Quantum Potential and Random Phase-Space Dynamics

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Abstract

We analyze limitations upon any kinetic theory inspired derivation of a probabilistic counterpart of the Schrödinger picture quantum dynamics. Neither dissipative nor non-dissipative stochastic phase-space processes based on the white-noise (Brownian motion) kinetics are valid candidates unless additional constraints (a suitable form of the energy conservation law) are properly incorporated in the formalism.

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Born’s statistical interpretation postulate in quantum theory may be viewed as a justification for seeking stochastic counterparts of the Schrödinger picture quantum dynamics, see e.g. [1] - [5] and references there-in. The pertinent random dynamics typically has been introduced and analyzed in the configuration space of the system, in terms of Markovian diffusion-type processes which are compatible both with the time evolution of the probability density $\rho = |\psi|^2$ and that of the wave function phase $S$ in the Madelung decomposition $\psi = \rho^{1/2} \exp(iS)$, the single-valuedness condition being implicit, [5].

In the corresponding hydrodynamical description of the Schrödinger dynamics, we encounter two local conservation laws: the continuity equation $\partial_t \rho = -\nabla \cdot (\mathbf{v} \rho)$ for the probability density $\rho$ and the Euler-type equation, controlling the space-time dependence of the current velocity $\mathbf{v}(\mathbf{x},t) = \frac{\hbar}{2m} \nabla S(\mathbf{x},y)$ (a gradient of the modified Hamilton-Jacobi equation for $S$):

$$ (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{m} \mathbf{F} - \nabla Q_q $$

where $Q_q = -\frac{\hbar^2}{2m^3} \frac{\Delta \rho^{1/2}}{\rho^{1/2}}$ is the familiar de Broglie-Bohm quantum potential, [8]. By $\mathbf{F}$ we indicate the external force field acting upon particles which in the conservative case coincides with $-\nabla V$ for a suitable potential $V(\mathbf{x})$, while the non-conservative case in our further discussion will be restricted to the Lorentz force example $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ for charge $e$.

The de Broglie-Bohm potential $Q_q$ is usually interpreted to have a pure quantum origin, hence one may be tempted to lend a status of an amusing curiosity to the fact that pressure-type potentials
of the very same (de Broglie-Bohm) functional form notoriously appear in the so-called moment equations (local conservation laws) associated with certain kinetic partial differential equations, like e.g. the Fokker-Planck-Kramers equation for the phase-space random transport.

Indeed, hydrodynamical conservation laws for dissipative and non-dissipative stochastic phase-space processes give rise to Euler-type and respective Hamilton-Jacobi equations where the pertinent contribution appears in the characteristic form \( \pm \nabla \left[ 2d^2(t)\frac{\Delta \rho^{1/2}}{\rho^{1/2}} \right] \). [7]

In particular, the free Brownian motion is known \([8, 3]\) to induce the current velocity \( \vec{v} = -D \nabla \rho \) which obeys the continuity equation (that trivially yields \( \partial_t \rho = D \Delta \rho \)) and the local momentum conservation law

\[
(\partial_t + \vec{v} \cdot \nabla) \vec{v} = -2D^2 \nabla \frac{\Delta \rho^{1/2}}{\rho^{1/2}}
\]

where \( D \) is the diffusion constant (set formally \( D \equiv \hbar/2m \) for comparison with Eq. (1)). Notice the negative sign on the right-hand-side of that equation, to be compared with the corresponding quantum mechanical law.

This deceivingly minor sign issue locates the standard Brownian motion in the framework of so-called Euclidean quantum mechanics \([9]\), and stays behind the "Brownian recoil principle" idea of Refs. \([2, 4]\), which was a proposed recipe to circumvent the breakdown of microscopic momentum and energy conservation laws (a typical though not much disputed feature of the Brownian motion).

Inspired by the recent paper \([10]\) let us consider those issues from a broader kinetic theory perspective. We investigate circumstances (including various constraints) under which the classical phase-space kinetics can approximate and eventually may become transformed into a stochastic counterpart of the Schrödinger picture quantum dynamics.

