Quantum chimera states

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We study a theoretical model of closed quasi-hermitian chain of spins which exhibits quantum analogues of chimera states, i.e. long life classical states for which a part of an oscillator chain presents an ordered dynamics whereas another part presents a disordered chaotic dynamics. For the quantum analogue, the chimera behavior deals with the entanglement between the spins of the chain. We discuss the entanglement properties, quantum chaos, quantum disorder and semi-classical similarity of our quantum chimera system. The quantum chimera concept is novel and induces new perspectives concerning the entanglement of multipartite systems.

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INTRODUCTION

Recently intriguing states exhibiting both ordered and disordered dynamics have been discovered in long range coupled sets of oscillators \([1]\) and have been highlighted in other coupled sets of classical (mechanical, electronic or opto-electronic) systems \([2,4]\). The particularity of such states is a part of the oscillator set exhibits an ordered dynamics (synchronized oscillations) whereas another part exhibits a disordered dynamics (oscillations without correlation between the oscillators) which can be considered as chaotic. This regime is not transient, these intriguing states have got a long and sometimes an infinite life duration: chaos does not spread to the whole set and the disordered part does not collapse to synchronized oscillations in a short time. These states have been called chimera, in reference to the mythological creature hybrid of a lion, a snake and a goat. An interesting simple example of chimera states have been studied in \([3]\). It consists of a closed chain of \(N\) oscillators with long range coupling of their phases:

\[
\dot{\theta}_i(t) = \omega - \nu \sum_{j=i-M}^{j+M} \sin(\theta_i(t) - \theta_j(t) + \alpha) \tag{1}
\]

where \(\theta_i\) is the phase of \(i\)-th oscillator, \(\omega\) and \(\nu\) are constant frequencies, \(\alpha\) is a constant angle and \(M \in \{2, \ldots, N/2 - 1\}\) is the range of coupling (the indices are taken modulo \(N\)).

In this paper we show that a simple quantum system, a closed chain of spins, offers quantum analogues of the chimera states. It is well known that spin chains can exhibit kinds of quantum disorder and of quantum chaos \([5]\). To involve a kind of chimera states, our model consists of a non-hermitian spin chain \([10,13]\) which can be assimilated to a spin chain in contact with an environment. This model is presented next section. Disorder and chaos in the model are discussed in the following sections. These notions, which are ambiguous in quantum mechanics, can be enlightened by our model.

THE MODEL

We consider a closed chain of \(N\) spins \(\frac{1}{2}\). Let \(\{\hat{I}_i\}_{i=1,\ldots,N}\) be the set of the observables defined by

\[
\hat{I}_i = \frac{\hbar \omega_i}{2} \sigma_z + \frac{\hbar \nu}{2M} \sin \alpha + \frac{\hbar \nu}{2M} \cos \alpha \sum_{j=i-M}^{i+M} (\sigma_{z,i} \otimes \sigma_{z,j} - \sigma_{z,i} \otimes \sigma_{z,j}) \tag{2}
\]

where \(\{\sigma_x,\sigma_y,\sigma_z\}\) are the Pauli matrices and \(\sigma_\pm = \sigma_x \pm i\sigma_y\) (the indices denote the spin on which the Pauli matrix acts as an operator; the indices are taken modulo \(N\)). \(M \in \{2, \ldots, N/2 - 1\}\) is the range of coupling between the spins, \(\nu\) is a constant frequency, \(\alpha\) is a constant angle and \(\omega_i\) is the Larmor frequency of the \(i\)-th spin in a local magnetic field. The observable \(\hat{I}_i\) is a quantum analogue of the equation (1) of the classical model. Indeed let \(|\theta,\phi\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + e^{i\phi} \sin \frac{\theta}{2} |\downarrow\rangle\) be the spin coherent state \([14]\), i.e. the quantum state closer to the classical spin state defined by the phase space point \((\theta, \phi)\) (\(\theta\) and \(\phi\) are the angles, which are the coordinates on the Bloch sphere). We have \(|\theta,\phi|\sigma_z|\theta,\phi\rangle = \cos \theta \) and \(|\theta,\phi|\sigma_z|\theta,\phi\rangle = e^{i\phi} \sin \theta\). Let \(|\theta\rangle = |\theta_1,0\rangle \otimes \ldots \otimes |\theta_N,0\rangle\) be the coherent state for the \(N\) spins of the chain with \(\phi_1 = \ldots = \phi_N = 0\). We have

