Maximum Matching on Trees in the Online Preemptive and the Incremental Dynamic Graph Models

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Abstract

We study the Maximum Cardinality Matching (MCM) and the Maximum Weight Matching (MWM) problems on trees in the Online Preemptive model, and in the Incremental Dynamic Graph model. In the Online Preemptive model, the edges of a graph are revealed one by one and the algorithm is required to always maintain a valid matching. On seeing an edge, the algorithm has to either accept or reject the edge. If accepted, then the adjacent edges are discarded, and all rejections are permanent. In this model, the complexity of the problems is settled for deterministic algorithms \cite{10, 13}. Epstein et al. \cite{4} gave a 5.356-competitive randomized algorithm for MWM, and also proved a lower bound of 1.693 for MCM. The same lower bound applies for MWM. In a dynamic graph, at each update step an edge is added or deleted from the graph. If only insertions are allowed, then the graph is said to be an Incremental Dynamic Graph. Gupta \cite{6} proved that for any $\epsilon \leq 1/2$, there exists an algorithm that maintains a $(1 + \epsilon)$-approximate MCM for an incremental bipartite graph in an “amortized” $O\left(\frac{\log^2 n}{\epsilon^4}\right)$ update time.

No better bounds are known even in special cases. In this paper we show that some of the results can be improved in the case of trees. In the online preemptive model, we present a 3-competitive barely random algorithm (that uses only $O(1)$ bits of randomness) for MWM on growing trees (where the new edge revealed will be incident on some vertex of the already seen tree), and also present a 64/33-competitive barely random algorithm (which uses only two bits of randomness) for MCM on trees.

Inspired by the above mentioned algorithm for MCM, we present a $3/2$-approximate (in expectation) randomized algorithm for MCM on trees with a “worst case” update time of $O(1)$, in the incremental dynamic graph model. No $(2 - \epsilon)$-approximation algorithms with $O(1)$ worst case update time are known in this model even for trees.
1 Introduction

The Maximum (Cardinality/Weight) Matching problem is one of the most extensively studied problems in Combinatorial Optimization. See Schrijver’s book [12] and references therein for a comprehensive overview of classic work. A matching $M \subseteq E$ is a set of edges such that at most one edge is incident on any vertex. Traditionally the problem was studied in the offline setting where the entire input is available to the algorithm beforehand. But over the last few decades it has been extensively studied in various other models where the input is revealed in pieces, like the vertex arrival model (adversarial and random), the edge arrival model (adversarial and random), streaming and semi-streaming models, the online preemptive model, etc. [9, 4, 3, 10, 5, 8]. In this paper, we study the Maximum Cardinality Matching (MCM) and Maximum Weight Matching (MWM) problems on trees in the Online Preemptive model, and in the Incremental Dynamic Graph model.

In the online preemptive model, the edges appear in online manner, and the algorithm is supposed to accept or reject an edge on arrival. If accepted, the algorithm can reject it later, and all rejections are permanent. The algorithm is supposed to maintain a valid matching at every stage. In this model, there is no restriction on how much memory an algorithm can use, or how much processing time the algorithm can take to process after an edge is revealed. There is a 5.828-competitive deterministic algorithm due to McGregor [10] for MWM, and a tight lower bound on deterministic algorithms due to Varadaraja [13]. Epstein et al. [4] gave a 5.356-competitive randomized algorithm for MWM, and also proved a 1.693 lower bound on the competitive ratio achievable by any randomized algorithm for MCM. No better lower bound is known for MWM.

In [2], the authors give a 28/15-competitive randomized algorithm for MCM on growing trees (defined below). We extend this in two ways. We present a 3-competitive barely random (which uses only $O(1)$ bits of randomness) algorithm (described in Section 2) for MWM on growing trees (where the new edge seen revealed will be incident on some vertex of the already revealed tree) in the online preemptive model. (Note that the randomized algorithm for MWM due to Epstein et al. [4] uses infinite bits of randomness, whereas our algorithm only uses $O(1)$ bits.) Although, growing trees is a very restricted class of graphs, there are a couple of reasons to study the performance of the algorithm on this class of input. Firstly, almost all lower bounds, including the one due to Varadaraja [13] for MWM is on growing trees. Secondly, even for this restricted class, the analysis is involved. We use the primal-dual technique for analyzing the performance of this algorithm. However, new ideas are needed for the generalizations in this paper. We show that this analysis is indeed tight by giving an example, for which the algorithm achieves the competitive ratio 3.

In [2], the authors gave the first randomized algorithm with competitive ratio less than 2 for MCM in the online preemptive model, also on growing trees. In Section 3 we extend their algorithm to give a 64/33-competitive barely random (which uses only two bits of randomness) algorithm for trees.

