METHODS OF MINIMIZATION OF CALCULATIONS IN HIGH ENERGY PHYSICS:
I. A Covariant Method of the Calculation of Amplitudes of Processes Involving Polarized Dirac Particles

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March 26, 2022

Abstract

The article deals with a number of the existing variants of direct calculation of amplitudes of processes with polarized Dirac particles. It is shown, that all of them are special cases of one and the same mathematical scheme. The advantages and disadvantages of this scheme are considered. A new variant of the method of calculation of amplitudes, keeping all the advantages of this scheme, but free from disadvantages, is proposed. In particular, this variant is suitable for the evaluation of amplitudes of processes with interfering diagrams and does not need any additional calculations, that are necessary in the general case. Expressions for the amplitudes of processes involving both massive and massless particles are presented. These expressions make it possible to create in an easy way the computer programs for automatic calculation of amplitudes. The existing computer programs for calculation of amplitudes, their limitations and disadvantages are briefly considered.

1 Introduction

In order to calculate the observables of the processes with the Dirac particles, especially for high-order diagrams and in the case of taking into account the polarization effects, it
is necessary to calculate the traces of products of great number of the Dirac $\gamma$-matrices. Thus, we face the problem of obtaining the analytical expressions for the physical quantities we are interested in.

One way to eliminate this problem is to calculate the amplitudes of the processes directly. This method allows us to reduce essentially the number of the Dirac $\gamma$-matrices in the expressions considered so also the number of the expressions themselves (in the situation when we have the interfering diagrams). The results obtained in this way depend only on 4-vectors and are suitable for numerical analysis. To calculate the squared amplitude (by means of which observables are expressed), we need to calculate only the squares of the modules of the complex numbers obtained. The numerical values obtained in this way are equivalent in principle to the values obtained in the classical way, but can have smaller errors caused by loss of accuracy at the calculations.

The appeal of such an approach has stimulated a lot of articles, where a number of different ways of calculation of amplitudes have been proposed. However all of them have the limitations. Moreover, there are the incorrect methods (for example [47] – [49], see subsection 6 of Section 6). These reasons and the fact that the expressions which have been obtained by different ways for the same amplitudes have different analytical form lead to a certain mistrust to the idea of direct calculation of amplitudes.

Up to now the methods of calculation of amplitudes have been separated into two classes: the covariant methods, i.e. the methods reduced to the evaluation of the traces of the products of the Dirac $\gamma$-matrices (see [1] – [25]) and the methods reduced to the multiplication of $\gamma$-matrices and bispinors which are written in the matrix form (see [28] – [36], [38] – [42]). However all of them are in fact the special cases of one and the same mathematical scheme.

In Section 2 the covariant methods of calculation are briefly considered. The general mathematical scheme of calculation of amplitudes is presented. It describes all the known methods. It is shown also how the results of any special method can be transformed into the results of another one.

In Section 3 the advantages and disadvantages of the general scheme are considered. In particular we consider carefully the additional calculations which are necessary to be performed in the general case when one has the interfering diagrams.

In Section 4 a new covariant method of calculation of amplitudes is proposed. This method keeps all the advantages of the general scheme but is free from its disadvantages.

In Section 5 the general expressions for the amplitudes of processes involving both massive and massless Dirac particles are presented. (Examples of calculation of the concrete processes can be found in [1].) Using the presented formulae one can create the computer program for automatic calculation of amplitudes.

In Section 6 the methods of calculation of amplitudes based on the multiplication of $\gamma$-matrices and bispinors which are written in the matrix form and some similar methods are considered. It is shown that all of them are the special cases of the general scheme considered.
In Section 7 the existing computer programs for automatic calculation of amplitudes, their limitations and drawbacks are briefly considered.

2 The general scheme of the covariant calculation of amplitudes

There is an even number \((2N)\) of fermions in initial and final state for any reaction with Dirac particles. Therefore every diagram contains \(N\) nonclosed fermion lines. The expression

\[
M_{if} = \bar{u}_f Qu_i
\]

(1)

corresponds to every line in the amplitude of the process, \(u_i, u_f\) are the Dirac bispinors for free particles. (For definiteness, we assume that fermions are particles. However the results obtained can be generalized to the case when both of the fermions are antiparticles or one fermion is a particle and another one is an antiparticle. This generalization is considered at the end of Section 4.)

\[
\bar{u} = u^+ \gamma^0,
\]

and \(Q\) is a matrix operator characterizing the interaction. The operator \(Q\) is expressed as a linear combination of the products of the Dirac \(\gamma\)-matrices (or their contractions with 4-vectors) and can have any number of free Lorentz indexes.

The formula (1) can be rewritten as

\[
M_{if} = \text{Tr}(Qu_i \bar{u}_f)
\]

however the expression for the operator \(u_i \bar{u}_f\) is not known. Because of this, in order to calculate \(M_{if}\) we use the following general scheme

\[
M_{if} = \bar{u}_f Qu_i = (\bar{u}_f Qu_i) \cdot \frac{\bar{u}_i Zu_f}{\bar{u}_i Zu_f} = \text{Tr}(Qu_i \bar{u}_i Zu_f \bar{u}_f) \frac{\bar{u}_i Zu_f}{\bar{u}_i Zu_f}
\]

\sim \frac{\text{Tr}(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{|\bar{u}_i Zu_f|} = \frac{\text{Tr}(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{\sqrt{\text{Tr}(Zu_i \bar{u}_i Zu_f \bar{u}_f)}} = M_{if}\]

(2)

where \(Z\) is an arbitrary \(4 \times 4\)-matrix,

\[
\bar{Z} = \gamma^0 Z^+ \gamma^0
\]

(symbol \(\sim\) stands for ”an equality up to a phase factor sign”).

The projection operators are substituted for \(u \bar{u}\) in (2). For a particle with mass \(m\) we have:

\[
u(p, n) \bar{u}(p, n) = \frac{1}{2} (\hat{p} + m)(1 + \gamma_5 \hat{n}) = \mathcal{P}(p, n)
\]

(3)
where \( \hat{a} = \gamma_\mu a^\mu \) for any 4-vector \( a \),

\[
p^2 = m^2, \quad n^2 = -1, \quad pn = 0, \quad \bar{u}u = 2m, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3
\]

(\( p \) is the 4-momentum of the particle, \( n \) is the 4-vector specifying the axis of the spin projections of the particle).

For a massless particle the projection operator is the following

\[
u_\pm(p)\bar{u}_\pm(p) = \frac{1}{2}(1 \pm \gamma_5)\hat{p} = \mathcal{P}_\pm(p)
\]

where

\[
p^2 = 0, \quad \bar{u}_\pm\gamma_\mu u_\pm = 2p_\mu
\]

(signs \( \pm \) correspond to the helicity of particle).

Actually the general scheme \( (2) \) describes all the known methods of calculation of amplitudes.

1. In the articles \( [2] - [9] \) one chooses 

\[
Z = 1.
\]

The results of the articles \( [10] \) reduce to the same choice for the calculation of amplitudes of type \( \bar{u}_\pm(p_f)u_\mp(p_i) \) for the processes with massless particles.

2. In \( [8], [9] \) one proposes 

\[
Z = \gamma_5.
\]

The results of the article \( [11] \) reduce to the same choice.

3. The results of \( [12], [13] \) reduce to

\[
Z = \gamma^0.
\]

The expressions obtained in the article \( [14] \) reduce to the same choice under the following restrictions:

(a) Dirac particles with equal mass in the center-of-mass frame are considered;

(b) the polarization state of the particles is the helicity, i.e. 4-vectors, which determine the axes of spin projections, are (see e.g. \( [60] \));

\[
n^\mu(p) = \frac{\lambda}{m} \left( \frac{p_0}{|p|}, \frac{p}{|p|} \right), \quad \lambda = \pm 1;
\]

\(^1\) We use the same metric as in the book \( [37] \):

\[
a^\mu = (a_0, \vec{a}), \quad a_\mu = (a_0, -\vec{a}), \quad ab = a_\mu b^\mu = a_0b_0 - \vec{a}\vec{b}
\]
(c) the particles have the opposite helicities.

4. The results of [14] also reduce to the choice

\[ Z = \gamma^0 \gamma_5 \hat{l} \]

[where \( l^\mu = \frac{1}{|\vec{p}| \sqrt{p_x^2 + p_y^2}} \left[ 0, p_x p_z, p_y p_z, -(p_x^2 + p_y^2) \right] \), \( l^2 = -1 \), \( (lp_i) = (lp_f) = 0 \)]

under the following restrictions:

(a) Dirac particles with equal mass in the center-of-mass frame are considered;
(b) the polarization state of the particles is the helicity;
(c) the particles have equal helicities.

5.1. In [15] it was proposed to choose for \( Z \)

\[ Z = \hat{l} \]

where \( l \) is an arbitrary 4-vector such that \( l^2 = -1 \). The same choice was proposed in [5] for the calculation of amplitudes of processes with massless particles.

5.2. In [16] it was proposed to choose for \( Z \)

\[ Z = \hat{q} \]

(where \( q \) is an arbitrary 4-vector such that \( q^2 = 0 \), it was proposed to choose \( q^\mu = (1, 1, 0, 0) \) for numerical calculations) under the following restrictions:

(a) the 4-vectors, which determine the axes of the spin projections, are

\[ n = \lambda \left[ \frac{1}{m^2} p - \frac{m}{(pq) q} \right], \; \lambda = \pm 1 \]

\((p \) is the 4-momentum of the corresponding particle),

(b) \[ \lambda_i = \lambda_f \]

5.3. In [9] it was also proposed that for the calculation of amplitudes of processes with massless particles

\[ Z = \hat{r} \]

where \( r \) is an arbitrary 4-momentum such that \( r^2 = m^2 \).
6. The results of papers [17] – [19] reduce to
\[ Z = 1 + \gamma^0 . \]

The results of the paper [20] reduce to the same choice under the restriction
\[ n_f = \left( (n_f)_0, \vec{n}_i - \frac{(n_i)_0}{(p_i)_0 + m_i} \vec{p}_i + \frac{(n_f)_0}{(p_f)_0 + m_f} \vec{p}_f \right) , \]
\[ (n_f)_0 = \frac{1}{m_f} \left[ (\vec{p}_f \vec{n}_i) - \frac{(n_i)_0}{(p_i)_0 + m_i} (\vec{p}_f \vec{p}_i) \right] . \]

7. The results of [20] also reduce to
\[ Z = (1 + \gamma^0) \gamma_5 \hat{l} \]
where
\[ n_f = \left( -(n_f)_0, \vec{n}_i + \frac{(n_i)_0}{(p_i)_0 + m_i} \vec{p}_i - \frac{(n_f)_0}{(p_f)_0 + m_f} \vec{p}_f \right) , \]
\[ (n_f)_0 = \frac{1}{m_f} \left[ (\vec{p}_f \vec{n}_i) - \frac{(n_i)_0}{(p_i)_0 + m_i} (\vec{p}_f \vec{p}_i) \right] . \]
where
\[ l^\mu = (0, -\vec{o}) , \quad \vec{o}^2 = 1 , \quad (\vec{o} \vec{n}_i) = \frac{(n_i)_0}{(p_i)_0 + m_i} (\vec{o} \vec{p}_i) . \]

