On conformal measures and harmonic functions
for group extensions

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Abstract
We prove a Perron-Frobenius-Ruelle theorem for group extensions of topological Markov chains based on a construction of \( \sigma \)-finite conformal measures and give applications to the construction of harmonic functions.

Keywords Group extension, conformal measures, harmonic functions

1 Introduction

The Perron-Frobenius-Ruelle theorem is a statement about the maximal eigenvalue of an operator \( L \) who preserves the cone of positive functions. Namely, it provides existence of a function \( f \) in this cone and \( v \) in its dual, such that \( L f = \rho f \) and \( L^* v = \rho v \), with \( \rho \) referring to the spectral radius of \( L \). The first result of this type was obtained by Perron in [21] as a byproduct of his analysis of periodic continued fractions. He proved that, for a strictly positive \( n \times n \)-matrix \( A \), the maximal eigenvalue \( \rho \) is simple. Moreover, his proof reveals that there exist strictly positive vectors \( x, y \in \mathbb{R}^n \) such that \( x^t A = \rho x^t \), \( A y = \rho y \) and that \( \rho^{-n} A^n \) converges to \( y \cdot x^t \). Even though there were many important contributions to the theory of positive operators in the following decades, e.g. by Doeblin-Fortet ([10]) or Birkhoff ([4]), whose methods are today standard tools in proving exponentially fast convergence of the iterates (see, e.g., [3, 17]), it was only at the end of the 60’s when Ruelle obtained an analog of Perron’s theorem for the one-dimensional Ising model with long range interactions from mathematical physics ([25, Theorem 3]). In the context of dynamical systems, as observed by Bowen, the result of Ruelle has the following formulation in terms of shift spaces. For a fixed \( k \in \mathbb{N} \), let

\[
\Sigma := \{(x_i : i \in \mathbb{N}) : x_i \in \{1, \ldots, k\} \quad \forall i \in \mathbb{N}\},
\]

\[
\theta : \Sigma \to \Sigma \quad (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)
\]

and suppose that \( \varphi : \Sigma \to (0, \infty) \) is a log-Hölder continuous function with respect to the shift metric (for details, see below). The associated operator, the Ruelle operator, is then defined by

\[
L_\varphi(f)(x) := \sum_{\theta(y) = x} \varphi(y) f(y).
\]
Ruelle’s theorem states that there exist a strictly positive Hölder continuous function $h$ and a probability measure $\mu$ such that $L^\rho(f) = \rho f \cdot L^\rho_\mu = \rho \mu$ and $\lim \rho^n L^\rho_\mu f = \int f \, d\mu \cdot h$. Furthermore, $\rho^{-n} L^\rho_\mu f \to \int f \, d\mu \cdot h$ converges exponentially fast, which implies, among many other things, that $h$ is unique and that the measure $\mu$ is exact (and, in particular, ergodic).

The aim of this note is to establish an analogue of the result for dynamical systems of the form

$$T: \Sigma \times G \to \Sigma \times G, (x, g) \mapsto (\theta x, g \psi(x)),$$

where $G$ is a discrete group $G$, $\Sigma$ a shift space with the b.i.p.-property as defined below and $\psi: \Sigma \to G$ a locally constant function. This kind of dynamical system is called group extension, or, as they first were considered by Rokhlin in [24], Rokhlin transformation. Even though one might be tempted to think of $\psi$ as a cocycle, the probably most fruitful approach is to consider $T$ as a kind of random walk on $G$. That is, by fixing a potential function $\varphi$ which only depends on the first coordinate, $\varphi(x)$ stands for the transition probability to go from $(x, g)$ to $(\theta x, g \psi(x))$, which is also reflected by the fact that the Ruelle operator $L^\rho_\psi$ associated to $T$ has many similarities to the Markov operator of a random walk.

In this setting, it is possible to obtain the following operator theorem which is the main result (Theorem 5.1) of this note. Under a technical condition (which is satisfied if, e.g., $\Sigma$ is compact), it is shown that there exists a Lipschitz continuous map $\mu \to \nu_\mu$ from the space of probability measures on $\Sigma \times G$ to the space of $\sigma$-finite, conformal measures, that is $\nu_\mu$ is $\sigma$-finite and $L^\rho_\psi(\nu_\mu) = \rho \nu_\mu$. This map is constructed using the method by Denker-Urbanski in [8]. By adapting ideas of Patterson ([20]) and Sullivan ([30]) from hyperbolic geometry, one obtains by variation of $\mu$ a family of strictly positive, $\rho$-harmonic functions. In here, we refer to $h: \Sigma \times G \to \mathbb{R}$ as harmonic or $\rho$-harmonic if $L^\rho(h) = \rho h$. In particular, theorem 5.1 gives rise to families of $\sigma$-finite, conformal measures $\{\nu_\mu\}$, $\rho$-harmonic, positive functions $\{h\}$, and $T$-invariant measures $\{h d\nu_\mu\}$. Furthermore, as the conformal measures are pairwise equivalent, the function $k(\mu, z) := (d\nu_\mu/d\nu)(z)$ for a fixed conformal reference measure $\nu$ is defined and, as shown in theorem 5.1, its logarithm is locally Lipschitz continuous with respect to the first coordinate and $T$-invariant with respect to the second.

It is important to point out that these families might not be one-dimensional. For example, the classical and general result of Zimmer in [33] (see [14] for a version for group extensions) states that ergodicity implies amenability. Hence, as $G$ not necessarily is amenable, $\nu_\mu$ might not be ergodic and the standard argument for uniqueness of conformal measures no longer is applicable. However, for the setting in here, a sharp criterium of classical flavour holds (Proposition 5.3). That is, $\nu_\mu$ is conservative and ergodic if and only if

$$\sum_{n=1}^{\infty} \rho^{-n} \sum_{T^n(x, id) = (x, id)} \prod_{k=0}^{n-1} \varphi(\theta^k(x)) = \infty.$$

Hence, it is of interest to analyse the families of conformal measures and harmonic functions if $T$ is non-ergodic. In order to have explicit examples at hand, we use the fact that a random walk with independent increments can be identified with a group extension. For $T$ associated with the random walk on $\mathbb{Z}^d$ or the free group $F_d$, the $d$-dimensional central limit theorem and the local limit theorem by Gerl and Woess in [12], respectively, allow to explicitly deter-
mine \( \mathcal{C} \). For these specific examples, it turns out that the family of conformal measures is one-dimensional for \( \mathbb{Z}^d \) and non-trivial for \( \mathbb{F}_d \).

For a further analysis of the general setting, these conformal measures are employed to construct a positive map from the space of functions \( \mathcal{C} \) whose logarithm is uniformly continuous to the space of harmonic functions \( \mathcal{H} \) satisfying a certain local Lipschitz condition (Theorem 6.1). Then, in order to at least roughly determine the behaviour of harmonic functions and \( \nu \) at infinity, further ideas from probability and ergodic theory are employed. Namely, for a given pair \((h,\nu)\) of a positive harmonic function and a conformal measure, \( h d\nu \) is invariant and therefore, the natural extension of \((T, h d\nu)\) is well-defined. Therefore, through Martingale convergence, it is possible to show (Corollary 6.3) for \( G \) non-amenable and under a symmetry condition that

\[
\nu_\mu(\{(x, \psi(\omega) \cdots \psi(\theta^{n-1}(\omega)) : x \in \Sigma\}) = o(n^n),
\]

for a.e. \( \omega \in \Sigma \) with respect to the equilibrium measure of \((\Sigma, \theta, \rho)\). These results also have a canonical application to the dimension theory of graph directed Markov systems, which is outlined in theorem 7.1.

## 2 Topological Markov chains

We begin with defining the basic object of our analysis, that is topological Markov chains and their group extensions. For a countable alphabet \( \mathcal{W} \) and a matrix \((a_{ij} : i, j \in \mathcal{W})\) with \( a_{ij} \in \{0, 1\} \) for all \( i, j \in \mathcal{W} \) and no rows and columns equal to 0, let the pair \((\Sigma, \theta)\) denote the associated one-sided topological Markov chain given by

\[
\Sigma := \{(w_k : k = 1, 2, \ldots) : w_k \in \mathcal{W}, a_{w_k w_{k+1}} = 1 \forall i = 0, 1, \ldots\},
\]

\[
\theta : \Sigma \to \Sigma, \theta : (w_k : k = 1, 2, \ldots) \mapsto (w_k : k = 2, 3, \ldots).
\]

A finite sequence \( w = (w_1 w_2 \ldots w_n) \) with \( n \in \mathbb{N} \), \( w_k \in \mathcal{W} \) for \( k = 1, 2, \ldots, n \) and \( a_{w_k w_{k+1}} = 1 \) for \( k = 1, 2, \ldots, n-1 \) is referred to as admissible or as word of length \( n \), the set of words of length \( n \) will be denoted by \( \mathcal{W}^n \) and the set

\[
[w] := \{(v_k) \in \Sigma : w_k = v_k \forall k = 1, 2, \ldots, n\}
\]

is referred to as a cylinder of length \( n \). Furthermore, \(|w|\) denotes the length of a word and \( \mathcal{W}^\infty = \bigcup_{n=1}^{\infty} \mathcal{W}^n \) the set of all admissible words. Since \( \theta^n : [w] \to \theta^n([w]) \) is a homeomorphism, observe that the inverse \( \tau_w : \theta^n([w]) \to [w] \) is well defined.

As it is well known, \( \Sigma \) is a Polish space with respect to the topology generated by cylinders and \( \Sigma \) is compact with respect to this topology if and only if \( \mathcal{W} \) is a finite set. Moreover, the topology generated by cylinders is compatible with the metric defined by, for \( r \in (0, 1) \) and \((w_k), (v_k) \in \Sigma,
\]

\[
d_r([w_k], [v_k]) := r^{\min(i : w_i \neq v_i)}. \]

Observe that with respect to this definition, the \( r^n \)-neighbourhood of \((w_k) \in \Sigma\) is given by the cylinder \([w_1 w_2 \ldots w_n]\) of length \( n \). Also recall that \( \Sigma \) is topologically transitive if for all
If $a, b \in \mathcal{W}$, there exists $n \in \mathbb{N}$ such that $\theta^n([a]) \cap [b] \neq \emptyset$ and is topologically mixing if for all $a, b \in \mathcal{W}$, there exists $N \in \mathbb{N}$ such that $\theta^n([a]) \cap [b] \neq \emptyset$ for all $n \geq N$. Moreover, a topological Markov chain is said to have big images and big preimages if there exists a finite set $\mathcal{S}_{\text{big}} \subset \mathcal{W}$ such that for all $v \in \mathcal{W}$, there exists $\beta_1, \beta_2 \in \mathcal{S}_{\text{big}}$ such that $(v\beta_1) \in \mathcal{W}^2$ and $(\beta_2 v) \in \mathcal{W}^2$. Finally, we say that a topological Markov chain satisfies the big images and preimages (b.i.p.) property if the chain is topologically mixing and has big images and preimages (see [27]). Note that the b.i.p. property coincides with the notion of finite irreducibility for topological mixing topological Markov chains as introduced by Mauldin and Urbanski ([18]).

