FUNCTIONAL LAW OF LARGE NUMBERS AND CENTRAL LIMIT THEOREM FOR SLOW-FAST MCKEAN-VLASOV EQUATIONS

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ABSTRACT. In this paper, we study the asymptotic behavior of a fully-coupled slow-fast McKean-Vlasov stochastic system. Using the non-linear Poisson equation on Wasserstein space, we first establish the strong convergence in the averaging principle of the functional law of large numbers type. In particular, the diffusion coefficient of the slow process can depend on the distribution of the fast motion. Then we consider the stochastic fluctuations of the original system around its average, and prove that the normalized difference will converge weakly to a linear McKean-Vlasov Ornstein-Uhlenbeck type process, which can be viewed as a functional central limit theorem. Extra drift and diffusion coefficients involving the expectation are characterized explicitly. Furthermore, the optimal rates of the convergence are also obtained.

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1. INTRODUCTION

Consider the following slow-fast McKean-Vlasov stochastic differential equation (SDE for short) in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$:

\[
\begin{aligned}
\mathrm{d}X^\varepsilon_t &= F(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}, Y^\varepsilon_t, \mathcal{L}_{Y^\varepsilon_t}) \mathrm{d}t + G(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}, \mathcal{L}_{Y^\varepsilon_t}) \mathrm{d}W^1_t, \quad X^\varepsilon_0 = \xi, \\
\mathrm{d}Y^\varepsilon_t &= \frac{1}{\varepsilon}c(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}, Y^\varepsilon_t, \mathcal{L}_{Y^\varepsilon_t}) \mathrm{d}t + \frac{1}{\varepsilon^2}b(X^\varepsilon_t, Y^\varepsilon_t, \mathcal{L}_{Y^\varepsilon_t}) \mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\sigma_1(\mathcal{L}_{X^\varepsilon_t}, Y^\varepsilon_t) \mathrm{d}W^1_t + \frac{1}{\varepsilon}\sigma_2(\mathcal{L}_{X^\varepsilon_t}, Y^\varepsilon_t, \mathcal{L}_{Y^\varepsilon_t}) \mathrm{d}W^2_t, \quad Y^\varepsilon_0 = \eta,
\end{aligned}
\]

where $d_1, d_2 \geq 1$, $F: \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_1}$, $G: \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_1}$, $c: \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_2}$, $b: \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_2}$, and $\sigma_1: \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_2} \otimes \mathbb{R}^{d_1}$ and $\sigma_2: \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \to \mathbb{R}^{d_2} \otimes \mathbb{R}^{d_1}$ are measurable functions, $W^1_t$, $W^2_t$ are $d_1$, $d_2$-dimensional independent standard Brownian motions both defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\xi, \eta$ are $d_1$, $d_2$-dimensional random variables, respectively. The small parameter $0 < \varepsilon \ll 1$ represents the separation of time scales between the slow component $X^\varepsilon_t$ (which can be thought of as the mathematical model for a phenomenon appearing at the natural time scale) and the fast motion $Y^\varepsilon_t$ (which can be interpreted as the fast varying environment). Here and throughout, we denote by $\mathcal{L}_\xi$ the distribution of a random

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variable $\varsigma$, and $\mathcal{P}_2(\mathbb{R}^d)$ ($d \geq 1$) the space of all square integrable probability measures over $\mathbb{R}^d$ equipped with the Wasserstein metric, i.e.,

$$W_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{P}(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \forall \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d),$$

where $\mathcal{P}(\mu_1, \mu_2)$ is the class of measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal $\mu_1$ and $\mu_2$.

The McKean-Vlasov SDE describes the limiting behavior of an individual particle involving within a system of particles interacting through their empirical measure, as the size of the population grows to infinity (the so-called propagation of chaos, see e.g. [31]). A distinct feature is that the coefficients in the equation depend not only on the solution process itself but also on its time marginal distributions. The pioneer work on such systems was indicated by Kac [19] in kinetic theory and McKean [23] in the study of non-linear partial differential equations (PDEs for short). So far, the McKean-Vlasov SDEs have been investigated in various aspects such as well-posedness, connection with non-linear Fokker-Planck equations, large deviation and numerical approximation, etc. We refer the readers to [1, 3, 30, 4, 10, 11, 12, 17, 24, 29] and the references therein. Meanwhile, multi-scale models have wide range of applications including climate weather interactions, intracellular biochemical reactions, geophysical fluid flows and stochastic volatility in finance, etc., and have been the central topic of study in science and engineering (see e.g. [2, 18, 25, 27, 28, 32, 34, 35]). In particular, multiple scales can leads to hysteresis loops in the bifurcation diagram and induce phase transitions of certain McKean-Vlasov equation as studied in [13, 14].

Due to the widely separated time scales and the cross interactions between the slow and fast motions, the multi-scale McKean-Vlasov equations turn out to be more difficult to deal with. Hence, a simplified equation which governs the evolution of the system over the long time scale is highly desirable. In this direction, the theory of averaging principle provides a good approximation for the slow component. Existing averaging results for the multi-scale McKean-Vlasov SDEs can be found in [5, 6, 7, 15, 16, 26]. However, the coefficients of the systems considered in these works are not allowed to rely on the distribution of the fast motion. Recently, a system of weakly interacting diffusions in a two-scale potential relying on the faster empirical measure was considered in [13], the combined mean field and diffusive limits were investigated. The authors in [22] considered the diffusion approximation of the multi-scale McKean-Vlasov SDEs by using a non-linear PDE as the corrector, where the coefficients depend on the distributions of both the slow component and the fast motion, yet only weak convergence is established therein.

In this paper, we shall first prove the strong convergence in the averaging principle for the system (1.1), see Theorem 2.1 below. More precisely, we show that as $\varepsilon \to 0$ the slow component $X^\varepsilon_t$ will converge in $L^2(\Omega)$ to $\bar{X}_t$ which satisfies the following McKean-Vlasov equation:

$$d\bar{X}_t = \bar{F}(\bar{X}_t, \mathcal{L}\bar{X}_t)dt + \bar{G}(\bar{X}_t, \mathcal{L}\bar{X}_t)dW^1_t, \quad \bar{X}_0 = \xi,$$

(1.2)
where the averaged coefficients are defined by

\[
\begin{align*}
\bar{F}(x, \mu) &:= \int_{\mathbb{R}^d} F(x, \mu, y, \zeta^\mu) \zeta^\mu(dy), \\
\bar{G}(x, \mu) &:= G(x, \mu, \zeta^\mu),
\end{align*}
\tag{1.3}
\]

and \(\zeta^\mu(dy)\) is the unique invariant measure of the following parameterized McKean-Vlasov equation: for fixed \(\mu \in \mathcal{P}_2(\mathbb{R}^d)\),

\[
dY^{\mu, \eta}_t = b(\mu, Y^{\mu, \eta}_t, \mathcal{L}_{Y^{\mu, \eta}_t}) dt + \sigma_1(\mu, Y^{\mu, \eta}_t, \mathcal{L}_{Y^{\mu, \eta}_t}) d\hat{W}^1_t
\]
\[
+ \sigma_2(\mu, Y^{\mu, \eta}_t, \mathcal{L}_{Y^{\mu, \eta}_t}) d\hat{W}^2_t,
\]
\[
Y^{\mu, \eta}_0 = \eta, \tag{1.5}
\]

where \(\hat{W}^1_t = \varepsilon^{-1} W^1_{\varepsilon^2 t}\) and \(\hat{W}^2_t = \varepsilon^{-1} W^2_{\varepsilon^2 t}\) are two new independent Brownian motions. We point out that for the systems considered in \([5, 6, 7, 15, 16, 26]\), where the coefficients do not depend on the distribution of the fast motion, the equation (1.5) then reduces to the classical Itô’s SDE (distribution-independent case), which is much easier to handle. Moreover, it is interesting to note that we allow the diffusion coefficient \(G\) in the slow process of system (1.1) to depend on the distribution of the fast variable, while it is well-known that in the theory of averaging principle of classical SDEs, the strong convergence may not hold when the diffusion coefficient depends on the fast variable, see Remark 2.2 below. This involves a delicate analysis on the convergence in the Wasserstein distance of the distribution of the fast process \(Y^{\varepsilon}_t\) to \(\zeta^{\bar{X}_t}\). Note that in the definitions (1.3) and (1.4) of the limit coefficients, we have freezed the \(\nu\)-measure variable of the coefficients \(F\) and \(G\) at the invariant measure \(\zeta^\mu\).

Next, we proceed to study the small fluctuations of the slow component \(X^{\varepsilon}_t\) around its average \(\bar{X}_t\), which is form of functional central limit type theorem. Namely, we are interested in the asymptotic behavior of the normalized difference

\[
Z^{\varepsilon}_t := \frac{X^{\varepsilon}_t - \bar{X}_t}{\varepsilon}
\]

as \(\varepsilon \to 0\). We shall show that as \(\varepsilon \to 0\), the deviation process \(Z^{\varepsilon}_t\) converges weakly to the solution of a linear Ornstein-Uhlenbeck type McKean-Vlasov SDE, see equation (2.12) and Theorem 2.3 below. In particular, the average effect of the drift term \(c\) in the original system (1.1) will appear in the limit (even though it does not appear in the averaged equation (1.2)). Meanwhile, several extra drift and diffusion terms arise, which are explicitly characterized in terms of the solution of an auxiliary non-linear Poisson equation on the Wasserstein space. We provide two interesting particular cases to illustrate the result, see Remark 2.4 and Example 2.5 below.

The rest of this paper is organized as follows. In Section 2, we state the main results. In Section 3, we prepare some auxiliary results about the non-linear Poisson equation on the whole space and some a priori estimates. Section 4 is devoted to the proof of the strong convergence in the averaging principle. In Section 5, we establish a fluctuation lemma and then give the proof of the functional central limit type theorem. Finally, an Itô’s formula is provided in the Appendix for the sake of complicity.
Notations. To end this section, we introduce some notations. Throughout this paper, the letter $C$ with or without subscript denotes positive constant whose value may changes from line to line. For brevity, we define

$$ L_1 := L_1(x, \mu, y, \nu) := \sum_{i=1}^{d_1} F_i(x, \mu, y, \nu) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d_1} (GG^*(x, \mu, \nu))_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad (1.6) $$

and

$$ L_2 := L_2(x, \mu, y, \nu) := \sum_{i=1}^{d_2} c_i(x, \mu, y, \nu) \frac{\partial}{\partial y_i}. \quad (1.7) $$

For a function $f(x, \mu, y, \nu)$ on $\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$, we say $f$ is Lipschitz continuous if there exists a positive constant $L$ such that for every $x_1, x_2 \in \mathbb{R}^{d_1}$, $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^{d_1})$, $y_1, y_2 \in \mathbb{R}^{d_2}$ and $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2})$,

$$ |f(x_1, \mu_1, y_1, \nu_1) - f(x_2, \mu_2, y_2, \nu_2)| \leq L(\|x_1 - x_2\| + |y_1 - y_2| + W_2(\mu_1, \mu_2) + W_2(\nu_1, \nu_2)). $$

Let us briefly recall the derivatives with respect to the measure variable introduced by Lions; see [9, Section 6] or [20] for more details. The idea is to consider the canonical lift of a real-valued function $f : \mathcal{P}_2(\mathbb{R}^{d}) \to \mathbb{R}$ into a function $F : L^2(\Omega) \ni X \mapsto f(L_X) \in \mathbb{R}$. Using the Hilbert structure of the space $L^2(\Omega)$, the function $f$ is said to be differentiable at $\mu \in \mathcal{P}_2(\mathbb{R}^{d})$ if its canonical lift $F$ is Fréchet differentiable at some point $X$ with $L_X = \mu$. By Riesz' representation theorem, the Fréchet derivative $D F(X)$, viewed as an element of $L^2(\Omega)$, can be given by a function $\partial_\mu f(\mu)(\cdot) : \mathbb{R}^{d} \mapsto \mathbb{R}^{d}$ such that

$$ D F(X) = \partial_\mu f(L_X)(X). $$

The function $\partial_\mu f(\mu)(x)$ is then called the Lions derivative ($L$-derivative for short) of $f$ at $\mu$. Similarly, we can define the higher order derivatives of $f$ at $\mu$.

Let $d_1, d_2 \geq 1$ and $k, \ell, m \in \mathbb{N} = \{0, 1, 2, \cdots\}$. We introduce the following spaces of functions.

- The space $C^{m, (k, \ell)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. A function $f(x, \mu, y, \nu)$ is in $C^{m, (k, \ell)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$ if for any $(\mu, y, \nu)$, the mapping $x \mapsto f(x, \mu, y, \nu)$ is in $C^{m}(\mathbb{R}^{d_1})$, and for any $(x, y, \nu)$, the mapping $\mu \mapsto f(x, \mu, y, \nu)$ is in $C^{m, (k, \ell)}(\mathcal{P}_2(\mathbb{R}^{d_1}))$, and for fixed $(x, \mu)$, the mapping $(y, \nu) \mapsto f(x, \mu, y, \nu)$ is in $C^{m, (k, \ell)}(\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$.

- The space $C^{k, (k, \ell)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. A function $f(\mu, y, \nu)$ is in $C^{k, (k, \ell)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$ if $f \in C^{k, (k, \ell)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, and we can find a version of $\partial^k f(\mu, y, \nu)(\tilde{x}_1, \cdots, \tilde{x}_k)$ such that the mapping $(y, \nu) \mapsto \partial^k f(\mu, y, \nu)(\tilde{x}_1, \cdots, \tilde{x}_k)$ is in $C^{k, (k, \ell)}(\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$.

- The space $C^{k, (k, \ell)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. A function $f(x, \mu, y, \nu)$ is in $C^{k, (k, \ell)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$ if $f \in C^{k, (k, \ell)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, and for every $x \in \mathbb{R}^{d_1}$, the mapping $(\mu, y, \nu) \mapsto \partial^k f(x, \mu, y, \nu)(\tilde{x}_1, \cdots, \tilde{x}_k)$ is in $C^{k, (k, \ell)}(\mathcal{P}_2(\mathbb{R}^{d_2}) \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. 

