Optimal values of bipartite entanglement in a tripartite system

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Abstract
For a general tripartite system in some pure state, an observer possessing any two parts will see them in a mixed state. By the Hughston-Jozsa-Wootters theorem, each basis set of local measurement on the third part will correspond to a particular decomposition of the bipartite mixed state into a weighted sum of pure states. It is possible to associate an average bipartite entanglement (\(S\)) with each of these decompositions. The maximum value of \(S\) is called the entanglement of assistance (\(E_A\)) while the minimum value is called the entanglement of formation (\(E_F\)). An appropriate choice of the basis set of local measurement will correspond to an optimal value of \(S\); we find here a generic optimality condition for the choice of the basis set. In the present context, we analyze the tripartite states \(W\) and \(GHZ\) and show how they are fundamentally different.

Keywords: Tripartite system; Bipartite entanglement; Optimality condition

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1. Introduction
Besides its fundamental importance in interpreting and understanding quantum mechanics, the quantum entanglement has attracted an immense interest in recent times because of its potential to play a significant role in modern technology. In addition, the quantum entanglement has now become a powerful tool to study the quantum many-body systems [1]. To be able to use the entanglement effectively and efficiently, it is necessary to characterize and quantify it by meaningful ways. There are many entanglement
measures proposed for this purpose. In the axiomatic approach, there are some conditions to be satisfied for an entanglement measure to be a proper monotone \[2, 3\]. For pure bipartite state, the von Neumann entropy is a suitable and widely used entanglement measure. Unfortunately it is not a good measure for non-pure or mixed bipartite state. For the mixed states, two popular entanglement measures are the concurrence \[4\] and negativity \[5\]. These two measures are reliable for distinguishing entangled states from separable states respectively for \(2 \times 2\) system, and \(2 \times 2\) and \(2 \times 3\) systems. Generalization of these measures to higher dimensions is possible, but most of the time they are not satisfactory and unique \[2, 6, 7\].

The entanglement of formation \((E_F^\infty) [8]\) for a mixed state is a more general concept than the concurrence. For \(2 \times 2\) system, the concurrence is monotonically related to \(E_F^\infty [9]\). Even though \(E_F^\infty\) is a good measure for any bipartite mixed state, it is extremely difficult to calculate. The reason can be understood from the definition of the quantity, as shown below,

\[
E_F^\infty(\rho) = \inf \left\{ \sum_i p_i S_i(\vert \psi_i \rangle) : p_i \geq 0, \sum_i p_i = 1; \rho = \sum_i p_i \vert \psi_i \rangle \langle \psi_i \vert \right\}. \tag{1}
\]

Here \(S_i\) can be any good entanglement measure of the pure bipartite state \(\vert \psi_i \rangle\) (e.g. it can be von Neumann entropy). Basically, to find \(E_F^\infty\) one has to do minimization of an average quantity over infinite possible decomposition of the given mixed state (\(\rho\)).

The entanglement of assistance \((E_A^\infty) [10]\) is another measure for an arbitrary bipartite mixed state \([11]\). Though it is not a proper monotone \([11]\), it got importance due to its possible application in quantum technology. It is defined as the maximum possible average entropy between two parties, as shown below,

\[
E_A^\infty(\rho) = \sup \left\{ \sum_i p_i S_i(\vert \psi_i \rangle) : p_i \geq 0, \sum_i p_i = 1; \rho = \sum_i p_i \vert \psi_i \rangle \langle \psi_i \vert \right\}. \tag{2}
\]

Quantifying and classifying multipartite entanglement is a very hard problem; it is now generally accepted that a single number is not enough for the purpose \([2]\). In this work we concentrate on a general tripartite system in a pure state. First we note that, according to the Hughston-Jozsa-Wootters theorem \([12]\), any finite decomposition compatible with a given mixed bipartite state can be created by a local measurement on a third part added
in a pure state with the bipartite system. For a general tripartite system 
\((A - B - C)\) in pure state, this theorem implies that, any local measure-
ment on a part \((C)\) will correspond to a particular decomposition of the
bipartite mixed state that the other two parts \((A - B)\) are in. It may be
here worth mentioning that all the quantum measurements in this work are
considered to be non-selective projective-type (von Neumann measurement).
One can associate an average entanglement \((\overline{S})\) with every decomposition
of a mixed state; very often this quantity is called the entanglement ‘localized’
between two parts of a tripartite system by doing a local measurement on the
third part. It is possible to optimize the quantity \(\overline{S}\) by choosing appropriate
measurement basis. The maximum and minimum values of \(\overline{S}\) are
termed respectively as the entanglement of assistance \((E_A)\) and the entangle-
ment of formation \((E_F)\) for the given tripartite pure state \(|\Psi\rangle\). These two quantities
are given by following equations,

