AUTOMORPHIC FUNCTIONS AS THE TRACE OF FROBENIUS

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Abstract. We prove that the trace of the Frobenius endofunctor of the category of automorphic sheaves with nilpotent singular support is isomorphic to the space of unramified automorphic functions, settling a conjecture from [AGKKRV1]. More generally, we show that traces of Frobenius-Hecke functors produce shtuka cohomologies.

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Introduction
0.1. Overview. This work is part of a series, following [AGKRRV1] and [AGKRRV2], attempting to understand the (unramified, function field) arithmetic Langlands conjectures via geometric and categorical techniques. We begin with an overview of the problems considered here.

0.1.1. Some of the most striking applications of geometric representation theory pass through the sheaves-functions dictionary of Grothendieck–Deligne.

Recall the setting: one has an algebraic stack $\mathcal{Y}$ over $\mathbb{F}_q$, assumed to be defined over $\mathbb{F}_q$, and an $\ell$-adic Weil sheaf $\mathcal{F}$ on $\mathcal{Y}$. For a rational point $y \in \mathcal{Y}(\mathbb{F}_q)$, one takes the trace of Frobenius on the stalk $y^*(\mathcal{F})$ to obtain an element of $\overline{\mathbb{Q}}_\ell$; this defines a function $\text{funct}(\mathcal{F}) : \mathcal{Y}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$.

One finds that many functions of interest in harmonic analysis over finite fields arise by this procedure, and that the perspective offered by sheaf theory provides deep insights into function theory.

For example, this is the case in the theory of automorphic functions (for function fields), whose realizations via $\ell$-adic sheaves exhibit explicit constructions of Langlands’ conjectures.

0.1.2. In this paper, we establish a categorical analogue of the sheaves-functions dictionary: instead of passing from sheaves to functions, we pass from categories (of sheaves) to vector spaces (of functions).

As in the previous setting, we decategorify using trace of Frobenius. However, whereas before we considered the trace of a Frobenius endomorphism of a vector space (and thus produced a scalar), we now consider the trace of a Frobenius endofunctor of a category (and thus produce a vector space).

Unlike the usual Grothendieck–Deligne paradigm, where one may take a general algebraic stack $\mathcal{Y}$, our results are specialized to spherical automorphic functions. The results of this paper establish conjectures from [AGKRRV1], which specify a relation between the category of automorphic sheaves and the vector space of automorphic functions via the trace of Frobenius.

0.1.3. We give a precise formulation of our main results below. However, by way of motivation, we highlight the following application, which illustrates how sheaf-theoretic considerations provide insights into the classical theory of automorphic functions.

In [AGKRRV1], we introduced a version of the geometric Langlands conjecture suitable for $\ell$-adic sheaves. From this conjecture, combined with the Trace Conjecture, proved in this paper, we deduced that the space of compactly supported spherical automorphic functions (denoted $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$ in the body of this text) can be described as

$$\text{(0.1)} \quad \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)) \simeq \Gamma(\text{LocSys}^\text{arthm}_G(X), \omega^\text{LocSys}^\text{arthm}_G(X)).$$
In the above formula, \( X \) is the smooth projective curve corresponding to our choice of function field and \( \text{LocSys}_G^{\text{arithm}}(X) \) is the algebraic stack over \( \mathbb{Q}_\ell \), defined in [AGKRRV1], parametrizing (unramified) Langlands parameters.\(^1\)

In the everywhere unramified function field setting, this conjecture provides an interesting alternative to Langlands’s original perspective: for general reductive \( G \), the above conjecture yields a complete description of the space of automorphic functions in terms of Galois data, whereas classical Langlands conjectures only concern \( L \)-packets.

**Remark 0.1.4.** The work [VLaf] of V. Lafforgue and its extension [Xue1] by C. Xue provide a decomposition of the space of (not necessarily unramified!) automorphic functions in terms of Langlands parameters. It should come as no surprise that our results are closely related to their work.

First, our work is also based on considering cohomologies of shtukas.

And second, our constructions show that \( \text{Funct}_c(\text{Bun}_G(F_q)) \) arises as global sections of some quasi-coherent sheaf \( \text{Drinf}_{\text{arithm}} \) on \( \text{LocSys}_G^{\text{arithm}}(X) \). (The conjecture expressed by formula \( (0.1) \) says that \( \text{Drinf}_{\text{arithm}} \) is the dualizing sheaf of \( \text{LocSys}_G^{\text{arithm}}(X) \).) The existence of \( \text{Drinf}_{\text{arithm}} \) recovers the spectral decomposition of \( \text{Funct}_c(\text{Bun}_G(F_q)) \) along the set of classical Langlands parameters, see Remark 3.4.2.

We refer to [AGKRRV1, Sect. 24] for further discussion of the relation to V. Lafforgue’s work.

**Remark 0.1.5.** The principle that geometric methods enrich Langlands’s conjectures is an old one, dating (at least) to [De] and [Dr]. But precise refinements of Langlands’s conjectures using geometric ideas have emerged recently. The present paper provides one example. Similarly, the Fargues–Scholze geometrization program (see [FS]) aims to produce a more robust form of the local Langlands conjectures.

### 0.2. Formulation of the main result

We now proceed to the statement of our main result.

**0.2.1. Notation.** Throughout the paper, we use algebraic geometry over the two fields \( k := F_q \) and \( e := \mathbb{Q}_\ell \) (where \( \ell \in F_q^\times \)).

When we work over \( F_q \), we generally work with algebraic stacks \( Y \) that are assumed to be defined over \( F_q \); we abuse notation somewhat in letting \( Y(F_q) \) denote the groupoid of \( F_q \)-points of the corresponding stack. We let \( \text{Frob}_Y : Y \to Y \) denote the geometric (\( q \)-)Frobenius morphism, whose stack of fixed-points \( Y^{\text{Frob}} \) is a discrete stack, and identifies with (the étale sheafification of) the groupoid \( \{ F_q \} \).

Let \( X \) be a smooth projective curve over \( F_q \), but assumed defined over \( F_q \). Let \( G/F_q \) be a reductive group, considered over \( F_q \) via its split form. Let \( \text{Bun}_G \) denote the moduli stack of principal \( G \)-bundles on \( X \).

We refer to Sect. 0.8 for further details on our conventions.

**0.2.2. Categorical trace.** Throughout this paper, all DG categories are enriched over the field \( e \). In particular, \( \text{Vect} = \text{Vect}_e \) (the DG category of chain complexes of \( e \)-vector spaces), and so on.

We remind that the category \( \text{DGCat} \) of presentable DG categories is equipped with a canonical symmetric monoidal structure \( \otimes \), the *Lurie tensor product*. On general grounds, this means one may speak about categorical duals and traces as follows.

If \( C \in \text{DGCat} \) is dualizable, there is another DG category \( C^\vee \) equipped with canonical unit and counit maps

\[
\text{u}_C : \text{Vect} \to C \otimes C^\vee
\]

and

\[
\text{ev}_C : C \otimes C^\vee \to \text{Vect}.
\]

For an endofunctor \( \Phi : C \to C \) of \( C \), we have \( \text{Tr}(\Phi, C) \in \text{Vect} \) defined as

\[
\text{Tr}(\Phi, C) := \text{ev}_C \left( (\Phi \otimes \text{Id})(\text{u}_C) \right).
\]

\(^1\)An alternative construction of \( \text{LocSys}_G^{\text{arithm}}(X) \) due to P. Scholze and X. Zhu, may be found in [Zhu]. Their construction proceeds along very different lines, but conjecturally produces an equivalent object.
We refer to [GKRV] for further discussion.

0.2.3. Categories of sheaves. We consider the category $\text{Shv}(\text{Bun}_G)$ of automorphic sheaves. Precisely, $\text{Shv}(\text{Bun}_G)$ is the category of ind-constructible $\mathcal{O}_\mathbb{F}_q$-sheaves on $\text{Bun}_G$. As in [AGKRRV1], we also consider its full DG subcategory

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$$

of objects with singular support in the global nilpotent cone.

The categories $\text{Shv}(\text{Bun}_G)$ and $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ have favorable finiteness properties: they are compactly generated and therefore dualizable in $\text{DGCat}$. Moreover, if we assume [AGKRRV1, Conjecture 14.1.8], the embedding $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)$ preserves compactness.

0.2.4. Pushforward with respect to the geometric Frobenius endomorphism (see Sect. 0.8.2) defines an auto-equivalence $(\text{Frob}_{\text{Bun}_G})^*$ of $\text{Shv}(\text{Bun}_G)$, which preserves the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

Hence, it makes sense to consider the categorical trace

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})^* , \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \in \text{Vect}.$$
0.3.4. Below, we will be considering functors $S : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$.

By the definition of $\text{Rep}(\hat{G})_{\text{Ran}}$, such functors amount to compatible systems of functors

$$S_I : \text{Rep}(\hat{G})_{\text{Ran}} \otimes \mathbb{I} \to \text{Shv}(X^I)$$

defined for $I \in \text{fSet}$.

In examples, the functors $S_I$ tend to be more familiar avatars of the functor $S$, so it is convenient to reference them.

0.3.5. On the one hand, we have a functor

$$Sht^T : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$$

constructed as the composition

$$\text{Rep}(\hat{G})_{\text{Ran}} \to \text{End}_{\text{DGCat}}(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)) -\circ (\text{Frob}_{\text{Bun}_G})^* \to \text{End}_{\text{DGCat}}(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \overset{T^T}{\to} \text{Vect}.$$ 

In other words, we take $V \in \text{Rep}(\hat{G})_{\text{Ran}}$, form the corresponding Hecke functor, compose with Frobenius, and take the trace of the resulting endofunctor of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

By construction, we have

$$Sht^T(1_{\text{Rep}(\hat{G})_{\text{Ran}}}) = \text{Tr}((\text{Frob}_{\text{Bun}_G})^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)).$$

Remark 0.3.6. To make the above more explicit, we describe the corresponding functors $Sht^T_I : \text{Rep}(\hat{G})_{\text{Ran}} \otimes \mathbb{I} \to \text{Shv}(X^I)$ for a finite set $I$ (see Sect. 0.3.4 just above).

Precomposing with $(\text{Frob}_{\text{Bun}_G})^*$, we obtain a functor

$$H(V, -) \circ (\text{Frob}_{\text{Bun}_G})^* : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{QLisse}(X)^{\otimes I},$$

and we can consider its parameterized trace

$$\text{Tr}(H(V, -) \circ (\text{Frob}_{\text{Bun}_G})^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \in \text{QLisse}(X)^{\otimes I}.$$ 

Unwinding the constructions, the resulting functor

$$\text{Rep}(\hat{G})_{\text{Ran}} \to \text{QLisse}(X)^{\otimes I},$$

followed by the embedding

$$\text{QLisse}(X)^{\otimes I} \hookrightarrow \text{Shv}(X^I),$$

is our $Sht^T_I$.

0.3.7. On the other hand, following [VLaf], to the data $(I \in \text{fSet}, V \in \text{Rep}(\hat{G})_{\text{Ran}})$ we can attach the compactly supported shtuka cohomology, which is an object

$$\text{Sht}(V) \in \text{Shv}(X^I),$$

see Sect. 3.1 in the main body of the paper. These functors satisfy the requisite compatibilities needed to define a functor

$$\text{Sht} : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}.$$ 

Example 0.3.8. For $I = \emptyset$, the functor $\text{Sht}_\emptyset$ amounts to a map $\text{Vect} \to \text{Vect}$, i.e., a vector space, which corresponds to $\text{Sht}(1_{\text{Rep}(\hat{G})_{\text{Ran}}})$. This vector space is $\text{Funct}_{c}(\text{Bun}_G(F_q))$.

0.3.9. Our second main result (it appears as Theorem 4.1.2 in the main body of the paper) asserts:

**Main Theorem 0.3.10.** There is a canonical equivalence

$$\text{Sht}^T \simeq \text{Sht}$$

of functors $\text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$.

In particular, Theorem 0.3.10 implies that the functors $\text{Sht}^T_I$ and $\text{Sht}_I$ are canonically isomorphic. This is exactly the assertion of the Shtuka Conjecture [AGKRRV1, Conjecture 22.5.7].

Theorem 0.2.6 stated above is obtained from Theorem 0.3.10 by evaluating both functors on the object $1_{\text{Rep}(\hat{G})_{\text{Ran}}} \in \text{Rep}(\hat{G})_{\text{Ran}}$. 
0.4. **How do we compute the trace?** In this subsection we explain a key ingredient that goes into the computation of the trace of the Frobenius endofunctor.

0.4.1. By way of motivation, let us try to compute the trace of an endofunctor of the form \( \phi_! \), where:

- \( \mathcal{Y} \) is an algebraic stack over \( \mathbb{C} \);
- \( \phi \) is its endomorphism;
- \( \phi_! \) is the direct image with compact supports functor on the category \( \text{Shv}^{\text{Betti}}(\mathcal{Y}) \) of all Betti sheaves on \( \mathcal{Y} \) with coefficients in \( \mathbb{C} \)-vector spaces (see, e.g., [AGKRRV1, Sects. G.1 and G.7]).

We claim that there is a canonical isomorphism

\[
\text{Tr}(\phi_!, \text{Shv}^{\text{Betti}}(\mathcal{Y})) \simeq C_c(\mathcal{Y}^\phi, \mathcal{E}_\mathcal{Y}^\phi),
\]

where \( \mathcal{Y}^\phi \) is the stack of \( \phi \)-fixed points on \( \mathcal{Y} \).

The computation proceeds as follows. First, one shows that the external tensor product functor

\[
\text{Shv}^{\text{Betti}}(\mathcal{Y}) \otimes \text{Shv}^{\text{Betti}}(\mathcal{Y}) \to \text{Shv}^{\text{Betti}}(\mathcal{Y} \times \mathcal{Y}).
\]

is an equivalence (see [GKRV, Corollary A.2.9]).

Thus, we can consider the object

\[
\text{ps-uy} := (\Delta_\mathcal{Y})!(\mathcal{E}_\mathcal{Y}) \in \text{Shv}^{\text{Betti}}(\mathcal{Y} \times \mathcal{Y})
\]

as an object of \( \text{Shv}^{\text{Betti}}(\mathcal{Y}) \otimes \text{Shv}^{\text{Betti}}(\mathcal{Y}) \).

Base change shows that the above object and the pairing

\[
\text{ev}_\mathcal{Y} : \text{Shv}^{\text{Betti}}(\mathcal{Y}) \otimes \text{Shv}^{\text{Betti}}(\mathcal{Y}) \to \text{Vect}, \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C_c(\mathcal{Y}, \mathcal{F}_1 \otimes \mathcal{F}_2)
\]

define an identification

\[
\text{Shv}^{\text{Betti}}(\mathcal{Y}) \simeq \text{Shv}^{\text{Betti}}(\mathcal{Y})^\vee.
\]

Using the above identification, we compute \( \text{Tr}(\phi_!, \text{Shv}^{\text{Betti}}(\mathcal{Y})) \) as pull-push along the following diagram, where we apply \( * \)-pullback along vertical arrows and \( ! \)-pushforward along horizontal ones:

\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \text{pt} \\
\downarrow \Delta_\mathcal{Y} & & \\
\mathcal{Y} \times \mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathcal{Y}
\end{array}
\]

By base change, we can replace this diagram by

\[
\begin{array}{ccc}
\mathcal{Y}^\phi & \longrightarrow & \mathcal{Y} \longrightarrow & \text{pt} \\
\downarrow & & \downarrow & \\
\mathcal{Y} & \longrightarrow & \text{pt}
\end{array}
\]

and pull-push along the latter diagram exactly gives \( C_c(\mathcal{Y}^\phi, \mathcal{E}_\mathcal{Y}^\phi) \).

\[\text{i.e., not necessarily ind-constructible.}\]
0.4.2. Let us now take $Y$ to be an algebraic stack over $\mathbb{F}_q$, defined over $\mathbb{F}_q$ (see Sect. 0.2.1), and $\phi = \text{Frob}_Y$. Instead of $\text{Shv}^{\text{Betti}}(Y)$, we take the category of ind-constructible sheaves $\text{Shv}(Y)$, as defined in [AGKRRV1, Sect. F.1].

Let us try to compute $\text{Tr}((\text{Frob}_Y)!$, $\text{Shv}(Y))$. Note that $(\text{Frob}_Y)! \cong (\text{Frob}_Y)^*$, since $\text{Frob}_Y$ is a proper morphism.

Note that $Y_{\text{Frob}} \cong Y(\mathbb{F}_q)$, viewed as a discrete algebraic stack. Hence, the counterpart of the right-hand side of (0.4) yields

$$\text{Funct}_c(Y(\mathbb{F}_q)).$$

In the example, of $Y = \text{Bun}_G$, we would thus obtain the space of unramified automorphic functions with compact supports.

However, the counterpart of the computation of the left-hand side of (0.4) along the lines of Sect. 0.4.1 invalid and, as a result, it is, in general, emphatically not true that $\text{Tr}((\text{Frob}_Y)!$, $\text{Shv}(Y))$ is isomorphic to $\text{Funct}_c(Y(\mathbb{F}_q))$.

The reason for the failure of the calculation is that the corresponding functor (0.6)

$$\text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Shv}(Y \times Y)$$

is fully faithful, but not an equivalence.

In particular, the object

$$\text{ps-u}_Y := (\Delta_Y)_!(\underline{\mathbb{F}}_q) \in \text{Shv}(Y \times Y)$$

does not belong to the essential image of (0.6).

Moreover, it is in general not true that the pairing

(0.7) $$\text{ev}_Y : \text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Vect}, \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C^c(Y, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

is the counit of any self-duality on $\text{Shv}(Y)$.

Remark 0.4.3. That said, the category $\text{Shv}(Y)$ is self-dual (at least when $Y$ is quasi-compact) by a different procedure, for which the counit is given by

$$\text{ev}_Y : \text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Vect}, \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C_\text{\underline{\text{\mathcal{C}}}}(Y, \mathcal{F}_1 \otimes \mathcal{F}_2).$$

The corresponding equivalence

$$(\text{Shv}(Y)^c)^\text{op} \to \text{Shv}(Y)^c$$

is given by the Verdier duality involution.

We refer to the resulting self-duality of $\text{Shv}(Y)$ as the Verdier self-duality.

When $Y$ is not quasi-compact, a variant of this construction is still applicable, but the dual of $\text{Shv}(Y)$ is the category $\text{Shv}(Y)_\text{co}$, see [AGKRRV2, Sects. C.2 and C.3].

0.4.4. Thus, as was mentioned above, it is not true, in general, that $\text{Tr}((\text{Frob}_Y)^*, \text{Shv}(Y))$ is isomorphic to $\text{Funct}_c(Y(\mathbb{F}_q))$.

We now take $Y = \text{Bun}_G$, but instead of all of $\text{Shv}(\text{Bun}_G)$, we take $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

A key observation is that if we restrict the pairing (0.7), the resulting pairing

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Vect}$$

is miraculously the counit of a self-duality. This is the content of the main result of [AGKRRV2], stated therein as Theorem 3.2.2.

Remark 0.4.5. The usage of the word “miraculous” in the above phrase is not accidental:

In [AGKRRV2, Theorem 3.3.3], we show that the self-duality of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ induced by (0.7) is related to the Verdier self-duality of Remark 0.4.3 via the miraculous functor, denoted $\text{Mir}_{\text{Bun}_G}$.

There is always a natural map $\text{Tr}((\text{Frob}_Y)^*, \text{Shv}(Y)) \to \text{Funct}_c(Y(\mathbb{F}_q))$, see (0.9) below.
0.4.6. The self-duality of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, induced by (0.7) paves a way to computing traces of endo-functors of this category.

However, in order to do so, we need one more ingredient: we need to have an explicit description of the unit of this self-duality. Another key takeaway from the paper [AGKRRV2] is such a description: There exists a particular object $R \in \text{Rep}(\hat{\mathcal{G}})_{\text{Ran}}$ (termed Beilinson’s spectral projector, see Sect. 0.5.2 below) such that when we apply it along the left (or right) factor of $\text{Bun}_G \times \text{Bun}_G$ to $\text{ps-u}_{\text{Bun}_G}$, the resulting object belongs to the essential image of the functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G \times \text{Bun}_G)$$

and is the unit of the above self-duality of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

0.5. Outline of the proof. We now give an overview of the proof of Theorem 0.3.10. It requires a few additional objects and some relations between them.

0.5.1. First, we recall the (non-algebraic) derived stack $\text{LocSys}_{\text{G}}^{\text{rest}}(X)$ over $\mathfrak{e}$ introduced in [AGKRRV1], the stack of local systems with restricted variation.

We have a naturally defined symmetric monoidal localization functor $\text{Loc} : \text{Rep}(\hat{\mathcal{G}})_{\text{Ran}} \to \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{rest}}(X))$, see Sect. 2.3.

0.5.2. We use Beilinson’s spectral projector, $R \in \text{Rep}(\hat{\mathcal{G}})_{\text{Ran}}$, see Sect. 0.4.6.

It has the following property vis-a-vis $\text{LocSys}_{\text{G}}^{\text{rest}}(X)$:

$$\text{Loc}(R) \simeq \mathcal{O}_{\text{LocSys}_{\text{G}}^{\text{rest}}(X)};$$

see Theorem 2.4.5 which is a restatement of [AGKRRV1 Theorem 12.7.4].

0.5.3. Using the result of [Xue2] mentioned above, we show in Corollary 3.1.4 that the functor $\text{Sht}$ factors canonically as a composition

$$\text{Rep}(\hat{\mathcal{G}})_{\text{Ran}} \xrightarrow{\text{Loc}} \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{rest}}(X)) \xrightarrow{\text{Sht}_{\text{Loc}}} \text{Vect}$$

for a certain functor $\text{Sht}_{\text{Loc}}$.

0.5.4. At this point, the assertion of Theorem 0.3.10 follows easily from the properties of $R$ mentioned above:

The property of $R$ mentioned in Sect. 0.4.6 implies that we have a canonical isomorphism

$$\text{Sht}_{\text{Tr}}(-) \simeq \text{Sht}(R \ast -);$$

this follows by essentially rerunning the calculation in Sect. 0.4.1.

Applying Sect. 0.5.3 we rewrite this further as

$$\text{Sht}_{\text{Loc}} \circ \text{Loc}(R \ast -).$$

Since Loc is monoidal and sends $R$ to the unit of $\text{QCoh}(\text{LocSys}_{\text{G}}^{\text{rest}}(X))$, we have

$$\text{Loc}(R \ast -) \simeq \text{Loc}(-).$$

Hence, combining, we obtain

$$\text{Sht}_{\text{Tr}}(-) \simeq \text{Sht}(R \ast -) \simeq \text{Sht}_{\text{Loc}} \circ \text{Loc}(R \ast -) \simeq \text{Sht}_{\text{Loc}} \circ \text{Loc}(-) \simeq \text{Sht}(-),$$

which is exactly the assertion of Theorem 0.3.10.
0.6. Relation to the classical sheaves-function dictionary. The Trace Conjecture as stated in [AGKRRV1] is somewhat stronger than Theorem 0.2.6 stated above, in that it specifies a particular morphism from \( \text{Tr}((\text{Frob}_{\text{Bun}_G})^\ast, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \) to \( \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)) \) that is supposed to be an isomorphism, see [AGKRRV1, Conjecture 22.3.7(b) and Remark 22.3.9].

In this subsection we describe the results of the present paper in this direction.

0.6.1. Let \( \mathcal{F} \) be a compact object of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \), equipped with a structure of weak Weil equivariance, i.e., a morphism
\[
\alpha : \mathcal{F} \to (\text{Frob}_{\text{Bun}_G})^\ast(\mathcal{F}),
\]
which is equivalent to the datum of a map
\[
\alpha' : (\text{Frob}_{\text{Bun}_G})^\ast(\mathcal{F}) \to \mathcal{F}.
\]
By functoriality of the categorical trace, we can attach to the pair \( (\mathcal{F}, \alpha) \) its class
\[
\text{cl}(\mathcal{F}, \alpha) \in \text{Tr}((\text{Frob}_Y)^\ast, \text{Shv}(Y)),
\]
Hence, applying Theorem 0.2.6, we can further attach to it an element of \( \text{Funct}_c(Y(\mathbb{F}_q)) \).

We wish to say that this function equals the function attached to \( (\mathcal{F}, \alpha') \) by the classical sheaves-function correspondence, i.e., by taking pointwise traces of the Frobenius on the stalks of \( \mathcal{F} \).

However, here is an issue: we do not know that \( \mathcal{F} \) is constructible as a sheaf on \( \text{Bun}_G \). So, the above pointwise trace operation is not a priori well-defined, as the stalks in question may be infinite-dimensional.

