SKELLAM AND TIME-CHANGED VARIANTS OF THE GENERALIZED FRACTIONAL COUNTING PROCESS

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Abstract. In this paper, we study a Skellam type variant of the generalized counting process (GCP), namely, the generalized Skellam process. Some of its distributional properties such as the probability mass function, probability generating function, mean, variance and covariance are obtained. Its fractional version, namely, the generalized fractional Skellam process (GFSP) is considered by time-changing it with an independent inverse stable subordinator. It is observed that the GFSP is a Skellam type version of the generalized fractional counting process (GFCP) which is a fractional variant of the GCP. It is shown that the one-dimensional distributions of the GFSP are not infinitely divisible. An integral representation for its state probabilities is obtained. We establish its long-range dependence property by using its variance and covariance structure. Also, we consider two time-changed versions of the GFCP. These are obtained by time-changing the GFCP by an independent Lévy subordinator and its inverse. Some particular cases of these time-changed processes are discussed by considering specific Lévy subordinators.

1. Introduction

The point processes with random time are of particular interest in the theory of stochastic processes due to their potential applications in areas such as finance, hydrology, econometrics, etc. The time fractional Poisson process (TFPP) and the space fractional Poisson process are two extensively studied fractional extensions of the Poisson process. These time-changed processes are obtained by choosing an independent stable subordinator and its inverse process as a time-change component in the Poisson process (see Laskin (2003), Beghin and Orsingher (2009), Meerschaert et al. (2011), Orsingher and Polito (2012) etc.)

Di Crescenzo et al. (2016) introduced and studied the generalized fractional counting process (GFCP) \( \{M^\alpha(t)\}_{t \geq 0}, 0 < \alpha \leq 1 \) whose state probabilities \( p^\alpha(n, t) = \text{Pr}\{M^\alpha(t) = n\} \) satisfy the following system of fractional differential equations:

\[
\partial_t^\alpha p^\alpha(n, t) = -\Lambda p^\alpha(n, t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j p^\alpha(n-j, t), \quad n \geq 0,
\]

with the initial conditions

\[
p^\alpha(n, 0) = \begin{cases} 
1, & n = 0, \\
0, & n \geq 1.
\end{cases}
\]

Here, \( \Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k \) for a fixed positive integer \( k \) and \( \partial_t^\alpha \) is the Caputo fractional derivative defined in (2.1). The GFCP performs \( k \) kinds of jumps of amplitude \( 1, 2, \ldots, k \) with positive rates \( \lambda_1, \lambda_2, \ldots, \lambda_k \), respectively.

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For $\alpha = 1$, the GFCP reduces to the generalized counting process (GCP) $\{M(t)\}_{t \geq 0}$. For $k = 1$, the GFCP and the GCP reduces to the TFPP and the Poisson process, respectively. It is known that (see Di Crescenzo et al. (2016))

$$M^\alpha(t) \overset{d}{=} M(Y_\alpha(t)),$$

where $\overset{d}{=}$ denotes equality in distribution. Kataria and Khandakar (2021c) studied some additional properties of these generalized processes which includes an application of the GCP in risk theory. It is shown by them that many known counting processes such as the Poisson process of order $k$, the Pólya-Aeppli process of order $k$, the negative binomial process, the convoluted Poisson process and their fractional versions etc. are particular cases of the GFCP.

It is important to note that the recently studied time-changed processes are mainly constructed by time-changing a point process with an independent subordinator and its inverse. A subordinator $\{D_f(t)\}_{t \geq 0}$ is a one-dimensional Lévy process which is characterized by the following Laplace transform (see Applebaum (2009), Section 1.3.2):

$$E(e^{-sD_f(t)}) = e^{-tf(s)},$$

where $f(s) = \int_0^\infty (1 - e^{-sx}) \mu(dx)$, $s > 0$, is called the Bernstein function. Here, $\mu$ is a non-negative Lévy measure that satisfies $\mu([0, \infty)) = \infty$ and $\int_0^\infty \min\{x, 1\} \mu(dx) < \infty$. It has non-decreasing sample paths and $D_f(0) = 0$ almost surely (a.s.). The first passage time of a subordinator is called the inverse subordinator. It is defined as

$$H_f(t) := \inf\{r \geq 0 : D_f(r) > t\}, \quad t \geq 0.$$

In this paper, we first introduce a Skellam type variant of the GCP. We call it the generalized Skellam process (GSP) and denote it by $\{S(t)\}_{t \geq 0}$. It is defined as follows:

$$S(t) := M_1(t) - M_2(t),$$

where $\{M_1(t)\}_{t \geq 0}$ and $\{M_2(t)\}_{t \geq 0}$ are independent generalized counting processes with positive rates $\lambda_j$’s and $\mu_j$’s, $j = 1, 2, \ldots, k$, respectively. For any integer $n$, we obtain its state probability $q(n, t) = Pr\{S(t) = n\}$ in the following form:

$$q(n, t) = e^{-(\Lambda + \bar{\Lambda})t} (\Lambda/\bar{\Lambda})^{n/2} I_{|n|} \left(2t \sqrt{\Lambda \bar{\Lambda}}\right),$$

where $\Lambda = \sum_{j=1}^k \lambda_j$ and $\bar{\Lambda} = \sum_{j=1}^k \mu_j$. Here, $I_{|n|}(\cdot)$ denotes the modified Bessel function of first kind defined in (2.5). Also, we obtain its probability generating function (pgf), characteristic function, mean, variance, covariance etc.

We consider a fractional version of the GSP, namely, the generalized fractional Skellam process (GFSP). It is denoted by $\{S^\alpha(t)\}_{t \geq 0}$, $0 < \alpha \leq 1$. It is defined as the GSP time-changed by an independent inverse stable subordinator, that is,

$$S^\alpha(t) := \begin{cases} S(Y_\alpha(t)), & 0 < \alpha < 1, \\ S(t), & \alpha = 1. \end{cases}$$
The GFSP is a Skellam type version of the GFCP. It is shown that the state probabilities \( q^\alpha(n,t) = \Pr\{S^\alpha(t) = n\}, \ n \in \mathbb{Z} \) of GFSP solves the following system of fractional differential equations:

\[
\partial_t^\alpha q^\alpha(n,t) = \Lambda (q^\alpha(n-1,t) - q^\alpha(n,t)) - \bar{\Lambda} (q^\alpha(n,t) - q^\alpha(n+1,t)),
\]

with \( q^\alpha(0,0) = 1 \) and \( q^\alpha(n,0) = 0, \ n \neq 0 \). We derive its \( r \)th factorial moment and obtain an integral representation for its probability mass function (pmf). It is shown that the GFSP exhibits the long-range dependence (LRD) property whereas its increment process has the short-range dependence (SRD) property. We have also proved that the one-dimensional distributions of GFSP are not infinitely divisible.

We consider a time-changed version of the GFCP, namely, the time-changed generalized fractional counting process-I (TCGFCP-I). It is obtained by time-changing the GFCP by an independent Lévy subordinator \( \{D_f(t)\}_{t \geq 0} \) such that \( \mathbb{E}(D_f^r(t)) < \infty \) for all \( r > 0 \). We establish a version of the law of iterated logarithm for it. For \( \alpha = 1 \), we show that the TCGFCP-I is equal in distribution to a limiting case of a suitable compound Poisson process. It is shown that the TCGFCP-I exhibits the LRD property under some suitable restrictions on the Lévy subordinator. Some particular cases of TCGFCP-I are discussed by taking specific Lévy subordinator such as the gamma subordinator, tempered stable subordinator and inverse Gaussian subordinator. We derive the Lévy measure and the associated system of governing differential equations for these particular cases. Another time-changed version of the GFCP, namely, the time-changed generalized fractional counting process-II (TCGFCP-II) is considered which is obtained by time-changing the GFCP by an independent inverse subordinator. Some particular cases of the TCGFCP-II are discussed.

2. Preliminaries

In this section, we give some known definitions and results about fractional derivatives, some special functions, inverse stable subordinator, the GCP and its fractional version. The results presented here will be used later.

2.1. Fractional derivatives. The Caputo fractional derivative is defined as (see Kilbas et al. (2006))

\[
\partial_t^\alpha f(t) := \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) \, ds, & 0 < \alpha < 1, \\
f'(t), & \alpha = 1.
\end{cases}
\]

(2.1)

For \( \gamma \geq 0 \), the Riemann-Liouville (R-L) fractional derivative is defined as (see Kilbas et al. (2006))

\[
D_t^\gamma f(t) := \begin{cases} 
\frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-m}} \, ds, & m - 1 < \gamma < m, \\
\frac{d^m}{dt^m} f(t), & \gamma = m,
\end{cases}
\]

(2.2)

where \( m \) is a positive integer.
The following relationship holds for the Caputo and R-L fractional derivatives (see Meerschaert and Straka (2013)):

$$\partial_t^\alpha f(t) = D_t^\alpha f(t) - f(0+) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}.$$  (2.3)

### 2.2. Some special functions.

Here, we briefly describe three special functions.

#### 2.2.1. Mittag-Leffler function.

The three-parameter Mittag-Leffler function is defined as (see Kilbas et al. (2006), p. 45)

$$E_{\alpha,\beta}^\gamma(x) := \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\Gamma(j + \gamma) x^j}{j! \Gamma(j + \alpha + \beta)}, \quad x \in \mathbb{R},$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.

It reduces to two-parameter Mittag-Leffler function for $\gamma = 1$. It further reduces to the Mittag-Leffler function for $\gamma = \beta = 1$.

The following result holds for $n \geq 0$ (see Kilbas et al. (2006), Eq. (1.8.22)):

$$E_{\beta,\gamma}^{(n)}(x) = n! E_{\beta,n\beta+\gamma}(x),$$  (2.4)

where $E_{\beta,\gamma}^{(n)}(\cdot)$ denotes the $n$th derivative of two-parameter Mittag-Leffler function.

#### 2.2.2. Bessel function.

The modified Bessel function of first kind is defined as (see Sneddon (1956), p. 114):

$$I_n(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+n}}{j!(j+n)!}, \quad x \in \mathbb{R}.$$

(2.5)

For any integer $n$, the following properties hold (see Sneddon (1956), p. 115):

$$\begin{cases}
I_n(x) = I_{-n}(x), \\
\frac{d}{dx} I_n(x) = \frac{1}{2} (I_{n-1}(x) + I_{n+1}(x)).
\end{cases}$$  (2.6)

#### 2.2.3. Wright function.

The Wright function is defined as (see Mainardi (2010)):

$$M_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j! \Gamma(1 - j\alpha - \alpha)}, \quad x \in \mathbb{R},$$

where $0 < \alpha < 1$.

