I. INTRODUCTION

Measuring top quark properties is one of the major tasks at the Large Hadron Collider (LHC). In particular, the large amount of $t\bar{t}$ events enables the search for exotic decay modes of the top quark. In line with this, top quark decays via flavor-changing neutral current (FCNC) decays of the top quark induced by a $Z'$ boson, namely $t \to c Z'$, based on a model of the gauged $L_\mu - L_\tau$ symmetry (the difference between the muon and tauon numbers) with vector-like quarks, which was introduced to explain the anomalous LHCb data. We illustrate that searching for $t \to c Z'$ via $Z' \to \mu^+ \mu^-$ with LHC Run 1 data can already probe a parameter region which is unexplored by B physics for the $Z'$ mass around $O(10)$ GeV or more. We further extend the model to very light $Z'$ with mass below 400 MeV, which is motivated by the muon $g - 2$ anomaly. Taking rare $B$ and $K$ meson decay data into account, we give upper limits on the $t \to c Z'$ branching ratio for the light $Z'$ case, and discuss about its observability at the LHC. We also scrutinize the possibility that the decay $K_L \to \pi^0 Z'$ with $Z' \to \nu \bar{\nu}$ may lead to apparent violation of the usual Grossman–Nir bound of $\mathcal{B}(K_L \to \pi^0 \nu \bar{\nu}) < 1.4 \times 10^{-9}$.

PACS numbers: 13.20.Eb 13.20.He 14.65.Ha 14.70.Pw
decays. We have pointed out in Ref. [22] that such a $Z'$ with mass around $m_{Z'}$ can evade $K^+ \to \pi^+ Z'$ searches, but cause $K_L \to \pi^0 Z' (\to \nu \bar{\nu})$ with rate exceeding the commonly perceived Grossman–Nir (GN) bound [23] of $B(K_L \to \pi^0 \nu \bar{\nu}) < 1.4 \times 10^{-9}$. Note that a very light $Z'$ of $L_{\mu} - L_{\tau}$ might explain [22] the PeV-scale cosmic neutrino spectrum observed by IceCube [25].

In this paper, we investigate how large the $t \to c Z'$ decay rate can be, based on the gauged $L_{\mu} - L_{\tau}$ model of Ref. [15]. We consider the two well-motivated $Z'$ mass ranges: (i) the heavy $Z'$ scenario with $m_b \lesssim m_z < m_t - m_e$, which is motivated by the $P_L^f$ and $R_K$ anomalies; (ii) the light $Z'$ scenario with $m_{Z'} \lesssim 400$ MeV, which is motivated by the muon $g - 2$ anomaly. The former was already sketched in Ref. [15]. It was pointed out that the right-handed $t c Z'$ coupling is unconstrained from $B$ and $K$ meson data and can lead to the $t \to c Z'$ decay with $\sim 1\%$ branching ratio. We revisit their result by updating $b \to s$ transition data with a correction to the $t \to c Z'$ formula. On the other hand, Scenario (ii) was not studied in Ref. [15], but clearly exhibits rather different phenomenology compared to Scenario (i). As the on-shell $Z'$ could be produced by $B$ and $K$ meson decays, the meson decay rates could be hugely enhanced, and even the right-handed $t c Z'$ coupling is constrained by data at one-loop level. Scenario (ii) is further divided into two categories: (ii-a) $2m_b < m_{Z'} < 400$ MeV; (ii-b) $m_{Z'} < 2m_b$. In Scenario (ii-b), the $Z'$ decays only into neutrinos, rendering $t \to c Z'$ searches at the LHC difficult, but it gives interesting implications for rare kaon decays as mentioned above.

This paper is organized as follows. We recapitulate in Sec. II the model of Ref. [15], then study $t \to c Z'$ in Scenario (i) for heavy $Z'$ motivated by the $b \to s$ anomalies. We then discuss the observability of $t \to c Z'$ decay at the LHC with $Z' \to \mu^+ \mu^-$. In Sec. III and IV we study $t \to c Z'$ for light $Z'$ motivated by the muon $g - 2$ anomaly. We consider Scenario (ii-a) in Sec. III where the $Z'$ mass is above dimuon threshold. We give formulas for FCNC $B$ and $K$ decays, collect relevant rare $B$ and $K$ decay data, then give upper limits on $t \to c Z'$ branching ratio. In Sec. IV, we consider Scenario (ii-b) where the $Z'$ is below the dimuon threshold. After giving upper limits on $t \to c Z'$ branching ratios, we discuss a special implication for rare kaon decay experiments, expanding the discussion of Ref. [22]. Sec. V is devoted to discussion and conclusions. In Appendix A we give the decay distribution for $B^0 \to K^{*0} Z' \to K \pi \mu^+ \mu^-$ four-body decay to estimate the efficiency at LHCB. In Appendix B loop functions used in our analysis are given.

### II. $P_L^f$- and $R_K$-Motivated $Z'$

#### A. Model

We first recapitulate the model introduced in Ref. [15]. A new Abelian gauge group $U(1)'$ is introduced that gauges the $L_{\mu} - L_{\tau}$ symmetry. This $U(1)'$ symmetry is spontaneously broken by the vacuum expectation value (v.e.v.) of a scalar field $\Phi$, which is charged under $U(1)'$ but singlet under the SM gauge group. The mass of $Z'$ is given then by $m_{Z'} = g' v_{\Phi}$, where $g'$ is the $U(1)'$ gauge coupling and $v_{\Phi} = \sqrt{\langle \Phi \rangle}$ is the v.e.v. of $\Phi$. We adopt the convention where the covariant derivative acting on $\Phi$ is given by $D_{\alpha} = \partial_{\alpha} + i g' Q_{\Phi} Z'_{\alpha}$, with $Q_{\Phi} = +1$ the $U(1)'$ charge of $\Phi$. The $Z'$ decays to leptons via

$$\mathcal{L} = - g' Z'_{\alpha} (\bar{\mu} \gamma^\alpha \mu + \bar{\nu}_L \gamma^\alpha \nu_L - \bar{\tau} \gamma^\alpha \tau - \bar{\nu}_{\tau L} \gamma^\alpha \nu_{\tau L}) .$$

We set the kinetic mixing between the $U(1)'$ and $U(1)_Y$ gauge bosons to be zero throughout this paper.

In order to induce the effective $Z'$ couplings to the SM quark currents, new vector-like quarks, which mix with the SM quarks, are introduced: $Q_L = (\bar{U}_L, D_L, U_R, D_R)$, which replicates one generation of SM quarks, and chiral partners $Q_R = (\bar{U}_R, D_R)$. Unlike the SM quarks, the new vector-like quarks are charged under $U(1)'$, with charges $Q_Q = +1$ for $Q = Q_L + Q_R$ and $Q_Q' = Q_D = -1$ for $U \equiv \bar{U}_L + U_R$ and $D \equiv \bar{D}_L + D_R$. The mass term for the vector-like quarks is given by

$$-\mathcal{L}_m = m_Q Q Q + m_{UU} \bar{U} U + m_D \bar{D} D,$n

where the three mass parameters are taken to be real without loss of generality. The Yukawa mixing term between the vector-like quarks and SM quarks is given by

$$-\mathcal{L}_{\text{mix}} = \Phi \sum_{i=1}^{3} \left( \bar{U}_R Y_{Qu_i} u_{iL} + \bar{D}_R Y_{Qd_i} d_{iL} \right)$$

$$+ \Phi' \sum_{i=1}^{3} \left( \bar{U}_L Y_{Uu_i} u_{iR} + \bar{D}_L Y_{Dd_i} d_{iR} \right) + \text{h.c.}$$

Here, $SU(2)_L$ symmetry imposes

$$Y_{Qu_i} = \sum_{j=1}^{3} V_{u,i}^* Y_{Qd_j},$$

for $i = 1, 2, 3$, where $V_{u,i,d}$ is an element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Integrating out the heavy vector-like quarks, one obtains the effective $Z'$ couplings to the SM quarks as

$$\mathcal{L}_{\text{eff}} = - Z'_{\alpha} \sum_{i,j=1}^{3} \left( g_{u,i,j}^L \bar{u}_{iL} \gamma^\alpha u_{jL} + g_{u,i,j}^R \bar{u}_{iR} \gamma^\alpha u_{jR} \right.$$

$$+ g_{d,i,j}^L \bar{d}_{iL} \gamma^\alpha d_{jL} + g_{d,i,j}^R \bar{d}_{iR} \gamma^\alpha d_{jR} \right),$$

for $i = 1, 2, 3$, where $V_{u,i,d}$ is an element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.
The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.

The effective couplings to $t \to cZ'$, for instance, are generated by the diagrams shown in Fig. 1.
where $x_W \equiv m_0^2/m_W^2$, and we have set $m_0^2/m_t^2 \rightarrow 0$. Note that our result is a factor of four smaller than the one shown in Ref. \[13\].

The first term in Eq. \[18\] is induced by the left-handed $t \rightarrow c$ current (first diagram in Fig. \[1\]), which is related to the left-handed $b \rightarrow s$ current by SU(2)$_L$ symmetry. Neglecting Cabibbo suppressed terms, Eq. \[5\] implies $Y_{Qt} \sim Y_{Qb}$, $Y_{Qc} \sim Y_{Qs}$. One may eliminate the $Y_{Qb}Y_{Qc}^*/m_{u}^2$ dependence in Eq. \[15\] by using $\Delta C_9^u$ [Eq. \[6\]] to rewrite the left-handed current contribution

$$B(t \rightarrow cZ')_{LH} \simeq \frac{(1-x')^2(1+2x')}{2(1-x_W^2)(1+2x_W)}|\Delta C_9^u|^2 v_\phi^2.$$  

(19)

Applying the lower bound, Eq. \[16\], and upper bound, Eq. \[15\], on $v_\phi$, we obtain the allowed range for left-handed current contribution:

$$0.7 \times 10^{-8} \lesssim B(t \rightarrow cZ')_{LH} \lesssim 0.8 \times 10^{-6},$$  

(20)

for a $Z'$ mass sufficiently below the kinematic threshold. Note that the lower limit assumes the central value of $\Delta C_9^u$ from the global fit in Eq. \[10\], while the upper limit is insensitive to $\Delta C_9^u$ as the dependence cancels out. The branching ratio can be slightly larger than the upper value quoted in Ref. \[13\], i.e. few $\times 10^{-7}$. This is because the $B_s$-mixing constraint on $v_\phi$, Eq. \[15\], is weakened due to the decoupling of $D$ quark effects, which is favored by the measured $R_{K^*}$ value.

On the other hand, the second term in Eq. \[15\], induced by the right-handed $t \rightarrow c$ current (second diagram in Fig. \[1\]), is not related to FCNCs in the down-type quark sector. Treating $m_U$, $Y_{Ut}$ and $Y_{Uc}$ as free parameters, the right-handed current contribution can be easily enhanced over the left-handed current contribution. To see how large it can be, we introduce a mixing parameter between the vector-like quark $U$ and $t_R$ or $c_R$ as

$$\delta_{Uq} \equiv \frac{Y_{Ut}v_\phi}{\sqrt{2m_U}}, \quad (q = t, c)$$  

(21)

and recast the right-handed current contribution as

$$B(t \rightarrow cZ')_{RH} \simeq \frac{(1-x')^2(1+2x')}{2(1-x_W^2)(1+2x_W)} \frac{v_\phi^2}{v_F^2} |\delta_{Ut}\delta_{Uc}|^2.$$  

(22)

For fixed values of $\delta_{Ut}$ and $\delta_{Uc}$, this can be enhanced by lowering the value of $v_\phi$, which is bounded from below by neutrino trident production [Eq. \[18\]]. Taking reasonably large mixing parameters $\delta_{Ut} \equiv \delta_{Uc} \simeq \lambda$ for illustration, Eq. \[15\] implies

$$B(t \rightarrow cZ')_{RH} \lesssim 3 \times 10^{-4} \quad (\delta_{Ut} = \delta_{Uc} \simeq \lambda).$$  

(23)

This is smaller than the value in Ref. \[13\], i.e. $\sim 1\%$, partially because of the correction factor $1/4$ in Eq. \[18\]. In the corrected formula, the $\sim 1\%$ branching ratio requires rather large mixing parameters: $\delta_{Ut} / \delta_{Uc} \sim 0.5$. 

---

GeV. This range is excluded by the constraint from the neutrino trident production, Eq. \[13\]. The case for the light $Z'$ will be discussed in the next section.

