On Doubly Warped Product Finsler Manifolds

E. Peyghan and A. Tayebi

November 1, 2011

Abstract

In this paper, we introduce horizontal and vertical warped product Finsler manifold. We prove that every C-reducible or proper Berwaldian doubly warped product Finsler manifold is Riemannian. Then, we find the relation between Riemmanian curvatures of doubly warped product Finsler manifold and its components, and consider the cases that this manifold is flat or has scalar flag curvature. We define the doubly warped Sasaki-Matsumoto metric for warped product manifolds and find a condition under which the horizontal and vertical tangent bundles are totally geodesic. Also, we obtain some conditions under which a foliated manifold reduces to a Reinhart manifold. Finally, we study an almost complex structure on the slit tangent bundle of a doubly warped product Finsler manifold.

Keywords: Doubly warped product manifold, Sasaki-Matsumoto lift metric, Vaisman connection, Reinhart manifold, Kähler structure.

1 Introduction.

In the Riemannian or semi-Riemannian cases, the doubly warped product of Riemannian (semi-Riemannian) manifolds was studied by many authors [1, 6, 7, 8, 19], and several application to theoretical physics were given. For instance in [7], Beem-Powell considered this product for Lorentzian manifolds. Also, Allison considered causality and global hyperbolicity of doubly warped product and null pseudo convexity of Lorentzian doubly warped product in [1].

This construction can be extended for Finslerian metrics with some minor restriction. In [2, 3], Asanov gave the generalization of the Schwarzschild metric in the Finslerian setting and obtain some models of relativity theory described through the warped product of Finsler metrics. Then, Shen used a construction of warped of Riemannian metrics at the vertical bundle, and obtained a Finslerian warped product metric [18]. Recently, Kozma-Peter-Varga used the Finsler fundamental functions to define their warped product [10]. Then they studied the relationships between the Cartan connection of the doubly warped product manifold and components of it.

1 2000 Mathematics subject Classification: 53C60, 53C25.
Let \((M_1, F_1)\) and \((M_2, F_2)\) be two Finsler manifolds with \(\text{dim} M_1 = n_1\) and \(\text{dim} M_2 = n_2\) and \(f_1 : M_1 \to \mathbb{R}^+\) and \(f_2 : M_2 \to \mathbb{R}^+\) be two smooth functions. Let \(\pi_1 : M_1 \times M_2 \to M_1\) and \(\pi_2 : M_1 \times M_2 \to M_2\) be the natural projection maps. The product manifold \(M_1 \times M_2\) endowed with the metric \(F : TM_1^0 \times TM_2^0 \to \mathbb{R}\) is considered,

\[
F(v_1, v_2) = \sqrt{f_2^2(\pi_2(v_2))F_1^2(v_1) + f_1^2(\pi_1(v_1))F_2^2(v_2)},
\]

where \(TM_1^0 = TM_1 - \{0\}\) and \(TM_2^0 = TM_2 - \{0\}\). The metric defined above is a Finsler metric. The product manifold \(M_1 \times M_2\) with the metric \(F(v) = F(v_1, v_2)\) for \((v_1, v_2) \in TM_1^0 \times TM_2^0\) defined above will be called the doubly warped product of the manifolds \(M_1\) and \(M_2\) and \(f_1\) and \(f_2\) will be called the warping functions. We denote this doubly warped by \(f_2M_1 \times_{f_1} M_2\). If either \(f_1 = 1\) or \(f_2 = 1\), but not both, then \(f_2M_1 \times_{f_1} M_2\) becomes a warped product of Finsler manifolds \(M_1\) and \(M_2\). If both \(f_1\) and \(f_2\), then we have a product manifold. If neither \(f_1\) nor \(f_2\) is constant, then we have a nontrivial (proper) doubly warped product manifold.

This paper is arranged as follows: In section 2, we give some of basic concepts related to Finsler manifolds. In section 3, we introduce the horizontal and vertical distributions on tangent bundle of a doubly warped product Finsler manifold and construct the Finsler connection on this manifold. Then, we prove that every C-reducible or proper Berwaldian doubly warped product Finsler manifold reduces to a Riemannian manifold. In section 4, for very two Finsler manifolds \((M_1, F_1)\) and \((M_2, F_2)\), we introduce the Riemannian curvature of doubly warped product Finsler manifold \((f_2M_1 \times_{f_1} M_2, F)\) and find the relation between it and Riemannian curvatures of its components \((M_1, F_1)\) and \((M_2, F_2)\). In the cases that \((f_2M_1 \times_{f_1} M_2, F)\) is flat or it has the scalar flag curvature, we obtain some results on its components. In section 5, the doubly warped Sasaki-Matsumoto metric \(G\) is introduced for the doubly warped product Finsler manifold. Then by using the Levi-Civita connection of this metric, we find some conditions under which \(HTM^0\) and \(VTM^0\) are totally geodesic. In section 6, we obtain the Vaisman connection of Riemannian foliated manifold \((TM^0, F_V, G)\) and show that it is a Reinhart space if and only if \((M_1, F_1)\) and \((M_2, F_2)\) are Riemannian manifolds. Finally, we define an almost complex structure on the slit tangent bundle of a doubly warped product Finsler manifold and show that this structure with the doubly warped Sasaki-Matsumoto metric construct an almost Hermitian structure. Then, we prove that \((TM^0, G, J)\) is a Kählerian manifold if and only if the doubly warped horizontal distribution \(HTM^0\) is integrable.

2 Preliminary

Let \(M\) be a \(n\)-dimensional \(C^\infty\) manifold. Denote by \(T_xM\) the tangent space at \(x \in M\), by \(TM = \cup_{x \in M} T_xM\) the tangent bundle of \(M\), and by \(TM^0 = TM - \{0\}\) the slit tangent bundle on \(M\). A Finsler metric on \(M\) is a function
$F : TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM^o$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$; (iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right] \big|_{s, t = 0}, \quad u, v \in T_xM.$$ 

Define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u, v) \right] \big|_{t = 0}, \quad u, v, w \in T_xM.$$ 

The family $C := \{C_y\}_{y \in TM}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian $[5][17]$. For $y \in T_xM$, define mean Cartan torsion $I_y$ by $I_y(u) := I_y(y)u$, where $I_y := g^{jk}C_{ijk}$, $g^{jk}$ is the inverse of $g_{jk}$ and $u = u^i \frac{\partial}{\partial x^i}$. By Deicke’s Theorem, $F$ is Riemannian if and only if $I_y = 0$.

Let $(M, F)$ be a Finsler manifold. Then for $y \in T_xM$, define the Matsumoto torsion $M_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \},$$

$h_{ij} := FF^{-1}_{ij} = g_{ij} - \frac{1}{e^2} g_{ip}g_{jq}g_{pq}$ is the angular metric. A Finsler metric $F$ is said to be $C$-reducible if $M_y = 0$ $[7]$ Matsumoto proves that every Randers metric satisfies that $M_y = 0$. Later on, Matsumoto-Hôjô proves that the converse is true too. It is remarkable that, a Randers metric $F = \alpha + \beta$ on a manifold $M$ is just a Riemannian metric $\alpha$ perturbated by a one form $\beta$ on $M$ $[25]$.

For a Finsler manifold $(M, F)$, a global vector field $G$ is induced by $F$ on $TM$, which in a standard coordinate $(x^i, y^j)$ for $TM$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2F^2(x, y) \frac{\partial}{\partial y^i},$ where

$$G^i := \frac{1}{4} g^{ik} \left\{ \frac{\partial^2}{\partial x^k \partial y^l} y^j - \frac{\partial}{\partial x^l} \left[ \frac{\partial F^2}{\partial y^j} \right] \right\}, \quad y \in T_xM.$$ 

The $G$ is called the spray associated to $(M, F)$. Then we can define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $B_y(u, v, w) := B^i_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^l}$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$ 

The $B$ is called the Berwald curvature. $F$ is called a Berwald metric if $B = 0$.

The Riemann curvature $R_y = R^i_{jk} dx^k \otimes \frac{\partial}{\partial x^j}|_z : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i_{jk} = 2 \frac{\partial G^i}{\partial x^j} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^i}{\partial y^k}. \quad (2.2)$$
For a flag $P = \text{span} \{y, u\} \subset T_xM$ with flagpole $y$, the flag curvature $K = K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}. \quad (2.3)$$

We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_xM$, the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle $TM_0$. If $K = \text{constant}$, then $F$ is said to be of constant flag curvature.

### 3 Doubly Warped Nonlinear Connection

Let $(M_1, F_1)$ and $(M_2, F_2)$ be two Finsler manifolds. Then the functions

$$\begin{align*}
(i) & \ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2_i(x, y)}{\partial y^i \partial y^j}, & (ii) & \ g_{\alpha\beta}(u, v) = \frac{1}{2} \frac{\partial^2 F^2_\alpha}{\partial u^\alpha \partial v^\beta},
\end{align*} \quad (3.4)$$

define a Finsler tensor field of type $(0, 2)$ on $TM^*_1$ and $TM^*_2$, respectively. Now let $(f, M_1 \times_f M_2, F)$ be a doubly warped Finsler manifold and let $x \in M$ and $y \in T_xM$, where $x = (x, u)$, $y = (y, v)$, $M = M_1 \times M_2$ and $T_xM = T_xM_1 \oplus T_uM_2$. Then by using (1.1) and (3.4), we conclude that

$$\left( g_{ab}(x, u, y, v) \right) = \left( \frac{1}{2} \frac{\partial^2 F^2(x, u, y, v)}{\partial y^a \partial y^b} \right) = \begin{bmatrix} f_{ij}^2 \ g_{ij} & 0 \\ 0 & f_{\alpha\beta}^2 \ g_{\alpha\beta} \end{bmatrix}, \quad (3.5)$$

where $y^a = (y^i, v^\alpha)$, $y^b = (y^j, v^\beta)$, $g_{ij} = f_{ij}^2 g_{ij}$, $g_{\alpha\beta} = f_{\alpha\beta}^2 g_{\alpha\beta}$, $g_{ij} = g_{\alpha\beta} = 0$, $i, j, \ldots \in \{1, \ldots, n_1\}$, $\alpha, \beta, \ldots \in \{1, \ldots, n_2\}$ and $a, b, \ldots \in \{1, \ldots, n_1 + n_2\}$.

