HAMILTON TYPE ENTROPY FORMULA ALONG THE RICCI FLOW ON SURFACES WITH BOUNDARY

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Abstract. In this article, we establish a monotonicity formula of Hamilton type entropy along Ricci flow on compact surfaces with boundary. We also study the relation between our entropy functional and the $W$-functional of Perelman type.

1. INTRODUCTION

The aim of this short note is to formulate a monotonicity of Hamilton type entropy along Ricci flow on compact surfaces with boundary. We further aim to investigate the relation between our entropy functional and the so-called $W$-functional of Perelman type.

1.1. Hamilton monotonicity. Let $(M, g(t))_{t \in [0, T)}$ be a manifold equipped with a time-dependent Riemannian metric. Hamilton [16] has introduced the notion of Ricci flow

$$\partial_t g = -2 \text{Ric}.$$ 

He [17] has studied the (normalized) Ricci flow on closed surfaces, and established some convergence results. Also, he has obtained a monotonicity of a certain entropy functional. We consider a Ricci flow $(S^2, g(t))_{t \in [0, T)}$ on the two dimensional sphere whose initial metric has positive scalar curvature $R > 0$, and volume $8\pi$. Notice that the positivity of $R$ is preserved, and the volume $v(S^2)$ evolves by $8\pi(1-t)$ (see e.g., [7, Lemma 4.4]). The Hamilton entropy functional is defined by

$$E(t) := \int_{S^2} R \log R \, dv + 8\pi \log(1-t),$$

which is non-negative (see e.g., [7, Lemma 4.5]). This type of entropy functional appears in the context of not only the Ricci flow theory but also minimal surface theory and Gauss curvature flow theory (see [3, Subsection 2.3], [15]).

The monotonicity in [17] can be stated as follows (see [17, Theorem 7.2], and cf. [7, Proposition 4.7], [8, Lemma 2.1], [9, Exercise 9.9]):

**Theorem 1.1** ([17]). In the above situation, let $f$ be a smooth function on $S^2$ solving

$$R + \Delta f = \frac{1}{1-t}.$$ 

Then we have

$$\frac{d}{dt} E(t) = -\int_{S^2} \left( R \| \nabla f - \nabla \log R \|^2 + 2 \left\| \frac{1}{2} R g + \nabla^2 f - \frac{1}{2(1-t)} g \right\|^2 \right) \, dv \leq 0.$$ 

In particular, $E(t)$ is non-increasing.

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1.2. Hamilton type monotonicity on surfaces with boundary. One of the purposes of this note is to generalize Theorem [14] for Ricci flow on compact surfaces with boundary. Ricci flow on manifolds with boundary has been investigated by several authors, which is not as much as that on manifolds without boundary (see e.g., [13], [23], [24], [26] for short time existence and uniqueness, [4], [5], [6], [10], [11], [12], [26] for convergence, [11], [24] for entropy formulas of Perelman type). Gianniotis [13] has established a quite general short time existence and uniqueness result (see [13, Theorem 1.2]). As summarized in [12], such a result holds for a given initial metric, mean curvature and induced metric on the boundary satisfying compatibility conditions. We keep in mind the setting of [13].

Let \((M, g(t))_{t \in [0, T)}\) be a Ricci flow on compact surface with boundary whose initial metric has positive scalar curvature \(R > 0\). We further assume a Neumann type boundary condition (1.2)

\[ R_\nu = 0 \]

for all \(t \in [0, T)\), here the left hand side means the derivative of the scalar curvature \(R\) in the direction of the outward unit normal vector \(\nu\) on the boundary \(\partial M\). The condition (1.2) together with the evolution formula of \(R\), strong maximum principle and parabolic Hopf lemma ensures that the positivity of \(R\) is preserved (cf. (2.1) below, and [2, Section 3]).

Remark 1.2. Li [20] investigated the (normalized) Ricci flow on compact surfaces with boundary under the Neumann type boundary condition (1.2), and obtained a short time existence result (see [20, Corollary 6]).

