Flows for rectangular matrix models

René Lafrance
and
Robert C. Myers

Department of Physics, McGill University, Montréal, Québec, Canada H3A-2T8

Abstract

Several new results on the multicritical behavior of rectangular matrix models are presented. We calculate the free energy in the saddle point approximation, and show that at the triple-scaling point, the result is the same as that derived from the recursion formulae. In the triple-scaling limit, we obtain the string equation and a flow equation for arbitrary multicritical points. Parametric solutions are also examined for the limit of almost-square matrix models. This limit is shown to provide an explicit matrix model realization of the scaling equations proposed to describe open-closed string theory.

lafrance@hep.physics.mcgill.ca
rcm@hep.physics.mcgill.ca
1 Introduction

In recent years, the study of random matrix models as a lattice regulator of two-dimensional euclidean gravity has shed light on the theory of non-critical strings. The discovery of double-scaling limit[1–3] has led to results to all orders in the genus expansion. The study of new matrix models may lead to further insights for the continuum theory of random surfaces.

Rectangular $N \times M$ matrix models differ from previously studied models in that they have two independent large $N$ parameters. In general, one finds that the critical behavior is characterized by two-dimensional singularities. There are two special cases in these models, described by a double-scaling limit: the vector model limit where $M$ remains finite, and the almost-square matrix limit where $P = N - M$ remains finite. These cases were solved in ref. [4], where a surface interpretation was also provided. The first limit leads to a phase of dense polymers, and the second, to random surfaces. Another class of critical points is described by a triple-scaling limit, which corresponds to independently taking both $N$ and $M$ to infinity in correlation with the approach of critical matrix couplings [7].

In this paper, we present new results for the multicritical behavior of rectangular matrix models. The remainder of this section introduces our notation. In section 2, we explicitly calculate the free energy for the simplest non-trivial potential in the saddle point approximation. This calculation is performed to ensure that the recursion formulae do not lead to spurious critical points. As expected, we verify that the triple-scaling calculations correctly produce the critical behavior of the free energy. In section 3 we solve the model in the triple-scaling limit, and present the string equation for arbitrary multicritical points. The partial differential equation arising from the potential independent recursion relations is also formulated as a flow equation [8]. Next, we examine the limit of almost-square matrices where $P$ remains finite. We show that in this limit these matrix models provide an explicit realization of the open-closed string equations proposed in ref. [9], with $P$ playing the role of the open string coupling constant. We briefly discuss the new results in section 5.

We now review the techniques developed in ref. [4] for the solution of rectangular matrix models. Given an ensemble of $N \times M$ matrices $T$ with complex entries, we want to study the partition function

$$Z = \int dT \exp(-2\beta \text{tr} V(T^\dagger T))$$

where $V(T^\dagger T) = \sum_{p=1}^{L} a_p (T^\dagger T)^p / (2p)$. This matrix ensemble is related to the tangent space of $U(N+M)/(U(N) \times U(M))$ [4]. One can “diagonalize” $T$ with the natural $U(N) \times U(M)$ action

$$T = V_1 \left( \begin{array}{c} X_M \\ 0 \end{array} \right) V_2$$

where $V_1 \in U(N), V_2 \in U(M)$ and $X_M = \text{diag}(x_1, \ldots, x_M)$. Thus the partition function

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Footnote: The model with $N = M$ was first studied in ref.'s [1, 4].
where we have set $y_i = x_i^2$. One observes that the partition function depends independently on $M$ and $P = N - M$. One could try to incorporate the $y^{N-M}$ as a logarithmic term in the potential, but there is no systematic way to proceed with such a potential. Rather ref. [7] developed recursion relations relating measures $d\mu[l] \equiv dy y^l \exp(-2\beta V(y))$ with different powers $l$. This provides a systematic framework to study all phases of these models.

