Conservative curved systems and free functional model

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Abstract. We introduce conservative curved systems over multiply connected domains and study relationships of such systems with related notions of functional model, characteristic function, and transfer function. In contrast to standard theory for the unit disk, characteristic functions and transfer functions are essentially different objects. We study possibility to recover the characteristic function for a given transfer function. As the result we obtain the procedure to construct the functional model for a given conservative curved system. We employ the functional model to solve some problems in perturbation theory.

0. Introduction

It is well known [1, 2] that there is a one-to-one correspondence between unitary colligations or (in terminology of system theory) conservative linear systems

$$\mathfrak{A} = \begin{pmatrix} T & N \\ M & L \end{pmatrix} \in \mathcal{L}(H \oplus \mathcal{M}, H \oplus \mathcal{M}), \quad \mathfrak{A}^*\mathfrak{A} = I, \quad \mathfrak{A}\mathfrak{A}^* = I$$

and operator valued functions $\Theta(z)$ of the Schur class

$$S = \{\Theta \in H^\infty(\mathcal{D}, \mathcal{L}(\mathcal{M}, \mathcal{M})) : ||\Theta||_\infty \leq 1\}.$$ 

The mapping defined by the formula $\Theta(z) = L^* + zN^*(I - zT^*)^{-1}M^*$, $|z| < 1$ is one of the directions of the above mentioned correspondence. The opposite direction is realized via functional model [1, 2], whose essential ingredients are Hardy spaces $H^2$ and $H^2_-$ (see [3]; note that $L^2 = H^2 \oplus H^2_-$). These two sides of the theory are equipollent: both have simple, clear and independent descriptions and we can easily change a point of view from unitary colligations to Schur class functions and back.

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We shall consider operator valued functions (or rather sets of operator valued functions $\Theta = (\Theta^+, \Xi_+, \Xi_-)$) of weighted Schur classes $S_\Xi$:

$$S_\Xi := \{ (\Theta^+, \Xi_+, \Xi_-) : \Theta^+ \in H^\infty(G_+, \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)), \forall \zeta \in C \forall n \in \mathcal{H}_+ \|\Theta^+(\zeta)n\|_{-\zeta} \leq \|n\|_{+\zeta} \}$$

where $G_+$ is a multiply-connected domain bounded by a rectifiable Jordan curve $C$ (we use also notation $G_-$ for $\text{Ext } C$), $\Xi_\pm$ are weights such that $\Xi_\pm, \Xi_\pm^{-1} \in L^\infty(C, \mathcal{L}(\mathcal{H}_\pm))$, $\Xi_\pm(\zeta) \geq 0$, $\zeta \in C$, and $\|n\|_{\pm\zeta} = (\Xi_\pm(\zeta)n, n)^{1/2}$, $n \in \mathcal{H}_\pm$.

At this moment we ought to look for a suitable generalization of conservative systems (=unitary colligations), but we find ourselves in a position to make a choice what features of unitary colligations to retain. First, we are going to keep conservatism, that means the characteristic function of such a system ought to be of weighted Schur class. Here we fork with [4], where general transfer functions, rather than conservative ones, is a subject of study. Second, we want to retain analyticity in both domains $G_+$ and $G_-$. In such a way we reserve to ourself possibility to exploit techniques that are typical for the boundary values problems (singular integral operators, the Riemann-Hilbert problem, etc) including connections to the stationary scattering theory [5] (e.g., the smooth methods of T.Kato). Thus we will use both Smirnov spaces $E^2(G_\pm)$ [2], which are analogues of the Hardy spaces $H^2$ and $H^2$. The requirement of analyticity in both domains conflicts with requirement of orthogonality: in general, the decomposition $L^2(C) = E^2(G_+) + E^2(G_-)$ is not orthogonal. The combination “analyticity in $G_+$ plus orthogonality” is a mainstream of development in the multiply-connected case starting with [6] and therefore at this point we fork with traditional way of generalization of Sz.-Nagy-Foiaş theory [6, 7, 8]. Nevertheless, our requirements are also substantial and descend from applications (see for details [9, 10, 11]): in [9] we studied the duality of spectral components for trace class perturbations of a normal operator with spectrum on a curve; the functional model from [10] goes back to [11], which is devoted to spectral analysis of linear neutral functional differential equations. Moreover, one can regard the present paper as a result of comparison and unification of the approaches from [9, 10]. If we examine our variant of functional model with more retrospective point of view, we can observe that it lies on intersection of evolution lines going back to [1, 2, 3, 4, 5, 6, 7, 8].

In order to formulate a main result, we need some background (see for details [9, 12, 13]). Consider pairs $\Pi = (\pi_+, \pi_-)$ of operators $\pi_\pm \in \mathcal{L}(L^2(C, \mathcal{H}_\pm), \mathcal{H})$ such that

$$\begin{align*}
(i)_1 & \quad (\pi_+^0 \pi_+)\mathbf{z} = z(\pi_+^0 \pi_+); \\
(ii)_1 & \quad (\pi_-^0 \pi_-)\mathbf{z} = z(\pi_-^0 \pi_-); \\
(iii) & \quad \text{Ran} \pi_+ \vee \text{Ran} \pi_- = \mathcal{H},
\end{align*}$$

where $P_\pm$ are projections such that $\text{Ran} P_\pm = E^2(G_\pm)$ and $\text{Ker} P_\pm = E^2(G_\mp)$. Operators $\pi_\pm : L^2(C, \Xi_\pm) \to \mathcal{H}$ are isometries if we regard them as operators acting from weighted $L^2$ with operator valued weights $\Xi_\pm = \pi_\pm^* \pi_\pm$. Note that in
this interpretation $\pi^\pm_k$ are the adjoint operators of $\pi_k$. We shall say that $\Pi$ is a free functional model of Sz.-Nagy-Foiaš type (see [16] for the unit disk case) and write $\Pi \in \text{Mod}$. However, in contrast to [16], in our axiomatics we use neither a unitary dilation nor orthogonal compliments.

It can easily be checked that

$$\Theta = (\pi^+_1, \pi^+_2, \pi^+_3, \pi^-_3) \in S_\Xi. \quad \text{(MtoC)}$$

Therefore we have defined the functor $F_{cm} : \text{Mod} \to \text{Cfn}$, where $\text{Ob(Cfn)} = S_\Xi$ and the symbol $\text{Cfn}$ denotes the category of characteristic functions. Conversely, for a given $\Theta \in S_\Xi$, it is possible to construct (up to unitarily equivalence) a functional model $\Pi \in \text{Mod}$ such that $\Theta = (\pi^+_1, \pi^+_2, \pi^+_3, \pi^-_3)$. i.e., there exists the inverse functor $F_{mc} = F_{cm}^{-1} : \text{Cfn} \to \text{Mod}$. Note that any characteristic function $\Theta \in S_\Xi$ can be transformed to the form $\Theta_p \oplus \Theta_u$, where $\Theta_p$ is the pure "part" of $\Theta$ and $\Theta_u$ is the unitary "part" of $\Theta$ (see Remark 1.3).

We define curved conservative systems in terms of the functional model. Let $\Pi \in \text{Mod}$. Define the model system $\hat{\Sigma} = F_{sm}(\Pi) := (\hat{T}, \hat{M}, \hat{N}, \hat{\Theta}_u, \hat{\Xi})$, where

$$\hat{T} \in \mathcal{L}(K_\Theta), \quad \hat{T}f := \mathcal{U}f - \pi^+_1 \widehat{\mathcal{M}}f;$$
$$\widehat{\mathcal{M}} \in \mathcal{L}(K_\Theta, \mathcal{H}_+), \quad \widehat{\mathcal{M}}f := \frac{1}{2\pi i} \int_C (\pi^+_1 f)(z) \, dz;$$
$$\hat{N} \in \mathcal{L}(\mathcal{H}_-, K_\Theta), \quad \hat{N}n := P_\Theta \pi^-_1 n; \quad \text{(MtoS)}$$
$$\hat{\Theta}_u \text{ is the unitary "part" of } \hat{\Theta} = (\pi^+_1, \pi^+_2, \hat{\Xi});$$
$$\hat{\Xi} := (\pi^+_2, \pi^+_3, \pi^-_3);$$

$f \in K_\Theta := \text{Ran } P_\Theta$, $P_\Theta := (I - \pi^-_1 P_\Theta \pi^+_1)(I - \pi^-_1 P_\Theta \pi^+_1)$, $n \in \mathcal{H}_-$, and the normal operator $\mathcal{U}$ with absolutely continuous spectrum lying on $C$ is uniquely determined by conditions $\mathcal{U}\pi^+_\pm = \pi^+_\pm z$. Note that the model operators $\hat{T}, \hat{M}, \hat{N}$ arise in a natural way and are the simplest in certain sense (see Remark 1.3).

A coupling $\Sigma = (T, M, N, \Theta_u, \Xi)$ is called a curved conservative system if there exists a functional model $\Pi$, an operator $W \in \mathcal{L}(H, K_\Theta \oplus K_u)$, and a normal operator $\hat{T}_u \in \mathcal{L}(K_u)$, $\sigma(\hat{T}_u) \subset C$ such that $W^{-1} \in \mathcal{L}(K_\Theta \oplus K_u, H)$,

$$(\hat{T} \oplus \hat{T}_u)W = WT, \quad \hat{M}W = M, \quad \hat{N} = NW, \quad \hat{\Theta}_u = \Theta_u, \quad \hat{\Xi} = \Xi. \quad \text{(Sys)}$$

We will use the symbol $\text{Sys}$ to denote the category of curved conservative systems. All curved conservative systems form $\text{Ob(Sys)}$. We shall mainly consider systems with $K_u = \{0\}$. The relations (MtoS) defines the functor $F_{sm} : \text{Mod} \to \text{Sys}$.

