One can readily check that these bases form MUB, i.e.,
\[
\langle b'; c' | b; c \rangle = \delta_{c,c'},
\]
\[
| \langle b', c' | b; c \rangle | = \frac{1}{\sqrt{d}} \quad b \neq b', \tag{2.9}
\]
Thus, these $d$ distinct sets of bases plus the computational basis (which is mutually unbiased to all of these sets) form the maximal number of MUB, that is $d + 1$. The proof of the last formula involves the well known [19] Gaussian sums.

Since Schwinger operators, Eq. (2.2) form an operator basis for $d$-dimensional Hilbert space, so do the set of operators $(X Z^b)^m$ together with $Z^b$ ($b = 0, 1, \cdots, d-1$).
Since we may write an arbitrary density operator as
\[ \rho = \frac{1}{d} \left( \sum_{m=1}^{d-1} \sum_{b=0}^{d-1} \text{Tr} \left[ \rho (XZ^b)^m \right] \left( (XZ^b)^m \right)^\dagger \right) + \sum_{i=0}^{d-1} \text{Tr} \left[ \rho Z^i \right] (Z^i)^\dagger \]. \hspace{1cm} (2.10)

We are now aiming to relate this form to the expression of density operators in terms of MUB’s states \(|b; c\rangle\). We shall first prove the following theorem.

**Completeness Theorem**

Let several sets of operators, each labelled by \(b\), constitute of \(d\) orthogonal, commuting unitary matrices, \(U_α(b)\ (α = 0, 1, 2, \ldots, d-1)\), over a \(d\) dimensional Hilbert space. We take, without loss of generality, \(U_0(b) = I\). Let \(|ν_b\rangle\ (ν = 0, 1, 2, \ldots, d-1)\), be a basis that diagonalize the \(d\) matrices of the \(b\) set, such that

\[ \langle ν_b | U_α(b) | ν_c' \rangle = W^α_ν(b) δ_{ν,ν'}, \hspace{1cm} (2.11) \]

\[ |W^α_ν(b)| = 1. \]

Then, the following (completeness) relation

\[ \frac{1}{d} \sum_α W^{α}_{ν_b}(b)(W^{α}_{ν_b}(b))^\ast = δ_{ν_b,ν'_b}, \hspace{1cm} (2.12) \]

holds.

**Proof**

Let’s define \(d\) vectors for each set \(b\),

\[ |α_b\rangle = \frac{1}{\sqrt{d}} \sum_{ν_b} W^α_{ν_b}(b) |ν_b\rangle \]

\[ \Rightarrow \langle ν_b | α_b \rangle = \frac{1}{\sqrt{d}} W^α_{ν_b}(b). \hspace{1cm} (2.13) \]

Now the orthogonality of \(U_α(b)\) implies

\[ d δ_{α,α'} = \text{Tr} \left[ U_α(b) U^{\dagger}_{α'}(b) \right] \]

\[ = \sum_{ν,ν'} \langle ν_b | U_α(b) | ν'_b \rangle \langle ν'_b | U^{\dagger}_{α'}(b) | ν_b \rangle \]

\[ = \sum_{ν} W^α_ν(b)(W^{α}_{ν')(b))^\ast = d \sum_{ν} \langle ν'_b | ν_b \rangle \langle ν_b | α_b \rangle. \]

\[ \therefore \langle α'|α \rangle = δ_{α,α'}. \hspace{1cm} (2.14) \]

Thus the sets of \(d\) vectors \(|α_b\rangle\) form a complete orthonormal basis for each \(b\). Hence we have the completeness relation

\[ \langle ν_b | ν'_b \rangle = δ_{ν_b,ν'_b} = \sum_{α_b} \langle ν_b | α_b \rangle \langle α_b | ν'_b \rangle \hspace{1cm} (2.15) \]

\[ = \frac{1}{d} \sum_α W^{α}_{ν_b}(b)(W^{α}_{ν_b}(b))^\ast \]