Let us first introduce some basic assumptions. Throughout this paper, we assume the following condition holds:

\( (H^{\sigma,b}) \): there exist constants \( c_2 > c_1 \geq 0 \) such that for every \( \mu \in \mathcal{P}_2(\mathbb{R}^{d_1}) \), \( y_1, y_2 \in \mathbb{R}^{d_2} \) and \( \nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2}) \),

\[
3\|\sigma_1(\mu, y_1, \nu_1) - \sigma_1(\mu, y_2, \nu_2)\|^2 + \|\sigma_2(\mu, y_1, \nu_1) - \sigma_2(\mu, y_2, \nu_2)\|^2 \\
+ 2\langle b(\mu, y_1, \nu_1) - b(\mu, y_2, \nu_2), y_1 - y_2 \rangle \leq c_1W_2(\nu_1, \nu_2)^2 - c_2|y_1 - y_2|^2.
\]

Recall that for fixed \( \mu \in \mathcal{P}_2(\mathbb{R}^{d_1}) \), \( Y_{t}^{\mu, \eta} \) satisfies the McKean-Vlasov equation (1.5). It turns out that the distribution of \( Y_{t}^{\mu, \eta} \) only depends on \( \eta \) through its distribution \( \mathcal{L}_\eta = \nu \). Thus, for a given measure \( \nu \in \mathcal{P}_2(\mathbb{R}^{d_2}) \), we can define a (non-linear) semigroup \( \{P_t^{\mu, \nu}\}_{t \geq 0} \) on \( \mathcal{P}_2(\mathbb{R}^{d_2}) \) by letting

\( P_t^{\mu, \nu} := \mathcal{L}_{Y_{t}^{\mu, \eta}} \text{ with } \mathcal{L}_\eta = \nu. \)

We say that a probability measure \( \zeta^\mu \) is an invariant measure of the McKean-Vlasov equation (1.5) or the process \( Y_{t}^{\mu, \eta} \) if

\( P_t^{\mu, \nu} \zeta^\mu = \zeta^\mu, \quad \forall t \geq 0. \)

Under assumption \( (H^{\sigma,b}) \), it is known that (see e.g. [33, Theorem 3.1]) there exists a unique invariant measure \( \zeta^\mu(dy) \) for the equation (1.5). Moreover, there exist constants \( C_0, \lambda_0 > 0 \) such that for every \( \nu \in \mathcal{P}_2(\mathbb{R}^{d_2}) \),

\[
W_2(P_t^{\mu, \nu}, \zeta^\mu) \leq C_0 e^{-\lambda_0 t} W_2(\nu, \zeta^\mu). \tag{2.1}
\]

The following is the first main result of this paper.

**Theorem 2.1.** (Strong convergence). Let \( (H^{\sigma,b}) \) holds. Assume that \( G \) and \( c \) are Lipschitz continuous, \( F \in C_b^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})) \) with \( \partial_x F(x, \mu, \cdots, \nu) \in C_b^1(\mathbb{R}^{d_2}) \) and \( \sigma_1, \sigma_2, b \in C_b^{2,(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})) \). Then we have for any \( T > 0 \),

\[
\sup_{t \in [0,T]} \mathbb{E}|X_{t}^\varepsilon - \tilde{X}_t|^2 \leq C_T \varepsilon^2,
\]

where \( X_{t}^\varepsilon \) and \( \tilde{X}_t \) satisfy the McKean-Vlasov equation (1.1) and (1.2), respectively, and \( C_T > 0 \) is a constant independent of \( \varepsilon \).

**Remark 2.2.** (i) We shall show that the averaged coefficients \( \bar{F} \) defined in (1.3) and \( \bar{G} \) defined in (1.4) are Lipschitz continuous with respect to \( (x, \mu) \) (see Lemma 4.2 below). Thus, there exists a unique strong solution \( \bar{X}_t \) to the averaged McKean-Vlasov equation (1.2).

(ii) In the theory of the averaging principle of classical SDEs (i.e., when the coefficients in system (1.1) do not depend on the distribution of the solution), counter example
is known which shows that the strong convergence does not hold when the diffusion coefficient $G$ in the slow variable depends on the fast motion $Y_t^\varepsilon$, see e.g. [21]. Here, we show that the diffusion coefficient $G$ can depend on the distribution $\mathcal{L}_{Y_t^\varepsilon}$ of the fast motion. This involves the convergence in Wasserstein distance of the distribution of $G$.

As a result, we have

$$ G(x, \mu, \nu) \equiv G(x, \mu). $$

(2.3)

Next, we proceed to identify the limit of the normalized difference

$$ Z_t^\varepsilon := \frac{X_t^\varepsilon - \bar{X}_t}{\varepsilon}, $$

(2.2)

as $\varepsilon \to 0$. For this, we assume

$$ G(x, \mu, \nu) \equiv G(x, \mu). $$

(2.3)

As a result, we have

$$ Z_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}) - \bar{F}(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \right] \, ds + \frac{1}{\varepsilon} \int_0^t \left[ G(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - G(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \right] \, dW_s^1, $$

where $\bar{F}(x, \mu)$ is given by (1.3). To introduce the limit $\bar{Z}_t$ of $Z_t^\varepsilon$, we need to consider the following Poisson equation on $\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$:

$$ \mathcal{L}_0(\mu, y, \nu) \Phi(x, \mu, y, \nu) = -[F(x, \mu, y, \nu) - \bar{F}(x, \mu)] =: -\delta F(x, \mu, y, \nu), $$

(2.4)

where $(x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$ are regarded as parameters, and for a test function $\varphi(y, \nu)$, the operator $\mathcal{L}_0$ is defined by

$$ \mathcal{L}_0 \varphi(y, \nu) := \mathcal{L}_0(\mu, y, \nu) \varphi(y, \nu) $$

$$ := \frac{1}{2} \text{Tr}(a(\mu, y, \nu) \cdot \partial^2_y \varphi(y, \nu)) + b(\mu, y, \nu) \cdot \partial_y \varphi(y, \nu) $$

$$ + \int_{\mathbb{R}^{d_2}} \left[ \frac{1}{2} \text{Tr}(a(\mu, \tilde{y}, \nu) \cdot \partial^2_{\tilde{y}} \varphi(y, \nu)(\tilde{y})) \right] + b(\mu, \tilde{y}, \nu) \cdot \partial_y \varphi(y, \nu)(\tilde{y}) \right] \nu(d\tilde{y}), $$

(2.5)

with $a(\mu, y, \nu) = [\sigma_1 \sigma_1^* + \sigma_2 \sigma_2^*](\mu, y, \nu)$. In fact, the operator $\mathcal{L}_0$ can be viewed as the infinitesimal generator of the parameterized McKean-Vlasov SDE (1.5). Note that the equation (2.4) is totally non-linear due to the existence of the integral part with respect to the measure $\nu$ in (2.5). According to Theorem 3.1 below, there exists a unique solution $\Phi(x, \mu, y, \nu)$ to the equation (2.4). Let us define

$$ c \cdot \partial_y \Phi(x, \mu) := \int_{\mathbb{R}^{d_2}} c(x, \mu, y, \zeta^\mu) \cdot \partial_y \Phi(x, \mu, y, \zeta^\mu) \zeta^\mu(dy), $$

(2.6)

$$ c \cdot \partial_y \Phi(x, \mu)(\tilde{x}) := \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_2}} c(\tilde{x}, \mu, \tilde{y}, \zeta^\mu) \cdot \partial_y \Phi(x, \mu, y, \zeta^\mu)(\tilde{y}) \zeta^\mu(dy) \zeta^\mu(dy), $$

(2.7)

$$ \sigma_1^* \cdot \partial_y \Phi(x, \mu)(\tilde{x}) := \int_{\mathbb{R}^{d_2}} \sigma_1^*(\mu, y, \zeta^\mu) \cdot \partial_x \partial_y \Phi(x, \mu, y, \zeta^\mu) \zeta^\mu(dy), $$

(2.8)

$$ \partial_y \Phi \cdot \sigma_1(x, \mu) := \int_{\mathbb{R}^{d_2}} \partial_y \Phi(x, \mu, y, \zeta^\mu) \cdot \sigma_1(\mu, y, \zeta^\mu) \zeta^\mu(dy), $$

(2.9)
\[
\frac{1}{2} (\partial_y \Phi \cdot \sigma_1)(\partial_y \Phi \cdot \sigma_1)^\ast(x, \mu) := \int_{\mathbb{R}^d} \left( \partial_y \Phi(x, \mu, y, \zeta^\mu) \cdot \sigma_1(\mu, y, \zeta^\mu) \right) \cdot \zeta^\mu(dy), \quad (2.10)
\]

and
\[
\frac{1}{2} (\partial_y \Phi \cdot \sigma_2)(\partial_y \Phi \cdot \sigma_2)^\ast(x, \mu) := \int_{\mathbb{R}^d} \left( \partial_y \Phi(x, \mu, y, \zeta^\mu) \cdot \sigma_2(\mu, y, \zeta^\mu) \right) \cdot \zeta^\mu(dy). \quad (2.11)
\]

Then the limit \( \bar{Z}_t \) for the deviation process \( Z_t^\varepsilon \) turns out to satisfy the following McKean-Vlasov SDE:
\[
d\bar{Z}_t = \partial_x \bar{F}(\bar{X}_t, L_{\bar{X}_t})d\bar{Z}_t + \bar{E} \left[ \partial_{\mu} \bar{F}(\bar{X}_t, L_{\bar{X}_t})d\bar{Z}_t \right] dt + \bar{E} \left[ c \cdot \partial_y \Phi(\bar{X}_t, L_{\bar{X}_t}) d\bar{Z}_t \right] dt + \bar{E} \left[ \partial_1 \bar{F}(\bar{X}_t, L_{\bar{X}_t}) \right] d\bar{Z}_t dt + \sqrt{\Sigma(\bar{X}_t, L_{\bar{X}_t})} dW_t, \quad Z_0 = 0, \quad (2.12)
\]

where \( \bar{W}_t \) is another Brownian motion independent of \( W_t^1 \), \( \bar{X}_t \) is the unique strong solution of the averaged equation (1.2), the process \( (\bar{X}_t, \bar{Z}_t) \) is defined on a copy of \((\bar{X}_t, \bar{Z}_t)\) on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) of the original probability space \((\Omega, \mathcal{F}, P)\), and \( \bar{E} \) is the expectation taken with respect to \( \bar{P} \). The diffusion coefficient \( \Sigma(x, \mu) \) is defined by
\[
\Sigma(x, \mu) = (\partial_y \Phi \cdot \sigma_1)(\partial_y \Phi \cdot \sigma_1)^\ast(x, \mu) - \partial_y \Phi \cdot \sigma_1 \cdot (\partial_y \Phi \cdot \sigma_1)^\ast(x, \mu) + (\partial_y \Phi \cdot \sigma_2)(\partial_y \Phi \cdot \sigma_2)^\ast(x, \mu). \quad (2.13)
\]

Note that the matrix \( \Sigma(x, \mu) \) is always positive semi-definite.

The following is the second main result of this paper.

**Theorem 2.3.** (Central limit theorem). Let assumptions \((H^a,b)\) and \((2.3)\) hold. Assume that \( G \in (C^{4,(1,3)}_b \cap C^{4,2,2}_b \cap C^{1,3,1}_b)(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^1)), c \in (C^{4,2,2}_b \cap C^{1,1,1}_b)(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^1) \times \mathcal{P}_2(\mathbb{R}^2)), \sigma_1, \sigma_2, b \in (C^{3,3,3}_b \cap C^{4,2,2}_b)(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^1) \times \mathcal{P}_2(\mathbb{R}^2)) \) and \( F \in (C^{4,2,2}_b \cap C^{1,3,1}_b)(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^1) \times \mathcal{P}_2(\mathbb{R}^2)). \) Then for any \( T > 0 \) and \( \varphi \in (C^{1,3}_b \cap C^{2,2}_b)(\mathcal{P}_2(\mathbb{R}^1)), \) we have
\[
\sup_{t \in [0,T]} \left| \varphi(\mathcal{L}_{Z_t^\varepsilon}) - \varphi(\mathcal{L}_{\bar{Z}_t}) \right| \leq C_T \varepsilon,
\]
where \( Z_t^\varepsilon \) satisfies the McKean-Vlasov equation (2.12), and \( C_T > 0 \) is a constant independent of \( \varepsilon \). In particular, we have for every \( \phi \in C^{4}_b(\mathbb{R}^1), \)
\[
\sup_{t \in [0,T]} \left| \mathbb{E}\phi(Z_t^\varepsilon) - \mathbb{E}\phi(\bar{Z}_t) \right| \leq C_T \varepsilon.
\]
Remark 2.4. (i) Note that the average effect of the term \( c \) in the system (1.1) does not appear in the averaging principle of the law of large number (i.e., it does not appear in equation (1.2)). However, in the central limit theorem it arises in the equation (2.12). The expectation term involving \( L \)-derivative in the measure argument of the solution \( \Phi \) in (2.12) is due to the effect of the dependence on the fast distribution in the coefficients.

(ii) The terms involving \( \sigma_1 \) in (2.12) seem to be new even for classical SDEs, which is due to the effect of the common noise.

Example 2.5. When \( G \equiv I_d \) (the identity matrix) in system (1.1), then the corresponding averaged equation (1.2) becomes

\[
d\bar{X}_t = \bar{F}(\bar{X}_t, \mathcal{L}_{\bar{X}_t})dt + dW_t.
\]

As a result, we have

\[
Z_t^\varepsilon := \frac{X_t^\varepsilon - \bar{X}_t}{\varepsilon} = \frac{1}{\varepsilon} \int_0^t \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}) - \bar{F}(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \right] ds
\]

\[
= \frac{1}{\varepsilon} \int_0^t \left[ \bar{F}(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - \bar{F}(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \right] ds
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \left[ F(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}) - \bar{F}(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) \right] ds =: \mathcal{I}_1(\varepsilon) + \mathcal{I}_2(\varepsilon).
\]

Let \( \bar{Z}_t \) be the limit of \( Z_t^\varepsilon \). Then, at least formally, we have by the mean value theorem that

\[
\mathcal{I}_1(\varepsilon) \to \int_0^t \left[ \partial_x \bar{F}(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) \bar{Z}_s + \mathbb{E}[\partial_{\mu} \bar{F}(\bar{X}_s, \mathcal{L}_{\bar{X}_s})(\bar{X}_s)\bar{Z}_s] \right] ds \quad \text{as} \quad \varepsilon \to 0.
\]

The limit for the second term \( \mathcal{I}_2(\varepsilon) \) is far from being obvious. We provide two cases to illustrate the result.

