Multi-Objective Event-triggered Consensus of Linear Multi-agent Systems

Amir Amini, Student Member, IEEE, Arash Mohammadi, Member, IEEE and Amir Asif, Senior Member, IEEE

Abstract

This paper proposes a distributed consensus algorithm for linear event-based heterogeneous multi-agent systems (MAS). The proposed scheme is event-triggered in the sense that an agent selectively transmits its information within its local neighbourhood based on a directed network topology under the fulfillment of certain conditions. Using the Lyapunov stability theorem, the system constraints and event-triggering condition are expressed in terms of several linear matrix inequalities (LMIs) to derive the consensus parameters. The objective is to design the transmission threshold and minimum-norm heterogeneous control gains which collectively ensure an exponential consensus convergence rate for the closed-loop systems. The LMI computed control gains are robust to uncertainty with some deviation from their nominal values allowed. The practicability of the proposed event-based framework is further studied by proving the Zeno behaviour exclusion. Numerical simulations quantify the advantages of our event-triggered consensus approach in second-order, linear and heterogeneous multi-agent systems.

Index Terms—Multi-agent Systems, Event-based Consensus, Multi-objective Control Design, Linear Matrix Inequalities.

I. INTRODUCTION

Among many cooperative behaviors in multi-agent systems (MAS), consensus has attracted considerable attention due to its vast application in sensor networks, unmanned aerial vehicles (UAV), and mobile robotic teams. Related works in this area primarily deal with the problem of distributed consensus, where all agents constantly transmit their information within the network [1]–[4]. Although various interesting practical features have previously been studied to solve consensus in MASs, continuous data transmission and continuous control input update are critical restrictions in practice. Therefore, in order to preserve the limited and valuable energy allocated to local microprocessors installed on each agent, strategies which decrease data transmission and control input updates are of great interest [5], [6]. In this regard, periodic time-triggered communication and control scheme were proposed in [7] and [8] to cope with consensus in first and second-order integrators. Event-triggered broadcasting strategies, however, offer superior performance as they allow agents to transmit information, irrespective of time interval, and only if a predefined triggering condition is satisfied. Motivated by early results in event-triggered control methods such as [9], the authors in [10] extended the procedure to address the consensus problem of first-order integrators. More
recently, researchers have studied the event-based consensus problem from several aspects. For instance, in [11] effective triggering rules are proposed to guarantee consensus of second-order agents in undirected networks. In [12], event-based output consensus problem in heterogeneous linear MASs is investigated again in undirected topologies. Moreover, the consensus problem of linear and nonlinear second-order MASs is addressed in [13]. In [14], an edge event hybrid-driven rule is proposed to ensure second-order leader-following consensus. The aforementioned methods are limited to integrators and are not capable of achieving event-based consensus in more general linear agents with directed topologies. At the same time, most existing event-based schemes addressing consensus, overlook control objectives and the closed-loop system is only guaranteed to be stable [15], [16]. The use of multi-objective optimization is essential in practical applications with multiple performance requirements [17]. Furthermore, a vast majority of the relevant works only design a common control gain and share it among all agents to acquire consensus [15], [16], [18]. Such a design approach is not completely distributed. In a fully distributed structure, each agent should be able to choose a specific control gain, according to its own dynamics and connectivity within the communication network. In addition, the aforementioned single-value control design methods are not capable of handling possible heterogeneity among multi-agent system dynamics. As fully discussed in [19], control design problem for consensus of heterogeneous MASs is nontrivial, and non-selection of control gains often lead to unstable system behavior under directed networks.

As a powerful design method, linear matrix inequality (LMI) optimization guarantees system stability for desired design objectives through convex optimization [20]. For a multi-objective problem such as minimum gain resilient heterogeneous control design in an event-based communication scheme as is being considered in this paper, an analytical solution to compute the design parameters is difficult (if not impossible) to derive. As discussed earlier, analytical solutions proposed in the literature need strong assumptions on agent dynamics or/and network topology, and no specific closed-loop performance is guaranteed. Formulating the problem within an LMI framework is a practically feasible solution to pursue in many control applications [21]–[25]. We note that deriving optimization matrix inequalities in a linear form is a non-trivial effort. Some suggested consensus approaches result in bilinear matrix inequalities (BMIs) that are even more difficult to solve [26], [27].

To address the aforementioned limitations, the paper investigates the problem of objective-based control design for event-triggered consensus in heterogeneous linear MASs. The main contributions of the paper are listed as follows: 1) To guarantee consensus, we couple the control gain design and event-triggering function to benefit from multi-objective optimization. The proposed algorithm has an exponential consensus convergence rate with robust minimum-norm control gains; 2) The proposed approach provides additional degrees of freedom by designing heterogeneous control gains for event-triggered multi-agent networks. This is a unique advancement over most of the existing works in event-triggered MASs where a common control gain is used by the agents; 3) To the best of our knowledge, the event-triggered consensus problem has not been considered previously in the context of heterogeneous second-order MASs with directed network topologies. This is the first instance of incorporating multi-objective LMI optimization in consensus problems for second
order event-triggered heterogeneous MASs.

The remaining paper is organized as follows. Section 2 introduces required preliminary concepts. The problem is stated in Section 3. In Section 4, we proceed to formulate the multi-objective event-based consensus problem. The algorithm to derive unknown design parameters within the LMI optimization is proposed in Section 4. We provide simulation examples in Section 5 to evaluate the capability of the algorithm. Finally, Section 6 concludes the paper.

II. Preliminaries and Graph Theory

Throughout the paper, we use normal alphabets to denote matrices or scalars, and bold letters to specify vectors. Notation $\mathbb{R}^{m \times n}$ refers to $(m \times n)$ real-valued matrices. In what follows we present necessary matrix notation and commonly-used operations for matrix $A = \{a_{ij}\} \in \mathbb{R}^{m \times n}$ with real entries $a_{ij}$, and $B = \{b_{ij}\} \in \mathbb{R}^{m \times n}$. $|A|$: Matrix with component-wise absolute value of $A$, i.e., $\{ |a_{ij}| \} \in \mathbb{R}^{m \times n}$; $\|A\|$: Frobenius norm of $A$; $A^T$: Transpose of $A$; $A^\dagger$: Pseudo inverse of $A$; $\lambda_{\min}(A)$: minimum eigenvalue of $A$; $\lambda_{\max}(A)$: maximum eigenvalue of $A$; $A > 0$: $A$ is symmetric positive definite, i.e., $x^T A x > 0, \forall x \in \mathbb{R}^n$; $A \geq 0$: $A$ is symmetric semi-positive definite, i.e., $x^T A x \geq 0, \forall x \in \mathbb{R}^n$; null($A$): Null space of $A$, i.e., $\{ x \in \mathbb{R}^n | A x = 0 \}$; $A \otimes B$: Kronecker product of $A$ and $B$; $A \circ B$: Hadamard product of $A$ and $B$; $a_{(i, \ast)}$: The $i$-th row of matrix $A$, i.e., $[a_{i1}, \ldots, a_{in}]$. Similarly, for vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, the term $u \leq v$ defines the component-wise inequality, i.e., $u_i \leq v_i, i = 1, \ldots, N$. Moreover, $\|u\|$ is the Euclidean norm of $u$, and $1_n$ defines the $(n \times 1)$ column vector with all elements equal to one.

In symmetric block matrices, the asterisk $\ast$ represents the lower triangle block which is induced by symmetry.

Lemma 1. Schur Complement [28].

Considering matrices $R$, $Q$, and $S$ with appropriate dimensions, the following two statements are equivalent.

$$R > 0, \quad Q - SR^{-1}S^T > 0 \iff \begin{bmatrix} Q & S \\ * & R \end{bmatrix} > 0.$$  

Lemma 2. S-procedure [28].

