Quadratic double ramification integrals and the noncommutative KdV hierarchy

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Abstract
In this paper we compute the intersection number of two double ramification (DR) cycles (with different ramification profiles) and the top Chern class of the Hodge bundle on the moduli space of stable curves of any genus. These quadratic DR integrals are the main ingredients for the computation of the DR hierarchy associated to the infinite-dimensional partial cohomological field theory given by $\exp(\mu^2 \Theta)$, where $\mu$ is a parameter and $\Theta$ is Hain’s theta class, appearing in Hain’s formula for the DR cycle on the moduli space of curves of compact type. This infinite rank DR hierarchy can be seen as a rank 1 integrable system in two space and one time dimensions. We prove that it coincides with a natural analogue of the Korteweg-de-Vries (KdV) hierarchy on a noncommutative Moyal torus.

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1. Introduction
The main idea of this paper comes from the observation that the double ramification (DR) cycle, that is, the class in the cohomology of the moduli space of stable curves representing the most natural compactification of the locus of smooth curves whose marked points support a principal divisor $[13]$, can be seen as a partial cohomological field theory (CohFT) $[17, 18]$ with an infinite-dimensional phase space.

In $[1, 2, 5]$ it was shown how to associate to any partial CohFT an integrable hierarchy of Hamiltonian systems of evolutionary PDEs in one space and one time dimensions. The number of dependent variables in these system of PDEs equals the dimension of the phase space of the partial CohFT. This integrable system is called the DR hierarchy and its properties and generalizations (including a quantization which exists in the case of actual CohFTs) where studied in $[3, 4, 6, 7]$.

This paper wants to answer the question: ‘what is the DR hierarchy associated to the infinite rank partial CohFT given by the DR cycle’? As the general construction of the DR hierarchy already involves intersection numbers of a partial CohFT with the top Hodge class and a DR cycle, choosing as CohFT a second DR cycle leads us directly to having to compute intersection
numbers of two different DR cycles and the top Hodge class. This is, of course, a question of its own geometric interest and, as we show in Chapter 2 of this paper, it has a very explicit answer.

In fact there is a natural deformation of the DR cycle which gives a one-parameter family of partial CohFTs. It comes from Hain’s formula [13], expressing the DR cycle (restricted to the moduli space of stable curves of compact type) as the $g$th power of (the pullback to the moduli space of stable curves of compact type) of the class of the theta divisor on the universal Jacobian. If we consider instead the exponential of such theta class, putting into play all of its powers, we get a more general but still very explicit infinite rank partial CohFT.

After showing in Chapter 3 how to trade the resulting infinite rank DR hierarchy in one space and one time dimensions as a rank 1 hierarchy in two space and one time dimensions, we set out to compute it explicitly. The main result of this paper is in Chapter 4, where we show that noncommutative integrable systems have never appeared before in the study of the KdV hierarchy, the natural generalization of the ordinary KdV hierarchy on a torus to that the DR hierarchy of our infinite rank partial CohFT coincides with the noncommutative Jacobian. If we consider instead the exponential of such theta class, any power of the theta class and any power of the psi class at one marked point. Note also that noncommutative integrable systems have never appeared before in the study of the cohomology ring of the moduli spaces and, in particular, the structure of the DR cycle, which became the object of an intensive research in recent years (see, for example, [15, 16, 19]).

2. Quadratic double ramification integrals

For a pair of nonnegative integers $(g, n)$ in the stable range, that is, satisfying $2g + 2 - n > 0$, let $\mathcal{M}_{g,n}$ be the moduli space of stable curves with genus $g$ and $n$ marked points labeled by the set $[n] := \{1, \ldots, n\}$. For integers $a_1, \ldots, a_n$, such that $\sum a_i = 0$, the DR cycle $\text{DR}_g(a_1, \ldots, a_n) \in H^*(\mathcal{M}_{g,n}, \mathbb{C})$ is the Poincaré dual to the pushforward to $\mathcal{M}_{g,n}$ of the virtual fundamental class of the moduli space of rubber stable maps from curves of genus $g$ with $n$ marked points to $\mathbb{P}^1$ relative to 0 and $\infty$ with ramification profile given by $(a_1, \ldots, a_n)$. Here ‘rubber’ means that we consider maps up to the $\mathbb{C}^*$-action in the target $\mathbb{P}^1$ and a positive (negative) coefficient $a_i$ indicates a pole (zero) at the $i$th marked point of order $a_i$ ($-a_i$), while $a_i = 0$ just indicates that the $i$th marked point is unconstrained. For future convenience we will also define the class $\text{DR}_g(a_1, \ldots, a_n)$ to vanish in case $\sum a_i \neq 0$.

Let us introduce the class $\Theta(a_1, \ldots, a_n) \in H^2(\mathcal{M}_{g,n}, \mathbb{C})$ defined by

$$\Theta(a_1, \ldots, a_n) := \sum_{j=1}^{n} \frac{a_j^2 \psi_j}{2} - \frac{1}{4} \sum_{h=0}^{g} \sum_{J \subset [n]} a_j^2 \delta_h^J, \quad (2.1)$$

if $\sum a_i = 0$ and zero otherwise, where $\psi_i$, $1 \leq i \leq n$, is the first Chern class of the $i$th tautological line bundle, and, for $J \subset [n]$ and $0 \leq h \leq g$ in the stable range $2h - 1 + |J| > 0$ and $2g - h - 1 + (n - |J|) > 0$, $a_j := \sum_{J \subset J} a_j$ and $\delta_h^J \in H^2(\mathcal{M}_{g,n}, \mathbb{C})$ is the class of the irreducible boundary divisor of $\mathcal{M}_{g,n}$ formed by stable curves with a separating node at which two stable components meet, one of genus $h$ and marked points labeled by $|J|$ and the other of genus $g - h$ and marked points labeled by the complement $J^c$ (naturally, $\delta_h^J = 0$ if at least one of the stability conditions $2h - 1 + |J| > 0$ and $2g - h - 1 + (n - |J|) > 0$ is not satisfied).

