The theory of a resonant level coupled to several conduction electron channels in equilibrium and out-of-equilibrium

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(Dated: March 23, 2022)

The spinless resonant level model is studied when it is coupled by hopping to one of the arbitrary number of conduction electron channels. The Coulomb interaction acts between the electron on the impurity and in the different channels. In case of repulsive or attractive interaction the conduction electrons are pushed away or attracted to ease or hinder the hopping by creating unoccupied or occupied states, respectively. In the screening of the hopping orthogonality catastrophe plays an important role. In equilibrium the weak and strong coupling limit the renormalizations are treated by perturbative, numerical and Anderson-Yuval Coulomb gas methods. In case of two leads the current due to applied voltage is treated in the weak coupling limit. The presented detailed study should help to test other methods suggested for non-equilibrium transport.

PACS numbers: 73.63.Kv, 72.10.Fk, 72.15.Qm

I. INTRODUCTION

In the recent years the quantum impurity problem out-of-equilibrium has attracted great interest. The most relevant realizations are the quantum dots connected to at least two metallic leads and short metallic wires containing magnetic impurities. In the impurity problem the exact methods play distinguished roles especially the Bethe Ansatz and conformal invariance. The generalization of these methods to out-of-equilibrium situations are the most challenging new directions. Mehta and Andrei are aiming to solve the Kondo problem on a dot with two contact attached. First a simple resonant level without spin was studied to test the new generalization of the Bethe Ansatz method. Their elegant suggestion is very provocative. In order to test this kind of new methods we perform a detailed study of that problem using different weak coupling perturbative methods combined with NRG. As the final goal we calculate the current flowing through the impurity when a finite voltage is applied on the contacts. The most challenging claim of Mehta and Andrei is that the current is a non-monotonic function of the strength of the Coulomb coupling between the electron on the dot and conduction electrons in the two leads.

In order to make the comparison more explicit we generalize the time-ordered scattering formalism for non-equilibrium in the next leading logarithmic order. In this way the current is calculated as a function of the applied voltage and the Coulomb coupling strength. Increasing the Coulomb coupling strength we find also a non-monotonic feature but the order of increasing and decreasing regions is the opposite to the finding of Mehta and Andrei.

The model to be treated is the following: A single impurity orbital is coupled to two reservoirs of Fermi gas via hopping but the two reservoirs have different chemical potentials $\mu_L$ and $\mu_R$ on left and right of the impurity in a one dimensional model. $\mu_L - \mu_R = eV$ is determined by the applied voltage $V$ ($e$ is the electronic charge). The Coulomb interaction acts between the electron on the impurity level and the conduction electrons at the impurity position. Thus the Hamiltonian has the form

$$H = H_0 + H_1 + H_2,$$

with

$$H_0 = \sum_{k,\alpha} (k - k_\alpha) v_F a^\dagger k_\alpha a_{k\alpha} + \varepsilon_d d^\dagger d,$$

where $k > 0$ and $k_L - k_R = eV/v_F$, $v_F$ is the Fermi velocity, $a^\dagger_{k\alpha}$ is the creation operator of the spinless Fermion in lead $\alpha = L/R$, while $\varepsilon_d$ is the energy of the local level and $d^\dagger$ is the creation operator for the electron on that site. The interaction term is

$$H_1 = U (d^\dagger d - \frac{1}{2}) \left( \sum_{\alpha = L,R} a^\dagger_{\alpha} a_{\alpha} - \frac{1}{2} \right),$$

where $U$ is the Coulomb coupling which in a physical case $U > 0$, $a_{\alpha} = \frac{1}{\sqrt{L}} \sum_k a_{k\alpha}$, and $L$ is the length of the chain. The existence of the subtraction of $1/2$ is not essential, they can be omitted and than $\varepsilon_d$ is shifted as $\varepsilon_d - U/2$ and a local potential $-U/2$ is acting on the electrons, but the latter one can be taken into account by changing the electron density of states in the leads at the position of the impurity.

The hybridization between the lead electrons and the localized electron is described by

$$H_2 = V_\alpha \sum_{\alpha} (d^\dagger a_{\alpha} + a_{\alpha}^\dagger d),$$

where $V_\alpha$ is the hybridization matrix element.

