Some Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integral

Imdat Iscan\(^1\), Sercan Turhan\(^2\) and Selahattin Maden\(^3\)

\(^1\)Department of Mathematics, Giresun University, Giresun, Turkey
\(^2\)Dereli Vocational High School, Giresun University, Giresun, Turkey
\(^3\)Department of Mathematics, Ordu University, Ordu, Turkey

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Abstract: In this paper, we gave the new general identity for differentiable functions. As a result of this identity some new and general inequalities for differentiable harmonically-convex functions are obtained.

Keywords: Harmonically-convex, Hermite-Hadamard-Fejer type inequality, fractional integral.

1 Introduction

The classical or the usual convexity is defined as follows,

**Definition 1.** A function \( f : I \rightarrow \mathbb{R} \), \( \emptyset \neq I \subseteq \mathbb{R} \), is said to be convex on \( I \) if inequality

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \).

A number of papers have been written on inequalities using the classical convexity and one of the most captivating inequalities in mathematical analysis is stated as follows,

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},
\]

(1)

where \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex mapping and \( a, b \in I \) with \( a \leq b \). Both the inequalities hold in reversed direction if \( f \) is concave. The inequalities stated in (1) are known as Hermite-Hadamard inequalities.

For more results on (1) which provide new proof, significantly extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [2,3,5,6,8,9,12,13,15,16] and the references there in.

The usual notion of convex function have been generalized in diverse manners. One of them is the so called harmonically s-convex functions and is stated in the definition below.

* Corresponding author e-mail: imdat.iscan@giresun.edu.tr
Definition 2. [5,7] Let $I \subset (0, \infty)$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically $s$-convex (concave), if

$$f \left( \frac{tx + (1-t)y}{t} \right) \leq (\geq) t f(y) + (1-t)f(x)$$

holds for all $x, y \in I$ and $t \in [0,1]$, and for some fixed $s \in (0,1]$.

It can be easily seen that for $s = 1$ in Definition 2 reduces to following Definition 3,

Definition 3. [6] A function $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically-convex function, if

$$f \left( \frac{tx + (1-t)y}{t} \right) \leq tf(y) + (1-t)f(x)$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality is reversed, then $f$ is said to be harmonically concave.

Proposition 1. [6] Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $f : I \rightarrow \mathbb{R}$ is function, then:

(i) if $I \subset (0,\infty)$ and $f$ is convex and nondecreasing function then $f$ is harmonically convex.
(ii) if $I \subset (0,\infty)$ and $f$ is harmonically convex and nonincreasing function then $f$ is convex.
(iii) if $I \subset (-\infty,0)$ and $f$ is harmonically convex and nondecreasing function then $f$ is convex.
(iv) if $I \subset (-\infty,0)$ and $f$ is convex and nonincreasing function then $f$ is harmonically convex.

For the properties of harmonically-convex functions and harmonically-s-convex function, we refer the reader to [1,5,6,7,8,10,11] and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for harmonically-convex and for harmonically-s-convex functions.

In [14], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1).

Theorem 1. Let $f : [a,b] \rightarrow \mathbb{R}$ be convex function. Then the inequality

$$f \left( \frac{a+b}{2} \right) \int_a^b g(x)dx \leq \frac{1}{2} \int_a^b f(x)g(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx,$$

(2)

holds, where $g : [a,b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2) see [15].

In [6], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically-s-convex functions as follows.

Theorem 2. [15] Let $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:

$$f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

(3)

In [11], Iscan and Wu represented Hermite-Hadamard’s inequalities for harmonically convex functions in fractional integral form as follows.
Theorem 3. [11] Let \( f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \) be a function such that \( f \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is harmonically-convex on \([a,b]\), then the following inequalities for fractional integrals hold:

\[
\int \frac{2ab}{a+b} f' \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma(\alpha+1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ J^\alpha_{1/a^-} \left( f \circ h \right)(1/b) + J^\alpha_{1/b^+} \left( f \circ h \right)(1/a) \right\} \leq \frac{f(a) + f(b)}{2},
\]

with \( \alpha > 0 \) and \( h(x) = 1/x \).

