NON-DEGENERATE NEAR-PARABOLIC RENORMALIZATION

ALEK KAPIAMBA

Abstract. Invariant classes under parabolic and near-parabolic renormalization have proved extremely useful for studying the dynamics of polynomials. The first such class was introduced by Inou-Shishikura to study quadratic polynomials; their argument has been extended to the unicritical cubic case by Chéritat and Yang. However, all of these classes are only applicable to maps which have a fixed point with multiplier close to one, though it is well-known that similar phenomena occur when the multiplier is close to any root of unity. In this paper we define the parabolic and near-parabolic renormalization operators in the general setting and construct invariant classes. Additionally, we compare the corresponding near-parabolic renormalizations when one root of unity is close to another.

Introduction

Let \( f(z) \) be a holomorphic function defined on a neighborhood of a fixed point \( z_0 \in \mathbb{C} \). The multiplier of the fixed point is the quantity \( \lambda = f'(z_0) \), and \( z_0 \) is called parabolic when \( \lambda \) is a root of unity. The parabolic fixed point is called simple when \( \lambda = 1 \) and it is called non-degenerate when \( f''(0) \neq 0 \).

When \( f(z) \) has a simple non-degenerate parabolic fixed point \( z_0 \) the local dynamics of \( f \) near \( z_0 \) are relatively tame, however for a holomorphic function \( h(z) \) which is a perturbation of \( f(z) \) the local dynamics of \( f \) can be drastically more wild. This phenomenon, called parabolic implosion, was first studied by Douady-Hubbard \([DH84; DH85]\) and Lavaurs \([Lav89]\) using Fatou coordinates and Lavaurs maps. While describing the Hausdorff dimension of the Mandelbrot set in \([Shi98]\), Shishikura introduced parabolic and near-parabolic renormalization, providing another lens to study parabolic implosion. The parabolic renormalization acts on maps \( f \) with a simple non-degenerate parabolic fixed point; Shishikura introduced a class of maps invariant under parabolic renormalization in \([Shi00]\). The near-parabolic renormalization, also called a cylinder renormalization by Yampolsky in \([Yam02]\), acts on maps \( h \) which are perturbations of maps with simple non-degenerate parabolic fixed points; Inou-Shishikura introduced a class of maps invariant under near-parabolic renormalization in \([Shi98]\). The Inou-Shishikura class has had several remarkable applications: it is used by Buff-Chéritat to prove the existence of quadratic Julia sets of positive measure \([BC12]\), by Cheraghi-Chéritat to partially resolve the Marmi-Moussa-Yoccoz conjecture \([CC15]\), and by Cheraghi-Shishikura to make progress towards the celebrated MLC conjecture \([CS15]\). There are numerous other applications of the Inou-Shishikura class, see for example \([AC18]\, [Che13]\, [Che17]\, [Che19]\, [SY16]\).

All maps in the Inou-Shishikura class have critical points of local degree two, so it is most commonly used to study the dynamics of quadratic polynomials. In \([Yan15]\), Yang has modified the argument of Inou-Shishikura to produce a class of maps, all with critical points of local degree three, which is invariant under parabolic and near-parabolic renormalization, allowing for generalization of the applications to unicritical cubic polynomials. Chéritat has
introduced smaller classes of maps which are invariant under parabolic and near-parabolic renormalization [Ché22]; Chéritat’s construction is flexible enough so that the classes can have critical points of arbitrary local degree. This allows for further generalization of the applications of the Inou-Shishikura class.

All of the applications of the above invariant classes are limited to studying perturbations of maps with simple non-degenerate parabolic fixed points. However, it is well-known that similar parabolic implosion phenomena occur in the general setting. For a holomorphic map $f(z)$ which has a parabolic fixed point $z_0$ the local dynamics of $f$ near $z_0$ remain relatively tame: if the multiplier of the fixed point is a $q$-th root of unity then orbits are attracted towards $z_0$ along $νq$ distinct directions and repelled away from 0 along $νq$ other distinct directions for some integer $ν ≥ 1$. The parabolic fixed point is called degenerate or non-degenerate if $ν > 1$ or $ν = 1$ respectively. The parabolic implosion in the degenerate case is studied in [Oud02]; there the bifurcation phenomenon is quite complicated and invariant classes under the corresponding near-parabolic renormalization have not been constructed. But in the general non-degenerate case the parabolic implosion is analogous to the simple case; Shishikura outlines the necessary modifications to generalize the analysis in [Shi98]. However, the construction of invariant classes for near-parabolic renormalization in [IS08] and [Yan15] are delicate and can not be immediately adapted to the general non-degenerate setting. Luckily the construction of Chéritat in [Ché22] is flexible enough to generalize, so for any root of unity we can construct a class of maps invariant under the corresponding near-parabolic renormalization, allowing for further extension of the applications of the Inou-Shishikura class.

There are two main goals of this paper. The first is to provide a reference for parabolic implosion in the non-degenerate setting; while Shishikura outlines the necessary generalizations in [Shi98] the reader must reconstruct the statements themselves. Moreover for a map $f(z)$ which has a non-degenerate parabolic fixed point with multiplier $e^{2πip/q}$, Shishikura studies perturbations of $f$ which have a fixed point with multiplier $e^{2πi(p+α)/q}$ with $α ∈ C$; this parameterization of the multiplier is ill-suited for near-parabolic renormalization, we present an alternative parameterization using the continued fraction of $p/q$ which is more agreeable. The second goal of this paper is to compare different near-parabolic renormalizations of a map which has a fixed point with multiplier close to two different roots of unity. For example, if $h(z)$ is a map with a fixed point of multiplier $e^{2πi/n}$ for some integer $n$ and some $α ∈ C$, then when $n$ is large we can view $h$ as a perturbation of a map with a fixed point of multiplier either 1 or $e^{2πi/n}$; we will show that we can directly relate the corresponding near-parabolic renormalizations.

This paper is organized as follows. In §1 and §2 we present the parabolic and near-parabolic renormalizations respectively in the general non-degenerate setting. In §3 we define classes invariant under these renormalization operators. In §4 we compare different parabolic and near-parabolic renormalizations.

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1. Parabolic renormalization

For an analytic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we define a petal for $f$ to be a Jordan domain $P \subset \text{Dom}(f)$ such that $f$ is univalent on $P$ and there exists a univalent map $\phi : P \to \mathbb{C}$, called a Fatou coordinate, which satisfies:

(1) For all $z \in P$, the following are equivalent:
   - $f(z) \in P$;
   - $\phi(z) + 1 \in \phi(P)$;
   - $\phi(f(z)) = \phi(z) + 1$.

(2) If both $w$ and $w + n$ belong to $\phi(P)$ for some integer $n \geq 0$, then $w + j \in \phi(P)$ for all integers $0 \leq j \leq n$.

(3) For any $w \in \mathbb{C}$, there exists $n \in \mathbb{Z}$ such that $w + n \in \phi(P)$.

If $f(P) \subset P$ or $P \subset f(P)$, then we will say that $P$ is an attracting or repelling petal respectively. The Fatou coordinate $\phi$ is unique up to post-composition with a translation.

For any $t \in \mathbb{R}$ and any $z_1, z_2 \in \mathbb{C}$ satisfying $\text{Re}(z_2 - z_1) > t \cdot \text{Im}(z_2 - z_1)$, we define the region

$$\Omega_t(z_1, z_2) := \{w \in \mathbb{C} : \text{Re}z_1 - t \cdot |\text{Im}(w - z_1)| < \text{Re}w < \text{Re}z_2 + t \cdot |\text{Im}(w - z_2)|\}.$$ 

We also allow $z_1$ or $z_2$ to be $-\infty$ or $\infty$ respectively, in those cases we ignore the inequality containing $z_1$ or $z_2$ respectively. We define a strip to be a connected subset of $\mathbb{C}$ which is bounded by two parallel lines, and we will say that a strip in $\Omega_t(z_1, z_2)$ is maximal if both components of its boundary non-trivially intersect $\partial \Omega_t(z_1, z_2)$.

Let us fix some reduced rational number $p/q \in [-1/2, 1/2]$ and an analytic function $f$ defined on a neighborhood of 0 such that

$$f(z) = e^{2\pi ip/q}z + O(z^2)$$

and

$$f^q(z) = z + a z^{q+1} + O(z^{q+2})$$

near $z = 0$ for some $a \neq 0$. In this case we will say that $f$ has a non-degenerate $p/q$-parabolic fixed point at 0.