**Potentials.** A further basic object for our analysis is a fixed, strictly positive function $\varphi : \Sigma \to \mathbb{R}$ which is referred to as a potential. This function might be seen as weight on the preimages of a point and in many applications, $\varphi$ is defined as the conformal derivative of an underlying iterated function system. For $n \in \mathbb{N}$ and $w \in \mathcal{W}^n$, set $\Phi_n := \prod_{k=0}^{n-1} \varphi \circ \theta^k$ and $\Phi_w := \Phi_n \circ \tau_w$. We refer to $\varphi$ as a potential of (locally) bounded variation if

$$\sup \left\{ \frac{\Phi_n(x)}{\Phi_n(y)} : n \in \mathbb{N}, w \in \mathcal{W}^n, x, y \in [w] \right\} < \infty.$$ 

From now on, for positive sequences $(a_n), (b_n)$ we will write $a_n \ll b_n$ if there exists $C > 0$ with $a_n \leq C b_n$ for all $n \in \mathbb{N}$, and $a_n \asymp b_n$ if $a_n \ll b_n \ll a_n$. For example, the above could be rewritten by $\Phi_{|w|}(x) \asymp \Phi_{|w|}(y)$ for all $w \in \mathcal{W}^\infty$ and $x, y \in [w]$. A further, stronger assumption on the variation is related to local Hölder continuity. Recall that the $n$-th variation of a function $f : \Sigma \to \mathbb{R}$ is defined by

$$V_n(f) = \sup \{|f(x) - f(y)| : x_i = y_i, i = 0, 1, 2, \ldots, n-1\}.$$ 

The function $f$ is referred to as a locally Hölder continuous function if there exists $0 < r < 1$ and $C \geq 1$ such that $V_n(f) \ll r^n$ for all $n \geq 1$. Moreover, we refer to a locally Hölder continuous function with $\|f\|_\infty < \infty$ as a Hölder continuous function. We now recall a well-known estimate. For $n \leq m$, $x, y \in [w]$ for some $w \in \mathcal{W}^m$, and a locally Hölder continuous function $f$,

$$|\sum_{k=0}^{n-1} f \circ \theta^k(x) - f \circ \theta^k(y)| \ll \frac{1}{1 - r} r^{m-n}.$$  

In particular, if $\log \varphi = f$ is locally Hölder continuous, then $\varphi$ is a potential of bounded variation. Moreover, as $r^{m-n} = d(\theta^n(x), \theta^n(y))$, there exists $C_\varphi \geq 1$ such that

$$|\Phi_w(x)/\Phi_w(y) - 1| \leq C_\varphi d(x, y) \quad \text{and} \quad \Phi_w(x)/\Phi_w(y) \leq C_\varphi$$

for all $w \in \mathcal{W}^\infty$ and $x, y \in [w]$.

**Conformal measures.** In here, due to the fact that the constructions canonically lead to $\sigma$-finite measures, we will make use of a slightly more general definition of conformality by allowing infinite measures. We refer to a $\sigma$-finite Borel measure $\mu$ as a $\varphi$-conformal measure if

$$\mu(\theta(A)) = \int_A \frac{1}{\varphi} d\mu$$
for all Borel sets $A$ such that $\theta|_A$ is injective. For $w = (w_1 \ldots w_n) \in \mathcal{W}^n$ and a potential of bounded variation, it then immediately follows that

$$\mu([w]) = \Phi_n(x) \mu(\theta([w_n]))$$

(2)

for all $x \in [w]$. Note that this estimate implies that $P_1(\theta, \varphi) = 0$ is a necessary condition for the existence of a conformal measure with respect to a potential of bounded variation. Moreover, if $\mu(\theta([w])) \approx 1$ (e.g., if $\mu$ is finite and $\theta$ has the big image property), we obtain that

$$\mu([w]) = \Phi_n(x)$$

(3)

for all $n \in \mathbb{N}$, $w \in \mathcal{W}^n$ and $x \in [w]$. Also note that a probability measure satisfying (3) is referred to as a $\varphi$-Gibbs measure.

**Ruelle’s operator, b.i.p. and Gibbs-Markov maps.** Ruelle’s operator is defined, for $f : \Sigma \rightarrow \mathbb{R}$ in a suitable function function space to be specified later, by

$$L_{\varphi}(f) = \sum_{v \in \mathcal{W}} 1_{\theta([v])} \cdot \varphi \circ \tau_v \cdot f \circ \tau_v.$$  

Furthermore, there is an associated action on the space of $\sigma$-finite Borel measures defined through $\int f \, dL_{\varphi}(v) := \int L_{\varphi}(f) \, dv$, for each continuous $f : \Sigma \rightarrow [0, \infty)$. We then have that $\nu$ is a $\varphi/\rho$ conformal measure if and only if $L_{\varphi}^*(\nu) = \rho \nu$. If, in addition, there is a measurable function $h : \Sigma \rightarrow [0, \infty)$ with $L_{\varphi}(h) = \rho h$, then $d\mu := hd\nu$ defines an invariant, $\sigma$-finite measure, that is $\mu = \mu \circ \theta^{-1}$. Moreover, for $\varphi' := \varphi h/(\rho h \circ \theta)$, we have $L_{\varphi'}(1) = 1$.

An important consequence of the b.i.p. property is a Perron-Frobenius-Ruelle theorem in case of an infinite alphabet $\mathcal{W}^1$ (see [18, 27]). That is, if $(\Sigma, \theta)$ has the b.i.p. property, $\log \varphi$ is Hölder continuous and $\|L_{\varphi}(1)\|_{\infty} < \infty$, then there exists a Gibbs measure $\mu$ and a Hölder continuous, strictly positive eigenfunction $h$ of $L_{\varphi}$, which is uniformly bounded from above and below. Moreover, in this situation, $(\Sigma, \theta, \mu)$ has the Gibbs-Markov property, that is $\mu$ is a Borel probability measure, for all $w \in \mathcal{W}^1$, $\mu$ and $\mu \circ \tau_w$ are equivalent, $\inf \{\mu(\theta([w])) : w \in \mathcal{W}^1\} > 0$ and there exists $0 < r < 1$ such that, for all $m, n \in \mathbb{N}, v \in \mathcal{W}^m, w \in \mathcal{W}^n$ with $(vw) \in \mathcal{W}^{m+n}$,

$$\sup_{x, y \in [w]} \left| \log \frac{d\mu \circ \tau_v}{d\mu}(x) - \log \frac{d\mu \circ \tau_v}{d\mu}(y) \right| \ll r^n.$$  

(4)

As it is well known, the action on the space of bounded continuous functions of the transfer operator with respect to $\mu$ coincides with $L_{\varphi/\rho}$ and, with $h$ referring to the function given by the Perron-Frobenius-Ruelle, $hd\mu$ is an invariant probability measure with exponential decay of correlations and associated transfer operator given by $L_{(\varphi h)/(\rho h \circ \theta)}$ (see [3, 27]).

Furthermore, several arguments in here are based on an inequality in the flavour of Doeblin-Fortet or Lasota-Yorke for arbitrary topological Markov chains $(\Sigma, \theta)$ and potentials $\varphi$ such that is $\log \varphi$ is locally Hölder continuous. For $f : \Sigma \rightarrow \mathbb{R}$, define

$$D(f) : \Sigma \rightarrow [0, \infty), (x_1, x_2, \ldots) \mapsto \sup_{y, \tilde{y} \in [x_1]} \left| \frac{f(y) - f(\tilde{y})}{d_r(y, \tilde{y})} \right|.$$
That is, \( D(f)(x) \) is the local Hölder coefficient of the function \( f \) restricted to \([a], \) with \( x \in [a] \).

Now assume that \( L^n_\varphi(f) \) is well-defined. Then, for \( x, y \) in the same cylinder,

\[
\left| L^n_\varphi(f)(x) - L^n_\varphi(f)(y) \right| \\
\leq C_\varphi L^n_\varphi(|f|)(x) d_r(x, y) + r^n L^n_\varphi(D(f))(y) d_r(x, y) \\
\leq C_\varphi d_r(x, y) L^n_\varphi(|f| + r^n D(f))(x) (5)
\]

If, in addition, for all \( a \in \mathcal{W}^1 \), either \( f(x) = 0 \) for all \( x \in [a] \) or \( |f(x)/f(y) - 1| \leq C_a d_r(x, y) \) for all \( x, y \in [a] \), set \( LD(f)(x) := 0 \) in the first case and \( LD(f)(x) := \sup|f(x)/f(y) - 1|d_r(x, y) : x, y \in [a] \) in the second case. By the same arguments,

\[
\left| L^n_\varphi(f)(x) - L^n_\varphi(f)(y) \right| \\
\leq C_\varphi d_r(x, y) L^n_\varphi(|f|)(x) + C_\varphi \sum_{v \in \mathcal{W}^n} \Phi_v(x) \left| f \circ \tau_v(x) \right| - 1 \\
\leq C_\varphi d_r(x, y) L^n_\varphi(|f| + r^n LD(f))(x) (6)
\]

### 3 Group extensions of topological Markov chains.

Fix a countable group \( G \) and a map \( \psi : \Sigma \to G \) such that \( \psi \) is constant on \([w] \) for all \( w \in \mathcal{W}^1 \).

Then, for \( X := \Sigma \times G \) equipped with the product topology of \( \Sigma \) and the discrete topology on \( G \), the group extension or \( G \)-extension \((X, T)\) of \((\Sigma, \theta)\) is defined by

\[
T : X \to X, (x, g) \to (\theta x, g \psi(x)).
\]

Note that \((X, T)\) is a topological Markov chain with respect to the alphabet \( \mathcal{W}^1 \times G \) and the following transition rule: \(((a, g), (b, h))\) is admissible if and only if \((ab) \in \mathcal{W}^2 \) and \( g\psi(a) = h \), where \( \psi(a) := \psi(x) \), for some \( x \in [a] \). Furthermore, set \( X^g := \Sigma \times \{g\} \) and

\[
\psi_n(x) := \psi(x) \psi(\theta x) \cdots \psi(\theta^{n-1} x)
\]

for \( n \in \mathbb{N} \) and \( x \in \Sigma \). Observe that \( \psi_n : \Sigma \to G \) is constant on cylinders of length \( n \) which implies that \( \psi_k(w) := \psi_k(x) \), for some \( x \in [w] \), \( k \leq n \) and \( w \in \mathcal{W}^n \), is well defined. If \( k = n \), we will write \( \psi_w := \psi_n(w) \). It is then easy to see that the finite words of \((X, T)\) can be identified with \( \mathcal{W}^\infty \times G \) by

\[
((w_0, \ldots, w_n), g) \equiv ((w_0, g), (w_1, g \psi_1(w)), \ldots, (w_n, g \psi_n(w))).
\]

Also observe that topologically transitivity of \((X, T)\) implies that \( \{\psi(a) : a \in \mathcal{W}^1\} \) is a generating set for \( G \) as a semigroup.

Throughout, we now fix a topological mixing topological Markov chain \((\Sigma, \theta)\), and a topological transitive \( G \)-extension \((X, T)\). Furthermore, we fix a (positive) potential \( \varphi : \Sigma \to \mathbb{R} \) with
distinguish between the Ruelle operator of \( \theta \) and \( a \) be written in calligraphic letters. That is, for \( v \) given by \( \phi \) \( \tau \), that is \( \tau_v(x, g) := (\tau_v(x), g\psi(v)^{-1}) \). In order to distinguish between the Ruelle operator of \( \theta \) and \( T \), these objects for the group extension will be written in calligraphic letters. That is, for \( a \in \mathcal{H}, \xi \in [a] \times \{id\}, (\eta, g) \in \mathcal{X}, \) and \( n \in \mathbb{N} \),

\[
\mathcal{L}(f)(\xi, g) := \sum_{v \in \mathcal{H}} \phi(\tau_v(\xi)) f \circ \tau_v(\xi, g).
\]

**Remark 3.1** In the context of topological transitivity, it is natural to ask whether \((X, T)\) is ergodic with respect to the product of the Gibbs measure on \( \Sigma \) and the counting measure. For example, a classical result of Zimmer in [33] (see also [14]) states that ergodicity of \((X, T)\) implies that \( G \) is amenable, that is, there exists a sequence \((K_n)\) of finite subsets of \( G \) with \( \bigcup_n K_n = G \) such that

\[
\lim_{n \to \infty} \frac{|gK_n \Delta K_n|}{|K_n|} = 0 \quad \forall g \in G,
\]

where \( \Delta \) refers to the symmetric difference and \( |\cdot| \) to the cardinality of a set. Moreover, it was shown in [29] for this class of extensions that \( P_G(T) = P_G(\theta) \) implies that \( G \) is amenable. Hence, if \( G \) is a non-amenable group, then \( P_G(T) < P_G(\theta) \) and \((X, T)\) is not ergodic. In particular, by bounded distortion, \( T \) has to be totally dissipative. For a further criterion for ergodicity, we also refer to corollary 5.3 below. Also note that the classical result of Varopoulos on recurrent groups motivates the conjecture that a group extension only can be ergodic if \( G \) is a finite extension of the trivial group, \( \mathbb{Z} \) or \( \mathbb{Z}^2 \).