In case of equilibrium it is useful to generalize the model to $N$ reservoirs instead of $L, R$, and then $\alpha$ runs from 1 to $N$. The hybridization matrix element is

$$V_\alpha \sum_{\alpha} (d^\dagger a_{\alpha} + a_{\alpha}^\dagger d).$$
through $\alpha = 0, 1, \ldots, N - 1$ and $\mu_\alpha = \mu$. Then the hybridization term in $H_2$ is chosen in a specific form
\begin{equation}
H_2 = V_0 \left( d^{\dagger} a_0 + a_0^{\dagger} d \right),
\end{equation}
indicating that only the electrons with $\alpha = 0$ are hybridizing while the others are taking part only in the Coulomb screening. Namely, only those electrons are hybridizing which have the symmetry of the localized orbital ($s$-like). As a result of the screening the electron gas is polarized depending on the occupation of the localized state and that polarizations lead to orthogonality catastrophes.

The model with $N = 1$ is known as a resonant level model and has been studied in great detail and the one with $N \geq 1$ has been introduced to study finite range interaction in 3D.

The goal of the present paper is to provide weak coupling results for $V \neq 0$. But before doing that the $V = 0$ equilibrium case is studied in the weak coupling limit by diagram technique and then to extend the results for stronger couplings Wilson’s numerical renormalization group technique (NRG)5,6 and Anderson-Yuval Coulomb gas method7 in order to check the validity of weak coupling results concerning a specific behavior. Namely at some stage of the calculation in the exponent of the renormalized quantities a combination
\begin{equation}
- \bar{\omega} U + \frac{1}{2} N(\bar{\omega} U)^2
\end{equation}
appears. For $U > 0$ that is changing sign at $\bar{\omega} U = \frac{2}{N}$ and that leads in changing to decreasing behavior but that crossover is well beyond the validity of the perturbation theory at least for $N = 2$.

In order to judge the real situation, an NRG study will be performed including the weak ($\bar{\omega} U \ll 1$) as well as strong coupling regions ($\bar{\omega} U \gtrsim \frac{2}{N}$) to get an insight whether the crossover indicated above is expected or it is likely artifact of the weak coupling theory. We also map the problem to a one-dimensional Coulomb model closely following the work of Anderson and Yuval, where the screening can even be in the strong coupling limit. All of these methods are suggesting a coherent picture of the crossover and they agree very well especially for $N = 4$.

The study of such a crossover is especially relevant as in the work of Mehta and Andre7 such a crossover is suggested in the current flowing in the non-equilibrium case $V \neq 0$ at $\bar{\omega} U \sim 2$. If we could find the crossover already in equilibrium then it is obvious to expect that in the non-equilibrium situation.

The paper is organized in the following way: In Section III we provide the analytical perturbative method up to next to leading logarithmic order introducing extra channels for screening, where the non-monotonic competition of the vertex and self-energy correction is already demonstrated in equilibrium. In Section IV the equilibrium calculation is extended to strong coupling by using Wilson’s numerical renormalization group technique and the result is compared to that of the analytical calculation. In Section V the Anderson-Yuval method is presented. In Section VI the time dependent scattering method is applied for non-equilibrium closely following the generalized version of Anderson’s poor man’s scaling in the next to leading order and the current is calculated. In Section VII the results are summarized. In Appendix A some cancellation due to Ward identities are discussed.

II. PERTURBATION THEORY: WEAK COUPLING LIMIT

The resonant level model is given by Eqs. (1), (2) and (3). That does not contain non-commuting terms, thus Kondo behavior is not expected in the weak coupling limit. In the strong coupling limit that model, however, can be mapped to an anisotropic Kondo model5,6,7 but that mapping is not considered here. The model shows strong similarities to the X-ray absorption10,11, as the strength of the interaction (invariant charge) between conduction electrons and the electron on the impurity level is scale invariant. The system shows scaling in terms of reduction of the conduction electron band width, $D$. In case $N = 1$ the scaling equations were derived by Schlottmann5 and those can be easily extended for arbitrary $N$.

