Cycles and Chaos in Fractional Families of Maps

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The skeleton of classical chaos comprises periodic orbits. To find periodic orbits is a relatively simple task in regular dynamics. In this paper we derive algebraic equations which define asymptotically periodic solutions in fractional and fractional difference maps.

I. INTRODUCTION

Presence of the power-law memory is a significant feature of many natural (biological, physical, etc.) and social systems. Continuous and discrete fractional calculus is the instrument to describe behavior of systems with the power-law memory. Existence of chaotic solutions is an inevitable property of nonlinear dynamics (regular and fractional). Behavior of fractional systems can be very different from the behavior of the corresponding systems with no memory.

In this paper we derive the equations that allow calculations of the coordinates of the asymptotically periodic sinks.

II. FRACTIONAL/FRACTIONAL DIFFERENCE MAPS

In this section we will omit an introduction of the basic definitions of fractional and fractional difference calculus and refer a reader to two relevant reviews \([32, 33]\). We will present only already widely accepted definitions of the Caputo fractional and fractional difference universal \(\alpha\)-family of maps.

The Caputo universal map of the order \(\alpha\) is defined (derived) as

\[
x_{n+1} = \sum_{k=0}^{m-1} b_k \frac{h^k}{k!} (n+1)^k - \frac{h^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{\alpha-1},
\]

where \(\alpha \in \mathbb{R}, \alpha \geq 0, m = \lceil \alpha \rceil, x_n = x(t = nh), t \) is time, \(n \in \mathbb{Z}, n \geq 0, b_k \in \mathbb{R} \) are constants, and \(G_K(x)\) is a function (could be nonlinear) with a parameter \(K\).

The \(h\)-difference Caputo universal \(\alpha\)-family of maps is
defined (derived) as

\[ x_{n+1} = \sum_{k=0}^{m-1} \frac{x_k}{k!} ((n+1)h)_k \]

\[ -\frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n-s-m+\alpha)^{(\alpha-1)} G_K(x_{s+m-1}), \quad (2) \]

where \( x_k = x(kh), \alpha \in \mathbb{R}, \alpha \geq 0, m = [\alpha], n \in \mathbb{Z}, \)
\( n \geq 0, c_k \in \mathbb{R} \) are constants, and \( G_K(x) \) is a function with a parameter \( K \). In both maps \( h \in \mathbb{R} \) and \( h > 0 \). The definition of the falling factorial \( t^{(\alpha)} \) is

\[ t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} t^{\alpha}, \quad t \neq -1, -2, -3, \ldots. \quad (3) \]

The falling factorial is asymptotically a power function:

\[ \lim_{t \to \infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^{\alpha}} = 1, \quad \alpha \in \mathbb{R}. \quad (4) \]

The \( h \)-falling factorial \( t_h^{(\alpha)} \) is defined as

\[ t_h^{(\alpha)} = h^{\alpha} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} = h^{\alpha} \left( \frac{t}{h} \right)^{(\alpha)}, \quad \frac{t}{h} \neq -1, -2, -3, \ldots. \quad (5) \]

In many papers, where particular forms of the Caputo universal map (Caputo logistic, with \( G_K(x) = x - Kx(1-x) \), and standard, with \( G_K(x) = K \sin(x) \), maps) are investigated, the authors assume \( h = 1 \).

### III. ASYMPTOTICALLY PERIODIC CYCLES FOR \( 0 < \alpha < 1 \)

When \( 0 < \alpha < 1 \), all forms of the universal \( \alpha \)-family of maps introduced in this paper, Eqs. (1) and (2), can be written in the form

\[ x_n = x_0 - \sum_{k=0}^{n-1} \tilde{G}(x_k) U(n-k). \quad (6) \]

