THE LOGARITHMIC SCHRÖDINGER OPERATOR AND ASSOCIATED DIRICHLET PROBLEMS

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Abstract. In this note, we study the integrodifferential operator \((I - \Delta)^{\log}\) corresponding to the logarithmic symbol \(\log(1 + |\xi|^2)\), which is a singular integral operator given by

\[
(I - \Delta)^{\log} u(x) = d_N \int_{\mathbb{R}^N} \frac{u(x) - u(x + y)}{|y|^N} \omega(|y|) \, dy,
\]

where \(d_N = \pi^{-\frac{N}{2}}\), \(\omega(r) = 2^{1 - \frac{N}{2}} r^\nu K_{\frac{N}{2}}(r)\) and \(K_{\nu}\) is the modified Bessel function of second kind with index \(\nu\). This operator is the Lévy generator of the variance gamma process and arises as derivative \(\partial_s |t|_s = 0\) of fractional relativistic Schrödinger operators at \(s = 0\).

In order to study associated Dirichlet problems in bounded domains, we first introduce the functional analytic framework and some properties related to \((I - \Delta)^{\log}\), which allow to characterize the induced eigenvalue problem and Faber-Krahn type inequality. We also derive a decay estimate in \(\mathbb{R}^N\) of the Poisson problem and investigate small order asymptotics \(s \to 0^+\) of Dirichlet eigenvalues and eigenfunctions of \((I - \Delta)^s\) in a bounded open Lipschitz set.

Keywords. Logarithmic symbol, Faber-Krahn inequality, gamma process, small order asymptotics.

1. Introduction and main result

The present paper is devoted to the study of the integrodifferential operator corresponding to the logarithmic symbol \(\log(1 + |\cdot|^2)\) and associated Dirichlet problems in domains. This symbol is known in the probability literature as the characteristic exponent of the symmetric variance gamma process in \(\mathbb{R}^N\) and can be seen as a subclass of increasing Lévy process [2]. As particular case of geometric stable processes \(\log(1 + |\cdot|^{2s})\) for \(s \in (0, 1)\), it plays an important role in the study of Markov process [5] and finds applications to many different fields such as engineering reliability, credit risk theory in structure models, option pricing in mathematical finance [3] and it is used to study the heavy-tailed financial models [23, 26, 32]. It was recently used in wave equation to model damping mechanism in \(\mathbb{R}^N\) (see [7]).

Let us emphasize that the associated operator \((I - \Delta)^{\log}\), which we call the logarithmic Schrödinger operator in the following, has been studied extensively in the literature from a probabilistic and potential theoretic point of view, see e.g. [20, 21, 29, 32, 33] and the references therein. The main purpose of the present paper is to give an account on functional analytic properties of this operator from a PDE point of view. So some of the results we present here are not new but are stated under somewhat different assumptions related to the concept of weak solutions. Moreover, we present proofs not relying on probabilistic techniques but instead on purely analytic methods which are to some extent simpler and more accessible to PDE oriented readers.

Integrodifferential operators of order close to zero are getting increasing interest in the study of linear and nonlinear integrodifferential equations, see for e.g. [5, 20, 21, 29, 32, 33] and the references therein. In particular, the logarithmic Schrödinger operator \((I - \Delta)^{\log}\) has the same singular local behavior as that of the logarithmic Laplacian \(L_\Delta\) studied in [5], while it eliminates the integrability problem of \(L_\Delta\) at infinity. We recall that for compactly supported Dini continuous...
functions $\varphi : \mathbb{R}^N \to \mathbb{R}$, the logarithmic Laplacian $L_\Delta$ is defined by

$$L_\Delta \varphi(x) = c_N \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\varphi(x)1_{B_\epsilon(x)}(y) - \varphi(y)}{|x-y|^N} \, dy + \rho_N \varphi(x),$$

(1.1)

with the constants $c_N := \frac{\Gamma(N/2)}{\pi^{N/2}}$ and $\rho_N := 2 \ln 2 + \psi(N/2) - \gamma$, see [8] for more details. Similarly as in [8], the starting point of the present paper is the observation

$$\lim_{s \to 0^+} (I - \Delta)^s u = u \quad \text{for} \quad u \in C^2(\mathbb{R}^N),$$

(1.2)

where for $s \in (0,1)$, the operator $(I - \Delta)^s$ stands for the relativistic Schrödinger operator which, for sufficiently regular function $u : \mathbb{R}^N \to \mathbb{R}$, is represented via hypersingular integral (see [31] page 548) and (10)

$$(I - \Delta)^s u(x) = u(x) + d_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \omega_s(|y|) \, dy,$$

(1.3)

where $d_{N,s} = \frac{\pi^{-(N+4s)}}{(1+s)}$ is a normalization constant and the function $\omega_s$ is given by

$$\omega_s(|y|) = 2^{1-\frac{N+2s}{2}}|y|^{\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N+2s}{2}} e^{-t} \frac{|y|^2}{4t} \, dt.$$

(1.4)

In particular, if $u \in C^2(\mathbb{R}^N)$, then $(I - \Delta)^s u(x)$ is well defined by (1.2) for every $x \in \mathbb{R}^N$. Here the function $K_\nu$ is the modified Bessel function of the second kind with index $\nu > 0$ and it is given by the expression

$$K_\nu(r) = \left(\frac{\pi}{2r}\right)^{\frac{\nu}{2}} e^{-r} \int_0^\infty e^{-rt} t^{-\frac{\nu}{2}} (1+t/2)^{-\frac{\nu}{2}} \, dt.$$

The normalization constant $d_{N,s}$ in (1.2) is chosen such that the operator $(I - \Delta)^s$ is equivalently defined via its Fourier representation given by

$$\mathcal{F}((I - \Delta)^s u)(\xi) = (1 + |\xi|^2)^s \mathcal{F}(u)(\xi), \quad \text{for a.e} \ \xi \in \mathbb{R}^N,$$

(1.5)

where $\mathcal{F}$ denotes the usual Fourier transform. It therefore follows from (1.2) that one may expect a Taylor expansion with respect to parameter $s$ of the operator $(I - \Delta)^s$ near zero for $u \in C^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ as

$$(I - \Delta)^s u(x) = u(x) + s(I - \Delta)^{\log} u(x) + o(s) \quad \text{as} \quad s \to 0^+,$$

where, the logarithmic Schrödinger operator $(I - \Delta)^{\log}$ appears as the first order term in the above expansion. Indeed, we have the following.

**Theorem 1.1.** Let $u \in C^\alpha(\mathbb{R}^N)$ for some $\alpha > 0$ and $1 < p \leq \infty$. Then

$$\frac{d}{ds} \bigg|_{s=0} [(I - \Delta)^s u](x) = d_N \int_{\mathbb{R}^N} \frac{u(x) - u(x+y)}{|y|^N} \omega(|y|) \, dy = \int_{\mathbb{R}^N} (u(x) - u(x+y)) J(y) \, dy,$$

(1.6)

for $x \in \mathbb{R}^N$, where $d_N := \pi^{-\frac{N}{2}} = - \lim_{s \to 0^+} \frac{d_{N,s}}{s}$, $J(y) = d_N \omega(|y|)$, and

$$\omega(|y|) = 2^{1-\frac{N}{2}}|y|^{\frac{N}{2}} K_{\frac{N}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N}{2}} e^{-t} \frac{|y|^2}{4t} \, dt.$$

(1.7)

Moreover,

(i) If $u \in L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$, then $(I - \Delta)^{\log} u \in L^p(\mathbb{R}^N)$ and

$$\frac{(I - \Delta)^s u - u}{s} \to (I - \Delta)^{\log} u \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{as} \quad s \to 0^+.$$

(ii) $\mathcal{F}((I - \Delta)^{\log} u)(\xi) = \log(1 + |\xi|^2) \mathcal{F}(u)(\xi)$, \text{ for almost every } \xi \in \mathbb{R}^N.
We note that in the particular case $N = 1$, it follows from the definition of $\omega$ in (1.7) (see also [15, (2.4)] and [32, Remark 4.5]) that $\omega(r) = \pi^{N/2} e^{-r}$ and

$$(I - \Delta)^{\log} u(x) = \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|} e^{-|x - y|} \, dy. \quad (1.8)$$

We note here that the operator in (1.8) appears in [25] and is identified as symmetrized Gamma process (see also [22, Example 1]). We stress however that the symbol of this operator is $\log(1 + \xi^2)$ and not $\log(1 + |\xi|)$ as claimed in [25, Page 183]. The representation of $J$ in (1.10) provides an explicit expression for the kernel of the variance Gamma process in $\mathbb{R}^N$ and gives the following asymptotics expansions

$$J(z) \sim \begin{cases} \pi^{-\frac{N}{2}} \frac{N}{2} \Gamma(\frac{N}{2}) |z|^{-N} & \text{as } |z| \to 0 \\ \pi^{-\frac{N-1}{2}} \frac{N-1}{2} \frac{N+1}{2} e^{-|z|} & \text{as } |z| \to \infty. \end{cases} \quad (1.9)$$

Indeed, these expansions follow directly from (1.7) and the asymptotics expansions of the modified Bessel function $K_{\nu}$ (see Section 2.1, (see also [32, Theorem 3.4 and 3.6] for other proof).

The Green function of the operator $(I - \Delta)^{\log}$ is given (see [15,21]) by

$$G(x) = \int_{0}^{\infty} q_t(x) \, dt \quad x \in \mathbb{R}^N, \quad (1.10)$$

where for $t > 0$, $q_t : \mathbb{R}^N \to \mathbb{R}$ is the density of the symmetry variance Gamma process i.e., for all $t > 0$ and $x \in \mathbb{R}^N$,

$$q_t(x) \geq 0, \quad \int_{\mathbb{R}^N} q_t(x) \, dx = 1 \quad \text{and} \quad \mathcal{F}(q_t)(\xi) = e^{-t \log(1 + |\xi|^2)}.$$

It follows from (1.7) that for any $t > 0$,

$$q_t(x) = \frac{2^{1-N}}{\pi^{N/2}} \frac{N}{2} \frac{1}{\Gamma(t)} \left( \frac{|x|}{2} \right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|), \quad (1.11)$$

and the Green function for $(I - \Delta)^{\log}$ then writes

$$G(x) = \frac{2^{1-N}}{\pi^{N/2}} \int_{0}^{\infty} \frac{1}{\Gamma(t)} \left( \frac{|x|}{2} \right)^{t-\frac{N}{2}} K_{t-\frac{N}{2}}(|x|) \, dt. \quad (1.12)$$

Using the asymptotics expansions for the modified Bessel function (see [21, Section 2], we have the following proposition.

**Proposition 1.2.** The function $G$ in (1.13) satisfies the asymptotics properties

$$G(x) \sim \begin{cases} c_N |x|^{-N} & \text{as } |x| \to 0 \\ c_N \frac{N}{2} \pi^{1/2} |x|^{-\frac{N}{2}} e^{-|x|} & \text{as } |x| \to \infty. \end{cases} \quad (1.13)$$

Moreover, for $f \in L^1(\mathbb{R}^N)$, the solution $u = G * f$ of the equation $(I - \Delta)^{\log} u = f$ in $\mathbb{R}^N$ satisfies

$$u(x) = \begin{cases} O(|x|^{-N}) & \text{as } |x| \to 0 \\ O(e^{-|x|}) & \text{as } |x| \to \infty. \end{cases} \quad (1.14)$$

The next task is the study in weak sense with the source function $f \in L^2(\Omega)$, the following related Dirichlet elliptic problem in open bounded set $\Omega \subset \mathbb{R}^N$

$$\begin{cases} (I - \Delta)^{\log} u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.15)$$
In order to settle the corresponding functional analytic framework and energy space related to integro-differential operator $(I - \Delta)^{\log}$, we introduce the following space

$$H^{\log}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \mathcal{E}_\omega(u, u) < \infty \right\}$$

where with $J$ as in (1.6), the bilinear form considered here is given by

$$\mathcal{E}_\omega(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))J(x - y) \, dx \, dy.$$

We shall see in Section 2 that $H^{\log}(\mathbb{R}^N)$ is a Hilbert space endowed with the scalar product

$$(u, v) \rightarrow \langle u, v \rangle_{H^{\log}(\mathbb{R}^N)} = \langle u, v \rangle_{L^2(\mathbb{R}^N)} + \mathcal{E}_\omega(u, u),$$

where $\langle u, v \rangle_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(x)v(x) \, dx$ with corresponding norm

$$\|u\|_{H^{\log}(\mathbb{R}^N)} = \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \mathcal{E}_\omega(u, u) \right)^{\frac{1}{2}}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of $\mathbb{R}^N$. Here and the following we identify the space $L^2(\Omega)$ with the space of functions $u \in L^2(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. We denote by $H^{\log}_0(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_{H^{\log}(\mathbb{R}^N)}$. We have, by the Riesz representation theorem that problem (1.15) admits a unique weak solution $u \in H^{\log}_0(\Omega)$ with

$$\mathcal{E}_\omega(u, v) = \int_\Omega f(x)v(x) \, dx \quad \text{for all } v \in H^{\log}_0(\Omega).$$

Moreover, if $f \in L^\infty(\Omega)$ and $\Omega$ satisfies a uniform exterior sphere condition, it follows from the Green function representation and the regularity estimates in [20][21][27] that $u \in C_0(\Omega) := \{ u \in C(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \}$.

We aim next to study the eigenvalue problem in bounded domain $\Omega \subset \mathbb{R}^N$ involving the logarithmic Schrödinger operator $(I - \Delta)^{\log}$, that is, we consider (1.15) with $f = \lambda u$. To avoid an a priori regularity assumption, we consider the eigenvalue problem (1.15) in weak sense. We call a function $u \in H^{\log}_0(\Omega)$ an eigenfunction of (1.15) corresponding to the eigenvalue $\lambda$ if

$$\mathcal{E}_\omega(u, \varphi) = \lambda \int_\Omega u \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega). \quad (1.16)$$

We then have the following characterisation of the eigenvalues and eigenfunctions for the operator $(I - \Delta)^{\log}$ in an open bounded set $\Omega$ of $\mathbb{R}^N$.