Definition 4. A function \( g : [a,b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is said to be harmonically symmetric with respect to \( 2ab/a + b \) if

\[
g(x) = g \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{x} \right)
\]

holds for all \( x \in [a,b] \).

Theorem 4. In [1] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:

Theorem 5. Suppose that \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be harmonically-convex function and \( a, b \in I \), with \( a < b \). If \( f \in L[a,b] \) and \( g : [a,b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( 2ab/a + b \), then

\[
\int \frac{2ab}{a+b} \left( \int_a^b \frac{g(x)f(x)}{x^2} \, dx \right) dx \leq \frac{f(a) + f(b)}{2} \left( \int_a^b \frac{g(x)}{x^2} \, dx \right)
\]

In [10] Iscan and Kunt represented Hermite-Hadamard-Fejer type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 6. Let \( f : [a,b] \rightarrow \mathbb{R} \) be harmonically convex function with \( a < b \) and \( f \in L[a,b] \). If \( g : [a,b] \rightarrow \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( 2ab/a + b \), then the following inequalities for fractional integrals hold:

\[
\int \frac{2ab}{a+b} \left[ J^\alpha_{1/a^-} \left( g \circ h \right)(1/b) + J^\alpha_{1/b^+} \left( g \circ h \right)(1/a) \right] \leq \frac{f(a) + f(b)}{2} \left[ J^\alpha_{1/a^-} \left( g \circ h \right)(1/b) + J^\alpha_{1/b^+} \left( g \circ h \right)(1/a) \right]
\]

with \( \alpha > 0 \) and \( h(x) = 1/x, x \in \left[ \frac{1}{b^2}, \frac{1}{a^2} \right] \).

Definition 5. Let \( f \in L[a,b] \). The right-hand side and left-hand side Hadamard fractional integrals \( J^\alpha_{a^+} f \) and \( J^\alpha_{b^-} f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a
\]

\[
J^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt \) and \( J^\alpha_{a^+} f(x) = J^\alpha_{b^-} f(x) = f(x) \).

Lemma 1. For \( 0 < \theta \leq 1 \) and \( 0 < a \leq b \) we have

\[
|a^\theta - b^\theta| \leq (b-a)^\theta.
\]
In [4] D. Y. Hwang found out a new identity and by using this identity, established a new inequalities. Then in [12] İ. İşcan and S. Turhan used this identity for GA-convex functions and obtain generalized new inequalities. In this paper, we established a new inequality similar to inequality in [12] and then we obtained some new and general integral inequalities for differentiable harmonically-convex functions using this lemma. The following sections, let the notion,

\[ L(t) = \frac{dL}{dt} \quad \text{and} \quad U(t) = \frac{dU}{dt} \quad \text{and} \quad H = H(a, b) = \frac{2ab}{a+b}. \]

## 2 Main result

Throughout this section, let \( \|g\|_\infty = \sup_{x \in [a,b]} |g(x)| \), for the continuous function \( g: [a, b] \to [0, \infty) \) be differentiable mapping \( P \), where \( a, b \in I \) with \( a \leq b \), and \( h: [a, b] \to [0, \infty) \) be differentiable mapping.

**Lemma 2.** If \( f' \in L[a, b] \) then the following inequality holds:

\[
|h(b) - 2h(a)| \left( \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right) = \frac{b-a}{4ab} \left\{ \int_0^1 [2h(L(t)) - h(b)] f'(L(t)) (L(t))^2 dt + \int_0^1 [2h(U(t)) - h(b)] f'(U(t)) (U(t))^2 dt \right\}.
\]

**Proof.** By the integration by parts, we have

\[ I_1 = \int_0^1 [2h(L(t)) - h(b)] d(f(L(t))) = [2h(L(t)) - h(b)] f(L(t))|_0^1 - \left( \frac{1}{a} - \frac{1}{b} \right) \int_0^1 f(L(t))h'(L(t)) (L(t))^2 dt \]

and

\[ I_2 = \int_0^1 [2h(U(t)) - h(b)] d(f(U(t))) = [2h(U(t)) - h(b)] f(U(t))|_0^1 - \left( \frac{1}{a} - \frac{1}{b} \right) \int_0^1 f(U(t))h'(U(t)) (U(t))^2 dt. \]