Now we consider the the spray coefficients of $F_1$, $F_2$ and $F$ as follows

$$\begin{align*}
G^i(x, y) & = \frac{1}{4} g^{ih} \left( \frac{\partial^2 F^2_i}{\partial y^h \partial x^j} y^j - \frac{\partial F^2_i}{\partial x^h} \right)(x, y), \quad (3.6) \\
G^\alpha(u, v) & = \frac{1}{4} g^{\alpha\gamma} \left( \frac{\partial^2 F^2_\alpha}{\partial v^\gamma \partial u^\beta} u^\beta - \frac{\partial F^2_\alpha}{\partial u^\gamma} \right)(u, v), \quad (3.7) \\
G^a(x, y) & = \frac{1}{4} g^{ab} \left( \frac{\partial^2 F^2}{\partial y^a \partial x^c} y^c - \frac{\partial F^2}{\partial x^b} \right)(x, y). \quad (3.8)
\end{align*}$$

Taking into account the homogeneity of both $F^2_1$ and $F^2_2$, we can derive from (3.6) and (3.7) that $G^i$ and $G^\alpha$ are positively homogeneous of degree two respect to $(y^i)$ and $(v^\alpha)$, respectively. Hence, by Euler’s theorem about the homogeneous functions, we conclude that

$$\frac{\partial G^i}{\partial y^j} y^j = 2G^i \quad \text{and} \quad \frac{\partial G^\alpha}{\partial v^\beta} v^\beta = 2G^\alpha.$$ 

By setting $a = i$ in (3.8), we have

$$G^i(x, u, y, v) = \frac{1}{4} g^{ih} \left( \frac{\partial^2 F^2_i}{\partial y^h \partial x^j} y^j + \frac{\partial^2 F^2_i}{\partial y^h \partial u^\alpha} v^\alpha - \frac{\partial F^2_i}{\partial x^h} \right). \quad (3.9)$$
Direct calculations give us

\[ \frac{\partial F^2}{\partial x^h} = f_2 \frac{\partial F^1}{\partial x^h} + \frac{\partial f_2}{\partial x^h} F_2^2 \]
\[ \frac{\partial^2 F^2}{\partial y^b \partial x^j} = f_2 \frac{\partial^2 F_1^2}{\partial y^b \partial x^j} \]
\[ \frac{\partial^2 F^2}{\partial y^b \partial u^\alpha} = \frac{\partial f_2}{\partial u^\alpha} \frac{\partial F_1^2}{\partial y^b} \]

Putting these equations together \( g^{ih} = \frac{1}{f_2} g^{ih} \) in (3.9) and using (3.6) imply that

\[ G^i(x, u, y, v) = G^i(x, y) + \frac{1}{4f_2^2} g^{ih} \left( \frac{\partial f_2}{\partial u^\alpha} \frac{\partial F_1^2}{\partial y^h} v^\alpha - \frac{\partial f_2}{\partial x^h} F_2^2 \right). \quad (3.10) \]

Similarly, by putting \( a = \alpha \) in (3.8) and using (3.7), we obtain

\[ G^\alpha(x, u, y, v) = G^\alpha(u, v) + \frac{1}{4f_1^2} g^{\alpha\gamma} \left( \frac{\partial f_1}{\partial x^\gamma} \frac{\partial F_2^2}{\partial u^\alpha} y^\gamma - \frac{\partial f_1}{\partial u^\gamma} F_1^2 \right). \quad (3.11) \]

Therefore we have \( G^\alpha = (G^i, G^\alpha) \), where \( G^i \) and \( G^\alpha \) are given by (3.8), (3.10) and (3.11), respectively. Now, we put

\[(i) \quad G^i_\alpha := \frac{\partial G^i}{\partial y^\alpha}, \quad (ii) \quad G^i_\beta := \frac{\partial G^i}{\partial v^\beta}, \quad (iii) \quad G^\alpha_\beta := \frac{\partial G^\alpha}{\partial v^\beta}. \quad (3.12)\]

Then we have

**Lemma 1.** The coefficients \( G^i_\alpha \) defined by (3.12) satisfy in the following

\[ \left( G^i_\alpha(x, u, y, v) \right) = \begin{bmatrix} G^i(x, u, y, v) & G^i(x, u, y, v) \\ G^\alpha(x, u, y, v) & G^\alpha(x, u, y, v) \end{bmatrix}, \quad (3.13) \]

where

\[ G^i_j(x, u, y, v) := \frac{\partial G^i}{\partial y^j} = G^i_j - \frac{1}{4f_2^2} g^{ih} \frac{\partial f_2}{\partial y^j} F_2^2 + \frac{1}{2f_2^2} \frac{\partial f_2}{\partial u^\alpha} v^\alpha \delta_j^i, \quad (3.14) \]
\[ G^\alpha_j(x, u, y, v) := \frac{\partial G^\alpha}{\partial y^j} = \frac{1}{4f_2^2} g^{\alpha\gamma} \left( \frac{\partial f_2}{\partial x^\gamma} \frac{\partial F_2^2}{\partial v^\gamma} - \frac{\partial f_2}{\partial u^\gamma} F_2^2 \right), \quad (3.15) \]
\[ G^i_\beta(x, u, y, v) := \frac{\partial G^i}{\partial v^\beta} = \frac{1}{4f_1^2} g^{\alpha\gamma} \left( \frac{\partial f_1}{\partial x^\gamma} \frac{\partial F_2^2}{\partial v^\gamma} - \frac{\partial f_2}{\partial u^\gamma} F_2^2 \right), \quad (3.16) \]
\[ G^\alpha_\beta(x, u, y, v) := \frac{\partial G^\alpha}{\partial v^\beta} = G^\alpha_\beta - \frac{1}{4f_1^2} g^{\alpha\gamma} \frac{\partial f_1}{\partial v^\gamma} F_2^2 + \frac{1}{2f_1^2} \frac{\partial f_1}{\partial x^\gamma} y^\gamma \delta_\beta^\alpha. \quad (3.17) \]

**Proof.** By using (3.10) and (3.12), we have

\[ \frac{\partial G^i}{\partial y^j} = \frac{\partial G^i}{\partial y^j} + \frac{1}{4f_2^2} g^{ih} \left( \frac{\partial f_2}{\partial x^\gamma} \frac{\partial F_2^2}{\partial y^h} v^\gamma - \frac{\partial f_2}{\partial u^\gamma} F_2^2 \right) + g^{ih} \frac{\partial f_2}{\partial u^\gamma} \frac{\partial^2 F^2}{\partial y^h \partial y^h} v^\gamma. \quad (3.18) \]
But from the part (i) of (3.4), we get \( \frac{\partial F^2}{\partial y^h} = 2g_{hk}y^k \). Hence, we have

\[
\frac{\partial g^{ih}}{\partial y^j} \frac{\partial F^2}{\partial y^h} = 2 \frac{\partial g^{ih}}{\partial y^j} g_{hk} y^k = 0.
\] (3.19)

By plugging (i) of (3.4) and (3.19) in (3.18) and using \( g_{ih} g_{hj} = \delta_{ij} \), we have (3.14). In a similar way, we can obtain (3.15)-(3.17).

Now, we are going to consider \( VTM^o \), the kernel of the differential of the projection map

\[
\pi := (\pi_1, \pi_2) : TM^o_1 \oplus TM^o_2 \to M_1 \times M_2,
\]

which is a well-defined subbundle of \( TT M^o \). Locally, \( VTM^o \) is spanned by the natural vector fields \( \left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^m} \right\} \) and it is called the **doubly warped vertical distribution** on \( TM^o \). Then, using the functions introduced by (3.14)-(3.17), the nonholonomic vector fields are defined as follows

\[
\begin{align*}
\delta^d_{dx^i} &:= \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j} - G^\alpha_i \frac{\partial}{\partial v^\alpha}, \quad \text{(3.20)} \\
\delta^d_{du^\alpha} &:= \frac{\partial}{\partial u^\alpha} - G^j_\alpha \frac{\partial}{\partial y^j} - G^\beta_\alpha \frac{\partial}{\partial v^\beta}. \quad \text{(3.21)}
\end{align*}
\]

This make it possible to construct a complementary vector subbundle \( HTM^o \) to \( VTM^o \) in \( TT M^o \), which is locally presented as follows

\[
HTM^o := \text{span} \{ \delta^d_{dx^1}, \ldots, \delta^d_{dx^n}, \delta^d_{du^1}, \ldots, \delta^d_{du^m} \}.
\]

\( HTM^o \) is called the **doubly warped horizontal distribution** on \( TM^o \). Thus the tangent bundle of \( TM^o \) admits the decomposition

\[
TT M^o = HTM^o \oplus VTM^o. \quad \text{(3.22)}
\]

**Proposition 1.** Let \((f_1, M_1 \times f_2, M_2, F)\) be a doubly warped product Finsler manifold. Then \( G = (G^i_j) \) is the nonlinear connection on \( TM = TM_1 \oplus TM_2 \). Further, we have

\[
\begin{align*}
\frac{\partial G^i_j}{\partial y^k} y^k + \frac{\partial G^i_j}{\partial v^\gamma} v^\gamma &= G^i_j \\
\frac{\partial G^\alpha_\beta}{\partial y^k} y^k + \frac{\partial G^\alpha_\beta}{\partial v^\gamma} v^\gamma &= G^\alpha_\beta.
\end{align*}
\]
Definition 1. Using decomposition (3.22), the doubly warped vertical morphism \( v^d : TTM^\circ \to VTM^\circ \) is defined by
\[
v^d := \frac{\partial}{\partial y^i} \otimes \delta^d y^i + \frac{\partial}{\partial v^\alpha} \otimes \delta^d v^\alpha,
\]
where
(i) \( \delta^d y^i := dy^i + G^i_j dx^j + G^i_\beta du^\beta \),
(ii) \( \delta^d v^\alpha := dv^\alpha + G^\alpha_j dx^j + G^\alpha_\beta du^\beta \). (3.23)

For this projective morphism, we have
\[
v^d\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^i}, \quad v^d\left(\frac{\partial}{\partial v^\alpha}\right) = \frac{\partial}{\partial v^\alpha}, \quad v^d\left(\frac{\delta^d}{\delta^d x^i}\right) = 0, \quad v^d\left(\frac{\delta^d}{\delta^d u^\alpha}\right) = 0.
\]

From the above equations, we get \((v^d)^2 = v^d\) and \(\ker(v^d) = HTM^\circ\). This mapping is called the doubly warped vertical projective.

Definition 2. Using decomposition (3.22), the doubly warped horizontal projective \( h^d : TTM^\circ \to HTM^\circ \) is defined by \( h^d = id - v^d \) or
\[
h^d := \frac{\delta^d}{\delta^d x^i} \otimes dx^i + \frac{\delta^d}{\delta^d u^\alpha} \otimes du^\alpha.
\]

For this projective morphism, we have
\[
h^d\left(\frac{\delta^d}{\delta^d x^i}\right) = \frac{\delta^d}{\delta^d x^i}, \quad h^d\left(\frac{\delta^d}{\delta^d u^\alpha}\right) = \frac{\delta^d}{\delta^d u^\alpha}, \quad h^d\left(\frac{\partial}{\partial y^i}\right) = 0, \quad h^d\left(\frac{\partial}{\partial v^\alpha}\right) = 0.
\]
Thus we result that \((h^d)^2 = h^d\) and \(\ker(h^d) = VTM^\circ\).

Definition 3. Using decomposition (3.22), the doubly warped almost tangent structure \( J^d : HTM^\circ \to VTM^\circ \) is defined by
\[
J^d : \frac{\partial}{\partial y^i} \otimes dx^i + \frac{\partial}{\partial v^\alpha} \otimes du^\alpha,
\]
or
\[
J^d\left(\frac{\delta^d}{\delta^d x^i}\right) = \frac{\partial}{\partial y^i}, \quad J^d\left(\frac{\delta^d}{\delta^d u^\alpha}\right) = \frac{\partial}{\partial v^\alpha}, \quad J^d\left(\frac{\partial}{\partial y^i}\right) = J^d\left(\frac{\partial}{\partial v^\alpha}\right) = 0.
\]

Thus we result that \(J^2 = 0\) and \(\ker J = ImJ = VTM^\circ\).

