KOLYVAGIN CLASSES VERSUS NON-CRISTALLINE DIAGONAL CLASSES

FRANCESCA GATTI AND VICTOR ROTGER

ABSTRACT. Let $E/\mathbb{Q}$ be an elliptic curve having multiplicative reduction at a prime $p$. Let $(g, h)$ be a pair of eigenforms of weight 1 arising as the theta series of an imaginary quadratic field $K$, and assume that the triple-product $L$-function $L(f, g, h, s)$ is self-dual and does not vanish at the central critical point $s = 1$. The main result of this article is a formula expressing the $p$-adic iterated integrals introduced in [DLR15] to the Kolyvagin classes associated by Bertolini and Darmon to a system of Heegner points on $E$.

1. INTRODUCTION

Let $E/\mathbb{Q}$ be an elliptic curve with square-free conductor $N_f$ and let $f \in S_2(N_f)$ be the normalized newform attached to it by modularity. Let $K$ be an imaginary quadratic field of discriminant $-D_K$ relatively prime to $N_f$. Let $\psi_g, \psi_h : \mathbb{A}_K^\times / K^\times \rightarrow \mathbb{C}^\times$ be two finite order Hecke characters of $K$ of conductors $c_g, c_h$ respectively and let $g := \theta(\psi_g)$ and $h := \theta(\psi_h)$ denote the theta series associated to them. Assume that the central characters of $g$ and $h$ are mutually inverse. Then

$$g \in M_1(N_g, \chi), \quad h \in M_1(N_h, \chi^{-1})$$

are weight one modular forms of level $N_g = D_K \cdot N_K/Q(\epsilon_g), N_h = D_K \cdot N_K/Q(\epsilon_h)$ respectively. For simplicity we assume throughout that

$$\text{gcd}(N_f, N_g, N_h) = 1 \quad (1.1)$$

and we set $N := \text{lcm}(N_f, N_g, N_h)$. Let $p$ be an odd prime number such that

$$p \mid N_f \quad \text{and} \quad p \text{ is inert in } K. \quad (1.2)$$

Hence $E$ has multiplicative reduction at $p$ and the completion $K_p$ of $K$ at $p$ is the unramified quadratic extension of $\mathbb{Q}_p$. Let

$$\varphi_{\text{Tate}} : \mathbb{Q}_p^\times / q_E^\mathbb{Z} \xrightarrow{\sim} E(\mathbb{Q}_p)$$

be Tate’s uniformization, with $q_E \in p\mathbb{Z}_p$, and assume throughout that

$$p \nmid \text{ord}_p(q_E). \quad (1.3)$$

Set

$$a := a_p(E) = a_p(f) \in \{\pm 1\}.$$
In other words, \( a = 1 \) (resp. \( a = -1 \)) according to whether \( E \) has split (non-split) multiplicative reduction at \( p \).

Let \( \alpha_g, \beta_g \) (resp. \( \alpha_h, \beta_h \)) denote the roots of the \( p \)-th Hecke polynomial of \( g \) (resp. of \( h \)). Note that \( \text{(1.2)} \) implies that
\[
\alpha_g = -\beta_g \quad \text{and} \quad \alpha_h = -\beta_h.
\]

Since the nebentype character of \( h \) is the inverse of that of \( g \), one either has
\[
(\alpha_h, \beta_h) = (1/\alpha_g, -1/\alpha_g) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1/\alpha_g, 1/\alpha_g).
\]
Throughout this article we fix the ordering of the pair \((\alpha_h, \beta_h)\) in such a way that
\[
(\alpha_g, \alpha_h) = -a.
\]

Fix throughout algebraic closures \( \overline{\mathbb{Q}} \) and \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q} \) and \( \mathbb{Q}_p \) respectively and an embedding \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \).

Let \( L \subset \overline{\mathbb{Q}} \) denote the number field generated by the traces of \( \psi_g \) and \( \psi_h \), together with the roots \( \alpha_g, \beta_g, \alpha_h, \beta_h \), and let \( L_p \) denote the completion of \( L \) within \( \overline{\mathbb{Q}}_p \).

Set \( g_\alpha(z) := g(z) - \beta_g g(pz) \) and define \( h_\alpha \) analogously. Let \( f, g_\alpha, h_\alpha \) be Hida families passing through \( f, g_h, h_h \) respectively. As explained in [DR14] and [DR17], associated to any choice of \( \Lambda \)-adic test vectors \((\tilde{f}, \tilde{g}_\alpha, \tilde{h}_\alpha)\) of tame level \( N \) there is a three-variable \( p \)-adic \( L \)-function
\[
L^g_p(\tilde{f}, \tilde{g}_\alpha, \tilde{h}_\alpha).
\]

This \( p \)-adic \( L \)-function interpolates the square-roots of the central values of the classical \( L \)-function
\[
L(\tilde{f} \otimes \tilde{g}_\alpha \otimes \tilde{h}_m, s) \quad \text{attached to the specializations of the choice of test vectors at classical points of weights} \quad k, \ell, m \quad \text{with} \quad k, \ell, m \geq 2 \quad \text{and} \quad \ell \geq k + m.
\]

Note that the point \((2, 1, 1)\) corresponding to our triple of modular forms \((f, g, h)\) lies outside the region of classical interpolation for \( L^g_p(\tilde{f}, \tilde{g}_\alpha, \tilde{h}_\alpha) \). As in [DLR15] and [GGMR19] we are interested in studying the \( p \)-adic \( L \)-value
\[
I_p(f, g_\alpha, h_\alpha) := L^g_p(\tilde{f}, \tilde{g}_\alpha, \tilde{h}_\alpha)(2, 1, 1)
\]
regarded as an element of \( \overline{\mathbb{Q}}_p^\times / L^\times \). As explained in [DR17] and [DLR15], different choices of \( \Lambda \)-adic test vectors yield the same \( p \)-adic value \( I_p(f, g_\alpha, h_\alpha) \) up to algebraic scalars in \( L^\times \). This entitles us to denote it simply \( I_p(f, g_\alpha, h_\alpha) \), dropping from the notation this choice.

The triple-product \( L \)-function \( L(f, g, h, s) = L(E, \rho, s) \) may be recast as the Hasse-Weil \( L \)-series of the twist of \( E \) by the tensor product \( \rho = \rho_g \otimes \rho_h \) of the two Artin representations associated to \( g \) and \( h \). In light of \( \text{(1.3)} \), the order of vanishing of \( L(f, g, h, s) \) at \( s = 1 \) is always even and one thus expects the central critical value \( L(E, \rho, 1) \) to be generically nonzero. We assume throughout that we fall indeed in this generic case, that is to say, we have

**Assumption 1.1.** \( L(E, \rho, 1) \neq 0 \).

The main purpose of this article is proving a formula relating the \( p \)-adic iterated integral \( I_p(f, g_\alpha, h_\alpha) \) to the Kolyvagin classes associated by Bertolini and Darmon in [BD97].

In order to recall briefly the latter, for every \( m \geq 1 \) let \( F_m / H \) be the layer of degree \( p^m \) within the anticyclotomic \( \mathbb{Z}_p \)-extension of \( H \). In [BD97] Bertolini and Darmon associated to a canonical collection of Heegner points \( \{ \alpha_m \in E(F_m) \} \) a global cohomology class \( K \in H^1(H, V_f) \) that is referred to in loc. cit. as the Kolyvagin class associated to \( E/H \).

Fix a prime \( p \) of \( H \) above \( p \) and denote \( \Phi_{m,p} \) the \( p \)-primary part of the group of connected components of the Néron model of \( E \) over the completion of \( F_m \) at the unique prime ideal above \( p \). Denote \( \Phi_{\infty,p} := \lim_{\rightarrow} \Phi_{m,p} \) the projective limit with respect to the natural projection maps \( \Phi_{m,p} \rightarrow \Phi_{m-1,p} \).
As explained in Lemma 2.1 there is a canonical isomorphism $\varphi : \Phi_{\infty,p} \rightarrow Z_p$ induced by Tate’s uniformization. Denote by $\bar{\alpha}_m$ the image of $\alpha_m$ in $\Phi_{m,p}$ and set $\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty,p}$. Define the period
\[ \Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p. \]

Note that $H^1(K_p, V_f)$ is naturally a $\text{Gal}(K_p/Q_p)$-module and we let $H^1(K_p, V_f)^\pm$ denote the $Q_p$-subspace on which $\text{Fr}_p$ acts with eigenvalue $\pm 1$.

The Kolyvagin class $K$ is not in general crystalline at $p$, that is to say, the local class $\text{res}_p(K) \in H^1(K_p, V_f)$ is not expected to lie in Bloch-Kato’s finite subspace $H^1_f(K_p, V_f)$. Nevertheless, one can show (cf. Proposition 5.11) that $\text{res}_p(K) \in H^1_f(K_p, V_f)$ and therefore there exists a local point
\[ Q_p^a \in (E(K_p) \otimes Q_p)^a \] such that $\delta_p(Q_p^a) = \text{res}_p(K)^a$.

where $\delta_p : E(K_p) \otimes Q_p \rightarrow H^1_f(K_p, V_f)$ stands for Kummer’s isomorphism. In spite of the notation we have chosen, beware that $Q_p^a$ is not expected to be the $a$-component of any local point $Q_p \in E(K_p) \otimes Q_p$, precisely because $\text{res}_p(K)$ does not lie in $H^1_f(K_p, V_f)$. In other words, while $\text{res}_p(K)^a$ is crystalline, the local class $\text{res}_p(K)^{-a}$ is not.

Let $c = \prod_{v \mid \infty} c_v(\hat{f}, \hat{g}, \hat{h}_a) \in L$ denote the product of local automorphic factors appearing in Prop. 2.1 (a) (iii) associated to the choice of test vectors at $(2, 1, 1)$, and define the algebraic $L$-value
\[ L^{alg}(E \otimes \rho, 1) := \frac{L(E \otimes \rho, 1)}{\pi^4(f, j)^2}. \]

It follows from the work of Harris and Kudla [HK91] that the above ratio lies in $L$ and is in fact non-zero by [1.1]. The following is the main theorem of this note.

**Theorem A.** For any choice of test vectors, we have
\[ I_p(f, g_a, h_a) = \frac{\sqrt{c} \cdot \sqrt{L^{alg}(E \otimes \rho, 1)}}{\Pi_p \cdot L_{g_a}} \times \log_p(Q_p^a) \quad (\text{mod } L^\times) \]

where
- $L_{g_a} \in K_p$ is a period on which $\text{Fr}_p$ acts as multiplication by $-1$ and only depends on $g_a$,
- $\log_p : E(K_p) \rightarrow K_p$ denotes the $p$-adic logarithm.

**Remark 1.2.** Period $L_{g_a}$ was first introduced in [DR16] and is well-defined only up to scalars in $L^\times$; cf. [3.7] for more details. Let $H_g$ be the number field cut out by the Artin representation attached to the adjoint of $g$ and fix a completion $H_{g,p}$ of $H_g$ at a prime above $p$. In [DR16] it is conjectured that $L_{g_a} = \log_p(u_{g_a}) \pmod{L^\times}$, where $\log_p : \text{Gal}(H_{g,p} \otimes L) \rightarrow H_{g,p} \otimes L$ is the usual $p$-adic logarithm and $u_{g_a} \in \text{Gal}(H_{g,[1/p]^\times} \otimes L$ is the so-called Gross–Stark unit attached to $g_a$ [DLR13 §1.2]. See [3.7] for the explicit definition of $L_{g_a}$.

The body of the paper is devoted to the proof of this result. In [3] we relate Bloch-Kato’s dual exponential map on $E$ to the group of connected components. This is an exercise in $p$-adic Hodge theory which is surely well-known to experts, but we included because we did not find precise references in the literature. We are most grateful to A. Iovita for his assistance in this section. In [4] we review the theory of Hida families and Selmer groups, and culminates with the description of an explicit basis of the relaxed Selmer group associated to the triple $(f, g, h)$. In [4] we exploit the fine work of Bertolini, Seveso and Venerucci in order to prove a formula relating the iterated integral $I_p(f, g_a, h_a)$ in terms of the basis described in the previous section. Finally, in the
last section we introduce Kolyvagin classes and prove our main result, which is found in Corollary

5.13

Theorem A is vaguely similar to the main theorem of [DLR15] §3, although both the statements and specially the proofs are ostensibly different. While in loc. cit. the motivic elements appearing in the statement (both the diagonal classes and Heegner points) are cristalline at \( p \), in our article they are not and this makes the calculations substantially different. Moreover, in [DLR15] §3 one makes crucial use of a factorization of \( p \)-adic \( L \)-series (which follows rather trivially from a comparison of critical \( L \)-values), while in our setting the analogous \( p \)-adic \( L \)-functions associated to \( K \) are not available (because in our scenario \( p \) remains inert in \( K \)). It would be of great interest to investigate whether the recent \( p \)-adic \( L \)-functions of Andreatta and Iovita [AI19] could be exploited in order to provide an alternative proof of our formula, although this does not appear to be a straight-forward project.

As we have already pointed out, one of the key ingredients of our proof are the results of Bertolini, Seveso and Venerucci in [BSV13]. In loc. cit. the authors study the exceptional zero phenomena appearing in the scenario of diagonal cycles at weights \((2,1)\) when \( \alpha_\varphi \alpha_h = \pm \alpha \). An arithmetic application of these results is obtained in [BSV15] and [DRb] to the theory of Stark-Heegner points on elliptic curves, in the setting where \( \alpha_\varphi \alpha_h = a \) and an improved \( p \)-adic \( L \)-function plays a relevant role. The present work focus instead in the opposite setting where \( \alpha_\varphi \alpha_h = -a \) and it is rather an improved diagonal cohomology class (cf. [BSV13] §8.3] and [4.2 of this note] which plays a prominent role.

2. \( p \)-adic Hodge theory for elliptic curves with multiplicative reduction

Let \( E/\mathbb{Q}_p \) be an elliptic curve of multiplicative reduction at a prime \( p \). Let 

\[
\chi_E : G_{\mathbb{Q}_p} \rightarrow \{ \pm 1 \}
\]

be the trivial character if \( E \) has split multiplicative reduction over \( \mathbb{Q}_p \), and the quadratic unramified character if \( E \) has non-split multiplicative reduction. For any \( G_{\mathbb{Q}_p} \)-module \( M \), let \( M(\chi_E) \) denote the twist of \( M \) by \( \chi_E \).

Tate’s uniformisation provides a \( G_{\mathbb{Q}_p} \)-equivariant isomorphism

\[
(2.1) \quad \varphi_{\text{Tate}} : \bar{\mathbb{Q}}_p^x(\chi_E)/q_E^\mathbb{Z} \xrightarrow{\cong} E(\bar{\mathbb{Q}}_p)
\]

for some \( q_E \in p\mathbb{Z}_p \). As in [1.3], assume throughout that \( p \nmid n := \text{ord}_p(q_E) \).

Let \( T = T_E := \varprojlim E[p^n] \) denote the Tate module associated to \( E \) and set \( V := T \otimes \mathbb{Q}_p \). Let \( E_0(\bar{\mathbb{Q}}_p) \) denote the set of points in \( E(\bar{\mathbb{Q}}_p) \) that stay nonsingular in the special fiber of \( E \) at \( p \). The module

\[
T^+ := \varprojlim E_0[p^n]
\]

fits in an exact sequence of \( \mathbb{Z}_p[G_{\mathbb{Q}_p}] \)-modules

\[
(2.2) \quad 0 \longrightarrow T^+ \overset{\iota}{\longrightarrow} T \overset{\pi}{\longrightarrow} T^- \longrightarrow 0,
\]

where \( T^- := T/T^+ \). The modules \( T^+ \) and \( T^- \) are free over \( \mathbb{Z}_p \) of rank one, and Tate’s uniformization \( (2.1) \) induces \( G_{\mathbb{Q}_p} \)-equivariant isomorphisms

\[
(2.3) \quad \mathbb{Z}_p(\chi_E)(1) \xrightarrow{\cong} T^+, \quad \mathbb{Z}_p(\chi_E) = \varprojlim_m (\bar{\mathbb{Q}}_p^x(\chi_E)/q_E^\mathbb{Z})/p^m \rightarrow \text{Hom}_p(\chi_E) \xrightarrow{\cong} T^-.
\]

More precisely, if we fix compatible systems

\[
\tilde{z} := (\tilde{z}^{(m)})_m \in \mathbb{Z}_p(\chi_E)(1), \quad \tilde{q} := (q_E^{1/p^m})_m \in \varprojlim (\bar{\mathbb{Q}}_p^x/q_E^\mathbb{Z})[p^m]
\]

...
then

\[ \varepsilon := \varphi_{\text{Tate}}(\hat{\varepsilon}), \quad q := \varphi_{\text{Tate}}(\hat{q}) \]

form a \( \mathbb{Z}_p \)-basis of \( T \). Moreover, if \( \chi_{\text{cyc}} : G_\mathbb{Q} \to \mathbb{Z}_p^\times \) denotes the cyclotomic character, then

1. \( \varepsilon \) is a basis of \( T^+ \) on which \( G_{\mathbb{Q}_p} \) acts as the character \( \chi_E \chi_{\text{cyc}} \);
2. \( \hat{q} := \pi(q) \) is a basis of \( T^- \) on which \( G_{\mathbb{Q}_p} \) acts via \( \chi_E \).

Let \( K \) be an imaginary quadratic field in which \( p \) is inert and let \( K_p \) denote the completion of \( K \) at \( p \). Notice that \( \chi_{E|G_{K_p}} = 1 \) and thus \( \mathbb{Z}_p \)-basis gives an exact sequence of \( \mathbb{Q}_p[G_{K_p}] \)-modules

\[ 0 \to V^+ \xrightarrow{\iota} V \xrightarrow{\pi} V^- \to 0 \]

such that \( \dim_{\mathbb{Q}_p} V^+ = \dim_{\mathbb{Q}_p} V^- = 1 \), \( G_{K_p} \) acts on \( V^+ \) via \( \chi_{\text{cyc}} \), and \( G_{K_p} \) acts trivially on \( V^- \).

2.1. The group of connected components. Fix a ring class field \( H \) of \( K \) of conductor \( c \) prime to \( p \) and for every integer \( m \geq 1 \), let \( H(p^m) \) be the ring class field of \( K \) of conductor \( c \cdot p^m \). The Galois group \( \text{Gal}(H(p^m)/H) \) is cyclic of order \( c_m := (p+1)p^{m-1} \). Let \( F_m \) be the intermediate field \( H \subseteq F_m \subseteq H(p^{m+1}) \) such that \( \text{Gal}(F_m/H) \) is cyclic of order \( p^m \). Since \( p \) is inert in \( K \), the prime ideal \( p\mathcal{O}_K \) splits completely in \( H \). Fix once and for all a prime \( p \) of \( H \); it ramifies totally in \( H(p^m) \) as \( p\mathcal{O}_{F_m} = p^{c_m} \). Let us denote \( F_{m,p} \) the completion of \( F_m \) at \( p_m \), \( \mathcal{O}_{m,p} \) its ring of integers and \( \mathbb{F}_{m,p} \) its residue field.

