Infra-red fixed points revisited.

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Abstract

We reconsider how Yukawa couplings may be determined in terms of a gauge coupling through the infra-red fixed point structure paying particular regard to the rate of approach to the fixed point. Using this we determine whether the fixed point structure of an underlying unified theory may play a significant rôle in fixing the couplings at the gauge unification scale. We argue that, particularly in the case of compactified theories, this is likely to be the case and illustrate this by a consideration of phenomenologically interesting theories. We discuss in particular what the infra-red fixed point structure implies for the top quark mass.
1 Introduction

The idea that there may be a stage of unification beyond the standard model has led to
the realisation of the importance of radiative corrections in determining the dimension
$\leq 4$ terms in the lagrangian. This, together with an assumption about unification of
couplings, has led to the successful prediction of the ratio of gauge couplings in the
minimal supersymmetric extension of the Standard Model with a gauge unification scale
$M_X \approx 10^{16}$GeV. The same ideas applied to the soft SUSY breaking terms leads to a
convincing description of the origin of electroweak breaking. In this paper we consider
again whether radiative corrections can also determine the Yukawa couplings and hence
the masses and mixing angles of the theory. The question is made more timely by the
evidence for a top quark with a mass of $O(174$GeV$)$ for a large mass is characteristic of
the expectations following from the infra-red fixed point structure of the Standard Model
and its extensions.

We start by reviewing the fixed point structure found in the Standard Model. Keeping
only the top Yukawa coupling, $h_t$, and the gauge couplings, $g_{3,2,1}$, we have

$$8\pi^2 \frac{d \ln(h_t)}{dt} = 8g_3^2 + \frac{3}{4}(3g_2^2 + g_1^2) + \frac{2}{3}g_1^2 - \frac{9}{2}h_t^2$$

where $t = \ln(\mu^2/\mu_0^2)$ and $\mu_0$ and $\mu$ are the initial and final scales at which the couplings are
determined.

If one ignores the smaller gauge couplings $g_2$ and $g_1$ the equation has a fixed point
structure which relates the top Yukawa coupling to the QCD coupling $g_i$. We have

$$8\pi^2 \frac{d \ln(h_t^2)}{dt} = g_3^2 - \frac{9}{2}h_t^2$$

giving the infra-red stable fixed point value \[1\]

$$(h_t^2)^* = \frac{2}{9}g_3^2$$  \hspace{1cm} (3)

However, as stressed by Chris Hill [2], this fixed point value is not reached for large
initial values of the top quark coupling because the range in $t$ as $\mu$ varies between between
the Planck scale and the electroweak scale is too small to cause the trajectories closely to
approach the fixed point. Rather Hill showed a “Quasi fixed point” governs the value of
$h_t$ for large initial values of $h_t$. To exhibit this it is useful to provide a complete analytic
solution to eq[2].

2 Analytic solution of the renormalisation group
equations

We first solve the general form of the renormalisation group equations for the case of a
single dominant Yukawa coupling and several gauge couplings

$$\frac{dg_i^2}{dt} = -\frac{h_ig_i^4}{(4\pi)^2}$$

$$\frac{dY_t}{dt} = Y_t(\sum_i r_i\tilde{\alpha}_i - sY_t)$$  \hspace{1cm} (4)
where

\[ \tilde{\alpha}_i = \frac{g_i^2}{(4\pi)^2} \]

\[ Y_i(t) = \frac{h_i^2}{(4\pi)^2} \]

\[ \beta_i = \tilde{\alpha}_i(0)b_i \]

The solution to these equations is

\[ \tilde{\alpha}_i(t) = \frac{\tilde{\alpha}_i(0)}{1 + \beta_i t} \]

\[ Y_i(t) = \frac{Y_i(0)E_1(t)}{1 + sY_i(0)F_1(t)} \]

where

\[ E_1(t) = \Pi_i (1 + \beta_i t)^{(B_i - 1)} \]

\[ F_1(t) = \int_0^t E_1(t')dt' \]

and

\[ B_i = \frac{r_i}{b_i} + 1 \]

### 2.1 Fixed point structure

In the Standard Model if we keep only the dependence on the largest gauge coupling, \(g_3\), the equations have an infra-red-fixed point. In this instance we have

\[ E_1(t) = (1 + \beta_3 t)^{(B_3 - 1)} \]

\[ F_1(t) = \frac{(1 + \beta_3 t)^{B_3}}{B_3^{\beta_3}} - \frac{1}{B_3^{\beta_3}} \]