Consider orthogonal polynomials on the positive real line

$$\int_0^\infty d\mu[l] \ P_n^{(l)}(y) P_m^{(l)}(y) = \delta_{n,m} h_n^{(l)} .$$

The recursion relations stepping in $l$ are

$$P_n^{(l)}(y) = P_n^{(l+1)}(y) + \phi_n^{(l)} P_{n-1}^{(l+1)}(y)$$

$$y P_n^{(l+1)}(y) = P_n^{(l)}(y) + \theta_n^{(l)} P_{n+1}^{(l)}(y) .$$

The above coefficients also enter into the standard recursion relation

$$y P_n^{(l)} = P_{n+1}^{(l)} + \left( \theta_{n+1}^{(l)} + \phi_n^{(l)} \right) P_n^{(l)} + \theta_n^{(l)} \phi_n^{(l)} P_{n-1}^{(l)} .$$

Elementary manipulations produce the potential independent relations

$$\phi_n^{(l)} - \phi_{n-1}^{(l+1)} = \theta_n^{(l+1)} - \theta_n^{(l)}$$

$$\phi_n^{(l-1)} \theta_n^{(l+1)} = \phi_n^{(l)} \theta_n^{(l)} .$$

One may also derive two potential dependent relations from

$$\frac{2n+l+1}{2\beta} h_n^{(l)} = \int d\mu[l] y V'(y) P_n^{(l)}(y) P_n^{(l)}(y)$$

$$\frac{n}{2\beta} h_{n-1}^{(l+1)} = \int d\mu[l] V'(y) \left[ P_n^{(l)}(y) P_n^{(l)}(y) + \theta_n^{(l)} P_n^{(l)}(y) P_{n-1}^{(l)}(y) \right] .$$

In the planar limit, $\beta \to \infty$ with $g = N/\beta$ and $q = M/\beta$ fixed, we assume $\theta^{(P \pm i)} \to \theta$ and $\phi^{(P \pm i)} \to \phi$. Then, eq. (3) is trivial while eqs. (4) take the form

$$g = \theta \partial_\theta U(\theta, \phi)$$

$$q = \phi \partial_\phi U(\theta, \phi)$$

where

$$U(\theta, \phi) = 2 \int_0^{2\pi} d\lambda \frac{1}{2\pi} V(e^{i\lambda} + \theta + \phi + e^{-i\lambda} \theta \phi) .$$

These techniques could also be applied to Penner-like models[10].
The critical points are identified as points where the two-dimensional map \((g, q) \mapsto (\theta, \phi)\) is singular. Such points occur at the vanishing of the jacobian determinant

\[
|J| = \left| \begin{array}{cc}
\frac{\partial g}{\partial q} & \frac{\partial g}{\partial q}
\end{array} \right| = 0 .
\]

(6)

Multicritical behavior is produced by demanding higher derivatives of linear combinations of \(g\) and \(q\) also vanish. The free energy may be determined from

\[
\theta^{(P)}_M \phi^{(P)}_M \approx \frac{Z_{M+1,P}Z_{M-1,P}}{Z_{M,P}^2} \approx \exp(-\partial^2_M F) .
\]

2 Saddle point approximation

We begin by calculating the free energy for the simplest non-trivial potential in the saddle point approximation using the techniques of ref. [11]. This calculation provides a test to verify that the triple-scaling analysis of the recursion relations correctly generates the critical behavior of the matrix model. There are known cases where recursion analysis leads to spurious results. For example, using multiple limits (more than two) of the scaling functions to describe eigenvalue densities on multiple intervals, generically yields false critical points [12]. Additional constraints producing identical minima for the potential on each interval are required. For rectangular matrices in the triple-scaling limit, given that different derivations lead to expressions for the free energy which differ by functions of \(s\) (defined below), one may suspect that the results are also spurious. Below the nonanalytic behavior of the free energy identified through the recursion relations is reproduced in the saddle point approximation for the simplest non-trivial case. This confirms that the triplescaling analysis of the recursion relations does not lead to spurious results.

The planar limit of rectangular matrix models with non-vanishing \(a_1\) and \(a_2\) was first solved in ref. [13]. For a general potential, the saddle point approximation is discussed in ref. [7]. We will briefly review the results below. The entire integrand of the partition function in eq. (2) may be written \(\exp(-\beta E)\) with

\[
E = 2 \sum_{i=1}^M V(y_i) - \frac{1}{\beta} \sum_{i,j=1}^M \ln |y_i - y_j| - \frac{N - M}{\beta} \sum_{i=1}^M \ln y_i .
\]

(7)

Eq. (7) has an interpretation as the energy of \(M\) charged particles on a line. The three contributions are: an external potential, \(2V(y_i)\); a Coulomb interaction between two particles; and an electrostatic repulsion away from the origin, with strength \(P = N - M\). At equilibrium, the particles are confined to an interval \([A, B]\) with \(0 \leq A \leq B\).

