STRONGLY-FIBRED ITERATED FUNCTION SYSTEMS AND THE BARNESLEY–VINCE TRIANGLE

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ABSTRACT. We review the theory of semiattractors associated with non-contractive Iterated Function Systems (IFSs) and demonstrate its applications on a concrete example. In particular, we present criteria for the existence of semiattractors due to Lasota and Myjak. We also discuss the Kieninger criterion which allows us to characterise when a semiattractor is strongly-fibred. Finally, we consider a specific example of a non-contractive IFS introduced by Barnsley and Vince. We find an invariant measure for this system which allows us to describe its semiattractor. The difficulty in analysing this IFS stems from the fact that it is neither eventually contractive nor contractive on average.

1. Introduction

Iterated function systems (IFS) provide a general framework for generating fractals. Traditionally, an IFS consists of a finite set of Banach contractions acting on a complete metric space. Almost from the beginning of the modern theory of IFSs, introduced by Hutchinson in 1981 and later developed by a number of researches (see [18] and the references therein for an overview), it was allowed for a possibility that at least some maps comprising an IFS were not contractive. The most widely accepted approach to study IFSs is to switch from the dynamics on sets to the dynamics on measures and consider the IFSs that are contractive on average, see for example, [2]. In this setting, invariant and asymptotically stable measures play the roles of invariant sets and attractors, respectively. Lasota and Myjak provided a unified framework to study IFSs which are contractive on average by developing the theory of semiattractors, see [16, 17]. Meanwhile, the concept of topological contractivity has started to crystallize in various works, see for example [10, 12], culminating in the Kieninger’s classification of invariant sets as point-fibred or strongly-fibred, see [13]. Very recently, point-fibred minimal invariant sets turned out to be attractors of IFSs consisting of weak contractions (which are not very far from Banach contractions, see for instance [11, 20]). On the other hand, strongly-fibred minimal invariant sets that are not point-fibred cannot be attractors of (weakly) contractive IFSs. However, such sets can be attractors of IFSs for which the chaos game algorithm works (see [3, 7, 6]). In order to avoid restricting the whole space on which an IFS acts to an invariant set, one often needs to extend the concept of an attractor. A proper natural extension is the concept of a semiattractor, as illustrated in this paper by analysing an example of a strongly-fibred triangle introduced by Barnsley and Vince, see [5].

2010 Mathematics Subject Classification. 28A80.
Key words and phrases. semiattractor, invariant measure, iterated function system, strongly-fibred IFS.
2. Preliminaries

In this section we will introduce the notion of an iterated function system (IFS) together with its probabilistic extension. We will also define a strict attractor, a semiattractor, and an invariant probability measure of an IFS.

Throughout this paper $X$ stands for a Polish space. However, let us note that many results stated here, namely those which do not involve measures and metrics, are valid for general topological spaces. In that setting one only needs to replace Hausdorff convergence with Vietoris convergence; e.g., [3].

The Lipschitz constant of $f : X \to X$ is denoted by $\text{Lip}(f)$. The lower Kuratowski limit of a sequence of sets $S_n \subseteq X$, $n \geq 1$, is the set

$$\text{Li} S_n = \{ y \in X : y = \lim_{n \to \infty} x_n \text{ for some sequence } x_n \in S_n \};$$

see [8]. By $\mathcal{M}_1(X)$ we denote the space of Borel probability measures $\mu$ on $X$. The support of $\mu$ is denoted by $\text{supp} \mu$.

An iterated function system (IFS) $F$ is a system of continuous maps $f_i : X \to X$, $i \in I$, where $I$ is a finite indexing set. We write $F = (X; f_i : i \in I)$. The associated Hutchinson operator $F : 2^X \to 2^X$ is given by

$$F(S) = \text{cl} \bigcup_{i \in I} f_i(S)$$

for $S \subseteq X$, where $2^X$ stands for the power set of $X$ and $\text{cl}$ denotes the topological closure of a set. To simplify notation we write $F(x)$ instead of $F(\{x\})$.

