INTERSECTION NUMBERS ON THE COMPACT VARIETY OF RATIONAL RULED SURFACES

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Abstract. We describe a natural action on the Quot scheme, $R_d$ compactifying the space of degree $d$ maps from $\mathbb{P}^1$ to the Grassmannian of lines. We identify the fixed points components for this action and the weights of the normal bundle of these components. We compute the degree of this Quot scheme under the generalized Plücker embedding by applying Atiyah-Bott localization formula.

1. Introduction.

The Quot scheme is a fine moduli space equipped with an universal element. It has been used many times as a smooth compactification of the space of morphisms of a fixed degree from a curve $C$ to a Grassmannian, [2]. It is known that when $C$ is of genus 0, the Quot scheme is irreducible and smooth. Recalling the notation of [4], we denote by $R_d$ the Quot scheme parametrizing quotients of rank 2 and degree $d$ of a trivial vector bundle $\mathcal{O}_{\mathbb{P}^1}^d$, and by $R_0^d$ the open set of morphisms. S.A. Strømme in [6] computes the Betti numbers of $R_d$ and gives a description with generators and relations of its cohomology ring. In particular, he gives a basis for the Picard group $A^1(R_d)$.

Here we compute the degree of the generalized Plücker embedding of the Quot scheme $R_d$. It is called generalized Plücker embedding because in some sense can be considered a generalization of the Plücker embedding given by the hyperplane class of the corresponding Grassmannian. In our case, we are considering the Grassmannian of lines in $\mathbb{P}^3$, which we will denote as $G(2,4)$, but similar computations can be obtained for a general Grassmannian $G(k,n)$ of $k$–planes in $\mathbb{C}^n$. In particular for $n – k = 1$, the Quot scheme is a projective space and therefore intersection theory here is well understood. The method that we use here, involves the geometry of the Quot scheme and the Bott residue formula.

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The Bott residue formula expresses the degree of certain zero-cycles classes on a smooth complete variety with an action of an algebraic torus in terms of local contributions supported on the components of the fixed point set. We describe a natural action on the Quot scheme. We identify the fixed points components for this action and the weights of the normal bundle of these components. We study the equivariant codimension one cohomology group of $R_d$ for this action.

**Notation.** We work over the field of complex numbers $\mathbb{C}$. Let $X$ be a projective homogeneous variety, then $A_dX$ and $A^nX$ can be taken to be the Chow homology and cohomology groups of $X$. We identify $A^nX$ with $A^{n-d}X$ by the Poincaré duality isomorphism. We use cup product $\cup$ for the product in $A^\ast X$. The moduli space $\overline{M}_{0,n}(X,\beta)$ parametrizes marked stable maps from genus 0 curves to $X$ in the cohomology class $\beta \in H^2(X,\mathbb{Z})$. Let $\gamma_i$ be cycles on $X$, if $\sum \text{codim}(\gamma_i) = \text{dim}(\overline{M}_{0,n}(X,d))$, the Gromov-Witten invariant $I_{0,n,d}(\gamma_1,\ldots,\gamma_n)$ is defined as the top degree class:

$$I_{0,n,d}(\gamma_1,\ldots,\gamma_n) = \int_{[\overline{M}_{0,n}(X,d)]^{\text{vir}}} \pi_1^\ast(\gamma_1) \cup \ldots \cup \pi_n^\ast(\gamma_n),$$

where $[\overline{M}_{0,n}(X,d)]^{\text{vir}}$ denotes the virtual fundamental class of $\overline{M}_{0,n}(X,d)$.

2. **The degree of the generalized Plücker embedding and the Vafa-Intriligator formula.**

When we fix the degree, $d$, and the rank, 2, of a locally free sheaf $E$ on $\mathbb{P}^1$, we are fixing its Hilbert polynomial,

$$P(t) = \chi(E(t)) = d + 2t + 2.$$

The moduli, $\text{Quot}_d(\mathbb{P}^1, P(t))$, of quotients with fixed Hilbert polynomial $P(t)$, is a fine moduli space which we will denote as $R_d$. We observe the quotient $\mathcal{O}_{\mathbb{P}^1}^d \to E \to 0$, determines a point $q \in \text{Quot}(\mathbb{P}^1, P(t))$ and a morphism $f_q : \mathbb{P}^1 \to G(2,4)$, by the universal property of the Grassmannian. By definition, there is an universal quotient

$$\mathcal{O}_{\text{Quot} \times \mathbb{P}^1}^d \to \mathcal{E}_{\text{Quot} \times \mathbb{P}^1}.$$

S.A. Strømme in [6] gives a basis for the Picard group $A^1(R_d)$, formed by the divisors:

$$\alpha = c_1(\pi_1^\ast(\mathcal{E} \otimes \pi_2^\ast\mathcal{O}_{\mathbb{P}^1}(d)) - c_1(\pi_1^\ast(\mathcal{E} \otimes \pi_2^\ast\mathcal{O}_{\mathbb{P}^1}(d - 1))),$$

$$\beta = c_1(\pi_1^\ast(\mathcal{E} \otimes \pi_2^\ast\mathcal{O}_{\mathbb{P}^1}(d - 1)),$$
where $E$ is the universal quotient over $R_d \times \mathbb{P}^1$, and $\pi_1, \pi_2$ are the projection maps over the first and second factors respectively. Let $e : R^0_d \times \mathbb{P}^1 \to G(2, 4)$ be the evaluation map and $T_1 \in H_0(G(2, 4), \mathbb{Z})$, $T_a \in H_1(G(2, 4), \mathbb{Z})$ be the classes of an hyperplane and $a$-plane respectively. Let $e : R^0_d \to G(2, 4)$ be the evaluation map and $T_1, T_a \in H^6(G(2, 4), \mathbb{Z})$ be the classes of a hyperplane and $a$-plane respectively. The hyperplane class determines the Plücker embedding of the Grassmannian $G(2, 4)$ as a quadric in $\mathbb{P}^5$ which corresponds to a variety of lines in $\mathbb{P}^3$. The $a$-plane $T_a$ corresponds to lines in $\mathbb{P}^3$ contained in a given plane. The following set of morphisms define Weil divisors on $R^0_d$:

$$D_i := \{ \varphi \in R^0_d | e(t, \varphi) \cap T_a \neq \emptyset \},$$

$$Y := \{ \varphi \in R^0_d | e(t_i, \varphi) \in T_1 \text{ for a fixed } t_i \in \mathbb{P}^1 \}.$$

These divisors extend to divisors on $R_d$.

