An isomorphism theorem for Alexander biquandles

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Abstract

We show that two Alexander biquandles $M$ and $M'$ are isomorphic iff there is an isomorphism of $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$-modules $h : (1 - st)M \to (1 - st)M'$ and a bijection $g : O_s(A) \to O_s(A')$ between the $s$-orbits of sets of coset representatives of $M/(1 - st)M$ and $M'/(1 - st)M'$ respectively satisfying certain compatibility conditions.

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1 Introduction

In [11], it was shown that two finite Alexander quandles of the same cardinality are isomorphic iff their $(1 - t)$ submodules are isomorphic as $\mathbb{Z}[t^{\pm 1}]$-modules. These finite Alexander quandles are useful as a source of knot invariants defined by counting homomorphisms in various ways – setwise, weighted by cocycles in various quandle cohomology theories, etc. (see [2] for more).

Alexander quandles have been generalized to Alexander biquandles [8]. Both quandles and biquandles are examples of algebraic structures with axioms derived from Reidemeister moves, the former with generators of the algebra corresponding to arcs and the latter with generators corresponding to semi-arcs in the knot diagram. The resulting non-associative algebraic structures are thus naturally suited for defining invariants of knots and links.

Biquandles have been studied in several recent papers such as [5], [3] and [10]. In particular, [5] lists a number of known types of finite biquandles, including Alexander biquandles. In this paper we give necessary and sufficient conditions for the existence of a biquandle homomorphism $f : M \to M'$ of Alexander biquandles which generalizes the main result from [11].

2 Biquandles

We begin with the definition of a biquandle.

Definition 1 A biquandle is a set $B$ with four binary operations $B \times B \to B$ denoted by

$$(a, b) \mapsto a^b, \; a^b, \; a_b, \; a_{\overline{b}}$$

respectively, satisfying:
1. For every pair of elements $a, b \in B$, we have
   
   (i) $a = a^{b_{a}}$,  
   (ii) $b = b_{a}^{a}$,  
   (iii) $a = a^{b}$, and  
   (iv) $b = b_{a}^{a}$.

2. Given elements $a, b \in B$, there are unique elements $x, y \in B$, possibly but not necessarily distinct, such that
   
   (i) $x = a^{b}$,  
   (ii) $a = x^{a}$,  
   (iii) $b = b_{a}$,  
   (iv) $y = a^{b}$, and  
   (v) $b = b_{y}$.

3. For every triple $a, b, c \in B$ we have:
   
   (i) $a^{bc} = a^{c_{a}b}$,  
   (ii) $c_{ba} = c_{a}b_{a}$,  
   (iii) $(b_{a})^{c_{a}b} = (b^{c})_{a}^{a}$,  
   (iv) $a^{bc} = a^{c_{a}b}$,  
   (v) $c_{ba} = c_{a}b_{a}$, and  
   (vi) $(b_{a})^{c_{a}b} = (b^{c})_{a}^{a}$.

4. Given an element $a \in B$, there are unique elements $x, y \in B$, possibly but not necessarily distinct, such that
   
   (i) $x = a_{x}$,  
   (ii) $a = x_{a}$,  
   (iii) $y = a^{y}$, and  
   (iv) $a = y_{a}$.

A biquandle is a type of invertible switch, i.e., a solution $S : X \times X \to X \times X$ to the (set-theoretic) Yang-Baxter equation

$$(S \times I)(I \times S)(S \times I) = (I \times S)(S \times I)(I \times S)$$

where $X$ is a set and $I : X \to X$ is the identity. The components of such a solution $S$ satisfy axiom (3), and if $S$ is invertible the components and the components of the inverse also satisfy axiom (1). An invertible switch $S(a, b) = (b_{a}, a^{b})$ then defines a biquandle if its component functions satisfy axioms (2) and (4). See [5] for more.

The biquandle axioms are motivated by the Reidemeister moves in knot theory – if we assign generators to each semi-arc in an oriented link diagram and consider the outbound semi-arcs at a crossing to be the results of the inbound semi-arcs operating on each other, with barred operations at negative crossings and unbarred operations at positive crossings, then the biquandle axioms are a set of minimal conditions required to make the resulting algebraic structure invariant under Reidemeister moves.