8. The results of papers [21], [22] reduce to
\[ Z = m + \hat{r} \]
\[ (r \text{ is an arbitrary 4-momentum such that } r^2 = m^2) , \text{ as this takes place in this paper} \]
\[ \text{for the 4-vectors, which determine the axes of the spin projections, one uses} \]
\[ n_i = \frac{m_i^2 p_f - (p_i p_f) p_f}{m_i \sqrt{(p_i p_f)^2 - m_i^2 m_f^2}} , \quad n_f = -\frac{m_f^2 p_i - (p_i p_f) p_f}{m_f \sqrt{(p_i p_f)^2 - m_i^2 m_f^2}} . \]

9. The results of [23] for the amplitudes of type \( \bar{u}_\pm(p_f) u_\pm(p_i) \) with massless particles reduce to
\[ Z = \hat{q}_1 \hat{q}_2 \]
where \( q_1, q_2 \) are arbitrary 4-vectors such that \( q_1^2 = q_2^2 = 0. \)
The results of the papers [24], [25] for the amplitudes of type $\bar{u}_\pm(p_f)u_\mp(p_i)$ with massless particles reduce to

$$Z = \hat{l}\hat{q}$$

where $l$, $q$ are arbitrary 4-vectors such that

$$l^2 = -1, \quad q^2 = 0, \quad (ql) = 0.$$ 

As this takes place one chooses $l^\mu = (0, 0, 1, 0)$, $q^\mu = (1, 1, 0, 0)$ for numerical calculations.

The results of the paper [15] reduce to the same choice for the calculation of amplitudes of processes with massive particles under the following restrictions:

(a) for the 4-vectors, which determine the axes of the spin projections, one uses

$$n = \lambda \left[ \frac{1}{m} p - \frac{m}{(pq)} q \right], \quad \lambda = \pm 1$$

($p$ is the 4-momentum of the corresponding particle),

(b) $\lambda_i = -\lambda_f$

The remaining methods of the calculation of amplitudes are considered in Section 6. We emphasize again that all this methods can be reduced to (2).

The results of the calculation of amplitudes for different $Z$ differ only by the phase factor. Really,

$$M_{if}(Z_1) = \frac{\text{Tr}(QP_iZ_1P_f)}{\sqrt{\text{Tr}(Z_1P_iZ_1P_f)}} = \frac{\text{Tr}(QP_iZ_1P_f)}{\sqrt{\text{Tr}(Z_1P_iZ_1P_f)}} \cdot \frac{\text{Tr}(Z_2P_iZ_2P_f)}{\text{Tr}(Z_2P_iZ_2P_f)}$$

$$= \frac{\text{Tr}(QP_iZ_1P_f) \cdot \text{Tr}(Z_2P_iZ_2P_f)}{\sqrt{\text{Tr}(Z_1P_iZ_1P_f) \cdot \text{Tr}(Z_2P_iZ_2P_f) \cdot \text{Tr}(Z_1P_iZ_1P_f) \cdot \text{Tr}(Z_2P_iZ_2P_f)}}$$

$$= \frac{\text{Tr}(QP_iZ_2P_f) \cdot \text{Tr}(Z_2P_iZ_1P_f)}{\sqrt{\text{Tr}(Z_1P_iZ_1P_f) \cdot \text{Tr}(Z_2P_iZ_2P_f) \cdot \text{Tr}(Z_1P_iZ_1P_f) \cdot \text{Tr}(Z_2P_iZ_2P_f)}}$$

$$= \frac{\text{Tr}(QP_iZ_2P_f)}{\sqrt{\text{Tr}(Z_2P_iZ_2P_f)}} \cdot \left[ \frac{\text{Tr}(Z_2P_iZ_1P_f)}{\text{Tr}(Z_1P_iZ_2P_f)} \right]^{1/2} = M_{if}(Z_2) \cdot e^{i\phi},$$

where

$$e^{i\phi} = \left[ \frac{\text{Tr}(Z_2P_iZ_1P_f)}{\text{Tr}(Z_1P_iZ_2P_f)} \right]^{1/2}.$$
In (5) we have used the identity:

\[ \text{Tr}(AP_iBP_f) \cdot \text{Tr}(CP_iDP_f) = \text{Tr}(Au_i\bar{u}_iBu_f\bar{u}_f) \cdot \text{Tr}(Cu_i\bar{u}_iDu_f\bar{u}_f) \]

\[ = (\bar{u}_fAu_i)(\bar{u}_iBu_f)(\bar{u}_fCu_i)(\bar{u}_iDu_f) \equiv (\bar{u}_fAu_i)(\bar{u}_iBu_f)(\bar{u}_fDu_i)(\bar{u}_iCu_f) \]

\[ = \text{Tr}(Au_i\bar{u}_iDu_f\bar{u}_f) \cdot \text{Tr}(Cu_i\bar{u}_iBu_f\bar{u}_f) = \text{Tr}(AP_iDP_f) \cdot \text{Tr}(CP_iBP_f) \tag{7} \]

where \( A, B, C, D \) are arbitrary \( 4 \times 4 \)-matrices.

3 Advantages and disadvantages of the general scheme

Let us consider the advantages and disadvantages of the general scheme of calculation of amplitudes (2) illustrating them by the elementary example with \( Z = 1 \).

The advantages are following:

1. The calculation of the amplitude of the process is analogous to the calculation of the squared amplitude of this process

\[ |M_{if}|^2 = (\bar{u}_fQu_i)(\bar{u}_iQu_f)^* = (\bar{u}_fQu_i)(\bar{u}_i\bar{Q}u_f) = \text{Tr}(Qu_i\bar{u}_i\bar{Q}u_f\bar{u}_f) = \text{Tr}(QP_i\bar{Q}P_f) \tag{8} \]

and reduces to the standard operation of the evaluation of the trace of a linear combination of the products of the Dirac \( \gamma \)-matrices.

As this takes place, for the calculation of amplitudes it is enough to make the replacement in (8)

\[ \bar{Q} \rightarrow \frac{Z}{\sqrt{\text{Tr}(ZP_iZP_f)}} \tag{9} \]

This observation allows us to supplement the computer algebra systems, using standard methods for calculation of cross sections (e.g. [62], [63]), by the programs for the calculation of the amplitudes.

For \( Z = 1 \) the replacement (9) takes the form

\[ \bar{Q} \rightarrow \frac{1}{\sqrt{[m_i\bar{m}_f + (p_ip_f)] [1 - (n_in_f)] + (p_in_f)(p_fn_i)}} \]

2. The expression for the amplitude of the process contains lesser number of the Dirac \( \gamma \)-matrices than the expression for the square of the modulus of the amplitude of the same process. Because of this, after the evaluation of traces of the \( \gamma \)-matrices the expression for the amplitude contains lesser number of terms than the expression for the square of the modulus of the same amplitude. The estimate of a gain for \( Z = 1 \)
is presented in Table 1. (Here we consider the processes involving massive particles. The maximum number of the $\gamma$-matrices in projection operators of the massive particles is contained in the terms $\hat{p}_i \gamma_5 \hat{n}_i$ and $\hat{p}_f \gamma_5 \hat{n}_f$. Therefore the maximum contribution from the projection operators in expressions considered is equal to 4 $\gamma$-matrices.)

| Number of $\gamma$-matrices in $Q$ , $N$ | 1 | 3 | 5 |
| Number of $\gamma$-matrices in the term of $|M_{if}|^2$, which gives the maximum contribution, $N_1 = 4 + 2N$ | 6 | 10 | 14 |
| Number of the terms after the trace evaluation, $(N_1 - 1)!!$ | 15 | 945 | 135135 |
| Number of $\gamma$-matrices in the term of $M_{if}$, which gives the maximum contribution$^2$, $N_2 = (4 + N) - 1$ | 4 | 6 | 8 |
| Number of the terms after the trace evaluation, $(N_2 - 1)!!$ | 3 | 15 | 105 |
| The estimate of a gain, $(N_1 - 1)!!/((N_2 - 1)!!)$ | 5 | 63 | 1287 |

Table 1: The estimate of a gain obtained at calculation of the amplitudes for $Z = 1$

It is follows from Table 1 that the more complicated the process considered (i.e. the greater number of the $\gamma$-matrices is contained in the operator $Q$ which describes the interaction) the greater gain is given by the method of the calculation of the amplitude in comparison with the standard method of the calculation of the square of the modulus of the amplitude. This is also true for any operator $Z$ in the general scheme (2).

Really, when the number of $\gamma$-matrices in operator $Q$ increases by $I$, their number in the numerator of the expression (2) for the amplitude increases only by $I$, (denominator does not change), but in the expression (8) for the squared amplitude the number of $\gamma$-matrices increases by $2I$ (as the formula (8) contains both the operator $Q$ and the operator $\bar{Q}$).

However, the scheme considered has two essential disadvantages in the general case:

1. There is a denominator in (2), hence the ambiguity of the type $0/0$ can appear during the calculations. For example if one chooses $Z = 1$ then the expressions for the amplitudes have the denominator

$$\sqrt{Tr[P(p_i,n_i)P(p_f,n_f)]} = \sqrt{[m_i m_f +(p_i p_f)][1 -(n_i n_f)] + (p_i n_f)(p_f n_i)}$$

which is equal to zero for

$$n_f = -n_i + \frac{(p_f n_i)}{m_i m_f +(p_i p_f)} (p_i + \frac{m_i}{m_f} p_f)$$

$^2$In this case $(4 + N)$ is an odd number, and consequently the term with $(4 + N)$ $\gamma$-matrices vanishes
2. All the expressions for the amplitudes [as it follows from (4)] are known up to a phase factor:

$$\mathcal{M}_{if} = M_{if} \frac{\bar{u}_i Z u_f}{|\bar{u}_i Z u_f|}.$$ (10)

Obviously this fact creates no problem, if we calculate the amplitude for alone diagram. Really

$$\begin{align*}
(M_{if})^* &= \frac{[Tr(QP_iZP_f)]^*}{\sqrt{Tr(ZP_iZP_f)}} = \frac{[Tr(Qu_i \bar{u}_i Z u_f \bar{u}_f)]^*}{\sqrt{Tr(ZP_iZP_f)}} \\
&= \frac{(\bar{u}_f \bar{Z} u_i)(\bar{u}_i Qu_f)}{\sqrt{Tr(ZP_iZP_f)}} = \frac{Tr(\bar{Z}P_i \bar{Q}P_f)}{\sqrt{Tr(ZP_iZP_f)}},
\end{align*}$$ (11)

$$M_{if}(M_{if})^* = \frac{Tr(QP_iZP_f) \cdot Tr(\bar{Z}P_i \bar{Q}P_f)}{Tr(ZP_iZP_f)}$$ (12)

$$= \frac{Tr(QP_i \bar{Q}P_f) \cdot Tr(\bar{Z}P_i ZP_f)}{Tr(ZP_iZP_f)} = Tr(QP_i QP_f) = |M_{if}|^2.$$ [In (12) we used the identity (10).]

