ERGODIC PROPERTIES AND THERMODYNAMIC FORMALISM OF
MARKOV MAPS INDUCED BY LOCALLY EXPANDING ACTIONS

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Abstract. For topologically mixing locally expanding semigroup actions generated by a finite collection of $C^{1+\alpha}$ conformal local diffeomorphisms, we provide a countable Markov partition satisfying the finite images and the finite cycle properties. We show that they admit inducing schemes and describe the tower constructions associated with them. An important feature of these towers is that their induced maps are equivalent to a subshift of countable type. Through the investigating the ergodic properties of induced map, we prove the existence of a liftable absolutely continuous stationary measure for the original action. We then establish a thermodynamic formalism for induced map and deduce the unicity of equilibrium state.

1. Introduction

In this article, we mainly describe an abstract inducing scheme for a locally expanding semigroup action generated by a finite collection of $C^{1+\alpha}$ (local) diffeomorphisms on a compact Riemannian manifold which parameterized by random walks. This scheme provides a symbolic representation over $(W,T,\tau)$, where $T$ is the induced map acting on the inducing domain $W$ and $\tau$ is the inducing time. This created symbolic dynamic allows to apply standard techniques from symbolic dynamics as well. This means that these actions can be partially described, in a symbolic way, as a subshift on a countable set of states. In particular, some basic facts about the original action can be achieved through the identifying the induced symbolic maps. This is a standard way of understanding partial facts about the initial dynamic. For instance, in the uniformly hyperbolic case, Bowen in [3] used a Markov partition as a tool to present a fairly complete description of ergodic properties of the dynamic. In [4], Bowen and Series described Markov partitions related to the action of certain groups of hyperbolic isometries on the boundary space. In this very special case of Fuchsian groups, not only a Markov map associated to the group was constructed, but also the induced map and the group were orbit equivalent. After Bowen, the scenario has been developed to the context of non-uniformly hyperbolic systems, see for instance [5,16,19,20]. In the non-uniformly hyperbolic setting, we cannot expect to have finite Markov partitions. However, in many cases it is possible to use the symbolic approach by finding a countable Markov partition, or a more general inducing scheme [20, 29]. The same strategy applied by Hofbauer [14]. He used a countable-state Markov model to study equilibrium states for piecewise monotonic interval maps.

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Inducing schemes also use the tower approach to model the system by a countable-state Markov shift \cite{20,21}. Every inducing scheme generates a symbolic representation by a tower which is well adapted to constructing absolutely continuous measures and equilibrium measures for a certain class of potential functions using the formalism of countable state Markov shifts. The projection of these measures from the tower are natural candidates for equilibrium measures for the original system.

Apart from the advantage of returning to the initial dynamic, studying the formalism of the induced map, as a typical example of countable Markov map, can also be interesting on its own. Sarig in series of papers, \cite{22–26}, has made many fundamental contributions to thermodynamics of countable Markov chains. He adapted transience, null recurrence, and positive recurrence for non-zero potential functions.

Here, we consider locally expanding semigroups generated by a finite family of $C^{1+\alpha}$ conformal (local) diffeomorphisms on a compact manifold $M$. However, a unified approach to the thermodynamic formalism for semigroup actions in the absence of probability measures invariant under all elements of the semigroup is still incomplete. Meanwhile, we show how the choice of the random walk in the semigroup settles the ergodic properties of the action. We construct a special countable Markov partition for such actions which is achieved primarily through a finite cover and satisfies the finite cycle property. Further, we generalize the notion of an inducing scheme to this kind of semigroup actions for generating a symbolic representation. We solve the liftability problem for towers associated with inducing schemes and prove the existence of an absolutely continuous stationary measure for the original action. Then, we use the symbolic approach to study equilibrium states of an appropriate class of real-valued potential functions. We also describe the thermodynamic formalism of the induced map through studying its Gibbs and equilibrium states.

2. Backgrounds and Statement of the Results

In this section we try to state the basic concepts and definition needed for our formalism. Throughout the paper $M$ is a compact connected smooth Riemannian manifold and $\lambda$ denotes the normalized Lebesgue measure.

2.1. Locally expanding semigroup actions. Given a finite set of continuous maps $g_i : M \to M$, $i = 1, 2, \ldots, N$, $N > 1$, and the finitely generated semigroup $\Gamma$ (under function composition) with the finite set of generators $F_1 = \{Id, g_1, \ldots, g_N\}$, we will write

$$\Gamma = \bigcup_{n \in \mathbb{N}_0} F_n,$$

where $F_0 = \{id\}$ and $g \in F_n$ if and only if $g = g_{i_1} \circ \cdots \circ g_{i_n}$, with $g_{i_j} \in F_1$. We will assume that the generator set $F_1$ is minimal, meaning that no function $g_j$, for $j = 1, \ldots, N$, can be expressed as a composition from the remaining generators. Observe that each element $g$ of $F_n$ may be seen as a word that originates from the concatenation of $n$ elements in $F^* = \{g_i : i = 1, 2, \ldots, N\}$. If $g$ is the concatenation of $n$ elements of $F^*$, we will write $|g| = n$. Take $P = (p_1, \ldots, p_N)$ a probability vector so that $p_i$ gives us the probability with which apply $g_i$. Consider the semigroup $\Gamma$ with the finite set of generators $F_1 = \{Id, g_1, \ldots, g_N\}$ and a row irreducible stochastic matrix $P = (p_{ij})_{N \times N}$ satisfying $p_{ij} \geq 0$, $\sum_{j=1}^N p_{ij} = 1$ for all $i \in \{1, \ldots, N\}$. Additionally assume that for any $i, j \in \{1, \ldots, N\}$, there exist $i_1, i_2, \ldots, i_k$ such that $p_{i_1}, p_{i_1i_2}, \ldots, p_{i_kj} > 0$. Here, we assume that $g_i, i = 1, \ldots, N$, are
$C^{1+\alpha}$ diffeomorphisms or local diffeomorphisms on $M$. Notice that a map $g : M \rightarrow M$ is a local diffeomorphism, if for each point $x$ in $M$, there exists an open set $U$ containing $x$, such that $g(U)$ is open in $M$ and $g|_U : U \rightarrow g(U)$ is a diffeomorphism.

Our focus is a random walk of $\Gamma$ in the phase space $M$ of the following nature (see [2]): specify a starting point $x \in M$ and a starting code $i_0 \in \{1, \ldots, N\}$. Choose a number $i_1 \in \{1, \ldots, N\}$ with the probability that $i_1 = j$ being $p_{i_0 j}$ and then define $x_1 = g_{i_1}(x_0)$. Then pick $i_2 \in \{1, \ldots, N\}$, with the probability that $i_2 = j$ being $p_{i_1 j}$, and go to the point $x_2 = g_{i_2}(x_1)$. Continue in this way to generate an orbit $\{x_n\}_{n=0}^\infty$. Notice that by having some entries in $(p_{ij})$ equal to zero, the allowable sequences in the random walk are restricted.

We say that a subset $A$ is (forward) invariant for the semigroup $\Gamma$ if $h(A) \subseteq A$ for all $h \in \Gamma$. For $x \in M$ the total forward orbit of $x$ is defined by

$$O^+_\Gamma(x) = \{h(x) : h \in \Gamma\}.$$ 

**Definition 2.1.** We say that the semigroup $\Gamma$ is locally expanding if for each $x \in M$ there exists a map $h \in \Gamma$ so that $m(Dh(x)) > 1$, where $m(A) = \|A^{-1}\|^{-1}$ denotes the co-norm of a linear transformation $A$.

**Remark 2.2.** The following facts come straightforwardly from the definition above.

(i) It is easily seen that, by compactness of $M$, if $\Gamma$ is locally expanding then there are a finite family of mappings $f_1, \ldots, f_k \in \Gamma$, finite collection of open balls $V_1, \ldots, V_k$ in $M$ and a constant $\sigma < 1$ such that $M = V_1 \cup \cdots \cup V_k$, the restriction $f_i|_{V_i}$ is a diffeomorphism onto its image and

$$||Df_i(x)^{-1}|| < \sigma \text{ for every } x \in V_i \text{ and } i = 1, \ldots, k.$$ 

(ii) By continuity of the mappings $x \mapsto ||Df_i(x)^{-1}||$, $i = 1, \ldots, k$, there is $r > 0$ such that the restriction $f_i|_{B_r(V_i)}$ is a diffeomorphism onto its image and

$$||Df_i(x)^{-1}|| < \sigma \text{ for every } x \in B_r(V_i) \text{ and } i = 1, \ldots, k,$$

where $B_r(V_i)$ stands for the $r$-neighborhood of the set $V_i$.

- **Conformality.** A smooth map $g$ is a conformal map if there exists a function $\alpha : M \rightarrow \mathbb{R}$ such that for all $x \in M$ we have that $Dg(x) = \alpha(x)\text{Isom}(x)$, where $\text{Isom}(x)$ denotes an isometry of $T_xM$. Clearly, $\alpha(x) = ||Dg(x)|| = m(Dg(x))$, for all $x \in M$.

Notice that every diffeomorphism or local diffeomorphism in dimension one is a conformal map.

### 2.2. Countable Markov partitions. Let $\Gamma$ be a semigroup of diffeomorphisms or local diffeomorphisms defined on a compact smooth manifold $M$. A countable collection $\{M_i : i \in I\}$ composed of closed subsets of $M$ together with a collection $\{h_i \in \Gamma : i \in I\}$ is called a countable Markov partition for $\Gamma$ if the following statements hold:

- $\lambda(M \setminus \bigcup_{i \in I} M_i) = 0$, and $M_i = \text{Cl}(\text{int}(M_i))$.
- $\text{int}(M_i) \cap \text{int}(M_j) = \emptyset$ whenever $i \neq j$.
- If $\text{int}(M_j) \cap h_i^{-1}(\text{int}(M_j)) \neq \emptyset$, then $h_i(M_i) \supset M_j$.

For a countable Markov partition $(M, \mathcal{H}) = (M_i, h_i)_{i \in I}$ we say that

- $(M, \mathcal{H})$ has the finite images property (FIP) if $\mathcal{B} = \{h_i(\text{int}(M_i)) : i \in I\}$ consists of finitely many open subsets $\{B_1, \ldots, B_N\}$ of $M$;
\(\diamond (\mathcal{M}, \mathcal{H})\) has the finite cycle property (FCP) if there exists \(n_1, \ldots, n_N \in I\) such that for all \(\ell \in I\), there exist \(i, j \in \{1, \ldots, N\}\) such that \(h_{n_i}(M_{n_i}) \supset M_{n_j}\) and \(h_{\ell}(M_{\ell}) \supset M_{n_j}\).

