MODULI SPACES OF TYPE B SURFACES WITH TORSION

PETER GILKEY

Abstract. We examine moduli spaces of locally homogeneous surfaces of Type B with torsion where the symmetric Ricci tensor is non-degenerate. We also determine the space of affine Killing vector fields in this context.

1. Introduction

Let $\nabla$ be a connection on the tangent bundle of a smooth manifold $M$ of dimension $m$. Let $\vec{x} = (x^1, \ldots, x^m)$ be a system of local coordinates on $M$. Adopt the Einstein convention and sum over repeated indices to expand $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$ where $\Gamma = (\Gamma_{ij}^k)$ are the Christoffel symbols of the connection. We say that $\nabla$ is torsion free if $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ or, equivalently, if $\Gamma_{ij}^k = \Gamma_{ji}^k$. The importance of the torsion free condition lies in the observation that $M$ is torsion free if and only if for every point $P$ of $M$, there exist coordinates centered at $P$ so that $\Gamma_{ij}^k(P) = 0$. Thus in the torsion free setting, one may normalize the coordinate system so that only the second and higher order derivatives of the connection 1-form play a role in defining local invariants of Weyl type. We also note that if $\nabla$ is a connection with torsion, then there is a naturally associated torsion free connection $\tilde{\nabla}$ with the same parametrized geodesics. For these reasons, torsion free connections have been studied extensively in the literature. There are, however, natural situations in which manifolds with torsion enter. We refer, for example, to work on torsion-gravity [4, 11, 20, 22, 36, 37, 45], on hyper-Kähler with torsion supersymmetric sigma models [23, 24, 25, 43], on contact geometries [1], on almost product manifolds [41], on non-integrable geometries [2, 8], on the non-commutative residue for manifolds with boundary [46], on Hermitian and anti-Hermitian geometry [30], on CR geometry [17], on Einstein–Weyl gravity at the linearized level [10], on Yang-Mills flow with torsion [30], on ESK theories [14], on double field theory [31], on BRST theory [26], and on the symplectic and elliptic geometries of gravity [12].

The curvature operator $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ and the Ricci tensor $\rho := \text{Tr}(Z \to R(Z, X)Y)$ have components:

\[
R_{ijk}^l = \partial_{x_i} \Gamma_{jk}^l - \partial_{x_j} \Gamma_{ik}^l + \Gamma_{in}^l \Gamma_{jk}^n - \Gamma_{jn}^l \Gamma_{ik}^n,
\]

\[
\rho_{jk} = \partial_{x_i} \Gamma_{jk}^i - \partial_{x_j} \Gamma_{ik}^i + \Gamma_{in}^i \Gamma_{jk}^n - \Gamma_{jn}^i \Gamma_{ik}^n.
\]

Since in this quite general setting, unlike the pseudo-Riemannian context, the Ricci tensor need not be symmetric, we introduce the symmetric Ricci tensor defining:

\[
\rho_s(X, Y) := \frac{1}{2} \left( \rho(X, Y) + \rho(Y, X) \right).
\]

One is interested in classifying such structures up to isomorphism, i.e. up to the action of the pseudo group of germs of diffeomorphisms. And it is natural to start

2010 Mathematics Subject Classification. 53C21.

Key words and phrases. Ricci tensor, moduli space, locally homogeneous affine manifold, connection with torsion.
with the locally homogeneous examples; \( \mathcal{M} := (M, \nabla) \) is said to be locally homogeneous if given any two points \( P \) and \( Q \) of \( M \), there is the germ of a diffeomorphism \( \Psi_{P,Q} \) taking \( P \) to \( Q \) which commutes with \( \nabla \). The case of surfaces is particularly tractable and is of interest in its own right; connections on surfaces have been used to construct new examples of pseudo-Riemannian metrics that do not have a Riemannian analog \cite{3, 10, 15, 35}. We shall assume henceforth that \( \mathcal{M} \) is not flat. The following result was established in the torsion free setting by Opozda \cite{12} and later extended by Arias-Marco and Kowalski \cite{3} to the case of surfaces with torsion.

**Theorem 1.1.** Let \( \mathcal{M} = (M, \nabla) \) be a locally homogeneous surface. Then at least one of the following three possibilities hold which describe the local geometry:

(A) There exists a coordinate atlas so the Christoffel symbols \( \Gamma_{ij}^{k} \) are constant.

(B) There exists a coordinate atlas so the Christoffel symbols have the form \( \Gamma_{ij}^{k} = (x^{\alpha})^{-1}C_{ij}^{k} \) for \( C_{ij}^{k} \) constant and \( x^{i} > 0 \).

(C) \( \nabla \) is the Levi-Civita connection of a metric of constant Gauss curvature.

These classes are not disjoint. While there are no surfaces which are both Type \( \mathcal{A} \) and Type \( \mathcal{C} \), there are surfaces which are both Type \( \mathcal{A} \) and Type \( \mathcal{B} \), and, up to isomorphism, there are are two surfaces which are Type \( \mathcal{B} \) and Type \( \mathcal{C} \) — the hyperbolic plane and the Lorentzian analogue. The remaining Type \( \mathcal{C} \) geometry is modeled on that of the round sphere and plays no role in our analysis.

As the Type \( \mathcal{A} \) setting was discussed in \cite{4} \cite{6} \cite{28}, we shall concentrate on the Type \( \mathcal{B} \) setting in this paper. We introduce some notational conventions:

**Definition 1.2.** Let \( \mathcal{W}_{\mathbb{R}}(2) := (\mathbb{R}^{2})^{\ast} \otimes (\mathbb{R}^{2})^{\ast} \otimes \mathbb{R}^{2} \); two indices are down and one index is up. If \( C \in \mathcal{W}_{\mathbb{R}}(2) \), let \( \Gamma_{ij}^{k} = (x^{\alpha})^{-1}C_{ij}^{k} \) be the Christoffel symbols of the associated connection \( \nabla = \nabla^{C} \) of Type \( \mathcal{B} \). If \( p + q = 2 \), let

\[
\mathcal{W}_{\mathbb{R}}(p, q) := \{ C \in \mathcal{W}_{\mathbb{R}}(2) : \text{signature}(\rho_{p,q}) = (p, q) \}.
\]

Let \( \mathcal{W}_{\mathbb{R}}(p, q) \) (resp. \( \mathcal{W}_{\mathbb{R}}^{0}(p, q) \)) be the associated moduli space where we identify two connections \( \nabla \) and \( \nabla \) if there exists a local diffeomorphism (resp. orientation preserving local diffeomorphism) of \( \mathbb{R}^{+} \times \mathbb{R} \) intertwining \( \nabla \) and \( \nabla \). Let \( \rho_{C} \) be the Ricci tensor of \( \nabla^{C} \). The components of \( \rho_{C} \) are then given by:

\[
\rho_{C,11} = (x^{1})^{-2}\{(C_{111} - C_{122}^{2} + 1)C_{212}^{1} + C_{112}^{1}(C_{222} - C_{221})\},
\rho_{C,12} = (x^{1})^{-2}\{(C_{222}^{2} + C_{121}^{1}C_{212}^{1} - C_{112}^{1}C_{221})\},
\rho_{C,21} = (x^{1})^{-2}\{-C_{211}^{1} + C_{121}^{1}C_{222}^{1} - C_{112}^{1}C_{221}\},
\rho_{C,22} = (x^{1})^{-2}\{(C_{111} - C_{122}^{2} - 1)C_{212}^{1} + C_{121}^{1}(C_{222} - C_{221})\}.
\]

**Definition 1.3.** The following groups will play an important role in our analysis:

\( \mathcal{G} := \{ T : (x^{1}, x^{2}) \to (mx^{1}, ax^{1} + bx^{2} + d) \text{ for } m > 0 \text{ and } b \neq 0 \} \),

\( \mathcal{H} := \{ T : (x^{1}, x^{2}) \to (mx^{1}, mx^{2} + d) \text{ for } m > 0 \} \subset \mathcal{G} \),

\( \mathcal{I} := \{ T : (x^{1}, x^{2}) \to (x^{1}, ax^{1} + bx^{2}) \text{ for } b \neq 0 \} \subset \mathcal{G} \),

\( \mathcal{I}^{+} := \{ T : (x^{1}, x^{2}) \to (x^{1}, ax^{1} + bx^{2}) \text{ for } b > 0 \} \subset \mathcal{I} \).

