Numerical radius and distance from unitary operators

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Abstract

Denote by \( w(A) \) the numerical radius of a bounded linear operator \( A \) acting on Hilbert space. Suppose that \( A \) is invertible and that \( w(A) \leq 1+\varepsilon \) and \( w(A^{-1}) \leq 1+\varepsilon \) for some \( \varepsilon \geq 0 \). It is shown that \( \inf\{\|A-U\| : U \text{ unitary}\} \leq c\varepsilon^{1/4} \) for some constant \( c > 0 \). This generalizes a result due to J.G. Stampfli, which is obtained for \( \varepsilon = 0 \). An example is given showing that the exponent \( 1/4 \) is optimal. The more general case of the operator \( \rho \)-radius \( w_{\rho}(\cdot) \) is discussed for \( 1 \leq \rho \leq 2 \).

1 Introduction and statement of the results

Let \( H \) be a complex Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle \) and the associated norm \( \| \cdot \| \). We denote by \( B(H) \) the C*-algebra of all bounded linear operators on \( H \) equipped with the operator norm
\[
\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.
\]
It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators \( A \) with \( \|A\| \leq 1 \) and \( \|A^{-1}\| \leq 1 \). More generally, if \( A \in B(H) \) is invertible
\[
\inf\{\|A-U\| : U \text{ unitary}\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).
\]
We refer to \cite{6} Theorem 1.3 and \cite{9} Theorem 1 for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if \( A \in B(H) \) is an invertible operator satisfying \( \|A\| \leq r \) and \( \|A^{-1}\| \leq r \) for some \( r \geq 1 \), then there exists a unitary operator \( U \in B(H) \) such that \( \|A-U\| \leq r-1 \).

The numerical radius of the operator \( A \) is defined by
\[
w(A) = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.
\]
Stampfli has proved in \cite{8} that numerical radius contractivity of \( A \) and of its inverse \( A^{-1} \), that is \( w(A) \leq 1 \) and \( w(A^{-1}) \leq 1 \), imply that \( A \) is unitary. We define a function \( \psi(r) \) for \( r \geq 1 \) by
\[
\psi(r) = \sup\{\|A\| : A \in B(H), w(A) \leq r, w(A^{-1}) \leq r\},
\]
the supremum being also considered over all Hilbert spaces \( H \). Then the conditions \( w(A) \leq r \) and \( w(A^{-1}) \leq r \) imply \( \max(\|A\|-1, 1-\|A^{-1}\|^{-1}) \leq \max(\|A\|-1, \|A^{-1}\|-1) \leq \psi(r)-1 \), hence the existence of a unitary operator \( U \) such that \( \|A-U\| \leq \psi(r)-1 \). We have the two-sided estimate
\[
r + \sqrt{r^2 - 1} \leq \psi(r) \leq 2r.
\]
The upper bound follows from the well-known inequalities \( w(A) \leq \|A\| \leq 2w(A) \), while the lower bound is obtained by choosing \( H = \mathbb{C}^2 \) and
\[
A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad y = \sqrt{r^2 - 1},
\]
in the definition of \( \psi \). Indeed, we have \( A = A^{-1} \), \( w(A) = \sqrt{1+y^2} = r \), and \( \|A\| = y + \sqrt{1+y^2} = r + \sqrt{r^2-1} \).

Our first aim is to improve the upper estimate.

**Theorem 1.1.** Let \( r \geq 1 \). Then

\[
\psi(r) \leq X(r) + \sqrt{X(r)^2 - 1}, \quad \text{with} \quad X(r) = r + \sqrt{r^2 - 1}.
\]

The estimate given in Theorem 1.1 is more accurate than \( \psi(r) \leq 2r \) for \( r \) close to 1, more precisely for \( 1 \leq r \leq 1.029055 \ldots \). It also gives \( \psi(1) = 1 \) (leading to Stampfli’s result) and the following asymptotic estimate.

