Secret Keys Assisted Private Classical Communication Capacity over Quantum Channels

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We prove a regularized formula for the secret key-assisted capacity region of a quantum channel for transmitting private classical information. This result parallels the work of Devetak on entanglement assisted quantum communication capacity [1]. This formula provides a new family protocol, the private father protocol, under the resource inequality framework that includes private classical communication without secret key assistance as a child protocol.

PACS numbers: 03.67.Hk, 03.67.Mn, 03.67.Pp
Keywords: Private channel capacity, father protocol, secret keys, and resource inequality.

I. INTRODUCTION

Secret keys, by definition, refer to common randomness available to a sender and receiver at distant locations while any other party has absolutely no information about it. Generating secret keys requires preserving secrecy from a third party [2]. An information-theoretic model in the classical setting is the “wiretap channel” [3], where the sender wants to communicate with one legitimate receiver while keeping the eavesdropper completely ignorant of the message sent. Private communication can be achieved via encryption once secret keys are generated. Secret keys are a valuable resource that can be used to achieve information transmission tasks.

The above scenario has a quantum analogue, where secret keys are generated over a quantum channel. The secret key generating protocol has been proposed by several authors [4, 5, 6], and in [7], it has been shown that the capacity of a quantum channel for transmitting private classical information is the same as the capacity of the same channel for generating secret keys. Furthermore, neither capacity is enhanced by public classical communication in the forward direction. This raises the interesting question of how these different resources interconvert in a quantum information protocol, and was partially answered in [1, 8, 9].

The formal treatment of quantitative interconversions between nonlocal information processing resources is studied in [1], wherein such an asymptotically faithful conversion is expressed as a resource inequality (RI). These resource inequalities are extremely powerful, and sometimes lead to new quantum protocols [8]. For example, they allow us to relate the family protocols to several well-known quantum protocols by direct application of teleportation or superdense coding, etc.

In this paper, we study the private classical communication capacity over a quantum channel assisted by a secret key. We show that secret keys are a useful nonlocal resource that can increase the private classical communication capacity over quantum channels; however, unlimited secret keys do not help. The trade-off between the rate of secret key consumption and the rate of increased private classical communication is presented quantitatively. Under the RI framework, our protocol can be understood as a “private father protocol” due to its similarity to the original father protocol. Furthermore, the unassisted private classical communication capacity [2] can be seen as a child protocol.

This paper is organized as follows. Section II contains definitions, notation, and relevant background material. Section III contains statements and proofs of our main result. In section IV we rewrite our result under the RI framework, and show how to recover the unassisted private classical capacity from ours. We conclude in section V.

II. NOTATION

Consider a classical-quantum system $XQ$ in the state described by an ensemble $\{p(x), \rho_x\}$ with $p(x)$ defined on $\mathcal{X}$ and $\rho_x$ being density operators on the Hilbert space $\mathcal{H}_Q$ of system $Q$. Such a state $\rho^{XQ}$ of systems $XQ$ can be represented by the “enlarged Hilbert space” (EHS) representation:

$$\rho^{XQ} = \sum_x p(x) |x_x\rangle \langle x_x| \otimes \rho_x^Q,$$

where $X$ is a dummy quantum system and $\{|x_x\rangle : x \in \mathcal{X}\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_X$ of system $X$. The reduced density operators of systems $X$ and $Q$ are $\rho^X = \text{Tr}_Q \rho^{XQ} = \sum_x p(x) |x_x\rangle \langle x_x|$, and $\rho^Q = \text{Tr}_X \rho^{XQ}$ respectively. The von Neumann entropy of the quantum state $\rho^Q$ is $H(Q)_\rho = -\text{Tr}(\rho^Q \log \rho^Q)$. (We will omit the subscript $\rho$ when the state is clear from the context.) Notice that the von Neumann entropy of the dummy quantum system $X$ is equal to the Shannon entropy of random variable $X$ whose probability distribution is $p(x)$. The conditional entropy is defined as

$$H(Q|X) = H(QX) - H(X).$$  (1)
It should be noted that conditioning on classical variables (systems) amounts to averaging; therefore (1) is also equal to

$$H(Q|X) = \sum_x p(x)H(Q)_{p_x}. \quad (2)$$

The mutual information is

$$I(X;Q) = H(X) + H(Q) - H(QX).$$

Next, we will briefly introduce definitions and properties of typical sequences and subspaces [10].

