On the automorphisms of the Drinfel’d double of a Borel Lie subalgebra

Michaël Bulois and Nicolas Ressayre

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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra with a Borel subalgebra $\mathfrak{b}$. Consider the semidirect product $I\mathfrak{b} = \mathfrak{b} \rtimes \mathfrak{b}^*$, where the dual $\mathfrak{b}^*$ of $\mathfrak{b}$ is equipped with the coadjoint action of $\mathfrak{b}$ and is considered as an abelian ideal of $I\mathfrak{b}$. We describe the automorphism group $\text{Aut}(I\mathfrak{b})$ of the Lie algebra $I\mathfrak{b}$. In particular we prove that it contains the automorphism group of the extended Dynkin diagram of $\mathfrak{g}$. In type $A_n$, the dihedral subgroup was recently proved to be contained in $\text{Aut}(I\mathfrak{b})$ by Dror Bar-Natan and Roland van der Veen in [Bv20] (where $I\mathfrak{b}$ is denoted by $I\mathfrak{u}_n$). Their construction is handmade and they asked for an explanation which is provided by this note. Let $\mathfrak{n}$ denote the nilpotent radical of $\mathfrak{b}$. We obtain similar results for $I\mathfrak{b} = \mathfrak{b} \rtimes \mathfrak{n}^*$ that is both an Inönü-Wigner contraction of $\mathfrak{g}$ and the quotient of $I\mathfrak{b}$ by its center.

1 Introduction

Given any complex Lie algebra $\mathfrak{a}$, one can form the associated Drinfeld double $(I\mathfrak{a}, \mathfrak{a})$. Here, $I\mathfrak{a} = \mathfrak{a} \rtimes \mathfrak{a}^*$ is the semidirect product of $\mathfrak{a}$ with its dual $\mathfrak{a}^*$ where $\mathfrak{a}^*$ is considered as an abelian ideal and $\mathfrak{a}$ acts on $\mathfrak{a}^*$ via the coadjoint action.

As mentioned in [Bv20], for applications in knot theory and representation theory, the most important case is when $\mathfrak{a} = \mathfrak{b}$ is the Borel subalgebra of some simple Lie algebra $\mathfrak{g}$. It is precisely the situation studied here. In addition to [Bv20], several examples of these algebras appear with variations in the literature. In [NW93], Nappi-Wittney use the case when $\mathfrak{g} = \mathfrak{sl}_2$ in conformal field theory. Several authors also consider $I\mathfrak{b} := \mathfrak{b} \rtimes \mathfrak{n}^*$ where $\mathfrak{n}$ is the derived subalgebra of $\mathfrak{b}$. It is the quotient of $I\mathfrak{b}$ by its center. Note that $\mathfrak{b} \rtimes \mathfrak{n}^*$ is a contraction of $\mathfrak{g}$ (see Section 2.1 for details). In [KZJ07], Knutson and Zinn-Justin meet this algebra for $\mathfrak{g} = \mathfrak{gl}_n$ in the associative setting, see below. In [Fei12, Fei11], Feigin uses $\mathfrak{b} \rtimes \mathfrak{n}^*$ in order to study degenerate flag varieties for $\mathfrak{g} = \mathfrak{sl}_n$. For a general semisimple Lie algebra $\mathfrak{g}$, in [PY12], Panyushev and Yakimova study the invariants of $\mathfrak{b} \rtimes \mathfrak{n}^*$ under the action of their adjoint group. Finally, in [PY13, Pho20], similar considerations are studied replacing $\mathfrak{b}$ by an arbitrary parabolic subalgebra of $\mathfrak{g}$.
The aim of this note is to give new interpretations of $I_b$ and $\overline{I_b}$ in the language of Kac-Moody algebras and to completely describe the automorphism groups of $I_b$ and $\overline{I_b}$.

Before describing this group, we introduce some notation. Let $r$ denote the rank of $\mathfrak{g}$ and $G$ the adjoint group with Lie algebra $\mathfrak{g}$. Let $B$ be the Borel subgroup of $G$ with $\mathfrak{b}$ as Lie algebra. Consider two abelian additive groups: the quotient $\mathfrak{g}/\mathfrak{b}$ and the space $\mathcal{M}_r(\mathbb{C})$ of $r \times r$-matrices.

An important ingredient is the extended Dynkin diagram of $\mathfrak{g}$. On Figure 1, these diagrams and their automorphism groups are shortly recalled (see Section 2.2). The notation $D(\ell)$ stands for the dihedral group of order $2\ell$, not to be confused with the Dynkin diagram of type $D\ell$.

The following is the main result of the paper.

**Theorem 1.** The neutral component $\text{Aut}(I_b)^\circ$ of the automorphism group $\text{Aut}(I_b)$ of the Lie algebra $I_b$ decomposes as

$$\mathbb{C}^* \rtimes \left( B \ltimes \frac{\mathfrak{g}}{\mathfrak{b}} \times \mathcal{M}_r(\mathbb{C}) \right).$$

The group of components $\text{Aut}(I_b)/\text{Aut}(I_b)^\circ$ is isomorphic to the automorphism group of the extended Dynkin diagram of $\mathfrak{g}$ and can be lifted to a subgroup of $\text{Aut}(I_b)$.

The details of how these subgroups act on $I_b$ are given in Section 3. Section 4 explain how the semidirect products are formed.

One of the amazing facts is that the extended Dynkin diagram of $\mathfrak{g}$ plays a crucial role in $\text{Aut}(I_b)$. On one hand, we explain this by constructing the extended Cartan matrix of $\mathfrak{g}$ in terms of $I_b$ in Section 3.1. On the other hand, this diagram is the Dynkin diagram of the untwisted affine Lie algebra constructed from the loop algebra of $\mathfrak{g}$. A second explanation is given by Theorem 3 that realizes $I_b$ as a subquotient of the affine Lie algebra associated to $\mathfrak{g}$.

More generally, $I_b$ is a degeneration $\lim_{\epsilon \to 0} \mathfrak{g}_+^\epsilon$ with $\mathfrak{g}_+^\epsilon \cong \mathfrak{g} \oplus \mathfrak{b}$ for $\epsilon \in \mathbb{C} \setminus \{0\}$. In Section 2 we explain how to interpret this degeneration in the affine Lie algebra setting. We also study the possible lifting of $\theta \in \text{Aut}(\tilde{D})$ to $\text{Aut}(\mathfrak{g}_+^\epsilon)$, see Section 3.5.

**Link with other works.** In [KZJ07], Knutson and Zinn-Justin defined a degeneration $\bullet$ of the standard associative product on $\mathcal{M}_n(\mathbb{C})$. Let $\mathfrak{b}$ denote the set of upper triangular matrices. Identifying the vector space $\mathcal{M}_n(\mathbb{C})$ with $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ in a natural way one gets

$$(R, L) \bullet (V, M) = (RV, RM + LV),$$

for any $R, V \in \mathfrak{b}$ and $L, M \in \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$. The Lie algebra of the group $(\mathcal{M}_n(\mathbb{C}), \bullet)^\times$ of invertible elements of this algebra is $\mathfrak{b} \ltimes \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$, where the product is defined similarly to that of $I_b$. Note also that a cyclic automorphism (corresponding in our setting to the cyclic automorphism of the extended Dynkin diagram of type $A_{n-1}$ and with the unexpected cyclic automorphism of $[Bv20]$) appears in [KZJ07]. Moreover [KZJ07, Proposition 2], which realizes $(\mathcal{M}_n(\mathbb{C}), \bullet)$ as a subquotient of $\mathcal{M}_n(\mathbb{C}[t])$, is similar to our Theorem 3.
| $\tilde{A}_1$ | $\tilde{A}_\ell \ (\ell \geq 2)$ | $\tilde{B}_\ell \ (\ell \geq 3)$ |
| --- | --- | --- |
| $\text{Aut}(\tilde{D}) = \mathbb{Z}/2\mathbb{Z}$ | $\text{Aut}(\tilde{D}) = D_{(\ell+1)}$ | $\text{Aut}(\tilde{D}) = \mathbb{Z}/2\mathbb{Z}$ |
| $\tilde{G}_2$ | $\tilde{C}_\ell \ (\ell \geq 2)$ | $\tilde{D}_\ell \ (\ell \geq 5)$ |
| $\text{Aut}(\tilde{D})$ is trivial | $\text{Aut}(\tilde{D}) = \mathbb{Z}/2\mathbb{Z}$ | $\text{Aut}(\tilde{D}) = D_{(4)}$ |
| $\tilde{D}_4$ | $\tilde{E}_7$ | $\tilde{F}_4$ |
| $\text{Aut}(\tilde{D}) = \mathcal{S}_3$ | $\text{Aut}(\tilde{D}) = \mathbb{Z}/2\mathbb{Z}$ | $\text{Aut}(\tilde{D})$ is trivial |

