An Improved Sequential Quadratic Programming Algorithm for Solving General Nonlinear Programming Problems✩

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Abstract

In this paper, a class of general nonlinear programming problems with inequality and equality constraints is discussed. Firstly, the original problem is transformed into an associated simpler equivalent problem with only inequality constraints. Then, inspired by the ideals of sequential quadratic programming (SQP) method and the method of system of linear equations (SLE), a new type of SQP algorithm for solving the original problem is proposed. At each iteration, the search direction is generated by the combination of two directions, which are obtained by solving an always feasible quadratic programming (QP) subproblem and a SLE, respectively. Moreover, in order to overcome the Maratos effect, the higher-order correction direction is obtained by solving another SLE. The two SLEs have the same coefficient matrices, and we only need to solve the one of them after a finite number of iterations. By a new line search technique, the proposed algorithm possesses global and superlinear convergence under some suitable assumptions without the strict complementarity. Finally, some comparative numerical results are reported to show that the proposed algorithm is effective and promising.

Keywords: general nonlinear programming, sequential quadratic programming, method of quasi-strongly sub-feasible directions, global convergence, superlinear convergence

2000 MSC: 49M37, 90C26, 90C30, 90C55

1. Introduction

In this paper we consider the following nonlinear programming problem

\[
\begin{aligned}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i \in I_1 := [1, 2, \ldots, m'], \\
& \quad f_i(x) = 0, \quad i \in I_2 := [m' + 1, m' + 2, \ldots, m],
\end{aligned}
\]

✩Guo and Bai's research is supported by the National Natural Science Foundation of China (Grant No. 11071158), Jian's research is supported by the National Natural Science Foundation of China (Grant No. 11271086) and the Natural Science Foundation of Guangxi (Grant No. 2011GXNSFD018002) as well as Innovation Group of Talents Highland of Guangxi higher School.

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Preprint submitted to Elsevier May 1, 2014
where \( f_i : \mathbb{R}^n \to \mathbb{R} \ (i \in \{0 \} \cup I_1 \cup I_2) \) are smooth functions. The feasible set and gradients of problem (1) are denoted as follows:

\[
\Omega := \{ x \in \mathbb{R}^n : f_i(x) \leq 0, \ i \in I_1; \ f_i(x) = 0, \ i \in I_2 \}, \text{ and } g_i(x) := \nabla f_i(x), \ i \in \{0 \} \cup I_1 \cup I_2.
\]

Sequential quadratic programming (SQP) algorithms have been widely studied by many authors during the past several decades, e.g., Refs.\[1, 2, 3, 4, 5, 6, 7\], and have been proved highly effective for solving problem (1). SQP algorithms generate iteratively the main search directions by solving the standard quadratic programming (QP) subproblem

\[
\min g_0(x)^T d + \frac{1}{2} d^T H d
\]

\[
\text{s.t. } f_i(x) + g_i(x)^T d \leq 0, \ i \in I_1,
\]

\[
 f_i(x) + g_i(x)^T d = 0, \ i \in I_2,
\]

where \( H \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix. Then one performs a line search which is a one dimensional minimization problem to determine a steplength, and obtain the next iteration point.

SQP algorithms may fail since the equality constraints of QP subproblem are hard to be satisfied in the process of iteration. Mayne and Polak\[8\] propose a new way for overcoming this difficulty. In their scheme, they consider the following related family of simpler problem

\[
\min F_c(x) := f_0(x) - c \sum_{i \in I_2} f_i(x)
\]

\[
\text{s.t. } f_i(x) \leq 0, \ i \in I,
\]

where \( I := I_1 \cup I_2 \) and parameter \( c > 0 \). Especially, \( F_c(x) = f_0(x) \) if \( I_2 = \emptyset \). We denote the feasible set of problem (2) by

\[
\Omega^+ := \{ x \in \mathbb{R}^n : f_i(x) \leq 0, \ i \in I \}.
\]

Moreover, they prove that the original problem (1) is equivalent to problem (2) when \( c \) is sufficiently large but finite. Note that problem (2) only has inequality constraints, so the corresponding QP subproblem has no equality constraints, and SQP algorithms will be always successful under some suitable conditions. More advantages and further applications of this technique can be seen in \[9, 10, 11\].

Recently, Guo propose an algorithm for solving problem (1) with \( I_2 = \emptyset \) in \[12\]. In this algorithm, the initial iteration point can be chosen arbitrarily. The main search direction is obtained by solving one QP subproblem and one (or two) system(s) of linear equations. The algorithm possesses global and superlinear convergence under some suitable assumptions without the strict complementarity. Furthermore, some comparative numerical results are reported to show that the algorithm is effective.

Inspired by the ideas in \[8, 12\], we propose a new SQP algorithm for solving problem (1). First, problem (1) is equivalently transformed into problem (2) (see Lemma 2.2). In order to overcome the inconstant of QP subproblem,
we consider a modified QP subproblem

$$\min \nabla F_c(x)^T d + \frac{1}{2} d^T H d$$

s.t. $f_i(x) + g_i(x)^T d \leq \varphi(x)$, \quad $i \in I^*(x)$,

$$f_i(x) + g_i(x)^T d \leq 0$$ \quad $i \in I^-(x)$,

where $\varphi(x) := \max(0, f_i(x), i \in I)$, $I^*(x) := \{i \in I : f_i(x) > 0\}$, $I^-(x) := \{i \in I : f_i(x) \leq 0\}$. QP subproblem (3) has the following advantages:

- subproblem (3) always has a feasible solution $d = 0$.
- subproblem (3) is a strictly convex program if $H$ is positive definite, so it always has a unique solution.
- $d$ is a solution of subproblem (3) if and only if it is a KKT point of subproblem (3).

In order to get the global convergence of the algorithm, the search direction is generated by the combination of two directions, which are obtained by solving QP subproblem (3) and a system of linear equations, respectively. For overcoming the Maratos effect, the higher-order correction direction is generated by solving another system of linear equations. The two systems of linear equations have the same coefficient matrices. The superlinear convergence is derived under the strong second-order sufficient conditions (SSOSC) without the strict complementarity. Moreover, for further comparing the performance of the method of strongly sub-feasible directions (MSSFD) with the method of quasi-strongly sub-feasible directions (MQSSFD), the technical of MQSSFD is adopted in our new algorithm. Finally, some comparative numerical results are reported to show that our new algorithm is promising. The main features of the proposed algorithm are summarized as follows:

- the initial iteration point is arbitrary, and the number of constraints satisfying constraint condition is monotone nondecreasing.
- the objective function of problem (2) is used directly as the merit function.
- the parameter $c$ is adjusted automatically only for a finite number of times (see Lemma 3.1).
- at each iteration, the search direction is generated by a combination of two directions, which are obtained by solving an always feasible QP subproblem and a system of linear equations, respectively.
- after finite iterations, the iteration points always lie in $\Omega^+$.
- under SSOSC without the strict complementarity, the proposed algorithm possesses global and superlinear convergence.