We introduce an entropy functional of Hamilton type as follows:

(1.3) \[ \mathcal{E}_\partial(t) := \int_M R \log R \, dv - \log \bar{R} \int_M R \, dv, \]

where \(\bar{R}\) denotes the average of \(R\) defined as

\[ \bar{R} := \frac{1}{v(M)} \int_M R \, dv. \]

We will verify that \(\mathcal{E}_\partial(t)\) is non-negative (see Proposition 2.3 below).

Remark 1.3. In virtue of the Gauss-Bonnet theorem, the entropy functional (1.1) also can be written in the form of (1.3). Similarly, for surfaces with boundary, if the volume with respect to the initial metric is \(4\pi \chi(M)\), then (1.3) can be written as

\[ \mathcal{E}_\partial(t) = \int_M R \log R \, dv \]

\[ + 4\pi \chi(M) \left( 1 - \frac{1}{2\pi \chi(M)} \int_{\partial M} \kappa \, ds \right) \log \left( \frac{(1 - t) + \frac{1}{2\pi \chi(M)} \int_0^t \int_{\partial M} \kappa \, ds \, d\xi}{1 - \frac{1}{2\pi \chi(M)} \int_{\partial M} \kappa \, ds} \right) \]

for the Euler characteristic \(\chi(M)\) of \(M\), and the geodesic curvature \(\kappa\) of \(\partial M\).

We are now in a position to state one of our main results.

Theorem 1.4. Let \((M, g(t))_{t \in [0, T)}\) be a Ricci flow on compact surface with boundary whose initial metric has positive scalar curvature. We further assume the Neumann type boundary condition (1.2). Let \(f\) be a smooth function on \(M\) solving the Neumann boundary problem

(1.4) \[
\begin{cases}
R + \Delta f = \bar{R} & \text{in } M, \\
 f_\nu = 0 & \text{on } \partial M.
\end{cases}
\]
Then we have
\begin{equation}
\frac{d}{dt} \mathcal{E}_\partial(t) = - \int_M \left( R \|\nabla f - \nabla \log R\|^2 + 2 \left\| \frac{1}{2} R g + \nabla^2 f - \frac{1}{2} \nabla R g \right\|^2 \right) dv \\
- 2 \int_{\partial M} \kappa \|\nabla_{\partial M} (f|_{\partial M})\|^2 ds.
\end{equation}
In particular, if \( \partial M \) is convex with respect to the initial metric (i.e., \( \kappa(0) \) is non-negative), then \( \mathcal{E}_\partial(t) \) is non-increasing.

**Remark 1.5.** The convexity of \( \partial M \) is preserved in our setting. Actually, we possess the following evolution formula of \( \kappa \) along Ricci flow (see e.g., [26, (3.22)], [11, Proposition 2.1]):
\[ \partial_t \kappa = \frac{1}{2} \kappa R - \frac{1}{2} R_v. \]
In particular, under the condition \( (1.2) \), we can solve it as follows:
\[ \kappa(t) = \kappa(0) \exp \left( \frac{1}{2} \int_0^t R d\xi \right). \]
This tells us the desired claim. Once the boundary \( \partial M \) becomes convex, the Gauss-Bonnet theorem and the positivity of \( R \) yield that \( \chi(M) \) must be positive.

**Remark 1.6.** We mention the critical point of \( \mathcal{E}_\partial(t) \). Let \( \partial M \) be convex. Suppose that the time derivative of \( \mathcal{E}_\partial(t) \) vanishes at time \( t_0 > 0 \). Then by Theorem 1.4 we see
\[ \nabla f = \nabla \log R, \quad \frac{1}{2} R g + \nabla^2 f = \frac{1}{2} \nabla R g, \quad \kappa \|\nabla_{\partial M} (f|_{\partial M})\|^2 = 0. \]
The first identity says that \( f \) agrees with \( \log R \) up to addition by a constant. The second one means that it is a gradient shrinking Ricci soliton. We now observe the third identity. In the case where \( \kappa \) is positive everywhere, \( f \) must be constant over \( \partial M \); in particular, \( R \) also has the same property due to the first identity and the Neumann boundary conditions \( (1.2) \) and \( (1.4) \). On the other hand, when \( \kappa = 0 \), the boundary \( \partial M \) is geodesic; in particular, we can consider the double of \( M \), and its universal cover is a smooth gradient shrinking Ricci soliton over \( S^2 \). It is well-known that every Ricci soliton on \( S^2 \) has constant scalar curvature (see e.g., [9, Corollary 9.11]). Thus \( M \) also has constant scalar curvature.

