Five dimensional gauge theories
and
Vertex operators

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\textit{To Grigori Olshanski, on his 64th birthday}

\begin{abstract}
We study supersymmetric gauge theories in five dimensions, using their relation to the K-theory of the moduli spaces of torsion free sheaves. In the spirit of the BPS/CFT correspondence the partition function and the expectation values of the chiral, BPS protected observables are given by the matrix elements and more generally by the correlation functions in some $q$-deformed conformal field theory in two dimensions. We show that the coupling of the gauge theory to the bi-fundamental matter hypermultiplet inserts a particular vertex operator in this theory. In this way we get a generalization of the main result of [6] to $K$-theory. The theory of interpolating Macdonald polynomials is an important tool in our construction.
\end{abstract}

\textsuperscript{0}On leave of absense
1 Introduction

1.1

We begin with a brief explanation of the historical motivation behind the problems studied in this paper. The reader may want to consult, for example, [33, 35, 40] for more details.

In [33], partition functions of certain 4-dimensional supersymmetric quantum gauge theories were given a mathematical definition as equivariant integrals over the instanton moduli spaces. The integrands depend on the matter content of the theory and represent characteristic classes of natural vector bundles over instanton moduli.

This description can be generalized [31] to 5-dimensional gauge theories on a product of the original 4-fold $X$ and a circle $S^1$. Integration in equivariant cohomology is then replaced by push-forwards in equivariant $K$-theory.

Since the early days of the studies of $S$-duality in gauge theory [17] and later in string theory the rich structure of conformal field theories in two dimensions was suspected to be within the realm of the four dimensional supersymmetric gauge theories. Partly this feeling was fueled by the constructions in [28, 29, 30] which realized affine Kac-Moody algebras and their deformations acting via some correspondences on the cohomology and the $K$-theory of the moduli spaces of gauge instantons on $\mathbb{R}^4$ and resolutions of its orbifolds.

It was proposed in [33] to interpret the chiral fields of the two dimensional conformal field theory describing the BPS sector of the four dimensional $N = 2$ supersymmetric theory, as well as its equivariant generalizations, the so-called $\Omega$-deformation, as the zero modes of the chiral tensor field in the theory on the M-theory or NS5 brane, which engineers the $N = 2$ theory, by wrapping on a Riemann surface, the Seiberg-Witten curve $\Sigma$. It was later realized in [24, 35] that the conformal field theory need not to live on the Seiberg-Witten curve, but rather some auxiliary curve $C$. The Seiberg-Witten curve $\Sigma$, an emergent geometry, is a ramified covering of $C$. This circle of ideas led to the concept of the BPS/CFT correspondence [34]. In recent years several interesting examples of the BPS/CFT correspondence were found. In [48, 11, 12] a class of four dimensional theories, the so-called $S$-class theories, labelled by a Riemann surface $C$ together with a choice of its pants decomposition, and a choice of the $ADE$ simple Lie group, was proposed. In [11] the evidence for the equality of the partition functions of the
Ω-deformed $S$-class theories of $A_1$ type with the conformal blocks of Liouville theory on $C$ was given, leading to the AGT conjecture. The general $A_k$ case is conjectured to be related to the Toda conformal field theories. In [37] a particular limit of the general Ω-deformed theory was conjectured to capture the spectrum of a quantum integrable system, which for the $S$-class four dimensional gauge theories is a version of quantum Hitchin system. The investigation of the five dimensional theories compactified on a circle should lead to a relativistic [31] (difference) version of the quantum Hitchin system, which is yet another reason to be interested in the story below.

1.2

In this paper we shall focus on the theories with the gauge groups which are the products of the unitary groups, say $U(r) \times U(r')$, with the matter in the bifundamental representation. We study the theory on $X \times S^1$ where $X$ is a complex surface, for example $\mathbb{C}^2$. In holomorphic description, the $U(r)$ instantons correspond to rank $r$ holomorphic bundles on $X$. More precisely, the moduli space of instantons is partly compactified to the moduli space $\mathcal{M}(r)$ of the torsion free rank $r$ sheaves $\mathcal{E}$ on a certain compactification $\tilde{X}$ of $X$, with a trivialization

$$\Phi : \mathcal{E}|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$$

over the compactification divisor $D = \tilde{X} \backslash X$.

On the product $\mathcal{M}(r) \times \mathcal{M}(r')$ of instanton moduli, there is a natural sheaf formed by Ext-groups between the bundles in questions (see Section 2.4 for precise definitions). It describes matter in the bifundamental representation.

As any $K$-theory class on the product, the extension bundle defines a Fourier-Mukai operator

$$\Phi_{\text{Ext}} : K_{G \times T}(\mathcal{M}(r)) \to K_{G \times T}(\mathcal{M}(r'))$$

where $G$ is the group of constant gauge transformations which act on $\mathcal{M}(r)$ by changing the trivialization, $\Phi \mapsto g \circ \Phi$, and $T = \mathbb{C}^\times \times \mathbb{C}^\times$ is the maximal torus of the group $GL(2, \mathbb{C})$. It acts by complex rotations of $X$.

1.3

The structure of $G \times T'$-equivariant cohomology and $G \times T'$-equivariant $K$-theory of $\mathcal{M}(r)$ greatly simplifies when $T'$ preserves a holomorphic symplectic
form on $X$. In this case, it was noted in [35] that $\Phi_{\text{Ext}}$ is a vertex operator for Nakajima’s Heisenberg algebra, see also [22, 24].

The correct fully $G \times T$-equivariant generalization was found for $r = r' = 1$ in [5, 6]. Here we generalize this result to $K$-theory.

The definition of the operator $\Phi_{\text{Ext}}$ is given in Section 2.4.4 and our formula for it in terms of certain deformed Heisenberg operators is the Corollary 1 in Section 4.15.

1.4

Equivariant $K$-theory does not permit a number of dimension-based arguments used in [6]. Therefore, in the present paper we take a very different approach. The computation of $\Phi_{\text{Ext}}$ is eventually reduced to a certain fundamental fact in the harmonic analysis for Cherednik algebra, namely the so-called Macdonald-Mehta-Cherednik identity [8, 10].

This identity computes the values of the Hermitian form

$$(f, g) \mapsto \langle f, \Theta g \rangle_\Delta$$

in the basis of Macdonald polynomials. Here $\langle \cdot, \cdot \rangle_\Delta$ is Macdonald Hermitian inner product associated to a root system and $\Theta$ is the theta function of the corresponding weight lattice. Our formula for $\Phi_{\text{Ext}}$ is obtained from the $GL(N)$ case as $N \to \infty$. In this limit, the vertex operators arise as a stabilization of the triple-product factorization of $\Theta$.

The same problem can be also approached from the angle of interpolation Macdonald polynomials, see Section 6. For brevity, in this note we use it in reverse and deduce an interesting formula for interpolation polynomials from the Macdonald-Mehta-Cherednik identity, see Proposition 2.

1.5

This paper was written in 2008, in time for Grigori Olshanski’s 60th birthday, but had only a very limited circulation. It was meant to be a part of a larger project which, regrettably, has been slow to materialize. We thought that this could be a good occasion to revisit our old manuscript.

While there has been a great progress since 2008 in the understanding of the issues considered here (see for example [2] and also [18] where an equivalent formula may be found), we feel that our basically elementary
reduction to Macdonald-Mehta-Cherednik formulas, as well as the connection to interpolation polynomials, could be of interest.

