Risk and Intertemporal Preferences over Time Lotteries

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Abstract

This paper studies relations among axioms on individuals’ intertemporal choices under risk. The focus is on Risk Averse over Time Lotteries (RATL), meaning that a fixed prize is preferred to a lottery with the same monetary prize but a random delivery time. Though surveys and lab experiments documented RATL choices, Expected Discounted Utility cannot accommodate any RATL. This paper’s contribution is two-fold. First, under a very weak form of Independence, we generalize the incompatibility of RATL with two axioms about intertemporal choices: Stochastic Impatience (SI) and No Future Bias. Next, we prove a representation theorem that gives a class of models satisfying RATL and SI everywhere. This illustrates that there is no fundamental conflict between RATL and SI, and leaves open possibility that RATL behavior is caused by Future Bias.

Keywords: Intertemporal choices, Time lotteries, Stochastic Impatience, Discount rate, Future Bias

1 Introduction

The study of intertemporal choices under risk has a long history. Traditionally, the Expected Discounted Utility (EDU) model is widely used to predict individuals’ intertemporal choices under risk thanks to its simplicity and tractable function form. EDU inherits the shortcomings of both Expected Utility and Discounted Utility, so absurdities caused by EDU have motivated the search for alternative models. Popular extensions of Discounted Utility include

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the quasi-hyperbolic discounting (Phelps and Pollak (1968)) and the generalized hyperbolic discounting (Loewenstein and Prelec (1992)), not to mention the numerous generalizations to Expected Utility.

An absurdity about EDU concerns time lotteries - those with fixed monetary prizes and random delivery time. Time lotteries arise naturally in practice, for example, when a manufacturer sets a fixed price for its product but is unsure about when the product will be sold, or when a passenger waits for a taxi that has fixed fees. Borrowing from the study of risk aversion over random monetary payoffs, we define a choice to be Risk Seeking over Time Lotteries (RSTL) if a time lottery is preferred to its certain counterpart – the lottery with the same monetary prize and a fixed delivery time as the expected delivery time of the time lottery. Otherwise, the choice is said to be Risk Averse over Time Lotteries (RATL).

It has been observed that EDU as well as commonly used Expected Utility models with alternative discount functions predict that decision makers are always RSTL (Chesson and Viscusi (2003), Onay and Öncüler (2007)). Whenever a preference is represented by $V(p) = \mathbb{E}_p[D(t)v(x)]$, where $D$ can be regarded as the discount function on time and $v$ is the evaluation of monetary prizes, the convexity of $D$ implies RSTL. On the other hand, surveys and lab experiments have found evidence in support of RATL in some situations (Chesson and Viscusi (2003); Onay and Öncüler (2007); DeJarnette et al. (2020)).

What is more puzzling is an incompatibility result by DeJarnette et al. (2020). They put forward the assumption of Stochastic Impatience (SI); this property requires that, when pairing two monetary prizes with two delivery times, individuals like to match the higher monetary prize with the earlier delivery time. Moreover, they relaxed the assumption of Stationarity in time to No Future Bias, which means the marginal rate of substitution of time for money is decreasing in time. Assuming Independence, No Future Bias, and other standard assumptions, DeJarnette et al. found that SI implies RSTL.

In this article, we demonstrate the incompatibility among a weaker notion of RATL, SI, and No Future Bias when Independence is further relaxed. Specifically, this illustrates the incompatibility of RATL, SI, and No Future Bias beyond Expected Utility and Generalized Local Bilinear Utility considered in DeJarnette et al. (2020). The weak independence we endow is strictly weaker than both the Axiom of Degenerate Independence (Grant et al. (1992)) and the Certainty Independence Axiom (Gilboa and Schmeidler (2004)).

Given that our incompatibility result does not assume Independence, we are motivated to reconcile SI with RATL by dropping No Future Bias. We prove a representation theorem that gives conditions for an Expected Utility model to satisfy SI and RATL everywhere. The representation theorem results in a functional form of

$$V(p) = \mathbb{E}_p[\phi(D(t)v(x))].$$
This functional form includes Generalized EDU considered in DeJarnette et al. (2020) as a special case, in which \( D(t) = \beta^t \). Moreover, compared to the standard model \( V(p) = \mathbb{E}_p[D(t)v(x)] \), the additional curvature provided by \( \phi \) governs risk attitude, which makes RATL easier to be attained. In particular, this representation theorem shows there is no fundamental contradiction between SI and RATL, even within Expected Utility. We conclude that RATL may be accommodated by allowing some Future Bias.

## 2 Related Literature

Chesson and Viscusi (2003) were the first to notice that EDU implies RSTL as long as the discount rate is positive. This result was generalized by Onay and Öncüer (2007), who demonstrated RSTL for Discounted Utility models with convex discount functions.

SI was first studied in DeJarnette et al. (2020), and in the same paper they proved the incompatibility of RATL with SI and No Future Bias in the Expected Utility models and Generalized Local Bilinear Utility (GLBU) models. They discussed two solutions to accommodate RATL. The first one is to drop “Intertemporal” Independence and keep SI; the Dynamic Ordinal Certainty Equivalent model (Selden (1978), Selden and Stux (1978)) may reconcile SI with RATL because of the “order of aggregation.” The second solution is to drop SI. In this case, the Generalized Expected Discounted Utility model

\[ V(p) = \mathbb{E}_p[\phi(\beta^t u(x))] \]

is able to satisfy RATL when the curvature of \( \phi \) is large enough.

The surveys and lab experiments conducted by Chesson and Viscusi (2003), Onay and Öncüer (2007) and DeJarnette et al. (2020) to test RATL usually let participants choose from a time lottery and its certain counterpart. Chesson and Viscusi (2003) conducted a survey among business owners and managers and found that a majority chose the risky option. The survey conducted by Onay and Öncüer (2007) among 146 business managers revealed a preference for the certain option, especially when there is a small probability of very early arrival. DeJarnette et al. (2020) run a lab experiment among 196 subjects, which concluded that the majority of subjects made the RATL choices in the majority of questions. As for SI, Lanier et al. (2020) found evidence in favor of SI, especially among those whose choices are in accordance with Impatience and Lottery Equivalence.

