A Brief Overview of the Sock Matching Problem

Bojana Pantić and Olga Bodroža-Pantić
Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Serbia
e-mail: dmi.bojana.pantic@student.pmf.uns.ac.rs
olga.bodroza-pantic@dmi.uns.ac.rs

Abstract. This short note deals with the so-called Sock Matching Problem. We define $B_{n,k}$ as the number of all the finite sequences $a_1, \ldots, a_{2n}$ of nonnegative integers which contain at least one occurrence of $k$ ($1 \leq k \leq n$) and for which $a_1 = 1$, $a_{2n} = 0$ and $|a_i - a_{i+1}| = 1$. The value $a_i$ can be interpreted as the number of unmatched socks being present after having drawn the first $i$ socks randomly out of the pile which initially contained $n$ pairs of socks. Here, establishing a link between this problem and with both some old and some new results, related to the number of restricted Dyck paths, we obtain a few valid forms of the sock matching theorem and prove that the probability for $k$ unmatched socks to appear (in the very process of drawing one sock at a time) approaches $1$ as the number of socks becomes large enough.

Key Words: Sock matching, Dyck path, Generating functions
AMS Subject Classification: 05A15, 05A16, 03B48, 00A69

1 Introduction

In simple terms, what is understood under The Sock Matching Problem [2] is the following procedure. Out of the laundry pile that contains exactly $n$ different pairs of socks socks are being drawn randomly, one at a time (so that in the end all the $2n$ socks get matched). In each move one tries to find the adequate pair among the drawn socks, in case it had already been obtained in the process. Furthermore, each of the two options: drawing a match for some sock or drawing a sock that has no match as of yet matches a single move, either one unit up or one unit to the right, on a Dyck path (a
path in an $n \times n$ grid starting from the lower left corner $(0, 0)$ and ending
in the upper right corner $(n, n)$ using merely moves up and to the right
without ever crossing the diagonal, see Fig. 1a) - this particular model of
the Dick paths was used in [2]).

It is a well known fact indeed that the number of all the Dick paths
of order $n$ is equivalent to the $n^{th}$ Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ (see
[8]).

Let us now formulate our Sock Matching Problem in somewhat math-
ematically stricter terms. In fact, let us focus upon the total number of
ways, labeled by $B_{n,k}$, to get at least $k$ unmatched socks at least once
during the matching process. Considering the aforementioned interpretation of
our problem that relies on the use of Dick paths, it is our task to determine
which ones out of these $C_n$ possibilities are those that present the paths
which hit or pass above the line $y = x + k$.

Admittedly there is a wide range of interpretations of the Catalan nu-
merals $C_n$. However, it is that which allows various authors to opt themselves
for the most suitable one. Here, we make use of the following terminology
from [8, 9]:

- Lattice paths, used in [4], consider the up- and down-steps. The
  former $(1, 1)$-steps represent the case when ”a sock with no match
  has been drawn”, whereas the latter $(1, -1)$-steps represent the case
  when ”a match has been made”. Here, a lattice path goes from $(0, 0)$
  to $(2n, 0)$ on the Cartesian plane without ever moving across the x-
  axis (though it is allowed to hit it), as shown in Fig. 1b);

- The number of planted plane trees (i.e. rooted trees which have been
  embedded in the plane so that the relative order of subtrees at each
  branch is part of its structure; ordered trees) with $n + 1$ nodes, see
  Fig. 2;

- Discrete random walks in a straight line with an absorbing barrier at
  0 (represented by the sequences $1 = c_1, c_2, c_3, \ldots, c_{2n+2} = 0$; where
  $c_i \geq 1$ for $i < 2n + 2$ and $| c_i - c_{i+1} | = 1$);

- The number of all the sequences $a_1, \ldots, a_{2n}$ of nonnegative integers
  with $a_1 = 1$, $a_{2n} = 0$ and $| a_i - a_{i+1} | = 1$ (Problem 6.19 ($u^5$) in
  [9]). (Note that the value $a_i$ can be interpreted as the exact number
  of unmatched socks which are present after drawing the $i^{th}$ sock. Let
  us also note that further on this interpretation shall be refer to as the
  one with the nonnegative sequences).

