Robust continuous-variable entanglement of microwave photons with cavity electromechanics

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We investigate the controllable generation of robust photon entanglement with a circuit cavity electromechanical system, consisting of two superconducting coplanar waveguide cavities (CPWC’s) capacitively coupled by a nanoscale mechanical resonator (MR). We show that, with this electromechanical system, two-mode continuous-variable entanglement of cavity photons can be engineered deterministically either via coherent control on the dynamics of the system, or through a dissipative quantum dynamical process. The first scheme, operating in the strong coupling regime, explores the excitation of the cavity Bogoliubov modes, and is insensitive to the initial thermal noise. The second one is based on the reservoir-engineering approach, which exploits the mechanical dissipation as a useful resource to perform ground state cooling of two delocalized cavity Bogoliubov modes. The achieved amount of entanglement in both schemes is determined by the relative ratio of the effective electromechanical coupling strengths, which thus can be tuned and made much larger than that in previous studies.

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I. INTRODUCTION

Circuit cavity electromechanics [1,2], the counterpart of cavity optomechanics [3,4] in the form of electrical circuits, describes the parametrical coupling between the motion of a micro or nanoscale MR and an electrical circuit. The underlying physics of cavity electromechanics is that the motion of the mechanical oscillator modulates the capacitance of the electrical circuit, thus creating parametrical coupling between these two systems. Compared to their optomechanical analogues, electromechanical systems have the advantages that these low-loss superconducting circuits are easily cooled to ultralow temperatures [5], and can be fabricated on a single chip using the standard optical lithographic techniques. Recent experimental and theoretical progress has shown that cavity optomechanics and electromechanical systems are pretty useful for macroscopic tests of the fundamental laws of quantum mechanics, or for other practical applications relevant with quantum phenomena [8–31]. Of particular interest is the generation of non-classical motional, photonic and hybrid quantum states for basic tests of quantum theory, as well as applications in quantum information processing [32–40].

In order to achieve entanglement of photons, one can use a three-mode optomechanical system, consisting of two optical target modes and a mechanical auxiliary mode. Several theoretical works have described such schemes for entanglement generation, which use the auxiliary mode to mediate an effective coherent interaction between the two target modes [47–51]. However, the entanglement generated in those protocols often suffers from the unavoidable decoherence and dissipation associated with such systems. For instance, the mechanical thermal noise and mechanical dissipation available in optomechanical system often play a negative role in the entanglement preparation process. The traditional method for beating such decoherence process often needs the strong photon-phonon interaction to exceed the decay of the photons and phonons. The achieved photon entanglement is often limited by the constraints on the magnitude of the optomechanical coupling strengths. It is thus appealing to present some new schemes for robust photon entanglement with such three-mode optomechanical or electromechanical systems.

In this work, we study the robust generation of photon entanglement with a circuit cavity electromechanical system consisting of two superconducting CPWC’s and a nanoscale or micro MR. In the proposed experimental setup, the superconducting cavities are capacitively coupled by a capacitor that incorporates the nanoscale or micro MR into its electrode plates, and is biased by a driving voltage. In this case, the cavity modes only couple with the mechanical mode and do not interact with each other. We show that, through suitably choosing the driving frequencies of the voltages, we can generate various linear operations between cavity photons and mechanical phonons on demand via the modulation of the coupling strength.

In particular, with this circuit electromechanical system we present two different protocols for preparing continuous-variable entangled states of microwave photons. The first protocol, operating in the strong coupling regime, relies on coherent control over the dynamics of the system and is insensitive to the initial thermal noise. In this scheme, we show that at selected time the Bogoliubov modes composed of the cavity modes only can be excited. Since these cavity Bogoliubov modes do not contain the mechanical mode, the scheme is hence robust against the mechanical noise. The second protocol is

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based on a dissipative quantum dynamical process, which exploits the mechanical dissipation as a resource and only needs high frequency low-Q mechanical oscillators. We show that at steady state robust entanglement of two cavity modes can be generated through tailoring the dissipative environment of the two target modes. In the present case, the reservoir-engineering scheme exploits the ground state cooling of two delocalized cavity Bogoliubov modes. The photon entanglement achieved in both schemes is determined by the ratio of the effective electromechanical coupling strengths, rather than their magnitudes. Therefore, the amount of entanglement obtained in these schemes can be far greater than that in previous works. These protocols may have promising applications for continuous-variable quantum information processing with cavity electromechanics.

