ON AN IDENTITY FOR THE PROJECTION OPERATORS OF AUTOMORPHIC FORMS

BIAO WANG

Abstract. In this note, we revisit an identity that Miller and Schmid showed in their article on a general Voronoi summation formula for \( GL(n, \mathbb{Z}) \) in 2009. For the proof, we mainly follow Cogdell and Piatetski-Shapiro’s ideas in their work on converse theorems for \( GL(n) \).

1. Introduction

The Voronoi summation formula is a basic tool in the study of analytic number theory. For example, it is a basic ingredient in the \( GL(2) \) and \( GL(3) \) subconvexity problem [9]. Miller and Schmid [7] first established the Voronoi formula for \( GL(3) \) over \( \mathbb{Q} \). Then it was generalized to \( GL(n) \) over \( \mathbb{Q} \) in [8] with another proof later found in [4]. In the appendix of [8], they showed an identity which is explicitly used by Jacquet, Piatetski-Shapiro, and Shahika to establish the functional equation of \( L \)-functions on \( GL(n) \times GL(m) \) for \( |n-m| > 1 \). In [5], Ichino and Templier discovered such an identity as well. And they used it as a basic ingredient to prove their Voronoi formula over number fields. Actually, this identity can be interpreted as an identity on the projection operator \( \mathbb{P}_m^n \) for \( 1 \leq m \leq n-2 \) and \( n \geq 4 \). We state it as follows.

Proposition 1.1. Let \( n \geq 4 \). Let \( F \) be a global field, and \( \mathfrak{A} \) the ring of adeles of \( F \). Let \( \mathcal{A}_{cusp}(GL_n) \) be the space of automorphic cusp forms on \( GL_n(\mathfrak{A}) \). Let \( \mathbb{P}_m^n \) be the projection operator defined by (6), and \( \bar{\mathbb{P}}_m^n \) the dual projection operator defined by (8). Then for \( 1 \leq m \leq n-2 \), restricted on \( \mathcal{A}_{cusp}(GL_n) \) the following identity holds:

\[
\mathbb{P}_m^n = \bar{\mathbb{P}}_m^n \circ \iota.
\]

Here \( \iota \) is defined by \( \iota \varphi(g) = \varphi(g^*) = \varphi^*(g^{-1}) \) for any \( \varphi \in \mathcal{A}_{cusp}(GL_n) \) and \( g \in GL_n(\mathfrak{A}) \).

Proposition 1.1 is essentially as same as Proposition A.1 in Miller and Schmid’s article [8]. In this note, we give a proof from the ideas in Cogdell and Piatetski-Shapiro’s work [2] on converse theorems for \( GL(n) \). In [2], they defined two functions \( U_\xi \) and \( V_\xi \). The Fourier-Whittaker expansion of \( \mathbb{P}_m^n U_\xi \) is well-known. Following the proof of [2, Lemma 2.3], we will find the Fourier-Whittaker expansion of \( \mathbb{P}_m^n V_\xi \) in section 2.3, which is the main part of this note. Then we will use them to prove the identity (1) in section 3.

An interesting problem is whether one can use the identity (1) to derive a Voronoi-type summation formula for \( GL(n) \times GL(m) \) for \( 2 \leq m \leq n-2 \) as [5]. We hope it could be solved in the following years.

2. The Fourier-Whittaker expansions of projections

2.1. The functions \( U_\xi \) and \( V_\xi \). Let \( n \geq 4 \) be an integer. Let \( F \) be a global field, \( \mathfrak{A} \) the ring of adeles of \( F \), and \( \psi \) a non-trivial additive character of \( F \backslash \mathfrak{A} \). Let \( (\pi, V_\pi) \) be an irreducible generic representation of \( GL_n(\mathfrak{A}) \). For any \( \xi \in V_\pi \), let \( W_\xi \) be the corresponding element in the Whittaker model \( W(\pi, \psi) \) of \( \pi \). Denote by \( U_n \) the group of upper triangular matrices in \( GL_n \) with unit diagonal.