**Corollary 1.2.** We have

\[
\psi(1+\epsilon) \leq 1 + \sqrt{8\epsilon + O(\epsilon^{1/2})}, \quad \epsilon \to 0.
\]

Our second aim is to prove that the exponent \( 1/4 \) in Corollary 1.2 is optimal. This is a consequence of the following result.

**Theorem 1.3.** Let \( n \) be a positive integer of the form \( n = 8k + 4 \). There exists a \( n \times n \) invertible matrix \( A_n \) with complex entries such that

\[
w(A_n) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad w(A_n^{-1}) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.
\]

Indeed, Theorem 1.3 implies that

\[
\psi\left(\frac{1}{\cos \frac{\pi}{n}}\right) \geq \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.
\]

Taking \( 1+\epsilon = 1/\cos \frac{\pi}{n} = 1 + \frac{\pi^2}{2n^2} + O\left(\frac{1}{n}\right) \), we see that the exponent \( \frac{1}{4} \) cannot be improved.

More generally, we can consider for \( \rho \geq 1 \) the \( \rho \)-radius \( w_\rho(A) \) introduced by Sz.-Nagy and Foiaş (see [5, Chapter 1] and the references therein). Consider the class \( C_\rho \) of operators \( T \in \mathcal{B}(H) \) which admit unitary \( \rho \)-dilations, i.e. there exist a super-space \( H \supset H \) and a unitary operator \( U \in \mathcal{B}(H) \) such that

\[
T^n = \rho PU^n P^*, \quad \text{for} \quad n = 1, 2, \ldots.
\]

Here \( P \) denotes the orthogonal projection from \( H \) onto \( H \). Then the operator \( \rho \)-radius is defined by

\[
w_\rho(A) = \inf\{\lambda > 0 ; \lambda^{-1}A \in C_\rho\}.
\]

From this definition it is easily seen that \( r(A) \leq w_\rho(A) \leq \rho\|A\| \), where \( r(A) \) denotes the spectral radius of \( A \). Also, \( w_\rho(A) \) is a non-increasing function of \( \rho \). Another equivalent definition follows from [5, Theorem 11.1]:

\[
w_\rho(A) = \sup_{h \in E_\rho} \left\{ (1 - \frac{1}{\rho}) |\langle Ah, h \rangle| + \sqrt{(1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 + (\frac{2}{\rho} - 1) \|Ah\|^2} \right\}, \quad \text{with}
\]

\[
E_\rho = \{h \in H ; \|h\| = 1 \text{ and } (1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 - (1 - \frac{2}{\rho}) \|Ah\|^2 \geq 0\}.
\]

Notice that \( E_\rho = \{h \in H ; \|h\| = 1\} \) whenever \( 1 \leq \rho \leq 2 \). This shows that \( w_1(A) = \|A\|, w_2(A) = w(A) \) and \( w_\rho(A) \) is a convex function of \( A \) if \( 1 \leq \rho \leq 2 \).

We now define a function \( \psi_\rho(r) \) for \( r \geq 1 \) by

\[
\psi_\rho(r) = \sup\{\|A\| ; A \in \mathcal{B}(H), w_\rho(A) \leq r, w_\rho(A^{-1}) \leq r\}.
\]

As before, the conditions \( w_\rho(A) \leq r \) and \( w_\rho(A^{-1}) \leq r \) imply the existence of a unitary operator \( U \) such that \( \|A-U\| \leq \psi_\rho(r)-1 \), and we have \( \psi_\rho(r) \leq \rho r \). We will generalize the estimate (1) from Theorem 1.1 by proving, for \( 1 \leq \rho \leq 2 \), the following result.
Theorem 1.4. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(r) \leq X_\rho(r) + \sqrt{X_\rho(r)^2 - 1},$$  \hspace{1cm} (2)

with $X_\rho(r) = \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2)^2 - 4r^2}}{2r}$.