II. COUPLING TWO SUPERCONDUCTING CPWC’S VIA A MR

![Diagram of two superconducting CPWC's](image)

FIG. 1. (Color online) The schematic of two superconducting CPWC’s capacitively coupled by a nanoscale MR, which is driven by a gate voltage $V_g$.

As shown in Fig.1, we consider an electromechanical system where two superconducting CPWC’s are capacitively coupled by a nanoscale or micro MR, which is driven by a gate voltage $V_g$. To implement this scheme, the superconducting CPWC’s are electrically connected with a capacitor biased by a driving voltage. The capacitor is formed by two parallel metal plates, one of which is replaced by a metallic membrane realizing the MR (drum resonator) $\bar{g}_0$. Alternatively, one can choose to use a micromechanical bulk dilatational resonator, as recently used in the experiment to couple with a phase qubit $\bar{g}_1$. The working frequency of these MR’s is in the range of GHz, and they can couple to the CPWC’s through interdigitated capacitors. In both cases, the capacitance $C$ of the capacitor is dependent on the MR displacement $X = \sqrt{\hbar/(2m_0\omega_m)}(\hat{b} + \hat{b}^\dagger)$, where $m$ is the mass of the MR, $\omega_m$ the mechanical vibration frequency, and $\hat{b}$ the annihilation operator for the MR. If we assume that the displacement is much smaller than the equilibrium distance $d$ between the metallic membrane and the metallic base electrode, then the capacitance approximately becomes $C = C_0(1 + X/d)$, where $C_0$ is the capacitance for the MR in equilibrium.

For a superconducting CPWC $\bar{g}_2$, the voltage at position $x$ is

$$V_j(x) = \sqrt{\hbar \omega_j/C_j}(\hat{a}_j^\dagger + \hat{a}_j)\cos(2\pi x/L_j), (j = 1, 2), \quad (1)$$

where $\omega_j$ is the resonant frequency, $C_j$ the total capacitance, $\hat{a}_j$ the annihilation operator, and $L_j$ the length for the $j$th CPWC respectively. With a coupling capacitance $C$ between these CPWC’s, the coupled interaction can be derived as

$$H_I = \frac{1}{2}C_0(1 + X/d)(V_1(0) + V_2(0) - V_g)^2. \quad (2)$$

We subsequently perform a rotating-wave approximation to simplify the coupled interaction. After neglecting rapidly oscillating and other higher order terms, the Hamiltonian describing the coupled system can be derived as

$$\mathcal{H} = \hbar \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \hbar \omega_m \hat{b}^\dagger \hat{b}\cos(2\pi x/L),$$

$$-\hbar g_1(t)(\hat{b} + \hat{b}^\dagger)(\hat{a}_1 + \hat{a}_1^\dagger) - \hbar g_2(t)(\hat{b} + \hat{b}^\dagger)(\hat{a}_2 + \hat{a}_2^\dagger)$$

where $g_j(t) = \frac{C_0}{d} \sqrt{\hbar \omega_j/C_j} V_g(t)$, and we have included the free Hamiltonian of the two CPWC’s and MR in the first three terms. The last two terms in Eq. (3) are the linear interaction between the CPWC’s and the MR. Up to now the result is valid for arbitrary driving voltage signals. Through adjusting the driving frequency of the voltage, we can generate various linear operations. For instance, the beam-splitter interaction between two cavity modes and the mechanical mode, as discussed in Ref. [53, 54] to realize intracavity state transfer, can be recovered from Hamiltonian (3) by choosing $V_g(t) = V_0 \cos(\omega_d t)$ and setting $\omega_d = \omega_1 - \omega_m = \omega_2 - \omega_m$. In this way, the modulation of the coupling strength provides an effective tool for controlling the interaction between the CPWC’s and the MR. In the following section, we will show how to prepare robust two-mode entangled states of the cavity photons through engineering the desired interaction between photons and phonons.