Returning to the first term of Eq. (2.10) - let \((XZ^b)^m\) correspond to \(U_α(b)\), i.e. the index \(m\) corresponds to \(α\) \((α > 0)\) and \(U_0(b) = I\). Since the sum in first term of Eq. (2.10) runs over \(m > 0\), for each \(b\), we add and subtract the identity. In the second term we have that \(Z^i\) goes over to \(Z_α - d\) orthonormal, commuting unitary matrices with \(Z_0 = I\). With these replacements we may write,

\[ \rho = \frac{1}{d} \sum_{m=0}^{d-1} \sum_{b=0}^{d-1} \text{Tr} \left[ \rho U_m(b) U^{\dagger}_m(b) - I \right] + \frac{1}{d} \sum_{m=0}^{d-1} \text{Tr} \left[ \rho Z_m \right] Z^m_m. \hspace{1cm} (2.16) \]

Expressing each of these operators via its spectral representation, e.g.,

\[ U_m(b) = \sum_{c=0}^{d-1} |b;c⟩⟨b;c|, \hspace{1cm} (2.17) \]

where \(|b;c⟩\) are given in [2, 3] and substituting into the expressions above we get, after utilizing the completeness theorem,

\[ ρ = \sum_{b,c} |b;c⟩⟨b;c|ρ|b;c⟩⟨b;c| + \sum_n |n⟩⟨n|ρ⟨n| - I \hspace{1cm} (2.18) \]

This is an expression for \(ρ\) in terms of probabilities. The number \(|⟨b;c|ρ|b;c⟩|\) corresponds to the probability to find the state \(|b;c⟩\) in \(ρ\). We see that, as shown in [3], the state of the system, i.e. \(ρ\), is obtainable via the \(d + 1\) measurements, where the measurement of a unitary operator is understood by the measurement of the two commuting hermitian operators,

\[ \hat{M}_1 = \hat{X} \hat{Z}^b + (\hat{X} \hat{Z}^b)^\dagger \]

\[ \hat{M}_2 = i(\hat{X} \hat{Z}^b - (\hat{X} \hat{Z}^b)^\dagger). \hspace{1cm} (2.19) \]

Each of these measurements yields the \(d - 1\) independent probabilities (since the sum of the probabilities adds to one). This gives \((d + 1)(d - 1) = d^2 - 1\) numbers that determine the density matrix. It should be noted that this holds since the operators are non-degenerate.

**III. ENTANGLEMENT ASSOCIATED WITH MUB**

We now wish to consider two \(d\) dimensional particle state. The Hilbert space dimensionality is now \(d^2\) while the operator space is \(d^4\) dimensional. Our aim is to utilize the single particle MUB given above to construct two particle entangled state bases that, thereby, may be viewed as associated with the MUB.

We first wish to state what we consider as a self evident sufficient condition for entanglement: Given single
particle (particle \(\mu\)) operators \(A\) and \(A'\) which do not commute: \(AA' = \alpha A'A\), and similarly, given the non-commuting operators \(B\) and \(B'\) for a second particle (particle \(\nu\)): \(BB' = \beta B'B\). Let \(\alpha, \beta\) be scalars with \(\alpha \beta = 1\).

The two two-particle operators \(AB\) and \(A'B'\) commute. We now assert that the common eigenfunction of these two operators is that of an entangled state - we are unaware of a formal proof of this which we consider as obvious and use it as a guide for constructing entangled states.