To this end we need to analyze phase-space probability densities \( f(\vec{r}, \vec{v}, t) \) whose dynamics is governed by the kinetic equation of the form

\[
(\partial_t + \vec{v} \cdot \nabla \vec{v} + \frac{1}{m} \vec{F} \cdot \nabla \vec{v}) f = C(f)
\]

where \( m \) is a mass parameter, and the term \( C(f) \), in analogy with the standard kinetic theory reasoning \([12, 13]\), is a substitute for the so-called collision integral and encompasses all details about the environment (surrounding medium) action and reaction upon the propagated particle.

Evaluation of moment equations (local conservation laws) associated with the kinetic equation (3) relies on the concrete choice of \( C(f) \) and especially on the microscopically conserved quantities. In particular, the mass (probability) conservation directly follows from \( \int C(f) d^3u = 0 \). The system total momentum is known to be conserved in the force-free case, only if \( \int \vec{v}^2 C(f) d^3u = \vec{0} \).

To elucidate the particular role of those "collision invariants" let us consider specific substitutes for the collision integral, which impose limitations on the number of respected microscopic conservation
Case 1: \( \int C(f)d^3u = 0 \) only

Let us consider the standard Brownian motion in external conservative force fields, \[1\]. We know a priori that the external noise intensity is determined by a parameter \( q = D\beta^2 \) where \( D = \frac{kT}{m\beta} \), while the friction parameter \( \beta \) is given by the Stokes formula \( m\beta = 6\pi \eta a \). Consequently, the effect of the surrounding medium on the motion of the particle is described by two parameters: friction constant \( \beta \) and bath temperature \( T \). Assumptions about the asymptotic (equilibrium) Maxwell-Boltzmann distribution and the fluid reaction upon the moving particle are here implicit.

The resulting (Markov) phase-space diffusion process is completely determined by the transition probability density \( P(\vec{x}, \vec{u}, t| \vec{x}_0, \vec{u}_0, t_0) \), which is typically expected to be a fundamental solution of the Kramers equation, so that an initially given phase-space distribution \( f(\vec{x}_0, \vec{u}_0, t_0) \) is propagated according to:

\[
\left( \partial_t + \vec{u} \cdot \vec{\nabla} + \frac{P}{m} \cdot \vec{\nabla} \right) f = C(f) = \left( q\vec{\nabla}^2 + \beta \vec{u} \cdot \vec{\nabla} \right) f
\]

First of all let us notice that in the present case \( \int C(f)d^3u = 0 \), while \( \int -\vec{u}C(f)d^3u \neq 0 \).

Accordingly, the continuity equation holds true for the marginal (spatial) probability density \( \rho = \int f d^3u \) while \( \frac{1}{\rho} \int \vec{v}C(f)d^3u = -\beta \vec{v}(\vec{x}, t) \) where \( \vec{v} = \frac{1}{\rho} \int \vec{v} f d^3u \). That has a devastating effect on the form of the corresponding moment equation.

The large friction (Smoluchowski) regime of the above phase-space random dynamics is a classic \[1, 1\]. In fact, by following the traditional pattern of the hydrodynamical formalism, \[12, 13\], we easily infer the closed system of two (which is special to Markovian diffusions !) local conservation laws for the Smoluchowski process taking place in the configuration space, \[4, 14\]:

\[
\partial_t \rho + \vec{\nabla} \cdot (\vec{v} \rho) = 0 \tag{5}
\]

\[
(\partial_t + \vec{v} \cdot \vec{\nabla}) \vec{v} = \vec{\nabla} (\Omega - Q) \tag{6}
\]

where \( \vec{v}(\vec{x}, t) = \frac{P}{m\beta} - D\frac{\vec{\nabla} \rho}{\rho} \) defines the so-called current velocity of Brownian particles and, when inserted to the continuity equation allows to obtain the Fokker-Planck equation, \[1\]. Here, the volume force (notice the positive sign) +\( \vec{\nabla} \Omega \) instead of \( -\vec{\nabla} V \) as should have been to comply with Eq. (1), reads:

\[
\Omega = \frac{1}{2} \left( \frac{P}{m\beta} \right)^2 + D \vec{\nabla} \cdot \left( \frac{P}{m\beta} \right) \tag{7}
\]
while the pressure-type contribution $-\nabla Q$ explicitly involves, \[ Q = 2D^2 \Delta \rho^{1/2} \rho^{1/2} \] (8)

where $\Delta = \nabla^2$ is the Laplace operator.

