Coupling Josephson qubits via a current-biased information bus

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Josephson qubits without direct interaction can be effectively coupled by sequentially connecting them to an information bus: a current-biased large Josephson junction treated as an oscillator with adjustable frequency. The coupling between any qubit and the bus can be controlled by modulating the magnetic flux applied to that qubit. This tunable and selective coupling provides two-qubit entangled states for implementing elementary quantum logic operations, and for experimentally testing Bell’s inequality.

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PACS. 03.65.Ud - Entanglement and quantum nonlocality (e.g. EPR paradox, Bell’s inequalities, GHZ states, etc.).

Superconducting circuits with Josephson junctions offer one of the most promising candidates for realizing quantum computation.1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17

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The large CBJJ works in the phase regime and describes restricted to the Hilbert space spanned by the two lowest states of this data bus: $|0\rangle$ and $|1\rangle$.

The eigenenergy $\hbar \omega_{LC}$ of the bus used in (2) is much higher than that of the qubits. Therefore, adiabatically eliminating such a bus yields a direct interbit coupling. However, the energy scale of our proposed data bus (i.e., the CBJJ oscillator) is $\omega_s/2\pi \sim 10$ GHz (2), which is of the same order of the Josephson energy (e.g., $E_J/\hbar \sim 13$ GHz (2)). Therefore, the quanta in the present bus can be excited or de-excited when the qubit is operated. The Hamiltonian of (1) clearly shows that the coupling between the chosen $k$th qubit and the bus can be turned on and off (2), when the threaded flux $\Phi_k$ differs from or equals to $(l' + 1/2)\Phi_0$, $l' = 0, 1, 2, \ldots$. For simplicity, hereafter we let $l, l' = 0$. Two qubits can be independently coupled by independently interacting with the bus sequentially when exciting/de-exciting the vibrational qatha of the bus. Under the usual rotating-wave approximation, the dynamics for such a coupling mechanism can be described by the following effective Hamiltonian

$$
\hat{H}_{k_b} = \hat{H}_k + \hat{H}_b + i\lambda_k \left[ \hat{\sigma}^{(k)}_c \hat{a} - \hat{\sigma}^{(k)}_c \hat{a}^\dagger \right],
$$

$$
\hat{H}_k = \frac{E_J^{(k)}}{2} \hat{\sigma}^{(k)}_z - \delta E_{C_k}^{(k)} \hat{\sigma}^{(k)}_c, \quad \hat{H}_b = \hbar \omega_b \left( \hat{n} + \frac{1}{2} \right),
$$

with $\hat{\sigma}_c = 2\pi \hat{\phi}_b / \Phi_0$ is the operator of charges on the CBJJ and $[\hat{\phi}_b, \hat{\phi}_b] = i\hbar$. $C_j^{(k)} = C_j^{(k)} + \delta E_{C_k}^{(k)}$, $E_J^{(k)} = 2e_J^{(k)} \cos(\pi \Phi_k / \Phi_0), C_k = C_k^{(g)} + \delta E_{C_k}^{(k)}$, and $C_b = C_b + \sum_{k=1} C_j^{(k)} / C_k$. $\Phi_0 = h/(2e)$ and $n_k$ are the flux quantum and the excess number of Cooper pairs in the superconducting box of the $k$th qubit, respectively. When the applied gate voltage $V_k$ is set near the degeneracy points ($n_g^{(k)} = (l + 1/2), l = 0, 1, 2, \ldots$), then only the two lowest-energy charge states, $|n_k = 0\rangle = |\uparrow_k\rangle$ and $|n_k = 1\rangle = |\downarrow_k\rangle$, play a role. The large CBJJ works in the phase regime and describes the motion of a “particle” with mass $m = C_b \Phi_0 / 2\pi^2$ in a potential $U(\phi_b) = -E_b \cos(\phi_b + I_b \phi_b / I_c)$ with $I_c = 2\pi E_b / \Phi_0$. For $I_b < I_c$, there exist a series of minima of $U(\phi_b)$. Near these points, $U(\phi_b)$ approximates a harmonic oscillator potential with characteristic frequency $\omega_b = \sqrt{8E_b^{(g)} E_b / \hbar^2 \left[ 1 - (I_b / I_c)^2 \right]^{1/4}}, E_b^{(g)} = e^2 / (2C_b)$. The approximate number of metastable quantum bound states (2) is $N_s = 2^{3/4} \sqrt{E_b} / E_b^{(g)} (1 - I_b / I_c)^{5/4}$. For a low bias current, the dynamics of the CBJJ can be safely restricted to the Hilbert space spanned by the two lowest states of this data bus: $|0\rangle$ and $|1\rangle$. 

