Generic Bell correlation between arbitrary local algebras in quantum field theory

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Abstract

We prove that for any two commuting von Neumann algebras of infinite type, the open set of Bell correlated states for the two algebras is norm dense. We then apply this result to algebraic quantum field theory — where all local algebras are of infinite type — in order to show that for any two spacelike separated regions, there is an open dense set of field states that dictate Bell correlations between the regions. We also show that any vector state cyclic for one of a pair of commuting nonabelian von Neumann algebras is entangled (i.e., non-separable) across the algebras—from which it follows that every field state with bounded energy is entangled across any two spacelike separated regions.
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I. Introduction

There are many senses in which the phenomenon of Bell correlation, originally discovered and investigated in the context of elementary nonrelativistic quantum mechanics [1, 2], is “generic” in quantum field theory models. For example, it has been shown that every pair of commuting nonabelian von
Neumann algebras possesses some normal state with maximal Bell correlation [3] (see also [4]). Moreover, in most standard quantum field models, all normal states are maximally Bell correlated across spacelike separated tangent wedges or double cones [3,5–8]. Finally, every bounded energy state in quantum field theory sustains maximal Einstein-Podolsky-Rosen correlations across arbitrary spacelike separated regions [3], and has a form of nonlocality that may be evinced by means of the state’s violation of a conditional Bell inequality [10]. (We also note that the study of Bell correlation in quantum field theory has recently borne fruit in the introduction of a new algebraic invariant for an inclusion of von Neumann algebras [7,8].)

Despite these numerous results, it remains an open question whether “most” states will have some or other Bell correlation relative to arbitrary spacelike separated regions. Our main purpose in this note is to verify that this is so: for any two spacelike separated regions, there is an open dense set of states which have Bell correlations across those two regions.

In section II we prove the general result that for any pair of mutually commuting von Neumann algebras of infinite type, a dense set of vectors will induce states which are Bell correlated across these two algebras. In section III we introduce, following [11], a notion of “nonseparability” of states that generalizes, to mixed states, the idea of an entangled pure state vector. We then show that for a pair of nonabelian von Neumann algebras, a vector cyclic for either algebra induces a nonseparable state. Finally, in section IV we apply these results to algebraic quantum field theory.

II. Bell correlation between infinite von Neumann algebras

Let \( \mathcal{H} \) be a Hilbert space, let \( \mathcal{S} \) denote the set of unit vectors in \( \mathcal{H} \), and let \( \mathcal{B}(\mathcal{H}) \) denote the set of bounded linear operators on \( \mathcal{H} \). We will use the same notation for a projection in \( \mathcal{B}(\mathcal{H}) \) and for the subspace in \( \mathcal{H} \) onto which it projects. If \( x \in \mathcal{S} \), we let \( \omega_x \) denote the state of \( \mathcal{B}(\mathcal{H}) \) induced by \( x \). Let \( \mathcal{R}_1, \mathcal{R}_2 \) be von Neumann algebras acting on \( \mathcal{H} \) such that \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \), and let \( \mathcal{R}_{12} \) denote the von Neumann algebra \( \{ \mathcal{R}_1 \cup \mathcal{R}_2 \}'' \) generated by \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Following [7], we set

\[
\mathcal{T}_{12} \equiv \left\{ (1/2)[A_1(B_1 + B_2) + A_2(B_1 - B_2)] : A_i = A_i^* \in \mathcal{R}_1, B_i = B_i^* \in \mathcal{R}_2, -I \leq A_i, B_i \leq I \right\},
\]

(1)
Elements of $T_{12}$ are called Bell operators for $\mathcal{R}_{12}$. For a given state $\omega$ of $\mathcal{R}_{12}$, let

$$\beta(\omega) \equiv \sup \{|\omega(R)| : R \in T_{12}\}.$$  \hspace{1cm} (2)

