ANTIPODAL POINT ARRANGEMENTS ON SPHERES, 
CLASSIFICATION OF NORMAL SYSTEMS AND HYPERPLANE 
ARRANGEMENTS

C.P. ANIL KUMAR

ABSTRACT. For any positive integer \( k \), we classify the antipodal point arrangements (refer 
to Definitions \[3.1,3.2,5.1,5.2\]) on the sphere \( \mathbb{F}_k^+ \) over a field \( \mathbb{F} \) with \( 1-ad \) structure (refer 
to Definition \[1.1\]), up to isomorphism, by associating a finite complete set of cycle invariants. The classification for dimension \( k = 2 \) is done in Theorem \[3.6\] and the classification 
for dimension \( k > 2 \) is done in Theorem \[5.6\]. Normal systems (refer to Definitions \[1.2,1.3\]) 
arise as coarse invariants during classification of hyperplane arrangements i.e. they classify 
hyperplane arrangements modulo translations. Theorem \[5.6\] in turn classifies the normal 
system associated to an hyperplane arrangement up to an isomorphism. With one more 
invariant, the concurrency arrangement sign function (refer to Definition \[6.1\]), we com-
pletely classify the isomorphism classes of hyperplane arrangements over the field of reals 
in the last Section \[6\] in Theorem \[6.6\].

1. Introduction

The main motivation to write this article arises during the characterization of isomorphism 
classes of hyperplane arrangements by associating a finite complete set of invariants. This 
characterization is done in Section \[6\] over the field of reals. Before we restate the relevant 
problem regarding classification of normal systems we need a few definitions.

Definition 1.1 (A Field with \( 1-ad \) Structure).
Let \( (\mathbb{F}, \leq) \) be a totally ordered field. We say \( \mathbb{F} \) has a \( 1-ad \) structure if in addition the total 
order satisfies the following properties.
- If \( x, y, z \in \mathbb{F} \) then \( x \leq y \Rightarrow x + z \leq y + z \).
- If \( x, y \in \mathbb{F} \) then \( x \geq 0, y \geq 0 \Rightarrow xy \geq 0 \).

Definition 1.2 (Normal System).
Let \( (\mathbb{F}, \leq) \) be a field with \( 1-ad \) structure. Let \( \{L_1, L_2, \ldots, L_n\} \) be a finite set of lines passing 
through the origin in \( \mathbb{F}^m \). Let \( U = \{\pm v_1, \pm v_2, \ldots, \pm v_n\} \) be a set of antipodal pairs of vectors 
on these lines. We say \( \{L_1, L_2, \ldots, L_n\} \) forms a normal system if the set 
\[ B = \{v_1, v_2, \ldots, v_n\} \]
is maximally linearly independent i.e. any subset of \( B \) of cardinality at most \( m \) is linearly 
independent.

Definition 1.3 (Convex Positive Bijection and Isomorphism between Two Normal Sys-
tems).
Let \( \mathbb{F} \) be a field with \( 1-ad \) structure. Let 
\[ \{L_1, L_2, \ldots, L_n\}, \{M_1, M_2, \ldots, M_n\} \]
be two finite sets of lines passing through the origin in \( \mathbb{F}^m \) both of them have the same 
cardinality \( n \) which form normal systems. Let 
\[ U_1 = \{\pm v_1, \pm v_2, \ldots, \pm v_n\}, U_2 = \{\pm w_1, \pm w_2, \ldots, \pm w_n\} \]

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be two sets of antipodal pairs of $\mathbb{F}$–vectors on these lines. We say a bijection $\delta : U_1 \rightarrow U_2$ is a convex positive bijection if
\[
\delta (-u) = -\delta (u), \quad u \in U_1
\]
and for any base $B = \{u_1, u_2, \ldots, u_m\} \subset U_1$ and a vector $u \in U_1$ we have
\[
u = \sum_{i=1}^{m} a_i u_i \text{ with } a_i > 0, 1 \leq i \leq m, \text{ if and only if },
\]
\[
\delta (u) = \sum_{i=1}^{m} b_i \delta (u_i) \text{ with } b_i > 0, 1 \leq i \leq m.
\]
We say two normal systems are isomorphic if there exists a convex positive bijection between their corresponding sets of antipodal pairs of normal $\mathbb{F}$–vectors.

Now we mention the relevant problem regarding classification of normal systems. In Article[2], the following open problem has been stated.

**Problem 1.4** (Classification of Normal Systems and Finding Representatives in Each Isomorphism Class).

Classify and enumerate the Normal Systems up to isomorphism by associating invariants which can be used to easily construct a family of normal systems representing each isomorphism class for every positive integer cardinality $n$ of the normal system.

Here in this article we classify normal systems up to isomorphism by associating a finite complete set of cycle invariants over a field $\mathbb{F}$ with 1–ad structure and thereby classify the hyperplane arrangements over the field of real numbers also using one more invariant the concurrency arrangement sign function. The enumeration problem of the number of isomorphism classes of normal systems and the problem of representing their isomorphism classes by a well defined list of representatives still remain open (refer to Question[7.1]). Now we mention a couple of definitions.

**Definition 1.5** (An Hyperplane Arrangement).

Let $m, n$ be positive integers. We say a set
\[
(H^m_n)^F = \{H_1, H_2, \ldots, H_n\}
\]
of $n$ hyperplanes in $\mathbb{F}^m$ form an hyperplane arrangement if
- For $1 \leq r \leq m, 1 \leq i_1 < i_2 < \ldots < i_r \leq n$ we have
  \[
  \text{dim}_F (H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}) = m - r.
  \]
- For $r > m, 1 \leq i_1 < i_2 < \ldots < i_r \leq n$ we have
  \[
  H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r} = \emptyset.
  \]

**Definition 1.6** (Isomorphism Between Two Hyperplane Arrangements).

Let $\mathbb{F}$ be a field with 1–ad structure. Let
\[
(H^m_n)^F_1 = \{H^1_1, H^1_2, \ldots, H^1_n\}, (H^m_n)^F_2 = \{H^2_1, H^2_2, \ldots, H^2_n\}
\]
be two hyperplane arrangements in $\mathbb{F}^m$. We say a map $\phi : (H^m_n)^F_1 \rightarrow (H^m_n)^F_2$ is an isomorphism between these two hyperplane arrangements if $\phi$ is a bijection between the sets $(H^m_n)^F_1, (H^m_n)^F_2$ in particular on the subscripts and given $1 \leq i_1 < i_2 < \ldots < i_{m-1} \leq n$ and lines
\[
L = H^1_{i_1} \cap H^1_{i_2} \cap \ldots \cap H^1_{i_{m-1}}, M = H^2_{\phi(i_1)} \cap H^2_{\phi(i_2)} \cap \ldots \cap H^2_{\phi(i_{m-1})}
\]
the order of intersection vertices on the lines $L, M$ agree via the bijection induced by $\phi$ again on the sets of cardinality $m$ (corresponding to the vertices on $L$) containing $\{i_1, i_2, \ldots, i_{m-1}\}$ and (corresponding to the vertices on $M$) containing $\{\phi(i_1), \phi(i_2), \ldots, \phi(i_{m-1})\}$. 
In Article [2] it has been proved that a normal system is a coarse invariant for an hyperplane arrangement i.e. if two hyperplane arrangements are isomorphic then their normal systems are isomorphic. Conversely if the associated normal systems of two hyperplane arrangements are isomorphic then the hyperplane arrangements are isomorphic modulo translations. Also we have observed in the same article in the case of dimension two that any two normal systems of the same cardinality are isomorphic. However in dimensions more than two there exists non-isomorphic normal systems which therefore excludes the possibility of any two hyperplane arrangements with these two normal systems being isomorphic.

1.1. Brief Survey and the Structure of the Paper. With relevance to antipodal point arrangements (refer to Definition 5.1) or normal systems, the theory of matroids is a well studied subject. Matroids are combinatorial abstractions of vector configurations and hyperplane arrangements. E. Katz [4] gives a survey of this theory aimed at algebraic geometers. Here in this article we study specific kind of antipodal pairs of vectors arranged on spheres, vector configurations, which are associated to normal systems that arise from hyperplane arrangements and classify them combinatorially. The method of associating cycle invariants as a combinatorial model to point arrangements in the plane has already been explored by authors J.E.Goodman and R.Pollack [3]. Also the slope problem mentioned in chapter 10, page 60 in M.Aigner and G.M.Ziegler [1], Proofs from THE BOOK, explains a similar method.

Section 2 defines a $k$-dimensional sphere $\mathbb{P}^{k+1}_F$ over a field $F$ with $1 - ad$ structure as a generalization of the $k$-dimensional sphere $S^k \subset \mathbb{R}^{k+1}$. Section 3 is devoted to the classification of antipodal point arrangements on $\mathbb{P}^{2+}_F$ in two dimensions. Theorem 3.6 states the classification theorem in the dimension two case. Section 4 revisits the two non-isomorphic examples of normal systems in dimension three that were mentioned in Article [2] and computes the combinatorial invariants. Section 5 is devoted to classification of antipodal point arrangements on $\mathbb{P}^{k+1}_F$ in higher dimensions for $k > 2$. Theorem 5.6 states the classification theorem in higher dimensions. In the last Section 6 we combinatorially classify hyperplane arrangements over the field of reals using associated normal system and the concurrency arrangement sign function.

2. The Analogue of Spheres over Fields with $1 - ad$ Structure

Over the field $\mathbb{R}$ of reals the $k$-dimensional sphere $S^k$ is defined as

$$S^k = \{(x_1, x_2, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} | \sum_{i=1}^{k+1} x_i^2 = 1\}.$$

We also observe that every line $L \subset \mathbb{R}^{k+1}$ passing through the origin meets the sphere in two distinct points on either side of the origin. With this observation we define the analogue of the sphere over an arbitrary field $F$ with $1 - ad$ structure.