Let \( E \) be the Néron model of \( E \) over \( \mathbb{Z}_p \), and let \( \tilde{E} := E \times_{\mathbb{Z}_p} \mathbb{F}_p \) denote its special fiber. For all \( m \geq 1 \), let \( \Phi_{m,p} \) denote the \( p \)-Sylow subgroup of the group

\[ \tilde{E}(\mathbb{F}_{m,p})/\tilde{E}_0(\mathbb{F}_{m,p}) \cong E(F_{m,p})/E_0(F_{m,p}) \]

of connected components of the base change of \( E \) to \( \mathcal{O}_{m,p} \).

**Lemma 2.1.** There are isomorphisms of \( \mathbb{Q}_p \)-modules

\[ T^- \cong \mathbb{Z}_p(\chi_E) \cong \Phi_{\infty,p}. \]

**Proof.** The first isomorphism is \( \mathbb{Z}_p(\chi_E) \cong \Phi_{\infty,p} \). Tate’s uniformisation provides a description of the group \( \mathbb{Z}_p(\chi_E) \) as

\[ \varphi_{\text{Tate}} : F_{m,p}^\times / q_{E}^\mathcal{O}_{m,p}(\chi_E) \cong (\mathbb{Z}/\text{ord}_{m,p}(q_E)\mathbb{Z})(\chi_E) \xrightarrow{\sim} E(F_{m,p})/E_0(F_{m,p}), \]

where \( \text{ord}_{m,p} \) is the discrete valuation on \( \mathcal{O}_{m,p} \). We have \( p\mathcal{O}_{F_m} = \prod_{p_m \mid p} p^{c_m}_m \), where \( p_m = (\pi_m) \) is the maximal ideal of \( \mathcal{O}_{m,p} \). Hence \( q_E = p^n\alpha = \pi_m^{p^n}\alpha' \) where \( \alpha, \alpha' \in \mathcal{O}_{F_{m,p}}^\times \), i.e.

\[ \text{ord}_{m,p}(q_E) = n \cdot p^m. \]

Under condition \( \mathbb{Z}_p(\chi_E) \), then \( \mathbb{Z}_p(\chi_E) \) gives the isomorphism

\[ \varphi_{\text{Tate}} : (\mathbb{Z}/np^{m}\mathbb{Z})(\chi_E) \cong (\mathbb{Z}/n\mathbb{Z})(\chi_E) \oplus (\mathbb{Z}/p^m\mathbb{Z})(\chi_E) \xrightarrow{\sim} E(F_{m,p})/E_0(F_{m,p}). \]

The lemma follows after taking \( p \)-primary parts and passing to the inverse limit. \( \square \)
2.2. The $G_{K_p}$-cohomology of $E$. Let $B_{\text{dR}} = \text{Frac}(B_{\text{dR}}^+)$ denote Fontaine’s algebra of de Rham periods. For a de Rham $G_{K_p}$-representation $W$, denote

$$D_{\text{dR}}(W) := (W \otimes _{Q_p} B_{\text{dR}})^{G_{K_p}}.$$ 

This Dieudonné module comes naturally equipped with a filtration, inherited from the filtration on $B_{\text{dR}}$, i.e.

$$\text{Fil}^j D_{\text{dR}}(W) := (W \otimes _{Q_p} \text{Fil}^j B_{\text{dR}})^{G_{K_p}},$$

and in particular we have $\text{Fil}^0 D_{\text{dR}}(W) = (W \otimes B_{\text{dR}}^+)^{G_{K_p}}$. The inclusion $\alpha : W \rightarrow W \otimes _{Q_p} B_{\text{dR}}$ induces in cohomology the map

$$\alpha_* : H^1(K_p, W) \rightarrow H^1(K_p, W \otimes B_{\text{dR}}).$$

Define

$$H^1_g(K_p, W) := \ker(\alpha_*) \subseteq H^1(K_p, W), \quad H^1_1(K_p, W) := H^1(K_p, W)/H^1_g(K_p, W).$$

Similarly, if $B_{\text{cris}}$ denotes the ring of crystalline periods, define

$$H^1_f(K_p, W) := \ker(H^1(K_p, W) \rightarrow H^1(K_p, W \otimes B_{\text{cris}})).$$

Let $\log_p : 1 + p\mathbb{Z}_p \rightarrow p\mathbb{Z}_p$ be the usual $p$-adic logarithm, $\exp_p$ denote its inverse and set $u := \exp(p) \in 1 + p\mathbb{Z}_p$. Let

$$(2.7) \quad \text{Art} : K_p^\times \rightarrow G^\text{ab}_{K_p}$$

denote the Artin map of local class field theory, where $G^\text{ab}_{K_p}$ denotes the Galois group of the maximal abelian extension $K^\text{ab}_p$ of $K_p$ over $K_p$. Recall that (2.7) gives an isomorphism once extended to the profinite completion $\hat{K}_p^\times$ of $K_p^\times$.

Decomposition

$$K_p^\times = p^\times \oplus (1 + p\mathcal{O}_{K_p}) \oplus (\mathcal{O}_{K_p}/p\mathcal{O}_{K_p})^\times,$$

corresponds via (2.7) to

$$G^\text{ab}_{K_p} \cong \text{Gal}(K^\text{nr}_p/K_p) \times \text{Gal}(K_{\infty}/K_p) \times \text{Gal}(K'_p/K_p),$$

where $K^\text{nr}_p$ is the maximal unramified extension of $K_p$, $K_{\infty}/K_p$ is a $\mathbb{Z}_p^2$-extension and $K'_p/K_p$ is finite. More precisely, via the Artin map, $p^\hat{\times} \cong \text{Gal}(K^\text{nr}_p/K_p)$. Fix a prime $p$ of $H$ above $p$, and denote

$$K_p(\mu_{p^m}) := \lim_{\rightarrow} K_p(\mu_{p^n}); \quad H(p^\infty)_p := \lim_{\rightarrow} H(p^m)_p,$$

where $p_m$ is the unique prime of $H(p^m)$ lying above $p$. Let

$$K_p \subseteq K_{\text{cyc},p} \subseteq K_p(\mu_{p^\infty}), \quad K_p \subseteq K_{\text{ant},p} \subseteq H(p^\infty)_p$$

be maximal sub-extensions such that $\text{Gal}(K_{\text{cyc},p}/K_p) \cong \text{Gal}(K_{\text{ant},p}/K_p) \cong \mathbb{Z}_p$. Then

$$K_{\infty} = K_{\text{cyc},p} \cdot K_{\text{ant},p}.$$

Fix an element $u_* \in 1 + p\mathcal{O}_{K_p}$ such that $\text{Fr}_p u_* = -u_*$ and $\{u, u_*\}$ is a $\mathbb{Z}_p$-basis of $1 + p\mathcal{O}_{K_p}$. Then

$$\text{Gal}(K_{\infty,p}/K_p) \cong \Gamma_{\text{cyc}} \times \Gamma_{\text{ant}},$$

where

1. $\Gamma_{\text{cyc}} := \text{Gal}(K_{\text{cyc},p}/K_p)$ is generated topologically by $\sigma_{\text{cyc}} := \text{Art}(u)$;
2. $\Gamma_{\text{ant}} := \text{Gal}(K_{\text{ant},p}/K_p)$ is generated topologically by $\sigma_{\text{ant}} := \text{Art}(u_*)$. 

Set also $\Gamma_{\text{nr}} := \text{Gal}(K_{\text{nr}}^p/K_p)$. Recall the cyclotomic character $\chi_{\text{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$; by an abuse of notation, we continue to denote $\chi_{\text{cyc}} : \Gamma_{\text{cyc}} \to 1 + p\mathbb{Z}_p$ its restriction to $\Gamma_{\text{cyc}}$. Let also $\chi_{\text{ant}}$ and $\chi_{\text{nr}}$ denote the characters of $\Gamma_{\text{ant}}$ and $\Gamma_{\text{nr}}$ such that

$$\chi_{\text{ant}}(\sigma_{\text{ant}}) = u, \quad \chi_{\text{nr}}(\text{Fr}_p) = u$$

respectively.

Note that

$$\{\xi_{\text{nr}} := \log_p(\chi_{\text{nr}}), \xi_{\text{cyc}} := \log_p(\chi_{\text{cyc}}), \xi_{\text{ant}} := \log_p(\chi_{\text{ant}})\}$$

is naturally a $\mathbb{Q}_p$-basis of $\text{Hom}_{\text{cont}}(G_{K_p}^{ab}, \mathbb{Q}_p)$. After composing with the isomorphism $\mathbb{Q}_p \xrightarrow{\text{Tate}} V_f|_{G_{K_p}}$ provided by Tate’s uniformization this further yields a $\mathbb{Q}_p$-basis of $H^1(K_p, V^-)$, which in turn may be regarded as a basis of $H^1(K_p, \Phi_{\infty, p} \otimes \mathbb{Q}_p)$ by means of Lemma 2.1.

For a class $\xi \in H^1(K_p, V^-)$, we denote $\bar{\xi}$ its image in the singular quotient $H^1_s(K_p, V^-)$.

**Lemma 2.2.** \{\xi_{\text{cyc}}, \xi_{\text{ant}}\} is a $\mathbb{Q}_p$-basis for $H^1_s(K_p, V^-)$.

**Proof.** By definition, the submodule $H^1_s(K_p, \mathbb{Q}_p) = H^1_f(K_p, \mathbb{Q}_p)$ of $H^1(K_p, \mathbb{Q}_p) = \text{Hom}_{\text{cont}}(G_{K_p}, \mathbb{Q}_p)$ is given by $H^1_f(K_p, \mathbb{Q}_p(\chi_E)) \cong \text{Hom}_{\text{cont}}(\Gamma_{\text{nr}}, \mathbb{Q}_p)$ and is generated by $\xi_{\text{nr}}$. The lemma follows because $H^1_s(K_p, \mathbb{Q}_p) = H^1_s(K_p, V^-) = H^1(K_p, V^-)/H^1_f(K_p, V^-)$.

Write the uniformizer of the elliptic curve $E/\mathbb{Q}_p$ as

$$(2.8) \quad q_E = p^n u^sx \in p^n\mathbb{Z}_p, \quad n \geq 1, s \in \mathbb{Z}_p, x \in \mu_{p-1},$$

so that $n = \text{ord}_p(q_E)$ and $ps = \log_p(q_E)$. Let

$$\pi_* : H^1(K_p, V) \to H^1(K_p, V^-)$$

be the morphism induced in $G_{K_p}$-cohomology by the projection $\pi : V \to V^-$. For a $\text{Gal}(K_p/\mathbb{Q}_p)$-module $M$, let us write $M^\pm$ for the eigenspace on which $\text{Fr}_p$ acts as $\pm 1$. Set $a := a_p(E)$ as in the introduction.

**Proposition 2.3.** Let $x_{\text{cyc}}, x_{\text{ant}} \in H^1_s(K_p, V)$ be elements such that

$$(2.9) \quad \pi_* x_{\text{cyc}} = n\xi_{\text{cyc}} - s\xi_{\text{nr}}, \quad \pi_* x_{\text{ant}} = \xi_{\text{ant}} \text{ in } H^1(K_p, V^-).$$

Then \{\bar{x}_{\text{cyc}}, \bar{x}_{\text{ant}}\} is a $\mathbb{Q}_p$-basis for $H^1_s(K_p, V)$ and $\pi_*$ descends to an isomorphism

$$\bar{\pi}_* : H^1_s(K_p, V) \xrightarrow{\cong} H^1_s(K_p, V^-), \quad \bar{x}_{\text{cyc}} \mapsto n \cdot \bar{\xi}_{\text{cyc}}, \quad \bar{x}_{\text{ant}} \mapsto \bar{\xi}_{\text{ant}}.$$ 

Moreover, the Frobenius eigenspaces $H^1_s(K_p, V)^a$ and $H^1_s(K_p, V)^{-a}$ are generated respectively by $\bar{x}_{\text{cyc}}$ and $\bar{x}_{\text{ant}}$.

**Proof.** Consider the long exact sequence

$$(2.10) \quad 0 \to V^- \xrightarrow{\partial^0} H^1(K_p, V^+) \xrightarrow{\pi} H^1(K_p, V) \xrightarrow{\rho} H^2(K_p, V^-) \to 0$$

induced in $G_{K_p}$-cohomology by (2.3). The connecting homomorphisms $\partial^0, \partial^1$ can be expressed in terms of $q_E$ as follows. Cup product and the trace map give a pairing

$$\langle \ , \ \rangle : H^1(K_p, V^+) \times H^1(K_p, V^-) \to H^2(K_p, V^+) \cong \mathbb{Q}_p$$

satisfying

$$\langle \delta_p(q_E), \xi \rangle = \xi(\text{Art}(q_E))$$

for all $\xi \in H^1(K_p, V^-)$. If we still call $\partial^1$ its composition with $H^2(K_p, V^+) \cong \mathbb{Q}_p$, then it coincides with the map $\langle \delta_p(q_E), \cdot \rangle$. Using the notation of (2.8), we have $\text{Art}(q_E) = \text{Fr}_p^* \sigma_{\text{cyc}}$ and thus
(1) \(\partial^1(\xi_{\text{cyc}}) = \xi_{\text{cyc}}(\text{Art}(q_E)) = \xi_{\text{cyc}}(\sigma_{\text{cyc}}^n) = \log_p(\chi_{\text{cyc}}(\sigma_{\text{cyc}}^n)) = s \log_p(u) = sp\);  
(2) \(\partial^1(\xi_{\text{ant}}) = \xi_{\text{ant}}(\text{Art}(q_E)) = \log_p(\chi_{\text{ant}}(\sigma_{\text{ant}}^0)) = 0\);  
(3) \(\partial^1(\xi_{\text{nr}}) = \xi_{\text{nr}}(\text{Art}(q_E)) = \xi_{\text{nr}}(\text{Fr}_p^n) = \log_p(\chi_{\text{nr}}(\sigma_{\text{cyc}})) = n \log_p(u) = np\).

Combining these computations with the exactness of (2.10) we deduce that the image of \(\pi_*\) is generated by \(\{n \cdot \xi_{\text{cyc}} - s \cdot \xi_{\text{nr}}, \xi_{\text{ant}}\}\), and this gives the existence of \(x_{\text{cyc}}, x_{\text{ant}}\) as in the statement.

From (2.10) and Lemma 2.4 we deduce that \(\tilde{\pi}_*(\bar{x}_{\text{cyc}}) = n \cdot \xi_{\text{cyc}}, \tilde{\pi}_*(\bar{x}_{\text{ant}}) = \xi_{\text{ant}}\), and that \(\tilde{\pi}_*\) is surjective. The map \(\iota_*\) of (2.10) restricts to an isomorphism \(\iota_* : H^1_f(K_p, V^+) \cong H^1_f(K_p, V)\), hence \(\tilde{\pi}_*\) is also injective.

Finally, in order to understand the action of Frobenius, note that \(H^1_s(K_p, V^-) \cong H^1_s(K_p, V^-)\) is naturally a \(\text{Gal}(K_p/\mathbb{Q}_p)\)-module. As explained in the proof of Lemma 2.4 \(\{\xi_{\text{cyc}}, \xi_{\text{ant}}\}\) is a basis of (2.11)

\[H^1_s(K_p, V^-) \cong \text{Hom}(\Gamma_{\text{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \text{Hom}(\Gamma_{\text{ant}}, \mathbb{Q}_p(\chi_E))\].

We have

\[
\xi_{\text{ant}}^{\text{Fr}_p}(\chi_{\text{ant}}) = \chi_{\text{E}}(\text{Fr}_p) \cdot \xi_{\text{ant}}(\text{Fr}_p \sigma_{\text{ant}} \text{Fr}_p^{-1}) = a \cdot \xi_{\text{ant}}(\sigma_{\text{ant}}) = -a \cdot \xi_{\text{ant}}(\sigma_{\text{ant}})
\]

and via (2.11) it follows that

\[H^1_s(K_p, V^-)^{-a} \cong \text{Hom}(\Gamma_{\text{ant}}, \mathbb{Q}_p(\chi_E))\]

is generated by \(\xi_{\text{ant}}\). Analogously, \(\bar{\xi}_{\text{cyc}}\) generates \(H^1_s(K_p, V^-)^{a} \cong \text{Hom}(\Gamma_{\text{cyc}}, \mathbb{Q}_p(\chi_E)).\)

\[\square\]

2.3. Bloch–Kato logarithm and dual exponential maps. We wish to describe explicitly the Bloch–Kato logarithm and the dual exponential maps relative to the representation \(V = V_f\) regarded as a \(G_{K_p}\)-module. First we need to study \(D_{\text{dR}}(V_{K_p}), D_{\text{dR}}(V_{K_p}^\vee)\) and their filtration, where \(V_{K_p} := V_{I^f_{G_{K_p}}}\). Let \(C_p\) be the completion of a fixed algebraic closure of \(\mathbb{Q}_p\), and let

\[\mathcal{R} := \lim_{\substack{\longrightarrow}} \mathcal{O}_{C_p}/p\mathcal{O}_{C_p}.
\]

Let \(W(\mathcal{R})\) be the ring of Witt vectors of \(\mathcal{R}\) and denote \([\cdot] : \mathcal{R} \rightarrow W(\mathcal{R})\) the Teichmuller lift. There is an isomorphism

\[\lim_{\substack{\longrightarrow}} \mathcal{O}_{C_p} \xrightarrow{\cong} \mathcal{R}, \quad (x^{(n)})_n \mapsto (x_n := x^{(n)} \mod p)_n.
\]

Recall the basis \(\tilde{\xi} := (\xi^{(m)})_m\) of \(Z_p(1)\) we introduced in §2.1. It can be regarded as an element of \(\lim_{\substack{\longrightarrow}} \mathcal{O}_{C_p} \cong \mathcal{R}\), and if \(\log_{q_E}\) denotes the branch of the \(p\)-adic logarithm such that \(\log_{q_E}(q_E) = 0\), then

\[t := \log_{q_E}([\tilde{\xi}])\]

generates the maximal ideal \(\text{Fil}^1 B_{\text{dR}}\) of \(B_{\text{dR}}^+ = \text{Fil}^0 B_{\text{dR}}\). Analogously, we can regard \(\tilde{q}\) as an element of \(\mathcal{R}\) and we define

\[\tilde{q}^\# := \log_{q_E}([\tilde{q}]/q_E) \in \text{Fil}^1 B_{\text{dR}}\]

Recall that the images \(\varepsilon, q\) of \(\tilde{\xi}, \tilde{q}\) under Tate’s uniformization form a basis of \(V\). By elementary calculations one obtains the following lemma.