By a result of Hain [13], we know that

$$\text{DR}_g(a_1, \ldots, a_n) \mid_{\mathcal{M}_{g,n}} = \frac{1}{g!} \Theta(a_1, \ldots, a_n)^g \mid_{\mathcal{M}_{g,n}^{\text{ct}}}, \quad (2.2)$$
where $\mathcal{M}_{g,n}^{ct}$ is the locus of stable curves with no nonseparating nodes. Moreover, always by [13], the class $\Theta(a_1, \ldots, a_n)|_{\mathcal{M}_{g,n}^{ct}}$ represents the pullback to $\mathcal{M}_{g,n}^{ct}$ of the theta divisor on the universal Jacobian over $\mathcal{M}_{g,n}^{ct}$, which implies the following relation in $H^*(\mathcal{M}_{g,n}^{ct}, \mathbb{C})$,

$$\Theta(a_1, \ldots, a_n)^{g+1}|_{\mathcal{M}_{g,n}^{ct}} = 0.$$  

(2.3)

Let $\lambda_i \in H^{2i}(\mathcal{M}_{g,n}^{ct}, \mathbb{C})$, $1 \leq i \leq g$, be the ith Chern class of the Hodge bundle on $\mathcal{M}_{g,n}^{ct}$. We have [11]

$$\lambda_i|_{\mathcal{M}_{g,n}^{ct}} = 0.$$  

(2.4)

The classes $\lambda_i$, $\text{DR}_g(a_1, \ldots, a_n)$ and $\Theta(a_1, \ldots, a_n)$ are algebraic, that is, they belong in fact to the Chow ring $A^*(\mathcal{M}_{g,n}^{ct})$. By the localization exact sequence, see, for example, [12, Section 1.8], for all $k$,

$$A_k(\mathcal{M}_{g,n}^{ct}) \rightarrow A_k(\mathcal{M}_{g,n}^{ct}) \rightarrow A_k(\mathcal{M}_{g,n}) \rightarrow 0,$$  

where $i$ and $j$ are the inclusion maps of $\mathcal{M}_{g,n}^{ct}$ into $\mathcal{M}_{g,n}$, and by equation (2.4), we deduce that, if $\alpha \in A_k(\mathcal{M}_{g,n}^{ct})$ is such that $\alpha|_{\mathcal{M}_{g,n}^{ct}} := j^* \alpha = 0$, then $\lambda_j \cdot \alpha = 0 \in A^*(\mathcal{M}_{g,n}^{ct})$. This allows to deduce from identities (2.2) and (2.3) the following relations in $H^*(\mathcal{M}_{g,n}^{ct}, \mathbb{C})$:

$$\lambda_j \cdot \text{DR}_g(a_1, \ldots, a_n) = \frac{1}{g!} \lambda_j \Theta(a_1, \ldots, a_n)^g,$$  

(5)

$$\lambda_j \Theta(a_1, \ldots, a_n)^{g+1} = 0.$$  

(6)

For two pairs of nonnegative integers $(g_1, n_1 + 1)$ and $(g_2, n_2 + 1)$ in the stable range, let $g! : \mathcal{M}_{g_1, n_1 + 1} \times \mathcal{M}_{g_2, n_2 + 1} \rightarrow \mathcal{M}_{g,n}$ be the map that glues two stable curves of genus $g_1$ and $g_2$ with marked points labeled by the sets $I = \{i_1, \ldots, i_{n_1}, n_1 + 1\}$ and $J = \{j_1, \ldots, j_{n_2}, n_2 + 1\}$, respectively, with $I \cup J = [n + 2]$, at their last marked points to form stable curve with a separating node with genus $g = g_1 + g_2$, and marked points labeled by $[n]$. It is easy to see from the definitions that we have

$$g! \Theta(a_1, \ldots, a_n) = \Theta(a_1, \ldots, a_{n_1}, -k) + \Theta(a_{j_1}, \ldots, a_{n_{j_2}}, k),$$  

(7)

where $k = \sum_{k=1}^{n_1} a_{i_k} = \sum_{k=2}^{n_2} a_{j_k}$ and the classes on the right-hand side are pulled back from each of the two factors in the product $\mathcal{M}_{g_1, n_1 + 1} \times \mathcal{M}_{g_2, n_2 + 1}$. By [8], we also have

$$g! \text{DR}_g(a_1, \ldots, a_n) = \text{DR}_{g_1}(a_{i_1}, \ldots, a_{i_{n_1}}, -k) \text{DR}_{g_2}(a_{j_1}, \ldots, a_{j_{n_2}}, k).$$

In the following we will denote by $\text{DR}_{g_1}(a_{i_1}, \ldots, a_{i_{n_1}}, -k) \boxtimes \text{DR}_{g_2}(a_{j_1}, \ldots, a_{j_{n_2}}, k)$ the push-forward $g! g^* \text{DR}_g(a_1, \ldots, a_n) \in H^*(\mathcal{M}_{g,n}^{ct}, \mathbb{C})$.

We have the following result on the intersection number of two DR cycles and the top Chern class of the Hodge bundle.

**Theorem 2.1.** Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $g \in \mathbb{Z}_{\geq 0}$, then

$$\int_{\mathcal{M}_{g,n}} \lambda_g \text{DR}_g(a_1, a_2, -a_1 - a_2) \text{DR}_g(b_1, b_2, -b_1 - b_2) = \frac{(a_1 b_2 - a_2 b_1)^{2g}}{2^{3g} g! (2g + 1)!!}.$$  

(2.8)

**Proof.** By equation (2.5), Theorem 2.1 is equivalent to the formula

$$\int_{\mathcal{M}_{g,n}} \lambda_g \Theta(a_1, a_2, -a_1 - a_2)^g \text{DR}_g(b_1, b_2, -b_1 - b_2) = \frac{(a_1 b_2 - a_2 b_1)^{2g}}{2^{3g} (2g + 1)!!}, \quad g \geq 0.$$  