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then

(i) Problem (1.15) admits an eigenvalue $\lambda_1(\Omega) > 0$ characterized by

$$\lambda_1(\Omega) = \inf_{\substack{u \in H^{\log}_0(\Omega) \setminus \{0\} \colon \|u\|_{L^2(\Omega)} = 1}} \frac{\mathcal{E}_\omega(u, u)}{\|u\|^2_{L^2(\Omega)}} = \inf_{u \in \mathcal{P}_1(\Omega)} \mathcal{E}_\omega(u, u), \quad (1.17)$$

with $\mathcal{P}_1(\Omega) := \{ u \in H^{\log}_0(\Omega) : \|u\|_{L^2(\Omega)} = 1 \}$ and there exists a positive function $\varphi_1 \in H^{\log}_0(\Omega)$, which is an eigenfunction corresponding to $\lambda_1(\Omega)$ and that attains the minimum in (1.17), i.e. $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $\lambda_1(\Omega) = \mathcal{E}_\omega(\varphi_1, \varphi_1)$.

(ii) The first eigenvalue $\lambda_1(\Omega)$ is simple, that is, if $u \in H^{\log}_0(\Omega)$ satisfies (1.16) with $\lambda = \lambda_1(\Omega)$, then $u = \alpha \varphi_1$ for some $\alpha \in \mathbb{R}$.

(iii) Problem (1.15) admits a sequence of eigenvalues $\{\lambda_k(\Omega)\}_{k \in \mathbb{N}}$ with

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \lambda_{k+1}(\Omega) \cdots,$$

with corresponding eigenfunctions $\varphi_k$, $k \in \mathbb{N}$ and $\lim_{k \to \infty} \lambda_k(\Omega) = +\infty$. Moreover, for any $k \in \mathbb{N}$, the eigenvalue $\lambda_k(\Omega)$ can be characterized as

$$\lambda_k(\Omega) = \inf_{u \in \mathcal{P}_k(\Omega)} \mathcal{E}_\omega(u, u) \quad (1.18)$$
where $\mathcal{P}_k(\Omega)$ is given by
\[
\mathcal{P}_k(\Omega) := \{ u \in H^1_0(\Omega) : \int_{\Omega} u \varphi_j \, dx = 0 \text{ for } j = 1, 2, \ldots, k-1 \text{ and } \| \varphi_k \|_{L^2(\Omega)} = 1 \}. 
\]

(iv) The sequence $\{ \varphi_k \}_{k \in \mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $\lambda_k(\Omega)$ form a complete orthonormal system of $L^2(\Omega)$ and an orthogonal system of $H^1_0(\Omega)$.

Using the $\delta$-decomposition technique introduced in [12], we provide a boundedness result of the eigenfunctions introduced in Theorem 1.3.

**Proposition 1.4.** Let $u \in H^1(\Omega)$ and $\lambda > 0$ satisfying (1.10). Then $u \in L^\infty(\Omega)$ and there exists a constant $C := C(N, \Omega) > 0$ such that
\[
\| u \|_{L^\infty(\Omega)} \leq C \| u \|_{L^2(\Omega)}.
\]

Our next result concerns the Faber-Krahn inequality for the logarithmic Schrödinger operator $(I - \Delta)^\log$. Here and in the following, we denote by $B^*$ the open ball in $\mathbb{R}^N$ centered at zero with radius determined such that $|\Omega| = |B^*|$

**Theorem 1.5** (Faber-Krahn inequality). Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and $\lambda_{1,\log}(\Omega)$ be the principal eigenvalue of $(I - \Delta)^\log$ in $\Omega$. Then
\[
\lambda_{1,\log}(\Omega) \geq \lambda_{1,\log}(B^*).
\]
Moreover, if equality occurs, $\Omega$ is a ball. Consequently, if $\Omega$ is a ball in $\mathbb{R}^N$, the first eigenfunction $\varphi_{1,\log}$ corresponding to $\lambda_{1,\log}(B)$ is radially symmetric.

Our last result concerns small order asymptotics $s \to 0^+$ of eigenvalues and corresponding eigenfunctions of the relativistic Schrödinger operator $(I - \Delta)^s$ on bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, which is an analogue, but a part of the result of the small order asymptotics $s \to 0^+$ proved in [12] for the fractional Laplacian.

**Theorem 1.6.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$, and $\lambda_{k,s}(\Omega)$ resp. $\lambda_{k,\log}(\Omega)$ be the $k$-th Dirichlet eigenvalue of $(I - \Delta)^s$ resp. of $(I - \Delta)^\log$ on $\Omega$. Then for $s \in (0,1)$, the eigenvalue $\lambda_{k,s}(\Omega)$ satisfies the expansion
\[
\lambda_{k,s}(\Omega) = 1 + s \lambda_{k,\log}(\Omega) + o(s) \quad \text{as} \quad s \to 0^+.
\]
Moreover, if $(s_n)_n \subset (0, s_0)$, $s_0 > 0$ is a sequence with $s_n \to 0$ as $n \to \infty$, then if $\psi_{1,s}$ is the unique nonnegative $L^2$-normalized eigenfunction of $(I - \Delta)^s$ corresponding to the principal eigenvalue $\lambda_{1,s}(\Omega)$, we have that
\[
\psi_s \to \psi_{1,\log} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad s \to 0^+,
\]
and after passing to a subsequence, we have that
\[
\psi_{k,s} \to \psi_{k,\log} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad s \to 0^+,
\]
where $\psi_{1,\log}$, resp. $\psi_{k,\log}$, $k \geq 2$ is the unique nonnegative $L^2$-normalized eigenfunction resp. a $L^2$-normalized eigenfunction corresponding to $\lambda_{1,\log}(\Omega)$ resp. to $\lambda_{k,\log}(\Omega)$.

The paper is organized as follows. In Section 2, we provide the proof of Theorem 1.1 and establish some properties of $(I - \Delta)^\log$ and functional spaces. In Section 3, we prove Theorem 1.3 and, using the $\delta$-decomposition technique introduced in [12], we give the proof of Proposition 1.4 on the $L^\infty$-bound of eigenfunctions and close the section with the proof of Theorem 1.5 on Faber-Krahn inequality. Section 4 is dedicated to the proof of Theorem 1.6 on small order asymptotics $s \to 0^+$ of the eigenvalues and corresponding eigenfunctions of $(I - \Delta)^s$. In section 5, we establish the proof of Proposition 1.2 concerning the decay of the solution of Poisson problem in $\mathbb{R}^N$. Finally, Section 6 collects some theorems that can be directly deduced from
known results in the literature.

**Notation:** We let $\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ denote the measure of the unit sphere in $\mathbb{R}^N$ and, for a set $A \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, we define $\delta_A(x) := \text{dist}(x, A^c)$ with $A^c = \mathbb{R}^N \setminus A$ and, if $A$ is measurable, then $|A|$ denotes its Lebesgue measure. Moreover, for given $r > 0$, let $B_r(A) := \{x \in \mathbb{R}^N : \text{dist}(x, A) < r\}$, and let $B_r(x) := B_r(\{x\})$ denote the ball of radius $r$ with $x$ as its center. If $x = 0$ we also write $B_r$ instead of $B_r(0)$. If $A$ is open, we denote by $C^k(A)$ the space of functions $u : \mathbb{R}^N \to \mathbb{R}$ which are $k$-times continuously differentiable and with compactly contained in $A$. If $f$ and $g$ are two functions, then, $f \sim g$ as $x \to a$ if $\frac{f(x)}{g(x)}$ converges to a constant as $x$ converges to $a$.

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2. Properties of the operator and Functional spaces

We commence this section with the establishment of the integral representation of the operator $(I - \Delta)^{\alpha}$ for a function $u \in C^\alpha(\mathbb{R}^N)$, that is, we provide the proof of Theorem 1.1. After that, we also provide some properties of the functional spaces related to $(I - \Delta)^{\alpha}$. We first introduce the following asymptotics approximations (see [28]) for the modified Bessel function $K_\nu$. It well-known that

$$K_\nu(r) \sim \begin{cases} 2\nu^{-1} \Gamma(\nu)|r|^{-\nu}, & r \to 0, \quad \nu \neq 0, \\ \log \frac{1}{r}, & r \to 0, \quad \nu = 0, \\ \sqrt{\pi/2} r^{-\frac{\nu}{2}} e^{-r}, & r \to +\infty, \end{cases}$$  

and the monotonicity (see [28, 10.37.1])

$$|K_\nu(r)| < |K_\mu(r)| \quad \text{for} \quad 0 \leq \nu < \mu. \quad (2.2)$$

Consequently,

$$\omega_s(r) \sim \begin{cases} \Gamma(N+2s)/2, & r \to 0, \\ 2^{-\frac{N}{2s}+\frac{s}{2}} r^{-\frac{N}{2s}+\frac{s}{2}} e^{-r}, & r \to +\infty. \end{cases}$$  

Note also that the functions $s \to \omega_s$ and $s \to d_{N,s}$ defined in [13] are continuous function of $s$ and we have that $\lim_{s \to 0^+} d_{N,s} = 0$ and, by dominated convergent theorem,

$$\omega(|y|) := \lim_{s \to 0^+} \omega_s(|y|) = 2^{1-\frac{N}{2s}} |y|^\frac{N}{2s} K_{\frac{N}{2s}}(|y|) = \int_0^\infty t^{-1+\frac{N}{2s}} e^{-t-\frac{|y|^2}{\pi t}} dt. \quad (2.4)$$

We now give the

**Proof of Theorem 1.1.** Let $u \in C^\alpha(\mathbb{R}^N)$ with $0 < \alpha < \min\{\frac{N}{2}, \frac{1}{2}\}$. Then, from the definition of $(I - \Delta)^\alpha$ in [13], the principal value can be dropped out and we have the different quotient

$$\frac{(I - \Delta)^\alpha u - u}{s} = \frac{d_{N,s}}{s} \int_{\mathbb{R}^N} \frac{u(x + y) - u(x)}{|y|^{N+2s}} \omega_s(|y|) \, dy = A_\varepsilon(s, x) + D_\varepsilon(s, x),$$

where $\varepsilon > 0$, with $A_\varepsilon(s, x)$ and $D_\varepsilon(s, x)$ given respectively by

$$A_\varepsilon(s, x) := \frac{d_{N,s}}{s} \int_{|y| < \varepsilon} \frac{u(x + y) - u(x)}{|y|^{N+2s}} \omega_s(|y|) \, dy,$$

$$D_\varepsilon(s, x) := \frac{d_{N,s}}{s} \int_{|y| \geq \varepsilon} \frac{u(x + y) - u(y)}{|y|^{N+2s}} \omega_s(|y|) \, dy.$$
First, from (1.4) and (1.7) and the fact that $|y|^{-2s} \leq \epsilon^{-2}$ for $|y| \geq \epsilon$ and $s \in (0, 1)$, we have by dominated convergent theorem that

$$D_\epsilon(s, x) = \frac{d_{N,s}}{s} \int_{|y| \geq \epsilon} \frac{u(x + y) - u(x)}{|y|^{N+2s}} \omega_s(|y|) \, dy \to D_\epsilon(0, x) \quad \text{as } s \to 0^+,$$

with

$$D_\epsilon(0, x) := d_N \int_{|x-y| \geq \epsilon} \frac{u(x) - u(y)}{|x-y|^N} \omega(|x-y|) \, dy = \int_{|x-y| \geq \epsilon} (u(x) - u(y)) J(x-y) \, dy.$$

Since next $u \in C^0(\mathbb{R}^N)$, it also follows that

$$A_\epsilon(s, x) = \frac{d_{N,s}}{s} \int_{|y| < \epsilon} \frac{u(x + y) - u(x)}{|y|^{N+2s}} \omega_s(|y|) \, dy \to A_\epsilon(0, x) \quad \text{as } s \to 0^+,$$

with

$$A_\epsilon(0, x) = d_N \int_{|y| < \epsilon} \frac{u(x) - u(x+y)}{|y|^N} \omega(|y|) \, dy = \int_{|x-y| < \epsilon} (u(x) - u(y)) J(x-y) \, dy.$$

We recall that $\lim_{s \to 0} d_{N,s} / s = -d_N$. It is easy to see that $A_\epsilon(0, x) \to 0$ as $\epsilon \to 0^+$, and from the thee fact that $u \in C^0(\mathbb{R}^N)$, we also infer that

$$\left| (I - \Delta)^{\alpha/2} u(x) - D_\epsilon(x, 0) \right| \leq C \int_{|y| < \epsilon} \min\{1, |y|^\alpha\} \, dy \to 0 \quad \text{as } \epsilon \to 0^+.$$

Since $u \in C^0(\mathbb{R}^N)$, setting $\kappa_{N,s,u} = \frac{d_{N,s}}{s} \Gamma((N + 2s)/2) \|u\|_{C^0(\mathbb{R}^N)} \omega_{N-1}$ it follows from (2.3) that

$$|A_\epsilon(s, x)| \leq \left| \frac{d_{N,s}}{s} \right| \int_{|y| < \epsilon} \frac{\|u\|_{C^0(\mathbb{R}^N)}}{|y|^{N+2s-\alpha}} \omega_s(|y|) \, dy \leq \kappa_{N,s,u} \frac{\epsilon^{\alpha-2s}}{\alpha - 2s}.$$

Consequently,

$$\|A_\epsilon(s, \cdot)\|_{L^p(B_x)} \leq \frac{k_{N,s,u}}{\alpha - 2s} \quad \text{for } 1 \leq p \leq \infty.$$