\[
E_F(|\Psi\rangle) = \inf \left\{ \sum_i p_i S_i(|\psi_i\rangle^{AB}) : p_i \geq 0, \sum_i p_i = 1; \right\}
\]

\[
|\Psi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle^C |\psi_i\rangle^{AB}
\]

\[
E_A(|\Psi\rangle) = \sup \left\{ \sum_i p_i S_i(|\psi_i\rangle^{AB}) : p_i \geq 0, \sum_i p_i = 1; \right\}
\]

\[
|\Psi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle^C |\psi_i\rangle^{AB}
\]

Here \(|\phi_i\rangle^C\)s form an orthonormal basis set of \(C\) and this set is used as a basis
set for local measurement. After measurement, parts \(A\) and \(B\) jointly assume
pure state \(|\psi_i\rangle^{AB}\) with probability \(p_i\). In general \(|\psi_i\rangle^{AB}\)s are not orthogonal
to each other (see next section).

Let us now deliberate little more on the two definitions of the entangle-
ment of formation (Eqs. (1) and (3)) and the two definitions of the entangle-
ment of assistance (Eqs. (2) and (4)). It is known that, a person possessing
two parts \((A\) and \(B)\) of a tripartite system (which is in a pure state \(|\Psi\rangle\)) will
only see a mixed state \(\rho^{AB} = \text{Tr}_C(|\Psi\rangle\langle\Psi|)\). Each basis set of measurement
of \(C\) corresponds to a particular decomposition of the mixed state \(\rho^{AB}\). The
number of pure states appearing in a decomposition can not exceed the basis
set dimension of \(C\) (say, \(D_C\)) [12]. On the other hand, the number of terms in
an unrestricted decomposition of a mixed bipartite state (without reference to $C$ or any pure tripartite state) can be in principle any large number. For a given pure tripartite state, to generate a decomposition of $\rho^{AB}$ where the number of terms is larger than $D_C$, one can add a suitable ancilla to the the third part $C$ and do a joint measurement \cite{12}. This discussion shows that, $E^\infty_F(\rho^{AB}) \leq E_F(\langle \Psi \rangle)$ and $E^\infty_A(\rho^{AB}) \geq E_A(\langle \Psi \rangle)$.

In general finding $E^\infty_F$ or $E^\infty_A$ for an arbitrary bipartite mixed state is difficult. As we mentioned before, in principle there can be any number of terms in an unrestricted decomposition of a mixed state. This makes even numerical calculations of the quantities really hard. It is though speculated that, to evaluate $E^\infty_F$ or $E^\infty_A$ for a mixed bipartite state $\rho$, it is enough to consider only a finite number of terms in the decomposition of $\rho$. For example, it was proved that it is sufficient to consider only $r^2$ terms in a decomposition to find $E^\infty_F$, where $r$ is the rank of the mixed state \cite{13}. In fact it turned out that, for $2 \times 2$ systems it is enough to consider only four states for the purpose \cite{14,15}. In this context, for a given tripartite system in some pure state, we derive here an optimality condition for the measurement basis set; when satisfied, the average entanglement ($\bar{S}$) associated with the decomposition corresponding to the measurement is optimal. This condition will not only be helpful in studying a given pure tripartite state, by using ancilla, it would also be helpful in finding the entanglement of formation and the entanglement of assistance of an arbitrary mixed state. The condition derived here can be used to verify whether a given solution (i.e., given decomposition of the mixed state under study) is optimal.

In the last part of this paper, we analyze two tripartite states $W$ and $GHZ$, and show how they are fundamentally different in the present context.

2. The optimality condition

Let $A$, $B$ and $C$ are three parts of a tripartite system in the pure state $\langle \Psi \rangle$. A local quantum measurement on $C$ by some basis set would result in $S (= A + B)$ assuming different pure states with appropriate probabilities.