(2.9)
Denote the left-hand side of equation (2.9) by \( f_g(\overline{a}, \overline{b}) \), where \( \overline{a} = (a_1, a_2, a_3) \), \( \overline{b} = (b_1, b_2, b_3) \) and \( a_3 = -a_1 - a_2, b_3 = -b_1 - b_2 \). Clearly, \( f_0(\overline{a}, \overline{b}) = 1 \), so suppose that \( g \geq 1 \). In order to compute the integral, let us express one of the classes \( \Theta(\overline{a}) \) using formula (2.1). Using relation (2.3) and the formula for the intersection of a psi class with a DR cycle [8], for any \( 1 \leq i \leq 3 \) we compute

\[
\int_{\mathcal{M}_{g,3}} \lambda_g \psi_i \Theta(\pi)^{g-1} \text{DR}_g(\overline{b})
= \frac{2g - 1}{2g + 1} \int_{\mathcal{M}_{g,3}} \lambda_g \Theta(\pi)^{g-1} \text{DR}_1(b_i, -b_i) \otimes \text{DR}_{g-1}(b_j, b_k, b_3)
- \frac{2b_j}{(2g + 1)b_i} \int_{\mathcal{M}_{g,3}} \lambda_g \Theta(\pi)^{g-1} \text{DR}_{g-1}(b_i, b_k, b_3) \otimes \text{DR}_1(b_j, -b_j)
- \frac{2b_k}{(2g + 1)b_i} \int_{\mathcal{M}_{g,3}} \lambda_g \Theta(\pi)^{g-1} \text{DR}_{g-1}(b_i, b_j, b_k) \otimes \text{DR}_1(b_k, -b_k)
= \left[ \frac{2g - 1}{2g + 1} \frac{b_i^2}{24} + \frac{2b_j}{(2g + 1)(b_j + b_k)} \frac{b_j^2}{24} + \frac{2b_k}{(2g + 1)(b_j + b_k)} \frac{b_k^2}{24} \right] f_{g-1}(\overline{a}, \overline{b})
= \frac{(2g + 1)b_i^2}{24(2g + 1)} f_{g-1}(\overline{a}, \overline{b}),
\]
where \( \{i, j, k\} = \{1, 2, 3\} \). As a result,

\[
\int_{\mathcal{M}_{g,3}} \lambda_g \left( \sum_{i=1}^{3} a_i^2 \psi_i \right) \Theta(\pi)^{g-1} \text{DR}_g(\overline{b}) = f_{g-1}(\overline{a}, \overline{b}) \sum_{i,j<k} a_i^2 \frac{(2g + 1)b_i^2 - 6b_jb_k}{24(2g + 1)}.
\]

Next we compute

\[
\int_{\mathcal{M}_{g,3}} \lambda_g \left( \frac{1}{2} \sum_{h=0}^{g} \sum_{J \subset [3]} a_J^2 \delta_h^J \right) \Theta(\pi)^{g-1} \text{DR}_g(\overline{b})
= \sum_{i,j<k} a_i^2 \int_{\mathcal{M}_{g,3}} \lambda_g \Theta(\pi)^{g-1} \text{DR}_1(b_i, -b_i) \otimes \text{DR}_{g-1}(b_j, b_k, b_3)
= f_{g-1}(\overline{a}, \overline{b}) \sum_{i=1}^{3} a_i^2 b_i^2 \frac{24}{24}.
\]

Summarizing the above computations we get

\[
f_g(\overline{a}, \overline{b}) = \frac{1}{2} \int_{\mathcal{M}_{g,3}} \lambda_g \left( \sum_{i=1}^{3} a_i^2 \psi_i - \frac{1}{2} \sum_{h=0}^{g} \sum_{J \subset [3]} a_J^2 \delta_h^J \right) \Theta(\pi)^{g-1} \text{DR}_g(\overline{b})
= \frac{1}{2} f_{g-1}(\overline{a}, \overline{b}) \left( \sum_{i,j<k} a_i^2 \frac{(2g + 1)b_i^2 - 6b_jb_k}{24(2g + 1)} - \sum_{i=1}^{3} a_i^2 b_i^2 \frac{24}{24} \right).
\]
that proves formula (2.9) and completes the proof of the theorem. □

3. An infinite rank partial CohFT and its DR hierarchy

Recall the following definition, which is a generalization first considered in [18] of the notion of CohFT from [17].

**Definition 3.1.** For a pair of nonnegative integers \((g, n)\) in the stable range \(2g - 2 + n > 0\), a partial CohFT is a system of linear maps \(c_{g,n} : V^\otimes n \to H^{even}(\overline{M}_{g,n}, \mathbb{C})\), where \(V\) is an arbitrary finite-dimensional \(\mathbb{C}\)-vector space, called the phase space, together with a special element \(e_1 \in V\), called the unit, and a symmetric nondegenerate bilinear form \(\eta \in (V^*)^\otimes 2\), called the metric, such that, chosen any basis \(e_1, \ldots, e_{\dim V}\) of \(V\), the following axioms are satisfied.

(i) The maps \(c_{g,n}\) are equivariant with respect to the \(S^\alpha_n\)-action permuting the \(n\) copies of \(V\) in \(V^\otimes n\) and the \(n\) marked points in \(\overline{M}_{g,n}\), respectively.

(ii) \(\pi^* c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g,n+1}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_1)\) for \(1 \leq \alpha_1, \ldots, \alpha_n \leq \dim V\), where \(\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}\) is the map that forgets the last marked point.

Moreover \(c_{0,3}(e_\alpha \otimes e_\beta \otimes e_1) = \eta(e_\alpha \otimes e_\beta) = \eta_{\alpha \beta}\) for \(1 \leq \alpha, \beta \leq \dim V\).

(iii) \(gl^* c_{g_1+g_2,n_1+n_2}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g_1,n_1+1}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_1)\eta\gamma^{\mu \nu} c_{g_2,n_2+1}(\otimes_{j=1}^m e_{\nu_j} \otimes e_{1})\) for \(2g_1 - 1 + n_1 > 0, 2g_2 - 1 + n_2 > 0\) and \(1 \leq \alpha_1, \ldots, \alpha_n \leq \dim V\), where \(I \sqcup J = \{1, \ldots, n\}\), \(|I| = n_1, |J| = n_2\), and \(gl^* : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}\) is the corresponding gluing map and where \(\eta_{\alpha \beta}\) is defined by \(\eta_{\mu \nu} \eta_{\alpha \beta} = \delta_{\mu \beta}\) for \(1 \leq \alpha, \beta \leq \dim V\).