Here, we introduce some geometrical objects of doubly warped Finsler manifold. In order to simplify the equations, we rewritten the basis of \( HTM^\circ \) and \( VTM^\circ \) as follows:
\[
\frac{\delta^d}{\delta^d x^a} = \frac{\delta^d}{\delta^d x^i} \delta^i_a + \frac{\delta^d}{\delta^d u^\alpha} \delta^\alpha_a,
\]
\[
\frac{\partial}{\partial y^a} = \frac{\partial}{\partial y^i} \delta^i_a + \frac{\partial}{\partial v^\alpha} \delta^\alpha_a.
\]
It is clear that $TTM^o = \text{span}\{ \frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^a} \}$. The Lie brackets of this basis is given by following

\[
\left[ \frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b} \right] = R^c_{\ ab} \frac{\partial}{\partial y^c}, \quad \left[ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b} \right] = G^c_{\ ab} \frac{\partial}{\partial y^c}, \quad \left[ \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right] = 0, \quad (3.24)
\]

where

\[
(i) \quad R^c_{\ ab} = \frac{\delta^d G^c_{\ de}}{\delta x^b}, \quad \frac{\delta^d G^c_{\ de}}{\delta x^a}, \quad (ii) \quad G^c_{\ ab} = \frac{\partial G^c_{\ ab}}{\partial y^c} \quad (3.25)
\]

**Corollary 1.** Let $(f_1 M_1 \times f_2 M_2, F)$ be a doubly warped product Finsler manifold. Then

\[
R^c_{\ iab} = (R^k_{\ ij}, R^k_{\ i\beta}, R^k_{\ \alpha\beta}, R^k_{\ ij}, R^k_{\ i\beta}, R^k_{\ \alpha\beta} \cdots)
\]

where

\[
R^k_{\ ij} := \frac{\delta^d G^k_{\ ij}}{\delta x^k}, \quad R^k_{\ i\beta} := \frac{\delta^d G^k_{\ \beta i}}{\delta x^k}, \quad R^k_{\ \alpha\beta} := \frac{\delta^d G^k_{\ \alpha\beta}}{\delta x^k}
\]

With a simple calculation, we have the following.

**Corollary 2.** Let $(f_2 M_1 \times f_1 M_2, F)$ be a doubly warped product Finsler manifold. Suppose that $G = (G^k_{\ ab})$ is the nonlinear connection on $TM$. Then

\[
G^c_{\ ab} = (G^k_{\ ij}, G^k_{\ i\beta}, G^k_{\ \alpha\beta}, G^k_{\ ij}, G^k_{\ i\beta}, G^k_{\ \alpha\beta} \cdots)
\]

where

\[
G^k_{\ ij} = \frac{\partial G^k_{\ ij}}{\partial y^j} = G^k_{\ ij} - \frac{1}{4f_j^2} \frac{\partial^2 G^k_{\ \alpha\beta}}{\partial y^j \partial y^\alpha \partial x^\beta} \frac{\partial f_j^2}{\partial x^\beta} = G^k_{\ ij},
\]

\[
G^k_{\ i\beta} = \frac{\partial G^k_{\ \beta i}}{\partial y^j} = -\frac{1}{4f_j^2} \frac{\partial G^k_{\ \beta i}}{\partial y^j} \frac{\partial f_j^2}{\partial x^\alpha} \frac{\partial F_i^2}{\partial x^\alpha} + \frac{1}{2f_j^2} \frac{\partial f_j^2}{\partial x^\beta} = G^k_{\ i\beta},
\]

\[
G^k_{\ \alpha\beta} = \frac{\partial G^k_{\ \alpha\beta}}{\partial y^j} = \frac{1}{2f_j^2} \frac{\partial G^k_{\ \alpha\beta}}{\partial y^j} = G^k_{\ \alpha\beta},
\]

\[
G^\gamma_{\ ij} = \frac{\partial G^\gamma_{\ ij}}{\partial y^j} = \frac{1}{2f_j^2} \frac{\partial G^\gamma_{\ ij}}{\partial y^j} \frac{\partial f_j}{\partial x^\alpha} = G^\gamma_{\ ij},
\]

\[
G^\gamma_{\ i\beta} = \frac{\partial G^\gamma_{\ i\beta}}{\partial y^j} = \frac{1}{4f_j^2} \frac{\partial G^\gamma_{\ i\beta}}{\partial y^j} \frac{\partial f_j}{\partial x^\alpha} \frac{\partial F_i^2}{\partial x^\alpha} + \frac{1}{2f_j^2} \frac{\partial f_j}{\partial x^\beta} = G^\gamma_{\ i\beta},
\]

\[
G^\gamma_{\ \alpha\beta} = \frac{\partial G^\gamma_{\ \alpha\beta}}{\partial y^j} = G^\gamma_{\ \alpha\beta} - \frac{1}{4f_j^2} \frac{\partial G^\gamma_{\ \alpha\beta}}{\partial y^j} \frac{\partial f_j}{\partial x^\alpha} \frac{\partial F_i^2}{\partial x^\alpha} = G^\gamma_{\ \alpha\beta}.
\]
Apart from $G_{ab}^c$, the functions $F_{ab}^c$ are given by

\[ F_{ab}^c = \frac{1}{2} \delta^c_{x^a} \left( \frac{\delta^d g_{ec}}{\delta x^d} + \frac{\delta^d G_{db}}{\delta x^d} - \frac{\delta^d g_{eb}}{\delta x^d} \right). \] (3.26)

**Corollary 3.** Let $(f_2 M_1 \times f_1 M_2, F)$ be a doubly warped product Finsler manifold. Then

\[ F_{ab}^c = (F_{ij}^k, F_{ij}^k, F_{ij}^k, F_{ij}^k, F_{ij}^k, F_{ij}^k, F_{ij}^k, F_{ij}^k), \]

where

\[ F_{ij}^k = F_{ij}^k - \frac{1}{2} g^{kh} \left( M_{ij}^h \frac{\partial^2 g_{hi}}{\partial y^r} + M_{ij}^h \frac{\partial g_{hi}}{\partial y^r} - M_{ij}^k \frac{\partial g_{hi}}{\partial y^r} \right) \] (3.27)

\[ F_{\alpha\beta}^k = \frac{1}{2} f_2 g^{kh} \left( \frac{\partial f_2^k}{\partial x^h} g_{\alpha\beta} - f_2^k G_{\alpha\beta} \frac{\partial g_{hi}}{\partial y^r} \right) = F_{\alpha\beta}^k \] (3.28)

\[ F_{ij}^\gamma = -\frac{1}{2} f_2 g^{\gamma\lambda} \left( \frac{\partial f_2^i}{\partial x^\lambda} g_{ij} - f_2^i G_{\lambda} \frac{\partial g_{ij}}{\partial y^r} \right) = F_{ij}^\gamma \] (3.30)

\[ F_{ij}^{\beta\gamma} = \frac{1}{2} f_2 g^{\gamma\lambda} \left( \frac{\partial f_2^i}{\partial x^\lambda} g_{ij} - f_2^i G_{\lambda} \frac{\partial g_{ij}}{\partial y^r} \right) = F_{ij}^{\beta\gamma} \] (3.31)

\[ F_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\lambda} \left( M_{\beta}^\mu \frac{\partial g_{\alpha\lambda}}{\partial x^\mu} + M_{\alpha}^\mu \frac{\partial g_{\lambda\beta}}{\partial x^\mu} - M_{\alpha}^\lambda \frac{\partial g_{\beta\lambda}}{\partial x^\mu} \right) \] (3.32)

and $F_{ij}^k = \frac{1}{2} g^{kh} (\frac{\delta g_{hi}}{\delta x^k} + \frac{\delta g_{hi}}{\delta x^k} - \frac{\delta g_{ij}}{\delta x^k})$, $F_{ij}^\gamma = \frac{1}{2} g^{\gamma\lambda} (\frac{\delta g_{hi}}{\delta x^\lambda} + \frac{\delta g_{hi}}{\delta x^\lambda} - \frac{\delta g_{ij}}{\delta x^\lambda})$, $M_i^r = \frac{1}{2 f_2} \frac{\partial f_2^i}{\partial x^r} \delta^r_i - \frac{1}{2 f_2} \frac{\partial f_2^i}{\partial x^r} \delta^r_i$, $F_{ij}^k = \frac{1}{2 f_2} \frac{\partial f_2^i}{\partial x^r} \delta^r_i$ and $M_i^a = \frac{1}{2 f_2} \frac{\partial f_2^i}{\partial x^r} \delta^r_i$.

**Proof.** By using (3.20), we obtain

\[ F_{ij}^k = \frac{1}{2} g^{kh} \left( \frac{\delta^d g_{hi}}{\delta x^k} \delta^d \frac{\delta^d g_{hi}}{\delta x^k} - \frac{\delta^d g_{ij}}{\delta x^k} \delta^d \frac{\delta^d g_{ij}}{\delta x^k} \right) \] (3.33)

Since $g_{ij}$ is a function with respect to $(x, y)$, then by (3.14) and (3.21) we get

\[ \frac{\delta^d g_{hi}}{\delta x^k} = \frac{\partial g_{hi}}{\partial x^r} - G_{ij} \frac{\partial g_{hi}}{\partial y^r} - M_{ij} \frac{\partial g_{hi}}{\partial y^r} \] (3.34)

Interchanging $i$ and $j$ in (3.33), gives us $\frac{\delta^d g_{hi}}{\delta x^i}$. Again, by interchanging $j$ and $h$ in (3.34), we obtain $\frac{\delta^d g_{ij}}{\delta x^j}$. By setting these equation in (3.33), we get (3.27). By similar calculations, we can prove the other relations. \(\square\)

**Lemma 2.** Let $(f_2 M_1 \times f_1 M_2, F)$ be a doubly warped product Finsler manifold. Then $\gamma \gamma F_{bc} = G_{bc}^\gamma$, where $F_{bc}$ and $G_{bc}^\gamma$ are defined by (3.20) and (i) of (3.12), respectively.
Proof. By using (i) of (3.4), we get

\[(i) \quad \frac{\partial g_{ij}}{\partial y^r} = \frac{\partial g_{ir}}{\partial y^j} = \frac{\partial g_{jr}}{\partial y^i}, \quad (ii) \quad \frac{\partial F^2_{ij}}{\partial y^r} = 2g_{ij}y^j. \quad (3.35)\]

Since \(g_{ij}\) is 0-positive homogenous, then by using Euler’s theorem and the part (i) of (3.35), we obtain

\[y^r \frac{\partial g_{ij}}{\partial y^r} = y^r \frac{\partial g_{ir}}{\partial y^j} = y^r \frac{\partial g_{jr}}{\partial y^i} = 0. \quad (3.36)\]