**Theorem 1.1.** For any neighborhood $V$ of zero and any $z_1, z_2 \in \mathbb{C}$, there exist attracting and repelling petals $P^f_{\text{att}}$ and $P^f_{\text{rep}}$ for $f^q$ inside $V$ with corresponding Fatou coordinates $\phi^f_{\text{att}}$ and $\phi^f_{\text{rep}}$ respectively such that

1. $0 \in \partial P^f_{\text{att}} \cap \partial P^f_{\text{rep}}$.
2. the union $\bigcup_{n=0}^{q-1} f^n(P^f_{\text{att}} \cup P^f_{\text{rep}})$ forms a punctured neighborhood of zero.
3. $\phi^f_{\text{att}}(P^f_{\text{att}}) = \Omega_1(z_1, \infty)$ and $\phi^f_{\text{rep}}(P^f_{\text{rep}}) = \Omega_1(-\infty, z_2)$.
4. For any $x \in [-1, 1]$ and $y \in \mathbb{R}$, if $|y|$ is sufficiently large then

$$f^j \circ (\phi^f_{\text{rep}})^{-1}(x + iy) \in P^f_{\text{rep}},$$

where $0 \leq j < q$ is an integer satisfying

$$j \equiv \begin{cases} -1 \mod q & \text{if } y > 0, \\ 0 \mod q & \text{if } y < 0. \end{cases}$$

**Proof.** See [Mil06, Chapter 10].
We can define an analytic extension of $\phi_{att}^f$ by

$$\rho^f(z) := \phi_{att}^f \circ f^{nq+j}(z) - n$$

for any $z \in \hat{\mathbb{C}}$ and integers $n \geq 0$, $0 \leq j < q$ such that $f^{nq+j}(z) \in P_{att}^f$. We can similarly define an analytic extension of $(\phi_{rep}^f)^{-1}$ by

$$\chi^f(w + n) := f^{nq}(z)$$

for any $z \in P_{rep}^f$, $w = \phi_{rep}^f(z)$, and integer $n \geq 0$ such that $z \in \text{Dom}(f^{nq})$.

The horn map for $f$ is defined by

$$H^f := \rho^f \circ \chi^f,$$

it is clear that this function is analytic and commutes with the translation by one.

**Proposition 1.2.** The domain of the horn map $H^f$ contains $\{w \in \mathbb{C} : \text{Im } w > \eta_0\}$ for some $\eta_0 > 0$. There exist constants $c_+^f$ and $c_-^f$ such that $H^f(w) - w$ tends to $c_+^f$ when $\text{Im } w$ tends to $\pm \infty$ respectively.

**Proof.** See [Shi00] and [Shi98, A.5] □

For any $\delta \in \mathbb{C}$ we set $T_\delta(w) = w + \delta$. We define a parabolic renormalization of $f$ to be a map of the form

$$R_\delta f := \text{Exp} \circ H^f \circ T_\delta \circ \text{Exp}^{-1}$$

for some $\delta \in \mathbb{C}$. Proposition 1.2 implies that $R_\delta f$ is defined on punctured neighborhoods of zero and infinity in $\hat{\mathbb{C}}$ and can be continuously extended by setting $R_\delta f(0) = 0$ and $R_\delta f(\infty) = \infty$. The map $R_\delta f$ is sometimes called at top parabolic renormalization of $f$ in contrast to the bottom parabolic renormalization

$$z \mapsto \frac{1}{R_\delta f(1/z)}$$

which enjoys many of the same properties. We will focus solely on top parabolic renormalizations; the bottom parabolic renormalizations can be studied symmetrically.

Let us note that the definition of parabolic renormalization given here differs from the definition in [Shi00] or [IS08], where there parabolic renormalization is defined as $R_\delta f$ for the unique choice of $\delta$ satisfying $(R_\delta f)'(0) = 1$. While that definition is well-suited for studying maps with simple parabolic fixed points, for our more general setting it is important to consider more possible values for the multiplier at zero. Let us also note that we could have defined $R_\delta f$ so that it is semi-conjugate to $T_\delta \circ H^f$ instead of $H^f \circ T_\delta$. This distinction is purely aesthetic; in this paper it is convenient to have the critical values of $R_\delta f$ not depend on $\delta$, in some cases it is instead convenient to have the domain of $R_\delta f$ not depend on $\delta$.

**2. Near-parabolic renormalization**

Let us now fix some rational $p/q \in [-1/2, 1/2]$ and analytic map $f$ with a non-degenerate $p/q$-parabolic fixed point at zero. In this section we will see that for some perturbations of $f$ the local dynamics near zero resembles that of $f$; in particular there still exist petals and
Fatou coordinates. Throughout this paper we will consider analytic maps in the compact-open topology with varying domains, i.e. a neighborhood of $f$ is a set of the form

$$\left\{ h : \text{Dom}(h) \to \mathbb{C} \mid h \text{ is analytic on } \text{Dom}(h) \supset K, \text{ and } |f(z) - h(z)| < \epsilon \text{ for all } z \in K \right\}$$

for some $\epsilon > 0$ and compact set $K \subset \text{Dom}(f)$. The precise local dynamics of a map close to $f$ depend on the continued fraction of $p/q$.

2.1. Modified continued fractions. We can associate to $p/q$ a (possibly empty) sequence $\langle (a_n, \varepsilon_n) \rangle_{n=1}^N$, called the modified continued fraction of $p/q$, as follows. First, let $x_0 = p/q$ and assume that $x_n \in [-1/2, 1/2]$ is defined for some $n \geq 0$. If $x_n \neq 0$, then there exists a unique $\varepsilon_{n+1} \in \{\pm 1\}$ and a unique $y_{n+1} \in (0, 1/2]$ such that $x_n = \varepsilon_{n+1} \cdot y_{n+1}$. In this case, there exists a unique integer $a_{n+1} \geq 2$ and a unique $x_{n+1} \in (-1/2, 1/2]$ such that

$$\frac{1}{y_{n+1}} = a_{n+1} + x_{n+1}.$$ 

We repeat this construction recursively to produce the modified continued fraction; the resulting sequence is guaranteed to be finite as $p/q$ is rational. For $N \geq 0$, we denote by $\mathbb{Q}_N$ the set of all rational numbers $[-1/2, 1/2]$ whose modified continued fraction has length $N$. In particular, $0/1$ is the unique element of $\mathbb{Q}_0$.

Assuming now that $p/q \in \mathbb{Q}_N$ for some $N \geq 0$, the approximates of $p/q$ are inductively defined as

$$p_{-1} = 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + \varepsilon_n p_{n-2} \quad \text{for } 1 \leq n \leq N,$$

$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + \varepsilon_n q_{n-2} \quad \text{for } 1 \leq n \leq N.$$ 

By construction, $p/q = p_N/q_N$. We will call $p'/q' = p_{N-1}/q_{N-1}$ the parent of $p/q$; by abuse of notation we will not differentiate between the fraction $p'/q'$ and the pair $(p', q')$ so that the parent $1/0$ of $0/1$ is well-defined.

We define the signature of $p/q$ to be

$$\mathcal{S}(p/q) := (-1)^N \prod_{n=1}^{N} \varepsilon_n,$$

using the convention that the empty product is equal to 1.

**Proposition 2.1.** $\mathcal{S}(x) = p'q - pq'$.

**Proof.** If $p/q = 0/1$, then $\mathcal{S}(x) = 1 = 1 \cdot 1 - 0 \cdot 0$. If $x \in \mathbb{Q}_N$ for some $N > 0$, then the parent $p''/q''$ of $p'/q'$ is defined and

$$p'q - pq' = p'(a_N q' + \varepsilon_N q'') - q'(a_N p' + \varepsilon_N p'')$$

$$= -\varepsilon_N (p'' q' - p' q'')$$

$$= \mathcal{S}(p/q)$$

by induction. \qed

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We define the Möbius transformation \( \mu_{\frac{p}{q}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) by
\[
\mu_{\frac{p}{q}}(z) := \frac{p + \mathcal{S}(\frac{p}{q}) \cdot z}{q + \mathcal{S}(\frac{p}{q}) \cdot z} = \frac{\varepsilon_1}{a_1 + \varepsilon_2} + \frac{\varepsilon_3}{a_2 + \varepsilon_4} + \cdots + \frac{\varepsilon_N}{a_N + \mathcal{S}(x) \cdot z}.
\]

2.2. Petals and Fatou coordinates. For some \( C > 1 \) and \( 0 < r < \frac{1}{C} \) we set
\[
A = A(r, C) := \{ x + iy : 0 < x < r, |y| < Cx^2 \}.
\]
For any \( w \in \mathbb{C} \) we set \( \exp(w) = e^{2\pi i w} \). Let \( h \) be an analytic map defined on a neighborhood of zero such that \( h(0) = 0 \) and \( h'(0) = \exp \circ \mu_{\frac{p}{q}}(\alpha) \) for some \( \alpha \in A \). Let us fix some point \( z_0 \) such that \( f^n(z_0) \) converges asymptotically to zero when \( n \to \infty \).