Figure 1: Extended Dynkin diagrams and their automorphisms
A generalization of $\mathfrak{b}$ is the following: fix a simple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Let $\mathfrak{n}_\mathfrak{p}(\cong \mathfrak{g}/\mathfrak{p})$ be the nilradical of a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Then $\mathfrak{q}_\mathfrak{p} := \mathfrak{p} \ltimes \mathfrak{n}_\mathfrak{p}$ is also a degeneration of $\mathfrak{g}$. In the study of semi-invariants of $\mathfrak{q}_\mathfrak{p}$ some data linked with the extended Dynkin diagram also make appearance in [Yak14 Theorem 5.5] (Borel case) and in [Pho20, Proposition 5.2.1] (general case). In type $A_{n-1}$, standard parabolics are characterized by an ordered partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n$. Transforming $\lambda$ into $\mu := (\lambda_k, \lambda_1, \ldots, \lambda_{k-1})$, the cyclic action of $\mathbb{Z}/n\mathbb{Z}$ coming from the symmetries of the extended Dynkin diagrams described in [Bv20] allows to write $\mathfrak{q}_{\mu\lambda} \cong \mathfrak{q}_{\mu\mu}$. This explains many symmetries noted in [Pho20], see (3.9) in loc. cit.

Motivation and story of the present work. In [Bv20], Bar-Natan and van der Veen constructed an “unexpected” cyclic automorphism of $\mathfrak{b}$ when $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. The first version of this work was an explanation of this automorphism by using affine Lie algebras. Simultaneously with this first version, A. Knutson mentioned to Bar-Natan his earlier work [KZJ07] with Zinn-Justin.

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2 The Lie algebras $\mathfrak{b}$, $\mathfrak{g}_\epsilon^+$ and $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$

2.1 Definitions of $\mathfrak{b}$ and $\mathfrak{g}_\epsilon^+$

Let $\mathfrak{g}$ be a complex simple Lie algebra with Lie bracket denoted by $[\ , \ ]$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{b}^-$ be the Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ which is opposite to $\mathfrak{b}$. Set $\mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^-$ viewed as a vector space. In this section, we define the Lie bracket $[\ , \ ]_\epsilon$ on $\mathcal{V}$ depending on the complex parameter $\epsilon$, interpolating between $\mathfrak{b}$ and the direct product $\mathfrak{g} \oplus \mathfrak{h}$.

Let $\mathfrak{n}$ and $\mathfrak{n}^-$ denote the derived subalgebras of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. Fix $\epsilon \in \mathbb{C}$. Define the skew-symmetric bilinear bracket $[\ , \ ]_\epsilon$ on $\mathcal{V}$ by

$$
[x, x']_\epsilon = [x, x'] \quad \forall x, x' \in \mathfrak{b}
$$

$$
y, y' \mapsto \epsilon[y, y'] \quad \forall y, y' \in \mathfrak{b}^-
$$

$$
[x, y]_\epsilon = \epsilon X + \frac{H}{2}, Y \quad \forall x \in \mathfrak{b}, y \in \mathfrak{b}^- \text{ where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^{-}
$$

Then $[\ , \ ]_\epsilon$ satisfies the Jacobi identity (see discussion after (3) for a proof). Endowed with this Lie bracket, $\mathcal{V}$ is denoted by $\mathfrak{g}_\epsilon^+$. The linear map

$$
\varphi_\epsilon : \mathfrak{b} \oplus \mathfrak{b}^- \longrightarrow \mathfrak{b} \oplus \mathfrak{b}^- \quad (x, y) \mapsto (x, \epsilon y) \quad \text{for any } x \in \mathfrak{b}, y \in \mathfrak{b}^-
$$
allows to interpret $g^\epsilon_+$ as an Inönü-Wigner contraction [IW53] of $g^1_+$. Indeed, for any nonzero $\epsilon$, we have

$$[X, Y]_\epsilon = \varphi_\epsilon^{-1}([\varphi_\epsilon(X), \varphi_\epsilon(Y)]_1) \quad \forall X, Y \in \mathcal{V}. \quad (1)$$

We now describe $g^1_+$. Using the triangular decomposition

$$g = n \oplus \mathfrak{h} \oplus n^-,$$  \quad (2)

one defines the injective linear map

$$\iota^1_{g_+} : \quad g = n \oplus \mathfrak{h} \oplus n^- \rightarrow \quad g^1_+ \quad (\xi, \alpha, \zeta) \mapsto (\xi + \frac{\alpha}{2}, \frac{\alpha}{2} + \zeta)$$

and checks that it is a Lie algebra homomorphism whose image is an ideal of $g^1_+$. Moreover, the image of

$$\iota^1_{\mathfrak{h}_+} : \quad \mathfrak{h} \rightarrow \quad g^1_+ \quad \alpha \mapsto (-\alpha, \alpha)$$

is the center of $g^1_+$ and, as Lie algebras,

$$g^1_+ = \iota^1_{g_+}(g) \oplus \iota^1_{\mathfrak{h}_+}(\mathfrak{h}). \quad (3)$$

Observe that we never used the Jacobi identity for $[\ , \ ]_1$ to prove the isomorphism (3). Hence, we can deduce from it that $[\ , \ ]_1$ satisfies the Jacobi identity. Then, the expression (1) implies that $[\ , \ ]_\epsilon$ satisfies the Jacobi identity for any nonzero $\epsilon$. Since this property is closed on the space of bilinear maps, it is satisfied by $[\ , \ ]_0$ too.

Consider now $I\mathfrak{b}$ with its Lie bracket $[\ , \ ]_{I\mathfrak{b}}$ defined as follows: $\mathfrak{b}^*$ is an abelian ideal on which $\mathfrak{b}$ acts by the coadjoint action. Denote by $\kappa : g \rightarrow g^*$ the Killing form on $g$. Since the orthogonal complement of $\mathfrak{b}$ with respect to $\kappa$ is $n$, $\mathfrak{b}^*$ identifies with $g/n$ as a $\mathfrak{b}$-module. Identify $g/n$ with $\mathfrak{b}^*$ in a canonical way (that is by $y \in \mathfrak{b}^- \mapsto y + n$) and denote by $\pi : g \rightarrow \mathfrak{b}^-$ the quotient map. Then $I\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ identifies with $\mathfrak{b} \oplus \mathfrak{b}^- = \mathcal{V}$. Let $[\ , \ ]_I$ denote the Lie bracket transferred to $\mathcal{V}$ from $[\ , \ ]_{I\mathfrak{b}}$. Let $x, x' \in \mathfrak{b}$ and $y, y' \in \mathfrak{b}^-$ and decompose $[x, y'] - [x', y]$ as $X + H + Y$ with respect to $g = n \oplus \mathfrak{h} \oplus n^-$. Then

$$[(x, y), (x', y')]_I = ([x, x'], H + Y). \quad (4)$$

We now describe $g^0_+$. The Lie bracket $[\ , \ ]_0$ on $\mathcal{V} = g^0_+$ is given by

$$[(x, y), (x', y')]_0 = ([x, x'], \frac{H}{2} + Y). \quad (5)$$

Comparing (4) and (5), one gets that the following linear map $\eta$ is a Lie algebra isomorphism between $g^0_+$ and $I\mathfrak{b}$:

$$\eta : \quad \mathcal{V} = \mathfrak{b} \oplus (\mathfrak{h} \oplus n^-) \rightarrow \quad \mathfrak{b} \oplus \mathfrak{b}^* = I\mathfrak{b} \quad (x, h, y) \mapsto (x, \kappa(2h + y, \square)).$$