On the other hand, using again (2.3) with $s = 0$, we infer that

$$|D_\epsilon(0, x)| \leq \int_{|x-y| \geq \epsilon} |u(x) - u(x+y)| J(y) \, dy$$

$$\leq 2\|u\|_{C^0(\mathbb{R}^N)} \left( \int_{|x-y| \geq \epsilon} |y|^{-N} \, dy + \int_{|y| \geq 1} e^{-|y|} \, dy \right)$$

$$\leq 2\|u\|_{C^0(\mathbb{R}^N)} \left( 2 \frac{1 - \epsilon^\alpha}{\alpha} + \omega_{N-1} \Gamma(N, 1) \right) = C_{N, \epsilon} \|u\|_{C^0(\mathbb{R}^N)}.$$

Therefore,

$$\|D_\epsilon(0, \cdot)\|_{L^{\infty}(\mathbb{R}^N \setminus B_x)} < \infty.$$

Next, by the Minkowski’s integral inequality, we have

$$\|D_\epsilon(0, \cdot)\|_{L^p(\mathbb{R}^N \setminus B_x)} \leq \left( \int_{\mathbb{R}^N \setminus B_x} \left( \int_{|y| \geq \epsilon} |u(x) - u(x+y)| J(y) \, dy \right)^p \, dx \right)^{1/p}$$

$$\leq \int_{\mathbb{R}^N \setminus B_x} \left( \int_{|y| \geq \epsilon} |u(x) - u(x+y)|^p \, dx \right)^{1/p} J(y) \, dy$$

$$\leq 2^{\frac{p-1}{p}} \|u\|_{L^p(\mathbb{R}^N \setminus B_x)} \int_{\mathbb{R}^N \setminus B_x} J(y) \, dy < \infty.$$

Therefore, we conclude that $D_\epsilon(0, \cdot) \in L^p(\mathbb{R}^N \setminus B_x)$ for all $1 \leq p \leq \infty$ and thus

$$\|D_\epsilon(s, \cdot) - D_\epsilon(0, \cdot)\|_{L^p(\mathbb{R}^N \setminus B_x)} \to 0 \quad \text{uniformly in } \epsilon \quad \text{as } s \to 0^+. \quad (2.5)$$
This allows to conclude for $x \in \mathbb{R}^N$ that
\[
\lim_{\epsilon \to 0^+} D_\epsilon(0,x) = \lim_{\epsilon \to 0^+} \int_{|y| \geq \epsilon} (u(x) - u(x + y)) J(y) \, dy = (I - \Delta)^{\log} u(x). \tag{2.6}
\]
Taking into account the above facts, we find with $1 \leq p < \infty$ that
\[
\left\| \frac{(I - \Delta)^s u - u}{s} - (I - \Delta)^{\log} u \right\|_{L^p(\mathbb{R}^N)} = \left\| A_\epsilon(s, \cdot) + D_\epsilon(s, \cdot) - (I - \Delta)^{\log} u \right\|_{L^p(\mathbb{R}^N)} \\
\leq \| A_\epsilon(s, \cdot) \|_{L^p(\mathbb{R}^N)} + \| D_\epsilon(s, \cdot) - (I - \Delta)^{\log} u \|_{L^p(\mathbb{R}^N)} \\
\leq \kappa_{N,s,u} \frac{\epsilon^{\frac{2p}{p} + \alpha - 2s}}{\alpha - 2s} + \| D_\epsilon(s, \cdot) - (I - \Delta)^{\log} u \|_{L^p(\mathbb{R}^N)}.
\]
Therefore, using (2.5) and (2.6), we have for every $1 \leq p < \infty$ that
\[
\limsup_{s \to 0^+} \left\| \frac{(I - \Delta)^s u - u}{s} - (I - \Delta)^{\log} u \right\|_{L^p(\mathbb{R}^N)} \leq \kappa_{N,s,u} \frac{\epsilon^{\frac{2p}{p} + \alpha}}{\alpha} \quad \text{for every } \epsilon > 0,
\]
where $\kappa_{N,s,u}$ is independent of $\epsilon$. The case $p = \infty$ follows by the same computation and
\[
\limsup_{s \to 0^+} \left\| \frac{(I - \Delta)^s u - u}{s} - (I - \Delta)^{\log} u \right\|_{L^\infty(\mathbb{R}^N)} \leq \kappa_{N,s,u} \frac{\epsilon^\alpha}{\alpha} \quad \text{for every } \epsilon > 0.
\]
Moreover, it follows from the arbitrariness of $\epsilon$ that
\[
\lim_{s \to 0^+} \left\| \frac{(I - \Delta)^s u - u}{s} - (I - \Delta)^{\log} u \right\|_{L^p(\mathbb{R}^N)} = 0 \quad \text{for every } 1 \leq p \leq \infty.
\]
This completes the proof of item (i). The proof of item (ii) is a particular case with $p = 2$. Moreover, using the continuity of the Fourier transform in $L^2(\mathbb{R}^N)$, we have that
\[
\mathcal{F}((I - \Delta)^{\log} u) = \lim_{s \to 0^+} \frac{\mathcal{F}((I - \Delta)^s u) - \mathcal{F}(u)}{s} = \lim_{s \to 0^+} \left( \frac{(1 + | \cdot |^2)^s - 1}{s} \right) \mathcal{F}(u) \\
= \log (1 + | \cdot |^2) \mathcal{F}(u) \quad \text{in } L^2(\mathbb{R}^N).
\]
We therefore infer that
\[
\mathcal{F}((I - \Delta)^{\log} u)(\xi) = \log (1 + | \cdot |^2) \mathcal{F}(u)(\xi), \quad \text{for almost every } \xi \in \mathbb{R}^N.
\]
The proof of Theorem 1.1 is henceforth completed. \hfill \Box

In the following, we let $L_0(\mathbb{R}^N)$ denotes the space
\[
L_0(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} : \| u \|_{L_0(\mathbb{R}^N)} < \infty \right\} \quad \text{with } \| u \|_{L_0(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \frac{|u(x)| e^{-|x|}}{(1 + |x|)^{N+1}} \, dx.
\]
Let $U$ be a measurable subset and $u : U \to \mathbb{R}$ be a measurable function. The modulus of continuity of $u$ at a point $x \in U$ is defined by
\[
\omega_{u,x,U} : (0, +\infty) \to [0, +\infty), \quad \omega_{u,x,U}(r) = \sup_{y \in U, \, |x - y| \leq r} |u(x) - u(y)|.
\]
The function $u$ is called Dini continuous at $x$ if
\[
\int_0^1 \frac{\omega_{u,x,U}(r)}{r} \, dr < \infty.
\]
Moreover, we call $u$ uniformly Dini continuous in $U$ for the uniform modulus of continuity
\[
\omega_{u,U}(r) := \sup_{x \in U} \omega_{u,x,U}(r) \quad \text{if } \int_0^1 \frac{\omega_{u,U}(r)}{r} \, dr < \infty.
\]
In the following proposition, we list some properties the operator $(I - \Delta)^{\log}$.\n
Proposition 2.1.  
(i) Let \( u \in L_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). If \( u \) is locally Dini continuous at some point \( x \in \mathbb{R}^N \), then the operator \((I - \Delta)^{\log} u\) is well defined by
\[
(I - \Delta)^{\log} u(x) = \int_{\mathbb{R}^N} (u(x) - u(y))J(x - y) \, dy.
\]
(ii) Let \( \varphi \in C^2_c(\mathbb{R}^N) \) for some \( \alpha > 0 \), there is \( C = C(N, \varphi) \) such that
\[
| (I - \Delta)^{\log} \varphi(x) | \leq C\|\varphi\|_{C^0(\mathbb{R}^N)} \frac{e^{-|x|}}{(1 + |x|^2)^{\alpha/2}}.
\]
In particular, for \( u \in L_0(\mathbb{R}^N) \), \((I - \Delta)^{\log} u\) defines a distribution via the map
\[
\varphi \mapsto \langle (I - \Delta)^{\log} u, \varphi \rangle = \int_{\mathbb{R}^N} u(I - \Delta)^{\log} \varphi \, dx.
\]
(iii) Let \( u \in L_0(\mathbb{R}^N) \) and \( r > 0 \) such that \( u \in C^\alpha(B_r(0)) \) for some \( \alpha > 0 \). Then there exists a constant \( C := C(N, \alpha) > 0 \) such that
\[
| (I - \Delta)^{\log} u(x) | \leq C(\|u\|_{C^\alpha(B_{r+0})} + \|u\|_{L^\infty(\mathbb{R}^N)}).
\]
(iv) If \( u \in C^\beta(\mathbb{R}^N) \) for some \( \beta > 0 \), then \((I - \Delta)^{\log} u \in C^{\beta - \epsilon}(\mathbb{R}^N) \) for every \( \epsilon > 0 \) such that
\[
| (I - \Delta)^{\log} u |_{\beta - \epsilon} \leq C\|u\|_{C^\beta(\mathbb{R}^N)}.
\]
(v) Let \( \varphi, \psi \in C^\infty_c(\Omega) \). Then we have the product rule
\[
(I - \Delta)^{\log}(\varphi \psi)(x) = \varphi(x)(I - \Delta)^{\log} \psi(x) + \psi(x)(I - \Delta)^{\log} \varphi(x) - \Lambda(\varphi, \psi).
\]
with
\[
\Lambda(\varphi, \psi) := \int_{\mathbb{R}^N} (\varphi(x) - \varphi(y))(\psi(x) - \psi(y))J(x - y) \, dy.
\]
If \( \rho_\epsilon, \epsilon > 0 \) is a family of mollified, then
\[
[(I - \Delta)^{\log}(\rho_\epsilon * \varphi)](x) = \rho_\epsilon * [(I - \Delta)^{\log} \varphi](x).
\]

Proof. Let \( x \in \mathbb{R}^N \). By splitting the integral and using the asymptotic of \( J \) in \((1.9)\), we have the following,
\[
| (I - \Delta)^{\log} u(x) | \leq \int_{B_1(x)} |u(x) - u(y)|J(x - y) \, dy + \int_{\mathbb{R}^N \setminus B_1(x)} (|u(x)| + |u(y)|)J(x - y) \, dy.
\]
\[
\leq \Gamma/N^2 |u|_{L^{\infty}(\Omega)} \int_{0}^{1} \frac{\omega_{N-r}(r)}{r} \, dr + C\|u\|_{L^\infty} \int_{\mathbb{R}^N \setminus B_1(x)} e^{-|y|} \, dy + C \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u(y)|e^{-|x-y|}}{|x-y|^2} \, dy.
\]
\[
\leq C(1 + |u|_{L^{\infty}(\mathbb{R}^N)}) + C \left( \int_{B_{1+2|x|}(0) \setminus B_1(x)} + \int_{\mathbb{R}^N \setminus B_{1+2|x|}(0)} \right) \frac{|u(y)|e^{-|x-y|}}{|x-y|^2} \, dy.
\]
\[
\leq C(1 + |u|_{L^{\infty}(\mathbb{R}^N)}) \cdot \left( \int_{B_{1+2|x|}(0)} |u(y)| \, dy + \int_{\mathbb{R}^N \setminus B_{1+2|x|}(0)} \frac{|u(y)|e^{-|x-y|}}{|x-y|^2} \, dy.\right.
\]
Now, since \( |x - y| \geq \frac{1}{2}(1 + |y|) \) for \( |y| \geq 1 + 2|x| \), it follows that
\[
| (I - \Delta)^{\log} u(x) | \leq C(1 + |u|_{L^{\infty}(\mathbb{R}^N)}) + \|u\|_{L_0(\mathbb{R}^N)} < \infty.
\]
This shows that \((I - \Delta)^{\log} u(x)\) is well-defined.
To prove \((ii)\), for \( x \in \mathbb{R}^N \), we use again \((1.9)\) and the representation
\[
(I - \Delta)^{\log} \varphi(x) = \frac{d_N}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x + y) - \varphi(x - y)}{|y|^N} \omega(|y|) \, dy.
\]
Put \( A := \|\varphi\|_{C^0(\mathbb{R}^N)} \). Note first that, since \( \varphi \in C^\alpha_c(\mathbb{R}^N) \), we have
\[
|2\varphi(x) - \varphi(x + y) - \varphi(x - y)| \leq A \min\{1, |y|^\alpha\}.
\]
Therefore, for any $x \in \mathbb{R}^N$, we have with $0 < r < 1$ that
\[
|\langle (I - \Delta)^{\log} \varphi, x \rangle| \leq \frac{d N}{2} \int_{\mathbb{R}^N} \frac{|2\varphi(x) - \varphi(x + y) - \varphi(x - y)|}{|y|^N} \omega(|y|) \, dy
\]
\[
\leq A \int_{\mathbb{R}^N} \min\{1, |y|^a\} \omega(|y|) \, dy
\]
\[
\leq CNA \left( \int_{B_r} |y|^{a - N} \, dy + \int_{B_1 \setminus B_r} \frac{1}{|y|^N} \, dy + \int_{\mathbb{R}^N \setminus B_1} e^{-|y|} \, dr \right)
\]
\[
\leq C(N, r, \alpha) A.
\]
Next, let $R > 0$ be such that $B_1(\text{supp } \varphi) \subset B_R(0)$. Let $x \in \mathbb{R}^N$ satisfying $|x| \geq \frac{|x|}{2} > R$, then $1 + |y| \leq \frac{|x|}{2}$ for $y \in B_1(\text{supp } \varphi)$ and $|x - y| \geq |x| - |y| \geq \frac{|x|}{2} + 1 \geq |x| + 1$. Moreover, since $\varphi(x) \equiv 0$ for $x \in \mathbb{R}^N \setminus B_R(0)$, it follows that
\[
|\langle (I - \Delta)^{\log} \varphi, x \rangle| \leq 2d N A \int_{\text{supp } \varphi} \frac{\omega(|x - y|)}{|x - y|^N} \, dy \leq CNA \int_{\text{supp } \varphi} \frac{e^{-|x - y|}}{|x - y|^\frac{N - a}{2}} \, dy
\]
\[
\leq CNA \int_{\text{supp } \varphi} \frac{e^{-\frac{|x|}{2}}}{(1 + |x|)^{\frac{N - a}{2}}} \, dy \leq C_N |\text{supp } \varphi| A \frac{e^{-\frac{|x|}{2}}}{(1 + |x|)^{\frac{N - a}{2}}}.
\]
Therefore, combining the above computations, we find that
\[
|\langle (I - \Delta)^{\log} \varphi, x \rangle| \leq C_N, A \frac{e^{-|x|}}{(1 + |x|)^{\frac{N - a}{2}}}
\]
for all $x \in \mathbb{R}^N$.