A bijection is easily established between the sets of the aforementioned
combinatorial objects. For instance, the Dyck path shown in Fig. 1a)
corresponds to both the path in Fig. 1b) (which can be obtained by rotating that very figure for $-45^\circ$ and then expanding it with the expansion coefficient of $\sqrt{2}$) as well as to the tree in Fig. 2. To be more precise, by wandering around that tree the vertical component of successive positions describes a path from 1 (the root of the tree) to 0. Consequently, in this particular example the corresponding discrete random walk would be 1, 2, 1, 2, 1, 2, 3, 2, 3, 2, 1, 0; whereas the nonnegative sequence, mentioned in the last interpretation, would be 1, 0, 1, 0, 1, 2, 1, 2, 1, 0.

The height of a Dyck path is the greatest distance from the diagonal to the path, the height of a lattice path the greatest distance from the $x$-axis to the path, whilst the height of a planted ordered tree is the number of nodes on a maximal simple path starting at a root. Clearly, the height of a Dyck path is $\max_i a_i$ in the nonnegative sequences interpretation, whereas the height of a corresponding planted plane tree is $\max_i c_i$ in the discrete random walks interpretation. Now, it is worthwhile realising that the latter value is for one greater than the former. To illustrate this point, take another look at Figures 1 and 2. There, the height of the presented Dyck path (Fig. 1) is 2 (at most 2 unmatched socks appear), as opposed to the height of the planted plane tree (Fig. 2) which is 3!

Bearing all this in mind, the outlined problem from the heading, i.e. the number $B_{n,k}$ may represent as follows:

- the number of Dyck paths of height at least $k$ (the ones that hit or cross the line $y = x + k$),

- the number of lattice paths of height at least $k$ (the ones that hit or cross the line $y = k$),
- the number of planted plane trees of height at least $k + 1$,
- the number of discrete random walks with $\max_i c_i \geq k + 1$,
- the number of all the nonnegative sequences containing the letter $k$.

Figure 2: a tree with its random walk

Contemporary researches related to the Dyck paths refer mainly to the number of restricted lattice paths, where crossing the $x$-axis is allowed. Ilić and Ilić in [4] gave the upper and lower bounds for this number in the form of binomial coefficients. Forging a link between this problem and an older paper [5] from 1985. H. Prodinger in [7] provides an explicit formula for those, as given in this theorem:

**Theorem 1.1** The number of random walks from $(0, 0)$ to $(2n, 0)$ with up-steps and down-steps of one unit each, under the condition that the path is placed between the lines $y = -h$ and $y = t$ is equal to

\[
\sum_{j \geq 0} \left[ \binom{2n}{n - j(h + t + 2)} - \binom{2n}{n - j(h + t + 2) - h - 1} - \binom{2n}{n - j(h + t + 2) - t - 1} + \binom{2n}{n - (j + 1)(h + t + 2)} \right]. \tag{1}
\]

In the special case, for $h = 0$, we obtain the number of all the sequences $a_1, \ldots, a_{2n}$ of nonnegative integers with $a_1 = 1$, $a_{2n} = 0$ and $|a_i - a_{i+1}| = 1$ and $a_i \leq t$ for $n \geq 1$, $t \geq 0$, which we label by $A_{n,t}$. Obviously, $A_{n,0} = 0$ and $A_{n,1} = 1$ for $n \geq 1$, $A_{n,t} = C_n$ for $t \geq n$. The value $A_{n,t}$ was already essentially obtained in [1] in the distant 1972, in the form of trigonometric functions. The authors of that paper used the rooted tree (planted plane tree) interpretation. As for convenience, we reformulate their results in the following theorem.
Theorem 1.2

\[ A_{n,t} = \frac{1}{t+2} \sum_{1 \leq j \leq \frac{t+1}{2}} 4^{n+1} \sin^2 \left( \frac{j\pi}{t+2} \right) \cos^{2n} \left( \frac{j\pi}{t+2} \right), \]  

where \( n \geq 1 \) and \( t \geq 1 \).

It is quite an interesting fact that this formula has been rediscovered many a time, as the authors of paper [1] clearly point out, and that above all Lagrange derived a formula in 1775, which essentially includes this as a special case.

