GAUSSIAN BOUNDS FOR THE INHOMOGENEOUS LANDAU EQUATION WITH HARD POTENTIALS

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Abstract. We consider weak solutions of the spatially inhomogeneous Landau equation with hard potentials ($\gamma \in (0, 1]$), under the assumption that mass, energy, and entropy densities are under control. In this regime, with arbitrary initial data, we show that solutions satisfy pointwise Gaussian upper and lower bounds in the velocity variable. This is different from the behavior in the soft potentials case ($\gamma < 0$), where Gaussian estimates are known not to hold without corresponding assumptions on the initial data. Our upper bounds imply weak solutions are $C^\infty$ in all three variables, and that continuation of solutions is governed only by the mass, energy, and entropy.

1. Introduction

The Landau equation is an integro-differential kinetic model arising in plasma physics. For $(t, x, v) \in [0, T] \times \mathbb{R}^d$, the solution $f(t, x, v) \geq 0$ satisfies

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left[\bar{a}^f(t, x, v)v_{\bar{a}^f}f + \bar{b}^f(t, x, v)\cdot \nabla_v f + \bar{c}^f(t, x, v)f\right],$$

with the nonlocal coefficients $\bar{a}^f(t, x, v) \in \mathbb{R}^{d \times d}$, $\bar{b}^f(t, x, v) \in \mathbb{R}^d$, and $\bar{c}^f(t, x, v) \in \mathbb{R}$ defined by

$$\bar{a}^f(t, x, v) := a_{d, \gamma} \int_{\mathbb{R}^d} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|}\right) |w|^\gamma f(t, x, v - w) \, dw,$$

$$\bar{b}^f(t, x, v) := b_{d, \gamma} \int_{\mathbb{R}^d} |w|^\gamma wf(t, x, v - w) \, dw,$$

$$\bar{c}^f(t, x, v) := c_{d, \gamma} \int_{\mathbb{R}^d} |w|^\gamma f(t, x, v - w) \, dw,$$

and $a_{d, \gamma}$, $b_{d, \gamma}$, $c_{d, \gamma}$ are positive constants. For $\gamma > -d$, the Landau equation arises as a limit of the Boltzmann equation as grazing collisions predominate. Although the case $d = 3$, $\gamma = -3$ (where (1.4) must be replaced by $\bar{c}^f = cf$) is the most directly relevant as a physical model (see e.g. [20]), the cases $\gamma \in (-d, 1]$ have also attracted a great deal of interest from a mathematical standpoint, partly motivated by the desire to better understand grazing collisions, which play a delicate role in the study of the non-cutoff Boltzmann equation. In this article, we are interested in the case of hard potentials, i.e. $\gamma \in (0, 1]$. (In fact, our results hold for any $\gamma \in (0, 2)$, but $\gamma \in (0, 1]$ is the case of interest in the literature.)

Functions of the form $e^{-c|v - v_0|^2}$ for $v_0 \in \mathbb{R}^d$ and $c, \alpha > 0$ (referred to as Maxwellsians) are equilibrium solutions of (1.1). The goal of this article is to prove, under relatively weak a priori assumptions, that solutions of (1.1) are bounded above and below by Maxwellsians, and that these

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estimates depend only on physically meaningful quantities. Define

\[ M_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \, dv, \quad \text{(mass density)} \]

\[ E_f(t, x) = \int_{\mathbb{R}^d} |v|^2 f(t, x, v) \, dv, \quad \text{(energy density)} \]

\[ H_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \log f(t, x, v) \, dv. \quad \text{(entropy density)} \]

These hydrodynamic quantities corresponding to \( f \) are physically observable at the macroscopic scale. We will assume throughout that for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \),

\[
0 < m_0 \leq M_f(t, x) \leq M_0, \quad E_f(t, x) \leq E_0, \quad \text{and} \quad H_f(t, x) \leq H_0, \tag{1.5}
\]

for some constants \( m_0, M_0, E_0, H_0 \). In the spatially homogeneous case (when \( f \) is independent of \( x \)), the mass, energy, and entropy are functions of \( t \) only, and it is known that mass and energy are conserved, and entropy is nonincreasing. These properties follow from the integral identities

\[
\int_{\mathbb{R}^d} Q(f, f) \, dv = \int_{\mathbb{R}^d} |v|^2 Q(f, f) \, dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \log f Q(f, f) \, dv \leq 0. \quad \text{(See, e.g., [27] Chapter 1, Section 2)}. \]

Therefore, assumption (1.5) would be unnecessary in the homogeneous case, as long the initial data has finite mass, energy, and entropy. But in the spatially inhomogeneous case considered here, the above integral identities only imply that the \( L^2 \) norms of \( M_f \) and \( E_f \) are conserved, and that the \( L^1 \) norm of \( H_f \) is nonincreasing. It is not clear whether these hydrodynamic quantities can blow up in \( L^\infty \) in finite time. Therefore, we include (1.5) as an assumption. We say a constant is universal if it depends only on \( d, \gamma, m_0, M_0, E_0, \) and \( H_0 \).

We will work with solutions satisfying

\[ G_f(t, x) := \int_{\mathbb{R}^d} |v|^\gamma+2 f(t, x, v) \, dv \leq G_0, \tag{1.6} \]

for some \( G_0 > 0 \). This ensures that the matrix \( \bar{a}^j \) in (1.2) is well-defined, but we will seek estimates that do not depend quantitatively on \( G_0 \). Assumption (1.6) would clearly be unnecessary in the case \( \gamma \leq 0 \).

We say that \( f \geq 0 \) satisfying (1.5) and (1.6) is a weak solution of (1.1) if \( f, \nabla_v f, \) and \( (\partial_t + v \cdot \nabla_x) f \) are in \( L^2_d \) ([0, T] \times \mathbb{R}^d), and

\[
\int_{[0, T] \times \mathbb{R}^d} \phi(\partial_t + v \cdot \nabla_x) f \, dv \, dx \, dt = \int_{[0, T] \times \mathbb{R}^d} \left[ -\bar{a}^j_{ij} \partial_i \phi \partial_{v^j} f + \phi (\bar{b}^j \cdot \nabla_v f + \bar{c}^j f) \right] \, dv \, dx \, dt.
\]

for any \( \phi \in H^1_d([0, T] \times \mathbb{R}^d) \), where repeated indices are summed over.

Our first main result gives pointwise Gaussian upper bounds for weak solutions:

**Theorem 1.1.** Let \( \gamma \in (0, 1) \), and let \( f \) be a bounded weak solution to the Landau equation (1.1) on \([0, T] \times \mathbb{R}^d\), satisfying (1.5) and (1.6). Then there exists a decreasing function \( K : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \)

with \( K(t) \to \infty \) as \( t \to 0^+ \), such that

\[
f(t, x, v) \leq K(t) e^{-\alpha |v|^2}, \quad (t, x, v) \in (0, T] \times \mathbb{R}^d,
\]

with \( K(t) \) and \( \alpha > 0 \) depending on universal constants and \( G_0 \).

Furthermore, there exist a decreasing function \( J \) and an increasing function \( \beta \) from \( \mathbb{R}_+ \rightarrow \mathbb{R}_+ \)

with \( J(t) \to \infty \) and \( \beta(t) \to 0 \) as \( t \to 0^+ \), such that

\[
f(t, x, v) \leq J(t) e^{-\beta(t)|v|^2}, \quad (t, x, v) \in (0, T] \times \mathbb{R}^d,
\]

with \( J(t) \) and \( \beta(t) \) depending on universal constants. In particular, \( J(t) \) and \( \beta(t) \) are independent of \( G_0 \).
Explicit expressions for $K(t)$, $J(t)$, and $\beta(t)$ are given below. Note that all three functions are independent of the time of existence $T$.

