PROPAGATION OF CHAOS FOR A CLASS OF FIRST ORDER MODELS WITH SINGULAR MEAN FIELD INTERACTIONS

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Abstract. Dynamical systems of \(N\) particles in \(\mathbb{R}^D\) interacting by a singular pair potential of mean field type are considered. The systems are assumed to be of gradient type and the existence of a macroscopic limit in the many particle limit is established for a large class of singular interaction potentials in the stochastic as well as the deterministic settings. The main assumption on the potentials is an appropriate notion of quasi-convexity. When \(D = 1\) the convergence result is sharp when applied to strongly singular repulsive interactions and for a general dimension \(D\) the result applies to attractive interactions with Lipschitz singular interaction potentials, leading to stochastic particle solutions to the corresponding macroscopic aggregation equations. The proof uses the theory of gradient flows in Wasserstein spaces of Ambrosio-Gigli-Savaree.

1. Introduction

Let \(F\) be an odd map \(\mathbb{R}^D \to \mathbb{R}^D\) and consider the following system of \(N\) stochastic differential equations (SDEs) on \(\mathbb{R}^D\) (Ito diffusions):

\[
\dot{x}_i(t) = \frac{1}{N-1} \sum_{j \neq i} F(x_i - x_j)dt + \sqrt{\frac{2}{\beta_N}} dB_i(t), \quad i = 1, 2, ..., N
\]

for a given parameter \(\beta_N \in [0, \infty]\), where \(B_i\) denotes \(N\) independent Brownian motions on \(\mathbb{R}^D\) and where the sum ranges over the \(N - 1\) indices where \(j \neq i\). We also allow the deterministic case \(\beta_N = \infty\) where the system above is an ordinary differential equation. When \(F\) is Lipschitz continuous these systems have strong solutions, which are unique, given appropriate initial conditions and determine the corresponding empirical measures

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}
\]

taking values in the space \(\mathcal{P}(\mathbb{R}^D)\) of probability measures on \(\mathbb{R}^D\). For general, possibly singular \(F\), such systems frequently arise as “first order mean field models” in statistical mechanics problems in mathematical physics, biology, numerical Monte-Carlo simulations and various other fields, where the \(x_i\)s represent the positions of \(N\) pair interacting particles (or individual/agents) on \(\mathbb{R}^D\) with \(F\) playing the role of the interaction force (see for example, [36] and references therein). A classical problem is to show that, under appropriate assumptions on \(F\) and the initial data, a deterministic macroscopic evolution emerges from the microscopic dynamics in
the “many particle limit” where $N \to \infty$, assuming that $\beta_N$ has a limit:

$$\lim_{N \to \infty} \beta_N = \beta \in [0, \infty]$$

This problem has been studied extensively in three different settings, ranging from the purely deterministic to the completely stochastic:

1. The evolution is deterministic (i.e. $\beta_N = \infty$) and the initial positions $x_i(t)$ are also deterministic. One then assumes that the corresponding empirical measures of $N$ particles at $t = 0$ have a definite limit $\mu_0$, as $N \to \infty$, i.e. they converge to a probability measure $\mu_0$ on $\mathbb{R}^D$, in a suitable topology.

2. The evolution is deterministic, but the initial positions $x_i(0)$ are taken to be independent random variables with identical distribution $\mu_0$.

3. The noise term is present (i.e. $\beta_N \neq \infty$) and the initial positions $x_i(0)$ are taken to be independent random variables with identical distribution $\mu_0$.

In the first, purely deterministic, case, the problem is to show that there exists a curve $\mu_t$ in $\mathcal{P}(\mathbb{R}^D)$ emanating from $\mu_0$ such that, for any positive time $t$,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)} \to \mu_t$$

in a given topology on $\mathcal{P}(\mathbb{R}^D)$ (then the corresponding deterministic particle system is usually said to have a mean field limit). In the second case the empirical measures of $N$-particle defines, at any given time $t$, a random measure and the problem is then to show that the convergence above holds in law for any $t > 0$ (then propagation of chaos is said to hold; a terminology introduced by Kac [48], inspired by the work of Boltzmann). For simplicity we will simply say that the system (1.1) has a macroscopic limit $\mu_t$ if the convergence above holds in all three situations (note that convergence in setting 1 implies convergence in the Setting 2.

As is well-known the setting above admits a pure PDE formulation, not involving any stochastic calculus and it is this analytic point of view that we will adopt here. Indeed, the laws of the SDEs (1.1) define a curve $\mu_t^{(N)}$ of probability measures on $(\mathbb{R}^D)^N$, which can be directly defined as the solution to a linear PDE on $(\mathbb{R}^D)^N$; the forward Kolmogorov equation (also called the linear Fokker-Planck equation). Accordingly, the empirical measure at time $t$ (formula 1.2) can be viewed as the random measure $\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$ on $((\mathbb{R}^D)^N, \mu_t^{(N)})$.

For a Lipschitz continuous force term $F$ the existence of a macroscopic limit goes back to the seminal work of McKean [57, 58] and there are by now various different approaches and developments (see for example [36, 79, 63] for a recent review on Setting 1 and 2 above see [42]). Moreover, the macroscopic limit $\mu_t$ may then be characterized as the unique weak solution of the following non-local drift-diffusion equation on $\mathbb{R}^D$:

$$\frac{d\mu_t}{dt} = \frac{1}{\beta} \Delta \mu_t - \nabla \cdot (\mu_t b[\mu_t]), \quad b[\mu_t](x) = \int_{\mathbb{R}^D} F(x - y) \mu(y)$$

(1.3)

(often called the McKean-Vlasov equation in the literature). However, in many naturally occurring models the interaction force $F$ is not locally Lipschitz and even unbounded around $x = 0$ and the main purpose of the present paper is to establish the existence of a macroscopic limit for a wide class of such singular $F$, including strongly singular repulsive $F$ when $D = 1$ and locally bounded (but not necessarily continuous) attractive $F$, for any dimension $D$. 

1.1. Statement of the main results. We will be concerned with the case when
the interaction force $F$ can be realized as a gradient:

$$F(x) = -\langle \nabla w \rangle (x)$$

(in the almost everywhere sense) for an even function $w$. In particular, this is always
the case in dimension $D = 1$. Then the SDEs $\text{(1.1)}$ are often called the (overdamped)
Langevin equation. Following [45, 42] we will say that the interaction is *weakly singular* if
$w(x-y)$ is continuous and *strongly singular* if its absolute value blows-up along the diagonal. In terms of the singularity along the diagonal our results
apply, when $D = 1$, to any singular repulsive $w$ in $L^1_{loc}$ (which is hence strongly singular) and when $D \geq 1$ to attractive locally Lipschitz continuous $w$.

We will denote by $\mathcal{P}_2(\mathbb{R}^D)$ the space of all probability measures on $\mathbb{R}^D$ with finite
second moments, endowed with its standard topology defined by weak convergence together with convergence of the second moments. In other words, this is the
topology on $\mathcal{P}_2(\mathbb{R}^D)$ determined by the Wasserstein $L^2$-metric on $\mathcal{P}_2(\mathbb{R}^D)$. The interaction potential $w$ induces, in the usual way, an energy type functional on the
space $\mathcal{P}_2(\mathbb{R}^D)$ defined by

$$E(\mu) = \frac{1}{2} \int_{\mathbb{R}^D \times \mathbb{R}^D} w(x-y) \mu(x) \otimes \mu(y),$$

assuming that $w \in L^1_{loc}$. The corresponding free energy functional $F_{\beta}(\mu)$ is then obtained by adding the scaled Boltzmann entropy $H(\mu)/\beta$ to $E(\mu)$. More generally, we will consider the case when a confining potential $V$ is included in the system $\text{(1.1)}$ (thus breaking the translational symmetry), i.e.

$$F(x-y) = -\langle \nabla w \rangle (x-y) - \langle \nabla V \rangle (x) - \langle \nabla V \rangle (y),$$

which amounts to replacing $w(x-y)$ with $w(x-y) + V(x) + V(y)$. We will say that
a lsc function $\psi(x)$ on $\mathbb{R}^D$ is *quasi-convex* if it is $\lambda-$convex, i.e. its distributional
Hessian is bounded form below by $\lambda I$, for some (possibly negative) number $\lambda$ and if, in
the negative case, $|\lambda|$ can be taken arbitrarily small as $|x| \to \infty$. For example, any polynomial on $\mathbb{R}^D$ with leading term of the form $C x^{2m}$ for $C > 0$ is quasi-convex in
our sense, as is any perturbation of $\psi(x)$ by a compactly supported smooth function
(see Section $\text{[4]}$).

**Theorem 1.1.** Consider the case when $D = 1$ and assume that $w(x)$ and $V(x)$ are
quasi-convex on $[0, \infty]$ and $\mathbb{R}$, respectively and that $w \in L^1_{loc}(\mathbb{R})$. Then, given any
limiting initial measure $\mu_0$ a macroscopic evolution $\mu_t$ emerges, as $N \to \infty$, in all
three settings considered above.

The main novelty of the previous theorem is the establishment of propagation of
chaos for very singular interactions in the completely stochastic Setting 3 above, for example the strongly singular power-laws considered in Theorem $\text{[12]}$ below. Moreover, in this general form the result also appears to be new in the deterministic
settings.

Before continuing some comments on the definition of the corresponding probability measures $\mu^{(N)}(t)$ on $\mathbb{R}^N$ are in order in the general non-smooth setting (see Section $\text{[2.4]}$ for more details). Concretely, but indirectly, these may be defined by
regularization: i.e. as the unique weak limit of the probability measures $\mu^{(N)}(t)$
obtained by replacing $w$ (and similarly $V$) with any sequence $w_\epsilon$ of smooth uniformly $\lambda-$convex functions on $[0, \infty[$ converging to $w$ as $\epsilon \to 0$, in a suitable way
(for example, increasing to \( w \)). From this point of view the previous theorem can be interpreted as saying that the limits \( \epsilon \to 0 \) and \( N \to \infty \) commute. In fact, the same proof reveals that we may as well let \( \epsilon \) depend on \( N \) as long as \( \epsilon_N \to 0 \) as \( N \to \infty \). More directly, the curve \( \mu^{(N)}(t) \), and the macroscopic limit \( \mu(t) \), will be intrinsically defined using the theory of gradient flows on Wasserstein spaces, following the approach in [1] [2]. In particular, the notion of generalized geodesics and generalized \( \lambda \)-convexity in Wasserstein spaces, introduced in [1], plays a key role in the proof.

An important feature of our approach is that the convergence in Theorem 1.1 will be established using direct variational arguments, realizing the limit \( \mu_t \) as the unique gradient flow of the corresponding free energy functional \( F_\beta \), in the sense of evolutionary variational inequalities (EVI)[2]. The corresponding Wasserstein gradient flow of \( F_\beta \) has previously been studied in [23, 25]. This approach bypasses the delicate issue whether the limit can be uniquely characterized as a weak solution of the McKean-Vlasov equation [1, 3] in the sense that for any \( \phi \in C^2_c(\mathbb{R}^D) \)

\[
\frac{d}{dt} \int_{\mathbb{R}^D} \mu_t(x) \phi(x) = \frac{1}{\beta} \int \mu_t(x) \Delta \phi(x) + \frac{1}{2} \int \mu_t(x) \otimes \mu_t(y) F(x-y) \cdot (\nabla \phi(x) - \nabla \phi(y)) ,
\]

holds in the distributional sense when \( t \in [0, \infty) \) with \( \mu_t \to \mu_0 \) weakly, as \( t \to 0 \) and similarly when a potential term \( V \) is added (this weak formulation is indeed equivalent to equation [1, 3] when \( F \) is continuous, since \( F \) is odd). But once \( \mu_t \) has been realized as an EVI gradient flow one can invoke the subdifferential calculus in [1] (which provides a rigorous framework for the Otto calculus [67]) to show that \( \mu_t \) is indeed a weak solution of the McKean-Vlasov equation [1, 3]. However, it should be stressed that, in general, there is no uniqueness result for weak solutions of such singular equations.

The following theorem illustrates our general convergence results, in any dimension, in the case when the interaction force \( F \) has a power-law singularity in \( \mathbb{R}^D \) in the sense that

\[
F(x) = f(x) |x|^{\alpha} \sigma(x)
\]

where \( f \) is a bounded \( C^2 \)-smooth function and \( \sigma(x) := x/|x| \). The power-law singularity is said to be repulsive if \( f > 0 \) and attractive if \( f < 0 \) and we will refer to the case when \( f \) is constant as the model case.

**Theorem 1.2.** In any dimension \( D \) a macroscopic limit emerges for the attractive power-laws with \( \alpha \leq 0 \). When \( D = 1 \) the results also holds for repulsive power-laws with \( \alpha \in [0, 2] \). More generally, for any \( D \) the results hold when a quasi-convex potential \( V \) is included.

The repulsive case \( \alpha = 1 \) (with \( f = 1 \)) corresponds to a logarithmic interaction potential and has been extensively studied in random matrix theory and free probability. The corresponding stochastic system [14] then coincides with Dyson’s Brownian motion, as recalled below. For \( \alpha > 1 \) the corresponding repulsive interaction potential is a long range potential, i.e. the model pair interaction potential is of the form \( 1/|x-y|^s \) (where \( s := \alpha - 1 \)) with \( s < D \). Such interaction potentials appear frequently in the physics litterature as pseudo/effective potentials [55]. In the model case the corresponding McKean-Vlasov equation can be formulated as a non-local porous medium type equation, coupled to a fractional Laplacian [10, 18].
The long range condition is precisely what is need to make sure that the weak McKean-Vlasov equation (1.5) makes sense, since it ensures that the integrand is in $L^1_{\text{loc}}(\mathbb{R}^{2D})$ (using that the the factor $\phi'(x) - \phi'(y)$ is comparable to $|x - y|$ as $x \to y$). In this respect the previous theorem appears to be optimal when applied to repulsive power-laws in 1D.

The main thrust of the results in Theorem 1.2 for attractive power-laws concern the range when $\alpha \in [-1, 2]$, i.e. the range where the power-singularity of the interaction force is proportional to $|x|^{\gamma}$ for $\gamma \in [0, 1]$ and thus not locally Lipschitz continuous. The corresponding McKean-Vlasov equations have been studied extensively in the subtle case when $\beta = \infty$, i.e. in the absence of diffusion. The equation in question is then often called the aggregation equation and it exhibits interesting concentration phenomena. For example, there is, in general, no uniqueness of weak solutions and classical solutions blow-up in a finite time (see [52] and references therein). The particular model case with $D = \gamma = 1$ is covered by the theory of scalar conservation laws where the blow-up of solutions corresponds to the classical phenomenon of shock formations and where uniqueness of weak solutions only holds for entropy solutions [11]. The application of the theory of Wasserstein gradient flows to the general aggregation was introduced in [25], which, as recalled above, provides a canonical solution $\mu_t$ of the equations for all times (generalizing the notion of entropy solutions, as recalled in Section 6.2).

The results above arose as a “spin-off effect” of the new approach to propagation of chaos introduced in the companion paper [8], motivated by complex geometry or more precisely by the construction of Kähler-Einstein metrics on complex algebraic manifolds. In [8] the focus was on interaction energies $E^{(N)}(x_1, \ldots, x_N)$ in $\mathbb{R}^D$ which are highly non-linear in the sense that they are not $m$–point interactions for any finite $m$. On the other hand the drawback with the general result in [8], which is applied to the construction of toric Kähler-Einstein metrics, is that it requires that $E^{(N)}(x_1, \ldots, x_N)$ be $\lambda$–convex on all of $\mathbb{R}^D$. In particular, when applied to pair interactions of the form (1.4) it forces $w(x)$ to be Lipschitz continuous around $x = 0$.

The main point of the present paper is to show how to circumvent this problem, when $D = 1$, by working directly on the quotient space $\mathbb{R}^N/S_N$. The condition that $D = 1$ then enters in the key convexity result Proposition 4.6 formulated in terms of optimal transportation. We also give a general formulation of the approach in [8] which gives a unified approach to establish the existence of a macroscopic limit $\mu_t$ in all three settings above. In fact, Theorem 1.1 will arise as a special case of a general convergence result concerning interaction energies $E^{(N)}(x_1, \ldots, x_N)$ which, when $D = 1$, are assumed $\lambda$–convex when $x_1 < x_2 < \ldots < x_N$, which in particular applies to $m$–point interaction energies.}

1.2. Comparison with previous results in the singular setting. As will be recalled below there has recently there has been remarkable progress on the case of singular forces $F$, but there are still comparatively few general results. One of the main problems is to single out a class of weak solutions of an appropriate singular version of the equation (1.3) where uniqueness holds and then to show (i) a suitable compactness/continuity result when $N \to \infty$ and (ii) that any limiting curve $\mu_t$ obtained in the first step is contained in the class of weak solutions where uniqueness holds (see, for example, the discussions in [36, 39]).