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We use the same letter $\psi$ to denote the character on $U_n(\A)$ defined by $\psi(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$ for $u = (u_{i,j}) \in U_n(\A)$. For any $\xi$ in the space $V_\pi$ of $\pi$ and any $g \in GL_n(\A)$, set

\begin{align}
U_\xi(g) &= \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\xi((\gamma)_1 g), \\
V_\xi(g) &= \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\xi((1_{n} \gamma)_1 g).
\end{align}

Then Cogdell and Piatetski-Shapiro [2] used the following basic lemma as their approach to proving the converse theorems.

**Lemma 2.1** ([2, Proposition 1]). $\pi$ is automorphic cuspidal if and only if

\begin{equation}
U_\xi(I_n) = V_\xi(I_n), \forall \xi \in V_\pi.
\end{equation}

**Remark 2.2.** For the projection operator $P^n_{n-2}$ defined by (6), Cogdell and Piatetski-Shapiro [2] actually showed that $\pi$ is automorphic cuspidal if and only if

\begin{equation}
P^n_{n-2}U_\xi(I_n) = P^n_{n-2}V_\xi(I_n)
\end{equation}

for all $\xi \in V_\pi$.

2.2. **Projection operators.** Let $1 \leq m \leq n - 2$, and let $Y_{n,m}$ be the unipotent radical of the standard parabolic subgroup of the partition $(m+1,1,\ldots,1)$ in $GL_n$. That is,

\begin{equation}
Y_{n,m} = \left\{ \begin{pmatrix}
I_{m+1} & * & 1 \\
* & \ddots & * \\
0 & \cdots & 1
\end{pmatrix} \right\} \subset GL_n.
\end{equation}

**Definition 2.3.** For $1 \leq m \leq n - 2$, the **projection operator** $P^n_m$ is defined by

\begin{equation}
P^n_m \phi(g) = \int_{Y_{n,m}(F) \backslash Y_{n,m}(\A)} \phi(yg)\psi^{-1}(y)dy
\end{equation}

for a function $\phi$ on $GL_n(\A)$ which is left invariant under $Y_{n,m}(F)$.

**Remark 2.4.** The definition 2.3 of $P^n_m$ comes from Jacquet and Liu [6, page 197] and differs slightly from the definition used by Cogdell in [3, §2.2.1].

Now, let $M_{(n-m-1) \times m}$ be the space of matrices of type $(n-m-1) \times m$, and

\begin{equation}
X_{n,m} = \left\{ \begin{pmatrix}
I_m \\
x & I_{n-m-1}
\end{pmatrix} : x \in M_{(n-m-1) \times m} \right\} \subset GL_n.
\end{equation}

Let $w_n = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}$ be the $n \times n$ matrix with unit opposite diagonal, and $w_{n,m} = \begin{pmatrix}w_m & w_{n-m}\end{pmatrix}$.

**Definition 2.5.** The **dual projection operator** $\bar{P}^n_m$ is defined by

\begin{equation}
\bar{P}^n_m \phi(g) = \int_{X_{n,m}(\A)} \left[ \int_{Y_{n,m}(F) \backslash Y_{n,m}(\A)} \phi(yxw_{n,m}g^t)\psi^{-1}(y)dy \right] dx
\end{equation}

for a function $\phi$ on $GL_n(\A)$ which is left invariant under $Y_{n,m}(F)$. Here for the convention of notations, we write $dg = dx$ for $g = \begin{pmatrix}
I_m \\
x & I_{n-m-1}
\end{pmatrix} \in X_{n,m}(\A)$ and $x \in M_{(n-m-1) \times m}(\A)$.

**Remark 2.6.** Our definition of $\bar{P}^n_m$ differs from the definition of Cogdell [3]. In [3], $\bar{P}^n_m = t \circ \bar{P}^n_m \circ t$. 

2.3. Fourier-Whittaker expansions of the projections. Let \( P_n = (GL_{n-1}^*) \) and \( Q_n = (^{1}GL_{n-1}) \). The following proposition gives the Fourier-Whittaker expansion of \( P_n U_\xi \).