Corollary 1.5. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(1+\varepsilon) \leq 1 + 4\sqrt{8(\rho - 1)}\varepsilon + O(\varepsilon^{1/2}), \quad \varepsilon \to 0.$$

We recover in this way for $1 \leq \rho \leq 2$ the recent result of Ando and Li [2, Theorem 2.3], namely that $w_\rho(A) \leq 1$ and $w_\rho(A^{-1}) \leq 1$ imply $A$ is unitary. The range $1 \leq \rho \leq 2$ coincides with the range of those $\rho \geq 1$ for which $w_\rho(\cdot)$ is a norm. Contrarily to [2], we have not been able to treat the case $\rho > 2$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case $\rho = 2$. The proof of Theorem 1.3 which shows the optimality of the exponent 1/4 in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfli [8], we have developed another (more complicated) approach in the case $\rho = 2$.

2 Proof of Theorem 1.4 about $\psi_\rho$

Let us consider $M = \frac{1}{2}(A + (A^*)^{-1})$; then

$$M^*M - 1 = \frac{1}{4}(A^* A + (A^* A)^{-1} - 2) \geq 0.$$  

This implies $\|M^{-1}\| \leq 1$. In what follows $C^{1/2}$ will denote the positive square root of the self-adjoint positive operator $C$. The relation $(A^* A - 2M^* M + 1)^2 = 4M^* M(M^* M - 1)$ yields

$$A^* A - 2M^* M + 1 \leq 2(M^* M)^{1/2}(M^* M - 1)^{1/2},$$

whence $A^* A \leq ((M^* M)^{1/2} + (M^* M - 1)^{1/2})^2$.

Therefore $\|A\| \leq \|M\| + \sqrt{\|M\|^2 - 1}$.

We now assume $1 \leq \rho \leq 2$. Then $w_\rho(\cdot)$ is a norm and the two conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply $w_\rho(M) \leq r$. The desired estimate of $\psi_\rho(r)$ will follow from the following auxiliary result.

Lemma 2.1. Assume $\rho \geq 1$. Then the assumptions $w_\rho(M) \leq r$ and $\|M^{-1}\| \leq 1$ imply $\|M\| \leq X_\rho(r)$.

Proof. The contractivity of $M^{-1}$ implies

$$\|u\| \leq \|Mu\|, \quad (\forall u \in H).$$  \hspace{1cm} (3)

As $w_\rho(M) \leq r$, it follows from a generalization by Durszt [4] of a decomposition due to Ando [1], that the operator $M$ can be decomposed as

$$M = \rho r B^{1/2}U C^{1/2},$$

with $U$ unitary, $C$ selfadjoint satisfying $0 < C < 1$, and $B = f(C)$ with $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$. Notice that $f$ is a decreasing function on the segment $[0,1]$ and an involution: $f(f(x)) = x$. Let $[\alpha, \beta]$ be the smallest segment containing the spectrum of $C$. Then $[\sqrt{\alpha}, \sqrt{\beta}]$ is the smallest segment
containing the spectrum of $C^{1/2}$ and $[\sqrt{f(\beta)}, \sqrt{f(\alpha)}]$ is the smallest segment containing the spectrum of $B^{1/2}$. We have
\[
\|u\| \leq \|Mu\| \leq \rho r \sqrt{f(\alpha)} \|C^{1/2}u\|, \quad (\forall u \in H).
\]
Choosing a sequence $u_n$ of norm-one vectors ($\|u_n\| = 1$) such that $\|C^{1/2}u_n\|$ tends to $\sqrt{\alpha}$, we first get $1 \leq \rho r \sqrt{\alpha f(\alpha)}$, i.e. $1 - (2 + \rho r^2 - \rho)\alpha + \rho^2 r^2 \alpha^2 \leq 0$. Consequently we have
\[
2 + \rho r^2 - \rho - \frac{\sqrt{(2 + \rho r^2)^2 - 4\rho r^2}}{2 \rho r^2} \leq \alpha \leq 2 + \rho r^2 - \rho + \frac{\sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2},
\]
and by $\alpha = f(f(\alpha))$
\[
2 + \rho r^2 - \rho - \frac{\sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2} \leq f(\alpha) \leq 2 + \rho r^2 - \rho + \frac{\sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2}.
\]
Similarly, noticing that $\|(M^*)^{-1}\| \leq 1$, $M^* = \rho r C^{1/2}U^*B^{1/2}$ and $C = f(B)$, we obtain
\[
2 + \rho r^2 - \rho - \frac{\sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2} \leq \beta \leq 2 + \rho r^2 - \rho + \frac{\sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2}.
\]
Therefore
\[
\|M\| \leq \rho r \|B^{1/2}\| \|C^{1/2}\| = \rho r \sqrt{f(\alpha)} \beta \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4\rho r^2}}{2 \rho r^2}.
\]
This shows that $\|M\| \leq X_\rho(r)$. \qed

