Dirac Equation in Gauge and Affine-Metric Gravitation Theories

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Abstract

We show that the covariant derivative of Dirac fermion fields in the presence of a
general linear connection on a world manifold is universal for Einstein’s, gauge and affine-
metric gravitation theories.

1

We follow the gauge approach to the description of gravitational interactions.
In reality, one observes three types of fields: gravitational fields, fermion fields and gauge
fields associated with internal symmetries. We are not concerned here with Higgs fields whose
dynamics remains elusive.

If the gauge invariance under internal symmetries is kept in the presence of a gravitational
field, Lagrangian densities of gauge fields must depend on a metric gravitational field only.

Different spinor models of fermion matter have been suggested. At present, all observable
fermion particles are Dirac fermions. Therefore, we restrict our consideration to Dirac fermion
fields in gravitation theory.

In the gauge gravitation theory, gravity is represented by pairs \((h, A_h)\) of gravitational fields
\(h\) and associated Lorentz connections \(A_h\). The connection \(A_h\) is usually identified with
both a connection on a world manifold \(X\) and a spinor connection on the the spinor bundle
\(S_h \to X\) whose sections describe Dirac fermion fields \(\psi_h\) in the presence of the gravitational
field \(h\). The problem arises when Dirac fermion fields are described in the framework of the
affine-metric gravitation theory. In this case, the fact that a world connection is some Lorentz
connection may result from the field equations, but it can not be assumed in advance. There are
models where the world connection is not a Lorentz connection. Moreover, it may happen
that a world connection is the Lorentz connection with respect to different gravitational fields
\([15]\). At the same time, a Dirac fermion field can be regarded only in a pair \((h, \psi_h)\) with a
certain gravitational field \(h\).
Indeed, one must define the representation of cotangent vectors to $X$ by the Dirac’s $\gamma$-matrices in order to construct the Dirac operator. Given a tetrad gravitational field $h(x)$, we have the representation

$$\gamma_h : dx^\mu \mapsto \tilde{dx}^\mu = h^\mu_a \gamma^a.$$  

However, different gravitational fields $h$ and $h'$ yield the nonequivalent representations $\gamma_h$ and $\gamma_{h'}$.

It follows that, fermion-gravitation pairs $(h, \psi_h)$ are described by sections of the composite spinor bundle

$$S \to \Sigma \to X \quad (1)$$

where $\Sigma \to X$ is the bundle of gravitational fields $h$ where values of $h$ play the role of parameter coordinates, besides the familiar world coordinates. [10, 13]. In particular, every spinor bundle $S_h \to X$ is isomorphic to the restriction of $S \to \Sigma$ to $h(X) \subset \Sigma$. Performing this restriction, we come to the familiar case of a field model in the presence of a gravitational field $h(x)$. The feature of the dynamics of field systems on the composite bundle (1) lies in the fact that we have the modified covariant differential of fermion fields which depend on derivatives of gravitational fields $h$.

As a consequence, we get the following covariant derivative of Dirac fermion fields in the presence of a gravitational field $h(x)$:

$$\tilde{D}_\lambda = \partial_\lambda - \frac{1}{2} A^{abc}_\mu (\partial_\lambda h^\mu_c + K^{\mu \nu}_\lambda h^\nu_c) I_{ab}, \quad (2)$$

$$A^{abc}_\mu = \frac{1}{2} (\eta^{ca} h^b_\mu - \eta^{cb} h^a_\mu),$$

where $K$ is a general linear connection on a world manifold $X$, $\eta$ is the Minkowski metric, and

$$I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$$

are generators of the spinor group $L_s = SL(2, \mathbb{C})$.

Let us emphasize that the connection

$$\tilde{K}^{ab}_\lambda = A^{abc}_\mu (\partial_\lambda h^\mu_c + K^{\mu \nu}_\lambda h^\nu_c) \quad (3)$$

is not the connection

$$K^k_{m\lambda} = h^k_\mu (\partial_\lambda h^\mu_m + K^{\mu \nu}_\lambda h^\nu_m) = K^{ab}_\lambda (\eta_{am} \delta^k_b - \eta_{bm} \delta^k_a)$$

written with respect to the reference frame $h^a = h^a_\lambda dx^\lambda$, but there is the relation

$$\tilde{K}^{ab}_\lambda = \frac{1}{2} (K^{ab}_\lambda - K^{ba}_\lambda). \quad (4)$$

If $K$ is a Lorentz connection $A_h$, then the connection $\tilde{K}$ (3) consists with $K$ itself.
The covariant derivative (2) has been considered by several authors [1, 8, 16]. The relation (4) correspond to the canonical decomposition of the Lie algebra of the general linear group. By the well-known theorem [6], every general linear connection being projected onto the Lie algebra of the Lorentz group yields a Lorentz connection.

