Separable bi-Hamiltonian systems with quadratic in momenta first integrals

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December 29, 2021

Abstract

Geometric separability theory of Gel’fand-Zakharevich bi-Hamiltonian systems on Riemannian manifolds is reviewed and developed. Particular attention is paid to the separability of systems generated by the so-called special conformal Killing tensors, i.e. Benenti systems. Then, infinitely many new classes of separable systems are constructed by appropriate deformations of Benenti class systems. All such systems can be lifted to the Gel’fand-Zakharevich bi-Hamiltonian form.
PACS numbers: 02.30.lk,02.40.-k,45.20.Jj
Keywords: integrability, separability, bi-Hamiltonian systems

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1 Introduction

The separation of variables for solving by quadratures the Hamilton-Jacobi (HJ) equations of related Liouville integrable dynamic systems with quadratic in momenta first integrals has a long history as a part of analytical mechanics. There are some milestones of that theory. First, in 1891 Stäckel initiated a program of classification of separable systems presenting conditions for separability of the HJ equations in orthogonal coordinates [1]-[3]. Then, in 1904 Levi-Civita found a test for the separability of a Hamiltonian dynamics in a given system of canonical coordinates [4]. The next was Eisenhart [5]-[7], who in 1934 inserted a separability theory in the context of Riemannian geometry, making it coordinate free and introducing the crucial objects of the theory, i.e. Killing tensors. This approach was then developed by Woodhouse [8], Klainin [9]-[10] and others. Finally, in 1992, Benenti [11] -[13] constructed a particular but very important subclass of separable systems, based on the so called special conformal Killing tensors.

The first constructive theory of separated coordinates for dynamical systems was made by Sklyanin [14]. He adopted the method of soliton theory, i.e. the Lax representation, to systematic derivation of separated coordinates. In that approach involutive functions appear as coefficients of characteristic equation (spectral curve) of Lax matrix. The method was successfully applied to separation of variables for many integrable systems [14]-[21].
Recently, a modern geometric theory of separability on bi-Poisson manifolds was constructed [22]-[29], related to the so-called Gel’fand-Zakharevich (GZ) bi-Hamiltonian systems [30],[31]. Obviously, it contains as a special case Liouville integrable systems with all constants of motion being quadratic in momenta functions. Indeed, Ibort et.al. [32] proved that the Benenti class of systems can be lifted to the GZ bi-Hamiltonian form.

In the following paper we construct in a systematic way a separability theory of all Liouville integrable systems on Riemannian manifolds, which are of the GZ type, including as a special case the Benenti class of systems. Actually, first we construct a quasi-bi-Hamiltonian theory on the so called $\omega_N$ manifolds and then we lift it to the related GZ bi-Hamiltonian systems on bi-Poisson manifolds. What is important, infinitely many classes of separable systems are constructed from appropriate deformations of the Benenti class of systems. In that sense we demonstrate the crucial role of this particular class of systems in the separability theory of dynamic systems on Riemannian manifolds.

The organization of the paper is as follows. In Section 2 we sketch the geometric separability theory of GZ bi–Hamiltonian systems, which was recently constructed and is a main tool used in the paper. In Section 3 we recall the basic facts about separable dynamics on Riemannian manifolds. Section 4 deals with a special case of separable systems, i.e. the so-called Benenti systems. We review this class of systems systematically as it plays a crucial role in a separability theory of Gel’fand-Zakharevich systems on Riemannian manifolds and is of a special importance for the theory developed in this paper. In Section 5, we construct the simplest new classes of separable systems being the so-called 1-hole deformations of the Benenti class. For this example we explain the main ideas of our approach as well as the methods of systematic construction of separable potentials and (quasi)-bi-Hamiltonian representations. In Section 6, we develop the approach to the case of arbitrary k-hole deformations of the Benenti class of systems, constructing a complete theory of separable GZ systems on Riemannian manifolds. Finally, in Section 7, we illustrate our theory by a few simple examples.

2 Gel’fand-Zakharevich bi-Hamiltonian systems and their separability on $\omega_N$ manifolds

2.1 Basic definitions

In this Section we sketch the basic concepts of the modern geometric separability theory to make other parts of the paper more clear to the reader. Let us first remind few basic facts from the Poisson geometry. Given a manifold $\mathcal{M}$, a Poisson operator $\pi$ on $\mathcal{M}$ is a mapping $\pi : T^*\mathcal{M} \to T\mathcal{M}$ that is fibre-preserving (i.e. $\pi|_{T^*_x\mathcal{M}} : T^*_x\mathcal{M} \to T_x\mathcal{M}$ for any $x \in \mathcal{M}$) and such that the induced bracket on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$

$$\{\ldots\}_\pi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) , \ \{F,G\}_\pi \overset{\text{def}}{=} \langle dF, \pi dG \rangle \quad (2.1)$$

(where $\langle\ldots,\rangle$ is the dual map between $T\mathcal{M}$ and $T^*\mathcal{M}$). It is skew-symmetric, satisfies Jacobi identity and the Leibniz rule and the symbol $d$ denotes the operator of exterior derivative. The
second order contravariant tensor field $\pi$ can always be interpreted as a bivector, $\pi \in \Lambda^2(\mathcal{M})$ and in a given coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$ we have

$$\pi = \sum_{i<j}^m \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \quad (2.2)$$

The function $C : \mathcal{M} \to \mathbb{R}$ is called the Casimir function of the Poisson operator $\pi$ if for an arbitrary function $F : \mathcal{M} \to \mathbb{R}$ we have $\{F, C\}_\pi = 0$ (or, equivalently, if $\pi dC = 0$). A linear combination $\pi_\xi = \pi_1 - \xi \pi_0$ ($\xi \in \mathbb{R}$) of two Poisson operators $\pi_0$ and $\pi_1$ is called a Poisson pencil if the operator $\pi_\xi$ is Poissonian for any value of the parameter $\xi$. In this case we say that $\pi_0$ and $\pi_1$ are compatible. Given a Poisson pencil $\pi_\xi = \pi_1 - \xi \pi_0$ we can often construct a sequence of vector fields $X_i$ on $\mathcal{M}$ that have a twofold Hamiltonian form (the so-called bi-Hamiltonian chain)

$$X_i = \pi_1 dh_i = \pi_0 dh_{i+1} \quad (2.3)$$

where $h_i : \mathcal{M} \to \mathbb{R}$ are called the Hamiltonians of the chain (2.3) and where $i$ is some discrete index. This sequence of vector fields may or may not truncate (depending on the existence of Casimir functions).

Let us consider a bi-Poisson manifold $(\mathcal{M}, \pi_0, \pi_1)$ of dim $M = 2n + m$ where $\pi_0, \pi_1$ is a pair of compatible Poisson tensors of rank $2n$. Moreover we assume that the Poisson pencil $\pi_\xi$ admits $m$, polynomial with respect to the pencil parameter $\xi$, Casimir functions of the form

$$h^{(j)}(\xi) = \sum_{i=0}^{n_j} h^{(j)}_i \xi^{n_j-i}, \quad j = 1, \ldots, m, \quad (2.4)$$

such that $n_1 + \ldots + n_m = n$ and $h^{(j)}_i$ are functionally independent. The collection of $n$ bi-Hamiltonian vector fields

$$\pi_\xi dh^{(j)}(\xi) = 0 \iff X_i^{(j)} = \pi_1 dh^{(j)}_i = \pi_0 dh^{(j)}_{i+1}, \quad i = 1, \ldots, n_j, \quad j = 1, \ldots, m, \quad (2.5)$$

is called the Gel’fand-Zakharevich system of the bi-Poisson manifold $\mathcal{M}$. Notice that each chain starts from a Casimir of $\pi_0$ and terminates with a Casimir of $\pi_1$. Moreover all $h^{(j)}_i$ pairwise commute with respect to both Poisson structures

$$X^{(j)}_i(h^{(k)}_l) = \langle dh^{(k)}_l, \pi_0 dh^{(j)}_{i+1} \rangle = \langle dh^{(k)}_l, \pi_1 dh^{(j)}_i \rangle = 0. \quad (2.6)$$

### 2.2 Geometric separability theory

The necessary condition for separability of Hamiltonian functions $h^{(j)}_i$ is a projectibility of the Poisson pencil $\pi_\xi$ onto a symplectic leaf $\mathcal{N}$ of $\pi_0$, that is a $2n$-dimensioal submanifold defined by fixed values of Casimir functions of $\pi_0 : h^{(1)}_0 = c_1, \ldots, h^{(m)}_0 = c_m$. Thus, $\mathcal{N}$ is a submanifold of a codimension $m$ in $\mathcal{M}$. The projection is done through the particular realization of the Marsden-Ratiu scheme [33], i.e. along the appropriate transversal distribution. Let $Z_i, i = 1, \ldots, m$ be some vector fields transversal to $\mathcal{N}$, spanning an involutive (integrable) distribution $Z$ in $\mathcal{M}$ of
a constant dimension $m$ (that is a smooth collection of $m$-dimensional subspaces $Z_x \subset T_x M$ at every point $x$ in $M$). The word ‘transversal’ means here that no vector field $Z_i$ is at any point tangent to the submanifold $N$ passing through this point. Hence, the tangent bundle $T M$ splits into a direct sum

$$T M = T N \oplus Z$$

(2.7)

(which means that at any point $x$ in $M$ we have $T_x M = T_x N \oplus Z_x$ with $s$ such that $x \in N$) and so does its dual

$$T^* M = T^* N \oplus Z^*,$$

(2.8)

where $T^* N$ is the annihilator of $Z$ and $Z^*$ is the annihilator of $T N$. It means that if $\alpha$ is a one form in $T^* N$ then $\alpha(Z_i) = 0$ for all $i = 1, \ldots, m$ and if $\beta$ is a one-form in $Z^*$ then $\beta$ vanishes on all vector fields tangent to $N$. Moreover, we assume that the vector fields $Z_i$ which span $Z$ are chosen in such a way that $dh_0^{(j)}$, $j = 1, \ldots, m$ is a basis in $Z^*$ that is dual to the basis $Z_i$ of the distribution $Z$,

$$\langle dh_0^{(j)}, Z_i \rangle = Z_i(h_0^{(j)}) = \delta_{ij},$$

(2.9)

(it is no restriction since for any distribution $Z$ transversal to $N$ we can choose its basis so that (2.9) is satisfied). The ‘orthogonality’ condition (2.9) together with the involutivity of the distribution indicates the relation

$$[Z_i, Z_j] = 0,$$

(2.10)

where $[,]$ stands for the commutator of vector fields.

**Theorem 1** Poisson tensor $\pi_1$ is projectible onto a symplectic leaf $N$ of $\pi_0$ along a transversal integrable distribution $Z$ if the normalized vector fields $Z_i$ locally generating $Z$ are symmetries of $\pi_0$ ($L_{Z_i} \pi_0 = 0$) and satisfy

$$L_{Z_i} \pi_1 = \sum_j Y_i^{(j)} \wedge Z_j,$$

(2.11a)

where

$$Y_i^{(j)} = \pi_0 d(Z_i(h_0^{(j)})) = [Z_i, \pi_1 dh_0^{(j)}] = [Z_i, X_1^{(j)}]$$

(2.11b)

and $L_{Z}$ means a Lie derivative in the direction of $Z$. The projection of $\pi_1$ onto $N$ along $Z$ is equivalent to the projection of the deformed Poisson tensor

$$\pi_{1D} = \pi_1 - \sum_j X_1^{(j)} \wedge Z_j, \quad X_1^{(j)} = \pi_1 dh_0^{(j)} = \pi_0 dh_1^{(j)}$$

(2.12)

onto its symplectic leaf $N$. Moreover, $\pi_1$ and $\pi_{1D}$ are compatible.

The proof as well as the details of the whole construction the reader can find in [29], [34] and [35]. Notice that

$$L_{Z_i} \pi_{1D} = 0,$$

so $Z_i$ are also symmetries of $\pi_{1D}$. 

5
Let us denote the projections of \( \pi_0, \pi_1 \) by \( \theta_0, \theta_1 \) and restrictions of \( (h_1^{(1)}, ..., h_m^{(m)})|_N \) to \( N \) by \( (H_1^{(1)}, ..., H_m^{(m)}) := (H_1, ..., H_n) \). Notice that the projection of the Poisson pencil \( \pi_\xi \) is a Poisson pencil \( \theta_\xi \) on \( N \) and in a generic case both Poisson tensors \( \theta_0, \theta_1 \) are nondegenerate. Hence, \( N \) is a bi-symplectic manifold as is endowed with two symplectic forms \( \omega_0, \omega_1 \) defined by

\[
\{F,G\}_{\theta_i} = \omega_i(X_F, X_G), \quad X_F = \theta_0 dF, \quad i = 0, 1.
\] (2.13)

This simply means that \( \omega_0 = \theta_0^{-1} \) and \( \omega_1 = \omega_0 \theta_1 \omega_0 \). Indeed

\[
\omega_0(X_F, X_G) = \omega_0(X_F, X_G) = \omega_0 \theta_0 dF, \theta_0 dG = \{F,G\}_{\theta_0},
\] (2.14a)

\[
\omega_1(X_F, X_G) = \omega_1(X_F, X_G) = \omega_0 \theta_1 \omega_0 X_F, X_G = \{F,G\}_{\theta_1}.
\] (2.14b)

Moreover, one can construct the tensor field \( N := \theta_1 \theta_0^{-1} = \theta_1 \omega_0 \), of type (1,1), called a recursion operator of \( N \) and its dual \( N^* = \omega_0 \theta_1 \). Notice that

\[
N \theta_0 = \theta_1, \quad N^* \omega_0 = \omega_1.
\] (2.15)

The important property of \( N \) is that its Nijenhuis torsion

\[
T(N)(X,Y) := [NX,NY] - N([NX,Y] + [X,NY] - N[X,Y])
\] (2.16)

vanishes as a consequence of the compatibility between \( \theta_0 \) and \( \theta_1 \) and hence implies that \( \omega_1 \) is closed [36]. Such manifolds are known as the so-called \( \omega N \) manifolds. The generic case means that 2\( n \)-dimensional \( \omega N \) manifold is endowed with a recursion operator \( N \) which has at every point \( n \) distinct double eigenvalues \( \lambda_1, ..., \lambda_n \), which are functionally independent on \( N \). Choosing \( \lambda_i \) as the canonical position coordinates, we can always supplement a set of local coordinates \( (\lambda_i, \mu_i) \) on \( N \) by the canonically conjugate momenta \( \mu_i \).

**Definition 2** A set of local coordinates \( (\lambda_i, \mu_i) \) on \( \omega N \) manifold \( N \) is called a set of Darboux-Nijenhuis (DN) coordinates if they are canonical with respect to \( \theta_0 \) and diagonalize the recursion operator with diagonal elements being its eigenvalues.

It means that in the \( (\lambda, \mu) \) coordinates

\[
\theta_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \Lambda_n & 0 \\ 0 & \Lambda_n \end{pmatrix},
\] (2.17)

where \( \Lambda_n = \text{diag}(\lambda_1, ..., \lambda_n) \), and their differentials span the \( T^*N \) which is an eigenspace of \( N^* \) (the adjoint of \( N \)), as

\[
N^* d\lambda_i = \lambda_i d\lambda_i, \quad N^* d\mu_i = \lambda_i d\mu_i, \quad i = 1, ..., n.
\] (2.18)
As well known, \( n \) functionally independent Hamiltonian functions \( H_i, i = 1, \ldots, n \) are said to be separable in the canonical coordinates \((\lambda, \mu)\) if there are \( n \) relations, called the separation conditions (Sklyanin [14]), of the form

\[
\varphi_i(\lambda^i, \mu^i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n,
\]

\[
\det \left[ \frac{\partial \varphi_i}{\partial H_j} \right] \neq 0,
\]

which guarantee the solvability of the appropriate Hamilton-Jacobi equations and involutivity of \( H_i \). A special case, when all separation relations (2.19) are affine in \( H_i \) is given by the set of equations

\[
\sum_{k=1}^n \phi_i^k(\lambda_i, \mu_i)H_k = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n,
\]

where \( \phi \) and \( \psi \) are arbitrary smooth functions of their arguments, are called the Stäckel separation conditions and the related dynamic systems are called the Stäckel separable ones.

**Theorem 3** [29] Let \( \mathcal{N} \) be a generic \( \omega \mathcal{N} \) manifold and let \((H_1, \ldots, H_n)\) be a set of \( n \) functionally independent Hamiltonians on \( \mathcal{N} \). Then, the foliation defined by \((H_1, \ldots, H_n)\) is separable in \( DN \) coordinates and the subspace spanned by \((dH_1, \ldots, dH_n)\) is invariant with respect to \( \mathcal{N}^* \) if there exist some functions \( \alpha_{ij} \) such that

\[
\mathcal{N}^*dH_i = \sum_{j=1}^n \alpha_{ij}dH_j, \quad i = 1, \ldots, n.
\]

Hence, the distribution defined by \((H_1, \ldots, H_n)\), spanned by the Hamiltonian vector fields \( X_{H_i} \), is invariant with respect to \( \mathcal{N} \).

The formula (2.21) can be written in the equivalent form

\[
\theta_1 dH_i = \sum_{j=1}^n \alpha_{ij} \theta_0 dH_j, \quad i = 1, \ldots, n,
\]

which will be called a quasi-bi-Hamiltonian representation of separable dynamics. Notice that the projection of bi-Hamiltonian chains (2.5) onto \( \mathcal{N} \) takes the form of (2.22). Indeed from (2.12) we get

\[
\theta_1 dH_i^{(j)} = \pi_1 D|_{\mathcal{N}} dh_i^{(j)}|_{\mathcal{N}} = (\pi_1 D dh_i^{(j)})|_{\mathcal{N}}
\]

\[
= (\pi_1 dh_i^{(j)})|_{\mathcal{N}} - \sum_{k=1}^m \left( Z_k(h_i^{(j)})X_1^{(k)} \right)|_{\mathcal{N}}
\]

\[
= (\pi_0 dh_i^{(j)}|_{\mathcal{N}}) - \sum_{k=1}^m \left( Z_k(h_i^{(j)})\pi_0 dh_1^{(k)} \right)|_{\mathcal{N}}
\]

\[
= \pi_0|_{\mathcal{N}} dh_i^{(j)}|_{\mathcal{N}} - \sum_{k=1}^m Z_k(h_i^{(j)})|_{\mathcal{N}} \pi_0|_{\mathcal{N}} dh_1^{(k)}|_{\mathcal{N}}
\]

\[
= \theta_0 dH_i^{(j)} - \sum_{k=1}^m Z_k(h_i^{(j)})|_{\mathcal{N}} \theta_0 dh_1^{(k)},
\]

where \( \pi|_{\mathcal{N}} \) means a plain restriction of \( \pi \) to \( \mathcal{N} \).
Theorem 4 [29] A n-tuple \((H_1, \ldots, H_n)\) of separable functions on \(\omega N\) manifold \(N\) is Stäckel separable iff additionally to the condition (2.21) we have

\[
N^*d\alpha_{ij} = \sum_{k=1}^{n} \alpha_{ik}d\alpha_{kj}, \quad i, j = 1, \ldots, n. \tag{2.24}
\]

Equivalently, a n-tuple \((h^{(1)}_1, \ldots, h^{(m)}_m)\) of functions from Gel’fand-Zakharevich bi-Hamiltonian chains on bi-Poisson manifold \(M\) is Stäckel separable iff a Poisson pencil \(\pi_\xi\) is projectible onto a symplectic leaf \(N\) of \(\pi_0\) and \(h^{(j)}_i\) functions are affine in Casimir coordinates \(c_i, i = 1, \ldots, m,\) i.e.

\[
Z_k(Z_l(h^{(j)}_i)) = 0 \tag{2.25}
\]

for all \(k, l, j = 1, \ldots, m\) and \(i = 1, \ldots, n_m\).