Following ref. [11], the saddle point solution of (4) entails the introduction of an eigenvalue density, \(\rho(z)\) which solves

\[
\frac{\partial V(y)}{\partial y} - \frac{P}{2\beta y} = \frac{M}{\beta} \int_{A}^{B} \frac{dz}{y - z} \rho(z) .
\]

(8)
One finds that $\rho(z)$ takes the form

$$\rho(z) = \frac{\beta}{\pi M} u(z) \sqrt{(B - z)(z - A)}$$  \hspace{1cm} (9)

with

$$u(z) = \sum_{k=1}^{L} h_k z^k$$

where

$$h_k = \frac{1}{2} \sum_{p=0}^{L-k} \frac{a_{p+k+2}}{4^p} \sum_{l=0}^{p} \left( \frac{p}{l} \right)^2 (\sqrt{B} + \sqrt{A})^{2l}(\sqrt{B} - \sqrt{A})^{2p-2l}.$$  \hspace{1cm} (10)

After integrating (8), the saddle point free energy becomes

$$F_0 = \beta E = M \int_{A}^{B} dy \rho(y) \left[ 2\beta V(y) - P \ln |y| - 2M \ln |y - z_0| \right] + \beta MV(z_0) - \frac{PM}{2} \ln z_0$$  \hspace{1cm} (11)

where $z_0$ is a constant of integration, which does not affect the final result.

Critical behavior corresponds to a nonanalytic dependence of the free energy in the coupling constants in $V$. For fixed couplings, this can be expressed as nonanalyticity in $M/\beta$ and $P/\beta$. Such behavior arises when the eigenvalue density acquires zeroes at the boundaries beyond those evident in (9). This saddle-point discussion is connected to the planar limit description (and eq. (6), in particular) by the relation

$$|J| = 4ABu(A)u(B).$$

We are now ready to calculate the free energy for the simplest non-trivial potential:

$$V(y) = \frac{a_1}{2} y + \frac{a_2}{4} y^2.$$  \hspace{1cm} (12)

Using eq.’s (9) and (10), the eigenvalue density is given by

$$\rho(z) = \frac{1}{2\pi M} \frac{\beta}{\sqrt{(B - z)(z - A)}} \left[ \left( \frac{a_1}{2} + \frac{a_2}{4}(A + B) \right) \frac{1}{z} + a_2 \right] \left( \sqrt{B} - \sqrt{A} \right).$$  \hspace{1cm} (13)

We evaluate (11), using (13) and setting $z_0 = (A + B)/2$, to obtain

$$\frac{F_0}{\beta^2} = \frac{(B - A)^2}{32} \left[ a_1^2 + \frac{5}{4} a_1 a_2 (A + B) + \frac{a_2^2}{32} (9A^2 + 9B^2 + 14AB) \right]$$

$$+ \frac{P}{8\beta} \left[ a_1 \left( \sqrt{B} - \sqrt{A} \right)^2 + \frac{a_2}{8} \left( 3B^2 + 3A^2 + 2AB - 4(A + B)\sqrt{AB} \right) \right]$$

$$+ \frac{M}{4\beta} \left[ \frac{a_1}{2} (A + B) + \frac{2M}{\beta} + \frac{P}{\beta} \right] - \frac{P}{\beta} \left[ \frac{2M}{\beta} + \frac{P}{\beta} \right] \log \frac{\sqrt{B} + \sqrt{A}}{2}$$

$$- \left( \frac{M}{\beta} \right)^2 \log \frac{B - A}{4} + \left( \frac{P}{2\beta} \right)^2 \log AB.$$  \hspace{1cm} (14)

The endpoints $A$ and $B$ are determined by

$$\frac{2M}{\beta} + \frac{P}{\beta} = \frac{a_1}{2} (A + B) + \frac{a_2}{8} (3A^2 + 3B^2 + 2AB)$$  \hspace{1cm} (15)

$$\frac{P}{\beta} = \sqrt{AB} \left( a_1 + \frac{a_2}{2} (A + B) \right).$$  \hspace{1cm} (16)
Given these expressions, it is not at all apparent which matrix potentials will yield nonanalytic behavior in the free energy. Our approach is to determine the critical point from the planar string equations (5), and then examine the free energy (14) for nonanalytic behavior using the relations

$$A = (\sqrt{\theta} - \sqrt{\phi})^2 \quad B = (\sqrt{\theta} + \sqrt{\phi})^2.$$  \hspace{1cm} (17)