The key step in the second statement of Theorem 1.1 is finding an upper bound for $G_f(t,x)$ that is independent of $G_0$ (Theorem 3.3). Since this upper bound does not depend on the initial data, it must blow up as $t \to 0$, which is why $\beta(t)$ in Theorem 1.1 degenerates as $t \to 0$.

By the estimate of [13], $f$ is locally Hölder continuous in its entire domain. The Gaussian bounds of Theorem 1.1 allow us to pass regularity of $f$ to regularity of the nonlocal coefficients (see [12] and [14]) and bootstrap Schauder estimates exactly as in [17] to conclude the solution is smooth:

**Corollary 1.2.** Any bounded weak solution $f$ of (1.1) on $[0,T] \times \mathbb{R}^2$, satisfying (1.5) and (1.6), is in $C^\infty((0,T] \times \mathbb{R}^2)$. Furthermore, all partial derivatives satisfy Gaussian upper bounds in $v$ that are uniform in $(t,x) \in [0,T] \times \mathbb{R}^d$ and depend only on universal constants, the order of the derivative, and $t_0$.

Theorem 1.1 and Corollary 1.2 make no decay assumption on the initial data. This is in contrast to the corresponding results for soft potentials ($\gamma < 0$), which require the initial data to satisfy Gaussian decay in $v$. (See [3] and [17].) Theorem 1.1 extends a result in [9] for the hard potentials case of the spatially homogeneous equation, which states that arbitrarily high moments of the solution are finite for $t > 0$, with an upper bound that may degenerate as $t \to 0$.

Our last result gives lower Gaussian bounds in $v$. Statement (a) is a propagating estimate, and statement (b) is a self-generating estimate analogous to Theorem 1.1.

**Theorem 1.3.** Let $f$ be as in Theorem 1.1

(a) There exist $\alpha, \mu > 0$ depending on universal constants and $G_0$, such that if $f(0,x,v) \geq ce^{-\alpha |v|^2}$ for all $(x,v) \in \mathbb{R}^2$, then

$$f(t,x,v) \geq ce^{-\mu t}e^{-\alpha |v|^2}, \quad (t,x,v) \in [0,T] \times \mathbb{R}^d.$$  

(b) There exist an increasing function $\delta$ and a decreasing function $\omega$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\delta(t) \to 0$ and $\omega(t) \to \infty$ as $t \to 0^+$, such that

$$f(t,x,v) \geq \delta(t)e^{-\omega(t)|v|^2}, \quad (t,x,v) \in (0,T] \times \mathbb{R}^d,$$

with $\delta(t)$ and $\omega(t)$ depending only on universal constants.

Theorem 1.3(b) is also false for soft potentials, where the optimal generating lower bounds have decay proportional to $e^{-\gamma|v|^{2+\gamma}}$. (See [13].)

1.1. Related work. The starting point of conditional regularity for (1.1) under the assumption (1.5) is the work of Golse-Imbert-Mouhot-Vasseur [13], who proved a Harnack inequality and local $C^\alpha$ estimate for weak solutions of the Landau equation under the assumption (1.5). (See also the earlier, related estimates by Pascucci-Polidoro [23] and Wang-Zhang [28].) In the case $\gamma > 0$, the estimate of [13] depends on a quantitative upper bound for $G_f(t,x)$. In the case of moderately soft potentials ($\gamma \in (-2,0)$), the present author, jointly with Cameron and Silvestre [3], used the local estimate of [13] to derive global estimates for weak solutions satisfying (1.5), and found a priori pointwise decay proportional to $(1+|v|)^{-1}$. It was also shown in [3] that polynomial decay in $v$ with exponent greater than $d+2$ cannot hold for arbitrary initial data, and in particular, Gaussian decay in $v$ cannot hold in general. It is not clear how to bridge this gap between the known and optimal a priori decay, but by our Theorem 1.1, there is no such gap when $\gamma \in (0,1]$.

In the same context of $\gamma \in (-2,0]$ and weak solutions satisfying (1.5), $C^\infty$ smoothing in all three variables was established by Henderson-Snelson [17], for initial data with Gaussian decay. A similar result holds for very soft potentials ($\gamma \in [-d,-2]$), with an additional higher integrability assumption on $f$. See also Imbert-Mouhot [19] for a smoothing result on a related kinetic model. In [13], Henderson-Snelson-Tarfulea derived pointwise lower bounds for solutions with mass and
energy densities bounded above (for $\gamma \in (-2, 0)$), which implies that the lower bound on the mass and the upper bound on the entropy can be removed from the criteria for smoothness and continuation. It should also be possible to remove these two assumptions from the results in the current paper, but we do not explore this here.

Global existence for the inhomogeneous Landau equation has only been shown for perturbative initial data. If $f_{in}$ is sufficiently close to Maxwellian equilibrium state $M$, a solution exists for all time and converges to $M$ as $t \to \infty$. This result was found by Guo [10] and later improved by many other authors, see [5, 22, 25, 4] and the references therein. Recently, Luk [21] constructed global solutions to (1.1) for $\gamma \in (-2, 0)$ with initial data close to the vacuum state $f \equiv 0$, and showed that these solutions scatter to zero as $t \to \infty$. Global existence with general initial data remains a challenging open issue (for any value of $\gamma$), but our results imply that a smooth solution exists for as long as the hydrodynamic quantities stay under control as in (1.5).

In the spatially homogeneous case, where the solution is assumed to be independent of $x$ and the equation has a parabolic structure, much more is known. In this regime, the Landau equation with hard potentials was studied in detail by Desvillettes-Villani [9], who showed global existence and smoothness of solutions with suitable initial data, as well as self-generating polynomial moments and Maxwellian lower bounds. The corresponding study for $\gamma = 0$ was done by Villani [26]. Global solutions are also known to exist for $\gamma \in [-2, 0)$, see Alexandre-Liao-Lin [11] and Wu [29]. Generally speaking, when $\gamma \leq 0$, decay of the solution in $v$ (such as higher moments or Gaussian upper bounds) is propagated forward in time, and when $\gamma > 0$, decay is self-generating. A similar dichotomy holds for the inhomogeneous Landau equation, as shown by our Theorem 1.1 and the results of [3]. For other existence and regularity results for the spatially homogeneous case, see [7, 8, 12, 14, 15, 24] and the references therein.

Regarding the long-time behavior of large-data solutions to (1.1), the famous paper of Desvillettes-Villani [10] found that a priori global solutions with sufficient smoothness and decay converge almost exponentially to Maxwellians. By the results in the current paper, any global solution satisfying (1.5) with $T = \infty$ satisfies the decay and smoothness hypotheses of [10] on $[t_0, \infty)$ for any $t_0 > 0$.

1.2. Notation. Sometimes, we will write $z = (t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Because of the symmetry properties of the equation, the most convenient sets on which to study local estimates are twisted cylinders of the form

$$Q_r(z_0) = \{z \in \mathbb{R}^{2d+1} : t_0 - r^2 < t \leq t_0, |x - x_0 - (t - t_0)v_0| < r^3, |v - v_0| < r\},$$

for some $z_0 = (t_0, x_0, v_0)$ and $r > 0$. We also write $Q_r = Q_r(0, 0, 0)$. We use the notation $A \lesssim B$ when $A \leq CB$ for a universal constant $C$, and $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

In this section, we extend the pointwise upper bounds of [3] to the case $\gamma \in (0, 1]$. The proofs are similar to [3], but we must pay careful attention to the dependence of all constants on $G_0$. We will assume $G_0 \geq 1$, since we want to find an upper bound for $G_f(t, x)$. First, we establish estimates on the coefficients in (1.1), relative to our assumptions on $f$:

\footnote{Only the Coulomb case ($\gamma = -3$) is considered in [10], but one expects that similar techniques can handle other values of $\gamma$, including $\gamma \in (0, 1]$.}
Lemma 2.1. Let $f$ satisfy (1.5) and (1.6). Then there exist universal constants $c_1, C_1, C_2, C_3$ such that
\[
\bar{a}_{ij}^f(t, x, v) e_i e_j \geq c_1 \begin{cases} (1 + |v|)^\gamma, & e \in \mathbb{S}^{d-1}, \\ (1 + |v|)^{\gamma+2}, & e \cdot v = 0, \end{cases}
\]
\[
\bar{a}_{ij}^f(t, x, v) e_i e_j \leq G_0 + C_1 \begin{cases} (1 + |v|)^{\gamma+2}, & e \in \mathbb{S}^{d-1}, \\ (1 + |v|)^\gamma, & e \cdot v = |v|, \end{cases}
\]
\[
|\bar{b}^f(t, x, v)| \leq C_2(1 + |v|)^{\gamma+1},
\]
\[
|\bar{c}^f(t, x, v)| \leq C_3(1 + |v|)^\gamma.
\]

Proof. For this proof, the dependence of $f$, $\bar{a}^f$, $\bar{b}^f$, and $\bar{c}^f$ on $t$ and $x$ is irrelevant, so we will write $f(v)$, etc.