Hence, in (2.21)

\[
Z_k(h^{(j)}_i) = \alpha^{(j)}_{i,k} \tag{2.26}
\]

are \(c\)-independent, so

\[
N^*dH^{(j)}_i = dH^{(j)}_i - \sum_{k=1}^{m} \alpha^{(j)}_{i,k}dH^{(k)}_1. \tag{2.27}
\]

In such a case the Stäckel separation conditions (2.20) take the form

\[
\sum_{j=1}^{m} \sigma^*_j(\lambda_i, \mu_i)h^{(j)}(\lambda_i) = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n, \tag{2.28}
\]

where \(h^{(j)}(\lambda_i)\) is a Casimir \(h^{(j)}(\xi)\) evaluated at \(\xi = \lambda_i\) and \(\sigma^*_j\) is a Casimir multiplier. For the majority of known integrable systems \(\psi_i(\lambda_i, \mu_i) = \psi(\lambda_i, \mu_i)\) and \(\sigma^*_j(\lambda_i, \mu_i) = \sigma_j(\lambda_i, \mu_i)\), and then, the separation conditions (2.28) can be presented in a compact form as \(n\) copies of the so-called separation curve

\[
\sum_{j=1}^{m} \sigma_j(\xi, \mu)h^{(j)}(\xi) = \psi(\xi, \mu), \quad (\xi, \mu) = (\lambda_i, \mu_i), \quad i = 1, \ldots, n. \tag{2.29}
\]

For the uniqueness of the representation (2.29) we chose the normalization condition \(\sigma_m = 1\).

Let us define the following matrix

\[
\alpha(\xi) = \begin{pmatrix}
Z_1(h^{(1)}(\xi)) & \cdots & Z_1(h^{(m)}(\xi)) \\
\cdots & \cdots & \cdots \\
Z_m(h^{(1)}(\xi)) & \cdots & Z_m(h^{(m)}(\xi))
\end{pmatrix}. \tag{2.30}
\]

Lemma 5 [29]:

(i) Determinant of \(\alpha(\xi)\) coincides with the minimal polynomial of \(N\)

\[
\det \alpha(\xi) = \sqrt{\det(\xi I - N)}. \tag{2.31}
\]
(ii) Casimir multipliers $\sigma^i_j(\lambda_i, \mu_i)$ are given by the relation

$$\alpha(\lambda_i)\sigma^i(\lambda_i, \mu_i) = 0,$$

(2.32)

where $\sigma^i = (\sigma^i_1, \ldots, \sigma^i_{m-1}, 1)^T$ and $\alpha(\lambda_i)$ is the matrix $\alpha(\xi)$ evaluated at $\xi = \lambda_i$.

Up to now we sketch the passage from degenerated Poisson pencil $\pi_\lambda$ to nondegenerate one $\theta_\lambda: \pi_\lambda \rightarrow \theta_\lambda$. According to the further applications, we are interested in the inverse passage: $\theta_\lambda \rightarrow \pi_\lambda$, as we would like to construct GZ bi-Hamiltonian chains from a given quasi-bi-Hamiltonian ones.

**Lemma 6** Let $\theta$ be a Poisson tensor on manifold $\mathcal{N}$ of dim $\mathcal{N} = m$, and let $\pi$ be a second order contravariant tensor field on $\mathcal{M} = \mathcal{N} \otimes \mathbb{R} \otimes \ldots \otimes \mathbb{R}$ of the same rank as $\theta$, i.e. with $k$ additional Casimir functions $\varphi_1, \ldots, \varphi_k$. Moreover, let the restriction of $\pi$ to $\mathcal{N}$ (the level set of $\varphi_1 = \ldots = \varphi_k = 0$) coincides with $\theta$. Then, the operator $\pi$ is also Poisson.

**Proof.** Let us complete the set of functions $\varphi_i$ with some functions $x_j$ which parametrize $\mathcal{N}$ submanifold, to a coordinate system $(x, \varphi)$ on $\mathcal{M}$. Then, the matrix of the operator $\pi$ has last $k$ rows and columns equal to zero, while the $m$ dimensional upper left block coincides with $\theta$, which is Poisson. ■

Let $\theta_0$ and $\theta_1$ be compatible Poisson tensors on $\mathcal{N}$. Then, let $\pi_0$ and $\pi_{1D}$ be Poisson extensions of $\theta_0$ and $\theta_1$ from $\mathcal{N}$ to $\mathcal{M}$, such as in Lemma 6, with a common set of Casimirs $\varphi_1, \ldots, \varphi_k$. Of course, $\pi_0$ and $\pi_{1D}$ are still compatible. Now, let $\pi_1$ be another Poisson extension of $\theta_1$ from $\mathcal{N}$ to $\mathcal{M}$ with a new set of Casimirs $\psi_1, \ldots, \psi_k$. The question is: when $\pi_1$ is compatible with $\pi_0$?

Assume that there are vector fields $Z_i, i = 1, \ldots, k$ transversal to the submanifold $\mathcal{N}$, such that $L_{Z_i}\pi_0 = 0, i = 1, \ldots, k$, and $\pi_1$ has the form

$$\pi_1 = \pi_{1D} + \sum_{i=1}^{k} X_i \wedge Z_i,$$

(2.33)

where $X_i$ are some vector fields.

**Lemma 7** A sufficient condition for compatibility of $\pi_0$ and $\pi_1$ is that $X_i$ are Hamiltonian vector fields with respect to $\pi_0$.

The Jacobi identity for a tensor $\pi$ is equivalent to the relation $[\pi, \pi]_S = 0$, where $[.,.]_S$ is a Schouten bracket. Hence, the compatibility of two Poisson tensors $\pi_0$ and $\pi_1$ is equivalent to the relation $[\pi_0, \pi_1]_S = 0$. Before we start the proof we will therefore remind some basic facts about Schouten bracket.

The Schouten bracket [37], [38] in general is a bilinear mapping $\Lambda^q(\mathcal{M}) \times \Lambda^r(\mathcal{M}) \rightarrow \Lambda^{q+r-1}(\mathcal{M})$ that with every $q$-vector and $r$-vector associates a $(q + r - 1)$-vector $[Q, R]_S$ (Schouten bracket of $Q$ and $R$) that is skewsymmetric in the sense that

$$[R, Q]_S = (-1)^{qr}[Q, R]_S$$

(2.34)
and that satisfies the Leibniz identity
\[
[Q, R \wedge P]_S = [Q, R]_S \wedge P + (-1)^{(q-1)r} R \wedge [Q, P]_S,
\] (2.35)
where \( P \) is any \( p \)-vector \( \in \Lambda^p(M) \). Using the above properties of the Schouten bracket one can show that if \( X, Y \) are some vector fields, then \( [X, Y]_S = [X, Y] = L_X Y \) is a usual Lie bracket (commutator) of vector fields and if moreover \( P \) is a bivector then
\[
[X \wedge Y, P]_S = Y \wedge [X, P]_S - X \wedge [Y, P]_S, \quad [X, P]_S = L_X P.
\] (2.36)

**Proof.** From properties of the Schouten bracket
\[
[\pi_0, \pi_1]_S = [\pi_0, \pi_{1D}] + \sum_{i=1}^{k} X_i \wedge Z_i)_S = \sum_{i=1}^{k} [\pi_0, X_i \wedge Z_i]_S
\]
\[
= \sum_{i=1}^{k} (Z_i \wedge L_{X_i} \pi_0 - X_i \wedge L_{Z_i} \pi_0) = \sum_{i=1}^{k} Z_i \wedge L_{X_i} \pi_0.
\] (2.27)
Hence, the most general condition for the compatibility is
\[
\sum_{i=1}^{k} Z_i \wedge L_{X_i} \pi_0 = 0,
\] (2.38)
with the strong solution \( L_{X_i} \pi_0 = 0, i = 1, ..., k \), which takes the place when \( X_i \) are Hamiltonian vector fields with respect to \( \pi_0 \), i.e. there are some functions \( f_1, ..., f_k \) such that
\[
X_i = \pi_0 df_i.
\] (2.39)

Notice that when in addition we impose the normalization conditions \( Z_i(\varphi_j) = \delta_{ij}, i, j = 1, ..., k \), and commutativity of \( X_i \) then \( \pi_1 d\varphi_j = X_j \) and \( X_j \) are bi-Hamiltonian vector fields
\[
\pi_0 df_i = \pi_1 d\varphi_i = X_i,
\] (2.40)
which is just our case (2.12) with \( \varphi_i = h_{0}^{(i)}, f_i = h_{1}^{(i)} \).

Finally, let us mention the problem of explicit construction of DN coordinates \((\lambda, \mu)\). Assume that \( N \) is parametrized by a set of coordinates \( \{v_i\}_{i=1}^{2n} \), not necessary canonical. The first part of the transformation \((\lambda, \mu) \rightarrow (v)\), i.e. \( \lambda \rightarrow v \) is reconstructed from the characteristic equation of \( N \)
\[
\sqrt{\det(\xi I - N)} = \det \alpha(\xi) = 0 \quad \Rightarrow \quad \lambda_i = \lambda_i(v), \quad i = 1, ..., n.
\] (2.41)
The reconstruction of missing part of the transformation: \( \mu \rightarrow v, \) in general case, is far from being trivial and is discussed in [24]-[29]. In the following we restrict ourselves to the canonical point transformations \((\lambda, \mu) \rightarrow (q, p)\), where reconstruction of the remaining \( n \) relations \( p_i = p_i(\lambda, \mu) \) from the first \( n \) ones \( q_i = q_i(\lambda) \) is standard.

In the next Sections we apply this theory to dynamic systems on Riemannian manifolds, constructing all those which can be lifted to the Gel’fand-Zakharevich bi-Hamiltonian chains.
3 Introduction to separable Riemannian dynamics

3.1 Hamiltonian representation

Let \((Q,g)\) be a Riemann (pseudo-Riemann) manifold with covariant metric tensor \(g\) and local coordinates \(q^1,\ldots,q^n\). Moreover, let \(G := g^{-1}\) be a contravariant metric tensor satisfying \(\sum_{j=1}^n g_{ij} G^{jk} = \delta^k_i\). The Levi-Civita connection components are defined by

\[
\Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^n G^{il} (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}), \quad \partial_i \equiv \frac{\partial}{\partial q^i}. \tag{3.1}
\]

The equations

\[
q^{it} + \Gamma^i_{jk} q^{jt} q^{kt} = G^{ik} \partial_k V(q), \quad i = 1,\ldots,n, \quad q_t \equiv \frac{dq}{dt} \tag{3.2}
\]
describe the motion of a particle in the Riemannian space with the metric \(g\). Eqs. (3.2) can be obtained by varying the Lagrangian

\[
\mathcal{L}(q,q_t) = \frac{1}{2} \sum_{i,j} g_{ij} q^i_t q^j_t - V(q) \tag{3.3}
\]

and are called Euler-Lagrange equations. Obviously, for \(G = I\) equations (3.2) reduce to Newton equations of motion.

One can pass in a standard way to the Hamiltonian description of dynamics, where the Hamiltonian function takes the form

\[
H(q,p) = \sum_{i=1}^n q^i_t \frac{\partial \mathcal{L}}{\partial q^i_t} - \mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n G^{ij} p_i p_j + V(q), \quad p_i :\equiv \frac{\partial \mathcal{L}}{\partial q^i_t} = \sum_j g_{ij} q^j_t \tag{3.4}
\]

and equations of motion are

\[
\begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dH = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = X_H \tag{3.5}
\]

\[
q^i_t = \frac{\partial H}{\partial p_i}, \quad p_{it} = -\frac{\partial H}{\partial q^i_t}.
\]

\(X_H\) denotes the Hamiltonian vector field and the whole dynamics takes place on the phase space \(\mathcal{N} = T^*G\) in local coordinates \((q^1,\ldots,q^n,p_1,\ldots,p_n)\).

Of special importance is the geodesic motion \(V(q) \equiv 0\), with Euler-Lagrange equations

\[
q^{it} + \Gamma^i_{jk} q^{jt} q^{kt} = 0, \quad i = 1,\ldots,n \tag{3.6}
\]

and Hamiltonian representation

\[
\begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dE = X_E, \quad E = \frac{1}{2} \sum_{i,j=1}^n G^{ij} p_i p_j. \tag{3.7}
\]
3.2 Separated coordinates

We are going to present a complete theory of a subclass of one-particle dynamics, containing Liouville integrable, separable, (quasi-)bi-Hamiltonian systems with \( n \) quadratic in momenta constants of motion. Stäckel [1]-[3] gave the first characterization of the Riemann (pseudo-Riemann) manifold \((Q,g)\) on which the equations of geodesic motion can be solved by separation of variables. He proved that if in a system of orthogonal coordinates \((\lambda, \mu)\) there exists a non-singular matrix \( \varphi = (\varphi^l_k(\lambda_k)) \), called a Stäckel matrix such that the geodesic Hamiltonians \( E_r \) are of the form

\[
E_r = \frac{1}{2} \sum_{i=1}^{n} (\varphi^{-1})^i_r \mu_i^2,
\]

then \( E_r \) are functionally independent, pairwise commute with respect to the canonical Poisson bracket and the Hamilton-Jacobi equation associated to \( E_1 \) is separable.

Then, Eisenhart [5]-[7] gave a coordinate-free representation for Stäckel geodesic motion introducing a special family of Killing tensors. As known, a \((1,1)\)-type tensor \( B = (B^i_j) \) (or a \((2,0)\)-type tensor \( B = (B^{ij}) \)) is called a Killing tensor with respect to \( g \) if

\[
\{ \sum (BG)^{ij} p_i p_j, E \}_{\theta_0} = 0 \quad \text{or} \quad \{ \sum (B)^{ij} p_i p_j, E \}_{\theta_0} = 0.
\]

He proved [5]-[7] that the geodesic Hamiltonians can be transformed into the Stäckel form (3.8) if the contravariant metric tensor \( G = g^{-1} \) has \((n-1)\) commuting independent contravariant Killing tensors \( A_r \) of a second order such that

\[
E_r = \frac{1}{2} \sum_{i,j} A^{ij}_r p_i p_j,
\]

admitting a common system of closed eigenforms \( \alpha_i \)

\[
(A^*_r - v^*_r G)\alpha_i = 0, \quad d\alpha_i = 0, \quad i = 1, ..., n,
\]

where \( v^*_r \) are eigenvalues of \((1,1)\) Killing tensor \( K_r = A_r g (K^*_r = gA^*_r) \).

From now on, separated canonical coordinates will be denoted by \((\lambda, \mu)\) and natural canonical coordinates by \((q, p)\). For \( n \) degrees of freedom, let us consider \( n \) Stäckel Hamiltonian functions in separated coordinates in the following form

\[
H_r = \frac{1}{2} \sum_{i=1}^{n} x^i_r G^{ij} \mu^2_i + V_r(\lambda) = \frac{1}{2} \mu^T K_r G \mu + V_r(\lambda), \quad r = 1, ..., n,
\]

where \( \mu = (\mu_1, ..., \mu_n)^T \), and \( V_r(\lambda) \) are appropriate potentials separable in \((\lambda, \mu)\) coordinates. For the integrable system (3.11) the Stäckel separation conditions (2.20) take the general form

\[
\sum_{k=1}^{n} \phi^k_i(\lambda^i)H_k = \frac{1}{2} f_i(\lambda^i) \mu^2_i + \gamma_i(\lambda^i) \quad i = 1, ..., n,
\]
where \( f_i, \gamma_i, \phi_i^k \) are arbitrary smooth functions of its argument and the normalization \( \phi_i^n = 1, \ i = 1, ..., n \) is assumed. To get the explicit form of \( H_k = H_k(\lambda, \mu) \) one has to solve the system of linear equations (3.12). The results are the following

\[
G^{ii} = (-1)^{i+1} \frac{f_i(\lambda^i) \det W^{i1}}{\det W}, \quad v_r^i = (-1)^{i+r} \frac{\det W^{ir}}{\det W^{i1}}, \quad (3.13)
\]

\[
V_r = \sum_{i=1}^{n} (-1)^{i+r} \gamma_i(\lambda^i) \frac{\det W^{ir}}{\det W}, \quad (3.14)
\]

where

\[
W = \begin{pmatrix}
\phi_1^1(\lambda^1) & \phi_1^2(\lambda^1) & \cdots & \phi_1^{n-1}(\lambda^1) & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_n^1(\lambda^n) & \phi_n^2(\lambda^n) & \cdots & \phi_n^{n-1}(\lambda^n) & 1
\end{pmatrix}, \quad (3.15)
\]

and \( W^{ik} \) is the \((n-1) \times (n-1)\) matrix obtained from \( W \) after we cancel its \( i \)th row and \( k \)th column. Then the Stäckel matrix \( \varphi \) is given by

\[
\varphi = \begin{pmatrix}
\phi_1^1(\lambda^1) & \phi_1^2(\lambda^1) & \cdots & \phi_1^{n-1}(\lambda^1) & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_n^1(\lambda^n) & \phi_n^2(\lambda^n) & \cdots & \phi_n^{n-1}(\lambda^n) & 1
\end{pmatrix}, \quad (3.16)
\]

In our further considerations we restrict to the case when \( \phi_i^k(\lambda_i) \) are monomials. Actually, we assume that

\[
\phi_i^k(\lambda_i) = \phi_i^k(\lambda_i) = (\lambda_i)^{m_k}, \quad m_n = 0 < m_{n-1} < ... < m_1 \in \mathbb{N}, \quad (3.17)
\]

which is a sufficient condition for the existence of bi-Hamiltonian representation of Gel’fand-Zakharevich type with separation conditions and the separation curve in the form

\[
\sum_{j=1}^{m} \sigma_j(\lambda^i) h^{(j)}(\lambda^i) = \frac{1}{2} f(\lambda^i) \mu_i^2 + \gamma(\lambda^i), \quad i = 1, ..., n, \quad (3.18a)
\]

\[
\sum_{j=1}^{m} \sigma_j(\xi) h^{(j)}(\xi) = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi). \quad (3.18b)
\]

### 3.3 Integration by quadratures

Let us consider a set of coordinates \( \{\lambda_i, \mu_i\}_{i=1}^{n} \) on \( T^*Q \) canonical with respect to \( \theta_0 \). One can try to linearize the system (3.11) through a canonical transformation \( (\lambda, \mu) \rightarrow (b, a) \) in the form

\[
b^j = \frac{\partial W}{\partial \lambda^j}, \quad \mu_i = \frac{\partial W}{\partial \lambda_i}, \quad \text{where } W(\lambda, a) \text{ is a generating function that solves the related Hamilton-Jacobi (HJ) equations}
\]

\[
E_r(\lambda, \frac{\partial W}{\partial \lambda}) = a_r, \quad r = 0, ..., n. \quad (3.19)
\]
The HJ equations (3.19) are nonlinear partial differential equations that are very difficult to solve. In general it is a hopeless task. However, there are rare cases when one can find a solution of (3.19) in the separable form

\[ W(\lambda, \mu) = \sum_{i=1}^{n} W_i(\lambda_i, \mu) \] (3.20)

that turns the HJ equations into a set of decoupled ordinary differential equations that can be solved by quadratures. Such \((\lambda, \mu)\) coordinates are called separated coordinates. In the \((b, a)\) coordinates the flow \(d/dt\) associated with every Hamiltonian \(H_j\) is trivial

\[ \frac{d a_i}{d t_j} = 0, \quad \frac{d b_i}{d t_j} = \delta_{ij}, \quad i, j = 1, \ldots, n \] (3.21)

and the implicit form of the trajectories \(\lambda(t_j)\) is given by

\[ b^i(\lambda, \mu) = \frac{\partial W}{\partial a_i} = \delta_{ij} t_j + \text{const}, \quad i, j = 1, \ldots, n. \] (3.22)

As explained in the previous section, separated coordinates are defined by \(n\) relations of the form

\[ \varphi_i(\lambda_i, \mu_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n \] (3.23)

joining each pair \((\lambda^i, \mu_i)\) of conjugate coordinates and all Hamiltonians \(H_i, \ i = 1, \ldots, n\). Fixing the values of the Hamiltonians \(H_j = \text{const} = a_j\) one obtains an explicit factorization of the Liouville tori given by the equations

\[ \varphi_i(\lambda^i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n \]

\[ \Downarrow \]

\[ \varphi_i(\lambda^i, \frac{d W_i}{d \lambda^i}, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n \] (3.24)

i.e. a decoupled system of ordinary differential equations.

