A one-level additive Schwarz preconditioner for a discontinuous Petrov-Galerkin method

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1 A discontinuous Petrov-Galerkin method for a model Poisson problem

Discontinuous Petrov-Galerkin (DPG) methods are new discontinuous Galerkin methods \cite{3, 4, 5, 6, 7, 8} with interesting properties. In this article we consider a domain decomposition preconditioner for a DPG method for the Poisson problem. Let $\Omega$ be a polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$), $\Omega_h$ be a simplicial triangulation of $\Omega$. Following the notation in \cite{8}, the model Poisson problem (in an ultraweak formulation) is to find $U \in U$ such that

$$b(\sigma, \tau) = l(\tau) \quad \forall \, \tau \in V,$$

where $U = [L_2(\Omega)]^d \times L_2(\Omega) \times H^1_0(\Omega) \times \mathbb{R}$, $V = H(\text{div}; \Omega_h) \times H^1(\Omega_h)$,

$$b(\sigma, \tau) = \int_{\Omega} \sigma \cdot \tau \, dx - \sum_{K \in \Omega_h} \int_K \hat{u} \tau \, dx + \sum_{K \in \Omega_h} \int_{\partial K} \hat{\sigma} \cdot n \, ds$$

$$- \sum_{K \in \Omega_h} \int_K \sigma \cdot \text{grad} \, v \, dx + \sum_{K \in \Omega_h} \int_{\partial K} v \hat{\sigma} \, n \, ds$$

for $\sigma = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U$ and $\tau = (\tau, v) \in V$, and $l(\tau) = \int_{\Omega} f v \, dx$.

Here $H^{-1/2}(\partial \Omega_h)$ (resp. $H^{-1/2}(\partial \Omega_h)$) is the subspace of $\prod_{K \in \Omega_h} H^{1/2}(\partial K)$ (resp. $\prod_{K \in \Omega_h} H^{-1/2}(\partial K)$) consisting of the traces of functions in $H^1(\Omega)$ (resp. traces of the normal components of vector fields in $H(\text{div}; \Omega)$, and $H(\text{div}; \Omega_h)$ (resp. $H^1(\Omega_h)$) is the space of piecewise $H(\text{div})$ vector fields (resp. $H^1$ functions). The

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inner product on $V$ is given by

$$
((\tau_1, v_1), (\tau_2, v_2))_V = \sum_{K \in \Omega_h} \int_K [\tau_1 \cdot \tau_2 + \div \tau_1 \div v_1 + v_1 v_2 + \grad v_1 \cdot \grad v_2] \, dx.
$$

The DPG method for the Poisson problem computes $v_h \in U_h$ such that

$$
b(u_h, v) = I(v) \quad \forall v \in V_h.
$$

Here the trial space $U_h(\subset U)$ is defined by

$$
U_h = \prod_{K \in \Omega_h} [P_m(K)]^d \times \prod_{K \in \Omega_h} P_m(K) \times \tilde{P}_{m+1}(\partial \Omega_h) \times P_m(\partial \Omega_h),
$$

$P_m(K)$ is the space of polynomials of total degree $\leq m$ on an element $K$, $\tilde{P}_{m+1}(\partial \Omega_h) = H_{\partial \Omega_h}^{1/2}(\partial \Omega_h) \cap \prod_{K \in \Omega_h} \tilde{P}_{m+1}(\partial K)$, where $\tilde{P}_{m+1}(\partial K)$ is the restriction of $P_{m+1}(K)$ to $\partial K$, and $P_m(\partial \Omega_h) = H_{\partial \Omega_h}^{-1/2}(\partial \Omega_h) \cap \prod_{K \in \Omega_h} P_m(\partial K)$, where $P_m(\partial K)$ is the space of piecewise polynomials on the faces of $K$ with total degree $\leq m$.