III. GENERATING CONTINUOUS-VARIABLE ENTANGLEMENT OF PHOTONS CONFINED IN THE CPWC’S

We now consider the case where the MR is driven by a gate voltage of the form $V_g(t) = V_0 \cos(\omega_0 t) + V_2 \cos(\omega_2 t + \phi)$, where $\phi$ is a fixed phase difference between the voltage components. If we choose the driving frequencies as $\omega_0^1 = \omega_1 + \omega_m, \omega_2^1 = |\omega_1 - \omega_m|$, corresponding to the blue sideband and red sideband driving for the MR, then under the rotating-wave approximation we can obtain the Hamiltonian in the interaction picture

$$\mathcal{H} = -\hbar \Theta_1(\hat{a}_1^\dagger \hat{b}^\dagger + \hat{a}_1 \hat{b}) - \hbar \Theta_2(\hat{a}_2^\dagger \hat{b} + \hat{a}_2 \hat{b}^\dagger), \quad (4)$$
where
\[ \Theta_j = \frac{C_0}{2d} \sqrt{\frac{\omega_j}{2m\omega_C}} V_j^2, \quad (j = 1, 2). \]

The Hamiltonian describes a system of three coupled harmonic oscillators with controllable coefficients. The first term describes simultaneous creation or annihilation of a photon in CPWC1 and a phonon and is responsible for entangling the CPWC1 and the MR, while the second term describes the exchange of excitation quanta between the CPWC2 and the motion. These terms together will lead the CPWC’s to be entangled with each other. In what follows, we will discuss two different schemes to realize this goal, one of which is based on coherent control on the dynamics of the system, while the other is through a dissipative quantum dynamical process.

A. Dynamical generation of photon entanglement via excitations of the cavity Bogoliubov modes

We first focus on the regime where the dissipative effects on the coherent dynamics can be neglected, i.e., the strong coupling regime, \( \{\Theta_1, \Theta_2\} \gg \{\kappa_1, \kappa_2, n_{th}\gamma_m, \gamma_m\} \), where \( \kappa_j \) is the \( j \)-th CPWC field decay rate, \( n_{th} \) is the thermal equilibrium occupation number for the mechanical mode at temperature \( T \), and \( \gamma_m \) is mechanical dissipation rate. This regime can be easily realized, since the coupling strength \( \Theta_j \) can be tuned by the classical driving amplitude \( V_j^2 \), and high-Q superconducting CPWC’s and MR’s can be conveniently fabricated in the laboratory. In this limit the coherent dynamics of the coupled system can be easily solved in the Heisenberg representation.

The Heisenberg equations of motion read
\[ \dot{\hat{a}}_1 = i\Theta_1 \hat{b}^\dagger, \]
\[ \dot{\hat{a}}_2 = i\Theta_2 \hat{b}, \]
\[ \dot{\hat{b}} = i\Theta_1 \hat{a}_1^\dagger + i\Theta_2 \hat{a}_2, \]
which would generate periodic dynamics provided that \( |\Theta_2| > |\Theta_1| \). After some straightforward derivations, we can obtain the time evolution of the operators as
\[ \hat{a}_1(t) = \frac{i\Theta_1}{\Theta} \hat{b}^\dagger(0) \sin \Theta t + \frac{\Theta_1}{\Theta^2} [1 - \cos \Theta t] \hat{a}_1^\dagger(0) + \frac{1}{\Theta^2} |\Theta_2|^2 |\Theta_1|^2 \]
\[ -|\Theta_1|^2 \cos \Theta t \hat{a}_1(0), \]
\[ \hat{a}_2(t) = \frac{\Theta_2}{\Theta} \hat{b}(0) \sin \Theta t - \frac{\Theta_1}{\Theta^2} [1 - \cos \Theta t] \hat{a}_1^\dagger(0) - \frac{1}{\Theta^2} |\Theta_1|^2 \]
\[ -|\Theta_2|^2 \cos \Theta t \hat{a}_2(0), \]
\[ \hat{b}(t) = \hat{b}(0) \cos \Theta t + \frac{1}{\Theta}[\Theta_2 \hat{a}_2(0) + i\Theta_1 \hat{a}_1^\dagger(0)] \sin \Theta t, \]
with \( \Theta = \sqrt{|\Theta_2|^2 - |\Theta_1|^2} \). In general these solutions describe tripartite entanglement among cavity modes and the mechanical oscillator. However, we find at the instant \( T_\pi = \pi/\Theta \) Eqs. (\ref{eq:8})-\( (\ref{eq:10}) \) become
\[ \hat{a}_1(T_\pi) = \frac{|\Theta_1|^2 + |\Theta_2|^2}{\Theta^2} \hat{a}_1(0) + \frac{2\Theta_1 \Theta_2}{\Theta^2} \hat{a}_2(0), \]
\[ \hat{a}_2(T_\pi) = -\frac{|\Theta_1|^2 + |\Theta_2|^2}{\Theta^2} \hat{a}_2(0) - \frac{2\Theta_1 \Theta_2}{\Theta^2} \hat{a}_1^\dagger(0), \]
\[ \hat{b}(T_\pi) = -\hat{b}(0). \]