There are two different kinds of vertex corrections depicted in Fig. 1a,b,c, where the solid lines stand for conduction electrons, the dotted line for electron on the impurity level and the interactions are indicated by dots ($U$) and crosses ($V$). In case of conduction electrons the channel indices are also indicated. The Hartree-Fock energy shift can be incorporated by $\varepsilon_d$. The self-energy of the electron on the impurity is depicted in Fig. 1d. In the calculation of the self-energy counter terms are introduced to eliminate the constant terms to keep $\varepsilon_d = 0$ unrenormalized. Closely following the earlier work6,12,13, the invariant charge for the Coulomb interaction takes the form
\begin{equation}
U_{\text{inv}} = \Gamma(\omega/D)d(\omega/D),
\end{equation}
where $\Gamma$ is the vertex function and
\begin{equation}
d(\omega/D) = G(\omega, D)(\omega - \varepsilon_d)
\end{equation}
can be determined perturbatively starting with 1 where $G$ is the renormalized one-electron Green’s function. The functions $\Gamma(\omega)$ and $d(\omega)$ are
\begin{align}
\Gamma(\omega) &= 1 + N \bar{\omega} U^2 \log\left(\frac{D}{\omega}\right) + \ldots \\
d(\omega) &= 1 - N \bar{\omega} U^2 \log\left(\frac{D}{\omega}\right) + \ldots,
\end{align}
where $\bar{\omega}$ is the conduction electron density of states, for spinless electrons in one of the channels $n = 1, \ldots, N - 1$. 


As the Coulomb interaction is independent of \( n \), thus the factor \( N \) occurs. As the consequence of the Ward identity relating the vertex correction and the self-energy depicted in Fig. 1 and d cancel out in \( U_{\text{inv}} \). \(^{6,10,11,13}\)

\[
\frac{d}{d[\omega]} (\Gamma(\omega)d(\omega)) = \frac{d}{d[\omega]} U_{\text{inv}}(\omega) = 0 ,
\]

thus \( U_{\text{inv}} = U \). The renormalization group gives

\[
d(\omega) = \left( \frac{\omega}{D} \right)^N \delta^2 U^2
\]

for arbitrary \( n \).

The hybridization contains vertex correction for \( n = 0 \) (Fig. 2)

\[
V(\omega) = V \left( 1 + U_0 \log \frac{D}{\omega} + \ldots \right) ,
\]

but it does not contain linear contribution in \( \log(D/\omega) \) in the order of \( U^2 \). The relevant invariant charge \( V_{\text{inv}}(\omega) \) is

\[
V_{\text{inv}}(\omega) = V(\omega) - \frac{1}{2} N(q_0 U)^2 \log \frac{D}{\omega} + \ldots
\]

That behavior will be further discussed in Sec. [VI].

The scaling regions, however, are not unlimited as the impurity level has its own width, \( \Gamma_{\text{imp}} \). There are two contributions to the level width \( \Gamma_{\text{imp}} \).

The hybridization broadens the impurity level just in case of the Anderson model and that is in the second order in \( V \)

\[
\Gamma_{\text{imp}}(\omega) = \pi q_0 V^2(\omega) = \pi q_0 V \left( \frac{\omega}{D} \right)^{-2q_0 U + N(q_0 U)^2} ,
\]

where also the effect of renormalization is taken into account. There is also a Korringa like broadening due to the creation of electron-hole pairs thus \( \Gamma_{\text{Korringa}} \sim U^2 \omega \), where \( \omega \) comes from the phase space. That is impor-
The renormalized hybridization as a function of \(g_0U\) for different channel numbers. The intermediate maximum can diverge only for \(g_0U = 1\), in all the other cases the increases are also rather moderate.

### III. NRG Approach for \(V = 0\)

In order to determine the region of validity of the weak coupling approach in equilibrium, we have performed numerical renormalization group (NRG) analysis for the \(N = 2\) and \(N = 4\) cases.

In Wilson’s NRG technique—after the logarithmic discretization of the conduction band—one maps the original Hamiltonian of an impurity problem to a semi-infinite chain with the impurity at the end of the chain. As a consequence of the logarithmic discretization the hopping amplitude along the chain decreases exponentially as \(t_n \sim \Lambda^{-n/2}\) where \(\Lambda > 1\) is a discretization parameter (we have used \(\Lambda = 2\) throughout the calculations) while \(n\) is the site index. The separation of energy scales provided by the exponentially vanishing hopping amplitude allows us to diagonalize the Hamiltonian iteratively to approximate the ground state and the excitation spectrum of the full chain. Since we know the eigenenergies and eigenvectors of the Hamiltonian, we can calculate dynamical quantities such as the density of states using the Lehman representation of the spectral function.