1.6

The interpolation Schur functions and, more generally, Macdonald polynomials, a subject pioneered by G. Olshanski, have been finding numerous applications in representation theory, algebraic geometry, and mathematical physics, the present paper being yet another example.

An important question, on which we touch here, is why a beautiful theory of such polynomials exists for $GL(n)$ and also $BC_n$, but not, for example, for the roots system $A_n$. In section 6, we discuss a very basic reason for this: the weight lattice $\mathbb{Z}^n$ of $GL(n)$ is an orthogonal direct sum of $\mathbb{Z}$'s.

1.7

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2 Background

An important player in what follows is the moduli space $\mathcal{M}(1) = \Pi_n \text{Hilb}_n$ which has different incarnations. For fixed $n \geq 0$ it is a moduli space of $n$ D0 branes bound to a single D4 brane in the IIA string theory [9], it is a moduli space of $U(1)$ instantons of charge $n$ on a noncommutative Euclidean space $\mathbb{R}^4$ [32], it is a resolution of singularities of the moduli space of charge $n$ $U(r)$ instantons on $\mathbb{R}^4$ for $r = 1$, [29], and it is a Hilbert scheme of $n$ points on $\mathbb{C}^2$. The larger family of varieties that includes moduli $\mathcal{M}(r)$ of torsion-free $\mathbb{C}[x,y]$-modules of rank $r > 1$ and more general Nakajima varieties also play an important role in supersymmetric gauge theories and the question that we study here for Hilbert schemes is interesting and important in that greater generality. In this paper, however, we focus on Hilbert schemes. Thus, from the gauge theory perspective, we restrict ourselves to abelian gauge theories on $\mathbb{C}^2 \cong \mathbb{R}^4$. 

5
2.1 Hilbert schemes of points

2.1.1

For $n = 0, 1, 2, \ldots$, let $\text{Hilb}_n$ denote the Hilbert scheme of $n$ points in the plane $\mathbb{C}^2$. By definition, a point $I \in \text{Hilb}_n$ is an ideal $I \subset \mathbb{C}[x, y] = \mathcal{O}_{\mathbb{C}^2}$ such that

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n.$$ 

It has a natural structure of complex algebraic variety and is known to be smooth and irreducible of dimension $2n$, see for example [30] for an introduction.

2.1.2

There is a universal subscheme

$$\mathfrak{Z} \subset \text{Hilb}_n \times \mathbb{C}^2$$

such that its intersection with $\{I\} \times \mathbb{C}^2$, $I \in \text{Hilb}_n$, is the subscheme of $\mathbb{C}^2$ defined by the ideal $I$. The sheaf of ideals $\mathcal{J}$ of $\mathfrak{Z}$ is called the universal ideal sheaf on $\text{Hilb}_n \times \mathbb{C}^2$.

2.1.3

The group $GL(2)$ acts naturally on $\mathbb{C}^2$ and, hence, on $\text{Hilb}_n$. In this paper, we will be working with the $GL(2)$-equivariant $K$-theory of $\text{Hilb}_n$. We also pick a maximal torus $T$ in $GL(2)$.

We denote by $q^{-1}, t^{-1}$ the torus weights on $\mathbb{C}^2$, that is

$$[\mathbb{C}^2] = q^{-1} + t^{-1}$$

in the representation ring of $G$. This means that torus weights in $\mathbb{C}[x, y]$ are \{q^it^j\}, $i, j \geq 0$.

2.1.4

The physics background for this problem is the following. Consider $\mathcal{N} = 1$ supersymmetric gauge theory in five dimensions, and compactify it on a circle $S = S^1$, of circumference $2\pi r$, with twisted boundary conditions, such that the remaining space $\mathbb{R}^4$ is rotated by an element $(q, t)$ of the maximal torus.
of the spin cover $Spin(4)$ of the rotation group. In addition the fermions in the vector multiplet are subject to the additional $SU(2)$ $R$-symmetry twist

\[
\begin{pmatrix}
(qt)^{\frac{1}{2}} & 0 \\
0 & (qt)^{-\frac{1}{2}}
\end{pmatrix}
\]

which is chosen so as to preserve some fraction of the original supersymmetry. The preserved supercharge can be interpreted as the equivariant de Rham differential acting in the space of equivariant forms on the loop space on the space of four dimensional gauge fields. The source of the loops is the circle $S$ of compactification of the five dimensional theory down to four dimensions.

Actually, the construction described above would lead to $|q| = |t| = 1$. However, by turning on some additional fields in the background of $N = 2$ supergravity one can make $q, t$ general complex numbers. One can also describe this background (the general $\Omega$-background) by lifting the theory to six dimensions, and then compactifying on a two-torus $T^2$. The flat $Spin(4)$ connection on $T^2$, in the limit where the torus collapses to a circle, becomes, effectively, a flat $Spin(4, \mathbb{C})$-connection, which can always be reduced to a $T$-connection. The pair $(q, t)$ is its holonomy.

The common lore states that the gauge theory in five dimensions is not well-defined as the quantum field theory unless it has a cutoff. However, there exist nontrivial fixed points at the strong coupling, which can be realized as limits of compactifications of M-theory on Calabi-Yau three folds, or via brane configurations. Another possible ultraviolet completion of the theory is given by its embedding in the $(2, 0)$ superconformal theory compactified on a circle $S'$.

The $(2, 0)$ theory is, in turn, a limit of the theory on a single M5 brane. The worldvolume $X$ of the M5 brane is six dimensional, and the normal bundle to $X$ is a rank 5 vector bundle $N$ over $X$. The rotation $(4)$ acts geometrically by rotating the fibers of $N$. A generic rotation of a five dimensional Euclidean space has an invariant one dimensional line $\mathbb{R}$.

The full setup therefore involves M-theory on the eleven-dimensional manifold, which is a product of a real line $\mathbb{R}$, a circle $S'$ and a rank 8 vector bundle $N \times X$ over $S$. The geometry of the vector bundle is determined by the two rotation angles which translate to $q$ and $t$ parameters of our story.
2.1.5

While Hilb$_n$ is not compact, the Euler characteristic (a virtual representation of $GL(2)$)

$$\chi_{\text{Hilb}}(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i \text{Ext}^i_{\text{Hilb}}(\mathcal{F}, \mathcal{G})$$ (5)

is well-defined as a $GL(2)$-module for any pair $\mathcal{F}, \mathcal{G}$ of coherent sheaves on Hilb$_n$. Namely, it is a direct sum of irreducibles with finite multiplicities.

To see this, one can use the natural map

$$\pi : \text{Hilb}_n \to (\mathbb{C}^2)^n/S(n)$$

that takes an ideal $I$ to the support of $\mathbb{C}[x, y]/I$, counting multiplicity. The map $\pi$ is proper. On the other hand, the center of $G$ acts on $\mathbb{C}^{2n}/S(n)$ with positive weights.