**Remark 1.7.** Cortissoz-Murcia [11] have obtained monotonicity formulas of Perelman type in a similar setting (see e.g., [11, Theorems 3.1 and 3.2], and also [24, Theorem 3.3]). One can observe that a similar boundary term to that in \( (1.5) \) appears. Also, Ni [21] has formulated monotonicity of Perelman type for linear heat equation on static manifolds with boundary (see [21, Corollary 3.1], and also [19, Problem 8.3]). In the same manner, a similar boundary term appears.

### 1.3. Perelman type monotonicity on surfaces with boundary.

Motivated by the question of Ni [21], Guo [14] has examined a relation between the Hamilton entropy functional and the so-called \( \mathcal{W} \)-functional introduced by Perelman [22] on closed surfaces. Inspired by [11], we investigate the relation between our entropy functional \( (1.3) \) and the \( \mathcal{W} \)-functional on compact surfaces with boundary.

We consider the setting in Subsection 1.2. We introduce a \( \mathcal{W} \)-functional of Guo type by
\[ \mathcal{W}_\partial(t) := \int_M \left( \tau (R - \|\nabla \log R\|^2) - \log R - \log \tau \right) R dv - 2 \log \tau \int_{\partial M} \kappa ds, \]
where
\[ \tau := T - t. \]
Remark 1.8. By the inequality (2.3) stated below, and the Gauss-Bonnet theorem, we possess the following relation between $E_\partial(t)$ and $W_\partial(t)$:

$$W_\partial(t) = \tau \left( \frac{d}{dt} E_\partial(t) \right) - E_\partial(t) - 4\pi \chi(M) \log \tau + \tau v(M) \frac{R^2}{2} - \log R \int_M R \, dv.$$ 

Our second main theorem is the following (cf. [14, Theorem 1.2]):

**Theorem 1.9.** Let $(M, g(t))_{t \in [0,T)}$ be a Ricci flow on compact surface with boundary whose initial metric has positive scalar curvature. We further assume the Neumann type boundary condition (1.2). Then we have

$$\frac{d}{dt} W_\partial(t) = 2\tau \int_M R \left\| \frac{1}{2} R g + \nabla^2 \log R - \frac{1}{2\tau} g \right\|^2 \, dv + 2\tau \int_{\partial M} \kappa \left( R \left\| \nabla_{\partial M} (\log R) \right\|_{\partial M}^2 + \frac{1}{\tau^2} \right) \, ds.$$

In particular, if $\partial M$ is convex with respect to the initial metric, then $W_\partial(t)$ is non-decreasing.

**Remark 1.10.** Let us discuss the critical point of $W_\partial(t)$. Let $\partial M$ be convex. Assume that its derivative vanishes at $t_0 > 0$. In view of Theorem 1.9,

$$\frac{1}{2} R g + \nabla^2 \log R = \frac{1}{2\tau} g, \quad \kappa = 0.$$ 

It follows that it is a gradient shrinking Ricci soliton with geodesic boundary. From the same discussion as in Remark 1.6, we can conclude that $M$ has constant scalar curvature.

2. Proof

Let us prove our main theorems.

2.1. Reilly formula. In this subsection, we recall the Reilly formula, which is a key ingredient of the proof of our main theorem. Let $(M, g)$ be a compact manifold with boundary. The second fundamental form of $\partial M$ is defined as

$$\Pi(X, Y) := g(\nabla_X v, Y)$$

for tangent vectors $X, Y$ on $\partial M$, where $\nabla$ denotes the Levi-Civita connection. Note that in the two dimensional case, the geodesic curvature $\kappa$ coincides with $\Pi(v, v)$ for a unit tangent vector $v$ of $\partial M$. Furthermore, the mean curvature $H$ is defined as the trace of $\Pi$. We possess:

**Theorem 2.1** ([25]). For all $f \in C^\infty(M)$, it holds that

$$\int_M (\Delta f)^2 - \text{Ric}(\nabla f) - \|\nabla^2 f\|^2 \, dv = \int_{\partial M} 2 f \Delta_{\partial M}(f|_{\partial M}) + f^2 H + \Pi(\nabla_{\partial M}(f|_{\partial M}), \nabla_{\partial M}(f|_{\partial M})) \, dv_{\partial M}.$$ 

For later convenience, we also recall the following well-known and useful formula, which is used in the standard proof of Reilly formula (see e.g., [18, Chapter 8]):

**Lemma 2.2.** For all $f \in C^\infty(M)$ we have

$$(\|\nabla f\|^2)_v = 2 f_v [\Delta f - \Delta_{\partial M}(f|_{\partial M}) - f_v H] + 2 g_{\partial M}(\nabla_{\partial M}(f|_{\partial M}), \nabla_{\partial M} f_v) - 2 \Pi(\nabla_{\partial M}(f|_{\partial M}), \nabla_{\partial M}(f|_{\partial M})).$$
2.2. Proof of Hamilton type monotonicity. Let \((M, g(t))_{t \in [0,T]}\) be as in Theorem 1.4.

We first verify the following (cf. [7, Lemma 4.5]):

**Proposition 2.3.** For all \(t \in [0,T)\) we have \(E_\partial(t) \geq 0\).

**Proof.** We see \(c \geq 1 - e^{-1}\) for all \(c > 0\), and hence

\[
E_\partial(t) = \int_M R \log \left(\frac{R}{\bar{R}}\right) \, dv \geq \int_M (R - \bar{R}) \, dv = 0.
\]

This proves the desired estimate. \(\square\)

Before we state the next assertion, we recall the following evolution formulas along Ricci flow (see e.g., [1, Corollaries 4.16 and 4.20]):

\[
(2.1) \quad \partial_t dv = -R \, dv, \quad \partial_t R = \Delta R + R^2.
\]

Let us calculate the following:

**Lemma 2.4.**

\[
\partial_t \left( \log \bar{R} \int_M R \, dv \right) = v(M) \bar{R}^2.
\]

**Proof.** By using (2.1) and integration by parts with (1.2), we see

\[
\partial_t \bar{R} = \frac{1}{v(M)^2} \left\{ v(M) \left( \int_M \partial_t R \, dv - \int_M R^2 \, dv \right) - \partial_t v(M) \int_M R \, dv \right\} = \bar{R}^2.
\]

It follows that

\[
\partial_t \left( \log \bar{R} \int_M R \, dv \right) = \frac{\partial_t \bar{R}}{R} \int_M R \, dv + \log \bar{R} \left( \int_M \partial_t R \, dv - \int_M R^2 \, dv \right)
= \frac{\partial_t \bar{R}}{R} \int_M R \, dv = v(M) \bar{R}^2.
\]

We arrive at the desired formula. \(\square\)

Lemma 2.4 yields the following:

**Lemma 2.5.**

\[
(2.2) \quad \frac{d}{dt} E_\partial(t) = \int_M \left( (\Delta R) \log R + R^2 \right) \, dv - v(M) \bar{R}^2
\]

\[
(2.3) \quad = \int_M \left( -\|\nabla \log R\|^2 + R \right) R \, dv - v(M) \bar{R}^2.
\]

**Proof.** The formulas (2.1), Lemma 2.4, and integration by parts with (1.2) imply

\[
\frac{d}{dt} E_\partial(t) = \int_M \left( (\partial_t R)(1 + \log R) - R^2 \log R \right) \, dv - \partial_t \left( \log \bar{R} \int_M R \, dv \right)
= \int_M \left( (\Delta R + R^2)(1 + \log R) - R^2 \log R \right) \, dv - v(M) \bar{R}^2
= \int_M \left( (\Delta R) \log R + R^2 \right) \, dv - v(M) \bar{R}^2,
\]

which is (2.2). The equality (2.3) follows from integration by parts with (1.2). \(\square\)