No Future Bias is the key axiom that separates our results from previous studies of RATL. This notion, first proposed by Prelec (2004) under the name “decreasing impatience,” has been controversial. One piece of evidence supporting No Future Bias is preference reversals (Green et al. (1994), Kirby and Herrnstein (1993)), which means subjects initially prefer a small but early reward over a large but late reward, and after equally delaying the two prizes,
the preference relation is reversed.

On the other hand, some recent lab experiments and surveys find evidence against No Future Bias, especially when the time delay between the first and the second test is small. Sayman and Öncüler (2009) and Attema et al. (2010) documented the converse of preference reversals: a smaller and sooner prize is more likely to be preferred to a larger and later prize after equally delaying the two prizes. Takeuchi (2011) took a non-parametric approach and found that most subjects are Future Biased.

When assuming separability between time and money, decision makers’ discount rate in a given time is independent of the amount of prize, and this is characterized by a discount function $D(t)$. In this case, No Future Bias is equivalent to a declining discount rate, also called “hyperbolic discounting.” Previous findings of time inconsistencies seemed so strong that hyperbolic discounting soon became popular. Two commonly used hyperbolic discount functions are the standard one, $D(t) = \frac{1}{1+kt}, k > 0$, and the quasi-hyperbolic discounting, $D(t) = \beta^t \delta, 0 < \delta < 1$, if $t > 0$. There have been numerous tests of hyperbolic discounting, yielding inconsistent and inconclusive results. Additional discussions can be found in Rubinstein (2003), Benhabib et al. (2010), and Halevy (2015).

Many utility representations have been proposed other than the standard one, $V(p) = \mathbb{E}_p[D(t)v(x)]$. Fishburn and Rubinstein (1982) used a “utility independence” condition which yields $V(p) = \mathbb{E}_p[D(t)v(x)+\omega(t)]$. Mu et al. (2021) proposed a utility representation on time lotteries, assuming a strong form of stationarity. However, the standard model always satisfies SI and the model of Fishburn and Rubinstein (1982) in general does not have a clean characterization of SI. The representation of DeJarnette et al. (2020), $V(p) = \mathbb{E}_p[\phi(\beta^t u(x))]$, has the advantage that SI is captured by the curvature of $\phi$. In fact, their model is a special case of ours; our model allows any reasonable discount function $D(t)$ in place of $\beta^t$, thus providing more flexibility. Many more papers, including Epstein (1983), Blavatskyv (2020), and Dillenberger et al. (2020), study preferences over lotteries over consumption streams.

### 3 Setup

Let $X \subset \mathbb{R}^+$ be a closed interval of monetary prizes and $T \subset \mathbb{R}^+ \cup \{0\}$ be a closed interval of delivery dates, both possibly unbounded. A pair $(x, t)$ means $x$ amount of money is received at time $t$. Denote by $\Delta(X \times T)$ the set of simple lotteries on $X \times T$ so these lotteries are random in both quantity and time. We will use $p$ to denote a lottery in $\Delta(X \times T)$. This paper studies preference relations $\succeq$ over $\Delta(X \times T)$.

We can fix the quantity dimension and only consider variation over the delivery time. For any $x \in X$, we say $p_x \in \Delta(X \times T)$ is a time lottery with prize $x$ if $p_x$ is supported
on \( \{x\} \times T \). Let
\[
\bar{t}_{px} := \sum t \cdot p_x (x, t)
\]
be the expected arrival time of a time lottery.

In analogy to the standard risk aversion over random monetary prizes, we introduce the notion of risk aversion in the time dimension.

**Definition 1** A preference relation \( \succeq \) over \( \Delta (X \times T) \) is **Risk Averse over Time Lotteries (RATL)** if for any time lottery \( p_x \), \( \delta(x, \bar{t}_{px}) \succeq p_x \). Similarly, a relation \( \succeq \) is **Risk Seeking over Time Lotteries (RSTL)** if for any time lottery \( p_x \), \( \delta(x, \bar{t}_{px}) \preceq p_x \).

By restricting attention to binary lotteries with two events of probability \( \frac{1}{2} \), we have a weaker property that plays an important role in the following discussions.

**Definition 2** We say \( \succeq \) satisfies **weak RATL** if for any \( x \in X \) and \( t_1, t_2 \in T \),
\[
\delta(x, t_1 + t_2) \succeq \frac{1}{2} \delta(x, t_1) + \frac{1}{2} \delta(x, t_2).
\]
Similarly, \( \succeq \) is **weak RSTL** if the above holds with \( \preceq \) in place of \( \succeq \).

## 4 Incompatibility Theorem

The standard Expected Discounted Utility (EDU) model evaluates a lottery \( p \in \Delta (X \times T) \) by
\[
V(p) = \mathbb{E}_p[\beta^t v(x)]
\]
where \( \beta \in (0, 1) \) and \( v \) is non-negative and strictly increasing. Despite the evidence of RATL in some situations, EDU predicts individuals are always RSTL. A slightly more general model has the functional form
\[
V(p) = \mathbb{E}_p[D(t)v(x)],
\]
where \( D \) is often called the discount function. Using Jensen’s inequality, we easily deduce that this model obeys RSTL if \( D \) is convex, which is the case of the generalized quasi-hyperbolic discount function \( D(t) = (1 + \alpha t)^{-\beta/\alpha}, \alpha, \beta > 0 \).

This discrepancy between the observed RATL behavior and no accommodation of RATL in commonly used theoretical models is further generalized by DeJarnette et al. (2020). They introduced the notion of Stochastic Impatience.

**Definition 3** A preference relation \( \succeq \) satisfies **Stochastic Impatience (SI)** if for any \( t_1 < t_2 \in T \) and \( x_1 > x_2 \in X \), we have
\[
\frac{1}{2} \delta(x_1, t_1) + \frac{1}{2} \delta(x_2, t_2) \succeq \frac{1}{2} \delta(x_1, t_2) + \frac{1}{2} \delta(x_2, t_1).
\]
In other words, individuals like to match the earlier arrival time with the higher prize.

Clearly, if $\succeq$ can be represented by

$$V(p) = E_p[D(t)v(x)],$$

where $D \in (0, 1]$ is monotone decreasing and $v$ is non-negative and monotone increasing, then $\succeq$ satisfies SI.