### 2.3. Inverse stable subordinator.

A stable subordinator $\{D_\alpha(t)\}_{t \geq 0}, 0 < \alpha < 1$, is a non-decreasing Lévy process. Its Laplace transform is given by $\mathbb{E} \left( e^{-sD_\alpha(t)} \right) = e^{-ts^\alpha}, s > 0$.

Its first passage time $\{Y_\alpha(t)\}_{t \geq 0}$ is called the inverse stable subordinator which is defined as

$$Y_\alpha(t) := \inf\{x \geq 0 : D_\alpha(x) > t\}.$$

The mean, variance and covariance of the inverse stable subordinator are given by (see Leonenko et al. (2014))

$$\mathbb{E} (Y_\alpha(t)) = t^\alpha \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)},$$  (2.7)

$$\text{Var} (Y_\alpha(t)) = \left( \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - \frac{1}{4} \right) t^{2\alpha},$$  (2.8)
\[ \text{Cov} \left( Y_\alpha(s), Y_\alpha(t) \right) = \frac{1}{\Gamma^2(\alpha + 1)} \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) + F(\alpha; s, t) \right), \quad 0 < s \leq t, \quad (2.9) \]

where \( F(\alpha; s, t) = \alpha t^{2\alpha} B(\alpha, \alpha + 1; s/t) - (ts)^\alpha \). Here, \( B(\alpha, \alpha + 1) \) and \( B(\alpha, \alpha + 1; s/t) \) denote the beta function and the incomplete beta function, respectively.

For fixed \( s \) and large \( t \), the following result holds (see Kataria and Khandakar (2021c), Eq. (2.5)):

\[ \text{Cov} \left( Y_\alpha(s), Y_\alpha(t) \right) \sim \frac{1}{\Gamma^2(\alpha + 1)} \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) - \frac{\alpha^2 s^{\alpha+1}}{(\alpha + 1)t^{1-\alpha}} \right). \quad (2.10) \]

### 2.4. The GCP and its fractional version

Here, we give some known results for the GCP and its fractional version, the GFCP (see Di Crescenzo et al. (2016), Kataria and Khandakar (2021c)).

The state probabilities \( p(n, t) = \Pr \{ M(t) = n \} \) of GCP are obtained by Di Crescenzo et al. (2016) which can be represented as follows:

\[ p(n, t) = \sum_{\Omega(k, n)} \prod_{j=1}^{k} \frac{(\lambda_j t)^{x_j}}{x_j!} e^{-\Lambda t}, \quad n \geq 0, \quad (2.11) \]

where \( \Omega(k, n) \) is the set \( \{ (x_1, x_2, \ldots, x_k) : x_1 + 2x_2 + \cdots + kx_k = n, \ x_j \in \mathbb{N}_0 \ \forall \ 1 \leq j \leq k \} \).

Its pgf \( G(u, t) = \mathbb{E} \left( u^M(t) \right) \) is given by

\[ G(u, t) = \exp \left( -\sum_{j=1}^{k} \lambda_j (1 - u^j) t \right), \quad |u| \leq 1 \quad (2.12) \]

and its characteristic function \( \psi(\xi, t) = \mathbb{E} \left( e^{\xi M(t)} \right), \ \omega = \sqrt{-1} \) is given by

\[ \psi(\xi, t) = \exp \left\{ -t \left( \Lambda - \sum_{j=1}^{k} e^{\omega \xi \lambda_j} \right) \right\}, \quad \xi \in \mathbb{R}. \quad (2.13) \]

The following limiting result holds for it:

\[ \lim_{t \to \infty} \frac{M(t)}{t} = \sum_{j=1}^{k} j \lambda_j, \quad \text{in probability.} \quad (2.14) \]

Its mean and variance are obtained by Di Crescenzo et al. (2016) in the following form:

\[ \mathbb{E}(M(t)) = r_1 t \quad \text{and} \quad \text{Var}(M(t)) = r_2 t, \]

where

\[ r_1 = \sum_{j=1}^{k} j \lambda_j \quad \text{and} \quad r_2 = \sum_{j=1}^{k} j^2 \lambda_j. \quad (2.15) \]

The mean and covariance of GFCP are given by (see Kataria and Khandakar (2021c))

\[ \mathbb{E}(M^\alpha(t)) = r_1 \mathbb{E}(Y_\alpha(t)), \quad (2.16) \]

\[ \text{Cov}(M^\alpha(s), M^\alpha(t)) = r_2 \mathbb{E}(Y_\alpha(s)) + r_1^2 \text{Cov}(Y_\alpha(s), Y_\alpha(t)), \quad 0 < s \leq t. \quad (2.17) \]
3. Generalized Skellam process and its fractional extension

In this section, we introduce a Skellam type variant of the GCP, namely, the generalized Skellam process (GSP). We denote it by \( \{S(t)\}_{t \geq 0} \) and define it as

\[
S(t) := M_1(t) - M_2(t),
\]

where \( \{M_1(t)\}_{t \geq 0} \) and \( \{M_2(t)\}_{t \geq 0} \) are independent generalized counting processes with positive rates \( \lambda_j \)'s and \( \mu_j \)'s, \( j = 1, 2, \ldots, k \), respectively.

For \( k = 1 \), the GSP reduces to the Skellam process introduced and studied by Barndorff-Nielsen et al. (2012). For \( \lambda_j = \lambda \) and \( \mu_j = \mu, j = 1, 2, \ldots, k \), the GSP reduces to the Skellam process of order \( k \) (see Gupta et al. (2020)).

Using (2.12), the pgf of GSP can be obtained as follows:

\[
G_S(u, t) = \exp \left\{ -t \sum_{j=1}^{k} (\lambda_j(1 - u^j) + \mu_j(1 - u^{-j})) \right\}. \tag{3.1}
\]

It satisfies the following differential equation:

\[
\frac{d}{dt} G_S(u, t) = \sum_{j=1}^{k} (\lambda_j(u^j - 1) + \mu_j(u^{-j} - 1)) G_S(u, t), \quad G_S(u, 0) = 1.
\]

**Remark 3.1.** On substituting \( \lambda_j = \lambda \) and \( \mu_j = \mu \) for all \( j = 1, 2, \ldots, k \) in (3.1), we get the pgf of Skellam process of order \( k \) (see Gupta et al. (2020), Eq. (39)).

Similarly, on using (2.13), the characteristic function of GSP can be obtained as

\[
\psi_S(\xi, t) = \exp \left\{ -t \left( \Lambda + \bar{\Lambda} - \sum_{j=1}^{k} e^{\omega \xi_j} \lambda_j - \sum_{j=1}^{k} e^{-\omega \xi_j} \mu_j \right) \right\},
\]

where \( \Lambda = \sum_{j=1}^{k} \lambda_j \) and \( \bar{\Lambda} = \sum_{j=1}^{k} \mu_j \). Thus, the Lévy measure of GSP is given by

\[
\Pi_S(dx) = \sum_{j=1}^{k} \lambda_j \delta_j dx + \sum_{j=1}^{k} \mu_j \delta_{-j} dx,
\]

where \( \delta_j \)'s are Dirac measures.

Let \( m_1 = \sum_{j=1}^{k} j (\lambda_j - \mu_j) \) and \( m_2 = \sum_{j=1}^{k} j^2 (\lambda_j + \mu_j) \). The mean, variance and covariance of GSP are given by

\[
\mathbb{E}(S(t)) = m_1 t, \quad \text{Var}(S(t)) = m_2 t, \quad \text{Cov}(S(s), S(t)) = m_2 \min\{s, t\}.
\]

The GSP exhibits the overdispersion as \( \text{Var}(S(t)) - \mathbb{E}(S(t)) > 0 \) for all \( t > 0 \).

The following definition of LRD and SRD property will be used (see Maheshwari and Vellaisamy (2016)):

**Definition 3.1.** Let \( s > 0 \) be fixed and \( \{X(t)\}_{t \geq 0} \) be a stochastic process such that

\[
\text{Corr}(X(s), X(t)) \sim c(s) t^{-\gamma}, \quad \text{as } t \to \infty,
\]

for some \( c(s) > 0 \). The process \( \{X(t)\}_{t \geq 0} \) has the LRD property if \( \gamma \in (0, 1) \) and SRD property if \( \gamma \in (1, 2) \).

**Remark 3.2.** For fixed \( s \) and large \( t \), the correlation function of GSP has the following asymptotic behaviour:

\[
\text{Corr}(S(s), S(t)) \sim \sqrt{s t}^{-1/2}.
\]

Thus, it exhibits the LRD property.
On using \(2.14\), we obtain the following limiting result for GSP:

\[
\lim_{t \to \infty} \frac{S(t)}{t} = \sum_{j=1}^{k} j(\lambda_j - \mu_j), \quad \text{in probability.}
\] (3.2)

**Theorem 3.1.** For any \(n \in \mathbb{Z}\), the state probability \(q(n, t) = \Pr\{S(t) = n\}\) of GSP is given by

\[
q(n, t) = e^{-(\Lambda + \bar{\Lambda})t} (\Lambda/\bar{\Lambda})^{n/2} I_{[n]} \left(2t \sqrt{\Lambda \bar{\Lambda}}\right).
\] (3.3)

**Proof.** As \(\{M_1(t)\}_{t \geq 0}\) and \(\{M_2(t)\}_{t \geq 0}\) are independent, we have

\[
q(n, t) = \sum_{m=0}^{\infty} \Pr\{M_1(t) = m + n\} \Pr\{M_2(t) = m\} \mathbb{I}_{\{n \geq 0\}}
+ \sum_{m=0}^{\infty} \Pr\{M_2(t) = m + |n|\} \Pr\{M_1(t) = m\} \mathbb{I}_{\{n < 0\}}
= \sum_{m=0}^{\infty} \left( \sum_{\Omega(k, m+n)} \prod_{j=1}^{k} \frac{(\lambda_j t x_j)^{x_j}}{x_j!} e^{-\lambda_j t} \right) \left( \sum_{\Omega(k, m)} \prod_{j=1}^{k} \frac{(\mu_j t x_j)^{x_j}}{x_j!} e^{-\lambda_j t} \right) \mathbb{I}_{\{n \geq 0\}}
+ \sum_{m=0}^{\infty} \left( \sum_{\Omega(k, m+|n|)} \prod_{j=1}^{k} \frac{(\lambda_j t x_j)^{x_j}}{x_j!} e^{-\lambda_j t} \right) \left( \sum_{\Omega(k, m)} \prod_{j=1}^{k} \frac{(\mu_j t x_j)^{x_j}}{x_j!} e^{-\lambda_j t} \right) \mathbb{I}_{\{n < 0\}}
\]

where we have used \(2.11\) in the last step. Here, \(\mathbb{I}_A\) denotes the indicator function for the set \(A\).