In this formula \( \tilde{G}(x) = h^{\alpha} G_K(x)/\Gamma(\alpha) \) and \( x_0 \) is the initial condition. In fractional maps Eq. (1)

\[ U_\alpha(n) = n^{\alpha-1}, \quad U_\alpha(1) = 1 \quad (7) \]

and in fractional difference maps, Eq. (2),

\[ U_\alpha(n) = (n+\alpha-2)^{(\alpha-1)}, \quad U_\alpha(1) = (\alpha-1)^{(\alpha-1)} = \Gamma(\alpha). \quad (8) \]

For \( n = lN + m \), where \( 0 < m < l + 1 \), Eq. (1) can be written as

\[ x_{lN+m} = x_0 - \sum_{k=0}^{lN+m-1} \tilde{G}(x_k) U_\alpha(lN+m-k) \]

\[ = x_0 - \sum_{n=1}^{lN+m} \tilde{G}(x_{IN+m-n}) U_\alpha(n) \]

\[ = x_0 - \sum_{j=1}^{lN-1} \sum_{k=0}^{N-1-j} \tilde{G}(x_{IN+m-lk-j}) U_\alpha(lk+j) \]

\[ - \sum_{j=1}^{m} \tilde{G}(x_{m-j}) U_\alpha(lN+j). \quad (9) \]

For \( 0 < m < l \)

\[ x_{lN+m+1} - x_{lN+m} \]

\[ = \sum_{j=1}^{lN-1} \sum_{k=0}^{N-1-j} \tilde{G}(x_{IN+m-lk-j}) U_\alpha(lk+j) \]

\[ - \sum_{j=1}^{N-1} \tilde{G}(x_{IN+m-lk}) [U_\alpha(lk) - U_\alpha(lk+1)] \]

\[ - \sum_{k=1}^{N-1} \tilde{G}(x_{IN+m-lk}) U_\alpha(lk+1) \]

\[ = \sum_{j=1}^{N-1} \sum_{k=0}^{N-1-j} \tilde{G}(x_{IN+m-lk-j}) U_\alpha(lk+j) \]

\[ + \sum_{k=1}^{N-1} \tilde{G}(x_{IN+m-lk}) U_\alpha(lk+1) \]

\[ - \tilde{G}(x_{IN+m}) U_\alpha(1) + \tilde{G}(x_{m}) U_\alpha(lN) + S_0, \quad (10) \]

where \( S_0 \) is a sum of a finite number of elements, which tend to zero as \( N \to \infty \). Let’s assume that in the limit \( N \to \infty \) the system converges to a period \( l \)-cycle

\[ x_{lim,m} = \lim_{N \to \infty} x_{Nl+m}, \quad 0 < m < l + 1, \quad (11) \]

and consider the limit of Eq. (10) as \( N \to \infty \).

\[ x_{lim,m+1} - x_{lim,m} = \lim_{N \to \infty} \sum_{j=1}^{l-1} \sum_{k=0}^{N-1-j} \tilde{G}(x_{IN-lk+m-j}) [U_\alpha(lk+j) \]

\[ - U_\alpha(lk+j+1)] + \lim_{N \to \infty} \sum_{k=1}^{N-1} \tilde{G}(x_{IN-lk+m}) [U_\alpha(lk) \]

\[ - U_\alpha(lk+1) - \tilde{G}(x_{lim,m}) U_\alpha(1). \quad (12) \]
Let’s find the limit (0 ≤ j < l and for j = 0 the sum will start from k = 1)
\[
\lim_{N \to \infty} \sum_{k=0}^{N-1} \tilde{G}(x_{N-lk+m-j}) U_\alpha(lk+j) - U_\alpha(lk+j+1)
\]
= \lim_{N_1 \to \infty, N-N_1 \to \infty} \sum_{k=0}^{N_1} \tilde{G}(x_{N-lk+m-j}) U_\alpha(lk+j)

= -U_\alpha(lk+j+1) + \lim_{N_1,N \to \infty} \sum_{k=N_1+1}^{N} \tilde{G}(x_{N-lk+m-j})
\times \left[U_\alpha(lk+j) - U_\alpha(lk+j+1)\right].
\] (13)