For a countable Markov partition \((\mathcal{M}, \mathcal{H}) = \{(M_i, h_i)\}_{i \in I}\) with the index set \(I\), consider a transition matrix \(A = (a_{ij})_{I \times I}\) which is defined by

\[
t_{ij} = \begin{cases} 
1 & \text{if} \ \text{int}(M_i) \cap h_{-1}^{-1}(\text{int}(M_j)) \neq \emptyset \\
0 & \text{otherwise}.
\end{cases}
\]

It is obvious that \(A = (t_{ij})_{I \times I}\) is a matrix of zeroes and ones with no zero columns or rows. Take

\[
\Sigma_A^+ := \{(i_0, i_1, \ldots) \in I^{\mathbb{N} \cup \{0\}} : \forall j \geq 0, \ t_{i_ji_{j+1}} = 1\}
\]

the corresponding one-sided topological Markov chain and \(\sigma_A : \Sigma_A^+ \rightarrow \Sigma_A^+\) the left shift map, \((\sigma_A(i))_j = i_{j+1}\), for each \(i = (i_0, i_1, \ldots) \in \Sigma_A^+\). We equip \(\Sigma_A^+\) with the topology generated by the base of cylinder sets

\[
[a_0, \ldots, a_{n-1}] := \{(i_0, i_1, \ldots) \in \Sigma_A^+ : i_j = a_j \text{ for } j = 0, \ldots, n-1\}
\]

for all \(n \in I\). We say that the one-sided topological Markov chain \(\Sigma_A^+\) is topologically mixing if one-sided topological Markov shift \((\Sigma_A^+, \sigma_A)\) is topologically mixing as a topological dynamical system.

2.3. Topologically mixing semigroup actions. For a locally expanding semigroup \(\Gamma\), take a finite family of mappings \(f_1, \ldots, f_k \in \Gamma\), finite collection of open balls \(V_1, \ldots, V_k\) in \(M\) and constants \(\sigma < 1\) and \(r > 0\) given by Remark 2.2. Take \(\eta > 0\) the Lebesgue number of the covering \(\{V_1, \ldots, V_k\}\). Assume \(\eta\) is small enough so that \(\eta < \frac{\sigma}{2}\). A finite word \(w = (i_1, \ldots, i_n)\) with \(i_j \in \{1, \ldots, k\}\) is an admissible word if there exists \(x \in M\) such that \(x \in V_{i_1}\) and \(f_{i_1} \circ \cdots \circ f_{i_1}(x) \in V_{i_{j+1}}\), for each \(j = 1, \ldots, n - 1\).

- Accessibility. For a given pair of points \(x, y \in M\), we say that \(y\) is accessible by \(x\), if there exists an admissible word \(w = (i_1, \ldots, i_n)\) such that \(f_{i_1}^n(x) = y\), where \(f_i^n := f_{i_1} \circ \cdots \circ f_{i_1}\), denoting by \(x \equiv_w y\). Given a ball \(B\) of radius \(\varepsilon\) and center \(y\), with \(\varepsilon < \eta\), if \(y\) is accessible by a point \(x \in M\), along an admissible word \(w = (i_1, \ldots, i_n)\), then the subset \(B_{i_1, \ldots, i_n} := f_{i_1}^{n-1}(B)\) is called an admissible \(n\)-cylinder, where \(f_{i_1}^{n-1}(B) = (f_{i_1}^{n})^{-1}(B)\). Notice that by Lemma 3.1, below, the map \(f_{i_1}^{n-1}(B)\) in the above definition is well defined on \(B\).

- Topologically mixing. A locally expanding semigroup \(\Gamma\) is topologically mixing if for each two small open sets \(U\) and \(V\) of \(M\) with \(\text{diam}(U)\) and \(\text{diam}(V)\) are less than \(\eta/2\), there exist an admissible word \(w = (i_1, \ldots, i_n)\) and an open ball \(B\) with radius \(\varepsilon\), such that \(B \supset V\) and the \(n\)-cylinder \(B_{i_1, \ldots, i_n} = f_{i_1}^{n-1}(B)\) is contained in \(U\); hence \(f_{i_1}^{n}(B_{i_1, \ldots, i_n}) \subset V\) which implies that \(f_{i_1}^{n}(U) \supset V\).

**Theorem A.** Let \(\Gamma\) be a locally expanding semigroup generated by \(C^{1+\alpha}\) conformal diffeomorphisms or local diffeomorphisms defined on a compact connected smooth manifold \(M\). If \(\Gamma\) is topologically mixing then \(\Gamma\) admits a countable Markov partition with the finite images and the finite cycle properties.

Theorem A is a essential modification of the Markov partition provided by the authors of this article, [9], to get FCP and BIP. The proof of Theorem A is given in Section 3.
2.4. Markov induced maps and towers for locally expanding semigroup actions.

For a locally expanding semigroup $\Gamma$, a tower is determined by a subset $W \subset M$, the base of the tower, a positive integer-valued function $\tau$ on $W$, the inducing time, the map $T: W \rightarrow W$, the induced map, and a countable partition $(\mathcal{M}, \mathcal{H})$ of $W$. The function $\tau$ is constant on each partition element $S$ and is a return time of $S$ to $W$ but it is not necessarily the first return time to the base. A crucial feature of the tower is that $\mathcal{M}$ is a generating partition for the induced map $T$ so that it is equivalent to a subshift on a countable set of states.

To be more precise, a transformation $T$ from some open and dense region $W$ is a Markov induced map for $\Gamma$ if there exist a countable Markov partition $(\mathcal{M}, \mathcal{H})$ and a return time function $\tau: S \rightarrow \mathbb{Z}^+$, where $S = \{\text{int}(M_i) : M_i \in \mathcal{M}\}$, such that for each $S \in S$, $\tau(\text{int}(M_i)) = |h_i|$, where $|h_i|$ is the length of $h_i$, and the inducing domain by $W := \bigcup_{M_i \in \mathcal{M}} \text{int}(M_i)$, and the induced map $T: W \rightarrow M$ by

$$T(x) := h_i(x), \text{ if } x \in \text{int}(M_i),$$

so that the following are satisfied:

(H1) (FIP and FCP) $(\mathcal{M}, \mathcal{H})$ has FIP and FCP properties;

(H2) (Extension) for each $M_i \in \mathcal{M}$ there exists a connected open set $U_{M_i} \supset M_i$, such that $h_i|_{U_{M_i}}$ is a diffeomorphism onto its image;

(H3) (Generating condition) the partition $\mathcal{M}$ is generating in the following sense: for any $i = (i_0, i_1, \ldots ) \in \Sigma^+_A$, the intersection

$$M_{i_0} \cap \left( \bigcap_{k \geq 1} (h_{i_0})^{-1} \circ \cdots \circ (h_{i_{k-1}})^{-1}(M_{i_k}) \right)$$

is singleton;

(H4) (Uniform expansion) there exists $0 < \sigma < 1$ such that for any $M_i \in \mathcal{M}$ and $x \in M_i$

$$\|DT(x)^{-1}\| < \sigma;$$

(H5) (Bounded distortion) there exists $K > 0$ such that for any $M_i \in \mathcal{M}$ and $x, y \in \text{int}(M_i)$

$$K^{-1} \leq \left| \frac{\det(DT(x))}{\det(DT(y))} \right| \leq K.$$ 

Then we say that $(\tau, (\mathcal{M}, \mathcal{H}))$ is an inducing scheme for $\Gamma$.

Theorem B. Let $\Gamma$ be a locally expanding topologically mixing semigroup of $C^{1+\alpha}$ conformal (local) diffeomorphisms on a compact connected manifold $M$. Then $\Gamma$ admits an inducing scheme satisfying conditions (H1), (H2), (H3), (H4) and (H5).

The proof of Theorem B is given in Section 4. The Authors in [8] study the connection between the topological entropy of the locally expanding semigroup action using the main properties of dynamical zeta function and in particular deduce the existence of absolutely continuous stationary measures. With the same motivation, we first prove the following.

Theorem C. Under the assumption of the previous theorem, there exists a $T$-invariant absolutely continuous ergodic measure $\mu_T$ for the induced map $T$. Moreover,

$$h_{\mu_T}(T) = \int_W \log J_T d\mu_T,$$

where $h_{\mu_T}(T)$ denotes the metric entropy of the map $T$. 
Establishing liftability of a given invariant measure could be a challenging task and one of the goals of this article is to address the liftability problem. Using this folklore method, we provide an absolutely continuous stationary measure for the semigroup \( \Gamma \) by lifting an absolutely continuous \( T \)-invariant measure (see subsection 5.1 for precise definitions).

**Corollary D.** There exists a liftable absolutely continuous ergodic stationary measure \( m \) for the semigroup \( \Gamma \).

We will prove Theorem C and Corollary D in Section 5. Finally, in Section 6, we prove that the induced map \( T \) is (measure-theoretic) isomorphic with a countable Markov subshift. We then establish a thermodynamic formalism for the induced map \( T \). In particular, we deduce the existence of Gibbs measures and equilibrium states for the induced Markov chain (see Theorem 6.2).

### 3. Countable Markov Partitions with the FIP and FCP

Dynamical systems that admit Markov partitions with finite or countable number of partition elements allow symbolic representations by topological Markov shifts with finite or countable alphabets. As a result these systems exhibit high level of chaotic behaviors of trajectories. Our aim in this section is to construct a countable Markov partition with the finite images and the finite cycle properties for locally expanding semigroups.

Let us fix a locally expanding semigroup \( \Gamma \) generated by a finite family of \( C^{1+\alpha} \) conformal diffeomorphisms or local diffeomorphisms on a compact manifold \( M \). Then, by Remark 2.2, there exist the mappings \( f_1, \ldots, f_k \in \Gamma \), a finite collection \( \{V_1, \ldots, V_k\} \) consists of finitely many open connected subsets of \( M \) and constants \( r > 0 \) and \( \sigma < 1 \) satisfying the following conditions:

1. the restriction \( f_i|_{B_r(V_i)} \) is a diffeomorphism onto its image,
2. \( \|Df_i(x)^{-1}\| < \sigma \) for every \( x \in B_r(V_i) \),
3. \( f_i|_{B_r(V_i)} \) is a conformal map,

where \( B_r(V_i) \) here stands for the \( r \)-neighborhood of the subset \( V_i \).

Take \( \eta > 0 \) the Lebesgue number of the covering \( \{V_1, \ldots, V_k\} \). Assume \( \eta \) is small enough so that \( \eta < \frac{r}{2} \).

The proof of Theorem A handled in this section via some auxiliary lemmas. Take the Lebesgue number \( \eta \) small enough so that \( \eta < \frac{r}{2} \). Suppose that \( w = (i_1, \ldots, i_n) \) is an admissible word for \( x \). For any \( 0 < \varepsilon < \frac{\eta}{2} \) the dynamical \( n \)-ball \( B(x, n, w, \varepsilon) \) defined as follows

\[
B(x, n, w, \varepsilon) := \{y \in M : d(f_w^n(x), f_w^n(y)) < \varepsilon, \text{ for every } 0 \leq j \leq n - 1\}.
\]

**Lemma 3.1.** Take a dynamical \((n+1)\)-ball of the form (3). Then \( f_w^n(B(x, n + 1, w, \varepsilon)) = B(f_w^n(x), \varepsilon) \), for any \((n+1)\)-admissible word of \( x \).