The effective $bsZ'$ coupling induces $B_s$ mixing, which provides an upper bound on $v_\phi$. The modification to the $B_s$ mixing amplitude is given by \[15\],

$$\frac{M_{12}^2}{M_{12}^2} \simeq 1 + (Y_{Qb}Y_{Qc}^*)^2 \left( \frac{v_F^2}{m_Q^2} + \frac{1}{16\pi^2}\frac{1}{m_Q^2} \right)$$

$$\times \left[ \frac{g_F^4}{16\pi^2\frac{1}{m_W^2}}(V_{tL}V_{tb})^2S_0 \right]^{-1},$$  

(14)

where $S_0 \simeq 2.3$ and the $D$ quark effects are decoupled, given the $b \rightarrow s$ transition data. It is useful to eliminate \[30\] the dependence on $Y_{Qb}Y_{Qc}^*/m_Q^2$ in terms of $\Delta C_9^u$ of Eq. \[9\]. Then, allowing BSM effects up to 15% \[15\], we find the upper bound

$$v_\phi \lesssim 5.6 \text{ TeV} \left( \frac{(34 \text{ TeV})^{-2}}{|\Delta C_9^u|} \right).$$  

(15)

We have neglected the $1/m_Q^2$ term in Eq. \[14\], which is numerically valid for $m_Q \lesssim 10$ TeV. For larger $m_Q$, the bound gets gradually stronger, e.g. $v_\phi \lesssim 5.4 \times 3.9$ TeV for $m_Q = 20 \ (50)$ TeV, with $\Delta C_9^u$ satisfying Eq. \[10\].

The constraint from kaon mixing can be avoided by assuming the mixing of $Q$ and $D$ quarks with $d$ quark is suppressed: $Y_{Qd} \simeq Y_{Dd} \simeq 0$. Although this assumption leads to $Y_{Qb} \simeq \lambda Y_{Qs}$ via Eq. \[5\], hence a new contribution to $D$ mixing, $B_s$ mixing still gives the strongest constraint \[15\]. We further set $Y_{Ud} \simeq 0$ to switch off the right-handed $c \rightarrow u$ current contribution to $D$ mixing, in order to pursue the possibility of large $t \rightarrow cZ'$ rate.

### D. Branching ratio for $t \rightarrow cZ'$

We now turn to $t \rightarrow cZ'$ decay. With the effective $tcZ'$ couplings in Eq. \[17\], the decay rate is given by

$$\Gamma(t \rightarrow cZ') = \frac{m_t}{32\pi} \lambda^{1/2}(1, x_c, x') \left[ (|g_{cl}|^2 + |g_{Rl}|^2) [1 + x_c - 2x'] \right. \left. + (1 - x_c^2)/x' - 12\text{Re}(g_{cl}g_{Rl}^*)\sqrt{x_c} \right],$$  

(16)

where $x_c \equiv m_c^2/m_t^2$, $x' \equiv m_2^2/m_2^2$ and

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2(xy + yz + zx).$$  

(17)

Taking the ratio with the $t \rightarrow bW$ rate, the $t \rightarrow cZ'$ branching ratio is given by

$$B(t \rightarrow cZ') \simeq \frac{(1-x')^2(1+2x')}{2(1-x_W^2)(1+2x_W)}$$

$$\times \left( |Y_{Qt}Y_{Qc}|^2 \frac{v_F^2}{4m_Q^2} + |Y_{Ut}Y_{Uc}|^2 \frac{v_F^2}{4m_U^2} \right),$$  

(18)
In Fig. 2 contours of $B(t \to cZ')_{RH}$ are given in the $(Y_{Ut}, Y_{Uc})$ plane for $m_{Z'} = 50$ GeV, $g' = 0.064$, $v_\phi = 780$ GeV, and $m_t = 2.5$ TeV. The vertical (horizontal) dashed lines mark the value of $Y_{Ut}$ ($Y_{Uc}$) above which the mixing parameter $\delta_{Ut}$ ($\delta_{Uc}$) exceeds $\lambda \approx 0.23$.

Before turning to search at LHC, we give the partial widths for $Z'$ decay

$$\Gamma(Z' \to \ell^+ \ell^-) = \frac{g'^2}{12\pi} m_{Z'} \left[ 1 - \frac{4m_\ell^2}{m_{Z'}^2} \right] \left[ 1 + \frac{2m_\ell^2}{m_{Z'}^2} \right],$$

$$\Gamma(Z' \to \nu \bar{\nu}) = \frac{g'^2}{24\pi} m_{Z'},$$

where $\ell = \mu, \tau$. The approximate branching ratios are

$$B_{\tau\tau} \simeq B_{\mu\mu} \simeq B_{\nu\nu} \simeq \frac{1}{3} \left(2m_\tau \ll m_{Z'}\right),$$

$$B_{\mu\mu} \simeq B_{\nu\nu} \simeq \frac{1}{2} \left(2m_\mu \ll m_{Z'} < 2m_\tau\right),$$

$$B_{\nu\nu} = 1 \left(m_{Z'} < 2m_\mu\right).$$

### E. $t \to cZ'$ search at LHC

The decay $t \to cZ'$ followed by $Z' \to \ell^+ \ell^-$ ($\ell = \mu, \tau$) can be searched for in $t\bar{t}$ events at the LHC. It is similar to $t \to qZ$ ($q = u, c$) decay, which has been searched for by the ATLAS $^{31}$ and CMS $^{32}$ experiments using $t\bar{t} \to Zq + Wb$ with leptonically decaying $Z$ and $W$, resulting in a final state with three charged leptons. The $t \to cZ'$ decay with heavy $Z'$ should be searched for in an analogous way, by modifying event selection criteria for an opposite sign charged lepton pair.

The current best limit on the $t \to qZ$ rate comes from the CMS analysis with the full Run 1 dataset $^{32}$, finding $B(t \to qZ) < 5 \times 10^{-4}$ at 95% C.L., while the ATLAS $^{31}$, based on the 20.3 fb$^{-1}$ dataset of the 8 TeV run, found $B(t \to qZ) < 7 \times 10^{-4}$ at 95% C.L. These limits should be improved with more data during the 13/14 TeV run of the LHC. The expected limits at the 14 TeV LHC with 300 fb$^{-1}$ (3000 fb$^{-1}$) data are $B(t \to qZ) < 2.7 \times 10^{-4}$ ($1.0 \times 10^{-4}$) for CMS $^{33,1}$ and $B(t \to qZ) < 2.2 \times 10^{-4}$ ($7 \times 10^{-5}$) for ATLAS $^{30,51}$.

For illustration, we attempt a reinterpretation of the CMS limits for $t \to cZ$ to the case for $t \to cZ'$ by a simple scaling of $Z$ and $Z'$ decay branching ratios into the charged leptons ($\ell = c, \mu$). An advantage of the $t \to cZ'$ search is the larger $Z'$ branching ratio, e.g., $B(Z' \to \mu^+ \mu^-) \simeq \frac{1}{3}$ for $m_{Z'} > 2m_\tau$, compared with $B(Z \to \ell^+ \ell^-) < 0.07$ (summed over $e$ and $\mu$). Multiplying the factor of $B(Z \to \ell^+ \ell^-)/B(Z' \to \ell^+ \ell^-) \approx 0.2$ to the current $^{32}$ CMS limits and future $^{33}$ CMS limits for $t \to cZ$, we infer sensitivities for $t \to cZ' (\to \mu^+ \mu^-)$ by CMS as

$$B(t \to cZ') \lesssim \begin{cases} 
10^{-4} & \text{("CMS" Run 1),} \\
5 \times 10^{-5} & \text{("CMS" 300 fb$^{-1}$),} \\
2 \times 10^{-5} & \text{("CMS" 3000 fb$^{-1}$),} 
\end{cases} \quad (26)$$

for the heavy $Z'$ with $m_{Z'} \sim O(10)$ GeV. The scaling of the ATLAS limits gives similar results. Therefore, the right-handed current induced $t \to cZ'$ might already be probed by Run 1 data (see Fig. 2), while the left-handed current contribution seems to be beyond the sensitivity of LHC, even with 3000 fb$^{-1}$ data (see Eq. (26)).

For light $Z'$ with $2m_\mu < m_{Z'} \lesssim 400$ MeV, the scaling factor is slightly reduced as $B(Z \to \ell^+ \ell^-)/B(Z' \to \ell^+ \ell^-) \approx 0.14$ due to larger $Z' \to \mu^+ \mu^-$ branching ratio ($\approx 1/2$). For such a light $Z'$, however, the search strategy needs to be changed. In particular, muon pairs produced by boosted $Z'$ bosons would be highly collimated, while the existing $t \to qZ$ search requires events with isolated charged leptons. Nevertheless, we adopt Eq. (26) for the light $Z'$ case as target values in the following analysis.

For the light $Z'$ with $m_{Z'} < 2m_\mu$, the $Z'$ decays only into neutrino pairs. Thus, the search at the LHC would be quite challenging.\(^3\)

---

\(^3\) In the Snowmass White Paper $^{33}$, a more optimistic value $\sim 10^{-5}$ is quoted as the CMS sensitivity for $t \to qZ$ with 300 fb$^{-1}$ data at the 14 TeV LHC. This was based on extrapolating from the 7 TeV result $^{33}$. The projections in Ref. $^{33}$, on the other hand, are based on Monte Carlo analysis.

\(^4\) The $t \to q \ell (q = u, c)$ decay with missing energy has been discussed based on dark matter models $^{34,52}$. 
FIG. 3. Lifetime of light $Z'$ with relevant constraints in the $(m_{Z'}, g')$ plane: solid lines are labeled contours for $\tau_{Z'}$, within $2\sigma$ [10], the gray-shaded region is excluded by neutrino trident production [22] and taken from Ref. [21]. The red crosses at $(m_{Z'}, g')=(135 \text{ MeV}, 10^{-3}), (219 \text{ MeV}, 1.1 \times 10^{-3}), (334 \text{ MeV}, 1.4 \times 10^{-3})$ indicate benchmark points adopted in our numerical study, as explained in text.

III. MUON $g-2$ AND $Z'$

In this and following sections, we consider the light $Z'$ scenarios motivated by muon $g-2$ anomaly. In Fig. 3 we give the parameter region (the blue band) in the $(m_{Z'}, g')$ plane that accounts [20] for the discrepancy, $\Delta \theta_\mu = (2.9 \pm 0.9) \times 10^{-9}$ [19], taking $2\sigma$ error range. The parameter space is strongly constrained [21] by neutrino trident production, the gray-shaded exclusion region. Thus, the $Z'$ boson of $L_\mu - L_\tau$ symmetry can explain the muon $g-2$ anomaly only if $m_{Z'} \lesssim 400 \text{ MeV}$, as given in Eq. 11. In this section, we consider the $t \to cZ'$ decay in the scenario of

$$2m_\mu < m_{Z'} \lesssim 400 \text{ MeV}, \quad \text{[Scenario (ii-a)]}$$

which permits the $Z' \to \mu^+ \mu^-$ decay.

The $Z'$ lifetime, $\tau_{Z'}$, estimated by summing up Eqs. (21), is also given in Fig. 3 as the black-solid contours. We see that $\tau_{Z'} \lesssim 0.1 \text{ fs}$ for the muon $g-2$ favored region above the dimuon threshold. The decay length of a light $Z'$ with energy $E_{Z'}$ is given by

$$\gamma c \tau_{Z'} \simeq 0.4 \mu \text{m} \left[ \frac{2}{N_{\text{eff}}} \right] \left[ \frac{10^{-3}}{g'} \right]^2 \left[ \frac{0.3 \text{ GeV}}{m_{Z'}} \right]^2 \left[ \frac{E_{Z'}}{10 \text{ GeV}} \right],$$

where $N_{\text{eff}} \simeq 2$ for $2m_\mu < m_{Z'} < 2m_\tau$, and $N_{\text{eff}} \simeq 1$ for $m_{Z'} < 2m_\mu$. Thus, for $m_{Z'} \gtrsim 2m_\mu$, the $Z'$ that is motivated by muon ($g-2$) decays promptly after production at colliders such as LHC and $B$ factories, with branching ratios approximately shown in Eq. (25). For $m_{Z'} < 2m_\mu$, the lifetime can be significantly longer for extremely light $Z'$, but its existence is simply felt as a missing energy (with no missing mass) in collider experiments, regardless of its decay.

A. FCNC $B$ decays

As the $Z'$ mass range in Eq. (27) is too low to explain the $P^0_K$ and $R_K$ anomalies, we treat rare $B$ meson decay data as providing constraints on the effective $bsZ'$ coupling. By SU(2)$_L$ symmetry, this also constrains the left-handed $tcZ'$ coupling. In the light $Z'$ scenario, rare $B$ meson decays provide rather strong constraints, and even the right-handed $tcZ'$ coupling becomes significantly constrained at one-loop level. We also discuss rare kaon decay constraints on the latter.