In our opinion, the advantage of the covariant derivative (2), consists in the fact that, being derived in the framework of the gauge gravitation theory, it may be also applied to the affine-metric gravitation theory and the conventional Einstein’s gravitation theory. We are not concerned here with the general problem of equivalence of metric, affine and affine-metric theories of gravity [3, 7]. At the same time, when $\nabla$ is the Levi-Civita connection of $h$, the Lagrangian density of fermion fields which utilizes the covariant derivative (2) comes to that in the Einstein’s gravitation theory. It follows that the configuration space of metric (or tetrad) gravitational fields and general linear connections may play the role of the universal configuration space of realistic gravitational models. In particular, one then can think of the generalized Komar superpotential as being the universal superpotential of energy-momentum of gravity [4].

2

We follow the geometric approach to field theory when classical fields are described by global sections of a bundle $Y \to X$ over a world manifold $X$. Their dynamics is phrased in terms of jet manifolds [1, 2, 4].

As a shorthand, one can say that the $k$-order jet manifold $J^kY$ of a bundle $Y \to X$ comprises the equivalence classes $j_x^k s$, $x \in X$, of sections $s$ of $Y$ identified by the first $k + 1$ terms of their Taylor series at a point $x$. Recall that a $k$-order differential operator on sections of a bundle $Y \to X$ is defined to be a bundle morphism of the bundle $J^kY \to X$ to a vector bundle over $X$.

In particular, given bundle coordinates $(x^\mu, y^i)$ of a bundle $Y$, the first order jet manifold $J^1Y$ of $Y$ is endowed with the coordinates $(x^\mu, y^i, y^i_\mu)$ where

$$y^i_\mu(j_x^1 s) = \partial_\mu s^i(x).$$

There is the 1:1 correspondence between the connections on the bundle $Y \to X$ and the global sections

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_i^i \partial_i)$$

of the affine jet bundle $J^1Y \to Y$. Every connection $\Gamma$ on $Y \to X$ yields the first order differential operator

$$D_\Gamma : J^1Y \to T^*X \otimes VY,$$

$$D_\Gamma = (y^i_\lambda - \Gamma_i^i)dx^\lambda \otimes \partial_i,$$

on $Y$ which is called the covariant differential relative to the connection $\Gamma$. We denote by $VY$ the vertical tangent bundle of $Y$. 

3
In the first order Lagrangian formalism, the first order jet manifold \( J^1Y \) of \( Y \) plays the role of the finite-dimensional configuration space of fields represented by sections \( s \) of a bundle \( Y \to X \). A first order Lagrangian density is defined to be an exterior horizontal density

\[
L : J^1Y \to \wedge^n T^*X \quad n = \text{dim } X,
\]

\[
L = \mathcal{L}(x^\mu, y^i, y^i_\mu)\omega \quad \omega = dx^1 \wedge ... \wedge dx^n,
\]
on the bundle \( J^1Y \to X \).

Note that, since the jet bundle \( J^1Y \to Y \) is affine, every polynomial Lagrangian density of field theory factors through

\[
L : J^1Y \xrightarrow{D} T^*X \otimes VY \to \wedge^n T^*X
\]

where \( D \) is the covariant differential relative to some connection on \( Y \).

3

Let us consider the gauge theory of gravity and fermion fields. By \( X \) is further meant an oriented 4-dimensional world manifold which satisfies the well-known topological conditions in order that gravitational fields and spinor structure can exist on \( X \). To summarize these conditions, we assume that \( X \) is not compact and that the tangent bundle of \( X \) is trivial.

Let \( LX \) be the principal bundle of oriented linear frames in tangent spaces to \( X \). In gravitation theory, its structure group \( GL^+(4, \mathbb{R}) \) is reduced to the connected Lorentz group \( L = SO(1, 3) \). It means that there exists a reduced subbundle \( L^hX \) of \( LX \) whose structure group is \( L \). In accordance with the well-known theorem, there is the 1:1 correspondence between the reduced \( L \) subbundles \( L^hX \) of \( LX \) and the global sections \( h \) of the quotient bundle

\[
\Sigma := LX/L \to X. \tag{5}
\]

These sections \( h \) describe gravitational fields on \( X \), for the bundle \( \Sigma \) is the 2-folder covering of the bundle of pseudo-Riemannian metrics on \( X \).