**Symmetric extensions.** In several interesting applications, group extensions are satisfying a certain notion of symmetry. In here, we will use a pathwise notion (as in [29]) in contrast to the more general notion in [13]. Namely, we say that \((\Sigma, \theta, \psi)\) is symmetric if there exists \( \mathcal{X}^1 \to \mathcal{X}^1, w \mapsto w^\dagger \) with the following properties.

(i) For \( w \in \mathcal{X}^1, (w^\dagger)^\dagger = w \).

(ii) For \( \nu, w \in \mathcal{X}^1, \) the word \((\nu w)\) is admissible if and only if \((w^\dagger \nu^\dagger)\) is admissible.

(iii) \( \psi(\nu^\dagger) = \psi(\nu)^{-1} \) for all \( \nu \in \mathcal{X}^1 \).

Moreover, we refer to \((\Sigma, \theta, \psi, \phi)\) as a symmetric group extension if \((\Sigma, \theta, \psi)\) is symmetric and, with \( \dagger : \mathcal{X}^\infty \to \mathcal{X}^\infty \) defined by \((w_1 \ldots w_n)^\dagger := (w_n^\dagger \ldots w_1^\dagger)\),

\[
\sup_{n \in \mathbb{N}} \sup_{x \in [w_i], y \in [w_j]} \Phi_n(x) \Phi_n(y) < \infty.
\]

**4 Conformal \( \sigma \)-finite measures**

As a first step towards a Ruelle theorem for group extensions, we now adapt ideas from [20, 8] in order to obtain invariant measures for the dual of the Ruelle operator. In contrast to [20, 8],
the method in here gives rise to conformal $\sigma$-finite measures, which seems to be advantageous as group extensions in many cases are totally dissipative dynamical systems and therefore might not admit finite invariant measures. We now fix $\xi \in \Sigma$ and, for $n \in \mathbb{N}$, set

$$Z^n(\xi) = \sum_{\theta^n(x) = \xi, \nu_n(x) = \id} \Phi_n(x) = Z^n_B(1_{X_{id}})(\xi, i d).$$

Since the construction relies on the divergence of a power series at its radius of convergence, recall that, for a sequence of positive real numbers $(a_n)$, the radius of convergence of $\sum_n a_n x^n$ is equal to $1/\rho$ where, by Hadamard’s formula,

$$\rho := \limsup_{n \to \infty} \sqrt[n]{a_n}.$$

We now ensure divergence at the radius of convergence by pointwise multiplication with a slowly diverging sequence as given by the following result. For the proof, we refer to [8].

**Lemma 4.1.** For a positive sequence $(a_n)$ with $\rho < \infty$, there exists a nondecreasing sequence $(b_n : n \in \mathbb{N})$ with $b_n \geq 1$ for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} b_n / b_{n+1} = 1$ and for all $s \geq 0,$

$$\sum_{n=1}^{\infty} b_n a_n s^{-n} \begin{cases} = \infty & s \leq \rho \\ < \infty & s > \rho. \end{cases}$$

Moreover, there exists a non-increasing sequence $(\lambda(n) : n \in \mathbb{N})$ with $\lambda(n) \geq 1$ and $\lambda(n) \to 1$ such that $b_n = \prod_{k=1}^{n} \lambda(k)$.

Now suppose that $\rho = \limsup_{n \to \infty} \sqrt[n]{Z^n(\xi)} < \infty$. Then, for $(b_n)$ given by Lemma 4.1 applied to $a_n = Z^n(\xi)$, we have that

$$\mathcal{P}(s) := \sum_{n \in \mathbb{N}} s^{-n} b_n Z^n(\xi).$$

diverges as $s \searrow \rho$. Furthermore, for $\rho < s < \infty$, set

$$m_{\mathbb{R}} := \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N}} s^{-n} b_n \sum_{\tau^n(z) = (\xi, i d)} \Phi_n(x) \delta_z,$$

where $\delta_z$ refers to the Dirac measure supported in $z$. Note that, by construction, $m_{\mathbb{R}}(X_{id}) = 1$ for all $s > \rho$. In order to construct a $\sigma$-finite, conformal measure, we consider an accumulation point $\nu$ of $(m_{\mathbb{R}})$ in the weak* topology, i.e. convergence of $\int f \, dm_{\mathbb{R}}$ to $\int f \, d\nu$ for every bounded and continuous function $f$. For ease of notation, we now identify $\Sigma$ with $X_{id}$ and, for $B \subset \Sigma$ with $T^k_{|B\times\{i d\}}$ invertible and $T^k(B \times \{i d\}) \subset X_{id}$, the restriction $T^k_{|B\times\{i d\}}$ with $\theta^k_{|B}$.

**Lemma 4.2.** Assume that, for $s_1 \searrow \rho$, there exists a probability measure $m$ on $\Sigma$ which is the weak* limit of $(m_{n_l}) : l \in \mathbb{N})$. Then, for each pair $(B, k)$ with $B \in \mathcal{B}(\Sigma)$, $k \in \mathbb{N}$ such that $T^k_{|B\times\{i d\}}$ is invertible and $T^k(B \times \{i d\}) \subset X_{id}$,

$$m(\theta^k(B)) = \int_{B} \rho^k / \Phi_k \, d m.$$
Proof. Suppose that $B$ is a cylinder, that is $B = [w]$ for some $w \in \mathcal{W}^m$ and $m > k$. Since $T^k$ is injective on $B \times \{id\}$, we have, for $s > \rho$, that

$$m_s(\theta^k(B)) = \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N} \cap \theta^n(B) \cap E_n} \frac{b_n \Phi_n(x)}{s^n} = \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N} \cap B \cap \theta^{-1}(E_n)} \frac{b_n \Phi_n(\theta^k x)}{s^n}$$

$$= \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N} \cap B \cap E_{n+k}} \frac{b_{n+k} \Phi_{n+k}(x)}{s^{n+k}} \frac{b_n s^k}{b_{n+k} \Phi_k(x)}$$

In particular this gives

$$\left| m_s(\theta^k(B)) - \int_B \frac{s^k}{\Phi_k(x)} \, dm_s \right| \leq \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N}} \left| \frac{b_n}{b_{n+k}} - 1 \right| \sum_{x \in B \cap E_{n+k}} \frac{b_{n+k} \Phi_{n+k}(x)}{s^{n+k}}$$

$$+ \frac{1}{\mathcal{P}(s)} \sum_{k=1}^{\infty} \sum_{n \in \mathbb{N} \cap B \cap E_n} b_n \Phi_{k-n}(\theta^n(x)) s^{k-n}.$$ 

By Lemma 4.1, it follows that $\lim \mathcal{P}(s_1) = \infty$, and hence the second term of the right hand side tends to zero as $l \to \infty$. Since $\lim_{n \to \infty} b_n/b_{n+k} = 1$, we then obtain that the first summand also tends to zero. Moreover, by applying the Portmanteau theorem to the open and closed set $[w]$, it follows that (8) holds for $[w]$. As $\mathcal{P}(\Sigma)$ is generated by cylinders, the lemma follows. \qed

As it seems to be impossible to show the existence of a weak* accumulation point of $(m_s)$ in full generality, the following condition is introduced.

**Definition 4.3.** We say that the group extension $(\Sigma, \theta, \mu)$ satisfies property (C) if there exists $(b_n)$ as in Lemma 4.1 and $(s_k)$ with $s_k \searrow \rho$ such that $(m_{s_k})$ converges weakly* to some probability measure on $X_{id}$ as $k \to \infty$.

In order to obtain criteria for property (C), recall that Prohorov’s theorem states that a sequence $(m_{g_k})$ has a weak* accumulation point if and only if for each $\varepsilon > 0$ there exists a compact set $K$ and $k_0 \in \mathbb{N}$ such that $m_{g_k}(K) \geq 1 - \varepsilon$ for all $k \geq k_0$, or in other words, if $(m_{g_k})$ is tight. In particular, if $\Sigma$ is a subshift of finite type, then the property is always satisfied. By lifting the limit from $\Sigma$ to $X$ as in [9, 7] we arrive at a conformal, not necessarily finite measure for $T$.

**Theorem 4.4.** Assume that $(\Sigma, \theta)$ satisfies the b.i.p. property, $\log \varphi$ is Hölder continuous, $\|L_\varphi(1)\|_\infty < \infty$ and that $(X, T)$ be a topologically transitive group extension with property (C). Then there exists a $\sigma$-finite, nonatomic, $(\rho/\varphi)$-conformal measure $\nu$ with $\nu(X_g) < \infty$, for each $g \in G$. Furthermore, there exists a sequence $(s_k)$ with $s_k \searrow \rho$, such that, for each non-negative, continuous function $f : X \to \mathbb{R}$,

$$\int f \, d\nu = \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n \in \mathbb{N}} b_n s_k^{-n} (\mathcal{L}_\varphi^n f)(\xi, id). \tag{9}$$

Before giving the proof, recall that the conditions on $(\Sigma, \theta, \varphi)$ are equivalent to the existence of a probability measure $\mu$ such that $(\Sigma, \theta, \mu)$ is a Gibbs-Markov map with the b.i.p. property. Hence, the above theorem holds in verbatim for topologically transitive group extensions of Gibbs-Markov maps, with $\mu$ playing the rôle of a reference measure.
Proof. By property (C), there exists \( s_k \searrow \rho \) and \( m \) such that \( m \) is the weak* -limit of \( (m_s : k \in \mathbb{N}) \). Using equation (3) in Lemma 4.2, we extend \( m \) to a measure \( \nu \) on \( \mathcal{B}(X) \) as follows. For \( b \in \mathcal{W}^1 \) and \( g \in G \), there exists by transitivity \( j \in \mathbb{N} \) and \( u \in \mathcal{W}^{j+1} \) with \( T^j([u, id]) = [b, g] \). The restriction of \( \nu \) on \([b, g] \) is now defined by

\[
\int_{[b, g]} f(x) d\nu(x, h) := \int_{[u]} f \circ \theta^j \rho^j / \Phi_j d\nu \]

for each bounded and continuous function \( f : \Sigma \to \mathbb{R} \). In particular, if \( f \) is supported on \([b] \), then by the same arguments as in the proof of Lemma 4.2,

\[
\int_{G_b} f(x) d\nu(x, h) = \int_{[u]} f \circ \theta^j \rho^j / \Phi_j d\nu \\
= \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n\in\mathbb{N}} b_n \rho^j s_k^n \sum_{x \in E_n \cap [u]} f \circ \theta^j (x)(\Phi_j(x))^{-1} \cdot \Phi_n(x) \\
= \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n>j} b_n^{-j} s_k^{-n} \sum_{(y, g) \in T^{-j-n}([\xi, id]) \cap [b, g]} \Phi_{n-j}(y) f(y).
\]

This proves equation (3). Finally, using the construction of \( \nu \) from \( m \) and the big preimages property, it easily can be seen that \( \nu(X_\rho) < \infty \) for each \( g \in G \). \( \square \)

We now collect several immediate consequences from conformality and the b.i.p.-property in the base.

**Proposition 4.5.** For the measure \( \nu \) given by Theorem 4.4, the following holds.

(i) If \( \lim_{n} \mathcal{L}^n(\xi) \rho^{-n} = 0 \), then \( \nu(X) = \infty \).

(ii) If \( L_{\varphi}(1) = 1 \), then \( d\nu \circ T^{-1} = \rho^{-1} d\nu \).

(iii) For \( w \in \mathcal{W}^n \), \( x \in [w] \) and \( g \in G \), we have \( \rho^n \nu([w, g]) \times \Phi_n(x) \nu(X_{g\psi_n(x)}) \).