Therefore

\[
\frac{I_1 + I_2}{2} = [h(b) - 2h(a)] \left( \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b-a}{2ab} \left\{ \int_0^1 f(L(t))h'(L(t)) (L(t))^2 dt + \int_0^1 f(U(t))h'(U(t)) (U(t))^2 dt \right\} \right).
\]

This complete the proof.

**Lemma 3.** For \( a, H, b > 0 \), we have

\[
\zeta_1(a, b) = \int_0^1 [2h(L(t)) - h(b)] (1-t) (L(t))^2 dt \tag{10}
\]

\[
\zeta_2(a, b) = \int_0^1 t (L(t))^2 [2h(L(t)) - h(b)] dt + \int_0^1 t(L(t))^2 [2h(U(t)) - h(b)] dt \tag{11}
\]

\[
\zeta_3(a, b) = \int_0^1 [2h(U(t)) - h(b)] (1-t) (U(t))^2 dt. \tag{12}
\]
Theorem 7. Let \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) be differentiable mapping \( I' \), where \( a, b \in I \) with \( a < b \). If the mapping \( |f'| \) is harmonically-convex on \([a, b]\), then the following inequality holds:

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b}{a} \int_a^b f(x)h'(x)dx \leq \frac{b-a}{4ab} \left[ \xi_1(a, b) |f'(a)| + \xi_2(a, b) |f'(H)| + \xi_3(a, b) |f'(b)| \right]
\]  
(13)

where \( \xi_1(a, b), \xi_2(a, b), \xi_3(a, b) \) are defined in Lemma 2.

Proof. Continuing equality (8) in Lemma 2

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b}{a} \int_a^b f(x)h'(x)dx \leq \frac{b-a}{4ab} \left\{ \int_0^1 \left[ 2h(L(t)) - h(b) \right] |f'(L(t))| (L(t))^2 |dt| + \int_0^1 \left[ 2h(U(t)) - h(b) \right] |f'(U(t))| (U(t))^2 |dt| \right\}
\]  
(14)

Using \( |f'| \) is harmonically-convex in (14).

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b}{a} \int_a^b f(x)h'(x)dx \leq \frac{b-a}{4ab} \left\{ \int_0^1 \left[ 2h(L(t)) - h(b) \right] \left\{ |t| f'(H)| + (1-t) |f'(a)| \right\} (L(t))^2 dt \right. \\
+ \left. \int_0^1 \left[ 2h(U(t)) - h(b) \right] \left\{ |t| f'(H)| + (1-t) |f'(b)| \right\} (U(t))^2 dt \right\}
\]  
(15)

by (15) and Lemma 2, this proof is complete.

Corollary 1. Let \( h(t) = \int_0^a \left( (x - \frac{1}{2})^\alpha - (\frac{1}{2} - x)^\alpha \right) g(x)dx \) for all \( 1/t \in [\frac{1}{b}, \frac{1}{a}] \), \( \alpha > 0 \) and \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \frac{2ab}{a+b} \) in Theorem 7, we obtain:

\[
\left| \left( \frac{f(a)+f(b)}{2} \right) \left[ \int_1^{1/b} g \circ \varphi(1/a) + J_{1/a}^{\alpha} g \circ \varphi(1/b) \right] - \left[ \int_1^{1/b} (f g \circ \varphi)(1/a) + J_{1/a}^{\alpha} (f g \circ \varphi)(1/b) \right] \right|
\]  
(16)

\[
\leq \frac{(b-a)^{\alpha+1} ||g||_\infty}{2^{\alpha+1}(ab)^{\alpha+1} \Gamma(\alpha+1)} \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)| \right]
\]