Let $P_d$ be the degree of $R_d$ by the morphism induced by the polarization given by the divisor $\alpha$, by lemma 3.2 of [5], $P_d$ is the degree of the top codimensional cohomology class given by the autointersection

$$P_d = \int_{[R_d]} Y^{4d+4} \cap [R_d] = \int_{[R_d]} \alpha^{4d+4} \cap [R_d].$$

This intersection number is computed in [7] via Quantum Cohomology. Note that in this case, the intersection is transversal in the Quot scheme compactification, and therefore the intersection number is the same than the given by integrating over the space of stable maps, that is the Gromov-Witten invariant $I_{0, 4d+4, d}(T_1, 4d+4, T_1)$. It can be obtained too by means of the formulas of Vafa and Intriligator, [2]. This does not happen when considering intersection numbers containing $D_i$.

**Vafa and Intriligator’s Formula:** Let $\zeta$ be a primitive $n$th root of $(-1)^k$ and assume that $0 \leq a_i \leq k$ and $a_1 + \ldots + a_N = \dim (\overline{M}_{0, 4d+4}(G(k, n), d))$. Then:

$$I_{0, 4d+4, d}(\gamma_{a_1}, \ldots, \gamma_{a_N}) = (-1)^k (\frac{k}{2}) n^{-k} \sum_{i_1 > \ldots > i_k} \sigma_{a_1} (\zeta^{i_1}) \ldots \sigma_{a_N} (\zeta^{i_k}) \prod_{j=1}^{k} \prod_{i \neq j} (\zeta^{i_j} - \zeta^{i_i})$$

where $\zeta^l = (\zeta^{i_1}, \ldots, \zeta^{i_k})$ and $\sigma_{a_i}$ are the elementary symmetric polynomials in $k$ variables.

In [3] it is proved that the top-codimensional classes involving the divisors $D_i$ have not enumerative meaning on the Quot scheme because there is an excess component of intersection contained in the boundary. Then the intersection is carried out in the Kontsevich compactification of stable maps, $\overline{M}_{0, d}(G(2, 4), d)$. 
The tool we are going to use for computing these degrees, is the Bott residue formula. For this purpose, we consider the equivariant action of a one-dimensional torus \( T = \mathbb{C}^* \) on the variety \( R_d \). First we study the varieties of fixed points under this action and we compute the equivariant Chern classes in the Chow equivariant rings of the normal bundles restricted to the varieties of fixed points and the line bundles corresponding to the divisors we intersect.

### Bott’s residues formula, (see [1]).

Let \( X \) be a smooth, complete variety of dimension \( n \) and let \( E \) be a \( T \)-equivariant vector bundle over \( X \). Then we have

\[
\int_X (p(E) \cap [X]) = \sum_{F \subset R^T} \pi_{F*} \left( \frac{p^T(E)}{c_{d_F}(N_F/X)} \right) \cap [F]_T.
\]

Here \( d_F \) denotes the codimension of the component \( F \) of fixed points in \( X \) and \( p^T(E) \) is the polynomial of degree \( n \) in the line bundles corresponding to the cycles expressing the product in the equivariant Chow ring of \( F \). The numerator will be a product of polynomials of degree 1 corresponding to the line bundles restricted to \( F \). The denominator will be a polynomial with the dimension of the normal bundle as degree.

In our case, we will apply the formula to compute the intersection

\[
\int_{[R_d]} \alpha^{4d+4} \cap [R_d].
\]

### 3. Varieties of fixed points under the \( \mathbb{C}^* \) action

We consider the diagonal action of the one-dimensional torus acting on the variety \( R_d \).

A point in \( R_d \) is given by a quotient: \( 0 \to N \to \mathcal{O}^4 \to E \to 0 \) in \( \mathbb{P}^1 \), where \( \chi(E(t)) = 2t + 2 + d \). Let \( \mathbf{w} = (w_0, w_1, w_2, w_3) \) be a quadruple of integral weights with \( w_0 < w_1 < w_2 < w_3 \). \( \mathbb{C}^* \) acts on each point:

\[
\forall t \in \mathbb{C}^*, \quad \mathbb{C}^* \times \mathbb{C}^4 \to \mathbb{C}^4 \quad t \cdot (x_0, x_1, x_2, x_3) = (t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3)
\]

Let \( 0 \to N \to \mathcal{O}^4_{R_d \times \mathbb{P}^1} \to E \to 0 \) be the universal exact sequence in \( R_d \times \mathbb{P}^1 \). This action induces an isomorphism \( \mathcal{O}^4 \to \mathcal{O}^4 \), such that the weight corresponding to the trivial sheaf \( \mathcal{O}_{R_d} \) is \( w_0 + w_1 + w_2 + w_3 \), since \( \pi_* \Lambda^4 \mathcal{O}^4_{R_d} \cong \mathcal{O}_{R_d} \), where \( \pi : R_d \times \mathbb{P}^1 \to R_d \).

Let \( E \) be of rank 2 with a nonzero torsion of degree \( d \). We suppose,

\[
E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}.
\]
The scheme $\text{Hilb}_{\mathbb{P}^1}^d$ will denote the Hilbert scheme of $d$ points in $\mathbb{P}^1$ which is isomorphic to $\mathbb{P}^d$ and $Z_d \subset \mathbb{P}^d \times \mathbb{P}^1$ will denote the incidence variety. The scheme $\text{Hilb}_{\mathbb{P}^1}^d$ parametrizes points $O_{\mathbb{P}^1} \rightarrow E \rightarrow 0$, with $E$ as in (9).

**Definition 3.1.** Let $\mathcal{P} = (P_i)_{1 \leq i \leq r}$ be a family of polynomials with rational coefficients. $\mathcal{P}$ is said to be a good partition of the polynomial $P$, if $\text{Hilb}_{\mathbb{P}^1}^d \neq \emptyset$ and $\mathcal{P} = \sum_{1 \leq i \leq r} P_i$.

**Examples of good partitions of the Hilbert polynomial of $E$.**

1. $P_1(t) = d$, $P_2(t) = 0$, $P_3(t) = t + 1$, $P_4(t) = t + 1$, is said to be a good partition of the polynomial $P(t) = 2t + 2 + d$.

2. Another good partition for $d$ odd would be:

   \[ P_{d-\frac{1}{2}} \times P_{d+\frac{1}{2}}, \quad P_{d+\frac{1}{2}} \times P_{d-\frac{1}{2}}, \quad P_{d}, \quad P_{\frac{d}{2}} \]

   $O_{\mathbb{P}^1} \rightarrow O_{Z_{d/2}} \oplus O_{Z_{d/2}} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \rightarrow 0$, is a boundary point corresponding to the partition (5).