When expressed in the product basis states of $C$ and $S$, the given tripartite state becomes,

$$|\Psi\rangle = \sum_{i,j=1,1}^{D_C,D_S} g_{i,j} |\xi_i\rangle^C |\phi_j\rangle^S.$$  \hspace{1cm} (5)
Here $|\xi_i^C\rangle$'s ($|\phi_i^S\rangle$'s) are some orthonormal basis vectors of the state space of $C$ ($S$) with dimensionality $D_C$ ($D_S$). This state is assumed to be normalized: $\sum_{i,j=1,1}^D g_{i,j} g_{i,j}^* = 1$. Let us now rewrite this state in the following special form,

$$|\Psi\rangle = \sum_{i=1}^D \sqrt{p_i} |\xi_i^C\rangle |\xi_i^S\rangle,$$  \hspace{1cm} (6)

with $p_i = \sum_{j=1}^{D_S} g_{i,j} g_{i,j}^*$ and $|\xi_i^S\rangle = \sum_{j=1}^{D_S} \frac{g_{i,j}}{\sqrt{p_i}} |\phi_j^S\rangle$. Here the summation runs over nonzero $p_i$'s, numbering $D \leq D_C$. In general, states $|\xi_i^S\rangle$'s are not orthogonal (but they all are normalized). The operational interpretation of the later expression of the state $|\Psi\rangle$ is that, if we perform a local quantum measurement on $C$ by the basis set $\{\xi^C\}$, the state $|\Psi\rangle$ will collapse and we will get $S$ in different pure states $|\xi_i^S\rangle$'s with corresponding probabilities $p_i$'s.

If $S_i$ is the von Neumann entropy of the pure bipartite state $|\xi_i^S\rangle$, then after the measurement, the average entropy (quantifying average entanglement) localized between $A$ and $B$ would be,

$$\bar{S}\{\xi^C\} = \sum_{i=1}^D p_i S_i.$$  \hspace{1cm} (7)

As both $p_i$'s and $S_i$'s depend on the choice of the basis set of measurement, $\{\xi^C\}$, the average entropy $\bar{S}$ will also depend on the choice of the basis set. We will now derive a condition for the choice of the basis set which optimizes $\bar{S}$.

We first note that, any basis set of measurement can be obtained from an (arbitrary) initial basis set $\{\xi^C\}$ by application of a series of elementary transformations (ETs). Here an ET is a small-angle orthonormal transformation (rotation) of any two basis states keeping others unchanged. We now derive first order change in $\bar{S}$ due to an ET. If $|\xi_i^C\rangle$ and $|\xi_j^C\rangle$ are any two initial basis states, then the two new basis states obtained by an ET would be,

$$|\xi_i^C\rangle = |\xi_i^C\rangle + \epsilon |\xi_j^C\rangle \text{ and } |\xi_j^C\rangle = |\xi_j^C\rangle - \epsilon |\xi_i^C\rangle.$$  \hspace{1cm} (8)

Here $\epsilon$ is the small angle (a parameter) whose higher order terms can be neglected. We may note that these two new basis states are orthogonal (i.e., $C\langle \xi_i^C | \xi_j^C \rangle = 0$) and normalized within first order approximation. Due to change in the basis states of measurement, corresponding probabilities and
states of $S$ would also change (cf. Eq. (6)). We will now relate these new probabilities and states with the older ones.

At this stage it is advantageous to express all the probabilities as the diagonal elements of a density operator (matrix), which is in our case the reduced density matrix (RDM) of $C$ (denoted by $\rho^C$). This RDM is given by $\rho^C = \text{Tr}_S(|\Psi\rangle\langle\Psi|) = gg^\dagger$; where $g = [g_{i,j}]$ is the matrix representing the tripartite state $|\Psi\rangle$ expressed in some product basis states of $C$ and $S$ (cf. Eq. (5)). Using this RDM, the probability associated with a basis state $|\xi\rangle$ of $C$ would be $p = C\langle\xi|\rho^C|\xi\rangle$. This allows us to write the new probabilities associated with the new basis states in Eq. (8) as,

$$p'_i = p_i + \epsilon k_{ij} \text{ and } p'_j = p_j - \epsilon k_{ji}, \quad (9)$$

with $k_{ij} = k_{ji} = C\langle\xi_{i}|\rho^C|\xi_{j}\rangle + C\langle\xi_{j}|\rho^C|\xi_{i}\rangle$.

