Admissible vectors and traces on the commuting algebra

Hartmut Führ
Institute for Biomathematics and Biometry, GSF National Research Center for Environment and Health, Ingolstädter Landstraße 1, D-85764 Neuherberg

Abstract. Given a representation of a unimodular locally compact group, we discuss criteria for associated coherent state expansions in terms of the commuting algebra. It turns out that for those representations that admit such expansions there exists a unique finite trace on the commuting algebra such that the admissible vectors are precisely the tracial vectors for that trace. This observation is immediate from the definition of the group Hilbert algebra and its associated trace. The trace criterion allows to discuss admissibility in terms of the central decomposition of the regular representation. In particular, we present a new proof of the admissibility criteria derived for the type I case. In addition we derive admissibility criteria which generalize the Wexler-Raz biorthogonality relations characterizing dual windows for Weyl-Heisenberg frames.

1. Introduction

Given a representation \((\pi, \mathcal{H}_\pi)\) of a unimodular, separable locally compact group \(G\), we want to discuss the existence and characterization of vectors giving rise to coherent state expansions on \(\mathcal{H}_\pi\). For this purpose, a vector \(\eta \in \mathcal{H}_\pi\) is called \textbf{bounded} if the coefficient operator

\[ V_\eta : \mathcal{H}_\pi \to L^2(G), \quad (V_\eta \varphi)(x) = \langle \varphi, \pi(x)\eta \rangle \]

is a bounded map. We are interested in inverting this operator, hence the following notion is natural: A pair of bounded vectors \((\eta, \psi)\) is called \textbf{admissible} if \(V_\eta^* V_\eta\) is the identity operator on \(\mathcal{H}_\pi\). Note that this property gives rise to the weak-sense inversion formula

\[ z = \int_G \langle z, \pi(x)\eta \rangle \pi(x)\psi \, d\mu_G(x), \]

which can be read as a continuous expansion of \(z\) in terms of the orbit \(\pi(G)\psi \subset \mathcal{H}_\pi\). Identities of this type are known as \textbf{coherent state expansion} in mathematical physics. A single vector \(\eta\) is called \textbf{admissible} if \((\eta, \eta)\) is an admissible pair. It is obvious from the definition that \((\eta, \psi)\) is admissible iff \((\psi, \eta)\) is. In such a case \(\eta\) is called the \textbf{dual vector} of \(\psi\).

Admissible vectors were first discussed almost exclusively in connection with irreducible, so-called \textbf{discrete series} or \textbf{square-integrable} representations [10]. The existence of admissible vectors for these representations is a fairly straightforward consequence of Schur’s Lemma, and slightly more complicated in the nonunimodular case. Recently exhaustive criteria for the existence and characterization of admissible vectors were established for the case that the regular representation \(\lambda_G\) of \(G\) is type I, using the Plancherel formula of the group. However, if we want to include discrete groups in this general discussion, the type I restriction is rather too rigid. Indeed, a discrete group \(G\) has a type I regular representation iff \(G\) itself is type I [11], and the latter is only the case if \(G\) is a finite extension of an abelian normal subgroup [14].

Let us now give a short survey of the paper. It initiated from the idea to replace the decomposition into irreducibles by the central decomposition of \(\lambda_G\), and to try to come up
The full group Hilbert algebra consists of all semifiniteness; we refer the reader to [6] for the definitions. To define the group Hilbert algebra, we let for operator. Note that these operators then lie in the natural trace on the von Neumann algebra \( VN \). As a result of this paper, Theorem 2.2, provides the key to this problem, by relating admissibility with criteria using the latter. This however requires an understanding of how admissible pairs for the fibre von Neumann algebras of the central decomposition (Proposition 3.1). As an application of this result we obtain a characterization of admissible pairs in the case that \( \lambda_G \) is type I (Theorem 3.2). A slightly weaker version of this result had been proved, by somewhat different arguments, in an earlier paper ([7, Theorem 1.6]). In the final section we sketch how the trace criterion gives rise to admissibility criteria in terms of certain orthogonality relations. As a special case we obtain the Wexler-Raz biorthogonality relations in Gabor analysis.