**Lemma 2.4.** Let

\[e_1 := q \otimes 1 - \varepsilon \otimes \frac{\tilde{q}^\#}{t} \in V \otimes \text{Fil}^0 B_{\text{dR}}, \quad e_2 := \varepsilon \otimes \frac{1}{t} \in V \otimes \text{Fil}^{-1} B_{\text{dR}}.\]
These elements are invariant with respect to the action of $G_{K_p}$, they form a $K_p$-basis of $\text{Fil}^0 \text{dR}(V_{K_p})$ and $e_1$ is a $K_p$-basis of $\text{Fil}^0 \text{dR}(V_{K_p})$. Moreover, there are isomorphisms

$$\iota_* : \text{dR}(V_{K_p}^+)/\text{Fil}^0 \text{dR}(V_{K_p}^+) \xrightarrow{\sim} \text{dR}(V_{K_p})/\text{Fil}^0 \text{dR}(V_{K_p}), \quad \varepsilon \otimes t^{-1} \mapsto e_2;$$

$$\pi_* : \text{Fil}^0 \text{dR}(V_{K_p}) \xrightarrow{\sim} \text{Fil}^0 \text{dR}(V_{K_p}^-), \quad e_1 \mapsto \tilde{q},$$

where, with a slight abuse of notation, $\tilde{q} := \tilde{q} \otimes 1$ in $(V^- \otimes B^+_{\text{dR}})^{G_{K_p}} = \text{Fil}^0 \text{dR}(V^-)$.

Kummer theory provides isomorphisms

$$\delta_p : O^\times_{K_p} \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K_p, \mathbb{Q}_p(1)), \quad E(K_p) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K_p, V),$$

which combined with (2.3) and Tate’s uniformization yield canonical identifications of $\mathbb{Q}_p[G_{K_p}]$-modules

$$O^\times_{K_p}(\chi_E) \otimes \mathbb{Q}_p \cong H^1_f(K_p, V^+) \cong H^1_f(K_p, V) = E(K_p) \otimes \mathbb{Q}_p.$$

Besides, the $p$-adic logarithm induces a morphism of $\mathbb{Q}_p[G_{K_p}]$-modules

$$\log_p : O^\times_{K_p}(\chi_E) \otimes \mathbb{Q}_p \longrightarrow K_p(\chi_E).$$

Via the identifications (2.12), it gives rise to the map

$$\log_E : E(K_p) \longrightarrow K_p(\chi_E),$$

which is the formal group logarithm on $E$ associated with the invariant differential $\omega_f$ on $E$ introduced in §3.1. Finally, (2.12) and the isomorphism

$$\text{dR}(V_{K_p})/\text{Fil}^0 \text{dR}(V_{K_p}) \xrightarrow{\sim} K_p(\chi_E), \quad e_2 \mapsto 1,$$

allow us to identify $\log_E$ with the composition of the Bloch–Kato logarithm

$$\log : H^1_f(K_p, V) \longrightarrow \text{dR}(V_{K_p})/\text{Fil}^0 \text{dR}(V_{K_p})$$

with the pairing with $\omega_f$.

For a $p$-adic de Rham representation $W$ of $G_{K_p}$, let

$$\gamma : \text{Fil}^0 \text{dR}(W) = (W \otimes_{\mathbb{Q}_p} B^+_{\text{dR}})^{G_{K_p}} \longrightarrow H^1(K_p, W \otimes_{\mathbb{Q}_p} B^+_{\text{dR}})$$

be the isomorphism defined by $x \mapsto \gamma(x)$, where $\gamma(x)$ is the cohomology class of the map

$$\sigma \mapsto x \cdot \log_p(\chi_{\text{cyc}}(\sigma)).$$

**Definition 2.5.** The Bloch–Kato dual exponential of $W$ is the map

$$\exp^* : H^1(K_p, W) \xrightarrow{\sigma} H^1(K_p, W \otimes_{\mathbb{Q}_p} B^+_{\text{dR}}) \xrightarrow{\gamma^{-1}} \text{Fil}^0 \text{dR}(W).$$

The map $H^1(K_p, W \otimes B^+_{\text{dR}}) \longrightarrow H^1(K_p, W \otimes B_{\text{dR}})$ is injective and thus $\ker(\exp^*) = H^1_{\text{dR}}(K_p, W)$. We shall regard the dual exponential associated to $W$ as the map

$$\exp^* : H^1(K_p, W) = H^1(K_p, W)/H^1_{\text{dR}}(K_p, W) \longrightarrow \text{Fil}^0 \text{dR}(W).$$

Similarly as in the case of the Bloch–Kato logarithm, the dual exponential for $W = K_p$ together with the identification $H^1(K_p, \mathbb{Q}_p(\chi_E)) = H^1(K_p, \mathbb{Q}_p(\chi_E))$ gives rise to an isomorphism of $K_p[G_{K_p}]$-modules

$$\exp^* : H^1(K_p, \mathbb{Q}_p(\chi_E)) \longrightarrow K_p(\chi_E).$$
It coincides with the dual exponential
\[(2.14) \quad \exp^* : H^1_s(\mathbb{K}_p, V) \rightarrow K_p(\chi_E) \]
via the identifications
\[H^1_s(\mathbb{K}_p, \mathbb{Q}_p(\chi_E)) \cong H^1_s(\mathbb{K}_p, V^-) \cong H^1_s(\mathbb{K}_p, V)\]
explained in the previous section, where
\[\text{Fil}^0 \mathcal{D}_{dR}(V_{K_p}) \cong K_p(\chi_E), \quad e_1 \mapsto 1.\]

It follows directly from the very definitions and the \(G_{\mathbb{Q}_p}\)-equivariance of the dual exponential map that
\[\exp^*(\xi_{\text{cyc}}) = 1 \quad \text{and} \quad z := \exp^*(\xi_{\text{ant}}) \in K_p^- .\]
The map \((2.14)\) and its ±-components can be explicitly described as follows.

**Proposition 2.6.** The dual exponential \(\exp^* : H^1_s(\mathbb{K}_p, V) \rightarrow K_p(\chi_E)\) is characterized by
\[\exp^*(\bar{x}_{\text{cyc}}) = n, \quad \exp^*(\bar{x}_{\text{ant}}) = z \in K_p^- .\]

**Proof.** By (the proof of) Proposition 2.3,
\[H^1_s(\mathbb{K}_p, \mathbb{Q}_p(\chi_E))^a \oplus H^1_s(\mathbb{K}_p, \mathbb{Q}_p(\chi_E))^{-a} \cong \text{Hom}(\Gamma_{\text{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \text{Hom}(\Gamma_{\text{ant}}, \mathbb{Q}_p(\chi_E)),\]
whose basis \(\{\xi_{\text{cyc}}, \xi_{\text{ant}}\}\) is compatible with the above decomposition.

The claim follows by using Lemma 2.4, Proposition 2.3 and the description of \((2.14)\) discussed before this proposition. \(\square\)

**Corollary 2.7.** There is a commutative diagram whose arrows are isomorphisms of \(G_{\mathbb{Q}_p}\)-modules:

\[
\begin{array}{ccc}
\text{Hom}_{\text{cont}}(\Gamma_{\text{cyc}}, \Phi_{\infty,p}) \oplus \text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty,p}) & \xrightarrow{\varphi} & H^1_s(\mathbb{K}_p, V) \\
\xrightarrow{\frac{1}{p} \text{ev}_{\text{cyc}}} & & \xrightarrow{\exp^*} \\
(\Phi_{\infty,p} \otimes \mathbb{Q}_p) \oplus (\Phi_{\infty,p} \otimes \mathbb{Q}_p) & \xrightarrow{\varphi} & K_p(\chi_E)
\end{array}
\]

where
\[(1) \quad \varphi := (\varphi_{\text{Tate}} * \bar{\pi})^{-1} ;
(2) \quad \varphi(\bar{q}, 0) = 1, \quad \varphi(0, \bar{q}) = z, \quad \text{where} \quad z := \exp^*(\xi_{\text{ant}}) \in K_p^- \quad \text{is as in Proposition 2.6} ;
(3) \quad \text{ev}_{\text{cyc}} \text{ denotes the valuation at } \sigma_{\text{cyc}} \in \Gamma_{\text{cyc}}, \text{ and analogously for ev}_{\text{ant}} .
\]
Moreover, \(\varphi\) decomposes with respect to the action of \(\text{Fr}_p\) as
\[(2.15) \quad \text{Hom}_{\text{cont}}(\Gamma_{\text{cyc}}, \Phi_{\infty,p}) \rightarrow H^1_s(\mathbb{K}_p, V)^a, \quad \text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty,p}) \rightarrow H^1_s(\mathbb{K}_p, V)^{-a} .\]

**Proof.** By definition we have
\[\xi_{\text{cyc}}(\sigma_{\text{cyc}}) = \varphi_{\text{Tate}}(\log_p(\chi_{\text{cyc}}(\sigma_{\text{cyc}}))) = \varphi_{\text{Tate}}(\log_p(\bar{u})) = \varphi_{\text{Tate}}(p) = p \cdot \bar{q} \]
and similarly, \(\xi_{\text{ant}}(\sigma_{\text{ant}}) = p \cdot \bar{q} .\) The statement follows by combining (the proof of) Proposition 2.6, Proposition 2.3 and Lemma 2.1. \(\square\)
3. Hida families and Selmer groups associated to \((f, g, h)\)

3.1. Filtrations and differentials. Let

\[ \varphi \in S_w(M, \varepsilon) \]

be an ordinary newform of weight \(w\), level \(M\) and character \(\varepsilon\) with coefficients in a finite extension \(L/\mathbb{Q}\). Fix an algebraic closure \(\bar{\mathbb{Q}}\) of \(\mathbb{Q}\) and an embedding \(L \subset \bar{\mathbb{Q}}\). Fix also a prime number \(p\), an algebraic closure \(\bar{\mathbb{Q}}_p\) of \(\mathbb{Q}_p\) and an embedding \(\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p\), with respect to which we assume \(\varphi\) is ordinary. Let \(L_p \subset \bar{\mathbb{Q}}_p\) denote the \(p\)-adic closure of \(L\).

Let \(V_\varphi\) denote the \(p\)-adic Galois representation attached to \(\varphi\) with coefficients in \(L_p\) and let \(\alpha_\varphi, \beta_\varphi\) be the roots of the Hecke polynomial \(x^2 + a_p(\varphi)x + \varepsilon(p)\) at \(p\) of \(V_\varphi\). Since \(\varphi\) is ordinary, one of these roots, say \(\alpha_\varphi\), is a \(p\)-adic unit. As a \(G_{\mathbb{Q}_p}\)-representation, \(V_\varphi\) is equipped with a decreasing filtration of \(L_p[G_{\mathbb{Q}_p}]\)-modules

\[ (3.1) \quad \text{Fil}^2(V_\varphi) = 0 \subseteq \text{Fil}^1(V_\varphi) \subseteq V_\varphi = \text{Fil}^0(V_\varphi). \]

Let

\[ \psi_\varphi : G_{\mathbb{Q}_p} \to L_p^\times \]

be the unramified character such that \(\psi_\varphi(F_{\mathbb{Q}_p}) = \alpha_\varphi\). The filtration (3.1) satisfies:

(i) \(V_\varphi^+ := \text{Fil}^1(V_\varphi)\) and \(V_\varphi^- := V_\varphi/\text{Fil}^1(V_\varphi)\) have dimension one over \(L_p\);

(ii) \(G_{\mathbb{Q}_p}\) acts on \(V_\varphi^+\) via \(\chi^{\omega_\varphi}_c \psi_\varphi^{-1}\);

(iii) \(G_{\mathbb{Q}_p}\) acts on \(V_\varphi^-\) via \(\psi_\varphi^*\). In particular, it is an unramified \(G_{\mathbb{Q}_p}\)-module.

When \(w \geq 2\) Faltings’ comparison theorem yields an isomorphism

\[ (3.2) \quad S_w(M)_{L_p}[\varphi] \xrightarrow{\cong} \text{Fil}^1(D_{\text{dR}}(V_\varphi)) \cong D_{\text{dR}}(V^-_\varphi), \]

where \(S_w(M)_{L_p}[\varphi]\) denotes the \(\varphi\)-isotypical component of \(S_w(M) \otimes L_p\). We define

\[ \omega_\varphi \in D_{\text{dR}}(V^-_\varphi) \]

to be the image of \(\varphi\) via (3.2).

Denote \(\varphi^* := \varphi \otimes \varepsilon^{-1} \in S_w(M, \varepsilon)\). Consider the pairing

\[ \langle , \rangle : V_\varphi \times V_{\varphi^*}(1) \to L_p \]

given by Poincaré’s duality. This in turn induces perfect pairings

\[ \langle , \rangle : D_{\text{dR}}(V_\varphi) \times D_{\text{dR}}(V_{\varphi^*}(1)) \to D_{\text{dR}}(L_p) = L_p \]

and

\[ \langle , \rangle : D_{\text{dR}}(V_\varphi^+) \times S_w(M)_{L_p}[\varphi^*] \to L_p. \]

Define

\[ \eta_\varphi \in D_{\text{dR}}(V_\varphi^+) \subseteq D_{\text{dR}}(V_\varphi) \]

to be the differential characterized by

\[ \langle \omega_\varphi, \eta_\varphi \rangle = 1. \]

If \(\varphi\) has weight \(w = 1\), then we will always assume that \(p \nmid M\). In this case, both \(\alpha_\varphi\) and \(\beta_\varphi\) are \(p\)-adic units, and throughout this paper we will also assume that

\[ \alpha_\varphi \neq \beta_\varphi. \]
In this setting, the filtration \([5.1]\) admits a splitting. More precisely, if we denote \(V^\alpha_\varphi, V^\beta_\varphi\) the eigenspaces for the action of \(\text{Fr}_p\) on \(V_\varphi\), with eigenvalues \(\alpha_\varphi, \beta_\varphi\) respectively, then

\[
V_\varphi = V^\alpha_\varphi \oplus V^\beta_\varphi, \quad V^+_\varphi = V^\beta_\varphi, \quad V^-_\varphi = V^\alpha_\varphi
\]
as \(\mathbb{Q}_p\)-modules. In this case, there is a pairing

\[
(3.3) \quad \text{D}_{\text{dR}}(V_\varphi) \times \text{D}_{\text{dR}}(V_\varphi) \to \text{D}_{\text{dR}}(L_p(\chi_\varphi)),
\]
and we let

\[
(3.4) \quad \omega_{\varphi,\alpha} \in \text{D}_{\text{dR}}(V^\alpha_\varphi), \quad \eta_{\varphi,\alpha} \in \text{D}_{\text{dR}}(V^\beta_\varphi)
\]
be the differentials attached by Ohta to the \(p\)-stabilized eigenform \(\varphi_\alpha\) as described in [DR16 §2]. They satisfy

\[
(3.5) \quad \langle \omega_{\varphi,\alpha}, \eta_{\varphi,\alpha} \rangle = g(\varepsilon) \in L_p,
\]
where \(g(\varepsilon)\) denotes the Gauss sum of the nebentype character \(\varepsilon\) of \(\varphi\).

Attached to a weight 1 modular form \(\varphi\) there is an Artin representation

\[
\rho_\varphi : G_\mathbb{Q} \to \text{GL}(V^\alpha_\varphi)
\]
where \(V^\alpha_\varphi\) is an \(L\)-vector space equipped with a (non-canonical) isomorphism \(j_\varphi : V^\alpha_\varphi \otimes_L L_p \cong V_\varphi\).

The choice of \(j_\varphi\) thus induces an \(L\)-structure on \(V_\varphi\) by \(V^L_\varphi := j_\varphi(V^\alpha_\varphi)\).

Let \(v^\alpha_\varphi\) be an \(L\)-basis of \(V^L_\varphi \cap V^\alpha_\varphi\) and \(v^\beta_\varphi\) an \(L\)-basis of \(V^L_\varphi \cap V^\beta_\varphi\). Let \(H\) be the number field cut out by \(\rho_\varphi\), fix once and for all an embedding \(H \subseteq \mathbb{Q}_p\) and denote \(H_p\) the \(p\)-adic completion of \(H\) inside \(\mathbb{Q}_p\). Finally, define periods

\[
(3.6) \quad \Omega_{\varphi,\alpha} \in H^{1/\alpha_\varphi}_p, \quad \Theta_{\varphi,\alpha} \in H^{1/\beta_\varphi}_p
\]
characterized as

\[
\Omega_{\varphi,\alpha} \otimes v^\alpha_\varphi = \omega_{\varphi,\alpha} \in \text{D}_{\text{dR}}(V^\alpha_\varphi), \quad \Theta_{\varphi,\alpha} \otimes v^\beta_\varphi = \eta_{\varphi,\alpha} \in \text{D}_{\text{dR}}(V^\beta_\varphi).
\]

Although the above periods depend on the choice of \(j_\varphi\), the ratio

\[
(3.7) \quad L_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}}
\]
is readily seen to be independent of this choice; it only depends on the choice of basis \(\{v^\alpha_\varphi, v^\beta_\varphi\}\) and is thus well defined up to scalars in \(L^\times\).

Let \(\Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]\) be the Iwasawa algebra and \(\mathcal{W} := \text{Spf}(\Lambda)\) be the weight space. It is the rigid analytic space whose \(A\)-points for any \(\mathbb{Z}_p\)-algebra \(A\) are

\[
\mathcal{W}(A) = \text{Hom}_{\text{cts}}(\mathbb{Z}_p - \text{alg}, \Lambda, A) = \text{Hom}_{\text{cts}}(1 + p\mathbb{Z}_p, \Lambda^\times).
\]

Recall that \(Z\) can be seen as a dense subset of \(\mathcal{W}(\mathbb{C}_p)\) identifying an integer \(k \in \mathbb{Z}\) with the character \(\nu_k : (x \mapsto x^k) \in \text{Hom}_{\text{cts}}(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times)\). More generally, given a Dirichlet character \(\epsilon\) of \(p\)-power conductor, we denote \(\nu_{k,\epsilon}\) the point \((x \mapsto \epsilon(x)x^k) \in \mathcal{W}(\mathbb{C}_p)\). A character \(\nu \in \mathcal{W}\) is called classical if it is of the form \(\nu = \nu_{k,\epsilon}\) with \(k \geq 2\) and \(k\) is called its weight. A classical point is cristalline if it is of the form \(\nu_{k,\omega,\nu}\), where \(\omega\) is the Teichmüller character.