In order to make a more precise comparison of our results to previous results we first briefly recall the general setting of first and second order mean field models (for
a general review see [42]). The latter models arise as Newton’s \( N \)-body equations on the phase space \( \mathbb{R}^{2D} \) (with a noise term). The first order models typically arise in mathematical physics as scaling limit of second order models containing friction/resistance, as well as instanton (tunneling) solutions in stochastic quantization [26]. See also [24] and references therein for applications to mathematical biology. The first, as well as second order, models have a rather different flavour depending on whether \( F \), viewed as a vector field on \( \mathbb{R}^D \)− \{0\}, is a gradient - as in our setting - or if it its divergence vanishes (the “incompressive” case). In the stochastic setting (\( \beta N < \infty \)) the first order gradient models that we consider here, i.e. overdamped Langevin processes, are widely used as a theoretical model for Monte-Carlo Markov schemes as used in numerics to simulate the Boltzmann-Gibbs measures associated to a sequence of the interaction energy/Hamiltonians \( E^{(N)} \) (i.e. the probability measures on \( \mathbb{R}^{DN} \) proportional to \( e^{-\beta N E^{(N)}(x)\,dx^\otimes N} \)). In particular, in the “zero temperature limit” where \( \beta N \to \infty \) such schemes are used to locate configurations with nearly minimal energy \( E^{(N)}(x) \) (for example, in the model case of power laws the interaction energy \( E^{(N)}(x_1,\ldots,x_N) \) is called the discrete Riesz s-energy and has been studied extensively in the mathematics litterature in connection to approximation theory [70]). In this numerical context the problem of propagation of chaos thus amounts to the question of whether a large-scale coherent structure should emerge in the numerical simulations, as \( N \to \infty \).

1.2.1. \( D = 1 \). When \( D = 1 \) propagation of chaos in the completely stochastic setting 3 has been established in increasing level of generality for the case \( \alpha = 1 \) of the repulsive logarithmic interaction potential \( w(t) = -\log t \) with \( V \) quadratic [78, 21]. One simplifying feature in the logarithmic setting is that the continuity property (i) discussed above automatically holds since the right hand side in the equation 1.5 is continuous wrt the weak topology on \( \mathcal{P}(\mathbb{R}) \). The proofs in [78, 21] also exploit further special features of the logarithmic interaction potential and the quadratic potential \( V \) leading to the uniqueness of weak solutions of the equation 1.5 (using a complexification argument; see Section 5). The results in particular apply to Dyson’s Brownian motion on the space of \( N \times N \) Hermitian matrices \( A \), where \( x_i \) represent the eigenvalues of \( A \) [35] and \( \beta N = \sqrt{N} \). The uniqueness of weak solutions under the assumption that the Fourier transform of \( V \) has exponential decay (in particular \( V \) is real analytic) was established in [17] Lemma 2.6] when \( \beta = \infty \) and in [38] when \( \beta < \infty \) (the uniqueness results in [17, 38] were used to establish a large deviation principle when \( V = 0 \) which in turn implies a stronger form of propagation of chaos when \( V = 0 \)). The propagation of chaos for a \( \lambda \)-convex potential \( V \) was claimed in [51], but there seems to be a gap in the proof. Indeed, the proof in [51] is based on the claim that any weak solution \( \mu_t \) to the corresponding equations 1.5 is uniquely determined by the initial data. However, the proof of the latter claim in [51] uses the formal Otto calculus, which in order to be rigorous would require further a priori regularity properties of \( \mu_t \). Instead, as explained above, the main point of our argument is that it directly produces a EVI gradient flow \( \mu_t \), which is a priori stronger than a weak solution to the equation 1.5 (see the discussion in the end of Section 5). In the more singular case of general repulsive power-laws with \( \alpha < 2 \) there seems to be no previous propagation of chaos results (the study of such very singular interactions was proposed in [62]). In the setting of second order models propagation of chaos in the stochastic setting with \( D = 1 \) and
α = 0 was settled very recently in [46] (see also the references in [46] concerning the deterministic case).

Our results also appear to be new in the purely deterministic setting in this generality (the mean field limit in the model case when α = 1 was established in [31] under stronger assumptions on the initial data, e.g. that ρ₀ be bounded and Lipschitz continuous, using the theory of viscosity solutions of non-local Hamilton-Jacobi equations). On the other hand, as shown in Section 7, in some situations the convergence in the deterministic setting could also be obtained from the stability results for Wasserstein gradient flows in [1], combined with some non-trivial Gamma-convergence results for singular discrete interaction energies [72] (but as far as we know this has not been noticed before in the literature).

1.2.2. D ≥ 1. In the case of globally λ–convex interactions (as in the case of attractive power-laws in Theorem 1.2) and βₙ = ∞ the convergence of the corresponding deterministic N–particle system towards the Wasserstein gradient flow µₜ was shown in [25], using the contractivity property of such gradient flows (see Section 7 for a comparison with the present setting). This was used in [25] to show that µₜ aggregates into a single Dirac mass in a finite time (starting from any compactly supported initial measure). The main novelty of Theorem 1.2, in the attractive case, is thus the possibility to add noise to the attractive particle system. For example, in the zero-temperature case βₙ → ∞ this yields a stochastic particle approximation scheme for constructing the Wasserstein gradient flow solution of the corresponding aggregation equation introduced in [25]. Such stochastic particle approximations have previously been obtained in the model case when D = 1 and α = 0, using the theory of scalar conservation laws and their entropy solutions (see [12] for β < ∞ and [43] when β = ∞). The underlying deterministic microscopic system is then the sticky particle system originating in cosmology [15] and its stochastic version, with βₙ → ∞, is called the adhesion model in cosmology.

Let us also briefly mention some further recent results on the general higher dimensional setting. When D = 2 the critical case of a power-law α = 1 (i.e. the Newtonian case) has been studied extensively in the divergence free case, notably in the case of the vortex model (where F is the Biot-Savart law F(x) = ±Jx/|x|^2 with J denoting rotation by 90 degrees) motivated by the 2D Euler and Navier-Stokes equations [59]. For example, partial results for the corresponding deterministic evolution (called the Helmholtz-Kirchhoff system) in the settings 1 and 2 were obtained in [74] [75] and propagation of chaos in the setting 3 was obtained in [65] [66] using Nash type estimates (for non-negative vorticity and µ₀ in L∞) and recently, in a stronger form, in [69], using the Fisher information. The results in [69], concerning propagation of chaos in the setting 3 (for D = 2), were extended to the gradient setting in [34] under the restriction α < 1 and a partial result concerning the critical case α = 1 (appearing in the Keller-Segel model for chemotaxis) was established in [32] (saying that propagation of chaos holds for some subsequence).

A completely different approach to mean field limits and propagation of chaos for first order gradient models without a noise term, i.e. in the setting 1 and 2 above, was introduced in [24], using stability properties of Wasserstein Lᵖ–distances and subtle estimates. Under some regularity assumptions, in particular that µₜ is in $P_1(\mathbb{R}^D) \cap L^p(\mathbb{R}^D)$ for $t \in [0, T]$, convergence in Setting 1 (and similarly in Setting 2) was established for any power law such that $α < D/p' - 1$, where $p'$ is the dual exponent to p (in particular, this means that $α < 0$ when $D = 1$). The arguments
in [24], where the use of Wasserstein distances in the singular case was first introduced and applied to the setting where the divergence of $F$ vanishes (in the case when $p = \infty$). See also [44, 45] for related work on second order models where it is assumed that the condition $\alpha < 1$ holds.

Let us finally point out that very recently the convergence of the deterministic Helmholtz-Kirchhow system studied in [74, 75] was finally settled in [30] (when $V = 0$ and under the assumption that the initial measure is Hölder continuous and that the initial energies are convergent as $N \to \infty$). The proof is based on a modulated energy method inspired by [73] and exploits the fact that the limiting equation $\mu_t$ is known to have a Hölder continuous density (when $V = 0$). The deterministic results in [30] also apply when $D = 1$, under the condition that the the Hölder regularity property of $\mu_t$ holds (which is an open problem).

1.3. Outline. We start in Section 2 by setting up the general theory of Wasserstein gradient flows from [1] that will be needed in the proof of the general convergence results given in Section 3. As we explain the proof of the latter result can be viewed as an analog in our setting of the stability of Wasserstein gradient flows on a Hilbert space, established in [1, 2]. General applications are discussed which which are then developed in the case of 1D translational invariant pair interactions in Section 4 (covering Theorem 1.1 above). In Section 5 it is shown that the corresponding Wasserstein gradient flows $\mu_t$ appearing in the macroscopic limit are (particular) weak solutions of the McKean-Vlasov equation (complementing some results in [23]). Some further applications, including the case $D \geq 1$ (covering Theorem 1.2 above) are developed in Section 6 and in the final Section 7 a comparison with the stability result in [1, 2] is made, in the deterministic setting.

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2. Preliminaries

2.1. Notation. Given a topological (Polish) space $Y$ we will denote the integration pairing between measures $\mu$ on $Y$ (always assumed to be Borel measures) and bounded continuous functions $f$ by

$$\langle f, \mu \rangle := \int f \mu$$

(we will avoid the use of the symbol $d\mu$ since $d$ will usually refer to a distance function on $Y$). In case $Y = \mathbb{R}^D$ we will say that a measure $\mu$ has a density, denoted by $\rho$, if $\mu$ is absolutely continuous wrt Lebesgue measure $dx$ and $\mu = \rho dx$. We will denote by $\mathcal{P}(\mathbb{R}^D)$ the space of all probability measures and by $\mathcal{P}_{ac}(\mathbb{R}^D)$ the subspace containing those with a density (which coincides with the space $\mathcal{P}^r(\mathbb{R}^D)$ of all regular measures as defined in [1], in this finite dimensional situation). The
Boltzmann entropy $H(\rho)$ taking values in $]-\infty, \infty[)$ is defined by

$$H(\rho) := \int_{\mathbb{R}^d} (\log \rho) \rho dx$$

More generally, given a reference measure $\mu_0$ on $Y$ the entropy of a measure $\mu$ relative to $\mu_0$ is defined by

$$H_{\mu_0}(\mu) = \int_{X^N} \left( \log \frac{\mu}{\mu_0} \right) \mu$$

if the probability measure $\mu$ on $X$ is absolutely continuous with respect to $\mu$ and otherwise $H(\mu) := \infty$. Given a lower semi-continuous (lsc, for short) function $V$ on $Y$ and $\beta \in [0, \infty]$ (the "inverse temperature") we will denote by $F_\beta^V$ the corresponding (Gibbs) free energy functional with potential $V$:

$$F_\beta^V(\mu) := \int_X V \mu + \frac{1}{\beta} H_{\mu_0}(\mu),$$

which coincides with $\frac{1}{\beta}$ times the entropy of $\mu$ relative to $e^{-V} \mu_0$.

2.2. Wasserstein spaces and metrics. We start with the following very general setup. Let $(X, d)$ be a given metric space, which is Polish, and denote by $\mathcal{P}(X)$ the space of all probability measures on $X$ endowed with the weak topology, i.e. $\mu_j \to \mu$ weakly in $\mathcal{P}(X)$ iff $\int_X \mu_j f \to \int_X \mu f$ for any bounded continuous function $f$ on $X$ (this is also called the narrow topology in the probability literature). The metric $d$ on $X$ induces $L^p$–type metrics on the $N$–fold product $X^N$ for any given $p \in [1, \infty[$:

$$d_p(x_1, \ldots, x_N; y_1, \ldots, y_N) := \left( \sum_{i=1}^N d(x_i, y_i)^p \right)^{1/p}$$

The permutation group $S^N$ on $N$ letters has a standard action on $X^N$, defined by $(\sigma, (x_1, \ldots, x_N)) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ and we will denote by $X^{(N)}$ and $\pi$ the corresponding quotient and quotient projection, respectively:

$$X^{(N)} := X^N / S^N, \quad \pi : X^N \to X^{(N)}$$

The quotient $X^{(N)}$ may be naturally identified with the space of all configurations of $N$ points on $X$. We will denote by $d_{(p)}$ the induced distance function on $X^{(N)}$, suitably normalized:

$$d_{X^{(N)},p}(x_1, \ldots, x_N; y_1, \ldots, y_N) := \inf_{\sigma \in S_N} \left( \frac{1}{N} \sum_{i=1}^N d(x_i, y_{\sigma(i)})^p \right)^{1/p}$$

The normalization factor $1/N^{1/p}$ ensures that the standard embedding of $X^{(N)}$ into the space $\mathcal{P}(X)$ of all probability measures on $X$:

$$X^{(N)} \hookrightarrow \mathcal{P}(X), \quad (x_1, \ldots, x_N) \mapsto \delta_N := \frac{1}{N} \sum \delta_{x_i}$$

(where we will call $\delta_N$ the empirical measure) is isometric when $\mathcal{P}(X)$ is equipped with the $L^p$–Wasserstein metric $d_{W^p}$ induced by $d$ (for simplicity we will also write $d_{W_p} = d_p$):

$$d_{W_p}(\mu, \nu) := \inf_\gamma \int_{X \times X} d(x, y)^p \gamma,$$
where $\gamma$ ranges over all coupleings between $\mu$ and $\nu$, i.e. $\gamma$ is a probability measure on $X \times X$ whose first and second marginals are equal to $\mu$ and $\nu$, respectively (see Lemma 2.2 below). We will denote $W^p(X, d)$ the corresponding $L^p$--Wasserstein space, i.e. the subspace of $P(X)$ consisting of all $\mu$ with finite $p$--th moments: for some (and hence any) $x_0 \in X$

$$\int_X d(x, x_0)^p \mu < \infty$$

We will also write $W^p(X, d) = \mathcal{P}_p(X)$ when it is clear from the context which distance $d$ on $X$ is used.

**Remark 2.1.** In the terms of the Monge-Kantorovich theory of optimal transport [80] $d^p_{W_p}(\mu, \nu) = \text{the optimal cost to for transporting $\mu$ to $\nu$ with respect to the cost functional $c(x, p) := d(x, y)^p$. Accordingly a coupling $\gamma$ as above is often called a transport plan between $\mu$ and $\nu$ and it said to be defined by a transport map $T$ if $\gamma = (I \times T)_*\mu$ where $T_*\mu = \nu$. In particular, if $X = \mathbb{R}^n$, $p = 2$ and $\mu$ has a density, then, by Brenier’s theorem [14], the optimal transport plan $\gamma$ is always defined by a unique transport $L^\infty_{loc}$--map $T(= T^{\nu}_\mu)$ of the form $T^{\nu}_\mu = \nabla(\phi(x) + |x|^2/2)$, where $\phi(x) + |x|^2/2$ is a convex function on $\mathbb{R}^n$ (optimizing the dual Kantorovich functional) and vice versa if $\nu$ has a density.

A key point will, in the following, be played by the following isometry properties:

**Lemma 2.2.** (Three isometries)

- The empirical measure $\delta_N$ defines an isometric embedding $(X^{(N)}, d_{(p)}) \rightarrow \mathcal{P}_p(X)$
- The corresponding push-forward map $(\delta_N)_*$ from $\mathcal{P}(X^{(N)})$ to $\mathcal{P}(\mathcal{P}(X))$ induces an isometric embedding between the corresponding Wasserstein spaces $W_q(X^{(N)}, d_{(p)})$ and $W_q(\mathcal{P}(X))$.
- The push-forward $\pi_*$ of the quotient projection $\pi : X^N \rightarrow X^{(N)}$ induces an isometry between the subspace of symmetric measures in $(W_q(X^N), \frac{1}{N!}d_p)$ and the space $(W_q(X^{(N)}, d_{(p)})$.

Let us also recall the following classical result, which is a weak version of Sanov’s theorem [28] Theorem 6.2.10:

**Lemma 2.3.** Let $\mu_0$ be a probability measure on $X$. Then $(\delta_N)_*\mu_0^\otimes N \rightarrow \delta_{\mu_0}$ in $\mathcal{P}(\mathcal{P}(X))$ weakly as $N \rightarrow \infty$.