Let us first consider the case when none of the $p_i$ and $p_j$ is zero. Now if $|\xi'_i\rangle$ and $|\xi'_j\rangle$ are the new states of $S$ corresponding to the two new basis states of $C$ (cf. Eq. (6)), then in the new scenario, the state $|\Psi\rangle$ can be rewritten as,

$$|\Psi\rangle = \sqrt{p'_i}|\xi'_i\rangle^C|\xi'_i\rangle^S + \sqrt{p'_j}|\xi'_j\rangle^C|\xi'_j\rangle^S + \cdots \quad (10)$$

Here we only focus on $i$-th and $j$-th states, as other terms are unchanged by the considered ET. Now using Eqs. (3) and (9) in the above expression and then comparing the terms associated with the initial basis states $|\xi_{i}\rangle$ and $|\xi_{j}\rangle$ from the two different expressions of $|\Psi\rangle$ (in Eqs. (6) and (10)), we get the following solutions for the new states of $S$:

$$|\xi'_{i}\rangle^S = |\xi_{i}\rangle^S + \epsilon (a_{ij}|\xi_{i}\rangle^S + b_{ij}|\xi_{j}\rangle^S) \quad \text{and} \quad (11)$$

$$|\xi'_{j}\rangle^S = |\xi_{j}\rangle^S - \epsilon (a_{ji}|\xi_{j}\rangle^S + b_{ji}|\xi_{i}\rangle^S). \quad (12)$$

Here $a_{ij} = -\frac{1}{2}k_{ij}p_i^{-1}$ and $b_{ij} = p_j^{1/2}p_i^{-1/2}$ in Eq. (11). By interchanging the indices $i$ and $j$ we get the similar terms in Eq. (12).

Let $\rho^A(\xi_i)$ and $\rho^A(\xi_j)$ are the RDMs of $A$ when $S$ is respectively in pure states $|\xi_i\rangle^S$ and $|\xi_j\rangle^S$. For example, $\rho^A(\xi_i) = \text{Tr}_B(|\xi_i\rangle^S \langle\xi_i|)$. Now if $Q = [Q_{lm}]$ and $R = [R_{lm}]$ are the matrices representing respectively the states $|\xi_i\rangle^S$ and $|\xi_j\rangle^S$ expressed in some product basis states of the parts $A$ and $B$, then in terms of these matrices, the RDMs of $A$ would be $\rho^A(\xi_i) = QQ^\dagger$ and
\( \rho^A(\xi_j) = RR^\dagger \). Similarly, the RDMs of \( A \) corresponding to the new states, given in Eqs. (11) and (12), would be,

\[
\rho^A(\xi'_i) = \rho^A(\xi_i) + \epsilon \left( 2a_{ij}\rho^A(\xi_i) + 2b_{ij}\Delta_{ij} \right) \quad \text{and} \quad (13)
\]

\[
\rho^A(\xi'_j) = \rho^A(\xi_j) - \epsilon \left( 2a_{ji}\rho^A(\xi_j) + 2b_{ji}\Delta_{ji} \right). \quad (14)
\]

Here \( \Delta_{ij} = \frac{1}{2}(QR^\dagger + RQ^\dagger) \), a Hermitian matrix. Let us here denote the first order changes in the RDMs in Eqs. (13) and (14) respectively as \( \epsilon \rho^A_{1}(ij) \) and \(-\epsilon \rho^A_{1}(ji)\); here we have

\[
\rho^A_{1}(ij) = 2a_{ij}\rho^A(\xi_i) + 2b_{ij}\Delta_{ij} \quad (15)
\]

\[
\rho^A_{1}(ji) = 2a_{ji}\rho^A(\xi_j) + 2b_{ji}\Delta_{ji}. \quad (16)
\]

It is worth mentioning that, as the trace (Tr) of any RDM is 1, we must have (through Eqs. (13) and (14)),

\[
\text{Tr} \rho^A_{1}(ij) = \text{Tr} \rho^A_{1}(ji) = 0. \quad (17)
\]

Now we can use the following relation (see Appendix A),

\[
\rho^A(\xi'_i) \log_2 \rho^A(\xi'_i) = \rho^A(\xi_i) \log_2 \rho^A(\xi_i) + \epsilon(\log_2 e)\rho^A_{1}(ij) + \epsilon \rho^A_{1}(ij) \log_2 \rho^A(\xi_i), \quad (18)
\]

to obtain the entropy corresponding to the new state \( |\xi'_i\rangle^S \). Tracing over both sides of this relation (Eq. (18)) and a similar relation for the state \( |\xi'_j\rangle^S \), we obtain respectively the following entropies for the new states of \( S \),

\[
S'_i = S_i - \epsilon \text{Tr} \rho^A_{1}(ij) \log_2 \rho^A(\xi_i) \quad \text{and} \quad (19)
\]

\[
S'_j = S_j + \epsilon \text{Tr} \rho^A_{1}(ji) \log_2 \rho^A(\xi_j). \quad (20)
\]