In recent years, algorithms for approximate MCM in dynamic graphs [1, 6, 7]
has been the focus of many studies due to their wide range of applications. The objective of these dynamic graph algorithms is to efficiently process an online sequence of update operations, such as edge insertions and deletions. It has to quickly maintain an approximate maximum matching despite an adversarial order of edge deletions and insertions. Dynamic graph problems are usually classified according to the types of updates allowed: decremental models allow only deletions, incremental models allow only insertions, and fully dynamic models allow both. We study MCM on trees in the Incremental model. Gupta \cite{6} proved that for any $\epsilon \leq 1/2$, there exists an algorithm that maintains a $(1 + \epsilon)$-approximate MCM on bipartite graphs in the incremental model in an “amortized” update time of $O\left(\frac{\log^2 n}{\epsilon^4}\right)$. We present a $3/2$-approximate (in expectation) randomized algorithm (described in Section 4) for MCM on trees in the incremental model with a “worst case” update time of $O(1)$. This algorithm is inspired by the randomized algorithm (discussed above) for MCM on trees in the online preemptive model.

**Online Preemptive Model vs. Incremental Dynamic Graph Model:**
There are two main differences between these models. Firstly, in the online preemptive model, once an edge is rejected/removed from the matching maintained by the algorithm, it cannot be added into its matching, whereas in the incremental dynamic graph model, rejected/removed edges can be added to the matching later on. Secondly, there is no restriction on how much time an algorithm in the online preemptive model can use to process a revealed edge, whereas in the incremental dynamic graph model, the algorithm is supposed to process the revealed edge fast. The term “fast” is used loosely, and is specific to any problem. For example, MCM on general graphs can be found in time $O(m\sqrt{n})$ when the entire input is available \cite{11}. But for dynamic graphs, every time an edge is inserted, the algorithm is expected to maintain a matching, approximate if not exact, in time lower than the time required by the optimal offline algorithm for MCM (say, for instance, in $O(polylog n)$ amortized time).

## 2 Barely Random Algorithms for MWM

In this section, we present a barely random algorithm (that uses only $O(1)$ bits of randomness) for MWM on growing trees in the online preemptive model. We describe the motivation for such an algorithm.

### 2.1 Motivation

McGregor’s deterministic algorithm \cite{10} for MWM is easy to describe – if the weight of the new edge is more than $(1 + \gamma)$ times the weight of the conflicting edges in current matching, then evict them and add the new edge. The algorithm is $(1 + \gamma)(2 + 1/\gamma)$-competitive, and attains the best competitive ratio of $3 + 2\sqrt{2} \approx 5.828$ for $\gamma = \frac{1}{\sqrt{2}}$. It achieves this competitive ratio for the following example. Start by presenting an edge of weight $x_0 = 1$ to the algorithm. This edge will be added to the matching. Assume inductively that after iteration
i, the algorithm’s matching has only the edge of weight \( x_i \). In iteration \( i + 1 \), give an edge of weight \( y_{i+1} = (1 + \gamma)x_i \) on one end point of \( x_i \). This edge will not be accepted in the algorithm’s matching. Give an edge of weight \( x_{i+1} = (1 + \gamma)x_i + \epsilon \) on the other end point of \( x_i \). This edge will be accepted in the algorithm’s matching, and \( x_i \) will be evicted. This process terminates for some large \( n \), letting \( x_{n+1} = (1 + \gamma)x_n \). The edge of weight \( x_{n+1} \) will not be accepted in the algorithm’s matching. The algorithm will hold only the edge of weight \( x_n \), whereas the optimum matching would include edges of weight \( y_1, \ldots, y_{n+1}, x_{n+1} \). It can be easily inferred that this gives the required lower bound on the competitive ratio.

Notice that the edges presented in the example crucially depended on \( \gamma \). To beat this, we use two algorithms with \( \gamma \) values \( \gamma_1 \) and \( \gamma_2 \) respectively, and choose one at random. We describe the algorithm next.

**Algorithm 1** Randomized Algorithm for MWM

1. Maintain two matchings \( M_1 \) and \( M_2 \). Let \( j = 1 \) with probability \( p \), and \( j = 2 \) otherwise.

2. On receipt of an edge \( e \):
   For \( i = 1, 2 \), if \( w(e) > (1 + \gamma_i)w(X(M_i, e)) \), then \( M_i = M_i \setminus X(M_i, e) \cup \{e\} \) (where \( X(M_i, e) \) is the set of conflicting edges to \( e \) in \( M_i \)).

3. Output \( M_j \).

Note that we cannot just output the best of two matchings because that could violate the constraints of the online preemptive model.

### 2.2 Analysis

We use the primal-dual technique to analyze the performance of this algorithm. The primal-dual technique used to analyze McGregor’s deterministic algorithm for MWM described in [2] is fairly straight forward. However the management becomes complicated with the introduction of randomness, and we are only able to analyze the algorithm in the very restricted setting of growing trees.