Let $T_0, T_1 \in \mathbb{R}^{n \times n}$ be symmetric matrices. If there exists a scalar $\tau \geq 0$ such that $T_0 - \tau T_1 > 0$, then the following inequalities on $T_0$ and $T_1$ are satisfied

$$x^T T_0 x > 0, \quad \forall x \neq 0 \quad \text{such that} \quad x^T T_1 x \geq 0. \quad (1)$$

The communication network of a MAS consisting of $N$ agents is modeled using a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, ..., N\}$ denotes the agent set, i.e., the $i$-th vertex indicates the $i$-th agent. The edge set $\mathcal{E}$ is defined as the Cartesian product of the two sets, i.e., $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If agent $j$ communicates its information to agent $i$, then the pair $(j, i)$ is an element of $\mathcal{E}$ denoted by $j \rightarrow i$ in graph representation. In a directed graph, $(j, i) \in \mathcal{E}$ is not equivalent to $(i, j) \in \mathcal{E}$. Term $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ denotes the weighted adjacency matrix for graph $\mathcal{G}$, where $a_{ii} = 0$, $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. The neighbor set of agent $i$ is defined
by $N_i = \{ j \in V \mid (i, j) \in E \}$. A directed graph contains a directed spanning tree if there exists a node in the graph which has directed paths to all other nodes. The Laplacian matrix corresponding to $G$ is defined as $L = \{ l_{ij} \} = D - A$, where $D = \text{diag} (\text{deg}_1, ..., \text{deg}_N)$, with $\text{deg}_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix has an eigenvalue of zero if and only if the directed network contains a directed spanning tree. Under this condition, all other eigenvalues have positive real components [29].

III. Problem Statement

Consider a multi-agent network system comprising of $N$ agents with the general linear dynamics given by

$$\dot{x}_i(t) = Ax_i(t) + B_i u_i(t), \quad 1 \leq i \leq N. \quad (2)$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector at time instant $t$, and $u_i(t) \in \mathbb{R}^m$ is the control input vector. Matrices $A$ and $B_i$, with appropriate dimensions, represent the system matrix and control input matrix, respectively.

Despite the common practice in event-based strategies (30, 16) that assumes identical agents across the network, we consider the input matrix $B_i$ to be different among agents. The agent model defined in (2) satisfies the controllability assumption for any pairs $(A, B_i)$. Moreover, the network configuration contains a directed spanning tree.

Remark 1. As a large class of mechanical systems, we note that the heterogeneous second-order MASs can be represented by (2).

A proposed distributed protocol $u_i(t)$ is said to solve the consensus problem if the following condition is fulfilled.

Definition 1. Given any initial condition, the consensus problem is solved if and only if the state disagreement norm of any two agents in the network asymptotically converges to zero [31], i.e.,

$$\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0, \quad \forall i, j \in V. \quad (3)$$

The agents share their information with the neighbors through a directed network to reach a common state value. However, in order to decrease the number of transmissions in the distributed scheme, an efficient event-triggering mechanism is of great interest. In the desired event-based strategy for data communication, agent $i$ observes its own state vector constantly. If a certain proposed condition with a designed threshold is violated, it transmits the state vector to its neighboring agents. Upon receiving the data, node $j$, a neighbor of agent $i$, updates its information regarding agent $i$ by incorporating the newest received data. Denoting $t_{i0}, t_{i1}, \ldots$ as the triggering time sequence of agent $i$, we define the most recently broadcasted information of agent $i$ for any interval between two consecutive triggering instants as follows

$$\hat{x}_i(t) = x_i(t_{ik}), \quad t \in [t_{ik}, t_{ik+1}). \quad (4)$$
In order to reach the consensus condition specified in Definition 1, the following distributed protocol for agent $i$ is proposed

$$\mathbf{u}_i(t) = (K_i + \Delta K_i(t)) \sum_{j \in N_i} (\hat{x}_i(t) - \hat{x}_j(t)).$$  \hspace{1cm} (5)$$

Matrix $K_i \in \mathbb{R}^{m \times n}$ is the nominal control gain to be designed for agent $i$, and $\Delta K_i(t)$ is the additive unknown norm-bounded, structured uncertainty in the controller parameter \[32\]. The control law proposed in (5) depends only on the last transmitted states. Such a control structure leads to an event-based control input update mechanism, meaning that the actuators receive input signals only on certain instants. Therefore, the number of control input updates are lower \[30\]. Another specific characteristic of the proposed event-triggered protocol (5) is assigning different control gains to individual agents, leading to a heterogeneous controller design. Performance degradation due to the sensitivity of the closed-loop system to inaccuracies in the control’s coefficients at the implementation stage is inevitable. Therefore, it is required to provide some level of robustness in the design of the control parameters. Noting that the perturbation in control parameters mainly happens due to inaccurate system modeling or round off errors, a practical solution considered here is to develop resilient control design techniques. Since a rapid non-conservative convergence rate is desirable in most applications, the control gain $K_i$’s, here, are supposed to assure a sufficient fast consensus with an exponential rate \[33\]. Unlike most work where convergence rate is neglected in the control design, we are therefore interested to incorporate conditions for exponential stabilizing with other design features. In the design of the proposed control law, we also minimize the norm of $K_i$’s in order to avoid large undesirable control inputs.

To summarize, we will design the event-triggered control law defined in (5) to reach consensus by assuring the stability of the closed-loop system as the primitive focus while satisfying these objectives: I) Minimizing the number of transmissions among the agents, leading to a lower number of control input updates; II) Efficient performance by designing heterogeneous control parameters; III) Robustness to a predefined level of uncertainty in the obtained control parameters; IV) Exponential rate of convergence, and; V) Minimizing control gains to decrease control input effort.

**IV. Problem formulation**

Let $e_i(t) = \hat{x}_i(t) - x_i(t)$ denote the measurement error between the most recently transmitted state and its instantaneous value for agent $i$. For the benefit of analyzing the MAS in a collective manner, we define $\mathbf{x}(t) = [x_1^T(t), \ldots, x_N^T(t)]^T$ as the stacked state vector, and $\hat{\mathbf{x}}(t) = [\hat{x}_1^T(t), \ldots, \hat{x}_N^T(t)]^T$ as the stacked vector for the last transmitted states. We also define $\mathbf{e}(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$ as the stacked measurement error vector, which is equivalent to

$$\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t).$$ \hspace{1cm} (6)$$

Now, we combine (2) with the proposed controller (5) to obtain the following augmented closed-loop system

$$\dot{\mathbf{x}}(t) = \left( A_{\mathbf{x}} + B(K + \Delta K(t))L_{\mathbf{d}} \right) \mathbf{x}(t) + B(K + \Delta K(t))L_{\mathbf{d}} \mathbf{e}(t),$$ \hspace{1cm} (7)$$
where \( L_{(\omega)} = L \otimes I_n \). The new variable \( A_{(\omega)} = I_n \otimes A \) is the global system matrix, and block-diagonal matrix \( B = \text{diag} (B_1, \ldots, B_N) \) is the global control input matrix. The unknown control gains are accumulated in matrix \( K = \text{diag} (K_1, \ldots, K_N) \) and their corresponding, possibly time-variant, perturbation are denoted by \( \Delta K(t) = \text{diag} (\Delta K_1(t), \ldots, \Delta K_N(t)) \). The latter matrix satisfies the following assumption.

**Assumption 1.** The following predefined upper bound (threshold) holds for \( \Delta K(t) \) in all time instants \( t \).

\[
\| \Delta K(t) \| \leq \delta. \tag{8}
\]

**A. System Transformation**

Before we proceed to present the event-triggering scheme and control design procedure, it is necessary to note that if \( A \) in (2) is unstable, then system matrix in (7) will be unstable since \( L_{(\omega)} \) always contains a zero eigenvalue [34]. Therefore, the goal is to reach consensus by establishing the stability of a transformed version of the system. Another reason for using this approach is to utilize the well-developed results in the Lyapunov method, which also provides a variety of performance indices beside ensuring the stability of the system. Hence, we convert the consensus problem at hand, i.e., the closed-loop system defined in (7), into an equivalent stability problem using an appropriate transformation. A proper state transformation, e.g., \( \tilde{x}(t) = T x(t) \), for achieving this objective needs to satisfy the following two conditions: (i) First, stability of such a transformed system, \( \tilde{x}(t) \), must be equivalent to the consensus problem for the closed-loop system defined by (7), and; (ii) All heterogeneous parameters \( B_i \) and \( K_i \) should be involved in the transformed system. It is worth mentioning that the transformations suggested in a majority of related works, such as [15] and [34], are incapable of meeting the latter challenge and, thus, heterogeneous control gain design is not applicable in such approaches.