Eq. (6) plays the role of the local momentum conservation law in the formalism. Let us however recall that $\vec{u}$ was not a microscopic "collision invariant" of the system. This fact, when combined with the large friction regime, enforces a marked difference in the local momentum conservation law in comparison with the standard Euler equation for a nonviscous fluid or gas:

$$ (\partial_t + \vec{v} \cdot \nabla) \vec{v} = -\nabla P \rho $$

(9)

where $P(\vec{x})$ stands for the pressure function (to be fixed by a suitable equation of state) and $\vec{F}$ is the very same (conservative $-\nabla V$) force acting upon particles as that appearing in the Kramers equation (3).

To have a glimpse of a dramatic difference between physical messages conveyed respectively by equations (6) and (9), it is enough to insert in (9) the standard equation of state $P(\vec{x}) = \alpha \rho^\beta$ with $\alpha, \beta > 0$ and choose $\vec{F} = -\omega^2 \vec{x}$ to represent the harmonic attraction in Eqs. (2) - (9), see also \[4\].

**Comment 1:** In view of the breakdown of microscopic momentum and energy conservation laws in the considered dissipative Brownian framework, one needs supplementary procedures that would compensate those failures on the level of local conservation laws. Markovian diffusion processes with the inverted sign of $\nabla (Q - \Omega)$ in the local momentum conservation law (6) i. e. respecting

$$ (\partial_t + \vec{v} \cdot \nabla) \vec{v} = \nabla (Q - \Omega) $$

(10)

instead of Eq. (6), were considered in Ref. \[4\] as implementations of the "third Newton law in the mean" and could have been related to the Schrödinger-type dynamics according to:

$$ i\partial_t \psi = -D\Delta \psi + \frac{1}{2mD} \Omega \psi, $$

(11)

see e.g. Refs. \[4, 14\]. Nonetheless, also under those premises, the volume force term $-\nabla \Omega$ in Eq. (11) does not in general coincide with the externally acting conservative force contribution (e.g. acceleration) $\frac{1}{m} \vec{F} = -\frac{1}{m} \nabla V$ akin to Eqs. (9) or (1).

**Comment 2:** Effects of external force fields acting upon particles are significantly distorted while passing to the local conservation laws in the large friction (Smoluchowski) regime. That becomes even more conspicuous in case of the Brownian motion of a charged particle in the constant magnetic field. In the Smoluchowski (large friction) regime, friction completely smothes out any rotational (due to the Lorentz force) features of the process. In the corresponding local momentum conservation law
there is no volume force contribution at all and merely the "pressure-type" potential $Q$ appears in a rescaled form, \[ Q = \frac{\beta^2}{\beta^2 + \omega_c^2} \cdot 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \] where $\beta$ is the (large) friction parameter and $\omega_c = \frac{eB}{m}$ is the rotational frequency of the charge $q_e$ particle in a constant homogeneous magnetic field $\overrightarrow{B} = (0, 0, B)$. Clearly, for moderate frequency values $\omega_c$ (hence the magnetic field intensity) and sufficiently large $\beta$ even this minor scaling remnant of the original Lorentz force would effectively disappear, yielding the free Brownian dynamics (2).

The last observation should be compared with results of Refs. [16] where frictionless stochastic processes were invoked to analyze situations present in magnetospheric environments. Specifically, one deals there with charged particles in a locally uniform magnetic field which experience stochastic electrical forcing. In the absence of friction, the rotational Lorentz force input should clearly survive when passing to the local conservation laws, in plain contrast with the Smoluchowski regime. By disregarding friction it is also possible to reproduce exactly the conservative external force acting upon particles in the local conservation laws, as originally suggested by [1].

**Case 2:** $\int C(f)d^3u = 0$ and $\int \overrightarrow{u}C(f)d^3u = \overrightarrow{0}$

Let us consider the frictionless phase-space dynamics in some detail. Clearly, that corresponds to dropping the frictional ($\beta$-dependent) contribution in the right-hand-side of Eq. (4). We immediately realize that the previous obstacle pertaining to the "collision invariant" $\overrightarrow{u}$ disappears. Indeed, now $\int \overrightarrow{u}C(f)d^3u = \overrightarrow{0}$.