**Theorem 1.** The competitive ratio of Algorithm 1 on growing trees is

\[
\max \left\{ \frac{1 + \gamma_1}{p}, \frac{1 + \gamma_2}{1 - p}, \frac{(1 + \gamma_1)(1 + \gamma_2)(1 + 2\gamma_1)}{p \cdot \gamma_1 + (1 - p)\gamma_2 + \gamma_1\gamma_2} \right\}.
\]

The primal and dual LPs used for the analysis are as follows.

| Primal LP | Dual LP |
|-----------|---------|
| \[ \max \sum_v \sum_e w_v x_e \] \[ \forall v : \sum_{v \in E} x_e \leq 1, x_e \geq 0 \] | \[ \min \sum_v y_v \] \[ \forall e : y_u + y_v \geq w_e, y_v \geq 0 \] |
We maintain both primal and dual variables along with the run of the algorithm. Consider a round in which an edge \( e \equiv (u, v) \) is revealed, where \( v \) is the new vertex. Before \( e \) is revealed, let \( e_1 \) and \( e_2 \) be the edges incident on \( u \) which belonged to \( M_1 \) and \( M_2 \) respectively. If such an \( e_i \) does not exist, then we may assume \( w(e_i) = 0 \). The primal and dual variables are updated as follows.

- If \( e \) is rejected by both matchings, we set the primal variable \( x_e = 0 \), and the dual variable \( y_v = 0 \).
- If \( e \) is added to \( M_1 \) only, then we set the primal variable \( x_e = p \), and the dual variable \( y_u = \max(y_u, \min((1 + \gamma_1)w(e), (1 + \gamma_2)w(e_2))) \), and \( y_v = 0 \).
- If \( e \) is added to \( M_2 \) only, then we set the primal variable \( x_e = 1 - p \), and the dual variable \( y_u = \max(y_u, \min((1 + \gamma_1)w(e_1), (1 + \gamma_2)w(e))) \), and \( y_v = 0 \).
- If \( e \) is added to both the matchings, then we set the primal variable \( x_e = 1 \), and the dual variables \( y_u = \max(y_u, (1 + \gamma_1)w(e)) \) and \( y_v = (1 + \gamma_1)w(e) \).
- If an edge \( e' \) is evicted from \( M_1 \) (or \( M_2 \)), we decrease its primal variable \( x_e \) by \( p \) (or \( 1 - p \) respectively), and the corresponding dual variables are unchanged.

We begin with three simple observations.

1. The cost of the primal solution is equal to the expected weight of the matching maintained by the algorithm.
2. The dual variables never decrease. Hence, if a dual constraint is feasible once, it remains so.
3. \( y_u \geq \min((1 + \gamma_1)w(e_1), (1 + \gamma_2)w(e_2)) \).

The idea behind the analysis is to prove a bound on the ratio of the dual cost and the primal cost while maintaining dual feasibility. By Observation 2, to ensure dual feasibility, it is sufficient to ensure feasibility of the dual constraint of the new edge. If the new edge \( e \) is not accepted in any \( M_i \), then \( w(e) \leq \min((1 + \gamma_1)w(e_1), (1 + \gamma_2)w(e_2)) \). Hence, the dual constraint is satisfied by Observation 3. Else, it can be seen that the dual constraint is satisfied by the updates performed on the dual variables.

The following lemma implies Theorem 1.

**Lemma 2.** \( \frac{\Delta_{Dual}}{\Delta_{Primal}} \leq \max \left\{ \frac{1+\gamma_1}{p}, \frac{1+\gamma_2}{1-p}, \frac{(1+\gamma_1)(1+\gamma_2)}{p(1+\gamma_1)(1+\gamma_2)} \right\} \) after every round.

We will use the following simple technical lemma.

**Lemma 3.** \( \frac{ax+b}{cx+d} \) increases with \( x \) iff \( ad - bc \geq 0 \).

**Proof.** (of the Lemma 2) There are four cases to be considered.
1. If edge $e$ is accepted in $M_1$, but not in $M_2$. Then $(1 + \gamma_1)w(e_1) < w(e) \leq (1 + \gamma_2)w(e_2)$. By Observation 3, before $e$ was revealed, $y_u \geq (1 + \gamma_1)w(e_1)$. After $e$ is accepted in $M_1$, $\Delta\text{Primal} = p(w(e) - w(e_1))$, and $\Delta\text{Dual} \leq (1 + \gamma_1)(w(e) - w(e_1))$. Hence,

$$\frac{\Delta\text{Dual}}{\Delta\text{Primal}} \leq \frac{(1 + \gamma_1)}{p}.$$ 

2. If edge $e$ is accepted in $M_2$, but not in $M_1$. Then $(1 + \gamma_2)w(e_2) < w(e) \leq (1 + \gamma_1)w(e_1)$. By Observation 3, before $e$ was revealed, $y_u \geq (1 + \gamma_2)w(e_2)$. After $e$ is accepted in $M_2$, $\Delta\text{Primal} = (1 - p)(w(e) - w(e_2))$, and $\Delta\text{Dual} \leq (1 + \gamma_2)(w(e) - w(e_2))$. Hence,

$$\frac{\Delta\text{Dual}}{\Delta\text{Primal}} \leq \frac{(1 + \gamma_2)}{1 - p}.$$ 