(i) When \( \sigma_2 = I_d \) and \( \sigma_1 = c \equiv 0 \) in system (1.1), then according to Theorem 2.3, we have

\[
\mathcal{I}_2(\varepsilon) \to \int_0^t \sqrt{\Sigma(\bar{X}_s, \mathcal{L}_{\bar{X}_s})} d\bar{W}_s,
\]

where \( \bar{W}_t \) is a new Brownian motion independent of \( W_t^1 \), and

\[
\Sigma(x, \mu) := (\partial_y \Phi)(\partial_y \Phi)^*(x, \mu).
\]

Thus the limit \( \bar{Z}_t \) satisfies the linear McKean-Vlasov equation

\[
d\bar{Z}_t = \partial_x \bar{F}(\bar{X}_t, \mathcal{L}_{\bar{X}_t}) \bar{Z}_tdt + \mathbb{E}[\partial_{\mu} \bar{F}(\bar{X}_t, \mathcal{L}_{\bar{X}_t})(\bar{X}_t)\bar{Z}_t] dt
\]

\[
+ \sqrt{\Sigma(\bar{X}_t, \mathcal{L}_{\bar{X}_t})} d\bar{W}_t.
\]

(ii) When \( \sigma_1 = I_d \) (with common noise) and \( \sigma_2 = c \equiv 0 \), then according to Theorem 2.3, we have

\[
\mathcal{I}_2(\varepsilon) \to \int_0^t \partial_x \partial_y \Phi(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) ds + \int_0^t \partial_{\mu} \Phi(\bar{X}_s, \mathcal{L}_{\bar{X}_s}) dW_t^1 + \int_0^t \sqrt{\Sigma(\bar{X}_s, \mathcal{L}_{\bar{X}_s})} d\bar{W}_s,
\]
where $\bar{W}_t$ is a new Brownian motion independent of $W_t^t$, and
\[
\Sigma(x,\mu) := (\partial_y \Phi)(\partial_y \Phi)^*(x,\mu) - \partial_y \Phi \cdot (\partial_y \Phi)^*(x,\mu).
\]
Thus the limit $\bar{Z}_t$ satisfies
\[
d\bar{Z}_t = \partial_x F(X_t, \mathcal{L}_{\bar{X}_t})\bar{Z}_t dt + \mathbb{E}\left[\partial_\mu F(X_t, \mathcal{L}_{\bar{X}_t})(\bar{X}_t)\bar{Z}_t\right] dt
+ \partial_y \Phi(X_t, \mathcal{L}_{\bar{X}_t}) dW_t^t + \sqrt{\Sigma(X_t, \mathcal{L}_{\bar{X}_t})} d\bar{W}_t.
\]

3. Poisson equation and auxiliary estimates

In this section, we first recall some results about the Poisson equation on the Wasserstein space. Then, we collect some a priori estimates that we shall use to prove our main results.

Consider the following Poisson equation on the whole space $\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_1})$:
\[
\mathcal{L}_0(\mu, y, \nu)U(x, \mu, y, \nu) = -f(x, \mu, y, \nu),
\]
where $(x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$ are parameters, and the operator $\mathcal{L}_0$ is defined by (2.5). In order to ensure the well-posedness of the equation (3.1), we need to assume that $f$ satisfies the following centering condition:
\[
\int_{\mathbb{R}^{d_2}} f(x, \mu, y, \zeta^\mu)\zeta^\mu(dy) = 0, \quad \forall (x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}),
\]
where $\zeta^\mu(dy)$ is the unique invariant measure of the frozen McKean-Vlasov equation (1.5). Furthermore, we need to consider the following de-coupled equation associated with (1.5):
\[
Y_t^{\mu,y,\nu} = y + \int_0^t b(\mu, Y_s^{\mu,y,\nu}, \mathcal{L}_{Y_s^{\mu,y,\nu}}) ds + \int_0^t \sigma_1(\mu, Y_s^{\mu,y,\nu}, \mathcal{L}_{Y_s^{\mu,y,\nu}}) dW_s^1
+ \int_0^t \sigma_2(\mu, Y_s^{\mu,y,\nu}, \mathcal{L}_{Y_s^{\mu,y,\nu}}) dW_s^2,
\]
with $\mathcal{L}_{\eta} = \nu$. The results below were proved in [22, Theorems 2.3, 2.4], which will be used frequently in the sequel.

**Theorem 3.1.** Let $(\mathbf{H}^{a,b})$ hold, $j, k, m, n \in \mathbb{N}$, and the function $f$ satisfy the centering condition (3.2).

(i) Assume that for every $(x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$, $a(\mu, \cdot, \cdot), b(\mu, \cdot, \cdot), f(x, \mu, \cdot, \cdot) \in C^{2m,m,2m}_{b}(\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. Then there exists a unique solution $U(x, \mu, \cdot, \cdot) \in C^{2m,m,2m}_{b}(\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$ to the equation (3.1), which also satisfies the centering condition (3.2) and is given by
\[
U(x, \mu, y, \nu) = \mathbb{E}\left(\int_0^\infty f(x, \mu, Y_t^{\mu,y,\nu}, \mathcal{L}_{Y_t^{\mu,y,\nu}}) dt\right),
\]
where $Y_t^{\mu,\eta}$ and $Y_t^{\mu,y,\nu}$ satisfy equations (1.5) and (3.3) with $\mathcal{L}_{\eta} = \nu$, respectively.
(ii) Assume that $a, b \in (C_{b}^{(m,k),2m,(m,m)} \cap C_{b}^{(n,k),2(m-n),(m-n,m-n)})(\mathcal{P}_{2}(\mathbb{R}^{d_{1}}) \times \mathbb{R}^{d_{2}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{2}}))$ and $f \in (C_{b}^{j,(m,k),2m,(m,m)} \cap C_{b}^{j,(n,k),2(m-n),(m-n,m-n)})(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}) \times \mathbb{R}^{d_{2}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{2}}))$ with $0 \leq n < m$. Then we have

$$U \in (C_{b}^{j,(m,k),2m,(m,m)} \cap C_{b}^{j,(n,k),2(m-n),(m-n,m-n)})(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}) \times \mathbb{R}^{d_{2}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{2}})).$$

Given a function $h(x, \mu, y, \nu)$, we denote by $\bar{h}(x, \mu)$ its average with respect to the invariant measure $\zeta^{\mu}(dy)$, i.e.,

$$\bar{h}(x, \mu) := \int_{\mathbb{R}^{d_{2}}} h(x, \mu, y, \zeta^{\mu}(dy)).$$

As a direct application of Theorem 3.1, we have the following result which illustrates the regularity of an averaged function.

**Lemma 3.2.** Assume that $(H^{a,b})$ holds and $\ell, m, k \in \mathbb{N}$. If for every $1 \leq n < m$, $a, b \in (C_{b}^{\ell,(m,k),2m,(m,m)} \cap C_{b}^{\ell,(n,k),2(m-n),(m-n,m-n)})(\mathbb{R}^{d_{1}} \times \mathbb{R}^{d_{2}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{2}}))$ and $h \in (C_{b}^{\ell,(m,k),2m,(m,m)} \cap C_{b}^{\ell,(n,k),2(m-n),(m-n,m-n)})(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}) \times \mathbb{R}^{d_{2}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{2}}))$. Then we have $\bar{h} \in C_{b}^{\ell,(m,k)}(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}))$. In particular,

(i) under the assumptions in Theorem 2.1, we have $\bar{F} \in C_{b}^{2,(1,1)}$;

(ii) under the assumptions in Theorem 2.3, we have $\bar{F} \in C_{b}^{4,(1,3)} \cap C_{b}^{4,(2,2)} \cap C_{b}^{4,(3,1)}$, and

$$c \cdot \partial_{y} \Phi, \sigma_{1} \cdot \partial_{y} \Phi, \partial_{y} \Phi \cdot \sigma_{1}, \delta F \cdot \Phi^{*} \in C_{b}^{3,(2,2)}, \quad c \cdot \partial_{\nu} \Phi \in C_{b}^{3,(2,2,3)},$$

where the above functions are defined by (2.6)-(2.11).

**Proof.** The conclusion that $\bar{h} \in C_{b}^{\ell,(m,k)}(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}))$ was proved in [22, Corollary 2.5]. Then under the assumptions in Theorem 2.1, we take $\ell = 2$ and $m = k = 1$ in the above, the assertion that $\bar{F} \in C_{b}^{2,(1,1)}(\mathbb{R}^{d_{1}} \times \mathcal{P}_{2}(\mathbb{R}^{d_{1}}))$ follows directly. Similarly, under the assumptions in Theorem 2.3, we deduce that $\bar{F} \in C_{b}^{4,(1,3)} \cap C_{b}^{4,(2,2)} \cap C_{b}^{4,(3,1)}$. Recall that $\Phi$ solves (2.4). By Theorem 3.1, we get $\Phi \in C_{b}^{4,(1,3),6,(3,3)} \cap C_{b}^{4,(1,3),4,(2,2)} \cap C_{b}^{4,(2,2),2,(1,1)}$. This together with the conditions on $c$ and $\sigma_{1}$ implies that $\bar{F} \in C_{b}^{3,(2,2)}$ and $c \cdot \partial_{\nu} \Phi \in C_{b}^{3,(2,2,3)}$. \hfill $\square$

Let $(X_{t}^{\xi}, Y_{t}^{\xi})$ satisfy the McKean-Vlasov equation (1.1). By using the similar arguments as [26, Lemma 3.1], we have the following moment estimates of the process $(X_{t}^{\xi}, Y_{t}^{\xi})$, the details of the proof are omitted.

**Lemma 3.3.** Let $(H^{a,b})$ hold. Assume that $F, G$ and $c$ are Lipschitz continuous. Then for any $T > 0$, there exists a positive constant $C_{T}$ such that

$$\sup_{0 < t < 1} \mathbb{E} \left[ \sup_{t \in [0,T]} |X_{t}^{\xi}|^{4} \right] \leq C_{T} (1 + \mathbb{E}|\xi|^{4} + \mathbb{E}|\eta|^{4}),$$

and

$$\sup_{0 < t < 1} \mathbb{E} |Y_{t}^{\xi}|^{4} \leq C_{T} (1 + \mathbb{E}|\xi|^{4} + \mathbb{E}|\eta|^{4}).$$

Recall that $Y_{t}^{\mu,n}$ is the unique strong solution of the equation (1.5). We have:
Lemma 3.4. Assume that \( (H^\sigma)^b \) holds. Then we have for any \( t > 0, \mu, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d) \) and \( \eta \in L^2(\Omega) \),
\[
\mathbb{E}|Y^\mu_{t_1} - Y^\mu_{t_2}|^2 \leq C_0 W_2(\mu_1, \mu_2)^2,
\]
and
\[
\mathbb{E}|Y^\mu_{t_1}|^2 \leq C_0 \left(e^{-(c_2-c_1)t} \mathbb{E}|\eta|^2 + W_2(\mu, \delta_0)^2\right),
\]
where \( C_0 > 0 \) is a constant independent of \( t \).

Proof. Using Itô’s formula and by \( (H^\sigma)^b \), we have that
\[
d\mathbb{E}|Y^\mu_{t_1} - Y^\mu_{t_2}|^2 = \mathbb{E}\left[2\langle Y^\mu_{t_1}, Y^\mu_{t_2} \rangle, b(\mu_1, Y^\mu_{t_1}, \mathcal{L}Y^\mu_{t_1}) - b(\mu_2, Y^\mu_{t_2}, \mathcal{L}Y^\mu_{t_2})\rangle + \|\sigma(\mu_1, Y^\mu_{t_1}, \mathcal{L}Y^\mu_{t_1}) - \sigma(\mu_2, Y^\mu_{t_2}, \mathcal{L}Y^\mu_{t_2})\|^2\right] dt
\]
which together with the comparison theorem implies
\[
\mathbb{E}|Y^\mu_{t_1} - Y^\mu_{t_2}|^2 \leq C_0 W_2(\mu_1, \mu_2)^2.
\]

In view of \( (H^\sigma)^b \), for every \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), \( y \in \mathbb{R}^d \) and \( \nu \in \mathcal{P}_2(\mathbb{R}^d) \), there exists \( C_0 > 0 \) such that
\[
2\langle y, b(\mu, y, \nu) \rangle + 3\|\sigma(\mu, y, \nu)\|^2 + 3\|\sigma(\mu, y, \nu)\|^2 \leq c_1 W_2(\nu, \delta_0)^2 - c_2 |y|^2 + C_0 W_2(\mu, \delta_0)^2.
\]
In the same way we get
\[
d\mathbb{E}|Y^\mu_{t_1}|^2 = \mathbb{E}\left[2\langle Y^\mu_{t_1}, b(\mu, Y^\mu_{t_1}, \mathcal{L}Y^\mu_{t_1}) \rangle + \|\sigma(\mu, Y^\mu_{t_1}, \mathcal{L}Y^\mu_{t_1})\|^2\right] dt
\]
which in turn yields that
\[
\mathbb{E}|Y^\mu_{t_1}|^2 \leq C_0 e^{-(c_2-c_1)t} \mathbb{E}|\eta|^2 + C_0 W_2(\mu, \delta_0)^2.
\]
Thus the desired results are proved. \( \square \)

Lemma 3.5. Assume that \( (H^\sigma)^b \) holds. Then there exists a positive constant \( C_0 \) such that for any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \),
\[
W_2(\zeta^\mu, \delta_0)^2 \leq C_0 W_2(\mu, \delta_0)^2.
\]

Proof. Using (2.1) and Lemma 3.4, we have
\[
W_2(\zeta^\mu, \delta_0)^2 \leq 2W_2(\zeta^\mu, P_t^\mu, \delta_0)^2 + 2W_2(P_t^\mu, \delta_0, \delta_0)^2
\]
\[
\leq C_0 e^{-2\lambda_0 t} W_2(\zeta^\mu, \delta_0)^2 + C_0 \mathbb{E}Y^\mu_{t_1} \mathbb{E}|Y^\mu_{t_1}|^2
\]
\[
\leq C_0 e^{-2\lambda_0 t} W_2(\zeta^\mu, \delta_0)^2 + C_0 W_2(\mu, \delta_0)^2,
\]
where \( C_0 > 0 \) is a constant independent of \( t \). Taking the limit \( t \to \infty \), the desired conclusion follows. \( \square \)
4. Strong convergence in the averaging principle

Using the technique of Poisson equation, we shall first establish a strong fluctuation estimate in Subsection 4.1. Then we prove the strong convergence of the slow-fast system (1.1) to the averaged system (1.2) in Subsection 4.2. The optimal rate of convergence follows as a byproduct.