Let \(n \geq 0\). On setting \(x_j = m_j\) and \(m = x + \sum_{j=1}^{k} (j - 1)m_j\), we get

\[
q(n, t) = e^{-(\Lambda + \bar{\Lambda})t} \sum_{x=0}^{\infty} \frac{e^{n+2x}}{(n+x)!x!} \left( \sum_{\sum_{i=1}^{k} m_i = n+x} (n+x)! \prod_{j=1}^{k} \frac{\lambda_j^{m_j}}{m_j!} \right) \left( \sum_{\sum_{i=1}^{k} m_i = x} x! \prod_{j=1}^{k} \frac{\mu_j^{m_j}}{m_j!} \right)
= e^{-(\Lambda + \bar{\Lambda})t} \sum_{x=0}^{\infty} \frac{e^{n+2x}}{(n+x)!x!} \Lambda^{n+x} \bar{\Lambda}^x
= e^{-(\Lambda + \bar{\Lambda})t} (\Lambda/\bar{\Lambda})^{n/2} I_{[n]} \left(2t \sqrt{\Lambda \bar{\Lambda}}\right),
\] (3.4)

where we have used multinomial theorem in the penultimate step and the definition \(2.6\) of modified Bessel function of first kind in the last step.

Similarly, for \(n < 0\), we get

\[
q(n, t) = e^{-(\Lambda + \bar{\Lambda})t} (\Lambda/\bar{\Lambda})^{n/2} I_{-n} \left(2t \sqrt{\Lambda \bar{\Lambda}}\right).
\] (3.5)

On combining \(3.4\) and \(3.5\), we get the required result. \(\square\)

On differentiating \(3.3\) and then using \(2.6\), we obtain the following result:

**Corollary 3.1.** The state probabilities of GSP satisfy the following system of differential equations:

\[
\frac{d}{dt} q(n, t) = \Lambda \left( q(n-1, t) - q(n, t) \right) - \bar{\Lambda} \left( q(n, t) - q(n+1, t) \right), \quad n \in \mathbb{Z},
\] (3.6)

with initial conditions \(q(0, 0) = 1\) and \(q(n, 0) = 0\), \(n \neq 0\).
Remark 3.3. On substituting $k = 1$ in (3.6), we get the governing system of differential equations for the pmf of Skellam process (see Kerss et al. (2014), Eq. (2.4)).

3.1. Generalized fractional Skellam process. Here, we consider a fractional version of the GSP. We call it the generalized fractional Skellam process (GFSP) and denote it by $\{S^\alpha(t)\}_{t \geq 0}$, $0 < \alpha \leq 1$. It is defined as

$$S^\alpha(t) := \begin{cases} S(Y_\alpha(t)), & 0 < \alpha < 1, \\ S(t), & \alpha = 1, \end{cases}$$

(3.7)

where the GSP $\{S(t)\}_{t \geq 0}$ is independent of the inverse stable subordinator $\{Y_\alpha(t)\}_{t \geq 0}$. It is important to note that the GFSP is a Skellam type version of the GFCP. For $k = 1$, the GFSP reduces to the fractional Skellam process of type II (see Kerss et al. (2014)). For $\lambda_j = \lambda$ and $\mu_j = \mu$, $j = 1, 2, \ldots, k$, the GFSP reduces to the fractional Skellam process of order $k$, introduced and studied by Kataria and Khandakar (2021b).

In the following result, we obtain the governing system of fractional differential equations for the state probabilities of GFSP.

Proposition 3.1. The state probabilities $q^\alpha(n, t) = \Pr\{S^\alpha(t) = n\}$, $n \in \mathbb{Z}$ of GFSP solves the following system of fractional differential equations:

$$\partial_t^\alpha q^\alpha(n, t) = \Lambda (q^\alpha(n - 1, t) - q^\alpha(n, t)) - \bar{\Lambda} (q^\alpha(n, t) - q^\alpha(n + 1, t)),$$

(3.8)

with initial conditions $q^\alpha(0, 0) = 1$ and $q^\alpha(0, n) = 0$, $n \neq 0$.

Proof. From (3.7), we have

$$q^\alpha(n, t) = \int_0^\infty q(n, u)h_\alpha(u, t)\, du,$$

(3.9)

where $q(n, .)$ is the pmf of $\{S(t)\}_{t \geq 0}$ and $h_\alpha(., t)$ is the probability density function (pdf) of $\{Y_\alpha(t)\}_{t \geq 0}$. Note that $q^\alpha(n, 0) = q(0, n)$ as $h_\alpha(u, 0) = \delta_0(u)$. On taking the R-L fractional derivative in (3.9), we get

$$D_t^\alpha q^\alpha(n, t) = -\int_0^\infty q(n, u)\frac{\partial}{\partial u}h_\alpha(u, t)\, du$$

$$= q(n, 0)h_\alpha(0+, t) + \int_0^\infty h_\alpha(u, t)\frac{d}{du}q(n, u)\, du$$

$$= q(n, 0)\frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + \int_0^\infty h_\alpha(u, t)\frac{d}{du}q(n, u)\, du,$$

(3.10)

where we have used the following results: $D_t^\alpha h_\alpha(u, t) = -\frac{\partial}{\partial u}h_\alpha(u, t)$ and $h_\alpha(0+, t) = t^{-\alpha}/\Gamma(1 - \alpha)$ (see Meerschaert and Straka (2013)).

On using (2.3) and (3.6) in (3.10), we get

$$\partial_t^\alpha q^\alpha(n, t) = \int_0^\infty h_\alpha(u, t)\frac{d}{du}q(n, u)\, du$$

$$= \int_0^\infty (\Lambda (q(n - 1, u) - q(n, u)) - \bar{\Lambda} (q(n, u) - q(n + 1, u))) h_\alpha(u, t)\, du,$$

which reduces to the required result on using (3.9). \qed

Remark 3.4. For $k = 1$, we get $\Lambda = \lambda_1$ and $\bar{\Lambda} = \mu_1$. So, in this case the system given in (3.8) reduces to the system of governing fractional differential equations for the state probabilities of fractional Skellam process of type II (see Kerss et al. (2014), Eq. (3.5)).
Remark 3.5. The pdf of inverse stable subordinator can be expressed in terms of the Wright function as follows (see Meerschaert et al. (2015), Eq. (4.7)):  

\[ h_\alpha(u,t) = t^{-\alpha} M_\alpha(ut^{-\alpha}). \]  

(3.11)

On substituting (3.3) and (3.11) in (3.9), we get an integral representation of the state probabilities of GFSP as follows:

\[ q^\alpha(n,t) = t^{-\alpha} \left( \frac{\Lambda}{\bar{\Lambda}} \right)^{n/2} \int_0^\infty e^{-(\Lambda+\bar{\Lambda})u} I_{|n|} \left( 2u \sqrt{\Lambda \bar{\Lambda}} \right) M_\alpha(ut^{-\alpha}) du, \quad n \in \mathbb{Z}. \]

Proposition 3.2. The pgf of GFSP is given by

\[ G_\alpha^S(u,t) = E_{\alpha,1} \left( \sum_{j=1}^k \left( \lambda_j (u^j - 1) + \mu_j (u^{-j} - 1) \right) t^\alpha \right). \]  

(3.12)

Proof. Using (3.1), we get

\[ G_\alpha^S(u,t) = \int_0^\infty G_\alpha^S(u,x) h_\alpha(x,t) dx \]

\[ = \int_0^\infty \exp \left\{ -x \sum_{j=1}^k \left( \lambda_j (1-u^j) + \mu_j (1-u^{-j}) \right) \right\} h_\alpha(x,t) dx \]

\[ = E_{\alpha,1} \left( \sum_{j=1}^k \left( \lambda_j (u^j - 1) + \mu_j (u^{-j} - 1) \right) t^\alpha \right). \]

This completes the proof. \(\square\)

The pgf of GFSP solves the following fractional differential equation:

\[ \partial_\alpha^t G_\alpha^S(u,t) = \sum_{j=1}^k \left( \lambda_j (u^j - 1) + \mu_j (u^{-j} - 1) \right) G_\alpha^S(u,t), \quad G_\alpha^S(u,0) = 1 \]

which is due to the fact that the Mittag-Leffler function is an eigenfunction of the Caputo fractional derivative.

Remark 3.6. On substituting \( \lambda_j = \lambda \) and \( \mu_j = \mu \) for all \( j = 1, 2, \ldots, k \) in (3.12), we get the pgf of fractional Skellam process of order \( k \) (see Kataria and Khandakar (2021b), Proposition 2.2).

Next we obtain the factorial moments of GFSP by using its pgf.

Proposition 3.3. The \( r \)th factorial moment of GFSP, that is, \( \Psi^\alpha(r,t) = \mathbb{E}(S^\alpha(t) \cdot S^\alpha(t) - 1) \cdot \cdots \cdot (S^\alpha(t) - r + 1)), \) \( r \geq 1, \) is given by

\[ \Psi^\alpha(r,t) = r! \sum_{n=1}^r \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \sum_{\sum m_\ell = r} \prod_{\ell=1}^n \left( \frac{1}{m_\ell!} \sum_{j=1}^k (j)_m \lambda_j + (-1)^m j^{(m)} \mu_j \right), \]

where \( j)_m = j(j-1) \cdots (j-m_\ell + 1) \) denotes the falling factorial and \( j^{(m)} = j(j+1) \cdots (j+m_\ell - 1) \) denotes the rising factorial.
Proof. Let \( \zeta(u) = \sum_{j=1}^{k} (\lambda_j(u^j - 1) + \mu_j(u^{-j} - 1)) \). On using the rth derivative of composition of two functions (see Johnson (2002), Eq. (3.3)) in (3.12), we get

\[
\Psi^\alpha(r, t) = \frac{\partial^r G^\alpha_s(u, t)}{\partial u^r} \bigg|_{u=1}
\]

\[
= \sum_{n=0}^{r} E_{\alpha, \alpha n+1}^n (t^\alpha \zeta(u)) \frac{1}{m!} \frac{n!}{(n-m)!} (-t^\alpha \zeta(u))^{n-m} \frac{d^r}{du^r} (t^\alpha \zeta(u))^m \bigg|_{u=1}
\]

\[
= \sum_{n=0}^{r} \frac{\ell^n}{\Gamma(n\alpha + 1)} \frac{d^r}{du^r} (\zeta(u))^m \bigg|_{u=1},
\]

where we have used (2.4) in the penultimate step. On using the following result (see Johnson (2002), Eq. (3.6))

\[
\frac{d^r}{dw^r} (g(w))^n = \sum_{m_1+m_2+\ldots+m_n=r} \frac{r!}{m_1! m_2! \ldots m_n!} g^{(m_1)}(w) g^{(m_2)}(w) \ldots g^{(m_n)}(w),
\]

we have

\[
\frac{d^r}{du^r} (\zeta(u))^m \bigg|_{u=1} = r! \sum_{\sum_{i=1}^{n} m_i = r} \prod_{\ell=1}^{n} \frac{1}{m_\ell!} \frac{d^{m_\ell}}{du^{m_\ell}} \zeta(u) \bigg|_{u=1}
\]

\[
= r! \sum_{\sum_{i=1}^{n} m_i = r} \prod_{\ell=1}^{n} \left( \frac{1}{m_\ell!} \sum_{j=1}^{k} \left( (j)m_\ell \lambda_j + (-1)^{m_\ell} j^{(m_\ell)} \mu_j \right) \right).
\]

As the right hand side of (3.14) vanishes for \( n = 0 \), the proof follows by substituting (3.14) in (3.13). \( \square \)

The mean, variance and covariance of GFSP are obtained by using Theorem 2.1 of Leonenko et al. (2014) as follows: Let \( 0 < s \leq t \). Then,

\[
\mathbb{E} (S^\alpha(t)) = m_1 \mathbb{E} (Y_\alpha(t)),
\]

\[
\text{Var} (S^\alpha(t)) = m_2 \mathbb{E} (Y_\alpha(t)) + m_1^2 \text{Var} (Y_\alpha(t)),
\]

\[
\text{Cov} (S^\alpha(s), S^\alpha(t)) = m_2 \mathbb{E} (Y_\alpha(s)) + m_1^2 \text{Cov} (Y_\alpha(s), Y_\alpha(t)).
\]

The GFSP exhibits overdispersion as \( \text{Var} (S^\alpha(t)) - \mathbb{E} (S^\alpha(t)) > 0 \) for all \( t > 0 \).