3. If edge $e$ is accepted in both the matchings, and $(1 + \gamma_1)w(e_1) \leq (1 + \gamma_2)w(e_2) < w(e)$. By Observation 3, before $e$ was revealed, $y_u \geq (1 + \gamma_1)w(e_1)$. After $e$ is accepted in both the matchings, $\Delta\text{Dual} \leq (1 + \gamma_1)(2w(e) - w(e_1))$. The change in primal cost is

$$\Delta\text{Primal} \geq w(e) - p \cdot w(e_1) - (1 - p) \cdot w(e_2)$$

$$\geq w(e) - p \cdot w(e_1) - (1 - p) \cdot \frac{w(e)}{1 + \gamma_2}$$

$$= \frac{p + \gamma_2}{1 + \gamma_2}w(e) - p \cdot w(e_1).$$

$$\Delta\text{Dual} \leq (1 + \gamma_1)\frac{2w(e) - w(e_1)}{\frac{p + \gamma_2}{1 + \gamma_2}w(e) - p \cdot w(e_1)}.$$ 

By Lemma 3, this value increases, for a fixed $w(e)$, with $w(e_1)$ if $\gamma_2 \leq \frac{p}{1 - 2p}$, and its worst case value is achieved when $(1 + \gamma_1)w(e_1) = w(e)$. Thus,

$$\frac{\Delta\text{Dual}}{\Delta\text{Primal}} \leq (1 + \gamma_1)\frac{2(1 + \gamma_1)(1 + \gamma_2) - (1 + \gamma_2)}{(p + \gamma_2)(1 + \gamma_1) - p(1 + \gamma_2)}$$

$$= (1 + \gamma_1)(1 + \gamma_2)\frac{1 + 2\gamma_1}{p \cdot \gamma_1 + (1 - p)\gamma_2 + \gamma_1\gamma_2}.$$ 

4. If $e$ is accepted in both the matchings, and $(1 + \gamma_2)w(e_2) \leq (1 + \gamma_1)w(e_1) < w(e)$. By Observation 3, before $e$ was revealed, $y_u \geq (1 + \gamma_2)w(e_2)$. After $e$ is accepted in both the matchings, $\Delta\text{Dual} \leq 2(1 + \gamma_1)w(e) - (1 + \gamma_2)w(e_2).$
The change in primal cost is
\[ \Delta \text{Primal} \geq w(e) - p \cdot w(e_1) - (1 - p) \cdot w(e_2) \]
\[ \geq w(e) - p \cdot \frac{w(e)}{1 + \gamma_1} - (1 - p)w(e_2) \]
\[ = \frac{1 - p + \gamma_1}{1 + \gamma_1} w(e) - (1 - p)w(e_2). \]

So, \[ \frac{\Delta \text{Dual}}{\Delta \text{Primal}} \leq (1 + \gamma_1) \frac{2(1 + \gamma_1)w(e) - (1 + \gamma_2)w(e_2)}{(1 - p + \gamma_1)w(e) - (1 - p)(1 + \gamma_1) \cdot w(e_2)}. \]

By Lemma 3 this value increases, for a fixed \( w(e) \), with \( w(e_2) \) if \( 2(1 + \gamma_1)^2(1 - p) - (1 - p + \gamma_1)(1 + \gamma_2) \geq 0 \), and its worst case value is achieved when \((1 + \gamma_2)w(e_2) = w(e)\). Again,

\[ \frac{\Delta \text{Dual}}{\Delta \text{Primal}} \leq (1 + \gamma_1)(1 + \gamma_2) \frac{1 + 2\gamma_1}{p \cdot \gamma_1 + (1 - p)\gamma_2 + \gamma_1 \gamma_2}. \]

The following theorem is an immediate consequence of Theorem 1.

**Theorem 4.** The randomized algorithm of MWM is \( 3 \)-competitive on growing trees, when \( p = \frac{1}{3} \), \( \gamma_1 = 0 \), and \( \gamma_2 = 1 \); and there exists an input where algorithm achieves this competitive ratio.

The input where the algorithm is \( 3 \)-competitive is as follows. Start by presenting an edge of weight \( x_0 = 1 \). It will be added to both \( M_1 \) and \( M_2 \). Assume inductively, that currently both matching only contain an edge of weight \( x_i \). Present an edge of weight \( y_{i+1} = x_i \) on one end point of \( x_i \). This edge will not be accepted in either of the matchings. Present an edge of weight \( x_{i+1} = 2 \cdot x_i + \epsilon \) on the other end point of \( x_i \). This edge will be accepted in both the matchings, and \( x_i \) will be evicted. For a sufficiently large value \( n \), let \( x_{n+1} = x_n \). So edge of weight \( x_{n+1} \) will not be accepted in either of the matchings. Both the matchings will hold only the edge of weight \( x_n \), whereas the optimum matching would include edges of weight \( y_1, \ldots, y_{n+1}, x_{n+1} \). The weight of the matching stored by the algorithm is \( 2^n \), whereas the weight of the optimum matching is \( \approx 3 \cdot 2^n \) (we have ignored the \( \epsilon \) terms here). This gives the competitive ratio \( 3 \).