The paper is organized into six sections. In Section 2, our new algorithm and its properties are presented. In Sections 3 and 4, we show that the proposed algorithm possesses global and superlinear convergence, respectively. In Section 5, some comparative numerical results are reported to show that the proposed algorithm is effective and promising. Some conclusions about the proposed algorithm are given in Section 6.

Throughout the paper we use the following notations for a point $x \in \mathbb{R}^n$ and an index subset $J \subseteq I$

$$\begin{cases}
    f_J(x) := \{f_i(x), i \in J\},
    g_J(x) := \{g_i(x), i \in J\},
    \bar{f}_J(x) := f_i(x), i \in I^+(x),
    \bar{f}_J(x) := f_i(x) - \varphi(x), i \in I^*(x),
    I_1(x) := \{i \in I_1 : \bar{f}_J(x) = 0\},
    I(x) := I_1(x) \cup I_2,
    I_0(x) := \{i \in I : \bar{f}_J(x) = 0\}.
\end{cases}$$
2. Description of algorithm

In this section, we start by giving some basic assumptions for problem (1).

Assumption 2.1. (i) The functions \( f_i(x) \) \( (i \in \{0 \cup I\}) \) are all continuously differentiable.

(ii) The gradient vectors \( \{g_i(x) : i \in I(x)\} \) are linearly independent for each \( x \in \mathbb{R}^n \).

To update the parameter \( c \) in problem (2), the matrices \( N(x) \), \( D(x) \) and multiplier vector \( \pi(x) \) are defined as follows:

\[
N(x) = (g_i(x), i \in I), \quad \pi(x) = -(N(x)^T N(x) + D(x))^{-1} N(x)^T g_0(x),
\]

\[
D(x) = \text{diag}(D_i(x), i \in I), \quad D_i(x) = \begin{cases} \|f_i(x)\|_p, & i \in I_1, \\ 0, & i \in I_2, \end{cases}
\]

where \( p \) is a positive parameter.

Note that \( D_i(x) > 0 \) for all \( i \in I \setminus I(x) \) in (4). By Assumption 2.1(ii), the following lemma holds immediately.

Lemma 2.1. Suppose that Assumption 2.1 holds. Then \((N(x)^T N(x) + D(x))\) is nonsingular and positive definite for all \( x \in \mathbb{R}^n \).

By Lemma 2.1 and 4, the relationship between problems (1) and (2) is shown in the following Lemma 2.2. Its proof can be referred to the one of Lemma 2.1(v) in [10].

Lemma 2.2. If \( c > |\pi_i(x)| \) for all \( i \in I_2 \), then \((x, \mu)\) is the KKT point of problem (1) if and only if the KKT point of problem (2), where \( \mu \) and \( \lambda \) satisfy

\[
\mu_i = \lambda_i, \quad \mu_i = \lambda_i - c, \quad i \in I_2.
\]

For the iteration point \( x^k \) and the parameter \( c_k \) of problem (2), QP subproblem (5) can be simplified as follows by the above notations

\[
\begin{align*}
\min & \quad \nabla F_{c_k}(x^k)^T d + \frac{1}{2} d^T H_k d \\
\text{s.t.} & \quad f_i(x^k) + g_i(x^k)^T d \leq 0, \quad i \in I.
\end{align*}
\]

When \( H_k \) is positive definite, \( d_k^0 \) is a solution of subproblem (6) if and only if there exists a corresponding KKT multiplier vector \( \lambda_k \) such that

\[
\begin{align*}
\nabla F_{c_k}(x^k) + H_k d_k^0 + \sum_{i \in I} \lambda_i^k g_i(x^k) &= 0, \\
f_i(x^k) + g_i(x^k)^T d_k^0 &\leq 0, \quad \lambda_i^k \geq 0, \quad \lambda_i^k (f_i(x^k) + g_i(x^k)^T d_k^0) = 0, \quad \forall i \in I.
\end{align*}
\]

Since \( d = 0 \) is a feasible solution of subproblem (6) and \( H_k \) is positive definite, it follows that

\[
\nabla F_{c_k}(x^k)^T d_k^0 + \frac{1}{2} (d_k^0)^T H_k d_k^0 \leq 0 \Rightarrow \nabla F_{c_k}(x^k)^T d_k^0 \leq 0,
\]

i.e., \( d_k^0 \) is a descent direction of \( F_{c_k}(x^k) \) at the iteration point \( x^k \).

Due to (7) and Lemma 2.2, the following lemma holds immediately.
Lemma 2.3. If \((d_0^k, \phi(x^k)) = (0, 0)\), then \(x^k\) is the KKT point of problem \((2)\). Furthermore, if \(c_i > |\mu_i|\) for all \(i \in I_2\), then \(x^k\) is the KKT point of problem \((1)\).

Again from \((7)\), it follows that \(d_0^k\) may not be a feasible direction of problem \((2)\) at the feasible iteration point \(x^k \in \Omega^+\). So a suitable strategy must be carried out to generate a feasible direction. Here, taking into account that \(x^k\) may be infeasible, we introduce the system of linear equations to get a unique solution \((d_i^k, h_i^k)\)

\[
\Gamma_k \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} H_k & N_k \\ N_k^T & -Q_k \end{pmatrix} \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ -(\|d_0^k\| + \phi(x^k)^\sigma) \sigma \end{pmatrix},
\]

where \(0 \in R^n, \sigma = (1, 1, \ldots, 1)^T \in R^n, \sigma \in (0, 1)\) and

\[
N_k = N(x^k) = (g_i(x^k), i \in I), \quad Q_k = \text{diag}(Q_k^i = [f_i(x^k)g_i(x^k) + g_i(x^k)^T d_0^k + |d_0^k|], i \in I).
\]

Then we consider the convex combination of \(d_0^k\) and \(d_i^k\)

\[
\hat{d}^k = (1 - \beta_k)d_0^k + \beta_k d_i^k,
\]

where \(\beta_k\) is the maximal value of \(\beta \in [0, 1]\) satisfying

\[
\nabla F_{c_i}(x^k)^T \hat{d}^k \leq \theta \nabla F_{c_i}(x^k)^T d_0^k + \phi(x^k)^\theta.
\]

Moreover, \((11)\) further implies that \(\beta_k\) is the optimal solution of linear programming

\[
\max \beta \quad \text{s.t.} \quad \beta \nabla F_{c_i}(x^k)^T d_i^k + (1 - \beta) \nabla F_{c_i}(x^k)^T d_0^k \leq \theta \nabla F_{c_i}(x^k)^T d_0^k + \phi(x^k)^\theta, \quad 0 \leq \beta \leq 1,
\]

where the positive parameter \(\theta < \sigma\). It is obviously that \((11)\) holds for \(\beta_k = 0\), since \(\phi(x^k) \geq 0\). The above linear programming further implies that \(\beta_k > 0\), otherwise, \(x^k\) is the KKT point of problem \((1)\) (see Lemma 2.3).