### 2.3. EVI gradient flows on the Wasserstein space.

Let $F$ be a lower semi-continuous function on a complete metric space $(M, d)$. In this generality there are, as explained in [1], various notions of weak gradient flows $u_t$ for $F$ (or “steepest descents”) emanating from an initial point $u_0$ in $M$. The strongest form of weak gradient flows on metric spaces discussed in [1] are defined by the property that $u_t$ satisfies the following *Evolution Variational Inequalities (EVI)* for some $\lambda \in \mathbb{R}$:

\[
\frac{1}{2} \frac{d}{dt} d^2(u_t, v) + F(u(t)) + \frac{\lambda}{2} d^2(\mu_t, \nu)^2 \leq F(v) \quad \text{a.e. } t > 0, \quad \forall v \in M : F(v) < \infty
\]
together with the initial condition $\lim_{t \to 0} u(t) = u_0$ in $(M, d)$. Then $u_t$ is uniquely determined by $u_0$, as shown in [1 Cor 4.3.3] and we shall say that $u_t$ is the gradient flow of $F$ emanating from $u_0$. As shown in [1] when $(M, d)$ is the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ such gradient flows can be constructed when $F$ has certain convexity properties that we next recall. Following [1] we first recall that a generalized geodesic $\mu$ connecting $\mu_0$ and $\mu_1$ in $\mathcal{P}_2(\mathbb{R}^d)$ with “base measure” $\nu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$ is the curve $\mu_s$ on $[0, 1]$ with values in $\mathcal{P}_2(\mathbb{R}^d)$ defined as the following family of push-forward measures:

$$\mu_s = ((1 - s)T_0 + sT_1)_* \nu$$

where $T_i$ is the optimal transport map (defined with respect to the cost function $|x - y|^2/2$) pushing forward $\nu$ to $\mu_i$ (compare Remark 2.1). More generally, to any triple $(\mu_0, \mu_1, \nu)$ of measures in $\mathcal{P}_2(\mathbb{R}^d)$ one can associate a corresponding notion of generalized geodesic $\mu_s$ in $\mathcal{P}_2(\mathbb{R}^d)$ (which may not be uniquely determined, unless $\nu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$), using transport plans instead of transport maps (see [1 Def 9.2.2]). Then $F$ is said to be $\lambda$-convex along generalized geodesics if, given any triple $(\mu_0, \mu_1, \nu)$ of measures in $\{ F < \infty \}$ there exists a corresponding generalized geodesic along which a certain convexity type inequality holds (see [1 Def 9.2.3]). Anyway, for our purposes it will be enough to consider generalized geodesics of the form appearing in formula (2.8) thanks to the following

**Proposition 2.4.** The notion of $\lambda$-convexity along generalized geodesics is preserved under $\Gamma$-convergence wrt the $\mathcal{P}_2(X)$-topology.

1. Let $F$ be a lsc function on $\mathcal{P}_2(\mathbb{R}^d)$ with the property that for any $\mu \in \{ F < \infty \}$ there exists a sequence $\mu_j$ converging to $\mu$ in $\mathcal{P}_2(\mathbb{R}^d)$ such that $F(\mu_j) \to F(\mu)$. Then $F$ is $\lambda$-convex along any generalized geodesic $\mu_s$ of the form (2.8) (i.e. with a base $\nu$ in $\mathcal{P}_{2,ac}(\mathbb{R}^d)$)

$$F(\mu_s) \leq (1 - s)F(\mu_0) + sF(\mu_1) - \frac{\lambda}{2}(1 - s)\lambda d(\mu_0, \mu_1)^2$$

2. If moreover $F$ is continuous on $\mathcal{P}(\mathbb{R}^d)$, wrt the weak topology, then the inequality above holds for $\mu_s$ iff it holds for the generalized geodesics $\mu_s^{(j)}$ with the same base $\nu$ obtained by fixing two sequences $\mu_0^{(j)}$ and $\mu_1^{(j)}$ in $\mathcal{P}_{2,ac}(\mathbb{R}^d)$ weakly converging to $\mu_0$ and $\mu_1$, respectively.

**Proof:** The first and second statement is the content of [1 Lemma 9.2.9] and [1 Prop 9.2.10], respectively and the last statement then follows by a standard compactness argument (just as in the proof of [1 Prop 9.1.3]).

We recall the notion of $\Gamma$-convergence used above, introduced by De Giorgi:

**Definition 2.5.** A one-parameter family of $F_h$ of functions on a topological space $\mathcal{P}$ is said to $\Gamma$-converges to a function $F$ on $\mathcal{P}$ if

$$\mu_h \to \mu \text{ in } \mathcal{P} \quad \Rightarrow \quad \liminf_{h \to \infty} F_h(\mu_h) \geq F(\mu)$$

and

$$\exists \mu_h \to \mu \text{ in } \mathcal{P} : \quad \lim_{h \to \infty} F_h(\mu_h) = F(\mu)$$

In the present setting $\mathcal{P} = \mathcal{P}_2(Y, d)$ and we will then say that $F_h \Gamma$-converges to $F$ strongly if, in the lower bound above holds also for every $\mu_h$ converging in the weak topology together with a uniform bound on the second moments (the terminology strong is non-standard).
We will need the following equivariant generalization of [H] Thm 11.2.1 which constructs an EVI gradient flow as a Minimizing Movement, i.e. as a limit of a time discretized Minimizing Movement scheme:

**Theorem 2.6.** Let $G$ be a compact group acting by isometries on $\mathbb{R}^d$ and $F$ a $G$-invariant lsc real-valued functional on $\mathcal{P}_2(\mathbb{R}^d)$ which is $\lambda-$convex along $G$-invariant generalized geodesics and satisfies the following coercivity property: there exist constants $\tau_*, C > 0$ and $\mu_* \in \mathcal{P}_2(\mathbb{R}^d)$ such that

$$F(\cdot) \geq -\frac{1}{\tau_*}d_2(\cdot, \mu_*)^2 - C$$

Then there is a unique solution $\mu_t$ to the EVI-gradient flow of $F$, emanating from any given $G$-invariant $\mu_0 \in \{F < \infty\}$ and $\mu_t$ remains $G$-invariant for any $t > 0$. Moreover, the flow is $\lambda-$contractive:

$$d_2(\mu_t, \nu_t) \leq e^{-\lambda t}d_2(\mu_0, \nu_0)$$

and

$$d_2(\mu_1(\tau), \mu_t) \leq C|\tau|^{1/2}F(\mu_0) \quad t \in [0, T]$$

where $\mu_1(\tau)$ is the corresponding minimizing movement with timestep $\tau$ and $C$ is a constant only depending on $\lambda, T$.

**Proof.** This is a straightforward generalization of the proof of [H] Thm 11.2.1. To see this we set $M := \mathcal{P}_2(\mathbb{R}^d)^G$ viewed as a closed subspace of the metric space $\mathcal{P}_2(\mathbb{R}^d)^G$. Restricting $d_2$ to $M$ gives a complete metric space $(M, d_2)$. By Theorem 4.0.4 in [H] we just have to verify the following two conditions on $(M, d_2)$: for any choice of $\mu_0, \mu_1$ and $\nu$ in $M$ there exists a curve $\gamma_t$ in $\mathcal{P}_2(\mathbb{R}^d)^G$ connecting $\mu_0$ and $\mu_1$ such that

- $d_2^2(\gamma_t, \mu)$ is $\lambda-$convex wrt $t$ for some $\lambda > 0$
- $F(\gamma_t)$ is $\lambda-$convex wrt $t$

As shown in [H] such a curve $\gamma_t$ exists in the space $\mathcal{P}_2(\mathbb{R}^d)$ and may be taken as a generalized geodesic connecting $\mu_0$ and $\mu_1$ with base $\nu$. Accordingly, all we have to do is to verify that if $\mu_0, \mu_1$ and $\nu$ are $G$-invariant, then the corresponding generalized geodesic may be taken to be $G$-invariant for all $t \in [0,1]$. But this follows from the dual Kantorovich formulation of the optimal transport problem [80].

To see this final point, note that by compactness $G$ can be embedded into $O(d, \mathbb{R})$, with the natural action of $O(d, \mathbb{R})$ around some fixed point. But for $A \in O(d, \mathbb{R})$, we have that $(\phi \circ A)^* = \phi^* \circ A$, where the star denotes Legendre transformation. □

**Remark 2.7.** We recall that it follows from the results in [H] that the EVI-gradient flow above has a number of further properties. For example, the flow defines a semigroup and is $1/2-$Hölder continuous as a map from any fixed time interval $[0, T]$ into $\mathcal{P}_2(\mathbb{R}^d)$ and Lipschitz continuous on any fixed bounded open time-interval. We also note that the corresponding curve $[\mu_t] \in W_2(\mathbb{R}^d/G)$ is the unique EVI-gradient flow of $F$ viewed as a function on $W_2(\mathbb{R}^d/G)$, as follows immediately from the the natural isometry between $W_2(\mathbb{R}^d/G)$ and $W_2(\mathbb{R}^d/G)$.

We briefly recall the construction of De Giorgi’s minimizing movement scheme in a general metric space $(M, d)$, which can be seen as a variational formulation of the (back-ward) Euler scheme [H Chapter 2]). Consider the fixed time interval $[0, T]$ and fix a (small) positive number $\tau$ (the “time step”). In order to define the “discrete flow” $u^\tau_j$ corresponding to the sequence of discrete times $t_j := j\tau$, where
$t_j \leq T$ with initial data $u_0$ one proceeds by iteration: given $u_j \in M$ the next step $u_{j+1}$ is obtained by minimizing the following functional on $(M,d) := W_2(\mathbb{R}^d)$:

$$u \mapsto J_{j+1}(u) := \frac{1}{2\tau}d(u,u_j)^2 + F(u)$$

Finally, one defines $u^\tau(t)$ for any $t \in [0,T]$ by setting $u^\tau(t_j) = u_j^\tau$ and demanding that $u^\tau(t)$ be constant on $[t_j,t_{j+1}]$ and right continuous (we are using a slightly different notation than the one in [1 Chapter 2]).

The following result goes back to McCann [56] (see also [1] for various elaborations):

**Lemma 2.8.** The following functionals are lsc and $\lambda-$convex along any generalized geodesics in $\mathcal{P}_2(\mathbb{R}^d)$:

- The “potential energy” functional $\mathcal{V}(\mu) := \int V\mu$, defined by a given lsc $\lambda-$convex and lsc function $V$ on $\mathbb{R}^d$ (and the converse also holds)
- The functional $\mu \mapsto \int V_N H^{\otimes N}$ defined by a given $\lambda-$convex function $V_N$ on $\mathbb{R}^{dN}$ and in particular the “interaction energy” functional $\mathcal{W}(\mu) := \int W(x-y)\mu(x) \otimes \mu(x)$ defined by a given lsc $\lambda-$convex function $W$ on $\mathbb{R}^d$.
- The Boltzmann entropy $H(\mu)$ (relative to $dx$) is lsc and convex along any generalized geodesics.

In particular, for any $\lambda-$convex function $V$ on $\mathbb{R}^d$ the corresponding free energy functional $F_\beta^V$ (formula 2.5) is $\lambda-$convex along generalized geodesics, if $\beta \in [0,\infty]$.

**Remark 2.9.** According to the Otto calculus [67] the EVI gradient flow on $\mathcal{P}_2(\mathbb{R}^d)$ of a sufficiently regular functional $F$ satisfies, if $\mu_t$ has has a smooth positive density $\rho_t$, the evolution equation

$$(2.11) \quad \frac{\partial \rho_t(x)}{\partial t} = \nabla_x \cdot (\rho v_t(x)), \quad v_t(x) = \nabla_x \frac{\partial F(\rho)}{\partial \rho} |_{\rho = \rho_t}$$

As shown in [1] these equations still hold in the weak sense of distributions under appropriate assumptions on $E$, for the EVI gradient flow solution $\mu_t$ (see Section 5). In particular, when $F$ is of the free energy form $F = E(\mu) + H(\mu)/\beta$

$$(2.12) \quad \frac{\partial \rho_t(x)}{\partial t} = \frac{1}{\beta} \Delta_x \rho_t(x) + \nabla_x \cdot (\rho v_t(x)), \quad v_t(x) = \nabla_x \frac{\partial E(\rho)}{\partial \rho} |_{\rho = \rho_t}$$

coincides with the McKean-Vlasov equation [13] when $E = \mathcal{W}$ for $W$ smooth. Moreover, when $E = \mathcal{V}$ the corresponding evolution equation is the linear Fokker-Planck equation associated to the potential $V$, as first shown when $V$ is smooth in the seminal work [17].

### 2.4. The defining laws on $X^N$ and the mean (free) energy $\mathcal{E}^{(N)}$ (and $\mathcal{F}_N$).

Let $X$ be the Euclidean space $\mathbb{R}^d$ and $V$ a smooth (and say coercive) function on $\mathbb{R}^D$. The SDE

$$(2.13) \quad dx = -\nabla V dt + dB/\beta$$

where $x(0)$ is a vector of iid variables with law $\mu_0$ (as before we allow the ODE case $\beta = \infty$) defines, for any fixed $T$, a probability measure $\eta_T$ on the space of all maps $[0,T] \to X$ (see for example [79] and reference therein). For $T$ fixed we can
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thus view $x(t)$ as an $X$-valued random variable on the latter probability space. Its law gives a curve of probability measures on $X$ of the form $\mu_t = \rho_t \, dx$, where the density $\rho_t$ satisfies the corresponding forward Kolmogorov equation:

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{\beta} \Delta \rho_t + \nabla \cdot (\rho_t \nabla V),$$

(also called the linear Fokker-Planck equation). Anyway, for our purposes we may as well forget about the SDE 2.13 and take the forward Kolmogorov equation 2.14 on $X$ as our the starting point. As recalled above (Remark 2.9) the latter evolution equation can be interpreted as the gradient-flow on the Wasserstein space $W_2((\mathbb{R}^D)^N)$, of the corresponding free energy functional.

In our setting we will take $V := E^{(N)}(x_1, ..., x_N)$ for a given symmetric function on $X := (\mathbb{R}^D)^N$. Following standard terminology in statistical mechanics we will call the corresponding (scaled) linear functional $E^{(N)}$ on $\mathcal{P}((\mathbb{R}^d)^N)$, defined by

$$E^{(N)}(\mu_N) := \frac{1}{N} \int_{(\mathbb{R}^d)^N} E^{(N)} \mu_N,$$

the mean energy. Similarly, the corresponding mean free energy $F^{(N)}$, at inverse temperature $\beta_N$, is defined by

$$F^{(N)}(\mu_N) := \frac{1}{N} \int_{(\mathbb{R}^d)^N} E^{(N)} \mu_N + \frac{1}{\beta_N N} H(\mu_N) := E^{(N)}(\mu_N) + \frac{1}{\beta_N} H^{(N)}(\mu_N)$$

where the scaled Boltzmann entropy $H^{(N)}(\mu_N)$ on $\mathcal{P}((\mathbb{R}^d)^N)$ is called the mean entropy.

More generally, we will allow $E^{(N)}$ to be singular, but we will make assumptions (such as $\lambda$–convexity modulo $S^N$) ensuring, by Theorem 2.6, that the Wasserstein gradient flow of $F_N$ is well-defined, giving a curve of probability measures $\mu^{(N)}(t)$ on $(\mathbb{R}^D)^N$. In the deterministic setting ($\beta = \infty$) it can be shown (see Section 4.3) that this approach is consistent with the classical notion of a strong solution, as long as such a solution exists (for example, when particles do not collide for $t > 0$). In the stochastic setting ($\beta < \infty$) we will not attempt to give any definition of a strong solution to the stochastic equation 2.13 since such solutions do not play any role in our proofs. On the other hand the present definition of $\mu^{(N)}(t)$ is stable under monotone regularizations of $E^{(N)}$ (preserving $\lambda$) and thus coincides with any other definition with similar stability properties. We refer to [2, 20, 21] for the notion of strong solutions under convexity assumptions as above and [77] for general results about strong solutions of SDEs with locally bounded drifts (see also Remark 4.13).

3. A GENERAL CONVERGENCE RESULT

In section we will give a more general formulation of the propagation of chaos result in [8, Theorem 1.1], by exploiting the $S^N$–symmetry (which will be crucial in the applications to strongly singular pair potentials). The present formulation also has the virtue of also applying in the purely deterministic setting. The proof could be given essentially by repeating the argument in [8]. But here we give a slightly different proof which can be seen as an analogue in our setting of the stability result of gradient flows on the Wasserstein space of a Hilbert space $Y$ in [4, 21] (recalled in Theorem 7.1 below). The new difficulties that arise in our setting is that
• The Gamma convergence of the corresponding functionals only holds in a restricted “relative” sense
• The space $Y$ is $\mathcal{P}_2(\mathbb{R}^D)$, which is not a Hilbert space (in particular, there are no general existence results for EVI gradient flows on $\mathcal{P}_2(Y)$, nor error estimates and convergence results for general minimizing movements).

These difficulties will be handled by exploiting the fact that the the limiting functional $F$ is linear wrt the ordinary affine structure on $\mathcal{P}_2(\mathbb{R}^D)$ and using the isometry properties of the embeddings in Lemma 2.2.