In [12], finite biquandles with cardinality $n$ are presented by $2n \times 2n$ block matrices composed of four $n \times n$ blocks which represent the operation tables of the four biquandle operations. Specifically, if $B = \{x_{1}, \ldots, x_{n}\}$ then the matrix of $B$ has four blocks $M = \begin{bmatrix} B_{1}^{1} & B_{2}^{1} \\ B_{3}^{1} & B_{4}^{1} \end{bmatrix}$ such that

$$B_{ij}^{l} = k \quad \text{where} \quad x_{k} = \begin{cases} (x_{l})_{i}^{(x_{j})} & l = 1 \\ (x_{l})_{i}^{(x_{j})} & l = 2 \\ (x_{l})_{i}^{(x_{j})} & l = 3 \\ (x_{l})_{i}^{(x_{j})} & l = 4 \end{cases}$$
Example 1  The *trivial* biquandle of order \( n \) is the set \( T = \{1, 2, \ldots, n\} \) with operations \( i^j = i \), \( i_j = i \), \( i^j = i \) and \( i_j = i \). It has matrix 

\[
 M = \begin{bmatrix}
 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
 2 & 2 & \ldots & 2 & 2 & 2 & \ldots & 2 \\
 \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
 n & n & \ldots & n & n & n & \ldots & n \\
 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
 2 & 2 & \ldots & 2 & 2 & 2 & \ldots & 2 \\
 \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
 n & n & \ldots & n & n & n & \ldots & n 
\end{bmatrix}.
\]

This matrix presentation was used in [12] to do a computer search in which all biquandles of order up to 4 were classified; matrix presentation of finite biquandles also makes symbolic computation with finite biquandles easy (see [4]). In [5] and [1], several examples of biquandle structures defined on groups and modules are given.

Example 2  The following definition comes from [1]. Let \( M \) be a module over a ring \( R \). Then 

\[
 x^g = C x + D y \quad \text{and} \quad x_y = A y + B x \quad \text{where} \quad A, B \in R \quad \text{are invertible,} \quad C = A^{-1} B^{-1} A (I - A), \quad \text{and} \quad D = I - A^{-1} B^{-1} A B \quad \text{defines an invertible switch on} \quad M \times M \quad \text{if the equation}
\]

\[
 [B, (A - I)(A, B)] = 0
\]

where \([X, Y] = XY - YX\) and \((X, Y) = X^{-1}Y^{-1}XY\) is satisfied. Thus, module theory is a source of biquandles.

Example 3  For a related example, let \( M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) considered as a \( \mathbb{Z}_2 \)-module and set

\[
 A = \begin{bmatrix}
 0 & 1 \\
 1 & 1 
\end{bmatrix}, \quad B = \begin{bmatrix}
 1 & 1 \\
 0 & 1 
\end{bmatrix}, \quad C = \begin{bmatrix}
 1 & 0 \\
 1 & 1 
\end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix}
 1 & 1 \\
 1 & 0 
\end{bmatrix}.
\]

Then \( M \) is a biquandle with

\[
 x^g = x_{\overline{g}} = C x + D y + \begin{bmatrix}
 1 \\
 1 
\end{bmatrix} \quad \text{and} \quad x_y = x^\overline{g} = A y + B x + \begin{bmatrix}
 1 \\
 1 
\end{bmatrix}.
\]

\( M \) has biquandle matrix (where \( x_1 = \begin{bmatrix}
 1 \\
 0 
\end{bmatrix}, \quad x_2 = \begin{bmatrix}
 0 \\
 1 
\end{bmatrix}, \quad x_3 = \begin{bmatrix}
 1 \\
 1 
\end{bmatrix} \) and \( x_4 = \begin{bmatrix}
 0 \\
 0 
\end{bmatrix} \)

\[
 \begin{bmatrix}
 3 & 1 & 2 & 4 & 4 & 1 & 3 & 2 \\
 2 & 4 & 3 & 1 & 2 & 3 & 1 & 4 \\
 1 & 3 & 4 & 2 & 3 & 2 & 4 & 1 \\
 4 & 2 & 1 & 3 & 1 & 4 & 2 & 3 \\
 4 & 1 & 3 & 2 & 3 & 1 & 2 & 4 \\
 2 & 3 & 1 & 4 & 2 & 4 & 3 & 1 \\
 3 & 2 & 4 & 1 & 1 & 3 & 4 & 2 \\
 1 & 4 & 2 & 3 & 4 & 2 & 1 & 3 
\end{bmatrix}.
\]

The counting invariant associated to this biquandle, \(|\text{Hom}(B(K), M)|\) where \( B(K) \) is a knot biquandle, distinguishes all of the Kishino knots from the unknot. See [12].