Using (3.36) and \(y^j F^k_{ij}\) in (3.27) imply that

\[y^j F^k_{ij} = g^k_{ij} y^j M^l_j \frac{\partial g_{li}}{\partial y^r}. \quad (3.37)\]

Direct calculation gives us

\[y^j M^l_j \frac{\partial g_{li}}{\partial y^r} = \frac{1}{2f_z^2} \frac{\partial f_z^2}{\partial u^\beta} v^\beta y^r \frac{\partial g_{li}}{\partial y^r} - \frac{1}{4 f_z^2} y^j \frac{\partial g_{rs}}{\partial y^r} \frac{\partial f_z^2}{\partial x^s} \frac{\partial g_{li}}{\partial y^r}. \]

Since \(g_{li}\) and \(g_{rs}\) are 0-positive homogenous, then from the above equation we conclude that

\[y^j M^l_j \frac{\partial g_{li}}{\partial y^r} = 0.\]

Therefore from (3.37), we derive

\[y^j F^k_{ij} = G^k_i. \quad (3.38)\]

From (3.15) and (3.28) and using (ii) of (3.35), (3.36) and 2-positive homogeneously of \(F^2_{ij}\) we get

\[v^\beta F^k_{ij} = \frac{1}{2f_z^2} \frac{\partial f_z^2}{\partial u^\beta} v^\beta \delta^k_i + \frac{1}{4 f_z^2} g^{kh} g^{rs} \frac{\partial g_{hi}}{\partial y^r} \frac{\partial f_z^2}{\partial x^s} F^2_{ij}. \quad (3.39)\]

On the other hand, we have

\[g^{kh} g^{rs} \frac{\partial g_{hi}}{\partial y^r} = g^{kh} g^{rs} \frac{\partial g_{hi}}{\partial y^r} = g^{kh} g^{rs} \frac{\partial g_{hi}}{\partial y^r} = -\frac{\partial g^{kh}}{\partial y^i}. \quad (3.40)\]

Setting (3.40) in (3.39) implies that

\[v^\beta F^k_{i\beta} = \frac{1}{2f_z^2} \frac{\partial f_z^2}{\partial u^\beta} v^\beta \delta^k_i - \frac{1}{4 f_z^2} g^{kh} \frac{\partial f_z^2}{\partial y^i} F^2_{ij} = G^k_i - G^k_i. \quad (3.41)\]

From (3.38) and (3.41), we get the following

\[y^\beta F^k_{i\beta} = y^j F^k_{ij} + v^\beta F^k_{i\beta} = G^k_i. \quad (3.42)\]
Similarly we obtain

\[ v^\beta F^\gamma_{i\beta} = G^\gamma_i, \quad v^\beta F^k_{\alpha\beta} = G^k_{\alpha}, \quad v^\beta F^\gamma_{i\beta} = G^\gamma_i, \]

\[ y^j F^k_{\alpha j} = 0, \quad y^j F^\gamma_{\alpha j} = G^\gamma_\alpha - G^\gamma_\alpha, \quad y^j F^\gamma_{ij} = 0 \]

These equations give us

\[ y^c F^k_{\alpha c} = y^j F^k_{\alpha j} + v^\beta F^k_{\alpha\beta} = G^k_{\alpha} \]

\[ y^c F^\gamma_{ic} = y^j F^\gamma_{ij} + v^\beta F^\gamma_{i\beta} = G^\gamma_i \]

\[ y^c F^\gamma_{ic} = y^j F^\gamma_{ij} + v^\beta F^\gamma_{i\beta} = G^\gamma_i. \]

This completes the proof.

The local components of doubly warped Cartan tensor field of Manifold \((f_1 M_1 \times f_2 M_2, F)\) is defined by

\[ C^a_{bc} = \frac{1}{2} g^{ae} \frac{\partial g_{bc}}{\partial y^e}. \]

From this definition, we conclude the following.

**Lemma 3.** Let \((f_1 M_1 \times f_2 M_2, F)\) be a doubly warped product Finsler manifold. Suppose that \(C^k_{ij}\) and \(C^\gamma_{\alpha\beta}\) be the local components of Cartan tensor field on \(M_1\) and \(M_2\), respectively. Then we have

\[ C^c_{ab} = \left( C^k_{ij}, C^k_{i\beta}, C^k_{\alpha j}, C^\gamma_{ij}, C^\gamma_{i\beta}, C^\gamma_{\alpha j}, C^\gamma_{\alpha\beta} \right), \]

where

\[ C^k_{ij} = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h} = C^k_{ij}, \quad C^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma \lambda} \frac{\partial g_{\alpha\beta}}{\partial y^\lambda} = C^\gamma_{\alpha\beta}, \]

and

\[ C^k_{i\beta} = C^k_{\alpha j} = C^k_{\alpha\beta} = C^\gamma_{ij} = C^\gamma_{i\beta} = C^\gamma_{\alpha j} = 0. \]

By using the Lemma 3 we can conclude the following.

**Corollary 4.** Let \((f_1 M_1 \times f_2 M_2, F)\) be a doubly warped product Finsler manifold. Then \((f_1 M_1 \times f_2 M_2, F)\) is a Riemannian manifold if and only if \((M_1, F_1)\) and \((M_2, F_2)\) are Riemannian manifolds.

Now, we are going to consider \(C\)-reducible doubly warped product Finsler manifold.

**Theorem 1.** Every \(C\)-reducible doubly warped product Finsler manifold \((f_1 M_1 \times f_2 M_2, F)\) is a Riemannian manifold.
Proof. We define the Matsumoto doubly warped tensor $M_{abc}$ as follows:

$$M_{abc} = C_{abc} - \frac{1}{n+1} \{ I_a h_{bc} + I_b h_{ac} + I_c h_{ab} \},$$  \hspace{1cm} (3.43)

where $I_a = g^{ik} C_{abc}$, $C_{abc} = g_{cd} C_{dab}^b$ and $h_{ab} = g_{ab} - \frac{1}{F^2} y_a y_b$ is the angular metric. By attention to (3.43) and the relations $C_{ijk} = f_1^2 C_{ijk}$ and $C_{\alpha\beta\gamma} = f_1^2 C_{\alpha\beta\gamma}$, we obtain

$$M_{\alpha jk} = -\frac{1}{n+1} \left\{ f_2^2 I_\alpha (y_{jk} - \frac{f_2^2}{F^2} y_j y_k) - \frac{f_2^2 f_2}{F^2} v_\alpha (I_j y_k + I_k y_j) \right\}. \hspace{1cm} (3.44)$$

Contracting (3.44) with $y^j y^k$ implies that

$$y^j y^k M_{\alpha jk} = -\frac{f_2^2 F_1^2}{(n+1)} \left( 1 - \frac{f_2^2 F_1^2}{F^2} \right) I_\alpha = -\frac{f_1^2 F_1^2 F_2^2}{(n+1) F^2} I_\alpha. \hspace{1cm} (3.45)$$

By assumption $M_{\alpha jk} = 0$, and then $I_\alpha = 0$, i.e., $(M_2, F_2)$ is a Riemannian manifold. By similar calculations, we can deduce that $(M_1, F_1)$ is a Riemannian manifold. This completes the proof.

Theorem 2. Every proper doubly warped product Finsler manifold $(f_j M_1 \times f_i M_2, F)$ with vanishing Berwald curvature is a Riemannian manifold.

Proof. The coefficients of Berwald curvature of a doubly warped product Finsler manifold $(f_j M_1 \times f_i M_2, F)$ are given by following:

$$B_{ijl}^k = B_{ijl}^k - \frac{1}{4 f_2^2} \partial^3 g^{kh} \partial f_2^2 \partial x^k F_2^2$$

$$B_{ijl}^k = -\frac{1}{4 f_2^2} \partial^2 g^{kh} \partial f_2^2 \partial F_2^2 \partial x^k \partial y^i \partial y^l \partial x^h \partial v^3$$

$$B_{ijl}^k = -\frac{1}{2 f_2^2} \partial g^{kh} \partial f_2^2 \partial x^h \partial x^i \partial x^l \partial x^k$$

$$B_{ijl}^k = -\frac{1}{2 f_2^2} \partial g^{kh} \partial f_2^2 \partial x^h \partial x^l$$

$$B_{ijl}^k = B_{ijl}^k - \frac{1}{4 f_2^2} \partial^3 g^{\nu \rho} \partial f_2^2 \partial F_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial x^k \partial F_1^2$$

$$B_{ijl}^k = -\frac{1}{4 f_2^2} \partial^2 g^{\nu \rho} \partial f_2^2 \partial F_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial y^i \partial y^l \partial x^k \partial F_1^2$$

$$B_{ijl}^k = -\frac{1}{2 f_2^2} \partial g^{\nu \rho} \partial f_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial y^i \partial y^l \partial x^k$$

$$B_{ijl}^k = -\frac{1}{2 f_2^2} \partial g^{\nu \rho} \partial f_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial x^k \partial F_1^2$$

$$B_{ijl}^k = -\frac{1}{2 f_2^2} \partial g^{\nu \rho} \partial f_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial x^k \partial F_1^2$$

$$B_{ijk}^k = -\frac{1}{2 f_2^2} \partial g^{\nu \rho} \partial f_2^2 \partial x^\nu \partial x^\rho \partial v^\lambda \partial u^\alpha \partial x^k \partial g_{jk}.$$
If \((f_2 M_1 \times f_1 M_2, F)\) is Berwaldian, then we have \(B^d_{abc} = 0\). By (3.49), we get

\[ C_{\alpha\beta\lambda} g^{kh} \frac{\partial f_1^2}{\partial x^h} = 0. \tag{3.56} \]

Multiplying (3.56) with \(g_{kr}\) implies that

\[ C_{\alpha\beta\lambda} \frac{\partial f_2^2}{\partial x^r} = 0. \tag{3.57} \]

By (3.57), if \(f_1\) is not constant then we result that \(C_{\alpha\beta\lambda} = 0\), i.e., \((M_2, F_2)\) is Riemannian. In the similar way, from (3.54) we conclude that if \(f_2\) is non constant then \((M_1, F_1)\) is Riemannian.

**Theorem 3.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold and \(f_1\) is constant on \(M_1\) \((f_2\) is constant on \(M_2\)). Then \((f_2 M_1 \times f_1 M_2, F)\) is Berwaldian if and only if \(M_1\) is Riemannian, \(M_2\) is Berwaldian and \(\frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} = 0\) \((M_2\) is Riemannian, \(M_1\) is Berwaldian and \(\frac{\partial g^{ij}}{\partial y^k} \frac{\partial f_1^2}{\partial x^i} = 0\)).