However, if we have the interfering diagrams, then the application of (2) leads to the fact that the expressions for the amplitudes corresponding to different channels are multiplied by different phase factors in the general case. In this situation the additional calculations are necessary. These calculations are considered below.

Let us consider the diagrams of the process of the general form, which proceeds in two different channels (see Fig. 1).

![Diagram](image.png)

Figure 1: The general form of the diagrams of the process proceeding in two interfering channels

The expression

$$M = (\bar{u}_f Qu_i) \cdot (\bar{u}_{f'} Ru_{i'}) = M_{if} \cdot M_{i'f'}$$ (13)

corresponds to the first diagram. The expression

$$M' = (\bar{u}_{f'} Su_i) \cdot (\bar{u}_f Tu_{i'}) = M_{i'f} \cdot M_{if}$$ (14)
corresponds to the second one, where $Q$, $R$, $S$, $T$ are arbitrary matrix operators characterizing the interaction.

There are three possibilities to calculate the amplitude of this process correctly:

(a) One can apply the Fierz transformation for the second diagram: $f \leftrightarrow f'$. However this method requires a large number of additional calculations.

(b) One can multiply both of the amplitudes by the same phase factors, for example by

$$\frac{\bar{u}_i Z_1 u_f}{|\bar{u}_i Z_1 u_f|} \cdot \frac{\bar{u}_{i'} Z_2 u_{f'}}{|\bar{u}_{i'} Z_2 u_{f'}|}.$$

In this case we have the expression for the first diagram

$$\frac{\text{Tr}(QP_i Z_1 P_f)}{\sqrt{\text{Tr}(Z_1 P_i Z_1 P_f)}} \cdot \frac{\text{Tr}(RP_{i'} Z_2 P_{f'})}{\sqrt{\text{Tr}(Z_2 P_{i'} Z_2 P_{f'})}}$$

and for the second diagram

$$\frac{\text{Tr}(SP_i Z_1 P_f T P_{i'} Z_2 P_{f'})}{\sqrt{\text{Tr}(Z_1 P_i Z_1 P_f)} \cdot \sqrt{\text{Tr}(Z_2 P_{i'} Z_2 P_{f'})}}.$$

It is obvious that the calculation of the amplitude for the second diagram is complicated enough. Besides, it is not convenient to use the expressions which have different structure for the calculation of amplitudes for the different diagrams.

(c) The third possibility is to calculate the relative phase for the first and second diagrams and to use the expression obtained for the phase correction of one of two amplitudes.

We will use the third possibility. Interference term has the form

$$M \cdot (M')^* = M_{i f} \cdot M_{i' f'} \cdot (M_{i f}')^* \cdot (M_{i' f'})^* = (\bar{u}_f Q u_i ) (\bar{u}_{f'} R u_{i'} ) (\bar{u}_i S u_{f'} ) (\bar{u}_{i'} T u_f ) .$$

Let us multiply the interference term by

$$\frac{(\bar{u}_i Z_1 u_f ) (\bar{u}_{i'} Z_2 u_{f'}) (\bar{u}_{i'} Z_3 u_{i'}) (\bar{u}_f Z_4 u_f )}{(\bar{u}_i Z_1 u_f ) (\bar{u}_{i'} Z_2 u_{f'}) (\bar{u}_{i'} Z_3 u_{i'}) (\bar{u}_f Z_4 u_f )} \cdot \frac{(\bar{u}_f Z_1 u_i ) (\bar{u}_i Z_3 u_{f'} ) (\bar{u}_{i'} Z_2 u_{f'} ) (\bar{u}_{i'} Z_4 u_{f'})}{(\bar{u}_f Z_1 u_i ) (\bar{u}_i Z_3 u_{f'} ) (\bar{u}_{i'} Z_2 u_{f'} ) (\bar{u}_{i'} Z_4 u_{f'})} \equiv 1$$

where $Z_1$, $Z_2$, $Z_3$, $Z_4$ are still arbitrary matrix operators.
As a result in this case we have

\[
M \cdot (M')^* \equiv \frac{Tr(QP_iZ_1P_f)}{Tr(Z_1P_iZ_1P_f)} \cdot \frac{Tr(RP_iZ_2P_f)}{Tr(Z_2P_iZ_2P_f)} \times \frac{Tr(\bar{Z}_3P_i\bar{S}P_f)}{Tr(Z_3P_iZ_3P_f)} \cdot \frac{Tr(\bar{Z}_4P_i\bar{T}P_f)}{Tr(Z_4P_iZ_4P_f)}
\]

\[
= \frac{Tr(QP_iZ_1P_f)}{\sqrt{Tr(Z_1P_iZ_1P_f)}} \cdot \frac{Tr(RP_iZ_2P_f)}{\sqrt{Tr(Z_2P_iZ_2P_f)}} \cdot \frac{Tr(\bar{Z}_3P_i\bar{S}P_f)}{\sqrt{Tr(Z_3P_iZ_3P_f)}} \cdot \frac{Tr(\bar{Z}_4P_i\bar{T}P_f)}{\sqrt{Tr(Z_4P_iZ_4P_f)}} \times \frac{Tr(\bar{Z}_3P_iZ_3P_f\bar{Z}_2P_iZ_4P_f)}{\sqrt{Tr(Z_1P_iZ_1P_f)Tr(Z_3P_iZ_3P_f)Tr(Z_2P_iZ_2P_f)Tr(Z_4P_iZ_4P_f)}}
\]

\[
= \mathcal{M}_{i'f'} \cdot (\mathcal{M}_{i'f'})^* \cdot (\mathcal{M}_{i'f})^* \cdot K
\]

where \( \mathcal{M}_{i'f'} \), \( \mathcal{M}_{i'f'} \), \( (\mathcal{M}_{i'f'})^* \), \( (\mathcal{M}_{i'f})^* \) are given by expressions analogous to (2) and (11), and the coefficient \( K \) is given by

\[
K = \frac{Tr(\bar{Z}_3P_iZ_3P_f\bar{Z}_2P_iZ_4P_f)}{\sqrt{Tr(Z_1P_iZ_1P_f)Tr(Z_3P_iZ_3P_f)Tr(Z_2P_iZ_2P_f)Tr(Z_4P_iZ_4P_f)}}.
\]

It is obvious that \( |K| = 1 \).

Thus we have to calculate the amplitude of process with interfering diagrams as follows

\[
\mathcal{M} + \mathcal{M}' = K \cdot \mathcal{M}_{i'f'} \cdot \mathcal{M}_{i'f'} + \mathcal{M}_{i'f'} \cdot \mathcal{M}_{i'f}.
\]

In particularly, if

\[
Z_1 = Z_2 = Z_3 = Z_4 = 1
\]

we have

\[
K = \frac{Tr(P_iP_fP_fP_f)}{\sqrt{Tr(P_iP_f)Tr(P_iP_f)Tr(P_iP_f)Tr(P_iP_f)}}.
\]

In the massless limit we have

\[
\bar{u}_\pm(p_f)Qu_\mp(p_i) \simeq \frac{Tr[Q\hat{p}_i\hat{p}_f(1 \mp \gamma_5)]}{2\sqrt{2}(p_ip_f)},
\]

\[
K = \frac{Tr[\hat{p}_i\hat{p}_f\hat{p}_i\hat{p}_f(1 \mp \gamma_5)]}{8\sqrt{(p_ip_f)(p_ip_f)(p_ip_f)(p_ip_f)}}.
\]

The formulae (18), (19) generalize the method of the calculation of amplitudes proposed in [10].
4 A new method of the calculation of amplitudes

All the disadvantages of the general scheme of the calculation of the amplitudes listed in the previous section can be eliminated with the help of a successful choice of the operator $Z$.

As a suitable choice we propose

$$Z = \mathcal{P} \quad \text{[see (3)]}$$

or

$$Z = \mathcal{P}_\pm \quad \text{[see (3)]}.$$

Really, in this case the unknown phase factor for each line of the diagram splits into two parts [see (10)]:

$$M_{if} = M_{if} \cdot \frac{\bar{u}_i \mathcal{P} u_f}{|\bar{u}_i \mathcal{P} u_f|} = M_{if} \cdot \frac{\bar{u}_i u \bar{u} u_f}{|\bar{u}_i u \bar{u} u_f|} = M_{if} \cdot \frac{\bar{u}_i u}{|\bar{u}_i u|} \cdot \frac{\bar{u} u_f}{|\bar{u} u_f|}.$$

Let us consider again the diagrams of the process shown in Fig. 1.

Calculating the amplitudes (13), (14), we have

$$\mathcal{M} = M_{if} \cdot M_{i'f'} \cdot \frac{\bar{u}_i u}{|\bar{u}_i u|} \cdot \frac{\bar{u} u_f}{|\bar{u} u_f|} \cdot \frac{\bar{u}_{i'} u'}{|\bar{u}_{i'} u'|} \cdot \frac{\bar{u} u_{f'}}{|\bar{u} u_{f'}|},$$

$$\mathcal{M}' = M_{if} \cdot M_{i'f'} \cdot \frac{\bar{u}_i u}{|\bar{u}_i u|} \cdot \frac{\bar{u} u_f'}{|\bar{u} u_f'|} \cdot \frac{\bar{u}_{i'} u'}{|\bar{u}_{i'} u'|} \cdot \frac{\bar{u} u_{f}}{|\bar{u} u_{f}|},$$

that is the amplitudes of the different diagrams are multiplied by the same phase factor, and therefore one can ignore it. This conclusion is right for any number of the interfering diagrams and for any number of the nonclosed fermion lines in the diagrams.