3.1. **Fixed points in $R_d$ under the $\mathbb{C}^*$-action.**

**Proposition 3.2.** The varieties of fixed points in $R_d$ under the $\mathbb{C}^*$-action are parametrized by,

\[
\begin{align*}
\mathbb{P}^d & \times \mathbb{P}^d \\
\mathbb{P}^{d-1} \times \mathbb{P}^1 & \\
\mathbb{P}^{d-2} \times \mathbb{P}^2 & \\
& \vdots \\
\mathbb{P}^{d+\frac{1}{2}} \times \mathbb{P}^{d-\frac{1}{2}} & \text{if } d \text{ odd} \\
\mathbb{P}^{\frac{d}{2}} \times \mathbb{P}^\frac{d}{2} & \text{if } d \text{ even}.
\end{align*}
\]

There are 12 components of each type.

**Proof.** Following the work of Bifet [3], we see that to study the components of fixed points under the $\mathbb{C}^*$-action, is equivalent to study good partitions for the Hilbert polynomial $P(t) = 2t + 2 + d$. 


(1) Corresponding to the partition,
\[ P_{d,0}(t) = d + 0 + t + 1 + t + 1, \]
we have,
\[ \text{Hilb}_{\mathbb{P}^1}^d \times \text{Hilb}_{\mathbb{P}^1}^0 \times \text{Hilb}_{\mathbb{P}^1}^{t+1} \times \text{Hilb}_{\mathbb{P}^1}^{t+1} \cong \mathbb{P}^d. \]
There are 12 of this kind.

(2) Corresponding to the partition,
\[ P_{b,a}(t) = b + a + t + 1 + t + 1, \quad b \geq a > 0, \quad b + a = d, \]
\[ \text{Hilb}_{\mathbb{P}^1}^b \times \text{Hilb}_{\mathbb{P}^1}^a \times \text{Hilb}_{\mathbb{P}^1}^{t+1} \times \text{Hilb}_{\mathbb{P}^1}^{t+1} \cong \mathbb{P}^b \times \mathbb{P}^a \]
There are \( \frac{d}{2} - 1 \) different components if \( d \) even, and \( \frac{d+1}{2} - 1 \) if \( d \) odd.
These are parametrized by:
\[ \mathbb{P}^{d-1} \times \mathbb{P}^1 \]
\[ \mathbb{P}^{d-2} \times \mathbb{P}^2 \]
\[ \vdots \]
\[ \left\{ \begin{array}{ll}
\mathbb{P}^{\frac{d+1}{2}} \times \mathbb{P}\frac{d-1}{2} & \text{if } d \text{ odd} \\
\mathbb{P}^{\frac{d}{2}} \times \mathbb{P}\frac{d}{2} & \text{if } d \text{ even.}
\end{array} \right. \]
For each one there are also \( \left( \frac{d}{2} \right) \times 2 \) components.
There are \( 12 \cdot \frac{d}{2} = 6d \) fixed point components if \( d \) even, and \( 12 \cdot \frac{d+1}{2} = 6 \cdot (d+1) \) if \( d \) odd.

□

The Euler characteristic of \( R_d \) is given by the formula (see corollary 1.4 of [6]),
\[ \chi(R_d) = \text{rk}_\mathbb{Z} A(R_d) = \binom{4}{2} \binom{d + 4 - 1}{d}. \]
Since the Chow ring of \( \mathbb{P}^d \) is \( \mathbb{Z}[h]/\langle h^d \rangle \) the contribution to the Euler characteristic of the component of fixed points of the first kind is \( d + 1 \) cycles. The contribution of the components of the second kind to the Euler characteristic is \( a \cdot b \), since \( A^*(\mathbb{P}^b \times \mathbb{P}^a) = \mathbb{Z}(H, h)/\langle H^b, h^a \rangle \).
Example: fixed points in $R_3$ under the $\mathbb{C}^*$-action. The Quot scheme $R_3$ parametrizes quotients with Hilbert polynomial, 

$$P_3(t) = 2t + 2 + 3.$$ 

There are two kinds of fixed points varieties in $R_3$ corresponding to the two partitions of the polynomial $P_3(t)$.

1. $P_{3,0}(t) = 3 + 0 + t + 1 + t + 1$

$$Hilb^3_{\mathbb{P}^1} \times Hilb^0_{\mathbb{P}^1} \times Hilb^1_{\mathbb{P}^1} \times Hilb^2_{\mathbb{P}^1} \cong \mathbb{P}^3.$$ 

There are $\binom{4}{2} \times 2 = 12$ of this kind, each one contributes 4 cycles, since the Chow ring of $\mathbb{P}^3$ is $A^*(\mathbb{P}^3) \cong \mathbb{Z}(h)/\langle h^4 \rangle$, here $h$ is the class of a hyperplane in $\mathbb{P}^3$.

2. $P_{2,1}(t) = 2 + 1 + t + 1 + t + 1$

$$Hilb^2_{\mathbb{P}^1} \times Hilb^1_{\mathbb{P}^1} \times Hilb^1_{\mathbb{P}^1} \times Hilb^2_{\mathbb{P}^1} \cong \mathbb{P}^2 \times \mathbb{P}^1.$$ 

There are $\binom{4}{2} \times 2 = 12$ of this kind, each one contributes 6 cycles in $A^*(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z}(H, h)/\langle H^3, h^2 \rangle$, where $H$ is the class of a hyperplane in $\mathbb{P}^2$ and $h$ in $\mathbb{P}^1$.

This number coincides with the Euler characteristic, since:

$$120 = \chi(R_3) = \binom{4}{2} \left( 3 + 2(4 - 2) - 1 \right)$$

There are 24 components of fixed points.

We have seen that the position of the trivial sheaves and the torsion sheaves is important. It determines different components up to isomorphism. It also determines the weights that act.

3.2. The $\mathbb{C}^*$-equivariant Chern classes. We shall associate to the component $Hilb^d_{\mathbb{P}^1} \times Hilb^0_{\mathbb{P}^1} \times Hilb^d_{\mathbb{P}^1} \times Hilb^2_{\mathbb{P}^1} \cong \mathbb{P}^d$ the quadruple $(d, 0, t+1, t+1)$ to simplify notation, and $c^T_1(\alpha|_{(d,0,t+1,t+1)}), c^T_2(\beta|_{(d,0,t+1,t+1)})$ will be the corresponding first $\mathbb{C}^*$-equivariant Chern classes in the equivariant Chow ring $A^*_e(\mathbb{P}^d)$.