Let $V' \{((\tau, v) \in V : \tau|_K \in [P_{m+2}(K)]^d, v|_K \in P_r(K) \forall K \in \Omega_h \}$ for some $r \geq m + d$. The discrete trial-to-test map $T_h : U_h \rightarrow V'$ is defined by

$$
(T_h u_h, v) = b(u_h, v), \quad \forall u_h \in U_h, \ x \in V',
$$

and the test space $V_h$ is $T_h U_h$.

We can rewrite (1) as $a_h(u_h, w) = l(T_h w)$ for all $w \in U_h$, where

$$
a_h(w, w) = b_h(w, T_h w) = (T_h w, T_h w)_V
$$

is an SPD bilinear form on $V_h \times V_h$, and we define an operator $A_h : U_h \rightarrow U_h'$ by

$$
\langle A_h w, w \rangle = a_h(w, w) \quad \forall w, w \in U_h.
$$

Our goal is to develop a one-level additive Schwarz preconditioner for $A_h$ (cf. [9]).

To avoid the proliferation of constants, we will use the notation $A \lesssim B$ (or $B \gtrsim A$) to represent the inequality $A \leq (\text{constant}) \times B$, where the positive constant only depends on the shape regularity of $\Omega_h$ and the polynomial degrees $m$ and $r$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

A fundamental result in [8] is the equivalence

$$
a_h(w, w) \approx \|\sigma\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\hat{u}\|_{H^1/2(\partial \Omega_h)}^2 + \|\hat{\sigma}\|_{H^{-1/2}(\partial \Omega_h)}^2
$$

that holds for all $w = (\sigma, u, \hat{u}, \hat{\sigma}) \in U_h$, where

$$
\|\hat{u}\|_{H^{1/2}(\partial \Omega_h)}^2 = \sum_{K \in \Omega_h} \|\hat{u}\|^2_{H^{1/2}(\partial K)} = \sum_{K \in \Omega_h} \inf_{\tilde{w} \in H^1(K) \Setminus \{0\}} \|\tilde{w}\|^2_{H^1(K)},
$$

$$
\|\hat{\sigma}\|_{H^{-1/2}(\partial \Omega_h)}^2 = \sum_{K \in \Omega_h} \|\hat{\sigma}\|^2_{H^{-1/2}(\partial K)} = \sum_{K \in \Omega_h} \inf_{q \in H(\text{div}; K) \Setminus \{0\}} \|q\|^2_{H(\text{div}; K)}.
$$
Therefore the analysis of domain decomposition preconditioners for $A_h$ requires a better understanding of the norms $\| \cdot \|_{H^{1/2}(\partial K)}$ and $\| \cdot \|_{H^{-1/2}(\partial K)}$ on the discrete spaces $\tilde{P}_{m+1}(\partial K)$ and $P_m(\partial K)$.

### 2 Explicit Expressions for the Norms on $\tilde{P}_{m+1}(\partial K)$ and $P_m(\partial K)$

**Lemma 1.** We have

$$\| \tilde{\xi} \|_{H^{1/2}(\partial K)}^2 \approx h_K \left( \| \tilde{\xi} \|_{L^2(\partial K)}^2 + \sum_{F \in \Sigma_K} |\tilde{\xi}|_{H^1(F)}^2 \right), \quad \forall \tilde{\xi} \in \tilde{P}_{m+1}(\partial K),$$

where $h_K$ is the diameter of $K$ and $\Sigma_K$ is the set of the faces of $K$.

**Proof.** Let $\mathcal{N}(K)$ be the set of nodal points of the $P_m$ Lagrange finite element associated with $K$ and $\mathcal{N}(\partial K)$ be the set of points in $\mathcal{N}(K)$ that are on $\partial K$.