Let us now solve explicitly the Hamilton-Jacobi equations (3.19) and related dynamic equations, with respect to the evolution parameter \(t_r\), written in \((\lambda, \mu)\) coordinates when separation conditions are given in the form (3.12), (3.17):

\[ \frac{1}{2} f(\lambda^i) \left( \frac{\partial W}{\partial \lambda^i} \right)^2 + \gamma(\lambda^i) = a_1(\lambda^i)^{m_1} + a_2(\lambda^i)^{m_2} + \ldots + a_n \equiv a(\lambda^i) \]

\[ \Downarrow \quad W = \sum_{i=1}^{n} W_i(\lambda_i, a) \]

\[ \frac{1}{2} f(\lambda^i) \left( \frac{d W_i}{d \lambda^i} \right)^2 + \gamma(\lambda^i) = a_1(\lambda^i)^{m_1} + a_2(\lambda^i)^{m_2} + \ldots + a_n \equiv a(\lambda^i) \]

\[ \Downarrow \]
\[ W_i(\lambda, a) = \int \sqrt{a(\xi) - \gamma(\xi)} d\xi \]

\[ W(\lambda, a) = \sum_{k=1}^{n} \int \sqrt{a(\xi) - \gamma(\xi)} d\xi \]

\[ b^i = \frac{\partial W}{\partial a_i} = \sum_{k=1}^{n} \int \frac{\xi^{m_i}}{\sqrt{2f(\xi)(a(\xi) - \gamma(\xi))}} d\xi \]

\[ \sum_{k=1}^{n} \int \frac{\xi^{m_i}}{\sqrt{\psi(\xi)}} d\xi = t_i \delta_{ri} + \text{const}_i, \quad i = 1, \ldots, n \]

(3.25)

where eqs.(3.25) are implicit solutions called the inverse Jacobi problem.

4 Systems of Benenti type

4.1 Separable geodesics

Among all Stäckel systems a particularly important subclass consists of those considered by Benenti [11], [12], [13] and constructed with the help of the so-called conformal Killing tensor. Let \( L = (L^i_j) \) be a second order mixed type tensor on \( Q \) and let \( \mathcal{T} : M \to \mathbb{R} \) be a function on \( M \) defined as \( \mathcal{T} = \frac{1}{2} \sum_{i,j=1}^{n} (LG)^{ij} p_ip_j \). If

\[ \{ \mathcal{T}, E \}_{\theta_0} = \kappa E, \quad \text{where} \quad \kappa = \{ \varepsilon, E \}_{\theta_0}, \quad \varepsilon = Tr(L), \]

then \( L \) is called a conformal Killing tensor with the associated potential \( \varepsilon = Tr(L) \). If we assume additionally that \( L \) has simple eigenvalues and its Nijenhuis torsion vanishes, then \( L \) is called a special conformal Killing tensor [39].

For the Riemannian manifold \((Q, g, L)\), geodesic flow has \( n \) constants of motion of the form

\[ E_r = \frac{1}{2} \sum_{i,j=1}^{n} A^{ij}_r p_ip_j = \frac{1}{2} \sum_{i,j=1}^{n} (K_r G)^{ij} p_ip_j, \quad r = 1, \ldots, n, \]

(4.2)

where \( A_r \) and \( K_r \) are Killing tensors of type \((2,0)\) and \((1,1)\), respectively. Moreover, as was shown by Benenti [11],[12], all the Killing tensors \( K_r \) with a common set of eigenvectors, are constructed from \( L \) by

\[ K_{r+1} = \sum_{k=0}^{r} \rho_k L^{r-k}, \]

(4.3)

where \( \rho_r \) are coefficients of the characteristic polynomial of \( L \)

\[ \det(\xi I - L) = \xi^n + \rho_1 \xi^{n-1} + \ldots + \rho_n, \quad \rho_0 = 1. \]

(4.4)
From (4.3) we immediately find that
\[ K_{r+1} - L K_r = \rho_r I, \quad -L K_n = \rho_n I, \quad K_1 = I. \] (4.5)

**Lemma 8** From (4.3) and (4.5) it follows that appropriate Killing tensors \( K_r \) are given by the following 'cofactor' formula
\[ \text{cof}(\xi I - L) = \sum_{i=0}^{n-1} K_{n-i} \xi^i, \] (4.6)
where \( \text{cof}(A) \) stands for the matrix of cofactors, so that \( \text{cof}(A)A = (\det A) I. \) Notice that \( K_1 = I, \) hence \( A_1 = G \) and \( E_1 \equiv E. \)

**Proof.**
\[
\begin{align*}
(\xi I - L) & \left( \sum_{i=0}^{n-1} K_{n-i} \xi^i \right) = \sum_{i=0}^{n-1} (K_{n-i} \xi^{i+1} - L K_{n-i} \xi^i) \\
& = -L K_n + (K_n - L K_{n-1}) \xi + \ldots + (K_2 - L K_1) \xi^{n-1} + K_1 \xi^n \\
& = I (\rho_n + \rho_{n-1} \xi + \ldots + \rho_1 \xi^{n-1} + \xi^n) \\
& = I \det(\xi I - L).
\end{align*}
\]

According to the above results, the functions \( E_r \), satisfy
\[ \{E_s, E_r\}_{\theta_0} = 0, \] (4.7)
and thus constitute a system of \( n \) constants of motion in involution with respect to the Poisson structure \( \theta_0 \). So, for a given metric tensor \( g \), the existence of a special conformal Killing tensor \( L \) is a sufficient condition for the geodesic flow on \( Q \) to be a Liouville integrable Hamiltonian system with all constants of motion quadratic in momenta.

It turns out that with the tensor \( L \) we can (generically) associate a coordinate system on \( Q \) in which the geodesic flows associated with all the functions \( E_r \) separate. Namely, let \( (\lambda^1(q), \ldots, \lambda^n(q)) \) be \( n \) distinct, functionally independent eigenvalues of \( L \), i.e. solutions of the characteristic equation \( \det(\xi I - L) = 0. \) Solving these relations with respect to \( q \) we get the transformation \( \lambda \rightarrow q \)
\[ q^i = \alpha_i(\lambda), \quad i = 1, \ldots, n. \] (4.8)

The remaining part of the transformation to the separation coordinates can be obtained as a canonical transformation reconstructed from the generating function \( W(p, \lambda) = \sum_i p_i \alpha_i(\lambda) \) in the following way
\[ \mu_i = \frac{\partial W(p, \lambda)}{\partial \lambda^i} \implies p_i = \beta_i(\lambda, \mu) \quad i = 1, \ldots, n. \] (4.9)

In the \( (\lambda, \mu) \) coordinates the tensor \( L \) is diagonal
\[ L = \text{diag}(\lambda^1, \ldots, \lambda^n) \equiv \Lambda_n, \] (4.10)
while the geodesic Hamiltonians have the following form \[22\]

\[E_r = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda^i} \mu_i^2, \quad r = 1, \ldots, n, \quad (4.11)\]

where

\[\Delta_i = \prod_{k=1, \ldots, n, k \neq i} (\lambda^i - \lambda^k), \quad (4.12)\]

\(\rho_r(\lambda)\) are symmetric polynomials (Vie\'t\'e polynomials) defined by the relation

\[\det(\xi I - \Lambda) = (\xi - \lambda^1)(\xi - \lambda^2) \ldots (\xi - \lambda^n) = \sum_{r=0}^{n} \rho_r \xi^{n-r} \quad (4.13)\]

and \(f_i\) are arbitrary smooth functions of one real argument. From (4.11) it immediately follows that in \((\lambda, \mu)\) variables the contravariant metric tensor \(G\) and all the Killing tensors \(K_r\) are diagonal

\[G^{ij} = \frac{f_i(\lambda^j)}{\Delta_i} \delta^{ij}, \quad (K_r)^i_j = -\frac{\partial \rho_r}{\partial \lambda^i} \delta^{ij}. \quad (4.14)\]

**Remark 9** When \(f_i(\lambda^i)\) is a polynomial of order \(\leq n\) the metric is flat, if the order of \(f\) is equal \(n + 1\) the metric is of constant curvature.

Moreover, as

\[
\Lambda^* d\lambda^i = \lambda^i d\lambda^i, \quad K^*_r d\lambda^i = -\frac{\partial \rho_r}{\partial \lambda^i} d\lambda^i = \nu^i_{r+1} d\lambda^i, \quad (4.15)
\]

then multiplying both sides of eq. (4.15) by \(G\) we get

\[\left( GK^*_r - \nu^i_{r+1} G \right) d\lambda_i = 0 \iff \left( A^*_r - \nu^i_{r+1} G \right) d\lambda_i = 0, \quad i = 1, \ldots, n, \quad (4.16)\]

i.e. the tensorial Eisenhart realization (3.10) of the St"ackel results.

To ensure that in the \((\lambda, \mu)\) coordinates the geodesic Hamiltonians (4.11) are separable it is sufficient to observe that in these coordinates they actually have the St"ackel form (3.8) with the related St"ackel matrix

\[
\varphi = \begin{pmatrix}
\frac{(\lambda^1)^{n-1}}{f_1(\lambda^1)} & \frac{(\lambda^2)^{n-2}}{f_1(\lambda^2)} & \ldots & \frac{\lambda^1}{f_1(\lambda^n)} & \frac{1}{f_1(\lambda^1)} \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
\frac{(\lambda^n)^{n-1}}{f_n(\lambda^n)} & \frac{(\lambda^n)^{n-2}}{f_n(\lambda^n)} & \ldots & \frac{\lambda^n}{f_n(\lambda^n)} & \frac{1}{f_n(\lambda^n)}
\end{pmatrix}. \quad (4.17)
\]

It means that St"ackel separation conditions (3.12) take the particular form

\[E_1(\lambda^i)^{n-1} + E_2(\lambda^i)^{n-2} + \ldots + E_n = \frac{1}{2} f_i(\lambda^i) \mu_i^2, \quad i = 1, \ldots, n. \quad (4.18)\]

For \(f_i(\lambda^i) = f(\lambda^i)\) eqs. (4.18) can be represented by \(n\) different copies \((\xi, \mu) = (\lambda^i, \mu_i), i = 1, \ldots, n\) of some curve

\[E_1 \xi^{n-1} + E_2 \xi^{n-2} + \ldots + E_n = \frac{1}{2} f(\xi) \mu^2 \quad (4.19)\]

called the separation curve of geodesic motion.

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4.2 (Quasi-)bi-Hamiltonian chains

The special conformal Killing tensor \( L \) can be lifted from \( Q \) to a \((1,1)\)-type tensor on \( \mathcal{N} = T^*Q \) where it takes the form

\[
N = \begin{pmatrix} L & 0 \\ F & L^* \end{pmatrix}, \quad F_j^i = \frac{\partial}{\partial q^i}(p^T L)_j - \frac{\partial}{\partial q^j}(p^T L)_i, \quad L^* = L^T, \tag{4.20}
\]

The lifted \((1,1)\) tensor \( N \) is Nijenhuis torsion free, like the \( L \) one, hence it defines a recursion operator on \( \mathcal{N} \). An important property of \( N \) is that when it acts on the canonical Poisson tensor \( \theta_0 \) it produces another Poisson tensor

\[
\theta_1 = N\theta_0 = \begin{pmatrix} 0 & L \\ -L^* & F \end{pmatrix}, \tag{4.21}
\]

compatible with the canonical one (actually \( \theta_0 \) is compatible with \( N^k\theta_0 \) for any integer \( k \)) making \( \mathcal{N} \) the \( \omega N \) manifold. It is now possible to show that the geodesic Hamiltonians \( E_r \) satisfy on \( \mathcal{N} = T^*Q \) the set of relations [32]

\[
\theta_1 dE_r = \theta_0 dE_{r+1} - \rho_r \theta_0 dE_1, \quad E_{n+1} = 0, \quad r = 1, \ldots, n. \tag{4.22}
\]

\[
N dE_r = dE_{r+1} - \rho_r dE_1, \quad N^* = \theta_0^{-1}\theta_1 = \begin{pmatrix} L^* & -F \\ 0 & L \end{pmatrix},
\]

which are a particular case of the quasi-bi-Hamiltonian chain (2.22) [40]. Obviously \( \{E_s, E_r\}_{\theta_1} = 0 \), as follows from (4.7) and (4.22). Notice that relation (4.5) is immediately reconstructed from (4.22). Indeed, according to

\[
\left( \frac{\partial E}{\partial p_1}, \ldots, \frac{\partial E}{\partial p_1} \right)^T = KGp, \quad p = (p_1, \ldots, p_n)^T, \tag{4.23}
\]

d\( p \) components of (4.22) are

\[
LK_r Gp = K_{r+1} Gp - \rho_r Gp \tag{4.24}
\]

On the other hand

\[
LK_r Gp = K_{r+1} Gp - \rho_r Gp \tag{4.25}
\]

\[
T_r \equiv \frac{1}{2} p^T LK_r Gp = \frac{1}{2} p^T K_{r+1} Gp - \frac{1}{2} \rho_r p^T Gp = E_{r+1} - \rho_r E_1,
\]

so, for \( r = 1 \) we get the relation

\[
\overline{L} \equiv \overline{L}_1 = E_2 - \rho_1 E_1 \tag{4.26}
\]
and immediate verification that \( L \) is a special conformal Killing tensor, as

\[
\{L, E_1\}_{\theta_0} = \{E_1, \rho_1\}_{\theta_0} E_1 = \kappa E_1, \quad \rho_1 = -TrL. \quad (4.27)
\]

As was presented in this section, tensors \( L^* \) and \( A^*_r \) have a common set of closed eigenforms (4.15), (4.16), hence obviously

\[
(G^{(s)} K_r^* - v^i G^{(s)}) d\lambda_i = 0 \iff (A^{(s)}_r^* - v^i G^{(s)}) d\lambda_i = 0, \quad i = 1, ..., n, \quad (4.28)
\]

where

\[
A^{(s)}_r = L^s A_r = K_r L^s G = K_r G^{(s)}, \quad s = \pm 1, \pm 2, \ldots . \quad (4.29)
\]

It means that \( A^{(s)}_r \) are contravariant Killing tensors of the metric \( G^{(s)} \) and functions

\[
E^{(s)}_r = \frac{1}{2} p^T A^{(s)}_r p, \quad r = 1, ..., n \quad (4.30)
\]

are in involution. Moreover, \( L \) is the special conformal Killing tensor for each \( G^{(s)} \). Indeed, from (4.24) we have

\[
L^{s+1} K_r G = L^s K_{r+1} G - \rho_r L^s G
\]

\[
\downarrow
\]

\[
LK_r L^s G = K_{r+1} L^s G - \rho_r L^s G
\]

\[
\downarrow
\]

\[
LK_r G^{(s)} = K_{r+1} G^{(s)} - \rho_r G^{(s)}
\]

\[
\downarrow
\]

\[
T^{(s)}_r = E^{(s)}_{r+1} - \rho_r E^{(s)}_1, \quad (4.31)
\]

where

\[
T^{(s)}_r = \frac{1}{2} p^T L A^{(s)}_r p. \quad (4.31a)
\]

Hence, for \( T^{(s)} = T^{(s)}_1 \), we find the condition (4.27)

\[
\{T^{(s)}_r, E^{(s)}_1\}_{\theta_0} = \{E^{(s)}_1, \rho_1\}_{\theta_0} E^{(s)}_1 = \kappa E^{(s)}_1. \quad (4.32)
\]

Let us denote by \( G^{(0)} \) a basic flat contravariant metric, which in \( \lambda \) coordinates takes the form

\[
\left(G^{(0)}\right)^{ij} = \frac{1}{\Delta_i} \delta^{ij}, \quad (4.33)
\]

i.e. \( f_i(\lambda^i) = 1, \quad i = 1, ..., n \) (4.14). It means that in the appropriate separation curve for geodesic motion (4.19) \( f(\xi) = 1 \). Moreover, the metric tensor \( G \) for which the separation curve (4.19) contains \( f(\xi) \) in the form of Laurent polynomial

\[
f(\xi) = \sum_{i=m_2}^{i=m_1} a_i \xi^i, \quad m_1, m_2 \in \mathbb{N}, \quad a_i = \text{const}, \quad (4.34)
\]
is constructed from the basic metric $G^{(0)}$ in the following way

$$G = f(L)G^{(0)}. \quad (4.35)$$

Now we pass to the bi-Hamiltonian representation. On the extended phase space $\mathcal{M} = T^*Q \times \mathbb{R}$, the extended geodesic Hamiltonians

$$e_r = E_r + cp_r, \quad r = 1, \ldots, n, \quad e_0 = c, \quad (4.36)$$

where $c$ is an additional coordinate, satisfy the following bi-Hamiltonian chain [32]

$$\begin{align*}
\pi_0 de_0 &= 0 \\
\pi_0 de_1 &= X_1 = \pi_1 de_0 \\
\pi_0 de_2 &= X_2 = \pi_1 de_1 \\
&\quad \vdots \\
\pi_0 de_n &= X_n = \pi_1 de_{n-1} \\
0 &= \pi_1 de_n
\end{align*} \quad (4.37)$$

with the Poisson operators $\pi_0$ and $\pi_1$

$$\begin{align*}
\pi_0 &= \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\pi_1 &= \begin{pmatrix} \theta_1 & \theta_0 de_1 \\ -(\theta_0 de_1)^T & 0 \end{pmatrix}.
\end{align*} \quad (4.38)$$

Both Poisson tensors $\pi_0$ and $\pi_1$ are compatible and degenerated. The Casimir of $\pi_0$ is $e_0$ and the Casimir of $\pi_1$ is $e_n$ and they start and terminate the bi-Hamiltonian chain (4.37). Of course $\{e_s, e_r\}_{\pi_0} = \{e_s, e_r\}_{\pi_1} = 0$. The projections of $\pi_0, \pi_1$ onto a symplectic leaf of $\pi_0$ ($c = \text{const}$) along $Z = \frac{\partial}{\partial c}$ reconstructs our nondegenerate Poisson tensors $\theta_0, \theta_1$. If we introduce the Poisson pencil $\pi_\xi = \pi_1 - \xi \pi_0$, the chain (4.37) can be written in a compact form

$$\pi_\xi d e(\xi) = 0, \quad e(\xi) = \sum_{r=0}^{n} e_{n-r} \xi^r, \quad (4.39)$$

where $e_\xi$ is a Casimir of the Poisson pencil $\pi_\xi$, depending polynomial on $\xi$, i.e. we deal with a Gel’fand-Zakharevich type system. The projection of (4.37) onto symplectic leaf of $\pi_0 : c = 0$ reconstructs the quasi-bi-Hamiltonian chain (4.22) on $\mathcal{N}$ in a compact form

$$\theta_\xi d(E(\xi)) + \rho(\xi) \theta_0 d(E_1) = 0$$

$$\Hh \quad (4.40)$$

where

$$E(\xi) = \sum_{r=0}^{n-1} E_{n-r} \xi^r, \quad \rho(\xi) = \det(\xi I - L) = \sum_{r=0}^{n} \rho_{n-r}(q) \xi^r. \quad (4.40a)$$

If we start with a bi-Hamiltonian chain (4.37) or a quasi-bi-Hamiltonian chain (4.22) written in the ‘physical’ (original) coordinates $(q, p)$ then we can usually find the functions $q(\lambda)$ that
constitute the first half of the transformation \((\lambda, \mu) \rightarrow (q, p)\) simply by taking the functions \(\rho_r(q)\) from the Hamiltonians \((4.36)\) or directly from \((4.4)\) and solve the system of equations \(\rho_r(q) = \rho_r(\lambda), \ r = 1, \ldots, n\) with respect to \(q\). The remaining functions \(p(\lambda, \mu)\) are calculated according to \((4.9)\).

