We discuss the condensation of relativistic spin-one fields at high chemical potential. This phenomenon leads to the spontaneous breaking of rotational invariance, together with the breaking of internal symmetries.

1 Introduction

There has been much interest recently in the phase structure of Quantum Chromodynamics (QCD) and QCD–like theories. An overview has been given at this workshop. Here the phase diagram not only depends on temperature and density, but also on the number of flavors $N_f$, and colors $N_c$. Most of our knowledge on the QCD phase diagram at finite temperature derives from lattice simulations. Lattice studies of the high density, low temperature region are however seriously obstructed by a complex fermion determinant. No such difficulties exist for $N_c = 2$, and hence predictions for two–color QCD can be compared with numerical solutions from the lattice. Such a theory with $N_f$ flavors has a global $SU(2N_f)$ symmetry. There are color–singlet diquark states which play the role of baryons. Interestingly, it has been predicted that at sufficiently high baryon chemical potential, spin-1 diquark states will condense. Indeed there appears to be preliminary evidence for this phenomenon from lattice calculations. With this physical application in mind we shall study a vector field transforming in the adjoint representation of a global $SU(2)$ group. The chemical potential is chosen along one of the generator’s direction (the “baryon charge” operator), and breaks the global symmetry explicitly to $U(1) \times \mathbb{Z}_2$. Apparently condensation of a vector field singles out a direction in space, the rotational $SO(3)$–invariance is spontaneously broken to an $SO(2)$ subgroup.

Introduction of a chemical potential explicitly breaks the Lorentz invariance, and in such a situation the relation between broken symmetries and Goldstone modes (gapless excitations) is of special interest. Here the particle–physics textbook folklore that every broken generator corresponds to a Goldstone boson does not apply. Instead, following Nielsen and Chadha, one should properly distinguish between two types of Goldstone modes, depending on whether in the long–wavelength limit their energy varies with an odd
(type I) or even (type II) power of momentum. The following inequality holds:

\[ N_{bg} \leq N_I + 2 \cdot N_{II} , \]

between the numbers of broken generators \( N_{bg} \), and type I/II Goldstone modes \( N_{I/II} \). In order to elucidate these aspects in case of the vector condensation, we proceed with constructing the simplest effective Lagrangian for a vector field.

### 2 A lagrangian model for vector condensation

We introduce a vector field \( A^a_\mu, a = 1, 2, 3 \) of mass \( m \) transforming under the adjoint representation of a global \( SU(2) \) symmetry. The Lagrangian, endowed with a potential of fourth order in the field, reads:

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{m^2}{2} A^a_\mu A^{a\mu} - \frac{\lambda}{4} (A^a_\mu A^{a\mu})^2 + \frac{\lambda'}{4} (A^a_\mu A^{a\nu})^2 ,
\]

with \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \), and metric convention \( \eta^{\mu\nu} = \text{diag}(+,-,-,-) \).

Here we confine ourselves to positive \( \lambda \) and \( \lambda' \). Stability of the potential furthermore demands \( \lambda > \lambda' \). The effect of a nonzero chemical potential associated to a given conserved charge - related to the generator \( B = T_3 \) - can be readily included by modifying the derivatives acting on the vector fields:

\[
\partial_\nu A_\rho \rightarrow \partial_\nu A_\rho - i [B_\nu , A_\rho] ,
\]

with \( B_\nu = \mu \delta_\nu B \equiv V_\nu B \) where \( V = (\mu , \vec{0}) \). The chemical potential thus resembles an external field in temporal direction. The kinetic term can then be cast in the form

\[
\mathcal{L}_{\text{kinetic}} = \frac{1}{2} A^a_\rho \mathcal{K}^{\rho\nu} A^b_\nu
\]

with

\[
\mathcal{K}^{\rho\nu} = \delta_{ab} \left[ g^{\rho\nu} \partial^2 - \partial^\rho \partial^\nu \right] - 4i \gamma_{ab} \left[ g^{\rho\nu} V \cdot \partial - \frac{V^\rho \partial^\nu + V^\nu \partial^\rho}{2} \right] + 2 \chi_{ab} [V \cdot V g^{\rho\nu} - V^\rho V^\nu] ,
\]

where

\[
\gamma_{ab} = \text{Tr} \left[ T^a [B, T^b] \right] , \quad \chi_{ab} = \text{Tr} \left[ [B, T^a] [B, T^b] \right] .
\]