Let

\[
\varphi = \sum a_n(\varphi)q^n \in \Lambda_\varphi[[q]]
\]
be a Hida family of tame level \(M\) and tame character \(\chi\), where \(\Lambda_\varphi\) is a finite flat extension of \(\Lambda\), and let \(\kappa : \mathcal{W}_\varphi \to \mathcal{W}\) be the weight map induced by the inclusion \(\Lambda \subseteq \Lambda_\varphi\). A point \(x \in \mathcal{W}_\varphi\)}
is called **classical** (resp. **crystalline***) if $\kappa(x)$ is, and we denote $W^c_\varphi$ (resp. $W^s_\varphi$) the set of classical (resp. crystalline) points of $W_\varphi$. For each $x \in W^c_\varphi$ with $\kappa(x) = \nu_k \epsilon$, the specialization 

$$\varphi_x := \sum_{n \geq 1} x(a_n(\varphi))q^n$$

of $\varphi$ at $x$ is the $q$-expansion of a classical $p$-ordinary eigenform of weight $k$ and character $\chi \epsilon \omega^{-k}$.

If $x \in W$ is a point of weight 1, meaning that $\kappa(x) = \nu_1 \epsilon$ for some Dirichlet character $\epsilon$ of $p$-power conductor, the specialization $\varphi_x$ may be a classical modular form or not. (Later we shall regard the $p$-stabilizations of $g$ and $h$ as weight 1 specializations of a Hida family, and these are classical by construction.)

Notice that the specializations of $\varphi$ at crystalline points have all nebentype $\chi$. If $x \in W^s_\varphi$ is a crystalline point of weight $k > 2$, then $\varphi_x$ is the ordinary $p$-stabilization of a newform $\varphi_x \in S_k(N, \chi)$. If $k = 2$, then $\varphi_x$ may be either old or new at $p$. We denote $\varphi_x := \varphi_x$ if it is new, while if $\varphi_x$ is old at $p$, we denote $\varphi_x$ the newform whose $p$-stabilization is $\varphi_x$. If no confusion is likely to arise, we may often denote these forms by an abuse of notation as $\varphi_k$ and $\varphi_k$.

Let $V_\varphi$ be the Galois representation of $G_Q$ attached to $\varphi$, a $\Lambda_\varphi$-module of rank 2. As a $G_{Q_p}$-representation there is again a filtration 

$$\text{Fil}^2(V_\varphi) = 0 \subseteq \text{Fil}^1(V_\varphi) \subseteq V_\varphi = \text{Fil}^0(V_\varphi)$$

where $V_\varphi^+ := \text{Fil}^1(V_\varphi)$ has dimension one, and the quotient $V_\varphi^- := V_\varphi / \text{Fil}^1(V_\varphi)$ is unramified. More precisely, $G_{Q_p}$ acts on $V_\varphi^-$ via the unramified character 

$$\psi_\varphi : G_{Q_p} \longrightarrow \Lambda_\varphi^\times$$

such that $\psi_\varphi(F_{p_i}) = a_p(\varphi)$.

If $x$ is crystalline and $L_p/Q_p$ is a finite extension containing the values of $x$, then the Galois representation $V_\varphi \otimes_{\Lambda_\varphi} L_p \cong V_{\varphi, x}$ is crystalline at $p$.

Fix now three cuspidal Hida families

$$f \in \Lambda_f[[q]], \quad g \in \Lambda_g[[q]], \quad h \in \Lambda_h[[q]].$$

Write $\Lambda_{fg} = \Lambda_f \hat{\otimes} \Lambda_g \hat{\otimes} \Lambda_h$ and $\mathcal{O}_{fg} := \Lambda_f[1/p] \hat{\otimes} \Lambda_g[1/p] \hat{\otimes} \Lambda_h[1/p]$. Let

$$W_{fg} = \text{Spf} \left( \Lambda_{fg} \right) = W_f \times W_g \times W_h$$

be the associated three-dimensional product of weight spaces. Denote $W^s_{fg}$ the set of crystalline points $(x, y, z) \in W^s_f \times W^s_g \times W^s_h$ of weights $(k, \ell, m) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ such that $k \equiv 2 \mod (p-1), \ell \equiv m \equiv 1 \mod 2(p-1)$. As above, by abuse of notation we may identify points $(x, y, z)$ of $W_{fg}$ with their weights $(k, \ell, m)$.

Set

$$\mathbb{V} := V_f \otimes V_g \otimes V_h \otimes \mathcal{O}_{fg} \Xi,$$

where $\Xi : G_{Q_p} \longrightarrow \mathcal{O}_{fg}^{\times}$ is the character defined in [BSVL] §4.6.2 and is characterized by the property that, if $\mathfrak{z} = (k, \ell, m) \in W^s_{fg}$ is a triple of crystalline points and $\rho_{\mathfrak{z}} := \nu_k \otimes \nu_\ell \otimes \nu_m : \mathcal{O}_{fg} \longrightarrow L_p$ is the corresponding specialisation map, then $\rho_{\mathfrak{z}}^* \Xi = \chi^{(4k-\ell-m)/2}$. We define a filtration on $\mathbb{V}$ by

$$\text{Fil}^p(\mathbb{V}) := \left[ \otimes_{i+j+k+l=q} \text{Fil}^i(V_f) \hat{\otimes} \text{Fil}^j(V_g) \hat{\otimes} \text{Fil}^l(V_h) \hat{\otimes} \mathcal{O}_{fg} \Xi \right].$$

Since $\mathbb{V} / \text{Fil}^2(\mathbb{V})$ has no $G_{Q_p}$-invariants, the natural inclusion

$$\text{Fil}^2(\mathbb{V}) = [V_f \otimes V^+_g \otimes V^+_h \otimes \mathcal{O}_{fg} \Xi] \oplus [V^+_f \otimes V_g \otimes V^+_h \otimes \mathcal{O}_{fg} \Xi] \oplus [V^+_f \otimes V^+_g \otimes V_h \otimes \mathcal{O}_{fg} \Xi] \hookrightarrow \mathbb{V}$$

is a field.
induces an injective morphism in cohomology
\[ H^1(Q_p, \Fil^2(V)) \to H^1(Q_p, V), \]
and henceforth we shall identify \( H^1(Q_p, \Fil^2(V)) \) with its image in \( H^1(Q_p, V) \). Defining
\begin{align*}
\Psi^f := V_f \oplus V_g \oplus V_h \otimes_{\O_{\text{rig}}} \Xi, \\
\Psi^g := V_f^+ \otimes V_g^- \otimes V_h^+ \otimes_{\O_{\text{rig}}} \Xi, \\
\Psi^h := V_f^+ \otimes V_g^- \otimes V_h^- \otimes_{\O_{\text{rig}}} \Xi,
\end{align*}
it follows that \( H^1(Q_p, \Fil^2(V)/\Fil^3(V)) \) decomposes as
\[ H^1(Q_p, \Psi^f) \oplus H^1(Q_p, \Psi^g) \oplus H^1(Q_p, \Psi^h). \]

For \( \varphi \in \{f, g, h\} \), we denote
\[ \pi_\varphi : H^1(Q_p, \Fil^2(V)) \to H^1(Q_p, \Psi^\varphi) \]
the natural projection map.

3.2. Selmer groups. Let \( W \) be a \( L_p[G_Q] \)-module. Given a prime number \( \ell \), define as customary
\[ H^1_f(Q_\ell, W) := \begin{cases} H^1_{ur}(Q_\ell, W) := H^1(Q_\ell^w/Q_{\ell}, W_\ell^w) & \ell \neq p \\
\ker \left( H^1(Q_p, W) \to H^1(Q_p, W \otimes_{Q_p} B_{\text{cris}}) \right) & \ell = p \end{cases} \]
and
\[ H^1(Q_\ell, W) := H^1(Q_\ell, W)/H^1_f(Q_\ell, W). \]
The Bloch–Kato Selmer group of \( W \) is
\[ \text{Sel}_p(W) := \{ x \in H^1(Q, W) | \text{res}_\ell(x) \in H^1_f(Q_\ell, W) \text{ for all } \ell \}, \]
where \( \text{res}_\ell : H^1(Q, W) \to H^1(Q_\ell, W) \) denotes the restriction map in Galois cohomology.

For each prime \( \ell \), we denote by \( \partial_\ell \) the composition
\[ \partial_\ell : H^1(Q, W) \xrightarrow{\text{res}_\ell} H^1_f(Q_\ell, W) \to H^1_s(Q_\ell, W). \]

The relaxed Selmer group is defined as
\[ \text{Sel}_{(p)}(W) := \{ x \in H^1(Q, W) | \text{res}_\ell(x) \in H^1_f(Q_\ell, W) \text{ for all } \ell \neq p \} \supset \text{Sel}_p(Q, W). \]

Let now \( (f, g, h) \) be the triple of eigenforms of weights \( (2, 1, 1) \) introduced at the beginning of the article. Recall Assumption \([12]\) imposed on \( V := V_f \otimes V_g \otimes V_h \) in the introduction, which implies
\[ \text{Sel}_p(V) = 0. \]
Indeed, as discussed in detail in \([5.1]\) the representation \( V \) decomposes as a direct sum
\[ V = [V_f \otimes V_{\psi_1}] \oplus [V_f \otimes V_{\psi_2}], \]
where, for \( i = 1, 2 \), the \( G_Q \)-representation \( V_{\psi_i} \) is induced by a ring class character \( \psi_i \) of the imaginary quadratic field \( K \). This decomposition induces factorisations
\[ L(E, \rho, s) = L(E/K, \psi_1, s) \cdot L(E/K, \psi_2, s), \quad \text{Sel}_p(V) = \text{Sel}_p(E \otimes \psi_1) \oplus \text{Sel}_p(E \otimes \psi_2). \]
Combining the factorisation of the complex \( L \)-function with Assumption \([12]\) we obtain that \( L(E/K, \psi_i, 1) \neq 0 \) for \( i = 1, 2 \), which, by \([BD97, \text{Theorem B}]\), implies that \( \text{Sel}_p(E \otimes \psi_1) = \text{Sel}_p(E \otimes \psi_2) = 0. \)

Lemma 3.1. There is an isomorphism
\[ \partial_p : \text{Sel}_{(p)}(V) \xrightarrow{\cong} H^1_s(Q_p, V). \]
Proof. This follows from Poitou–Tate duality as in [MR04, Theorem 2.3.4], where a similar statement is proved, under the (irrelevant) assumption that \( p \nmid N_f \). □

The characteristic polynomial for the action of \( \text{Fr}_p \) on \( V_g \) is \( x^2 - a_p(g)x + \chi(p) = x^2 - \varepsilon(p) \) and we may thus write the eigenvalues of \( \text{Fr}_p \) as \( \alpha_g = \lambda, \beta_g = -\lambda \) for some root of unity \( \lambda \). The same holds for \( h \) and since its nebentype is the inverse of that of \( g \) we have

\[
(\alpha_h, \beta_h) = (1, -1/\lambda) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1, 1/\lambda).
\]

Using the notation of [3.1] denote \( \{v_\alpha^g, v^g_\beta\} \) and \( \{v^h_\alpha, v^h_\beta\} \) a pair of \( L \)-bases for \( V_g \) and \( V_h \) consisting of eigenvectors with eigenvalue \( \alpha_g, \beta_g, \alpha_h, \beta_h \) respectively. Denote \( V_\alpha^g, V_\beta^g \), etc. the corresponding eigenspaces. Set \( V_{gh}^{\alpha} := V_g^\alpha \otimes V_h^\alpha \) and \( V_{gh}^{\alpha^*} := V_f \otimes V_{gh}^{\alpha^*} \), and likewise for the remaining pairs of eigenvalues. There is a decomposition of \( L_p[G_{Q_p}] \)-modules

\[
(3.9) \quad V := V_{fgh} = V^{\alpha} \oplus V^{\alpha^*} \oplus V^{\beta} \oplus V^{\beta^*}.
\]

Lemma 3.2. Let \( \triangle, \heartsuit \in \{\alpha, \beta\} \). Then

i) the Bloch–Kato logarithm gives an isomorphism

\[
\log_{\triangle \heartsuit} : H^1_f(Q_p, V^{\triangle \heartsuit}) \cong H^1_f(Q_p, V^+_f \otimes V_{gh}^{\triangle \heartsuit}) \xrightarrow{\cong} L_p;
\]

ii) the Bloch–Kato dual exponential gives an isomorphism

\[
\exp^{\triangle \heartsuit} : H^1_f(Q_p, V^{\triangle \heartsuit}) \cong H^1_f(Q_p, V^-_f \otimes V_{gh}^{\triangle \heartsuit}) \xrightarrow{\cong} L_p;
\]

Proof. Let \( W := V_{gh}^\triangle \). By [DRb] Lemma 2.4.1

\[
(3.10) \quad H^1_f(Q_p, V^\triangle) = \left\{ \begin{array}{ll}
H^1_f(Q_p, V^+_f \otimes V_{gh}^{\triangle \heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = +1 \\
H^1_f(Q_p, V^-_f \otimes V_{gh}^{\triangle \heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = -1.
\end{array} \right.
\]

Moreover, in the case in which \( \triangle \cdot \heartsuit \cdot a = -1 \), we have

\[
H^1_f(Q_p, V^+_f \otimes V_{gh}^{\triangle \heartsuit}) = H^1_f(Q_p, V^+_f \otimes V_{gh}^{\triangle \heartsuit})
\]

(see [DRa Example 3.1.4]). Since \( V^+_f \cong Q_p(1)(\chi_E) \), i) follows from the fact that

\[
V^+_f \otimes V_{gh}^{\triangle \heartsuit} = L_p(\zeta_{\text{cyc}})
\]

where \( \zeta : G_{Q_p} \longrightarrow Q_p^\times \) is the unramified character given by \( \zeta(\text{Fr}_p) = \triangle \cdot \heartsuit \cdot a \) (for more details, see e.g. [DRa Example 3.1.4]). The statement on the singular quotients is proven similarly using (3.10) and the fact that

\[
V^-_f \otimes V_{gh}^{\triangle \heartsuit} = L_p(\zeta).
\]

Set \( L^4_p = L_p \oplus L_p \oplus L_p \oplus L_p \). We obtain from the previous lemma isomorphisms

\[
\text{log} : H^1_f(Q_p, V) \longrightarrow L^4_p, \quad \text{exp}^* : H^1_f(Q_p, V) \longrightarrow L^4_p,
\]

where \( \text{log} = \log^{\alpha \alpha} \oplus \log^{\alpha \beta} \oplus \log^{\beta \alpha} \oplus \log^{\beta \beta} \) and \( \text{exp}^* = \exp_{\alpha \alpha}^* \oplus \exp_{\alpha \beta}^* \oplus \exp_{\beta \alpha}^* \oplus \exp_{\beta \beta}^* \).

Combining Lemma 3.1 with Lemma 3.2, we obtain the following result on the \( L_p \)-structure of the relaxed Selmer group attached to \( V \).

Corollary 3.3. There are isomorphisms

\[
\text{Sel}(p)(V) \xrightarrow{\partial_p} H^1_f(Q_p, V) \xrightarrow{\exp^*} L^4_p.
\]
We now turn to pin down a particular basis of $\text{Sel}_{[p]}(V)$ which is canonical up to multiplication by elements in $L^\times$. In order to do this, we need a more precise description of the finite and singular parts of $H^1(Q_p, V^{\triangledown\triangle})$ in terms of $H^1(K_p, V_f)$. Note that the latter space is equipped with a natural action of $\text{Gal}(K_p/Q_p)$.

In what follows, for any $\text{Gal}(K_p/Q_p)$-module $M$ we denote $M^\pm$ the subspace of $M$ on which $\text{Fr}_p$ acts as multiplication by $\pm 1$. Note that $\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h \in \{\pm 1\}$ and thus it makes sense to consider $M^{\alpha_g\alpha_h}$, etc.

**Lemma 3.4.** Let $\triangledown, \triangle \in \{\alpha, \beta\}$. There are canonical isomorphisms of $L_p$-vector spaces

$$H^1_f(Q_p, V_f \otimes V_{gh}^{\triangledown\triangle}) \cong H^1_f(K_p, V_f)^{\triangledown\triangle} \otimes V_{gh}^{\triangledown\triangle}$$

and

$$H^1_f(Q_p, V_f \otimes V_{gh}^{\triangledown\triangle}) \cong H^1_s(K_p, V_f)^{\triangledown\triangle} \otimes V_{gh}^{\triangledown\triangle}.$$ 

**Proof.** Let $\chi$ be the quadratic character of $G_{Q_p}$ with $\chi(\text{Fr}_p) = -1$. We describe the isomorphisms in the case in which $E$ has split multiplicative reduction at $p$ and $\triangle \cdot \triangledown = -1$. In this setting,

$$H^1_f(Q_p, V^{\diamond\Box}) = H^1_f(Q_p, V_f^+ \otimes V_{gh}^{\diamond\Box}) \quad \text{by Lemma 3.2}$$

$$= H^1_f(Q_p, Q_p(1) \otimes L_p(\chi)) \quad \text{by Lemma 3.1 (ii)}$$

$$= H^1_f(K_p, Q_p(1) \otimes L_p(\chi))^{G_{Q_p}} \quad \text{by the inflation-restriction exact sequence}$$

$$= (H^1_f(K_p, Q_p(1) \otimes L_p(\chi))^{G_{Q_p}})$$

$$= (H^1_f(K_p, V_f) \otimes L_p(\chi))^{G_{Q_p}} \quad \text{because } \chi|_{G_{K_p}} = 1$$

by (2.12).