1. $B \to K^{(*)}Z'$ formulas

The light $Z'$ can be produced directly in $B \to K^{(*)}Z'$ decays, with $Z' \to \mu^+ \mu^- /\nu \bar{\nu}$. For the $bsZ'$ couplings of Eq. (6), the branching ratio is given by

$$\mathcal{B}(B \to K Z') = \frac{|g_{L}^{B} + g_{R}^{B}|^{2}}{64\pi} \frac{\beta_{Z}^{3}}{m_{Z'}^{2} \Gamma_{B}} \left[ f_{+}^{BK}(m_{Z'}) \right]^{2},$$

where $f_{+}^{BK}$ is the $B \to K$ form factor and

$$\beta_{X Y Z} \equiv \lambda_{x y z}^{1/2}(1, m_{Y}^{2}/m_{X}^{2}, m_{Z}^{2}/m_{X}^{2}),$$

with $\lambda(x, y, z)$ defined in Eq. (17). For $B \to K^* Z'$, the branching ratio can be expressed as

$$\mathcal{B}(B \to K^* Z') = \frac{\beta_{Z}^{3}}{16\pi m_{B}^{2} \Gamma_{B}} \left[ |H_{0}|^{2} + |H_{+}|^{2} + |H_{-}|^{2} \right],$$

where the helicity amplitudes $H_{0,\pm}$ are given by

$$H_{0} = (g_{L}^{B} - g_{R}^{B}) \left[ -\frac{1}{2}(m_{B} + m_{K'}) \xi A_{1}(m_{Z'}^{2}) \right],$$

$$H_{\pm} = \frac{1}{2}(g_{L}^{B} - g_{R}^{B})(m_{B} + m_{K'}) A_{1}(m_{Z'}^{2}) \pm (g_{L}^{B} + g_{R}^{B}) \frac{m_{K'} m_{Z'}}{m_{B} + m_{K'}} \sqrt{\xi^{2} - 1} V(m_{Z'}^{2}),$$

with $\xi \equiv (m_{B}^{2} - m_{K'}^{2} - m_{Z'}^{2})/(2m_{K'} m_{Z'})$, and $A_{1}, A_{2}, V$ are $B \to K^{*}$ form factors. For the form factor numerical values, we adopt the fit formulas from light-cone sum

\footnote{We imply both $B^{0} \to K^{0}Z'$ and $B^{\pm} \to K^{\pm}Z'$. A similar convention applies to $B \to K^{\ast} Z'$.}
rule calculations \cite{41, 42}. As the $Z'$ couples to the muon through the vector current, there is no new physics contribution to $B_s \to \mu^+\mu^-$. For later convenience, we provide numerical expressions of the $B^+ \to K^+Z'$ and $B^0 \to K^0Z'$ branching ratios for $m_{Z'} \lesssim 400$ MeV:

$$
B(B^+ \to K^+Z') \approx 2.2 \times 10^{12} |g_a^{L} + R_b^R|^2 \left( \frac{300 \text{ MeV}}{m_{Z'}} \right)^2, \\
B(B^0 \to K^0Z') \approx 2.4 \times 10^{12} |g_a^{L} - R_b^R|^2 \left( \frac{300 \text{ MeV}}{m_{Z'}} \right)^2 + 1.4 \times 10^{10} |g_a^{L} + R_b^R|^2. \tag{33}
$$

Note that the $(m_B/m_{Z'})^2$ enhancement comes from the longitudinally polarized $Z'$. 

2. $B \to K^{(*)}\mu^+\mu^-$ data

With \(~50\%\) $Z'$ decaying into muon pairs, $B \to K^{(*)}Z'$ decays would leave footprints in the dimuon mass ($q^2 \equiv m_{\mu\mu}$) spectra of $B \to K^{(*)}\mu^+\mu^-$ decays. As the SM prediction \cite{43} is not reliable for $q^2 < 1$ GeV$^2$, one challenge for low-mass new boson search is the estimation of SM background. Instead, one could take a data-based approach \cite{44} by searching for a narrow peak in the dimuon spectrum. With full 3.0 fb$^{-1}$ Run 1 data, the LHCb experiment has performed \cite{45} such a dedicated search for a new hidden-sector boson $B' \to K'^0\chi$ with $\chi \to \mu^+\mu^-$. Scanning the dimuon spectrum for 214 MeV $\lesssim m_{\mu\mu} \lesssim 4350$ MeV and finding no evidence for a signal, upper limits of $O(10^{-9})$ on $B(B^0 \to K'^0\chi)B(\chi \to \mu^+\mu^-)$ are set for most of the $\chi$ range with $\tau_{\chi} \leq 100$ ps.

As the LHCb analysis \cite{45} assumed $\chi$ to be scalar, the upper limits could be different for the $Z'$ case due to difference in efficiency. We estimated the ratio of efficiency for vector vs scalar boson, based on information in supplemental material of Ref. \cite{45}, and confirmed that the change is rather small, varying within 0–20\% in the mass range of our interest. (See Appendix A for detail.) Hence, we can apply directly the limits in Ref. \cite{45}:

$$
B(B^0 \to K'^0\chi)B(\chi \to \mu\mu) \lesssim (0.8-6.3) \times 10^{-9}, \quad \text{(LHCb)} \tag{34}
$$
at 95\% C.L. for 214 MeV $\lesssim m_{\chi} \lesssim 400$ MeV, with $\tau_{\chi} \ll 1$ ps. As the width of $Z'$ is very small, we neglect the interference between the $Z'$ and SM contributions.

The LHCb result greatly improves the previous limit set by Belle \cite{46}:

$$
B(B^0 \to K'^0X)B(X \to \mu\mu) \lesssim (2.3-5.0) \times 10^{-8}, \quad \text{(Belle)} \tag{35}
$$
at 90\% C.L. for a vector boson $X$ with mass in 212 MeV $\lesssim m_X \lesssim 300$ MeV. But the Belle result complements LHCb for the range 212 MeV $\lesssim m_X \lesssim 214$ MeV just above the dimuon threshold of 211.3 MeV. There are no existing results for the dedicated search of low-mass new bosons in the $B \to K\mu^+\mu^-$ mode. We stress the importance of search in this mode, as the two decay modes are complementary in probing the chiral structure of $bsZ'$ couplings: the $B \to KZ'$ rate depends on the vector-like combination $g_a^L + R_b^R$, while the $B \to K^*Z'$ rate is sensitive to the axial-vector combination $g_a^L - R_b^R$, as can be read from Eq. \ref{33}.

In a previous study \cite{22}, published before the advent of the LHCb analysis \cite{43}, we attempted at constraining the $Z'$ effect using existing LHCb data for $B^+ \to K^+\mu^+\mu^-$. We chose the 1 fb$^{-1}$ result \cite{47} instead of the 3 fb$^{-1}$ one \cite{48}, as the latter provides the dimuon spectrum only for $q^2 > 0.1$ GeV$^2$ (316 MeV$^2$), which covers only half of the $Z'$ mass range in Scenario (ii-a). The 1 fb$^{-1}$ result, however, gives the spectrum for $q^2 > 0.05$ GeV$^2$, which can probe $m_{Z'}$ down to 224 MeV. In contrast to $B \to K^+\mu^+\mu^-$, the photon peak is absent in $B^+ \to K^+\mu^+\mu^-$, and the measured $q^2$ spectrum \cite{47} is rather flat in the low $q^2$ range, with average differential branching ratio $\frac{d\mathcal{B}}{dq^2} = (2.41 \pm 0.22) \times 10^{-8}$ GeV$^{-2}$ in 1 GeV$^2 < q^2 < 6$ GeV$^2$. Treating this as background, we subtracted it from the measured value of $d\mathcal{B}/dq^2 = (2.85 \pm 0.30) \times 10^{-8}$ GeV$^{-2}$ in the lowest $q^2$ bin of 0.05 GeV$^2 < q^2 < 2.00$ GeV$^2$. We then estimated the allowed range of the $Z'$ contribution in this bin: $\Delta\mathcal{B}(B^+ \to K^+\mu^+\mu^-) = (0.86 \pm 0.59) \times 10^{-8}$. At 2$\sigma$, this reads \cite{22}

$$
B(B^+ \to K^+Z')B(Z' \to \mu^+\mu^-) \lesssim 2.0 \times 10^{-8}, \quad \text{("LHCb")} \tag{36}
$$
for 224 MeV $\lesssim m_{Z'} \lesssim 1414$ MeV.

3. $B \to K^{(*)}\nu\bar{\nu}$ data

In Scenario (ii-a), the other \(~50\%) of $Z'$ bosons decay into neutrino pairs, resulting in $B \to K^{(*)}\nu\bar{\nu}$. Sensitivities of experimental searches for $B \to K^{(*)}\nu\bar{\nu}$ by the BaBar \cite{49} and Belle \cite{50} experiments are still above the SM level. For our purpose, the BaBar result \cite{49} is useful, as model-independent constraints on BSF effects are given for spectra of $s_B \equiv m_{\nu\bar{\nu}}^2/m_B^2$ bin by bin. From Fig. 6 of Ref. \cite{49}, the first bin $0 < s_B < 0.1$, or $0 < m_{\nu\bar{\nu}} \lesssim 1670$ MeV, gives the constraints $\Delta\mathcal{B}(B^+ \to K^+\nu\bar{\nu}) = (0.35_{-0.15}^{+0.60}) \times 10^{-5}$ and $\Delta\mathcal{B}(B^+ \to K^{+*}\nu\bar{\nu}) = (-0.11_{-0.4}^{+1.9}) \times 10^{-5}$. The other two decay modes with $K^0$ or $K^{*0}$ give weaker limits. The $K^+$ channel favors nonzero BSF effects due to the observation of a small excess over the expected background. But the probability to observe such an excess in the signal region is 8.4\%, hence is not significant. The above limits at 2$\sigma$ imply, for $0 < m_{Z'} \lesssim 1670$ MeV,

$$
0.05 < 10^7 B(B^+ \to K^+Z')B(Z' \to \nu\bar{\nu}) < 1.55, \\
10^5 B(B^+ \to K^{+*}Z')B(Z' \to \nu\bar{\nu}) < 3.7. \quad \text{(BaBar)} \tag{37}
$$

Given the $Z'$ branching ratios $B_{\mu\mu} \sim B_{\nu\nu} \sim 1/2$, the excess in $B^+ \to K^+\nu\bar{\nu}$ is not compatible with the LHCb
limit on $B^+ \rightarrow K^+ Z^\prime(\rightarrow \mu^+ \mu^-)$, Eq. (36). In Scenario (ii-a), we therefore treat the BaBar limits just as a reference, except when the $Z'$ mass is close to the dimuon threshold and the $B \rightarrow K^{(*)}\mu^+\mu^-$ limits do not apply.

B. $t \rightarrow cZ'$ via left-handed current

The $B \rightarrow K^+ Z'$ rate is sensitive to the combination $g_{cb}^L - g_{cb}^R$, while its dependence on $g_{cb}^L + g_{cb}^R$ is weaker, as can be seen from Eq. (33). Hence, the limits on $B^0 \rightarrow K^{*0}Z'(\rightarrow \mu^+ \mu^-)$ by LHCb, Eq. (34), draw an ellipse extending along the $g_{cb}^L = g_{cb}^R$ direction on the $(g_{cb}^L, g_{cb}^R)$ plane for each $m_{Z'}$ value. The resulting constraints on the $bsZ'$ couplings are $|g_{sb}^L|, |g_{sb}^R| \lesssim (2-7) \times 10^{-10}$ for 214 MeV $\lesssim m_{Z'} \lesssim 400$ MeV.

The constraints on the $bsZ'$ couplings are improved if the limit on $B^0 \rightarrow K^+ Z'(\rightarrow \mu^+ \mu^-)$ extracted from the LHCb data, Eq. (35), is further imposed: $|g_{sb}^L|, |g_{sb}^R| \lesssim 1 \times 10^{-10}$ for 224 MeV $\lesssim m_{Z'} \lesssim 400$ MeV, which is rather stable w.r.t. the limit. The limit implies

$$m_Q \gtrsim 670 \text{ TeV} \sqrt{|Y_{Qb} Y_{Qb}| \left(\frac{m_{Z'}}{300 \text{ MeV}}\right) \left(10^{-3} \text{ g}^*\right)}, \quad (38)$$

which is an order of magnitude larger than in Scenario (i) [Eq. (11)]. Assuming the SU(2)$_L$ relation $g_{cb}^L \approx g_{cb}^R$, then, we obtain bounds on the left-handed current contribution to the $t \rightarrow cZ'$ branching ratio:

$$B(t \rightarrow cZ')_{\text{LH}} \lesssim (3-4) \times 10^{-15}, \quad (39)$$

for 224 MeV $\lesssim m_{Z'} \lesssim 400$ MeV. These values would be too small to measure even with the high-luminosity LHC upgrade. [See Eq. (26) for a naive expectation.]