The Lie group \( \mathcal{G} \) is the group of affine transformations which preserve \( \mathbb{R}^{+} \times \mathbb{R} \). The group \( \mathcal{G} \) is generated by the subgroups \( \mathcal{H} \) and \( \mathcal{I} \); \( \mathcal{I}^{+} \) is the connected component of the identity in \( \mathcal{I} \). If \( T \in \mathcal{I} \), then \( T \) acts by homotheties and translations; such a \( T \) preserves any Type \( \mathcal{B} \) connection. Let \( \nabla^{e} \) be the flat Euclidean connection on \( \mathbb{R}^{+} \times \mathbb{R} \) with vanishing Christoffel symbols; \( \nabla^{C} = \nabla^{e} + (x^{1})^{-1}C \). Let \( T \in \mathcal{I} \). Since \( T\nabla^{e} = \nabla^{T}T \) and \( T^{\ast}(x^{1}) = x^{1} \), \( T^{\ast}(\nabla^{C}) = \nabla^{T^{\ast}C} \) where \( T^{\ast}C \) is defined by the usual linear action on \( \mathcal{W}_{\mathbb{R}}(2) \). More specifically,

\[
(T^{\ast}C)(\partial_{x^{1}}, \partial_{x^{2}}, dx^{k}) := C(T\partial_{x^{1}}, T\partial_{x^{2}}, Tdx^{k}).
\]
Definition 1.4. If $X$ is a smooth vector field on $M$, let $\Xi^t_X$ be the local flow defined by $X$. We say that $X$ is an affine Killing vector field if $\Xi^t_X \nabla = \nabla$ or, equivalently, (see Kobayashi-Nomizu [34, Chapter VI]), the Lie derivative $L^R_X$ of $\nabla$ by $X$. We say that $X$ is a Killing vector field if $C$ is a Lie algebra of germs of affine Killing vector fields at $P$ and let $\mathfrak{X}_C(P)$ be the space of germs of diffeomorphisms of $\mathbb{R}^+ \times \mathbb{R}$ at $P$ so $\Phi^C := \Phi^{-1} \circ \nabla^C \circ \Phi$ is again a connection of Type $B$. Since the geometry is homogeneous, the particular point $P$ which is chosen is irrelevant. Let $\mathfrak{h} := \text{Span}_K \{x^i \partial_{x_i} + x^2 \partial_{x_2}, \partial_{x_2} \}$ be the Lie algebra of $\mathcal{H}$. If $C \in \mathcal{W}_B(2)$, and $P \in \mathbb{R}^+ \times \mathbb{R}$, then $\mathcal{H} \subset \mathfrak{X}_C(P)$ and $\mathfrak{h} \subset \mathfrak{X}_C(P)$.

The following is the main result of this paper. It shows the moduli spaces $\mathcal{M}^+(p,q)$ are real analytic manifolds and determines the affine Killing vector fields for any $C \in \mathcal{W}(p,q)$.

Theorem 1.5. Let $p + q = 2$.

1. If $C \in \mathcal{W}_B(p,q)$ is not of Type $C$, then $\mathcal{X}(C, P) = \mathcal{G}$, $\mathfrak{X}(P) = \mathfrak{h}$, and $\nabla^C$ is not of Type $A$.
2. $\mathcal{M}^+(p,q)$ may be identified with $\mathcal{B}^+(p,q)/\mathcal{T}^+.$
3. $\mathcal{M}^+(p,q)$ has a natural real-analytic structure.
4. $Z^+_B(p,q) \to \mathcal{B}^+(p,q) \times \mathcal{T}^+$ is a trivial $\mathcal{T}^+$ principal bundle.

Assertion (1) shows that only the linear action is important when considering the local isomorphism type of a Type $C$ geometry if the symmetric Ricci tensor is assumed non-degenerate. Examples of $\mathcal{F}$ show that this can fail of $\rho_{s,C}$ is permitted to have rank 1 even in the torsion free setting; the assumption that $\rho_{s,C}$ is non-singular is essential. If instead of considering the more general case of Type $B$ connections with torsion, we restrict to the subset of Christoffel symbols satisfying the symmetry $C_{ij}^k = C_{ji}^k$, the same arguments hold and we obtain some (but not all) of the results of $\mathcal{F}$ in the Type $B$ setting using an entirely different approach.

We note that $\mathcal{F}$ relied on the complete description of $\mathcal{X}(C, P)$ for all torsion $C$ which was given in $\mathcal{B}$; the discussion of $\mathcal{B}$ shows that the possible algebras of Killing vector fields in the setting of torsion is vastly more complicated and thus an approach based on the exhaustive classification of $\mathcal{F}$ is unlikely to be successful in the setting of connections with torsion.

Here is a brief guide to the remainder of this paper. Assertion 1 will be proved in Section $\mathcal{F}$. In Section $\mathcal{G}$ we use the linear fractional transformations over the complex numbers or over the para-complex numbers to discuss the orientation preserving isometries of the hyperbolic plane and of the Lorentzian analogue. In Section $\mathcal{H}$ we use the action of $\mathcal{T}^+$ to put the symmetric Ricci tensor in normal form. We then examine these normal forms to show $\mathcal{X}(C, P) = \mathcal{H}$ in Section $\mathcal{I}$ and in Section $\mathcal{J}$. We use this analysis to establish the remaining assertions of Theorem 1.5 (1) in Section $\mathcal{K}$ and in Section $\mathcal{L}$. The second assertion of Theorem 1.5 is immediate from the first assertion. The remaining assertions are established in Section $\mathcal{M}$. In Section $\mathcal{N}$ we discuss corresponding results for the unoriented moduli spaces $\mathcal{W}_B(p,q)$.

2. Reducing the structure group

The following metric tensors will play a central role in our analysis:

Definition 2.1. Let 

\[ g_\pm := \frac{dx^1 \otimes dx^1 \pm dx^2 \otimes dx^2}{(x^1)^2} \quad \text{and} \quad g_0 := \frac{dx^1 \otimes dx^2 + dx^2 \otimes dx^1}{(x^1)^2}. \]
The associated Levi-Civita connections $\nabla^\pm$ and $\nabla^0$ are of Type $B$; their non-zero Christoffel symbols and their Ricci tensors are given by

$$\Gamma_{\pm;11}^1 = \Gamma_{\pm;12}^2 = \Gamma_{\pm;21}^2 = -\frac{1}{2r}, \quad \Gamma_{\pm;22}^1 = \pm \frac{1}{2r}, \quad \Gamma_{0,11}^1 = -\frac{1}{2r},$$

$$\rho_{\pm} := \frac{1}{(x^1)^2} \begin{pmatrix} -1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \quad \rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\text{SO}(g)$ be the 3-dimensional Lie group of orientation preserving isometries of the metric $g \in \{ g_+, g_-, g_0 \}$. In Section 2.1 we study $\text{SO}(g_{\pm})$ and in Section 2.2 we study $\text{SO}(g_0)$. In Section 2.3 we use the action of the group $I^+$ to normalize the Ricci tensor to be, modulo sign, $g_+, g_-$, or $g_0$. In the remaining sections, we use the results of these two sections to establish the first assertion of Theorem 1.3 by considering these 3 cases seriatim.