Therefore, at this instant the mechanical motion is decoupled from the cavity modes and returns to its initial state. Moreover, at the time \( T_\pi \) the two cavity modes are entangled with each other.

To be more specific, we introduce the unitary operator
\[ \hat{S}(\zeta) = e^{\zeta \hat{a}_1 \hat{a}_2^\dagger - \zeta^\dagger \hat{a}_2^\dagger \hat{a}_1}, \quad \zeta = \tan^{-1} \left[ 2r/(1 + r^2) \right], \quad r = |\Theta_2/\Theta_1|. \]

Then we find Eqs. (\ref{eq:11}) and (\ref{eq:12}) can be rewritten as the delocalized cavity Bogoliubov mode operators
\[ \hat{a}_1(T_\pi) = \cosh \zeta \hat{a}_1(0) + \sin \zeta \hat{a}_2^\dagger(0) \hat{S}^\dagger, \]
\[ \hat{a}_2(T_\pi) = -\cosh \zeta \hat{a}_2(0) + \sin \zeta \hat{a}_1^\dagger(0) \hat{S}^\dagger, \]
\[ \hat{b}(T_\pi) = -\hat{b}(0). \]

These results imply that at the instant \( T_\pi \) the Bogoliubov mode composed of the cavity modes only will be excited. Since these cavity Bogoliubov modes do not contain the mechanical mode, the scheme is hence robust against the mechanical noise.

In the Schrödinger picture, the time evolution operator of the total system corresponds to \( \hat{U}(T_\pi) = \hat{S}(\zeta)^\dagger \otimes I_m \), where \( I_m \) is the identity operator for the mechanical mode. Thus, if initially the MR density matrix is a thermal state at temperature \( T \) given by
\[ \rho_m(0) = \left( 1 - e^{-\hbar \omega_m/k_B T} \right) e^{-H_m/k_B T} \]
where \( k_B \) is the Boltzmann constant, and \( H_m = \hbar \omega_m (\hat{b}^\dagger \hat{b} + 1/2) \), then \( \rho_m(T_\pi) = \rho_m(0) \) in the Schrödinger representation. Moreover, at the time \( T_\pi \), the two cavity modes will be prepared in a two-mode squeezed state if the initial state is the vacuum state for both cavity modes, \( |00\rangle_c \). In particular, using the factored form of the two-mode squeeze operator \( \hat{S}(\zeta) \),
\[ \hat{S}(\zeta) = \cosh \zeta e^{-\frac{r}{2} \sinh 2 \zeta \tanh \zeta} e^{-\frac{r}{2} \sinh 2 \zeta \tanh \zeta} \ln \cosh \zeta e^{rac{r}{2} \sinh 2 \zeta \tanh \zeta} \]

at the time \( T_\pi \) the state of the two cavity modes is
\[ |\psi\rangle_c = \frac{1}{\cosh \zeta} \sum_{n=0}^{\infty} (\tanh \zeta)^n |n, n\rangle_c \]
\[ = \frac{1 - r^2}{1 + r^2} \sum_{n=0}^{\infty} \left( \frac{2r}{1 + r^2} \right)^n |n, n\rangle_c. \]

The state \( |\psi\rangle_c \) is a two-mode squeezed state of the photon fields in the two cavities, which exhibits Einstein-Podolsky-Rosen (EPR) entanglement. The mechanical motion of the MR plays a fundamental role in establishing the entanglement, nevertheless the initial motional state does not affect the efficiency of the scheme. The degree of squeezing (squeeze parameter \( \zeta \)) and amount
of entanglement are determined by the ratio of $\Theta_2$ to $\Theta_1$, which can be controlled on demand through tuning the driving signals. When the squeezed state is generated at the time of $T_x$, we switch off the couplings between the CPWC's and the MR. Then the squeezed state can be preserved until the cavity fields are coupled out.