Let’s consider the second sum in the last expression. Because \(x\) is converging, it must be bounded and we assume that \(G(x)\) is also bounded on a bounded domain: \(|\tilde{G}(x_{N-lk+m-j})| < C_1\). The terms \(U_\alpha(lk+j) - U_\alpha(lk+j+1)\) are of the order \((lk+j)^{\alpha-2}\) and the series
\[
S_{j+1} = \sum_{k=0}^{\infty} \left[U_\alpha(lk+j) - U_\alpha(lk+j+1)\right], \quad 0 < j < l,
\]
\[\hat{S}_1 = \sum_{k=1}^{N} \left[U_\alpha(lk) - U_\alpha(lk+1)\right]
\] (14)
are converging. This implies that for every \(\varepsilon > 0\) there exists a \(N_1\) such that
\[
\sum_{k=N_1+1}^{N} \left[U_\alpha(lk+j) - U_\alpha(lk+j+1)\right] < \frac{\varepsilon}{2C_1}
\] (15)
for every \(N_1 > N\) and every \(N > N_1\) and the limit of the sum is zero.

Now, let’s consider the first sum of the last expression in Eq. (13). Because \(\tilde{G}(x_{N-lk+m-j})\) is converging for \(N >> N_1\) to \(\tilde{G}(x_{lN,m-j})\) for \(0 \leq j \leq m\) or to \(\tilde{G}(x_{lN,m-j+1})\) for \(m \leq j < l\), for every \(\varepsilon > 0\) there exists a large \(N\) such that \(G(x_{N-lk+m-j}) = \tilde{G}(x_{lN,m-j}) + \varepsilon(k)\) or \(\tilde{G}(x_{N-lk+m-j}) = \tilde{G}(x_{lN,m-j+1}) + \varepsilon(k)\), where \(|\varepsilon(k)| < \frac{\varepsilon}{2S_{j+1}}\) for every \(k \leq N_1\). This implies that for every \(N_1\) and \(\varepsilon > 0\) there exists a \(N\) such that the considered sum deviates less than \(\varepsilon/2\) and the total expression Eq. (13) deviates less than \(\varepsilon\) from
\[
\tilde{G}(x_{lN,m-j})S_{j+1}, \quad 0 < j < m,
\]
\[
\tilde{G}(x_{lN,m-j+1})S_{j+1}, \quad m \leq j < l,
\]
\[
\tilde{G}(x_{lN,m})\hat{S}_1, \quad j = 0.
\] (16)

This proves that expressions Eq. (16) are the limits as \(N \to \infty\) in Eq. (13).

Finally, Eq. (12) can be written as
\[
x_{lN,m+1} - x_{lN,m} = S_1 \tilde{G}(x_{lN,m}) + \sum_{j=1}^{N-1} S_{j+1} \tilde{G}(x_{lN,m-j})
\]
\[
+ \sum_{j=m}^{l-1} S_{j+1} \tilde{G}(x_{lN,m-j+1}), \quad 0 < m < l
\] (17)
where
\[
S_1 = -U_\alpha(1) + \sum_{k=1}^{\infty} \left[U_\alpha(lk) - U_\alpha(lk+1)\right].
\] (18)

It is easy to see that
\[
\sum_{j=1}^{l} S_j = 0.
\] (19)

To complete the system of equations that defines \(l\) variables \(x_{lN,m}, 1 \leq j \leq l\), we have to add one more equation to the system of \(l-1\) equations Eq. (17). Let’s consider the total of all \(l\) cycle limiting points, which is the limiting value of the following sum:
\[
\sum_{m=1}^{l} x_{lN+m} = lx_0
\]
\[
- \sum_{m=1}^{l} \sum_{j=1}^{N-1} \tilde{G}(x_{lN+m-lk-j}) U_\alpha(lk+j)
\]
\[
- \sum_{m=1}^{l} \tilde{G}(x_{lN+1}) U_\alpha(lN+j).
\] (20)