For 214 MeV $\lesssim m_{Z'} \lesssim 224$ MeV, the $B^+ \rightarrow K^+ Z'(\rightarrow \mu^+ \mu^-)$ limit does not apply, and bounds on the $t \rightarrow cZ'$ rate are weakened:

$$B(t \rightarrow cZ')_{\text{LH}} \lesssim (2-4) \times 10^{-13}, \quad (40)$$

for 214 MeV $\lesssim m_{Z'} \lesssim 224$ MeV. In the narrow interval 212 MeV $\lesssim m_{Z'} < 214$ MeV, the LHCb limits on $B^0 \rightarrow K^{*0}Z'(\rightarrow \mu^+ \mu^-)$ are taken over by the Belle limits, Eq. (35), which give an order of magnitude weaker bounds on $B(t \rightarrow cZ')_{\text{LH}}$ than the case for 214 MeV $\lesssim m_{Z'} \lesssim 224$ MeV. The remaining spot of 211.3 MeV $\lesssim m_{Z'} < 212$ MeV, just above the dimuon threshold, is still constrained by the BaBar limits on $B \rightarrow K^{(*)}\mu\nu$, Eq. (37), leading to $B(t \rightarrow cZ')_{\text{LH}} \lesssim 4 \times 10^{-12}$, which is further diluted by a small $Z'$ branching ratio $B(Z' \rightarrow \mu^+ \mu^-) \lesssim 10\%$. Note that the excess in $B^+ \rightarrow K^+ \mu\nu$ does not necessarily imply a nonzero $g_{cb}^L \neq g_{cb}^R$, as $g_{cb}^L$ alone can still explain the excess. These values of $B(t \rightarrow cZ')_{\text{LH}}$ would be still too small for measurements at the LHC.

In deriving the limits on the left-handed current contribution to $B(t \rightarrow cZ')$, we assumed the SU(2)$_L$ relation $g_{cb}^L \approx g_{cb}^R$. This is valid for $Y_{Qt} \sim Y_{Qb}$, but does not hold in general. More precisely, the SU(2)$_L$ relation is given by Eq. (3): $Y_{Qb} = V_{ub} Y_{Qc} + V_{ub} Y_{Qt} \sim A \lambda^2 Y_{Qc} + Y_{Qt}$, and $Y_{Qs} = V_{us} Y_{Qc} + V_{us} Y_{Qt} \sim Y_{Qc} - A \lambda^2 Y_{Qc}$, where $A \approx 0.81$ [1], and $Y_{Qs}$ is taken to be zero to avoid $D$ meson constraints. A remarkable deviation from our assumption occurs when $Y_{Qc}/Y_{Qb} \sim \lambda^2$: $Y_{Qs}$ vanishes for $Y_{Qc} \sim A \lambda^2 Y_{Qc}$, hence, $g_{cb}^L \propto Y_{Qs} Y_{Qb} \simeq 0$ (and $g_{cb}^R \simeq 0$). This allows a large $g_{cb}^L$ without violating the $b \rightarrow sZ'$ (and $s \rightarrow dZ'$) constraints. Yet, this implies $Y_{Qd} \sim \lambda Y_{Qc}$ with $Y_{Qb} \sim Y_{Qt}$, hence, $g_{cb}^L \propto Y_{Qb} Y_{Qb} \sim \lambda g_{ct}^L$, which would be constrained by the measurement of the $B^+ \rightarrow \pi^+ \mu^+\mu^-$ decay by LHCb [51, 52], as well as $B-B$ mixing. We do not pursue such an extreme case in this paper.

Before moving on, we briefly mention the $B_\tau$ meson mixing constraint. For light $Z'$, the local $(sb)(sh)$ box operator construction in usual renormalization group analysis is not valid, as the $Z'$ remains a dynamical degree of freedom at the $m_B$ scale. Here, we simply recover the momentum dependence in the $Z'$ propagator in the usual heavy $Z'$ formula and set the $Z'$ momentum to the $B_\tau$ mass scale. To see the impact of this constraint, for simplicity, we only include the left-handed $bsZ'$ coupling effects. Employing the unitarity gauge and the vacuum insertion approximation [40], we find that the $B_\tau$ meson mixing amplitude is modified as

$$\frac{M_{12}}{M_{12}^{\text{SM}}} \simeq 1 - \frac{(g_{cb}^L)^2 m_{Z'}^2}{m_{B_\tau}^2 - m_{Z'}^2 \left(1 - \frac{5 m_{B_\tau}^2}{8 m_{Z'}^2}\right)} \times \left[\frac{g_\tau^2}{16\pi^2} (V_{ts} V_{tb})^2 S_0\right]^{-1}. \quad (41)$$

Allowing 15% BSM effect, we obtain $|g_{sb}^L| \lesssim 2 \times 10^{-6}(m_{Z'}/300 \text{ MeV})$, four orders of magnitude weaker than the constraint from $B \rightarrow K^{(*)}Z'(\rightarrow \mu^+ \mu^-)$.

C. Right-handed $tcZ'$ coupling: loop-induced down-quark sector FCNCs

The right-handed $tcZ'$ coupling induces FCNCs in down-type quark sector at one-loop level. More precisely, the SU(2)$_L$ singlet vector-like quark $U$, responsible for the effective right-handed $tcZ'$ coupling, mediates the diagram in Fig. 4, leading to extra contributions to effective $sZ'$ and $bsZ'$ couplings. Assisted by the SM charged current couplings of the $W$ boson to left-handed quarks, not only $tcZ'$, but also flavor-diagonal $ttZ'$ and $ccZ'$ contribute. These contributions are loop, chirality and CKM suppressed. There are thus no significant impacts for the heavy $Z'$. However, for light $Z'$, the loop-induced FCNC decays give meaningful constraints, as the meson decay rates are hugely enhanced due to on-shell nature of the $Z'$, compensating these suppressions.

In estimating the loop-induced FCNC couplings, for simplicity, we set $Y_{Qd} = Y_{Qs} = 0 \ (i = 1, 2, 3)$ to turn off the tree-level FCNC couplings in the down-type quark sector. We further set $Y_{t\nu u} = 0$ to avoid constraints from
\[ D \text{ meson decays and mixing, for sake of maximizing the } t \rightarrow cZ' \text{ decay rate. We are then left with the Yukawa mixing couplings for the vector-like quark } U \text{ to right-handed top or charm quarks, } Y_{U\ell} \text{ and } Y_{Uc}. \]

Working in the 't Hooft-Feynman gauge, we calculate the diagram of Fig. 4 as well as similar contributions with the would-be Nambu-Goldstone bosons. We neglect the external momenta as usual, but keep all internal momenta, including the one for the vector-like quark \( U \). We then obtain the loop-induced effective couplings \[ \mathcal{L}_{\text{eff}} \supset \Delta g_{d_d^j}^2 \lambda_{d_d^j}^\alpha \lambda_{d_d^j}^\alpha + \text{h.c.}, \] where \( j = 1, 2, 3 \), and
\[ \Delta g_{d_d^j} = \frac{g' v^2}{32 \pi^2 v^2} \left[ V_{Ud_d^j} V_{Ud_d^j}^* \kappa_{tt} f_{tt} + (V_{Ud_d^j} V_{Ud_d^j}^* \kappa_{tc}) f_{tc} + (V_{Ud_d^j} V_{Ud_d^j}^* \kappa_{cc} f_{cc}) \right], \]
with
\[ \kappa_{uu} = Y_{Uu} Y_{Uu}^* m_u/m_u, \]
and
\[ f_{tt} \simeq -\frac{3 m_W^2}{m_t^2 - m_W^2} \left( 1 - \frac{m_W^2}{m_t^2 - m_W^2} \log \frac{m_t^2}{m_W^2} \right) + \log \frac{m_t^2}{m_W^2}, \]
\[ f_{tc} \simeq 1 + \log \frac{m_t^2}{m_c^2} + \frac{3 m_W^2}{m_t^2 - m_W^2} \log \frac{m_t^2}{m_W^2}, \]
\[ f_{cc} \simeq 4 \log \frac{m_t^2}{m_c^2} + \log \frac{m_t^2}{m_c^2} - 3. \]

The expressions for these loop functions are in the large \( m_t^2 \) limit. Exact expressions used in our numerical study are given in Appendix B.

In the loop-induced \( bsZ' \) and \( bdZ' \) couplings, Eq. (43), the top-top loop contribution dominates due to chiral factor, so long as \( Y_{Uc}/Y_{U\ell} \) is not too large. Then, \( \Delta g_{g_b}^L/\Delta g_{g_b}^R \simeq V_{Ud_d^j} V_{Ud_d^j}^* \sim \lambda \), hence, the ratio \( B(b \rightarrow dZ')/B(b \rightarrow sZ') \sim \lambda^2 \approx 0.05 \) is SM-like. For large \( Y_{Uc} \) (e.g. \( Y_{Uc}/Y_{U\ell} \gtrsim 4 \) for \( m_U = 2 \text{ TeV} \)), the top-charm loop contribution becomes dominant. Even in this case, \( \Delta g_{g_b}^L/\Delta g_{g_b}^R \simeq V_{Ud_d^j} V_{Ud_d^j}^* \sim \lambda \), hence, the ratio \( B(b \rightarrow dZ')/B(b \rightarrow sZ') \) is still SM-like. Thus, the better measured \( b \rightarrow s \) decays are more suitable to watch \( Z' \) effect than \( b \rightarrow d \). We consider \( b \rightarrow sZ' \) and \( s \rightarrow dZ' \) decays below.

D. FCNC \( K \) decays

1. \( K \rightarrow \pi Z' \) formulas

One can obtain \( B(K \rightarrow K^{(*)}Z') \) for loop-induced \( bsZ' \) coupling by replacing \( g_{g_b}^L \rightarrow \Delta g_{g_b}^L, g_{g_b}^R \rightarrow 0 \) in Eqs. (30) and (31). Then, \( B(K \rightarrow K^{(*)}Z') \simeq B(K \rightarrow K Z') \), as can be seen from Eq. (33). Hence, the LHCb limits on \( B \rightarrow K^{0} \chi(-\mu^+\mu^-) \), Eq. (34), give the strongest constraint in most \( m_{Z'} \) ranges of Scenario (ii-a).

The loop-induced \( sdZ' \) coupling causes \( K \rightarrow \pi Z' \) for \( m_{Z'} < m_K - m_\pi \), leading to \( K \rightarrow \pi \mu^+\mu^- \) and \( K \rightarrow \pi \nu \bar{\nu} \) decays. The branching ratios for \( K^+ \rightarrow \pi^+ Z' \) and \( K_L \rightarrow \pi^0 Z' \) decays are given by
\[ B(K^+ \rightarrow \pi^+ Z') = \frac{|\Delta g_{d_d^j}^L|^2}{64 \pi} \frac{m_K^3}{m_{Z'}^3} \left[ f_{K^+\pi^+}^2 (m_{Z'}^2) \right]^2, \]
\[ B(K_L \rightarrow \pi^0 Z') = \frac{[\text{Im}(\Delta g_{d_d^j}^L)]^2}{64 \pi} \frac{m_K^3}{m_{Z'}^3} \left[ f_{K_L}^2 (m_{Z'}^2) \right]^2, \]
where \( \beta_{K\pi Z'} \) is defined by Eq. (30), and \( f_{K^+\pi^+}^2 \) and \( f_{K_L}^2 \) are the \( K \rightarrow \pi Z' \) form factors. For the latter, we adopt the result of Ref. [30], which is based on the partial NNLO calculation with isospin-breaking effects in chiral perturbation theory. In estimating the \( K_L \rightarrow \pi^0 Z' \) rate, we took \( |K_L| \simeq |K^{(*)}| \sqrt{2} \) with the phase convention where \( CP|K^{(*)}| = -|K^{(*)}| \), neglecting CP violation in kaon mixing. The branching ratio for \( K_S \rightarrow \pi^0 Z' \) can be obtained from the one for \( K_L \rightarrow \pi^0 Z' \) with the replacements: \( \text{Im}(\Delta g_{d_d^j}^L) \rightarrow \text{Re}(\Delta g_{d_d^j}^L) \) and \( \beta_{K\pi Z'} \rightarrow \tau_{K_S} \).

