Regular basis and R-matrices for the $\widehat{su}(n)_k$ Knizhnik-Zamolodchikov equation

L.K. Hadjiivanov
Theoretical Physics Division, Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

Ya.S. Stanev
Dipartimento di Fisica, Università di Roma ”Tor Vergata”, I.N.F.N. – Sezione di Roma ”Tor Vergata”, Via della Ricerca Scientifica 1, I-00133 Roma, Italy

and

I.T. Todorov
Theoretical Physics Division, Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

Abstract
Dynamical $R$-matrix relations are derived for the group-valued chiral vertex operators in the $SU(n)$ WZNW model from the KZ equation for a general four-point function including two step operators. They fit the exchange relations of the $U_q(sl_n)$ covariant quantum matrix algebra derived previously by solving the dynamical Yang-Baxter equation. As a byproduct, we extend the regular basis introduced earlier for $SU(2)$ chiral fields to $SU(n)$ step operators and display the corresponding triangular matrix representation of the braid group.

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\footnote{1}{e-mail address: lhadji@inrne.bas.bg}
\footnote{2}{On leave of absence from the Institute for Nuclear Research and Nuclear Energy, BG-1784 Sofia, Bulgaria; e-mail address: stanev@roma2.infn.it}
\footnote{3}{e-mail address: todorov@inrne.bas.bg}
1 Introduction

There are, essentially, two different approaches to the Wess-Zumino-Novikov-Witten (WZNW) model - a model describing the conformally invariant dynamics of a closed string moving on a compact Lie group $G$ \cite{1, 2}. The axiomatic approach \cite{3} relies on the representation theory of Kac-Moody current algebras and on the Sugawara formula for the stress-energy tensor. The resulting chiral conformal block solutions of the Knizhnik-Zamolodchikov (KZ) equation are multivalued analytic functions which span a monodromy representation of the braid group \cite{4, 5}. Surprisingly (at least at first sight), the associated symmetry was related to the recently discovered quantum groups \cite{6, 7, 8, 9, 10, 11}. This relation was explained in some sense by the second, canonical approach to the problem \cite{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23}. The Poisson-Lie symmetry \cite{24} of the WZNW action \cite{13, 15, 16} indeed gives rise to quantum group invariant quadratic exchange relations \cite{13, 17, 25} at the quantum level.

In spite of continuing efforts \cite{26, 27, 23}, the correspondence between the two approaches is as yet only tentative: there is still no consistent operator formulation of the chiral WZNW model to date that would reproduce the known conformal blocks. The objective of the present paper is to provide a step in filling this gap.

The first problem we are addressing is to find the precise correspondence between the monodromy representation of the braid group \cite{4, 5} and the $R$-matrix exchange relations \cite{13, 15, 17} among step operators – i.e., field operators transforming under the defining $n$-dimensional ("quark") representation of $SU(n)$. To this end we consider the four-point "conformal block" of two step operators sandwiched between a pair of primary chiral fields transforming under arbitrary irreducible representations (IR) of $SU(n)$, only restricted by the condition that the resulting space of $SU(n)$ invariants is non-empty. In fact, for a given initial and final states $|p\rangle$ and $\langle p'|$ (labeled by the highest weights of $SU(n)$ IR – see Section 2), the space $\mathcal{F}(p, p')$ of invariant tensors of the type

$$\mathcal{F}(p, p') = \{ \langle p'|\varphi^A_1 \varphi^B_2 |p\rangle, A, B = 1, \ldots, n \} \quad (1.1)$$

(the subscripts 1 and 2 replacing the world sheet variables and discrete quantum numbers other than the $SU(n)$ indices $A, B$) can be either 0, 1 or 2-dimensional. We concentrate on the most interesting 2-dimensional case that
includes the antisymmetric tensor product of $\varphi_1$ and $\varphi_2$) but also write down the 1-dimensional ("anyonic") braid relations.

A new result of the paper is the extension to $\widehat{su}(n)$ step operators of the "regular basis" (introduced originally for $\widehat{su}(2)$ blocks [1] as a counterpart of a distinguished basis of quantum group, $U_q(sl_2)$, invariants [28]). Moreover, we demonstrate that the Möbius invariant amplitude for the $\widehat{su}(2)$ and the $\widehat{su}(n)$ theory coincide and so do the "normalized braid matrices" $q^{1/2}B$. This allows to extend earlier results on the Schwarz finite monodromy problem for the $\widehat{su}(2)$ KZ equation to the $\widehat{su}(n)$ case.

We start, in Section 2, with some background material including various forms of the KZ equation. Special attention is devoted to two bases of $SU(n)$ invariants (whose properties and interrelations are spelled out in Appendix A). They appear as prototypes of the $s$-channel basis and regular basis of solutions of the KZ equation studied in Sections 3 and 4, respectively. The standard notion of a chiral vertex operator (CVO) [4] and its zero modes’ counterpart [25, 17, 18, 20, 21, 22, 23] are applied in Section 3 for studying the braid properties of the "physical solutions" of the KZ equation. It is the regular basis introduced in Section 4 that is appropriate to also include its logarithmic solutions.

## 2 KZ equation for a 4-point conformal block

We label the IR of $SU(n)$ by their shifted highest weights (see [23]):

\[
p_{i+1} = p_i - p_{i+1} = \lambda_i + 1 \quad (\lambda_i \in \mathbb{Z}_+) \quad \text{where} \quad \sum_{i=1}^n p_i = 0. \tag{2.1}
\]

Let the highest weight $p'$ belong to the tensor product of $p = (p_1, \ldots, p_n)$ with a pair of "quark" IR (with $\lambda_i = \delta_{i1}$). The basic object of our study will be the four-point block

\[
w^{AB}_{z_1 p, p'}(z_1, z_2, z_3, z_4; p, p') \equiv w^{AB}_{\alpha \beta}(z_1, z_2, z_3; p, p') = \langle 0 | \Phi^\alpha_{p'}(z_1) \varphi^A_\alpha(z_2) \varphi^B_\beta(z_3) \Phi_p(z_4) | 0 \rangle \tag{2.2}
\]

where $\varphi^A_\alpha(z)$ is the fundamental chiral ("quark", or "group valued") field ($A, \alpha$ are $SU(n)$ and $U_q(sl_n)$ indices, respectively), $\Phi_p(z)$ is a (primary) chiral field carrying weight $p$ (whose tensor indices are omitted), and $p^* = (-p'_1, \ldots, -p'_n)$ is the weight conjugate to $p'$. Let $v^{(i)}$ be the shift of weight under the application of an $SU(n)$ step operator $\varphi^i_\alpha(z)$:

\[
(v^{(i)}|p) = p_i, \quad (v^{(i)}|v^{(j)}) = \delta_{ij} - \frac{1}{n}, \quad \sum_{i=1}^n v^{(i)} = 0. \tag{2.3}
\]
Then $p'$ and $p$ satisfy
\[ p' - p = v(i) + v(j) \equiv v(m) + v(m'), \quad m = \min (i, j), \quad m' = \max (i, j) \quad (2.4) \]
where we assume that $p,p'$ and $p + v(m)$ are dominant weights.