If $\omega = \omega_x|_{\mathcal{R}_{12}}$ for some $x \in \mathcal{S}$, we write $\beta(x)$ to abbreviate $\beta(\omega_x|_{\mathcal{R}_{12}})$. From (2), it follows that the map $\omega \to \beta(\omega)$ is norm continuous from the state space of $\mathcal{R}_{12}$ into $[1, \sqrt{2}]$. Since the map $x \to \omega_x|_{\mathcal{R}_{12}}$ is continuous from $\mathcal{S}$, in the vector norm topology, into the (normal) state space of $\mathcal{R}_{12}$, in the norm topology, it also follows that $x \to \beta(x)$ is continuous from $\mathcal{S}$ into $[1, \sqrt{2}]$. If $\beta(\omega) > 1$, we say that $\omega$ violates a Bell inequality, or is Bell correlated. In this context, Bell’s theorem [1] is the statement that a local hidden variable model of the correlations that $\omega$ dictates between $\mathcal{R}_1$ and $\mathcal{R}_2$ is only possible if $\beta(\omega) = 1$. Note that the set of states $\omega$ on $\mathcal{R}_{12}$ that violate a Bell inequality is open (in the norm topology) and, similarly, the set of vectors $x \in \mathcal{S}$ that induce Bell correlated states on $\mathcal{R}_{12}$ is open (in the vector norm topology).

We assume now that the pair $\mathcal{R}_1, \mathcal{R}_2$ satisfies the Schlieder property. That is, if $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$ such that $AB = 0$, then either $A = 0$ or $B = 0$. Let $V \in \mathcal{R}_1$ and $W \in \mathcal{R}_2$ be nonzero partial isometries. Suppose that the initial space $V^*V$ of $V$ is orthogonal to the final space $VV^*$ of $V$; or, equivalently, that $V^2 = 0$. Similarly, suppose $W^2 = 0$. Consider the projections

$$E = V^*V + VV^*, \quad F = W^*W + WW^*.$$  \hspace{1cm} (3)

We show that there is a Bell operator $\overline{R}$ for $\mathcal{R}_{12}$ such that $\overline{R}y = \sqrt{2}y$ for some unit vector $y \in EF$, and $\overline{R}(I - E)(I - F) = (I - E)(I - F)$.

Let

$$A_1 = V + V^* \quad B_1 = W + W^*$$

$$A_2 = i(V^* - V) \quad B_2 = i(W^* - W)$$

$$A_3 = [V, V^*] \quad B_3 = [W, W^*].$$  \hspace{1cm} (4)

Note that $A_i^2 = E$, the $A_i$ are self-adjoint contractions in $\mathcal{R}_1$, $A_i E = E A_i = A_i$, and $[A_1, A_2] = 2i A_3$. Similarly, $B_i^2 = F$, the $B_i$ are self-adjoint contractions in $\mathcal{R}_2$, $B_i F = F B_i = B_i$, and $[B_1, B_2] = 2i B_3$. If we let $\overline{R}$ denote the Bell operator constructed from $A_i, B_i$, a straightforward calculation shows that (cf. [1])

$$R^2 = EF - \frac{1}{4}[A_1, A_2][B_1, B_2] = EF + A_3 B_3.$$  \hspace{1cm} (5)

Note that $P \equiv VV^* \neq 0$ is the spectral projection for $A_3$ corresponding to eigenvalue 1, and $Q \equiv WW^* \neq 0$ is the spectral projection for $B_3$ corresponding to eigenvalue 1. Since $\mathcal{R}_1, \mathcal{R}_2$ satisfy the Schlieder property, there
is a unit vector $y \in PQ$, and thus $A_3B_3y = y$. Since $PQ < EF$, it follows from \( R \) that $R^2y = 2y$. Thus, we may assume without loss of generality that $Ry = \sqrt{2}y$. (If $Ry \neq \sqrt{2}y$, then interchange $B_1, B_2$ and replace $A_1$ with $-A_1$. Note that the resulting Bell operator $R' = -R$ and $R'y_0 = \sqrt{2}y_0$, where $y_0 \equiv (\sqrt{2}y - Ry)/\|\sqrt{2}y - Ry\| \in EF$.)