Remark that a notion of infinite rank partial CohFT, that is, a partial CohFT with an infinite-dimensional phase space \(V\), requires some care. One needs to clarify what is meant by the matrix \(\eta\) and to make sense of the, a priori infinite, sum over \(\mu\) and \(\nu\), both appearing in Axiom (iii). One possibility is demanding that the image of the linear map \(V^\otimes (n-1) \to H^*(\overline{M}_{g,n}, \mathbb{C}) \simeq V^*\) induced by \(c_{g,n} : V^\otimes n \to H^*(\overline{M}_{g,n}, \mathbb{C})\) is contained in \(H^*(\overline{M}_{g,n}, \mathbb{C}) \simeq \eta^*(V)\), where \(\eta^* : V \to V^*\) is the injective map induced by the bilinear form \(\eta\). Then in Axiom (iii), instead of using an undefined bilinear form \(\eta^{\alpha \beta}\) on \(V^*\), one can use the bilinear form on \(\eta^* (V)\) induced by \(\eta\). This solves the problem with convergence.

A useful special case is the following. Let the basis \(\{e_{\alpha_i}\}_{\alpha \in I}\) of \(V\) be countable and, for any \((g, n)\) in the stable range and each \(e_{\alpha_1}, \ldots, e_{\alpha_m} \in V\), let the set \(\{\beta \in \overline{M}_{g,n+1} \otimes e_1 \otimes e_{\beta} \neq 0\}\) be finite. In particular this implies that the matrix \(\eta_{\alpha \beta}\) is row- and column-finite (each row and each column have a finite number of nonzero entries), which is equivalent to \(\eta^2 (V) \subseteq \text{span}(\{e_{\alpha_i}\}_{\alpha \in I})\), where \(\{e_{\alpha_i}\}_{\alpha \in I}\) is the dual ‘basis’. Let us further demand that the injective map \(\eta^2 : V \to \text{span}(\{e_{\alpha_i}\}_{\alpha \in I})\) is surjective too, that is, that a unique two-sided row- and column-finite matrix \(\eta^{\alpha \beta}\), inverse to \(\eta_{\alpha \beta}\), exists (it represents the inverse map \(\eta^2^{-1} : \text{span}(\{e_{\alpha_i}\}_{\alpha \in I}) \to V\)). Then the equation appearing in Axiom (iii) is well defined with the double sum only having a finite number of nonzero terms for each boundary divisor.

We will now construct an example of such infinite rank partial CohFT.
PROPOSITION 3.2. Let μ be a formal parameter. The classes $c_{g,n}(\otimes_{a=1}^{n} e_{a}) := \exp(\mu^{2}\Theta(a_{1}, \ldots, a_{n}))$ form an infinite rank partial CohFT with a phase space $V = \text{span}\{e_{a}\}_{a \in \mathbb{Z}}$, where the unit is $e_{0}$ and the metric, written in the basis $\{e_{a}\}_{a \in \mathbb{Z}}$, is $\eta_{ab} = \delta_{a+b,0}$.

Proof. Axioms (i) and (ii) follow directly from the definition of the classes $\Theta(a_{1}, \ldots, a_{n})$. For Axiom (iii) note that, for fixed $e_{a_{1}}, \ldots, e_{a_{n-1}}$, we have $c_{g,n}(\otimes_{a=1}^{n-1} e_{a} \otimes e_{n}) = 0$ unless $b = -\sum_{i=1}^{n-1} a_{i}$. Moreover we have $g^{ab} = \delta_{a+b,0}$ and equation (2.7) implies Axiom (iii) where the double sum consists of just one term. □

In [1] it was shown how to associate an integrable system of evolutionary Hamiltonian PDEs, called the DR hierarchy, to any CohFT and in [2] it was remarked how a partial CohFT is sufficient for the construction to work. Let us see how such construction generalizes to the infinite rank partial CohFT introduced in Proposition 3.2. Recall from [1, 5] that the DR Hamiltonians are the generating series

$$
\begin{equation}
g_{a,d} := \sum_{g \geq 0, n \geq 1 \atop 2g-1+n > 0} \frac{(-\varepsilon^{2})^{g}}{n!} \sum_{b_{1}, \ldots, b_{n} \in \mathbb{Z}} \left( \int_{\text{DR}_{g}(b_{1}, \ldots, b_{n})} \lambda_{g} \psi_{1}^{d} c_{g,n+1}(e_{a} \otimes \otimes_{j=1}^{n} e_{a_{j}}) \right) \prod_{j=1}^{n} p_{b_{j}} e^{-ib_{j}x},
\end{equation}
$$

(3.1)

for $a \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$, seen as formal power series in the formal variables $\varepsilon, \mu, e^{ix}, p_{b}^{a}, a, b \in \mathbb{Z}$.

Thanks to the fact that the intersection numbers appearing in equation (3.1) vanish unless $\sum b_{j} = -b$ and that, by formula (2.5), the class $\lambda_{g} \text{DR}_{g}(b_{1}, \ldots, b_{n})$ is a polynomial in $b_{1}, \ldots, b_{n}$ homogeneous of degree $2g$, the above generating functions can be expressed uniquely (see, for example, [5]) as a degree 0 differential polynomial, that is, a formal power series in $\varepsilon, \mu$ and the new formal variables $u_{k}^{a}, a \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$, of degree 0 with respect to the grading $\deg u_{k}^{a} = k$, $\deg \varepsilon = -1$. The relation between the new variables $u_{k}^{a}$ and the old ones $p_{b}^{a}, e^{ix}$ is given by the formula $u_{k}^{a} = \partial_{x}^{k}(\sum_{b \in \mathbb{Z}} p_{b}^{a} e^{ibx})$. The expression of the operator $\partial_{x}$ in the new variables $u_{k}^{a}$ is given by

$$
\partial_{x} = \sum_{a \in \mathbb{Z} \atop j \geq 0} u_{j+1}^{a} \frac{\partial}{\partial u_{j}^{a}}.
$$

Specifically, thanks to the fact that the intersection numbers appearing in equation (3.1) vanish unless $\sum a_{j} = -a$, we obtain $g_{a,d} \in \mathbb{C}[u_{k}^{a}]^{0}[u_{k}^{a}, \varepsilon, \mu]^{0}$, where we put the superscript [0] to denote the space of differential polynomials of degree 0.