Let $T^n_{X,\delta}$ denote the set of typical sequences associated with some random variable $X$ such that for the probability distribution $p$ defined on the set $X$

$$T^n_{X,\delta} = \left\{ x^n : \forall x \in X, \left| \frac{N(x|x^n)}{n} - p(x) \right| \leq \delta \right\},$$

where $N(x|x^n)$ is the number of occurrences of $x$ in the sequence $x^n := x_1 \cdots x_n$ of length $n$.

Assume the density operator $\rho$ of system $Q$ has the following spectral decomposition: $\rho = \sum_i \rho_i |i\rangle \langle i|$. Then we can define the typical projector as

$$\Pi^n_{Q,\delta} = \sum_{y^n \in T^n_{Q,\delta}} |y^n\rangle \langle y^n|, \quad (3)$$

where $|y^n\rangle$ is a state in $H^n_Q$. For a collection of states $\{\rho_x, x \in X\}$, the conditional typical projector is defined as

$$\Pi^n_{Q|x,\delta}(x^n) = \bigotimes_x \Pi^n_{Q|x,\delta}, \quad (4)$$

where $I_x = \{i : x_i = x\}$ is the indicator and $\Pi^n_{Q|x,\delta}$ denotes the tensor product of the typical projector of the density operator $\rho^n_x$ in the positions given by the set $I_x$ with the identity everywhere else.

Fixing $\delta > 0$, we will need the following properties of typical subspaces and conditionally typical subspaces:

$$\text{Tr} \sigma^n_{x^n} \Pi^n_{Q|x,\delta}(x^n) \geq 1 - \epsilon \quad (5)$$

$$\text{Tr} \sigma^n_{x^n} \Pi^n_{Q,\delta(|x|+1)} \geq 1 - \epsilon \quad (6)$$

$$\text{Tr} \Pi^n_{Q|x,\delta} \leq \alpha \quad (7)$$

$$\Pi^n_{Q|x,\delta}(x^n) \Pi^n_{Q|x,\delta}(x^n) \leq \beta^{-1} \Pi^n_{Q|x,\delta}(x^n) \quad (8)$$

where $\alpha = 2^{n[H(Q) + \epsilon \delta]}$ and $\beta = 2^{n[H(Q;X) - \epsilon \delta]}$ for $\epsilon = 2^{-n(c'\delta)^2}$ and some constants $c$ and $c'$.

Finally, we need some facts about trace distances (taken from [10]). The trace distance between two density operators $\rho$ and $\sigma$ can be defined as

$$\|\rho - \sigma\|_1 = \text{Tr} |\rho - \sigma|,$$

where $|A|$ $\equiv \sqrt{A^\dagger A}$ is the positive square root of $A^\dagger A$. The monotonicity property of trace distance is

$$\|\rho^{RB} - \sigma^{RB}\|_1 \geq \|\rho^{B} - \sigma^{B}\|_1. \quad (9)$$

### III. MAIN RESULT

#### A. Classical-quantum channels

We begin by defining our private classical communication protocol for a $\{c \rightarrow qq\}$ channel from sender Alice to receiver Bob and eavesdropper Eve. The channel is defined by the map $W : x \rightarrow \sigma^n_{BE}$, with $x \in X$ and the state $\sigma^n_{BE}$ defined on a bipartite quantum system $BE$; Bob has access to subsystem $B$ and Eve has access to subsystem $E$. Alice’s task is to transmit, by some large number $n$ uses of the channel $W$, one of $\{0,1\}^n$ equiprobable messages to Bob so that he can identify the message with high probability while at the same time Eve receives almost no information about the message. In addition, Alice and Bob are given some private strings (secret keys), picked uniformly at random from the set $\{0,1\}^n$, before the protocol begins. The inputs to the channel $W^\otimes n$ are classical sequences $x^n \in X^n$ with probability $p^n(x^n)$. The outputs of $W^\otimes n$ are density operators $\sigma^n_{BE} = \sigma^n_{BE} \otimes \cdots \otimes \sigma^n_{E}$ living on some Hilbert space $H_B^n E^n$.