The next lemma shows the solvability of \((8)\). Its proof is elementary in view of \([i \in I: Q_i^k = 0] \subseteq I_0(x^k) \subseteq I(x^k)\).

Lemma 2.4. Suppose that Assumption \((2.7)\) holds and \(H_k\) is positive definite. Then \(\Gamma_k\) defined in \((8)\) is nonsingular and \((8)\) has a unique solution.

Lemma 2.5. Suppose that Assumption \((2.7)\) holds. Then

(i) \(\nabla F_{c_i}(x^k)^T \hat{d}^k \leq -\frac{1}{2} \theta (d_0^k)^T H_k d_0^k + \phi(x^k)^\theta\).

(ii) \(g_i(x^k)^T \hat{d}^k \leq -\beta_k (|d_0^k| + \phi(x^k)^\sigma), \forall i \in I_0(x^k)\).

proof: (i) Since \(d = 0\) is a feasible solution of subproblem \((6)\), and from \((11)\), it holds that

\[
\nabla F_{c_i}(x^k)^T \hat{d}^k \leq \theta \nabla F_{c_i}(x^k)^T d_0^k + \phi(x^k)^\theta \leq -\frac{1}{2} \theta (d_0^k)^T H_k d_0^k + \phi(x^k)^\theta.
\]
(ii) First, from (7) and (8), it follows that
\[ g_i(x^k)^T d_i^k \leq 0, \quad g_i(x^k)^T d_i^1 = -||d^1_0|| - \varphi(x^k)^\gamma, \quad \forall i \in I_0(x^k). \]
Then we obtain that by (10)
\[ g_i(x^k)^T d_i^k \leq -\beta_k(||d^p_0|| + \varphi(x^k)^\gamma) \text{ for all } i \in I_0(x^k). \]

By Lemma 2.5 we know that \( \hat{d}^k \) is an improved direction. In order to overcome the Maratos effect and avoid the strict complementarity condition as well as reduce the computational cost, a suitable higher-order correction direction should be introduced by an appropriate approach. Here, we introduce the following system of linear equations to yield the higher-order correction direction \( d_2^k \)
\[
\Gamma_k \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ -(||d^p_0|| + \varphi(x^k)^\gamma)\sigma - F(x^k + d_1^k) \end{pmatrix},
\]
where \( \sigma \in (2, 3) \) and
\[
F(x^k + d_1^k) = (f_i(x^k + d_1^k) - f_i(x^k) - g_i(x^k)^T d_{1i}^k, \quad i \in I).
\]
Note that the term \( \varphi(x^k) \) is introduced in our paper, the relationship between \( d_1^k \) and \( d_2^k \) will be different from the traditional form \( ||d_2^k|| = O(||d_1^k||^2) \) \([15, 16, 17]\), the details can be seen in Lemma 4.1.

We are now ready to present our algorithm for solving problem (1) as follows.

**Algorithm 2.1.**

Parameters: \( p, \epsilon, \gamma, \gamma_0 > 0, \ c_{-1} > 0, \ \rho > 1, \ 0 < \theta < \sigma, \ \sigma, \eta, \alpha, \hat{\alpha} \in (0, 1), \ \tau \in (2, 3). \)
Data: \( x^0 \in R^n, \) a symmetric positive definite matrix \( H_0 \in R^{nxn}, \) and \( k := 0. \)

**Step 1.** Update parameter \( c_k; \) Compute \( c_k \ (k = 1, 2, \ldots) \) by
\[
c_k = \begin{cases} 
\max\{s_k, \ c_k-1 + \gamma\}, & \text{if } s_k > c_{k-1}, \\
\ c_k-1, & \text{if } s_k \leq c_{k-1},
\end{cases}
\]
(14)
**Step 2.** Solve QP subproblem: Solve QP subproblem (6) to get a solution. If \((d_0^k, \varphi(x^k)) = (0, 0), \) then \( x^k \) is the KKT point of problem (1) and stop; otherwise, go to Step 3.

**Step 3.** Solve system of linear equations: Solve (12) to get a solution \((d_2^k, h_k^k), \) and let \( d^k = d_0^k + d_2^k. \)

**Step 4.** Let \( t = 1. \) (a) If
\[
\begin{align*}
F_{\alpha}(x^k + td^k) \leq & F_{\alpha}(x^k) + a\nabla F_{\alpha}(x^k)^T d^k_0 + \rho(1 - \alpha)\varphi(x^k)^\gamma, \\
f_i(x^k + td^k) \leq & \max(0, \varphi(x^k) - \alpha||d_0^p|| + \varphi(x^k)^\gamma), \quad i \in I, \\
|F(x^k + td^k)| \geq & |F(x^k)|,
\end{align*}
\]
is satisfied, then let \( t_k = t, \) and go to Step 7; otherwise, go to (b).

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Assumption 3.1. For this purpose, the following assumption is necessary.

(ii) There exist positive constants \( a \) and \( b \) such that

\[
\sup_{k} \left( x^{k+1} - x^{k} \right) \leq a \left( \|d^{k} - \hat{d}^{0}\| \right)^{2}, \quad \forall \, d \in \mathbb{R}^{n}, \, \forall \, k.
\]

Denote the active set for QP subproblem \( P \) by

\[
L(x^{k}) \equiv \{ i \in I : \ f_{i}(x^{k}) + g_{i}(x^{k})^{T} d_{0}^{k} = 0 \}.
\]
Suppose that $x^*$ is a given accumulation point of \((x^k)\). In view of $I^r(x^k)$, $I^l(x^k)$ and $L(x^k)$ are subsets of the finite set $I$, by Lemma 2.7 iii), we can assume that there exists an infinite index set $K$ such that

\[
x^k \to x^*, \quad I^r(x^k) \equiv I^r, \quad I^l(x^k) \equiv I^l, \quad L(x^k) \equiv L, \quad \varphi(x^k) \to \varphi(x^*), \quad \forall k \in K.
\]

(18)\hfill

**Lemma 3.1.** Suppose that Assumptions 2.7 and 3.1(i) hold. Then there exists an index $k_1 > 0$ such that $c_k = c_{k_1} \triangleq c$ for all $k \geq k_1$.

The detailed proof of this lemma can be found in [10]. Due to Lemma 3.1, we assume that $c_k \equiv c$ for all $k$ in the rest of this paper. The results given in the following lemma are very important in the subsequent analysis.

**Lemma 3.2.** Suppose that Assumptions 2.7 and 3.1 hold. Then

(i) The sequence $\{d^k\}_{k=1}^{\infty}$ is bounded.

(ii) There exists a constant $r_0 > 0$ such that $\|\Gamma^{-1}_k\| \leq r_0$ for all $k$.

(iii) The sequence $\{d^k\}_{k=1}^{\infty}$, $\{d^k\}_{k=1}^{\infty}$, $\{h^k\}_{k=1}^{\infty}$ and $\{h^k\}_{k=1}^{\infty}$ are all bounded.