Given any $\tilde{\xi} \in \tilde{P}_{m+1}(\partial K)$, we define $\tilde{\xi}_s \in P_{m+1}(K)$ by

$$\tilde{\xi}_s(p) = \begin{cases} \tilde{\xi}(p) & \text{if } p \in \mathcal{N}(\partial K), \\ \tilde{\xi}_s & \text{if } p \in \mathcal{N}(K) \setminus \mathcal{N}(\partial K), \end{cases}$$

where $\tilde{\xi}_s$ is the mean value of $\tilde{\xi}$ over $\partial K$. Since $\tilde{\xi}_s = \tilde{\xi}$ on $\partial K$, we have

$$\| \tilde{\xi} \|_{H^{1/2}(\partial K)} = \inf_{w \in H^1(K), w|_{\partial K} = \tilde{\xi}} \| w \|_{H^1(K)} \leq \| \tilde{\xi}_s \|_{H^1(K)}. \quad (7)$$

Suppose $w \in H^1(K)$ satisfies $w = \tilde{\xi}$ on $\partial K$. It follows from (6) and the trace theorem with scaling that

$$\| \tilde{\xi}_s \|_{L^2(K)} \lesssim h_K \| \tilde{\xi} \|_{L^2(\partial K)} = h_K \| w \|_{L^2(\partial K)} \lesssim \| w \|_{H^1(K)}, \quad (8)$$

and, by standard estimates,

$$|\tilde{\xi}_s|_{H^1(K)}^2 = |\tilde{\xi} - \tilde{\xi}_s|_{H^1(K)}^2 \lesssim h_K^{-1} \| \tilde{\xi} - \tilde{\xi}_s \|_{L^2(\partial K)} \lesssim h_K^{-1} \| w - w_{\partial K} \|_{L^2(\partial K)} \lesssim \| w \|_{H^1(K)}^2. \quad (9)$$

Combining (7)–(9), we have $\| \tilde{\xi} \|_{H^{1/2}(\partial K)} \approx \| \tilde{\xi}_s \|_{H^1(K)}$. The lemma then follows from (6), the equivalence of norms on finite dimensional spaces and scaling. \qed

**Lemma 2.** We have

$$\| \xi \|_{H^{-1/2}(\partial K)}^2 \approx h_K \| \xi \|_{L^2(\partial K)}^2 + h_K^{-d} \left( \int_{\partial K} |\xi| \, ds \right)^2, \quad \forall \xi \in P_m(\partial K).$$
Proof. We begin with the reference simplex $\hat{K}$. Let $RT_m(\hat{K})$ be the $m$-th order Raviart-Thomas space (cf. [2]). Given any $\zeta \in P_m(\partial \hat{K})$, we introduce a (nonempty) subspace $RT_m(\hat{K}, \zeta) = \{ q \in RT_m(\hat{K}) : q \cdot n = \zeta \text{ on } \partial \hat{K} \text{ and } \text{div} \ q \in P_0(\hat{K}) \}$ of $RT_m(\hat{K})$.

Let $\zeta^* \in RT_m(\hat{K}, \zeta)$ be defined by

$$\zeta^* = \min_{q \in RT_m(\hat{K}, \zeta)} \|q\|_{L^2(\hat{K})}.$$  

Then the map $\hat{S} : P_m(\partial \hat{K}) \rightarrow RT_m(\hat{K})$ that maps $\zeta$ to $\zeta^*$ is linear and one-to-one, and we have $(\hat{S}\zeta) \cdot n = \zeta$ on $\partial \hat{K}$, $\text{div}(\hat{S}\zeta) \in P_0(\hat{K})$ and

$$\|\hat{S}\zeta\|_{L^2(\hat{K})} \approx \|\zeta\|_{L^2(\partial K)} \quad \forall \zeta \in P_m(\partial \hat{K}). \tag{10}$$