The claim follows, because $V_{gh}^{\triangledown\triangle} = L_p(\chi)$ as $G_{Q_p}$-modules and the subspace of $G_{Q_p}$-invariants of $H^1_f(K_p, V_f) \otimes V_{gh}^{\triangledown\triangle}$ is $H^1_f(K_p, V_f)^{\triangledown\triangle} \otimes V_{gh}^{\triangledown\triangle}$. Using similar computations as above, one also verifies that

$$H^1_s(Q_p, V^{\diamond\Box}) \cong (H^1_f(K_p, V_f) \otimes V_{gh}^{\diamond\Box}/(H^1_f(K_p, V_f) \otimes V_{gh}^{\diamond\Box})^{G_{Q_p}} \cong (H^1_s(K_p, V_f) \otimes V_{gh}^{\diamond\Box})^{G_{Q_p}}.$$ 

The remaining cases are proven similarly. \hfill \square

**Remark 3.5.** The logarithm maps of Lemma 3.2 are twisted versions of (2.13). More precisely, $\log_{\alpha\beta}$ may be accordingly recast as

$$H^1_f(Q_p, V_f \otimes V_{gh}^{\alpha\beta}) = E(K_p)^{\beta\alpha} \otimes E^{\alpha\beta}_{gh} \quad \text{for } x \otimes v_{gh}^{\alpha\beta} \quad \text{and } K_p(\chi_E)^{\beta\alpha} \otimes K_p(\chi_E)^{\alpha\beta} \otimes L_p \cong L_p$$

Using Proposition 2.6 we can define the following generators $k^+$ and $k^-$ of $K_p(\chi_E)^+$ and $K_p(\chi_E)^-$ respectively:

$$k^+ := \begin{cases} 1 & \text{if } a = +1; \\ z & \text{if } a = -1; \end{cases} \quad \text{and} \quad k^- := \begin{cases} z & \text{if } a = +1; \\ 1 & \text{if } a = -1. \end{cases}$$

Using 3.5 and 3.6, the latter pairing is

$$\langle v_{gh}^{\alpha\beta}, \eta_{gh}, \omega_{h}\rangle = \frac{1}{\Omega_{h\alpha}(\Theta_{h\alpha})} \langle \omega_{gh}, \eta_{gh}, \omega_{h}\rangle = \frac{1}{\Omega_{gh}} \otimes 1 \in K_p^{\alpha\beta} \otimes L_p,$$
and the right-most isomorphism of (3.11) is given by
\[ K_p(\chi_E)^{\beta_\alpha} \otimes K_p^{\alpha_\beta} \xrightarrow{x \otimes y} K_p(\chi_E)^{+} \xrightarrow{\exp} \mathbb{Q}_p \xrightarrow{k^+} 1. \]

Analogously, the dual exponential maps of Lemma 3.2 are twisted versions of (2.11), and \( \exp^{*}_{\alpha_\beta} \) may be recast under the identification provided by Lemma 3.3 as
\[ \exp^{*} \colon H^1_s(K_p, V) \xrightarrow{x \otimes v_g^\alpha \otimes v_h^\beta} L_p \xrightarrow{\exp^*} \exp^*(x) \cdot (v_g^\alpha v_h^\beta, \eta_g, \omega_h). \]

We will still denote by
\[ \exp^{*}_p : H^1_s(K_p, V_f)^\pm \xrightarrow{\exp} \mathbb{Q}_p \]
the composition of (2.11) with the isomorphism \( K_p(\chi_E)^\pm \cong \mathbb{Q}_p \) sending \( k^\pm \) to 1.

**Corollary 3.6.** Define \( X_+ \in H^1_s(K_p, V_f) \) such that \( \exp^*_p(X_+) = 1 \). The \( p \)-relaxed Selmer group \( \text{Sel}_{(p)}(V) \) admits a basis
\[ \{ \xi^{\alpha_\alpha}, \xi^{\alpha_\beta}, \xi^{\beta_\alpha}, \xi^{\beta_\beta} \} \]
characterized as
\[ \partial_{p}^{\alpha_\alpha} = X_{\alpha_\alpha} \otimes v_g^{\alpha} \otimes v_h^{\alpha}, \quad \partial_{p}^{\alpha_\beta} = X_{\alpha_\beta} \otimes v_g^{\alpha} \otimes v_h^{\beta}, \]
\[ \partial_{p}^{\beta_\alpha} = X_{\beta_\alpha} \otimes v_g^{\beta} \otimes v_h^{\alpha}, \quad \partial_{p}^{\beta_\beta} = X_{\beta_\beta} \otimes v_g^{\beta} \otimes v_h^{\beta}. \]

**Proof.** By Corollary 3.3 and using the decomposition (3.9), there are isomorphisms
\[ H^1_s(K_p, V_f)^\beta \otimes V_{gh}^{\alpha} \xrightarrow{\partial_{p}^{\alpha_\beta}} H^1_s(K_p, V_f)^{\alpha_\beta} \otimes V_{gh}^{\beta} \]
By Lemma 3.3 each of the four components in (3.13) is isomorphic to
\[ H^1_s(K_p, V_f)^{\beta_\alpha} \otimes V_{gh}^{\alpha}, \quad H^1_s(K_p, V_f)^{\alpha_\beta} \otimes V_{gh}^{\beta} \]

respectively. These are in turn isomorphic to \( V_{gh}^{\alpha} \otimes V_{gh}^{\beta} \) via \( \exp^{*}_{\alpha_\alpha} \oplus \exp^{*}_{\alpha_\beta} \oplus \exp^{*}_{\beta_\alpha} \oplus \exp^{*}_{\beta_\beta} \).

4. A SPECIAL VALUE FORMULA FOR THE TRIPLE PRODUCT \( p \)-ADIC \( L \)-FUNCTION IN RANK 0

The aim of this section is to describe the \( p \)-adic \( L \)-value \( L_p(f, g_\alpha, h_\alpha) \) in terms of the basis \( \{ \xi^{\alpha_\alpha}, \xi^{\alpha_\beta}, \xi^{\beta_\alpha}, \xi^{\beta_\beta} \} \) of \( \text{Sel}_{(p)}(V) \) appearing in Corollary 3.6. This section lies within the framework of the exceptional setting of [BSVb] and we recall here the notation and the main results from loc.cit. that we shall use.

4.1. The triple product \( p \)-adic \( L \)-function. If \( \varphi = \sum a_n(\varphi)q^n \) is a modular form of weight \( w \), level \( M \) and nebentype character \( \chi_\varphi \) and \( p \nmid M \), the Hecke polynomial at \( p \) of \( \varphi \) is
\[ x^2 - a_p(\varphi)x + \chi_\varphi(p)p^{w-1} = (x - \alpha_\varphi)(x - \beta_\varphi), \]
where we label the eigenvalues so that \( \text{ord}_p(\alpha_\varphi) \leq \text{ord}_p(\beta_\varphi) \). Recall that if \( \varphi \) is ordinary at \( p \), then \( \alpha_\varphi = \beta_\varphi \) is a \( p \)-adic unit. If \( p \mid M \) we have \( \alpha_\varphi = a_p(\varphi) \).

Since \( g \) has weight \( w = 1 \), both \( \alpha_g \) and \( \beta_g \) are \( p \)-adic units. Recall that we are assuming that \( \alpha_g \neq \beta_g \), thus \( g \) has two different ordinary \( p \)-stabilisations. We denote \( g_\alpha \) and \( g_\beta \) the stabilisation satisfying \( U_pg_\alpha = \alpha_g g_\alpha \) and \( U_pg_\beta = \beta_g g_\beta \) respectively. The same holds for \( h_\alpha \) and \( h_\beta \), we denote
similarly its $p$-stabilisations. Since $E$ has multiplicative reduction at $p$, the level of the newform is divisible by $p$ and it is $p$-stabilised, meaning that $f_{\alpha} = f$.

Let $f = \sum_{n \geq 1} a_n(f) \in \Lambda_f[[q]]$, $g \in \Lambda_g[[q]]$, $h \in \Lambda_h[[q]]$ be the Hida families passing through $f, g, h$, respectively, where $\Lambda_f, \Lambda_g$ and $\Lambda_h$ are finite flat extensions of the Iwasawa algebra $\Lambda$. These Hida families have tame level $N_f/p, N_g, N_h$ and tame character $1, \chi, \hat{\chi}$ respectively.

Let $N := \text{lcm}(N_f/p, N_g, N_h)$. A test vector for $(f, g, h)$ is a triple of Hida families $(\hat{f}, \hat{g}, \hat{h})$ of tame level $N$ such that $\hat{f}$ is of the form $\sum \lambda_d f(q^d) \in \Lambda_f[[q]]$ with $\lambda_d \in \Lambda_f$ where $d$ runs over the divisors of $N/N_f$, and similarly for $\hat{g}$ and $\hat{h}$.

Recall from §3.1 the set $W^*_{fgh}$ of cristalline points. Denote $W^f_{fgh} := \{(k, \ell, m) \in W^*_{fgh} \mid k \geq \ell + m\}$ and define $W^g_{fgh}$ and $W^h_{fgh}$ analogously. Triplets of points in $W^f_{fgh} \cup W^g_{fgh} \cup W^h_{fgh}$ are called unbalanced, and points in the complement $W^{\text{bal}}_{fgh} := W^*_{fgh} \setminus (W^f_{fgh} \cup W^g_{fgh} \cup W^h_{fgh})$ are called balanced.

Let

$$L^f_p(f, g, h), L^g_p(f, g, h), L^h_p(f, g, h) : W_f \times W_g \times W_h \to \mathbb{C}_p$$

be the triple product $p$-adic $L$-functions attached to $(\hat{f}, \hat{g}, \hat{h})$ constructed in [DR14].

4.2. Exceptional cases and improved Euler systems. As introduced in §3.1, let

$$\mathcal{V} = V_f \otimes V_g \otimes V_h \otimes O_{\text{fgh}} \Xi$$

and let

$$\kappa := \kappa(f, g, h) \in H^1(Q, \mathcal{V})$$

be the diagonal class constructed in [BSVb] and [DRA]. By [BSVb Corollary 4.7.1] and [DRA Proposition 3.5.7], the local class

$$\kappa_p := \text{res}_p(\kappa) \in H^1(Q_p, \mathcal{V})$$

belongs to $H^1(Q_p, \text{Fil}^2(\mathcal{V}))$. Moreover, for $\varphi \in \{f, g, h\}$, the $p$-adic $L$-function $L^\varphi_p(\hat{f}, \hat{g}, \hat{h})$ can be recast as the image of $\kappa_p$ under the idoneous Perrin-Riou’s $\Lambda$-adic logarithm: more precisely, [BSVb Theorem A], [DRA Theorem 2.29] assert that

$$L^\varphi_p(\hat{f}, \hat{g}, \hat{h}) = L^\varphi(\text{res}_p(\kappa)),$$

where $L^\varphi : H^1(Q_p, \text{Fil}^2(\mathcal{V})) \to O_{\text{fgh}}$ is the homomorphism described in [BSVb Proposition 4.6.2], [DRA Proposition 3.5.6].

Let $\mathcal{S}$ denote the surface cut out by the equation $k = 2 + \ell - m$ in $W_f \times W_g \times W_h$ and let $F_{\mathcal{S}}$ be the fraction field of the ring of Iwasawa functions on $\mathcal{S}$.

Define the two-variable meromorphic Iwasawa function

$$\mathcal{E}_g := \mathcal{E}_g(f, g, h) := 1 - \frac{a_p(g_\ell)}{\chi(p)a_p(f_{\ell-m+2})a_p(h_m)} \in F_{\mathcal{S}}.$$ 

Recall from the introduction that we have $p \parallel N_f$, $p \nmid N_g N_h$, and $\alpha_g \alpha_h = -a$. More explicitly, we have

$$\beta_g = -\alpha_g, \quad (\alpha_h, \beta_h) = \begin{cases} (-1/\alpha_g, 1/\alpha_g) & \text{if } a = +1; \\ (1/\alpha_g, -1/\alpha_g) & \text{if } a = -1. \end{cases}$$

Hence

$$\mathcal{E}_g(2, 1, 1) = 1 = 1 - \frac{\alpha_g \beta_h}{a} = 0.$$
This fact forces the vanishing of the class \( \kappa(2,1,1) \), as explained in \cite{BSVb} (cf. also Prop. 4.1 below).

Let \( \rho_S : \mathcal{F}_{fgh} \to \mathcal{F}_S \) be the map taking a function \( F(k, \ell, m) \) to its restriction \( F(2 + \ell - m, \ell, 1) \) to the plane \( S \). Denote \( \mathcal{V}|_S := \mathcal{V} \otimes_{\mathcal{F}_{fgh}, \rho_S} \mathcal{F}_S \) and let

\[
\kappa|_S := \rho_{\mathcal{S}, \star}(\kappa) \in H^1(\mathbb{Q}, \mathcal{V}|_S)
\]
denote the restriction of \( \kappa \) to \( S \).

**Proposition 4.1.** There exists a global cohomology class

\[
\kappa^*_g \in H^1(\mathbb{Q}, \mathcal{V}|_S)
\]
satisfying, for each \( \ell, m \in \mathbb{Z}_{\geq 1} \):

\[
(4.3) \quad \kappa(2 + \ell - m, \ell, m) = \mathcal{E}_g(2 + \ell - m, \ell, m) \cdot \kappa^*_g(2 + \ell - m, \ell, m).
\]

**Proof.** This is shown in \cite{BSVb, §8.3} \( \blacksquare \)

Let \( C \subset S \) denote the curve given on which the set of points \((\ell + 1, \ell, 1)\) for \( \ell \in \mathbb{Z}_{\geq 1} \) is dense.

**Proposition 4.2.** There exists an analytic function \( \mathcal{L}_p^f \cdot \star \) on \( C \) satisfying, for each \( \ell \in \mathbb{Z}_{\geq 1} \):

\[
\mathcal{L}_p^f \cdot \star(\ell) = \frac{\langle w_N(f_{\ell+1})^{|p\rangle}, h_g \rangle}{\langle w_N(f_{\ell+1})^{|p\rangle}, w_N(f_{\ell+1})^{|p\rangle} \rangle} \mathcal{E}_g(\ell + 1, \ell, 1) \cdot \mathcal{L}_p^f \cdot \star(\ell) \mod L^\infty.
\]

Moreover,

\[
(4.4) \quad \mathcal{L}_p^f \cdot \star(1) = \frac{1}{2(1 - 1/p)} \exp^\star(\pi \beta \partial_p \kappa^*_g(2, 1, 1)) = \frac{1}{2(1 - 1/p)} \exp^\star(\partial_p \kappa^*_g(2, 1, 1)).
\]

**Proof.** This is proved in \cite{BSVb} Lemma 8.6 + equation (170)]. In particular, combining Theorem \( A \), part 3 of Proposition 8.2 and Lemma 8.6 of loc. cit. it follows that the factor \( \lambda_w \) in equation (170) is \( 1/2(1 - 1/p) \). \( \blacksquare \)

**Remark 4.3.** By (4.2) the factor \( 1 - \frac{a_p(g_{\ell}) \alpha_h}{a_p(f_{\ell+1})} \) does not vanish in a neighborhood of \((2,1,1)\) in \( C \).

4.3. \( I_p(f,g_\alpha,h_\alpha) \) in terms of the basis. In this section we finally obtain, in Theorem 4.7, the formula for \( I_p(f,g_\alpha,h_\alpha) \). Recall from the introduction the Galois representation

\[
V = V_f \otimes V_g \otimes V_h.
\]

Note that the character \( \Xi \) introduced in §3.1 specializes at \((2,1,1)\) to the trivial character, and thus the specialization of the improved \( \Lambda \)-adic cohomology class of Prop. 4.1 yields a global cohomology class

\[
\kappa^*_g(2, 1, 1) \in \text{Sel}(\mathbb{Q})(V).
\]

Recall the periods

\[
(4.5) \quad \Omega_{g_\alpha} \in K_p^{1/\alpha_\nu}, \quad \Theta_{g_\alpha} \in K_p^{1/\beta_\eta}, \quad \mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}} \in K_p^{\beta_\eta/\alpha_\nu}
\]

introduced in §3.1.
Proposition 4.4. We have

$$\kappa_g^*(2,1,1) = \Theta_{g_a} \Theta_{h_a} \frac{2(1 - 1/p)\sqrt{c\sqrt{L(E \otimes \rho, 1)}}}{\pi^2(f,f)} \cdot \xi^{\beta\beta},$$

where $c \in L^\times$ is the product of the local terms appearing in [DLR15, Proposition 2.1 (iii)].

Proof. According to [BSVb, 8.3] one has $\text{res}_p(\kappa_g^*(2,1,1)) \in H^1(\mathbb{Q}_p, \text{Fil}^2(V))$. Recall that

$$H^1(\mathbb{Q}_p, \text{Fil}^2(V)/\text{Fil}^3(V)) = H^1(\mathbb{Q}_p, V^{-} \otimes V^{\beta}),$$

and hence $\partial_M\kappa_g^*(2,1,1) \in H^1(\mathbb{Q}_p, V)$ lies in $H^1(\mathbb{Q}_p, V^{\beta\beta})$. Then, by definition of the basis of $\text{Sel}_{(\rho)}(V)$ described in Corollary 3.6,

$$\kappa_g^*(2,1,1) = \frac{\exp_{\beta\beta}(\partial_M\kappa_g^*(2,1,1))}{\exp_{\beta\beta}(\partial_M\xi^{\beta\beta})} \cdot \xi^{\beta\beta}. $$

We need to compute the numerator and denominator of the previous formula. By (4.4),

$$\exp_{\beta\beta}(\partial_M\kappa_g^*(2,1,1)) = 2(1 - 1/p)\{w_N(f), h\}$$

$$= 2(1 - 1/p)\{w_N(f), w_N(f)\}$$

$$= 2(1 - 1/p)\frac{\sqrt{c\sqrt{L(E \otimes \rho, 1)}}}{\pi^2(f,f)}.$$ 

where the last equality is given by [Ich08]. Besides,

$$\exp_{\beta\beta}(\partial_M\xi^{\beta\beta}) = (\exp_{\beta\beta}(\partial_M\xi^{\beta\beta}) \otimes \eta f \omega_{g_a} \omega_{h_a})$$

$$= \exp_{\beta\beta}(X_{\beta\beta})(\eta^\beta \omega_h \omega_{g_a} \otimes \omega_{h_a})$$

$$= \frac{1}{\Theta_{g_a} \Theta_{h_a}} (\eta_{g_a} \otimes \eta_{h_a} \omega_{g_a} \otimes \omega_{h_a})$$

$$= \frac{1}{\Theta_{g_a} \Theta_{h_a}}.$$ 

The proposition follows. \qed

Let $\Lambda_{\text{cyc}} = \mathbb{Z}_p[[j + 1]]$ denote the usual Iwasawa algebra regarded as the ring of bounded analytic functions on an open disc centered at $j = -1$, and let $\chi_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \Lambda_{\text{cyc}}^\times$ denote the $\Lambda$-adic cyclotomic character characterized by the property that $\nu_j(\chi_{\text{cyc}}) = \chi_{\text{cyc}}^{-1}$ for all $j \in \mathbb{Z}$.

Recall the three-variable Iwasawa algebra $\Lambda_{fgh}$ and set

$$\Lambda_{fgh} = \Lambda_{\text{cyc}} \otimes \Lambda_{\text{cyc}}, \quad \mathbb{V}_{fgh} := \text{Sp}(\Lambda_{fgh}) = \mathcal{W}_{fgh} \times \mathbb{W}, \quad \mathbb{W}_{fgh}^\circ := \mathcal{W}_{fgh}^\circ \times \mathbb{W}^\circ.$$

Let $\Psi : G_{\mathbb{Q}_p} \rightarrow \Lambda_{\text{cyc}}^\times$ denote the unramified character taking $\text{Fr}_p$ to $\frac{a_p(g)}{a_p(f)a_p(h)\chi(p)}$. Define

$$\mathbb{M}^g := \Lambda_{fgh}(\Psi), \quad \mathbb{M}^\prime := \mathbb{M}^g \otimes \Lambda_{\text{cyc}}(\chi_{\text{cyc}})$$

and note that $\mathbb{M}^g$ is the unramified twist of the local Galois representation $\mathbb{V}^g$ introduced in (3.8).