0.6.2. To overcome to issue mentioned above, for the duration of this subsection we will assume [AGKRRV1, Conjecture 14.1.8]. This conjecture has several equivalent formulations:

- The category \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) is generated by objects that are compact in the ambient category \( \text{Shv}(\text{Bun}_G) \);
- Compact objects of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) are compact as objects of \( \text{Shv}(\text{Bun}_G) \);
- Compact objects of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) are constructible as objects of \( \text{Shv}(\text{Bun}_G) \);
- Compact objects of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) are bounded below in the natural t-structure on \( \text{Shv}(\text{Bun}_G) \).

An analog of [AGKRRV1, Conjecture 14.1.8] is known when our ground field \( k \) has characteristic 0, and \( \text{Shv}(-) \) is either the category of ind-holonomic D-modules or the category of ind-constructible Betti sheaves (see [AGKRRV1, Theorems 16.4.3 and 16.4.10]). Furthermore, we believe that we can prove it when \( G = GL_n \).

To summarize, this conjecture is a rather plausible statement.

0.6.3. Assuming [AGKRRV1, Conjecture 14.1.8], we obtain that the embedding (0.2) admits a continuous right adjoint.

Hence, by functoriality of categorical trace, we obtain a map
\[
(0.8) \quad \text{Tr}((\text{Frob}_{\text{Bun}_G})^\ast, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \to \text{Tr}((\text{Frob}_{\text{Bun}_G})^\ast, \text{Shv}(\text{Bun}_G)).
\]

0.6.4. For any quasi-compact algebraic stack \( Y \) over \( \overline{\mathbb{F}_q} \) and defined over \( \mathbb{F}_q \), there is a canonical local term map (see [AGKRRV1, Sect. 22.2]),
\[
(0.9) \quad \text{LT}_Y : \text{Tr}((\text{Frob}_Y)^\ast, \text{Shv}(Y)) \to \text{Funct}(Y(\mathbb{F}_q)),
\]
where \( \text{Funct}(\cdot) \) stands for the (classical) vector space \( e \)-valued functions.

In addition, in loc. cit., we extended this construction to non-quasi-compact stacks (such as \( \text{Bun}_G \)). In this setting, the above map takes the form
\[
\text{LT}_Y : \text{Tr}((\text{Frob}_Y)^\ast, \text{Shv}(Y)) \to \text{Funct}_c(Y(\mathbb{F}_q)),
\]
where \( \text{Funct}_c(Y(\mathbb{F}_q)) \subset \text{Funct}(Y(\mathbb{F}_q)) \) is the subspace of compactly supported \( e \)-valued functions.
Remark 0.6.5. By functoriality of traces, any (possibly lax) Weil sheaf
\((\mathcal{F} \in \text{Shv}(\mathcal{Y}), \alpha : \mathcal{F} \to (\text{Frob}_\mathcal{Y})_*\mathcal{F}))\)
on defines an element
\(\text{cl}(\mathcal{F}, \alpha) \in \text{Tr}((\text{Frob}_\mathcal{Y})_*\mathcal{F}, \text{Shv}(\mathcal{Y})).\)

According to [AGKRRV1, Sect. 22.2], the value of the map \(\text{LT}_\mathcal{Y}\) on \(\text{cl}(\mathcal{F}, \alpha)\) produces the corresponding Grothendieck–Deligne function, denoted \(\text{funct}(\mathcal{F}).\)

0.6.6. By the above, we obtain a map
\[(0.10) \quad \text{Tr}((\text{Frob}_{\text{Bun}_G})_*\text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \quad \longrightarrow \quad \text{LT}_{\text{Bun}_G} \quad \Longrightarrow \quad \text{Tr}((\text{Frob}_{\text{Bun}_G})_*\text{Shv}(\text{Bun}_G)) \quad \longrightarrow \quad \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)),\]

where we note that the vector space \(\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))\) is the space of compactly supported unramified automorphic functions.

0.6.7. We can now state our third main (it appears as Theorem 5.2.3 in the main body of the paper):

**Main Theorem 0.6.8.** The map \((0.10)\) equals the isomorphism of Theorem 0.3.6.

**Corollary 0.6.9.** The map \((0.10)\) is an isomorphism.

0.6.10. Informally, one can view Corollary 0.6.9 as saying that “there are enough weak Weil sheaves on \(\text{Bun}_G\) with nilpotent singular support to recover all automorphic functions, and any relations between the automorphic functions defined by such sheaves have categorical origins.”

Of course, the fact that we are considering sheaves with nilpotent singular support is crucial here. If we did not have the singular support condition, we would obviously have enough sheaves to recover all functions. However, the relations imposed by sheaves would not match the relations on functions.

**Remark 0.6.11.** The phrase in quotation makes in the previous remark would be a correct assertion if the phrase inside the quotation marks was understood in the derived sense.

More precisely, for a (compactly generated) category \(\mathcal{C}\) with an endofunctor \(\Phi\), the trace object \(\text{Tr}(\Phi, \mathcal{C}) \in \text{Vect}\) is computed as the geometric realization of a certain canonically defined simplicial object of \(\text{Vect}\); let us denote it \(\text{Tr}(\Phi, \mathcal{C})^\bullet\).

In particular, we have a map \(\text{Tr}(\Phi, \mathcal{C})^0 \to \text{Tr}(\Phi, \mathcal{C})\) (here the superscript 0 denotes the space of 0-simplices), and hence a map
\[H^0(\text{Tr}(\Phi, \mathcal{C})^0) \to H^0(\text{Tr}(\Phi, \mathcal{C})).\]

Now, the image of the latter map is the span of the classes
\(\text{cl}(c, \alpha), \quad c \in \mathcal{C}^0, \alpha : c \to \Phi(c).\)

However, it is not true, in general, that \(H^0(\text{Tr}(\Phi, \mathcal{C}))\) is spanned by the above classes: higher cohomologies of higher simplices can also contribute.

0.7. **Organization of the paper.** We now describe how the present paper is structured.

0.7.1. In Sect. 1 we review the formalism of Hecke functors acting on \(\text{Shv}(\text{Bun}_G)\).

In particular, we introduce the Ran version of the category \(\text{Rep}(\check{G})\), denoted \(\text{Rep}(\check{G})_{\text{Ran}}\), which is a monoidal category that acts on \(\text{Shv}(\text{Bun}_G)\) by integral Hecke functors.

This section does not contain any new results.
0.7.2. In Sect. 2 we review some notions associated with the stack of local systems with restricted variation, introduced in [AGKRRV1], and denoted $\text{LocSys}_{\check{G}}^{\text{restr}}(X)$.

Apart from the definition of $\text{LocSys}_{\check{G}}^{\text{restr}}(X)$, the main points are:

(i) Description of the dual category $\text{QCoh}(\text{LocSys}_{\check{G}}^{\text{restr}}(X))^\vee$ as the category of compatible collections of functors

$$\text{Rep}(\check{G})^\otimes I \to \text{QLisse}(X)^\otimes I, \quad I \in \text{fSet};$$

(ii) The localization functor

$$\text{Loc} : \text{Rep}(\check{G})_{\text{Ran}} \to \text{QCoh}(\text{LocSys}_{\check{G}}^{\text{restr}}(X));$$

(iii) Construction of Beilinson’s spectral projector, which is an explicit object $R \in \text{Rep}(\check{G})_{\text{Ran}}$, one of whose main properties is the isomorphism

$$\text{Loc}(R) \simeq \mathcal{O}_{\text{LocSys}_{\check{G}}^{\text{restr}}(X)}.$$

(iv) Corollary 2.3.3 which asserts that a functor $S : \text{Rep}(\check{G})_{\text{Ran}} \to \text{Vect}$ factors (uniquely) through the localization functor $\text{Loc}$ exactly when the functors $S_I$ are valued in $\text{QLisse}(X)^\otimes I \subset \text{Shv}(X)$.

The entirety of the material of this section is a reformulation of the results in Parts I and II of [AGKRRV1].

0.7.3. In Sect. 3 we review the shtuka construction and some of its variants.

First, we introduce the functor $\text{Sht} : \text{Rep}(\check{G})_{\text{Ran}} \to \text{Vect}$.

We then quote the main result from [Xue2], which we interpret as saying that the functors $\text{Sht}_I$ take values in $\text{QLisse}(X)^\otimes I \subset \text{Shv}(X)$. Using (iv) above, this yields the existence of the functor $\text{Sht}_{\text{Loc}}$ from above.

0.7.4. In Sect. 4 we formulate and prove the main result of this paper, Theorem 4.1.2 (which is the same as Theorem 0.3.10 above). As particular cases, this statement contains both the (unrefined) Trace Conjecture and the Shtuka Conjecture.

The argument is the one outlined above.

0.7.5. In Sects. 5 and 6 we assume the validity of [AGKRRV1] Conjecture 14.1.8, and we show that the isomorphism

$$\text{Tr}((\text{Prob}_s, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \simeq \text{Funct}_c(\text{Bun}_G(F_q))$$

of Theorem 4.1.4 is induced by the local term map (0.10), as in the statement of Theorem 0.6.8.

0.8. Notations and conventions. The notations in this paper largely follow those of [AGKRRV1] and [AGKRRV2].

0.8.1. Algebraic geometry. There will be disjoint “two algebraic geometries” at play in this paper: one on the automorphic side, and another on the spectral side.

On the automorphic side, our algebraic geometry will be over the ground field $k$, which in this paper is $\mathbb{F}_q$. Our algebro-geometric objects will be either schemes or algebraic stacks locally of finite type over $k$. In this paper we will not need more general prestacks. Moreover, the algebraic geometry that we consider over $k$ is classical (i.e. not derived).

On the spectral side, our algebraic geometry will be over the field of coefficients $\mathcal{E} := \mathbb{Q}_\ell$, see below. We will consider just one algebro-geometric object over $\mathcal{E}$—the (pre)stack $\text{LocSys}_{\check{G}}^{\text{restr}}(X)$ (see Sect. 2), but it will play quite a prominent role. Importantly, the algebraic geometry we consider over $\mathcal{E}$ is derived: by default, all schemes, stacks, etc. over $\mathcal{E}$ are derived.
0.8.2. **Frobenius endomorphism.** Let $\mathcal{Y}$ be an algebraic stack over $\mathbb{F}_q$, but defined over $\mathbb{F}_{q'}$. In this case, we can consider the geometric Frobenius endomorphism of $\mathcal{Y}$, denoted

$$\text{Frob}_Y : \mathcal{Y} \to \mathcal{Y}.$$ 

Thus, whenever we refer to the Frobenius endomorphism of $\mathcal{Y}$, we will assume that $\mathcal{Y}$ is defined over $\mathbb{F}_q$. This is the case of our curve $X$, the reductive group $G$, and the stack $\text{Bun}_G$ of principal $G$-bundles on $X$.

0.8.3. **Higher algebra.** We will work with DG categories over the field of coefficients $e := \mathbb{Q}_\ell$.

All our conventions and notations regarding DG categories are imported from [AGKRRV2, Sects. 0.5.2-0.5.3].

There will be two kinds of sources that feed into higher algebra, i.e., the sources of DG categories. One will be various categories produced out of $\ell$-adic sheaves on the automorphic side. Another will be categories of quasi-coherent sheaves on the spectral side (specifically, the category of quasi-coherent sheaves on $\text{LocSys}^\text{const}_G(X)$).

0.8.4. **Sheaves.** For a scheme $S$ of finite type, we let $\text{Shv}(S)^{\text{constr}}$ denote the category of constructible $\mathbb{Q}_\ell$-adic sheaves on $S$, viewed as a (small) DG category over the field of coefficients $e = \mathbb{Q}_\ell$.

We let $\text{Shv}(S)$ denote the (cocomplete) DG category $\text{Ind}(\text{Shv}(S)^{\text{constr}})$. We extend the assignment

$$S \mapsto \text{Shv}(S)$$

from schemes to algebraic stacks by the procedure explained in [AGKRRV2 Sect. A.1].

For a given stack $\mathcal{Y}$, we will denote by

$$e_\mathcal{Y}, \omega_\mathcal{Y} \in \text{Shv}(\mathcal{Y})$$

the constant and dualizing sheaves, respectively.

If $\mathcal{Y}$ is smooth, inside $\text{Shv}(\mathcal{Y})$ we consider a full subcategory

$$\text{QLisse}(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y}),$$

defined as in [AGKRRV1 Sect. 1.2].

0.8.5. **Singular support.** Let $\mathcal{Y}$ be an algebraic stack and $N$ a conical Zariski-closed subset of $T^*(\mathcal{Y})$. We will denote by

$$\text{Shv}_N(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y})$$

the corresponding full subcategory, defined as in [AGKRRV1 Sects. E.5 and F.6].

If $\mathcal{Y}$ is smooth and $N$ is the zero-section, usually denoted $\{0\}$, we will also use the notation $\text{QLisse}(\mathcal{Y})$ for $\text{Shv}_{\{0\}}(\mathcal{Y})$.

0.8.6. **Functors (co)defined by kernels.** In a few places in this paper we will make reference to functors defined or codefined by a kernel. We refer the reader to [AGKRRV2 Sect. B], where these notions are introduced.

Following loc. cit., given a functor $F : \text{Shv}(\mathcal{Y}_1) \to \text{Shv}(\mathcal{Y}_2)$, defined or codefined by a kernel, we will denote by

$$\text{Id}_Z \boxtimes F : \text{Shv}(Z \times \mathcal{Y}_1) \to \text{Shv}(Z \times \mathcal{Y}_2)$$

the corresponding functor for an algebraic stack $Z$ (thought of as a stack of parameters).
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1. The Hecke action

In this section we will recall the pattern of Hecke action of the category $\text{Rep}(\tilde{G})$ on $\text{Shv}(\text{Bun}_G)$, and some related formalism. The section contains no original material.

1.1. Hecke functors.

1.1.1. In this paper, by the phenomenon of Hecke action we will understand a system of functors, defined for every finite set $I$ and every algebraic stack $Z$ (thought of as a stack of parameters):

$$(1.1) \quad \text{Id}_Z \boxtimes H : \text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(Z \times \text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G \times X^I).$$

For a fixed $V \in \text{Rep}(\tilde{G}) \otimes I$, we will denote by $\text{Id}_Z \boxtimes H(V, -)$ the resulting functor $\text{Shv}(Z \times \text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G \times X^I)$.

When $Z = \text{pt}$, we will simply write $H$ (resp., $H(V, -)$).

1.1.2. The Hecke functors (1.1) are associative in the following sense: we have a natural commutative diagram of functors

$$
\begin{array}{ccc}
\text{Rep}(\tilde{G}) \otimes I \otimes \text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(Z \times \text{Bun}_G) & \xrightarrow{\text{mult}} & \text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(Z \times \text{Bun}_G) \\
\text{Id} \otimes (\text{Id} \boxtimes H) & \downarrow & \downarrow \text{Id} \boxtimes H \\
\text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(Z \times \text{Bun}_G \times X^I) & \xrightarrow{(\text{Id} \times \Delta_{X^I})} & \text{Shv}(Z \times \text{Bun}_G \times X^I \times X^I),
\end{array}
$$

where

$$\text{mult} : \text{Rep}(\tilde{G}) \otimes I \otimes \text{Rep}(\tilde{G}) \otimes I \to \text{Rep}(\tilde{G}) \otimes I$$

is the tensor product functor and $\Delta_{X^I} : X^I \to X^I \times X^I$ is the diagonal embedding. The data of associativity of (1.1) comes additionally with higher coherence for higher powers of $\text{Rep}(\tilde{G}) \otimes I$.

We can rephrase this as saying that the category $\text{Shv}(Z \times \text{Bun}_G \times X^I)$ is a module category for the monoidal category $\text{Rep}(\tilde{G}) \otimes I$, and the action is $\text{Shv}(X^I)$-linear (i.e. it is a module category for the monoidal category $\text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(X^I)$), where the symmetric monoidal structure on $\text{Shv}(X^I)$ is given by $\otimes$.
1.1.3. The functors \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) are naturally compatible with maps between finite sets. Namely, for a map \( \psi : I \to J \), we have a datum of commutativity for the diagram
\[
\begin{array}{c}
\text{Rep(\tilde{G})}^{\otimes I} \otimes \text{Shv}(Z \times \text{Bun}_G) \\
\downarrow \text{Id} \otimes \text{mult}^{\psi} \\
\text{Rep(\tilde{G})}^{\otimes J} \otimes \text{Shv}(Z \times \text{Bun}_G)
\end{array} \rightarrow \begin{array}{c}
\text{Shv}(Z \times \text{Bun}_G \times X^J) \\
\downarrow (\text{Id} \times \Delta^{\psi})^J \\
\text{Shv}(Z \times \text{Bun}_G \times X^J)
\end{array}
\]
where \( \text{mult}^{\psi} : \text{Rep(\tilde{G})}^{\otimes I} \to \text{Rep(\tilde{G})}^{\otimes J} \) is the functor given by the symmetric monoidal structure on \( \text{Rep(\tilde{G})} \), and \( \Delta^{\psi} : X^J \to X^I \) is the diagonal map defined by \( \psi \).

The above data of commutativity are endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

Moreover, this data is compatible with the associativity described in Sect. 1.1.2. Namely, the functor
\[
(\text{Id} \times \Delta^{\psi})^I : \text{Shv}(Z \times \text{Bun}_G \times X^I) \to \text{Shv}(Z \times \text{Bun}_G \times X^I)
\]
is a functor of \( \text{Shv}(X^I) \otimes \text{Rep(\tilde{G})}^{\otimes I} \)-module categories.

1.1.4. A feature of the functors \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) is that they are functors that are both defined and codefined by kernels; see Sect. 0.8.6 for what this means. In practical terms, this implies that for a map \( Z_1 \to Z_2 \) between algebraic stacks, we have a datum of commutativity for the diagrams
\[
\begin{array}{c}
\text{Shv}(Z_1 \times \text{Bun}_G) \\
\downarrow \\
\text{Shv}(Z_2 \times \text{Bun}_G)
\end{array} \rightarrow \begin{array}{c}
\text{Shv}(Z_1 \times \text{Bun}_G \times X^I) \\
\downarrow \\
\text{Shv}(Z_2 \times \text{Bun}_G \times X^I)
\end{array}
\]
where the vertical arrows are given by either \( \triangleleft \)- or \( ! \)- pushforwards, and also for the diagrams
\[
\begin{array}{c}
\text{Shv}(Z_1 \times \text{Bun}_G) \\
\uparrow \\
\text{Shv}(Z_2 \times \text{Bun}_G)
\end{array} \rightarrow \begin{array}{c}
\text{Shv}(Z_1 \times \text{Bun}_G \times X^I) \\
\uparrow \\
\text{Shv}(Z_2 \times \text{Bun}_G \times X^I)
\end{array}
\]
where the vertical arrows are given by either \( ! \)- or \( * \)- pullbacks.

Moreover, this datum of commutativity is functorial in \( V \in \text{Rep(\tilde{G})}^{\otimes I} \), and is compatible with the datum of commutativity of the diagrams \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \).

1.2. The ULA property of the Hecke action.

1.2.1. Another key feature of the functors \( \text{Id}_Z \boxtimes \mathbb{H} \) is that for \( \mathcal{F} \in \text{Shv}(Z \times \text{Bun}_G)_{\text{constr}} \) and \( V \in (\text{Rep(\tilde{G})}^{\otimes I})^c \), the object
\[
(\text{Id}_Z \boxtimes \mathbb{H})(V, \mathcal{F}) \in \text{Shv}(Z \times \text{Bun}_G \times X^I)
\]
is ULA with respect to the projection
\[
Z \times \text{Bun}_G \times X^I \to X^I.
\]
1.2.2. Let us denote by $\text{Id}_X \boxtimes \mathcal{H}^I$ the functor
\[
\text{Rep}(\mathcal{G}) \otimes \mathbf{Sh}(\mathcal{Z} \times \text{Bun}_G) \to \mathbf{Sh}(\mathcal{Z} \times \text{Bun}_G \times X^I)
\]
defined as
\[
(\text{Id}_X \boxtimes \mathcal{H}^I)(V, \mathcal{F}) := (\text{Id}_X \boxtimes \mathcal{H})(V, \mathcal{F}) \otimes p^I_!(\mathbf{e}_X).
\]
Note that for a fixed $I$, the difference between $\text{Id}_X \boxtimes \mathcal{H}$ and $\text{Id}_X \boxtimes \mathcal{H}^I$ amounts to a cohomological shift by $2|I|$ since $\mathbf{e}_X \simeq \omega_X[-2|I|]$.

1.2.3. The ULA property of the objects $(\text{Id}_X \boxtimes \mathcal{H})(V, \mathcal{F})$ implies that we have canonical isomorphisms
\[
(\text{Id}_X \boxtimes \mathcal{H})(V, \mathcal{F}) \otimes p_2^!(M) \simeq (\text{Id}_X \boxtimes \mathcal{H}^I)(V, \mathcal{F}) \otimes p_2^!(M), \quad M \in \mathbf{Sh}(X^J).
\]
Furthermore, for a map of finite sets $\psi : I \to J$, we have a data of commutativity for the diagram
\[
\begin{array}{ccc}
\text{Rep}(\mathcal{G}) \otimes \mathbf{Sh}(\mathcal{Z} \times \text{Bun}_G) & \to & \mathbf{Sh}(\mathcal{Z} \times \text{Bun}_G \times X^I) \\
\text{mult}^\psi \otimes \text{Id} & & (\text{Id}_X \boxtimes \mathcal{H})^* \\
\end{array}
\]
endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

1.2.4. Thus, we can regard the functors $\text{Id}_X \boxtimes \mathcal{H}^I(V, -)$ also as defined and codetermined by kernels, and they have the formal properties parallel to those of the functors $\text{Id}_X \boxtimes \mathcal{H}(V, -)$.

1.3. Hecke action on $\mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G)$.

1.3.1. One of the main actor in this paper is the full subcategory
\[
\mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \subset \mathbf{Sh}(\text{Bun}_G).
\]

The following result, essentially due to [NY], describes the behavior of this subcategory under the Hecke functors (this is stated as [AGKRRV] Theorem 14.2.4):

**Theorem 1.3.2.** The Hecke functor $\mathcal{H}$ for $I = \{\ast\}$ sends the full subcategory
\[
\mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \subset \mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}(\text{Bun}_G)
\]
to the full subcategory
\[
\mathbf{Sh}_{\text{Nilp} \times \{0\}}(\text{Bun}_G \times X) \subset \mathbf{Sh}(\text{Bun}_G \times X).
\]

1.3.3. Recall also (see [AGKRRV] Theorem F.9.7 combined with Corollary E.4.7) that for any algebraic stack $\mathcal{Y}$ and a conical half-dimensional Zariski-closed subset $N \subset T^*(\mathcal{Y})$, the (a priori fully faithful) functor
\[
\mathbf{Sh}_N(\mathcal{Y}) \otimes \mathbf{QLisse}(X) \to \mathbf{Sh}_{N \times \{0\}}(\mathcal{Y} \times X)
\]
is an equivalence.

Thus, from Theorem 1.3.2 we obtain:

**Corollary 1.3.4.** The Hecke functor $\mathcal{H}$ for $I = \{\ast\}$ sends the full subcategory
\[
\mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \subset \mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}(\text{Bun}_G)
\]
to the full subcategory
\[
\mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \otimes \mathbf{QLisse}(X) \subset \mathbf{Sh}(\text{Bun}_G \times X).
\]

Iterating, from Corollary 1.3.4 we further obtain:

**Corollary 1.3.5.** The Hecke functors $\mathcal{H}$ map the full subcategory
\[
\mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \subset \mathbf{Rep}(\mathcal{G}) \otimes \mathbf{Sh}(\text{Bun}_G)
\]
to the full subcategory
\[
\mathbf{Sh}_{\text{Nilp}}(\text{Bun}_G) \otimes \mathbf{QLisse}(X) \subset \mathbf{Sh}(\text{Bun}_G \times X^I).
\]
1.3.6. Note that for a scheme or stack $\Y$, we can consider $\QLisse(\Y)$ as a full subcategory of $\Shv(\Y)$ in two different ways.

One is the tautological embedding

$$\tag{1.5} \QLisse(\Y) \hookrightarrow \Shv(\Y), \quad \mathcal{L} \mapsto \mathcal{L};$$

it endows $\QLisse(\Y)$ with a symmetric monoidal structure induced by the $!$-tensor product on $\Shv(\Y)$.

We also have a different embedding:

$$\tag{1.6} \QLisse(\Y) \hookrightarrow \Shv(\Y), \quad \mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{I}_\Y;$$

it endows $\QLisse(\Y)$ with a symmetric monoidal structure induced by the $*$-tensor product on $\Shv(\Y)$.