We shall express the four-point block $w$ (2.2) in terms of a conformally invariant amplitude $F$ setting
\[ w_{\alpha\beta}^{AB}(z; p, p') = D(z; p, p') F_{\alpha\beta}^{AB}(\eta; p, p') \quad (2.5) \]
where the cross ratio $\eta$ and the prefactor $D(z; p, p')$ are given by
\[ \eta = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad z_{ij} = z_i - z_j, \quad (2.6) \]
\[ D(z; p, p') = \left( \frac{z_{24}}{z_{12} z_{14}} \right)^{\Delta(p')} \left( \frac{z_{13}}{z_{14} z_{34}} \right)^{\Delta(p)} z_{23}^{-2\Delta} \eta^{\Delta' - \Delta} (1 - \eta)^{\Delta_a}. \]

Here
\[ 2h \Delta(p) \equiv C_2(p) = \frac{1}{n} \sum_{r < s} p_{rs}^2 - \frac{n(n^2 - 1)}{12}, \quad \Delta = \frac{n^2 - 1}{2nh}, \quad (2.7) \]
\[ \Delta' \equiv \Delta(p + v(m')) = \Delta(p) + \frac{p_{mm'}}{h} + \frac{n - 1}{2nh}, \quad \Delta_a = \frac{(n + 1)(n - 2)}{nh} \]
(For the current algebra $\widehat{su}(n)_k$ the height $h$ is given by $h = k + n$, $k$ being the level.) Note that
\[ 2p_{mm'} = h (\Delta(p') - \Delta(p)) - (p_{mm'} + \delta_{mm'}) + \frac{2 - n}{n}. \]

**Remark 2.1** The prefactor, a product of powers of the differences $z_{ij}$, is determined by the overall scale dimension of $w_{\alpha\beta}^{AB}(z; p, p')$ and by infinitesimal $L_1$ invariance, i.e.,
\[ \sum_{a=1}^{4} z_a^\nu \left( z_a \frac{\partial}{\partial z_a} + (\nu + 1) \Delta_a \right) D(z; p, p') = 0 \quad \text{for} \quad \nu = 0, \pm 1, \]
\[ \Delta_1 = \Delta(p^{*r}) \equiv \Delta(p'), \quad \Delta_2 = \Delta_3 = \Delta, \quad \Delta_4 = \Delta(p) \quad (2.8) \]
(see (2.7)), up to a monomial in $\eta$ and $1 - \eta$. Our choice (2.6) corresponds to extracting the leading singularities so that $F_{\alpha\beta}^{AB}(\eta; p, p')$ should be finite and nonzero at both $\eta = 0$ and $\eta = 1$.

Let $C_{ab} = t_a . t_b$ be the polarized Casimir invariant $(a, b = 1, \ldots, 4)$. The generators $t_a$ of the representation $(a)$ of $SU(n)$ are normalized in such a
way that if \((a)\) refers to an IR of weight \(p\), then \(t_a^2\) coincides with \(C_2(p)\) of (2.7); in our case, with \(C_a := t_a^2\), we have

\[
C_4 = C_2(p) = \frac{1}{n} \sum_{i<j} (p^2_{ij} - (j - i)^2) = \frac{1}{n} \sum_{i<j} p^2_{ij} - \frac{n(n^2 - 1)}{12}. \tag{2.9}
\]

\(SU(n)\) invariance of the Wightman function implies

\[
(t_1 + t_2 + t_3 + t_4) w_{\alpha\beta}(z; p, p') = 0 = \left( C_a + \sum_{b \neq a} C_{ab} \right) w_{\alpha\beta}(z; p, p'). \tag{2.10}
\]

The KZ equation

\[
\left( \hbar \frac{\partial}{\partial z_a} - \sum_{b \neq a} \frac{C_{ab}}{z_{ab}} \right) w_{\alpha\beta}(z; p, p') = 0, \quad 1 \leq a \leq 4, \tag{2.11}
\]

yields (choosing, say, \(a = 2\) and using (2.10))

\[
\left( \hbar \frac{d}{d\eta} - \frac{\Omega_{12}}{\eta} + \frac{\Omega_{23}}{1 - \eta} \right) F_{\alpha\beta}(\eta; p, p') = 0 \tag{2.12}
\]

where

\[
\Omega_{12} = C_{12} + p_m + \delta_{mm'} + \frac{n^2 + n - 4}{2n}, \quad \Omega_{23} = C_{23} + \frac{1}{n} + 1 = P_{23} + 1, \tag{2.13}
\]

\(P_{23}\) being the permutation operator for the two factors in the tensor product \(\mathbb{C}^n \otimes \mathbb{C}^n\) of fundamental IR \((P_{23}^2 = 1)\). \(\Omega_{12}\) and \(\Omega_{23}\) satisfy

\[
\Omega_{12} \Omega_{23} \Omega_{12} = (p - 1) \Omega_{12}, \quad \Omega_{23} \Omega_{12} \Omega_{23} = (p - 1) \Omega_{23} \tag{2.14}
\]

\[
p = p_{mm'} + \delta_{mm'} \in \mathbb{N}, \quad \Omega_{12}^2 = p_{mm'} \Omega_{12}, \quad \Omega_{23}^2 = 2 \Omega_{23}
\]

– see Appendix A.

The value \(p = 1\) is special and we shall consider it separately.

For \(p \geq 2\) both \(\Omega_{12}\) and \(\Omega_{23}\) are nontrivial (nonnegative) operators with a single eigenvalue zero. In this case the space \(\mathcal{F}(p, p')\) \((\equiv \mathcal{F}^{AB})\) is conveniently spanned by the eigenvectors \(I_0 = (I_0^{AB})\) and \(I_1 = (I_1^{AB})\) of \(\Omega_{12}\) and \(\Omega_{23}\), respectively, corresponding to eigenvalue 0 (cf. [11] and Appendix A below):

\[
\Omega_{12} I_0 = 0 = \Omega_{23} I_1, \quad I_1 = (P_{23} - 1) I_0. \tag{2.15}
\]

We shall set

\[
F_{\alpha\beta}^{AB}(\eta; p, p') = I_0^{AB} (1 - \eta) f_{\alpha\beta}^0(\eta) + I_1^{AB} \eta f_{\alpha\beta}^1(\eta). \tag{2.16}
\]
Inserting (2.14) into (2.12), we find the following first order system for \( f^\ell \equiv f^\ell_{\alpha\beta}(\eta) \), \( \ell = 0, 1 \):

\[
h(1 - \eta) \frac{df^0}{d\eta} = (h - 2)f^0 + (p - 1)f^1, \quad h \eta \frac{df^1}{d\eta} = (p - h)f^1 - f^0. \tag{2.17}
\]

It yields a hypergeometric (HG) equation for each \( f^\ell \), \( \ell = 0, 1 \):

\[
\eta(1 - \eta) \frac{d^2 f^\ell}{d\eta^2} + \left(1 + \ell - \frac{p}{h} - (3 - \frac{p + 2}{h})\eta\right) \frac{df^\ell}{d\eta} = \left(1 - \frac{1}{h}(1 - \frac{p + 1}{h})\right)f^\ell. \tag{2.18}
\]

For \( p = 1 \) we have, in view of (2.14), either \( m = m' \) implying

\[
P_{23}S_0 = S_0 = I_0, \quad I_1 = 0, \quad S_1 = S_0, \quad F(\eta) = Ks\eta^2, \tag{2.19}
\]

or \( m' = m + 1 \), \( p_{m,m+1} = 1 \), when

\[
P_{23}S_1 = -S_1, \quad S_0 = 0 = I_0 = I_1, \quad F(\eta) = Ks\eta^2 \tag{2.20}
\]

(see Appendix A). In both cases \( \dim \mathcal{F}(p,p') \leq 1 \). In particular, for \( p = p^{(0)} \), \( m = m' - 1 = 1 \) and for \( p' = p^{(0)} \), \( m' = m + 1 = n \) we are dealing with a 3-point function.

It is remarkable that the KZ equation (2.12) for the Möbius invariant amplitude \( F_{\alpha\beta}(\eta; p,p') \) is in fact independent of the group label \( n \) – as made manifest by the system (2.17). Only the prefactor \( D(\varpi; p,p') \) (2.6) carries an \( n \)-dependence.