The authors of [2] derive a recurrence formula, to which they refer to as the so-called Sock Matching Theorem, for the numbers \( B_{n,k} \). Additionally, they make a proposition that the probability \( P_{n,k} \) for a Dyck path to reach the line \( y = x + k \) approaches 1 as the number \( n \) becomes large enough, i.e. \( \lim_{n \to \infty} \frac{B_{n,k}}{C_n} = 1 \). However, we noticed some inaccuracies in these proofs (for more details on that see [6], page 3 and 4). For the purpose of providing a valid formula for \( B_{n,k} \), with the accompanying proofs, we shall present two equivalent alternatives in Section 2, using more than just the mentioned authors’ idea. In Section 3 we give an explicit expression for \( B_{n,k} \). In Section 4 we prove the mentioned theorem about the asymptotic behavior of the ratio \( B_{n,k}/C_n \) when \( n \) converges to infinity.

2 The sock matching theorem

Theorem 2.1 (The Sock Matching Theorem - I alternative)

The sequence \( B_{n,k} \) whose \( n \)th term represents the number of Dyck paths of order \( n \) which hit or cross the line \( y = x + k \) is determined by the following recurrence formula:

\[ B_{n,k} = \sum_{i=1}^{n} (B_{i-1,k-1}C_{n-i} + C_{i-1}B_{n-i,k} - B_{i-1,k-1}B_{n-i,k}). \]  

Proof (The first one): Let \((i, i)\) be the first point on the line \( y = x \) which the Dyck path visits after \((0,0)\). Further, we take three possibilities into consideration: the line hits \( y = x + k \) before \((i, i)\) (Case 1), the line hits \( y = x + k \) after \((i, i)\) (Case 2), and the line hits \( y = x + k \) both before and after \((i, i)\) (Case 3).

The number of paths in the first case is \( B_{i-1,k-1}C_{n-i} \). Namely, the number of ways to hit \( y = x + k \) between \((0,0)\) and \((i, i)\) without hitting \( y = x \) is the same as the number of ways to get from \((0,1)\) to \((i - 1, i)\).
hitting $y = x + k$ but not crossing $y = x + 1$, which is $B_{i-1,k-1}$ and the number of ways to get from $(i, i)$ to $(n, n)$ without crossing $y = x$ is $C_{n-i}$.

Similarly, the numbers in the second and third case are $C_{i-1}B_{n-i,k}$ and $B_{i-1,k-1}B_{n-i,k}$, respectively. □

**Proof (The second one):** There is, however, yet another approach which may be made in order to obtain the formula (3) which makes use of the recurrence relation satisfied by the numbers $A_{n,k}$ derived in [1]:

$$A_{n,k+1} = A_{n-1,k+1}A_{0,k} + A_{n-2,k+1}A_{1,k} + \ldots + A_{0,k+1}A_{n-1,k},$$  

where $n \geq 1$ and $k \geq 0$; with the initial conditions for $A_{n,0} = 0$, when $n \geq 1$ and for $A_{0,k} \equiv 1$, when $k \geq 0$. To be more specific, as

$$B_{n,k} = C_n - A_{n,k-1},$$  

where $n \geq 1$ and $k \geq 1$ (evidently, $B_{n,1} = C_n$) making the necessary substitutions in (4) and with the use of the well-known recurrence relation for Catalan numbers

$$(C_0 = 1, \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \text{ for } n \geq 0)$$

we obtain the desired relation.

□

**Theorem 2.2 (The Sock Matching Theorem - II alternative)**

$$B_{n,k} = \sum_{j=0}^{n-1} (B_{j,k}C_{n-j-1} + C_jB_{n-j-1,k-1} - B_{j,k}B_{n-j-1,k-1}).$$  

(6)

**Proof:** Similarly to the previous alternative we take the point $(i, i)$ into consideration, only this time as the last point on the line $y = x$ that the Dyck path visits before $(n, n)$. Seen from this perspective, the corresponding numbers for the three cases would be exactly the values $B_{j,k}C_{n-j-1}$, $C_jB_{n-j-1,k-1}$ and $B_{j,k}B_{n-j-1,k-1}$.

By the way, the proof for the second formula (6) could, obviously, be obtained from (3) by a fairly simple substitution: $i = n - j$. □

### 3 The explicit formula for $B_{n,k}$

We now give the explicit expression for the values of $B_{n,k}$.