(iv) If the extension is symmetric, then

\[ \nu(X_\rho) = \nu(X_{g^{-1}}), \nu([w^1, \psi_w^{-1} g^{-1} \psi_w]) = \nu([w, g]). \]

**Proof.** The first assertion follows from (3) applied to \( f = 1 \). In order to prove part 2, note that \( L_{\varphi}(1) = 1 \) implies that \( \mathcal{L}_{\varphi}(1) = 1 \). Hence, for \( f \in L^1(\nu) \), we have

\[
\frac{1}{\mathcal{P}(s)} \sum_{n\in\mathbb{N}} b_n s^{-n} (\mathcal{L}_{\varphi}^n f \circ T)(\xi, id) = \frac{1}{\mathcal{P}(s)} \sum_{n\in\mathbb{N}} b_n s^{-n} (\mathcal{L}_{\varphi}^{n-1} f)(\xi, id) \\
= \frac{s^{-1}}{\mathcal{P}(s)} \sum_{n\in\mathbb{N}} b_n b_{n-1} s^{-n+1} (\mathcal{L}_{\varphi}^{n-1} f)(\xi, id).
\]

Since \( \mathcal{P}(s) \to \infty \) as \( s \to \rho \) and \( \lim_{n} b_n/b_{n-1} = 1 \) as \( s \to \infty \), we obtain that \( \int f \circ T \, d\nu = \rho^{-1} \int f \, d\nu \).

Part 3 is a consequence of conformality and the b.i.p. property. Namely, by (2),

\[ \rho^n \nu([w, g]) \times \Phi_n(x) \nu(\{\theta^n([w]) \times \{g\psi_n(x)\}\}) \leq \Phi_n(x) \nu(X_{g\psi_n(x)}). \]
Furthermore, by the big images property, there exists \( a \in \mathcal{A}_{\text{bip}} \) such that \( |a| \in \mathcal{B}^{|w|}(w) \). Hence, it remains to show that \( v(|a, h|) \approx v(X_h) \) for all \( h \in G \). By the big preimages property, for each \( y \in \Sigma \), there exists \( b \in \mathcal{A}_{\text{bip}} \) such that \( y \in \theta([b]) \). Hence, by transitivity of \( T \), there exists a finite word \( w \) such that \( awb \) is admissible and \( \psi_{awb} = id \). Hence, \( v(|a, h|) \geq v(|awb, h|) = v(\theta([b]) \times \{h\}) \) with respect to a constant only depending on \( awb \), which implies that

\[
|\mathcal{A}_{\text{bip}}|v(|a, h|) \gg \sum_{b \in \mathcal{A}_{\text{bip}}} v(\theta([b]) \times \{h\}) \geq v(X_h).
\]

The proof of the remaining assertion relies on a similar argument. For each \( w \in \mathcal{W}^\infty \) with \( \psi_w = g \) and \( \xi \in \theta^{[w]}(w) \), there exists by transitivity a finite word \( u \) such that such that \( wu \) is admissible, \( \xi \in \theta^{[wu]+[w]}(w^u) \) and \( \psi_u = id \). As \( \mathcal{A}_{\text{bip}} \) is finite, \( u \) can be chosen from a finite set. Hence, by the definition of \( v \) and the symmetry of \( \varphi \), we have \( v(X_{g^{-1}}) \ll v(X_g) \) which implies that \( v(X_{g^{-1}}) \approx v(X_g) \). The second assertion follows from this and part 3. \( \square \)

5 The Ruelle-Perron-Frobenius theorem for group extensions

In order to prove the existence of eigenfunctions for the Ruelle operator, we make use of a well-known idea from hyperbolic geometry (see [20, 30]): As the reference point for the construction in theorem 4.4 was chosen arbitrarily, there exists a family \( \{v_\xi : \xi \in X\} \) of conformal measures. It is then relatively easy to show that \( \xi \mapsto d\nu_\xi/d\nu \) defines an eigenfunction, provided that \( \{v_\xi\} \) is a family of pairwise equivalent measures. In here, this approach is partially generalized by constructing a conformal measure \( \nu_\mu \) for a given probability measure \( \mu \) on \( X \). In order to do so, recall that the Vaserstein distance \( W \) of two probability measures \( \mu, \tilde{\mu} \) is a metric compatible with the weak convergence and is equal to, by Kantorovich’s duality,

\[
W(\mu, \tilde{\mu}) = \sup \left\{ \int f d(\mu - \tilde{\mu}) : D(f) \leq 1 \right\},
\]

where \( D(f) := \sup |f(\xi) - f(\tilde{\xi})|/d(\xi, \tilde{\xi}) : \xi, \tilde{\xi} \in X \) denotes the Lipschitz coefficient with respect to the metric defined by \( d((x, g), (y, g)) = d_r(x, y) \) and \( d((x, g), (x, h)) = 1 \) for \( g \neq h \). In the following theorem, \( v \) refers to the \( \sigma \)-finite, conformal measure on \( X \) given by theorem 4.4 with respect to some fixed base point in \( X_{id} \).

Theorem 5.1. Let \((X, T)\) be a topologically transitive group extension with property (C) of a Gibbs-Markov map with the b.i.p. property. Then there exists a sequence \( (s_k) \) with \( s_k \searrow \rho \) such that for each \( \mu \in \mathcal{M}(X) \),

\[
v_\mu := \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n \in \mathbb{N}} b_n s_k^{-n} (\mathcal{L}_\mu^R)^*(\mu)
\]

exists. Furthermore, \( \{v_\mu : \mu \in \mathcal{M}(X)\} \) is a family of pairwise equivalent measures and for the Radon-Nikodym derivative \( \mathcal{K} : \mathcal{P}(X) \times X \to \mathbb{R}, (\mu, z) \mapsto (d\nu_\mu/d\nu)(z) \), we have the following.

(i) There exists \( D > 0 \) such that for \( \nu \)-a.e. \( z \in X, g \in G \) and probability measures \( \mu_1, \mu_2 \) supported on \( X_g \),

\[
|\log \mathcal{K}(\mu_1, z) - \log \mathcal{K}(\mu_2, z)| \leq DW(\mu_1, \mu_2).
\]
(ii) For all $\mu \in \mathcal{M}(X)$, we have $\mathcal{H}(\mathcal{L}_\phi^*(\mu), z) = \rho \mathcal{H}(\mu, z)$ for $v$-a.e. $z$.

(iii) For each $\mu \in \mathcal{M}(X)$, the map $\mathcal{H}(\mu, \cdot)$ is $T$-invariant, that is $\mathcal{H}(\mu, z) = \mathcal{H}(\mu, T(z))$ for $v$-a.e. $z \in X$. In particular, if $T$ is ergodic with respect to $v$, then $\nu_\mu$ is a multiple of $v$, $\mathcal{H}(\mu, z)$ is constant with respect to $z$ and $\{\nu_\mu : \mu \in \mathcal{M}(X)\}$ is one-dimensional.

Remark 5.2 Before giving the proof, we discuss a relation to Ruelle’s operator theorem. Namely, by considering the restriction $X \to \mathcal{M}_0(X)$, $x \to \nu_x := \nu_{\delta_x}$, part (ii) of the above implies that $h_z : x \mapsto \mathcal{H}(\delta_x, z)$ satisfies $\mathcal{L}_\phi(h_z) = \rho h_z$. Hence, the above gives rise to the construction of a family of $\sigma$-finite, conformal measures $\{\nu_\mu : \mu \in \mathcal{M}(X)\}$ and a family of eigenfunctions $\{h_z : z \in X\}$. If $v$ is ergodic, these families are one dimensional, that is, they are subsets of $\{(tv : t > 0) \text{ and } (x \to tv_X(X_{id}) : t > 0)\}$, respectively.

Proof. We begin with the construction of $\nu_\mu$ for the case that $\mu$ is a Dirac measure $\delta_\zeta$. So assume that $\zeta \in E(\zeta) := \bigcup_{n \in \mathbb{N}} T^{-n}([\zeta, id])$, for $\zeta \in \Sigma$ and define, for a non-negative, continuous function $f : X \to \mathbb{R}$ and $s > \rho$,

$$m^s_\zeta(f) := \sum_{n \in \mathbb{N}} b_n s^{-n}(\mathcal{L}_\phi^nf)(\zeta),$$

where $\mathcal{P}(s)$ is given by (7). It follows from property (C) that $m^s_\zeta$ restricted to functions on $X_{id}$ defines a tight family of measures, and hence that, for a suitable subsequence $(s_{k_j} : j \in \mathbb{N})$ of $(s_k)$ given by property (C),

$$m_\zeta(f) := \lim_{j \to \infty} \sum_{n \in \mathbb{N}} b_n s_{k_j}^{-n}(\mathcal{L}_\phi^nf)(\zeta)$$

exists for each non-negative, continuous function $f : X \to \mathbb{R}$. In particular, $m_\zeta$ defines a measure. Since $E(\zeta)$ is countable it is moreover possible to choose the subsequence $(s_{k})$ such that the limit in (11) exists for all $\zeta \in E(\zeta)$ and $f$ non-negative and continuous. Moreover, as $\lim_{n} b_n/b_{n+1} = 1$ for each $k \in \mathbb{N}$, it follows that

$$m_\zeta(f) = \rho^{-k} \sum_{\nu \in \mathcal{W}^k, \zeta \in \mathcal{W}^k(\nu)} \Phi_k(\tau_\nu(\zeta)) m_{\nu(\zeta)}(f) = \rho^{-k} \mathcal{L}_\phi^k m_\zeta(f).$$

Hence, for $\zeta$ with $T^n(\zeta) = (\zeta, id)$, it follows from (12) that, for each Borel set $A$, $m_{\zeta(\zeta)}(A) = \nu(A) \geq \rho^{-n} \Phi_\zeta m_\zeta(A)$. On the other hand, it follows from transitivity that there exist $\nu \in \mathcal{W}^m$ and $m \in \mathbb{N}$ such that $\tau_\nu(\zeta)$ and $(\zeta, id)$ are in the same cylinder. Hence, by combining the above argument with bounded distortion, we obtain that

$$\rho^{-m} \Phi_\zeta^{-1} m_\zeta(A) \gg \nu(A) \geq \rho^{-n} \Phi_\zeta m_\zeta(A).$$

In particular, the measures are equivalent and the Radon-Nikodym derivative $\mathcal{H}(\zeta, \cdot) := dm_\zeta/d\nu$ exists and is a.s. strictly positive.

We now prove that $m_\zeta(A) \asymp m_\zeta(A)$ whenever the second coordinates coincide, that is $\zeta_1, \zeta_2 \in E(\zeta) \cap X_g$ for some $g \in G$. In order to do so, assume that $\zeta_2 \in [a, g]$ for some $a \in \mathcal{S}_\zeta$. By the b.i.p.-property, there exist $b \in \mathcal{S}_\zeta$ and $h \in G$ such that $\zeta_1 \in T([b, h])$ and by transitivity a finite word $w$ such that $awb$ is admissible with $\psi_{awb} = id$. As above, it follows that
$\Phi_{awb}(x) m_{\zeta_2}(A) \ll m_{\zeta_1}(A)$ for any $x \in [awb]$. Hence, as $\mathcal{A}_{\text{bip}}$ is finite, $m_{\zeta_1}(A) \ll m_{\zeta_1}(A)$ with respect to a constant which does not depend on $\zeta_1$ and $a \in \mathcal{A}_{\text{bip}}$.

