Cooling through parametric modulations and phase-preserving quantum measurements

Sreenath K. Manikandan\textsuperscript{1,}\textsuperscript{*} and Sofia Qvarfort\textsuperscript{1,2,}\textsuperscript{†}

\textsuperscript{1}Nordita, KTH Royal Institute of Technology and Stockholm University, Hannes Alfven’s väg 12, SE-106 91 Stockholm, Sweden
\textsuperscript{2}Department of Physics, Stockholm University, AlbaNova University Center, SE-106 91 Stockholm, Sweden

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We propose a cooling protocol that uses phase-preserving quantum measurements and phase-dependent modulations of the trapping potential at parametric resonance to cool a quantum oscillator to near its quantum-mechanical ground-state. The sequential measurements and feedback provide a definite phase reference and stabilize the oscillator in the long-time limit. The protocol is robust against moderate amounts of dissipation and phase errors in the feedback loop. Our work has implications for the cooling of mechanical resonators and the integration of quantum refrigerators into quantum circuits.

Introduction — Recent advances in fabricating and integrating devices in the nano-scale have made it possible to realize several candidate physical systems where quantum mechanical behaviors are readily observed. Examples of this include superconducting quantum circuits [1, 2], ultra-cold atoms [3, 4], ion traps [5], electron-spin qubits in semi-conductor platforms [6], and nanomechanical oscillators [7], to name just a few. In addition, it is also exciting that controlled manipulation of quantum information has been demonstrated in recent years across various quantum platforms. Notable achievements include the ability to prepare desired quantum mechanical states on demand [8, 9], perform gate operations [1, 10, 11], as well as quantum-limited measurements [12], and real-time feedback control [13].

Cooling has been one of the most significant challenges for harvesting quantum effects in the nano-scale, since temperatures of the millikelvin (mK) regime is a necessity in almost all known quantum platforms presently available [14]. One family of system where cooling is of particularly importance are mechanical oscillators in the quantum regime. These range from, to name just a few, moving-end mirror Fabry–Pérot cavities and clamped membrane oscillators [7, 15], levitated systems [16, 17], quantum LC circuits [18], and hybrid optomechanical systems [19]. Achieving the quantum-mechanical ground-state of oscillators via cooling is central to the ability to explore various fundamental physics questions, such as sensing weak forces and gravitational effects [20–24], maintaining long-enough coherence times for information processing tasks [25], and probing fundamental physics [26].

A number of protocols have recently been developed and made possible the remarkable achievements of near-ground-state and ground-state cooling. They include, among others, resolved-sideband cooling [27, 28], velocity damping [29, 30], Doppler cooling [31], and coherent scattering [32]. One such cooling method, known as parametric feedback-cooling, has been especially successful in cooling optically levitated systems [33–36] and clamped nanobeam systems [37].

Many of these cooling principles were originally developed for classical systems, but the recent progresses in various experimental quantum physics platforms have made it possible to extend such cooling principles to the quantum regime. Motivated by this, the present Letter investigates a possible quantum regime of parametric cooling of a simple harmonic oscillator, aided by quantum measurements. In particular, we show that modulations of the harmonic potential at parametric resonance can reduce the mean quanta of quantum states with a fixed phase reference. In order to cool down arbitrary quantum states of the oscillator such as thermal states, we generalize the above idea to propose a cyclic cooling protocol combining phase-preserving (heterodyne) quantum measurements and conditional parametric modulations of the harmonic potential (see Fig. 1). These parametric modulations are modelled with Mathieu’s equation [22, 38], which was first discussed by

\* sreenath.k.manikandan@su.se
\† sofia.qvarfort@fysik.su.se
Mathieu in 1868 [39]. It describes the behavior of a diverse family system ranging from the quantum pendulum to a child on a swing. The inherent instabilities found in the solutions to Mathieu’s equation are mitigated in this protocol by the sequential quantum measurements and the phase feedback control, which change the effective description of the dynamics. This is further substantiated by the fact that the mean quanta of the oscillator under this protocol tends to one in the long-time limit. In addition, we show that our cooling protocol is robust against moderate amounts of dissipation and significant phase errors in the feedback loop. The Letter is concluded by discussing the quantum advantages and limitations for our cooling principle, as well as potential experimental realizations.

**Dynamics** — The Hamiltonian describing the driven quantum oscillator has the following form,

\[ \dot{H}(t) = \dot{H}_0 + 2m\omega_0 f(t) \dddot{x} = \dot{H}_0 + h f(t) (\hat{a}^\dagger + \hat{a})^2, \]

where,

\[ \dot{H}_0 = \frac{1}{2} m \omega_0^2 \dddot{x}^2 + \frac{1}{2m} \dddot{p}^2 = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \]

is the free Hamiltonian for the quantum oscillator and \( f(t) \) is the function for the driving amplitude, where we have used \( \dddot{x} = \sqrt{\hbar/2\omega_0 m}(\dddot{a}^\dagger + \dddot{a}) \) and \( \dddot{p} = i\sqrt{\hbar \omega_0/2}(\dddot{a}^\dagger - \dddot{a}) \). Here, \( m \) is the mass of the oscillator and \( \omega_0 \) is the frequency of the mode. In this work, we consider the following sinusoidal driving profile \( f(t) = \lambda \cos(\omega_p t + \phi_p) \), where \( \lambda \) is the driving amplitude, \( \omega_p \) is the drive frequency and \( \phi_p \) is the phase. When \( \omega_p = 2\omega_0 \), the drive is referred to as parametric[40].

