The aim of the paper is the consideration of particle collisions in the vicinity of the horizon of rotating black holes. Existence of geodesics for massive and massless particles arising from the region inside of the gravitational radius in ergosphere leads to the different possibilities to get very high energy in the centre mass frame of two particles. The classification of all such geodesics on the basis of the proved theorem for extremal spherical orbits is given. Case of the unlimited growth of energy for the situation when one of the particles (the critical one) is moving along the “white hole” geodesic with the close to the upper limit angular momentum while the other particle is on the usual geodesic and the case of the unlimited negative angular momentum of the first particle are considered.

Key words: black holes, Kerr metric, particle collisions, geodesics.

1. Introduction

In the paper of Bañados, Silk and West [1] the existence of the special resonance in the vicinity of the horizon of the rotating black hole with extremal angular momentum was shown leading to the unlimited growth of the energy in the centre of mass frame. The physical reason of the appearance of such a resonance is simple. One of the particles called the critical one is strongly “rotated” in the ergosphere of the rotating black hole so that its radial component is close to zero. The other noncritical particle has the radial...
component close to the velocity of light. So the relative velocity of two particles also is close to the velocity of light and unlimited growing Lorentz factor leads to the unlimited growth of the energy in the centre of mass frame.

In our papers [2]–[4] the similar effect was considered for the nonextremal Kerr’s black holes due to the opinion that extremal black holes are absent in Nature due to gravitational radiation of such objects [5]. It was shown that critical particles can arise due to multiple collisions when noncritical particle arriving to the horizon from the external space after collision with another noncritical particle acquires the critical angular momentum after the intermediate collision.

At last in our papers [6, 7] it was shown that growing without limit high energies in the centre of mass frame in collisions of two particles can occur at any point of the ergosphere if one of the particles has the large negative angular momentum. However there was a problem to get such a large momentum.

A new phenomenon studied by us in [8] is the role of white hole geodesics in collisions of particles in the ergosphere of the black hole. As it is known from the text books [9] the geodesic completeness leads necessarily to the existence together with usual geodesics arriving from the external space the geodesics coming from the inside of the gravitational radius called by us the “white hole geodesics”. It occurs that all geodesics with negative energies relative to the infinity [10] playing important role in the Penrose process [11] of getting the energy from the rotating black hole belong to this class.

The present paper is the continuation of [8]. Differently from [8] where only the equatorial geodesics were considered here we prove the theorem leading to the classification of all such lines in the general case. Massless particles on white hole geodesics are also considered. The unlimited growth of the energy of collision in the centre of mass frame is considered for the case when one of the colliding particles is moving along the white hole geodesic. It is shown that both effects considered in our previous papers can occur without additional hypotheses on getting the critical momentum or large in absolute value angular momentum.

The system of units $\hbar = c = 1$ is used.
2. The white hole geodesics in Kerr’s metric

The Kerr’s metric [12] in Boyer-Lindquist coordinates [13] has the form:

$$ds^2 = dt^2 - \frac{2Mr}{\rho^2} (dt - a \sin^2 \theta \, d\varphi)^2 - \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - (r^2 + a^2) \sin^2 \theta \, d\varphi^2,$$

(1)

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2,$$

(2)

$M$ is the black hole mass, $aM$ its angular momentum. Let $0 \leq a \leq M$. The event horizon of the Kerr’s black hole corresponds to the coordinate

$$r = r_H \equiv M + \sqrt{M^2 - a^2}. \quad (3)$$

The static limit surface is defined by the value

$$r = r_1 \equiv M + \sqrt{M^2 - a^2} \cos^2 \theta. \quad (4)$$

The region of space-time between the static limit and the event horizon is called ergosphere.

The geodesic equations in Kerr’s metric [11] have the form (see [14], Sec. 62 or [15], Sec. 3.4.1)

$$\rho^2 \frac{dt}{d\lambda} = -a \left( aE \sin^2 \theta - J \right) + \frac{r^2 + a^2}{\Delta} P, \quad \rho^2 \frac{d\varphi}{d\lambda} = -aE + \frac{J}{\sin^2 \theta} + \frac{aP}{\Delta},$$

(5)

$$\rho^2 \frac{dr}{d\lambda} = \sigma_r \sqrt{R}, \quad \rho^2 \frac{d\theta}{d\lambda} = \sigma_\theta \sqrt{\Theta},$$

(6)

$$R = P^2 - \Delta \left[ m^2 r^2 + (J - aE)^2 + Q \right], \quad P = (r^2 + a^2) E - aJ,$$

$$\Theta = Q - \cos^2 \theta \left[ a^2 (m^2 - E^2) + \frac{J^2}{\sin^2 \theta} \right].$$