**proof:** (i) Due to the fact that $d = 0$ is a feasible solution of subproblem (6) and $d_0^1$ is an optimal solution, we have

\[
\nabla F_c(x^k)^T d_0^k + \frac{1}{2}(d_0^k)^T H d_0^k \leq 0.
\]

(19)\hfill

By Assumption 3.1 and the continuity of $\nabla F_c(x^k)$, there exists a constant $\bar{c} > 0$ such that $\|\nabla F_c(x^k)\| \leq \bar{c}$ for all $k$. Combining (19) with (17), we get

\[
-\bar{c} ||d_0^k|| + \frac{1}{2} a ||d_0^k||^2 \leq 0,
\]

which implies that $\{d_0^k\}$ is bounded for all $k$.

(ii) Suppose by contradiction that there exists an infinite index set $K$ such that

\[
\|\Gamma^{-1}_k\| \to \infty, \quad k \in K.
\]

(20)\hfill

Without loss of generality, we assume that there exists an infinite index set $K' \subseteq K$ such that

\[
x^k \to x^*, \quad \Gamma_k \to \Gamma_* = \begin{pmatrix} H & N_* \\ N_*^T & -Q^* \end{pmatrix}, \quad d_0^k \to d_0^*, \quad \varphi(x^k) \to \varphi(x^*), \quad I_0(x^k) \to I_0(x^*), \quad \forall k \in K',
\]

where

\[
N_* = (g_i(x^*), \quad i \in I), \quad Q^* = \text{diag}(Q^*_i) = [\hat{f}(x^*)]^T [\hat{f}(x^*) + g_i(x^*)^T d_0^*] + ||d_0^*||, \quad i \in I).
\]

It holds that $Q^*_i \geq 0$ for all $i \in I$, and $Q^*_i > 0$ for all $i \in I \setminus I_0(x^*)$. Similar to the proof of Lemma 2.4, we can conclude that $\Gamma_*$ is nonsingular. So $\|\Gamma^{-1}_k\| \to \|\Gamma^{-1}_*\|$ for $k \in K'$, which contradicts (20), thus the conclusion (ii) holds.

(iii) Taking into account (8), (12) and (10), we can obtain the boundedness of $\{d^1_{1k}\}, \{d^1_{2k}\}, \{h^1_{k}\}$ and $\{h^2_{k}\}$ by employing the result of parts (i) and (ii).

Similar to the analysis of Lemma 3.3 in [12], we can obtain the following results.
Lemma 3.3. Suppose that Assumptions 2.1 and 3.1 are satisfied. Then
\( (i) \lim_{k \to \infty} (d_k^0, \varphi(x^k)) = (0, 0), \lim_{k \to \infty} d_k^x = \lim_{k \to \infty} d_k^\lambda = 0 \) and \( \lim_{k \to \infty} h_k = 0. \)
\( (ii) \lim_{k \to \infty} ||x^{k+1} - x^k|| = 0. \)

Theorem 3.1. Suppose that Assumptions 2.1 and 3.1 hold. Then Algorithm 2.1 either stops at the KKT point \( x_k \) of problem (1) after a finite number of iterations or generates an infinite sequence of points such that each accumulation point \( x^* \) of \( \{x_k\} \) is the KKT point of problem (1). Furthermore, there exists an index set \( K \) such that \( \{(x^k, \lambda^k) : k \in K\} \) converges to the KKT pair \( (x^*, \lambda^*) \) of problem (3) and the KKT pair \( (x^*, \mu^*) \) of problem (7), respectively, where \( x^* = (A^*_L, 0_{I_1L}) \) and \( \mu^* = (\mu^*_L, 0_{I_1L}). \)

\textbf{proof:} By Lemma 2.7(iii), we assume without loss of generality that there exists an infinite subset \( K \) such that (18) holds. Let matrix \( A_k = (g_i(x^k), \ i \in I) \). From Lemma 3.3(i), it follows that \( L \subseteq I_0(x^*) = \{i \in I : f_i(x^*) = 0\} \), which together with Assumption 2.1 shows that \( A_k^T A_k \) is nonsingular for \( k \in K \) large enough, since \( A_k \xrightarrow{k \to \infty} A_* \), where \( A_* \triangleq (g_i(x^*), \ i \in I) \).

From (7) and Lemma 3.3(i), we have for \( k \in K \) large enough,
\[ A_k^T = -(A_k^T A_k)^{-1} A_k^T (F_k(x^k) + H_k d_k^0) \rightarrow -(A_*^T A_*)^{-1} A_*^T F(x^*) \triangleq A_*^T. \]
Denote the multiplier vector \( \lambda^* = (A_*^T, 0_{I_1L}) \), then \( \lim_{k \to \infty} \lambda^k = \lambda^* \). Passing to the limit \( k \in K (k \to \infty) \) in (7), it follows that
\[ \nabla F(x^*) + N_c \lambda^* = 0, \quad f_i(x^*), \lambda^*_i \geq 0, \quad f_i(x^*), \lambda^*_i = 0, \quad i \in I, \]
which shows that \( (x^*, \lambda^*) \) is the KKT pair of problem (3). By the definition of \( c_k \) and Lemma 3.1 we have \( c > \max_{i \in I} c_i(x^*) \). So from Lemma 2.2 we can conclude that \( (x^*, \mu^*) \) is the KKT pair of problem (1) with \( \mu^*_i = A_i - c, \ i \in I_2; \mu^*_i = \lambda^*_i - c, \ i \in I_1; \mu^*_i = 0, \ i \in I \setminus I \). Obviously, \( \lim_{k \to \infty} (x^k, \lambda^k) = (x^*, \lambda^*) \) and \( \lim_{k \to \infty} (x^k, \mu^k) = (x^*, \mu^*). \)
The proof is completed. \( \square \)

4. Rate of convergence

In this section we further discuss the strong and superlinear convergence of Algorithm 2.1. For these purposes, we make the following assumption.