Let $\zeta_1, \ldots, \zeta_{N_m}$ be a basis of $P_m(\partial \hat{K})$ and $1 = \phi_1, \ldots, \phi_{N_m} \in H^{1/2}(\partial \hat{K})$ satisfy $\det \left[ \int_{\partial \hat{K}} \zeta_i \phi_j \, d\hat{s} \right]_{1 \leq i,j \leq N_m} \neq 0$. We define the map $\hat{Q} : H(\text{div}; \hat{K}) \rightarrow P_m(\partial \hat{K})$ by

$$\int_{\partial \hat{K}} (\hat{Q}q) \phi_j \, d\hat{s} = \langle q \cdot n, \phi_j \rangle_{H^{-1/2}(\partial \hat{K}) \times H^{1/2}(\partial \hat{K})} \quad \text{for } 1 \leq j \leq N_m.$$  

It follows from the definition of $\hat{Q}$ that $\|\hat{Q}q\|_{L^2(\partial \hat{K})} \lesssim \|q\|_{H(\text{div}; \hat{K})}$ for all $q \in H(\text{div}; \hat{K})$, and $\hat{Q}q = \zeta$ if $q \cdot n = \zeta \in P_m(\partial \hat{K})$, in which case

$$\|\hat{S}\zeta\|_{L^2(\hat{K})} \lesssim \|\zeta\|_{L^2(\partial \hat{K})} = \|\hat{Q}q\|_{L^2(\partial \hat{K})} \lesssim \|q\|_{H(\text{div}; \hat{K})}. \tag{11}$$

Moreover, since $\phi_1 = 1$, we have

$$\int_{\hat{K}} \text{div}(\hat{S}\zeta) \, d\hat{x} = \int_{\partial \hat{K}} (\hat{Q}q) 1 \, d\hat{s} = \langle q \cdot n, 1 \rangle_{H^{-1/2}(\partial \hat{K}) \times H^{1/2}(\partial \hat{K})} = \int_{\hat{K}} \text{div} q \, d\hat{x}$$

and hence

$$\|\text{div}(\hat{S}\zeta)\|_{L^2(\hat{K})} \lesssim \|\text{div} q\|_{L^2(\hat{K})}. \tag{12}$$

Now we turn to a general simplex $K$. It follows from (10)–(12) and standard properties of the Piola transform for $H(\text{div})$ (cf. [10]) that there exists a linear map $S : P_m(\partial K) \rightarrow RT_m(K)$ with the following properties:

(i) $(S\zeta) \cdot n = \zeta$ and hence

$$\|\zeta\|_{H^{-1/2}(\partial K)} = \inf_{q \in H(\text{div}; K), q \cdot n |_{\partial K} = \zeta} \|q\|_{H(\text{div}; K)} \leq \|S\zeta\|_{H(\text{div}; K)} \quad \forall \zeta \in P_m(\partial K),$n

(ii) for any $q \in H(\text{div}; K)$ such that $q \cdot n = \zeta$, we have

$$\|S\zeta\|_{H(\text{div}; K)} \lesssim \|q\|_{H(\text{div}; K)};$$

(iii) $\text{div}(S\zeta) \in P_0(K)$ and hence

$$\int_{\hat{K}} \text{div}(S\zeta) \, d\hat{x} = \int_{\partial \hat{K}} \zeta \, d\hat{s} \quad \text{or} \quad \|\text{div}(S\zeta)\|^2_{L^2(\hat{K})} = \left( \int_{\partial \hat{K}} \zeta \, d\hat{s} \right)^2 / |K|.$$
Properties (i)–(iv) then imply

\[ h_K^{-d} \| S\xi \|_{L^2(K)}^2 \approx h_K^{-(d-1)} \| \xi \|_{L^2(\partial K)}^2. \]

(iv) we have

\[ \| \xi \|_{H^{-1/2}(\partial K)}^2 \approx \| S\xi \|_{H^{1/2}(\partial K)} \approx h_K \| \xi \|_{L^2(\partial K)}^2 + h_K^{-d} \left( \int_{\partial K} \xi \, ds \right)^2. \]

\[ \Box \]