Let us discuss the non-dissipative random dynamics in two phase-space dimensions instead of six. The motion is governed by a transition density $P(x, u, t|x_0, u_0, t_0)$ which uniquely defines the corresponding time homogeneous phase-space Markovian diffusion process. Here, the function $P$ is the fundamental solution of the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -um\frac{\partial P}{\partial x} + q\frac{\partial^2 P}{\partial u^2} \tag{13}$$

in the form first given by Kolmogorov [17]:

$$P(x, u, t|x_0, u_0, t_0 = 0) = \frac{1}{2\pi t} \exp \left[ -\frac{(u - u_0)^2}{4qt} - \frac{3(x - x_0 - \frac{u + u_0}{2}t)^2}{qt^3} \right]. \tag{14}$$

Here $q$ stands for the noise intensity parameter which may take an arbitrary non-negative value (in contrast to the dissipative case where fluctuation-dissipation relations set a connection of $q$ with $\beta$ and $T$).
We are interested in passing to a hydrodynamical picture, following the traditional recipes \[12, 13\]. To this end we need to propagate certain initial probability density and investigate effects of the random dynamics. Let us choose most obvious (call it natural) example of:

\[
 f_0(x, u) = \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini})^2}{2a^2}\right) \left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2b^2}\right).
\]  

so that at time \(t\) we have

\[
 f(x, u, t) = \int P(x, u, t | x_0, u_0, t_0 = 0) f_0(x_0, u_0) \, dx_0 du_0.
\]

Since \(P(x, u, t | x_0, u_0, t_0)\) is the fundamental solution of the Kramers equation, the joint density \(f(x, u, t)\) is also the solution and can be written in the familiar, [11], form of

\[
 W(R,S) =  \left(\frac{1}{4\pi^2(eg - h^2)}\right)^{\frac{1}{2}} \exp\left[-\frac{gR^2 - 2hRS + eS^2}{2(eg - h^2)}\right]
\]

for \(f(x, u, t) = W(R, S)\). However, in the present case functional entries are adopted to the frictionless motion and read as follows:

\[
 S = u - u_{ini} \\
 R = x - x_{ini} - u_{ini}t \\
 e = a^2 + b^2t^2 + \frac{2}{3}qt^3 \\
 g = b^2 + 2qt \\
 h = b^2t + qt^2.
\]

The marginals \(\rho(x, t) = \int f(x, u, t) \, du\) and \(\rho(u, t) = \int f(x, u, t) \, dx\) are

\[
 \rho(x, t) = \left(\frac{1}{2\pi f}\right)^{\frac{1}{2}} \exp\left(-\frac{R^2}{2f}\right) = \left(\frac{1}{2\pi (a^2 + b^2t^2 + \frac{2}{3}qt^3)}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini} - u_{ini}t)^2}{2(a^2 + b^2t^2 + \frac{2}{3}qt^3)}\right)
\]

and

\[
 \rho(u, t) = \left(\frac{1}{2\pi g}\right)^{\frac{1}{2}} \exp\left(-\frac{S^2}{2g}\right) = \left(\frac{1}{2\pi (b^2 + 2qt)}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2(b^2 + 2qt)}\right)
\]

Let us introduce an auxiliary (reduced) distribution

\[
 \tilde{W}(S|R) = \frac{W(S, R)}{\int W(S, R) \, dS} = \left(\frac{1}{2\pi \left( g - \frac{h^2}{e} \right)}\right)^{\frac{1}{2}} \exp\left(-\frac{|S - \frac{b}{2} R|^2}{2(g - \frac{h^2}{e})}\right)
\]
where in the denominator we recognize the marginal spatial distribution $\int W(S,R) dS = \rho$.