However, it follows tautologically that the two symmetric monoidal structures on $\QLisse(\Y)$ coincide. Moreover, the operations

$$\QLisse(\Y) \otimes \Shv(\Y) \xrightarrow{\text{mult}} \Shv(\Y)$$

and

$$\QLisse(\Y) \otimes \Shv(\Y) \xrightarrow{\text{mult}} \Shv(\Y)$$

define the same monoidal action of $\QLisse(\Y)$ on $\Shv(\Y)$.

We will apply this discussion to $\Y = X^I$.

Remark 1.3.7. Note also that when $\Y$ is smooth of dimension $n$, the embeddings (1.5) and (1.6) differ by the cohomological shift $[2n]$. Yet they should not be confused.

1.3.8. An assertion parallel to Corollary 1.3.3 holds for the $\mathcal{H}^I$ functors. Namely, these functors send

$$\Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv_{\text{Nilp}}(\Bun_G) \subset \Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv(\Bun_G)$$

to

$$\Shv_{\text{Nilp}}(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{I}} \subset \Shv(\Bun_G \times X^I),$$

where we will think of the embedding $\QLisse(X)^\otimes_{\mathcal{I}} \hookrightarrow \Shv(X^I)$ as given by (1.6).

1.3.9. Thus, we can think of the Hecke action on $\Shv_{\text{Nilp}}(\Bun_G)$ either by means of the functors

$$\tag{1.7} \mathcal{H} : \Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv_{\text{Nilp}}(\Bun_G) \rightarrow \Shv_{\text{Nilp}}(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{I}},$$

when we think of $\QLisse(X^I)$ as embedded into $\Shv(X^I)$ via (1.3), or, equivalently, as

$$\tag{1.8} \mathcal{H}' : \Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv_{\text{Nilp}}(\Bun_G) \rightarrow \Shv_{\text{Nilp}}(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{I}},$$

when we think of $\QLisse(X^I)$ as embedded into $\Shv(X^I)$ via (1.6).

The functors (1.7) and (1.8) are canonically isomorphic. Thus, in what follows we will not distinguish notationally between $\mathcal{H}$ and $\mathcal{H}'$, when applied to objects from $\Shv_{\text{Nilp}}(\Bun_G)$, and just use the notation

$$\tag{1.9} \mathcal{H} : \Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv_{\text{Nilp}}(\Bun_G) \rightarrow \Shv_{\text{Nilp}}(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{I}}.$$

For a map of finite sets $\psi : I \rightarrow J$, we have a data of commutativity for the diagram

$$\begin{array}{ccc}
\Rep(\tilde{G})^\otimes_{\mathcal{I}} \otimes \Shv_{\text{Nilp}}(\Bun_G) & \xrightarrow{\mathcal{H}} & \Shv(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{I}} \\
\Id \otimes \text{mult}^\psi & \Downarrow & \Id \times \text{mult}^\psi \\
\Rep(\tilde{G})^\otimes_{\mathcal{J}} \otimes \Shv_{\text{Nilp}}(\Bun_G) & \xrightarrow{\mathcal{H}} & \Shv(\Bun_G) \otimes \QLisse(X)^\otimes_{\mathcal{J}},
\end{array}$$

where in the right vertical arrow the functor

$$\text{mult}^\psi : \QLisse(X)^\otimes_{\mathcal{I}} \rightarrow \QLisse(X)^\otimes_{\mathcal{J}},$$

4Recall that our conventions are such that the default pullback functor is $!$-pullback, and therefore, by definition, lisse sheaves are those that are locally $!$-pulled back from a point. As explained below, it will be important to also consider the usual notion of lisse sheaves, i.e. those which are locally $*$-pulled back from a point.


is given by the symmetric monoidal structure on $\text{QLisse}(X)$.

These data of commutativity are endowed with a homotopy coherent system of compatibilities for compositions of maps of finite sets.

1.4. The category $\text{Rep}(\hat{G})_{\text{Ran}}$. We will now introduce a device that allows us to express the Hecke action on $\text{Shv(Bun}_C)$ in terms of a single monoidal category, the Ran version of $\text{Rep}(\hat{G})$, denoted $\text{Rep}(\hat{G})_{\text{Ran}}$.

We refer the reader to [AGKRRV1, Sect. 11] for a detailed discussion of this construction.

1.4.1. Let $\mathcal{C}$ be a symmetric monoidal DG category. We define a new symmetric monoidal DG category $\mathcal{C}_{\text{Ran}}$ by the following construction.

Let $\text{TwArr}(\text{fSet})$ be the category of twisted arrows on $\text{fSet}$, see [GKRV, Sect. 1.2.2].

The category $\mathcal{C}_{\text{Ran}}$ is the colimit over $\text{TwArr}(\text{fSet})$ of the functor

$$(1.11) \quad \text{TwArr}(\text{fSet}) \rightarrow \text{DGCat}$$

that sends

$$((I \rightarrow J) \mapsto \mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J)).$$

Here for a map

$$(1.12) \quad \begin{array}{ccc}
I_1 & \longrightarrow & J_1 \\
\phi_I & \downarrow & \phi_J \\
I_2 & \longrightarrow & J_2,
\end{array}$$

in $\text{TwArr}(\text{fSet})$, the corresponding functor

$$\mathcal{C}^{\otimes I_1} \otimes \text{Shv}(X^{J_1}) \rightarrow \mathcal{C}^{\otimes I_2} \otimes \text{Shv}(X^{J_2})$$

is given by the tensor product functor along the fibers of $\phi_I$

$$(1.13) \quad \text{mult}^{\phi_I} : \mathcal{C}^{\otimes I_1} \rightarrow \mathcal{C}^{\otimes I_2}$$

and the functor

$$(1.14) \quad (\Delta_{\phi_J} : \text{Shv}(X^{J_1}) \rightarrow \text{Shv}(X^{J_2}),$$

where $\Delta_{\phi_J} : X^{J_2} \rightarrow X^{J_1}$ is the diagonal map induced by $\phi_J$.  

1.4.2. The functor $\mathcal{C}_{\text{Ran}}$ is naturally right-lax symmetric monoidal. Therefore, the colimit $\mathcal{C}_{\text{Ran}}$ carries a natural symmetric monoidal structure. Explicitly, this symmetric monoidal structure can be described as follows. For

$$V_1 \otimes M_1 \in \mathcal{C}^{\otimes I_1} \otimes \text{Shv}(X^{J_1}) \text{ and } V_2 \otimes M_2 \in \mathcal{C}^{\otimes I_2} \otimes \text{Shv}(X^{J_2}),$$

the tensor product of their images in $\mathcal{C}_{\text{Ran}}$ is the image of the object

$$(V_1 \otimes V_2) \otimes (M_1 \boxtimes M_2) \in \mathcal{C}^{\otimes (I_1 \sqcup I_2)} \otimes \text{Shv}(X^{J_1 \sqcup J_2}).$$

We will denote the resulting monoidal operation on $\mathcal{C}_{\text{Ran}}$ by

$$\forall_1, V_2 \mapsto V_1 \star V_2.$$

We denote the unit object by $1_{\mathcal{C}_{\text{Ran}}}$. It arises from $(\text{Id} : \emptyset \rightarrow \emptyset) \in \text{TwArr}(\text{fSet})$ and the corresponding map

$$\text{Vect} = \mathcal{C}^{\otimes \emptyset} \otimes \text{Shv}(X^{\emptyset}) \rightarrow \mathcal{C}_{\text{Ran}}.$$
1.4.3. Let \( (\psi : I \to J) \in \text{TwArr}(\text{fSet}) \) be given.

We denote by
\[
\text{ins}_\psi : \mathcal{C}^I \otimes \text{Shv}(X^J) \to \mathcal{C}_{\text{Ran}}
\]
the corresponding functor.

In the important special case \( \psi = \text{Id}_I : I \to I \), we use the notation \( \text{ins}_I \) in place of \( \text{ins}_{\text{Id}_I} \).

1.4.4. We will apply the above discussion to the case \( \mathcal{C} = \text{Rep}(\check{G}) \).

We denote the resulting (symmetric monoidal) category by \( \text{Rep}(\check{G})_{\text{Ran}} \).

1.5. A Ran version of the Hecke action.

1.5.1. We now claim that the datum of the functors (1.1) together with the compatibilities (1.2) allow to define an action of \( \text{Rep}(\check{G})_{\text{Ran}} \) on \( \text{Shv}(Z \times \text{Bun}_G) \).

Namely, for \( (I \xrightarrow{\psi} J) \in \text{TwArr}(\text{Set}) \) and \( V \otimes M \in \text{Rep}(\check{G})^I \otimes \text{Shv}(X^I) \),

we let the corresponding endofunctor of \( \text{Shv}(Z \times \text{Bun}_G) \) be the composition
\[
\begin{align*}
\text{Shv}(Z \times \text{Bun}_G) \xrightarrow{\text{Id}_Z \boxtimes H(V,-)} \text{Shv}(Z \times \text{Bun}_G \times X^I) \xrightarrow{- \otimes \psi_I^*(-) \star (M)} \\
\text{Shv}(Z \times \text{Bun}_G \times X^I) \xrightarrow{\Delta_{(\psi,-)}} \text{Shv}(Z \times \text{Bun}_G \times \text{Bun}_G). 
\end{align*}
\]

1.5.2. Note, however, that using (1.3), and the fact that \( X \) is proper, we can rewrite the expression in (1.15) as
\[
\begin{align*}
\text{Shv}(Z \times \text{Bun}_G) \xrightarrow{\text{Id}_Z \boxtimes H(V,-)} \text{Shv}(Z \times \text{Bun}_G \times X^I) \xrightarrow{- \otimes \psi_I^*(-) \star (M)} \\
\text{Shv}(Z \times \text{Bun}_G \times X^I) \xrightarrow{\Delta_{(\psi,-)}} \text{Shv}(Z \times \text{Bun}_G \times \text{Bun}_G).
\end{align*}
\]

1.5.3. The interpretation of the Hecke action via (1.15) implies that it commutes with \(!\)-pullbacks and \(\Delta\)-pushforwards along maps \( f : Z_1 \to Z_2 \). And the interpretation of the Hecke action via (1.16) implies that it commutes with *-pullbacks and \(!\)-pushforwards along maps \( f : Z_1 \to Z_2 \).

This implies that for a given \( V \in \text{Rep}(\check{G})_{\text{Ran}} \) its Hecke action is a functor both defined and codefined by a kernel, (see Sect. 0.8.6 for what this means).

We will denote the resulting endofunctor of \( \text{Shv(Bun}_G) \) by \( H_V \), and of \( \text{Shv}(Z \times \text{Bun}_G) \) by \( \text{Id}_Z \boxtimes H_V \) (see Sect. 0.8.6 for the \( \otimes \) notation).

We will refer to endofunctors of \( \text{Shv(Bun}_G) \) (or, more generally, \( \text{Shv}(Z \times \text{Bun}_G) \)) that arise in this way as integral Hecke functors.

1.5.4. For later use, we introduce the following notation. For \( V \in \text{Rep}(\check{G})_{\text{Ran}} \) we let
\[
\mathcal{X}_V \in \text{Shv(Bun}_G \times \text{Bun}_G)
\]
denote the object equal to
\[
(\text{Id}_{\text{Bun}_G} \boxtimes H_V)(\text{ps-u}_{\text{Bun}_G}),
\]
where
\[
\text{ps-u}_{\text{Bun}_G} := (\Delta_{\text{Bun}_G})(\mathcal{L}_{\text{Bun}_G}).
\]

1.5.5. By Theorem 1.3.2, the action of \( \text{Rep}(\check{G})_{\text{Ran}} \) on \( \text{Shv(Bun}_G) \) preserves the full subcategory \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv(Bun}_G) \).

1.6. The dual category of \( \text{Rep}(\check{G})_{\text{Ran}} \). For what follows we will need to recall some constructions pertaining to duality on \( \text{Rep}(\check{G})_{\text{Ran}} \).

We refer the reader to [ACKRRV1] Sects. 11.3 and 11.4 for a detailed discussion.
1.6.1. Let \( C \) be a general symmetric monoidal DG category. Assume that \( C \) is dualizable, and that for every \( (I \to J) \in \text{TwArr}(\text{fSet}) \), the functor
\[
\text{mult}^\psi : C^\otimes I \to C^\otimes J
\]
is such that the dual functor \((C^\otimes J)^\vee \to (C^\otimes I)^\vee\) admits a left adjoint.

In this case, one shows that the category \( C_{\text{Ran}} \) is dualizable (see, e.g., [GR, Chapter 1, Proposition 6.3.4]).

1.6.2. The dual category \((C_{\text{Ran}})^\vee\) is the category of continuous functors \( C_{\text{Ran}} \to \text{Vect} \), and hence it can be described as
\[
(1.17) \quad \lim_{(I \to J) \in \text{TwArr}(\text{fSet})^{op}} (C^\otimes I \otimes \text{Shv}(X^J))^\vee,
\]
where the limit is formed using the functors dual to the ones used in the formation of the colimit in Sect. 1.4.1.

Using the Verdier self-duality on \( \text{Shv}(X^J) \), we can rewrite
\[
(1.18) \quad (C_{\text{Ran}})^\vee \simeq \lim_{(I \to J) \in \text{TwArr}(\text{fSet})^{op}} \text{Maps}_{\text{DGCat}}(C^\otimes I, \text{Shv}(X^J)),
\]
where \( \text{Maps}_{\text{DGCat}}(\cdot, \cdot) \) stands for the DG category of continuous \( e \)-linear functor between two objects of \( \text{DGCat} \).

Explicitly, the transition functors in (1.18) are defined as follows. For a morphism in \( \text{TwArr}(\text{fSet}) \) given by (1.12), the corresponding functor
\[
\text{Maps}_{\text{DGCat}}(C^\otimes I_2, \text{Shv}(X^{J_2})) \to \text{Maps}_{\text{DGCat}}(C^\otimes I_1, \text{Shv}(X^{J_1}))
\]
is given by precomposition (1.13) and postcomposition with
\[
(\Delta_{\phi^J})^! : \text{Shv}(X^{J_2}) \to \text{Shv}(X^{J_1}),
\]
which is the functor dual to (1.14), under the Verdier self-duality of \( \text{Shv}(X^J) \).

Remark 1.6.3. Suppose for a moment that \( C \) is rigid (see [GR, Chapter 1, Sect. 9.1] for what this means). In this case, we have a natural identification \( C^\vee \simeq C \), and we can further rewrite the right-hand side in (1.18) as
\[
\lim_{(I \to J) \in \text{TwArr}(\text{fSet})^{op}} C^\otimes I \otimes \text{Shv}(X^J),
\]
where the limit is formed using the functors \textit{right adjoint} to the ones used in the formation of the colimit in Sect. 1.4.1. Hence, the above limit is isomorphic to the colimit
\[
\colim_{(I \to J) \in \text{TwArr}(\text{fSet})} C^\otimes I \otimes \text{Shv}(X^J)
\]
(see [GR, Chapter 1, Proposition 2.5.7]), i.e., to \( C_{\text{Ran}} \) itself.

This implies that for \( C \) rigid, the category \( C_{\text{Ran}} \) is naturally self-dual. However, we will not use this self-duality for the purposes of this paper.

\footnote{In fact, the category \( C_{\text{Ran}} \) is itself rigid, see [AGKRRV1, Sect. 11.3].}
Consider the \((\infty,2)\)-category
\[ \text{DGCat}^{\text{fSet}} := \text{Funct}(\text{fSet}, \text{DGcat}). \]

There will be several DG categories of interest in this paper that will arise as
\[ \text{Maps}_{\text{DGcat}^{\text{fSet}}}(\mathcal{C}_1, \mathcal{C}_2) \]
for some particular \( \mathcal{C}_1, \mathcal{C}_2 \subset \text{DGcat}^{\text{fSet}} \).

Concretely, an object of \( \text{Maps}_{\text{DGcat}^{\text{fSet}}}(\mathcal{C}_1, \mathcal{C}_2) \) is a collection of functors between DG categories
\[ \mathcal{C}_1(I) \to \mathcal{C}_2(I), \quad I \in \text{fSet} \]
that make the diagrams
\[
\begin{array}{ccc}
\mathcal{C}_1(I) & \longrightarrow & \mathcal{C}_2(I) \\
\varepsilon_1(\psi) & \downarrow & \varepsilon_2(\psi) \\
\mathcal{C}_1(J) & \longrightarrow & \mathcal{C}_2(J)
\end{array}
\]
commute for \( \psi : I \to J \), along with a homotopy coherent system of higher compatibilities.

Here are the first few objects of \( \text{DGcat}^{\text{fSet}} \) that we will need.

One is the object denoted \( \mathcal{C} \otimes \text{fSet} \) and defined by
\[ I \in \text{fSet} \leadsto \mathcal{C} \otimes I \in \text{DGcat}, \]
where the functoriality is furnished by the symmetric monoidal structure on \( \mathcal{C} \).

Another object, denoted \( \text{Shv}^I(\text{X}^{\text{fSet}}) \), is defined by
\[ I \in \text{fSet} \leadsto \text{Shv}(\text{X}^I) \in \text{DGcat}, \]
where for \( \psi : I \to J \), the corresponding functor \( \text{Shv}(\text{X}^I) \to \text{Shv}(\text{X}^J) \) is \( (\Delta_\psi)^I \).

Consider the category
\[ \text{Maps}_{\text{DGcat}^{\text{fSet}}}(\mathcal{C} \otimes \text{fSet}, \text{Shv}^I(\text{X}^{\text{fSet}})). \]

Note that this category identifies with the limit \((1.18)\) (see e.g. [GKRV, Lemma 1.3.12]).

Thus, to summarize, we obtain a canonical equivalence
\[ (1.19) \quad \text{Maps}_{\text{DGcat}^{\text{fSet}}}(\mathcal{C} \otimes \text{fSet}, \text{Shv}^I(\text{X}^{\text{fSet}})) \simeq (\mathcal{C}_{\text{ran}})^\vee. \]

Explicitly, given
\[ S : \mathcal{C}_{\text{ran}} \to \text{Vect}, \]
the corresponding system of functors
\[ S_I : \mathcal{C} \otimes I \to \text{Shv}(\text{X}^I) \]
is recovered as follows:

We precompose \( S \) with \( \text{ins}_I \) to obtain a functor
\[ \mathcal{C} \otimes \text{Shv}(\text{X}^I) \to \text{Vect}. \]

By Verdier duality, the datum of the latter functor is equivalent to the datum of a functor \( S_I \):
\[ S \circ \text{ins}_I(c \otimes \mathcal{M}) = C(\text{X}^I, S_I(c) \otimes \mathcal{M}), \quad c \in \mathcal{C} \otimes I, \mathcal{M} \in \text{Shv}(\text{X}^I). \]
1.6.8. Vice versa, the pairing

$$\mathcal{C}_{\text{Ran}} \otimes \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\mathcal{C}^{\otimes \text{fSet}}, \text{Shv}'(X^{\text{Set}})) \to \text{Vect}$$

is explicitly given as follows:

For an object \( \{S_I\} \in \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\mathcal{C}^{\otimes \text{fSet}}, \text{Shv}'(X^{\text{Set}})) \), the corresponding functor

$$S : \mathcal{C}_{\text{Ran}} \to \text{Vect}$$

is such that for \((I \xrightarrow{x} J) \in \text{TwArr}(\text{fSet})\), the resulting functor

$$\mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J) \xrightarrow{\text{ins}_x} \mathcal{C}_{\text{Ran}} S \to \text{Vect},$$

equals

$$\mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J) \xrightarrow{\text{mult} \otimes \text{Id}} \mathcal{C}^{\otimes J} \otimes \text{Shv}(X^J) \xrightarrow{\text{ev}_{X^J}} \text{Vect},$$

where \( \text{ev}_{X^J} \) is the Verdier duality pairing on \( \text{Shv}(X^J) \), i.e.,

$$\text{Shv}(X^J) \otimes \text{Shv}(X^J) \xrightarrow{\Delta} \text{Shv}(X^J) \xrightarrow{C} \text{Vect}.$$

2. Quasi-coherent sheaves on \( \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \)

Although the statement of the Trace Conjecture does not involve Langlands duality, we will need some of its ingredients for the proof. Indeed, one of the key tools in the proof will be the category of quasi-coherent sheaves on the (pre)stack \( \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \), classifying local systems with restricted variation with respect to the Langlands dual group \( \check{G} \) of \( G \).

2.1. The (pre)stack \( \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \)

We start by recalling the definition of the prestack

$$\text{LocSys}_{\text{restr}}^{\text{estr}}(X),$$

following [AGKRRV1, Sect. 1.4].

2.1.1. For a test affine (derived) scheme \( S \), we let \( \text{Maps}(S, \text{LocSys}_{\text{restr}}^{\text{estr}}(X)) \) be the space of right t-exact symmetric monoidal functors

$$\text{Rep}(\check{G}) \to \text{QCoh}(S) \otimes \text{QLisse}(X).$$

According to [AGKRRV1, Theorem 1.4.5], the prestack \( \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \) can be written as the quotient \( \mathbb{Z}/\check{G} \), where \( \mathbb{Z} \) is a disjoint union of formal affine schemes locally almost of finite type (over the field of coefficients \( e \)).

2.1.2. The main results of this paper will be based on considering the (symmetric monoidal) DG category

$$\text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X)).$$

We will now explain a certain feature that this category possesses, which is a consequence of properties of \( \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \) as a prestack.

(i) First, according to [AGKRRV1, Lemma 7.3.2 and Sect. 7.9.1], the diagonal map

$$\Delta_{\text{LocSys}_{\text{restr}}^{\text{estr}}(X)} : \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \to \text{LocSys}_{\text{restr}}^{\text{estr}}(X) \times \text{LocSys}_{\text{restr}}^{\text{estr}}(X)$$

is affine, so the functor

$$(\Delta_{\text{LocSys}_{\text{restr}}^{\text{estr}}(X)})^* : \text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X)) \to \text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X) \times \text{LocSys}_{\text{restr}}^{\text{estr}}(X))$$

is continuous.

(ii) Second, according to [AGKRRV1, Corollary 7.1.8(b) and Sect. 7.9.1], the DG category

$$\text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X))$$

is dualizable. By [GR, Chapter 3, Proposition 3.1.7], this implies that for any prestack \( \mathfrak{y} \) over \( e \), the functor of external tensor product

$$\text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X)) \otimes \text{QCoh}(\mathfrak{y}) \to \text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{estr}}(X) \times \mathfrak{y})$$

is an equivalence.
In particular, we can view

\[(\Delta_{\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)})^* (O_{\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)})\]

as an object of

\[
\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)) \otimes \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)).
\]

(iii) And third, according to [AGKRRV1 Proposition 7.5.4 and Sect. 7.9.1], the above object

\[(\Delta_{\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)})^* (O_{\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)}) \in \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)) \otimes \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\]

defines the unit of a self-duality on \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\).

**Remark 2.1.3.** Points (i) and (ii) above mean that the prestack \(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)\) is semi-passable in the terminology of [AGKRRV1 Sect. 7.4.3].

Point (iii) is the combination of [AGKRRV1 Lemma 7.4.2], which says that \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\) is compact, and \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\) is not compact, so the functor of global sections

\[
\Gamma(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X), -) : \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)) \to \text{Vect},
\]

the right adjoint to (2.1), is not continuous.

However, due to the self-duality of \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\) of Sect. 2.1.2(iii), we can consider the functor dual to (2.1), which is a functor

(2.2) \[
\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)) \to \text{Vect},
\]

denoted \(\Gamma(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X), -)\).

We refer the reader to [AGKRRV1 Sects. 7.6 and 7.7] for a more detailed discussion of this functor.

**2.1.5. The tautological objects.** For a finite set \(I\) and an object \(V \in \text{Rep}(\hat{G})^\otimes I\), let

\[
\text{Ev}(V) \in \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)) \otimes QLisse(X)^\otimes I
\]

be the corresponding tautological object:

For \(S \to \text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)\) corresponding to a symmetric monoidal functor

\[
\Phi_S : \text{Rep}(\hat{G}) \to \text{QCoh}(S) \otimes QLisse(X),
\]

the pullback of \(\text{Ev}(V)\) to \(S\), viewed as an object in \(\text{QCoh}(S) \otimes QLisse(X)^\otimes I\) equals the value on \(V\) of the composition

\[
\text{Rep}(\hat{G})^\otimes I \overset{\otimes I}{\to} (\text{QCoh}(S) \otimes QLisse(X))^\otimes I \to \text{QCoh}(S) \otimes QLisse(X)^\otimes I,
\]

where the second arrow is given by the \(I\)-fold tensor product functor

\[
\text{QCoh}(S)^\otimes I \to \text{QCoh}(S).
\]

**2.2. Description of the category** \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\). The prestack \(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X)\) was defined using the symmetric monoidal categories \(\text{Rep}(\hat{G})\) and \(QLisse(X)\). Therefore, it would not be very surprising to have a description of the category \(\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{restr}}(X))\) purely in terms of functors between the above two categories.