3 A Domain Decomposition Preconditioner

Let \( \Omega \) be partitioned into overlapping subdomains \( \Omega_1, \ldots, \Omega_J \) that are aligned with \( \Omega_h \). The overlap among the subdomains is measured by \( \delta \) and we assume (cf. [11]) there is a partition of unity \( \theta_1, \ldots, \theta_J \in C^\infty(\hat{\Omega}) \) that satisfies the usual properties:

\[ \theta_j \geq 0, \sum_{j=1}^J \theta_j = 1 \text{ on } \bar{\Omega}, \quad \theta_j = 0 \text{ on } \Omega \setminus \Omega_j, \text{ and} \]

\[ \| \nabla \theta_j \|_{L^\infty(\Omega)} \lesssim \delta^{-1} \quad \forall 1 \leq j \leq J. \quad (13) \]

We take the subdomain space to be \( U_j = \{ \psi \in U_h : \psi = 0 \text{ on } \Omega \setminus \Omega_j \} \). Let \( \psi = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U_h \). Then \( \psi \in U_j \) if and only if (i) \( \sigma \) and \( u \) vanish on every \( K \) outside \( \Omega_j \) and (ii) \( \hat{u} \) and \( \hat{\sigma}_n \) vanish on \( \partial K \) for every \( K \) outside \( \Omega_j \). We define \( a_j(\cdot, \cdot) \) to be the restriction of \( a_h(\cdot, \cdot) \) on \( U_j \times U_j \). Let \( A_j : U_j \rightarrow U_j' \) be defined by

\[ (A_j \psi_j, \psi_j) = a_j(\psi_j, \psi_j) \quad \forall \psi_j, \psi_j \in U_j. \quad (14) \]

It follows from (3) that

\[ a_j(\psi_j, \psi_j) \approx \| \sigma_j \|_{L^2(\Omega_j)}^2 + \| u_j \|_{L^2(\Omega_j)}^2 + \| \hat{u}_j \|_{H^{1/2}(\partial \Omega_j, h)}^2 + \| \hat{\sigma}_{jn,j} \|_{H^{-1/2}(\partial \Omega_j, h)}^2, \quad (15) \]

where \( \psi_j = (\sigma_j, u_j, \hat{u}_j, \hat{\sigma}_{jn,j}) \in U_j \). \( \Omega_j, h \) is the triangulation of \( \Omega_j \) induced by \( \Omega_h \) and the norms \( \| \cdot \|_{H^{1/2}(\partial \Omega_j, h)} \) and \( \| \cdot \|_{H^{-1/2}(\partial \Omega_j, h)} \) are analogous to those in (4) and (5).

Let \( I_j : U_j \rightarrow U_h \) be the natural injection. The one-level additive Schwarz preconditioner \( B_h : U_h' \rightarrow U_h \) is defined by

\[ B_h = \sum_{j=1}^J I_j A_j^{-1} I_j'. \]

Lemma 3. We have

\[ \lambda_{\min}(B_h A_h) \gtrsim \delta^2. \]

Proof. Let \( I_{h,1}, I_{h,2}, I_{h,3} \) and \( I_{h,4} \) be the nodal interpolation operators for the components \( \prod_{K \in \Omega_h} [P_m(K)]^d \), \( \prod_{K \in \Omega_h} P_m(K), \hat{P}_{m+1}(\partial \Omega_h) \) and \( P_m(\partial \Omega_h) \) of \( U_h \) respectively. Given any \( \psi = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U_h \), we define \( \psi_j \in U_j \) by

\[ \psi_j = (I_{h,1}(\theta_j \sigma), I_{h,2}(\theta_j u), I_{h,3}(\theta_j \hat{u}), I_{h,4}(\theta_j \hat{\sigma}_n)). \]
Then we have $w = \sum_{j=1}^{J} \psi_j$ and, in view of (14) and (15),

$$
\langle A_j \psi_j, \psi_j \rangle \approx \| I_{h, 1}(\theta_j \sigma) \|^2_{L^2(\Omega_j)} + \| I_{h, 2}(\theta_j u) \|^2_{L^2(\Omega_j)} + \| I_{h, 3}(\theta_j \hat{u}) \|^2_{H^{1/2}(\partial \Omega_{jh})} + \| I_{h, 4}(\theta_j \hat{\sigma}_h) \|^2_{H^{-1/2}(\partial \Omega_{jh})},
$$