The solutions presented in the previous section do not make explicit the fixed point structure. To do this it is necessary to eliminate the functions \(E_1\) and \(F_1\) in eq(\ref{eq:6}) giving

\[ \frac{Y_i(t)}{\tilde{\alpha}_3(t)} = \frac{Y_i(0)}{\tilde{\alpha}_3(0)} \frac{(\frac{\alpha_3(0)}{\alpha_3(t)})^{B_3}}{1 + sY_i(0)(\frac{\alpha_3(0)}{\alpha_3(t)})B_3 - 1} \]

After some algebra this reduces to

\[ \frac{Y_i(t)}{\tilde{\alpha}_3(t)} = \left( \frac{Y_i}{\tilde{\alpha}_3} \right)^* \frac{1}{1 + (\frac{\alpha_3(t)}{\alpha_3(0)})B_3(\frac{Y_i}{\tilde{\alpha}_3})^*(\frac{\tilde{\alpha}_3(0)}{Y_i(0)} - \frac{\alpha_3}{Y_i})^*} \]
where the star superscript denotes the fixed point value

\[
\left( \frac{Y_t}{\alpha_3} \right)^* = \frac{B_3 b_3}{s}
\]  

(12)

In this form it is clear that as \( t \to \infty \) the ratio of the Yukawa coupling to the gauge coupling tends to its fixed point value because \( \frac{\alpha_3(t)}{\alpha_3(0)} B_3 \to 0 \). In the case of the Standard Model

\[
b_3 = -7
\]

\[
r = 8
\]

\[
s = \frac{9}{2}
\]  

(13)

giving \( \left( \frac{Y_t}{\alpha_3} \right)^{SM}_S = \frac{2}{9} \) and \( B_3 = -\frac{1}{17} \). The smallness of the power \( B_3 \) in eq(11) together with the slow evolution of the QCD coupling in the range \( M_X \) to \( m_t \) means that in practice the fixed point is not reached. If we take \( M_X = O(10^{16} \text{GeV}) \) then \( \frac{\alpha_3(t)}{\alpha_3(0)} B_3 \approx 0.8 \) so we are quite far away from the true fixed point. This means that, in general, one will be sensitive to the initial value of the top Yukawa coupling. For example if \( \frac{Y_t(0)}{\alpha_3(0)} \gg \left( \frac{Y_t}{\alpha_3} \right)^* \) then it will remain at its fixed point. On the other hand if \( \frac{Y_t(0)}{\alpha_3(0)} \gg \left( \frac{Y_t}{\alpha_3} \right)^* \) then we may see from eq(11) that the value at low scales are relatively insensitive to the initial value giving a “Quasi-fixed-point” value which is really just the fixed point value radiatively corrected by known gauge boson contributions

\[
\left( \frac{Y_t}{\alpha_3} \right)^{QFP} = \frac{\left( \frac{Y_t}{\alpha_3} \right)^*}{1 - \left( \frac{\alpha_3(t)}{\alpha_3(0)} \right) B_3}
\]  

(14)

Again using \( M_X = O(10^{16} \text{GeV}) \) we find the Quasi-fixed-point gives a value for the top quark Yukawa coupling and hence the top quark mass approximately twice the true fixed point value, 220GeV rather than 110GeV\(^1\). In addition to the QCD gauge corrections just discussed which correct for the fact that the RG flow is over a relatively small distance there are also significant corrections due to the \( SU(2) \otimes U(1) \) gauge interactions. In this case the renormalisation group equations eq(11) do not have an exact infra-red-fixed point and we must use the full solution to the renormalisation group equations eq(6). Including these effects the quasi fixed point for the top pole mass is 240GeV.