3.1. The assumptions on $E^{(N)}$. Set $X = \mathbb{R}^n$ and denote by $d$ the Euclidean distance function on $X$. In the following $E^{(N)}$ will denote a lsc symmetric, i.e. $S_N-$invariant, sequence of functions in $L^1_{\text{loc}}(X^N)$ and we will make the following assumptions, where $\mathcal{E}^{(N)}$ denotes the corresponding mean energies (formula 2.13):

1. (“Convergence of the mean energies”) There exists a lsc functional $E(\mu)$ on $\mathcal{P}_2(X)$ with the property that $\{E < \infty\}$ is dense in $\mathcal{P}_2(X)$, $\{E < \infty\} \cap \{H < \infty\} \neq \emptyset$, and such that for any sequence of symmetric probability measures $\mu^{(N)}(x)$ on $X$ satisfying $\Gamma_N := (\delta_N)_* \mu^{(N)} \rightarrow \Gamma$ weakly in $\mathcal{P}(\mathcal{P}(X))$ and with a uniform bound on the second moments we have

$$\int_{\mathcal{P}(X)} E(\mu) \Gamma(\mu) \leq \liminf_{N \rightarrow \infty} \mathcal{E}^{(N)}(\mu^{(N)})$$

and for any $\mu \in \mathcal{P}_2(X)$ we have that

$$\limsup_{N \rightarrow \infty} \mathcal{E}^{(N)}(\mu^{\otimes N}) \leq E(\mu)$$

2. (“Convexity of the mean energies”): The mean energy functional $\mathcal{E}^{(N)}$ on $\mathcal{P}(X^N)^{SN}$ is $\lambda-$convex along generalized geodesics in $\mathcal{P}(X^N)^{SN}$.

3. (“Coercivity”) There exists a constant $C$ such that

$$E^{(N)}(x_1, \ldots, x_N) \geq -C \frac{|x_1|^2 + \cdots + |x_1|^2}{N} - C$$

(or equivalently, that $\mathcal{E}^{(N)} \geq -C((\delta_N)_* d^2(\cdot, \Gamma_0) - C$ for a fixed element $\Gamma_0$ in $\mathcal{P}_2(\mathcal{P}_2(X))$)

Lemma 3.1. The functional $E(\mu)$ is $\lambda-$convex along generalized geodesics in $\mathcal{P}_2(X)$ and coercive.

Proof. We first observe that taking $\mu^{(N)} = \mu^{\otimes N}$ and using Sanov’s theorem (or Lemma 2.3) gives

$$\lim_{N \rightarrow \infty} \mathcal{E}^{(N)}(\mu^{\otimes N}) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{X^N} E^{(N)}(\mu^{\otimes N}) = E(\mu)$$

Fix $\nu, \mu_0, \mu_1 \in \mathcal{P}_2(X)$, and let $\mu_t$ be the generalized geodesic with base measure $\nu$. But then $\mu_t^{\otimes N}$ is the (symmetric) generalized geodesic with base $\nu^{\otimes N}$ connecting $\mu_0^{\otimes N}, \mu_1^{\otimes N}$, and the convexity statement follows from the convexity assumption on $\mathcal{E}^{(N)}$. The coercivity of $E$ follows from the coercivity assumption on $\mathcal{E}^{(N)}$ by letting $\Gamma_0 = \delta_{\mu_0}$ and writing

$$\mathcal{E}^{(N)}(\mu^{\otimes N}) \geq -C d_2(\delta_{N}, \mu^{\otimes N}, \delta_{\mu})^2 - C \rightarrow -C d_2(\mu, \mu_*)^2 - C,$$

By Sanov’s theorem and the isometry properties of Lemma 2.2. \qed
3.2. Formulation of the general convergence results.

**Theorem 3.2.** Let $E^{(N)}$ be a sequence of functions on $(\mathbb{R}^d)^N$ satisfying the assumptions above and let $\mu^{(N)}$ be a sequence of symmetric probability measures on $(\mathbb{R}^d)^N$ such that

$$\Gamma_N := (\delta_N)^* \mu^{(N)} \to \delta_{\mu_0}$$

in $W_2(\mathcal{P}_2(\mathbb{R}^d))$ as $N \to \infty$, where $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Further assume that $\mu^{(N)} \in \{F_{\beta}^{(N)} < \infty\}$. Then the EVI gradient flow solution $\mu^{(N)}(t)$ of the corresponding forward Kolmogorov equation [2.14] at inverse temperature $\beta_N$ on $(\mathbb{R}^d)^N$ with initial data $\mu^{(N)}(0) = \mu^{(N)}$ satisfies

$$\Gamma(t) := (\delta_N)^* \mu^{(N)}(t) \to \delta_{\mu_t},$$

in $W_2(\mathcal{P}_2(\mathbb{R}^d))$ as $N \to \infty$, where $\mu_t$ is the EVI gradient flow on $\mathcal{P}_2(\mathbb{R}^d)$ of the corresponding free energy type functional $F_{\beta}(\mu)$.

3.3. The proof of Theorem 3.2. In the proof we will need a modified form of strong Gamma-convergence that we call **Gamma convergence relative to the subset $\mathcal{D} \subset \mathcal{P}$** defined by only requiring that the equality in the second condition in Definition [2.14] holds for $\mu \in \mathcal{D}$ (and similarly for strong Gamma-convergence). We will also say that a functional $\mathcal{F}$ on $\mathcal{P}_2(Y,d)$ admits minimizing movements relative to $\mathcal{D} \subset \mathcal{P}_2(Y,d)$ if for any given $\Gamma_0 \in \mathcal{D}$ there exists a continuous curve $\Gamma(t) \in \mathcal{D}$ emanating from $\Gamma_0$, which can be realized as the limit, as $t \to 0$, of time discrete minimizing movements $\Gamma_\tau(t) \in \mathcal{D}$, where $\Gamma_\tau(t)$ is assumed to be uniquely determined by $\Gamma_0$.

Now set $Y := \mathcal{P}_2(\mathbb{R}^D)$. By embedding $\mathcal{P}_2((\mathbb{R}^D)^N/S_N)$ isometrically into $\mathcal{P}_2(Y)$, using the push-ward map $(\delta_N)^*$ we can and identify the mean free energies $\mathcal{F}(\mathcal{N})$ with functionals on $\mathcal{P}_2(Y)$, extended by $\infty$ to all of $\mathcal{P}_2(Y)$. We will take $\mathcal{D}$ to be subset of all Dirac measures $\Gamma$ in $\mathcal{P}_2(Y)$, i.e. $\Gamma = \delta_\mu$ for some $\mu \in Y$.

**Lemma 3.3.** If Assumption 1 in Section 2.1 holds, then the mean free energies $\mathcal{F}(\mathcal{N})$ strongly Gamma-converges to the lsc linear functional $\mathcal{F}(\Gamma)$ on $\mathcal{P}_2(Y)$, relative to $\mathcal{D}$.

**Proof.** The lower bound follows directly from Assumption 1 together with the fact that the mean entropy functionals satisfy the lower bound in the Gamma convergence (by subadditivity [69], see also Theorem 5.5 in [40] for generalizations). To prove the upper bound, we first consider the case $\beta < \infty$ and fix an element $\Gamma$ of the form $\delta_\mu$. We may then take the approximating sequence to be for the form $(\delta_N)_t \mu_t \in \mathcal{D}$. Then the required convergence follows from Assumption 1 together with the basic property $\mathcal{H}(\mu_t) = H(\mu_t)$. When $\beta = \infty$, since it may happen that $H(\mu) < \infty$, we must first regularize the measure $\mu$. Fix an arbitrary measure $\mu_0$ such that $\mathcal{F}(\mu) < \infty$, $H(\mu) < \infty$, let $\mu_1 = \mu$, and for $t \in [0,1]$ let $\mu_t$ be the displacement interpolation. Then $H(\mu_t) < \infty$ for all $t \in [0,1]$ (see Lemma 2.15), and by the above argument it then holds that $\mathcal{F}(\mu_t) \to \mathcal{F}(\mu_1)$. By $\lambda$-convexity and lower semicontinuity, $t \to \mathcal{F}(\mu_t)$ is continuous, and thus $\mathcal{F}(\mu_t) \to \mathcal{F}(\mu)$ as $t \to 1$. By a diagonal argument we can then find a sequence $t_N$ such that $\mathcal{F}(\mu_t^N) \to \mathcal{F}(\mu)$, completing the proof. \hfill $\Box$

**Lemma 3.4.** Let $F$ be a lsc functional on a metric space $(Y,d)$ with the property that at for any given element $y_0$ in $(Y,d)$ there is an EVI gradient flow (with
parameter $\lambda$) emanating from $y_0$ and which can be realized as a limit of discrete minimizing movements. Denote by $F$ the lsc linear functional on $\mathcal{P}_2(Y,d)$ associated to $\Gamma$. Then, for any given element $\Gamma_0$ of the form $\Gamma_0 = \delta_{y_0}$ in $\mathcal{P}_2(Y,d)$ there is an EVI gradient flow (with parameter $\lambda$) emanating from $\Gamma_0$, namely $\Gamma_0 := \delta_{y_t}$, where $y_t$ is the EVI gradient flow of $F$ emanating from $y_0$. In particular, $F$ admits minimizing movements relative to $\mathcal{D}$.

Proof. This is an abstract version of the third point in Lemma 4.9 below and is proved in exactly the same way using that $F$ is linear. \qed

Now we discretize time, with mesh $\tau$, and consider the minimizing movements $\Gamma^*_N(t)$ and $\Gamma^*(t)$ of $\mathcal{F}^{(N)}$ and $\mathcal{F}$, respectively. By Lemma 5.8 we have that, for any fixed $\Gamma \in \mathcal{P}_2(Y)$

\[
\mathcal{J}^{(N)}(\cdot) := \frac{1}{2\tau}d(\cdot, \Gamma)^2 + \mathcal{F}^{(N)}(\cdot) \rightarrow \mathcal{J}(\cdot) := \frac{1}{2\tau}d(\cdot, \Gamma)^2 + \mathcal{F}
\]

in the sense of relative strong Gamma-convergence, as $N \to \infty$. But then it follows from basic properties of Gamma-convergence, using that $\mathcal{J}^{(N)}(\cdot)$ is uniformly coercive and compactness properties in the Wasserstein space, that the corresponding minimizers converge. Hence, starting with $\Gamma_0 := \delta_{\mu_0}$ it follows by induction that

\[
\lim_{N \to \infty} \Gamma^*_N(t) = \Gamma^*(t),
\]

where $\Gamma^*(t) \in \mathcal{D} \subset \mathcal{P}_2(Y)$ at any fixed time $t$.

To conclude the proof of Theorem 5.2 we just have to make sure that the error terms appearing when comparing $\Gamma^*_N(t)$ with $\Gamma_N(t)$, when $\tau \to 0$ can be uniformly controlled as $N \to \infty$.

Step 1: First assume that $\mu_0 \in \{ F_{\beta} < \infty \}$ and $\limsup F_{\beta}^{(N)}(\mu^{(N)}) < \infty$. By Theorem 2.6 the gradient flow $\mu_t$ of $F$ emanating from a given $\mu_0$ exists and is uniquely determined. We let $\Gamma_t := \delta_{\mu_t}$ be the corresponding flow on $\mathcal{P}_2(\mathcal{P}_2(X))$. Consider the fixed time interval $[0,T]$ and fix a small time step $\tau > 0$. Denote by $\mu^t(\tau)$ the discretized minimizing movement of $F(\mu)$ with time step $\tau$ and set $\Gamma^*_t := \delta_{\mu^t(\tau)}$. For any fixed $t \in [0,T]$ we then have, by the triangle inequality,

\[
d(\Gamma_N(t), \Gamma^*_N(t)) \leq d(\Gamma_N(t), \Gamma^*_N(t)) + d(\Gamma(t), \Gamma^*(t)) + d(\Gamma^*_N(t), \Gamma^*(t))
\]

By the isometry property in Lemma 2.2 and the assumed convexity properties we have, by \[\text{Theorem 4.0.4.0.7, p.79}], that $d(\Gamma_N(t), \Gamma^*_N(t)) \leq C\tau^{1/2}$ (uniformly in $N$) and $d(\Gamma(t), \Gamma^*(t)) \leq C\tau^{1/2}$. Moreover, by \[\text{Lemma 5.3, lim}_{N \to \infty} d(\Gamma^*_N(t), \Gamma^*(t)) = 0\] for any fixed $\tau$. Hence, letting first $N \to \infty$ and then $\tau \to 0$ gives $\lim_{N \to \infty} d(\Gamma_N(t), \Gamma(t)) = 0$, which concludes the proof.

Step 2: The case when $F(\mu_0) < \infty$

Set $\nu_{0}^{(N)} := \mu_0^{\otimes N}$ and denote by $\nu_t^{(N)}$ the EVI gradient flow of $\mathcal{F}^{(N)}$ emanating from $\nu_{0}^{(N)}$. Let $\epsilon_N(t) := N^{-1}d_2(\nu_t^{(N)}, \mu_t^{(N)})$. By Lemma 2.2 and the isometry properties in Lemma 2.2 $\epsilon_N(0) \to 0$. Hence, by the $\lambda$-contractivity in Theorem 2.6 $\epsilon_N(t) \leq e^{-\lambda t}\epsilon_N(0) \to 0$ for any fixed positive $t > 0$. But then the desired convergence follows from the previous step, using the triangle inequality.

Step 3: The case of a general $\mu_0 \in \mathcal{P}_2(\mathbb{R}^D)$

By assumption there exists a sequence $\mu_{j,0}$ in $\mathcal{P}_2(\mathbb{R}^D)$ such that $d(\mu_{j,0}, \mu_0) \leq 1/j$ and $F(\mu_{j,0}) < \infty$. We define $\nu_j^{(N)}(t)$ as before, up to replacing $\mu_0 := \nu_{\infty}$ with $\nu_j$.
and then set $\Gamma_{j,N}(t) := (\delta_N)_j \nu_j^{(N)}(t)$. By the triangle inequality we have, for any fixed $j$,

$$d(\Gamma_N(t), \Gamma(t)) \leq d(\Gamma_{j,N}(t), \Gamma_j(t)) + d(\Gamma(t), \Gamma_j(t)) + d(\Gamma_N(t), \Gamma_{j,N}(t)),$$

where the first term tends to zero as $N \to \infty$ by the previous step. By construction the second term satisfies $d(\Gamma(0), \Gamma_j(0)) \leq 1/j$ and hence, by $\lambda$-contractivity, $d(\Gamma(t), \Gamma_j(t)) \leq e^{-\lambda t}/j$. Similarly, using the triangle inequality again, the third term satisfies

$$d(\Gamma_N(0), \Gamma_{j,N}(0)) \leq N^{-1}d_2(\mu^{(N)}(0), \mu_0^{(N)}_\otimes) + 1/j := \epsilon_N + 1/j$$

where $\epsilon_N \to 0$ as $N \to \infty$ (as in Step 1). Hence, by $\lambda$-contractivity, $d(\Gamma_N(t), \Gamma_{j,N}(t)) \leq e^{-\lambda t}(\epsilon_N + 1/j)$. Accordingly, letting first $N \to \infty$ and then $j \to \infty$ concludes the proof.

3.4. General structure of the applications. For applications to the purely deterministic Setting 1 (described in the introduction of the paper) one simply take $\mu^{(N)}$ to be the normalized $S_N$-orbit in $X^N$ of the Dirac measure supported at $(x_1(0), ..., x_N(0))$. In the Setting 2 and 3 in the introduction of the paper one takes $\mu^{(N)} = \mu_0^{(N)}$. Before developing these applications in some particular settings in Sections 4, 6 we make some remarks about more general situations where the assumptions in Section 3.1 hold. First of all, as shown in [8], the assumptions hold when

$$\frac{1}{N} E^{(N)}(x_1, x_2, ..., x_N) = E(\delta_N) + o(1),$$

when $E$ is uniformly Lipschitz continuous and $\lambda$-convex and the error term tends to zero, as $N \to \infty$ (for $x_i$ uniformly bounded). But the main point in the present paper is that the assumptions are also satisfied in some naturally occurring very singular situations. For example, Assumption 1 is satisfied if one starts with a “polynomial” functional $E(\mu)$ on $\mathcal{P}(X)$, i.e.

$$E(\mu) = \sum_{m=1}^M \int_{X^m} w_m \mu \otimes \mu$$

where $w_m$ are assumed upper semi-continuous functions $X^m \to [-\infty, \infty]$ in $L^1_{loc}(X^m)$, which are smooth (or continuous) on the open subset of configurations in $X^m$ where no two points coincide. Then one can then define a “renormalized” interaction $N$-particle interaction $E^{(N)}(x_1, ..., x_N)$ by setting

$$E^{(N)}(x_1, ..., x_N) := \frac{1}{N^{(m-1)}} \sum_{m=1}^M \sum_I w_m(x_{i_1}, ..., x_{i_m}),$$

where the inner sum runs over all multiindices $I = (i_1, ..., i_m)$ of length $m$ and with the property that no two indices of $I$ coincide. Then $E^{(N)}(x_1, ..., x_N)$ is finite for generic configurations, or more precisely on the complement of the fixed point locus of the $S_N$-action on $X^N$ (but the equality (3.1) does not hold as the right hand side is identically $\infty$ if some $w_m$ takes the value $\infty$). Moreover, it can be shown that the Assumption 1 is valid (as discussed below). However, the main issue is the convexity of the corresponding mean free energy, which in particular implies that $E^{(N)}(x_1, ..., x_N)$ must be $\lambda$-convex on the interior of any fundamental domain $\Lambda$ for the $S_N$-action on $X^N$. As it turns out the latter condition is, in fact, also sufficient.
for the convexity assumption 2 two hold when $D = 1$ (as is shown precisely as in the proof of Proposition 4.6 below). This is the reason that we will mainly consider the one dimensional setting in Section 4.