2. The group Hilbert algebra, traces and admissible pairs

Throughout the paper, \( G \) denotes a separable unimodular locally compact group, and \( (\pi, \mathcal{H}) \) a (unitary, strongly continuous) representation of \( G \). \( \lambda_G \) is the left regular representation, acting on \( L^2(G) \) by \( (\lambda_G(x)f)(y) = f(x^{-1}y) \). We denote the commuting algebra of \( \lambda_G \) as \( VN_r(G) \), the right group von Neumann algebra.

Our definitions and notations regarding the group Hilbert algebra are taken from [5]. In order to define the group Hilbert algebra, we let for \( f, g \in L^2(G) \), and additionally either \( f \) or \( g \) in \( C_c(G) \),

\[
U_f(g) = g \ast f.
\]

The (full) group Hilbert algebra consists of all \( f \in L^2(G) \) for which \( U_f \) extends to a bounded operator. Note that these operators then lie in \( VN_r(G) \). Writing \( f^*(x) = f(x^{-1}) \), we note that \( V_f = \alpha U_f^\ast = U_{f^\ast} \). Hence the bounded vectors are precisely the elements of the full Hilbert algebra. We note in passing that the full Hilbert algebra contains the dense subspace \( L^1(G) \cap L^2(G) \) of \( L^2(G) \).

Let us recall the definition of a trace. Given a von Neumann algebra \( A \), we let \( A^+ \) denote the cone of positive elements. A mapping \( tr : A^+ \to \mathbb{R}^+ \cup \{\infty\} \) is called a trace if it satisfies the following two properties:

- \( tr(S + \alpha T) = tr(S) + \alpha tr(T) \), for all \( S, T \in A^+, \alpha \in \mathbb{R}^+ \). Note the conventions \( \alpha \infty = \infty \) for \( \alpha > 0 \) and \( 0 \infty = 0 \).
- \( tr(UTU^\ast) = tr(T) \), for all \( T \in A^+ \) and all unitary \( U \in A \).

Further relevant properties, that traces may or may not have, are faithfulness, normality and semifiniteness; we refer the reader to [5] for the definitions. \( tr \) is finite if \( tr(\text{Id}_H) = 1 \). A trace \( tr \) uniquely extends to a linear functional on the two-sided ideal

\[
\mathcal{M}_{tr} = \{ S \in A : tr(|S|) < \infty \} ,
\]

for finite traces this is obviously \( A \) itself. We will denote the extension by \( tr \) as well. If the trace is normal, the associated linear functional is ultra-weakly continuous [5, III.6, Proposition 1].

The group Hilbert algebra induces a faithful, normal and semifinite trace on \( VN_r(G)^+ \), by letting for \( T \in VN_r(G)^+ \)

\[
tr(T) = \begin{cases} \|f\|^2 & T = U_f^\ast U_f \text{ for a bounded vector } f \\ \infty & \text{otherwise} \end{cases}
\]

We note that for bounded vectors \( f, g \in L^2(G) \), \( V_f^\ast V_g \) is in \( \mathcal{M}_{tr} \), with

\[
tr(V_f^\ast V_g) = \langle f, g \rangle \quad \text{(1)}
\]
For our arguments it will sometimes be convenient to assume that $\mathcal{H}_\pi = \mathcal{H} \subset L^2(G)$ is a closed leftinvariant subspace, on which $\pi$ acts by left translation. This is not a restriction, thanks to the following lemma which collects a few facts about admissible pairs. We expect most of these statements to be widely known.