We describe the dynamics governed by the Hamiltonian in Eq. (1) using the solutions developed in Refs. [22, 38, 41] (re-visited in Supplemental Note 1). In what follows, we briefly summarize the solutions. Since the Hamiltonian in Eq. (1) is quadratic in its operator arguments, the evolution of a Gaussian state is captured fully by considering only the evolution of the first and second moments. Defining \( \mathcal{X} = (\hat{a}, \hat{a}^\dagger) \), the solution to the evolution of the vector of first moments reads,

\[ \mathcal{X}(t) = \mathcal{U}(t) \mathcal{X}(0) \mathcal{U}^\dagger(t) = \mathbf{S}(t) \mathcal{X}, \]

where \( \mathcal{U}(t) \) is the time-evolution operator given by

\[ \mathcal{U}(t) = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^t dt' H(t') \right], \]

and \( \mathbf{S}(t) \) is a \( 2 \times 2 \) symplectic matrix given by

\[ \mathbf{S}(t) = \mathcal{T} \exp \left[ \Omega \int_0^t dt' H(t) \right], \]

for which \( \mathcal{T} \) indicates time-ordering of the exponential, \( \Omega \) is the symplectic form, defined in this basis as \( \Omega = i \text{diag}(-1, 1) \), and \( H(t) \) is the Hamiltonian matrix, defined by \( \dot{H}(t) = \frac{1}{2} \hat{X}^\dagger H(t) \hat{X} \). The \( H \) matrix corresponding to the Hamiltonian in Eq. (1) reads,

\[ H(t) = \begin{pmatrix} 1 + 2f(t)/\omega_0 & 2f(t)/\omega_0 \\ 2f(t)/\omega_0 & 1 + 2f(t)/\omega_0 \end{pmatrix}. \]

The time-evolution generated by the Hamiltonian matrix in Eq. (6) can be written as a Bogoliubov transformation of the first moments with,

\[ \mathbf{S}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta^*(t) & \alpha^*(t) \end{pmatrix}, \]

where \( \alpha(t) \) and \( \beta(t) \) are Bogoliubov coefficients which satisfy the normalisation conditions \( |\alpha(t)|^2 - |\beta(t)|^2 = 1 \) and \( \alpha(t)\beta^*(t) - \alpha^*(t)\beta(t) = 0 \). This means, for example, that the operator \( \hat{a}(t) \) evolves as,

\[ \hat{a}(t) = \alpha(t) \hat{a} + \beta(t) \hat{a}^\dagger. \]

The time-dependent coefficients \( \alpha(t) \) and \( \beta(t) \) can be written as (see Appendix B in [38]),

\[ \alpha(t) = \frac{1}{2} \left[ P(t) - i Q(t) + \frac{d}{dt} (i P(t) + Q(t)) \right], \]

\[ \beta(t) = \frac{1}{2} \left[ P(t) + i Q(t) + \frac{d}{dt} (i P(t) - Q(t)) \right], \]

where the functions \( P(t) \) and \( Q(t) \) are both solutions to the following differential equation:

\[ \ddot{y} + (\omega_0^2 + 4\omega_0 f(t)) y = 0, \]

where \( P(t) \) can be obtained by using the initial conditions \( P(0) = 1 \) and \( P(0) = 0 \), and where \( Q(t) \) can similarly be obtained by setting \( Q(0) = 0 \) and \( Q(0) = 1 \). These initial conditions follow from the fact that we require that \( \mathbf{S}(t = 0) = I \).

**Modulation at parametric resonance** — When the frequency modulation occurs at twice the free frequency \( \omega_p = 2\omega_0 \), Eq. (10) takes the form of Mathieu’s differential equation [39]:

\[ \frac{d^2 y}{dx^2} + [a - 2\epsilon \cos(2x + \phi_p)] y = 0. \]

In our case \( x = \omega_p t, a = 1 \), and \( \epsilon = -2\lambda/\omega_0 \). Mathieu’s equation is usually defined without the phase \( \phi_p \), such that when \( \phi_p = 0 \), the solutions can be represented as Mathieu’s functions of the first kind: \( ce_n(t, \lambda) \) and \( sc_n(t, \lambda) \). The solutions have no analytic form, but are periodic with \( 2\pi \). We also note that at \( a = 1 \), the solutions are fundamentally unstable [42]. As a result, the system can only be operated at parametric resonance over short time-scales and with weak driving strengths, which is what we assume for the cooling protocol. A similar issue arises when solving the dynamics of classical oscillators with a time-modulated potential [43]. We discuss this further in Supplemental Note 2. For non-zero \( \phi_p \), the solutions can be expressed by linear combinations
of $c_{\eta}(t, \lambda)$ and $s_{\eta}(t, \lambda)$. While Eq. (11) does not allow for an exact analytical solution, an approximate solution can be obtained when $\lambda/\omega_0 \ll 1$ using a two time-scale method (see Supplemental Note 2 1). It reads, to first order in $\lambda/\omega_0$ [22, 38],

$$
P(t) \approx \cos(\omega_0 t) + \frac{\lambda}{\omega_0} \left[ \cos(\omega_0 t) \cos(\phi_p) - \cos(\omega_0 t + \phi_p) - \omega_0 t \sin(\omega_0 t + \phi_p) \right], 
$$

$$
Q(t) \approx \sin(\omega_0 t) + \frac{\lambda}{\omega_0} \left[ \sin(\omega_0 t) \cos(\phi_p) - \omega_0 t \cos(\omega_0 t + \phi_p) \right]. 
$$