4. Applications to singular pair interactions in 1D

In the following it will be convenient to use the following (non-standard) terminology: a continuous function $\psi(x)$ on a convex domain of $\mathbb{R}^D$ is quasi-convex if it is $\lambda-$convex, i.e. it can be written as $\psi(x) = \phi(x) + \lambda|x|^2/2$ for some convex function $\phi$ and if, for $|x|$ sufficiently large, $\psi(x) = \phi(x) + o(|x|^2)$ for some (possibly different) convex function $\phi$.

4.1. Setup. Let $w(s)$ be a quasi-convex real-valued function on $]0, \infty[$ such that there exist positive constants $A$ satisfying

$$\lim_{s \to 0} \inf w(s) \geq -A.$$ 

Extend $w$ to a lsc function $w : \mathbb{R} \to ]-\infty, \infty]$ by demanding that $w(-s) = w(s)$ for $s \neq 0$ and $w(0) := \lim_{s \to 0} w(s)$. We define the corresponding pair interaction function by

$$W(x, y) := w(x-y)(= w(|x-y|))$$

which is called repulsive (attractive) if $w(s)$ is decreasing (increasing) on $]0, \infty[$. Given a quasi-convex function $V(x)$ we define the corresponding $N-$point interaction energy by

$$(4.1) \quad E_{WV}^{(N)}(x_1, x_2, \ldots, x_N) := \frac{1}{N} - \frac{1}{2} \sum_{i \neq j} w(x_i - x_j) + V(x_i)$$

Remark 4.1. Note that even if $\lambda > 0$, the function $W(x, y)$ is at best only 0-convex due to translation invariance. Since a $\lambda$-convex function for $\lambda > 0$ is also 0-convex, we will to simplify notation in the sequel implicitly that assume $\lambda \leq 0$.

We will consider the general setting of an $N-$dependent inverse temperature $\beta_N$ such that

$$\lim_{N \to \infty} \beta_N := \beta \in ]0, \infty[.$$ 

Then the corresponding SDEs can be formally written as

$$(4.2) \quad dx_i(t) = -\frac{1}{(N-1)} \sum_{j \neq i} (\nabla w)(x_i - x_j) dt - (\nabla V)(x_i) dt + \sqrt{\frac{2}{\beta_N}} dB_i(t),$$

(see Section 2.4).

Lemma 4.2. There exists a sequence of continuous functions $w_R(t)$ on $]0, \infty[$ increasing to $w_R$ such that $w_R$ is quasi-convex, with a $\lambda$ independent of $R$ and such that $w_R = a_R t + o(t^2)$ for some $a_R \in \mathbb{R}$ when $|x| \geq R$.

Proof. The only issue is the $\lambda-$convexity close to $t = 0$, for $t \geq 0$ and it may be obtained as follows. First assume that $\lambda = 0$ and fix and number $\epsilon > 0$. Let $w^{(\epsilon)}$ be the convex function on $[0, \infty]$ coinciding with $w$ on the complement of $[0, \epsilon]$ and on $[\epsilon, \infty]$ it coincides with the affine function defined by the left tangent line of $w$ at $t = \epsilon$. By convexity $w^{(\epsilon)}$ increases to $w$ as $\epsilon \to 0$. In general, if $w$ is $\lambda-$convex for some $\lambda \leq 0$ we can decompose $w(t) := \phi(t) - \lambda t^2/2$ where the first term is convex. Replacing $\phi(t)$ by $\phi^{(\epsilon)}(t)$ as above and defining $\phi^{(\epsilon)} := w^{(\epsilon)}(t) - \lambda t^2/2$ then gives
a sequence of continuous $\lambda$-convex functions on $[0, \infty]$ increasing to $w$. Similarly, by decomposing $w(t) = \phi(t) + o(t^2)$ for $|t| \geq R$ for a (possibly different) convex function $\phi$ we can replace $\phi$ on $[R, \infty]$ with the corresponding affine function which is tangent to $\phi$ as $t$ increases to $R$.

Remark 4.3. In the case when $w$ is decreasing on $[0, \infty]$ we could simply take $w_R(t) := w(t + 1/R)$ above.

We fix a sequence of quasi-convex continuous functions $w_R$ and $V_R$ which are bounded from above and increase to $w$ and $V$, respectively, where $R$ will be referred to as the “truncation parameter” (such sequences exist by the assumption on lower semi-continuity).

Example 4.4. (power-laws) Our setup applies in particular to the repulsive power-laws

$$w(|x|) \sim |x|^s \quad s \in [-1,0],$$

whose role in the case $s = 0$ it played by the repulsive logarithmic potential $w(|x|) = -\log|x|$ and for $s > 0$ by

$$w(x) \sim -|x|^s, \quad s \in [0,1]$$

The results also apply in the following cases of attractive potentials (see Section 6.2 for $D > 1$):

$$w(x) \sim |x|^s \quad s \in [1,\infty]$$

Similarly, the assumptions are satisfied by linear combinations of interactions whose asymptotics as $|x| \to 0$ and $|x| \to \infty$ are comparable to the power-laws as above. In particular, this is the case for the interactions used in applications to swarming and flocking models, which are usually taken to be repulsive at a short distances and attractive at large distances (see [24] and references therein). For example, this is the case for the Morse potential used in swarming models: $w(|x|) = C_R e^{-|x|/l_R} - C_A e^{-|x|/l_A}$ which is clearly $\lambda$-convex on $[0, \infty]$ for some (possibly negative) $\lambda$. When $C_R/C_A > 1$ and $l_R/l_A < 1$ it is repulsive/attractive at small/large distances.

4.2. Propagation of chaos in the large $N$-limit and convexity.

Proposition 4.5. The functional

$$(4.3) \quad E_{W,V}(\mu) := \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} W \mu \otimes \mu + \int V \mu := W(\mu) + V(\mu) \in [\infty, \infty]$$

is well-defined, and lsc on $\mathcal{P}_2(\mathbb{R})$ and satisfies the coercivity property $\exists \Gamma$. Moreover,

$$(4.4) \quad \frac{1}{N} \int E_{W,V}^{(N)}(x_1, x_2, ..., x_N) \mu^{\otimes N} = E_{W,V}(\mu),$$

if $E_{W,V}(\mu) < \infty$ and in general

$$(4.5) \quad E_{W,V}(\mu) = \lim_{R \to \infty} E_{W_R,V_R}(\mu)$$
where $E_{W_R, V_\mu}(\mu)$ is continuous along any sequence $\mu_j$ converging weakly in $\mathcal{P}(\mathbb{R})$ with a uniform bound on the second moments. In particular, taking $\mu = \delta_N(x_1, \ldots, x_N)$ we have

$$E_{W_R, V_\mu}(\delta_N(x_1, \ldots, x_N)) + O\left(\frac{C_R}{N}\right) = \frac{1}{N} E_{W_R, V_\mu}^{(N)}(x_1, x_2, \ldots, x_N)$$

Proof. To simplify the notation we assume that $V = 0$, but the general case is similar. First note that the fact that $E_W(\mu)$ is well-defined is trivial in case $\mu$ has compact support since then $W \geq -C$ on the support of $\mu$. Formula (4.4) then follows immediately from the definition, using the Fubini-Tonelli theorem to interchange the order of integration. In the general case we note that fixing $\delta > 0$ and setting $U_\delta := \{(x, y) : |x - y| > \delta\}$ gives $\int_{U_\delta} W \mu \otimes \mu \geq -C_\delta \int |x - y|^2 \mu \otimes \mu \geq 2C_\delta \int \{|x|^2 + |y|^2\} \mu \otimes \mu := C_\delta' > \infty$ since $\mu \in \mathcal{P}_2(\mathbb{R})$. Hence, $E_W(\mu) := \int_{\mathbb{R} \times \mathbb{R}} W \mu \otimes \mu := \int_{U_\delta} W \mu \otimes \mu$ is well-defined, since $W \geq A_\delta$ on $U_\delta$. The convergence (4.5) as $R \to \infty$ then follows from the monotone and dominated convergence theorems. We note that for any fixed $R$ the functional $E_{W_R}(\mu)$ is continuous along any sequence $\mu_j$ converging weakly in $\mathcal{P}(\mathbb{R})$ with a uniform bound on the second moments, as follows from the fact that $W_R$ is continuous and bounded from above on any compact subset $[-k, -k]$ of $\mathbb{R}$ together with the simple tail estimate

$$\int_{|x| \geq k} W_R \mu_j \leq \sup_{|x| \geq k} \frac{W_R(x)}{|x|^2} \int |x|^2 \mu_j,$$

which by the quasi-convexity assumption tends to zero, as $k \to \infty$, uniformly in $j$.

Finally, since $E_W(\mu)$ is an increasing sequence of continuous functionals $E_{W_R}(\mu)$ it follows that $E_W(\mu)$ is lower semi-continuous on $\mathcal{P}_2(\mathbb{R})$. To prove the last statement we note that, by definition, the error term in question comes from the missing diagonal terms in the definition of $E_{W_R}^{(N)}(x_1, x_2, \ldots, x_N)$ corresponding to $i = j$ i.e. from

$$\frac{1}{N} \frac{1}{N-1} \sum_{i=1}^N w_R(x_i - x_i) = \frac{1}{N} \frac{N}{N-1} w_R(0) = O\left(\frac{C_R}{N}\right),$$

which concludes the proof. \qed

**Proposition 4.6.** (convexity). The mean energy functional

$$E_{W, V}(\mu)^N := \frac{1}{N} \int_{\mathbb{R}^N} E_{W, V}^{(N)} \mu^N,$$

restricted to the subspace of symmetric probability measures $\mathcal{P}_2(\mathbb{R}^N)^{S_N}$ in $\mathcal{P}_2(\mathbb{R}^N)$, is $\min(\lambda, 0)$-convex along generalized geodesics with symmetric base $\nu_N$. In particular, the functional $E_{W, V}(\mu)$ is $lsc$ and $\min(\lambda, 0)$-convex along generalized geodesics in $\mathcal{P}_2(\mathbb{R})$ and satisfies the coercivity condition (2.10).

**Proof.** We first claim that it is enough to prove the $\lambda-$convexity for generalized geodesics in $\mathcal{P}_2(\mathbb{R}^N)^{S_N}$ such that the base $\nu_N$ is also in $\mathcal{P}_2(\mathbb{R}^N)^{S_N}$ for $t \geq 0$. Next, we observe that when $E_{W, V}^{(N)}$ continuous and bounded the corresponding mean energy functional on $\mathcal{P}(\mathbb{R}^N)$ is continuous wrt the weak topology. Now, given a generalized geodesic $\mu_s$ in $\mathcal{P}_2(\mathbb{R}^N)$ with a base $\nu$
in $\mathcal{P}_2(\mathbb{R}^N)_{abs}$ we can approximate the end points weakly by measures with finite entropy (by a simple convolution and truncation argument). By the convexity of the entropy, the corresponding generalized geodesics $\mu^{(j)}$ have finite entropy for any fixed $s \in [0,1]$ and in particularly $\mu^{(j)}$ is a curve in $\mathcal{P}_2(\mathbb{R}^N)_{abs}$. By Prop [2,4] this proves the claim above.

We will write $x := (x_1, \ldots, x_N)$ etc. Let $\mu_0^{(N)}$, $\mu_1^{(N)}$ and $\nu^{(N)}$ be three given symmetric measures in $\mathcal{P}_{2,ac}(\mathbb{R}^N)$ and denote by $T_0$ and $T_1$ the optimal maps pushing forward $\nu^{(N)}$ to $\mu_0^{(N)}$ and $\mu_1^{(N)}$, respectively. Let $T_t := (1-t)T_0 + tT_1$ so that $\mu_t := T_t\nu^{(N)}$ is the corresponding generalized geodesic. The key point of the proof is the following

Claim: (a) $T_t$ commutes with the $S_N$–action and (b) $T_t$ preserves order, i.e. $x_i < x_j$ iff $T(x)_i < T(x)_j$.

The first claim (b) follows directly from Kantorovich duality [14] [50]. Indeed, $T_i$ (for $i \in \{0,1\}$) is an optimal transport map iff we can write $T_i = \nabla \phi_i$ where the convex function $\phi_i$ on $\mathbb{R}^N$ minimizes the Kantorovich functional $J_i$ corresponding to the two $S_N$–invariant measures $\mu_i^{(N)}$ and $\nu^{(N)}$. But then it follows from general principles that the minimizer can also be taken $S_N$–invariant. To prove the claim (b) we will use the well-known fact that any optimal map $T$ is cyclical monotone and in particular for any $x$ and $x'$ in $\mathbb{R}^N$

$$|x - T(x)|^2 + |x' - T(x')|^2 \leq |x - T(x')|^2 + |x' - T(x)|^2$$

(as follows from the fact that $T$ is the gradient of a convex function). In particular, denoting by $\sigma := (ij) \in S_N$ the map on $\mathbb{R}^N$ permuting $x_i$ and $x_j$ we get,

$$|x - T(x)|^2 + |\sigma x - T(\sigma x)|^2 \leq |x - T(\sigma x)|^2 + |\sigma x - T(x)|^2$$

But since (by (a)) $T\sigma = \sigma T$ and $\sigma$ acts as an isometry on $\mathbb{R}^N$ the left hand side above is equal to $2|x - T(x)|^2$ and similarly, since $\sigma^{-1} = \sigma$ the right hand side is equal to $2|\sigma x - T(x)|^2$. Hence setting $y := T(x)$ gives

$$|x - y|^2 \leq |\sigma x - y|^2$$

Finally, expanding the squares above and using that $\sigma = (ij)$ gives $-2(x_iy_i + x_jy_j) \leq -2(x_jy_i + x_iy_j)$ or equivalently: $(x_i - x_j)(y_i - y_j) \geq 0$, which means that $x_i < x_j$ iff $y_i < y_j$ and that concludes the proof of (b).

Now, by the previous claim the map $T_t$ preserves the fundamental domain

$$\Lambda := \{x : x_1 < x_2 < \ldots < x_N\}$$

for the $S_N$–action on $\mathbb{R}^N$. But, by assumption, on the subset $\Lambda$ the function $E^{(N)}_\Lambda$ is convex and this is enough to run the usual argument to get convexity of the mean energy on the subspace of symmetric measures. Indeed, we can decompose

$$\int_{X^N} E^{(N)} \mu^{(N)}_t = \sum \int_{\sigma(\Lambda)} E^{(N)}(T_t)_* \nu^{(N)}$$

(4.8)

(using that $(T_t)_* \nu^{(N)}$ does not charge null sets, since it has a density). For any fixed $\sigma$ the integral above is equal to $\int_{\sigma(\Lambda)} T_t E^{(N)} \nu^{(N)}$ (since $T_t$ preserves $\sigma(\Lambda)$) which, by the $S_N$–invariance of $\nu^{(N)}$ and $E^{(N)}$ in turn is equal to $\int_\Lambda T_t E^{(N)} \nu^{(N)}$. But since $E^{(N)}$ is convex on $\Lambda$ and $T_t$ preserves $\Lambda$ the function $T_t^* E^{(N)}$ is convex in $t$ for any fixed $x \in \Lambda$ and hence, by the decomposition \[ \int_{X^N} E^{(N)} \mu^{(N)}_t \] is convex.
wrt \( t \), as desired. Finally, the convexity of \( E(\mu) \) follows immediately by taking \( \mu^{(N)} \) to be a product measure \( \mu^{\otimes N} \) and using formula \( 4.3 \).

**Remark 4.7.** The first convexity statement may appear to contradict the second point in Lemma 2.8 which seems to force \( E^{(N)} \) to be convex on all of \( \mathbb{R}^N \) (which will not be the case in general). But the point is that we are only integrating against symmetric measures. As for the convexity of \( E_{W,V}(\mu) \) is is indeed well-known that it holds precisely when the symmetric function \( w(x) \) is convex on \( [0,\infty[ \) (see [4] [23]). This fact can be proved more directly by using that, in this special case, \( T(x) = (f(x_1), f(x_2), ..., f(x_N)) \) clearly preserves order since \( f \), being the derivative of a convex function, is clearly increasing.