Returning to our two-particle Hilbert space. The discussion of the previous section implies that the \(d^4\) unitary orthogonal operators,

\[
(X^{m_1} Z^{l_1})_\mu (X^{m_2} Z^{l_2})_\nu, \quad m_1, l_1, m_2, l_2 = 0, 1, \ldots, d - 1,
\]

form a complete orthonormal operator basis. Restricting our consideration to \(m_1, l_1, m_2, l_2 \in \mathbb{F}_d\), allow us to reconstruct and classify the operators in (3.1) as follows

\[
\begin{align*}
(X^{b_1})_\mu (X^{b_2})_\nu, \quad (d - 1)^2 & \text{ operators} \\
Z^{b_1}_\mu (X^{b_2})_\nu, \quad d \cdot (d - 1) & \text{ operators} \\
(X^{b_1})_\mu Z^{b_2}_\nu, \quad d \cdot (d - 1) & \text{ operators} \\
Z^{b_1}_\mu Z^{b_2}_\nu, \quad d^2 & \text{ operators},
\end{align*}
\]

here \(b_1, b_2 = 0, 1, \ldots, d - 1\) and \(m_1, m_2 = 1, \ldots, d - 1\). This construction is carried out with an exact analogy to the one-particle case (c.f. Eq. (2.3)). One readily verifies that the above \(d^4\) operators are orthogonal and thus span the operator space.

First, let us note that these operators are diagonalized in the one-particle MUB basis that was introduced in section II. Alas, these bases do not form an MUB set on the two-particle Hilbert space. Thus, for example, the basis which diagonalize \(Z^{b_1}_\mu (X^{b_2})_\nu\) is \(\{|n\}_\mu |b_1; c\}_\nu\), while the operators \(Z^{b_1}_\mu Z^{b_2}_\nu\) are diagonalized in \(\{|n\}_\mu |n\}_\nu\). These bases are clearly not MUB. However, in this construction we do find sets of MUB for the two-particle Hilbert space that is directly constructed from the one-particle MUB. The basis which diagonalize \(Z^{b_1}_\mu (X^{b_2})_\nu\) (i.e., \(\{|n\}_\mu |b_1; c\}_\nu\)) and the basis that operators \((X^{b_1})_\mu Z^{b_2}_\nu\) are diagonalized in \(\{|b_1; c\}_\mu |n\}_\nu\) are both MUB. Another set of MUB constructed with these MUB bases are diagonalized \((X^{b_1})_\mu (X^{b_2})_\nu\) \(\{|b_1; c\}_\mu |b_2; c'\}_\nu\) and the basis that diagonalize \(Z^{b_1}_\mu Z^{b_2}_\nu\).

We show that some of the above operators are diagonalized by entangled states that form two-particle MUB and have a close relation to a single-particle MUB.

To show this, we rewrite the first set in Eq. (3.2) as

\[
\left(X^{s} Z^{b} \right)_\mu (Z^{m} Z^{l})_\nu^{m_1},
\]

where the parameters \(s, m, b, l\) are defined as \(s m_1 \equiv m_2\), \(s m \equiv b_1\) and \(b_2 \equiv bm_2 - sm_1\). This set, for fixed \(s, m, b, l\), contains \(d(d - 1)^2\) orthogonal, commuting operators. (Allowing \(s, m, b, l\) to vary, the sets contain \([d(d - 1)]^2\) orthogonal operators.) Now consider \(d\) orthogonal operators extracted from the last set of Eq. (3.2) in the following way: We partition the set into two sets by \((r, s) = (1, 2, \ldots, d - 1)\)

\[
\begin{align*}
Z^{r} Z^{s}_\mu (d - 1)^2 & \text{ operators} \\
Z^{r}_\mu (d - 1) & \text{ operators}, \\
Z^{s}_\mu (d - 1) & \text{ operators},
\end{align*}
\]

these, with the identity operator, add up to the set of \(d^2\) orthogonal, commuting unitary operators. For fixed \(s\), the \((d - 1)\) operators \(Z^{r} Z^{s}_\mu (d - 1)^2\) commute with the \((d - 1)\) operators given in Eq. (3.3) defined above to form, with the identity, \(d^2\) orthogonal commuting operators in our \(d^2\) dimensional Hilbert. Thus these define the representations wherein they are diagonal. In terms of the computational basis for the two particles, these states are given by

\[
|b, s; c_1, c_2\rangle_{\mu, \nu} = \frac{1}{\sqrt{d}} \sum_{n} \omega^{s c_1 n (n-1) - c_1 n} |n\rangle_{\mu} |sn + c_2\rangle_{\nu}.
\]