4.1. Strong Fluctuation estimate. Given a function \( f(x, \mu, y, \nu) \) on \( \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \), the following result gives an estimate for the fluctuations of the process \( f(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \) over the time interval \( [0, t] \).

**Lemma 4.1.** Let \((\mathcal{H}^b)\) hold. Assume that \( F, G \) and \( c \) are Lipschitz continuous and \( \sigma_1, \sigma_2, b \in C_b^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})) \). Then for every \( f \in C_b^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})) \) satisfying (3.2) and \( \partial_x f(x, \mu, \cdot, \nu) \in C_b^1(\mathbb{R}^{d_2}) \), we have

\[
\mathbb{E} \left| \int_0^t f(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \, ds \right|^2 \leq C_0 \epsilon^2,
\]

where \( C_0 > 0 \) is a constant independent of \( \epsilon \).

**Proof.** Since \( f \) satisfies (3.2), by the assumptions on the coefficients and Theorem 3.1, there exists a unique solution \( \psi \in C_b^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})) \) to the following Poisson equation:

\[
\mathcal{L}_0(\mu, y, \nu)\psi(x, \mu, y, \nu) = -f(x, \mu, y, \nu),
\]

where \((x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})\) are regarded as parameters, and the operator \( \mathcal{L}_0 \) is defined by (2.5). Moreover, we have \( \partial_x \psi(x, \mu, \cdot, \nu) \in C_b^1(\mathbb{R}^{d_2}) \). Using Itô’s formula (see Lemma 6.1 below or [11, Proposition 2.1]), we deduce that

\[
\psi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon})
\]

\[
= \psi(\xi, \mathcal{L}_\xi, \eta, \mathcal{L}_\eta) + \int_0^t \left[ \mathcal{L}_1(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon})\psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) + \frac{1}{\epsilon}\mathcal{L}_2(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon})\psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) + \frac{1}{\epsilon^2}\mathcal{L}_0(\mathcal{L}_{X_s^\epsilon}, \mathcal{L}_{Y_s^\epsilon})\psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \right] ds + M_t^1 + \frac{1}{\epsilon}M_t^2 + \frac{1}{\epsilon^2}M_t^3
\]

\[
+ \frac{1}{\epsilon} \int_0^t \text{Tr} \left( (G\sigma_1^*)(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \partial_x \partial_y \psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \right) ds
\]

\[
+ \mathbb{E} \left( \int_0^t F(\tilde{X}_s^\epsilon, \mathcal{L}_{\tilde{X}_s^\epsilon}, \tilde{Y}_s^\epsilon, \mathcal{L}_{\tilde{Y}_s^\epsilon}) \cdot \partial_x \psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) ds \right)
\]

\[
= \frac{1}{2} \text{Tr} \left( G^*(\tilde{X}_s^\epsilon, \mathcal{L}_{\tilde{X}_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \cdot \partial_x \left[ \partial_x \psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) \right] \right)
\]

\[
+ \frac{1}{\epsilon} c(\tilde{X}_s^\epsilon, \mathcal{L}_{\tilde{X}_s^\epsilon}, \tilde{Y}_s^\epsilon, \mathcal{L}_{\tilde{Y}_s^\epsilon}) \cdot \partial_x \psi(X_s^\epsilon, \mathcal{L}_{X_s^\epsilon}, Y_s^\epsilon, \mathcal{L}_{Y_s^\epsilon}) ds,
\]

where the operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are defined by (1.6) and (1.7), respectively, the process \((\tilde{X}_s^\epsilon, \tilde{Y}_s^\epsilon)\) is a copy of the original process \((X_s^\epsilon, Y_s^\epsilon)\) defined on a copy \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) of the
original probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and for \(i = 1, 2, 3\), \(M_i^t\) are martingales defined by

\[
M_1^t := \int_0^t \partial_x \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot G(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) dW_s^1,
\]

\[
M_2^t := \int_0^t \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \sigma_1(\mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) dW_s^1,
\]

\[
M_3^t := \int_0^t \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \sigma_2(\mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) dW_s^2.
\]

Multiplying \(\varepsilon^2\) from both sides of (4.2), taking expectation and in view of the equation (4.1), we obtain

\[
\mathbb{E} \left| \int_0^t f(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) ds \right|^2 \leq C_1 \left[ \varepsilon^4 \mathbb{E} |\psi(\xi, \mathcal{L}_t, \eta, \mathcal{L}_t)|^2 + \varepsilon^4 \mathbb{E} |\psi(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon)|^2 \right.
\]

\[
+ \varepsilon^4 \mathbb{E} |M_1^t|^2 + \varepsilon^2 \mathbb{E} |M_2^t|^2 + \varepsilon^2 \mathbb{E} |M_3^t|^2 \left. \right]
\]

\[
+ C_1 \varepsilon^4 \mathbb{E} \left| \int_0^t \mathcal{L}_1(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) ds \right|^2
\]

\[
+ C_1 \varepsilon^2 \mathbb{E} \left| \int_0^t \mathcal{L}_2(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) ds \right|^2
\]

\[
+ C_1 \varepsilon^2 \mathbb{E} \left| \int_0^t \text{Tr} \left( (G \sigma_1^\varepsilon)(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \partial_x \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \right) ds \right|^2
\]

\[
+ C_1 \varepsilon^4 \mathbb{E} \left| \int_0^t F(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon)(\tilde{X}_s^\varepsilon) \right.
\]

\[
\left. + \frac{1}{2} \text{Tr} \left( GG^*(\tilde{X}_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \partial_y \left[ \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon)(\tilde{X}_s^\varepsilon) \right] \right) ds \right|^2
\]

\[
+ C_1 \varepsilon^2 \mathbb{E} \left| \int_0^t c(\tilde{X}_s^\varepsilon, \mathcal{L}X_s^\varepsilon, \tilde{Y}_s^\varepsilon, \mathcal{L}Y_s^\varepsilon) \cdot \partial_y \psi(X_s^\varepsilon, \mathcal{L}X_s^\varepsilon, Y_s^\varepsilon, \mathcal{L}Y_s^\varepsilon)(\tilde{Y}_s^\varepsilon) ds \right|^2
\]

\[=: \sum_{i=1}^6 \mathcal{U}_i(\varepsilon).\]

By Lemma 3.3, we derive that

\[
\mathbb{E} |\psi(\xi, \mathcal{L}_t, \eta, \mathcal{L}_t)|^2 + \mathbb{E} |\psi(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon)|^2 \leq C_1 (1 + \mathbb{E} |\xi|^2 + \mathbb{E} |\eta|^2 + \mathbb{E} |X_t^\varepsilon|^2 + \mathbb{E} |Y_t^\varepsilon|^2) < \infty.
\]

At the same time, using the Burkholder-Davis-Gundy inequality, we get

\[
\mathbb{E} |M_1^t|^2 + \mathbb{E} |M_2^t|^2 + \mathbb{E} |M_3^t|^2 \leq C_1 \int_0^t (1 + \mathbb{E} |X_s^\varepsilon|^2 + \mathbb{E} |Y_s^\varepsilon|^2) ds < \infty.
\]
Consequently, we have
\[ U_1(\varepsilon) \leq C_1 \varepsilon^2. \]  
(4.3)
Using the assumptions on the coefficients and the regularity of \( \psi \) again, and by Lemma 3.3, we arrive at
\[ \sum_{i=2}^{6} U_i(\varepsilon) \leq C_2 \varepsilon^2 \int_0^t \left( 1 + E|X_s^\varepsilon|^4 + E|Y_s^\varepsilon|^4 \right) ds \leq C_2 \varepsilon^2, \]
which together with (4.3) implies the desired result. The proof is thus finished. \( \square \)

4.2. Proof of Theorem 2.1. Throughout this subsection, we assume that the conditions in Theorem 2.1 hold. Recall that \( X_t^\varepsilon \) and \( \tilde{X}_t \) satisfy the McKean-Vlasov equation (1.1) and (1.2), respectively. In order to prove the strong convergence of \( X_t^\varepsilon \) to \( \tilde{X}_t \), we first give the following lemma, which indicates that the averaged function \( \bar{F} \) and \( \bar{G} \) are Lipschitz continuous.

**Lemma 4.2.** Let \( \bar{F} \) and \( \bar{G} \) are defined by (1.3) and (1.4), respectively. Then for any \( x_1, x_2 \in \mathbb{R}^{d_1} \) and \( \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^{d_1}) \), we have
\[ |\bar{F}(x_1, \mu_1) - \bar{F}(x_2, \mu_2)| \leq C_0 \left( |x_1 - x_2| + W_2(\mu_1, \mu_2) \right), \]  
(4.4)
\[ \|\bar{G}(x_1, \mu_1) - \bar{G}(x_2, \mu_2)\| \leq C_0 \left( |x_1 - x_2| + W_2(\mu_1, \mu_2) \right), \]  
(4.5)
where \( C_0 > 0 \) is a constant.

**Proof.** By the definition of \( \bar{F} \) and the Lipschitz continuity of \( F \), we have
\[ |\bar{F}(x_1, \mu_1) - \bar{F}(x_2, \mu_2)| = \left| \int_{\mathbb{R}^{d_2}} F(x_1, \mu_1, y, \zeta^{\mu_1}) \zeta^{\mu_1}(dy) - \int_{\mathbb{R}^{d_2}} F(x_2, \mu_2, y, \zeta^{\mu_2}) \zeta^{\mu_2}(dy) \right| \]
\[ \leq \left| \int_{\mathbb{R}^{d_2}} (F(x_1, \mu_1, y, \zeta^{\mu_1}) - F(x_2, \mu_2, y, \zeta^{\mu_2})) \zeta^{\mu_1}(dy) \right| \]
\[ + \left| \int_{\mathbb{R}^{d_2}} F(x_2, \mu_2, y, \zeta^{\mu_2}) \zeta^{\mu_1}(dy) - \int_{\mathbb{R}^{d_2}} F(x_2, \mu_2, y, \zeta^{\mu_2}) \zeta^{\mu_2}(dy) \right| \]
\[ \leq C_0 \left( |x_1 - x_2| + W_2(\mu_1, \mu_2) + W_2(\zeta^{\mu_1}, \zeta^{\mu_2}) \right). \]  
(4.6)
Using the definition of Wasserstein distance, (2.1) and Lemma 3.4, we get
\[ W_2(\zeta^{\mu_1}, \zeta^{\mu_2})^2 \leq 3W_2(\zeta^{\mu_1}, \mathcal{L}_{Y_{t,0}}^{\mu_1})^2 + 3W_2(\zeta^{\mu_2}, \mathcal{L}_{Y_{t,0}}^{\mu_2})^2 + 3W_2(\mathcal{L}_{Y_{t,0}}^{\mu_1}, \mathcal{L}_{Y_{t,0}}^{\mu_2})^2 \]
\[ \leq C_0 e^{-2\lambda_0 t} \left( W_2(\zeta^{\mu_1}, \delta_0)^2 + W_2(\zeta^{\mu_2}, \delta_0)^2 \right) + C_0 E|Y_{t,0}^{\mu_1} - Y_{t,0}^{\mu_2}|^2 \]
\[ \leq C_0 e^{-2\lambda_0 t} \left( W_2(\zeta^{\mu_1}, \delta_0)^2 + W_2(\zeta^{\mu_2}, \delta_0)^2 \right) + C_0 W_2(\mu_1, \mu_2)^2. \]

Letting \( t \to \infty \), we obtain
\[ W_2(\zeta^{\mu_1}, \zeta^{\mu_2})^2 \leq C_0 W_2(\mu_1, \mu_2)^2. \]
This together with (4.6) yields (4.4). Similarly, we deduce that
\[ \|\bar{G}(x_1, \mu_1) - \bar{G}(x_2, \mu_2)\| = \|G(x_1, \mu_1, \zeta^{\mu_1}) - G(x_2, \mu_2, \zeta^{\mu_2})\| \]
\[ \leq C_0 \left( |x_1 - x_2| + W_2(\mu_1, \mu_2) + W_2(\zeta^{\mu_1}, \zeta^{\mu_2}) \right), \]
which in turn implies that (4.5) holds. Thus the proof is completed. \( \square \)
Now, we are in the position to give:

**Proof of Theorem 2.1.** In view of (1.1) and (1.2), we have

\[
X_t^\varepsilon - \tilde{X}_t = \int_0^t \left[ F(X_s^\varepsilon, L_{X_s^\varepsilon}, Y_s^\varepsilon, L_{Y_s^\varepsilon}) - \tilde{F}(\tilde{X}_s, L_{\tilde{X}_s}) \right] ds \\
+ \int_0^t \left[ G(X_s^\varepsilon, L_{X_s^\varepsilon}, L_{Y_s^\varepsilon}) - \tilde{G}(\tilde{X}_s, L_{\tilde{X}_s}) \right] dW_s^1 \\
= \int_0^t \left[ F(X_s^\varepsilon, L_{X_s^\varepsilon}, Y_s^\varepsilon, L_{Y_s^\varepsilon}) - \tilde{F}(X_s^\varepsilon, L_{X_s^\varepsilon}) \right] ds \\
+ \int_0^t \left[ \tilde{F}(X_s^\varepsilon, L_{X_s^\varepsilon}) - \tilde{F}(\tilde{X}_s, L_{\tilde{X}_s}) \right] ds \\
+ \int_0^t \left[ \tilde{G}(X_s^\varepsilon, L_{X_s^\varepsilon}) - \tilde{G}(\tilde{X}_s, L_{\tilde{X}_s}) \right] dW_s^1 \\
+ \int_0^t \left[ G(X_s^\varepsilon, L_{X_s^\varepsilon}, L_{Y_s^\varepsilon}) - \tilde{G}(X_s^\varepsilon, L_{X_s^\varepsilon}) \right] dW_s^1.
\]