We also note that, as would be expected [see (16)]

$$K \equiv 1$$

if

$$Z_1 = Z_2 = Z_3 = Z_4 = \mathcal{P}$$

or

$$Z_1 = Z_2 = Z_3 = Z_4 = \mathcal{P}_\pm$$

since projection operators have the following properties:

$$\bar{\mathcal{P}} = \mathcal{P} , \quad \mathcal{P} A \mathcal{P} = Tr[\mathcal{P} A] \cdot \mathcal{P} \quad \text{[see (10)]} \quad (20)$$

$$\bar{\mathcal{P}}_\pm = \mathcal{P}_\pm , \quad \mathcal{P}_\pm A \mathcal{P}_\pm = Tr[\mathcal{P}_\pm A] \cdot \mathcal{P}_\pm \quad \text{[see (10)]} \quad (21)$$

Really

$$\bar{\mathcal{P}} = \gamma^0 \mathcal{P} + \gamma^0 = \gamma^0 (\bar{u} u) + \gamma^0 = \gamma^0 (u u^0 + u^0) + \gamma^0$$

$$= \gamma^0 [(\gamma^0 + u^0) u^0 + u^0] + \gamma^0 = \gamma^0 [\gamma^0 uu^0] + \gamma^0 = uu^0 + \gamma^0 = uu^0 = \bar{u} u = \mathcal{P}.$$
\[ \mathcal{P} \mathcal{A} \mathcal{P} = (u)_\alpha (\bar{u})_\beta (A)^{\beta \rho} (u)_{\rho \delta} = [(\bar{u})_\beta (A)^{\beta \rho} (u)_{\rho \delta}] (u)_\alpha (\bar{u})_\delta = \text{Tr}[\mathcal{P} A] \cdot \mathcal{P}. \]

Therefore we can calculate the amplitude of the process with interfering diagrams as follows

\[ M + M' = \frac{\text{Tr}(Q \mathcal{P}_i \mathcal{P} f)}{\sqrt{\text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P} P_f)}} \cdot \frac{\text{Tr}(R \mathcal{P}_v \mathcal{P} P_v)}{\sqrt{\text{Tr}(\mathcal{P} P_v) \text{Tr}(\mathcal{P} P_v')}} \]

\[ + \frac{\text{Tr}(S \mathcal{P}_i \mathcal{P} P_f)}{\sqrt{\text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P} P_f)}} \cdot \frac{\text{Tr}(T \mathcal{P}_v \mathcal{P} P_f)}{\sqrt{\text{Tr}(\mathcal{P} P_v) \text{Tr}(\mathcal{P} P_f)}} \]

\[ = \frac{\text{Tr}(Q \mathcal{P}_i \mathcal{P} P_f) \cdot \text{Tr}(R \mathcal{P}_v \mathcal{P} P_v) + \text{Tr}(S \mathcal{P}_i \mathcal{P} P_f) \cdot \text{Tr}(T \mathcal{P}_v \mathcal{P} P_f)}{\sqrt{\text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P} P_v) \text{Tr}(\mathcal{P} P_f) \text{Tr}(\mathcal{P} P_f')}}. \tag{22} \]

[where for the denominators in (22) we used the identity

\[ \text{Tr}(\bar{P} \mathcal{P}_i \mathcal{P} P_f) = \text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P} P_f), \]

which follows from (24) – (23)].

This expression enables us to calculate the amplitude numerically. The complex numbers obtained in this way may be used to calculate the squared amplitude.

Note that in (22) and in the analogous formulae used further we shall use the equality sign instead of the symbol \(\simeq\), since now there does not exist any trouble with the phase factors. (However one cannot do this when the phase factor is not the same for all terms! – see, for example, subsection 5 of Section 6.)

Thus, if we use the projection operator for \(Z\) then in addition to the two advantages of the general scheme (3) considered in Section 3 we obtain one more advantage:

3. For the processes with interfering diagrams the number of the expressions which are necessary to be calculated is reduced since it is not necessary to calculate the interfering terms.

If in individual cases under numerical calculations the denominator in (22) vanishes, it is sufficient to change an arbitrary projection operator \(\mathcal{P}\) entering (22) (i.e. to change arbitrary 4-vectors entering it). In this case the amplitudes for the different diagrams are multiplied by one and the same phase factor. Really, under the replacement of \(\mathcal{P}\) by \(\mathcal{P}'\), we obtain [using (3) and (3)]:

\[ \mathcal{M}_{if} = \frac{\text{Tr}(Q \mathcal{P}_i \mathcal{P} P_f)}{\sqrt{\text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P} P_f)}} = \frac{\text{Tr}(Q \mathcal{P}_i \mathcal{P}' P_f)}{\sqrt{\text{Tr}(\mathcal{P} P_i) \text{Tr}(\mathcal{P}' P_f)}} \cdot \left[ \frac{\text{Tr}(\mathcal{P} P_i \mathcal{P} P_f)}{\text{Tr}(\mathcal{P} P_i \mathcal{P} P_f) \text{Tr}(\mathcal{P} P_f)} \right]^{1/2}, \]
As this takes place we have [see (7)]

\[
\frac{\text{Tr}(P^iP_jP'_f)}{\text{Tr}(PP'_f)} = 1
\]

\[
\frac{\text{Tr}(P^iP'_jP'_f)}{\text{Tr}(PP'_f)} = 1
\]

\[
\frac{\text{Tr}(P^iP'_jP'_f)}{\text{Tr}(PP'_f)} = 1
\]

Note that if the operators, characterizing the interaction, contain the product of \(\gamma\)-matrices the number of which is greater than the number of \(\gamma\)-matrices involved in the projection operator, then the calculation of the amplitude for a single diagram is easier than the calculation of the squared amplitude.

However, for the processes with interfering diagrams the method of the calculation of amplitudes is easier in any case, because we need not calculate the interference terms.

As it was already mentioned, the method considered can be generalized easily to the case of the reaction with participation of antiparticles. To do this it is sufficient to substitute the projection operators of antiparticles in place of the operators of particles in (2). For example, if we are interested in the value \(\bar{v}_f Qu_j\), where \(v_f\) is the bispinor of a free antiparticle, then

\[
\bar{v}_f Qu_j = \frac{\text{Tr}(Qu_i\bar{u}_iZv_j\bar{v}_f)}{\sqrt{\text{Tr}(Zu_i\bar{u}_iZv_j\bar{v}_f)}}
\]

where

\[
v(p, n)\bar{v}(p, n) = \frac{1}{2}(-m + \hat{p})(1 + \gamma_5\hat{n})
\]

\[
p^2 = m^2, \quad n^2 = -1, \quad pn = 0, \quad \bar{v}v = -2m
\]

for a massive antiparticle, or

\[
v_\pm(p)\bar{v}_\pm(p) = \frac{1}{2}(1 \mp \gamma_5)\hat{p}
\]

\[
p^2 = 0, \quad \bar{v}_\pm\gamma_\mu v_\pm = 2p_\mu
\]

for a massless antiparticle.

As always, we use (3) or (4) instead of Z.
5 The formulae for the calculation of the amplitudes of the processes involving the polarized Dirac particles

In this Section the expressions for the amplitudes of the processes with both the massive and massless Dirac particles are presented. Using the formulae (2) – (4) and (23) – (25), we obtain:

\[ \bar{u}_\pm(p_f)Qu_\mp(p_i) = \bar{u}_\pm(p_f)Qv_\pm(p_i) = \bar{v}_\mp(p_f)Qv_\pm(p_i) = \bar{v}_\pm(p_f)Qu_\mp(p_i) \]

\[ = \frac{Tr[Q\hat{p}_i\hat{q}\hat{p}_f(1 \mp \gamma_5)]}{4\sqrt{(qp_i)(qp_f)}}. \quad (26) \]

Here

\[ Z = \frac{1}{2}(1 \mp \gamma_5)\hat{q} = \mathcal{P}_\mp, \quad q^2 = 0. \]

The massless 4-vector \( q \) can be arbitrary, but it must be the same for all nonclosed fermion lines of the diagrams considered.

\[ \bar{u}_\pm(p_f)Qu_\mp(p_i) = \bar{u}_\pm(p_f)Qv_\pm(p_i) = \bar{v}_\mp(p_f)Qv_\pm(p_i) = \bar{v}_\pm(p_f)Qu_\mp(p_i) \]

\[ = \frac{Tr[Q\hat{p}_i\hat{r}\hat{l}\hat{p}_f(1 \mp \gamma_5)]}{4\sqrt{[(rp_i) \pm m(lp_i)][(rp_f) \mp m(lp_f)]}}. \quad (27) \]

Here

\[ Z = \frac{1}{2}(m \mp \hat{r})(1 + \gamma_5\hat{l}) = \mathcal{P}, \quad r^2 = m^2, \quad l^2 = -1, \quad rl = 0. \]

The same remark as for the vector \( q \) in (26) is takes place for the 4-vectors \( r \) and \( l \).

In the latter case we can not use the easier operator \( \mathcal{P}_\mp \) for \( Z \), since in this case the numerator and the denominator are equal to zero. However to simplify the calculations we can require \( m = 0, \quad r^2 = 0 \) in (27):

\[ \bar{u}_\pm(p_f)Qu_\mp(p_i) = \bar{u}_\pm(p_f)Qv_\mp(p_i) = \bar{v}_\mp(p_f)Qv_\pm(p_i) = \bar{v}_\pm(p_f)Qu_\mp(p_i) \]

\[ = \frac{Tr[Q\hat{p}_i\hat{r}\hat{p}_f(1 \mp \gamma_5)]}{4\sqrt{(rp_i)(rp_f)}}. \quad (28) \]

This formula generalizes the method of the calculation of amplitudes proposed in [24] – [25]. Further

\[ \bar{u}_\pm(p_f)Qu(p_i, n_i) = \bar{v}_\mp(p_f)Qu(p_i, n_i) = \frac{Tr[Q(m_i \pm \hat{p}_i\hat{n}_i + \hat{p}_i \pm m_i\hat{n}_i)\hat{q}\hat{p}_f(1 \mp \gamma_5)]}{4\sqrt{2[(qp_i) \pm m_i(qn_i)](qp_f)}}. \quad (29) \]
\[ u_{\pm}(p_f)Qv(p_i, n_i) = \bar{v}_{\pm}(p_f)Qv(p_i, n_i) = \frac{Tr \left[ Q(-m_i \pm \hat{p}_i \hat{n}_i + \hat{p}_i \mp n_i)\hat{q}\hat{p}_f(1 \mp \gamma_5) \right]}{4\sqrt{2} \left[ (qpi) \mp m_i(qni) \right] (qpf) } , \]

\[ \bar{u}(p_f, n_f)Qu_{\pm}(p_i) = \bar{u}(p_f, n_f)Qv_{\pm}(p_i) = \frac{Tr \left[ Q(1 \mp \gamma_5)\hat{p}_i\hat{q}(m_f \mp \hat{p}_f \hat{n}_f + \hat{p}_f \mp m_f \hat{n}_f) \right]}{4\sqrt{2} (qpi) \left[ (qpf) \mp m_f(qnf) \right] } , \]

\[ \bar{v}(p_f, n_f)Qu_{\pm}(p_i) = \bar{v}(p_f, n_f)Qv_{\pm}(p_i) = \frac{Tr \left[ Q(1 \mp \gamma_5)\hat{p}_i\hat{q}(-m_f \mp \hat{p}_f \hat{n}_f + \hat{p}_f \mp m_f \hat{n}_f) \right]}{4\sqrt{2} (qpi) \left[ (qpf) \mp m_f(qnf) \right] } . \]

As noted above, \( Q \) is the matrix operator which characterizes the interaction. It is a linear combination of the products of the Dirac \( \gamma \)-matrices (or their contractions with 4-vectors).

If in the course of numerical calculations the denominators in (26) – (36) become zero for some values of vectors \( p_i, n_i \) and \( p_f, n_f \), then it suffice to change the values of arbitrary 4-vectors \( q \) or \( r, l \) contained in these formulae (simultaneously for all the lines of the diagrams being considered).

To simplify the calculations one can substitute the vectors of the problem instead of arbitrary vectors \( q \) or \( r, l \) but this may cause an ambiguity of the type \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) which can not be resolved.