### 2.1. Isometry groups of the metrics $g_{\pm}$.

Let $(u, v) = (x^2, x^1)$ to put things in a more standard form. The orientation preserving isometry groups $\text{SO}(g_{\pm})$ of the metrics $g_{\pm} := v^{-2}(du^2 \pm dv^2)$ can be expressed in terms of linear fractional transformations. Let

$$\text{id} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \iota_- := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \iota_+ := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\iota_\pm^2 = \pm \text{id}$. Let $C_\pm := \text{Span}_\mathbb{R}\{\text{id}, \iota_\pm\} \subset M_2(\mathbb{R})$; $C_-$ is isomorphic to the complex numbers and $C_\pm$ is isomorphic to the para-complex numbers. Let $z_{\pm} := u + iv_\pm \in C_\pm$. If $z_\pm \neq 0$, then $z_\pm$ is invertible; $z_\pm$ is invertible if and only if $u \neq \pm v$. Let

$$T_{\pm,A}(z_{\pm}) := \frac{az_{\pm} + b}{cz_{\pm} + d} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).$$

Note that $T_{\pm,A}$ is not defined on all of $C_{\pm}$ but only on the open dense subset where $cz_{\pm} + d$ is invertible. Let

$$\Re\{z_{\pm}\} := u, \quad \Im\{z_{\pm}\} := v, \quad \tilde{z}_\pm := u - iv_\pm, \quad dz_{\pm} := du + idv_\pm, \quad d\tilde{z} := du - idv_\pm, \quad \text{PSL}(2, \mathbb{R}) := \text{SL}(2, \mathbb{R})/\{\pm \text{id}\},$$

$$\partial_{z_\pm} := \frac{i}{2}(\partial_u id + \iota_\pm \partial_v), \quad \partial_{\tilde{z}_\pm} := \frac{i}{2}(\partial_u id + \iota_\pm \partial_v).$$

We have $z_{\pm} \tilde{z}_\pm = (u^2 - v^2) \text{id}$.

#### Lemma 2.2. Adopt the notation established above.

1. Let $A$ and $B$ belong to $\text{SL}(2, \mathbb{R})$. Then $T_{\pm,A} \circ T_{\pm,B} = T_{\pm,A \circ B}$.
2. Let $w_{\pm} := T_{\pm,A}(z_{\pm})$. Then
   
   (a) $\Re(w_{\pm}) = \Re(z_{\pm})(cz_{\pm} + d)^{-1}(c\tilde{z}_\pm + d)^{-1}$,
   
   (b) $dw_{\pm} = (cz_{\pm} + d)^{-2}dz_{\pm}$ and $d\tilde{w}_{\pm} = (c\tilde{z}_\pm + d)^{-2}d\tilde{z}_\pm$,

   (c) $\partial_{w_{\pm}} = (cz_{\pm} + d)^2\partial_{z_{\pm}}$ and $\partial_{\tilde{w}_{\pm}} = (c\tilde{z}_\pm + d)^2\partial_{\tilde{z}_\pm}$.

3. If $A \in \text{SL}(2, \mathbb{R})$, then $T_{\pm,A}^* g_{\pm} = g_{\pm}$.
4. The map $A \rightarrow T_{\pm,A}$ identifies $\text{PSL}(2, \mathbb{R})$ with the group of orientation preserving isometries of $g_{\pm}$.

#### Proof. Although this result appears in the literature (see, for example, [13]), we give the proof as it is entirely elementary to keep our discussion as self-contained as possible and to establish notation for further use. We prove the first assertion.
by computing:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}, \quad A\tilde{A} = \begin{pmatrix} a\tilde{a} + b\tilde{c} & a\tilde{b} + b\tilde{d} \\ c\tilde{a} + d\tilde{c} & c\tilde{b} + d\tilde{d} \end{pmatrix}, \]

\[ (T_{\pm, A} \circ T_{\pm, B})z_\pm = \frac{a\tilde{a}z_\pm + b\tilde{c} + b\tilde{z}_\pm + d}{c\tilde{a}z_\pm + d\tilde{c} + d\tilde{z}_\pm} = \frac{a(\tilde{a}z_\pm + \tilde{b}) + b(\tilde{c}z_\pm + \tilde{d})}{c(\tilde{a}z_\pm + \tilde{b}) + d(\tilde{c}z_\pm + \tilde{d})} = \frac{(a\tilde{a} + b\tilde{c})\tilde{z}_\pm + (ab + bd)}{(c\tilde{a} + d\tilde{c})\tilde{z}_\pm + (cb + dd)} = T_{\pm, A\tilde{A}}z_\pm. \]

We prove Assertion (2a) by computing:

\[ 2\Im(w_\pm)\epsilon_\pm = T_{\pm, A}z_\pm - T_{A, z_\pm} = \frac{(a\tilde{a}z_\pm + b\tilde{b} + (a\tilde{z}_\pm + b\tilde{c}))}{(c\tilde{a}z_\pm + d\tilde{c} + (c\tilde{z}_\pm + d\tilde{d}))} = 2\Im(z_\pm). \]

We use the quotient rule to prove Assertion (2b) by noting:

\[ dw_\pm = \frac{a(cz_\pm + d) - (az_\pm + b)c}{(cz_\pm + d)^2}dz_\pm = (cz + d)^{-2}dz_\pm. \]

Assertion (2c) then follows by duality. Note that

\[ dz_\pm \otimes \partial z_\pm + d\partial z_\pm \otimes d\partial z_\pm = (du \otimes dv) \otimes (du \otimes dv) = 2(du \otimes dv) = 2(du \otimes dv) \otimes dv \otimes du. \]

We use this identity to express

\[ g_\pm \otimes \partial = \frac{dz_\pm \otimes \partial z_\pm + d\partial z_\pm \otimes dz_\pm}{2\Im(z_\pm)}. \]

Assertion (3) then follows from the identities of Assertion (2) since

\[ \frac{dw_\pm \otimes \partial w_\pm + d\partial w_\pm \otimes w_\pm}{\Im(z_\pm)^2} = \frac{dz_\pm \otimes \partial z_\pm + d\partial z_\pm \otimes dz_\pm}{(cz_\pm + d)^2(z_\pm + d)^2} = \frac{dz_\pm \otimes \partial z_\pm + d\partial z_\pm \otimes z_\pm}{\Im(z_\pm)^2}. \]

We recover the subgroup \( \mathcal{H} \) of Definition 1.3 by considering the transformations:

\[ T_{\pm, A} : z_\pm \rightarrow \begin{cases} mz_\pm & \text{if } A = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \left(\sqrt{m}\right)^{-1} \end{pmatrix}, \\ z_\pm + d & \text{if } A = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \end{cases} \]

These act transitively on \( \mathbb{R}^+ \times \mathbb{R} \) by homotheties and translations. But we also recover additional 1-parameter subgroups:

\[ T_{-\theta}(z_-) := \frac{\cos \theta z_- + \sin \theta}{-\sin \theta z_- + \cos \theta} \in SO(g_-) \text{ for } A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \]

\[ T_{+\theta}(z_+) := \frac{\cosh(t)z_+ + \sinh(t)}{\sinh(t)z_+ + \cosh(t)} \in SO(g_+) \text{ for } A = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \]
Note that
\[
T_{-\theta}(\tau_-) = \frac{\cos \theta \tau_- + \sin \theta}{-\sin \theta \tau_- + \cos \theta} = \frac{(\cos \theta \tau_- + \sin \theta)(\sin \theta \tau_- + \cos \theta)}{(-\sin \theta \tau_- + \cos \theta)(\sin \theta \tau_- + \cos \theta)} = \tau_-,
\]
\[
T_{+\tau}(\tau_+) = \frac{\cosh \tau_+ + \sinh t}{\sinh \tau_+ + \cosh t} = \frac{(\cosh \tau_+ + \sinh t)(\sinh \tau_+ + \cosh t)}{(\sinh \tau_+ + \cosh t)(\sinh \tau_+ + \cosh t)} = \tau_+.
\]

Thus the transformations \( T_{+\theta} \) and \( T_{-\tau} \) preserve \( \tau_{\pm} \). Since \( dT_{\theta}(t) \) is a rotation subgroup and \( dT_1(t) \) is a hyperbolic rotation group on the tangent space of \( \mathbb{C}_{\pm} \) at \( \tau_{\pm} \), we obtain the full group of orientation preserving isometries.

\[\Box\]

**Remark 2.3.** A bit of caution must be observed here since we must regard \( \tau \) as defined on \( \mathbb{R}^2 \) minus the real axis and the elements of \( \text{PSL}(2, \mathbb{R}) \) are not defined everywhere but only have dense ranges and domains. This will play no role in what follows so we suppress this technicality in the interests of brevity and simplicity.

2.2. The isometry group of the metric \( g_0 \). Again, we let \( (u, v) = (x^2, x^1) \) so \( g_0 = v^{-2}(du \otimes dv + dv \otimes du) \). Let \( \Phi_c(u, v) := (u, v)/(1 - cv) \).