Theorem 3.3. The first equivariant Chern classes of the fixed points components are

1. $c^T_1(\alpha|_{(t+1,t+1,d,0)}) = h + w_0 + w_1$, $c^T_1(\beta|_{(t+1,t+1,d,0)}) = d w_0 + d w_1 + d w_2$,

2. $c^T_2(\alpha|_{(t+1,d,t+1,0)}) = h + w_0 + w_2$, $c^T_2(\beta|_{(t+1,d,t+1,0)}) = d w_0 + d w_1 + d w_2$,

3. $c^T_3(\alpha|_{(d,0,t+1,t+1)}) = h + w_0 + w_3$, $c^T_3(\beta|_{(d,0,t+1,t+1)}) = d w_0 + d w_2 + d w_3$, 


Proof. We consider the universal quotient in \( R_d \times \mathbb{P}^1 \) restricted to the fixed point component \((d,0,t+1,t+1)\). It is enough to consider one component of fixed points by the symmetry of the computations.

\[
0 \to \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}(-1,-d) \oplus \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}^4 \oplus 0 \oplus \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1} \to 0.
\]
Let $h$ denote the positive generator of the projective space $\mathbb{P}^d$.

Since $Z_d$ is $\pi_1$-flat, the restriction of the coherent sheaf $B_{d-1}$ to the fixed point component $(d, 0, t + 1, t + 1)$ is well defined by,

$$B_{d-1}|_{(d, 0, t+1, t+1)} = \pi_1^*[(O_{Z_d} \oplus 0 \oplus O_{\mathbb{P}^4 \times \mathbb{P}^1} \oplus O_{\mathbb{P}^4 \times \mathbb{P}^1}) \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1)] =$$

$$\pi_1^*O_{Z_d} \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1) \oplus \pi_1^*\pi_2^*O_{\mathbb{P}^1}(d - 1) \oplus \pi_1^*\pi_2^*O_{\mathbb{P}^1}(d - 1),$$

where

$$\pi_1 \pi_2^*O_{\mathbb{P}^1}(d - 1) = O_{\mathbb{P}^d}^d.$$

We consider the exact sequence defining the sheaf $O_{Z_d}$:

$$0 \rightarrow O_{\mathbb{P}^d \times \mathbb{P}^1}(-1, -d) \rightarrow O_{\mathbb{P}^d \times \mathbb{P}^1} \rightarrow O_{Z_d} \rightarrow 0,$$

and we tensorize with the line bundle $\pi_2^*O_{\mathbb{P}^1}(d - 1)$ and take the long exact sequence associated to the pushforward $\pi_1^*$:

$$0 \rightarrow \pi_1^*(O_{\mathbb{P}^d \times \mathbb{P}^1}(-1, -d) \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1)) \rightarrow \pi_1^*(O \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1)) \rightarrow$$

$$\rightarrow \pi_1^*(O_{Z_d} \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1)) \rightarrow 0,$$

The vanishing of $R^1\pi_1^*(O_{\mathbb{P}^d \times \mathbb{P}^1}(-1, -d))$ implies that

$$\pi_1^*(O_{Z_d} \otimes \pi_2^*O_{\mathbb{P}^1}(d - 1)) = O_{\mathbb{P}^d}^d.$$ Therefore the rank of $B_{d-1}|_{(d, 0, t+1, t+1)}$ is $2d + d = 3d$ and the restriction of $\beta_d$ to the component $(d, 0, t + 1, t + 1)$ is $O_{\mathbb{P}^d}$. Since $\mathcal{O}_{\mathbb{P}^d}^d = O_{\mathbb{P}^d}$ the corresponding weight is $dw_0 + dw_2 + dw_3$.

The first equivariant Chern class $\alpha_d|(d, 0, t+1, t+1)$ is defined by

$$\alpha_d|(d, 0, t+1, t+1) = c_1(B_d|(d, 0, t+1, t+1)) - c_1(B_{d-1}|(d, 0, t+1, t+1)),$$

with

$$B_d|(d, 0, t+1, t+1) = \pi_1^*[O_{Z_d} \otimes O_{\mathbb{P}^4 \times \mathbb{P}^1} \oplus O_{\mathbb{P}^4 \times \mathbb{P}^1}) \otimes \pi_2^*O_{\mathbb{P}^1}(d)]$$

$$= \pi_1^*O_{Z_d} \otimes \pi_2^*O_{\mathbb{P}^1}(d) \oplus \pi_1^*\pi_2^*O_{\mathbb{P}^1}(d) \oplus \pi_1^*\pi_2^*O_{\mathbb{P}^1}(d).$$

We have the following exact sequence

$$0 \rightarrow \pi_1^*(O_{\mathbb{P}^d \times \mathbb{P}^1}(-1, 0)) \rightarrow \pi_1^*(O \otimes \pi_2^*O_{\mathbb{P}^1}(d)) \rightarrow \pi_1^*(O_{Z_d} \otimes \pi_2^*O_{\mathbb{P}^1}(d)) \rightarrow 0.$$
The rank of $B_d|_{(d,0,t+1,t+1)}$ is $2(d+1) + d = 3d + 2$ and the weight $dw_0 + (d+1)w_2 + (d+1)w_3$, therefore the restriction of $\alpha_d$ to $(d,0,t+1,t+1)$ is

$$h + w_2 + w_3$$

We now compute the restrictions of the divisors $\alpha_d$ and $\beta_d$ to the components of fixed points of the second kind. Consider the following incidence variety

$$\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^l \supset Z_b \times Z_a$$

and the restriction of the coherent sheaf $B_{d-1}$ to the fixed point component $(b,a,t+1,t+1)$,

$$B_{d-1}|_{(b,a,t+1,t+1)} = \pi_{12*}[(O_{Z_b} \oplus O_{Z_a} \oplus O_{p^b \times \mathbb{P}^a \times \mathbb{P}^l_1} \oplus O_{p^a \times \mathbb{P}^b \times \mathbb{P}^l_1}) \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1)]$$

where $b + a = d$, $b \geq a > 0$, and

$$\pi_{12*}(O_{p^b \times \mathbb{P}^a \times \mathbb{P}^l_1}(0,-1,-a) \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1)) = \pi_{12*}O_{p^a \times \mathbb{P}^b \times \mathbb{P}^l_1}(0,-1,b-1).$$

Since $R^1\pi_{12*}O_{p^b \times \mathbb{P}^a \times \mathbb{P}^l_1}(0,-1,b-1) = 0$, the following exact sequence stands:

$$0 \to O(0,-1) \otimes O^b \to O^d \to \pi_{12*}(O_{Z_b} \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1)) \to 0 \text{ over } \mathbb{P}^b \times \mathbb{P}^a.$$ 