The lower bounds for $\bar{a}_{ij}^f$ are proven in [9, Proposition 4] in the case $d = 3$. The same conclusion for arbitrary dimension $d$ and $\gamma \in [-d, 0)$ is established in [24, Lemma 3.1], with an argument that does not use the sign of $\gamma$ in any essential way. As such, we omit the proof of the lower bounds.

For the upper bounds on $\bar{a}_{ij}^f$, let $e \in \mathbb{S}^{d-1}$ be arbitrary. From (1.2), we have
\[
\bar{a}_{ij}^f(v) e_i e_j \lesssim \int_{\mathbb{R}^d} |v - z|^{\gamma+2} f(z) \, dz 
\]
\[
\lesssim \int_{\mathbb{R}^d} (|v|^{\gamma+2} + |z|^{\gamma+2}) f(z) \, dz 
\]
\[
\lesssim M_0(1 + |v|^{\gamma+2}) + G_0.
\]

Next, if $e \in \mathbb{S}^{d-1}$ is parallel to $v$, then
\[
\bar{a}_{ij}^f(v) e_i e_j \lesssim \int_{\mathbb{R}^d} \left(1 - \frac{(v - z) \cdot e}{|v - z|} \right)^2 |v - z|^{\gamma+2} f(z) \, dz 
\]
\[
= \int_{\mathbb{R}^d} (|z|^2 - (z \cdot e)^2) |v - z|^{\gamma} f(z) \, dz 
\]
\[
\lesssim \int_{\mathbb{R}^d} (|v|^2|z|^2 + |z|^{\gamma+2}) f(z) \, dz 
\]
\[
\lesssim E_0(1 + |v|)^\gamma + G_0.
\]

For $\bar{b}^f$, since $\gamma \leq 1$, we have
\[
|\bar{b}^f(v)| \leq \int_{\mathbb{R}^d} |v - z|^{\gamma+1} f(z) \, dz \lesssim \int_{\mathbb{R}^d} (|v|^{\gamma+1} + |z|^{\gamma+1}) f(z) \, dz \lesssim M_0(1 + |v|)^{\gamma+1} + E_0.
\]

The bound on $\bar{c}^f$ follows by a similar calculation. \qed

Next, we quote a theorem of [13] that gives a local $L^\infty$ estimate for weak solutions of
\[
\partial_t g + v \cdot \nabla_x g = \nabla_v \cdot (A \nabla_v g) + B \cdot \nabla_v g + s
\]
where $A$, $B$, and $s$ are bounded and measurable, and $A$ is uniformly elliptic. When we apply this estimate, the constant will depend on $G_0$ via the upper ellipticity constant $\Lambda$ (see Lemma 2.1). To determine the dependence on $\Lambda$, which is not explicitly stated in [13], we must follow the proof in that article and keep track of the constant $\Lambda$ at every step. This straightforward but tedious task is outlined in Appendix A.

Theorem 2.2. Let $g$ be a weak solution to (2.1) in $Q_1$. Then there holds
\[
\|g\|_{L^\infty(Q_{1/2})} \leq C \left(\|g\|_{L^2(Q_1)} + \|s\|_{L^\infty(Q_1)}\right),
\]
where \( C = C'(1 + \Lambda)^P \) for some \( P > 0 \) depending only on the dimension \( d \), and \( C' \) depending only on \( d \) and \( \lambda \).

**Proof.** See [13] Theorem 12. The form of \( C \) is justified in Appendix A below. \( \square \)

As in [3], we can apply scaling techniques to (2.2) and derive an improved pointwise estimate:

**Proposition 2.3.** Let \( g \) be a weak solution of (2.1) in \( Q_R \) for some \( R > 0 \), with

\[
0 < \lambda I \leq A(t, x, v) \leq \Lambda, \quad (t, x, v) \in Q_R,
\]

\[
|B(t, x, v)| \leq \Lambda/R, \quad (t, x, v) \in Q_R,
\]

\( s \in L^\infty(Q_R) \).

Then the estimate

\[
g(0, 0, 0) \leq C \left( \|g\|_{L^2_t L^{\frac{4}{d+2}}_x(Q_R)} \|s\|_{L^\infty(Q_R)}^{\frac{d}{d+2}} + R^{-d} \|g\|_{L^\infty_t L^1_x(Q_R)} \right)
\]

holds, with \( C = C'(1 + \Lambda)^P \) as in Theorem 2.2.

**Proof.** See [3], Proposition 3.2 and Lemma 3.3. \( \square \)

To apply Proposition 2.3 to the Landau equation, since the ellipticity constants of \( \bar{a}^f_{ij} \) degenerate as \(|v| \to \infty\), we need a change of variables in \( v \) that produces an equation with universal ellipticity constants in a small cylinder. This change of variables was first introduced in [3] for the case \( \gamma \in (-2, 0)\).

**Lemma 2.4.** Let \( z_0 = (t_0, x_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \) be such that \(|v_0| \geq 2\), let \( S \) be the linear transformation such that

\[
Se = \begin{cases}
|v_0|^{1+\gamma/2}e, & e \cdot v_0 = 0 \\
|v_0|^{\gamma/2}e, & e \cdot v_0 = |v_0|,
\end{cases}
\]

and define

\[
S_{z_0}(t, x, v) := (t + t_0, x + tx_0, v + tv_0).
\]

Then there exists a radius

\[
r_1 = c_1|v_0|^{-1-\gamma/2} \min \left(1, \sqrt{t_0/2} \right),
\]

with \( c_1 \) universal, such that:

(a) There exists a constant \( C > 0 \) independent of \( v_0 \in \mathbb{R}^d \setminus B_2 \) such that for all \( v \in B_{r_1} \),

\[
C^{-1}|v_0| \leq |v_0 + Sv| \leq C|v_0|.
\]

(b) If \( f \) is a weak solution of the Landau equation (1.1) satisfying (1.5) and (1.6), then \( f_{z_0}(t, x, v) := f(S_{z_0}(t, x, v)) \) satisfies

\[
\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \nabla_v \cdot (A(z) \nabla_v f_{z_0}) + B(z) \cdot \nabla_v f_{z_0} + C(z) f_{z_0} \tag{2.3}
\]

in \( Q_1 \), and the coefficients

\[
A(z) = S^{-1} \bar{a}^f(S_{z_0}(z)) S^{-1}, \quad B(z) = S^{-1} \bar{b}^f(S_{z_0}), \quad C(z) = \bar{c}^f(S_{z_0}(z))
\]

satisfy

\[
\lambda I \leq A(z) \leq \Lambda I,
\]

\[
|B(z)| \leq |v_0|^{1+\gamma/2},
\]

\[
|C(v)| \leq |v_0|^{\gamma},
\]

with \( \Lambda \lesssim G^P \), and \( \lambda \) and the bounds on \( B(z) \) and \( C(z) \) depending only on universal constants.
Proof. To prove (a), since \( v \in B_{r_1} \), we have
\[
|Sv| \leq c_1 (1 + |v_0|)^{1+\gamma/2} |v| \leq c_1,
\]
so that \( |v_0 + Sv| \approx |v_0| \) for \( |v_0| \geq 2 \).