In order to check the above analysis, we exploit the total variance $V = \langle (\Delta \hat{n})^2 + (\Delta \hat{v})^2 \rangle$ of a pair of EPR-like operators $\hat{u} = \hat{X}_1 - \hat{X}_2$, and $\hat{v} = \hat{P}_1 + \hat{P}_2$, with $X_j = (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}$, and $P_j = -i(\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2}$, $j = 1, 2$. According to Ref. 59, a two-mode Gaussian state is entangled if and only if $V < 2$. For the two-mode squeezed vacuum state $\hat{S}^\dagger(\zeta)|00\rangle_z$, the total variance $V = \langle (\Delta \hat{n})^2 + (\Delta \hat{v})^2 \rangle = 2e^{-2\kappa_t}$ implying this state exhibits EPR entanglement. In Fig. 2, the quantity $V$ is plotted versus the scaled time $\Theta t$ for different values of the initial thermal phonon number $n_{th}$ and parameter $r$. Fig. 2(a) shows the total variance $V$ as a function of the scaled time $\Theta t$ under different values of the thermal phonon number $n_{th}$ for a fixed parameter $r$. From Fig. 2(a) it can be found that the total variance $V$ is unaffected by thermal noise at the instant $T_x$, i.e., at half period it is independent on the initial thermal phonon number $n_{th}$. This result can be understood since our discussions hold under the conditions that the couplings of the system to the thermal reservoirs can be neglected. In such a case the coherent dynamics is purely governed by the Hamiltonian (11) and thermal effects enter only as an initial condition for the mechanical mode. Fig. 2(b) displays $V$ versus $\Theta t$ under different values of the ratio $r$ with a fixed mean phonon number $n_{th}$. We find that at half period the two CPWC’s are steered into a two-mode squeezed state starting from a thermal state for the MR. The degree of squeezing and the total variance for the two-mode squeezed state are determined by the parameter $r$. Therefore, the amount of entanglement can be tuned and made large by the ratio of the effective electromechanical coupling strengths. These numerical results are in accordance with the analytical conclusions above.

### B. Robust photon entanglement at steady state through a dissipative dynamical process

In the previous section, we have discussed how to prepare photon entanglement via coherent control on the evolution of the hybrid system. Though the protocol seems promising, the experimental implementation of this scheme requires stringent conditions, i.e., the coupling to the environment reservoirs should be neglected, thus requiring very high quality factors for both the CPWC’s and MR’s. For superconducting CPWC’s, this condition can be fulfilled since high-Q superconducting stripline cavities are easy to be fabricated in the laboratory. However, as for the mechanical oscillator, in particular which has to be incorporated into a capacitor in order to couple to the CPWC’s, this requirement is too demanding. In addition, high frequency MR’s are required when the quantum regime is entered. However at present these GHz mechanical oscillators are plagued by very low quality factors. It is known that MR performance degrades considerably as the oscillating frequency increases. In this section, we will present an alternative scheme which exploits the mechanical dissipation as a useful resource and only needs high frequency low-Q mechanical oscillators.

In what follows, the system-environment interaction is assumed Markovian, and then is described by a master equation in Lindblad form. We assume that the CPWC’s couple with the vacuum bath, but the MR couples with a thermal bath. Then the time evolution of the density operator $\hat{\rho}$ for the whole system is described by the master equation

$$\frac{d\hat{\rho}}{dt} = -i\frac{\hbar}{\Theta_1} [\mathcal{H}, \hat{\rho}] + \mathcal{L}_{c_1} \hat{\rho} + \mathcal{L}_{c_2} \hat{\rho} + \mathcal{L}_m \hat{\rho},$$

where

$$\mathcal{L}_{c_1} \hat{\rho} = \frac{\kappa_1}{2} (2\hat{a}_j \hat{\rho} \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j \hat{\rho} - \hat{a}_j^\dagger \hat{a}_j \hat{\rho}^\dagger \hat{a}_j),$$

$$\mathcal{L}_m \hat{\rho} = \frac{\gamma_m}{2} (n_{th} + 1)(2\hat{b}^\dagger \hat{b} \hat{\rho} - \hat{b} \hat{b}^\dagger \hat{\rho} - \hat{b} \hat{b}^\dagger \hat{\rho}^\dagger \hat{b}^\dagger) + \frac{\kappa_m}{2} n_{th} (2\hat{b} \hat{b} \hat{\rho} - \hat{b} \hat{b}^\dagger \hat{\rho} - \hat{b} \hat{b}^\dagger \hat{\rho}^\dagger \hat{b}^\dagger).$$