**Proof.** Obviously, by definition of a dynamical ball and due to assumption (b) above, we get

\[
d(f_w^j, f_w^n(z), f_w^n(w)) \leq \sigma^{n-j} d(z, w)
\]

for every \( z, w \in B(f_w^n(x), \varepsilon) \) and every \( 0 \leq j \leq n \), where \( f_w^{-n} = (f_w^n)^{-1} \). The inclusion \( f_w^n(B(x, n + 1, w, \varepsilon)) \subseteq B(f_w^n(x), \varepsilon) \) is an immediate consequence of definition of a dynamical ball. To prove the converse, consider the inverse map \( f_w^{-n} : B(f_w^n(x), \varepsilon) \to B(x, \varepsilon) \). Given any \( y \in B(f_w^n(x), \varepsilon) \), let \( z = f_w^{-n}(y) \). Then, \( f_w^n(z) = y \) and, by (4),

\[
d(f_w^j, f_w^n(z)) \leq \sigma^{n-j} d(f_w^n(z), f_w^n(x)) \leq d(y, f_w^n(x)) < \varepsilon
\]
for every $0 \leq j \leq n$. This shows that $z \in B(x, n + 1, w, \varepsilon)$. □

For a given pair of points $x, y \in M$, let $y$ be accessible by $x$ along an admissible word $w = (i_1, \ldots, i_n)$ of $x$, i.e. $f^j_w(x) = y$. Given a ball $B$ of radius $\varepsilon$ and center $y$, by Lemma 3.1, one gets that $f^{-n}_w(B) = B(x, n + 1, w, \varepsilon)$. In this case, we say that $B$ is reachable from the $(n + 1)$-ball $B(x, n + 1, w, \varepsilon)$.

Notice that, here, the ball of radius $r$ is meant with respect to the Riemannian distance $d(x, y)$ on $M$. Given any path-connected domain $D \subseteq M$, we define the inner distance $d_D(x, y)$ between two points $x$ and $y$ in $D$ to be the infimum of the lengths of all curves joining $x$ to $y$ inside $D$. For any curve $\gamma$ inside $D$, we denote by $d_D(\gamma)$ the length of $\gamma$. For a ball $B$, the notation $D(B)$ stands for the set of all diameters of $B$. We recall that a diameter of $B$ is a geodesic curve $\gamma$ inside $B$ which connects two points $x$ and $y$ from the boundary $\partial B$. We need the following auxiliary lemma.

**Lemma 3.2.** There is $K > 0$ such that for any open ball $B = B(y, \varepsilon)$, with $\varepsilon \leq \frac{\eta}{2}$, any $n \in \mathbb{N}$ and any admissible $n$-word $w = (i_1, \ldots, i_n)$ such that $B$ is accessible by an $(n + 1)$-ball $B_{n+1} = B(x, n + 1, w, \varepsilon)$ around the point $x$, one has that

$$\min_{\gamma \in D(B)} d_{B_{n+1}}(\gamma_n) \geq K \max_{\gamma \in D(B)} d_{B_{n+1}}(\gamma_n),$$

where, for each curve $\gamma \in D(B)$, $\gamma_n$ is the lifting of $\gamma$ inside $B(x, n + 1, w, \varepsilon)$, i.e. $f^j_w(\gamma_n) = \gamma$.

**Proof.** For any $z \in B$, take the points $z_j$, $1 \leq j \leq n$, such that $z_n = z$, $z_{j-1} = f^{-1}_j(z_j)$ and let $a_j(z_j) := m(Df^{-1}_j(z_j)) = \|Df^{-1}_j(z_j)\|$. By conformality of the mappings $f_j |_{B_r(V_j)}$, $j = 1, \ldots, k$, one has that $m(Df^{-n}_w(z)) = \|Df^{-n}_w(z)\| = \prod_{j=1}^n a_j(z_j)$. Take $a_{i_1, \ldots, i_n}(z) := \prod_{j=1}^n a_j(z_j)$, we conclude that

$$\frac{\max_{\gamma \in G(B)} d_{B_{n+1}}(f^{-n}_w \gamma)}{\min_{\gamma \in G(B)} d_{B_{n+1}}(f^{-n}_w \gamma)} \leq \frac{\sup\{a_{i_1, \ldots, i_n}(z) : z \in B\}}{\inf\{a_{i_1, \ldots, i_n}(z) : z \in B\}}.$$

Let $F_j(x) := \log(a_j(x))$, for $x \in f_j(B_r(V_j))$ and $j = 1, \ldots, k$, where $a_j(x) = m(Df^{-1}_j(x))$. Since $f_j$ is $C^{1+\alpha}$, the mapping $F_j$ is Hölder continuous, hence there exist constants $C \geq 1$ and $\alpha > 0$ such that for any $x, y \in f_j(B_r(V_j))$, $j = 1, \ldots, k$, one has that $|F_j(x) - F_j(y)| \leq Cd(x, y)^\alpha$. Hence, for every $z, u \in B$, taking the points $z_j$ and $u_j$ with $z_n = z$, $u_n = u$, $z_{j-1} = f^{-1}_j(z_j)$ and $u_{j-1} = f^{-1}_j(u_j)$, $j = 1, \ldots, n$, one has that

$$|\log a_{i_1, \ldots, i_n}(z) - \log a_{i_1, \ldots, i_n}(u)| = \sum_{j=1}^n |\log a_i(z_j) - \log a_i(u_j)|$$

$$\leq \sum_{j=1}^n |\log a_i(z_j) - \log a_i(u_j)|$$

$$= \sum_{j=1}^n |F_i(z_j) - F_i(u_j)| \leq C \sum_{j=1}^n d(z_j, u_j)^\alpha$$

$$\leq C(2\gamma)^\alpha \sum_{j=1}^{n} \sigma^{(n-j)\alpha} \leq \frac{C(2\gamma)^\alpha}{1 - \sigma^\alpha},$$

where $\gamma = \max\{\text{diam}(f_j(B_r(V_j))) : j = 1, \ldots, k\}$. It is enough to take $K := \exp(C(2\gamma)^\alpha)$. □
Let us take a finite covering $\mathcal{B} = \{B_1, \ldots, B_N\}$ of open balls of $M$ with centers $p_j, j = 1, \ldots, N$ and radius $\varepsilon$ with $0 < \varepsilon \leq \eta$, where $\eta < \frac{\varepsilon}{2}$ is the Lebesgue number of the covering $\{V_1, \ldots, V_k\}$. If the point $p_j$ is accessible by a point $x \in M$ along an admissible word $w = (i_1, \ldots, i_n)$ of $x$ then by Lemma 3.1, $f_w^{-n}(B_j) = B(x, n+1, \varepsilon)$, where $B(x, n+1, \varepsilon)$ is a dynamical $(n+1)$-ball around $x$. Put $M_{n,j,w}(x) := \text{Cl}(f_w^{-n}(B_j))$. If there is no confusion, we will abbreviate $M_{n,j,w}(x)$ by $M_{n,j,w}$. Briefly, we say that $M_{n,j,w}$ is a closed dynamical ball which is achieved through the collection $\mathcal{B}$. Let us take $\Omega$ the family of all admissible finite words with the alphabets $\{1, \ldots, k\}$.

**Proposition 3.3.** The set

$$\mathcal{V} := \{M_{n,j,w} : n \in \mathbb{N}, j \in \{1, \ldots, N\}, w \in \Omega \text{ with } M_{n,j,w} \ni B_j\}$$

of all closed dynamical balls achievable through the collection $\mathcal{B}$ satisfies the following conditions:

1) $\mathcal{V}$ is a covering of $M$;
2) for every $x \in M$ and $\delta > 0$, there is an element $M_{n,j,w} \in \mathcal{V}$ such that $x \in M_{n,j,w}$ and $\text{diam}(M_{n,j,w}) < \delta$;
3) there is a constant $K_1 > 0$, does not depend on the indices $n, j$ and the word $w$, such that for any $M_{n,j,w} \in \mathcal{V}$, $\text{diam}(M_{n,j,w})^d \leq K_1 \lambda(M_{n,j,w})$, where $d = \dim(M)$ and $\lambda$ denotes the normalized Lebesgue measure on $M$.

Each covering $\mathcal{V}$ enjoying all the conditions (1), (2) and (3) of Proposition 3.3 is called a Vitali covering of $M$.

**Proof.** We first claim that for each $\ell \in \mathbb{N}$, the collection

$$\mathcal{V}_\ell := \{M_{n,j,w} \in \mathcal{V} : n \geq \ell, j = 1, \ldots, N\}$$

is a covering of $M$. Indeed, for each $x \in M$ and for any $n \geq \ell$, there exists an admissible word $w \in \Omega$ of the length $n$. Since $\mathcal{B}$ is a covering of $M$ there is $j \in \{1, \ldots, N\}$, $f_w^n(x) \in B_j$ and hence, $x \in f_w^{-n}(B_j)$, that is $x$ is contained in $M_{n,j,w}$. Now, since $\mathcal{V} = \bigcup_{\ell=1}^{\infty} \mathcal{V}_\ell$, so the first condition holds. Condition (2) is an immediate consequence of inequality (4) given in Lemma 3.1. Finally, the last condition is a consequence of Lemma 3.2. □

By the classical Vitali Theorem [10] if $\mathcal{V}$ is a Vitali covering for a set $A \subset \mathbb{R}^m$, then there is in $\mathcal{V}$ a sequence of pairwise disjoint elements whose union exhausts all of $A$ but a Lebesgue null set. The next result is a reformulation of Vitali Covering Theorem in our context.

**Proposition 3.4.** Under the above assumptions, for any open set $U$ of $M$, there exists a finite or countably infinite disjoint subcollection $\{M_j\} \subseteq \mathcal{V}$ such that $\lambda(U \Delta \bigcup_j M_j) = 0$.

**Proof of Theorem A.** Take the semigroup $\Gamma$. We show that $\Gamma$ admits a countable Markov partition with the finite images and the finite cycle properties if it is topologically mixing.