For (b), the equation \( (2.3) \) satisfied by \( f_{z_0} \) follows by direct computation. The uniform ellipticity of \( A(z) \) (with constants independent of \( |v_0| \)) is the only subtle part of this lemma, and is the reason we must take \( |v| \lesssim |v_0|^{-1-\gamma/2} \). The proof can be found in \cite[Lemma 4.1]{3}. For the bound on \( B(z) \), Lemma \ref{lem:2.1} and conclusion (a) imply
\[
|B(z)| \lesssim \| S^{-1} \| \| \delta f^{(S_{z_0}(z))} \| \lesssim (1 + |v_0|)^{1+\gamma/2}.
\]
The bound on \( C(z) \) follows similarly. \( \square \)

Next, we find global upper bounds for any solution, that depend only on universal constants and \( G_0 \). The proof also gives some polynomial decay, but we will not make any use of this.

**Theorem 2.5.** Let \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R}_+ \) be a bounded weak solution of \( (1.1) \) satisfying \( (1.5) \) and \( (1.6) \). Then
\[
f(t, x, v) \leq C \left( 1 + t^{-d/2} \right) (1 + |v|)^{-1-\gamma},
\]
with \( C \lesssim G_0^P \) for some \( P \) depending on dimension.

The \( P \) in this theorem is not the same as the \( P \) in Theorem 2.4.

**Proof.** Define
\[
K := \sup_{[0, T] \times \mathbb{R}^d} \min\{t^{d/2}, 1\} f(t, x, v).
\]
We may assume \( K \geq 1 \). Let \( \varepsilon > 0 \), and choose \( z_0 = (t_0, x_0, v_0) \in [0, T] \times \mathbb{R}^d \) such that \( f(t_0, x_0, v_0) > (K - \varepsilon) \max\{t_0^{-d/2}, 1\} \). Define \( r_0 = \min\{1, \sqrt{v_0}\}/2 \), and note that \( r_0^{-d} \approx (1 + t_0^{-d/2}) \).

If \( |v_0| \leq 2 \), Lemma \ref{lem:2.1} and Proposition \ref{prop:2.3} with \( g(z) = f(z_0 + z) \) and \( s(z) = \bar{c}^2(z_0 + z) f(z_0 + z) \) in \( Q_{r_0}(z_0) \) imply
\[
f(t_0, x_0, v_0) \leq CG_0^P \left( M_0^{2/(d+2)} K^{d/(d+2)} + r_0^{-d} M_0 \right) \lesssim G_0^P r_0^d K^{d/(d+2)}. \tag{2.4}
\]

If \( |v_0| \geq 2 \), let \( r_1 = c_1 r_0 |v_0|^{-1-\gamma/2} \) and \( f_{z_0} \) be as in Lemma \ref{lem:2.4}. By Lemma \ref{lem:2.4}b), \( f_{z_0} \) solves \( (2.3) \) in \( Q_{r_1} \), and
\[
0 < \lambda I \leq A(z) \leq \Lambda I,
\]
\[
|B(z)| \lesssim |v_0|^{1+\gamma/2},
\]
\[
|C(z)| \lesssim |v_0|^{\gamma},
\]
in \( Q_{r_1} \), where \( \Lambda \lesssim G_0^P \) and all other constants are universal. By Lemma \ref{lem:2.4}a) and the definition of \( K \),
\[
\|C f_{z_0}\|_{L^\infty(Q_{r_1})} \lesssim K r_0^{-d} |v_0|^{\gamma} \tag{2.6}
\]
Let \( Q_{r_1, r_1} \) be the image of \( Q_{r_1} \) under \( z \mapsto S_{z_0}(z) \), and note that
\[
\|f_{z_0}\|_{L^\infty_{r_2} L^1(Q_{r_1})} = \det(S^{-1}) \|f\|_{L^\infty_{r_2} L^1(Q_{r_2})} = |v_0|^{-(d-1)(2+\gamma)/2+\gamma/2} \|f\|_{L^\infty_{r_2} L^1(Q_{r_2})} \leq |v_0|^{-(1+\gamma+d(2+\gamma)/2)} G_0, \tag{2.7}
\]
\[
\|f_{z_0}\|_{L^\infty_{r_2} L^1(Q_{r_1})} \lesssim K r_0^{-d} |v_0|^{\gamma} \tag{2.6}
\]
where the last inequality comes from (1.6) and Lemma 2.4(a). By (2.6), we can apply Proposition 2.3 in $Q_{r_1}$ with $g = f_{z_0}$ and $s = C(z) f_{z_0}$ to obtain

$$f(t_0, x_0, v_0) \leq C \left( \| f_{z_0} \|_{L^{\infty}(Q_{r_1})} + r_0^{-d} |v_0|^{d(2+\gamma)/2} \right),$$

and since this is true for any $\phi$, we have

$$f(t_0, x_0, v_0) \leq C G_0^{(P+1)(d+2)/2} r_0^{-d} (1 + |v_0|)^{-1-\gamma}, \quad \text{(2.8)}$$

using (2.4) and (2.7).

By our choice of $(t_0, x_0, v_0)$, (2.4) and (2.8) imply

$$(K - \varepsilon) a_i f_i \leq C G_0^{(P+1)(d+2)/2} r_0^{-d},$$

and since this is true for any $\varepsilon > 0$, we have $K \leq C G_0^{(P+1)(d+2)/2}$. Applying (2.8) again, we conclude $f(t_0, x_0, v_0) \leq C G_0^{(P+1)(d+2)/2} r_0^{-d} (1 + |v_0|)^{-1-\gamma}$. \[ \square \]

### 3. Gaussian Decay

In this section, we show that all bounded solutions have Gaussian decay. The proof relies on the maximum principle for $H^1$ weak solutions of the linear Landau equation, which can be found in, e.g., the appendix of [4]. Given a function $f$ satisfying (1.5) and (1.6), a solution $\phi$ to the linear Landau equation satisfies

$$\partial_t \phi + v \cdot \nabla_x \phi = \nabla_v \cdot (\bar{a} f \nabla_v \phi) + \bar{b} f \cdot \nabla_v \phi + \bar{c} f \phi.$$

Note that if $\phi$ is smooth, this is equivalent to

$$\partial_t \phi + v \cdot \nabla_x \phi = \text{tr}(\bar{a} f D_v^2 \phi) + \bar{c} f \phi,$$

since $\bar{b} f_i = -\partial_{v_i} \bar{a} f_{ij}$.

First, we show Gaussians with appropriate decay constants are super- or sub-solutions of the linear Landau equation for large velocities:

**Lemma 3.1.** Let $f$ be a bounded function satisfying (1.5) and (1.6), and let $\bar{a} f$ and $\bar{c} f$ be defined by (1.2) and (1.4) respectively.

(a) There exists a universal constant $C$ such that, if $\alpha = \frac{C}{G_0}$, then the function

$$\phi(v) := e^{-\alpha |v|^2}$$

satisfies

$$\bar{a}_{ij} f_{ij} \phi + \bar{c} f \phi \leq -C G_0^{-1} |v|^{\gamma+2} \phi,$$

for $|v| \geq C G_0^{1/2}$.

(b) There exists $C$ universal such that, if $\alpha = C G_0$, then $\phi(v)$ defined as in (a) satisfies

$$\bar{a}_{ij} f_{ij} \phi + \bar{c} f \phi \geq C G_0^2 |v|^{\gamma+2} \phi,$$

for $|v| \geq C G_0^{1/2}$.