\[
|\theta,\phi\rangle \hat{I}_i |\theta,\phi\rangle = \frac{\hbar \omega_i \cos \theta_i}{2} + \frac{\hbar \nu}{2M} \sum_{j=i-M}^{i+M} \sin(\theta_i - \theta_j + \alpha) \tag{3}
\]

which is similar to the classical first integral \(I_i = \dot{\theta}_i + \frac{\nu}{2M} \sum_{j=i-M}^{i+M} \sin(\theta_i - \theta_j + \alpha)\) if we assimilate \(\omega \cos \theta_i / \frac{2}{\alpha}\) to \(\dot{\theta}_i\). We note that \(\dot{\hat{I}_i}, \dot{\hat{I}_j} \neq \delta_{ij}\) like \(\{\hat{I}_i, \hat{I}_j\} \neq \delta_{ij}\) (with \(\{,\}\) the Poisson bracket). We use the observables \(\{\hat{I}_i\}_i\) to define the Hamiltonian of our quantum analogue of
the system (1):

\[
H = \sum_{i=1}^{N} \hat{I}_i = \sum_{i=1}^{N} \frac{\hbar \omega_i}{2} \sigma_{z_i} + \frac{N \hbar \nu}{2M} \sin \alpha + \frac{\hbar \nu \sin \alpha}{2M} \sum_{i=1}^{N} \sum_{j=i-M}^{i+M} (\sigma_{+i} \otimes \sigma_{+j} + \sigma_{z_i} \otimes \sigma_{z_j}) \tag{5}
\]

Note that this quantum system is analogue to the system (1) in the sense of (3). However it is not the quantization of the model (1) and this last is not the classical limit of the system (4). \(H\) is not hermitian but its spectrum is real since the matrix representation of \(H\) is upper diagonal with real values on the diagonal \((\pm \frac{\hbar \omega_i}{2} + k \frac{\nu}{4M} \sin \alpha, k \in \mathbb{Z})\). We say that \(H\) is quasi-hermitian [15]. \(H\) can be viewed as the effective Hamiltonian of a long range coupling Heisenberg spin chain in contact with an environment. Such a system is described by the quantum master equation for its density matrix \(\rho\):

\[
\frac{i\hbar}{\tau} \dot{\rho} = [H_0, \rho] - \frac{\hbar}{2}(K\rho + \rho K) + i\Gamma \rho \Lambda^\dagger \tag{6}
\]

with \(H_0 = \sum_i \frac{\hbar \omega_i}{2} \sigma_{z_i} + \sum_{i,j} J_{ij}(\sigma_{z_i} \otimes \sigma_{z_j} + \sigma_{x_i} \otimes \sigma_{x_j} - \sigma_{y_i} \otimes \sigma_{y_j})\) and \(K = -2\sum_{i,j} J_{ij}(\sigma_{z_i} \otimes \sigma_{y_j} + \sigma_{y_i} \otimes \sigma_{z_j})\). In the conditions where the quantum jump \(\Gamma\|\Lambda\) is neglected, the chain is governed by the effective Hamiltonian \(H = H_0 - \frac{i}{2} K\) which coincides with (4) by taken \(J_{ij} = \frac{\hbar \nu \sin \alpha}{2M}\) if \(|j - i| \leq M\) and \(i \neq j\) \((J_{ij} = 0\) if \(|j - i| > M\) or \(i = j\)). The frequencies \(\omega_i\) in the classical model can be chosen equal to a same value but it is physically more significant to randomly choose \(\omega_i\) in an interval \([0, \omega_\alpha]\) describing the local magnetic field perturbed by the effects of the environment (and corresponding to the chaotic distribution of the values \(\{\theta_i\}\) in the classical model).