Given a sequence \( \beta_N \in ]0,\infty[ \) converging to \( \beta \in ]0,\infty[ \) we recall that \( \mathcal{F}^{(N)} \) denotes the corresponding mean free energy functional \( 4.10 \). Similarly, we define the the corresponding (macroscopic) free energy functional on \( \mathcal{P}(\mathbb{R}) \) by

\[
F_{\beta}(\mu) := E_{W,V}(\mu) + \frac{1}{\beta} H(\mu)
\]

Combining the previous proposition with Theorem 3.2 shows that the EVI-gradient flows \( \mu_t \) of \( F_{\beta_N} \) and \( F_{\beta}^{(N)} \) on \( \mathcal{P}_2(\mathbb{R}) \) and \( \mathcal{P}_2(\mathbb{R}^N)^{\times N} \), respectively exist for appropriate initial measures.

**Theorem 4.8.** Let \( W \) and \( V \) be a two-point interaction energy and potential as in Section 4.2 and denote by \( \mu_t^{(N)} \) the corresponding probability measures on \( \mathbb{R}^N \) evolving according to the forward Kolmogorov equation associated to the stochastic process \( 4.4 \). Assume that at the initial time \( t = 0 \)

\[
\lim_{N \to \infty} \delta_N \ast \mu_t^{(N)} = \delta_{\mu_0}
\]

in the \( L^2 \)–Wasserstein metric. Then, at any positive time

\[
\lim_{N \to \infty} \delta_N \ast \mu_t^{(N)} = \delta_{\mu_t}
\]

in the \( L^2 \)–Wasserstein metric, where \( \mu_t \) is the EVI-gradient flow on \( \mathcal{P}_2(\mathbb{R}) \) of the free energy functional \( F_{\beta} \), emanating from \( \mu_0 \).

**Proof.** Assumptions 2 and 3 in Section 3 have been verified above and we just need the verify Assumption 1 in order to apply Theorem 3.2. The upper bound \( 4.3 \) follows precisely as before, using formula \( 4.3 \). In order to verify the lowerer bound we fix the truncation parameter \( R > 0 \) and observe that, since \( E_{W,V} \geq E_{W_R,V_R}^{(N)} \), formula \( 4.3 \) gives

\[
E_{W,V}^{(N)}(\mu_N)/N \geq \int E_{W_R,V_R}(\delta_N(x_1, ..., x_N)) \mu_N + C_R/N
\]

But

\[
\int E_{W_R,V_R}(\delta_N(x_1, ..., x_N)) \mu_N = \int_{\mathcal{P}(\mathbb{R})} E_{W_R,V_R}(\mu) \ast \delta_N \ast \mu_N \to \int_{\mathcal{P}(\mathbb{R})} E_{W_R,V_R}(\mu) \Gamma
\]

as \( N \to \infty \), by the continuity properties of the functional \( E_{W_R,V_R}(\mu) \) (Prop 4.2). Hence,

\[
\liminf_{N \to \infty} \frac{1}{N} \int_{\mathbb{R}^N} E^{(N)}(\mu^{(N)}) \geq \int E_{W_R,V_R}(\mu) \Gamma
\]

for any \( R > 0 \). Finally, letting \( R \to \infty \) and using the monotone convergence theorem concludes the proof.
4.3. Deterministic mean field limits for strongly singular pair interactions. Let us next specialize to deterministic “strongly singular” interactions, i.e. the case when $\beta_N = \infty$ and $w(s)$ blows up as $s \to 0$. In this case the corresponding EVI gradient flows on $\mathbb{R}^N$ in the deterministic case $\beta_N = \infty$, are induced by the classical solutions to the corresponding system of ODEs.

**Lemma 4.9.** Fix a positive integer $N > 0$. Let $w$ be as in Section 4.2 and assume moreover that $w$ is $C^2$—smooth on $[0, \infty[$ with $w(s) \to \infty$ as $s \to 0$ and that $|\nabla w|$ remains bounded as $s \to \infty$. Then

- there is a unique smooth solution to the corresponding ODE on $\mathbb{R}^N$ obtained by setting $\beta_N = \infty$ in equation 4.2:
  \[
  dx_i(t)/dt = -\sum_{j \neq i}(\nabla w)(x_i - x_j)
  \]
  if the initial condition satisfies $x_i(0) \neq x_i(0)$ when $i \neq j$ (then this condition is preserved for any $t > 0$).
- the corresponding curve of symmetric probability measures
  \[
  \mu^{(N)}(x_1(t), \ldots, x_N(t)) := \frac{1}{N!} \sum_{\sigma \in S_N} \delta_{x_{\sigma(i)}(t), \ldots, x_{\sigma(N)}(t)}
  \]
  on $\mathbb{R}^N$ (i.e. the normalized Dirac measure supported on the $S_N$—orbit in $\mathbb{R}^N$ of $(x_1(t), \ldots, x_N(t))$ coincides with the unique EVI gradient flow solution to the corresponding mean energy functional $E^{(N)}$ on $\mathcal{P}(\mathbb{R}^N)^{S_N}$, with initial data $\mu_0(x_1(0), \ldots, x_N(0))$).
- if the initial coordinates $x_i(0)$ are viewed as $N$ iid random variables on $\mathbb{R}$ with distribution $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, then the law $(x_1(t), \ldots, x_N(t))_{\ast} \mu_0^{\otimes N}$ of the corresponding random variable $x_1(t), \ldots, x_N(t)$, for a given positive time $t$, gives the unique EVI gradient flow solution of the corresponding mean energy functional $E$ on $\mathcal{P}(\mathbb{R}^N)$ with initial data $\mu_0^{\otimes N}$.

**Proof.** Set $x := (x_1, \ldots, x_N)$, where we may without loss of generality assume that $x_1 < \ldots < x_N$, i.e. that $x$ is a point in the fundamental domain $\Lambda$ for the $S_N$—action. By the standard Cauchy-Lipschitz existence result for ODE’s with Lipschitz continuous drift there exists $T > 0$ and a solution $x(t)$ for $t \in [0, T]$ where $x(t)$ stays in the open convex set $\Lambda$. Moreover, since the flow $x_t$ on $\Lambda$ is the gradient flow (wrt the Euclidean metric) of the smooth function $E^{(N)}(\cdot) := E_W^{(N)}$ on $\Lambda$ it follows that

\[
E^{(N)}(x(t)) \leq E^{(N)}(x(0)) =: A < \infty.
\]

Moreover, we claim that

\[
\|x(t)\| \leq B
\]

for some constant $B$ only depending on the initial data and $T$. Accepting this for the moment it then follows from the bounds 4.11, 4.12 and the singularity assumption on $w(0)$ that there is a positive constant $\delta$ such that $|x_i(t) - x_j(t)| \geq \delta$ for any $(i, j)$ such that $i \neq j$ and any $t \in [0, T]$. Hence, by restarting the flow again the short-time existence result translates into a long-time existence result, i.e. $T = \infty$. Before establishing the claimed bound 4.12 it may be worth pointing out that the bound in question is not needed in the case when $w(s) \geq -C$ as $s \to \infty$ (indeed, the $+\infty$ singularity appearing when two particles merge can then not be compensated by letting some particles go off to infinity), but this problem could a priori appear
if, for example, \( w(s) = -\log s \). To prove the bound 4.12 we observe that it will be enough to prove the second point in the statement of the lemma. Indeed, since the second moments of the EVI gradient flow \( \mu^{(N)}(x_1(t), \ldots, x_N(t)) \) are uniformly bounded on any fixed time interval this will imply the desired uniform bound 4.12 on \( \mathbb{R}^N \).

We next turn to the proof of the second point. We recall that by the convexity result in Proposition 4.10 together with the general result Theorem 2.6 the functional \( E \) admits an EVI gradient flow \( \mu^{(N)}(t) \) on \( \mathcal{P}(\mathbb{R}^N)_2 \) emanating from any given symmetric measure \( \mu^{(N)}(0) \in \mathcal{P}(\mathbb{R}^N)_{\mathcal{S}_N} \). Here we will take \( \mu^{(N)}(0) \) to be as in formula 4.11. Setting \( Y = \mathbb{R}^N/\mathcal{S}_N \) we can identify the space \( \mathcal{P}(\mathbb{R}^N)_{\mathcal{S}_N} \) with \( \mathcal{P}(Y)_2 \) and \( \mu^{(N)}(t) \) with an EVI gradient flow \( \mu^{(N)}(t) \) on \( \mathcal{P}(Y)_2 \) emanating from \( \mu^{(N)}(0) = \delta_y \), where \( y \) is the point \( [x(0), \ldots, x(N)] \in Y \). We claim that \( \mu^{(N)}(t) \) is of the form \( \delta_y(t) \) for a curve \( y(t) \) in \( \mathcal{P}(Y)_2 \). Indeed, since \( \mu^{(N)}(t) \) arises as limit of minimizing movements it is enough to establish the claim for the minimizing movement corresponding to any given time discretization. But in the latter situation the corresponding functionals

\[
J_{j+1}(\cdot) := \frac{1}{2\tau} d(\cdot, \mu_t)^2 + E(\cdot) - E(\mu_t)
\]

appearing in formula ?? (with \( F = E \)) are linear (wrt the ordinary affine structure) at each time step if \( \mu_{t_j} \) is assumed to be a Dirac mass, i.e of the form \( \delta_{y_{t_j}} \) (since \( \mathcal{E} \) and \( d^2(\cdot, \delta_{y_{t_j}}) \) are both linear). Hence, by Choquet’s theorem, any optimizer is of the form \( \delta_{y_{t_{j+1}}} \) for some point \( y_{t_{j+1}} \in Y \) and thus, by induction, this proves the claim. Next we apply the general isometric embedding

\[
Y \to \mathcal{P}_2(Y), \quad y \mapsto \delta_y
\]

which clearly has the property that \( E(y) = E(x^{(N)}) = \mathcal{E}(\delta_y) \). The EVI gradient flow \( \delta_y(t) \) on \( \mathcal{P}(Y)_2 \) thus gives rise to an EVI gradient flow \( y(t) \) on \( Y \). Finally, the curve \( y(t) \in \mathbb{R}^N/\mathcal{S}_N \) may be identified with a curve \( x(t) \) in the domain \( \Lambda \subset \mathbb{R}^N \), which concludes the proof of the second point.

The last point can be proved in a similar manner by approximating the initial measure with a sum of Dirac masses. Alternatively, one can use that the result is well-known when \( E \) is smooth on \( \mathbb{R}^N \) and the initial symmetric measure \( \nu_0 \) on \( \mathbb{R}^N \) has a smooth density (indeed, then \( y(t), \nu_0 \) satisfies a transport equation which defines the unique EVI gradient flow solution of \( \mathcal{E} \) emanating from \( \nu_0 \)). The general case then follows by approximation using the stability property of EVI gradient flows [11].

**Remark** 4.10. By Theorem 2.6 one can, in fact, start the EVI gradient flow of \( \mathcal{E}^{(N)} \) from any symmetric measure \( \mu^{(N)}(0) \) and in particular from any measure of the form \( \mu^{(N)}(x_1(0), \ldots, x_N(0)) \), given an arbitrary configuration \( x_1(0), \ldots, x_N(0) \). Since, \( \mathcal{E}^{(N)}(\mu^{(N)}(t)) < \infty \) when \( t > 0 \) this corresponds to a classical solution \( (x_1(t), \ldots, x_N(t)) \) of mutually distinct points when \( t > 0 \).

We can now give a purely deterministic version of Theorem 4.8. We also establish the corresponding propagation of chaos result when the initial particle positions are taken as random iid variables:
Theorem 4.11. Assume that $w$ defines a strongly singular two-point interaction and denote by $x^{(N)}(t) := (x_1,\ldots,x_N)(t)$ the solution of the system of ODEs on $\mathbb{R}^N$ in the previous lemma.

- If

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(0)} \to \mu_0
\]

in $\mathcal{P}_2(\mathbb{R})$ for a given measure $\mu_0$ in $\mathcal{P}_2(\mathbb{R})$. Then, for any positive time $t$,

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)} \to \mu_t,
\]

where $\mu_t$ is the EVI gradient flow solution of the corresponding functional $E_W$ on $\mathcal{P}_2(\mathbb{R})$.

- Similarly, if the $x_i(0)$'s are viewed as $N$ iid random variables with distribution $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, then the corresponding random measure $\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}$ converges to $\mu_t$ in law, as $N \to \infty$ (and hence propagation of chaos holds).

Proof. To prove the first point we continue with the notation in the previous lemma and note that

\[
\Gamma_N := (\delta_N)_*(\mu^{(N)}(x^{(N)})) = \delta_{x^{(N)}},
\]

i.e. $\Gamma_N$ is the delta measure on $\mathcal{P}(\mathcal{P}(\mathbb{R}^N))$ supported on the measure $\delta_{x^{(N)}}$ (as follows directly from the definitions). By assumption $\Gamma_N(0) \to \delta_{\mu_0}$ in $\delta_{x^{(N)}}$ in $\mathcal{P}_2(\mathcal{P}(\mathbb{R}^N))$ as $N \to \infty$. But then it follows, precisely as in the proof of Theorem 4.8 that $\Gamma_N(t) \to \delta_{\mu_t}$ in $\mathcal{P}_2(\mathcal{P}(\mathbb{R}^N))$ for any fixed $t > 0$. Finally, using the relation 4.14 again concludes the proof of the first point in the theorem, using the very definition of $\Gamma_N(t)$ (and the $S_N$-invariance of the empirical measure $\delta_N$). As for the last point it follows immediately from the last point in the previous lemma combined with Theorem 4.8 (applied to the case $\beta_N = \infty$). \hfill \Box

4.4. An alternative approach not involving the $S_N$-action. In this section we point out that when $D = 1$ it is possible to dispense with the $S_N$-action and thus bypass the use of the $G$-equivariance in Theorem 2.6. On the other hand, one virtue of the more geometric approach used above is that it may turn out to be useful when $D > 1$. Indeed, given an interaction energy $E^{(N)}$ the Assumption 2 in Section 3.1 can be replaced by the convexity of $E^{(N)}$ along any class of “interpolating curves” satisfying the two conditions appearing in the proof of Theorem 2.6 (as emphasized in the introduction of [1] this is all that it is needed to develop the general theory of gradient flows in [1]). However, finding such interpolating curves for a given singular interaction $E^{(N)}$ appears to be a highly non-trivial task.

Denote by $\Lambda_N \subset \mathbb{R}^N$ the interior of the standard fundamental domain for the $S_N$-action on $\mathbb{R}^N$. Its closure $\overline{\Lambda_N}$ is the closed convex subset $\{x_1 \leq x_2 \leq \ldots \leq x_N\}$ that inherits a (geodesically) complete metric from the Euclidean metric on $\mathbb{R}^N$. The starting point is the following basic

Lemma 4.12. The quotient map induces an isometry $\Phi$ between the the Euclidean space $\overline{\Lambda_N}$ and the metric quotient space $\mathbb{R}^N/S^N$.

Proof. For completeness we give the simple proof. The quotient map induces a surjective map $\Phi$ from $\overline{\Lambda_N}$ to $\mathbb{R}^N$ and it is, by continuity, enough to prove
that $\Phi$ is an isometry on the dense open subset $\Lambda_N$ of $\overline{\Lambda_N}$. But, by definition, 
\[d(\Phi(x_1, \ldots, x_N), \Phi(y_1, \ldots, y_N)^2 = \inf_{\sigma \in S_N} \sum_i |x_i - y_{\sigma(i)}|^2,\]
which, by monotonicity, is realized when $\sigma$ is the identity, as desired.

Using the isometry in the previous lemma we can identify the space $P_2(\mathbb{R}^N)^{SN}$ with the space of all probability measures on $\Lambda_N$. Concretely, this means that a measure $\mu_N$ on $\Lambda_N$ is identified with the average of its $S_N$--orbit in $\mathbb{R}^N$. In this way the EVI gradient flow $\mu_N(t)$ of the mean energy functional $E^{(N)}$ on $P_2(\mathbb{R}^N)^{SN}$ considered above can be identified with the EVI gradient flow of a functional $\tilde{E}^{(N)}$ on $P_2(\mathbb{R}^N)$, emanating from a measure supported on $\Lambda_N$, where $\tilde{E}^{(N)}$ is the linear functional associated to the function $\tilde{E}_N := \chi_{\Lambda_N} + E(t)$, where $\chi_{\Lambda_N}$ is the indicator function of $\Lambda_N$, i.e. equal to zero on $\Lambda_N$ and infinity on its complement.

By our assumptions $\tilde{E}^{(N)}$ is $\lambda$--convex on all of $\mathbb{R}^N$ and hence, by Prop 4.5, $\tilde{E}^{(N)}$ satisfies the assumptions in Theorem 4.6 with $G$ trivial.