where

\[
C_1(\alpha) = \int_0^1 (1-t) [(1+t)^\alpha - (1-t)^\alpha] (L(t))^2 dt
\]

\[
C_2(\alpha) = \int_0^1 t [(1+t)^\alpha - (1-t)^\alpha] [(L(t))^2 + (U(t))^2] dt
\]

\[
C_3(\alpha) = \int_0^1 (1-t) [(1+t)^\alpha - (1-t)^\alpha] (L(t))^2 dt
\]
Specially in (16) and using Lemma 1, for \(0 < \alpha \leq 1\) we have:

\[
\left\| \frac{f(a) + f(b)}{2} \right\| \left[ J_{1/b}^{a} g \circ \varphi (1/a) + J_{1/a}^{a} g \circ \varphi (1/b) \right] - \left[ J_{1/b}^{a} (f g \circ \varphi)(1/a) + J_{1/a}^{a} (f g \circ \varphi)(1/b) \right] \leq \frac{(b-a)^{\alpha+1} \|g\|_{\infty}}{2(ab)^{\alpha+1} \Gamma (\alpha + 1)} \left[ C_1 (\alpha) |f' (a)| + C_2 (\alpha) |f' (H)| + C_3 (\alpha) |f' (b)| \right]
\]

where

\[
C_1 (\alpha) = \int_{0}^{1} (1-t)^{\alpha} (L(t))^2 dt, \quad C_2 (\alpha) = \int_{0}^{1} t^{\alpha+1} \left[ (L(t))^2 + (U(t))^2 \right] dt, \quad C_3 (\alpha) = \int_{0}^{1} (1-t)^{\alpha} (U(t))^2 dt.
\]

Proof. By left side of inequality (15) in Theorem 7, when we write \(h(t) = \frac{1}{(\sqrt{b} - \alpha - (\sqrt{a} - 1))} g \circ \varphi(x)dx\) for all \(x \in [1/b, 1/a]\) and \(\varphi(x) = 1/x\), we have

\[
\Gamma (\alpha) \left( \frac{f(a) + f(b)}{2} \right) \left[ J_{1/b}^a g \circ \varphi (1/a) + J_{1/a}^a g \circ \varphi (1/b) \right] - \Gamma (\alpha) \left[ J_{1/b}^a (f g \circ \varphi)(1/a) + J_{1/a}^a (f g \circ \varphi)(1/b) \right].
\]

On the other hand, right side of inequality (15), with

\[
\Psi (x, a, b) = \left( x - \frac{1}{b} \right)^{\alpha-1} + \left( \frac{1}{a} - x \right)^{\alpha-1}
\]

\[
\leq \frac{b-a}{4ab} \left\{ \int_{0}^{1/a} \frac{1}{2} \int_{L(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx - \int_{1/b}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx \right\} \left\{ t |f' (H)| + (1-t) |f' (a)| \right\} (L(t))^2 dt
\]

\[
+ \int_{0}^{1/a} \frac{1}{2} \int_{L(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx - \int_{1/b}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx \right\} \left\{ t |f' (H)| + (1-t) |f' (b)| \right\} (U(t))^2 dt
\]

Since \(g(x)\) is symmetric to \(x = \frac{2ab}{a+b}\), we have

\[
2 \int_{L(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx - \int_{1/b}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx = \int_{L(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx
\]

and

\[
2 \int_{U(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx - \int_{1/b}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx = \int_{U(t)}^{1/a} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx
\]

for all \(t \in [0, 1]\). By (18)–(20), we have

\[
\left\| \frac{f(a) + f(b)}{2} \right\| \left[ J_{1/b}^a g \circ \varphi (1/a) + J_{1/a}^a g \circ \varphi (1/b) \right] - \left[ J_{1/b}^a (f g \circ \varphi)(1/a) + J_{1/a}^a (f g \circ \varphi)(1/b) \right] \leq \frac{b-a}{4ab \Gamma (\alpha)} \left\{ \int_{0}^{1/L(t)} \left[ \Psi (x, a, b) \right] g \circ \varphi(x)dx \right\} \left\{ t |f' (H)| + (1-t) |f' (a)| \right\} (L(t))^2 dt
\]
Corollary 2. In Corollary 1,