Dynamic inconsistency observed in experiments initiated search for alternative models, and many forms of hyperbolic discount functions emerged. Prelec (2004) suggested it is one common property behind these models that drives dynamic inconsistency, which he termed as “decreasing impatience.” We instead use the name coined by DeJarnette et al. (2020).

**Definition 4** A preference relation $\succeq$ satisfies **No Future Bias** if for all $x, y \in X$, $s, t \in T$ with $t < s$ and $\tau > 0$,

$$\delta(x, t) \sim \delta(y, s) \implies \delta(x, t + \tau) \preceq \delta(y, s + \tau).$$

For some $x, y, s, t$ as above, we say the choice has **Future Bias** if the above $\preceq$ is replaced by $\succ$.

DeJarnette et al. (2020) proved an incompatibility result among SI, No Future Bias, and RATL. The following are standard assumptions on $\succeq$ we will use later.

**Axiom 0** Completeness and Transitivity.

**Axiom 1** Outcome Monotonicity: For all $x, y \in X$ and $s \in T$, if $x > y$ then $\delta(x, s) \succ \delta(y, s)$.

**Axiom 2** Impatience: For all $x \in X$ and $s, t \in T$, if $t < s$ then $\delta(x, t) \succ \delta(x, s)$.

**Axiom 3** Continuity: For all $p \in \Delta$, the sets $\{q \in \Delta : p \succeq q\}$ and $\{q \in \Delta : q \succeq p\}$ are weakly closed.

**Theorem 0** (Theorem 2 in DeJarnette et al. (2020)) Suppose that $\succeq$ admits a GLBU representation, i.e. there exists a function $u$ defined on $X \times T$ and a constant $\pi(\frac{1}{2})$ such that whenever a lottery $p = \frac{1}{2}\delta(x, t) + \frac{1}{2}\delta(x', t')$ with $u(x, t) \geq u(x', t')$, $p$ is evaluated by

$$V(p) = \pi(\frac{1}{2})u(x, t) + \left(1 - \pi(\frac{1}{2})\right)u(x', t').$$

Then Axioms 0-2, SI, and No Future Bias imply weak RSTL.

GLBU is a common feature in a wide scope of behavioral models. However, we demonstrate that if we stick to No Future Bias, then the incompatibility between SI and RATL extends to more general models. More specifically, we relax GLBU to a weaker axiom.
Definition 5  A preference relation $\succeq$ is said to satisfy **Weak Certainty Independence (WCI)** if for any 

$$p = \delta(x,t), q = \delta(y,s), r = \delta(z,u),$$

we have

$$p \succeq q \implies \frac{1}{2}p + \frac{1}{2}r \succeq \frac{1}{2}q + \frac{1}{2}r.$$ (1)

WCI is strictly weaker than the Certainty Independence introduced by Gilboa and Schmeidler (2004), which allows $p, q$ in (1) to be any lottery not necessarily certain. Moreover, WCI is strictly weaker than the Axiom of Degenerate Independence studied by Grant et al. (1992), which does not require $r$ to be a certain prize. Clearly, if a model admits a GLBU representation, then it satisfies WCI.

Our incompatibility result is as follows:

**Theorem 1** Suppose $\succeq$ satisfies Axioms 0-3, SI, and No Future Bias. Then for any $(x, t)$ in $\text{int}(X \times T)$, there exists $s > 0$ such that for any $\tau \in (0, s)$, we have

$$\frac{1}{2}\delta(x,t-\tau) + \frac{1}{2}\delta(x,t+\tau) \succeq \delta(x,t).$$

In other words, SI and No Future Bias imply weak RSTL in the local sense.

Although the conclusion of Theorem 1 is local in contrast to everywhere RSTL in Theorem 0, our incompatibility theorem includes substantially more models than DeJarnette’s et al. To see this, we note that WCI means a binary lottery $\frac{1}{2}\delta(x,t) + \frac{1}{2}\delta(y,s)$ is evaluated by

$$f(u(x, t), u(y, s))$$

for some utility function $u$ evaluating fixed prizes and some $f$ that aggregates these two possibilities. Besides monotonicity, there is no restrictions on $f$ in the models covered by Theorem 1. On the other hand, GLBU requires $f$ takes the form

$$f(m, n) = a \max u(m, n) + (1 - a) \min u(m, n)$$

for some constant $a \in (0, 1)$.

The flexibility of $f$ resulting from relaxing GLBU to WCI allows us to apply the incompatibility result to more models. In reference point models, individuals’ evaluation of an outcome can depend on other factors such as their evaluation of other possible outcomes. For example, Loomes and Sugden (1986) proposed a disappointment model in which individuals compare the actual outcome to a prior expectation. When adapting to our setting, their model becomes

$$V(p) = \mathbb{E}_p \left[ u(x, t) + R(u(x, t) - \bar{u}) \right],$$
where \( \bar{u} \) denotes the prior expectation and \( R \) is an increasing function satisfying \( R(0) = 0 \). This disappointment model rarely satisfies GLBU. However, with many natural choices of \( \bar{u} \) (e.g. \( \bar{u} \) is a constant or \( \bar{u} = \mathbb{E}_p u \)), the disappointment model satisfies WCI.

5 Accommodation of RATL with SI

Given that our incompatibility result that says No Future Bias and SI cannot be accommodated with RATL under a very weak form of Independence, we are motivated to drop No Future Bias to accommodate RATL with SI.

To begin with, the following is an example that satisfies SI and RATL in a strict sense everywhere.

**Example 1** Suppose \( \succeq \) can be represented by

\[
V(p) = \mathbb{E}_p - \left[ -\log \left( d^a v(x) \right) \right]^b,
\]

where \( a > 1, b \in \left( \frac{1}{a}, 1 \right), d \in (0, 1) \) and \( v \) is monotone increasing and \( \text{Range}(v) \in (0, 1) \). Then \( \succeq \) satisfies Axioms 0-3, and Independence. Moreover, it satisfies strict SI and strict RATL, in the sense that

\[
\frac{1}{2} \delta(x_1, t_1) + \frac{1}{2} \delta(x_2, t_2) \succ \frac{1}{2} \delta(x_1, t_2) + \frac{1}{2} \delta(x_2, t_1)
\]

for any \( t_1 < t_2, x_1 > x_2 \) and

\[
\delta(x, \bar{t}_p) \succ p_x
\]

for any time lottery \( p_x \).