Assumption 4.1. (i) The functions \( f_i(x) (i \in \{0\} \cup I) \) are all second-order continuously differentiable.
(ii) The KKT pair \( (x^*, \mu^*) \) of problem (1) satisfies the strong second-order sufficient conditions, i.e.,
\[ d^T \nabla^2_{x^*} L(x^*, \mu^*) d > 0, \quad \forall \ d \in R^n, \ d \neq 0, \quad g_i(x^*) d = 0, \quad i \in I^*_1, \]
where \( \nabla^2_{x^*} L(x^*, \mu^*) = \nabla^2 f_0(x^*) + \sum_{i \in I} \mu^*_i \nabla^2 f_i(x^*), \ I^*_1 = \{i \in I_1 : \mu^*_i > 0\} \cup I_2. \)
\begin{remark}

Similar to the proof of Lemma 4.2, we can conclude that \((x^*, \lambda^*)\) satisfying
\begin{equation}
\lambda^*_i = \mu^*_i, \; i \in I_1; \; \lambda^*_i = \mu^*_i + c, \; i \in I_2
\end{equation}
is the KKT point of problem (\textbf{3}). Moreover, \(\{i \in I : \lambda^*_i > 0\} = \{i \in I_1 : \mu^*_i > 0\} \cup I_2\), which implies that KKT pair \((x^*, \lambda^*)\) of problem (\textbf{3}) also satisfies the strong second-order sufficiency conditions, i.e.,
\begin{equation*}
d^T \nabla^2_{xx} L_c(x^*, \lambda^*) d > 0, \; \forall \; d \in \mathbb{R}^n, \; d \neq 0, \; g_i(x^*)^T d = 0, \; i \in I^*_c,
\end{equation*}
where \(\nabla^2_{xx} L_c(x^*, \lambda^*) = \nabla^2 F_c(x^*) + \sum_{i \in I} \lambda^*_i \nabla^2 f_i(x^*), \; I^*_c = \{i \in I : \lambda^*_i > 0\}.
\end{remark}

Under the stated assumptions, we have the following theorem.

\begin{theorem}
Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\begin{enumerate}[(i)]
\item \(\text{Lemma 4.1.}\) Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\item \(\text{Theorem 4.1.}\) Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\end{enumerate}
\end{theorem}

\begin{proof}
(i) From the proof of Theorem 3.1 and part (i), one can conclude that each accumulation point of sequences \(\{x^k\}\) and \(\{\mu^k\}\) is the KKT multiplier for problem (\textbf{3}) and problem (\textbf{1}) associated with \(x^*\), respectively. Togethering with the uniqueness of the KKT multiplier, this furthermore implies that part (ii) holds.
\end{proof}

\begin{lemma}
Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\begin{enumerate}[(i)]
\item \(\text{Lemma 4.1.}\) Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\item \(\text{Theorem 4.1.}\) Suppose that Assumptions 2.1, 3.1 and 4.1 hold. Then
\end{enumerate}
\end{lemma}

\begin{proof}
(i) In view of \(F(x^k + d^k_0) = O(||d^k_0||^2)\), the proof is elementary from (12) and Lemma 4.3.1.
\end{proof}

\begin{assumption}
Suppose that the KKT pair \((x^*, \lambda^*)\) and matrix \(H_k\) satisfy
\begin{equation}
\|\nabla^2_{xx} L_c(x^*, \lambda^*) - H_k d^k_0\| = o(||d^k_0||),
\end{equation}
where \(\nabla^2_{xx} L_c(x^*, \lambda^*) = \nabla^2 F_c(x^*) + \sum_{i \in I} \lambda^*_i \nabla^2 f_i(x^*) = \nabla^2 L(x^*, \mu^*).\)
\end{assumption}
Theorem 4.2. Suppose that Assumptions $2.1, 3.1, 4.1$ and $4.2$ hold. Then the inequalities in (15) always hold for $t = 1$ and $k$ large enough.

Proof: We assume that $t = 1$ and $k$ large enough in the whole process of proof. First of all, we discuss the second and the last inequalities of (15).

For $i \notin I_0(x^k)$, i.e., $f_i(x^k) < 0$. In view of $(x^k, d_0^k, d_2^k, \varphi(x^k)) \rightarrow (x^*, 0, 0, 0) (k \rightarrow \infty)$, we have $d_k = d_0^k + d_2^k \rightarrow 0 (k \rightarrow \infty)$. So we can conclude that the second inequalities and the last inequality of (15) are both satisfied.

For $i \in I_0(x^k)$, it holds that $f_i(x^k) = 0$. On one hand, since $\lim_{k \rightarrow \infty} f_i(x^k) = f_i(x^*) = 0$ and $\lim \varphi(x^k) = 0$ as well as (9), it follows that $Q_i^k \rightarrow 0$ and $Q_i^k = o(\|f_i(x^k) + g_i(x^k)^T d_0^k\| + o(\|d_0^k\|^3))$. On the other hand, we have from (12) and Lemma 4.1 i)

$$g(x^k)^T d_2^k = -\|d_0^k\| + \varphi(x^k)^T - f_i(x^k + d_0^k) + f_i(x^k) + g_i(x^k)^T d_0^k + o(\|f_i(x^k) + g_i(x^k)^T d_0^k\| + o(\|d_0^k\|^3) + o(\varphi(x^k)^T).$$

(22)

Then we obtain by Taylor expansion and (22)

$$f_i(x^k + d^k) = \begin{cases} -\|d_0^k\|^T - f_i(x^k) + g_i(x^k)^T d_0^k + o(\|f_i(x^k) + g_i(x^k)^T d_0^k\| + O(\|d_0^k\|^3), & \text{if } i \in I_0(x^k) \cap I^r; \\ -\|d_0^k\|^T - \varphi(x^k)^T - \varphi(x^k) - f_i(x^k) + g_i(x^k)^T d_0^k + o(\|f_i(x^k) + g_i(x^k)^T d_0^k\|) + O(\|d_0^k\|^3) + o(\varphi(x^k)^T) & \text{if } i \in I(x^k) \cap I^r. \end{cases}$$

(23)

By $\tau \in (2, 3)$, the first equality of (23) implies that $f_i(x^k + d^k) \leq 0$ for all $i \in I_0(x^k) \cap I^r$, i.e., the second inequalities of (15) hold for $i \in I_0(x^k) \cap I^r$ and the third inequality of (15) holds.

Again from (23), $\tau \in (2, 3)$ as well as $\alpha \in (0, \frac{1}{2})$, we have for $i \in I_0(x^k) \cap I^r$

$$f_i(x^k + d^k) - \max[0, \varphi(x^k) - \alpha(\|d_0^k\| + \varphi(x^k)^T)] = -(1 - \alpha)(\|d_0^k\| + \varphi(x^k)^T - f_i(x^k) + g_i(x^k)^T d_0^k) + o(\|f_i(x^k) + g_i(x^k)^T d_0^k\| + o(\varphi(x^k)^T) + O(\|d_0^k\|^3) \leq 0.

Summarizing the above analysis, we have proved that the second and the last inequalities of (15) are satisfied for $t = 1$ and $k$ large enough.