Replacing $\mathfrak{b}^-$ and $\mathfrak{b}^*$ by $n^-$ and $n^*$ respectively, one defines $g^\epsilon$ and one gets the isomorphisms $g \simeq g^\epsilon$ (for any $\epsilon \neq 0$) and $g^0 \simeq I\mathfrak{b}$.
2.2 The affine Kac-Moody Lie algebra

The untwisted affine Kac-Moody Lie algebra \( \mathfrak{g}^{KM} \) is constructed from the simple Lie algebra \( \mathfrak{g} \). We refer to [Kum02, Chapters I and XIII] for the basic properties of \( \mathfrak{g}^{KM} \). Denote by \( \mathfrak{z}(\mathfrak{g}^{KM}) \) the one dimensional center of \( \mathfrak{g}^{KM} \). Consider the Borel subalgebra \( \mathfrak{b}^{KM} \) of \( \mathfrak{g}^{KM} \) and its derived subalgebra \( \mathfrak{n}^{KM} \). By killing the semi-direct product and the central extension from the construction of \( \mathfrak{g}^{KM} \), one gets

\[
\tilde{\mathfrak{g}} := [\mathfrak{g}^{KM}, \mathfrak{g}^{KM}]/\mathfrak{z}(\mathfrak{g}^{KM}) \\
\cong \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g},
\]

and

\[
\tilde{\mathfrak{b}} := (\mathfrak{b}^{KM} \cap [\mathfrak{g}^{KM}, \mathfrak{g}^{KM}])/\mathfrak{z}(\mathfrak{g}^{KM}) \subset \tilde{\mathfrak{g}} \\
\tilde{\mathfrak{n}} := (\mathfrak{n}^{KM} \cap [\mathfrak{g}^{KM}, \mathfrak{g}^{KM}])/\mathfrak{z}(\mathfrak{g}^{KM}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}].
\]

Identify \( \mathfrak{g} \) with the subspace \( \mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}} \). Note that \( \mathfrak{g}^{KM}/\mathfrak{z}(\mathfrak{g}^{KM}) = \tilde{\mathfrak{g}} + \mathbb{C}d \) where \( d \) acts as the derivation with respect to \( t \).

We consider the set of (positive) roots \( \Phi^+ \) (resp. \( \tilde{\Phi}^+ \)) of \( \mathfrak{g} \) (resp. \( \mathfrak{g}^{KM} \)) and the set of simple roots \( \Delta \) (resp. \( \tilde{\Delta} \)) with respect to \( \mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g} \) (resp. \( \mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{KM}) \subset \mathfrak{b}^{KM} \subset \mathfrak{g}^{KM} \)). We recall the following classical facts:

\[
\mathfrak{n}^{KM} \cong \tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \tilde{\Phi}^+} \tilde{\mathfrak{g}}_\alpha
\]

where \( \tilde{\mathfrak{g}}_\alpha \cong \mathfrak{g}^{KM}_\alpha \) is the root space associated to \( \alpha \). Moreover, \( \tilde{\mathfrak{n}} \) is generated, as a Lie algebra by the subspaces \( (\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}} \). The identification of \( \tilde{\Delta} \) with \( \{ \alpha \in \tilde{\Delta} | \alpha(d) = 0 \} \) yields the above-described embedding \( \mathfrak{g} \subset \tilde{\mathfrak{g}} \). Denoting by \( \delta \) the indivisible positive imaginary root in \( \tilde{\Phi} \), we have

\[
\tilde{\Phi} = \{ n\delta + \alpha | \alpha \in \Phi \cup \{0\}, n \in \mathbb{Z} \} \setminus \{0\} \\
\tilde{\Delta} = \Delta \cup \{ \alpha_0 + \delta \}
\]

where \( \alpha_0 \) is the lowest root of \( \Phi \). Note that \( \tilde{\mathfrak{g}}_{\alpha_0} = t^n \mathfrak{h} \ (n \in \mathbb{Z}) \), using the notation \( \tilde{\mathfrak{g}}_0 := \mathfrak{h} \).

Finally, the extended Dynkin diagram can be reconstructed from the combinatorics of \( \tilde{\Delta} \) in \( \tilde{\Phi} \). Indeed, the nodes correspond to the elements of \( \tilde{\Delta} \) and the non-diagonal entries \( a_{\alpha, \beta} \) of the generalized Cartan matrix (encoding the arrows of the diagram) are \( a_{\alpha, \beta} = -\max\{n \in \mathbb{N} | \beta + n\alpha \in \tilde{\Phi} \} \) by Serre relations.

We list in Figure 11 the extended Dynkin diagram \( \tilde{\mathcal{D}}_{\mathfrak{g}} \) in each simple type. The black node corresponds to the simple root \( \alpha_0 + \delta \). We also provide the automorphism group of \( \tilde{\mathcal{D}}_{\mathfrak{g}} \). Note that by the definition of \( \mathfrak{g}^{KM} \) given in [Kum02, §1.1], any \( \theta \in \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}}) \) provides an automorphism \( \theta^{KM} \in \text{Aut}(\mathfrak{g}^{KM}) \) stabilizing both \( \mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{KM}) \) and \( \mathfrak{b}^{KM} \) and permuting the generators \( e_\alpha, f_\alpha \ (\alpha \in \tilde{\Delta}) \) via \( \theta^{KM}(e_\alpha) = e_{\theta(\alpha)} \) and \( \theta^{KM}(f_\alpha) = f_{\theta(\alpha)} \). Since \( \mathfrak{z}(\mathfrak{g}^{KM}) \) and \( [\mathfrak{g}^{KM}, \mathfrak{g}^{KM}] \) are characteristic in \( \mathfrak{g}^{KM} \), i.e. stabilized by any automorphism of Lie algebra, this yields an automorphism \( \tilde{\theta} \in \text{Aut}(\tilde{\mathfrak{g}}) \). Note that some choices have to be made for \( \theta^{KM}(d) \), but the automorphism \( \tilde{\theta} \) only depends on \( \theta \) and on the \( e_\alpha, f_\alpha \ (\alpha \in \tilde{\Delta}) \), since those elements generate \( \tilde{\mathfrak{g}} \).
In a first version of this paper, it was claimed that ð is C[t]-linear. In fact, it is unclear whether this result holds in general. However, we have the following

**Lemma 2.** Under above notation, there exists \( \lambda \in \{\pm 1\} \) such that

\[
\forall x \in \tilde{g}, \ \tilde{\theta}(tx) = \lambda t\tilde{\theta}(x).
\]

In particular, the automorphism \( \tilde{\theta} \in \text{Aut}(\tilde{g}) \) stabilizes \( t\tilde{n} \).

Moreover, \( \lambda = 1 \) whenever the order of \( \theta \) is odd.

**Proof.** Note that, since \( \tilde{\theta} \) comes from an element \( \theta^{\text{KM}} \in \text{Aut}(g^{\text{KM}}) \), its action on the semi-group \( \tilde{\Phi}^+ \) stabilizes the semi-group of positive imaginary roots \( \mathbb{N}^* \delta \) and thus fixes its generator \( \delta \). In particular, in the additive group \( \tilde{\Phi} \cup \{0\} \), we have \( \tilde{\theta}(\cdot + \delta) = \delta + \tilde{\theta}(\cdot) \). Defining \( \Psi \) on \( \tilde{g} \) via \( \Psi(x) = \tilde{\theta}^{-1}(t^{-1}\tilde{\theta}(tx)) \), we thus get that \( \Psi_\alpha := \tilde{\Psi}|_{\tilde{g}_\alpha} \) is an invertible linear map on \( \tilde{g}_\alpha \) for any \( \alpha \in \tilde{\Phi} \cup \{0\} \). Since \( \dim \tilde{g}_\alpha = 1 \) for \( \alpha \in \tilde{\Phi} \setminus \mathbb{Z}\delta \), we can thus define \( \lambda_\alpha \) as the element of \( C^* \) such that \( \Psi_\alpha = \lambda_\alpha \text{Id}_{\tilde{g}_\alpha} \).