For the potential (12), the vanishing of the jacobian determinant becomes

$$|J| = a_1^2 + 4a_1a_2(\theta_c + \phi_c) + 4a_2^2(\theta_c^2 + \theta_c\phi_c + \phi_c^2) = 0 \hspace{1cm} (18)$$

Setting $g_c = 1 = \theta_c$ and $\phi_c = y^2$ with $0 \leq y \leq 1$, eq.'s (5,18) yield

$$a_1 = 2\frac{1 + y + y^2}{2y + 1} \quad a_2 = -\frac{1}{2y + 1} \quad q_c = \frac{y^3(2 + y)}{(2y + 1)} \hspace{1cm} (19)$$

One easily confirms that for these couplings the eigenvalue density $\rho(z)$ has an extra zero at $z = B = (1 + y)^2$.

To study the singularity, one expands (5) around these critical values

$$\Delta g = -\frac{2}{2y + 1}(\Delta \phi - y\Delta \theta) - \frac{1}{2y + 1}(\Delta \theta^2 + 2\Delta \theta\Delta \phi)$$

$$\Delta q = \frac{2y}{2y + 1}(\Delta \phi - y\Delta \theta) - \frac{1}{2y + 1}(\Delta \phi^2 + 2\Delta \phi\Delta \theta)$$

where $\Delta$ is used to denote $\Delta x = x - x_c$. One observes that both $\Delta g$ and $\Delta q$ are proportional to $\Delta \phi = \Delta \phi - y\Delta \theta$, and that $\Delta \hat{g} = \Delta q + y\Delta g$ has no linear variation. Further for generic rectangular matrices (i.e., $0 < y < 1$), it is convenient to define

$$\hat{g} = g - \frac{2y + 1}{3y(y + 1)}\hat{\phi} \quad \hat{\theta} = \theta + \frac{2y + 1}{3y(y + 1)}\hat{\phi}.$$  

With these choices, the planar equations become

$$\Delta \hat{g} = -\frac{2}{2y + 1}\Delta \hat{\phi} + \frac{2(y^2 + y + 1)}{3y(2y + 1)(y + 1)}\Delta \hat{\theta}\Delta \hat{\phi} + \frac{(y - 1)(y + 2)}{9y^2(y + 1)^2}\Delta \phi^2$$

$$\Delta \hat{q} = -\frac{1}{2y + 1}\left[3y(y + 1)\Delta \hat{\theta}^2 - \frac{y^2 + y + 1}{3y(y + 1)}\Delta \phi^2\right] \hspace{1cm} (20)$$

The quadratic form of the singularity in (20) suggests $\Delta \hat{q} \approx -\delta^2t$, where $\delta = \beta^{-2/5}$ just as for quadratic singularities in double-scaling[1–3]. Then eq. (20) yields $\Delta \hat{g} = -\delta s$, $\Delta \hat{\theta} = -\sum \delta^p f_p$ and $\Delta \hat{\phi} = -\sum \delta^p h_p$, where $f_p$ and $h_p$ are solved for in terms of $s$ and $t$.

Upon inserting these scalings in eq. (14), the planar free energy becomes

$$F_0 = \frac{12y(y + 1)}{5(2y + 1)^2} \left[\frac{(y^2 + y + 1)(2y + 1)^2}{36y^2(y + 1)^2}s^2 + \frac{2y + 1}{3y(y + 1)}t\right]^{5/2} \hspace{1cm} (21)$$
This expression differs from the planar contribution to free energy presented for this model in ref. [7], only by rescalings of \( s \) and \( t \), which were made there (and also in the next section) to simplify the results. Without such rescalings, both expressions would be identical. This demonstrates that the recursion relations successfully determine the correct critical behavior for the matrix model. Note that non-universal contributions to \( F_0 \) were dropped from (21). In fact, some of these terms are actually divergent, being proportional to \( \beta^{\alpha/5} \) with \( \alpha > 0 \), but they are analytic in both \( s \) and \( t \). Similar divergent analytic terms appear in the scaling analysis of the saddle point approximation for the hermitian matrix model [14]. A finite term analytic in \( s \) was also dropped. In general, one might expect additional finite terms which are nonanalytic in \( s \) [], but they do not arise for this critical point.