First let us focus on the physically relevant case of \(U > 0\) and \(N = 2\). To compare the numerical data with the weak coupling results, we have calculated the impurity density of states for different values of the interaction strength \(U\). The results are shown in Fig. 4. The numerical data validates the weak coupling results for \(U/D \leq 0.3\). In our NRG calculation we considered a flat band with constant density of states \(g_0 = 1/2D\), where \(D\) stands for the half bandwidth. In the lower panel of Fig. 4, the renormalized value of the hybridization, \(V_{\text{ren}}\) is shown as a function of the interaction strength \(U\). In NRG calculation, we have defined \(V_{\text{ren}}\) from the finite size spectrum directly. The finite size spectrum as a function of iteration number crosses over from the initial fixed point to the strong coupling one characterized by single particle phase shifts \(\delta = \pi/2\) at around \(M^*\). \(M^*\) is determined by the renormalized hybridization, \(\Delta_{\text{ren}} = \pi V^2_{\text{ren}} g_0 \sim \Lambda^{-M^*/2}\). We take \(M = M^*\) when the energy of the first excited state exceeds 90% of its fixed point value.

To answer the question whether an intermediate maximum appears outside the weak coupling limit or not, we have performed calculation with very large values of the interaction strength up to \(U/D = 5.0\). The results are shown in Fig. 5.
Our conclusion is that even for $N = 2$ such a non-monotonic behavior is found but the position of the maximum as well as the shape of the curve for large $U$ differs essentially from those obtained by weak coupling calculation. It still remains a question whether for case of many channels the weak coupling calculation is reliable or not. To treat many channels with NRG is very challenging but to see the tendency with increasing channel number, we performed the numerical analysis of the case $N = 4$. The results are plotted in Fig. 5 as well. Our data suggests that already for $N = 4$ the position of the turning point as well as the decay of the curve at large $U$ is reproduced by the weak coupling calculation with a much better accuracy than in case of $N = 2$.

IV. ANDERSON-YUVAL APPROACH

In most of the physical cases the Coulomb interaction $U$ dominates over the hopping term $V_0$. To overcome that difficulty, Anderson and Yuval\textsuperscript{9} introduced a path integral method for the Kondo problem where the interaction $U$ is described in terms of phase shifts while the hopping is treated as perturbation. The similarity between the Kondo and the present problem can be exploited in the following way: The complex time axis is divided into intervals at the short time cutoff $\tau$ and as it is shown in Fig. 6 where the solid lines represent the time interval when the impurity level is occupied and the light ones stand for unoccupied levels. The conduction electrons can join the time line at the end points of the intervals where hopping $V_0$ takes place while they can touch the time line at any points due to the Coulomb interaction. Those are indicated by dashed lines which are labeled according to the channel indices $a_i$.

![FIG. 5: (Color online) Comparison between weak coupling RG and NRG approach: The renormalized value of the hybridization, $V_{\text{ren}}$, as a function of the interaction strength $U$ for $N = 2$ and $N = 4$.](image)

![FIG. 6: The solid lines represents the time interval when the impurity level is occupied and the light ones stand for unoccupied levels. The conduction electrons can join the time line at the end points of the intervals where hopping $V_0$ takes place while they can touch the time line at any points due to the Coulomb interaction. Those are indicated by dashed lines which are labeled according to the channel indices $a_i$.](image)
or expressed in terms of phase $z$ (phase shifts) they are
\[
\frac{dy}{d\ln \tau} = y \left(1 - z_0 - \frac{1}{2} \sum_\alpha z_\alpha^2 \right)
\]
\[
\frac{dz_\alpha}{d\ln \tau} = 2(\delta_\alpha 0 + z_\alpha) y^2.
\]
Here the scaling is carried out by increasing $\tau$ to reduce the electronic bandwidth. The term $1/2y$ in the first line of Eq. (21) originates in the explicit factor $\tau_0^{1/2}$ in the definition of $y$ (13) and disappears from the corresponding scaling equation of $V$.

The system of equations Eqs. (21) must be solved for initial values $y(\tau_0) = 1$ but $z_\alpha$ can be arbitrary. The fugacity can either increase or decrease exponentially depending on the quantity $(z_0 + 1/2) \sum_\alpha z_\alpha^2$. Similar expressions were obtained in Ref. 7 by matching the perturbative results with expression in terms of phase shifts.