Following the standard hydrodynamical picture method \cite{12, 13, 14} we define local (configuration space conditioned) moments: $\langle u \rangle_x = \int u \tilde{W} du$ and $\langle u^2 \rangle_x = \int u^2 \tilde{W} du$. From (20) it follows that

$$\langle u \rangle_x = u_{ini} + \frac{h}{e} R = u_{ini} + \frac{b^2 t + qt^2}{a^2 + b^2 t^2 + \frac{2}{3} qt^3} [x - x_{ini} - u_{ini} t] \quad (21)$$

$$\langle u^2 \rangle_x - \langle u \rangle_x^2 = \left( g - \frac{h^2}{e} \right) = \frac{qt^3 \left( 2b^2 + qt \right) + 3a^2 \left( b^2 + 2qt \right)}{3a^2 + t^2 (3b^2 + 2qt)} \quad (22)$$

The continuity (0-th moment) and the momentum conservation (first moment) equations come out in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\langle u \rangle_x \rho) = 0 \quad (23)$$

$$\frac{\partial}{\partial t} (\langle u \rangle_x \rho) + \frac{\partial}{\partial x} (\langle u^2 \rangle_x \rho) = 0 \quad (24)$$

These equations yield the local momentum conservation law in the form (set $v(x,t) = \langle u \rangle_x$):}

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \frac{\partial}{\partial x} \right) \langle u \rangle_x = -\frac{1}{w} \frac{\partial P_{kin}}{\partial x}$$

$$\quad (25)$$

where we encounter the standard \cite{13} textbook notion of the pressure function

$$P_{kin}(x,t) = \left[ \langle u^2 \rangle_x - \langle u \rangle_x^2 \right] w(x,t). \quad (26)$$

The marginal density $\rho$ obeys $\nabla \rho \frac{\Delta \rho^{1/2}}{\rho^{1/2}}$ and that in turn implies

$$-\frac{1}{w} \frac{\partial P_{kin}}{\partial x} = 2 \left( eg - h^2 \right) \nabla \left[ \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \right]. \quad (27)$$

As a consequence, the local conservation law takes the form (we ultimately set $<u>_{x,t} = v$):

$$\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v = -\nabla P_{kin} \rho = 2 \left( eg - h^2 \right) \nabla \left[ \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \right] = + \nabla Q \quad (28)$$

where (we point out the plus sign in the above, see e.g. Eq. (10))

$$eg - h^2 = a^2 b^2 + 2a^2 qt + \frac{2}{3} b^2 qt^3 + \frac{1}{3} q^2 t^4 \equiv D^2(t) \quad (29)$$

and by adopting the notation $D^2(t) = eg - h^2$ we get $-\frac{1}{\rho} \frac{\partial P_{kin}}{\partial x} = + \nabla Q$ with the functional form of $Q(x,t)$ given by Eq. (8). Here, instead of a diffusion constant $D$ we insert the (positive) time-dependent function $D(t)$.
With those notational adjustments, we recognize in Eq. (28) a consistent Euler form of the local momentum conservation law, in case of vanishing volume forces (c.f. Eqs. (6), (8), (10) for comparison).

A carefully executed, tedious calculation allows to demonstrate, [7], that an analogous result holds true in case of a harmonic attraction and for a nonconservative example of the Lorentz force in action. Both volume forces appear undistorted (that was not the case in the large friction regime) in the corresponding local momentum conservation laws. Indeed, we recover a universal relationship:

\[
\left[ \partial_t + \vec{v} \cdot \nabla \right] \vec{v} = \frac{\vec{F}}{m} + 2d^2(t) \nabla \left[ \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \right] 
\]

where \( \vec{F} \) denotes external force acting on the particle, and \( d = (\det C)^{1/2} \) where \( C \) is the covariance matrix of random variables (vectors) \( \vec{S} \) and \( \vec{R} \) (defined for each system) and \( n \) stands for the dimension of configuration space of an appropriate system.

1. free particle: \( \vec{R} = x, F \equiv 0, n = 1 \)
2. charged particle in a constant magnetic field: \( \vec{R} = (x, y), \vec{F} = e \langle \vec{w} \rangle_\varphi \times \vec{B}, n = 2 \)
3. harmonically bound particle: \( \vec{R} = x, F = -m\omega^2 x, n = 1 \)

In case of harmonic and magnetic confinement, we need to have identified parameter range regimes that allow for a positivity of the time dependent coefficient \( d(t) \equiv D(t) \) (in the force-free case it is positive with no reservations), in the pressure-type contribution acquiring a characteristic form of \( -\nabla_w \varphi \omega^2 = +\nabla Q \). Indeed, only by means of a proper balance between \( q \) and \( \omega_c \) we can achieve a positivity of the coefficient \( d^2 \) in case of the charged particle in a magnetic field:

\[
d^2(t) = eg - h^2 - k^2 = a^2 b^2 - 8q^2 + \frac{4 b^2 q t}{\omega_c^2} + \frac{8 q^2 t^2}{\omega_c^2} + 2a^2 q t + \frac{8 q^2}{\omega_c^2} \cos(t \omega_c) - \frac{4 b^2 q}{\omega_c^3} \sin(t \omega_c).
\]