In the case of the minimal supersymmetric standard model (MSSM) we have \( b_3, r, s \)

\[
b_3 = -3
\]

\[
r = \frac{16}{3}
\]

\[
s = 6
\]  

(15)

resulting in a larger value for \( \left( \frac{Y_t}{\alpha_3} \right)^*_{MSSM} = \frac{7}{18} \) and \( B_3, B_3 = -\frac{7}{5} \). In this case \( \frac{\alpha_3(t)}{\alpha_3(0)} B_3 \approx 0.46 \) giving a somewhat closer approach to the true fixed point; the Quasi-fixed-point is approximately a third greater than the true fixed point at 155sin\( \beta \) where tan\( \beta \) is the ratio of the two Higgs vacuum expectation values of the MSSM. The effects of including \( SU(2) \otimes U(1) \) corrections is illustrated for the case of the MSSM by the graph of Fig[1]. In this graph we plot the value of \( (m_t/sin\beta) \) versus the ratio of the gauge to Yukawa couplings

\(^1\)All masses quoted here are pole masses, related to running masses by a simple correction \[4\].
evaluated at the unification scale normalised by their fixed point value. The effect of the quasi fixed point is clear from the graph because the region of large $\tilde{Y}_t(0)$ gives values of $m_t$ focused in a small region (for $\tilde{\alpha}_3(0)/\tilde{Y}_t(0) < 0.1(\tilde{\alpha}_3)^*_{MSSM}$ we find $205 < \frac{m_t}{\sin \beta} < 210 \text{GeV}$).

Of course the interesting question is whether the top mass is determined by this (quasi) fixed point behaviour. Until the value of $\sin \beta$ is determined this is not known but clearly the large value predicted by the quasi-fixed-point is needed to accommodate a top mass of $O(174 \text{GeV})$.

3 Are “Unified” fixed points relevant to the determination of quark and lepton masses and mixing angles?

As we have seen the quasi-infra-red fixed point may be of importance in determining the top quark mass in supersymmetric theories. The difficulty in being more definite lies partly with the uncertainty in $\beta$ and partly in the fact that (cf. Fig. 1) the value of the Yukawa coupling at low energies is not completely insensitive to the initial value. The former difficulty will be eliminated when (and if) the structure of the low-energy effective supersymmetric theory is determined experimentally and $\tan \beta$ measured. We shall consider here whether the fixed point structure of the underlying theory beyond the Standard Model sheds light on the initial value at the gauge unification scale and hence the latter difficulty. The obvious attraction of this would be that the couplings would be determined simply in terms of the gauge couplings by the dynamics and knowledge of the gauge group and multiplet content would suffice to determine the couplings.

At first sight it appears that the infra-red structure of the theory beyond the Standard Model will play no significant role in the determination of the parameters at the gauge
unification scale because the domain over which the renormalisation group flow is relevant is too small to give appreciable approach to any infra-red fixed points. This may be seen clearly in the solution of Section 2.1. From eq(11) we see that the important parameter determining whether the fixed point is reached or not is \( (\alpha(t)/\alpha(0))^B \). Given that the maximum range of \( t \) over which the renormalisation flow can be computed runs only from the compactification scale, \( \Lambda_c \), (or the Planck scale, \( M_P \), which is expected to be slightly larger) to the gauge unification scale, \( M_X \approx 10^{16} \text{GeV} \), we see that \( t \leq \ln((\Lambda_c/M_X)^2) \approx 9 \), much less than the value \( t \approx 65 \) that is relevant in the flow from the gauge unification scale to the electroweak breaking scale! As a result we might expect that \( (\alpha(M_X)/\alpha(M_C))^B \) is not small and that the infra-red-fixed point plays a negligible role in determining the couplings at the gauge unification scale.

However we shall argue that there are reasons why this conclusion is likely be wrong in many extensions of the Standard Model. Firstly the value of \( (\alpha(M_X)/\alpha(M_C))^B \) depends on the exponent \( B \) which, as we shall demonstrate by specific examples, is likely to be much larger than in the Standard Model or MSSM. Secondly the ratio of couplings under the exponent depends not only on \( t \) but also on the beta function above gauge unification. Again, as we shall demonstrate by considering realistic examples, this is expected to be larger than that of the MSSM. These effects act in the same direction and can lead to the fixed points structure of the theory beyond the Standard Model being more important than in the Standard Model or the MSSM. There is an even more compelling reason for the importance of the fixed point structure which applies to theories with a stage of compactification. In this case the evolution of couplings is much faster (following a power law rather than a logarithmic evolution) leading to a very small value for \( (\alpha(M_X)/\alpha(M_P))^B \). We shall illustrate these points by several representative examples.