**Definition 2.1.** The $k$-dimensional sphere in $\mathbb{P}^{k+1}_F$ is defined as

$$\mathbb{P}^{k+1}_F = \{(x_1, x_2, \ldots, x_{k+1}) | [(x_1, x_2, \ldots, x_{k+1})] = \{\lambda(x_1, x_2, \ldots, x_{k+1}) | \lambda \in F^+ \text{i.e.} \lambda > 0\}, \quad 0 \neq (x_1, x_2, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}\}$$

We say points

$$[(x_1^i, x_2^i, \ldots, x_{k+1}^i)] \in \mathbb{P}^{k+1}_F, 1 \leq i \leq k + 1$$

are linearly independent if

$$\text{Det}[x_j^i]_{(k+1) \times (k+1)} \neq 0.$$

This does not depend on the choice of the representatives. Similarly we say

$$[(x_1^{k+2}, x_2^{k+2}, \ldots, x_{k+1}^{k+2})] \in \mathbb{P}^{k+1}_F.$$
is a positive combination of
$$[(x_1^i, x_2^i, \ldots, x_{k+1}^i)] \in \mathbb{PF}_F^{k+1}, 1 \leq i \leq k+1$$
if there exists \(\lambda_i > 0, 0 \leq i \leq k+1\) such that
$$f(x_1^i, x_2^i, \ldots, x_{k+1}^i) = \sum_{i=1}^{k+1} \lambda_i (x_1^i, x_2^i, \ldots, x_{k+1}^i).$$

Again here sign of \(\lambda_i\) does not depend on the choice of representatives.

Now we define antipodes on a sphere \(\mathbb{PF}_F^1\).**

**Definition 2.2 (Definition of Antipodes).** We say points

$$[(x_1, x_2, \ldots, x_k, x_{k+1})], [(y_1, y_2, \ldots, y_k, y_{k+1})] \in \mathbb{PF}_F^k$$
are antipodes if there exists \(\lambda < 0, \lambda \in \mathbb{F}\) such that 
\( (x_1, x_2, \ldots, x_{k+1}) = \lambda(y_1, y_2, \ldots, y_{k+1}) \).

We denote
$$-[(x_1, x_2, \ldots, x_{k+1})] = [(y_1, y_2, \ldots, y_{k+1})].$$

**Definition 2.3 (Sphere Variety).**

Let \(\mathbb{F}\) be a field with \(1 – ad\) structure. Let \(n \geq 0\) be a positive integer. Let
$$V_n(f) = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{F}^{n+1} \mid f(x_1, x_2, \ldots, x_n, x_{n+1}) = 0\}$$
is called a sphere variety if every line \(L = \{\lambda(x_1, x_2, \ldots, x_n, x_{n+1}) \mid \lambda \in \mathbb{F}\}\) for a point \((x_1, x_2, \ldots, x_n, x_{n+1}) \in (\mathbb{F}^{n+1})^*\) intersects \(V_n(f)\) in two points
$$\lambda_1(x_1, x_2, \ldots, x_n, x_{n+1}), \lambda_2(x_1, x_2, \ldots, x_n, x_{n+1})$$
with \(\lambda_1 \lambda_2 < 0\).

**Note 2.4.** If such an algebraic set \(V_n(f)\) exists then it has to be irreducible for \(n \geq 1\) and \(V_n(f)\) is a variety (refer to the proof of Theorem 2.5).

Now we prove a theorem on existence of sphere varieties over certain fields \(\mathbb{F}\) with \(1 – ad\) structure.

**Theorem 2.5.** Let \(\mathbb{F}\) be a field with \(1 – ad\) structure. Define for every \(n \geq 1\) a polynomial \(f_1\) of degree 2 as follows if it exists and satisfies the following. For \(n \geq 2\), let
$$f_{n-1}[x_1, x_2, \ldots, x_{n-1}] = f_{n-1}^2[x_1, x_2, \ldots, x_{n-1}] + f_{n-1}^1[x_1, x_2, \ldots, x_{n-1}] + f_{n-1}^0[x_1, x_2, \ldots, x_{n-1}]$$
be a polynomial of degree 2, with homogeneous decomposition into \(f_{n-1}^2, f_{n-1}^1, f_{n-1}^0\). Suppose \(f_{n-1}\) has no solutions in \(\mathbb{F}^{n-1}\) and the homogeneous polynomial equation
$$f_{n-1}^2[x_1, x_2, \ldots, x_{n-1}] = 0$$
has only origin in \(\mathbb{F}^{n-1}\) as a solution. Let
$$f_n^2(x_1, x_2, \ldots, x_n) = x_n^2 f_{n-1}^2(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n})$$
be its homogenization. Suppose in addition the range of \(f_n^2(x_1, x_2, \ldots, x_n)\) for \((x_1, x_2, \ldots, x_n) \in (\mathbb{F}^n)^*\) is contained in the subset of squares in \(\mathbb{F}\). Then the zero set \(V(f_n^2 - 1)\) defined by the equation \(f_n^2 = 1\) is a sphere variety.

**Proof.** The proof is immediate. We only prove that \(f_n^2 - 1\) is irreducible for \(n \geq 2\). Suppose it is reducible then we have
$$f_n^2 - 1 = g(x_1, x_2, \ldots, x_n)h(x_1, x_2, \ldots, x_n)$$
where both \(g, h\) are linear polynomials in \(x_1, x_2, \ldots, x_n\) with nonzero constant coefficients. Over fields with \(1 – ad\) structure not every ray passing through the origin intersects one of these two zero sets \(g = 0, h = 0\) which are affine hyperplanes. Hence we arrive at a contradiction. \(\blacksquare\)
Example 2.6.

- Let $F = \mathbb{Q} \cap \mathbb{R}$.
- Let $F = K \subset \mathbb{R}$ where $K$ is the smallest extension of $\mathbb{Q}$ which is positively quadratically closed (refer to Definition 2.7 and Theorem 2.8).

For $n \geq 1$, let $f_n^2(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2$, let $f_n^1 \equiv 0$, $f_n^0 \equiv 1$.

The examples where such polynomials exist are the fields with $1 - ad$ structure which are positively quadratically closed.

2.1. Positively Quadratically Closed Fields. We begin with the definition of a positively quadratically closed field.

Definition 2.7. We say a field $F$ with $1 - ad$ structure is positively quadratically closed if for every $\lambda > 0, \lambda \in F, \pm \sqrt{\lambda} \in F$.

Theorem 2.8. Let $F$ be a field with $1 - ad$ structure. Then there exists a positively quadratically closed field $\mathbb{F}_{PQC}$ with $1 - ad$ structure containing $F$.

Proof. Let $(F, \leq)$ be a field with $1 - ad$ structure. Consider the field $F_1$ to be the compositum of quadratic extensions obtained by adjoining square roots of positive elements of $F$.

$$F_1 = \bigvee_{d>0} F[\sqrt{d}].$$

Then we can extend the total order on $F$ to $F_1$ as follows. The extension is done step by step. Consider $F[\sqrt{d}]$ with $d > 0$. Then we define

$$a + bv\sqrt{d} > 0$$

if $a > 0, b > 0$ or if $a > 0, b < 0$ and $a^2 > b^2d$ or if $a < 0, b > 0$ and $b^2d > a^2$ or if $a = 0, b > 0$. Also we define

$$a + bv\sqrt{d} < 0$$

if $a < 0, b < 0$ or if $a = 0, b < 0$. We define $a + b\sqrt{d} = 0$ if and only if $a = 0, b = 0$ or we can use trichotomy property to obtain this conclusion. We define $a + b\sqrt{d} = c + ev\sqrt{d}$ if $(a - c) + (b - e)\sqrt{d} > 0$. Having extended the total order to $F[\sqrt{d}]$ we prove that the total order is extendable to the field $F_1$. Otherwise by Zorn’s lemma there exists a maximal element $K$ such that $K \subset F_1$ for which the total order is extendible. Now if $F_1 \neq K$ then there exists $d \in F$ such that $\sqrt{d} \notin K$. So we can extend the total order to the field $K[\sqrt{d}]$ which is a contradiction to maximality. Hence we have $F_1 = K$.

Now we construct a field $F_i$ inductively which contains the square roots of all positive elements of $F_{i-1}$ for $i > 1$. Then consider the field given by

$$F_{PQC} = \bigcup_{i \geq 1} F_i$$

This field is positively quadratically closed. This is also a field with $1 - ad$ structure.

Note 2.9. If $F$ is a quadratically positively closed field then every line passing through the origin in the affine space $A^n_F$ has a pair of antipodal unit vectors. i.e. the sphere $S^{n-1}_F$ is complete with respect to the lines passing through the origin or every such line intersects the sphere in a pair of antipodal unit vectors.
3. Antipodal Point Arrangements on the $2-$Sphere $\mathbb{PF}^2_+$

Now we define antipodal point arrangements on the $2-$sphere $\mathbb{PF}^2_+$.

**Definition 3.1** (Antipodal Point Arrangement on the $2-$Sphere $\mathbb{PF}^2_+$).

We say a set $\mathcal{P}_n = \{\pm P_1, \pm P_2, \ldots, \pm P_n\} \subset \mathbb{PF}^2_+$ of points is a point arrangement on the sphere if three points of $\mathcal{P}_n$ are linearly dependent then some two of them are antipodal.

**Definition 3.2** (Isomorphism Between two Antipodal Point Arrangements on the $2-$Sphere $\mathbb{PF}^2_+$).

Two point arrangements

$$\mathcal{P}_n = \{\pm P_1, \pm P_2, \ldots, \pm P_n\}, \mathcal{Q}_m = \{\pm Q_1, \pm Q_2, \ldots, \pm Q_m\} \subset \mathbb{PF}^2_+$$

are isomorphic if $n = m$ and there is a bijection $\phi : \mathcal{P}_n \rightarrow \mathcal{Q}_m$ between the two sets such that the following occurs.

- $\phi(-A) = -\phi(A)$ for all $A \in \mathcal{P}_n$.
- for any $A, B, C, D \in \mathcal{P}_n$ if $D$ is a positive combination of $A, B, C$ if and only if $\phi(D)$ is a positive combination of $\phi(A), \phi(B), \phi(C)$.

3.1. Algebraic Symbols Associated to a Four Antipodal Point Arrangement on the Sphere $\mathbb{PF}^2_+$

We begin this section with the standard arrangement.

3.1.1. The Standard Arrangement and its Associated Symbols. The arrangement $S_4$ consists of four antipodal pairs of points given by

$$S_4 = \{x = [(1,0,0)], x = [(-1,0,0)], y = [(0,1,0)], -y = [(0,-1,0)], z = [(0,1,0)], -z = [(0,-1,0)], P \text{ a point in } I-\text{Octant}, -P \text{ its antipode point in } VII-\text{Octant} \} \subset \mathbb{PF}^2_+$$

There are twenty four symbols that we associate to this standard arrangement. Before we actually describe these symbols we mention four important aspects.

1. A symbol is of the form

$$a \rightarrow (b, c, d)$$

2. We say that it is compatible or associated to an antipodal point arrangement if $a, b, c, d$ represent elements of the arrangement in $\mathbb{PF}^2_+$ such that $a$ is a positive combination of $b, c, d$.