**Proof.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a Berwaldian manifold and \(f_1\) is constant on \(M_1\). Then from (3.54) we result that \(C_{ijk} = 0\), i.e., \((M_1, F_1)\) is Riemannian. Also, (3.53) gives us \(\frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_1^2}{\partial u^\alpha} = 0\). Differentiating this equation with respect to \((v^\beta)\) we deduce \(\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\lambda \partial v^\beta} \frac{\partial f_1^2}{\partial u^\alpha} = 0\) and consequently \(\frac{\partial^2 g^{\alpha\gamma}}{\partial v^\lambda \partial v^\beta} \frac{\partial f_1^2}{\partial u^\alpha} = 0\). Setting this equation in (3.51) we derive \(B^\gamma_{\alpha\beta\lambda} = 0\), i.e., \((M_2, F_2)\) is Berwaldian. In the similar way, we can prove the converse of this assertion.

**Corollary 5.** Let \((M_1 \times f_1 M_2, F)\) be a proper warped product Finsler manifold. Then \((M_1 \times f_1 M_2, F)\) is Berwaldian if and only if \(M_2\) is Riemannian, \(M_1\) is Berwaldian and \(\frac{\partial g^{ij}}{\partial y^k} \frac{\partial f_1^2}{\partial x^i} = 0\).

### 4 Riemannian Curvature of a Doubly Warped Product Manifold

The Riemannian curvature of a doubly warped product Finsler manifold \((f_2 M_1 \times f_1 M_2, F)\) with respect to Berwald connection is given by

\[ R^a_{\ bcd} = \frac{\delta^d F^a_{bc}}{\delta x^d} - \frac{\delta^d F^a_{bd}}{\delta x^c} + F^a_{de} F^e_{bc} - F^a_{ce} F^e_{bd}. \tag{4.58} \]

For the definition of Berwald connection see [22] and [23].

**Lemma 4.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold. Then

\[ R^a_{\ cd} = \gamma^b R^a_{\ bcd}, \]

where \(R^a_{\ cd}\) and \(\gamma^b R^a_{\ bcd}\) are given by (3.25) and (4.58).
By using Corollary 3 and Lemma 2, we obtain
\[
y^b R^i_{b kl} = y^b \frac{\delta^i Fr^i_{bkk}}{\delta^i x^l} - y^b \frac{\delta^i Fr^i_{bkl}}{\delta^i x^k} + y^b F^i_{tec} F^e_{bk} - y^b F^i_{ke} F^e_{kl}. \tag{4.59}
\]

By using Corollary 3 and Lemma 2 we obtain
\[
y^b \frac{\delta^i Fr^i_{bkl}}{\delta^i x^l} = \frac{\delta^i G^i_k}{\delta^i x^l} + F^i_{jk} G^j_k + F^i_{\beta k} G^\beta_k, \quad y^b F^i_{te} F^e_{bk} = F^i_{th} G^h_k + F^i_{\gamma k} G^\gamma_k. \tag{4.60}
\]

Interchanging \(i\) and \(j\) in (4.60) implies that
\[
y^b \frac{\delta^i Fr^i_{bkl}}{\delta^i x^l} = \frac{\delta^i G^i_k}{\delta^i x^l} + F^i_{jk} G^j_k + F^i_{\beta k} G^\beta_k, \quad y^b F^i_{ke} F^e_{kl} = F^i_{kh} G^h_k + F^i_{\gamma k} G^\gamma_k. \tag{4.61}
\]

Plugging (4.60) and (4.61) in (4.59), give us \(y^b R^i_{b kl} = R^i_{kl}\). In the similar way, we can obtain this relation for another indices. □

Using (4.58), we are going to compute the Riemannian curvature of a doubly warped product Finsler manifold.

**Lemma 5.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold. Then the Riemannian curvature of a doubly warped product Finsler manifold is given by following

\[
R^i_{j kl} = R^i_{j kl} - A_{(kl)} \left\{ M^i_j \frac{\partial F^i_{jk}}{\partial y^\gamma} + \frac{\delta^i M^i_j}{\delta^i x^l} + F^i_{th} M^h_{jk} + M^i_h F^h_{jk} - M^i_h M^h_{jk} \right. \\
+ \left. \frac{1}{4 f_1 f_2} g^{ik} g^{\gamma \delta} (g_{kl} \frac{\partial f^2}{\partial u^\delta} - f^2 G^\alpha_{\gamma} \frac{\partial g_{\delta k}}{\partial y^\gamma}) (g_{jk} \frac{\partial f^2}{\partial u^\gamma} - f^2 G^\beta_{\delta} \frac{\partial g_{k j}}{\partial y^\delta}) \right\}. \tag{4.62}
\]

\[
R^i_{\alpha kl} = A_{(kl)} \left\{ \frac{1}{2 f_1 f_2} \frac{\delta^l}{\delta^l x^k} \frac{\partial f^2}{\partial u^\alpha} \delta^k_j - f^2 G^\alpha_{\gamma} \frac{\partial g_{\gamma k}}{\partial y^\gamma} \right. \\
- \left. f^2 G^\alpha_{\gamma} \frac{\partial g_{\delta k}}{\partial y^\delta} \right\} + \frac{1}{4 f_1 f_2} (F^i_{rl} - M^i_k) \frac{\partial f^2}{\partial u^\alpha} \delta^k_j \\
- \frac{1}{4 f_1 f_2} f^2 G^\alpha_{\delta} \frac{\partial g_{\delta k}}{\partial y^\delta} (\frac{\partial f^2}{\partial u^\beta} \delta^k_j) + \frac{1}{4 f_1 f_2} f^2 G^\alpha_{\delta} \frac{\partial g_{\delta k}}{\partial y^\delta} (\frac{\partial f^2}{\partial u^\beta} \delta^k_j) \right\}. \tag{4.63}
\]

\[
R^i_{j \beta \gamma} = A_{(\beta \gamma)} \left\{ \frac{\delta^l}{\delta^l u^\alpha} \left( \frac{1}{2 f_2} \frac{\partial f^2}{\partial u^\delta} \delta^k_j - f^2 G^\alpha_{\gamma} \frac{\partial g_{\delta k}}{\partial y^\delta} \right) + \frac{1}{4 (f_2)^2} \frac{\partial f^2}{\partial u^\alpha} \delta^k_j \right. \\
- \left. f^2 G^\alpha_{\gamma} \frac{\partial g_{\delta k}}{\partial y^\delta} \right\}. \tag{4.64}
\]
\[ R^\gamma_{\alpha \beta} = R^\gamma_{\alpha \beta} \quad \text{and sub-} \quad \text{traction.} \]

Proof. By (4.38), we have

\[ R^j_{i kl} = \frac{\delta^i}{\delta x^j} F^j_{i kl} - \frac{\delta^j}{\delta x^i} + F^j_{ik} F^k_{jl} - F^i_{jk} F^j_{kl} \quad \text{(4.69)} \]

By using (3.27), we get

\[ F^i_{jk} = F^i_{jk} - M^i_{jk}. \]
Since $F^i_{jk}$ is a function with respect to $(x, y)$, then by (3.14) and (3.20) we derive that

$$
\frac{\delta^d F^i_{jk}}{\delta x^l} = \delta F^i_{jk} + M^r_i \frac{\partial F^i_{jk}}{\partial y^r} - \delta^d M^i_{jk}.
$$

(4.70)

Interchanging $k$ and $l$ in (4.70) implies that

$$
\frac{\delta^d F^i_{jl}}{\delta x^k} = \delta F^i_{jl} + M^r_i \frac{\partial F^i_{jl}}{\partial y^r} - \delta^d M^i_{jl}.
$$

(4.71)

By plugging (3.27), (3.28), (3.30), (4.70) and (4.71) in (4.69), we can obtain (4.62). In the similar way, we can obtain this relation for another indices.

**Theorem 4.** Let $(f_2 M_1 \times f_1 M_2, F)$ be a flat doubly warped product Finsler manifold. Then

(i) if $(M_1, F_1)$ is Riemannian then the components of the Riemannian Curvature of $M_1$ are as follows:

$$
R^i_{j kl} = \frac{||grad f_2||^2}{f_1^2} (\delta^i_l g_{jk} - \delta^i_k g_{jl}).
$$

(4.72)

(ii) if $(M_2, F_2)$ is Riemannian then the components of the Riemannian Curvature of $M_2$ are as follows:

$$
R_{\alpha \beta \lambda} = \frac{||grad f_1||^2}{f_2^2} (\delta^\alpha_\lambda g_{\alpha \beta} - \delta^\alpha_\beta g_{\alpha \lambda}).
$$

(4.73)

**Proof.** Since the proof of (ii) similar to (i), then we only prove (i). Let $(M_1, F_1)$ be a Riemannian manifold. Then $g_{ij}$ is a function of $(x)$, only. Therefore we have $M^i_{jk} = 0$. Also, the function $F^i_{jk}$ independent of $(y)$. By using (4.62), we conclude that

$$
R^i_{j kl} = R^i_{j kl} - \frac{1}{4 f_1^2 f_2^2} (\delta^i_l g_{jk} - \delta^i_k g_{jl}) g^{\alpha \gamma} \frac{\partial f_2^2}{\partial u^\alpha} \frac{\partial f_2^2}{\partial u^\gamma}.
$$

But we have

$$
g^{\alpha \gamma} \frac{\partial f_2^2}{\partial u^\alpha} \frac{\partial f_2^2}{\partial u^\gamma} = 4 f_2^2 ||grad f_2||^2.
$$

Hence the above equation rewritten as follows

$$
R^i_{j kl} = R^i_{j kl} - \frac{||grad f_2||^2}{f_1^2} (\delta^i_l g_{jk} - \delta^i_k g_{jl}).
$$

(4.74)

Since $(f_2 M_1 \times f_1 M_2, F)$ is a flat manifold, then we have $R^i_{j kl} = 0$. Therefore, (4.74) gives us (4.72).
Now, let \((M_1, F_1)\) is a Riemannian manifold and \(f_1\) is a scalar function on \(M_1\). Then from (4.72), we have
\[
R_{j^j k^l} = K_1 \|\text{grad} f_2\|^2 (\delta^i_j g_{jk} - \delta^i_k g_{jl}),
\]
where \(K_1\) is constant. Since \(\text{grad} f_2\) is independent of \(x\), then it is a constant function on \(M_1\). The similar argument is hold, if \((M_2, F_2)\) is Riemannian and \(f_2\) is constant on \(M_2\). Therefore we have the following corollary.

**Corollary 6.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a flat doubly warped product Finsler manifold. Then

(i) if \((M_1, F_1)\) is a Riemannian manifold and \(f_1\) is constant on \(M_1\), then \(M_1\) is a space of positive constant curvature \(K_1 \|\text{grad} f_2\|^2\);

(ii) if \((M_2, F_2)\) is a Riemannian manifold and \(f_2\) is constant on \(M_2\), then \(M_2\) is a space of positive constant curvature \(K_2 \|\text{grad} f_1\|^2\).

By the corollaries 4 and 6 we conclude the following.

**Corollary 7.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a flat doubly warped product Riemannian manifold. Then

(i) if \(f_1\) is constant on \(M_1\), then \(M_1\) has positive constant curvature \(K_1 \|\text{grad} f_2\|^2\) and \(M_2\) is a flat manifold;

(ii) if \(f_2\) is constant on \(M_2\), then \(M_2\) has positive constant curvature \(K_2 \|\text{grad} f_1\|^2\) and \(M_1\) is a flat manifold.