### 3 Barely Random Algorithm for MCM on trees

In this section, we present a barely random algorithm for MCM on trees in the online preemptive model. It is an extension of the algorithm described in [2] for MCM on growing trees.
Algorithm 2 Randomized Algorithm for Trees

1. The algorithm maintains four matchings: $M_1, M_2, M_3,$ and $M_4$. Pick $k \in \text{u.a.r.} \ [4]$.

2. On receipt of an edge $e$, the processing happens in two phases.
   (a) **The augment phase.** The new edge $e$ is added to each $M_i$ in which there are no edges adjacent to $e$.
   (b) **The switching phase.** For $i = 2, 3, 4,$ in order, $e$ is added to $M_i$ (if it was not added in the previous phase) and the conflicting edges (denoted by $X(M_i, e)$) are discarded, provided it decreases the quantity $\sum_{j \in [4], i \neq j} |X(M_i \cap M_j, e)| = |X(M_i, e)| |M_i \cap M_j|$.

3. Output matching $M_k$.

Note that in the switching phase, the expected size of the matching stored by the algorithm might decrease. For example, consider two disjoint edges $e_1$ and $e_2$ that have been revealed. Each of them will belong to all four matchings. So the expected size of the matching stored by the algorithm is 2. Now, if an edge $e$ is revealed between $e_1$ and $e_2$, then $e$ will be added to $M_2$ and $M_3$. The expected size of the matching is now 1.5. The important thing to notice here is that the decrease is not too much, and we are able to prove that the competitive ratio of the algorithm still remains below 2.

We will say that a vertex (an edge) is **covered** by a matching $M_i$ if there is an edge in $M_i$ which is incident on (adjacent to) the vertex (edge). We also say that an edge is covered by a matching $M_i$ if it belongs to $M_i$. We begin with the following observations.

- After an edge is revealed, its end points are covered by all four matchings.
- An edge $e$ that does not belong to any matching has four edges incident on its end points such that each of these edges belong to a distinct matching. This holds when the edge is revealed, and does not change subsequently.
- Every edge is covered by at least three matchings.

An edge is called **internal** if there are edges incident on both its end points which belong to some matching. An edge is called a **leaf edge** either if one of its end point is a leaf or if all the edges incident on one of its end point do not belong to any matching. An edge is called **bad** if its end points are covered by only three matchings.

We begin by proving some properties about the algorithm. The key structural lemma that keeps “influences” of bad edges local is given below.

**Lemma 5.** At most five consecutive vertices on a path can have bad edges incident on them.
According to the Lemma 5, there can be at most four consecutive internal bad edges or at most five bad leaf edges incident on five consecutive vertices of a path. The proof of this lemma is in the Appendix A.

**Theorem 6.** The randomized algorithm for finding MCM on trees is $\frac{64}{33}$-competitive.

The primal-dual technique is used to analyze the performance of this algorithm. Here are the well known Primal and Dual formulations of the matching problem. The primal formulation is known to be optimum for bipartite graphs. For general graphs, odd set constraints have to be added. But they are not needed in this paper.

| Primal LP | Dual LP |
|-----------|---------|
| $\max \sum x_e$ | $\min \sum y_v$ |
| $\forall v : \sum_{e \in v} x_e \leq 1$ | $\forall e : y_u + y_v \geq 1$ |
| $x_e \geq 0$ | $y_v \geq 0$ |

Once all edges have been seen, we distribute the primal charge among the dual variables, and use the primal-dual framework to infer the competitive ratio. If end points of every edge are covered with four matchings, then the distribution of dual charge is easy. However we do have bad edges, and would like the edges in matchings to contribute more to the end-points of these edges. Then, the charge on the other end-point would be less and we need to balance this through other edges. Details follow.

**Lemma 7.** There exists an assignment of the primal charge to the dual variables such that the dual constraint for each edge $e \equiv (u, v)$ is satisfied at least $\frac{33}{64}$ in expectation, i.e. $\mathbb{E}[y_u + y_v] \geq \frac{33}{64}$.

**Proof.** Root the tree at an arbitrary vertex. For any edge $e \equiv (u, v)$, let $v$ be the parent vertex, and $u$ be the child vertex. The dual variable assignment is done after the entire input is seen, as follows.

**Dual Variable Management:** An edge $e$ will distribute its primal weight between its end-points. The exact values are discussed below. In general, we look to transfer all of the primal charge to the parent vertex. But this does not work and we need a finer strategy. This is detailed below.