The listed formulae are simple enough and may be used for creating the computer programs for automatic calculation of the amplitudes. Besides the presence of the multipliers
(1 ± γ₅) in all the formulae allows us to use the formulae of the Fierz transformations (see, e.g. [61]), if it is necessary, to simplify the calculations:

\[
[(1 \pm \gamma_5)\gamma_{\mu}]_{ij} [(1 \pm \gamma_5)\gamma_{\nu}]_{kl} = 2[(1 \pm \gamma_5)\gamma_{\mu}]_{il} [(1 \pm \gamma_5)\gamma_{\nu}]_{kj} \tag{37}
\]
\[
[(1 \pm \gamma_5)\gamma_{\mu}]_{ij} [(1 \pm \gamma_5)\gamma_{\nu}]_{kl} = -[(1 \pm \gamma_5)\gamma_{\mu}]_{il} [(1 \pm \gamma_5)\gamma_{\nu}]_{kj} \tag{38}
\]

\(i, j, k, l\) are indices that label the components of \(4 \times 4\)-matrices.

6 Methods of the calculation of the amplitudes based on multiplication of \(\gamma\)-matrices and bispinors which are written in the matrix form and methods based on transformation of bispinors

Throughout this Section we use the chiral representation of the \(\gamma\)-matrices:

\[\gamma^0 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} O & \sigma^k \\ -\sigma^k & O \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}, \]

\[\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\]

In this representation the projection operator for a massive particle has the form:

\[\mathcal{P}(p, n) = \frac{1}{2}(\hat{p} + m)(1 + \gamma_5\hat{n}) = \frac{1}{2}.
\]

\[
\begin{pmatrix}
(m + [p]_{0z} - i[p]_{xy}) & p_-n_+ - p_+n_- & p_+ - mn_- & -p_+ + mn_+

(p_+n_+ - p_-n_-) & m - [p]_{0z} + i[p]_{xy} & -p_+ + mn_- & p_+ - mn_+

(p_+ + mn_+) & p_- + mn_+ & m + [p]_{0z} + i[p]_{xy} & p_+n_+ - p_-n_+

(p_- + mn_-) & p_- + mn_- & p_-n_+ - p_-n_- & m - [p]_{0z} - i[p]_{xy}
\end{pmatrix}
\]

where for any 4-vectors \(a\) and \(b\)

\[a_\pm = a_0 \pm a_z, \quad a_\pm = a_x + ia_y, \quad [ab]_{0z} = a_0b_z - a_zb_0, \quad [ab]_{xy} = a_xb_y - a_yb_x, \]

\[p_0 = E = \sqrt{m^2 + \vec{p}^2}\]
The projection operator for a massive antiparticle:

\[
P_\alpha(p, n) = \frac{1}{2}(\hat{p} - m)(1 + \gamma_5 \hat{n}) = \frac{1}{2}.
\]

\[
\begin{pmatrix}
-m + [pm]_{0z} - i[pm]_{xy} & p_- n^*_\perp - p^*_\perp n_- & p_- + mn_- & -p^*_\perp - mn^*_\perp \\
p_+ n_\perp - p_\perp n_+ & -m + [pm]_{0z} + i[pm]_{xy} & -p_\perp - mn_\perp & p_+ + mn_+
\end{pmatrix}
\]

As noted above, the calculations of the amplitudes by multiplication of \(\gamma\)-matrices and bispinors which are written in the matrix form are reduced to the general scheme (2). Let us illustrate this by several examples.

1.1. Let us consider

\[
u(p, n) = \frac{P(p, n)}{\sqrt{Tr [P(p, n)P(r, l)]}} \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]

\[
= \frac{1}{2\sqrt{(p_0 + m)(1 + n_z) - p_z n_0}} \begin{pmatrix}
p_- + m(1 - n_-) + [pm]_{0z} - i[pm]_{xy} \\
(m + p_+) n_\perp - p_\perp (1 + n_+) \\
p_+ + m(1 + n_+) + [pm]_{0z} + i[pm]_{xy} \\
(m + p_-) n_\perp + p_\perp (1 - n_-)
\end{pmatrix},
\]

where \(r^\mu = (1, 0, 0, 0)\), \(l^\mu = (0, 0, 0, 1)\).

\[
u'(p, n) = \frac{P(p, n)}{\sqrt{Tr [P(p, n)P(r, l')]}} \begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix}
\]

\[
= \frac{1}{2\sqrt{(p_0 + m)(1 - n_z) + p_z n_0}} \begin{pmatrix}
(m + p_-) n^*_\perp - p^*_\perp (1 + n_-) \\
p_+ + m(1 - n_+) - [pm]_{0z} + i[pm]_{xy} \\
(m + p_+) n^*_\perp + p^*_\perp (1 - n_+) \\
p_- + m(1 + n_-) - [pm]_{0z} - i[pm]_{xy}
\end{pmatrix},
\]

where \(r^\mu = (1, 0, 0, 0)\), \(l'^\mu = (0, 0, 0, -1)\). Note that

\[
u'(p, n) = e^{i\varphi(p, n)} u(p, n),
\]

where

\[
e^{i\varphi(p, n)} = \frac{(p_0 + m)n^*_\perp - p^*_\perp n_0}{\sqrt{(p_0 + m)(1 - n_z) + p_z n_0} \sqrt{(p_0 + m)(1 + n_z) - p_z n_0}}.
\]
Similarly:

\[
v(p, n) = \frac{P_a(p, n)}{\sqrt{\text{Tr} [P_a(p, n) P(r, l)]}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
v(p, n) = \frac{1}{2\sqrt{(p_0 - m)(1 + n_z) - p_z n_0}} \begin{pmatrix} p_0 - m(1 + n_z) + [pn]_0z - i[pn]_{xy} \\ -(m + p_+)n_\perp - p_\perp(1 + n_+) \\ p_+ - m(1 + n_+) + [pn]_0z + i[pn]_{xy} \\ -(m + p_-)n_\perp + p_\perp(1 - n_-) \end{pmatrix}
\]

\[
(45)
\]

\[
v'(p, n) = \frac{P_a(p, n)}{\sqrt{\text{Tr} [P_a(p, n) P(r, l')]}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]

\[
v'(p, n) = \frac{1}{2\sqrt{(p_0 - m)(1 - n_z) + p_z n_0}} \begin{pmatrix} -(m + p_-)n_\perp^* - p_\perp^*(1 + n_-) \\ p_+ - m(1 - n_-) - [pn]_0z + i[pn]_{xy} \\ -(m + p_+)n_\perp^* + p_\perp^*(1 - n_-) \\ p_0 - m(1 - n_-) + [pn]_0z - i[pn]_{xy} \end{pmatrix}
\]

\[
(46)
\]

Note that

\[
v'(p, n) = e^{i\phi(p, n)} v(p, n)
\]

where

\[
e^{i\phi(p, n)} = \frac{(p_0 - m)n_\perp^* - p_\perp n_0}{\sqrt{(p_0 - m)(1 - n_z) + p_z n_0} \sqrt{(p_0 - m)(1 + n_z) - p_z n_0}}
\]

\[
(48)
\]

Obviously the bispinors (41) – (42) for particles and the bispinors (45) – (46) for antiparticles satisfy as the Dirac equation so also the equation for the vector \(n\) determining the axis of the spin projections, since when constructing the bispinors there were used the corresponding projection operators (39), (40). Besides one can easily verify that these bispinors satisfy the normalization conditions

\[
\bar{u}(p, n) u(p, n) = \bar{u}'(p, n) u'(p, n) = 2m
\]

\[
\bar{v}(p, n) v(p, n) = v'(p, n) v'(p, n) = -2m
\]
It follows from (41):
\[
\bar{u}(p_f, n_f)Qu(p_i, n_i) = Tr[Qu(p_i, n_i)\bar{u}(p_f, n_f)]
\]
\[
= Tr\left[\frac{P(p_i, n_i)}{\sqrt{Tr[P(p_i, n_i)P(r, l)]}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{P(p_f, n_f)}{\sqrt{Tr[P(p_f, n_f)P(r, l)]}} \right]
\]
(49)
i.e. the calculation of the amplitude in this case reduces to the general scheme (2) if

\[
Z = P(r, l) = \frac{1}{2}(1 + \gamma^0)(1 - \gamma^5\gamma^3) .
\]

Similarly
\[
\bar{u}'(p_f, n_f)Qu'(p_i, n_i)
\]
\[
= Tr\left[\frac{P(p_i, n_i)}{\sqrt{Tr[P(p_i, n_i)P(r, l')]} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \frac{P(p_f, n_f)}{\sqrt{Tr[P(p_f, n_f)P(r, l')]} )} \right]
\]
(50)
i.e. the calculation of the amplitude in this case reduces to the general scheme (2) if

\[
Z = P(r, l') = \frac{1}{2}(1 + \gamma^0)(1 + \gamma^5\gamma^3) .
\]

And, as it follows from (43) – (44), the expressions obtained for the amplitudes differ by the phase factor.

Similarly, for the amplitude
\[
\bar{u}(p_f, n_f)Qu'(p_i, n_i)
\]
we have
\[
Z = P(r, l) \cdot e^{i\varphi(p_i, n_i)}
\]
and so on.

Note that until the present time the bispinors of the general form (41) – (42), (45) – (46) have not been used for the calculation of the amplitudes of processes, however their special forms described below are widely used.
1.2. Let us consider the form of the bispinors (41) – (42), (45) – (46) in the situation when the polarization state of particles and antiparticles is the helicity, i.e.

\[ n^\mu(p) = \frac{\lambda}{m} \left( |\vec{p}|, \frac{p_0}{|\vec{p}|} \right), \quad \lambda = \pm 1. \]

In this case the bispinors take the form

\[
\begin{align*}
 u(p, +) &= \frac{1}{2\sqrt{(p_0 + m) |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix}
 (p_0 + m - |\vec{p}|) (|\vec{p}| + p_z) \\
 (p_0 + m - |\vec{p}|) p_\perp \\
 (p_0 + m + |\vec{p}|) (|\vec{p}| + p_z) \\
 (p_0 + m + |\vec{p}|) p_\perp
\end{pmatrix} \\
 &= \frac{1}{\sqrt{2 |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix}
 \sqrt{p_0 - |\vec{p}|} (|\vec{p}| + p_z) \\
 \sqrt{p_0 - |\vec{p}|} p_\perp \\
 \sqrt{p_0 + |\vec{p}|} (|\vec{p}| + p_z) \\
 \sqrt{p_0 + |\vec{p}|} p_\perp
\end{pmatrix}.
\]

In (51) we used the identity:

\[
\frac{p_0 + m \pm |\vec{p}|}{\sqrt{2(p_0 + m)}} = \sqrt{p_0 \pm |\vec{p}|}.
\]