Given a section \( h \) of \( \Sigma \), let \( \Psi^h \) be an atlas of \( LX \) such that the corresponding local sections \( z^h_\xi \) of \( LX \) take their values into \( L^hX \). With respect to \( \Psi^h \) and a holonomic atlas \( \Psi^T = \{ \psi^T_\xi \} \) of \( LX \), a gravitational field \( h \) can be represented by a family of \( GL_4 \)-valued tetrad functions

\[
h_\xi = \psi^T_\xi \circ z^h_\xi, \quad dx^\lambda = h^\lambda_a(x)h^a.
\]

By the Lorentz connections \( A_h \) associated with a gravitational field \( h \) are meant the principal connections on the reduced subbundle \( L^hX \) of \( LX \). They give rise to principal connections on \( LX \) and to spinor connections on the \( L_s \)-lift \( P_h \) of \( L^hX \).

There are different ways to introduce Dirac fermion fields. We follow the algebraic approach.
Given a Minkowski space $M$, let $Cl_{1,3}$ be the complex Clifford algebra generated by elements of $M$. A spinor space $V$ is defined to be a minimal left ideal of $Cl_{1,3}$ on which this algebra acts on the left. We have the representation

$$\gamma : M \otimes V \rightarrow V$$

of elements of the Minkowski space $M \subset Cl_{1,3}$ by Dirac’s matrices $\gamma$ on $V$.

Let us consider a bundle of complex Clifford algebras $Cl_{1,3}$ over $X$ whose structure group is the Clifford group of invertible elements of $Cl_{1,3}$. Its subbundles are both a spinor bundle $S_M \rightarrow X$ and the bundle $Y_M \rightarrow X$ of Minkowski spaces of generating elements of $Cl_{1,3}$. To describe Dirac fermion fields on a world manifold $X$, one must require $Y_M$ to be isomorphic to the cotangent bundle $T^*X$ of $X$. It takes place if there exists a reduced $L$ subbundle $L^hX$ such that

$$Y_M = (L^hX \times M)/L.$$  

Then, the spinor bundle

$$S_M = S_h = (P_h \times V)/L_s$$

is associated with the $L_s$-lift $P_h$ of $L^hX$. In this case, there exists the representation

$$\gamma_h : T^*X \otimes S_h = (P_h \times (M \otimes V))/L_s \rightarrow (P_h \times \gamma(M \times V))/L_s = S_h$$

of cotangent vectors to a world manifold $X$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. As a shorthand, one can write

$$\dd x^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a.$$  

Given the representation (8), we shall say that sections of the spinor bundle $S_h$ describe Dirac fermion fields in the presence of the gravitational field $h$. Indeed, let

$$A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A^a_{\lambda \mu} I_a A^B B A^B \psi B \partial_\lambda)$$

be a principal connection on $S_h$. Given the corresponding covariant differential $D$ and the representation $\gamma_h$ (8), one can construct the Dirac operator

$$D_h = \gamma_h \circ D : J^1 S_h \rightarrow T^*X \otimes V S_h \rightarrow V S_h,$$  

$$\dd y^A \circ D_h = h^\lambda_a \gamma^a B (y^B - \frac{1}{2} A^a_{\lambda \mu} I_a A^B B y^B)$$

on the spinor bundle $S_h$.

Different gravitational fields $h$ and $h'$ define nonequivalent representations $\gamma_h$ and $\gamma_{h'}$. It follows that a Dirac fermion field must be regarded only in a pair with a certain gravitational field. There is the 1:1 correspondence between these pairs and sections of the composite spinor bundle (8).
By a composite bundle is meant the composition

\[ Y \to \Sigma \to X. \tag{10} \]

of a bundle \( Y \to X \) denoted by \( Y_\Sigma \) and a bundle \( \Sigma \to X \). It is coordinatized by \((x^\lambda, \sigma^m, y^i)\) where \((x^\mu, \sigma^m)\) are coordinates of \( \Sigma \) and \( y^i \) are the fiber coordinates of \( Y_\Sigma \). We further assume that \( \Sigma \) has a global section.

The application of composite bundles to field theory is founded on the following [9]. Given a global section \( h \) of \( \Sigma \), the restriction \( Y_h \) of \( Y_\Sigma \) to \( h(X) \) is a subbundle of \( Y \to X \). There is the 1:1 correspondence between the global sections \( s_h \) of \( Y_h \) and the global sections of the composite bundle \( (10) \) which cover \( h \). Therefore, one can think of sections \( s_h \) of \( Y_h \) as describing fermion fields in the presence of a background parameter field \( h \), whereas sections of the composite bundle \( Y \) describe all the pairs \((s_h, h)\). The configuration space of these pairs is the first order jet manifold \( J^1Y \) of the composite bundle \( Y \).