- If $e$ does not belong to any matching, then it does not contribute to the value of dual variables.
- If $e$ belongs to a single matching then, depending on the situation, one of $0, 2\epsilon, 3\epsilon, 4\epsilon, \text{ or } 5\epsilon$ of its primal charge will be assigned to $u$ and rest will be assigned to $v$. The constant $\epsilon$ may be chosen to be $1/16$.
- If $e$ belongs to two matchings, then at most $6\epsilon$ of its primal charge will be assigned to $u$ as required. The rest is assigned to $v$.
- If $e$ belongs to three or four matchings, then its entire primal charge is assigned to $v$. 
We will show that $y_u + y_v \geq 2 + \epsilon$ for such an edge, when summed over all four matchings. The value of $\epsilon$ is chosen later.

For the sake of analysis, if there are bad leaf edges incident on both end points of an internal edge, then we analyze it as a bad internal edge. We need to do this because a bad leaf edge might need to transfer its entire primal charge to the vertex on which there are edges which do not belong to any matching. Note that the end points of the internal edge would still be covered by three matchings, even if we consider that the bad leaf edges do not exist on its end points. The analysis breaks up into eight cases.

**Case 1.** Suppose $e$ does not belong to any matching. There must be a total of at least 4 edges incident on $u$ and $v$ besides $e$, each belonging to a distinct matching. Of these 4, at least a total of 3 say $e_1$, $e_2$, and $e_3$, must be between some children of $u$ and $v$, to $u$ and $v$ respectively. The edges $e_1$, $e_2$, and $e_3$, each assign a charge of at least $1 - 5\epsilon$ to $y_u$ and $y_v$, respectively. Therefore, $y_u + y_v \geq 3 - 15\epsilon \geq 2 + \epsilon$.

**Case 2.** Suppose $e$ is a bad leaf edge that belongs to a single matching, and internal edges are incident on $v$. This implies that, there are two edges $e_1$ and $e_2$ from $v$’s child vertex to $v$, which belongs to single matching, and another edge $e_3$, also belonging to single matching, from $v$ to its parent vertex. The edge $e$ assigns a charge of $1$ to $y_v$. If $e_1$ assigns a charge of $1 + \epsilon$ or $1 - 2\epsilon$ or $1 - \epsilon$ to $y_v$, then $e_2$ assigns $\epsilon$ or $2\epsilon$ or $3\epsilon$ to $y_v$. If $e_2$ assigns $\epsilon$ or $2\epsilon$ or $3\epsilon$ to $y_v$, respectively to $y_u$. In either case, $y_u + y_v = 2 + \epsilon$. The key fact is that $e_1$ could not have assigned $5\epsilon$ to its child vertex. Since, then, by Lemma 5 $e$ cannot be a bad edge.

**Case 3.** Suppose $e$ is a bad leaf edge that belongs to a single matching, and internal edges are incident on $u$. This implies that, there are two edges $e_1$ and $e_2$ from $u$’s children to $u$, each belonging to a single distinct matching. The edge $e$ assigns a charge of $1$ to $y_u$. Both $e_1$ and $e_2$ assign a charge of at least $1 - 4\epsilon$ to $y_u$. In either case, $y_u + y_v \geq 3 - 8\epsilon \geq 2 + \epsilon$. The key fact is that neither $e_1$ nor $e_2$ could have assigned more than $4\epsilon$ to their corresponding child vertex. Since, then, by Lemma 5 $e$ cannot be a bad edge.

**Case 4.** Suppose $e$ is an internal bad edge. This implies (by Lemma 5) that, there is an edge $e_1$ from a $u$’s child vertex to $u$, which belongs to a single matching. Also, there is an edge $e_2$, from $v$ to its parent vertex (or from a $v$’s child vertex to $v$), which also belongs to a single matching. The edge $e$ assigns its remaining charge (1 or $1 - \epsilon$ or $1 - 2\epsilon$ or $1 - 3\epsilon$ or $1 - 4\epsilon$) to $y_v$. If $e_1$ assigns a charge of $1 + \epsilon$ or $1 - 2\epsilon$ or $1 - \epsilon$ or $1 - 4\epsilon$ to $y_u$, then $e_2$ assigns $2\epsilon$ or $3\epsilon$ or $4\epsilon$ or $5\epsilon$ respectively to $y_u$. In either case, $y_u + y_v = 2 + \epsilon$. The key fact is that $e_1$ could not have assigned $5\epsilon$ to its child vertex. Since, then, by Lemma 5 $e$ cannot be a bad edge.

**Case 5.** Suppose $e$ is not a bad edge, and it belongs to a single matching. Then either there are at least two edges $e_1$ and $e_2$ from $u$’s or $v$’s children to $u$ or $v$ respectively, or $e_1$ on $u$ and $e_2$ on $v$, each belonging to single matching, or one edge $e_3$ from $u$’s or $v$’s child vertex to $u$ or $v$, respectively, which belongs to two matchings, or one edge $e_4$ from $u$’s or $v$’s child vertex to $u$ or $v$, respectively, which belongs to single matching, and one edge $e_5$ from $v$ to its parent vertex which belongs to two matchings. In either case, $y_u + y_v \geq 3 - 10\epsilon \geq 2 + \epsilon$. 