Let \( \alpha, \beta \in \tilde{\Phi} \cup \{0\}, x_\alpha \in \tilde{g}_\alpha, x_\beta \in \tilde{g}_\beta \). By \( \mathbb{C}[t] \)-bilinearity of the bracket, we get

\[
\Psi_{\alpha+\beta}([x_\alpha, x_\beta]) = \tilde{\theta}^{-1}(t^{-1}[\tilde{\theta}(tx_\alpha), \tilde{\theta}(tx_\beta)]) = [\Psi_{\alpha}(x_\alpha), x_\beta].
\]

(6)

For \( \alpha = 0, x_\alpha = h \in \mathfrak{h} \) and \( \beta \in \tilde{\Phi} \setminus \mathbb{Z}\delta \), we get

\[
\lambda_\beta \beta(h)x_\beta = \Psi_{\beta}(\beta(h)x_\beta) = \beta(\Psi_0(h))x_\beta.
\]

(7)

In particular, \( \Psi_0 \) induces on \( \mathfrak{h}^* \) a linear map \( ^t\Psi_0 \) sending \( \beta \) to \( \lambda_\beta \beta \) for each \( \beta \in \mathfrak{h} \subset \tilde{\Phi} \setminus \mathbb{Z}\delta \). If \( \beta, \gamma \in \Delta \) correspond to connected diagrams of the Dynkin diagram of \( \mathfrak{g} \), then \( \beta, \gamma \) and \( \beta + \gamma \) are eigenvectors of \( ^t\Psi_0 \) so \( \lambda_\beta = \lambda_\gamma \). By connectivity of the Dynkin diagram, we get that the \( \lambda_\beta \) (\( \beta \in \Delta \)) are all equal to a single value \( \lambda \). Since \( \Delta \) generates \( \mathfrak{h}^* \), we get \( \Psi_0 = \lambda \text{Id}_{\tilde{g}_0} \).

For any \( \beta \in \tilde{\Phi} \setminus \mathbb{Z}\delta \), we can choose \( h \in \mathfrak{h} \) such that \( \beta(h) \neq 0 \). Applying (7) yields \( \lambda_\beta \beta(h)x_\beta = \beta(\lambda h)x_\beta \), that is \( \lambda_\beta = \lambda \).

When \( \alpha = -\beta \in \Delta, n \in \mathbb{Z} \), we get \( \Psi_{n\delta}(tn[x_\alpha, x_{-\alpha}]) = [\Psi_{\alpha}(x_\alpha), tn.x_{-\alpha}] = \lambda tn[x_\alpha, x_{-\alpha}] \).

Since the \( tn[\tilde{g}_\alpha, \tilde{g}_{-\alpha}] (\alpha \in \Delta) \) generate \( \tilde{g}_{n\delta} \), this yields \( \Psi_{n\delta} = \lambda \text{Id}_{\tilde{g}_{n\delta}} \). Finally, we have proved that \( \Psi = \lambda \text{Id}_\delta \) and this yields the first assertion of the Lemma.

Let \( m \) be the order of \( \theta \). The first assertion of the lemma can be rewritten as \( t^{-1}\tilde{\theta}t = \lambda \tilde{\theta} \) where \( t^{\pm 1} \) denotes the multiplication by \( t^{\pm 1} \) in \( \tilde{g} \). This identity to the power \( m \) yields \( \lambda^m = 1 \).

In the setting of [Kum02, Chapter XIII], the Cartan involution \( \omega \) of \( \tilde{g} \) sending each generator \( e_\alpha (\alpha \in \Delta) \) to \( -f_\alpha \) is given by

\[
\omega(t^i x) = t^{-i}\tilde{\omega}(x) \quad (i \in \mathbb{Z}, x \in \mathfrak{g})
\]

where \( \tilde{\omega} \) is the Cartan involution of \( \mathfrak{g} \). As a consequence, \( \omega t = t^{-1} \). Also, \( \omega \circ \tilde{\theta} \circ \omega(e_\alpha) = \omega \circ \tilde{\theta} \circ (-f_\alpha) = -\omega(f_{\theta(\alpha)}) = e_{\theta(\alpha)} \neq \tilde{\theta}(e_\alpha) \) and the same computation gives \( \omega \circ \tilde{\theta} \circ \omega(f_\alpha) = \tilde{\theta}(f_\alpha) \) so \( \omega \theta \omega = \tilde{\theta} \). Then conjugating \( t^{-1}\tilde{\theta}t = \lambda \tilde{\theta} \) by the involution \( \omega \) yields \( t\theta t^{-1} = \lambda \tilde{\theta} \). It follows from these equalities that \( \lambda^2 = 1 \). Hence \( \lambda \in \{\pm 1\} \) with \( \lambda = 1 \) if \( m \) is odd.

Finally, \( \tilde{\theta} \) permutes the generators of \( \tilde{n} \): \( (e_{\alpha})_{\alpha \in \Delta} \). Hence \( \tilde{\theta} \) stabilizes \( \tilde{n} \) and \( \tilde{\theta}(t\tilde{n}) = \pm t\tilde{n} = t\tilde{n} \)

\( \Box \)
Remark. We also checked in several cases, including the cyclic automorphism in type A, that $\lambda = 1$. In such cases, $\tilde{\theta}$ then also stabilizes $(t - \epsilon)\tilde{n}$ for any $\epsilon \in \mathbb{C}$.

2.3 Realization of $\mathfrak{g}_+^\epsilon$

The Lie algebras $\tilde{b}$ and $\tilde{n}$ decompose as

\[ \tilde{b} = \mathbb{C}[t]b \oplus t\mathbb{C}[t]n^-, \]
\[ \tilde{n} = \mathbb{C}[t]n \oplus t\mathbb{C}[t]b^- . \]

Moreover, $(t - \epsilon)\tilde{n}$ is an ideal of $\tilde{b}$, and $\tilde{b}/((t - \epsilon)\tilde{n})$ is a Lie algebra.

Theorem 3. Let $\epsilon \in \mathbb{C}$. The Lie algebras $\mathfrak{g}_+^\epsilon$ and $\tilde{b}/(t - \epsilon)\tilde{n}$ are isomorphic. Similarly, $\mathfrak{g}^\epsilon$ is isomorphic to $\tilde{b}/(t - \epsilon)\tilde{b}$.

Proof. From Section 2.1, we have $\mathfrak{g}_+^1 = b \oplus b^-$ as vector spaces. Elements of $\mathfrak{g}_+^1$ will be written as couples with respect to this decomposition.

Set $\tilde{g}_+^1 := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}_+^1$ and extend $\iota_0^1$ to an injective $\mathbb{C}[t^{\pm 1}]$-linear map $\tilde{g} \to \tilde{g}_+^1$. Consider the subspace $\mathfrak{w} := \mathbb{C}[t]b \oplus t\mathbb{C}[t]b^-$ that is a Lie subalgebra of $\tilde{g}_+^1$. If $\epsilon \neq 0$, the Inönü-Wigner contraction $(\mathbb{I})$ on $\tilde{g}_+^1$ with respect to the decomposition $b \oplus b^-$ gives rise to $\mathfrak{g}_+^\epsilon$ $(\epsilon \in \mathbb{C})$. We easily deduce that the linear map

\[ \mathfrak{g}_+^\epsilon \to \mathfrak{w}/(t - \epsilon)\mathfrak{w} \]
\[(x,y) \mapsto x + ty + (t - \epsilon)w \quad \text{for any } x \in b \text{ and } y \in b^- , \quad (8) \]

is a Lie algebra isomorphism. For $\epsilon = 0$, it is still a linear isomorphism and, by continuity, a Lie algebra homomorphism.