3
3 Alexander biquandles

In this section we give necessary and sufficient conditions for two Alexander biquandles to be isomorphic. We begin with a definition from [8].

**Definition 2** Let $M$ be a module over the ring $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables. Then $M$ is a biquandle with operations

\[
x^y = tx + (1 - st)y, \quad x_y = sx, \quad x^\varphi = t^{-1}x + (1 - s^{-1}t^{-1})y, \quad \text{and} \quad x_\varphi = s^{-1}x.
\]

Such a biquandle is called an Alexander biquandle.

As expected, we have:

**Definition 3** A homomorphism of Alexander biquandles is a map $f : M \to M'$ satisfying

\[
f(x^y) = f(x)^{f(y)}, \quad f(x_y) = f(x)^{f(y)}, \quad f(x^\varphi) = f(x)^{f(\varphi)}, \quad \text{and} \quad f(x_\varphi) = f(x)^{f(\varphi)}
\]

or equivalently

\[
f(tx + (1 - st)y) = tf(x) + (1 - st)f(y), \quad f(sx) = sf(x),
\]

\[
f(t^{-1}x + (1 - s^{-1}t^{-1})y) = t^{-1}f(x) + (1 - s^{-1}t^{-1})f(y), \quad \text{and} \quad f(s^{-1}x) = s^{-1}f(x).
\]

**Example 4** If $s, t$ are invertible in $\mathbb{Z}_n$ then $\mathbb{Z}_n$ has the structure of an Alexander biquandle with the operations above. For example, $\mathbb{Z}_3$ with $s = 2$ and $t = 1$ has $(1 - st) = 2$ and biquandle matrix

\[
\begin{bmatrix}
3 & 2 & 1 & 3 & 2 & 1 \\
1 & 3 & 2 & 1 & 3 & 2 \\
2 & 1 & 3 & 2 & 1 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3
\end{bmatrix}
\]

where we use $\mathbb{Z}_3 = \{1, 2, 3\}$.

If $f(0) = 0$, then for $f : M \to M'$ to be a homomorphism of Alexander biquandles, it suffices for $f$ to satisfy the first two equations in definition 3.

**Lemma 1** Let $f : M \to M'$ be a function which satisfies $f(0) = 0 \in M'$ and

\[
f(tx + (1 - st)y) = tf(x) + (1 - st)f(y) \quad \text{and} \quad f(sx) = sf(x).
\]

Then $f$ is a homomorphism of Alexander biquandles.

**Proof.** We must show that

\[
f(t^{-1}x + (1 - s^{-1}t^{-1})y) = t^{-1}f(x) + (1 - s^{-1}t^{-1})f(y) \quad \text{and} \quad f(s^{-1}x) = s^{-1}f(x).
\]

The second is easy:

\[
s^{-1}f(x) = s^{-1}f(s^{-1}x) = s^{-1}sf(s^{-1}x) = f(s^{-1}x).
\]
The condition that $f(0) = 0$ implies
\[
    f(tx) = f(tx + (1 - st)0) = tf(x) + (1 - st)f(0) = tf(x)
\]
and
\[
    f((1 - st)y) = f(t0 + (1 - st)y) = tf(0) + (1 - st)f(y) = (1 - st)f(y).
\]
Then we also have
\[
    t^{-1}f(x) = t^{-1}f(t(t^{-1}x)) = t^{-1}tf(t^{-1}x) = f(t^{-1}x).
\]
Moreover, if $y = (1 - st)z = tw$ then $f(-y) = -f(y)$ since
\[
    f(-y) + f(y) = f(t(-w)) + f((1 - st)z)
    = tf(-w) + (1 - st)f(z)
    = f(t(-w) + (1 - st)z)
    = f(-y + y) = f(0) = 0.
\]
But then
\[
    f(t^{-1}x + (1 - s^{-1}t^{-1})y) = f(t(t^{-2}x) + (1 - st)(-s^{-1}t^{-1}y))
    = tf(t^{-2}x) + (1 - st)f(-s^{-1}t^{-1}y)
    = t^{-1}f(x) + f(-s^{-1}t^{-1}(1 - st)y)
    = t^{-1}f(x) - s^{-1}t^{-1}f((1 - st)y)
    = t^{-1}f(x) + (1 - st)(-s^{-1}t^{-1})f(y)
    = t^{-1}f(x) + (1 - s^{-1}t^{-1})f(y)
\]
as required.