The example lies in a large class of utility functions that satisfy SI and RATL everywhere. We prove a representation theorem that characterizes this class of functions in Expected Utility.

**Axiom 4** Independence: For all \( p, q, r \in \Delta(X \times T) \) and \( \lambda \in (0, 1) \),

\[
p \succeq q \Leftrightarrow \lambda p + (1 - \lambda) r \succeq \lambda q + (1 - \lambda) r
\]

**Axiom 5** Double Cancellation: For any \( x_1, x_2, x_3 \in X \) and \( t_1, t_2, t_3 \in T \), if \( \delta(x_1, t_1) \succeq \delta(x_2, t_2) \), and \( \delta(x_2, t_3) \succeq \delta(x_3, t_1) \), then \( \delta(x_1, t_3) \succeq \delta(x_3, t_2) \).

**Theorem 2** \( \succeq \) satisfies Axioms 0-5, RATL, and SI if and only if \( \succeq \) can be represented by

\[
V(p) = \mathbb{E}_p[\phi(D(t)v(x))]
\]
and the following are satisfied:

1. $\phi$ is strictly increasing, and continuous on $\text{Range}(D \cdot v)$. In addition, $\phi \circ \exp$ is convex.
2. $D > 0$ is strictly decreasing and continuous on $T$.
3. $v > 0$ is strictly increasing and continuous on $X$.
4. $\phi(D(\cdot)v(x))$ is concave for each $x \in X$.

What’s more, $\ln D$ and $\ln v$ are unique up to positive linear transformations. After fixing a choice of $D$ and $v$, $\phi$ is unique up to positive linear transformations.

**Remark 3**

Given the representation

$$V(p) = \mathbb{E}_p[\phi(D(t)v(x))],$$

RATL arises from two parts: Future Bias and curvature of $\phi$. Due to the separability of time with monetary prizes, Future Bias is equivalent to increasing discount rate. Hence, for any $\tau > 0$, the discount from $\delta(x,t)$ to $\delta(x,t+\tau)$ is less than the discount from $\delta(x,t+\tau)$ to $\delta(x,t+2\tau)$.

However, Future Bias alone is usually not enough to explain RATL:

$$V(p) = \mathbb{E}[e^{-t-\frac{\tau^2}{2}}v(x)]$$

satisfies Future Bias, but not RATL. Only if the discount function $D$ is concave can $V(p) = \mathbb{E}[D(t)v(x)]$ satisfies RATL, but this is usually not desirable.

Risk attitude is capture by $\phi$. The more concave $\phi$ is, the more risk averse decision makers are and the more likely they are RATL. However, SI requires $\phi$ to be more convex than logarithmic functions. Example 1 shows that $\phi(x) := -(\log x)^b$ is at the right level of convexity that is able to accommodate both SI and RATL.

Despite discarding No Future Bias and adopting everywhere RATL in the representation theorem, it is important to note that we are not arguing No Future Bias is completely false, or RATL should hold everywhere. Admittedly, there is evidence in support of No Future Bias and lab experiments do document some RSTL choices when there is a small probability of a very early arrival. However, our representation shows that RATL and SI are not fundamentally conflicting in Expected Utility. Hence, decision makers’ occasional RATL choices may be due to Future Bias but not the failure of Expected Utility or SI. This opens the door for future study.
Appendix

A Key lemmas about SI and RATL

Assuming Independence and Continuity, preferences on \( \Delta (X \times T) \) can be represented by

\[
V(p) = \mathbb{E}_p u(x, t)
\]

for some \( u \) defined on \( X \times T \). Within the Expected Utility framework, RATL and RSTL relations can be passed to the curvature of \( u \).

**Lemma 1** In the Expected Utility model in which a preference \( \succeq \) is represented by

\[
V(p) = \mathbb{E}_p u(x, t),
\]

\( \succeq \) is RATL if and only if \( u(x, \cdot) \) is concave for any \( x \in X \). Similarly, \( \succeq \) is RSTL if and only if \( u(x, \cdot) \) is convex for any \( x \).

This claim follows directly from Jensen’s inequality.

There is a neat characterization of SI within Expected Utility: time and monetary prizes are complimentary. This has been observed by [DeJarnette et al. (2020)](https://example.com), but here we give a rigorous proof.

**Proposition 1** Suppose \( \succeq \) can be represented by

\[
V(p) = \mathbb{E}_p u(x, t),
\]

where \( u \) is twice differentiable. Then the relation \( \succeq \) satisfies SI if and only if \( \frac{\partial^2 u}{\partial x \partial t} \leq 0 \) on \( \text{int}(X \times T) \).

**Proof.** \( \implies \) Stochastic Impatience implies that for any \( x_0 \in \text{int}(X) \), \( t_0 \in \text{int}(T) \), and small increments \( \Delta x > 0 \) and \( \Delta t > 0 \), we have

\[
u(x_0 + \Delta x, t_0 - \Delta t) + u(x_0 - \Delta x, t_0 + \Delta t) \geq u(x_0 + \Delta x, t_0 + \Delta t) + u(x_0 - \Delta x, t_0 - \Delta t). \quad (2)
\]