Here we used Eq. (17) to get these relations. Let us now denote the first order changes in entropies in Eqs. (19) and (20) as \(-\epsilon S^1_{ij}\) and \(\epsilon S^1_{ji}\) respectively, with,

\[
S^1_{ij} = \text{Tr} \rho^A_{1}(ij) \log_2 \rho^A(\xi_i) \quad (21)
\]

\[
S^1_{ji} = \text{Tr} \rho^A_{1}(ji) \log_2 \rho^A(\xi_j). \quad (22)
\]

Now using these new entropies in Eqs. (19) and (20) along with the new probabilities in Eq. (9), we get the new average entropy (cf. Eq. (7)):

\[
S' = \sum_{i=1}^{D} p'_i S'_i = \bar{S} + \epsilon \bar{S}_1, \quad (23)
\]
where, \( \bar{S}_1 = k_{ij}S_i - p_iS^1_{ij} - k_{ji}S_j + p_jS^1_{ji} \). This \( \bar{S}_1 \) is the first order change in the average entropy due to an ET of the \( i \)th and \( j \)th basis states.

Before we set the optimality condition, let us now check the special cases when both \( p_i \) and \( p_j \) are or one of them is zero. When \( p_i = p_j = 0 \), then \( k_{ij} = k_{ji} = 0 \). Therefore, \( p'_i = p'_j = 0 \) (see Eq. (9)). Which implies that \( \bar{S}_1 \) is zero. On the other hand, when \( p_i \neq 0 \) and \( p_j = 0 \), we gave again \( k_{ij} = k_{ji} = 0 \). From Eq. (9) we have \( p'_i = p_i \) and \( p'_j = 0 \). From Eq. (11), we also have \( |\xi'_i\rangle^S = |\xi_i\rangle^S \). This implies that, \( \bar{S}_1 = 0 \). In these two special cases the first order change in average entropy due to any elementary transformation (ET) of the two basis states concerned is always zero; therefore we need not consider these cases to determine whether some basis set of measurement is optimal.

So, the desired optimality condition can be obtained by equating the first order change in the average entropy due to an ET of any two basis states for which corresponding probabilities are nonzero. This condition is given by the following equation \( \bar{S}_1 = 0 \) (cf. Eq. (23)) or,

\[
k_{ij}S_i - p_iS^1_{ij} = k_{ji}S_j - p_jS^1_{ji},
\]

(24)

for all \( i \) and \( j \) for which corresponding probabilities are nonzero.

3. Study of W and GHZ states

There are two types of genuine pure tripartite states (which can not be converted to each other by the SLOCC operation [16]), namely \( W \) and \( GHZ \) states. These states are defined as follows,

\[
|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)
\]

(25)

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)
\]

(26)

Main difference between these two states is, the entanglement in \( |W\rangle \) is robust in the sense that when one qubit is traced out other two qubits remain entangled, on the contrary, the entanglement in \( |GHZ\rangle \) is fragile due to the fact that when one qubit is traced out other two qubits become unentangled. In what follows, we will study the average entanglement \( \bar{S} \) between two qubits for both the tripartite states. In particular we show that, for \( |W\rangle \) state, \( 0 < E_F < E_A < 1 \), on the contrary for \( |GHZ\rangle \) state, \( E_F = 0 \) and
$E_A = 1$, i.e. those two quantities attain their extreme possible values. (The quantities $E_F$ and $E_A$ are defined in Eqs. (3) and (4) respectively.)

For $W$ and $GHZ$ states, we will now first see that if measurement basis set is the set of eigenstates of the RDM of $C$ (i.e., $\rho^C$), then the corresponding average entropy 'localized' in $S$ ($= A + B$) is optimal. We note that when measurement is performed by eigenstates of RDM ($\rho^C$), $k_{ij} = k_{ji} = 0$ and Eq. (24) reduces to

$$\text{Tr} \Delta_{ij} \log_2 \rho^A(\xi_i) = \text{Tr} \Delta_{ji} \log_2 \rho^A(\xi_j),$$

(27)

where $\{\xi\}$ is the set of the eigenstates of the RDM of $S$ (i.e., $\rho^S$).