In the following we focus on the regime where $\gamma_m \gg \{\Theta_1, \Theta_2\} \gg \{\kappa_1, \kappa_2, n_{th} \gamma_m\}$. The condition $\gamma_m \gg \{\Theta_1, \Theta_2\}$ corresponds to strong mechanical damping for the MR, i.e., very low quality factors, while $n_{th} \gamma_m \ll \gamma_m$ implies near zero temperature for the mechanical mode, thus requiring ground state cooling of the MR. In effect, in the regime of large mechanical frequency (in the GHz range) and at cryogenic temperature, the thermal phonon number is nearly zero, i.e., $n_{th} = (e^{\hbar \omega_{th}/k_B T} - 1)^{-1} \approx 0$, which corresponds to coupling with the vacuum bath for
the MR. Under this regime the master equation (21) then can be approximated as

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \hat{\rho}] + \frac{\gamma_m}{2}(2\hat{b}\hat{\rho}\hat{b} - \hat{b}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b}) \tag{21}$$

We now introduce the phonon number representation for the density operator \(\hat{\rho}\), i.e., \(\hat{\rho} = \sum_{m,n=0}^{\infty} \rho_{mn}|m\rangle\langle n|\), where \(\rho_{mn}\) are the density-matrix elements in the basis of the phonon number states \(|n\rangle\), \(n = 0, 1, 2, \ldots\), and are still operators with respect to the cavity fields. Under the condition of strong mechanical damping, the populations of the highly excited motional states can be neglected. Therefore, we consider only the matrix elements \(\rho_{mn}\) inside the subspace \(|0\rangle, |1\rangle\) of the phonon numbers.

In this case, the master equation (21) leads to the following set of coupled equations of motion for the density-matrix elements

$$\begin{align*}
\rho_{00} &= -i\rho_{01}(\Theta_1\hat{a}_1^\dagger + \Theta_2\hat{a}_2^\dagger) + i(\Theta_1\hat{a}_1 + \Theta_2\hat{a}_2)\rho_{11} + \gamma_m\rho_{01}(22) \\
\rho_{11} &= -i\rho_{10}(\Theta_1\hat{a}_1 + \Theta_2\hat{a}_2^\dagger) + i(\Theta_1\hat{a}_1^\dagger + \Theta_2\hat{a}_2)\rho_{01} - \gamma_m\rho_{11} \\
\rho_{01} &= -i\rho_{10}(\Theta_1\hat{a}_1 + \Theta_2\hat{a}_2^\dagger) + i(\Theta_1\hat{a}_1^\dagger + \Theta_2\hat{a}_2)\rho_{11} - \gamma_m\rho_{10}(23)
\end{align*}$$

In the regime of strong damping rate \(\gamma_m\), the elements \(\rho_{01}\) and \(\rho_{11}\) can be adiabatically eliminated from the above equations, leading to

$$\rho_{01} = \frac{2i\Theta}{\gamma_m}(\mathcal{D}^\dagger \rho_{11} - \rho_{00} \mathcal{D}^\dagger) \tag{25}$$

where

$$\mathcal{D} = \Theta_2\hat{a}_2 + \Theta_1\hat{a}_1^\dagger \tag{26}$$

is the cavity Bogoliubov mode operator. The reduced density operator for the CPWC’s can be approximated as \(\hat{\rho}_c = \text{Tr}_m(\hat{\rho}) \approx \rho_{00} + \rho_{11}\). Replacing (23) into (22) and (21), and adding up them, after neglecting higher-order terms we obtain the evolution of the cavity modes with an effective master equation

$$\frac{d\hat{\rho}_c}{dt} = \frac{\Gamma_c}{2}(2\mathcal{D}\hat{\rho}_c\mathcal{D}^\dagger - \mathcal{D}^\dagger\mathcal{D}\hat{\rho}_c - \hat{\rho}_c\mathcal{D}^\dagger\mathcal{D}) \tag{27}$$