Further

\[
\begin{align*}
 u(p, -) &= \frac{1}{2\sqrt{(p_0 + m) |\vec{p}| (|\vec{p}| - p_z)}} \begin{pmatrix}
 (p_0 + m + |\vec{p}|) (|\vec{p}| - p_z) \\
 (p_0 + m + |\vec{p}|) (-p_\perp) \\
 (p_0 + m - |\vec{p}|) (|\vec{p}| - p_z) \\
 (p_0 + m - |\vec{p}|) (-p_\perp)
\end{pmatrix} \\
 &= \frac{1}{\sqrt{2 |\vec{p}| (|\vec{p}| - p_z)}} \begin{pmatrix}
 \sqrt{p_0 + |\vec{p}|} (|\vec{p}| - p_z) \\
 \sqrt{p_0 + |\vec{p}|} (-p_\perp) \\
 \sqrt{p_0 - |\vec{p}|} (|\vec{p}| - p_z) \\
 \sqrt{p_0 - |\vec{p}|} (-p_\perp)
\end{pmatrix}.
\]

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\[ u'(p, +) = \frac{1}{2 \sqrt{(p_0 + m) |p| (|p| - p_z)}} \left( \begin{array}{c} (p_0 + m - |p|) p_+^* \\ (p_0 + m - |p|) (|p| - p_z) \\ (p_0 + m + |p|) p_+^* \\ (p_0 + m + |p|) (|p| - p_z) \end{array} \right) \]

\[ = \frac{1}{\sqrt{2 |p| (|p| - p_z)}} \left( \begin{array}{c} \sqrt{p_0 - |p|} p_+^* \\ \sqrt{p_0 - |p|} (|p| - p_z) \\ \sqrt{p_0 + |p|} p_+^* \\ \sqrt{p_0 + |p|} (|p| - p_z) \end{array} \right) \]

\[ u'(p, -) = \frac{1}{2 \sqrt{(p_0 + m) |p| (|p| + p_z)}} \left( \begin{array}{c} (p_0 + m + |p|) (|p| - p_z) \\ (p_0 + m + |p|) (|p| + p_z) \\ (p_0 + m - |p|) (|p| - p_z) \\ (p_0 + m - |p|) (|p| + p_z) \end{array} \right) \]

\[ = \frac{1}{\sqrt{2 |p| (|p| + p_z)}} \left( \begin{array}{c} \sqrt{p_0 + |p|} (-p_+^*) \\ \sqrt{p_0 + |p|} (|p| + p_z) \\ \sqrt{p_0 - |p|} (-p_+^*) \\ \sqrt{p_0 - |p|} (|p| + p_z) \end{array} \right) \]

As this takes place we have

\[ u'(p, \pm) = \pm u(p, \pm) \cdot \frac{p_+^*}{\sqrt{p_x^2 + p_y^2}} = \pm u(p, \pm) \cdot \frac{p_+^*}{\sqrt{p}} \cdot \frac{p_+^*}{\sqrt{p}}. \]

Further

\[ v(p, +) = \frac{1}{2 \sqrt{(p_0 - m) |p| (|p| - p_z)}} \left( \begin{array}{c} (p_0 - m + |p|) (|p| - p_z) \\ (p_0 - m + |p|) (|p| - p_z) \\ (p_0 - m - |p|) (|p| - p_z) \\ (p_0 - m - |p|) (|p| - p_z) \end{array} \right) \]

\[ = \frac{1}{\sqrt{2 |p| (|p| - p_z)}} \left( \begin{array}{c} \sqrt{p_0 - |p|} (|p| - p_z) \\ \sqrt{p_0 - |p|} (-p_+) \\ - \sqrt{p_0 + |p|} (|p| - p_z) \\ - \sqrt{p_0 + |p|} (-p_+) \end{array} \right) \]

In (55) we used the identity:

\[ \frac{p_0 - m \pm |p|}{\sqrt{2(p_0 - m)}} = \pm \sqrt{p_0 \pm |p|}. \]
As this takes place we obtain

\[ v(p, -) = \frac{1}{2 \sqrt{(p_0 - m) |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix} (p_0 - m - |\vec{p}|) (|\vec{p}| + p_z) \\ (p_0 - m - |\vec{p}|) p_+ \\ (p_0 - m + |\vec{p}|) (|\vec{p}| + p_z) \\ (p_0 - m + |\vec{p}|) p_+ \end{pmatrix} \]

\[ = \frac{1}{\sqrt{2 |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix} -\sqrt{p_0 - |\vec{p}|} (|\vec{p}| + p_z) \\ -\sqrt{p_0 - |\vec{p}|} p_+ \\ \sqrt{p_0 + |\vec{p}|} (|\vec{p}| + p_z) \\ \sqrt{p_0 + |\vec{p}|} p_+ \end{pmatrix}, \tag{56} \]

\[ v'(p, +) = \frac{1}{2 \sqrt{(p_0 - m) |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix} (p_0 - m + |\vec{p}|) (-p_+^*) \\ (p_0 - m + |\vec{p}|) (|\vec{p}| + p_z) \\ (p_0 - m - |\vec{p}|) (-p_+^*) \\ (p_0 - m - |\vec{p}|) (|\vec{p}| + p_z) \end{pmatrix} \]

\[ = \frac{1}{\sqrt{2 |\vec{p}| (|\vec{p}| + p_z)}} \begin{pmatrix} \sqrt{p_0 + |\vec{p}|} (-p_+^*) \\ \sqrt{p_0 + |\vec{p}|} (|\vec{p}| + p_z) \\ -\sqrt{p_0 - |\vec{p}|} (-p_+^*) \\ -\sqrt{p_0 - |\vec{p}|} (|\vec{p}| + p_z) \end{pmatrix}, \tag{57} \]

\[ v'(p, -) = \frac{1}{2 \sqrt{(p_0 - m) |\vec{p}| (|\vec{p}| - p_z)}} \begin{pmatrix} (p_0 - m - |\vec{p}|) p_+^* \\ (p_0 - m - |\vec{p}|) (|\vec{p}| - p_z) \\ (p_0 - m + |\vec{p}|) p_+^* \\ (p_0 - m + |\vec{p}|) (|\vec{p}| - p_z) \end{pmatrix} \]

\[ = \frac{1}{\sqrt{2 |\vec{p}| (|\vec{p}| - p_z)}} \begin{pmatrix} -\sqrt{p_0 - |\vec{p}|} p_+^* \\ -\sqrt{p_0 - |\vec{p}|} (|\vec{p}| - p_z) \\ \sqrt{p_0 + |\vec{p}|} p_+^* \\ \sqrt{p_0 + |\vec{p}|} (|\vec{p}| - p_z) \end{pmatrix}. \tag{58} \]

As this takes place we obtain

\[ v'(p, \pm) = \mp v(p, \pm) \cdot \frac{p_+^*}{\sqrt{p_x^2 + p_y^2}} = \mp v(p, \pm) \cdot \frac{p_+^*}{|\vec{p}|}. \]

(The signs \(\pm\) in notation of the bispinors correspond to the helicity of particles or antiparticles.)

The bispinors (51) – (58) have been considered in a number of papers.
In particular, in [26] the bispinors (51), (52) were considered.
In [27] the bispinors (51), (54) were considered.

In [28] it is proposed to use the bispinors (51), (54), (56), (57) [in this paper the bispinors (54), (57) have the opposite sign].

In [29] it is proposed to use the bispinors (51) – (58) [in this paper the bispinors (55), (57) have the opposite sign].

In [30] the bispinors (51), (54), (56), (57) are used [in this paper the bispinor (57) has the opposite sign].

In [31] there are presented the expressions for the amplitudes that can be obtained by means of the formulae (52), (53), (56), (57) written in the center-of-mass frame of the fermions considered.

The bispinors described in this subsection are also widely used in calculations others authors.

1.3. Let us consider a special form of the bispinors (51) – (58) at \( m = 0 \) (\( |\vec{p}| = p_0 \)):

\[
\begin{align*}
\quad u_+(p) = v_-(p) &= \frac{1}{\sqrt{p_+}} \begin{pmatrix}
0 \\
p_+ \\
p_\perp
\end{pmatrix} , \\
\quad u_-(p) = v_+(p) &= \frac{1}{\sqrt{-p_-}} \begin{pmatrix}
p_-
-\quad -p_\perp \\
0 \\
0
\end{pmatrix} , \\
\quad u'_+(p) = v'_-(p) &= \frac{1}{\sqrt{-p_-}} \begin{pmatrix}
0 \\
p_\perp
0
\end{pmatrix} , \\
\quad u'_-(p) = v'_+(p) &= \frac{1}{\sqrt{p_+}} \begin{pmatrix}
-p_\perp \\
p_+
0
\end{pmatrix} .
\end{align*}
\quad (59)\quad (60)\quad (61)\quad (62)
\]

In a number of papers it has been proposed to use the bispinors (59) – (62).

In [32] it was proposed to use the bispinors (59), (62).

The method proposed in [33] is reduced to the application of the bispinors of the form (60), (61) [bispinor (61) was used with the additional factor \( i \)].

In [34] it was proposed to use the bispinors of the form (59), (62) [here the bispinor (62) was used with the opposite sign].

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The method proposed in [35] reduces to the application of a special form of the bispinors (59) – (62) at \( p_y = 0 \).

These methods are widely used in calculations.

1.4. For \( n^\mu = \frac{\lambda}{m} (\tfrac{p_x p_z}{p_0 + m}, \tfrac{p_y p_z}{p_0 + m}, m + \tfrac{p_z^2}{p_0 + m}) \), \( \lambda = \pm 1 \)

the formulae (41), (42), (45), (46) imply

\[
    u(p, +) = \frac{1}{\sqrt{2(p_0 + m)}} \begin{pmatrix}
    p_+ + m \\
    -p_{\perp} \\
    p_+ + m \\
    p_{\perp}
\end{pmatrix},
\]

(63)

\[
    u'(p, -) = \frac{1}{\sqrt{2(p_0 + m)}} \begin{pmatrix}
    -p_{\perp}^* \\
    p_+ + m \\
    p_{\perp}^* \\
    p_+ + m
\end{pmatrix},
\]

(64)

\[
    v(p, -) = \frac{1}{\sqrt{2(p_0 + m)}} \begin{pmatrix}
    -p_- - m \\
    p_{\perp} \\
    p_+ + m \\
    p_{\perp}
\end{pmatrix},
\]

(65)

\[
    v'(p, +) = \frac{1}{\sqrt{2(p_0 + m)}} \begin{pmatrix}
    -p_{\perp}^* \\
    p_+ + m \\
    -p_{\perp}^* \\
    -p_- + m
\end{pmatrix},
\]

(66)

The bispinors (63) – (66) are considered in [36], [37], [38] (in [37] and [38] the bispinor (66) has the opposite sign).

In [39] the bispinors (63), (64) are used for a special case when the particles are moving on \( xy \) plane, i.e. \( p^\mu = (p_0, p_x, p_y, 0) \) and \( n^\mu = \lambda(0, 0, 0, 1) \), \( \lambda = \pm 1 \).