The time-dependent coefficient for the frictionless harmonic attraction reads:

\[
d^2(t) = eg - h^2 = \frac{-q^2 + 2 b^2 q t \omega^2 + 2 q^2 t^2 \omega^2 + 2a^2 b^2 \omega^4 + 2a^2 q t \omega^4 + q^2 \cos(2 t \omega) + q \omega (-b^2 + a^2 \omega^2) \sin(2 t \omega)}{2 \omega^4}
\]

and a proper balance between \( q \) and \( \omega \) needs to be maintained again to secure a positivity of \( d^2(t) \).

In Ref. [7] we have investigated the above expressions in the low noise intensity regime and for rather short duration times of the pertinent stochastic processes. That was motivated by the major conceptual input of [16, 7] that undamped random flights in external force fields may have physical
relevance when dissipative time scales are much longer than the time duration of processes of interest, including the particle life-time. Under those premises, we can view the noise intensity parameter $q$ as a book-keeping label and investigate leading terms in all expressions encompassing small $q$/short time $t$ effects in the hitherto considered random dynamics. In particular, it is obvious that by neglecting all $q$-dependent terms we readily arrive at the leading term: $d^2(t) \rightarrow a^2b^2 \equiv D^2$. After adopting this notation, Eq. (31) acquires a conspicuous quantum mechanical form in the leading order of $q$-dependent series expansion.

To elucidate the previous observation, let us explicitly associate the $q \ll 1$ local momentum conservation law with that known to be appropriate for quantum harmonic oscillator. We impose the following condition on the dispersion parameters $a = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $b = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$ of initial distributions $\rho_0(x)$ and $\rho_0(v)$ respectively

$$d^2(0) = a^2b^2 \equiv \left(\frac{\hbar}{2m}\right)^2.$$  

This condition is obviously equivalent to imposing a priori the Heisenberg uncertainty relation $a(mb) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sqrt{\langle mv^2 \rangle - \langle mv \rangle^2} = \frac{\hbar}{2}$. Upon an additional demand $b^2 = a^2\omega^2$ which is an identity for the choice of $b^2 = \frac{\hbar}{2m\omega^2}$, we recover

$$\rho(x,t) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{3}{2}} \exp \left(\frac{m\omega}{\hbar} (x - x_{ini} \cos \omega t)^2\right)$$

which directly corresponds to the quantum evolution of the coherent state of the harmonic oscillator.

Let us point out that the Planck constant entered the formalism through an additional assumption about an initial form (widths) of the phase-space probability density, see Eq. (33).

To conclude, we have demonstrated that neither large friction nor frictionless cases may be regarded as valid probabilistic phase-space motion ancestors for a consistent stochastic counterpart of the quantum Schrödinger picture dynamics.

The formal reason for those failures is rooted in violations of the microscopic conservation laws as expressed through the non-vanishing "collision integrals". However, the frictionless case appears to be sufficiently close (at least in the small $q$ and short duration time $t$ regime) to the quantum dynamics in its hydrodynamical description. Large friction dynamics, even if augmented by the concept of the "Brownian recoil principle", is incapable of reproducing correctly the external force effects in the local momentum conservation laws.

An interesting point, worth further exploration, is that in the frictional case only "collision invariant" has been respected, while in case of the frictionless dynamics, two basic "collision invariants" were in usage. It is rather obvious that in case of the Brownian motion inspired kinetics, the "collision integral" $\int v^2 C(f)d^3u$ is non-vanishing. Therefore, any kinetic framework exploiting $\overline{v^2}$ or its analog
(cf. [13] as a microscopically conserved quantity (that is the case in Ref. [10], albeit this restriction is not used at all in major derivations) must definitely depart from the "plain" white noise scenario. The usage of additional constraints that are capable of compensating violations of the microscopic energy conservation law, is unavoidable.

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