The regions for attractive interaction and large enough repulsive one will be treated separately. In the first case $(-z_0 - 1/2 \sum_\alpha z_\alpha^2) < 0 \ (U < 0, z_0 > 0)$. The solution is $y/y_0 = (\tau/\tau_0)^{-z_0/2} \sum_\alpha z_\alpha^2$ thus $y$ is decreasing, therefore $z_\alpha (\alpha = 0, \ldots)$ are slowly varying thus the $\tau$ dependence can be ignored. Thus
\[
V_{\text{inv}} = V_0 \left(\frac{\tau}{\tau_0}\right)^{-z_0/2} \sum_\alpha z_\alpha^2.
\]

The situation is different for repulsive interaction ($z_0 < 0$). There are two regions in that case. In the first case $(-z_0 - 1/2 \sum_\alpha z_\alpha^2) > 0$ and $y$ is increasing. Then Eq. (22) is valid as far as $y < 1$ is satisfied. There is, however, a crossover where $(-z_0 - 1/2 \sum_\alpha z_\alpha^2) = 0$ to the second region where $y$ decreases again and the screening dominates. The larger $N$ the stronger the decrease is.

The crossover between the increasing and decreasing regions is at $z_0 = -1/2 \sum_\alpha z_\alpha^2$.

As $z_\alpha$-s are very slightly renormalized thus the unrenormalized values can be used and $\delta_\alpha$ is independent of $\alpha (\delta < 0)$. Thus the crossover is at $z_\alpha = 2/N$ and then for
\[
N = 2: \quad \delta = \frac{\pi}{2}: \quad \delta U \rightarrow \infty
\]
\[
N = 4: \quad \delta = \frac{\pi}{4}: \quad \delta U = \frac{1}{2}.
\]

Comparing with the results of NRG in case $N = 2$ the turning point is at $\delta U \rightarrow \infty$ thus the agreement is not complete but at least it could be argued that it is inside the accuracy of the scaling equation. The weak coupling scaling result is very poor as it was expected (see Fig. 4). For $N = 4$ all the methods give very similar results.

The general solution of the scaling equations can be searched in the form $C_\alpha(\tau) = (C_\alpha)_{\text{initial}} \zeta(\tau)$. Then
\[
\frac{d\zeta}{d\ln \tau} = -2y^2 \zeta
\]
\[
\frac{dy}{d\ln \tau} = \left(1 - \frac{1}{2} \sum_\alpha C_\alpha^2 \right) y.
\]

The scaling trajectories are
\[
4y^2(\tau) - \zeta^2(\tau) \sum_\alpha C_\alpha^2 + 4 \ln \zeta(\tau) = 4y^2(0) - \sum_\alpha C_\alpha^2.
\]

During the renormalization $y$ is fast increasing and the scaling is stopped where it reaches unity. Meanwhile $\zeta$ decreases slowly and its renormalized value can be extracted from Eq. (23). The result in leading order in $y$ reads as
\[
\ln \zeta(\tau) = -\frac{y^2(\tau) - y_0^2}{\frac{1}{2} - z_0 - \frac{1}{2} \sum_\alpha z_\alpha^2}.
\]

Outside that region the long time approximation for the conduction electron Green’s function cannot be applied.

V. WEAK COUPLING APPROACH FOR OUT OF EQUILIBRIUM

Considering theoretical methods two ways can be followed: the Keldysh Green’s function method or the calculation of the scattering amplitude by time-ordered perturbation theory. Here the second method will be followed, where the initial conduction electron states can be arbitrary non-equilibrium states and for the intermediate and final states the actual non-equilibrium distributions are taken into account. That method has been earlier applied in the leading logarithmic approximation16,17,18, which is a generalization of Anderson’s poor man scaling19.

Here the extension of that method is presented to next leading order. For equilibrium first the Kondo model was treated by that way20. The basic idea is to calculate the development of the initial $|i\rangle$ state to the final $|f\rangle$ one, but in second order the renormalization of the norms of those states must be corrected also. Thus the scattering matrix element to be considered is