When \(N \to \infty\) the last term in this equation goes to zero and the terms on the first line of this equation are also finite. This implies that the limiting value of the sum on the second line
\[
S = \sum_{k=0}^{N-N_1} \sum_{j=1}^{l} U_\alpha(lk+j) \sum_{m=1}^{l} \tilde{G}(x_{lN+m-lk-j})
\]
\[
+ \sum_{k=N-N_1}^{N} \sum_{j=1}^{l} U_\alpha(lk+j) \sum_{m=1}^{l} \tilde{G}(x_{lN+m-lk-j})
\] (21)
also must be finite. For a finite value of \(N_1\), the second sum in the last expression is finite. For large values of \(N-k, \sum_{m=1}^{l} \tilde{G}(x_{lN+m-lk-j})\) is converging to some constant value \(S_G\).
\[
S_G = \lim_{N \to \infty} \sum_{j=1}^{l} \tilde{G}(x_{lN+m-lk-j}).
\] (22)

If \(S_G > 0\) then there exists a \(N_1\) such that for every \(k\) in the first sum \(\tilde{G}(x_{lN+m-lk-j}) > C_1 > 0\); if \(S_G < 0\) then there exists a \(N_1\) such that for every \(k\) in the first sum \(\tilde{G}(x_{lN+m-lk-j}) < C_2 < 0\). In both cases the first sum is diverging when \(N \to \infty\). The only case in which the limiting value of \(S\) is finite is when \(S_G = 0\):
\[
\sum_{j=1}^{l} \tilde{G}(x_{lN,m}) = 0.
\] (23)

The system of \(l\) equations, Eq. (17) and Eq. (23), is a system that defines all \(l\) values \(x_{lN,m}, 1 \leq m \leq l\) of an asymptotic \(l\)-cycle.
IV. PERIOD 3 CYCLES

Calculations of the periodic points start with the calculations of coefficients $S_1$, $S_2$, and $S_3$.

A. Fractional maps

In fractional maps functions $U_\alpha(n)$ are defined by Eq. (7). For $S_1$ we may write

$$S_1 = S_{1,1} + S_{1,2} ,$$

where the finite series

$$S_{1,1} = -1 + \sum_{k=1}^{N} [(3k)^{\alpha-1} - (3k+1)^{\alpha-1}]$$

(24)

with large $N$ (to tabulate values of $S_i$ we used $N = 20000$) can be directly calculated with a high (machine) accuracy. To calculate the infinite series

$$S_{1,2} = \sum_{k=N+1}^{\infty} [(3k)^{\alpha-1} - (3k+1)^{\alpha-1}]$$

(25)

in each term we factor out $(3k)^{\alpha-1}$ and develop the difference into a Taylor series. At the end, the expression to calculate $S_{1,2}$ can be written as

$$S_{1,2} = (1 - \alpha)3^{\alpha-2} \left\{ \zeta_N(2 - \alpha) + \alpha - 2 \right\} \zeta_N(3 - \alpha)$$

$$+ \frac{\alpha - 3}{9} \left[ \zeta_N(4 - \alpha) + \frac{\alpha - 4}{12} \zeta_N(5 - \alpha) \right] + O(N^{\alpha-5}) ,$$

(26)

where

$$\zeta_N(m - \alpha) = \zeta(m - \alpha) - \sum_{k=1}^{N} k^{\alpha-m} ,$$

(27)

and we used a fast method for calculating values of the Riemann $\zeta$-function. In a similar way, the expressions for $S_2$ and $S_3$ can be written as

$$S_2 = \sum_{k=0}^{N} [(3k+1)^{\alpha-1} - (3k+2)^{\alpha-1}]$$

$$+ (1 - \alpha)3^{\alpha-2} \left\{ \zeta_N(2 - \alpha) + \alpha - 2 \right\} \zeta_N(3 - \alpha)$$

$$+ \frac{\alpha - 3}{27} \left[ 7\zeta_N(4 - \alpha) + \frac{5(\alpha - 4)}{4} \zeta_N(5 - \alpha) \right] + O(N^{\alpha-5})$$