3 Triple-scaling limit

In this section, we reexamine the triple-scaling analysis of ref. [7]. We formulate the potential independent relations in terms of flow equations, and this allows us to determine the general string equation for this class of multicritical points.

In these models since there are two parameters \( N \) and \( M \) which diverge separately as \( \beta \to \infty \), it is natural to begin with a general scaling ansatz for two linearly independent combinations of \( P \) and \( M \)

\[
\frac{AM + CP}{\beta} = \left[ \frac{AM + CP}{\beta} \right]_c - \beta^{\nu - 1} t \quad \quad \frac{BM + DP}{\beta} = \left[ \frac{BM + DP}{\beta} \right]_c - \beta^{\hat{\nu} - 1} s .
\]

In the scaling limit then, finite differences become derivatives: \( \partial_M = -\beta^{\nu} A \partial_t - \beta^{-\nu} B \partial_s \), and \( \partial_P = -\beta^{-\nu} C \partial_t - \beta^{-\hat{\nu}} D \partial_s \). The planar limit analysis suggests \( \hat{\nu} = 3 \nu \). As in the previous section, we set \( \theta_c = 1 \) and \( \phi_c = y^2 \), and define \( \delta = \beta^{-\nu} \). We then introduce

\[
\theta^{(P+1)}_{M+n} = 1 - \exp \left[ -\delta (An + Cl) \partial_t - \delta^3 (Bn + Dl) \partial_s \right] \sum_{q=2}^{\infty} \delta^q h_q(t, s) \quad \quad \phi^{(P+1)}_{M+n} = y^2 - \exp \left[ -\delta (An + Cl) \partial_t - \delta^3 (Bn + Dl) \partial_s \right] \sum_{q=2}^{\infty} \delta^q k_q(t, s) \quad (22)
\]

First we determine the scaling limit of eq. (3), which are independent of the matrix potential. Inserting (22), leads to \( k_2(t, s) = y (h_2(t, s) + g_2(s)) \) and a partial differential equation

\[
\partial_s h_2 = \frac{C^2}{4T} \left( \frac{y - 1}{y^2} \partial_t \left[ h_2^2 - \frac{C^2}{6} \frac{y + 1}{y} \partial_t^2 h_2 \right] - \frac{C^2}{2T} \frac{y^2}{y^2} g_2 \partial_t h_2 - \frac{1}{2} \partial_s g_2 \right) \quad (23)
\]

where \( T = AD - BC \). It was noted in ref. [7] that neglecting the terms involving \( g_2 \), eq. (23) is the KdV equation with \( s \) and \( t \) playing the roles of the time and space coordinates, respectively. Presently, we also point out that eq. (23) has the form of a flow equation. This connection is made clear by first introducing \( h(t, s) = h_2(t, s) + g_2(s)/2 \), and then
using (some of) the freedom to rescale $s \to \alpha s$, $t \to \beta t$, $h \to \gamma h$, and $g_2 \to \eta g_2$. Thus eq. (23) can be expressed as

$$\partial_s h = \partial_t \left(2g_2(s) R_1[h] + R_2[h]\right)$$

where $R_1[h] = -h/4$ and $R_2[h] = (3h^2 - \partial_t^2 h)/16$ are the first and second Gel’fand-Dikii differential polynomials\[15].

Eq. (24) may be compared to the flow equations which are usually discussed in the context of matrix models and the KdV hierarchy\[8]. Given the string equation for the general model interpolating between multicritical points

$$t = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \mu_k R_k[h],$$

the generalized KdV equations arise as flow equations for $h(\{\mu_k\}, t)$

$$\frac{\partial h}{\partial \mu_k} = \partial_t R_{k+1}[h].$$

If the first two couplings in the string equation (23) were correlated as $\mu_1 = s$ and $\frac{\partial \mu_0}{\partial s} = 2g_2(s)$, eq. (26) would lead to a flow equation of the form given in eq. (24). We reiterate that these results follow without restricting the matrix potential in any way. The only assumption is to fix the ratio of the scaling exponents, $\nu/\nu = 3$ (and $q \geq 2$). We have not found any other choices of this ratio which lead to interesting results.

