On $k$-Bend and Monotonic $\ell$-Bend Edge Intersection Graphs of Paths on a Grid

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Abstract

If a graph $G$ can be represented by means of paths on a grid, such that each vertex of $G$ corresponds to one path on the grid and two vertices of $G$ are adjacent if and only if the corresponding paths share a grid edge, then this graph is called EPG and the representation is called EPG representation. A $k$-bend EPG representation is an EPG representation in which each path has at most $k$ bends. The class of all graphs that have a $k$-bend EPG representation is denoted by $B_k$. $B^m_\ell$ is the class of all graphs that have a monotonic $\ell$-bend EPG representation, i.e. an $\ell$-bend EPG representation, where each path is ascending in both columns and rows.

It is trivial that $B^m_\ell \subseteq B_k$ for all $k$. Moreover, it is known that $B^m_k \nsubseteq B_k$, for $k = 1$. By investigating the $B_k$-membership and the $B^m_\ell$-membership of complete bipartite graphs we prove that the inclusion is also proper for $k \in \{2, 3, 5\}$ and for $k \geq 7$. In particular, we derive necessary conditions for this membership that have to be fulfilled by $m$, $n$ and $k$, where $m$ and $n$ are the number of vertices on the two partition classes of the bipartite graph. We conjecture that $B^m_k \nsubseteq B_k$ holds also for $k \in \{4, 6\}$.

Furthermore, we show that $B_k \nsubseteq B^m_{2k-9}$ holds for all $k \geq 5$. This implies that restricting the shape of the paths can lead to a significant

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increase of the number of bends needed in an EPG representation. So far no bounds on the amount of that increase were known. We prove that $B_1 \subseteq B_m^3$ holds, providing the first result of this kind.

**Keywords:** paths on a grid, EPG graph, (monotonic) bend number, complete bipartite graph

1 Introduction and Definitions

In 2009 Golumbic, Lipshteyn and Stern [16] introduced edge intersection graphs of paths on a grid. If a graph $G$ can be represented by means of paths on a grid, such that each vertex of $G$ corresponds to one path on the grid and two vertices of $G$ are adjacent if and only if the corresponding paths share a grid edge, then this graph is called edge intersection graph of paths on a grid (EPG) and the representation is called EPG representation. Here the term edge intersection of paths refers to the fact that the paths share a grid edge.

A $k$-bend EPG representation or $B_k$-EPG representation is an EPG representation in which each path has at most $k$ bends. A graph that has a $B_k$-EPG representation is called $B_k$-EPG and the class of all $B_k$-EPG graphs is denoted by $B_k$. We consider the following natural ordering of grid lines: the columns increase from the left to the right and the rows increase from the bottom to the top. A path on a grid is called monotonic, if it is ascending in both columns and rows, i.e. it has the shape of a staircase that is going upwards from the left to the right. The graphs that have a $B_\ell$-EPG representation in which each path is monotonic are called $B_\ell^m$-EPG and the class of all these graphs is denoted by $B_\ell^m$. The bend number $b(G)$ of a graph $G$ is the minimum $k$ such that $G$ is $B_k$-EPG. The monotonic bend number $b^m(G)$ of graph $G$ is defined as the minimum $\ell$ such that $G$ is $B_\ell^m$-EPG. Note that already Golumbic, Lipshteyn and Stern [16] showed that each graph is $B_k$-EPG and $B_\ell^m$-EPG for some $k$ and $\ell$.

As described in [16] the motivation for investigating EPG graphs was initially related to applications from circuit layout setting and chip manufacturing. In the knock-knee circuit layout model the wires can be seen as paths on a grid which can cross and bend at a grid point but are not allowed to share a grid edge, see [8, 19]. The wires can be put in multiple layers each of them being a grid and such that the wires of each layer do not share a grid edge. In this setting the minimum number of layers needed to accommodate all wires would be equal to the chromatic number of the corresponding graph. Consider now that a so-called transition hole is needed, whenever a wire bends. If a large number of transition holes is included, whenever a wire bends. If a large number of transition holes is included, the layout area and consequently, the cost of the chip, may increase. Therefore, it might be
desirable to find a circuit layout setting which minimizes the largest number of bends used in each wire. In our notation this corresponds to finding the minimum $k$ such that the corresponding graph is in $B_k$.

Similar graph classes known in the literature include edge intersection graphs of paths on a tree (EPT) (see [15]), vertex intersection graph of paths on a tree (VPT) (see [14]) and vertex intersection graphs of paths on a grid (VPG) (see [1]). In this paper we will only deal with EPG graphs.

There has been a lot of research on EPG graphs since their introduction. One of the topics of interest is the recognition problem of $B_k$-EPG graphs, i.e. to determine for a given $k$ and a given graph whether this graph is in $B_k$ ($B_k^m$). Currently it is known that the recognition problem is NP-hard for $B_1$ (Heldt, Knauer and Ueckerdt [18]), $B_2^m$ (Cameron, Chaplick and Hoàng [9]), $B_2$ and $B_3^m$ (Pergel and Rzążewski [20]).

Recently a number of results on combinatorial optimization problems on specific $B_k$-EPG graphs have been published. Subject of investigation are certain NP-hard combinatorial optimization problems which turn out to be tractable, i.e. polynomially solvable or approximable within a guaranteed approximation ratio, for $B_k$-EPG graphs, see [5, 6, 7, 12]. Thus, the computation of the bend number and the monotonic bend number of graphs or related upper bounds is a relevant research question in this context. However, this appears to be a challenging task, considering that even the recognition of $B_k$ ($B_k^m$) graphs is NP-hard for $k = 1$ and $k = 2$, as mentioned above.

A related and more viable line of research is the determination of (upper bounds on) the (monotonic) bend number of special graph classes. Among the first graph class for which an upper bound on the bend number was given were planar graphs. The first upper bound was 5 and it was obtained in 2009 by Biedl and Stern [4]. In 2012 Heldt, Knauer and Ueckerdt [17] improved the bound to 4 and also showed that 2 is an upper bound on the bend number of outerplanar graphs. Çela and Gaar [10] showed recently that 2 is also an upper bound on the monotonic bend number of outerplanar graphs. Moreover, they give a full characterization of any maximal outerplanar graph and any cactus[1] with (monotonic) bend number equal to 0, 1 and 2 in terms of forbidden induced subgraphs.

Also other graph classes were considered. Recently Francis and Lahiri [13] proved that Halin graphs are in $B_2^m$ and Deniz, Nivelle, Ries and Schindl [11] provided a characterization of split graphs for which there exists a $B_1$-EPG representation which uses only L-shaped paths on the grid, i.e. paths consisting of a vertical top-bottom segment followed by a horizontal left-right

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[1] A connected graph is called a cactus iff any two simple cycles in it share at most one vertex.
Another line of research on EPG graphs concerns the mutual relationship between the classes \( B_k \) and the classes \( B_k^m \). Our paper is a contribution in this direction. The chains of inclusions \( B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \) and \( B_0^m \subseteq B_1^m \subseteq B_2^m \subseteq \ldots \) trivially hold. Furthermore, \( B_0 = B_0^m \subseteq B_1^m \) and \( B_k^m \subseteq B_k \), for every \( k \), are obvious. In [18] Heldt, Knauer and Ueckerdt dealt with the question whether the complete bipartite graph \( K_{m,n} \) on \( m \) and \( n \) vertices in the two partition classes is in \( B_k \). They identified several sufficient conditions which have to be fulfilled by \( m \), \( n \) and \( k \) to guarantee that \( K_{m,n} \) is in \( B_k \) or \( K_{m,n} \) is not in \( B_k \). They used this kind of results to prove that \( B_k \not\subseteq B_{k+1} \) holds for every \( k \geq 0 \). In this paper, we will derive new results of this type, especially for the monotonic case. It is still not known whether \( B_k^m \not\subseteq B_{k+1}^m \) also holds.

The relationship between \( B_k \) and \( B_k^m \) has already been considered in the literature. Golumbic, Lipshteyn and Stern [16] conjectured that \( B_1^m \not\subseteq B_1 \), which was confirmed in [12]. In this paper, we show that \( B_k^m \not\subseteq B_k \) also holds for \( k \in \{2, 3, 5\} \) and \( k \geq 7 \), while the cases \( k = 4 \) and \( k = 6 \) remain open.

Furthermore, we are interested in the gap between the bend number \( b(G) \) and the monotonic bend number \( b^m(G) \) of a graph. More precisely we pose the question whether there exists a function \( f: \mathbb{N} \rightarrow \mathbb{N} \) such that \( b^m(G) \leq f(b(G)) \) holds for every graph \( G \). As a first step towards answering this question we show that \( B_k \not\subseteq B_{2k-4}^m \) holds for any \( k \in \mathbb{N} \), \( k \geq 5 \), which implies the existence of graphs for which \( b^m(G) \geq 2k - 8 \) and \( b(G) \leq k \), for any \( k \in \mathbb{N} \), \( k \geq 5 \). Moreover, we show that \( b(G) \leq 1 \) implies \( b^m(G) \leq 3 \).