**Lemma 2.1** (a) For any bounded vector $\eta, V_\eta$ intertwines $\pi$ with $\lambda_G$.

(b) If $\pi$ has an admissible pair $(\eta, \psi)$, then both $V_\eta$ and $V_\psi$ are topological embeddings into $L^2(G)$. Conversely, given a bounded vector $\eta$ such that $V_\eta$ is a topological embedding, a dual vector for $\eta$ is given by $\psi = (V_\eta^* V_\eta)^{-1} \eta$. $\psi$ is the unique dual vector with minimal norm.

(c) There exists an admissible pair $(\eta, \psi)$ iff there exists an admissible vector.

(d) If there exists an admissible pair, $\pi$ is unitarily equivalent to a subrepresentation of $\lambda_G$.

**Proof.** Part (a) is immediate, and (d) then follows from (c). The first statement of (b) is obvious. For the existence of a dual vector, we observe that $S = V_\eta V_\eta^*$ is a strictly positive operator with bounded inverse. Hence $V_{S^{-1} \eta} = V_\eta \circ S^{-1}$ is bounded, i.e., $S^{-1} \psi$ is a bounded vector. The computation

$$V_\eta^* V_\eta = V_{S^{-1} \eta}^* V_\eta = S^{-1} V_\eta^* V_\eta = \text{Id}_{\mathcal{H}_\pi} ,$$

shows that $\psi$ is a dual vector. A similar calculation shows $V_{S^{-1/2} \eta}^* V_{S^{-1/2} \eta} = \text{Id}_{\mathcal{H}_\pi}$, i.e., (c).

Hence, for the proof of minimality of $\|\psi\|$ (which is the only thing left to show), we may assume that $\mathcal{H}_\pi = \mathcal{H} \subset L^2(G)$, and that $\pi$ is left translation on $\mathcal{H}$. Then the commuting algebra $\pi(G)'$ is readily identified as the reduced von Neumann algebra $\{pTP : T \in VN_r(G)\} \subset VN_r(G)$, where $p$ denotes the projection onto $\mathcal{H}$.

The set of all dual windows is an affine subspace, since the difference of two dual windows is in the linear subspace

$$W = \{ x \in H_\pi \text{ bounded vector} : V_x^* V_\eta = 0 \} .$$

Hence a dual window of minimal norm is necessarily unique. Now (I) entails for $x \in W$

$$\langle x, \psi \rangle = tr(V_x^* V_\psi) = tr(V_x^* V_{S^{-1} \eta}) = tr(V_x^* V_\eta S^{-1}) = tr(0) = 0 ,$$

hence $\psi \perp W$, and $\|\psi\|$ is indeed minimal. \qed

The characterization of admissible vectors in terms of the trace requires one more piece of notation: Given a particular trace $tr$ on a von Neumann algebra $\mathcal{A}$, we call a pair of elements $(\eta, \psi)$ of the underlying Hilbert space **tracial** if

$$\forall T \in \mathcal{A}^+ : tr(T) = \langle T\eta, \psi \rangle .$$

**Theorem 2.2** Let $\mathcal{H} \subset L^2(G)$ be a closed, leftinvariant subspace, with associated leftinvariant projection $p$, and let $\pi$ denote the restriction of $\lambda_G$ to $\mathcal{H}$.

(a) There exists an admissible pair for $\mathcal{H}$ iff $tr(p) < \infty$.

(b) For all pairs $(\eta, \psi) \in \mathcal{H} \times \mathcal{H}$ of bounded vectors: $(\eta, \psi)$ is admissible iff $(\eta, \psi)$ is tracial for $\pi(G)'$.

**Proof.** For part (a), first assume that there exists an admissible vector $\eta$. Hence $p = V_\eta^* V_\eta = U_\eta^* U_\eta$, and thus $tr(p) = \|\eta\|^2 < \infty$.