Quantum gates. — The physical characteristic (e.g., the eigenfrequency) of the bus in the present circuit does not need to be changed, once it is set up beforehand. It still undergoes a free evolution $U_0(t)$ ruled by a non-zero $\hat{H}_b$ during the operational delay, i.e., the time interval when the qubits do not evolve because their Hamiltonians are temporarly set to zero (when $\Phi_k = \Phi_0 / 2, V_k = e / C_k^{(k)})$. 

FIG. 1: A pair of SQUID-based charge qubits, located on the left of the dashed line, coupled to a large CBJJ on the right, which acts as an information bus. The circuit is divided into two parts, the qubits and the bus. The dashed line only indicates a separation between these. The controllable gate voltage $V_k (k = 1, 2)$ and external flux $\Phi_k$ are used to manipulate the qubits and their interactions with the bus. The bus current remains fixed during the operations.
Before and after the desired operations, the bus should remain in its ground state \(|0_b\rangle\). In principle, the time-evolutions listed in Table I are sufficient to implement any desired operation for manipulating the quantum information stored in the present circuit. In fact, any single-qubit rotation, including the typical Hadamard gate: \(H_g = [\hat{\sigma}_z^{(k)} + \hat{\sigma}_x^{(k)} + \hat{\sigma}_z^{(k)}]/\sqrt{2}\), on the kth qubit can be easily realized by selectively using \(\hat{U}_1^{(k)}(t)\) and \(\hat{U}_3^{(k)}(t)\). Any single-qubit operation is not influenced by the free evolution of the bus during the time delay between successive operations.

In order to realize two-qubit gates, we must be able to couple distant qubits via their sequential interactions to the bus. We set the bias current \(I_b\) such that \(\hbar\omega_b = 2\epsilon^{(1)}\) and then perform a three-step process. First, we couple the first (control) qubit to the bus by switching off its applied flux \(\Phi_1\) and produce the evolution \(\hat{U}_2^{(1)}(t_1)\). After a duration \(t_1\) determined by \(\sin(2\zeta t_1)/h = -1\), the control qubit is decoupled from the bus. This process implements the evolutions: \(|0_0\rangle|0_1\rangle \rightarrow |0_0\rangle|0_1\rangle\) and \(|0_0\rangle|1_1\rangle \rightarrow e^{-i\omega_D t_1}|1_0\rangle|0_1\rangle\).

Next, we let the second (target) qubit work at a non-degenerate point \((V_2 \neq e/C_g^{(2)})\) and couple it to the bus by switching off its applied flux \(\Phi_2\). After a duration \(t_2\) determined by
\[
\sin\left(\frac{\zeta_2 t_2}{h^2 \Delta_2}\right) = \cos\left(\frac{\zeta_2 t_2}{2h} + \frac{\zeta_2^2 t_2}{2h^2 \Delta_2}\right) = 1,
\]
the target qubit is backed to its degenerate point \((V_2 = e/C_g^{(2)})\) and decoupled from the bus. This sequence of operations generate the evolutions:
\[
\begin{align*}
|0_0\rangle|0_2\rangle & \rightarrow e^{-i\xi}|0_0\rangle|0_2\rangle, |0_0\rangle|1_2\rangle \rightarrow e^{-i\xi}|0_0\rangle|1_2\rangle, \\
|1_0\rangle|0_2\rangle & \rightarrow i e^{-i(\xi + \theta_2)} (\cos \eta_2|1_0\rangle|0_2\rangle + \sin \eta_2|1_0\rangle|1_2\rangle), \\
|1_0\rangle|1_2\rangle & \rightarrow i e^{-i(\xi + \theta_2)} (\sin \eta_2|1_0\rangle|0_2\rangle - \cos \eta_2|1_0\rangle|1_2\rangle),
\end{align*}
\]
with \(\xi = \omega_b (t_1 + t_2 + \tau_3)/2 + \zeta_2^2 t_2/(2h^2 \Delta_2)\), and \(t_g = \sum_{s=1}^3 t_s + \sum_{s=1}^2 t_s\). Finally, we couple again the control qubit to the bus and perform the evolution \(\hat{U}_3^{(2)}(t_3)\) with \(\sin(2\zeta t_3/h) = 1\), yielding evolutions: \(|0_0\rangle|0_1\rangle \rightarrow |0_0\rangle|0_1\rangle\) and \(|1_0\rangle|0_1\rangle \rightarrow e^{-i\omega_D t_3}|1_0\rangle|0_1\rangle\). In practice, the free evolutions \(\hat{U}_0^{(1)}(\tau_1)\) and \(\hat{U}_0^{(2)}(\tau_2)\) exist during the time delays between the first (second) and second (third) pulses. If the delays are further set such that the total duration \(t_g\) satisfies the condition \(\sin \omega_b t_g = 1\), then the above three-step process with two delays yields a quantum operation \(U(t_g) = \hat{U}_3^{(2)}(t_3)\hat{U}_0^{(2)}(t_2)\hat{U}_0^{(1)}(t_1)\).