Set $\tilde{b}_0^- := \iota_0^1(b^-) = \{(h,h)| h \in \mathfrak{h}\} \oplus n^-$. Observe that $tb^-_0$ is contained in $\mathfrak{w}$. Indeed, for any $h \in \mathfrak{h}$, the element $t(h,h) = t(h,0) + t(0,h)$ belongs to $\mathbb{C}[t]b \oplus t\mathbb{C}[t]b^-$. In particular, one gets a linear map induced by the inclusions of $b$ and $tb^-_0$ in $\mathfrak{w}$:

\[ b \oplus tb^-_0 \to \mathfrak{w} . \]

One can easily check that it induces a linear isomorphism $b \oplus tb^-_0 \to \mathfrak{w}/(t - \epsilon)\mathfrak{w}$. Setting $\tilde{b}_m := (b \oplus tb^-_0)_{\text{Lie}}$, the Lie subalgebra of $\mathfrak{w}$ generated by $b \oplus tb^-_0$, we thus get a Lie algebra isomorphism.

\[ \tilde{b}_m /((t - \epsilon)\mathfrak{w} \cap \tilde{b}_m) \to \mathfrak{w}/(t - \epsilon)\mathfrak{w} . \quad (9) \]

Since, $b = \{(h,0)| h \in \mathfrak{h}\} \oplus \iota_0^1(n)$ and $\langle \iota_0^1(n) \oplus \iota_0^1(tb^-) \rangle_{\text{Lie}} = \iota_0^1(\langle n \oplus tb^- \rangle_{\text{Lie}}) = \iota_0^1(\tilde{n})$, we have

\[ \tilde{b}_m = \{(h,0)| h \in \mathfrak{h}\} \oplus \iota_0^1(\tilde{n}) \cong \iota_0^1(\tilde{b}) \cong \tilde{b} , \quad (10) \]

the middle Lie algebra isomorphism being the identity on $\iota_0^1(\tilde{n})$ and sending $(h,0)$ to $\frac{1}{2}(h,h)$ for each $h \in \mathfrak{h}$. Moreover, $(t - \epsilon)\mathfrak{w} \cap \tilde{b}_m = (t - \epsilon)\iota_0^1(\tilde{n})$. Indeed, $(t - \epsilon)\iota_0^1(\tilde{n})$ is contained in $(t - \epsilon)\mathfrak{w} \cap \tilde{b}_m$, and $b \oplus tb^-_0$ is complementary to $(t - \epsilon)\iota_0^1(\tilde{n})$ in $\tilde{b}_m$. 

8
We finally get the desired Lie isomorphism

\[
\tilde{b}/(t - \epsilon)\tilde{n} \cong \tilde{b}_0/(t - \epsilon)\tilde{n}_0 \cong \mathfrak{n}/(t - \epsilon)\mathfrak{n} \cong \mathfrak{g}^+.
\]

In addition, we can make explicit the isomorphism of Theorem 3:

\[
\gamma_{\epsilon} : \mathfrak{g}^+ \cong \tilde{b}/(t - \epsilon)\tilde{n}
\]

\[
\begin{align*}
(x, 0) &\mapsto x & \text{if } x \in \mathfrak{n} \\
(0, y) &\mapsto ty & \text{if } y \in \mathfrak{n}^{-} \\
(a, b) &\mapsto (a - \epsilon b) + 2t b & \text{if } a, b \in \mathfrak{h}
\end{align*}
\]

and its inverse map is induced by

\[
\theta : \tilde{b} \rightarrow \mathcal{V}
\]

\[
\begin{align*}
P x &\mapsto P(\epsilon)x & \text{if } x \in \mathfrak{n} \\
t R y &\mapsto R(\epsilon)y & \text{if } y \in \mathfrak{n}^{-} \\
Q h &\mapsto \left(\frac{Q\epsilon + Q(0)}{2} h, \frac{Q\epsilon - Q(0)}{2} h\right) & \text{if } h \in \mathfrak{h} (\epsilon \neq 0) \\
&\mapsto (Q(0) h, \frac{1}{2} Q'(0) h) & \text{if } h \in \mathfrak{h} (\epsilon = 0)
\end{align*}
\]

Note that, in order to prove Theorem 3, we could alternatively have checked directly that \(\theta\) is a surjective Lie algebra homomorphism from \(\tilde{b}\) onto \(\mathfrak{g}^+\) with kernel \((t - \epsilon)\tilde{n}\).

3 Some subgroups of Aut(\(Ib\))

3.1 The roots of \(Ib\)

From Sections 2.1 and 2.3, we can interpret the Lie algebra \(Ib\) in the Kac-Moody world via the isomorphism

\[
Ib \rightarrow \tilde{b}/t\tilde{n}
\]

\[
(x, y) \mapsto x + ty \quad \left(\begin{array}{c}
x \in \mathfrak{b}, \\
y \in \mathfrak{b}^{-} \cong \mathfrak{g}/\mathfrak{n} \cong \mathfrak{b}^\ast
\end{array}\right)
\]

From now on, this identification will be made systematically. In particular, we write \(Ib = \mathfrak{b} \oplus t\mathfrak{b}^-\). We first describe some basic properties of \(Ib\) in this language.

**Lemma 4.** 1. The subalgebra \(\mathfrak{c} := \mathfrak{h} \oplus t\mathfrak{h}\) is a Cartan subalgebra of \(Ib\). Namely, \(\mathfrak{c}\) is abelian and equal to its normalizer.

2. Under the action of \(\mathfrak{c}\), \(Ib\) decomposes as

\[
Ib = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} t\mathfrak{g}_\alpha.
\]

For \(\alpha \in \Phi^+\), \(\mathfrak{c}\) acts on \(\mathfrak{g}_\alpha\) with the weight \((\alpha, 0) \in \mathfrak{h}^\ast \times t\mathfrak{h}^\ast\). For \(\alpha \in \Phi^-\), \(\mathfrak{c}\) acts on \(t\mathfrak{g}_\alpha\) with the weight \((\alpha, 0) \in \mathfrak{h}^\ast \times t\mathfrak{h}^\ast\). Here, we identified \(\mathfrak{c}^\ast\) with \(\mathfrak{h}^\ast \times t\mathfrak{h}^\ast\) in a natural way.
3. The set of ad-nilpotent elements of $Ib$ is \( \tilde{n}/t\tilde{n} = n \oplus tb^- \).

4. The center of $Ib$ is \( \mathfrak{z}(Ib) = th \).

5. The derived subalgebra of $Ib$ is \([Ib; Ib] = \tilde{n}/t\tilde{n}\).

Proof. 1-2) The fact that $c$ is abelian and the decomposition in $h$-eigenspaces are clear from the definition of $\tilde{g}$. The action of $th$ is zero since it sends $\tilde{n}$ to $t\tilde{n}$ that vanishes itself in $Ib$. The decomposition of $Ib$ in weight spaces under the action of $c$ follows. Then this decomposition also implies that $c$ is its own normalizer in $Ib$.

3) The elements of $\tilde{n}/t\tilde{n}$ are clearly ad-nilpotent. From 2), an element with nonzero component in $h$ is not ad-nilpotent.

4) Since $th$ acts as $0$ on $\tilde{n}/t\tilde{n}$ and on $h$, we have $th \subset \mathfrak{z}(Ib)$. The decomposition in weight spaces implies the converse inclusion.

5) The inclusion $[Ib, Ib] \subset \tilde{n}/t\tilde{n}$ is clear. On the other hand we deduce from the weight space decomposition that the subspaces $(\tilde{g}_\alpha)_{\alpha \in \Delta}$ belong to $[Ib, Ib]$. Since they generate $\tilde{n}$ in $\tilde{g}$, the result follows.

It follows from Lemma 4 and Theorem 3 that $\overline{Ib} \cong Ib/th \cong g^0_+ / \mathfrak{z}(g^0_+) \cong g^0_+$. Then it is straightforward from Lemma 4 and its proof that

- $h$ is a Cartan subalgebra of $\overline{Ib}$.
- The non-zero $h$-weights (resp. weight spaces) on $\overline{Ib}$ coincide with the non-zero $c$-weights (resp. weight space) on $Ib$ via projection. In particular $\Phi(\overline{Ib}) \cong \Phi(Ib) \cong \Phi$.
- $[\overline{Ib}, \overline{Ib}] = \tilde{n}/t\tilde{n}$.

From Lemma 4(2), the set $\Phi(Ib)$ of nonzero weights of $c$ acting on $Ib$ identifies with $\Phi$. It is also useful to embed $\Phi(Ib)$ in $\tilde{\Phi}$ by

\[
\varphi : \quad \Phi(Ib) \quad \longrightarrow \quad \tilde{\Phi}
\]

\[
\alpha \in \Phi^+ \quad \longmapsto \quad \alpha
\]

\[
\alpha \in \Phi^- \quad \longmapsto \quad \delta + \alpha
\]

Indeed, the weight space $(Ib)_\alpha$ identifies with $\tilde{g}_{\varphi(\alpha)}$, for any $\alpha \in \Phi(Ib)$. In particular, for $\alpha, \beta \in \Phi \cup \{0\}$, we have $[Ib_{\varphi^{-1}(\alpha)}, Ib_{\varphi^{-1}(\beta)}] \subset Ib_{\varphi^{-1}(\alpha + \beta)}$ with equality when $\alpha, \beta, \alpha + \beta \notin \{0, \delta\}$. Set also $\Delta(Ib) = \varphi^{-1}(\Delta) = \Delta \cup \{\alpha_0\}$.