(28)

and

$$S_3 = \sum_{k=0}^{N} [(3k+2)^{\alpha-1} - (3k+3)^{\alpha-1}]$$

$$+ (1 - \alpha)3^{\alpha-2} \left\{ \zeta_N(2 - \alpha) + \alpha - 2 \right\} \zeta_N(3 - \alpha)$$

$$+ \frac{\alpha - 3}{9} \left[ 19\zeta_N(4 - \alpha) + \frac{65(\alpha - 4)}{12} \zeta_N(5 - \alpha) \right] + O(N^{\alpha-5})$$

(29)

To calculate values of $S_i$ in this paper (see Tables I and II) we used $N = 20000$ and a fast method to calculate the $\zeta$-function. To estimate the accuracy of our computations, we also calculated the value of $\Sigma S_i$, whose deviation from zero represents the absolute error.

| $\alpha$ | $S_1$ | $S_2$ | $S_3$ | $\Sigma S_i$ |
|---------|-------|-------|-------|-------------|
| 0.01    | -8503346.6023882 | 2479663.62e-14 |
| 0.05    | -8451510.5933434 | 2518076.62e-14 |
| 0.1     | -8384473.5818399 | 2566074.64e-14 |
| 0.15    | -8314875.5700882 | 2613993.54e-14 |
| 0.2     | -8242640.5580876 | 2661764.32e-14 |
| 0.25    | -8167694.5453837 | 2709315.13e-14 |
| 0.3     | -8089959.5333390 | 2765699.55e-14 |
| 0.35    | -8009359.5205915 | 2808445.67e-14 |
| 0.4     | -7925818.5075961 | 2849857.73e-14 |
| 0.45    | -7839259.4943524 | 2895718.59e-14 |
| 0.5     | -7749096.4808670 | 2940931.10e-14 |
| 0.55    | -7656783.4671835 | 2985398.20e-14 |
| 0.6     | -7560713.4531899 | 3029016.21e-14 |
| 0.65    | -7461323.4389847 | 3071676.18e-14 |
| 0.7     | -7358540.4245274 | 3113265.13e-14 |
| 0.75    | -7252290.4098625 | 3153665.11e-14 |
| 0.8     | -7142504.3949752 | 3192752.10e-14 |
| 0.85    | -7029113.3798715 | 3230398.13e-14 |
| 0.9     | -6912052.3645582 | 3266470.27e-14 |
| 0.95    | -6791256.3490427 | 3308030.50e-14 |
| 0.99    | -6691891.3364903 | 3326988.21e-13 |

B. Fractional difference maps

In fractional difference maps functions $U_\alpha(n)$ are defined by Eq. (3). Let’s consider formula for $S_2$ (formulae for $S_1$ and $S_3$ may be obtained in a similar way). Each term in the sum $S_2$ can be written as

$$U_\alpha(3k+1) - U_\alpha(3k+2) = \frac{\Gamma(3k + \alpha)}{\Gamma(3k + 1)} - \frac{\Gamma(3k + \alpha + 1)}{\Gamma(3k + 2)}$$

$$= (1 - \alpha) \ast \frac{\Gamma(3k + \alpha)}{\Gamma(3k + 2)} .$$

(30)