From the above computations, we have that $|\langle (I - \Delta)^{\log} u, \varphi \rangle| \leq C_{N, \alpha} \|\varphi\|_{C^a(B_R(0))}\|u\|_{L_0(\mathbb{R}^N)}$ and if the sequence $\{u_n\}_n$ converges to $u$ in $L_0(\mathbb{R}^N)$ as $n \to \infty$ then
\[
|\langle (I - \Delta)^{\log} u_n - (I - \Delta)^{\log} u, \varphi \rangle| \leq C_{N, \alpha} A \|u_n - u\|_{L_0(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.
\]

Proof of (iii). This follows from (i) and the inequality
\[
|2u(x) - u(x + y) - u(x - y)| \leq \|u\|_{C^a(B_{r/2}(0))}\|y\|^a
\]
for $y \in B_{r/2}(0)$.

Proof of (iv). Let $0 < r < 1$ be small. We have the following estimate of the difference,
\[
|\langle (I - \Delta)^{\log} u, x_1 \rangle - \langle (I - \Delta)^{\log} u, x_2 \rangle| \leq d_N (I_1 + I_2)
\]
where $I_1$ and $I_2$ are given by
\[
I_1 := \int_{B_r} \frac{|u(x_1) - u(x_1 + y)| + |u(x_2) - u(x_2 + y)|}{|y|^N} \omega(|y|) \, dy
\]
\[
I_2 := \int_{\mathbb{R}^N \setminus B_r} \frac{|u(x_1) - u(x_2)| + |u(x_1 + y) - u(x_2 + y)|}{|y|^N} \omega(|y|) \, dy
\]
For $I_1$, we use the inequality $|u(x_1) - u(x_1 + y)| \leq \|u\|_{C^a(B^N)} \|y\|^\beta$ to get
\[
I_1 \leq 2\|u\|_{C^a(B^N)} \int_{B_r} |y|^\beta \omega(|y|) \, dx \leq \frac{2\|N - 1\|N/2}{\beta} \|u\|_{C^a(B^N)} \|y\|^\beta
\]
For $I_2$, we use $|u(x_1) - u(x_2)| + |u(x_1 + y) - u(x_2 + y)| \leq 2\|u\|_{C^a(B^N)} |x_1 - x_2|$ and,
\[
I_2 \leq 2|x_1 - x_2|^2 \|u\|_{C^a(B^N)} \left( \int_{B_1 \setminus B_r} \frac{\omega(|y|)}{|y|^N} \, dy + \int_{\mathbb{R}^N \setminus B_1} \frac{\omega(|y|)}{|y|^N} \, dy \right)
\]
\[
\leq 2|x_1 - x_2|^2 \|u\|_{C^a(B^N)} \left( \Gamma(N/2) \int_{B_1 \setminus B_r} \frac{1}{|y|^N} \, dy + \int_{\mathbb{R}^N \setminus B_1} \frac{e^{-|y|}}{|y|^N} \, dy \right)
\]
\[
\leq 2|x_1 - x_2|^2 \|u\|_{C^a(B^N)} \omega_{N - 1} \left( \Gamma(N/2) \log \frac{1}{r} + \Gamma(N, 1) \right)
\]
where we have used the inequality \( \log(\rho) \leq \frac{\epsilon}{\epsilon} \) for \( \epsilon > 0 \) and \( \rho \geq 1 \) (see [16]). Therefore, taking \( r = |x_1 - x_2| \), we end with
\[
|\{I - \Delta\}^{\log^r}u(x_1) - (I - \Delta)^{\log^r}u(x_2)| \leq C(N, \beta, \epsilon)\|u\|_{C^\beta(\mathbb{R}^N)}|x_1 - x_2|^{\beta - \epsilon}.
\]

Proof of (v). This easily follows by integrating the following equality
\[
(\varphi(x)\psi(x) - \varphi(y)\psi(y)) = (\varphi(x) - \varphi(y))\psi(x) + (\psi(x) - \psi(y))\varphi(x) - (\varphi(x) - \varphi(y))(\psi(x) - \psi(y)),
\]
while the second statement is an application of Fubini’s theorem. This completes the proof of Proposition [24].

We next list some properties for functions belonging to the space \( H^{log}(\mathbb{R}^N) \).

**Lemma 2.2.** The following assertions hold true

1. If \( u \in H^{log}(\mathbb{R}^N) \), then \( |u|, u^\pm \in H^{log}(\mathbb{R}^N) \) with \( \|u\|_{H^{log}(\mathbb{R}^N)}, \|u^\pm\|_{H^{log}(\mathbb{R}^N)} \leq \|u\|_{H^{\psi^2}} \).
2. The space \( C^{0,\alpha}_c(\mathbb{R}^N) \) is a subset of \( H^{log}(\mathbb{R}^N) \) for any \( \alpha > 0 \).
3. If \( \varphi \in C^{0,\alpha}_c(\mathbb{R}^N) \) and \( u \in H^{log}(\mathbb{R}^N) \), then \( \varphi u \in H^{log}(\mathbb{R}^N) \) and there is a constant \( C := C(N, \varphi) > 0 \) such that
\[
\|\varphi u\|_{H^{log}(\mathbb{R}^N)}^2 \leq C\|u\|_{H^{log}(\mathbb{R}^N)}^2.
\]

**Proof.** It straightforward to see by integrating the inequality
\[
|u(x)| - |u(y)| \leq |u(x) - u(y)|
\]
that \( \mathcal{E}_\omega(|u|, |u|) \leq \mathcal{E}_\omega(u, u) \) and \( \|u\|_{H^{log}(\mathbb{R}^N)} \leq \|u\|_{H^{\psi^2}} \). Using also the inequality
\[
2(u^+(x) - u^+(y))(u^-(x) - u^-(y)) = -2(u^-(x)u^+(y) + u^+(y)u^+(x)) \leq 0 \quad \text{for} \ x, y \in \mathbb{R}^N,
\]
it follows that
\[
\mathcal{E}_\omega(u, u) = \mathcal{E}_\omega(u^+, u^+) + \mathcal{E}_\omega(u^-, u^-) - 2\mathcal{E}_\omega(u^+, u^-) \geq \mathcal{E}_\omega(u^+, u^+) + \mathcal{E}_\omega(u^-, u^-),
\]
proving clearly that the first item holds. Now, for the second item, we use \( u \in C^{0,\alpha}_c(\mathbb{R}^N) \) be such that \( \text{supp } u \subset B_r, r > 0 \), without loss of generality we may assume that \( 0 < r < 1 \) such that we can directly apply the asymptotics in [16]. We therefore have
\[
\mathcal{E}_\omega(u, u) = \frac{1}{2} \int_{B_r} \int_{B_r} |u(x) - u(y)|^2 J(x, y) \, dx \, dy + \int_{B_r} u^2(x) \int_{\mathbb{R}^N \backslash B_r} J(x - y) \, dy \, dx
\]
\[
\leq C_1 \int_{B_r} \int_{B_r} |x - y|^{2\alpha - N} \, dx \, dy + C_2 \int_{B_r} u^2(x) \left( \int_{B_r \backslash B_{r/2}} |x - y|^{-N} \, dy \right)
\]
\[
+ \int_{\mathbb{R}^N \backslash B_1} e^{-|x-y|/d} \, dx \leq C \frac{|B_r(0)|}{2\alpha} r^{-2\alpha} + C_3,
\]
where the constants \( C := C(N) > 0, C_2 := C_2(r, N) > 0 \) and \( C_3 := C_3(r, N) > 0 \). The second item is proved. We next prove item 3. Let \( u \in H^{log}(\mathbb{R}^N) \) and \( \varphi \in C^{0,\alpha}_c(\mathbb{R}^N) \) with \( \text{supp } \varphi \subset B_r \), for \( 0 < r < 1 \). Then using the inequality
\[
|\varphi(x)u(x) - \varphi(y)u(y)|^2 \leq 2(|u(x) - u(y)|^2|\varphi(x)|^2 + |u(y)|^2|\varphi(x) - \varphi(y)|^2),
\]
we get
\[
\mathcal{E}_\omega(u, u) \leq \int_{B_r} \int_{B_r} |\varphi(x)|^2 |u(x) - u(y)|^2 J(x, y) \, dx \, dy
\]
\[
+ 2 \int_{B_r} u^2(x) \int_{B_r} |\varphi(x) - \varphi(y)|^2 J(x - y) \, dy \, dx
\]
\[
+ C \int_{B_r} |\varphi(x)u(x)|^2 \left( \int_{B_r \backslash B_{r/2}} |x - y|^{-N} \, dy \right) + \int_{\mathbb{R}^N \backslash B_1} e^{-|x-y|/d} \, dx
\]
sequence in $L^\infty$. 

Proof. Since $\|u\|_{L^2(\mathbb{R}^N)} \leq C_\varphi \|u\|_{L^2(\mathbb{R}^N)}$, we have that $\varphi u \in H^{log}(\mathbb{R}^N)$ and item 3 is proved. 

We recall the space $\mathcal{H}_0^0(\Omega)$, corresponding to the analytical framework for the logarithmic Laplacian $L_\Delta$ introduced in [8], see also [12], given by

$$
\mathcal{H}_0^0(\Omega) = \left\{ u \in L^2(\mathbb{R}^N) : u \equiv 0 \text{ on } \Omega^c \text{ and } \int_{x,y \in \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^N} dxdy < \infty \right\}. 
$$

Here $\Omega^c = \mathbb{R}^N \setminus \Omega$, and the map

$$(u, v) \mapsto \langle u, v \rangle_{\mathcal{H}_0^0(\Omega)} := \frac{C_N}{2} \int_{x,y \in \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^N} dxdy,$$

is a scalar product on $\mathcal{H}_0^0(\Omega)$. The space $\mathcal{H}_0^0(\Omega)$ is a Hilbert space with induced norm $\|\cdot\|_{\mathcal{H}_0^0(\Omega)} = \langle \cdot, \cdot \rangle_{\mathcal{H}_0^0(\Omega)}$. Moreover, The space $C_c^2(\Omega)$ is dense in $\mathcal{H}_0^0(\Omega)$ and

the embedding $\mathcal{H}_0^0(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

We have the following Lemma

**Lemma 2.3.**

(i) the space $H^{log}(\mathbb{R}^N)$ is a Hilbert space and, $H^m(\mathbb{R}^N) \subset H^{log}(\mathbb{R}^N)$ for all $m > 0$.

(ii) If $\Omega \subset \mathbb{R}^N$ is an open set with finite measure then we have the following Poincaré inequality with $C := C(N, \Omega)$

$$
\|u\|_{L^2(\Omega)} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 J(x-y) dxdy, \quad u \in \mathcal{H}_0^{log}(\Omega) \tag{2.8}
$$

(iii) If $\Omega \subset \mathbb{R}^N$ is bounded, then there a constant $C_j := C(N, \Omega)$, $j = 1, 2$ such that

$$
C_1 \mathcal{E}_\omega(u, u) \leq \|u\|^2_{\mathcal{H}_0^{log}(\Omega)} \leq C_2 \mathcal{E}_\omega(u, u) \tag{2.9}
$$

(iv) The space $C_c^\infty(\Omega)$ is dense in $\mathcal{H}_0^{log}(\Omega)$ and

the embedding $\mathcal{H}_0^{log}(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

**Proof.** Let $\{u_n\}_n \subset H^{log}(\mathbb{R}^N)$ be a Cauchy sequence. Then $\{u_n\}_n$ is in particular a Cauchy sequence in $L^2(\mathbb{R}^N)$ and hence there exists a $u \in L^2(\mathbb{R}^N)$ such that $u_n \to u$ as $n \to \infty$. Passing to a subsequence we get that $u_n \to u$ a.e in $\mathbb{R}^N$ as $n \to \infty$ and by Fatou Lemma we have

$$
\mathcal{E}_\omega(u, u) \leq \liminf_{n \to \infty} \mathcal{E}_\omega(u_n, u_n) \leq \sup_{n \in \mathbb{N}} \mathcal{E}_\omega(u_n, u_n) < \infty,
$$

showing that $u \in H^{log}(\mathbb{R}^N)$. Apply once more Fatou Lemma it follows that

$$
\|u_n - u\|^2_{H^{log}(\mathbb{R}^N)} = \|u_n - u\|^2_{L^2(\mathbb{R}^N)} + \mathcal{E}_\omega(u_n - u, u_n - u) \leq \liminf_{n \to \infty} \|u_n - u_m\|^2_{H^{log}(\mathbb{R}^N)};
$$

for $n, m \in \mathbb{N}$. The claim follows since $\{u_n\}_n$ is a Cauchy sequence in $H^{log}(\mathbb{R}^N)$.

By Plancherel thereon the norm in $H^{log}(\mathbb{R}^N)$ is also given via Fourier representation

$$
\|u\|^2_{H^{log}(\mathbb{R}^N)} = \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \log(1 + |\xi|^2) |\mathcal{F}(u)(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
$$

Therefore, using the standard inequality $\log \rho \leq \frac{\rho - 1}{m}$ for $\rho \geq 1$ for $m > 0$ (see e.g. [16]) one see that the space $H^{log}(\mathbb{R}^N)$ is larger than any Sobolev space $H^m(\mathbb{R}^N) := W^{m,2}(\mathbb{R}^N)$. In fact if
$u \in H^m(\mathbb{R}^N)$ then the proof of (i) is completed by the following inequality,

$$\|u\|_{H^m(\mathbb{R}^N)}^2 = \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \log(1 + |\xi|^2)|\mathcal{F}(u)(\xi)|^2 d\xi$$

$$\leq \|u\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{m} \int_{\mathbb{R}^N} (1 + |\xi|^2)^m|\mathcal{F}(u)(\xi)|^2 d\xi \leq C_m \|u\|_{H^m(\mathbb{R}^N)}^2.$$  

(2.10)

The Poincaré inequality in (ii) follows from [11, Lemma 2.9] and [17] if $\Omega$ is bounded or bounded in one direction. We provide the proof here for $\Omega \subset \mathbb{R}^N$ with continuous boundary.