4.3 Separable potentials

What potentials can be added to geodesic Hamiltonians \(E_r\) without destroying their separability within the above schema? It turns out that there exists a sequence of basic separable potentials \(V_r^{(k)}, \ k = \pm 1, \pm 2, \ldots\), which can be added to geodesic Hamiltonians \(E_r\) such that the new Hamiltonians

\[
H_r(q, p) = E_r(q, p) + V_r^{(k)}(q), \quad r = 1, \ldots, n,
\]

are in involution with respect to \(\theta_0\) and \(\theta_1\) and are still separable in the same coordinates \((\lambda, \mu)\). It means that \(H_r\) follow the quasi-bi-Hamiltonian chain \((4.22)\)

\[
N^*dH_r = dH_{r+1} - \rho_r dH_1, \quad H_{n+1} = 0, \quad r = 1, \ldots, n,
\]

while for potentials we have

\[
L^*dV_r = dV_{r+1} - \rho_r dV_1, \quad r = 1, \ldots, n,
\]

\[
\Downarrow
\]

\[
dV_{r+1} = \sum_{k=0}^{r} \rho_k \left( L^* \right)^{r-k} dV_1
\]

\[
\Downarrow \ (4.3)
\]

\[
dV_{r+1} = K_{r+1}^* dV_1 \implies d(K_{r+1}^* dV_1) = 0.
\]

The relations \((4.44)\) were derived for the first time by Benenti [11], [12].

**Theorem 10** The basic separable potentials \(V_r^{(m)}\) are given by the following recursion relation [23], [27]

\[
V_r^{(m+1)} = V_{r+1}^{(m)} - \rho_r V_1^{(m)},
\]

and its inverse

\[
V_r^{(-m-1)} = V_{r-1}^{(-m)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(-m)}, \quad V_r^{(0)} = -\delta_{r,n}.
\]

**Proof.** The proof is inductive. We show it for positive potentials. Assuming that potentials \(V_r^{(m)}\) fulfil condition \((4.43)\), we prove that potentials \(V_r^{(m+1)}\) fulfil the same condition. The condition \((4.43)\) is true for the first nontrivial potentials \(V_r^{(m)} = \rho_r\), which are coefficients of characteristic polynomials of a special conformal Killing tensor \(L\) [32]

\[
d\rho_{r+1} = L^* d\rho_r + \rho_r d\rho_1, \quad r = 1, \ldots, n.
\]
Notice that (4.47) is a special case of the condition (2.24) and follows from the Stäckel separability. Then we have

$$L^*dV^{(m+1)} + \rho_r dV_1^{(m+1)}$$

$$= L^*d(V_r^{(m)} - \rho_r V_1^{(m)}) - \rho_r d(V_2^{(m)} - \rho_1 V_1^{(m)})$$

$$= L^*dV_r^{(m+1)} - \rho_r L^*dV_1^{(m)} - V_1^{(m)}(L^*d\rho_r + \rho_r d\rho_1)$$

$$+ \rho_r (dV_2^{(m)} - \rho_1 dV_1^{(m)})$$

$$= L^*dV_r^{(m+1)} - \rho_r L^*dV_1^{(m)} - V_1^{(m)}d\rho_{r+1} + \rho_r L^*dV_1^{(m)}$$

$$= L^*dV_r^{(m+1)} - V_1^{(m)}d\rho_{r+1} = dV^{(m)} - \rho_r + 1 dV_1^{(m)} - V_1^{(m)}d\rho_{r+1}$$

$$= d(V_r^{(m+1)} - \rho_r + 1 V_1^{(m)}) = dV^{(m+1)}.$$

Notice that $V_r^{(m)} = -\delta_{r,n-m}$, $m = 0, \ldots, n - 1$, $V_r^{(-1)} = \rho_r$, $V_r^{(-1)} = \rho_n$. Such a notation will be useful in the case of deformed Benenti systems.

Now, again the extended Hamiltonians $h_r : \mathcal{N} \times \mathbb{R} \to \mathbb{R}$

$$h_r = H_r + c\rho_r \quad (4.48)$$

satisfy the bi-Hamiltonian chain (compare with (4.37))

$$\pi_0 dh_0 = 0$$

$$\pi_0 dh_1 = X_1 = \pi_1 dh_0$$

$$\pi_0 dh_2 = X_2 = \pi_1 dh_1$$

$$\vdots$$

$$\pi_0 dh_n = X_n = \pi_1 dh_{n-1}$$

$$0 = \pi_1 dh_n$$

with $\pi_0$ as in (4.38),

$$\pi_1 = \begin{pmatrix}
\theta_1 & \theta_0 dh_1 \\
- (\theta_0 dh_1)^T & 0
\end{pmatrix} \quad (4.50)$$

and $\{h_s, h_r\}_{\pi_0} = \{h_s, h_r\}_{\pi_1} = 0$. If we use the following notation

$$H(\xi) = E(\xi) + V(\xi), \quad V(\xi) = \sum_{j=0}^{n-1} V_{n-j}(q)\xi^j,$$

then the recursion formulas (4.45) and (4.46) can be written in the compact form [41]

$$V(\xi)^{(k+1)} = \lambda V(\xi)^{(k)} - \det(\xi I - L)V_1^{(k)} \quad (4.51)$$

and

$$V(\xi)^{(-k-1)} = \frac{1}{\xi} \left( V(\xi)^{(-k)} - \frac{\det(\xi I - L)}{\det L} V_n^{(-k)} \right) \quad (4.52)$$
and our bi-Hamiltonian chain (4.49) is given by

\[ \pi_\xi dh(\xi) = 0, \quad h(\xi) = H(\xi) + cp(\xi), \]

while the corresponding quasi-bi-Hamiltonian (4.42) chain takes the form

\[ \theta_\xi dH(\xi) + \rho(\xi)\theta_0 dH_1(q,p) = 0, \quad \rho(\xi) = \frac{\partial h(\xi)}{\partial c} \]

\[ (N^* - \xi I) dH(\xi) + \rho(\xi) dH_1(q,p) = 0. \]

In \((\lambda, \mu)\) coordinates the full (i.e. with a non-zero potential part) Hamiltonians (4.48) of bi-Hamiltonian chain (4.49) attain the form

\[ h_r(\lambda, \mu, c) = -\sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda^i} \left[ \frac{1}{2} f_i(\lambda^i) \mu^2 + \gamma_i(\lambda^i) \right] + cp_r(\lambda), \quad r = 1, ..., n. \]

\[ (4.55) \]

**Lemma 11** Nontrivial potentials \(V_r^{(n-1+k)}\) and \(V_r^{(-k)}\) \(k = 1, 2, ..., \) enter the separation curve

\[ H_1 \xi^{n-1} + H_2 \xi^{n-2} + ... + H_n = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi) \]

as \(\gamma(\xi) = -\xi^{n-1+k}, -\xi^{-k}\), hence \(\gamma_i(\lambda^i) = -(\lambda^i)^{n-1+k}\) and \(\gamma_i(\lambda^i) = -(\lambda^i)^{-k}\), respectively.

**Proof.** Potentials \(V_r^{(n)} = \rho_r\) are coefficients of characteristic equation of the special conformal Killing tensor \(L\)

\[ \xi^n + \rho_1 \xi^{n-1} + ... + \rho_n = 0. \]

\[ (4.57) \]

Then, we define \(V_r^{(n+k)}\) potentials by a generating equation

\[ \xi^{n+k} + V_1^{(n+k)} \xi^{n-1} + ... + V_n^{(n+k)} = 0. \]

\[ (4.58) \]

The recursion formula (4.45) is reconstructed as follows. From (4.58) we have

\[ \xi^{n+k+1} + V_1^{(n+k)} \xi^n + ... + V_n^{(n+k)} \xi = 0. \]

\[ (4.59) \]

Elimination of \(\xi^n\) via (4.57) leads to the form

\[ \xi^{n+k+1} + (V_2^{(n+k)} - \rho_1 V_1^{(n+k)}) \xi^{n-1} + ... + (V_n^{(n+k)} - \rho_{n-1} V_1^{(n+k)}) \xi - \rho_n V_1^{(n+k)} = 0 \]

\[ (4.60) \]

\[ \downarrow \]

\[ V_r^{(n+k+1)} = V_{r+1}^{(n+k)} - \rho_r V_1^{(n+k)}. \]

For the inverse potentials we have from (4.57)

\[ \frac{1}{\rho_n} \xi^{n-1} + \frac{\rho_1}{\rho_n} \xi^{n-2} + ... + \frac{\rho_{n-1}}{\rho_n} \xi^{-1} + \xi^{-1} = 0, \]

\[ (4.61) \]
so \( V^{(-1)}_r = \frac{\rho_{r-1}}{\rho_n} \). Then, we define \( V^{(-k)} \) potentials by a generating equation

\[
V^{(-k)}_1 \xi^{n-1} + \ldots + V^{(-k)}_n \xi^{-k} = 0. \tag{4.62}
\]

The recursion formula (4.46) is reconstructed as follows. From (4.62) we have

\[
V^{(-k)}_1 \xi^{n-2} + \ldots + V^{(-k)}_n \xi^{-1} + \xi^{-k-1} = 0. \tag{4.63}
\]

Substituting (4.61) to eliminate \( \xi^{-1} \) we get

\[
V^{(-k)}_1 \xi^{n-2} + \ldots + V^{(-k)}_n (-\frac{1}{\rho_n} \xi^{n-1} - \ldots - \frac{\rho_{n-1}}{\rho_n}) + \xi^{-k-1} = 0
\]

\[
\downarrow
\]

\[
V^{(-k-1)}_r = V^{(-k)}_r - \frac{\rho_{r-1}}{\rho_n} V^{(-k)}_n. \tag{4.64}
\]

\section{One-hole deformation of Benenti systems}

Separable systems on Riemannian manifolds considered by Benenti belong to important but very particular subclass of such systems. In this context, a question about classification of all separable systems on Riemannian manifolds, with \( n \) quadratic in momenta constants of motion, arises. The classification can be made with respect to the admissible forms of St"ackel separability conditions. The right hand side of the conditions (3.12) is always the same for the class of systems considered

\[
\text{r.h.s.} = \frac{1}{2} f_i(\lambda^i) \mu_i^2 + \gamma_i(\lambda^i) = \psi(\lambda^i, \mu_i), \tag{5.1}
\]

so different classes of separable systems are described by different forms of the l.h.s. of St"ackel conditions. In the simplest Benenti case, it is given by the following polynomial form

\[
\text{l.h.s.} = H_1 \xi^{n-1} + H_2 \xi^{n-2} + \ldots + H_n, \quad \xi = \lambda^1, \ldots, \lambda^n. \tag{5.2}
\]

We will show, that all other classes, given by some polynomial in \( \lambda \), are appropriate deformations of the Benenti class.

First, let us define a 1-hole deformation of the Benenti class. Consider the following separability condition

\[
\bar{H}_1 \xi^{(n+1)-1} + \bar{H}_2 \xi^{(n+1)-2} + \ldots + \bar{H}_{n+1} = \psi(\xi, \mu), \quad \bar{H}_{n+1} = 0 \tag{5.3}
\]

and the Benenti separability condition with the same \( \psi \) representation

\[
H_1 \xi^{n-1} + H_2 \xi^{n-2} + \ldots + H_n = \psi(\xi, \mu). \tag{5.4}
\]
Notice that all Benenti systems are classified by different forms of $\psi$, i.e. by $f_i(\lambda^i)$ and $\gamma_i(\lambda^i)$. A missing monomial (a hole) in (5.3) is $\tilde{H}_n \xi^{(n+1)-n_1}$. Using the characteristic equation of a conformal Killing tensor $L$ of the Benenti system (5.4)

$$\xi^n + \rho_1 \xi^{n-1} + ... + \rho_n = 0,$$

for the elimination of $\xi^n$, equation (5.3) can be transformed to the form

$$(\tilde{H}_2 - \rho_1 \tilde{H}_1) \xi^{n-1} + ... + (\tilde{H}_{n+1} - \rho_n \tilde{H}_1) = \psi(\xi, \mu)$$

(5.5)

and hence, comparing it with (5.4) we find

$$H_r = \tilde{H}_{r+1} - \rho_r \tilde{H}_1, \quad r = 1, ..., n,$$

(5.6)

with the inverse

$$\tilde{H}_{r+1} = H_r - \frac{\rho_r}{\rho_{n_1-1}} H_{n_1-1}, \quad r = 0, ..., n,$$

(5.7)

where $H_0 = 0, \rho_0 = 1$.

Notice, that formula (5.7) applies separately to the geodesic and the potential parts, i.e.

$$\tilde{E}_{r+1} = E_r - \frac{\rho_r}{\rho_{n_1-1}} E_{n_1-1},$$

(5.8a)

$$\tilde{V}_{r+1} = V_r - \frac{\rho_r}{\rho_{n_1-1}} V_{n_1-1}, \quad r = 0, ..., n.$$

(5.8b)

5.1 Geodesic part

Let us first look onto geodesic Hamiltonians

$$\tilde{E}_r = \frac{1}{2} \rho^T (\rho_{n_1-1} K_{r-1} - \rho_{r-1} K_{n_1-1}) \frac{1}{\rho_{n_1-1}} G p, \quad r = 1, ..., n + 1.$$  

(5.9)

Using the known relation for the Benenti chain

$$\rho_r I = K_{r+1} - L K_r$$

we get

$$\tilde{E}_1 = -\frac{1}{\rho_{n_1-1}} E_{n_1-1} = -\frac{1}{\rho_{n_1-1}} \frac{1}{2} \rho^T K_{n_1-1} G p$$

$$= \frac{1}{2} \rho^T G p \implies \tilde{G} = -\frac{1}{\rho_{n_1-1}} K_{n_1-1} G$$

(5.10)

and

$$\tilde{E}_r = \frac{1}{2} \rho^T [\frac{1}{\rho_{n_1-1}} (K_{n_1} K_{r-1} - K_{n_1-1} K_r)] p$$

$$= \frac{1}{2} \rho^T \tilde{K}_r \tilde{G} p \implies \tilde{K}_r = K_r - K_{r-1} K_{n_1} (K_{n_1-1})^{-1}.$$  

(5.11)

Although we know from the construction that $\tilde{E}_r$ are in involution, as they are Stäckel geodesics, but here we prove it in a coordinate free way.
Lemma 12 $\vec{K}_r$ are Killing tensors of the metric $\vec{G}$ which share the common set of eigenfunctions, i.e. $\vec{E}_r$ are in involution.

Proof. From involutivity of Benenti Hamiltonians we have

\[
0 = \{H_r, H_s\}_{\theta_0} = \{E_r + V_r, E_s + V_s\}_{\theta_0} = \{E_r, V_s\}_{\theta_0} + \{V_r, E_s\}_{\theta_0}
\]

(5.12)

Then,

\[
\{\vec{E}_{r+1}, \vec{E}_{s+1}\}_{\theta_0} = \{E_r, -\frac{\rho_s}{\rho_{n+1}} E_{n+1-1}\}_{\theta_0} + \{E_s, -\frac{\rho_r}{\rho_{n+1}} E_{n+1-1}\}_{\theta_0} + \{E_r, \rho_{n+1} \}_{\theta_0} \frac{\rho_s}{\rho_{n+1}} E_{n+1-1} + \{E_s, \rho_{n+1} \}_{\theta_0} \frac{\rho_r}{\rho_{n+1}} E_{n+1-1}
\]

(5.13)

\[
\Rightarrow \{E_r, V_s\}_{\theta_0} = \{E_s, V_r\}_{\theta_0} \Rightarrow \{E_r, \rho_s\}_{\theta_0} = \{E_s, \rho_r\}_{\theta_0}.
\]

Then,

\[
\{\vec{E}_{r+1}, \vec{E}_{s+1}\}_{\theta_0} = 0.
\]

\[
\square
\]

5.2 Deformed potentials

Let us analyze the basic deformed potentials. The first potentials are the following. From (5.8b) and the Benenti potentials we have $\vec{V}_r^{(m)} = -\delta_{r-1,n-m}$ for $m < n + 1$, $m \neq (n + 1) - n_1$ and

\[
\vec{V}_r^{(n+1) - n_1} = \frac{\rho_r - 1}{\rho_{n+1}}, \quad \vec{V}_r^{(n+1)} = \rho_r - \frac{\rho_r - 1}{\rho_{n+1}}, \quad \ldots.
\]

(5.14)

Notice that $\vec{V}_r^{(m)} = 0$ for $m \geq n + 1$ and $\vec{V}_r^{(n+1) - n_1} = 1$.

Lemma 13 Nontrivial basic potentials $\vec{V}_r^{(n+1) - n_1}, \vec{V}_r^{(n+k)}$ and $\vec{V}_r^{(-k)} \ k = 1, 2, \ldots$ enter the separation curve

\[
\vec{H}_1 \xi^n + \vec{H}_2 \xi^{n-1} + \ldots + \vec{H}_{n+1} = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi), \quad \vec{H}_{n_1} = 0,
\]

(5.15a)

as $\gamma(\xi) = -\xi^{(n+1) - n_1}, -\xi^{n+k}, -\xi^{-k}$.

Proof. We will show the following generating equations for the potentials considered

\[
\xi^{n+k} + \vec{V}_1^{(n+k)} \xi^n + \ldots + 0 \xi^{n+1-n_1} + \ldots + \vec{V}_r^{(n+k)} = 0, \quad k = 1, 2, \ldots,
\]

(5.15a)

\[
\xi^{-k} + \vec{V}_1^{(-k)} \xi^n + \ldots + 0 \xi^{n+1-n_1} + \ldots + \vec{V}_r^{(-k)} = 0, \quad k = 1, 2, \ldots,
\]

(5.15b)

\[
\vec{V}_1^{(n+1-n_1)} \xi^n + \ldots + \xi^{n+1-n_1} + \ldots + \vec{V}_r^{(n+1-n_1)} = 0.
\]

(5.15c)
For the first two equations we have \((m > n \text{ or } m < 0)\)
\[
\xi^m + \tilde{V}^{(m)}_1 \xi^n + ... + \tilde{V}^{(m)}_{n+1} = 0, \quad \tilde{V}^{(m)}_{n+1} = 0
\]
\[
\updownarrow
\]
\[
\xi^m + \tilde{V}^{(m)}_1 (-\rho_1 \xi^{n-1} - ... - \rho_n) + ... + \tilde{V}^{(m)}_{n+1} = 0
\]
\[
\updownarrow
\]
\[
\xi^m + (\tilde{V}^{(m)}_2 - \rho_1 \tilde{V}^{(m)}_1) \xi^{n-1} + ... + (\tilde{V}^{(m)}_{n+1} - \rho_n \tilde{V}^{(m)}_1) = 0
\]
\[
\updownarrow
\]
\[
\xi^m + V^{(m)}_1 \xi^{n-1} + ... + V^{(m)}_n = 0
\]
which reveal the known deformation relations (5.8b)
\[
V^{(m)}_r = \tilde{V}^{(m)}_{r+1} - \rho_r \tilde{V}^{(m)}_1 \iff \tilde{V}^{(m)}_{r+1} = V^{(m)}_r - \frac{\rho_r}{\rho_{n+1-1}} V^{(m)}_n
\quad (5.16)
\]
between nontrivial basic potentials from Benenti class and respective deformed potentials. For the last case (5.15c) we have
\[
\tilde{V}^{(n+1-n_1)}_1 \xi^n + ... + \xi^{n+1-n_1} + ... + \tilde{V}^{(n+1-n_1)}_{n+1} = 0,
\]
\[
\updownarrow
\]
\[
\xi^n + \frac{\tilde{V}^{(n+1-n_1)}_2}{\tilde{V}^{(n+1-n_1)}_1} \xi^{n-1} + ... + \frac{1}{\tilde{V}^{(n+1-n_1)}_1} \xi^{n+1-n_1} + ... + \frac{\tilde{V}^{(n+1-n_1)}_{n+1}}{\tilde{V}^{(n+1-n_1)}_1} = 0
\]
\[
\updownarrow
\]
\[
\xi^n + \rho_1 \xi^{n-1} + ... + \rho_n = 0
\]
\[
\downarrow
\]
\[
\tilde{V}^{(n+1-n_1)}_r = \frac{\rho_{r-1}}{\rho_{n+1-1}}
\]
which is a special case of deformations (5.16) related to the trivial Benenti potential \(V^{(n+1-n_1)}_r = -\delta_{r,n+1-1}\). □

Alternatively, the basic potentials \(\tilde{V}^{(m)}_r, m > n + 1\) can be constructed recursively as in the Benenti case.