In order to achieve the consensus condition defined in Definition [11] the Laplacian matrix is first nominated as the transformation matrix, i.e., \( \tilde{x}(t) = L \otimes I_n x(t) \). Using Laplacian \( L \) as the transformation matrix, however, would result in a singular system, since \( L \) is not full ranked and thus the set of state disagreement emerging in \( \tilde{x}(t) \) would be linearly dependent. Since dealing with singular systems brings several technical and analytical difficulties [35], we remove one row of \( L \) to design the transformation matrix. The proposed solution will eliminate system redundancy and provides a reduced full-rank system. Therefore, we let \( \tilde{L} \in \mathbb{R}^{(N-1) \times N} \) denote a matrix which is obtained by removing one arbitrary row of the Laplacian matrix. The proposed state transformation is, therefore, given by

\[
x_i(t) = \hat{L}_{(\omega)} x_i(t), \tag{9}
\]

where \( \hat{L}_{(\omega)} = \tilde{L} \otimes I_n \).

**Lemma 3.** It follows from (9) that \( x_i(t) = 0 \) if and only if \( x_i(t) = \cdots = x_N(t) \). The consensus condition in Definition [11] is satisfied when \( x_i(t) = 0 \).

**Proof.** If \( x_i(t) = 0 \), then according to (9) we have \( \hat{L}_{(\omega)} x(t) = 0 \), which means \( x(t) \) belongs to the null space of \( \hat{L}_{(\omega)} \), i.e., \( x(t) \in \text{null} (\hat{L}_{(\omega)}) \). Since the row sum of \( L \), and similarly \( \tilde{L} \), is zero, the null space of \( \hat{L}_{(\omega)} \) is...
We define the disagreement vector for agent $B$. Event-triggering scheme decides on its own triggering time. The primary goal here is to determine the maximum stable value for $\hat{x}(t)$, where $\hat{x}(t)$ is the consensus vector to which all $x_i(t)$, $(1 \leq i \leq N)$, converge. Therefore, it is concluded that $x_i(t) = \ldots = x_N(t) = \hat{x}(t)$. Accordingly, the consensus equation defined in (9) is satisfied. The statements are bidirectional in the sense that if $x_i(t) = \ldots = x_N(t) = \hat{x}(t)$, then $\hat{x}(t) = 0$ holds.

According to Lemma 3, the consensus problem for system (7) is equivalent to the stability problem of the system expressed in terms of transformation (9). It is worth mentioning that the consensus vector $\hat{x}(t)$ may be constant or time-varying depending on the dynamics of the MAS.

**Lemma 4.** $\hat{L}_{(n)} A_{(n)} = A_{(N-1)} \hat{L}_{(n)}$, where $A_{(N-1)} = I_{N-1} \otimes A$.

**Proof.** $\hat{L}_{(n)} A_{(n)} = (\hat{L} \otimes I_n)(I_n \otimes A) = (\hat{L}_{(n)} I_n)(I_n A) = (I_{N-1} \hat{L}) \otimes (AI_n) = (I_{N-1} \otimes A)(\hat{L} \otimes I_n) = A_{(N-1)} \hat{L}_{(n)}$.

Using Lemma 4, the closed-loop system given in (7) is transformed to the following reduced order structure,

$$\dot{x}_i(t) = (A_{(N-1)} + K + \Delta K) x_i(t) + (A + \Delta K) e_i(t), \quad (10)$$

where $K = \hat{L}_{(n)} B K L$, $\Delta K = \hat{L}_{(n)} B \Delta K(t) L$, and $e_i(t) = \hat{L}_{(n)} e(t)$, with $L = L_{(n)} \hat{L}_{(n)}$. The reduced measurement error for the closed-loop system given in (11) is $e_i(t) = \hat{x}_i(t) - x_i(t)$, where $\hat{x}_i(t) = \hat{L}_{(n)} \hat{x}(t)$.

**Remark 2.** Without loss of generality and for the sake of brevity in notation, we remove row $N$ from the Laplacian matrix $L$, to derive $\hat{L}$.

The exponential stability for system (10) is defined below.

**Definition 2.** Given damping coefficient $\zeta > 0$, system (10) is $\zeta$-exponentially stable if there exists a positive scalar $c$ such that $x_i(t)$ satisfies the following condition

$$\|x_i(t)\| \leq ce^{-\zeta t}\|x_i(0)\|, \quad t \geq 0. \quad (11)$$

In the following section, we proceed to introduce and formulate the event-triggering mechanism.

**B. Event-triggering scheme**

We define the disagreement vector for agent $i$ as $\hat{x}_i(t) = l^{(n)}_{(i, \bullet)} \hat{x}(t)$, with $l^{(n)}_{(i, \bullet)} = l_{(i, \bullet)} \otimes I_n$. In fact, $\hat{x}_i(t)$ provides the instantaneous disagreement between the last transmitted state corresponding to agent $i$ and the last received states from its neighbors. Let $\hat{x}(t) = [\hat{x}_1^T(t), \ldots, \hat{x}_N^T(t)]^T$ denote the stacked disagreement vector. Given $t^i_k$, the next triggering instant for agent $i$ is, therefore, determined from the following condition

$$t^i_{k+1} = \inf \{ t > t^i_k : h(e_i(t), \hat{x}_i(t), \phi) \geq 0 \}, \quad (12)$$

where $h(e_i(t), \hat{x}_i(t), \phi) = \|e_i(t)\| - \phi \|\hat{x}_i(t)\|$, and real-valued scalar $\phi > 0$ is the transmission threshold to be determined. Note that the triggering function given in (12) is asynchronous, i.e., each agent independently decides on its own triggering time. The primary goal here is to determine the maximum stable value for $\phi$, which provides the minimum number of transmissions for a particular network configuration with guaranteed
control performances. Between two consecutive events for agent $i$, the triggering function is non-positive, i.e., $h_i \leq 0$. Thus, we consider the following component-wise inequality derived based on (12)

$$e^{[N]} \leq \phi \hat{X}^{[N]}_i,$$

where $e^{[N]} = [\|e_i(t)\|, \ldots, \|e_N(t)\|]^T$, and $\hat{X}^{[N]}_i = [\|\hat{X}_i(t)\|, \ldots, \|\hat{X}_N(t)\|]^T$. In order to merge the design of maximum possible transmission $\phi$ with desired control objectives, the event-triggering condition (13) needs to be expressed as a function of the system’s state variables, i.e., $x_i(t)$ and $e_i(t)$. In this regard, the following two Lemmas are introduced to transform (13) into the required structure.

**Lemma 5.** If a certain value $\phi$ satisfies (13), the following entry-wise inequality is also satisfied

$$\hat{L} e^{[N]} \leq \phi |\hat{L}| \hat{X}^{[N]}_i.$$  

**Proof.** Component $i$ in $\hat{L} e^{[N]}$ is computed as $l_{(i, \bullet)} e^{[N]}_i (1 \leq i \leq N - 1)$. Therefore, the entries of $e^{[N]}$ are multiplied by exactly one positive and at least one negative value in a network containing a directed spanning tree. According to the Euclidean normed-space properties, the absolute value of row vector $l_{(i, \bullet)}$ lies within an upper bound, i.e.,

$$l_{(i, \bullet)} e^{[N]}_i \leq \phi |l_{(i, \bullet)}| \hat{X}^{[N]}_i, \quad 1 \leq i \leq N - 1.$$  

Expanding (15) for all rows of $\hat{L}$ results in (13). □

**Lemma 6.** Denote matrix $M = \{ m_{ij} \} = \{ l_{ij} + \alpha_j l_{i2} \}, \ 1 \leq i \leq N-1 \text{ and } 1 \leq j \leq N-1$, as the correlation matrix, with $\alpha = [\alpha_1, \ldots, \alpha_{N-1}] = l_{(N, \bullet)} \hat{L}^\dagger$. If a certain value $\phi$ satisfies the following entry-wise inequality

$$\psi_e \leq \psi_{\hat{X}},$$

it also satisfies inequality (14). The undefined vectors in (16) are $\psi_e = \left[ ||l^{(n)}_{(1, \bullet)} e(t)||, \ldots, ||l^{(n)}_{(N-1, \bullet)} e(t)|| \right]^T$, and $\psi_{\hat{X}} = \left[ ||\phi m^{(n)}_{(1, \bullet)} \hat{X}_1(t)||, \ldots, ||\phi m^{(n)}_{(N-1, \bullet)} \hat{X}_N(t)|| \right]^T$, with $m^{(n)}_{(i, \bullet)} = m_{(i, \bullet)} \otimes I_n$.