For item (iii), we use the asymptotics in (1.9) to get

$$\|u\|_{L^2(\Omega)}^2 = \frac{1}{2} \int |x - y| \omega(|x - y|) \, dx \, dy \leq C_1 \int \int |x - y|^N \omega(|x - y|) \, dx \, dy.$$

Next, using Poincaré inequality for $\mathcal{H}^0_0(\Omega)$ again with (1.9) we get that

$$\mathcal{E}(u, u) = \frac{d_N}{2} \int \int |x - y|^N \omega(|x - y|) \, dx \, dy$$

$$\leq \Gamma N \int \int \frac{|u(x) - u(y)|^2}{|x - y|^N} \omega(|x - y|) \, dx \, dy + 2 \int_\Omega |u(x)|^2 \int_{\Omega \cap \{|x - y| \geq 1\}} \omega(|x - y|) \, dy \, dx$$

$$+ \int_\Omega |u(x)|^2 \int_{\mathbb{R}^N \setminus \Omega} \omega(|x - y|) \, dx \, dy \leq C_2 \|u\|_{\mathcal{H}^0_0(\Omega)}$$

with

$$C_2 := C \left(1 + \sup_{x \in \Omega} \left( \int_{\mathbb{R}^N \setminus \Omega} \omega(|x - y|) \, dy + \int_{\Omega \cap \{|x - y| \geq 1\}} \omega(|x - y|) \, dy \right) \right) < \infty.$$  

The proof of (iv) follows from [8, Theorem 3.1] and (iii) since the space $C_K^\infty(\Omega)$ is dense in $\mathcal{H}^0_0(\Omega)$ and the embedding $\mathcal{H}^0_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. The proof ends here. $\square$

As a consequence of the Poincaré inequality, we have for bounded $\Omega$ with continuous boundary that the space $\mathcal{H}^0_0(\Omega)$ can be identified by

$$\mathcal{H}^0_0(\Omega) = \left\{ u \in H^0(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.$$  

and it is a Hilbert space endowed with the scalar product $(v, w) \mapsto \mathcal{E}(v, w)$ and the corresponding norm $\|u\|_{\mathcal{H}^0_0(\Omega)} = \sqrt{\mathcal{E}(u, u)}$. 


3. Eigenvalue problem

In this section, we provide the proof of Theorem 1.3, proposition 1.4 and Theorem 1.5 concerning the study of the Dirichlet eigenvalue problem in bounded open set \( \Omega \).

\[
\begin{align*}
(I - \Delta)^{\log} u &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.
\end{align*}
\] (3.1)

We start with the

**Proof of Theorem 1.3.** Let \( \Psi : \mathcal{H}_0^{\log}(\Omega) \to \mathbb{R} \) be the functional defined by

\[
\Psi(u) := \mathcal{E}_\omega(u,u) = \|u\|^2_{\mathcal{H}_0^{\log}(\Omega)}.
\]

We use the direct method of minimization. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a minimizing sequence for \( \Psi \) in \( \mathcal{P}_1(\Omega) := \{ u \in \mathcal{H}_0^{\log}(\Omega) : \|u\|_{L^2(\Omega)} = 1 \} \), that is

\[
\lim_{n \to \infty} \Psi(u_n) = \inf_{u \in \mathcal{P}_1(\Omega)} \Psi(u) \geq 0 > -\infty.
\]

Then by the definition of \( \Psi \), the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( \mathcal{H}_0^{\log}(\Omega) \) and up to subsequence, there exists \( u_0 \in \mathcal{H}_0^{\log}(\Omega) \) such that thanks to (2.9),

\[
u_n \to u_0 \quad \text{weakly in } \mathcal{H}_0^{\log}(\Omega) \quad \text{(3.2)}
\]

\[
u_n \to u_0 \quad \text{strongly in } L^2(\Omega). \quad \text{(3.3)}
\]

It follows from (3.3) that \( \|u_0\|_{L^2(\Omega)} = 1 \) and that \( u_0 \in \mathcal{P}_1(\Omega) \). Using the lower-semi-continuity of the norm in \( \mathcal{H}_0^{\log}(\Omega) \), we deduce that

\[
\inf_{u \in \mathcal{P}_1(\Omega)} \Psi(u) = \lim_{n \to \infty} \Psi(u_n) \geq \Psi(u_0) \geq \inf_{u \in \mathcal{P}_1(\Omega)} \Psi(u).
\]

This yields that \( \Psi(u_0) = \inf_{u \in \mathcal{P}_1(\Omega)} \Psi(u) \) and, the first eigenvalue is \( \lambda_1(\Omega) = \Psi(u_0) \), with the corresponding eigenfunction \( \varphi_1 = u_0 \in \mathcal{P}_1(\Omega) \). By the Lagrange multipliers theorem, there exists \( \lambda \in \mathbb{R} \) such that

\[
\mathcal{E}_\omega(\varphi_1, v) = \langle \mathcal{E}'(\varphi_1), v \rangle = \lambda \int_{\Omega} \varphi_1 v \, dx \quad \text{for all } v \in \mathcal{H}_0^{\log}(\Omega). \quad \text{(3.4)}
\]

Taking in particular \( v = \varphi_1 \), we find that \( \lambda = \lambda_1(\Omega) = \mathcal{E}_\omega(\varphi_1, \varphi_1) \). We next show that \( \varphi_1 \) does not change sign in \( \Omega \). Indeed, since \( \mathcal{E}_\omega(|v|, |v|) \leq \mathcal{E}_\omega(v, v) \) for \( v \in \mathcal{H}_0^{\log}(\Omega) \), it follows that \( |\varphi_1| \in \mathcal{P}_1(\Omega) \) and by the definition of \( \lambda_1(\Omega) \) we have that

\[
\lambda_1(\Omega) = \mathcal{E}_\omega(|\varphi_1|, |\varphi_1|),
\]

showing that \( \varphi_1 \) does not change sign in \( \Omega \). We may assume that \( \varphi_1 \) is nonnegative. Suppose then that \( \varphi_1(x_0) = 0 \) for some \( x_0 \in \Omega \). Then

\[
0 = \lambda_1(\Omega) \varphi_1(x_0) = -d_N \int_{\mathbb{R}^N} \frac{\varphi_1(x_0)}{|x - y|} \omega(|x - y|) \, dy < 0
\]

which yields a contradiction. Therefore \( \varphi_1 > 0 \) in \( \Omega \) and (i) is proved.

We prove (ii) via contradiction. Suppose that there exists a function \( v \in \mathcal{P}_1(\Omega) \) satisfying \( (I - \Delta)^{\log} v = \lambda_1 v \) with \( v \neq \alpha \varphi_1 \) for every \( \alpha \in \mathbb{R} \). Then \( w := v - \alpha \varphi_1 \) satisfies also \( (I - \Delta)^{\log} w = \lambda_1 w \). But since \( \varphi_1 > 0 \) in \( \Omega \), by choosing \( \alpha = \frac{v(x_0)}{\varphi_1(x_0)} \), \( x_0 \in \Omega \), it follows that \( w \) vanishes at \( x_0 \in \Omega \) and therefore must change sign. This contradicts (i) and thus the eigenvalue \( \lambda_1(\Omega) \) is simple.

We prove (iii) by induction. We first note that, if follows from the simplicity of \( \lambda_1(\Omega) \) in (ii) that \( \lambda_1(\Omega) < \lambda_2(\Omega) \). By the same construction as in the case \( k = 1 \), we get a sequence
of eigenfunctions \( \varphi_2, \ldots, \varphi_k \in \mathcal{H}_0^{\text{log}}(\Omega) \) and eigenvalues \( \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega), \) \( k \in \mathbb{N} \) with the properties that

\[
\lambda_j(\Omega) = \inf_{u \in \mathcal{P}_j(\Omega)} \mathcal{E}_\omega(u, u) = \mathcal{E}_\omega(\varphi_j, \varphi_j), \quad j = 1, \cdots, k \quad \text{and}
\]

\[
\mathcal{E}_\omega(\varphi_j, v) = \lambda_j(\Omega) \int_\Omega \varphi_j v \, dx \quad \text{for all} \quad v \in \mathcal{H}_0^{\text{log}}(\Omega).
\]

Next, we define \( \lambda_{k+1}(\Omega) \) as in \( \text{(1.18)} \), that is

\[
\lambda_{k+1}(\Omega) = \inf_{u \in \mathcal{P}_{k+1}(\Omega)} \mathcal{E}_\omega(u, u).
\]

By the same argument as above, the value \( \lambda_{k+1}(\Omega) \) is attained by a function \( \varphi_{k+1} \in \mathcal{P}_{k+1}(\Omega) \) and by the Lagrange multipliers theorem, there exists \( \lambda \in \mathbb{R} \) such that

\[
\mathcal{E}_\omega(\varphi_{k+1}, v) = \lambda \int_\Omega \varphi_{k+1} v \, dx \quad \text{for all} \quad v \in \mathcal{P}_{k+1}(\Omega).
\]

Taking in particular \( v = \varphi_{k+1} \) in \( \text{(1.15)} \), we get that \( \lambda = \lambda_{k+1}(\Omega) \). Moreover, for \( j = 1, \cdots, k \), it follows from the definition of \( \mathcal{P}_{k+1}(\Omega) \) and taking \( v = \varphi_j \) in \( \text{(3.5)} \), we find that

\[
\mathcal{E}_\omega(\varphi_{k+1}, \varphi_j) = 0 = \lambda_j(\Omega) \int_\Omega \varphi_{k+1} \varphi_j \, dx.
\]

In other to conclude that \( \varphi_{k+1} \) is an eigenfunction corresponding to eigenvalue \( \lambda_{k+1}(\Omega) \), we need to show that \( \text{(3.5)} \) holds for all \( v \in \mathcal{H}_0^{\text{log}}(\Omega) \). To see this we write \( \mathcal{H}_0^{\text{log}}(\Omega) = \text{span}\{\varphi_1, \cdots, \varphi_k\} \oplus \mathcal{P}_{k+1}(\Omega) \) such that any \( v \in \mathcal{H}_0^{\text{log}}(\Omega) \) can be written as \( v = v_1 + v_2 \) with \( v_1 \in \text{span}\{\varphi_1, \cdots, \varphi_k\} \) and \( v_2 \in \mathcal{P}_{k+1}(\Omega) \). It follows from \( \text{(3.5)} \) with \( v \) replaced by \( v_2 = v - v_1 \in \mathcal{P}_{k+1}(\Omega) \) that

\[
0 = \mathcal{E}_\omega(\varphi_{k+1}, v_2) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1} v_2 \, dx
\]

\[
= \mathcal{E}_\omega(\varphi_{k+1}, v) - \mathcal{E}_\omega(\varphi_{k+1}, v_1) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1} (v - v_1) \, dx
\]

\[
= \mathcal{E}_\omega(\varphi_{k+1}, v) - \lambda_{k+1}(\Omega) \int_\Omega \varphi_{k+1} v \, dx,
\]

where we used equality in \( \text{(3.6)} \). This shows that \( \text{(3.5)} \) holds for all \( v \in \mathcal{H}_0^{\text{log}}(\Omega) \). We have just constructed inductively an \( L^2(\Omega) \)-normalized sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) in \( \mathcal{H}_0^{\text{log}}(\Omega) \) and a nondecreasing sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \in \mathbb{R} \) such that \( \text{(1.15)} \) holds and such that \( \varphi_k \) is an eigenfunction of \( \text{(1.15)} \) corresponding to \( \lambda = \lambda_k(\Omega) \) for every \( k \in \mathbb{N} \). Moreover, we have by construction that \( \{\varphi_k\}_{k \in \mathbb{N}} \) form an orthogonal system in \( L^2(\Omega) \). To complete the proof of \( (iii) \), it remains to show that \( \lim_{k \to +\infty} \lambda_k(\Omega) = +\infty \). Suppose by contradiction that

\[
\mathcal{E}_\omega(\varphi_k, \varphi_k) = \lambda_k(\Omega) = c_0 \in \mathbb{R} \quad \text{as} \quad k \to +\infty \quad \text{for every} \quad k \in \mathbb{N}.
\]

Then the sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) is bounded in \( \mathcal{H}_0^{\text{log}}(\Omega) \) and, up to subsequence, there is \( \varphi_0 \in \mathcal{H}_0^{\text{log}}(\Omega) \) such that

\[
\varphi_k \to \varphi_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to +\infty.
\]

It follows in particular that \( \{\varphi_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( L^2(\Omega) \). But orthogonality in \( L^2(\Omega) \) implies that \( \|\varphi_k - \varphi_j\|_{L^2(\Omega)} = 2 \) for every \( k \) and \( j \), which leads to a contradiction.

For the proof of assertion \( (iv) \), the orthogonality follows from \( (iii) \). we then need to show that the sequence of eigenfunctions \( \{\varphi_k\}_{k \in \mathbb{N}} \) is a basis for both \( L^2(\Omega) \) and \( \mathcal{H}_0^{\text{log}}(\Omega) \). Let suppose by contradiction that there exists a nontrivial \( u \in \mathcal{H}_0^{\text{log}}(\Omega) \) with

\[
\|u\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \int_\Omega \varphi_k u \, dx = 0 \quad \text{for any} \quad k \in \mathbb{N}.
\]

(3.7)
Since we have that $\lim_{k \to +\infty} \lambda_k(\Omega) = +\infty$, there exists an integer $k_0 > 0$ such that
\[
\Psi(u) < \lambda_{k_0}(\Omega) = \inf_{v \in \mathcal{P}_{k_0}(\Omega)} \Psi(v).
\]
This implies that $u /\in \mathcal{P}_{k_0}(\Omega)$ and, by the definition of $\mathcal{P}_{k_0}(\Omega)$, we have that $\int_{\Omega} \varphi_j u \, dx \neq 0$ for some $j \in \{1, \cdots, k_0 - 1\}$. This contradicts (3.7). We conclude that $\mathcal{H}^{\log}_0(\Omega)$ is contained in the $L^2$-closure of the span of $\{\varphi_k : k \in \mathbb{N}\}$. Since $\mathcal{H}^{\log}_0(\Omega)$ is dense in $L^2(\Omega)$, we conclude that the span of $\{\varphi_k : k \in \mathbb{N}\}$ is dense in $L^2(\Omega)$, and hence, the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$. This complete the proof of Theorem 1.3.