For the state $|W\rangle$, two eigenstates of $\rho^S$ corresponding to the non-zero eigenvalues are $|00\rangle$ and $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$. It is easy to check that $2 \times 2$ matrices representing these two states satisfy the condition given in Eq. (27). Therefore the basis set of measurement $\{|1\rangle, |0\rangle\}$ (the eigenstates of $\rho^C$) is optimal. It is easy to find the corresponding optimal value of average entanglement ($\bar{S}$) by noting the Schmidt decomposition of the state:

$$|W\rangle = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}(|10\rangle^S + |01\rangle^S)\right)|0\rangle^C + \frac{1}{\sqrt{3}}|00\rangle^S|1\rangle^C.$$ Now recognizing that the entropies of the two states of $S$ appearing in the decomposition are 1 and 0 respectively, and corresponding probabilities are $2/3$ and $1/3$, we get the optimal value of $\bar{S}$ to be $2/3$.

For the state $|GHZ\rangle$, two eigenstates of $\rho^S$ corresponding to the non-zero eigenvalues are $|00\rangle$ and $|11\rangle$. It is again easy to check that $2 \times 2$ matrices representing these two states satisfy the condition given in Eq. (27). Therefore the basis set of measurement $\{|1\rangle, |0\rangle\}$ (the eigenstates of $\rho^C$) is optimal. It is again easy to find the corresponding optimal value of average entanglement ($\bar{S}$) by noting the Schmidt decomposition of the state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}|00\rangle^S|0\rangle^C + \frac{1}{\sqrt{2}}|11\rangle^S|1\rangle^C.$$ Now recognizing that the entropies of the two states of $S$ appearing in the decomposition are both zero, we get the optimal value of $\bar{S}$ to be 0.

Though the optimality test can tell us whether a basis set of measurement is optimal, it can not tell us whether the corresponding average entropy is a maximum (i.e., $E_A$) or minimum (i.e., $E_F$). Fortunately, for any $2 \times 2 \times 2$ pure tripartite state, it is possible to calculate $E_F$ and $E_A$ (cf. Eqs. (3) and (4)) exactly. Now we will do the detail exact calculation for both the states ($W$ and $GHZ$) to verify the above results as well as to find $E_F$ and $E_A$ for those two states.
To proceed, let us take the most general set of measurement basis as \( \{ \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle, \sin \frac{\theta}{2} |0\rangle - e^{i\phi} \cos \frac{\theta}{2} |1\rangle \} \) (two diagonally opposite points on Bloch sphere), where \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \). If we now denote this basis set by \( \{|\xi_1\rangle^C, |\xi_2\rangle^C\} \), then the W state can be written as,

\[
|W\rangle = \frac{1}{\sqrt{3}} \sqrt{|a|^2 + 2|b|^2} |\xi_1\rangle^C (a'|00\rangle^S + b'|10\rangle^S + b'|01\rangle^S) + \frac{1}{\sqrt{3}} \sqrt{|c|^2 + 2|d|^2} |\xi_2\rangle^C (c'|00\rangle^S + d'|10\rangle^S + d'|01\rangle^S)
\]

(28)

Here \( a' = \frac{a}{\sqrt{|a|^2 + 2|b|^2}}, \quad b' = \frac{b}{\sqrt{|a|^2 + 2|b|^2}}, \quad c' = \frac{c}{\sqrt{|c|^2 + 2|d|^2}} \) and \( d' = \frac{d}{\sqrt{|c|^2 + 2|d|^2}} \). In terms of \( \theta \) and \( \phi \), these parameters are given by \( a = e^{-i\phi} \sin \frac{\theta}{2}, \quad b = \cos \frac{\theta}{2}, \quad c = -e^{-i\phi} \cos \frac{\theta}{2} \) and \( d = \sin \frac{\theta}{2} \). The operational interpretation of the above expression of the W state (Eq. (28)) is that if one does quantum measurement on \( C \) by the basis set \( \{|\xi_1\rangle^C, |\xi_2\rangle^C\} \), then \( S \) will be found in the state \( |\xi_1\rangle^S \) with probability \( p_1 = \frac{1}{3}(|a|^2 + 2|b|^2) \) and in the state \( |\xi_2\rangle^S \) with probability \( p_2 = \frac{1}{3}(|c|^2 + 2|d|^2) \); here \( |\xi_1\rangle^S = a'|00\rangle^S + b'|10\rangle^S + b'|01\rangle^S \) and \( |\xi_2\rangle^S = c'|00\rangle^S + d'|10\rangle^S + d'|01\rangle^S \). It is easy to find \( 2 \times 2 \) RDMs of \( A \) (i.e., \( \rho^A \)) for both the states \( |\xi_1\rangle^S \) and \( |\xi_2\rangle^S \). From these RDMs, one can get corresponding entropies and the average entropy thereafter (see Appendix B). This average entropy \( \bar{S} \) is only a function of \( \theta \); \( \phi \) does not appear in its expression. The minima and maxima can be found from \( \frac{\partial \bar{S}}{\partial \theta} = 0 \). We find that global maximum corresponds to \( \theta = 0 \) (or equivalently \( \theta = \pi \)) and global minimum corresponds to \( \theta = \frac{\pi}{2} \). These can be seen clearly from Fig. 1. We may note that, when \( \theta = 0 \) or \( \pi \), within a global phase factor, the measurement basis set becomes \( \{|1\rangle, |0\rangle\} \). This asserts that the eigenstates of \( \rho^C \) are the optimal basis set of measurement. For the W state, the maximum and minimum possible values of \( \bar{S} \) are \( 2/3 \) (\( \simeq 0.67 \)) and \( \log_2 3 - \frac{2\sqrt{3}}{3} \log_2 \left( \frac{3+\sqrt{5}}{2} \right) \) (\( \simeq 0.55 \)) respectively. So for this tripartite state \( E_F \simeq 0.55 \) and \( E_A \simeq 0.67 \).