Taking expectation from both sides of the above equality, we get that there exists a constant $C_0 > 0$ such that for every $t \in [0, T]$,

\[
\mathbb{E}|X_t^\varepsilon - \tilde{X}_t|^2 \leq C_0 \mathbb{E} \left| \int_0^t \left[ F(X_s^\varepsilon, L_{X_s^\varepsilon}, Y_s^\varepsilon, L_{Y_s^\varepsilon}) - \tilde{F}(X_s^\varepsilon, L_{X_s^\varepsilon}) \right] ds \right|^2 \\
+ C_0 \int_0^t \mathbb{E}|\tilde{F}(X_s^\varepsilon, L_{X_s^\varepsilon}) - \tilde{F}(\tilde{X}_s, L_{\tilde{X}_s})|^2 ds \\
+ C_0 \int_0^t \mathbb{E}|\tilde{G}(X_s^\varepsilon, L_{X_s^\varepsilon}) - \tilde{G}(\tilde{X}_s, L_{\tilde{X}_s})|^2 ds \\
+ C_0 \int_0^t \mathbb{E}|G(X_s^\varepsilon, L_{X_s^\varepsilon}, L_{Y_s^\varepsilon}) - \tilde{G}(X_s^\varepsilon, L_{X_s^\varepsilon})|^2 ds \\
=: \mathcal{J}_1(\varepsilon) + \mathcal{J}_2(\varepsilon) + \mathcal{J}_3(\varepsilon) + \mathcal{J}_4(\varepsilon). \tag{4.7}
\]

In what follows, we estimate the above four terms one by one. To control the first term, note that by the definition of $\tilde{F}(x, \mu)$, we have

\[
\int_{\mathbb{R}^d_2} \left[ F(x, \mu, y, \zeta^\mu) - \tilde{F}(x, \mu, y, \zeta^\mu) \right] \zeta^\mu(dy) = 0.
\]

Moreover, by the assumptions that $\sigma_1, \sigma_2, b \in C_b^{(1, 2), (1, 1)}$, $F \in C_b^{(2, 1), (2, 1)}$ and Lemma 3.2, we have

\[
F(x, \mu, y, \nu) - \tilde{F}(x, \mu) \in C_b^{(2, 1), (2, 1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})).
\]

This together with the assumption $\partial_x F(x, \mu, \cdot, \nu) \in C_b^1(\mathbb{R}^{d_2})$ and Lemma 4.1 yields

\[
\mathcal{J}_1(\varepsilon) \leq C_1 \varepsilon^2. \tag{4.8}
\]
For the second and third terms, using Lemma 4.2 we deduce that
\[
\mathcal{J}_2(\varepsilon) + \mathcal{J}_3(\varepsilon) \leq C_2 \int_0^t \left[ \mathbb{E}|X_s^\varepsilon - \bar{X}_s|^2 + \mathcal{W}_2(\mathcal{L}_{X_s^\varepsilon}, \mathcal{L}_{\bar{X}_s})^2 \right] ds
\]

\[
\leq C_3 \int_0^t \mathbb{E}|X_s^\varepsilon - \bar{X}_s|^2 ds.
\]

(4.9)

As for \( \mathcal{J}_4(\varepsilon) \), by the definition of \( \hat{G}(x, \mu) \) and the assumption on \( G \), we have

\[
\mathcal{J}_4(\varepsilon) \leq C_4 \int_0^t \mathcal{W}_2(\mathcal{L}_{Y_{s/t}^\varepsilon}^{\mu, \eta} |_{\mu = \mathcal{L}_{X_s^\varepsilon}}, \zeta |_{\mathcal{L}_{X_s^\varepsilon}}) ds + C_4 \int_0^t \mathcal{W}_2(\mathcal{L}_{\tilde{Y}_s^\varepsilon}, \mathcal{L}_{Y_{s/t}^\varepsilon}^{\mu, \eta} |_{\mu = \mathcal{L}_{X_s^\varepsilon}}) ds
\]

\[
=: \mathcal{J}_{4,1}(\varepsilon) + \mathcal{J}_{4,2}(\varepsilon),
\]

(4.10)

where \( Y_{t}^{\mu, \eta} \) is the unique strong solution of the equation (1.5). Applying (2.1), (3.5) and Lemma 3.3, we get

\[
\mathcal{J}_{4,1}(\varepsilon) = C_4 \int_0^t \mathcal{W}_2(\mathcal{L}_{Y_{s/t}^{\mu, \eta}} |_{\mu = \mathcal{L}_{X_s^\varepsilon}}, \zeta |_{\mathcal{L}_{X_s^\varepsilon}}) ds
\]

\[
\leq C_4 \int_0^t e^{-\frac{2\beta_0 s}{\varepsilon}} \cdot \mathcal{W}_2(\zeta |_{\mathcal{L}_{X_s^\varepsilon}}, \nu) ^2 ds
\]

\[
\leq C_4 \int_0^t e^{-\frac{2\beta_0 s}{\varepsilon}} \cdot (1 + \mathcal{W}_2(\mathcal{L}_{X_s^\varepsilon}, \delta_0)^2) ds \leq C_4 \varepsilon^2.
\]

(4.11)

To control \( \mathcal{J}_{4,2}(\varepsilon) \), for every \( t \geq 0 \), we let \( \tilde{Y}_t^\varepsilon = Y_{t/\varepsilon}^\varepsilon \). Then it is easy to see that \( \tilde{Y}_t^\varepsilon \) satisfies the following equation:

\[
\tilde{Y}_t^\varepsilon = \eta + \varepsilon \int_0^t c(X_{s/t}^\varepsilon, \mathcal{L}_{X_{s/t}^\varepsilon}, \tilde{Y}_s^\varepsilon, \mathcal{L}_{\tilde{Y}_s}) ds + \int_0^t b(X_{s/t}^\varepsilon, \tilde{Y}_s, \mathcal{L}_{\tilde{Y}_s}) ds
\]

\[
+ \int_0^t \sigma_1(\mathcal{L}_{X_{s/t}^\varepsilon}, \tilde{Y}_s, \mathcal{L}_{\tilde{Y}_s}) d\tilde{W}_s^1 + \int_0^t \sigma_2(\mathcal{L}_{X_{s/t}^\varepsilon}, \tilde{Y}_s, \mathcal{L}_{\tilde{Y}_s}) d\tilde{W}_s^2.
\]

Note that

\[
\mathcal{W}_2(\mathcal{L}_{Y_{t/\varepsilon}^\varepsilon}, \mathcal{L}_{Y_{t/\varepsilon}^{\mu, \eta}} |_{\mu = \mathcal{L}_{X_t^\varepsilon}})^2 = \mathcal{W}_2(\mathcal{L}_{\tilde{Y}_t^\varepsilon}, \mathcal{L}_{Y_{t/\varepsilon}^{\mu, \eta}} |_{\mu = \mathcal{L}_{X_t^\varepsilon}})^2 \leq \left[ \mathbb{E}|\tilde{Y}_{t/\varepsilon}^\varepsilon - Y_{t/\varepsilon}^{\mu, \eta}|^2 \right]_{\mu = \mathcal{L}_{X_t^\varepsilon}}.
\]

(4.12)

By Itô’s formula and the assumption \( (H^{a,b}) \), we deduce that for every \( t > 0 \),

\[
d\mathbb{E}|\tilde{Y}_{t}^\varepsilon - Y_{t}^{\mu, \eta}|^2 = \mathbb{E}[2(\tilde{Y}_{t}^\varepsilon - Y_{t}^{\mu, \eta}, b(\mathcal{L}_{X_{t/2}^\varepsilon, \tilde{Y}_{t}^\varepsilon, \mathcal{L}_{\tilde{Y}_{t}^\varepsilon}) - b(\mu, Y_{t}^{\mu, \eta}, \mathcal{L}_{Y_{t}^{\mu, \eta}}))
\]

\[
+ \|\sigma_1(\mathcal{L}_{X_{t/2}^\varepsilon}, \tilde{Y}_{t}^\varepsilon, \mathcal{L}_{\tilde{Y}_{t}^\varepsilon}) - \sigma_1(\mu, Y_{t}^{\mu, \eta}, \mathcal{L}_{Y_{t}^{\mu, \eta}})\|^2
\]

\[
+ \|\sigma_2(\mathcal{L}_{X_{t/2}^\varepsilon}, \tilde{Y}_{t}^\varepsilon, \mathcal{L}_{\tilde{Y}_{t}^\varepsilon}) - \sigma_2(\mu, Y_{t}^{\mu, \eta}, \mathcal{L}_{Y_{t}^{\mu, \eta}})\|^2]dt
\]

\[
\leq - (c_2 - c_1) \mathbb{E}|\tilde{Y}_{t}^\varepsilon - Y_{t}^{\mu, \eta}|^2 dt + C_4 \mathcal{W}_2(\mathcal{L}_{X_{t/2}^\varepsilon}, \mu)^2 dt
\]

\[
+ C_4 \varepsilon^2 (1 + \mathbb{E}|X_{t/2}^\varepsilon|^2 + \mathbb{E}|Y_{t/2}^\varepsilon|^2) dt.
\]
which together with the comparison theorem implies
\[ E|\hat{Y}_t^\varepsilon - Y_t^\mu_{\eta}|^2 \leq C_4 \int_0^t e^{-(c_2 - c_1)(t-s)} \cdot \mathcal{W}_2(\mathcal{L}X_{t,s}^\varepsilon, \mu)^2 ds \]
\[ + C_4 \varepsilon^2 \int_0^t e^{-(c_2 - c_1)(t-s)} \cdot (1 + E|X_{t,s}^\varepsilon|^2 + E|Y_{t,s}^\varepsilon|^2) ds. \]
As a result, we have
\[
\left[ E|\hat{Y}_t^{\varepsilon}_{t/s^2} - Y_{t/s^2}^{\mu_{\eta}}|^2 \right]_{\mu = \mathcal{L}X_t^\varepsilon} \\
\leq C_4 \int_0^{\varepsilon^2} e^{-(c_2 - c_1)(t-s)} \cdot \mathcal{W}_2(\mathcal{L}X_{t,s}^\varepsilon, \mathcal{L}X_t^\varepsilon)^2 ds \\
+ C_4 \varepsilon^2 \int_0^{\varepsilon^2} e^{-(c_2 - c_1)(t-s)} \cdot (1 + E|X_{s}^\varepsilon|^2 + E|Y_{s}^\varepsilon|^2) ds \\
\leq C_4 \int_0^t e^{-(c_2 - c_1)(\frac{t-s}{\varepsilon^2})} \cdot E|X_{s}^\varepsilon - X_t^\varepsilon|^2 ds \\
+ C_4 \int_0^t e^{-(c_2 - c_1)(\frac{t-s}{\varepsilon^2})} \cdot (1 + E|X_{s}^\varepsilon|^2 + E|Y_{s}^\varepsilon|^2) ds \\
\leq C_4 \int_0^t e^{-(c_2 - c_1)(\frac{t-s}{\varepsilon^2})} \cdot \frac{t-s}{\varepsilon^2} ds + C_4 \varepsilon^2 \leq C_4 \varepsilon^2,
\]
where the constant $C_4$ is independent of $\varepsilon$. Taking this back into (4.12), we arrive at
\[ J_{4,2}(\varepsilon) \leq C_4 \varepsilon^2. \quad (4.13) \]
Substituting (4.13) and (4.11) into (4.10) gives
\[ J_4(\varepsilon) \leq C_4 \varepsilon^2. \]
This together with (4.7), (4.8) and (4.9) yields
\[ E|X_t^\varepsilon - \bar{X}_t|^2 \leq C_5 \varepsilon^2 + C_5 \int_0^t E|X_{s}^\varepsilon - \bar{X}_s|^2 ds, \]
which in turn implies the desired assertion by Gronwall’s inequality. \hfill \Box

5. Functional central limit type theorem

In this section, we study the functional central limit type theorem for the system (1.1). Recall that assumption (2.3) holds, i.e., the diffusion coefficient $G$ in the slow equation does not depend on the distribution of the fast motion. Note that in this case, we get
\[ \bar{G}(x, \mu) = G(x, \mu). \]
We shall first derive some weak fluctuation estimates in Subsection 5.1. Then we give the proof of Theorem 2.3 in Subsection 5.2.
5.1. Weak Fluctuation estimates. Recall that $Z_t^\varepsilon$ and $\bar{Z}_t$ are defined by (2.2) and (2.12), respectively. By definition, we can write

$$
dZ_t^\varepsilon = \frac{1}{\varepsilon}\delta F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})dt$$

$$+ \left(\frac{1}{\varepsilon}[\bar{F}(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) - F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})]dt + \frac{1}{\varepsilon}[G(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) - G(X_t, \mathcal{L}_{X_t})]dW_t^1\right),$$

where $\delta F$ is defined by (2.4). For the sake of convenience, we define

$$\mathcal{L}_3 := \mathcal{L}_3(x, \mu, y, \nu) := \sum_{i=1}^{d_1} \left[ F_i(x, \mu, y, \nu) - \bar{F}_i(x, \mu) \right] \frac{\partial}{\partial z_i}, \quad (5.1)$$

and

$$\mathcal{L}_4^\varepsilon := \mathcal{L}_4^\varepsilon(x, \mu, \tilde{x}, \tilde{\mu}) := \frac{1}{\varepsilon} \sum_{i=1}^{d_1} \left[ \bar{F}_i(x, \mu) - \bar{\bar{F}}_i(\tilde{x}, \tilde{\mu}) \right] \frac{\partial}{\partial z_i}$$

$$+ \frac{1}{2\varepsilon^2} \sum_{i,j=1}^{d_1} \left[ \left[ G(x, \mu) - G(\tilde{x}, \tilde{\mu}) \right]\left[ G(x, \mu) - G(\tilde{x}, \tilde{\mu}) \right]^* \right]_{ij} \frac{\partial^2}{\partial z_i \partial z_j}$$

$$+ \frac{1}{\varepsilon} \sum_{i,j=1}^{d_1} \left[ G(x, \mu)\left[ G(x, \mu) - G(\tilde{x}, \tilde{\mu}) \right]^* \right]_{ij} \frac{\partial^2}{\partial x_i \partial z_j}. \quad (5.2)$$

Let $f(t, x, \mu, y, \nu, z, \pi)$ be a function satisfying the centering condition, i.e., for every fixed $(t, x, \mu, z, \pi) \in \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$,

$$\int_{\mathbb{R}^{d_2}} f(t, x, \mu, y, \nu, z, \pi) \zeta^\mu(dy) = 0. \quad (5.3)$$

Then, we consider the following Poisson equation:

$$\mathcal{L}_3(\mu, \nu) \psi(t, x, \mu, y, \nu, z, \pi) = -f(t, x, \mu, y, \nu, z, \pi), \quad (5.4)$$

where $(t, x, \mu, z, \pi)$ are regarded as parameters. We have the following fluctuation estimate for the process $f(t, X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon})$.