In order to prove the opposite direction, for each $b \in \mathcal{A}_{\text{bip}}$ choose $x_b \in [b]$. Also note that, for each $v \in \mathcal{W}^1$ with $\zeta_1 \in \theta([v] \times [g])$, there exists $b(v) \in \mathcal{A}_{\text{bip}}$ such that $vb(v)$ is admissible. As $\varphi(\tau_v(\zeta_1)) = \varphi(\tau_v((xb(v), g)))$, we have by the above that

$$m_{\zeta_1}(A) = \rho^{-1} \sum_{v: x_b \in \theta([v] \times [g])} \varphi(\tau_v(\zeta_1)) m_{\tau_v(\zeta_1)}(A)$$

$$= \rho^{-1} \sum_{v: x_b \in \theta([v] \times [g])} \varphi(\tau_v((xb(v), g))) m_{\tau_v((xb(v), g))}(A)$$

$$\leq \sum_{b \in \mathcal{A}_{\text{bip}}} m_{((xb,v),g)}(A) \ll \mathcal{A}_{\text{bip}} | m_{\zeta_2}(A).$$

Hence, $m_{\zeta_1}(A) \times m_{\zeta_2}(A)$ which implies that

$$\sup \left\{ \frac{\mathbb{E}((x_1, g), (x_2, g) \in E(\xi), g \in G, z \in X)}{\mathbb{E}((x_1, g), (x_2, g) \in E(\xi), g \in G)} \right\} < \infty. \quad (13)$$

In order to extend $\mathbb{L}(\cdot, \cdot)$ to a globally defined function, we now show that $\zeta \mapsto \mathbb{L}(\zeta, z)$ is log Hölder. For $k, n \in \mathbb{N}, \zeta_1, \zeta_2 \in [a, g] \cap E(\xi), a \in \mathcal{W}^1, b \in \mathcal{W}^k$ with $k \leq n$ and $h \in G$, we obtain by (5) that

$$| \mathcal{L}_\varphi^n(1_{[b,h]})(\zeta_1) - \mathcal{L}_\varphi^n(1_{[b,h]})(\zeta_2) | \leq C_\varphi d(\zeta_1, \zeta_2) \mathcal{L}_\varphi^n(1_{[b,h]})(\zeta_1), \quad (14)$$

with $C_\varphi$ only depending on the Hölder constant of $\varphi$. Hence,

$$| m_{\zeta_1}^z([b,h]) - m_{\zeta_2}^z([b,h]) | \leq C_\varphi m_{\zeta_1}^z([b,h]) d(\zeta_1, \zeta_2)$$

and, by taking the limit,

$$| m_{\zeta_1}([b,h]) - m_{\zeta_2}([b,h]) | \leq C_\varphi m_{\zeta_1}([b,h]) d(\zeta_1, \zeta_2).$$

Since cylinder sets are generating the Borel algebra and are stable under intersections it follows by taking the limit as $[b,h] \to z \in X$ that $| dm_{\zeta_1}/dm_{\zeta_2}(z) - 1 | \ll d(\zeta_1, \zeta_2)$ for $\nu$-a.e. $z \in X$. Furthermore, as $\zeta_1, \zeta_2 \in X_{\varphi}$, it follows from (13) that $dm_{\zeta_1}/dm_{\zeta_2} = 1$. Hence, $|\log(dm_{\zeta_1}/dm_{\zeta_2}(z))| \ll d(\zeta_1, \zeta_2)$, which proves that the function $\zeta \mapsto \log \mathbb{L}(\zeta, z)$ is Lipschitz continuous on $E(\xi) \cap [a, g]$ with respect to a Lipschitz coefficient which is independent from $z$ and $[a, g]$. By a further application of (13), there is a uniform bound for $|\log(dm_{\zeta_1}/dm_{\zeta_2}(z))|$ which is independent from $z$ and $g$. As $E(\xi)$ is dense by transitivity, there exists a unique locally Lipschitz continuous extension of $\zeta \mapsto \log \mathbb{L}(\zeta, z)$ to $X$. By taking the exponential of this extension, we obtain a globally defined function which, for ease of notation, will also be denoted by $\mathbb{L}(\cdot, \cdot)$. As the function has the same regularity as the one defined on $E(\xi)$, we have shown that there exists $D > 0$ such that, for all $g \in G, \zeta_1, \zeta_2 \in X_{\varphi}$ and $\nu$-a.e. $z \in X$,

$$| \log \mathbb{L}(\zeta_1, z) - \log \mathbb{L}(\zeta_2, z) | \leq D d(\zeta_1, \zeta_2).$$

In order to obtain the representation (10), note that the construction of $m_{\zeta}$ through (11) extends to all $\zeta \in X$ by the estimate (14) and the fact that $E(\zeta)$ is dense in $X$. The next step is
to verify that (11) extends to an arbitrary Borel probability measure \( \mu \) on \( X \). In analogy to the above, define

\[
M_n(f) := \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N}} b_n s^{-n} \int f d(L_{\phi}^n)^*(\mu) = \frac{1}{\mathcal{P}(s)} \sum_{n \in \mathbb{N}} b_n s^{-n} \int L_{\phi}^n(f) d\mu = \int m_{\zeta}(f) d\mu(\zeta),
\]

where the last equality follows from monotone convergence. By a further application of monotone convergence, it follows that \( \lim_k M_n(f) = \int m_{\zeta}(f) d\mu \) which proves that (11) defines a measure and that \( \nu_{\mu}(f) := \int m_{\zeta}(f) d\mu \). Moreover, as

\[
\int f d\nu_{\mu} = \int m_{\zeta}(f) d\mu = \int f(z) \kappa(\zeta, z) d\nu(z) d\mu(\zeta),
\]

it follows that \( d\nu_{\mu} / d\nu = \int \kappa(\zeta, z) d\mu(\zeta) \), which will also be denoted by \( \kappa(\mu, z) \), by a slight abuse of notation. This finishes the proof of the existence of \( \nu_{\mu} \). Part 1 of the theorem then follows from the definition of \( W \) through Kantorovich’s duality.

In order to prove part 2, note that (10) implies that, for \( \zeta \in X \) and each positive and continuous function \( f \), that

\[
\int f(z) \kappa(\zeta, z) d\nu(z) = \int f d\nu_{\zeta} = \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n \in \mathbb{N}} b_n s_k^{-n} (L_{\phi}^n(f))(\zeta) \quad (16)
\]

\[
= \sum_{\nu \in \mathcal{W}} \varphi \circ \tau_\nu(\zeta) \lim_{k \to \infty} \frac{1}{\mathcal{P}(s_k)} \sum_{n \in \mathbb{N}} b_n s_k^{-n} (L_{\phi}^{n-1}(f))(\tau_\nu(\zeta))
\]

\[
= \sum_{\nu \in \mathcal{W}} \varphi \circ \tau_\nu(\zeta) \rho^{-1} \int f d\nu_{\tau_\nu(\zeta)} = \sum_{\nu \in \mathcal{W}} \varphi \circ \tau_\nu(\zeta) \rho^{-1} \int f\kappa(\tau_\nu(\zeta), \cdot) d\nu
\]

\[
= \rho^{-1} \int f(z) L_{\phi}(\kappa(\cdot, z))(\zeta) d\nu(z)
\]

where the last identity follows from monotone convergence. Hence, by (15),

\[
\rho \int f \kappa(\cdot, \cdot) d\nu = \int f(z) \int L_{\phi} \kappa(\cdot, z) d\mu(\zeta) d\nu(z)
\]

\[
= \int f(z) \int \kappa(\cdot, z) dL_{\phi}^*(\mu) d\nu(z) = \int f(\kappa(L_{\phi}^*(\mu), \cdot)) d\nu.
\]

As \( f \) is arbitrary, \( \rho \kappa(\mu, z) = \kappa(\mu, z) \) almost surely, which is part 2 of the theorem.

For the proof of part 3, note that \( X \) is a Besicovitch space and that each \( \nu_{\mu} \) is conformal. Therefore, we have for \( \nu \)-a.e. \(( (w_1), g ) \in X \), that

\[
\kappa(\mu, (w_1), g) = \lim_{n \to \infty} \frac{\nu_{\mu}(\{ (w_1 \ldots w_n), g \})}{\nu(\{ (w_1 \ldots w_n), g \})} = \lim_{n \to \infty} \rho^{-1} \int_{\{ (w_1 \ldots w_n), g \}} f d\nu_{\mu} / \int_{\{ (w_1 \ldots w_n), g \}} f d\nu.
\]

It hence follows from continuity of \( \varphi \) that \( \kappa \) is \( T \)-invariant in the second coordinate. The second statement is a standard application of the ergodic theorem.
We now give a brief characterization of the measures given by above theorem in case of an ergodic extension (as e.g. in example 1 below for \( d = 1, 2 \)). For ease of exposition, we assume that the base transformation is a Gibbs-Markov map with respect to the invariant probability \( \mu \) on \( \Sigma \). In this situation, the product measure \( \mu_G \) of \( \mu \) and the counting measure on \( G \) clearly is \( 1/\varphi \)-conformal and \( T \)-invariant, i.e. \( \mu_G = \mu_G \circ T^{-1} \). However, note that \( \mu_G \) in many cases is totally dissipative, e.g. if \( G \) is non-amenable (33[14]).

If \( T \) is conservative with respect to \( \mu_G \), then \( T \) also is ergodic and \( \sum_n Z^n_\varphi(\xi) = \infty \) (see 3[1] and the proof of proposition 5.3 below). In particular, \( \rho \) and \( \sum \) are both \( 1/\varphi \)-conformal, as observed by Sullivan, \( d\nu/d\mu_G \) exists, is \( T \)-invariant and hence constant. This then implies that the measures \( \nu_* \) are again all multiples of the product measure \( \mu_G \). If \( T \) is conservative with respect to \( \nu \) and \( \rho \leq 1 \), then the same arguments show that the measures \( \nu_* \) are again all multiples of \( \nu \). In this situation, a result by Jaerisch (14) shows that the invariant measure \( h(x, z) d\nu(x) \) is unique and is the product of another measure on \( \Sigma \) and counting measure on \( G \).

As a corollary of the existence of \( \rho^{-1} L_\varphi \)-invariant functions as shown in Remark 5.2, one obtains the following criterion of classical flavor for ergodicity.

**Proposition 5.3.** The map \( T \) is either conservative or totally dissipative with respect to \( \nu \). If \( T \) is conservative, then \( T \) is ergodic. Furthermore, \( T \) is conservative and ergodic if and only if

\[
\sum_n \rho^{-n} Z^n(\xi) = \infty.
\]

**Proof.** Observe that \( T \) is a transitive topological Markov chain and that it follows from

\[
(d\nu/d\nu \circ T)(x, g) = \rho \phi(x)
\]

that \( d\nu/d\nu \circ T \) is a potential of bounded variation. Hence, \( (T, \nu) \) is a Markov fibered system with the bounded distortion property as in [3]. In particular, \( (T, \nu) \) either is totally dissipative or conservative and if \( (T, \nu) \) is conservative, then it is ergodic. Note that \( \rho^{-1} L_\varphi \) acts as the transfer operator on \( L^1(\nu) \). It hence follows from the definition of the transfer operator that, for all \( W \) measurable and \( n \in \mathbb{N} \),

\[
\int 1_W \rho^{-n} L_\varphi^n(x, g) d\nu(x, g) = \int 1_W \circ T^n 1_{X_{id}} d\nu = \nu(T^{-n}(W) \cap X_{id}).
\]

Now assume that \( \sum_n \rho^{-n} Z^n(\xi) = \infty \). It follows from bounded variation and transitivity that the sum diverges for all \( \xi \in \Sigma \). For \( W := \{z \in X_{id} : T^n(z) \notin X_{id} \forall n \geq 1\} \), we hence have that \( \nu(W) = 0 \). Hence, the first return map

\[
T_{X_{id}} : X_{id} \to X_{id}, \ (x, id) \to T^{n_x}(x, g), \ n_x := \min\{n \geq 1 : T^n(x, id) \in X_{id}\}
\]

is well defined. By substituting \( \nu \) with an equivalent, invariant measure given by the above theorem, an application of Poincaré’s recurrence theorem gives that \( T_{X_{id}} \) is conservative. It is then easy to see that \( T \) also is conservative, and hence ergodic. The remaining assertion is a consequence of the standard result in ergodic theory, that \( T \) is ergodic and conservative if and only if \( \sum_n \rho^{-n} L_\varphi^n(f) \) diverges for all \( f \geq 0, \int f d\nu > 0 \) (see [1] Prop. 1.3.2)).

\[
\Box \quad \Box
\]
6 Harmonic functions

By applying theorem 5.1 to Dirac measures, it is possible to construct a map \( \Theta : \mathcal{C} \to \mathcal{H} \) from a subspace of the continuous functions to a subspace of \( \rho \)-harmonic functions. In here, we refer to \( f : X \to \mathbb{R} \) as \( \rho \)-harmonic if \( \mathcal{L}_\rho(f) = \rho f \). In order to define \( \mathcal{C} \), fix a reference point \( \xi_0 \in X_{id} \) and set \( v_\xi := v_{\xi_0} \). The space \( \mathcal{C} \) is now defined by

\[
\mathcal{C} := \left\{ f : X \to \mathbb{R} : v_\xi(|f|) < \infty, \lim_{n \to \infty} C_n(f) = 0 \right\},
\]

where

\[
C_n(f) := \inf \left\{ C : |f(z_1) - f(z_2)| \leq C|f(z_1)| \forall z_1, z_2 \in \{w,g\}, w \in \mathcal{W}^n, g \in G \right\}.
\]

The space might be alternatively characterised as the space of log-uniformly continuous functions with an integrability condition. Namely, if \( C_n(f) < \infty \), then for \( |w, g| \), \( w \in \mathcal{W}^n \) and \( g \in G \) either \( f|_{|w,g|} = 0 \) or \( f(z) \neq 0 \) for \( z \in \{w,g\} \). In particular, with 0/0 = 1, it follows that

\[
|f(z_1)/f(z_2) - 1| < C_n(f), \quad \forall z_1, z_2 \in \{w,g\}.
\]

Hence, if \( C_n(f) < 1 \), then \( f(z_1)/f(z_2) > 0 \), that is the sign of \( f \) is constant on \( \{w,g\} \). These arguments show that \( f \in \mathcal{C} \) if and only if \( \log f_+ \) and \( \log f_- \) are uniformly continuous, with \( f_\pm \) referring to the strictly positive and negative parts of \( f \) and \( |f| \neq 0 \) is a union of cylinders of length \( n \), for some \( n \) depending on \( f \).