The collections $\mathcal{V}$ and $\mathcal{V}_\ell$, provided by Proposition 3.3, are Vitali coverings of $M$. Take $\ell$ large enough so that $\text{diam}(\mathcal{V}_\ell) < \beta$, where $\beta$ is the Lebesgue number of the covering $\mathcal{B}$. Since $\Gamma$ is topologically mixing, hence for each $j = 1, \ldots, N$ there is a closed dynamical ball $M_{n,j,i,j,w} \in \mathcal{V}_\ell$, with $n_j \in \mathbb{N}$, $i_j \in \{1, \ldots, N\}$ and $w_j \in \Omega$, such that for any $j = 1, \ldots, N - 1$,

$$M_{n_j,i_j,w_j} \subset B_j, \quad f_{w_j}^{n_j}(\text{int}(M_{n_j,i_j,w_j})) = B_{j+1}, \quad \text{and } f_{w_N}^{n_N}(\text{int}(M_{n_N,i_N,w_N})) = B_1.$$
Put $U = M \setminus \bigcup_{B_i \in \mathcal{B}} \partial B_i$ and take $\tilde{U} = U \setminus \bigcup_{j=1}^N M_{n_j,i,j,w_j}$ which are both open subsets of $M$. Applying Proposition 3.4 to $\tilde{U}$, one can obtain a countably infinite subfamily $\tilde{\mathcal{M}} = \{M_i : i \in I\} \subseteq \mathcal{V}$, for some countable index set $I$, of disjoint closed dynamical balls such that $\lambda(\tilde{U} \triangle \bigcup_{i \in I} M_i) = 0$. Moreover, the elements of $\tilde{\mathcal{M}}$ are contained in $\tilde{U}$. In particular, for each $i \in I$, $\operatorname{int}(M_i) \cap \partial B_i = \emptyset$ and $\operatorname{int}(M_i) \cap M_{n_j,i,j,w_j} = \emptyset$. Note that for each $M_i \in \tilde{\mathcal{M}}$, there exists some $n_i \in \mathbb{N}$, $j_i \in \{1, \ldots, N\}$ and $w_i \in \Omega$ in such a way that $M_i = M_{n_i,j_i,w_i}$ and $f_{w_i}^n(\operatorname{int}(M_i)) = B_{j_i}$. For simplicity, put $h_i = f_{w_i}^n$. Take $\tilde{\mathcal{H}} := \{h_i : i \in I\} \subset \Gamma$ corresponding to $\tilde{\mathcal{M}}$. Clearly, $h_i(\operatorname{int}(M_i)) \in \mathcal{B}$. It is easily seen that the collection $\mathcal{M} = \tilde{\mathcal{M}} \cup \{M_{n_j,i,j,w_j} : j = 1, \ldots, N\}$ together with $\mathcal{H} = \tilde{\mathcal{H}} \cup \{f_{w_j}^n : j = 1, \ldots, N\}$ is a Markov partition for $\Gamma$ with the finite images and finite cycle properties, where $M_{n_j,i,j,w_j}$ and $f_{w_j}^n$ given by (6). Indeed, suppose that $h_j(\operatorname{int}(M_j)) \cap \operatorname{int}((M_i)) \neq \emptyset$. Since $h_j(\operatorname{int}(M_j)) \in \mathcal{B}$ and $M_i$ doesn’t intersect $\bigcup_{B_i \in \mathcal{B}} \partial B_i$, one has that $M_i \subset h_j(M_j)$, that is $(\mathcal{M}, \mathcal{H})$ satisfies the Markovian property, see Figure 1. Moreover, for each $i$, since $\mathcal{B}$ is a cover of $M$ and $\operatorname{diam}(M_i)$ is less than the Lebesgue number of $\mathcal{B}$, hence there exists $j \in \{1, \ldots, N\}$ with $M_i \subset B_{j+1}$, furthermore, by (6), $f_{w_j}^n(\operatorname{int}(M_{n_j,i,j,w_j})) = B_{j+1}$ and hence $f_{w_j}^n(M_{n_j,i,j,w_j}) \supset M_i$. Also, $h_i(\operatorname{int}(M_i)) = B_j$, for some $B_j \subset \mathcal{B}$ and therefore, $h_i(M_i) \supset M_{n_j,i,j,w_j}$. Thus the Markov partition $(\mathcal{M}, \mathcal{H})$ satisfies the finite images and finite cycle properties. Since $\mathcal{M}$ and $\mathcal{H}$ are countable, they can be reindexed with the positive integers, giving us the positive integers $b_j$, $j = 1, \ldots, N$, for which $M_{b_j} := M_{n_j,i,j,w_j} \in \mathcal{M}$, $b_j := f_{w_j}^n \in \mathcal{H}$ and $h_{b_j}(\operatorname{int}M_{b_j}) = B_{j+1}$, for $j = 1, \ldots, N-1$ and $h_{b_N}(\operatorname{int}M_{b_N}) = B_1$. Now the proof of Theorem A is finished.

Figure 1 illustrates a countable Markov partition with the finite images and the finite cycle properties. The red closed balls stand for the closed dynamical balls $M_{b_j}$.

![Figure 1. Countable Markov partition with BIP and FCP](image)

### 4. Inducing scheme

In this section, we prove the existence of inducing schemes for locally expanding topologically mixing semigroups. Fix any locally expanding topologically mixing semigroup $\Gamma$ of $C^{1+\alpha}$ conformal (local) diffeomorphisms on a compact connected manifold $M$. By Theorem A, the semigroup $\Gamma$ admits a Markov partition $(\mathcal{M}, \mathcal{H}) = \{(M_i, h_i)\}_{i \in \mathbb{N}}$ with the finite images and the finite cycle properties. In particular, $h_i(\operatorname{int}(M_i)) \in \mathcal{B} = \{B_1, \ldots, B_N\}$. In the rest of this section, we fix elements $M_{b_j} \in \mathcal{M}$ with the corresponding maps $h_{b_j}$, $j = 1, \ldots, N$, such that $h_{b_j}(\operatorname{int}M_{b_j}) \in \mathcal{B}$ and they satisfy the finite cycle property. This means that for indices
Moreover, for each $x_j \in \mathbb{N}$, $j = 1, \ldots, N$, we have

$$h_{b_j}(\text{int} M_{b_j}) = B_{j+1},$$

for $j = 1, \ldots, N - 1$ and $h_{b_N}(\text{int} M_{b_N}) = B_1$,

see the proof of Theorem A. Let us take the induced map $T : W \to M$ by $T(x) = h_i(x)$, if $x \in \text{int}(M_i)$, where $W = \bigcup_{M_i \in M} \text{int}(M_i)$ is the inducing domain. We obtain a bounded distortion formulation for the induced map $T$. This observation together with Theorem A guarantee the existence of an inducing scheme for the semigroup $\Gamma$ satisfying conditions $(H1)$, $(H2)$, $(H3)$, $(H4)$ and $(H5)$ in Section 2.

**Remark 4.1.** The following two facts hold.

i) By the finite images and the finite cycle properties, for each $i = 1, \ldots, N$, one has that

$$T^N(\text{int}(M_{b_i})) \supseteq \bigcup_{\ell = 1}^{\infty} \text{int}(M_{i\ell}).$$

Notice that the subset $\bigcup_{\ell = 1}^{\infty} \text{int}(M_{i\ell})$ is an open and dense subset of $M$ with the full Lebesgue measure.

ii) Each $M_j \in M$ is a closed dynamical $n$-ball, for some $n \in \mathbb{N}$. Define inductively closed dynamical $(n + n_i_1 + \cdots + n_i_\ell)$-balls, $\ell \in \mathbb{N}$, by: $M_{i_0} := M_j$, $h_{i_0} := h_j$, and for any $\ell \geq 1$, let

$$M_{i_0, \ldots, i_\ell} := h_{i_\ell}^{-1}(M_{i_0, \ldots, i_{\ell-1}}) \text{ with } M_{i_0, \ldots, i_{\ell-1}} \subset h_{i_{\ell}}(M_{i_{\ell}}),$$

where $n_{ij}$ is the length of $h_{ij}$. Hence, we have the following chain of expanding maps:

$$M_{i_0, \ldots, i_\ell} \overset{h_{i_\ell}}{\to} M_{i_0, \ldots, i_{\ell-1}} \cdots \overset{h_{i_0}}{\to} \text{int}(M_{i_0})$$

with $h_{i_0}(\text{int}(M_{i_0})) \subseteq \mathcal{B}$. In particular,

$$\text{diam}(M_{i_0, \ldots, i_\ell}) \leq \sigma^{n_{i_\ell}} \text{diam}(M_{i_0, \ldots, i_{\ell-1}}).$$

It is not hard to see that there exists a one to one correspondence between the dynamical balls $M_{i_0, \ldots, i_\ell}$ and the cylinder sets $[i_0, i_1, \ldots, i_\ell]$ of $\Sigma_A^+$.

The following formulation of the bounded distortion property is satisfied.

**Lemma 4.2.** There exists $K_1 > 1$ such that for any dynamical ball $M_{i_0, \ldots, i_\ell}$ given by (7) in Remark 4.1 and for each $y, z \in \text{int}(M_{i_0, \ldots, i_\ell})$ one has

$$K_1^{-1} \leq \frac{|\text{det}(DT^j(y))|}{|\text{det}(DT^j(z))|} \leq K_1, \ j = 0, \ldots, \ell + 1.$$

**Proof.** By construction, for each dynamical ball $M_{i_0, \ldots, i_\ell}$ there is an admissible word $\omega = (i_0, i_1, \ldots, i_\ell)$ with $i_j \in \mathbb{N}$, $j = 0, \ldots, \ell$, such that

$$h_{i_0} \circ h_{i_1} \circ \cdots \circ h_{i_\ell}(\text{int}(M_{i_0, \ldots, i_\ell})) = B_m \in \mathcal{B}, \text{ for some } m \in \{1, \ldots, N\}$$

and hence $T^{\ell+1}(\text{int}(M_{i_0, \ldots, i_\ell})) = B_m$. In particular,

$$\text{int}(M_{i_0, \ldots, i_\ell}) = h_{i_\ell}^{-1} \circ \cdots \circ h_{i_0}^{-1}(B_m).$$

Moreover, for each $x \in \text{int}(M_{i_0, \ldots, i_\ell})$ and all $1 \leq j \leq \ell$ one has that

$$h_{i_{\ell-j}} \circ \cdots \circ h_{i_j}(x) \in h_{i_{\ell-j-1}}^{-1} \circ \cdots \circ h_{i_0}^{-1}(B_m).$$
Also, by construction, for each $0 \leq j \leq \ell$ there exists an admissible finite word $w^{(j)}$ of the alphabets $\{1, \ldots, k\}$ with the length $n_j$ such that $h_{ij} = f_{w^{(j)}_i}$, see the proof of Proposition 3.3 for more details. Take $n = n_0 + \ldots + n_\ell$. Then there is an admissible finite word $w = (j_1, \ldots, j_n)$ of the alphabets $\{1, \ldots, k\}$ with the length $n$ such that $h_{i_0} \circ h_{i_1} \circ \cdots \circ h_{i_n} = f_w^n$, where $f_w^n = f_{j_1} \circ \cdots \circ f_{j_n}$. Finally, by construction, for each $1 \leq t \leq n$, one has that

$$\text{diam}(f_{j_{n-t}}^{-1} \circ \cdots \circ f_{j_1}^{-1}(B_m)) \leq \sigma \text{diam}(f_{j_{n-t-1}}^{-1} \circ \cdots \circ f_{j_1}^{-1}(B_m)),$$

where $0 < \sigma < 1$ is given by Remark 2.2.