(i) If \( \alpha = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx - \int_a^b \frac{g(x)}{x^2} f(x) dx \right| \leq \frac{(b - a)^2}{4(ab)^2} \|g\|_{\infty} \left[ C_1(1) |f'(a)| + C_2(1) |f'(H)| + C_3(1) |f'(b)| \right]
\]

(23)

where for \( a, b, H > 0 \), we have

\[
C_1(1) = \int_0^1 (1 - t) t (L(t))^2 dt
\]

\[
C_2(1) = \int_0^1 t^2 \left[ (L(t))^2 + (U(t))^2 \right] dt
\]

\[
C_3(1) = \int_0^1 (1 - t) t (U(t))^2 dt
\]
(ii) If \( g(x) = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (16):  

\[
\left\| \frac{f(a) + f(b)}{2} - \frac{(ab)^{\alpha}}{2(b-a)\alpha} \left[ J_{1/b^+}^\alpha (f\circ\varphi) (1/2) + J_{1/a^-}^\alpha (f\circ\varphi) (1/2) \right] \right\| \leq \frac{(b-a)}{2a^2+2ab} \left[ C_1 (\alpha) \left| f^\prime (a) \right| + C_2 (\alpha) \left| f^\prime (H) \right| + C_3 (\alpha) \left| f^\prime (b) \right| \right].
\]  

(iii) If \( g(x) = 1 \) and \( \alpha = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):  

\[
\left\| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right\| \leq \frac{(b-a)}{4(ab)} \left[ C_1 (1) \left| f^\prime (a) \right| + C_2 (1) \left| f^\prime (H) \right| + C_3 (1) \left| f^\prime (b) \right| \right].
\]  

Theorem 8. Let \( f : I \subseteq \mathbb{R}\setminus\{0\} \rightarrow \mathbb{R} \) be differentiable mapping \( f' \), where \( a, b \in I \) with \( a < b \). If the mapping \( |f'|^q \) is harmonically-convex on \([a, b]\), then the following inequality holds:  

\[
\left\| h(b) - 2h(a) \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h^\prime (x) \, dx \right\| \leq \frac{b-a}{4ab} \left\{ \eta_1^{1/4} \times \eta_2^{1/4} + \eta_3^{1/4} \times \eta_4^{1/4} \right\}
\]  

where  

\[
\eta_1 = \int_0^1 \left| 2h(L(t)) - h(b) \right| \, dt,
\]

\[
\eta_2 = \int_0^1 \left( 2h(L(t)) - h(b) \right) \, dt \times \left( \left( L(t) \right)^{2q} \left| f^\prime (a) \right|^q + \left( 1 - t \right) \left( L(t) \right)^{2q} \left| f^\prime (H) \right|^q \right),
\]

\[
\eta_3 = \int_0^1 \left| 2h(U(t)) - h(b) \right| \, dt,
\]

\[
\eta_4 = \int_0^1 \left( 2h(U(t)) - h(b) \right) \, dt \times \left( \left( U(t) \right)^{2q} \left| f^\prime (a) \right|^q + \left( 1 - t \right) \left( U(t) \right)^{2q} \left| f^\prime (H) \right|^q \right).
\]

Proof. Continuing from (14) in Theorem 7, we use Hölder Inequality and we use that \( |f'|^q \) is harmonically-convex. Thus this proof is complete.