A probabilistic IFS $F_p = (X; (f_i, p_i) : i \in I)$ is the IFS $F = (X; f_i : i \in I)$ together with a vector of probabilities $(p_i)_{i \in I}$, $p_i \in [0,1]$, $\sum_{i \in I} p_i = 1$. The associated Markov operator $M : \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ is given by

$$M\mu = \sum_{i \in I} p_i \cdot \mu \circ f_i^{-1}$$

for $\mu \in \mathcal{M}_1(X)$, where $\mu \circ f_i^{-1}$ is the push-forward measure through $f_i$.

We are now ready to define a strict attractor ([3, 5]) and a semiattractor ([17, 21]).

Definition 2.1. Let $F = (X; f_i : i \in I)$ be an IFS and $F$ the associated Hutchinson operator; $F^n$ stands below for the $n$-fold composition of $F$. A nonempty closed set $A_* \subseteq X$ is

(i) an invariant set if $F(A_*) = A_*$;

(ii) a minimal invariant set if $A_*$ is an invariant set and there is no nonempty closed invariant set $C$ that is a proper subset of $A_*$, i.e., $C \subseteq A_*$;

(iii) a strict attractor if there exists an open $U \supseteq A_*$ such that

$$F^n(S) \to A_* \text{ for all nonempty compact } S \subseteq U$$

where the convergence takes place with respect to the Hausdorff metric (see [3, 14]); the largest open $U$ in (1) is called the basin of $A_*$ and denoted by $B(A_*)$;

(iv) a semiattractor, when $\bigcap_{x \in X} \text{Li} F^n(x) = A_*$. 

By an attractor (in a given sense) of the probabilistic IFS $F_p$, we understand an attractor of the underlying IFS $F$, regardless of the choice of probabilities. It is worth noting that, in general, the semiattractor may be unbounded, see [16, Example 6.2].
The relation between various kinds of attractors is summarized in the next theorem.

**Theorem 2.2** (21 [17, 4]). Let $\mathcal{F} = (X; f_i : i \in I)$ be an IFS and $A_\ast \subseteq X$ a nonempty closed set.

(i) A strict attractor $A_\ast$ is a compact invariant set which is unique within its basin $\mathcal{B}(A_\ast)$.

(ii) A semiattractor $A_\ast$ of $\mathcal{F}$ is the smallest invariant set, i.e., $A_\ast \subseteq C$ for every nonempty closed invariant set $C$. Even more, $A_\ast \subseteq C$ for every nonempty closed set $C$ with $F(C) \subseteq C$.

(iii) If $A_\ast$ is a strict attractor with basin $\mathcal{B}(A_\ast)$, then $F(\mathcal{B}(A_\ast)) \subseteq \mathcal{B}(A_\ast)$ and $A_\ast$ is a compact semiattractor of the restricted IFS $\mathcal{F}|\mathcal{B}(A_\ast) = (\mathcal{B}(A_\ast); f_i|_{\mathcal{B}(A_\ast)} : i \in I)$.

(iv) If $A_\ast$ is a compact semiattractor of $\mathcal{F}$, then $A_\ast$ is a strict attractor of the restricted IFS $\mathcal{F}|A_\ast = (A_\ast; f_i|_{A_\ast} : i \in I)$.

Let us now introduce the notion of an invariant measure for an IFS (see [15, 16, 17, 14, 22]).

**Definition 2.3.** Let $\mathcal{F}_p$ be a probabilistic IFS and $M$ the associated Markov operator; $M^n$ stands below for the $n$-fold composition of $M$. We say that a probabilistic measure $\mu_\ast \in M_1(X)$ is

(i) an invariant measure, when $M\mu_\ast = \mu_\ast$,

(ii) an asymptotically stable measure, when $M^n\mu \rightarrow \mu_\ast$ for every probabilistic measure $\mu \in M_1(X)$, where $\rightarrow$ denotes the weak convergence of measures (see [9]).

In case (ii) we say that $M$ itself is asymptotically stable.

Observe that any asymptotically stable measure is a unique invariant measure. The reason for this is that Markov operators induced by probabilistic IFSs are weakly continuous, see [21, Section 3 p.483].

### 3. Existence of Semiattractors

In this section we provide two existence criteria for semiattractors using the higher iterates of an IFS. The first criterion is purely topological one.