**Lemma 5.**

1. The derived subalgebra of $Ib^{(1)} := [Ib, Ib]$ is

\[
Ib^{(2)} = th \oplus \bigoplus_{\alpha \in \Phi(Ib) \setminus \Delta(Ib)} (Ib)_\alpha
\]

2. Assume that $g$ is not $\mathfrak{sl}_2$. For $\alpha, \beta \in \Delta(Ib)$ ($\alpha \neq \beta$), the corresponding entry of the generalized Cartan Matrix of $g^{KM}$ is given by

\[
a_{\alpha, \beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \Phi(Ib)\}.
\]
Proof. 1) Recall that \( \tilde{n} \) is generated as a Lie algebra by the \( (\tilde{g}_\alpha)_{\alpha \in \tilde{\Delta}} \). Thus, for weight reasons, the \( (\tilde{g}_\alpha)_{\alpha \in \tilde{\Phi} \setminus \tilde{\Delta}} \) are root spaces included in \([\tilde{n}, \tilde{n}]\). Since \( \tilde{\Delta} \) is a linearly independent set, they are in fact the only root spaces not contained in \([\tilde{n}, \tilde{n}]\). Taking a quotient, this yields \( \bigoplus_{\alpha \in \Phi(Ib) \setminus \Delta(Ib)} (Ib)_\alpha = Ib(2) \).

2) Recall that the statement is valid if we replace \( \Phi(Ib) \) by \( \tilde{\Phi} \), see Section \( 2.2 \). It is thus sufficient to show that 
\[
\beta + n\alpha \in \tilde{\Phi} \Rightarrow \beta + n\alpha \in \Phi(Ib).
\]

When \( \alpha, \beta \in \Delta \), the statement is clear since \( \Phi^+ \subseteq \Phi(Ib) \).

If \( \beta = \delta + \alpha_0 \), then \( \beta + n\alpha \in \tilde{\Phi} \) means that \( \alpha_0 + n\alpha \in \Phi \). Expressing \( \alpha_0 \) as a linear combination of simple roots, one gets only negative coefficients. Since \( g \) is not \( sl_2 \), some of them remain negative in the expression of \( \alpha_0 + n\alpha \), so this root has to lie in \( \Phi^- \). Thus \( \beta + n\alpha \in \Phi(Ib) \).

If \( \alpha = \delta + \alpha_0 \), then \( \beta + n\alpha \in \tilde{\Phi} \) means that \( \beta + n\alpha_0 \in \Phi \). For height reasons, we must have \( n \in \{0, 1\} \). Then, \( \beta + n\alpha \in \Phi(Ib) \). \( \square \)

Remark. One can observe that the first assertion of Lemma 5 is similar to 
\[
[n, n] = \bigoplus_{\alpha \in \Phi^+ \setminus \Delta} b_\alpha.
\]

3.2 The adjoint subgroup of \( Aut(Ib) \)

Let \( G \) be the adjoint group with Lie algebra \( g \). Let \( T \) and \( B \) be the connected subgroups of \( G \) with Lie algebras \( h \) and \( b \). Consider now \( b^- \cong g/n \) equipped with the addition as an abelian algebraic group. The adjoint action of \( B \) on \( g \) stabilizes \( n \) and induces a linear action on \( b^- \cong g/n \) by group isomorphisms. We can construct the semidirect product:
\[
IB := B \rtimes b^-.
\]

By construction the Lie algebra of \( IB \) identifies with \( Ib \). The adjoint action of \( IB \) on \( Ib \) is given by
\[
IB \times Ib \longrightarrow Ib
\]
\[
((b, f), x + ty) \longrightarrow b \cdot x + tb \cdot (y + [f, x] + n) \quad \text{for } b \in B, x \in b \text{ and } f, y \in b^-,
\]

where \( y + [f, x] + n \) is viewed as an element of \( g/n \cong b^- \) and where \( \cdot \) denotes the \( B \)-action on \( b \) and on \( b^- \). It induces a group homomorphism
\[
Ad : IB \longrightarrow Aut(Ib)
\]

with kernel \( Z(IB) \cong (1, b) \). In particular, one gets:

Lemma 6. The image \( Ad(IB) \) is isomorphic to \( B \rtimes g/b \).
Note also that $\text{Ad}(IB) = H \ltimes (N \ltimes \mathfrak{g}/\mathfrak{b})$ where $N$ and $H$ are the connected subgroups of $B$ with respective Lie algebras $\mathfrak{n}$ and $\mathfrak{h}$. Since $\mathfrak{n} + t\mathfrak{b}^-$ is the set of ad-nilpotent elements of $IB$, we get the following result from (11).

**Lemma 7.** 1. The group of elementary automorphisms $\text{Aut}_e(IB) = \exp \text{ad}(\mathfrak{n} + t\mathfrak{b}^-)$ coincides with $N \ltimes \mathfrak{g}/\mathfrak{b}$.

2. $\text{Ad}(IB) = \exp \text{ad}(IB)$

### 3.3 A unipotent subgroup of $\text{Aut}(IB)$

Let $\mathfrak{a}$ be a Lie algebra. We consider the derived subalgebra $\mathfrak{a}^{(1)} := [\mathfrak{a}, \mathfrak{a}]$, the center $\mathfrak{z} := \mathfrak{z}(\mathfrak{a})$ and the quotient Lie algebra $\bar{\mathfrak{a}} := \mathfrak{a}/\mathfrak{z}$.

Any linear map $u \in \text{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z})$, defines a linear map $\bar{u} : \mathfrak{a} \rightarrow \mathfrak{a}$ where $X \mapsto X + u(X)$.

Since $u$ takes values in $\mathfrak{z}$ and vanishes on $\mathfrak{a}^{(1)}$, we have

$$[\bar{u}(X), \bar{u}(Y)] = [X + u(X), Y + u(Y)] = [X, Y] = [X, Y] + u([X, Y]) = \bar{u}([X, Y]).$$

In other words, $\bar{u}$ is a morphism of Lie algebras.

On the other hand, any $\theta \in \text{Aut}(\mathfrak{a})$ stabilizes the center of $\mathfrak{a}$, and hence it induces an automorphism of $\bar{\mathfrak{a}}$. This yields a natural group homomorphism

$$R : \text{Aut}(\mathfrak{a}) \rightarrow \text{Aut}(\bar{\mathfrak{a}}).$$

**Lemma 8.** Assume that $\mathfrak{z}(\mathfrak{a}) \subset \mathfrak{a}^{(1)}$. Under previous notation, we have an exact sequence of groups

$$0 \rightarrow \text{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z}) \rightarrow \text{Aut}(\mathfrak{a}) \xrightarrow{R} \text{Aut}(\bar{\mathfrak{a}})$$

where $\text{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z})$ is seen as the additive vector group.

We denote

$$U := \{ \bar{u} \mid u \in \text{Hom}(IB/IB^{(1)}, \mathfrak{z}(IB)) \}.$$  \hspace{1cm} (13)

This lemma, together with Lemma [4] implies the following results

**Corollary 9.** 1. $(U, \circ)$ is a normal subgroup of $\text{Aut}(IB)$ of dimension $(\dim \mathfrak{h})^2$

2. $R(\text{Aut}(IB)) = \text{Aut}(IB)/U \subset \text{Aut}(\overline{IB})$.

We will see in Lemma [14] that the last inclusion is actually an equality (i.e. the sequence of Lemma [8] is a short exact sequence for $\mathfrak{a} = IB$)
Proof of Lemma We have

\[(\bar{u} \circ \bar{v})(X) = (X + v(X)) + u(X + v(X)) = X + u(X) + v(X) = \bar{u} + \bar{v}(X)\]

where the middle equality is due to \(v(X) \in \mathfrak{z} \subset a^{(1)} \subset \text{Ker}(u)\). So the map \(u \mapsto \bar{u}\) is semi-group homomorphism from \((\text{Hom}(a/a^{(1)}, \mathfrak{z}), +)\) to \((\text{End}(a), \circ)\). Since \((\text{Hom}(a/a^{(1)}, \mathfrak{z}), +)\) is actually a group, its image is contained in \(\text{Aut}(a)\).