with \(\Gamma_c = 4\Theta^2/\gamma_m\). This master equation has the form of the standard engineering reservoir scheme, which describes ground state cooling of the cavity Bogoliubov mode \(\mathcal{D}\). The only pure steady state of the system is the eigenstate \(|\psi\rangle\) of the operator \(\mathcal{D}\) with zero eigenvalue, ensuring that there is no further eigenstate \(|\phi\rangle\) of \(\mathcal{D}\) such that \(|\mathcal{D}\mathcal{D}^\dagger|\phi\rangle = 0\). For the operator \(\mathcal{D} = \Theta_2\hat{a}_2 + \Theta_1\hat{a}_1^\dagger\), we will find that this condition cannot be satisfied. From the eigenvalue equation \(\mathcal{D}|\psi\rangle = 0\), and the relation \(\mathcal{D} = \mathcal{S}\hat{a}_2\mathcal{S}^\dagger\), with \(\mathcal{S}(\zeta) = e^{\zeta\hat{a}_2^\dagger\hat{a}_1^\dagger - \zeta\hat{a}_1\hat{a}_2}\), \(\zeta = \tanh^{-1}|\Theta_1/\Theta_2|\), one can readily find that

$$|\psi\rangle_c = \mathcal{S}|\mu, 0\rangle_c \tag{28}$$

is a steady state of the master equation (27), but not the only one. Here \(|\mu, 0\rangle_c\) denotes an arbitrary state for the first CPWC mode and the vacuum state for the second one.

In order to steer the system into the two-mode squeezed state \(\mathcal{S}(0, 0)\), we need another dissipative process together with the described one, leading to the effective master equation

$$\frac{d\hat{\rho}_c}{dt} = \frac{\Gamma_c}{2}(2\mathcal{D}\hat{\rho}_c\mathcal{D}^\dagger - \mathcal{D}^\dagger\mathcal{D}\hat{\rho}_c - \hat{\rho}_c\mathcal{D}^\dagger\mathcal{D} \tag{29})$$

where \(\mathcal{D} = \mathcal{S}\hat{a}_1\mathcal{S}^\dagger\) is the other delocalized cavity Bogoliubov mode operator. This master equation describes simultaneous ground state cooling of the system in the transformed picture with the basis \(\mathcal{D}, \mathcal{D}^\dagger\). In fact, one can find that \(\mathcal{D}\mathcal{S}(0, 0)\), \(\mathcal{D}\mathcal{S}(0, 0)\), \(\mathcal{S}(0, 0)\), \(\mathcal{S}(0, 0)\), and \(\mathcal{S}(0, 0)\) are the density-matrix elements in the basis \(|\mu, 0\rangle, |\mu, 1\rangle\) of the operator \(\mathcal{D}\), \(\mathcal{D}^\dagger\)

$$\mathcal{H}' = -\hbar\Theta_1(\hat{a}_1^\dagger\hat{b} + \hat{a}_1\hat{b}^\dagger) - \hbar\Theta_2(\hat{a}_2^\dagger\hat{b} + \hat{a}_2\hat{b}^\dagger) \tag{30}$$

following the same reasoning as that for the \(\mathcal{D}\) operator, with the driving frequencies chosen as \(\omega_{1d} = |\omega_m - \omega_1|\), and \(\omega_{2d} = \omega_2 + \omega_m\). However, for the case of just one MR, we cannot get the master equation (29) to realize simultaneous cooling of both modes \(\mathcal{D}\) and \(\mathcal{D}^\dagger\), since both \(\mathcal{H}\) and \(\mathcal{H}'\) cannot be possessed simultaneously only through adjusting the driving frequencies for one MR. In order to have both cooling processes, in \(\mathcal{D}\) and \(\mathcal{D}^\dagger\), we can employ a stroboscopic cooling scheme. In this approach, the system evolves during a time \(t\) in \(N\) cycles of duration \(\delta t = t/N\), while the driving parameters alternate between the ones with respect to \(\mathcal{D}\) and those of \(\mathcal{D}^\dagger\). The stroboscopic limit is valid provided that the time interval \(\delta t\) is much smaller than \(1/\Gamma\), in which case the effective dynamics of the system is just as that described by the master equation (29). Alternatively, one can couple the CPWC’s with two MR’s, each of which is driven by a bichromatic microwave signal to induce sidebands in the CPWC-MR coupling. In this case, one can realize the Hamiltonian \(\mathcal{H}'\) for one MR, and simultaneously have the Hamiltonian \(\mathcal{H}'\) for the other. In the regime of strong mechanical damping for both MR’s, one can exploit the engineering reservoir scheme to get the effective master equation (29).