Let $\theta : \Lambda_{fgh} \rightarrow \Lambda_{fgh}$ denote the homomorphism taking a function $F(k, \ell, m, j)$ to its restriction $F(k, \ell, m, (\ell - k - m)/2)$. As shown in the proof of [BSVb, Prop. 4.6.2], there is an isomorphism

$$\theta : \mathbb{M}^{\prime \otimes \theta} \cong \mathbb{V}^g.$$
Define also the $\Lambda$-adic Dieudonné modules
\[
\mathcal{D}^\varphi := (M^\varphi \otimes \hat{\mathbb{Q}}_p)^{G_\varphi}, \quad \overline{\mathcal{D}}^\varphi := \mathcal{D}^\varphi \otimes \Lambda_{\text{cyc}},
\]
where $\hat{\mathbb{Q}}_p$ is the ring of integers of the $p$-adic completion $\hat{\mathbb{Q}}_p$ of the maximal unramified extension of $\mathbb{Q}_p$.

As it directly follows from the above definitions, the specialization of $\overline{\mathcal{M}}^\varphi$ at a point $\underline{x} = (k, \ell, m, j) \in \overline{W}_{\text{fg}}$ is
\[
\overline{M}^\varphi_{\underline{x}} := M^\varphi \otimes \underline{x} L_p = L_p(\Psi(k,\ell,m,\ell,m,-m,j)) = V^+_{f_k} \otimes V^-_{g_r} \otimes V^+_{h_m} (2 - k - m - j).
\]

Bloch–Kato’s logarithm and dual exponential maps give rise to an isomorphism
\[
\begin{align*}
\log_{\alpha\beta} : H^1(\mathbb{Q}_p, L_p(\Psi(k,\ell,m,\ell,m,-m,j))) &\longrightarrow D_{\text{dR}}(L_p(\Psi(k,\ell,m,\ell,m,-m,j))) \quad \text{if } j < 0, \\
\exp_{x}^* : H^1(\mathbb{Q}_p, L_p(\Psi(k,\ell,m,\ell,m,-m,j))) &\longrightarrow D_{\text{dR}}(L_p(\Psi(k,\ell,m,\ell,m,-m,j))) \quad \text{if } j \geq 0.
\end{align*}
\]

In particular, at $\underline{x} = (2,1,1,1)$ this isomorphism becomes the map
\[
\log_{\alpha\beta} : H^1(\mathbb{Q}_p, V^{\alpha\beta}) \longrightarrow L_p,
\]
which is shown (cf. Remark 3.5).

**Proposition 4.5.** There exists a unique homomorphism of $\Lambda_{\text{fg}}$-modules
\[
\tilde{L}_g : H^1(\mathbb{Q}_p, \overline{M}^\varphi) \longrightarrow \overline{\mathcal{D}}^\varphi
\]
such that for all $\underline{Z} \in H^1(\mathbb{Q}_p, \overline{M}^\varphi)$ and $\underline{x} = (k, \ell, m, j) \in \overline{W}_{\text{fg}}$ with
\[
1 - \Psi_{w(\text{Fr}_p)p}^{-j-1} \neq 0,
\]
we have
\[
\tilde{L}_g(\underline{Z}(\underline{x})) = \left(1 - \frac{p^j}{\Psi_{w(\text{Fr}_p)}}\right) \left(1 - \Psi_{w(\text{Fr}_p)p}^{-j-1}\right) \cdot \begin{cases} 
\frac{(-1)^{j+1}}{(j-1)!} \log_{\underline{x}}(\underline{Z}(\underline{x})) & j < 0; \\
\frac{1}{j!} \exp_{x}^*(\underline{Z}(\underline{x})) & j \geq 0.
\end{cases}
\]

**Proof.** The proposition as stated here is [BSVb] Proposition 4.6.1 for $g$, and it is a consequence of [LZ14].

By the proof of [BSVb] Proposition 4.6.2, the relation between $\tilde{L}_g$ and the logarithm $L_g$ that appears in [LZ14] is given, for all $\underline{Z} \in H^1(\mathbb{Q}_p, \overline{M}^\varphi)$ by
\[
L_g(\theta_*(\underline{Z}))(k, \ell, m) = \langle \tilde{L}_g(\underline{Z})(k, \ell, m, (\ell - k - m)/2), \omega_{\ell} \eta_{g_r} \omega_{h_m} \rangle.
\]
Here, if $\underline{x} = (k, \ell, m, (\ell - k - m)/2)$, then
\[
\omega_{\ell} \eta_{g_r} \omega_{h_m} \in \overline{\mathcal{D}}^\varphi \otimes \underline{x} L_p \cong D_{\text{dR}}(V^+_{f_k} \otimes V^-_{g_r} \otimes V^+_{h_m} (\ell - k - m + 4)/2)
\]
are the differentials defined as in [BSVb] (228) and are natural generalizations of the ones introduced in §3.1.

When $j$ is a fixed integer, note that the Euler-like factors appearing above vary analytically. As it will suffice or our purposes, set $j = -1$ and recall the plane $S$ parametrized by points of weights $(2 + \ell - m, \ell, m)$ in $W_{\text{fg}}$. By a slight abuse of notation we also regard $S$ as the plane in $W_{\text{fg}}$ parametrizing points of weights $(2 + \ell - m, \ell, m, -1)$.

Define the homomorphism
\[
\tilde{L}_g^S : H^1(\mathbb{Q}_p, \overline{M}^\varphi_{\mid S}) \longrightarrow \overline{\mathcal{D}}^\varphi_{\mid S}
\]
of $\mathcal{O}_S$-modules given by

\begin{equation}
\tilde{\mathcal{L}}_g^* = \mathcal{E}_g \times \tilde{\mathcal{L}}_{g|S}.
\end{equation}

In light of the above proposition, for any $Z \in H^1(\mathbb{Q}_p, \mathbb{M}_{1|S}^\rho)$ and $\ell \geq 1$ one then has

\begin{equation}
\tilde{\mathcal{L}}_g^*(Z)(\ell) = (1 - \frac{a_p(g\ell)}{a_p(f_{\ell+1})^p}) \cdot (\log(\ell + 1, 1, -1)) (Z(\ell + 1, f, 1, -1), \omega_{\ell+1}, \eta_{g\ell}, \omega_{h\alpha}).
\end{equation}

**Proposition 4.6.** We have

\[ I_p(f, g\alpha, h\alpha) = 1 - \frac{1}{p} \log_{\alpha\beta}(\pi_{\alpha\beta}\kappa_g^*(2, 1, 1)) \mod L^\times. \]

**Proof.** To avoid notational clutter, we continue to denote $S$ the plane in $\mathbb{W}_g$ containing the points $(2 + m, \ell, m, -1)$ as a dense subset. Let $\tilde{\kappa}_{g,p}^* \in H^1(\mathbb{Q}_p, \mathbb{M}_{1|S}^\rho)$ be any lift of $\text{res}_p(\kappa_g^*)$, whose existence is assured by the fact that the map

\[ \theta_* : H^1(\mathbb{Q}_p, \mathbb{W}^\rho) \otimes_{\theta} \mathcal{O}_{fg} \rightarrow H^1(\mathbb{Q}_p, \mathbb{W}^\rho) \]

induced by (4.6) is an isomorphism (cf. the proof of [BSVb, Proposition 4.62]).

Combining Proposition 4.5 with (4.1), (4.3), (4.9) we deduce

\[ \mathcal{L}_g^p(f, g\alpha, h\alpha) = \mathcal{L}_{g|S}(\text{res}_p(\kappa_g^*)) = \mathcal{E}_{g|S} \cdot \tilde{\mathcal{L}}_{g|S}(\text{res}_p(\kappa_g^*)) \]

\[ = \mathcal{E}_{g|S} \cdot (\tilde{\mathcal{L}}_{g|S}(\tilde{\kappa}_{g,p}^*), \omega_{\ell+1}, \eta_{g\ell}, \omega_{h\alpha}). \]

By (4.10), (4.11) and (4.12), the latter quantity is equal to

\[ (\tilde{\mathcal{L}}_g^*(\tilde{\kappa}_{g,p}^*), \omega_{\ell+1}, \eta_{g\ell}, \omega_{h\alpha}) = (1 - \frac{a_p(g\ell)}{a_p(f_{\ell+1})^p}) \cdot \log_{\alpha\beta}(\pi_{\alpha\beta}\kappa_g^*(\ell + 1, 1, 1)). \]

Since $I_p(f, g\alpha, h\alpha) = \mathcal{L}_g^p(f, g\alpha, h\alpha)(2, 1, 1)$ by definition, the proposition follows by using (4.2). \hfill \Box

Define the local point

\[ P_{\alpha\beta} \in E(K_p)^{\alpha\beta} \]

as the one satisfying

\begin{equation}
\pi_{\alpha\beta} \text{ res}_p(\xi^{\beta\beta}) = \delta_p(P_{\alpha\beta}) \otimes v_g^\alpha \otimes v_h^\beta \in \left( H^1_f(K_p, V_f) \otimes V_{gh}^{\alpha\beta} \right)^{G_{ap}},
\end{equation}

where $\delta_p : E(K_p) \rightarrow H^1_f(K_p, V_f)$ is the Kummer map and $E(K_p)^{\alpha\beta} := E(K_p)^{\alpha\beta} \otimes L_p$.

**Theorem 4.7.** We have

\[ I_p(f, g\alpha, h\alpha) = \frac{2(1 - 1/p)^2 \sqrt{c}}{\pi^2(f, f)} \times \sqrt{L(E \otimes \rho, 1)} \times \frac{\log_p(P_{\alpha\beta})}{\mathcal{L}_{g\alpha}} \mod L^\times. \]
Proof. By Proposition \ref{thm:prop}, \( \kappa_\pi^*(2, 1, 1) = \frac{\Theta_{g_\pi} \Theta_{h_\pi} 2(1 - 1/p)\sqrt{cL(E \otimes \rho, 1)}}{\pi^2(f, f)} \xi^{\beta \beta} \). It thus follows from Proposition \ref{thm:pcc} that

\[
I_p(f, g_\alpha, h_\alpha) = \left(1 - \frac{1}{p}\right) \log_{\alpha \beta}(\pi_{\alpha \beta} \kappa_\pi^*(2, 1, 1))
\]

\[
= \frac{\Theta_{g_\pi} \Theta_{h_\pi} \sqrt{c} (1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2(f, f)} \log_{\alpha \beta}(\pi_{\alpha \beta} \xi^{\beta \beta})
\]

\[
= \frac{\Theta_{g_\pi} \Theta_{h_\pi} \sqrt{c} (1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2(f, f)} \log_p(P_{\alpha \beta})^\alpha \otimes \eta_{h_\alpha} \omega_{h_\alpha}
\]

\[
= \frac{\Theta_{g_\pi} \Theta_{h_\pi} \sqrt{c} (1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2(f, f)} \log_p(P_{\alpha \beta}).
\]

The theorem follows in light of \eqref{eq:principal}. \hfill \Box

5. A special value formula for \( I_p(f, g_\alpha, h_\alpha) \) in terms of Kolyvagin classes

The aim of this section is to relate Theorem \ref{thm:main} to Bertolini-Darmon’s Kolyvagin classes constructed in \cite{BD97} §6.

5.1. Decomposition of the representation \( V \) and consequences. The underlying reason why the \( p \)-adic \( L \)-value \( L_p(\hat{f}, g_\pi, h_\pi)(2, 1, 1) \) may be related to Kolyvagin classes relies on the decomposition of the Galois representation \( V_g \otimes V_h \) as the direct sum of a pair of 2-dimensional representations induced from characters of an imaginary quadratic field.

More precisely, recall that \( g \) and \( h \) are theta series of two finite order Hecke characters \( \psi_g, \psi_h : G_K \rightarrow \mathbb{C}^\times \). As explained in the introduction, there is a decomposition

\[
(5.1) \quad V_g \otimes V_h \cong V_{\psi_1} \oplus V_{\psi_2}
\]

where \( \psi_1 := \psi_g \psi_h, \psi_2 := \psi_g \psi_h' \) and \( V_{\psi_i} := \text{Ind}_{G_K}^{G_\mathbb{Q}}(\psi_i) \).

For \( i = 1, 2 \), \( \psi_i \) is a ring class character, i.e. it factors through the absolute Galois group of a number field \( H_i \) which is a ring class field of \( K \) of some conductor \( c_i \). Recall we have assumed that \( p \) does not divide the conductor of \( \psi_g \) and \( \psi_h \), hence that the characters \( \psi_1 \) and \( \psi_2 \) are unramified at \( p \), and this implies that \( p \nmid c_1 c_2 \). The Artin representation

\[
\rho = \rho_g \otimes \rho_h : G_\mathbb{Q} \rightarrow \text{GL}(V_g \otimes L_p V_h)
\]

thus factors through \( \text{Gal}(H/K) \), where \( H \) is the ring class field of \( K \) of conductor \( c := \text{len}(c_1, c_2) \). Moreover, since the prime \( p \) is inert in \( K \) and \( p \nmid c \), the principal ideal \( pQ_K \) splits completely in \( H \). Fix once and for all a prime \( p \) of the field \( H \) dividing \( p \) and denote \( H_p \) the completion of \( H \) at \( p \). Then \( H_p = K_p \) and, in particular, the restriction of \( \psi_i \) to the decomposition group \( G_{K_p} = G_{H_p} \) is trivial for \( i = 1, 2 \).

Recall that \( \text{Fr}_p \) acts on \( V_g \) with eigenvalues \( \alpha_g = \lambda, \beta_g = -\lambda \) and on \( V_h \) with eigenvalues

\[
(\alpha_h, \beta_h) = \begin{cases} 
(1/\lambda, 1/\lambda) & \text{if } a = +1; \\
(1/\lambda, -1/\lambda) & \text{if } a = -1,
\end{cases}
\]

so we always have \( \alpha_g \alpha_h = -\alpha \). As in the previous sections, choose an \( L \)-basis of eigenvectors \( v_\alpha^g, v_\beta^g \) of \( V_g \) and likewise a basis \( v_\alpha^h, v_\beta^h \) of \( V_h \).
Note also that Fr$_p$ acts on $V_{\psi_1}$ and $V_{\psi_2}$ with eigenvalues $\pm 1$.

**Lemma 5.1.** There is a $G_Q$-equivariant isomorphism

$$\Psi : V_g \otimes V_h \cong V_{\psi_1} \oplus V_{\psi_2}.$$  

Moreover, $V_{\psi}$ admits an $L$-basis $\{v^+_1, v^-_1\}$ of eigenvectors with relative eigenvalues $\{+1, -1\}$ satisfying:

- if $a = +1$, then
  $$\Psi(v^a_g \otimes v^a_h) = v^+_1 - v^-_2, \quad \Psi(v^\beta_g \otimes v^\beta_h) = v^+_1 + v^-_2, \quad \Psi(v^\alpha_g \otimes v^\alpha_h) = v^+_1 + v^-_2.$$  

- If $a = -1$, then
  $$\Psi(v^a_g \otimes v^a_h) = v^+_1 + v^-_2, \quad \Psi(v^\beta_g \otimes v^\beta_h) = v^+_1 - v^-_2, \quad \Psi(v^\alpha_g \otimes v^\alpha_h) = v^+_1 - v^-_2.$$  

**Proof.** Let $\psi$ be a finite order Hecke character of $K$, let $\varphi := \theta(\psi)$ and denote Fr$_p \in G_Q \setminus G_K$ a Frobenius element at $p$. As explained in [DLR17 §2.2], we can choose a basis $\{u_\varphi, v_\varphi\}$ of $V_\varphi$ such that, if

$$\rho_\varphi : G_Q \longrightarrow GL(V_\varphi)$$

is the $\ell$-adic representation attached to $\varphi$, then $v_\varphi = \rho_\varphi(Fr_p)u_\varphi$ and with respect to this basis,

$$\rho_\varphi(\sigma) = \begin{pmatrix} \psi(\sigma) & 0 \\ 0 & \psi'(\sigma) \end{pmatrix} \text{ for } \sigma \in G_K; \quad \rho_\varphi(\tau) = \begin{pmatrix} 0 & \eta(\tau) \\ \eta'(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_Q \setminus G_K.$$

Here $\psi'$ denotes the character defined by $\psi'(\sigma) := \psi(Fr_p \sigma Fr_p^{-1})$, and $\eta$ is a function on $G_Q \setminus G_K$. Recall that the characteristic polynomial for the action of Fr$_p$ on $V_\varphi$ is $x^2 - a_p(\varphi)x + \chi_\varphi(p) = x^2 - \varepsilon_\varphi(p)$, where $\chi_\varphi = \chi_K \varepsilon_\varphi$ and $\varepsilon_\varphi$ is the central character of $\varphi$. We can always choose $\{u_\varphi, v_\varphi\}$ such that $\rho_\varphi(Fr_p) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$, with $\zeta^2 = \varepsilon_\varphi(p)$. Then a basis of eigenvectors for the action of Fr$_p$ on $V_\varphi$ is

$$\{v^\varphi_\zeta = u_\varphi + v_\varphi, \quad v^\varphi_{-\zeta} = u_\varphi - v_\varphi\}$$

with eigenvalue $\zeta$, $-\zeta$ respectively. Using this notation we obtain the basis

$$(u_g, v_g), \quad (u_h, v_h), \quad (u_1, v_1), \quad (u_2, v_2)$$

of $V_g, V_h, V_{\psi_1}, V_{\psi_2}$ where we denoted $u_i := u_{\theta(\psi_i)}$ and $v_i := v_{\theta(\psi_i)}$. The corresponding matrix $\rho_\varphi(Fr_p)$ is

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/\lambda \\ 1/\lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Consider the basis

$$(u_g \otimes u_h, u_g \otimes v_h, v_g \otimes u_h, v_g \otimes v_h)$$

of $V_g \otimes V_h$, and the basis

$$(u_1, v_1, u_2, v_2)$$

of $V_{\psi_1} \oplus V_{\psi_2}$. Comparing the matrices above, we conclude that the isomorphism $\Psi$ is given by the rule

$$u_g \otimes u_h \mapsto u_1, \quad u_g \otimes v_h \mapsto u_2, \quad v_g \otimes u_h \mapsto v_2, \quad v_g \otimes v_h \mapsto v_1.$$
Moreover, since the eigenvalues of \( \text{Fr}_p \) acting on \( V_g \) and \( V_h \) are \( \lambda, -\lambda, 1/\lambda, 1/\lambda \), a basis of eigenvectors for \( V_g \otimes V_h \) is

\[
\begin{align*}
\psi_g^\lambda \otimes \psi_h^1/\lambda &= (u_g + v_g) \otimes (u_h + v_h), \\
\psi_g^{-\lambda} \otimes \psi_h^{1/\lambda} &= (u_g - v_g) \otimes (u_h + v_h), \\
\psi_g^\lambda \otimes \psi_h^{-1/\lambda} &= (u_g + v_g) \otimes (u_h - v_h), \\
\psi_g^{-\lambda} \otimes \psi_h^{-1/\lambda} &= (u_g - v_g) \otimes (u_h - v_h),
\end{align*}
\]

with eigenvalues \( +1, -1, -1, +1 \) respectively. On the other hand, a basis of eigenvectors for \( V_{\psi_i} \oplus V_{\psi_2} \) is

\[
\begin{align*}
(v_1^+ := u_1 + v_1, & \quad v_1^- := u_1 - v_1, & \quad v_2^+ := u_2 + v_2, & \quad v_2^- := u_2 - v_2)
\end{align*}
\]

In conclusion,

\[
\begin{align*}
\psi_g^\lambda \otimes \psi_h^1/\lambda \to v_1^+ + v_2^+, & \quad \psi_g^{-\lambda} \otimes \psi_h^{1/\lambda} \to v_1^- - v_2^-, & \quad \psi_g^\lambda \otimes \psi_h^{-1/\lambda} \to v_1^- + v_2^-, & \quad \psi_g^{-\lambda} \otimes \psi_h^{-1/\lambda} \to v_1^+ - v_2^+.
\end{align*}
\]

Recall that

\[ V_{\psi_i} = \text{Ind}_{G_K}^{G_Q}(\psi_i) = \{ v : G_Q \to L(\psi) \mid v(\sigma \tau) = \psi_i(\sigma)v(\tau) \forall \sigma \in G_K, \tau \in G_Q \} = L \oplus L \cdot \text{Fr}_p. \]

The proof of the previous lemma shows that we may choose the vectors \( v_i^\pm \) such that

\[
v_i^+(1) = v_i^-(1) = 1.
\]

Any other choice of basis would be of the form \((a_i v_i^+, b_i v_i^-)\) for some scalars \( a_i, b_i \in L^\times \), and would yield the same formula for \( I_p(f, g_a, h_a) \) up to a scalar in \( L^\times \). This is fine because that is precisely the ambiguity we are working with in our framework, as explained in the introduction.