**Theorem 3.1.** Let $\mathcal{F} = (X; f_i : i \in I)$ be an IFS. If, for some positive integer $k$, $A_\ast$ is a semiattractor of $\mathcal{F}^k = (X; f_{ij} : j \in I^k)$, then $A_\ast$ is a semiattractor of $\mathcal{F}$.

**Proof.** Let us first show that

\[ \bigcap_{x \in X} \text{Li} F^n(x) \supseteq A_\ast \neq \emptyset. \]  

(2)

Denote by $G$ the Hutchinson operator induced by the IFS $\mathcal{F}^k$. Then $G = F^k$. Let $z \in A_\ast = \bigcap_{x \in X} \text{Li} G^n(x)$. Fix also $x \in X$. For every $0 \leq j \leq k - 1$ there exists a sequence

\[ y_n^{(j)} \in G^n(y), \quad \text{where } y = f_i^1(x), \]

such that $z = \lim_{n \to \infty} y_n^{(j)}$. (Conventionally $f_i^0(x) = x$.) Putting

\[ x_{nk+j} := y_n^{(j)} \in F^{nk+j}(x) \quad \text{for } 0 \leq j \leq k - 1, n \geq 1, \]
yields $z = \lim_{n \to \infty} x_n, x_n \in F^n(x)$. Therefore $z \in \text{Li } F^n(x)$. Being $x \in X$ arbitrary, $z \in \bigcap_{x \in X} \text{Li } F^n(x)$.

At this stage, by (2), we know that $F$ admits a semiattractor. To finish the proof it is enough to prove the inclusion that is opposite to (2). Indeed, we have

$$\bigcap_{x \in X} \text{Li } F^n(x) \subseteq \bigcap_{x \in X} \text{Li } F^{kn}(x) \subseteq \bigcap_{x \in X} \text{Li } G^n(x) = A_\ast.$$  

□

Using the above theorem, we can now improve the Lasota–Myjak criterion [17, Theorem 6.1] for the existence of semiattractors and asymptotically stable measures.

**Theorem 3.2 (Higher iterate Lasota–Myjak criterion).** Let $F_p = (X; (f_i, p_i) : i \in I)$ be a probabilistic IFS with positive probabilities $p_i > 0$ that consists of Lipschitz maps $f_i$. Let $F_p^k = (X; (f_{i_k} p_k) : k \in I^k)$, $k \geq 1$, be its $k$th iterate, where $i_k = (i_1, \ldots, i_k)$, $f_{i_k} = f_{i_1} \circ \cdots \circ f_{i_k}$, $p_k = \prod_{l=1}^k p_{i_l}$. Suppose that $F_p^k$ is average contractive, i.e.,

$$\sum_{i \in I^k} p_i \cdot \text{Lip}(f_i) < 1.$$  

Then we have

(a) $F_p$ and $F_p^k$ admit semiattractors and these semiattractors coincide;

(b) the Markov operator induced by $F_p^k$ coincides with $M^k$ where $M$ is the Markov operator induced by $F_p$ and both operators are asymptotically stable and share the same unique invariant measure; moreover, the invariant measure supports the semiattractors in (a).

**Proof.** Part (a). By the average contractivity, there exists at least one $i_k \in I^k$ such that $\text{Lip}(f_{i_k}) < 1$. Then by [17, Theorem 3.2] (which can be also found as [21, Theorem 6.3]) we know that $F_p^k$ admits a semiattractor. (Note that we could have used [17, Theorem 6.1] as well.) This semiattractor is also the semiattractor of $F_p$ according to Theorem 3.1.