**Proof.** Based on the reverse triangle inequality in the Euclidean normed-space, we conclude that

$$l_{(i, \bullet)} e^{[N]} \leq ||l^{(n)}_{(i, \bullet)} e(t)||, \quad 1 \leq i \leq N - 1,$$

which is equivalent to $\hat{L} e^{[N]} \leq \psi_e$ considering all rows. On the other hand, the sub-additivity property in the Euclidean normed-space proves that

$$\|\phi l^{(n)}_{(i, \bullet)} \hat{X}_i(t)\| \leq \phi \|l_{(i, \bullet)}| \hat{X}^{[N]}_i, \quad 1 \leq i \leq N - 1.$$  

Since $\hat{X}_i(t)$ is formed by the row space of Laplacian matrix, a certain component in $\hat{X}_i(t)$, e.g., $\hat{X}_N(t)$, is always dependent on the other components; meaning that $\hat{X}_N(t)$ can be written as a linear combination of $\hat{X}_i(t)$ to $\hat{X}_{N-1}(t)$, i.e., $\hat{X}_N(t) = \alpha_1 \hat{X}_1(t) + \cdots + \alpha_{N-1} \hat{X}_{N-1}(t)$. Thus, the coefficients $\alpha_i$ are calculated as $\alpha = l_{N, \bullet} \hat{L}^\dagger$. By substituting $\hat{X}_N(t)$ with its linear equivalent value, inequality (18) reduces to

$$\|\phi m^{(n)}_{(i, \bullet)} \hat{X}_i(t)\| \leq \phi \|l_{(i, \bullet)}| \hat{X}^{[N]}_i, \quad 1 \leq i \leq N - 1.$$  

We conclude from (19) that \( \hat{\psi}_x \leq \phi |\hat{L}|^{\mathbb{N}} \). Under the assumption outlined in inequality (16), the outcome of (17) and (19) is the following sequence of inequalities

\[
\hat{L}e^{\mathbb{N}} \leq \psi_e \leq \hat{\psi}_x \leq \phi |\hat{L}|^{\mathbb{N}},
\]

which complete the proof for Lemma 6.

In conclusion, Lemma 6 states that any obtained value for \( \phi \) which satisfies (16), also satisfies (14). Furthermore, inequality (14) is equivalent to the triggering condition defined in (13) according to Lemma 5.

Inequality (16) is favorable in the sense that it can be expressed as a global quadratic constraint in the form of \( e^T e(t) \leq \hat{x}_r^T(t) \Phi^2 \hat{x}_r(t) \), where \( \Phi = \phi \Phi_{(N-1)n} \), and \( \Phi_{(n)} = M \otimes I_n \). Replacing \( \hat{x}_r(t) \) with \( e_r(t) + x_r(t) \), the equivalent condition is given below.

\[
e^T e(t) \leq (e_r(t) + x_r(t))^T M_{(n)}^T \Phi^2 M_{(n)} (e_r(t) + x_r(t)).
\]

Inequality (21) represents the event-triggering constraint that is expected to appear in the convex optimization framework. Once the feasible transmission threshold \( \phi \) is obtained through optimization, the desired event condition defined in (12) is exploited to determine the triggering moments for each agent. Inequality (21) is, therefore, guaranteed according to Lemma 5 and Lemma 6.

**Remark 3.** Although transmission threshold \( \phi \) seems to depend only on \( x_r(t), e_r(t), \) and \( M_{(n)} \) from (21), we will see that \( \phi \) is also affected by other design parameters emerging in the optimization problem.

**C. Main Result**

The following theorem computes the minimum-norm resilient heterogeneous control gain \( K_i \)'s and maximum transmission threshold \( \phi \) used in our event-based consensus algorithm.

**Theorem 1.** The optimum values for the transmission threshold \( \phi \) and control gains \( K_i \)'s \( (1 \leq i \leq N) \) are computed from

\[
\phi = \sqrt{\tau_3 \gamma - 1}, \quad \text{and} \quad K_i = B_i^T \Phi^{-1} \Theta_i,
\]

which are conditioned on the existence of matrices \( \Theta_i \in \mathbb{R}^{n \times n} (1 \leq i \leq N) \), symmetric positive definite matrix \( \Phi \in \mathbb{R}^{n \times n} \), and positive scalars \( \tau_j \ (1 \leq j \leq 3) \). Under such conditions, the following minimization derives the minimum-valued positive scalars \( \gamma, \mu, \) and \( v_i \) for \( (1 \leq i \leq N) \)

\[
\min_{\Theta_i, \gamma, \tau_j, \Phi, \mu, v_i} \gamma + \mu + \sum_{i=1}^{N} v_i, \quad \text{for} \ (1 \leq i \leq N),
\]

subject to:

\[
\begin{bmatrix}
\Pi_1 & \Pi_2 \\
* & \Pi_3
\end{bmatrix} < 0,
\begin{bmatrix}
\mu I & I \\
* & \Phi
\end{bmatrix} > 0,
\begin{bmatrix}
-\gamma & \Theta^T \\
* & -I
\end{bmatrix} < 0,
\]

(23)
where

\[
\Pi_1 = \begin{bmatrix}
\pi_{11} & \Xi L \\
* & \tau_2 \delta^2 L^T L - \tau_2 I
\end{bmatrix},
\quad \Pi_2 = \begin{bmatrix}
P\hat{L}_{(n)} B & P\hat{L}_{(n)} B & \tau_3 M_{(n)}^T \\
0 & 0 & \tau_3 M_{(n)}^T
\end{bmatrix},
\quad \Pi_3 = \text{diag} (-\tau_1 I, -\tau_2 I, -\Gamma),
\]

\[
\Theta = \text{diag} (\Theta_1, \ldots, \Theta_N),
\quad \Upsilon = \text{diag} (v_1 I_n, \ldots, v_N I_n),
\]

with

\[
\pi_{11} = A_{(N-n)}^T P + PA_{(N-n)} + 2\zeta P + \tau_2 \delta^2 L^T L + \Xi L + L^T \Xi,
\]

\[
\Xi = \left( \hat{L} \otimes 1_n 1_n^T \right) \circ \left( 1_{N-n} \otimes [\Theta_1, \ldots, \Theta_N] \right),
\]

\[
\Gamma = \gamma I_{n(n-1)}, \quad \text{and} \quad P = I_{N-n} \otimes \mathcal{P}.
\]

In the above terms, parameters \(\{\delta, \zeta\}\) are constants with known positive values. The designed parameters stabilizes the system defined in (10) at the \(\zeta\)-exponential rate \(||x_r(t)|| < ce^{-\zeta t}||x_r(0)||\), where \(c = \sqrt{\lambda_{\max}(\mathcal{P})\lambda_{\min}^{-1}(\mathcal{P})}\).

The control objectives defined in section III are simultaneously guaranteed.