A similar calculation for the GHZ state is also done (see Appendix C); the global minimum and maximum occur at \( \theta = 0 \) (or equivalently at \( \theta = \pi \)) and at \( \theta = \frac{\pi}{2} \) respectively. This minimum and maximum values of the average entropy \( \bar{S} \) are respectively 0 and 1 (see Fig. 1). So for this tripartite state \( E_F = 0 \) and \( E_A = 1 \).
4. Conclusion

Any mixed bipartite state can be decomposed in innumerable ways. By the Hughston-Jozsa-Wootters theorem, any finite decomposition can be seen as a result of some local measurement on a separate part added in a pure state with the bipartite system. Since each of these decompositions can be assigned an average entanglement, by choosing an appropriate measurement basis set we will be able to get an optimal value of the average quantity. For a given tripartite system in some pure state, we have derived here an optimality condition for the measurement basis set; when satisfied, the average entanglement associated with the decomposition due to the measurement will be optimal. By the use of ancilla, this optimality condition would also be helpful in finding entanglement of formation and entanglement of assistance for a given bipartite mixed state (without any reference to tripartite state). This condition will also be of use in checking whether a given measurement basis set is optimal.

In the second part, we have studied two inequivalent tripartite states (W and GHZ); in particular, we showed that the set of eigenvectors of the RDM of a qubit is an optimal basis set of measurement (for both the states). To verify the result and to find the maximum and minimum possible average

Figure 1: The average entropy ($\bar{S}$) as a function of rotation angle $\theta$. 

[Graph showing the relationship between average entropy and rotation angle for W and GHZ states.]
bipartite entanglement ($E_A$ and $E_F$ respectively), we have also done the
detailed exact calculations for both the states. The values of $E_A$ and $E_F$
found for the states show why they are inequivalent.

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Appendix A

The expansion of the operator $\rho^A(\xi') \log_2 \rho^A(\xi')$ in the power series of the
parameter $\epsilon$ is difficult as $\rho^A_1(ij)$ and $\rho^A(i)$ do not commute in general.
Ultimately since we only need to know the trace of this operator (to find
entropy), and as Tr MN = Tr NM for any two compatible finite matrices $M$
and $N$, we can proceed in the following way to get the first order change in the
operator.

If $\lambda_l$ is the $l$-th eigenvalue of $\rho^A(i)$ and $\lambda^1_l$ is the expectation value of
$\rho^A_1(ij)$ in the corresponding eigenstate of $\rho^A(i)$, then the expectation value of the
operator $\rho^A(\xi') \log_2 \rho^A(\xi')$ in the eigenstate would be $(\lambda_l + \epsilon \lambda^1_l) \log_2 \lambda_l (1 + \epsilon \lambda^1_l)$. This equals to $\lambda_l \log_2 \lambda_l + \epsilon (\log_2 e) \lambda^1_l + \epsilon \lambda^1_l \log_2 \lambda_l$, using $\log_2 (1 + \epsilon \lambda^1_l) = \epsilon (\log_2 e) \lambda^1_l$. This eventually suggests the operator relation we use in the
text. This operator relation is not ill-defined due to the last term since when
$\lambda_l$ is zero, $\lambda^1_l$ also becomes zero (this can be understood by singular value
decomposition of the matrix $Q$; see form of $\rho^A$ and $\rho^A_1$ in the main text).