The Bogoliubov coefficients in Eq. (9) can therefore be approximated as,

$$
\alpha(t) \approx e^{-i\omega_0 t} - \frac{i}{\omega_0} e^{i\phi_p} \sin(\omega_0 t), \\
\beta(t) \approx -i\lambda t e^{-i(\omega_0 t + \phi_p)}. 
$$

We see that when $\lambda \to 0$, we are left with the free evolution $e^{-i\omega_0 t}$ encoded in $\alpha(t)$. To better understand the error of the approximate solutions, we may examine the Bogoliubov identity $|\alpha(t)|^2 - |\beta(t)|^2 = 1$. We do so in Supplemental Note 2 3, where we also compare these solutions with a perturbative expansion of $\hat{U}(t)$ in Eq. (4) (see Supplemental Note 2 2). We find that the perturbative expansion of $\hat{U}(t)$ produces an error that grows linearly in time, while the error of the approximate solutions to Mathieu’s equation oscillates in time. We therefore proceed with the results obtained from Mathieu’s equation.

Phase control for cooling — Equipped with the analytical tools to make predictions about the dynamics, we now proceed to derive the necessary phase relationship for cooling. By “cooling” in this context, we are referring to gradually decreasing the mean quanta of the oscillator. It can be noticed from Eq. (8) that the mean quanta in the oscillator changes as,

$$
\langle \hat{n}(t) \rangle = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = |\alpha(t)|^2 \langle \hat{a}^\dagger \hat{a} \rangle + |\beta(t)|^2 \langle \hat{a}^\dagger \hat{a} \rangle + \alpha^*(t) \beta(t) \langle \hat{a} \rangle^2 + \beta^*(t) \alpha(t) \langle \hat{a}^\dagger \rangle^2. 
$$

Since the first two terms are generically positive, in order to allow for cooling, the quantum oscillator must be initialized in states of non-vanishing cross-terms in Eq. (14), with appropriate phase reference such that the mean quanta in the oscillator decreases over time. A possible phase relationship necessary for cooling can be derived from Eq. (14) by considering the coherent state basis. For an initial coherent state $|\xi\rangle = |r e^{i\phi}\rangle$, the mean quanta evolves as,

$$
\langle \hat{n}(t) \rangle_{|\xi\rangle} = (|\alpha(t)|^2 + |\beta(t)|^2) |\xi|^2 + |\beta(t)|^2 \\
+ \alpha^*(t) \beta(t) (\xi^*)^2 + \alpha(t) \beta^*(t) \xi^2. 
$$

Using the approximate expressions in Eq. (13), we find that the number of quanta becomes, to first order in $\lambda$:

$$
\langle \hat{n}(t) \rangle_{|\xi\rangle} \approx r^2 \left[ 1 + \frac{\lambda}{\omega_0} (\cos(\phi_p) - \cos(2\omega_0 t + \phi_p)) - 2\omega_0 t \sin(2\phi + \phi_p) \right] + \mathcal{O}(\lambda^2). 
$$

Here, the last term inside the square brackets in Eq. (16) is proportional to $\omega_0 t$, which means that it will either increase or decrease the mean quanta in the oscillator as a function of time as determined by the phase relation $2\phi + \phi_p$. When $2\phi + \phi_p = \pi/2$, we see that the last term in Eq. (16) is maximized, producing an initial cooling effect at the rate $2\lambda r^2$. An analogous scenario occurs for the child in a swing problem, where a child crunches and stretches at twice the natural frequency of the swing to increase the amplitude of the oscillations. There, the initial phase difference between the child and the swing is important for deciding whether the amplitude decreases or increases at short time-scales.

Cooling cycles — We now generalize the above result to a two-step feedback cooling cycle that can be used to cool down arbitrary quantum initial states of the oscillator — such as thermal states — for which parametric modulations alone would be insufficient to decrease the mean quanta in the oscillator (see Supplemental Note 4). By a cycle, we refer to a fixed duration of time $t_c$ for which the potential is modulated in time. A generic two-step cooling cycle in such scenarios can be described as follows:

- **Step 1.** We first measure the quantum oscillator in the coherent state basis (phase-preserving quantum measurements) [44]. Let the measurement outcome be a coherent state denoted as, $|\xi\rangle = |r e^{i\phi}\rangle$, which has a definite amplitude ($r$) and a phase ($\phi$).

- **Step 2.** We then apply a conditional (feedback) modulation of the trapping potential. For each measurement outcome $|\xi\rangle = |r e^{i\phi}\rangle$, the phase of the frequency modulation $\phi_p$ is chosen such that we insert the optimal choice for the driving phase $\phi_p = \pi/2 - 2\phi$.