(7)

(8)

Here $E = \text{const}$ is the energy (relative to the infinity) of the moving particle, $J$ is the conserved projection of its angular momentum on the axis of rotation of the black hole, $m$ is the rest mass of the moving particle. For a particle with $m \neq 0$ the parameter $\lambda = \tau/m$, where $\tau$ is the proper time. $Q$ is the Carter constant. For the movement in equatorial plane $Q = 0$. The constants $\sigma_r, \sigma_\theta = \pm 1$ define the direction of movement in coordinates $r, \theta$. 
As it follows from (6) the parameters characterizing any geodesic must satisfy the conditions

\[ R \geq 0, \quad \Theta \geq 0. \] (9)

The condition of movement of the particle “forward” in time must be valid, so

\[ \frac{dt}{d\lambda} \geq 0. \] (10)

The conditions (9), (10) lead to the following inequalities for possible values of the energy \( E \) and the angular momentum projection \( J \) for the test particle at the point with coordinates \((r, \theta)\) for the fixed value of \( \Theta \geq 0 \) [7]:

Outside the ergosphere \( r^2 - 2rM + a^2 \cos^2 \theta > 0 \) one has

\[ E \geq \frac{1}{\rho^2} \sqrt{(m^2 \rho^2 + \Theta)(r^2 - 2rM + a^2 \cos^2 \theta)}, \quad J \in [J_-(r, \theta), J_+(r, \theta)], \] (11)

\[ J_\pm(r, \theta) = \frac{\sin \theta}{r^2 - 2rM + a^2 \cos^2 \theta} \left[ -2rMaE \sin \theta \right. \]

\[ \pm \sqrt{\Delta (\rho^4 E^2 - (m^2 \rho^2 + \Theta)(r^2 - 2rM + a^2 \cos^2 \theta))} \]. (12)

Inside the ergosphere \( r_H < r < r_1(\theta) \) one has

\[ r^2 - 2rM + a^2 \cos^2 \theta < 0, \quad J \leq J_+(r, \theta) \leq J_-(r, \theta), \] (13)

and the value of the particle energy can be any, as positive as negative. For \( r \) close to the horizon \( r_H \) from (12), (13) (for \( \theta \neq 0, \pi \)) one obtains

\[ J \leq J_H = \frac{2Mr_H E}{a}. \] (14)

So \( J_H \) is the upper frontier for the angular momentum projection of the particle with the energy \( E \) close to the event horizon of the black hole.

Earlier in paper [10] it was shown that geodesics with negative relative to the infinity energy in ergosphere originate and terminate at \( r = r_H \). And what is about the geodesics with positive energy also originating and terminating at \( r = r_H \)? Let us consider this problem?

Introduce the effective potential for the radial movement

\[ V_{\text{eff}} = -\frac{R}{2\rho^4}. \] (15)
Then due to (6),
\[
\frac{1}{2} \left( \frac{dr}{d\lambda} \right)^2 + V_{\text{eff}} = 0, \quad \frac{d^2 r}{d\lambda^2} = -\frac{\partial V_{\text{eff}}}{\partial r}. \tag{16}
\]

To find the answer on the posited question it is sufficient to find the regions of energy values of particles such that
\[
r > r_H, \quad V_{\text{eff}}(r) = 0 \Rightarrow \frac{\partial V_{\text{eff}}}{\partial r} > 0. \tag{17}
\]

Really the condition (17) means that in the upper in \( r \) point of the trajectory the acceleration \( d^2 r / d\lambda^2 \) due to (16) is directed in the direction of decrease of the radial coordinate i.e. to the event horizon. To find the boundary values of the parameters for such “white hole” geodesics consider the system of equations
\[
V_{\text{eff}}(r) = 0, \quad V'_r \equiv \frac{\partial V_{\text{eff}}}{\partial r} = 0. \tag{18}
\]

This system defines the orbits with constant value of the radial coordinate — the so called spherical orbits [16].

For lightlike geodesics (\( m = 0 \)) the system of equations (18) leads to expressions
\[
\frac{J}{E} = \frac{r^2(3M-r) - a^2(r+M)}{a(r-M)}, \quad \frac{a^2 Q}{E^2} = \frac{r^3(4Ma^2 - r(r-3M)^2)}{(r-M)^2}. \tag{19}
\]

For the nonrotating black hole (\( a = 0 \)) the spherical photon orbits are circular with values
\[
J = \pm 3\sqrt{3}ME, \quad r = 3M. \tag{20}
\]

In the region \( 2M < r < 3M \) inside the radius of the circular photon orbit of the Schwarzschild metric all lightlike geodesics with \( |J|/ME > 3\sqrt{3} \) originate and terminate at the gravitational radius \( r_H = 2M \).