## 3 s-channel basis of solutions. Braid relations

The expansion of the basic fields \( \varphi^A_\alpha(z) \) into chiral vertex operators (CVO), \( \varphi^A_\alpha(z) = \varphi^A_i(z) a^i_\alpha \) \([20, 21, 23]\) gives rise to an expansion of \( f^\ell_{\alpha\beta} \) into \( s \)-channel conformal blocks. We have

\[
f^\ell_{\alpha\beta}(\eta) = \sum_{\lambda=0}^1 s^\ell_\lambda(\eta) s^\lambda_{\alpha\beta}, \quad S^0_{\alpha\beta} = \langle p'|a^m_\alpha a^{m'}_\beta |p\rangle, \quad S^1_{\alpha\beta} = \langle p'|a^m_\alpha a^{m'}_\beta |p\rangle \tag{3.1}
\]

for

\[
\varphi^A_\alpha(z) = c_i^A \varphi^A_i(z) a^i_\alpha. \tag{3.2}
\]

Here \( a^i_\alpha \), (which, together with \( q^{\rho ij} \), generate a quantum matrix algebra of \( SL(n) \) type \([34, 23]\)) satisfy

\[
a^i_\alpha a^j_\alpha = a^j_\alpha a^i_\alpha, \quad a^i_\alpha a^j_\beta = q^{\epsilon\alpha\beta} a^j_\beta a^i_\alpha \quad \text{and, for } i \neq j, \alpha \neq \beta, \quad (3.3a)
\]

\[
\rho(p_{ij})[p_{ij} - 1]a^i_\alpha a^j_\beta = [p_{ij}] a^i_\beta a^j_\alpha - q^{\epsilon\beta\rho ij} a^i_\alpha a^j_\beta \quad \text{(} p(p)\rho(-p) = 1 \text{)}.
\]

6
We are using the notation of Section 2.3 in [23]

\[ \epsilon_{\beta \alpha} = 1 = -\epsilon_{\alpha \beta} \] for \( \alpha < \beta \), \( \epsilon_{\alpha \alpha} = 0 \); \[ [p] := \frac{q^p - q^{-p}}{q - q^{-1}} ; \quad q = e^{-\frac{2\pi i}{\hbar}} = q^{-1}. \]

The factor \( \rho(p_{ij}) \) constraint by the last equation in (3.2a) reflects the "gauge freedom" in the solution of the dynamical Yang-Baxter equation [32, 33, 34, 23]. The \( SU(n) \) tensor operators \( c_i^A \) are generators of the undeformed counterpart of the matrix algebra (see Appendix A); in particular,

\[ [c_i^A, c_j^B] = 0 = [c_i^A, c_j^B] ; \quad \text{and, for } i \neq j, A \neq B, \]

\[ r(p_{ij}) (p_{ij} - 1) c_j^A c_i^B = p_{ij} c_j^B c_i^A - c_i^A c_j^B \quad (r(p)r(-p) = 1). \]

The function \( s^0(\eta) \) is characterized as the analytic around the origin solution of the system (2.17), resp. of the HG equation (2.18) satisfying \( K_0 := s^0(0) > 0 \). It is given by the HG series

\[ s^0(\eta) = K_0 \frac{F(\alpha, \beta; 1 - \alpha + \beta; \eta)}{B(1 - \alpha, \beta)} \int_0^1 t^{\beta - 1} (1 - t)^{-\alpha} (1 - t\eta)^{-\alpha} dt \]

(3.4)

where \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) is the beta-function, \( p' - p \) is given by [24], \( p_{mn'} = p \geq 2 \) and \( \alpha = 1 - \frac{1}{\hbar}, \beta = 1 - \frac{p+1}{\hbar} \). For \( p = h - 1, s^0(\eta) = K_0 \) (\( \lim_{\beta \to 0} \frac{t^{\beta - 1}}{\Gamma(\beta)} = \delta(t) \)). From (2.17) one gets

\[ s^0(\eta) = \frac{\alpha - 1}{1 - \alpha + \beta} K_0 F(\alpha, \beta; 2 - \alpha + \beta; \eta). \]

(3.5)

The second solution, \( s^1(\eta) \), is obtained requiring that

\[ \eta^{\Delta(p + (m')) - \Delta(p + (m))} s^0_1(\eta) = \eta^{-\frac{\Delta}{2}} s^0_1(\eta) \]

(3.6)

is analytic around \( \eta = 0 \); we find

\[ s^0_1(\eta) = K_1 \eta^{\alpha - \beta} F(\alpha, 2\alpha - \beta; 1 + \alpha - \beta; \eta), \]

(3.7)

\[ s^1_1(\eta) = \frac{\alpha - \beta}{2\alpha - 1 - \beta} K_1 \eta^{\alpha - \beta - 1} F(\alpha - 1, 2\alpha - 1 - \beta; \alpha - \beta; \eta). \]

(3.8)

We shall now derive the braid relation among the above conformal blocks under the exchange of two step operators \( \varphi \).

The four-point blocks \( w_{\alpha \beta}^(A)(z_1, z_2, z_3, z_4; p, p') \) (2.3) are single valued in the domain

\[ O_4 = \{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4; |z_i| > |z_{i+1}| , i = 1, 2, 3; |\arg z_i| < \pi , i = 1, 2, 3, 4 \} \]

(3.9)
We consider the analytic continuation $\hat{w}_{\alpha_\beta}^A(z_1, z_3, z_2, z_4; p, p')$ of $w_{\alpha_\beta}^A(z_1, z_2, z_3, z_4; p, p')$ along paths in the homotopy class of a particular curve

$$C(2, 3) : [0, 1] \times (\mathbb{C}^4 \setminus \{z_i = z_j, i \neq j\}) \to \mathbb{C}^4 \setminus \{z_i = z_j, i \neq j\}$$ (3.10a)

which interchanges $z_2$ and $z_3$ and whose internal points belong to $\mathcal{O}_4$ whenever the end points belong to its boundary (3.9); e.g., for $z_i(t) = e^{i\epsilon_i(t)}$, $i = 1, 2, 3, 4$,

$$\zeta_1(0) = x_1 - i\epsilon, \quad \zeta_{2,3}(0) = x_{2,3}, \quad \zeta_4(0) = x_4 + i\epsilon,$$

$$x_i \in \mathbb{R}, \quad \epsilon > 0, \quad x_1 > x_2 > x_3 > x_4, \quad x_{14} < \pi,$$ (3.10b)

we define $C(2, 3)$ by

$$\zeta_1(t) = \zeta_1, \quad \zeta_{2,3}(t) = e^{-i\frac{\pi}{2}t}(x_{2,3} \cos \frac{\pi t}{2} + ix_{3,2} \sin \frac{\pi t}{2}), \quad \zeta_4(t) = \zeta_4; \quad 0 \leq t \leq 1$$ (3.10c)

(see Proposition 1.3 of [23]). The cross ratio $\eta$ and the prefactor $D \equiv D(z; p, p')$ (2.3) then change according to the law

$$z_{23} \to e^{-i\pi z_{23}} \Rightarrow \eta \to \frac{1}{\eta}, \quad D \to \eta^{\frac{n+1}{h}} \eta^{\frac{n+1}{h}} D \left(\frac{1}{\eta} = e^{-i\frac{\pi}{\eta}}\right)$$ (3.11)

where we have used (2.7) to derive the relations

$$\Delta(p) + \Delta(p') - 2\Delta' = \frac{p}{h} - \frac{1}{nh}, \quad 2\Delta - \Delta_{(a)} = \frac{1}{h} + \frac{1}{nh}.$$ (3.12)