3. If we give an anticlockwise local orientation to the plane $a^\perp$ with the direction ray $a \in \mathbb{PF}^2_+$ representing the thumb then ignoring signs the line cycle is given by

$$(bdc), \text{ and not by } (bcd).$$

4. The ordered triple $(b, c, d)$ where $a$ is a positive combination of $b, c, d$ has negative determinant.

The associated symbols for the standard arrangement are given by

$$P \rightarrow (y, x, z), P \rightarrow (x, z, y), P \rightarrow (z, y, x),$$

$$x \rightarrow (-y, P, -z), x \rightarrow (P, -z, -y), x \rightarrow (-z, -y, P),$$

$$z \rightarrow (-x, P, -y), z \rightarrow (P, -y, -x), z \rightarrow (-y, -x, P),$$

$$y \rightarrow (-z, P, -x), y \rightarrow (-x, -z, P), y \rightarrow (P, -x, -z),$$

$$-x \rightarrow (-P, y, z), -x \rightarrow (z, -P, y), -x \rightarrow (y, z, -P),$$

$$-z \rightarrow (-P, x, y), -z \rightarrow (x, y, -P), -z \rightarrow (y, -P, x),$$

$$-y \rightarrow (-P, z, x), -y \rightarrow (x, -P, z), -y \rightarrow (z, x, -P),$$

$$-P \rightarrow (-x, -y, -z), -P \rightarrow (-z, -x, -y), -P \rightarrow (-y, -z, -x).$$
In the above symbols the triples are all negatively oriented i.e. given \( P \) is in the first octant the triples have determinant negative.

### 3.1.2. The Symmetry Group on Four Elements and its Action on Symbols.

Here we explore the symmetry involved in the above set of 24 compatible symbols. We state the following theorem on the action of the symmetry group \( S_4 \) on the set of symbols and describe the transitive orbits.

**Theorem 3.3.** The group \( S_4 \) acts on the set

\[
S = \{ p \rightarrow (q, r, s) \mid p, q, r, s \in \{ \pm a, \pm b, \pm c, \pm d \} \text{ such that } \{ \pm p \} \cap \{ \pm q \} = \{ \pm p \} \cap \{ \pm r \} = \\
\{ \pm p \} \cap \{ \pm s \} = \{ \pm q \} \cap \{ \pm r \} = \{ \pm q \} \cap \{ \pm s \} = \{ \pm r \} \cap \{ \pm s \} = \emptyset \}
\]

of all symbols with the action given by

- \( p \rightarrow (q, r, s) \) Apply (12) to get \( -p \rightarrow (-r, -q, -s) \)
- \( p \rightarrow (q, r, s) \) Apply (23) to get \( r \rightarrow (-q, p, -s) \)
- \( p \rightarrow (q, r, s) \) Apply (34) to get \( s \rightarrow (-q, -r, p) \)
- \( p \rightarrow (q, r, s) \) Apply (14) to get \( -p \rightarrow (-s, -r, -q) \).

- The set \( S \) has 384 elements. Then each transitive orbit of an element under the action of \( S_4 \) contains 24 elements. There are 16 orbits.
- There are 8 orbits (192 elements satisfying property 4) that arise as compatible symbols associated to concrete four antipodal point arrangements.
- Each transitive orbit is the set of all compatible symbols corresponding to one fixed four antipodal pairs of points of the point arrangement on the sphere \( \mathbb{F}^2\mathbb{F}_p^+ \) provided one of the symbols in the orbit is compatible.
- Moreover the action of \( S_4 \) on the set \( S \) is free.

**Proof.** We have \( \#(S) = 4 \cdot 3! \cdot 2^4 = 384 \). We observe that the action is compatible with the relations

\[
(12)(23)(12) = (23)(12)(23), (23)(34)(23) = (34)(23)(34), \\
(34)(14)(34) = (14)(34)(14), (12)(34) = (34)(12), (14)(23) = (23)(14), \\
(12)^2 = (23)^2 = (34)^2 = (41)^2 = \text{identity}.
\]

So we have an action of the symmetric group \( S_4 \) on the set \( S \) of symbols. The set of compatible symbols, as a transitive orbit, obtained by the action of \( S_4 \) on the compatible symbol \( P \rightarrow (y, x, z) \) is precisely the above given 24 compatible symbols of the standard arrangement in Section 3.1.1. Similarly for every transitive orbit if one of the symbols is compatible then all the remaining 23 symbols of the orbit are compatible. The rest of the proof of the theorem is immediate. 

### 3.1.3. The Standard Arrangement and the Dictionary of Line-Cycles.

Here we associate line cycles to the points of the standard arrangement. Later we use this as a local dictionary for an antipodal arrangement on \( \mathbb{F}^2\mathbb{F}_p^+ \) to characterize the arrangement up to an isomorphism.

Consider the standard four antipodal point arrangement \( S \) given by

- \( P_1 = x = [(1, 0, 0)], -P_1 = -x = [(-1, 0, 0)] \),
- \( P_2 = y = [(0, 1, 0)], -P_2 = -y = [(0, -1, 0)] \),
- \( P_3 = z = [(0, 1, 0)], -P_3 = -z = [(0, -1, 0)] \),
- \( P_4 = P \) a point in \( I - \text{Octant}, -P_4 = -P \) its antipode point in \( VII - \text{Octant} \) in \( \mathbb{F}^2\mathbb{F}_p^+ \).
The compatible 24 symbols (an $S_4$ transitive orbit) gives rise to the following dictionary of line cycles at each point with $(x, y, z)$ denoting a positively oriented basis of the arrangement.

\[
\begin{align*}
\tau_4^+ &= (213) \text{ at } P_4, \\
\tau_4^- &= (231) \text{ at } -P_4, \\
\tau_3^+ &= (142) \text{ at } P_3, \\
\tau_3^- &= (124) \text{ at } -P_3, \\
\tau_2^+ &= (341) \text{ at } P_2, \\
\tau_2^- &= (314) \text{ at } -P_2, \\
\tau_1^+ &= (243) \text{ at } P_1, \\
\tau_1^- &= (234) \text{ at } -P_1.
\end{align*}
\]

Now we prove a theorem that given the dictionary of line cycles there is a unique way to recover back the 24 compatible symbols an $S_4$ orbit of the arrangement which is compatible with the standard arrangement.

We state the theorem as follows.

**Theorem 3.4.** Let $\mathcal{P}_4 = \{\pm P_1, \pm P_2, \pm P_3, \pm P_4\} \subset \mathbb{RP}_F^2$ be any four antipodal point arrangement on the sphere. Suppose the line cycles are given by

\[
\begin{align*}
\tau_4^+ &= (213) \text{ at } P_4, \\
\tau_4^- &= (231) \text{ at } -P_4, \\
\tau_3^+ &= (142) \text{ at } P_3, \\
\tau_3^- &= (124) \text{ at } -P_3, \\
\tau_2^+ &= (341) \text{ at } P_2, \\
\tau_2^- &= (314) \text{ at } -P_2, \\
\tau_1^+ &= (243) \text{ at } P_1, \\
\tau_1^- &= (234) \text{ at } -P_1.
\end{align*}
\]

Then the map $\delta : \mathcal{P}_4 \rightarrow S_4$ given by

\[
\begin{align*}
\delta : P_1 &\rightarrow x, -P_1 \rightarrow -x, P_2 \rightarrow y, -P_2 \rightarrow -y, \\
P_3 &\rightarrow z, -P_3 \rightarrow -z, P_4 \rightarrow P, -P_4 \rightarrow -P
\end{align*}
\]

is an isomorphism i.e. it is a convex positive bijection. Also $-\delta$ is an isomorphism. The $S_4$ invariant set of 24 compatible symbols are given by

\[
\begin{align*}
P_4 &\rightarrow (P_2, P_1, P_3), P_4 \rightarrow (P_1, P_3, P_2), P_4 \rightarrow (P_3, P_2, P_1), \\
P_1 &\rightarrow (-P_2, P_1, -P_3), P_1 \rightarrow (P_4, -P_3, -P_2), P_1 \rightarrow (-P_3, -P_2, P_4), \\
P_3 &\rightarrow (-P_1, P_4, -P_2), P_3 \rightarrow (P_4, -P_2, -P_1), P_3 \rightarrow (-P_2, -P_1, P_4), \\
P_2 &\rightarrow (-P_3, P_4, -P_1), P_2 \rightarrow (-P_1, -P_3, P_4), P_2 \rightarrow (P_4, -P_1, -P_3), \\
-P_1 &\rightarrow (-P_4, P_2, P_3), -P_1 \rightarrow (P_3, -P_4, P_2), -P_1 \rightarrow (P_2, P_3, -P_4), \\
-P_3 &\rightarrow (-P_4, P_1, P_2), -P_3 \rightarrow (P_1, P_2, -P_4), -P_3 \rightarrow (P_2, -P_4, P_1), \\
-P_2 &\rightarrow (-P_4, P_3, P_1), -P_2 \rightarrow (P_1, P_4, P_3), -P_2 \rightarrow (P_3, P_1, -P_4), \\
-P_4 &\rightarrow (-P_1, -P_2, -P_3), -P_4 \rightarrow (-P_3, -P_1, -P_2), -P_4 \rightarrow (-P_2, -P_3, -P_1).
\end{align*}
\]

**Proof.** Let us denote

\[
\{P_1 = x, -P_1 = -x, P_2 = y, -P_2 = -y, P_3 = z, -P_3 = -z, P_4 = P, -P_4 = -P\}.
\]

The octant views are given in Figure 3 based on the point $P$ lying in various octants with respect to a positively oriented system $(x, y, z)$.

However first we show that $(P_1, P_2, P_3)$ is positively oriented i.e. its determinant is positive and the symbol $P_4 \rightarrow (P_2, P_1, P_3)$ is compatible. Apriori we do not know the orientation of $(P_1, P_2, P_3)$ and the compatibility signs of the symbols.