It is clear that the map \(u \mapsto \bar{u}\) is injective and, since \(u\) takes values in \(\mathfrak{z}\), that \(R(\bar{u}) = \text{Id}_{\bar{a}}\).

In order to prove exactness of the sequence at \(\text{Aut}(a)\), there remains to prove the implication

\[\forall \theta \in \text{Aut}(a), R(\theta) = \text{Id}_{\bar{a}} \Rightarrow ((\theta - \text{Id})(\mathfrak{a}) \subset \mathfrak{z}) \text{ and } ((\theta - \text{Id})_{a^{(1)}} = 0)\]

The first property is immediate. The second one follows from the fact that, for such a \(\theta\), we have \(\theta([X, Y]) \in [X + \mathfrak{z}, Y + \mathfrak{z}] = [X, Y]\).

\[\square\]

### 3.4 The loop subgroup

**Lemma 10.** The following map is an injective group homomorphism

\[
\begin{align*}
\mathbb{C}^* & \longrightarrow \text{Aut}(Ib) \\
\tau & \longmapsto \left(\begin{array}{l}
\delta_\tau : Ib \longrightarrow Ib \\
x & \longmapsto x \quad \text{if } x \in b \\
\tau y & \longmapsto \tau ty \quad \text{if } y \in b^-
\end{array}\right).
\end{align*}
\]

We denote by \(D \subset \text{Aut}(Ib)\) the image of this map.

**Proof.** It is a straightforward check on \(b \ltimes t b^-\) that the \(\delta_\tau\) are automorphisms of \(Ib\). \(\square\)

**Remark.** The map \(\delta_\tau\) corresponds to the variable changing \(t \mapsto \tau t\) in the \(\mathbb{C}[t]\)-Lie algebra \(\mathfrak{b}/t\mathfrak{n}\). Moreover, the Lie algebra of \(D\) acts on \(Ib\) like \(\mathbb{C}d\) where \(d\) is the derivation involved in the definition of \(\mathfrak{g}^{KM}\).

### 3.5 Automorphisms stabilizing the Cartan subalgebra

For any \(\alpha \in \Delta(Ib)\), fix generators \(e_\alpha\) of \(\tilde{\mathfrak{g}}\), \(\alpha \in \tilde{\Delta}\) giving rise to elements \(X_\alpha \in Ib_\alpha\) in the corresponding root space \((Ib)_\alpha\). Set

\[
\Gamma := \left\{ \theta \in \text{Aut}(Ib) \left| \begin{array}{l}
\theta(\mathfrak{h}) \subset \mathfrak{h} \\
\theta(\{X_\alpha : \alpha \in \Delta(Ib)\}) = \{X_\alpha : \alpha \in \Delta(Ib)\}
\end{array} \right. \right\}.
\]

Note that, since \(c\) is the sum of \(\mathfrak{h}\) with \(\mathfrak{z}(Ib)\) and since the center is characteristic, the elements of \(\Gamma\) also stabilize \(c\).

**Proposition 11.** The group \(\Gamma\) is isomorphic to the automorphism group of the affine Dynkin diagram of \(\mathfrak{g}\).
Proof. By construction, $\Gamma$ induces an action on $\Delta(Ib)$. By Lemma 3(2), we have for $g \in \Gamma$ and $\alpha, \beta \in \Delta(Ib)$:

$$a_{\alpha,\beta} = -\max\{n|(ad X_\alpha)^n(X_\beta) \neq 0\} = -\max\{n|g((ad X_\alpha)^n(X_\beta)) \neq 0\} = -\max\{n|(ad X_{g(\alpha)})^n(X_{g(\beta)}) \neq 0\} = a_{g(\alpha),g(\beta)}.$$ 

Hence $g$ actually induces an automorphism of the extended Dynkin diagram and we thus obtain a group homomorphism

$$\Theta : \Gamma \to Aut(\tilde{D}_g).$$

We claim that $\Theta$ is surjective. Indeed, fix an automorphism $\theta$ of the group $\tilde{D}_b$. As was mentioned in Section 2.2, there exists $\tilde{\theta} \in Aut(\tilde{g})$ which stabilizes both $\mathfrak{h}$ and $\mathfrak{b}$ and which permutes the generators \{e_\alpha : \alpha \in \tilde{\Delta}\} and thus $\tilde{\Delta} \cong \Delta(Ib)$ as $\theta$ does. By Lemma 2, $\tilde{\theta}$ stabilizes $\tilde{\mathfrak{n}}$, so induces the desired element of $Aut(\mathfrak{b}/t\mathfrak{n})$.

We now prove that $\Theta$ is injective. Let $\theta$ in its kernel. By the definition of the group $\Gamma$, $\theta$ stabilizes $\mathfrak{h}$. Since the restrictions of the elements of $\Delta(Ib)$ span $\mathfrak{h}^*$, the restriction of $\theta$ to $\mathfrak{h}$ has to be the identity. In particular, $\theta$ acts trivially on $\Phi(Ib)$ and stabilizes each root space $(Ib)_\alpha$ for $\alpha \in \Phi(Ib)$. But $\theta$ stabilizes the set $\{X_\alpha : \alpha \in \Delta(Ib)\}$. Hence $\theta$ acts trivially on each $\tilde{g}_\alpha$ for $\alpha \in \Delta(Ib)$. Since $\tilde{\mathfrak{n}}$ is generated by the $(\tilde{g}_\alpha)_{\alpha \in \Delta(Ib)}$, the restriction of $\theta$ to $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is the identity map. Finally, $\theta$ is trivial and $\Theta$ is injective. \hfill $\Box$

Remark.

1. [Bv20] Theorem 2] is the construction of an explicit order $n$ automorphism of $gl_{n+1}$. We can also interpret this automorphism in terms of the isomorphism $gl_{n+1} \cong \mathfrak{b}/(t-\epsilon)\mathfrak{n}$ of Theorem 3. Indeed, let $\theta$ be the cyclic automorphism of the extended Dynkin diagram in type $A_k$ and let $\tilde{\theta}$ be the automorphism of $\tilde{g}$ associated to $\theta$ as in Section 2.2. By Lemma 2 and the subsequent remark, $\tilde{\theta}$ induces an automorphism of $\mathfrak{b}/(t-\epsilon)\mathfrak{n}$. Moreover, it is easily checked that the action on layer 1 in [Bv20] is a cyclic permutation of the generators $(e_\alpha)_{\alpha \in \tilde{\Delta}}$.

2. Consider the trivial vector bundle $\mathcal{V} := \mathcal{V} \times \mathbb{A}^1$ over $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[\epsilon])$. The Lie bracket $[\ , \]_\epsilon$ endows $\mathcal{V}$ with a structure of a Lie algebra bundle meaning that $[\ , \]_\epsilon$ can be seen as a section of the vector bundle $\bigwedge^2 \mathcal{V}^* \otimes \mathcal{V}$ satisfying the Jacobi identity. Consider the group $\text{Aut}(\mathcal{V}, [\ , \]_\epsilon)$ consisting of automorphisms of the vector bundle $\mathcal{V}$ respecting the Lie bracket pointwise. Let $\theta \in \text{Aut}(\tilde{D})$ and assume that the $\tilde{\theta} \in \text{Aut}(\tilde{g})$ is $\mathbb{C}[t]$-linear (i.e. $\lambda = 1$ in Lemma 2). Then it is easy to check that $\tilde{\theta}$ induces an element of $\text{Aut}(\mathcal{V}, [\ , \]_\epsilon)$. In other words, $\theta$ lifts to a $\mathbb{A}^1$-family of automorphisms over the $\mathbb{A}^1$-family of Lie algebras $\mathcal{V}$.

---

1If $\tilde{g}$ is $sl_2$, Lemma 3(2) does not apply. However, any permutation of $\tilde{\Delta}$ is an automorphism of the extended Dynkin diagram in this case.
4 Description of Aut(Ib)

In this section, we describe the structure of

\[
\text{Aut}(Ib) = \{ g \in \text{GL}(Ib) : \forall X, Y \in Ib \quad g([X,Y]) = [g(X), g(Y)] \}
\]

in terms of the subgroups \( U, \text{Ad}(IB), D \) and \( \Gamma \) introduced in Section 3.