Conversely, if $tr(p) < \infty$, then $p = U_\eta^* U_\eta$, for some bounded vector $\eta \in L^2(G)$. Now the computation

$$V_{p\eta}^* V_{p\eta} = p V_{p\eta}^* V_{p\eta} = p$$

shows that $p\eta \in \mathcal{H}$ is admissible.
For (b), we compute, for any $T = V_g^*V_g$ with $g$ bounded, and for any pair $(\eta, \psi)$ of bounded vectors,
\[
\langle T\eta, \psi \rangle = \langle \eta * g^* g, \psi \rangle = \langle \eta * g^*, \psi * g^* \rangle = \langle g * \eta^*, g * \psi^* \rangle = \langle g, g * \psi^* \eta \rangle = \langle g, (V^*_\eta V_\psi)g \rangle .
\]
Hence, assuming that $(\eta, \phi)$ are admissible, we obtain
\[
\langle T\eta, \psi \rangle = \|g\|^2 = tr(T) ,
\]
as desired. Conversely, assuming traciality of $(\eta, \psi)$, the above calculation yields
\[
\|g\|^2 = tr(T) = \langle T\eta, \psi \rangle = \langle g, V^*_\eta V_\psi g \rangle ,
\]
for all bounded vectors. By polarization this leads to
\[
\langle h, g \rangle = \langle h, V^*_\eta V_\psi g \rangle ,
\]
for all bounded vectors $h, g$, and since these are dense, $V^*_\eta V_\psi = \text{Id}_\mathcal{H}$ follows. \qed

**Remark 2.3** The equivalent conditions from part (a) imply in particular that $\pi(G)'$ is a finite von Neumann algebra. However, finiteness of $\pi(G)'$ is not sufficient, as the following equivalences show:
\[
VN_r(G) \text{ is finite} \iff G \text{ is an SIN-group} \quad (2)
\]
\[
tr(Id_{L^2(G)}) < \infty \iff G \text{ is discrete} \quad (3)
\]
Here (2) is [5, 13.10.5], whereas (3) follows combining Theorem 2.2 (a) with [7, Proposition 0.4]. Recall that SIN-groups are defined by having a conjugation-invariant neighborhood-base at unity. Clearly this class comprises the locally compact abelian groups, hence for any nondiscrete LCA group $VN_r(G)$ is finite, but $tr(Id_{L^2(G)}) = \infty$.

### 3. Application to the central decomposition

In this section we consider the central decomposition of the regular representation and its use for the characterization of admissible pairs. In particular we recover the characterization obtained in [7] for the case that $\lambda_G$ is type I. The following facts concerning the central decomposition of $\lambda_G$ can be found in [5, 18.7.7, 18.7.8].

Let $\hat{G}$ denote the space of quasi-equivalence classes of factor representations of $G$, endowed with the natural Borel structure. Then there exists a standard positive measure $\nu_G$ on $\hat{G}$, and a measurable field of factor representations $\rho_{\sigma} \in \mathcal{A}_\sigma$, for $\nu_G$-almost every $\sigma$, such that
\[
\lambda_G \simeq \int_{\hat{G}} \oplus \rho_{\sigma} d\nu_G(\sigma) .
\]
The operator effecting the unitary equivalence is called the **Plancherel transform**. Moreover, the direct integral provides a decomposition of $VN_r(G)$ and the natural trace:
\[
VN_r(G) = \int_{\hat{G}} \mathcal{A}_\sigma d\nu_G(\sigma)
\]
for a measurable family of von Neumann algebras $\mathcal{A}_\sigma$ on $\mathcal{H}_\sigma$, as well as
\[
tr(T) = \int_{\hat{G}} tr(\sigma(T)) d\nu_G(\sigma) ,
\]
Admissible vectors and traces on the commuting algebra

when \((T_\sigma)_{\sigma \in \hat{G}}\) denotes the operator field corresponding to \(T\) under the central decomposition, and \(tr_\sigma\) is a faithful normal, semifinite trace on the factor \(A_\sigma\), which exists for \(\nu_G\)-almost every \(\sigma\). In particular, \(\nu_G\)-almost every \(A_\sigma\) is of type I or II.