2.2 Matrix integrals

Using the ADHM construction of the moduli spaces of instantons and torsion free sheaves, the Euler characteristics (5) can be written as an integral over $n \times n$ matrices (see [23, 26, 27] for similar integrals). This integral can be further reduced, using supersymmetric localization, to the $n$-fold contour integral:

$$\chi_{\text{Hilb}_n}(\mathcal{F}, \mathcal{G}) = \frac{1}{n!(2\pi i)^n} \oint \prod_{i=1}^n \frac{dx_i/x_i}{T(w, x_i; qt)} \frac{f(x)g(x^{-1})}{\tilde{\Delta}_{q,t}(x)}$$

$$\tilde{\Delta}_{q,t}(x) = \Gamma'(1)^n \prod_{1 \leq i \neq j \leq n} \Gamma(x_i/x_j),$$

$$\Gamma(y) = \frac{(1 - y)(1 - qty)}{(1 - qy)(1 - ty)},$$

where the contours are the circles $|x_i| = 1$, $|w| \sim 1$ and we assumed $|q|, |t| \ll 1$. Finally, the $K$-theory classes $\mathcal{F}$ and $\mathcal{G}$ are represented by the symmetric functions $f(x)$ and $g(x)$, with the dual bundle $\mathcal{G}^\vee$ corresponding to the symmetric function $g^\vee(x) = g(x^{-1})$. The $K$-theory of Hilb$_n$ is generated by the
tensor functors of the tautological rank \( n \) vector bundle \( V = \mathcal{O}/I \). Write the Chern character of \( V \) as

\[
Ch(V) = \sum_{i=1}^{n} x_i
\]  

(7)

An irreducible representation \( \mu \) of \( GL(n) \) corresponding to a length \( \ell(\mu) \leq n \) partition \( \mu \) gives rise to the associated vector bundle \( V_\mu \) over \( \text{Hilb}_n \), which can be constructed using, e.g. the Young symmetrizer. Its Chern character \( Ch(V_\mu) \) is represented by the Schur function \( s_\mu(x) \):

\[
s_\mu(x) = \frac{\text{Det}_{i,j} \| x_i^{\mu_j-j+n} \|}{\text{Det}_{i,j} \| x_i^{-j+n} \|}
\]  

(8)

The contour integral (6) has to be supplemented with the prescription, that it can be computed by residues, the poles corresponding to the \( T \)-fixed points in \( \text{Hilb}_n \). The latter are labeled by the size \( |\lambda| = n \) partitions \( \lambda \). The possible poles coming from the singularities of \( g(x^{-1}) \) for \( |x_i| < 1 \) should be dropped.

The pole corresponding to the partition \( \lambda \) is at (up to a \( S(n) \)-permutation):

\[
x_\square = w q^{i_\square - \frac{1}{2} t^{j_\square - \frac{1}{2}}}, \quad \square \in \lambda
\]  

(9)

where \( \square = (i_\square, j_\square) \) with \( 1 \leq i_\square \leq \ell(\lambda), 1 \leq j_\square \leq \lambda_i \).

2.3 From matrix to Macdonald

Note that a symmetric function of \( x_i \)'s can be expressed as a function of the power-sums \( p_k = \sum_{i=1}^{n} x_i^k \). One can take, for example, \( p_k \) with \( 1 \leq k \leq n \), as the generators. The measure \( \tilde{\Delta}_{q,t} \) in (6) can be reexpanded:

\[
\tilde{\Delta}_{q,t}(x) = \exp \sum_{k=1}^{\infty} \frac{1}{k} \left( -(1-q^k)(1-t^k)p_k p_{-k} + n \right)
\]  

(10)

We can also expand the \( T \)-factors in (6)

\[
\prod_{i=1}^{n} T(w, x_i; Q^2)^{-1} = \exp \sum_{k=1}^{\infty} \frac{Q^k}{k} (p_kw^{-k} + p_{-k}w^k)
\]  

(11)

Now define, for \( k > 0 \):

\[
\alpha_k = \frac{(w\sqrt{qt})^k}{1-q^k} - (1-t^k)p_k
\]  

(12)
Evaluation of \( \alpha_k \) on \((x_i)_{i=1}^n\) corresponding to a particular pole \((9)\) gives:

\[
\alpha_k = (w \sqrt{qt})^k \sum_{i=1}^{\infty} g^k \lambda_i t^{k \lambda_i} = \sum_{i=1}^{\infty} \xi_i^k
\]

(13)

Now the idea is to pass from the variables \(x_i\), which have the meaning of the positions of D0 branes on \(S \times \mathbb{R}\), to the variables \(\xi_i\). The formula \((12)\) (but not \((13)\)) makes sense for all \(k \neq 0\). In terms of \(\alpha_k\) the product of the measure factors in \((6)\) reads simply as:

\[
(qt)_{q,t} \Delta_{q,t}(x) \prod_{i=1}^{n} T(w, x_i; qt)^{-1} = \exp \sum_{k=1}^{\infty} \frac{1}{k} \left( n - \frac{1 - q^k}{1 - t^{-k}} \alpha_k \alpha_{-k} \right)
\]

(14)

where

\[
(x)_{q_1,q_2} = \prod_{i,j=0}^{\infty} (1 - xq_1^i q_2^j)
\]

(15)

Now, if instead of \((13)\) with an infinite number of \(\xi_i\)’s we take a finite number \(N\) of \(\xi_i\)’s, then, up to a divergent constant, \((14)\) defines a measure on symmetric functions \(F(\xi)\), and a hermitian form on symmetric polynomials \(P(\xi)\):

\[
\langle P, Q \rangle = \frac{1}{(2\pi i)^N N!} \int \prod_{a=1}^{N} d\xi_a \xi_a P(\xi) Q(\xi^{-1}) \Delta_{q,t}(\xi)
\]

\[
\Delta_{q,t}(\xi) = \exp \sum_{k=1}^{\infty} \frac{1 - q^k}{k} (N - \alpha_k \alpha_{-k}) = \prod_{n=1}^{\infty} \prod_{1 \leq a \neq b \leq N} \frac{1 - t^n \xi_a / \xi_b}{1 - qt^n \xi_a / \xi_b}
\]

(16)

This is the scalar product leading to Macdonald polynomials \([25]\).

### 2.4 The Ext bundle

2.4.1

The tangent bundle to \(\operatorname{Hilb}_n\) may be expressed in terms of the universal sheaf \(J\) as follows:

\[
T \operatorname{Hilb} = \chi_{C^2}(\mathcal{O}) - \chi_{C^2}(J,J),
\]

\[
= \chi_{C^2}(\mathcal{O}_3, \mathcal{O}) + \chi_{C^2}(\mathcal{O}, \mathcal{O}_3) - \chi_{C^2}(\mathcal{O}_3, \mathcal{O}_3)
\]

(17)
where the subscripts mean that we take the push-forward along the $\mathbb{C}^2$-factor,

$$\mathcal{O}_3 = \mathcal{O}/J$$

is the structure sheaf of the universal subscheme, and the push-forward in the second line of (17) is well defined because $\mathfrak{3}$ is proper in the $\mathbb{C}^2$-direction.

2.4.2

It is natural to generalize (17) to a bundle $E$ on the product of two Hilbert schemes with fiber

$$E(I, J) = \chi_{\mathbb{C}^2}(\mathcal{O}) - \chi_{\mathbb{C}^2}(I, J)$$

$$= \text{Ext}^1_{\mathbb{P}^1}(I, J(-1))$$

(18)

over a point

$$\mathfrak{3} \in \text{Hilb}_n \times \text{Hilb}_m.$$ 

Here $(-1)$ denotes the twist by the line $\mathbb{P}^2 \setminus \mathbb{C}^2$ at infinity of $\mathbb{C}^2$.