Consider the following choices. $(\pm P_1, \pm P_2, \pm P_3)$. Out of these

$(P_1, P_2, P_3), (-P_1, -P_2, P_3), (P_1, -P_2, -P_3), (-P_1, P_2, P_3)$

have the same sign of the determinant and the remaining

$(-P_1, -P_2, -P_3), (P_1, P_2, -P_3), (-P_1, P_2, P_3), (P_1, -P_2, P_3)$

have the same sign of the determinant.
Figure 1. Four Point Arrangements on the Sphere $S^2$

If the second set of determinants are positive then we argue as follows using Theorem 3.3. Suppose $P_4 \rightarrow (P_2, P_1, P_3)$ is compatible then we have $-P_4 \rightarrow (P_2, P_3, P_1)$ is compatible. Hence $\tau_1 = (243)$ which is invalid. Suppose $P_4 \rightarrow (P_2, P_1, P_3)$ is compatible then we have $-P_4 \rightarrow (P_2, P_3, P_1)$ is compatible. Hence $\tau_3 = (142)$ which is invalid. Suppose $P_4 \rightarrow (P_2, P_1, P_3)$ is compatible then we have $-P_4 \rightarrow (P_2, P_3, P_1)$ is compatible. Hence $\tau_3 = (142)$ which is invalid.

If the first set of determinants are positive then we argue as follows using Theorem 3.3. We have $P_4 \rightarrow (-P_2, -P_1, P_3), P_4 \rightarrow (-P_2, P_1, -P_3), P_4 \rightarrow (P_2, -P_1, -P_3)$ also give invalid line cycles. Hence we conclude that $(P_1, P_2, P_3)$ is positively oriented and the symbol $P_4 \rightarrow (P_2, P_3, P_1)$ is compatible.

This proves the theorem. We also note that the overall total flip given by

$$\nu : P_4 \rightarrow S_4, \nu : P_1 \rightarrow -x, -P_1 \rightarrow x, P_2 \rightarrow -y, -P_2 \rightarrow y, P_3 \rightarrow -z, P_3 \rightarrow z, P_4 \rightarrow -P, -P_4 \rightarrow P$$

is also an isomorphism i.e. a convex positive bijection. Using these line cycles we can write down all the $S_4$ invariant set of 24 compatible symbols.

There are other isomorphisms from $P_4$ to $S_4$ as well and below we describe all of them via the automorphism group $Aut(S_4)$.

3.1.4. Automorphism Group of the Standard Antipodal Point Arrangement.

Here we compute the automorphism group of the standard antipodal point arrangement.

**Theorem 3.5.** Let $S_4$ be the standard arrangement. Then

$$Aut(S_4) = S_4 \oplus (\mathbb{Z}/2\mathbb{Z})$$

**Proof.** We have the twenty four compatible symbols of the standard arrangement given in Section 3.1.3. If $p \rightarrow (q, r, s)$ is one such compatible symbol then the map

$$\delta : S_4 \rightarrow S_4, \delta : p \rightarrow -p, -P \rightarrow -p, y \rightarrow q, -y \rightarrow -q, x \rightarrow r, -x \rightarrow -r, z \rightarrow s, -z \rightarrow -s$$

is an isomorphism i.e. a convex positive bijection. Using these line cycles we can write down all the $S_4$ invariant set of 24 compatible symbols.
is an automorphism. We also have if $\phi$ is an automorphism then $-\phi$ is also an automorphism and moreover either $\phi(P) \rightarrow (\phi(y), \phi(x), \phi(z))$ or $-\phi(P) \rightarrow (-\phi(y), -\phi(x), -\phi(z))$ gives rise to a compatible symbol and the other one is not a compatible symbol. Hence we get

$$Aut(S_4) = S_4 \oplus (\mathbb{Z}/2\mathbb{Z}).$$

This proves the theorem. \hspace{1cm} \blacksquare

3.2. An Isomorphism Theorem for Antipodal Point Arrangements on the Two Dimensional Sphere $\mathbb{P}^2_F$.

Now we prove an isomorphism theorem for antipodal point arrangements on the sphere $\mathbb{P}^2_F$ which can be generalized to higher dimensions in Theorem 5.6.

3.2.1. Localization to Antipodal Point Subarrangements.

Let $\mathcal{P}_n = \{\pm P_1, \pm P_2, \ldots, \pm P_n\} \subset \mathbb{P}^2_F$ be an antipodal point arrangement. We introduce an equivalence relation $\sim$ on $S_n$ as follows. Let $g, h \in S_n$ then $g \sim h$ if $g = h$ or $g = h^{-1}$. This is an equivalence relation with reflexive, symmetric and transitive properties. The equivalence classes being $[\tau, \tau^{-1}]$. Any element of order at most two is an equivalence class containing just one element. Remaining equivalence classes has two elements. The antipode map $a : \mathbb{P}^2_F \rightarrow \mathbb{P}^2_F$ has negative determinant. The line cycles $\tau^+_i$ for $P_i, \tau^-_i$ for $-P_i$ associated to a pair of antipodes in $\mathcal{P}_n$ are mutually inverses of each other as there is a reflection about the origin is involved. So we actually obtain

$$(n-1) \text{ cycles with } \tau^-_i = (\tau^+_i)^{-1} \in S_{n-1} \{1, 2, \ldots, i-1, i+1, \ldots, n\}, 1 \leq i \leq n.$$ 

Now we consider the local scenario by restricting to just four antipodal pairs. The restriction map $|_{\text{local } A}$ and inverse map $(\ast)^{-1}$ commutes. We observe that

For any four subset $A \subset \{1, 2, \ldots, n\}$, $(\tau^-_i)|_{\text{local } A} = ((\tau^+_i)^{-1})|_{\text{local } A} = ((\tau^+_i)|_{\text{local } A})^{-1}, i \in A$

3.2.2. The Main Isomorphism Theorem in Two Dimensions. Now we state the theorem as follows.

**Theorem 3.6.** The following two assertions hold true.

1. The line cycles of the antipodal pairs of points of a point arrangement $\mathcal{P}_n \subset S^2$ determines the collection of local $S_4$-invariant set of compatible symbols for every four subset of antipodal pairs of points in $\mathcal{P}_n$.
2. Let $\mathcal{P}_n^1 = \{\pm P_1, \pm P_2, \ldots, \pm P_1\}, \mathcal{P}_n^2 = \{\pm P_2, \pm P_2, \ldots, \pm P_2\}$ be two point arrangements. Let $(\tau^+_i)_j$ be the line cycle associated to $P_i$ and $(\tau^-_i)_j$ be the line cycle associated to $-P_i$ for $j = 1, 2, 1 \leq i \leq n$. There exists a convex positive bijection (an isomorphism) $\delta : \mathcal{P}_n^1 \rightarrow \mathcal{P}_n^2$. If and only if there exists
   - a permutation $\pi \in S_n$ and
   - a sign vector $\mu = (\mu(1), \mu(2), \ldots, \mu(n)) \in (\mathbb{Z}/2\mathbb{Z})^n = \{1\}^n$

with the property that
   
   (a) either
   $$\begin{align*}
   (\tau^+_{\mu(i)}|_{\pi(i)})_j &= \pi(\tau^+_i)_{\pi^{-1}}, \\
   (\tau^-_{\mu(i)}|_{\pi(i)})_j &= \pi(\tau^-_i)_{\pi^{-1}}, 1 \leq i \leq n
   \end{align*}$$
   
   (b) or an overall total flip (here we can choose $-\mu$ in place of $\mu$)
   $$\begin{align*}
   (\tau^+_{\mu(i)}|_{\pi(i)})_j &= [\pi(\tau^+_i)_{1\pi^{-1}]^{-1} = \pi(\tau^-_i)_{\pi^{-1}}, \\
   (\tau^-_{\mu(i)}|_{\pi(i)})_j &= [\pi(\tau^-_i)_{1\pi^{-1}]^{-1} = \pi(\tau^+_i)_{1\pi^{-1}}, 1 \leq i \leq n
   \end{align*}$$

where
   - $[\mu(i) \ast (+)] = +, [\mu(i) \ast (-)] = -$ if $\mu(i) = +.$
• \([\mu(i)\ast (+)] = -, \mu(i)\ast (-) = + \) if \(\mu(i) = -\).

**Proof.** We prove the second assertion first. Suppose \(\delta: \mathcal{P}_n^1 \rightarrow \mathcal{P}_n^2\) is an isomorphism. Then the permutation \(\pi\) and the signed vector \(\mu\) are defined by the equation
\[
\delta(P_i) = \mu(i)P_{\pi(i)}, \delta(-P_i) = -\mu(i)P_{\pi(i)}, 1 \leq i \leq n.
\]

Now we define \(\pi,\mu\) the property 2a is satisfied. If we choose for \(\mu\) the following definition
\[
\delta(P_i) = -\mu(i)P_{\pi(i)}, \delta(-P_i) = \mu(i)P_{\pi(i)}, 1 \leq i \leq n
\]
then \(\pi,\mu\) satisfies the property 2b. This proves one way implication.

Now we prove the other way implication where we are given the permutation \(\pi\) and the signed vector \(\mu\) and changing \(\mu\) to \(-\mu\) if necessary we assume that the property 2a holds. First we localize to any two corresponding four antipodal point arrangements
\[
\{\pm P_i, \pm P_j, \pm P_k, \pm P_l\}, \{\pm P_{\pi(i)}, \pm P_{\pi(j)}, \pm P_{\pi(k)}, \pm P_{\pi(l)}\}.
\]

Since property 2a holds and the restriction map and the inverse map commutes with respect to localization there is an isomorphic way to identify these two arrangements using local line cycles via the local chart as the standard arrangement \(S_4\) using Theorem 3.4. Using this chart we conclude that locally there exists an isomorphism of the four antipodal point arrangements given by
\[
\delta: P_i \rightarrow \mu(i)P_{\pi(i)}, -P_i \rightarrow -\mu(i)P_{\pi(i)}, P_j \rightarrow \mu(j)P_{\pi(j)}, -P_j \rightarrow -\mu(j)P_{\pi(j)}
\]
\[
P_k \rightarrow \mu(k)P_{\pi(k)}, -P_k \rightarrow -\mu(k)P_{\pi(k)}, P_l \rightarrow \mu(l)P_{\pi(l)}, -P_l \rightarrow -\mu(l)P_{\pi(l)}
\]

These local isomorphisms patch up and extend uniquely to an isomorphism defined as
\[
\delta(P_i) = \mu(i)P_{\pi(i)}, \delta(-P_i) = -\mu(i)P_{\pi(i)}, 1 \leq i \leq n.
\]

We also observe that \(-\delta: \mathcal{P}_n^1 \rightarrow \mathcal{P}_n^2\) is an isomorphism. This proves the isomorphism theorem in two dimensions.