The expansion (2.16) into $SU(n)$ tensor invariants changes under the combined action of analytic continuation along the path $C(2, 3)$ and permutation of the indices $A, B$ as follows:

$$I_0^{AB}(1 - \eta) f_{\alpha\beta}^0(\eta) + I_1^{AB} \eta f_{\alpha\beta}^1(\eta) \quad \Rightarrow \quad$$

$$\Rightarrow -\frac{1}{\eta} \left((I_0^{AB} + I_1^{AB})(1 - \eta) f_{\alpha\beta}^0(\frac{1}{\eta}) + I_1^{AB} f_{\alpha\beta}^1(\frac{1}{\eta})\right).$$ (3.13)

Inserting further the expansion (3.1) for $f_{\alpha\beta}^\ell(\eta), \lim_{\ell} f_{\alpha\beta}^\ell(\eta)$, we finally, using the relation

$$F(\alpha, \beta; \gamma; \frac{1}{\eta}) = e^{-i\pi \alpha} \frac{\Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \alpha)} \eta^\alpha F(\alpha, 1 + \alpha - \gamma; 1 + \alpha - \beta; \eta) +$$

$$+e^{-i\pi \beta} \frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\alpha) \Gamma(\gamma - \beta)} \eta^\beta F(1 + \beta - \gamma, \beta; 1 + \beta - \alpha; \eta),$$ (3.14)

we deduce (setting $k(p) := \frac{K}{K_0}$)

$$D s_\lambda^\ell(\eta) \quad \Rightarrow \quad D s_\lambda^\ell(\eta) B_\lambda^N, \quad B = \begin{pmatrix} B_0^0 & B_0^1 \\ B_1^0 & B_1^1 \end{pmatrix} = \frac{q}{|p|} \left(\frac{q}{|p|} - k(p)b_p \right),$$ (3.15)
where \( B (\equiv B_3^2) \) is the braid matrix corresponding to the exchange \( \hat{2} \hat{3} \) in the \( s \)-channel basis and

\[
\begin{align*}
b_p &= \frac{\Gamma(1 + \alpha - \beta)\Gamma(\alpha - \beta)}{\Gamma(2\alpha - \beta)\Gamma(1 - \beta)} = \frac{\Gamma(1 + \frac{p}{h})\Gamma(\frac{p}{h})}{\Gamma(1 + \frac{p+1}{h})\Gamma(\frac{p+1}{h})} \\
\Rightarrow b_p b_{-p} &= \frac{\sin \pi \frac{p+1}{h} \sin \pi \frac{p+1}{h}}{\sin^2 \pi \frac{p}{h}} = \frac{|p + 1||p - 1|}{[p]^2}.
\end{align*}
\]

(3.16)

Note that in line with the remark at the end of Section 2 the product \( q^\frac{1}{h} B \) of the braid matrix with a scalar phase factor is independent of \( n \).

As we already mentioned, the case \( m = m' \) is much simpler. Indeed, the space \( \mathcal{F}(p, p') \) (1.1), for \( p, p + v(m) \) and \( p' = p + 2v(m) \) dominant, is one dimensional and so is the space of quantum group invariants \( (S_{\alpha\beta}^0 = S_{\alpha\beta}^1 = q^{\epsilon_{\alpha\beta}} S_{\beta\alpha}^0 \equiv S_{\alpha\beta}, \text{ cf. } (3.1) \) and (3.3a)). Due to the first equation (3.3b), the skewsymmetric \( SU(n) \) invariant \( I_{1AB}^1 \) is zero, see Appendix A, and \( I_{0AB}^1 \equiv I_{AB}^1 \) is symmetric, hence, \( \Omega_{23} I = 2I \), cf. (2.13) and (2.15). The analogs of (2.5), (2.16) and (3.1) read then

\[
u_{\alpha\beta}^{AB}(z; p, p') = D(z; p, p') I_{AB}^1 s(\eta) S_{\alpha\beta}
\]

(3.17)

and the KZ equation reduces to a first order equation for \( s(\eta) \):

\[
h \frac{d}{d\eta} s(\eta) = -\frac{2}{1 - \eta} s(\eta), \quad \text{i.e., } s(\eta) = K(1 - \eta)\frac{\hat{2}}{h}.
\]

(3.18)

Since in this case

\[
z_{23} \rightarrow e^{-i\pi} z_{23} \quad \Rightarrow \quad 1 - \eta \rightarrow e^{-i\pi \frac{1 - \eta}{\eta}}, \quad D \rightarrow q^{\frac{a+1}{h}} \eta^\frac{a}{h} D,
\]

(3.19)

we get simply

\[
D s(\eta) \rightarrow q^{1-\frac{a}{h}} D s(\eta).
\]

(3.20)

All this fits perfectly the operator exchange relations,

\[
\varphi_i^B(z_3) \varphi_j^A(z_2) = \varphi_i^A(z_2) \varphi_j^B(z_3) \hat{R}(p)_{ij}^{ij'}
\]

(3.21)

(related in [23] to the characteristic properties of the intertwining quantum matrix algebra). In particular, the last equation (3.3a) corresponds to the choice

\[
k(p) = \frac{\Gamma(\frac{1+p}{h})\Gamma(-\frac{p}{h})}{\Gamma(\frac{1-p}{h})\Gamma(\frac{p}{h})} \rho(p), \quad \rho(p)\rho(-p) = 1 \quad (= k(p)k(-p))
\]

(3.22)
of the ratio of the normalization constants $\frac{\hat{K}_1}{\hat{K}_0}$. The resulting $4 \times 4$ dynamical $R$-matrix $\hat{R}(p)$ reads [23]

$$
\hat{R}(p) = \begin{pmatrix}
\hat{R}(p)_{mm}^{mm'} & \hat{R}(p)_{mm}^{mm'} & \hat{R}(p)_{mm}^{mm'} & \hat{R}(p)_{mm}^{mm'} \\
\hat{R}(p)_{m'm'}^{mm} & \hat{R}(p)_{m'm'}^{mm} & \hat{R}(p)_{m'm'}^{mm} & \hat{R}(p)_{m'm'}^{mm} \\
\hat{R}(p)_{mm'}^{m'm'} & \hat{R}(p)_{mm'}^{m'm'} & \hat{R}(p)_{mm'}^{m'm'} & \hat{R}(p)_{mm'}^{m'm'} \\
\hat{R}(p)_{m'm'}^{m'm'} & \hat{R}(p)_{m'm'}^{m'm'} & \hat{R}(p)_{m'm'}^{m'm'} & \hat{R}(p)_{m'm'}^{m'm'} \\
\end{pmatrix}
$$

$$
= \frac{1}{q^n} \begin{pmatrix}
q & 0 & \frac{p}{|p|} & 0 \\
0 & \frac{|p-1|}{|p|} & 0 & 0 \\
\frac{|p+1|}{|p|} & \rho(-p) & -\frac{p}{|p|} & 0 \\
0 & 0 & 0 & q \\
\end{pmatrix}.
$$

(3.23)

The braid matrices $B_1^s$ and $B_3^s$ corresponding to the exchanges $\hat{1} \hat{2}$ and $\hat{3} \hat{4}$, respectively, are diagonal with eigenvalues

$$
\varepsilon_t q^{\Delta(p+v(t)) - \Delta(p) - \Delta} = \varepsilon_t q^{m - \frac{m+1}{2}},
$$

(3.24a)

$$
(t = m, m', \varepsilon_m = 1, \varepsilon_{m'} = -1) - \text{for } B_3^s, \text{ and}
$$

$$
\varepsilon_t q^{\Delta(p+v'(t)) - \Delta(p') - \Delta} = \varepsilon_t q^{p' + \frac{n^2 + n - 4}{2m}}
$$

(3.24b)

$$(p' = p + v'(t) = p + v(m) + v(m')) - \text{for } B_1^s. \text{ For } n = 2 \text{ and } p = (p_1, p_2) = (1, -1) = p' \Leftrightarrow p_{12} = 2 = p'_{12} - \text{i.e., for the four-point function of 4 spin-1/2 operators, the braid matrices } B_1^s \text{ and } B_3^s \text{ coincide:}
$$

$$
B_1^s = B_3^s = \begin{pmatrix}
q^\frac{1}{2} & 0 \\
0 & q^\frac{1}{2} \\
\end{pmatrix}.
$$

(3.24c)

Remark 3.1 The choice $\rho(p) = \pm q^{vp} \sqrt{\frac{|p+1|}{|p-1|}}$, $\nu$ real, for the remaining freedom in the normalization guarantees the unitarity of $B$ (3.15) (and of $\hat{R}(p)$) for $p + 1 < h$, however, it violates the property of the elements of the matrix $q^{\frac{1}{2}}B$ to belong to the cyclotomic field $\mathbb{Q}(q)$ which, as discussed in Section 5, could be useful.