The flag curvature of a Finsler metric which plays the central role in Finsler geometry, is called a Riemannian quantity because it is a natural extension of sectional curvature in Riemannian geometry. For a Finsler manifold \((M, F)\), the flag curvature is a function \(K(P, y)\) of tangent planes \(P \subset T_x M\) and directions \(y \in P\). The Finsler metric \(F\) is said to be of scalar flag curvature if the flag curvature \(K(P, y) = K(x, y)\) is independent of flags \(P\) associated with any fixed flagpole \(y\) [17].

**Theorem 5.** Let \((M_1, F_1)\) be a Riemannian manifold and \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler space of scalar flag curvature \(\lambda_1(x, u, y, v)\). Then \((M_1, F_1)\) has constant curvature \(K_1\) if and only if
\[
\lambda_1(x, u, y, v) = K_1 - \frac{\|\text{grad} f_2\|^2}{f_1^2}.
\]

**Proof.** Since \((f_2 M_1 \times f_1 M_2, F)\) is a space of scalar flag curvature \(\lambda_1(x, u, y, v)\), then we have
\[
R_{{j^j k^l}} = \lambda_1(x, u, y, v)(\delta^i_j g_{jk} - \delta^i_k g_{jl}). \tag{4.75}
\]
By setting (4.75) in (4.74), we obtain

\[ R_{i j k l} = \left[ \lambda_1(x, u, y, v) + \frac{||\text{grad} f||^2}{f_1^2} \right] (\delta_i g_{jk} - \delta_i g_{jl}). \]  

(4.76)

By using (4.76), the proof is completes. □

Similarly, we have the following.

**Theorem 6.** Let \((M_2, F_2)\) be a Riemannian manifold and \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler space of scalar flag curvature \(\lambda_2(x, u, y, v)\). Then \((M_2, F_2)\) has constant curvature \(K_2\) if and only if

\[ \lambda_2(x, u, y, v) = K_2 - \frac{||\text{grad} f_1||^2}{f_2^2}. \]

**Corollary 8.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Riemannian manifold of the constant curvature \(\lambda\). Then

(i) if \(f_1\) is constant on \(M_1\), then \(M_1\) and \(M_2\) have constant curvatures \(\lambda + ||\text{grad} f_2||^2\) and \(\lambda\), respectively;

(ii) if \(f_2\) is constant on \(M_2\), then \(M_1\) and \(M_2\) have constant curvatures \(\lambda\) and \(\lambda + ||\text{grad} f_1||^2\), respectively.

## 5 Doubly Warped Sasaki-Matsumoto Metric

Let \((M, F)\) be a Finsler manifold. It is well known that there are several ways to associate the slit tangent bundle \(TM^\circ\) of \(M\) with Riemannian metrics which are naturally induced by the Finsler metric \(F\). The most well-known such metric is the Sasaki-Matsumoto lift

\[ G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \]

to the \(TM^\circ\) (see [13, 14, 15, 16]). Now, let \((f_1 M_1 \times f_2 M_2, F)\) be a doubly warped product Finsler manifold. Then the doubly warped Sasaki-Matsumoto metric can introduced as follows

\[ G = f_2^2 g_{ij} dx^i \otimes dx^j + f_1^2 g_{\alpha \beta} du^\alpha \otimes du^\beta + f_2^2 g_{ij} \delta^d y^i \otimes \delta^d y^j + f_1^2 g_{\alpha \beta} \delta^d v^\alpha \otimes \delta^d v^\beta, \]

(5.77)

where \(\delta^d y^i\) and \(\delta^d v^\alpha\) are defined by (3.23).

**Proposition 2.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold. Then the Levi-Civita connection \(\nabla^d\) on the Riemannian manifold
\((T^{\mu}, G^i)\) is locally expressed as follows:

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial^d_{\mu}}{\partial y^\alpha} = F^\mu_{ij} \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{1}{2} R^\mu_{ij} \frac{\partial}{\partial y^\alpha} + \frac{1}{2} \nabla^\mu_{ij} \frac{\partial^d_{\mu}}{\partial y^\alpha} + \frac{1}{2} R^\gamma_{ij} \frac{\partial}{\partial y^\alpha}. \tag{5.78}
\]

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial^d_{\mu}}{\partial y^\alpha} = (C^s_{ij} + \frac{g_{ij}}{2} g^{ks} R^s_{ki}) \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{f_2^d}{2 f_1^d} g_{ij} g^{\gamma \mu} R^s_{\mu i} \frac{\partial^d_{\mu}}{\partial y^\alpha} \tag{5.79}
\]

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial y^\beta} = \frac{f_2^d}{2 f_2^d} g_{\lambda \beta} g^{ks} R^\lambda_{ks} \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{1}{2} g_{\lambda \beta} g^{\mu \gamma} R^\lambda_{\mu i} \frac{\partial^d_{\mu}}{\partial y^\alpha} \tag{5.80}
\]

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial^d_{\mu}}{\partial y^\beta} = \frac{f_2^d}{2 f_1^d} g_{\lambda \beta} g^{ks} R^\lambda_{ks} \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{1}{2} g_{\lambda \beta} g^{\mu \gamma} R^\lambda_{\mu i} \frac{\partial^d_{\mu}}{\partial y^\alpha} \tag{5.81}
\]

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial^d_{\mu}}{\partial y^\gamma} = \frac{f_2^d}{2 f_2^d} g_{\lambda \beta} g^{ks} R^\lambda_{ks} \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{1}{2} g_{\lambda \beta} g^{\mu \gamma} R^\lambda_{\mu i} \frac{\partial^d_{\mu}}{\partial y^\alpha} \tag{5.82}
\]

\[
\nabla^d_{\frac{\partial}{\partial x^\alpha}} \frac{\partial^d_{\mu}}{\partial y^\gamma} = \frac{f_2^d}{2 f_1^d} g_{\lambda \beta} g^{ks} R^\lambda_{ks} \frac{\partial^d_{\mu}}{\partial x^\alpha} + \frac{1}{2} g_{\lambda \beta} g^{\mu \gamma} R^\lambda_{\mu i} \frac{\partial^d_{\mu}}{\partial y^\alpha} \tag{5.83}
\]
\[ \nabla^d \frac{\partial}{\partial y^j} = \frac{g^{ks}}{2} (G^r_{kj} g_{ri} + G^r_{kr} g_{rj} - \delta^d_{ks} \delta^r_{ij} + C^{\gamma}_{ij} \frac{\partial}{\partial y^s}) \delta^d_{ks} + \frac{\partial}{\partial y^s} \] (5.86)

\[ \nabla^d \frac{\partial}{\partial y^j} = \frac{1}{2f_1} g^{ks} (f_2^2 G^r_{kj} g_{ri} + f_2^2 G^r_{kr} g_{rj} - \delta^d_{ks} \delta^r_{ij} + C^{\gamma}_{ij} \frac{\partial}{\partial y^s}) \delta^d_{ks} \] (5.87)

\[ \nabla^d \frac{\partial}{\partial y^j} = \frac{1}{2f_1} g^{ks} (-\delta^d_{ks} f_1 g_{ka} \gamma + f_2^2 G^r_{ka} g_{rj} - \delta^d_{ks} \delta^r_{ij} + C^{\gamma}_{ij} \frac{\partial}{\partial y^s}) \delta^d_{ks} \] (5.88)

The Levi-Civita connection \( \nabla^d \) induces a connection \( \nabla \) on \( VT^M \), i.e.,

\[ \nabla_X v^d Y = v^d(\nabla^d_X Y), \] (5.89)

for \( X, Y \in \Gamma(TT^M) \). Then we have

**Lemma 6.** we have

\[ \nabla^d \frac{\partial}{\partial y^j} = F^r_{ij} \frac{\partial}{\partial y^s} + F^r_{ij} \frac{\partial}{\partial y^s}, \quad \nabla^d \frac{\partial}{\partial y^j} = F^r_{ij} \frac{\partial}{\partial y^s} + F^r_{ij} \frac{\partial}{\partial y^s} \] (5.90)

\[ \nabla^d \frac{\partial}{\partial y^j} = F^r_{ij} \frac{\partial}{\partial y^s} + F^r_{ij} \frac{\partial}{\partial y^s}, \quad \nabla^d \frac{\partial}{\partial y^j} = F^r_{ij} \frac{\partial}{\partial y^s} + F^r_{ij} \frac{\partial}{\partial y^s} \] (5.91)

\[ \nabla^d \frac{\partial}{\partial y^j} = C^s_{ij} \frac{\partial}{\partial y^s} \frac{\partial}{\partial y^j} = C^s_{ij} \frac{\partial}{\partial y^s} \frac{\partial}{\partial y^j} = 0 \] (5.92)

**Proof.** By using \( 5.79, 5.81, 5.84-5.88 \) and \( 5.89 \) it is sufficient to proof the following equation:

\[ \frac{1}{2} g^{ce} \frac{\delta d g_{ca}}{\delta x^b} + g_{dc} G^d_{ba} - g_{da} G^d_{bc} = F^c_{ab}. \] (5.93)

By using (i) of \( 3.12 \), we derive

\[ g_{dc} G^d_{ba} - g_{da} G^d_{bc} = g_{dc} \frac{\partial g_{da}^d}{\partial y^a} - g_{da} \frac{\partial g_{dc}^d}{\partial y^b} + g_{dc} \frac{\partial g_{da}^d}{\partial y^a} - g_{da} \frac{\partial g_{dc}^d}{\partial y^b} \] (5.94)

where \( G_c := g_{dc} G^d \). By direct calculations and using \( 3.89 \) and \( 3.78 \), we deduce that

\[ \frac{\partial}{\partial y^b} (\frac{\partial G_c}{\partial y^a} - \frac{\partial G_a}{\partial y^c}) = \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^c}. \] (5.95)
Putting (5.95) in (5.94) implies that
\[ g_{de} G^d_{ba} - g_{da} G^d_{be} = \frac{\delta^d}{\delta X^b} - \frac{\delta^d}{\delta X^e}, \]  
(5.96)

By plugging (5.96) in the left side of equation (5.93) and using (3.26), we obtain the right side of equation (5.93). □

We say that the vertical distribution \( VTM^0 \) is totally geodesic in \( TTM^0 \) if
\[ \nabla_d \frac{\partial}{\partial y} \in \Gamma(VTM^0). \]

Similarly, we say that the horizontal distribution \( HTM^0 \) is totally geodesic in \( TTM^0 \) if
\[ \nabla_d \frac{\partial}{\partial u} \in \Gamma(HTM^0). \]

For more details, see [21].

**Proposition 3.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold. Then \( VTM^0 \) is totally geodesic if and only if \( F_{ab} = G_{ab} \).