For a curved conservative system $\Sigma$ define the transfer function

$$F_{ts}(\Sigma) := (\Upsilon(z), \Theta_u, \Xi), \text{ where } \Upsilon(z) := M(T - z)^{-1}N. \quad \text{(Tfn) + (StoT)}$$

The following relation between transfer and characteristic functions is central in our theory

$$\Upsilon(z) = \begin{cases} \Theta^-(z) - \Theta^+(z)^{-1}, & z \in G_+ \cup \rho(T); \\ -\Theta^-(z), & z \in G_-, \end{cases} \quad \text{(CtoT)}$$
where the operator valued functions $\Theta_{\pm}^-(z)$ are defined by

$$
\Theta^-_(\zeta) := (\pi_+^\dagger \pi_-)(\zeta) = \Xi_+ (\zeta)^{-1}\Theta_+ (\zeta)^\ast \Xi_- (\zeta), \quad \zeta \in C.
$$

The relation (CtoT) gives the expression for transfer functions in terms of a characteristic functions and uses both metric and analytic properties of a characteristic function. The relations (StoT) and (CtoT) define the functors $F_{ts}: \text{Sys} \to \text{Tfn}$ and $F_{ts}: \text{Cfn} \to \text{Tfn}$, respectively. Thus we have arrived to the following commutative diagram

$$
\begin{array}{ccc}
\text{Mod} & \xrightarrow{F_{cm}} & \text{Cfn} \\
\downarrow F_{sm} & & \downarrow F_{tc} \\
\text{Sys} & \xrightarrow{F_{ts}} & \text{Tfn}
\end{array}
$$

We have defined the notion of conservative curved system. Linear similarity (instead of unitarily equivalence for unitary colligations) is a natural kind of equivalence for conservative curved systems and duality is a substitute for orthogonality. Another distinctive feature of conservative curved systems is that we distinguish notions of characteristic and transfer function. In Section 1 we give a more detailed survey of categories $\text{Mod}, \text{Cfn}, \text{Sys}, \text{Tfn}$ and related functors. The fundamental relation $(\Phi&F)$ from that Section explains a functorial character of the transformation $F_{cm}, F_{sm}, F_{tc}, F_{ts}$.

As we can now see, characteristic functions and conservative curved systems are not on equal terms: first of them plays leading role because the definition of conservative curved system depends on the functional model, which, in turn, is uniquely determined by the characteristic function. But, surprisingly, the conservative curved systems is a comparatively autonomous notion and one of the aims of this paper is to “measure” a degree of this autonomy.

Examining the diagram (C&F) we see that it is an interesting problem to recover a functional model $\Pi \in \text{Mod}$ for a given conservative curved systems $\Sigma \in \text{Sys}$. We have $F_{sm} = F_{cm} \circ F_{tc} \circ F_{ts}$. Since the functor $F_{cm}$ is evidently invertible, the main problem is to invert $F_{tc}$. In fact, the projections $P_\pm$ are singular integral operators and we can regard (CtoT) as a system of singular integral equations with unknown $\Theta_+$ and given $\Upsilon, \Xi_\pm$. The main result of Section 2 (and of the paper on the whole) is the uniqueness of a solution of this system.

**Theorem A.** If $\Theta_1, \Theta_2 \in \text{Cfn}$ and $F_{ct}(\Theta_1) = F_{ct}(\Theta_2) \in \mathcal{N}(G_+ \cup G_-, \mathcal{L}(\mathcal{N}_-, \mathcal{N}_+))$, then $\Theta_1 = \Theta_2$.

In other words, if the transfer function $\Upsilon$ is a function of the Nevanlinna class and the weights $\Xi_\pm$ are fixed, then the pure ”part” $\Theta_\uparrow^\ast$ of the characteristic function is uniquely determined. Here $\mathcal{N}(G_+ \cup G_-, \mathcal{L}(\mathcal{N}_-, \mathcal{N}_+))$ is the Nevanlinna class of operator valued functions that admit representation of the form $\Upsilon(z) = 1/\delta(z) \Omega(z)$, where $\delta \in H^\infty(G_+ \cup G_-)$ and $\Omega \in H^\infty(G_+ \cup G_-, \mathcal{L}(\mathcal{N}_-, \mathcal{N}_+))$. 
We also describe a procedure how to find a characteristic function for given transfer function. Knowing the characteristic function we can construct the functional model $\Pi$ and obtain explicit formulas for conversion of the simple system $\Sigma$ into the model system $\hat{\Sigma}$. As a result we give purely intrinsic description of simple conservative curved systems with the transfer function of the Nevanlinna class. Note that conservative curved systems arise for certain problems of operator theory and we reap the benefit of functional model when we are able to determine that some set of operators $(T, M, N)$ is a curved conservative system \[9, 10\].

In Section 3 we employ the functional model to establish duality of spectral components of trace class perturbations of normal operators with spectrum on a curve. The statement is the same as in [9], but now we extend that assertion to the multiply connected case. One of our purposes is to demonstrate how and why the functional model for conservative curved systems can be successfully used in perturbation theory.

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1. Categories and functors

In this Section we survey (mainly without proofs) basic facts concerning the categories $\text{Mod}$, $\text{Cfn}$, $\text{Sys}$, $\text{Tfn}$ and related functors (see [9, 12, 13] for details). We shall use the language of categories and functors for short and for more systematic character of it (anyway we need some notation for our classes of objects and their transformations). We deal with the categories of functional models $\text{Mod}$, characteristic functions $\text{Cfn}$, conservative curved systems $\text{Sys}$, and transfer functions $\text{Tfn}$. Let $X \in \{\text{Mod}, \text{Cfn}, \text{Sys}, \text{Tfn}\}$, e.g., $X = \text{Mod}$. Define objects of category $X$

$$\text{Ob}(X) := \{O : O \text{ satisfies the condition } (X)\}.$$  

(Ob)

We regard two models $\Pi_1, \Pi_2 \in \text{Ob}(\text{Mod})$ as equal if there exists an unitary operator $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_{2\pm} = V\pi_{1\pm}$.

We regard two systems $\Sigma_1, \Sigma_2 \in \text{Ob}(\text{Sys})$ as equal if there exists an invertible operator $W \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$T_2 W = W T_1, \quad M_2 W = M_1, \quad N_2 = N_1 W, \quad \Theta_{2\pm} = \Theta_{1\pm}, \quad \Xi_{2\pm} = \Xi_{1\pm}.$$  

Remark 1.1. In this paper we deal with generalization of functional models from [9, 10]. In [9] we made use of models with $\Xi_{\pm} = \pi_{\pm}^* \pi_{\pm} = \delta_{\pm}(\zeta) I$, $\zeta \in \mathbb{C}$, where $\delta_{\pm}(\zeta)$ were scalar bounded positive functions. In [10] D. Yakubovich developed his theory using models such that $\Xi_+ = I$ and $\Xi_- = (\Theta^+ \Theta^+)^{-1}$, though he did not introduce the weights $\Xi_{\pm}$ in explicit form.

Remark 1.2. A characteristic function $\Theta \in \text{Ob}(\text{Cfn})$ (for short we shall often use $\Theta \in \text{Cfn}$) is called $\Xi$-pure if the (multiple valued character-automorphic) operator valued function $\Theta_0^+ = \chi_- \Theta^+ \chi_-^{-1}$ satisfies the condition $\forall z \in G_+ \forall n \in \mathfrak{N}_+, n \neq 0 \quad ||\Theta_0^+(z)n||_{\mathfrak{N}_-} < ||n||_{\mathfrak{N}_+}$, where $\chi_{\pm}$ are outer (character-automorphic) operator
valued functions such that \( \chi_{\pm}^+ \chi_{\pm} = \Xi_\pm \). A characteristic function \( \Theta \in \text{Cfn} \) is called a \( \Xi \)-unitary constant if \( \Theta^\perp = \Theta^+ \Theta^\perp = (\Theta^+)^{-1} \in H^\perp(\mathbb{C}^2, L(\mathfrak{M}_-, \mathfrak{M}_+)) \). For any \( \Theta \in \text{Cfn} \), there exist \( \eta_{\pm} \) such that \( \eta_{\pm}^\perp, \eta_{\pm}^\perp \in H^\perp(\mathbb{C}^2, L(\mathfrak{M}_-, \mathfrak{M}_+)) \) and the characteristic function \( \Theta_{pu} = (\eta_{\pm}^\perp)^\perp \eta_{\pm}, \eta_{\pm}^\perp \in \text{Cfn} \) can be represented in the form \( \Theta_{pu} = \Theta_p \oplus \Theta_u \), where \( \Theta_p \) is \( \Xi \)-pure and \( \Theta_u \) is a \( \Xi \)-unitary constant. If \( \Theta'_{pu} = \Theta'_p \oplus \Theta'_u \) is another such representation for some \( \eta'_{\pm} \), then the operator valued functions \( \psi_{\pm} = \eta_{\pm}^\perp \eta'_{\pm} \) admit the decompositions \( \psi_{\pm} = \psi_{\pm}^p \oplus \psi_{\pm}^u : \mathfrak{M}_{\pm}^p \oplus \mathfrak{M}_{\pm}^u \). This enable us to consider characteristic functions with equal \( \Xi \)-pure and/or \( \Xi \)-unitary "parts".