From now on, we will show that the first inequality of (15) holds. First of all, by Taylor expansion and Lemma 4.1 i), we have

$$\Delta_k \triangleq \nabla F_i(x^k)^T d^k + \frac{1}{2}(d^k)^T \nabla^2 F_i(x^k)d^k - \alpha \nabla F_i(x^k)^T d_0^k - \rho(1 - \alpha)\varphi(x^k)^T + o(\|d_0^k\|^2)$$

$$= \nabla F_i(x^k)^T(d_0^k + d_2^k) + \frac{1}{2}(d_0^k)^T \nabla^2 F_i(x^k)d_0^k - \alpha \nabla F_i(x^k)^T d_0^k - \rho(1 - \alpha)\varphi(x^k)^T + o(\|d_0^k\|^2) + o(\varphi(x^k)^T).$$

(24)

Then we get by the KKT conditions (7) and Lemma 4.1 i)

$$\nabla F_i(x^k)^T(d_0^k + d_2^k) = -\nabla F_i(x^k)^T H_k d_0^k - \sum_{i \in I(x^k)} \lambda_i g_i(x^k)^T(d_0^k + d_2^k) + o(\|d_0^k\|^2) + o(\varphi(x^k)^T).$$

(25)

For $i \in L(x^k) \subseteq I_0(x^k)$, it follows that $f_i(x^k) + g_i(x^k)^T d_0^k = 0$. From (23) and Lemma 4.1 i) as well as $\varphi(x^k) = o(\varphi(x^k)^T)$, we have

$$f_i(x^k + d^k) = \begin{cases} -\|d_0^k\|^T - \varphi(x^k)^T + o(\|d_0^k\|^3) + o(\varphi(x^k)^T), & i \in L(x^k); \\ f_i(x^k) + g_i(x^k)^T(d_0^k + d_2^k) + \frac{1}{2}(d_0^k)^T \nabla^2 f_i(x^k)d_0^k + o(\|d_0^k\|^2) + o(\varphi(x^k)^T), \end{cases}$$

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which further imply that

\[- \sum_{e \in L(x^k)} \lambda_e^* g_e(x^k)^T (d_0^e + d_2^e) = \sum_{e \in L(x^k)} \lambda_e^* f_i(x^k) + \frac{1}{2} \sum_{e \in L(x^k)} \lambda_e^* (d_0^e)^T \nabla^2 f_i(x^k) d_0^e + o(||d_0^e||^2) + O(\varphi(x^k)^r).\]  

(26)

Substituting (26) into (25), we have

\[\nabla F_e(x^k)^T (d_0^e + d_2^e) = -(d_0^e)^T H_k d_0^e + \frac{1}{2} \lambda_e^* f_i(x^k) + \frac{1}{2} (d_0^e)^T (\nabla^2 f_i) d_0^e - \frac{1}{2} \lambda_e^* \rho \varphi(x^k) o(||d_0^e||^2) + O(\varphi(x^k)^r),\]

which combined with (24) gives

\[\Delta_k = -(d_0^e)^T H_k d_0^e + \frac{1}{2} \lambda_e^* f_i(x^k) + \frac{1}{2} (d_0^e)^T (\nabla^2 f_i) d_0^e - \frac{1}{2} \lambda_e^* \rho \varphi(x^k) o(||d_0^e||^2) + O(\varphi(x^k)^r).\]

Due to \(\lambda_e^* f_i(x^k) \leq 0\) and \(\alpha \in (0, 1)\) as well as \(\theta < \sigma\), it follows from Assumptions 3.1 and 4.2 that

\[\Delta_k \leq (\alpha - \frac{1}{2}) \rho ||d_0^e||^2 + o(||d_0^e||^2) - \rho (1 - \alpha) \varphi(x^k) o(||d_0^e||^2) \leq 0,\]

i.e., the first inequality of (15) holds. The whole proof is completed. \(\square\)

**Theorem 4.3.** Under all above mentioned assumptions, \(\varphi(x^{k+1}) \equiv 0\) after a finite number of iterations, i.e., \(x^{k+1} \in \Omega^+\) for \(k\) large enough.

According to Theorem 4.2 and (23), the above theorem holds directly. Moreover, based on Theorems 4.2 and 4.3, the superlinear convergence of Algorithm 2.1 is given in Theorem 4.4 by Theorem 2.2.3 in [17].

**Theorem 4.4.** Suppose that Assumptions 2.1, 3.1, 4.1 and 4.2 are all satisfied. Then Algorithm 2.1 is superlinearly convergent, i.e., \(||x^{k+1} - x^*|| = o(||x^k - x^*||)\).

5. Numerical experiments

In this section, in order to show the effectiveness of our proposed algorithm, some classical problems in [18, 19] are tested and the corresponding comparative numerical results are reported in the following parts. The algorithm is implemented by using MATLAB R2008a on Windows XP platform, and on a PC with 2.53GHz CPU.

During the numerical experiments, the identity matrix \(E_n\) is selected as the initial Lagrangian Hessian, and the approximation Hessian matrix \(H_k\) is updated by BFGS formula [20]

\[H_{k+1} = H_k - \frac{H_k s^T s^k H_k}{s^T s^k} + \frac{\tilde{s}^k (\tilde{s}^k)^T}{(\tilde{s}^k)^T \tilde{s}^k} (k \geq 0),\]

where

\[s^k = x^{k+1} - x^k, \qquad \tilde{s}^k = y^k + \alpha_k (y^k s^k + A_k s^k), \quad \gamma_k = \min(||d_0^e||^2, \kappa \in (0, 1)), \quad A_k = (g_i(x^k), i \in L(x^k)), \quad y^k = \nabla f(x^k, A_k), \quad \tilde{y}^k = \nabla L(x^k, A_k), \quad \tilde{y}^k = \nabla F(x^k, A_k) + \sum_{i \in I} d_i^e g_i(x^k),\]

\[H_{k+1} = H_k - \frac{H_k s^k s^T s^k H_k}{s^T s^k} + \frac{\tilde{s}^k (\tilde{s}^k)^T}{(\tilde{s}^k)^T \tilde{s}^k} (k \geq 0),\]
\[ \alpha_k = \begin{cases} 
0, & \text{if } (s_k^h)^T y_k^h \geq \mu \|s_k^h\|^2, \mu \in (0, 1), \\
1, & \text{if } 0 \leq (s_k^h)^T y_k^h < \mu \|s_k^h\|^2, \\
1 + \frac{\gamma_k \|s_k^h\|^2 - (s_k^h)^T y_k^h}{\gamma_k \|s_k^h\|^2}, & \text{otherwise.}
\end{cases} \]

The parameters are selected as follows:

\[ \alpha = \hat{\alpha} = \eta = \kappa = \mu = c_{-1} = 0.5, \theta = 0.4, \sigma = 0.6, \rho = 2, \tau = 2.5, \epsilon = 0.5^3, p = 2, \gamma_0 = 2, \gamma = 1. \]

The algorithm stops if the termination criterions \( \|d_k^0\| \leq \bar{\varepsilon} \) and \( \varphi(x) = 0 \) are both satisfied.