**Lemma 14** The basic separable potentials \(\tilde{V}^{(m)}_r, m > n + 1\) are given by the following recursion relation
\[
\tilde{V}^{(m+1)}_r = \tilde{V}^{(m)}_{r+1} - \tilde{V}^{(m+1)}_r \tilde{V}^{(m)}_1 - \tilde{V}^{(m+1-n_1)}_r \tilde{V}^{(m)}_{n+1},
\quad (5.21)
\]
where \(\tilde{V}^{(n+1)-n_1}_r\) and \(\tilde{V}^{(n+1)}_r\) are given by (5.13).
Proof. The potentials $\tilde{V}_r^{(n+1)-n_1}$ enter the separation curve in the form (5.15c), while the potentials $\tilde{V}_r^{(n+1)}, \tilde{V}_r^{(m)}, \tilde{V}_r^{(m+1)}$ enter the separation curve in the form (5.15a) with $k = 1, m - n, m + 1 - n$. The recursion formula (5.21) is reconstructed as follows. Multiplying equation (5.15a) for $k = m - n$ by $\xi$ we have
\[
\xi^{m+1} + \tilde{V}_1^{(m)} \xi^{n+1} + \ldots + \tilde{V}_{n_1+1}^{(m)} \xi^{n+1-n_1} + \ldots + \tilde{V}_{n+1}^{(m)} \xi = 0.
\]
Substituting $\xi^{n+1}$ from (5.15a) for $k = 1$ and $\xi^{n+1-n_1}$ from (5.15c) we get
\[
\xi^{m+1} + \tilde{V}_1^{(m)} \left( -\tilde{V}_1^{(n+1)} \xi^n - \ldots - \tilde{V}_{n+1}^{(n+1)} \right) + \ldots + \\
+ \tilde{V}_{n_1+1}^{(m)} \left( -\tilde{V}_1^{(n+1-n_1)} \xi^n - \ldots - \tilde{V}_{n+1}^{(n+1-n_1)} \right) + \ldots + \tilde{V}_{n+1}^{(m)} \xi = 0.
\]
A comparison with the separation curve for the potential $\tilde{V}_r^{(m+1)}$ (eq. (5.15a) with $k = m+1-n$) reveals the formula (5.21).

Of course, all Hamiltonian functions $\tilde{H}_r$ are in involution with respect to $\theta_0$ as the proof of Lemma 6 is valid when we change $E_i \rightarrow H_i$ and $\tilde{E}_i \rightarrow \tilde{H}_i$.

5.3 Quasi-bi-Hamiltonian representation

The invariance of the subspace spanned by $(d\tilde{H}_1, \ldots, d\tilde{H}_{n+1})$ with respect to $N^*$ (2.21) is proved in the following theorem.

**Theorem 15** Hamiltonian functions $\tilde{H}_r$ fulfill the following quasi-bi-Hamiltonian chain
\[
d\tilde{H}_{r+1} = N^* d\tilde{H}_r + \alpha_r d\tilde{H}_1 + \beta_r d\tilde{H}_{n+1+1}
\]
\[
\Downarrow
\]
\[
\theta_0 d\tilde{H}_{r+1} = \theta_1 d\tilde{H}_r + \alpha_r \theta_0 d\tilde{H}_1 + \beta_r \theta_0 d\tilde{H}_{n+1+1},
\]
where $\alpha_r = \tilde{V}_r^{(n+1)}, \beta_r = \tilde{V}_r^{(n+1)-n_1}$.

**Proof.** We use the property of the Benenti chain
\[
dH_{r+1} = N^* dH_r + V_r^{(n)} dH_1, \quad (5.18a)
\]
\[
d\rho_{r+1} = L^* d\rho_r + V_r^{(n)} d\rho_1, \quad V_r^{(n)} = \rho_r, \quad (5.18b)
\]
and recursion relations (5.7), (5.8b). Hence, we have

\[
\text{r.h.s.}(5.17) = N^* d \bar{H}_r + \alpha_r d \bar{H}_1 + \beta_r d \bar{H}_{n1+1}
\]
\[
= N^* d(H_{r-1} - \frac{\rho_{r-1}}{\rho_{n1-1}} H_{n1-1}) + \left( \rho_r - \frac{\rho_{r-1} \rho_{n1}}{\rho_{n1-1}} \right) d(-\frac{1}{\rho_{n1-1}} H_{n1-1}) \\
+ \frac{\rho_{r-1}}{\rho_{n1-1}} (H_{n1} - \frac{\rho_{n1}}{\rho_{n1-1}} H_{n1-1}) \\
= N^* dH_{r-1} - \frac{\rho_{r-1}}{\rho_{n1-1}} N^* dH_{n1-1} - \rho_{r-1} H_{n1-1} N^* d(\frac{1}{\rho_{n1-1}}) \\
- \frac{1}{\rho_{n1-1}} H_{n1-1} N^* d\rho_{r-1} + \frac{\rho_{r-1}}{\rho_{n1-1}} dH_{n1} - \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} d\rho_{n1} \\
- \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) \\
= N^* dH_{r-1} + \rho_{r-1} dH_1 - \rho_{n1-1} N^* d(\frac{1}{\rho_{n1-1}}) \\
- \frac{1}{\rho_{n1-1}} H_{n1-1} N^* d\rho_{r-1} - \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} d\rho_{n1} - \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) \\
= dH_r - \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) + \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} (N^* d\rho_{r-1} - d\rho_{n1}) \\
- \frac{1}{\rho_{n1-1}} H_{n1-1} N^* d\rho_{r-1} \\
= dH_r - \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) - \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} d\rho_{1} - \frac{1}{\rho_{n1-1}} H_{n1-1} N^* d\rho_{r-1} \\
= dH_r - \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) - \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} d\rho_{1} - \frac{1}{\rho_{n1-1}} H_{n1-1} N^* d\rho_{r-1} + \rho_{r-1} d\rho_{1} \\
= dH_r - \rho_r d \left( \frac{H_{n1-1}}{\rho_{n1-1}} \right) - \frac{\rho_{r-1}}{\rho_{n1-1}^2} H_{n1-1} d\rho_{r} \\
= d \left( H_r - \frac{\rho_r}{\rho_{n1-1}} H_{n1-1} \right) = d \bar{H}_{r+1} = l.h.s.(5.17)
\]

Notice that from (5.17) it follows that also $\{\bar{H}_r, \bar{H}_s\}_{\theta_1} = 0$ is valid. Of course, formula (5.17) works separately for $\bar{E}_r$ and $\bar{V}_r$ in the form

\[
d\bar{E}_r+1 = N^* d\bar{E}_r + \alpha_r d\bar{E}_1 + \beta_r d\bar{E}_{n1+1}, \tag{5.19a}
\]
\[
d\bar{V}_r+1 = L^* d\bar{V}_r + \alpha_r d\bar{V}_1 + \beta_r d\bar{V}_{n1+1}. \tag{5.19b}
\]

Repeating recursive transformation (5.19b) we end up with the equivalent form

\[
d\bar{V}_r+1 = A_r d\bar{V}_1 + B_r d\bar{V}_{n1+1}, \tag{5.20a}
\]

where

\[
A_r = K_{r+1}^* - \frac{\rho_{n1}}{\rho_{n1-1}} K_r^*, \quad B_r = \frac{1}{\rho_{n1-1}} K_r^* .
\]
In analogy to the Benenti case (4.23), (4.24), the $d_p$ part of (5.19a) gives us immediately the analog of formula (4.5) for the one-hole case

$$\tilde{K}_{r+1} = L\tilde{K}_r + \alpha_r I + \beta_r \tilde{K}_{n_1+1}.$$  \hfill (5.21)

Let us introduce $\tilde{L}$ function as it was done for the Benenti case

$$\tilde{L} = \frac{1}{2} p^T L \tilde{G} p.$$  \hfill (5.22)

Then, from (5.21) we find

$$\tilde{L} = \tilde{E}_2 - \alpha_1 \tilde{E}_1 - \beta_1 \tilde{E}_{n_1+1}$$  \hfill (5.23)

and

$$\{\tilde{L}, \tilde{E}_1\}_\theta = \{\tilde{E}_1, \alpha_1\}_\theta \tilde{E}_1 + \{\tilde{E}_1, \beta_1\}_\theta \tilde{E}_{n_1+1} = \kappa_0^0 \tilde{E}_1 + \kappa_{n_1} \tilde{E}_{n_1+1}.$$  \hfill (5.24)

Thus, for $\tilde{G}$, $L$ is not a conformal Killing tensor.

### 5.4 Bi-Hamiltonian representation

Adding two Casimir coordinates $c_1, c_2$ (with respect to $\theta_0$) and extending Hamiltonians $\tilde{H}_r$ to the form

$$\tilde{h}_r = \tilde{H}_r + \alpha_r c_1 + \beta_r c_2$$  \hfill (5.25)

one can put the quasi-bi-Hamiltonian chain (5.17) from $T^*Q$ into a bi-Hamiltonian chain on $T^*Q \times \mathbb{R} \times \mathbb{R}$, being a composition of two bi-Hamiltonian sub-chains

$$\tilde{\pi}_0 d\tilde{h}_0 = 0$$

$$\tilde{\pi}_0 d\tilde{h}_1 = \tilde{X}_1 = \tilde{\pi}_1 d\tilde{h}_0$$

$$\vdots$$

$$\tilde{\pi}_0 d\tilde{h}_{n_1-1} = \tilde{X}_{n_1-1} = \tilde{\pi}_1 d\tilde{h}_{n_1-2}$$

$$0 = \tilde{\pi}_1 d\tilde{h}_{n_1-1}$$

$$\tilde{\pi}_0 d\tilde{h}_{n_1} = 0$$

$$\tilde{\pi}_0 d\tilde{h}_{n_1+1} = \tilde{X}_{n_1+1} = \tilde{\pi}_1 d\tilde{h}_n$$

$$\vdots$$

$$\tilde{\pi}_0 d\tilde{h}_{n+1} = \tilde{X}_{n+1} = \tilde{\pi}_1 d\tilde{h}_n$$

$$0 = \tilde{\pi}_1 d\tilde{h}_{n+1},$$

where $\tilde{h}_0 = c_1$, $\tilde{h}_{n_1} = c_2$ and

$$\tilde{\pi}_0 = \begin{pmatrix} \theta_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\pi}_1 = \begin{pmatrix} \theta_1 \\ -\theta_0 d\tilde{h}_1^T \\ -\theta_0 d\tilde{h}_{n_1+1}^T \end{pmatrix} \begin{pmatrix} \theta_0 d\tilde{h}_1 & \theta_0 d\tilde{h}_{n_1+1} \\ 0 \end{pmatrix}.$$
\(\tilde{\pi}_0\) and \(\tilde{\pi}_1\) are compatible Poisson tensors on \(T^* Q \times \mathbb{R} \times \mathbb{R}\) as in \((q,p,c)\) coordinates \(\tilde{\pi}_1\) tensor has the form (2.33)

\[
\tilde{\pi}_1 = \tilde{\pi}_{1D} + \sum_{i=1}^{k} X_i \wedge Z_i,
\]

where

\[
\tilde{\pi}_{1D} = \begin{pmatrix}
\theta_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad Z_1 = (0,\ldots,0,1,0)^T, \quad Z_2 = (0,\ldots,0,1)^T,
\]

\[X_1 = \pi_0 dh_1 = (\theta_0 dh_1,0)^T, \quad X_2 = \pi_0 dh_{n+1} = (\theta_0 dh_{n+1},0)^T\]

and vector fields \(Z_i\) and \(X_i\) fulfil condition from Lemma 7. The reduction of both chains onto the symplectic leaf \(c_1 = c_2 = 0\) of \(\tilde{\pi}_0\) along \(Z_1 = \frac{\partial}{\partial c_1}, Z_2 = \frac{\partial}{\partial c_2}\) reconstructs immediately the quasi-bi-Hamiltonian chain (5.17).

If we introduce the Poisson pencil \(\tilde{\pi}_\xi = \tilde{\pi}_1 - \xi \tilde{\pi}_0\) both chains (5.26a), (5.26b) can be written in a compact form

\[
\tilde{\pi}_\xi dh(\xi) = 0, \quad \tilde{h}(\xi) = (\tilde{h}^{(1)}(\xi), \tilde{h}^{(2)}(\xi))^T,
\]

where

\[
\tilde{h}^{(1)}(\xi) = h_0 \xi^{n_1-1} + \ldots + h_{n_1-1}, \quad \tilde{h}^{(2)}(\xi) = h_{n_1} \xi^{n+1-n_1} + \ldots + h_{n+1}
\]

and the operation is applied to each component of the vector. The quasi-bi-Hamiltonian chain (5.17) takes the form

\[
\tilde{\theta}_\xi d\tilde{H}(\xi) + \tilde{\alpha}(\xi) \left(\theta_0 d\tilde{H}^{(1)}\right) = 0
\]

\[
\downarrow
\]

\[
(N^* - \xi I) d\tilde{H}(\xi) + \tilde{\alpha}(\xi) d\tilde{H}^{(1)} = 0,
\]

where

\[
\tilde{\theta}_\xi = \tilde{\theta}_1 - \xi \tilde{\theta}_0, \quad \tilde{H}^{(1)} = (\tilde{H}_1, \tilde{H}_{n+1})^T, \quad \tilde{H}_1^{(1)} = H_1 \xi^{n_1-2} + \ldots + H_{n_1-1}, \quad \tilde{H}_1^{(2)} = H_{n_1+1} \xi^{n-n_1} + \ldots + H_{n+1}
\]

and

\[
\tilde{\alpha}(\xi) = \begin{pmatrix}
\frac{\partial \tilde{h}^{(1)}(\xi)}{\partial c_1} & \frac{\partial \tilde{h}^{(2)}(\xi)}{\partial c_1} \\
\frac{\partial \tilde{h}^{(1)}(\xi)}{\partial c_2} & \frac{\partial \tilde{h}^{(2)}(\xi)}{\partial c_2}
\end{pmatrix}.
\]

The Stäckel conditions for extended Hamiltonians \(\tilde{h}_r\) take the form (3.18b)

\[
c_1 \xi^{(n+1)} + \tilde{h}_1 \xi^{(n+1)-1} + \ldots + \tilde{h}_{n_1-1} \xi^{(n+1)-(n_1-1)}
\]

\[+ c_2 \xi^{(n+1)-n_1} + \tilde{h}_{n_1+1} \xi^{(n+1)-(n_1+1)} + \ldots + \tilde{h}_{n+1}
\]

\[= \xi^{n+2-n_1} \tilde{h}^{(1)}(\xi) + \tilde{h}^{(2)}(\xi) = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi),
\]

where \((\xi,\mu) = (\lambda_i,\mu_i), \ i = 1,\ldots,n.\)
6 k-hole deformations of Benenti systems

6.1 Deformation procedure

Here we extend the results of the previous section onto the general k-hole case. Let us start with the separability condition

\[ \tilde{H}_1 \xi^{(n+k)-1} + \tilde{H}_2 \xi^{(n+k)-2} + \cdots + \tilde{H}_{n+k} = \psi(\xi, \mu), \tag{6.1} \]

with k-holes in \( \xi^{(n+k)-n_1}, \xi^{(n+k)-n_2}, \ldots, \xi^{(n+k)-n_k}, \ 1 < n_1 < \ldots < n_k < n + k, \ k \in \mathbb{N}, \) i.e. \( \tilde{H}_{n_1} = \tilde{H}_{n_2} = \cdots = \tilde{H}_{n_k} = 0, \) and the separability condition for Benenti systems with the same \( \psi \)

\[ H_1 \xi^{n-1} + H_2 \xi^{n-2} + \cdots + H_n = \psi(\xi, \mu). \tag{6.2} \]

As for the basic potentials

\[ \xi^{n+k} + V_1^{(n+k)} \xi^{n-1} + \cdots + V_n^{(n+k)} = 0, \]

substituting this relation to (6.1) for \( \xi^{(n+k)-1}, \ldots, \xi^n \) we get a deformation of the chain (6.1) to the Benenti case (6.2)

\[ H_r = \tilde{H}_{r+k} - V_r^{(n+k-1)} \tilde{H}_1 - V_r^{(n+k-2)} \tilde{H}_2 - \cdots - V_r^{(n)} \tilde{H}_k, \quad r = 1, \ldots, n, \tag{6.3} \]

where \( \tilde{H}_{n_1} = \cdots = \tilde{H}_{n_k} = 0 \) and \( V_r^{(m)} \) are appropriate basic Benenti potentials.

**Lemma 16 Deformation of the Benenti case (6.2) to the chain (6.1), i.e. the inverse formula to the (6.3) one, is given by a following determinant form**

\[ \tilde{H}_r = \begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}. \tag{6.4} \]

**Proof.** First, we select from (6.3) \( k \) equations containing \( \tilde{H}_{n_1}, \ldots, \tilde{H}_{n_k} \)

\[ H_{n_1-k} = -V_{n_1-k}^{(n+k-1)} \tilde{H}_1 - \cdots - V_{n_1-k}^{(n)} \tilde{H}_k, \]

\[ \vdots \]

\[ H_{n_k-k} = -V_{n_k-k}^{(n+k-1)} \tilde{H}_1 - \cdots - V_{n_k-k}^{(n)} \tilde{H}_k. \]

The solution with respect to \( \tilde{H}_i, i = 1, \ldots, k \) is given by a determinant form

\[ \tilde{H}_i = \frac{W_i}{W}, \quad i = 1, \ldots, n, \]
where
\[
W = (-1)^k \begin{vmatrix}
V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\
\vdots & \ddots & \vdots \\
V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)}
\end{vmatrix}
\]
and
\[
W_i = (-1)^{k+i} \begin{vmatrix}
H_{n_1-k} & V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n_k-k} & V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)}
\end{vmatrix}
\]
with the column \((V_{n_1-k}^{(n+k-i)}, \ldots, V_{n_k-k}^{(n+k-i)})^T\) missing. Substituting this result to (6.3) we get
\[
\widetilde{H}_r = \frac{H_{r-k}W + V_{r-k}^{(n+k-1)}W_1 + \ldots + V_{r-k}^{(n)}W_k}{W}
\]
\[
\begin{vmatrix}
H_{r-k} & V_{r-k}^{(n+k-1)} & \cdots & V_{r-k}^{(n)} \\
H_{n_1-k} & V_{n_1-k}^{(n+k-1)} & \cdots & V_{n_1-k}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n_k-k} & V_{n_k-k}^{(n+k-1)} & \cdots & V_{n_k-k}^{(n)}
\end{vmatrix}
\]
\[
= \begin{vmatrix}
H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\
H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k}
\end{vmatrix}
\]
\[
= \begin{vmatrix}
\rho_{n_1-1} & \cdots & \rho_{n_1-k} \\
\vdots & \ddots & \vdots \\
\rho_{n_k-1} & \cdots & \rho_{n_k-k}
\end{vmatrix}
\]
The last step is valid due to the fact that \(V_i^{(n)} = \rho_i\), the form of the recursion formula for Benenti basic potentials (4.45) and the properties of determinants. It allow us to replace the arbitrary potential \(V^{(n+k-i)}\) in determinants by the \(V^{(n)} = \rho\) one. For each recursive step we have
\[
\begin{vmatrix}
\cdots & V_{n_1-k}^{(n+k-i)} & \cdots & \rho_{n_1-k} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & V_{n_k-k}^{(n+k-i)} & \cdots & \rho_{n_k-k}
\end{vmatrix}
= \begin{vmatrix}
\cdots & V_{n_1-k+1}^{(n+k-i-1)} - \rho_{n_1-k}V_1^{(n+k-i-1)} & \cdots & \rho_{n_1-k} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & V_{n_k-k+1}^{(n+k-i-1)} - \rho_{n_k-k}V_1^{(n+k-i-1)} & \cdots & \rho_{n_k-k}
\end{vmatrix}
\]
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The formula (6.4) applies separately to the geodesic and the potential parts.