**Remark 1.3.** For a functional model \( \Pi \in \text{Mod} \), one can define two subspaces \( \mathcal{D}_+ := \text{Ran} \, \pi_+ \mathfrak{P}_+ \mathfrak{P}_+^\perp \) and \( \mathcal{H}_+ := \text{Ran} \, (I - \pi_+ \mathfrak{P}_+ \mathfrak{P}_+^\perp) \). It is easily shown that \( \mathcal{D}_+ \subset \mathcal{H}_+ \) and \( \forall \, z \in \mathfrak{M}_+ \, \mathcal{D}_+ \subset (\mathcal{U} - z)^{-1} \mathcal{D}_+ \), \( \mathcal{H}_+ \subset (\mathcal{U} - z)^{-1} \mathcal{H}_+ \). In particular, \( \mathcal{D}_+ \subset \mathcal{U} \mathcal{D}_+ \), \( \mathcal{H}_+ \subset \mathcal{U} \mathcal{H}_+ \). Whence we see that the operator \( \mathcal{U} \) is a normal dilation of the operator \( \tilde{T} = \text{Pol} \mathcal{U} | \mathcal{K}_\Theta \). After straightforward computation we get \( \tilde{T} \mathcal{F} = \mathcal{U} \mathcal{F} - \pi_+ \mathcal{M} \mathcal{F} \). Taking into account that \( \text{Ran} \, \mathcal{P}_\pm = \mathcal{E}^2(\mathfrak{M}_\pm) \) and \( \text{Ker} \, \mathcal{P}_\pm = \mathcal{E}^2(\mathfrak{M}_\pm) \), we obtain \( \tilde{M} \mathcal{F} = (\pi_+^\perp \mathcal{U} \mathcal{F})(\infty), \tilde{M} : \mathcal{K}_\Theta \to \mathfrak{M}_+ \) and

\[
(\tilde{T} - z)^{-1} \mathcal{F} = (\mathcal{U} - z)^{-1}(\mathcal{F} - \pi_+ \mathcal{N}), \quad n = \begin{cases} \Theta^+(z)^{-1}(\pi_+^\perp f)(z), & z \in \mathfrak{M}_+ \cap \rho(\tilde{T}) \\ (\pi_+^\perp f)(z), & z \in \mathfrak{M}_- \\ \end{cases}
\]

Assuming the similar conditions for the dual model (see below) and using the duality \( \langle \tilde{N} \mathcal{F}, g \rangle = \langle \mathcal{N} \mathcal{F}, \tilde{M} g \rangle \), \( g \in \mathcal{K}_{\mathcal{U} \Theta} \), \( n \in \mathfrak{M}_- \), we obtain \( \tilde{N} \mathcal{F} = \mathcal{P}_\Theta \pi_- \mathcal{N} \). If we hold the condition \( \text{Ran} \, \mathcal{P}_\pm = \mathcal{E}^2(\mathfrak{M}_+ \mathfrak{M}_-) \) and \( \text{Ker} \, \mathcal{P}_\pm = \mathcal{E}^2(\mathfrak{M}_+ \mathfrak{M}_-) \), we have merely \( \tilde{M} : \mathcal{K}_\Theta \to \mathcal{L}^2(\mathfrak{M}_+ \mathfrak{M}_-) \) and nothing more. Note that it is possible to consider the functional model under those more weak assumptions, but then we lose one of advantages of our functional model, namely, its ability for automatic, easy and effective computation (see Section 3).

**Remark 1.4.** It can be shown that

\[
\check{r}_{nz} = (\tilde{T} - z)^{-1} \tilde{N} n = \mathcal{P}_\Theta \pi_+ \check{m} \quad \mathcal{C}^\perp - z, \quad m = \begin{cases} -\Theta^+(z)^{-1} n, & z \in \mathfrak{M}_+ \cap \rho(\tilde{T}) \\ n, & z \in \mathfrak{M}_- \end{cases}
\]

If \( \rho(\tilde{T}) \cap \mathfrak{M}_+ \neq \emptyset \), then \( \check{r}_{nz} = \mathcal{K}_\Theta \). In this connection, we call a system \( \Sigma \in \text{Sys} \) simple if \( \rho(\tilde{T}) \cap \mathfrak{M}_+ \neq \emptyset \) and \( \forall z \in \rho(\tilde{T}) \check{r}_{nz} = H \), where

\[
r_{nz} := (T - z)^{-1} \check{N} n, \quad n \in \mathfrak{M}_-, \quad z \in \rho(T).
\]

We postpone to introduce morphisms for our categories and at first consider functors \( \mathcal{F}_{X,Y} : \text{Ob}(X) \to \text{Ob}(Y) \), where \( X, Y \in \{\text{Mod}, \text{Cfn}, \text{Sys}, \text{Ffn}\} \). In the Introduction we have defined the functors \( \mathcal{F}_{cm}, \mathcal{F}_{sm}, \mathcal{F}_{lc}, \mathcal{F}_{as} \). By means of these functors we can translate relationships from language of one category into language of a parallel category. For instance, it is possible to give a complete description of the spectrum of a model operator \( \tilde{T} \) in terms of its characteristic...
function. Another example is the existence of the one-to-one correspondence between regular factorizations of a characteristic function and invariant subspaces of the operator $\hat{T}$.

Together with functors $F_{Y,X}$ we consider transformations $\Phi^X_\eta : \text{Ob}(X) \to \text{Ob}(X)$, where $\eta = (\varphi, \eta_+, \eta_-) \in C\mathcal{M}_\eta$. The latter means $\varphi : G_{1+} \to G_{2+}$ is a conformal mapping and $\eta_{\pm}, \eta_{\pm}^{-1} \in H^\infty(G_{2+}, \mathcal{L}(\mathfrak{H}_\pm))$.

For the category Mod, we set $\Pi_2 = \Phi^\text{Mod}_\eta(\Pi_1) := (\pi_{1+}C\varphi\eta_+, \pi_{1-}C\varphi\eta_-)$, where the operator $(C\varphi f)(\zeta) := \sqrt{\varphi'(\zeta)}f(\varphi(\zeta))$, $\zeta \in C_2$, $f \in L^2(C_2, \mathfrak{H})$ is unitary.

For the category Cfn we set $\Theta_2 = \Phi^\text{Cfn}_\eta(\Theta_1) := (\eta_1^{-1}(\Theta_1^+ \circ \varphi^{-1}) \eta_+, \eta_+(\Xi_{1+} \circ \varphi^{-1}) \eta_+, \eta^+_*(\Xi_{1-} \circ \varphi^{-1}) \eta_-)$. In fact, we have made use of this transformation in Remark 1.2.

For the category Sys we set $\Sigma_2 = \Phi^\text{Sys}_\eta(\Sigma_1) := (T_2, M_2, N_2, \Theta_{2u}, \Xi_2)$, where $T_2 = \varphi(T_1)$,

\[ M_2 f = -\frac{1}{2\pi i} \int_{C_1} \sqrt{\varphi'(\zeta)} \eta_+(\varphi(\zeta))^{-1} [M_1(T_1 - \cdot)^{-1} f]_-(\zeta) d\zeta, \]
\[ N_2^* g = -\frac{1}{2\pi i} \int_{C_1} \sqrt{\varphi'(\zeta)} \eta_-(\varphi(\zeta))^* [N_1^*(T_1^* - \cdot)^{-1} g]_-(\zeta) d\zeta, \]

and $\Xi_{2\pm} = \eta_{\pm}^*(\Xi_{1\pm} \circ \varphi^{-1}) \eta_{\pm}$.

The following important identity indicates the connection between functors and the transformations $\Phi^X_\eta$.

\[ F_{Y,X} \circ \Phi^X_\eta = \Phi^Y_\eta \circ F_{Y,X}. \quad (\Phi&F) \]

This relation can be verified for $X = \text{Mod}, Y = \text{Cfn}$ and $X = \text{Mod}, Y = \text{Sys}$. In particular, $F_{sm} \Phi^\text{Mod}_{\eta} = \Phi^\text{Sys}_{\eta} F_{sm}$ results from the identities

\[ \tilde{T}_2 P_{2\Theta} = P_{2\Theta} T_2, \quad \tilde{M}_2 P_{2\Theta} = M_2, \quad \tilde{N}_2 = P_{2\Theta} N_2, \]

where $\Pi_2 = \Phi^\text{Mod}_\eta(\Pi_1)$, $(T_2, M_2, N_2) = \Phi^\text{Sys}_\eta(\tilde{T}_1, \tilde{M}_1, \tilde{N}_1)$, $(\tilde{T}_1, \tilde{M}_1, \tilde{N}_1) = F_{sm}(\Pi_1)$, and $(\tilde{T}_2, \tilde{M}_2, \tilde{N}_2) = F_{sm}(\Pi_2)$.

Remark 1.5. The linear similarity $\tilde{T}_2 P_{2\Theta} = P_{2\Theta} \varphi(\tilde{T}_1)$ explains why we deal with equivalence classes of conservative curved systems rather than merely with systems. Note that only at first sight the linear similarity arises because of weights $\Xi_{\pm}$. Actually, the main reason is the following observation. Let $H = H_1 \oplus H_2'' = H_1 \oplus H_2^\prime$ and the subspace $H_1$ be an invariant under an operator $T$. Then in the corresponding triangular representations

\[ T = \begin{pmatrix} T_1 & \ast \\ 0 & T_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_1 & \ast \\ 0 & T_2'' \end{pmatrix} \]

the operators $T_2'$ and $T_2''$ are similar.
Conversely, we define the transformation $\Phi^T_n$ using $(\Phi & F)$ with $X = C_f$ and $Y = T_n$ as a definition. This transformation is well defined because of the implication $F_{le}(\Theta_1) = F_{le}(\Theta_2) \implies F_{le}(\Phi^C_n(\Theta_1)) = F_{le}(\Phi^C_n(\Theta_2))$, which holds for $\varphi' \in \cup_{p>1}L^p(C_1)$. There are the explicit formulas

$$
\Upsilon_2(z) = v_2^-(z), \quad v_2(z) n = F_n^-(z)((v_1^-(z) n)(\cdot)],
$$

$$
v_1(z) n = F_n^-(z)((\Upsilon_1(z) n)(\cdot)], \quad z \in G_{2-},
$$

where $(\Upsilon_1(z) n)_\pm(z)$ are the boundary values of the vector valued functions $\Upsilon_1(z) n$ from the domains $G_{\pm}$, $F_n^\psi(u)(z) := [P_n(\eta(\cdot)(u \circ \varphi^{-1})(\cdot))](z)$, and $A^\psi(z) := A(z)^\psi$. Taking into account that $P_{\pm}$ is actually a singular integral operator, we see that the transformation $F_n^\psi$ is defined for $\varphi' \in \cup_{p>1}L^p(C_1)$ and $u \in \cap_{q>1}L^q(C_{\varphi 1}, \mathcal{R})$.