**Theorem 2.2.** For any sufficiently large integer \( n_0 \geq 0 \) and any sufficiently large \( M_0 \geq 0 \), there exists a unique choice of \( P_{f_{\text{att}}}^f, P_{f_{\text{rep}}}^f, \phi_{f_{\text{att}}}^f, \) and \( \phi_{f_{\text{rep}}}^f \) as in Theorem 1.1 and an analytic map
\[
\phi_{h,f}^f : \Omega_1 \left( -1, \frac{1}{\alpha} - M_0 - 1 \right) \to \text{Dom}(h^q)
\]
when \( h \) is sufficiently close to \( f \) satisfying:

1. \( \phi_{h,f}^f(w) \) tends to 0 or a nonzero fixed point \( \sigma_{h,f}^f \) of \( h^q \) when \( \text{Im} w \) tends to \( +\infty \) or \( -\infty \) respectively.
2. \( \phi_{h,f}^f(0) = h^{n_0q}(z_0) \).
3. For any maximal strip \( S \) in \( \text{Dom}(\phi_{h,f}^f) \):
   (a) \( \phi_{h,f}^f \) maps \( S \) univalently onto a petal for \( h^q \) which has Fatou coordinate \( (\phi_{h,f}^f)^{-1} \).
   (b) The sets \( \phi_{h,f}^f(S), h \circ \phi_{h,f}^f(S), \ldots, h^{q-1} \circ \phi_{h,f}^f(S) \) are pair-wise disjoint.
   (c) \( \phi_{h,f}^f \) is the unique analytic map defined on \( S \) satisfying conditions (1), (2), and (3a) above.
4. \( \phi_{h,f}^f \) depends continuously and holomorphically on \( h \).
5. \( \phi_{h,f}^f \to (\phi_{f_{\text{att}}}^f)^{-1} \) and \( \phi_{h,f}^f \circ T_{1/\alpha} \to (\phi_{f_{\text{rep}}}^f)^{-1} \) when \( h \to f \).
6. if \( w^h \in \text{Dom}(\phi_{h,f}^f) \) satisfies \( w^h \to \infty \) and \( w^h - \frac{1}{\alpha} \to \infty \) when \( h \to f \), then \( \phi_{h,f}^f(w^h) \to 0 \).

**Proof.** For the \( q = 1 \) case this is proved in [Shi00], the necessary adjustments for \( q > 1 \) are given in [Shi98, A.5].

We will call a specific choice of \((z_0, n_0, M_0)\) for \( \phi_{h,f}^f, P_{f_{\text{att}}}^f, \) and \( P_{f_{\text{rep}}}^f \) as in Theorem 2.2 a **petal normalization**. As a consequence of Theorem 2.2, we can observe that varying \( f \) while keeping the same petal normalization does not affect \( \phi_{h,f}^f \):

**Corollary 2.3.** If \( g \) is another analytic map with a non-degenerate \( p/q \)-parabolic fixed point at 0 sufficiently close to \( f \), then \( \phi_{h,g}^f = \phi_{h,f}^f \) for any \( h \) sufficiently close to \( g \) when we take the same petal normalization for each map.
Let us now fix some choice of normalization \((z_0, n_0, M_0)\) for \(f\). We also fix some maximal strip \(S \subset \text{Dom}(\varphi^{h,f})\). We will say that the tilt \(t\) of \(S\) is the reciprocal of the slope of the lines bounding \(S\), so \(S\) is continuously parameterized by \(t \in [-1, 1]\). We have the following additional control on the geometry of \(\varphi^{h,f}(S)\):

**Proposition 2.4.** There exist constants \(M, M' > 0\) such that if \(h\) is sufficiently close to \(f\) and if \(S\) has tilt \(t\), then there is a branch of \(\log\) defined on \(\varphi^{h,f}(S)\) and some \(t' \in \mathbb{R}\) with \(|t' - t| \leq M \arg \alpha\) such that

\[
\sup_{w \in S} \inf_{z \in \gamma} \left| \frac{\log \varphi^{h,f}(w) - z}{2\pi i} \right| < M',
\]

where

\[
\gamma = \left\{ \frac{\log \alpha}{2\pi i} - is + s'(t' + i) : s, s' \geq 0 \text{ and } s \cdot s' = 0 \right\}.
\]

**Proof.** If \(q = 1\) then this follows directly from [BC12, Appendix A]; while only the case where \(\alpha \in \mathbb{R}\) and \(t = 0\) is considered there the same arguments can be applied for general \(\alpha\) and \(t\). The adjustments for general \(q\) are made in [Shi98, A.5].

### 2.3. Horn maps.

We will say that a point \(w \in S\) is **petal-entering** or **petal-exiting** in \(S\) if

\[
w - \Re \frac{1}{3\alpha} \notin S \quad \text{or} \quad w + \Re \frac{1}{3\alpha} \notin S
\]

respectively. Let us note that the petal-entering and petal-exiting points are separated.
Proposition 2.5. If $h$ is sufficiently close to $f$, then no point in $S$ is both petal-entering and petal-exiting.

Proof. As $S \subset \text{Dom}(\varphi^{h,f})$ is a maximal strip, it follows from the definition that $x \in \mathbb{R}$ belongs to $S$ if and only if

$$0 < x - 1 < \Re \frac{1}{\alpha} + t \cdot \Im \frac{1}{\alpha} - M_0$$

for some $t \in [-1, 1]$. The definition of $A$ implies that

$$\Re \frac{1}{\alpha} + t \cdot \Im \frac{1}{\alpha} - M > (1 - C \cdot \Re \alpha) \Re \frac{1}{\alpha} - M_0.$$ 

Thus if $\alpha$ is sufficiently small then the width of $S \cap \mathbb{R}$ is larger than $\Re \frac{2}{3\alpha}$; this implies the proposition. \qed

For any petal-entering point $w \in S$ and any $z$ so that $h^{nq+j}(z) = \varphi^{h,f}(z)$ for some integers $0 \leq n < \Re \frac{1}{3\alpha}$ and $0 \leq j < q$, we define

$$\rho^{h,f}(z) := w - n.$$ 

For any petal-exiting point $w \in S$ and any $0 \leq n < \Re \frac{1}{3\alpha}$ so that $z = \varphi^{h,f}(w) \in \text{Dom}(h^{nq})$, we define

$$\chi^{h,f} \left( w - \frac{1}{\alpha} + n \right) := h^{nq} \circ \varphi^{h,f}(z).$$ 

It follows from these definitions that $\rho^{h,f}$ and $\chi^{h,f}$ are analytic maps which semi-conjugate $h^q$ and $T_1$, and it follows from Theorem 2.2 that $\rho^{h,f} \to \rho^f$ and $\chi^{h,f} \to \chi^f$ when $h \to f$. The horn map for $h$ relative to $f$ is defined by

$$H^{h,f} := \rho^{h,f} \circ \chi^{h,f}.$$ 

We extend $H^{h,f}$ to a $T_1$-invariant domain using the functional equation $T_1 \circ H^{h,f} = H^{h,f} \circ T_1$.

Proposition 2.6. The horn map $H^{h,f}$ is both well-defined and analytic. There is some $\eta_0 > 0$ which does not depend on $h$ or $S$ so that the domain of $H^{h,f}$ contains \{ $w \in \mathbb{C} : \Im w > \eta_0$ \} when $h$ is close enough to $f$. There exist constants $c^h_{+}$ and $c^h_{-}$ such that $H^{h,f}(w) - w$ tends to $c^h_{\pm}$ when $\Im w$ tends to $\pm \infty$ respectively. Moreover, $H^{h,f} \to H^f$ when $h \to f$.

Proof. See [Shi00] and [Shi98, A.5]. \qed

While our definition of the horn map depends on our choice of $S$, it follows from the definition that for any $w$ in the domain the value of $H^{h,f}$ does not change when we vary the tilt of $S$ a small amount. Using the compactness of $[-1, 1]$, we can therefore restrict the domain of $H^{h,f}$ so that it does not depend on the tilt of $S$. Proposition 2.6 guarantees that $H^{h,f}$ still converges to $H^f$ when $h \to f$ after this restriction.