Let $h$, $H$ denote the positive generators of the projective spaces $\mathbb{P}^a$ and $\mathbb{P}^b$, respectively. The first Chern class of the bundle $\pi_{12*}(O_{Z_b} \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1))$ is $bh$. For computing the first Chern class of the subbundle $\pi_{12*}(O_{Z_b} \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1))$, we see that

$$\pi_{12*}(O_{p^b \times \mathbb{P}^a \times \mathbb{P}^l_1}(0,-1,-b) \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1)) = \pi_{12*}O_{p^a \times \mathbb{P}^b \times \mathbb{P}^l_1}(-1,0,a-1),$$

and

$$R^1\pi_{12*}O_{p^b \times \mathbb{P}^a \times \mathbb{P}^l_1}(-1,0,a-1) = 0,$$ 

therefore

$$0 \to O(-1,0) \otimes O^a \to O^d \to \pi_{12*}(O_{Z_b} \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1)) \to 0 \text{ on } \mathbb{P}^b \times \mathbb{P}^a.$$ 

By symmetry with the previous case, the first Chern class of the bundle $\pi_{12*}(O_{Z_b} \otimes \pi_{12*}O_{\mathbb{P}^l}(d-1))$ is $aH$. It follows that

$$c_1(B_{d-1}|_{(b,a,t+1,t+1)}) = aH + bh,$$ 

and its weight is $bw_0 + aw_1 + dw_2 + dw_3$. Finally, the restriction of $\beta_d$ to the component $(b,a,t+1,t+1)$, isomorphic to $\mathbb{P}^b \times \mathbb{P}^a$, is

$$aH + bh + bw_0 + aw_1 + dw_2 + dw_3$$

The restriction of $B_d$ to $(b,a,t+1,t+1)$ is given by

$$B_d|_{(b,a,t+1,t+1)} = \pi_{12*}[(O_{Z_b} \oplus O_{Z_a} \oplus O \oplus O) \otimes \pi_{12*}O_{\mathbb{P}^l}(d)].$$
We consider again the universal exact sequence bundle for the components of the first kind.

The Equivariant Chern classes of the Normal bundle in the equivariant Chow ring of the fixed points component are, for the first kind of components:

\[ \pi_{12*}(O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, -1, -a)) \otimes \pi_{3*}O_{\mathbb{P}^1}(d)) = \pi_{12*}O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, -1, b) \] and

\[ R^1\pi_{12*}O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, -1, b) = 0. \]

Therefore we have the exact sequence:

\[ 0 \to O(0, -1) \otimes O^{b+1} \to O^{d+1} \to \pi_{12*}(O_{Z_a} \otimes \pi_{3*}O_{\mathbb{P}^1}(d)) \to 0 \text{ over } \mathbb{P}^b \times \mathbb{P}^a. \]

Again from the fact that

\[ \pi_{12*}(O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 0, -b) \otimes \pi_{3*}O_{\mathbb{P}^1}(d)) = \pi_{12*}O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 0, a), \]

and that \( R^1\pi_{12*}O_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 0, a) = 0, \)

it follows that there is an exact sequence,

\[ 0 \to O(-1, 0) \otimes O^{a+1} \to O^{d+1} \to \pi_{12*}(O_{Z_b} \otimes \pi_{3*}O_{\mathbb{P}^1}(d)) \to 0 \text{ over } \mathbb{P}^b \times \mathbb{P}^a. \]

The first Chern class of \( B_d|_{(b, a, t + 1, t + 1)} \) is \((a + 1)H + (b + 1)h\) and its weight \( bw_0 + aw_1 + (d + 1)w_2 + (d + 1)w_3, \) therefore the restriction of \( \alpha_d \) to \((b, a, t + 1, t + 1)\) is,

\[ H + h + w_2 + w_3 \]

\[ \Box \]

4. The normal bundle.

**Theorem 4.1.** The Equivariant Chern classes of the Normal bundle in the equivariant Chow ring of the fixed points component are, for the first kind of components:

\[ c^T_{3d+4}(N_{\mathbb{P}^4/R_d}) = (h + (w_2 - w_0))^{d+1} \cdot (h + (w_3 - w_0))^{d+1} \cdot (w_0 - w_1) \]

\[ \cdot (h - (w_0 - w_1))^{d-1}(w_2 - w_1) \cdot (w_3 - w_1), \]

and for the second kind of components:

\[ c^T_{3d+4}(N_{\mathbb{P}^b \times \mathbb{P}^a/R_d}) = (H + w_2 - w_0)^{b+1} \cdot (H + w_3 - w_0)^{b+1} \]

\[ \cdot (h + w_2 - w_0)^{a+1} \cdot (-H + h + w_0 - w_1)^{b-a-1} \cdot (h + w_0 - w_1)^{a+1} \]

\[ \cdot (H + w_1 - w_0)^{b+1} \cdot (H - h + w_1 - w_0)^{a-b-1}, \]

for \( b - a \geq 0. \)

**Proof.** We consider the normal bundle of the fixed points component in \( R_d. \)

We need to compute the weight of the normal bundle for each component of fixed points, and its equivariant Chern class. We first study the normal bundle for the components of the first kind.

We consider again the universal exact sequence
$0 \to \mathcal{N} \to \mathcal{O}_{R_d \times \mathbb{P}^1}^4 \to \mathcal{E} \to 0$.

The tangent space to the variety $R_d$ is $\mathcal{T}_{R_d} \cong \pi_* \text{Hom}(\mathcal{N}, \mathcal{E})$ (see §7.7.1 of [6]), where $\pi : R_d \times \mathbb{P}^1 \to R_d$. If we restrict it to a component of fixed points of the first kind:

$$0 \to \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_4 \to \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}^4 \to \mathcal{E}_{\mathbb{P}^d \times \mathbb{P}^1} \to 0,$$

where $\bigoplus_{i=1}^4 \mathcal{Y}_i$ is the kernel of the quotient map $\mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}^4 \to \mathcal{E}_{\mathbb{P}^d \times \mathbb{P}^1} \to 0$. The restriction of the normal bundle to the component of fixed points yields

$$0 \to \mathcal{T}_{\mathbb{P}^d} \to \mathcal{T}_R \to \mathcal{N}_{\mathbb{P}^d} \to 0,$$

$\mathcal{N}_{\mathbb{P}^d/R_d} \cong \pi_* \Theta_{\mathbb{P}^d}(\mathcal{Y}_1, \mathcal{O}_{\mathbb{P}^1}/\mathcal{Y}_j)$. Let us suppose that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O}$, it is enough to consider one component of fixed points by the symmetry of the computations, therefore,

$$0 \to \mathcal{O}(-1, -d) \oplus \mathcal{O} \oplus 0 \oplus 0 \to \mathcal{O}^4 \to \mathcal{O}_{\mathbb{P}^d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O} \to 0,$$

and $\mathcal{T}_{\mathbb{P}^d} \cong \pi_* \text{Hom}(\mathcal{O}(-1, -d) \oplus \mathcal{O}, \mathcal{O}_{\mathbb{P}^d} \oplus 0 \oplus \mathcal{O} \oplus \mathcal{O})$.