An $(n, R, R_s, \epsilon)$ secret key-assisted private channel code consists of

- An encryption map $f : \{0,1\}^n \times \{0,1\}^{nR_s} \rightarrow \{0,1\}^{nR}$, i.e. $f$ generates an index random variable $K$ uniformly distributed in $\{0,1\}^{nR}$ based on the classical message embodied in the random variable $M$ and the shared secret key embodied in the random variable $S$. Furthermore, $f(m,s_1) \neq f(m,s_2)$ for $s_1 \neq s_2$ and $f(m_1,s) \neq f(m_2,s)$ for $m_1 \neq m_2$.

- An encoding map $E : \{0,1\}^{nR} \rightarrow X^n$. Alice encodes the index $k$ as $E(k)$ and sends it through the channel $W_E^{\otimes n}$, generating the state

$$T^n_{A,B} = \frac{1}{2^nR_s} \sum_{x \in \{0,1\}^n} |s\rangle \langle s| A^n \otimes |s\rangle \langle s| B^n \otimes$$

$$\frac{1}{2^nR} \sum_{m \in \{0,1\}^n} \sigma^{BE}_{E(f(m,s))} \quad (10)$$

- A decoding operation, and

- A decoding POVM $\{A_{k'}\}_{k' \in \{0,1\}^n}$, where $A_{k'}$ is a positive operator acting on $B$ and taking on values $k'$. Bob need to infer the index $k$ through the POVM;

- A decryption map $g : \{0,1\}^n \times \{0,1\}^{nR_s} \rightarrow \{0,1\}^{nR}$, where $g(m,s), s = m, \forall s, m$. This allows Bob to recover Alice’s message as $m' = g(k', s)$ based on $k'$ and $s$;

such that

$$\|\tilde{T}^{BE} - \tau^B \otimes \sigma^E\|_1 \leq \epsilon, \quad (11)$$

where $\tilde{T}^{BE}$ is the state of the subsystem $BE$ after Bob’s decoding operation, and

$$\tau^B = \frac{1}{2^nR} \sum_m |m\rangle \langle m| B^n$$
contains the private classical information that is decoupled from Eve’s state $\sigma^E$.

A rate pair $(R, R_s)$ is called achievable if for any $\epsilon, \delta > 0$ and sufficiently large $n$ there exists an $(n, R - \delta, R_s + \delta, \epsilon)$ private channel code. The private capacity region $C_{PF}(W)$ is a two-dimensional region in the $(R, R_s)$ plane with all possible achievable rate pairs $(R, R_s)$.

We now state our main theorem.

**Theorem 1** The private channel capacity region $C_{PF}(W)$ is given by

$$C_{PF}(W) = \bigcup_{n=1}^{\infty} \frac{1}{n} \tilde{C}_{PF}^{(1)}(W^{\otimes n}),$$

where the notation $\tilde{z}$ means the closure of a set $Z$ and $\tilde{C}_{PF}^{(1)}(W)$ is the set of all $R_s \geq 0$, $R \geq 0$ such that

$$R \leq I(X; B)_{\sigma} - I(X; E)_{\sigma} + R_s \quad (13)$$

$$R \leq I(X; B)_{\sigma}, \quad (14)$$

where $B|X$ is given by $W$ and $\sigma$ is of the form

$$\sigma^{XBE} = \sum_{x} p(x)\sigma(x)^{X} \otimes \sigma^{BE}.$$
We now invoke Proposition 1. Choose $R = I(X; B) - 2(c + c'\delta)\delta$. There exists a POVM $(\Lambda_k)_{k' \in \{0,1\}^n}$ acting on $B$ such that for all $k$,

$$\mathbb{E} \sum_{k'} |\pi(k'|k) - \delta(k,k')| \leq \epsilon . \quad (23)$$

After Bob performs the POVM, the state (10) becomes

$$\hat{\Upsilon} = \frac{1}{2nR_c} \sum_s |s \rangle \langle A_s| \otimes |s \rangle \langle B_s| \otimes \frac{1}{2nR_c} \sum_{m,k'} \pi(k'|f(m,s)) |k'| \langle k'| \otimes \sigma^E_{X_f(m,s)} ,$$

which is close to

$$\tilde{\Upsilon}_0 = \frac{1}{2nR_c} \sum_s |s \rangle \langle A_s| \otimes |s \rangle \langle B_s| \otimes \frac{1}{2nR_c} \sum_{m} |f(m,s)|f(m,s) \rangle \langle f(m,s)| \otimes \sigma^E_{X_f(m,s)} ,$$

in the sense that $\mathbb{E} ||\hat{\Upsilon} - \tilde{\Upsilon}_0||_1 \leq \epsilon$ by condition (23).