Our parameterization of the multiplier $h'(0)$ was made precisely so that we can compute $c^h_{+}$.

Proposition 2.7. $c^h_{+} = c^f_{+} = \frac{1 - \mathcal{G}(p/q)}{2} = \begin{cases} 0 & \text{if } \mathcal{G}(p/q) = +1, \\ 1 & \text{if } \mathcal{G}(p/q) = -1, \end{cases}$
Proof. It suffices to compute $c_{h,f}^+$. Let us fix a choice of $S$ and set $P_0 = \varphi_{h,f}(S)$. We denote by $S_{0,\mathrm{att}}$ and $S_{0,\mathrm{rep}}$ to be images under $\varphi_{h,f}$ of all the petal-entering and petal-exiting points in $S$ respectively. For all $0 \leq j < q$ we set $P_j, S_{j,\mathrm{att}},$ and $S_{j,\mathrm{rep}}$ to be the components of $h^{-j}(P_0), h^{-j}(S_{0,\mathrm{att}}),$ and $h^{-j}(S_{0,\mathrm{rep}})$ respectively which have 0 on their boundary. The sets $P_j$ are pair-wise disjoint Jordan domains by Theorem 2.2. We define Fatou coordinates on $P_j$ by
\[ \phi_{j,\mathrm{att}} = (\varphi_{h,f})^{-1} \circ h^j \quad \text{and} \quad \phi_{j,\mathrm{rep}} = T_{-1/\alpha} \circ (\varphi_{h,f})^{-1} \circ h^j. \]
We note that $\alpha \in A$ implies
\[ q\Re \mu_{p/q}(\alpha) - p > 0, \]
so orbits under $h^q$ near zero travel counter-clockwise around 0. Thus for $0 \leq m < q$ satisfying
\[ mp \equiv -1 \mod q \]
and taking the indices modulo $q$, the orbit of a point in $S_{0,\mathrm{rep}}$ close to zero must enter, in order, the sets
\[ S_{0,\mathrm{rep}}, S_{m,\mathrm{att}}, S_{m,\mathrm{rep}}, S_{2m,\mathrm{att}}, S_{2m,\mathrm{rep}}, \ldots, S_{qm,\mathrm{att}}. \]
Let $p'/q'$ denote the parent of $p/q$. Using Proposition 2.1 we can compute
\[ m = \frac{1 - \mathcal{G}(p/q)}{2}q + \mathcal{G}(p/q)q' = \begin{cases} \frac{q'}{q} & \text{if } \mathcal{G}(p/q) = +1, \\ \frac{q-q'}{q} & \text{if } \mathcal{G}(p/q) = -1. \end{cases} \]
Let $\hat{H}$ be the map in the repelling Fatou coordinate induced by the first non-trivial return to $S_{0,\mathrm{rep}}$ under $h^q$. More precisely, for any $z \in S_{0,\mathrm{rep}}$ and minimal integer $n > 0$ such that $h^nq(z) \in S_{0,\mathrm{rep}}$ and $h^{n'}q(z) \notin S_{0,\mathrm{rep}}$ for some $0 \leq n' < n$, we set
\[ \hat{H}(\phi_{0,\mathrm{rep}}(z)) = \phi_{0,\mathrm{rep}}(h^nq(z)) - n. \]
We can compute
\[ \lim_{\im w \to +\infty} \hat{H}(w) = w = -\frac{2\pi i}{\log(h^q)'(0)} = -\frac{1}{q\mu_{p/q}(\alpha) - p} = -\frac{q}{\alpha} - \mathcal{G}(p/q)q', \]
using the branch of log with imaginary part in $(-\pi, \pi]$. For all $0 \leq j < q$, let $H_j$ be the map in Fatou coordinates induced by the orbit under $h^q$ of points traveling from $S_{j,\mathrm{rep}}$ to $S_{j+m,\mathrm{att}}$. That is for any $z \in S_{j,\mathrm{rep}}$ and minimal integer $n > 0$ such that $h^nq(z) \in S_{j+m,\mathrm{rep}}$, we set
\[ H_j(\phi_{j,\mathrm{rep}}(z)) = \phi_{j+m,\mathrm{att}}(h^nq(z)) - n. \]
As $\phi_{j,\mathrm{rep}} = T_{-1/\alpha} \circ \phi_{j,\mathrm{att}}$ for all $j$, if $z \in S_{0,\mathrm{rep}}$ is sufficiently close to 0 and $w = \phi_{0,\mathrm{rep}}(z)$, then
\[ \hat{H}(w) = T_{-1/\alpha} \circ H_{(q-1)m} \circ \cdots \circ T_{-1/\alpha} \circ H_0(w). \]
Now let us fix some $z_0 \in S_{0,\mathrm{rep}}$ and set $w = \phi_{0,\mathrm{rep}}(z)$. For all $0 \leq j < q$, we let $z_j \in S_{j,\mathrm{rep}}$ be the point such that $h^j(z_j) = z_0$, so $\phi_{j,\mathrm{rep}}(z_j) = w$. We assume that $H_0(w)$ is defined, so there exists some minimal integer $n > 0$ such that $z_m := h^nq(z_0) \in S_{m,\mathrm{att}}$. We set $z'_m = h^m(z'_m) \in S_{0,\mathrm{att}}$, and for all $0 \leq j < q$ we let $z'_j \in S_{j,\mathrm{att}}$ be the point in $S_{j,\mathrm{att}}$ such that $h^{j'}(z'_j) = z'_m$, so $\phi_{j,\mathrm{att}}(z'_j) = \phi_{m,\mathrm{att}}(z'_m)$. If $m+j < q$, then
\[ h^{n+j}(z_j) = h^{nq}(z_0) = z'_m, \]
so $h^{nq}(z_j) = z'_{m+j}$ and
\[ H_j(w) = \phi_{j+m,\mathrm{att}}(z'_{m+j}) - n = \phi_{m,\mathrm{att}}(z'_m) - n = H_0(w). \]
If \( m + j \geq q \), then
\[
h^{(n+1)q}(z_j) = h^{(n+1)q-j}(z_0) = h^{(q-j)}(z'_m) = z'_{m+j-q},
\]
so
\[
H_j(w) = \phi_{m+j,\text{att}}(z'_{m+j-q}) - n - 1 = \phi_{m,\text{att}}(z'_m) - n - 1 = H_0(w) - 1.
\]
Note that there are exactly \( m \) choices of \( 0 \leq j < q \) such that \( m + j \geq q \).

Now we fix some \( z \in S_0,\text{rep} \) and \( w = \phi_0,\text{rep}(z) \) such that \( H^{h,f}(w) \) is defined. It follows from the definition of horn map that if \( z \) is close enough to 0 then
\[
H^{h,f}(w) = \phi_{0,\text{att}} \circ h^m \circ \phi_{m,\text{att}}^{-1} \circ H_0(w) = H_0(w).
\]

We can therefore compute
\[
-\frac{q}{\alpha} - \mathcal{G}(p/q)q' = \lim_{\text{Im } w \to +\infty} \tilde{H}(w) - w = -\frac{q}{\alpha} + \sum_{j=0}^{q-1} \left( \lim_{\text{Im } w \to +\infty} H_j(w) - w \right) = -\frac{q}{\alpha} - m + qc^{h,f}_+.
\]
Hence
\[
c^{h,f}_+ = \frac{m - \mathcal{G}(p/q)q'}{q} = \frac{1 - \mathcal{G}(p/q)}{2}.
\]

One of the key features of the horn map \( H^{h,f} \) is its relationship to high iterates of \( h \).