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Case 6. Suppose $e$ is a bad edge that belongs to two matchings, and internal edge is incident on $u$ or $v$. This implies that there is an edge $e_1$, from $u$'s child vertex to $u$ or from $v$ to its parent vertex which belongs to a single matching. The edge $e$ assigns a charge of 2 to $y_v$, and the edge $e_1$ assigns a charge of $\epsilon$ to $y_u$ or $y_v$ respectively. Thus, $y_u + y_v = 2 + \epsilon$.

Case 7. Suppose $e$ is not a bad edge and it belongs to two matchings. This means that either there is an edge $e_1$ from a $u$'s child vertex to $u$, which belongs to at least one matching, or there is an edge from $v$'s child vertex to $v$ that belongs to at least one matching, or there is an edge from $v$ to its parent vertex which belongs to two matchings. The edge $e$ assigns a charge of 2 among $y_u$ and $y_v$. The neighboring edges assign a charge of $\epsilon$ to $y_u$ or $y_v$ (depending on which vertex it is incident), to give $y_u + y_v \geq 2 + \epsilon$.

Case 8. Suppose, $e$ belongs to 3 or 4 matchings, then trivially $y_u + y_v \geq 2 + \epsilon$.

From the above conditions, the best value for the competitive ratio is obtained when $\epsilon = \frac{1}{16}$, yielding $E[y_u + y_v] \geq \frac{33}{64}$.

Lemma 7 implies that the competitive ratio of the algorithm is at most $\frac{64}{33}$.

4 Algorithm for MCM on trees in the Incremental Dynamic Graph Model

In this section, we present a randomized algorithm for MCM on trees in the incremental dynamic graph model. It is inspired by the randomized algorithm for MCM on trees described in Section 3. In the online preemptive model, we cannot add edges in the matching which were discarded earlier, which results in the existence of bad edges. But in the incremental dynamic graph model, there is no such restriction, and we show how we can store a small amount of extra information to ensure that there are no bad edges. As a consequence, we get a better approximation ratio, and also, the analysis also becomes significantly simpler. Details follow.
Algorithm 3 Randomized Algorithm for MCM on Trees

1. The algorithm maintains three matchings: $M_1$, $M_2$, and $M_3$. Pick $k \in u.a.r.$
   [3].

2. When an edge $e$ is inserted, the processing happens in two phases.
   (a) The augment phase. The new edge $e$ is added to each $M_i$ in which there are no edges adjacent to $e$.
   (b) The switching phase. For $i = 2, 3$, in order, $e$ is added to $M_i$ (if it was not added in the previous phase) and the conflicting edges (denoted by $X(M_i, e)$) are discarded, provided it decreases the quantity
   \[ \sum_{j \in [3], j \neq i} |X(M_i \cap M_j, e)| = |X(M_i, e)| |M_i \cap M_j| \]
   For every edge $e'$ discarded from $M_i$, add edges on other end point of $e'$ in $M_j$ ($\forall j \neq i$) to $M_i$ if possible.

3. Output the matching $M_k$.

Note that end points of every edge will be covered by all three matchings.

**Theorem 8.** The algorithm for finding MCM on trees is $\frac{3}{2}$-approximate in expectation, with a worst case update time $O(1)$.

We again use the primal-dual technique is used to analyze the performance of this algorithm.

**Lemma 9.** There exists an assignment of the primal charge amongst the dual variables such that the dual constraint for each edge $e \equiv (u, v)$ is satisfied at least $\frac{2}{3}$rd in expectation.

**Proof.** Root the tree at an arbitrary vertex. For any edge $e \equiv (u, v)$, let $v$ be the parent vertex, and $u$ be the child vertex. The dual variable assignment is done at the end of input, as follows.

- If $e$ does not belong to any matching, then it does not contribute to the value of dual variables.
- If $e$ belongs to a single matching, then its entire primal charge is assigned to $v$ as $y_v = 1$.
- If $e$ belongs to two matchings, then its entire primal charge is assigned equally amongst $u$ and $v$, as $y_u = 1$ and $y_v = 1$.
- If $e$ belongs to three matchings, then its entire primal charge is assigned to $v$ as $y_v = 3$.

The analysis breaks up into three cases.

**Case 1.** Suppose $e$ does not belong to any matching. There must be a total of
at least 2 edges incident amongst \( u \) and \( v \) besides \( e \), each belonging to a distinct matchings, from their respective children. Therefore, \( y_u + y_v \geq 2 \).

**Case 2.** Suppose \( e \) belongs to a single matching. Then either there is an edge \( e' \) incident on \( u \) or \( v \) which belongs to a single matching, from their respective children, or there is an edge \( e'' \) incident on \( u \) or \( v \) which belongs to two matchings. In either case, \( y_u + y_v \geq 2 \).

**Case 3.** Suppose \( e \) belongs to two or three matchings, then \( y_u + y_v \geq 2 \) trivially. 

Lemma 9 implies that the approximation ratio of the algorithm is at most \( \frac{3}{2} \) in expectation.

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A Proof of Lemma 5

We crucially use the following lemma to prove Lemma 5.