2. \( K \rightarrow \pi \mu^+\mu^- \) data

In SM, the \( K^+ \rightarrow \pi^+ \mu^+\mu^- \) decay is dominated by long-distance effects via one-photon exchange \( K^+ \rightarrow \pi^+ \gamma \). The decay can be described by chiral perturbation theory with the dimuon invariant mass spectrum \( dt/dz \propto |W(z)|^2 \), where \( z = m_{\mu^+\mu^-}/m_K \) and \( W(z) \) is the \( K \rightarrow \pi \gamma \) form factor, the most precise value for \( B(K^+ \rightarrow \pi^+ \mu^+\mu^-) \) by a single measurement comes from the NA48/2 experiment [54] at the CERN SPS. The measured \( z \)-spectrum in the whole kinematic range of \( 4m_{\mu^+\mu^-}/m_K \leq z \leq (1 - m_{\mu^+}/m_K)^2 \), corresponding to 211 MeV \( \lesssim m_{\mu^+\mu^-} \lesssim 345 \text{ MeV} \), is reasonably described by various form factor models. In particular, the measured \( z \)-spectrum does not exhibit significant excesses over the
fit curve by the linear form factor model. Here, we attempt a simple data-based approach to extract reasonable sizes for possible $Z'$ effects.

In a previous study [22], we focused on the largest upward deviation from the fit curve in the $z$-spectrum of NA48/2, which is located in the $z \in (0.32, 0.34)$ bin, corresponding to $m_{\mu\mu} \in (279, 288)$ MeV. Subtracting the fit value from the measured one, we read the allowed range for an extra contribution: $\Delta(d\varepsilon/dz) \simeq (2.5 \pm 1.5) \times 10^{-24}$ GeV. This corresponds to the deviation of the branching ratio in $m_{\mu\mu} \in (279, 288)$ MeV: $\Delta B(K^+ \to \pi^+\mu^+\mu^-) \simeq (9.4 \pm 5.6) \times 10^{-10}$. Allowing 2σ range, we estimate the limit on the $Z'$ contribution: $B(K^+ \to \pi^+Z')B(Z' \to \mu^+\mu^-) \lesssim 2.1 \times 10^{-9}$ for 279 MeV $\lesssim m_{Z'} \lesssim 288$ MeV. The constraint is tighter for other $Z'$ mass values in 211 MeV $\lesssim m_{\mu\mu} \lesssim 354$ MeV. For instance, we obtain

$$B(K^+ \to \pi^+Z')B(Z' \to \mu\mu) \leq 1.1 \times 10^{-9} \quad \text{(NA48/2)}$$

(47)

for 327 MeV $< m_{Z'} \lesssim 335$ MeV, and

$$B(K^+ \to \pi^+Z')B(Z' \to \mu\mu) \leq 1.2 \times 10^{-9} \quad \text{(NA48/2)}$$

(48)

for $2m_{\mu} < m_{Z'} \lesssim 221$ MeV.

For $K_L \to \pi^+\mu^+\mu^-$, the current best limit comes from KTeV [50] at Fermilab, giving the 90% C.L. limit

$$B(K_L \to \pi^+\mu^+\mu^-) < 3.8 \times 10^{-10}. \quad \text{(KTeV)}$$

(49)

This is above the SM prediction of $(1.29^{+0.24}_{-0.23}) \times 10^{-11}$ [52]. We thus impose the above limit on $B(K_L \to \pi^+Z')B(Z' \to \mu^+\mu^-)$ for $2m_{\mu} < m_{Z'} < 350$ MeV, covered by the kinematic selection of the KTeV analysis.

For $K_{S} \to \pi^+\mu^+\mu^-$ mode was measured by NA48/1 [58] at CERN SPS, giving $B(K_{S} \to \pi^+\mu^+\mu^-) = (2.9^{+3.5}_{-1.2}(\text{stat}) \pm 0.2(\text{syst})) \times 10^{-9}$. This was used as input in the SM prediction of $K_{S} \to \pi^+\mu^+\mu^-$ [57] to control the indirect CP violating contribution. For the possible $Z'$ contribution, isospin symmetry implies $B(K_{S} \to \pi^0Z') \lesssim (\tau_{K_{S}}/\tau_{K^+})B(K^+ \to \pi^0Z') \approx 0.007 \times B(K^+ \to \pi^0Z')$. Given that the experimental sensitivity on the $K^+ \to \pi^+Z'(\to \mu^+\mu^-)$ branching ratio is around $10^{-9}$, the above isospin relation constrains the $K_{S} \to \pi^0Z'(\to \mu^+\mu^-)$ branching ratio to be within $10^{-11}$, which may be beyond the sensitivity of NA48/1 data.

3. $K \to \pi\nu\bar{\nu}$ data

For $K^+ \to \pi^+\nu\bar{\nu}$ decay, the E949 experiment [59] at BNL, together with its predecessor E787, reported $B(K^+ \to \pi^+\nu\bar{\nu}) = (1.7_{-1.0}^{+1.5}) \times 10^{-10}$, which is consistent with SM prediction of $(8.25 \pm 0.64) \times 10^{-11}$ [57]. The measurement error is large and E787/E949 [60] also reported the 90% C.L. upper limit of $B(K^+ \to \pi^+\nu\bar{\nu}) < 3.35 \times 10^{-10}$. We remark, however, that the experimental analyses utilized limited intervals for the pion momentum $p_{\pi^+}$, or equivalently the neutrino pair mass $m_{\nu\bar{\nu}}$, to avoid blinding backgrounds from $K^+ \to \pi^+\pi^0$ and $K^+ \to \pi^+\pi^-\pi^+\pi^0\pi^0$, one is the $\pi\nu\bar{\nu}(1)$ region, where $211$ MeV $< p_{\pi^+} < 229$ MeV, or $0 < m_{\nu\bar{\nu}} \lesssim 116$ MeV; the other is the $\pi\nu\bar{\nu}(2)$ region, where $140$ MeV $< p_{\pi^+} < 199$ MeV, or $152$ MeV $\lesssim m_{\nu\bar{\nu}} \lesssim 261$ MeV. The kinematic selection of the $K^+ \to \pi^+\nu\bar{\nu}$ experiments has an interesting implication for $K_L \to \pi^0\nu\bar{\nu}$ search [22], as we will discuss in the next section.

The E787/E949 data have been used also for a dedicated search [60] of a two-body decay $K^+ \to \pi^+P^0\bar{\nu}$ with $P^0 \to \nu\bar{\nu}$, where $P^0$ is a hypothetical short-lived particle. The upper limits on $B(K^+ \to \pi^+P^0)B(P^0 \to \nu\bar{\nu})$ were given for the mass ranges of $0 \leq m_{P^0} \lesssim 125$ MeV or $150$ MeV $\lesssim m_{P^0} \lesssim 260$ MeV, which correspond to $\pi\nu\bar{\nu}(1)$ or $\pi\nu\bar{\nu}(2)$ regions, respectively. In the mass range which is relevant to Scenario (ii-a), the 90% C.L. upper limits increases almost monotonically with mass within

$$B(K^+ \to \pi^+P^0)B(P^0 \to \nu\bar{\nu}) \lesssim (0.4-5) \times 10^{-9}, \quad \text{(E949)}$$

(50)

for $2m_{\mu} < m_{P^0} \lesssim 260$ MeV.

To facilitate the discussion in Scenario (ii-b), we also quote 90% C.L. upper limits for typical $P^0$ masses below the dimuon threshold by setting $B(P^0 \to \nu\bar{\nu}) = 1$. For $0 < m_{P^0} \lesssim 125$ MeV, the strongest (weakest) bound is attained for $m_{P^0} \approx 95$ (125) MeV with $B(K^+ \to \pi^+P^0) \lesssim 5 \times 10^{-11}$ (4x $10^{-9}$). The bound is rather stable for $0 \leq m_{P^0} \lesssim 40$ MeV with $B(K^+ \to \pi^+P^0) \lesssim 10^{-10}$. For $150$ MeV $\lesssim m_{P^0} < 2m_{\mu}$, the strongest (weakest) bound is attained for $m_{P^0} \approx 190$ (150) MeV with $B(K^+ \to \pi^+P^0) \lesssim 4 \times 10^{-10}$ (10^{-8}).

For the pocket 125 MeV $\lesssim m_{P^0} \lesssim 150$ MeV around $\pi^0$ mass, the upper limit can still be obtained by using the $\pi^0 \to \nu\bar{\nu}$ search in $K^+ \to \pi^+\pi^0$ by E949 [61]: $B(\pi^0 \to \nu\bar{\nu}) < 2.7 \times 10^{-7}$ at 90% C.L. In this search, charged pions with momentum in 198 MeV $< p_{\pi^+} < 212$ MeV were selected, corresponding to 112 MeV $\lesssim m_{\nu\bar{\nu}} \lesssim 155$ MeV, hence the $\pi^0$-pocket can be fully covered. Combining with $B(K^+ \to \pi^+\pi^0) \approx 20.7\%$ [1], one has

$$B(K^+ \to \pi^+Z') < 5.6 \times 10^{-8}, \quad \text{(E949)}$$

(51)

at 90% C.L. for $112 \lesssim m_{Z'} \lesssim 155$ MeV.

The $K_L \to \pi^0\nu\bar{\nu}$ decay has been searched for by the E391a experiment [62] at the KEK proton synchrotron, setting the 90% C.L. upper limit

$$B(K_L \to \pi^0\nu\bar{\nu}) < 2.6 \times 10^{-8}, \quad \text{(E391a)}$$

(52)

without any particular cut on $m_{\nu\bar{\nu}}$. This is far above the SM prediction of $(2.60 \pm 0.37) \times 10^{-11}$ [57]. Therefore, we impose Eq. (42) on $B(K_L \to \pi^0Z')B(Z' \to \nu\bar{\nu})$ for $m_{Z'} < m_{K_L} - m_{\nu\bar{\nu}} \approx 363$ MeV. Note that in Scenario (ii-a), where $m_{Z'} > 2m_{\mu}$, the KTeV limit, Eq. (49), gives stronger constraint in general, as $B_{\mu\mu} \sim B_{\nu\bar{\nu}} \sim 1/2$. 

| Mode                  | Experiment | $m_{Z'} = 334$ MeV | $m_{Z'} = 219$ MeV | $m_{Z'} = 135$ MeV | Comment               |
|----------------------|------------|------------------|-----------------|-----------------|----------------------|
| $B^0 \to K^{*0}Z(\to \mu^+\mu^-)$ | LHCb [44]  | $< 4.41 \times 10^{-9}$ | $< 6.29 \times 10^{-9}$ | – | See Eq. (53). |
| $B^+ \to K^{*+}Z(\to \bar{\nu}\nu)$ | BaBar [49] | $(0.05, 1.55) \times 10^{-5}$ | $(0.05, 1.55) \times 10^{-5}$ | $(0.05, 1.55) \times 10^{-5}$ | See Eq. (57). |
| $K^+ \to \pi^+ Z(\to \mu^+\mu^-)$ | NA48/2 [54] | $\leq 1.1 \times 10^{-9}$ | $\leq 1.2 \times 10^{-9}$ | – | See Eqs. (44), (48). |
| $K_L \to \pi^0 Z(\to \mu^+\mu^-)$ | KTeV [56]  | $< 3.8 \times 10^{-10}$ | $< 3.8 \times 10^{-10}$ | – | See Eq. (49). |
| $K^+ \to \pi^+ Z(\to \nu\bar{\nu})$ | E787/E949 [60, 61] | $\leq 5 \times 10^{-10}$ | $< 5.6 \times 10^{-8}$ | – | See Eqs. (30), (31). |
| $K_L \to \pi^0 Z(\to \nu\bar{\nu})$ | E391a [62] | $< 2.6 \times 10^{-8}$ | $< 2.6 \times 10^{-8}$ | $< 2.6 \times 10^{-8}$ | See Eq. (52). |

TABLE I. Summary of $B$ and $K$ decay constraints for the three benchmark points in scenarios (ii-a) and (ii-b): $m_{Z'} = 334$, 219, 135 MeV. The numbers shown in third to fifth column are the allowed ranges for each branching ratios, used in Figs. 5-6. See text, and in particular the referred equations (last column), for detail.

There are no existing constraints on $K_S \to \pi^0\nu\bar{\nu}$, where the branching ratio is suppressed by $\tau_{K_S}/\tau_{K_L} \ll 1$ compared to $K_L \to \pi^0\nu\bar{\nu}$.

E. \( t \to cZ' \) via right-handed current

We can now combine all $B$ and $K$ decay data to constrain the right-handed current contribution $B(t \to cZ')_{RH}$. For illustration, we take the two benchmark points (shown by red crosses in Fig. 9):

- $m_{Z'} = 334$ MeV, $g' = 1.4 \times 10^{-3}$, $m_U = 2$ TeV,
- $m_{Z'} = 219$ MeV, $g' = 1.1 \times 10^{-3}$, $m_U = 2$ TeV.