4 Regular basis of solutions of the KZ equation. Triangular braid matrices

As noted in the Introduction, if we allow for unphysical values of $p$ such that, say, $p = h$, then the braid matrix $B_1$ (or $B_3$) is no longer diagonalizable,
the corresponding solution of the HG KZ equation having a logarithmic singularity. We shall now introduce a regular basis of solutions which remains meaningful also for such exceptional values of $p$. To this end we first introduce the $U_q(sl_n)$ counterpart of the basis $I_0, I_1$ of $SU(n)$ invariants, setting

$$T^0_{\alpha \beta} = S^0_{\alpha \beta} = \langle p | a^m_{\alpha} a^{m'}_{\beta} | p \rangle \quad (m < m'), \quad T^1_{\alpha \beta} = -T^0_{\alpha \beta} A^{\alpha' \beta'} = T^0_{\beta \alpha} - q^{\beta \alpha} T^0_{\alpha \beta} \quad \text{(4.1)}$$

where $A = (A^{\alpha' \beta'})$ is the quantum antisymmetrizer of $\mathfrak{U}$ satisfying $A^2 = [2, A]$. We shall set

$$f^\ell_{\alpha \beta}(\eta) = \frac{1}{\lambda} f^\lambda_{\alpha \beta} \quad (= \frac{1}{\lambda} s^\lambda_{\alpha \beta} \text{ for } p < h) \quad \text{(4.2)}$$

(cf. (3.1)), and will demonstrate that the solution of the system (2.17) is given by the contour integrals

$$f^0_0(\eta) = N_0 \int_0^1 t^{\frac{p-1}{h} - \ell} (1 - t)^{\frac{1}{h} - 1 + \ell} (t - \eta)^{\frac{1}{h} - 1} dt =$$

$$= N_0 B \left( \frac{1}{h}, \ell + \frac{1}{h}; 1 - \eta \right),$$

$$f^1_1(\eta) = N_1 \int_0^\eta t^{\frac{p-1}{h} - \ell} (1 - t)^{\frac{1}{h} - 1 + \ell} (\eta - t)^{\frac{1}{h} - 1} dt =$$

$$= N_1 B \left( \frac{1}{h}, 1 - \ell + \frac{p - 1}{h}; 1 - \ell + \frac{p - 1}{h} \right),$$

$$\quad \text{(4.3)}$$

$(\ell = 0, 1)$. The functions $f^\lambda_{\alpha \beta}(\eta)$ are chosen in such a way that if $B_2$ is the braid matrix associated with the exchange $\hat{23}$, then

$$(B_2)^\lambda_{\alpha \beta} = T^\mu_{\alpha \beta} \hat{R}^\alpha_{\alpha' \beta'}, \quad q^\frac{1}{h} \hat{R}^\alpha_{\alpha' \beta'} = q m_{\alpha' \beta'} - A^{\alpha' \beta'} \quad \text{(4.4)}$$

in other words, analytic continuation along the path (3.11) combined with a permutation of the indices $A, B$ is equivalent to the action of the constant (Jimbo) $\hat{R}$-matrix.

Equating the expansion (4.2) with (3.1) and using (A.19), (A.20), we can find the relation with the $s$-channel basis of KZ solutions:

$$f^\ell_{\alpha \beta}(\eta) = f^0_0(\eta) - \frac{[p - 1]}{[p]} f^1_1(\eta), \quad s^\ell_{\alpha \beta}(\eta) = \rho(p) \frac{[p - 1]}{[p]} f^\ell_{\alpha \beta}(\eta) \quad \text{(4.5)}$$

While the integral representations (13) defining $f^\ell_\lambda$ make perfect sense for all $p > \ell$, the amplitudes $s^\ell_{\alpha \beta}$ are ill defined for $p = h$. The $\hat{23}$ braid matrix determined from (4.4) and (A.18),

$$B_2 = \frac{q^\frac{1}{h}}{0 \ -q} \quad \text{(4.6)}$$

\[11\]
agrees with (3.23). Indeed, it follows from (3.23) and (A.19) that
\[ B_2 \begin{pmatrix} \frac{1}{[p-1][p]} & 0 \\ \rho(p)\frac{[p-1]}{[p]} & \rho(p) \end{pmatrix} \begin{pmatrix} S^0 \\ S^1 \end{pmatrix} = \]
\[ = \begin{pmatrix} \frac{1}{[p-1][p]} & 0 \\ \rho(p)\frac{[p-1]}{[p]} & \rho(p) \end{pmatrix} \frac{q^p}{[p]} \begin{pmatrix} q^p & \rho(p)[-p] \\ -q^p & \rho(p)[p + 1] \end{pmatrix} \begin{pmatrix} S^0 \\ S^1 \end{pmatrix} \]
which is consistent with (4.6) due to the \( q \)-number identities
\[ q[p] - [p - 1] = q^p, \quad [p + 1] - q^p = q[-p], \quad [p - 1] + q^p = q[p]. \] (4.7)

**Remark 4.1** If we identify \( \Phi_p \) in (2.2) with another \( SU(n) \) step operator (setting \( p_{12} = 2, p_{i+1} = 1 \) for \( 2 \leq i \leq n - 1 \)), then we can speak of a monodromy representation of the braid group \( B_3 \) on 3 strands with generators \( B_2 \) and
\[ B_3 = \frac{1}{q^n} \begin{pmatrix} 1 & 0 \\ -q & q \end{pmatrix} \] (4.8)

corresponding to the braiding \( \hat{\gamma} \):
\[ z_{34} \rightarrow e^{-i\pi}z_{34}, \quad \eta \rightarrow e^{-i\pi/2} \frac{1}{1 - \eta}, \quad D \rightarrow q^{n+1} (1 - \eta) \frac{1}{2} D, \] (4.9)
\[ (1 - \eta)I_0 \rightarrow -\frac{1}{1 - \eta} I_0, \quad \eta I_1 \rightarrow -\eta \frac{1}{1 - \eta} (I_0 + I_1). \]

Note that in this case \( p' \) determined by \( p'_{12} = 2 = p'_{23}, p'_{i+1} = 1 \) for \( 3 \leq i \leq n - 1 \) corresponds to the Young tableau \( \begin{array}{c} p' \\ 1 \end{array} \) and hence is, in general, different from \( p \), except for \( n = 2 \) when \( \Phi_{p'} = \Phi_{p''} \) is another step operator. In the \( n = 2 \) case we are actually dealing with a special representation of the braid group \( B_4 \), for which \( B_1 = B_3 \).