Definitely, what we have is

$$(12) \quad \mathcal{N}_{\mathbb{P}^d/R_d} \cong \pi_* \mathcal{O}_{\mathbb{P}^d} \otimes \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}(1, d) \oplus (\pi_* \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1}(1, d))^2 \oplus (\pi_* \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1})^2.$$

Since $\pi_* \mathcal{O}_{\mathbb{P}^d}$ is a bundle of rank $d$, the rank of the normal bundle is $3d + 4$. Therefore we need to compute, $c_{3d+4}(\mathcal{N}_{\mathbb{P}^d/R_d})$ in the equivariant Chow ring of $\mathbb{P}^d$, (see §1.8 of [1]),

$$A^*_S(\mathbb{P}^d) = \mathbb{Z}(h, t) / \prod_{i=0}^d (h + w_i t).$$

It will be a polynomial of degree $3d + 4$ in the variable $h$. The fiber of the normal bundle is isomorphic to

$$\text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}).$$

and the weights are by [3]:

$$\text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}) \oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^d}).$$
INTERSECTION NUMBERS ON THE COMPACT VARIETY OF RATIONAL RULED SURFACES

Given a $T$–equivariant vector bundle $E 	o X$ we get a canonical decomposition $E = \bigoplus_{\chi \in T} E^\chi$ where $E^\chi$ denotes the eigensubbundle consisting of vectors in $E$ on which $T$ acts with the character $\chi$.

The $i$–esima equivariant Chern class of a $T$–equivariant bundle of rank $r$ over $\mathbb{P}^d$ is such that the action of $T$ over each fiber is given by the character. It is by §2.2.1, [1]:

$$c^T_i = \sum_{j=0}^i \binom{r-j}{i-j} c_j(E_{\chi_{i-j}}) \chi_{i-j}.$$  

This formula relates the equivariant Chern classes of a bundle with the usual Chern classes. Thus, the problem is reduced to compute the usual Chern classes of the normal bundle and by using the Whitney formula,

$$c^T_{3d+4} (\mathcal{N}_{\mathbb{P}^d/R_d}) = c^T_2 (\pi_* \mathcal{O}_{Z_d}) \cdot c^T_{d+1} (\pi_* \mathcal{O}(1, d)) \cdot c^T_{d+1} (\pi_* \mathcal{O}(1, d)) \cdot c^T_1 (\pi_* \mathcal{O}) \cdot c^T_1 (\pi_* \mathcal{O}).$$

For computing the Chern classes of the equivariant subbundle $\mathcal{O}_d$ is required a little more work. For this purpose, we consider the exact sequence

$$0 \to \mathcal{O}_{d}(-1, -d) \to \mathcal{O} \to \mathcal{O}_d \to 0 \quad \text{on } \mathbb{P}^d \times \mathbb{P}^1,$$

$$0 \to \pi_* \mathcal{O}_{d}(-1, -d) \to \pi_* \mathcal{O} \to \pi_* \mathcal{O}_d \to R^1 \pi_* \mathcal{O}_{d}(-1, -d) \to 0.$$  

It follows $\pi_* \mathcal{O}_{d}(-1, -d) = 0$, since the fibers of this bundle are isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^1}(-d))$ which are 0-dimensional vectorial spaces.

$$R^1 \pi_* \mathcal{O}(-1, -d) = R^1 \pi_* (\pi^* \mathcal{O}_{1}(-d) \otimes \pi^* \mathcal{O}_{d}(-1)) = R^1 \pi_* \pi^* \mathcal{O}_{d}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$$  

and $R^1 \pi_* \pi^* \mathcal{O}_{d}(-1) \cong \mathcal{O}_{d-1}^d$ by Serre duality. Definitely, we have

$$0 \to \mathcal{O} \to \pi_* \mathcal{O}_d \to \mathcal{O}_d^d \otimes \pi^* \mathcal{O}_{d}(-1) \to 0 \quad \text{on } \mathbb{P}^d$$  

and therefore, by applying the Whitney formula, the total Chern class of the bundle is

$$c_i (\pi_* \mathcal{O}_d) = \prod_{i=0}^{d-1} (1 - t)$$

Now we can compute the Chern equivariant class of each equivariant subbundle of $E$. Let $E_{w_1}$ denote the eigensubbundle consisting of vectors in $\mathcal{N}_{\mathbb{P}^d/R_d}$ on which $C^*$ acts with weight $w_1 - w_j$.

$$c^T_{d} (\pi_* \mathcal{O}_d) = (w_0 - w_1) (h - (w_0 - w_1))^{d-1}$$  

$$c^T_{d+1} (E_{w_2 - w_0}) = (h + w_2 - w_0)^{d+1},$$  

$$c^T_{d+1} (E_{w_3 - w_0}) = (h + w_3 - w_0)^{d+1},$$  

$$c^T_{1} (E_{w_2 - w_1}) = (w_2 - w_1),$$  

$$c^T_{1} (E_{w_3 - w_1}) = (w_3 - w_1).$$
We have,
\[ c_{3d+4}^T(\mathcal{N}_{\mathbb{P}^d/R_d}) = (h + (w_2 - w_0))^{d+1} \cdot (h + (w_3 - w_0))^{d+1} \cdot (w_0 - w_1) \cdot (a - (w_0 - w_1))^{d-1} \cdot (w_2 - w_1) \cdot (w_3 - w_1). \]

**We now study** \( c_d(\mathcal{N}_{F/R_d}) \), **for the components of fixed points of the second kind.**

We consider the varieties of fixed points isomorphic to \( \mathbb{P}^b \times \mathbb{P}^a \), with \( b + a = d, \ b \geq a > 0 \).