The $Z'$ mass values are chosen such that the LHCb limit for $B^0 \to K^{*0}\chi(\to \mu^+\mu^-)$, Eq. (54), is weakened to be tolerant for a possible large $t \to cZ'$ rate: $m_{Z'} = 219$ MeV is one of the points which give the weakest limit in the whole range of 214 MeV $\leq m_{Z'} \leq 400$ MeV, while $m_{Z'} = 334$ MeV gives the weakest limit in the high mass region 260 MeV $\leq m_{Z'} \leq 400$ MeV, where the E949 limits for $K^+ \to \pi^+ P^0(\to \nu\bar{\nu})$ do not apply. The two benchmark points are phenomenological representatives of $B$ and $K$ decay constraints. The 95% C.L. upper limits by LHCb [44] are

$$B(B^0 \to K^{*0}\chi)B(\chi \to \mu^+\mu^-) < \begin{cases} 4.41 \times 10^{-9} & (m_{\chi} = 334 \text{ MeV}), \\
6.29 \times 10^{-9} & (m_{\chi} = 219 \text{ MeV}). \end{cases}$$ (53)

We neglect again the changes in efficiencies from scalar boson case. For loop-induced $bsZ'$ coupling, where $g_{bs}^2 = 0$, the changes are indeed extremely small, up to 6% in the mass range of our interest (See Appendix A). The $B$ and $K$ constraints are summarized in Table I.

In Fig. 5 (left), we give contours of $B(t \to cZ')_{RH}$ as black-solid lines in the $(Y_{U_U}, Y_{U_C})$ plane for $m_{Z'} = 334$ MeV benchmark point. The meson decay constraints are imposed by taking into account the $Z'$ branching ratios: $B_{\mu\mu} \approx 48\%$, $B_{\nu\nu} \approx 52\%$. The pink-shaded region is allowed by the LHCb bound on $B^0 \to K^{*0}\chi(\to \mu^+\mu^-)$ in Eq. (53). The light-green-shaded regions are favored by the mild excess in BaBar data for $B^+ \to K^{\ast} \nu\bar{\nu}$ at 2$\sigma$ [Eq. (47)]. The semi-transparent dark-gray shaded region represents 2$\sigma$ exclusion by the NA48/2 data for $K^+ \to \pi^+\mu^+\mu^-$, Eq. (17), from our illustration of the limit on the $Z'$ effect. The purple-solid lines are 90% C.L. exclusion by KTeV data for $K_L \to \pi^0\mu^+\mu^-$, Eq. (49).

All the data have better sensitivity for $Y_{U_T}$ than for $Y_{U_C}$, as the top-top loop contribution dominates the loop-induced effective couplings, Eq. (43), due to $m_t/m_c$ chiral enhancement. The LHCb limit for $B^0 \to K^{*0}\chi(\to \mu^+\mu^-)$ provides the strongest constraint, excluding the BaBar region that could account for the $B^+ \to K^+\nu\bar{\nu}$ excess. Nevertheless, the LHCb limit accommodates $B(t \to cZ')_{RH} \lesssim 10^{-6}$ along the funnel regions, extending towards larger $Y_{U_C}$. However, the NA48/2 limit for $K^+ \to \pi^+\mu^+\mu^-$ eventually cuts down these funnels. We obtain $B(t \to cZ')_{RH} \lesssim 2 \times 10^{-5}$.

We remark that $m_{Z'} = 334$ MeV is close to the kinematical limit $m_{K^+} - m_{\pi^+} \approx 354$ MeV of $K^+ \to \pi^+ Z'$, and the NA48/2 data is less constraining than generic cases. This can be seen from the velocity factor in Eq. (40): $\beta_{K^+\pi^+Z'} \approx 0.26$ for $m_{Z'} = 334$ MeV, leading to the suppression of $B(K^+ \to \pi^+ Z')$ by $\beta_{K^+\pi^+Z'} \approx 0.018$, compared with, e.g. $\beta_{K^+\pi^+Z'} \approx 0.31$ (0.080) for $m_{Z'} = 219$ (300) MeV.

For $m_{Z'} > m_{K^+} - m_{\pi^+}$, the two-body decay $K^+ \to \pi^+ Z'$ is kinematically forbidden, and the NA48/2 data loses constraining power, hence $B(t \to cZ')_{RH}$ can be arbitrary large along the funnels. However, the funnels imply some degree of fine-tuning between $Y_{U_U}$ and $Y_{U_C}$ with cancelled contributions to $b \to sZ'$. Furthermore, $Y_{U_C}$ should not be too large to maintain perturbativity.

A similar plot for the $m_{Z'} = 219$ MeV benchmark point is given in Fig. 5 (right), where $B_{\mu\mu} \approx 28\%$, $B_{\nu\nu} \approx 72\%$. In this case, the E949 limit for $K^+ \to \pi^+ P^0(\to \nu\bar{\nu})$ enters: $B(K^+ \to \pi^+ P^0)B(P^0 \to \nu\bar{\nu}) \lesssim 5 \times 10^{-10}$ at 90% C.L. [60], shown as semi-transparent dark-gray shaded exclusion region. This surpasses the NA48/2 limit for $K^+ \to \pi^+\mu^+\mu^-$, Eq. (48), shown by the light-gray solid

---

6 We thank M. Williams for providing us the precise upper values used in our study.
ellipse in the figure. The E949 limit fully excludes the funnel regions, and we obtain $B(t \to cZ')_{\text{RH}} \lesssim 0.8 \times 10^{-6}$.

The E949 limit gets stronger towards $m_{Z'} = 2m_{\mu}$ and generically excludes the funnel regions for $2m_{\mu} < m_{Z'} < 230$ MeV, leading to $B(t \to cZ')_{\text{RH}} \lesssim 10^{-6}$. Remarkably, this limit on $B(t \to cZ')_{\text{RH}}$ holds even in the pocket 211.3 MeV $\lesssim m_{Z'} < 212$ MeV, where neither LHCb nor Belle limits for $B^0 \to K^{*0}Z'(- \mu^+\mu^-)$ apply.

We have used $m_{\nu} = 2$ TeV, but obtained similar results for other $m_{\nu}$ values. This is because both $g_{\Delta}^{(s)}$ and $\Delta g_{\Delta}^{(d)}$ are proportional to $m_{\mu}^{-2}$, up to logarithmic dependence, multiplied by a quadratic form in $Y_{Ut}$ and $Y_{Uc}$ [See Eqs. (7) and (9)]. Thus, changing $m_{\nu}$ simply results in rescaled $Y_{Ut}$ and $Y_{Uc}$ values. A similar argument applies to the dependence on the U(1)' coupling $g'.$

As we took $Y_{Ut} = 0$ to avoid $D$ meson constraints, one might think that $Y_{Ut} > Y_{Uc} > Y_{Ut}$ is the natural ordering of these Yukawa couplings. Taking $Y_{Ut} = 1$ and $Y_{Uc} = \lambda$, we plot in Fig. [left] $B(t \to cZ')_{\text{RH}}$ as a function of $m_{\nu}$ for $m_{Z'} = 334$ MeV, with $B$ and $K$ constraints overlaid. In this case, the LHCb constraint is severe, implying $B(t \to cZ')_{\text{RH}} \lesssim 0.6 \times 10^{-10}$. A similar plot for $m_{Z'} = 219$ MeV is given in Fig. [right], where the LHCb limit implies $B(t \to cZ')_{\text{RH}} \lesssim 2 \times 10^{-10}$. The pocket 211.3 MeV $\lesssim m_{Z'} < 212$ MeV is still constrained by E949, giving $B(t \to cZ')_{\text{RH}} \lesssim 3 \times 10^{-9}$.

In short, for the hierarchical Yukawa couplings $Y_{Ut} = 1$, $Y_{Uc} = \lambda$, we obtain $B(t \to cZ')_{\text{RH}} \lesssim O(10^{-9})$ in the whole mass range of Scenario (ii-a), namely $2m_{\mu} < m_{Z'} \lesssim 400$ MeV. These values seem beyond reach of the LHC. On the other hand, if $Y_{Ut}$ and $Y_{Uc}$ are treated as free parameters, $B(t \to cZ')_{\text{RH}}$ can be much larger. In particular, $B(t \to cZ')_{\text{RH}} \sim O(10^{-5})$ is possible in the funnel regions for appropriately high $Z'$ masses such that the E949 limit is weakened. This may be within reach at the future LHC, as shown in Eq. (26), but a fine-tuned correlation between $Y_{Ut}$ and $Y_{Uc}$ would be needed. Note that our projection for LHC sensitivities is rather naive. A careful collider study should be done to judge the actual sensitivity for $t \to cZ'$ at the LHC.

Surveying other $m_{Z'}$ cases, we find the constraints on $B(t \to cZ')_{\text{RH}}$ in Scenario (ii-a) can be classified into the following three categories.

- $330$ MeV $\lesssim m_{Z'} \lesssim 400$ MeV: $B(t \to cZ')_{\text{RH}} \lesssim 10^{-6}$ is possible along the funnel regions allowed by hidden-sector boson search of LHCb in $B^0 \to K^{*0}\chi(- \mu^+\mu^-)$. In particular, if $m_{Z'} \gtrsim m_{K^+} - m_{\nu} \approx 354$ MeV, the $K^+$ decay constraints can be avoided, and $B(t \to cZ')_{\text{RH}}$ may be arbitrary large for large $Y_{Uc}$, up to perturbativity and associated fine-tuning of $Y_{Ut}$.

- $230$ MeV $\lesssim m_{Z'} \lesssim 330$ MeV:
In this case, the $Z'$ bosons are just felt as missing energy in collider experiments, but for $m_{Z'} = 334$ MeV and $g' = 1.4 \times 10^{-3}$ with hierarchical Yukawa couplings $Y_{Ut} = 1, Y_{Uc} = \lambda$, the NA48/2 limit is stronger towards $m_{Z'} < 2m_{\mu}$.

- $2m_{\mu} < m_{Z'} \lesssim 230$ MeV:
  The E949 limit gets stronger towards $m_{Z'} = 2m_{\mu}$, fully excluding the funnel regions, giving $B(t \to cZ')_{RH} \lesssim 10^{-6}$, even in the 211.3 MeV $\lesssim m_{Z'} < 212$ MeV pocket, where neither LHCb nor Belle limits for $B^0 \to K^{0*}Z' \to \mu^+\mu^-$ apply.

We have ignored the interference of $Z'$ effect with SM contribution, as the $Z'$ width is tiny. We remark, however, that for $m_{Z'} > 2m_{\pi}$, an absorptive part of $K^+ \to \pi^+\gamma^*(\rightarrow \mu^+\mu^-)$ induced by the $\pi\pi$ loop may invalidate the simple separation of SM and $Z'$ contributions to $K^+ \to \pi^+\mu^+\mu^-$. This might affect the second mass range, in particular the position where the funnels are cut down by the NA48/2 limit.

IV. A LIGHT $Z'$ THAT EVADES GROSSMAN-NIR BOUND

In this section, we study the scenario where

$$m_{Z'} < 2m_{\mu} \quad [\text{Scenario (ii-b)].}$$

In this case, the $Z'$ bosons decay exclusively into neutrino pairs and are just felt as missing energy in collider experiments. As such, it would be more challenging to search for $t \to cZ'$ at the LHC. Nevertheless, we estimate for completeness the allowed ranges for $t \to cZ'$ branching ratios via left- or right-handed current in this scenario. The relevant formulas and meson decay constraints were already summarized in the previous section.

An interesting outcome of this scrutiny is the possibility, pointed out by us previously [22], that an invisible $Z'$ boson could evade the commonly accepted Grossman-Nir (GN) bound [23] of $B(K_L \to \pi^0\nu\bar{\nu}) \lesssim 1.4 \times 10^{-9}$.

A. $t \to cZ'(\to \nu\bar{\nu})$ via left-handed current

We use BaBar data on $B \to K^{(*)}\nu\bar{\nu}$ in Eq. (37) to constrain the tree-level effective $bsZ'$ couplings $g^L_{sb}$ and $g^R_{sb}$. The $2\sigma$ range for $B^+ \to K^{+}\nu\bar{\nu}$ imposes

$$0.16 \times 10^{-9} \lesssim |g^L_{sb} + g^R_{sb}(\frac{100 \text{ MeV}}{m_{Z'}})| \lesssim 0.88 \times 10^{-9},$$

while the $B^+ \to K^{+}\nu\bar{\nu}$ data mainly constrains the other combination of the $bsZ'$ couplings,

$$|g^L_{sb} - g^R_{sb}(\frac{100 \text{ MeV}}{m_{Z'}})| \lesssim 1.3 \times 10^{-9}. \quad (56)$$

Combining the two constraints, we get

$$|g^L_{sb}|, |g^R_{sb}| \lesssim 1.1 \times 10^{-9}(\frac{m_{Z'}}{100 \text{ MeV}}), \quad (57)$$

for $m_{Z'} < 2m_{\mu}$. Note that the excess in the $B^+ \to K^{+}\nu\bar{\nu}$ data does not necessarily imply a nonzero $g^L_{sb}$, as $g^L_{sb} = 0$ can still explain the excess by a nonzero $g^R_{sb}$.