**Theorem 3.1**

$$B_{n,k} = \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \frac{2n+2}{n+1-j(k+1)} \right) - 4 \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{2n}{n-j(k+1)} \right).$$  

(7)
Proof (The first one): Recall that the value $A_{n,t}$ for the lower bound $h = 0$ is known from (1). Further, setting the upper bound to be $t = k - 1$ it follows directly from (5) that

$$B_{n,k} = \frac{1}{n+1}(\frac{2n}{n}) - \sum_{j \geq 0} \left( \binom{2n}{n-j(k+1)} - \binom{2n}{n-j(k+1)-1} \right)$$

$$- \left( \frac{2n}{n-j(k+1)-k} \right) + \binom{2n}{n-(j+1)(k+1)} \right). \quad (8)$$

After some minor algebraic simplifications of the above expression we have

$$B_{n,k} = \sum_{j \geq 1} \left[ \binom{2n}{n+1-j(k+1)} - 2 \binom{2n}{n-j(k+1)} + \binom{2n}{n-1-j(k+1)} \right], \quad (9)$$

which coincides with the formula presented in [1] for the number of planted plane trees with $n+1$ nodes whose height is greater than $k$. Now, since

$$\binom{2n+2}{n+1-j(k+1)} = \binom{2n+1}{n-j(k+1)} + \binom{2n+1}{n+1-j(k+1)}$$

$$= \binom{2n}{n-1-j(k+1)} + 2 \binom{2n}{n-j(k+1)} + \binom{2n}{n+1-j(k+1)},$$

the expression (7) is easily derivable from (9). □

Proof (The second one): Let us now commence from formula (2). By applying $\sin^2 \alpha = 1 - \cos^2 \alpha$ to it we obtain the following

$$A_{n,t} = \frac{4^{n+1}}{t+2} \left( \sum_{1 \leq j \leq \frac{t+1}{2}} \cos^{2n} \left( \frac{j \pi}{t+2} \right) - \sum_{1 \leq j \leq \frac{t+1}{2}} \cos^{2n+2} \left( \frac{j \pi}{t+2} \right) \right), \quad (10)$$

where $n \geq 1$. Further, using one of the notations for the representation of trigonometric power sums, namely the one with binomial coefficients, from [3], we have

$$\sum_{j=0}^{N-1} \cos^{2m} \left( \frac{j \pi}{N} \right) = 2^{1-2m}N \left( \binom{2m-1}{m-1} + \sum_{p=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{2m}{m-pN} \right), \quad (11)$$

where $m, N \in \mathbb{N}$ and $m \geq N$. Now, making the necessary substitutions, i.e. $N = t + 2$ and for $m$ at first $m = n$ and then $m = n + 1$, a brief
The simplification process leads to

\[ A_{n,t} = 4 \left[ \binom{2n-1}{n-1} + \sum_{j \geq 1} \binom{n-1}{n-j(t+2)} \right] 
- \left[ \binom{2n+1}{n} + \sum_{j \geq 1} \binom{n+1}{n+1-j(t+2)} \right] \]  

(12)

Once again, utilising (5) and yet again simplifying the obtained expression we eventually come to the desired formula (7). \(\Box\)

| \(B_{n,k}\) | \(k = 1\) | 2 | 3 | 4 | 5 | 6 | 7 |
|---------|-----------|---|---|---|---|---|---|
| \(B_{1,k}\) | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(B_{2,k}\) | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| \(B_{3,k}\) | 5 | 4 | 1 | 0 | 0 | 0 | 0 |
| \(B_{4,k}\) | 14 | 13 | 6 | 1 | 0 | 0 | 0 |
| \(B_{5,k}\) | 42 | 41 | 20 | 8 | 1 | 0 | 0 |
| \(B_{6,k}\) | 132 | 131 | 100 | 43 | 10 | 1 | 0 |
| \(B_{7,k}\) | 429 | 428 | 365 | 196 | 64 | 12 | 1 |
| \(B_{8,k}\) | 1430 | 1429 | 1302 | 820 | 336 | 89 | 14 |
| \(B_{9,k}\) | 4862 | 4861 | 4606 | 3265 | 1581 | 528 | 118 |
| \(B_{10,k}\) | 16796 | 16795 | 16284 | 12615 | 6984 | 2755 | 780 |
| \(B_{11,k}\) | 58786 | 58785 | 57762 | 47840 | 2755 | 780 |
| \(B_{12,k}\) | 208012 | 208011 | 205964 | 179355 | 13244 | 4466 |
| \(B_{13,k}\) | 742990 | 742989 | 738804 | 667875 | 52839 | 23276 |
| \(B_{14,k}\) | 2574439 | 2574438 | 2566238 | 2462022 | 1857278 | 113620 |
| \(B_{15,k}\) | 9064845 | 9064844 | 9054841 | 9180616 | 667875 | 52839 |