Eqs. (4.3), on the other hand, allow to relate the normalization constants \( K_\lambda \) of (3.4)-(3.8) and \( N_\lambda \) of (4.3). We find, comparing (3.8) with the second equation (4.3) for \( \ell = 1 \) and (4.3)
\[ \frac{p}{p - 1} K_1 = \rho(p) \frac{[p - 1]}{[p]} N_1 B_1 \left( \frac{1}{h}, \frac{p - 1}{h} \right) \Rightarrow K_1 = \rho(p) N_1 B_1 \left( \frac{1}{h}, -\frac{p}{h} \right); \] (4.10)

similarly, from (3.4), the first equation (4.3) and (4.3) we deduce
\[ K_0 = N_0 B_1 \left( \frac{1}{h}, \frac{p}{h} \right) \Rightarrow k(p) = \frac{K_1}{K_0} = \rho(p) \frac{\Gamma(-\frac{p}{h}) \Gamma(1 + \frac{p}{h})}{\Gamma(\frac{1-p}{h}) \Gamma(\frac{p}{h})} N_1. \] (4.11)

Comparing with (3.22), we find
\[ N_0 = N_1 \left( = \frac{K_0}{B_1(h, \frac{p}{h})} \right). \] (4.12)
Inserting (3.4), (3.7) and (4.3) into the last equation (4.5) for ℓ = 0 is equivalent to the following relation for HG functions [35]:

\[
\left(\frac{1}{N_0 B(\frac{1}{h}, \frac{1}{h})}\right) s_0^0(\eta) = (1 - \eta)^\frac{2}{h} - 1 F\left(1 - \frac{p}{h}, 1 - \frac{p}{h}; 1 - \frac{p}{h}; \eta\right) = \\
\frac{\Gamma\left(\frac{p}{h}\right) \Gamma\left(\frac{p}{h}\right)}{\Gamma\left(\frac{1}{h}\right) \Gamma\left(\frac{p+1}{h}\right)} F(1 - \frac{1}{h}, 1 - \frac{p+1}{h}; 1 - \frac{p}{h}; \eta) + \\
+ \frac{\Gamma\left(-\frac{p}{h}\right) \Gamma\left(\frac{p}{h}\right)}{\Gamma\left(\frac{1}{h}\right) \Gamma\left(1 - \frac{p}{h}\right)} \eta^p F(1 - \frac{1}{h}, 1 + \frac{p-1}{h}; 1 + \frac{p}{h}; \eta) .
\] (4.13)

Note that the poles appearing for \( p = h \) in the right hand side of (4.13) cancel.

5 Concluding remarks

The regular basis \( f_\lambda \) (4.3) of solutions of the KZ equation (dual to the basis \( \mathcal{I}^\lambda \) of \( U_q(sl_n) \) invariants) is characterized by the following properties.

(i) The functions \( f_\lambda^\ell , \ell = 0, 1 \), are well defined for all \( p > \ell \) including the value \( p = h \) (for which \( s_\lambda^\ell \) blow up).

(ii) The braid matrices \( B_2 \) (4.6) and \( B_3 \) for \( p = 2 \) (see (4.8)) are upper and lower triangular, respectively. Unlike their \( s \)-channel basis counterparts (as (3.13)), they have no singularities at \( p = h \). One could say that the \( s \)-basis is ill defined since it pretends to diagonalize the (non-diagonalizable for \( p = h \)) braid matrix \( B_3 \).

(iii) The elements of the braid matrices \( q^\frac{h}{2}B_a \) belong to the cyclotomic field \( \mathbb{Q}(q) \). This remark has been exploited in [38] to solve the Schwarz problem (of classifying the cases when the monodromy representation of the braid group gives rise to a finite matrix group). In fact, due to the observation that the braid groups for \( \widehat{su}(n)_k \) and \( \widehat{su}(2)_{k+n-2} \) essentially coincide, the results of [38] readily extend to the \( \widehat{su}(n) \) case.

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Appendix A. Bases and operators in the space of $SU(n)$ and of $U_q(sl_n)$ invariant tensors

A basis in $F(p,p')$ is conveniently expressed in terms of an $SU(n)$ intertwining matrix $c = (c^A_i)$ whose entries are subject to quadratic exchange relations

$$c^B_i c^A_j = c^A_j c^B_j \hat{R}^c(p)^{i'j'}_{ij}, \quad p_i c^A_j = c^A_j (p_i + v^{(j)}_i), \quad v^{(j)}_i = \delta^{(j)}_i - \frac{1}{n} \quad (A.1)$$

and the determinant condition (in which the two $\varepsilon$-s are totally antisymmetric for $r(p) = 1$ in (A.4) below)

$$\varepsilon^{i_1 \ldots i_n} c^{A_1}_{i_1} \ldots c^{A_n}_{i_n} = D_1(p) \varepsilon^{A_1 \ldots A_n}, \quad D_1(p) = \prod_{1 \leq i < j \leq n} p_{ij}. \quad (A.2)$$

The matrix $\hat{R}^c(p)$ satisfies the dynamical Yang-Baxter equation \cite{32,33,34}

$$\hat{R}^c_{12} (p) \hat{R}^c_{23} (p - v_1) \hat{R}^c_{12} (p) = \hat{R}^c_{23} (p - v_1) \hat{R}^c_{12} (p) \hat{R}^c_{23} (p - v_1)$$

and the ice condition $\hat{R}^c(p)^{ij}_{kl} = a^{ij}(p)\delta^{(k)}_i \delta^{(l)}_j + b^{ij}(p)\delta^{(k)}_l \delta^{(i)}_j$. The solution (a special case – for $q = 1$ – of the quantum dynamical $R$-matrix given below) depends on a non-zero function $r(p_{ij})$:

$$\hat{R}^c(p)^{ii}_{jj} = 1 \quad (\Leftrightarrow [c^A_i, c^B_i] = 0), \quad \left( \begin{array}{ll} \hat{R}^c(p)^{mm'}_{mm'} & \hat{R}^c(p)^{mm'}_{mm'} \\ \hat{R}^c(p)^{mm'}_{mm'} & \hat{R}^c(p)^{mm'}_{mm'} \end{array} \right) = \frac{1}{p} \left( \begin{array}{cc} 1 & \frac{p - 1}{r(p)} \\ \frac{p - 1}{r(-p)} & 1 \end{array} \right) \quad (A.3)$$

$$(m \neq m', \ p = p_{mm'}). \quad \text{The involutivity of } \hat{R}^c(p) \quad \text{implied by} \quad (A.4) \quad \text{fixes its determinant to } -1:\n
(\hat{R}^c(p))^2 = I \quad \Leftrightarrow \quad r(p)r(-p) = 1 \quad \Leftrightarrow \quad \det \hat{R}^c(p) = -1. \quad (A.5)$$

We consider the Fock-type representation of the intertwining matrix algebra with an $SU(n)$-invariant vacuum such that

$$c^A_i |0\rangle = 0, \quad i > 1, \quad \langle 0|c^A_i = 0, \quad j < n \quad (|0\rangle = |p^{(0)}\rangle, \quad p_{ij}^{(0)} = j - i). \quad (A.6)$$

The meaning of the above relations is that $c^A_i$, acting on a ket vector $|p\rangle$, adds a box to the $i$-th row of the Young tableau associated with the $SU(n)$ IR of highest weight $p$ (a more general statement, valid for $U_q = U_q(sl_n)$ and generic values of $q$ is proven in Section 3.1 of \cite{23}).