In this subsection, we will provide such a description, following [AGKRRV1].
2.2.1. Recall the category $\text{DGCat}^{\text{fSet}}$, see Sect. 1.6.4, and the object
$$\text{QLisse}(X)^{\otimes \text{fSet}} \in \text{DGCat}^{\text{fSet}},$$
see Sect. 1.6.6.

Note that we could also consider the object $\text{QLisse}(X^{\text{fSet}})$ of $\text{DGCat}^{\text{fSet}}$:
$$I \in \text{fSet} \rightsquigarrow \text{QLisse}(X^I) \in \text{DGCat}.$$

We have a naturally defined map in $\text{DGCat}^{\text{fSet}}$

$$(2.3) \quad \text{QLisse}(X)^{\otimes \text{fSet}} \to \text{QLisse}(X^{\text{fSet}}).$$

However, from [AGKRRV1, Theorem E.9.9 and Corollary E.4.7], we obtain:

**Lemma 2.2.2.** The map $(2.3)$ is an isomorphism.

2.2.3. Consider now the DG category $\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G}^{\otimes \text{fSet}}), \text{QLisse}(X)^{\otimes \text{fSet}})$, and the following functor, to be denoted

$$(2.4) \quad \text{coLoc} : \text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X))) \to \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G}^{\otimes \text{fSet}}), \text{QLisse}(X)^{\otimes \text{fSet}}).$$

Namely, $\text{coLoc}$ sends $F \in \text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X)))$ to the collection of functors

$$\mathcal{F}_I : \text{Rep}(\hat{G})^{\otimes I} \to \text{QLisse}(X)^{\otimes I}$$

defined by

$$\mathcal{F}_I(V) := (\Gamma \text{Hom}((\text{LocSys}_{\text{restr} \hat{G}}(X), -) \otimes \text{Id}_{\text{QLisse}(X)^{\otimes I}})(\mathcal{F} \otimes \text{Ev}(V)),$$

where $\Gamma$ is as in Sect. 2.1.4 and $\text{Ev}(V)$ is as in Sect. 2.1.5, and we view $\mathcal{F} \otimes \text{Ev}(V)$ as an object of $\text{LocSys}_{\text{restr} \hat{G}}(X) \otimes \text{QLisse}(X)^{\otimes I}$.

2.2.4. The following is a reformulation of one of the main results of the paper [AGKRRV1]:

**Theorem 2.2.5.** The functor $\text{coLoc}$ is an equivalence.

**Proof.** Recall the category $\text{Rep}(\hat{G})^{\otimes X\text{-lisse}} = \text{coHom}(\text{Rep}(\hat{G}), \text{QLisse}(X))$, see [AGKRRV1] Sects. 8.2.4 and 8.4.2.

According to [AGKRRV1] Lemma 8.2.8(b)], the category

$$\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{QLisse}(X)^{\otimes \text{fSet}})$$

identifies with the category

$$\text{Maps}_{\text{DGCat}}(\text{coHom}(\text{Rep}(\hat{G}), \text{QLisse}(X)), \text{Vect}).$$

Consider now the functor

$$(2.5) \quad \text{coHom}(\text{Rep}(\hat{G}), \text{QLisse}(X)) \to \text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X)))$$

of [AGKRRV1] Equation (8.8)].

Unwinding the definitions, we obtain that the composite functor

$$\text{Maps}_{\text{DGCat}}(\text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X))), \text{Vect}) \cong \text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X))) \overleftarrow{\text{coHom}(\text{Rep}(\hat{G}), \text{QLisse}(X))} \cong \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{QLisse}(X)^{\otimes \text{fSet}}) \cong \text{Maps}_{\text{DGCat}}(\text{coHom}(\text{Rep}(\hat{G}), \text{QLisse}(X)), \text{Vect})$$

identifies with the dual of $(2.5)$.

Now, [AGKRRV1] Theorem 8.3.7 says that $(2.5)$ is an equivalence. Hence, so is $\text{coLoc}$. \qed

2.3. **Localization.** In this subsection we will recall how $\text{Rep}(\hat{G})_{\text{Ran}}$ is related to $\text{QCoh}((\text{LocSys}_{\text{restr} \hat{G}}(X)))$. 
2.3.1. Consider the symmetric monoidal category $\text{Rep}(\tilde{G})_{\text{Ran}}$, see Sect. 1.4.1. We are going to construct a symmetric monoidal functor

$$\text{Loc} : \text{Rep}(\tilde{G})_{\text{Ran}} \to \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X))$$

that plays a key role in this work.

For an individual 

$$\left( I \xrightarrow{\psi} J \right) \in \text{TwArr}(\text{fSet}),$$

the corresponding functor

$$\text{Loc}_{I \xrightarrow{\psi} J} : \text{Rep}(\tilde{G}) \otimes I \otimes \text{Shv}(X_J) \to \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X))$$

sends $V \in \text{Rep}(\tilde{G})$ to the functor $\text{Shv}(X_J) \to \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X))$ equal to the composition

$$\text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)) \otimes \text{QLisse}(X_J) \otimes \text{Shv}(X_J) \xrightarrow{\text{mult} \otimes \text{Id}} \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)) \otimes \text{Shv}(X_J) \xrightarrow{\text{Id} \otimes \text{C}_J \cdot c(X_J)} \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)),$$

where the third arrow using the canonical action of $\text{QLisse}(X_J) \otimes J \simeq \text{QLisse}(X_J) \otimes \text{Shv}(X_J)$, see Sect. 1.3.6.

It is easy to see that the functors $\text{Loc}_{I \xrightarrow{\psi} J}$ indeed combine to define a functor, to be denote $\text{Loc}$, as in (2.6). Moreover, this functor carries a naturally defined symmetric monoidal structure.

2.3.2. Consider the dual functor

$$\text{Loc}^\vee : \text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)) \to (\text{Rep}(\tilde{G})_{\text{Ran}})^\vee.$$

Recall now that the category $\text{QCoh}(\text{LocSys}_G^{\text{restr}}(X))$ is self-dual (see Sect. 2.1.2(iii)), and that the category $(\text{Rep}(\tilde{G})_{\text{Ran}})^\vee$ can be described as $\text{Maps}_{\text{DGCat}}^{\text{fSet}}(C \otimes \text{fSet}, \text{Shv}(X^{\text{fSet}}))$ (see (1.19)).

Thus, we can view $\text{Loc}^\vee$ as a functor

$$\text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)) \to \text{Maps}_{\text{DGCat}}^{\text{fSet}}(C \otimes \text{fSet}, \text{Shv}(X^{\text{fSet}})).$$

Unwinding the definitions, we obtain that $\text{Loc}^\vee$ identifies with the composition

$$\text{QCoh}(\text{LocSys}_G^{\text{restr}}(X)) \xrightarrow{\text{coloc}} \text{Maps}_{\text{DGCat}}^{\text{fSet}}(C \otimes \text{fSet}, \text{QLisse}(X^{\text{fSet}}) \otimes \text{fSet}) \simeq \text{Maps}_{\text{DGCat}}^{\text{fSet}}(C \otimes \text{fSet}, \text{QLisse}(X^{\text{fSet}})) \to \text{Maps}_{\text{DGCat}}^{\text{fSet}}(C \otimes \text{fSet}, \text{Shv}(X^{\text{fSet}})),$$

where the last arrow is given by the (fully faithful) functor (1.5).

Hence, combining with Theorem 2.2.5, we obtain:

**Corollary 2.3.3.**

(a) The functor $\text{Loc}^\vee$ is fully faithful.

(b) An object $S \in (\text{Rep}(\tilde{G})_{\text{Ran}})^\vee$ lies in the essential image of $\text{Loc}^\vee$ if and only if the corresponding family of functors $\{S_I\}$

$$\text{Rep}(\tilde{G}) \otimes I \to \text{Shv}(X^I)$$

takes values in $\text{QLisse}(X^I) \subset \text{Shv}(X^I)$. 

2.3.4. Let
\[(\text{Rep}(\hat{G})_{\text{Ran}})^{\vee}_{\text{QLisse}} \subset (\text{Rep}(\hat{G})_{\text{Ran}})^{\vee}\]
be the full subcategory that under the equivalence (1.19) corresponds to
\[\text{Maps}_{\text{DGCat}^{\text{Set}}}((\text{Set}^\vee, \text{QLisse}(X^{\text{Set}})) \subset \text{Maps}_{\text{DGCat}^{\text{Set}}}((\text{Set}^\vee, \text{Shv}(X^{\text{Set}})),\]
where the embedding $\text{QLisse}(X^{\text{Set}}) \hookrightarrow \text{Shv}(X^{\text{Set}})$ is (1.5).

Thus, Corollary 2.3.3 can be reformulated as follows:

**Corollary 2.3.5.** The functor $\text{Loc}^{\vee}$ defines an equivalence

$$\text{QCoh}(\text{LocSys}_{\text{restr}}^{\text{G}}(X)) \rightarrow (\text{Rep}(\hat{G})_{\text{Ran}})^{\vee}_{\text{QLisse}}.$$ 

2.4. Beilinson’s spectral projector. In this subsection we will consider a certain object $R \in \text{Rep}(\hat{G})_{\text{Ran}}$, that will play a crucial role in the proof of the main results in this paper.

2.4.1. Let $R_{\text{Rep}(\hat{G})_{\text{Ran}}} \in \text{Rep}(\hat{G})_{\text{Ran}} \otimes \text{Rep}(\hat{G})_{\text{Ran}}$
be the object introduced in [AGKRRV1, Sect. 11.5].

The quickest way to define it is as the value on the unit object

$$1_{\text{Rep}(\hat{G})} \in \text{Rep}(\hat{G})_{\text{Ran}}$$
of the right adjoint to the monoidal operation

$$\text{Rep}(\hat{G})_{\text{Ran}} \otimes \text{Rep}(\hat{G})_{\text{Ran}} \rightarrow \text{Rep}(\hat{G})_{\text{Ran}}.$$ 

Since the above right adjoint carries a right-lax symmetric monoidal structure, the object $R_{\text{Rep}(\hat{G})_{\text{Ran}}}$ is naturally a commutative algebra in $\text{Rep}(\hat{G})_{\text{Ran}} \otimes \text{Rep}(\hat{G})_{\text{Ran}}$.

**Remark 2.4.2.** As was mentioned in Remark 1.6.3, the symmetric monoidal category $\text{Rep}(\hat{G})_{\text{Ran}}$ is rigid. The object $R_{\text{Rep}(\hat{G})_{\text{Ran}}}$ is the unit of the canonical self-duality defined for any rigid symmetric monoidal category.

2.4.3. We define the commutative algebra object

$$R_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)} \in \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X)) \otimes \text{Rep}(\hat{G})_{\text{Ran}}$$
to be

$$(\text{Loc} \otimes \text{Id})(R_{\text{Rep}(\hat{G})_{\text{Ran}}}).$$

2.4.4. Consider also the commutative algebra object

$$\text{(Id} \otimes \text{Loc})(R_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)}) =$$

\[= (\text{Loc} \otimes \text{Loc})(R_{\text{Rep}(\hat{G})_{\text{Ran}}}) \in \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X)) \otimes \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X)).\]

The following is [AGKRRV1] Theorem 12.7.4:

**Theorem 2.4.5.** The image of $(\text{Id} \otimes \text{Loc})(R_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)})$ under

$$\text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X)) \otimes \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X)) \rightarrow \text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X) \times \text{LocSys}_{\text{G}}^{\text{restr}}(X)),$$

identifies canonically with

$$(\Delta_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)})^* \circ (\mathcal{O}_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)}),$$
as commutative algebra objects.

**Remark 2.4.6.** In other words, Theorem 2.4.5 says that we have a canonical isomorphism

$$(\text{Loc} \otimes \text{Loc})(R_{\text{Rep}(\hat{G})_{\text{Ran}}}) \simeq (\Delta_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)})^* \circ (\mathcal{O}_{\text{LocSys}_{\text{G}}^{\text{restr}}(X)}),$$
as commutative algebra objects in $\text{QCoh}(\text{LocSys}_{\text{G}}^{\text{restr}}(X) \times \text{LocSys}_{\text{G}}^{\text{restr}}(X))$. 

2.4.7. Denote
\[ R := (\Gamma_!(\text{LocSys}_{\text{restr}}^G(X), -) \otimes \text{Id})(\text{R}_{\text{LocSys}_{\text{restr}}^G(X)}) \in \text{Rep}(\hat{G})_{\text{Ran}}, \]
where \( \Gamma_! \) is as in Sect. 2.1.4.

In other words,
\[ R = ((\Gamma_!(\text{LocSys}_{\text{restr}}^G(X), -) \circ \text{Loc}) \otimes \text{Id})(\text{R}_{\text{Rep}(\hat{G})_{\text{Ran}}}). \]

Recall (see [AGKRRV1, Sect. 7.6.1]) that the functor \( \Gamma_! \) carries a canonically defined (non-unital) right-lax symmetric monoidal structure. Hence, \( R \) is naturally a commutative algebra in \( \text{Rep}(\hat{G})_{\text{Ran}} \).

2.4.8. We claim:

**Corollary 2.4.9.** The object \( \text{Loc}(R) \in \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \) identifies canonically with \( \mathcal{O}_{\text{LocSys}_{\text{restr}}^G(X)} \), as commutative algebra objects.

**Proof.** Applying Theorem 2.4.5, it suffices to show that
\[ (\Gamma_!(\text{LocSys}_{\text{restr}}^G(X), -) \otimes \text{Id}) \circ (\Delta_{\text{LocSys}_{\text{restr}}^G(X)})^* (\mathcal{O}_{\text{LocSys}_{\text{restr}}^G(X)}) \simeq \mathcal{O}_{\text{LocSys}_{\text{restr}}^G(X)} \]
as commutative algebra objects.

We claim that
\[ (\Gamma_!(\text{LocSys}_{\text{restr}}^G(X), -) \otimes \text{Id}) \circ (\Delta_{\text{LocSys}_{\text{restr}}^G(X)})^* \simeq \text{Id} \]
as symmetric monoidal endofunctors of \( \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \).

This is a feature of any semi-rigid category, see [AGKRRV1, Appendix C]. Indeed, the identification as plain endofunctors follows by passage to the dual functors in
\[ \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \simeq \text{Vect} \otimes \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \]
\[ \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \otimes \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)) \overset{\mathcal{O}_{\text{LocSys}_{\text{restr}}^G(X)}}{\longrightarrow} \text{QCoh}(\text{LocSys}_{\text{restr}}^G(X)), \]
where we identify the dual of \( \text{mult}_{\text{QCoh}(\text{LocSys}_{\text{restr}}^G(X))} \) with its adjoint (see [AGKRRV1, Lemma C.3.5]).

The compatibility with the symmetric monoidal structures follows by unwinding the definition of the symmetric monoidal structure on \( \Gamma_!(\text{LocSys}_{\text{restr}}^G(X), -) \) in [AGKRRV1, Sect. C.3.8].

\( \square \)

3. The reciprocity law for shtuka cohomology

The main result of this section is Corollary 3.1.4 which asserts that the functor \( \text{Sht} \) introduced below factors through the localization functor \( \text{Loc} \). We will deduce it from a theorem of C. Xue on lisseness of shtuka cohomology.

As an application, we construct an object \( \text{Drinf} \in \text{QCoh}(\text{LocSys}_{\text{restr}}^G) \) that encodes the cohomology of shtuka moduli spaces.

3.1. **Functorial shtuka cohomology.** In this subsection we will interpret shtuka cohomology as a functor
\[ \text{Sht} : \text{Rep}(\hat{G})_{\text{Ran}} \rightarrow \text{Vect}, \]
i.e., as an object of the category \( (\text{Rep}(\hat{G})_{\text{Ran}})\vee \).
3.1.1. Recall the functor
\[ \mathcal{V} \in \text{Rep}(\hat{G})_{\text{Ran}} \twoheadrightarrow \mathcal{X}_V \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G), \]
see Sect. 1.5.3.

Let \( \text{Graph}_{\text{FrobBun}_G} : \text{Bun}_G \rightarrow \text{Bun}_G \times \text{Bun}_G \) be the graph of Frobenius, i.e., the map
\[ \text{Bun}_G \xrightarrow{\Delta_{\text{Bun}_G}} \text{Bun}_G \times \text{Bun}_G \xrightarrow{\text{FrobBun}_G \times \text{Id}} \text{Bun}_G \times \text{Bun}_G. \]

We define the functor \( Sht : \text{Rep}(\hat{G})_{\text{Ran}} \rightarrow \text{Vect} \) as the composition
\[ \text{Rep}(\hat{G})_{\text{Ran}} \xrightarrow{V} \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{(\text{Graph}_{\text{FrobBun}_G})^*} \text{Shv}(\text{Bun}_G) \xrightarrow{C \cdot (\text{Bun}_G, -)} \text{Vect}. \]

3.1.2. We will prove:

**Theorem 3.1.3.** The object \( Sht \in (\text{Rep}(\hat{G})_{\text{Ran}})^\vee \) belongs to the full subcategory \( (\text{Rep}(\hat{G})_{\text{Ran}})^\vee_{\text{QLisse}} \subset (\text{Rep}(\hat{G})_{\text{Ran}})^\vee \).

Applying Corollary 2.3.5, from Theorem 3.1.3, we obtain:

**Corollary 3.1.4.** The functor \( Sht : \text{Rep}(\hat{G})_{\text{Ran}} \rightarrow \text{Vect} \) factors as \( Sht_{\text{Loc}} \circ \text{Loc} \) for a uniquely defined functor \( Sht_{\text{Loc}} : \text{QCoh}(\text{LocSys}_{\text{restr}}\hat{G}(X)) \rightarrow \text{Vect} \).

3.2. **Proof of Theorem 3.1.3 and relation to the usual shtuka cohomology.** Once we relate the functor \( Sht \) to shtuka cohomology, the proof of Theorem 3.1.3 will be almost immediate from a recent theorem of C. Xue [Xue2].

3.2.1. For a finite set \( I \), consider the functor
\[ \text{Rep}(\hat{G}) \otimes I \rightarrow \text{Shv}(X^I), \]
to be denoted \( Sht_I \), that sends \( V \in \text{Rep}(\hat{G}) \otimes I \) to
\[ (3.1) \quad Sht_I(V) := (p_2) \circ (\text{Graph}_{\text{FrobBun}_G})^* \circ (\text{Id}_{\text{Bun}_G} \boxtimes H^1(V, -)) \circ \Delta_{\text{Bun}_G}. \]

The above functor \( Sht_I \) is the usual functor of (compactly supported) shtuka cohomology, studied by [VLaf].

3.2.2. We now quote the following crucial result of [Xue2]:

**Theorem 3.2.3.** The functor \( Sht_I \) takes values in \( \text{QLisse}(X^I) \subset \text{Shv}(X^I) \).

3.2.4. Let us show how Theorem 3.2.3 implies the assertion of Theorem 3.1.3.

By base change (and using the fact that \( X \) is proper), we obtain that for \( I \in \text{fSet} \), the functor
\[ \text{Rep}(\hat{G}) \otimes I \rightarrow \text{Shv}(X^I) \]
is given by sending
\[ V \in \text{Rep}(\hat{G}) \otimes I, M \in \text{Shv}(X^I) \rightarrow C \left( X^I, (Sht_I(V) \otimes \omega_{X^I}) \otimes M \right). \]

Thus, the object of \( \text{Maps}_{\text{DGCat}_{\text{fSet}}}(\text{Rep}(\hat{G}) \otimes \text{fSet}, \text{Shv}(X^{\text{fSet}})) \) corresponding to \( Sht \) is given by
\[ V \in \text{Rep}(\hat{G}) \otimes I \mapsto Sht_I(V) \otimes \omega_{X^I} \simeq Sht_I(V)[2|I|], \]
see Sect. 1.6.7.

This object belongs to \( \text{QLisse}(X^I) \) by Theorem 3.2.3 as required.
Remark 3.2.5. Note that by Theorem 3.2.3 the expression $C \left( X^t, (\text{Sh}_t(V) \otimes \omega_{X^t}) \otimes \mathcal{M} \right)$, which appears above, can be also rewritten as

$$C \left( X^t, \text{Sh}_t(V) \otimes \mathcal{M} \right),$$

see Sect. 3.2.3.

Remark 3.2.6. Along with the object $\text{Sh}^t(X^{\text{fSet}}) \in \text{DGCat}^{\text{fSet}}$, we can consider the object, denoted $\text{Sh}^t(X^{\text{fSet}})$, whose value on $I$ is again the category $\text{Sh}(X^I)$, but now for $I \to J$ we use the functor

$$(\Delta_\psi)^* : \text{Sh}(X^I) \to \text{Sh}(X^J).$$

As in Sect. 3.3.6 we have the fully faithful embeddings

$$\text{Sh}^t(X^{\text{fSet}}) \hookrightarrow \text{QLisse}(X^{\text{fSet}}) \hookrightarrow \text{Sh}^t(X^{\text{fSet}}),$$

given by (3.5) and (1.6).

By its construction, the system of functors $\{\text{Sh}_t\}$ is naturally an object of the category $\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}}))$, and we can view Theorem 3.2.3 as saying that it actually belongs to the essential image of the functor

$$\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{QLisse}(X^{\text{fSet}})) \hookrightarrow \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}})).$$

Now, the object of $\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}}))$ corresponding to $\text{Sht}$ equals the image of the resulting object $\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{QLisse}(X^{\text{fSet}}))$ under

$$\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{QLisse}(X^{\text{fSet}})) \hookrightarrow \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}})).$$

Thus, we denote by the same symbol $\{\text{Sh}_t\}$ the above objects of the categories

$$\text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}})) \hookrightarrow \text{Maps}_{\text{DGCat}^{\text{fSet}}}(\text{Rep}(\hat{G})^{\otimes \text{fSet}}, \text{Sh}^t(X^{\text{fSet}})).$$

Example 3.2.7. We have a canonical isomorphism

$$\text{Sht}(1_{\text{Rep}(\hat{G})_{\text{Ran}}}) \simeq \text{Sh}_q(\mathbf{e}) = C_{\mathbf{e}} \left( \text{Bun}_{\mathcal{G}}, (\text{Graph}_{\text{FrobBun}_{\mathcal{G}}})^* \circ (\Delta_{\text{Bun}_{\mathcal{G}}})_! (\mathbf{e}_{\text{Bun}_{\mathcal{G}}}) \right) \cong \text{Funct}_c(\text{Bun}_{\mathcal{G}}(F_q))$$

where the last isomorphism is by base-change.

3.3. Drinfeld’s sheaf.

3.3.1. By Corollary 3.1.4 there is a canonical functor $\text{Sht}_{\text{Loc}} : \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X)) \to \text{Vect}$, i.e.,

$$\text{Sht}_{\text{Loc}} \in \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X))^{\vee}.$$

As $\text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X))$ is canonically self-dual (cf. Sect. 2.1.2(iii)), to $\text{Sht}_{\text{Loc}}$ there corresponds an object

$$\text{Drinf} \in \text{QCoh}(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X)).$$

3.3.2. Let us observe some basic properties of $\text{Drinf}$.

First, we have

$$\Gamma_t(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X), \text{Drinf}) \simeq \text{Sht}_{\text{Loc}}(\mathcal{O}_{\text{LocSys}_{\mathcal{G}}^{\text{str}}(X)}).$$

by duality.

As

$$\mathcal{O}_{\text{LocSys}_{\mathcal{G}}^{\text{str}}} = \text{Loc}(1_{\text{Rep}(\hat{G})_{\text{Ran}}}),$$

we deduce from Example 3.2.7 that there is a canonical isomorphism

$$\Gamma_t(\text{LocSys}_{\mathcal{G}}^{\text{str}}(X), \text{Drinf}) \simeq \text{Sht}(1_{\text{Rep}(\hat{G})_{\text{Ran}}} = \text{Funct}_c(\text{Bun}_{\mathcal{G}}(F_q)).$$

*This object is named after V. Drinfeld, since it was his idea, upon learning about V. Lafforgue’s work, that shtuka cohomology should be encoded by a quasi-coherent sheaf on the stack of Langlands parameters.*
3.3.3. More generally, recall from Sect. [2.3.1] that for any $\mathcal{F} \in \text{QCoh}(\text{LocSys}_{G}^{\text{extr}}(X))$ and a finite set $I$, one defines a functor

$$\mathcal{F}_I : \text{Rep}(\mathcal{G})^{\otimes I} \to \text{QLisse}(X^{I}).$$

By construction, for $\mathcal{F} = \text{Drinf}$, the functor $\text{Drinf}_I$ coincides with $\text{Sht}_I$. In this manner, we see that $\text{Drinf}$ encodes cohomology of shtuka moduli spaces.