For \( B = T^3 \) we have

\[
\gamma_{ab} = \frac{i}{2} \epsilon^{abc} , \quad \chi_{11} = \chi_{22} = -\frac{1}{2} , \quad \chi_{33} = 0 .
\]
The chemical potential induces a “magnetic-type” mass term for the vectors at tree-level. The symmetries of the potential are more easily understood using the following Euclidean notation:

\[ \phi^a_M = (A^1_M, A^2_M) , \quad \psi_M = A^3_M \, , \]

with \( A_M = (iA_0, \hat{A}) \) and metric signature \((+ , + , + , +)\). In these variables the potential reads:

\[ V_{\text{vector}} = \frac{m^2}{2} (|\vec{\phi}_0|^2 + \psi_M^2) + \frac{m^2 - \mu^2}{2} |\vec{\phi}_I|^2 + \frac{\lambda}{4} (|\vec{\phi}_M|^2 + \psi_M^2)^2 - \frac{\lambda'}{4} [\vec{\phi}_M \cdot \vec{\phi}_N + \psi_M \psi_N]^2 \]

(9)

with \( I = X, Y, Z \) while \( M, N = 0, X, Y, Z \) and repeated indices are summed over. At zero chemical potential \( V_{\text{vector}} \) is invariant under the \( SO(4) \) Lorentz transformations while only the \( SO(3) \) symmetry is manifest at non zero \( \mu \).

As noted above, the chemical potential explicitly breaks the global \( SU(2) \) symmetry to \( U(1) \times Z_2 \). Note that for the special parameter choice \( \lambda' = 0 \), the potential alone is invariant under an \( SO(6) \) rotation group.

It is apparent, that due to the presence of the term proportional to \((m^2 - \mu^2)\), we have to distinguish the cases \( \mu \leq m \) and \( \mu > m \).

2.1 The Symmetric Phase: \( 0 < \mu \leq m \)

Here the \( SO(4) \) Lorentz and \( SU(2) \) symmetries are explicitly broken to \( SO(3) \) and \( U(1) \) respectively by the chemical potential. All fields have a vanishing expectation value in the vacuum: \( < \vec{\phi}_M >= < \psi_M >= 0 \). The curvatures of the potential on the vacuum (the masses) are:

\[ M^2_{\phi_0} = M^2_{\psi_M} = m^2 , \quad M^2_{\phi_I} = m^2 - \mu^2 \, . \]

(10)

We obtain the dispersion relations by diagonalizing the 12 by 12 matrix \( K \). This leads to 3 physical vectors (i.e. each of the following states has 3 components) with the dispersion relations:

\[ E_{\phi^\pm} = \pm \mu + \sqrt{\vec{p}^2 + m^2} \, , \quad E_{\psi} = \sqrt{\vec{p}^2 + m^2} \, . \]

(11)

We observe that for \( \mu \rightarrow m \) the 3 physical components associated with \( E_{\phi^\pm} \) will become massless signaling an instability. Indeed, for higher values of the chemical potential larger than \( m \) a vector condensation sets in.
2.2 The Spin-Flavor Broken Phase: \( \mu > m \)

In this phase at the global minimum of the potential we still have \( \langle \varphi_0^a \rangle = \langle \psi_M \rangle = 0 \). However there emerges a condensate:

\[
\langle \varphi^1_X \rangle = \sqrt{\frac{\mu^2 - m^2}{\lambda - \lambda'}}.
\] (12)

We have a manifold of equivalent vacua which are obtained rotating the chosen one under a \( Z_2 \times U(1) \times SO(3) \) transformation. The choice of the vacuum partially locks together the Lorentz group and the internal symmetry while leaving unbroken only the subgroup \( Z_2 \times SO(2) \). Two generators associated to the Lorentz rotations are now spontaneously broken together with the \( U(1) \) generator.

To proceed with the calculation of the dispersion relation of the vector states in the broken phase we first have to evaluate the curvatures of the potential on the new vacuum. The explicit formulas can be found elsewhere.