Part (b). It is readily seen that $M^k$ is the Markov operator of $F_p^k$, when $M$ is the Markov operator of $F_p$. Further, by the average contractivity, $M^k$ is asymptotically stable according to [15, Proposition 12.8.1] (which can be also found as [21, Fact 3.2]). Thus $M^k$ admits a unique invariant measure $\mu_\ast$ supporting the semiattractor of $F_p^k$, see [17, Theorem 6.1]. Hence $M$ is asymptotically stable with $\mu_\ast$ as its invariant measure, due to the standard trick concerning contractive fixed points of higher iterates, see [18, Remark 1.2(3)]. As before, $\mu_\ast$ supports the semiattractor of $F_p$. □

**Remark 3.3.** Part (a) of Theorem 3.2 can be deduced from the proof of Part (b).

Later on we will apply the above criterion when $X = \mathbb{R}^2$ and the maps comprising an IFS are affine. Therefore, let us recall how one can compute the Lipschitz constant $\text{Lip}(f)$ for an affine map $f$ on $\mathbb{R}^d$ equipped with the 2-norm, $\| \cdot \|_2$. If $f(x) = Ax + B$ for $x \in \mathbb{R}^d$, then

$$\text{Lip}(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|_2}{\|x - y\|_2} = \|A\|_2 = \sigma_{\text{max}}(A),$$  

where $\sigma_{\text{max}}(A)$ denotes the maximum singular value of $A$. 

where $\sigma_{\text{max}}(A)$ is the maximum singular value of $A$, see, for example, [19, Chap. 5.2].

4. STRONGLY-FIBRED IFS

We introduce the classification of invariant sets of IFSs due to B. Kieninger, see [13, 5].

Let $A_* \subseteq X$ be a compact minimal invariant set (strict attractor, semiattractor) of an IFS $F$. We define a coding map $\pi : I^\infty \to 2^{A_*}$ by the formula

$$\pi(i_1i_2\ldots) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(A_*) \quad \text{for} \quad (i_n)_{n=1}^{\infty} \in I^\infty.$$  

A set $\pi(i_1i_2\ldots)$ is called a fibre of $A_*$ with the address $i_1i_2\ldots$; it is a nonempty compact subset of $A_*$. In consequence, $A_*$ is a union of fibres,

$$A_* = \pi(I^\infty) = \bigcup_{(i_n)_{n=1}^{\infty} \in I^\infty} \pi(i_1i_2\ldots).$$

Let us remark that if $A_*$ is a strict attractor of $F$ and we replace in (3) $A_*$ with any compact set $C \supseteq A_*$ such that $F(C) \subseteq C \subseteq B(A_*)$, then the resulting set will coincide with the fibre as defined in (3), see [3, Proposition 1]. Note that in [13], the author considers more general framework and he takes $C = X$ where $X$ is a compact Hausdorff space.

**Definition 4.1.** We say that a compact minimal invariant set $A_*$ of an IFS $F$ is:

(i) **point-fibred**, if all its fibres are singletons;
(ii) **strongly-fibred**, if for every $a \in A_*$ and open $V \ni a$ there exists $(i_n)_{n=1}^{\infty} \in I^\infty$ such that $\pi(i_1i_2\ldots) \subseteq V$.

From the above definition, it is clear that a minimal invariant set which is point-fibred, is also strongly-fibred. Further, let us note that if an IFS is weakly contractive, then its strict attractor is point-fibred. And vice-versa, if $A_*$ is a point-fibred minimal invariant set of the IFS $F$, then $F$ can be remetrized to an IFS of weakly contractive maps on $A_*$, see [1, 20]. For a systematic account of basic types of weakly contractive IFSs we refer the reader to [18]. An IFS with a strongly-fibred but not point-fibred minimal invariant set is not weakly contractive under any remetrization. For recent generalizations of strong-fibredness we refer the reader to [7] and [11].

**Theorem 4.2 (The Kieninger criterion).** Let $F = (X; f_i : i \in I)$ be an IFS. Let $A_*$ be either a compact semiattractor or a strict attractor of $F$. Suppose that $A_*$ admits a singleton fibre, i.e., there exists $(i_n)_{n=1}^{\infty} \in I^\infty$ such that $\pi(i_1i_2\ldots) = \{a_*\}$ for some $a_* \in A_*$. Then $A_*$ is strongly-fibred.