**Proposition 2.7 ([3, Lemma 2.2]).** For any \( \xi \in V_\pi \), \( U_\xi \) is left invariant under \( P_n(F) \) and

\[
P_n U_\xi(g) = \sum_{\gamma \in U_m(F) \setminus GL_m(F)} W_\xi \left( (\gamma \ i_{n-m}) \ g \right).
\]

On the other hand, we set

\[
V_\xi^m(g) := V_\xi(\alpha_m g)
\]

for all \( g \in GL_n(k) \), where

\[
\alpha_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then \( V_\xi^m \) is left invariant under \( Q_n(F) \), where \( Q_n = \alpha_m^{-1} Q_n \alpha_m \). To find the Fourier-Whittaker expansion of \( P_n V_\xi^m \), it suffices to find it for \( P_n V_\xi^m \).

First, we rewrite the expression of \( V_\xi^m(g) \) as the same form as \( U_\xi(g) \). Let \( \tilde{W}_\xi(g) := W_\xi(w_n g') \) be the dual of \( W_\xi \). Then \( \tilde{W}_\xi \in \mathcal{W}(\pi, \psi^{-1}) \), where \( \pi \) is the contragradient representation of \( \pi \).

**Lemma 2.8.** In terms of \( \tilde{W}_\xi(g) \), \( V_\xi^m(g) \) can be rewritten as follows

\[
V_\xi^m(g) = \sum_{\gamma \in U_m(F) \setminus GL_{m-1}(F)} \tilde{W}_\xi \left( (\gamma \ i_{n-m}) \ w_n \alpha_m g' \right).
\]

**Proof.** Here we follow the ideas in the proof of the expansion of \( V_\xi(g) \) in [1, §3.2]. Notice first that

\[
(\begin{pmatrix} 1 & \gamma \\ 1 \end{pmatrix} \alpha_n) = \alpha_{n-1} \left( \begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix} \right) \alpha_n^{-1}, \quad \text{and} \quad \alpha_{n-1} = w_n \left( \begin{pmatrix} w_n & -1 \\ 1 & 0 \end{pmatrix} \right).
\]

Then by the definition of \( V_\xi^m \), we have

\[
V_\xi^m(g) = \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} W_\xi \left( (\gamma) \alpha_m g \right)
\]

\[
= \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} W_\xi \left( (\gamma) \alpha_n^{-1} \alpha_m \right)
\]

\[
= \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} \tilde{W}_\xi \left( w_n \left[ (\gamma) \alpha_n^{-1} \alpha_m \right] \right)
\]

\[
= \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} \tilde{W}_\xi \left( w_n \alpha_n^{-1} \left( \begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix} \right) \alpha_n^{-1} \alpha_m g' \right)
\]

\[
= \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} \tilde{W}_\xi \left( \left( \begin{pmatrix} w_n & -1 \\ 1 & 0 \end{pmatrix} \right) \alpha_n^{-1} \alpha_m g' \right)
\]

\[
= \sum_{\gamma \in U_n(F) \setminus GL_{n-1}(F)} \tilde{W}_\xi \left( (\gamma) \ w_n \alpha_m g' \right).
\]

Here, the last equality follows since the transformation \( \gamma \mapsto w_{n-1} \gamma w_{n-1}^{-1} \) permutes the set of right cosets \( U_n(F) \gamma \) in \( GL_{n-1}(F) \).

Now, we follow the ideas in the proof of [2, Lemma 2.3] to deduce the Fourier-Whittaker expansion of \( P_n V_\xi^m \).