**Lemma 5.1.** Let $(\mathcal{H}^{q,b})$ hold. Assume that $F, G$ and $c$ are Lipschitz continuous and $c_1, c_2, b \in C_0(\mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon})$.

Then for $f \in C_0^1(\mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon})$ satisfying $\partial_x f, \partial_y f(t, x, \mu, \nu, z, \pi) \in C_0^1(\mathbb{R}^{d_2})$ and (5.3), we have

$$\mathbb{E}\left( \int_0^t f(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$\leq C_0 \varepsilon^2 + C_0 \varepsilon \mathbb{E}\left( \int_0^t \mathcal{L}_3 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_4^\varepsilon \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \text{Tr}((G_1^\varepsilon)(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon}))ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_5 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_6 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_7 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_8 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_9 \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_{10} \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_{11} \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right)$$

$$+ \mathbb{E}\left( \int_0^t \mathcal{L}_{12} \psi(s, X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Y_s^\varepsilon, \mathcal{L}_{Y_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_{Z_s^\varepsilon})ds \right),$$

where $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \mathcal{H}_4 \subset \mathcal{H}_5 \subset \mathcal{H}_6 \subset \mathcal{H}_7 \subset \mathcal{H}_8 \subset \mathcal{H}_9 \subset \mathcal{H}_{10} \subset \mathcal{H}_{11} \subset \mathcal{H}_{12}$ and $\mathcal{L}_i$ is the operator defined by

$$\mathcal{L}_i := \mathcal{L}_i(x, \mu, y, \nu) := \sum_{i=1}^{d_1} \left[ F_i(x, \mu, y, \nu) - \bar{F}_i(x, \mu) \right] \frac{\partial}{\partial z_i}, \quad (5.5)$$

$$i = 1, 2, \ldots, 12.$$
+ \mathbb{E} \left( \int_0^t \text{Tr}(\sigma_1(\mathcal{L}_{X_s}, Y_{s}^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}) \frac{[G(X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}) - G(\bar{X}_s, \mathcal{L}_{X_s})]}{\varepsilon} \times \partial_y \partial_z \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon})) ds \right) \\
+ \mathbb{E} \left( \int_0^t c(\bar{X}_s, \mathcal{L}_{X_s}, \bar{Y}_s, \mathcal{L}_{Y_s}) \cdot \partial_y \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon})(\bar{Y}_s^{\varepsilon}) ds \right) \\
+ \mathbb{E} \left( \int_0^t \delta F(\bar{X}_s, \mathcal{L}_{X_s}, \bar{Y}_s, \mathcal{L}_{Y_s}) \cdot \partial_y \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon})(\bar{Z}_s^{\varepsilon}) ds \right) \right), \quad (5.5)

where \mathcal{L}_2 and \mathcal{L}_3 are given by (1.7) and (5.1), respectively, and \( C_0 > 0 \) is a constant independent of \( \varepsilon \).

**Remark 5.2.** Note that under the above assumptions and according to Theorem 3.1, we have \( \psi \in C^0_b(\mathbb{R}^{d_2}) \) and \( \partial_x \psi, \partial_y \psi(t, x, \mu, \nu, z, \pi) \in C^1_b(\mathbb{R}^{d_2}) \). Then we can compute that

\[
\mathbb{E} \left( \int_0^t \text{Tr}(\sigma_1(\mathcal{L}_{X_s}, Y_{s}^{\varepsilon}, \mathcal{L}_{Y_s}) \frac{[G(X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}) - G(\bar{X}_s, \mathcal{L}_{X_s})]}{\varepsilon} \times \partial_y \partial_z \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon})) ds \right) \\
\leq C_0 \int_0^t \mathbb{E} \left[ (1 + |Y_s| + W_2(\mathcal{L}_{X_s}, \delta_0) + W_2(\mathcal{L}_{Y_s}, \delta_0)) \left( |Z_s^{\varepsilon}| + (\mathbb{E}|Z_s^{\varepsilon}|^2)^{1/2} \right) \right] ds \\
\leq C_0 \int_0^t (1 + \mathbb{E}|X_s^{\varepsilon}|^2 + \mathbb{E}|Y_s^{\varepsilon}|^2 + \mathbb{E}|Z_s^{\varepsilon}|^2) ds < \infty.
\]

Meanwhile, the other terms on the right hand side of (5.5) can be controlled similarly. Therefore, we have

\[
\mathbb{E} \left( \int_0^t f(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon}) ds \right) \leq C_0 \varepsilon. \quad (5.6)
\]

However, the homogenization effects of the terms involving expectations in (5.5) will appear when we investigate the central limit theorem, so we keep them for later use.

**Proof.** Let \( \psi \) be the solution of the Poisson equation (5.4). Then by Itô’s formula, we deduce that

\[
\psi(t, X_t^{\varepsilon}, \mathcal{L}_{X_t}^{\varepsilon}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t}^{\varepsilon}, Z_t^{\varepsilon}, \mathcal{L}_{Z_t}^{\varepsilon}) = \psi(0, \xi, \mathcal{L}_\xi, \eta, \mathcal{L}_\eta, 0, \delta_0) \\
+ \int_0^t \left( \partial_s + \mathcal{L}_1 + \mathcal{L}_4 \right) \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon}) ds \\
+ \frac{1}{\varepsilon} \int_0^t \text{Tr}(\sigma_1^*) (X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}) \cdot \partial_y \partial_z \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon}) ds \\
+ \frac{1}{\varepsilon} \int_0^t \left( \mathcal{L}_2 + \mathcal{L}_3 \right) \psi(s, X_s^{\varepsilon}, \mathcal{L}_{X_s}^{\varepsilon}, Y_s^{\varepsilon}, \mathcal{L}_{Y_s}^{\varepsilon}, Z_s^{\varepsilon}, \mathcal{L}_{Z_s}^{\varepsilon}) ds
\]
\[
+ \frac{1}{\varepsilon^2} \int_0^t \mathcal{L}_0(\mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi) \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) ds \\
+ \frac{1}{\varepsilon^2} \int_0^t \text{Tr} \left( \sigma_1(\mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi) [G(X \xi, \mathcal{L}_X \xi) - G(\tilde{X} \xi, \mathcal{L}_X \xi)]^* \times \partial_\mu \partial_\nu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \right) ds \\
+ \tilde{\mathbb{E}} \left( \int_0^t F(\tilde{X} \xi, \mathcal{L}_X \xi, \tilde{Y} \xi, \mathcal{L}_Y \xi) \cdot \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi)(\tilde{X} \xi) \right. \\
+ \frac{1}{2} \text{Tr} \left( GG^*(\tilde{X} \xi, \mathcal{L}_X \xi) \cdot \partial_\xi \left[ \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi)(\tilde{X} \xi) \right] \right) \\
+ \frac{1}{\varepsilon} \left[ F(\tilde{X} \xi, \mathcal{L}_X \xi, \tilde{Y} \xi, \mathcal{L}_Y \xi) \right. \\
\left. \times \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi)(\tilde{X} \xi) \right] \\
+ \frac{1}{\varepsilon} [\tilde{F}(\tilde{X} \xi, \mathcal{L}_X \xi) - \tilde{F}(\tilde{X} \xi, \mathcal{L}_X \xi)] \cdot \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi)(\tilde{X} \xi) \right) ds \right) \\
\tag{5.7}
\]

where the operators \( \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \) and \( \mathcal{L}_4 \) are given by (2.5), (1.6), (1.7), (5.1) and (5.2), respectively, the process \((\tilde{X} \xi, \tilde{X} \xi, \tilde{Y} \xi, \tilde{Z} \xi)\) is a copy of the original process \((X \xi, X \xi, Y \xi, Z \xi)\) defined on a copy \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) of the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and for \(i = 1, 2, 3, 4\), \(M_i\) are martingales defined by

\[
M_1^1 := \int_0^t \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \cdot G(X \xi, \mathcal{L}_X \xi) dW_1^1, \\
M_2^1 := \int_0^t \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \cdot \sigma_1(\mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi) dW_1^1, \\
M_3^1 := \int_0^t \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \cdot \sigma_2(\mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi) dW_2^1, \\
M_4^1 := \int_0^t \partial_\mu \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \cdot \left[ G(X \xi, \mathcal{L}_X \xi) - G(\tilde{X} \xi, \mathcal{L}_X \xi) \right] dW_1^1.
\]

Multiplying \(\varepsilon^2\) and taking expectation on both sides of (5.7), and in view of (5.4), we obtain

\[
\mathbb{E} \left( \int_0^t f(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) ds \right) \\
= \varepsilon^2 \mathbb{E} \left[ \psi(0, \xi, \mathcal{L}_X \xi, \eta, \mathcal{L}_Y \xi, 0, \mathcal{L}_Z \xi) - \psi(t, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) \right] \\
+ \varepsilon^2 \mathbb{E} \left( \int_0^t \left( \partial_\mu + \mathcal{L}_1 + \mathcal{L}_4 \right) \psi(s, X \xi, \mathcal{L}_X \xi, Y \xi, \mathcal{L}_Y \xi, Z \xi, \mathcal{L}_Z \xi) ds \right)
\]
Let us first deal with the term involving \( G \) and the regularity of \( \psi \). We have that for some \( C_1 > 0 \),

\[
\mathbb{E} \left( \int_0^t \mathcal{L}^\varepsilon \psi(s, X^\varepsilon_s, L_{X^\varepsilon_s}, Y^\varepsilon_s, Z^\varepsilon_s, L_{Z^\varepsilon_s}) \right) \leq C_1 \mathbb{E} \left( \int_0^t \frac{1}{\varepsilon^2} \left[ \frac{\|G(X^\varepsilon_s, L_{X^\varepsilon_s}) - F(X^\varepsilon_s, L_{X^\varepsilon_s})\|}{\varepsilon} \right]^2 \right)
\]

Let us first deal with the term involving \( \mathcal{L}^\varepsilon \). Using the assumption on \( F \) and by Lemma 4.2, we have that \( \bar{F} \) satisfies the Lipschitz condition. This, together with the assumption on \( G \) and the regularity of \( \psi \), yields that for some \( C_1 > 0 \),

\[
\mathbb{E} \left( \int_0^t \mathcal{L}^\varepsilon \psi(s, X^\varepsilon_s, L_{X^\varepsilon_s}, Y^\varepsilon_s, Z^\varepsilon_s, L_{Z^\varepsilon_s}) \right) \leq C_1 \mathbb{E} \left( \int_0^t \frac{1}{\varepsilon^2} \left[ \frac{\|G(X^\varepsilon_s, L_{X^\varepsilon_s}) - F(X^\varepsilon_s, L_{X^\varepsilon_s})\|}{\varepsilon} \right]^2 \right)
\]
\[ + (1 + |X_s^\varepsilon| + \mathcal{W}_2(L_{X_s}, \delta_0)) \frac{\|G(X_s^\varepsilon, L_{X_s}) - G(X_s, L_{X_s})\|}{\varepsilon} ds \] 

\[ \leq C_1 \int_0^t (1 + \mathbb{E}|X_s^\varepsilon|^2 + \mathbb{E}|Z_s^\varepsilon|^2) ds < \infty. \quad (5.8) \]

Similarly, we have

\[ \mathcal{V}_4(\varepsilon) + \mathcal{V}_5(\varepsilon) \leq C_2 \varepsilon^2 \mathbb{E} \mathbb{E} \left( \int_0^t \left[ \left| \frac{\bar{F}(X_s^\varepsilon, L_{X_s}) - \bar{F}(\bar{X}_s, L_{X_s})}{\varepsilon} \right| + \frac{\|G(X_s^\varepsilon, L_{X_s}) - G(\bar{X}_s, L_{X_s})\|^2}{\varepsilon^2} \right] ds \right) \]

\[ \leq C_2 \varepsilon^2 \int_0^t (1 + \mathbb{E}|Z_s^\varepsilon|^2) ds \leq C_2 \varepsilon^2. \]

This together with (5.8) and the assumptions on coefficients implies that

\[ \sum_{i=1}^{5} \mathcal{V}_i(\varepsilon) \leq C_3 \varepsilon^2 + C_3 \varepsilon^2 \int_0^t (1 + \mathbb{E}|X_s^\varepsilon|^2 + \mathbb{E}|Y_s^\varepsilon|^2) ds \leq C_3 \varepsilon^2, \]

which in turn yields the desired conclusion. Thus the proof is completed. \[\square\]