We first consider some phenomenologically interesting extensions of the Standard Model which have infra-red-fixed points for the ratio of gauge to Yukawa couplings. These examples apply in uncompactified theories or, if there is a stage of compactification, below the compactification scale. Following from the discussion of Section 2.1, realisation of this idea requires a theory with a single gauge group or a product of identical gauge groups coupling to the quark fields for otherwise the gauge couplings will (in all probability) have different \( \beta \) functions and hence will not have infra-red-fixed points for the ratio of gauge to Yukawa couplings\(^{3}\). We will consider several examples of such a theory namely supersymmetric\(^{4}\) \( SU(5) \), \( SO(10) \) and \( SU(3) \otimes SU(3) \otimes SU(3) \). The latter is the simplest such extension of the Standard Model and we consider it first.

### 3.1 SU\(^{(3)}\)^3

The group SU\(^{(3)}\)^3 has many attractions as a non-Grand-Unified extension of the Standard Model. Provided the multiplet content is chosen symmetrically, the gauge couplings above the SU\(^{(3)}\)^3 unification scale, \( M_X \), evolve together. As a result it offers an example of a theory which can be embedded in the superstring which preserves the success of the unification predictions for gauge couplings even if, as has been found to be usually the case above the Planck scale unknown gravitational effects are important.\(^{2}\)

\(^3\)Also, more seriously, theories with more than one gauge coupling will spoil the success of the minimal unification of gauge couplings due to the additional running of gauge couplings above \( M_X \).

\(^4\)We restrict our attention to supersymmetric theories to preserve the success of the unification predictions for gauge couplings and quark and lepton masses but the general structure considered here applies equally well to non-supersymmetric theories.
case in the compactified string theories so far analysed, the unification or compactification scale is much higher than $10^{16} \text{GeV}$. It also has a relatively simple breaking pattern taking it to the Standard Model using just the fundamental representations of Higgs fields and this means it fits nicely into superstrings built using level-1 Kac-Moody algebras. Indeed there are 3-generation examples known of semi-realistic compactified string theories with the gauge group $SU(3)^3$.

For our purposes here it is not necessary to know the string origin (if any) for, as we stressed above, the fixed point structure determines the couplings of the theory. All that is needed is the multiplet content and the remaining (discrete) symmetry structure which dictates the allowed couplings. The light multiplet structure after symmetry breaking is just that of the MSSM. Shafi et. al. [6] have shown how this can naturally come from a model with $SU(3)_3$ symmetry through discrete symmetries. The multiplet content before symmetry breaking we take to consist of $n_g$ families contained in $n_g$ copies of $I$ representations where $I = ((1, 3, \bar{3}) + (\bar{3}, 1, 3) + (3, \bar{3}, 1))$. In addition there are two further copies of $I$ representations which contain the Higgs fields. The renormalisation group equations for this theory are given by

$$
\frac{d \tilde{\alpha}_i}{dt} = -6 \tilde{\alpha}_i^2
$$

$$
\frac{d Y_t}{dt} = (16 \tilde{\alpha} - 9 Y_t) Y_t
$$

where in the second equation we have used the fact that all three gauge couplings are equal, $\tilde{\alpha}_i = \tilde{\alpha}$, (assuming equal initial values). Note that the gauge couplings are not asymptotically free due to the profusion of matter fields. This we believe is a very common feature of extensions of the Standard Model which include all the fields necessary to break the symmetry fully. We will return to the implications of this shortly.

Applying the results derived above we see that there is indeed a fixed point given by $(\frac{Y_t}{\tilde{\alpha}})_{SU(3)_3} = \frac{22}{9}$ and $B = \frac{11}{3}$, much larger than in the cases considered above. The important factor determining the rate of approach to the fixed point is $(\frac{\alpha(M_X)}{\alpha(M_C)})^B$ and because above $M_X$ the theory is not asymptotically free the larger $B$ makes this factor smaller. In addition the factor also decreases because the running coupling evolves faster due to the larger $\beta$ function. If we take the conservative view and evolve from the compactification scale $O(10^{18} \text{GeV})$ to the gauge unification scale $O(10^{16} \text{GeV})$, normalising the gauge coupling to the MSSM gauge unification value $\alpha^{-1}(M_X) \approx 24$, we find $(\frac{\alpha(M_X)}{\alpha(M_C)})^B \approx 0.48$. Thus, the fixed point in the evolution between $10^{18} \text{GeV}$ to $10^{16} \text{GeV}$ plays a role as important in the $SU(3)^3$ theory as does in the MSSM evolving between $10^{16} \text{GeV}$ and $10^{2} \text{GeV}$! Following the analysis of Section 2.1 we expect the Yukawa coupling, for large initial values at the compactification scale, to be some 30% larger than the fixed point value giving, from eq(14)