(16)

The following bounds for the first two terms on the right-hand side of (16) are straightforward:

$$
\| I_{h, 1}(\theta_j \sigma) \|^2_{L^2(\Omega_j)} \lesssim \| \sigma \|^2_{L^2(\Omega_j)} \quad \text{and} \quad \| I_{h, 2}(\theta_j u) \|^2_{L^2(\Omega_j)} \lesssim \| u \|^2_{L^2(\Omega_j)}.
$$

(17)

We will use Lemma 1 and Lemma 2 to derive the following bounds

$$
\| I_{h, 3}(\theta_j \hat{u}) \|^2_{H^{1/2}(\partial \Omega_{jh})} \lesssim \delta^{-2} \| \hat{u} \|^2_{H^{1/2}(\partial \Omega_{jh})},
$$

(18)

$$
\| I_{h, 4}(\theta_j \hat{\sigma}_h) \|^2_{H^{-1/2}(\partial \Omega_{jh})} \lesssim \delta^{-2} \| \hat{\sigma}_h \|^2_{H^{-1/2}(\partial \Omega_{jh})}.
$$

(19)

Let $K \in \Omega_{j, h}$. It follows from Lemma 1, (13) and standard discrete estimates that

$$
\| I_{h, 3}(\theta_j \hat{u}) \|^2_{H^{1/2}(\partial K)} \approx h_k \left( \| I_{h, 3}(\theta_j \hat{u}) \|^2_{L^2(\partial K)} + \sum_{F \subseteq \Sigma K} \| I_{h, 3}(\theta_j \hat{u}) \|^2_{H^1(F)} \right)
$$

$$
\lesssim h_k \| \hat{u} \|^2_{L^2(\partial K)} + h_k \sum_{F \subseteq \Sigma K} \| \nabla \theta_j \|^2_{L^2(\Omega)} \| \hat{u} \|^2_{L^2(F)} + \| \theta_j \|^2_{L^2(\Omega)} \| \hat{u} \|^2_{H^1(F)}
$$

$$
\lesssim h_k \| \hat{u} \|^2_{L^2(\partial K)} + h_k \delta^{-2} \| \hat{u} \|^2_{L^2(\partial K)} + h_k \sum_{F \subseteq \Sigma K} \| \hat{u} \|^2_{H^1(F)} \lesssim \delta^{-2} \| \hat{u} \|^2_{H^{1/2}(\partial K)}.
$$

Summing up this estimate over all the simplexes in $\Omega_{j, h}$ yields (18).

Similarly, it follows from Lemma 2 and (13) that

$$
\| I_{h, 4}(\theta_j \hat{\sigma}_h) \|^2_{H^{-1/2}(\partial K)} \approx h_k \| I_{h, 4}(\theta_j \hat{\sigma}_h) \|^2_{L^2(\partial K)} + \left( \int_{\partial K} I_{h, 4}(\theta_j \hat{\sigma}_h) \, ds \right)^2
$$

$$
\lesssim h_k \| \hat{\sigma}_h \|^2_{L^2(\partial K)} + h_k \delta^{-2} \| \hat{\sigma}_h \|^2_{L^2(\partial K)} + h_k \delta^{-2} \left( \int_{\partial K} \hat{\sigma}_h \, ds \right)^2
$$

$$
\lesssim h_k \| \hat{\sigma}_h \|^2_{L^2(\partial K)} + h_k \delta^{-2} \| \hat{\sigma}_h \|^2_{L^2(\partial K)} + h_k \delta^{-2} \left( \int_{\partial K} \hat{\sigma}_h \, ds \right)^2 \lesssim \delta^{-2} \| \hat{\sigma}_h \|^2_{H^{-1/2}(\partial K)}.
$$

where $\theta^K$ is the mean value of $\sigma_j$ over $K$. Summing up this estimate over all the simplexes in $\Omega_{j, h}$ gives us (19).