Next we would like to examine the string equations which arise from the potential dependent relations (4). For a quadratic potential (12) with the critical values (19) and $\nu = 1/5$, one finds $g_2 = s/2$ and

$$t = \frac{s^2}{4} R_0[h] + \frac{3}{2} s R_1[h] - \frac{9}{2} \left(\frac{y + 1}{y - 1}\right)^2 R_2[h]$$

where $R_0[h] = 1/2$ and we have used the remaining freedom in rescaling variables to simplify these results.\[7] Comparing eq.'s (27) and (23), one sees that $\mu_0$ and $\mu_1$ have precisely the dependence on $s$ to be compatible with the flow equation (24).

Ref. \[3] also presents results for the critically tuned quartic potential which produces a third order multicritical point. Recasting those results in the form of eq. (23) by a rescaling, one finds that $g_2 = (s/6)^{1/2}$ and

$$t = 4 \left(\frac{s}{6}\right)^3 R_0[h] + \frac{3}{2} s R_1[h] - \frac{27}{2} \left(\frac{y + 1}{y - 1}\right)^4 R_3[h]$$

where $R_3[h] = -(10h^3 - 10h^2 \partial_t h - 5(\partial_t h)^2 + \partial_t^2 h)/64$. We have also carried out the triple scaling analysis for the $k = 4$ multicritical point, which is produced by tuning a sextic

\[5\]This string equation (27) is the same as that presented in ref. [3], but with slightly different rescalings, and a distinct combination of $h_2$ and $g_2$ for the scaling function.
potential. The results are similar to those above with \( g_2 \propto s^{1/3} \) and in eq. (28), \( s^{3/2} \rightarrow s^{4/3} \) and \( R_3[h] \rightarrow R_4[h] \), as well as changing numerical factors. For higher order critical points, we found it impractical to explicitly carry out the complete triple-scaling analysis, but the form of the general string equation is clear from the above examples. It takes the form of eq. (25) with the only nonvanishing coefficients being \( \mu_0, \mu_1 \) and \( \mu_k \). The values of \( \mu_0 \) and \( \mu_1 \) are fixed by the analysis of the potential independent equations (24). So it remains only to fix \( g_2 \) and \( \mu_k \). The full triple-scaling analysis is unnecessary to determine these coefficients, but rather they can be extracted from the planar limit alone. At the \((k+1)\)’th multicritical point, we find

\[
g_2 = \left[ \frac{(k-1)!k!}{2(2k-1)!} s \right]^{1/k}
\]

and the string equation becomes

\[
t = \frac{k}{k+1} s g_2 R_0[h] + \frac{3}{2} s R_1[h] - \frac{3}{2} \left( \frac{y+1}{y-1} \right)^{2k} R_{k+1}[h].
\] (29)

The string equation is that of the conventional \((k+1)\)’th order multicritical point but perturbed by the operators \( R_1 \) and \( R_0 \). The new scaling parameter \( s \) governs the strength of the perturbations.

4 Parametric solutions

A separate class of novel critical points is governed by coupled differential and finite difference equations. These occur in the special limits where only a single large \( N \) parameter diverges. At present, we have no further remarks on the case of the vector models. We wish to point out the connection of the almost square matrix models, in which \( P = N - M \) remains a finite parameter, to the open–closed string equations proposed by ref. [9]. The simplest critical point was discussed in ref. [7] where the scaling function satisfies the Painlevé II equation with a constant

\[
\frac{1}{2} \partial_s k_1^{(P)} - k_1^{(P)} k_1^{(P)} + s k_1^{(P)} = P + \frac{1}{2}
\]

(\( k_1^{(P)} \) and \( s \) will be defined below). Comparing to ref. [9] suggests that \( P \) plays the role of an open string coupling constant. In these parametric models, the potential independent recursion relations give rise to a finite difference relation

\[
k_1^{(P+1)} - \partial_s k_1^{(P)} = k_1^{(P+1)} + \partial_s k_1^{(P+1)}.
\] (30)