**Proof.** Let $\alpha$ be a positive constant to be chosen later. Since $\phi$ is radial, for any $v \neq 0$ we have

$$\partial_{ij} \phi = \frac{\partial_{rr} \phi}{|v|^2} v_i v_j + \frac{\partial_r \phi}{|v|} \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) = \left[ 4 \alpha^2 |v|^2 - 2 \alpha \right] v_i v_j - 2 \alpha \left( \delta_{ij} - \frac{v_i v_j}{|v|^2} \right) e^{-\alpha |v|^2},$$

and Lemma 2.4 implies

$$\bar{a}_{ij} f_{ij} \phi \leq \left[ (4 \alpha^2 |v|^2 - 2 \alpha) C_1 G_0 |v|^{\gamma} - 2 \alpha C_1 |v|^{\gamma+2} \right] e^{-\alpha |v|^2}$$

$$= \left[ (4 \alpha^2 C_1 G_0 - 2 \alpha C_1) |v|^{\gamma+2} - 2 \alpha C_1 G_0 |v|^\gamma \right] e^{-\alpha |v|^2},$$

and

$$\bar{c} f \phi \leq (2 \alpha) C_1 G_0 |v|^\gamma e^{-\alpha |v|^2}.$$
Therefore, we have \( \frac{\partial}{\partial t} \partial_{ij} \phi \leq -CG_0^{-1} |v|^{\gamma+2} \). With the bound on \( \partial^f \) in Lemma 2.1, this implies

\[
\frac{\partial}{\partial t} \partial_{ij} \phi + \partial^f \phi \leq \left[-CG_0^{-1} |v|^{\gamma+2} + C|v|^{\gamma}\right] \phi(v).
\]

This right-hand side is bounded by \(-CG_0^{-1} |v|^{\gamma+2}\) if

\[
|v| \geq CG_0^{1/2},
\]

for some (new) universal constant \( C \), which establishes (a).

For (b), the upper and lower bounds in Lemma 2.1 imply

\[
\frac{\partial}{\partial t} \partial_{ij} \phi \geq \left[(4\alpha^2|v|^2 - 2\alpha)c_1|v|^{\gamma} - 2\alpha C_1 G_0 |v|^{\gamma+2}\right] e^{-\alpha|v|^2}
\]

\[
\geq \left[(4\alpha^2 c_1 - 2\alpha C_1 G_0) |v|^{\gamma+2} - 2\alpha c_1 |v|^{\gamma}\right] e^{-\alpha|v|^2},
\]

so that if \( \alpha = C_1 G_0/(4c_1) \), we have

\[
\frac{\partial}{\partial t} \partial_{ij} \phi \geq CG_0^{2} |v|^{\gamma+2} - C_0 |v|^{\gamma} \geq CG_0^{2} |v|^{\gamma+2},
\]

provided \( |v| \geq CG_0^{-1/2} \). Since \( \partial^f \phi \geq 0 \), we are done.

\[\square\]

We are ready to prove the first assertion of Theorem 1.1.

**Theorem 3.2.** For any weak solution \( f \) of (1.1) satisfying (1.5) and (1.6), we have

\[
f(t, x, v) \leq CG_0^p \left(e^{Ct^{-\gamma}} + 1\right) e^{-\alpha|v|^2}
\]

for all \( t > 0 \), with \( P \) as in Theorem 2.3, \( \alpha \) from Lemma 3.1(a), and \( C \) universal.

**Proof.** Let \( p = 2/\gamma \) and \( \psi(t, x, v) = e^{C_0 t^{-p} - \alpha|v|^2} \), with \( C_0 \) to be determined. For \( |v| \geq CG_0^{1/2} \), Lemma 3.1 implies

\[-\partial_t \psi - v \cdot \nabla_x \psi + f(t, x, v) + \partial^f \psi \leq \psi \left(C_0 t^{-p-1} - CG_0^{-1} |v|^{\gamma+2}\right),
\]

which is nonpositive whenever

\[
|v| \geq \max\{CG_0^{1/2}, (pc_0G_0 t^{-p-1}/C^{1/(\gamma+2)}) \} \geq CC_0^{1/(\gamma+2)} G_0^{1/2} t^{-(p+1)/(\gamma+2)}.
\]

On the other hand, let \( |v|^2 \leq CC_0^{1/(\gamma+2)} G_0^{1/2} t^{-(p+1)/(\gamma+2)} \). Since \( p = 2/\gamma \), we have \( 2(p+1)/(\gamma+2) = p \). Therefore, since \( \alpha = CG_0^{-1} \), we have

\[
\psi(t, x, v) = \exp(C_0 t^{-p} - \alpha|v|^2) \geq \exp(C_0 t^{-p} - CC_0^{2/(\gamma+2)} t^{-p}),
\]

so for \( C_0 \) sufficiently large (depending only on universal constants), this right-hand side approaches \( \infty \) as \( t \to 0 \).

Let \( C_1 \leq C_0^p \) be the constant from Theorem 2.3. For \( t \in (0, 1] \), we have \( 2C_1 \psi(t, x, v) \geq 2C_1 e^{C_0 t^{-p}} \geq C_1 (1 + t^{-d/2}) \geq f(t, x, v) \) in

\[
\{(t, x, v) : 0 < t \leq 1, |v|^2 \leq CC_0^{2/(\gamma+2)} G_0 t^{-p}\}.
\]

Therefore, we have

\[
(-\partial_t - v \cdot \nabla_x + \partial^f) [f - 2C_1 \psi]_+ \geq 0, \quad t \in (0, 1] \times \mathbb{R}^{2d},
\]

and \( [f(0, x, v) - 2C_1 \psi(0, x, v)]_+ = 0 \) for all \((x, v) \in \mathbb{R}^{2d}\). The maximum principle implies \( [f - 2C_1 \psi]_+ = 0 \) in \((0, 1] \times \mathbb{R}^{2d}\), so that

\[
f(t, x, v) \leq 2C_1 e^{C_0 t^{-p}} e^{-\alpha|v|^2}, \quad t \in (0, 1].
\]

Next, with \( R_0 = CG_0^{1/2} \) as in Lemma 3.1(a), choose \( C_2 \) such that \( C_2 e^{-\alpha R_0^2} \geq 2C_1 \geq \|f\|_{L^\infty([1,T] \times \mathbb{R}^{2d})} \). Since \( \alpha R_0^2 \) is bounded independently of \( G_0 \), we can choose \( C_2 \leq C_1 \). Then we can apply the maximum principle to \( [f - (C_0^p + C_2) e^{-\alpha|v|^2}]_+ \) on \([1, T] \times \mathbb{R}^{2d}\) and conclude the proof. \[\square\]
We now show that the \((\gamma + 2)\) moment of \(f\) is bounded independently of \(G_0\) on \([t_0, T]\) for any \(t_0 > 0\). (We are seeking a bound that does not depend quantitatively on the \((\gamma + 2)\) moment of the initial data, so we cannot hope for a bound that is uniform in \(t \in [0, T]\).)