As in previous sections, denote

\[
V := V_f \otimes V_g \otimes V_h, \quad V_1 := V_f \otimes V_{\psi_1}, \quad V_2 := V_f \otimes V_{\psi_2}
\]

The map \( \Psi \) of Lemma 5.1 induces the isomorphism

\[
(5.3) \quad \Psi : V \xrightarrow{\cong} V_1 \oplus V_2.
\]

There is then a decomposition of the Selmer group

\[
(5.4) \quad \Psi_* : \text{Sel}_p(V) \xrightarrow{\cong} \text{Sel}_p(V_1) \oplus \text{Sel}_p(V_2),
\]

and the analogous decomposition holds for the relaxed and the strict Selmer groups associated to \( V \). By Artin’s formalism, decomposition (5.3) yields the factorization

\[
L(E, \rho, s) = L(E, \psi_1, s)L(E, \psi_2, s)
\]

decomposition of classical \( L \)-series. As explained in §3.2 Assumption 1.1 on the analytic rank of \( E \otimes \rho \) implies that

\[
\text{Sel}_p(V_1) = \text{Sel}_p(V_2) = 0.
\]

Denote \( V_i^\pm \), the eigenspace of \( V_{\psi_i} \) on which \( \text{Fr}_p \) acts as \( \pm 1 \), and let

\[
V_i^\pm := V_f \otimes V_{\psi_i}.
\]

Lemma 5.2. There are isomorphisms

\[
(5.5) \quad H^1_i(Q_p, V_i^\pm) \cong H^1_i(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm \quad \text{and} \quad H^1_s(Q_p, V_i^\pm) \cong H^1_s(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm.
\]

The Bloch–Kato logarithm and dual exponential yield isomorphisms

\[
(5.6) \quad H^1_i(Q_p, V_i^\pm) \xrightarrow{\log^\pm} L_p, \quad H^1_s(Q_p, V_i^\pm) \xrightarrow{\exp^\pm} L_p.
\]
Moreover, for \( i \in \{1, 2\} \), there are isomorphisms
\[
\text{Sel}_{(p)}(V_i) \xrightarrow{\partial_p} H^1_{dR}(\mathbb{Q}_p, V_i) \approx \mathbb{Z}_p^2.
\]

**Proof.** The isomorphisms are obtained as in the proofs of Lemma 3.4, Lemma 3.2 and Corollary 3.3 using (1.4).

Similarly as in Remark 3.5, the Bloch–Kato maps (3.4) are related to the logarithm and the dual exponential of the representation \( V_f \) as follows. Since \( V_{\psi_i} \) is the Galois representation attached to the theta series \( \theta(\psi_i) \), (3.3) yields a pairing
\[
\langle \ , \ : D_{dR}(V_{\psi_i}^+) \times D_{dR}(V_{\psi_i}^-) \rightarrow D_{dR}(L_p(\chi_i))
\]
where \( \chi_i \) is the Nebentype character of \( \theta(\psi_i) \). Since \( V_{\psi_i} \) is unramified, we have
\[
D_{dR}(V_{\psi_i}^\pm) = (V_{\psi_i}^\pm \otimes K_p)^G = V_{\psi_i}^\pm \otimes K_p.
\]
One can introduce differentials \( \omega_i^\pm \in D_{dR}(V_{\psi_i}^\pm) \) as in (3.4), and elements \( \Omega_i^\pm \in K_p \) characterised by the following relation
\[
v_i^\pm \otimes \Omega_i^\pm = \omega_i^\pm.
\]
Thus (5.8) in turn induces a pairing
\[
\langle \ , \ : V_{\psi_i}^+ \times D_{dR}(V_{\psi_i}^-) \rightarrow L_p
\]
by setting
\[
\langle v_i^+, \omega \rangle := \frac{1}{\Omega_i^+}(\omega_i^+, \omega)
\]
for all \( \omega \in D_{dR}(V_{\psi_i}^-) \). Hence we have
\[
\exp_j^+(x \otimes v) = \exp^+_{V_j}(x) \cdot \langle v, \omega_i^\pm \rangle
\]
for any \( x \in H^1_{dR}(K_p, V_f)^\pm \), \( v \in V_{\psi_i}^\pm \).

Recall from Lemma 5.1 the basis \( \{v_i^+, v_i^-\} \) for \( V_{\psi_i} \), and from Corollary 3.6 the local classes \( X_\pm \in H^1_{dR}(K_p, V_f)_L \) such that \( \exp_j^+(X_\pm) = 1 \).

**Corollary 5.3.** For \( i \in \{1, 2\} \), \( \text{Sel}_{(p)}(V_i) \) admits a basis \( \{\xi_i^+, \xi_i^-\} \) characterized as
\[
\partial_p \xi_i^\pm = X_\pm \otimes v_i^\pm.
\]

under the identifications given in (5.5).

**Proof.** The statement follows after applying Lemma 5.2 and using the same argument as in the proof of Corollary 3.6.

**Proposition 5.4.** Let
\[
\Psi : \text{Sel}_{(p)}(V) \rightarrow \text{Sel}_{(p)}(V_1) \oplus \text{Sel}_{(p)}(V_2)
\]
denote the isomorphism induced by (5.3).

- If \( a = +1 \) then
  \[
  \Psi_a \xi_{\alpha} = \xi_1^- - \xi_2^-, \quad \Psi_a \xi_{\alpha \beta} = \xi_1^+ + \xi_2^+, \quad \Psi_a \xi_{\beta}^\alpha = \xi_1^+ - \xi_2^-, \quad \Psi_a \xi_{\beta}^\beta = \xi_1^- + \xi_2^+.
  \]
- If \( a = -1 \) then
  \[
  \Psi_a \xi_{\alpha} = \xi_1^+ + \xi_2^+, \quad \Psi_a \xi_{\alpha \beta} = \xi_1^- - \xi_2^-, \quad \Psi_a \xi_{\beta}^\alpha = \xi_1^+ + \xi_2^-, \quad \Psi_a \xi_{\beta}^\beta = \xi_1^- - \xi_2^+.
  \]
Proof. Combining (5.7) and Lemma 3.1 we obtain a commutative diagram
\[
\begin{array}{c}
\text{Sel}_p(V) \xrightarrow{\Psi} \text{Sel}_p(V_1) \oplus \text{Sel}_p(V_2) \\
\partial_p \uparrow \quad \partial_p \oplus \partial_p \\
\text{H}^1_p(\mathbb{Q}_p, V) \xrightarrow{\Psi} \text{H}^1_p(\mathbb{Q}_p, V_1) \oplus \text{H}^1_p(\mathbb{Q}_p, V_2)
\end{array}
\]

where each arrow is an isomorphism. Moreover, by Lemma 3.4 and Lemma 5.2 we also have the commutative diagram
\[
\begin{array}{c}
\text{H}^1_p(\mathbb{Q}_p, V) \xrightarrow{\Psi} \text{H}^1_p(\mathbb{Q}_p, V_1) \oplus \text{H}^1_p(\mathbb{Q}_p, V_2) \\
\downarrow \\
(\text{H}^1_p(K_p, V_f) \otimes V_{gh})^{G_{op}} \xrightarrow{\Psi} (\text{H}^1_p(K_p, V_f) \otimes V_1)^{G_{op}} \oplus (\text{H}^1_p(K_p, V_f) \otimes V_2)^{G_{op}}
\end{array}
\]

where each arrow is an isomorphism and the bottom horizontal arrow is \(1 \otimes \Psi\). By definition,

\[\partial_p \xi^\alpha = X_{\alpha} \otimes v_\alpha^\beta \otimes v_h^\beta \in (\text{H}^1_p(K_p, V_f) \otimes V_{gh}^{\alpha \beta})^{G_{op}} \subseteq (\text{H}^1_p(K_p, V_f) \otimes V_{gh})^{G_{op}}.\]

By (1.4) we have

\[X_{\alpha} = \begin{cases} 
X_+ & \text{if } \alpha_g \cdot \alpha_h = +1; \\
X_- & \text{if } \alpha_g \cdot \alpha_h = -1,
\end{cases}\]

hence we deduce

\[(1 \otimes \Psi)(\partial_p \xi^\alpha) = \begin{cases} 
X_{\alpha} \otimes (v_1^+ + v_2^+) = \partial_p \xi_1^+ + \partial_p \xi_2^+ & \text{if } \alpha_g \cdot \alpha_h = +1; \\
X_{\alpha} \otimes (v_1^- - v_2^-) = \partial_p \xi_1^- - \partial_p \xi_2^- & \text{if } \alpha_g \cdot \alpha_h = -1.
\end{cases}\]

Similar computations prove the remaining cases.

\[\square\]

Lemma 5.5. Let \(\delta_p : E(K_p) \otimes \mathbb{Q}_p \longrightarrow \text{H}^1_p(K_p, V_f)\) denote Kummer’s map.

- If \(a = +1\), there is a local point \(P^+ \in E(K_p)^+ \otimes \mathbb{Q}_p\) such that
  \[
  \pi_1^+ \text{res}_p(\xi_1^-) = \delta_p P^+ \otimes v_1^+; \quad \pi_2^+ \text{res}_p(\xi_2^-) = \delta_p P^+ \otimes v_2^+.
  \]

- If \(a = -1\), there is a local point \(P^- \in E(K_p)^- \otimes \mathbb{Q}_p\) such that
  \[
  \pi_1^- \text{res}_p(\xi_1^-) = \delta_p P^- \otimes v_1^-; \quad \pi_2^- \text{res}_p(\xi_2^-) = \delta_p P^- \otimes v_2^-.
  \]

Proof. We prove the claim when \(a = -1\), as the other case works similarly. By Proposition 5.3

\[
\Psi \text{res}_p(\xi^\beta) = \text{res}_p(\xi_1^+) = \text{res}_p(\xi_2^+) \in \text{H}^1_p(\mathbb{Q}_p, V_1) \oplus \text{H}^1_p(\mathbb{Q}_p, V_2)
\]

Write

\[
\pi_{\alpha \beta} \text{res}_p(\xi^\beta) = Q^- \otimes v_\alpha^\beta \otimes v_h^\beta \in \text{H}^1_p(K_p, V_f)^- \otimes V_{gh}^{\alpha \beta}.
\]

Then, by Proposition 5.3 and (5.5)

\[
(5.12) \quad \Psi, \pi_{\alpha \beta} \text{res}_p(\xi^\beta) = Q^- \otimes (v_1^- - v_2^-) = Q^- \otimes v_1^- - Q^- \otimes v_2^-
\]

in

\[
(\text{H}^1_p(K_p, V_f) \otimes (V_{\psi_1} \oplus V_{\psi_2}^-))^{G_{op}} \cong \text{H}^1_p(K_p, V_f^-) \oplus \text{H}^1_p(K_p, V_f^-).
\]
On the other hand, by definition of the basis $\xi^\pm$ and using (5.5),
\[ \pi_1 \res_p(\xi^+_1) - \pi_2 \res_p(\xi^-_2) \in H^1_p(Q_p, V^-_1) \oplus H^1_f(Q_p, V^-_2) \cong (H^1_f(K, V_f) \otimes (V^\pm_1 \oplus V^\pm_2))^G_{\rho_\psi}. \]
More precisely, we can write
\[ (5.13) \quad \pi_1 \res_p(\xi^+_1) - \pi_2 \res_p(\xi^-_2) = P^+_1 \otimes v^+ - P^-_2 \otimes v^-. \]
for points $P^\pm_i \in E(K_i)^-$. Hence, comparing (5.13) and (5.12) we deduce that $Q_\pm = P^+_1 = P^-_2$. □

**Theorem 5.6.** Let $P^{\pm} \in E(K_i)^{\pm}$ be the local points of Lemma 5.2. Then
\[ I_p(f, g, h) = \frac{\sqrt{c} \cdot 2(1 - 1/p)^2 \sqrt{L(E \otimes \rho_1, 1)}}{\pi^2(f, f)} \times \frac{1}{L_{g_{a}} \times \log_p(P^a)} \mod L^{\times}. \]

**Proof.** Combine Theorem 4.7 with (4.12) and Lemma 5.5 and use the relation between the Bloch–Kato logarithms given in (3.11).

### 5.2. Kolyvagin classes

In this section we recall briefly the construction and main properties of the Kolyvagin classes defined in [BD97] §6.

Assume for simplicity that $\mathcal{O}_K = \{\pm 1\}$. Let $K$ be a ring class field of $K$ of conductor prime to $p$, and recall the fields $H \subseteq F_{m-1} \subseteq H(p^m)$ defined in (2.1). For each $m$, fix a generator $\sigma_m$ of $G_m := \text{Gal}(F_m/H)$ such that the image of $\sigma_m$ via the projection
\[ G_m \to G_m / \text{Gal}(F_m/F_{m-1}) \cong G_{m-1} \]
equals $\sigma_{m-1}$. Moreover, the elements $\{\sigma_m\}_m$ can be chosen so that $\sigma_{\text{ant}} = (\sigma_m|_{G_{F_m,p}})_m$, where $\sigma_{\text{ant}}$ is the generator of $\Gamma_{\text{ant}}$ we fixed in (2.2). In [BD96], Bertolini and Darmon constructed a collection of points $\{\tilde{\alpha}_m \in E(H(p^m))\}_m$ (in the notation of [BD96], $\tilde{\alpha}_m = \alpha_m(1)$) such that
\[ N_{H(p^m)/H} \tilde{\alpha}_m = 0. \]
For each $m \geq 1$, let
\[ \alpha_m := N_{H(p^{m+1})/F_m} \tilde{\alpha}_{m+1} \in E(F_m). \]
To each point $\alpha_m$, in [BD97] the authors attached a Kolyvagin class $K_m \in H^1(H, E[p^m])$ in the following way. The Kolyvagin derivative operator is the element $D_m := \sum_{i=1}^{p^m-1} i\sigma^i_m \in \mathbb{Z}[G_m]$.

**Lemma 5.7.** The Kolyvagin derivative satisfies the following equality in $\mathbb{Z}[G_m]$
\[ (\sigma_m - 1)D_m = p^m - N_{F_m/H}. \]

**Proof.** Note that $D_n = -\sum_{i=1}^{p^n} \sigma^i_m - 1$. Then $(\sigma_m - 1)D_n = -\sum_{i=1}^{p^n} (\sigma_m - 1) = -N_{F_m/H} + p^n$. □

Define $P_m := D_m \alpha_m \in E(F_m)$.

**Lemma 5.8.**

1. The point $P_m \in E(F_m)$ only depends on the choice of the generator $\sigma_m$ of $G_m$ up to the multiplication by an element $a_m \in \{1, \ldots, p^m - 1\}$. Moreover, if we choose the elements $\{\sigma_m\}_m$ to be compatible in the sense of (5.14), then $a_{m-1} \equiv a_m \mod p^{m-1}$, i.e. $(a_m)_m \in \lim_{m} (\mathbb{Z}/p^m) = \mathbb{Z}_p^\times$.

2. The class $[P_m]$ of $P_m$ in $E(F_m)/p^m E(F_m)$ is fixed by $G_m = \text{Gal}(F_m/H)$.

**Proof.**

1. Let $\sigma'_m$ any generator of $G_m \cong \mathbb{Z}/p^m\mathbb{Z}$ and define $D'_m := \sum i(\sigma'_m)^i$. Then there exists an element $\sigma^*_m \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ such that $\sigma^*_m = \sigma^m_m$, where $a_m \in \{1, \ldots, p^m - 1\}$. Then $D'_m = \sum i\sigma^*_m = a_m^{-1} D_m$. 