**Proposition 2.9.** For \( V_\xi^m \) defined in (10), we have that

\[
P_n V_\xi^m(g) = \sum_{\gamma \in U_m(F) \setminus GL_m(F)} \int_{X_{n,m}(k)} \tilde{W}_\xi \left( (\gamma \ i_{n-m}) \ x w_n \alpha_m g' \right) dx.
\]

\[\square\]
Proof. By the definition of $\mathbb{P}_m^n$ in (6), we have
\begin{equation}
\mathbb{P}_m^n V^n_\xi(g) = \int_{Y_{n,m}(F)\setminus Y_{n,m}(A)} V^n_\xi(yg)\psi^{-1}(y)dy.
\end{equation}

Then by Lemma 2.8, we can rewrite $V^n_\xi(yg)$ as follows:
\begin{align*}
V^n_\xi(yg) &= \sum_{\gamma \in U_{n-1}(F)\setminus GL_{n-1}(F)} \overline{W}_\xi \left( (\gamma_1) w_n \alpha_m (yy') \right) \\
&= \sum_{\gamma \in U_{n-1}(F)\setminus GL_{n-1}(F)} \overline{W}_\xi \left( (\gamma_1) \cdot w_n \alpha_m \left( w_m w_{n-m} \right)^{-1} \cdot \left( w_m w_{n-m} \right) y' \right) \\
&= \sum_{\gamma \in U_{n-1}(F)\setminus GL_{n-1}(F)} \overline{W}_\xi \left( (\gamma_1) \left( w_m w_{n-m} \right) y' \right),
\end{align*}
where $w_n \alpha_m \left( w_m w_{n-m} \right)^{-1} := \begin{pmatrix} \gamma_0 & 1 \end{pmatrix}$
and $\gamma_0 \in GL_{n-1}(F)$ is absorbed by $\gamma$ into the sum.

For $y$, we have the following decomposition:
\begin{equation}
y = \begin{pmatrix} I_m \ u \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y' \\ 0 \end{pmatrix},
\end{equation}
where $u \in U_{n-m}(F\setminus A)$ and $y' \in M_{m\times(n-m-1)}(F\setminus A)$. Here we use the notations $U_n(F\setminus A)$ and $M_{m\times\ell}(F\setminus A)$ as shorthand for $U_n(F)\setminus U_n(A)$ and $M_{m\times\ell}(F)\setminus M_{m\times\ell}(A)$, respectively. It follows that
\begin{equation}
\begin{pmatrix} w_m w_{n-m} \end{pmatrix} y' \begin{pmatrix} w_m w_{n-m} \end{pmatrix}^{-1} = \begin{pmatrix} I_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' & 1 \end{pmatrix},
\end{equation}
where $x' \in M_{(n-m-1)\times m}(F\setminus A)$ is a permutation of $-y'$.

Notice that $\psi(y) = \psi^{-1}(w_n u' w_{n-m}^{-1})$. Making change of variables $u \mapsto w_n u' w_{n-m}$ and $y' \mapsto x'$, we get
\begin{equation}
\mathbb{P}_m^n V^n_\xi(g) = \int_{x' \in M_{(n-m-1)\times m}(F\setminus A)} \int_{u \in U_{n-m}(F\setminus A)} \sum_{\gamma \in U_{n-1}(F)\setminus GL_{n-1}(F)} \overline{W}_\xi \left( (\gamma_1) \begin{pmatrix} I_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \end{pmatrix} \begin{pmatrix} w_m w_{n-m} \end{pmatrix} y' \right) \psi(u)dudx'.
\end{equation}

Next, we integrate over the last column of $u$.
Decompose $u$ into the product of three matrices:
\begin{equation}
\begin{pmatrix} I_m \ u \end{pmatrix} = \begin{pmatrix} I_{n-2} & 1 & u_{n-1} \\ 1 & u_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & 1 \end{pmatrix},
\end{equation}
where $u' \in U_{n-m-1}(F\setminus A)$ satisfies $\psi(u) = \psi(u_{n-1}) \psi(u')$.