\[
T_{fi}(\omega) = \frac{\langle f | H_{\text{int}} \sum_{n=0}^{\infty} \left(\frac{1}{\omega - H_0} H_{\text{int}}\right)^n | i \rangle}{\sum_{n=0}^{\infty} \langle f | \sum_{n=0}^{\infty} \left(\frac{1}{\omega - H_0} H_{\text{int}}\right)^n | i \rangle^{1/2}}.
\]
value will be \( \langle n_d \rangle = 1/2 \) for \( \varepsilon_d = 0 \) but in the general case \( \varepsilon_d \neq 0 \) it can depend on eV. The diagrams of the numerator up to \( \sim U^3 \) order are shown in Fig.7 (i)-(vi) where the diagrams should be decorated by the direction of the lines in all possible ways. The diagram of the self-energy is shown in Fig.7 (vii). In logarithmic approximation only the diagrams linear in \( \log \frac{D}{\pi \varkappa} \) are contributing to the scaling equations thus the relevant vertex corrections are (ii),(v) and (vi) while (iii) and (iv) are not as these provide \( \log^2 \frac{D}{\pi \varkappa} \). The type (ii) diagrams with the parallel and antiparallel lines cancel each other. As the logarithmic terms in diagrams (v),(vi) and (vii) come from closed electron loops which are independent of the actual value of \( \mu_L \) and \( \mu_R \) thus to the logarithmic term the left and right contacts contribute separately which should be independent of the applied voltage. That simplification is not sustained in higher order contributions where the left and right lines simultaneously occur. The self-energy correction in (vi) contributes by adding that to either of the incoming and outgoing \( d \)-lines. One of those corrections is cancelled by the denominator in Eq.26. As it is well known from the spinless fermionic case e.g. the X-ray absorption problem the remaining diagram is cancelled by (v). Thus single logarithmic term does not remain. That is similar to the equilibrium case, see Eq.9 thus the invariant coupling \( U_{\text{inv}} = \text{const} \). (For the details see Appendix A.)

In the following the renormalization of the hybridization depicted in Fig.1 and 2 is crucial where e.g. \( |i\rangle = d^\dagger|0\rangle \) and \( |f\rangle = a_{\kappa\alpha}^\dagger|0\rangle \). Keeping terms up to \( \sim V U^2 \). After taking the denominator in Eq.26 into account the final form of \( T_{fi} \) is

\[
\langle 0|a_{\kappa\alpha} T d^\dagger|0\rangle = V_\alpha \left( 1 + U \theta_0 \log \frac{D - \varepsilon_d}{\omega - \varepsilon_d} - \frac{1}{2} U^2 \theta_0^2 \log \frac{D}{\omega - \varepsilon_d} \right), \tag{27}
\]

where the first correction is due to the vertex depicted in Fig.2 while the second one arises from the self-energy on the leg of diagram reduced by the denominator by a factor 1/2. This result agrees with Eq.13 \( (N = 2) \) at eV = 0. Here taking the special case \( \varepsilon_d = 0 \) the voltage eV serves as a low energy cutoff.

As it has been mentioned earlier considering the \( d \)-level there is a steady state occupation \( n_d \). That value is determined from the balance of the in and outflow of the conduction electrons. To determine it for \( \varepsilon_d \neq 0 \) two other quantities must be known, namely the changes in the level position and the spectral function of the \( d \)-level, \( \varepsilon_d \) and \( \vartheta_d(\varepsilon) \) due to the applied voltage. That calculation can be carried out numerically in a self-consistent way.

The probability of scattering of an electron coming from the left (L) or right (R) into the \( d \)-level is denoted by \( W_{L/R}^+ \) while the opposite process by \( W_{L/R}^- \). These quantities are

\[
W_{L/R}^+ = (1 - n_d)2\pi \theta_0 \int V_{L/R}^2(\varepsilon) \vartheta_d(\varepsilon) f_{L/R}(\varepsilon) d\varepsilon, \tag{28}
\]

and

\[
W_{L/R}^- = n_d 2\pi \theta_0 \int V_{L/R}^2(\varepsilon) \vartheta_d(\varepsilon) (1 - f_{L/R}(\varepsilon)) d\varepsilon, \tag{29}
\]

where \( V_{L/R}(\varepsilon) \) are determined from renormalization group equations with the appropriate infrared cutoffs and \( f_{L/R}(\varepsilon) = f(\varepsilon \pm eV/2) \) is the Fermi distribution function for the leads in presence of the voltage. Those will be taken at zero temperature \( T = 0 \).
The steady state is determined by
\[\frac{d}{dt}n_d = W_L^+ + W_R^+ - W_L^- - W_R^- = 0.\] (30)

This equation combined with Eqs. (28) (29) gives
\[n_d = \frac{\int \varrho_d(\varepsilon) \left[ V_R^2(\varepsilon) f(\varepsilon + eV/2) + V_L^2(\varepsilon) f(\varepsilon - eV/2) \right] d\varepsilon}{\int \varrho_d(\varepsilon) \left[ V_L^2(\varepsilon) + V_R^2(\varepsilon) \right] d\varepsilon}.\] (31)