**Proof.** To derive the stability conditions for the closed-loop system defined in (10), we consider the Lyapunov function candidate

\[
V(t) = x^T_r(t)Px_r(t).
\]

(26)

Now, consider the following inequality

\[
\dot{V}(t) + 2\zeta V(t) < 0,
\]

(27)

where the time derivative of \(V(t)\) is defined as \(\dot{V}(t)\). The condition defined in (27) is equivalent to \(V(t) < V(0)e^{-2\zeta t}\). Considering (25), we obtain

\[
\lambda_{\min}(\mathcal{P})||x_r(t)||^2 \leq V(t) < V(0)e^{-2\zeta t} \leq \lambda_{\max}(\mathcal{P})e^{-2\zeta t}||x_r(0)||^2,
\]

which results in \(||x_r(t)|| < ce^{-\zeta t}||x_r(0)||\), with the \(c\) defined in Theorem 1. Therefore, the condition given in (27) is the sufficient constraint to ensure \(\zeta\)-exponential stability according to Definition 2. Now \(\dot{V}(t)\) is expanded according to the reduced closed-loop system (10) as follows

\[
\dot{V}(t) = \dot{x}^T_r(t)Px_r(t) + x^T_r(t)\dot{P}x_r(t)
\]

\[
= \left( A_x x_r(t) + \mathcal{A} x_r(t) + \hat{L}_{(n)} B \sigma_1 + \hat{L}_{(n)} B \sigma_2 \right)^T P x_r(t) + x^T_r(t) P \left( A_x x_r(t) + \mathcal{A} x_r(t) + \hat{L}_{(n)} B \sigma_1 + \hat{L}_{(n)} B \sigma_2 \right),
\]

where \(A_x = (A_{(N-n)} + \mathcal{A})\), \(\sigma_1 = \Delta_K(t)L x_r(t)\), and \(\sigma_2 = \Delta_K(t)L e_r(t)\). Defining \(\Omega = [x^T_r(t), e^T_r(t), \sigma_1^T, \sigma_2^T]^T\), one can rearrange (27) in terms of \(\Omega\) to obtain the following matrix structure

\[
\Omega^T \begin{bmatrix}
A^T P + PA_x + 2\zeta P & \mathcal{A} P & P \hat{L}_{(n)} B & P \hat{L}_{(n)} B \\
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{bmatrix} \Omega < 0.
\]

(28)

To formulate proper quadratic conditions with respect to \(\sigma_1\) and \(\sigma_2\), we use the upper bound norm for control uncertainties based on Assumption 4

\[
\sigma_1^T \sigma_1 = x^T_r(t)L^T \Delta^2_K(t)L x_r(t) \leq \delta^2 x^T_r(t)L^T L x_r(t),
\]

(29)
\[ \sigma_2^T \sigma_2 = e_i^T(t)L^T \Delta_k^2(t)\mathbb{L}e_i(t) \leq \delta^2 e_i^T(t)L^T \mathbb{L}e_i(t). \]  

(30)

The performance-related constraints derived in (21), (29), and (30) are required to be included in the stability constraint (28). Repeatedly using Lemma 2, the aforementioned constraints along with the new slack variables \( \tau_1, \tau_2 \) and \( \tau_3 \) appear in the following integrated matrix inequality

\[
\bar{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} & P\hat{L}(\mathbb{L}) & P\hat{L}(\mathbb{L}) \\ * & \pi_{22} & 0 & 0 \\ * & * & -\tau_1 I & 0 \\ * & * & * & -\tau_2 I \end{bmatrix} < 0,
\]

(31)

with

\[
\pi_{11} = A_x^TP + PA_x + 2\zeta P + \tau_1 \delta^2 L^T \mathbb{L} \mathbb{L} + \tau_3 M_{(n)}^T \Phi^2 M_{(n)},
\]

\[
\pi_{12} = PA + \tau_3 M_{(n)}^T \Phi^2 M_{(n)}, \quad \text{and}
\]

\[
\pi_{22} = \tau_2 \delta^2 L^T \mathbb{L} \mathbb{L} - \tau_3 I + \tau_3 M_{(n)}^T \Phi^2 M_{(n)}.
\]

Now, we apply Lemma 11 to obtain the inequality below

\[
\begin{bmatrix} A_x^TP + PA_x + 2\zeta P + \tau_1 \delta^2 L^T \mathbb{L} & PA & P\hat{L}(\mathbb{L}) & P\hat{L}(\mathbb{L}) & \tau_3 M_{(n)}^T \Phi \\ * & -\tau_3 I + \tau_2 \delta^2 L^T \mathbb{L} & 0 & 0 & \tau_3 M_{(n)}^T \Phi \\ * & * & -\tau_1 I & 0 & 0 \\ * & * & * & -\tau_2 I & 0 \\ * & * & * & * & -\tau_3 I \end{bmatrix} < 0.
\]

(32)

Pre and post-multiplying (32) with the positive definite matrix \( Q = \text{diag}(I, I, I, I, \Phi^{-1}) \) results in inequality

\[
\begin{bmatrix} A_x^TP + PA_x + 2\zeta P + \tau_1 \delta^2 L^T \mathbb{L} & PA & P\hat{L}(\mathbb{L}) & P\hat{L}(\mathbb{L}) & \tau_3 M_{(n)}^T \Phi \\ * & -\tau_3 I + \tau_2 \delta^2 L^T \mathbb{L} & 0 & 0 & \tau_3 M_{(n)}^T \Phi \\ * & * & -\tau_1 I & 0 & 0 \\ * & * & * & -\tau_2 I & 0 \\ * & * & * & * & -\tau_3 \Phi^{-2} \end{bmatrix} < 0.
\]

(33)

The matrix inequality derived in (33) is not linear as long as the optimization variables are multiplied by each other or their inverse form exists. Hence, the blocks containing \( PA_x, PA, \) and \( \tau_3 \Phi^{-2} \) need to be handled in such a way that the ultimate inequality turns into a linear structure. To this end, we first expand \( PA \) according to \( P \) defined in (25). Using Hadamard and Kronecker product, it is straightforward to derived the equation below

\[
PA = \left( \hat{L} \otimes 1_n 1_n^T \right) \circ \left( 1_{N-n} \otimes [\mathcal{P}B_1K_i, \ldots, \mathcal{P}B_nK_i] \right) L.
\]

Defining \( \Theta_i = \mathcal{P}B_iK_i, (1 \leq i \leq N) \), as alternative variables, the inequality (33) becomes linear with respect to \( \Theta_i \)'s, thus the term \( \Xi \) given in (25) is obtained. The same procedure is applicable for handling \( \tau_3 \Phi^{-2} \). Defining \( \Gamma = \tau_3 \Phi^{-2} = \gamma I_{n(N-n)} \), the maximization problem over \( \Phi \) is converted to an equivalent convex minimization problem over \( \Gamma \). The resulting inequality will now be a linear one with respect to \( \Gamma \) and \( \tau_3 \). Note that \( \tau_3 \) is not multiplied by any other variables except \( \Phi \). Finally, to make the obtained control gains implementable,
we minimize the size of $K_i$’s by adding appropriate constraints. As the two components which are used to
calculate control gains, we restrict the norm of $\Theta_i$’s and $P^{-1}$ by setting the following minimization conditions
on $\mu$ and $\Upsilon$ [37]

$$P^{-1} \prec \mu I, \quad \mu > 0,$$

and

$$\Theta_i^T \Theta_i < \upsilon_i I, \quad \upsilon_i > 0.$$  \hspace{1cm} (34)

The Schur complement lemma is enough to derive the two corresponding LMIs in (33). Once the optimization
problem is solved, unknown variables $\tau_3$, $P$, $\Theta_i$ and $\gamma$ are obtained. The Control gains and event-triggering
threshold coefficient are consequently derived from (22).

**Remark 4.** We note that, design parameters can alternatively be obtain ed directly from (32) by solving
BMIs. However, this approach is more computationally challenging and provides no guarantee of global
optimization.

**D. Zeno Behavior Exclusion**

From an implementation point of view, in an event-triggering scheme, there must always be a finite number
of triggering instants within a given finite time interval. Otherwise, the triggering mechanism would exhibit
Zeno behavior [38]. It is essential to prove that the time interval between any two events are strictly positive
for all agents. The following theorem provides the lower bound on the interval between two consecutive
triggering instants.

**Theorem 2.** Considering system (2), control law (5), event-triggering function (12), and design parameters
(22), the inter-event interval for agent $i$ is strictly positive and lower bounded by the following term

$$t_{i,k+1} - t_i \geq \frac{1}{\|A\|} \ln \left( \frac{\phi \|A\| \|\hat{X}_i(t_i)\|}{\mathcal{F}_i} + 1 \right),$$

where

$$\mathcal{F}_i = \max_{t \in [t_i, t_{i,k+1}]} \|A\hat{x}_i(t) + B_i(K_i + \Delta K_i(t))\hat{X}_i(t)\|,$$

and $\|\hat{X}_i(t)\| < \delta_{c_i}$. Positive-valued scalar $\delta_{c_i}$ is the
given stopping threshold for agent $i$, $(1 \leq i \leq N)$.