The DR Hamiltonians are defined as the local functionals $g_{a,d} := f g_{a,d} dx$, which denote the equivalence classes of $g_{a,d}$ in the quotient vector space $(\mathbb{C}[u_{k}^{a}]^{0}[u_{k}^{a}, \varepsilon, \mu]/(\text{Im}(\partial_{x} \otimes \mathbb{C})))^{0}$. Note that, with respect to the formal variables $p_{b}^{a}, e^{ix}$, the symbol $\int g_{a,d} dx$ represents the coefficient of $e^{ix}$ in the formal power series $g_{a,d}$.

A result of [1] says that the DR Hamiltonians mutually commute,

$$
\{g_{a_{1},d_{1}, a_{2},d_{2}}\} = 0, \quad a_{1}, a_{2} \in \mathbb{Z}, \quad d_{1}, d_{2} \in \mathbb{Z}_{\geq 0},
$$

(3.2)

with respect to the Poisson brackets of two local functionals $\mathcal{F}_{1} = f f_{1} dx, \mathcal{F}_{2} = f f_{2} dx$, with $f_{1}, f_{2} \in \mathbb{C}[u_{k}^{a}]^{0}[u_{k}^{a}, \varepsilon, \mu]^{0}$, given by

$$
\{\mathcal{F}_{1}, \mathcal{F}_{2}\} = \int \left( \sum_{a_{1}, a_{2} \in \mathbb{Z}} \frac{\delta \mathcal{F}_{1}}{\delta u^{a_{1}}} \delta_{a_{1}+a_{2},0} \partial_{x} \left( \frac{\delta \mathcal{F}_{2}}{\delta u^{a_{2}}} \right) \right) dx,
$$

(3.3)

where $\frac{\delta \mathcal{F}}{\delta u^{a}} = \sum_{k \geq 0} (-\partial_{x})^{k} \frac{\delta \mathcal{F}}{\delta u^{a_{2}}}$ for $\mathcal{F} = \int f dx$ and $f \in \mathbb{C}[u_{k}^{a}]^{0}[u_{k}^{a}, \varepsilon, \mu]^{0}$.
Now we make the following observation, specific for the infinite rank partial CohFT we are dealing with. For fixed \( d \in \mathbb{Z}_{\geq 0} \), let us collect the DR Hamiltonian densities \( g_{a,d} \), for all \( a \in \mathbb{Z} \), into a single generating function \( g_d := \sum_{a \in \mathbb{Z}} g_{a,d} e^{-iay} \) by use of the external formal variable \( e^y \). Because the classes \( c_{g,n+1}(e_a \otimes \otimes_{j=1}^n e_{a_j}) \) are polynomials in \( a_1, \ldots, a_n \) of top degree \( 2g \), where in particular the coefficient of \( \mu^{2j} \) is a polynomial of degree \( 2j \), and \( \sum a_j = -a \), we can consider the formal change of variables

\[
 u_{k_1,k_2} = \partial_y^{k_2} \left( \sum_{a \in \mathbb{Z}} u_{k_1}^a e^{iay} \right) = \partial_x^{k_1} \partial_y^{k_2} \left( \sum_{a,b \in \mathbb{Z}} p_{a,b} e^{iay+ibx} \right)
\]

and express \( g_d \) uniquely as a differential polynomial in these new variables, specifically \( g_d \in \mathbb{C}[[u_{*,*}, \varepsilon, \mu]]^{(0,0)} \), where \( \deg u_{k_1,k_2} = (k_1,k_2) \), \( \deg \varepsilon = (-1,0) \), \( \deg \mu = (0,-1) \). Naturally, we have

\[
 \partial_x = \sum_{k_1,k_2 \geq 0} u_{k_1+1,k_2} \frac{\partial}{\partial u_{k_1,k_2}}, \quad \partial_y = \sum_{k_1,k_2 \geq 0} u_{k_1,k_2+1} \frac{\partial}{\partial u_{k_1,k_2}}.
\]

We will denote \( u_{0,0} \) simply by \( u \).

The DR Hamiltonian densities \( g_{a,d} \) can be recovered from \( g_d \) by the formula \( g_{a,d} = \int (g_d e^{iay}) dy \), which extracts the coefficient of \( e^{-iay} \) from \( g_d \). Hence \( \mathcal{F}_{a,d} = \int \int (g_d e^{iay}) dx dy \). This suggests to restrict our attention to the Hamiltonians \( \mathcal{F}_{0,d} \), whose densities depend on \( e^y \) through \( u_{*,*} \) only. These are the simplest and most commonly considered kind of local functionals.

Let \( \mathcal{F}_d = \mathcal{F}_{0,d} = \int \int g_d \, dx dy \) be the equivalence class of \( g_d \) in the quotient vector space \( (\mathbb{C}[[u_{*,*}, \varepsilon, \mu]]/ (\text{Im}(\partial_x) \oplus \text{Im}(\partial_y) \oplus \mathbb{C}))^{(0,0)} \). Then, from equation (3.2), we deduce

\[
 \{ \mathcal{F}_{d_1}, \mathcal{F}_{d_2} \} = 0, \quad d_1, d_2 \in \mathbb{Z}_{\geq 0}, \tag{3.4}
\]

where the Poisson bracket of two local functionals \( \mathcal{F}_1 = \int \int f_1 \, dx dy, \mathcal{F}_2 = \int \int f_2 \, dx dy \), with \( f_1, f_2 \in \mathbb{C}[[u_{*,*}, \varepsilon, \mu]]^{(0,0)} \) is given by

\[
 \{ \mathcal{F}_1, \mathcal{F}_2 \} = \int \int \left( \frac{\delta \mathcal{F}_1}{\delta u} \delta_x \left( \frac{\delta \mathcal{F}_2}{\delta u} \right) \right) dx dy, \tag{3.5}
\]

where \( \frac{\delta}{\delta u} \) is the sum

\[
 \frac{\delta}{\delta u} = \sum_{k_1,k_2 \geq 0} (-\partial_x)^{k_1} (-\partial_y)^{k_2} \partial f / \partial u_{k_1,k_2} \text{ for } \mathcal{F} = \int \int f \, dx dy \text{ and } f \in \mathbb{C}[[u_{*,*}, \varepsilon, \mu]]^{(0,0)}.
\]