\[\square\]

**Lemma 2** Let $f : M \to M'$ be a homomorphism of biquandles. Then $f(0) \in \text{Ker}(\phi)$ where \(\phi : M' \to M'\) is given by $\phi(x) = (1 - s)x$.

**Proof.** Since $f$ is a homomorphism of biquandles, we have $f(x_y) = f(x)f(y)$ for all $x, y \in M$. In particular, $f(0) = f(s0) = f(0_y) = f(0)f(y) = sf(0) = (1 - s)f(0) = 0 \in M'$.

**Lemma 3** Let $g_z : M' \to M'$ be defined by $g_z(x) = x + z$. Then $g_z$ is an isomorphism of Alexander biquandles if $z \in \text{Ker}(\phi)$ where $\phi : M' \to M'$ is given by $\phi(x) = (1 - s)x$.

**Proof.** For any $z \in M'$, $g_z(x) = x + z$ is bijective. Thus, we must show that $(1 - s)z = 0$ implies that $g_z$ is a homomorphism of biquandles. That is, we must compare

1. $g_z(ta + (1 - st)b)$ with $tg_z(a) + (1 - st)g_z(b)$,
2. $g_z(sa)$ with $sg_z(a)$,
3. $g_z(t^{-1}a + (1 - s^{-1}t^{-1})b)$ with $t^{-1}g_z(a) + (1 - s^{-1}t^{-1})g_z(b)$ and
4. $g_z(s^{-1}a)$ with $s^{-1}g_z(a)$

5
where \( a, b \in M' \).

For (1) we see that
\[
g_z(ta + (1 - st)b) = ta + (1 - st)b + z
\]
and
\[
tg_z(a) + (1 - st)g_z(b) = ta + tz + (1 - st)b + (1 - st)z
\]
so subtracting yields
\[
-ta - (1 - st)b - z + ta + tz + (1 - st)b + (1 - st)z = -z + tz + (1 - st)z = -z + tz + z - stz = t(1 - s)z = 0.
\]

For (2) we see that
\[
g_z(sa) = sa + z \quad \text{and} \quad sg_z(a) = s(a + z)
\]
so subtracting yields
\[
sa + z - s(a + z) = sa + z - sa - sz = z - sz = (1 - s)z = 0.
\]

Finally, by lemma [1] we are done. \(\blacksquare\)

Not every biquandle isomorphism \( f : M \rightarrow M' \) sends \( 0 \in M \) to \( 0 \in M' \), but in light of lemmas [2] and [3] we may replace any isomorphism \( f : M \rightarrow M' \) which does not with \( f' = g(-f(0)) \circ f \), and then \( f'(0) = 0 \).

Let us denote the orbit of a subset \( X \subseteq M \) under multiplication by \( s \) by
\[
O_s(X) = \{ s^i x \mid i \in \mathbb{Z}, x \in X \}.
\]

We can now state our main result.

**Theorem 4** Two Alexander biquandles \( M \) and \( M' \) are isomorphic as biquandles iff they satisfy

(i) There is an isomorphism of \( \mathbb{Z}[s^{\pm 1}, t^{\pm 1}] \)-modules \( h : (1 - st)M \rightarrow (1 - st)M' \) and

(ii) For every set of coset representatives \( A \) of \( (1 - st)M \) in which the class of \( (1 - st)M \) is represented by \( 0 \in M \), there is a corresponding set of coset representatives \( A' \) of \( (1 - st)M' \) and a bijection \( g : O_s(A) \rightarrow O_s(A') \) such that \( g(A) = A' \) and
\[
(1 - st)g(\alpha) = h((1 - st)\alpha) \quad \text{and} \quad g(s\alpha + \omega) = sg(\alpha) + h(\omega)
\]
for every \( \alpha \in A \) and \( \omega \in (1 - st)M \).