**Proof.** By using the definition of totally geodesic, we deduce that \( VTM^0 \) is totally geodesic if and only if
\[ \nabla_d \frac{\partial}{\partial y} \in \Gamma(VTM^0), \quad \nabla_d \frac{\partial}{\partial u} \in \Gamma(VTM^0), \]  
(5.97)
\[ \nabla_d \frac{\partial}{\partial y} \in \Gamma(VTM^0), \quad \nabla_d \frac{\partial}{\partial u} \in \Gamma(VTM^0). \]  
(5.98)

Since \( G^d \) is parallel with respect to \( \nabla_d \), then we have
\[ G(\nabla_d \frac{\partial}{\partial y}) \frac{\partial}{\partial y} + G(\nabla_d \frac{\partial}{\partial u}) \frac{\partial}{\partial u} = 0, \]  
(5.99)
\[ G(\nabla_d \frac{\partial}{\partial y}) \frac{\partial}{\partial y} + G(\nabla_d \frac{\partial}{\partial u}) \frac{\partial}{\partial u} = 0. \]  
(5.100)

By using (5.80) and (5.90), we obtain
\[ \frac{1}{2} g^{kl}(\frac{\delta^d}{\delta x^k} g_{ij} + G^r_{kj} g_{ri} + G^r_{ki} g_{rj}) = g^{lh}(G^r_{ih} - F^r_{ih}) g_{sj}, \]  
(5.101)
\[ \frac{1}{2} g^{\mu\nu}(\frac{\delta^d}{\delta x^\mu} f^2_{ij} g_{\nu j} + f^2_{ij} G^r_{\mu j} g_{ri} + f^2_{ij} G^r_{\nu i} g_{rj}) = f^2_{ij} g^{\lambda\nu}(G^r_{i\lambda} - F^r_{i\lambda}) g_{sj}. \]  
(5.102)

Putting the above equations in (5.80) give us
\[ \nabla_d \frac{\partial}{\partial y} \in \Gamma(VTM^0) \quad \text{if and only if} \quad G^r_{ih} = F^r_{ih} \quad \text{and} \quad G^r_{i\lambda} = F^r_{i\lambda}. \]

Therefore \( \nabla_d \frac{\partial}{\partial y} \in \Gamma(VTM^0) \) if and only if \( G^r_{ih} = F^r_{ih} \) and \( G^r_{i\lambda} = F^r_{i\lambda} \). Similarly, we obtain the other relations. □
Corollary 9. Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Riemannian manifold. Then \(\nabla VTM^o\) is totally geodesic distribution.

Proposition 4. Let \((f_2 M_1 \times f_1 M_2, F)\) be a doubly warped product Finsler manifold. Then \(\nabla HTM^o\) is totally geodesic if and only if \((M_1, F_1)\) and \((M_2, F_2)\) are Riemannian manifolds and \(R^g_{ij} = R^g_{i\beta} = R^g_{i\beta} = R^g_{\alpha\beta} = 0\).

Proof. By definition, \(\nabla HTM^o\) is totally geodesic if and only if
\[
\nabla_{\delta^d} \delta^d \delta^d u^\alpha \in \Gamma(HTM^o), \quad \nabla_{\delta^d} \delta^d u^\alpha \in \Gamma(HTM^o),
\]
(5.103)
\[
\nabla_{\delta^d} \delta^d \delta^d u^\alpha \in \Gamma(HTM^o), \quad \nabla_{\delta^d} \delta^d u^\alpha \in \Gamma(HTM^o).
\]
(5.104)

(5.78) implies that \(\nabla_{\delta^d} \delta^d \delta^d u^\alpha \in \Gamma(HTM^o)\) if and only if \(R^g_{ij} = 0\) and
\[
-C^g_{ij} + \frac{1}{2} R^g_{ij} = 0.
\]
(5.105)

Interchanging \(i\) and \(j\) in the above equation gives us
\[
-C^g_{ji} + \frac{1}{2} R^g_{ji} = 0.
\]
(5.106)

It is remarkable that \(C^g_{ij}\) and \(R^g_{ij}\) are symmetric and skew-symmetric tensors with respect to \(i\) and \(j\), respectively. Then (5.105)+ (5.106), implies that \(C^g_{ij} = 0\), i.e., \((M_1, F_1)\) is a Riemannian manifold. In the similar way, we can prove another relations. \(\square\)

6 Doubly Warped Vaisman Connection

In this section, the Riemannian manifold \((TM^d, G)\) is considered for which \(M = f_2 M_1 \times f_1 M_2\) and \(G\) is given by [5.77] and the vertical foliation \(\mathcal{F}_V\) (i.e., \(TTM^o = HTM^o \oplus VTM^o\)) on it. Also, we consider the notation from [4] and [20], related to foliated manifolds entitled Vaisman connection. The Vaisman connection \(\nabla^v\) on the Riemannian foliated manifold \((TM^o, \mathcal{F}_V, G)\), is uniquely defined by the following conditions:

(i) if \(Y \in \Gamma(VTM^o)\) (respectively \(\in \Gamma(HTM^o)\)), then \(\nabla^v_X Y \in \Gamma(VTM^o)\) (respectively \(\in \Gamma(HTM^o)\)) for every \(X\);

(ii) if \(X, Y, Z \in \Gamma(VTM^o)\) (\(\Gamma(HTM^o)\)), then \((\nabla^v_X G)(Y, Z) = 0\);

(iii) \(v^d(T(X, Y)) = 0\) if at least one of the arguments is in \(\Gamma(VTM^o)\) and \(h^d(T(X, Y)) = 0\) if at least one of the arguments is in \(\Gamma(HTM^o)\).
Now, we are going to compute the Vaisman connection on the Riemannian foliated manifold \((TM^c, F_V, G)\).

**Proposition 5.** Let \((f_2, M_1 \times f_1, M_2, F)\) be a doubly warped product Finsler manifold. Then the Vaisman connection \(\nabla^v\) on \((TM^c, F_V, G)\), is locally expressed with respect to the adapted local basis \(\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta u^\alpha}, \frac{\partial}{\partial y^j}\}\) as follows:

\[
\frac{\partial}{\partial y^j} = G_{ij}^k \frac{\partial}{\partial y^k} + G_{ij}^\gamma \frac{\partial}{\partial u^\gamma}, \quad \frac{\partial}{\partial y^b} = G_{ab}^k \frac{\partial}{\partial y^k} + G_{ab}^\gamma \frac{\partial}{\partial u^\gamma} (6.107)
\]

\[
\frac{\partial}{\partial y^j} = G_{ij}^\gamma \frac{\partial}{\partial y^j} + G_{ij}^\gamma \frac{\partial}{\partial u^\gamma}, \quad \frac{\partial}{\partial y^b} = G_{ab}^\gamma \frac{\partial}{\partial y^j} + G_{ab}^\gamma \frac{\partial}{\partial u^\gamma} (6.108)
\]

\[
\frac{\partial}{\partial y^j} = F_{ij}^k \frac{\partial}{\partial y^k} + F_{ij}^\gamma \frac{\partial}{\partial u^\gamma}, \quad \frac{\partial}{\partial y^b} = F_{ab}^k \frac{\partial}{\partial y^k} + F_{ab}^\gamma \frac{\partial}{\partial u^\gamma} (6.109)
\]

\[
\frac{\partial}{\partial y^j} = C_{ij}^k \frac{\partial}{\partial y^k} + C_{ij}^\gamma \frac{\partial}{\partial u^\gamma}, \quad \frac{\partial}{\partial y^b} = C_{ab}^k \frac{\partial}{\partial y^k} + C_{ab}^\gamma \frac{\partial}{\partial u^\gamma} (6.110)
\]

\[
\frac{\partial}{\partial y^j} = \frac{\partial}{\partial y^b} = 0, (6.111)
\]

\[
\frac{\partial}{\partial y^j} = \frac{\partial}{\partial y^b} = 0. (6.112)
\]

**Proof.** From the condition (i) of Vaisman connection, we have

\[
\frac{\partial}{\partial y^j} = A_{ij}^k \frac{\partial}{\partial y^k} + A_{ij}^\gamma \frac{\partial}{\partial u^\gamma}, \quad \frac{\partial}{\partial y^b} = B_{ij}^k \frac{\partial}{\partial y^k} + B_{ij}^\gamma \frac{\partial}{\partial u^\gamma} (6.113)
\]

By using (6.113) and the condition (ii) of Vaisman connection, we get

\[
0 = V(T(\frac{\delta}{\delta y^j} \frac{\partial}{\partial y^j})) = (A_{ij}^k - G_{ij}^k) \frac{\partial}{\partial y^k} + (A_{ij}^\gamma - G_{ij}^\gamma) \frac{\partial}{\partial u^\gamma} - B_{ij}^k \frac{\delta}{\delta y^k} + B_{ij}^\gamma \frac{\delta}{\delta u^\gamma}.
\]

The above equation \(A_{ij}^k = G_{ij}^k\), \(A_{ij}^\gamma = G_{ij}^\gamma\) and \(B_{ij}^k = B_{ij}^\gamma = 0\). Therefore, we obtain the first equation of (6.107) and the first equation of (6.112). Similarly, we can obtain the another relations.

The Lemma [6] and Proposition [5] give us the following.

**Theorem 7.** Let \((f_2, M_1 \times f_1, M_2, F)\) be a doubly warped product Finsler manifold. Then the Levi-Civita and the Vaisman connections on the foliated manifold \((TM^c, F_V, G)\) induce the same connection on the structural bundle if and only if \(F_{ab}^e = G_{ab}^e\).

Therefore, we conclude the following.

**Corollary 10.** Let \((f_2, M_1 \times f_1, M_2, F)\) be a doubly warped product Riemannian manifold. Then the Levi-Civita and the Vaisman connections on the foliated manifold \((TM^c, F_V, G)\) induce the same connection on the structural bundle.
**Definition 4.** A Riemannian foliated manifold with the Riemannian metric $G$ is called a Reinhart space if and only if

$$(\nabla^v_X G)(Y, Z) = 0,$$  \hspace{1cm} (6.114)

for all the sections $X$ of the structural bundle and $Y, Z$ sections of the transversal bundle, where the covariant derivative is taken with respect to the Vaisman connection of the manifold [25].

**Theorem 8.** Let $(f_1 M_1 \times f_2 M_2, F)$ be a doubly warped product Finsler manifold. The foliated manifold $(TM^\circ, F_V, G)$ is a Reinhart space if and only if $(M_1, F_1)$ and $(M_2, F_2)$ are Riemannian manifolds.