**Proof.** Consider an interval $t \in [t_i, t_{i,k+1})$ for agent $i$. Based on the event-triggering mechanism discussed in
section IV-B, $e_i(t_i) = 0$. Then, $e_i(t)$ evolves from zero with the following dynamics until $t_{i,k+1}$ is determined
by (12) and $\phi$ from Theorem 1.

$$\frac{d}{dt} \|e_i(t)\| \leq \|\hat{x}_i(t)\| \leq \|A(\hat{x}_i(t) - e_i(t)) + B_i(K_i + \Delta K_i(t))\hat{X}_i(t)\| \leq \|A\| \|e_i(t)\| + \mathcal{F}_i(t),$$

where $\mathcal{F}_i(t) = \|A\hat{x}_i(t) + B_i(K_i + \Delta K_i(t))\hat{X}_i(t)\|$. One can solve (37) for $e_i(t)$ as follows

$$\|e_i(t)\| \leq \int_{t_k}^{t} \mathcal{F}_i(\eta)e^\|\|A\|\| t - \eta \, d\eta, \quad t \in [t_i, t_{i,k+1}).$$

(37)
Algorithm 1: Proposed Event-based Consensus

Input: Adjacency Weighted Matrix $A = \{a_{ij}\}$, Agents’ dynamics given in (2).

Output: Multi-objective Event-triggered Consensus.

Parameter Design: (D1 – D5)

I. Initialization

D1. Transformation Matrix: Remove $N^{th}$ row of Laplacian matrix $L$ in order to determine the reduced Laplacian matrix, $\hat{L}$.

D2. System Transformation: Derive reduced system (10).

D3. Correlation Matrix: Using Lemma 6 determine correlation matrix $M_{\langle n \rangle}$.

II. Optimization

D4. Solving the LMIs: Using convex optimization solvers solve the LMIs (23) for given parameters $\{\delta, \zeta\}$.

D5. Feasibility Verification: If a solution exists for (23) obtain $\phi$, and $K_i$’s from (22). Otherwise, change parameters $\{\delta, \zeta\}$, and repeat step D4.

Event-triggered Consensus: (C1 – C3)

C1. Initialization: Initialize consensus process by allowing all agents to transmit their initial states $x_i(0)$ to neighbours.

C2. Execution: Using $K_i$’s derived in Step D4, the states of agent $i$ in (2) are excited by local controller given in (5). Triggering condition (12) is responsible to determine the next state transmission to neighbours for agent $i$ as the states evolves to reach consensus.

C3. Consensus Achievement: Agent $i$ repeats Step C2 until convergence is achieved for the disagreement state vector, i.e., $\|\hat{X}_i(t)\| < \delta_{ci}$.

Incorporating (37) with the event-triggering mechanism $\|e_i(t)\| \leq \phi \|\hat{X}_i(t)\|$, the next triggering instant for agent $i$ does not happen until the right-hand side of (37) evolves from zero to reach $\phi \|\hat{X}_i(t_{i,k})\|$. Then it follows from (37) that $\frac{d}{dt} \|e_i(t)\| \leq \|A\| \|e_i(t)\| + F_i^k$, or equivalently

$$\|e_i(t)\| \leq \|A\| \left( \phi \|\hat{X}_i(t_{i,k})\| + F_i^k \right).$$ (38)

The next event is triggered at $t = t_{i,k+1}$ when

$$\|e_i(t_{i,k+1})\| = \phi \|\hat{X}_i(t_{i,k})\| \leq \frac{\|A\|}{\|A\|} \left( \phi \|\hat{X}_i(t_{i,k+1})\| + F_i^k \right),$$ (39)

which simplifies to (36). Observe that during the consensus process $\hat{X}_i(t_{i,k}) > \delta_{ci}$, thus the right hand side of (36) is strictly positive and $t_{i,k+1} - t_{i,k} > 0$.

The event-based consensus algorithm is summarized in Algorithm 1.
V. Simulations

A. Deterministic example

Consider a network of six second-order heterogeneous agents with the following dynamics:

\[ \ddot{m}_i r_i(t) = [1 + \Delta_{u_i}] u_i(t), \quad 1 \leq i \leq 6, \] (40)

where \( r_i(t) \in \mathbb{R} \) and \( \bar{m}_i > 0 \) defines, respectively, the position and inertia of agent \( i \). Inequality \( |\Delta_{u_i}| < 1 \) represents the uncertainty in the control input \( u_i(t) \in \mathbb{R} \) due to dis-adjustment of actuators. Equation (40) can be rearranged as \((\bar{m}_i/(1 + \Delta_{u_i})) \dot{r}_i(t) = u_i(t)\). As suggested in [39], the term \((\bar{m}_i/(1 + \Delta_{u_i}))\), denoted as \( \bar{m}_i \), is treated as the new inertia for agent \( i \). The state space representation for (40) is, therefore, given by

\[ \dot{r}_i(t) = v_i(t) \]
\[ m_i v_i(t) = u_i(t), \quad 1 \leq i \leq 6 \] (41)

where \( v_i(t) \in \mathbb{R} \) denotes the velocity of agent \( i \). Unlike previous work where the inertia \( m_i \)'s and uncertainty \( \Delta_{u_i} \)'s in the second-order MASs are not considered separately for each agent, i.e., \( m_i = 1 \) are assumed, model (41) considers a more practical scenario with heterogeneous inertias. Among various choices suggested to simulate heterogeneous inertias in literature, we consider \( m_i = 0.8 + 0.1i \), \( 1 \leq i \leq 6 \), as in [39]. The state space representation for (41) with respect to (2) is, hence, given by \( x_i(t) = [r_i(t), v_i(t)]^T \), \( A = [0, 1; 0, 0] \), \( B_1 = [0, 0.9]^T \); \( B_2 = [0, 1.0]^T \); \( B_3 = [0, 1.1]^T \); \( B_4 = [0, 1.2]^T \); \( B_5 = [0, 1.3]^T \); and \( B_6 = [0, 1.4]^T \). The directed network configuration corresponded to the (41) is described by the asymmetric Laplacian matrix

\[ L = [3, 0, 0, -1, -1, -1; 0, 2, 0, 0, -1, -1; 0, 0, 2, -1, 0, -1; 0, -1, 0, 2, 0, -1; -1, -1, 0, -1, 3, 0; -1, -1, -1, 0, 0, 3]. \]

To solve the consensus problem using Theorem 1, we initialize the LMI optimization with \( \zeta = 0.4 \) and \( \delta = 0.02 \). Using the YALMIP parser and SDPT3 solver, we solve (22) with the aforementioned values for the optimization and system parameters [40]. The parameters obtained from the LMI optimization (22) are

\[ P = \begin{bmatrix} 0.0608 & -0.0363 \\ -0.0363 & 0.1020 \end{bmatrix}, \quad \tau_3 = 0.1312, \quad \gamma = 5.032. \] (42)

Using (22), the control gains are derived as follows:

\[ K_1 = [-0.1187, -0.2952], \quad K_2 = [-0.1933, -0.2953], \]
\[ K_3 = [-0.1965, -0.4292], \quad K_4 = [-0.1391, -0.2523], \quad K_5 = [-0.2413, -0.3472], \quad K_6 = [-0.1842, -0.1659]. \]

The maximum transmission threshold is also calculated from (22) as \( \phi = 0.1614 \). In order to observe the state trajectories of the closed-loop heterogeneous multi-agent system (40) with the designed parameters, we pick initial values for \( x_i(0) = [i + 5, i - 2]^T \), \( 1 \leq i \leq 6 \). Moreover, the time-varying uncertainty in the control gains is assumed to be \( \Delta K_i(t) = \frac{1}{\sqrt{2}} \sin(t) [0.02, 0.02] \) for all agents. Recall that \( \| \Delta K(t) \| \leq \delta \). Computed with discretization intervals \( T_s = 10^{-3} \text{sec} \), the state trajectories of the six agents are shown in Figure 11(a). We further define the convergence criteria for the consensus process as \( \delta_{ci} = 5 \times 10^{-3} \). Figure 11(b) plots the control inputs as defined in (5) with respect to the aforementioned free or obtained parameters. Figure 2 is included to verify that the obtained parameters, i.e., \( K_i \)'s and \( \phi \), are capable of ensuring \( \zeta \)-exponential convergence among the agents for \( \zeta = 0.4 \) and \( \delta = 0.02 \).
The simulation results show that it takes 12412 iterations to achieve consensus in this experiment. However, the six agents, respectively, transmit their information on 807, 781, 317, 222, 311, and 140 occasions during the consensus process. The average number of data transmission per agent in this configuration is, therefore, 429.67 times. We also define a criteria to measure the efficiency of the event-based method in terms of the saved amount of transmission in percentage, denoted by $\overline{ST}$. The $\overline{ST}$ index is expressed as a function of the average number of transmissions per agent, defined as $\overline{AT}$, and total number of iterations to reach consensus, denoted by $\overline{TI}$, i.e,

$$\overline{ST}_\% = (1 - \overline{AT}/\overline{TI}) \times 100.$$  \hspace{1cm} (43)

Using the definition given in (43), 96.54% of the total possible transmissions are saved in this example. Another crucial performance-related factor in a comparison is the amount of control force consumption during the process. In order to consider the effect of control input expense, we measure the well-known input cost function $J_u = \sum_{i=1}^{N} \int_0^\infty u_i(t)^T u_i(t) dt$ in our analysis [41]. In the current experiment, the control cost is calculated as $J_u = 57.0516$. We will use the value of $J_u$ to compare the control expense in consensus processes.