Consider the second-order Taylor series of \( f \) at \( (x_0, t_0) \) and let \( \Delta x = \Delta t \rightarrow 0 \). By slight abuse of notation, we write \( u_x(x_0, t_0) \) as \( u_x \) and employ similar abbreviations for other derivatives.
at \((x_0, t_0)\). Rewrite Equation \([2]\) as

\[
\begin{align*}
&u(x_0, t_0) + \Delta x \cdot u_x - \Delta t \cdot u_t + \frac{1}{2} \left( u_{xx} (\Delta x)^2 + u_{tt} (\Delta t)^2 - 2u_{xt} (\Delta x) (\Delta t) \right) \\
&+ u(x_0, t_0) - \Delta x \cdot u_x + \Delta t \cdot u_t + \frac{1}{2} \left( u_{xx} (\Delta x)^2 + u_{tt} (\Delta t)^2 - 2u_{xt} (\Delta x) (\Delta t) \right) + o((\Delta x)^3) \\
\geq & \quad u(x_0, t_0) + \Delta x \cdot u_x + \Delta t \cdot u_t + \frac{1}{2} \left( u_{xx} (\Delta x)^2 + u_{tt} (\Delta t)^2 + 2u_{xt} (\Delta x) (\Delta t) \right) \\
&+ u(x_0, t_0) - \Delta x \cdot u_x - \Delta t \cdot u_t + \frac{1}{2} \left( u_{xx} (\Delta x)^2 + u_{tt} (\Delta t)^2 + 2u_{xt} (\Delta x) (\Delta t) \right) + o((\Delta x)^3)
\end{align*}
\]

Equivalently,

\[ u_{xt} \leq 0. \]

\[ \iff \] For any \(t_1 < t_2\) in \(T\) and \(x_1 > x_2\) in \(X\), we have

\[
\begin{align*}
u(x_1, t_1) + u(x_2, t_2) - u(x_1, t_2) - u(x_2, t_1) \\
= & \quad (u(x_1, t_1) - u(x_1, t_2)) + (u(x_2, t_2) - u(x_2, t_1)) \\
= & - \int_{t_1}^{t_2} u_t (x_1, t) \, dt + \int_{t_1}^{t_2} u_t (x_2, t) \, dt \\
= & \int_{t_1}^{t_2} (u_t (x_2, t) - u_t (x_1, t)) \, dt.
\end{align*}
\]

Since \(u_{xt} \leq 0\), the integrand of the last line \(u_t (x_2, t) - u_t (x_1, t) \geq 0\) so the integral is non-negative. \(\blacksquare\)

**B Analysis of Example II**

We shall prove that \(\gtrsim\) in Example II satisfies strict SI and strict RATL.

To check that \(\gtrsim\) satisfies strict SI, we adopt Proposition II (use strict SI and strict inequality in the second order derivative in place of SI and weak inequality). We have

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial x} \left( - \left[ - \log \left( d^a v(x) \right) \right]^b \right) \\
= \frac{\partial}{\partial t} \left( - \left[ - t^a \log d - \log v(x) \right]^b \right) \\
= \frac{\partial}{\partial x} \left( - b \left[ - t^a \log d - \log v(x) \right]^{b-1} \left( - a t^{-a-1} \log d \right) \right) \\
= - b(b - 1) \left[ - t^a \log d - \log v(x) \right]^{b-2} \left( - a t^{-a-1} \log d \right) \left( - \frac{1}{v(x)} v'(x) \right) \\
< 0
\]

because \(b(b - 1) < 0, -t^a \log d - \log v(x) > 0, -a t^{-a-1} \log d > 0\) and \(\frac{1}{v(x)} v'(x) < 0\).
As for RATL, we have

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \left( - \left[ - \log \left( d^a v(x) \right) \right]^b \right) \right) = \frac{\partial}{\partial t} \left( - b \left[ - t^a \log d - \log v(x) \right]^{b-1} \left( - a t^{a-1} \log d \right) \right) = - b(b - 1) \left[ - t^a \log d - \log v(x) \right]^{b-2} \left( - a t^{a-1} \log d \right)^2 - b \left[ - t^a \log d - \log v(x) \right]^{b-1} \left( - a(a - 1) t^{a-2} \log d \right) = - b \left[ - t^a \log d - \log v(x) \right]^{b-2} \left( - a t^{a-1} \log d \right) \left[ -(b - 1) a t^{a-1} \log d + [- t^a \log d - \log v(x)] (a - 1) t^{-1} \right].
\]

Since

\[- b \left[ - t^a \log d - \log v(x) \right]^{b-2} \left( - a t^{a-1} \log d \right) < 0\]

and

\[ - (b - 1) a t^{a-1} \log d + [- t^a \log d - \log v(x)] (a - 1) t^{-1} > - (b - 1) a t^{a-1} \log d + (- t^a \log d) (a - 1) t^{-1} = t^{a-1} (-(b - 1) a - (a - 1)) \log d = t^{a-1} (- a b + 1) \log d > 0, \]

we have

\[ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \left( - \left[ - \log \left( d^a v(x) \right) \right]^b \right) \right) < 0. \]

By Lemma \( \text{□} \) (with strict RATL and strictly concave in place of RATL and concave), \( \succeq \) satisfies strict RATL.

C Proof of Theorem \( \text{□} \)

Lemma 2 Suppose \( \succeq \) satisfies Continuity, Impatience and Outcome Monotonicity. Fix \( (x, t) \in \text{int}(X \times T) \). Then there exists \( s > 0 \) such that for any \( \tau \in (0, s) \), there exists \( y < x \) in \( X \) satisfying \( \delta_{(y, t - \tau)} \sim \delta_{(x, t)} \).

Proof. Let \( w = \min X \) be the worst possible monetary prize. Let \( S := \{ c : \delta_{(w, c)} \succeq \delta_{(x, t)} \} \) and \( L := \{ c : \delta_{(w, c)} \succeq \delta_{(x, t)} \} \). Then Continuity and Impatience imply that \( S \) and \( L \) are closed intervals and \( S \cup L = X \). By monotonicity, \( \delta_{(w, t)} \preceq \delta_{(x, t)} \) so \( t \in L \).

Case 1: \( S \neq \emptyset \). Let \( d := \max S = \min L \). Then \( \delta_{(w, d)} \sim \delta_{(x, t)} \). We must have \( d < t \); otherwise, Impatience and Outcome Monotonicity would imply

\[ \delta_{(x, t)} \succeq \delta_{(x, d)} \succ \delta_{(w, d)}, \]
a contradiction.