**Remark 4.13.** In the deterministic setting $\beta_N = \infty$ this means that one can identify the EVI gradient flow of $E^{(N)}$ on $\mathbb{R}^N/S^N$ with the gradient flow of $\tilde{E}^{(N)}$ the Euclidean space $\mathbb{R}^N$, defined in terms of the classical theory of gradient flows on Hilbert spaces (the multivalued subdifferential of $E^{(N)}$ at the boundary of $\Lambda_N$ then gets a contribution from the normal cone of the boundary of $\Lambda_N$). Similarly, the identifications above also show that when $\beta_N < \infty$ the EVI gradient flow of the corresponding mean energy functional coincides with the laws of the strong solution of the corresponding SDEs constructed in [20] using the theory multivalued montone maps (see also [2]).

5. **Realization of $\mu_t$ as a weak solution of the McKean-Vlasov equation.**

In this section we consider the relations between the EVI gradient flow $\mu_t$ appearing in Section 4 and weak solutions of the corresponding McKean-Vlasov equation, generalizing some results in [1] [23] to the present setting (and bypassing a gap in the argument in [23] in the case $\beta = \infty$). We conclude by briefly pointing out some subtle regularity problems of the flows in question. First recall that, in a general metric space $(M, d)$ the metric slope $|dF|/(\mu)$ of a functional $F$ on $M$ at $\mu \in M$ such that $F(\mu) < \infty$ is defined by
\[|dF|/(\mu) := \limsup_{\nu \to \mu} \frac{|F(\nu) - F(\mu)|^+}{d(\mu, \nu)}\]
and if $F(\mu) = \infty$, then $|dF|/(\mu) := \infty$. As explained in [1] the subdifferential calculus on the Wasserstein $P_2(\mathbb{R}^D)$ is considerably simpler in the case when $F$ satisfies the following assumption:
\begin{equation}
|dF|/(\mu) < \infty \implies \mu \in P_{abs}(\mathbb{R}^D)
\end{equation}
(which we will assume below). In particular, this is always the case for $F = F_\beta$ with $\beta < \infty$, where $F_\beta$ denotes the free energy functional [2,3]. Now the subdifferential $(\partial F)(\mu)$ at $\mu$ of a $\lambda$--convex functional on $P_2(\mathbb{R}^D)$ satisfying the assumption [5.1] may be defined as the convex subset of all $\xi \in L^2(\mathbb{R}^d, \mu)$ such that
\[F(\nu) \geq F(\mu) + \langle \xi, \nabla \phi_{\mu, \nu} \rangle_{L^2(\mathbb{R}^D, \mu)} + \lambda/2 d_2(\mu, \nu)^2, \forall \nu \in P_2(\mathbb{R}^D)\]
where $T_{\mu,\nu} := I + \nabla \phi_{\mu,\nu}$ is the unique optimal $L^\infty_{\text{loc}}$-map transporting $\mu$ to $\nu$ (see [10.1.1 B]). The corresponding minimal subdifferential $(\partial F)^0(\mu)$ is defined as the unique element in $(\partial F)(\mu) \subset L^2(\mathbb{R}^d, \mu)$ with minimal norm.

**Theorem 5.1.** Let $F$ be a lsc real-valued functional on $P_2(\mathbb{R}^d)$ which is $\lambda-$convex along generalized geodesics and satisfies the coercivity property [5.10]. Then the corresponding EVI gradient flow $\mu_t$ emanating from a given $\mu_0$ has the property that 
\[|dF|(<\mu) < \infty \text{ for } t > 0 \text{ and } \mu_t \text{ is a weak solution on } \mathbb{R}^d\times[0,\infty[ \text{ of the continuity equation} \]
\[\frac{d}{dt} \mu_t = -\nabla \cdot (\mu_t v_t), \]
where the time-dependent Borel vector fields $v_t$ with the property that $v_t = -(\partial^0 F)(\mu_t)$ for a.e. $t > 0$.

**Remark 5.2.** The result applies without the assumption [5.1] along $\mu_t$, but with a more elaborate definition of $\partial^0 F$ by [1] Thm 11.1.3, Thm 11.2.1. Briefly, one first defines the extended subdifferential $(\partial^0 F)(\mu)$ consisting of transport plans and then defines $(\partial F)(\mu)$ as the subset realized by transport maps (thus corresponding to vector fields) [1] 10.3.12. In this general setting the previous theorem then says that for a.e. $t$ the minimal transport plan is realized by the transport map defined by the vector field $v_t$.

In order to apply the general theory above we will need the following

**Lemma 5.3.** Assume that $V$ and $W$ are $\lambda-$convex and $C^1$ on $\mathbb{R}$ and $[0,\infty]$, respectively. Given $\mu \in P_2(\mathbb{R})$ such that $|dE_{W,V}|(\mu) < \infty$ and $\mu \in P_{\text{abs}}(\mathbb{R})$ the minimal subdifferential $\omega_{\mu} := (\partial^0 E_{W,V})(\mu) \in L^2(\mathbb{R}, \mu)$ satisfies
\[\int \omega_{\mu} \psi \mu = \frac{1}{2} \int W'(x - y)\left(\psi(x) - \psi(y)\right) \mu(x) \otimes \mu(y) + \int V' \psi \mu(<\infty)\]
for any $\psi$ in $\omega_{\mu}$.

**Proof.** This is proved essentially as in [23] Lemma 3.7 (see also [1] Thm 10.4.11) for the the general higher dimensional case, under the stronger assumption that when $w$ is differentiable on all of $\mathbb{R}^D$. But for completeness we recall the proof. First note that using [1] Prop 10.4.2 (concerning the case $W = 0$) we may as well assume, by linearity, that $V = 0$ and by replacing $w(x)$ with $w(x) - \lambda|x|^2/2$ we may as well assume that $w$ is convex (using that, by the Cauchy-Schwartz inequality $x\psi \in L^2(\mu)$). Now, assume that $\psi \in L^\infty_{\text{loc}}(\mathbb{R}) \cap L^2(\mu)$ and consider the family of $L^\infty-$maps $T_t(x) := x + t\psi(x)$ on $\mathbb{R}$ for $t \geq 0$. On one hand a direct calculation gives for $W := E_{W,0}$ that
\[\lim_{t \to 0^+} \frac{W((T_t)_* \mu) - W(\mu)}{t} = \frac{1}{2} \int W'(x - y)(\psi(x) - \psi(y)) \mu(x) \otimes \mu(y)\]
using that, by convexity, $(w((x - y) + t(\psi(x) - \psi(y))) - w(x - y))/t$ is defined and nondecreasing in $t$ for a.e. $(x, y)$ such that $x > y$ and also using that $W'$ is odd on $\mathbb{R} \setminus \{0\}$. Indeed, applying the monotone convergence theorem then gives the previous equality, since we have assumed that $\mu \in P_{\text{abs}}(\mathbb{R})$. On the other hand it follows directly from the definition of the metric slope that
\[\lim_{t \to 0^+} \frac{W((T_t)_* \mu) - W(\mu)}{t} \leq |dW|(\mu) \|\psi\|_{L^2(\mathbb{R}, \mu)}\]
But, since the RHS in formula (5.4) is linear wrt $\psi$ it then follows from the Riesz representation theorem that there exists a unique element $\omega_\mu \in L^2(\mathbb{R}, \mu)$ satisfying formula (5.3) with $\|\omega_\mu\|_{L^2(\mathbb{R}, \mu)} \leq |d\mathcal{W}|(\mu)$. Finally, to verify that $\omega_\mu \in \partial \mathcal{W}$ we fix $\nu \in \mathcal{P}(\mathbb{R})$. Since $\mu \in \mathcal{P}_{\text{abs}}(\mathbb{R})$ there exists a unique transport map $T := T_{\mu, \nu} := I + \nabla \phi_{\mu, \nu}$ such that $T_\ast \nu = \nu$. Setting $\psi := \nabla \phi_{\mu, \nu}$ and using the convexity of $\mathcal{W}$ on the Wasserstein space thus gives

$$
\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \lim_{t \to 0^+} \frac{\mathcal{W}(T_\ast \mu) - \mathcal{W}(\mu)}{t} = \langle \omega_\mu, \nabla \phi_{\mu, \nu} \rangle_{L^2(\mathbb{R}^d, \mu)}
$$

showing that $\omega_\mu \in \partial \mathcal{W}$, as desired. \qed

As a consequence, when $\beta > 0$, the limiting curve $\mu_\beta$ appearing in Theorem 1.1 is a weak solution of the McKean -Vlasov equation (1.3) in the following sense

**Proposition 5.4.** Let $V$ be a $\lambda$-convex function on $\mathbb{R}$ and let $w$ be as in the previous proposition and denote by $\mu_\beta(t)$ the EVI gradient flow $\mu_\beta(t) (\mathcal{P}_2(\mathbb{R}))$ of the corresponding free energy functional $F_\beta$ (formula (2.3)) emanating from a given $\mu(0) \in \mathcal{P}_2(\mathbb{R})$. Then, given any $\phi \in C^2_c(\mathbb{R})$, $\mu_\beta(t)$ is a distributional solution of the following equation on $[0, \infty[$:

$$
\frac{d}{dt} \int_\mathbb{R} \mu_\beta(t) \phi(x) = \frac{1}{\beta} \int \mu_\beta(t) \phi''(x) + \frac{1}{2} \int W'(x-y) (\phi'(x) - \phi'(y)) \mu_\beta(t) \otimes \mu_\beta(t) + \int V'(x) \phi(x) \mu_\beta(t)
$$

Proof. Since $F_\beta(\mu_\beta) < \infty$, for $t > 0$ and $\beta < \infty$ the assumption 5.1 holds along the curve $\mu_\beta(t)$, when $t > 0$ and hence it follows from [1] Thm 11.1.3, Thm 11.2.1 that the density $\rho_\beta(t)$ of $\mu_\beta(t)$ is a weak solution on $\mathbb{R} \times [0, \infty[$ of the following equation

$$
\frac{d}{dt} \rho_\beta(t) = \frac{1}{\beta} \Delta \rho_\beta(t) + \nabla (\rho_\beta(t) v_\beta(t)),
$$

where $v_\beta$ is the curve of Borel vector fields defined by $v_\beta(x) = -\partial^0 E_{W,V}(\mu_\beta)$. The result then follows from the previous proposition. \qed

In order to consider the case $\beta = \infty$ we will use the following general stability result:

**Lemma 5.5.** (stability wrt $\beta$). Let $\mu_\beta(t)$ the EVI gradient flows in the previous proposition, emanating from a given $\mu(0) \in \mathcal{P}_2(\mathbb{R})$, independent of $\beta$. Further assume that $E(\mu) < \infty$ for all compactly supported measures with an $L^\infty$ density. Then $\mu_\beta(t) \to \mu_\infty(t)$ in $\mathcal{P}_2(\mathbb{R})$, as $\beta \to \infty$.

Proof. According to [1] Thm 11.2.1 we just have to verify that $F_\beta$ gamma-converges towards $F_\infty (= E)$ on $\mathcal{P}_2(\mathbb{R})$. First observe that in order to verify the lim inf inequality appearing in the definition of Gamma convergence it is, by a standard diagonal argument, enough to verify it for the dense subset of all elements in $\mathcal{P}_2(\mathbb{R})$ with finite entropy. But for any such element $\mu$, $F_\beta(\mu) \to F_\infty(\mu)$ trivially.

We next turn to the verification of the lim sup inequality in the definition of Gamma convergence. Note that if $E(\mu) = \infty$ or $H(\mu) < \infty$, there is nothing to show. Further, since gamma-convergence is stable under continuous perturbations, we may as well assume that $E(\mu)$ is convex. So assume that $E(\mu) < \infty$ and $H(\mu) = \infty$. Then we need to find a recovery sequence $\mu_\beta \to \mu$ such that $H(\mu_\beta)/\beta \to 0$ and $E(\mu_\beta) \to E(\mu)$. To this end let $\mu_0 = dx|_{[0,1]}$ be the Lebesgue measure on the unit
interval, and let $\mu_1 = \mu$. For $t \in [0, 1]$ we let $\mu_t$ be the displacement interpolation. Then $E(\mu)$ is a convex function of $t$, and it follows that $E(\mu) \geq \limsup_{t \to 1} E(\mu_t)$ since $E(\mu_0) < \infty$. Hence, it is enough to show that for some choice $t = t(\beta)$, $\lim_{\beta \to \infty} t(\beta) = 1$, it holds that $H(\mu_t(\beta))/\beta \to 0$ as $\beta \to \infty$. We claim that $H(\mu_t) < \infty$ for all $t \in [0, 1]$, from which the result is immediate. To show the claim, note that since $\mu_0$ is supported on a convex set, there is a unique Brenier map $\nabla \phi_1$ transporting $\mu_0$ to $\mu_1$, and it holds that the displacement interpolation is given by $\phi_t = (1-t)x^2/2 + t\phi_1$. But then $MA(\phi_t) = \nabla^2 \phi_t = (1-t) + t \nabla^2 \phi_1 \geq 1 - t$. Thus an $L^\infty$ bound follows for the density $\mu_t = \rho_t dx$, since $\rho_t(x) = \rho_0(\nabla \phi_t)/MA(\phi_t(x)) \leq 1/(1-t)$. Hence $H(\mu_t) = \int \rho_t \log \rho_t dx \leq -\int \rho_t \log(1-t)dx = -\log(1-t).$ \hfill $\Box$

**Remark 5.6.** The previous lemma appears as Theorem 3.6 in [23] (in the case of model power laws). However, the verification of the Gamma convergence of the functionals $F_\beta := E + H/\beta$ towards $E$ was not provided in the proof in [23]. This convergence problem is a special case of the following general problem: consider a measure $\mu_0$ on a topological space $X$ (for example $\mathbb{R}^D$) and denote by $H_{\mu_0}(\mu)$ the entropy of a probability measure relative to $\mu_0$. Let $E(\mu)$ be defined by a singular integral operator on $P(X)$ with lower semi-continuous kernel $W(x, y)$. Showing that $F_\beta := E + H_{\mu_0}/\beta$ Gamma converges towards $E$, as $\beta \to \infty$, appears to be a rather subtle problem in general and requires an appropriate compatibility between $\mu_0$ and the kernel $W(x, y)$. For example, when $X = \mathbb{R}^2$ or $X = \mathbb{R}$ and $W(x, y) = -\log |x-y|$ the convergence can be shown to hold when $\mu_0$ is sufficiently regular in the potential theoretic sense (i.e. $\mu_0$ has regular asymptotical behaviour in the sense of [11] or satisfies a Bernstein-Markov property in the sense of [9]). We shall not develop this further here, but just point out that the regularity property in question does hold when $\mu_0$ is, for example, Lebesgue measure $dx$ on $\mathbb{R}$ (for non-trivial reasons).

However, in general, it is not enough to assume that $E(\mu_0) < \infty$. In general, as in the proof of the previous lemma, in order to verify the Gamma convergence it is enough to verify the following condition

- For any $\mu$ such that $E(\mu) < \infty$ there exists a sequence $\mu_j$ converging weakly to $\mu$ such that $\mu_j$ is absolutely continuous with respect to $\mu_0$ and satisfies $E(\mu_j) \to E(\mu)$

From the statistical mechanical point of view this condition appears naturally in the large deviation theory of the $N$–particle Gibbs measures corresponding to the pair interaction $W(x, y)$ when $\beta_N \to \infty$ [22] Hypothese 4, page 2373]. In the model case of power laws and $\mu_0 = dx$ on $X = \mathbb{R}^D$ the regularizations $\mu_j$ above can alternatively be defined using a convolution of $\mu_0$, by exploiting the convexity of the corresponding functional $E$ wrt the usual affine structure (which is a classical, but non-trivial fact) [22].

With the previous lemma in place we get the following

**Proposition 5.7.** When $\alpha \leq 1$ Proposition 5.6 holds for $\beta = \infty$, as well.

**Proof.** This follows immediately from combining Prop 5.4 with the stability property of Lemma 5.5 using that, in this particular case, $W^\prime(x-y)(\psi(x) - \psi(y)$ defines a continuous function on $\mathbb{R}^2$, since $\psi$ is assumed smooth. \hfill $\Box$

**Remark 5.8.** Recall that for any finite measure $\mu$ on $\mathbb{R}$ the limit

$$H_\mu(x) := \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{\mu(y)}{x-y}$$

(5.5)
exists for a.e. \( x \) in \( \mathbb{R} \) \cite[Theorem, page 1085]{33} and defines the Hilbert transform \( H_\mu(x) \) of \( \mu \). Formally, the weak McKean-Vlasov equation for a repulsive logarithmic interaction \((\alpha = 1)\) is equivalent to a weak solution to the continuity equation with drift \( \nu_t = H_\mu \). However, in general \( H_\mu \) is not in \( L^1_{\text{loc}}(\mathbb{R}) \), even if \( f \in L^1(\mathbb{R}) \) and hence some further a priori regularity on \( \mu_t \) is needed in order to even make sense of the corresponding continuity equation (which requires that \( H_\mu \in L^1(\mu) \)); see the discussion in \cite{37}. For example if \( \mu_t \) is in \( L^p(\mathbb{R}) \) for some \( p > 1 \) then so is \( H_\mu \) by Riesz classical theorem.