**Proposition 2.8.** If \( w, w' \in \text{Dom}(\varphi^{h,f}) \) satisfy
\[
w' = H^{h,f}\left(w - \frac{1}{\alpha}\right) + n
\]
for some integer \( n \), then there exist non-negative integers \( m, m' \) and \( 0 \leq j < q \) such that \( m - m' = n \) and
\[
h^{mq+j} \circ \varphi^{h,f}(w) = h^{mq'} \circ \varphi^{h,f}(w').
\]
If we can choose \( S \) so that \( w' \) is petal-exiting in \( S \) then \( m \cdot m' = 0 \). If \( |\text{Im } w| \) is sufficiently large, then \( m \cdot m' = 0 \) and
\[
j \equiv \begin{cases} -1 \mod q & \text{if } \text{Im } w > 0, \\ 0 \mod q & \text{if } \text{Im } w < 0. \end{cases}
\]

**Proof.** Fixing some choice of \( S \), the definition of the horn map implies that there exists some integers \( n_0, 0 \leq n_1 < \text{Re} \frac{2}{3\alpha} \), and \( 0 \leq j < q \) such that \( w + n_0 \) is petal-entering in \( S \) and
\[
H^{h,f}\left(w + n_0 - \frac{1}{\alpha}\right) = w_0 - n_1,
\]
where \( w_0 \) is petal-entering in \( S \) and
\[
h^{n_1q+j} \circ \varphi^{h,f}\left(w + n_0 - \frac{1}{\alpha}\right) = \varphi^{h,f}(w_0).
\]
Hence
\[
w' = w_0 + n - n_0 - n_1.
\]
As $\varphi^{h,f}$ semi-conjugates $T_1$ to $h^q$, it follows that if $n - n_0 - n_1 \geq 0$ then
\begin{align*}
  h^{mq+j} \circ \varphi^{h,f}(w) &= \varphi^{h,f}(w') \quad \text{if } n \geq 0 \text{ and } \\
  h^{mq+j} \circ \varphi^{h,f}(w) &= \varphi^{h,f}(w') \quad \text{if } n < 0.
\end{align*}

Similarly, if $n - n_0 - n_1 < 0$ then there exists some integer $0 \leq m \leq |n_0| + n_1$ so that
\begin{align*}
  h^{(n+m)q+j} \circ \varphi^{h,f}(w) &= h^m \circ \varphi^{h,f}(w') \quad \text{if } n \geq 0 \text{ and } \\
  h^{mq+j} \circ \varphi^{h,f}(w) &= h^{(m-n)q} \circ \varphi^{h,f}(w') \quad \text{if } n < 0.
\end{align*}

If we choose $S$ so that $w'$ is petal-entering, then the fact that $w_0$ is petal-entering combined with Proposition 2.5 implies that $n - n_0 - n_1 > 0$. As there are unique branches of $h^{-mq}$ near 0 and $\sigma^{h,f}$ which fix 0 and $\sigma^{h,f}$ respectively, if $|\text{Im } w|$ and $|\text{Im } w'|$ are both sufficiently large then we can apply $h^{-mq}$ to the above equations when $n - n_0 - n_1 < 0$. The computation of $j$ in this case follows from Part (4) of Theorem 1.1. \qed

2.4. Renormalization. We define the near-parabolic renormalization of $h$ relative to $f$ to be the the function
\[ R_f h := \text{Exp} \circ H^{h,f} \circ T_{-1/\alpha} \circ \text{Exp}^{-1}. \]

Proposition 2.6 implies that this map is defined on punctured neighborhoods of zero and infinity in $\mathbb{C}$ and can be continuously extended by setting $R_f h(0) = 0$ and $R_f h(\infty) = \infty$. Moreover, $R_f h \to R_{\delta} f$ when $h \to f$ and $n - 1/\alpha \to 1$. By Proposition 2.7, we can compute the multiplier $(R_f h)'(0) = \text{Exp}(-1/\alpha)$. The map $R_f h$ depends on our choice of petal normalization, but in the next section we will consider classes of maps which have a canonical choice of petal normalization. Just as for parabolic renormalization, we could instead consider the bottom near-parabolic renormalization
\[ w \mapsto \frac{1}{R_f h(1/z)}. \]

where the dynamics near-zero correspond to the dynamics of $h$ near $\sigma^{h,f}$. We will focus solely on the top near-parabolic renormalization $R_f h$; the bottom near-parabolic renormalization can be studied symmetrically.

While we have only defined the near-parabolic renormalization for $\alpha \in A$, we can similarly study the case where $-\alpha \in A$. Indeed, setting $f^*(z) = \overline{f(z)}$ and $h^* = \overline{h(z)}$ we can see that $f^*$ has a non-degenerate $-p/q$-parabolic fixed point at zero and the derivative at zero of $h^*$ is
\[ \text{Exp} \circ \mu_{-p/q}(-\alpha). \]

In this case we can define $R_f h = R_f h^*$ when $-\alpha \in A$ and $h$ is sufficiently close to $f$. We will restrict our attention to the $\alpha \in A$ case; the $-\alpha \in A$ case can be studied similarly.

2.5. Lavaurs maps. While we focus primarily on the parabolic and near-parabolic renormalizations, we end this section by presenting a common semi-conjugate formulation of parabolic implosion. A Lavaurs map for $f$ is defined to be a function of the form
\[ L^f_\delta := \chi^f \circ T_\delta \circ \rho^f \]
for some $\delta \in \mathbb{C}$. It follows from this definition that $\rho^f$ and $\chi^f \circ T_\delta$ both semi-conjugate the Lavaurs map $L^f_\delta$ to $H^f \circ T_\delta$. As an analogue of Theorem 2.8, the following result shows how the Lavaurs map relates to high iterates of $h$. 

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Theorem 2.9. For any sequence of integers \( \langle k_n \rangle_{n=1}^\infty \), if \( h = h_n \) and \( \alpha = \alpha_n \) satisfy \( h_n \to f \) and \( k_n - 1/\alpha_n \to \delta \) for some \( \delta \in \mathbb{C} \) when \( n \to \infty \), then \( h_n^{k_n} \to L_f^\delta \) when \( n \to \infty \).

Proof. See [Dou94, Proposition 18.2].  

3. Invariant classes

Let us fix some integer \( d \geq 2 \) and consider the class \( \mathcal{F} \) of all analytic maps \( f : \text{Dom}(f) \to \hat{\mathbb{C}} \) satisfying

1. \( \text{Dom}(f) \) is an open subset of \( \mathbb{C} \) containing \( \{0, \infty\} \);
2. \( f(0) = 0, f(\infty) = \infty, \) and \( f'(0) = 1; \) and
3. the restriction \( f : f^{-1}(\mathbb{C}^*) \to \mathbb{C}^* \) is a branched covering whose unique critical value is 1, and all critical points are of local degree \( d \).

For example, \( \mathcal{F} \) contains the unicritical polynomial

\[
G(z) := 1 - \left( \frac{d - z}{d} \right)^d.
\]

For any analytic map \( f \) and any \( w \in \mathbb{C} \), we denote by \( f \times w \) the map \( z \mapsto f(\text{Exp}(w) \cdot z) \). For any class of analytic maps \( \mathcal{G} \) we similarly denote \( \mathcal{G} \times w := \{ f \times w : f \in \mathcal{G} \} \).

We fix some \( f_0 \in \mathcal{F} \), some rational \( p/q \in [-1/2, 1/2] \), and set \( f = f_0 \times p/q \).

Proposition 3.1. \( f \) has a non-degenerate \( p/q \)-parabolic fixed point at zero.

Proof. By construction \( f'(0) = \text{Exp}(p/q) \). The uniqueness of the critical value of \( f \) implies that the parabolic fixed point is non-degenerate, see for example [Mil06] or [Shi00].  

For any choice of \( P^f_{\text{att}} \), the parabolic basin of \( f \) is the set

\[
B^f = \bigcup_{m \geq 0} f^{-m}(P^f_{\text{att}}).
\]

We denote by \( B^f_0 \) the component of \( B^f \) containing the unique critical value of \( f \), called the immediate parabolic basin of \( f \).

Proposition 3.2. The restriction of \( f^q \) to \( B^f_0 \) is analytically conjugate to the restriction of \( G \) to \( B^G_0 \).

In particular, Proposition 3.2 implies that there is a unique critical point \( c_{p^f} \) of \( f^q \) contained in \( B^f_0 \). We now fix another holomorphic map \( h \) close to \( f \). If \( h \) is close enough to \( f \), then \( h^q \) has a unique critical point \( c_{p^h,f} \) close to \( c_{p^f} \). Conjugating by a linear map, we can ensure that \( 1 := h^q(c_{p^h,f}) \).

Let us now assume that \( h(0) = 0 \) and \( h'(0) = \text{Exp} \circ \mu_{p/q}(\alpha) \) for some \( \alpha \in A = A(r, C) \).

Proposition 3.3. For any sufficiently large \( M \geq 1 \), if \( h \) is close enough to \( f \) then we can take the petal normalization \((1, 0, M)\) for \( \varphi^{h,f} \).