Then it can be checked that \((c_1, c_2) = 0, 1, 2, \ldots, d - 1\), we shall eschew the label \(\mu, \nu\) below

\[
Z^{r} Z^{s}_\mu |b, s; c_1, c_2\rangle = \omega^{c_2} |b, s; c_1, c_2\rangle
\]

\((X^{r})_\mu (X^{s})_\nu |b, s; c_1, c_2\rangle = \omega^{c_1} |b, s; c_1, c_2\rangle, \quad (3.6)
\]

c = sbc_2. These states are entangled states (in accord with our guide), they are orthonormal and each span the \(d^2\) dimensional Hilbert space. Since each operator set is labelled with the two indices, \(s\) and \(b\), we have \(d(d - 1)\) distinct sets which constitutes a \(d^2\) dimensional base composed of entangled states. Within these sets, \((d - 1)\) of them are MUB. By direct calculation one obtains,

\[
|\langle b, s; c_1, c_2|b', s'; c_1', c_2'\rangle| = \delta_{c_1, c_2} = b = b', \quad s = s',
\]

\[
= \delta_{c_2, c_1'} = \frac{1}{\sqrt{d}} \quad s = s', \quad b \neq b',
\]

\[
= \frac{1}{d} \quad s \neq s', \quad b = b'.
\]

Thus, bases with distinct \(s\) and \(b\) are MUB (i.e. \(d\)-1 entangled state bases) while common \(s\) and \(b\) defines a complete orthonormal \(d^2\) dimensional basis.

Next, we wish to relate these entangled state to a one-particle MUB. To convey the idea we focus on the set defined by \(s = 1\), and without loss of generality we consider \(c_2 = 0\):

\[
|b, s = 1; c_1, c_2 = 0\rangle_{\mu, \nu} = \frac{1}{\sqrt{d}} \sum_{n} \omega^{s c_1 n (n-1) - c_1 n} |n\rangle_{\mu} |n\rangle_{\nu}.
\]

Suppose that the \(\mu\) particle is found in one of the states which belong to the MUB that were introduced in section II. Then, the reduced state for the \(\nu\) particle is (up to normalization)

\[
\mu_{(b_1; c|b, s = 1; c_1, c_2 = 0)_{\mu, \nu}} = \frac{1}{\sqrt{d}} [b - b_1; c_1 - c]_{\nu}.
\]
This is the exact meaning of associating entangled states with single particle MUB. For the case $b_1 > b$, the basis label for particle 2, $b_2 = b - b_1$, is negative and should be defined as modulus $d$. Note that for an odd dimension $d$, $b_1$ is always different from $b_2$. This ensures that, after measurement, states of the two particles belong to different MUB. The projection given in Eq. (3.9) corresponds to Hermitian operators which were introduced in Eq. (2.11).

Finally, we gather the operators that provide a basis for the $d^2$-dimensional Hilbert space to state reconstruction. For this, we may use the $d^2$ unitary operators given in Eq. (3.1) as is rewritten via Eqs. (3.2), (3.3), and (3.4) to express an arbitrary two-particle density matrix as

$$
\rho = \sum_{b, s} \sum_{m, n} \frac{1}{d} \left[ \rho \left( X_{\mu} (X^* Z^{sb})_{\nu} (Z^*_{\mu} Z^{-1} )^{m} \right) \right]^{n} \times \left[ X_{\mu} (X^* Z^{sb})_{\nu} (Z^*_{\mu} Z^{-1} )^{m} \right]^{n+1} + \sum_{s, n} \left( tr \rho \left[ (Z^*)_{\mu} (Z^{-1})_{\nu} \right] \right)^{n} \times \left( (Z^*)_{\mu} (Z^{-1})_{\nu} \right)^{n+1} + \sum_{n, m} \sum_{s} \left( tr \rho Z^{n}_{\mu} (X Z^{m})_{\nu} \right)^{n} \times \left( Z^{n}_{\mu} (X Z^{m})_{\nu} \right)^{n+1} + \sum_{n} \left( tr \rho Z^{n}_{\mu} \right)^{n} \times \mu = \nu - I. \quad (3.10)
$$