The evolutionary PDEs generated via the above Poisson structure by the DR Hamiltonians \( \mathcal{F}_d \) are all compatible and have the form

\[
 \frac{\partial u}{\partial \mathcal{F}_d} = \partial_x \frac{\delta \mathcal{F}_d}{\delta u}, \quad d \in \mathbb{Z}_{\geq 0}. \tag{3.6}
\]

4. The noncommutative KdV hierarchy and the main theorem

The classical construction of the KdV hierarchy as the system of Lax equations

\[
 \frac{\partial L}{\partial t_n} = \frac{e^{2n}}{(2n+1)!!} \left[ \left( L^{n+1/2} \right)_+ L \right], \quad n \geq 1,
\]

where \( L = \partial_x^2 + 2 \varepsilon^{-2} u \), admits generalizations, called noncommutative KdV hierarchies, where one doesn’t have the pairwise commutativity of the \( x \)-derivatives of the dependent variable. In what follows we will work with a specific example from the class of noncommutative KdV hierarchies.
The graded algebra of differential polynomials in two space dimensions introduced in Section 3, \( \mathbb{C}[[u_*, \varepsilon, \mu]] \), where \( \deg u_{k_1, k_2} = (k_1, k_2) \), \( \deg \varepsilon = (-1, 0) \), \( \deg \mu = (0, -1) \), can be endowed with the following graded associative Moyal star-product. Let \( f, g \in \mathbb{C}[[u_*, \varepsilon, \mu]] \), with \( \deg f = (i_1, i_2) \), \( \deg g = (j_1, j_2) \), then define

\[
f \ast g := f \exp \left( \frac{i \mu}{2} \left( \frac{\partial_x}{\partial y} - \frac{\partial_x}{\partial y} \right) \right) g = \sum_{n \geq 0} \sum_{k_1+k_2=n} \frac{(-1)^{k_1}(i \mu)^n}{2^n k_1! k_2!} (\partial_{x_1}^k \partial_{y_1}^k f)(\partial_{x_2}^k \partial_{y_2}^k g),
\]

with \( \deg (f \ast g) = (i_1 + j_1, i_2 + j_2) \).

Let us consider the algebra of pseudo-differential operators of the form

\[
\sum_{i \leq n} a_i \ast \partial^i_x, \quad n \in \mathbb{Z}, \quad a_i \in \mathbb{C}[[u_*, \varepsilon, \mu]][\varepsilon, \varepsilon^{-1}].
\]

Consider the operator \( L := \partial_x^2 + 2\varepsilon^{-2} \mu \). The noncommutative KdV hierarchy with respect to the Moyal star-product (4.1) is defined by (see, for example, [10, 14])

\[
\frac{\partial L}{\partial t_n} = \frac{\varepsilon^{2n}}{(2n + 1)!} \left[ \left( L^{n+1/2} \right)_+ L \right]_+, \quad n \geq 1,
\]

where we put the subscript \( \ast \) in the notation for the commutator in order to emphasize that it is taken with respect to the noncommutative product \( \ast \). The first equation of the hierarchy is

\[
\frac{\partial u}{\partial t_1} = \frac{1}{2} \partial_x (u \ast u) + \frac{\varepsilon^2}{12} u_{xxx}.
\]

**Theorem 4.1.** The flows \( \frac{\partial}{\partial t_d} \), \( d \geq 1 \), of the DR hierarchy (3.6) are given by the noncommutative KdV hierarchy (4.4).

**Proof.** We prove the theorem in two steps.

**Step 1.** Let us prove that the flow \( \partial / \partial t_1 \) of the DR hierarchy (3.6) is given by

\[
\frac{\partial u}{\partial t_1} = \frac{1}{2} \partial_x (u \ast u) + \frac{\varepsilon^2}{12} u_{xxx}.
\]

For this we have to compute the integrals

\[
\int_{\mathcal{A}_{g,n+1}} \lambda_g \psi_1 \Theta(0, a_1, \ldots, a_n)^k DR_g(0, b_1, \ldots, b_n) = (2g - 2 + n) \int_{\mathcal{A}_{g,n}} \lambda_g \Theta(a_1, \ldots, a_n)^k DR_g(b_1, \ldots, b_n).
\]

Relation (2.6) implies that integral (4.4) is nonzero only if \( n = 3 \) and \( k = g \) or if \( n = 2, k = 0 \) and \( g = 1 \). In the second case integral (4.4) is equal to \( b_1^2 / 12 \), which gives the second term on the right-hand side of equation (4.3).