2. The class $[P_m]$ of $P_m$ in $E(F_m)/p^m E(F_m)$ is fixed by $G_m = \text{Gal}(F_m/H)$.
(2) It suffices to prove that the class of \(D_m \alpha_m\) is fixed by the generator \(\sigma_m\) of \(G_m\), i.e. that 
\[\sigma_n D_m \alpha_n - D_n \alpha_n \in p^n E(F_n).\] Combining (5.16) and (5.15) we obtain 
\[(\sigma_m - 1) D_m \alpha_m = p^m \alpha_m - N_{F_m/H} \alpha_m = p^m \alpha_m.\]

\[\square\]

The inflation-restriction and Kummer exact sequences yield the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (E(H)/p^m E(H)) & \rightarrow & H^1(H, E[p^m]) & \rightarrow & H^1(H, E)[p^m] & \rightarrow & 0 \\
0 & \rightarrow & (E(F_m)/p^m E(F_m))^G_m & \rightarrow & H^1(F_m, E[p^m])^G_m & \rightarrow & H^1(F_m, E)[p^m] & \rightarrow & 0 \\
& & & \downarrow \text{Inf} & & & \downarrow \text{Res} & & \\
& & & H^2(F_m/H, E(F_m)[p^m]) & & & & & \\
\end{array}
\]

whose rows and columns are exact. Let \(C\) be a constant annihilating \(E(F_m)[p^m]\), which can be chosen to be independent of \(m\) by [BD97] Lemma 6.3. By Lemma 5.8 we can consider the element 
\([p_m] \in (E(F_m)/p^m E(F_m))^{G_m}\), and we define

\[K_m := C K^*_m \in H^1(H, E[p^m]),\]

where \(K_m^*\) is a preimage via \(\text{Res}\) of \(-C \delta_p[p_m]\). For each \(m\), denote

\[G_m := \text{Gal}(F_m/K) \supseteq G_m := \text{Gal}(F_m/H)\]

and fix a complete set \(S = S_m = \{\sigma_1^{(m)}, \ldots, \sigma_k^{(m)}\}\) of representatives for the \(G_m\)-cosets in \(G_m\). We require these sets to satisfy the following compatibility condition. Let \(\pi_m : G_m \rightarrow G_{m-1}\) denote the projection induced by the inclusion \(F_{m-1} \subseteq F_m\). Then

\[\pi_m(\sigma_i^{(m)}) = \sigma_i^{(m-1)}\]

for all \(i\) and all \(m\). In other words, under this compatibility condition we can consider a complete set \(\tilde{S} = \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m\}\) of representatives of \(G_H\)-cosets in \(G_K\) such that, under the projection \(\tilde{\pi}_m : G_K \rightarrow \text{Gal}(F_m/K)\) we have

\[\tilde{\pi}_m(\tilde{\sigma}_i) = \sigma_i^{(m)}\]

**Lemma 5.9.** The classes \(K_m\) form a compatible system

\[K := (K_m)_m \in \lim_{\rightarrow m} H^1(H, E[p^m]) = H^1(H, T_p E) \subset H^1(H, V_f).\]

**Proof.** The norm map \(N_{F_{m+1}/F_m} : F_{m+1} \rightarrow F_m\) induces a map \(N_{F_{m+1}/F_m} : E(F_{m+1}) \rightarrow E(F_m)\). By [BD96] §2.45, we have

\[N_{F_{m+1}/F_m}(\alpha_{m+1}) = \alpha_m.\]

Consider the composition

\[f_{m+1} : E(F_{m+1})/p^{m+1} E(F_{m+1}) \xrightarrow{N_{F_{m+1}/F_m}} E(F_m)/p^{m+1} E(F_m) \rightarrow E(F_m)/p^m E(F_m)\]
where the second morphism is the projection given by the inclusion \( p^{m+1}E(F_m) \subseteq p^m E(F_m) \).

Hence (5.17) implies that
\[
(5.18) \quad f_{m+1}([\alpha_{m+1}]) = [\alpha_m].
\]

Consider the following commutative diagram
\[
\begin{array}{ccc}
E(F_{m+1})/p^{m+1}E(F_{m+1}) & \xrightarrow{\delta_p} & H^1(F_{m+1}, E[p^{m+1}]) \\
\downarrow f_{m+1} & & \downarrow f_{m+1, *} \\
E(F_m)/p^mE(F_m) & \xrightarrow{\delta_p} & H^1(F_m, E[p^m]).
\end{array}
\]

Here \( f_{m+1, *} \) is the map induced in cohomology by \( f_{m+1} \), i.e. it is the composition
\[
H^1(F_{m+1}, E[p^{m+1}]) \xrightarrow{N_{F_{m+1}/F_m}^{-1}} H^1(F_m, E[p^{m+1}]) \xrightarrow{P} H^1(F_m, E[p^m]),
\]
where the last map is the composition with the multiplication by \( p \) map \( E[p^{m+1}] \xrightarrow{P} E[p^m] \). Then (5.18) implies that
\[
f_{m+1, *}[D_{m+1}\alpha_{m+1}] = \delta_p[D_m\alpha_m].
\]

Recall the groups \( \Phi_{m, p} \) and \( \Phi_{\infty, p} \) defined in §2. By Lemma 2.2 and Proposition 2.3, there is an injection
\[
\text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p) \subseteq H^1_s(K_p, V_f).
\]

Recall the choice \( \sigma_m \) of generator of \( G_m \); if we still denote \( \sigma_m \) its restriction to \( \text{Gal}(F_{m, p}/H_p) = \text{Gal}(F_{m, p}/K_p) \), then \( \Gamma_{\text{ant}} \) is generated by \( \sigma_{\text{ant}} := (\sigma_m)_m \).

Recall the map \( \partial_p : H^1(H, V_f) \to H^1_s(K_p, V_f) \) onto the singular quotient.

**Proposition 5.10.** The class \( K \) lies in \( \text{Sel}_{(p)}(H, V_f) \) and \( \partial_p K \in \text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p) \). Moreover, if \( \bar{\alpha}_m \) denotes the image of \( \alpha_m \) in \( \Phi_{m, p} \) and \( \bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty, p} \), then
\[
\partial_p K(\sigma_{\text{ant}}) = \bar{\alpha}.
\]

**Proof.** [BD97] Proposition 6.9, 1.2. □

5.3. \( I_p(f, g_\alpha, h_\alpha) \) and Kolyvagin classes. Recall the ring class characters \( \psi_1, \psi_2 \) appearing in the decomposition (5.1). Since they are ring class characters unramified at \( p \), they factor through the Galois group \( \text{Gal}(H/K) \), where \( H \) is a ring class field of conductor prime to \( p \).

Using the Kolyvagin classes described in §5.2 in this section we define elements \( K^{\psi_i} \) in the relaxed Selmer groups \( \text{Sel}_{(p)}(K, V_f \otimes \psi_i) \) for \( i = 1, 2 \). The aim of this section is to compare the classes \( K^{\psi_1}, K^{\psi_2} \) with the local points in \( E(K)^{\pm} \) appearing in Theorem 5.6 in order to obtain a formula for the value \( I_p(f, g_\alpha, h_\alpha) \) in terms of these Kolyvagin classes. Using the notation of the previous sections, consider the dual exponential
\[
\text{exp}^\#_K : H^1_s(K_p, V_f)^\pm \to \mathbb{Q}_p
\]
of (5.12). As explained in §2 Tate’s uniformisation induces the isomorphism \( \varphi : \Phi_{\infty, p} \xrightarrow{\sim} \mathbb{Z}_p \) (in the notation of Lemma 2.2 \( \varphi := \varphi_{\text{Tate}} \) maps \( \bar{q} \) to 1). Recall the point \( \bar{\alpha} \in \Phi_{\infty, p} \) defined in Proposition 5.10, the sign
\[
a := a_p(E) \in \{ \pm 1 \}
\]
and the period
\[
\Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p.
\]
Proposition 5.11. We have

\[ \partial_p \mathbf{K} \in H^1_s(K_p, V_f)^{-a} \quad \text{and} \quad \exp_{-a}^* \partial_p \mathbf{K} = \frac{\text{ord}_p(q_E)}{p} \Pi_p. \]

In particular

\[ \text{res}_p(\mathbf{K})^a := \text{res}_p(\mathbf{K}) + a \text{res}_p(\mathbf{K}^{Fr_p}) \]

is crystalline, i.e. lies in \( H^1_f(\mathbb{Q}_p, V_f)^a \).

Proof. Recall we can regard \( \text{Hom}_{cont}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p) \) as a subspace of \( H^1_s(K_p, V_f) \). By (2.15) \( Fr_p \) acts on \( \text{Hom}_{cont}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p) \) as multiplication by \(-a\). Hence \( \text{res}_p(\mathbf{K}) \) belongs to \( H^1_s(K_p, V_f)^{-a} \) by Proposition 5.11 and the formula for \( \exp_{-a}^* \partial_p \mathbf{K} \) follows from Corollary 5.3. □

For \( i = 1, 2 \) let

\[ \text{Tr}_i = \sum_{\sigma \in S} \psi_i(\sigma) \cdot H^1(H, V_f) \longrightarrow H^1(K, V_f \otimes \psi_i) \]

denote the trace map onto the \( \psi_i \)-isotypic component and consider the isomorphism given by Shapiro’s Lemma

(5.19) \[ \text{Sh} : H^1(K, V_f \otimes \psi_i) \longrightarrow H^1(\mathbb{Q}, V_f \otimes \text{Ind}_{G_K}^{G_Q}(\psi_i)) =: H^1(\mathbb{Q}, V_i). \]

Define

\[ \mathbf{K}^\psi_i := \text{Sh}(\text{Tr}_i(\mathbf{K})) \in H^1(\mathbb{Q}, V_i). \]

It follows from Proposition 5.11 that \( \mathbf{K}^\psi_i \) lies in \( \text{Sel}(\mathbb{Q}_p)(V_i) \).

Frobenius element \( Fr_p \in G_{Q_p} \) acts on \( H^1(K_p, V_f) \) as an involution and we may consider the decomposition in \( \pm \)-eigenspaces

\[ H^1(K_p, V_f \otimes L_p) = H^1(K_p, V_f \otimes L_p)^\pm \oplus H^1(K_p, V_f \otimes L_p)^{-}. \]

As one readily verifies, Shapiro’s isomorphism restricts to

(5.20) \[ \text{Sh}_p : H^1(K_p, V_f \otimes L_p)^\pm \longrightarrow H^1(\mathbb{Q}_p, V_f \otimes V_i^\pm). \]

Corollary 5.12. The Kolyvagin class \( \mathbf{K}^\psi_i \) satisfies

\[ \mathbf{K}^\psi_i = \frac{h \cdot \text{ord}_p(q_E) \Pi_p}{p} \cdot \xi_i^{-a}. \]

Proof. Since \( \psi_i|_{G_{K_p}} = 1 \), the restriction of \( \text{Tr}_i \) to \( H^1(H_p, V_f \otimes L_p) = H^1(K_p, V_f \otimes L_p) \) is multiplication by \( h = [H : K] \). Hence \( \text{res}_p(\mathbf{K}^\psi_i) = h \cdot \text{Sh}_p(\text{res}_p(\mathbf{K})). \)

By Proposition 5.11 we have \( \partial_p \mathbf{K} \in H^1_s(K_p, V_f)^{-a} \), and by (5.20), it follows that \( \partial_p \mathbf{K}^\psi_i \in H^1_s(\mathbb{Q}_p, V_i^{-a}). \) Recall the choice of basis \( \xi_i^\pm \) made in Corollary 5.3. We may thus write

\[ \mathbf{K}^\psi_i = \frac{\exp_{-a}^* (\partial_p \mathbf{K}^\psi_i)}{\exp_{-a}^* (\partial_p \xi_i^{-a})} \cdot \xi_i^{-a}. \]

By definition of the basis \( \{ \xi_i^+, \xi_i^- \} \) and by (5.4), the denominator in the above expression is \( \exp^*(X_{-a}) \langle v_i^{-a}, \omega_i^{-a} \rangle = \langle v_i^{-a}, \omega_i^{-a} \rangle \). Let

\[ R \otimes v_i^{-a} \in H^1_s(K_p, V_f) \otimes V_i^{-a} \]
denote the image of $\partial_p \mathbf{K}^{\psi_i}$ via the isomorphism (5.3). Then

$$K^{\psi_i} = \frac{\exp^* (R)(v^{-a}_i, \omega^{-a}_i)}{(v^{-a}_i, \omega^{-a}_i)} \cdot \xi^{-a}_i = \exp^* (R) \cdot \xi^{-a}_i.$$ 

In order to compute the dual exponential of $R$, we need the following explicit expression for $\text{Sh}_p$. Recall that, as a $L_p [G_{Q_p}]$-module,

$$V_{\psi_i} = \text{Ind}_{G_{Q_p}}^{G_{K_p}} (1) = \{ v : G_{Q_p} \rightarrow L_p \mid v(\sigma \tau) = v(\tau) \forall \sigma \in G_{K_p}, \tau \in G_{Q_p} \}.$$ 

Consider the map $\text{ev} : V_{\psi_i} \rightarrow L_p, \quad v \mapsto v(1)$. It is an equivariant $G_{K_p}$-morphism which is compatible with the inclusion $G_{K_p} \hookrightarrow G_{Q_p}$, so it induces a morphism

$$\text{Sh}_p^{-1} : H^1 (Q_p, V_f \otimes V_{\psi_i}) \rightarrow H^1 (K_p, V_f \otimes L_p) = H^1 (K_p, V_f \otimes \psi_i),$$

which is the inverse of Shapiro’s isomorphism (5.19) restricted to $G_{K_p}$. More explicitly, if $\xi : G_{Q_p} \rightarrow V_f \otimes V_{\psi_i}$ represents a class in $H^1 (Q_p, V_f \otimes V_{\psi_i})$, then

$$\text{Sh}_p^{-1} (\xi) := (\text{id} \otimes \text{ev}) \ast \xi |_{G_{K_p}}$$

represents its image via (5.21). Recall that $R \otimes v_i^{-a}$ is the image of $h \cdot \partial_p \mathbf{K}$ via the composition

$$H^1_s (K_p, V_f \otimes L_p) -a \xrightarrow{\text{Sh}} H^1_s (Q_p, V_f \otimes V_{\psi_i}^{-a}) \xrightarrow{\text{Res}} H^1_s (K_p, V_f \otimes V_{\psi_i}^{-a}) \cong H^1_s (K_p, V_f) -a \otimes V_{\psi_i}^{-a}.$$ 

We conclude that the element $R \otimes v_i^{-a}$ is represented by the cocycle

$$h \cdot \text{Res}(\text{Sh}_p (\partial_p \mathbf{K})) : G_{K_p} \rightarrow V_f \otimes V_{\psi_i}^{-a}$$

satisfying

$$R \cdot (v_i^{-a} (1)) = h \cdot \partial_p \mathbf{K}.$$ 

In other words, by (5.2), we have

$$\exp^* (R) = h \cdot \exp^* (\partial_p \mathbf{K}).$$ 

The statement follows by applying Proposition 5.11. \hfill \Box

**Corollary 5.13.** We have

$$I_p (f, g_\alpha, h_\alpha) = \sqrt{\pi} \cdot 2p (1 - 1/p)^2 \cdot \sqrt{L (E \otimes 1, 1)} \cdot \frac{1}{\text{ord}_p (q_E)} \cdot \frac{1}{\lambda_{q_\alpha}} \times \log_p (Q^a_p) \mod L^x,$$

where $Q^a_p \in E (K_p)^a$ is characterized by $\delta_p (Q^a_p) = \text{res}_p (\mathbf{K})^a \in H^1_f (K_p, V_f)^a$.

**Proof.** By (5.20), if $\pi_a : H^1 (Q_p, V_i) \rightarrow H^1 (Q_p, V_i^a)$ denotes the natural projection, then

$$(5.23) \quad \pi_a \text{res}_p (\mathbf{K}^{\psi_i}) = h \cdot \text{Sh}_p (\text{res}_p (\mathbf{K})^a) \in H^1_f (Q_p, V_i^a).$$

Arguing as in the proof of Corollary 5.12 write

$$A \otimes v_i^a \in H^1_f (K_p, V_f)^a \otimes V_{\psi_i}^a$$

for the image of (5.23) via the isomorphism (5.5). Then

$$A = \frac{h \cdot \text{res}_p (\mathbf{K})^a}{v_i^a (1)} = h \cdot \text{res}_p (\mathbf{K})^a.$$ 

**Kolyvagin Classes vs Diagonal Classes**
Combining this with Corollary 5.12 and Lemma 5.5, we obtain
\[ h \cdot \text{res}_p(K)^a \otimes v^a_i = \pi_a \text{res}_p(K^i_p) = \frac{h \text{ord}_p(q_E) \Pi_p}{p} \delta_p(P^a) \otimes v^a_i. \]

Then
\[ \delta_p(P^a) = \frac{p}{\text{ord}_p(q_E) \Pi_p} \cdot \text{res}_p(K)^a, \]

and the thesis follows by applying Theorem 5.6.

\[ \square \]

REFERENCES

[AI19] F. Andreatta and A. Iovita. Katz type $p$-adic $L$-functions for primes $p$ non-split in the CM field. Preprint, 2019.

[BD96] M. Bertolini and H. Darmon. Heegner points on Mumford-Tate curves. Invent. Math., 126(3):413–456, 1996.

[BD97] M. Bertolini and H. Darmon. A rigid analytic Gross–Zagier formula and arithmetic applications. Annals of Mathematics, 146(1):111–147, 1997.

[BSVa] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Balanced diagonal classes and rational points on elliptic curves. In Heegner points, Stark–Heegner points and diagonal classes.

[BSVb] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Reciprocity laws for balanced diagonal classes. In Heegner points, Stark–Heegner points and diagonal classes.

[DLR15] H. Darmon, A. Lauder, and V. Rotger. Stark points and $p$-adic iterated integrals attached to modular forms of weight one. Forum Math. Pi, 3:e8, 95, 2015.

[DLR17] H. Darmon, A. Lauder, and V. Rotger. First order $p$-adic deformations of weight one newforms. In $L$-functions and automorphic forms, volume 10 of Contrib. Math. Comput. Sci., pages 39–80. Springer, Cham, 2017.

[DRa] H. Darmon and V. Rotger. $p$-adic families of diagonal cycles. In Heegner points, Stark–Heegner points and diagonal classes.

[DRb] H. Darmon and V. Rotger. Stark–Heegner points and diagonal classes. In Heegner points, Stark–Heegner points and diagonal classes.

[DR14] H. Darmon and V. Rotger. Diagonal cycles and Euler systems I: A $p$-adic Gross–Zagier formula. Ann. Sci. Éc. Norm. Supér. (4), 47(4):779–832, 2014.

[DR16] H. Darmon and V. Rotger. Elliptic curves of rank two and generalised Kato classes. Research in the Mathematical Sciences, 3(1):27, Aug 2016.

[DR17] H. Darmon and V. Rotger. Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin $L$-functions. J. Amer. Math. Soc., 30(3):601–672, 2017.

[GGM19] F. Gatti, X. Guitart, M. Masdeu, and V. Rotger. Special values of triple-product $p$-adic $L$-functions and non-crystalline diagonal classes. Preprint, 2019.

[HK91] M. Harris and S. Kudla. The central critical value of a triple product $L$-function. Ann. of Math. (2), 133(3):605–672, 1991.

[Ich08] A. Ichino. Trilinear forms and the central values of triple product $L$-functions. Duke Math. J., 145(2):281–307, 2008.

[LZ14] D. Loeffler and S.L. Zerbes. Iwasawa theory and $p$-adic $L$-functions over $\mathbb{Z}_p^2$-extensions. Int. J. Number Theory, 10(8):2045–2095, 2014.

[MR04] B. Mazur and K. Rubin. Kolyvagin systems. Mem. Amer. Math. Soc., 168(799):viii+96, 2004.

F. G.: DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT POLITÈCNICA DE CATALUNYA, C. JORDI GIRONA 1-3, 08034 BARCELONA, SPAIN

Email address: francesca.gatti@upc.edu

V. R.: IMTech, UPC and CENTRE DE RECERCA MATEMÀTIQUES, C. JORDI GIRONA 1-3, 08034 BARCELONA, SPAIN

Email address: victor.rotger@upc.edu