The same relation was also developed for scaling functions satisfying a string equation in the mKdV hierarchy with coupling constants which differ by one[9]. To confirm the connection of these parametric critical points to open–closed strings, we consider the next multicritical point.
This multicritical point can be produced by tuning a cubic potential, \( V(y) = \frac{a_1}{2}y^2 + \frac{a_2}{4}y^4 + \frac{a_3}{6}y^6 \). Square matrices correspond to \( q_c = q_e = 1 \), where the latter is a choice of normalization as is \( \theta_c = 1 \). These multicritical points are related to approach of zeroes in the eigenvalue density to the origin. So first we choose \( \phi_c = 1 \), which by eq. (17) yields \( A = 0 \) and \( B = 4 \). Now eq.’s (5) and (6) fix \( a_1 = -2 + 2a_3 \) and \( a_2 = 1 - 4a_3 \). Then, the eigenvalue density is given by

\[
\rho(z) = \frac{\beta}{\pi M} \sqrt{(4 - z)z (1 - 2a_3 + a_3 z)}. 
\]

Choosing \( a_3 = 1/2 \) yields the form \( \rho(z) \propto z^{3/2} \) at the origin.

For a scaling solution, we define \( \delta \equiv \beta^{-\nu} \) and we use the following scaling ansatz:

\[
M = 1 - \delta s \quad \text{and} \quad \theta(l) = \theta(l/M) + \phi(l) \quad \text{where} \quad \phi(l) = \frac{8}{3} \frac{1}{\nu}. 
\]

Derivatives do not replace finite differences in \( l \) because \( P \) does not scale. Given eq. (31) without any further assumptions, the potential independent relations (3) yield eq. (30) which relates the scaling function at different values of \( P \). The identical equation is found in ref. [9] to relate solutions of the unitary matrix models with different numbers of flavours of ‘quarks’[16]. The scaling of the potential dependent relations (4) yields \( \nu = 1/5, \gamma = 4, \)

\[
h_2^{(P)}(s) = \frac{\partial_s k_1^{(P)} - k_1^{(P)^2}}{4} \quad \text{(32)}
\]

and the scaling function \( k_1^{(P)} \) satisfies

\[
sk_1^{(P)} - k_1^{(P)^5} + \frac{5}{3} k_1^{(P)}(\partial_s k_1^{(P)})^2 + \frac{5}{3} k_1^{(P)^2} \partial_s^2 k_1^{(P)} - \frac{1}{6} \partial_s^4 k_1^{(P)} = P + \frac{1}{2} \quad \text{(33)}
\]

where we have shifted \( k_1^{(P)} \rightarrow \alpha k_1^{(P)} \) and \( s \rightarrow s/\alpha \) with \( \alpha = [8/3]^{1/5} \). This is the second equation in the mKdV hierarchy, where again \( P \) appears as the open string coupling in the constant term.

Using eq. (31), one obtains the free energy

\[
\partial_s^2 F^{(P)} = -\frac{1}{2}(k_1^{(P)^2} + \partial_s k_1^{(P)}) \quad \text{(34)}
\]

Here, the specific heat is proportional to the Miura map of the scaling function. Defining \( u \equiv k_1^{(P)^2} + \partial_s k_1^{(P)} \), using eq. (33) may express the string equation in terms of the specific heat

\[
u R^2[u] - \frac{1}{2} R[u] R'[u] + \frac{1}{4}(R'[u])^2 = P^2 \quad \text{(35)}
\]
where $R = R_2[u] - s$. This is the original form of the scaling equations proposed for open-closed strings with the open string coupling equal to $P$ \cite{9,17}. For $P = 0$, the perturbative solution of eq. \eqref{35} is simply $R[u] = 0$, which in the present case is the Painlevé I equation. As well though, eq. \eqref{35} provides a nonperturbative solution for two-dimensional gravity\cite{17}. The rectangular matrix models provide an explicit matrix model realization of these scaling equations extended to a nonvanishing open string coupling constant.

5 Discussion

Rectangular matrix models display a rich variety of multicritical behaviors. Underlying this diversity is the fact that these models have two independent large $N$ parameters, which leads to multicritical behavior governed by two-dimensional singularities.