Now we prove the first assertion. The local cycles of four antipodal subarrangements determine the \(S_4\) invariant set of 24 compatible symbols using Theorem 3.4. Hence the first assertion follows and we can write down all the compatible symbols of the given arrangement. ■

### 4. Examples of two Non-isomorphic Normal Systems in Three Dimensions over Rationals: Revisited

Consider the normal systems whose associated sets of antipodal vectors are given by
\[
\mathcal{U}_1 = \{\pm u_i \mid 1 \leq i \leq 6\}, \mathcal{U}_2 = \{\pm v_i \mid 1 \leq i \leq 6\}
\]
with \(\mathcal{U}_1 \cap \mathcal{U}_2 = \{\pm u_1, \pm u_2, \pm u_3, \pm u_4, \pm u_5\} = \{\pm v_1, \pm v_2, \pm v_3, \pm v_4, \pm v_5\}\) where
\[
u_1 = (1, 0, 0) = v_1, u_2 = (0, 1, 0) = v_2, u_3 = (0, 0, 1) = v_3,
\]
\[
u_4 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = v_4, u_5 = \left(\frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right) = v_5, u_6 = \left(\frac{6}{11}, \frac{6}{11}, \frac{7}{11}\right), v_6 = \left(\frac{2}{11}, \frac{6}{11}, \frac{9}{11}\right).
\]

Here below we find out line cycles of each point with respect to the given notation.

We have proved that these two are non-isomorphic normal systems by associating graphs of compatible pairs mentioned in Article 2. For example, from the 15 equations below for \(\mathcal{U}_1\) the vertex \(-u_1, u_2\) has degree one and is only compatible with \(\{u_4, -u_6\}\). From the 15 equations below for \(\mathcal{U}_2\) we observe that there is no vertex of degree one as we observe that if a vertex of the associated graph of compatible pairs has a positive degree then the degree is at least two.

Now we mention the following \(\binom{6}{3} = 15\) equations for \(\mathcal{U}_1\).

1. \(3u_4 = u_1 + 2u_2 + 2u_3 = (1, 2, 2)\).
(2) \(9u_5 = u_1 + 4u_2 + 8u_3 = (1, 4, 8)\).
(3) \(11u_6 = 6u_1 + 6u_2 + 7u_3 = (6, 6, 7)\).
(4) \(12u_4 = 3u_1 + 4u_2 + 9u_5 = (4, 8, 8)\).
(5) \(5u_1 + 21u_4 = 2u_2 + 22u_6 = (12, 14, 14)\).
(6) \(88u_6 = 41u_1 + 20u_2 + 63u_5 = (48, 48, 56)\).
(7) \(u_1 + 9u_5 = 4u_3 + 6u_4 = (2, 4, 8)\).
(8) \(11u_6 = 3u_1 + u_3 + 9u_4 = (6, 6, 7)\).
(9) \(9u_1 + 9u_5 = 10u_3 + 22u_6 = (12, 12, 24)\).
(10) \(9u_5 = 2u_2 + 6u_3 + 3u_4 = (1, 4, 8)\).
(11) \(18u_4 = 6u_2 + 5u_3 + 11u_6 = (6, 12, 12)\).
(12) \(54u_5 = 18u_2 + 41u_3 + 11u_6 = (6, 24, 48)\).
(13) \(44u_6 = 13u_1 + 30u_4 + 9u_5 = (24, 24, 28)\).
(14) \(123u_4 = 26u_2 + 45u_5 + 66u_6 = (41, 82, 82)\).
(15) \(13u_3 + 27u_4 = 27u_5 + 11u_6 = (9, 18, 31)\).

Then we have by actual computation the line cycles are given as

\[
\begin{align*}
(\tau_1^+) &= (24653) \text{ at } u_1, \\
(\tau_1^-) &= (23564) \text{ at } -u_1, \\
(\tau_2^+) &= (13546) \text{ at } u_2, \\
(\tau_2^-) &= (16453) \text{ at } -u_2, \\
(\tau_3^+) &= (16452) \text{ at } u_3, \\
(\tau_3^-) &= (12546) \text{ at } -u_3, \\
(\tau_4^+) &= (15326) \text{ at } u_4, \\
(\tau_4^-) &= (16235) \text{ at } -u_4, \\
(\tau_5^+) &= (16432) \text{ at } u_5, \\
(\tau_5^-) &= (12346) \text{ at } -u_5, \\
(\tau_6^+) &= (15324) \text{ at } u_6, \\
(\tau_6^-) &= (14235) \text{ at } -u_6.
\end{align*}
\]

Now we mention the following \(\binom{9}{0}\) = 15 equations for \(U_2\).

(1) \(3v_4 = v_1 + 2v_2 + 2v_3 = (1, 2, 2)\).
(2) \(9v_5 = v_1 + 4v_2 + 8v_3 = (1, 4, 8)\).
(3) \(11v_6 = 2v_1 + 6v_2 + 9v_3 = (2, 6, 9)\).
(4) \(12v_4 = 3v_1 + 4v_2 + 9v_5 = (4, 8, 8)\).
(5) \(27v_4 = 5v_1 + 6v_2 + 22v_6 = (9, 18, 18)\).
(6) \(88v_6 = 7v_1 + 12v_2 + 81v_5 = (16, 48, 72)\).
(7) \(v_1 + 9v_5 = 4v_3 + 6v_4 = (2, 4, 8)\).
(8) \(v_1 + 11v_6 = 3v_3 + 9v_4 = (3, 6, 9)\).
(9) \(v_1 + 27v_5 = 6v_3 + 22v_6 = (4, 12, 24)\).
(10) \(9v_5 = 2v_2 + 6v_3 + 3v_4 = (1, 4, 8)\).
(11) \(11v_6 = 2v_2 + 5v_3 + 6v_4 = (2, 6, 9)\).
(12) \(18v_5 = 2v_2 + 7v_3 + 11v_6 = (2, 8, 16)\).
(13) \(v_1 + 44v_6 = 18v_4 + 27v_5 = (9, 24, 36)\).
(14) \(66v_6 = 2v_2 + 21v_4 + 45v_5 = (12, 6, 54)\).
(15) \(v_3 + 11v_6 = 3v_4 + 9v_5 = (2, 6, 10)\).
Then we have by actual computation the line cycles are given as
\[(\tau_1^+)_2 = (24653) \text{ at } v_1, (\tau_1^-)_2 = (23564) \text{ at } -v_1,\]
\[(\tau_2^+)_2 = (13564) \text{ at } v_2, (\tau_2^-)_2 = (14653) \text{ at } -v_2,\]
\[(\tau_3^+)_2 = (14652) \text{ at } v_3, (\tau_3^-)_2 = (12564) \text{ at } -v_3,\]
\[(\tau_4^+)_2 = (16532) \text{ at } v_4, (\tau_4^-)_2 = (12356) \text{ at } -v_4,\]
\[(\tau_5^+)_2 = (14632) \text{ at } v_5, (\tau_5^-)_2 = (12364) \text{ at } -v_5,\]
\[(\tau_6^+)_2 = (15326) \text{ at } v_6, (\tau_6^-)_2 = (12354) \text{ at } -v_6.\]

5. Antipodal Point Arrangements on Higher Dimensional Spheres and Classification of Normal Systems

Here we mainly associate combinatorial invariants to antipodal point arrangements to classify them and hence classify the normal systems up to isomorphism. These combinatorial invariants turn out to be oriented cycles of points of the orthogonally projected arrangements along small subarrangements. We begin with the required definitions.

**Definition 5.1** (Antipodal Point Arrangement on the \(k\)-Sphere \(\mathbb{P}F^k\)).

We say a set \(P_n = \{\pm P_1, \pm P_2, \ldots, \pm P_n\} \subset \mathbb{P}F^k\) of points is a point arrangement on the sphere if for any \(1 \leq i_1 < i_2 < \ldots < i_{k+1} \leq n\) the points

\[P_{i_1}, P_{i_2}, \ldots, P_{i_{k+1}}\]

are linearly independent.

**Definition 5.2** (Isomorphism Between two Antipodal Point Arrangements on the \(k\)-Sphere \(\mathbb{P}F^k\)).

Two point arrangements

\[P_n = \{\pm P_1, \pm P_2, \ldots, \pm P_n\}, Q_m = \{\pm Q_1, \pm Q_2, \ldots, \pm Q_m\} \subset \mathbb{P}F^k\]

are isomorphic if \(n = m\) and there is a bijection \(\phi : P_n \rightarrow Q_n\) between the two sets such that the following occurs.

- \(\phi(-A) = -\phi(A)\) for all \(A \in P_n\).
- for any \(A, A_i \in P_n, 1 \leq l \leq k + 1, A\) is a positive combination of \(A_i, 1 \leq l \leq k + 1\) if and only if \(\phi(A)\) is a positive combination of \(\phi(A_i), 1 \leq l \leq k + 1\).

Now we prove the existence of orthogonal projections onto subspaces of \(\mathbb{F}^m\) where \(\mathbb{F}\) is a field with \(1 - ad\) structure.

5.1. Existence of Orthogonal Projections over Fields with \(1 - ad\) Structure. Let \(\mathbb{F}\) be a field with \(1 - ad\) structure. Let \(v^i = (x^i_1, x^i_2, \ldots, x^i_n)^t, 1 \leq i \leq k\) be a finite set of linear independent vectors in \(\mathbb{F}^n\) for \(k \leq n\). Define a linear transformation \(T\) given as follows.

\[T : \mathbb{F}^m \rightarrow \mathbb{F}^k \text{ where } [T]_{k \times n} = [x^j_i]_{1 \leq i \leq n, 1 \leq j \leq k}\]

Now we have \(\text{ker}(T) = \langle v^i : 1 \leq i \leq k \rangle^\perp\). We have row rank of \(T\) is \(k\). Since

\[\text{row} - \text{rank}(T) = \text{col} - \text{rank}(T), \text{Rank} + \text{Nullity} = n\]

we have \(\dim(\ker(T)) = n - k\).