A similar protocol with linear feedback has been discussed as an engine in [45]. In contrast, the protocol proposed here only requires the phase information [46]. We now examine the resulting average cooling for a quantum oscillator initialized in a thermal state at inverse temperature $\beta' = 1/k_B T$, given by,

$$
\hat{\theta}_{th} = \frac{e^{-\beta' H_0}}{Z}, \text{ where } Z = \text{tr} \{e^{-\beta' H_0}\}. 
$$

For an initial thermal state, the probability of obtaining a specific coherent state $|\xi\rangle = |r e^{i\phi}\rangle$ as the outcome of the measurement is given by the corresponding Husimi Q-function [47]: $Q(\xi) = Q(r, \phi) = \frac{1}{\pi (r+1)} e^{-r^2}$. Here
be, distribution, and the erasure cost for this example would using Landauer’s eraser principle [48]. The phase that the feedback phase reference per cycle can be obtained that the computational cost of erasing the memory of simulations, and find excellent agreement. We also note ulation process [45]. In Fig. 2(a), we compare this corresponding analytical prediction. (b) Shows a single realization of the cooling protocol for an isolated system with sequential measurements and drive. We start with a coherent state with mean quanta equal to $\tau^2 = 80$, and the duration between measurements is set to $t_c = 15\pi/\omega_0$. Here the black dots indicates the average quanta in the coherent state obtained from measurements, and the red dots indicate the average quanta at the end of the drive. The dashed blue line indicates variation in the mean quanta as the drive is turned on in each cycle, and the underlying green line is the corresponding prediction from solving the dynamics by analytical methods. When the measurement results in a state near the ground-state, the parametric modulations cease to cool the system (see Supplemental Note 5). (c) Shows the average cooling as a function of time when the system interacts with the environment. Here $n_b$ is the thermal quanta in the oscillator corresponding to the ambient temperature, $k_B T_b = 10 \omega_0$. In the absence of any dissipation, the oscillator can be cooled down to the single quantum level. (d) Shows the effects of uncertainty in phase control for different value of the phase spread $\Delta \phi$, in the absence of dissipation. Each simulation starts with a coherent state with mean quanta, $r^2 = 10$. The blue curves indicates corresponding theory predictions.

$\bar{n} = \text{Tr}\{\hat{b}_th\hat{n}\} = (e^{\frac{2\lambda}{\alpha}\bar{n}} - 1)^{-1}$. We find to first order in $\lambda$ that $\langle \hat{n}(t) \rangle_{\text{th}} = 1 + \bar{n}(1 - 2\lambda t) + O(\lambda^2)$. This means that the average cooling power at early times is $2\lambda \bar{n}$. The added one quantum of noise comes from the measurement process [45]. In Fig. 2(a), we compare this analytical predictions for $\langle \hat{n}(t) \rangle_{\text{th}}$ with numerical simulations, and find excellent agreement. We also note that the computational cost of erasing the memory of the feedback phase reference per cycle can be obtained using Landauer’s eraser principle [48]. The phase that results from measuring a thermal state obeys a uniform distribution, and the erasure cost for this example would be, $k_B T_m \log 2\pi$, where $T_m$ is the temperature of the memory.

Sequential cooling cycles — Now we consider applying a sequence of cooling cycles to a quantum oscillator, such that Step 1 and Step 2 above are repeated several times in sequence. For each cycle, the initial phase-preserving quantum measurements can be modeled by Kraus operators, $K(\alpha) = \frac{1}{\sqrt{2}} |\alpha \rangle \langle \alpha|$ sampling coherent states according to the corresponding Husimini Q function at the beginning of each cycle. Since the dynamics is Gaussian, we can approximate the sampling distribution as (in units where $\hbar = m = 1$) [47],

$$Q(\text{Re}, \text{Im}) = \mathcal{M}\mathcal{N}[\mu, \Sigma](\text{Re}, \text{Im}),$$

where $\mathcal{M}\mathcal{N}[\mu, \Sigma](\text{Re}, \text{Im})$ is a multivariate normal distribution with mean $\mu = \left[\frac{\sqrt{\omega_0/2}}{\langle \hat{p}\rangle/\sqrt{2\Sigma_{pp}}} \right]$ and a variance matrix defined as,

$$\Sigma = \frac{1}{2} \begin{bmatrix} \sigma_{xx} + 1/2 & \sigma_{xp} \\ \sigma_{px} & \sigma_{pp} + 1/2 \end{bmatrix},$$

where $\sigma_{ij}$ are the elements of the familiar covariance matrix of observables $\hat{i}, \hat{j}$ defined as $\sigma_{ij} = \langle \hat{i} - \langle \hat{i} \rangle, \hat{j} - \langle \hat{j} \rangle \rangle$. By first modulating the potential and then sampling from the $Q$ function to generate a new state, we are able to model the sequential cooling cycles. In simulations, we assume that the measured coherent state is aligned to the $\phi = 0$ axis via a unitary rotation such that $\phi_p = \pi/2$. In Fig. 2(b), we show an example of such a
quantum trajectory. The quantum oscillator is initialized in a coherent state with mean quanta, \( r^2 = 80 \), and the cycle duration is set to \( t_c = 15\pi/\omega_0 \) where the parametric modulations of the trapping potential is performed. An added benefit of this approach is that by incorporating sequential measurements, we improve the stability of the oscillator in the long-time limit. This is because, when measurements and feedback operations are accounted for, the coarse-grained dynamics of the oscillator is no-longer described by the Mathieu’s equations, but by an appropriately modified master equation that includes both the drive, and the quantum measurement backaction. Similar techniques have also been employed in the past to improve stability of otherwise unstable quantum oscillator systems [49].