Using what we termed the “completeness” theorem that is proven in section II we rewrite this as (we eschew the particle number label viz. $\mu, \nu$ as being obvious in the expression below), where

$$
\rho = -\frac{1}{d} + \sum_{b, s} \sum_{c_1, c_2} \langle b, s; c_1, c_2 | \rho | b, s; c_1, c_2 \rangle \times | b, s; c_1, c_2 \rangle_{\mu, \nu} \langle b, s; c_1, c_2 | \times (d - 1) \sum_{n, n'} \langle n, n' | \rho | n, n' \rangle \times | n \rangle_{\mu} \otimes | n' \rangle_{\nu} + \sum_{b=0}^{d} \sum_{n, c} \langle b; c | \rho | n \rangle \times | b; c \rangle_{\mu} \otimes | b; c \rangle_{\nu} + \sum_{b=0}^{d} \sum_{n, c} \langle n | b; c | \rho | b; c \rangle \times | n \rangle_{\mu} \otimes | n \rangle_{\nu}, \quad (3.11)
$$

where $s = 1, 2, \cdots, d - 1, b, n, n', c_1, c_2 = 0, 1, \cdots, d - 1$. In the above expression, $\rho$ is given in terms of probabilities. Thus Eq. (3.10) presents an operational scheme for two-particle state reconstruction which is based on Schwinger operators of one-particle. For example, the probability of finding the system whose state is $\rho$ in the (entangled) state $| b; s; c_1, c_2 \rangle$ two readings are required: that pertaining to measuring $X_{\mu} (X^* Z^{sb})_{\nu}$, and one that gives the value of measuring $Z^*_{\mu} Z^{-1}$. We consider these two readings as one measurement as it refers to the same sample. With this in mind an arbitrary density matrix is given in terms of $d^2 + d + 1$ measurements: The first term implies $d(d - 1)$ measurements, the second one and the third and fourth $d$ measurements each. The total number of measurements exceeds the optimal number, i.e., $d^2 + 1$, since not all the bases in this construction are MUB. However, the number of measurements in this scheme is less than one would have needed by considering only single-particles MUB, that is, $(d + 1)^2 = d^2 + 2d + 1$ measurements.

IV. CONCLUSIONS AND REMARKS

Mutual unbiased bases for one particle states in $d$ dimensions may be characterized by states that diagonalize a set of $d$ commuting orthogonal unitary operators which, in turn, is one of $d + 1$ such sets. The central attribute of these is that a measurement ascertaining the particle to be in a state $| u \rangle$ of one such basis implies that it to be with equal likelihood in any of the states $| v \rangle$ of any of the other bases. Maximally entangled state of two particles central attribute is that partial tracing of the coordinates of one of the particles (i.e. performing non selective measurements on it) leaves the other particle with equal likelihood in any state. We relate these two notions by constructing $d(d - 1)$ entangled states base vectors that form an MUB as well as relate to the one-particle MUB. We also gave an operational content for this interpretation. We showed that an arbitrary two particle density matrix accounting for Hilbert space dimensionality $d^2$, is accountable via $d^2 + d + 1$ measurements where $d(d - 1)$ of the measurements involve entangled states which we related to the single particle MUB.