Tabular 1: Numerical values of \(B(n,k)\) for small values of \(n\) and \(k\)
4 Asymptotic behavior

Theorem 4.1 The probability $P_{n,k}$ of reaching a given fixed $k$ ($k \geq 1$) approaches 1 as $n$ approaches infinity, i.e.

$$\lim_{n \to \infty} P_{n,k} = \lim_{n \to \infty} \frac{B_{n,k}}{C_n} = 1.$$ 

Proof: Firstly, notice that the theorem trivially holds for $k = 1$. Thus, we shall assume that $k \geq 2$. Exploiting (5) and (2) some more we have

$$\lim_{n \to \infty} P_{n,k} = \lim_{n \to \infty} \frac{\sum_{1 \leq j \leq \frac{k}{2}} 4^{n+1} \sin^2 \left( \frac{j\pi}{k+1} \right) \cos^{2n} \left( \frac{j\pi}{k+1} \right)}{\frac{1}{n+1} \frac{(2n)!}{n!n!}}.$$

Since $0 < \sin^2 \left( \frac{j\pi}{k+1} \right) < 1$ and $0 < \cos \left( \frac{j\pi}{k+1} \right) \leq \cos \left( \frac{\pi}{k+1} \right)$ for every $j$ such that $1 \leq j \leq \frac{k}{2}$ we have

$$1 \geq \lim_{n \to \infty} P_{n,k} \geq 1 - \lim_{n \to \infty} \frac{\sum_{1 \leq j \leq \frac{k}{2}} 4^{n+1} \frac{k}{2} \cos^{2n} \left( \frac{\pi}{k+1} \right)}{\frac{1}{n+1} \frac{(2n)!}{n!n!}}.$$

Applying the Stirling’s approximation we obtain the following

$$\lim_{n \to \infty} \frac{\sum_{1 \leq j \leq \frac{k}{2}} 4^{n+1} \frac{k}{2} \cos^{2n} \left( \frac{\pi}{k+1} \right)}{\frac{1}{n+1} \frac{(2n)!}{n!n!}} = \lim_{n \to \infty} \frac{2k(n+1)\sqrt{n} \cos^{2n} \left( \frac{\pi}{k+1} \right)}{k+1}.$$

Bearing in mind that $\cos^{2n} \left( \frac{\pi}{k+1} \right)$ approaches zero faster than $\frac{1}{(n+1)\sqrt{n}}$, it follows immediately that the last limit is zero, leaving $\lim_{n \to \infty} P_{n,k} = 1$. □

Acknowledgment

We wish to express our sincerest gratitude towards Dragan Stevanović for pointing out references [7] and [3], as well as towards the anonymous referee for the thoughtful and constructive remarks which helped improve the text.

Research supported by Grants OI 174018 and III 46005 of the Ministry of Education and Science of the Republic of Serbia.
References

[1] N. G. de Bruijn, D. E. Knuth, S. O. Rice, *The average height of planted plane trees, in Graph theory and computing*, edited by R.C.Read, Academic Press, New York 15-22(1972)

[2] S. Gilliand, C. Johnson, S. Rush, D. Wood, *The sock matching problem*, Involve, 7(5) (2014), 691–697.

[3] C. M. da Fonseca, M. L. Glasser, V. Kowalenko, *Basic trigonometric power sums with applications*, Ramanujan J. 42 (2017), 401–428.

[4] A. Ilić, A. Ilić, *On the number of restricted Dyck paths*, Filomat, 25(3) (2011), 119–201.

[5] W. Panny, H. Prodinger, *The expected height of paths for several notions of height*, Studia Scientiarum Mathematicarum Hungarica 20 (1985), 119–132.

[6] B. Pantić, O. Bodroža-Pantić, *A Brief Overview of the Sock Matching Problem*, arXiv:1609.08353v1 [math.CO] 27 Sep 2016

[7] H. Prodinger, *The number of restricted lattice paths revisited*, Filomat, 26 (6) (2012), 1133–1134.

[8] R. P. Stanley, *Enumerative Combinatorics*, Vol. I, Cambridge University Press, Cambridge, 2002.

[9] R. P. Stanley, *Catalan Addendum to Enumerative Combinatorics*, Volume 2, version of 25 May 2013, http://www-math.mit.edu/~rstan/ec/catadd.pdf 2013.