It is also interesting to study how the performance indices are affected if agents are intentionally allowed to transmit at a higher rate. To this end, we manually reduce the initially obtained $\phi = 0.1614$ to let agents benefit from receiving more data from their neighbors. The results are summarized in Table II. All other parameters remain the same as denoted previously. According to Table II, when the agents are allowed to transmit more data with lower values of $\phi$, the save transmission $\overline{ST}$ is reduced as a result of higher average transmission per agent $\overline{AT}$.

However, transmitting more information to the neighbors does not necessarily result in a faster consensus after a certain value as illustrated by $\overline{TI}$. Moreover, the control cost $J_u$ constantly gets reduced as a result of more communication. In other words, more transmission helps achieving smoother trajectories and control
TABLE I: Consensus performance, decreasing $\phi$ with $\{\delta = 0.02, \zeta = 0.4\}$

| Transmission threshold, $\phi$ | $\overline{T}$ | $\overline{A}$ | $\overline{ST}_{\%}$ | $J_u$ |
|-------------------------------|----------------|----------------|----------------------|-------|
| 0.12                          | 11920          | 461.17         | 96.13                | 50.18 |
| 0.08                          | 12651          | 742.00         | 94.13                | 44.05 |
| 0.04                          | 13272          | 1439.00        | 89.15                | 39.06 |
| 0.00                          | 13586          | 13586          | 0                    | 34.72 |

TABLE II: Consensus performance, increasing $\delta$ with $\zeta = 0.4$

| Control uncertainty $\delta$ | $\phi$    | $\overline{T}$ | $\overline{A}$ | $\overline{ST}_{\%}$ | $J_u$ |
|-------------------------------|-----------|----------------|----------------|----------------------|-------|
| 0.010                         | 0.1612    | 19184          | 699.83         | 96.35                | 55.26 |
| 0.020                         | 0.1614    | 12412          | 429.67         | 96.54                | 57.05 |
| 0.030                         | 0.1616    | 10400          | 221.33         | 97.87                | 61.81 |
| 0.040                         | 0.1617    | 12438          | 191.67         | 98.45                | 65.81 |
| 0.050                         | 0.1617    | 14369          | 366.50         | 97.44                | 66.78 |

inputs, but not necessarily at a faster convergence rate. In conclusion, once the $\phi$ is derived from (22), it is always possible to run the algorithm at a lower $\phi$, as a trade-off between $\overline{AT}$ and $J_u$.

In this part, we investigate how different choices of constraint-related values $\{\zeta, \delta\}$, will affect the consensus performance. Table III shows the effect of $\delta$ being increased until the LMIs (23) become infeasible while $\zeta$ is fixed. According to Table III a larger uncertainty $\delta$ leads to a slightly larger threshold $\phi$ in the feasibility region of (23). On the other hand, an increment in $\delta$ always results in relatively larger control gains. These two contrasting outcomes, result in a parabolic trend in $\overline{T}$. Therefore, it is difficult to predict $\overline{AT}$ and $\overline{T}$ beforehand. Greater uncertainty in control gains, results in a more fluctuating control input with huge jumps, thus $J_u$ is ascending.

In the next experiment we focus on the effect of $\zeta$ on consensus performance. The results are summarized in Table III. As we increase the value of the $\zeta$, larger $K_i$’s are obtained from the optimization LMI’s to accelerate the convergence process. Consequently, the actuators are forced to implement greater control

Fig. 2: Exponential stability for $x_i(t)$
Fig. 3: Monte Carlo results for $N = \{8, 12, 16\}$; (a): $\overline{T\!I}$ and $\overline{A\!T}$, varying $\zeta$ with fixed $\delta = 0.01$, (b): Control cost $J_u$, varying $\zeta$ with fixed $\delta = 0.01$, (c): $\overline{T\!I}$ and $\overline{A\!T}$, varying $\delta$ with fixed $\delta = 0.4$, (d): Control cost $J_u$, varying $\delta$ with fixed $\delta = 0.4$.

| damping coefficient $\zeta$ | $\phi$  | $\overline{T\!I}$ | $\overline{A\!T}$ | $\overline{S\!T\%}$ | $J_u$ |
|-----------------------------|---------|-------------------|-------------------|----------------------|-------|
| 0.10                        | 0.1601  | 24998             | 424.83            | 98.30                | 40.59 |
| 0.20                        | 0.1607  | 20916             | 720.17            | 96.55                | 45.69 |
| 0.30                        | 0.1611  | 16466             | 739.83            | 95.50                | 50.71 |
| 0.40                        | 0.1614  | 12412             | 429.67            | 96.53                | 57.05 |
| 0.50                        | 0.1616  | 9249              | 199.00            | 97.84                | 63.20 |
| 0.60                        | 0.1617  | 9064              | 354.50            | 96.08                | 69.29 |

input with more fluctuation which results in more control cost. In conclusion, as $\zeta$ is increased the $\overline{T\!I}$ constantly gets reduced at the expense of more $J_u$, while $\overline{S\!T}$ remains relatively the same.

B. Monte-Carlo Simulations

All scenarios in section V-A were based on a single network of six agents. As the adjacency matrix $A$ affects the feasibility region of (23), a Monte-Carlo simulation is included to study heterogeneous event-based consensus in randomly generated networks including a spanning tree. The inertias are also generated.
randomly as $m_i = 1 + 0.1 \times_i$, where $\times_i$ is a Gaussian random variable with zero mean and unit variance. In the first Monte-Carlo experiment the control uncertainty $\delta = 0.01$ is fixed, and $\zeta$ is gradually increased from 0.2 to 0.5 for different network sizes with $N = \{8, 12, 16\}$ (Figures 3(a) and 3(b)). In the second scenario, $\delta$ is gradually increased with fixed $\zeta = 0.4$ (Figures 3(c) and 3(d)). The following facts are observed through the two scenarios. (i) According to Figure 3(a), the value of save transmission $\overline{ST}$ is reduced as the network is expanded; (ii) As shown in Figure 3(a), $\overline{T1}$ and $\overline{AT}$ are more close to each other in larger networks as compared to smaller configurations; (iii) As observed in Figure 3(b), increasing the value of $\zeta$ results in a faster consensus convergence rate; (iv) As shown in Figure 3(c), the parabolic pattern is detectable for $\overline{T1}$ and $\overline{AT}$ over randomly generated networks of different size; (v) As illustrated in Figure 3(d), the control cost is raised when $K_i$’s are subjected to a larger uncertainty.

VI. Conclusion

The paper addresses the problem of event-based consensus with predefined objectives for a class of heterogeneous (multi-agent systems) MASs configured in directed networks. The augmented closed-loop system is transformed to an equivalent reduced system. The Lyapunov stability theorem is then used to incorporate the control objectives (exponential convergence rate, resilient control design, and minimum gain design) within an event-triggered function. The heterogeneous control gains and the transmission threshold are co-designed by solving an LMI-based optimization problem. It is also proved that the triggering mechanism does not exhibit the Zeno behavior. The effectiveness of the proposed algorithm is studied through simulations for heterogeneous second-order MASs. In future, we are interested in applying the proposed algorithm to even-based state estimation problems in sensor networks.