This gate can produce entanglement between qubits and also realize any quantum computation, accompanied by single-qubit rotations.

**Testing Bell’s inequality.**—Entanglement is a key ingredient for computational speedup in quantum computation. Historically, Bell’s inequalities were seen as an entanglement test: its violation implies that entanglement must exist. For a two-qubit entangled state \(|\psi_{\epsilon}\rangle\), the Clauser, Horne, Shimony and Holt (CHSH) form of Bell’s inequality: \(f(|\psi_{\epsilon}\rangle) \leq 2\) is usually tested by experimentally measuring the CHSH function \(f(\psi_{\epsilon}) = |E(\theta_1, \theta_2) + E(\theta'_1, \theta'_2) + E(\theta_1, \theta'_2) - E(\theta'_1, \theta'_2)|\).

Here, \(\theta_k\) are controllable classical variables and \(E(\theta_1, \theta_2)\) is the correlation for the outcomes of separately projected measurements of two qubits. A number of experimental tests \(\Box\) of Bell’s inequality have already been performed by using entangled photons and atoms. We now show that a desired entangled state can be created in a repeatable way and thus Bell’s inequality can also be tested experimentally by using this circuit.

We begin with an initial state \(|\psi_{\epsilon}\rangle = |0_0\rangle|1_1\rangle|\downarrow_2\rangle = |0_0\rangle \otimes (|0_1\rangle + |1_1\rangle) \otimes (|0_2\rangle + |1_2\rangle) /2\) with two qubits decoupled from the bus but working at their non-degenerate points (i.e., \(V_k \neq e/C_g^{(k)}\)). After applying a Hadamard gate \(H_g^{(2)}\) to the second qubit, the system evolves to the state
\[
|\psi_{\epsilon}^{(2)}(\theta_1, \theta_2, \theta_2)\rangle = U_{1}^{(1)}(\theta_1)|\Upsilon_1^{(2)}(\theta_2)\Upsilon_1^{(1)}(\theta_2)|\psi_{\epsilon}\rangle,
\]
where \(|\psi_{\epsilon}^{(2)}(\theta_1, \theta_2, \theta_2)\rangle = \cos \delta E^{(k)} t_k/h\). The corresponding correlation function is \(E(\theta_1, \theta_2, \theta_2) = \cos \delta E^{(k)} t_k/h\).
\[ \langle \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) | \sigma_z^{(1)} \otimes \sigma_z^{(2)} | \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) \rangle = \sin \beta_2 \sin \theta_1 - \sin \beta_2 \cos \theta_1 \cos \theta_2. \]

For certain chosen sets of angles: \( \theta_k = \{-\pi/8, 3\pi/8\} \), the CHSH function becomes
\[
f(|\psi_d^{(12)}(\beta_2))| = \sqrt{2} |\sin \beta_2 (\sin \beta_2 + 1)|. \tag{6}
\]

It is easy to numerically check that Bell’s inequality, \( f(|\psi_d^{(12)}(\beta_2))| \leq 2 \), is violated when \( E_2^{(2)}/\delta E_2^{(2)} < 0.776 \), which can be easily satisfied for this charge-qubit system.

Experimentally, the above procedure can be repeated many times at each of the four sets of angles and thus the correlation function \( E(\theta_1, \theta_2, \beta_2) = \sum_{\text{diff}} \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) \rangle / N_{\text{tot}} \), with \( N_{\text{same}}(\theta_1, \theta_2) \) being the number of events with two qubits being found in the same (different) logic states and \( N_{\text{tot}} = N_{\text{same}}(\theta_1, \theta_2) + N_{\text{diff}}(\theta_1, \theta_2) \) being the total experimental times for the same \( \theta_1 \) and \( \theta_2 \). Finally, Bell’s inequality can be tested by calculating the experimental CHSH function: \( f(|\psi_d^{(12)}(\beta_2))| = \sum_{\text{diff}} \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) \rangle / \sum_{\text{same}} \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) \rangle - \sum_{\text{diff}} \psi_d^{(12)}(\theta_1, \theta_2, \beta_2) \rangle \).