**Lemma 2.4.** If \( T \in \text{SO}(g_0) \), then \( T = H \circ \Phi_c \) for some \( c \) and for some \( H \in \mathcal{H} \).

**Proof.** Consider the intertwining map \( U(u, v) := (u, -v^{-1}) = (u, w) \); \( U \) is isometric, i.e. \( (u, v) = (u, -w^{-1}) \). Since \( dv = w^{-2}dw = uv^2dw \),
\[
g_0 = \frac{dx \otimes dy + dy \otimes dx}{v^2} = du \otimes dv + dv \otimes du.
\]
This shows that \( g_0 \) is flat. If \( T \in \text{SO}(g_0) \), let \( T_1 := U \circ T \circ U \in \text{SO}(du \otimes dv + dv \otimes du) \). We then have \( T_1(u, w) = (mu + d, m^{-1}w + c) \). Consequently,
\[
T = U T_1 U : (u, v) \rightarrow (u, -v^{-1}) \rightarrow (mu + d, m^{-1}v^{-1} + c) \rightarrow (mu + d, -(m^{-1}v^{-1} + c)^{-1}).
\]
We take \( c = 0 \) to recover the group \( \mathcal{H} ; (u, v) \rightarrow (mu + d, mv) \). And working modulo the action of \( \mathcal{H} \), we may set \( m = 1 \) and \( d = 0 \). This yields
\[
(u, v) \rightarrow (u, -v/(-1 + cv) = v/(1 - cv). \]  \[\Box\]

2.3. Normalizing the symmetric Ricci tensor. Following Definition 1.3, let \( T_{a,b} : (x^1, x^2) \rightarrow (x^1, ax^1 + bx^2) \) for \( b > 0 \) belong to \( \mathcal{I}^+ \). If \( C \in \mathcal{W}_p(q) \), let \( \rho_{a,C} \)
be the associated Ricci tensor and let \( \rho_{a,C,ij} \) be the components of the Ricci tensor. Note that \( (x^1)^2 \rho_{a,C,ij} \) is constant.

**Lemma 2.5.** Let \( C \in \mathcal{W}_p(q) \).

1. Let \( (p, q) \in \{(2,0), (0,2)\} \). If \( C \in \mathcal{W}_p(q) \), then there exists a unique \( (\lambda(C), a(C), b(C)) \in \mathbb{R}^3 \) so that \( \rho_{a,C}(C) = \lambda(C)g_+ \). The function \( C \rightarrow (\lambda(C), a(C), b(C))(C) \) is a real analytic map from \( \mathcal{W}_p(q) \) to \( \mathbb{R}^3 \).

2. Let \( (p, q) = (1,1) \) and let \( \mathcal{O}_p(1,1) := \{C \in \mathcal{W}_p(1,1) : \rho_{c,22} \neq 0\} \).

(a) If \( C \in \mathcal{O}_p(1,1) \), then there exists a unique \( (\lambda(C), a(C), b(C)) \in \mathbb{R}^3 \) so that \( \rho_{a,C}(C) = \lambda(C)g_+ \). The function \( C \rightarrow (\lambda(C), a(C), b(C))(C) \) is a real analytic map from \( \mathcal{O}_p(1,1) \) to \( \mathbb{R}^3 \).

(b) If \( C \in \mathcal{W}_p(1,1) - \mathcal{O}_p(1,1) \), then there exists a unique \( \varepsilon(C) \) \( \varepsilon(C) + 1 \) and a unique \( (a(C), b(C)) \in \mathbb{R}^2 \) so that \( \rho_{a,C}(C) = \varepsilon g_0 \). The function \( \varepsilon(C), a(C), b(C)) \) is continuous on \( \mathcal{W}_p(1,1) - \mathcal{O}_p(1,1) \).

**Proof.** Because \( T_{a,b}(x^1, x^2) = (x^1, ax^1 + bx^2) \) for \( b > 0 \), we have that:
\[
(T_{a,b})_x(dx^1) = dx^1, \quad (T_{a,b})_x(dx^2) = adx^1 + bdx^2,
\]
\[
(T_{a,b})_x(\partial_{x^1}) = \partial_{x^1} - ab^{-1}\partial_{x^2}, \quad (T_{a,b})_x(\partial_{x^2}) = b^{-1}\partial_{x^2}.
\]
Suppose that $\rho_{s, C, 22} \neq 0$. We apply Equation (1.a) to compute the Ricci tensor. For the moment, let $a(C)$ and $b(C)$ be arbitrary. We compute:

$$\rho_{s, C}((T_{(a(C), b(C))}, \partial z_1, T_{(a(C), b(C))}, \partial z_2) = b(C)^{-1}\{\rho_{s, C, 12} - a(C)\rho_{s, C, 22}\}.$$ 

To ensure that $\rho_{s, C, 12} = 0$, we must therefore take

$$a(C) := \frac{\rho_{s, C, 12}}{\rho_{s, C, 22}} \in \mathbb{R}.$$ 

Thus $a(C)$ is uniquely determined and is a real analytic function of $C$. We set $C_1(C) := T_{(a(C), 1)}C_1$. We now examine

$$\rho_{s, C_1}((T_{(0, \delta)}, \partial_1, (T_{(0, \delta)}, \partial_1),
\rho_{s, C_1}((T_{(0, \delta)}, \partial_2, (T_{(0, \delta)}, \partial_2) = b^{-2}\rho_{s, C_1}(\partial_1, \partial_2).$$

Suppose first that $(p, q) \in \{(2, 0), (0, 2)\}$. Then $\rho_{s, C}$ and hence $\rho_{s, C_1}$ are definite so $\rho_{s, C_1, 11}$ and $\rho_{s, C_1, 22}$ have the same sign and are non-zero. We set

$$b_1(C) := \frac{1}{\rho_{s, C_1, 11}}^{1/2} \text{ and } C_2 := T_{(0, b_1(C))}C_1.$$ 

We have that $\rho_{s, C_2}$ is diagonal and $\rho_{s, C_2, 11} = \pm \rho_{s, C_2, 22}$. Thus $\rho_{s, C_2} = \lambda(C)g_+ \text{ if } C \in \mathbb{W}_B(p, q)$ for $(p, q) \in \{(2, 0), (0, 2)\}$ and $\rho_{s, C_2} = \lambda(C)g_- \text{ if } C \in \mathbb{C}_B(1, 1)$. We use the group law to define $(a(C), b(C))$ so that $T_{(a(b), b(C))} = T_{(a, b)}T_{(0, b(C))}$. The parameters are uniquely determined and vary real analytically with $C$; Assertion (1) and Assertion (2a) now follow.

We complete the proof by establishing Assertion (2b). Suppose that $\rho_{s, C, 22} = 0$. We compute

$$\rho_{s, C}(T_{(a, 1)}, (T_{(a, z_1), \partial z_1}) = (x^1)^{-2}\{\rho_{s, C, 11} - 2a\rho_{s, C, 12}\}$$

so $a(C)$ is uniquely determined by requiring that $P_{s, C, 11} = 0$. Choosing $b$ appropriately, we can then use $T_{(0, b(C))}$ to choose $C_2$ so $P_{s, C, 22} = \pm 1$ and complete the proof of Assertion (2b). 