$$
Y_t(M_X) \approx 4.2
$$

In terms of the MSSM fixed point value plotted in Fig 1 this corresponds to $(\frac{\tilde{\alpha}(M_X)}{Y_t(M_X)})/(\tilde{\alpha})_{MSSM} \approx 0.08$ which means the low energy value will be very close to the quasi fixed point value. The effect of this on the final prediction for the top quark mass may be quantified by noting that if the initial value of $(\frac{\tilde{\alpha}(M_P)}{Y_t(M_P)})/(\tilde{\alpha})_{MSSM}$ is $x$ then

$$
x' = \frac{Y_t(M_X)}{Y_t(M_X)} \frac{(\tilde{\alpha})_{MSSM}}{(\tilde{\alpha})_{MSSM}}
$$
\[
\begin{array}{|c|c|c|c|}
\hline
n & \left(\frac{\tilde{\alpha}}{\alpha}\right)^* \left(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B & \left(\frac{\tilde{\alpha}}{\alpha}\right)^* \left(\frac{1}{\alpha(M_C)}\right)^B & \left(\frac{\tilde{\alpha}}{\alpha}\right)^* \left(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B \\
\hline
0 & 2.44 & 0.48 & 0.16 \\
2 & 3.78 & 0.22 & 0.10 \\
4 & 5.11 & 0.02 & 0.08 \\
\hline
\end{array}
\]

Table 1: $SU(3)^3$

\[
= x \left(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B + \left(\frac{\tilde{\alpha}}{\alpha}\right)^* \left(1 - \left(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B\right) \left(\frac{\tilde{\alpha}}{\alpha}\right)^* \left(\frac{1}{\alpha(M_C)}\right)^B
\]

Small $x'$ means the Yukawa coupling is in the domain of attraction of the quasi fixed point of the MSSM. If the second term is the largest then we see we are already in this domain since $(\frac{\tilde{\alpha}}{\alpha})(M_X)/\left(\frac{\tilde{\alpha}}{\alpha}(M_C)^B\right) << 1$. If the first term is largest there is still a focusing effect taking the coupling towards the domain of attraction because eq(17) gives approximately $x' = x\left(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B$. To illustrate the effect note that if $x \leq 1/2$ at $M_X$ the range of values for the top mass is $186 GeV \leq m_t \leq 210 GeV$; if $x \leq 1/2$ at $M_C$ after focusing the range of values becomes $195 GeV \leq m_t \leq 210 GeV$.

Let us consider how stable is our result to changes in the structure of the $SU(3)^3$ theory. In many string compactifications there arise additional states in conjugate representations which acquire mass at the stage of gauge symmetry breaking and do not appear in the low energy theory. Let us consider the effect of such states by adding to our theory $n$ copies of chiral superfields in $(I + \bar{I})$ representations. First we consider the case that their Yukawa couplings are small and so the only effect of these fields is to change the gauge beta function $b_i = 6 + 6n$. This in turn affects the position of the fixed point giving $(\frac{\tilde{\alpha}}{\alpha})^* = \frac{22+6n}{9}$. Now the systematic effect of adding additional matter is to increase $Y_t$, driving it closer to the quasi fixed point. It also changes (reduces) B while increasing the rate of change of the running coupling, these changes going in opposite directions in the determination of $(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B$. Putting this together gives $(\frac{\tilde{\alpha}(M_X)}{\alpha(M_C)}\right)^B = 0.22, 0.02$ for $n = 2, 4$ respectively showing that the effect of additional matter is to speed up the approach to the fixed point. The results are summarised in Table 5. In particular note that for the case of $n=4$ or more the ratio of couplings are within 1% of the fixed point value.