The feature of the dynamics of field systems on composite bundles consists in the following. Every connection \( A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_\lambda^i \partial_i) + d\sigma^m \otimes (\partial_m + A_m^i \partial_i) \) on the bundle \( Y_\Sigma \) yields the horizontal splitting

\[ VY = VY_\Sigma \oplus (Y \times V\Sigma), \]

\[ \dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A_m^i \sigma^m) \partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i). \]

Building on this splitting, one can construct the first order differential operator

\[ \tilde{D} : J^1Y \to T^*X \otimes VY_\Sigma, \]

\[ \tilde{D} = dx^\lambda \otimes (y^\lambda_i - \tilde{A}_\lambda^i - A_m^i \sigma^m) \partial_i, \tag{11} \]

on the composite bundle \( Y \). This operator possesses the following property.

Given a global section \( h \) of \( \Sigma \), let \( \Gamma \) be a connection on \( \Sigma \) whose integral section is \( h \), that is, \( \Gamma \circ h = J^1h \). It is readily observed that the differential \( (11) \) restricted to \( J^1Y_h \subset J^1Y \) comes to the familiar covariant differential relative to the connection

\[ A_h = dx^\lambda \otimes [\partial_\lambda + (A_m^i \partial_m h^m + \tilde{A}_\lambda^i) \partial_i] \]

on \( Y_h \). Thus, it is \( \tilde{D} \) that we may utilize in order to construct a Lagrangian density

\[ L : J^1Y \xrightarrow{\tilde{D}} T^*X \otimes VY_\Sigma \to \wedge^n T^*X \]

for sections of the composite bundle \( Y \).
In gravitation theory, we have the composite bundle

\[ LX \to \Sigma \to X \]

where \( \Sigma \) is the quotient bundle (3) and

\[ LX_\Sigma := LX \to \Sigma \]

is the \( L \)-principal bundle.

Let \( P_\Sigma \) be the \( L_s \)-principal lift of \( LX_\Sigma \) such that

\[ P_\Sigma/L_s = \Sigma, \quad LX_\Sigma = r(P_\Sigma). \]

In particular, there is the imbedding of the \( L_s \)-lift \( P_\h \) of \( L^h X \) onto the restriction of \( P_\Sigma \) to \( h(X) \).

Let us consider the composite spinor bundle (1) where \( S_\Sigma = (P_\Sigma \times V)/L_s \) is associated with the \( L_s \)-principal bundle \( P_\Sigma \).

Let us provide the principal bundle \( LX \) with a holonomic atlas \( \{ \psi^T_\xi, U_\xi \} \) and the principal bundles \( P_\Sigma \) and \( LX_\Sigma \) with associated atlases \( \{ z^s_\xi, U_\xi \} \) and \( \{ z_\xi = r \circ z^s_\xi \} \). With respect to these atlases, the composite spinor bundle is endowed with the bundle coordinates \( (x^\lambda, \sigma^\mu_a, \psi^A) \) where \( \sigma^\mu_a \) are the matrix components of the group element \( (\psi^T_\xi \circ z_\xi)(\sigma), \sigma \in U_\xi, \pi_{\Sigma X}(\sigma) \in U_\xi \). Given a section \( h \) of \( \Sigma \), we have

\[ (\sigma^\mu_a \circ h)(x) = h^\lambda_a(x), \]

where \( h^\lambda_a(x) \) are the tetrad functions (3).

Let us consider the bundle of Minkowski spaces

\[ (LX \times M)/L \to \Sigma \]

associated with the \( L \)-principal bundle \( LX_\Sigma \). Since \( LX_\Sigma \) is trivial, it is isomorphic to the pullback \( \Sigma \times T^*X \) which we denote by the same symbol \( T^*X \). Then, one can define the bundle morphism

\[ \gamma_\Sigma : T^*X \otimes S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \to (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma, \]

\[ \hat{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a, \]

over \( \Sigma \). When restricted to \( h(X) \subset \Sigma \), the morphism (12) comes to the morphism \( \gamma_h \) (8).
We use this morphism in order to construct the total Dirac operator on the composite spinor bundle $S$ (1).