**Proof.** Fix $a \in A_*$ and open $V \ni a$. Recall that $F^m(a_*) \to A_*$ when $m \to \infty$ (using Theorem 2.2 (iv), when $A_*$ is a semiattractor). Hence $F^m(a_*) \cap V \neq \emptyset$ for sufficiently large $m \geq 1$. Observe that

$$F^m(a_*) = \{f_{j_1} \circ \cdots \circ f_{j_m}(a_*): j_1, \ldots, j_m \in I\}.$$
Therefore there exist \( j_1, \ldots, j_m \in I \) such that \( f_{j_1} \circ \cdots \circ f_{j_m}(a_*) \in V \). Finally, we arrive at

\[
\pi(j_1 \ldots j_m i_1 i_2 \ldots) = \bigcap_{n=1}^{\infty} f_{j_1} \circ \cdots \circ f_{j_m} \circ f_{i_1} \circ \cdots \circ f_{i_n}(A_*)
\]

\[
= f_{j_1} \circ \cdots \circ f_{j_m} \left( \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(A_*) \right)
\]

\[
= f_{j_1} \circ \cdots \circ f_{j_m}(\{a_*\}) \subseteq V.
\]

To pass from (4) to (5), we note that the sets are nested and compact. The proof is complete as \( V \) was an arbitrary open set with \( V \cap A_* \neq \emptyset \).

Note that the above criterion is valid for a general topological space \( X \). It could be deduced directly from \cite[Proposition 4.3.16]{13} in the case where the singleton fibre \( \pi(i_1 i_2 \ldots) \) has a constant address, \( i_1 = i_2 = \ldots \). For a metric space \( X \) the reverse criterion is true and is due to E. Matias (see \cite[Proposition 4.1, Acknowledgements]{6}): If \( A_* \) is an invariant set which is strongly-fibred, then it admits at least one singleton fibre.

5. Strongly-fibred triangle

In this section an IFS which was introduced in \cite{5} will be analysed from the perspective of the Lasota–Myjak theory as discussed in Section 3.

Let \( \Delta \subseteq \mathbb{R}^2 \) be the filled triangle with vertices \((0,0), (1,0)\) and \((0,1)\) on the Euclidean plane. Let \( f_i : \mathbb{R}^2 \to \mathbb{R}^2, \ f_i(x, y) = A_i \left( \frac{x}{y} \right) + \left( \begin{array}{c} 0 \\ i-1 \end{array} \right) \) for \((x, y) \in \mathbb{R}^2, i = 1, 2\), where

\[
A_1 = \left( \begin{array}{cc} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right), \quad A_2 = \left( \begin{array}{cc} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{array} \right).
\]

Clearly, \( \Delta \) is an invariant set for \( \mathcal{F} \), i.e., \( \Delta = f_1(\Delta) \cup f_2(\Delta) \), see Figure 1. However, as it has been stated in \cite{5}, the IFS \( \mathcal{F} = (\mathbb{R}^2; f_1, f_2) \) does not admit a strict attractor. On the other hand, its restriction \( \mathcal{F}_\Delta = (\Delta; f_1, f_2) \) has \( \Delta \) as a strict attractor which is strongly-fibred.

![Figure 1](image-url)

**Figure 1.** The action of \( f_1 \) and \( f_2 \) on the strongly-fibred triangle \( \Delta \). \( f_1 \) maps the vertices of \( \Delta \) as follows: \((0,0) \mapsto (0,0), (1,0) \mapsto (1,0), (0,1) \mapsto (1/2, 1/2)\). \( f_2 \) maps the vertices of \( \Delta \) as follows: \((0,0) \mapsto (0,1), (1,0) \mapsto (0,0), (0,1) \mapsto (1/2, 1/2)\).
We will now show that for \( f \) since both maps let us show that \( F \). Hence we have \( \mu(B) = \frac{\mathcal{L}(B \cap \Delta)}{\mathcal{L}(\Delta)} = 2\mathcal{L}(B \cap \Delta) \) for Borel \( B \subseteq \mathbb{R}^2 \),

is the unique invariant measure of the probabilistic IFS \( \mathcal{F}_p = (\mathbb{R}^2; (f_i, p_i) : i \in \{1, 2\}) \) with \( p_1 = p_2 = \frac{1}{2} \).