6.2 Deformed geodesic motion

Let us first look onto \( n \) geodesic Hamiltonians \( \tilde{E}_r, \ r = 1, ..., n + k, \ r \neq n_1, ..., n_k \). Introducing the abbreviation

\[
\varphi(n_1, ..., n_k) = \begin{vmatrix}
\rho_{n_1-1} & \cdots & \rho_{n_1-k} \\
\cdots & \cdots & \cdots \\
\rho_{n_k-1} & \cdots & \rho_{n_k-k}
\end{vmatrix},
\]

one finds

\[
\tilde{E}_r = \frac{1}{\varphi} \begin{vmatrix}
E_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\
E_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\
\cdots & \cdots & \cdots & \cdots \\
E_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k}
\end{vmatrix}.
\]

Using the known relations for Benenti chain

\[
\rho_r I = K_{r+1} - LK_r
\]

and the property of determinants we get

\[
\tilde{E}_r = \frac{1}{2} p^T \begin{vmatrix}
K_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\
K_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\
\cdots & \cdots & \cdots & \cdots \\
K_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k}
\end{vmatrix} \frac{1}{\varphi} G_p
\]

\[
= \frac{1}{2} p^T \begin{vmatrix}
K_{r-k} & K_r & \cdots & K_{r-k+1} \\
K_{n_1-k} & K_{n_1} & \cdots & K_{n_1-k+1} \\
\cdots & \cdots & \cdots & \cdots \\
K_{n_k-k} & K_{n_k} & \cdots & K_{n_k-k+1}
\end{vmatrix} \frac{1}{\varphi} G_p
\]

\[
= (-1)^k \frac{1}{2} p^T (K_r D_0 - K_{r-1} D_1 + \cdots + (-1)^k K_{r-k} D_k) \frac{1}{\varphi} G_p,
\]

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From (6.4) the deformed potentials are (geodesic motion is included as a special case). Let us first consider basic potentials. Here we present the proof in the coordinate free form but on the level of full Hamiltonians and

and \( K_m \) in determinant calculations are treated as symbols not matrices.

Then,

\[
\tilde{E}_1 = \frac{1}{2} p^T \tilde{G} p \implies \tilde{G} = (-1)^k \frac{1}{\varphi} D_0 G,
\]

and

\[
\tilde{E}_r = \frac{1}{2} p^T \tilde{K}_r \tilde{G} p \implies \tilde{K}_r = K_r - K_{r-1} D_1 D_0^{-1} + \ldots + (-1)^k K_{r-k} D_k D_0^{-1},
\]

where

\[
D_0 = \begin{vmatrix} K_{n_1-1} & \cdots & K_{n_1-k} \\ \vdots & \ddots & \vdots \\ K_{n_k-1} & \cdots & K_{n_k-k} \end{vmatrix}.
\]

Again we know from the construction that \( \tilde{E}_r \) are in involution, as they are Stäckel geodesics. Here we present the proof in the coordinate free form but on the level of full Hamiltonians (geodesic motion is included as a special case). Let us first consider basic potentials.

### 6.3 Basic deformed potentials

From (6.4) the deformed potentials are

\[
\tilde{V}_r^{(m)} = \frac{1}{\varphi(n_1, \ldots, n_k)} \begin{vmatrix} V_{r-k}^{(m)} & \rho_{r-1} & \cdots & \rho_{r-k} \\ V_{n_1-k}^{(m)} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \vdots & \ddots & \vdots & \vdots \\ V_{n_k-k}^{(m)} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix},
\]

so using the recursion formula for the Benenti potentials and the properties of the determinants we have \( \tilde{V}_r^{(m)} = -\delta_{r-k,n-m} \), \( m < n + k \), \( m \neq (n + k) - n_i \), \( i = 1, \ldots, k \),

\[
\tilde{V}_r^{(n+k)-n_i} = (-1)^{i+1} \frac{1}{\varphi} \begin{vmatrix} \rho_{r-1} & \cdots & \rho_{r-k} \\ \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \vdots & \ddots & \vdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix},
\]

with the row \( (\rho_{n_1-1}, \ldots, \rho_{n_i-k}) \) missing,

\[
\tilde{V}_r^{(n+k)} = \frac{1}{\varphi} \begin{vmatrix} \rho_r & \rho_{r-1} & \cdots & \rho_{r-k} \\ \rho_{n_1} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \vdots & \ddots & \vdots & \vdots \\ \rho_{n_k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix},
\]

Notice that \( \tilde{V}_r^{(m)} = 0 \) for arbitrary \( m \geq n + k \) and \( \tilde{V}_r^{(n+k)-n_j} = \delta_{ij} \), \( i, j = 1, \ldots, k \).
As in the 1-hole case, one can show that nontrivial basic potentials \( \tilde{V}_r^{(n+k-n_i)}, i = 1, ..., k, \tilde{V}_r^{(n+k-1+l)} \) and \( \tilde{V}_r^{(-l)} l = 1, 2, ... \) fulfill the following generating equations

\[
\xi^{n+k-1+l} + \tilde{V}_1^{(n+k-1+l)} \xi^{(n+k)-1} + ... + \tilde{V}_{n+k}^{(n+k-1+l)} = 0, \quad (6.14a)
\]

\[
\xi^{(-l)} + \tilde{V}_1^{(-l)} \xi^{(n+k)-1} + ... + \tilde{V}_{n+k}^{(-l)} = 0, \quad (6.15b)
\]

\[
\xi^{n+k-n_i} + \tilde{V}_1^{(n+k-n_i)} \xi^{(n+k)-1} + ... + \tilde{V}_{n+k}^{(n+k-n_i)} = 0, \quad (6.14c)
\]

and hence enter separation curve

\[
\tilde{H}_1 \xi^{(n+k)-1} + \tilde{H}_2 \xi^{(n+k)-2} + ... + \tilde{H}_{n+k} = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi), \quad \tilde{H}_{n_i} = 0, \quad (6.15)
\]

as \( \gamma(\xi) = -\xi^{(n+k)-n_i}, -\xi^{n+k}, -\xi^{-k} \), where \( i = 1, ..., k, \ l = 1, 2, ... \).

Also as in the 1-hole case, it is not difficult to show that basic potentials \( \tilde{V}_r^{(m)}, m > n + k \) can be constructed recursively by the following recursion relation

\[
\tilde{V}_r^{(m+1)} = \tilde{V}_r^{(m)} - \tilde{V}_r^{(n+k)} \tilde{V}_1^{(m)} - \sum_{i=1}^{k} \tilde{V}_r^{(n+k-n_i)} \tilde{V}_i^{(m)} \tilde{V}_n^{(m)}, \quad (6.16)
\]

where \( \tilde{V}_r^{(n+k-n_i)}, i = 1, ..., k \) and \( \tilde{V}_r^{(n+k)} \) are given by (6.12) and (6.13).

### 6.4 Deformed Hamiltonian functions

For further applications we present now a recursion formula for deformed Hamiltonian functions understood in the following sense. Having the Hamiltonian functions for the \( k \)-hole case we construct the Hamiltonian functions for \((k + 1)\)-hole case, respectively. Let us consider the separability condition

\[
\tilde{H}_1 \xi^{(n+k+1)-1} + \tilde{H}_2 \xi^{(n+k+1)-2} + ... + \tilde{H}_{n+k+1} = \psi(\xi, \mu), \quad (6.17)
\]

with \((k + 1)\)-holes in \( \xi^{(n+k+1)-(n+1)}, ..., \xi^{(n+k+1)-(n_k+1)}, \xi^{(n+k+1)-n_k+1}, 1 < n_1 < ... < n_k < n+k+1 < n + k + 1, k \in \mathbb{N}, i.e. \( \tilde{H}_{n_1} = ... = \tilde{H}_{n_k} = \tilde{H}_{n_k+1} = 0 \), and the separability condition

\[
\tilde{H}_1 \xi^{(n+k)-1} + \tilde{H}_2 \xi^{(n+k)-2} + ... + \tilde{H}_{n+k} = \psi(\xi, \mu), \quad (6.18)
\]

with \( k \)-holes in \( \xi^{(n+k)-n_1}, \xi^{(n+k)-n_2}, ..., \xi^{(n+k)-n_k}, 1 < n_1 < ... < n_k < n + k, k \in \mathbb{N}, i.e. \( \tilde{H}_{n_1} = \tilde{H}_{n_2} = ... = \tilde{H}_{n_k} = 0 \), and the same \( \psi \). It means that the first \( k \) holes are fixed. From (6.14a) with \( l = 1 \) we have

\[
\xi^{n+k} + \tilde{V}_1^{(n+k)} \xi^{(n+k)-1} + ... + \tilde{V}_{n+k}^{(n+k)} = 0. \quad (6.19)
\]

Substituting (6.19) to the first term of (6.17) we get

\[
(\tilde{H}_2 - \tilde{V}_1^{(n+k)} \tilde{H}_1) \xi^{(n+k)-1} + ... - \tilde{V}_1^{(n+k)} \tilde{H}_1 \xi^{(n+k)-n_k+1} + ... + (\tilde{H}_{n+k} - \tilde{V}_1^{(n+k)} \tilde{H}_1) \xi^{(n+k)-n_k} = \psi(\xi, \mu) \quad (6.20)
\]
Then, from (6.22) we have

\[ \tilde{H}_r = \Pi_{r+1} - \tilde{V}^{(n+k)}_r, \quad r = 1, \ldots, n + k, \]  

(6.21)

where \( \Pi_{n+1} = \ldots = \Pi_{n+k+1} = 0 \), \( \tilde{V}^{(n+k)}_{n+1} = \ldots = \tilde{V}^{(n+k)}_{n+k+1} = 0 \). The inverse of (6.21)

\[ \Pi_{r+1} = \tilde{H}_r - \frac{\tilde{V}^{(n+k)}_r}{\tilde{V}^{(n+k)}_{n+k+1}} \tilde{H}_{n+k+1} \]  

(6.22)

is our final recursion formula. The formula is applicable separately for geodesic Hamiltonian functions \( \tilde{E}_r \) and potentials \( \tilde{V}_r \).

**Lemma 17** Functions \( \Pi_r \) are in involution with respect to the Poisson tensor \( \theta_0 \).

**Proof.** The proof is inductive. The 1-hole case was proved in Section 3. From the involutivity of \( k \)-holes Hamiltonians \( \tilde{H}_r \) we prove the involutivity of \( \Pi_r \) Hamiltonians from \( (k + 1) \)-holes case. From involutivity of \( \tilde{H}_r \) Hamiltonians we have

\[ 0 = \{ \tilde{H}_r, \tilde{H}_s \} = \{ \tilde{E}_r + \tilde{V}_r, \tilde{E}_s + \tilde{V}_s \} = \{ \tilde{E}_r, \tilde{V}_s \} + \{ \tilde{V}_r, \tilde{E}_s \} \]

\[ \Rightarrow \{ \tilde{H}_r, \tilde{V}_s \} = \{ \tilde{H}_s, \tilde{V}_r \} \Rightarrow \{ \tilde{H}_r, \tilde{V}^{(n+k)}_s \} = \{ \tilde{H}_s, \tilde{V}^{(n+k)}_r \} \]

Then, from (6.22) we have

\[
\{ \Pi_{r+1}, \Pi_{s+1} \} = \left\{ \tilde{H}_r - \frac{\tilde{V}^{(n+k)}_r}{\tilde{V}^{(n+k)}_{s+1}} \tilde{H}_{s+1}, \tilde{H}_s \right\} + \left\{ \tilde{H}_s - \frac{\tilde{V}^{(n+k)}_s}{\tilde{V}^{(n+k)}_{r+1}} \tilde{H}_{r+1}, \tilde{H}_r \right\}
\]

\[ = \left\{ \tilde{H}_r, \tilde{V}^{(n+k)}_{s+1} \right\} \frac{\tilde{V}^{(n+k)}_s}{\tilde{V}^{(n+k)}_{s+1}} 2 \tilde{H}_{s+1} + \left\{ \tilde{V}^{(n+k)}_s, \tilde{H}_s \right\} \frac{\tilde{V}^{(n+k)}_r}{\tilde{V}^{(n+k)}_{s+1}} 2 \tilde{H}_{s+1} + \]

\[ + \left\{ \tilde{V}^{(n+k)}_r, \tilde{H}_s \right\} \frac{\tilde{V}^{(n+k)}_s}{\tilde{V}^{(n+k)}_{r+1}} 2 \tilde{H}_{r+1} + \]

\[ + \left\{ \tilde{H}_{r+1}, \tilde{V}^{(n+k)}_s \right\} \frac{\tilde{V}^{(n+k)}_r}{\tilde{V}^{(n+k)}_{r+1}} 2 \tilde{H}_{r+1} = 0. \]

\[ \blacksquare \]

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6.5 Quasi-bi-Hamiltonian representation

Now we demonstrate that the subspace span by \((d\tilde{H}_1, ..., d\tilde{H}_{n+k})\) is invariant with respect to \(N^*\) (Theorem 3).

**Theorem 18** Hamiltonian functions \(\tilde{H}_r\) belong to the following quasi-bi-Hamiltonian chain

\[
d\tilde{H}_{r+1} = N^* d\tilde{H}_r + \alpha_r^0 d\tilde{H}_1 + \sum_{i=1}^{k} \alpha_r^{ni} d\tilde{H}_{ni+1} \quad \uparrow \quad (6.23)
\]

\[
\theta_0 d\tilde{H}_{r+1} = \theta_1 d\tilde{H}_r + \alpha_r^0 \theta_0 d\tilde{H}_1 + \sum_{i=1}^{k} \alpha_r^{ni} \theta_0 d\tilde{H}_{ni+1},
\]

where \(\alpha_s^r = \tilde{V}^{(n+k)-s}_r\). Of course formula (6.23) works separately for the geodesic Hamiltonians \(\tilde{E}_r\) and potentials \(\tilde{V}_r\)

\[
d\tilde{E}_{r+1} = N^* d\tilde{E}_r + \alpha_r^0 d\tilde{E}_1 + \sum_{i=1}^{k} \alpha_r^{ni} d\tilde{E}_{ni+1}, \quad (6.24a)
\]

\[
d\tilde{V}_{r+1} = L^* d\tilde{V}_r + \alpha_r^0 d\tilde{V}_1 + \sum_{i=1}^{k} \alpha_r^{ni} d\tilde{V}_{ni+1}. \quad (6.24b)
\]

**Proof.** The proof is inductive. Assuming that (6.23) and (6.24b) are valid for the \(k\)-hole case, with holes at positions \(\xi^{(n+k)-ni}, i = 1, ..., k\), we prove the validity of (6.23) for the \((k+1)\)-hole case with an extra hole at the position \(\xi^{(n+k+1)-nk+1}\). We take the \(k\)-hole and \((k+1)\)-hole separability conditions as in (6.17) and (6.18). Our assumptions are as follows

\[
d\tilde{H}_{r+1} = N^* d\tilde{H}_r + \alpha_r^0 d\tilde{H}_1 + \sum_{i=1}^{k} \alpha_r^{ni} d\tilde{H}_{ni+1}, \quad (6.25a)
\]

\[
d\alpha_r^{ni} = L^* d\alpha_r^{ni} + \alpha_r^{0} d\alpha_1^{ni} + \sum_{j=1}^{k} \alpha_r^{nj} d\alpha_{nj+1}^{ni}, \quad (6.25b)
\]

where \(i, j = 0, ..., k, n_0 = 0\). Notice that (6.25b) is a particular case of the condition (2.24) and follows from the fact that in a Stäckel case \(\alpha_{ij}\) are particular basic separable potentials. From (6.22) and (6.16) we find the following useful relations

\[
\tilde{H}_{r+1} = \tilde{H}_r - \frac{\alpha_r^0}{\alpha_r^{ni+1}} \tilde{H}_{ni+1}, \quad \tilde{H}_1 = -\frac{1}{\alpha_r^{ni+1}} \tilde{H}_{ni+1}, \quad r = 1, ..., n + k, \quad (6.26a)
\]

\[
\alpha_r^{ni+1} = \alpha_r^{ni} - \frac{\alpha_r^0}{\alpha_r^{ni+1}} \alpha_r^{ni+1}, \quad r = 1, ..., n + k, \quad (6.26b)
\]

\[
\alpha_r^{nk+1} = \frac{\alpha_r^0}{\alpha_r^{nk+1}}, \quad (6.26c)
\]
\( \pi_{r+1} = \alpha_{r+1}^0 + \alpha_r^0 \sum_{i=1}^k \alpha_{n_{k+1}-1}^{n_i} \alpha_{n_{i+1}}^0 - \frac{\alpha_r^0}{\alpha_{n_{k+1}-1}^{n_i}} \alpha_{n_{k+1}}^0 - \sum_{i=1}^k \alpha_{n_i}^{n_i} \alpha_{n_{i+1}}^0 \) \hspace{1cm} (6.26d)

where \( \tilde{H}_{n_i} = \alpha_{n_i}^0 = 0, i = 1, ..., k, \alpha_{n_i}^{n_i} = \delta_{ij} \). Expressing \( \tilde{H}, \pi \) by \( \tilde{H}, \alpha \) from (6.26a)-(6.26d) and using assumptions (6.25a), (6.25b) we find, after long but straightforward calculations, that

\[
N^* d\Pi_r + \alpha_r^0 d\Pi_1 + \sum_{i=1}^k \alpha_{n_{k+1}-1}^{n_i+1} d\Pi_{n_i+2} + \alpha_{n_{k+1}-1}^{n_{k+1}+1} d\Pi_{n_{k+1}+1}
\]

\[
= d\tilde{H}_r - \frac{\alpha_{n_{k+1}-1}^{n_{k+1}+1}}{\alpha_{n_{k+1}-1}^{n_{k+1}+1}} d\alpha_r^0 - \alpha_r^0 d \left( \frac{\tilde{H}_{n_{k+1}+1}^{n_{k+1}+1}}{\alpha_{n_{k+1}-1}^{n_{k+1}+1}} \right)
\]

\[
= d \left( \tilde{H}_r - \frac{\alpha_r^0}{\alpha_{n_{k+1}-1}^{n_{k+1}+1}} \tilde{H}_{n_{k+1}+1} \right) = d\Pi_{r+1}.
\]

Using the known relation for the Benenti potentials

\[ K_{r+1} = \sum_{k=0}^r \rho_k L^{r-k}, \]

one finds from (6.25b) that

\[
d\tilde{V}_{r+1} = \left[ (L^*)^r + \alpha_r^0 (L^*)^{r-1} + ... + \alpha_r^0 \right] d\tilde{V}_1 + \left[ \alpha_r^0 (L^*)^{r-1} + ... + \alpha_r^0 \right] d\tilde{V}_{n_{k+1}} + ... + \left[ \alpha_r^0 (L^*)^{r-1} + ... + \alpha_r^0 \right] d\tilde{V}_{n_{k+1}} + ... + \left[ \alpha_r^0 (L^*)^{r-1} + ... + \alpha_r^0 \right] d\tilde{V}_{n_{k+1}} + ...
\]

\[ = A_r^0 d\tilde{V}_1 + A_r^{n_1} d\tilde{V}_{n_{k+1}+1} + ... + A_r^{n_{k}} d\tilde{V}_{n_{k+1}}, \] \hspace{1cm} (6.27)

where

\[
A_r^0 = \frac{1}{\varphi} \begin{vmatrix} K_r & K_{r-1} & \cdots & K_{r-k} \\ \rho_{n_1} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n_k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix},
\]

\[
A_r^{n_i} = (-1)^{i+1} \frac{1}{\varphi} \begin{vmatrix} \rho_{n_1} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n_k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}
\]

with the row \((\rho_{n_1-1}, ..., \rho_{n_k-k})\) missing in \(A_r^{n_i}\) and again \(K_i\) in the calculation of determinants are treated as symbols not matrices.