Let be the component of fixed points defined by the universal quotient in \( \mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1 \):
\[ 0 \to \mathcal{O}(-1, 0, -b) \oplus \mathcal{O}(0, -1, -a) \oplus 0 \to \mathcal{O} \to \mathcal{O}_{\mathcal{Z}_b} \oplus \mathcal{O}_{\mathcal{Z}_a} \oplus \mathcal{O} \oplus \mathcal{O} \to 0 \]
\[ \mathcal{N}_{\mathbb{P}^b \times \mathbb{P}^a/R_d} = \pi_{12*} \Hom(\mathcal{O}(-1, 0, -b), \mathcal{O}_{\mathcal{Z}_a}) \oplus \pi_{12*} \Hom(\mathcal{O}(0, -1, -a), \mathcal{O}_{\mathcal{Z}_b}) \oplus \pi_{12*} \Hom(\mathcal{O}(0, -1, -a), \mathcal{O})^2 \oplus \pi_{12*} \Hom(\mathcal{O}(0, -1, -a), \mathcal{O})^2. \]

We denote \( E_{w_2-w_0} \) the subbundle \( \pi_{12*}((\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b)) \) on which \( \mathbb{C}^* \) acts with weight \( w_2 - w_0 \). We compute its equivariant Chern classes:
\[ c_{b+1}^T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b))) = (H + (w_2 - w_0))^{b+1}, \]
\[ c_{b+1}^T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b))) = (H + (w_3 - w_0))^{b+1}, \]
\[ c_{a+1}^T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a))) = (h + (w_2 - w_0))^{a+1}, \]
\[ c_{a+1}^T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a))) = (h + (w_3 - w_1))^{a+1}. \]

\( \mathbb{C}^* \) acts on \( \pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}(0, 1, a)) \) with weight \( w_0 - w_1 \). For computing its equivariant Chern class \( c_f^T(E_{w_0-w_1}) \) we consider the exact sequence,
\[ 0 \to \mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 0, -b) \otimes \mathcal{O}(0, 1, a) \to \mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a) \to \mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}(0, 1, a) \to 0 \]
(14) \[ 0 \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 1, -b + a)) \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a)) \to \pi_{12*}(\mathcal{O}_{\mathcal{Z}_b} \otimes \mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a)) \to R^1 \pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(-1, 1, -b + a)) \to 0 \]
\[ \pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, 1, a)) \cong \mathcal{O}(0, 1)^{a+1} \]

We suppose \( b > a \), thus we have
Therefore, by applying Whitney formula to (16), we get that
\[(18)\]
In the case \(a = b\), we have that
\[(15)\]
In either case, by applying Whitney formula to (14), we see that
\[c^T(E_{w_0-w_1}) = c^T_0(\pi_{12*}(\mathcal{O}_{Z_a} \otimes \mathcal{O}(0, 1, a))) = (-H + h + w_0 - w_1)^{b-a-1} \cdot (h + w_0 - w_1)^{a+1}.\]
\(\mathbb{C}^*\) acts on \(\pi_{12*}(\mathcal{O}_{Z_a} \otimes \mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b))\) with weight \(w_1 - w_0\). We now compute \(c^T_0(\pi_{12*}(\mathcal{O}_{Z_a} \otimes \mathcal{O}(1, 0, b)))\). We consider also the case in which \(a = b\). Consider the exact sequence
\[0 \to \mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(0, -1, -a) \otimes \mathcal{O}(1, 0, b) \to \mathcal{O}(1, 0, b) \to \mathcal{O}_{Z_a} \otimes \mathcal{O}(1, 0, b) \to 0,
\]
\[0 \to \pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a)) \to \pi_{12*}(\mathcal{O}(1, 0, b)) \to \pi_{12*}(\mathcal{O}_{Z_a} \otimes \mathcal{O}(1, 0, b)) \to 0\]
\[(16)\]
We observe in this case
\[R^1\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a)) \cong \mathcal{O}(1, -1) \otimes \pi_{12*}(\mathcal{O}_{\mathbb{P}^1}(a - b - 2)) = 0,
\]
\[\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a)) \cong \mathcal{O}(1, -1) \otimes \mathcal{O}^{b-a+1},
\]
\[\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b)) \cong \mathcal{O}(1, 0) \otimes \mathcal{O}^{b+1},
\]
\[\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a))\] is a bundle with total Chern class:
\[c_T(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a))) = ((H - h) t + 1)^{b-a+1},\]
\[c_T^1(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, -1, -b-a))) = ((H - h) t + w_1 - w_0)^{b-a+1},\]
\[c_T^1(\pi_{12*}(\mathcal{O}_{\mathbb{P}^b \times \mathbb{P}^a \times \mathbb{P}^1}(1, 0, b))) = (H t + w_1 - w_0)^{b+1},\]
By applying Whitney formula to (16), we get that
\[c_{a}(E_{w_1-w_0}) = \frac{(H + w_1 - w_0)^{b+1}}{(H - h + w_1 - w_0)^{b-a+1}},\]
therefore,
\[c_{3d+4}^T(N_{\mathbb{P}^b \times \mathbb{P}^a}/R_d) = c_{b+1}^T(E_{w_2-w_0}) \cdot c_{b+1}^T(E_{w_3-w_0}) \cdot c_{a+1}^T(E_{w_2-w_0}) \cdot c_{a+1}^T(E_{w_3-w_1}) \cdot c_{a}^T(E_{w_1-w_0}),\]
that is, for $b - a \geq 1$,
\[
c_T^{3d+4}(N_{\mathbb{P}^a \times \mathbb{P}^a}/R_d) = (H + w_2 - w_0)^{b+1} \cdot (H + w_3 - w_0)^{b+1} \cdot (h + w_2 - w_0)^{a+1},
\]
\[
(\mathbf{H}+h+w_0-w_1)^{b-a-1} \cdot (h+w_0-w_1)^{a+1} \cdot (H+w_1-w_0)^{b+1} \cdot (H-h+w_1-w_0)^{a-b-1},
\]
and for $a = b$,
\[
c_T^{3d+4}(N_{\mathbb{P}^a \times \mathbb{P}^a}/R_d) = (H+w_2-w_0)^{a+1} \cdot (H+w_3-w_0)^{a+1} \cdot (h+w_2-w_0)^{a+1} \cdot (h+w_0-w_1)^{a+1} \cdot (H+h+w_1-w_0)^{-1}.
\]
\[
□
\]

5. Appendix A: Calculation of Plücker degree of $R_3$.

We want to compute the degree of $R_3$ by the morphism induced by the divisor $\alpha$, that is, the generalized Plücker embedding $\mathbb{P}^3$. The intersection we compute, is
\[
P_3 = \int_{R_3} (\alpha^{\mathbf{16}} \cap [R_3]).
\]
We apply Bott residue formula. We have 24 summands, one for each component of fixed points. We know what the denominator is (3), and the restrictions of $\alpha$ to each subvariety of fixed points. The 24 summands corresponding to the 24 components of fixed points are:

1. $\mathbb{Hilb}_{p_2}^0 \times \mathbb{Hilb}_{p_1}^3 \times \mathbb{Hilb}_{p_1}^{l+1} \times \mathbb{Hilb}_{p_1}^{l+1}$
\[
\frac{(h + w_2 + w_3)^{16}}{(h + w_2 - w_1)^4(h + w_3 - w_1)^4(w_1 - w_0)(h - w_1 + w_0)^2(w_2 - w_0)(w_3 - w_0)}
\]