**Remark 1.6.** The transformation $F_n^\psi$ is an analogue of the well known in complex analysis the Faber transformation \cite{17, 18}. The classic Faber transformation for a simple connected domain $G_{\varphi}$ is the transformation $F_n^\psi$ such that $\eta \equiv I$, $\varphi$ is a conformal map of the external domains normed at the infinity, and the projection $P_{\pm}$ is replaced by $P_{\pm}$.

If $z \in G_{2+}$, we need some extra assumptions to write an explicit formula. It suffices to admit the existence of boundary values of operator valued function $\Theta_1^+(z)^{-1}$. In this case $\Upsilon_2(z)$ is the analytic continuation of

$$
\Upsilon_2(\varphi(z))_+ = \Upsilon_2(\varphi(z))_+ + \eta_+^0(\varphi(z)) (\Upsilon_1(z)_+ - \Upsilon_1(z)_-) \eta_-(\varphi(z)), \quad \zeta \in C_1
$$

into the domain $G_{2+}$.

The transformations $\Phi^X_n$ is useful if we want to transfer an object $O \in \text{Ob}(X)$ to a more simple form, e.g., we can pass to a circular domain or get rid of weights on some connected component of the boundary. For simple connected domains, in such a way we can reduce our problems to the case of the unit circle and get rid of the weights $\Xi$ completely.

**Remark 1.7.** In terms of the transformations $\Phi^T_n$, one can define $\Xi$-pure transfer functions. Under the assumption $\rho(T) \cap G_{\varphi} \neq \emptyset$ we shall call a transfer function $\Upsilon \in \text{Ob}(T_n)$ $\Xi$-pure if $\forall \eta \in CM_\varphi \Phi^T_n(T)(z) n \equiv 0 \Rightarrow n = 0$. This definition agrees with the corresponding definition for characteristic functions. Therefore there exists a decomposition $\Upsilon = \Upsilon_p \oplus \Upsilon_u$ in the same sense as in Remark \cite{12} (moreover, $\Upsilon_u \equiv 0$). Note that even in the simplest case of unitary colligations an unitary matrix $\left(\begin{array}{cc} T & N \\ M & L \end{array}\right)$ is recovered from the triple $(T, M, N)$ only up to the unitary part $L_u$ of the operator $L$, where $L_u := L|\text{Ker}(I-L^*L) : \text{Ker}(I-L^*L) \longrightarrow \text{Ker}(I-LL^*)$.

The following properties are basic for the transformations $\Phi_p^X$:

$$
\Phi^X_p = \text{id}_X, \quad \Phi^X_{p\eta_2} \circ \Phi^X_{p\eta_1} = \Phi^X_{p\eta_2 \circ \eta_1}, \quad (\Phi_p)
$$
where \( \eta_{32} \circ \eta_{21} := (\varphi_{32} \circ \varphi_{21}, \eta_{1+} \cdot (\eta_{2+} \circ \varphi_{32}^{-1}), \eta_{1-} \cdot (\eta_{2-} \circ \varphi_{32}^{-1})) \). For the category \( \text{Mod} \), these properties are obvious. For the rest categories they follow from \( (\Phi & F) \).

**Remark 1.8.** In the case of the categories \( \text{Cfn} \) and \( \text{Sys} \) it is possible to give direct proofs, which do not employ \( (\Phi & F) \).

At this point we have already prepared to introduce morphisms for our categories. Let \( O_1, O_2 \in \text{Ob}(X) \). We shall say that a triple \( m^X_{O_1 O_2} = (O_1, O_2, \eta) \) is a morphism in the category \( X \) if there exists \( \eta \in \text{CM}_\eta \) such that \( O_2 = \Phi^X_\eta(O_1) \). In other words, \( \text{Mor}(O_1, O_2) := \{m^X_{O_1 O_2} : \exists \eta \in \text{CM}_\eta \} \). The identity \( m_{O_1 O_2} = m_{O_1 O_2} \circ m_{O_1 O_2} \) follows from \( (\Phi & F) \). Taking into account \( (\Phi & F) \), it is easily shown that the transformation \( F_{YX} : \text{Ob}(X) \to \text{Ob}(Y) \) can be extend to morphisms and thus \( F_{YX} \) is a covariant functor.

Now we pass to the important and useful notion of duality. In language of categories and functors this means that there exists the transformation \( F_{YX} : \text{Ob}(X) \to \text{Ob}(X) \) such that \( F_{YX} \circ \Phi^X_\eta = \Phi^Y_{\eta_\circ} \circ F_{YX} \) and \( F_{YX} \circ \Phi^X_\eta = \text{id}_X \), where \( \eta_\circ = (\varphi^\sim, \eta_-^{-1}, \eta_+^{-1}) \).

This transformation can be extended to morphisms and therefore \( F_{YX} \) is a contravariant functor in the category \( X \). In particular, we have

\[
\begin{align*}
\Theta_s &= F_{cm}(\Theta) := (\Theta^+; \Xi_3), \quad \Xi_{3\pm} = \Xi_3^{-1}; \\
\Pi_s &= F_{sm}(\Pi) := (f, \pi_* v) \in \mathcal{H}, \quad f \in \mathcal{H}, \quad v \in L^2(C, \mathcal{H}); \\
\Sigma_s &= F_{ss}(\Sigma) := (T^*, N^*, M^*, \Theta_u^*, \Xi_s); \\
\Upsilon_s &= F_{sc}(\Upsilon) := (T^*, \Theta_u^*, \Xi_s),
\end{align*}
\]

where \( <u, v> := \frac{1}{2\pi} \int_C (u(z), v(z))_H dz, \quad u \in L^2(C, \mathcal{H}), \quad v \in L^2(C, \mathcal{H}) \).

For the functors \( F_{cm}, F_{sm}, F_{sc} \), the identity \( F_{YX} \circ F_{YX} = F_{YX} \circ F_{YX} \) can easily be checked.

## 2. From system to model

Our aim is to construct the functional model for a given conservative curved system. The main step on this way is Theorem A, which, in fact, shows that under certain conditions all our categories \( \text{Mod}, \text{Cfn}, \text{Sys}, \text{Tfn} \) are diverse views of the only one entity. We divide the proof into several parts, which we arrange as separate assertions.

The main hypothesis of Theorem A is \( \Upsilon \in \mathcal{N}(G_+ \cup G_-, L(\mathcal{H}_-, \mathcal{H}_+)) \). Note that for this it suffices to assume \( T - U \in \mathcal{S}_1, \quad U^* U = U U^*, \quad \sigma(U) \subset C, \quad \sigma_+(T) \subset C, \quad M, N \in \mathcal{S}_2 \), and \( C^{1+\varepsilon} \) smooth of the curve \( C \) (see [3] [19]). But instead of \( \Upsilon \in \mathcal{N}(G_+ \cup G_-, L(\mathcal{H}_-, \mathcal{H}_+)) \) we shall consider the more weaker condition, which consists in the existence of boundary values for an operator valued function \( \Theta^+(z)^{-1} \).

**Proposition 2.1.** Suppose \( \Theta \in \text{Cfn} \) and \( \Theta^+(\zeta)^{-1} \) possesses boundary values a.e. on \( C \). Then the operator valued function \( \Delta^+(\zeta) := (I - \Theta^+(\zeta) \Theta^-(\zeta))^{1/2} \) can be expressed in terms of the transfer function \( \Upsilon(z) \).
Proof. The identity $\Theta^-(\zeta) - \Theta^+(\zeta)^{-1} = \Upsilon(\zeta)_+ - \Upsilon(\zeta)_-$, $\zeta \in C$ follows from (CtoT). Then, taking into account that $\Theta^-(\zeta)$ is adjoint to $\Theta^+(\zeta) : \mathcal{N}_{+,\zeta} \to \mathcal{N}_{-,\zeta}$ (see (Cfn)), it suffices to prove the following Lemma.

Lemma 2.2. Suppose $L \in \mathcal{L}(H_1, H_2)$, $L^{-1} \in \mathcal{L}(H_2, H_1)$, and $\|L\| \leq 1$. Let $L = |L|U$ be the polar decomposition of $L$. Then $|\Theta^-(\zeta)| = \|L\| - 1$ and $U^*|\Theta^-(\zeta)|U = B(|L|^{-1} - |L|)^{-1} |\Theta^-(\zeta)|U$.

Proof. $\psi(z) = \frac{1}{2} \sqrt{2 + z - \sqrt{z^2 + 4z}}$. □ □

Let $\Pi \in \operatorname{Mod}$ and $\tau_+ := ((\Delta^+)^{-1}(\pi_+^1 - \Theta^+\pi_+^1))^\dagger$. Define the unitary operator $W_{NF} : \mathcal{H} \to \mathcal{H}_{NF}$ by the rule $W_{NF}f = (f_\pi, f_\tau) := (\pi_+^1 f, \tau_+^1 f)$, where $\mathcal{H}_{NF} := L^2(C, \Xi_+) \oplus \overline{\operatorname{clos}} L^2(C, \Xi_-)$ is endowed with the inner product $(W_{NF}f, W_{NF}g) := (f_\pi, g_\pi)L^2(C, \Xi_+) + (f_\tau, g_\tau)L^2(C, \Xi_-)$. Obviously, $W_{NF}^{-1}(f_\pi, f_\tau) = \pi_+ f_\pi + \tau_+ f_\tau$. Define also $K_{\Theta NF} := W_{NF}K_{\Theta} W_{NF}^\dagger$.