First we show that \( \mu \) is an invariant probability measure of \( \mathcal{F}_p \). Let \( B \) be a Borel subset of \( \mathbb{R}^2 \). Let us write \( B \) as \( B = B_1 \cup B_2 \cup B_3 \) where \( B_1 \subseteq f_1(\Delta) \), \( B_2 \subseteq f_2(\Delta) \) and \( B_3 \subseteq \mathbb{R}^2 \setminus \Delta \). Then we have

\[
\begin{align*}
  f_i^{-1}(B_j) \subseteq \mathbb{R}^2 \setminus \text{int } \Delta & \quad \text{for } i \in \{1, 2\}, j \in \{1, 2, 3\} \text{ and } i \neq j, \\
  \text{where } \text{int } \text{ denotes the interior of a set.}
\end{align*}
\]

Hence for \( i \in \{1, 2\}, j \in \{1, 2, 3\} \) and \( i \neq j \) we have

\[
0 \leq \mu(f_i^{-1}(B_j)) = 2\mathcal{L}(f_i^{-1}(B_j) \cap \Delta) \leq 2\mathcal{L}(\mathbb{R}^2 \setminus \text{int } \Delta \cap \Delta) = 0.
\]

We will now show that for \( i = 1, 2 \),

\[
\mu(f_i^{-1}(B_i)) = 2\mu(B_i).
\]

Since both maps \( f_1 \) and \( f_2 \) are affine linear with \( \det A_1 = \det A_2 = \frac{1}{2} \), we have

\[
\mu(f_i^{-1}(B_i)) = 2\mathcal{L}(f_i^{-1}(B_i) \cap \Delta) = 2\mathcal{L}(f_i^{-1}(B_i)) = 4\mathcal{L}(B_i) = 4\mathcal{L}(B_i \cap \Delta) = 2\mu(B_i).
\]

We also note that \( \mu(f_1(\Delta) \cap f_2(\Delta)) = 0 \). This implies that \( \mu(f_j^{-1}f_i(\Delta)) = 0 \) for \( i \neq j \).

Taking into account the above observations, we have

\[
\begin{align*}
  M \mu(B) &= \frac{1}{2} \mu(f_1^{-1}(B)) + \frac{1}{2} \mu(f_2^{-1}(B)) \\
  &= \frac{1}{2} \mu(f_1^{-1}(B_1) \cup f_1^{-1}(B_2) \cup f_1^{-1}(B_3)) + \frac{1}{2} \mu(f_2^{-1}(B_1) \cup f_2^{-1}(B_2) \cup f_2^{-1}(B_3)) \\
  &= \frac{1}{2} \mu(f_1^{-1}(B_1)) + \frac{1}{2} \mu(f_2^{-1}(B_2)) \\
  &= \mu(B_1) + \mu(B_2) = \mu(B).
\end{align*}
\]

So far we know that \( \Delta \) is an invariant set of \( \mathcal{F} \) and \( \mu \) is an invariant probability measure of \( \mathcal{F}_p \). To prove that \( \Delta \) is, indeed, a semiattractor of \( \mathcal{F} \) and \( \mu \) is unique, let us show that \( \mathcal{F}_p = (\mathbb{R}^2; (f_i \circ f_j, 1/4) : i, j \in \{1, 2\}) \) is contractive on average.

In order to do this, let us compute the Lipschitz constants of the maps which comprise \( \mathcal{F}_p \). Using the maximum singular values of \( A_i, A_j \), we get

\[
\begin{align*}
  \text{Lip}(f_i \circ f_2) = \|A_iA_2\|_2 &= \frac{1}{\sqrt{2}}, i = 1, 2, \\
  \text{Lip}(f_i \circ f_1) &= \|A_iA_1\|_2 = \frac{\sqrt{3\sqrt{17} + 13}}{4}, i = 1, 2.
\end{align*}
\]

Hence we have

\[
\sum_{i,j=1}^{2} \text{Lip}(f_i \circ f_j) < 4
\]
and so $\mathcal{F}^2 p$ is contractive on average. Using the Lasota-Myjak criterion (Theorem 3.2), we conclude that $\mu$ is a unique invariant probability measure for $\mathcal{F} p$ and $\Delta = \text{supp } \mu$ is a semiattractor of $\mathcal{F}$.