**Proof.** \((\Rightarrow)\) Suppose \( f : M \rightarrow M' \) is an isomorphism of biquandles, and without loss of generality suppose \( f(0) = 0 \). Then \( f \) commutes with multiplication by powers of \( s, t \) and \( (1 - st) \) and satisfies
\[
f(x + y) = f(x) + f(y) \quad \forall x, y \in (1 - st)M
\]
since \( x, y \in (1 - st)M \) implies \( x = tx', y = (1 - st)y' \) and then
\[
\begin{align*}
f(x + y) &= f(tx' + (1 - st)y') \\
&= tf(x') + (1 - st)f(y') \\
&= f(tx') + f((1 - st)y') \\
&= f(x) + f(y).
\end{align*}
\]
Hence, \( h = f|_{(1 - st)M} \) is an isomorphism of \( \mathbb{Z}[s^{\pm 1}, t^{\pm 1}] \)-modules.

Now, let \( A = \{ \alpha_i \mid i \in I \} \) for some indexing set \( I \) be a set of coset representatives for \( M/(1 - st)M \) and define \( g = f|_{O_s(A)} \). The image of \( g \) is \( f(O_s(A)) = O_s(f(A)) \) since \( f \) commutes with \( s \), and thus \( g \) is bijective. Then every element of \( M \) has the form \( x = \alpha_i + (1 - st)\omega \) for some \( \alpha_i \in A \) and \( (1 - st)\omega \in (1 - st)M \), and we have
\[
\begin{align*}
f(x) &= f(tt^{-1}\alpha_i + (1 - st)\omega) \\
&= tf(t^{-1}\alpha_i) + (1 - st)f(\omega) \\
&= f(tt^{-1}\alpha_i) + f((1 - st)\omega) \\
&= g(\alpha_i) + h((1 - st)\omega).
\end{align*}
\]
In particular,
\[
f(\alpha_i + (1 - st)M) = g(\alpha_i) + h((1 - st)M) = g(\alpha_i) + (1 - st)M';
\]
that \( A' = \{ g(\alpha_i) \mid i \in I \} \) is a set of coset representatives of \( M'/(1 - st)M' \) then follows from the bijectivity of \( f \) and the fact that if \( g(\alpha_i) = g(\alpha_j) + (1 - st)m' \), then
\[
\begin{align*}
\alpha_i &= f^{-1}f(\alpha_i) \\
&= f^{-1}(g(\alpha_j) + (1 - st)m') \\
&= \alpha_j + f^{-1}(1 - st)m'
\end{align*}
\]
and hence \( \alpha_i = \alpha_j \).

For any \( \alpha \in A \) we have
\[
(1 - st)g(\alpha) = 0 + (1 - st)f(\alpha)
\]
\[
= tf(0) + (1 - st)f(\alpha)
\]
\[
= f(t0 + (1 - st)\alpha) \\
= f((1 - st)\alpha) \\
= h((1 - st)\alpha).
\]
Moreover, if \( s\alpha + \omega \in O_s(A) \) where \( \alpha \in A \) and \( \omega = (1 - st)\omega' \in (1 - st)M \) then
\[
g(s\alpha + \omega) = f(s\alpha + \omega)
\]
\[
= f(tt^{-1}\alpha + (1 - st)\omega') \\
= tf(st^{-1}\alpha) + (1 - st)f(\omega') \\
= tf(st^{-1}\alpha) + f((1 - st)\omega') \\
= sf(\alpha) + f(\omega) \\
= sg(\alpha) + h(\omega)
\]
as required.