Let \( M_0 \) be the infimum over all \( M \geq 1 \) such that we can take the petal normalization \((1, 0, M')\) for any \( h \) close to \( f \) and all \( M' \geq M \). We define the canonical petal normalization relative to \( f \) to be \((1, 0, M_0 + 1)\). For the rest of this article we will always make this canonical choice of \( \varphi^{h,f} \); we denote \( \Omega^{h,f} = \text{Dom}(\varphi^{h,f}) \).
We define the \((p/q, 0)\)-fiber renormalization of \(f_0\) to be the map
\[
\mathcal{R}_{p/q,0}f_0 := \mathcal{R}_f = \text{Exp} \circ H_f \circ \text{Exp}^{-1}.
\]
We will also call \(\mathcal{R}_{p/q,0}f_0\) a \(p/q\)-parabolic fiber renormalization. For \(h = f_0 \Join \mu_{p/q}(\alpha)\) with \(\alpha \in \mathcal{A}\), we similarly define the \((p/q, \alpha)\)-fiber renormalization of \(f_0\) to be the map
\[
\mathcal{R}_{p/q,\alpha}f_0 := (\mathcal{R}_h f) \Join \frac{1}{\alpha} = \text{Exp} \circ H_{f,h} \circ \text{Exp}^{-1}.
\]
We will also call \(\mathcal{R}_{p/q,\alpha}f_0\) a near-\(p/q\)-parabolic fiber renormalization. We consider these fiber renormalization operators as acting on \(f_0\) instead of \(f\) or \(h\) as the functions \(f_0, \mathcal{R}_{p/q,0}f_0,\) and \(\mathcal{R}_{p/q,\alpha}f_0\) all have 0/1-parabolic fixed points at zero. Our first observation is that the class \(\mathcal{F}\) is invariant under \(\mathcal{R}_{p/q,0}f_0\).

**Proposition 3.4.** \(\mathcal{R}_{p/q,0}f_0 \in \mathcal{F}\).

**Proof.** The proof in the case \(p/q = 0/1\) and \(d = 2\) is given in [Shi00], the argument in the general case is identical. \(\square\)

It is proved in [DH85] that \(B^G\), the parabolic basin of the polynomial \(G\), is a Jordan domain. Consequently, \(\text{Dom}(\mathcal{R}_0G)\) is the union of two open Jordan domains, one containing 0 and the other containing \(\infty\) (see Figure 2). Let \(\text{Dom}_0(\mathcal{R}_0G)\) be the connected component of \(\text{Dom}(\mathcal{R}_0G)\) containing 0; by the Riemann mapping theorem there exists some \(R_G > 0\) and...
univalent map \( \phi_G : D_{R_G} \to \text{Dom}_0(R_0 G) \) satisfying \( \phi_G(0) = 0 \) and \( \phi'_G(0) = 1 \). For \( 0 \leq \epsilon < 1 \), let \( S_\epsilon \) be the set of univalent maps \( \phi : D_{(1-\epsilon)R_G} \to \mathbb{C} \) satisfying \( \phi(0) = 0 \) and \( \phi'(0) = 1 \). We define the class of maps

\[
\mathcal{F}_\epsilon := \{ R_0 G \circ \phi_G \circ \phi^{-1} : \phi \in S_\epsilon \}.
\]

It follows from the Koebe distortion theorem that \( S_\epsilon \), and consequently \( \mathcal{F}_\epsilon \), is compact with respect to the compact-open topology for all \( 0 \leq \epsilon < 1 \).

By abuse of notation, for any classes \( G, G' \) of analytic maps we will write \( G \subset G' \) if every map in \( G \) has a restriction in \( G' \). In particular, \( \mathcal{F}_{\epsilon'} \subset \mathcal{F}_\delta \) for any \( 0 \leq \epsilon' \leq \epsilon \).

**Proposition 3.5.** \( \mathcal{R}_{p/q,0}(\mathcal{F}) \subset \mathcal{F}_0 \subset \mathcal{F} \).

*Proof.* For \( p/q = 0/1 \), this is proved in [Shi00] and [LY14]. The same argument can be applied for general \( p/q \). \( \square \)

While \( \mathcal{F}_\epsilon \) is not contained in \( \mathcal{F} \) for \( \epsilon > 0 \), if \( \epsilon \) is sufficiently small then \( z = 1 \) is the unique critical value of every \( f_0 \in \mathcal{F}_\epsilon \). For any rational \( p/q \), if \( \epsilon > 0 \) is sufficiently small then we can define the canonical petal normalization for \( f_0 \propto p/q \) with \( f_0 \in \mathcal{F}_\epsilon \) identically to the \( f_0 \in \mathcal{F}_0 \) case. Hence the fiber renormalizations \( \mathcal{R}_{p/q,0} f_0 \) is defined for sufficiently small \( \alpha \in A \cup \{0\} \).

The class \( \mathcal{F}_\epsilon \) is also invariant under parabolic fiber renormalization:

**Theorem 3.6.** For any rational \( p/q \in [-1/2, 1/2] \), if \( \epsilon > 0 \) is sufficiently small then there exists \( 0 < \epsilon' < \epsilon \) satisfying

\[
\mathcal{R}_{p/q,0}(\mathcal{F}_\epsilon) \subset \mathcal{F}_{\epsilon'}.
\]

*Proof.* For \( p/q = 0/1 \), this theorem is the main result in [Ché22]. For general \( p/q \) the same argument can be applied. \( \square \)

The key feature of Theorem 3.6 is that we can pick \( \epsilon' < \epsilon \). As a consequence, the following lemma shows that the parabolic fiber renormalization operators are contracting on \( \mathcal{F}_\epsilon \):

**Proposition 3.7.** For any \( 0 < \epsilon' < \epsilon < 1 \) there exists a complete metric \( d \) on \( \mathcal{F}_{\epsilon'} \) such that if \( \mathcal{R} : \mathcal{F}_\epsilon \to \mathcal{F}_{\epsilon'} \) is a holomorphic operator, then

\[
d(\mathcal{R}(f_1), \mathcal{R}(f_2)) < \frac{1 - \epsilon}{1 - \epsilon'} d(f_1, f_2)
\]

for all \( f_1, f_2 \in \mathcal{F}_{\epsilon'} \). Moreover, convergence in this metric implies convergence in the compact-open topology.

*Proof.* See [IS08, Main Theorem 2]. \( \square \)

Moreover having \( \epsilon' < \epsilon \) in Theorem 3.6 allows us to perturb and produce classes invariant under \( \mathcal{R}_{p/q,\alpha} \) for \( \alpha \neq 0 \).

**Theorem 3.8.** For any rational \( p/q \in [-1/2, 1/2] \) and any \( C > 0 \), if \( \epsilon > 0 \) is sufficiently small then there exist \( r > 0 \) and \( 0 < \epsilon' < \epsilon \) such that

\[
\mathcal{R}_{p/q,\alpha}(\mathcal{F}_\epsilon) \subset \mathcal{F}_{\epsilon'}
\]

for all \( \alpha \in A(r, C) \).

*Proof.* For \( p/q = 0/1 \) this is proved in [IS08] for a slightly different class of maps, the same argument can be applied in this setting. \( \square \)
As a consequence, we can again apply Proposition 3.7 and see that the fiber renormalizations are contracting on \( \mathcal{F}_r \).

From the near-parabolic fiber renormalization operators, we can recover the near-parabolic renormalization operators. For any rational \( p/q \in [-1/2, 1/2], \ C > 0, \) sufficiently small \( \varepsilon > 0, \) and sufficiently small \( r > 0 \) we can define the near-\( p/q \)-parabolic renormalization \( \mathcal{F}_r \cong \mu_{p/q}(A(r, C)) \) to be the operator

\[
\tilde{f} \mapsto \mu_{p/q}(\alpha) \mapsto (\mathcal{R}_{p/q, \alpha} f) \times (-1/\alpha).
\]

It follows from Theorem 3.8 that this operator is hyperbolic.