2.4. Reduction to the general linear group (Case 1). Let $C$ define a Type $B$ structure which is not of Type $C$ with $\rho_{s, C, 22} \neq 0$ and with $\rho_{s, C}$ non-degenerate. Suppose that $\Phi$ is the germ of a diffeomorphism so that $\nabla^C := \Phi^*(\nabla^C)$ is again of Type $B$. We must show $\Phi \in \mathcal{G}$. Let $\nabla^\pm$ be the Levi-Civita connection of the metrics $g_\pm$. Denote the associated Christoffel symbols by $\Gamma^k_{ij} := \tilde{\gamma}^k_{ij}$, they are of Type $B$ and are given in Definition [23]. Pursuant to the discussion in Section [23], we can use the action of $T^\pm$ to assume $\rho_{s, C} = \lambda g_\pm$ and $\rho_{s, C} = \tilde{\lambda} g_\pm$ for $\epsilon = \pm 1$. This implies that $\Phi^*(g_\epsilon) = \frac{\lambda}{\tilde{\lambda}}g_\epsilon$. Since the metric $g_\epsilon$ is homogeneous and has non-vanishing Gauss curvature, $g_\epsilon$ does not admit a non-trivial homothety. Therefore $\lambda = \tilde{\lambda}$ and thus $\Phi$ is an isometry of $g_\epsilon$. By composing with the action $(x^1, x^2) \rightarrow (x^1, -x^2)$ if necessary, we can assume $\Phi$ preserves the orientation and thus $\Phi = T_A$ for $A \in \text{SL}(2, \mathbb{R})$ is given by a linear fractional transformation over the complex numbers or over the para-complex numbers as appropriate, i.e.

$$w := Tz = \frac{az + b}{cz + d} \text{ for } ad - bc = 1.$$ 

Decompose $\nabla^C = \nabla^\epsilon + \frac{1}{2}(C - C_\epsilon)$. The action by pull-back here is the linear action on $C - C_\epsilon$ since $T$ preserves $\nabla^\epsilon$. If $X$ and $Y$ are tangent vectors and if $Z^*$ is a cotangent vector, then

$$\{T(C - C_\epsilon)\}(X, Y, Z^*) = \frac{\tilde{\gamma}(z)}{\tilde{\gamma}(Tz)}(C - C_\epsilon)(TX, TY, TZ^*).$$
We shall suppress the subscripts on \( z_k \) and \( w_{\pm} \) to simplify the notation and use Lemma 2.2 to express objects in the \( z \) coordinate system in terms of the \( w \) coordinate system to express

\[
\frac{1}{3!} (w) = (cz + d)(c\bar{z} + d) \frac{1}{3!} (z), \quad \partial_z = (cz + d)^{-2} \partial_w, \quad dz = (cz + d)^2 dw.
\]

We may then express

\[
\{T^*(C - C_\sigma)\}(\partial_z, \partial_z, dz) = (cz + d)^{-1}(c\bar{z} + d)(C - C_\sigma)(\partial_w, \partial_w, dw),
\]

\[
\{T^*(C - C_\sigma)\}(\partial_z, \partial_z, dz) = (cz + d)(c\bar{z} + d)^{-1}(C - C_\sigma)(\partial_w, \partial_w, dw),
\]

\[
\{T^*(C - C_\sigma)\}(\partial_z, \partial_z, dz) = (cz + d)(c\bar{z} + d)^{-1}(C - C_\sigma)(\partial_w, \partial_w, dw),
\]

\[
\{T^*(C - C_\sigma)\}(\partial_z, \partial_z, dz) = (cz + d)^3(c\bar{z} + d)^{-3}(C - C_\sigma)(\partial_w, \partial_w, dw).
\]

The remaining complex Christoffel symbols are given by conjugation. By hypothesis \( T^*(C - C_\sigma) \) and \( (C - C_\sigma) \) are constant and do not vanish identically since \( C \) is not Type \( C \). Thus at least one of these equations is non-trivial so \( cz + d = c(z + d) \) for all \( z \) and for some \( \sigma \); consequently \( c = 0 \). This implies \( Tz = (az + b)/d \) and hence \( T \in \mathcal{G} \) as desired. This completes the proof in this special case.

2.5. Reduction to the general linear group (Case 2). Let \( C \) define a Type \( B \) structure which is not of Type \( C \) with \( \rho_{g,C,22} = 0 \) and with \( \rho_{s,C} \) is non-degenerate. We can change coordinates to assume \( \rho_{s,C} = \lambda g_0 \). Let \( \Phi \) be the germ of a diffeomorphism so that \( \nabla^C \Phi := \Phi^*(\nabla^C) \) is again of Type \( B \). We argue as before to assume that \( \Phi \) is an isometry of \( g_0 \) and that \( \Phi^*(C - C_0) \) is again of Type \( B \). We apply Lemma 2.4 to assume that \( \Phi(u,v) = (u,v/(1 - cv)) \) where \( (u,v) = (x^2, x^1) \). We must show \( c = 0 \). Let \( \epsilon_{iik} = \delta_{1,i} + \delta_{1,j} - \delta_{1,k} \) and let \( w = v/(1 - cv) \).

\[
\frac{1}{w} = (1 - cv) \frac{1}{w}, \quad dw = (1 - cv)^{-2} dv, \quad \partial_w = (1 - cv)^2 \partial_v,
\]

\[
(\Phi^*(C - C_0))(X, Y, Z^*) = \frac{v}{w}(C - C_0)(\Phi, X, Y, \Phi, Z^*),
\]

\[
(\Phi^*(C - C_0))ijk = (1 - cv) \frac{1}{w^2} (C - C_0)ijk.
\]

Thus either \( C = C_0 \) and \( C \) is of Type \( C \) or \( c = 0 \) and \( \Phi \in \mathcal{G} \). This completes the proof that \( \mathcal{X}(C, P) = H \).

2.6. Affine Killing vector fields and Type \( A \) geometry. The Lie algebra of affine Killing vector fields is the Lie algebra of the group of germs of diffeomorphisms which preserve \( \nabla^C \). Thus the fact that \( \mathcal{X}(C, P) = \mathfrak{h} \) is the Lie algebra of \( H \) follows from the fact that \( \mathcal{X}(C, P) = H \).

2.7. The pull-back of a Type \( B \) structure. We complete the proof of Theorem 3 (1) by showing that if \( \rho_{s,C} \) is non-degenerate, then \( C \) is not Type \( A \). Assume, to the contrary, that there exists the germ of a diffeomorphism \( \Phi \) so that \( \Phi^*(\nabla^C) \) is of Type \( A \). We use Lemma 2.2 to assume \( \rho_{s,C} = \lambda g_0 \) for \( \lambda \neq 0 \) and suitably chosen \( \epsilon \). Since \( \Phi^*\nabla^C \) is Type \( A \), \( \rho_{g^*\nabla^C} \) has constant coefficients and thus is flat. Since \( \Phi^*(\rho_{g^*}) = \rho_{g^*\nabla^C} \), we conclude \( \rho_{g^*\nabla^C} \) is flat. Since the metrics \( g_{\pm} \) have non-zero Gauss curvature, they are not flat. This implies that \( \rho_{g^*\nabla^C} = \lambda g_0 \). Let \( U(u,v) := (u, v - \epsilon^{-1}) \). Then \( U^*(du \otimes dv + dv \otimes du) = g_0 \) so \( \Phi^*U^*(du \otimes dv + dv \otimes du) = \Phi^*(\lambda g_0) \). The metrics \( du \otimes dv + dv \otimes du \) and \( \rho_{g^*\nabla^C} \) have constant coefficients. This implies \( T := U \circ \Phi \) is linear. Since \( U \) is idempotent, \( \Phi = U \circ T \). Thus \( T^*U^*\nabla^C \) is Type \( A \). Since linear maps preserve Type \( A \) structures, we may apply \( (T^{-1})^* = (T^*)^{-1} \) to see \( U^*\nabla^C \) is Type \( A \). Thus without loss of generality we may assume \( \Phi = U \).

Let \( \nabla^e \) be the Levi-Civita connection of the hyperbolic metric \( du \otimes dv + dv \otimes du \); this is the usual flat connection on Euclidean space. Let \( \nabla^0 \) be the Levi-Civita
connection of $g_0$ as given in Definition 4.1. We then have $U$ intertwines $\nabla^c$ and $\nabla^0$. We express $\nabla^C = \nabla^0 + \frac{C-C_0}{\nu}$ and $U^*\nabla^C = \nabla^c + A$ where $A$ is constant. Set $w = -\frac{1}{2}$. We compute

$$A_{ij}^k = U^* \left( \frac{C-C_0}{\nu} \right)_{ij}^k = v_1^{1+2\epsilon_ij} (C-C_0)_{ij}^k.$$ 

This implies $C = C_0$ so $\nabla^C = \nabla^{C_0}$ is the Levi-Civita connection of $g_0$. This is false as $\rho_{g_0} = 0$ and $\rho_{g_0} = \lambda g_0$. Theorem \ref{thm:principal_bundle} (1).