The rest of the paper is organized as follows. Section 2 deals with the (monotonic) bend number of \( K_{m,n} \). First we review some results from the literature on the bend number of \( K_{m,n} \), where \( m \leq n \). In particular, we discuss a theorem from [18] and point out that the proof of the theorem does not work out for \( m = 4 \) and \( m = 5 \). Further, we show that the statement of the theorem holds for \( m = 4 \), while we don’t know whether it holds for \( m = 5 \). However, we only exploit the statement of the theorem for \( m \geq 7 \) in our later work. In Section 2.2, we derive two inequalities on \( m \), \( n \) and \( k \) which have to be fulfilled if \( K_{m,n} \) is in \( B_k^m \). In Section 2.3, we show that \( b^m(K_{m,n}) \leq 2m - 2 \) for every \( m, n \in \mathbb{N} \), \( m \leq n \), which implies that \( b^m(G) \leq 2m - 2 \) holds for every graph \( G \) that is an induced subgraph of \( K_{m,n} \). Moreover, we show that this upper bound on \( b^m(K_{m,n}) \) is best possible, i.e. for each \( m \in \mathbb{N} \) there exists an \( n_m \in \mathbb{N} \), \( n_m \geq m \), such that \( b^m(K_{m,n_m}) = 2m - 2 \). An analogous behavior of \( b(K_{m,n}) \) has been already shown in literature (see [18]). However, we will see that this maximum bend number is attained already for smaller values of \( n_m \) in the monotonic case.
In Section 3.1 we present a graph which is in $B_2$ and not in $B_m^2$ in order to prove $B_m^2 \nsubseteq B_k$ for $k = 2$. In Section 3.2 we use the results of Section 2.2 to prove that $B_m^2 \nsubseteq B_k$ also for $k \in \{3, 5\}$ and $k \geq 7$, thus answering an open question posed in [16] for almost all values of $k$.

Finally, in Section 4 we investigate the relationship between $B_k$ and $B_m^\ell$ for $\ell > k$. In Section 4.1 we show that for odd $k \geq 5$ there is a graph in $B_k$ which is not in $B_m^2 k - 8$ and for even $k \geq 5$ there is a graph in $B_k$ which is not in $B_m^{2k-9}$. Then in Section 4.2 we prove that $B_1 \subseteq B_3^m$, giving the first result of this kind. We summarize our results and discuss some open questions in Section 5.

**Terminology and notation.** Finally, we settle the terminology and the notations used throughout the paper. The crossings of two grid lines are called grid points. The part of a grid line between two consecutive grid points is called a grid edge. A grid edge can be horizontal or vertical.

A path on a grid consists of two grid points, called the end points of the path, and a number of consecutive grid edges connecting the end points. If the two end-points lay on different vertical grid lines, we call the left-most point the start point and the other one the terminal point. Otherwise, we call the lower point the start point and the other one the terminal point. A turn of a path on the grid is called bend and a grid point, at which the path turns, is called a bend point.

The part of a path between two consecutive bend points is called a segment. Also the part of the path from the start point to the first bend point is called a segment. This is called the first segment of the path. Analogously, the part of the path from the last bend point to the terminal point is also called a segment. This is the last segment of the path. We consider the intermediate segments in their natural order: the segment of the path following the first one is the second segment, and so on.

The grid points contained in a segment of a path which are neither bend points nor end points of that path build the interior of that segment. Clearly any segment consists either entirely of horizontal grid edges or entirely of vertical grid edges. We call such segments horizontal and vertical segments, respectively. Paths without bends correspond to (horizontal or vertical) segments.

We say that two paths on a grid intersect, if they have at least one common grid edge. If two segments $S_1$, $S_2$ lie on the same grid line but do not intersect (if considered as paths), then we call them aligned; such a pair $(S_1, S_2)$ is called an alignment. Figure 1(a) depicts two aligned segments $S_1$ and $S_2$.

A pair $(S_1, S_2)$ of segments is called a crossing if one of the two segments
lies on a horizontal grid line, the other segment lies on a vertical grid line, and there is no grid point which belongs to the interior of both segments. Figure 1(b) depicts a crossing \((S_1, S_2)\) with grid point \(x\) belonging to the interior of both segments.

A pair \((S_1, S_2)\) of segments is called a pseudocrossing if one of the two segments lies on a horizontal grid line, the other segment lies on a vertical grid line, and there is no grid point which belongs to the interior of each of the segments. Figure 1(c)-(e) depict different pseudocrossings.

Given a set \(P\) of pairwise non-intersecting paths on a grid we define the alignments (crossings, pseudocrossings) of \(P\) as the set of all alignments (crossings, pseudocrossings) \((S_1, S_2)\) for which there exist two distinct paths \(P_1, P_2 \in P\) such that \(S_i\) is a segment of \(P_i\), for \(i \in \{1, 2\}\). Figure 1(f) depicts two paths \(P_1\) and \(P_2\) containing two alignments (a horizontal one and a vertical one) and two pseudocrossings.

![Figure 1](image)

Figure 1: (a) An alignment \((S_1, S_2)\). (b) A crossing \((S_1, S_2)\). (c)-(e) Different pseudocrossings \((S_1, S_2)\). (f) Two paths \(P_1\) and \(P_2\) containing two alignments and two pseudocrossings.

Finally, notice that in an EPG representation of a graph \(G\) with vertex set \(V\) we will denote by \(P_v\) the path on the grid corresponding to the vertex \(v \in V\).

# 2 Complete Bipartite Graphs

The aim of this section is to summarize existing results on the \(B_k\)-EPG representation of complete bipartite graphs and derive new upper and lower bounds on their (monotonic) bend number. We start by investigating some results from the literature in Section 2.1. Then we derive two Lower-Bound-Lemmas in Section 2.2. Eventually, in Section 2.3, we give an upper bound on the monotonic bend number of \(K_{m,n}\) for every \(m, n \in \mathbb{N}, m \leq n\). The results obtained in this section will be used in Section 3.2 where the relationship between \(B_k^n\) and \(B_k\) for \(k \geq 3\) is investigated.
Throughout this section we consider the complete bipartite graph $K_{m,n}$ with $m \leq n$. We denote the two partition classes of $K_{m,n}$ by $A$ and $B$, where $|A| = m$ and $|B| = n$. In an EPG representation we denote the set of all paths that correspond to vertices of $A$ and $B$ by $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively; so $\mathcal{P}_A = \{P_v : v \in A\}$ and $\mathcal{P}_B = \{P_w : w \in B\}$.

2.1 Upper Bounds on the Bend Number

First of all notice that the bend number of $K_{m,n}$ for $m \in \{0, 1, 2\}$ is known. The trivial case $m = 0$ corresponds to a graph without any edges and hence $b(K_{0,n}) = b^m(K_{0,n}) = 0$, for all $n \in \mathbb{N}$.

The other trivial case $m = 1$ corresponds to a star graph with $n + 1$ vertices. A $B_0$-EPG representation of this graph consists of a horizontal path $P$ with $n$ grid edges to represent the central vertex, and the pairwise different grid edges of $P$ represent the other vertices. Thus $b(K_{1,n}) = b^m(K_{1,n}) = 0$, for all $n \in \mathbb{N}$.

The bend number of $K_{2,n}$ has been determined by Asinowski and Suk [2] for all $n \in \mathbb{N}$: $b(K_{2,n}) = 2$ if and only if $n \geq 5$, $b(K_{2,n}) = 1$ if and only if $2 \leq n \leq 4$, and $b(K_{2,n}) = 0$ if and only if $n \leq 1$. The EPG representations for $K_{2,n}$ given in [2] are monotonic, therefore $b^m(K_{2,n}) = b(K_{2,n})$ holds for all $n \in \mathbb{N}$.

The more general case $m \geq 3$ has been considered by Heldt, Knauer and Ueckerdt in [18]. We first discuss the following result of these authors.

**Theorem 2.1** (Heldt, Knauer, Ueckerdt [18]). If $m \geq 4$ is even and $n = \frac{1}{4}m^3 - \frac{1}{2}m^2 - m + 4$, then $K_{m,n}$ is in $B_{m-1}$ but not in $B_{m-2}$. If $m \geq 7$ is odd and $n = \frac{1}{4}m^3 - m^2 + \frac{3}{4}m$, then $K_{m,n}$ is in $B_{m-1}$ but not in $B_{m-2}$.

The above theorem makes no statement for the cases $m = 3$ and $m = 5$. However, in [18] the authors claim that the statement for odd $m$ holds also for $m = 5$ (see [18, Theorem 4.4,]). But the proof provided in [18] is not correct for $m = 5$ and we do not know whether the statement is true in this case. Also for the case $m = 4$ the proof provided in [18] is not correct, however in this case the statement is true as argued below.

To be more precise, in [18] on the one hand the authors provide a $B_{m-1}$-EPG representation for $K_{m,n}$ for $m \geq 3$ and $n$ defined as in [Theorem 2.1] i.e. a constructive proof for one part of [18, Theorem 4.4,]. On the other hand the Lower-Bound-Lemma I [18, Lemma 4.1] is used in order to show that $K_{m,n}$ is not in $B_{m-2}$ for $n$ defined as in [Theorem 2.1]. This Lower-Bound-Lemma I states that

$$(k + 1)(m + n) \geq mn + \sqrt{2k(m + n)}$$

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holds for every $B_k$-EPG representation of $K_{m,n}$ with $n \geq m \geq 3$. Further they observe that for $n$ defined as in Theorem 2.1 the inequality $n \geq (m - 1)^2$ holds, while the inequality of the Lower-Bound-Lemma I is not fulfilled for $n \geq (m - 1)^2$ and $k = m - 2$, thus negating the membership of the corresponding graphs in $B_{m-2}$. However, for $n$ defined as in Theorem 2.1 the inequality $n \geq (m - 1)^2$ holds only if $m \geq 6$. Thus, the proof provided for [18, Theorem 4.4] only works for $m \geq 6$.