Let us note that for the average contractivity of $\mathcal{F}^2 p$, it is enough to assume that $p_1 < \frac{4-2\sqrt{2}}{\sqrt{3} \sqrt{1+13-2\sqrt{2}}} \approx 0.53$. So according to Theorem 3.2 and the fact that $\Delta$ is a semiattractor of $\mathcal{F}$, there is a unique invariant probability measure for $\mathcal{F} p$, where $p_1 < 0.53$, supported on $\Delta$. This gives rise to a natural question if such invariant measure is singular or absolutely continuous with respect to Lebesgue measure. One can also ask, given $p_1 \in [0.53, 1]$, does there exist a natural number $k$ such that $\mathcal{F}^k p$ is contractive on average?

Finally, we address the question how the semiattractor $\Delta$ is fibred by the IFS $\mathcal{F}$. Let us look at two particular fibres, $\pi(i^2)$, $i = 1, 2$, where $i$ is a constant infinite sequence of $i$’s. By simple calculation

$$A^n_1 = \left( \begin{array}{cc} 1 & \sum_{k=1}^{n} \frac{1}{2^k} \\ 0 & \frac{1}{2^n} \end{array} \right) = \left( \begin{array}{cc} 1 & 1 - 2^{-n} \\ 0 & 2^{-n} \end{array} \right)$$

we get that $f^n_0(\Delta)$ is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1 - 2^{-n}, 2^{-n})$, from which it follows that $\pi(i^2 T) = [0, 1] \times \{0\}$. Thus $\Delta$ is not point-fibred. Next, since $f^2_2$ is a contraction, we get $\pi(2^2) = \{(\frac{1}{2}, \frac{3}{2})\}$. According to the Kieninger criterion (Theorem 4.2) $\Delta$ is strongly-fibred by $\mathcal{F}$. To complete the picture of fibre structure of $\Delta$, let us note that all fibres are either points or intervals, see [5, Figure 7]. To see this, observe that each fibre

$$\pi(i_1 i_2 \ldots) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(\Delta)$$

is an intersection of nested triangles whose area is shrinked by factor $1/2$ downward the nest, i.e.,

$$\mathcal{L}(f_{i_1} \circ \cdots \circ f_{i_{n+1}}(\Delta)) = \frac{1}{2} \cdot \mathcal{L}(f_{i_1} \circ \cdots \circ f_{i_n}(\Delta))$$

Therefore, $\pi(i_1 i_2 \ldots)$ is a nonempty, compact and convex subset of the plane whose area is $0$, that is, either a point, or an interval.

The analysis of the above example was not straightforward because the IFS at hand was neither contractive on average ([14 Theorem 2.60]) nor eventually contractive ([14 Example 2.28]). Fortunately, the Markov operator turned out to be eventually average-contractive. The mere existence of the semiattractor could have been easily inferred from the contractivity of $f_2^2$ due to [17 Theorem 3.2] (which is [21 Theorem 6.3]). However, we would have not known if $\Delta$ is that semiattractor.

Techniques, similar to the ones we used in the above example, can be applied to other non-contractive affine IFSs. For instance, we have the following example.

**Example 5.1.** Let $\mathcal{F} p = (\mathbb{R}^2; (f_i, 1/2) : i \in \{1, 2\})$, where $f_1(x, y) = (x/2 + 1/2, y)$, $f_2(x, y) = (x, y/2)$ for $(x, y) \in \mathbb{R}^2$. We proceed as before. Noting that

$$\text{Lip}(f_i \circ f_i) = 1, \quad \text{Lip}(f_{3-i} \circ f_i) = 1/2, \quad i = 1, 2,$$

we get that the Markov operator $M$ induced by $\mathcal{F} p$ is asymptotically stable. Thus $\mathcal{F} p$ admits a unique invariant probability measure $\mu_*$ and a semiattractor $A_*$. It is easy to see that $A_* = \{(1, 0)\}$, because the common fixed point of $f_1$ and $f_2$ is
a minimal invariant set, and a fortiori $\mu_* = \delta_{(1,0)}$, the Dirac measure supported at the point $(1,0)$.

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