For the rotating Kerr’s black hole the radii of equatorial photonic circular orbits \( Q = 0 \) can be found from (19) as roots of the equation \( r(r-3M)^2 - 4Ma^2 = 0 \) and out the event horizon they are equal to [17]
\[
r_{\pm} = 2M \left[ 1 + \cos \left( \frac{2}{3} \arccos \frac{\pm a}{M} \right) \right]. \tag{21}
\]

The corresponding values of \( r_{\pm} \) and \( J/ME \) are represented on Fig. 1. One can see that the obtained values of \( J/ME \) coincide with the limiting values of
Figure 1: The radii (to the left) and angular momenta (to the right) of circular photon orbits in Kerr’s metric.

\[ J_{\pm}(r_{\pm}, \pi/2)/ME \] calculated using formulas (12). That is why if in the region \( r_H < r < r_+ \) the values of \( J/ME \) are larger than the values given by the curve (1) on Fig. 1 to the right then such geodesics originate and terminate on the gravitational radius \( r_H \). If in the region \( r_H < r < r_- \) the values of \( J/ME \) are less than the value given by the curve (2) on Fig. 1 to the right then such geodesics originate and terminate on the gravitational radius \( r_H \).

Note that due to (14) close to the horizon one has \( J/ME < 2r_H/a \).

3. Properties of spherical orbits in Kerr’s metric

As it was shown earlier the problem of boundary values of the parameters outside of the event horizon for geodesics originating and terminating on \( r = r_H \) is identical to the search of boundary values of parameters for spherical orbits. In case of massive particles \( m \neq 0 \) the minimal value of the energy for the spherical orbits limits the maximal value of the energy for such white hole geodesics.

The system of equations (18) defining the spherical orbits can be written as

\[ R(r, m, E, J, Q) = 0, \quad \frac{\partial R(r, m, E, J, Q)}{\partial r} = 0. \]  (22)

So there is no dependence of the effective potential on the coordinate \( \theta \).

For fixed \( m \neq 0 \) the conditions (22) define (probably non uniquely) the values of the energy and of the projection of the angular momentum as functions of the radial coordinate \( r \) and the Carter constant \( Q \):

\[ E = E(r, Q), \quad J = J(r, Q). \]  (23)
Differentiating the equalities (18) with respect to $r$ due to (23) one obtains

\[
\frac{\partial V}{\partial r} + \frac{\partial V}{\partial E} \frac{\partial E}{\partial r} + \frac{\partial V}{\partial J} \frac{\partial J}{\partial r} = 0, \quad (24)
\]

\[
\frac{\partial^2 V}{\partial r^2} + \frac{\partial V'}{\partial E} \frac{\partial E}{\partial r} + \frac{\partial V'}{\partial J} \frac{\partial J}{\partial r} = 0. \quad (25)
\]

If one looks for the extremal on angular momentum spherical orbit for fixed value of $Q$, i.e. $\partial J/\partial r = 0$ then from (24) one obtains

\[
V = 0, \quad V' = 0, \quad \frac{\partial J(r, Q)}{\partial r} = 0 \Rightarrow \frac{\partial V}{\partial E} \frac{\partial E}{\partial r} = 0. \quad (26)
\]

One can show that for any test particle in Kerr’s metric

\[
V = 0, \quad V' = 0, \quad \frac{\partial J(r, Q)}{\partial r} = 0 \Rightarrow \frac{\partial V}{\partial E} \neq 0. \quad (27)
\]

So the extremal in angular momentum spherical orbit is also extremal in the energy value: $\partial E/\partial r = 0$.

If one looks for the extremal in energy spherical orbit for fixed values of $Q$, i.e. $\partial E/\partial r = 0$ then from (24) one has

\[
V = 0, \quad V' = 0, \quad \frac{\partial E(r, Q)}{\partial r} = 0 \Rightarrow \frac{\partial V}{\partial J} \frac{\partial J}{\partial r} = 0. \quad (28)
\]

Elementary but long calculations give

\[
V = 0, \quad V' = 0, \quad \frac{\partial E(r, Q)}{\partial r} = 0, \quad \frac{\partial V}{\partial J} = 0 \Rightarrow \quad a \neq 0, \quad m \neq 0, \quad r = 6M, \quad E = \frac{2\sqrt{2}}{3} m, \quad J = -\frac{\sqrt{2}}{3} ma, \quad Q = 12m^2M^2. \quad (29)
\]

If one is not interested in this exception then the extremal in the energy spherical orbit is also extremal in the angular momentum projection: $\partial J/\partial r = 0$.