5.2. **Proof of Theorem 2.3.** Throughout this subsection, we assume that the conditions in Theorem 2.3 hold. Let \( \bar{X}_t^s,\xi \) be the unique solution of the equation (1.2) starting from the initial data \( \xi \in L^2(\Omega) \) at time \( s \), and \( Z_t^s,\xi,\vartheta \) be the unique solution of the equation (2.12) with the initial value \( \vartheta \in L^2(\Omega) \) at time \( s \). Namely, for \( t \geq s \),

\[ d\bar{X}_t^s,\xi = \bar{F}(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) dt + G(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) dW_t^1, \quad \bar{X}_s^s,\xi = \xi, \]

and

\[ dZ_t^s,\xi,\vartheta = \partial_x \bar{F}(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) Z_t^s,\xi,\vartheta dt + \mathbb{E} \left[ \partial_x G(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi})(\bar{X}_t^s,\xi) Z_t^s,\xi,\vartheta \right] dt \]

\[ + \frac{c \cdot \partial_x \Phi(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) dt + \mathbb{E} \left[ c \cdot \partial_x \Phi(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi})(\bar{X}_t^s,\xi) \right] dt \]

\[ + \frac{\sigma_x \cdot c \cdot \partial_x \Phi(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) \cdot G(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) dt \]

\[ + \partial_x G(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) Z_t^s,\xi,\vartheta dt W_t^1 \]

\[ + \mathbb{E} \left[ \partial_x G(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi})(\bar{X}_t^s,\xi) Z_t^s,\xi,\vartheta \right] dW_t^1 \]

\[ + \mathbb{E} \left[ \partial_x \Phi(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi}) \right] dW_t^1 + \sqrt{\Sigma(\bar{X}_t^s,\xi, L_{\bar{X}_t^s,\xi})} d\tilde{W}_t, \quad \bar{Z}_t^s,\xi,\vartheta = \vartheta, \]

where the positive semi-definite matrix \( \Sigma(x, \mu) \) is given by (2.13). For fixed \( T > 0 \) and function \( \varphi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \), we consider the following Cauchy problem on \([0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \).
By Itô’s formula, we derive
\[
\begin{align*}
\mathcal{P}_2(\mathbb{R}^d_1) \times \mathcal{P}_2(\mathbb{R}^d_1) : \\
\partial_t u(t, \xi, \theta) + \mathbb{E}[\tilde{F}(\xi, \theta) \cdot \partial_\mu u(t, \xi, \theta)(\xi)] \\
+ \frac{1}{2} \mathbb{E}\left[\text{Tr}\left(GG^*(\xi, \theta) \cdot \partial_\mu [\partial_\mu u(t, \xi, \theta)(\xi)]\right)\right] \\
+ \mathbb{E}\left[\partial_\mu \tilde{F}(\xi, \theta) \partial_\theta + \mathbb{E}[\partial_\mu \tilde{F}(\xi, \theta)(\xi, \tilde{\theta})] + c \cdot \partial_\theta \Phi(\xi, \theta) + \mathbb{E}\left[c \cdot \partial_\theta \Phi(\xi, \theta)(\xi, \tilde{\theta})\right]\right] \\
+ \frac{1}{2} \mathbb{E}\left[\text{Tr}(\partial_\theta \Phi^* \cdot \sigma_1(\xi, \theta) \cdot \partial_\mu [\partial_\theta u(t, \xi, \theta)(\theta)])\right] \\
+ \mathbb{E}\left[\text{Tr}(\partial_\theta \Phi^* \cdot \sigma_1(\xi, \theta) \cdot \partial_\mu [\partial_\theta u(t, \xi, \theta)(\theta)])\right] \\
+ \mathbb{E}\left[\text{Tr}(\delta F^* \cdot \phi^* \cdot \partial_\theta [\partial_\mu u(t, \xi, \theta)(\theta)])\right] = 0,
\end{align*}
\]  
\tag{5.9}
\]

Then by Lemma 3.2, we have $\tilde{F} \in (C_b^{4,(1,3)} \cap C_b^{4,(2,2)} \cap C_b^{4,(3,1)})(\mathbb{R}^d_1 \times \mathcal{P}_2(\mathbb{R}^d_1)), c \cdot \partial_\theta \Phi, \sigma_1^* \cdot \partial_\theta \delta_\theta \Phi, \delta F^* \cdot \phi^* \in C_b^{3,(2,2)}(\mathbb{R}^d_1 \times \mathcal{P}_2(\mathbb{R}^d_1))$ and $c \cdot \partial_\theta \Phi(x, \mu)(\tilde{x}) \in C_b^{3,(2,2)}(\mathbb{R}^d_1 \times \mathcal{P}_2(\mathbb{R}^d_1) \times \mathbb{R}^d_1)$. Therefore, there exists a unique solution $u \in C_b^{1,(2,1),(3,1)}([0, T] \times \mathcal{P}_2(\mathbb{R}^d_1) \times \mathcal{P}_2(\mathbb{R}^d_1))$ to the equation (5.9) by [8, Theorem 7.2], which is given by
\[
u(t, \xi, \theta) := \varphi(\mathcal{L}_{Z_\tau}^{\xi, \theta}).
\]  
\tag{5.10}
\]

Now, we are in the position to give:

**Proof.** Let $u(t, \xi, \theta)$ be defined by (5.10). Then we have
\[
\mathcal{K}(\varepsilon) := \varphi(\mathcal{L}_{Z_\tau}^{\xi, \theta}) - \varphi(\mathcal{L}_{Z_\tau}) = u(T, \mathcal{L}_{Z_\tau}^{\xi, \theta}) - u(0, \mathcal{L}, \delta_0).
\]

By Itô’s formula, we derive
\[
\mathcal{K}(\varepsilon) = \mathbb{E}\left(\int_0^T \partial_t u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1}) + F(X^\varepsilon, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Y^\varepsilon}, \mathcal{L}_{Y^\varepsilon_1}) \cdot \partial_\mu u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1})(X^\varepsilon_1)
\right.
\]
\[
+ \frac{1}{2} \text{Tr}\left(GG^*(X^\varepsilon, \mathcal{L}_{X^\varepsilon_1}) \cdot \partial_\mu [\partial_\mu u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1})(X^\varepsilon_1)]\right)dt
\]
\[
+ \frac{1}{2} \mathbb{E}\left(\int_0^T [F(X^\varepsilon, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Y^\varepsilon}) - \tilde{F}(X^\varepsilon_1, \mathcal{L}_{X^\varepsilon_1})] \cdot \partial_\mu u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1})(Z^\varepsilon_1)dt\right)
\]
\[
+ \mathbb{E}\left(\int_0^T \frac{\tilde{F}(X^\varepsilon_1, \mathcal{L}_{X^\varepsilon_1}) - \tilde{F}(X^\varepsilon_1, \mathcal{L}_{X^\varepsilon_1})}{\varepsilon} \cdot \partial_\mu u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1})(Z^\varepsilon_1)dt\right)
\]
\[
+ \mathbb{E}\left(\int_0^T \tilde{F}(X^\varepsilon_1, \mathcal{L}_{X^\varepsilon_1}) \cdot \partial_\mu u(t, \mathcal{L}_{X^\varepsilon_1}, \mathcal{L}_{Z^\varepsilon_1})(Z^\varepsilon_1)dt\right) = 0.
\]
Consequently, it follows by (5.3). Then we define
\[
\bar{\Phi}(t, x, \mu, y, \nu, z, \pi) = \Phi(x, \mu, y, \nu) \cdot \partial_\pi u(t, \mu, \pi)(z),
\]
and get
\[
\mathcal{L}_0(\mu, \nu, \pi)\bar{\Phi}(t, x, \mu, y, \nu, z, \pi) = -[F(x, \mu, y, \nu) - \bar{F}(x, \mu)] \cdot \partial_\pi u(t, \mu, \pi)(z).
\]
Moreover, we have \([F - \bar{F}] \cdot \partial_\pi u \in C^{1,2,1,1,2,2,1,1}_b\) and \(\partial_\pi F(x, \mu, \nu) \in C^1_b(\mathbb{R}^{d_2})\). Consequently, it follows by (5.5) that
\[
\frac{1}{\varepsilon} \mathbb{E}\left( \int_0^T [F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) - \bar{F}(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})] \cdot \partial_\pi u(t, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Z_t^\varepsilon})(Z_t^\varepsilon)dt \right)
\]
\[
\leq C_0 \varepsilon + C_0 \left[ \mathbb{E}\left( \int_0^T (\mathcal{Z}_2 + \mathcal{Z}_3) \bar{\Phi}(t, X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon})dt \right)
\right.
\]
\[
+ \mathbb{E}\left( \int_0^T \text{Tr}\left( (G\sigma_1^\varepsilon)(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) \cdot \partial_x \partial_y \bar{\Phi}(t, X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon}) \right)dt \right)
\]
\[
+ \mathbb{E}\left( \frac{\varepsilon}{\varepsilon} \mathbb{E}\left( \int_0^T \text{Tr}\left( (\mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) \cdot \partial_\pi \bar{\Phi}(t, X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}, Z_t^\varepsilon, \mathcal{L}_{Z_t^\varepsilon})(\bar{Z}_t^\varepsilon)dt \right) \right)
\).}

This together with (5.11) yields
\[
\mathcal{K}(\varepsilon) \leq C_0 \varepsilon + \mathbb{E}\left( \int_0^T \partial_t u(t, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Z_t^\varepsilon}) + F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) \cdot \partial_\mu u(t, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Z_t^\varepsilon})(Z_t^\varepsilon)dt \right)
\]
\[
+ \frac{1}{2} \text{Tr}\left( G G^\varepsilon(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) \cdot \partial_x \left[ \partial_\mu u(t, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Z_t^\varepsilon})(X_t^\varepsilon) \right] \right)dt
\]
\[
+ \mathbb{E}\left( \int_0^T \text{Tr}\left( \frac{\bar{F}(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) - \bar{F}(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})}{\varepsilon} \cdot \partial_\pi u(t, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Z_t^\varepsilon})(Z_t^\varepsilon)dt \right)
\)
\[
+ \frac{1}{2} \mathbb{E}\left( \int_0^T \text{Tr}\left( \frac{[G(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) - G(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon})][G(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}) - G(X_t, \mathcal{L}_{X_t})]}{\varepsilon^2} dt \right)
\).
\[
\times \partial_z \left[ \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(Z_t^\epsilon) \right] dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \left( \mathcal{L}_2 + \mathcal{L}_3 \right) \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon}) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \text{Tr} \left( (G\sigma_1^\epsilon)(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_x \partial_y \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon}) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \text{Tr} \left( \sigma_1(\mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \left[ \frac{G(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}) - G(X_t, \mathcal{L}_{X_t})}{\varepsilon} \right] \\
\times \partial_x \partial_y \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon}) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T c(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, \tilde{Y}_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_x \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon})(\tilde{Y}_t^\epsilon) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \delta F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, \tilde{Y}_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_x \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon})(\tilde{Z}_t^\epsilon) dt \right). 
\] 
(5.12)

For the last term on the right hand side of the above inequality, it is easy to see that for every fixed \((x, y, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d\), the function \((t, \tilde{x}, \mu, \tilde{y}, \nu, \tilde{z}, \pi) \mapsto \delta F(\tilde{x}, \mu, \tilde{y}, \nu) \cdot \partial_x \tilde{\Phi}(t, x, \mu, y, \nu, z, \pi)(\tilde{z})\) satisfies the centering condition. Thus by (5.6) we have

\[
\mathbb{E}\mathbb{E}\left( \int_0^T \delta F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, \tilde{Y}_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_x \tilde{\Phi}(t, X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}, Z_t^\epsilon, \mathcal{L}_{Z_t^\epsilon})(\tilde{Z}_t^\epsilon) dt \right) \leq C_0 \varepsilon. 
\]

Substituting this into (5.12) and in view of (5.9), we derive

\[
\mathcal{H}(\varepsilon) \\
\leq C_0 \varepsilon + \mathbb{E}\left( \int_0^T \left[ F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) - F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}) \right] \cdot \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(X_t^\epsilon) dt \right) \\
+ \mathbb{E}\left( \int_0^T \left[ \frac{\tilde{F}(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}) - \tilde{F}(X_t, \mathcal{L}_{X_t})}{\varepsilon} \right] \cdot \partial_x F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}) Z_t^\epsilon - \mathbb{E}[\partial_x F(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon})(X_t^\epsilon) \tilde{Z}_t^\epsilon] \right) \\
\times \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(Z_t^\epsilon) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \left[ c \cdot \partial_y \Phi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) - c \cdot \partial_y \Phi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}) \right] \cdot \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(Z_t^\epsilon) dt \right) \\
+ C_0\mathbb{E}\mathbb{E}\left( \int_0^T \left[ c(\tilde{X}_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, \tilde{Y}_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_y \Phi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon})(\tilde{Y}_t^\epsilon) - c \cdot \partial_y \Phi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon})(\tilde{X}_t^\epsilon) \right] \\
\times \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(Z_t^\epsilon) dt \right) \\
+ C_0\mathbb{E}\left( \int_0^T \left[ (G\sigma_1^\epsilon)(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \cdot \partial_x \partial_y \Phi(X_t^\epsilon, \mathcal{L}_{X_t^\epsilon}, Y_t^\epsilon, \mathcal{L}_{Y_t^\epsilon}) \right] \\
\times \partial_x u(t, \mathcal{L}_{X_t^\epsilon}, \mathcal{L}_{Z_t^\epsilon})(Z_t^\epsilon) dt \right).
\]
Similarly, according to (2.11) and the estimate (5.6), we also have
\[ \mathcal{H}_2(\varepsilon) + \mathcal{H}_4(\varepsilon) + \mathcal{H}_5(\varepsilon) + \mathcal{H}_7(\varepsilon) \leq C_2 \varepsilon. \]
In what follows, we estimate the remaining three terms one by one. Let us first handle \( \mathcal{H}_2(\varepsilon) \) and \( \mathcal{H}_6(\varepsilon) \). Using the mean value theorem and Theorem 2.1, we deduce that
\[
\mathcal{H}_2(\varepsilon) + \mathcal{H}_6(\varepsilon) \leq C_3 \int_0^T \left( \mathbb{E}(X_t^\varepsilon - \bar{X}_t^\varepsilon)^2 \cdot (1 + \mathbb{E}|Z_t^\varepsilon|^4) \right) dt \leq C_3 \varepsilon.
\]
As for $K(\varepsilon)$, using the same technique as above, by the mean value theorem and (5.6) again, we get that
\[
K(\varepsilon) \leq C_4 \mathbb{E} \left( \int_0^T \left\| \partial_y \Phi(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon) \sigma_1(\mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon) \left[ \frac{G(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon) - G(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon)}{\varepsilon} \right] \right\| dt \right)
\]
\[+ C_4 \mathbb{E} \left( \int_0^T \text{Tr} \left[ \left[ \partial_y \Phi(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon) \sigma_1(\mathcal{L}X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}Y_t^\varepsilon) - \overline{\partial_y \Phi} \sigma_1(X_t^\varepsilon, LX_t^\varepsilon) \right] \right. \] 
\[\left. \times \left[ \partial_x G(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon) Z_t^\varepsilon + \mathbb{E} \left[ \partial_x G(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon) (X_t^\varepsilon) \bar{Z}_t^\varepsilon \right] \right] \right. \] 
\[\left. \times \partial_z \left[ \partial_x u(t, \mathcal{L}X_t^\varepsilon, \mathcal{L}Z_t^\varepsilon) (Z_t^\varepsilon) \right] \right) dt \right)
\leq C_4 \int_0^T \left( \mathbb{E} |X_t^\varepsilon - X_t|^2 \right)^{\frac{1}{2}} \cdot \left( 1 + \mathbb{E} |Y_t^\varepsilon|^4 + \mathbb{E} |Z_t^\varepsilon|^4 \right) dt + C_4 \varepsilon \leq C_4 \varepsilon.
\]
Combining the above computations, we arrive at the desired conclusion. Thus the proof is completed. □