For $m = 4$ we have $n = 8$, and the construction in [18] proves that $K_{4,8}$ is in $B_3$. Furthermore, by applying the Lower-Bound-Lemma I for $m = 4$, $n = 6$ and $k = 2$ we get that $K_{4,6}$ is not in $B_2$. This implies that also $K_{4,8}$ is not in $B_2$. Therefore, the statement of Theorem 2.1 is also true for $m = 4$.

If $m = 5$ the construction in [18] yields that $K_{5,10}$ is in $B_4$. If we use the Lower-Bound-Lemma I, then we get that $K_{5,11}$ is not in $B_3$ and that the bend number of $K_{5,10}$ is at least 3. Therefore, the bend number of $K_{5,10}$ could be either 3 or 4.

2.2 Lower-Bound-Lemmas

In order to investigate the relationship between $B_k^m$ and $B_k$ for large values of $k$, we first derive a Lower-Bound-Lemma for $B_k^m$-EPG representations similarly to the Lower-Bound-Lemma I for $B_k$-EPG representations from [18]. To this end, we use an auxiliary result from [18, Lemma 4.6].

**Lemma 2.2** (Heldt, Knauer, Ueckerdt [18]). Let $3 \leq m \leq n$. Consider $K_{m,n}$ and denote by $A$ the subset of vertices of cardinality $m$ in the partition of the vertex set of $K_{m,n}$. Consider further a $B_k$-EPG representation of $K_{m,n}$ and denote by $P_A$ be the set of the paths on the grid corresponding to the vertices of $A$ in this representation. Let $c$ be the total number of crossings of $P_A$. Then, the following inequality holds:

$$n(2m - k - 2) \leq 2c + 2(k + 1)m.$$ 

In the following we derive inequalities on $m$, $n$ and $k$ which hold whenever a $K_{m,n}$ is in $B_k^m$. The following lemma is a first step towards such a result. Note that $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$ and $\lceil x \rceil$ is the least integer greater than or equal to $x$ for any real number $x$.

**Lemma 2.3.** Let $3 \leq m \leq n$. Consider $K_{m,n}$ and denote by $A$ the subset of vertices of cardinality $m$ in the partition of the vertex set of $K_{m,n}$. Consider further a $B_k$-EPG representation of $K_{m,n}$ and denote by $P_A$ be the set of the paths on the grid corresponding to the vertices of $A$ in this representation. Let
a, c and p be the total number of alignments, crossing and pseudocrossings of \( P_A \), respectively. Then, the following inequality holds:

\[
\begin{align*}
    n \left( m - \left\lceil \frac{k + 1}{2} \right\rceil \right) & \leq a + 2c + p.
\end{align*}
\]

**Proof.** Let \( w \) be a vertex of \( B \). For each vertex \( v \in A \) we denote by \( e^w_v \) a fixed but arbitrarily chosen common grid edge of \( P_v \) and \( P_w \). Such an edge exists, because \( P_w \) intersects \( P_v \) since \( w \) is adjacent to all vertices of \( A \). The grid edges \( e^w_v \) for all \( v \in A \) are pairwise disjoint, because the vertices of \( A \) are not adjacent to each other.

We order the vertices \( A = \{ v_1, \ldots, v_m \} \) in such a way that \( e^w_{v_i} \) precedes \( e^w_{v_{i+1}} \) in the path \( P_w \), for all \( i \in \{ 1, 2, \ldots, m - 1 \} \). Let \( x_w, y_w \) and \( z_w \) be the number of indices \( i \in \{ 1, \ldots, m - 1 \} \) such that \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie on the same segment of \( P_w \), on consecutive segments of \( P_w \), and neither on the same nor on consecutive segments of \( P_w \), respectively. Then, clearly \( x_w + y_w + z_w = m - 1 \) holds.

If \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie neither on the same nor on consecutive segments of \( P_w \), then the subpath of \( P_w \) between (and not including) the two segments of \( P_w \) containing \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) contains at least one segment and does not contain any \( e^w_{v_{i'}} \) for \( i' \in \{ 1, \ldots, m \} \). Let us call such a subpath a free subpath of \( P_w \). Since \( P_w \) has at most \( k + 1 \) segments and each free subpath is preceded and also succeeded by a segment containing \( e^w_{v_i} \) for some \( i' \in \{ 1, \ldots, m \} \), the number of free subpaths is at most \( \left\lfloor \frac{k}{2} \right\rfloor \) and hence \( z_w \leq \left\lfloor \frac{k}{2} \right\rfloor \) holds. To summarize up to now we have shown that

\[
\begin{align*}
    m - \left\lceil \frac{k + 1}{2} \right\rceil = m - 1 - \left\lfloor \frac{k}{2} \right\rfloor \leq m - 1 - z_w = x_w + y_w
\end{align*}
\]

(1)

holds.

It remains to determine an upper bound on \( x_w + y_w \). Towards this end, let \( S^w_i \) be the segment of \( P_{v_i} \) that contains \( e^w_{v_i} \) for \( i \in \{ 1, 2, \ldots, m \} \). Now we consider the pairs \( (S^w_i, S^w_{i+1}) \), \( i \in \{ 1, 2, \ldots, m - 1 \} \).

We denote by \( a_w \) the number of indices \( i \in \{ 1, \ldots, m - 1 \} \) such that \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie on the same segment of \( P_w \) and the pair \( (S^w_i, S^w_{i+1}) \) is an alignment. It is easy to see that if \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie on the same segment of \( P_w \), then the corresponding segments \( S^w_i \) and \( S^w_{i+1} \) of \( P_{v_i} \) and \( P_{v_{i+1}} \) lie on the same grid line and therefore \( (S^w_i, S^w_{i+1}) \) is an alignment. Thus, \( a_w = x_w \) holds.

Furthermore, let \( c_w (P_w) \) denote the number of \( i \in \{ 1, \ldots, m - 1 \} \) such that \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie on consecutive segments of \( P_w \) and the pair \( (S^w_i, S^w_{i+1}) \) is a crossing (pseudocrossing). If \( e^w_{v_i} \) and \( e^w_{v_{i+1}} \) lie on consecutive segments of
then one of the corresponding segments $S_i^w$ and $S_{i+1}^w$ is horizontal and the other one is vertical. Hence $(S_i^w, S_{i+1}^w)$ is either a crossing or a pseudocrossing. Therefore, $c_w + p_w = y_w$ holds.

As a result, we can use (1) to deduce that

$$m - \left\lceil \frac{k + 1}{2} \right\rceil \leq x_w + y_w = a_w + c_w + p_w$$

holds. Summing this up over all vertices $w \in B$ yields

$$n \left( m - \left\lceil \frac{k + 1}{2} \right\rceil \right) \leq \sum_{w \in B} (a_w + c_w + p_w).$$

It remains to determine an upper bound on $\sum_{w \in B} (a_w + c_w + p_w)$. Towards this end, let $a_B = \sum_{w \in B} a_w$, $c_B = \sum_{w \in B} c_w$ and $p_B = \sum_{w \in B} p_w$. Clearly, an alignment (crossing, pseudocrossing) $(S_i^w, S_{i+1}^w)$, for $w \in B$ and for $i \in \{1, 2, \ldots, m - 1\}$, is an alignment (crossing, pseudocrossing) of $\mathcal{P}_A$, since $S_i^w$ is a segment of $P_v$, for $i \in \{1, 2, \ldots, m\}$. This implies that $a_B \leq a$ and $p_B \leq p$ because the alignments and pseudocrossings counted in $a_B$ and $p_B$ are pairwise distinct due to the fact that the paths in $\mathcal{P}_A$ are pairwise non-intersecting and also the paths in $\mathcal{P}_B$ are pairwise non-intersecting.

The crossings counted in $c_B$ are not necessarily pairwise distinct because a crossing $(S_i^w, S_{i+1}^w)$ can also appear as a crossing $(S_i^{w'}, S_{i+1}^{w'})$, for some $w, w' \in B$, $w \neq w'$ and some $i, j \in \{1, 2, \ldots, m - 1\}$, see Figure 2. (Notice that in this case the vertices $v_i$ and $v_j$ coincide.) However, the same crossing cannot be counted more than twice in $c_B$ because the paths in $\mathcal{P}_B$ are pairwise non-intersecting, so $c_B \leq 2c$ holds.

![Figure 2: The crossings $(S_i^w, S_{i+1}^w)$ and $(S_j^{w'}, S_{j+1}^{w'})$ coincide.](image)

Finally, we can deduce

$$n \left( m - \left\lceil \frac{k + 1}{2} \right\rceil \right) \leq \sum_{w \in B} (a_w + c_w + p_w) = a_B + c_B + p_B \leq a + 2c + p.$$
The next lemma gives bounds on the number of alignments, crossings and pseudocrossings.

