Spacetime Variation of Lorentz-Violation Coefficients at Nonrelativistic Scale

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When the Standard-Model Extension (SME) is applied in curved spacetime, the Lorentz-violation coefficients must depend on spacetime position. This work describes some of the consequences of this spacetime variation. We focus on effects that appear at a nonrelativistic scale and extract sensitivity of completed experiments to derivatives of SME coefficient fields.

1. Introduction

Within the SME in Minkowski spacetime, Lorentz violation is controlled by a set of tensor coefficients that do not vary with spacetime position.\textsuperscript{1} In curved spacetime, however, Lorentz-violation coefficients that vary with spacetime position must be considered.\textsuperscript{2} In this work, we describe some of the consequences of the variation of these coefficients, particularly at a nonrelativistic scale.\textsuperscript{3}

2. Nonrelativistic Hamiltonian

Our framework is the minimal SME in weakly-curved spacetime with nonzero $a_\mu$ and $b_\mu$ coefficient fields, which is described by an action that is a combination of conventional, Lorentz-symmetric terms and a set of terms $\delta S$ that includes the effects of nonzero and spacetime-varying SME coefficients $a_\mu$ and $b_\mu$:

$$\delta S = \int d^4x \left\{ -\bar{\psi} \left( \delta \gamma^\mu - \frac{i}{2} h_\mu^{\alpha} + \frac{i}{2} h^{\alpha}_5 \delta \gamma^\mu \right) (a_\mu \gamma^\alpha + b_\mu \gamma_5 \gamma^\alpha) \psi \right\} .$$

To extract nonrelativistic effects, we apply the following procedure: (1) Perform a field redefinition to ensure a hermitian hamiltonian, and therefore
unitary time evolution. (2) Apply the Euler-Lagrange equations to the resulting action, then solve for \( i\partial_0 \psi = H \psi \) to extract the relativistic \( 4 \times 4 \) hamiltonian \( H \). (3) Perform a Foldy-Wouthuysen transformation, resulting in a nonrelativistic \( 4 \times 4 \) hamiltonian \( H_{NR} \).

The full result of these calculations is too long to include in this work, but typical behavior appears in the term

\[
H_{NR} \supset +1 \left\{ \begin{array}{c}
-mh^{m0} - \frac{i\partial_0 b_k}{4m} \varepsilon_p^m j^k \left( \delta^{kp} - \hat{h}^{kp} \right) \frac{pm}{m}
\end{array} \right\}.
\]

This term acts on nonrelativistic matter like certain terms in the SME nonrelativistic hamiltonian for Minkowski spacetime:

\[
H_{NR,Mink} \supset +1 \left\{ \begin{array}{c}
-mc_0^m + mc_{m0}^m \frac{pm}{m}
\end{array} \right\}.
\]

We may exploit this similarity to interpret the weakly-curved-spacetime factors as effective versions of the Minkowski-spacetime coefficients:

\[
(mc_{m0}^m + mc_{m0}^m)^{\text{eff}} = -mh^{m0} - \frac{i\partial_0 b_k}{4m} \varepsilon_p^m j^k \left( \delta^{kp} - \hat{h}^{kp} \right)
\]

We can then use the results of studies of the Minkowski-spacetime SME to interpret the effects of spacetime-varying coefficients. When the full calculation is done, we find that all terms that result from spacetime-varying \( a_\mu \) and \( b_\mu \) may be interpreted as effective values of Minkowski-spacetime coefficients \( a_\mu \), \( b_\mu \), \( c_\mu\nu \), and \( d_\mu\nu \).

Many derivatives \( \partial_0 a_\mu \) and \( \partial_\mu b_\nu \) contribute to multiple effective coefficients. In Table 1, the largest contribution of each derivative to the nonrelativistic hamiltonian is displayed.

| Minkowski-spacetime coefficient | Weakly-curved-spacetime coefficient | Intuitive equivalent |
|---------------------------------|------------------------------------|---------------------|
| \( mc_{m0}^m + mc_{m0}^m \)     | \( -\frac{i\partial_0 b_k}{4m} \varepsilon_p^m j^k \) | \( \text{curl}(\hat{b}) \) |
| \(-mc_{m0}^m \)                 | \( -\frac{i\partial_0 b_k}{4m} \varepsilon_p^m j^k \) | \( \text{curl}(\hat{a}) \) |
| \( md_{a0} \)                   | \( \frac{1}{4m} \varepsilon_p^m \left( \partial_0 a_0 - \partial_0 a_1 \right) \) | \( \text{grad}(a_0) - \partial_0 \hat{a} \) |
| \( md_{b0} \)                   | \( +\frac{1}{4m} \varepsilon_p^m \left( \partial_0 a_0 - \partial_0 a_1 \right) \) | \( i\partial_0 \hat{b}_k \) |
| \( md_{q} \) \( d_{ik} \)     | \( +\frac{1}{4m} \varepsilon_p^m \left( \partial_0 a_0 - \partial_0 a_1 \right) \) | \( i\text{curl}(\hat{b}) \) |
3. Analysis

3.1. Hermiticity

The Foldy-Wouthuysen transformation is unitary, and therefore a hermitian relativistic hamiltonian $H$ is guaranteed to yield a hermitian nonrelativistic hamiltonian $H_{NR}$. However, many individual terms are nonhermitian, including terms that are hermitian in the Minkowski spacetime limit.