### 3.2 $SU(5)$

In our next example we will consider an $SU(5)$ model. Matter is arranged in three generations in $I = \{\psi^{xy}(10) + \phi_x(5)\}$ representations together with another copies of chiral superfields in $(I + \bar{I})$ representations plus a Higgs sector made up of a (complex) adjoint, $\Sigma(24)$, to break $SU(5)$ and a set of Weinberg-Salam 5plets $H_1(5) + H_2(\bar{5})$. Keeping only the Yukawa coupling leading to the top quark mass the renormalization group equations are given by:

\[
\frac{d\tilde{\alpha}}{dt} = (3 - 4n)\tilde{\alpha}^2
\]

\footnote{In the Tian Yau three generation theory $n=6$.

8
\[
\frac{d\tilde{\alpha}}{dt} = (\frac{7}{2} - 2n)\tilde{\alpha}^2
\]

\[
\frac{dY}{dt} = (\frac{27}{4}\tilde{\alpha} - 14Y)Y
\]

The gauge coupling is asymptotically free only for \(n=0\). Then, the fixed points are given by:

\[
\left(\frac{Y_t}{\tilde{\alpha}}\right) = 2\left(\frac{48}{5}\tilde{\alpha} - 9Y_t\right)Y_t
\]

For \(n=0\), \(B = -\frac{27}{5}\) and evolving from the compactification scale to the gauge unification scale we get \((\frac{\alpha(M_X)}{\alpha(M_C)})^B \approx 0.62\). It can be seen that in this model, the fixed point structure plays a slightly less important role than in the SU(3)\(^3\) case. However for \(n \neq 0\) the approach to the fixed point is more rapid as is shown in Table (3.2) and essentially reaches the fixed point for \(n \geq 8\). In all cases the fixed point value is in the domain of attraction of the quasi fixed point of the MSSM.

3.3 \(SO(10)\)

One of the advantages of \(SO(10)\) over SU(5) Grand Unification is that only one 16-dimensional spinor representation of \(SO(10)\) is needed to accommodate all fermions (including the right handed neutrino) of one generation. Unlike SU(5), \(SO(10)\) is a group of rank 5 with the extra diagonal generator of \(SO(10)\) being B - L as in the left-right symmetric groups. Because of this there exist several intermediate symmetries through which \(SO(10)\) can descend to the \(SU(3) \otimes SU(2) \otimes U(1)\) group. For this paper we consider the simplest route which is given by the following stages of symmetry breaking:

\[
SO(10) \rightarrow SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1) \rightarrow SU(3) \otimes U(1)
\]

where we have also displayed the minimal set of Higgs representations needed to achieve the breaking pattern. The renormalization group equations for this model are given by:

\[
\frac{d\tilde{\alpha}}{dt} = (\frac{7}{2} - 2n)\tilde{\alpha}^2
\]

\[
\frac{dY}{dt} = (\frac{27}{4}\tilde{\alpha} - 14Y)Y
\]
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
n & (\frac{\alpha}{\alpha}) & (\frac{\alpha}{\alpha}) & (\frac{\alpha}{\alpha}) \\
\hline
0 & 0.23 & 0.91 & 1.70 \\
2 & 0.52 & 0.80 & 0.75 \\
4 & 0.80 & 0.69 & 0.49 \\
8 & 1.38 & 0.48 & 0.28 \\
\hline
\end{tabular}
\caption{SO(10)}
\end{table}

approach to the fixed point is slower than in the other examples and the fixed point value of the top coupling larger. For large n, however, the fixed point structure is still likely to be important in determining the low energy parameters.

### 3.4 Compactified (string) models

As we mentioned above, in compactified theories the evolution of couplings above the compactification scale is much faster (following a power law rather than a logarithmic evolution) leading to a very small value for \((\frac{\alpha(M_C)}{\alpha(M_P)})^B\). To illustrate this consider first the simple case of a dimension D=5 theory in which one dimension is a circle of radius \(R = \Lambda_c^{-1}\). We are interested in computing the one loop corrections to the effective action, for example a loop with two external gauge bosons at a scale Q. Then one finds for the polarisation tensor the form \[8\]

\[\Pi_{\mu\nu}(Q) = i(Q^\mu Q^\nu - Q^2 g^{\mu\nu}) \Pi(Q)\] (22)

where

\[\Pi(Q) \approx \beta_0 \sum_{n=0}^{n=\Lambda_s} \int \frac{d^4 P}{(2\pi^4)} \frac{1}{P^2 + n^2 \Lambda_c^2} \frac{1}{(P + Q)^2 + n^2 \Lambda_c^2}\] (23)