In gauge theory, a bundle like $E$ is related to matter in bifundamental representation of two gauge groups. Here we have the $U(1) \times U(1)$ bifundamental matter.

2.4.3

Any $K$-theory class $\mathcal{E}$ on the product of two varieties $X \times Y$, defines an operator

$$\Phi_\mathcal{E} : K(X) \to K(Y)$$

by the formula

$$\Phi_\mathcal{E}(\mathcal{F}) = p_{X*}(\mathcal{E} \otimes p_Y^*\mathcal{F}).$$

Here

$$p_X : X \times Y \to Y$$

is the projection along $X$, and similarly for $p_Y$. 

11
2.4.4

Our goal in this paper is to describe the operators

$$\Phi_{\Lambda^k E}, \ k = 1, 2, \ldots,$$

where $\Lambda^k E$ denote the exterior powers of $E$. It will be convenient to package them into a generating function

$$W(m) = \sum (-m)^k \Phi_{\Lambda^k E}.$$ (19)

The argument $m$ of this generating function is related to the mass $\mu$ of the bifundamental matter in gauge theory, via

$$m = e^{r\mu}$$ (20)

2.4.5

Concretely, we want to describe them as operators on symmetric functions using the identification

$$K_{GL(2)}(\text{Hilb}_n) \cong (\text{symmetric functions of degree } n) \otimes K_{GL(2)}(\text{pt}),$$

that follows from the work of Haiman [16] and Bridgeland, King, and Reid [4], see below.

In terms of symmetric functions, this task may be phrased as computing matrix coefficients of a certain operator in the basis of Macdonald polynomials.

2.5 Hermitian inner product in $K$-theory

2.5.1

For any $Z$, the Euler form (5) on $K(Z)$ is sesquilinear, but is not symmetric or antisymmetric. This may be fixed if the canonical line bundle $K_Z$ has a square root $K_Z^{1/2}$. For example, for the Hilbert scheme, the canonical class is a pure character

$$K_{\text{Hilb}_n} = (\Lambda^2 C^2)^{-n}$$

and its square root may be extracted on a double cover of $GL(2)$. 
2.5.2

One defines
\[ \langle \mathcal{F}, \mathcal{G} \rangle_Z = \chi_Z \left( \mathcal{F}, K_Z^{1/2} \otimes \mathcal{G} \right), \quad \mathcal{F}, \mathcal{G} \in K(Z) \quad (21) \]

where \( K_Z \) is the canonical line bundle of \( Z \). By Serre duality
\[ \langle \mathcal{F}, \mathcal{G} \rangle_Z = (-1)^{\text{dim}Z} \langle \mathcal{G}, \mathcal{F} \rangle_Z^\vee. \]

In this paper, we will meet only even-dimensional varieties, so the form (21) will be an Hermitian form for us.

2.5.3

We will take adjoints for maps between \( K \)-theories with respect to the form (21). Because the form is symmetric, we don’t need to distinguish between left and right adjoints. In particular, one has
\[ (\Phi_{\mathcal{E}})^* = \Phi_{\mathcal{E}_{\text{adjoint}}} \]

where
\[ \mathcal{E}_{\text{adjoint}} = \mathcal{E}^\vee \otimes K^{1/2}_X \otimes K^{1/2}_Y. \]

Here \( \mathcal{E}^\vee \) is the dual bundle and the square roots \( K^{1/2}_X, K^{1/2}_Y \) are pulled back to \( X \times Y \) along the two projections.

3 Factorization of the Ext operator

3.1

The first step in our computation is completely abstract and general. We consider
\[ j^{\otimes k} \in K_{GL(2) \times S(k)} \left( \text{Hilb} \times (\mathbb{C}^2)^k \right). \]

This is \( k \)-th tensor power of \( j \), except we take the \( \otimes \)-product along Hilb and the exterior \( \boxtimes \)-product along \( \mathbb{C}^2 \).

In other words, we consider \( \text{Hilb} \times \mathbb{C}^2 \) as a variety over Hilb and take exterior tensor product in that category. The symmetric group \( S(k) \) acts on \( j^{\otimes k} \) permuting the factors.
3.2

For any topological space \( Z \) we define:

\[
e^Z = \text{pt} \coprod_{k=1}^{\infty} Z^k / S(k)
\]  

(22)

For any \( Z \), we define \( K(Z^k / S(k)) \) as \( K_{S(k)}(Z^k) \) with Euler form

\[
\chi_{Z^k / S(k)} = (\chi_{Z^k})^{S(k)}
\]  

(23)

and set

\[
K(e^Z) \overset{\text{def}}{=} \bigoplus_{k \geq 0} K(Z^k / S(k)).
\]

For any \( \mathcal{F}, \mathcal{G} \in K(Z) \) we then have

\[
\chi_{Z^k / S(k)}(\mathcal{F}^{S(k)}, \mathcal{G}^{S(k)}) = S^k \chi_{Z}(\mathcal{F}, \mathcal{G}).
\]

Here \( Z \) can be a variety over any base so, in particular, we have

\[
\chi_{C^{2k} / S(k)}(\mathcal{F}^{S(k)}, \mathcal{G}^{S(k)}) = S^k \chi_{C^2}(\mathcal{I}, \mathcal{J}),
\]  

(24)

in \( K_{GL(2)}(\text{Hilb}) \).

3.3

Define

\[
V : K_{GL(2)}(\text{Hilb}) \to K_{GL(2)}(e^{C^2})
\]

(25)

by

\[
V = \bigoplus_{k \geq 0} \Phi \mathcal{F}_k^S.
\]

We also define the grading operators \( L_0 \) in the domain and target of (25) as follows: it acts in

\[
K_{GL(2)}(\text{Hilb}_n), \quad K_{GL(2)}(C^{2n} / S(n))
\]

with the eigenvalue \( n \).

From (24) and definitions, we have

\[
V^* \left( \frac{m}{\sqrt{qt}} \right)^{L_0} V = \sum_k m^k \Phi S_k \chi_{C^2}(\mathcal{I}, \mathcal{J}),
\]

where the star \( * \) denotes the adjoint with respect to (21).
3.4

From
\[
\chi_{C^2}(J, J) = \chi_{C^2}(\emptyset) - E(J, J)
\]
we conclude
\[
S^k \chi_{C^2}(J, J) = \sum_{l=0}^{k} (-1)^l S^{k-l} \chi_{C^2}(\emptyset) \otimes \Lambda^l E.
\]

(26)

Now the convenience of having a generating function \([19]\) becomes clear. We compute, cf. \([15]\)
\[
\sum_{k \geq 0} m^k S^k \chi_{C^2}(\emptyset) = 1/(m)_{q,t},
\]
Therefore, formula \([26]\) gives
\[
W(m) = (m)_{q,t} \sum_k m^k \Phi_{S^k \chi_{C^2}(J, J)}.
\]

Thus we have proven the following

**Theorem 1.** We have
\[
W(m) = (m)_{q,t} \varphi^* \left( \frac{m}{\sqrt{\text{qt}}} \right)^{L_0} \varphi.
\]

Note that the argument was completely abstract and applies to many more general moduli of sheaves.