Proposition 2.3. Suppose $\Theta_1, \Theta_2 \in \text{Cfn}$; $\Theta_1^+(\zeta)^{-1}, \Theta_2^+(\zeta)^{-1}$ possesses boundary values a.e. on $C$, and $\Upsilon = \mathcal{F}_{ct}(\Theta_1) = \mathcal{F}_{ct}(\Theta_2)$. Then $K_{\Theta^1 NF} = K_{\Theta^2 NF}$.

Proof. Take the vectors $\hat{r}_{nz}$ as in Remark [4]. By direct calculations, we have

$$\pi_+^1 \hat{r}_{nz} = \frac{(\Upsilon n)(\zeta)_+ - \Upsilon(z) n}{\zeta - z}, \quad \tau_+^1 \hat{r}_{nz} = \frac{-\Delta^+(\zeta) n}{\zeta - z}.$$  

Since $\forall z \in \rho(\mathcal{T}) \hat{r}_{nz} = K_{\Theta NF}$, it remains to make use of Prop. [4] □

Proposition 2.4. Let $\Theta_1 = (\Theta^+_1, \Xi)$ and $\Theta_2 = (\Theta^+_2, \Xi)$. Let $\chi_\pm$ be outer (character-automorphic) operator valued functions such that $\chi_\pm^2 \chi_\pm = \Xi_\pm$. Suppose $K_{\Theta_{1 NF}} = K_{\Theta_{2 NF}}$. Then there exists a unitary operator $U \in \mathcal{L}(\mathcal{N}_-)$ such that $\chi_- \Theta_2^+ \chi_+^{-1} = U \chi_- \Theta_1^+ \chi_+^{-1}$.

Proof. It is obvious that $W_{k NF} \pi_{k+} = I$, $k = 1, 2$. Hence, $W_{1 NF} \pi_{1+} = W_{2 NF} \pi_{2+}$. On the other hand, the subspace $\mathcal{H}_{NF+} := K_{\Theta_{1 NF}} + E^2(G_+, \mathcal{N}_+)$ are invariant under the operator of multiplication by the independent variable $z$. Therefore $z|\mathcal{H}_{NF+}$ is a subnormal operator with the normal spectrum on the curve $C$. By [5], there exists the Wold-Kolmogorov decomposition of the space $\mathcal{H}_{NF+}$ with respect to the operator $z$. On account of the uniqueness of this decomposition, we get for the pure part $W_{1 NF} \pi_{1-} E^2(G_+, \mathcal{N}_-) = W_{2 NF} \pi_{2-} E^2(G_+, \mathcal{N}_-)$. Therefore, $W_{1 NF} \pi_{1-} \chi_-^{-1} E^2(G_+, \mathcal{N}_-) = W_{2 NF} \pi_{2-} \chi_-^{-1} E^2(G_+, \mathcal{N}_-)$, where $E^2_\alpha(G_+, \mathcal{N}_+) = \chi_- E^2(G_+, \mathcal{N}_-)$ is the subspace of character-automorphic functions corresponding to the unitary representation $\alpha$ of the fundamental group of the domain $G_+$. In view that $W_{k NF} \pi_{k-} \chi_-^{-1} E^2_\alpha(G_+, \mathcal{N}_-) = 1, 2$ are isometries, by the generalized Beurling theorem, there exists an unitary operator $U \in \mathcal{L}(\mathcal{N}_-)$ such that
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\[ W_{1NF} \pi_1 - \chi_1^{-1} = W_{2NF} \pi_2 - \chi_2^{-1} U. \] Whence we have

\[
\chi_1 \Theta_2^\dagger \chi_1^{-1} = \chi_1 - \pi_1 - \pi_2 - \chi_1^{-1} = (\pi_2 - \chi_2^{-1})^* \pi_2 + \chi_1^{-1} =
\]

\[
(W_{2NF} W_{1NF} \chi_1^{-1} U^{-1})^* W_{2NF} W_{1NF} \chi_1^{-1} U^{-1} =
\]

\[
U(\pi_1 - \chi_1^{-1})^* \pi_1 + \chi_1^{-1} = U \chi_1^* \pi_1 + \chi_1^{-1} = U \chi_1 \Theta_2^* \chi_1^{-1}.
\]

**Remark 2.1.** In the proof we have made use of the similarity of the bundle shift \(z|E^2(G_+, \mathfrak{R})\) to the trivial shift \(z|E^2(G_+, \mathfrak{R})\) \[6\]. This assertion is equivalent to the fact of triviality of any analytic vector bundles over multiply connected domains (see \[6\], which in the scalar case goes back to \[20\]). We avoid to deal with multiply valued analytic functions and use the technique of weights instead of the traditional uniformization technique or analytic vector bundles. Note that the use of weights \(\Xi_+\) can be effective for the simple connected domains too (see Remarks \[27\] and \[5\]).

**Theorem A.** If \(\Theta_1, \Theta_2 \in \text{Cln} \) and \(\mathcal{F}_{cl}(\Theta_1) = \mathcal{F}_{cl}(\Theta_2) \in \mathcal{N}(G_+ \cup G_-, \mathcal{L}(\mathfrak{M}_-, \mathfrak{M}_+))\), then \(\Theta_1 = \Theta_2\).

**Proof of Theorem A.** Without loss of generality we can assume that the transfer function \(\Upsilon = \mathcal{F}_{cl}(\Theta_1) = \mathcal{F}_{cl}(\Theta_2)\) is \(\Xi\)-pure (see Remark \[17\]). By Prop. \[2.4\] and Prop. \[2.2\] we obtain \(\chi_1 \Theta_2^* \chi_1^{-1} = U \chi_1 \Theta_2^* \chi_1^{-1}\). Whence it follows the one-valuedness of the operator valued function \(\chi_1^{-1} U \chi_1\). Whence we can see that the operator \(U\) commutes with the \(\mathcal{W}^\star(\alpha)\) generated by the unitary representation \(\alpha\). By the theorem of Bungart \[21\], the analytic vector bundles corresponding to the operator valued character-automorphic function \(\chi_1\) is trivial with respect to the group of invertible operators \(G(\alpha)\) of the algebra \(\mathcal{W}^\star(\alpha)\). That means there exists character-automorphic function \(\chi(z)\) with values in \(G(\alpha)\) such that this function trivialize the analytic vector bundle. Therefore the operator valued function \(\eta_1 = \chi_1^{-1} \chi\) is one-valued and \(\eta_1, \eta_2^{-1} \in H^\infty(G_+ \cup G_-)\).

Let \(\eta = (\bar{z}, \bar{\eta}_1\eta_2)\), \(\Theta_1 = \Phi^{\text{Cln}}(\Theta_1)\), and \(\Theta_2 = \Phi^{\text{Cln}}(\Theta_2)\). Then \(\mathcal{F}_{cl}(\Theta_1) = \mathcal{F}_{cl}(\Theta_2)\). Therefore the transfer function \(\Upsilon = \mathcal{F}_{cl}(\Theta_1) = \mathcal{F}_{cl}(\Theta_2)\) corresponds to both characteristic functions \(\Theta_1\) and \(\Theta_2\). We have

\[
U \Theta_1^+ = U \eta_1^{-1} \Theta_1^+ = U \chi_1^{-1} \chi_1^{-1} U \chi_1 \chi_1^{-1} =
\]

\[
\chi_1^{-1} (U \chi_1 \Theta_2^* \chi_1^{-1} \chi_1^{-1} U \chi_1 \chi_1^{-1} \chi_1^{-1} \chi_1^{-1} = \eta_2^{-1} \Theta_2^+ = \Theta_2^+.\]

Since \(\Xi_1 = \Xi_2 = \chi^* \chi\) and \(U \chi^* \chi = \chi^* \chi U\), we also have

\[
\Theta_1 U^* = \Xi_1 \Theta_1^+ \chi^* U^* = \Xi_1 (U \Theta_1^+)^* \chi^* =
\]

\[
= \Xi_1 \Theta_2^+ \chi^* \chi = \Theta_2 U^*.
\]

Then, taking into account (CtoT), we get \(\Upsilon_2(z) = \Upsilon_1(z)^* U = \Upsilon_2(z)^* U\). Whence, \(\Upsilon_2(z)(U - I) = 0\). Since, the transfer function \(\Upsilon_2\) is \(\Xi\)-pure, we get \(U = I\) (see Remark \[1.7\]). Thus, \(\Theta_1^+ = \Theta_2^+\). This completes the proof of Theorem A. \(\square\)
Remark 2.2. If \( \dim \mathcal{H}_- < \infty \), we can make use of the more weak assertion from [22] instead of Bungart’s theorem [21].

Analyzing the above proof we can reveal the restoration procedure of the characteristic function \( \Theta \) for a given transfer function \( \Upsilon \). In the same way as in the proofs of Prop. [23] and Prop. [24] we define \( \mathcal{H}_{NF+} = \mathcal{K}_{\Theta NF+} E^2(G_+, \mathcal{H}_+) \), where \( \mathcal{K}_{\Theta NF+} = \cup_{z \in \rho(T)} \hat{p}_{z} \). Clearly, \( \mathcal{H}_{NF+} \subset \mathcal{H}_{NF} \).