Now admissibility is easily translated to traciality in the fibres. In the following, \((\hat{\psi}_\sigma)_{\sigma \in \hat{G}}\) denotes the Plancherel transform of \(\psi \in L^2(G)\).

**Proposition 3.1** Let \(\pi\) denote the restriction of \(\lambda_G\) to a closed, leftinvariant subspace \(\mathcal{H} \subset L^2(G)\). Let \(P\) denote the projection onto \(\mathcal{H}\), then \(P\) decomposes into a measurable field of projections \(\hat{P}_\sigma\), and \(\pi(G)'\) decomposes under the central decomposition into the von Neumann algebras \(\mathcal{C}_\sigma = \hat{P}_\sigma A_\sigma \hat{P}_\sigma\).

(a) For bounded \(\eta, \psi \in \mathcal{H}\), we have
\[
(\eta, \psi) \text{ is admissible for } \mathcal{H} \iff (\hat{\eta}_\sigma, \hat{\psi}_\sigma) \text{ is tracial for } \mathcal{C}_\sigma \ (\nu_G\text{a.e.})
\]

(b) \(\mathcal{H}\) has an admissible pair of vectors iff \(\int_{\hat{G}} tr(\hat{P}_\sigma) d\nu_G(\sigma) < \infty\). In particular, almost all \(\mathcal{C}_\sigma\) are finite von Neumann algebras.

The (potential) use of the proposition consists in the fact that we only need to characterize tracial pairs for factor representations. Unfortunately, we are not aware of any explicit criteria for tracial vectors associated to type II factors. For type I factors, though, they are easily derived, as the proof of the next theorem shows. Note that if \(\lambda_G\) is type I, the fibre spaces in the central decomposition are just the Hilbert-Schmidt spaces \(B_2(\mathcal{H}_\sigma)\), where \(\sigma\) runs through the unitary dual, and \(\lambda_G\) decomposes into left action on \(B_2(\mathcal{H}_\sigma)\) via \(\sigma\) [5, 18.8].

**Theorem 3.2** Let \(G\) be unimodular with \(\lambda_G\) type I. Let \(\mathcal{H} \subset L^2(G)\) be a leftinvariant subspace. Then there exists a measurable field of projections \(\hat{P}_\sigma\) on \(\mathcal{H}_\sigma\) such that
\[
P \simeq \int_{\hat{G}} 1 \otimes \hat{P}_\sigma \, d\nu_G(\sigma) .
\]

(a) \((\eta, \psi)\) is admissible \(\iff \) for \(\nu_G\)-almost every \(\sigma \in \hat{G}\) : \(\hat{\psi}_\sigma^* \hat{\eta}_\sigma = \hat{P}_\sigma\).

(b) There exists an admissible vector for \(\mathcal{H}\) iff
\[
\nu_\mathcal{H} = \int_{\hat{G}} \text{rank}(\hat{P}_\sigma) d\nu_G(\sigma) < \infty.
\]

**Proof.** The existence of the \(\hat{P}_\sigma\) follows from the type I property. In the following it is convenient to use tensor-product notation for rank-one operators, i.e., \(x \otimes y\) denotes the operator \(z \mapsto \langle z, y \rangle x\). Given a fixed \(\sigma \in \hat{G}\), the elements of \(\mathcal{K} = B_2(\mathcal{H}_\sigma) \circ \hat{P}_\sigma\) can be written uniquely as \(\eta = \sum_{i \in I} \eta_i \otimes e_i\), where \((e_i)_{i \in I}\) is a fixed orthogonal basis of \(\hat{P}(\mathcal{H}_\sigma)\). It follows that \(\mathcal{K}\) is conveniently identified with \(\mathcal{H}_\sigma \otimes \ell^2(I)\). In this identification the left action of \(\sigma\) on \(\mathcal{K}\) becomes \(\sigma \otimes 1\). Moreover, the commuting algebra is easily identified with \(1 \otimes B_2(\ell^2(I))\), and its trace is the usual operator trace.