4 Universal sheaf as an operator

4.1

Our next step is the identification of the operator \(\varphi\) defined in \([25]\). The description we seek goes via the identification
\[
K_{GL(2)}(\text{Hilb}) \xrightarrow{\varphi} K_{GL(2)}(e^{C^2})
\]
\[
\sim \downarrow \Lambda \xrightarrow{\sim} \Lambda
\]

(27)
where \( \Lambda \) denotes symmetric functions over
\[
R = \mathbb{Z}[q, t] \left[ \frac{1}{1 - q^k t^l} \right],
\]
and we similarly extend the scalars in the top row of (27). The vertical identifications in (27) are as follows.

### 4.2

The map
\[
K_{GL(2) \times S(n)}(\mathbb{C}^{2n}) \to K_{GL(2) \times S(n)}(pt) \cong \Lambda_n
\]  \tag{28}

is given simply by taking a global section. Here \( \Lambda_n \) denotes the space of symmetric functions of degree \( n \) and the isomorphism with \( K_{GL(2) \times S(n)}(pt) \) takes a Schur function \( s_\lambda \) to the corresponding irreducible representation \( \{ \lambda \} \) of the symmetric group (viewed as a trivial \( GL(2) \)-module). In other words,
\[
s_\lambda \leftrightarrow \{ \lambda \} \otimes \mathcal{O}_0
\]
where \( \mathcal{O}_0 \) is the skyscraper sheaf at the origin \( 0 \in \mathbb{C}^{2n} \).

### 4.3

The map (28) is an isomorphism of algebras with respect to induction of representations. Concretely, the product of two equivariant sheaves
\[
\mathcal{F}_i \in K_{GL(2) \times S(n_i)}(\mathbb{C}^{2n_i})
\]
is defined to be
\[
\mathcal{F}_1 \cdot \mathcal{F}_2 = \mathbb{C}S(n_1 + n_2) \otimes_{\mathbb{C}S(n_1) \otimes \mathbb{C}S(n_2)} \mathcal{F}_1 \boxtimes \mathcal{F}_2.
\]

### 4.4

The map (28) does not preserve Hom’s, that is, it induces a new Hermitian form on \( \Lambda \). It is best described in terms of the tensor inverse
\[
Kosz = \bigoplus (-1)^i \Lambda^i (\mathbb{C}^{2n})^\vee,
\]

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of the polynomial representation $S^*(\mathbb{C}^{2n})^\vee$. It appears in the Koszul resolution
\[
\text{Kosz} \otimes \mathcal{O}_{\mathbb{C}^{2n}} \to \mathcal{O}_0
\]
of the skyscraper sheaf at the origin. The Hermitian form on induced from the orbifold $\exp(\mathbb{C}^2)$ is
\[
\langle M, M' \rangle = (qt)^{n/2} \text{Hom}_{S(n)}(M, M' \otimes \text{Kosz}^\vee)
\]
for any two $S(n)$-modules $M$ and $M'$.

4.5

In particular, the power-sum functions $p_\mu$ satisfy
\[
\text{Hom}_{S(n)}(p_\mu, M') = \text{tr}_{M'} \sigma_\mu
\]
where $\sigma_\mu$ is a permutation with cycle type $\mu$. The trace of $\sigma_\mu$ in $\text{Kosz}$ is easy to compute and we get
\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z(\mu) g(\mu)
\]
(29)

where $z(\mu)$ is the order of the centralizer of the conjugacy class $\mu$ and
\[
g(\mu) = \prod_i (q^{\mu_i/2} - q^{-\mu_i/2})(t^{\mu_i/2} - t^{-\mu_i/2})
\]

4.6

The isomorphism $\mathcal{B}$ on the left in (27) is the map constructed by Haiman and Bridgeland-King-Reid. Without going in the details of this construction, it suffices to say that it sends the skyscraper sheaves at the torus fixed points to Macdonald polynomials.

The torus fixed points of $\text{Hilb}_n$ are the ideals of the form
\[
I_\lambda = (x_1^{\lambda_1}, x_2^{\lambda_2}y, x_3^{\lambda_3}y^2, \ldots)
\]
where $\lambda$ is a partition of $n$. The symmetric function corresponding to $\mathcal{O}_{I_\lambda}$ is
\[
H_\lambda = t^{n(\lambda)} \Upsilon J_\lambda(q, t^{-1}),
\]
(30)
where $J_\lambda$ is the integral form of the Macdonald polynomial, as defined in Macdonald’s book [25], $\Upsilon$ is an algebra automorphism defined by (cf. (12)):

$$\Upsilon p_k = (1 - t^{-k})^{-1} p_k,$$

and

$$n(\lambda) = \sum (i - 1) \lambda_i.$$

4.7

The map $B$ comes from an equivalence of derived categories and, hence, it is an isometry of $K$-groups. Concretely this means that $H_\mu$ are orthogonal with respect to (29)

$$\langle H_\lambda, H_\mu \rangle = \delta_{\lambda,\mu} \prod_{\text{cotangent weights } w} (w^{1/2} - w^{-1/2}),$$

where $w$ ranges over all torus weights in the cotangent space to Hilb$_n$ at $I_\mu$. The $T$-character of this space is well-known to be

$$\sum_{\square \in \lambda} q^a(\square) + 1 t^{-l(\square)} + q^{-a(\square)} t^{l(\square) + 1},$$

where $a(\square)$ and $l(\square)$ denote the arm and the leg of a square, as usual.

4.8

The line bundle

$$\mathcal{O}(1) = \Lambda^{\text{top}} (\mathcal{O}/I)$$

is the ample generator of the Picard group of the Hilbert scheme. The operator

$$\mathcal{F} \xrightarrow{T} \mathcal{F}(1) = \mathcal{F} \otimes \mathcal{O}(1)$$

plays an important role in the theory. By construction

$$T H_\lambda = q^{n(\lambda')} t^{n(\lambda)} H_\lambda.$$

This eigenvalue is the determinant of the torus action in $\mathcal{O}/I_\lambda$. 

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4.9

By definition,

\[ \mathcal{V} \mathcal{O}_{I_\mu} = \bigoplus_{k>0} (I_\mu)^{\otimes k} \]

Therefore

\[ \langle [\lambda] \otimes \emptyset, \mathcal{V} \mathcal{H}_\mu \rangle = (qt)^{|\lambda|/2} S^\lambda(I_\mu) \] (33)

where \( S^\lambda \) denotes the Schur functor

\[ S^\lambda(A) = \text{Hom}_{S(n)}([\lambda], A^{\otimes n}), \quad n = |\lambda|. \]

4.10

In particular, the map

\[ \Lambda \xrightarrow{\langle \cdot, \mathcal{V} \mathcal{H}_\mu \rangle} \mathbb{R} \]

is an antilinear algebra homomorphism. That is, there is a point \( x_\mu \) in the spectrum of \( \Lambda \) such that

\[ \langle f, \mathcal{V} \mathcal{H}_\mu \rangle = (qt)^{-\deg f/2} \bar{f}(x_\mu). \] (34)

The prefactor here is just to avoid fractional powers later.

4.11

The point \( x_\mu \) may be described very concretely. From definitions

\[ p_k(x_\mu) = (qt)^{k/2} (p_k, \mathcal{V} \mathcal{H}_\mu) \]

\[ = (1 - q^k)(1 - t^k) p_k(I_\mu) \]

\[ = p_k (\{ \text{generators of } I_\mu \}) - p_k (\{ \text{relations} \}) . \] (35)

where \( p_k(I_\mu) \) is the sum of \( k \)-th powers of of the torus weights in \( I_\mu \) and the arguments in \( p_k (\{ \text{generators of } I_\mu \}) \) are the torus weights of generators in a minimal free resolution of \( I_\mu \).