3 The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of $n \times n$ matrices $A = DBD$, defined for $n = 8k + 4$, by
\[
D = \text{diag}(e^{i\pi/2n}, \ldots, e^{(2k-1)i\pi/2n}, \ldots, e^{(2n-1)i\pi/2n}),
B = I + \frac{1}{2n \pi} E, \quad \text{where } E \text{ is a matrix whose entries are defined as } e_{ij} = 1 \text{ if } 3k + 2 \leq |i - j| \leq 5k + 2, \quad e_{ij} = 0 \text{ otherwise}.
\]
We first remark that $\|A\| = \|B\| = 1 + \frac{1}{8\sqrt{n}}$. Indeed, $B$ is a symmetric matrix with non negative entries, $Be = (1 + \frac{1}{8\sqrt{n}})e$ with $e^T = (1,1,1,\ldots,1)$. Thus $\|B\| = r(B) = 1 + \frac{1}{8\sqrt{n}}$ by the Perron-Frobenius theorem.

Consider now the permutation matrix $P$ defined by $p_{ij} = 1$ if $i = j + 1$ modulo $n$ and $p_{ij} = 0$ otherwise and the diagonal matrix $\Delta = \text{diag}(1,\ldots,1,-1)$. Then $P^{-1}DP = e^{i\pi/n} \Delta D$ and $P^{-1}EP = E$, whence $(P \Delta)^{-1}AP \Delta = e^{2i\pi/n} A$. Since $P \Delta$ is a unitary matrix, the numerical range $W(A) = \{ (Au, u); \|u\| = 1 \}$ of $A$ satisfies $W(A) = W((P \Delta)^{-1}AP \Delta) = e^{2i\pi/n} W(A)$. This shows that the numerical range of $A$ is invariant by the rotation of angle $2\pi/n$ centered in 0, and the same property also holds for the numerical range of $A^{-1}$.

We postpone the proof of the estimates $\|\frac{1}{2}(A + A^*)\| \leq 1$ and $\|\frac{1}{2}(A^{-1} + (A^{-1})^*)\| \leq 1$ to later sections. Using these estimates, we obtain that the numerical range $W(A)$ is contained in the half-plane $\{ z; \text{Re} \ z \leq 1 \}$, whence in the regular $n$-sided polygon given by the intersection of the half-planes $\{ z; \text{Re}(e^{2i\pi k/n} z) \leq 1 \}$, $k = 1,\ldots,n$. Consequently $w(A) \leq 1/\cos(\pi/n)$. The proof of $w(A^{-1}) \leq 1/\cos(\pi/n)$ is similar.
3.1 Proof of $\|\frac{1}{2}(A+A^*)\| \leq 1$.