By assumption, the mapping $g_i := \log |\det Df_i|_{V_i}$, for all $1 \leq i \leq k$, is $\alpha$-Hölder and thus for all $z, y \in V_i$, it holds that

$$|g_i(z) - g_i(y)| \leq C_0 d(z, y)^\alpha,$$

for some constants $C_0 > 0$ and $\alpha > 0$ (independent of $i$).

By these facts, it is easy to see that for each $z, y \in \text{int}(M_{i_0, \ldots, i_\ell})$ the following holds:

$$d(f_w^t(z), f_w^t(y)) \leq \sigma^{n-t}(\text{diam}(B)) \leq 2\varepsilon \sigma^{n-t},$$

where $f_w^t = f_{j_1} \circ \cdots \circ f_{j_t}$, $1 \leq t \leq n$. Therefore

$$\log \left| \frac{\det Df_w^t(z)}{\det Df_w^t(y)} \right| = \sum_{k=1}^t |g_{j_{t-k+1}}(f_w^{k-1}(z)) - g_{j_{t-k+1}}(f_w^{k-1}(y))|$$

$$\leq \sum_{k=1}^t C_0 d(f_w^{k-1}(z), f_w^{k-1}(y))^\alpha$$

$$\leq C_0(2\varepsilon)^\alpha \sum_{k=1}^t \sigma^{\alpha(n-t)}$$

$$\leq C_0(2\varepsilon)^\alpha \sum_{k=0}^\infty \sigma^{k\alpha}.$$ 

Take $K_1 = \exp(C_0(2\varepsilon)^\alpha \sum_{k=0}^\infty \sigma^{k\alpha})$. Then by this choice of $K_1$, the inequality of the lemma holds.

By Theorem A, the previous lemma and our construction, Theorem B can be followed.

**Remark 4.3.** The following two facts hold.

i) Let $h_j(\text{int}(M_j)) = B_i$, for some $B_i \in \mathcal{B}$. If $\text{int}(M_k) \subset B_i$ then $h_j^{-1}(\text{int}(M_k)) \subset \text{int}(M_j)$. We say that $h_j^{-1}|_{\text{int}(M_k)} : \text{int}(M_k) \to h_j^{-1}(\text{int}(M_k)) \subset \text{int}(M_j)$ is an inverse branch of $T$. More generally, for any $n \geq 1$, we call an inverse branch of $T^n$ a composition

$$h_{i_n}^{-1} \circ \cdots \circ h_{i_1}^{-1}|_{\text{int}(M_k)} : \text{int}(M_k) \to h_{i_n}^{-1} \circ \cdots \circ h_{i_1}^{-1}(\text{int}(M_k)) \subset \text{int}(M_{i_n}),$$

where $(i_1, \ldots, i_n)$ is an admissible word, $M_k \in \mathcal{M}$ with $h_{i_1}^{-1}(\text{int}(M_k)) \subset \text{int}(M_{i_1})$ and $h_{i_j}^{-1} \circ \cdots \circ h_{i_1}^{-1}(\text{int}(M_k)) \subset \text{int}(M_{i_j})$, for $1 \leq j \leq n$.

Take $\bar{W} := W \setminus \bigcup_{k \geq 0} T^{-k}(M \setminus W)$, where $W = \bigcup_{i \in \mathbb{N}} \text{int}(M_i)$ and $T^{-k}(M \setminus W)$ is the union of all inverse branches of $T^k$ on $M \setminus W$. Then, $\lambda(\bar{W}) = 1$. 

ii) Clearly, for each \( x \in \overline{W} \) there is a sequence \( \{M_{i_0,\ldots,i_{\ell-1}}\} \) of dynamical balls given by part (ii) of Remark 4.1 such that \( x \in \text{int}(M_{i_0,\ldots,i_{\ell-1}}) \). For simplicity we denote these dynamical balls by \( M_\ell(x) = M_{i_0,\ldots,i_{\ell-1}} \).

5. Ergodic properties of Markov induced maps and liftability

Consider the induced map \( T \) of locally expanding topologically mixing semigroup \( \Gamma \). The first statement of Theorem C is followed by the next result.

**Proposition 5.1.** There are \( C_0 > 0 \) and a \( T \)-invariant absolutely continuous probability measure \( \mu_T = \rho_T \lambda \) with \( C_0^{-1} \leq \rho_T \leq C_0 \), where \( \lambda \) denotes the normalized Lebesgue measure. In particular, \( \mu_T \) is ergodic.

**Proof.** Let us take \( J_T(x) = |\text{det}DT(x)| \), for each \( x \in W \). By Lemma 4.2, the following is satisfied: for each \( y, z \in \text{int}(M_{i_0,\ldots,i_{\ell-1}}) \) one has that

\[
K_1^{-1} \leq \frac{J_T(y)}{J_T(z)} \leq K_1.
\]

Now by the previous inequality and the change of variable formula there is a constant \( K_2 > 0 \) such that for any dynamical ball \( M_{i_0,\ldots,i_{\ell-1}} \) given by (7), with \( T^\ell(\text{int}(M_{i_0,\ldots,i_{\ell-1}})) = B_m \), for some \( B_m \in B \), if \( A_1, A_2 \subset B_m \) are two measurable subsets then

\[
K_2^{-1} \frac{\lambda(A_1)}{\lambda(A_2)} \leq \frac{\lambda(T^{-\ell}(A_1))}{\lambda(T^{-\ell}(A_2))} \leq K_2 \frac{\lambda(A_1)}{\lambda(A_2)},
\]

where \( T^{-\ell} \) is the corresponding inverse branch defined as the previous remark. Indeed, fixing \( x \in \text{int}(M_{i_0,\ldots,i_{\ell-1}}) \), it comes out that

\[
\frac{\int_{T^{-\ell}(A_1)} J_T d\lambda}{\int_{T^{-\ell}(A_2)} J_T d\lambda} \leq K_1^2 \frac{\lambda(T^\ell(x))\lambda(T^{-\ell}(A_1))}{\lambda(T^\ell(x))\lambda(T^{-\ell}(A_2))},
\]

and with the same argument we prove the other inequality of (9) with \( K_2^2 = K_2 \).

For each \( \ell \in \mathbb{N} \), consider the collection \( \mathcal{W}_\ell \) composed of all dynamical balls \( M_{i_0,\ldots,i_{\ell-1}} \) given by (7). Clearly \( \mathcal{W}_\ell \) is a countable partition of \( W \).

We claim that there exists \( C_0 > 0 \) such that for each \( \ell \in \mathbb{N} \) and any measurable subset \( A \) of \( W \) one has that

\[
C_0^{-1} \lambda(A) \leq \lambda(T^{-\ell}(A)) \leq C_0 \lambda(A).
\]

In fact, for each inverse branch as above, we set \( A_2 = B_m \) and \( A_1 = A \cap M_k \), for each \( M_k \subset B_m \). Then \( \lambda(T^{-\ell}(A)) \) is the sum of the terms \( \lambda(T^{-\ell}(A \cap M_k)) \), with \( M_k \subset B_m \) and \( B_m \in B \), over all inverse branches of \( T^\ell \), see Remarks 4.1 and 4.3. It follows from these observations and (9) that the claim holds.

Using (10) we will see that every accumulation point \( \mu_T \) of the sequence

\[
\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} T_i^* \lambda
\]

is a \( T \)-invariant probability absolutely continuous with respect to \( \lambda \), with density \( \rho_T \) bounded from zero and from infinity. Indeed, by (10), for every \( n \), the density \( \rho_n = d\mu_n/d\lambda \) satisfies \( C_0^{-1} \leq \rho_n \leq C_0 \) and the same holds for the density of the accumulation point \( \mu_T \).
Additionally, by statement (i) of Remark 4.1 and absolute continuity of $\mu_T$, we observe that the measure $\mu_T$ is ergodic, see the proof of Theorem 1 of [27] for more details. \hfill \Box

Take the absolutely continuous invariant probability measure $\mu_T$ from the previous proposition, then the entropy of $T$ with respect to the probability measure $\mu_T$ satisfies (11) below. It gives the last statement of Theorem C.

**Proposition 5.2.** If $T : W \to W$ is a Markov induced map of a topologically mixing locally expanding semigroup generated by $C^{1+\alpha}$ conformal (local) diffeomorphisms defined on a compact connected smooth manifold $M$ then

$$h_{\mu_T}(T) = \int_W \log J_T d\mu_T. \tag{11}$$

**Proof.** We can apply the approach used in Proposition 4.3 of [1] to our setting. First notice that the measure $\mu_T$ is ergodic, moreover, by construction, the countable Markov partition $\mathcal{M} = \{M_i : i \in \mathbb{N}\}$ is generating. Thus, we can apply Shannon-McMillan-Breiman theorem for the generating partition $\mathcal{M}$. We must show that for the partition $\mathcal{M}$, $H(\mathcal{M}) < \infty$. Indeed, let

$$b_n := \mu_T(\{\tau = n\}),$$

where $\tau$ is the return time function defined by $\tau(M_i) = |h_i|$, where $M_i \in \mathcal{M}$ and $|h_i|$ is the length of admissible word corresponding to $h_i$. Also, by construction, the return time function $\tau$ is integrable and $\sum_{n=1}^{\infty} n b_n < \infty$. If $b_n > \frac{1}{n^2}$ then $b_n \log(b_n) < 2b_n \log(n)$. Since the function $-x \log(x)$ is increasing near zero, hence for $b_n \leq \frac{1}{n^2}$ one has that $b_n \log(b_n) \leq \frac{2 \log(n)}{n^2}$. Therefore

$$-\sum_{n=1}^{\infty} b_n \log b_n \leq -\left( \sum_{n:b_n \leq \frac{1}{n^2}} b_n \log b_n + \sum_{n:b_n > \frac{1}{n^2}} b_n \log b_n \right) \leq \sum_{n=1}^{\infty} \frac{2 \log(n)}{n^2} + \sum_{n=1}^{\infty} 2b_n \log n < \infty.$$ 

Take a generic point $x \in \tilde{W}$. One has that

$$h_{\mu_T}(T) = h_{\mu_T}(T, \mathcal{M}) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_T(\mathcal{M}_n(x)) = \lim_{n \to \infty} -\frac{1}{n} \log \lambda(\mathcal{M}_n(x)),$$

where $\mathcal{M}_n(x)$ is defined as part (ii) of Remark 4.3. Notice that the last equality comes from the fact that $\lambda$ and $\mu_T$ are equivalent measures with uniformly bounded densities. We observe that each $\mathcal{M}_n(x)$ is equal to some $M_{i_0, \ldots, i_n}$, with $x \in \text{int}(M_{i_0, \ldots, i_n})$, by Remarks 4.1 and 4.3. Hence, we have

$$\lambda(W) = \lambda(\tilde{W}) = \int_{T^{-n}(\tilde{W})} J_T^n d\lambda,$$

where $J_T^n(x) = |\text{det}DT^n(x)|$. By the distortion estimates obtained in the proof of Proposition 5.1 we conclude that

$$K_1^{-1} \leq \lambda(T^{-n}(W)) J_T^n(x) \leq K_1.$$ 

By the above inequality we deduce that

$$\lim_{n \to \infty} -\frac{1}{n} \log \lambda(\mathcal{M}_n(x)) = \lim_{n \to \infty} \frac{1}{n} \log J_T^n(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J_T(T^i(x)) = \int_W \log J_T d\mu_T,$$
where the last equality holds by Birkhoff’s ergodic theorem.