Set $s = t - d$. For any $\tau \in (0, s)$, we let $S_{\tau} := \{ y : \delta_{(y,t-\tau)} \gtrsim \delta_{(x,t)} \}$ and $L_{\tau} := \{ y : \delta_{(y,t-\tau)} \lesssim \delta_{(x,t)} \}$. Then $S_{\tau}$ and $L_{\tau}$ are closed intervals with $S_{\tau} \cup L_{\tau} = X$. By Impatience, $x \in S_{\tau}$. What’s more, $\delta_{(w,t-\tau)} \lesssim \delta_{(w,d)} \sim \delta_{(x,t)}$ so $w \in L_{\tau}$. We set $y_{\tau} := \min S_{\tau} = \max L_{\tau}$. Then it follows that $\delta_{(y_{\tau},t-\tau)} \sim \delta_{(x,t)}$. Similar to before, Impatience and Outcome Monotonicity imply $y_{\tau} < x$.

**Case 2:** $S = \emptyset$. Set $s = t - \min T$. For any $\tau \in (0, s)$, we let $S_{\tau} := \{ y : \delta_{(y,t-\tau)} \gtrsim \delta_{(x,t)} \}$ and $L_{\tau} := \{ y : \delta_{(y,t-\tau)} \lesssim \delta_{(x,t)} \}$. Then $x \in S_{\tau}$. In addition, $S = \emptyset$ implies $\delta_{(x,t)} > \delta_{(w,t-\tau)}$. Thus, $w \in L_{\tau}$. We set $y_{\tau} = \min S_{\tau} = \max L_{\tau}$. Then $\delta_{(y_{\tau},t-\tau)} \sim \delta_{(x,t)}$ and $y_{\tau} < x$. ■

Fix $(x, t) \in \text{int}(X \times T)$, and let $s$ be as in the previous lemma. For any $\tau \in (0, s)$, choose $y < x$ in $X$ satisfying $\delta_{(y,t-\tau)} \sim \delta_{(x,t)}$. No Future Bias implies

$$\delta_{(x,t+\tau)} \gtrsim \delta_{(y,t)}.$$

By WCI, we are able to add the same term on both sides:

$$\frac{1}{2} \delta_{(x,t+\tau)} + \frac{1}{2} \delta_{(x,t-\tau)} \gtrsim \frac{1}{2} \delta_{(y,t)} + \frac{1}{2} \delta_{(x,t-\tau)}. \quad (3)$$

By SI,

$$\frac{1}{2} \delta_{(y,t)} + \frac{1}{2} \delta_{(x,t-\tau)} \gtrsim \frac{1}{2} \delta_{(x,t)} + \frac{1}{2} \delta_{(y,t-\tau)}. \quad (4)$$

The two terms on the right hand side can be combined into

$$\frac{1}{2} \delta_{(x,t)} + \frac{1}{2} \delta_{(y,t-\tau)} \sim \delta_{(x,t)} \quad (5)$$

using WCI.

Equation (3), (4), and (5) then imply

$$\frac{1}{2} \delta_{(x,t+\tau)} + \frac{1}{2} \delta_{(x,t-\tau)} \gtrsim \delta_{(x,t)}$$

as desired.

**D Proof of Theorem 2**

We begin by writing multiplicative forms into additive forms.

**Lemma 3** $\gtrsim$ satisfies Axioms 0-5, RATL, and SI if and only if $\gtrsim$ can be represented by

$$V(p) = \mathbb{E}_p[\phi(D(t) + v(x))]$$

and the following are satisfied:
Lemma 1. \( \phi \) is strictly increasing, convex and continuous on \( \text{Range}(D + v) \).

\((2') D \) is strictly decreasing and continuous on \( T \).

\((3') v \) is strictly increasing and continuous on \( X \).

\((4') \phi(D(\cdot) + v(x)) \) is concave for each \( x \in X \).

What’s more, such \( D \) and \( v \) are unique up to positive linear transformations. After fixing a choice of \( D \) and \( v \), \( \phi \) is unique up to positive linear transformations.

**Proof.** \( \iff \) Suppose we have such a representation of \( \succeq \). Completeness, Transitivity, and Independence are clearly satisfied. Since \( \phi \) is strictly increasing and \( D \) is strictly decreasing in time, we have Impatience. Outcome Monotonicity follows from strictly increasing \( \phi \) and strictly increasing \( v \). Continuity of \( \succeq \) follows from the continuity of \( D \) and \( v \). By Lemma [1] concavity of \( \phi(D(\cdot) + v(x)) \) implies RATL.

To check Double Cancellation, let \( x_1, x_2, x_3 \in X \) and \( t_1, t_2, t_3 \in T \) such that \( \delta(x_1, t_1) \succeq \delta(x_2, t_2) \), and \( \delta(x_2, t_3) \succeq \delta(x_3, t_1) \). We notice that because of monotonicity of \( \phi \),

\[ \phi(D(t_1) + v(x_1)) \geq \phi(D(t_2) + v(x_2)) \]

implies

\[ D(t_1) + v(x_1) \geq D(t_2) + v(x_2). \]

Similarly,

\[ D(t_3) + v(x_2) \geq D(t_1) + v(x_3) \]

follows from \( \delta(x_2, t_3) \succeq \delta(x_3, t_1) \). Combining the last two equations, we get

\[ D(t_3) + v(x_1) \geq D(t_2) + v(x_3), \]

which is equivalent to \( \delta(x_1, t_3) \succeq \delta(x_3, t_2) \).

The last property we want to check is SI. If \( x_1 > x_2 \) and \( t_1 < t_2 \), then

\[ [D(t_1) + v(x_1)] + [D(t_2) + v(x_2)] = [D(t_1) + v(x_2)] + [D(t_2) + v(x_1)]. \]

What’s more, \([D(t_1) + v(x_1)] \geq [D(t_1) + v(x_2)], [D(t_2) + v(x_1)] \geq [D(t_2) + v(x_2)] \). Convexity of \( \phi \) implies

\[ \phi(D(t_1) + v(x_1)) + \phi(D(t_2) + v(x_2)) \geq \phi(D(t_1) + v(x_2)) + \phi(D(t_2) + v(x_1)). \]

\( \implies \) Suppose \( \succeq \) satisfies Axioms 0-5, RATL, and SI. Let \( X^* \) denote \( \{\delta(x,t) : (x, t) \in X \times T\} \), viewed as a subset of \( \Delta(X \times T) \). \( X^* \) has the preference relation \( \succeq_{X^*} \) inherited from \( \Delta(X \times T) \).