4. Comparing Renormalizations

Fixing some rational \( p_0/q_0, p_1/q_1 \in [-1/2, 1/2], \) integer \( n_1 \gg 0, \) and \( \alpha_1 \in A = A(r, C), \) let \( h_0 \) be an analytic map satisfying

\[
h_0(0) = 0 \text{ and } h'_0(0) = \text{Exp} \circ \mu_{p_0/q_0}(\alpha_0),
\]

where

\[
\alpha_0 = \frac{1}{n_1 - \mu_{p_1/q_1}(\alpha_1)}.
\]

Thus if \( h_0 \) is sufficiently close to some \( f_0 \in \mathcal{F} \times p_0/q_0, \) then the near-parabolic renormalization \( h_1 := \mathcal{R}_{f_0} h_0 \) is defined. If additionally \( \alpha_1 \) is close to zero then \( h_0 \) is close to \( f_1 := \mathcal{R}_{p_1/q_1} f_0. \)

Setting

\[
p/q = \mu_{p_0/q_0}\left(\frac{1}{n_1 - p_1/q_1}\right),
\]

if \( h_0 \) is sufficiently close to some \( g_0 \in \mathcal{F} \times p/q \) then the near-parabolic renormalization \( \mathcal{R}_{g_0} h_0 \) is also defined. Fixing such a \( g_0, \) we denote \( g_1 = \mathcal{R}_{f_0} g_0. \)

**Theorem 4.1.** There exists a compact set \( X \subset \mathbb{C} \) and an integer \( M \geq 0 \) such that if \( h_0 \) is sufficiently close to \( f_0 \) and if \( h_1 \) is sufficiently close to \( f_1, \) then there are functions

\[
\varphi^{h_1, f_1} : \Omega^{h_1, f_1} \to \text{Dom}(h_0^{-Mq_0}) \text{ and }
\]

\[
\Phi^{h_1, f_1} : \Omega^{h_1, f_1} \to \Omega^{h_0, f_0} \cup X
\]

such that:

1. \( \varphi^{h_1, f_1} \) is analytic and

\[
\varphi^{h_1, f_1}(w) = \begin{cases} 
\varphi^{h_0, f_0} \circ \Phi^{h_1, f_1}(w) & \text{if } \Phi^{h_1, f_1}(w) \in \Omega^{h_0, f_0}, \\
h_0^{-Mq_0} \circ \varphi^{h_0, f_0} \circ T_M \circ \Phi^{h_1, f_1}(w) & \text{if } \Phi^{h_1, f_1}(w) \in X
\end{cases}
\]

for a continuous branch of \( h_0^{-Mq_0}. \)

2. \( \varphi^{h_1, f_1}(0) = 1; \)

3. for \( w \) close to 0, \( \text{Exp} \circ \Phi^{h_1, f_1}(w) = \varphi^{h_1, f_1}(w); \)

4. \( \varphi^{h_1, f_1} \circ T_1 = h_0^\alpha \circ \varphi^{h_1, f_1}; \)

5. if \( h_0 \) is sufficiently close to \( g_0 \) then \( \varphi^{h_1, f_1} = \varphi^{h_0, g_0} \) on \( \Omega^{h_1, f_1} \); and

6. for any compact set \( Y \subset \text{Dom}(H^{h_1}), \) if \( g_0 \) is sufficiently close to \( f_0 \) and if \( h_0 \) is sufficiently close to \( g_0, \) then \( H^{h_1, f_1}(w) - H^{h_0, g_0}(w) \in \mathbb{Z} \) for all \( w \in Y. \)
Proof. Let us fix some $\eta > 0$ so that

$$\left| H^{f_0}(w) - (w + c_1^{f_0}) \right| < \frac{1}{4q_1}$$

when $\text{Re } w > \eta$. For all integers $m$ we define

$$W_m := \{ w \in \mathbb{C} : mn_1 \leq \text{Re } w \leq (m+1)n_1 \} \cap \Omega_2((\eta + 9)i, n_1 + (\eta + 9)i).$$

Note that we can choose $\eta$ independent of $h_0$ so that $W_0 \subset \Omega^{h_0,f_0}$.

Increasing $\eta$ if necessary, we can assume that if $h_0$ and $h_1$ are sufficiently close to $f_0$ and $f_1$ respectively then for any integer $m$ there exists a unique continuous branch of $(H^{h_0,f_0} \circ T^{-1}_{-\frac{i}{\alpha_0}})^m$ defined on $W_m$ satisfying

$$\left| \left( H^{h_0,f_0} \circ T^{-1}_{-\frac{i}{\alpha_0}} \right)^m(w) - (w - n_1) \right| < 3$$

for all $w \in W_m$. Using these branches, we now observe that

$$(H^{h_0,f_0} \circ T^{-1}_{-\frac{i}{\alpha_0}})^m(W_m) \subset \Omega^{h_0,f_0}$$

for any integer $m$. Indeed the above inclusion holds automatically for $m = 0$, and for $m > 0$ we have

$$\text{Im } w > \eta + 9 + \frac{n_1}{2}(|m| - 1) - 3|m|$$

and

$$-3|m| < \text{Re } w < n_1 + 3|m|.$$
for all \( w \in (H^{ho,fo} \circ T_{-\frac{1}{\alpha_0}})^m(W_m) \), hence
\[
(H^{ho,fo} \circ T_{-\frac{1}{\alpha_0}})^m(W_m) \subset \Omega_{\frac{1}{2}}(\eta_i, n_1 + \eta_i) \subset \Omega^{ho,fo}
\]
provided \( \eta \) was chosen sufficiently large and \( n_1 \gg 0 \).

Let us now fix some maximal strip \( S \) in \( \Omega^{h_1,f_1} \) and let \( \tilde{S} \) be a component of \( \text{Exp}^{-1}(\varphi^{h_1,f_1}(S)) \).

For any \( w \in S \) with \( w' := w + 1 \in S \) and \( \tilde{w}, \tilde{w}' \in \tilde{S} \) with \( \text{Exp}(\tilde{w}) = \varphi^{h_1,f_1}(w) \), \( \text{Exp}(\tilde{w}') = \varphi^{h_1,f_1}(w') \), it follows from the definition of \( f_1 \) that there exists an integer \( \ell \) such that
\[
T_{1} \circ (T_{n_1} \circ H^{ho,fo} \circ T_{-\frac{1}{\alpha_0}})^{q_1}(\tilde{w}) = \tilde{w}'.
\]
This integer depends continuously on \( w \), hence it is constant for all choices of \( w \). Let \( \tilde{\sigma}^{h_1,f_1} \) be the unique element of \( \text{Exp}^{-1}(\sigma^{h_1,f_1}) \) on the boundary of \( \tilde{S} \), so \( \tilde{w} \) and \( \tilde{w}' \) both converge towards \( \tilde{\sigma}^{h_1,f_1} \) when \( \text{Im} w \to -\infty \). If \( h_1 \) is sufficiently close to \( f_1 \) then \( \text{Im} \tilde{\sigma}^{h_1,f_1} > \eta \), so we must have \( \ell = -q_1 c_+^{-} - p_1 \). Proposition 2.8 implies that if both \( \tilde{w} \) and \( \tilde{w}' \) belong to \( \Omega^{ho,fo} \) then
\[
h_0^{(\ell + n_1 q_0 + q_1)} \circ \varphi^{ho,fo}(\tilde{w}) = \varphi^{ho,fo}(\tilde{w}'),
\]
where \( 0 \leq s < q \) satisfies \( s \equiv -1 \mod q_0 \). Letting \( p_0/q_0 \) denote the parent of \( p_0/q_0 \), we can compute
\[
s = \begin{cases} 
q_0 & \text{if } \mathcal{S}(p_0/q_0) = +1, \\
q_0 - q_0' & \text{if } \mathcal{S}(p_0/q_0) = -1.
\end{cases}
\]
Using Proposition 2.7 and the definition of \( p/q \), we can compute
\[
(\ell + n_1 q_0 + q_1) q_0 + q_1 = n_1 q_0 q_1 - p_1 q_0 + q_1 q_0' \mathcal{S}(p_0/q_0) = q.
\]