Regarding integral (4.4) for \( n = 3 \) and \( k = g \), by Theorem 2.1, we have

\[
\sum_{g \geq 0} \sum_{a_1, a_2, b_1, b_2 \in \mathbb{Z}} \frac{(-\varepsilon^2 \mu^2)^g}{g!} \int_{\mathcal{A}_{g,4}} \lambda_g \psi_1 \Theta(0, -a_1 - a_2, a_1, a_2)^g DR_g(0, -b_1 - b_2, b_1, b_2) p_{b_1}^{a_1} p_{b_2}^{a_2}
\]

\[
= \sum_{g \geq 0} \sum_{a_1, a_2, b_1, b_2 \in \mathbb{Z}} \frac{(-\varepsilon^2 \mu^2)^g}{2g(2g)!} (a_1 b_1 - a_2 b_2)^g p_{b_1}^{a_1} p_{b_2}^{a_2}
\]

\[
= \sum_{g \geq 0} \sum_{k_1+k_2=2g} \sum_{a_1, a_2, b_1, b_2 \in \mathbb{Z}} \frac{(-1)^{k_1} (-\varepsilon^2 \mu^2)^g}{2g k_1! k_2!} (a_1 b_1)^{k_1} (a_2 b_2)^{k_2} p_{b_1}^{a_1} p_{b_2}^{a_2}
\]
\[= \sum_{g \geq 0} \sum_{k_1 + k_2 = 2g} \frac{(-1)^{k_2}(-\varepsilon^2 \mu^2)^g}{2^g k_1! k_2!} u_{k_1, k_2} u_{k_2, k_1}\]

\[= u \ast u.\]

Equation (4.3) is then proved.

**Step 2.** Let us now check that all other flows \(\frac{\partial}{\partial t_d}\) of the DR hierarchy, for \(d \geq 2\), are described by the noncommutative KdV hierarchy.

Let \(f \in \mathbb{C}[[u \ast, \varepsilon, \mu]]\) and let \(\deg f = (\deg_x f, \deg_y f)\) so that, in particular,

\[\deg_x u_{a,b} = a, \quad \deg_y u_{a,b} = b, \quad \deg_x \varepsilon = \deg_y \mu = -1.\]

We see that both the flows of the DR hierarchy (3.6) and the flows of the noncommutative KdV hierarchy (4.2) have the form

\[\frac{\partial u}{\partial t_d} = P_d(u \ast, \mu, \varepsilon) = \sum_{i \geq 0} P_{d,i}(u \ast, \mu) \varepsilon^i, \quad d \geq 1, \quad (4.5)\]

where \(P_{d,i}(u \ast, \mu)\) are polynomials in the variables \(u \ast, \mu\) satisfying

\[P_1 = \frac{1}{2} \partial_x (u \ast u) + \frac{\varepsilon^2}{12} u_{xxx}, \quad (4.6)\]

\[P_{d,0} = \partial_x \left( \frac{u^{d+1}}{(d+1)!} \right), \quad (4.7)\]

\[\deg_x P_{d,i} = i + 1, \quad \deg_y P_{d,i} = 0. \quad (4.8)\]

It remains to check that a hierarchy of commuting flows of the form (4.5), satisfying properties (4.6)–(4.8), is unique. This is guaranteed by the following lemma.

**Lemma 4.2.** Suppose that \(P(u \ast)\) is a polynomial in the variables \(u \ast\) of degrees \(\deg_x P = d \geq 2, \deg_y P = q \geq 0\), and such that the flows

\[\frac{\partial u}{\partial t} = uu_x, \quad \frac{\partial u}{\partial \tau} = P(u \ast),\]

commute. Then \(P = 0\).

**Proof.** Without loss of generality we can assume that \(P\) is homogeneous with respect to an auxiliary gradation given by \(\overline{\deg} u_{a,b} := 1\). So we assume that \(\overline{\deg} P = k \geq 1\).

For a vector \(\pi = (a_1, \ldots, a_k) \in \mathbb{Z}^k\) denote \(|\pi| := \sum a_i\). Let

\[\mathcal{P}_k := \{\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}_{\geq 0}^k | \lambda_1 \geq \cdots \geq \lambda_k\},\]

\[\mathcal{P}_{k,q} := \{\lambda \in \mathcal{P}_k | |\lambda| = q\}.\]

The set \(\mathcal{P}_{k,q}\) is endowed with the lexicographical order. We will use the standard notation

\[m_p(\lambda) := \sharp \{1 \leq i \leq k | \lambda_i = p\}, \quad \lambda \in \mathcal{P}_k, \quad p \geq 0.\]

The sequence \((m_0(\lambda), m_1(\lambda), \ldots)\) uniquely determines \(\lambda\) that justifies the notation

\[\lambda = (0^{m_0(\lambda)} 1^{m_1(\lambda)} \cdots).\]
Introduce a basis \( f_\lambda, \lambda \in \mathcal{P}_k \), in the space of symmetric polynomials \( \mathbb{C}[x_1, \ldots, x_k]^S_k \) by

\[
f_\lambda(x_1, \ldots, x_k) := \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(k)}^{\lambda_k}, \quad \lambda \in \mathcal{P}_k.
\]

For \( \lambda \in \mathcal{P}_k \) we call a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_k] \) \( \lambda \)-symmetric, if it is invariant with respect to the permutation of any pair of variables \( x_i \) and \( x_j \), \( i \neq j \), such that \( \lambda_i = \lambda_j \). We also introduce a notation for the symmetrization of a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_k] \) with respect to the variables \( x_1, \ldots, x_\alpha \), \( 1 \leq \alpha \leq k \):

\[
\text{Sym}_{x_1, \ldots, x_\alpha} f := \frac{1}{\alpha!} \sum_{\sigma \in S_\alpha} f(x_{\sigma(1)}, \ldots, x_{\sigma(\alpha)}, x_{\alpha+1}, \ldots, x_k).
\]

Making the substitutions \( u(x, y) = \sum_{\alpha \in \mathbb{Z}} u^\alpha(y) e^{i\alpha x} \) and \( t \mapsto \frac{1}{\tau} \), the flows \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \tau} \) can be written in the form

\[
\frac{\partial u^\alpha}{\partial t} = \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}} \alpha_1 u^{\alpha_1} u^{\alpha_2}, \quad \alpha \in \mathbb{Z},
\]

\[
\frac{\partial u^\alpha}{\partial \tau} = \sum_{\lambda \in \mathcal{P}_k} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}} P_\lambda(\alpha_1, \ldots, \alpha_k) u^{\alpha_1}_{\lambda_1} \cdots u^{\alpha_k}_{\lambda_k}, \quad \alpha \in \mathbb{Z},
\]

where \( P_\lambda \) is a \( \lambda \)-symmetric polynomial in \( \alpha_1, \ldots, \alpha_k \) of degree \( d \). Here \( u^\alpha := \frac{\partial}{\partial y} u^\alpha \).