Then \( \succeq_{X^*} \) satisfies Outcome Monotonicity, Impatience, Continuity and Double Cancellation.
By Debreu (1959) or Fishburn (1970), there exist a continuous function $D$ on $T$ and a continuous function $v$ on $X$ such that $\succeq_X$ can be represented by $D(t) + v(x)$. What’s more, $D$ and $v$ are unique up to positive linear transformations. Outcome Monotonicity and Impatience of $\succeq_X$ implies that $v$ is strictly increasing and $D$ is strictly decreasing.

We fix a choice of $D$ and $v$. Let $Y := \text{Range}(D + v)$. Since $D$ and $v$ are continuous functions on closed intervals, $Y$ is a connected interval. $\succeq$ induces a preference relation $\succeq_Y$ on $\Delta Y$, the set of simple lotteries on $Y$, as follows. For $p \in \Delta (X \times T)$, define $p'$ by

$$p'(a) = p(\{(x,t) : D(t) + v(x) = a\})$$

for $a \in Y$. Let $p' \succeq_Y q'$ if and only if $p \succeq q$. Independence guarantees the well-definedness of $\succeq_Y$. It can be easily checked that $\succeq_Y$ is a preference relation and inherits Independence and Continuity from $\succeq$. Thus, there exists a continuous function $\phi : Y \to \mathbb{R}$ such that $\succeq_Y$ can be represented by

$$V(p') = \mathbb{E}_{p'} \phi.$$

What’s more, $\phi$ is unique up to positive linear transformations. It follows that $\succeq$ can be represented by

$$V(p) = \mathbb{E}_p[\phi(D(t) + v(x))].$$

Since $v$ is strictly increasing, Outcome Monotonicity implies that $\phi$ is strictly increasing.

By Lemma 1, RATL implies concavity of $\phi(D(\cdot) + v(x))$ for each $x \in X$.

We then demonstrate convexity of $\phi$ follows from SI. Let $A = \sup v - \inf v > 0$, possibly infinite. We first show that for any $0 < \varepsilon < A$, $\phi(b + \varepsilon) - \phi(b)$ is strictly increasing in $b$ whenever $b, b + \varepsilon \in \text{Range}(D + v)$. This is because by SI, for any $x_1, x_2$ such that $v(x_1) = v(x_2) + \varepsilon$, we have

$$\phi(D(t_1) + v(x_1)) - \phi(D(t_1) + v(x_2)) \geq \phi(D(t_2) + v(x_1)) - \phi(D(t_2) + v(x_2))$$

for any $t_1 < t_2$. That is $\phi(\cdot + v(x_2) + \varepsilon) - \phi(\cdot + v(x_2))$ is increasing in $\text{Range}(D)$. As $x_1, x_2$ ranges in $X$, $v(x_2)$ covers $\text{Range}(v) \cap (\text{Range}(v) - \varepsilon)$ so $\phi(\cdot + \varepsilon) - \phi(\cdot)$ is increasing in $\text{Range}(D + v) \cap (\text{Range}(D + v) - \varepsilon)$.

We claim that for any rational $a \in (0, 1)$ and any $c_2 > c_1$, $c_1, c_2 \in \text{Range}(D + v)$, we have

$$a \phi(c_1) + (1 - a) \phi(c_2) \geq \phi(ct_1 + (1 - a)c_2).$$

Let $a = \frac{p}{q}$, where $p, q$ are positive integers such that $\delta := \frac{c_2 - c_1}{q} < A$. Then $ac_1 + (1 - a)c_2 = c_1 + (q - p)\delta$, and

$$\phi(k\delta + c_1) - \phi((k - 1)\delta + c_1).$$
is increasing in $k$. Thus,

$$\frac{1}{q-p} \left[ \phi(ac_1 + (1-a)c_2) - \phi(c_1) \right]$$

$$= \frac{1}{q-p} \sum_{k=1}^{q-p} \left[ \phi(k\delta + c_1) - \phi((k-1)\delta + c_1) \right]$$

$$\leq \phi(ac_1 + (1-a)c_2) - \phi(ac_1 + (1-a)c_2 - \delta)$$

$$\leq \frac{1}{p} \sum_{k=q-p+1}^{q} \left[ \phi(k\delta + c_1) - \phi((k-1)\delta + c_1) \right]$$

$$= \frac{1}{p} \left[ \phi(c_2) - \phi(ac_1 + (1-a)c_2) \right]$$

Reorganizing this, we get

$$a\phi(c_1) + (1-a)\phi(c_2) \geq \phi(ac_1 + (1-a)c_2).$$

We then remove the assumption that $a$ is rational in the last equation by continuity of $\phi$. Suppose $a$ can be approximated by a sequence of rationals $\{a_i\} \subset (0,1)$. For each $i$, we have

$$a_i\phi(c_1) + (1-a_i)\phi(c_2) \geq \phi(a_ic_1 + (1-a_i)c_2).$$

Let $i \to \infty$. Since $\phi$ is continuous, we still have

$$a\phi(c_1) + (1-a)D(c_2) \geq \phi(ac_1 + (1-a)c_2)$$

in the limit. In conclusion, $\phi$ is convex on $Range(D+v)$. ■

Now we go back to the proof of Theorem 2

$\implies$ Suppose $\succeq$ satisfies Axioms 0-5, RATL, and SI. By Lemma 3, there exist $\phi^*$, $D^*$ and $v^*$ such that $\succeq$ can be represented by

$$V(p) = \mathbb{E}_p[\phi^*(D^*(t) + v^*(x))]$$

such that properties (1')-(4') in the statement of that lemma hold. We let

$$\phi = \phi^* \circ \ln$$

$$D = \exp(D^*)$$

$$v = \exp(v^*)$$
then
\[ V(p) = \mathbb{E}_p[\phi(D(t)v(x))], \]

(1)-(4) in Theorem 2 hold. The uniqueness of \( D, v, \) and \( \phi \) also follows from the uniqueness of \( D^*, v^*, \) and \( \phi^* \).

\[ \iff \quad \text{Conversely, we assume } \succeq \text{ can be represented by} \]
\[ V(p) = \mathbb{E}_p[\phi(D(t)v(x))] \]

such that (1)-(4) in Theorem 2 are true. We set
\[ \phi^* = \phi \circ \exp \]
\[ D^* = \ln D \]
\[ v^* = \ln v. \]

Then (1')-(4') in Lemma 3 follow from (1)-(4). Hence by Lemma 3, Axioms 0-5, RATL, and SI are satisfied.