First of all, some notations used in the following tables are defined in Table 1.

| Prob | The number of test problem in [18]. |
|------|------------------------------------|
|  \( n \) | The number of variables of test problem. |
|  \( I_1/I_2 \) | The number of equality and inequality constraints, respectively. |
|  \( N_0 \) | The number of objective function evaluations. |
|  \( N_f \) | The number of all constraint functions evaluations. |
|  \( N_{io} \) | The number of iterations out of the feasible set. |
|  \( N_{ii} \) | The number of iterations within the feasible set. |
|  \( N_f + N_{io} \) | The number of all iterations. |
|  \( \text{CPU} \) | The CPU time (second). |

Table 1: Definitions of some notations

In order to show the computational efficiency of Algorithm 2.1 (shorted by ALGO 2.1), which is compared with other types of algorithms, including SQP algorithms and systems of linear equations (SLE) algorithms. The statistics of these algorithms are given in Table 2. The “Feasible” (or “Infeasible”) in Table 2 means that the initial iteration point have to be feasible (or can be chosen arbitrarily) for the solving problem.

| Author | Types of algorithm (shorted by) | Types of solving problem |
|--------|--------------------------------|--------------------------|
| Jin and Wang [21] | Feasible SQP (JW-FSQP) | nonlinear inequality constrained programming |
| Qi and Yang [2] | Infeasible SQP (QY-IFSQP) | nonlinear inequality constrained programming |
| Wang, Chen and He [22] | Infeasible SLE (WCH-IFSLE) | nonlinear equality and inequality constrained programming |
| Guo [12] | Infeasible SQP (G-IFSQP) | nonlinear inequality constrained programming |
| Jian, Ke, Zheng and Tang [23] | Infeasible SQP (JKZT-IFSQP) | nonlinear inequality constrained programming |
| Yang, Li and Qi [24] | Feasible SLE (YLQ-FSLE) | nonlinear inequality constrained programming |
| Gu and Zhu [5] | Infeasible SQP (GZ-IFSQP) | nonlinear equality and inequality constrained programming |
| Gill, Murray and Saunders [25] | SNOPT | nonlinear equality and inequality constrained programming |

Table 2: Description of some comparative algorithms

In Table 3 we compare the number of \( N_f \) and \( F_v \) required by ALGO 2.1 with those required by JW-FSQP. The test problems are chosen from [18], and initial iteration points are all feasible except Prob 030. The optimality tolerance is the same as in [21]. The results in Table 3 show that the number of iterations of ALGO 2.1 is much smaller than that of JW-FSQP for most test problems. From the viewpoints of \( N_f \) and \( F_v \), we can conclude that ALGO 2.1 is more effective than JW-FSQP.
In Table 4, we further compare the number of Ni, Nf0, Nf and Fv required by ALGO 2.1 with those required by QY-IFSQP. The optimality criterions and the starting iteration points for the test problems are the same as in [2]. “Point (a)” (or “Point (b)”) in Table 4 denotes that the corresponding initial point is “feasible” (or “infeasible”). Note that Nf refers to the number of evaluations of \( f_i \) in [2], however, Nf denotes the number of all constraint functions evaluations in our paper. The Ni column is displayed as the total number of iterations. Only if the initial iteration points are chosen as (a), \( N_i = N_{ii} \), otherwise, \( N_i = N_{io} + N_{ii} \). For example, “5+12” means that the algorithm generates a feasible point after five iterations, and after another twelve iterations the algorithm produces an approximately optimal solution. For the test problems, from the viewpoints of Ni and Nf0, the results show that ALGO 2.1 is obviously better than QY-IFSQP for Prob 034, 043, 065 and 100 for point (a). For point (b), again from the viewpoints of Ni, Nf0 and Fv, ALGO 2.1 is competitive with QY-IFSQP for most of test problems except Prob. 037 and 043.

Note that all test problems in Tables 3 and 4 only have inequality constraints. In order to show the performance of ALGO 2.1 for solving problems with equality constraints, ALGO 2.1 is further compared with WCH-IFSLE, GZ-IFSQP and SNOPT, respectively. The test problems and stopping criterions as well as initial iteration points are the same as in [22] and [5], respectively.

For comparing the performance of ALGO 2.1 with WCH-IFSLE and GZ-IFSQP as well as SNOPT, we use performance profiles as described in Dolan and Moré’s paper [26]. Our profiles for figures are based on the number of iterations. The function \( \rho(\tau) \) is the (cumulative) distribution function for the performance ratio within a factor \( \tau \in \mathbb{R} \). The value of \( \rho(\tau) \) is the probability that the solver will win over the rest of the solvers. The corresponding results of performance are shown in Figure 1. From Figure 1 it is obviously that the performance of ALGO 2.1 is better than that of WCH-IFSLE, i.e., ALGO 2.1 has the most wins compare with WCH-SLE. Moreover, our algorithm is competitive with SNOPT (which is a well-known SQP algorithm for solving nonlinear constrained programming) although the performance of GZ-IFSQP is better than ALGO 2.1.

Note that the above test problems are relatively small. In order to show the more clearly effectiveness of ALGO...
Table 4: Comparative numerical results of ALGO 2.1 and QY-IFSQP

| Prob | n/|I₀|/|I₂| | Point | Ni | N₀ | N₁ | F₀ | F₁ | CPU |
|------|-----|-----|-----|-----|-----|-----|
| 034  | 3/8/0 (a) ALGO 2.1 | 24 | 25 | 443 | −0.83403244521568 | 0.17 |
|      | QY-IFSQP 34 | 161 | 162 | −0.83403244524796 | − |
|      | (b) ALGO 2.1 | 5+12 | 18 | 374 | −0.83403244522367 | 0.13 |
|      | QY-IFSQP 13 | 32 | 33 | −0.83403244526530 | − |
| 035  | 3/4/0 (a) ALGO 2.1 | 12 | 13 | 119 | 0.11111111111111 | 0.05 |
|      | QY-IFSQP 10 | 46 | 46 | 0.11111111111111 | − |
|      | (b) ALGO 2.1 | 1+9 | 11 | 84 | 0.11111111111111 | 0.03 |
|      | QY-IFSQP 9 | 11 | 12 | 0.11111111111111 | − |
| 036  | 3/7/0 (a) ALGO 2.1 | 7 | 8 | 114 | −3299.99999999996 | 0.02 |
|      | QY-IFSQP 3 | 7 | 7 | −3300 | − |
|      | (b) ALGO 2.1 | 1+4 | 6 | 77 | −3299.99999999997 | 0.03 |
|      | QY-IFSQP 5 | 10 | 10 | −3300 | − |
| 037  | 3/8/0 (a) ALGO 2.1 | 23 | 24 | 467 | −3455.999999999965 | 0.17 |
|      | QY-IFSQP 12 | 34 | 34 | −3456.000000000001 | − |
|      | (b) ALGO 2.1 | 34+35 | 70 | 1189 | −3455.999999999998 | 0.58 |
|      | QY-IFSQP 29 | 113 | 119 | −3456.000000000001 | − |
| 043  | 4/3/0 (a) ALGO 2.1 | 12 | 13 | 81 | −44 | 0.06 |
|      | QY-IFSQP 14 | 29 | 29 | −44.00000000000001 | − |
|      | (b) ALGO 2.1 | 67+8 | 76 | 1270 | −44 | 0.56 |
|      | QY-IFSQP 19 | 60 | 65 | −44.0000000000001 | − |
| 044  | 4/10/0 (a) ALGO 2.1 | 20 | 21 | 448 | −14.99999999935652 | 0.17 |
|      | QY-IFSQP 7 | 13 | 13 | −15 | − |
|      | (b) ALGO 2.1 | 4+5 | 10 | 190 | −14.99999999999756 | 0.09 |
|      | QY-IFSQP 8 | 18 | 21 | −15 | − |
| 065  | 3/7/0 (a) ALGO 2.1 | 8 | 9 | 126 | 0.95352885680478 | 0.03 |
|      | QY-IFSQP 13 | 38 | 39 | 0.95352885680478 | − |
|      | (b) ALGO 2.1 | 1+13 | 15 | 217 | 0.95352885680478 | 0.11 |
|      | QY-IFSQP 15 | 41 | 44 | 0.95352885680188 | − |
| 066  | 3/8/0 (a) ALGO 2.1 | 10 | 11 | 175 | 0.51816327418156 | 0.06 |
|      | QY-IFSQP 8 | 8 | 8 | 0.51816327418154 | − |
|      | (b) ALGO 2.1 | 2+13 | 16 | 252 | 0.51816327418154 | 0.11 |
|      | QY-IFSQP 14 | 26 | 28 | 0.51816327418153 | − |
| 100  | 7/4/0 (a) ALGO 2.1 | 20 | 21 | 193 | 682.5663838261504 | 0.14 |
|      | QY-IFSQP 28 | 96 | 96 | 680.6300573744018 | − |
|      | (b) ALGO 2.1 | 7+14 | 22 | 258 | 682.5663838261520 | 0.16 |
|      | QY-IFSQP 26 | 90 | 92 | 680.6300573743961 | − |
Figure 1: The left figure shows the performance of ALGO 2.1 and WCH-IFSLE, the right figure shows the performance of ALGO 2.1 and GZ-IFSQP as well as SNOPT.