We now prove Theorem 1.4.
Proof of Theorem 1.4. Let \((M, g(t))_{t \in [0,T]}\) and \(f\) be as in Theorem 1.4. Using (2.3), we have
\[
\frac{d}{dt} \mathcal{E}_\delta(t) = \int_M \left(-R \|\nabla \log R\|^2 + R^2\right) \, dv - v(M) \overline{R}^2.
\]
We also deduce
\[
R^2 = (\Delta f)^2 - 2\overline{R} \Delta f + \overline{R}^2
\]
from (1.4). By integration by parts with \(f\), we have
\[
\int_M R^2 \, dv = \int_M (\Delta f)^2 \, dv + v(M) \overline{R}^2.
\]
Combining (2.4) and (2.5), we obtain
\[
\frac{d}{dt} \mathcal{E}_\delta(t) = \int_M \left(-R \|\nabla \log R\|^2 + (\Delta f)^2\right) \, dv.
\]
On the other hand, by integration by parts with \(f\), and by (1.4),
\[
\int_M R \|\nabla f\|^2 - 2(\Delta f)^2 + R \|\nabla \log R\|^2 \, dv
= \int_M R \|\nabla f\|^2 + 2g(\nabla f, \nabla \Delta f) + R \|\nabla \log R\|^2 \, dv
= \int_M R \|\nabla f \cdot \nabla \log R\|^2 \, dv.
\]
Furthermore, Theorem 2.3 yields
\[
\int_M 2(\Delta f)^2 - R \|\nabla f\|^2 - 2 \|\nabla^2 f\|^2 \, dv = 2 \int_{\partial M} \Pi(\nabla_{gM}(f|_{gM}), \nabla_{gM}(f|_{gM})) \, dv_{gM}.
\]
Summing up (2.7) and (2.8), we see
\[
\int_M R \|\nabla \log R\|^2 - 2 \|\nabla^2 f\|^2 \, dv = \int_M R \|\nabla f - \nabla \log R\|^2 \, dv
+ 2 \int_{\partial M} \Pi(\nabla_{gM}(f|_{gM}), \nabla_{gM}(f|_{gM})) \, dv_{gM}.
\]
Moreover, summing up (2.6) and (2.9) implies
\[
\frac{d}{dt} \mathcal{E}_\delta(t) + \int_M R \|\nabla f - \nabla \log R\|^2 \, dv + 2 \int_{\partial M} \Pi(\nabla_{gM}(f|_{gM}), \nabla_{gM}(f|_{gM})) \, dv_{gM}
= \int_M (\Delta f)^2 - 2 \|\nabla^2 f\|^2 \, dv = -2 \int_M \|\nabla^2 f - \frac{\Delta f}{2} g\|^2 \, dv.
\]
We finally apply (1.4) to the right hand side. This completes the proof.

2.3. Proof of Guo type monotonicity. Let \((M, g(t))_{t \in [0,T]}\) be as in Theorem 1.9. In order to prove Theorem 1.9, we prepare several lemmas. We start with the following:

Lemma 2.6.
\[\partial_\nu v = \frac{R}{2} \nu.\]

Proof. Let us consider a (time-independent) local coordinate \(x_1\) on \(\partial M\), and take the time derivative of \(g(\nu, \partial_1) = 0\). From the two dimensional Ricci flow equation and (1.2),
\[0 = (\partial_t g)(\nu, \partial_1) + g(\partial_\nu, \partial_1) = (-Rg)(\nu, \partial_1) + g(\partial_\nu, \partial_1) = g(\partial_\nu, \partial_1),\]
and hence \(\partial_\nu v\) is orthogonal to \(\partial M\). Next, we take the time derivative of \(g(\nu, \nu) = 1\). Then
\[0 = (\partial_t g)(\nu, \nu) + 2g(\partial_\nu, \nu) = (-Rg)(\nu, \nu) + 2g(\partial_\nu, \nu) = -R + 2g(\partial_\nu, \nu).
\]
We arrive at the desired claim. 

Having at hand Lemma 2.6 we obtain the following:

**Lemma 2.7.**

$$\left(\Delta R\right)_\nu = 0.$$ 

**Proof.** From (2.1), (1.2) and Lemma 2.6 we deduce

$$\left(\Delta R\right)_\nu = (d(\Delta R))(\nu) = (d(\partial_t R))(\nu) = (\partial_t (dR))(\nu) = -dR(\partial_t \nu) = -\frac{R}{2}R_\nu = 0.$$ 

We complete the proof. 