Now for $i = 1, 2$, let $A_i = (I - E) + A_1$ and $B_i = (I - E) + B_1$. It is easy to see that $A_i^2 = I$ and $B_i^2 = I$, so the $A_i$ and $B_i$ are again self-adjoint contractions in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. If we let $\tilde{R}$ denote the corresponding Bell operator, a straightforward calculation shows that

$$\tilde{R} = (I - E)(I - E) + (I - E)B_1 + A_1(I - E) + R.$$  

(6)

Since the $\sqrt{2}$ eigenvector $y$ for $R$ lies in $EF$, we have $\tilde{R}y = Ry = \sqrt{2}y$. Furthermore, since $A_i(I - E) = 0$ and $B_i(I - E) = 0$, we have $\tilde{R}(I - E)(I - F) = (I - E)(I - F)$ as required.

A special case of the following result, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are type $I_{\infty}$ factors, was proved as [12, Prop. 1]. Recall that $\mathcal{R}$ is said to be of infinite type just in case the identity $I$ is equivalent, in $\mathcal{R}$, to one of its proper subprojections.

**Proposition 1.** Let $\mathcal{R}_1, \mathcal{R}_2$ be von Neumann algebras acting on $\mathcal{H}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$, and $\mathcal{R}_1, \mathcal{R}_2$ satisfy the Schlieder property. If $\mathcal{R}_1, \mathcal{R}_2$ are of infinite type, then there is an open dense subset of vectors in $\mathcal{S}$ which induce Bell correlated states for $\mathcal{R}_{12}$.

Note that the hypotheses of this proposition are invariant under isomorphisms of $\mathcal{R}_{12}$. Thus, by making use of the universal normal representation of $\mathcal{R}_{12}$ [13, p. 458], in which all normal states are vector states, it follows that the set of states Bell correlated for $\mathcal{R}_1, \mathcal{R}_2$ is norm dense in the normal state space of $\mathcal{R}_{12}$.

**Proof of the proposition:** Since $\mathcal{R}_1$ is infinite, there is a properly infinite projection $P \in \mathcal{R}_1$ [13, Prop. 6.3.7]. Since $P$ is properly infinite, we may apply the halving lemma [13, Lemma 6.3.3] repeatedly to obtain a countably infinite family $\{P_n\}$ of mutually orthogonal projections such that $P_n \sim P_{n+1}$ for all $n$ and $\sum_{n=1}^{\infty} P_n = P$. (Halve $P$ as $P_1 + F_1$; then halve $F_1$ as $P_2 + F_2$, and so on. Now replace $P_1$ by $P - \sum_{n=2}^{\infty} P_n$; cf. [13, Lemma 6.3.4].) Let $P_0 \equiv I - P$. For each $n \in \mathbb{N}$, let $V_n$ denote the partial isometry with initial space $V_n^*V_n = P_n$ and final space $V_nV_n^* = P_{n+1}$. By the same reasoning, there is a countable family $\{Q_n\}$ of mutually orthogonal projections in $\mathcal{R}_2$.
and partial isometries $W_n$ with $W_n^*W_n = Q_n$ and $W_nW_n^* = Q_{n+1}$. For each $n \in \mathbb{N}$, let

$$
A_{1,n} = V_{n+1} + V_{n+1}^* \quad B_{1,n} = W_{n+1} + W_{n+1}^*,
A_{2,n} = i(V_{n+1}^* - V_{n+1}) \quad B_{2,n} = i(W_{n+1}^* - W_{n+1}),
$$

and let

$$
E_n = V_{n+1}^*V_{n+1} + V_{n+1}V_{n+1}^* = P_{n+1} + P_{n+2},
F_n = W_{n+1}^*W_{n+1} + W_{n+1}W_{n+1}^* = Q_{n+1} + Q_{n+2}.
$$

Define $\tilde{A}_{i,n}$ and $\tilde{B}_{i,n}$ as in the discussion preceding this proposition, let $\tilde{R}_n$ be the corresponding Bell operator, and let the unit vector $y_n \in E_nF_n$ be the $\sqrt{2}$ eigenvector for $\tilde{R}_n$.