5.1. Stationary Measures. A crucial problem here is to decode back the information obtained for \( T \) into information about the original semigroup \( \Gamma \). It also requires a relation between \( T \)-invariant measures on \( W \) and stationary measures of \( \Gamma \). An \( T \)-invariant measure \( \mu_T \) on \( W \) with integrable inducing time \( \tau \) can be lifted to the tower, thus producing a stationary measure for \( \Gamma \) called the lift of \( \mu_T \). In what follows, we introduce the lifted measure. First, we recall that the partition \( M \) satisfies the finite images property. Thus, \( h_i(\text{int}(M_i)) \in \mathcal{B} = \{B_1, \ldots, B_N\} \), where \( \mathcal{B} \) is a finite covering of open balls of \( M \) with radius \( \varepsilon, 0 < \varepsilon \leq \frac{\eta}{2} \), where \( \eta < \frac{\varepsilon}{2} \) is the Lebesgue number of the covering \( \{V_1, \ldots, V_k\} \). We may assume that \( \max\{\text{diam}(V_i) : i = 1, \ldots, k\} < \eta \).

Consider a transition matrix \( A = (a_{ij})_{i,j=1}^k \) which is defined by

\[
a_{ij} = \begin{cases} 1 & \text{if } f_i(V_j) \cap V_j \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}
\]

Notice that, by the choice of \( r \), if \( f_i(V_j) \cap V_j \neq \emptyset \) then \( f_i(B_r(V_i)) \supset V_j \). Write \( \Omega \) for the finite set of symbols \( \{1, \ldots, k\} \). It is obvious that \( A = (a_{ij})_{i,j=1}^k \) is a matrix of zeroes and ones with no zero columns or rows. Take

\[
\Sigma_+^A := \{i = (i_0, i_1, \ldots) : i_j \in \Omega, \ \forall j \in \mathbb{N} \cup \{0\}, \text{ and } a_{i_j,i_{j+1}} = 1\}
\]

and

\[
\Sigma_A := \{i = (\ldots, i_{-1}, i_0, i_1, \ldots) : i_j \in \Omega, \ \forall j \in \mathbb{Z}, \text{ and } a_{i_j,i_{j+1}} = 1\}
\]

the corresponding one-sided and two-sided topological Markov chains, respectively, and \( \sigma_A^+ : \Sigma_+^A \to \Sigma_+^A \) and \( \sigma_A : \Sigma_A \to \Sigma_A \) the left shift maps. We equip \( \Sigma_+^A \) and \( \Sigma_A \) with the topology generated by the base of cylinder sets

\[
C_{[a_k, \ldots, a_m]} := \{i \in \Sigma_+^A : i_j = a_j \text{ for } j = k, \ldots, m\}, \text{ where } k, m \in \mathbb{N} \cup \{0\} \text{ and } k \leq m.
\]

and

\[
C_{[a_k, \ldots, a_m]} := \{i \in \Sigma_A : i_j = a_j \text{ for } j = k, \ldots, m\}, \text{ where } k, m \in \mathbb{Z} \text{ and } k \leq m.
\]

Let \( \prod = (\pi_{ij})_{i,j=1}^k \), \( \pi_{ij} \in [0,1] \) be a right stochastic matrix, (i.e., \( \forall i \sum_j \pi_{ij} = 1 \)) such that \( \pi_{ij} = 0 \) iff \( a_{ij} = 0 \). Let \( p \) be its eigenvector with non-negative components that corresponds to the eigenvalue 1:

\[
\forall i, \ p_i \geq 0, \text{ and } \sum_i \pi_{ij}p_i = p_j.
\]

We can always assume \( \sum_i p_i = 1 \). By the Perron-Frobenius theorem, there is a \( \sigma_A \)-invariant ergodic measure \( \nu \) corresponding to \( \prod \) on \( \Sigma_A \). Suppose that \( \nu^+ \) be the restriction of \( \nu \) on the subset \( \Sigma_+^A \) through the natural projection of \( \Sigma_A \) on \( \Sigma_+^A \) and let \( \nu_i^+ \) be the restriction of \( \nu^+ \) on the cylinder \( C_{i,0}^+ := \{i \in \Sigma_+^A : i_0 = i\} \).

Notice that since the semigroup \( \Gamma \) is topologically mixing, hence \( \Sigma_+^A \) is transitive. Also for any stochastic matrix \( \prod \) there exists at least one vector \( p \) satisfying (13). Such a vector is unique whenever the shift is transitive and \( \pi_{ij} \neq 0 \Leftrightarrow a_{ij} \neq 0 \) (as in our case).

Consider a discrete Markov process \( \prod(\Gamma) \) on the space \( \mathcal{I} = \{1, \ldots, k\} \times M \). The transition probability from a point \( (i, x) \) to a point \( (j, f_i(x)) \) equals \( \pi_{ij} \). Denote by \( \mathcal{B} \) the Borel sigma-algebra on \( M \). For any measure \( \hat{\mu} \) on the space \( \mathcal{I} \), it is natural to denote its stochastic image
\[ (f \ast \hat{\mu})_j := \sum_i \pi_{ij} (f_j) \ast \hat{\mu}_i, \]

where \( \hat{\mu}_i \) is the restriction of the measure \( \hat{\mu} \) to \( I_i = \{ i \} \times M \).

As in [15], a measure \( \hat{\mu} \) on the space \( \mathcal{I} \) is stationary if \( f \ast \hat{\mu} = \hat{\mu} \). Roughly speaking, stationary measures, just like the case of full shift, can be characterized as measures generating invariant measures for the corresponding skew-products. More precisely, a measure \( \eta \) is stationary if and only if \( \eta^+ \) defined by

\[ \eta^+ = \sum_{i=1}^k \nu_i^+ \times \eta_i \]

is invariant for the skew-product \( S^+(i, x) = (\sigma_A^+ (i), f_{i_0} (x)) \), with \( i \in \Sigma_A^+ \) and \( x \in M \), where \( \eta_i \) is the normalized measure define by \( \eta_i = \eta |_{I_i} / \eta(I_i) \), see [11].

Now, we try to build an absolutely continuous stationary measure. We recall the measure \( \mu_T \) from Proposition 5.1 which is an ergodic \( T \)-invariant absolutely continuous measure equivalent to Lebesgue. Take

\[ Q_{\mu_T} := \sum_{w \in AD} |w| \mu_T(C_w), \]

where \( AD \) is the collection of all finite admissible words from the alphabets \( \{ 1, \ldots, k \} \) and for any admissible word \( w \), \( |w| \) denotes the length of \( w \). Notice that by construction and condition (H4), \( Q_{\mu_T} < \infty \). We define a lifted measure \( m \) on the space \( \mathcal{I} \) as

\[ m(\{ i \} \times E) := \frac{p_i}{Q_{\mu_T}} \sum_{\ell=0}^{\infty} \sum_{w \in AD_i} \sum_{|M_k| > \ell} \mu_T(f_w^{-\ell} (E) \cap M_k), \]

where \( E \in \mathcal{B} \) is a measurable set, \( AD_i \) is the collection of all finite admissible words with length \( \ell \). Also, \( M_k \) is an element of countable Markov partition \( (M_i, h_i)_{i \in I} \). Then

\[ (f \ast m)_j(\{ i \} \times E) := \frac{1}{Q_{\mu_T}} \sum_{i=1}^k \pi_{ij} p_i \sum_{\ell=0}^{\infty} \sum_{w \in AD_i} \sum_{|M_k| > \ell} \mu_T(f_w^{-\ell} (E) \cap M_k). \]

The following result is immediate. In particular, Corollary D is verified by it.

**Proposition 5.3.** The measure \( m \) is an absolutely continuous stationary measure. In particular, \( m \) is ergodic.

6. **Markov extensions, Gibbs measures and equilibrium states**

Consider any locally expanding topologically mixing finitely generated semigroup \( \Gamma \) of \( C^{1+\alpha} \) conformal (local) diffeomorphisms on a compact connected manifold \( M \). By Theorem A, the semigroup \( \Gamma \) admits a Markov partition \( (\mathcal{M}, \mathcal{H}) = \{(M_i, h_i)\}_{i \in \mathbb{N}} \) with the finite images and the finite cycle properties, hence, \( h_i(\text{int}(M_i)) \in \mathcal{B} = \{ B_1, \ldots, B_N \} \). Take the elements \( M_{b_j} \in \mathcal{M} \) with the corresponding maps \( h_{b_j}, j = 1, \ldots, N \), such that \( h_{b_j}(\text{int}M_{b_j}) \in \mathcal{B} \) and they satisfy the finite cycle property. This means that for indices \( b_j \in \mathbb{N} \), \( j = 1, \ldots, N \), we have

\[ h_{b_j}(\text{int}M_{b_j}) = B_{j+1}, \text{ for } j = 1, \ldots, N - 1 \text{ and } h_{b_N}(\text{int}M_{b_N}) = B_1. \]
Let us take the induced map \( T : W \to M \) by \( T(x) = h_i(x) \), if \( x \in \text{int}(M_i) \), where \( W = \bigcup_{M_i \in \mathcal{M}} \text{int}(M_i) \) is the inducing domain.

In this section we apply results of Mauldin and Urbanski [17, 18] and of Sarig [22–24, 26], Yuri [29] and Buzzi and Sarig [4] to establish the existence and uniqueness of equilibrium measures for the induced map \( T \).