**Lemma 2.4.** Consider two paths $P_1$, $P_2$ in a $B_k$-EPG representation that do not intersect (i.e. have no grid edge in common). Let $a$, $c$ and $p$ be the number of alignments, crossings and pseudocrossings of $\{P_1, P_2\}$, respectively. If one path starts horizontally and the other one starts vertically, then

(a) $c + p \leq 2 \left\lfloor \frac{k + 1}{2} \right\rfloor \left\lceil \frac{k + 1}{2} \right\rceil + \left\lceil \frac{k + 1}{2} \right\rceil - \left\lfloor \frac{k + 1}{2} \right\rfloor$ and

(b) if the paths are monotonic $a + c \leq k + 1$ holds.

If both paths start horizontally or both paths start vertically, then

(c) $c + p \leq 2 \left\lfloor \frac{k + 1}{2} \right\rceil \left\lfloor \frac{k + 1}{2} \right\rceil$ and

(d) if the paths are monotonic $a + c \leq k$ holds.

**Proof.** First we consider (a) and (c). In a crossing or a pseudocrossing $(S_1, S_2)$ of $\{P_1, P_2\}$ one of the segments is horizontal and the other one is vertical. Notice that a path that starts with a horizontal segment has at most $\left\lceil \frac{k + 1}{2} \right\rceil$ horizontal and at most $\left\lfloor \frac{k + 1}{2} \right\rfloor$ vertical segments, whereas a path that starts with a vertical segment has at most $\left\lfloor \frac{k + 1}{2} \right\rfloor$ horizontal and at most $\left\rceil \frac{k + 1}{2} \right\rceil$ vertical segments. If one of the paths starts horizontally and the other path starts vertically this implies that

\[
c + p \leq 2 \left\lfloor \frac{k + 1}{2} \right\rceil \left\lfloor \frac{k + 1}{2} \right\rceil + \left\lceil \frac{k + 1}{2} \right\rceil - \left\lfloor \frac{k + 1}{2} \right\rfloor = 2 \left\lfloor \frac{k + 1}{2} \right\rceil \left\lfloor \frac{k + 1}{2} \right\rceil + \left( \left\lceil \frac{k + 1}{2} \right\rceil - \left\lfloor \frac{k + 1}{2} \right\rfloor \right)^2,
\]

where we can omit the square because the squared value is either 0 or 1, and hence (a) holds. With the same arguments we obtain

\[
c + p \leq 2 \left\lfloor \frac{k + 1}{2} \right\rceil \left\lfloor \frac{k + 1}{2} \right\rceil
\]

for paths that start in the same direction. Thus (c) is satisfied.

Next we consider (b) so assume the paths are monotonic. It is easy to see that each segment of $P_1$ cannot cross two or more segments of $P_2$ and
cannot be aligned with two or more segments of $P_2$. Furthermore, whenever a segment of $P_1$ crosses a segment of $P_2$, it cannot be aligned with another segment of $P_2$. Moreover, whenever a segment of $P_1$ is aligned with a segment of $P_2$, it cannot cross another segment of $P_2$. Hence each segment of $P_1$ can be part of at most one crossing or alignment. This implies (b) as $P_1$ has at most $k + 1$ segments.

In order to prove (d) assume without loss of generality that both paths start horizontally. The arguments of (b) imply that each segment of each of the paths can appear in at most one crossing or one alignment. We distinguish two cases. If one of the paths starts in a lower grid line than the other, then the first segment of this path can neither be aligned to nor cross the other path. Therefore, alignments and crossings can only occur on the remaining $k$ segments of the path and hence $a + c \leq k$ holds. If both paths start on the same grid line, then let without loss of generality the first segment of $P_1$ lie to the left of the first segment of $P_2$. It is easy to see that the second segment of $P_1$ can neither be aligned to nor cross any segment of $P_2$. Therefore, also in this case we have $a + c \leq k$. This proves (d).

Next we combine the bounds on the number of crossings derived in Lemma 2.4 with Lemma 2.2 in the following result.

**Lemma 2.5.** Let $3 \leq m \leq n$. In every $B^m_k$-EPG representation of $K_{m,n}$

$$n(2m - k - 2) \leq k(m - 1)m + \frac{1}{2}m^2 + 2(k + 1)m$$

holds.

**Proof.** Let $c$ denote the number of crossings of the paths in $P_A$. Every $B^m_k$-EPG representation is also a $B_k^*$-EPG representation. Therefore, it follows from Lemma 2.2 that

$$n(2m - k - 2) \leq 2c + 2(k + 1)m$$

holds for every $B^m_k$-EPG representation of $K_{m,n}$. Now we give an upper bound on $c$. Let $\ell$ be the number of paths in $P_A$ which start with a horizontal segment. Then, $m - \ell$ paths of $P_A$ start with a vertical segment. Since the paths in $P_A$ are pairwise non-intersecting, the number $c$ of crossings of $P_A$ can be calculated as $c = \sum_{\{v,v'\} \subseteq A} c_{v,v'}$, where $c_{v,v'}$ is the number of crossings of $\{P_v, P_{v'}\}$.

If both $P_v$ and $P_{v'}$ start with a horizontal (vertical) segment, then $c_{v,v'} \leq k$ by Lemma 2.4(d). If one of the paths $P_v$ and $P_{v'}$ starts with a horizontal segment and the other one starts with a vertical segment, then $c_{v,v'} \leq k + 1$.
by Lemma 2.4(b). Notice that there are exactly \( \ell(m - \ell) \) pairs of paths \( P_v \) and \( P_{v'} \) with the latter property and \( \binom{m}{2} - \ell(m - \ell) \) pairs of paths \( P_v \) and \( P_{v'} \) both starting with a horizontal (vertical) segment. In total we get

\[
c = \sum_{\{v, v'\} \subseteq A} c_{v,v'} \leq k \binom{m}{2} + \ell(m - \ell).
\]

Since \( \ell(m - \ell) \leq \left( \frac{m}{2} \right)^2 \) for all \( 0 \leq \ell \leq m \) we get

\[
c \leq k \binom{m}{2} + \frac{m^2}{4} = \frac{1}{2} \left( k(m - 1)m + \frac{1}{2}m^2 \right),
\]

which in combination with (2) completes the proof.

Next we combine the bounds on the number of crossings derived in Lemma 2.4 and Lemma 2.3 as follows.

**Lemma 2.6.** Let \( 3 \leq m \leq n \). In every \( B^m_k \)-EPG representation of \( K_{m,n} \)

\[
n \left( m - \left\lfloor \frac{k + 1}{2} \right\rfloor \right) \leq \binom{m}{2} \left[ \frac{k + 1}{2} \right] \left[ \frac{k + 1}{2} \right] + \frac{1}{4} m^2 \left( 1 + \left\lfloor \frac{k + 1}{2} \right\rfloor - \left\lfloor \frac{k + 1}{2} \right\rfloor \right)
\]

holds.

**Proof.** We combine Lemma 2.3 and Lemma 2.4 by proceeding analogously as in the proof of Lemma 2.5.

In particular, let \( a, c \) and \( p \) be the number of alignments, crossings and pseudocrossings of \( P_A \), respectively. As done in the proof of Lemma 2.5 we can compute \( c \) as the sum of the number of crossings \( c_{v,v'} \) of \( \{P_v, P_{v'}\} \) over all pairs \( \{v, v'\} \subseteq A \). Similarly we write \( p \) and \( a \) as the sum of the number of pseudocrossings \( p_{v,v'} \) (alignments \( a_{v,v'} \)) of \( \{P_v, P_{v'}\} \) over all pairs \( \{v, v'\} \subseteq A \). Thus, we obtain

\[
a + 2c + p = \sum_{\{v, v'\} \subseteq A} a_{v,v'} + \sum_{\{v, v'\} \subseteq A} c_{v,v'} \leq \sum_{\{v, v'\} \subseteq A} (a_{v,v'} + c_{v,v'}) + \sum_{\{v, v'\} \subseteq A} (c_{v,v'} + p_{v,v'}). \]

Then, we use Lemma 2.4(b) and (d) to bound each summand of the first sum from above and Lemma 2.4(a) and (c) to bound each summand of the second sum from above. Then we transform the sum of these upper bounds analogously as in the proof of Lemma 2.5 and finally use Lemma 2.3 to bound \( a + 2c + p \) from below. This completes the proof.

To summarize Lemma 2.5 and Lemma 2.6 provide inequalities on \( m, n \) and \( k \) which hold whenever a \( K_{m,n} \) with \( 3 \leq m \leq n \) is in \( B^m_k \). These inequalities are used in Sections 2.3 and 3.2.
2.3 Upper Bounds on the Monotonic Bend Number

In [18] a lot of work has been done to determine the bend number of $K_{m,n}$ in dependence of $m$ and $n$. In particular, it was proven that $b(K_{m,n}) = 2m - 2$ for $m \geq 3$ and $n \geq m^4 - 2m^3 + 5m^2 - 4m + 1$. We deduce a similar result for the monotonic case.

We first generalize a result of [4]. There it was shown by slightly modifying a construction of [16] that $K_{m,n} \in B_{2m-2}$ for all $n$. We modify the construction of [4] and give an analogous result for the monotonic case.

**Theorem 2.7.** It holds that $K_{m,n} \in B_{2m-2}$.

**Proof.** In order to prove this, it is enough to give a $B_{2m-2}$-EPG representation of $K_{m,n}$, which can be found in Figure 3. Each vertex of $K_{m,n}$ belonging to the partition class $A$ of size $m$ is represented in the grid by a path consisting of just one horizontal segment. Each of the $n$ vertices of the other partition class $B$ is represented in the grid by a staircase with $2m - 2$ bends. The staircases have pairwise empty intersections.