In order to define \( \mathcal{H} \), recall that \( d_r \) refers to the shift metric on \( X \), with \( r \in (0,1) \) adapted to the Hölder continuity of \( \log \rho \). In order to be able to not only consider positive \( \rho \)-harmonic functions, the following coefficients for the local regularity of a function \( f : X \to \mathbb{R} \) are useful.

\[
D_r(f) := \sup \left\{ \frac{|f(x) - f(z)|}{d_r(x, z)} : d_r(x, z) < 1 \right\}
\]

\[
LD(f) := \sup \left\{ \left| \frac{f(z_1)}{f(z_2)} - 1 \right| : d_r(z_1, z_2) < 1 \right\}
\]

The space \( \mathcal{H} \) is now defined through a control of the local Lipschitz constant \( D_r(f) \) as follows.

\[
\mathcal{H}^+ := \left\{ \{f : X \to [0,\infty) : \mathcal{L}_\rho(f) = \rho f, LD(f) < \infty \right\},
\]

\[
\mathcal{H} := \left\{ \{f : X \to \mathbb{R} : \mathcal{L}_\rho(f) = \rho f, \exists h \in H^+ s.t. D_r(f) \leq h(x) \forall x \in X \right\}.
\]

The map \( \Theta \) is then defined by, for \( f \in \mathcal{C} \),

\[
\Theta(f)(z) := v_\xi(f) = \int \kappa(\delta_z, y) f(y) d\nu_\xi(y).
\]

Based on a slightly more involved version of the argument used in the proof of log-Hölder continuity of \( \kappa \) in theorem 5.1, we are now in position to prove that \( \Theta \) is well defined and that \( LD \) is always bounded by

\[
C_\rho := \sup \left\{ \left| \frac{\Phi_{\rho_\xi, r}(z_1)}{\Phi_{\rho_\xi, r}(z_2)} - 1 \right| : d_r(z_1, z_2) < 1 \right\} < \infty.
\]

**Theorem 6.1.** The map \( \Theta : \mathcal{C} \to \mathcal{H} \) is well defined. If \( f \in \mathcal{H} \) and \( f \geq 0 \), then \( LD(f) \leq C_\rho \) and, in particular, \( f \in \mathcal{H}^+ \).
Proof. Suppose that \( f \in \mathcal{C} \). By applying the arguments in \([15]\) to \( f \) shows that \( \mathcal{L}_\nu(\Theta(f)) = \rho \Theta(f) \). Hence, it remains to obtain a bound on \( D_\lambda(f) \). For ease of notation, set \( f_v := f \circ \tau_v \), for \( v \in W^k \) and \( n \in \mathbb{N} \). Suppose that \( z_1, z_2 \in [w, g] \) with \( w \in W^k \), \( g \in G \) and that \( n \) is sufficiently large such that for all \( v \in W^k \), either \( f_v(z_1) = f_v(z_2) = 0 \) or \( f_v(z_1), f_v(z_2) \neq 0 \). Setting \( 0/0 := 1 \) and \( A_n := \sup_{v \in W^k} \frac{|f_v(z_1)|}{f_v(z_2)} - 1 \), we obtain by a similar argument as in \([5]\) that

\[
\left| \mathcal{L}^n_\nu(f)(z_1) - \mathcal{L}^n_\nu(f)(z_2) \right| \\
\leq \sum_{v \in W^k} \left| \Phi_{n,v}(z_1) - \Phi_{n,v}(z_2) \right| f_v(z_1) + \sum_{v \in W^k} \left| \Phi_{n,v}(z_1) \right| f_v(z_2) - f_v(z_2)) |f_v(z_1) - f_v(z_2)| \\
\leq C_n d_{r}(z_1, z_2) \cdot \mathcal{L}^n_\nu(|f|)(z_1) + A_n \cdot \mathcal{L}^n_\nu(|f|)(z_2).
\]

Since \( \lim_{n \to \infty} A_n = 0 \), we have

\[
|\Theta(f)(z_1) - \Theta(f)(z_2)| = |v_{z_1}(f) - v_{z_2}(f)| \\
\leq C_n d_{r}(z_1, z_2) v_{z_1}(|f|) = C_n d_{r}(z_1, z_2) \Theta(|f|)(z_1).
\]

Hence, \( D_\lambda(f) \leq C_n \Theta(|f|)(z_1) \). By dividing with \( \Theta(|f|) \) and substituting \( f \) with \( |f| \), the same argument shows that \( LD(\Theta(|f|)) \leq C_n \). In particular, \( \Theta(|f|) \in \mathcal{H}^+ \). Now assume that \( f \in \mathcal{H} \) and \( f \geq 0 \). Then there exists \( \hat{h} \geq 0 \) with \( D_\lambda(h) \leq \hat{h} \) and \( \mathcal{L}_\nu(\hat{h}) = \rho \hat{h} \). By similar arguments,

\[
|h(z_1) - h(z_2)| = \rho^{-n} \left| \mathcal{L}^n_\nu(h)(z_1) - \mathcal{L}^n_\nu(h)(z_2) \right| \\
\leq \rho^{-n} \left( C_n d_{r}(z_1, z_2) \mathcal{L}^n_\nu(h)(z_1) + n \cdot d_{r}(z_1, z_2) \cdot \mathcal{L}^n_\nu(D_{\lambda}(h))(z_2) \right) \\
\leq C_n d_{r}(z_1, z_2) \left( h(z_1) + n \cdot \hat{h}(z_2) \right).
\]

Since \( n \) is arbitrary and \( r \in (0, 1) \), \( LD(h) < C_n \). \( \Box \) \( \Box \)

The classical Martin boundary of a random walk on a group is a quotient of the space of paths, where two paths \((g_k), (h_k)\) in \( G \) are identified if \( \lim_k K(\cdot, g_k) = \lim_k K(\cdot, h_k) \), where \( K \) refers to the Martin kernel (see, e.g., \([32]\) ). In the context of group extensions, the natural candidate for a path in \( G \) is given by \((\psi_k(x))_k\), for some \( x \in \Sigma \), whereas the function \((z, g) \mapsto v_z(X_g)/v_0(X_g)\) might serve as the analogue of the Martin kernel.

Here, the situation is different. Assume that \((x, g) = ((w_k), g) \in X \). Using the conformality of \( v \) in proposition \([4, 5]\) we have by theorem \([5, 1]\) that, for \( f_n := 1_{[w_1 \ldots w_n, g]}/v_0([w_1 \ldots w_n, g]) \),

\[
v_z(X_{g\psi_n(x)})/v_0(X_{g\psi_n(x)}) = \Theta(f_n)(z) \xrightarrow{n \to \infty} \kappa(z, x, g).
\]

### 6.1 Natural extensions and immediate implications

In order to obtain information on the asymptotic behavior of elements of \( \mathcal{H} \), we now employ ideas from the theory of Markov processes, which are similar but somehow dual to the ones for Markov maps. Namely, in order to obtain a stochastic process associated with \((X, T)\), we
consider the process with transition probability \( (dm \circ \tau_x / dm)(x) \) for transitions from \( x \) to \( \tau_x(x) \), where \( m \) is an \( T \)-invariant measure. Hence, the appropriate object are the left-infinite sequences with respect to an invariant measure \( h \) constructed from \( m \). That is, the stochastic process is the left half of the natural extension of \((X, T, m)\) whose construction in case of an underlying shift space we recall now. Set

\[
Y := \{(w_i, g_i) : i \in \mathbb{Z} : w_i \in \mathcal{W}, g_i \in G, a_{w_i}w_{i+1} = 1, g_{i+1} = g_i \psi(w_i)\},
\]

\[
S : Y \to Y, ((w_i, g_i)) \to ((w'_i, g'_i)), \text{ with } w'_i = w_{i+1}, g'_i = g_{i+1} \forall i \in \mathbb{Z}.
\]

In other words, \( S \) is the left shift on the two sided shift space \( Y \). The cylinder sets of \( Y \) are given by,

\[
\{((w_0, g_0) \cdots (w_n, g_n))\}_k := \{((v_i, h_i)) \in Y : (v_{k+j}, h_{k+j}) = (w_j, g_j), \text{ for } j = 0, \ldots, n\}.
\]

If \( m \) is \( T \)-invariant, then \( \hat{m}(\{(w_0, g_0) \cdots (w_n, g_n))\}_k) := m(\{(w_0, w_1, \ldots, w_n)\}) \) defines a measure \( \hat{m} \) on \( Y \). As it easily can be seen, we then have, for

\[
\pi : Y \to X, ((w_i, g_i)) \to ((w_i : i \geq 0), g_0),
\]

that \( \pi \circ S = T \circ \pi, \hat{m} = m \circ \pi^{-1} \), \( S \) is invertible, \( \hat{m} \) is \( S \)-invariant and \((Y, S, \hat{m})\) is minimal in the sense that the \( \sigma \)-algebra \( \mathcal{F} \) generated by the cylinder sets of \( Y \) is generated by \( \{S^n(\pi^{-1}(\mathcal{B})) : n \in \mathbb{Z}\} \), with \( \mathcal{B} \) referring to the \( \sigma \)-algebra generated by the cylinder sets of \( X \). In particular, \((Y, S, \mathcal{F}, \hat{m})\) is the natural extension of \((X, T, \mathcal{B}, m)\) (see, e.g., [6]).

Observe that there are several canonical choices for the invariant measure \( m \). Either \( \mu \) is \( \theta \)-invariant and \( m \) is the product \( \mu_A \) of \( \mu \) and the counting measure on \( G \), or \( dm = hd\nu_0 \), for some \( h \in \mathcal{H}^+ \). However, in both cases, it is possible to identify martingales with respect to the filtration \( (\mathcal{F}_n : n \in \mathbb{N}) \), where \( \mathcal{F}_n := S^n \circ \pi^{-1}(\mathcal{B}) \). We begin with the analysis of \((Y, S)\) with respect to \( \hat{m}_G \).

**Proposition 6.2.** Suppose that \( \mu \) is \( \theta \)-invariant and that \( h \in \mathcal{H}^+ \). Then, for \( \hat{m}_G \), a.e. \( z \in Y \),

\[
h_\infty(z) := \lim_{n \to \infty} \rho^{-n} h \circ \pi \circ S^{-n}(z)
\]

exists. If \( \rho < 1 \), then \( h_\infty = 0 \), and if \( \rho = 1 \), then \( h_\infty = h_\infty \circ S \) and \( h_\infty < \infty \) a.s.