3.4. Some remarks. We now provide some additional remarks on the object $\text{Drinf}$.

**Remark 3.4.1.** Recall (see [AGKRRV1 Sect. 24.1]) that the stack $\text{LocSys}_{G}^{\text{arithm}}(X)$ is defined as

$$(\text{LocSys}_{G}^{\text{extr}}(X))^{\text{Frob}}.$$

Let $\iota$ denote the forgetful map

$$\text{LocSys}_{G}^{\text{arithm}}(X) \to \text{LocSys}_{G}^{\text{extr}}(X).$$

A priori, $\text{LocSys}_{G}^{\text{arithm}}(X)$ is a formal stack (i.e., a quotient of a formal affine scheme by an action of $G$), but [AGKRRV1 Theorem 24.1.4] says that $\text{LocSys}_{G}^{\text{arithm}}(X)$ is actually an algebraic stack.

One can show that the object $\text{Drinf}$ can be a priori obtained as $\iota_{*}(\text{Drinf}^{\text{arithm}})$ for a canonically defined object

$$\text{Drinf}^{\text{arithm}} \in \text{QCoh}(\text{LocSys}_{G}^{\text{arithm}}(X)).$$

This additional structure on $\text{Drinf}$ encodes the equivariance of the objects

$$\text{Sht}_I(V) \in \text{Shv}(X^{I}), \quad V \in \text{Rep}(\mathcal{G})^{\otimes I}$$

with respect to the partial Frobenius maps acting on $X^{I}$. See also Sect. [4.3.6].

**Remark 3.4.2.** The object $\text{Drinf}^{\text{arithm}}$ allows to recover the spectral decomposition of the space of automorphic functions along classical Langlands parameters, established in [VLaf] for the cuspidal subspace and extended in [Xue1] to the entire space.

Namely, by [6.2], we have:

$$\Gamma(\text{LocSys}_{G}^{\text{arithm}}(X), \text{Drinf}^{\text{arithm}}) \simeq \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)).$$

Set

$$\mathcal{A} := \Gamma(\text{LocSys}_{G}^{\text{arithm}}(X), \mathcal{O}_{\text{LocSys}_{G}^{\text{arithm}}(X)}),$$

this is is a commutative DG algebra over $\mathfrak{e}$ that lives in non-positive cohomological degrees. Set

$$\text{LocSys}_{G}^{\text{arithm, coarse}}(X) := \text{Spec}(\mathcal{A});$$

this is an affine (derived) scheme over $\mathfrak{e}$.

Let $\mathcal{A}^0$ denote the 0-th cohomology of $\mathcal{A}$, so that $\text{Spec}(\mathcal{A}^0)$ is the classical affine scheme $\text{cl}^{\mathcal{A}^{\text{arithm, coarse}}(X)}$ underlying $\text{LocSys}_{G}^{\text{arithm, coarse}}(X)$

Now, one can show that the set of $\mathfrak{e}$-points of $\text{Spec}(\mathcal{A}^0)$ is in bijection with isomorphism classes of semi-simple Frobenius-equivariant $G$-local systems on $X$, see [AGKRRV1 Theorem 4.6.5]. I.e., one can view $\text{cl}^{\mathcal{A}^{\text{arithm, coarse}}(X)}$ as the scheme of classical Langlands parameters.

By construction, $\mathcal{A}$ acts on the space of global sections of any object of $\text{QCoh}(\text{LocSys}_{G}^{\text{arithm}}(X))$. In particular, we obtain an action of $\mathcal{A}$ on $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$. However, since $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$ sits in cohomological degree 0, this action factors through an action of $\mathcal{A}^0$ on $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$.

Thus, we can view $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$ as global sections of a canonically defined quasi-coherent sheaf on $\text{cl}^{\mathcal{A}^{\text{arithm, coarse}}(X)}$. This indeed may be viewed as a spectral decomposition of $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$ over classical Langlands parameters.

Furthermore, one can show that $\mathcal{A}^0$ is a quotient of V. Lafforgue’s algebra of excursion operators. So the above action of $\mathcal{A}^0$ recovers the action of the excursion algebra on $\text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$, established in [Xue1].
Remark 3.4.3. The above construction of the object Drinf (resp., Drinf_{arithm}) was specific to the *everywhere unramified* situation. In a subsequent publication, we will show this construction can be generalized to allow for level structure.

I.e., given a divisor $D \subset X$ defined over $\mathbb{F}_q$, one can construct objects

$$\text{Drinf}_D \in \text{LocSys}_G^{\text{restr}}(X - D)$$

and

$$\text{Drinf}_{\text{arithm}}^D \in \text{LocSys}_{G}^{\text{arithm}}(X - D)$$

that encode shtuka cohomology with level structure.

What is for now a far-fetched goal is to interpret Drinf_{arithm} also as categorical trace, see Sect. 4.5.8 for what we mean by that.

4. Calculating the trace

In this section we will prove the main result of this paper, Corollary 4.1.4, which asserts that the space of (compactly supported) automorphic functions can be obtained as the (categorical) trace of Frobenius on the category of automorphic sheaves with nilpotent singular support.

4.1. Traces of Frobenius-Hecke operators.

4.1.1. Recall that the $\text{Rep}(\hat{G})_{\text{Ran}}$-action on $\text{Shv}(\text{Bun}_G)$ preserves its subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$. Therefore, we obtain a functor

$$\text{Rep}(\hat{G})_{\text{Ran}} \to \text{Maps}_{\text{DGCat}}(\text{Shv}_{\text{Nilp}}(\text{Bun}_G), \text{Shv}_{\text{Nilp}}(\text{Bun}_G))$$

sending $V$ to the functor $V \star -$.

Note also that the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$ is preserved by the endofunctor $(\text{Frob}_{\text{Bun}_G})^*$, see [AGKRRV1, Sect. 22.3.3].

We define a functor

$$\text{Sht}^\text{Tr} : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$$

as the functor

$$V \mapsto \text{Tr}(H_V \circ (\text{Frob}_{\text{Bun}_G})^* \circ \text{Shv}_{\text{Nilp}}(\text{Bun}_G))$$

In other words, we compose $H_V$ with pushforward along the geometric Frobenius $\text{Frob}_{\text{Bun}_G}$ endomorphism of $\text{Bun}_G$ and form the trace (as an endofunctor of the (dualizable) DG category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$).

Our main theorem asserts:

**Theorem 4.1.2.** There is a canonical isomorphism of functors $\text{Sht} \simeq \text{Sht}^\text{Tr} : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$.

We will prove this result in Sect. 4.2.

4.1.3. For the moment, we assume Theorem 4.1.2 and deduce further results from it.

First, observe that by definition, we have

$$\text{Sht}^\text{Tr}(1_{\text{Rep}(\hat{G})_{\text{Ran}}}) = \text{Tr}((\text{Frob}_{\text{Bun}_G})^* \circ \text{Shv}_{\text{Nilp}}(\text{Bun}_G))$$

On the other hand, by Example 3.2.7 we have

$$\text{Sht}(1_{\text{Rep}(\hat{G})_{\text{Ran}}}) = \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$$

Hence, we obtain:

**Corollary 4.1.4.** There exists a canonical isomorphism in Vect

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})^* \circ \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \simeq \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)).$$
4.1.5. By (1.19), the functor $\text{Sht}^T_I$ corresponds to a system of functors $\{\text{Sht}^T_I\}$

$$\text{Sht}^T_I : \text{Rep}(\tilde G)^{\otimes I} \to \text{Shv}(X^I).$$

From Theorem 4.1.2 we immediately obtain:

**Corollary 4.1.6.** For an individual finite set $I$, the functors $\text{Sht}_I$ and $\text{Sht}^T_I$ are canonically isomorphic.

4.1.7. We will use the following construction.

Let $C$ be a dualizable DG category. Let $\text{Vect}^{u_C} C \xrightarrow{\mu_C} C \xrightarrow{\epsilon_C} C$ be the unit and counit of the duality, respectively.

Let $T : C \to C \otimes D$ be a functor, where $D$ is another DG category. We can consider the relative trace object $\text{Tr}(T, C) \in D$ defined as the composition

$$\text{Vect}^{u_C} C \xrightarrow{\mu_C} C \xrightarrow{\epsilon_C} C \otimes D \xrightarrow{\text{ev}_C} D.$$

4.1.8. Let $I$ be a finite set. Recall that we have a canonically defined functor

$$\text{H} : \text{Rep}(\tilde G)^{\otimes I} \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{QLisse}(X^I),$$

see (1.9).

In particular, for $V \in \text{Rep}(\tilde G)^{\otimes I}$ we can consider the object

$$\text{Tr}(\text{H}(V, -) \circ (\text{Frob}_{\text{Bun}_G})^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \in \text{QLisse}(X^I),$$

and this operation defines a functor

$$(4.2) \quad \text{Rep}(\tilde G)^{\otimes I} \to \text{QLisse}(X^I).$$

Unwinding the definitions, it is easy to see that the functor $\text{Sht}^T_I$ of (4.1) identifies with the composition of (4.2) and the embedding $\text{QLisse}(X^I) \to \text{Shv}(X^I)$ of (1.5).

4.1.9. Hence, from Corollary 4.1.6 we obtain:

**Corollary 4.1.10.** For an individual finite set $I$, the functor

$$\text{Sht}_I : \text{Rep}(\tilde G)^{\otimes I} \to \text{QLisse}(X^I)$$

of (4.1) identifies canonically with the functor

$$V \mapsto \text{Tr}(\text{H}(V, -) \circ (\text{Frob}_{\text{Bun}_G})^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)).$$

Corollary 4.1.10 is the **Shtuka Conjecture** from [AGKRRV1] (Conjecture 22.5.7 in loc.cit.).

4.1.11. Finally, we note that Theorem 4.1.2 may be tautologically be reformulated in terms of Drinfeld sheaves.

First, as was noted above, the functors $\text{Sht}^T_I$ take values in $\text{QLisse}(X^I)$. Therefore, by Corollary 2.3.5 $\text{Sh}^T_I$ factors uniquely as $\text{Sh}^T_I \circ \text{Loc}$ for a uniquely defined functor

$$\text{Sh}^T_I : \text{QCoh}(\text{LocSys}_{\tilde G}^{\text{restr}}(X)) \to \text{Vect}.$$

As in Sect. 3.3 by self-duality of $\text{Q Coh}(\text{LocSys}_{\tilde G}^{\text{restr}}(X))$, to $\text{Sh}^T_I$ there corresponds an object $\text{Drinf}^T_I \in \text{Q Coh}(\text{LocSys}_{\tilde G}^{\text{restr}}(X))$.

From Theorem 4.1.2 we obtain:

**Corollary 4.1.12.** The objects $\text{Drinf}$ and $\text{Drinf}^T_I$ of $\text{Q Coh}(\text{LocSys}_{\tilde G}^{\text{restr}})$ are canonically isomorphic.
4.2. Proof of Theorem 4.1.2

4.2.1. Recall the object $R \in \text{Rep}(\hat{G})_{\text{Ran}}$ of Sect. 2.4. The proof of Theorem 4.1.2 will be based on the following result.

**Theorem 4.2.2.** *There is a canonical isomorphism* 
$$Sht^{\text{Tr}}(-) \simeq \text{Sht}(R \star -)$$
*as functors* $\text{Rep}(\hat{G})_{\text{Ran}} \to \text{Vect}$.

**Remark 4.2.3.** As the proof of Theorem 4.2.2 will show, we will rather construct an isomorphism 
$$Sht^{\text{Tr}}(-) \simeq \text{Sht}(- \star R),$$
and we will swap $\text{Sht}(- \star R)$ for $\text{Sht}(R \star -)$ using the fact that the category $\text{Rep}(\hat{G})_{\text{Ran}}$ is symmetric monoidal.

Our preference of one over the other is purely notational.

4.2.4. We postpone the proof of Theorem 4.2.2 to Sect. 4.4. It is essentially a calculation using the results of [AGKRRV2], which we recall in Sect. 4.3.

For the present, we assume Theorem 4.2.2 and deduce Theorem 4.1.2 from it.

4.2.5. The argument is straightforward at this point using Theorem 3.1.3. Recall that Corollary 3.1.4 provides a factorization $\text{Sht} = \text{Sht}^{\text{Loc}} \circ \text{Loc}$.

Now recall that $\text{Loc}$ is monoidal and sends $R$ to the structure sheaf by Corollary 2.4.9. Therefore, 
$$\text{Loc}(R \star -) \simeq \text{Loc}(-)$$
as functors $\text{Rep}(\hat{G})_{\text{Ran}} \to \text{QCoh}(\text{LocSys}_{G}^{\text{red}}(X))$.

By Theorem 4.2.2, we obtain
$$\text{Sht}^{\text{Tr}} \simeq \text{Sht}(R \star -) \simeq \text{Sht}^{\text{Loc}} \circ \text{Loc}(R \star -) \simeq \text{Sht}^{\text{Loc}} \circ \text{Loc}(-) \simeq \text{Sht}$$
as desired.

4.3. Self-duality for $\text{Shv}_{\text{Nilp}}(\text{Bun}_{G})$. To prove Theorem 4.2.2, we use the explicit description of the dual of $\text{Shv}_{\text{Nilp}}(\text{Bun}_{G})$ from [AGKRRV2]. We review these results below.

4.3.1. Consider the object $\mathcal{K}_{R} \in \text{Shv}(\text{Bun}_{G} \times \text{Bun}_{G})$, where $R \in \text{Rep}(\hat{G})_{\text{Ran}}$ was defined in Sect. 2.4 and where the notation $\mathcal{K}_{-}$ is an in Sect. 1.5.4.

A priori, it is defined as an object of the category $\text{Shv}(\text{Bun}_{G} \times \text{Bun}_{G})$. However, we have the following key result, see [AGKRRV2] Sect. 3.1.2 and Proposition 3.1.4):

**Theorem 4.3.2.** *The object $\mathcal{K}_{R}$ lies in the essential image of the fully faithful functor* 
$$\text{Shv}_{\text{Nilp}}(\text{Bun}_{G}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_{G}) \to \text{Shv}(\text{Bun}_{G} \times \text{Bun}_{G})$$.

**Remark 4.3.3.** In [AGKRRV2] Sect. 3.1.2], the object $\mathcal{K}_{R}$ was denoted $\text{ps-u}_{\text{Bun}_{G},\text{Nilp}}$.

4.3.4. For a stack $\mathcal{Y}$ let 
$$\text{ev}_{\mathcal{Y}} : \text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \to \text{Vect}$$
be the functor given by 
$$\mathcal{F}_{1}, \mathcal{F}_{2} \mapsto C_{c}(\mathcal{Y}, \mathcal{F}_{1} \otimes \mathcal{F}_{2}).$$

**Warning:** in general, the pairing $\text{ev}_{\mathcal{Y}}$ is *not* perfect, i.e., it does not define a self-duality on $\text{Shv}(\mathcal{Y})$. 
4.3.5. Take $Y = \text{Bun}_G$, and we restrict the pairing $\text{ev}^Y_{\text{Bun}_G}$ to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)$.

We now quote the following result (see [AGKRRV2 Theorem 3.2.2]):

**Theorem 4.3.6.** The object $X_r \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ together with the pairing

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \rightarrow \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{\text{ev}^Y_{\text{Bun}_G}} \text{Vect}$$

define an identification

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^\dagger \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

**Remark 4.3.7.** The trace calculation involved in the proof of Theorem 4.2.2 uses the explicit description of the dual of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, provided by Theorem 4.3.6. This may be the most conceptually non-trivial place in the paper:

There is a more obvious candidate for the dual of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, namely the category denoted $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$, see [AGKRRV2 Corollary 2.6.5]. The two descriptions of the dual are related by a non-trivial operation, namely the *miraculous functor* $\text{Mir}_{\text{Bun}_G}$, which defines an equivalence $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \rightarrow \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, see [AGKRRV2 Corollary 2.9.5].

However, if we used the description of the dual of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ as $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$, we would not be able to directly relate the functor $\text{Sht}^{\text{Fl}}$ to shtukas, as defined in [VLa]. Rather, we would encounter co-shtukas. The latter is a functor $\text{Rep}(\hat{G})_{\text{Ran}} \rightarrow \text{Vect}$ given by

$$\text{Rep}(\hat{G})_{\text{Ran}} \xrightarrow{\land_{\text{co}}} \mathcal{X}_V \rightarrow \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co1}} \xrightarrow{\text{Graph}_{\text{Frob}} \hat{\text{Bun}}_G} \text{Shv}(\text{Bun}_G)_{\text{co}} \xrightarrow{C_4(\text{Bun}_G, -)} \text{Vect},$$

where:

- $\text{Shv}(\text{Bun}_G)_{\text{co}}$ is the category defined in [AGKRRV2 Sect. 2.5];
- $\text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}}$ is as in [AGKRRV2 Sect. C.4.5];
- $C_4(\text{Bun}_G, -) : \text{Shv}(\text{Bun}_G)_{\text{co1}} \rightarrow \text{Vect}$ is the functor of renormalized chains, see [AGKRRV2 Sect. C.3.4];
- $\mathcal{X}_V := (\text{Id}_{\text{Bun}_G} \otimes \text{H}_V)(\mathfrak{u}_{\text{Bun}_G, \text{co1}})$, where $\mathfrak{u}_{\text{Bun}_G, \text{co1}} \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co1}}$

is as in [AGKRRV2 Sect. C.4.6].

The problem with this approach is that co-shtukas seem to be a rather unwieldy objects, compared to shtukas.

4.4. **Proof of Theorem 4.2.2**

4.4.1. Let us be given a dualizable DG category $C$ and functors $T, S : C \rightarrow C$.

Note that we can compute $\text{Tr}(S \circ T, C) \subset \text{Vect}$ as the composition

$$\text{Vect} \xrightarrow{\text{ev}_C} C^\vee \otimes C \xrightarrow{T^\vee \otimes S} C^\vee \otimes C \xrightarrow{\text{ev}_{C, C} \otimes \text{Id}} \text{Vect}.$$ 

4.4.2. For $V \in \text{Rep}(\hat{G})_{\text{Ran}}$, we deduce from Theorem 4.3.6 and the above observation that we can compute $\text{Sht}^{\text{Fl}}(V)$ as the composition

$$\text{Vect} \xrightarrow{\land_{\text{co}}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{((\text{Frob}_{\text{Bun}_G})^\dagger)^\vee \otimes \text{H}_V} \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{C_4(\text{Bun}_G, -)} \text{Vect}.$$ 

Here $((\text{Frob}_{\text{Bun}_G})^\dagger)^\vee$ is the dual functor to $(\text{Frob}_{\text{Bun}_G})^\dagger$ with respect to the duality of Theorem 4.3.6.
4.4.3. To proceed further, we need to compute $((\text{Frob}_{\text{Bun}}G)_*)^\vee$.

Note that the functor $(\text{Frob}_{\text{Bun}}G)_*$ also preserves the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$ (this is established in the course of the proof of [AGRRV1, Lemma 22.3.2]).

Lemma 4.4.4. In the above notation, the functor $((\text{Frob}_{\text{Bun}}G)_*)^\vee : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is canonically isomorphic to $(\text{Frob}_{\text{Bun}}G)_*$.

Proof.

For an algebraic stack $Y$, the pullback functor $(\text{Frob}_Y)_*$ is a self-equivalence of $\text{Shv}(Y)$. Hence, its right adjoint, which is $(\text{Frob}_Y)_!$, is the inverse of $(\text{Frob}_Y)_*$.

Take $Y = \text{Bun}_G$. Since the functors $(\text{Frob}_{\text{Bun}}G)_*$ and $(\text{Frob}_{\text{Bun}}G)_*$ both preserve $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$, we obtain that $(\text{Frob}_{\text{Bun}}G)_*|_{\text{Shv}_{\text{Nilp}}(\text{Bun}_G)}$ and $(\text{Frob}_{\text{Bun}}G)_*|_{\text{Shv}_{\text{Nilp}}(\text{Bun}_G)}$ are mutually inverse equivalences.

The assertion of the lemma follows now formally from the fact that counit of the self-duality on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, i.e., the functor $\text{ev}_{\text{Bun}_G}$, is mapped to itself by $(\text{Frob}_{\text{Bun}}G)_* \otimes (\text{Frob}_{\text{Bun}}G)_*$. To check this, we have to show that

\[
C_c(\text{Bun}_G, (\text{Frob}_{\text{Bun}}G)_*(-)) \simeq C_c(\text{Bun}_G, -).
\]

The latter is true for any algebraic stack

\[
C_c(Y, (\text{Frob}_Y)_*(-)) \simeq C_c(Y, (\text{Frob}_Y)_* \circ (\text{Frob}_Y)^*(-)) \overset{\text{Frob}_Y \text{ is proper}}{\simeq} \simeq C_c(Y, (\text{Frob}_Y)_* \circ (\text{Frob}_Y)^*(-)) \simeq C_c(Y, -).
\]

Remark 4.4.5. The isomorphism $7$ implies that $(\text{Frob}_Y)_!$ is also an equivalence. Hence, its right adjoint, i.e., $(\text{Frob}_Y)^!$ is its inverse. From here we obtain

\[
(\text{Frob}_Y)^! \simeq ((\text{Frob}_Y)_!)^{-1} \simeq ((\text{Frob}_Y)_!)_! \simeq (\text{Frob}_Y)^*.
\]

4.4.6. Combining Sect. 4.4.2 and Lemma 4.4.4, we obtain that for $V \in \text{Rep}(\tilde{G})_{\text{Ran}}$, we can compute $\text{Sh}(\text{V})$ as the composition:

\[
\begin{align*}
\text{Vect} & \xrightarrow{\text{Spec}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\text{(Frob}_{\text{Bun}}G)_* \otimes \text{H}_V} \\
& \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\phi} \text{Shv}(\text{Bun}_G) \overset{C_c(\text{Bun}_G, -)}{\to} \text{Vect},
\end{align*}
\]

which is the same as

\[
\begin{align*}
\text{Vect} & \xrightarrow{\text{Spec}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\otimes} \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{\text{Id}_{\text{Bun}_G} \otimes \text{H}_V} \\
& \to \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{(\text{Frob}_{\text{Bun}}G \times \text{id})_*} \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{\Delta_{\text{Bun}_G}^\vee} \text{Shv}(\text{Bun}_G) \xrightarrow{C_c(\text{Bun}_G, -)} \text{Vect}.
\end{align*}
\]

\({}^7\)Note that for an algebraic stack, the Frobenius morphism is proper and radial, but not necessarily schematic.
4.4.7. Note that
\[(\text{Id}_{\text{Bun}_G} \boxtimes H_V)(\mathcal{X}_R) \simeq \mathcal{X}_{V \times R} \simeq \mathcal{X}_{R \times V}.
\]
Therefore, we can compute \(\text{Sht}^\text{Tr}(\mathcal{V})\) as the composition
\[
\text{Vect} \xrightarrow{\mathcal{X}_{R \times V}} \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{(\text{Frob}_{\text{Bun}_G} \times \text{id})^\ast} \text{Shv}(\text{Bun}_G) \xrightarrow{\Delta_{\text{Bun}_G}^\ast} \text{Shv}(\text{Bun}_G) \xrightarrow{C_x(Bun_G, -)} \text{Vect}.
\]
This coincides with
\[
\text{Vect} \xrightarrow{\mathcal{X}_{R \times V}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{(\text{Graph}_{\text{Frob}_{\text{Bun}_G}})^\ast} \text{Shv}(\text{Bun}_G) \xrightarrow{C_x(Bun_G, -)} \text{Vect},
\]
which is the same as \(\text{Sht}(R \times V)\), by definition.
\[\square\text{Theorem 4.2.2}\]

4.5. **Interpretation as enhanced trace.** The contents of this section are an extended remark and are not necessary for the rest of the paper.

4.5.1. Recall (see [AGKRRV1, Theorem 14.3.2]) that the category \(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)\) carries a monoidal action of \(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X))\).

Moreover, the action of \((\text{Frob}_{\text{Bun}_G})^\ast\) is compatible with the action of the monoidal automorphism of \(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X))\), where \(\text{Frob}\) is the automorphism of \(\text{LocSys}_{\text{restr}}^\text{arithm} G(X)\), induced by the Frobenius endomorphism \(\text{Frob}_X\) of \(X\).