Appendix B

For the $W$ state, when $S (=A+B)$ is in the bipartite state $|\xi_1\rangle^S = a'|00\rangle^S + b'|10\rangle^S + b'|01\rangle^S$, the RDM for the part $A$ is given by,

$$
\rho^A(\xi_1) = \begin{pmatrix}
|a'|^2 + |b'|^2 & a'(b')^* \\
(b'(a')^*)^* & |b'|^2
\end{pmatrix},
$$

with $\lambda^\pm_1 = \frac{1}{2} \left[ 1 \pm \frac{\sin^2 \frac{\theta}{2}}{1 + \cos^2 \frac{\theta}{2}} \sqrt{1 + 3 \cos^2 \frac{\theta}{2}} \right]$ being its two eigenvalues expressed
in terms of the parameter $\theta$. The entropy of this state is given by $S_1 = \frac{1}{2} \left[ 1 \pm \frac{\sin^2 \frac{\theta}{2}}{1 + \cos^2 \frac{\theta}{2}} \sqrt{1 + 3 \cos^2 \frac{\theta}{2}} \right] \log_2 \left[ 1 \pm \frac{\sin^2 \frac{\theta}{2}}{1 + \cos^2 \frac{\theta}{2}} \sqrt{1 + 3 \cos^2 \frac{\theta}{2}} \right]$.
−λ_1^+ log_2 λ_1^+ − λ_1^- log_2 λ_1^- . Similarly, in case of the state \( |\xi_2\rangle^S = c'|00\rangle^S + d'|10\rangle^S + d'|01\rangle^S \), the entropy is given by \( S_2 = −λ_2^+ log_2 λ_2^+ − λ_2^- log_2 λ_2^- \), where \( λ_2^± = \frac{1}{2} \left[ 1 ± \frac{cos^2 θ}{1+sin^2 \frac{θ}{2}} \sqrt{1 + 3sin^2 \frac{θ}{2}} \right] \). These two states come with probabilities \( p_1 = \frac{1}{3}(1+cos^2 \frac{θ}{2}) \) and \( p_2 = \frac{1}{3}(1+sin^2 \frac{θ}{2}) \) respectively when quantum measurement is done on part \( C \) by the basis set \( \{ |\xi_1\rangle^C, |\xi_2\rangle^C \} \). The average entropy localized in \( S \) due to measurement on \( C \) is therefore, \( \bar{S} = p_1 S_1 + p_2 S_2 \). We may here note that, \( \bar{S} \) depends only on the parameter \( θ \).

Appendix C

For the GHZ state, when a measurement is done on \( C \) by the basis set \( \{ |\xi_1\rangle^C, |\xi_2\rangle^C \} \), \( S \) will be found in the bipartite state \( |\xi_1\rangle^S = a|00\rangle^S + b|11\rangle^S \) with probability \( p_1 = \frac{1}{2} \) and in the state \( |\xi_2\rangle^S = c|00\rangle^S + d|11\rangle^S \) with probability \( p_2 = \frac{1}{2} \). Here, \( a = \cos \frac{θ}{2}, b = e^{-iφ}sin \frac{θ}{2}, c = sin \frac{θ}{2} \) and \( d = -e^{-iφ}\cos \frac{θ}{2} \). When \( S \) is in \( |\xi_1\rangle^S \), the RDM for the part \( A \) is given by,

\[
\rho^A(\xi_1) = \begin{pmatrix}
|a|^2 & 0 \\
0 & |b|^2 
\end{pmatrix},
\]

with eigenvalues \( λ_{11} = \cos^2 \frac{θ}{2} \) and \( λ_{12} = \sin^2 \frac{θ}{2} \) expressed in terms of the parameters \( θ \). The entropy of this state is given by \( S_1 = −λ_{11} log_2 λ_{11} − λ_{12} log_2 λ_{12} \). Similarly, the entropy for the state \( |\xi_2\rangle^S \) is given by \( S_2 = −λ_{21} log_2 λ_{21} − λ_{22} log_2 λ_{22} \); where, \( λ_{21} = \sin^2 \frac{θ}{2} \) and \( λ_{22} = \cos^2 \frac{θ}{2} \). The average entropy localized in \( S \) due to measurement on \( C \) is therefore, \( \bar{S} = p_1 S_1 + p_2 S_2 \) or \( −(\cos^2 \frac{θ}{2} log_2 \cos^2 \frac{θ}{2} + \sin^2 \frac{θ}{2} log_2 \sin^2 \frac{θ}{2}) \).

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