First, by $\begin{pmatrix} \gamma_1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 1 & u_{n-1} \\ 1 & u_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-2} & 1 & \gamma_{n-1,n-1} u_{n-1} \\ 1 & \gamma_{n-1,n-1} u_{n-1} \end{pmatrix} \begin{pmatrix} \gamma_1 \end{pmatrix}$, we have
\begin{equation}
\overline{W}_\xi \left( (\gamma_1) \begin{pmatrix} I_m \ u \end{pmatrix} g \right) \psi(-\gamma_{n-1,n-1} u_{n-1}) \overline{W}_\xi \left( (\gamma_1) \begin{pmatrix} I_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ 0 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u'_{1} \\ 0 \end{pmatrix} g \right).
\end{equation}

Integrating over $u_{n-1}$ on $F\setminus A$, we get $\gamma_{n-1,n-1} = 1$. So we can write $\gamma$ as
\begin{equation}
\gamma = \begin{pmatrix} \gamma' \\ 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & 1 \\ 1 & 1 \end{pmatrix}
\end{equation}
\[ \begin{pmatrix} I_{m-2} & \vdots & 1 \\ v & 1 \end{pmatrix} \begin{pmatrix} I_m & u_{m+1} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & 1 \\ u_{n-2} & \vdots & 1 \end{pmatrix} = \begin{pmatrix} I_m & u_{m+1} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & 1 \\ u_{n-2} & \vdots & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & v \\ 1 & 1 \end{pmatrix}. \]

This implies that

\[ \tilde{W}_\xi \left( \begin{pmatrix} \gamma' \\ I_2 \end{pmatrix} \begin{pmatrix} I_m & u_m' \\ v & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ v & 1 \end{pmatrix} \right) g = \psi(-u_{m+1} v_{m+1} - \cdots - u_{n-2} v_{n-2}) \tilde{W}_\xi \left( \begin{pmatrix} \gamma' \\ I_2 \end{pmatrix} \begin{pmatrix} I_{n-2} & v \\ 1 & 1 \end{pmatrix} \begin{pmatrix} I_m & u_m' \\ v & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ v & 1 \end{pmatrix} \right). \]

Integrating over \( u_{m+1}, \ldots, u_{n-2} \), we get \( v_{m+1} = \cdots = v_{n-2} = 0 \). Thus, after integrating over the last column of \( u \), we get that

\[ \mathbb{P}_m V^\xi_m(g) = \int_{u' \in U_{n-m-1}(F)} \int_{x' \in M_{(n-m-1) \times m}(F)} \sum_{\gamma \in U_{n-2}(F) \backslash GL_{n-2}(F)} \sum_{v \in F^m} \tilde{W}_\xi \left( \begin{pmatrix} \gamma' \\ I_2 \end{pmatrix} \begin{pmatrix} I_m & u_m' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ 0 & 1 \end{pmatrix} \right) \int_{x' \in M_{(n-m-1) \times m}(F)} dx' \psi(u') du'. \]

Notice that

\[ \begin{pmatrix} I_m & u_m' \\ u_1 & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ x' & 1 \end{pmatrix} = \begin{pmatrix} I_m & u_m' \\ u_1 & 1 \end{pmatrix}. \]

By a change of variables \( u' x' \mapsto x' \) and collapsing \( v \) with the last row of \( x' \),

\[ \mathbb{P}_m V^\xi_m(g) = \int_{u' \in U_{n-m-1}(F)} \int_{x' \in M_{(n-m-2) \times m}(F)} \int_{x_{n-1} \in H^m} \sum_{\gamma \in U_{n-2}(F) \backslash GL_{n-2}(F)} \tilde{W}_\xi \left( \begin{pmatrix} \gamma' \\ I_2 \end{pmatrix} \begin{pmatrix} I_m & u_m' \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ 0 & 1 \end{pmatrix} \right) dx_{n-1} dx' \psi(u') du'. \]

Then interchanging the matrices on \( (x', x_{n-1}) \) and \( u' \) and making a change of variables again, we get that

\[ \mathbb{P}_m V^\xi_m(g) = \int_{u' \in U_{n-m-1}(F)} \int_{x' \in M_{(n-m-2) \times m}(F)} \int_{x_{n-1} \in H^m} \sum_{\gamma \in U_{n-2}(F) \backslash GL_{n-2}(F)} \tilde{W}_\xi \left( \begin{pmatrix} \gamma' \\ I_2 \end{pmatrix} \begin{pmatrix} I_m & u_m' \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} I_m & w_m \\ 0 & 1 \end{pmatrix} \right) dx_{n-1} dx' \psi(u') du'. \]