In this case, following [GKRV, Sect. 3.8.2], to the pair \((\text{Shv}_{\text{Nilp}}(\text{Bun}_G), (\text{Frob}_{\text{Bun}_G})^\ast)\) we can attach its enhanced trace,
\[
\text{Tr}_{\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X))}^\text{enh}((\text{Frob}_{\text{Bun}_G})^\ast, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \in \text{HH}_\ast(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X)), \text{Frob}^\ast),
\]
where \(\text{HH}_\ast(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X)), \text{Frob}^\ast)\) is the (symmetric monoidal) DG category of Hochschild chains on the (symmetric) monoidal category \(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X))\) with respect to the (symmetric) monoidal endofunctor \(\text{Frob}^\ast\).

Furthermore, we have
\[
\text{HH}_\ast(\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X)), \text{Frob}^\ast) \simeq \text{QCoh}((\text{LocSys}_{\text{restr}}^\text{arithm} G(X))^{\text{Frob}}),
\]
see [AGKRRV1] Sect. 7.10.4.

4.5.2. Recall that we denote
\[
\text{LocSys}_{\text{G}}^\text{arithm}(X) := (\text{LocSys}_{\text{G}}^\text{restr} (X))^{\text{Frob}},
\]
and by \(\iota\) the forgetful map
\[
\text{LocSys}_{\text{G}}^\text{arithm}(X) \to \text{LocSys}_{\text{G}}^\text{restr} (X),
\]
see [AGKRV1] Sect. 24.1].

Thus, we can interpret the above object \(\text{Tr}_{\text{QCoh}(\text{LocSys}_{\text{restr}}^\text{arithm} G(X))}^\text{enh}((\text{Frob}_{\text{Bun}_G})^\ast, \text{Shv}_{\text{Nilp}}(\text{Bun}_G))\) as an object, denoted
\[
\text{Drinf}^{\text{Tr}, \text{arithm}} \in \text{QCoh}(\text{LocSys}_{\text{G}}^\text{arithm}(X)).
\]

We have
\[(4.4) \text{Drinf}^{\text{Tr}} \simeq \iota_\ast(\text{Drinf}^{\text{Tr}, \text{arithm}});\]
this assertion is essentially [GKRV Theorem 4.4.4], with a generalization supplied by [AGKRRV1] Theorem 7.10.6].
4.5.3. Recall the objects
\[ \text{Ev}(V) \in \text{Qcoh}(\text{LocSys}^\text{restr}_G(X)) \otimes \text{QLisse}(X)^{\otimes I}. \]

By construction, the corresponding objects
\[ (\iota^* \otimes \text{Id})(\text{Ev}(V)) \in \text{Qcoh}(\text{LocSys}^\text{arithm}_G(X)) \otimes \text{QLisse}(X)^{\otimes I} \]
carry a structure of equivariance with respect to the partial Frobenius endomorphisms acting along \( X^I \).

By (4.4) and the projection formula, we have
\[ (4.5) \quad \text{Sht}^I_{\text{Tr}}(V) \simeq (\Gamma(V \otimes \text{Drinf}^\text{Tr,arithm}) \circ \iota^*)(\iota^* \otimes \text{Id})(\text{Ev}(V) \otimes \text{Drinf}^\text{Tr,arithm})). \]

This interpretation shows that the objects \( \text{Sht}^I_{\text{Tr}}(V) \in \text{QLisse}(X)^{\otimes I} \subset \text{Shv}(X^I) \)
carry a natural structure of equivariance with respect to the partial Frobenius endomorphisms on \( X^I \).

Remark 4.5.4. Recall that \( \text{LocSys}^\text{arithm}_G(X) \) is actually a quasi-compact algebraic stack, see Remark 3.4.1. This implies that in formula (4.5), we have
\[ \Gamma(\text{LocSys}^\text{arithm}_G(X), -) \simeq \Gamma(\text{LocSys}^\text{arithm}_G(X), -). \]

4.5.5. Having proved Theorem 4.1.2 and hence Corollary 4.1.6, we obtain that the objects \( \text{Sht}^I_{\text{Tr}}(V) \in \text{Shv}(X^I) \)
also carry a natural structure of equivariance with respect to the partial Frobenius endomorphisms on \( X^I \).

However, it is not difficult to show that this structure identifies with the structure of partial Frobenius equivariance on shtukas, defined in [VLaf] and [Xue1]. The matching between the two is essentially [GKRV, Lemma 4.5.4].

4.5.6. Now, given an object \( \mathcal{F} \in \text{Qcoh}(\text{LocSys}^\text{restr}_G(X)) \), the datum required to exhibit it as
\[ \iota_*(\mathcal{F}^\text{arithm}) \]
for some \( \mathcal{F}^\text{arithm} \in \text{LocSys}^\text{arithm}_G(X) \) is equivalent to a compatible collection of data of partial Frobenius equivariance on the associated functors
\[ \mathcal{F}_I : \text{Rep}(\hat{G})^{\otimes I} \to \text{QLisse}(X)^{\otimes I}, \quad V \mapsto (\Gamma(V \otimes \text{Drinf}^\text{Tr,arithm}) \circ \iota^*)(\iota^* \otimes \text{Id})(\text{Ev}(V) \otimes \mathcal{F}). \]

4.5.7. As was mentioned in Sect. 4.5.5, the structure of partial Frobenius equivariance on the functors \( \text{Sht}_I \) is priori defined in [VLaf] and [Xue1], thereby producing an object
\[ \text{Drinf}^\text{arithm} \in \text{Qcoh}(\text{LocSys}^\text{arithm}_G(X)) \]
that encode the cohomology of shtukas with level structure, thereby producing an object
\[ \text{Drinf}^\text{arithm} \in \text{Qcoh}(\text{LocSys}^\text{arithm}_G(X)). \]

Thus, the assertion in Sect. 4.5.6 can interpreted as an isomorphism
\[ \text{Drinf}^\text{arithm} \simeq \text{Drinf}^\text{Tr,arithm}, \]
which induces the isomorphism
\[ \text{Drinf} \simeq \text{Drinf}^\text{Tr} \]
of Theorem 1.1.2 by applying \( \iota_* \).

4.5.8. Furthermore, as was mentioned in Remark 3.4.3, the corresponding structure of partial Frobenius equivariance can be constructed also on the functors
\[ \text{Sht}_{I,D} : \text{Rep}(\hat{G})^{\otimes I} \to \text{QLisse}(X - D)^{\otimes I} \]
that encode the cohomology of shtukas with level structure, thereby producing an object
\[ \text{Drinf}^\text{arithm}_D \in \text{Qcoh}(\text{LocSys}^\text{arithm}_G(X - D)). \]

What we do not have at the moment is the interpretation of this object \( \text{Drinf}^\text{arithm}_D \) as enhanced categorical trace (nor of the functors \( \text{Sht}_{I,D} \) as just categorical traces).
5. Local terms

5.1. A hypothesis. In order to simplify the exposition, for the duration of this section, we will assume [AGKRRV1, Conjecture 14.1.8]. This conjecture says that the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is generated by objects that are compact as objects of the ambient category $\text{Shv}(\text{Bun}_G)$.

Since we know that $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is compactly generated as a DG category (see [AGKRRV1, Theorem 16.1.1]), this conjecture is equivalent to the statement that the forgetful functor

$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G)$

preserves compactness, or, equivalently, that it admits a continuous right adjoint.

5.2. Formulation of the problem.

5.2.1. In this section we will construct four maps $\text{LT}_{\text{naive}}, \text{LT}_{\text{true}}, \text{LT}_{\text{Serre}}, \text{LT}_{\text{Sht}} : \text{Tr}(\text{Frob}_{\text{Bun}_G}^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \to \text{Funct}(\text{Bun}_G(\mathbb{F}_q))$ that we refer to as local term morphisms.

At this point, we have only encountered the last of these four morphisms: by definition, $\text{LT}_{\text{Sht}}$ is the isomorphism of Corollary 4.1.4.

5.2.2. Assuming the construction of the other three morphisms, we can state the main result of this section and the next:

**Theorem 5.2.3.** The four morphisms $\text{LT}_{\text{naive}}, \text{LT}_{\text{true}}, \text{LT}_{\text{Serre}}, \text{LT}_{\text{Sht}}$ are equal.

**Remark 5.2.4.** By Corollary 4.1.4, the source and target of each local term map is in $\text{Vect}^\mathbb{Q}$, so equality (as opposed to homotopy) is the relevant notion.

5.2.5. As $\text{LT}_{\text{Sht}}$ is an isomorphism, from Theorem 5.2.3 we deduce:

**Corollary 5.2.6.** Each of the four local term morphisms is an isomorphism.

In the case of $\text{LT}_{\text{naive}}$ (see below), this corollary recovers the Trace Conjecture as formulated in [AGKRRV1, Conjecture 22.3.7].

We now proceed to the construction of the local term morphisms.

5.3. Naive and true local terms. In what follows, we fix an algebraic stack $\mathcal{Y}$, which plays the role of $\text{Bun}_G$. We will add certain additional assumptions to $\mathcal{Y}$ as we proceed.

The material in this subsection closely follows [AGKRRV1, Sect. 22.1-22.2], to which we refer the reader for more detail.

5.3.1. Naive local term. Any algebraic stack $\mathcal{Y}$ has the property that $\text{Shv}(\mathcal{Y})$ is compactly generated and every compact object is $!$-extended from some quasi-compact open in $\mathcal{Y}$ (see [AGKRRV1, Sect. F.1.1]).

Suppose $y \in \mathcal{Y}(\mathbb{F}_q)$ is given. In this case, the functor $y^* : \text{Shv}(\mathcal{Y}) \to \text{Vect}$ preserves compact objects and intertwines $(\text{Frob}_y)^*$, with the identity functor (by 4.3). Therefore, functoriality of traces yields a map

$\text{Tr}((\text{Frob}_y)^*, \text{Shv}(\mathcal{Y})) \to \text{Tr}(\text{Id}, \text{Vect}) = e$.

In the quasi-compact case, this yields a map

$\text{LT}_{\text{naive}} : \text{Tr}((\text{Frob}_y)^*, \text{Shv}(\mathcal{Y})) \to \text{Funct}(\mathcal{Y}(\mathbb{F}_q))$.

**Remark 5.3.2.** By definition, this construction satisfies the compatibility referenced in Remark 0.6.5.
5.3.3. True local term. Suppose first that $Y$ is quasi-compact. Suppose in addition that $Y$ is locally of the form $Z/H$, where $Z$ is a scheme of finite type and $H$ is an affine algebraic group.

As in [AGKRRY2 Sect. A.4], these assumptions imply that $\text{Shv}(Y)$ is canonically self-dual (via Verdier duality), with pairing, denoted $e_{Y}$:

$$\text{Shv}(Y) \otimes \text{Shv}(Y) \subset \text{Shv}(Y \times Y) \xrightarrow{\Delta_Y} \text{Shv}(Y)$$

The unit for this duality, denoted $u_{\text{Shv}(Y)} \in \text{Shv}(Y) \otimes \text{Shv}(Y)$ is obtained by applying the right adjoint to the embedding

$$\text{Shv}(Y) \otimes \text{Shv}(Y) \xhookrightarrow{\boxtimes} \text{Shv}(Y \times Y)$$

to $(\Delta_Y)_*(\omega_Y)$.

In what follows we will not distinguish notationally between $u_{\text{Shv}(Y)}$ and its image under the fully faithful functor (5.1). Thus, by adjunction we obtain a map

$$u_{\text{Shv}(Y)} \rightarrow (\Delta_Y)_*(\omega_Y).$$

From here, we obtain a map

$$\text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)) \simeq C_{\bullet}(Y, \Delta_Y \circ (\text{Frob}_Y)_\ast (\text{Id}))((u_{\text{Shv}(Y)})) \simeq C_{\bullet}(Y, \Delta_Y \circ (\text{Frob}_Y \times \text{Id}) \ast (u_{\text{Shv}(Y)})) \rightarrow$$

$$\rightarrow C_{\bullet}(Y, \Delta_Y \circ (\text{Frob}_Y \times \text{Id}) \ast (\Delta_Y)_\ast(\omega_Y)) \simeq C_{\bullet}(y_{\text{Frob}}, \omega_{y_{\text{Frob}}}) \simeq \text{Funct}(Y(F_q))$$

whose composition we denote by

$$\text{LT}_Y^{\text{true}} : \text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)) \rightarrow \text{Funct}(Y(F_q)).$$

5.3.4. We have (see [GV Theorem 0.4]):

**Theorem 5.3.5.**

(a) The maps $\text{LT}_Y^{\text{true}}$ and $\text{LT}_Y^{\text{naive}}$ are canonically homotopic.

(b) For a schematic map $f : Y_1 \rightarrow Y_2$, the diagram

$$\begin{array}{ccc}
\text{Tr}((\text{Frob}_{Y_1})_*, \text{Shv}(Y_1)) & \xrightarrow{\text{LT}_Y^{\text{true}}} & \text{Funct}(Y_1(F_q)) \\
\downarrow & & \downarrow \\
\text{Tr}((\text{Frob}_{Y_2})_*, \text{Shv}(Y_2)) & \xrightarrow{\text{LT}_Y^{\text{true}}} & \text{Funct}(Y_2(F_q))
\end{array}$$

is commutative, where the left vertical arrow is induced by the functor $f : \text{Shv}(Y_1) \rightarrow \text{Shv}(Y_2)$, and the right vertical arrow is given by pushforward.

(c) The commutative diagram in point (b) is compatible with the identification of point (a) with the commutative diagram

$$\begin{array}{ccc}
\text{Tr}((\text{Frob}_{Y_1})_*, \text{Shv}(Y_1)) & \xrightarrow{\text{LT}_Y^{\text{naive}}} & \text{Funct}(Y_1(F_q)) \\
\downarrow & & \downarrow \\
\text{Tr}((\text{Frob}_{Y_2})_*, \text{Shv}(Y_2)) & \xrightarrow{\text{LT}_Y^{\text{naive}}} & \text{Funct}(Y_2(F_q)).
\end{array}$$
5.3.6. Let now \( Y \) be not necessarily quasi-compact. We will consider the poset of quasi-compact open substacks

\[ \mathcal{U} \hookrightarrow Y, \]

and the corresponding functors \( j_i : \text{Shv}(\mathcal{U}) \to \text{Shv}(Y) \).

The category \( \text{Shv}(Y) \) is compactly generated by the essential images of \( j_i|_{\text{Shv}(\mathcal{U})} \). Furthermore, the induced map

\[ \text{colim} \text{Tr}((\text{Frob}_U)_*, \text{Shv}(\mathcal{U})) \to \text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)) \]

is an isomorphism (see [AGKRRV1, Sect. 22.1.11]).

Using the commutative diagrams in Theorem 5.3.5(b), this allows to define the maps

\[ \text{LT}_{\text{naive}}^Y \text{ and } \text{LT}_{\text{true}}^Y \]

from \( \text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)) \) to

\[ \text{colim} \text{Funct}(\mathcal{U}(\mathbb{F}_q)) \simeq \text{Funct}_c(Y(\mathbb{F}_q)). \]

Moreover, by Theorem 5.3.5(a), these two maps are canonically homotopic.

5.3.7. The case of \( \text{Bun}_G \). We now specialize to the case of \( \text{Bun}_G \). We consider the full subcategory

(5.2)

\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G). \]

As was mentioned earlier, it is preserved by the endofunctor \( (\text{Frob}_{\text{Bun}_G})_* \). By the hypothesis in Sect. 5.1, the embedding (5.2) admits a continuous right adjoint. Hence, the functoriality of the categorical trace construction yields a map

\[ \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \to \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}(\text{Bun}_G)). \]

Composing with the maps \( \text{LT}_{\text{naive}}^\text{Bun}_G \) and \( \text{LT}_{\text{true}}^\text{Bun}_G \), we obtain two maps

\[ \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \Rightarrow \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)), \]

that we will denote \( \text{LT}_{\text{naive}}^\text{Bun}_G \) and \( \text{LT}_{\text{true}}^\text{Bun}_G \), respectively.

However, the identification between \( \text{LT}_{\text{naive}}^\text{Bun}_G \) and \( \text{LT}_{\text{true}}^\text{Bun}_G \) (see Sect. 5.3.6) gives rise to an identification between \( \text{LT}_{\text{naive}}^\text{Bun}_G \) and \( \text{LT}_{\text{true}}^\text{Bun}_G \).

Remark 5.3.8. Let us explain the practical implication of the equality \( \text{LT}_{\text{naive}}^\text{Bun}_G = \text{LT}_{\text{Bun}_G}^\text{Sht} \), stated in Theorem 5.2.3.

Let \( F \) be an an object of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^c \) equipped with a weak Weil structure, i.e., a map

(5.3)

\[ \alpha : F \to (\text{Frob}_{\text{Bun}_G})_*(F). \]

To such a pair \( (F, \alpha) \), we can attach its class

\[ \text{cl}(F, \alpha) \in \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \]

(see [GKRV Sect. 3.4.3]). Thus, using Corollary 4.1.4, to \( (F, \alpha) \) we can attach an element of

\[ \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)), \]

i.e., a compactly supported automorphic function.

Now, the content of Theorem 5.2.3 is that the above element of \( \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)) \) equals the function attached to \( F \), viewed as a weak Weil sheaf via \( \alpha \), by the usual sheaf-function correspondence, i.e., by taking pointwise traces of the Frobenius.

5.4. Serre local term. In this subsection we will define the last remaining map in Sect. 5.2.3, denoted \( \text{LT}_{\text{Serre}}^\text{Bun}_G \).
5.4.3. Suppose that \( \mathcal{Y} \) be an algebraic stack, and let \( N \subset T^*(\mathcal{Y}) \) be a conical Zariski-closed subset. Consider the (fully faithful) embedding

\[
\text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \to \text{Shv}(\mathcal{Y} \times \mathcal{Y}).
\]

Denote

\[
\text{ps-u}_g := \Delta(\mathfrak{e}_g) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}),
\]

and let

\[
\text{ps-u}_{g,N} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})
\]

be obtained by applying to \( \text{ps-u}_g \) the right adjoint to the functor \((5.4)\). We will not distinguish notationally between \( \text{ps-u}_{g,N} \) and its image along \((5.4)\).

The counit of the adjunction defines a map

\[
\text{ps-u}_{g,N} \to \text{ps-u}_g.
\]

5.4.4. The map \((5.5)\) gives rise to a natural transformation

\[
(\text{ev}_g \otimes \text{Id}) \circ (\text{Id} \otimes \text{ps-u}_{g,N}) = ((\text{C}_c(\mathcal{Y}, -) \circ \Delta_g^*) \boxtimes \text{Id}_g) \circ (\text{Id}_g \boxtimes \text{ps-u}_{g,N}) \to ((\text{C}_c(\mathcal{Y}, -) \circ \Delta_g^*) \boxtimes \text{Id}_g) \circ (\text{Id}_g \boxtimes \text{ps-u}_g) \simeq \text{Id}
\]

as endofunctors of \( \text{Shv}_N(\mathcal{Y}) \) (see Sect. \((5.5.4)\) for the notation \( \text{ev}_g \)).

Recall, following \cite{AGKRRV2} Definition 5.5.4, that the pair \((\mathcal{Y}, N)\) is said to be Serre if the natural transformation \((5.4)\) is an isomorphism. If this is the case, then the data of

\[
\text{ps-u}_{g,N} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \text{ and } \text{ev}_g : \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \to \text{Vect}
\]

define the unit and counit of a self-duality on \( \text{Shv}_N(\mathcal{Y}) \).

5.4.5. Suppose that \((\mathcal{Y}, N)\) is Serre. As with the true local term morphism, we define

\[
\text{LT}^{\text{Serre}}_{\mathcal{Y}, N} : \text{Tr}((\text{Frob}_g)_*, \text{Shv}_N(\mathcal{Y})) \to \text{Funct}_c(\mathcal{Y}(\mathbb{F}_q))
\]

as the composition

\[
\text{Tr}((\text{Frob}_g)_*, \text{Shv}_N(\mathcal{Y})) \simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N}))
\]

\[
\simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N})) \simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N})) \simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N})) = \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N}))
\]

\[
\simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N})) \simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N})) \simeq \text{C}_c(\mathcal{Y}, \Delta_g^* \circ ((\text{Frob}_g)_* \otimes \text{Id})(\text{ps-u}_{g,N}))
\]

5.4.4. We take \( \mathcal{Y} = \text{Bun}_G \) and \( N = \text{Nilp} \). We will denote the endofunctor \( \mathcal{H}_R \) by \( \text{P}_{\text{Nilp, Bun}_G} \) (cf. \cite{AGKRRV2} Sect. 1.6.1).

Similarly, for a stack \( \mathcal{Z} \), we will write \( \text{Id}_\Sigma \boxtimes \text{P}_{\text{Nilp, Bun}_G} \) instead of \( \text{Id}_\Sigma \boxtimes \mathcal{H}_R \).

Using this notation, we have

\[
\mathcal{X}_R := (\text{Id}_{\text{Bun}_G} \boxtimes \text{P}_{\text{Nilp, Bun}_G})(\text{ps-u}_{\text{Bun}_G}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G).
\]

5.4.5. We now quote the following result, see \cite{AGKRRV2} Proposition 2.4.6:

**Theorem 5.4.6.** For any algebraic stack \( \mathcal{Z} \), the endofunctor of \( \text{Shv}(\mathcal{Z} \times \text{Bun}_G) \) given by

\[
\text{Id}_\Sigma \boxtimes \text{P}_{\text{Nilp, Bun}_G}
\]

identifies with the precomposition of the embedding

\[
\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\mathcal{Z} \times \text{Bun}_G).
\]

with its right adjoint.

---

\( \text{It is this assertion that uses (and is equivalent to) the validity of } \cite{AGKRRV2} \text{ Conjecture 14.1.8.} \)
In what follows we will denote by $\text{Id}_Z \boxtimes \varepsilon$ the natural transformation
\begin{equation}
(5.7) \quad \text{Id}_Z \boxtimes P_{\text{Nilp}, \text{Bun}_G} \to \text{Id}_{\text{Shv}(\mathbb{Z} \times \text{Bun}_G)}
\end{equation}
equal to the counit of the above adjunction.

When $Z = \text{pt}$, we will simply write
\[ \varepsilon : P_{\text{Nilp}, \text{Bun}_G} \to \text{Id}_{\text{Shv}(\text{Bun}_G)}. \]

5.4.7. Iterating, from Theorem 5.4.6 we obtain:

**Corollary 5.4.8.** The endofunctor $P_{\text{Nilp}, \text{Bun}_G} \boxtimes P_{\text{Nilp}, \text{Bun}_G}$ of $\text{Shv}(\text{Bun}_G \times \text{Bun}_G)$ identifies with the precomposition of the embedding
\begin{equation}
(5.8) \quad \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G \times \text{Bun}_G).
\end{equation}
with its right adjoint.

We will denote by $\varepsilon \boxtimes \varepsilon$ the resulting natural transformation
\[ P_{\text{Nilp}, \text{Bun}_G} \boxtimes P_{\text{Nilp}, \text{Bun}_G} \to \text{Id}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}. \]

5.4.9. Recall that Theorem 4.3.2 says that the object $K_R$ belongs to the essential image of the functor $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G \times \text{Bun}_G)$.

This formally implies that the map
\[ (P_{\text{Nilp}, \text{Bun}_G} \boxtimes \text{Id}_{\text{Bun}_G})(K_R) \xrightarrow{\varepsilon \boxtimes \text{Id}_{\text{Bun}_G}} K_R \]
is an isomorphism.

In other words, we obtain an identification
\[ K_R \cong (P_{\text{Nilp}, \text{Bun}_G} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(\text{ps-u}_{\text{Bun}_G}). \]

By Corollary 5.4.8 the resulting map
\[ K_R \xrightarrow{\varepsilon \boxtimes \text{ps-u}_{\text{Bun}_G}} \text{ps-u}_{\text{Bun}_G} \]
identifies $K_R$ with the value on $\text{ps-u}_{\text{Bun}_G}$ of the right adjoint to (5.8).

Hence, in the notations of Sect. 5.4.1 above, we have
\[ K_R \cong \text{ps-u}_{\text{Bun}_G, \text{Nilp}}. \]

5.4.10. Combining with Theorem 4.3.6 we conclude that the pair $(\text{Bun}_G, \text{Nilp})$ is Serre.

Thus, we obtain a well-defined map
\[ \text{LT}^\text{Serre} := \text{LT}^\text{Serre}_{\text{Bun}_G, \text{Nilp}}. \]

5.4.11. At this point, all the terms in Theorem 5.2.3 have been defined. We have already seen that $\text{LT}^\text{naive} = \text{LT}^\text{true}$.