Table 5: Comparative numerical results of ALGO 2.1, G-IFSQP, JKZT-IFSQP and YLQ-FSLE

| Prob        | n/|I1|/|I2| | Algorithm | Ni | Ni0 | Nf | Fv  | CPU  |
|-------------|------------------------|------------------|--------------|-------------------------------------------------|------|------|-----|------|------|
| Svanberg-10 | 10/30/0                | ALGO 2.1         | 14           | 15     | 1140  | 15.731517 | 0.38 |
|             |                        | G-IFSQP          | 17           | 18     | 1200  | 15.731517 | 0.44 |
|             |                        | JKZT-IFSQP       | 28           | 28     | 1753  | 15.731533 | –    |
|             |                        | YLQ-FSLE         | 36           | 227    | 258   | 15.731517 | –    |
| Svanberg-30 | 30/90/0                | ALGO 2.1         | 23           | 24     | 6210  | 49.142526 | 2.67 |
|             |                        | G-IFSQP          | 25           | 26     | 5490  | 49.142526 | 2.09 |
|             |                        | JKZT-IFSQP       | 27           | 27     | 4975  | 49.142545 | –    |
|             |                        | YLQ-FSLE         | 101          | 777    | 864   | 49.142526 | –    |
| Svanberg-50 | 50/150/0               | ALGO 2.1         | 29           | 30     | 13050 | 82.581912 | 5.59 |
|             |                        | G-IFSQP          | 33           | 34     | 11550 | 82.581912 | 5.95 |
|             |                        | JKZT-IFSQP       | 37           | 37     | 11762 | 82.581928 | –    |
|             |                        | YLQ-FSLE         | 108          | 881    | 968   | 82.581912 | –    |
| Svanberg-80 | 80/240/0               | ALGO 2.1         | 38           | 39     | 33120 | 132.749819| 13.80|
|             |                        | G-IFSQP          | 42           | 43     | 24720 | 132.749819| 15.38|
|             |                        | JKZT-IFSQP       | 47           | 47     | 24100 | 132.749830| –    |
|             |                        | YLQ-FSLE         | 190          | 1666   | 1835  | 132.749819| –    |
| Svanberg-100| 100/300/0              | ALGO 2.1         | 42           | 43     | 43200 | 166.197171| 23.09|
|             |                        | G-IFSQP          | 55           | 56     | 39600 | 166.197171| 30.53|
|             |                        | JKZT-IFSQP       | 46           | 46     | 27880 | 166.197199| –    |
|             |                        | YLQ-FSLE         | 178          | 1628   | 1782  | 166.197171| –    |
The corresponding results are given in Tables 5 and 6. In Table 5, the performance of ALGO 2.1 is compared with G-IFSQP, JKZT-IFSQP and YLQ-FSLE, respectively. The initial iteration points are feasible and the stopping criterions are the same as that reported in [12]. From the results in Table 5 in viewpoints of NII and NF0, it follows that ALGO 2.1 is more effective than G-IFSQP and JKZT-SQP as well as YLQ-SLE for solving “Svanberg” problems, respectively.

Moreover, ALGO 2.1 is compared with G-IFSQP for “Svanberg” problems with infeasible initial iteration point, i.e. $x_0 = (10, \ldots, 10)^T$. The optimality thresholds are the same as in [12], and the comparative results are given in Table 6. For Prob Svanberg-20, Svanberg-30 and Svanberg-40, although Nio and Fv of ALGO 2.1 are the same as that of G-IFSQP, Nii of ALGO 2.1 is less than that of G-IFSQP. For Prob Svanberg-50 and Svanberg-200, ALGO 2.1 can enter into the feasible region more quickly than G-SQP. In view of Nio, Nii and Fv, it holds that ALGO 2.1 is more competitive than G-IFSQP. Furthermore, the comparative results of ALGO 2.1 and G-IFSQP in Tables 5 and 6 further imply that the efficiency of MQSSFD is higher than that of MSSFD. And this also proves that the conclusions in [14] is correct.
6. Conclusions

In this paper, inspired by the ideas in [8, 12], an improved SQP algorithm with arbitrary initial iteration point for solving problem (1) is proposed. Firstly, problem (1) is equivalently transformed into an associated simpler problem (2). At each iteration, the search direction is generated by solving an always QP subproblem and one (or two) SLE (s). The two SLEs have the same coefficient matrices. After a finite number of iterations, the iteration points always lie in the feasible region of problem (2), and we only need to solve the one SLE. In the process of iteration, the feasibility of the iteration points is monotone increasing. Under some mild assumptions without the strict complementary, our algorithm possesses global and superlinear convergence. Some comparative numerical results in Section 5 show that our algorithm is effective and promising.

Acknowledgements

The authors would like to thank the associated editor and the one anonymous referee for taking the time to provide detailed and highly valuable comments, which significantly improved the quality of our manuscript. The first author would also like to thank Dr. Li-Ping Tang and Dr. Jing Zhang for their help to revise English language errors in the manuscript.

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