3. The topology of the moduli spaces

3.1. Principal bundles. Let $G$ be a real analytic Lie group which acts in a real analytic fashion on a real analytic manifold $N$. Let $G_P := \{ g \in G : gP = P \}$ be the isotropy group of the action. The action is said to be fixed point free if $G_P = \{ \text{id} \}$ for all $P$. The action is said to be proper if given points $P_n \in N$ and $g_n \in G$ with $P_n \to P \in N$ and $g_nP_n \to \tilde{P} \in N$, we can choose a convergent subsequence so $g_{n_k} \to g \in G$. We refer to \cite{5, 27} for the proof of the following result; see also the discussion in \cite{28}.

**Lemma 3.1.** Let the action of $G$ on $N$ be fixed point free, proper, and real analytic. Then there is a natural real analytic structure on the quotient space $N/G$ so that $G/N \to N/G$ is a principal $G$ bundle.

3.2. The action of $\mathcal{I}^+$ on $\mathcal{W}_B(p,q)$. We have already shown that we may identify the moduli space $\mathcal{W}_B^+(p,q)$ with $\mathcal{W}_B(p,q)/\mathcal{I}^+$. Consequently, Assertion 2 and Assertion 3 of Theorem \ref{thm:moduli_spaces} will follow from Lemma 3.1 and from the following result.

**Lemma 3.2.** Let $p + q = 2$.

1. $\mathcal{I}^+$ acts without fixed points on $\mathcal{W}_B(p,q)$.
2. The action of $\mathcal{I}$ on $\mathcal{W}_B(p,q)$ is real analytic.
3. The action of $\mathcal{I}$ on $\mathcal{W}_B(p,q)$ is proper.

**Proof.** Since there exist a unique $(\lambda, a, b)$ so that $T_{a,b}\rho_{s,C} = \lambda g_0$, it follows that the action of $\mathcal{I}^+$ on $\mathcal{W}_B(p,q)$ is fixed point free and Assertion 1 follows. Assertion 2 is immediate from the definition.

We choose a slightly more convenient parametrization of $\mathcal{I}$ to prove Assertion 3. For $b \neq 0$, set $S_{a,b}(x^1, x^2) := (x^1, b^{-1}(-ax^1 + x^2))$. Then:

$$\begin{align*}
(S_{a,b})_*(dx^1) &= dx^1, & (S_{a,b})_*(dx^2) &= b^{-1}(-adx^1 + dx^2), \\
(S_{a,b})_*(\partial x^1) &= \partial x^1 + a\partial x^2, & (S_{a,b})_*(\partial x^2) &= b\partial x^2. 
\end{align*}$$

(3.a)

We fix $x_1 = 1$ and regard $\rho_{s,C}$ as a function of $C$. We then have

$$\begin{align*}
(S_{a,b}\rho_{s,C})_{11} &= \rho_{s,C,11} + 2a\rho_{s,C,12} + a^2\rho_{s,C,22}, \\
(S_{a,b}\rho_{s,C})_{12} &= b(\rho_{s,C,12} + a\rho_{s,C,22}), & (S_{a,b}\rho_{s,C})_{22} &= b^2\rho_{s,C,22}.
\end{align*}$$

Suppose that

$$C_n \to C_\infty \quad \text{and} \quad \tilde{C}_n := S_{a_n,b_n}C_n \to \tilde{C}_\infty.$$ 

(3.b)

We wish to show there exists a convergent subsequence so $S_{a_n,b_n} \to S \in \mathcal{I}$. We have equivalently

$$\tilde{C}_n \to C_\infty \quad \text{and} \quad C_n = S_{a_n,b_n}^{-1}\tilde{C}_n \to C_\infty.$$ 

Clearly there exists a subsequence so $S_{a_n,b_n} \to S$ if and only if there exists a subsequence so $S_{a_n,b_n}^{-1} \to S^{-1}$. Thus the roles of $C_\ast$ and $\tilde{C}_\ast$ are entirely equivalent.
We compute:

\[
\begin{align*}
\rho_{s,\tilde{C}_{\infty},22} &= \lim_{n \to \infty} b_n^2 \rho_{s,C_n,22}, \\
\rho_{s,\tilde{C}_{\infty},12} &= \lim_{n \to \infty} \{b_n (\rho_{s,C_n,12} + a_n \rho_{s,C_n,22})\}, \\
\rho_{s,\tilde{C}_{\infty},11} &= \lim_{n \to \infty} \{\rho_{s,C_n,11} + 2a_n \rho_{s,C_n,12} + a_n^2 \rho_{s,C_n,22}\}.
\end{align*}
\] (3.c), (3.d), (3.e)

Note that we can conjugate Equation (3.b) replacing \(b\) provides a contradiction as and 

\[
\text{Case 1: } \rho_{s,\tilde{C}_{\infty},22} \neq 0 \text{ and } \rho_{s,\tilde{C}_{\infty},22} \neq 0. \quad \text{Equation (3.a) implies } \lim_{n \to \infty} b_n^2 \text{ exists.}
\]

Thus by passing to a subsequence if necessary, we may assume that 
\[b_n \to b \neq 0.\]

Examine Equation (3.d) then implies \(\lim_{n \to \infty} a_n \text{ exists.}\)

\[
\text{Case 2a. } \rho_{s,\tilde{C}_{\infty},22} \neq 0 \text{ and } \rho_{s,\tilde{C}_{\infty},22} = 0. \quad \text{We may assume that } \rho_{s,\tilde{C}_{\infty}} = \lambda \theta_{\tilde{C}_{\infty}}
\]

and 
\[\rho_{s,\tilde{C}_{\infty}} = \lambda \theta_{\tilde{C}_{\infty}}.\]

Since 
\[\lim_{n \to \infty} \rho_{s,C_n,22} = \rho_{s,\tilde{C}_{\infty},22} = \lambda \neq 0 \text{ and } \rho_{s,\tilde{C}_{\infty},22} = 0,
\]

Equation (3.c) implies \(b_n \to 0.\) By Equation (3.d), \(a_n \to \infty.\) Equation (3.d) then provides a contradiction as 
\[\rho_{s,C_n,22} \to \lambda = 0 \text{ and } \rho_{s,C_n,12} \text{ and } \rho_{s,C_n,11} \text{ are bounded.}\]

\[
\text{Case 2b. } \rho_{s,\tilde{C}_{\infty},22} = 0 \text{ and } \rho_{s,\tilde{C}_{\infty},22} \neq 0. \quad \text{We interchange the roles of } \{C_n, C_{\infty}\}
\]

and \(\{\tilde{C}_n, \tilde{C}_{\infty}\}\) and use the argument of Case 2a.

\[
\text{Case 3. } \rho_{s,\tilde{C}_{\infty},22} = \rho_{s,\tilde{C}_{\infty},22} = 0. \quad \text{By Lemma 2.5 we may assume that } \rho_{s,\tilde{C}_{\infty}} = \varepsilon \theta_{\tilde{C}_{\infty}}
\]

and 
\[\rho_{s,\tilde{C}_{\infty}} = \varepsilon \theta_{\tilde{C}_{\infty}}.\]

Let \(\Psi(\alpha, \beta; C) : \mathbb{R}^2 \times \mathcal{W}_\mathcal{S}(2) \to \mathbb{R}^2\)

be defined by setting:

\[
\Psi(\alpha, \beta; C) := \left((S_{\alpha,\beta} \rho_{s,C})_{12}, (S_{\alpha,\beta} \rho_{s,C})_{11}\right) = (\beta \rho_{s,C,12} + \alpha \beta \rho_{s,C,22}, \rho_{s,C,11} + 2\alpha \rho_{s,C,12} + \alpha^2 \rho_{s,C,22})
\]

We fix \(C\) and compute the Jacobian with respect to \((a, b)\):

\[
\det \Psi' = \det \left( \begin{array}{cc} \beta \rho_{s,C,22} & \rho_{s,C,22} + \alpha \rho_{s,C,22} \\ 2\rho_{s,C,12} + 2\alpha \rho_{s,C,22} & 0 \end{array} \right) = -2(\rho_{s,C,12} + \alpha \rho_{s,C,22})^2.
\]