Discussion.— Two types of noise, fluctuations of the applied gate voltage \( V_k \) and bias current \( I_b \), must be considered in the present qubit-bus system. For simplicity, these two environmental noises are treated as two separate Boson baths with Ohmic spectral densities and assumed to be weakly coupled to the qubit and CBJJ, respectively. The Hamiltonian of the 4th qubit coupling to the bus, containing these fluctuations, can be written as
\[
\hat{H} = \hat{H}_{kb} + \sum_{j=1,2} \sum_{\omega_j} \hbar \omega_j \hat{a}_{j}^\dagger \hat{a}_{\omega_j} = \frac{e C_g^{(k)}}{C_k} \hat{\sigma}_z^{(k)} (\hat{R}_1 + \hat{R}_1^\dagger) - \frac{\hbar}{2 C_0 \omega_b} \left( \hat{a}^\dagger \hat{R}_2 + \hat{a} \hat{R}_2^\dagger \right), \tag{7}
\]

Here, \( \hat{a}_{j} \) is the Boson operator of the \( j \)-th bath and \( \omega_j \) its coupling strength. The relaxation and decoherence rates of our qubit-bus system can also be calculated by using the well-established Bloch-Redfield formalism. Under the usual secular approximation, the relaxation and decoherence rates are characterized by two dimensionless coupling parameters, \( \alpha_V = (C_g^{(k)}/C_0)^2 R_V/R_K \) and \( \alpha_I = \text{Re}(\gamma)/|C_0 \omega_b| \), which describe the couplings of the voltage fluctuations to the qubit and the bias-current fluctuations to the bus, respectively. Here, \( R_K = h/2e^2 \approx 25.8 \text{ k}\Omega \) is the quantum of resistance, \( R_V \) is the Ohmic resistor of the voltage, and \( \text{Re}(\gamma) \) is the dissipative part of the admittance of the current bias. If the qubit decouples from the bus, \( \alpha_V (\alpha_I) \) characterizes the decoherence and relaxation of the qubit (bus). It has been estimated in Ref. \( \dagger \) that the dissipation for a single SQUID-based charge qubit is sufficiently weak (\( \alpha_V \approx 10^{-6} \)), which allows, in principle, for \( 10^6 \) coherent single-qubit manipulations. However, for a single CBJJ the dimensionless parameter \( \alpha_I \) only reaches \( 10^{-3} \) for typical experimental parameters : \( 1/\text{Re}(\gamma) \sim 100 \text{ k}\Omega \), \( C_0 \sim 6 \text{ pF} \), \( \omega_b/2\pi \sim 10 \text{ GHz} \). This implies that the quantum coherence of the present qubit-bus system is mainly limited by the bias-current fluctuations. Fortunately, the impedance of the above CBJJ can be engineered to be \( 1/\text{Re}(\gamma) \sim 500 \text{ k}\Omega \). This lets \( \alpha_I \) reach up to \( 10^{-5} \) and allow about \( 10^9 \) coherent manipulations of the qubit-bus system.

In summary, we have proposed a scheme for coupling two SQUID-based charge qubits by sequentially using their interactions with a common large Josephson junction biased by a fixed current. Each interaction is tunable by controlling the external flux applied to the chosen SQUID-based charge qubit. The proposed circuit allows the possibility of implementing elementary quantum logic operations, including arbitrary single-qubit gates and universal two-qubit gates. The created two-qubit entangled states can be used to test Bell’s inequality.

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Indeed, such a possibility requires that the Josephson energies of two junctions in a SQUID-loop to be equal or have a very small relative difference $|\delta \varepsilon_j / \varepsilon_j|$ in their coupling energies. Experimentally [19], this has been reached with $|\delta \varepsilon_j / \varepsilon_j| \sim 1\%$. This implies that another interaction term $\delta H \approx \delta \varepsilon_j \sin(\pi \Phi_0 / \Phi_0) \sin \left[ \frac{\phi_k - C^{(k)} \phi_b}{C_k} \right]$, due to the different critical currents of the two junctions, can be safely neglected and thus the system can be described by the Hamiltonian (1).

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