**Proof.** Let $X = X^i \partial/\partial y^i + X^\alpha \partial/\partial v^\alpha \in \Gamma(VTM^\circ)$ and $Y = Y^j \partial/\partial x^j + Y^\beta \partial/\partial u^\beta$, $Z = Z^k \partial/\partial x^k + Z^\gamma \partial/\partial u^\gamma$ belong to $\Gamma(HTM^\circ)$, also $\nabla^v$ be the Vaisman connection on $(TM^\circ, F_V, G^d)$. By (6.112), we obtain

$$(\nabla^v_X G)(Y, Z) = X^i \frac{\partial}{\partial y^i} (Y^j Z^k f^2_{jk}) + X^i \frac{\partial}{\partial y^i} (Y^\beta Z^\gamma f^2_{1\beta\gamma})$$

$$+ X^\alpha \frac{\partial}{\partial v^\alpha} (Y^j Z^k f^2_{jk}) + X^\alpha \frac{\partial}{\partial v^\alpha} (Y^\beta Z^\gamma f^2_{1\beta\gamma})$$

$$- X^i \frac{\partial Y^j}{\partial y^i} Z^k f^2_{jk} - X^i \frac{\partial Y^\beta}{\partial y^i} Z^\gamma f^2_{1\beta\gamma} - X^\alpha \frac{\partial Y^j}{\partial v^\alpha} Z^k f^2_{jk}$$

$$- X^\alpha \frac{\partial Y^\beta}{\partial v^\alpha} Z^\gamma f^2_{1\beta\gamma} - X^i Y^j \frac{\partial Z^k}{\partial y^i} f^2_{1\beta\gamma} = 2X^i Y^j Z^k f^2_{1\beta\gamma} + 2X^\alpha Y^\beta Z^\gamma f^2_{1\alpha\beta\gamma}.$$
or

\[
J\left(\frac{\delta^d}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta^d}{\delta x^i},
\]

\[
J\left(\frac{\delta^d}{\delta u^\alpha}\right) = -\frac{\partial}{\partial v^\alpha}, \quad J\left(\frac{\partial}{\partial v^\alpha}\right) = \frac{\delta^d}{\delta u^\alpha}.
\]

It is easy to see that \(J^2 = -I\), i.e., \(J\) is an almost complex structure on \(T M^\circ\). Also, simple calculations give us \(G(JX, JY) = G(X, Y)\), where \(X, Y \in \Gamma(TM^\circ)\). It means that \(G\) is almost Hermitian with respect to \(J\). The almost symplectic structure associated to the almost Hermitian structure \((G, J)\) is defined by

\[
\Omega(X, Y) := G(X, JY), \quad \forall X, Y \in \Gamma(TM^\circ).
\]

By using \((7.115)\) and the above equation, we obtain

\[
\Omega\left(\frac{\delta^d}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\delta^d}{\delta x^i}, J\left(\frac{\partial}{\partial y^j}\right)\right)
\]

\[
= G\left(\frac{\delta^d}{\delta x^i}, \frac{\delta^d}{\delta x^j}\right)
\]

\[
= f_2^2 g_{ij}.
\]

Similarly, we get the following

\[
\Omega\left(\frac{\delta^d}{\delta u^\alpha}, \frac{\partial}{\partial v^\beta}\right) = f_1^2 g_{\alpha\beta}
\]

and

\[
\Omega\left(\frac{\delta^d}{\delta x^i}, \frac{\delta^d}{\delta x^j}\right) = \Omega\left(\frac{\delta^d}{\delta x^i}, \frac{\delta^d}{\delta u^\alpha}\right) = \Omega\left(\frac{\delta^d}{\delta x^i}, \frac{\partial}{\partial v^\beta}\right)
\]

\[
= \Omega\left(\frac{\delta^d}{\delta u^\alpha}, \frac{\delta^d}{\delta u^\alpha}\right) = \Omega\left(\frac{\delta^d}{\delta u^\alpha}, \frac{\partial}{\partial v^\beta}\right)
\]

\[
= \Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\beta}\right)
\]

\[
= \Omega\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\beta}\right) = 0.
\]

Therefore, we can rewrite \(\Omega\) as follows:

\[
\Omega = f_2^2 g_{ij} dx^i \wedge \delta^d y^j + f_1^2 g_{\alpha\beta} du^\alpha \wedge \delta^d v^\beta.
\]

By direct calculations, it is result that \(\Omega = d\omega\), where

\[
\omega = f_2^2 y^i g_{ij} dx^j + f_1^2 v^\beta g_{\alpha\beta} du^\alpha.
\]

Thus \(\Omega\) is a close form. By attention to these explanations, we can conclude the following theorem.
Theorem 9. \((T^\alpha, G, J)\) is an almost Kählerian manifold.

Consequently, the Kähler structure on \((T^\alpha, J, G)\) is equivalent to the integrability condition of \(J\). The integrability of \(J\) is equal to the vanishing of tensor field \(N_J\), which is given by following

\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad (7.115)
\]

where \(X, Y \in \Gamma(T^\alpha)\). By (7.115), in computing \(N_J\), the following equations are presented

\[
\begin{align*}
N_J(\frac{\delta^d}{\delta x^i}, \frac{\delta^d}{\delta y^j}) &= -N_J(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = -R_{ij}^k \frac{\partial}{\partial y^k} - R_{ij}^\gamma \frac{\partial}{\partial \nu^\gamma} \quad (7.116) \\
N_J(\frac{\delta^d}{\delta x^\alpha}, \frac{\delta^d}{\delta x^\beta}) &= -N_J(\frac{\partial}{\partial \nu^\alpha}, \frac{\partial}{\partial \nu^\beta}) = -R_{\alpha\beta}^k \frac{\delta^d}{\delta y^k} - R_{\alpha\beta}^\gamma \frac{\delta^d}{\delta \nu^\gamma} \quad (7.117) \\
N_J(\frac{\delta^d}{\delta y^i}, \frac{\partial}{\partial \nu^l}) &= -N_J(\frac{\delta^d}{\delta x^k}, \frac{\partial}{\partial \nu^l}) = -R_{ik}^\gamma \frac{\delta^d}{\delta y^k} - R_{ik}^\gamma \frac{\delta^d}{\delta \nu^\gamma} \quad (7.118) \\
N_J(\frac{\delta^d}{\delta x^\alpha}, \frac{\partial}{\partial \nu^l}) &= -N_J(\frac{\delta^d}{\delta x^\alpha}, \frac{\partial}{\partial \nu^l}) = -R_{\alpha\beta}^k \frac{\delta^d}{\delta y^k} - R_{\alpha\beta}^\gamma \frac{\delta^d}{\delta \nu^\gamma} \quad (7.119) \\
N_J(\frac{\delta^d}{\delta x^i}, \frac{\partial}{\partial y^l}) &= -R_{ij}^k \frac{\delta^d}{\delta x^k} - R_{ij}^\gamma \frac{\delta^d}{\delta y^\gamma} \quad (7.120) \\
N_J(\frac{\delta^d}{\delta x^\alpha}, \frac{\partial}{\partial y^l}) &= -R_{\alpha\beta}^k \frac{\delta^d}{\delta x^k} - R_{\alpha\beta}^\gamma \frac{\delta^d}{\delta y^\gamma}. \quad (7.121)
\end{align*}
\]

Thus we have the following.

Lemma 7. The complex structure \(J\) defined by (7.115) is integrable if and only if \(R^a_{bc} = 0\), where \(a, b, c = 1, \ldots, n_1 + n_2\).

On the other hand, \(R^a_{bc} = 0\) is equivalent to the integrability of \(HT^\alpha\).

Therefore, using Theorem 9 and Lemma 7, we conclude the following.

Theorem 10. \((T^\alpha, G, J)\) is a Kählerian manifold if and only if the doubly warped horizontal distribution \(HT^\alpha\) is integrable.

References

[1] D. E. Allison, Pseudoconvexity in Lorentzian doubly warped product, Geom. Dedicata. 39 (1991), 223-227.

[2] G. S. Asanov, Finslerian extensions of Schwarzschild metric, Fortschr. Phys. 40 (1992), 667-693.

[3] G. S. Asanov. Finslerian metric functions over the product \(R \times M\) and their potential applications, Rep. Math. Phys. 41 (1998), 117-132.
[4] V. Balan and A. Manea, Leafwise 2-jet cohomology on foliated Finsler manifold, Balkan Society of Geometres Proceeding (BSGP) 16, Geometry Balkan Press, (2009), 28-41.

[5] D. Bao, S.S. Chern and Z. Shen, An Introduction to Riemannian-Finsler Geometry, Springer-Verlag, 2000.

[6] J. K. Beem, P. Ehrlich and T. G. Powell, Warped product manifolds in relativity, in: Selected Studies: A Volume Dedicated to the Memory of Albert Einstein, (North-Holland, Amsterdam), (1982), 41-56.

[7] J. B. Beem and T. G. Powell, Geodesic completeness and maximality in Lorentzian warped products, Tensor (N. S.), 39(1982), 31-36.

[8] A. Gebarowski, On conformally recurrent doubly warped products, Tensor (N. S.), 57 (1996), 192-196.

[9] Y. Ichijyô, Finsler spaces modeled on a Minkowski space, J. Math. Kyoto Univ. 16 (1976), 639-652.

[10] L. Kozma, I. R. Peter and C. Varga, Warped product of Finsler-manifolds, Ann. Univ. Sci. Pudapest 44, 157 (2001).

[11] M. Matsumoto, On C-reducible Finsler spaces, Tensor, N. S. 24 (1972), 29-37.

[12] M. Matsumoto and S. Hôjô, A conclusive theorem for C-reducible Finsler spaces, Tensor. N. S. 32 (1978), 225-230.

[13] E. Peyghan and A. Tayebi, Finslerian Complex and Kahlerian Structures, Journal of Non-Linear Analysis: RWA, 11 (2010), 3021-3030.

[14] E. Peyghan and A. Tayebi, A Kähler structures on Finsler spaces with non-zero constant flag curvature, J. Math. Phys. 51, 022904, (2010).

[15] E. Peyghan and A. Tayebi, Killing vector fields of horizontal Liouville type, C. R. Acad. Sci. Paris, Ser. 1, 349 (2011), 205-208.

[16] E. Peyghan, A. Tayebi and C. Zhong, Foliations on the tangent bundle of Finsler manifolds, Science in China, Series A: Math, (2011), preprint.

[17] Z. Shen, Differential Geometry of Spray and Finsler Spaces, Kluwer Academic Publishers, 2001.

[18] Z. Shen, Finsler manifolds of constant positive curvature, Contemporary Mathematics, 196(1995), 83-93.

[19] B. Unal, Doubly warped products, Diff. Geom. Appl. 15(2001), 253-263.

[20] I. Vaisman, Cohomology and Differential Forms, Marcel Dekker Inc., New York, 1973.
[21] B. Y. Wu, *Some results on the geometry of tangent bundle of Finsler manifolds*, Publ. Math. Debrecen. 71(2007), 185-193.

[22] A. Tayebi, A. Azizpour and E. Esrafilian, *On a family of connections in Finsler geometry*, Publ. Math. Debrecen. 72(2008), 1-15.

[23] A. Tayebi and B. Najafi, *Shen’s process on Finslerian connections*, Bull. Iran. Math. Society. Vol. 36, No. 2 (2010), 57-73.

[24] A. Tayebi and E. Peyghan, *On Ricci tensors of Randers metrics*, Journal of Geometry and Physics. 60(2010), 1665-1670.

[25] A. Tayebi and E. Peyghan, *On a class of Riemannian metrics arising from Finsler structures*, C. R. Acad. Sci. Paris, Ser. I, 349 (2011), 319-322.

Esmail Peyghan
Department of Mathematics, Faculty of Science
Arak University
Arak 38156-8-8349, Iran
Email: epeyghan@gmail.com

Akbar Tayebi
Department of Mathematics, Faculty of Science
Qom University
Qom, Iran
Email: akbar.tayebi@gmail.com