![Figure 3: A $B_{2m-2}$-EPG representation of $K_{m,n}$.](image)

Note that **Theorem 2.7** implies that $b^m(G) \leq 2m - 2$ holds for every graph $G$ that is an induced subgraph of $K_{m,n}$. Furthermore, **Theorem 2.7** shows that for fixed $m$ and varying $n$, $b(K_{m,n}) \leq b^m(K_{m,n}) \leq 2m - 2$ holds. Hence, the upper bound on the number of bends needed for an EPG representation of $K_{m,n}$ with $3 \leq m \leq n$ is the same, namely $2m - 2$, no matter whether all kind of bends or only monotonic bends are allowed. This fact is even more surprising if we take into account **Theorem 4.1** which states the existence of graphs for which the gap between the bend number and the monotonic bend number can be arbitrarily large.
However, it turns out that the upper bound on $b^m(K_{m,n})$ is already reached for a smaller $n$ than the upper bound on $b(K_{m,n})$. In particular, the above stated result from [18] implies that $b(K_{m,n}) = 2m - 2$ for $n \geq N_1$ for some $N_1 \in \Theta(m^4)$. As a consequence of the next result it follows that $b^m(K_{m,n}) = 2m - 2$ for $n \geq N_2$ already for some $N_2 \in \Theta(m^3)$.

**Theorem 2.8.** Let $3 \leq m$. If $n \geq 2m^3 - \frac{1}{2}m^2 - m + 1$ then $K_{m,n} \not\in B_{2m-3}$.

**Proof.** Suppose, in order to derive a contradiction, that $K_{m,n} \in B_{2m-3}$. By applying Lemma 2.5 for $k = 2m - 3$ we get that

$$n(2m - (2m - 3) - 2) \leq (2m-3)(m-1)m + \frac{1}{2}m^2 + 2(2m-2)m$$

Then, doing the maths operations we have that $n \leq 2m^3 - \frac{1}{2}m^2 - m$ has to hold. This contradicts $n \geq 2m^3 - \frac{1}{2}m^2 - m + 1$.

### 3 Relationship between $B_k^m$ and $B_k$

It is an open question of [16] to determine the relationship between $B_k^m$ and $B_k$ for $k \geq 1$. Obviously $B_k^m \subseteq B_k$ holds for every $k$. In [16] Golumbic, Lipshteyn and Stern conjectured that $B_1^m \not\subseteq B_1$. This conjecture was confirmed by Cameron, Chaplick and Hoàng in [9] by showing that the graph $S_3$, which was known to be in $B_1$ from [16], is not in $B_1^m$. The graph $S_3$ is isomorphic to the subgraph induced by the vertices \{a, b, c, d, e, f\} in the graph represented in Figure 8(a).

In this section we consider the question whether $B_k^m \not\subseteq B_k$ holds also for $k \geq 2$. We first consider the case $k = 2$ in Section 3.1 and then the remaining cases $k \geq 3$ in Section 3.2. The case distinction is due to the different methods used in the investigations.

#### 3.1 Relationship between $B_2^m$ and $B_2$

The aim of this section is to prove that $B_2^m \not\subseteq B_2$ holds. For this purpose we show that the graph $H_1$ represented in Figure 4 is in $B_2$ but not in $B_2^m$. $H_1$ is defined as follows.

**Definition 3.1.** The graph $H_1$ depicted in Figure 4 is constructed in the following way. The vertices \{u, v\} and \{a_1, \ldots, a_{50}\} form a $K_{2,50}$. Furthermore, for every $1 \leq j < 50$ the vertices \{a_j, a_{j+1}\} and \{b_1, \ldots, b_{50,j}\} form a $K_{2,50}$. Additional to that for every $1 \leq j < 50$ and for every $1 \leq i < 50$ there is the graph $H_2$ of Figure 4(b) placed between the vertices $b_{i,j}$ and $b_{i+1,j}$.
The next result follows from a proof of Heldt, Knauer and Ueckerdt given in [17]. In Proposition 1 of [17] they use a similar construction in order to prove that there is a planar graph with treewidth at most 3 which is not in $B_2$. Their construction builds also on the graph $H_1$ (called $G$ in their paper) but the graph suspended between any two vertices $b_{i,j}, b_{i+1,j}$, for $1 \leq i, j < 50$, (called $H$ in their paper) is a 29-vertex graph different from $H_2$. In the first part of the proof of Proposition 1 Heldt, Knauer and Ueckerdt prove some properties of $B_2$-EPG representations of the subgraph of $H_1$ as summarized in the following lemma.

**Lemma 3.2** (Heldt, Knauer, Ueckerdt [17]). *In any $B_2$-EPG representation of the graph $H_1$ depicted in Figure 4 there exist two indices $i$ and $j$, $1 \leq i, j \leq 49$, with the following properties:

(a) the paths $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$ consist of three segments each,
(b) there is a segment $S_j$ of the path $P_{a_j}$ which completely contains one end segment of $P_{b_{i,j}}$ and one end segment of $P_{b_{i+1,j}}$,
(c) there is a segment $S_{j+1}$ of the path $P_{a_{j+1}}$ which completely contains the other end segments of $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$,
(d) $S_j$ and $S_{j+1}$ are either both vertical segments or both horizontal segments.

With this auxiliary result we are able to prove the following lemma.

**Lemma 3.3.** The graph $H_1$ is not in $B_2^m$.

**Proof.** Suppose, in order to derive a contradiction, that $H_1$ is in $B_2^m$. Every $B_2^m$-EPG representation is a $B_2$-EPG representation as well, therefore [Lemma 3.2] holds also for any $B_2^m$-EPG representation of $H_1$. Assume without loss of generality that the center segment (i.e. the second segment) of $P_{b_{i,j}}$ is a horizontal segment, that it is above the center segment of $P_{b_{i+1,j}}$, and that the segment $S_j$ of $P_{a_j}$ is on the left side of the segment $S_{j+1}$ of $P_{a_j+1}$. Then the positioning of the segments of the paths has to look like in Figure 5.

![Figure 5: A part of the hypothetical $B_2^m$-EPG representation of $H_1$.](image)

Each vertex $c_\ell$, $1 \leq \ell \leq 6$, of the copy of $H_2$ between $b_{i,j}$ and $b_{i+1,j}$ is adjacent to both $b_{i,j}$ and $b_{i+1,j}$, but neither to $a_j$ nor to $a_{j+1}$. Therefore, each of the six paths $P_{c_1}, \ldots, P_{c_6}$ has to share a grid edge with the center segments of both $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$. As a result, $P_{c_6}$ starts with a first horizontal segment intersecting the center segment of $P_{b_{i+1,j}}$, continues with a second vertical segment and ends with a third horizontal segment intersecting the center segment of $P_{b_{i,j}}$, for every for $1 \leq i \leq 6$.

Now consider the vertices $c_1$, $c_3$ and $c_5$. They are pairwise nonadjacent, so $P_{c_1}, P_{c_3}, P_{c_5}$ are non-intersecting. Therefore, the three vertical segments of these paths are disjoint and can be ordered from the left to the right. Let $P_L$, $P_M$ and $P_R$ be the path in $\{P_{c_1}, P_{c_3}, P_{c_5}\}$ with the left-most, the middle and the right-most center segment, respectively. In the following we say that a path $P_{c_i}$ lies to the left of, to the right of and on another path $P_{c_j}$ if the center segment of $P_{c_i}$ lies to the left of, to the right of and on the center segment of $P_{c_j}$ for some $1 \leq i \neq j \leq 6$, respectively.

Next take a closer look at the paths $P_{c_4}$ and $P_{c_6}$. Each of them intersects each of the three paths $P_L$, $P_M$ and $P_R$, since both vertices $c_4$, $c_6$ are adjacent
to each of $c_1$, $c_3$ and $c_5$. Since $c_4$ and $c_6$ are not adjacent to each other, $P_{c_4}$ and $P_{c_6}$ do not intersect and hence the vertical segments of $P_{c_4}$ and $P_{c_6}$ are disjoint. Assume without loss of generality that $P_{c_4}$ is to the left of $P_{c_6}$.

If $P_{c_4}$ lies to the right of or on $P_L$, then $P_{c_6}$ cannot intersect $P_L$ on the first or second segment of $P_L$, because $P_{c_6}$ is to the right of $P_{c_4}$ and does not intersect $P_{c_4}$. Therefore, $P_{c_6}$ intersects $P_L$ on its third segment. This implies that $P_{c_6}$ lies to the left of or on $P_M$. But $P_{c_4}$ is to the left of $P_{c_6}$, which is to the left of or on $P_M$. Thus, no point of $P_{c_4}$ can lie to the right of the center segment of $P_M$, which implies that $P_{c_4}$ does not intersect $P_R$, a contradiction. Analogously, it follows that $P_{c_6}$ cannot lie to the left of or on $P_R$.

As a result $P_{c_4}$ lies to the left of $P_L$ and $P_{c_6}$ lies to the right of $P_R$. $P_{c_4}$ has to intersect $P_R$, so the third segment of $P_L$ and $P_M$ are completely contained in the third segment of $P_{c_4}$. Similarly $P_{c_6}$ has to intersect $P_L$, so the first segment of $P_M$ and $P_R$ are completely contained in the first segment of $P_{c_6}$.

For an illustration of this configuration see Figure 6.