From (25) one obtains

\[
V = 0, \quad V' = 0, \quad \frac{\partial E}{\partial r} = 0, \quad \frac{\partial J}{\partial r} = 0 \Rightarrow \frac{\partial^2 V}{\partial r^2} = 0. \quad (30)
\]

If for fixed parameters of the geodesic $(m, E, J, Q)$ and for small deviation of $r$ corresponding to the spherical orbit the force $d^2r/d\lambda^2$ appears tending
to lead the value of \( r \) to the initial value, then such orbits are called steady. From (16) and (18) it follows that the spherical orbit is steady if

\[
V_{\text{eff}}(r) = 0, \quad \frac{\partial V_{\text{eff}}}{\partial r} = 0, \quad \frac{\partial^2 V_{\text{eff}}}{\partial r^2} > 0.
\]  

(31)

So due to (30) all extremal in energy (with exception (29)) spherical orbits are also limiting stable spherical orbits.

One can formulate this as the **theorem on extremal orbits.** The spherical orbit in the field of the Kerr’s black hole extremal in the value of the angular momentum projection (in energy with the exception of the case (22)) for the fixed value of the Carter constant is also extremal in the value of the energy (the angular momentum projection) and it is the stable circular orbit.

For \( Q = 0 \) and \( \theta = \pi/2 \) one obtains the evident **consequence.** The circular orbit in Kerr’s metric, extremal in the value of the energy (angular momentum) is also extremal in the value of the angular momentum (energy) and is the stable circular orbit.

For the Newton gravitational law the equations (23) in usual units have the form

\[
E = mc^2 - \frac{GmM}{2r}, \quad J = \pm m\sqrt{GMr}.
\]  

(32)

All circular orbits are stable. Extremal values in energy and angular momentum occur only on the boundaries for \( r \to \infty \) or \( r \to 0 \).

The case of the Schwarzschild metric was considered by us in [8] where it was shown that for particle energies \( E < m2\sqrt{2}/3 \), the geodesics out of the event horizon of the black hole originate and terminate on the gravitational radius \( r = 2M \).

Solving the system of equations (18) for Kerr’s metric after long algebraic transformations one obtains

\[
\varepsilon \equiv \frac{E}{m} = \frac{x^3(x - 2) + AH_\pm}{x^2\sqrt{x^3(x - 3) + 2AH_\pm}}, \quad l \equiv \frac{J}{mM} = \frac{(x^2 + A^2)H_\pm - 2Ax^3}{x^2\sqrt{x^3(x - 3) + 2AH_\pm}}.
\]  

(33)

where

\[
x = \frac{r}{M}, \quad A = \frac{a}{M}, \quad q = \frac{Q}{m^2M^2}, \quad H_\pm = \pm \sqrt{x^5 + A^2q^2 + qx^3(3 - x) - Aq}.
\]  

(34)

For \( Q = 0 \) formulas (33) give known expressions for equatorial orbits of the Kerr’s metric [17]. The results [8] obtained for this case from formulas (33)
follow from the proved by us theorem and known parameters of limiting circular stable orbits. For example from the value of the energy $E = \frac{m}{\sqrt{3}}$ of the stable limiting orbit of the black hole with $a = M$ it follows that all equatorial geodesics with the energy $E < \frac{m}{\sqrt{3}}$ out of the horizon originate and terminate on $r = r_H$.

Differentiating (33) one obtains

$$\frac{\partial \varepsilon}{\partial q} = \frac{\pm A(x^3 - AH \pm)}{2x^2 \sqrt{x^3(x - 3)} + 2AH \pm \sqrt{x^5 + A^2q^2 + qx^3(3 - x)}}. \quad (35)$$

For arbitrary geodesics with $E < m$ the Carter’s constant $Q \geq 0$ due to (6), (8). In this case positive values of $J$ for spherical orbits are possible only for the choice of $H_+$ in (33). Then from (35) one obtains $\partial \varepsilon/\partial q > 0$ for $r > r_H$ and so for the extremal spherical orbits ($\partial \varepsilon/\partial x = 0$)

$$\frac{d\varepsilon}{dq} = \varepsilon \frac{d\varepsilon}{dx} \frac{dq}{d\varepsilon} + \frac{\partial \varepsilon}{\partial q} > 0. \quad (36)$$

On the boundaries of the region of existence of spherical orbits with $H_+ > 0$ for $r > r_H$ one can see that the energy cannot be larger than the corresponding values for equatorial circular orbits. That is why the limiting energy values of energies for massive particles found for equatorial geodesics in [8] are the same for the nonequatorial movement with $J > 0$, particularly all geodesics with the energy $E < \frac{m}{\sqrt{3}}$ and $J > 0$ out of the horizon of the Kerr’s black hole with $a = M$ originate and terminate on $r = r_H$.