For the countable Markov partition \((\mathcal{M}, \mathcal{H}) = (\{M_i, h_i\})_{i \in \mathbb{N}}\), we recall the transition matrix \( A = (t_{ij})_{\mathbb{N} \times \mathbb{N}} \) which is defined by
\[
t_{ij} = \begin{cases} 
1 & \text{if } \text{int}(M_i) \cap h_i^{-1}(\text{int}(M_j)) \neq \emptyset \\
0 & \text{otherwise}.
\end{cases}
\]

It is obvious that \( A = (t_{ij})_{\mathbb{N} \times \mathbb{N}} \) is a matrix of zeroes and ones with no zero columns or rows. Take
\[
\Sigma_A^+ := \{(i_0, i_1, \ldots) \in \mathbb{N}^{\mathbb{N}\cup\{0\}} : \forall j \geq 0, \ t_{ij_{j+1}} = 1\}
\]
the corresponding one-sided topological Markov chain and \( \sigma_A : \Sigma_A^+ \to \Sigma_A^+ \) the left shift map, \( (\sigma_A)^j(i) = i_{j+1} \), for each \( i = (i_0, i_1, \ldots) \in \Sigma_A^+ \). Then \( (\Sigma_A^+, \sigma_A) \) is called the induced Markov chain corresponding to the induced map \( T \). For a potential \( \varphi : \Sigma_A^+ \to \mathbb{R} \) we denote by
\[
\var_n(\varphi) := \sup\{|\varphi(i) - \varphi(j)| : i, j \in \Sigma_A^+, \ i_\ell = j_\ell, \ 0 \leq \ell \leq n - 1\}
\]
the \( n \)-th variation of \( \varphi \). We say that \( \varphi \) has summable variations if
\[
\sum_{n=2}^{\infty} \var_n(\varphi) < \infty
\]
and \( \varphi \) is locally Hölder continuous if there exist \( C > 0 \) and \( 0 < \theta < 1 \) such that for all \( n \geq 2 \),
\[
\var_n(\varphi) \leq C\theta^n.
\]

**Remark 6.1.** By definition, it is easily seen that if a potential \( \varphi \) is locally Hölder continuous then it has summable variations.

Now, let \( \varphi \) be a potential for \((\Sigma_A^+, \sigma_A)\). A Gibbs measure for \( \varphi \) is an invariant probability measure \( \mu_\varphi \) such that for some global constants \( P \) and \( B \) and every cylinder \([a_0, \cdots, a_{n-1}]\),
\[
\frac{1}{B} \leq \frac{\mu_\varphi([a_0, \cdots, a_{n-1}])}{e^{\varphi(a_0, \cdots, a_{n-1})}} \leq B,
\]
for all \( i \in [a_0, \cdots, a_{n-1}] \), where \( \varphi_n(i) = \sum_{k=0}^{n-1} \varphi \circ \sigma_A^k(i) \). For \( \ell \in \mathbb{N} \), let
\[
Z_n(\varphi, \ell) := \sum_{\sigma_A^\ell(i) = 1, i_\ell = \ell} \exp(\varphi_n(i)).
\]
The Gurevic-Sarig pressure of \( \varphi \) is the number
\[
\mathcal{P}(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \ell).
\]
This notion introduced by Gurevic in [12, 13] which is a generalization of the notion of topological entropy \( h_G(\sigma_A) \) for countable Markov chains, so that \( \mathcal{P}(0) = h_G(\sigma_A) \). Sarig [22] proved that the limit in (14) exists for all \( \ell \in \mathbb{N} \) and is independent of \( \ell \) if the potential \( \varphi \) has summable variations. Moreover, he generalized the variational principle for countable
Markov chains which is referred to as variational principle for the Gurevic-Sarig pressure: if \( \varphi \) has summable variations and \( \sup(\varphi) < \infty \) then
\[
\mathcal{P}(\varphi) = \sup\{h_\mu(\sigma_A) + \int \varphi d\mu\} < \infty,
\]
where the supremum is taken over all \( \sigma_A \)-invariant Borel probability measures on \( \Sigma_A^+ \) such that \( \int \varphi d\mu < \infty \). An equilibrium measure is an invariant Borel probability measures for which the supremum is attained.

The one-sided topological Markov chain \((\Sigma_A^+, \sigma_A)\) has the big images and preimages (BIP) property if
\[
\exists b_1, \ldots, b_N \text{ such that } \forall a \in I \exists i, j \text{ such that } t_{b_i} a t_{ab_j} = 1.
\]
For \((W, T)\) given by (2.4), we say that \( \psi \) is a piecewise H"older continuous potential of \((W, T)\) if the restriction of \( \psi \) to the interior of any element of the partition \( \mathcal{M} \) is Hölder continuous, i.e. for all \( x, y \) in the interior of the same element of \( \mathcal{M} \),
\[
|\psi(x) - \psi(y)| \leq Kd(x, y)^\alpha
\]
for some \( \alpha > 0, K < \infty \).

**Theorem 6.2.** Every topologically mixing locally expanding semigroup \( \Gamma \) generated by a finite collection of \( C^{1+\alpha} \) conformal (local) diffeomorphisms on a compact and connected smooth manifold \( M \) admits a Markov inducing scheme \((\tau, (\mathcal{M}, \mathcal{H}))\) with the induced map \( T \) for which the following statements hold:

1. Let \((\Sigma_A^+, \sigma_A)\) be the induced Markov chain corresponding to the induced map \( T \). Let \( W \) be the domain of \( T \). Then \((W, T)\) is isomorphic to \((\Sigma_A^+, \sigma_A)\) outside some Lebesgue negligible set. Moreover, \((\Sigma_A^+, \sigma_A)\) is topologically mixing and satisfies the BIP property.

2. Assume \( \varphi \) is a potential of \((\Sigma_A^+, \sigma_A)\) having summable variations. Then \( \sigma_A \) admits a unique Gibbs measure \( \mu_\varphi \) if \( \mathcal{P}(\varphi) < \infty \) and \( \sum_{n=1}^{\infty} \text{var}_n(\varphi) < \infty \). Moreover, \( \varphi \) is positive recurrent. If in addition, the entropy \( h_{\mu_\varphi}(\sigma_A) < \infty \), then \( \mu_\varphi \) is the unique Gibbs and equilibrium measure.

3. Let \( \psi \) be a piecewise Hölder continuous potential of the induced map \( T \) with finite Gurevic-Sarig pressure and finite entropy. Also let \( \sup(\psi) < \infty \). Then \( \psi \) admits a unique equilibrium state \( \mu_\psi \). Furthermore, \( \mu_\psi \) is ergodic.

The proof of Theorem 6.2 handled in this section via some auxiliary lemmas and technical propositions.

**Proposition 6.3.** The one-sided topological Markov chain \((\Sigma_A^+, \sigma_A)\) satisfies the BIP property given by (15).

**Proof.** Consider the semigroup \( \Gamma \) with a Markov partition \((\mathcal{M}, \mathcal{H})\) which satisfies the finite images and the finite cycle properties and let \( \mathcal{A} \) be the corresponding transition matrix given by (12). Note that Theorem A guarantees the existence of such Markov partition. By the finite cycle property, there exist integer numbers \( b_\ell, \ell = 1, \ldots, N - 1 \), so that \( M_{b_\ell} \subset B_{\ell+1}, M_{b_N} \subset B_N \) and \( h_{b_N}(\text{int}(M_{b_N})) = B_1 \). Given \( a \in \mathbb{N} \), since \( \mathcal{B} = \{B_1, \ldots, B_N\} \) is an open cover of \( M \), there exists \( i \in \{1, \ldots, N\} \) such that \( M_a \subset B_i \). Hence, \( h_{b_{i-1}}(M_{b_{i-1}}) \supset B_i \supset M_a \) which ensures that \( t_{b_{i-1}} a = 1 \). Also, \( h_a(\text{int}(M_a)) = B_j \) for some \( j \in \{1, \ldots, N\} \). Then by the finite cycle property, \( h_a(M_a) \supset M_{b_j} \), and hence \( t_{ab_j} = 1 \). \( \square \)
It is not hard to see that there exists a one to one correspondence between the dynamical balls $M_{i_0,\ldots,i_\ell}$ and the cylinder sets of $\Sigma_A^+$. Hence, according to Remark 4.1, we get the next result.

**Proposition 6.4.** One-sided topological Markov shift $(\Sigma_A^+, \sigma_A)$ is topologically mixing.

By Propositions 6.4 and 6.3, the one-sided topological Markov shift $(\Sigma_A^+, \sigma_A)$ is topologically mixing and satisfies the BIP property. Hence, one can get the next result which is a restating of the main results in [22,23] in our context.

**Theorem 6.5.** Assume $\varphi$ is a potential of $(\Sigma_A^+, \sigma_A)$ having summable variations. Then $\sigma_A$ admits a unique Gibbs measure $\mu_\varphi$ if $P(\varphi) < \infty$ and $\sum_{n=1}^\infty \text{var}_n(\varphi) < \infty$. Moreover, $\varphi$ is positive recurrent. If in addition, the entropy $h_{\mu_\varphi}(\sigma_A) < \infty$, then $\mu_\varphi$ is the unique Gibbs and equilibrium measure. Furthermore, if $\sup \varphi < \infty$ then $\mu_\varphi$ is strongly mixing and hence it is ergodic.

Part (2) of Theorem 6.2 can be followed from the previous theorem. The rest of this section is devoted to the proof of the other assertions.

Buzzi and Sarig [7] proved that potentials with summable variations on topologically transitive countable Markov shifts have at most one equilibrium measure. Then they applied this result to non-Markov, multidimensional piecewise expanding maps (with the finite elements) by using the connected Markov diagram introduced in [6] under the following assumption: the total pressure of the system should be greater than the topological pressure of the boundary of some reasonable partition separating almost all orbits. In our construction, in Proposition 6.6 below, we relax this assumption and construct a measure-theoretic conjugation between the Markov induced map and a countable Markov shift with the BIP property. Notice that, in our setting, $(W, S, T)$ is a piecewise expanding map with a partition $S$ which is an infinite collection of non-empty, pairwise disjoint open subsets of $M$ with dense union. However, the proof of the proposition essentially has the same strategy as that of [7].

**Proposition 6.6.** Let $\mu_\varphi$ be the unique Gibbs ergodic measure given by Theorem 6.5. Then there is a continuous bi-measurable bijection map $\zeta$ from a $\mu_\varphi$-full subset of $\Sigma_A^+$ to a (Lebesgue) full subset of $M$ such that $\zeta \circ \sigma_A = T \circ \zeta$.