REFERENCES

[1] A. Amini, A. Azarbahram, and M. Sojoodi, “H∞ consensus of nonlinear multi-agent systems using dynamic output feedback controller: an lmi approach,” Nonlinear Dynamics, vol. 85, no. 3, pp. 1865–1886, 2016.
[2] Z. Li, W. Ren, X. Liu, and M. Fu, “Consensus of multi-agent systems with general linear and lipschitz nonlinear dynamics using distributed adaptive protocols,” IEEE Trans. Autom. Control, vol. 58, no. 7, pp. 1786–1791, 2013.
[3] A. Mohammadi and A. Asif, “Distributed consensus innovation particle filtering for bearing/range tracking with communication constraints,” IEEE Trans. Signal Process., vol. 63, no. 3, pp. 620–635, 2015.
[4] S. Zhu, C. Chen, X. Ma, B. Yang, and X. Guan, “Consensus based estimation over relay assisted sensor networks for situation monitoring,” IEEE J. Sel. Top. Signal Process., vol. 9, no. 2, pp. 278–291, 2015.
[5] G. Battistelli, L. Chisci, and D. Selvi, “Distributed averaging of exponential-class densities with discrete-time event-triggered consensus,” IEEE Trans. Control Network Syst., 2016.
[6] A. Amini, A. Mohammadi, and A. Asif, “Event-based consensus for a class of heterogeneous multi-agent systems: An lmi approach,” in ICASSP, 2017.
[7] G. Xie, H. Liu, L. Wang, and Y. Jia, “Consensus in networked multi-agent systems via sampled control: fixed topology case,” in American Control Conference, pp. 3902–3907, IEEE, 2009.
[8] W. Ren and Y. Cao, “Convergence of sampled-data consensus algorithms for double-integrator dynamics,” in CDC 2008. 47th IEEE Conference on, pp. 3965–3970, IEEE, 2008.
[9] P. Tabuada, “Event-triggered real-time scheduling of stabilizing control tasks,” IEEE Trans. Autom. Control, vol. 52, no. 9, pp. 1680–1685, 2007.
[10] D. V. Dimarogonas and K. H. Johansson, “Event-triggered control for multi-agent systems,” in CDC/CCC 2009. Proceedings of the 48th IEEE Conference on, pp. 7131–7136, IEEE, 2009.

[11] M. Cao, F. Xiao, and L. Wang, “Event-based second-order consensus control for multi-agent systems via synchronous periodic event detection,” IEEE Trans. Autom. Control, vol. 60, no. 9, pp. 2452–2457, 2015.

[12] W. Hu, L. Liu, and G. Feng, “Output consensus of heterogeneous linear multi-agent systems by distributed event-triggered/self-triggered strategy,” IEEE Trans. Cybern., vol. PP, no. 99, pp. 1–11, 2016.

[13] H. Li, X. Liao, T. Huang, and W. Zhu, “Event-triggering sampling based leader-following consensus in second-order multi-agent systems,” IEEE Trans. Autom. Control, vol. 60, no. 7, pp. 1998–2003, 2015.

[14] M. Cao, F. Xiao, and L. Wang, “Second-order leader-following consensus based on time and event hybrid-driven control,” Syst. Control Lett., vol. 74, pp. 90–97, 2014.

[15] N. Mu, X. Liao, and T. Huang, “Event-based consensus control for a linear directed multiagent system with time delay,” IEEE Trans. Circuits Syst. II Express Briefs, vol. 62, no. 3, pp. 281–285, 2015.

[16] D. Yang, W. Ren, X. Liu, and W. Chen, “Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs,” Automatica, vol. 69, pp. 242–249, 2016.

[17] R. T. Marler and J. S. Arora, “Survey of multi-objective optimization methods for engineering,” Struct. Multidiscip. Optim., vol. 26, no. 6, pp. 369–395, 2004.

[18] X. Zhou, P. Shi, C.-C. Lim, C. Yang, and W. Gui, “Event based guaranteed cost consensus for distributed multi-agent systems,” J. Franklin Inst., vol. 352, no. 9, pp. 3546–3563, 2015.

[19] J. Mei, W. Ren, and J. Chen, “Consensus of second-order heterogeneous multi-agent systems under a directed graph,” in 2014 ACC, pp. 802–807, IEEE, 2014.

[20] C. Scherer, P. Gahinet, and M. Chilali, “Multiobjective output-feedback control via lmi optimization,” IEEE Trans. Autom. Control, vol. 42, no. 7, pp. 896–911, 1997.

[21] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, “A truncated prediction approach to consensus control of lipschitz nonlinear multi-agent systems with input delay,” IEEE Trans. Control Netw. Syst.

[22] H. Li, X. Jing, and H. R. Karimi, “Output-feedback-based control for vehicle suspension systems with control delay,” IEEE Trans. Ind. Electron., vol. 61, no. 1, pp. 436–446, 2014.

[23] P. Badri, A. Amini, and M. Sojoodi, “Robust fixed-order dynamic output feedback controller design for nonlinear uncertain suspension system,” Mech. Syst. Sig. Process., vol. 80, pp. 137–151, 2016.

[24] L. Jiang, W. Yao, Q. Wu, J. Wen, and S. Cheng, “Delay-dependent stability for load frequency control with constant and time-varying delays,” IEEE Trans. Power Syst., vol. 27, no. 2, pp. 932–941, 2012.

[25] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, “A truncated prediction approach to consensus control of lipschitz nonlinear multi-agent systems with input delay,” IEEE Trans. Control Network Syst., 2016.

[26] H. Zhao and J. H. Park, “Dynamic output feedback consensus of continuous-time networked multiagent systems,” Complexity, vol. 20, no. 5, pp. 35–42, 2015.

[27] X. Zhou, P. Shi, C.-C. Lim, C. Yang, and W. Gui, “Event based guaranteed cost consensus for distributed multi-agent systems,” J. Franklin Inst., vol. 352, no. 9, pp. 3546–3563, 2015.

[28] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear matrix inequalities in system and control theory. SIAM, 1994.

[29] Z. Li, G. Wen, Z. Duan, and W. Ren, “Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs,” IEEE Trans. Autom. Control, vol. 60, no. 4, pp. 1152–1157, 2015.

[30] W. Hu, L. Liu, and G. Feng, “Consensus of linear multi-agent systems by distributed event-triggered strategy,” IEEE Trans. Cybern., vol. 46, no. 1, pp. 148–157, 2016.

[31] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proceedings of the IEEE, vol. 95, no. 1, pp. 215–233, 2007.

[32] M. S. Mahmoud, Resilient control of uncertain dynamical systems, vol. 303. Springer Science & Business Media, 2004.

[33] A. D. Ames, K. Galloway, K. Sreenath, and J. W. Grizzle, “Rapidly exponentially stabilizing control lyapunov functions and hybrid zero dynamics,” IEEE Trans. Autom. Control, vol. 59, no. 4, pp. 876–891, 2014.

[34] Y. Liu and Y. Jia, “h∞ consensus control of multi-agent systems with switching topology: a dynamic output feedback protocol,” Int. J. Control, vol. 83, no. 3, pp. 527–537, 2010.
[35] J. Y. Ishihara and M. H. Terra, “On the lyapunov theorem for singular systems,” *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1926–1930, 2002.

[36] V. Phat, Y. Khongtham, and K. Ratchagit, “Lmi approach to exponential stability of linear systems with interval time-varying delays,” *Linear Algebra Appl.*, vol. 436, no. 1, pp. 243–251, 2012.

[37] D. Šiljak and D. Stipanovic, “Robust stabilization of nonlinear systems: the lmi approach,” *Mathematical problems in Engineering*, vol. 6, no. 5, pp. 461–493, 2000.

[38] V. S. Dolk, P. Tesi, C. De Persis, and W. Heemels, “Event-triggered control systems under denial-of-service attacks,” *IEEE Trans. Control Network Syst.*, 2016.

[39] J. Mei, W. Ren, and J. Chen, “Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph,” *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2019 – 2034, 2016.

[40] J. Lofberg, “Yalmip: A toolbox for modeling and optimization in matlab,” in *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pp. 284–289, IEEE, 2005.

[41] L. Yu and J. Chu, “An lmi approach to guaranteed cost control of linear uncertain time-delay systems,” *Automatica*, vol. 35, no. 6, pp. 1155–1159, 1999.