In the logarithmic case the uniqueness of weak solutions \( \mu_t \) to the McKean-Vlasov equation was established in \cite{21} for \( V \) quadratic and in \cite{17, 38} under the assumption that the Fourier transform of \( V \) has exponential decay (in particular, \( V \) is real-analytic). The proofs in \cite{21} exploit that the Hilbert transform \( H_\mu(x) \) is, for almost any \( x \), the boundary value along the real axes of the Cauchy transform \( G_\mu(z) \) of \( \mu \), which defines a holomorphic function on the upper half plane. Then a weak solution in the sense of the previous proposition corresponds to a strong solution for a complex Burger type equation in the upper half plane. However, the uniqueness of weak solutions seems to be open when \( V \) is only assumed smooth and \( \lambda \)-convex (in \cite{38} the uniqueness in the class of solutions \( \mu_t \) in \( L^p(\mathbb{R}) \) for \( p \geq 2 \) is established). Moreover, in the case of attractive power-laws with \( \alpha = 0 \) uniqueness of weak solutions fails, in general (see Section 6.2).

Let us also recall that in the model case of repulsive power-law with exponent \( \alpha \in [0, 2] \) the corresponding drift \( V_t \) can, at least formally, be written as minus the gradient of the fractional Laplacian \((\alpha)\) of \( \mu \). Certain weak solutions of the corresponding evolutions are constructed in \cite{10} when \( D = 1 \) and in \cite{18} when \( D \geq 1 \), using an elaborate regularization scheme involving several parameters. However, a key point of the Wassestein gradient flow approach is that the constructed limiting curve \( \mu_t \) is an EVI gradient flow and thus automatically uniquely determined.

**Remark 5.9.** Let now \( W \) be a repulsive power law with \( \alpha \in [1, 2] \) (and take, for example, \( V(x) = Cx^2 \)). In \cite[Remark 2.2]{23} it is claimed that \( E_{W,V}(\mu) < \infty \) implies that \( \mu \) is absolutely continuous w.r.t. \( dx \), for \( t > 0 \), and as a consequence it is claimed in \cite[Remark 3.10]{23} that \( \mu_t(t) \) is a weak solution in the sense of Prop 5.4 also for \( \beta = \infty \) (since \( E_{W,V} < \infty \) along the EVI gradient flow). But the first claim appears to be incorrect. Indeed, in the logarithmic case it is well-known that there are measures \( \mu \) which are not absolutely continuous w.r.t. \( dx \), but with the property \( E_{W,V}(\mu) < \infty \), for example measures with sufficiently small Haussdorff dimension (such as the standard Cantor set). A similar counter example applies when \( \alpha > 1 \).

On the other hand, it may very well be that using further properties of the EVI gradient flow (for example, that the metric slopes are finite for \( t > 0 \)) one can establish the first claim, or directly the second claim, for any repulsive powerlaw. Alternatively, it seems likely that, assuming that \( \mu_0 \in L^p(\mathbb{R}^D) \) for some \( p > 1 \) one can show that \( \mu_t \in L^p(\mathbb{R}^D) \), by a regularization procedure, as shown in \cite{18} in a related context (when \( \alpha \leq 1 \) this follows from the viscosity approach in \cite{10}, when \( V = 0 \)). But we will not go further into this here.

**6. Further applications**

In this section we give some further examples illustrating the general structure of the applications discussed in Section 4.
6.1. **Variants of strongly singular power-laws in 1D.** The next example generalizes the power-law in Section 4 to pair interaction which are not translationally invariant:

**Example 6.1.** Given a function \( g(x, y) \) which is positive, bounded and with bounded first and second partial deriviatives consider the following pair interaction potential:

\[
W(x, y) := \frac{g(x, y)}{|x - y|^s},
\]

and let \( E^{(N)}(x_1, x_2, \ldots, x_N) \) be the corresponding pair interaction energy. Then the assumptions in Section 3 are satisfied, as follows essentially by the same arguments as in Section 4. We recall that such interactions appear, for example, as spin type Hamiltonians in the mathematical physics litterature. To briefly explain this consider \( N \) particles located at \( x_1, \ldots, x_N \) with internal spin type degrees of freedom \( S_1, \ldots, S_N \) taking values in the unit-sphere in \( \mathbb{R}^n \). The corresponding spin type Hamiltonian may be defined by

\[
H(x_1, \ldots, x_N; S_1, \ldots, S_N) := \sum_{i,j} J_{x_i x_j} (1 - S_i \cdot S_j),
\]

for a given function \( J_{xy} \) on \( \mathbb{R}^2 \), the spin interaction. These models have been studied extensively in lattice models where the positions \( x_1, \ldots, x_N \) are fixed once and for all and put on a lattice, for example, \( x_i = i \) in the 1D setting. The case \( n = 1 \) and \( J_{xy} = |x - y|^{-\alpha} \) is then usually called the \( \alpha \)-Ising model (while the general case \( n \geq 1 \) is called the \( O(n) \)-model); see [6, Section 4] for the corresponding static mean field limit and [33] for a similar stochastic dynamical lattice model with long range interactions, where a McKean-Vlasov type equation appears in the macroscopic limit. The present setting applies to the opposite setting where the positions \( x_1, \ldots, x_N \) are free while \( S_i = S(x_i) \) for a given smooth map \( S \) from \( \mathbb{R}^n \) into the unit-sphere in \( \mathbb{R}^n \), for \( n \geq 2 \). One then sets \( g(x, y) = (1 - S(x) \cdot S(y)) \) in formula 6.1. For example, this happens when the system is coupled to a strong exterior magnetic field \( B(x) \), effectively forcing \( S(x) \) to be parallel to \( B \).

Let us also give some 1D examples which are not pair interactions:

**Example 6.2.** Consider the following 1D \( N \)-particle interaction energy:

\[
E^{(N)}(x_1, x_2, \ldots, x_N) := \frac{1}{N^3} \sum_{i,j,k} \frac{1}{(|x_i - x_j| + |x_k - x_i| + |x_k - x_j|)} \quad s \in [0, 1[.
\]

where the sum runs over all indices \( i, j, k \) in \([1, N]\) such that \( i, j \) and \( k \) do not all coincide. This means that \( E^{(N)}(x_1, x_2, \ldots, x_N) \) blows up precisely when at least three points merge (but remains bounded if only two points merge). If \( x_i < x_j < x_k \) then the function appearing above is clearly convex and in \( L^1_{\text{loc}} \) and hence the assumptions in Section 3 are satisfied, as discussed in Section 4.

6.2. **Globally quasi-convex interactions in any dimension.** Next, we give some extensions from the case \( D = 1 \) to higher dimensions.

**Theorem 6.3.** Let \( w_m, m = 1, \ldots, M \) be quasi-convex funtions on \((\mathbb{R}^D)^m\) and \( V(x) \) a quasi-convex function on \( \mathbb{R}^D \). Assume that at the initial time \( t = 0 \)

\[
\lim_{N \to \infty} (\delta_N)_* \mu^{(N)}_t = \delta_{\mu_0}
\]
in the $L^2$–Wasserstein metric. Then, at any positive time
\[ \lim_{N \to \infty} (\delta_N)_{*, \mu}^{(N)} = \delta_{\mu_t} \]
in the $L^2$–Wasserstein metric, where $\mu_t$ is the EVI-gradient flow on $P_2(\mathbb{R})$ of the corresponding free energy functional $F_\beta$, emanating from $\mu_0$.

**Proof.** This is proved essentially as in the proof Theorem 4.8. In fact, the proof is even simpler by the assumption of global semi-convexity. One can then approximate each $w_m$ on all of $\mathbb{R}^D$ with a sequence $w_m^{(\phi)}$ of quasi-convex functions (with the same $\lambda$) increasing to $w_m$ and such that $|w_R(x)| = o(|x|^2)$ as $|x| \to \infty$, for in fixed $R$. Indeed, by the quasi-convexity assumption it is enough to treat the case when $w_m(= \phi)$ is convex. In that case we can, for example, take
\[ \phi^{(R)} := \inf_{y \in \mathbb{R}^D} R|x - y| + \phi(y). \]
It is then shown, precisely as in the proof of Theorem 4.8, that the corresponding $N$–partical energy $E^{(N)}(x_1, \ldots, x_N)$ satisfies the assumptions in Section 3.1. □

In particular the previous theorem applies to the non-smooth weakly singular attractive pair interactions
\[ W(x, y) := w(|x - y|) = |x - y|^{1+\alpha}, \ 0 \leq \alpha < 1. \]
In the purely stochastic setting such pair interactions appear, for example, in the study of granular media and swarming models (see [25] and references therein). The purely deterministic setting for such pair interaction has been studied in depth in [25]. For example it is shown that, given any initial $\mu_0$ with compact support the corresponding EVI gradient flow $\mu_t$ of $E$ is a Dirac mass, i.e. $\mu_t = \delta_{x(t)}$, when $t \geq T$ for some finite time $T$. But as far as we know there are no results concerning such weakly singular attractive interactions in the purely stochastic setting ($\beta_N < \infty$), apart from the case when $\alpha = 1$ and $D = 1$ where the theory of scalar conservation laws applies [12] (the case of smooth potentials with polynomial growth is studied in [36, 54]).

**Remark 6.4.** For the attractive pair interactions of the form (6.2) it is shown in [25] that the corresponding minimal subdifferential is given by
\[ (\partial^0 W)(\mu) := \int_{\{x \neq y\}} (\nabla_x W)(x, y)\mu(y), \]
which is a point-wise well-defined function. Setting $(\nabla w)(0) := 0$ (which, by symmetry, coincides with the minimal subdifferential of $w(x)$ at 0) this means that $(\partial^0 W)(\mu) = \mu * \nabla w$.

By Theorem 5.1 the macroscopic limit $\mu_t$ is a weak solution of the corresponding equation
\[ \frac{d}{dt}\mu_t = \frac{1}{\beta} \Delta \mu_t + \nabla \cdot (\mu_t(\partial^0 W)(\mu_t)) \]
When $\beta = \infty$ it is well-known that a weak solution is not unique, in general. For example, when $D = 1$ and $\alpha = 0$ the map $\mu \mapsto \mu * \nabla w$ gives a correspondence between weak solutions $\mu_t$ as above and weak solution $u_t$ of the scalar conservation law (Burger’s equation)
\[ \partial_t u_t = \frac{1}{\beta} \partial_x^2 u_t + \partial_x(u_t^2)/2 \]
satisfying \(|u| \leq 1, \partial_x u \geq 0\). By the general theory of scalar conservation laws a weak solution \(u_t\) is uniquely determined if it is an entropy solution. Moreover, \(u_t\) is an entropy solution iff \(\mu_t\) is the EVI gradient flow of \(E(\mu)\) (as follows, for example, from the stability of entropy solutions and EVI gradient flows when \(\beta \to \infty\); see [11] for further results).

7. Relations to stability properties of gradient flows in the deterministic setting

Let us start by stressing that, in the completely deterministic setting (i.e. Setting 1 in the introduction of the paper) the theory of Wasserstein gradient flows has certainly been used before to establish mean field limits going beyond the classical setting when \(F\) is locally Lipchitz continuous. Indeed, as explained in [25], as soon as the energy functional \(E(\mu)\) is lsc and has the property that

- \(E(\mu)\) is \(\lambda\)-convex along generalized geodesics in the \(L^2\)-Wasserstein space \(P_2(\mathbb{R}^D)\)
- The gradient flow of \(E(\mu)\) preserves particles, i.e. it preserves discrete measures of the form \(\delta_x\)

then the existence of a mean field limit follows directly from the \(\lambda\)-contractivity of the gradient flow of \(E\) on \(P_2(\mathbb{R}^D)\). In particular, as shown in [25], these assumption are satisfied if \(w\) is convex on all of \(\mathbb{R}^D\) and even (and in particular locally Lip continuous), for example when \(W(x) = |x|^{1+\alpha}, \alpha \geq 0\). Such a potential is always attractive. However, even if the pair interaction potential \(w\) is Lipschitz continuous, i.e. \(F\) is bounded, the property of preservation of particles fails when \(F\) is repulsive. Moreover, in the strongly singular case the gradient flow of \(E\) never preserves particles (since \(E(\mu) = \infty\) when \(\mu\) is discrete). In our approach this problem is bypassed by instead working with the \(N\)-particle mean energy functional \(E_N\) on the Wasserstein space \(P_2(\mathbb{R}^D)\). On the other hand in the purely deterministic setting, there is also an alternative approach using the following stability result in [11] (Thm 11.1.2) (see also [21]) for gradient flows on \(P_2(X)\) for \(X\) the Euclidean space \(\mathbb{R}^D\) (or more generally a Hilbert space).

**Theorem 7.1.** [11] (Stability) Suppose that \(\Phi_N\) and \(\Phi\) are functionals on \(P_2(X)\) which are \(\lambda\)-convex along generalized geodesics and such that \(\Phi\) is uniformly coercive. Let \(\mu_N(t)\) and \(\mu(t)\) be the corresponding EVI gradient flows in \(P_2(X)\) emanating from \(\mu_{N,0}\) and \(\mu_0\), respectively. If \(\mu_{N,0} \to \mu\) in \(P_2(X)\), as \(N \to \infty\) and \(\limsup_{N \to \infty} \Phi_N(\mu_{N,0}) < \infty\), then \(\mu_{N,0}(t) \to \mu(t)\) in \(P_2(X)\) for any positive time \(t\).

**Remark 7.2.** If \(\Phi(\mu) < \infty\) one can dispense with the assumption \(\limsup_{N \to \infty} \Phi_N(\mu_{N,0}) < \infty\) in the previous theorem, using the definition of \(\Gamma\)-convergence together with the contractivity property, as in the Step 2 in the proof of Theorem 3.2. Moreover, when \(D = 1\) it is enough to assume that \(\Phi_N\) and \(\Phi\) are \(\lambda\)-convex along ordinary geodesics in \(P_2(\mathbb{R})\), using that the latter space is Euclidean.

Now set \(D = 1\) and consider a sequence of symmetric, i.e. \(S_N\)-invariant, lsc functions \(E_N\) on the \(N\)-particle space \(\mathbb{R}^N\) which are \(\lambda\)-convex on the fundamental domain \(\{x_1 < x_2 < \ldots < x_N\}\) of the \(S_N\)-action, as in Section [11]. Using the embedding \(\delta_N\) of \(\mathbb{R}^N/S_N\) in \(P_2(\mathbb{R})\) we can identify \(E_N\) with a sequence of functionals on \(P_2(\mathbb{R})\) set to be equal to \(\infty\) on the complement of \(\delta_N(\mathbb{R}^N/S_N)\). As
observed in Section 4.4 $E(N)$ is convex on the quotient space $\mathbb{R}^N/S^N$ and hence, since $\delta_N$ is an isometry and its image is geodesically closed, the corresponding function on $\mathcal{P}_2(\mathbb{R})$ is also $\lambda-$convex. In order to apply the previous theorem to pair interactions one can then invoke the following result from [72, Prop 2.8, Remark 2.19] (similar results are used to establish large deviations of the corresponding Gibbs measures; see [2] and references therein):

**Proposition 7.3.** [72] Let $E(N)$ be the $N-$point interaction energy on $(\mathbb{R}^D)^N$ associated to a translationally invariant radial pair interaction $W(x,y)(:=w(|x-y|))$ such that $W \in L^1(\mathbb{R}^{2D})$ and $w(x)$ is monotone in the radial direction and positive when $|x| \leq 1$. Then the corresponding functionals on $\mathcal{P}_2(\mathbb{R}^D)$ converge to $E_W(\mu)$.

Assuming that the initial measure $\mu_0$ has the property that $E(\mu_0) < \infty$ Theorem 7.1 combined with the previous proposition thus implies the existence of a mean field limit $\mu_t$ of the corresponding deterministic systems, as in Section 4.11. But the advantage of our general convergence results in Theorem 3.2 when applied to the purely deterministic setting, is that it allows $E(\mu_0) = \infty$ and moreover there is no need to establish the $\Gamma-$convergence of the interaction energies $E(N)$. Indeed, the convergence assumption 1 in Section 3.1 for the corresponding mean energy functional $E(N)$ on $\mathcal{P}_2(\mathbb{R}^D)$ is almost trivially satisfied for any pair interaction (or more generally, for any $m-$point interaction).

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