**Proposition 3.4.** The GFSP exhibits the LRD property.

**Proof.** From (3.15) and (3.16), we get

\[
\text{Corr} (S^\alpha(s), S^\alpha(t)) = \frac{m_2 \mathbb{E} (Y_\alpha(s)) + m_1^2 \text{Cov} (Y_\alpha(s), Y_\alpha(t))}{\sqrt{\text{Var} (S^\alpha(s))} \sqrt{m_2 \mathbb{E} (Y_\alpha(t)) + m_1^2 \text{Var} (Y_\alpha(t))}}.
\]

On using (2.7), (2.8) and (2.10) for fixed \( s \) and large \( t \), we get

\[
\text{Corr} (S^\alpha(s), S^\alpha(t)) \sim \frac{m_2 \Gamma^2(\alpha + 1) \mathbb{E} (Y_\alpha(s)) + m_1^2 \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) - \frac{\alpha^2 s^{\alpha+1}}{\Gamma(\alpha+1)} \right)}{\Gamma^2(\alpha + 1) \sqrt{\text{Var} (S^\alpha(s))} \sqrt{\frac{m_2 \mathbb{E} (Y_\alpha(s))}{\Gamma(\alpha+1)} + \frac{2m_1^2 s^{\alpha+1}}{\Gamma(2\alpha+1)} - \frac{m_1^2 s^{2\alpha}}{\Gamma^2(\alpha+1)}}}
\]

\[
\sim c_0(s) t^{-\alpha},
\]
where
\[ c_0(s) = \frac{m_2 \Gamma^2(\alpha + 1) \mathbb{E}(Y_\alpha(s)) + m_2^2 \Gamma^2(\alpha + 1)}{\Gamma^2(\alpha + 1) \sqrt{\text{Var}(S_\alpha(s))} \sqrt{\frac{2m_2^2}{\Gamma(2\alpha + 1)}}} \].

As \(0 < \alpha < 1\), it follows that the GFSP has the LRD property. \(\square\)

**Remark 3.7.** For a fixed \(h > 0\), the increment of GFSP is defined as
\[ Z_\alpha^h(t) := S_\alpha(t + h) - S_\alpha(t) \].

It can be shown that the increment process \(\{Z_\alpha^h(t)\}_{t \geq 0}\) exhibits the SRD property. The proof follows similar lines to that of Theorem 1 of Maheshwari and Vellaisamy (2016).

**Proposition 3.5.** The one-dimensional distributions of GFSP are not infinitely divisible.

**Proof.** Using the self-similarity property of \(\{Y_\alpha(t)\}_{t \geq 0}\), we get \(S_\alpha(t) \overset{d}{=} S(t^\alpha Y_\alpha(1))\). Thus,
\[
\lim_{t \to \infty} \frac{S_\alpha(t)}{t^\alpha} = \lim_{t \to \infty} \frac{S(t^\alpha Y_\alpha(1))}{t^\alpha} = Y_\alpha(1) \lim_{t \to \infty} \frac{S(t^\alpha Y_\alpha(1))}{t^\alpha Y_\alpha(1)} = Y_\alpha(1) \sum_{j=1}^k j(\lambda_j - \mu_j),
\]
where we have used (3.2).

Suppose on contrary \(S_\alpha(t)\) is infinitely divisible. Then, by using Proposition 2.1 of Steutel and van Harn (2004), it follows that \(S_\alpha(t)/t^\alpha\) is infinitely divisible. It is known that the limit of a sequence of infinitely divisible random variables is infinitely divisible (see Steutel and van Harn (2004), Proposition 2.2). This implies that \(Y_\alpha(1)\) is infinitely divisible. This leads to a contradiction as \(Y_\alpha(1)\) is not infinitely divisible (see Vellaisamy and Kumar (2018)). \(\square\)

### 4. GFCP time-changed by a Lévy subordinator

In this section, we consider a time-changed version of the GFCP. We call it the time-changed generalized fractional counting process-I (TCGFCP-I) and denote it by \(\{Z_\alpha^f(t)\}_{t \geq 0}\), \(0 < \alpha \leq 1\).

Let \(\{D_f(t)\}_{t \geq 0}\) be a Lévy subordinator such that \(\mathbb{E}(D_f^r(t)) < \infty\) for all \(r > 0\). We define the TCGFCP-I as
\[ Z_\alpha^f(t) := M^\alpha(D_f(t)) \tag{4.1} \]
where the GFCP \(\{M^\alpha(t)\}_{t \geq 0}\) is independent of \(\{D_f(t)\}_{t \geq 0}\).

For \(\alpha = 1\), the TCGFCP-I reduces to a time-changed version of the GCP, namely, the time-changed generalized counting process-I (TCGCP-I) \(\{Z_f(t)\}_{t \geq 0}\), that is,
\[ Z_f(t) := M(D_f(t)) \tag{4.2} \]

For \(k = 1\), the TCGFCP-I reduces to TCFPP-I, a time-changed version of the TFPP which is introduced and studied by Maheshwari and Vellaisamy (2019). Also, for \(k = 1\), the TCGCP-I reduces to a time-changed version of the Poisson process which is introduced and studied by Orsingher and Toaldo (2015) with the condition that the involved subordinator has finite moments of all order.
On taking $\lambda_j = \lambda$ for all $j = 1, 2, \ldots, k$, the GFCP and GCP reduces to the time fractional Poisson process of order $k$ (TFPPoK) and Poisson process of order $k$ (PPoK) (see Kataria and Khandakar (2021c)), respectively. For such choice of $\lambda_j$’s, the TCGFCP-I and TCGCP-I reduces to a time-changed version of the TFPPoK and PPoK (see Sengar et al. (2020)), respectively. For $\lambda_j = \lambda(1-\rho)\rho^{j-1}/(1-\rho^k)$, $0 \leq \rho < 1$, $j = 1, 2, \ldots, k$, the GFCP and GCP reduces to the fractional Pólya-Aeppli process of order $k$ (FPAPoK) and Pólya-Aeppli process of order $k$ (PAPoK) (see Kataria and Khandakar (2021c)), respectively. In this case, the TCGFCP-I and TCGCP-I reduces to a time-changed version of the FPAPoK and PAPoK, respectively.

For $\lambda_j = \lambda(1-\rho)\rho^{j-1}$, $0 \leq \rho < 1$, for all $j \geq 1$ with $k \to \infty$, the GFCP and GCP reduces to the fractional Pólya-Aeppli process (FPAP) and Pólya-Aeppli process (PAP) (see Kataria and Khandakar (2021c)), respectively. Thus, the TCGFCP-I and TCGCP-I reduces to a time-changed version of the FPAP and PAP, respectively. Also, when $\lambda_j = \beta_{j-1} - \beta_j$, $j \geq 1$ where the sequence $\{\beta_j\}_j \in \mathbb{Z}$ is such that $\beta_j = 0$ for all $j < 0$ and $\beta_j > \beta_{j+1} > 0$ for all $j \geq 0$ with $\lim_{j \to \infty} \beta_{j+1}/\beta_j < 1$, the GFCP and GCP reduces to the convoluted fractional Poisson process and convoluted Poisson process, respectively.

Thus, in this case, the TCGFCP-I and TCGCP-I reduces to a time-changed version of the convolutional fractional Poisson process and convolutional Poisson process, respectively.

**Theorem 4.1.** The pmf $p_f(n, t) = \Pr\{Z_f(t) = n\}$ of TCGCP-I is given by

$$p_f(n, t) = \sum_{k} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \mathbb{E}(e^{-\Lambda D_f(t)} D_f^{s_k}(t)),$$ \hspace{1cm} (4.3)

where $s_k = x_1 + x_2 + \cdots + x_k$ and $\Omega(k, n)$ is as given in (2.11).

**Proof.** Let $h_f(x, t)$ be the pdf of $D_f(t)$, and recall that $p(n, x)$ denotes the pmf of GCP. From (4.2), we get

$$p_f(n, t) = \int_{0}^{\infty} p(n, x) h_f(x, t) \, dx$$

$$= \int_{0}^{\infty} \sum_{\Omega(k, n)} \prod_{j=1}^{k} \frac{(\lambda_j x_j)^{x_j}}{x_j!} e^{-\Lambda x} h_f(x, t) \, dx, \hspace{1cm} \text{(using (2.11))} \hspace{1cm} (4.4)$$

$$= \sum_{\Omega(k, n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \mathbb{E}(e^{-\Lambda D_f(t)} D_f^{s_k}(t)).$$

This completes the proof. \hfill \Box

Note that

$$\sum_{n=0}^{\infty} p_f(n, t) = \sum_{n=0}^{\infty} \sum_{s_k=0}^{n} \mathbb{E}(e^{-\Lambda D_f(t)} D_f^{s_k}(t)) \sum_{x_1 + x_2 + \cdots + x_k = s_k \atop x_1 + 2x_2 + \cdots + kx_k = n} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}$$

$$= \sum_{s_k=0}^{\infty} \mathbb{E}(e^{-\Lambda D_f(t)} D_f^{s_k}(t)) \sum_{n=s_k \atop x_1 + x_2 + \cdots + x_k = s_k \atop x_1 + 2x_2 + \cdots + kx_k = n} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!}$$
\[
\begin{align*}
&= \sum_{s_k=0}^{\infty} \mathbb{E} \left( e^{-\Lambda_f(t)} D_{f}^{s_k}(t) \right) \sum_{x_1+x_2+\cdots+x_k=s_k} \frac{k}{x_j!} \prod_{j=1}^{k} \lambda_j^{x_j} \\
&= \sum_{s_k=0}^{\infty} \frac{\Lambda_s}{s_k!} \mathbb{E} \left( e^{-\Lambda_f(t)} D_{f}^{s_k}(t) \right), \quad \text{(using multinomial theorem)} \\
&= \int_{0}^{\infty} h_f(x,t) e^{-\Lambda x} \sum_{s_k=0}^{\infty} \frac{(\Lambda x)^{s_k}}{s_k!} dx = \int_{0}^{\infty} h_f(x,t) dx = 1.
\end{align*}
\]
Thus, \( p_f(n,t) \) is indeed a pmf.