Define on \(\mathbb{F}^m\) with \(m > 0\) a positive integer,

\[< v = (x_1, x_2, \ldots, x_m), w = (y_1, y_2, \ldots, y_m) >_{\mathbb{F}^m} = \sum_{i=1}^{m} x_iy_i\]

This is a symmetric bilinear form with the property that

- \(< v, v >_{\mathbb{F}^m} \geq 0\) for \(v \in \mathbb{F}^m\).
- \(< v, v >_{\mathbb{F}^m} = 0 \iff v = 0\).
Then for \( w_1 \in \mathbb{F}^n, w_2 \in \mathbb{F}^k \)
\[
<Tw_1, w_2>_{\mathbb{F}^k} = w_2^tTw_1 = w_1^tT^tw_2 = <w_1, T^tw_2>_{\mathbb{F}^n}.
\]
Now we observe that if \( w_1 \in \text{Ker}(T) \iff < w_1, T^tw_2 >_{\mathbb{F}^n} = 0 \) for all \( w_2 \in \mathbb{F}^k \). So we conclude that
\[
\text{Ker}(T) = \text{Span} < v^j : 1 \leq i \leq k >.
\]
So we conclude that
\[
\text{Ker}(T) \oplus \text{Range}(T^t) = \mathbb{F}^n.
\]
Now we define the orthogonal projections as \( P = P_{\text{Ker}(T)} = 0, P_{\text{Range}(T^t)} = \text{Id}, Q = Q_{\text{Ker}(T)} = \text{Id}, Q_{\text{Range}(T^t)} = 0. \)
Now these projections satisfy the following relations. \( I = P + Q \)
\[
<w_1, w_2>_{\mathbb{F}^n} = <w_1, Pw_2>_{\mathbb{F}^n}, <Qw_1, w_2>_{\mathbb{F}^n} = <w_1, Qw_2>_{\mathbb{F}^n} \quad \text{for} \quad w_1, w_2 \in \mathbb{F}^n.
\]
This proves the existence of orthogonal projections.

5.2. Dimension Reduction and Multiple Orthogonally Projected Antipodal Arrangements along Small Subarrangements.

We begin with a definition.

**Definition 5.3** (Orthogonally Projected Antipodal Point Arrangements). Let \( \mathbb{F} \) be a field with \( 1 - \text{ad} \) structure. Let \( \mathcal{P}_n = \{ \pm P_1, \ldots, \pm P_n \} \) be an antipodal point arrangement in the \( k \)-dimensional sphere \( \mathbb{P}^{k+}_{\mathbb{F}} \). Let \( \mathcal{A} = \{ \pm P_1, \pm P_2, \ldots, \pm P_i \} \subset \mathcal{P}_n \) be an antipodal point subarrangement with \( 1 \leq r \leq k - 2 \). We can orthogonally project using \( Q^A \) the subarrangement \( \mathcal{P}_n \setminus \mathcal{A} \) to the space orthogonal to the space spanned by vectors of \( \mathcal{A} \) to obtain an antipodal point arrangement
\[
\mathcal{P}^A_{n-r} = \{ P^A_j = Q^A(P_j), -P^A_j = Q^A(-P_j) \mid 1 \leq j \leq n, j \neq i, 1 \leq l \leq r \}
\]
in the \( k - r \) dimensional sphere \( \mathbb{P}^{(k-r)+}_{\mathbb{F}} \).

Now we prove a theorem on the signs.

**Theorem 5.4** (Sign of the Combination does not change after Projection).

Let \( \mathcal{P}_n = \{ \pm P_1, \ldots, \pm P_n \} \) be an antipodal point arrangement in the \( k \)-dimensional sphere \( \mathbb{P}^{k+}_{\mathbb{F}} \). Let \( \mathcal{A} = \{ \pm P_{i} \} \). Let \( \mathcal{P}^A_{n-1} \) denote the projected arrangement. Suppose
\[
P_i = [(x^i_1, x^i_2, \ldots, x^i_k, x^i_{k+1})], -P_i = -[(x^i_1, x^i_2, \ldots, x^i_k, x^i_{k+1})], 1 \leq i \leq n.
\]
Suppose we have
\[
(x^j_1, x^j_2, \ldots, x^j_k, x^j_{k+1}) = \sum_{i=1}^{k} \lambda_i (x^i_1, x^i_2, \ldots, x^i_k, x^i_{k+1}) + \lambda_n (x^n_1, x^n_2, \ldots, x^n_k, x^n_{k+1})
\]
for some \( j \notin \{i_1, i_2, \ldots, i_k, n\} \), \( \lambda_i, \lambda_n \in \mathbb{F}^* \). Suppose we have
\[
P^A_j = \sum_{i=1}^{k} \beta_i P^A_{i}
\]
then
\[
\text{sign}(\lambda_i) = \text{sign}(\beta_i), 1 \leq l \leq k.
\]

**Proof.** This theorem is immediate. \( \blacksquare \)

Now we prove an isomorphism theorem about signs for antipodal point arrangements on spheres \( \mathbb{P}^{k+}_{\mathbb{F}} \).
\textbf{Theorem 5.5} (An isomorphism theorem).

Let \( \mathbb{F} \) be a field with \( 1 \)–ad structure. Let \( \mathcal{P}_n^j = \{ \pm P_1^j, \pm P_2^j, \ldots, \pm P_n^j \} \) be two antipodal point arrangements in \( \mathbb{P}^k_\mathbb{F} \) for \( j = 1, 2 \). Let
\[
\pm P_1^j, \pm P_2^j, \ldots, \pm P_n^j, \pm P_{k+1}^j
\]
be \( k + 2 \) antipodal pairs of points of the arrangement. Let \( A = \{ i_1, i_2, \ldots, i_{k+1} \} \). With respect to the set \( A \) let
\[
P^j_i = \sum_{r=1}^m (\lambda_{ir}^j)_A P_{ir}^j, j = 1, 2.
\]

Suppose we have
\[
\text{sign}( (\lambda_{ir}^j)_A^1 ) = \text{sign}( (\lambda_{ir}^j)_A^2 ) \quad \text{for any such choice}.
\]

Then the map
\[
\delta : P^j_i \rightarrow P^2_i, \delta : -P^1_i \rightarrow -P^2_i, 1 \leq i \leq n
\]
is an isomorphism of antipodal point arrangements \( \mathcal{P}_n^j \).

\textit{Proof.} This theorem is immediate and \( \delta \) is a convex positive bijection. \( \blacksquare \)

\subsection{5.3. Line Cycle Invariants Associated to Points of the Projected Arrangements.}

Let \( \mathbb{F} \) be a field with \( 1 \)–ad structure. Let \( \mathcal{P}_n = \{ \pm P_1, \pm P_2, \ldots, \pm P_n \} \) be an antipodal point arrangement in \( \mathbb{P}^k_\mathbb{F} \). Let
\[
A = \{ \pm P_{j_1}, \pm P_{j_2}, \ldots, \pm P_{j_{k-2}} \} \subset \mathcal{P}_n
\]
be a subset of cardinality \( k - 2 \). Then consider the projected arrangement \( \mathcal{P}_n^A = \mathbb{P}^2_\mathbb{F} \subset \mathbb{P}^2_\mathbb{F} \). These arrangements on the two dimensional spheres give rise to anticlockwise oriented line cycles at each point of \( \mathcal{P}_n^A \) denoted as follows.
\[
(\tau_j^+)^A \in S_{n-k+1}( \{ 1, 2, \ldots, n \} \setminus \{ j_1, j_2, \ldots, j_{k-2}, j \} ) \text{ at } P^A_j,

(\tau_j^-)^A \in S_{n-k+1}( \{ 1, 2, \ldots, n \} \setminus \{ j_1, j_2, \ldots, j_{k-2}, j \} ) \text{ at } -P^A_j
\]
both of which are \((n - k + 1)\)–cycles which are mutual inverses of each other for \( 1 \leq j \leq n, j \neq j_l, 1 \leq l \leq k - 2 \).

Now we prove the following isomorphism theorem for line cycle invariants.

\textbf{Theorem 5.6.} Let \( \mathbb{F} \) be a field with \( 1 \)–ad structure. Let \( \mathcal{P}_n = \{ \pm P_1, \pm P_2, \ldots, \pm P_n \} \) be an antipodal point arrangement on \( \mathbb{P}^k_\mathbb{F} \). The line cycle invariants of antipodal pairs
\[
P_j, -P_j, 1 \leq j \leq n
\]
given by mutually inverse cycles
\[
(\tau_j^+)^A \in S_{n-k+1}( \{ 1, 2, \ldots, n \} \setminus \{ j_1, j_2, \ldots, j_{k-2}, j \} ) \text{ at } P^A_j,

(\tau_j^-)^A \in S_{n-k+1}( \{ 1, 2, \ldots, n \} \setminus \{ j_1, j_2, \ldots, j_{k-2}, j \} ) \text{ at } -P^A_j
\]
after projection along the small subarrangement
\[
A = \{ \pm P_{j_1}, \pm P_{j_2}, \ldots, \pm P_{j_{k-2}} \} \subset \mathcal{P}_n \setminus \{ \pm P_j \}
\]
given by
\[
\mathcal{P}_n^A = \{ \pm P_i^A | i \in \{ 1, 2, \ldots, n \} \setminus \{ j_1, j_2, \ldots, j_{k-2} \} \}
\]
for all such possible choices of \( A \) determines the antipodal point arrangement up to isomorphism. i.e. For \( l = 1, 2 \) let
\[
\mathcal{P}_n^l = \{ \pm P_1^l, \pm P_2^l, \ldots, \pm P_n^l \} \subset \mathbb{P}^k_\mathbb{F}
\]
be two antipodal point arrangements with the line cycle invariants of antipodal pairs
\[
P_j^l, -P_j^l, 1 \leq j \leq n
\]
given by mutually inverse cycles
\[(\tau_j^{+})^A \in S_{n-k+1}(\{1, 2, \ldots, n\}\setminus\{j_1, j_2, \ldots, j_{k-2}, j\}) \text{ at } (P^j)^A,\]
\[(\tau_j^{-})^A \in S_{n-k+1}(\{1, 2, \ldots, n\}\setminus\{j_1, j_2, \ldots, j_{k-2}, j\}) \text{ at } - (P^j)^A\]

after projection along the small subarrangement
\[A = \{\pm P^j_{j_1}, \pm P^j_{j_2}, \ldots, \pm P^j_{j_{k-2}}\} \subset P_n \setminus \{\pm P^j\}.\]

Then they are isomorphic if and only if there exists

1. a permutation \(\pi \in S_n\) and
2. a sign vector \(\mu \in (\mathbb{Z}/2\mathbb{Z})^n = \{\pm\}^n\)

such that for all invariant line cycles either
\[\left(\tau_{\pi(j)}^{[\mu(j)\ast(+)\pi]}(A)\right)_2 = \pi(\tau_j^+)A_1 \pi^{-1},\]

which is equivalent to
\[\left(\tau_{\pi(j)}^{[\mu(j)\ast(-)\pi]}(A)\right)_2 = \pi(\tau_j^-)A_1 \pi^{-1}\]

holds or with a total flip the following holds. (Also we could flip the sign of \(\mu\))
\[\left(\tau_{\pi(j)}^{[\mu(j)\ast(-)\pi]}(A)\right)_2 = \pi(\tau_j^+)A_1 \pi^{-1},\]

which is equivalent to
\[\left(\tau_{\pi(j)}^{[\mu(j)\ast(+)\pi]}(A)\right)_2 = \pi(\tau_j^-)A_1 \pi^{-1}\]

where

1. \(\pi(A) = \{\pm P_{\pi(j_1)}, \pm P_{\pi(j_2)}, \ldots, \pm P_{\pi(j_{k-2})}\}\).
2. \([\mu(j) \ast (+)] = +, [\mu(j) \ast (-)] = - \text{ if } \mu(j) = +.
   \bullet [\mu(j) \ast (+)] = -, [\mu(j) \ast (-)] = + \text{ if } \mu(j) = -.