Now, let $x$ be any unit vector in $\mathcal{H}$. Since $\sum_{i=0}^{n} P_i \leq I - E_n$, we have $(I - E_n) \to I$ in the strong-operator topology. Similarly, $(I - F_n) \to I$ in the strong-operator topology. Therefore if we let

$$
x_n \equiv \frac{(I - E_n)(I - F_n)x}{\|(I - E_n)(I - F_n)x\|},
$$

we have

$$
x = \lim_n (I - E_n)(I - F_n)x = \lim_n x_n.
$$

Note that the inner product $\langle x_n, y_n \rangle = 0$, and thus

$$
z_n \equiv (1 - n^{-1})^{1/2}x_n + n^{-1/2}y_n
$$

is a unit vector for all $n$. Since $\lim_n z_n = x$, it suffices to observe that each $z_n$ is Bell correlated for $\mathcal{R}_{12}$. Recall that $\tilde{R}_n(I - E_n)(I - F_n) = (I - E_n)(I - F_n)$, and thus $\tilde{R}_nx_n = x_n$. A simple calculation then reveals that

$$
\beta(z_n) \geq \langle \tilde{R}_nz_n, z_n \rangle = (1 - n^{-1}) + n^{-1}\sqrt{2} > 1.
$$

\[\Box\]

### III. Cyclic vectors and entangled states

Proposition 1 establishes that Bell correlation is generic for commuting pairs of infinite von Neumann algebras. However, we are given no information about the character of the correlations of particular states. We provide a partial remedy for this in the next proposition, where we show that any vector cyclic for $\mathcal{R}_1$ (or for $\mathcal{R}_2$) induces a state that is not classically correlated; i.e., it is “nonseparable.”
Again, let $\mathcal{R}_1, \mathcal{R}_2$ be von Neumann algebras on $\mathcal{H}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2'$. Recall that a state $\omega$ of $\mathcal{R}_{12}$ is called a normal product state just in case $\omega$ is normal, and there are states $\omega_1$ of $\mathcal{R}_1$ and $\omega_2$ of $\mathcal{R}_2$ such that

$$\omega(AB) = \omega_1(A)\omega_2(B), \quad (13)$$

for all $A \in \mathcal{R}_1, B \in \mathcal{R}_2$. Werner [11], in dealing with the case of $\mathcal{B}(\mathbb{C}^n) \otimes \mathcal{B}(\mathbb{C}^n)$, defined a density operator $D$ to be classically correlated — the term separable is now more commonly used — just in case $D$ can be approximated in trace norm by convex combinations of density operators of form $D_1 \otimes D_2$. Although Werner’s definition of nonseparable states directly generalizes the traditional notion of pure entangled states, he showed that a nonseparable mixed state need not violate a Bell inequality; thus, Bell correlation is in general a sufficient, though not necessary condition for a state’s being nonseparable. On the other hand, it has since been shown that nonseparable states often possess more subtle forms of nonlocality, which may be indicated by measurements more general than the single ideal measurements which can indicate Bell correlation [14]. (See [12, 15] for further discussion.)

In terms of the linear functional representation of states, Werner’s separable states are those in the norm closed convex hull of the product states of $\mathcal{B}(\mathbb{C}^n) \otimes \mathcal{B}(\mathbb{C}^n)$. However, in case of the more general setup — i.e., $\mathcal{R}_1 \subseteq \mathcal{R}_2'$, where $\mathcal{R}_1, \mathcal{R}_2$ are arbitrary von Neumann algebras on $\mathcal{H}$ — the choice of topology on the normal state space of $\mathcal{R}_{12}$ will yield in general different definitions of separability. Moreover, it has been argued that norm convergence of a sequence of states can never be verified in the laboratory, and as a result, the appropriate notion of physical approximation is given by the (weaker) weak-* topology [14, 17]. And the weak-* and norm topologies do not generally coincide even on the normal state space [18].