We find in general three states with null curvature. An exception occurs for the choice \( \lambda' = 0 \) in which case there are five zero curvature states. To explain this behavior we note that for \( \lambda' = 0 \) the potential possesses an enhanced \( SO(6) \) global symmetry which breaks to \( SO(5) \) when the vector field condenses. The associated five states would correspond to the ordinary Goldstone modes in
the absence of an explicit Lorentz breaking. Now the symmetry breaking pattern of our theory is $SO(3) \times U(1) \times Z_2 \to SO(2) \times Z_2$ and we find the three expected gapless excitations irrespective of the parameters of the potential.

In Fig.1 we show how the mass gaps evolve as a function of the chemical potential $\mu$. In the unbroken phase, for $\mu \leq m$ all gaps are threefold degenerate and the splitting of the states of different baryon charge is clearly visible. For $\mu > m$ the modes with nonvanishing gap split further. The dashed lines are twofold degenerate ($SO(2)$–vectors) while the solid lines correspond to $SO(2)$ scalar states. There are three gapless modes, whose dispersion relations however show some subtleties.

2.3 Dispersion relations and Goldstone counting

The gapless states also fall into a doublet forming an $SO(2)$ vector and an $SO(2)$ singlet. Their dispersion relations at small momenta have the form

$$E^2_{\varphi^+} = v^2_{\varphi^+} p^2 + O(p^4), \quad E^2_{\varphi^0} = v^2_{\varphi^0} p^2 + O(p^4),$$

where we introduced the “superfluid velocities” $v_{\varphi^+}, v_{\varphi^0}$. One can easily show that these velocities can be expressed directly in terms of the curvatures in the directions orthogonal to the gapless modes as follows:

$$v^2_{\varphi^+} = v^2_{\varphi^0} = 2 \frac{M_{\varphi^+}^2}{M_{\varphi^0}^2 + 4\mu^2}.$$  \hfill{(14)}

It now turns out, that always $M_{\varphi^+}^2 \neq 0$. Therefore the scalar state will always have a linear dispersion relation $E \propto p$. Not so the $SO(2)$ vector states. In the case of enhanced potential symmetry $SO(6)$, the curvature $M_{\varphi^+}^2$ vanishes and with it the velocity $v^2_{\varphi^+}$. The dispersion relation hence becomes a quadratic one, $E \propto p^2$, the $SO(2)$ state turns into a type II Goldstone boson. We emphasize, that were Lorentz invariance unbroken, the curvatures $M_{\varphi^+}^2$ would precisely be masses and there would be a Goldstone for each flat direction of the potential. Obviously the chemical potential prevents the emergence of extra Goldstone bosons in the case $\lambda^\prime = 0$. Simply the full Lagrangian (kinetic plus potential terms) does not share the potential’s larger symmetry. Still, the additional flat directions of the potential have a physical effect: they are responsible for turning some of the Goldstones into type II modes. This appears to be a perfectly general phenomenon for the type of kinetic term that arises at finite chemical potential. Finally we count: for $\lambda^\prime \neq 0$ there are three broken generators and three type I Goldstones, hence $N_{bg} = N_I$, the standard situation. For $\lambda^\prime = 0$ we have still three broken generators, but on
the right hand side of the Nielsen–Chadha inequality \( [1] \) we count \( 1 \times \text{type I} + 2 \times 2 \text{ type II} = 5 \). This is larger than the number of broken generators, which appears to be a novel observation. We note that five is precisely the number of broken generators of the potential symmetry group, which would correspond to the breaking pattern \( SO(6) \to SO(5) \).

We finally point out that the choice of coupling constants discussed here gives rise to the “polar” phase with a real order parameter. In principle, an interaction with negative \( \lambda' \) might give rise to “ferromagnetic” type phases characterized by a complex order parameter and a different symmetry breaking pattern \( [2] \).

3 Where to find vector condensates – some examples

We already mentioned the condensation of vector fields in two–color QCD, which has been the main focus of our work. In three color QCD at high quark chemical potential the 2SC phase allows a spin-1 diquark condensate with albeit smaller gap than the spin-0 condensate. This case has been recently investigated \( [3] \). We also note that spin-1 condensation is expected to give rise to a rich phenomenology of vortices \( [4] \), which have been discussed in the context of atomic Bose–Einstein condensates.

In the color–flavor locked phase of color superconducting quark matter, the possibilities of condensation of vector states in presence of a nonvanishing isospin chemical potential are as yet unexplored.

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