The commutator of the flows \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \tau} \) is given by

\[
\frac{\partial}{\partial t} \frac{\partial u^\alpha}{\partial \tau} - \frac{\partial}{\partial \tau} \frac{\partial u^\alpha}{\partial t} = \sum_{\beta \in \mathbb{Z}, b \geq 0} \frac{\partial}{\partial y} \left( \sum_{\beta_1, \beta_2 \in \mathbb{Z}} P_\lambda(\beta_1, \ldots, \beta_k) u^{\beta_1}_{\lambda_1} \cdots u^{\beta_k}_{\lambda_k} \right) u^{\alpha_2} - \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}} \frac{\partial}{\partial y} \left( \sum_{\lambda \in \mathcal{P}_k, \sum \beta_i = \alpha_1} P_\lambda(\beta_1, \ldots, \beta_k) u^{\beta_1}_{\lambda_1} \cdots u^{\beta_k}_{\lambda_k} \right) u^{\alpha_1} = \sum_{\lambda \in \mathcal{P}_{k+1, q}} \sum_{\sum \alpha_i = 1} Q_\lambda(\alpha_1, \ldots, \alpha_k+1) u^{\alpha_1}_{\lambda_1} \cdots u^{\alpha_{k+1}}_{\lambda_{k+1}},
\]

where a polynomial \( Q_\lambda \) is \( \lambda \)-symmetric.

Let \( \tilde{\lambda} := \max\{\lambda \in \mathcal{P}_{k,q} | P_\lambda \neq 0\} \) and \( \tilde{\lambda} := (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k, 0) \in \mathcal{P}_{k+1, q} \). We see that the sum in line (4.9) has the form

\[
\sum_{\sum \alpha_i = 1} Q_\lambda(\alpha_1, \ldots, \alpha_k+1) u^{\alpha_1}_{\lambda_1} \cdots u^{\alpha_{k+1}}_{\lambda_{k+1}} + \sum_{\lambda < \tilde{\lambda}} \sum_{\sum \alpha_i = 1} Q_\lambda(\alpha_1, \ldots, \alpha_k+1) u^{\alpha_1}_{\lambda_1} \cdots u^{\alpha_{k+1}}_{\lambda_{k+1}}.
\]
and
\[ Q_\lambda(\alpha_1, \ldots, \alpha_{k+1}) = \text{Sym}_{\alpha_{s+1}, \ldots, \alpha_{k+1}} \left[ \sum_{i=1}^{k} (\alpha_i P_\lambda(\alpha_1, \ldots, \alpha_i + \alpha_{k+1}, \ldots, \alpha_k) - \alpha_i P_{\lambda}(\alpha_1, \ldots, \alpha_k)) \right] + \sum_{i=1}^{s} \alpha_{k+1} P_\lambda(\alpha_1, \ldots, \alpha_i + \alpha_{k+1}, \ldots, \alpha_k) - \alpha_{k+1} P_{\lambda}(\alpha_1, \ldots, \alpha_k) \],

where \( s := \lfloor 1 \leq i \leq k \rfloor \).

We decompose the polynomial \( P_\lambda \) in the following way:
\[ P_\lambda(\alpha_1, \ldots, \alpha_k) = \sum_{\pi \in \mathbb{Z}_{\geq 0}, \mu \in P_{k-s}} C_{\pi, \mu} \alpha_1^{a_1} \cdots \alpha_s^{a_s} f_\mu(\alpha_{s+1}, \ldots, \alpha_k), \quad C_{\pi, \mu} \in \mathbb{C}. \]

Let
\[ P^{(p)}_\lambda(\alpha_1, \ldots, \alpha_k) := \sum_{\pi \in \mathbb{Z}_{\geq 0}, \mu \in P_{k-s-d-p}} C_{\pi, \mu} \alpha_1^{a_1} \cdots \alpha_s^{a_s} f_\mu(\alpha_{s+1}, \ldots, \alpha_k), \quad p \geq 0, \]
\[ P^{(p, \mu)}_\lambda(\alpha_1, \ldots, \alpha_k) := \left( \sum_{\pi \in \mathbb{Z}_{\geq 0}, \|\pi\|=p} C_{\pi, \mu} \alpha_1^{a_1} \cdots \alpha_s^{a_s} \right) f_\mu(\alpha_{s+1}, \ldots, \alpha_k), \quad p \geq 0, \quad \mu \in P_{k-s-d-p}. \]

Let \( \tilde{d} := \max\{ p \geq 0 | P^{(p)}_\lambda \neq 0 \} \) and \( \tilde{\mu} = (0^{m_0}1^{m_1} \ldots) := \max\{ \mu \in P_{k-s-d-d} | P^{(\tilde{d}, \mu)}_\lambda \neq 0 \} \). Note that
\[ \text{Sym}_{\alpha_{s+1}, \ldots, \alpha_{k+1}}(f_\mu(\alpha_{s+1}, \ldots, \alpha_k)\alpha_{k+1}) = f_{\tilde{\mu}}(\alpha_{s+1}, \ldots, \alpha_{k+1}), \]
where \( \tilde{\mu} := (0^{m_0}1^{m_1}2^{m_2} \ldots) \). It is now easy to see that
\[ Q^{(\tilde{d}, \tilde{\mu})}_\lambda(\alpha_1, \ldots, \alpha_{k+1}) = (d + s - 1 + \sum_{i \geq 2} m_i) \left( \sum_{\pi \in \mathbb{Z}_{\geq 0}, \|\pi\| = \tilde{d}} C_{\pi, \tilde{\mu}} \alpha_1^{a_1} \cdots \alpha_s^{a_s} \right) f_{\tilde{\mu}}(\alpha_{s+1}, \ldots, \alpha_{k+1}), \]
which is nonzero, because \( d \geq 2 \) and \( P^{(\tilde{d}, \tilde{\mu})}_\lambda \neq 0 \). This contradicts the assumption that the flows \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \tau} \) commute. The lemma is proved. \( \square \)

This completes the proof of the theorem. \( \square \)

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