It is necessary to verify the model through numerical simulations. To provide an example, here we consider the two-MR case, where the dynamics of the system can be simulated by the following master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \hat{\rho}] + \mathcal{L}_{c_1}\hat{\rho} + \mathcal{L}_{c_2}\hat{\rho} + \mathcal{L}_{m_1}\hat{\rho} + \mathcal{L}_{m_2}\hat{\rho} \tag{31}$$
Regarding the experimental feasibility of the proposals, currently available experimental setups of cavity electromechanics [2, 8] are promising platforms for realizing the schemes. We consider superconducting CPWC’s with the fundamental frequency of $2\pi \times 10$ GHz, whose damping rate can be as low as $\kappa /2\pi \approx 10$ kHz given a quality factor $Q = 10^6$ from recent circuit QED experiments. The vacuum-fluctuations-induced voltage between the central conductor and the ground plane of the CPWC’s is typically of order of $\mu$V. As for the nanoscale MR’s, we can choose to utilize an aluminum membrane integrated into a capacitor [2] for the first scheme, or a piezoelectric dilatation resonator [7] for the second one, which comprises a piezoelectric thin film of aluminum nitride, sandwiched between two aluminum metal electrodes. The superconducting CPWC’s and nano MR’s can be fabricated on a single chip with wafer-scale optical lithographic techniques. For a nearly circular membrane with a diameter of 15 $\mu$m and a thickness of 100 nm [2], drum-like modes are allowed to resonate freely. The fundamental mode is $\omega_m/2\pi = 10.69$ MHz, giving a zero-point motion of 4.1 fm and a damping rate $\gamma_m/2\pi = 30$ Hz [2]. For the first scheme, with the chosen parameters $V^2_x = 10$ V, $V_x = 1.01 V_c^2$, $d = 50$ nm, $C_0 = 40$ fF, we get $\Theta \sim 3$ MHz, and the operation time for generating the target state is about $T_{x} \sim 1$ $\mu$s. This time is much shorter than the photon life time, and the decoherence time for the mechanical mode with about $10^3$ phonons. With regard to the second scheme, the piezoelectric dilatation resonator is particularly suitable [8], which has 6 GHz frequency and very strong damping rate $\gamma_m/2\pi \approx 23$ MHz ($Q \approx 260$). At the temperature $T \approx 25$ mK, the number of thermal phonons in the mechanical mode is less than 0.07. If we assume that $V_x = 1$ V, $V_x^2 = 2 V_c^2$, $d = 50$ nm, $C_0 = 25$ fF, then we have $\Theta \sim 17$ MHz. The time for reaching the stationary state is about $4/\Gamma_c \sim 0.5$ $\mu$s.

Finally, We discuss how to measure the entanglement between the resonators. To implement this task one can use the experimental state tomography technique realized recently to detect a two-mode squeezed state in the microwave domain [62]. In the experiment, all four quadrature components $X_1, X_2, P_1, P_2$ of a two-mode squeezed state are measured in a two-channel heterodyne setup using amplitude detectors. Then, the full covariance matrix can be determined via analyzing two-dimensional phase space histograms for all possible pairs of quadratures.

V. CONCLUSIONS

To conclude, we have studied the robust generation of photon entanglement with an electromechanical system, in which two CPWC’s are capacitively coupled by a MR. With this cavity electromechanical system, we have
presented two different schemes to generate two-mode continuous-variable entangled states of microwave photons confined in the cavities. The first scheme is based on coherent control over the dynamics of the system to selectively induce excitations of the cavity Bogoliubov modes. The second one is based on a dissipative quantum dynamical process, which exploits the mechanical dissipation as a useful resource to implement ground state cooling of the cavity Bogoliubov modes. These protocols may have interesting applications in quantum information processing with electromechanical systems.

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Note added: After completing this work, we became aware of two related works on the arXiv, which exploit the quite same ideas to generate photon entanglement with optomechanical systems. These works have already been published \cite{19,20}.

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