A weak-operator dense subspace of \(B(\ell^2(I))\) is spanned by the operators \(e_{i,k} = \delta_i \otimes \delta_k\), where \(\delta_i \in \ell^2(I)\) denotes the usual Kronecker-\(\delta\) concentrated at \(i\). Now, given \(\hat{\eta}_\sigma = \sum_{i \in I} \eta_i \otimes e_i\) and \(\hat{\psi}_\sigma = \sum_{i \in I} \psi_i \otimes e_i\), we compute
\[
tr(e_{i,k}) = \delta_{i,k}
\]
and
\[
\langle (1 \otimes e_{i,k}) \eta, \psi \rangle = \langle \eta_i, \psi_k \rangle ,
\]
whence we obtain the following traciality condition

\[
(\hat{\eta}_\sigma, \hat{\psi}_\sigma) \text{ tracial } \iff \forall i,k : \langle \eta_i, \psi_k \rangle = \delta_{i,k}
\]

\[
\iff \left( \sum_{i \in I} \psi_i \otimes e_i \right)^* \left( \sum_{i \in I} \eta_i \otimes e_i \right) = \hat{P}_\sigma,
\]

which proves part (a). Part (b) follows easily from (a), see [7].

We wish to point out that the admissibility criteria, however abstract they may appear, have been made explicit for certain classes of representations, in particular for multiplicity-free representations. See [9] for a discussion of quasiregular representations of semidirect product groups, and [8] for a treatment of Weyl-Heisenberg frames with integer sampling ratio.

Another interesting class of representations are the factor subrepresentations of the regular representation, i.e., the atoms in the central decomposition, and the elements of their quasi-equivalence classes. These representations were already considered in [13], though not with a view to constructing admissible vectors.

**Corollary 3.3** Let \( \pi \) be a factor representation.

(a) \( \pi \) has admissible vectors iff \( \pi \) is equivalent to a subrepresentation of \( \lambda_G \), and \( \pi(G)' \) is a finite von Neumann algebra. In particular, \( \pi \) has either type I or II, and there exists a faithful, finite, normal trace \( tr \) on \( \pi(G)' \), unique up to normalization.

(b) The trace on \( \pi(G)' \) can be normalized in such a way that the following equivalence holds:

\( \langle \eta, \psi \rangle \text{ is admissible } \iff \langle \eta, \psi \rangle \text{ is tracial} \).

### 4. Checking admissibility using biorthogonality relations

While the discussion of the type I case shows that the characterization of admissible vectors via the trace on the commutant can be used to some effect, in the general case the merits are much less obvious. In this section we sketch a procedure to arrive at more concrete necessary and sufficient conditions for admissible pairs, in terms of certain scalar products. We will then demonstrate that the Wexler-Raz biorthogonality relations are a special instance of this approach. For the formulation of the admissibility conditions, we require

- A family \((T_i)_{i \in I} \subset \pi(G)'\) spanning a weak-operator dense subspace of \( \pi(G)' \). Recall that the density requirement means that for each \( S \in \pi(G)' \) there exists a net \((S_j)_{j \in J}\) in the span such that for all pairs \( y,z \in \mathcal{H}_\pi \) we have \( \langle S_j y, z \rangle \to \langle Sy, z \rangle \).
- An admissible pair \((\eta_0, \psi_0)\).

Then for a pair of bounded vectors \((\eta, \psi)\) we have the following equivalence:

\[
(\eta, \psi) \text{ is admissible } \iff \forall i \in I : \langle T_i \eta, \psi \rangle = \langle T_i \eta_0, \psi_0 \rangle.
\] (4)

The proof of the condition is immediate from the assumptions and Theorem 2.2. The criterion is explicit as soon as the \( T_i \) and the admissible pair \((\eta_0, \psi_0)\) are known explicitly. Clearly, generators are preferable which provide particularly simple relations.