![Figure 6: The only possible placement of paths $P_{c_4}$, $P_{c_6}$ and $\{P_L, P_M, P_R\} = \{P_{c_1}, P_{c_3}, P_{c_5}\}$ in the hypothetical $B_2^m$-EPG representation of $H_1$.](image)

Now consider the path $P_{c_2}$. Note that if $P_{c_2}$ intersects the middle segment of $P_M$, then $P_{c_2}$ is on $P_M$, in which case $P_{c_2}$ also intersects the first and third segments of $P_M$. So we can conclude that if $P_{c_2}$ intersects $P_M$, then $P_{c_2}$ intersects either the first segment of $P_M$ or the third segment of $P_M$. But the former is completely contained in $P_{c_6}$ and the latter is completely contained in $P_{c_4}$, which means that $P_{c_2}$ intersects either $P_{c_4}$ or $P_{c_6}$, which is a contradiction to the fact that $c_2$ is nonadjacent to $c_4$ and $c_6$. We can therefore conclude that $P_{c_2}$ does not intersect $P_M$. Since $c_2$ is adjacent to both $c_1$ and $c_3$, this means that $P_M = P_{c_5}$ and $\{P_L, P_R\} = \{P_{c_1}, P_{c_3}\}$. But now it is not possible for $P_{c_2}$ to intersect both $P_L$ and $P_R$ without intersecting $P_M$. Hence, $H_1$ cannot have a $B_2^m$-EPG representation.

After proving that $H_1$ is not in $B_2^m$, we observe that $H_1$ is in $B_2$ and obtain the following theorem.

**Theorem 3.4.** It holds that $B_2^m \subset \not= B_2$. 

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Proof. The fact that $B_2^m \subseteq B_2$ follows by definition. In order to see that strict inclusion holds, we consider the graph $H_1$ depicted in Figure 4.

![Figure 4](image1)

Figure 4: A $B_2$-EPG representation of the graph $H_1$ of Figure 4. Every gray area represents the $B_2$-EPG representation of $H_2$ depicted in (b).

We have already seen in Lemma 3.3 that the graph $H_1$ is not in $B_2^m$. So it is enough to show that $H_1$ is in $B_2$. To this end, consider a $B_2$-EPG representation of $H_1$ given in Figure 7.

![Figure 7](image2)

Figure 7: (a) A $B_2$-EPG representation of the graph $H_1$ of Figure 4. Every gray area represents the $B_2$-EPG representation of $H_2$ depicted in (b).

Summarizing $B_m \subseteq B_k$ holds for $k = 1$ as shown in [9] and also for $k = 2$ as shown in this paper.

3.2 Relationship between $B^m_k$ and $B_k$ for $k \in \{3, 5\}$ and $k \geq 7$

In this section we use the results from Section 2 in order to investigate the relationship between $B^m_k$ and $B_k$ for $k \in \{3, 5\}$ and $k \geq 7$.

We start with $k = 3$ and prove that $B_3^m \subseteq B_3$ holds. To this end we use a result of [18] to show that a particular graph is in $B_3$, and then use results of Section 2 to prove that this graph is not in $B_3^m$. 

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Lemma 3.5. It holds that $B^m_3 \subsetneq B_3$.

Proof. Since $B^m_3 \subseteq B_3$ obviously holds, it is enough to show that $B^m_3 \subsetneq B_3$. Heldt, Knauer, Ueckerdt [18] showed that $b(K_{3,36}) = 3$, hence $K_{3,36}$ belongs to $B_3$. Now assume that $K_{3,36}$ is in $B^m_3$. Then by Lemma 2.6 we have

$$36 \left( 3 - \left\lfloor \frac{4}{2} \right\rfloor \right) \leq 3 \left( 2 \left\lfloor \frac{4}{2} \right\rfloor \left\lceil \frac{4}{2} \right\rceil + 3 \right) + \frac{1}{4} \cdot 3^2,$$

That is, $36 \leq 35.25$, a contradiction. Hence, $K_{3,36}$ is not in $B^m_3$. \hfill \square

Now we know that $B^m_k \subsetneq B_k$ holds for $k \leq 3$. Next we show $B^m_5 \subsetneq B_5$. Similarly as in the case of $k = 3$ we use a result of [18] to show that a particular graph is in $B_5$ and then use results of Section 2 to prove that this graph is not in $B^m_5$. 

Lemma 3.6. It holds that $B^m_5 \subsetneq B_5$.

Proof. Since $B^m_5 \subseteq B_5$ obviously holds, it is enough to show $B^m_5 \neq B_5$.

Heldt, Knauer, Ueckerdt [18] showed that $K_{m,n} \in B_{2m-3}$ if $n \leq m^4 - 2m^3 + \frac{5}{2}m^2 - 2m - 4$ (see Theorem 4.5 in [18]). For $m = 4$ this implies that $K_{4,156} \in B_5$. Assume that $K_{4,156} \in B^m_5$. Then, by Lemma 2.5 we get

$$156(2 \cdot 4 - 5 - 2) \leq 5 \cdot 3 \cdot 4 + \frac{1}{2} \cdot 4^2 + 2 \cdot 6 \cdot 4$$

That is, $156 \leq 116$, a contradiction. So $K_{4,156} \notin B^m_5$ but $K_{4,156} \in B_5$, hence $B^m_5 \neq B_5$. \hfill \square

Finally we show $B^m_k \subsetneq B_k$ for $k \geq 7$. To this end, we use Lemma 2.5 and Theorem 2.1.

Lemma 3.7. It holds that $B^m_k \subsetneq B_k$ for $k \geq 7$.

Proof. We first prove the statement for odd $k$. Theorem 2.1 implies that $K_{k+1, \frac{1}{4}(k+1)^3 - \frac{1}{4}(k+1)^2 - (k+1) + 4} = K_{k+1, \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}} \in B_k$ for $k \geq 3$. Suppose, in order to derive a contradiction, that this graph is in $B^m_k$. Then, by Lemma 2.5 with $m = k + 1$ and $n = \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}$ it follows that

$$\left( \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4} \right) (2(k+1) - k - 2) \leq k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2$$

\[ \Leftrightarrow \]

$$k \left( \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4} \right) \leq k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2}$$

\[ \Leftrightarrow \]

$$k^2 - 3k^3 - 19k^2 - 9k - 10 \leq 0,$$
which is a contradiction for \( k \geq 7 \). Hence, for odd \( k \geq 7 \) there is a graph in \( B_k \) which is not in \( B^m_k \) and therefore \( B^m_k \not\subseteq B_k \) holds for odd \( k \geq 7 \).

Now consider the complementary case of even \( k \). Theorem 2.1 implies that the graph \( K_{k+1,\frac{1}{4}(k+1)^3-(k+1)^2+\frac{1}{4}(k+1)} = K_{k+1,\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{4}k} \in B_k \) for \( k \geq 6 \). Suppose, in order to derive a contradiction, that this graph is in \( B^m_k \). Then, by Lemma 2.5 with \( m = k + 1 \) and \( n = \frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{4}k \) we get

\[
\left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) (2(k+1) - k - 2) \leq k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2 
\]

\[
\iff k \left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) \leq k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2}
\]

\[
\iff k^4 - 5k^3 - 16k^2 - 20k - 10 \leq 0,
\]

which is a contradiction for \( k \geq 8 \). Hence, for even \( k \geq 8 \) there is a graph in \( B_k \) which is not in \( B^m_k \). Therefore \( B^m_k \not\subseteq B_k \) for even \( k \geq 8 \) and this completes the proof.

Lemma 3.5, Lemma 3.6 and Lemma 3.7 imply the following theorem.

**Theorem 3.8.** It holds that \( B^m_k \not\subseteq B_k \) for \( k = 3, k = 5 \) and \( k \geq 7 \).

Summarizing, in Theorem 3.4 and Theorem 3.8 we have shown that \( B^m_k \not\subseteq B_k \) for \( k \in \{2, 3, 5\} \) and for \( k \geq 7 \), addressing herewith a question raised in [16]. Recall that \( B^m_1 \not\subseteq B_1 \) was already shown in [9]. Thus, the only open cases are \( k = 4 \) and \( k = 6 \). We conjecture that \( B^m_k \not\subseteq B_k \) holds also for these two remaining cases.

**Conjecture 3.9.** \( B^m_k \not\subseteq B_k \) holds also for \( k = 4 \) and \( k = 6 \).

However, this remains an open question.

## 4 Relationship between \( B_k \) and \( B^m_\ell \) for \( \ell > k \)

Recall that the inclusions chains \( B_i \subseteq B_{i+1} \) and \( B^m_i \subseteq B^m_{i+1} \) trivially hold for all \( i \in \mathbb{N} \). In other words the size of the classes of graphs that have a (monotonic) \( k \)-bend EPG representation increase with increasing \( k \). Also the relationships \( B_0 = B^m_0 \) and \( B_0 \subseteq B^m_1 \) are trivial. Moreover, \( B^m_k \not\subseteq B_k \) holds for almost all \( k \in \mathbb{N} \), as shown in Section 3.