Let \( \mathcal{H} = \mathcal{H}_{NF+} \oplus \mathcal{V} \mathcal{H}_{NF+} \). Clearly, \( \mathcal{H} \) is an isometry. Then there exists the Wold decomposition of the subspace \( \mathcal{H}_{NF+} \) with regard to the operator \( \mathcal{H} \mathcal{H}_{NF+} \) (see [1]). We put \( \mathfrak{M} = \mathcal{H}_{NF+} \oplus \mathcal{V} \mathcal{H}_{NF+} \). Taking into account the vector representation of Hardy spaces for multiply connected domains (see, e.g., [23]), we see \( \dim \mathfrak{M} = n \cdot \dim \mathfrak{H}_- \). Let \( V \in \mathcal{L}(\mathfrak{H}_-, \mathfrak{M}) \) be a unitary operator. Define

\[
W f := \sum_{k=-\infty}^{+\infty} \mathcal{U}_0^k V^{-1} \mathcal{P}_\mathfrak{M} V^{-k} f, \quad W : \mathcal{H}_{NF} \to \mathcal{L}^2(\mathcal{T}, \mathfrak{H}_-),
\]

where \( \mathcal{U}_0 \) is the operator of multiplication by the independent variable in \( \mathcal{L}^2(C, \mathfrak{H}_-) \).

Similarly, let \( \chi_0 = \chi(z) \) in the space \( \mathcal{L}^2(C, \mathfrak{H}_-) \) and \( \mathfrak{M}_0 = E^2_0(G_+, \mathfrak{H}_-) \oplus \mathcal{V} \mathcal{E}_0(G_+, \mathfrak{H}_-) \), where the unitary representation \( \alpha \) is corresponded to the outer operator valued function \( \chi_- \) defined by \( \chi_- \chi_- = \chi_- \). Let \( V_0 \in \mathcal{L}(\mathfrak{H}_-, \mathfrak{M}_0) \) be a unitary operator. It can be shown that

\[
\mathcal{L}^2(C, \mathfrak{H}_-) = \bigoplus_{k=-\infty}^{+\infty} \mathcal{V}_0^k \mathfrak{M}_0, \quad \mathcal{E}^2_0(G_+, \mathfrak{H}_-) = \bigoplus_{k=0}^{+\infty} \mathcal{V}_0^k \mathfrak{M}_0.
\]

We define also

\[
W_0 f := \sum_{k=-\infty}^{+\infty} \mathcal{U}_0^k V_0^{-1} \mathcal{P}_\mathfrak{M}_0 V_0^{-k} f, \quad W_0 : \mathcal{L}^2(C, \mathfrak{H}_-) \to \mathcal{L}^2(\mathcal{T}, \mathfrak{H}_-).
\]

It is easily shown that \( W^* \) is an isometry, \( W_0 \) is an unitary operator, and \( W_0 = \mathcal{U}_0 W_0 \). Besides, we have \( W_0 \mathcal{E}^2_0(G_+, \mathfrak{H}_-) = \mathcal{H}^2(\mathcal{D}, \mathfrak{H}_+) \) and \( W \mathcal{H}_{NF+} = \mathcal{H}^2(\mathcal{D}, \mathfrak{H}_+) \), where \( \mathcal{H}_{NF+}^p := \bigoplus_{k=0}^{+\infty} \mathcal{V}^k \mathfrak{M} \) and the subspace \( \mathcal{H}_{NF+}^p \) reduces the operator \( \mathcal{U} \).

Let an unitary operator \( Y : \mathfrak{H}_- \to \mathfrak{H}_- \) be a solution of the linear equation \( \mathcal{U} W^* Y W_0 \mathfrak{M} = W^* \mathcal{M} Y W_0 \mathfrak{M} \), where \( \mathcal{M} \) is the operator of multiplication by the univalent constant \( \mathcal{Y} \) in \( \mathcal{L}^2(\mathcal{T}, \mathfrak{H}_+) \). Such a solution necessarily exists if \( \mathcal{Y} \) is a transfer function. Note that the equation is equivalent to the Riccati equation \( Y^* \mathcal{U} W^* Y \mathfrak{M} = W_0 z W_0^* \mathfrak{M} \). If \( Y \) is a solution of the equation, the identity \( \mathcal{U} W^* \mathcal{M} Y W_0 = W^* \mathcal{M} Y W_0 z \) can easily be extended to the whole space \( \mathcal{L}^2(\mathcal{T}, \mathfrak{H}_+) \).

We set $\pi_{0+} := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : L^2(C, \mathcal{H}_+) \to \mathcal{H}_{NF}$ and $\pi_{0-} := W^*m_Y W_0 : L^2(C, \mathcal{H}_-) \to \mathcal{H}_{NF}$. It can easily be checked that $\pi_{0+}^* \pi_{0-} = I$, $U \pi_{0-} = \pi_{0-} U$, and $\pi_{0-} E_2^d(G_, \mathcal{H}_-) = \mathcal{H}_{NF, +}^D$. A mapping $\pi_{0-}^*$ satisfying these conditions is determined up to a constant unitary operator $U$ such that $\pi_{0-}^* = \pi_{0-} U^*$.

Let $\Theta_U := \chi_1^{-1} U \Theta_0^+ \chi_+$ and $\Theta_U := (\chi_+^{-1} \chi_+)^{-1} \Theta_0^+ (\chi_+^{-1} \chi_-)$, where $\Theta_0^+ = \pi_{0-}^* \pi_{0+}$. According to Theorem A the unitary operator $U$ is uniquely determined by the conditions $\Upsilon_+(\zeta) = \Theta_U^+(\zeta) - \Theta_U^+(\zeta)^{-1}$, $\Upsilon_-(\zeta) = - \Theta_U^-(\zeta)$, $\zeta \in C$. Note that these conditions are linear. Thus we can complete recover the characteristic function $\Theta$ for a given transfer function $\Upsilon$.

**Remark 2.3.** For simple-connected domains, we can suggest a more direct procedure [13], which essentially exploits the relation $(\Phi & F)$ and reduces the problem to the case of the unit disk and unitary colligations.

**Remark 2.4.** If the absolutely continuous spectrum of the operator $T$ is rich, e.g., $\text{clos} \Delta^+(\zeta) \mathcal{H}_- = \mathcal{H}_-$, then, by Lemma 2.2, we can restore the characteristic function $\Theta^+$ from the relation $\Upsilon^+ (\zeta) = \Theta^+_U (\zeta) - \Theta^+_U (\zeta)^{-1} = \Upsilon_+ (\zeta) - \Upsilon_-(\zeta)$, $\zeta \in C$.

**Remark 2.5.** We have established the Theorem A under the assumption that the weights $\Xi_1$, $\Xi_2$ are equal. If this assumption is dropped, we lose the uniqueness and the characteristic functions $\Theta_1^+$, $\Theta_2^+$ satisfy some relations (see [14]).

Note that in his paper D. Yakubovich considered only the case of $\Xi$-inner operator valued functions, i.e., he assumed (in our terms) that $\Theta^- = (\Theta^+)^{-1}$.

We pass to the problem to construct the functional model for a given simple (see Remark 1.3) conservative curved system. Let $\Sigma = (T, M, N, \Theta, \Xi)$. So, we can consider its transfer function $\Upsilon$. If $\Upsilon(z) \in \mathcal{N}(G_+ \cup G_-, L(\mathcal{H}_-, \mathcal{H}_+))$, then by the above we are able to restore the corresponding characteristic function $\Theta$. Thus we need to verify that certain construction (=the functional model) agrees with the system $\Sigma$. Such a kind of problem arises in applications [11] and was studied in [10] for the case of $\Xi$-inner functions.

The operator $W$, which realizes a similarity of a system $\Sigma$ to the model system $\overline{\Sigma}$, can easily be calculated: $W = W_{NF}^{-1} V$, where $V \in L(H, \mathcal{K}_{NF})$ and $\forall f \in H$

\[
(V f)_+ := -[M(T - \zeta)^{-1} f]_-, \\
(V f)_- := (\Delta^+)^{-1} \Theta^+ ([M(T - \zeta)^{-1} f]_+ - [M(T - \zeta)^{-1} f]_-).
\]

(Sim)

Therefore we need to verify accordance of the operator $V$ and the functional model.

**Proposition 2.5.** Suppose a system $\Sigma = (T, M, N, \Theta, \Xi)$ is simple and there exists $\Theta \in \text{Cfn}$ such that $\Upsilon = \mathcal{F}_{all}(\Theta)$ and $\Theta^+(z)^{-1}$ possesses boundary values a.e. on $C$, where $\Upsilon(z) = M(T - z)^{-1} N$. Suppose that $\forall f \in H$ there exist $[M(T - \zeta)^{-1} f]_\pm$ a.e. on the curve $C$. Let $V : H \to \mathcal{K}_{NF}$ be defined by formulas (Sim). If $||V|| < \infty$ and $||V^{-1}|| < \infty$. Then $\Sigma \in \text{Sys}$. 

Proof. It follows from the identity $V r_{nz} = \hat{r}_{nz}$. □

More symmetric variant of this assertion can be formulated if we take into account the dual model.

**Proposition 2.6.** Under the hypotheses of Proposition and let the operator $V : H \to K_{+NF}$ be defined by formulas those dual to (Sim). If $||V|| < \infty$ and $||V_*|| < \infty$. Then $\Sigma \in \text{Sys}$. 