1.5. For  

\[
    n^\mu = \frac{\lambda}{m} (\tfrac{m^2}{p_0 + p_z}, \tfrac{p_x p_y p_z}{p_0 + p_z}, \tfrac{m^2}{p_0 + p_z}) \), \( \lambda = \pm 1 \)
\]

the formulae (41), (42), (45), (46) imply

\[
    u(p, +) = \frac{1}{\sqrt{p_+}} \begin{pmatrix}
    m \\
    0 \\
    p_+ \\
    p_{\perp}
\end{pmatrix},
\]

(67)

\footnote{In the rest frame of a particle (antiparticle) the vector \( n^\mu \) takes the form \( \lambda(0, 0, 0, 1) \).}

\footnote{In the rest frame of a particle (antiparticle) the vector \( n^\mu \) takes the form \( \lambda(0, 0, 0, 1) \).}
Dirac particles (antiparticles), the polarization state of which is the helicity:

\[ u'(p, -) = \frac{1}{\sqrt{p_+}} \left( \begin{array}{c} -p'_\perp \\ p_+ \\ 0 \\ m \end{array} \right), \quad (68) \]

\[ v(p, -) = \frac{1}{\sqrt{p_+}} \left( \begin{array}{c} -m \\ 0 \\ p_+ \\ p'_\perp \end{array} \right), \quad (69) \]

\[ v'(p, +) = \frac{1}{\sqrt{p_+}} \left( \begin{array}{c} -p'_\perp \\ p_+ \\ 0 \\ -m \end{array} \right). \quad (70) \]

The bispinors (67) – (71) are used in [10], [52]. Note that in the massless limit these bispinors take the form (59), (52).

2. In the papers [11], [12], [13] it is proposed to use the following bispinors for massive Dirac particles (antiparticles), the polarization state of which is the helicity:

\[ u''(p, +) = \frac{1}{\sqrt{2 |p| |p_\perp|}} \left( \begin{array}{c} \sqrt{p_0 - |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ \sqrt{p_0 - |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \\ \sqrt{p_0 + |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ \sqrt{p_0 + |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \end{array} \right), \quad (71) \]

\[ u''(p, -) = \frac{1}{\sqrt{2 |p| |p_\perp|}} \left( \begin{array}{c} -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \\ \sqrt{p_0 - |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ \sqrt{p_0 - |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \end{array} \right), \quad (72) \]

\[ v''(p, +) = \frac{1}{\sqrt{2 |p| |p_\perp|}} \left( \begin{array}{c} \sqrt{p_0 - |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \\ -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ \sqrt{p_0 - |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \end{array} \right), \quad (73) \]

\[ v''(p, -) = \frac{1}{\sqrt{2 |p| |p_\perp|}} \left( \begin{array}{c} \sqrt{p_0 - |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ \sqrt{p_0 - |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \\ -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| + p_z) p'_\perp}} \\ -\sqrt{p_0 + |p| \sqrt{(|\vec{p}| - p_z) p'_\perp}} \end{array} \right). \quad (74) \]
(In [41] the bispinors for antiparticles have the opposite sign, and in [43] the bispinors for antiparticles have an additional factor $i$).

As this takes place we have [see (51) – (52), (55) – (56)]:

$$u''(p, \pm) = \pm u(p, \pm) \cdot \sqrt{\frac{p^*_l}{|p_\perp|}}, \quad v''(p, \pm) = \pm v(p, \pm) \cdot \sqrt{\frac{p^*_l}{|p_\perp|}}.$$ 

Thus if we use the bispinors in form (71) – (74) then the calculation of the amplitudes of processes is reduced to the general scheme (2). For example for the amplitude

$$\bar{u}''(p_f, +)Q u''(p_i, +)$$

we have

$$Z = \mathcal{P}(r, l) \cdot \sqrt{\frac{(p_i)_l^*}{|(p_i)_\perp|}} \cdot \sqrt{\frac{(p_f)_l}{|(p_f)_\perp|}},$$

where $r^\mu = (1, 0, 0, 0)$, $l^\mu = (0, 0, 0, 1)$.

3. In the paper [44] it is proposed to use the bispinor for a massive particle in the form:

$$u(p, n) = \frac{\mathcal{P}(p, n)}{\sqrt{(pq) - m(nq)}} \frac{1}{\sqrt{q_+}} \begin{pmatrix} 0 \\ 0 \\ q_+ \\ q_\perp \end{pmatrix},$$

where

$$q^\mu = (q_0 = 1 + |\alpha|^2, q_x = 2Re \alpha, q_y = 2Im \alpha, q_z = 1 - |\alpha|^2)$$

is a massless 4-vector, $\alpha$ is an arbitrary complex number and $\mathcal{P}(p, n)$ has the form (39).

In this paper it was shown that

$$\bar{u}(p_f, n_f)Q u(p_i, n_i) = \frac{Tr \left[ Q\mathcal{P}(p_i, n_i)\mathcal{P}_+(q)\mathcal{P}(p_f, n_f) \right]}{\sqrt{Tr \left[ \mathcal{P}(p_i, n_i)\mathcal{P}_+(q) \right] Tr \left[ \mathcal{P}(p_f, n_f)\mathcal{P}_+(q) \right] }},$$

where $\mathcal{P}_+(q) = \frac{1}{2}(1 + \gamma_5)\hat{q}$.

So the methods of the calculation of amplitudes discussed above in this Section are described by the general scheme (2). Moreover, most of them are the special cases of the covariant method proposed in Section 4. However the defects of these methods are complicated calculations, bulky and noncovariant form of their results.

In conclusion let us consider two methods based on bispinor transformation. These methods are similar to the ones discussed above, but do not use the explicit form of the bispinors and projection operators.
4. In the paper [45] it is proposed to transform the bispinor for a massive particle as follows

\[ u(p, n) = \frac{P(p, n)}{\sqrt{(pq) + m(nq)}} u_-(q) = \frac{P(p, n)}{\sqrt{Tr[P(p, n)P_-(q)]}} u_-(q) . \tag{75} \]

In fact, the expression for the bispinor is given by (75) up to a phase factor. Really, the right-hand side of Eq. (75) can be written as

\[ \frac{P(p, n)}{\sqrt{Tr[P(p, n)P_-(q)]}} u_-(q) = \frac{u(p, n)\bar{u}(p, n)}{|\bar{u}(p, n)u_-(q)|} u_-(q) = \frac{\bar{u}(p, n)u_-(q)}{|\bar{u}(p, n)u_-(q)|} u(p, n) . \]

Further

\[ u(p, n) = \frac{(\hat{p} + m)(1 + \gamma_5\hat{n})}{2\sqrt{(pq) + m(nq)}} u_-(q) = \frac{\hat{p} + m}{4(pq)\sqrt{(pq) + m(nq)}} u_-(q) \]

\[ = \frac{(\hat{p} + m) \left[ \frac{1}{2}(1 - \gamma_5)\hat{q} + i\frac{1}{2}(1 - \gamma_5)\hat{q}\hat{l} \right] P(p, n)}{4(pq)\sqrt{(pq) + m(nq)}} u_-(q) \]

\[ = \frac{(\hat{p} + m) \left[ P_-(q) + iP_-(q)\hat{l} \right] P(p, n)}{4(pq)\sqrt{(pq) + m(nq)}} u_-(q) \tag{76} \]

\[ = \frac{(\hat{p} + m) \left\{ Tr[P_-(q)P(p, n)] + iTr[P_-(q)\hat{l}P(p, n)] \right\}}{4(pq)\sqrt{(pq) + m(nq)}} u_-(q) \]

\[ = \frac{(\hat{p} + m) \left\{ [(pq) + m(nq)] + [(pl)(nq) - (pq)(nl) - i\varepsilon(p, n, q, l)]\hat{l} \right\}}{4(pq)\sqrt{(pq) + m(nq)}} u_-(q) , \]

where \( q \) is an arbitrary massless 4-vector; \( l \) is an arbitrary 4-vector such that \( l^2 = -1 \), \( q\hat{l} = 0 \) [to obtain (76), we used a variation of the identity (21)]

\[ P_\pm(q)Au_\pm(q) = Tr[P_\pm(q)A] \cdot u_\pm(q) \]

where \( A \) is an arbitrary matrix operator.

As it follows from (75), the above presented transformation of the bispinor leads to the method of the calculation of the amplitudes given by a special case of formula (33).
5. In the paper [46] it is proposed to transform the bispinor for a massive particle as follows

\[ u(p, n) = \frac{\bar{u}_+(q^p)u_-(q^m)}{m} u_+(q^p) + u_-(q^m) \]  

(77)

where

\[ q^p = \frac{1}{2}(p + mn), \quad q^m = \frac{1}{2}(p - mn), \quad (q^p)^2 = (q^m)^2 = 0, \quad (q^p q^m) = \frac{m^2}{2}. \]

This transformation can be obtained by the method similar to the one discussed in previous subsection. Really,

\[ u(p, n) = \frac{\mathcal{P}(p, n)}{\sqrt{Tr[\mathcal{P}(p, n)\mathcal{P}^-(q^m)]}} u_-(q^m) = \frac{\hat{q}^p + m}{m} u_-(q^m) = \frac{1}{2}(1 + \gamma_5)\hat{q}^p + m u_-(q^m) \]

\[ = \left[ \frac{u_+(q^p)\bar{u}_+(q^p)}{m} + 1 \right] u_-(q^m) = \frac{\bar{u}_+(q^p)u_-(q^m)}{m} u_+(q^p) + u_-(q^m). \]

Because of this, the transformation of the bispinor considered leads to the method of the calculation of the amplitudes given by a special case of formula (33).

However, to calculate \( \bar{u}_+(q^p)u_-(q^m) \) the authors of [46] use the method which was proposed in the papers [24] – [25]. (This method was discussed in subsection 10 of Section 2.) In this case we have up to a phase factor [see (28)]

\[ \bar{u}_+(q^p)u_-(q^m) \sim \frac{Tr[(1 - \gamma_5)\hat{q}^m\hat{q}\hat{q}^p]}{4\sqrt{(qq^p)(qq^m)}} \]

\[ = [(q^p)_y + i(q^p)_z] \sqrt{(q^m)_0 - (q^m)_x} - [(q^m)_y + i(q^m)_z] \sqrt{(q^p)_0 - (q^p)_x} \]

where \( l^\mu = (0, 0, 1, 0), \quad q^\mu = (1, 1, 0, 0) \).

As a result, the first term in (77) obtains the phase factor [see (10)]

\[ \frac{\bar{u}_-(q^m)l\hat{q}u_+(q^p)}{|\bar{u}_-(q^m)l\hat{q}u_+(q^p)|} \]

and the bispinor under consideration does not satisfy the Dirac equation.