We next give the

**Proof of Proposition 1.4.** We work here with the $\delta$-decomposition of the nonlocal operators as described in [13] Theorem 3.1. For this, let $\Omega \subset \mathbb{R}^N$ be open and bounded set of $\mathbb{R}^N$. For $\delta > 0$, we let $J_\delta := 1_{B_\delta} J$ and $K_\delta := J - J_\delta$. Note that for $u, v \in \mathcal{H}_0^{\log}(\Omega)$,
\[
\mathcal{E}_\delta(u, v) = \mathcal{E}_\omega(u, v) + \frac{dN}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))K_\delta(x - y) \, dx \, dy
\]
where the $\delta$-dependent quadratic form $\mathcal{E}_\delta$ is given by
\[
(u, v) \mapsto \mathcal{E}_\delta(u, v) = \frac{dN}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))J_\delta(x - y) \, dx \, dy,
\]
the function $K_\delta \in L^1(\mathbb{R}^N)$ and the constant $\kappa_\delta$ is
\[
\kappa_\delta = \int_{B_1 \setminus B_\delta} K_\delta(z) \, dz > \int_{B_1 \setminus B_\delta} \frac{1}{|z|^N} \, dz = -c_N \ln \delta \to +\infty \quad \text{as} \quad \delta \to 0.
\]
Next, let $c > 0$ be a constant to be chosen later. Consider the function $w_c = (u - c)^+ : \Omega \to \mathbb{R}$. Then $w_c \in \mathcal{H}_0^{\log}(\Omega)$ by Lemma 2.2 see also [18] Lemma 3.2. Moreover, for $x, y \in \mathbb{R}^N$ we have that $(u(x) - u(y))(w_c(x) - w_c(y)) \geq (w_c(x) - w_c(y))^2$. Indeed,
\[
\begin{align*}
(u(x) - u(y))(w_c(x) - w_c(y)) &= ([u(x) - c] - [u(y) - c])(w_c(x) - w_c(y)) \\
&= \frac{[(u(x) - c)w_c(x) + [u(x) - c]w_c(y)] - [u(x) - c]w_c(y) - w_c(x)[u(y) - c]}{2} \\
&= w_c^2(x) + w_c^2(y) - 2w_c(x)w_c(y) + [u(x) - c]^{-} w_c(y) + w_c(x)[u(y) - c]^{-} \\
&\geq w_c^2(x) + w_c^2(y) - 2w_c(x)w_c(y) = (w_c(x) - w_c(y))^2.
\end{align*}
\]
This implies that
\[
\mathcal{E}_\delta(w_c, w_c) = \frac{dN}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (w_c(x) - w_c(y))^2J_\delta(x - y) \, dx \, dy
\]
\[
\leq \frac{dN}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(w_c(x) - w_c(y))J_\delta(x - y) \, dx \, dy
\]
\[
= \mathcal{E}_\omega(u, w_c) - \kappa_\delta(u, w_c)_{L^2(\Omega)} + \langle K_\delta * u, w_c \rangle_{L^2(\Omega)}
\]
\[
\leq (\lambda - \kappa_\delta)\langle u, w_c \rangle_{L^2(\Omega)} + \|K_\delta * u\|_{L^\infty(\mathbb{R}^N)}(1, w_c)_{L^2(\Omega)}
\]
(3.8)
Note that $\kappa_\delta \to +\infty$ as $\delta \to 0$. Hence, we may fix $\delta > 0$ such that $\lambda + \kappa_\delta < -1$. Moreover, with this choice of $\delta$, together with the trivial inequality $u(x)w_c(x) \geq cw_c(x)$ for $x \in \Omega$, we infer that
\[
\mathcal{E}_\delta(w_c, w_c) \leq \int_{\Omega} \|K_\delta * u\|_{L^\infty(\mathbb{R}^N)} - c)w_c \, dx
\]
\[
\leq \int_{\Omega} (cN, \delta)\|u\|_{L^2(\mathbb{R}^N)} - c)w_c \, dx.
\]
(3.9)
The quantity $c_{N, \delta} \|u\|_{L^2(\mathbb{R}^N)}$ is obtained in the following computation using Hölder’s (or Young’s) inequality combined with the asymptotics in \(1.9\),

\[
\|k_\delta \ast u\|_{L^\infty(\mathbb{R}^N)} \leq c_{N, \delta} \|u\|_{L^2(\mathbb{R}^N)}.
\]

We then deduce from \(3.3\) with $c > c_{N, \delta} \|u\|_{L^2(\mathbb{R}^N)}$ that

\[
0 \leq \mathcal{E}_\delta^c(w_c, w_c) \leq 0,
\]

which implies that $\mathcal{E}_\delta^c(w_c, w_c) = 0$. Consequently, $w_c = 0$ in $\Omega$ by the Poincaré type inequality. But then $u(x) \leq c$ a.e. in $\Omega$, and therefore

\[
u(x) \leq c \max_{\mathbb{R}^N} u.
\]

Repeating the above argument for $-u$ in place of $u$, we conclude that

\[
\|u\|_{L^\infty(\Omega)} \leq c \|u\|_{L^2(\mathbb{R}^N)}.
\]

This completes the proof of Proposition 1.4.

For the proof of Theorem 1.5, we first state a Polya-Szegö type inequality for $(I - \Delta)^{\log}$.

**Lemma 3.1.** Let $u^*$ be the symmetric radial decreasing rearrangement of $u$. Then,

\[
\mathcal{E}_\omega(u^*, u^*) \leq \mathcal{E}_\omega(u, u).
\]

Moreover, the equality occurs for radial decreasing functions. Here,

**Proof.** By a changes of variable, we write the kernel $J$ as

\[
J(z) = d_N |z|^{-N} \omega(|z|) = 4(\frac{\pi}{2})^{\frac{N}{2}} \int_0^\infty e^{-t|z|^2} t^{\frac{N}{2} - 1} e^{-\frac{t}{2}} dt.
\]

Then by Fubuni’s theorem, we write the quadratic form as

\[
\mathcal{E}_\omega(u, u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 J(x, y) \, dx \, dy = 2(\frac{\pi}{2})^{\frac{N}{2}} \int_0^\infty G(t, u) \, t^{\frac{N}{2} - 1} e^{-\frac{t}{2}} dt,
\]

where,

\[
G(t, u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 e^{-t|x-y|^2} \, dx \, dy.
\]

Noticing that

\[
(e^{-t|z|^2})^* = e^{-t|z|^2}, \quad \text{for all} \quad t \geq 0,
\]

It follows from \(11\) corollary 2.3 and Theorem 9.2 see also \(13\) Theorem A1] that

\[
G(t, u^*) \leq G(t, u) \quad \text{for all} \quad t \geq 0.
\]

This gives that

\[
\mathcal{E}_\omega(u^*, u^*) \leq \mathcal{E}_\omega(u, u) \quad \text{for} \quad u \in H^{\log}(\mathbb{R}^N).
\]

The proof of Lemma 3.1 is completed.

**Proof of Theorem 1.5.** This is a direct consequence of lemma 3.1 and the characterization of the first eigenvalue $\lambda_{1, \log}(\Omega)$ of $(I - \Delta)^{\log}$ in $\Omega$. Since we know by Theorem 1.3 that the first eigenfunction $\varphi_{1, \log}$ corresponding to $\lambda_{1, \log}(\Omega)$ is unique and strictly positive in $\Omega$, we have thanks to Lemma 3.1 that

\[
\lambda_{1, \log}(\Omega) = \frac{\mathcal{E}_\omega(\varphi_{1, \log}, \varphi_{1, \log})}{\|\varphi_{1, \log}\|_{L^2(\Omega)}} \geq \frac{\mathcal{E}_\omega(\varphi^*_{1, \log}, \varphi^*_{1, \log})}{\|\varphi^*_{1, \log}\|_{L^2(B^*)}} = \inf_{u \in H_{\log}^0(B^*)} \frac{\mathcal{E}_\omega(u, u)}{\|u\|_{L^2(B^*)}^2} = \lambda_{1, \log}(B^*),
\]

where we have used (see \(6\) Lemma 3.3) the fact that

\[
\int_{\Omega} |u|^2 \, dx = \int_{B^*} |u^*|^2 \, dx.
\]
This gives the proof of (1.19). For the equality, if we suppose that \( \lambda_{1, \log}(\Omega) = \lambda_{1, \log}(B^*) \) with \( |\Omega| = |B^*| \), then we must have the following equality

\[
\mathcal{E}_L(\varphi_{1, \log}, \varphi_{1, \log}) = \mathcal{E}_L(\varphi^*_{1, \log}, \varphi^*_{1, \log})
\]

and by [13, Lemma A2] we deduce that the first eigenfunction \( \varphi_{1, \log} \) has to be proportional to a translate of a radially symmetric decreasing function such that the level set

\[
\Omega_0 := \{ x \in \mathbb{R}^N : \varphi_{1, \log} > 0 \}
\]

is a ball. Since \( \varphi_{1, \log} > 0 \) in \( \Omega \) by definition and it is unique, it follows that \( \Omega \) must coincide with \( \Omega_0 \) and has to be a ball. The proof of Theorem 1.5 is then completed. \( \square \)

4. Small order Asymptotics

This section is dedicated to the proof of Theorem 1.6. We first introduce some notions and preliminary lemmas that shall be used. For \( 0 < s < 1 \), we introduce the Sobolev space (see [31, 34])

\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \omega_s(|x - y|) dx dy < \infty \right\}
\]

with corresponding norm given by

\[
\|u\|_{H^s(\mathbb{R}^N)} = \left( \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \omega_s(|x - y|) dx dy \right)^{\frac{1}{2}}
\]

\[
= \left( \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set. We will use the fact that (see [31])

the space \( C^2_c(\Omega) \) is dense in \( H^s_0(\Omega) \),

where the space \( H^s_0(\Omega) \) is the completion of \( C^\infty_c(\Omega) \) with respect to the norm \( \| \cdot \|_{H^s(\mathbb{R}^N)} \). We start with the following Dirichlet eigenvalue problem

\[
\begin{cases}
(I - \Delta)^s u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(4.1)

where \( \Omega \) is a bounded Lipschitz open set of \( \mathbb{R}^N \). We define the first Dirichlet eigenvalue of \((I - \Delta)^s\) in \( \Omega \) by

\[
\lambda_{1,s}(\Omega) = \inf_{u \in C^2_c(\Omega)} \sup_{\|u\|_{L^2(\Omega)}=1} \frac{\mathcal{E}_{\omega,s}(u, u)}{|\omega|^2 L^2(\Omega)} = \inf_{u \in C^2_c(\Omega)} \mathcal{E}_{\omega,s}(u, u),
\]

(4.2)

where the quadratic form \((u, v) \mapsto \mathcal{E}_{\omega,s}(u, v)\) is defined by

\[
\mathcal{E}_{\omega,s}(u, v) = \int_{\Omega} u(x)v(x) dx - \frac{d_{\mathbb{R}^s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \omega_s(|x - y|) dx dy
\]

\[
= \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)| |\mathcal{F}(v)(\xi)| d\xi.
\]

By the Courant-Fischer minimax principle, the eigenvalues \( \lambda_{k,s}(\Omega) \), \( k \in \mathbb{N} \) can be characterized equivalently as

\[
\lambda_{k,s}(\Omega) = \inf_{V \subset C^2_c(\Omega) \setminus \{0\} \atop \dim V = k} \max_{\|v\|_{L^2(\Omega)}=1} \mathcal{E}_{\omega,s}(v, v) = \inf_{V \subset C^2_c(\Omega) \setminus \{0\} \atop \dim V = k} \max_{\|v\|_{L^2(\Omega)}=1} \mathcal{E}_{\omega,s}(v, v).
\]

(4.3)
Lemma 4.2. Let $\lambda_{k,s}(\Omega) = E_{\omega,s}(\psi_{k,s}, \psi_{k,s}) \geq E_s(\psi_{k,s}, \psi_{k,s}) \geq \inf_{v \in C^2_c(\Omega), \|v\|_{L^2(\Omega)} = 1} E_s(v, v) = \lambda^F_{1,s}(\Omega),$

where $\psi_{k,s}$ is a $L^2$-normalized eigenfunction of $(I - \Delta)^s$ corresponding to $\lambda_{k,s}(\Omega)$ and $\lambda^F_{1,s}(\Omega)$ is the first Dirichlet eigenvalue of the fractional Laplacian $(-\Delta)^s$ in $\Omega$ with

$$E_s(u, v) := \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \ dx \ dy.$$ 

We need the following elementary estimates and inequalities.