**QUANTUM CHIMERA STATES**

We consider the eigenstates and the biorthogonal eigenstates of \(H\) respectively:

\[
H|\chi_n\rangle = \chi_n|\chi_n\rangle \quad H^\dagger|\chi_n^\dagger\rangle = \chi_n|\chi_n^\dagger\rangle \tag{7}
\]

with \(\chi_n \in \mathbb{R}\) and \(|\chi_n^\dagger|\chi_p\rangle = \delta_{np}\). In order to enlighten the similarity of these eigenstates to chimera states, we consider the Husimi distribution [16] \(h^\chi_{\theta n}(\theta) = |\langle \theta | 0 | \rho^\chi_{\theta n} | 0 \rangle|\) where \(\rho^\chi_{\theta n} = \text{tr}_i|\chi_n\rangle \langle \chi_n|\) is the density matrix of the spin \(i\) when the chain is in the state \(|\chi_n\rangle\) \((\text{tr}_i\) denotes the partial trace over all spin spaces except the \(i\)-th). \(h^\chi_{\theta n}(\theta)\) measures the probability of similarity between the mixed quantum state \(\rho^\chi_{\theta n}\) and the classical spin state characterized by an angle \(\theta\) with the \(z\)-axis. To complete the analysis we consider also the up population \(p^\chi_{\theta n} = \langle \uparrow | \rho^\chi_{\theta n} | \uparrow \rangle\) (the occupation probability of the state up by the spin \(i\)), the coherence of the spin \(i\) \(c^\chi_{\theta n} = |\langle \uparrow | \rho^\chi_{\theta n} | \downarrow \rangle|\), and the linear entropy \(S^\chi_{\theta n} = 1 - \text{tr}(\rho^\chi_{\theta n})^2\) (the entanglement measure of the spin \(i\) with the other spins).

A typical eigenstate is shown figure [11]. We observe its similarity with the classical states of the model (1) studied in [8]: a part of the spin chain presents a large entropy and the other one, a zero (or a small) entropy. But in contrast with the classical case where the entropy measures the disorder, in this quantum context the entropy measures the entanglement. In comparison, the computation of the same quantities for different models of chaotic or random spin chains or glasses [6, 8, 17] shows eigenstates with a large entanglement which is uniform on the chain (or with small variations between nearest neighbour spins). These models do not involve states with both some spins highly entangled and the other ones totally not entangled as in figure [11]. The “green region” of the Husimi distribution (the entangled region) corresponds to the “chaotic part” of the chain and the region where the Husimi distribution shows spins “aligned” with the up or the down directions (the non entangled region) corresponds to the
regular part of the chain. The chain is closed and other eigenstates present an entangled region centered on other spins. Moreover, in contrast with the classical case, the green region is not necessarily connected as in figure 2. The quantum states like figures 1 and 2 can be considered as quantum chimera states. Note that the present model like the chaotic or the random models [4, 6, 17] presents also few totally regular (non entangled) states.