(⇐) Suppose that $h : (1 - st)M \to (1 - st)M'$ is an isomorphism of $\mathbb{Z}[s^\pm 1, t^\pm 1]$-modules, $A$ and $A'$ respectively are sets of coset representatives for $M/(1 - st)M$ and $M'/(1 - st)M'$ with $0 \in M$ as the representative in $A$ of the coset $0 + (1 - st)M \subseteq M$, and that $g : O_s(A) \to O_s(A')$ is a bijection satisfying $g(A) = A'$,

$$(1 - st)g(\alpha) = h((1 - st)\alpha) \quad \text{and} \quad g(s\alpha + \omega) = sg(\alpha) + h(\omega)$$

for all $\alpha \in A$ and $s\alpha + \omega \in O_s(A)$ with $\omega \in (1 - st)M$. In particular,

$$(1 - st)g(0) = h((1 - st)0) = h(0) = 0$$

implies that $g(0) = 0 \in A'$.

Now define $f : M \to M'$ by setting

$$f(s^i\alpha + \omega) = g(s^i\alpha) + h(\omega) = s^i g(\alpha) + h(\omega), \quad \alpha \in A, \ \omega \in (1 - st)M.$$ 

To see that $f$ is well-defined, note that every element of $M$ can be written as $x = \alpha + \omega$ in a unique way with $\alpha \in A$, $\omega \in (1 - st)M$. So, if $x = \alpha + \omega = s^i\alpha' + \omega' + \omega''$ with $s^i\alpha' + \omega' \in O_s(A)$, then $s^i\alpha' + \omega' = \alpha + \omega - \omega'' \in O_s(A)$ and we have

$$g(s^i\alpha' + \omega') = g(\alpha + \omega - \omega'') = g(\alpha) + h(\omega - \omega')$$

so that

$$g(s^i\alpha' + \omega') + h(\omega'') = g(\alpha) + h(\omega - \omega'') + h(\omega'') = g(\alpha) + h(\omega) + h(\omega'').$$

Define $k : A \to A'$ by $k = g|_A$. Then $k$ is bijective, and $f(\alpha + \omega) = k(\alpha) + h(\omega)$ for every $\alpha \in A, \omega \in (1 - st)M$. Then $f$ is bijective, since $f$ is setwise the cartesian product of the bijective maps $k$ and $h$.

Now if $x = \alpha + \omega$ we have

$$f(sx) = f(s\alpha + s\omega) = g(s\alpha) + h(s\omega) = s(g(\alpha) + h(\omega)) = sf(x).$$

It follows that $s^i g(x) = g(s^i x)$ for every $i \in \mathbb{Z}$.

Note that $\alpha = s^{-1}\alpha - s^{-1}(1 - st)\alpha$ for any $\alpha \in M$. Then if $x = \alpha_1 + \omega_1$ and $y = \alpha_2 + \omega_2$ with $\alpha_i \in A$ and $\omega_i \in (1 - st)M$, we have

$$f(tx + (1 - st)y) = f(t(\alpha_1 + \omega_1) + (1 - st)(\alpha_2 + \omega_2)) = f((t\alpha_1 + tw_1 + (1 - st)\alpha_2 + (1 - st)\omega_2) = f(s^{-1}\alpha_1 - s^{-1}(1 - st)\alpha_1 + tw_1 + (1 - st)\alpha_2 + (1 - st)\omega_2) = g(s^{-1}\alpha_1 + h(-s^{-1}(1 - st)\alpha_1 + tw_1 + (1 - st)\alpha_2 + (1 - st)\omega_2) = s^{-1}g(\alpha_1) - s^{-1}h((1 - st)\alpha_1 + th(\omega_1) + h((1 - st)\alpha_2 + (1 - st)h(\omega_2) = s^{-1}g(\alpha_1) - s^{-1}(1 - st)g(\alpha_1 + th(\omega_1)h((1 - st)\alpha_2 + (1 - st)h(\omega_2) = tg(\alpha_1) + th(\omega_1) + (1 - st)g(\alpha_2) + (1 - st)h(\omega_2) = tf(x) + (1 - st)f(y).}$$
and $f$ is an isomorphism of biquandles.

If $s = 1$ then $O_s(A) = A$ and $s\alpha + \omega \in O_s(A)$ implies $\omega = 0$, so the condition that $g(s\alpha + \omega) = sg(\omega) + h(\omega)$ is automatic. It is then possible to show (see [11]) that if $|M| = |M'| < \infty$ and condition (i) is satisfied, then for every choice of coset representatives $A$ of $M/(1-st)M$ there exists a corresponding set of coset representatives $A'$ of $M'/(1-st)M'$ so that $(1-st)g(\alpha) = h((1-st)\alpha)$. If $s \neq 1$, that is, if $M$ and $M'$ are Alexander biquandles which are not Alexander quandles, then condition (i) and $|M| = |M'|$ are not sufficient for $M$ to be isomorphic to $M'$ as biquandles, as the next example demonstrates.