**Lemma 4.2.** For $s \in (0, 1)$ and $r > 0$ we have

$$\left| \frac{(1 + r^2)^s - 1}{s} \right| \leq 2 \left(1 + r^4 \right)$$

and

$$\left| \frac{(1 + r^2)^s - 1}{s} - \log(1 + r^2) \right| \leq 2s \left(1 + r^4 \right).$$

Consequently, for every $u \in C^2_c(\Omega)$ and $s \in (0, 1)$, we have

$$|E_s(u, u) - \|u\|_{L^2(\Omega)}^2| \leq 2s \left(\|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \right)$$

and

$$|E_s(u, u) - \|u\|_{L^2(\Omega)}^2 - sE_s(u, u)| \leq 2s^2 \left(\|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \right).$$

**Proof.** For fix $r > 0$, let $h_r(s) = (1 + r^2)^s$. Then we have

$$h'_r(r) = (1 + r^2)^r \ln(1 + r^2) \quad \text{and} \quad h''_r(r) = (1 + r^2)^r \ln^2(1 + r^2).$$

Consequently, since $(1 + r^2)^s \leq (1 + r^2)$ for $s \in (0, 1)$ and $\ln(1 + r^2) \leq (1 + r^2)$,

$$\left| \frac{(1 + r^2)^s - 1}{s} - \log(1 + r^2) \right| \leq \frac{\ln(1 + r^2)}{s} \int_0^s (1 + r^2)^r \ d\tau \leq \ln(1 + r^2)(1 + r^2)^s \leq 2 \left(1 + r^4 \right)$$

where in the last step we used that $(1 + r^2)^2 \leq 2(1 + r^4)$ for $r > 0$. Hence (4.4) holds. Moreover, by Taylor expansion,

$$h_r(s) = 1 + s \ln(1 + r^2) + \ln^2(1 + r^2) \int_0^s (1 + r^2)^r (s - \tau) \ d\tau$$

and therefore

$$\left| \frac{(1 + r^2)^s - 1}{s} - \log(1 + r^2) \right| \leq \frac{\ln^2(1 + r^2)}{s} \int_0^s (1 + r^2)^r (s - \tau) \ d\tau \leq s(1 + r^2)^s \ln^2(1 + r^2).$$

But since $\ln^2(1 + r^2) \leq (1 + r^2)$ and $(1 + r^2)^s \leq (1 + r^2)$ for $s \in (0, 1)$, (4.5) follows. Next, let $u \in C^2_c(\Omega)$ and $s \in (0, 1)$. By (4.4) and Fourier transform for $E_s$, we have

$$|E_s(u, u) - \|u\|_{L^2(\Omega)}^2| \leq \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s - 1 \ |\hat{u}(\xi)|^2 \ d\xi$$

$$\leq 2s \int_{\mathbb{R}^N} \left(1 + |\xi|^4\right) |\hat{u}(\xi)|^2 \ d\xi \leq 2s \left(\|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \right).$$

Thus (4.6) follows. Moreover, by (4.5) we have

$$|E_s(u, u) - \|u\|_{L^2(\Omega)}^2 - sE_s(u, u)| \leq \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s - 1 - s \ln(1 + |\xi|^2) \ |\hat{u}(\xi)|^2 \ d\xi$$

$$= s \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s - 1 - \ln(1 + |\xi|^2) \ |\hat{u}(\xi)|^2 \ d\xi.$$
\[
|\mathbf{R}_s(u)| \leq 2s^2 \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} \right) \to 0 \quad \text{as } s \to 0^+ \text{ uniformly in } u \in S_V.
\]

Consequently,
\[
\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \leq \max_{u \in S_V} \mathcal{E}_\omega(u,u) - 1 \leq C.
\]

Hence (4.8) holds. Moreover, setting \( R_s(u) = \frac{\mathcal{E}_{\omega,s}(u,u) - 1}{s} - \mathcal{E}_\omega(u,u) \) for \( u \in C^2_c(\Omega) \), we deduce from (4.11) that
\[
\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \leq \max_{u \in S_V} \mathcal{E}_\omega(u,u) + \max_{u \in S_V} |R_s(u)|
\]

while, by (4.7),
\[
|R_s(u)| \leq 2s^2 \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} \right) \to 0 \quad \text{as } s \to 0^+ \text{ uniformly in } u \in S_V.
\]

Consequently,
\[
\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \leq \max_{u \in S_V} \mathcal{E}_\omega(u,u).
\]

Since \( V \) was chosen arbitrarily, the characterization of the Dirichlet eigenvalues of \((I - \Delta)^{\log}\) given in (4.3) implies that
\[
\limsup_{s \to 0^+} \frac{\lambda_{k,s}(\Omega) - 1}{s} \leq \inf_{V \subseteq C^2_c(\Omega)} \max_{u \in V} \mathcal{E}_\omega(u,u) = \lambda_{k, \log}(\Omega).
\]

This shows that the inequality in (4.9) holds. It follows directly from (4.8) that
\[
\limsup_{s \to 0^+} \lambda_{k,s}(\Omega) \leq 1 \quad \text{for all } k \in \mathbb{N}.
\]

From Remark 4.1 we have that \( \lambda_{k,s}(\Omega) \geq \lambda^+_{k,s}(\Omega) \). It therefore follows from [12, Lemma 2.8] that
\[
\liminf_{s \to 0^+} \lambda_{k,s}(\Omega) \geq 1 \quad \text{for all } k \in \mathbb{N}.
\]

This proves (4.10) and the proof of Lemma 4.3 is completed. \( \square \)

Lemma 4.4. Let \( k \in \mathbb{N} \). If \( \psi_{k,s} \in H^s_0(\Omega) \) denote an \( L^2 \)-normalized eigenfunction of \((I - \Delta)^s\), then the set
\[
\{ \psi_{k,s} : s \in (0,1) \}
\]

is uniformly bounded in \( H^s_0(\Omega) \) and therefore relatively compact in \( L^2(\Omega) \).
Proof. To ease notation, we set $\psi_s \equiv \psi_{k,s}$, the $k$-th $L^2$-normalized eigenfunction corresponding to $\lambda_{k,s}(\Omega)$, $k \in \mathbb{N}$. By (4.10), there exists a constant $C = C(N, \Omega, k) > 0$ such that

$$C \geq \frac{\lambda_{k,s}(\Omega) - 1}{s} = \frac{\mathcal{E}_{\omega,s}(\psi_s, \psi_s) - 1}{s} = \int_{\mathbb{R}^N} \frac{1}{s}(1 + |\xi|^2)^{s - 1} |\psi_s(\xi)|^2 \, d\xi = \int_0^1 \int_{\mathbb{R}^N} \log(1 + |\xi|^2) |\psi_s(\xi)|^2 (1 + |\xi|^2)^{s} \, d\xi \, dt \\
\geq \frac{1}{2} \int_0^1 \int_{\mathbb{R}^N} \log(1 + |\xi|^2) |\psi_s(\xi)|^2 \, d\xi \, dt = \frac{1}{2} \mathcal{E}_{\omega}(\psi_s, \psi_s).$$

Therefore, there exist a constant $M := M(\Omega, k, N) > 0$ such that

$$\sup_{s \in (0, 1)} \|\psi_s\|_{L^2(\Omega)} \leq M \tag{4.13}$$

We conclude from (4.13) that $\psi_s$ remains uniformly bounded in $\mathcal{H}^{log}_{0}(\Omega)$ for $s \in (0, 1)$. Consequently $\{\psi_{s,k} : s \in (0, 1)\}$ is uniformly bounded in $\mathcal{H}^{log}_{0}(\Omega)$ and relatively compact in $L^2(\Omega)$ since we have from (2.9) that $\mathcal{H}^{log}_{0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. \hfill \square

We now give the

Proof of Theorem 1.6. The proof follows the idea in article [12, Theorem 2.1] by the author combined with [8, Theorem 3.5]. It then suffices, in view of Lemma 4.3, to consider an arbitrary sequence $(s_n)_n \subset (0, 1)$ with $\lim_{n \to \infty} s_n = 0$, and to show that, after passing to a subsequence,

$$\lim_{n \to \infty} \lambda_{k,s_n}(\Omega) - 1 = \lambda_{k,log}(\Omega) \quad \text{for } k \in \mathbb{N}. \tag{4.14}$$

Let $\{\psi_{k,s_n} : k \in \mathbb{N}\}$ be an orthonormal system of eigenfunctions corresponding to the Dirichlet eigenvalue $\lambda_{k,s_n}(\Omega)$ of $(I - \Delta)^{s_n}$. By Lemma 4.4 it follows that, for every $k \in \mathbb{N}$, the sequence of functions $\psi_{k,s_n}, n \in \mathbb{N}$ is bounded in $\mathcal{H}^{log}_{0}(\Omega)$ and relatively compact in $L^2(\Omega)$. Consequently, we may pass to a subsequence such that, for every $k \in \mathbb{N}$,

$$\psi_{k,s_n} \to \psi_{k,log}^* \text{ weakly in } \mathcal{H}^{log}_{0}(\Omega) \text{ and } \psi_{k,s_n} \to \psi_{k,log}^* \text{ strongly in } L^2(\Omega) \quad \text{as } n \to \infty. \tag{4.15}$$

Moreover, by Lemma 4.3 we may, after passing again to a subsequence if necessary, assume that, for every $k \in \mathbb{N}$,

$$\lambda_{k,s_n}(\Omega) - 1 \to \lambda_{k,log}^* \in \left[ - \infty, \lambda_{k,log}(\Omega) \right] \quad \text{as } n \to \infty. \tag{4.16}$$

To prove then (4.14), it now suffices to show that

$$\lambda_{k,log}(\Omega) = \lambda_{k,log}^* \quad \text{for } k \in \mathbb{N}. \tag{4.17}$$

It follows from (4.15) that

$$\|\psi_{k,log}^*\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \langle \psi_{k,log}^*, \psi_{\ell,log}^* \rangle_{L^2(\Omega)} = 0 \quad \text{for } k, \ell \in \mathbb{N}, \, \ell \neq k. \tag{4.18}$$

Moreover, for $v \in C^2_c(\Omega)$ and $n \in \mathbb{N}$, we have from Theorem 1.3 that

$$\mathcal{E}_{\omega,s_n}(\psi_{k,s_n}, v) = \lambda_{k,s_n}(\Omega) \langle \psi_{k,s_n}, v \rangle_{L^2(\Omega)} \tag{4.19}$$

and therefore, rearranging (4.19), it follows from (i) in Theorem 1.1 with $p = 2$ that

$$\lim_{n \to \infty} \frac{\lambda_{k,s_n}(\Omega) - 1}{s_n} \langle \psi_{k,s_n}, v \rangle_{L^2(\Omega)} = \lim_{n \to \infty} \frac{1}{s_n} \mathcal{E}_{\omega,s_n}(\psi_{k,s_n}, v) - \langle \psi_{k,s_n}, v \rangle_{L^2(\Omega)} = \lim_{n \to \infty} \langle \psi_{k,s_n}, (I - \Delta)^{s_n} v - v \rangle_{L^2(\Omega)} \tag{4.20}$$

$$= \langle \psi_{k,log}^*, (I - \Delta)^{log} v \rangle_{L^2(\Omega)} = \mathcal{E}_{\omega}(\psi_{k,log}^*, v).$$
Since moreover $\langle \psi_{k,sn}, v \rangle_{L^2(\Omega)} \to \langle \psi_{k,\log}^*, v \rangle_{L^2(\Omega)}$ as $n \to \infty$ for any $k \in \mathbb{N}$ and $v \in C_0^2(\Omega)$, in particular, for $k = 1$, we may choose $v \in C_0^2(\Omega)$ such that $\langle \psi_{1,\log}^*, v \rangle_{L^2(\Omega)} > 0$. It follows from (4.16) and (4.20) that $\lambda_1^*$ satisfies $-\infty < \lambda_1^* \leq \lambda_{1,\log}(\Omega)$ and

$$E_\omega(\psi_{1,\log}^*, v) = \lambda_1^*(\psi_{1,\log}^*, v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^{\log}(\Omega).$$

(4.21)

Thus $\psi_{1,\log}^*$ is an eigenfunction of $(I - \Delta)^{\log}$ corresponding to the eigenvalue $\lambda_1^*$. Since $\lambda_1^* \leq \lambda_{1,\log}(\Omega)$, it follows from the definition of the principal eigenvalue (1.17) that $\lambda_1^* = \lambda_{1,\log}(\Omega)$ and then $\lambda_{1,\log}(\Omega) = \lambda_1^* = \lim_{s \to 0^+} \frac{\lambda_{1,s}(\Omega) - 1}{s}$. From the uniqueness of the first eigenfunction, we get that $\psi_{1,\log}^* = \psi_{1,\log}$ is the nonnegative $L^2$-normalized eigenfunction of $(I - \Delta)^{\log}$ corresponding to $\lambda_{1,\log}(\Omega)$. In short, we have just shown that as $s \to 0^+$,

$$\lambda_{1,s}(\Omega) \to \lambda_{1,\log}(\Omega) \quad \text{and} \quad \psi_{1,s} \to \psi_{1,\log} \quad \text{in } L^2(\Omega).$$

This completes the proof for $k = 1$. Now for $k \geq 2$, it still follows from (4.16) and (4.20) that

$$E_\omega(\psi_{k,\log}^*, v) = \lambda_k^*(\psi_{k,\log}^*, v)_{L^2(\Omega)} \quad \text{for all } v \in C_0^2(\Omega),$$

(4.22)

where $\psi_{k,\log}^*$ is a Dirichlet eigenfunction of $(I - \Delta)^{\log}$ corresponding to $\lambda_k^*$, now with

$$\lambda_k^* \in [\lambda_{1,\log}(\Omega), \lambda_{k,\log}(\Omega)].$$

Next, for fixed $k \in \mathbb{N}$ we consider $E_k^\log := \text{span}\{\psi_{1,\log}^*, \psi_{2,\log}^*, \ldots, \psi_{k,\log}^*\}$, which is a $k$-dimensional subspace of $H_0^{\log}(\Omega)$ by (4.18). Since

$$\lambda_1^* \leq \lambda_2^* \leq \ldots \leq \lambda_k^*$$

as a consequence of (4.20) and since $\lambda_{i,s,n}(\Omega) \leq \lambda_{j,s,n}(\Omega)$ for $1 \leq i \leq j \leq k$, $n \in \mathbb{N}$, we have the following estimate for every $v = \sum_{i=1}^k \alpha_i \psi_{i,\log}^* \in E_k^\log$ with $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$:

$$E_\omega(v, v) = \sum_{i,j=1}^k \alpha_i \alpha_j E_\omega(\psi_{i,\log}^*, \psi_{j,\log}^*) = \sum_{i,j=1}^k \alpha_i \alpha_j \lambda_i^* \langle \psi_{i,\log}^*, \psi_{j,\log}^* \rangle_{L^2(\Omega)}$$

$$= \sum_{i=1}^k \alpha_i^2 \lambda_i^* \|\psi_{i,\log}^*\|^2_{L^2(\Omega)} \leq \lambda_k^* \sum_{i=1}^k \alpha_i^2 = \lambda_k^* \|v\|^2_{L^2(\Omega)}.$$

(4.24)

(4.25)

The characterization in (4.33) now yields that

$$\lambda_{k,\log}(\Omega) \leq \max_{v \in E_k^\log, \|v\|_{L^2(\Omega)} = 1} E_\omega(v, v) \leq \lambda_k^*.$$

Since also $\lambda_k^* \leq \lambda_{k,\log}(\Omega)$ by (4.16), the equality in (4.17) follows. We thus conclude that (4.14) holds and also (4.20) follows. Moreover, the statement (4.22) of the theorem follows a posteriori from the equality $\lambda_k^* = \lambda_{k,\log}(\Omega)$, since we have already seen that $\psi_{k,s,n} \to \psi_{k,\log}^*$ in $L^2(\Omega)$, the proof is thus finished here.