After these preliminaries we shall introduce (for $p > 1$) a basis of eigenvectors of $\Omega_{12}$ (and hence, of $C_{12}$), setting

$$S^{AB}_0 = T^{AB}_{mm'} := \langle p' | c^A_{m'} c^B_{m'} | p \rangle, \quad S^{AB}_1 = T^{AB}_{m'm} := \langle p' | c^A_{m} c^B_{m'} | p \rangle. \quad (A.7)$$
Indeed, for $p' = p + v(i) + v(j)$ we have

$$C_{12}T_{ij} = \frac{1}{2} \left( |p + v(j)|^2 - |p'|^2 - \frac{n^2 - 1}{n} \right) T_{ij} = -\left( p_i + \delta_{ij} + \frac{n^2 + n - 4}{2n} \right) T_{ij}.$$  

(A.8)

Hence, in view of (2.13),

$$\Omega_{12} S_0 = 0, \quad \Omega_{12} S_1 = p_{mm'} S_1 \quad \Rightarrow \quad \Omega_{12}^2 = p_{mm'} \Omega_{12}.$$  

(A.9)

The spectrum of $\Omega_{23}$, on the other hand, is found from the last equation in (2.13) which yields $\Omega_{23}^2 = 2\Omega_{23}$. We have

$$\Omega_{23} S_0 = (P_{23} + \mathbb{1}) S_0 \quad \Rightarrow \quad \Omega_{23} (P_{23} - \mathbb{1}) S_0 = 0, \quad \Omega_{23} (P_{23} + \mathbb{1}) S_0 = 2(P_{23} + \mathbb{1}) S_0.$$  

(A.10)

This suggests the introduction of the basis (2.11):

$$I_0 := S_0, \quad I_1 := (P_{23} - \mathbb{1}) S_0 \quad (\Rightarrow \quad P_{23} I_0 = I_0 + I_1, \quad P_{23} I_1 = -I_1).$$  

(A.11)

In order to express, conversely, $S_1$ in terms of $I_0$, $I_1$ we use (A.1), (A.4) to derive

$$(P_{23} S_0)^{AB} = T_{m'm}^{BA} = \frac{1}{p} (T_{m'm}^{AB} + r(p)(p - 1) T_{m'm}^{AB}),$$  

(A.12)

or

$$S_1^{AB} (\equiv T_{m'm}^{AB}) = r(-p)\left( \frac{p}{p - 1} I_1^{AB} + I_0^{AB} \right), \quad I_1^{AB} = \frac{p - 1}{p} (r(p) S_1^{AB} - S_0^{AB}).$$  

(A.13)

The action of $\Omega_{ab}$ on $I_\ell$ is given by (2.13) and by

$$\Omega_{23} I_0 = 2 I_0 + I_1, \quad \Omega_{12} I_1 = p I_1 + (p - 1) I_0.$$  

(A.14)

These relations imply also the last two equations (2.14).

We proceed to displaying the interrelations between the $U_q$ invariants $S^\lambda$ (3.1) and $T^\lambda$ (4.1) derived from the properties of the quantum matrix algebra generated by $a_\alpha^i$ and $q^{p_i}$ with basis exchange relations

$$\hat{R}(p)^{ij}_{ij'} a_{i'}^\alpha a_{j'}^\beta = a_i^\alpha a_j^\beta \hat{R}^{\alpha'\beta'}_{\alpha\beta}, \quad q^{p_i} a_i^\alpha = a_i^\alpha q^{p_i + \delta_i^j - \frac{1}{n}}$$  

(A.15)

(plus a suitable determinant condition, see [34, 23]) where $\hat{R}$ is the (Jimbo) $U_q n^2 \times n^2$ $R$-matrix appearing in (4.4). The dynamical $R$-matrix $\hat{R}(p)$ again satisfies (A.3) while the involutivity property (A.5) is replaced by a more general Hecke algebra condition implying

$$q^\frac{1}{n} \hat{R}(p) = q \mathbb{1} - A(p), \quad A(p)^2 = [2] A(p).$$  

(A.16)
The third equation in (3.3a) corresponds to the solution
\[ A(p)_{ij}' = \frac{[p_{ij} - 1]}{[p_{ij}]} \left( \delta_i^i \delta_j^j - \rho(p_{ij}) \delta_j^i \delta_i^j \right), \quad i \neq j, \quad A_{ij}' = 0 = A_{ij}'' \] (A.17)

\( (\rho(p_{ij}) \rho(p_{ji}) = 1) \). It follows from the definition (4.1) that
\[ q^\frac{1}{2} T_{\alpha \beta}^0 \hat{R}_{\alpha \beta}^{i' j'} = q T_{\alpha \beta}^0 + T_{\alpha \beta}^1, \quad q^\frac{1}{2} T_{\alpha \beta}^{i' j'} \hat{R}_{\alpha \beta}^{i' j'} = -q T_{\alpha \beta}^1. \] (A.18)

Eqs. (3.1), (4.1), (4.4) and (A.15)-(A.17), on the other hand, imply
\[ T_{\alpha \beta}^1 = \frac{[p - 1]}{[p]} \left( \rho(p) S_{\alpha \beta}^1 - S_{\alpha \beta}^0 \right), \quad p = p_{mm'} \quad (m \neq m'). \] (A.19)

Conversely, \( S_{\alpha \beta}^\lambda \) are expressed in terms of \( T_{\alpha \beta}^\lambda \) by
\[ S_{\alpha \beta}^0 = T_{\alpha \beta}^0, \quad \rho(p) S_{\alpha \beta}^1 = T_{\alpha \beta}^0 + \frac{[p]}{[p - 1]} T_{\alpha \beta}^1. \] (A.20)

As noted above, relations (A.11), (A.13) appear as the \( q \to 1 \) limit of (A.19), (A.20) (for \( T^0 (= S^0) \to I_0 (= S_0) \), resp. \( T^1 \to I_1, \quad S^1 \to S_1 \), and \( \rho(p) \to r(p) \)).

References

[1] Witten, E.: Non-abelian bosonization in two dimensions, Commun.
Math. Phys. 92 (1984) 455-472.

[2] Gepner, D. and Witten, E.: String theory on group manifolds, Nucl.
Phys. B278 (1986) 493.

[3] Knizhnik, V.G. and Zamolodchikov, A.B.: Current algebra and Wess-
Zumino model in two dimensions, Nucl. Phys. B247 (1984) 83-103.
Todorov, I.T.: Infinite Lie algebras in 2-dimensional conformal field
theory, in: H.-D. Doebner and T. Palev (eds), Proceedings of the
XIII International Conference on Differential Geometric Methods in
Physics, Shumen, Bulgaria, 1984, World Scientific, Singapore, 1986,
pp. 297-347; Current algebra approach to conformal invariant two-
dimensional models, Phys. Lett. B153 (1985) 77-81.

[4] Tsuchiya, A. and Kanie, Y.: Vertex operators in the conformal field
theory on \( \mathbb{P}_1 \) and monodromy representations of the braid group, Lett.
Math. Phys. 13 (1987) 303-312; Conformal field theory and solvable
lattice models, Adv. Studies in Pure Math. 16 (1988) 297; Errata, ibid
19 (1990) 675.
[5] Kohno, T.: Monodromy representations of braid groups and Yang-Baxter equations, *Ann. Inst. Fourier* **37** (1987) 139-160.

[6] Alvarez–Gaumé, L., Gomez, C. and Sierra, G.: Hidden quantum symmetries in rational conformal field theories, *Nucl. Phys. B319* (1989) 155; Quantum group interpretation of some conformal field theories, *Phys. Lett. B220* (1989) 142; Duality and quantum groups, *Nucl. Phys. B330* (1990) 347.

[7] Felder, G., Fröhlich, J. and Keller, G.: Braid matrices and structure constants for minimal conformal models, *Commun. Math. Phys.* **124** (1989) 647-664.

[8] Ganchev, A. and Petkova, V.: $U_q(sl(2))$ invariant operators and minimal theories fusion matrices, *Phys. Lett. B233* (1989) 374. Furlan, P., Ganchev, A. and Petkova, V.: Quantum groups and fusion rules multiplicities, *Nucl. Phys. B343* (1990) 205-227.