Example 5  Let $M = \mathbb{Z}_8$ with $s = 3$ and $t = 5$, and let $M' = \mathbb{Z}_8$ with $s = 5$ and $t = 3$. Then $(1-st) = 6$ and $(1-st)M = (1-st)M' = \{0, 2, 4, 6\}$. Moreover, $h : (1-st)M \to (1-st)M'$ defined by $h(x) = -x$ satisfies $h(3x) = 5h(x)$ and $h(5x) = 3h(x)$ since $5(2) \equiv -3(2) \mod 8$.

Now, let $A = \{0, 1\}$ so that $O_s(A) = \{0, 1, 3\}$. Then in order to satisfy
\[
6g(1) = h((6)1) = h(6) = 2
\]
we must have either $g(1) = 3$ or $g(1) = 7$. If $g(1) = 3$ then
\[
g(sx) = sg(x) \quad \Rightarrow \quad g(3) = g(3(1)) = 5g(1) = 5(3) = 7
\]
but then
\[
g(\alpha + \omega) = g(\alpha) + h(\omega) \quad \Rightarrow \quad g(3) = g(1 + 2) = g(1) + h(2) = 3 + 6 = 1 \neq 7.
\]
Similarly, $g(1) = 7$ implies $g(3(1)) = 5g(1) = 5(7) = 3$ while $g(1 + 2) = g(1) + h(2) = 7 + 6 = 5 \neq 3$.

Thus, for our choice of coset representatives $A$ there is no bijection $g : O_s(A) \to O_s(A')$ satisfying $(1-st)g(\alpha) = h((1-st)\alpha)$ and $g(s\alpha + \omega) = sg(\alpha) + h(\omega)$ for all $\alpha, \omega \in A$ and $\omega \in (1-st)M$, and hence $M$ and $M'$ are non-isomorphic Alexander biquandles. Our Maple computations confirm this result.

4  Future Research

There remains much to be done in the study of biquandles and Alexander biquandles in particular. Computation of the Yang-Baxter homology groups for Alexander biquandles might shed additional light on the homology of quandles. It is conjectured that the knot biquandle is a complete invariant of virtual knots up to vertical mirror image (see [5] and [9]); if this is true, then it should be possible to derive nearly all other invariants of knots from the biquandle of a knot, potentially illuminating the relationships between the invariants in the process.

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References

[1] S. Budden and R. Fenn. The equation \([B, (A - 1)(A, B)] = 0\) and virtual knots and links. *Fund. Math.* **184** (2004) 19-29.

[2] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. *Trans. Am. Math. Soc.* **355** (2003) 3947-3989.

[3] J. S. Carter, M. Elhamdadi and M. Saito. Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles. *Fund. Math.* **184** (2004) 31-54.

[4] C. Creel and S. Nelson. Symbolic computation with finite biquandles. In preparation.

[5] R. Fenn, M. Jordan-Santana, and L. Kauffman. Biquandles and virtual links. *Topology Appl.* **145** (2004) 157-175.

[6] L.H. Kauffman. Virtual knot theory. *European J. Combin.* **20** (1999) 663-690.

[7] N. Kamada and S. Kamada. Abstract link diagrams and virtual knots. *J. Knot Theory Ramifications* **9** (2000) 93-106.

[8] L. H. Kauffman and D. Radford. Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links. *Contemp. Math.* **318** (2003) 113-140.

[9] L. H. Kauffman and D. Hrencecin. Biquandles for virtual knots. [arXiv:math/0703216](http://arxiv.org/abs/math/0703216)

[10] L. H. Kauffman and V. O. Manturov. Virtual biquandles. *Fund. Math.* **188** (2005) 103-146.

[11] S. Nelson. Classification of finite Alexander quandles. *Topology Proceedings* **27** (2003) 245-258.

[12] S. Nelson and J. Vo. Matrices and finite biquandles. *Homology Homotopy Appl.* **8** (2006) 51-73.