As in the one-hole case (5.21), the \(d_p\) part of of (5.19a) gives us the analog of formulae (4.5) for the \(k\)-hole case

\[ \tilde{K}_{r+1} = L\tilde{K}_r + \alpha_r^0 I + \sum_{i=1}^k \alpha_r^{n_i} \tilde{K}_{n_i+1}. \] \hspace{1cm} (6.28)

Then, from (6.28) we find that for \(\tilde{L}\) function (5.22) the following relation holds

\[ \tilde{L} = \tilde{E}_2 - \alpha_0^0 \tilde{E}_1 - \sum_{i=1}^k \alpha_r^{n_i} \tilde{E}_{n_i+1}, \] \hspace{1cm} (6.29)
\[ \{ \tilde{L}, \tilde{E}_1 \}_{\theta_0} = \{ \tilde{E}_1, \alpha_1^0 \}_{\theta_0} \tilde{E}_1 + \sum_{i=1}^{k} \{ \tilde{E}_1, \alpha_i^1 \}_{\theta_0} \tilde{E}_{n_i+1} \]

\[ = \kappa^0 \tilde{E}_1 + \sum_{i=1}^{k} \kappa^i \tilde{E}_{n_i+1}. \quad (6.30) \]

so obviously, \( L \) is not a conformal Killing tensor for \( \tilde{G} \) given by (6.8).

Notice, that although the number \( k \) can be arbitrary large, nevertheless, the maximal number of nonvanishing terms \( \alpha_r^{\nu_i} d\tilde{H}_{n_i+1} \) in (6.23) is lower or equal to \( n \). In fact, if \( n_i \) and \( n_{i+1} \) are two successive numbers, i.e. \( n_{i+1} = n_i + 1 \), then \( \alpha_r^{\nu_i} d\tilde{H}_{n_i+1} = \alpha_r^{\nu_i} d\tilde{H}_{n_i+1} = 0 \), as from construction \( \tilde{H}_{n_i+1} = 0 \). Hence, for a string of successive numbers \( n_i - s_i + 1, n_i - s_i + 2, \ldots, n_i \), only the term with \( n_i \) is nonzero in formula (6.23), as

\[ \tilde{H}_{n_i-s_i+1} = \tilde{H}_{n_i-s_i+2} = \ldots = \tilde{H}_{n_i} = 0. \]

Thus, assume that in the sequence

\[ \tilde{H}_1 \xi^{(n+k)-1} + \tilde{H}_2 \xi^{(n+k)-2} + \ldots + \tilde{H}_{n+k} \]

we have \( l \) strings of holes, where the \( i \)-th string has \( s_i \) holes and \( s_1 + \ldots + s_l = k \). Then, the quasi-bi-Hamiltonian chain (6.23) takes the form

\[ d\tilde{H}_{r+1} = N^r d\tilde{H}_r + \alpha_r^{\nu_0} d\tilde{H}_1 + \alpha_r^{\nu_1} d\tilde{H}_{n_1+1} + \ldots + \alpha_r^{\nu_l} d\tilde{H}_{n_l+1}, \quad (6.31) \]

where \( n_0 = 0 \).

6.6 Gel’fand-Zakharevich bi-Hamiltonian representation

Now we lift our quasi-bi-Hamiltonian representation, constructed in a previous subsection, into a GZ form. Adding \( l+1 \) Casimir coordinates \( c_i \) (with respect to \( \theta_0 \)) and extending Hamiltonians \( \tilde{H}_r \) to the form affine in \( c_i \)

\[ \tilde{h}_r = \tilde{H}_r + \sum_{i=1}^{l+1} \alpha_r^{\nu_i-1} c_i, \quad (6.32) \]

one can transform the quasi-bi-Hamiltonian chain on \( T^*Q \) into a bi-Hamiltonian chain on \( T^*Q \times \mathbb{R} \times \ldots \times \mathbb{R} \), being a composition of \( l+1 \) bi-Hamiltonian sub-chains

\[ \tilde{\pi}_0 d\tilde{h}_{n_i} = 0 \]

\[ \tilde{\pi}_0 d\tilde{h}_{n_{i+1}} = \tilde{\pi}_1 d\tilde{h}_{n_i} \]

\[ \vdots \]

\[ \tilde{\pi}_0 d\tilde{h}_{n_{i+1}-s_{i+1}} = \tilde{\pi}_1 d\tilde{h}_{n_{i+1}-s_{i+1}-1} \]

\[ 0 = \tilde{\pi}_1 d\tilde{h}_{n_{i+1}-s_{i+1}}. \quad (6.33) \]
where $i = 0, 1, ..., l$, $\tilde{h}_0 = c_1$, $\tilde{h}_{n_i} = c_{i+1}$, $\tilde{h}_{n_{i+1} - s_{i+1}} = \tilde{h}_{n_{i+1}} = \tilde{h}_{n+k}$ and

$$\tilde{\pi}_0 = \begin{pmatrix} \theta_0 & 0 & 0 & \cdots & 0 \\ 0 & \theta_1 & 0 & \cdots & 0 \\ 0 & \cdots & \theta_1 & \cdots & 0 \\ 0 & \cdots & \cdots & \theta_1 \end{pmatrix},$$

$$\tilde{\pi}_1 = \begin{pmatrix} \theta_1 \theta_0 \tilde{h}_1 & \theta_0 \tilde{h}_{n_1+1} & \cdots & \theta_0 \tilde{h}_{n_{l+1}} \\ -(\theta_0 \tilde{h}_1)^T & -(\theta_0 \tilde{h}_{n_1+1})^T & & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -(\theta_0 \tilde{h}_{n_{l+1}})^T \end{pmatrix}.$$

Again $\tilde{\pi}_0$ and $\tilde{\pi}_1$ are compatible Poisson structures as, according to Lemma 7, $\tilde{\pi}_1$ takes the form

$$\tilde{\pi}_1 = \tilde{\pi}_1D + \sum_{i=1}^{l+1} X_i \wedge Z_i,$$

where $Z_i = \frac{\partial}{\partial c_i}, X_i = \tilde{\pi}_0 \tilde{h}_{n_{i+1}}, L_{\tilde{Z}} \tilde{\pi}_0 = 0$. The reduction of $l + 1$ chains onto the symplectic leaf $c_1 = ... = c_{l+1} = 0$ of $\tilde{\pi}_0$ along the distribution $Z = (\frac{\partial}{\partial c_1}, ..., \frac{\partial}{\partial c_{l+1}})$ reconstructs immediately the quasi-bi-Hamiltonian chain (6.23).

If we introduce the Poisson pencil $\tilde{\pi}_\xi = \tilde{\pi}_1 - \xi \tilde{\pi}_0$, $l + 1$ chains (6.33) can be written in a compact form

$$\tilde{\pi}_\xi d\tilde{h}(\xi) = 0, \quad \tilde{h}(\xi) = (\tilde{h}^{(1)}(\xi), \tilde{h}^{(2)}(\xi), ..., \tilde{h}^{(l+1)}(\xi))^T,$$

where

$$\tilde{h}^{(i)}(\xi) = \tilde{h}_{n_i} \xi^{n_{i+1} - n_i - s_{i+1}} + ... + \tilde{h}_{n_{i+1} - s_{i+1}} \quad i = 1, 2, ..., l + 1.$$

The quasi-bi-Hamiltonian chain (5.17) takes the form

$$\tilde{\theta}_\xi d\tilde{H}(\xi) + \tilde{\alpha}(\xi) \tilde{\theta}_0 \tilde{H}(1)(q,p) = 0$$

$$\uparrow$$

$$(N^* - \xi I) d\tilde{H}(\xi) + \tilde{\alpha}(\xi) d\tilde{H}(1)(q,p) = 0,$$

where

$$\tilde{H}^{(i)}(\xi) = \tilde{H}_{n_i+1} \xi^{n_{i+1} - n_i - s_{i+1} + 1} + ... + \tilde{H}_{n_{i+1} - s_{i+1}} \quad i = 1, 2, ..., l + 1,$$

$$\tilde{H}(1) = (\tilde{H}_1, \tilde{H}_{n_1+1}, ..., \tilde{H}_{n_{l+1}})^T$$

and

$$\tilde{\alpha}(\xi)_j^i = \frac{\partial \tilde{h}^{(j)}(\xi)}{\partial c_i}, \quad i, j = 1, ..., l + 1.$$
The Stäckel separation conditions for extended Hamiltonians \( \tilde{h}_r \) take the form (3.18b)

\[
c_1 \xi^{(n+k)} + \tilde{h}_1 \xi^{(n+k)-1} + \ldots + \tilde{h}_{n_1-s_1-1} \xi^{(n+k)-(n_1-s_1-1)} + c_2 \xi^{(n+k)-n_1} + \tilde{h}_{n_1+1} \xi^{(n+k)-(n_1+1)} + \ldots + \tilde{h}_{n_1+1} \\
= \sum_{i=1}^{l-1} \xi^{(n+k)-n_{i+1}-s_{i+1}} \tilde{h}_i^{(i)}(\xi) = \frac{1}{2} f(\xi)\mu^2 + \gamma(\xi), \quad (\xi, \mu) = (\lambda_i, \mu_i), \quad i = 1, \ldots, n.
\]

(6.36)

Remark 19 Systems considered in this paper, although obtained through the deformation procedure on the level of Hamiltonian functions, are far from being trivial generalizations of Benenti systems. There is no obvious relations between solutions of a given Benenti system and all its deformations. In each case we have a different inverse Jacobi problem to solve. Notice, that the common feature of appropriate deformed systems is the same set of separated coordinates, determined by the related Benenti system.

7 Examples

7.1 Henon-Heiles system

Let us consider the integrable case of the Henon-Heiles system

\[
(q^1)_{tt} = -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c \\
(q^2)_{tt} = -q^1 q^2,
\]

(7.1)

with the corresponding Lagrangian

\[
L = \frac{1}{2}(q^1)_t^2 + \frac{1}{2}(q^2)_t^2 - (q^1)^3 - \frac{1}{2} q^1 (q^2)^2 + cq^1.
\]

(7.1a)

The bi-Hamiltonian chain is of the following form [42]

\[
\pi_0 dh_0 = 0 \\
\pi_0 dh_1 = X_1 = \pi_1 dh_0 \\
\pi_0 dh_2 = X_2 = \pi_1 dh_1 \\
0 = \pi_1 dh_2,
\]

(7.2)

where

\[
h_0 = c, \\
h_1 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + (q^1)^3 + \frac{1}{2} q^1 (q^2)^2 - cq^1 \\
= E_1 + V_1(q) + \rho_1(q) c = H_1 + \rho_1(q)c, \\
h_2 = \frac{1}{2} q^2 p_1 p_2 - \frac{1}{2} q^1 p_2^2 + \frac{1}{16} (q^2)^4 + \frac{1}{4} (q^1)^2 (q^2)^2 - \frac{1}{4} c(q^2)^2 \\
= E_2 + V_2(q) + \rho_2(q) c = H_2 + \rho_2(q)c,
\]

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Notice that
\[ G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & -q^1 \end{pmatrix}. \]

The quasi-bi-Hamiltonian chain is
\[ dH_{r+1} = N^*dH_r + \rho_r dH_1, \quad \rho_r = \frac{\partial h_r}{\partial c}, \quad r = 1, 2, \]
where \( N^* = \theta_0^{-1}\theta_1. \)

The transformation to separated coordinates \((\lambda, \mu)\) takes the form
\[ q^1 = \lambda^1 + \lambda^2, \quad q^2 = 2\sqrt{-\lambda^1\lambda^2}, \]
\[ p_1 = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \quad p_2 = \sqrt{-\lambda^1\lambda^2} \left( \frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1} \right), \]
and the separability conditions for \(H_i\) respectively \(h_i\) are reconstructed from separation curves
\[ H_1 \xi + H_2 = \frac{1}{2} \xi \mu^2 + \xi^4, \]
\[ c\xi^2 + h_1 \xi + h_2 = \frac{1}{2} \xi \mu^2 + \xi^4. \]

**1-hole deformation of the Henon-Heiles.**

The only possibility is \( n_1 = 2. \) Then,
\[ \tilde{G} = -\frac{1}{\rho_1}G = \begin{pmatrix} \frac{1}{q} & 0 \\ 0 & \frac{1}{q} \end{pmatrix}, \quad \tilde{K}_2 = 0, \quad \tilde{K}_3 = -K_2^2 = \begin{pmatrix} -\frac{1}{4}(q^2)^2 & \frac{1}{2}q^1q^2 \\ \frac{1}{2}q^1q^2 & -\frac{1}{4}(q^2)^2 - (q^1)^2 \end{pmatrix}, \]
\[ \tilde{V}_1 = -\frac{1}{\rho_1}V_1, \quad \tilde{V}_2 = 0, \quad \tilde{V}_3 = V_2 - \frac{\rho_2}{\rho_1}V_1, \]
and
\[ \alpha_r = \rho_r - \rho_{r-1}\rho_2\rho_1^{-1}, \quad \beta_r = \rho_{r-1}\rho_1^{-1}. \]

The quasi-bi-Hamiltonian chain takes the form
\[ d\tilde{H}_{r+1} = N^*d\tilde{H}_r + \alpha_r d\tilde{H}_1 + \beta_r d\tilde{H}_{n_1+1}, \quad r = 1, 2, 3. \]
while two bi-Hamiltonian sub-chains are
\[ \tilde{\pi}_0 dh_0 = 0 \quad \tilde{\pi}_0 dh_2 = 0 \]
\[ \tilde{\pi}_0 dh_1 = \tilde{X}_1 = \tilde{\pi}_1 dh_0 \quad \tilde{\pi}_0 dh_3 = \tilde{X}_3 = \tilde{\pi}_1 dh_2 \]
\[ 0 = \tilde{\pi}_1 dh_1 \quad 0 = \tilde{\pi}_1 dh_3, \]

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where
\[
\begin{align*}
\tilde{h}_0 &= c_1, \\
\tilde{h}_1 &= \frac{1}{2} \frac{1}{q^1} p^2_1 + \frac{1}{2} \frac{1}{q^1} p^2_2 + (q^1)^2 + \frac{1}{2} (q^2)^2 - c_1 \left[ q_1 + \frac{1}{4} q^1 (q^2)^2 \right] - c_2 \frac{1}{q^1} \\
\tilde{h}_2 &= c_2, \\
\tilde{h}_3 &= -\frac{1}{8} \frac{2}{q^1} p^2_1 + \frac{1}{2} q^2 p_1 p_2 - \frac{1}{16} (q^2)^4 + \frac{1}{16} c_1 \frac{(q^2)^4}{q^1} + \frac{1}{4} c_2 \frac{(q^2)^2}{q^1}
\end{align*}
\]
and
\[
\tilde{\pi}_0 = \begin{pmatrix} \theta_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\pi}_1 = \begin{pmatrix} \theta_1 & \theta_0 \tilde{h}_1 & \theta_0 \tilde{h}_3 \\ - (\theta_0 \tilde{h}_1)^T & 0 \\ - (\theta_0 \tilde{h}_3)^T \end{pmatrix}.
\]

The transformation to separated coordinates is given by (7.4) and the separability conditions for \( H_i \) respectively \( h_i \) are represented by separation curves
\[
\tilde{H}_1 \xi^2 + \tilde{H}_3 = \frac{1}{2} \xi \mu^2 + \xi^4, \quad (7.8a)
\]
\[
c_1 \xi^3 + \tilde{h}_1 \xi^2 + c_2 \xi + \tilde{h}_3 = \frac{1}{2} \xi \mu^2 + \xi^4. \quad (7.8b)
\]
The related Euler-Lagrange equations (3.2) are
\[
q^1 q^2_1 + \frac{1}{2} (q^1)^2 - \frac{1}{2} (q^2)^2 = -2q^1 + \left[ 1 - \frac{1}{4} (q^1)^{-2} (q^2)^2 \right] c_1 - (q^1)^{-2} c_2, \quad (7.9)
\]
\[
q^1 q^2_1 + q_1^3 q^2 = -q^2 + \frac{1}{2} (q^1)^{-1} q^2 c_1.
\]

**2-hole deformation of the Henon-Heiles.**

The only possibility is 2-hole string \( n_1 = 2, n_2 = n_1 + 1 = 3 \). Then,
\[
\tilde{G} = \frac{1}{\rho^1_2 - \rho_2} G = \begin{pmatrix} 4 \frac{(q^1)^2 + q^2}{(q^1)^2 + q^2} & 0 \\ 0 & \frac{4}{(q^1)^2 + q^2} \end{pmatrix}, \quad \tilde{K}_2 = \tilde{K}_3 = 0,
\]
\[
\tilde{K}_4 = K_2^3 = \begin{pmatrix} \frac{-1}{3} q_1 (q^2)^2 & \frac{1}{8} (q^2)^3 + \frac{1}{2} (q^1)^2 + q^2 \\ \frac{1}{4} (q^2)^3 + \frac{1}{2} (q^1)^2 + q^2 & \frac{-1}{3} q_1 (q^2)^2 - (q^1)^3 \end{pmatrix},
\]
\[
\tilde{V}_1 = -\frac{1}{\rho^1_2 - \rho_2} V_1, \quad \tilde{V}_2 = \tilde{V}_3 = 0, \quad \tilde{V}_4 = V_2 - \frac{\rho_1 \rho_2}{\rho^1_2 - \rho_2} V_1,
\]
\[
\alpha_1 = \rho_1 - \frac{\rho_2 \rho_1}{\rho^1_2 - \rho_2}, \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = \frac{\rho^3_2}{\rho^1_2 - \rho_2},
\]
\[
\beta_1 = -\frac{1}{\rho^1_2 - \rho_2}, \quad \beta_2 = \beta_3 = 0, \quad \beta_4 = \frac{\rho_2 \rho_1}{\rho^1_2 - \rho_2}.
\]
and two bi-Hamiltonian chain are

\[ \tilde{\pi}_0 \tilde{d}h_0 = 0 \]
\[ \tilde{\pi}_0 \tilde{d}h_1 = \tilde{X}_1 = \tilde{\pi}_1 \tilde{d}h_0 \]
\[ 0 = \tilde{\pi}_1 \tilde{d}h_1 \]
\[ \tilde{\pi}_0 \tilde{d}h_3 = 0 \]
\[ \tilde{\pi}_0 \tilde{d}h_4 = \tilde{X}_4 = \tilde{\pi}_1 \tilde{d}h_3 \]
\[ 0 = \tilde{\pi}_1 \tilde{d}h_4 \tag{7.10} \]

where

\[ \tilde{\pi}_0 = \begin{pmatrix} \theta_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\pi}_1 = \begin{pmatrix} \theta_1 & \theta_0 \tilde{d}h_1 & \theta_0 \tilde{d}h_4 \\ -\theta_0 \tilde{d}h_1^T & -\theta_0 \tilde{d}h_4^T & 0 \end{pmatrix} \]

The transformation to separated coordinates is given by (7.4) and the separation curves for \( H_i \) respectively \( h_i \) are

\[ \tilde{H}_1 \xi^3 + \tilde{H}_4 = \frac{1}{2} \xi \mu^2 + \xi^4, \tag{7.11a} \]
\[ c_1 \xi^4 + \tilde{h}_1 \xi^3 + c_2 \xi + \tilde{h}_4 = \frac{1}{2} \xi \mu^2 + \xi^4. \tag{7.11b} \]