Remark 4.1. On substituting \( \lambda_j = \lambda, j = 1, 2, \ldots, k \) in (4.3), we get
\[
\left. p_f(n,t) \right|_{\lambda_j=\lambda} = \sum_{\Omega(k,n)} \frac{\lambda^k}{x_1!x_2!\ldots x_k!} \mathbb{E} \left( e^{-k\Lambda_f(t)} D_{f}^{s_k}(t) \right), \quad n \geq 0,
\]
which agrees with the pmf of a time-changed PPoK (see Sengar et al. (2020), Eq. (7)).

Remark 4.2. From (4.4), the pmf of TCGCP-I can alternatively be expressed as
\[
\begin{align*}
&= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{(-1)^{s_k}}{\Lambda_s} \frac{d^s_k}{dx^s_k} \int_{0}^{\infty} e^{-\Lambda x v} h_f(x,t) dx \bigg|_{v=1} \\
&= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{(-1)^{s_k}}{\Lambda_s} \frac{d^s_k}{dx^s_k} e^{-tf(\Lambda v)} \bigg|_{v=1}.
\end{align*}
\]
For \( k = 1 \), the above expression reduces to the pmf of a time-changed Poisson process (see Orsingher and Toaldo (2015), Eq. (2.4)).

Next, we obtain the pgf \( G_f(u,t) = \mathbb{E} \left( u^{Z_f(t)} \right) \) of TCGCP-I.

Proposition 4.1. The pgf of TCGCP-I is given by
\[
G_f(u,t) = \exp \left\{ -tf \left( \sum_{j=1}^{k} \lambda_j (1 - u^j) \right) \right\}, \quad |u| \leq 1. \quad (4.5)
\]

Proof. Using (2.12), we get
\[
\begin{align*}
G_f(u,t) &= \int_{0}^{\infty} G(u,x) h_f(x,t) dx \\
&= \int_{0}^{\infty} \exp \left( -\sum_{j=1}^{k} \lambda_j (1 - u^j) x \right) h_f(x,t) dx \\
&= \exp \left\{ -tf \left( \sum_{j=1}^{k} \lambda_j (1 - u^j) \right) \right\}.
\end{align*}
\]
This completes the proof. \( \square \)

The pgf of TCGCP-I satisfies the following differential equation:
\[
\frac{d}{dt} G_f(u,t) = -f \left( \sum_{j=1}^{k} \lambda_j (1 - u^j) \right) G_f(u,t), \quad G_f(u,0) = 1.
\]

Remark 4.3. On putting \( k = 1 \) in (4.5), we get the pgf of a time-changed Poisson process (see Orsingher and Toaldo (2015), Eq. (2.2)).

Remark 4.4. We note that the TCGCP-I is equal in distribution to \( \{X'_f(t)\}_{t \geq 0} \) where \( X'_f(t) = \sum_{j=1}^{k} jN_j(D_f(t)) \), a time-changed process introduced by Zuo et al. (2021). Here, for each \( 1 \leq j \leq k, \) \( \{N_j(t)\}_{t \geq 0} \) is a Poisson process with intensity \( \lambda_j \) which is independent of the Lévy subordinator \( \{D_f(t)\}_{t \geq 0} \). This holds due to the fact that \( M(t) \overset{d}{=} \sum_{j=1}^{k} jN_j(t) \) (see Kataria and Khandakar (2021c)).

The distribution of jumps of the process \( \{X'_f(t)\}_{t \geq 0} \) is given by Zuo et al. (2021), Eq. (4.2) as follows:

\[
\Pr\{X'_f(h) = n\} = \begin{cases} 1 - hf(\Lambda) + o(h), & n = 0, \\ -h \sum_{\Omega(k,n)} f^{(s_k)}(\Lambda) \prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} + o(h), & n \geq 1. \end{cases}
\] (4.6)

Next, we obtain a version of the law of iterated logarithm for TCGFCP-I.

**Theorem 4.2.** Let \( f \) be a Bernstein function associated with Lévy subordinator \( \{D_f(t)\}_{t \geq 0} \) such that \( \lim_{x \to 0+} f(\lambda x)/f(x) = \lambda^\theta, \lambda > 0 \) which implies that it is regularly varying at \( 0^+ \) with index \( 0 < \theta < 1 \). Also, let

\[
g(t) = \frac{\log \log t}{\phi(t^{-1} \log \log t)}, \quad t > e,
\]

where \( \phi \) is the inverse of \( f \). Then,

\[
\liminf_{t \to \infty} \frac{Z^\alpha_f(t)}{(g(t))^\alpha} = \frac{d}{\sum_{j=1}^{k} j\lambda_j Y_\alpha(1)\theta^\alpha (1 - \theta)^{\alpha(1-\theta)/\theta}}.
\] (4.7)

**Proof.** From (1.2) and (1.1), we get

\[
Z^\alpha_f(t) \overset{d}{=} M(Y_\alpha(D_f(t))) \overset{d}{=} M(D^\alpha_f(t)Y_\alpha(1)),
\]

where we have used the self-similarity property of \( \{Y_\alpha(t)\}_{t \geq 0} \). Thus,

\[
\liminf_{t \to \infty} \frac{Z^\alpha_f(t)}{(g(t))^\alpha} = \liminf_{t \to \infty} \frac{M(D^\alpha_f(t)Y_\alpha(1))}{(g(t))^\alpha}
\]

\[
= \liminf_{t \to \infty} \left( \frac{M(D^\alpha_f(t)Y_\alpha(1))}{D^\alpha_f(t)Y_\alpha(1)} \right) \left( \frac{D^\alpha_f(t)Y_\alpha(1)}{(g(t))^\alpha} \right)
\]

\[
= \frac{d}{\sum_{j=1}^{k} j\lambda_j Y_\alpha(1) \left( \liminf_{t \to \infty} \frac{D_f(t)}{g(t)} \right)^\alpha}, \quad \text{(using (2.14))}
\]

\[
= \frac{d}{\sum_{j=1}^{k} j\lambda_j Y_\alpha(1) \theta^\alpha (1 - \theta)^{\alpha(1-\theta)/\theta}}.
\]

where the fact that \( D_f(t) \to \infty \) as \( t \to \infty \) a.s., is used in the penultimate step. Also, the last step follows from the following law of iterated logarithm of Lévy subordinator (see Bertoin (1996), Theorem 14, p. 92):

\[
\liminf_{t \to \infty} \frac{D_f(t)}{g(t)} = \theta(1 - \theta)^{(1-\theta)/\theta}, \quad \text{a.s.}
\]
This completes the proof. □

**Remark 4.5.** On substituting \( k = 1 \) in (4.7), we get the law of iterated logarithm for TCFPP-I (see Maheshwari and Vellaisamy (2019), Theorem 3.5). Also, on taking \( \lambda_j = \lambda, j = 1, 2, \ldots, k \) in (4.7), we get the law of iterated logarithm for a time-changed version of the TFPoK as follows:

\[
\lim_{t \to \infty} \frac{Z^\alpha(t)}{(g(t))^\alpha} = \frac{d}{2} \lambda Y_\alpha(1) \theta^\alpha (1 - \theta)^{(1-\theta)/\theta}.
\]

Moreover, on substituting \( \lambda_j = \lambda(1 - \rho) \rho^{j-1}/(1 - \rho^k), 0 \leq \rho < 1, j = 1, 2, \ldots, k \) in (4.7), we get the law of iterated logarithm for a time-changed version of FPAPoK as follows:

\[
\lim_{t \to \infty} \frac{Z^\alpha(t)}{(g(t))^\alpha} = \frac{d}{1 - \rho^k} (1 + \rho + \cdots + \rho^{k-1} - k \rho^k) Y_\alpha(1) \theta^\alpha (1 - \theta)^{(1-\theta)/\theta}.
\]

Orsingher and Toaldo (2015) showed that the Poisson process time-changed by a Lévy subordinator can be obtained as the limit of a suitable compound Poisson process. A similar result holds true for TCGCP-I.

**Theorem 4.3.** Let \( m \) be a fixed positive integer and \( \{X_j\}_{j \geq 1} \) be a sequence of independent and identically distributed random variables such that

\[
\text{Pr}\{X_1 = n\} = \frac{1}{u(m)} \int_0^\infty \text{Pr}\{M(s) = n\} \mu(ds), \quad n \geq m,
\]

where \( u(m) = \int_0^\infty \text{Pr}\{M(s) \geq m\} \mu(ds) \). Then, for \( t > 0 \), we have

\[
\lim_{m \to 0} Z_m(t) \overset{d}{=} Z_f(t),
\]

where \( Z_m(t) = X_1 + X_2 + \cdots + X_{N(tu(m)/\Lambda)} \).