Bob applies the decryption map $g$ to his system $B$, resulting in a state $\tilde{\Upsilon}^{A,B,\tilde{F}}$. By the monotonicity of trace distance [9], we have

$$\mathbb{E} ||\tilde{\Upsilon}^{B,E} - \tilde{\Upsilon}_0^{B,E}||_1 \leq \epsilon ,$$

where

$$\tilde{\Upsilon}_0^{B,E} = \frac{1}{2nR_c} \sum_m |m \rangle \langle m| \otimes \frac{1}{2nR_c} \sum_s \sigma^E_{X_f(m,s)} .$$

By the Markov inequality, $\operatorname{Pr}\{\operatorname{not} \ell_0\} \leq \sqrt{\epsilon}$, where $\ell_0$ is the logic statement

$$||\tilde{\Upsilon}^{B,E} - \tilde{\Upsilon}_0^{B,E}||_1 \leq \sqrt{\epsilon} . \quad (24)$$

By the union bound,

$$\operatorname{Pr}\{\operatorname{not} (\ell_0 \& \ell_1 \& \ldots \& \ell_{|m|})\} \leq \sum_{i=0}^{2^nR} \operatorname{Pr}\{\operatorname{not} \ell_i\} \leq \epsilon + \sqrt{\epsilon} .$$

Hence there exists a specific choice of $\{X_{f(m,s)}\}$, say $\{x_{f(m,s)}\}$, for which all these conditions are satisfied. Consequently,

$$||\tilde{\Upsilon}^{B,E} - \tau^B \otimes \sigma^E||_1 \leq ||\tilde{\Upsilon}^{B,E} - \tilde{\Upsilon}_0^{B,E}||_1 + ||\tilde{\Upsilon}_0^{B,E} - \tau^B \otimes \sigma^E||_1 \leq 2\epsilon + 20\sqrt{\epsilon} .$$

as claimed. \hfill \Box

**Proof of converse**

We shall prove that, for any $\delta, \epsilon > 0$ and sufficiently large $n$, if an $(n, R, R_s, \epsilon)$ secret key assisted private channel code has rate $R$ then (13) hold.
The private classical communication protocol is shown in Fig. 2

\[ nR = H(K) \]
\[ = I(K; K') + H(K|K') \]
\[ \leq I(K; K') + 1 + n\epsilon \log |\mathcal{X}|, \]
where the last inequality follows from Fano’s inequality:

\[ H(K|K') \leq 1 + \Pr(K \neq K') nR, \]
and \( \Pr(K \neq K') \leq \epsilon \) is guaranteed by the HSW theorem. Hence,

\[ I(K; K') \leq I(K; B^n) \]
\[ \leq I(X^n; B^n), \]
where the first inequality follows from the data processing inequality while the second inequality comes from the Markov condition \( K \to X^n \to B^n E^n \). We then have

\[ R - \delta \leq \frac{1}{n} I(X^n; B^n), \]
where without loss of generality \( \epsilon \leq \frac{\delta}{\log |\mathcal{X}|} \) and \( n \geq \frac{2}{\delta} \). This proves (14).

On the other hand,

\[ I(M; M') = I(M; M'|S) + I(S; MS) \]
\[ \leq I(K; M'|S) + H(S) \]
\[ \leq I(K; B^n|S) + H(S) \]
\[ \leq I(X^n; B^n|S) + H(S) \]
where (28) follows from (27), (29) follows from data processing inequality, and (30) follows from the Markov condition \( K \to X^n \to B^n E^n \). Furthermore, (24) guarantees that

\[ \epsilon \geq I(M; E^n|S) \]
\[ = I(K; E^n|S) \]
\[ \geq I(X^n; E^n|S). \]
Combining (30) and (33) gives

\[ I(M; M') \leq I(M; S; M'|S) \]
\[ \leq I(X^n; B^n|S) - I(X^n; E^n|S) \]
\[ + H(S) + \epsilon. \]