Corollary 3. Let \( h(t) = \int_{1/t}^{1/a} \left( x + \frac{1}{2} \right)^{\alpha-1} \left( \frac{1}{2} + x \right)^{\alpha-1} (g \circ \varphi) (x) \, dx \) for all \( t \in [a, b] \) and \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \frac{2ab}{a+b} \) in Theorem 8, we obtain:  

\[
\left\| \frac{f(a) + f(b)}{2} - \left[ J_{1/b^+}^\alpha (g \circ \varphi) (1/2) + J_{1/a^-}^\alpha (g \circ \varphi) (1/2) \right] \right\| \leq \frac{(b-a)}{2a+\alpha+b} \left\| g \right\|_{\infty} \left( 2^{2(\alpha-1)} - 1 \right)^{1/4} \left[ C_1 (\alpha, q) \left| f^\prime (a) \right|^q + C_2 (\alpha, q) \left| f^\prime (H) \right|^q + C_3 (\alpha, q) \left| f^\prime (b) \right|^q \right]^{1/4}
\]  

\[\]
where for \( q > 1 \)

\[
C_1(\alpha, q) = \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] t (L(t))^{2q} \, dt
\]

\[
C_2(\alpha, q) = \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] (1 - t) \left( (L(t))^{2q} + (U(t))^{2q} \right) \, dt
\]

\[
C_3(\alpha, q) = \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] t (U(t))^{2q} \, dt.
\]

**Proof.** Continuing from (22) of Corollary 1 and (26) in Theorem 8,

\[
\left| \left( \frac{f(a) + f(b)}{2} \right) [\xi_1] - [\xi_2] \right| \leq \frac{(b - a)^{\alpha + 1}}{2^{\alpha + 1} \Gamma(\alpha + 1)} \left\{ \ell_1 \times \ell_2 + \ell_1 \times \ell_3 \right\} \leq \frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma(\alpha + 1)} (\xi_0)^{\frac{1}{\lambda} + \frac{1}{q}} [\xi_2 + \xi_3]
\]

where

\[
\xi_0 = \frac{2^{\alpha + 1} - 2}{\alpha + 1},
\]

\[
\xi_1 = J_{ab}^\alpha g(b) + J_{b}^\alpha g(a),
\]

\[
\xi_2 = J_{a}^\alpha (fg)(b) + J_{b}^\alpha (fg)(a),
\]

\[
\ell_1 = \left( \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] \, dt \right)^{1 - \frac{1}{q}},
\]

\[
\ell_2 = \left( \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] (1 - t) \left( (L(t))^{2q} + (U(t))^{2q} \right) \, dt \right)^{\frac{1}{q}},
\]

\[
\ell_3 = \left( \int_0^1 [(1 + t)^\alpha - (1 - t)^\alpha] t (U(t))^{2q} \, dt \right)^{\frac{1}{q}}.
\]

By the power-mean inequality \( (a' + b')^{2^{1-r}} (a + b)^r \) for \( a > 0, b > 0, r < 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) we have

\[
\frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma(\alpha + 1)} (\xi_0)^{\frac{1}{\lambda} + \frac{1}{q}} [\xi_4 + \xi_5] \leq \frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma(\alpha + 1)} \left( \frac{2^2 (2^\alpha - 1)}{\alpha + 1} \right)^{\frac{1}{q}} \left[ \frac{1}{0} (\xi_4 + \xi_5 + \xi_3) \, dt \right]^{\frac{1}{q}},
\]

where

\[
\xi_4 = [(1 + t)^\alpha - (1 - t)^\alpha] (L(t))^{2q} |f'(a)|^q,
\]

\[
\xi_2 = [(1 + t)^\alpha - (1 - t)^\alpha] (1 - t) \left( (L(t))^{2q} + (U(t))^{2q} \right) |f'(H)|^q,
\]

\[
\xi_3 = [(1 + t)^\alpha - (1 - t)^\alpha] t (U(t))^{2q} |f'(b)|^q.
\]
Corollary 4. When $\alpha = 1$ and $g(x) = 1$ is taken in Corollary 3, we obtain:

\[
\left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{(b-a)} \int_a^b \frac{f(x)}{x^2} \, dx \leq \left( \frac{b-a}{2^{2+\frac{1}{q}} (ab)} \right) \left[ C_1 (1, q) \left| f'(a) \right|^q + C_2 (1, q) \left| f'(H) \right|^q + C_3 (1, q) \left| f'(b) \right|^q \right]^{\frac{1}{q}} .
\]

(30)

This proof is complete.

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