**Proof.** Consider the symbolic dynamic of $(M; M, T)$ which is the left-shift $\sigma_M$ on: $$\Sigma(M) = \{(M_{i_0}, M_{i_1}, \ldots) : (i_0, i_1, \ldots) \in \Sigma_A^+, \text{ and } \forall i_j, M_{i_j} \in M\}.$$ There exists a one-to-one correspondence between the elements of the Markov chain $\Sigma_A^+$ and the symbol space $\Sigma(M)$. We equip $\Sigma(M)$ by the following distance $$d((M_{i_0}, M_{i_1}, \ldots), (M_{j_0}, M_{j_1}, \ldots)) = 2^{-n},$$ where $n$ is the smallest integer such that $M_{i_{n-1}} \neq M_{j_{n-1}}$ or $n = \infty$. Define $$\pi : \Sigma(M) \to \bigcup_{M_i \in M} M_i, \quad \pi(M_{i_0}, M_{i_1}, \ldots) = \bigcap_{n \geq 0} M_{i_0,\ldots,i_n},$$ where $M_{i_0,\ldots,i_n}$ is the dynamical ball defined by (7). By (8), the mapping $\pi$ is continuous. Moreover, putting $\Delta := \pi^{-1}(\partial M)$, one gets $$\pi : \Sigma(M) \setminus \bigcup_{k \geq 0} \sigma_{M}^+(\Delta) \to \bigcup_{M_i \in M} M_i \setminus \bigcup_{k \geq 0} T^{-k} \partial M,$$
is bi-measurable, injective and surjective. In particular, $\pi \circ \sigma_\mathcal{M} = T \circ \pi$. Note that if $\pi(M_{i_0}, M_{i_1}, \ldots) = x$ then $x \in M_{i_0}$, $h_{i_0}(x) \in M_{i_1}$, and $h_{i_n-1} \circ \cdots \circ h_{i_0}(x) \in M_{i_n}$, for each $n \in \mathbb{N}$, i.e. the sequence $(M_{i_0}, M_{i_1}, \ldots)$ is the itinerary of $x$. Therefore, $\sigma_\mathcal{M}(M_{i_0}, M_{i_1}, \ldots)$ is the itinerary of $T(x) = h_{i_0}(x)$. These observations ensure that $\sigma_\mathcal{M}(\Delta) \subset \Delta$, i.e. $\Delta$ is forward invariant. Now, we define the mapping

$$
\xi : \Sigma_\mathcal{A}^+ \to \Sigma(\mathcal{M}), \ (i_0, i_1, \ldots) \mapsto (M_{i_0}, M_{i_1}, \ldots).
$$

Clearly, $\xi$ is continuous. Let $\tilde{\Delta} := \xi^{-1}(\Delta)$. Then

$$
\xi : \Sigma_\mathcal{A}^+ \setminus \bigcup_{k \geq 0} \sigma_{\mathcal{A}}^{-k}(\tilde{\Delta}) \to \Sigma(\mathcal{M}) \setminus \bigcup_{k \geq 0} \sigma_{\mathcal{M}}^{-k}(\Delta), \ (i_0, i_1, \ldots) \mapsto (M_{i_0}, M_{i_1}, \ldots)
$$

is a bijective bi-measurable map. Now, take

$$
\zeta := \pi \circ \xi.
$$

$\zeta$ is the desired map from $\Sigma_\mathcal{A}^+ \setminus \bigcup_{k \geq 0} \sigma_{\mathcal{A}}^{-k}(\tilde{\Delta})$ to $\Sigma(\mathcal{M}) \setminus \bigcup_{k \geq 0} T^{-k}\partial\mathcal{M}$ defined by

$$(i_0, i_1, \ldots) \mapsto \bigcap_{n \geq 0} M_{i_0, ..., i_n}.$$ 

Consider the Gibbs measure $\mu_\varphi$ as in Theorem 6.5. Since $\mu_\varphi$ is $\sigma_\mathcal{A}$-invariant, $\mu_\varphi(\sigma_\mathcal{A}^+(\tilde{\Delta})) = \mu_\varphi(\sigma_\mathcal{A}(\tilde{\Delta}))$, hence one has

$$
\mu_\varphi\left(\bigcap_{n \geq 0} \sigma_{\mathcal{A}}^n(\tilde{\Delta})\right) \neq \mu_\varphi(\sigma_\mathcal{A}(\tilde{\Delta})).
$$

The set $\bigcap_{n \geq 0} \sigma_{\mathcal{A}}^n(\tilde{\Delta})$ has measure 0 or 1 as it is $\sigma_\mathcal{A}$-invariant and $\mu_\varphi$ is ergodic. Since its complement is a nonempty open set and has positive measure by the fact that $\mu_\varphi$ is Gibbs measure, one gets $\mu_\varphi(\tilde{\Delta}) = 0$. \hfill \Box

Assume $\psi$ is a piecewise Hölder continuous potential of $(\bigcup_{M_i \in \mathcal{M}} \text{int}(M_i), T)$. This means that the restriction of $\psi$ to the interior of any element of $\mathcal{M}$ is Hölder continuous, i.e. for all $x, y$ in the interior of the same element of $\mathcal{M}$,

$$
|\psi(x) - \psi(y)| \leq K d(x, y)^\alpha
$$

for some $\alpha > 0$, $K < \infty$. Define $\varphi : \Sigma_\mathcal{A}^+ \to \mathbb{R}$ by

$$
(16) \quad \varphi((i_0, i_1, \ldots)) = \lim_{n \to \infty} \inf \psi(\text{int} M_{i_0, ..., i_n})
$$

and note that

$$
\varphi((i_0, i_1, \ldots)) = \psi(\pi(M_{i_0}, M_{i_1}, \ldots)) \text{ whenever } \pi(M_{i_0}, M_{i_1}, \ldots) \notin \partial\mathcal{M}.
$$

Thus $\varphi := \psi \circ \pi \circ \xi = \psi \circ \zeta$ whenever $\pi(M_{i_0}, M_{i_1}, \ldots) \notin \partial\mathcal{M}$. It is easy to check that $\varphi$ is Hölder continuous if $\psi$ is piecewise Hölder continuous. Moreover, due to (8), if $\psi$ is piecewise Hölder continuous then $\psi$ is locally Hölder continuous and since there exists a one-to-one correspondence between the cylinders of the Markov chain $\Sigma_\mathcal{A}^+$ and the admissible cylinders of $(\bigcup_{M_i \in \mathcal{M}} \text{int}(M_i), T)$, hence $\varphi$ is locally Hölder continuous and thus by Remark 6.1 it has summable variations.

In the following, we assume that $\psi$ is piecewise Hölder continuous and $\sup(\psi) < \infty$ and recall $\varphi$ from (16). By the above argument and these observations, $\sup(\varphi) < \infty$ and $\varphi$ has summable variations, hence it admits a unique Gibbs measure $\mu_\varphi$ by Theorem 6.5. Furthermore, $\mu_\varphi(\tilde{\Delta}) = 0$. Now, let $\mu_\psi := \mu_\varphi \circ \zeta^{-1}$. Then $\mu_\psi(\bigcup_{k \geq 0} T^{-k}\partial\mathcal{M}) = 0$ and
therefore, \( \zeta : (\Sigma^+_A, \mu_\varphi) \to (\bigcup_{M_i \in \mathcal{M}} \text{int}(M_i), \mu_\varphi) \) is a measure-theoretic isomorphism. Hence, \( \mathcal{P}_{\sigma_A}(\varphi) = \mathcal{P}_T(\psi) \). In particular, \( h_{\mu_\varphi}(\sigma_A) = h_{\mu_\varphi}(T) \). Suppose \( h_{\mu_\varphi}(T) < \infty \) and \( \mathcal{P}_T(\psi) < \infty \). By these facts and Theorem 6.5 one has
\[
    h_{\mu_\varphi}(T) + \int \psi d\mu_\varphi = h_{\mu_\varphi}(\sigma_A) + \int \varphi d\mu_\varphi = \mathcal{P}_{\sigma_A}(\varphi) = \mathcal{P}_T(\psi).
\]

Notice that \( \tilde{M} = M \setminus \bigcup_{M_i \in \mathcal{M}} \text{int}(M_i) \) has zero Lebesgue measure. \( T \) is a Markov induced map and \( \mu_\varphi \) is a weak Gibbs measure in the sense of [29], hence, by [28, Thm. 2.2] or [29, Thm. 6], \( T \) satisfies the variational principle. Hence \( \mu_\varphi \) is an equilibrium state and the proof of Theorem 6.2 is finished.

6.1. Example. Here, we illustrate an example of a locally expanding semigroup composed of \( C^{1+\alpha} \) diffeomorphisms which fits in our situation.

**Example 6.7.** Let \( S^2 \) be the unit sphere in \( \mathbb{R}^3 \). Take the points \( p_i \in S^2, i = 1, \ldots, k \), and \( \alpha > 0 \) so that the open balls \( V_i, i = 1, \ldots, k \), with center \( p_i \) and radius \( \alpha \) covers the sphere \( S^2 \). Take \( r > 0 \) so that \( 2\alpha < r \) and let \( \beta > 0 \) be the Lebesgue number of the covering \( \{V_1, \ldots, V_k\} \) with \( \beta < r/2 \). Fix \( 0 < \varepsilon < \beta/6 \).

Let \( g_i \in \text{Diff}^{1+\alpha}(S^2), i = 1, \ldots, k \), be Morse-Smale diffeomorphisms so that each \( g_i \) has a unique repelling fixed point at \( p_i \) and a unique sink at some point \( q_i \in S^2 \setminus B_r(V_i) \). Moreover, we assume that the mappings \( g_i \) are conformal maps on \( B_r(V_i) \) and there exist \( \frac{1}{2} < \sigma < 1 \) such that \( \|Dg_i(y)^{-1}\| < \sigma \) for every \( y \in B_r(V_i) \). Take the translation \( R_i, i = 1, \ldots, k \), so that \( R_i(p_i) = p_{i+1} \) for \( i = 1, \ldots, k-1 \), and \( R_k(p_k) = p_1 \). Now we define diffeomorphisms \( f_i \in \text{Diff}^{1+\alpha}(S^2) \) by \( f_i := R_i \circ g_i, i = 1, \ldots, k \). A little thinking shows that the semigroup generated by \( F = \{f_1, \ldots, f_k\} \) is a locally expanding semigroup with the topologically mixing property. Indeed, let \( U \) and \( V \) be two open subset of \( S^2 \) whose diameters are less than \( \varepsilon \). Let \( x \in U \) and \( w = (i_0, i_1, \ldots, i_n) \) be an admissible word such that \( x \in V_{i_0} \) and \( f_{i_1} \circ \cdots \circ f_{i_0}(x) \in V_{i_{j+1}} \) for \( j = 0, \ldots, n-1 \). Assume that \( f_{i_n} \circ \cdots \circ f_{i_0}(x) = y \). Then \( f_{n-1}^{-1}(B_x(y)) = C(x) \subset U \) provided that \( n \) is large enough, where \( B_x(y) \) is a ball with center \( y \) and radius \( \varepsilon \), \( f_{n}^{-1} = (f_{n}^{-1})^{-1} \) and \( f_{n}^{-1} = f_{n-1} \circ \cdots \circ f_{i_0} \). Since \( \text{diam}(V) < \varepsilon < \beta \), hence \( V \subset V_j \) for some \( j \in \{1, \ldots, k\} \).

By the choice of \( f_i, i = 1, \ldots, k \), there is an admissible word \( w' \) of the length \( \ell \) such that \( f_{\ell + 1}^{-1}(B_x(y)) \supset V_{j'} \). Let us take \( \rho = w'w' \) the concatenation of \( w \) and \( w' \). Then \( f_{\rho}^{-1}(C(x)) \supset V \).

By this observation, the semigroup \( \Gamma \) generated by \( F = \{f_1, \ldots, f_k\} \) is topologically mixing.

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