2. $\mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1}$
\[
\frac{(h + w_0 + w_2)^{16}}{(h + w_0 - w_1)^4(h + w_2 - w_1)^4(w_1 - w_3)(h - w_1 + w_3)^2(w_2 - w_3)(w_0 - w_3)}
\]

3. $\mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1}$
\[
\frac{(h + w_0 + w_3)^{16}}{(h + w_0 - w_1)^4(h + w_3 - w_1)^4(w_1 - w_0)(h - w_1 + w_2)^2(w_0 - w_2)(w_3 - w_2)}
\]

4. $\mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1} \times \mathbb{Hilb}_{p_2}^{l+1}$
\[
\frac{(h + w_1 + w_3)^{16}}{(h + w_1 - w_2)^4(h + w_3 - w_2)^4(w_2 - w_0)(h - w_2 + w_0)^2(w_1 - w_0)(w_3 - w_0)}
\]
(5) \[ \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{0} \times \text{Hilb}_{p_1}^{0} \]
\[(h + w_0 + w_1)^{16} \]

(6) \[ \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{0} \times \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_0 + w_3)^{16} \]

(7) \[ \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_1 + w_3)^{16} \]

(8) \[ \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_2 + w_3)^{16} \]

(9) \[ \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_1 + w_3)^{16} \]

(10) \[ \text{Hilb}_{p_1}^{0} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{3} \]
\[(h + w_1 + w_2)^{16} \]

(11) \[ \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{0} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{3} \]
\[(h + w_0 + w_3)^{16} \]

(12) \[ \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{3} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_0 + w_1)^{16} \]

(13) \[ \text{Hilb}_{p_1}^{2} \times \text{Hilb}_{p_1}^{2} \times \text{Hilb}_{p_1}^{t+1} \times \text{Hilb}_{p_1}^{t+1} \]
\[(h + w_1 - w_0)^{16} \]

\[(h + w_0)^{2}(H + 2h + w_0 - w_1)(h + w_2 - w_0)^2(h + w_3 - w_0)^2(H + w_3 - w_1)^{2}(H + w_2 - w_1)^{2} \]
\begin{align}
\text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \times \text{Hilb}^2_{p_1} \times \text{Hilb}^{t+1}_{p_1} \\
(H + h + w_1 + w_3)^{16} \\
(h + w_2 - w_0)^2(H + 2h + w_0 - w_2)(h + w_1 - w_0)^2(h + w_3 - w_0)^2(H + w_1 - w_2)^3(H + w_3 - w_2)^3 \\
\text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \times \text{Hilb}^2_{p_1} \times \text{Hilb}^{t+1}_{p_1} \\
(H + h + w_1 + w_2)^{16} \\
(h + w_3 - w_0)^2(H + 2h + w_0 - w_3)(h + w_2 - w_0)^2(h + w_1 - w_0)^2(H + w_2 - w_3)^3(H + w_1 - w_0)^3 \\
\text{Hilb}^{t+1+1}_{p_1} \times \text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \times \text{Hilb}^2_{p_1} \\
(H + h + w_0 + w_2)^{16} \\
(h + w_3 - w_1)^2(H + 2h + w_1 - w_3)(h + w_2 - w_1)^2(h + w_3 - w_1)^2(H + w_0 - w_3)^3(H + w_2 - w_3)^3 \\
\text{Hilb}^2_{p_2} \times \text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \times \text{Hilb}^{t+1+1}_{p_1} \\
(H + h + w_2 + w_3)^{16} \\
(h + w_4 - w_1)^2(H + 2h + w_1 - w_4)(h + w_2 - w_1)^2(h + w_3 - w_1)^2(H + w_0 - w_4)^3(H + w_3 - w_1)^3 \\
\text{Hilb}^{t+1+1}_{p_1} \times \text{Hilb}^2_{p_1} \times \text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \\
(H + h + w_0 + w_3)^{16} \\
(h + w_0 - w_2)^2(H + 2h + w_2 - w_0)(h + w_0 - w_2)^2(h + w_3 - w_2)^2(H + w_0 - w_1)^3(H + w_3 - w_1)^3 \\
\text{Hilb}^{t+1+1}_{p_1} \times \text{Hilb}^2_{p_1} \times \text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \\
(H + h + w_0 + w_3)^{16} \\
(h + w_0 - w_3)^2(H + 2h + w_2 - w_0)(h + w_0 - w_3)^2(h + w_3 - w_3)^2(H + w_1 - w_0)^3(H + w_3 - w_0)^3 \\
\text{Hilb}^{t+1+1}_{p_1} \times \text{Hilb}^2_{p_1} \times \text{Hilb}^1_{p_1} \times \text{Hilb}^{t+1}_{p_1} \\
(H + h + w_0 + w_1)^{16} \\
(h + w_2 - w_1)^2(H + 2h + w_1 - w_2)(h + w_0 - w_1)^2(h + w_1 - w_3)^2(H + w_0 - w_2)^3(H + w_1 - w_2)^3
\end{align}
\[ (23) \]
\[ \text{Hilb}_{\mathbb{P}^2}^{l+1} \times \text{Hilb}_{\mathbb{P}^1}^2 \times \text{Hilb}_{\mathbb{P}^2}^{l+1} \times \text{Hilb}_{\mathbb{P}^1}^1 \]
\[ (H + h + w_0 + w_2)_{16} (h + w_1 - w_3)(H + 2h + w_3 - w_1)(h + w_0 - w_3)2(h + w_2 - w_3)^2(H + w_0 - w_1)^3(H + w_2 - w_3)^3 \]

\[ (24) \]
\[ \text{Hilb}_{\mathbb{P}^2}^{l+1} \times \text{Hilb}_{\mathbb{P}^1}^2 \times \text{Hilb}_{\mathbb{P}^2}^{l+1} \times \text{Hilb}_{\mathbb{P}^1}^1 \]
\[ (H + h + w_1 + w_2)_{16} (h + w_0 - w_3)(H + 2h + w_3 - w_0)(h + w_2 - w_3)^2(h + w_1 - w_3)^2(H + w_1 - w_0)^3(H + w_2 - w_0)^3 \]

Once we have all the summands, we take the direct image by the morphism \( \pi_1 : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3 \) and \( \pi_{12} : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1 \) for the first kind of components and for the second kind of components respectively. The only terms surviving in the first case are those in \( h^3 \), and in the second case the terms in \( H^2 h \). We have used Maple program to make the computations. This degree is 128. This result coincides with the one obtained by means of the Vafa-Intriligator formula, [2] and indeed it follows easily from Vafa-Intriligator formula that the degree \( P_d \) coincides with \( 2^{d+1} \).

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