For the next proposition, then, we will suppose that the separable states of $\mathcal{R}_{12}$ are those normal states in the weak-* closed convex hull of the normal product states. Note that $\beta(\omega) = 1$ if $\omega$ is a product state, and since $\beta$ is a convex function on the state space, $\beta(\omega) = 1$ if $\omega$ is a convex combination of product states [8, Lemma 2.1]. Furthermore, since $\beta$ is lower semicontinuous in the weak-* topology [8, Lemma 2.1], $\beta(\omega) = 1$ for any separable state. Conversely, any Bell correlated state must be nonseparable.

We now introduce some notation that will aid us in the proof of our result. For a state $\omega$ of the von Neumann algebra $\mathcal{R}$ and an operator $A \in \mathcal{R}$, define the state $\omega^A$ on $\mathcal{R}$ by

$$\omega^A(X) \equiv \frac{\omega(A^*XA)}{\omega(A^*A)}, \quad (14)$$
if \( \omega(A^*A) \neq 0 \), and let \( \omega^A = \omega \) otherwise. Suppose now that \( \omega(A^*A) \neq 0 \) and \( \omega \) is a convex combination of states:

\[
\omega = \sum_{i=1}^{n} \lambda_i \omega_i. \tag{15}
\]

Then, letting \( \lambda_i^A \equiv \omega(A^*A)^{-1}\omega_i(A^*A)\lambda_i \), \( \omega^A \) is again a convex combination

\[
\omega^A = \sum_{i=1}^{n} \lambda_i^A \omega_i^A. \tag{16}
\]

Moreover, it is not difficult to see that the map \( \omega \to \omega^A \) is weak-\( \ast \)-continuous at any point \( \rho \) such that \( \rho(A^*A) \neq 0 \). Indeed, let \( \mathcal{O}_1 = N(\rho^A : X_1, \ldots, X_n, \epsilon) \) be a weak-\( \ast \) neighborhood of \( \rho^A \). Then, taking \( \mathcal{O}_2 = N(\rho : A^*A, A^*X_1A, \ldots, A^*X_nA, \delta) \) and \( \omega \in \mathcal{O}_2 \), we have

\[
|\rho(A^*X_iA) - \omega(A^*X_iA)| < \delta, \tag{17}
\]

for \( i = 1, \ldots, n \), and

\[
|\rho(A^*A) - \omega(A^*A)| < \delta. \tag{18}
\]

By choosing \( \delta < \rho(A^*A) \neq 0 \), we also have \( \omega(A^*A) \neq 0 \), and thus

\[
|\rho^A(X_i) - \omega^A(X_i)| < O(\delta) \leq \epsilon, \tag{19}
\]

for an appropriate choice of \( \delta \). That is, \( \omega^A \in \mathcal{O}_1 \) for all \( \omega \in \mathcal{O}_2 \) and \( \omega \to \omega^A \) is weak-\( \ast \) continuous at \( \rho \).

Specializing to the case where \( \mathcal{R}_1 \subseteq \mathcal{R}_2' \), and \( \mathcal{R}_{12} = \{\mathcal{R}_1 \cup \mathcal{R}_2\}'' \), it is clear from the above that for any normal product state \( \omega \) of \( \mathcal{R}_{12} \) and for \( A \in \mathcal{R}_1 \), \( \omega^A \) is again a normal product state. The same is true if \( \omega \) is a convex combination of normal product states, or the weak-\( \ast \) limit of such combinations. We summarize the results of this discussion in the following lemma:

**Lemma.** For any separable state \( \omega \) of \( \mathcal{R}_{12} \) and any \( A \in \mathcal{R}_1 \), \( \omega^A \) is again separable.