**Theorem 3.3.** With \(f\) as above, for any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon\), depending only on universal constants and \(\varepsilon\), such that

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C_\varepsilon \left( 1 + t^{-1/(1-\gamma/2-\varepsilon)} \right), \quad t \in (0, T], \ x \in \mathbb{R}^d.
\]

**Proof.** Let \(t_* \in (0, 1]\) be arbitrary. By Theorem 3.2, \(f\) is bounded by \(K e^{-\alpha |v|^2}\) on \([t_*, T]\), with \(\alpha = C/G_0\) and \(K \leq e^{C t_*^{-2/\gamma} G_0^\beta}\). We will interpolate between this pointwise Gaussian decay and the energy bound. For \(p > 1\) to be chosen later, let \(q\) be such that \(1/p + 1/q = 1\). For \(t \geq t_*\), we have

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq K^{1/p} \int_{\mathbb{R}^d} e^{-\alpha |v|^2/p} f^{1/q} |v|^{\gamma+2/p} |v|^{2/q} \, dv
\]

\[
\leq K^{1/p} \left( \int_{\mathbb{R}^d} e^{-\alpha |v|^2} |v|^{p\gamma+2} \, dv \right)^{1/p} \left( \int_{\mathbb{R}^d} |v|^2 f \, dv \right)^{1/q}
\]

\[
\leq K^{1/p} C_p \alpha^{-(d+2)/(2p) - \gamma/2}
\]

\[
\leq e^{(C t_*^{-2/\gamma})/p} G_0^{\gamma/2} E_0 C_p (C G_0)^{(d+2)/(2p) + \gamma/2},
\]

with

\[
C_p = \left( \int_{\mathbb{R}^d} e^{-|w|^2} |w|^{p\gamma+2} \, dw \right)^{1/p} = \left( C_d \int_0^\infty e^{-r^p r^{\gamma+2} + 1} \, dr \right)^{1/p} = \left( C_d \Gamma \left( \frac{p\gamma + d}{2} + 1 \right) \right)^{1/p}.
\]

It follows from Stirling’s approximation that \(\lim_{x \to \infty} \Gamma(ax + b)^{1/x} / x^a\) exists for any \(a, b > 0\). Therefore, we have \(C_p \lesssim p^{\gamma/2}\). Choosing

\[
p = \max \{ C t_*^{-2/\gamma}, [(d + 2)/2 + P]/\varepsilon \},
\]

we obtain

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C e^{1} E_0 G_0^{(2+\varepsilon)/\gamma} p^{\gamma/2} \leq C t_*^{-1} G_0^{1/2 + \varepsilon},
\]

with \(C_\varepsilon\) depending on universal constants and \(\varepsilon\). Let \(\mu = \gamma/2 + \varepsilon < 1\). Then we finally have

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C t_*^{-1} G_0^\mu, \quad t \geq t_*.
\]

(3.1)

To apply (3.1) iteratively, we need to wait a short amount of time at each step. Let \(t_0 \in (0, 1]\) and \(n \in \mathbb{N}\) be arbitrary, and define the following sequence of times:

\[
t_{1,n} = 2^{-n+1} t_0, \ t_{2,n} = 2^{-n+2} t_0, \ t_{3,n} = 2^{-n+3} t_0, \ldots, t_{n-1,n} = t_0/2, \ t_{n,n} = t_0.
\]

Apply (3.1) to \(f\) with \(t_* = t_{1,n}\):

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C t_0^{-1} 2^{n-1} G_0^\mu =: G_{1,n}, \quad t \geq t_{1,n}.
\]

Using the new bound \(G_{1,n}\) for \(G_{f}(t, x)\), we apply (3.1) to \(f(t_{1,n} + t, x, v)\) with \(t_* = t_{2,n} - t_{1,n} = 2^{-n+1} t_0\) and obtain

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C t_0^{-1} 2^{n-1} G_{1,n}^\mu = (C t_0^{-1})^{1+\mu} 2^{(n-1)(1+\mu)} G_0^{\varepsilon^2} =: G_{2,n}, \quad t \geq t_{2,n}.
\]
Continuing, we apply (3.1) to \( f(t_{2,n} + t, x, v) \) with \( t_* = t_{3,n} - t_{2,n} = 2^{-n+2}t_0 \):

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C \varepsilon t_0^{-1} 2^{n-2} G_{2,n}^{\mu} \\
= (C_0 t_0^{-1})^{1+\mu^2} 2^{n-2} 2^{(n-1)(\mu+\mu^2)} G_0^{\mu} =: G_{3,n}, \quad t \geq t_{3,n}.
\]

Repeating this \( n \) times, we have

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq (C_0 t_0^{-1})^{1+\mu^2+\mu^{n-1}} C_n G_0^{\mu^n} =: G_{n,n}, \quad t \geq t_0,
\]

where

\[
C_n = 2 \cdot 2^2 \mu \cdot 2^3 \mu^2 \cdots 2^{(n-2)\mu} \cdot 2^{(n-1)\mu^2}.
\]

Let the number of steps \( n \to \infty \), and we have

\[
G_n \to 2 \sum_{k=1}^{\infty} k \mu^{k-1} = 2^{1/(1-\mu)^2}.
\]

This implies

\[
\int_{\mathbb{R}^d} |v|^{\gamma+2} f(t, x, v) \, dv \leq C(C_0 t_0^{-1})^{1/(1-\mu)}, \quad t \geq t_0,
\]

as desired. For general \( t_0 \in [0, T] \), we proceed as above, replacing \( t_0 \) with \( \min\{1, t_0\} \), and conclude the statement of the theorem. \( \square \)

We can now prove the second assertion of Theorem 1.1. Theorem 3.3 implies \( G(t, x) \leq C_0 (1 + (t_0/2)^{-1/2}) \) on \([t_0/2, T] \times \mathbb{R}^d \), for some \( \varepsilon \in (0, 1/2) \). Applying Theorem 3.2 to \( f(t_0/2 + t, x, v) \) with this new bound for \( G(t, x) \), we conclude

\[
f(t_0, x, v) \leq C(t_0/2)^{-P/(1-\gamma/2-\varepsilon)} e^{C(t_0/2)^{-2/\gamma}} e^{-C t_0^{1/(1-\gamma/2-\varepsilon)} |v|^2},
\]

as desired.

Finally, we derive Gaussian lower bounds for \( f \).

**Proof of Theorem 1.3** Throughout this proof, we will write

\[
L = -\partial_t - v \cdot \nabla_x + a_{ij}^{(f)} \partial_{v_i} v_j + \tilde{c}^f.
\]

As usual, we sum over repeated indices.

(a) Let \( \psi_1(t, x, v) = e^{-\mu t} e^{-\alpha |v|^2} \), with \( \mu > 0 \) to be determined. For \( |v| \geq R_0 \) := \( CG_0^{-1/2} \), Lemma 3.1(b) implies

\[
L \psi_1 \geq (\mu + CG_0^2) |v|^{\gamma+2} \psi_1 \geq 0.
\]

(Note that \( \tilde{c}^f \psi_1 \geq 0 \).) For \( |v| \leq R_0 \), it follows from Lemma 2.1 that

\[
a_{ij}^{(f)} \partial_{v_i} \psi_1 \geq -CG_0 (1 + |v|)^{2+\gamma} |D^2 \psi_1| \geq -CG_0 R_0^{\gamma+4} |\psi_1|,
\]

which is bounded from below by some constant depending on \( R_0 \). Choosing \( \mu \) sufficiently large, we have \( L \psi_1 \geq 0 \) in \((0, T] \times \mathbb{R}^d \), and the conclusion follows from applying the maximum principle to \( e^{\psi_1 - f} \).

(b) For any \( t_1 \in (0, T] \) and \( x \in \mathbb{R}^d \), the hydrodynamic bounds 1.4 imply for any \( r > 0 \),

\[
m_0 \leq \int_{\mathbb{R}^d} f(t_1, x, v) \, dv \leq \| f(t_1, x, \cdot) \|_{L^{\infty}(B_r)} C d!^d + E_0 r^{-2}.
\]

Clearly, there exists \( r > 0 \) such that \( r = \| f(t_1, x, \cdot) \|_{L^{\infty}(B_r)}^{-1/(d+2)} E_0^{1/(d+2)} \). Theorem 2.4 implies \( r \gtrsim t_1^{d/(2(d+2))} \). With this choice of \( r \), we have \( \| f(t_1, x, \cdot) \|_{L^{\infty}(B_r)} \gtrsim m_0^{(d+2)/2} E_0^{d/2} \). Applying a scaled version of the Harnack inequality 13 Theorem 4, we have for any \( t_2 > t_1 \),

\[
\inf_{x \in \mathbb{R}^d} f(t_2, x, v) \geq \delta, \quad \text{for all } x \in \mathbb{R}^d,
\]

with \( \delta > 0 \) depending on \( t_2 - t_1, r, \) and universal constants.
Now, let $t_0 > 0$ be arbitrary, and apply (3.2) with $t_1 = t_0/2$ and $t_2 = t_0$. The constants $r$ and $\delta$ depend only on universal constants and $t_0$, and they are nondecreasing as $t_0$ increases. (This can be seen either by tracking the dependencies in (3.2) or by applying the same reasoning to $f(t' + t, x, v)$ for any $t' > 0$.) We conclude

$$\inf_{v \in B_r} f(t, x, v) \geq \delta, \quad \text{for all } t \geq t_0, \ x \in \mathbb{R}^d,$$

with $\delta, r > 0$ depending on universal constants and $t_0$.