Let us now derive admissibility criteria in the context of Weyl-Heisenberg frames. These frames are obtained by picking a window function \( \eta \in L^2(\mathbb{R}) \) and translating it along a time-frequency lattice \( \Gamma \). The shifts are described in terms of the operators

\[
T_x : f \mapsto f(\cdot - x) , \ M_\omega : f \mapsto e^{2\pi i \omega \cdot f}.
\]
Admissible vectors and traces on the commuting algebra

For the following we fix $\alpha, \beta > 0$, with $\alpha \beta \leq 1$. This is a well-known necessary and sufficient condition for the existence of Weyl-Heisenberg frames. The sufficiency is proved by the admissible vector $\eta_0$ given below, for necessity confer, among others, [1, 2, 12]. Given $\eta \in L^2(\mathbb{R})$, we wish to decide whether the family
\[ \{ M_{am}T_{\beta n}\eta : m, n \in \mathbb{Z} \} \]
constitutes a frame of $L^2(\mathbb{R})$. Recall that the latter property means that the coefficient map
\[ T_{f,\alpha,\beta} : f \mapsto \langle f, M_{am}T_{\beta n}\eta \rangle_{m,n \in \mathbb{Z}} \]
defines a topological embedding $L^2(\mathbb{R}) \hookrightarrow \ell^2(\mathbb{Z}^2)$. As in the proof of Lemma 2.1 we see that $f$ generates a Weyl-Heisenberg frame iff there exists a dual window $g$ generating a Weyl-Heisenberg frame and satisfying $T_{g,\alpha,\beta}^* T_{f,\alpha,\beta} = \text{Id}$.

Since the time-frequency shifts $M_{am}T_n$ do not constitute a group of operators, the group-theoretic interpretation of the problem requires a slight detour in the form of the next lemma.

**Lemma 4.1** Define the group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{T}$, with group law
\[ (m, n, z)(m', n', z') = (m + m', n + n', zz' e^{-2\pi i \alpha \beta m' n}) . \]
$G$ acts on $L^2(\mathbb{R})$ via the representation
\[ \pi(m, n, z) = M_{am}T_{\beta n}z . \]
For all $(f, g) \in L^2(\mathbb{R})$ with $T_{f,\alpha,\beta}, T_{g,\alpha,\beta}$ bounded, $f$ generates a Weyl-Heisenberg frame with dual window $g$ iff $(f, g)$ is an admissible pair for $\pi$.

**Proof.** The statements concerning $G$ and $\pi$ are immediate from the definitions. For the last statement, observe that
\[
V_g^* V_f h = \int_{\mathbb{T}} \sum_{m,k \in \mathbb{Z}} \langle h, \pi(m, k, z)f \rangle \pi(m, k, z)g \, dz
= \int_{\mathbb{T}} \sum_{m,k \in \mathbb{Z}} \langle h, \pi(m, k, 0)f \rangle \pi(m, k, 0)g \, dz
= T_{g,\alpha,\beta}^* T_{f,\alpha,\beta} .
\]

**Remark 4.2** The representation $\pi$ is type I iff $\alpha \beta$ is rational. For the only-if part confer [2, Remark 2.], whereas the if-part follows from the fact that the group itself is type I if $\alpha \beta$ is rational (a straightforward application of Mackey’s theory). In the case where $1/(\alpha \beta) \in \mathbb{Z}$, there exist admissibility criteria which employ the so-called Zak transform; here the representation is even multiplicity-free. See [8] for an interpretation of the Zak transform criterion in the light of Theorem 3.2

Following the general procedure sketched above, we now observe that
- $\eta_0 = \sqrt{\alpha} \chi_{[0,\beta)}$ is an admissible vector [3].
- The commuting lattice
  \[ \Lambda_c = \{ M_{m/\beta}T_{n/\alpha} : m, n \in \mathbb{Z} \} \]
generates a weak-operator dense subspace of $\pi(G)'$ ([3, Appendix 6.1]).
Admissible vectors and traces on the commuting algebra