5. Decay Estimates

This section deals with the proof of Proposition 1.2 concerning the decay estimates at infinity and at zero of the solution $u$ corresponding to Poisson problem,

$$(I - \Delta)^{\log} u = f \quad \text{in } \mathbb{R}^N.$$  

(5.1)

The fundamental solution of equation (5.1) can be given in term of the Green function $G : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ (see (1.10)) defined by

$$G(x) = \int_0^\infty \frac{1}{1^t} \int_0^\infty p_s(x)s^{-1}e^{-s} \, dsdt,$$
We have in the sense of distributional that $\mathcal{F}(G)(\xi) = \frac{1}{\log(1+|\xi|^2)}$, $\xi \in \mathbb{R}^N \setminus \{0\}$. Indeed, for $\varphi \in \mathcal{S}$, we have by Fubini’s theorem that

$$
\int_{\mathbb{R}^N} G(\xi) \mathcal{F}(\varphi)(\xi) \, d\xi = \int_0^\infty \frac{1}{\Gamma(t)} \int_{\mathbb{R}^N} p_s(\xi) \mathcal{F}(\varphi)(\xi) \, d\xi s^{t-1} e^{-s} \, ds dt
$$

$$
= \int_{\mathbb{R}^N} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-s(1+|\xi|^2)} s^{t-1} \, ds dt \varphi(\xi) \, d\xi
$$

$$
= \int_{\mathbb{R}^N} \int_0^\infty (1+|\xi|^2)^{-t} \, dt \varphi(\xi) \, d\xi = \int_{\mathbb{R}^N} \frac{1}{\log(1+|\xi|^2)} \varphi(\xi) \, d\xi,
$$

and then

$$
\mathcal{F}^{-1}\left(\frac{1}{\log(1+|\xi|^2)}\right)(x) = G(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.
$$

We then define the solution $u$ of equation (5.1) for a $f \in \mathcal{C}^\infty_c(\mathbb{R}^N)$ by

$$
u(x) = [G * f](x) = \int_{\mathbb{R}^N} G(x-y)f(y) \, dy \quad \text{for } x \in \mathbb{R}^N.
$$

(5.2)

This follows from the property of Fourier transform and convolution since

$$
\mathcal{F}(u) = \mathcal{F}(G)\mathcal{F}(f) \quad \text{and} \quad \log(1+|\xi|^2)\mathcal{F}(u) = \log(1+|\xi|^2)\mathcal{F}(G)\mathcal{F}(f) = \mathcal{F}(f).
$$

We now give the

**Proof of Proposition 1.2.** For $|x|$ small, We split the integral representation of $G$ in two pieces as follows

$$
G_1(x) = \frac{2^{1-N}}{\pi N/2} \int_0^\infty \frac{1}{\Gamma(2t)} \left(\frac{|x|}{2}\right)^{t-2} K_{t-\frac{N}{2}}(|x|) \, dt
$$

and

$$
G_2(x) = \frac{2^{1-N}}{\pi N/2} \int_0^\infty \frac{1}{\Gamma(2t)} \left(\frac{|x|}{2}\right)^{t-2} K_{t-\frac{N}{2}}(|x|) \, dt.
$$

Since $t \leq \frac{N}{2}$, it follows from the asymptotics property (2.1) for $K_\nu$ (see (13)) that as $|x| \to 0$,

$$
K_{t-\frac{N}{2}}(|x|) \sim 2^{t-\frac{N}{2}-1}\Gamma\left(t - \frac{N}{2}\right)|x|^{-t+\frac{N}{2}} \begin{cases} 2^{-t}\log \left(\frac{1}{|x|}\right) & \text{if } t < \frac{N}{2}, \\ 2^{-t} & \text{if } t = \frac{N}{2}, \\ \Gamma\left(t - \frac{N}{2}\right) & \text{if } t > \frac{N}{2}, \end{cases}
$$

Plugging the above approximations in $G_1$, we end up with

$$
G_1(x) \sim \begin{cases} 2^{1-N} \log \frac{1}{|x|} & \text{as } |x| \to 0 \quad \text{if } t = \frac{N}{2}, \\ 2^{N} |x|^{-N} \int_0^\infty \frac{\Gamma(N/2-t)}{4\Gamma(t)} dt & \text{as } |x| \to 0 \quad \text{if } t < \frac{N}{2}, \\ \frac{2^N}{\pi N/2} \int_0^\infty \frac{\Gamma(N/2-t)}{4\Gamma(t)} dt & \text{as } |x| \to 0 \quad \text{if } t > \frac{N}{2}, \end{cases}
$$

(5.3)

where we have used that since $N > 2t$, $|x|^{-N+2t} \sim |x|^{-N}$ as $|x| \to 0$. Since also $t < \frac{N}{2}$, we have

$$
\int_0^{\frac{N}{2}} \frac{\Gamma(N/2-t)}{4\Gamma(t)} dt < \infty.
$$

Now, for $t > \frac{N}{2}$, again by using (2.1), we have

$$
K_{t-\frac{N}{2}}(|x|) \sim 2^{t-\frac{N}{2}-1}\Gamma\left(t - \frac{N}{2}\right)|x|^{-t+\frac{N}{2}} \quad \text{as } |x| \to 0.
$$

Taken the above approximations into account, we get the approximation for $G_2$,

$$
G_2(x) \sim \frac{2^{-N}}{\pi N/2} \int_0^\infty \frac{\Gamma(t-N/2)}{4\Gamma(t)} dt \quad \text{as } |x| \to 0
$$

(5.4)
First observe that if \( f(t) \) is smooth and \( \frac{t - N}{2} \) is sufficiently large, we have that \( \int_{t}^{\infty} \frac{\Gamma(t - \frac{N}{2})}{4t(t-\frac{2}{3})} dt < \infty \). Therefore, combining the approximations of \( G_1 \) in (5.3) and \( G_2 \) in (5.4) we get

\[
|x|^N G(x) \sim \frac{2^N}{\pi^{N/2}} \int_{0}^{\frac{N+1}{2}} \frac{\Gamma(N/2 - t)}{4\Gamma(t)} dt \quad as \quad |x| \to 0.
\]

We next investigate the case with the modulus of \( |x| \) large. From the asymptotics property (2.1) we have for all \( t \geq 0 \) that

\[
|x|^{-\frac{N}{2}} K_{\frac{N}{2}}(|x|) \sim \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |x|^{-\frac{N+1}{2} + t} e^{-|x|} \quad as \quad |x| \to \infty
\]

\[
\sim \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} |x|^{-\frac{N+1}{2}} e^{-|x|} \quad as \quad |x| \to \infty.
\]

From this, we infer that

\[
G(x) \sim 2^{-\frac{N+1}{2}} \frac{\Gamma(N/2 - t)}{\pi^{N/2}} |x|^{-\frac{N+1}{2} + t} e^{-|x|} \int_{0}^{\infty} \frac{1}{2\Gamma(t)} dt \quad as \quad |x| \to \infty.
\]

Noticing that \( \lim_{t \to +\infty} \frac{1}{2\Gamma(t)} = 0 \) and \( \lim_{t \to +\infty} \frac{1}{2\Gamma(t)} = 1 \), the above integral is finite and

\[
\int_{0}^{\infty} \frac{1}{2\Gamma(t)} dt \sim 1.
\]

We therefore infer that

\[
G(x) \sim 2^{-\frac{N+1}{2}} \frac{\Gamma(N/2 - t)}{\pi^{N/2}} |x|^{-\frac{N+1}{2} + t} e^{-|x|} \quad as \quad |x| \to \infty.
\]

For \( f \in L^1(\mathbb{R}^N) \), we write

\[
u(x) = \int_{\mathbb{R}^N} G(x-y)f(y) dy = \int_{\mathbb{R}^N} G(y)f(x-y) dy
\]

First observe that if \( f \geq 0 \), we have that

\[
u(x) \geq \int_{B(x,|x|)} G(x-y)f(y) dy \geq C e^{-|x|} \int_{B(x,|x|)} f(y) dy.
\]

Since \( B(x,|x|) \to \mathbb{R}^N \) as \( |x| \to \infty \) and \( f \in L^1(\mathbb{R}^N) \), we see that \( u(x) = O(e^{-|x|}) \) as \( |x| \to \infty \).

Moreover, Since \( G(x) \) decays as \( e^{-|x|} \) at infinity, there exists a constant \( M > 0 \) such that

\[
C > 0 \quad is \quad a \quad positive \quad constant.
\]

We then write

\[
e^{x|x|} u(x) = e^{\frac{|x|}{2}} f(x) = \int_{\mathbb{R}^N} e^{y|x|} G(y)f(x-y) dy.
\]

Thus,

\[
|e^{x|x|} u(x)| \leq \int_{\mathbb{R}^N} e^{y|x|} G(y)f(x-y) dy \leq \| e^{\frac{|x|}{2}} f \|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} f(x-y) dy \leq C \| f \|_{L^1(\mathbb{R}^N)}.
\]

This allows to conclude that \( u(x) \) decays as \( e^{-|x|} \) at infinity, that is

\[
u(x) = O(e^{-|x|}) \quad as \quad |x| \to \infty.
\]

As before, there exists \( \delta > 0 \) such that

\[
\| x^N u(x) \|_{L^\infty(\mathbb{R}^N)} \leq C \quad for \quad |x| < \delta.
\]

Therefore,

\[
\| x^N u(x) \| \leq C \int_{\mathbb{R}^N} f(x-y) dy \leq C \| f \|_{L^1(\mathbb{R}^N)}.
\]
This allows to conclude that
\[ u(x) = O(|x|^{-N}) \quad \text{as} \quad |x| \to 0. \]
This completes the proof of Theorem 1.2. \( \square \)

6. Additional remarks

We present in this section some results concerning the logarithmic Schrödinger operator \((I - \Delta)^{\log}\) that can be directly deduced from known results in the literature. For this fact, we introduce the following space \(V_\omega(\Omega)\), being the space of all functions \(u \in L^2_{loc}(\mathbb{R}^N)\) such that
\[
\rho(u, \Omega) := \int_\Omega \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^N} \omega(|x - y|) \, dx \, dy < \infty.
\]
Then the quantity \(E_\omega(u, v)\) is well defined for \(u \in \mathcal{H}^{\log}_0(\Omega)\) and \(v \in V_\omega(\Omega)\) (see [18, Lemma 3.1]). The proof of the following results on the maximum principle for the operator \((I - \Delta)^{\log}\) on an open set \(\Omega\) of \(\mathbb{R}^N\) can be deduced from [18].

**Theorem 6.1.**

(i) (Strong maximum principle) Let \(\Omega \subset \mathbb{R}^N\) be a bounded subset and \(u \in L^0(\mathbb{R}^N)\) be a continuous function on \(\overline{\Omega}\) satisfying
\[
(I - \Delta)^{\log} u \geq 0 \quad \text{in} \quad \Omega, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.
\]
Then \(u > 0\) in \(\Omega\) or \(u \equiv 0\) a.e. in \(\mathbb{R}^N\).

(ii) (Weak maximum principle) Let \(u \in V_\omega(\Omega)\) with \((I - \Delta)^{\log} u \geq 0\) in \(\Omega\) weakly and \(u \geq 0\) in \(\mathbb{R}^N \setminus \Omega\). Then \(u \geq 0\) in \(\mathbb{R}^N\).

(iii) (Small volume maximum principle) There exists \(\delta > 0\) such that for every open bounded set \(\Omega\) of \(\mathbb{R}^N\) with \(|\Omega| \leq \delta\) and every function \(u \in V_\omega(\Omega)\) satisfying
\[
(I - \Delta)^{\log} u \geq c(x)u \quad \text{in} \quad \Omega \quad \text{and} \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,
\]
with \(c \in L^\infty(\mathbb{R}^N)\), then \(u \geq 0\) in \(\mathbb{R}^N\).

We recall that, \(u \in V_\omega(\Omega)\) satisfies \((I - \Delta)^{\log} u \geq 0\) in \(\Omega\) weakly means,
\[
E_\omega(u, \varphi) \geq 0 \quad \text{for all nonnegative} \quad \varphi \in C^\infty_c(\Omega).
\]

Next, consider the following semilinear elliptic problem involving the operator \((I - \Delta)^{\log}\) in a bounded set \(\Omega\) of \(\mathbb{R}^N\),
\[
(I - \Delta)^{\log} u = f(x, u) \quad \text{in} \quad \Omega \quad u = 0 \quad \text{on} \quad \mathbb{R}^N \setminus \Omega, \tag{6.1}
\]
where \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is continuous. The following result on the radially symmetry of the solution can deduced from [19].

**Theorem 6.2.** Assume that \(f\) is locally Lipschitz with respect to the second variable and radially symmetric and strictly decreasing in \(r = |x|\). Then every positive solution of \((6.1)\) is radially symmetric and strictly decreasing in \(|x|\).

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