Using the SU(2)$_L$ relation $g^L_{cL} \approx g^L_{sb}$, we obtain the upper limit on the left-handed current contribution to the $t \to cZ'$ branching ratio,

$$B(t \to cZ')_{LH} \lesssim 4 \times 10^{-12} \quad (58)$$

for $m_{Z'} < 2m_{\mu}$. Note that the limit does not depend on $m_{Z'}$, as it cancels out in the final expression.

B. $t \to cZ'(\to \nu\bar{\nu})$ via right-handed current

We constrain the right-handed current contribution to the $t \to cZ'$ branching ratio by using $B^+ \to K^{+}\nu\bar{\nu}$ and $K^+ \to \pi^0\nu\bar{\nu}$ data. As discussed in the previous section, the E949 constraint from $K^+ \to \pi^+P^0(\to \nu\bar{\nu})$ search can
be avoided, if the $Z'$ mass falls into the $\pi^0$-mass window, i.e. $125$ MeV \(\lesssim m_{Z'} \lesssim 150\) MeV. Although this mass window is still constrained by $\pi^0 \to \nu\bar{\nu}$, searched in $K^+ \to \pi^+\pi^0$ [Eq. (51)], the limit is rather weak compared to the $K^+ \to \pi^+ P^0 (\to \nu\bar{\nu})$ limits outside the $\pi^0$-window. [See explanation below Eq. (52).] To allow for the possibility of a large $t \to cZ'$ rate, we take the following benchmark point:

- $m_{Z'} = 135$ MeV, $g' = 10^{-3}$, $m_U = 2$ TeV,

which is shown by a red cross in Fig. 3. The $B$ and $K$ decay constraints are summarized in Table I. As discussed in the previous section, this particular choice of $g'$ and $m_U$ does not affect the final result for $B(t \to cZ')_{\text{RH}}$.

In Fig. 7, we give contours of $B(t \to cZ')_{\text{RH}}$ as black-solid lines in the $(Y_{Ut}, Y_{Uc})$ plane. The $B$ and $K$ decay constraints are overlaid with the shadings and line styles as in Fig. 4 [right]. The semi-transparent dark-gray shaded region is excluded by the E949 limit on $\pi^0 \to \nu\bar{\nu}$ at 90\% C.L. [Eq. (51)]. In the present case, the E391a constraint on $K_L \to \pi^0\nu\bar{\nu}$ [Eq. (52)], shown as blue-solid lines, also plays a role. The green shaded regions, favored by the mild $B^+ \to K^+\nu\bar{\nu}$ excess in BaBar data [in Eq. (57)], are compatible with other constraints in most parts of the shown range. This is in contrast to Scenario (ii-a), where the constraints from $B^0 \to K^{*0}\chi(\to \mu^+\mu^-)$ and $K^+ \to \pi^+ P^0 (\to \nu\bar{\nu})$ exclude the BaBar regions. In this benchmark point, we obtain $B(t \to cZ')_{\text{RH}} \lesssim 5 \times 10^{-5}$.

Fixing the Yukawa couplings to $Y_{Ut} = 1$, $Y_{Uc} = \lambda$, we plot $B(t \to cZ')_{\text{RH}}$ in Fig. 8 as a function of $m_U$ with the same $m_{Z'}$ and $g'$ values. For this case, the BaBar excess favors a nonzero but small $t \to cZ'$ rate within $6 \times 10^{-9} \lesssim B(t \to cZ')_{\text{RH}} \lesssim 5 \times 10^{-7}$.

C. Apparent violation of Grossman-Nir bound

The light $Z'$ has an interesting implication for kaon decay experiments [22].

From isospin symmetry, the branching ratio $B(K_L \to \pi^0\nu\bar{\nu})$ is connected with $B(K^+ \to \pi^+\nu\bar{\nu})$ by a model-independent relation, known as the Grossman-Nir (GN) bound [23].

$$B(K_L \to \pi^0\nu\bar{\nu}) \lesssim 4.3 \times B(K^+ \to \pi^+\nu\bar{\nu}),$$

where the overall factor of 4.3 comes from $\tau_{K_L}/\tau_{K^+} \approx 4.1$ and isospin-breaking effects. Plugging in the 90\% C.L. upper limit of $B(K^+ \to \pi^+\nu\bar{\nu}) < 3.35 \times 10^{-10}$ by E949 [60], the GN bound leads to

$$B(K_L \to \pi^0\nu\bar{\nu}) \lesssim 1.4 \times 10^{-9}. \quad \text{("GN") (60)}$$

This is an order of magnitude stronger than the direct limit on $K_L \to \pi^0\nu\bar{\nu}$ by E391a, Eq. (52). In Figs. 5 and 8, this commonly accepted GN bound is shown by the orange-dashed lines.

There are two ongoing experiments in search for $K \to \pi\nu\bar{\nu}$ decays. The NA62 experiment [62] at CERN aims at measuring of order $100 K^+ \to \pi^+\nu\bar{\nu}$ events, while the KOTO experiment [61] at J-PARC aims at $3\sigma$ measurement of $K_L \to \pi^0\nu\bar{\nu}$ at SM rate. KOTO has already
To keep generality, we recover the dependence on $Z$, where the isospin breaking factor $m$ does not apply, and therefore, without such a selection, KOTO is already entering the domain of New Physics.

FIG. 9. Maximally allowed value as a function of $m_{Z'}$, for ratio $B(K_L \rightarrow \pi^0 Z')/B(K^+ \rightarrow \pi^+ Z')$, given in Eq. (61).

reached the sensitivity of E391a [Eq. (52)], but folklore is that KOTO can start to probe New Physics effects only after Eq. (60) is breached.

We have argued, however, that the kinematic selection in $K^- \rightarrow \pi^- \nu \bar{\nu}$ searches (including both E949 and NA62) makes them insensitive to the possible existence of a light new boson $X$, produced in $K^- \rightarrow \pi^- X^0$, if $m_{X^0} \sim m_{\pi}$ or larger than $2m_{\pi}$. If so, the usual GN bound of Eq. (60) does not apply, and therefore, without such a selection, KOTO is already entering the domain of New Physics.

A relation similar to Eq. (59) still holds for the light $Z'$ contribution, with a slight modification in the overall coefficient. Taking the ratio of the $K_L \rightarrow \pi^0 Z'$ and $K^+ \rightarrow \pi^+ Z'$ branching ratios in Eq. (40), we obtain the light $Z'$ version of the GN bound:

$$\frac{B(K_L \rightarrow \pi^0 Z')}{B(K^+ \rightarrow \pi^+ Z')} = \frac{\tau_{K_L}}{\tau_{K^+}} \frac{1}{r(m_{Z'}^2)} \left| \frac{g_{d_s}^L + g_{d_s}^R}{g_{d_s}^L + g_{d_s}^R} \right|^2 \leq \frac{\tau_{K_L}}{\tau_{K^+}} \frac{1}{r(m_{Z'}^2)},$$

where the isospin breaking factor $r(m_{Z'}^2)$ is defined by

$$1 \frac{1}{r(m_{Z'}^2)} = \frac{m_{K^0}^2}{m_{K^+}^2 + \beta_{K^+} \beta_{K^0} m_{Z'}^2} \left[ \left( \frac{f_{K^0}^{\pi^+}}{f_{K^+}^{\pi^0}} \right)^2 \left( \frac{m_{Z'}^2}{m_{Z'}^2} \right)^2 \right].$$

To keep generality, we recover the dependence on $g_{d_s}^R$, assuming the interaction form of Eq. (61). The form factor ratio is known to be $q^2$-independent up to next-to-leading order in chiral perturbation theory, with numerical value $f_{K^+}^{\pi^+} / f_{K^0}^{\pi^0}(q^2) = 1.0238 \pm 0.0022$. The genuine GN bound of Eq. (61) is then given by

$$\frac{B(K_L \rightarrow \pi^0 Z')}{B(K^+ \rightarrow \pi^+ Z')} \leq 4.122 \left( \frac{1.003}{r(m_{Z'}^2)} \right),$$

where $r(0) = 1.003$ is taken as reference. The right-hand side, the “GN coefficient”, depends on the $Z'$ mass, as illustrated in Fig. 9.

For $125 \text{ MeV} \lesssim m_{Z'} \lesssim 150 \text{ MeV}$, plugging in the 90% C.L. upper limit on $B(K^+ \rightarrow \pi^+ Z')$ by E949 [Eq. (51)], we obtain $B(K_L \rightarrow \pi^0 Z') \lesssim 2.3 \times 10^{-7}$. The direct bound on $K_L \rightarrow \pi^0 \nu \bar{\nu}$ by E391a, Eq. (52), is indeed stronger than this true GN bound.

The above argument is general and applicable to any weakly interacting light boson, or short-lived invisibly decaying boson, that couples to $s \rightarrow d$ currents. The bound of Eq. (61) holds for a massive vector boson that couples to the $s \rightarrow d$ currents in the form of Eq. (62). On the other hand, for the $L_\mu - L_\tau$ gauge boson with loop-induced $sdZ'$ coupling of Eq. (13), we obtain $|\text{Im}(\Delta g_{d_s}^L)/g_{d_s}^L| \sim |\text{Im}(V_{tL} V_{dL})/(V_{tL} V_{dL}^*)| \approx 0.15$, as long as $Y_{tL} \gtrsim Y_{uL}$, due to top-bottom dominance in loop. Thus, the GN bound of Eq. (61) cannot be saturated in this case.

The argument can be further extended to three-body kaon decays where the final state contains a pair of new massive invisible particles ($\chi$), i.e., $K \rightarrow \pi \chi \chi$. If the mass of $\chi$ is larger than $m_{\pi}$, the decay is allowed only if the invariant mass of the $\chi$ pair satisfies $2m_{\pi} < 2m_{\pi^0} < m_{K^0} - m_{\pi}$. In this case, the $\pi^+ \pi^- \chi$ momentum is always outside the signal regions of the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ experiments, hence, the usual GN bound does not apply. An interesting candidate is the very light neutralino in the minimal supersymmetric standard model, which was discussed in Ref. [64], although the analysis needs to be updated in light of recent LHC results. (See Ref. [65] for recent assessment of the light neutralino.)

V. DISCUSSION AND CONCLUSIONS

The so-called $P_5'$ and $R_K$ anomalies in $b \rightarrow s$ transitions, as revealed by LHCb data, suggest the possible existence of a new massive gauge boson $Z'$ coupling to left-handed $b \rightarrow s$ current, which in turn implies $tcZ'$ coupling. Motivated by this, we studied the top FCNC decay $t \rightarrow cZ'$ based on the gauged $L_\mu - L_\tau$ model with vector-like quarks that mix with SM quarks. The model can also be applied to address the muon $g - 2$ anomaly, which turns out to allow only a very light $Z'$ due to neutrino scattering data. The situation is mutually exclusive with the $b \rightarrow s$ anomalies. We studied how large the $t \rightarrow cZ'$ rate can be in three well-motivated scenarios: (i) heavy $Z'$ with $m_{Z'} \lesssim m_{Z^0}; m_{Z^0} - m_c$, motivated by the $P_5'$ and $R_K$ anomalies; (ii-a) light $Z'$ with $2m_{\pi} < m_{Z^0} \lesssim 400$ MeV, motivated by the muon $g - 2$ anomaly; (ii-b) the $(g - 2)_{\mu}$-motivated $Z'$ with $m_{Z'} < 2m_{\mu}$.

In Scenario (i), using a global fit result of $b \rightarrow s$ data as well as $B_s$ meson mixing constraint, we find that the left-handed current contribution to branching ratio $B(t \rightarrow cZ'_{LH})$ can be as large as $10^{-6}$. We also find that the right-handed current contribution $B(t \rightarrow cZ'_{RH})$, which is not constrained by $B$ data, can be as large as $O(10^{-4})$ with reasonably large mixing (around the Cabibbo angle) between vector-like quark $U$ and $t, c$. The left-handed case would be beyond the reach of even the high-luminosity LHC upgrade, while the right-handed case might be accessible already with LHC Run 1 data. (See Eq. (20) for our naive projection based on $t \rightarrow qZ$...
results.]