**Proof.** To determine the arrangement up to isomorphism we do the following. Let
\[P_{i_1}, P_{i_2}, \ldots, P_{i_{k+1}}, P_I\]

be \(k + 2\) points of the arrangement. With respect to \(A\) define the coefficients \(\lambda^l_{ij}\) by letting
\[P_I = \sum_{j=1}^{k+1} \lambda^l_{ij} P_j\]

To determine the arrangement up to isomorphism using Theorem 5.5 we need to determine the signs of \(\lambda^l_{ij}, 1 \leq j \leq m\) using the combinatorial invariants.

Now when \(k = 2\) we know that the line cycles give rise to the compatible set of \(S_4\)-invariant set of 24 symbols locally for all the subarrangements using Theorem 3.6. These determine the signs and hence the antipodal point arrangement on \(\mathbb{P}_F^{2+}\) is determined up to isomorphism.

Now we consider a general value of \(k\). Now using various orthogonal projections along subsets of \(A\) and repeated application of Theorem 5.4 we can recover the signs of the coefficients \(\lambda^l_{ij}\) from the combinatorial line cycle invariants. Now we use Theorem 5.5.

The rest of Theorem 5.6 also follows. ■
6. Concurrency Arrangement Sign Function and Classification of Hyperplane Arrangements over Reals

Here in this section we assume that the field \( F \) is the field \( \mathbb{R} \) of reals. We have already done the classification of a normal system which gives a coarse invariant for an hyperplane arrangement. With one more invariant which is the concurrency arrangement sign function we completely classify the hyperplane arrangements upto isomorphism over the field \( \mathbb{R} \) of reals.

We begin with a definition.

**Definition 6.1 (Concurrency Arrangement Sign Function).**

Let

\[
(H^m_n)^R = \{ H_i : \sum_{j=1}^{m} a_{ij} x_j = c_i, 1 \leq i \leq n \},
\]

be an hyperplane arrangement in \( \mathbb{R}^m \). Let

\[
\{ L_i = \lambda(a_{i1}, a_{i2}, \ldots, a_{im}) \mid \lambda \in \mathbb{R}, 1 \leq i \leq n \}
\]

be the associated normal system with

\[
U = \{ \pm (a_{i1}, a_{i2}, \ldots, a_{im}) \in \mathbb{R}^m \mid 1 \leq i \leq n \}
\]

be a set of antipodal pairs of vectors on the lines of the normal system. Let

\[
\mathcal{P}_n = \{ \pm[a_{i1}:a_{i2}:\ldots:a_{im}] \mid 1 \leq i \leq n \} \subseteq \mathbb{P}^{(m-1)+}
\]

be the associated antipodal point arrangement on the \( m-1 \) dimensional sphere. For every \( 1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n \) consider the hyperplane \( M_{\{i_1,i_2,\ldots,i_{m+1}\}} \) passing through the origin in \( \mathbb{R}^n \) in the variables \( y_1, y_2, \ldots, y_n \) whose equation is given by

\[
M_{\{i_1,i_2,\ldots,i_{m+1}\}}(y_1, y_2, \ldots, y_n) = 
\begin{vmatrix}
  a_{i_11} & a_{i_12} & \cdots & a_{i_1(m-1)} & a_{i_1m} & y_1 \\
  a_{i_21} & a_{i_22} & \cdots & a_{i_2(m-1)} & a_{i_2m} & y_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  a_{i_{m-1}1} & a_{i_{m-1}2} & \cdots & a_{i_{m-1}(m-1)} & a_{i_{m-1}m} & y_{i_{m-1}} \\
  a_{i_m1} & a_{i_m2} & \cdots & a_{i_m(m-1)} & a_{i_mm} & y_m \\
  a_{i_{m+1}1} & a_{i_{m+1}2} & \cdots & a_{i_{m+1}(m-1)} & a_{i_{m+1}m} & y_{i_{m+1}}
\end{vmatrix} = 0
\]

Then the associated concurrency arrangement of hyperplanes passing through the origin in \( \mathbb{R}^n \) is given by

\[
(C^m_n)^R = \{ M_{\{i_1,i_2,\ldots,i_{m+1}\}} \mid 1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n \}.
\]

For this concurrency arrangement associated to an hyperplane arrangement the sign function is defined as

\[
S : (C^m_n)^R \rightarrow \{ \pm 1 \},
\]

\[
M_{\{i_1,i_2,\ldots,i_{m+1}\}} \rightarrow \text{sign}(M_{\{i_1,i_2,\ldots,i_{m+1}\}}(c_1, c_2, \ldots, c_n))
\]

**Definition 6.2 (Concurrency Arrangement Sign Function Induced by a Convex Positive Bijection).**

Let

\[
(H^m_n)^R = \{ H_i^k : \sum_{j=1}^{m} a_{ij}^k x_j = c_i^k, 1 \leq i \leq n \}, k = 1, 2
\]

be two hyperplane arrangements of \( n \) hyperplanes in the euclidean space \( \mathbb{R}^m \). Let

\[
\mathcal{U}_k = \{ \pm (c_i^k = (a_{i1}^k, a_{i2}^k, \ldots, a_{im}^k)) \in \mathbb{R}^m \mid 1 \leq i \leq n \}, k = 1, 2
\]
be a set of antipodal pairs of vectors on the lines of the normal systems. Let $\delta : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ be a convex positive bijection. Then this isomorphism $\delta$ gives rise to a concurrency arrangement of the second hyperplane arrangement for the choice of normals
\[ \delta(v_1^1), \delta(v_1^2), \ldots, \delta(v_1^n) \]
of the hyperplanes given by
\[ \left( (\mathcal{C}_{m+1}^n)_{1/2}^\mathbb{R} \right)^\delta = \{ M^\delta_{\{\pi(i_1),\pi(i_2),\ldots,\pi(i_{m+1})\}} \mid 1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n \} \]
where
\[ M^\delta_{\{\pi(i_1),\pi(i_2),\ldots,\pi(i_{m+1})\}}(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)}) = \det \begin{pmatrix}
\delta(v^1_{i_1}) & y_{\pi(i_1)} \\
\delta(v^1_{i_2}) & y_{\pi(i_2)} \\
\vdots & \vdots \\
\delta(v^1_{i_{m+1}}) & y_{\pi(i_{m+1})}
\end{pmatrix} = 0 \]
and a sign function for the corresponding choice of constants $d_i \in \{ \pm c_i^2 \}$, $1 \leq i \leq n$ with $d_{\pi(i)} = c_{\pi(i)}^2$ if $\delta(v^1_{i}) = v^1_{\pi(i)}$ and $d_{\pi(i)} = -c_{\pi(i)}^2$ if $\delta(v^1_{i}) = -v^1_{\pi(i)}$ for a permutation $\pi \in S_n$ again induced by $\delta$. We define the induced concurrency arrangement sign function of the second hyperplane arrangement by
\[ S^2 : \left( (\mathcal{C}_{m+1}^n)_{1/2}^\mathbb{R} \right)^\delta \rightarrow \{ \pm 1 \} \text{ with } M^\delta_{\{\pi(i_1),\pi(i_2),\ldots,\pi(i_{m+1})\}} \rightarrow \text{sign}(M^\delta_{\{\pi(i_1),\pi(i_2),\ldots,\pi(i_{m+1})\}}(d_{\pi(1)}, d_{\pi(2)}, \ldots, d_{\pi(n)})). \]

**Definition 6.3 (Infinity Arrangement).**
Let $\mathcal{H}_m^\mathbb{R}$ be an hyperplane arrangement. We say $\mathcal{H}_m^\mathbb{R}$ is an infinity arrangement if there exists a permutation $\sigma \in S_n$ such that the hyperplane $H_{\sigma(1)}$ is an hyperplane at infinity with respect to the arrangement
\[ \{H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(n-1)}\}. \]

Now we prove a lemma.

**Lemma 6.4.** Let $H_i : \sum_{i=1}^m a_{ij}x_j = c_i$, $1 \leq i \leq m+1$ be $(m+1)$ hyperplanes forming the simplex $\Delta^{m+1}H_1H_2\ldots H_{m+1}$. Suppose the outward normal for each face of the simplex is given by
\[ (a_{i1}, a_{i2}, \ldots, a_{im}) \in \mathbb{R}^m, 1 \leq i \leq m+1 \]
Let $P_1, P_2, \ldots, P_{m+1}$ be the opposite vertices of the simplex with respect to planes $H_1, H_2, \ldots, H_{m+1}$ respectively. If the outward normal orientation of the simplex and the orientation $[P_1 P_2 \ldots P_{m+1}]$ on the vertices are the same orientation for the simplex then the determinant
\[ \det \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1m} & c_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
| & | & \ddots & | & | \\
a_{m1} & a_{m2} & \cdots & a_{mm} & c_m \\
a_{(m+1)1} & a_{(m+1)2} & \cdots & a_{(m+1)m} & c_{m+1}
\end{pmatrix} \]
is positive.

**Proof.** Consider Figure 2 when dimension $m = 2$. We prove this lemma as follows. Without loss of generality let us assume we have a standard simplex with $P_1$ the origin, $P_2 = (1, 0, \ldots, 0), P_3 = (0, 1, \ldots, 0), P_{m+1} = (0, 0, \ldots, 1)$. The induced orientation on the face
Figure 2. Dimension $m = 2$ for a Triangle

$P_2 P_3 \ldots P_{m+1}$ agrees with the outward pointing normal $(1, 1, \ldots, 1)$. The equations for the hyperplanes are given by

$H_1 : \sum_{i=1}^{m} x_i = 1, H_{j+1} : -x_j = 0, 1 \leq j \leq m$

The above determinant reduces to

$\det \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} = (-1)^m (-1)^m = 1 > 0$

Note 6.5.

(1) In Lemma 6.4 the determinant sign changes if we change any two rows i.e. the orientation $P_1 P_2 \ldots P_{m+1}$ does not induce outward normal. Also the determinant sign changes if we keep the order of the rows same but change any row vector to its negative vector.