### 7.2 7-th order stationary KdV

Let us consider the so-called first Newton representation of the seventh-order stationary flow of the KdV hierarchy \([43],[22]\). It is a Lagrangian system of second order Newton equations

\[ q_1' = -10(q^1)^2 + 4q^2 \]
\[ q_2' = -16q^1q^2 + 10(q^1)^3 + 4q^3 \]
\[ q_3' = -20q^1q^3 - 8(q^2)^2 + 30(q^1)^2q^2 - 15(q^1)^4 + c \]
\[ \tag{7.12} \]

with the corresponding Lagrangian

\[ \mathcal{L} = q_1^3 q_1^3 + \frac{1}{2} (q_2^1)^2 + 4q_2^2q_3^3 - 10(q^1)^2q^3 - 8q^1(q^2)^2 + 10(q^1)^3q^2 - 3(q^1)^5 + cq^1, \tag{7.12a} \]

so that it can be cast in a Hamiltonian form. In fact, the above system can be represented as a bi-Hamiltonian vector field \( X_1 \) belonging to a bi-Hamiltonian chain of the form

\[ \pi_0 \tilde{d}h_0 = 0 \]
\[ \pi_0 \tilde{d}h_1 = X_1 = \pi_1 \tilde{d}h_0 \]
\[ \pi_0 \tilde{d}h_2 = X_2 = \pi_1 \tilde{d}h_1 \]
\[ \pi_0 \tilde{d}h_3 = X_3 = \pi_1 \tilde{d}h_2 \]
\[ 0 = \pi_1 \tilde{d}h_3, \tag{7.13} \]
\[ h_0 = c \]

\[ h_1 = p_1 p_3 + \frac{1}{2} p_2^2 + 10(q^1)^2 q^3 - 4q^2 q^3 + 8q^1(q^2)^2 - 10(q^1)^3 q^2 + 3(q^1)^5 - cq^1 \]

\[ = E_1 + V_1(q) + \rho_1(q) c = H_1 + \rho_1(q)c \]

\[ h_2 = \frac{\pi^1}{8}(q^2)^2 p_3^2 - \frac{1}{8} q^1 p_2^2 + \frac{1}{8} p_1 p_2 - \frac{1}{2} q^1 p_1 p_3 + 2(q^1)^2(q^2)^2 + \frac{5}{2}(q^1)^4 q^2 \\
- \frac{5}{4}(q^1)^6 - 2(q^2)^3 + (q^3)^2 - 6q^1 q^2 q^3 + \frac{1}{4} c(q^1)^2 + \frac{1}{4} c q^2 \\
= E_2 + V_2(q) + \rho_2(q) c = H_2 + \rho_2(q)c \]

\[ h_3 = \frac{1}{8}(q^2)^2 p_3^2 + \frac{1}{8}(q^1)^2 p_2^2 + \frac{1}{8} p_1 p_2 + \frac{1}{4} q^1 p_1 p_2 + \frac{1}{2} q^2 p_1 p_3 - \frac{1}{4} q^1 q^2 p_2 p_3 \\
- \frac{1}{2} q^3 p_2 p_3 - 3(q^1)^3(q^2)^2 + q^1(q^2)^3 + \frac{5}{2}(q^1)^5 q^2 + 2q^1(q^3)^2 \\
+ \frac{5}{4}(q^1)^4 q^3 + (q^2)^2 q^3 - (q^1)^2 q^2 q^3 - \frac{1}{4} c q^1 q^2 - \frac{1}{4} c q^3 \\
= E_3 + V_3(q) + \rho_3(q) c = H_3 + \rho_3(q)c \]

with the corresponding operators \( \pi_0 \) and \( \pi_1 \)

\[ \Pi_0 = \begin{pmatrix} \theta_0 & 0_{6 \times 1} \\ 0_{1 \times 6} & 0 \end{pmatrix} , \quad \theta_0 = \begin{pmatrix} 0_3 & I_3 \\ -I_3 & 0_3 \end{pmatrix} \]

\[ \Pi_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & -q^2 & q^2 & 0 \\ 0 & 0 & 0 & 2q^3 & q^2 & q^1 \\ -q^1 & -q^2 & -2q^3 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{pmatrix} \begin{bmatrix} 2X_1 \\ \cdot \end{bmatrix} \]

and \( X_1 = \theta_0 dh_1, \quad d = \left( \frac{\partial}{\partial q^1}, \frac{\partial}{\partial p_2} \right)^T \), so that the operator \( \theta_1 \) in the corresponding quasi-bi-Hamiltonian chain is of the form

\[ \theta_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & q^1 & -1 & 0 \\ 0 & 0 & 0 & -q^2 & q^2 & 0 \\ 0 & 0 & 0 & 2q^3 & q^2 & q^1 \\ -q^1 & -q^2 & -2q^3 & 0 & p_2 & p_3 \\ 1 & 0 & -q^2 & -p_2 & 0 & 0 \\ 0 & 1 & -q^1 & -p_3 & 0 & 0 \end{pmatrix} . \]

From the form of \( h_1 \) one can directly see that the inverse metric tensor \( G \) expressed in \((q,p)\) variables has in this example an anti-diagonal form

\[ G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]
while the conformal Killing tensor $L$ has the form

$$L = \frac{1}{2} \begin{pmatrix} q^1 & -1 & 0 \\ q^2 & 0 & -1 \\ 2q^3 & q^2 & q^1 \end{pmatrix}$$

and hence $K_1 = I$, $A_1 = G$,

$$K_2 = \frac{1}{2} \begin{pmatrix} -q^1 & -1 & 0 \\ q^2 & -2q^1 & -1 \\ 2q^3 & q^2 & -q^1 \end{pmatrix}, \quad K_3 = \frac{1}{4} \begin{pmatrix} q^2 & q^1 & 1 \\ -q^1 q^2 - 2q^3 & (q^1)^2 & 0 \\ (q^2)^2 & -q^1 q^2 - 2q^3 & q^2 \end{pmatrix},$$

$$A_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 & q^1 \\ -1 & -2q^1 & q^2 \\ -q^1 & q^2 & 2q^3 \end{pmatrix}, \quad A_3 = \frac{1}{4} \begin{pmatrix} 1 & q^1 & 0 \\ q^1 & (q^1)^2 & q^2 \\ q^2 & -q^1 q^2 - 2q^3 & (q^2)^2 \end{pmatrix}.$$

The quasi-bi-Hamiltonian chain is given by (7.3) with $r = 1, 2, 3$, where

$$\rho_1 = -q^1, \quad \rho_2 = \frac{1}{4} (q^1)^2 + \frac{1}{2} q^2, \quad \rho_3 = \frac{1}{4} q^1 q^2 - \frac{1}{4} q^3.$$

The transformation $(\lambda, \mu) \to (q, p)$ is constructed from the relations

$$q^1 = \lambda^1 + \lambda^2 + \lambda^3, \quad \frac{1}{4} (q^1)^2 + \frac{1}{2} q^2 = \lambda^1 \lambda^2 + \lambda^1 \lambda^3 + \lambda^2 \lambda^3, \quad \frac{1}{4} q^1 q^2 + \frac{1}{4} q^3 = \lambda^1 \lambda^2 \lambda^3,$$

(the explicit formulas are given in [22]) and the separation curves for $H_i$ respectively $h_i$ are

$$H_1 \xi^2 + H_2 \xi + H_3 = \frac{1}{8} \mu^2 + 16 \xi^7, \quad (7.14a)$$

$$c \xi^3 + h_1 \xi^2 + h_2 \xi + h_3 = \frac{1}{8} \mu^2 + 16 \xi^7. \quad (7.14b)$$

### 1-hole deformation.

There are two admissible cases of one-hole deformation: $n_1 = 2, 3$. Here we illustrate the case $n_1 = 3$. The deformed Hamiltonians are the following

$$\tilde{H}_1 = -\frac{1}{\rho_2} H_2 = \frac{1}{2} p^T \tilde{G} p + \tilde{V}_1,$$

$$\tilde{H}_2 = H_1 - \frac{\rho_1}{\rho_2} H_2 = \frac{1}{2} p^T \tilde{A}_2 p + \tilde{V}_2,$$

$$\tilde{H}_3 = 0,$$

$$\tilde{H}_4 = H_3 - \frac{\rho_3}{\rho_2} H_2 = \frac{1}{2} p^T \tilde{A}_4 p + \tilde{V}_4,$$

where

$$\tilde{G} = -\frac{1}{\rho_2} K_2 G = \frac{2}{(q^1)^2 + 2q^2} \begin{pmatrix} 0 & 1 & q^1 \\ 1 & \frac{1}{2} q^1 & -q^2 \\ q^1 & -q^2 & -\frac{1}{2} q^3 \end{pmatrix}, \quad \tilde{A}_1 = \tilde{G},$$

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\[ \tilde{A}_2 = -\frac{1}{\rho_2} (K^2_2 - K_1 K_3) G \]

\[ \tilde{A}_3 = 0, \quad \tilde{A}_4 = \frac{1}{\rho_2} K_3^2 G = \frac{1}{4} \frac{1}{(q^1)^2 + 2q^2} \]

\[ \begin{pmatrix}
(q^1)^2 + 2q^2 & (q^1)^3 - 2q^3 \\
* & q^1 [(q^1)^3 - 2q^1 q^2 - 4q^3]
\end{pmatrix} \begin{pmatrix}
2(q^2)^2 - (q^1)^2 q^2 - 2q^1 q^3 \\
-(q^1)^3 q^2 - 2(q^1)^2 q^3 - 2q^2 q^3 \\
2(q^2)^3 + (q^1)^2 q^2 + 4q^1 q^2 q^3 + 4(q^3)^2
\end{pmatrix} \]

the respective potentials are

\[ \tilde{V}_1 = \frac{5(q^1)^6 - 8(q^1)^2 (q^2)^2 - 10(q^1)^4 q^2 + 8(q^2)^3 - 4(q^3)^2 + 24q^1 q^2 q^3}{(q^1)^2 + 2q^2}, \]

\[ \tilde{V}_2 = \frac{[(q^1)^7 - 3(q^1)^5 q^2 - 5(q^1)^4 q^3 + 2(q^1)^3 (q^2)^2 + 4(q^1)^2 q^2 q^3 - 4q^1 (q^2)^3 - 2(q^1)^3 q^2 + 4(q^2)^2 q^3][(q^1)^2 + 2q^2] + 24q^1 q^2 q^3}{(q^1)^2 + 2q^2}, \]

\[ \tilde{V}_3 = 0, \]

\[ \tilde{V}_4 = \frac{[2(q^1)^5 (q^2)^2 - 3(q^1)^3 (q^2)^3 + 2(q^1)^3 (q^3)^2 - q^1 q^2 (q^3)^2 + 4(q^1)^4 q^2 q^3 - 5(q^1)^2 (q^2)^2 q^3 + (q^3)^3][(q^1)^2 + 2q^2] + 24q^1 q^2 q^3}{(q^1)^2 + 2q^2}, \]

and (*) are elements making matrix symmetric. The quasi-bi-Hamiltonian chain takes the form (7.6) with \( r = 1, \ldots, 4 \)

\[ \alpha_r = \rho_r - \frac{\rho_{r-1} \rho_2}{\rho_2}, \quad \beta_r = \frac{\rho_{r-1}}{\rho_2}, \]

while the related bi-Hamiltonian chain splits onto two sub-chains of the form

\[ \begin{align*}
\tilde{\pi}_0 \tilde{d} \tilde{h}_0 &= 0 \\
\tilde{\pi}_0 \tilde{d} \tilde{h}_1 &= \tilde{X}_1 = \tilde{\pi}_1 \tilde{d} \tilde{h}_0 \\
\tilde{\pi}_0 \tilde{d} \tilde{h}_2 &= \tilde{X}_2 = \tilde{\pi}_1 \tilde{d} \tilde{h}_1 \\
\tilde{\pi}_0 \tilde{d} \tilde{h}_3 &= 0 \\
\tilde{\pi}_0 \tilde{d} \tilde{h}_4 &= \tilde{X}_4 = \tilde{\pi}_1 \tilde{d} \tilde{h}_3 \\
0 &= \tilde{\pi}_1 \tilde{d} \tilde{h}_4,
\end{align*} \] (7.16)

where

\[ \tilde{h}_0 = c_1, \quad \tilde{h}_3 = c_2, \quad \tilde{h}_i = \frac{1}{2} p^T \tilde{A}_i p + \tilde{V}_i + \alpha_i c_1 + \beta_i c_2, \quad i = 1, 2, 4. \]

Separation curves for respective Hamiltonian functions are

\[ \tilde{H}_1 \xi^3 + \tilde{H}_2 \xi^2 + \tilde{H}_4 = \frac{1}{8} \mu^2 + 16 \xi^7, \]

\[ c_1 \xi^4 + \tilde{h}_1 \xi^3 + \tilde{h}_2 \xi^2 + c_2 \xi + \tilde{h}_4 = \frac{1}{8} \mu^2 + 16 \xi^7. \]
2-hole deformation.

There are three admissible cases of 2-hole deformation. One related to 3-Casimir Poisson pencil: \( n_1 = 2, n_2 = 4 \) and two cases related to 2-Casimir Poisson pencils, i.e. \( n_1 = 2, n_2 = 3 \) and \( n_1 = 3, n_2 = 4 \). Here we present the first case. The deformed Hamiltonians are the following

\[
\begin{align*}
\tilde{H}_1 &= \frac{1}{\rho_1 \rho_2 - \rho_3} H_2 = \frac{1}{2} p^T \tilde{G} p + \tilde{V}_1, \\
\tilde{H}_2 &= 0, \\
\tilde{H}_3 &= H_1 + \frac{\rho_2 - \rho_1^2}{\rho_1 \rho_2 - \rho_3} H_2 = \frac{1}{2} p^T \tilde{A}_3 p + \tilde{V}_3, \\
\tilde{H}_4 &= 0, \\
\tilde{H}_5 &= H_3 + \frac{\rho_1 \rho_3}{\rho_1 \rho_2 - \rho_3} H_2 = \frac{1}{2} p^T \tilde{A}_5 p + \tilde{V}_5,
\end{align*}
\]

where

\[
\tilde{A}_1 = \tilde{G} = \frac{1}{\rho_1 \rho_2 - \rho_3} K_1 K_2 G = \frac{2}{q_1^3 - q_1 q_2 + q_3} \begin{pmatrix} 0 & 1 & q_1^4 \\ 1 & \frac{1}{2} q_1^4 & -q_2^2 \\ q_1^4 & -q_2^2 & -\frac{1}{2} q_3^4 \end{pmatrix},
\]

\( \tilde{A}_2 = 0, \quad \tilde{A}_3 = \frac{1}{\rho_1 \rho_2 - \rho_3} K_2 (K_3 - K_2^2) G, \quad \tilde{A}_4 = 0, \quad \tilde{A}_5 = \frac{1}{\rho_1 \rho_2 - \rho_3} K_2 K_3^2 G, \)

\[
\begin{align*}
\tilde{V}_1 &= \frac{4}{q_3 - q_1 q_2 - q_1^3} V_2, \quad \tilde{V}_2 = 0, \quad \tilde{V}_3 = V_1 + \frac{2q_2 - 3q_1^2}{q_3 - q_1 q_2 - q_1^3} V_2, \\
\tilde{V}_4 &= 0, \quad \tilde{V}_5 = V_3 + \frac{q_1 (q_1 q_2 + q_3)}{q_3 - q_1 q_2 - q_1^3} V_2.
\end{align*}
\]

The quasi-bi-Hamiltonian chain takes the form

\[
d\tilde{H}_{r+1} = N^* d\tilde{H}_r + \alpha_0^r d\tilde{H}_1 + \alpha_2^r d\tilde{H}_3 + \alpha_4^r d\tilde{H}_5, \quad r = 1, ..., 5,
\]

\[
\alpha_0^r = \rho_r + \frac{\rho_r - 2 \rho_2 \rho_3 - \rho_r - 1 \rho_3^2}{\rho_1 \rho_2 - \rho_3}, \quad \alpha_2^r = \frac{\rho_r - 1 \rho_2 - \rho_r - 2 \rho_3}{\rho_1 \rho_2 - \rho_3}, \quad \alpha_4^r = \frac{\rho_r - 2 \rho_3 - \rho_r - 1}{\rho_1 \rho_2 - \rho_3}
\]

and the respective bi-Hamiltonian chain splits into three sub-chains

\[
\begin{align*}
\tilde{\pi}_0 d\tilde{h}_0 &= 0, & \tilde{\pi}_0 d\tilde{h}_2 &= 0, & \tilde{\pi}_0 d\tilde{h}_4 &= 0, \\
\tilde{\pi}_0 d\tilde{h}_1 &= \tilde{X}_1 = \tilde{\pi}_1 d\tilde{h}_0, & \tilde{\pi}_0 d\tilde{h}_3 &= \tilde{X}_3 = \tilde{\pi}_1 d\tilde{h}_2, & \tilde{\pi}_0 d\tilde{h}_5 &= \tilde{X}_5 = \tilde{\pi}_1 d\tilde{h}_4, \\
0 &= \tilde{\pi}_1 d\tilde{h}_1, & 0 &= \tilde{\pi}_1 d\tilde{h}_3, & 0 &= \tilde{\pi}_1 d\tilde{h}_5.
\end{align*}
\]

where

\[
\tilde{\pi}_0 = \begin{pmatrix} \theta_0 & 0 & 0 & 0 \\ 0 & \theta_0 & 0 & 0 \\ 0 & 0 & \theta_0 & 0 \\ 0 & 0 & 0 & \theta_0 \end{pmatrix}, \quad \tilde{\pi}_1 = \begin{pmatrix} \theta_1 & \theta_1 & \theta_1 & \theta_1 \\ -(\theta_0 d\tilde{h}_1)^T & -(\theta_0 d\tilde{h}_3)^T & -(\theta_0 d\tilde{h}_5)^T & 0 \\ -(\theta_0 d\tilde{h}_1)^T & -(\theta_0 d\tilde{h}_3)^T & -(\theta_0 d\tilde{h}_5)^T & 0 \\ -(\theta_0 d\tilde{h}_1)^T & -(\theta_0 d\tilde{h}_3)^T & -(\theta_0 d\tilde{h}_5)^T & 0 \end{pmatrix}.
\]

and

\[
\tilde{h}_0 = c_1, \quad \tilde{h}_2 = c_2, \quad \tilde{h}_4 = c_3, \quad \tilde{h}_i = \tilde{H}_i + \alpha_i^0 c_1 + \alpha_i^2 c_2 + \alpha_i^4 c_3, \quad i = 1, 3, 5.
\]
The respective separation curves are
\[ \bar{H}_1 \xi^4 + \bar{H}_3 \xi^2 + \bar{H}_5 = \frac{1}{8} \mu^2 + 16 \xi^7, \]  
\[ (7.20a) \]
\[ c_1 \xi^5 + \bar{h}_1 \xi^4 + c_2 \xi^3 + \bar{h}_3 \xi^2 + c_3 \xi + \bar{h}_5 = \frac{1}{8} \mu^2 + 16 \xi^7. \]  
\[ (7.20b) \]

8 Summary

We have presented a geometric separability theory of Liouville integrable systems with \( n \) quadratic in momenta constants of motion
\[ H_i(q, p) = p^T A_i(q) p + V_i(q), \quad i = 1, ..., n \]  
\[ (8.1) \]
and with separation curves of polynomial type
\[ H_1 \xi^{m_1} + ... + H_n \xi^{m_n} = \frac{1}{2} f(\xi) \mu^2 + \gamma(\xi), \quad m_n = 0 < m_{n-1} < ... < m_1 \in \mathbb{N}, \]  
\[ (8.2) \]
where \( f(\xi) \) and \( \gamma(\xi) \) are Laurent polynomials of \( \xi \). Such systems can be lifted to the class of Gel’fand-Zakharevich bi-Hamiltonian systems, defined by linear Poisson pencils and their Casimirs: polynomial functions in pencil parameter. First we briefly summarized the geometric separability theory of all GZ bi-Hamiltonian systems. Then, we reviewed with details a special class of such systems introduced by Benenti and characterized by separation curves of the form (8.2) with \( m_k = n - k \). Finally, we proved in a very systematic way that all remaining separable systems, characterized by separation curves of the form (8.2), are constructed by the appropriate deformations of related Benenti systems. The most fundamental and surprising result of the presented paper can be formulated in the following way. If, for a given canonical coordinate system \((q_i, p_i), i = 1, ..., n\), we have a pair of objects, i.e. contravariant metric tensor \( G^{(0)} \) and related special conformal Killing tensor \( L \), then we can construct systematically, in these coordinates, all Liouville integrable and separable Hamiltonian systems (8.1) with respective separation curves of the form (8.2) and explicit form of transformation to separated coordinates. Observe, that according to what was said in Section 4, the separation curve of geodesic motion for \( G^{(0)} \) is the following
\[ E_1 \xi^{n-1} + E_2 \xi^{n-2} + ... + E_n = \frac{1}{2} \mu^2. \]  
\[ (8.3) \]
So, the passage from the system (8.3) to the systems (8.2) is constructive and determined completely by the pair \((G^{(0)}, L)\).

Acknowledgments

The author would like to thanks prof. Franco Magri for many stimulating discussions on the subject. This research was supported in part by KBN grant No. 5P03B 004 20.
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