Putting (2), (3) and (16)–(19) together we find $\sum_{j=1}^{J} \langle A_j \psi_j, \psi_j \rangle \lesssim \delta^{-2} \langle A_h \psi, \psi \rangle$, which implies $\lambda_{\min}(B_h A_h) \gtrsim \delta^2$ by the standard theory of additive Schwarz preconditioners [11].  

Combining Lemma 3 with the standard estimate $\lambda_{\max}(B_h A_h) \lesssim 1$, we obtain the following theorem.

...
**Theorem 1.** We have
\[
\kappa(B_h A_h) = \frac{\lambda_{\max}(B_h A_h)}{\lambda_{\min}(B_h A_h)} \leq C \delta^{-2},
\]
where the positive constant $C$ depends only on the shape regularity of $\Omega_h$ and the polynomial degrees $m$ and $r$.

**Remark 1.** Theorem 1 is also valid for DPG methods based on tensor product finite elements.

### 4 Numerical results

We solve the Poisson problem on the square $(0,1)^2$ with exact solution $u = \sin(\pi x_1) \sin(\pi x_2)$ and uniform square meshes. The trial space is based on $Q_1$ polynomials for $\sigma$ and $u$, $P_2$ polynomials for $\hat{u}$, and $P_1$ polynomials for $\hat{\sigma}_n$. We use bicubic polynomials for the space $V^r$ in the construction of the trial-to-test map $T_h$.

The number of conjugate gradient iterations required to reduce the residual by $10^{10}$ are given in Table 1 for four overlapping subdomains. The linear growth of the number of iterations for the unpreconditioned system is consistent with the condition number estimate $\kappa(A_h) \lesssim h^{-2}$ in [8]. Note that in this case the boundary of every subdomain has a nonempty intersection with $\partial \Omega$ and it is not difficult to use a discrete Poincaré inequality to show that the estimate in Theorem 1 can be improved to $\kappa(B_h A_h) \lesssim |\ln h| \delta^{-1}$. This is consistent with the observed growth of the number of iterations for the preconditioned system as $\delta$ decreases.

| $h$ | $\delta$ | unpreconditioned | preconditioned |
|-----|---------|-----------------|----------------|
| $2^{-2}$ | $2^{-2}$ | 496 | 14 |
| $2^{-3}$ | $2^{-3}$ | 1556 | 17 |
| $2^{-4}$ | $2^{-4}$ | 3865 | 20 |
| $2^{-5}$ | $2^{-5}$ | 8793 | 27 |
| $2^{-6}$ | $2^{-6}$ | 20 | |
| $2^{-7}$ | $2^{-7}$ | 18 | |

In Table 2 we display the results for $h = 2^{-5}$ and various subdomain sizes $H$ with $\delta = H/2$. The estimate $\kappa(B_h A_h) \lesssim \delta^{-2} \approx H^{-2}$ is consistent with the observed linear growth of the number of iterations for the preconditioned system as $H$ decreases. Such a condition number estimate for the one-level additive Schwarz preconditioner is known to be sharp for standard finite element methods [1].
Table 2 Number of iterations with $h = 2^{-5}$ and various subdomain sizes $H$ with $\delta = H/2$.

| $h$   | $H$   | Unpreconditioned | Preconditioned |
|-------|-------|------------------|-----------------|
| $2^{-5}$ | $2^{-1}$ | 18793            | 15              |
| $2^{-2}$ |              | 25               |                 |
| $2^{-3}$ |              | 45               |                 |
| $2^{-4}$ |              | 89               |                 |

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