Suppose 
\[\rho_{s,C,1} = \varepsilon \theta_{\tilde{C}_{\infty}} \text{ for } \varepsilon = \pm 1. \quad \text{Then } \Psi(0,1; C_1) = (\varepsilon, 0) \text{ and } \det \Psi(0,1; C_1) = -2.
\]

Let \(\varepsilon > 0\) be given. The inverse function shows that there exists \(\delta = \delta(\varepsilon) > 0\) so that \(|C - C_1| < \delta,\) then there exists a unique \((\alpha, \beta)\) with 
\[|\varepsilon| < \varepsilon \text{ so that } \Psi(\alpha, \beta; C, \tilde{C}_{\infty}) = (\lambda, 0), \text{ i.e.}\]

\[
(S_{\alpha,C},(\beta(C) \rho_{s,C})_{12} = \varepsilon \text{ and } (S_{\alpha,C},(\beta(C) \rho_{s,C})_{22} = 0 \text{ for } |C - C_1| < \delta.
\]

Assume Equation (3.1) holds with \(\rho_{s,C,\tilde{C}_{\infty}} = \varepsilon \theta_{\tilde{C}_{\infty}}, \) and 
\[\rho_{s,\tilde{C}_{\infty}} = \tilde{\varepsilon} \theta_{\tilde{C}_{\infty}}. \quad \text{We may then choose } (\alpha_n, \beta_n) \to (0,1) \text{ and } (\tilde{\alpha}_n, \tilde{\beta}_n) \to (0,1) \text{ so}\]

\[
(S_{\alpha_n,\beta_n} C_n)_{11} = 0, \quad (S_{\alpha_n,\beta_n} C_n)_{12} = \varepsilon, \quad (S_{\alpha_n,\beta_n} \tilde{C}_{\tilde{C}_n})_{11} = 0, \quad (S_{\alpha_n,\beta_n} \tilde{C}_{\tilde{C}_n})_{12} = \tilde{\varepsilon}.
\]

Thus by replacing 
\[\{C_n, C_{\infty}, \tilde{C}_n, \tilde{C}_{\infty}, S_{a_n,b_n}\}\]

by

\[
\{S_{\alpha_n,\beta_n} C_n, S_{\alpha_n,\beta_n} S_{\alpha_n,\beta_n}^{-1} C_{\tilde{C}_n}, S_{\alpha_n,\beta_n} S_{\alpha_n,\beta_n}^{-1} \tilde{C}_{\tilde{C}_n}, S_{\alpha_n,\beta_n} S_{\alpha_n,\beta_n}^{-1} \tilde{C}_{\tilde{C}_n}, S_{a_n,b_n}\}\].
\]
we may replace Equation (3.1) by

\[ C_n \rightarrow C_\infty, \quad \tilde{C}_n := S_{a_n,b_n} C_n \rightarrow \tilde{C}_\infty, \quad \rho_{s,C_n} = \varepsilon g_0, \quad \rho_{a\tilde{C}_n} = \tilde{\varepsilon} g_0, \]

\[ (\rho_{s,C_n})_{11} = 0, \quad (\rho_{s,C_n})_{12} = \varepsilon, \quad (\rho_{a\tilde{C}_n})_{11} = 0, \quad (\rho_{a\tilde{C}_n})_{12} = \tilde{\varepsilon}. \]

We then get the equations:

\[ 0 = \lim_{n \rightarrow \infty} b_n^2 \rho_{s,C_n,22}, \quad \varepsilon = \lim_{n \rightarrow \infty} b_n (\varepsilon + a_n \rho_{s,C_n,22}), \quad 0 = \lim_{n \rightarrow \infty} (2a_n \varepsilon + a_n^2 \rho_{s,C_n,22}). \quad (3.f, g, h) \]

**Case 3a:** \( \{a_n\} \) is a bounded sequence. We pass to a subsequence to ensure \( a_n \rightarrow a \). Since \( \rho_{s,C_n,22} \rightarrow 0 \), we use Equation (3.1) to conclude \( a_n \rightarrow 0 \). Equation (3.2) then shows that \( \lim_{n \rightarrow \infty} b_n \rightarrow \frac{\varepsilon}{2} \). This completes the proof in this special case.

**Case 3b:** \( \|a_n\| \rightarrow \infty \). Because \( \lim_{n \rightarrow \infty} a_n(2\varepsilon + a_n \rho_{s,C_n,22}) = 0 \), we have that \( \lim_{n \rightarrow \infty} 2\varepsilon + a_n \rho_{s,C_n,22} = 0 \). We substitute this into Equation (3.1) to conclude that \( \lim_{n \rightarrow \infty} b_n(\varepsilon - 2\varepsilon) = \tilde{\varepsilon} \). Thus \( \lim_{n \rightarrow \infty} b_n \) exists and is non-zero. We use Equation (3.3) to compute:

\[ \tilde{C}_{12} = \lim_{n \rightarrow \infty} \left( S_{a_n,b_n} C_n \right)_{12} = \lim_{n \rightarrow \infty} \{b_n C_{12}^1 + a_n b_n C_{22}^1\}, \]

\[ \tilde{C}_{11} = \lim_{n \rightarrow \infty} \left( S_{a_n,b_n} C_n \right)_{11} = \lim_{n \rightarrow \infty} \{C_{11}^1 + 2a_n C_{12}^1 + a_n^2 C_{22}^1\}, \]

\[ \tilde{C}_{22} = \lim_{n \rightarrow \infty} \left( S_{a_n,b_n} C_n \right)_{22} = \lim_{n \rightarrow \infty} \{C_{12}^1 + a_n C_{22}^1 - a_n(C_{12}^1 + a_n C_{22}^1)\}. \quad (3.i, j, k) \]

We examine Equation (3.i) and the sum of Equation (3.j) and Equation (3.k) to see that \( \lim_{n \rightarrow \infty} a_n C_{22}^1 \) and \( \lim_{n \rightarrow \infty} a_n(C_{11}^1 + C_{22}^1) \) exist. Since \( \|a_n\| \rightarrow \infty \), \( \lim_{n \rightarrow \infty} C_{22}^1 = 0 \) and \( \lim_{n \rightarrow \infty} |C_{11}^1 + C_{22}^1| = 0 \). We obtain similarly that \( \lim_{n \rightarrow \infty} (C_{21}^1 + C_{22}^1) = 0 \). Consequently \( C_{22}^1 = 0 \) and \( C_{21}^1 = -C_{22}^1 \).

We impose these relations and compute

\[ (x^1)^2 \rho_{s,C,22} = -2(C_{22}^2)^2 \quad \text{and} \quad (x^1)^2 \rho_{s,C,12} = C_{22}^2 - C_{21}^2 C_{22}^1. \]

Since \( \rho_{s,C,22} = 0 \) by hypothesis, we have \( C_{22}^2 = 0 \) and hence \( \rho_{s,C,12} = 0 \) which is false. Thus this case does not in fact arise.

**3.3. The proof of Theorem 1.5 (4).** Since \( \mathcal{I}^+ \) is a contractible Lie group, it follows that any \( \mathcal{I}^+ \) principal bundle is trivial. The fact that \( W_B(p,q) \rightarrow \mathcal{M}_B(p,q) \) is a trivial principal bundle also follows for \( (p,q) \in \{(2,0),(0,2)\} \) from Lemma 2.5.

**3.4. The unoriented moduli space.** Let \( \mathcal{I}_C := \{ T_{a,b} \in \mathcal{I} : T_{a,b} C = C \} \) be the isotropy group associated to an element \( C \in W_B(p,q) \). Modulo conjugation, we can assume \( a = 0 \) and thus if the isotropy group is non-trivial, \( C \) is invariant under the coordinate transformation \( (x^1, x^2) \rightarrow (x^1, -x^2) \). This yields the relations \( C_{11}^1 = C_{12}^1 = C_{21}^1 = C_{22}^2 = 0 \). We compute:

\[ \rho_{C} = (x^1)^2 \left( (C_{11}^1 - C_{12}^1) C_{21}^1 + (C_{11}^1 - C_{12}^1 - 1) C_{22}^1 \right). \]

We rescale \( x^2 \) to put \( \rho_{C} \) in diagonal form. This normalizes \( C \) and we obtain a smooth real-analytic 3-dimensional submanifold \( \mathcal{S} \) of \( W_B \). Since \( W_B \) is 6 dimensional, the normal bundle of \( \mathcal{S} \) in \( W_B \) is 3-dimensional and the fiberwise quotient by \( \mathbb{Z}_2 \) shows that \( W_B \) is not smooth but has a \( \mathbb{Z}_2 \) orbifold singularity along the image of \( \mathcal{S} \); the link being \( \mathbb{R}P^2 \). By contrast, the corresponding submanifolds of
the torsion free moduli space $\mathcal{Z}_B^+$ are 2-dimensional since $C_{12}^2 = C_{21}^2$ and since the ambient manifold is 4-dimensional, the quotient is smooth and the projection $\mathcal{Z}_B^+ \to \mathcal{Z}_B$ is a ramified $\mathbb{Z}^2$ covering with links $\mathbb{RP}^1 = S^1$. 