Observe that Aut(Ib) is a Zariski closed subgroup of the linear group GL(Ib).

**Theorem 12.** We have the following decompositions

\[
\text{Aut}(Ib) = \Gamma \ltimes (D \ltimes (\text{Ad}(IB) \times U)),
\]

\[
\text{Aut}(\overline{Ib}) = \Gamma \ltimes (D \ltimes (\text{Ad}(IB))).
\]

In particular, the neutral component is \( \text{Aut}(Ib)^0 = D \ltimes (\text{Ad}(IB) \times U) \) and \( \Gamma \cong \text{Aut}(\overline{D_g}) \) can be seen as the component group of \( \text{Aut}(Ib) \).

The result is a consequence of the lemmas provided below. Indeed, by Lemma 14 the four subgroups generate Aut(Ib). By Corollary 11 and Lemma 13 below, the subgroup generated by \( U \) and Ad(IB) is a direct product \( U \times \text{Ad}(IB) \). Then the structure of Aut(Ib) follows from Lemma 15. That of Aut(\overline{Ib}) follows the same lines, using Corollary 12. Note that we have identified \( \Gamma, \text{Ad}(IB) \) and \( D \) with their image under \( R \), via Lemma 8.

Since \( D, \text{Ad}(IB) \) and \( U \) are connected and \( \Gamma \) is discrete, \( \text{Aut}(Ib) = \bigsqcup_{g \in R} g\text{Ad}(IB)U \) is a finite disjoint union of irreducible subsets of the same dimension. They are thus the irreducible components of Aut(IB) and the remaining statements of Theorem 12 follow.

**Lemma 13.** The subgroups \( U \) and \( \text{Ad}(IB) \) are normal in Aut(Ib). Moreover, \( U \cap \text{Ad}(IB) = \{\text{Id}\} \).

**Proof.** Recall that \( \text{Ad}(IB) \) is generated by the exponentials of \( \text{ad}(x) \) with \( x \in Ib \). Then for any \( \theta \in \text{Aut}(Ib) \),

\[
\theta \text{Ad}(IB)\theta^{-1} = \theta \exp(Ib)\theta^{-1} = \exp(\theta(Ib)) = \exp(Ib) = \text{Ad}(IB).
\]

Let \((b, f) \in IB \) and \( h \in \mathfrak{h} \). Then \( \text{Ad}(b, f)(h) = b \cdot h + t b \cdot ([f, h] + n) \). Assuming that \( \text{Ad}(b, f) = \bar{u} \in U \), we have \( \text{Ad}(b, f)(\mathfrak{h}) \subset \mathfrak{h} + \mathfrak{z} \) so \( \text{Ad}(b)(\mathfrak{h}) \subset \mathfrak{h} \), that is \( b \) belongs to the normalizer of \( \mathfrak{h} \) in \( B \), which turns to be \( T \). In particular, \( b \cdot [f, h] \subset n^- \) and \( \text{Ad}(b, f)(\mathfrak{h}) \subset \mathfrak{h} + (n + tn^-) \). Hence \( u = 0 \) and finally \( \text{Ad}(IB) \cap U = \{\text{Id}\} \). \( \square \)

**Lemma 14.** We have \( \text{Aut}(Ib) = \Gamma \text{DAd}(IB)U \) and \( \text{Aut}(\overline{Ib}) = \Gamma \text{DAd}(IB) \).

**Proof.** Let \( \theta \in \text{Aut}(Ib) \). Since the two Cartan subalgebras \( \mathfrak{c} \) and \( \theta(\mathfrak{c}) \) are Ad-conjugate (see [Bou75], §3, n° 2, th. 1), there exists \( \theta_1 \in \text{Ad}(IB)\theta \) which stabilizes \( \mathfrak{c} \).

Then \( \theta_1(\mathfrak{h}) \) is complementary to the center \( t \mathfrak{h} = \theta_1(t\mathfrak{h}) \) in \( \mathfrak{c} \). Thus, there exists \( \theta_2 \in U\theta_1 \) such that \( \theta_2 \) stabilizes \( \mathfrak{h} \).
Since \( \theta_2 \) stabilizes \( c \), it acts on \( \Phi(IB) \). Moreover, \( IB^{(1)} = [Ib, Ib] \) and \( IB^{(2)} = [Ib^{(1)}, Ib^{(1)}] \) are characteristic and stabilized by \( \theta_2 \). So, Lemma \[3\] implies that \( \theta_2 \) stabilizes \( \Phi(IB) \setminus \Delta(IB) \) and hence \( \Delta(IB) \). Arguing as in the proof of Proposition \[11\] we show that the induced permutation is actually an automorphism of the extended Dynkin diagram. Thus there exists \( \theta_3 \in \Gamma \theta_2 \) with the additional property that the induced permutation on \( \Delta(IB) \) and thus on \( \Phi(IB) \) are trivial. Then \( \theta_3 \) acts on each \( (Ib)_a \) for \( a \in \Delta(IB) \).

Since \( \Delta \) is a basis of \( h^* \), one can find \( h \in H \subset B \subset IB \) such that \( \text{Ad}(h) \circ \theta_3 \) acts trivially on each \( (Ib)_a \) for \( a \in \Delta \). Moreover, \( D \) acts trivially on these roots spaces and with weight 1 on \( (Ib)_{a_0} \). This yields \( \theta_4 \in D \text{Ad}(H) \Gamma U \text{Ad}(IB) \theta \) which acts trivially on \( h \) and on each \( (Ib)_a \), \( a \in \Delta(IB) \).

Recall now that \( \bar{n}/t\bar{n} \) is generated by the spaces \( ((Ib)_a)_{a \in \Delta(IB)} \). Since \( \theta_4 \) acts trivially on \( \bar{n} \) and on \( h \), it has to be trivial. As a consequence, \( \theta \in \text{Ad}(IB) U \Gamma \text{Ad}(H) D = \Gamma D \text{Ad}(IB) U \), the last equality following from Lemma \[13\] and Corollary \[9\].

Recalling that \( \Phi(IB) = \Phi(\overline{IB}) \), the same proof applies for \( \overline{IB} \) instead of \( Ib \), replacing \( c \) by \( h \) and skipping step from \( \theta_1 \) to \( \theta_2 \). \[\square\]

**Lemma 15.** The intersections \( D \cap (\text{Ad}(IB) \times U) \) and \( \Gamma \cap (D \ltimes (\text{Ad}(IB) \times U)) \) are the trivial group \( \{\text{Id}\} \). Moreover, \( (D \ltimes (\text{Ad}(IB) \times U)) \) is normal in \( \text{Aut}(IB) \).

**Proof.** Let \( \tau \in C^* \), \( b \in B \), \( f \in g/n \) and \( u \in \text{Hom}(Ib/[Ib, Ib], z(IB)) \) such that the associated elements \( \delta_\tau \in D \), \( (b, f) \in IB \) and \( \bar{u} \in U \) (see Section \[3\]) satisfy \( \delta_\tau = \text{Ad}(b, f) \circ \bar{u} \). For \( x \in b \), we have

\[
x = \delta_\tau(x) = (\text{Ad}(b, f) \circ \bar{u})(x) = \text{Ad}(b, f)(x + u(x)) = b \cdot x + (b \cdot u(x) + tb \cdot ([f, x] + n)).
\]

In particular, \( b \cdot x = x \) and, whenever \( x \in n \), \( b \cdot [f, x] = 0 \) in \( g/n \). So \( b \in B \) centralizes \( b \) and \( \text{ad}_g f \) normalizes \( n \). As a consequence, \( b = 1_B \), \( f \) is 0 in \( g/b \) and \( u = 0 \). Thus the only element of \( D \cap (\text{Ad}(IB) \times U) \) is the trivial one.

Since \( [Ib, Ib] \) is characteristic in \( Ib \), we have a natural group morphism \( p : \text{Aut}(Ib) \to \text{Aut}(Ib/[Ib, Ib]) \). From the description of \( [Ib, Ib] \) in Lemma \[3\] it is straightforward that \( D \), \( \text{Ad}(IB) \) and \( U \) are included in \( \text{Ker}(p) \) while \( p_{II} \) is injective. From Lemma \[14\] we then deduce that \( D \ltimes (\text{Ad}(IB) \times U) = \text{Ker}(p) \) and the desired properties follow. \[\square\]

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