Next, we show $f(t, x, \cdot)$ is bounded below by a Gaussian. Define

$$\psi_2(t, x, v) = e^{-\rho[1+(t-t_0)^{-1}]|v|^2},$$

with $\rho > 0$ to be determined. Letting $\Omega = \{ t \geq t_0, x \in \mathbb{R}^d, |v| \geq r/2 \}$, the function $\psi_2$ can be extended smoothly by 0 on the part of $\partial \Omega$ with $t = t_0$ (since $|v| \geq r/2$ in $\Omega$). By (3.3), we have $f \geq \delta \psi_2$ on $\partial \Omega$. It remains to show $\psi_2$ is a subsolution in $\Omega$. Let $\omega(t) = \rho[1+(t-t_0)^{-1}]$. By the calculations of Lemma 3.1(b), we have for $|v| \geq r/2$,

$$\tilde{a}_{ij}^t \partial_{v_i v_j} \psi_2 \geq \left(4\omega^2(t)c_1 - 2C(t_0)\omega(t) \right) |v|^{\gamma+2} - 2\omega(t)c_1 |v|^{\gamma} \psi_2,$$

where $C(t_0)$ is given by Theorem 3.3. For $\rho$ large enough (depending on universal constants, $t_0$, and $r$) we have $\tilde{a}_{ij}^t \partial_{v_i v_j} \psi_2 \geq C \omega^2(t) |v|^{\gamma+2} \psi_2$ in $\Omega$. This implies

$$L \psi_2 \geq \left[ \omega'(t)|v|^2 + C \omega^2(t) |v|^{\gamma+2} \right] \psi_2.$$

Since $\omega'(t) = -\rho(t-t_0)^2$, we can choose $\rho$ sufficiently large, depending on universal constants, $t_0$, and $r$, such that $\omega^2(t) |v|^{\gamma+2} \geq \omega'(t)|v|^2$ and $L \psi_2 \geq 0$ for $|v| \geq r/2$. Applying the maximum principle in $\Omega$, we conclude

$$f(t, x, v) \geq \delta e^{-\rho[1+(t-t_0)^{-1}]|v|^2}, \quad t \geq t_0.$$

The constants $\delta = \delta_{t_0}$ and $\rho = \rho_{t_0}$, depending on $t_0$, may be useful to know in other contexts. Using this, we conclude $f(t, x, v) \geq \delta_{t_0} e^{-\rho_{t_0}[1+(t_0/2)^{-1}]|v|^2}$ for $t \geq t_0$, as desired. \hfill $\square$

### Appendix A. Dependence of Local Estimates on Ellipticity Constants

In [13], a Harnack inequality and local $C^\alpha$ estimate are proven for kinetic Fokker-Planck equations of the form (2.1). We are concerned only with the local $L^\infty$ estimate (Theorem 2.2 above), which does not require the full strength of the Harnack inequality. In this appendix, we estimate the dependence of the constant on $\Lambda$, $\lambda$, and $||B||_{L^\infty}$. The dependence on $\lambda$ and $||B||_{L^\infty}$ is not relevant for the present article, but may be useful to know in other contexts. For simplicity, we will assume $\Lambda, ||B||_{L^\infty} \geq 1$ and $\lambda \leq 1$.

The proof of the $L^\infty$ estimate (Theorem 12 in [13]) proceeds in the following steps:

**Step 1: Global regularity estimate (13, Lemma 10).** Starting with an equation of the form

$$\begin{align*}
(\partial_t + v \cdot \nabla) g &= \nabla_v \cdot (A \nabla_v g) + \nabla_v \cdot H_1 + H_0, \quad (A.1)
\end{align*}$$

with $H_0, H_1 \in L^2(\mathbb{R}^{2d+1})$ and $g, H_0, H_1$ supported in $(t, x, v) \in \mathbb{R} \times \mathbb{R}^d \times B_{r_0}(0)$, one integrates against $2g$, and using the Poincaré inequality in $v$, obtains the estimate

$$\lambda ||\nabla_v g||_{L^2}^2 \leq \frac{C}{\lambda} \left( ||H_0||_{L^2}^2 + ||H_1||_{L^2}^2 \right),$$

where $L^q = L^q(\mathbb{R}^{2d+1})$. Applying the hypoelliptic estimate of [2], and using $(1 + |v|^2) \leq 1 + r_0^2$ and $||A||_{L^\infty} \leq \Lambda$, gives

$$||D_{t,v}^2 g||_{L^2} + ||D_{t,v}^2 g||_{L^2} \lesssim \Lambda (1 + r_0^2) \left( ||\nabla_v g||_{L^2} + ||H_1||_{L^2}^2 + ||H_0||_{L^2}^2 \right),$$

which, combined with the above estimate for $||\nabla_v g||_{L^2}$ and the Sobolev embedding $H^4(\mathbb{R}^{2d+1}) \subset L^p(\mathbb{R}^{2d+1})$, yields

$$||g||_{L^p} \leq C \left( ||H_1||_{L^2}^2 + ||H_0||_{L^2}^2 \right), \quad (A.2)$$

In the equation (A.1), $g$ is a function of $(t, x, v)$, and $A$ and $H_0, H_1$ are $\mathbb{R}^{d\times d}$ matrices and vectors, respectively.
with \( p = (6d + 1)/(6d + 1) \) and the constant \( C \) proportional to \( \frac{\Lambda}{\lambda^2} (1 + r_0^2) \).

**Step 2: Caccioppoli inequality ([13] Lemma 11).** Considering subsolutions of (2.1) defined in a cylinder \( Q_{r_1} \) and integrating the equation against \( 2g\Psi^2 \) over \( \mathcal{R} := [t_1, 0] \times \mathbb{R}^{2d} \), with \( t_1 \in (-r_1^2, 0) \) and \( \Psi \) a smooth, compactly supported cutoff with \( 0 \leq \Psi \leq 1 \), one has (using Young’s inequality)

\[
\int_{\mathcal{R}} \left( g^2 \Psi^2 \right) + 2\lambda \int_{\mathcal{R}} \|\nabla v g\|^2 \Psi^2 \leq \int_{\mathcal{R}} g^2 [(\partial_t + v \cdot \nabla x)(\Psi^2)] + 8(\Lambda + \|B\|_{\Lambda^\infty})^2 \lambda^{-1} (\|\nabla_x \Psi\|^2 + \Psi^2) + 2 \int g \Psi^2 + \lambda \int |\nabla v g|^2 \Psi^2.
\]

For \( r_0 \in (0, r_1) \), choose \( \Psi \) such that \( \Psi \equiv 1 \) in \( Q_{r_0} \), \( \Psi(t, x, v) = 0 \) for \( t = 0 \), and supp \( \Psi \subset Q_{r_1} \), and obtain

\[
\|g\|_{L^2_t L^2_{x,v}(Q_{r_0})}^2 + \|\nabla v g\|_{L^2_t(Z_{Q_{r_0}})}^2 \leq C \left( C_{0.1} \|g\|_{L^2_t(Q_{r_1})}^2 + \|s\|_{L^2_t(Q_{r_1})}^2 \right),
\]

with \( C \lesssim \left( \frac{\Lambda + \|B\|_{\Lambda^\infty}}{\lambda} \right)^2 \) and \( C_{0.1} \) depending on \( r_0 \) and \( r_1 \) (via derivatives of \( \Psi \)).