We now divide \(E\) into two parts. We set \(N = \int_M R \log R \, dv\), \(R = \log R \int_M R \, dv\) such that \(N = E + R\). In view of Remark 1.8, \(W\) can be expressed as

\[W = \frac{\partial}{\partial t} N = \frac{\partial}{\partial t} R + \frac{4\pi \chi(M) \log \tau}{\tau^2};\]

in particular,

\[\frac{d}{dt} W = \frac{d^2}{dt^2} N - \frac{2}{\tau} \frac{d}{dt} N + \frac{4\pi \chi(M)}{\tau^2}.\]

Therefore, it suffices to calculate the first two derivatives of \(N\). Thanks to Lemmas 2.2 and 2.7, we see the following (cf. [14, Theorem 1.2 and Lemma 2.1]):

**Lemma 2.8.**

\[\frac{d}{dt} N = \int_M \left( (\Delta R) \log R + R^2 \right) \, dv,\]

\[\frac{d^2}{dt^2} N = 2 \int_M R \left( \frac{1}{2} Rg + \nabla^2 \log R \right)^2 \, dv + 2 \int_{\partial M} \kappa R \|\nabla_{\partial M} (\log R)\|_{\partial M}^2 \, ds\]

\[\quad = 2 \int_M R \left( \frac{1}{2} Rg + \nabla^2 \log R - \frac{1}{2\tau} g \right)^2 \, dv + 2 \frac{d}{dt} N + \frac{4\pi \chi(M)}{\tau^2} \]

\[\quad + 2 \int_{\partial M} \kappa \left( R \|\nabla_{\partial M} (\log R)\|_{\partial M}^2 + \frac{1}{\tau^2} \right) \, ds.\]

**Proof.** The first one follows from the same calculation as in the proof of (2.2). We consider the second one (2.12). This has been obtained by [14, Theorem 1.2 and Lemma 2.1] for closed surfaces. By the same calculation (without integration by parts), one can verify

\[\frac{d^2}{dt^2} N = \int_M \Delta (\Delta R + R^2) \log R + \frac{(\Delta R)^2}{R^2} + 3R\Delta R + R^3 \, dv.\]

In what follows, taking care of the boundary terms, we make use of integration by parts with (1.2). First, we do it for the first term. It holds that

\[\frac{d^2}{dt^2} N = \int_M (\Delta R + R^2) \Delta \log R + \frac{(\Delta R)^2}{R} + 3R\Delta R + R^3 \, dv\]

\[= \int_M (\Delta R + R^2) \left( \frac{\Delta R}{R} - \frac{\|\nabla R\|^2}{R^2} \right) + \frac{(\Delta R)^2}{R} + 3R\Delta R + R^3 \, dv.\]
Here we used Lemma 2.7. Further, by applying the integration by parts to the term \( \| \nabla R \|^2 \),
\[
\frac{d^2}{dt^2} N_\partial(t) = \int_M 2(\Delta R)(\Delta \log R) + \Delta R \| \nabla \log R \|^2 + 5R\Delta R + R^3 \, dv.
\]
We now apply integration by parts to the second term. Then from Lemma 2.2 we derive
\[
\frac{d^2}{dt^2} N_\partial(t) = \int_M 2(\Delta R)(\Delta \log R) + R\Delta \| \nabla \log R \|^2 + 5R\Delta R + R^3 \, dv
- \int_{\partial M} R(\| \nabla \log R \|^2)_{\nu} \, ds
= \int_M 2(\Delta R)(\Delta \log R) + R\Delta \| \nabla \log R \|^2 + 5R\Delta R + R^3 \, dv
+ 2 \int_{\partial M} \kappa R \| \nabla \partial M (\log R) \|_{\partial M}^2 \, ds.
\]
The rest is same as [14], and that is left to the reader. \( \square \)

**Remark 2.9.** In view of (2.12), if \( \partial M \) is convex, then \( N_\partial(t) \) is also convex.

We are now in a position to prove Theorem 1.9.

**Proof of Theorem 1.9.** Substituting (2.12) into (2.10), we complete the proof. \( \square \)

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