Hence

\[ nR = H(M) \]
\[ = I(M; M') + H(M|M') \]
\[ \leq I(M; M') + 1 + n\epsilon \log |\mathcal{X}|, \]
where (39) follows from the Fano’s inequality. Choosing \( \epsilon \leq \frac{\delta}{\log |\mathcal{X}|} \) and \( n \geq \frac{2}{\delta} \), (30) and (39) give

\[ R - \delta \leq \frac{1}{n} [I(X^n; B^n|S) - I(X^n; E^n|S) + H(S)] \]
\[ = \frac{1}{n} [I(X^n; B^n|S) - I(X^n; E^n|S)] + R_s, \]
where \( R_s = \frac{H(S)}{n} \). Since we can write

\[ \frac{1}{n} [I(X^n; B^n|S) - I(X^n; E^n|S)], \]
as the average with respect to the distribution of \( S \), and \( K_s \to X^n_s \to B^n X^n \) holds for each \( s \), we can choose a particular value of \( s \) that maximizes (13). \( \square \)

B. Generic quantum channels

Suppose now that Alice and Bob are connected by a noisy quantum channel \( \mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B) \), where \( \mathcal{B}(\mathcal{H}) \) denotes the space of bounded linear operators on \( \mathcal{H} \). Let \( U_N : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{BE}) \) be an isometry extension of \( \mathcal{N} \) that includes the unobserved environment \( \mathcal{E} \) which is completely under the control of the eavesdropper Eve. Theorem 1 then can be rewritten as the following

**Theorem 2** The private channel capacity region \( C_{PF}(\mathcal{N}) \) is given by

\[ C_{PF}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n} \tilde{C}_{PF}^{(1)}(\mathcal{N}^\otimes n), \]
where the notation \( Z \) means the closure of a set \( Z \) and \( \tilde{C}_{PF}^{(1)}(\mathcal{N}) \) is the set of all \( R_s \geq 0, R \geq 0 \) such that

\[ R \leq I(A; B)_\sigma - I(A; E)_\sigma + R_s \]
\[ R \leq I(A; B)_\sigma, \]
where \( \sigma \) is of the form

\[ \sigma^{ABE} = U_N^{A'} BE(\psi^{AA'}), \]
for some pure input state \( |\psi\rangle^{AA'} \) whose reduced density operator \( \rho^{A'} = \sum_x p(x) p_x \) and \( U_N : A' \to BE \) is an isometric extension of \( \mathcal{N} \).
With the spectral decomposition of the input state $\rho_A' = \sum_x p(x) \rho_x$, each $U_N$ induces a corresponding \{c \rightarrow qq\} channel. Therefore, the results of the previous section can be directly applied here.

IV. PRIVATE FATHER PROTOCOL

In this section, we will phrase our result using the theory of resource inequalities developed in [8]. The channel $N : A \rightarrow B$ assisted by some rate $R_s$ of secret key shared between Alice and Bob was used to enable a rate $R$ of secret communication between Alice and Bob. This is written as

$$\langle N \rangle + R_s[cc]^* \geq I(A;B)[c \rightarrow c]^*. \quad (46)$$

This resource inequality holds iff $(R_s, R) \in C_{PF}(N)$, with $C_{PF}(N)$ given in Theorem 2. The “if” direction, i.e. the direct coding theorem, followed from the “corner points”

$$\langle N \rangle + I(A;E)[cc]^* \geq I(A;B)[c \rightarrow c]^*. \quad (47)$$

This resource inequality (47) is called the private father protocol due to its similarity of the father protocol in [8].

We can recover the unassisted private channel capacity result in [5]:

$$\langle N \rangle \geq I_c(A)B[c \rightarrow c]^*. \quad (48)$$

This resource inequality can be obtained by appending the following noiseless resource inequality

$$[c \rightarrow c]^* \geq [cc]^* \quad (49)$$

to the output of (47).

V. CONCLUSION

In this paper, we have found a regularized expression for the secret keys assisted capacity region $C_{PF}(N)$ of a quantum channel $N$ for transmitting private classical information. Our result shows that secret key are a valuable nonlocal resource for transmitting private information. One interesting problem is to investigate how secret keys can be applied in other quantum protocols. For example, it might be plausible that the entanglement generation protocol could be boosted by secret keys. However, the result seems unlikely. In particular, it is impossible to construct a secret key-assisted entanglement generation protocol by simply coherifying the protocol proposed in this paper. Another open problem is to obtain a single-letterized formula of Theorem 1.

Acknowledgment

We are grateful to Igor Devetak for valuable discussions. MH and ZL were supported by NSF grant no. 0524811. TAB was supposed by NSF grant no. 0448658.

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