In Scenario (ii-a), we find $\mathcal{B}(t \to cZ')_{\text{RH}}$ to be extremely tiny, below $10^{-11}$, due to rare $B$ decay constraints. In this scenario, even the right-handed current contribution is constrained by rare $B$ and $K$ decays via one-loop effects. We find that $\mathcal{B}(t \to cZ')_{\text{RH}} \lesssim 10^{-6}$ is allowed only at the cost of fine-tuning the relation between $Y_{\ell t}$ and $Y_{\ell c}$. Nevertheless, in such cancellation regions, $\mathcal{B}(t \to cZ')_{\text{RH}}$ may be larger than $\mathcal{O}(10^{-5})$ for $330$ MeV $\lesssim m_{Z'} \lesssim 400$ MeV. Our naive projection based on $t \to qZ$ results suggests that this could be within reach of the ATLAS and CMS experiments with $300^{-1}$ data at the (13-)14 TeV LHC. However, a careful collider study is needed to find the true sensitivity, as the search strategy needs to be changed from the $t \to qZ$ case.

Scenario (ii-b) can accommodate larger $t \to cZ'$ branching ratios for the right-handed current contribution: $\mathcal{B}(t \to cZ')_{\text{RH}} \lesssim 5 \times 10^{-5}$. This case, however, would be more challenging for collider search, as the $Z'$ decays exclusively into neutrinos (but with little missing mass). Such a light $Z'$ is interesting instead for rare kaon decay experiments, and could even lead to observation of New Physics beyond the so-called Grossman-Nir bound, or $\mathcal{B}(K_L \to \pi^0 + \text{nothing}) > 1.4 \times 10^{-9}$. If this happens, our prediction is that it occurs via $K_L \to \pi^0 X^0$ with unobserved $m_{X^0} \sim m_{\pi^0}$, with our $Z'$ motivated by muon $g-2$ as a candidate. We remark that the $Z'$ in Scenarios (ii-a) and (ii-b) may also be probed by the future neutrino beam facility LBNE [21] via neutrino trident production [21]. And certainly LHCb and Belle II experiments should pursue further “bump” searches in $B \to K^{(*)} \mu^+ \mu^-$ and $B \to K^{(*)} \mu \bar{\nu}$ decays.

In this paper, we assumed a particular $Z'$ model to study $t \to cZ'$ decay. In the model, the right-handed $tcZ'$ coupling correlates with the $ttZ'$ and $ccZ'$ couplings. In particular, the $ttZ'$ coupling is strongly constrained by loop-induced decays from chiral $m_t/m_c$ enhancement, and the $tcZ'$ coupling in turn is also constrained indirectly. This correlation is not general, and the meson decay constraints might be relaxed in some other $Z'$ models where the right-handed $tcZ'$ coupling is independent from the $ttZ'$ coupling.

The muon $g-2$ anomaly implies that the $U(1)'$ symmetry breaking scale $v_{\phi}$ is around the electroweak scale of 246 GeV or below. The mass of the new Higgs boson $\phi$ behind the spontaneous breaking of the $U(1)'$ symmetry, is hence expected to be below 1 TeV and within reach at the LHC. The mixing of the vector-like $U$ quark with top via the $\phi$-Yukawa interaction leads to effective $t\phi$ coupling. Thus, the $\phi$ can be produced via gluon fusion $gg \to \phi$, followed by $\phi \to Z'Z' \to 4\mu/2\mu2\nu$, as pointed out in Ref. [22]. The effective $tt\phi$ coupling, however, is highly suppressed compared to the SM top Yukawa coupling, due to the constraints from $B^0 \to K^{*0}Z' \to \mu^+\mu^-$ as discussed in Sec. [11] hence the $gg \to \phi$ cross section is too small to be observed at the LHC [26]. Instead, the effective $tcc$ couplings, generated in a similar way as $tcZ'$, may offer another $\phi$ production mechanism, i.e. $t \to c\phi$ in $t\bar{t}$ events at the LHC. This gives rise to a striking signature, namely two collimated dimuons in $pp \to t\bar{t} \to bWc\phi(\to Z'Z')$ with $Z'Z' \to (\mu^+\mu^-)(\mu^+\mu^-)$. This interesting possibility will be pursued elsewhere [26].

**Acknowledgement.** KF is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, No. 15J01079. WSH is supported by the Academic Summit grant MOST 103-2745-M-002-001-ASP of the Ministry of Science and Technology, as well as by grant NTU-EPR-103R8915. MK is supported under NSC 102-2112-M-033-007-MY3. MK thanks Y. Chao, A. Mauri, J. Taineau and M. Williams for valuable discussions. WSH thanks T. Blake and M. Pepe-Altarelli for correspondence. KF thanks useful correspondence with S. Gori, and the NTUHEP group for hospitality during exchange visits.

**Appendix A: Efficiency for $B^0 \to K^{*0}Z' \to K\pi\mu^+\mu^-$**

In order to estimate the efficiency for the $B^0 \to K^{*0}Z' \to K\pi\mu^+\mu^-$ decay at the LHCb, we need information of the angular distribution for this decay. In the narrow width approximation, the normalized differential decay width for $B^0 \to K^{*0}Z' \to K\pi\mu^+\mu^-$ is given by

$$
\frac{1}{16\pi} \frac{d\Gamma}{d\cos\theta_K d\cos\theta_\ell d\phi} = \frac{9}{16\pi(1+2m_\mu^2/m_{Z'})^2} \left|\frac{H_0}{|H_0|^2 + |H_+|^2 + |H_-|^2}\right|^2 
\times \left\{ -\beta^2 \left|\frac{H_0}{|H_0|^2 \cos^2 \theta_K \cos^2 \theta_\ell}
+ \frac{1}{4}(|H_+|^2 + |H_-|^2) \sin^2 \theta_K \sin^2 \theta_\ell + \Xi(\theta_K, \theta_\ell, \phi)\right.\right.
\left.\left.\left.+ |H_0|^2 \cos^2 \theta_K + \frac{1}{2}(|H_+|^2 + |H_-|^2) \sin^2 \theta_\ell\right)\right\},
$$

(A1)

where $\beta = \sqrt{1 - 4m_\mu^2/m_{Z'}}$, and the helicity amplitudes $H_{0,\pm}$ are given in Eq. (32). The $\phi$-dependence enters solely through the function

$$
\Xi(\theta_K, \theta_\ell, \phi) = -\frac{1}{4} \sin 2\theta_K \sin 2\theta_\ell \left\{ \cos \phi \left[ \text{Re}(H_0 H_0^*) + \text{Re}(H_0 H_-^*) \right]
- \sin \phi \left[ \text{Im}(H_0 H_0^*) - \text{Im}(H_0 H_-^*) \right]\right\} + \frac{1}{2} \sin^2 \theta_K \sin^2 \theta_\ell
\times \left[ \cos 2\phi \text{Re}(H_+ H_+^*) + \sin 2\phi \text{Im}(H_+ H_+^*) \right].
$$

(A2)

We follow the LHCb convention [70] for the definition of decay angles: $\theta_\ell$ is the angle between direction of $\mu^-$ and direction opposite to $\bar{B}^0$ in the $Z'$ rest frame; $\theta_K$ is the angle between direction of $K^-$ and direction opposite to $\bar{B}^0$ in the $K^{*0}$ rest frame; $\phi$ is the angle between $Z' \to \mu^+\mu^-$ decay plane and $\phi$ is the angle between $Z' \to K^- \pi^+$ decay plane in the $\bar{B}^0$ rest frame.
FIG. 10. Ratio of efficiencies between vector-boson \( Z' \) and scalar-boson \( \chi \) for \( B^0 \rightarrow K^{*0} Z' (\chi) \rightarrow K^- \pi^+ \mu^+ \mu^- \) events collected by LHCb. Solid line is for \( g^L_{sb} \neq 0, g^R_{sb} = 0 \) and dashed line is for \( g^L_{sb} = g^R_{sb} \). The other two cases of \( g^L_{sb} = 0, g^R_{sb} \neq 0 \) and \( g^L_{sb} = -g^R_{sb} \) behave similarly to the solid line.

If a new scalar-boson \( \chi \) mediates the four-body decay instead of the vector-boson \( Z' \), the angular distribution simply behaves as \( \Delta t \Delta c \Delta \theta \Delta \phi \propto \cos^2 \theta_K \). This is the case assumed in the LHCb search \( ^{14} \) for hidden-sector bosons \( \chi \) in \( B^0 \rightarrow K^{*0} \chi \). In order to convert the limits on \( B^0 \rightarrow K^{*0} \chi \) into the \( Z' \) case, one needs to know the ratio of the efficiencies between the \( \chi \) and \( Z' \) cases. Ref. \( ^{14} \) (see Supplemental Material) provides this information in the form of ratio between integrals of the trigonometric functions appearing in Eq. \( ^{(11)} \) and integral of \( \cos^2 \theta_K \), taking into account the efficiency. Using this information, we obtain the ratio of efficiencies for \( Z' \) to \( \chi \), as shown in Fig. 10. For \( g^R_{sb} = 0 \), corresponding to the loop-induced coupling discussed in Sec. III, the change in efficiencies from the scalar case is within 6% for \( m_{Z'} \leq 400 \) MeV. If we allow general chiral structure for \( bsZ' \) coupling, the change is still small, within 20% for \( m_{Z'} \leq 400 \) MeV.

Appendix B: Loop functions for effective couplings

The loop functions given in Eq. \( ^{(45)} \) are approximate formulas in the large \( m_{\ell} \) limit. In our numerical study, we use the following expression:

\[
f_{qq'} = -4m_W^2 m_{\ell}^2 I_{qq'}^1 + (2m_W^2 + m_{\ell}^2) m_U^2 I_{qq'}^2 - 2m_W^2 I_{qq'}^0,
\]

where \( q, q' = t, c \), and

\[
I_{qq'}^1 = \int \frac{d^4k}{i(2\pi)^4} \frac{16\pi^2}{(k^2 - m_q^2)(k^2 - m_{q'}^2)(k^2 - m_{U}^2)(k^2 - m_{W}^2)},
\]

\[
I_{qq'}^2 = \int \frac{d^4k}{i(2\pi)^4} \frac{16\pi^2}{(k^2 - m_q^2)(k^2 - m_{q'}^2)(k^2 - m_{U}^2)(k^2 - m_{W}^2)} = \frac{2}{(m_{U}^2 - m_{W}^2)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[ \frac{\ln \frac{\beta_{qq'}}{\alpha_{qq'}} - (1 - x_1 - x_2)(m_{U}^2 - m_{W}^2)}{\beta_{qq'}} \right],
\]

\[
I_{qq'}^0 = \int \frac{d^4k}{i(2\pi)^4} \frac{16\pi^2}{(k^2 - m_q^2)(k^2 - m_{q'}^2)(k^2 - m_{U}^2)(k^2 - m_{W}^2)} = \frac{6}{m_{U}^2 - m_{W}^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[ \frac{1 - x_1 - x_2 - \frac{\alpha_{qq'}}{m_{U}^2 - m_{W}^2} \ln \frac{\beta_{qq'}}{\alpha_{qq'}}}{\alpha_{qq'}} \right],
\]

with

\[
\alpha_{qq'} = x_1 m_q^2 + x_2 m_{q'}^2 + (1 - x_1 - x_2) m_{W}^2, \quad \beta_{qq'} = x_1 m_q^2 + x_2 m_{q'}^2 + (1 - x_1 - x_2) m_{U}^2.
\]

[1] K.A. Olive et al. [Particle Data Group], Chin. Phys. C 38, 090001 (2014).
[2] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 111, 191801 (2013) [arXiv:1308.1207 [hep-ex]].
[3] LHCb Collaboration, LHCb-CONF-2015-002; R. Aaij et al. [LHCb Collaboration], JHEP 1602, 104 (2016) [arXiv:1512.04442 [hep-ex]].
[4] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 113, 151601 (2014) [arXiv:1406.6882 [hep-ex]].
[5] S. Descotes-Genon, J. Matias, J. Virto, Phys. Rev. D 88, 074002 (2013) [arXiv:1307.5683 [hep-ph]].
[6] W. Altmannshofer and D.M. Straub, Eur. Phys. J. C 73, 2646 (2013) [arXiv:1308.1501 [hep-ph]].
[7] F. Beaujean, C. Bobeth, D. van Dyk, Eur. Phys. J. C 74, 2807 (2014) [Erratum Eur. Phys. J. C 74, 3179 (2014)] [arXiv:1310.2178 [hep-ph]].
[8] R.R. Horgan, Z. Liu, S. Meinel, M. Wingate, Phys. Rev. Lett. 112, 212003 (2014) [arXiv:1310.3887 [hep-ph]].
[9] T. Hurth and F. Mahmoudi, JHEP 1404, 097 (2014) [arXiv:1312.5267 [hep-ph]].
[10] R. Alonso, B. Grinstein, J.M. Camalich, Phys. Rev. Lett. 113, 241802 (2014) [arXiv:1407.7044 [hep-ph]].