**Proof.** The pgf of \( Z_m(t) \) can be written as

\[
\mathbb{E}(u^{Z_m(t)}) = \exp \left( -tu(m)(1 - \mathbb{E}(u^{X_1})) \right)
\]

\[
= \exp \left( -tu(m) \sum_{n=m}^\infty (1 - u^n) \text{Pr}\{X_1 = n\} \right)
\]

\[
= \exp \left( -tu(m) \sum_{n=m}^\infty (1 - u^n) \frac{1}{u(m)} \int_0^\infty \text{Pr}\{M(s) = n\} \mu(ds) \right)
\]

\[
= \exp \left( -t \int_0^\infty \sum_{n=m}^\infty (1 - u^n) \text{Pr}\{M(s) = n\} \mu(ds) \right).
\]

On letting \( m \to 0 \), we get

\[
\lim_{m \to 0} \mathbb{E}(u^{Z_m(t)}) = \exp \left( -t \int_0^\infty \sum_{n=0}^\infty (1 - u^n) \text{Pr}\{M(s) = n\} \mu(ds) \right)
\]

\[
= \exp \left( -t \int_0^\infty \left( 1 - e^{-s} \sum_{j=1}^k \lambda_j (1 - u^j) \right) \mu(ds) \right)
\]

\[
= \exp \left( -tf \left( \sum_{j=1}^k \lambda_j (1 - u^j) \right) \right).
\]
This completes the proof. □

4.1. Dependence structure of TCGFCP-I. Let $0 < s \leq t < \infty$ and assume that

$$l_1 = r_1/\Gamma(\alpha + 1), \quad l_2 = r_2/\Gamma(\alpha + 1), \quad d = \alpha l_1^2 B(\alpha, \alpha + 1),$$

where $r_1$ and $r_2$ are given in (2.13). From (2.16), the mean of TCGFCP-I is obtained as follows:

$$E(Z^\alpha_f(t)) = E(E(M^\alpha(D_f(t))|D_f(t))) = l_1E(D^\alpha_f(t)).$$

On substituting (2.7) and (2.9) in (2.17), we get

$$E(M^\alpha(s)M^\alpha(t)) = l_2s^\alpha + ds^{2\alpha} + \alpha l_1^2 t^{2\alpha} B(\alpha, \alpha + 1; s/t).$$

Thus,

$$E(Z^\alpha_f(s)Z^\alpha_f(t)) = E(E(M^\alpha(D_f(s))M^\alpha(D_f(t))|D_f(t)))
= l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) + \alpha l_1^2E(D^\alpha_f(t)B(\alpha, \alpha + 1; D_f(s)/D_f(t))).$$

Hence, the covariance of TCGFCP-I is given by

$$\text{Cov}(Z^\alpha_f(s), Z^\alpha_f(t)) = l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) - l_1^2E(D^\alpha_f(s))E(D^\alpha_f(t)) + \alpha l_1^2E(D^\alpha_f(t)B(\alpha, \alpha + 1; D_f(s)/D_f(t))). \quad (4.8)$$

On putting $s = t$ in (4.8), we get its variance as follows:

$$\text{Var}(Z^\alpha_f(t)) = E(D^\alpha_f(t)) \left(l_2 - l_1^2E(D^\alpha_f(t))\right) + 2dE(D^{2\alpha}_f(t)). \quad (4.9)$$

Next, we show that the TCGFCP-I has the LRD property provided the associated Lévy subordinator satisfies certain asymptotic conditions.

**Theorem 4.4.** Let $E(D^\alpha_f(t)) \sim k_it^{\theta}$ for $i = 1, 2$ such that $0 < \theta < 1$, $k_1 > 0$ and $k_2 \geq k_1^2$. Then, the TCGFCP-I exhibits the LRD property.

**Proof.** For fixed $s$ and large $t$, the following asymptotic result holds (see Maheshwari and Vellaisamy (2019), Theorem 3.3):

$$\alpha E(D^\alpha_f(t)B(\alpha, \alpha + 1; D_f(s)/D_f(t))) \sim E(D^\alpha_f(s))E(D^\alpha_f(t - s)).$$

On using it in (4.8), we get

$$\text{Cov}(Z^\alpha_f(s), Z^\alpha_f(t)) \sim l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) - l_1^2E(D^\alpha_f(s))E(D^\alpha_f(t)) + \alpha l_1^2E(D^\alpha_f(t)B(\alpha, \alpha + 1; D_f(s)/D_f(t)))
\sim l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) - l_1^2E(D^\alpha_f(s))k_1(t^\theta - (t - s)^\theta)
\sim l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) - l_1^2E(D^\alpha_f(s))k_1s\theta t^{\theta - 1},$$

where we have used $E(D^\alpha_f(t)) \sim k_1t^\theta$ in the penultimate step.

Similarly, from (4.9), we get

$$\text{Var}(Z^\alpha_f(t)) \sim l_2k_1t^\theta - l_1^2k_1^2t^{2\theta} + 2dk_2t^{2\theta}
\sim (2dk_2 - k_1^2l_1^2)t^{2\theta}.$$

For large $t$, we have

$$\text{Corr}(Z^\alpha_f(s), Z^\alpha_f(t)) \sim \frac{l_2E(D^\alpha_f(s)) + dE(D^{2\alpha}_f(s)) - l_1^2E(D^\alpha_f(s))k_1s\theta t^{\theta - 1}}{\sqrt{\text{Var}(Z^\alpha_f(s))}(2dk_2 - k_1^2l_1^2)t^{2\theta}}
\sim c_1(s)t^{-\theta},$$
where
\[ c_1(s) = \frac{l_2 \mathbb{E} \left( D_1^\alpha(s) \right) + d \mathbb{E} \left( D_2^{2\alpha}(s) \right)}{\sqrt{\text{Var} \left( Z_1^\alpha(s) \right) (2dk_2 - k_1^2l_1^2)}}. \]

As \( 0 < \theta < 1 \), the proof follows. \( \square \)

**Remark 4.6.** Along the similar lines it can be shown that TCGCP-I exhibits the LRD property.

### 4.2. Some special cases of the TCGCP-I

Here, we discuss three special cases of the TCGCP-I by taking three specific Lévy subordinators, namely, the gamma subordinator, the tempered stable subordinator (TSS) and the inverse Gaussian subordinator (IGS) as a time-change component in the GCP.

#### 4.2.1. GCP time-changed by gamma subordinator.

The pdf \( g(x, t) \) of a gamma subordinator \( \{Z(t)\}_{t \geq 0} \) is given by
\[ g(x, t) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-ax}, \quad x > 0, \]
where \( a > 0 \) and \( b > 0 \). Its associated Bernstein function is \( f_1(s) = b \log(1 + s/a) \), \( s > 0 \) (see Applebaum (2009), p. 55).

On taking \( f_1 \) as the Bernstein function in (4.2), we get the GCP time-changed by an independent gamma subordinator as
\[ Z_{f_1}(t) := M(Z(t)), \quad t \geq 0. \]  
(4.10)

On using (4.6), the distribution of its jumps is obtained in the following form:
\[ \Pr\{Z_{f_1}(h) = n\} = \begin{cases} 1 - bh \log(1 + \Lambda/a) + o(h), & n = 0, \\ -bh \sum_{\Omega(k,n)} \frac{(-1)^{s_k-1}(s_k - 1)!}{(a + \Lambda)^{s_k}} \prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} + o(h), & n \geq 1. \end{cases} \]  
(4.11)

**Remark 4.7.** On taking \( k = a = b = 1 \) in (4.11), we get the distribution of jumps of gamma-Poisson process (see Orsingher and Toaldo (2015), Eq. (4.16)).

From (4.5), the pgf of \( Z_{f_1}(t) \) is given by
\[ \mathcal{G}_{f_1}(u, t) = \left( 1 + \frac{1}{a} \sum_{j=1}^{k} \lambda_j (1 - u^j) \right)^{-bt}. \]

**Proposition 4.2.** The Lévy measure of \( Z_{f_1}(t) \) is given by
\[ \Pi_{f_1}(dx) = \sum_{n=1}^{\infty} \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{b \Gamma(s_k)}{(a + \Lambda)^{s_k}} \delta_n(dx). \]  
(4.12)

**Proof.** The Lévy measure for gamma subordinator is given by \( \mu_Z(ds) = bs^{-1}e^{-as}ds \). Using a result (see Sato (1999), Theorem 30.1, p. 197), the Lévy measure of \( Z_{f_1}(t) \) is obtained as follows:
\[ \Pi_{f_1}(dx) = \int_{0}^{\infty} \sum_{n=1}^{\infty} p(n, s) \delta_n(dx) \mu_Z(ds). \]
\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\Omega(k,n)} \frac{\lambda_j}{x_j!} b_{\delta_n}(dx) \int_0^\infty s^{s_k-1}e^{-(\Lambda+a)s}ds, \]

where we have used (2.11). This gives the required result. \(\Box\)

**Remark 4.8.** On taking \(k = 1\) in (4.12), we get

\[ \Pi_{f_1}(dx) \bigg|_{k=1} = \sum_{n=1}^{\infty} \frac{b}{n} \left( \frac{\lambda_1}{\lambda_1+a} \right)^n \delta_n(dx), \]

which is the Lévy measure of negative binomial process (see Beghin and Vellaisamy (2018)).

**Proposition 4.3.** Let \(\gamma \geq 1\) and \(\kappa(x) := \Gamma'(x)/\Gamma(x)\) be the digamma function. Then, the pmf \(p_{f_1}(n,t) = \Pr\{Z_{f_1}(t) = n\}, n \geq 0\) solves the following equation:

\[ D^\gamma_t p_{f_1}(n,t) = b D^{\gamma-1}_t (\log(a) - \kappa(bt)) p_{f_1}(n,t) + b \int_0^\infty p(n,x) \log(x) D^{\gamma-1}_t g(x,t) dx, \]

where \(D^\gamma_t\) is the R-L fractional derivative defined in (2.2).

**Proof.** From (4.10), we have

\[ p_{f_1}(n,t) = \int_0^\infty p(n,x) g(x,t) dx. \tag{4.13} \]

The following result holds for the pdf of gamma subordinator (see Beghin and Vellaisamy (2018), Lemma 2.2):

\[ D^\gamma_t g(x,t) = b D^{\gamma-1}_t (\log(ax) - \kappa(bt)) g(x,t), \quad x > 0, \]
\[ g(x,0) = 0. \]

Taking the R-L fractional derivative in (4.13) and using the above result, we get

\[ D^\gamma_t p_{f_1}(n,t) = \int_0^\infty p(n,x) D^\gamma_t g(x,t) dx \]
\[ = b \int_0^\infty p(n,x) D^{\gamma-1}_t (\log(ax) - \kappa(bt)) g(x,t) dx \]
\[ = b D^{\gamma-1}_t \log(a) \int_0^\infty p(n,x) g(x,t) dx + b \int_0^\infty p(n,x) \log(x) D^{\gamma-1}_t g(x,t) dx \]
\[ - b D^{\gamma-1}_t \kappa(bt) \int_0^\infty p(n,x) g(x,t) dx. \]

The proof follows on using (4.13). \(\Box\)

### 4.2.2. GCP time-changed by tempered stable subordinator

Let \(0 < \theta < 1\) be the stability index and \(\eta > 0\) be the tempering parameter of a TSS \(\{D_{\eta,\theta}(t)\}_{t \geq 0}\). Its associated Bernstein function \(f_2(s)\) is given by

\[ f_2(s) = (\eta + s)^\theta - \eta^\theta, \quad s > 0. \tag{4.14} \]

On taking \(f_2\) as the Bernstein function in (4.2), we get the GCP time-changed by an independent TSS as

\[ Z_{f_2}(t) := M(D_{\eta,\theta}(t)), \quad t \geq 0. \tag{4.15} \]
On using (4.6), the distribution of its jumps is obtained in the following form:

\[
\Pr \{ \mathcal{Z}_f(h) = n \} = \begin{cases}
1 - h ((\eta + \Lambda)^\theta - \eta^\theta) + o(h), & n = 0, \\
- h \sum_{\Omega(k,n)} (\theta)_s_k (\eta + \Lambda)^{\theta - s_k} \frac{(-\lambda_j)^{x_j}}{x_j!} + o(h), & n \geq 1,
\end{cases}
\]

(4.16)

where \((\theta)_s_k = \theta(\theta - 1) \cdots (\theta - s_k + 1)\).