Iterating the previous steps on \( u' \) from (19) to (27) until getting rid of \( u' \), we can finally reach that

\[ \mathbb{P}_m V^\xi_m(g) = \sum_{\gamma \in U_{m}(F) \backslash GL_{m}(F)} \int_{M_{(n-m-1) \times n}(F)} \tilde{W}_\xi \left( \begin{pmatrix} \gamma \\ I_{n-m} \end{pmatrix} \begin{pmatrix} I_m & w_{n-m} \\ x & 1 \end{pmatrix} \begin{pmatrix} I_m & w_{n-m} \\ 0 & 1 \end{pmatrix} \right) dx, \]

which is (13). This completes the proof of Proposition 2.9. \( \Box \)
3. Proof of Proposition 1.1

In this section, we prove Proposition 1.1 by the Fourier-Whittaker expansions of $\mathbb{P}_n U_\xi$ and $\mathbb{P}_m V_\xi$.

Suppose $\varphi \in \mathcal{A}_{\text{cusp}}(GL_n)$ is an automorphic cusp forms on $GL_n(\mathbb{A})$ and $g \in GL_n(\mathbb{A})$. By Lemma 2.1, we have $\varphi(g) = U_\varphi(g) = V_\varphi(g)$. Since $\varphi$ is automorphic, we have $V_\varphi(g) = V_{\varphi}^m(g)$ as well. It follows that

$$\mathbb{P}_m \varphi(g) = \mathbb{P}_m U_\varphi(g) = \mathbb{P}_m V_\varphi^m(g).$$

(29)

Let $\widetilde{\varphi} = \psi \varphi$, then $\widetilde{W}_\varphi \in \mathcal{W}(\pi, \psi^{-1})$ is the Whittaker function of $\varphi$. By Proposition 2.7, we get the Fourier-Whittaker expansion of $\mathbb{P}_m \varphi(g)$ as follows

$$\mathbb{P}_m \varphi(g) = \sum_{\gamma \in U_m(\mathbb{F}) \backslash GL_m(\mathbb{F})} \widetilde{W}_\varphi \left( \left( \begin{array}{cc} \gamma & I_{n - m} \\ 0 & 1 \end{array} \right) g \right).$$

(30)

On the other hand, by the definition (8) of $\mathbb{P}_m \varphi(g)$, we have

$$\mathbb{P}_m \varphi(g) = \int_{X_{n, m}(\mathbb{A})} \mathbb{P}_m \varphi(xw_{n,m}g') dx.$$

(31)

Plugging (30) into (31) and interchanging the sum and the integration, we get that

$$\mathbb{P}_m \varphi(g) = \sum_{\gamma \in U_m(\mathbb{F}) \backslash GL_m(\mathbb{F})} \int_{X_{n, m}(\mathbb{A})} \widetilde{W}_\varphi \left( \left( \begin{array}{cc} \gamma & I_{n - m} \\ 0 & 1 \end{array} \right) xw_{n,m}g' \right) dx.$$

(32)

By Proposition 2.9, the right hand side of (32) is equal to the Fourier-Whittaker expansion of $\mathbb{P}_m V_\varphi^m(g)$, which means that

$$\mathbb{P}_m \varphi(g) = \mathbb{P}_m V_\varphi^m(g).$$

(33)

Hence, from (29) and (33) we get that

$$\mathbb{P}_m \varphi(g) = \mathbb{P}_m \widetilde{\varphi}(g),$$

(34)

which gives us the desired identity (1). This completes the proof of Proposition 1.1.

Remark 3.1. From the proof above, one can see that the identity (1) in Proposition 1.1 corresponds to Cogdell’s identity $\mathbb{P}_m U_\xi(I_{m+1}) = \mathbb{P}_m V_\xi(I_{m+1})$ in [3, Proposition 5.1], which is related to the functional equation of the Eulerian integrals [3, Theorem 2.1] for $GL(n) \times GL(m)$.

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Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260-2900, USA
E-mail address: bwang32@buffalo.edu