The rest of this section is devoted to the proof of the equality $\text{LT}^\text{Serre} = \text{LT}^\text{Sht}$.

Finally, we show $\text{LT}^\text{Serre} = \text{LT}^\text{true}$ in Sect. 6.

5.5. **Comparison of $\text{LT}^\text{Serre}$ and $\text{LT}^\text{Sht}$**.

5.5.1. We begin by constructing upgraded versions of the two local term morphisms in question.

More precisely, we will show that there exist natural transformations
\[ \text{LT}^\text{Serre}_{\mathcal{F}}, \text{LT}^\text{Sht}_{\mathcal{F}} : \text{Shv}^\text{Ty} \to \text{Shv} \]
of functors
\[ \text{Rep}((\mathcal{G})_{\text{Ran}}) \to \text{Vect} \]
with the property that when we evaluate either of these natural transformations on $1_{\text{Rep}(\mathcal{G})_{\text{Ran}}}$, we obtain the relevant local term map
\[ \text{LT}^\text{Ty} : \text{Tr}((\text{Frob}_{\text{Bun}_G})^*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \to \text{Shv}^\text{Ty}(1_{\text{Rep}(\mathcal{G})_{\text{Ran}}}) \to \text{Shv}(1_{\text{Rep}(\mathcal{G})_{\text{Ran}}}) \cong \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)). \]
5.5.2. The natural transformation $\tilde{\LT}^{\Sh}$ (in fact, an isomorphism) has been already defined: this is the isomorphism arising from Theorem 4.1.2.

5.5.3. We will now construct $\tilde{\LT}^{\Serre}$.

Recall the isomorphism $\Sh^\Tr(-) \simeq \Sh(R \star -)$ of Theorem 4.2.2. Thus, we can interpret the sought-for map $\tilde{\LT}^{\Serre}$ as a natural transformation

$$\Sh(R \star -) \to \Sh(-).$$

5.5.4. By construction, the functors $\Sh(-)$ and $\Sh(R \star -)$ send $V \in \Rep(\tilde{\mathcal{G}})_{\Ran}$ to the vector space obtained by applying

$$C_\cdot (\text{Bun}_G, (\text{Graph}_{\text{Frob}} \cup \text{Bun}_G)^* (-)) : \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \to \text{Vect}$$

to the objects

$$(\text{Id}_{\text{Bun}_G} \boxtimes H_V)(\text{ps-u}_{\text{Bun}_G}) \text{ and } (\text{Id}_{\text{Bun}_G} \boxtimes H_{R \star V})(\text{ps-u}_{\text{Bun}_G}),$$

respectively.

The sought-for natural transformation (5.9) is induced by the natural transformation of functors

$$\text{Id}_{\text{Bun}_G} \boxtimes H_{R \star V} \to \text{Id}_{\text{Bun}_G} \boxtimes H_V$$
given by

$$\text{Id}_{\text{Bun}_G} \boxtimes H_{R \star V} \simeq (\text{Id}_{\text{Bun}_G} \boxtimes H_R) \circ (\text{Id}_{\text{Bun}_G} \boxtimes H_V) =$$

$$= (\text{Id}_{\text{Bun}_G} \boxtimes \mathbb{P}_{\text{Nilp}, \text{Bun}_G}) \circ (\text{Id}_{\text{Bun}_G} \boxtimes H_V) \xrightarrow{(\text{Id}_{\text{Bun}_G} \boxtimes \varepsilon) \circ \text{id}} \text{Id}_{\text{Bun}_G} \boxtimes H_V,$$

where $\text{Id}_{\text{Bun}_G} \boxtimes \varepsilon$ is as in (5.7).

5.5.5. We will prove:

**Theorem 5.5.6.** There is a canonical isomorphism

$$\tilde{\LT}^{\Serre} \simeq \tilde{\LT}^{\Sh}$$

of natural transformations

$$\Sh^\Tr \to \Sh.$$

Clearly, Theorem 5.5.6 implies the isomorphism $\tilde{\LT}^{\Serre} \simeq \tilde{\LT}^{\Sh}$. The proof of Theorem 5.5.6 will be given in Sect. 5.7.

5.6. An algebra structure on $R$. For the proof of Theorem 5.5.6 we need to digress and discuss some properties related to the commutative algebra structure on the object $R \in \Rep(\tilde{\mathcal{G}})_{\Ran}$, see Sect. 2.4.7.

5.6.1. In what follows we will need one more compatibility property of the commutative algebra structure on $R$.

Let $Z$ be an algebraic stack. Consider the endofunctor $(\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G})$ of $\text{Shv}(Z \times \text{Bun}_G)$, see Sect. 5.4.4.

On the one hand, the algebra structure on $R$ yields a map

$$m : (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}) \circ (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}) \to (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}).$$

On the other hand, we have the map

$$\text{Id}_Z \boxtimes \varepsilon \circ (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}) \xrightarrow{(\text{Id}_Z \boxtimes \varepsilon) \circ \text{id}} (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}) \to (\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}),$$

where $\text{Id}_Z \boxtimes \varepsilon$ is as in (5.7). (The map (5.11) is in fact an isomorphism and equals the structure on $\text{Id}_Z \boxtimes \mathbb{P}_{\text{Nilp,Bun}_G}$ of idempotent endofunctor.)

**Proposition 5.6.2.** The maps (5.10) and (5.11) are canonically homotopic.
5.6.3. Before we prove Proposition 5.6.2 let us quote its corollary that we will use in the proof of Theorem 5.5.6.

Note that for $\mathcal{V} \in \text{Rep}(\hat{G})_{\text{Ran}}$ we have a tautological identification
\begin{equation}
(\text{Id}_{\text{Bun}_G} \boxtimes \text{P}^\text{Nilp}_G)(\mathcal{V}) \simeq \mathcal{K}_{R \ast \mathcal{V}}
\end{equation}
as objects of $\text{Shv}(\text{Bun}_G \times \text{Bun}_G)$.

For $\mathcal{W} \in \text{Rep}(\hat{G})_{\text{Ran}}$, let $\varepsilon_{\mathcal{W}} : (\text{Id}_Z \boxtimes \text{P}^\text{Nilp}_G)(\mathcal{W}) \to \mathcal{K}_{\mathcal{W}}$ denote the value of the natural transformation $\text{Id}_Z \boxtimes \varepsilon$ on $\mathcal{K}_{\mathcal{W}}$.

**Corollary 5.6.4.** For $\mathcal{V} \in \text{Rep}(\hat{G})_{\text{Ran}}$, the diagram
\begin{equation}
\begin{array}{ccc}
(\text{Id}_{\text{Bun}_G} \boxtimes \text{P}^\text{Nilp}_G) & \circ & (\text{Id}_{\text{Bun}_G} \boxtimes \text{P}^\text{Nilp}_G)(\mathcal{V}) \\
\downarrow \sim & & \downarrow \sim \\
(\text{Id}_{\text{Bun}_G} \boxtimes \text{P}^\text{Nilp}_G)(\mathcal{V}) & \xrightarrow{\varepsilon_{\mathcal{V}} \circ} & \mathcal{K}_{R \ast \mathcal{V}}
\end{array}
\end{equation}
commutes.

5.6.5. **Proof of Proposition 5.6.2.** Both the source and the target functor vanish on the subcategory $\ker(\text{Id}_Z \boxtimes \text{P}^\text{Nilp}_G)$. Hence, it is sufficient to establish the commutativity when evaluated on the full subcategory $\text{Shv}(Z) \otimes \text{Shv}^\text{Nilp}(\text{Bun}_G) \subset \text{Shv}(Z \times \text{Bun}_G)$.

Since all the functors involved act only on the second factor, it is sufficient to show that the map
$$\text{Id} \simeq (\text{P}^\text{Nilp}_G \circ \text{P}^\text{Nilp}_G)|_{\text{Shv}^\text{Nilp}(\text{Bun}_G)} \to \text{P}^\text{Nilp}_G$$
is the identity map.

However, this follows from [AGKRRV1, Theorem 14.3.2]:

This theorem implies that the action of $\text{Rep}(\hat{G})_{\text{Ran}}$ on $\text{Shv}^\text{Nilp}(\text{Bun}_G)$ factors uniquely through an action of $\text{QCoh}(\text{LocSys}_{\hat{G}}(X))$ via the functor Loc.

Now the result follows from the fact that the isomorphism
$$\text{Loc}(R) \simeq \mathcal{O}_{\text{LocSys}^\text{rub}}(X)$$
of Corollary 2.4.9 respects the (commutative) algebra structures.

\[\square\] [Proposition 5.6.2]

5.7. **Proof of Theorem 5.5.6**

5.7.1. **Proof of Theorem 5.5.6 Step 0.** We begin by introducing some notation.

Throughout the argument, we will replace $\text{Sh}^\text{tr}$ with $\text{Sh}(R \ast -)$. In particular, we consider our natural transformations as mapping
$$\hat{\text{LT}}^\text{Serre}, \hat{\text{LT}}^\text{Sht} : \text{Sh}(R \ast -) \to \text{Sh}.$$

For a fixed object $\mathcal{V} \in \text{Rep}(\hat{G})_{\text{Ran}}$, we denote the corresponding maps by
$$\hat{\text{LT}}_{\mathcal{V}}^\text{Serre}, \hat{\text{LT}}_{\mathcal{V}}^\text{Sht} : \text{Sh}(R \ast \mathcal{V}) \to \text{Sh}(\mathcal{V}).$$

To unburden the notation, we will use $- \otimes -$ to denote $\otimes_{\text{LocSys}^\text{rub}}(X)$ and $\mathcal{O}$ to denote $\mathcal{O}_{\text{LocSys}^\text{rub}}(X)$. 

\[\square\] [Proposition 5.6.2]
5.7.2. Proof of Theorem 5.5.6, Step 1. Observe that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Maps}_{\text{DGCat}}(\text{QCoh}(\text{LocSys}_{G}(X), \text{Vect}) & \xrightarrow{F \mapsto F(\cdot \otimes -)} & \text{Maps}_{\text{DGCat}}(\text{QCoh}(\text{LocSys}_{G}(X), \text{Vect}) \\
& & \\
\text{Loc}^\vee & \xrightarrow{\sim} & \text{Loc}^\vee
\end{array}
\]

Here we have used the isomorphism \( \text{Loc}(R) \simeq \mathcal{O} \). The vertical arrows are fully faithful by Corollary 2.3.3(a), and the top horizontal arrow is tautologically isomorphic to the identity. Therefore, the bottom arrow is fully faithful when restricted to the essential image of \( \text{Loc}^\vee \).

The (isomorphic!) functors \( \text{Sht} \) and \( \text{Sht}(\cdot \otimes -) \) lie in the essential image of \( \text{Loc}^\vee \) by Theorem 3.1.3. Therefore, it suffices to identify the natural transformations \( \tilde{\text{LT}}_{\text{Sht}} \) and \( \tilde{\text{LT}}_{\text{Sht}}^{\text{Serre}} \) after precomposing with \( \text{R} \otimes \text{Id} \).

That is, it suffices to identify the two induced natural transformations

\[
\text{Sht}(\text{R} \ast (\text{R} \ast \mathcal{V})) \to \text{Sht}(\text{R} \ast \mathcal{V}),
\]

i.e., the two maps

\[
\tilde{\text{LT}}_{\text{R} \otimes \mathcal{V}}^{\text{true}}, \tilde{\text{LT}}_{\text{R} \otimes \mathcal{V}}^{\text{Serre}} : \text{Sht}(\text{R} \ast (\text{R} \ast \mathcal{V})) \to \text{Sht}(\text{R} \ast \mathcal{V}), \quad \mathcal{V} \in \text{Rep}(\tilde{G})_{\text{Ran}}.
\]

5.7.3. Proof of Theorem 5.5.6, Step 2. Let \( m : \text{R} \ast \text{R} \to \text{R} \) denote the multiplication for the algebra structure on \( \text{R} \).

We obtain a map

\[
m_{\mathcal{V}} : \text{Sht}(\text{R} \ast \text{R} \ast \mathcal{V}) \to \text{Sht}(\text{R} \ast \mathcal{V}).
\]

We claim that there is a natural identification

\[
m_{\mathcal{V}} \simeq \tilde{\text{LT}}_{\text{R} \otimes \mathcal{V}}^{\text{true}}
\]

of morphisms \( \text{Sht}(\text{R} \ast \text{R} \ast \mathcal{V}) \to \text{Sht}(\text{R} \ast \mathcal{V}) \).

Indeed, the compatibility of the isomorphism \( \text{Loc}(\text{R}) \simeq \mathcal{O} \) with algebra structures implies that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Loc}(\text{R}) \otimes \text{Loc}(\text{R} \ast \mathcal{V}) & \xrightarrow{\sim} & \text{Loc}(\text{R} \ast \mathcal{V}) \\
& \xrightarrow{\sim} & \text{Loc}(\text{R} \ast \mathcal{V}) \\
\text{Loc}(\text{R} \ast \mathcal{V}) & \xrightarrow{\text{Id}} & \text{Loc}(\text{R} \ast \mathcal{V})
\end{array}
\]

where the left vertical arrow is the canonical isomorphism obtained by identifying \( \text{Loc}(\mathcal{V}) \simeq \mathcal{O} \). Applying \( \text{Sht}_{\text{Loc}} \) and the definition yields the claim.

5.7.4. Proof of Theorem 5.5.6, Step 3. By the above, it suffices to show that there are natural identifications

\[
m_{\mathcal{V}} \simeq \tilde{\text{LT}}_{\text{R} \otimes \mathcal{V}}^{\text{true}}
\]

of morphisms \( \text{Sht}(\text{R} \ast \text{R} \ast \mathcal{V}) \to \text{Sht}(\text{R} \ast \mathcal{V}) \).

Now, \( \text{Sht}_{\text{Loc}}^{\text{true}} \) is obtained by applying \( C_{c}(\text{Bun}_{G}, (\text{Graph}_{\text{ProBun}_{G}})^{-}(\cdot)) \) to the commutative diagram of Corollary 5.6.3.

6. Comparison of \( \text{LT}_{\text{true}} \) and \( \text{LT}^{\text{Serre}} \)

6.1. Statement of the result.
6.1.1. Let $Y$ be a quasi-compact algebraic stack, and let $N$ be a conical Zariski-closed subset of $T^*(Y)$. We will assume that the subcategory
\[(6.1) \text{Shv}_N(Y) \hookrightarrow \text{Shv}(Y)\]
is generated by objects that are compact in $\text{Shv}(Y)$ (in [AGKRRV2] Sect. A.5.2 this property of $(Y, N)$ was termed “constraccessible”).

Assume that $N$ is Frobenius-invariant, so the endofunctor $(\text{Frob}_Y)_*$ of $\text{Shv}(Y)$ preserves the subcategory $\text{Shv}_N(Y) \subset \text{Shv}(Y)$, see [AGKRRV1] Sect. 22.3.1 and Lemma 22.3.2.

In this case, the embedding (6.1) induces a map
\[(6.2) \text{Tr}((\text{Frob}_Y)_*, \text{Shv}_N(Y)) \to \text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)).\]

Let us denote by
\[\text{LT}_Y^{\text{true}}: \text{Tr}((\text{Frob}_Y)_*, \text{Shv}_N(Y)) \to \text{Funct}(Y(F_q))\]
the composition of (6.2) with the map
\[\text{LT}_Y^{\text{true}}: \text{Tr}((\text{Frob}_Y)_*, \text{Shv}(Y)) \to \text{Funct}(Y(F_q)),\]
defined in Sect. 5.3.4.

6.1.2. Assume now that the pair $(Y, N)$ is Serre (see Sect. 5.4.2 for what this means).

Recall that in this case, we also have the map
\[\text{LT}_Y^{\text{Serre}}: \text{Tr}((\text{Frob}_Y)_*, \text{Shv}_N(Y)) \to \text{Funct}(Y(F_q)).\]

6.1.3. Recall now the miraculous endofunctor $\text{Mir}_Y$, see [AGKRRV2] Sect. 5.6.2]. Following [AGKRRV2] Definition 5.6.6, we will say that the pair $N$ is miraculous-compatible if the endofunctor $\text{Mir}_Y$ preserves the subcategory (6.1).

The main result of this section reads:

**Theorem 6.1.4.** Assume that $N$ is miraculous-compatible. Then the maps $\text{LT}_Y^{\text{true}}$ and $\text{LT}_Y^{\text{Serre}}$
\[\text{Tr}((\text{Frob}_Y)_*, \text{Shv}_N(Y)) \Rightarrow \text{Funct}(Y(F_q))\]
are equal.

**Remark 6.1.5.** The assertion of Theorem 6.1.4 is far from tautological: it says that two ways to map $\text{Tr}((\text{Frob}_Y)_*, \text{Shv}_N(Y))$ to $\text{Funct}(Y(F_q))$, corresponding to two different self-dualities on $\text{Shv}_N(Y)$ coincide.

A somewhat analogous problem arises when we calculate the trace of the identity endofunctor on the category $\text{QCoh}(Z)$, where $Z$ is a smooth proper scheme. There are two ways to calculate the trace that correspond to two choices of self-duality data on $\text{QCoh}(Z)$: the naive self-duality and Serre self-duality. Each calculation yields Hodge cohomology of $Z$, i.e.,
\[\oplus_i \Gamma(Z, \Omega^i(Z))[[i]].\]

However, the resulting two identifications are different, and the difference is given by the Todd class of $Z$. This observation lies at the core of a proof of the Grothendieck-Riemann-Roch theorem via categorical traces, see [KP].
Let us explain how Theorem 6.1.4 implies the equality $LT^{true} = LT^{Serre}$ (the issue here is the fact that $Bun_G$ is not quasi-compact).

Let $\mathcal{Y}$ be a not necessarily quasi-compact algebraic stack, and let $N \subset T^*(\mathcal{Y})$ be a conical Zariski-closed subset. We will recall some definitions from [AGKRRV2, Sect. C.1].

An open substack $\mathcal{U} \rightarrow \mathcal{Y}$ is said to be cotruncative if for every quasi-compact open $\mathcal{U}' \subset \mathcal{Y}$, the open embedding $\mathcal{U} \cap \mathcal{U}' \rightarrow \mathcal{U}'$ is such that the functor $j_* : Shv(\mathcal{U} \cap \mathcal{U}') \rightarrow Shv(\mathcal{U}')$ admits a right adjoint as a functor defined by a kernel.

An open substack $\mathcal{U} \rightarrow \mathcal{Y}$ is said to be universally $N$-cotruncative if it is cotruncative, and for every stack $Z$ and $N|_Z \subset T^*(Z)$, the functor $(id \times j)^* : Shv(Z \times \mathcal{U}) \rightarrow Shv(Z \times \mathcal{Y})$ sends $Shv_{N|_Z \times N}(Z \times \mathcal{U}) \subset Shv(Z \times \mathcal{U})$ to $Shv_{N|_Z \times N}(Z \times \mathcal{Y}) \subset Shv(Z \times \mathcal{Y})$.

Recall that $\mathcal{Y}$ is said to be truncatable if we can write $\mathcal{Y}$ as a union of quasi-compact cotruncative open substacks. Finally, recall that $\mathcal{Y}$ is said to be universally $N$-truncatable if we can write $\mathcal{Y}$ as a filtered union of quasi-compact universally $N$-cotruncative open substacks.

Assume that $N$ is Frobenius-invariant. By Sect. 5.3.6 we have a well-defined map

$$LT^{true}_Y : Tr((\text{Frob}_Y)^*, Shv(\mathcal{Y})) \rightarrow \text{Funct}_c(\mathcal{Y}(\overline{F}_q))$$

Let us make the following assumptions:

- $\mathcal{Y}$ is universally $N$-truncatable;
- For every quasi-compact universally $N$-cotruncative open substack $\mathcal{U} \subset \mathcal{Y}$, we have:
  - The pair $(\mathcal{U}, N|_\mathcal{U})$ is constraccessible;
  - The pair $(\mathcal{U}, N|_\mathcal{U})$ is Serre;
  - $N|_\mathcal{U}$ is miraculous-compatible.

It follows that in this case, the subcategory $Shv_N(\mathcal{Y})$ is generated by objects that are compact in $Shv(\mathcal{Y})$, and that the pair $(\mathcal{Y}, N)$ is Serre. So, the map

$$LT^{Serre}_{\mathcal{Y}, N} : Tr((\text{Frob}_Y)^*, Shv_N(\mathcal{Y})) \rightarrow \text{Funct}_c(\mathcal{Y}(\overline{F}_q))$$

is also well-defined by Sect. 5.4.3. Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\colim_{\mathcal{U}} Tr((\text{Frob}_\mathcal{U})^*, Shv_N(\mathcal{U})) &\rightarrow & \colim_{\mathcal{U}} \text{Funct}_c(\mathcal{U}(\overline{F}_q)) \\
\downarrow & & \downarrow \\
Tr((\text{Frob}_\mathcal{Y})^*, Shv_N(\mathcal{Y})) &\rightarrow & \text{Funct}_c(\mathcal{Y}(\overline{F}_q)).
\end{array}$$

It now follows formally from Theorems 6.1.4 and 5.3.3 b) that the maps $LT^{true}_{\mathcal{Y}, N}$ and $LT^{Serre}_{\mathcal{Y}, N}$ are canonically homotopic.

We apply the above discussion to $\mathcal{Y} = Bun_G$ and $\text{Nilp} = N$. The conditions in Sect. 6.1.7 are satisfied by [AGKRRV2, Theorem 1.7.3], [AGKRRV1, Conjecture 14.1.8 and Lemma F.8.10], [AGKRRV2, Corollary 5.8.4] and [AGKRRV2, Proposition 2.8.10], respectively. This implies the desired equality

$$LT^{true} = LT^{Serre}.$$
6.2.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack. We start by constructing a natural transformation
\begin{equation}
C_c(\mathcal{Y}, \Delta_\mathcal{Y} \circ (\text{Id} \otimes \text{Mir}_\mathcal{Y})(-)) \to C_\bullet(\mathcal{Y}, \Delta_\mathcal{Y}(-)),
\end{equation}
as functors $\text{Shv}(\mathcal{Y} \times \mathcal{Y}) \to \text{Vect}$.

6.2.2. It suffices to specify the value of the natural transformation (6.3) on compact objects. Thus, we fix a compact object $Q \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})$.

We note that the map
\begin{equation}
C \cdot \Delta(\mathcal{Y}, \text{Mir}_\mathcal{Y})(Q) \to C \cdot (\mathcal{Y}, \text{Mir}_\mathcal{Y}(Q)),
\end{equation}
is an isomorphism for $Q$. So we need to construct a map
\begin{equation}
C_c(\mathcal{Y}, \Delta_\mathcal{Y} \circ (\text{Id} \otimes \text{Mir}_\mathcal{Y})(Q)) \to C_c(\mathcal{Y}, \Delta_\mathcal{Y}(Q)),
\end{equation}
functorial in $Q \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})$.

In what follows we will use the notation $u_\mathcal{Y} := (\Delta_\mathcal{Y}^* \omega_\mathcal{Y}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})$.

We start with the map
\[ \Omega \boxtimes \mathcal{D}^{\text{Verdier}}(\mathcal{Y}) \to u_{\mathcal{Y} \times \mathcal{Y}}, \]
given by Verdier duality. Applying the transposition $\sigma_{2,3}$, we interpret it as a map
\begin{equation}
(\Omega \boxtimes \mathcal{D}^{\text{Verdier}}(\mathcal{Y}))^{\sigma_{2,3}} \to u_\mathcal{Y} \boxtimes u_\mathcal{Y}.
\end{equation}

By definition
\begin{equation}
(\text{Id} \otimes \text{Mir}_\mathcal{Y})(u_\mathcal{Y}) \simeq ps-u_\mathcal{Y}
\end{equation}
Applying the functor $\text{Id} \otimes \text{Id} \otimes \text{Mir}_\mathcal{Y} \otimes \text{Id}$ to the map (6.5), we obtain a map
\begin{equation}
((\text{Id} \otimes \text{Mir}_\mathcal{Y})(Q) \boxtimes \mathcal{D}^{\text{Verdier}}(\mathcal{Y}))^{\sigma_{2,3}} \to u_{\mathcal{Y} \times \mathcal{Y}} \boxtimes ps-u_{\mathcal{Y}}.
\end{equation}

Applying the functor
\[(p_{1,2}) \circ (\Delta_\mathcal{Y} \times \text{id} \times \text{id})^* \circ \sigma_{2,3} : \text{Shv}(\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}) \to \text{Shv}(\mathcal{Y} \times \mathcal{Y}),\]
from (6.7) we obtain a map
\[ C_c(\mathcal{Y}, \Delta_\mathcal{Y} \circ (\text{Id} \otimes \text{Mir}_\mathcal{Y})(Q)) \boxtimes \mathcal{D}^{\text{Verdier}}(\mathcal{Y}) \to u_\mathcal{Y}.
\]
Applying Verdier duality again, we obtain the desired map (6.4).