If electron-hole symmetry holds thus \(\varepsilon_d = 0\), then \(n_d = 1/2\). The next step is to determine the self-energy of the \(d\)-electron. The \(d\)-electron propagator is
\[\langle 0|dH \frac{1}{\omega - H_0} d^{|0}\rangle\]
which can be simply developed because the occupied \(d\)-level determines the time flow. The self-energy corrections appear also in the normalization. The effect of hybridization is just to give an extra broadening of the \(d\)-level to be considered later. Without hybridization the self-energy is
\[\Sigma(\omega + i\delta) = U^2 \varrho_0^2 \int_{-D+\mu}^{D+\mu} d\xi' \int_{-D+\mu}^{D+\mu} d\xi'' \left( 1 - f(\xi'') \right) f(\xi') \frac{1}{\omega + \xi' - \xi'' - \varepsilon_d + i\delta}.\] (32)

In the equilibrium calculation the term proportional to \(\sim 2D\log 2\) is eliminated by the applied counter term to keep \(\varepsilon_d\) unrenormalized while \(\text{Im}\Sigma(\omega)\) is a Korringa type of relaxation. To be noted that the voltage does not occur as the energy goes directly into the electron-hole creation of the same electrode. As we mentioned at the end of Sec II, that broadening is less important at small energies. The hybridization of the \(d\)-level gives the essential part of the broadening just like
\[\Gamma(\varepsilon) = 2\pi \varrho_0 (V_L^2(\varepsilon) + V_R^2(\varepsilon)),\] (35)
where the voltage dependent hybridization strength must be used.

The the \(d\)-electron spectral function is
\[\varrho_d(\omega) = \frac{\Gamma(\omega)/2}{\pi (\omega - \varepsilon_d)^2 + (\Gamma(\omega)/2)^2}.\] (36)

With the help of these quantities we are ready to calculate the current through the impurity:
\[I = \frac{1}{2} \frac{d}{dt} (N_R - N_L) = W_L^- + W_L^+ - W_R^- - W_R^+.\] (37)

Combining this equation with the expression of \(W_{L/R}^\pm\) given in Eqs. (28) (29) the current takes the form
\[I = n_d 2\pi \varrho_0 \int \left[ V_R^2(\varepsilon) - V_L^2(\varepsilon) \right] \varrho_d(\varepsilon) d\varepsilon - 2\pi \varrho_0 \int \left[ f(\varepsilon - eV/2)V_R^2(\varepsilon) - f(\varepsilon + eV/2)V_L^2(\varepsilon) \right] \varrho_d(\varepsilon) d\varepsilon.\]

The numerical calculation goes as follows: for a fixed value of \(eV\) we discretize the energy interval \(\omega_i \in [-D +


\[ \mu, D + \mu \] and calculate the renormalized hybridization \( V_{L/R}(eV, \omega_i) \) and the impurity self-energy. The latter is evaluated in such a way that the renormalized d-level position zero \( \tilde{\varepsilon}_d = 0 \). By performing a sum over \( \omega_i \) we cab calculate at the level occupation \( n_d(eV) \) and the current \( I(eV) \) for the given value of the voltage. The result is shown in Fig.8 for \( U/D = 0, \ldots, 0.10 \).

In that regime of \( U \) the weak coupling RG gave good results in equilibrium (presented in Sec. III) therefore one expects reliable results in out-of-equilibrium as well. As it is shown in Fig.8 the current increases with increasing interaction strength. The results can be interpreted as follows: with increasing \( U \) the impurity spectral function gets broadened since the hybridization is enhanced. In the linear regime \((eV < \Delta)\) the \( U \)-dependence drops out since the increase of coupling to the leads \( \Gamma(eV, U) \) is cancelled by the decrease of the height of the d-level spectral function which scale as \( \rho_d(\omega = 0) \sim 1/\Gamma(eV, U) \). For larger values of the voltage \( eV > \Delta \) the d-level is experienced not only at the peak of the spectral function \( (\rho_d(\omega = 0)) \) thus the current is not linear in \( eV \) any more. For \( eV \gg \Delta \) the complete d-level contributes to the current thus it saturates and the asymptotic value is proportional to \( \Gamma(eV, U) \).