**Remark 4.9.** On taking \(k = 1\) in (4.16), we get the distribution of jumps of relativistic Poisson process (see Orsingher and Toaldo (2015), Eq. (4.11)).

From (4.5), its pgf is given by

\[
G_{f_2}(u, t) = \exp \left\{ -t \left( \left( \eta + \sum_{j=1}^{k} \lambda_j (1 - u^j) \right)^\theta - \eta^\theta \right) \right\}.
\]

**Proposition 4.4.** The Lévy measure of \(\mathcal{Z}_f(t)\) is given by

\[
\Pi_{f_2}(dx) = \frac{\theta}{\Gamma(1 - \theta)} \sum_{n=1}^{\infty} \sum_{\Omega(k,n)} \lambda_j^{x_j} \frac{\Gamma(s_k - \theta)}{x_j!} (\Lambda + \eta)^{s_k - \theta} \delta_n(dx).
\]

The proof of Proposition 4.4 follows similar lines to that of Proposition 4.2 by using the Lévy measure of TSS, that is, \(\mu_{D_{\eta,\theta}}(ds) = \theta s^{-\theta - 1} e^{-\eta s} ds / \Gamma(1 - \theta)\).

**Proposition 4.5.** The pmf \(p_{f_2}(n, t) = \Pr \{ \mathcal{Z}_f(t) = n \}\) satisfies the following system of differential equations:

\[
\left( \eta^\theta - \frac{d}{dt} \right)^{1/\theta} p_{f_2}(n, t) = (\eta + \Lambda)p_{f_2}(n, t) - \sum_{j=1}^{\min(n,k)} \lambda_j p_{f_2}(n - j, t), \quad n \geq 0.
\]

**Proof.** From (4.15), we have

\[
p_{f_2}(n, t) = \int_0^\infty p(n, x) h_{\eta,\theta}(x, t) dx,
\]

(4.17)

where \(h_{\eta,\theta}(\cdot, t)\) is the pdf of TSS.

Let \(\delta_0(x)\) denote the Dirac delta function. On using the fact that \(\lim_{x \to 0} h_{\eta,\theta}(x, t) = \lim_{x \to \infty} h_{\eta,\theta}(x, t) = 0\) and the following result (See Beghin (2015), Eq. (15)):

\[
\frac{\partial}{\partial x} h_{\eta,\theta}(x, t) = -\eta h_{\eta,\theta}(x, t) + \left( \eta^\theta - \frac{\partial}{\partial t} \right)^{1/\theta} h_{\eta,\theta}(x, t),
\]

with the initial conditions \(h_{\eta,\theta}(x, 0) = \delta_0(x)\) and \(h_{\eta,\theta}(0, t) = 0\) in (4.17), we get

\[
\left( \eta^\theta - \frac{d}{dt} \right)^{1/\theta} p_{f_2}(n, t) = \int_0^\infty p(n, x) \left( \eta h_{\eta,\theta}(x, t) + \frac{\partial}{\partial x} h_{\eta,\theta}(x, t) \right) dx
\]

\[
= \eta p_{f_2}(n, t) - \int_0^\infty h_{\eta,\theta}(x, t) \frac{d}{dx} p(n, x) dx
\]

\[
= \eta p_{f_2}(n, t) - \int_0^\infty \left( -\Lambda p(n, x) + \sum_{j=1}^{\min(n,k)} \lambda_j p(n - j, x) \right) h_{\eta,\theta}(x, t) dx,
\]

where we have used (1.1) with \(\alpha = 1\) in the last step. The proof follows on using (4.17).  \(\square\)
If $\theta^{-1} = m \geq 2$ is an integer then the pmf $p_{f_2}(n, t)$ solves
\[
\sum_{i=1}^{m} (-1)^i \binom{m}{i} \eta^{(1-i/m)} \frac{d^i}{dt^i} p_{f_2}(n, t) = \Lambda p_{f_2}(n, t) - \sum_{j=1}^{\min\{n,k\}} \lambda_j p_{f_2}(n-j, t). \tag{4.18}
\]

Further, on putting $k = 1$ in (4.18), we get the system of differential equations that governs the state probabilities of Poisson process time-changed by TSS (see Kumar et al. (2011), Remark 4.1).

4.2.3. GCP time-changed by inverse Gaussian subordinator. Let \{Y(t)\}_{t \geq 0} be an IGS whose pdf is given by (see Applebaum (2009), Eq. (1.27))
\[
q(x, t) = (2\pi)^{-1/2} t^3/2 \exp \left\{ \delta t - \frac{1}{2} (\delta^2 t^2 x^{-1} + \gamma^2 x) \right\}, \quad x > 0,
\]
where $\delta > 0$ and $\gamma > 0$. Its associated Bernstein function is
\[
f_3(s) = \delta \left( \sqrt{2s + \gamma^2} - \gamma \right), \quad s > 0. \tag{4.19}
\]

On taking $f_3$ as the Bernstein function in (4.2), we get the GCP time-changed by an independent IGS as
\[
Z_{f_3}(t) := M(Y(t)), \quad t \geq 0. \tag{4.20}
\]

On using (4.6), the distribution of its jumps is obtained in the following form:
\[
\Pr\{Z_{f_3}(h) = n\} = \begin{cases} 
1 - h\delta \left( \sqrt{2\Lambda + \gamma^2} - \gamma \right) + o(h), & n = 0 \\
-\delta h \sum_{\Omega(k,n)} 2^{s_k} \left( \frac{1}{2} \right)^{s_k} (2\Lambda + \gamma^2)^{\frac{1}{2} - s_k} \prod_{j=1}^{k} \frac{(-\lambda_j)^{x_j}}{x_j!} + o(h), & n \geq 1.
\end{cases}
\]
where $(\frac{1}{2})_{s_k}$ denotes the falling factorial.

From (4.5), the pgf of $Z_{f_3}(t)$ is given by
\[
G_{f_3}(u, t) = \exp \left\{ -t\delta \left( \sqrt{2 \sum_{j=1}^{k} \lambda_j (1-u^j) + \gamma^2} - \gamma \right) \right\}.
\]

Proposition 4.6. The Lévy measure of $Z_{f_3}(t)$ is given by
\[
\Pi_{f_3}(dx) = \frac{\delta}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(s_k - 1/2)}{x_j!} \prod_{j=1}^{k} \frac{\lambda_{x_j}}{(\Lambda + \gamma^2/2)^{s_k-1/2}} \delta_n(dx).
\]

The proof of Proposition 4.4 follows similar lines to that of Proposition 4.2 by using the Lévy measure of IGS, that is, $\mu_Y(ds) = \delta e^{-\gamma^2 s^2/2} ds / \sqrt{2\pi s^3}$.

Proposition 4.7. The pmf $p_{f_3}(n, t) = \Pr\{Z_{f_3}(t) = n\}$ satisfies the following system of differential equations:
\[
\left( \frac{d^2}{dt^2} - 2\delta^2 \frac{d}{dt} \right) p_{f_3}(n, t) = 2\delta^2 \left( \Lambda p_{f_3}(n, t) - \sum_{j=1}^{\min\{n,k\}} \lambda_j p_{f_3}(n-j, t) \right), \quad n \geq 0. \tag{4.21}
\]
Proof. From (4.20), we have
\[ p_{f_3}(n, t) = \int_0^\infty p(n, x)q(x, t) \, dx. \] (4.22)

On taking derivatives, we get
\[ \frac{d}{dt}p_{f_3}(n, t) = \int_0^\infty p(n, x)\frac{\partial}{\partial t}q(x, t) \, dx \]
and
\[ \frac{d^2}{dt^2}p_{f_3}(n, t) = \int_0^\infty p(n, x)\frac{\partial^2}{\partial t^2}q(x, t) \, dx. \]

On using the fact that \( \lim_{x\to\infty} q(x, t) = \lim_{x\to0} q(x, t) = 0 \) and the following result for the pdf of IGS (see Vellaisamy and Kumar (2018), Eq. (3.3)):

\[
\frac{\partial^2}{\partial t^2}q(x, t) - 2\delta\gamma \frac{\partial}{\partial t}q(x, t) = 2\delta^2 \frac{\partial}{\partial x}q(x, t)
\]

in (4.22), we get
\[
\left( \frac{d^2}{dt^2} - 2\delta\gamma \frac{d}{dt} \right) p_{f_3}(n, t) = \int_0^\infty p(n, x)\left( \frac{\partial^2}{\partial t^2} - 2\delta\gamma \frac{\partial}{\partial t} \right) q(x, t) \, dx
\]
\[
= 2\delta^2 \int_0^\infty p(n, x)\frac{\partial}{\partial x}q(x, t) \, dx
\]
\[
= -2\delta^2 \int_0^\infty q(x, t) \frac{d}{dx}p(n, x) \, dx
\]
\[
= -2\delta^2 \int_0^\infty \left( -\Lambda p(n, x) + \sum_{j=1}^{\min\{n,k\}} \lambda_j p(n - j, x) \right) q(x, t) \, dx,
\]
where we have used (1.1) with \( \alpha = 1 \). The proof is complete on using (4.22). \( \square \)

Remark 4.10. Taking \( \lambda_j = \lambda \) for all \( j = 1, 2, \ldots, k \) in (4.21), we get the system of differential equations that governs the state probabilities of a time-changed PPoK (see Sengar et al. (2020), Theorem 5.1). For \( k = 1 \) in (4.21), we get the corresponding result for the Poisson process time-changed by IGS (see Kumar et al. (2011), Proposition 2.1).

5. GFCP time-changed by inverse subordinator

Here, we consider another time-changed version of the GFCP by using the inverse subordinator. The first passage time of Lévy subordinator \( \{D_f(t)\}_{t \geq 0} \) is called the inverse subordinator. It is defined as

\[ H_f(t) := \inf \{ r \geq 0 : D_f(r) > t \}, \quad t \geq 0. \]

Note that \( \mathbb{E} \left( H_f^r(t) \right) < \infty \) for all \( r > 0 \) (see Aletti et al. (2018), Section 2.1).