References

[1] I. Agricola, S. Chiossi, and A. Fino, “Solvmanifolds with integrable and non-integrable $G_2$ structures”, J. Diff. Geo. and Appl. 25 (2007), 125–135.

[2] I. Agricola, S. Chiossi, T. Friedrich, J. Höll, “Spinorial description of SU(3) and G2 manifolds”, J. Geom. Phys. 98 (2015), 535-555.

[3] T. Arias-Marco and O. Kowalski, “Classification of locally homogeneous affine connections with arbitrary torsion on 2-manifolds”, Monatsh. Math. 153 (2008), 1–18.

[4] X. Bekaert and K. Morand, “Connections and dynamical trajectories in generalized Newton-Cartan gravity I. An intrinsic view”, J. Math. Physics 57 (2016), 022507.

[5] W. Boothby, “An introduction to differentiable manifolds and Riemannian geometry”, Academic Press (1976), New York.

[6] M. Brozos-Vázquez, E. García-Río, and P. Gilkey, “Homogeneous affine surfaces: Killing vector fields and gradient Ricci solitons”, http://arxiv.org/abs/1512.05515.

[7] M. Brozos-Vázquez, E. García-Río, and P. Gilkey, “Homogeneous affine surfaces: Moduli spaces”, http://arxiv.org/abs/1604.06610 To appear J. Math. Anal. and Appl.

[8] S. Bunk, “A method of deforming $G$-structures”, J. Geom. and Phys. 15, 72–80.

[9] E. Calviño-Louzao, E. García-Río, P. Gilkey, and R. Vázquez-Lorenzo, “The geometry of modified Riemannian extensions”, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465 (2009), 2023–2040.

[10] E. Calviño-Louzao, E. García-Río, and R. Vázquez-Lorenzo, “Riemann Extensions of Torsion-Free Connections with Degenerate Ricci Tensor”, IJGMMP 10 (2013), 1350021.

[11] R. Cartas-Fuentevilla, “Fully torsion-based formulation for curved manifolds”, IJGMMP 10 (2013), 1350021.

[12] R. Cartas-Fuentevilla, J. Solano-Altamirano, and P. Enriquez-Silverio, “Post-Riemannian approach for the symplectic and elliptic geometries of gravity”, J. Phys. A. 44 (2011), 195206.

[13] F. Catoni, R. Cannata, V. Catoni, P. Zampetti, “Lorentz surfaces with constant curvature and their physical interpretation”, Italian Physical Society Nuovo Cim. B120 (2005), 37–52; DOI: 10.1393/ncb/i2004-10129-3; arXiv [math-ph/0508012]. New York, 1996.

[14] R. da Rocha, L. Fabbri, J. da Silva, R. Calvalcanti, and J. Silva-Neto, “Flag-dipole spinor fields in ESK gravities”, J. Math. Phys. 54 (2013), 102505.

[15] A. Derdzinski, “Noncompactness and maximum mobility of type III Ricci-flat self-dual neutral Walker four-manifolds”, Q. J. Math. 62 (2011), 363–395.

[16] S. Deser, S. Erti, and D. Grumiller, “Canonical bifurcation in higher derivative, higher spin theories”, J. Phys. A. 46 (2013), 214018.

[17] G. Dileo and A. Lotta, “Some Einstein nil manifolds with skew torsion arising in CR geometry”, IJGMMP 12 (2015), 1560017.

[18] S. Dumitrescu, “Locally homogeneous rigid geometric structures on surfaces” Geom. Dedicata 160 (2012), 71–90.

[19] V. Dzhunushaliev, “Cosmological constant and Euclidean space from nonperturbative quantum torsion”, J. Math. Phys. 53 (2012), 1550008.

[20] L. Fabbri, “A discussion on the most general torsion-gravity with electrodynamics for Dirac spinor matter fields”, IJGMMP 12 (2015), 1560099.

[21] S. Fedoruk and A. Smilga, “Comments on HKT supersymmetric sigma models and their Hamiltonian reduction”, J. Math. Phys. 57 (2016), 033504.

[22] S. Gallot, D. Hulin, J. Lafontaine, “Riemannian Geometry 3rd ed”, Springer Universitext (2014).
[28] P. Gilkey, “The moduli space of Type $A$ surfaces with torsion and non-singular symmetric Ricci tensor”, arXiv:1605.06698 (to appear J. Geometry and Physics).

[29] J. Gauntlett, D. Martelli, and D. Waldram, “Superstrings with intrinsic torsion”, Phys. Rev. D 69 (2004), 086002.

[30] J. Gegenberg, A. Day, H. Liu, and S. Seahra, “An instability of hyperbolic space under the Yang-Mills flow”, J. Math. Phys. 55 (2014), 042501.

[31] O. Hohm and B. Zwiebach, “Towards an invariant geometry of double field theory”, J. Math. Phys. 54 (2013), 032303.

[32] S. Ivanov, “Connections with torsion, parallel spinors, and geometry of spin(7) manifolds”, Math. Research Letters 11 (2004), 171–186.

[33] M. Kassuba, “Eigenvalue estimates for Dirac operators in geometries with torsion”, Ann. Glob. Anal. Geom. 37 (2010), 33–71.

[34] S. Kobayashi and K. Nomizu, “Foundations of Differential Geometry vol. I and II”, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc.,

[35] O. Kowalski and M. Sekizawa, “The Riemann extensions with cyclic parallel Ricci tensor”, Math. Nachr. 287 (2014), 955–961.

[36] R. Lompay and A. Petrov, “Covariant differential identities and conservation laws in metric-torsion theories of gravity. I. General consideration”, J. Math. Phys. 54 (2013), 062504.

[37] R. Lompay and A. Petrov, “Covariant differential identities and conservation laws in metric-torsion theories of gravity. II. Manifestly generally covariant theories”, J. Math. Phys. 54 (2013), 102504.

[38] M. Manev, “A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics”, J. Geom. Phys. 61 (2011), 248–259.

[39] M. Manev, “Natural connection with totally skew-symmetric torsion on almost contact manifolds with B-metric”, IJMMP 9 (2012), 125044.

[40] M. Manev and K. Gribachev, “A connection with parallel totally skew-symmetric torsion on a class of almost hyper complex manifolds with Hermitian and anti-Hermitian metrics”, IJMMP 8 (2011), 115–131.

[41] D. Mejerov, “Natural connection with totally skew-symmetric torsion on Riemannian almost product manifolds”, IJMMP 09 (2012), 1250003.

[42] B. Opozda, “A classification of locally homogeneous connections on 2-dimensional manifolds”, J. Diff. Geo. Appl. 21 (2004), 173–198.

[43] A. Smilga, “Supercharges in the hyper-Kähler with torsion supersymmetric sigma models”, J. Math. Phys. 54 (2012), 112105.

[44] C. Stadtmüller, “Adapted connections on metric contact manifolds”, J. Geom. Phys. 62 (2012), 2170–2187.

[45] S. Vignolo, S. Carloni, and L. Fabbri, “Torsion gravity with non minimally coupled sermonic field: some cosmological models”, Phys. Rev. D. 91 (2015), 043528.

[46] J. Wang, Y. Wang, and C. Yang, “Dirac operators with torsion and the non-commutative residue for manifolds with boundary”, J. Geom. and Phys. 81 (2014), 92–111.