(2) This lemma is useful in fixing the signs of the sign map coherently (refer to 11, 12) under isomorphisms as mentioned in Theorem 6.6

Now we prove the following isomorphism theorem.

**Theorem 6.6.** Let

$$(\mathcal{H}_n^m)_k^R = \{H_k^i : \sum_{j=1}^{m} a_{ij}^k x_j = c_{ij}^k, 1 \leq i \leq n\}, k = 1, 2$$

be two hyperplane arrangements of $n$ hyperplanes in the euclidean space $\mathbb{R}^m$. With the notations in Definition 6.2, we have, $(\mathcal{H}_n^m)_k^R, k = 1, 2$ are isomorphic if and only if

(1) There exists $\delta : U_1 \longrightarrow U_2$ a convex positive bijection and

(2) The concurrency arrangement sign maps

$S_1^1 : (C_{(m+1)}^n)^R \longrightarrow \{\pm 1\}, S_2^2 : (C_{(m+1)}^n)^R \delta \longrightarrow \{\pm 1\}$

are such that for all $M_{\{i_1, i_2, \ldots, i_{m+1}\}} \in (C_{(m+1)}^n)^R$

(a) Either

$S_1^1(M_{\{i_1, i_2, \ldots, i_{m+1}\}}) = S_2^2(M_{\{\pi(i_1), \pi(i_2), \ldots, \pi(i_{m+1})\}})$
(b) Or

\[ S^1(M_{i_1,i_2,\ldots,i_{m+1}}) = -S^2(M_{\pi(i_1),\pi(i_2),\ldots,\pi(i_{m+1})}) \]

Proof. In the concurrency arrangement associated to a hyperplane arrangement over reals the constant coefficients corresponding to points in a convex cone \( C \) and its opposite cone \(-C\) are all isomorphic and all the remaining cones correspond to different isomorphism classes under isomorphisms which are identity on subscripts i.e. the corresponding automorphism group is \( \mathbb{Z}/2\mathbb{Z} \). This is proved in Article [2] in Theorem 10 using Lefschetz Fixed Point Theorem.

To prove the forward implication we observe the following in a sequence of steps.

1. Since the given hyperplane arrangements are isomorphic there exists a convex positive bijection \( \delta : U_1 \rightarrow U_2 \).

2. Assume without loss of generality that \( \delta \) induces trivial permutation on the subscripts upon relabelling the second hyperplane arrangement. Also assume that upon relabelling the antipodal pairs of vectors in \( U_2 \) and changing the signs of constant coefficients in \( (H_n^m)_{\mathbb{R}} \) coherently, if necessary, we have

\[ \delta(v_i) = v_i, 1 \leq i \leq n \]

and we have two concurrency arrangements

\[ (C_{(m+1)}^n)_{\mathbb{R}}^1, (C_{(m+1)}^n)_{\mathbb{R}}^2 \]

such that \((c_1^1, c_2^1, \ldots, c_n^1), (c_1^2, c_2^2, \ldots, c_n^2)\) give rise to isomorphic arrangements by an isomorphism which is trivial on subscripts.

3. Let \((c_1^1, c_2^1, \ldots, c_n^1) \in C, (c_1^2, c_2^2, \ldots, c_n^2) \in D\) where \( C, D \) represent convex cones in the concurrency arrangements \((C_{(m+1)}^n)_{\mathbb{R}}^1, (C_{(m+1)}^n)_{\mathbb{R}}^2\) respectively.

4. Since they are isomorphic by an isomorphism which is identity on subscripts, the boundary codimension one hyperplanes giving rise to \( C \) and \( D \) have the same set of subscripts which correspond to the set of \( m \)-dimensional simplex polyhedrals present in the isomorphic arrangements \((H_n^m)_{\mathbb{R}}^1, (H_n^m)_{\mathbb{R}}^2\).

5. This pairing between \( C \) and \( D \) extends to a bijection of all the cones in \((C_{(m+1)}^n)_{\mathbb{R}}^1\) and \((C_{(m+1)}^n)_{\mathbb{R}}^2\) such that each bijective pair \( \tilde{C}, \tilde{D} \) of cones correspond to isomorphic arrangements under isomorphism which is trivial on subscripts.

6. Now the important assumption we make is that, both the arrangements \((H_n^m)_{\mathbb{R}}^1, (H_n^m)_{\mathbb{R}}^2\) are infinity arrangements with permutations \( \sigma_1, \sigma_2 \) both being identity (refer to Definition [6.3]). Such an isomorphic bijective pair \( (\tilde{C}, \tilde{D}) \) of cones exists. We actually choose one of the four pairs \((\tilde{C}, \tilde{D}), (\tilde{C}, \tilde{D}), (-\tilde{C}, \tilde{D}), (-\tilde{C}, \tilde{D})\) for further reasoning to stick to condition [a]. This we do it mainly to make some observations independent of the value \( k = 1, 2 \), i.e., independent of the two arrangements.

7. For \( 1 \leq i_1 < i_2 < \ldots < i_m < i_{m+1} \leq n \), let \( \Delta^m H_{i_1}^k H_{i_2}^k \ldots H_{i_{m+1}}^k, k = 1, 2 \) denote simplices in the respective arrangements. Let \( P_{i_j}^k \) be the vertex opposite to \( H_{i_j}^k \) in the simplex \( \Delta^m H_{i_1}^k H_{i_2}^k \ldots H_{i_{m+1}}^k, k = 1, 2, 1 \leq j \leq m + 1 \). The orientation \([P_{i_1}^k P_{i_2}^k \ldots P_{i_{m+1}}^k], k = 1, 2 \) induce similar orientation on each face with respect to outward pointing normals of both the simplices i.e.,

(a) either orientation induced by \([P_{i_1}^k P_{i_2}^k \ldots P_{i_{m+1}}^k] \) on a face \( H_{i_j}^k \) agrees with the outward pointing normal for both \( k = 1, 2 \),

(b) or orientation induced by \([P_{i_1}^k P_{i_2}^k \ldots P_{i_{m+1}}^k] \) on a face \( H_{i_j}^k \) disagrees with the outward pointing normal for both \( k = 1, 2 \).
If $w_{i_l}^k \in \{\pm v_{i_l}^k\}, 1 \leq l \leq (m + 1), k = 1, 2$ are the outward pointing normals of these two simplices $\Delta^m H_{i_1}^k H_{i_2}^k \cdots H_{i_{m+1}}^k, k = 1, 2$ then there exists positive constants $\lambda_{i_l}^k, 1 \leq l \leq m + 1, k = 1, 2$ such that

$$\sum_{l=1}^{m+1} \lambda_{i_l}^k w_{i_l}^k = 0, \lambda_{i_l}^k > 0$$

Hence we conclude that we have $\delta(w_{i_1}^1) = w_{i_2}^2, 1 \leq l \leq m + 1$ uniformly for any such choice $1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n$.

Now to prove the condition (a) on the sign maps $S^1, S^2$ it is enough to prove for these two isomorphic infinity arrangements $(\mathcal{H}_n^m)_1, (\mathcal{H}_n^m)_2$.

Both values $S^1(M_{i_1,i_2,\ldots,i_{m+1}}), S^2(M_{i_1,i_2,\ldots,i_{m+1}})$ represent same signed values because of the following. We refer to Figure 2 and use previous Lemma 6.4.

The signs of $S^k, k = 1, 2$ on the respective simplices depend on the number of sign changes from $w_{i_l}^k$ to $v_{i_l}^k, 1 \leq l \leq m + 1$ and the orientation $[P_{i_1}^k P_{i_2}^k \cdots P_{i_{m+1}}^k]$ induced by the vertices with respect to outward pointing normal of the simplices $\Delta^m H_{i_1}^k H_{i_2}^k \cdots H_{i_{m+1}}^k$ which does not depend on whether $k = 1, 2$.

Now using Lemma 6.4 we conclude that the signs $S^k, k = 1, 2$ are equal.

This proves one way implication.

Now we prove the converse in a sequence of steps.

(1) Suppose there exists a convex positive bijection such that the condition on the sign functions hold. Here we assume by choosing $-\delta$ for $\delta$ if necessary that Condition (a) holds.

(2) Now by applying similar changes to the constants in the cones $C, D$ we can assume that one of them, the first one, is an infinity arrangement with the permutation $\sigma_1$ being identity (refer to Definition 6.3) whose convex cone is $\tilde{C}$ and that Condition (a) holds. Let the convex cone for the other be $\tilde{D}$.

(3) Now tracing back the argument, about signs and the normals, again using Lemma 6.4 we conclude that the second arrangement is also an infinity arrangement with its associated permutation $\sigma_2$ being identity as well.

(4) This argument, about tracing back signs and normals, goes through using Lemma 6.4 for one of the pairs ($\tilde{C}, \tilde{D}$), ($-\tilde{C}, \tilde{D}$), ($\tilde{C}, -\tilde{D}$), ($-\tilde{C}, -\tilde{D}$), because we observe the following.

In a hyperplane arrangement, if we have a hyperplane such that, the outward normal of all simplices with this hyperplane face is same then this hyperplane is a hyperplane at infinity.

(5) Now both are infinity arrangements and there exists a convex positive bijection. For one of the pairs of cones in $(\pm \tilde{C}, \pm \tilde{D})$ we conclude using the proof of the forward implication of the main theorem in Article [2] that these two are isomorphic.

(6) Now we reverse the changes applied to the constants performed in the first one and hence on the second one to conclude that both the given initial arrangements are isomorphic.

This completes the proof of this theorem.
7. Open Questions: The Enumeration Problem and The Problem of a Complete List of Representatives

We have solved the classification problem of isomorphism classes of hyperplane arrangements and the classification problem of isomorphism classes of normal systems in any dimension. Now we mention the remaining two questions which are still open.

**Question 7.1.**

Let $F$ be a field with $1-ad$ structure. Let $n, m$ be positive integers.

1. (Enumeration Problem): Enumerate the isomorphism classes of normal systems in $F^m$ of cardinality $n$.
2. (Representation Problem): Construct a complete list of representatives for the list of isomorphism classes of normal systems in $F^m$ of cardinality $n$.

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CENTER FOR STUDY OF SCIENCE, TECHNOLOGY AND POLICY # 18 & 19, 10TH CROSS, MAYURA STREET, PAPANNA LAYOUT, NAGASHETTYHALLI, RMV II STAGE, BENGALURU - 560094 KARNATAKA,INDIA

E-mail address: akcp1728@gmail.com