**Thermal environment** — We now proceed to investigate the effects of the system interacting with a thermal bath that we treat as the environment. To this end, we consider a scenario of performing multiple cycles of cooling, while the quantum oscillator is undergoing collisional interactions with the reservoir modes. Following [50], we model these interactions with the reservoir using an adiabatic Markovian master equation, resulting in the following dynamical equations for the first and second moments [50],

\[
\dot{x}(t) = p(t) - \gamma x(t)/2, \quad \dot{p}(t) = -\omega(t)^2 x(t) - \gamma p(t)/2,
\]

\[
\dot{\sigma}_{xx}(t) = -\gamma \sigma_{xx}(t) + \frac{2\bar{n}_B + 1}{2\omega(t)} + 2\sigma_{xp}(t),
\]

\[
\dot{\sigma}_{xp}(t) = -\gamma \sigma_{xp}(t) + \sigma_{pp}(t) - \sigma_{xp}(t)\omega(t)^2, \quad \sigma_{xp}(t) = \gamma \sigma_{px}(t),
\]

\[
\dot{\sigma}_{pp}(t) = -\gamma \sigma_{pp}(t) + \frac{(2\bar{n}_B + 1)\omega(t)}{2} - 2\sigma_{xp}(t)^2, \quad \sigma_{pp}(t) = \gamma \sigma_{pp}(t),
\]

where \( \gamma \) is the dissipation rate, \( \bar{n}_B \) is the thermal occupation of a reservoir mode at frequency \( \omega_B = \omega_0/2 \), having a temperature \( k_B T_B = 10\omega_0 \), and \( \omega(t) = \omega_0 \sqrt{1 + 4f(t)/\omega_0} \). In Fig. 2(c), we demonstrate that our cooling protocol is able to cool down the quantum oscillator below the ambient temperatures on an average, even when moderate amounts of dissipation are accounted for. We also show that in the absence of any dissipation, our cooling protocol is able to reach the single quantum limit. This is a fundamental limitation of our current protocol since the phase-preserving quantum measurements add one quantum of noise on average [43]. At this stage, our cooling protocol may be combined with other well-known ground state cooling principles that work well in the low quantum limit—such as linear feedback cooling [51].

**Ideal cycle length** — Since each measurement in the coherent-state basis costs an average of a single quantum to perform, the average cycle length must reduce the mean quanta by at least a single quantum to achieve overall cooling. To lowest order in \( \lambda \), the difference in quanta \( \Delta n(t) = \langle \hat{n}(0) \rangle - \langle \hat{n}(t) \rangle \) when driving at the ideal phase offset \( \phi = \pi/2 \) is given by \( \Delta n(t) = 2\lambda t\sqrt{2\omega_0 - 2\sin(2\omega_0 t)}\omega_0 + O(\lambda^2) \). For a single cooling cycle to be effective, we require that \( \Delta n(t) \geq 1 \). Ignoring the oscillating term, we find \( \omega_0 \geq 1/(2\lambda^2) \). For example, given \( r \leq 100 \) and \( \lambda = 0.02\omega_0 \), we must modulate the potential for \( t\omega_0 \geq 5 \times 10^{-3} \) to remove at least one quantum.

**Phase inaccuracy** — The protocol relies on the ability to adjust the phase of the modulation to \( \phi_0 + \phi_p = \pi/2 \). Errors arise when the driving phase cannot be set exactly to this value, due to, for example, latency in the feedback loop. To model this scenario, we consider several realizations of an individual cooling cycle where the driving phase \( \phi_p \) is sampled around the ideal driving phase \( \phi_0 \) from a normal distribution of standard deviation \( \Delta \phi \). The sampling probability distribution \( P(\phi_p) \) is given by \( P(\phi_p) = \frac{1}{\Delta \phi \sqrt{2\pi}} \exp \left[-(\phi_p - \phi_0)^2/2\Delta \phi^2\right] \). The result is a smeared phase function centered around \( \phi_0 \). In Fig. 2(d), we demonstrate that our cooling protocol is robust against significant phase errors up to twenty percent of the ideal phase, which is an advantage for our protocol for experimental realizations.

**Physical realization** — The protocol described here requires only two ingredients: the ability to modulate the trapping frequency at parametric resonance and the ability to perform phase-preserving quantum measurements. In general, an interaction that modulates the trapping potential can be generated by imposing an electrostatic force or external strong optical field to the mechanical mode [52]. In levitated systems, such modulations can be straightforwardly implemented, where the percentage change of the trapping potential is known as the modulation depth \( G \) [43]. In this work, \( G \) is related to the driving amplitude \( \lambda \) as \( G = 4\lambda/\omega_0 \), which for \( \lambda/\omega_0 = 0.01 \) is \( G = 0.04 \) or 4%. In hybrid traps, modulation depths as high as 5% are possible [43], while in optical tweezers, around 0.4% is more common [53]. Beyond optical and hybrid traps, candidate systems include magnetically levitated magnets [54, 55], diamagnets [56], and superconducting spheres [57]. The second ingredient, phase-preserving measurements, may be performed by for example pulsing light through the cavity when the system is in the unresolved sideband regime [58, 59], or using stroboscopic heterodyne measurements. In addition, superconducting quantum circuits also offer novel methodologies to perform phase-preserving quantum measurements dynamically [60].