For example, for \( I_{\square} \), we have two generators of weights \( q, t \) and one relation of weight \( qt \), so

\[ p_k(x_\square) = q^k + t^k - (qt)^k . \]
4.12

Now we can state the key result which gives a formula for the operator \( V \) in terms of a Heisenberg algebra action on \( \Lambda \).

We define the creation operators as multiplication by

\[ \alpha_{-n} = \frac{p_n}{(1-q^n)(1-t^n)}, \quad n > 0. \]

We define the annihilation operators as adjoint operators

\[ \alpha_n = \alpha_{-n}^*. \]

It follows that

\[ [\alpha_n, \alpha_m] = \delta_{n+m} n g(n)^{-1}. \]

or, in other words,

\[ \alpha_n = n (qt)^{n/2} \frac{\partial}{\partial p_n}, \quad n > 0. \]

4.13

We also define the corresponding holomorphic/singular parts of a deformed boson

\[ \phi_{\pm}(z) = \sum_{n>0} \alpha_n z^{-n}. \]

These satisfy

\[ \phi_{\pm}(z)^* = -\phi_{\mp}(1/z) \]

and

\[ [\phi_-(z), \phi_+(w)] = \log \left( \frac{\sqrt{qt} w/z}{q,t} \right) \quad (36) \]

where the function \((x)_{q,t}\) was defined in \((15)\).

4.14

Theorem 2.

\[ V = (-1)^{L_0} T e^{\phi_+(1)} e^{\phi_-(\sqrt{q})} \]

Note that in Theorem 2 the operator \( T \) appears after the vertex operator, that is, it is applied on the orbifold side, and not on the Hilbert scheme side where it acts naturally.

Proof of Theorem 2 will be given in Section 5.
4.15

**Corollary 1.** We have

\[
W(m) = \left( \frac{m}{\sqrt{qt}} \right)^{L_0} e^{\phi_+(1) - \phi_+(qt/m)} e^{\phi_-(\sqrt{q}t) - \phi_-(\sqrt{q}/m)}
\]

This follows from Theorems 1 and 2 and commutation relations.

5 **Proof of Theorem 2**

5.1

Our strategy is to reduce Theorem 2 to Macdonald-Mehta-Cherednik identities for $GL(N)$. In particular, the vertex operators will arise as a stabilization of theta functions as $N$ grows to infinity. Here it will be important that theta functions for the weight lattice of $GL(N)$ have a factorization, namely the Jacobi triple product.

The MMC identities are best understood in the context of $SL(2, \mathbb{Z})$ action by automorphisms of Cherednik algebras, see [7, 8, 19]. The same $SL(2, \mathbb{Z})$ action plays a very important role in supersymmetric gauge theories. In the present note, we skip a proper discussion of this topic and go straight for the formulas.

5.2

We can use the operator (31)

\[
\Upsilon : \Lambda \to \Lambda
\]

to pull back the Hermitian form

\[
\langle f, g \rangle_\Upsilon = \langle \Upsilon f, \Upsilon g \rangle.
\]

Then

\[
\langle f, \Upsilon^\top J_\lambda(q, t^{-1}) \rangle_\Upsilon = t^{-n(\lambda)} \bar{f}(q^{\lambda_1}, q^{\lambda_2}t, q^{\lambda_3}t^2, \ldots)
\]

where

\[
\Upsilon^\top = \Upsilon^{-1} \Upsilon.
\]

Indeed, it is enough to check (37) on the generators $p_k$ in which case it reduces to (35).
After replacing $t$ by $t^{-1}$, the Hermitian form $\langle \cdot, \cdot \rangle_\Upsilon$ is the limit of the normalized Macdonald inner product for $GL(N)$ as $N \to \infty$, that is
\[
\langle f, g \rangle_\Upsilon = \lim_{N \to \infty} \frac{\langle \tilde{f} \tilde{g} \rangle_N}{\langle 1 \rangle_N}
\]
where, cf. (16),
\[
\langle g \rangle_N = \frac{1}{N!} \int_{|x_i|=1} g(x) \Delta(x) \mathrm{dHaar}
\]
and, cf. (10),
\[
\Delta(x) = \prod_\alpha \frac{(x^\alpha; q)_\infty}{(x^\alpha/t; q)_\infty}.
\]
Here $\alpha$ ranges over all roots of $GL(N)$.

In Cherednik theory, it is well-known that the operator that takes Macdonald polynomials $J_\lambda$ to evaluation at $q^{\lambda+\rho}$ as in (37) it the theta function of the weight lattice. So Theorem 2 becomes, basically, a matter of taking the limit of the theta function as $N \to \infty$. For $GL(N)$, it is easy because of the factorization of the theta function.

Cherednik’s formula for the root system $A_N$ says, in coordinates,
\[
\frac{1}{N!} \int dx \cdot P_\mu(x) P_{\nu}(x^{-1}) \Theta(x) \Delta(x; q, t) =
q^{\frac{\mu^2}{2} + \frac{\nu^2}{2} + n(\nu)} P_\nu(q^{-\mu}) \prod_{i<j} \frac{(q^{\mu_i+\nu_j+tj-i}; q)_\infty}{(q^{\mu_i+\nu_j+tj-i+1}; q)_\infty}.
\]
(38)
Here $t = q^k$ for $k$ a positive integer, $N > |\mu|, |\nu|$, and
\[
f(q^{-\mu}) = f(q^{-\mu_1}, q^{-\mu_2} t, q^{-\mu_3} t^2, \ldots), \quad \mu^2 = \sum_i \mu_i^2.
\]
Our notation differs from formula 1 of [8], in that we use the parameters $q, t$, for the Macdonald polynomials instead of $q^2, t^2$, and

$$\Delta(x; q, t) = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(x_i x_j^{-1} t; q)_\infty}. $$

Secondly, our definition of $f(q^\nu)$ differs by a factor of $t^{-N/2}$, which is canceled by $t^{(\rho, \nu) - n(\nu)}$, and $\Theta$ is the theta function on the $\mathbb{Z}^N$ lattice,

$$\Theta = \prod_{i=1}^N \vartheta(x_i; q), \quad \vartheta(x) = \sum_n x^n q^{n^2/2}. $$

Using the full lattice does not affect the integral since $N$ is large enough.

By the Jacobi triple product formula,

$$\Theta(x) = (q; q)_\infty^N \Theta_+(x) \Theta_+(x^{-1}), $$

where

$$\Theta_+ = \exp \left( \sum_{k>0} \frac{(-1)^k}{q^{k/2} - q^{-k/2}} p_k \right). $$

Now divide the left hand side by $c_N(q; q)_\infty^N$, where $c_N$ is given by ([25] chapter VI, (9.7)),

$$c_N = \frac{1}{N!} [\Delta]_1 $$

and $[\Delta]_1$ denotes the constant term in $x_i$. Taking the limit as $N \to \infty$ gives

$$\left( \omega \Gamma_-(\frac{q^{1/2}}{1-q}) \omega \cdot P_\mu, \omega \Gamma_-(\frac{q^{1/2}}{1-q}) \omega \cdot P_\nu \right)_{q,t}, $$

where

$$\omega p_k = (-1)^{k-1} p_k $$

is the standard involution on symmetric functions and

$$(p_\mu, p_\nu)_{q,t} = \delta_{\mu, \nu} \delta(\mu) \prod_k \frac{1 - q^{\mu_k}}{1 - t^{\mu_k}}. $$

is the standard Macdonald symmetric inner-product.