4. The energy of particles collision in the centre of mass frame

One can find the energy $E_{\text{c.m.}}$ of two colliding particles with masses $m_1$ and $m_2$ in the centre of mass frame by taking the square of the formula

$$(E_{\text{c.m.}}, 0, 0, 0) = p^i_{(1)} + p^i_{(2)}, \quad (37)$$

where $p^i_{(n)}$ are 4-momenta of particles $(n = 1, 2)$. Taking the square of (37) and using (33), (6) and $p^i_{(n)}p_{(n)i} = m_n^2$ one obtains [18]:

$$E_{\text{c.m.}}^2 = m_1^2 + m_2^2 + \frac{2}{\rho^2} \left[ \frac{P_1P_2 - \sigma_{1r}\sqrt{R_1} \sigma_{2r}\sqrt{R_2}}{\Delta} - \frac{(J_1 - aE_1 \sin^2 \theta)(J_2 - aE_2 \sin^2 \theta)}{\sin^2 \theta} - \sigma_{1\theta} \sqrt{\Theta_1} \sigma_{2\theta} \sqrt{\Theta_2} \right]. \quad (38)$$
Taking the limit for $r \to r_H$, one obtains for falling $\sigma_1r = \sigma_2r = 1$ particles in equatorial ($\theta = \pi/2$) plane [19]:

$$
E_{c.m.}^2(r \to r_H) = \frac{(J_{1H}J_2 - J_{2H}J_1)^2}{4M^2(J_{1H} - J_1)(J_{2H} - J_2)}
+ m_1^2 \left[ 1 + \frac{J_{2H} - J_2}{J_{1H} - J_1} \right] + m_2^2 \left[ 1 + \frac{J_{1H} - J_1}{J_{2H} - J_2} \right],
$$

(39)

So the energy of the collision in the centre of mass frame is increasing without limit if $r \to r_H$ and the angular momentum for the fixed relative to infinity energy of one of the particles (the critical) tend to the limiting value close to the event horizon $J_H$ (see (14)). However, excluding the case of the extremal rotating black hole $a = M$ particles with the angular momentum $J_H$ cannot achieve the horizon if they fall from the infinity. The effective potential energy creates the barrier so high energies of collisions in the centre of mass frame can be achieved only in decays or multiple collisions [2, 3], when one has the increasing of the angular momentum of one of the falling particles $J \to J_H$. Other possibility of the unlimited growth of energy of collisions close to the event horizon is the collision with the particles moving in the opposite direction moving along the white hole geodesics [8]. However as one can see from the Penrose diagram for eternal black holes for the Schwarzschild case for $r \to r_H$ such collisions could occur in the infinite past.

There is another possibility of collisions with unlimited energy. From (38) in the ergosphere in the limit $J_2 \to -\infty$ one obtains

$$
E_{c.m.}^2 \approx J_2 \frac{r^2 - 2rM + a^2\cos^2\theta}{\rho^2\Delta\sin^2\theta} \left( \sigma_1r\sqrt{J_{1+} - J_1} - \sigma_2r\sqrt{J_{1-} - J_1} \right)^2 \to +\infty.
$$

(40)

The energy of collisions is increasing without limit (for fixed energy of particles relative to infinity) when the angular momentum of one of the particles in the ergosphere tends to $-\infty$. Due to the fact that particle with $J_2 \to -\infty$ cannot arrive to ergosphere there is a necessity of multiple collisions [6, 7] or existence of strong magnetic fields.

However if the collision of the particle coming from the infinity occurs with the particle moving along the white hole geodesic for which due to (13) there are no limitations from below on the angular momentum then the energy of collisions in the centre of mass frame can be large without limit at any point of the ergosphere.
For white hole geodesics close to the horizon the upper limit of the permitted angular momentum of particles tend to $J_H$ (see (14)). The energy of collision (39) close to the horizon of the particle falling from the infinity with that moving along the white hole geodesic with such angular momentum will be also high. There is no necessity of multiple collisions.

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