Since the $(\ell,j)$-entry of $A$ is $e^{i(\ell+j-1)\frac{\pi}{n}}(\delta_{\ell,j} + \frac{e^{i\pi}}{2\frac{n}{|\pi|}})$, the matrix $\frac{1}{2}(A+A^*)$ is a real symmetric matrix whose $(i,j)$-entry is $\cos\left((i+j-1)\frac{\pi}{n}\right)\left(\delta_{i,j} + \frac{e^{i\pi}}{2\frac{n}{|\pi|}}\right)$. It suffices to show that, for every $u = (u_1, \cdots, u_n)^T \in \mathbb{R}^n$, we have $\|u\|^2 - \text{Re}(Au, u) \geq 0$. Let $E = \{(i,j) ; 1 \leq i,j \leq n, 3k + 2 \leq |i-j| \leq 5k + 2\}$. The inequality which has to be proved is equivalent to

$$\sum_{i=1}^{n} 2\sin^2\left((i-\frac{1}{2})\frac{\pi}{n}\right) u_i^2 - \frac{1}{2n^2\pi^2} \sum_{i,j \in E} \cos\left((i+j-1)\frac{\pi}{n}\right) u_i u_j \geq 0.$$ 

Setting $v_j = u_j \sin\left((j-\frac{1}{2})\frac{\pi}{n}\right)$, this may be also written as follows

$$2\|v\|^2 - \langle Mv, v \rangle + \frac{1}{2n^2\pi^2} \langle Ev, v \rangle \geq 0, \quad (v \in \mathbb{R}^n).$$

(4)

Here $M$ is the matrix whose entries are defined by

$$m_{ij} = \frac{1}{2n^2\pi^2} \cot\left((i-\frac{1}{2})\frac{\pi}{n}\right) \cot\left((j-\frac{1}{2})\frac{\pi}{n}\right), \quad \text{if } (i,j) \in E, \quad m_{ij} = 0 \quad \text{otherwise}.$$ 

We will see that the Frobenius (or Hilbert-Schmidt) norm of $M$ satisfies $\|M\|_F \leq \sqrt{9/32} < 3/4$. A fortiori, the operator norm of $M$ satisfies $\|M\| \leq 3/4$. Together with $\|E\| = n/4$, this shows that $\|M\| + \frac{1}{2n^2\pi^2}\|E\| \leq \frac{7}{8}$. Property (3) is now verified.

It remains to show that $\|M\|_F^2 \leq \frac{9}{32}$. First we notice that $m_{ij} = m_{ji} = m_{n+1-i,n+1-j}$, and $m_{ii} = 0$. Hence, with $E' = \{(i,j) \in E ; i < j \text{ and } i+j \leq n+1\}$,

$$\|M\|_F^2 = 2 \sum_{i<j} |m_{ij}|^2 \leq 4 \sum_{(i,j) \in E'} |m_{ij}|^2.$$ 

We have, for $(i,j) \in E'$,

$$2j \leq i+j+5k+2 \leq n+5k+3 = 13k+7, \quad \text{thus } 3k+3 \leq j \leq \frac{13k+7}{2},$$

$$2i \leq i+j-3k-2 \leq n-3k-1 = 5k+3, \quad \text{thus } 1 \leq i \leq \frac{5k+3}{2}.$$ 

This shows that

$$\frac{3\pi}{10} \leq \frac{3k+2}{16k+8} \pi \leq (j-\frac{1}{2})\frac{\pi}{n} \leq \frac{13k+6}{16k+8} \pi \leq \pi - \frac{3\pi}{10}, \quad \text{hence } |\cot\left((j-\frac{1}{2})\frac{\pi}{n}\right)| \leq \cot \frac{3\pi}{10} \leq \frac{3}{7}.$$ 

We also use the estimate $\cot\left((i-\frac{1}{2})\frac{\pi}{n}\right) \leq n/(\pi(i-\frac{1}{2}))$ and the classical relation $\sum_{i \geq 1}(i-1/2)^{-2} = \pi^2/2$ to obtain

$$\|M\|_F^2 \leq 4 \sum_{(i,j) \in E'} |m_{ij}|^2 \leq \frac{4}{n^2 \pi^2} \sum_{i \geq 1} \frac{1}{(i-1/2)^2} (2k+1) \frac{9}{4} = \frac{9}{32}.$$ 