This means that in general the minimum number of bends needed for an EPG representation of a graph increases when the representing paths on the grid are required to be monotonic. Analogously, in general the minimum
number of bends needed for an EPG representation of a graph decreases as compared to the minimum number of bends needed in a monotonic EPG representation. Quantifying the magnitude of such an increase (decrease) arises as a natural question in this context. More generally, it would be interesting to investigate the existence of non-trivial functions \( f, g : \mathbb{N} \to \mathbb{N} \) such that \( b^m(G) \leq f(b(G)) \) and \( b(G) \leq g(b^m(G)) \) holds for all graphs \( G \), or only for all \( G \) belonging to some particular class of graphs.

To the best of our knowledge questions of this kind have not been addressed in the literature so far. In this section we present some related results. In particular, in Section 4.1 we show that the increase (decrease) of the number of bends as mentioned above cannot be bounded by one, in general. More precisely, by combining the results of Theorem 4.1 and Theorem 2.7 with some result known in the literature we show that none of the inclusions \( B_k \subseteq B_{m}^{n} \) holds, a result not known so far in the literature. Then in Section 4.2 we show that \( B_1 \subseteq B_3^{m} \) holds.

4.1 Relationship between \( B_k \) and \( B_{2k-9}^{m} \)

**Theorem 4.1.** Let \( k \geq 5 \). If \( k \) is odd, then there is a graph which is in \( B_k \) but not in \( B_{2k-8}^{m} \). If \( k \) is even, there is a graph which is in \( B_k \) but not in \( B_{2k-9}^{m} \).

**Proof.** Consider first the case where \( k \) is odd. In this case \( G_k := K_{k+1, \frac{k(k+1)}{4} - \frac{k}{4}(k+1)^2 - \frac{(k+1)}{4}} = K_{k+1, \frac{k^3}{4} + \frac{k^2}{4} - \frac{k}{4} + \frac{1}{4}} \) is in \( B_k \) for \( k \geq 3 \).

Assume that \( G_k \) belongs to \( B_{2k-8}^{m} \) for \( k \geq 5 \). Then, Lemma 2.5 implies

\[
\left( \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4} \right) \leq (2k - 8)(k+1) + \frac{1}{2}(k+1)^2 + 2(2k - 7)(k+1)
\]

\[
\Leftrightarrow 8 \left( \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4} \right) \leq 2k^3 - \frac{3}{2}k^2 - 17k - \frac{27}{2}
\]

\[
\Leftrightarrow 7k^2 + 14k + 71 \leq 0,
\]

which is a contradiction for \( k \geq 0 \). So \( G_k \) is not in \( B_{2k-8}^{m} \). Hence, for odd \( k \geq 5 \), there is a graph in \( B_k \) which is not in \( B_{2k-8}^{m} \).

Consider now the case where \( k \) is even. \( G_k' := K_{k+1, \frac{1}{4}(k+1)^3 - \frac{k}{2}(k+1)^2 + \frac{3}{4}(k+1)} = K_{k+1, \frac{k^3}{4} + \frac{k^2}{4} - \frac{k}{4} + \frac{1}{4}} \) ∈ \( B_k \) for \( k \geq 6 \). If we assume that \( G_k' \) is in \( B_{2k-9}^{m} \) for \( k \geq 6 \), we obtain the following inequality by applying
Lemma 2.5

\[
\left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) 9 \leq (2k - 9)(k + 1)k + \frac{1}{2}(k + 1)^2 + 2(2k - 8)(k + 1)
\]
\[
\Leftrightarrow 9 \left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) \leq 2k^3 - \frac{5}{2}k^2 - 20k - \frac{31}{2}
\]
\[
\Leftrightarrow k^3 + k^2 + 62k + 62 \leq 0,
\]

which is a contradiction for \(k \geq 0\). Hence, \(G'_k\) is in \(B_k\) but not in \(B_{2k-9}\) for even \(k \geq 6\).

\[\square\]

Theorem 4.1 reveals that \(B_k \not\subseteq B_{2k-8}\) for odd \(k \geq 5\) and that \(B_k \not\subseteq B_{2k-9}\) for even \(k \geq 5\). Thus, restricting the paths of the EPG representation to be monotonic is a significant limitation. Theorem 4.1 clearly implies that \(B_k \subseteq B_{k+1}\) does not hold in general.

We can also settle the question whether \(B_{k+1}^m \subseteq B_k\) holds in general. Indeed, in [18] it was proven that \(b(K_{m,n}) = 2m - 2\) for \(m \geq 3\) and \(n \geq m^4 - 2m^3 + 5m^2 - 4m + 1\). Hence, in particular, \(K_{m,m^4-2m^3+5m^2-4m+1}\) is in \(B_{2m-2}\), but it is not in \(B_{2m-3}\). On the other hand, Theorem 2.7 implies that \(K_{m,m^4-2m^3+5m^2-4m+1}\) is in \(B_{2m-2}\), so \(B_{2m-2}^m \not\subseteq B_{2m-3}\) for all \(m \geq 3\). Thus, \(B_{k+1}^m \subseteq B_k\) does not hold in general.

### 4.2 Relationship between \(B_1\) and \(B_3^m\)

As mentioned at the beginning of Section 4 in general the minimum number of bends needed for an EPG representation of a graph increases when the paths on the grid are required to be monotonic. In order to quantify the amount of this increase we would like to find the minimum \(\ell\) such that \(B_k \subseteq B_{\ell}^m\). Theorem 4.1 shows that \(2k - 9\) is a lower bound for \(\ell\), i.e. \(\ell \geq 2k - 9\) for \(k \geq 5\).

In the following we focus on small values of \(k\). Since \(B_0 = B_0^m\) holds, 1 is the smallest value of \(k\) for which \(\ell\) and/or bounds on it are not known. In the following we show that \(B_1 \subseteq B_3^m\), i.e. 3 is an upper bound on the minimum value of \(\ell\) for which \(B_1 \subseteq B_3^m\).

**Theorem 4.2.** The inclusion \(B_1 \subseteq B_3^m\) holds.

**Proof.** Let \(G\) be a graph in \(B_1\). We show that \(G\) is in \(B_3^m\) by presenting a monotonic \(B_3\)-EPG representation of \(G\). The latter is constructed by transforming a \(B_1\)-EPG representation of \(G\) into a \(B_3^m\)-EPG representation of \(G\) as described below. The transformation is illustrated by means of an example;
Figures 8(a) and 8(b) show a graph $G$ and a $B_1$-EPG representation of it, respectively, whereas Figure 11 shows the corresponding $B_3^{ep}$-EPG representation obtained as a result of the transformation mentioned above.

Let $R$ be an arbitrary $B_1$-EPG representation of $G$. We place another copy of the same $B_1$-EPG representation to the top right of $R$, see Figure 9, and then step by step modify both the original $B_1$-EPG representation and its copy as described below. At any point in time during this modification process we denote by $R_1$ and $R_2$ the current modified $B_1$-EPG representation and the current modified copy of the original $B_1$-EPG representation, respectively. For a vertex $v$ of $G$ we denote by $P_v$, $P_v^1$ and $P_v^2$ the path corresponding to $v$ in $R$, $R_1$ and $R_2$, respectively. At the beginning of the modification process $R_1$ and $R_2$ coincide with the original $B_1$-EPG representation and its copy, respectively, as in Figure 9.

![Figure 8](image_url)  
(a) A graph $G$. (b) The $B_1$-EPG representation $R$ of $G$.

![Figure 9](image_url)  
Figure 9: The grid with the two copies $R_1$ and $R_2$ of $R$ shown in Figure 8(b).

Now consider the vertices of $G$ one by one in an arbitrary order and for
every vertex perform the modifications described below. Let \( v \) be the currently considered vertex. The modification of \( R_1 \) is driven by the horizontal segment of the path \( P_v \), if any, whereas the modification of \( R_2 \) is driven by the vertical segment of \( P_v \), if any. If \( P_v \) has a horizontal segment, we modify \( R_1 \) as follows. We introduce a new vertical grid line \( L_v \) directly to the left of the vertical grid line containing the right end point of the horizontal segment of \( P^1_v \) in \( R_1 \) and shorten the horizontal segment of \( P^1_v \) to end in \( L_v \) instead of ending at the original right end point. Then, if the path \( P_v \) contains a vertical segment which starts at the original right end point of the horizontal segment mentioned above, we modify \( P^1_v \) in \( R_1 \) by shifting its vertical segment to lie on \( L_v \).

If \( P_v \) has a vertical segment, we modify \( R_2 \) as follows. We introduce a new horizontal grid line \( L_v^- \) directly beneath the horizontal grid line containing the lower end point of the vertical segment of \( P^2_v \) in \( R_2 \) and extend the vertical segment of \( P^2_v \) until \( L_v^- \). Then, if the path \( P_v \) contains a horizontal segment which starts at the original lower end point of the vertical segment mentioned above, we modify \( P^2_v \) in \( R_2 \) by shifting its horizontal segment to lie on \( L_v^- \).

An example of the modified grid and paths and the final \( R_1, R_2 \) for the graph in Figure 8 (a) can be seen in Figure 10.