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Using the commutation relations, we have

\[
\left( \omega \Gamma_\mu \left( \frac{q^{1/2}}{1-q} \right) \Gamma_\nu \left( \frac{q^{1/2}}{1-t} \right) \omega \cdot P_\mu, P_\nu \right)_{q,t} =

q^{\nu^2 + \nu^2} t^{-n(\nu)} P_\nu(q^{-\mu-\rho}) \prod_i \left( \frac{(t^i; q)_\infty}{(t; q)_\infty} \prod_{j>i} \frac{(q^{\nu_j-\nu_i} t^{j-i}; q)_\infty}{(q^{\nu_j-\nu_i} t^{j-i-1}; q)_\infty} \right) =

q^{\nu^2 + \nu^2} t^{-n(\nu)} P_\nu(q^{-\mu-\rho}). \quad (40)
\]

For the remainder of the paper define a symmetric inner-product by

\[
(p_\mu, p_\nu)_{q,t} = \delta_{\mu,\nu} z(\mu) \prod_k (1 - q^k)(1 - t^k).
\]

Given an expression \( f \), let \( f_{[k]} \) denote the result of substituting \( x^k \) in for \( x \) for all arguments in \( f \), and define the vertex operators on \( \Lambda \) by

\[
\Gamma_\mu = \exp \left( \sum_{k>0} f_{[k]} \frac{\partial}{\partial p_k} \right), \quad \Gamma_\nu = \exp \left( \sum_{k>0} f_{[k]} \frac{\partial}{\partial p_k} \right),
\]

so that

\[
e^{\phi_+(z)} = \Gamma_\mu \left( -\frac{z}{(1-q)(1-t)} \right), \quad e^{\phi_-(z)} = \Gamma_\nu \left( z^{-1} \sqrt{qt} \right)
\]

Now we find the matrix elements of the right hand side in theorem 2

\[
\left\langle H_\mu, (-1)^{L_0} T \ e^{\phi_+(1)} e^{\phi_-(\sqrt{qt})} H_\nu \right\rangle =

\pm \left( \omega T^{-1} H_\mu, (qt)^{-L_0/2} e^{\phi_+(1)} e^{\phi_-(\sqrt{qt})} H_\nu \right)_{q,t} =

\pm q^* t^* \left( H_\mu, e^{\phi_+(1)} e^{\phi_-(\sqrt{qt})} H_\nu \right)_{q,t} =

\pm q^* t^* \left( \Upsilon J_\mu(q, t^{-1}), \Gamma_\mu \left( -\frac{1}{(1-q)(1-t)} \right) \Gamma_\nu (1) \Upsilon J_\nu(q, t^{-1}) \right)_{q,t} =

\pm q^* t^* \left( \Upsilon J_\mu(q^{-1}, t), \Upsilon \Gamma_\mu \left( \frac{t^{-1}}{1-q} \right) \Gamma_\nu \left( \frac{1}{1-t^{-1}} \right) J_\nu(q^{-1}, t) \right)_{q,t} =

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\[ \pm q^* t^* \left( J_\mu, \Gamma_- \left( \frac{t^{-1}}{1 - q^{-1}} \right) \Gamma_+ \left( \frac{1}{1 - t^{-1}} \right) J_\nu \right) \bigg|_{q,t}^{q=q^{-1}} = \]

\[ \pm q^* t^* \left( J_\mu, \omega \Gamma_- \left( \frac{q^{1/2}}{1 - q} \right) \Gamma_+ \left( \frac{q^{1/2}}{1 - t} \right) \omega J_\nu \right) \bigg|_{q,t}^{q=q^{-1}}. \]  

(41)

Here we have left out the sign and the powers of \(q, t\) outside the expression for simplicity, and have made use of the following facts about symmetric polynomials,

\[ \Gamma_\pm(f)x^{L_0} = x^{L_0} \Gamma_\pm(f x^{\pm 1}), \quad \Gamma_\pm(f)\omega = \omega(-1)^{L_0} \Gamma_\pm(-f)(-1)^{L_0}, \]

\[ (\Upsilon f, \Upsilon g) = t^{2\deg(f)}(f, g)_{q,t}, \quad J_\mu(q, t^{-1}) = q^* t^* J_\mu(q^{-1}, t). \]

The last formula follows from the fact that \(P_\mu\) is real, \(\overline{P_\mu} = P_\mu\).

Inserting the integral form

\[ J_\mu = \prod_{\square \in \mu} (1 - q^{\alpha(\square)} t^{\ell(\square) + 1}) P_\mu, \]

into Eq. (41), and using the formula (40), we get

\[ \pm q^* t^* J_\mu(q^{-\nu - \rho}) \bigg|_{q=q^{-1}}. \]

By the definition of \(H_\mu\), we have

\[ \overline{H_\mu}(x_\nu) = t^* J_\mu(q^{-\nu - \rho}) \bigg|_{q=q^{-1}}. \]

Using this relation, keeping track of the signs and powers of \(q, t\), we end up with

\[ \langle H_\mu, \forall H_\nu \rangle, \]

from which the theorem follows. \(\square\)

6 Interpolation polynomials

6.1

In this section, we interpret our results in terms of interpolation Macdonald polynomials, studied by the G. Olshanski and one of the authors,
F. Knop, S. Sahi, E. Rains, and many others, see for example [20, 42, 43] and also [39] for an elementary introduction to the subject. Interpolation Macdonald polynomials are remarkable inhomogeneous symmetric functions defined by Newton-style interpolation conditions. As it turns out, the original orthogonal Macdonald polynomials are their terms of top degree.

Macdonald polynomials generalize characters of semisimple Lie groups, that is, of $G = GL(N)$ in the case at hand, while Macdonald-Ruijsenaars commuting difference operators [44] generalize the center $Z(U_g)$ of the universal enveloping algebra of $G$, or more precisely the radial parts of the invariant differential operators on $G$. There is a natural Fourier pairing

$$Z(U_g) \otimes \mathbb{C}[G]^G \to \mathbb{C}$$

between differential operators and functions on the group that applies an operator to a function and evaluates the result at a special point, in this case, the identity element.

This may be used to construct a distinguished linear basis of $Z(U_{gl}(N))$ formed by elements of a given order that vanish in as many irreducible representations as possible, see [41] for a comprehensive discussion. The Harish-Chandra isomorphism sends this basis to the basis of interpolation Schur functions $s^\ast_{\lambda}$, that becomes the basis $P^\ast_{\mu}$ of interpolation Macdonald polynomials under the deformation.

### 6.2

Concretely, the interpolation Macdonald polynomials $P^\ast_{\mu}$ are the unique, up to multiple, symmetric polynomials of degree $|\mu|$ such that

$$P^\ast_{\mu}(q^\lambda t^\rho) = 0, \quad \mu \nsubseteq \lambda,$$

where $q^\lambda t^\rho = (q^{\lambda_1}, q^{\lambda_2}t^{-1}, q^{\lambda_3}t^{-2}, \ldots)$. These can be normalized to be monic, and, in fact, to satisfy

$$P^\ast_{\mu} = P_{\mu} + \ldots$$

where $P_{\mu}$ are the monic orthogonal Macdonald polynomials and dots stand for terms of lower degree.