**Conclusions** — We have proposed a protocol that uses phase-dependent parametric modulations of the trapping potential and phase-preserving quantum measurements to cool down a quantum harmonic oscillator near its quantum-mechanical ground state. We have demonstrated that our protocol is robust against moderate amounts of dissipation and inaccuracies in the phase control. The degree of robustness suggests that our cooling principle can be used to achieve near quantum ground-state cooling, especially when experimental capabilities are limited. There are a number of applications for this protocol, such as aiding the cooling of mechanical oscillators in the quantum regime towards the ground-state. Our cooling protocol may also be combined with various quantum refrigerator schemes proposed based on fundamental thermodynamic principles [61–65], to further ex-
explore quantum enhanced cooling in the nanoscale.

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Data availability statement — The code used to generate the figures shown in this work can be found in the following online GitHub repository.

[1] G. Wendin, Reports on Progress in Physics 80, 106001 (2017).
[2] M. Kjaergaard, M. E. Schwartz, J. Braunmüller, P. Krantz, J. I.-J. Wang, S. Gustavsson, and W. D. Oliver, Annual Review of Condensed Matter Physics 11, 369 (2020).
[3] M. Saffman, T. G. Walker, and K. Molmer, Reviews of modern physics 82, 2313 (2010).
[4] M. Tomza, K. Jachymski, R. Gerritsma, A. Negretti, T. Calarco, Z. Idziaszek, and P. S. Julienne, Reviews of modern physics 91, 035001 (2019).
[5] D.-I. Cho, S. Hong, M. Lee, T. Kim, et al., Micro and Nano Systems Letters 3, 1 (2015).
[6] G. Burkard, T. D. Ladd, J. M. Nichol, A. Pan, and J. R. Petta, arXiv preprint arXiv:2112.08863 (2021).
[7] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Reviews of Modern Physics 86, 1391 (2014).
[8] M. Hofheinz, E. Weig, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. O’connell, H. Wang, J. M. Martinis, and A. Cleland, Nature 454, 310 (2008).
[9] D. Matsukevich and A. Kuzmich, Science 306, 663 (2004).
[10] Y. Makhlin, G. Schön, and A. Shnirman, Reviews of modern physics 73, 357 (2001).
[11] K. Reuer, J.-C. Besse, L. Wernli, P. Magnard, P. Kurpiers, G. J. Norris, A. Walraff, and C. Eichler, Physical Review X 12, 011008 (2022).
[12] H. M. Wiseman and G. J. Milburn, Quantum measurement and control (Cambridge university press, 2009).
[13] R. Vijay, C. Macklin, D. Slichter, S. Weber, K. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Nature 490, 77 (2012).
[14] F. Giazotto, T. T. Heikkilä, A. Luukkani, A. M. Savin, and J. P. Pekola, Reviews of Modern Physics 78, 217 (2006).
[15] F. Marquardt and S. M. Girvin, Physics 2, 40 (2009).
[16] J. Millen, T. S. Monteiro, R. Pettit, and A. N. Vamivakas, Reports on Progress in Physics 83, 026401 (2020).
[17] C. Gonzalez-Ballestero, M. Aspelmeyer, L. Novotny, R. Quidant, and O. Romero-Isart, Science 374, eaaz3027 (2021).
[18] U. Vool and M. Devoret, International Journal of Circuit Theory and Applications 45, 897 (2017).
[19] B. Rogers, N. L. Gullo, G. De Chiara, G. M. Palmia, and M. Paternostro, Quantum Measurements and Quantum Metrology 2 (2014).
[20] S. Qvarfort, A. Serafini, P. F. Barker, and S. Bose, Nature Communications 9, 3690 (2018).
[21] M. Rademacher, J. Millen, and Y. L. Li, Advanced Optical Technologies 9, 227 (2020).
[22] S. Qvarfort, A. D. K. Plato, D. E. Bruschi, F. Schneider, D. Braun, A. Serafini, and D. Rätzel, Physical Review Research 3, 013150 (2021).
[23] D. C. Moore and A. A. Geraci, Quantum Science and Technology 6, 014008 (2021).
[24] S. Qvarfort, D. Rätzel, and S. Stoppyra, New Journal of Physics (2021).
[25] K. Stannigel, P. Komar, S. Habraken, S. Bennett, M. D. Lukin, P. Zoller, and P. Rabl, Physical Review Letters 109, 013603 (2012).
[26] H. Ulbricht, Testing fundamental physics by using levitated mechanical systems, in Molecular Beams in Physics and Chemistry: From Otto Stern’s Pioneering Exploits to Present-Day Feats (Springer International Publishing, Cham, 2021) pp. 303–332.
[27] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. Allman, C. Cicak, A. J. Sirois, J. D. Whitaker, K. W. Lehnert, and R. W. Simmonds, Nature 475, 359 (2011).
[28] J. Chan, T. M. Alegre, A. H. Safavi-Naeini, J. T. Hill, A. Krause, S. Gröblacher, M. Aspelmeyer, and O. Painter, Nature 478, 89 (2011).
[29] T. Li, in Fundamental Tests of Physics with Optically Trapped Microspheres (Springer, 2013) pp. 81–110.
[30] F. Tebbenjohanns, M. Frimmer, A. Militaru, V. Jain, and L. Novotny, Physical Review Letters 122, 223601 (2019).
[31] P. Barker, Physical Review Letters 105, 073002 (2010).
[32] U. Delić, M. Reisenbauer, K. Dare, D. Grass, V. Vuletić, N. Kiesel, and M. Aspelmeyer, Science 367, 892 (2020).
[33] J. Gieseler, B. Deutsch, R. Quidant, and L. Novotny, Physical Review Letters 109, 103603 (2012).
[34] V. Jain, J. Gieseler, C. Moritz, C. Dellago, R. Quidant, and L. Novotny, Physical Review Letters 116, 243601 (2016).
[35] F. Tebbenjohanns, M. L. Mattana, M. Rossi, M. Frimmer, and L. Novotny, Nature 595, 378 (2021).
[36] L. Magrini, P. Rosenzweig, C. Bach, A. Deutschmann-Olej, S. G. Hofer, S. Hong, N. Kiesel, A. Kugi, and M. Aspelmeyer, Nature 595, 373 (2021).
[37] J. Guo, R. Norte, and S. Gröblacher, Physical Review Letters 123, 223602 (2019).
[38] S. Qvarfort, A. Serafini, A. Xuereb, D. Braun, D. Rätzel, and D. E. Bruschi, Journal of Physics A: Mathematical and Theoretical 53, 075304 (2020).
[39] E. Mathieu, Journal de Mathematiques Pures et Appliques 13, 137 (1868).
[40] The addition of the second term in Eq. (1) allows us to redefine $\omega_0 \rightarrow \omega_0 \sqrt{1 + 4f'(t)/\omega_0}$, such that we obtain a new time-dependent frequency. However, doing so also requires us to treat the mode operators $\hat{a}$ and $\hat{a}^\dagger$ as time-dependent operators. For now, we will treat the last term
in Eq. (1) as a separate interaction term which causes the initially defined operators $\hat{a}^\dagger, \hat{a}$ to evolve, and this helps in making analytical predictions for the main results of the paper. Compared numerical simulations are carried out in the former approach, where the quantum oscillator is time-evolved with a time-dependent frequency, $\omega(t) = \omega_0 \sqrt{1 + 4f(t)/\omega_0}$.