**Proof.** Additionally to the identity $V r_{nz} = \hat{r}_{nz}$ we need to make use of the identity $(V r_{nz}, V r_{*mw}) = (r_{nz}, r_{*mw})$. □

**Remark 2.6.** For simple-connected domains and scalar weights, we can present another description for conservative curved systems [9 12 13]:

$$(T, M, N) = (W_0 \varphi(T_0)W_0^{-1}, M_0 \psi_+(T_0)W_0^{-1}, W_0 \psi_-(T_0)N_0),$$

where $\mathfrak{M}_0 = \left( \begin{array}{cc} T_0 & N_0 \\ M_0 & L_0 \end{array} \right)$ is a simple unitary colligation, $W_0 \in \mathcal{L}(H_0, H)$, $W_0^{-1} \in \mathcal{L}(H, H_0), \varphi : \mathbb{D} \to G_+$ is a conformal map, $\psi_+ = \sqrt{\varphi}/(\eta_+ \circ \varphi), \psi_- = \sqrt{\varphi}/(\eta_- \circ \varphi)$, and $\eta_\pm, \eta_-^{-1} \in H^\infty(G_+, \mathcal{L}(\mathfrak{M}_\pm))$.

**Remark 2.7.** Note also that $T$ is the main operator of a system $\Sigma \in \text{Sys}$ iff $T$ is similar to a c.n.u. $G_+$-contraction. An operator $T_0 \in \mathcal{L}(K_0)$ is called a $G_+$-contraction if there exists a Hilbert space $H_0$ and a normal operator $U_0 \in \mathcal{L}(H_0)$ such that $K_0 \subset H_0, (T_0 - z)^{-1} = P_0(U_0 - z)^{-1}|K_0, z \in G_+, \sigma(U_0) \subset C$, where $P_0$ is the orthogonal projection onto the subspace $K_0$. A $G_+$-contraction $T_0$ is called a c.n.u. $G_+$-contraction if the operator $T_0$ have no nontrivial subspaces $K_0 \subset K_0$ such that the operator $T_0|K_0'$ is normal and $\sigma(T_0|K_0') \subset C$.

3. Applications to perturbation theory

In this Section we employ the functional model to establish duality of spectral components [9]. Note that the duality problem was parental for our variant of generalized Sz.-Nagy-Foiaş’s model.

Recall the definitions of the spectral components [21 9 11 25]:

$\tilde{M}(S) = \{ f \in H : \forall g \in H \ ((S - \zeta)^{-1}f, g)_+ = ((S - \zeta)^{-1}f, g)_-, \zeta \in C \};$

$\tilde{N}_+= (S) = \{ f \in H : \forall g \in H \ ((S - z)^{-1}f, g)_+ \in E^2(G_+) \};$

$\tilde{D}_+(S) = \{ f \in H : \forall g \in H \ ((S - z)^{-1}f, g)_+ \in D(G_+) \},$

where $D(G_+) = \{ f : f = \varphi/\delta, \varphi, \delta \in H^\infty(G_+), \delta$ is an outer function $\}$ is the Nevanlinna-Smirnov space [9]. Various combinations of the above linear subspaces are considered: $N = N_+ \cap N_-, \tilde{N}M = \tilde{N} \cap \tilde{M}, \tilde{D}M = \tilde{D} \cap \tilde{M}$. The closure $X(S) = \text{clos} \tilde{X}(S)$, where $\tilde{X} \in \{ \tilde{M}, \tilde{D}_+, \tilde{D}_+, \tilde{N}, \tilde{N}_+, \tilde{N} \}$, is called a (weak) spectral component of the operator $S$. Note that the absolutely continuous $N(S)$ and singular $M(S)$ subspaces are generalizations of the corresponding notions for
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self-adjoint operators \( \mathcal{A} \) and for contractions of \( C_{11}, C_{00} \) classes \([1]\). Concerning
of a spectral meaning of these components we refer the reader to \([2,3,4,5]\). The main theorem from \([6]\) is the following.

**Theorem B.** Suppose \( C \) is a simple closed \( C^{4+\epsilon} \)-smooth curve and \( U,S \in \mathcal{L}(H) \)
are operators such that \( U^*U = UU^*, \) \( S - U \in \mathfrak{S}_1, \) \( \sigma(U) \subset C, \) \( \sigma_+(S) \subset C. \) Then

1) \( N(S)^\perp = M(S^*) ; \) 2) \( N_\pm(S)^\perp = DM_\mp(S^*) ; \) 3) \( NM_\pm(S)^\perp = D_\mp(S^*) . \)

In the present paper we extend this result to the case of multiply connected domains. Note that particular cases of this Theorem play an important role in non-selfadjoint scattering theory and in the theory of extreme factorizations of \( J \)-contraction-valued functions (\( J \)-outer-inner and \( A \)-singular-regular factorizations) (see the corresponding references in \([6]\)).

As in \([8]\) we are going to solve the duality problem by means of the functional model. Since the spectral components appear in the duality relations in a symmetrical way, the both Smirnov classes \( E^2(G_+) \) and \( E^2(G_-) \) ought to be tantamount ingredients of the functional model. By the same reason, the standard trick to involve into a construction the Riemann surface (= double of \( G_+ \), see, e.g., \([8,23]\)) does not work in our case because we lose all geometric information concerning the domain \( G_- \).

We shall use generalized model of S.Naboko \([15,24,14]\). The key observation was done in \([15]\) (see also \([24,14]\)), where it was revealed the possibility to study effectively by means of Sz.-Nagy-Foiaş’s model perturbations of certain form. It is obvious that any trace class perturbation of a self-adjoint operator can be represented in the form \( L = (L_R + i|L_I|) + |L_I|^{1/2}\mathcal{R}|L_I|^{1/2}, \) where \( L_R = \frac{1}{2}(L + L^*), \) \( L_I = \frac{1}{2i}(L - L^*) \) and \( |L_I|^{1/2} \in \mathfrak{S}_2. \) It is an exercise \([14]\) to verify that any trace class perturbation of an unitary operator can be represented in the form \( S = T + (I - T T^*)^{1/2}\mathcal{R}(I - T^*T)^{1/2}, \) where \( T \) is a contraction and \( (I - T T^*)^{1/2}, (I - T^*T)^{1/2} \in \mathfrak{S}_2. \) Therefore any trace class perturbation of self-adjoint or unitary operator can be studied using Naboko’s approach. In our setting these perturbations look like \( S = T + N\mathcal{R}M, \) where \( (T,M,N) \) is a conservative curved system and \( \mathcal{R} \) is a bounded linear operator. It is a non-trivial problem to prove that any trace class perturbation of a normal with the spectrum lying on a curve \( C \) admits such representation. It was the main obstacle, which we surmounted in \([6]\). For \( C^{4+\epsilon} \)-smooth curves we have established the following theorem.

**Theorem.** Suppose \( C \) is a simple closed \( C^{4+\epsilon} \)-smooth curve and \( U,S \in \mathcal{L}(H) \) are operators such that \( U^*U = UU^*, \) \( S - U \in \mathfrak{S}_1, \) \( \sigma(U) \subset C, \) \( \sigma_+(S) \subset C. \) Then the operator \( S \) admits the representation \( S = \varphi(T_0) + N_0\mathcal{R}M_0, \) where \( \varphi \) is a conformal mapping of the unit disk \( \mathbb{D} \) onto the domain \( G_+ \), \( \mathfrak{A}_0 = \begin{pmatrix} T_0 & N_0 \\ M_0 & L_0 \end{pmatrix} \) is a simple unitary colligation, \( T_0^{-1} \in \mathcal{L}(H), \) \( M_0, N_0 \in \mathfrak{S}_2, \) \( \mathcal{R} \in \mathcal{L}(\mathfrak{N}) \) \( (\) without loss of generality it can be assumed that \( \mathfrak{R}_+ = \mathfrak{R}_- = \mathfrak{R}. \)

**Remark 3.1.** In this connection note that this representation lead us to the model with the scalar weights \( \Xi_+ = |\varphi'|I, \) \( \Xi_- = \frac{1}{|\varphi'|}I \) \( (\) cf. Remark \([2,1]. \)
In the theorem was proved under assumption that the domain $G_+$ is simple-connected. Actually, it can be extended to the case of multiply connected domains.

Proposition 3.1. Suppose $C$ is a simple closed $C^{4+\varepsilon}$-smooth curve and $U, S \in L(H)$ are operators such that $U^*U = UU^*$, $S - U \in \mathcal{G}_1$, $\sigma(U) \subset C, \sigma_c(S) \subset C$. Then there exists $\Pi \in \text{Mod}$ and an invertible operator $W$ such that $S = W^{-1} \tilde{S}W = W^{-1}(\hat{T} + \tilde{\mathcal{N}} \times \tilde{\mathcal{M}})W$, $(\hat{T}, \tilde{\mathcal{M}}, \tilde{\mathcal{N}}) = \mathcal{F}_\text{sm}(\Pi)$, $\tilde{\mathcal{M}}, \tilde{\mathcal{N}} \in \mathcal{G}_2$, $\times \in L(\mathfrak{H})$.

Sketch of the proof. Let $C = (\cup_{k=-1}^{n} C_k) \cup C_0, G_- = (\cup_{k=1}^{n} C_k) \cup G_{0-}$, where $G_{k-} := \text{Int} C_k$ and $\infty \in G_{0-} := \text{Ext} C_0$. Using spectral projections from Riesz-Danford calculus, we can construct the operator $\oplus_{k=-n}^{n} S_k$ such that $S_k - U_k \in \mathcal{G}_1, U_k \in \mathcal{G}_2, \sigma(U_k) \subset C_k, \sigma_c(S_k) \subset C$ and the operator $\oplus_{k=0}^{n} S_k$ is similar to the operator $S$. Note that $U_k$ are not necessary the spectral parts of the operator $U$ corresponding to the components $C_k$ (naive candidates to realize similarity can have nonzero index; recall that $\text{ind} V := \dim \text{Ran} V - \dim \text{Ker} V$). In fact, $U_k$ may be obtained using those spectral parts by adding or removing eigenvectors to the corresponding subspace. Then we can reduce our problem to the simple-connected case: $S_k = T_k + N_k \times M_k$. We need only to note that the models of the operators $T_k$ corresponding to the curves $C_k$ and $C$ are similar. □

Thus we have reduced the duality problem to operators of the form $\tilde{S} = \hat{T} + \tilde{\mathcal{N}} \times \tilde{\mathcal{M}}$. Note that, since Naboko's model is a superstructure on Sz.-Nagy-Foiaş's model, we made an extra effort in Section 1 (as well as in \textsc{[4]} \textsc{[12]} \textsc{[13]}) to develop our variant of generalized Sz.-Nagy-Foiaş's model as simple as possible with the aim to focus entirely upon the superstructure by automating calculations on the level of the underlying model.