Here \(\Lambda_c\) is the compactification scale and \(\Lambda_s\) is the scale beyond which the theory changes, for example the string scale, and above which it makes no sense to use an effective field theory approximation. The sum includes the Kaluza Klein modes with mass, \(n\Lambda_c\), less than or equal to \(\Lambda_s\). One would think that the appearance of such massive states would invalidate the use of mass independent renormalisation group equations but this is not the case for the sum of their contributions just serves to change the original propagator in 4D to one in 5D leaving the original renormalisation group equation intact. At scales \(Q << 1/R\) evaluation of eq(23) gives

\[\Pi(Q) \approx -\frac{\beta_0}{(4\pi)^2} \frac{(\ln(QR)^2 - 2(\Lambda_s R - 1))}{\Lambda_c^2 (P + Q)^2 + n^2 \Lambda_c^2}\] (24)

The important point to note is that the integration over the range \(\Lambda_c < P < \Lambda_s\) generates a power, \((\Lambda_s/\Lambda_c)^2\) instead of \(\log(\Lambda_s/\Lambda_c)\). This happens because in this range the theory is effectively 5 dimensional and so the loop contribution is approximately

\[\Pi(Q) \approx i\beta_0 \frac{(\Lambda_s R - 1)}{(4\pi)^2}\] (25)
As a result the effect of running in this region is enhanced

\[ \alpha^{-1}(t_c) = \alpha^{-1}(0) + \frac{b}{4\pi} \left( \frac{M_P}{\Lambda_c} \right)^{D-4} \quad (26) \]

where \((D - 4)\) is the number of compactified dimensions. Even for a small difference between the compactification and Planck scale the change from logarithmic to power law evolution will make the factor \((\frac{\alpha(M_c)}{\alpha(M_p)})^B\) very small implying that the infrared fixed point structure will dominate the determination of couplings.

Of course the discussion so far has been much oversimplified for compactification on a circle does not lead to realistic theories. However the effect persists, if somewhat ameliorated, in realistic compactifications. This has been most thoroughly studied in the context of superstring theories constructed via orbifold compactification in which the terms we have been discussing are known as threshold effects of the Kaluza Klein modes \([3, 10, 11]\). The simplest example, the symmetric orbifold models, have Kaluza Klein modes which fall into \(N=4, N=2\) and \(N=1\) supermultiplets. The former do not renormalise the gauge couplings but the latter two do giving

\[ \alpha^{-1}(M_s) = \alpha^{-1}(0) - \frac{\tilde{b}_a}{4\pi} \ln(|\eta(iT)|^4(T + \bar{T})) + c_a \quad (27) \]

where \(c_a\) are moduli independent constants coming from the \(N = 1\) sector and and \(\tilde{b}_a\) is the beta function of the gauge group factor in the \(N=2\) sector. \(T\) is the expectation value of a moduli field which sets the scale of compactification, the toroidal radius being given by \(R^2 = Re(T)\). The function \(\eta(T)\) is the Dedekind eta-function \((\eta(T)q^{1/24}\Pi_n(1 - q^n), q(T) = exp(-2\pi T))\) We see therefore that the naive expectations are changed in orbifold compactification due to the different multiplet structure of the massive excitations. The effects do persist however, albeit somewhat reduced. To see this explicitly let us expand eq(27) for large \(T\) corresponding to large radius of compactification\(\footnote{6}\). This gives

\[ \alpha_a^{-1}(t_c) \approx \alpha^{-1}(0) + \frac{\tilde{b}_a}{4\pi} \left( \frac{M_P}{\Lambda_c} \right)^2 \quad (28) \]

It may be seen that, up to the Green Schwarz term and for modular weight -1 fields, this is just the result of eq(26) with \(two\) compactified dimensions corresponding to the sector with \(N=2\).