The key mathematical idea is that the product $AB$ of hermitian operators $A$ and $B$ is hermitian if and only if $A$ and $B$ commute with each other. For example, fermion momentum operators $p_j$ commute with the SME coefficient $b_0$ in Minkowski spacetime since $b_0$ does not vary with spacetime position. However, in curved spacetime, $\partial_j b_0 \neq 0$, and therefore $[p_j, b_0] \neq 0$. As a result, the term $-\gamma^j \gamma^5 \tfrac{b_0 p_j}{m}$ is hermitian in Minkowski spacetime but nonhermitian in curved spacetime. However, the combination $\gamma^j \gamma^5 \left( -b_0 p_j + \frac{1}{2} i \partial_j b_0 \right)$, which appears in the nonrelativistic hamiltonian, is hermitian in both Minkowski and curved spacetimes.

All terms that appear in the Minkowski hamiltonian also appear in the curved-spacetime hamiltonian as part of a combination like this, confirming the hermiticity of the nonrelativistic hamiltonian.

3.2. Sensitivity of Completed Experiments

We may exploit the correspondence between derivatives of SME coefficients in curved spacetime and effective values of coefficients in Minkowski coefficients to extract bounds on the former from published analysis of the latter. For example, the neutron-associated coefficient $\tilde{b}_X$ has the bound $|\tilde{b}_X| \lesssim 10^{-33}$ GeV. In Minkowski spacetime, $\tilde{b}_X$ receives a contribution from $m d_{XT}$. In curved spacetime, the effective value of $m d_{XT}$ receives a contribution from the curl of $a$: $(m d_{XT})_{\text{eff}} \supset \frac{1}{2m} (\partial_Y a_Z - \partial_Z a_Y)$. Therefore, we find that the $Z$ component of the curl of $\tilde{a}$ has the bound $|\partial_Y a_Z - \partial_Z a_Y| \lesssim 10^{-33}$ GeV$^2$ for neutrons.

Similar analysis yields the bounds from completed experiments that are summarized in Table 2. Bounds expressed in parentheses require somewhat stronger assumptions than those expressed without parentheses.

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Table 2. Maximal sensitivity to derivatives of SME coefficients from already-completed experiments.

| Weakly-curved-spacetime coefficient | Electron | Proton | Neutron |
|-------------------------------------|----------|--------|---------|
| $\partial_X a_T$                     | $10^{-29}$ | $[5]$  | $10^{-26}$ | [12] |
| $\partial_Y a_T$                     | $10^{-29}$ | $[5]$  | $10^{-26}$ | [12] |
| $\partial_Z a_T$                     | $10^{-29}$ | $[5]$  | $10^{-27}$ | [12] |
| $\partial_X a_T$                     | $10^{-34}$ | $[5]$  | $10^{-33}$ | [4]  |
| $\partial_Y a_T$                     | $10^{-34}$ | $[5]$  | $10^{-33}$ | [4]  |
| $\partial_Z a_T$                     | $10^{-32}$ | $[5]$  | $10^{-28}$ | [9]  |
| $\delta J K \partial J a_T$          | $(10^{-21})$ | $[6]$  | $(10^{-11})$ | [10] |
| $\partial_X b_T$                     | $(10^{-34})$ | $[5]$  | $(10^{-33})$ | [4]  |
| $\partial_Y b_T$                     | $(10^{-34})$ | $[5]$  | $(10^{-33})$ | [4]  |
| $\partial_Z b_T$                     | $(10^{-32})$ | $[5]$  | $(10^{-28})$ | [9]  |
| $\partial_X a_T$                     | $(10^{-26})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Y a_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Z a_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_X a_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Y a_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Z a_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_X b_T$                     | $(10^{-26})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Y b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-27})$ | [12] |
| $\partial_Z b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-27})$ | [12] |
| $\partial_X b_T$                     | $(10^{-21})$ | $[7]$  | $10^{-29}$ | [10] |
| $\partial_Y b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Z b_T$                     | $(10^{-29})$ | $[7]$  | $10^{-20}$ | [11] |
| $\partial_X b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Y b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-26})$ | [12] |
| $\partial_Z b_T$                     | $(10^{-29})$ | $[7]$  | $10^{-20}$ | [11] |
| $\partial_X b_T$                     | $(10^{-29})$ | $[5]$  | $(10^{-27})$ | [12] |

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