Note that a special case of the transformation considered, namely the situation when a 4-vector, specifying the axis of the spin projections, has form

\[ n = \lambda \left[ \frac{1}{m} p - \frac{m}{(pq^m)} q^m \right], \quad \lambda = \pm 1 \]

was proposed in [25].
6. Incorrect methods of calculation of the amplitudes

In [47] for deriving of expressions for amplitudes incorrect identity (15), writing without any substantiation, is used

\[ \vec{c}' = -(ss') + \frac{(ps')(p's)}{(pp') + mm'} \]  \hspace{1cm} (15)

where \( s_0 = \frac{\vec{p}\vec{c}}{m}, \quad s' = \vec{c} + \frac{s_0}{p_0 + m}\vec{p} \); \( s'_0 = \frac{\vec{p}'\vec{c}'}{m'}, \quad s' = \vec{c}' + \frac{s'_0}{p'_0 + m'}\vec{p}' \).

Authors of [48] propose to use bispinors for massive Dirac particles in the form

\[ u_+(p) = \begin{bmatrix} \frac{i}{\sqrt{2(pq)}} \left( \begin{array}{c} -q_- \\ q_- \\ \frac{1}{\sqrt{k_-}} \\ k_- \end{array} \right) \\ \frac{i}{\sqrt{k_-}} \left( \begin{array}{c} k_- \\ -q_- \\ m \sqrt{2(pq)}q_- \\ q_- \end{array} \right) \end{bmatrix}, \]

\[ u_-(p) = \begin{bmatrix} \frac{i}{\sqrt{k_-}} \left( \begin{array}{c} k_- \\ -q_- \\ m \sqrt{2(pq)}q_- \end{array} \right) \\ \frac{i}{\sqrt{2(pq)}} \left( \begin{array}{c} -q_- \\ q_- \\ k_- \end{array} \right) \end{bmatrix} \]

where

\[ p = k + \frac{m^2}{2(pq)}q, \quad p^2 = m^2, \quad q^2 = 0, \]

\[ k = p - \frac{m^2}{2(pq)}q, \quad k^2 = 0. \]

It is stated, that \( q \) is an arbitrary massless vector. However from elementary requirement

\[ \bar{u}_+(p)\gamma_5u_+(p) = \bar{u}_-(p)\gamma_5u_-(p) = 0 \]

we have

\[ \frac{q_x}{q_0 - q_z} = \frac{p_x}{p_0 - p_z} \]

i.e. vector \( q \) is not arbitrary. Besides, 4-vector, which determines the axes of the spin projections, is not defined in this paper.

In paper [49] there are mistakes in phase factors of coefficients \( C_{\lambda\lambda'} \), given by equations (10) therein. As a consequence method gives incorrect results. For example, using this
method we have
\[
[\bar{u}(p, +)\gamma_5 u(p', +)][\bar{u}(p, +)\gamma_5 u(p', +)]^* = \frac{|\bar{p}| |\bar{p'}| + (\bar{p}p')}{|\bar{p}| |\bar{p'}| (pp' + mm')^2 \sqrt{p_z^2 + p_y^2}}.
\]
\[
\{ (p_0p'_0 - |\bar{p}| |\bar{p'}| + mm')\sqrt{p_z^2 + p_y^2} [p_0(p\bar{p'}) - p'_0p'\bar{p}']^2
\]
\[
+ (p_0p'_0 + |\bar{p}| |\bar{p'}| + mm')\sqrt{p_z^2 + p_y^2} [||\bar{p}| |\bar{p'}| - (\bar{p}p')]^2 m^2
\]
\[
- 2(mp_0' + m'p_0)p_x [p_0(p\bar{p'}) - p'_0p'\bar{p}'] [||\bar{p}| |\bar{p'}| - (\bar{p}p')] m \}
\]

However, calculation by the classical method gives another result:
\[
|\bar{u}(p, +)\gamma_5 u(p', +)|^2 = -\frac{|mm' - p_0p'_0 + |\bar{p}| |\bar{p'}|]|(\bar{p}p') + |\bar{p}| |\bar{p'}|}.
\]

7 The computer programs for the calculation of the amplitudes

A number of computer programs for the calculation of the amplitudes of processes with polarized Dirac particles (antiparticles) has been developed in the last few years. However all of them have these or those disadvantages and limitations. In particular, they do not allow us to calculate the amplitudes of processes involving massive Dirac particles (antiparticles) with arbitrary polarization states. Let us consider these programs briefly.

1. The program CHANEL (see [50]) is a part of the computer system GRACE (see [51]). This program allows us to calculate the amplitudes of processes with massive Dirac particles (antiparticles), the polarization state of which is the helicity, i.e. the 4-vectors specifying the axes of the spin projection are
\[
n^\mu(p) = \frac{\lambda}{m} \left( |\bar{p}| \frac{p_0}{|\bar{p}|} \right), \quad \lambda = \pm 1
\]
[where $p^\mu = (p_0, \vec{p})$ is the 4-momentum of the corresponding particle (antiparticle)].

The program uses the method of calculation, which is analogous to the one considered in subsection 5 of Section 6:

(a)
\[
\lambda = +1
\]
\[
(q^\nu_{+1}) = \frac{p_0 + |\bar{p}|}{2|\bar{p}|} (|\bar{p}|, \bar{p}) , \quad (q^\nu_{-1}) = \frac{p_0 - |\bar{p}|}{2|\bar{p}|} (|\bar{p}|, -\bar{p}) ,
\]

32
\[ u(p, \lambda = +1) = \frac{\mathcal{P}(p, \lambda = +1)}{\sqrt{\text{Tr}[\mathcal{P}(p, \lambda = +1) \mathcal{P}(q^p_{+1})]}} u_+(q^p_{+1}) = \frac{q^m_{+1} + m}{m} u_+(q^p_{+1}) \]

\[ = \frac{1}{2}(1 - \gamma_5)q^m_{+1} + m \frac{u_+(q^p_{+1}) u_+(q^m_{+1})}{m} + 1 \] \[ u_+(q^p_{+1}) \]

\[ = \frac{1}{2}(1 - \gamma_5)q^m_{+1} + m \frac{u_+(q^p_{+1}) u_+(q^m_{+1})}{m} + 1 \] \[ u_+(q^p_{+1}) \]

Introducing the notation

\[ q_1 = q^p_{+1} = q^m_{-1}, \quad q_2 = q^m_{+1} = q^p_{-1} \]

and combining (78) and (79) we obtain

\[ u(p, \lambda) = \frac{u_-(q_2) u_+(q_1)}{m} u_-(q_2) + u_+(q_1) . \] \[ (80) \]

Further we have [up to phase factor – see also (28)]

\[ \bar{u}_-(q_2) u_+(q_1) \simeq \text{Tr} \left[ q^\dagger \hat{q} \hat{q} \hat{q}_2 (1 + \lambda \gamma_5) \right] = \lambda m \frac{\text{Tr} \left[ q^\dagger \hat{q} \hat{q}_p (1 - \lambda \gamma_5) \right]}{4 \sqrt{(qr)^2 - (q^p_r)^2}} \]

\[ = m \lambda p_y - i p_z \sqrt{p^2_y + p^2_z} . \]

Here

\[ q_1 = \frac{p_0 + |\vec{p}|}{2} (r + l_p), \quad q_2 = \frac{p_0 - |\vec{p}|}{2} (r - l_p) . \]
\[ r^\mu = (1, 0, 0, 0), \quad l_p^\mu = (0, \vec{p}, |\vec{p}|) , \]

\( q, l \) are 4-vectors such that

\[ q^2 = 0, \quad l^2 = -1, \quad (ql) = 0 \]

[in this case one chooses \( q^\mu = (1, 1, 0, 0), \quad l^\mu = (0, 0, 1, 0) \)].

However in this case the first term in (80) obtains the phase factor

\[ \lambda \frac{\bar{u}_{\lambda}(q_1) \hat{q} \hat{l} u_{-\lambda}(q_2)}{|\bar{u}_{\lambda}(q_1) \hat{q} \hat{l} u_{-\lambda}(q_2)|} . \]

As a result, the bispinor (80) does not satisfy the Dirac equation.

2. The program COMPUTE (see [52]) use the method of the calculation of the amplitudes of processes with massive Dirac particles (antiparticles) proposed in [40] and considered in subsection 1.5 of Section 6.

Remind that in this case the polarization state of particles (antiparticles) is defined by the vectors

\[ n^\mu = \frac{\lambda}{m} (p_0 - \frac{m^2}{p_0 + p_z}, p_x, p_y, p_z + \frac{m^2}{p_0 + p_z}) , \quad \lambda = \pm 1 \]

[where \( p^\mu = (p_0, p_x, p_y, p_z) \) is the 4-momentum of the corresponding particle (antiparticle)].

3. The program HELAS (see [53]) is a part of the computer system MadGraph (see [54]). It calculates the amplitudes of processes with massive Dirac particles (antiparticles), the polarization state of which is the helicity by the method proposed in [26]. This method is considered in subsection 1.2 of Section 6.

4. There are two variants for the program HIP. The first variant (see [55]) calculates the amplitudes of processes with massless Dirac particles (antiparticles) by the method proposed in [25] and considered in subsection 10 of Section 2.

The second one (see [56]) calculates the amplitudes of processes with massive Dirac particles (antiparticles), the polarization state of which is the helicity, by the method proposed in [26] and considered in subsection 1.2 of Section 6.

5. The program SPINORP (see [57]) calculates the amplitudes of processes with massless Dirac particles (antiparticles) by the method proposed in [33] and considered in subsection 1.3 of Section 6.
6. The program WPHAST 1.0 (see [58]) calculates the amplitudes of processes involving massive Dirac particles (antiparticles) by the method proposed in [16] and considered in subsections 5.2 and 10 of Section 2.

Recall that in this case the polarization state of particles (antiparticles) is defined by the vectors

\[ n = \lambda \left[ \frac{1}{m^\mu} \left( p - \frac{m}{(pq)^\mu} q \right) \right], \quad \lambda = \pm 1 \]

where \( p \) is the 4-momentum of the corresponding particle (antiparticle) and \( q \) is an arbitrary 4-vector such that \( q^2 = 0 \) and in numerical calculations one chooses \( q^\mu = (1, 1, 0, 0) \).

7. In [59] there was proposed the program for the calculation of the amplitudes of processes with massive Dirac particles (antiparticles) by the method proposed in [36] and considered in subsection 1.4 of Section 6.

Recall that in this case the polarization state of particles (antiparticles) is defined by the vectors

\[ n^\mu = \frac{\lambda}{m} \left( p^\mu, \frac{p_x p_z}{p_0 + m}, \frac{p_y p_z}{p_0 + m}, m + \frac{p_z^2}{p_0 + m} \right), \quad \lambda = \pm 1 \]

where \( p^\mu = (p_0, p_x, p_y, p_z) \) is the 4-momentum of the corresponding particle (antiparticle).

However, in addition to programs described above some authors use the computer algebra systems to calculate the amplitudes of processes for concrete problems. For example, the author of [28] uses the system MAPLE for the calculation of the amplitudes of processes involving massive Dirac particles (antiparticles), the polarization state of which is the helicity, by the method considered in subsection 1.2 of Section 6.

Thus, at present there are no computer programs for the calculation of the amplitudes of processes involving massive Dirac particles (antiparticles) which have arbitrary polarization states. The programs of this sort may be created with help of the formulae presented in Section 5.

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