**Proof.** Set \( W_n := \rho^{-n} h \circ \pi \circ S^{-n} \). Since \( \hat{m}_G \) is \( S \)-invariant, \( \int f \circ T g d\mu_G = \int f \mathcal{L}(g) d\mu_G \) and \( \mathcal{L}_\psi(h) = \rho h \), we have for all \( A \in \mathcal{B} \) that

\[
\int_{S^n(\pi^{-1}(A))} \mathcal{E}(W_n+1|\mathcal{F}_n) d\hat{m}_G
\]

\[
= \rho^{-n-1} \int 1_A \circ \pi \circ S^{-n} h \circ \pi \circ S^{-n-1} d\hat{m}_G
\]

\[
= \rho^{-n+1} \int 1_A \circ T h d\mu_G = \rho^{-n} \int 1_A h d\mu_G
\]

\[
= \rho^{-n} \int 1_A \circ \pi \circ S^{-n} h \circ \pi \circ S^{-n} d\mu_G = \int_{S^n(\pi^{-1}(A))} W_n d\hat{m}_G.
\]
Hence, $\mathbb{E}(W_{n+1} | \mathcal{F}_n) = W_n$ and $(W_n, \mathcal{F}_n)$ is a positive martingale. In particular, $h_\infty := \lim_n W_n$ by Doob’s convergence theorem. As it easily can be verified, we have $h_\infty = \rho h_\infty \circ S$. Furthermore, by Fatou’s Lemma and the martingale property, $\int_{\mathbb{R}} h_\infty d\tilde{\mu} \leq \int_{\mathbb{R}} h d\mu$ for all measurable sets $A \subset \mathcal{X}$, which implies that $h_\infty < \infty$ a.s.

By applying the proposition to $\Theta(1_{X_0})$, we obtain the decay of $\nu$ along $\mu$-a.e. path as $n \to -\infty$. If the extension is symmetric, the result also transfers to paths with $n \to \infty$.

**Corollary 6.3.** If $\rho < 1$ and $\mu_G$ is invariant, then, for $\mu_G$-a.e. $(\{w_i\}, g) \in Y$,

$$\lim_{n \to \infty} \nu_0(X_{g, w_1, \ldots, w_n}) / \rho^n = 0, \quad \lim_{n \to \infty} \frac{\nu_0([w_{-n}, \ldots, w_{-1}, g])}{\mu([w_{-n}, \ldots, w_{-1}])} = 0.$$  

Moreover, if the group extension is symmetric, then for $\mu$-a.e. $x \in \Sigma$ and $g \in G$,

$$\lim_{n \to \infty} \nu_0(X_{\psi_n(x)}) / \rho^n = 0, \quad \lim_{n \to \infty} \frac{\nu_0([w_1, \ldots, w_n, \psi_n(x) g \psi_n(x)^{-1}])}{\mu([w_1, \ldots, w_n])} = 0.$$  

**Proof.** The first two assertions follow from $\nu_{1, x, g}(X_{1, id}) = \nu_{0, x, g}^{-1}$ and (iii) of proposition 4.5 whereas the last two assertions are a consequence of the fact that $Y \to Y$, $(\{w_i\}, g) \to (\{w_i^+\}, g)$ is a non-singular automorphism.

By considering the natural extension of the invariant version of $\nu_0$, we obtain a further convergence. That is, as the measure $dm_h := hd\nu_0$ is $T$-invariant, there exists a unique extension to an invariant, $\sigma$-finite, $S$-invariant measure $\hat{m}_h$ on $Y$. The analogue of proposition 6.2 is as follows.

**Proposition 6.4.** Suppose that $f, h \in \mathcal{H}$, $h > 0$ such that $\|f / h\|_\infty < \infty$. Then, for $\hat{m}_h$-a.e. $z \in Y$,

$$\Xi_h(f)(z) := \lim_{n \to \infty} \frac{f \circ \pi \circ S^{-n}(z)}{h \circ \pi \circ S^{-n}(z)}$$  

exists, $\Xi_h(f) \circ S = \Xi_h(f)$ and $\mathbb{E}_{\hat{m}_h}(\Xi_h(f) | \mathcal{F}_0) = f \circ \pi / h \circ \pi$. Moreover, for the signed invariant measure $\hat{m}_f$, we have $d\hat{m}_f / d\hat{m}_h = \Xi_h(f)$.

**Proof.** The proof that $(f \circ \pi \circ S^{-n} / h \circ \pi \circ S^{-n} | \mathcal{F}_n)$ is a bounded martingale and is the same as above and therefore omitted. Hence, $\Xi_h(f)$ is well defined and by bounded convergence, we have for $A \in \mathcal{B}$ and $k \in \mathbb{N}$ that

$$\int_{S^k \pi^{-1}(A)} \Xi_h(f) d\hat{m}_h = \lim_{n \to \infty} \int_A 1_A \circ \pi \circ S^{-k} \frac{f \circ \pi \circ S^{-n}}{h \circ \pi \circ S^{-n}} d\hat{m}_h = \lim_{n \to \infty} \int_A 1_A \circ T^{-k} f d\nu_0 = \int_A f d\nu_0 = m_f(A) = \hat{m}_f(S^k \pi^{-1}(A)).$$

Since $\mathcal{F}$ is generated by $\{\mathcal{F}_n : n \in \mathbb{N}\}$, we have $\Xi_h(f) d\hat{m}_h = d\hat{m}_f$. The remaining assertion in the conditional expectation is a consequence of the above for $k = 0$. 

\[\boxed{}\]
7 Applications and examples

The construction of conformal measures has the following application to conformal graph directed Markov systems. In order to have a zero of the pressure function, we have to assume that there exists \( h > 0 \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \| L_{\psi}^n(1) \| = 1, \quad \| L_{\psi}^n \| < \infty
\]  

are satisfied. It follows from standard arguments that the expression on the left hand side of (17), seen as a function of \( h \), is continuous and strictly decreasing to 0 on its domain of definition. Hence, if there exists \( h' \) such that the left hand side of (17) is finite and greater than or equal to 1 and (18) holds, then there exists a zero of the pressure function. In the context of graph directed Markov systems, this property is known as strong regularity (see [19]). Furthermore, if \( |\mathcal{W}| < \infty \), then this is true for \( h' = 0 \), and in particular there always exists a zero of the pressure function in this case.

Now let \( \delta \) be given by (17) and set \( \rho_{\delta} := \exp(P_{G}(\theta, \varphi^\delta)) \geq 1 \). It then follows from the Ruelle-Perron-Frobenius theorem for systems with the b.i. p. property (see, e.g., [27]) that there exists a \( \rho_{\delta}/\varphi^\delta \)-conformal probability measure \( \mu_{\delta} \) and a Hölder continuous function \( h_{\delta} \) with \( L_{\psi}^n(h_{\delta}) = \rho_{\delta} h_{\delta} \) such that \( \theta \) has the Gibbs-Markov property with respect to the invariant measure given by \( h_{\delta} d \mu_{\delta} \). As an application of Theorem 4.4 and Proposition 4.5 we obtain that there exists a \( \sigma \)-finite measure \( \nu \) on \( X \) which is \( 1/\varphi^\delta \)-conformal, and which satisfies, for \( w \in \mathcal{W}^n \) and \( x \in [w] \),

\[
\nu([w, g]) \asymp \varphi^\delta_n(x) \nu(X_{g \psi_n(x)}).
\]  

Theorem 7.1. Assume that the group extension is symmetric and that property (C), (17) and (18) are satisfied. Then, for \( \mu_{\delta} \)-a.e. \((w_k) \in \Sigma\),

\[
\lim_{n \to \infty} \frac{\log \nu([w_1 \cdots w_n, i d])}{\log \varphi_n(x)} = \delta + \frac{P_{G}(\theta, \varphi^\delta)}{\int (\log \varphi) h_{\delta} d \mu_{\delta}}.
\]

Moreover, the group \( G \) is amenable if and only if the above limit is equal to \( \delta \). If \( G \) is non-amenable, then, for \( \mu_{\delta} \)-a.e. \((w_k) \in \Sigma\),

\[
\lim_{n \to \infty} \rho_{\delta} \frac{\nu([w_1 \cdots w_n, i d])}{(\varphi_n(x))^\delta} = 0.
\]

Before giving the proof, we sketch a straight forward application to conformal dynamical systems. Namely, if \( \Sigma \) is given by a conformal iterated function system, the inverse branch \( \tau_w \) corresponds to a conformal map and \( \Phi_{\mid \mathcal{W}} \circ \tau_w \) to its conformal derivative. In this situation, the above limit can be identified with the \( \nu \)-dimension \( \dim_{\nu} \) of the support of \( \mu_{\delta} \). Hence, with \( H(h_{\delta} d \mu_{\delta}) \) referring to the entropy of \( h_{\delta} d \mu_{\delta} \), it follows from the variational principle that

\[
\dim_{\nu}(\text{supp}(\mu_{\delta})) = \delta + \frac{P_{G}(\theta, \varphi^\delta)}{\int (\log \varphi) h_{\delta} d \mu_{\delta}} = 2 \delta + \frac{H(h_{\delta} d \mu_{\delta})}{\int (\log \varphi) h_{\delta} d \mu_{\delta}}.
\]

Moreover, note that in many regular situations, \( \delta \) is equal to the Hausdorff dimension \( \dim(K) \) of the attractor \( K \) of the iterated function system. In this situation, the amenability of \( G \) is equivalent to \( \dim_{\nu}(\text{supp}(\mu_{\delta})) = \dim(K) \).
of theorem\cite{7,2}. By symmetry and proposition \cite{6,2}, \( \lim_{n} (\log v(X_{\varphi_n(x)})) / n = \log \rho \delta \). Hence, by \cite{19},

\[
\log(v([w_n, i d])) = \delta + \lim_{n \to \infty} \frac{\log v(X_{\varphi_n(x)})}{\log \Phi_n(x)} = \delta + \lim_{n \to \infty} \frac{P_G(\theta, \varphi^n)}{\log \Phi_n(x)}. 
\]

The above limit exists by application of the ergodic theorem. The amenability criterion is an immediate corollary of Kesten’s criterion for group extensions in \cite{29}, where it is shown that \( P_G(\theta, \varphi^n) = 0 \) if and only if \( G \) is amenable. For the remaining assertion, note that \( \rho \delta < 1 \) by non-amenability. The assertion then follows from corollary \cite{6,3}.

In order to have concrete examples of the \( \sigma \)-finite measure at hand, we give two examples from probability theory, where known local limit theorems give rise to explicit expressions.

**Example 1** The first example is Polya’s random walk on \( \mathbb{Z}^d \). Choose \((p_i \in (0, 1) : i \in \{ \pm 1, \ldots, \pm d \})\) with \( \sum_{i=1}^{d} (p_i + p_{-i}) = 1 \) and consider the random walk on \( \mathbb{Z}^d \) with transition probabilities \( P(\varphi e_i) = p_{\varphi i} \), where \( e_i \) refers to the \( i \)-th element of the canonical basis of \( \mathbb{Z}^d \).