As in the fractional case, we will split the total into two sums:

$$S_2 = S_{2,1} + S_{2,2} ,$$

(31)
where the finite series
\[
S_{2,1} = \sum_{k=0}^{N} \frac{(1 - \alpha) \Gamma(3k + \alpha)}{(3k + 1)!}
\]  
(33)
can be directly calculated with a high accuracy. To calculate the infinite series
\[
S_{2,2} = \sum_{k=N+1}^{\infty} \frac{(1 - \alpha) \Gamma(3k + \alpha)}{\Gamma(3k + 2)}
\]  
(34)
we will use the following approximation (36):
\[
\Gamma(z + a) = \Gamma(z + b) \left\{ 1 + \frac{(a + b - 1)(a - b)}{2z} + \frac{1}{12z^2} \left( a - b \right) \right\} \times [3(a + b - 1)^2 - (a - b + 1)] + O(z^{-3}).
\]  
(35)
Then, we obtain the following expression to calculate \(S_2\):
\[
S_2 = (1 - \alpha) \sum_{k=0}^{N} \frac{\Gamma(3k + \alpha)}{(3k + 1)!}
\]  
+ \( (1 - \alpha)^3 \alpha^2 - 2 \left\{ \zeta_N(2 - \alpha) + \frac{\alpha - 2}{6} \left( \alpha + 1 \right) \zeta_N(3 - \alpha) \right\}
\]  
+ \( \frac{(\alpha - 3)(3\alpha^2 + 5\alpha + 4)}{36} \zeta_N(4 - \alpha) \} + O(N^{\alpha - 4}).
\]  
(36)
The following expressions were used to calculate \(S_1\) and \(S_3\):
\[
S_1 = -\Gamma(\alpha) + (1 - \alpha) \sum_{k=1}^{N} \frac{\Gamma(3k + 1 + \alpha)}{(3k)!}
\]  
+ \( (1 - \alpha)^3 \alpha^2 - 2 \left\{ \zeta_N(2 - \alpha) + \frac{\alpha - 2}{6} \left( \alpha - 1 \right) \zeta_N(3 - \alpha) \right\}
\]  
+ \( \frac{(\alpha - 3)(3\alpha^2 - 7\alpha + 4)}{36} \zeta_N(4 - \alpha) \} + O(N^{\alpha - 4}).
\]  
(37)
and
\[
S_3 = (1 - \alpha) \sum_{k=0}^{N} \frac{\Gamma(3k + 1 + \alpha)}{(3k + 2)!}
\]  
+ \( (1 - \alpha)^3 \alpha^2 - 2 \left\{ \zeta_N(2 - \alpha) + \frac{\alpha - 2}{6} \left( \alpha + 3 \right) \zeta_N(3 - \alpha) \right\}
\]  
+ \( \frac{(\alpha - 3)(3\alpha^2 + 17\alpha + 28)}{36} \zeta_N(4 - \alpha) \} \}
\]  
+ O(N^{\alpha - 4}).
\]  
(38)
The results of calculations are given in Table II.
VI. BIFURCATIONS AND ASYMPTOTICALLY PERIODIC POINTS IN LOGISTIC $\alpha$-FAMILIES OF MAPS $0 \leq \alpha \leq 1$

Information on the first bifurcations and cycle 2 sink points in the fractional/fractional difference standard, with $G_K(x) = K \sin(x)$, and logistic, with $G_K(x) = x - Kx(1-x)$, families of maps for $0 \leq \alpha \leq 2$ can be found in [32, 33] and references therein. Here we will consider only the logistic families of maps and present more detailed results.

TABLE III. S values for period 2 cycles (fractional maps).