3.2 Proof of $\|\frac{1}{2}(A^{-1}+A^{-1})^*\| \leq 1$.

We start from

$$(A^{-1})^* = D(1 + \frac{1}{2n^2\pi^2} E)^{-1} D = D^2 - \frac{1}{2n^2\pi^2} DED + \frac{1}{4n^2} DE^2(1 + \frac{1}{2n^2\pi^2} E)^{-1} D,$$

and we want to show that $\|u\|^2 - \text{Re}(A^{-1}u, u) \geq 0$. As previously, we set $v_j = u_j \sin\left((j-\frac{1}{2})\frac{\pi}{n}\right)$. The inequality $\|\frac{1}{2}(A^{-1}+A^{-1})^*\| \leq 1$ is equivalent to

$$2\|v\|^2 - \langle (M_1 + M_2 + M_3 + M_4)v, v \rangle \geq 0, \quad (v \in \mathbb{R}^n).$$
Here the entries of the matrices $M_p$, $1 \leq p \leq 4$, are given by

$$(m_1)_{ij} = -\frac{1}{2n^{3/2}} \cot((i-\frac{1}{2})\frac{\pi}{n}) \cot((j-\frac{1}{2})\frac{\pi}{n}) e_{ij},$$

$$(m_2)_{ij} = \frac{1}{2n^{3/2}} e_{ij},$$

$$(m_3)_{ij} = \frac{1}{4n^3} \cot((i-\frac{1}{2})\frac{\pi}{n}) \cot((j-\frac{1}{2})\frac{\pi}{n}) f_{ij},$$

$$(m_4)_{ij} = -\frac{1}{4n^3} f_{ij},$$

e_{ij}$ and $f_{ij}$ respectively denoting the entries of the matrices $E$ and $F = E^2(1 + \frac{1}{2n^{3/2}} E)^{-1}$. Noticing that $M_1 = -M$, we have $\|M_1\| \leq \frac{3}{4}$, $\|M_2\| = \frac{1}{8}\sqrt{n}$, $\|F\| \leq \frac{n^2/16}{1-1/(8\sqrt{n})} \leq \frac{n^2}{14}$ and $\|M_4\| = \frac{1}{4n^3} \|F\|$. Now we use

$$\|M_3\|^2 \leq \|M_3\|_{\infty} \leq \frac{1}{16n} \max_{i,j} |f_{ij}|^2 \sum_{i,j} |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 |\cot((j-\frac{1}{2})\frac{\pi}{n})|^2,$$

together with

$$\sum_{i,j} |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 |\cot((j-\frac{1}{2})\frac{\pi}{n})|^2 = \left( \sum_{i=1}^{n} |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 \right)^2 \leq 4 \left( \sum_{i=1}^{n/2} |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 \right)^2 \leq n^4,$$

to obtain

$$\|M_3\| \leq \frac{1}{4n} \max_{i,j} |f_{ij}|.$$

Using the notation $\|E\|_\infty := \max\{\|Eu\|_\infty : u \in \mathbb{C}^n, \|u\|_\infty \leq 1\}$ for the operator norm induced by the maximum norm in $\mathbb{C}^d$, it holds $\|E\|_\infty = n/4$, whence $\frac{1}{2n^{3/2}} E\|_\infty \leq 1/8$ and thus $(1 + \frac{1}{2n^{3/2}} E)^{-1}\|_\infty \leq \frac{1}{1-1/8} = \frac{8}{7}$. This shows that

$$\max_{i,j} |f_{ij}| \leq \|(1 + \frac{1}{2n^{3/2}} E)^{-1}\|_\infty \max_{i,j} |e_{ij}|^2 \leq \frac{2n}{7},$$

by denoting $e_{ij}^2$ the entries of the matrix $E^2$ and noticing that $\max_{i,j} |e_{ij}| = n/4$. Finally, we obtain $\|M_3\| \leq \frac{1}{14}$ and $\|M_1 + M_2 + M_3 + M_4\| \leq \frac{3}{4} + \frac{1}{8} + \frac{1}{14} + \frac{1}{16} < 1$.

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