The structure of the perturbation $\tilde{\mathcal{N}} \times \tilde{\mathcal{M}}$ permits us to compute the resolvent of the operator $\tilde{S}$:

$$(\tilde{S} - z)^{-1}f = (U - z)^{-1}(f + (\pi_+ \times_+^* + \pi_- \times_-^*) n(z)), \quad f \in \mathcal{K}_\Theta,$$

where $n(z) = \Theta_+^*(z)^{-1}(\pi_+^* f)(z), \quad \times_+^* = -I - \Theta_+^* \times, \quad \times_-^* = \times, \quad \Theta_+^* = -\times + \Theta + \Theta^* \Theta_+ \times, \quad \Theta_+ = I - \Theta^* \times \Theta$. Using this formula we obtain the descriptions of the spectral components in terms of the functional model:

$$\tilde{\mathcal{M}}(\tilde{S}) = \{ f \in \mathcal{K}_\Theta : \Theta_+^*(\zeta)^{-1}(\pi_+^* f)(\zeta) = \Theta_+^*(\zeta)^{-1}(\pi_+^* f)(\zeta) \},$$

$$\tilde{\mathcal{N}}_+^\times(\tilde{S}) = \{ f \in \mathcal{K}_\Theta : \pi_+^* f \in \Theta_+^*(z) E^2(G_\times, \mathfrak{H}) \},$$

$$\tilde{\mathcal{D}}_\times(\tilde{S}) = \{ f \in \mathcal{K}_\Theta : \pi_+^* f \in \Theta_+^*(z) E^2(G_\times, \mathfrak{H}) \},$$

where $\Theta^\pm(z) = \Theta^\pm_+(z) \Theta^\pm_\times(z)$ is the inner-outer factorization \textsc{[11]}. Since there exists operators $W$ and $U$ such that $W^{-1} \tilde{S}W - U \in \mathcal{G}_1, U^*U = UU^*$, $\sigma(U) \subset C$ and $\tilde{\mathcal{M}}, \tilde{\mathcal{N}} \in \mathcal{G}_2$, we have $(\Theta^\pm_+)^{-1} \in \mathcal{N}(G_\times, L(\mathfrak{H}))$. Besides, if the curve $C$ is $C^{2+\varepsilon}$ smooth then $\Theta^\pm_+ \in H^\infty(G_\times, L(\mathfrak{H}))$. We "lift" spectral components up to
the level of the “dilation” space $\mathcal{H}$:

\[
M(\Pi, \kappa) = \{ f \in \mathcal{H} : \Theta^+_{\kappa}(\zeta)^{-1}(\pi^+_\kappa f)(\zeta) = \Theta^-_{\kappa}(\zeta)^{-1}(\pi^-_\kappa f)(\zeta) \},
\]
\[
N_{\pm}(\Pi, \kappa) = \{ f \in \mathcal{H} : P_{\pm}(\kappa^*_\Pi \pm \kappa_- \pm \pi^+_\kappa f = 0 \},
\]
\[
D_{\pm}(\Pi, \kappa) = \{ f \in \mathcal{H} : \pi^+_\kappa f \in \Theta^\pm_{\kappa} E^2(G_{\pm}, \mathcal{R}) \}.
\]

For them we have

\[
\tilde{M}(\hat{S}) = M(\Pi, \kappa) \cap K\Theta, \quad \tilde{N}_{\pm}(\hat{S}) = P_\Theta N_{\pm}(\Pi, \kappa), \quad \tilde{D}_{\pm}(\hat{S}) = D_{\pm}(\Pi, \kappa) \cap K\Theta.
\]

The possibility itself of such lifting depends heavily on the structure of the perturbation in question. The above mentioned properties $(\Theta^\pm_{\kappa})^{-1} \in N(G_{\pm}, L(\mathcal{R}))$ and $\Theta^\pm_{\kappa} \in H^\infty(G_{\pm}, L(\mathcal{R}))$ are also crucial for this lifting.

Using the dual model one can prove that $N(\Pi, \kappa)^\perp = M(\Pi^*, \kappa^*)$, $N_{\pm}(\Pi, \kappa)^\perp = D_{\mp}(\Pi^*, \kappa^*)$, and $N_{\pm}(\Pi, \kappa)^\perp = D_{\mp}(\Pi^*, \kappa^*)$. Then we can rewrite the proof from [9] almost literally. We need only to use some notions from [9] in extended meaning, e.g., in Prop.4.9(4) [9] we need to replace the subspaces $E_{\Theta^\pm_{\kappa^*}}$ by $E_{\Omega^\pm_{\kappa^*}}$, where $\Omega^\pm$ is multipliers that counteracts a possible multiple-valuedness of $\Theta^\pm_{\kappa^*}$. If the boundary of the domain $G_{\pm}$ is $C^{2+\epsilon}$ smooth, the operator valued functions $\Omega^\pm$ can be chosen $C^{2+\epsilon}$ smooth too. The latter has as consequence [9, 13] that the weak smooth vectors are dense in the subspaces $N_{\pm}(\hat{S})$.

References

[1] Szőkefalvi-Nagy B., Foiaş C., Harmonic analysis of operators on Hilbert space. North-Holland, Amsterdam-London, 1970.

[2] Brodskiy M. S., Unitary operator nodes and their characteristic functions, Uspehi mat. nauk, 33 (1978), no. 4, 141–168.

[3] Duren P. L., Theory of $H^p$ spaces, Pure Appl. Math., vol. 38, Academic Press, New York–London, 1970.

[4] Bart H., Gohberg I., Kaashoek M.A., Minimal factorization of matrix and operator functions. Operator Theory: Adv. and Appl., 1979.

[5] Yafaev D. R., Mathematical scattering theory: General theory, Translations of Mathematical Monographs, 105 (1992), AMS, Providence, Rhode Island.

[6] Abrahamse M. B., Douglas R. G. A class of subnormal operators related to multiply connected domains, Adv. in Math., 19 (1976), 106–148.

[7] Ball J. A., Operators of class $C_{00}$ over multiply connected domains, Michigan Math. J., 25 (1978), 183–195.

[8] Pavlov B. S., Fedorov S. I. Group of shifts and harmonic analysis on Riemann surface of genus one, Algebra i Analiz, 1 (1989), no.2, 132–168.

[9] Tikhonov A. S., Functional model and duality of spectral components for operators with continuous spectrum on a curve. Algebra i Analiz, 14 (2002), no.4, 158–195.

[10] Yakubovich D. V., Linearly similar model of Sz.-Nagy – Foias type in a domain. Algebra i Analiz, 15 (2003), no.2, 180–227.
[11] Verduyn Lunel S. M., Yakubovich D. V.  *A functional model approach to linear neutral functional differential equations*, Integr.Equ.Oper.Theory, 27 (1997), 347–378.

[12] Tikhonov A.S., *Free functional model related to simply-connected domains*. Operator Theory: Adv. and Appl., Vol.154 (2004), 405-415.

[13] Tikhonov A.S., *Transfer functions for "curved" conservative systems*. Operator Theory: Adv. and Appl., Vol.153 (2004), 255-264.

[14] Makarov N. G., Vasyunin V. I., *A model for noncontraction and stability of the continuous spectrum*. Lect. Notes in Math., 864 (1981), 365–412.

[15] Naboko S. N., *Functional model for perturbation theory and its applications to scattering theory*. Trudy Mat. Inst. Steklov 147 (1980), 86–114.

[16] Nikolski N. K., Vasyunin V. I., *Elements of spectral theory in terms of the free functional model. Part I: Basic constructions*, Holomorphic spaces (eds. Sh. Axler, J. McCarthy, D. Sarason), MSRI Publications 33 (1998), 211–302.

[17] Suetin P. K., *Series of Faber polynomials*. Nauka, Moscow, 1984.

[18] Gaier D., *Lectures on approximation in the complex domain*. Birkhauser, Basel-Boston, 1980.

[19] Tikhonov A. S. *Boundary values of operator-valued functions and trace class perturbations*. Rom. J. of Pure and Appl. Math., 47 (2002), no.5–6, 761–767.

[20] Sarason D. E. *The $H^p$ spaces of an annulus*. Mem.Amer.Math.Soc., 56 (1956).

[21] Bungart L. *On analytic fiber bundles. 1*, Topology, 7 (1968), 55–68.

[22] Grauert H. *Analytische Faserungen über holomorph vollständigen Räumen*. Math.Ann., 135 (1958), 263–273.

[23] Fedorov S. I., *On harmonic analysis in multiply connected domain and character-automorphic Hardy spaces*. Algebra i Analiz, 9 (1997), no.2, 192–239.

[24] Naboko S. N., *On the spectral analysis of nonselfadjoint operators*. Dokl. Akad. Nauk SSSR 232 (1977), no. 1, 36–39; English transl., Soviet Math. Dokl. 18 (1977), no. 1, 32–36.

[25] Makarov N. G., *Canonical subspaces of almost unitary operators*. A. Haar Memorial Conference. Vol. 1, 2 (Budapest, 1985), Colloq. Math. Soc. János Bolyai, vol. 49, North-Holland, Amsterdam–New York, 1985, 611–621.

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