[41] F. Schneiter, S. Qvarfort, A. Serafini, A. Xuereb, D. Braun, D. Rätzle, and D. E. Bruschi, Physical Review A 101, 033834 (2020).

[42] C. M. Bender, S. Orszag, and S. A. Orszag, Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory, Vol. 1 (Springer Science & Business Media, 1999).

[43] T. W. Penny, A. Pontin, and P. F. Barker, Physical Review A 104, 023502 (2021).

[44] E. Arthurs and J. L. Kelly, Bell System Technical Journal 44, 725 (1965).

[45] S. K. Manikandan, C. Elouard, K. W. Murch, A. Aufèves, and A. N. Jordan, arXiv preprint arXiv:2107.13234 (2021).

[46] While a linear feedback using both amplitude and phase information can be implemented in practice, simultaneously implementing this feedback may be limited in certain experimental platforms.

[47] K. Husimi, Proceedings of the Physico-Mathematical Society of Japan. 3rd Series 22, 264 (1940).

[48] R. Landauer, IBM journal of research and development 5, 183 (1961).

[49] A. Levy, L. Diósi, and R. Kosloff, Physical Review A 93, 052119 (2016).

[50] H. Leitch, N. Piccione, B. Bellomo, and G. De Chiara, AVS Quantum Science 11, 044041 (2019).

[51] F. Tebbenjohanns, Linear feedback cooling of a levitated nanoparticle in free space, Ph.D. thesis, ETH Zurich (2020).

[52] M. Blencowe, Physics Reports 395, 150 (2004).

[53] J. Vovrosh, M. Rashid, D. Hempston, J. Bateman, M. Paternostro, and H. Ulbricht, JOSA B 34, 1421 (2017).

[54] T. Wang, S. Lourette, S. R. O’Kelley, M. Kayci, Y. Band, D. F. J. Kimball, A. O. Sushkov, and D. Budker, Physical Review Applied 11, 044041 (2019).

[55] A. Vinante, P. Falferi, G. Gasbarri, A. Setter, C. Timberlake, and H. Ulbricht, Physical Review Applied 13, 064027 (2020).

[56] C. W. Lewandowski, T. D. Knowles, Z. B. Etienne, and B. D’Urso, Phys. Rev. Applied 15, 014050 (2021).

[57] M. G. Latorre, A. Paradkar, G. Higgins, and W. Wieczorek, arXiv preprint arXiv:2109.15071 (2021).

[58] M. R. Vanner, I. Pikovski, G. D. Cole, M. Kim, C. Brukner, K. Hammerer, G. J. Milburn, and M. Aspelmeyer, Proceedings of the National Academy of Sciences 108, 16182 (2011).