**VI. CONCLUSION**

The resonant level model studied has very different behavior for attractive and repulsive interaction. This difference can be understood using the site representation for conduction electrons in the strong \( U \) limit by the following argument:

(i) **in case of attractive interaction** the particle on the d-level attracts electrons to pile up around the impurity occupying the next site thus the electron on the d-level is blocked for hopping to the conduction band.

(ii) **in case of repulsive interaction** the site next to the occupied d-level is empty thus that electron can easily jump to the conduction band.

All the methods predict that increasing the strength of an attractive interaction the hopping rate \( V \) is reduced and for strong enough coupling it even goes to zero (see Fig.5 for \( U \rho_0 < -0.25 \)). The effect of orthogonality catastrophe reducing the hopping is less relevant because that have been already reduced by the vertex correction.

In the repulsive case for large \( U \) the orthogonality catastrophe (self-energy correction) is reducing essentially the hopping rate. Thus the effective hopping \( V_{\text{eff}} \) is first enhanced by the effect described above and can reach a maximum which is followed by a reduction due to the orthogonality catastrophe. The position of the maximum can be pushed to lower energies by increasing the number of the screening channels \( N \) thus the perturbative result becomes more and more reliable. In case of \( N = 2 \) the latter method leads to a pronounced maximum but the NRG indicates only slowly varying bump. In the Anderson-Yuval approach the maximum is even pushed to infinity. Thus the existence of the maximum is supported only by the NRG.

Considering the time ordered scattering formalism the results are in accordance with the expectation of the weak coupling result for \( N = 2 \). The increasing \( U \) results in increasing current as first \( V \) is increased but for larger \( U \) due to the orthogonality catastrophe the current is essentially reduced (see Fig.9). As the NRG does not give a sharp crossover for the hopping rate \( V_{\text{eff}} \) thus the corresponding effect in the current must be less pronounced in reality. Of course, for \( N \gg 2 \) the crossover must clearly exist. Unfortunately, in the work of Mehta and Andrei the crossover is in the range of reduced current, which is just the opposite what is expected on the ground of the physical picture established and results obtained by different methods for the hopping rate.

Very detailed further studies of the Bethe Ansatz
The diagrams of the numerator of Eq. (26) up to $\sim U^3$ order are shown in Fig. 7(i)-(vi) and the diagram of the self-energy is shown in Fig. 7(vii). As noted earlier, the logarithmic terms come from diagrams (v),(vi) and (vii). Note that the self-energy correction in (vi) (See also Fig. 10b) contributes by adding that to either of the incoming or outgoing $d$-lines. One of those corrections is cancelled by the diagram shown in Fig. 7(vii) in the denominator in Eq. (26). Therefore the two relevant diagrams are those depicted in Fig. 10. The contribution of Fig. 10b can be written as

$$\frac{1}{\omega - \varepsilon_d} U^2 \frac{1}{g_0} \int_{-D+\mu}^{D+\mu} d\xi' \int_{-D+\mu}^{D+\mu} d\xi'' (1 - f(\xi'')) f(\xi') \frac{1}{\omega + \xi' - \xi'' - \varepsilon_d}.$$  \hfill (A1)

To get the purely logarithmic term we can now expand the integral in linear order in $\sim (\omega - \varepsilon_d)$ and get the form

$$U^2 \frac{1}{g_0} \int_{-D+\mu}^{D+\mu} d\xi' \int_{-D+\mu}^{D+\mu} d\xi'' (1 - f(\xi'')) f(\xi') \frac{1}{(\omega + \xi' - \xi'' - \varepsilon_d)^2}.$$ \hfill (A2)

On the other hand, the contribution of Fig. 10a for small frequencies can be evaluated and the $d$-electron lines indicated by arrows occur twice just like the denominator squared in Eq. (A2). That Ward identity ensures the cancellation of diagram shown in Fig. 7(v) by one of the leg one in Fig. 7(vi). This means that no logarithmic term survives in second order thus the invariant coupling $U_{\text{inv}} = \text{const}$.

Acknowledgments

The authors acknowledge the fruitful discussions with N. Andrei, P. Mehta, J. von Delft, J. Kroha, P. Wölfle, P. Schmitteckert and G. Zaránd. This work was supported by Projects OTKA D048665, T048782, TS049881 and T046303. A.Z. is grateful to the Humboldt Foundation to support his stay in Munich. L.B. acknowledges the support of the János Bolyai Scholarship.

APPENDIX A: CANCELLATION OF THE LOGARITHMIC TERMS IN THE RENORMALIZATION OF THE COULOMB INTERACTION

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