This random walk has an equivalent description through the following group extension. Let \( \Sigma \) be the full shift with \( 2d \) symbols \( \{-d, \ldots, -1, 1, \ldots, d \} \) and \( \varphi \) the locally constant function defined by \( \varphi|_{\{ \pm i \}} := p_{\pm i} \). Note that \( \sum_{i=1}^{d} (p_i + p_{-i}) = 1 \) implies \( L_\varphi(1) = 1 \). Moreover, it is well known that the measure defined by \( \mu([i_1 \ldots i_n]) := p_{i_1} \cdots p_{i_n} \) is \( \bar{\theta} \)-invariant, ergodic and \( 1/\varphi \)-conformal. The associated group extension is defined through

\[
\psi : \Sigma \to \mathbb{Z}^d \quad (i_1 i_2 \cdots) \mapsto \begin{cases} e_{i_1} : i_1 > 0 \\ -e_{-i_1} : i_1 < 0 \end{cases}.
\]

As \( \Sigma \) is the full shift and \( \varphi \) is constant on cylinders, it follows from the construction that \( v_{(x,g)} = v_{(y,g)} \) for all \( x, y \in \Sigma \) and \( g \in G \). Therefore, we only will write \( v_{\varphi} \) for \( v_{(x,g)} \). In order to apply known local limit theorems from probability theory, observe that

\[
\mathcal{L}^\rho_n(1_{X_{t \varphi}})(x, g) = \sum_{u \in \mathcal{W}^n : \psi_n(u) = g} \phi_n(\tau_w(x)) = P(X_n = g),
\]

where \( X_n = h \) refers to the random walk at time \( n \) started in the identity with distribution \((p_i)\) and \( P \) to the probability of the associated Markov process. By the local limit theorem for Polya’s random walk \cite{31 theorem 13,12}, we have that, for \((k_1, \ldots, k_d) \in \mathbb{Z}^d \) and \( n \in \mathbb{N} \) such that \( n - (k_1 + \cdots + k_d) \) is even,

\[
P(X_n = (k_1, \ldots, k_d)) \sim C n^{-d/2} \left( 2 \sum_{i=1}^{d} \sqrt{p_{i} p_{-i}} \right)^n \prod_{i=1}^{d} \left( \sqrt{p_i / p_{-i}} \right)^{k_i}.
\]

Hence, \( \rho = 2 \sum_{i=1}^{d} \sqrt{p_i p_{-i}} \) and, with \( \lambda_i := \sqrt{p_i / p_{-i}} \),

\[
\mathcal{L}^\rho_n(1_{X_{(k_1, \ldots, k_d)}})(x, i d) \sim C n^{-d/2} \rho^n \prod_{i=1}^{d} \lambda_i^{-k_i}.
\]

Recall that a random walk is called symmetric if \( p_i = p_{-i} \) for all \( i = 1, \ldots, d \). The estimate then implies that \( \rho = 1 \) if and only if the random walk is symmetric. Furthermore, by proposition

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the term $n^{-d/2}$ implies that the group extension is ergodic and conservative with respect to $v$ if and only if $d = 1$ or $d = 2$. It is remarkable that this conclusion is independent of symmetry. In order to determine $v_{id}$ explicitly, note that the local limit theorem implies that

$$v_{id}(X_{(k_1, \ldots, k_d)}) = \lim_{k \to \infty} \frac{\sum_{n \in \mathbb{N}} b_n s_k^{-n}(\mathcal{L}_d^{id})_1(X_{(k_1, \ldots, k_d)})(x, i)}{\sum_{n \in \mathbb{N}} b_n s_k^{-n}(\mathcal{L}_d^{id})_1(X_{(i, i, \ldots, i)})(x, i)} = \prod_{i=1}^d \lambda_{ki}.$$ 

Using conformality then gives that, for a cylinder $[(i_1, \ldots, i_n), z]$ in $\Sigma \times \mathbb{Z}^d$, 

$$v_{id}([(i_1 \ldots i_n), z]) = \rho^{-n} p_{i_1} \cdots p_{i_n} v_{id}(X_{z+\psi(a(i_1 \ldots i_n)}) = \rho^{-n} p_{i_1} \cdots p_{i_n} v_{id}(X_n) \prod_{k=1}^n \lambda_{i_k}^{-1} = \rho^{-n} v_{id}(X_n) \prod_{k=1}^n \sqrt{p_{i_k} p_{-i_k}} = \frac{1}{2^n} v_{id}(X_n) \prod_{k=1}^n \sqrt{p_{i_k} p_{-i_k}}.$$ 

In particular, the last term in (20) reveals the local symmetry

$$v_{id}([(i_1 \ldots i_k \ldots i_n), z]) = v_{id}([(i_1 \ldots -i_k \ldots i_n), z]), \quad (k \in 1, \ldots, n),$$

whereas globally, the measure is multiplicative with respect to the last component, that is

$$v_{id}([(i_1 \ldots i_n), z_1 + z_2]) = v_{id}([(i_1 \ldots i_n), z_1]) v_{id}([(i_2 \ldots i_n), z_2]).$$

Furthermore, (20) implies that the the function $\mathcal{L}_d^{id}$ from Theorem 5.1 is given by

$$\mathcal{L}_d^{id}(\delta_{(\lambda, \bar{g})}, (y, h)) = \frac{dV_g}{dV_{id}}(y, h) = v(X_g).$$

These considerations might be summarized as follows. If $\varphi$ is symmetric, then $\rho = 1$ and $v(X_g) = 1$ for all $g \in \mathbb{Z}^d$. If $\varphi$ is not symmetric, then $\rho < 1$ and $|v_{id}(X_g) : g \in \mathbb{Z}^d|$ neither is bounded from below nor from above. Moreover, the function $h$ defined by $h(x, g) := v_g(X_{id})$ is an $\mathcal{L}_d^{id}$-proper function by remark 5.2. Therefore, $dm := hdv$ is $T$-invariant. However, as it easily can be verified, $m(X_g) = 1$ for all $g \in \mathbb{Z}^d$ and, in particular, $m$ is the measure associated to the symmetric random walk with transition probabilities $P(\pm e_i) = \sqrt{p_{i_k} p_{-i_k}}/(2 \sum_k \sqrt{p_{i_k} p_{-i_k}})$.

**Example 2** In this example, we replace the group $\mathbb{Z}^d$ with the free group $F_d$ with $d$ generators $g_1, \ldots, g_d$. As above, the transition probabilities are given by $P(g_{i \pm l}) = p_{i \pm l}$, where $g_{i-1} := g_i^{-1}$. The construction of the associated group extension then has to be adapted only by changing $\psi$ to

$$\psi : \Sigma \to F_d, \quad (i_1 i_2 \cdots) \mapsto g_{i_1}.$$ 

As above, we now apply a local limit theorem. The result of Gerl and Woess in [12] is applicable in full generality, however, for ease of exposition, we restrict ourselves to the special case where $q := \sqrt{p_{i_k} p_{-i_k}}$ does not depend on $i$. Then, by (5.3) and (5.4) in [12], we have that $\rho = 2q \sqrt{2d - 1}$ and that

$$\lim_{n \to \infty} \frac{P(X_n = g_{i_1} \cdots g_{i_k})}{P(X_n = id)} = \left(1 + \frac{d - 1}{d} k\right) (2d - 1)^{-k/2} \prod_{i=1}^k \lambda_{i_k}.$$ 

(21)
for \( n \) and \( k \) even and \( g_{i_1} \cdots g_{i_k} \) in reduced form, that is \( i_l \neq i_{l+1} \), for \( l = 1, \ldots, n - 1 \). Also note that there is a misprint in equation (5.4) in [12]. In there, one has to replace \( d/l(d-1) \) in the first factor by its inverse as in (21). As above, the right hand side in (21) is equal to \( v_{id}(X_{g_{i_1} \cdots g_{i_k}}) \). Using the identities for \( q \) and \( \rho \) and setting \( C_k := 1 + k(d-1)/d \), this gives that

\[
v_{id}(X_{g_{i_1} \cdots g_{i_k}}) = C_k(2/\rho)^k \prod_{i=1}^k q \lambda_{-i} = C_k(2/\rho)^k \prod_{i=1}^k p_{-i}.
\]

Since the identity requires that \( g = g_{i_1} \cdots g_{i_k} \) is in reduced form, we have to introduce the following operations on finite words in order to obtain a formula for arbitrary cylinders. For \( w = (i_1 \ldots i_n) \in \mathcal{W}^n \), there exists a unique \( k \leq n \) and a word \( (j_1 \ldots j_k) \in \mathcal{W}^k \) such that \( \psi_{n}(w) = g_{j_1} \cdots g_{j_k} \) is in reduced form. We will refer to \( \tau(w) := (j_1 \ldots j_k) \) as the \textit{active part} of \( w \), whereas the word which is obtained by deleting the entries of \( \tau(w) \) from \( w \) is referred to as the \textit{inactive part} \( i(w) \) of \( w \). Note that \( \psi_{k}(\tau(w)) = \psi_{n}(w) \) and \( \psi_{n-k}(i(w)) = i(d) \). Moreover, for a given word \( v = (i_1 \ldots i_n) \in \mathcal{W}^n \), we will refer to \( \kappa(v) := (-i_n, \ldots, -i_2, -i_1) \) as the \textit{inverse word} of \( v \). For ease of notation, we will also make use of the Bernoulli measure on \( \Sigma \) defined through

\[
\mu([i_1 \ldots i_n]) = p_{i_1} \cdots p_{i_n}.
\]

As it will be shown below, the measure of a cylinder \([w, g]\), for \( w \in \mathcal{W}^n \) and \( g \in G \) and the function \( \kappa \) given by theorem [5.1] depend on possible cancelations of the concatenation of the path to \( g \in G \) and \( w \). So, let \( v_g \in \mathcal{W}^m \) be given by \( \psi_{m}(v_g) = g \) and \( \psi_{n}(v_g) = \emptyset \), that is \( v_g \) is given by the reduced form of \( g \). With \( k := |\tau(v_g w)| \), the conformality of \( v_{id} \) implies that

\[
v_{id}([w, g]) = \rho^{-n} \mu([w])v_{id}(X_{g_{\tau(w)}}(v_g)) = \rho^{-n} \mu([w])C_k(2/\rho)^k \mu([\kappa(\tau(v_g w)])
\]

in case that \( \tau(v_g w) \neq \emptyset \). If \( \tau(v_g w) = \emptyset \), then \( v_{id}([w, g]) = \rho^{-n} \mu([w]) \) by the same arguments.

The identity now allows to determine the function \( \kappa \) explicitly. That is, for \( g_1, g_2 \in G \), \( x \in \Sigma \) and \( (w_n) \) with \( w_n \in \mathcal{W}^n \) and \( x = \lim_{n \to \infty} [w_n] \), we have \( v_{id} \) as, that

\[
\kappa(g_1, (x, g_2)) = \lim_{n \to \infty} \frac{v_{id}([w_n, g_1, g_2])}{v_{id}([w_n, g_2])} = \lim_{n \to \infty} \frac{v_{id}([w_n, g_1^{-1} g_2])}{v_{id}([w_n, g_2])} = \lim_{n \to \infty} \frac{C_{[\tau(v_{g_1}^{-1} g_2 w_n)]} \left(\frac{2}{\rho}\right)^{|\tau(v_{g_1}^{-1} g_2 w_n)| - |\tau(v_g w_n)|} \mu([\kappa(\tau(v_{g_1}^{-1} g_2 w_n))])}{\mu([\kappa(\tau(v_g w_n))])}.
\]

Observe that total dissipativity implies that \( \psi_{n}(x) : n \in \mathbb{N} = (\psi_{n}(w_n)) : n \in \mathbb{N} \) will almost surely only return finitely many times to a finite subset of \( G \). Hence, the first term in the product converges to 1 whereas the second and third eventually are constant. By setting \( k_{g_1, g_2} (x) := \lim_{n \to \infty} \mathbb{P}^{g_1} g_2 \psi_{n}(x) | g_1 \psi_{n}(x) | g_2 \psi_{n}(x) | 0 \), analyzing the cancelations in \( v_{g_1} v_{g_2} w_n \) and using that \( q^2 = p_1 p_{-1} \), it follows that

\[
\kappa(g_1, (x, g_2)) = (2/\rho)^{k_{g_1, g_2}(x)} \lim_{n \to \infty} \frac{\mu([\kappa(\tau(v_{g_1}^{-1} g_2 w_n))])}{\mu([\kappa(\tau(v_g w_n))])}
= \frac{(2/\rho)^{k_{g_1, g_2}(x)} \mu([v_{g_2}])}{q^{k_{g_1, g_2}(x)}} = (2d-1)^{k_{g_1, g_2}(x)}/2 \sqrt{\mu([v_{g_1}])/\mu([v_{g_1}])}.
\]
The regularity of $h$ can now be analyzed through $k_{g_1, g_2}$. In order to do so, for each open subset $U$ of $X$ and $g = g_1 \in G$, observe that $k_{g, g_2}(U)$ is equal to $\{-|g|, 2 - |g|, \ldots, |g| - 2, |g|\}$. This implies that

$$\sup_{z, \tilde{z} \in U} k(g, z) = (2d - 1)|g| \sqrt{\frac{\mu|v_g|}{\mu|kv_g|}}.$$ 

In particular, the fluctuations of $h(g, \cdot)$ only depend on $|g|$ and the quantity $\mu|v_g|/\mu|kv_g|$, which measures the asymmetry of the random walk. If the random walk is symmetric, that is $p_i = p_{-i} = 1/2d$ for all $i = 1, \ldots, d$, then this simplifies to

$$\rho = \sqrt{2d - 1/d}, \quad v_{id}(X_g) = C_{|g|}(2d - 1)^{-\frac{|g|}{2}}, \quad k(g_1, (x, g_2)) = (2d - 1)^{-\frac{k(g_1, g_2)}{2}}.$$ 

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