Of course the fixed point structure in general also changes above the compactification scale. We have seen that the relevant \(\beta\) function is that of the \(N=2\) sector which in general may differ from the theory below the compactification scale. The Yukawa coupling evolution also changes. The wave function threshold corrections of an untwisted field \(A_j\) associated with the \(j\)-th of the three internal compactified planes of the orbifold has been computed \([1]\)

\[ Y_j = 2\tilde{\gamma}_j \ln(|\eta(iT_j)|^4(T_j + \bar{T}_j)) + y_j \quad (29) \]

where \(y_j\) is a moduli-independent constant and the coefficient \(\tilde{\gamma}_j\) is the anomalous dimension of the \(A_j\)-field in the corresponding \(N = 2\) supersymmetric theory. Since this field belongs to an \(N = 2\) vector supermultiplet \(\tilde{\gamma}_j = -\tilde{b}_j/2\) where \(\tilde{b}_j\) is the corresponding \(\beta\)

\footnote{6}{For states of modular weight -1 and vanishing Green Schwarz term \(\tilde{b} = b\).}

\footnote{7}{Note that in string theories the string scale \(\Lambda_s\) is somewhat larger than the compactification scale \(\Lambda_c\) by a factor \(\leq O(10)\)}
function coefficient of any gauge subgroup that transforms \( A_j \) non-trivially in the embedding \( N = 2 \) theory. The Yukawa coupling comes from the term in the superpotential \( W = 2A_1 A_2 A_3 \). Including the wave function normalisations the tree level physical Yukawa coupling is \( \lambda_{123} = \sqrt{2} g \) the relation to the gauge coupling corresponding to the fact the fields belong to \( N=2 \) supermultiplets. Including the effects of string threshold corrections (i.e. the Kaluza Klein modes) gives

\[
Y_{123}(M_s) = \frac{2\tilde{\alpha}_a}{1 + 4\pi \alpha_a(M_s)(y_1 + y_2 + y_3)}
\]

After a little algebra this may be rewritten in the form of eq(11) with

\[
t = (|\eta(iT_j)|^4(T_j + \bar{T}_j))
\]

\[
B \quad = \quad 1
\]

\[
\left(\frac{Y_{123}}{\tilde{\alpha}}\right)_{\text{orbifold}}^* \quad = \quad 2
\]

The fixed point corresponds to the \( N=2 \) value \( \lambda = \sqrt{2} g \) when the initial \( N=2 \) breaking terms \( y_i \) become irrelevant. The rate of approach is just determined by \( \alpha(t)/\alpha(0) \). Due to the power law evolution this may be very small in theories with a lot of matter and a non-asymptotically free coupling in the \( N = 2 \) sector. For example with the same \( \beta \) function as in the \( SU(3)^3 \) case, normalising the coupling at \( t_s \) to be the unified coupling 1/24, we would get the tree level coupling, \( \tilde{\alpha}(0) \) to be of \( O(1) \) for \( M_c/M_s=1/5! \) In this case a reasonable estimate for \( (\alpha(M_X)/\alpha(M_S))^B \) is 1/24 implying the fixed point is very closely approached.

Of course the value of the couplings at \( M_s \) must be fed into the RG equations for the theory below the compactification scale to take it to \( M_X \) where the MSSM RG equations take over. To quantify this we simply take the focusing formula from eq(17). Using eq(31) shows \( x(M_S) \approx 24x(0) \) so the range of values in the domain of attraction of the MSSM fixed point is very large. If \( x(M_s) \leq 1/2 \) then after focusing the range of values becomes 209 \( \leq m_t \leq 210 \).

## 4 Summary

In conclusion we have found that the infra-red fixed point structure of the unified theory beyond the Standard Model is likely to play a very important and in many cases a dominant role in the determination of ratios of the Yukawa to gauge couplings. This means that these couplings may be determined simply from a knowledge of the multiplet content and gauge group structure without needing to know their value in the underlying “Theory of Everything”. (In string theories this may avoids the difficult question of determining the moduli dependence of the couplings and the values of the moduli). The most obvious prediction is for the top quark mass and we have argued that it is very likely to be very close to its quasi fixed point value. However many further predictions for the effective low-energy theory follow from the IR fixed point structure corresponding to the appearance of the hidden symmetries of the RG equations. For example unrelated Yukawa couplings will be driven to be equal at the fixed point if they involve chiral superfields carrying the same gauge quantum numbers as in our \( SU(3)^3 \) example discussed above, the RG fixed point structure will make them equal. Similar conclusions follow for other fixed point
structures not discussed here relating the soft SUSY breaking mass terms of the MSSM [12]. It may therefore be hoped that a study of this type of Infra Red fixed points will shed light on the values of all the parameters of the Standard Model and not just the top quark mass predictions presented here.

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