**Proposition 2.** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be nonabelian von Neumann algebras such that \( \mathcal{R}_1 \subseteq \mathcal{R}_2' \). If \( x \) is cyclic for \( \mathcal{R}_1 \), then \( \omega_x \) is nonseparable across \( \mathcal{R}_{12} \).
Proof. From [7, Lemma 2.1], there is a normal state $\rho$ of $R_{12}$ such that $\beta(\rho) = \sqrt{2}$. But since all normal states are in the (norm) closed convex hull of vector states [13, Thm 7.1.12], and since $\beta$ is norm continuous and convex, there is a vector $v \in S$ such that $\beta(v) > 1$. By the continuity of $\beta$ (on $S$), there is an open neighborhood $O$ of $v$ in $S$ such that $\beta(y) > 1$ for all $y \in O$. Since $x$ is cyclic for $R_1$, there is an $A \in R_1$ such that $Ax \in O$. Thus, $\beta(Ax) > 1$ which entails that $\omega_{Ax} = (\omega_x)^A$ is a nonseparable state for $R_{12}$. This, by the preceding lemma, entails that $\omega_x$ is nonseparable.

Note that if $R_1$ has at least one cyclic vector $x \in S$, then $R_1$ has a dense set of cyclic vectors in $S$ [13]. Since each of the corresponding vector states is nonseparable across $R_{12}$, Proposition 2 shows that if $R_1$ has a cyclic vector, then the (open) set of vectors inducing nonseparable states across $R_{12}$ is dense in $S$. On the other hand, since the existence of a cyclic vector for $R_1$ is not invariant under isomorphisms of $R_{12}$, Proposition 4 does not entail that if $R_1$ has a cyclic vector, then there is a norm dense set of nonseparable states in the entire normal state space of $R_{12}$. (Cf. the analogous discussion preceding the proof of Proposition 1.) Indeed, if we let $R_1 = B(C^2) \otimes I$, $R_2 = I \otimes B(C^2)$, then any entangled state vector is cyclic for $R_1$; but, the set of nonseparable states of $B(C^2) \otimes B(C^2)$ is not norm dense [13, 20]. However, if in addition to $R_1$ or $R_2$ having a cyclic vector, $R_{12}$ has a separating vector (as is often the case in quantum field theory), then all normal states of $R_{12}$ are vector states [13, Thm. 7.2.3], and it follows that the nonseparable states will be norm dense in the entire normal state space of $R_{12}$.

IV. Applications to algebraic quantum field theory

Let $(M, g)$ be a relativistic spacetime and let $A$ be a unital $C^*$-algebra. The basic mathematical object of algebraic quantum field theory (see [17, 21, 22]) is an association between precompact open subsets $O$ of $M$ and $C^*$-subalgebras $A(O)$ of $A$. (We assume that each $A(O)$ contains the identity $I$ of $A$.) The motivation for this association is the idea that $A(O)$ represents observables that can be measured in the region $O$. With this in mind, one assumes

1. Isotony: If $O_1 \subseteq O_2$, then $A(O_1) \subseteq A(O_2)$.

2. Microcausality: $A(O') \subseteq A(O)'$. 

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Here $O'$ denotes the interior of the set of all points of $M$ that are spacelike to every point in $O$.

In the case where $(M, g)$ is Minkowski spacetime, it is assumed in addition that there is a faithful representation $x \rightarrow \alpha_x$ of the translation group of $M$ in the group of automorphisms of $\mathfrak{A}$ such that

3. **Translation Covariance**: $\alpha_x(\mathfrak{A}(O)) = \mathfrak{A}(O + x)$.

4. **Weak Additivity**: For any $O \subseteq M$, $\mathfrak{A}$ is the smallest $C^*$-algebra containing $\cup_{x \in M} \mathfrak{A}(O + x)$.

The class of physically relevant representations of $\mathfrak{A}$ is decided by further desiderata such as — in the case of Minkowski spacetime — a unitary representation of the group of translation automorphisms which satisfies the spectrum condition. Relative to a fixed representation $\pi$, we let $\mathfrak{R}_\pi(O_1), \mathfrak{R}_\pi(O_2)$ denote the von Neumann algebra $\pi(\mathfrak{A}(O))''$ on the representation space $\mathcal{H}_\pi$. In what follows, we consider only nontrivial representations (i.e., $\dim \mathcal{H}_\pi > 1$), and we let $S_\pi$ denote the set of unit vectors in $\mathcal{H}_\pi$.