6.2.3. Let us take $\Omega := (\text{Graph}_{\text{Frob}_Y})(\omega_\mathcal{Y}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})$.

Then the right-hand side in (6.3) identifies with
\[ C_\bullet(\mathcal{Y}^\text{Prob}, \omega_{\mathcal{Y}^\text{Prob}}) \simeq C(\mathcal{Y}^\text{Prob}, \omega_{\mathcal{Y}^\text{Prob}}). \]

Since $\text{Frob}_Y$ is a proper map,
\[(\text{Id} \otimes \text{Mir}_\mathcal{Y})(\text{Graph}_{\text{Frob}_Y})(\omega_\mathcal{Y}) \simeq (\text{Graph}_{\text{Frob}_Y})(\mathcal{Y}).\]

Hence, the left-hand side in (6.3) identifies with
\[ C_c(\mathcal{Y}^\text{Prob}, \mathcal{E}_{\mathcal{Y}^\text{Prob}}). \]
6.2.4. We will prove:

**Theorem 6.2.5.** The diagram

\[
\begin{array}{ccc}
C_\bullet((Y, \Delta_y^\circ (\text{Id} \otimes \text{Mir}_Y \circ (\text{Graph}_{Y_{\text{Prob}}})_* (\omega_Y))) & \cong & C_\bullet((Y, \Delta_y^\circ (\text{Graph}_{Y_{\text{Prob}}})_* (\omega_Y))) \\
\cong & & \cong \\
\text{C}(\mathcal{Y}^{\text{Prob}}) & \xrightarrow{\sim} & \text{C}(\mathcal{Y}^{\text{Prob}}, \omega_Y) \\
\cong & & \cong \\
\text{Funct}(\mathcal{Y}(\mathbb{F}_q)) & \xrightarrow{\text{id}} & \text{Funct}(\mathcal{Y}(\mathbb{F}_q))
\end{array}
\]

commutes.

The proof will be given in Sect. 6.4. We will presently show how Theorem 6.2.5 implies Theorem 6.1.4.

6.3. **Proof of Theorem 6.1.4**

6.3.1. Let ev$_Y$ denote the pairing

\[
C_\bullet((Y, \Delta_y^\circ (- \boxtimes -)) : \text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \to \text{Vect}.
\]

The assumption that Shv$_N(Y)$ is generated by objects compact in the ambient category Shv(Y) implies that the restriction of ev$_Y$ to

\[
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \subset \text{Shv}(Y) \otimes \text{Shv}(Y)
\]

defines an identification

\[
\text{Shv}_N(Y) \cong \text{Shv}_N(Y)^{\vee},
\]

see [ACKR12, Sect. A.4.4].

The unit of this duality, to be denoted u$_Y$, is obtained by applying to u$_Y$ the right adjoint to the fully faithful embedding

\[
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \hookrightarrow \text{Shv}(Y \times Y),
\]

see [ACKR12, Corollary 5.4.5].

In what follows, we will not distinguish notationally between u$_{y,N}$ and its image along (6.4). The counit of the adjunction gives rise to a map

\[
u_{y,N} \to u_Y.
\]

6.3.2. The assumption that N is miraculous-compatible implies that the endofunctor Id$_Y \boxtimes \text{Mir}_Y$ of Shv(Y $\times$ Y) preserves the subcategory

\[
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \subset \text{Shv}(Y \times Y).
\]

The resulting endofunctor of Shv$_N(Y) \otimes$ Shv$_N(Y)$ identifies with Id $\otimes$ Mir$_Y$.

By [ACKR12 Corollary 5.6.10(c)], the assumption that the pair (Y, N) is Serre implies that the above endofunctor Id $\otimes$ Mir$_Y$ intertwines the duality data given by the pair (u$_{y,N}, \text{ev}_Y$) with one given by (ps-u$_{y,N}, \text{ev}_Y$).

In particular, we have a canonical isomorphism

\[
\text{ps-u}_{y,N} \cong (\text{Id} \otimes \text{Mir}_Y)(u_{y,N})
\]

and a datum of commutativity for the diagram

\[
\begin{array}{ccc}
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) & \xrightarrow{\text{Id} \otimes \text{Mir}_Y} & \text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \\
\text{ev}_Y & & \text{ev}_Y \\
\text{Vect} & \xrightarrow{\text{Id}} & \text{Vect},
\end{array}
\]
i.e., an isomorphism of functors

\[(6.13)\quad \ev'_y \circ (\Id \otimes \Mir_y) \simeq \ev_y, \quad \Shv_N(y) \otimes \Shv_N(y) \xrightarrow{\simeq} \Vect.\]

6.3.3. In particular, for an endofunctor \(F\) of \(\Shv_N(y)\), we have a commutative diagram

\[
\begin{array}{ccc}
\Tr(F, \Shv_N(y)) & \xrightarrow{\text{id}} & \Tr(F, \Shv_N(y)) \\
\downarrow & & \downarrow \\
\ev'_y \circ (F \otimes \Mir_y)(ps-u_{y,N}) & \xrightarrow{\sim} & \ev'_y \circ (F \otimes \Mir_y)(u_{y,N}) \\
\end{array}
\]

\[(6.14)\]

\[
\begin{array}{ccc}
\ev'_y \circ (F \otimes \id)(ps-u_{y,N}) & \xrightarrow{\sim} & \ev'_y \circ (F \otimes \Mir_y)(u_{y,N}) \\
\end{array}
\]

6.3.4. We now use the following two observations:

(i) The diagram

\[
\begin{array}{ccc}
ps-u_{y,N} & \xrightarrow{\sim} & (\Id \otimes \Mir_y)(u_{y,N}) \\
\downarrow & & \downarrow \\
ps-u_y & \xrightarrow{\sim} & (\Id \otimes \Mir_y)(u_y) \\
\end{array}
\]

commutes. This follows tautologically from the constructions.

(ii) The isomorphism \[(6.13)\] is canonically homotopic to the restriction of the natural transformation along the embedding \[(6.11)\]. This follows from [AGKRRV 2, Proposition 5.7.2].

Concatenating, we obtain a commutative diagram

\[
\begin{array}{ccc}
\ev'_y \circ ((\Frob_y)^* \otimes \id)(ps-u_{y,N}) & \xrightarrow{\sim} & \ev_y \circ ((\Frob_y)^* \otimes \id)(u_{y,N}) \\
\downarrow & & \downarrow \\
C_c(y, -) \circ (\Delta_y)^* \circ (\Frob_y \times \id)_*(ps-u_y) & \xrightarrow{\sim} & C^*_c(y, -) \circ (\Frob_y \times \id)_*(u_y). \\
\end{array}
\]

6.3.5. Hence, concatenating with \[(6.14)\] for \(F = (\Frob_y)^*\), we obtain a commutative diagram

\[
\begin{array}{ccc}
\Tr((\Frob_y)^*, \Shv_N(y)) & \xrightarrow{\text{id}} & \Tr((\Frob_y)^*, \Shv_N(y)) \\
\downarrow & & \downarrow \\
\ev'_y \circ ((\Frob_y)^* \otimes \id)(ps-u_{y,N}) & \xrightarrow{\sim} & \ev_y \circ ((\Frob_y)^* \otimes \id)(u_{y,N}) \\
\downarrow & & \downarrow \\
C_c(y, -) \circ (\Delta_y)^* \circ (\Frob_y \times \id)_*(ps-u_y) & \xrightarrow{\sim} & C^*_c(y, -) \circ (\Frob_y \times \id)_*(u_y) \\
\end{array}
\]

6.3.6. Finally, we note that the commutative diagram \[(6.8)\] can be rephrased as

\[
\begin{array}{ccc}
C_c(y, -) \circ (\Delta_y)^* \circ (\Frob_y \times \id)_*(ps-u_y) & \xrightarrow{\sim} & C^*_c(y, -) \circ (\Frob_y \times \id)_*(u_y) \\
\downarrow & & \downarrow \\
C_c(y^\Frob, \omega_{y\Frob}) & \xrightarrow{\sim} & C(y^\Frob, \omega_{y\Frob}) \\
\downarrow & & \downarrow \\
\text{Funct}(y(\mathbb{F}_q)) & \xrightarrow{\text{id}} & \text{Funct}(y(\mathbb{F}_q)). \\
\end{array}
\]
Thus, concatenating with the commutative diagram in Sect. 5.3.5 above, we obtain a commutative diagram

\[
\begin{array}{cccc}
\text{Tr}((\text{Frob})_*, \text{Shv}_N(\mathcal{F})) & \xrightarrow{id} & \text{Tr}((\text{Frob})_*, \text{Shv}_N(\mathcal{F})) \\
\downarrow & & \downarrow \\
\text{ev}_y \circ ((\text{Frob})_* \odot \text{id})(\text{ps-u}_{y,Y,N}) & \xrightarrow{\text{id} \odot \text{id}} & \text{ev}_y \circ ((\text{Frob})_* \odot \text{id})(\text{u}_Y,N)) \\
\end{array}
\]

Thus, concatenating with the commutative diagram in Sect. 6.3.5 above, we obtain a commutative diagram

\[
\begin{array}{cccc}
C_*(\mathcal{Y}, -) \circ (\Delta y)^* \circ (\text{Frob}_Y \times \text{id})_* (\text{ps-u}_y) & \xrightarrow{\text{id} \circ \text{id}} & C_*(\mathcal{Y}, -) \circ (\Delta y)^! \circ (\text{Frob}_Y \times \text{id})_*(\text{u}_Y) \\
\downarrow & & \downarrow \\
C_*(\mathcal{Y}^{\text{Prob}}, \mathcal{E}_{\mathcal{Y}^{\text{Prob}}}) & \cong & C_*(\mathcal{Y}^{\text{Prob}}, \omega_{\mathcal{Y}^{\text{Prob}}}) \\
\end{array}
\]

in which the left composite vertical arrow is LT\textsubscript{Serre}\textsubscript{N}, and the right composite vertical arrow is LT\textsubscript{true}\textsubscript{N}.

This provides the sought-for identification between LT\textsubscript{Serre}\textsubscript{N} and LT\textsubscript{true}\textsubscript{N}.

### 6.4. Proof of Theorem 6.2.5

6.4.1. First, we recall the local version of the Grothendieck-Lefschetz trace formula, following [GV].

Let \( \mathcal{X} \) be a quasi-compact algebraic stack, and let \( \mathcal{F} \in \text{Shv}(\mathcal{X}) \) be a constructible sheaf, equipped with a weak Weil structure, i.e., a morphism

\[
\alpha : \mathcal{F} \to (\text{Frob}_Y)_*(\mathcal{F}),
\]

or equivalently, a morphism

\[
\alpha^L : \text{Frob}_Y^*(\mathcal{F}) \to \mathcal{F}.
\]

On the one hand, we attach to the pair \((\mathcal{F}, \alpha^L)\) a function \(\text{funct}(\mathcal{F})^{\text{naive}} \in \text{Funct}(\mathcal{Y}(\mathcal{F}_q))\) by the standard procedure of taking the trace of Frobenius on \(\ast\)-fibers of \(\mathcal{F}\) at \(\mathcal{F}_q\)-points of \(\mathcal{F}\). I.e., for a Frobenius-invariant point

\[
\text{pt} \xrightarrow{i_{\mathcal{X}}} \mathcal{X},
\]

we consider the endomorphism

\[
i_{\mathcal{X}}^!(\mathcal{F}) \simeq (\text{Frob}_Y \circ i_{\mathcal{X}})^!(\mathcal{F}) \simeq i_{\mathcal{Y}}^* \circ \text{Frob}_Y^!(\mathcal{F}) \xrightarrow{\alpha^L} i_{\mathcal{Y}}^!(\mathcal{F}),
\]

and we set the value of \(\text{funct}(\mathcal{F})^{\text{naive}}\) at \(y \in \mathcal{Y}(\mathcal{F}_q)\) to be the trace of the above endomorphism.

On the other hand, we can attach to \((\mathcal{F}, \alpha)\) a function \(\text{funct}(\mathcal{F})^{\text{true}} \in \text{Funct}(\mathcal{Y}(\mathcal{F}_q))\), defined as follows.

Consider the canonical maps

\[
\mathcal{F} \boxtimes \mathcal{D}^\text{Verdier}(\mathcal{F}) \to (\Delta y)_*(\omega_{\mathcal{Y}}) \text{ and } \mathcal{E}_{\mathcal{Y}} \to \mathcal{F} \boxtimes \mathcal{D}^\text{Verdier}(\mathcal{F}).
\]

From the first of these maps we produce the map

\[
\xrightarrow{\text{id} \circ \text{id}} (\text{Frob}_Y \times \text{id})_*(\mathcal{F} \boxtimes \mathcal{D}^\text{Verdier}(\mathcal{F})) \to (\text{Frob}_Y \times \text{id})_*((\Delta y)_*(\omega_{\mathcal{Y}})) \simeq (\text{Graph}_{\text{Frob}_Y})_*(\omega_{\mathcal{Y}}).
\]

The function \(\text{funct}(\mathcal{F})^{\text{true}}\), viewed as an element of

\[
C_*(\mathcal{Y}^{\text{Prob}}, \omega_{\mathcal{Y}^{\text{Prob}}}) \simeq C_*(\mathcal{Y}, (\Delta y)_* (\text{Graph}_{\text{Frob}_Y})_*(\omega_{\mathcal{Y}})),
\]
corresponds to the map
\[ \mathbb{E}_y \rightarrow \mathcal{F} \otimes D_{\text{Verdier}}(\mathcal{F}) = \Delta^!_y(\mathcal{F} \otimes D_{\text{Verdier}}(\mathcal{F})) \xrightarrow{(\text{6.16})} \Delta^!_y \circ (\text{Graph}_{\text{Frob}_y})_*(\omega_y). \]

The local Grothendiek-Lefschetz trace formula says:

**Theorem 6.4.2.** The functions \( \text{funct}(\mathcal{F})_{\text{naive}} \) and \( \text{funct}(\mathcal{F})_{\text{true}} \) are equal.

**Remark 6.4.3.** When \( \mathcal{F} \) is compact, the assertion of Theorem 6.4.2 is a particular case of that of Theorem 5.3.5. Namely, the functions \( \text{funct}(\mathcal{F})_{\text{naive}} \) and \( \text{funct}(\mathcal{F})_{\text{true}} \) are the values of the maps \( \text{LT}_{\text{naive}}^y \) and \( \text{LT}_{\text{true}}^y \) on the element
\[ \text{cl}(\mathcal{F}, \alpha) \in \text{Tr}(\text{Frob}_y^*, \text{Shv}(\mathcal{Y})), \]
respectively.

For \( \mathcal{F} \) which is constructible but not compact, the assertion of Theorem 6.4.2 can be obtained by proving a version of Theorem 5.3.5 for the renormalized version of the category \( \text{Shv}(\mathcal{Y}) \), namely, one obtained as the ind-completion of the constructible subcategory of \( \text{Shv}(\mathcal{Y}) \).

**6.4.4.** We precede the proof of Theorem 6.2.5 by the following observation.

Let \( \mathcal{Q} \) be a constructible object of \( \text{Shv}(\mathcal{Y} \times \mathcal{Y}) \). Note that the procedure in Sect. 6.2.2 defines a map
\[ (6.16) \psi : C_{\cdot}(\mathcal{Y}, \Delta^!_y \circ (\text{Id} \otimes \text{Miry})(\mathcal{Q})) \rightarrow C(\mathcal{Y}, \Delta^!_y(\mathcal{Q})). \]

It is easy to see that the map (6.16) equals the composition of the value of the natural transformation (6.3), followed by the canonical map
\[ (6.17) C_{\cdot}(\mathcal{Y}, \Delta^!_y(\mathcal{Q})) \rightarrow C(\mathcal{Y}, \Delta^!_y(\mathcal{Q})). \]

**6.4.5.** We are ready to launch the proof of Theorem 6.2.5. We apply the observation in Sect. 6.4.4 to
\[ \mathcal{Q} := (\text{Graph}_{\text{Frob}_y})_*(\omega_y). \]

Note that in this case, the map (6.17) is an isomorphism, as the corresponding map identifies with
\[ C_{\cdot}(\mathcal{Y}^\text{Prob}, \omega_{\text{Frob}}) \rightarrow C(\mathcal{Y}^\text{Prob}, \omega_{\text{Frob}}). \]

We need to show that a certain map
\[ \text{Funct}(\mathcal{Y}(\mathcal{F}_q)) = C_{\cdot}(\mathcal{Y}^\text{Prob}, \mathbb{E}_y^\text{Prob}) \rightarrow C(\mathcal{Y}^\text{Prob}, \omega_{\text{Frob}}) \simeq \text{Funct}(\mathcal{Y}(\mathcal{F}_q)) \]
equals the identity, where the middle arrow is the result of the construction in Sect. 6.4.4 applied to the above choice of \( \mathcal{Q} \).

We interpret the above map as a functional
\[ (6.18) \text{Funct}(\mathcal{Y}(\mathcal{F}_q)) \otimes \text{Funct}(\mathcal{Y}(\mathcal{F}_q)) \simeq C_{\cdot}(\mathcal{Y}_y^\text{Prob}, \mathbb{E}_y^\text{Prob}) \otimes C_{\cdot}(\mathcal{Y}_y^\text{Prob}, \mathbb{E}_y^\text{Prob}) \rightarrow \mathfrak{e}, \]
and we wish to show that this functional is given by
\[ \begin{align*}
    f_1, f_2 & \mapsto \sum_{y \in \mathcal{Y}(\mathcal{F}_q)} \frac{1}{|\text{Aut}(y)(\mathcal{F}_q)|} \cdot f_1(y) \cdot f_2(y).
\end{align*} \]
6.4.6. Let us unwind the construction of the functional \((6.15)\). We start with the map \((6.7)\) for the above choice of \(Q\). This is a map
\[
(6.19) \quad \left(\left(\text{Graph}_{\text{Prob}}(\xi_y) \boxtimes \text{Graph}_{\text{Prob}}(\xi_y)\right)^{\sigma_{2,3}}\right) \to (\Delta y)_*(\omega y) \boxtimes (\Delta y)_*(\xi_y).
\]
We apply to this map the functor
\[
(p_{2,3})_* \circ (\Delta y \times \text{id} \times \text{id})^* \circ \sigma_{2,3} \colon \text{Shv}(y \times y \times y \times y) \to \text{Shv}(y \times y),
\]
and we obtain a map
\[
C_c(y_{\text{Prob}},\xi_y) \rightarrow (\Delta y)_*(\omega y).
\]
We apply to the latter map the adjunction
\[
C_c(y,\omega) \colon \text{Shv}(y \times y) \cong \text{Vec}(\Delta y)_*(\omega y),
\]
and we obtain the desired pairing
\[
C_c(y_{\text{Prob}},\xi_y) \rightarrow \text{Vec}(\Delta y)_*(\omega y).
\]

6.4.7. Let us unwind the construction of the functional \((6.15)\). We start with the map \((6.7)\) for the above choice of \(Q\). This is a map
\[
(6.20) \quad \left(\left(\text{Graph}_{\text{Prob}}(\xi_y) \boxtimes (\Delta y)_*(\omega y)\right)\right)^{\sigma_{2,3}} \to \left(\left(\text{Graph}_{\text{Prob}}(\xi_y) \boxtimes (\Delta y)_*(\omega y)\right)\right)^{\sigma_{2,3}}.
\]
From this morphism we obtain an element of
\[
C \left(\left(\text{Graph}_{\text{Prob}}(\xi_y) \boxtimes (\Delta y)_*(\omega y)\right)\right)^{\sigma_{2,3}} \cong \text{Vec}(\Delta y)_*(\omega y).
\]
by the following procedure.
We apply to \((6.20)\) the functor
\[
(p_{2,3})_* \circ (\Delta y \times \text{id} \times \text{id})^* \circ \sigma_{2,3} \colon \text{Shv}(y \times y \times y \times y) \to \text{Shv}(y \times y),
\]
and we obtain a map
\[
(\Delta y)_*(\xi_y) \rightarrow C \left(\left(\text{Graph}_{\text{Prob}}(\xi_y) \boxtimes (\Delta y)_*(\omega y)\right)\right) \cong \text{Vec}(\Delta y)_*(\omega y).
\]
Applying the adjunction
\[
(\Delta y)_*(\xi_y) : \text{Vec} \cong \text{Shv}(y \times y) : C(\Delta y)_*(\xi_y),
\]
we obtain the desired element of
\[
(6.21) \quad C(\Delta y)_*(\xi_y) \cong \text{Vec}(\Delta y)_*(\omega y).
\]
We wish to show that the resulting function is the characteristic function of the diagonal, i.e., its value on a \((y_1,y_2) \in y(\mathbb{F}_q) \times y(\mathbb{F}_q)\) equals the cardinality of the set of isomorphisms between the corresponding two points of the groupoid \(y(\mathbb{F}_q)\).

6.4.8. However, unwinding the definitions, we obtain that the map \((6.20)\) identifies with the map \((6.15)\) for \(F = (\Delta y)_*(\xi_y)\) and \(\alpha\) being the tautological map \(\alpha_{\text{taut}}\)
\[
(\Delta y)_*(\xi_y) \simeq (\text{Frob}_{y \times y})_*(\Delta y)_*(\xi_y) \simeq (\text{Frob}_{y \times y})_* \circ (\Delta y)_*(\xi_y).
\]
From here, we obtain that the element in \((6.21)\) constructed above equals
\[
\text{func}^{\text{true}}((\Delta y)_*(\xi_y), \alpha_{\text{taut}}).
\]
Applying Theorem \(6.4.2\) we obtain that the above element equals
\[
\text{func}^{\text{naive}}((\Delta y)_*(\xi_y), \alpha_{\text{taut}}).
\]
Now, the classical Grothendieck-Lefschetz trace formula about the compatibility of the assignment
\[
(\mathcal{F}, \alpha) \mapsto \text{func}^{\text{naive}}(\mathcal{F}, \alpha)
\]
with the \(!\)-pushforward functor implies that the above function equals the direct image (=sum along the fibers) of the constant function along the map
\[
y(\mathbb{F}_q) \to (y \times y)(\mathbb{F}_q) \simeq y(\mathbb{F}_q) \times y(\mathbb{F}_q),
\]
as required.
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References

[AGKRRV1] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support, arXiv:2010.01906

[AGKRRV2] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, Duality for automorphic sheaves with nilpotent singular support, arXiv:2012.07665

[De] P. Deligne, La formule de dualité globale, SGA 4, tome 3, Lecture Notes in Mathematics, vol. 305 (1973), 481–587.

[Dr] V. Drinfeld, Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2), American Journal of Mathematics (1981), 85–114.

[FS] L. Fargues and P. Scholze, Geometrization of the local Langlands correspondence, arXiv:1902.13459

[Ga] D. Gaitsgory, From geometric to function-theoretic Langlands (or how to invent shtukas), arXiv:1606.09608

[GKR V] D. Gaitsgory, D. Kazhdan, N. Rozenblyum, Y. Varshavsky, A toy model for the Drinfeld-Lafforgue shtuka construction, arXiv:1908.05420

[GR] D. Gaitsgory and N. Rozenblyum, A study in derived algebraic geometry, Vol. 1: Correspondences and Duality, Mathematical surveys and monographs 221 (2017), AMS, Providence, RI.

[GV] D. Gaitsgory, Y. Varshavsky, Local terms for the categorical trace of Frobenius, arXiv:2012.14257

[KP] G. Kondyrev and A. Prihodko, Equivariant Riemann-Roch via formal deformation theory, arXiv:1906.00172

[VLa] V. Lafforgue, Champs affines et paramétrisation de Langlands globale, JAMS 31 (2018), 719–891.

[NY] D. Nadler and Z.-W. Yun, Spectral action in Betti geometric Langlands, arXiv:1611.04978

[Xue1] C. Xue, Finiteness of cohomology groups of stacks of shtukas as modules over Hecke algebras, and applications, arXiv:1811.09513

[Xue2] C. Xue, Smoothness of cohomology sheaves of stacks of shtukas, arXiv:2012.12834

[Zhu] X. Zhu, Coherent sheaves on the stack of Langlands parameters, arXiv:2008.02998.