| $\alpha$ | $S_1$ | $S_2$ | $\Sigma S_1$ |
|-------|------|------|---------|
| 0.01  | 0.6915452 | 0.6915452 | 1.3830904 |
| 0.05  | 0.6850718 | 0.6850718 | 1.3701436 |
| 0.1   | 0.6768319 | 0.6768319 | 1.3536538 |
| 0.15  | 0.6684265 | 0.6684265 | 1.3351730 |
| 0.2   | 0.6598547 | 0.6598547 | 1.3162994 |
| 0.25  | 0.6511517 | 0.6511517 | 1.2967034 |
| 0.3   | 0.6422900 | 0.6422900 | 1.2763800 |
| 0.35  | 0.6331341 | 0.6331341 | 1.2549682 |
| 0.4   | 0.6238908 | 0.6238908 | 1.2328516 |
| 0.45  | 0.6144789 | 0.6144789 | 1.2100378 |
| 0.5   | 0.6048086 | 0.6048086 | 1.1865172 |
| 0.55  | 0.5951502 | 0.5951502 | 1.1623004 |
| 0.6   | 0.5852340 | 0.5852340 | 1.1373640 |
| 0.65  | 0.5751509 | 0.5751509 | 1.1118068 |
| 0.7   | 0.5649016 | 0.5649016 | 1.0855982 |
| 0.75  | 0.5544874 | 0.5544874 | 1.0588358 |
| 0.8   | 0.5439096 | 0.5439096 | 1.0315192 |
| 0.85  | 0.5331700 | 0.5331700 | 1.0036300 |
| 0.9   | 0.5222073 | 0.5222073 | 0.9751376 |
| 0.95  | 0.5112285 | 0.5112285 | 0.9460875 |
| 0.99  | 0.5022549 | 0.5022549 | 0.9165048 |

TABLE IV. S values for T=2 cycles (fractional difference maps).

A. $T = 2$ cycles

In case $T = 2$, the system of equations Eq.(17) and Eq.(23) can be written as

$$\begin{cases} 
(1 - K)(x_{lim,1} + x_{lim,2}) + K(x_{lim,1}^2 + x_{lim,2}^2) = 0, \\
x_{lim,1} - x_{lim,2} = S_2 \frac{1}{K \alpha} (x_{lim,1} - x_{lim,2})[1 - K + (x_{lim,1} + x_{lim,2})]^\alpha.
\end{cases}$$

Two fixed point solutions with $x_{lim,1} = x_{lim,2}$ are $x_{lim,1} = 0$, stable for $K < 1$, and $x_{lim,1} = (K - 1)/K$.

The $T = 2$ sink is defined by the equation

$$x_{lim,1}^2 - \left( \frac{\Gamma(\alpha)}{S_2 K h^\alpha} + \frac{K - 1}{K} \right) x_{lim,1} + \frac{\Gamma(\alpha)}{2(S_2 K h^\alpha)^2} + \frac{(K - 1) \Gamma(\alpha)}{S_2 K^2 h^\alpha} = 0,$$

which has solutions

$$x_{lim,1} = \frac{K_{C1S} + K - 1 + \sqrt{(K - 1)^2 - K_{C1S}^2}}{2K},$$

where (see [50])

$$K_{C1S} = \frac{\Gamma(\alpha)}{S_2 h^\alpha}$$

is a first bifurcation point of the standard families of maps, defined when

$$K \geq 1 + \frac{\Gamma(\alpha)}{S_2 h^\alpha} = 1 + K_{C1S} \text{ or } K \leq 1 - \frac{\Gamma(\alpha)}{S_2 h^\alpha} = 1 - K_{C1S}.$$
Conditions Eq. (46) are valid for all logistic $\alpha$-families of maps considered in this paper. Here we’ll consider $K > 0$ and $h \leq 1$. It is easy to show, and this is done in [34], that $S_{2}$ is less than $U_{\alpha}(1)/2$, which is either $1/2$ or $\Gamma(\alpha)/2$ (also $\Gamma(\alpha) > 0.885$ for $\alpha > 0$). Then $\Gamma(\alpha)/(S_{2}h^{\alpha}) > 1$ and we may ignore the second of the inequalities in Eq. (46). Note that the fixed point $x = (K - 1)/K$ is stable when

$$1 \leq K < K_{C1} = 1 + \frac{\Gamma(\alpha)}{S_{2}h^{\alpha}} = 1 + K_{C1}. \quad (47)$$

Below we present the table (Table V) of fixed point - $T=2$ cycle bifurcation points for fractional and fractional difference maps and a corresponding graph, Fig. 1, which is a part of previously reported [33] 2D bifurcation diagram.