Now we construct a \( B_3^m \)-EPG representation of \( G \) with a path \( Q_v \) for every vertex \( v \) in the following way. If the path \( P_v \) consists of a single horizontal segment, we define \( Q_v \) as the horizontal segment of \( P^1_v \) in \( R_1 \) and call this segment the lower segment of \( Q_v \). If the path \( P_v \) consists of a single vertical segment, we define \( Q_v \) as the vertical segment of \( P^2_v \) in \( R_2 \) and call this segment the upper segment of \( Q_v \). If the path \( P_v \) contains a horizontal and a vertical segment, then the path \( Q_v \) starts with the horizontal segment of \( P^1_v \) in \( R_1 \); this segment is called the lower segment of \( Q_v \). Further the path \( Q_v \) continues with a vertical segment lying on the vertical grid line \( L_v \) and ending at the intersection of \( L_v \) and \( L_v^- \). This intersection is the upper end point of this segment. Starting at this grid point \( Q_v \) proceeds with a horizontal segment lying on \( L_v^- \) until it reaches the vertical grid line containing the vertical segment of \( P^2_v \) in \( R_2 \). Finally \( Q_v \) ends with the vertical segment of \( P^2_v \) in \( R_2 \); this segment is called the upper segment of \( Q_v \). The result of this construction for the graph given in Figure 8 (a) and its \( B_1 \)-EPG representation \( R \) is depicted in Figure 11.

Observe that this construction has the following properties. If \( P_v \) contains two segments, then \( Q_v \) contains 4 segments, the lower one being the horizontal segment of \( P^1_v \) in \( R_1 \) and the upper one being the vertical segment of \( P^2_v \) in \( R_2 \). The two remaining segments, a vertical and a horizontal one, are contained in the two additionally introduced grid lines that are used by no other path, because every path \( Q_v \) uses only the additional grid lines \( L_v \).
and \( L_v^- \) introduced exclusively for the vertex \( v \). If \( P_v \) consists of one horizontal (vertical) segment, then \( Q_v \) consists also of one horizontal (vertical) segment which coincides with the corresponding segment of \( P^1_v \) (\( P^2_v \)) in \( R_1 \) (\( R_2 \)) and is a lower (upper) segment. It is easy to see that every path \( Q_v \) in this construction is monotonic and bends at most 3 times.

What is left to show is that the above construction indeed leads to an EPG representation of \( G \), i.e. that any two paths \( Q_v \) and \( Q_v' \) intersect if and only if the vertices \( v \) and \( v' \) are adjacent in \( G \). To this end, it is enough to show that two paths \( Q_v \) and \( Q_{v'} \) intersect, if and only if the paths \( P_v \) and \( P_{v'} \) intersect in the original \( B_1 \)-EPG representation \( R \).

Assume \( Q_v \) and \( Q_{v'} \) intersect. First consider the case that at least one of \( Q_v \) and \( Q_{v'} \) consists of only one segment. Assume without loss of generality that \( Q_v \) consists of one horizontal segment. Due to the properties of the construction this segment of \( Q_v \) is a lower segment and hence the unique segment of \( P^1_v \) in \( R_1 \). Consequently, again due to the properties of the construction, the segment of \( Q_{v'} \) intersecting \( Q_v \) is the horizontal segment of \( P^1_{v'} \) in \( R_1 \).
Figure 11: The obtained $B_3^m$-EPG representation of the graph given in Figure 8(a).

Hence $P^1_v$ and $P^1_{v'}$ intersect in the final $R_1$ on their horizontal segments. By construction this is only the case if $P_v$ and $P_{v'}$ intersect on their horizontal segments in $R$, because during the update of $R_1$ only vertical segments of paths are moved into new grid lines in such a way that no new intersections are created.

Now assume that both paths $Q_v$ and $Q_{v'}$ consist of more than one segment. There are no intersections of the paths in any additionally introduced grid lines because every additionally introduced grid line is related to one vertex and the additionally introduced grid line related to different vertices are different. Moreover, by construction every additionally introduced vertical grid line contains at most one segment of the path $P^1_v$ in $R_1$ representing the vertex $v$ to which the line is related. Analogously every additionally introduced horizontal grid line contains at most one segment of the path $P^2_v$ in $R_2$ representing the vertex $v$ to which the line is related. These considerations together with the fact that $R_1$ and $R_2$ do not share any grid lines imply that the intersection of $Q_v$ and $Q_{v'}$ involves either the lower segments of each path, or it involves the upper segments of each path. Consequently,
according to the properties of the construction, the paths \( Q_v \) and \( Q_{v'} \) intersect in their lower segments (in \( R_1 \)) or in their upper segments (in \( R_2 \)). In both situations we can proceed as in the previous case.

Next we show the other direction of the equivalence, that is we assume that \( P_v \) and \( P_{v'} \) intersect in the original \( B_1 \)-EPG representation \( R \) of \( G \) and show that also \( Q_v \) and \( Q_{v'} \) intersect. By construction, if \( P_v \) and \( P_{v'} \) intersect in a horizontal grid line, then the modified paths \( P^1_v \) and \( P^1_{v'} \) intersect in a horizontal grid line in \( R_1 \) at all times. Thus, the properties of the construction imply the intersection of the lower segments of \( Q_v \) and \( Q_{v'} \). Analogously, if \( P_v \) and \( P_{v'} \) intersect in a vertical grid line, then the modified paths \( P^2_v \) and \( P^2_{v'} \) intersect in a vertical grid line in \( R_2 \) at all times, and the properties of the construction imply the intersection of the upper segments of \( Q_v \) and \( Q_{v'} \).

Notice that it is an open question whether the result of Theorem 4.2 is the best possible, that is whether \( \ell = 3 \) is really the minimum \( \ell \) such that \( B_1 \subseteq B_m^\ell \) or whether even \( B_1 \subseteq B_m^2 \) holds.

We conclude this section with a few comments related to the size of the grid in EPG representations, that is the number of horizontal and vertical grid lines used by the paths in the EPG representation. Recently this question was investigated by Biedl, Derka, Dujmović and Morin [3]. The size of the \( B_3^m \)-EPG representation obtained by the construction in the proof of Theorem 4.2 depends on the size of the \( B_1 \)-EPG representation of the graph; in the worst case the constructed \( B_3^m \)-EPG representation uses twice as many horizontal grid lines and twice as many vertical grid line as compared to the original \( B_1 \)-EPG representation and an additional horizontal and vertical grid line for every vertex. This gives rise to the natural question whether the construction given in the proof of Theorem 4.2 is the best possible with respect to the size of the grid. Currently we cannot answer this question.

When considering the dependency of the grid size of the \( B_3^m \)-EPG representation on the grid size of the starting \( B_1 \)-EPG representation in the construction given in [3], another natural question arises. What is the smallest possible size of the grid in a \( B_1 \)-EPG representation of a \( B_1 \)-EPG graph? In the small EPG representations dealt with in [3] no fixed number of bends is considered, so the question above is also open.

### 5 Conclusions and Open Problems

In this paper, we investigated the relationship between the classes \( B_k \) and \( B_\ell^m \) for different values of \( k, \ell \in \mathbb{N} \).

In particular, we considered the bend number and the monotonic bend number of complete bipartite graphs. We extended the already known result
by specifying a particular graph which is in $B \overrightarrow{K}$.

Gave an almost complete answer to the open question on the correctness of $f, g$.

In this context the existence and the identification of non-trivial functions $1$ except for $k > 3$.

However, we could deal with some specific problems related to that question.

Additionally, we considered the relationship of $B_k$ and $B^m_\ell$ for $\ell > k$.

In this context the existence and the identification of non-trivial functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $b^m(G) \leq f(b(G))$ and $b(G) \leq g(b^m(G))$ holds for any graph $G$ (or for any graph belonging to some particular class of graphs) is a general question the answer of which seems to be out of reach at the moment.

However, we could deal with some specific problems related to that question.

In particular, we showed that for every $k \geq 5$ there is a graph in $B_k$ which is not in $B^m_{2k-9}$, proving that $B_k \not\subseteq B^m_{2k-9}$ holds. In terms of the function $f$ above this implies $f(x) \geq 2x - 8$ for all $x \geq 5$, $x \in \mathbb{N}$. Furthermore, we deduced that $B^m_{k+1} \subseteq B_k$ does not hold in general by providing a graph that is $B^m_{2m-2}$ but not in $B^m_{2m-3}$ for every $m \geq 3$. This implies that $g(2x) \geq 2x$ for all $x \geq 2$, $x \in \mathbb{N}$ for the above function $g$.

Further, we showed that $B_1 \subseteq B^m_2$, but we do not know whether this result is the best possible, i.e. whether there is a graph in $B_1$ which is not in $B^m_2$ or whether $B_1 \subseteq B^m_2$ holds.

Another natural question which seems to be simple but has not been answered yet concerns the inclusion $B^m_{k} \subseteq B^m_{k+1}$. We conjecture this inclusion to be strict, that is we conjecture that $B^m_{k} \not\subseteq B^m_{k+1}$ holds. A possible approach to prove this conjecture for a given $k \in \mathbb{N}$ would be to specify a particular pair of natural numbers $(m, n)$ with $3 \leq m \leq n$ for which (a) some Lower-Bound-Lemma implies $K_{m,n} \not\in B^m_{k}$ and (b) a $B^m_{k+1}$-EPG representation can be constructed. The identification of such a pair $(m, n)$, $3 \leq m \leq n$, would clearly prove the existence of a complete bipartite graph $K_{m,n}$ with monotonic bend number equal to $k$ for any $k \geq 2$.

Finally, the size of (monotonic) EPG representations is another subject of interest. In particular, it would be interesting to determine the mini-
minimum number of grid lines needed for a $B_k$-EPG representation and $B^m_\ell$-EPG representation of a graph $G$ with $b(G) \leq k$ and $b^m(G) \leq \ell$, respectively.

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