After the transformations that take $P_{\mu}$ to $H_{\mu}$, the evaluation condition in (43) becomes

$$H^\ast_{\mu}(\oplus x_{\lambda}) = 0,$$

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where $p_k(\otimes x_\lambda) = -p_k(x_\lambda)$. In formulas, this minus sign appears as the mismatch between the denominator in

$$p_k(q^\lambda t^{-\rho}) = \frac{1}{1 - t^k} \left( p_k \left( \{ \text{generators of } I_\lambda \} \right) - p_k \left( \{ \text{relations} \} \right) \right)$$

and the similar, but not identical factor in (31).

6.3

The pairing (42) becomes the following Fourier pairing

$$(f, H_\mu)_{\text{Fourier}} = f(\otimes x_\mu) H_\mu(\otimes x_\varnothing) .$$

(44)

It is easy to relate it to the Hermitian pairing (34). A basic property of interpolation Macdonald polynomials is their orthogonality

$$(H^*_\mu, H^*_\lambda)_{\text{Fourier}} = 0 , \quad \mu \neq \lambda ,$$

with respect to (44), see [39]. Formally, this orthogonality is equivalent to the binomial formula of [38]. The orthogonality implies, in particular, the symmetry of Fourier pairing (44), which is one of the cornerstones of Macdonald-Cherednik theory, see e.g. [7].

The essence of Macdonald-Mehta-Cherednik identities, and also of Theorem 2, is that the multiplication by the theta function of the weight lattice takes Macdonald Hermitian inner product to a Fourier-type pairing. One can then use a Wiener-Hopf factorization of the theta function, or equivalently the factorization in Theorem 2 into raising and lowering operators, to produce an operator that takes an orthogonal basis for one form to an orthogonal basis for another. This means it takes orthogonal Macdonald polynomials to interpolation Macdonald polynomials. Concretely, we have the following

**Proposition 2.**

$$H^*_\mu = T^{-1} e^{\sum_{n>0} (-1)^{n+1} \frac{\partial^n}{\partial \rho^n} \ T \ H_\mu} .$$

(45)

Conversely, Theorem 2 may be deduced from (45). In fact, this was our original proof, before we realized the equivalence to the MMC identity. Other statements in the literature can be seen equivalent to (45), in particular Proposition 3.5.17 in [17], which goes back to [13].
6.4

From this point of view, the form of the vanishing condition in (43)

\[ P_\mu^*(q^\lambda t^\rho) = 0, \quad \lambda \notin \mu + (\mathbb{Z}_{\geq 0})^\infty, \]

may be traced to the support

\[ (\mathbb{Z}_{\geq 0})^N \subset \mathbb{Z}^N = \text{weight lattice of } GL(N) \]  \hspace{1cm} (46)

of the Wiener-Hopf, that is, triple-product factorization of the theta-function.

Here, by a Wiener-Hopf factorization of a multivariate function we mean its factorization as a function of one of its arguments. Thus, for example, the support of a Wiener-Hopf factorization of a theta-function \( \Theta_L \) of a general lattice \( L \) is a just some half-space of \( L \). A much smaller support in (46) is a special feature of \( GL(N) \), and also of \( BC_n \), which makes the theory of interpolation polynomials much richer for these root system.

This explains why one has a very good theory of interpolation polynomials for \( GL(N) \) but not, for example, for \( SL(N) \).

7 Applications

One application of our result is the calculation of the partition \( \mathbb{Z} \)-functions of the five dimensional \( A_r \)-type quiver \( U(1)^{r+1} \) gauge theories. The latter are characterized by \( r + 1 \) coupling constants \( q_0, q_1, \ldots, q_r \), and \( r + 1 \) masses \( \mu_0, \mu_1, \mu_2, \ldots, \mu_r \).

In this section we change the notations \( (q, t) \mapsto (q_1, q_2) \). The partition function is given by:

\[ Z(\mu, q; q_1, q_2) = \text{Tr} \prod_{i=0}^{r} q_i^{L_0} W(m_i) \]  \hspace{1cm} (47)

We should stress that (47) is obtained using the five dimensional gauge theory considerations. The formalism described in this paper allowed to repackage the instanton sum into the trace of a product of vertex operators, a typical expression for the correlation function of the chiral operators in a \( q \)-deformed conformal field theory on a torus. The expression (47) can be now used to test the duality between the IIA string theory and the M-theory, in that the gauge theory we are discussing in five dimensions is obtained by twisted
compactification of the \((2,0)\)-theory with some defects on an elliptic curve. Let us discuss this in some more detail now.

Note that for \(r = 0\) the theory we are discussing is the \(U(1) N = 2^*\) theory, i.e. the theory with the adjoint hypermultiplet. The latter can be viewed geometrically as the compactification of the \((2,0)\) theory on the elliptic curve with twisted boundary conditions on both the worldvolume and the transverse space of the theory (i.e. the twist involves the \(R\)-symmetry group \(Spin(5)\)). The noncompact part of the worldvolume, the Euclidean space \(\mathbb{R}^4 \approx \mathbb{C}^2\), with the coordinates 1234 is twisted with the parameters \((q_1, q_2)\), while the transverse \(\mathbb{R}^5 \approx \mathbb{C}^2 \oplus \mathbb{R}^1\), with the coordinates 56789 is twisted with the parameters \((m/\sqrt{q_1 q_2}, m^{-1}/\sqrt{q_1 q_2}) \oplus 1\). The trace calculation (47) for \(r = 0\) gives, cf. [35]:

\[
Z_{\text{inst}}(\mu, q; q_1, q_2) = \text{Tr} q^{L_0} W(m) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n m^n} \frac{(m^n - q_1^n)(m^n - q_2^n)}{(1 - Q^n)(1 - q_1^n)(1 - q_2^n)}
\]

(48)

where

\[
Q = \frac{qm}{\sqrt{q_1 q_2}}
\]

In this way we prove the conjecture of [36] modulo the redefinition \(m \mapsto mq_1 q_2\). Actually, the expression (48) does not compute the full partition function of the five dimensional gauge theory. Indeed, the latter must agree with the partition function of the \((2,0)\) theory compactified on a torus, and enjoy some modular properties, as a consequence. It is clear what is the missing piece. The formula (48) is obtained by analyzing the effects of instantons, yet there is also a purely perturbative contribution, which is \(q\)-independent.

It is not hard to compute it, with the result:

\[
Z_{\text{pert}}(\mu; q_1, q_2) = \exp \sum_{n=1}^{\infty} \frac{q_1^n q_2^n}{n} \frac{1 - m^n}{(1 - q_1^n)(1 - q_2^n)}
\]

(49)

Then \(Z = Z_{\text{pert}} Z_{\text{inst}}\) is equal to the partition function

\[
Z = \text{Tr}_\mathcal{H} \left( (-1)^F q^{L_0} q^{-L_0} q_1^{J_{12} - \frac{1}{2}(J_{56} + J_{78})} q_2^{J_{34} - \frac{1}{2}(J_{56} + J_{78})} m^{J_{56} - J_{78}} \right)
\]

(50)

of the free \((2,0)\) six dimensional tensor multiplet computed as a trace over the Hilbert space \(\mathcal{H}\) obtained by quantization on \(S^1 \times \mathbb{R}^4\). It would be interesting to extend this analysis to the case of general \(A_r\) quiver theories.
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