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Appendix A: Operator count at finite dimensionality

The numerator involves two counts. The first shows that our groupings of the operators retains all of them and each just once. The second involves the algebraic addition and subtractions of operators to obtain groupings each of which involves $d^2 - 1$ (not including the identity) orthogonal and commuting operators eligible for the application of the completion theorem, section II. We begin with the first demonstration: Eq. (3.2) contains $d^2(d - 1)^2$ orthonormal operators. Next we add the two sets of Eq. (4.4) that contain $d^2 - 1$ orthonormal operators. Now we add the two sets that constitute the third and fourth lines of Eq. (3.2) these contain $2d^2(d - 1)$ orthonormal operators. The sum total of these with the
identity added is $d^4$, QED.

The groupings of sets of $d^2-1$ of commuting (and orthogonal) sets amendable for the applicability of the completion theorem is done as follows: first we add to each of the $d(d-1)$ commuting operators labelled by distinct $b$ and $s$ the set of $d-1$ operators $(Z^s\mu Z^{-\mu})^n$, $n=1, 2, \cdots, d-1$. The combined set is composed of $d^2-1$ orthogonal and commuting operators (to which, when supplemented with the identity, we associate the $d^2$ dimensional orthonormal basis $(b, s; c_1, c_2)$). To compensate we subtract this added set and then combine this subtracted set with the rest of the operators. Now we add to it $-d(d-1)$ $\sum_{n=1}^{d} (Z_n^s + Z_n^{-s})$, these together (supplemented with the identity) form $d-1$ sets each with $d^2$ commuting and orthogonal unitary operators. Correcting for this subtraction we add to each of $d$ sets $(b = 0, 1, \cdots, d-1)$ of the form $\sum_{n=0, m=1}^{d^2} Z_n^s (XZ^b)_m^n$ to get again sets each of which is made up (with the identity) of $d^2$ commuting and orthogonal unitary operators.

**Appendix B: Entanglement of MUB states at finite dimensionality**

Our method for the finite dimensional may be applied to the case where the states’ label are finite field variables. This is the case for dimensionality $d$, with $d = p^n$, $p$ an odd prime [11, 12, 13, 14]. The MUB are given by, using our notation, $(\omega_p = e^{2\pi i/p})$:

$$|b; c\rangle = \frac{1}{\sqrt{d}} \sum_{c \in F_p} \omega_p^{tr[\frac{b}{2}n^2+cn]} |n\rangle_{\mu}|n\rangle_{\nu},$$  \hspace{1cm} (4.2)

with $b_1 + b_2 = b$, $c_1 + c_2 = c$. Thus projecting the first particle to (any of the) $b_1$, $c_1$ labelled state results in projecting the second particle to $|b_2; c_2\rangle$ state.

$$\mu(b_1; c_1|b; c\rangle_{\mu\nu} = \frac{1}{\sqrt{d}} |b_2; c_2\rangle_{\nu}. \hspace{1cm} (4.3)$$

Here $b, c, n \in F_q$, $F_d$ is Galois field with $d$ elements, $|n\rangle$ vector in the computational basis (labelled with an element of the field); and $tr[\alpha] = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{n-1}}$. The trace $tr$, is a mapping with $\alpha \in F_d$: $\alpha \in F_p$. Now a basic property of trace is: $tr[\alpha + \beta] = tr[\alpha] + tr[\beta]$. (The factor $1/2$ in the exponent means the solution of $2x = 1 \in F_d$.) Using our entanglement scheme (cf. [15, 16]), i.e. entangling within the computational basis, the entangled state corresponding to the generic MUB state Eq. (4.1) is,

$$|b; c\rangle_{\mu\nu} = \frac{1}{\sqrt{d}} \sum_{c \in F_p} \omega_p^{tr[\frac{b}{2}n^2+cn]} |n\rangle_{\mu}|n\rangle_{\nu}, \hspace{1cm} (4.2)$$

$$\omega_p = e^{2\pi i/p}.$$  \hspace{1cm} (4.3)

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