Let

$$\tilde{A} = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_\lambda^B \partial_B) + d\sigma^\mu_a \otimes (\partial_\mu + A^B_a \partial_B)$$

be a principal connection on the bundle $S_\Sigma$ and $\tilde{D}$ the corresponding differential (11). We have the first order differential operator

$$D = \gamma_\Sigma \circ \tilde{D} : J^1S \to T^*X \otimes VS_\Sigma \to VS_\Sigma,$$

$$\psi^A \circ D = \sigma^\lambda_a \sigma^A_B (\psi^B_\lambda - \tilde{A}^B_\lambda - A^B_a \sigma^\mu_\lambda \partial_B) \sigma^a$$

on $S$. One can think of it as being the total Dirac operator since, for every section $h$, the restriction of $D$ to $J^1S_h \subset J^1S$ comes to the Dirac operator $D_h$ (9) relative to the connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (\tilde{A}^B_\lambda + A^B_a \partial_\lambda h^a_B) \partial_B]$$

on the bundle $S_h$.

In order to construct the differential $\tilde{D}$ (11) on $J^1S$ in explicit form, let us consider the principal connection on the bundle $LX_\Sigma$ which is given by the local connection form

$$\tilde{A} = (\tilde{A}^{ab}_\mu dx^\mu + A^{abc}_\mu d\sigma^\mu_c) \otimes I_{ab}, \quad (13)$$

$$\tilde{A}^{ab}_\mu = \frac{1}{2} K^\nu_\lambda \sigma^\lambda_c (\eta^{ca} \sigma^b_\nu - \eta^{cb} \sigma^a_\nu),$$

$$A^{abc}_\mu = \frac{1}{2} (\eta^{ca} \sigma^b_\mu - \eta^{cb} \sigma^a_\mu), \quad (14)$$

where $K$ is a general linear connection on $TX$ and (14) corresponds to the canonical left-invariant free-curvature connection on the bundle

$$GL^+(4, \mathbb{R}) \to GL^+(4, \mathbb{R})/L.$$

Accordingly, the differential $\tilde{D}$ relative to the connection (13) reads

$$\tilde{D} = dx^\lambda \otimes [\partial_\lambda - \frac{1}{2} A^{abc}_\mu (\sigma^\mu_c + K^\nu_\lambda \sigma^\nu_c) I_{ab} A_B \psi^B \partial_A]. \quad (15)$$

Given a section $h$, the connection $\tilde{A}$ (13) is reduced to the Lorentz connection $\tilde{K}$ (3) on $L^hX$, and the differential (13) leads to the covariant derivatives of fermion fields (2).

We utilize the differential (13) in order to construct a Lagrangian density of Dirac fermion fields. Their Lagrangian density is defined on the configuration space $J^1(S \oplus S^+) \subset \Sigma$

$$(x^\mu, \sigma^\mu_a, \psi^A, \psi^+_A, \sigma^\mu_\lambda, \psi^+_A, \psi^+_\lambda, \psi^+_A)$$.
It reads

\[ L_\psi = \left\{ \frac{i}{2} \left[ \psi_A^+ (\gamma^0 \gamma^\lambda)^A_B (\psi^B_\lambda) - \frac{1}{2} A^{\mu} \sigma^{\mu}_{c\lambda} (\sigma^{\mu}_{c\lambda} + K_{\nu\lambda} \sigma^{\nu}_{c\lambda}) I_{ab}^C \psi_C \right] - \left[ \psi_A^+ - \frac{1}{2} A^{\mu} \sigma^{\mu}_{c\lambda} (\sigma^{\mu}_{c\lambda} + K_{\nu\lambda} \sigma^{\nu}_{c\lambda}) (\psi^+ A^{\mu} I_{ab}^C \psi_C \right] \right\} - m \psi_A^+ (\gamma^0)^A_B \psi_B \sigma^{-1} \omega \]  

(16)

where

\[ \gamma^\mu = \sigma^\mu \gamma^a, \quad \sigma = \det(\sigma^{\mu}), \]

and

\[ \psi_A^+ (\gamma^0)^A_B \psi_B \]

is the Lorentz invariant fiber metric in the bundle \( S \oplus S^* \Sigma \).

One can easily verify that

\[ \frac{\partial L_\psi}{\partial K^{\mu}_{\nu\lambda}} + \frac{\partial L_\psi}{\partial K^{\mu}_{\nu\lambda}} = 0. \]

Hence, the Lagrangian density (16) depends on the torsion of the general linear connection \( K \) only. In particular, it follows that, if \( K \) is the Levi-Civita connection of a gravitational field \( h(x) \), after the substitution \( \sigma^\nu_{c} = h^\nu_{c}(x) \), the Lagrangian density (16) comes to the familiar Lagrangian density of fermion fields in the Einstein’s gravitation theory.

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