6. Appendix

In this section, we give an Itô formula under the case of common noises. Let $X_t$ and $Y_t$ are two $d$-dimensional Itô processes, i.e.,
\[
\begin{align*}
\text{d}X_t &= f(t)\text{d}t + g_1(t)\text{d}W_t + g_2(t)\text{d}B_t, \quad X_0 = \xi, \\
\text{d}Y_t &= b(t)\text{d}t + \sigma_1(t)\text{d}W_t + \sigma_2(t)\text{d}B_t, \quad Y_0 = \eta,
\end{align*}
\]
where $W_t$ and $B_t$ are two independent Brownian motion, and $f(t), b(t), g_i(t), \sigma_i(t)(i = 1, 2)$ are progressively measurable processes with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, such that for every $T > 0$,
\[
\mathbb{E} \left[ \int_0^T \left( |f(t)|^2 + \|g_1(t)\|^4 + \|g_2(t)\|^4 + |b(t)|^2 + \|\sigma_1(t)\|^4 + \|\sigma_2(t)\|^4 \right) dt \right] < \infty.
\]
Then we have:

Lemma 6.1. Assume that $u \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ and for every compact set $K \subset \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,
\[
\sup_{(x,\mu,\nu,\nu)\in K} \left[ \int_{\mathbb{R}^d} \left( |\partial_{\mu} u(x, \mu, y, \nu)(\bar{x})|^2 + \|\partial_\mu \partial_{\mu} u(x, \mu, y, \nu)(\bar{x})\|^2 \right) \mu(d\bar{x}) \right]
\]
\[
+ \int_{\mathbb{R}^d} \left( |\partial_{\nu} u(x, \mu, y, \nu)(\bar{y})|^2 + \|\partial_{\nu} \partial_{\nu} u(x, \mu, y, \nu)(\bar{y})\|^2 \right) \nu(d\bar{y}) \right] < \infty.
\]
Then for $\mu_t = \mathcal{L}X_t$ and $\nu_t = \mathcal{L}Y_t$, we have $\mathbb{P}$-a.s.,
\[
\begin{align*}
u_t = u(\xi, \mathcal{L}_\xi, \eta, \mathcal{L}_\eta) + \int_0^t f(s) \cdot \partial_x u(X_s, \mu_s, Y_s, \nu_s) ds
\end{align*}
\]
Following the idea of the proof of Itô’s formula in [10, Proposition 5.102], we assume without loss of generality that the derivatives of $u$ are bounded, and $f(t), b(t), g_i(t), \sigma_i(t)(i = 1, 2)$ are continuous and satisfy

$$E\left[ \sup_{0 \leq t \leq T} (|f(t)|^2 + ||g_1(t)||^4 + ||g_2(t)||^4 + |b(t)|^2 + ||\sigma_1(t)||^4 + ||\sigma_2(t)||^4) \right] < \infty.$$  

For every $k \geq 1$, let $(X_t^k, Y_t^k)$ be independent copies of $(X_t, Y_t)$, namely,

$$dX_t^k = f^k(t)dt + g_1^k(t)dW_t^k + g_2^k(t)dB_t^k, \quad X_0^k = \xi^k;$$
$$dY_t^k = b^k(t)dt + \sigma_1^k(t)dW_t^k + \sigma_2^k(t)dB_t^k, \quad Y_0^k = \eta^k.$$  

Denote by $\mu_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{X_t^k}$ and $\nu_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{Y_t^k}$ the empirical measures of $(X_t^k)_{1 \leq k \leq N}$ and $(Y_t^k)_{1 \leq k \leq N}$, respectively. Then we define

$$u^N(x_1, \ldots, x_N, y_1, \ldots, y_N) = u\left(\frac{1}{N} \sum_{k=1}^N \delta_{x_k}, \frac{1}{N} \sum_{k=1}^N \delta_{y_k}\right).$$

Using the classical Itô’s formula, we derive

$$u(\mu_t^N, \nu_t^N) = u^N(X_t^1, \ldots, X_t^N, Y_t^1, \ldots, Y_t^N)$$

where the process $(\tilde{X}_t, \tilde{f}(t), \tilde{g}_1(t), \tilde{g}_2(t), \tilde{Y}_t, \tilde{b}(t), \tilde{\sigma}_1(t), \tilde{\sigma}_2(t))$ is a copy of the original process $(X_t, f(t), g_1(t), g_2(t), Y_t, b(t), \sigma_1(t), \sigma_2(t))$ defined on a copy $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
\[= u^N(\xi^1, \ldots, \xi^N, \eta^1, \ldots, \eta^N) + \int_0^t \frac{1}{N} \sum_{k=1}^N \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s) \cdot f^k(s) ds\]

\[+ \frac{1}{N} \sum_{k=1}^N \int_0^t \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s) \cdot g^k_1(s) dW^k_s + \frac{1}{N} \sum_{k=1}^N \int_0^t \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s) \cdot g^k_2(s) dB^k_s\]

\[+ \frac{1}{2N} \int_0^t \sum_{k=1}^N \text{Tr}([g^k_1 g^{k*}_1 + g^k_2 g^{k*}_2](s) \cdot \partial_\nu \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s)) ds\]

\[+ \frac{1}{2N^2} \int_0^t \sum_{k=1}^N \text{Tr}([\sigma_1^k \sigma_1^{k*} + \sigma_2^k \sigma_2^{k*}](s) \cdot \partial_\nu^2 \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s, X^k_s)) ds\]

\[+ \frac{1}{N} \sum_{k=1}^N \int_0^t \partial_\nu u(\mu^N_s, \nu^N_s)(Y^k_s) \cdot b^k(s) ds\]

\[+ \frac{1}{N} \sum_{k=1}^N \int_0^t \partial_\nu u(\mu^N_s, \nu^N_s)(Y^k_s) \cdot \sigma_1^k(s) dW^k_s + \frac{1}{N} \sum_{k=1}^N \int_0^t \partial_\nu u(\mu^N_s, \nu^N_s)(Y^k_s) \cdot \sigma_2^k(s) dB^k_s\]

\[+ \frac{1}{2N} \int_0^t \sum_{k=1}^N \text{Tr}([\sigma_1^k \sigma_1^{k*} + \sigma_2^k \sigma_2^{k*}](s) \cdot \partial_\nu \partial_\nu u(\mu^N_s, \nu^N_s)(Y^k_s)) ds\]

\[+ \frac{1}{2N^2} \int_0^t \sum_{k=1}^N \text{Tr}([\sigma_1^k \sigma_1^{k*} + \sigma_2^k \sigma_2^{k*}](s) \cdot \partial_\nu^2 u(\mu^N_s, \nu^N_s)(Y^k_s, Y^k_s)) ds\]

\[+ \frac{1}{N^2} \int_0^t \sum_{k=1}^N \text{Tr}([g^k_1 \sigma_1^{k*} + g^k_2 \sigma_2^{k*}](s) \cdot \partial_\nu \partial_\mu u(\mu^N_s, \nu^N_s)(X^k_s, Y^k_s)) ds.\]

Taking expectation from both sides of the above equality (the stochastic integrals have zero expectation due to the properties of martingale) gives

\[E u(\mu_t^N, \nu_t^N) = E u(\mu_0^N, \nu_0^N) + \int_0^t E[\partial_\mu u(\mu^N_s, \nu^N_s)(X^1_s) \cdot f^1(s)] ds\]

\[+ \frac{1}{2} \int_0^t E[\text{Tr}([g^1_1 g^{1*}_1 + g^1_2 g^{1*}_2](s) \cdot \partial_\nu \partial_\mu u(\mu^N_s, \nu^N_s)(X^1_s))] ds\]

\[+ \frac{1}{2N} \int_0^t E[\text{Tr}([g^1_1 g^{1*}_1 + g^1_2 g^{1*}_2](s) \cdot \partial_\nu^2 u(\mu^N_s, \nu^N_s)(X^1_s, X^1_s))] ds\]

\[+ \int_0^t E[\partial_\nu u(\mu^N_s, \nu^N_s)(Y^1_s) \cdot b^1(s)] ds\]

\[+ \frac{1}{2} \int_0^t E[\text{Tr}([\sigma_1^1 \sigma_1^{1*} + \sigma_2^1 \sigma_2^{1*}](s) \cdot \partial_\nu \partial_\nu u(\mu^N_s, \nu^N_s)(Y^1_s))] ds\]

\[+ \frac{1}{2N} \int_0^t E[\text{Tr}([\sigma_1^1 \sigma_1^{1*} + \sigma_2^1 \sigma_2^{1*}](s) \cdot \partial_\nu^2 u(\mu^N_s, \nu^N_s)(Y^1_s, Y^1_s))] ds\]
where we have used that \((Y_t^k, X_t^k)_{1 \leq k \leq N}\) are independently identically distributed. We know from [10] that \(\mathbb{P}\text{-a.s.}\)

\[
\mathcal{W}_2(\mu_t^N, \mu_t) \to 0, \quad \mathcal{W}_2(\nu_t^N, \nu_t) \to 0, \quad \text{as } N \to \infty.
\]

Taking limit \(N \to \infty\) yields

\[
u(\mu_t, \nu_t) = u(\mathcal{L}_0, \mathcal{L}_0) + \int_0^t \mathbb{E}[\partial_u u(\mu_s, \nu_s)(X_s) \cdot f(s)] ds
\]

\[
+ \frac{1}{2} \int_0^t \mathbb{E}[\text{Tr}(g_1 g_1^* + g_2 g_2^*)(s) \cdot \partial_x \partial_u u(\mu_s, \nu_s)(X_s))] ds
\]

\[
+ \int_0^t \mathbb{E}[\partial_x u(\mu_s, \nu_s)(Y_s) \cdot b(s)] ds + \frac{1}{2} \int_0^t \mathbb{E}[\text{Tr}(\sigma_1 \sigma_1^* + \sigma_2 \sigma_2^*)(s) \cdot \partial_y \partial_x u(\mu_s, \nu_s)(Y_s))] ds.
\]

(6.2)

Then we define the function:

\[
U(t, x, y) = u(x, \mu_t, y, \nu_t).
\]

For any \(t \geq 0\) and \(h > 0\), by (6.2) we compute

\[
U(t + h, x, y) - U(t, x, y) = u(x, \mu_{t+h}, y, \nu_{t+h}) - u(x, \mu_t, y, \nu_t)
\]

\[
= \int_t^{t+h} \mathbb{E}[\partial_u u(\mu_s, \nu_s)(X_s) \cdot f(s)] ds
\]

\[
+ \frac{1}{2} \int_t^{t+h} \mathbb{E}[\text{Tr}(g_1 g_1^* + g_2 g_2^*)(s) \cdot \partial_x \partial_u u(\mu_s, \nu_s)(X_s))] ds
\]

\[
+ \int_t^{t+h} \mathbb{E}[\partial_x u(\mu_s, \nu_s)(Y_s) \cdot b(s)] ds
\]

\[
+ \frac{1}{2} \int_t^{t+h} \mathbb{E}[\text{Tr}(\sigma_1 \sigma_1^* + \sigma_2 \sigma_2^*)(s) \cdot \partial_y \partial_x u(\mu_s, \nu_s)(Y_s))] ds.
\]

Consequently, we have that \(U\) is differentiable in \(t\) and

\[
\partial_t U(t, x, y) = \mathbb{E}[\partial_u u(\mu_t, y, \nu_t)(X_t) \cdot f(t)] + \mathbb{E}[\partial_v u(\mu_t, y, \nu_t)(Y_t) \cdot b(t)]
\]

\[
+ \frac{1}{2} \mathbb{E}[\text{Tr}(g_1 g_1^* + g_2 g_2^*)(t) \cdot \partial_x \partial_u u(\mu_t, y, \nu_t)(X_t))]
\]

\[
+ \frac{1}{2} \mathbb{E}[\text{Tr}(\sigma_1 \sigma_1^* + \sigma_2 \sigma_2^*)(t) \cdot \partial_y \partial_x u(\mu_t, y, \nu_t)(Y_t))].
\]

(6.3)

Applying the classical Itô formula to \(U\), we get

\[
U(t, X_t, Y_t) = U(0, \xi, \eta) + \int_0^t \partial_t U(s, X_s, Y_s) ds + \int_0^t f(s) \cdot \partial_x U(s, X_s, Y_s) ds
\]

\[
+ \frac{1}{2} \int_0^t \text{Tr}(g_1 g_1^* + g_2 g_2^*)(s) \cdot \partial_x^2 U(s, X_s, Y_s)) ds
\]

30
\[
+ \int_0^t \partial_x U(s, X_s, Y_s) \cdot g_1(s) dW_s + \int_0^t \partial_x U(s, X_s, Y_s) \cdot g_2(s) dB_s \\
+ \int_0^t b(s) \cdot \partial_y U(s, X_s, Y_s) ds + \frac{1}{2} \int_0^t \text{Tr}([\sigma_1^* \sigma_1^* + \sigma_2^* \sigma_2^*](s) \cdot \partial_y^2 U(s, X_s, Y_s)) ds \\
+ \int_0^t \partial_y U(s, X_s, Y_s) \cdot \sigma_1(s) dW_s + \int_0^t \partial_y U(s, X_s, Y_s) \cdot \sigma_2(s) dB_s \\
+ \int_0^t \text{Tr}([g_1 \sigma_1^* + g_2 \sigma_2^*](s) \cdot \partial_y \partial_x U(s, X_s, Y_s)) ds,
\]

which together with (6.3) implies the desired result. \[\square\]

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