DISORDER AND ENTANGLEMENT

Disorder does not have the same status for quantum or classical systems. It is the entanglement which is involved by the quantum chaos and not the disorder. It must be interesting to measure these two physical concepts globally. The average linear entropy \( S^{\rho} = \frac{1}{N} \sum_{i=1}^{N} S^{\rho_i} \) is a measure of the mean entanglement of the chain in the state \( \rho_n \). If each spin is in a pure state, the linear entropy \( 1 - \text{tr}(\rho^{\chi}) \) of the average state \( \langle \rho^{\chi} \rangle = \frac{1}{N} \sum_{i=1}^{N} \rho_i^{\chi} \) is a measure of the disorder because it is zero if all the pure states are equal and is large if the pure states are strongly different. But if the spins are in mixed states, \( 1 - \text{tr}(\rho^{\chi}) \) includes also the entanglement entropy of the chain. We propose then as a measure of the quantum disorder \( D^{\chi} = 1 - \text{tr}(\rho^{\chi}) - \langle S^{\rho} \rangle \). We have represented figure 3 the entanglement and disorder distribution for the chimera model in comparison with chaotic and regular models. The totally regular systems present eigenstates concentrated on the zero entanglement axis (the larger disordered states being with half of the spins in the pure state up and the other half in the pure state down). The chaotic systems present eigenstates concentrated in the neighbourhood of the zero disorder axis. The chimera model presents a distribution of its eigenvectors clearly between these two cases, characterizing its hybrid nature.

CHAOTIC BEHAVIOR

A last question concerns the chaotic nature of the quantum chimera model. Quantum chaos is an ambiguous concept since in classical dynamics the chaos is strongly linked to the non-linear effects whereas the quantum dynamics is fundamentally a linear theory. A commonly used criterion of quantum chaos for spin systems is the level spacing distribution (LSD) of the spectrum [6]. A regular system presents a LSD as Dirac picks, a (pseudo)-random system presents a LSD as a Poisson distribution (characterizing the disorder of the energy levels without correlation) and a chaotic system presents a LSD as a Wigner-Dyson distribution (characterizing the disorder of the energy levels with correlations). With this definition of quantum chaos, the chimera system [4] is neither chaotic, its LSD is Dirac picks if \( \omega_1 = \ldots = \omega_N \) or a Poisson distribution if \( \{\omega_i\}_i \) are randomly chosen in \([0, \omega_0]\). This can be a manifestation of the hybrid nature of the system or an indication that the LSD criterion is not completely pertinent for non-hermitian Hamiltonians.
Another criterion of quantum chaos [18] concerns the dynamical behavior of a chosen state $\psi_0$ with respect to its survival probability $p_{\text{surv}}(t) = |\langle \psi_0 | e^{-itH} | \psi_0 \rangle|^2$ (with $H$ denoting the Hamiltonian). $\psi_0$ is a scattering state if its survival probability grows on and on with an almost chaotic models [6–9]. But for these cases $p_0$ is a bound state if its survival probability is constant, or presents periodic or quasiperiodic oscillations.

$\psi_0$ is a scattering state if its survival probability falls quickly and definitively to zero. $\psi_0$ is a chaotic state if its survival probability chaotically oscillates with globally a slow decrease to zero with erratic resurgences of non-zero probabilities. These behaviours can be enlightened by considering the cumulated survival probability $p_{\text{cum}}(t) = \int_0^t p_{\text{surv}}(t')dt'$. For a bound state the cumulated survival probability grows linearly, for a scattering state it quickly increases until a maximal value and then remains constant, for a chaotic state it grows on and on but not linearly. A chaotic quantum system is then a system exhibiting some chaotic states. Let $|\psi_0\rangle$ be a state with the spins in states up or down (without superposition) relatively disordered, for example $|\psi_0\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$. Such a state is close to a chaotic state as shown by its survival probability and its cumulated survival probability drawn figure[4]. We see that the survival probability seems “chaotically” oscillate with a global decrease and with erratic resurgences. The cumulated survival probability grows on and on with an almost linear growth. This ambiguous behavior is certainly the manifestation of the nature of the chimera system which is a hybrid of a both chaotic and regular system.

The system defined by the Hamiltonian (4) exhibits hybrid behaviors between a chaotic and a regular system. The chimera states of spin chains, presenting both highly entangled regions and totally not entangled regions (which are stable because the chimera states are eigenstates), could be very interesting for quantum information protocols. We could imagine transports of information using the couplings of the chain from a region to another one with manipulations taking advantage of the radical difference of the entanglement amplitudes.

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