Proposition 2.4 implies that if \( h_1 \) is sufficiently close to \( f_1 \) then we can choose \( \tilde{S} \) so that
\[
\tilde{S} \subset \left\{ w \in \mathbb{C} : -1 < \text{Re } w < \text{Re } \frac{1}{\alpha_0} - M_0 \right\} \cup \left( \bigcup_{m \in \mathbb{Z}} W_m \right),
\]
where \((1, 0, M_0)\) is the canonical petal normalization for perturbations of \( f_0 \). For any \( w \in S \) and \( \tilde{w} \in \tilde{S} \) with \( \text{Exp}(\tilde{w}) = \varphi^{h_1,f_1}(w) \), we define
\[
\Phi^{h_1,f_1}(w) := \begin{dcases} 
\tilde{w} & \text{if } -1 < \text{Re } \tilde{w} < \text{Re } \frac{1}{\alpha_0} - M_0, \\
(H^{ho,fo} \circ T_{-\frac{1}{\alpha_0}})^m(\tilde{w}) & \text{if } \tilde{w} \in W_m,
\end{dcases}
\]
\[
\tilde{\varphi}^{h_1,f_1}(w) := \varphi^{ho,fo} \circ \Phi^{h_1,f_1}(w).
\]
While the map \( \Phi^{h_1,f_1} \) is discontinuous, it follows from Proposition 2.8 that \( \tilde{\varphi}^{h_1,f_1} \) is analytic on \( \Omega^{h_1,f_1} \). It follows from the above and analytic continuation that \( \tilde{\varphi}^{h_1,f_1} \) semi-conjugates \( T_1 \) to \( h_0^q \). Varying the tilt of \( S \) analytically extends \( \tilde{\varphi}^{h_1,f_1} \) to all of \( \Omega^{f_1,h_1} \). As \( \varphi^{h_1,f_1}(0) = 1 \), \( \Phi^{h_1,f_1}(0) \) is an integer. Translating \( \tilde{S} \) by an integer, we can ensure \( \Phi^{h_1,f_1}(0) = 0 \); however let us note that after this modification we are no longer guaranteed that the image of \( \Phi^{h_1,f_1} \) is contained in \( \Omega^{ho,fo} \). Proposition 2.4 guarantees that there is some compact set \( X \subset \mathbb{C} \) which does not depend on \( h_0 \) such that the image of \( \Phi^{h_1,f_1} \) is contained in \( X \cup \Omega^{ho,fo} \). Thus we can extend \( \varphi^{ho,fo} \) to the image of \( \Phi^{h_1,f_1} \) so that \( \tilde{\varphi}^{h_1,f_1} \) keeps the desired properties.

If \( h_0 \) is sufficiently close to \( g_1 \), which forces \( \eta_1 \) to be small, then we can choose \( S \) so that \( \tilde{S} \) is contained in an infinite strip in \( \Omega^{ho,fo} \). Thus \( \tilde{\varphi}^{h_1,f_1} \) maps \( S \) univalently onto a petal for \( h_0^q \) which contains \( h_0^{m_0,q_0}(cv^{f_0}) \); it follows from the uniqueness in Part (3c) of Theorem 2.2 that
\[ \tilde{\varphi}^{h_1,f_1} = \varphi^{h_0,g_0} \] on \( S \) in this case, so the same holds on \( \text{Dom}(\tilde{\varphi}^{h_1,f_1}) \) by analytic continuation. For any compact set \( Y \subset \text{Dom}(H^{f_1}) \) the horn map \( H^{f_1} \) is induced by finitely many iterates of \( f_1 \). These iterates of \( f_1 \) are induced by finitely many iterates of \( T_k \circ H^{f_0} \circ T_\delta \), where the integer \( k \) is allowed to vary in some uniformly bounded set for each iterate. Thus the horn map \( H^{h_1,f_1} \) is similarly induced by finitely many iterates of \( h_1 \) when \( h_1 \) is close to \( f_1 \), which are induced by finitely many iterates of \( T_k \circ H^{h_0,f_0} \circ T_{n_1-1/\alpha_1} \) when \( h_0 \) is close to \( f_0 \). Proposition 2.8 implies that this corresponds to finitely many iterates of \( \tilde{\varphi}^{(n_1+k)q_0+j} \), and the fact that \( k \) and \( j \) are uniformly bounded across these iterates implies \((n_1+k)q_0+j < q \) when \( h_0 \) is close to \( f_0 \). If \( h_0 \) is sufficiently close to \( g_0 \), so \( \tilde{\varphi}^{h_1,f_1} = \varphi^{h_0,g_0} \), it follows that the finitely many iterates of \( h_0^{(n_1+k)q_0+j} \) which induce \( H^{h_1,f_1} \) similarly induce \( H^{h_0,g_0} \). Note that it follows from our definition of the extended attracting Fatou coordinates \( \rho^{h_1,f_1} \) and \( \rho^{h_0,g_0} \) that the corresponding horn maps may differ by an integer; in particular the number of iterates of \( h_0^q \) which induce \( H^{h_1,f_1} \) is not necessarily the same number of iterates of \( h_0^q \) which induce \( H^{h_0,g_0} \). \( \square \\

**Corollary 4.2.** For any compact set \( X \subset \text{Dom}(R_{0,f_1}) \), if \( g_0 \) is sufficiently close to \( f_0 \) then \( R_{0,g_0} = R_{0,g_1} \) on \( X \).

**Proof.** Let us assume \( \frac{1}{\alpha_1} \) is a positive integer, so \( R_{g_0,h_0} \to R_{0,g_0} \) and \( R_{g_1,h_1} \to R_{0,g_1} \) when \( h_0 \to g_0 \). As \( R_{g_0,h_0} = R_{g_1,h_1} \) on \( X \) when \( g_0 \) is sufficiently close to \( f_0 \) by Proposition 4.1, the corollary follows immediately. \( \square \\

It follows from Proposition 4.1 that we can view \( \tilde{\varphi}^{h_1,f_1} \) and \( R_{f_1,h_1} \) as extensions of \( \varphi^{h_0,g_0} \) and \( R_{g_0,h_0} \) respectively to the case where \( h_0 \) is not close to \( g_0 \). One subtle yet important aspect of Proposition 4.1 is that how close \( h_1 \) needs to be to \( f_1 \) does not depend on how close \( h_0 \) is to \( f_0 \). As a consequence, we get the following uniformity for certain fiber renormalizations.

**Corollary 4.3.** For any \( N \geq 0 \), we can analytically extend the definition of \( R_{p/q,\alpha} \) so that the constants \( r, C, \delta, \) and \( \delta' \) in Theorem 3.8 are uniform over all \( p/q \in \mathbb{Q}_N \).

**Proof.** Keeping our notation from earlier in this section and fixing some \( f \in \mathcal{F} \), let us assume

\[
 f_0 = f \times p_0/q_0, \quad g_0 = f \times p/q, \quad \text{and} \quad h_0 = f \times \mu_{p_0/q_0}(\alpha_0).
\]

Thus

\[
 (R_{g_0,h_0}) \times \frac{1}{\alpha_1} = R_{p/q,\alpha_1} f \quad \text{and} \quad (R_{f_0,h_0}) \times \frac{1}{\alpha_0} = R_{p/q,\alpha_0} f.
\]

Unpacking the definitions, if \( \alpha_0 \) is sufficiently small then

\[
 R_{f_1,h_1} = R_{(R_{p_0/q_0,\alpha_0} f) \times p_1/q_1} ((R_{p_0/q_0,\alpha_0} f) \times \mu_{p_1/q_1}(\alpha_1))
\]

\[
 = R_{(R_{p_0/q_0,\alpha_0} f) \times p_1/q_1} ((R_{p_0/q_0,\alpha_0} f) \times \mu_{p_1/q_1}(\alpha_1))
\]

\[
 = (R_{p_1/q_1,\alpha_1} R_{p_0/q_0,\alpha_0} f) \times \frac{1}{\alpha_1}.
\]

Hence

\[
 R_{p/q,\alpha_1} f = R_{p_1/q_1,\alpha_1} R_{p_0/q_0,\alpha_0} f
\]
on any compact subset of $\text{Dom}(R_{p/q_1}f_1)$ when $\alpha_0$ and $\alpha_1$ are sufficiently small. It follows that when $n_1$ is sufficiently large, we can extend the definition of $R_{p/q,\alpha}$ so that the constants $r, C, \delta$, and $\delta'$ depend only on $p_0/q_0$ and $p_1/q_1$.

For $N > 0$ and any sequence $\{p_k/q_k\}_{k=1}^{\infty}$ in $\mathbb{Q}_N$ with respective modified continued fractions $\{(a_{n,k}, \varepsilon_{n,k})\}_{n=1}^{\infty}$, either $\{p_k/q_k\}_{k=1}^{\infty}$ is contained in a finite subset of $\mathbb{Q}_N$ or there exist some integer $1 \leq m \leq N$ so that $a_{m,k} \to \infty$ when $k \to \infty$. By induction on $N$, the above implies that the constants $r, C, \delta$, and $\delta'$ can be chosen uniformly over the entire sequence, hence these constants can be chosen uniformly over $\mathbb{Q}_N$. □

A natural question which arises from Corollary 4.3 is whether the constants in Theorem 3.8 can be chosen uniformly over all rational $p/q$. To answer this question, we would need to understand the behavior of the fiber renormalizations $R_{p/q,0}$ and $R_{p/q,0}$ when $p/q$ tends toward an irrational number.

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