**Proposition 3.** Let $\{\mathfrak{A}(O)\}$ be a net of local algebras over Minkowski spacetime. Let $\pi$ be any representation in the local quasiequivalence class of some irreducible vacuum representation (e.g. superselection sectors in the sense of Doplicher-Haag-Roberts [23] or Buchholz-Fredenhagen [24]). If $O_1, O_2$ are any two open subsets of $M$ such that $O_1 \subseteq O_2'$, then the set of vectors inducing Bell correlated states for $\mathfrak{R}_\pi(O_1), \mathfrak{R}_\pi(O_2)$ is open and dense in $S_\pi$.

**Proof.** Let $O_3, O_4$ be precompact open subsets of $M$ such that $O_3 \subseteq O_1, O_4 \subseteq O_2$, and such that $O_3 + N \subseteq O_4'$ for some neighborhood $N$ of the origin. In an irreducible vacuum representation $\phi$, local algebras are of infinite type [25, Prop. 1.3.9], and since $O_3 + N \subseteq O_4'$, the Schlieder property holds for $\mathfrak{R}_{\phi}(O_3), \mathfrak{R}_{\phi}(O_4)$ [27]. If $\pi$ is any representation in the local quasiequivalence class of $\phi$, these properties hold for $\mathfrak{R}_{\pi}(O_3), \mathfrak{R}_{\pi}(O_4)$ as well. Thus, we may apply Proposition 1 to conclude that the set of vectors inducing Bell correlated states for $\mathfrak{R}_{\pi}(O_3), \mathfrak{R}_{\pi}(O_4)$ is dense in $S_\pi$. Finally, note that any state Bell correlated for $\mathfrak{R}_{\pi}(O_3), \mathfrak{R}_{\pi}(O_4)$ is Bell correlated for $\mathfrak{R}_{\pi}(O_1), \mathfrak{R}_{\pi}(O_2)$.

**Proposition 4.** Let $(M, g)$ be a globally hyperbolic spacetime, let $\{\mathfrak{A}(O)\}$ be the net of local observable algebras associated with the free Klein-Gordon field [22], and let $\pi$ be the GNS representation of some quasi-free Hadamard
state \([23]\). If \(O_1, O_2\) are any two open subsets of \(M\) such that \(O_1 \subseteq O_2\), then the set of vectors inducing Bell correlated states for \(\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)\) is open and dense in \(S_\pi\).

**Proof.** The regular diamonds (in the sense of \([28]\)) form a basis for the topology on \(M\). Thus, we may choose regular diamonds \(O_3, O_4\) such that \(O_3 \subseteq O_1\) and \(O_4 \subseteq O_2\). The nonfiniteness of the local algebras \(\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)\) is established in \([28\, Thm. 3.6.g]\), and the split property for these algebras is established in \([28\, Thm. 3.6.d]\). Since the split property entails the Schlieder property, it follows from Proposition 1 that the set of vectors inducing Bell correlated states for \(\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)\) [and thereby Bell correlated for \(\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)\)] is dense in \(S_\pi\). \(\square\)

There are many physically interesting states, such as the Minkowski vacuum itself, about which Propositions 3 and 4 are silent. However, Reeh-Schlieder type theorems entail that many of these physically interesting states are induced by vectors which are cyclic for local algebras, and thus it follows from Proposition 2 that these states are nonseparable across any spacelike separated pair of local algebras. In particular, although there is an upper bound on the Bell correlation of the Minkowski vacuum (in models with a mass gap) that decreases exponentially with spacelike separation \([6\, Prop. 3.2]\), the vacuum state remains nonseparable (in our sense) at all distances. On the other hand, since nonseparability is only a necessary condition for Bell correlation, none of our results decide the question of whether the vacuum state always retains some Bell correlation across arbitrary spacelike separated regions.

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