**Step 3: Gain of integrability ([13] Theorem 6]).** Letting \( Q_{\text{int}} = Q_{r_0} \), \( Q_{\text{ext}} = Q_{r_1} \), and \( Q_{\text{mid}} = Q_{(r_0+r_1)/2} \), define cutoffs \( \chi_1 \) with \( \chi_1 = 1 \) in \( Q_{\text{int}} \) and \( \chi_1 = 0 \) outside \( Q_{\text{mid}} \), and \( \chi_{1/2} \) with \( \chi_{1/2} = 1 \) in \( Q_{\text{mid}} \) and \( \chi_{1/2} = 0 \) outside \( Q_{\text{ext}} \). For \( g \) a nonnegative solution of (2.1), the truncated function \( g \chi_1 \) is a subsolution of (2.1) with

\[
H_1 = (-A \nabla v \chi_1) g \chi_{1/2}, \quad H_0 = (B \chi_1 - A \nabla v \chi_1) \cdot \nabla v g \chi_{1/2} + g \chi_{1/2} (\partial_t + v \cdot \nabla x) \chi_1 + s \chi_1.
\]

One has

\[
\|H_1\|_{L^2_{t,x,v}}^2 \lesssim \Lambda^2 \|\nabla v \chi_1\|_{L^\infty}^2 \|g\|_{L^2_t(Q_{\text{ext}})}^2,
\]

\[
\|H_0\|_{L^2_{t,x,v}}^2 \lesssim (\|B\|_{L^\infty}^2 + \Lambda^2 \|\nabla v \chi_1\|_{L^\infty}^2) \|\nabla v g\|_{L^2_t(Q_{\text{mid}})}^2 + (\|\partial_t + v \cdot \nabla x\| \chi_1) \|g\|_{L^2_t(Q_{\text{ext}})}^2 + \|s\|_{L^2_t(Q_{\text{ext}})}^2.
\]

Using (A.3) to estimate \( \|\nabla v g\|_{L^2_t(Q_{\text{mid}})}^2 \), we have, after collecting terms,

\[
\|H_0\|_{L^2_{t,x,v}}^2 + \|H_1\|_{L^2_{t,x,v}}^2 \lesssim (\Lambda^2 + \|B\|_{L^\infty}^2)^2 \Lambda^{-2} \left[ (1 + \|\nabla v \chi_1\|_{L^\infty}^2) \|s\|_{L^2_{t,x,v}}^2 + \|\partial_t + v \cdot \nabla x\| \chi_1 \|g\|_{L^2_t(Q_{\text{ext}})} \right].
\]

Estimating \( \|\nabla v \chi_1\|_{L^\infty} \) in terms of \( r_0 \) and \( r_1 \), and applying (A.2), one has

\[
\|g\|_{L^2_{t,x,v}(Q_{\text{int}})}^2 \leq C \left( C_{0.1}^2 \|g\|_{L^2_{t,q}(Q_{\text{ext}})}^2 + C_{0.1} \|s\|_{L^2_{t,q}(Q_{\text{ext}})}^2 \right)
\]

with \( C \lesssim \Lambda^4 + \|B\|_{L^\infty} \Lambda^{-4} \) and \( C_{0.1} \) as in Step 2.

**Step 4: De Giorgi iteration ([13] Theorem 12]).** For any \( g > 0 \), the goal is to show the existence of \( \kappa \in (0, 1] \) such that if \( \|s\|_{L^\infty(Q_{1/2})} \leq \kappa \) and \( \|g\|_{L^2(Q_{1/2})} \leq \kappa \), then \( \|g\|_{L^\infty(Q_{1/2})} \leq \frac{1}{2} \). Taking \( \kappa = 1 \) and applying this result to \( g/\kappa^{-1}\|g\|_{L^2(Q_{1/2})} + \|s\|_{L^\infty(Q_{1/2})} \) will imply the estimate (2.2) with constant proportional to \( \kappa^{-1} \).

Define radii \( r_n = \frac{1}{2}(1 + 2^{-n}) \) and constants \( C_n = \frac{1}{2}(1 - 2^{-n}) \) for all integers \( n \geq 0 \). Considering \( g_n = (g - C_n)^+ \), which is a subsolution of (2.1) in \( Q_{r_n} \) with source term \( s\chi_{(g \geq C_n)} \), and proceeding as in Step 2 with a suitable cutoff \( \Psi_n \) in \( Q_{r_n} \), one obtains

\[
U_n := \|g_n\|_{L^2_{t,x,v}(Q_{r_n})}^2 \leq C \left( 4^n \|g_n\|_{L^2(Q_{r_{n-1}})}^2 + 2 \int_{Q_{r_{n-1}}} g_n s \right),
\]
with $C \lesssim \left( \frac{\Lambda + \|B\|_{L^\infty}}{\lambda} \right)^2$. Applying Hölder’s inequality in both terms on the right-hand side, and using the fact that $g_{n-1} \geq C_n - C_{n-1} = 2^{-n-1}$ whenever $g_n \geq 0$, one concludes

$$U_n \leq C2^{4n} \left[ \|g_{n-1}\|_{L^p(Q_{r_n-1})}^2 U_{n-1}^{-\frac{2}{p}} + \|s\|_{L^\infty(Q_{r_n})} \|g_{n-1}\|_{L^p(Q_{r_n-1})} U_{n-1}^{-\frac{2}{p}} \right], \quad (A.4)$$

with the same $C$. Next, from Step 3 we have

$$\|g_{n-1}\|_{L^p(Q_{r_n-1})}^2 \lesssim \Lambda (\Lambda^4 + \|B\|_{L^\infty}^4) \lambda^{-4} \left( 8^n \|g_{n-1}\|_{L^2(Q_{r_n-2})}^2 + 4^n \int_{Q_{r_{n-2}}} \lambda^2 \chi_{g_{n-1} \geq 0} \right),$$

with $p$ as above. Estimating both terms on the right-hand side in a manner similar to the above, we have

$$\|g_{n-1}\|_{L^p(Q_{r_n-1})}^2 \lesssim \Lambda (\Lambda^4 + \|B\|_{L^\infty}^4) \lambda^{-4} 2^{4n}(1 + g^2) U_{n-2}.$$ 

Here, we take $g = 1$. Plugging this into (A.4) and using $U_{n-1} \leq U_{n-2}$ and $U_{n-2} \leq \kappa \leq 1$ gives

$$U_n \lesssim \Lambda (\Lambda^6 + \|B\|_{L^\infty}^6) \lambda^{-6} 2^{6n} U_{n-2}^{-\frac{2}{p}}.$$

Renaming $V_n = U_{2n}$, $\alpha = \frac{3}{2} - \frac{1}{p} > 1$, and

$$\beta = \Lambda (\Lambda^6 + \|B\|_{L^\infty}^6) \lambda^{-6} 2^8,$$

we have $V_n \leq \beta^n V_{n-1}^\alpha$, which, applied iteratively, gives

$$V_n \leq \beta^n V_{n-1}^\alpha \leq \left( \frac{\beta(\alpha - 1)^2 V_0}{\alpha} \right)^n.$$ 

If

$$V_0 = \|g\|_{L^p(Q_{r_n})}^2 \leq \beta^{-\alpha - n} =: \kappa,$$

then $U_{2n} = V_n \to 0$ as $n \to \infty$, and $U_{\infty} = \|(g - \frac{1}{2})_+\|_{L^2(Q_{1/2})}^2 = 0$, so $g \leq \frac{1}{2}$ in $Q_{1/2}$. This implies the constant in the $L^\infty$ estimate is proportional to

$$\kappa^{-1} = \beta^{-\alpha - n} \lesssim (\Lambda^7 + \Lambda \|B\|_{L^\infty}^7) 3(6d+4)(2d+1),$$

so we may take a value (not necessarily optimal) of $P = 21(6d+4)(2d+1)$ in Theorem 2.2.

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