Hence, after verifying that
\[ \langle M_{m/\beta} T_{n/\alpha} \eta_0, \eta_0 \rangle = \alpha \beta \delta_{m,0} \delta_{n,0}, \]
we obtain the Wexler-Raz biorthogonality relations as a special case of (4):

**Corollary 4.3** Let \( g, \gamma \) be such that \( T_{g;\alpha,\beta}, T_{\gamma;\alpha,\beta} \) are bounded. Then \( \gamma \) is a dual window for \( g \) iff
\[ \langle M_{m/\beta} T_{n/\alpha} \gamma, g \rangle = \alpha \beta \delta_{m,0} \delta_{n,0}. \]  

**Remark 4.4** A more general “Wexler-Raz-relation” is
\[ T_{f;\alpha,\beta} T_{g;\alpha,\beta} h = \frac{1}{\alpha \beta} T_{h;1/\beta,1/\alpha} T_{g;1/\beta,1/\alpha} f \] 
proved for suitable \( f, g, h \) in [4]. (6) is easily seen to imply (5). It is not clear whether (6) has a counterpart in the general setting.

**Concluding remarks**

Von Neumann algebra techniques have been used previously for establishing criteria for the existence of cyclic and/or admissible vectors, see for instance [1, 12]. In particular the coupling constant has proved to be a powerful tool for existence results, see [2, 12]. However, these techniques seem to be of limited use for the explicit construction of admissible vectors. By contrast, this paper aims at providing criteria for these vectors, though it is clear that much remains to be done to make these criteria work. The authors of [4] used the trace on the commuting algebra in the Weyl-Heisenberg frame context, but did not point out the close connection to admissibility.

**References**

1. L. Baggett, *Processing a radar signal and representations of the discrete Heisenberg group*, Coll. Math. 60/61 (1990), 195-203.
2. M.B. Bekka, *Square integrable representations, von Neumann algebras and an application to Gabor analysis*, Preprint, 2002.
3. I. Daubechies and A. Grossmann, *Painless nonorthogonal expansions*, J. Math. Phys. 27 (1986), 1271-1283.
4. I. Daubechies, H.J. Landau and Z. Landau, *Gabor time-frequency lattices and the Wexler-Raz identity*, J. Fourier Analysis and Applications 1 (1995), 437-478.
5. J. Dixmier, *C*-Algebras*. North Holland, Amsterdam, 1977.
6. J. Dixmier, *Von Neumann Algebras*. North Holland, Amsterdam, 1981.
7. H. Führ, *Admissible vectors for the regular representation*, Proc. AMS 130 (2002), 2959-2970.
8. H. Führ, *Plancherel transform criteria for Weyl-Heisenberg frames with integer oversampling*, submitted, available as [math.FA/0206309](http://arxiv.org/abs/math.FA/0206309).
9. H. Führ and M. Mayer, *Continuous wavelet transform from semidirect products: Cyclic representations and Plancherel measure*, J. Fourier Analysis and Applications, to appear.
10. A. Grossmann, J. Morlet and T. Paul, *Transforms associated to square integrable group representations I: General results*, J. Math. Phys. 26 (1985), 2473-2479.
11. E. Kaniuth, *Der Typ der regulären Darstellung diskreter Gruppen*, Math. Ann. 182 (1969), 334-339.
12. M. Rieffel, *Von Neumann algebras associated to pairs of lattices in Lie groups*, Math. Ann. 257 (1981), 403-418.
13. J. Rosenberg, *Square-integrable factor representations of locally compact groups*, Trans. AMS 261 (1978), pp. 1-33.
14. E. Thoma, *Eine Charakterisierung diskreter Gruppen vom Typ I*, Invent. Math. 6 (1968), 190-196.