References

Attema, A. E., H. Bleichrodt, K. I. M. Rohde, and P. P. Wakker, “Time-tradeoff sequences for analyzing discounting and time inconsistency,” Management Science, 2010, 56 (11), 2015–2030. https://doi.org/10.1287/mnsc.1100.1219.

Benhabib, J., A. Bisin, and A. Schotter, “Present-bias, quasi-hyperbolic discounting, and fixed costs,” Games and Economic Behavior, 2010, 69 (2), 205–223. https://doi.org/10.1016/j.geb.2009.11.003.

Blavatskyy, P., “Expected discounted utility,” Theory and Decision, 2020, 88 (2), 297–313. http://dx.doi.org/10.2139/ssrn.3432022.

Chesson, H. W. and W. K. Viscusi, “Commonalities in Time and Ambiguity Aversion for Long-Term Risks,” Theory and Decision, 2003, 54 (1), 57–71. https://doi.org/10.1023/A:1025095318208.

Debreu, G., “Topological methods in cardinal utility theory,” 1959. Available at https://elischolar.library.yale.edu/cgi/viewcontent.cgi?article=1298&context=cowles-discussion-paper-series.
DeJarnette, P., D. Dillenberger, D. Gottlieb, and P. Ortoleva, “Time lotteries and stochastic impatience,” *Econometrica*, 2020, 88 (2), 619–656. https://doi.org/10.3982/ECTA16427.

Dillenberger, D., D. Gottlieb, and P. Ortoleva, “Stochastic impatience and the separation of time and risk preferences,” 2020. Available at http://pietroortoleva.com/papers/SIvsTimeRisk.pdf.

Epstein, L. G., “Stationary cardinal utility and optimal growth under uncertainty,” *Journal of Economic Theory*, 1983, 31 (1), 133–152. https://doi.org/10.1016/0022-0531(83)90025-X.

Fishburn, P. C., *Utility Theory for Decision Making*, New York: Wiley, 1970.

— and A. Rubinstein, “Time preference,” *International Economic Review*, 1982, pp. 677–694. https://doi.org/10.2307/2526382.

Gilboa, I. and D. Schmeidler, “Maxmin expected utility with non-unique prior,” in “Uncertainty in economic theory” 2004, pp. 141–151. https://doi.org/10.1016/0304-4068(89)90018-9.

Grant, S., A. Kajii, and B. Polak, “Many good choice axioms: when can many-good lotteries be treated as money lotteries?,” *Journal of Economic Theory*, 1992, 56 (2), 313–337. https://doi.org/10.1016/0022-0531(92)90085-V.

Green, L., N. Fristoe, and J. Myerson, “Temporal discounting and preference reversals in choice between delayed outcomes,” *Psychonomic Bulletin & Review*, 1994, 1 (3), 383–389. https://doi.org/10.3758/BF03213979.

Halevy, Y., “Time consistency: Stationarity and time invariance,” *Econometrica*, 2015, 83 (1), 335–352. https://doi.org/10.3982/ECTA10872.

Kirby, K. N. and R. J. Herrnstein, “Preference reversals due to myopic discounting of delayed reward,” *Psychological science*, 1995, 6 (2), 83–89. https://doi.org/10.1111/j.1467-9280.1995.tb00311.x.

Lanier, J., B. Miao, J. K.-H Quah, and S. Zhong, “Intertemporal Consumption with Risk: A Revealed Preference Analysis,” Available at SSRN 3168361, 2020. http://dx.doi.org/10.2139/ssrn.3168361.

Loewenstein, G. and D. Prelec, “Anomalies in Intertemporal Choice: Evidence and an Interpretation,” *The Quarterly Journal of Economics*, 1992, 107 (2), 573–597. https://doi.org/10.2307/2118482.
Loomes, G. and R. Sugden, “Disappointment and dynamic consistency in choice under uncertainty,” *The Review of Economic Studies*, 1986, 53 (2), 271–282. https://doi.org/10.2307/2297651.

Mu, X., L. Pomatto, P. Strack, and P. Tamuz, “Monotone additive statistics,” *arXiv preprint arXiv:2102.00618*, 2021.

Onay, S. and A. Öncüler, “Intertemporal Choice Under Timing Risk: An Experimental Approach,” *Journal of Risk and Uncertainty*, 2007, 34 (2), 99–121. https://doi.org/10.1007/s11166-007-9005-x.

Phelps, E. S. and R. A. Pollak, “On Second-Best National Saving and Game-Equilibrium Growth,” *The Review of Economic Studies*, 1968, 35 (2), 185–199. https://doi.org/10.2307/2296547.

Prelec, D., “Decreasing impatience: a criterion for Non-stationary time preference and “hyperbolic” discounting,” *The Scandinavian Journal of Economics*, 2004, 106 (3), 511–532. https://doi.org/10.1111/j.0347-0520.2004.00375.x.

Rubinstein, A., “Economics and psychology”? The case of hyperbolic discounting,” *International Economic Review*, 2003, 44 (4), 1207–1216. https://doi.org/10.1111/1468-2354.t01-1-00106.

Sayman, S. and A. Öncüler, “An investigation of time inconsistency,” *Management Science*, 2009, 55 (3), 470–482. https://doi.org/10.1287/mnsc.1080.0942.

Selden, L., “A new representation of preferences over” certain x uncertain” consumption pairs: The” ordinal certainty equivalent” hypothesis,” *Econometrica: Journal of the Econometric Society*, 1978, pp. 1045–1060. https://doi.org/10.2307/1911435.

_ and I. E. Stux, *Consumption Trees, OCE Utility and the Consumption/Savings Decision* 1978. Available at https://www8.gsb.columbia.edu/researcharchive/articles/25696.

Takeuchi, K., “Non-parametric test of time consistency: Present bias and future bias,” *Games and Economic Behavior*, 2011, 71 (2), 456–478. https://doi.org/10.1016/j.geb.2010.05.005.