Lemma 10. (a) If an edge e belongs only to $M_4$ at the end of input, then bad edges cannot be incident on both its end points.

(b) Also, if an edge e was added to $M_4$ only in the switching phase, then e cannot be a bad edge.
Proof. There are two cases to consider.

1. Suppose $e$ was added to $M_4$ only when it was revealed. Then on one of its end point either there should be two edges incident (other than $e$), such that each of them belongs to a single matching, or there should be one edge which belongs to two matchings. In either case, the edges incident on that end point of $e$ should have neighboring edges which belong to some matching (by description of algorithm). And hence, these edges cannot be bad.

2. Suppose $e$ was added to $M_4$ as well as some other matching when it was revealed. If $e$ belonged to three matchings when it was revealed, then its neighboring edge will have its end points covered by at least four matching edges, and this number can never go below four. If $e$ belonged to two matchings when it was revealed, then it
   – (a) either has one neighboring edge which belongs to two matchings,
   – (b) or one neighboring edge on each of its end points, each belonging to distinct matching,
   – (c) or two neighboring edges on one of its end points, such that both of them belong to distinct matchings.

In Case (a), this neighboring edge should have a neighboring edge on its other end point which belongs to some matching, and hence it cannot be a bad edge. In Case (b), each of these edge should have at least two neighboring edges of their own on their respective other end point, which belong to certain matching. Hence, both these edges cannot be bad. In Case (c), both these edges should have neighboring edges of their own on their respective other end point, which belong to certain matching. Hence, both these edges cannot be bad.

For the second part of lemma, if edge $e$ added to $M_4$ in the switching phase, then it means that $e$ will have three neighboring edges $e_1, e_2$, and $e_3$, belonging to $M_1, M_2$, and $M_3$, respectively. This is because $e$ will be added to $M_4$ in the switching phase only if it is not added to $M_2$ or $M_3$ in the switching phase, which means there are edges which belong only to $M_2$ and $M_3$ respectively.

Proof. (of Lemma 5) There are two cases to consider.

1. Suppose if there is a bad leaf edge $e$ which belongs to $M_4$. If $e$ is added to $M_4$ in the switching phase, then $e$ cannot be a bad edge (by part (b) of Lemma 10). So, $e$ has to be added to $M_4$ in the augment phase for it to be a bad leaf edge in future.
   - If $e$ was added to $M_4$ alone when revealed, then it must have neighbors $e_1$ and $e_2$ such that both of them do not belong to $M_4$. Then, they must have had neighboring edges $e'_1$ and $e'_2$ respectively which belonged to $M_4$ (at some stage). Suppose $e''_1$ (and/or $e''_2$) switches $e'_1$ (and/or $e'_2$ respectively) out of $M_4$, then $e''_1$ (and/or $e''_2$ respectively) cannot be a bad edge (by part (b) of Lemma 10). Otherwise, the Lemma holds due to part (a) of Lemma 10.
• If \( e \) was added to two matchings (\( M_4 \) being one of them) when it was revealed, and finally has only one internal neighboring edge \( e_1 \), then \( e_1 \) will have a neighboring edge \( e_2 \) on its other end point. Either \( e_2 \) belongs to \( M_4 \) or its neighboring edge \( e'_2 \) on other end point belongs to \( M_4 \). The lemma holds if finally \( e'_2 \) belongs to \( M_4 \) (by part (a) of Lemma 10) or if finally the neighboring edge \( e''_2 \) of \( e'_2 \) belongs to \( M_4 \) (by part (b) of Lemma 10). (The proof for this case will also work for the case when \( e \) was revealed first as a single disconnected edge, and then \( e_1 \) was revealed on one of its end points.)

• If \( e \) was added to two or three matchings (\( M_4 \) being one of them) when it was revealed, and finally has two internal neighboring edges \( e_1 \) and \( e_2 \), then \( e_1 \) and \( e_2 \) must have neighboring edges \( e'_1 \) and \( e'_2 \) respectively which belong to \( M_4 \) (at some stage). Suppose \( e''_1 \) (and/or \( e''_2 \)) switches \( e'_1 \) (and/or \( e'_2 \), respectively) out of \( M_4 \), then \( e''_1 \) (and/or \( e''_2 \), respectively) cannot be a bad edge (by part (b) of Lemma 10). Otherwise, the Lemma holds due to part (a) of Lemma 10.

2. Let \( e_1 \) and \( e_2 \) be two bad internal edges which do not belong to \( M_4 \). Then, they must have had neighboring edges \( e'_1 \) and \( e'_2 \) respectively which belonged to \( M_4 \) (at some stage). Suppose \( e''_1 \) (and/or \( e''_2 \)) switches \( e'_1 \) (and/or \( e'_2 \), respectively) out of \( M_4 \), then \( e''_1 \) (and/or \( e''_2 \), respectively) cannot be a bad edge (by part (b) of Lemma 10). Otherwise, the Lemma holds due to part (a) of Lemma 10.