We define a time-changed version of the GFCP by time-changing it with an independent inverse subordinator as follows:

\[ \tilde{Z}_f^\alpha(t) := M^\alpha(H_f(t)), \quad t \geq 0. \]

We call the process \( \{ \tilde{Z}_f^\alpha(t) \}_{t \geq 0} \) as the time-changed generalized fractional counting process-II (TCGFCP-II).
For $\alpha = 1$, the TCGFCP-II reduces to a time-changed version of the GCP, namely, the time-changed generalized counting process-II (TCGCP-II), that is,

$$
\bar{Z}_f(t) := M(H_f(t)), \quad t \geq 0.
$$

(5.1)

The pmf $\bar{p}_f(n, t) = \Pr\{\bar{Z}_f(t) = n\}$ of TCGCP-II is given by

$$
\bar{p}_f(n, t) = \sum_{\Omega(k, n)} \prod_{j=1}^k \frac{\lambda_j^{x_j}}{x_j!} \mathbb{E}\left(e^{-\Lambda H_f(t)} H_j^{x_j}(t)\right), \quad n \geq 0.
$$

The proof of the above result follows similar lines to that of Theorem 4.1.

Let $l_1$, $l_2$ and $d$ be as given in Section 4.1. The mean, variance and covariance of TCGFCP-II are given by

(i) $\mathbb{E}\left(\bar{Z}_f^\alpha(t)\right) = l_1 \mathbb{E}\left(H_f^\alpha(t)\right)$,

(ii) $\text{Var}\left(\bar{Z}_f^\alpha(t)\right) = \mathbb{E}\left(H_f^\alpha(t)\right) (l_1 \mathbb{E}\left(H_f^\alpha(t)\right)) + 2d \mathbb{E}\left(H_f^{2\alpha}(t)\right)$,

(iii) $\text{Cov}\left(\bar{Z}_f^\alpha(s), \bar{Z}_f^\alpha(t)\right) = l_2 \mathbb{E}\left(H_f^\alpha(s)\right) + d \mathbb{E}\left(H_f^{2\alpha}(s)\right) - l_1^2 \mathbb{E}\left(H_f^\alpha(s)\right) \mathbb{E}\left(H_f^\alpha(t)\right) + \alpha l_1^2 \mathbb{E}\left(H_f^{2\alpha}(t)B(\alpha, \alpha + 1; H_f(s)/H_f(t))\right), \quad 0 < s \leq t.$

The proof of (i)-(iii) follows similar lines to the corresponding results of TCGFCP-I (see Section 4.1). Thus, the proofs are omitted.

Next we discuss two particular cases of the TCGCP-II.

5.1. GCP time-changed by the inverse TSS. The inverse TSS $\{\mathcal{L}_{\eta, \theta}(t)\}_{t \geq 0}$ is defined as the first passage time of TSS $\{\mathcal{D}_{\eta, \theta}(t)\}_{t \geq 0}$, $0 < \theta < 1$, $\eta > 0$, that is,

$$
\mathcal{L}_{\eta, \theta}(t) := \inf\{r \geq 0 : \mathcal{D}_{\eta, \theta}(r) > t\}, \quad t \geq 0.
$$

In (5.1), if we choose the Bernstein function $f_2$ which is given in (4.14) then we get the GCP time-changed by an independent inverse TSS. Thus,

$$
\bar{Z}_{f_2}(t) := M(\mathcal{L}_{\eta, \theta}(t)), \quad t \geq 0.
$$

(5.2)

**Proposition 5.1.** The pmf $\bar{p}_{f_2}(n, t) = \Pr\{\bar{Z}_{f_2}(t) = n\}$ solves the following system of differential equations:

$$
\left(\eta + \frac{d}{dt}\right)^\theta \bar{p}_{f_2}(n, t) = \left(\eta^\theta \bar{p}_{f_2}(n, t) - t^{-\theta} E_{1,1-\theta}^{1-\theta}(-\eta t)p(n, 0) + p(n, x)l_{\eta, \theta}(x, t)\right)_{x=0} - \Lambda \bar{p}_{f_2}(n, t) + \sum_{j=1}^{\min(n, k)} \lambda_j \bar{p}_{f_2}(n - j, t), \quad n \geq 0.
$$

Proof. From (5.2), we have

$$
\bar{p}_{f_2}(n, t) = \int_0^\infty p(n, x)l_{\eta, \theta}(x, t) \, dx,
$$

(5.3)

where $l_{\eta, \theta}(\cdot, t)$ is the pdf of $\mathcal{L}_{\eta, \theta}(t)$.

The following result holds (See Kumar et al. (2019), Eq. (25)):

$$
\frac{\partial}{\partial x} l_{\eta, \theta}(x, t) = - \left(\eta + \frac{\partial}{\partial t}\right)^\theta l_{\eta, \theta}(x, t) + \eta^\theta l_{\eta, \theta}(x, t) - t^{-\theta} E_{1,1-\theta}^{1-\theta}(-\eta t) \delta_0(x),
$$

(5.4)
where $\delta_0(x) = l_{n,\theta}(x, 0)$. From (5.3) and (5.4), we get
\[
\left( \eta + \frac{d}{dt} \right)^\theta \bar{p}_{f_3}(n, t) = \int_0^\infty p(n, x) \left( \eta^\theta l_{n,\theta}(x, t) - t^{-\theta} E_{1,1-\theta}(-\eta t)\delta_0(x) - \frac{\partial}{\partial x} l_{n,\theta}(x, t) \right) \, dx \\
= \eta^\theta \bar{p}_{f_3}(n, t) - t^{-\theta} E_{1,1-\theta}(-\eta t) \int_0^\infty p(n, x)\delta_0(x) \, dx \\
+ p(n, x)l_{n,\theta}(x, t)|_{x=0} + \int_0^\infty l_{n,\theta}(x, t)\frac{d}{dx} p(n, x) \, dx \\
= \eta^\theta \bar{p}_{f_3}(n, t) - t^{-\theta} E_{1,1-\theta}(-\eta t)p(n, 0) + p(n, x)l_{n,\theta}(x, t)|_{x=0} \\
+ \int_0^\infty \left( -\Lambda p(n, x) + \sum_{j=1}^{\min\{n,k\}} \lambda_j p(n - j, x) \right) l_{n,\theta}(x, t) \, dx,
\]
where we have used $\lim_{x \to \infty} l_{n,\theta}(x, t) = 0$ (see Alrawashdeh et al. (2017), Lemma 4.6). On using (5.3), we get the required result. 

\[ \square \]

5.2. GCP time-changed by the first passage time of IGS. The first passage time \( \{H(t)\}_{t \geq 0} \) of the IGS \( \{Y(t)\}_{t \geq 0} \) is defined as
\[
H(t) \vDash \inf\{r \geq 0 : Y(r) > t\}, \quad t \geq 0.
\]
The function $f_3$ given in (4.19) is the associated Bernstein function for an IGS. In (5.1), if we choose the Bernstein function $f_3$ then we get the GCP time-changed by an independent \( \{H(t)\}_{t \geq 0} \), that is,
\[
\bar{Z}_{f_3}(t) := M(H(t)), \quad t \geq 0. \tag{5.5}
\]

**Proposition 5.2.** The pmf $\bar{p}_{f_3}(n, t) = \Pr\{\bar{Z}_{f_3}(t) = n\}$, $n \geq 0$ solves the following system of differential equations:
\[
\delta \left( \gamma^2 + 2\frac{d}{dt} \right)^{1/2} \bar{p}_{f_3}(n, t) = (\delta \gamma - \Lambda) \bar{p}_{f_3}(n, t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j \bar{p}_{f_3}(n-j, t) - \delta \gamma \text{Erf} \left( \gamma \sqrt{t/2} \right) p(n, 0),
\]
where $\text{Erf}(\cdot)$ is the error function.

**Proof.** Let $h(\cdot, t)$ be the pdf of $H(t)$. From (5.1), we have
\[
\bar{p}_{f_3}(n, t) = \int_0^\infty p(n, x)h(x, t) \, dx. \tag{5.6}
\]
On using the following result in (5.6) (see Wyłomańska et al. (2016), Eq. (2.22)):
\[
\frac{\partial}{\partial x} h(x, t) = -\delta \left( \gamma^2 + 2\frac{d}{dt} \right)^{1/2} h(x, t) + \delta \gamma h(x, t) - \delta \sqrt{2/\pi t} e^{-\gamma^2/2\delta_0(x)},
\]
where the initial condition is $h(x, 0) = \delta_0(x)$, we get
\[
\delta \left( \gamma^2 + 2\frac{d}{dt} \right)^{1/2} \bar{p}_{f_3}(n, t) = \int_0^\infty p(n, x) \left( \delta \gamma h(x, t) - \delta \sqrt{2/\pi t} e^{-\gamma^2/2\delta_0(x)} - \frac{\partial}{\partial x} h(x, t) \right) dx \\
= \delta \gamma \bar{p}_{f_3}(n, t) - \delta \sqrt{2/\pi t} e^{-\gamma^2/2\delta_0(x)} p(n, 0) \\
+ p(n, 0)h(0, t) + \int_0^\infty h(x, t)\frac{d}{dx} p(n, x) \, dx \\
= \delta \gamma \bar{p}_{f_3}(n, t) - \delta \sqrt{2/\pi t} e^{-\gamma^2/2\delta_0(x)} p(n, 0) + p(n, 0)h(0, t).
\[
+ \int_0^\infty \left( -\Lambda p(n, x) + \sum_{j=1}^{\min\{n, k\}} \lambda_j p(n - j, x) \right) h(x, t) dx \\
= \delta \gamma \bar{\rho} f_3(n, t) - \delta \sqrt{2/\pi t} e^{-\gamma^2 t/2} p(n, 0) + p(n, 0) h(0, t) \\
- \Lambda \bar{\rho} f_3(n, t) + \sum_{j=1}^{\min\{n, k\}} \lambda_j \bar{\rho} f_3(n - j, t),
\]
where we have used (5.6). The proof follows on using the following result (see Vellaisamy and Kumar (2018), Proposition 2.2):
\[
\lim_{x \to 0} h(x, t) = h(0, t) = \delta e^{-\gamma^2 t/2} \left( \sqrt{2/\pi t} - \gamma e^{\gamma^2 t/2} \text{Erf} \left( \gamma \sqrt{t}/2 \right) \right).
\]

6. Concluding remarks

In this paper, we introduce and study the GSP and a fractional version of it, namely, the GFSP. The GSP and GFSP are Skellam type variants of the GCP and GFCP, respectively. Some distributional properties such as the pmf, pgf, mean, variance and covariance are derived for these processes. It is shown that the GSP and GFSP exhibits the LRD property. We obtain the systems of differential equations that govern their state probabilities. Two time-changed versions of the GFCP, namely, TCGFCP-I and TCGFCP-II are considered by time-changing it by an independent Lévy subordinator and its inverse. We obtain a version of the law of iterated logarithm for the TCGFCP-I. Some particular cases of these time-changed processes are considered by choosing specific Lévy subordinators such as the gamma subordinator, the TSS, the IGS and their inverse. For these particular cases, we obtain the governing system of differential equations for their state probabilities. It is known that the GCP has application in risk theory (see Kataria and Khandakar (2021c)). We expect the TCGCP-I to have potential application in risk theory as it exhibit the LRD property.

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