| $\alpha$ | Fractional | Fr. difference |
|----------|------------|----------------|
| 0.01     | 144.7832   | 2.006956       |
| 0.05     | 29.12069   | 2.035265       |
| 0.1      | 15.05894   | 2.071773       |
| 0.15     | 10.30884   | 2.109569       |
| 0.2      | 7.957356   | 2.148699       |
| 0.25     | 6.568304   | 2.189207       |
| 0.3      | 5.658249   | 2.231144       |
| 0.35     | 5.021497   | 2.274561       |
| 0.4      | 4.555365   | 2.319508       |
| 0.45     | 4.209246   | 2.366410       |
| 0.5      | 3.930169   | 2.414214       |
| 0.55     | 3.715490   | 2.464086       |
| 0.6      | 3.544610   | 2.515717       |
| 0.65     | 3.407708   | 2.569168       |
| 0.7      | 3.297843   | 2.624505       |
| 0.75     | 3.209999   | 2.681793       |
| 0.8      | 3.140484   | 2.741101       |
| 0.85     | 3.086546   | 2.802501       |
| 0.9      | 3.046122   | 2.860666       |
| 0.95     | 3.017659   | 2.931873       |
| 0.99     | 3.002712   | 2.986185       |

TABLE V. Fixed point - $T=2$ cycle bifurcation points for fractional (middle column) and fractional difference (right column) maps.

Poincaré plots (return maps) present a good instrument to analyze cyclic behavior and chaos in fractional/fractional difference maps. Return maps for fractional difference logistic map with $\alpha = 0.75$, various values of $K$, and the initial point $x_{0} = 0.3$, obtained after 500000 iterations, are given in Figs. 2–4. Calculated using Eq. (44), $T = 2$ sink is stable in Fig. 2 and unstable in Figs. 3 and 4. It is easy to see that the return map in Fig. 2 converges to the $T = 2$ point calculated using Eqs. (17) and (23). This confirms correctness of the equations and our calculations. The chaotic return map in Fig. 2 looks like a multi-scroll attractor of a dissipative system. Similarity of the behavior of fractional systems to the behavior of dissipative systems was noticed by many authors and one of the first examples of this similarity can be found in [39].

VII. CONCLUSION

The main idea of this paper is to use cyclic sinks to analyze the periodic behavior and chaos in discrete fractional systems. The first step of this analysis is an algorithm...
FIG. 3. The Poincaré plot (500000 iterations) for fractional difference logistic map with $\alpha = 0.75$, $K = 3.3$, and the initial point $x_0 = 0.3$. Stable $T = 4$ sink marked by the plus signs. The asymptotically unstable $T = 2$ sink is marked by the stars and the unstable fixed point $(K - 1)/K$ by the circle.

FIG. 4. The Poincaré plot (500000 iterations) for fractional difference logistic map with $\alpha = 0.75$, $K = 3.4$, and the initial point $x_0 = 0.3$. The asymptotically unstable $T = 2$ sink is marked by the stars and the unstable fixed point $(K - 1)/K$ by the circle.

to calculate the cyclic points. In this paper we derived equations, Eq. (17) and Eq. (23), which define asymptotically cyclic points. These equations contain coefficients ($\sum S_i$) which are the same for all maps. We calculated (and tabulated) these coefficients for period $T = 2$ and $T = 3$ cycles. Similar calculations can be used to calculate $S_i$ for any periodic cycles.

We also constructed Poincaré plots (return maps) for the logistic fractional difference family of maps. The plots confirm correctness of the analysis done in this paper. We propose and plan to use periodic sinks to analyze chaotic behavior of fractional systems. In the following papers we plan to add higher order unstable cyclic points to Poincaré plots of fractional chaotic systems in order to investigate regularities in chaotic attractor-like structures of Poincaré plots of fractional systems.

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