Based on the analysis of the recursion relations alone, the connection of various critical points to singular behavior of the actual matrix model is not always clear. For the triple-scaling points, we have verified the validity of critical behavior by recovering the same nonanalytic planar contribution from a saddle point analysis. The multicritical points for the triple-scaling limit are governed by string equations which are simply expressed in terms of the Gel’fand-Dikii differential polynomials \cite{15}. The usual $k$’th order string equation in the KdV hierarchy is perturbed by the $R_1$ and $R_0$ operators. The corresponding coupling $\mu_1$ and $\mu_0$ are functions of the new scaling parameter, and a flow equation expressing this dependence naturally arises. The effect of the $R_0$ can be absorbed in a renormalization of the cosmological constant to $\tau = t + X s^{k/(k-1)}$, where $X$ is some numerical constant. For small $s$, as well as the usual genus expansion in $\tau^{-(2k+1)/k}$, there is also an expansion in $s/\tau^{(k-1)/2}$ at each genus due to the perturbation of $R_1$ in the string equation. In ref. \cite{7}, it was conjectured that the new dynamics uncovered by the triple scaling analysis should be related to a gas of punctures arising from the distinction between $M$ and $N$ loops in the surface interpretation\cite{4}. In the present context then, it appears that tuning for triple scaling involves tuning these punctures to behave as a linear combination of the $R_0$ and $R_1$ operators. One might expect that a more subtle scaling would lead to more complex perturbations by linear combinations involving higher order Gel’fand-Dikii polynomials, but as yet we have been unsuccessful in producing such a tuning.

The present reformulation of the triple scaling results provides some insight into the interpretation of the dual expansion for large $|s|$, which was noted in ref. \cite{7}. The $R_1$ perturbation expansion in $|s|/\tau^{(k-1)/2}$ is expected to have a finite radius of convergence, and beyond that point the solution should be expanded in terms of $\tau/|s|^{k/(k-1)}$. In this domain, the model is expected to be in the neighbourhood of the “topological” model governed by $R_1[h]$. (This crossover behavior can be explicitly seen for the $k = 2$ and $3$ critical points, at least in the planar limit where the string equations are easily solved.) For this interpretation to be applicable, one should consider the real root of the planar equation which vanishes as $|s| \rightarrow \infty$. This leads to the expected expansion, $h \approx -\frac{5}{3} \frac{\tau}{s} + \ldots$. 

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Note that in this phase, the full expansion is not entirely a perturbation expansion in $\tau/|s|^{k/(k-1)}$ which is the planar contribution of $R_0[h]$, but rather the derivative terms in $R_k[h]$ produce higher genus contributions as well. This is not the complete story though. For large $|s|$, the planar equation may have another real root which yields an expansion $h \propto s^{1/(k-1)} + \ldots$ (here if $(k-1)$ is even, one requires $s > 0$, and then there is a second root at $-s^{1/(k-1)}$). These were the dual expansions presented in ref. [7]. In these cases, both $R_1$ and $R_k$ are equally important in determining the behavior of the perturbative expansion. The interpretation of these extra strongly coupled phases remains unclear. Similar strongly coupled domains may also separate the $R_1$ and $R_k$ phases, if there is a gap between the regions of convergence of the expansions in $|s|/\tau^{(k-1)/k}$ and $\tau/|s|^{k/(k-1)}$.

Finally, we have examined the parametric solutions for almost-square matrices. By examining the $k = 2$ critical point, we confirmed that these solutions provide an explicit matrix model realization of the scaling equations for open-closed strings [9], with $P = N - M$ playing the rôle of the open string coupling constant. This is a natural position for $P$ to appear in, since in the limit of almost-square matrices, the surface interpretation associates $P$ with introducing punctures or boundaries [4]. The scaling equations, which naturally arise with our scaling ansatz (51), are those of the mKdV hierarchy but with a constant $P + \frac{1}{2}$. The same equations naturally arise in the double-scaling limit of unitary matrices coupled to $C = P + \frac{1}{2}$ flavors of quarks [16]. We emphasize though that the physics of these models is not identical. In the present case, the free energy is simply related to the Miura map of the scaling function (54). For the unitary matrix models, one has simply $\partial^2_{s} F^{(P)} = -\frac{1}{2} k(s)^2$, where $k(s)$ is the corresponding scaling function [16]. Thus even though one has a simple map between solutions in the KdV and mKdV hierarchies [4], the physics of unitary and rectangular matrix models remains distinct.

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