[9] Todorov, I.T.: Quantum groups as symmetries of chiral conformal algebras, in: H.-D. Doebner and J.-D. Hennig (eds), *Quantum Groups*, Lecture Notes in Physics **370**, Springer, Berlin, 1990, pp. 231-277. Hadjiivanov, L.K., Paunov, R.R. and Todorov, I.T.: Quantum group extended chiral $p$-models, *Nucl. Phys. B356* (1991) 387-438.

[10] K. Gawędzki, K.: Geometry of Wess-Zumino-Witten models of conformal field theory, *Nucl. Phys. (Proc. Suppl.)* **18B** (1990) 78-91.

[11] Stanev, Ya.S., Todorov, I.T. and Hadjiivanov, L.K.: Braid invariant rational conformal models with a quantum group symmetry, *Phys. Lett. B276* (1992) 87-94. Stanev, Ya.S. and Todorov, I.T.: Monodromy representations of the mapping class group $B_n$ for the $su_2$ Knizhnik-Zamolodchikov equation, in: H. Grosse and L. Pittner (eds), *Low Dimensional Models in Statistical Physics and Quantum Field Theory*, Springer, Berlin, 1996, pp. 201-222.

[12] Babelon, O.: Extended conformal algebra and the Yang-Baxter equation, *Phys. Lett. B215* (1988) 523-529. Blok, B.: Classical exchange algebras in the Wess-Zumino-Witten model, *Phys. Lett. B233* (1989) 359-362.

[13] Faddeev, L.D.: On the exchange matrix for WZNW model, *Commun. Math. Phys.* **132** (1990) 131-138. Alekseev, A. and Shatashvili, S.: Quantum groups and WZNW models, *Commun. Math. Phys.* **133** (1990) 353-368.
[14] Balog, J., Dabrowski, L. and Fehér, L.: Classical $r$-matrix and exchange algebra in WZNW and Toda theories, *Phys. Lett.* **B244** (1990) 227-234.

[15] Faddeev, L.D.: Quantum symmetry in conformal field theory by Hamiltonian methods, in: J. Fröhlich et al. (eds), *New Symmetry Principles in Quantum Field Theory*, Gargèse lectures, 1991, Plenum Press, New York, 1992, pp. 159–175.

Alekseev, A.Yu., Faddeev, L.D. and Semenov-Tian-Shansky, M.A.: Hidden quantum groups inside Kac-Moody algebras, *Commun. Math. Phys.* **149** (1992) 335-345.

[16] Gawędzki, K.: Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory, *Commun. Math. Phys.* **139** (1991) 201-213.

[17] Falceto, F. and Gawędzki, K.: Lattice Wess-Zumino-Witten model and quantum groups, *Journ. Geom. Phys.* **11** (1993) 251-279.

[18] Chu, M., Goddard, P., Halliday, I., Olive, D. and Schwimmer, A.: Quantization of the Wess-Zumino-Witten model on a circle, *Phys. Lett.* **B266** (1991) 71-81.

Chu, M. and Goddard, P.: Quantisation of the $SU(N)$ WZW model at level $k$, *Nucl. Phys.* **B445** (1995) 145-168, [hep-th/9407116](http://arxiv.org/abs/hep-th/9407116).

[19] Furlan, P., Hadjiivanov, L.K. and Todorov, I.T.: Canonical approach to the quantum WZNW model, ICTP Trieste and ESI Vienna preprint IC/95/74, ESI 234 (1995).

[20] Furlan, P., Hadjiivanov, L.K. and Todorov, I.T.: Operator realization of the $SU(2)$ WZNW model, *Nucl. Phys.* **B474** (1996) 497-511, [hep-th/9602101](http://arxiv.org/abs/hep-th/9602101).

[21] Furlan, P., Hadjiivanov, L.K. and Todorov, I.T.: A quantum gauge group approach to the 2D $SU(n)$ WZNW model, *Int. J. Mod. Phys.* **A12** (1997) 23-32, [hep-th/9610202](http://arxiv.org/abs/hep-th/9610202).

[22] Caneschi, L. and Lysiansky, M.: Chiral quantization of the WZW $SU(n)$ model, *Nucl. Phys.* **B505** (1997) 701-726.

[23] Furlan, P., Hadjiivanov, L.K., Isaev, A.P., Ogievetsky, O.V., Pyatov, P.N. and Todorov, I.T.: Quantum matrix algebra for the $SU(n)$ WZNW model, IHES Bures-sur-Yvette preprint IHES/P/00/11, [hep-th/0003210](http://arxiv.org/abs/hep-th/0003210) (submitted to Commun. Math. Phys.).

[24] Semenov-Tian-Shansky, M.A.: Dressing transformations and Poisson group actions, *Publ. RIMS, Kyoto Univ.* **21** (1985) 1237-1260.
[25] Alekseev, A.Yu. and Faddeev, L.D.: \((T^*G)_t\): A toy model for conformal field theory, Commun. Math. Phys. 141 (1991) 413-422.

[26] Dubois-Violette, M. and Todorov, I.T.: Generalized cohomologies and the physical subspace of the \(SU(2)\) WZNW model, Lett. Math. Phys. 42 (1997) 183-192, hep-th/9704069. Generalized homologies for the zero modes of the \(SU(2)\) WZNW model, Lett. Math. Phys. 48 (1999) 323-338, math.QA/9905071.

[27] Dubois-Violette, M., Furlan, P., Hadjiivanov, L.K., Isaev, A.P., Pyatov, P.N. and Todorov, I.T.: A finite dimensional gauge problem in the WZNW model, in: H.-D. Doebner et al. (eds), Quantum Theory and Symmetries, Proceedings of the International Symposium in Goslar, Germany, 1999, World Scientific, Singapore, 2000, pp. 331-349, hep-th/9910206.

[28] Furlan, P., Stanev, Ya.S. and Todorov, I.T.: Coherent state operators and \(n\)-point invariants for \(U_q(sl(2))\), Lett. Math. Phys. 22 (1991) 307-320.

[29] Faddeev, L.D., Reshetikhin, N.Yu. and Takhtajan, L.A., Quantization of Lie groups and Lie algebras, Algebra i Analiz 1:1 (1989) 178 (English translation: Leningrad Math. Journ. 1 (1990) 193).

[30] Jimbo, M.: A \(q\)-analogue of \(U(gl(N + 1))\), Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986) 247-252.

[31] Drinfeld, V.G.: Quantum groups, in: Proceedings of the International Congress of Mathematicians, Berkeley, 1986, Academic Press, 1986, vol.1, pp. 798-820.

[32] Gervais, J.-L. and Neveu, A.: Novel triangle relation and absence of tachyons in Liouville string field theory, Nucl. Phys. B238 (1984) 125-141.

[33] Isaev, A.P.: Twisted Yang-Baxter equations for linear quantum (super) groups, Journ. Phys. A: Math. Gen. 29 (1996) 6903-6910, q-alg/9511006.

[34] Hadjiivanov, L.K., Isaev, A.P., Ogievetsky, O.V., Pyatov, P.N. and Todorov, I.T.: Hecke algebraic properties of dynamical \(R\)-matrices. Application to related matrix algebras, Journ. Math. Phys. 40 (1999) 427-448, q-alg/9712026.

[35] Bateman, H. and Erdelyi, A.: Higher Transcendental Functions, vol. 1, McGraw-Hill Book Co., New York, 1953.
[36] Fulton, W.: *Young Tableaux: With Applications to Representation Theory and Geometry*, Cambridge Univ. Press, New York, 1997.

[37] Kac, V.G.: *Infinite Dimensional Lie Algebras*, Cambridge Univ. Press, 1990.

[38] Stanev, Ya.S. and Todorov, I.T.: On the Schwarz problem for the $\hat{su}_2$ Knizhnik-Zamolodchikov equation, Lett. Math. Phys. **35** (1995) 123-134.