[59] L. A. Kanari-Naish, J. Clarke, S. Qvarfort, and M. R. Vanner, arXiv preprint arXiv:2109.08525 (2021).

[60] P. Campagne-Ibarcq, P. Six, L. Bretheau, A. Sarlette, M. Miraahi, P. Rouchon, and B. Huard, Physical Review X 6, 011002 (2016).

[61] B. Karimi and J. Pekola, Physical Review B 94, 184503 (2016).

[62] A. Levy and R. Kosloff, Physical Review Letters 108, 070604 (2012).

[63] S. K. Manikandan, F. Giazotto, and A. N. Jordan, Physical Review Applied 11, 054034 (2019).

[64] S. K. Manikandan, A. Jussain, and A. N. Jordan, Physical Review B 102, 235427 (2020).

[65] A. Fornieri, G. Timossi, P. Virtanen, P. Solinas, and F. Giazotto, Nature Nanotechnology 12, 425 (2017).

[66] J. Wei and E. Norman, Journal of Mathematical Physics 4, 575 (1963).

[67] D. E. Bruschi, A. R. Lee, and I. Fuentes, Journal of Physics A: Mathematical and Theoretical 46, 165303 (2013).

[68] H. R. Lewis Jr and W. Riesenfeld, Journal of mathematical physics 10, 1458 (1969).

[69] We here also correct a few typos found in the original derivations.

[70] I. Kovacic, R. Rand, and S. Mohamed Sah, Applied Mechanics Reviews 70 (2018).

[71] A. Serafini, M. Lostaglio, S. Longden, U. Shackerley-Bennett, C.-Y. Hsieh, and G. Adesso, Phys. Rev. Lett. 124, 010602 (2020).

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**Supplemental note 1. Derivation of the dynamics**

In this appendix, we connect the derivation of the solutions for the dynamics, which were first presented in [38], with a more intuitive solution using a Lie algebra method [66]. We identify a set of operators that is closed under commutation, which allows us to set up differential equations that, when solved, provide the exact solution to the dynamics. We then show that these solutions can be mapped to those derived in [38]. The solutions and the derivations build on methods also developed in Ref [67]. In addition, we note that the dynamics of this form may also be treated using the exact Lewis–Riesenfeld solutions [68].

We start by identifying the elements of the Lie algebra that generate the time-evolution induced by the Hamiltonian in Eq. (1). The elements are

\[
\begin{align*}
\hat{a}^\dagger \hat{a},
\hat{a}^2 + \hat{a}^\dagger^2,
i (\hat{a}^\dagger^2 - \hat{a}^2).
\end{align*}
\]  

(S1.1)

It can be verified that the algebra is closed under commutation. The corresponding symplectic matrices in the $\begin{pmatrix} \hat{a}, \hat{a}^\dagger \end{pmatrix}$
basis, which we call $A_0$, $A_+$, and $A_-$ are given by

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_+ = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_- = 2i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (S1.2)$$

The symplectic matrix that encodes the evolution of the system is given by

$$S(t) = T\exp \left[ \Omega \int_0^t dt' \, H(t') \right]. \quad (S1.3)$$

We then differentiate this matrix with respect to time $t$. We find

$$\frac{d}{dt} S(t) = \Omega H(t) S(t). \quad (S1.4)$$

We then multiply the expression by $S^{-1}(t)$ on the right-hand side to find

$$S(t) S^{-1}(t) = \Omega H(t). \quad (S1.5)$$

Then, we make the following ansatz for the solution to $S(t)$:

$$S(t) = e^{j_0 \Omega A_0} e^{J_+ \Omega A_+} e^{J_- \Omega A_-}, \quad (S1.6)$$

where $J_0$, $J_+$, and $J_-$ are time-dependent coefficients that we wish to find. The equivalent Hilbert space ansatz is

$$\hat{U}(t) = e^{-iJ_0 \hat{a} \hat{a}^\dagger} e^{-iJ_+ (\hat{a}^\dagger \hat{a} + i\hat{\alpha})} e^{-iJ_- [i(\hat{a}^2 - \hat{\alpha}^2)]}. \quad (S1.7)$$

We note that the operators in Eq. (S1.7) are equivalent to single-mode squeezing and a phase rotation with $\hat{a} \hat{a}^\dagger$. The connection between the Hilbert space picture and the phase-space picture is

$$\hat{U}(t) \hat{X} \hat{U}(t) = S(t) \hat{X}, \quad (S1.8)$$

where $\hat{X} = (\hat{a}, \hat{a}^\dagger)^T$, as in the main text. We then differentiate the ansatz in Eq. (S1.6) to find

$$\dot{S}(t) = \dot{j}_0 \Omega A_0 + \dot{J}_+ \Omega A_+ e^{-J_- \Omega A_-} + \dot{J}_- e^{J_0 \Omega A_0} e^{J_+ \Omega A_+} e^{J_- \Omega A_-} + \dot{j}_- e^{J_0 \Omega A_0} e^{J_+ \Omega A_+} e^{-J_- \Omega A_-} e^{-J_0 \Omega A_0}. \quad (S1.9)$$

Multiplying by $S^{-1}(t)$ on the right, we find

$$\dot{S}(t) S^{-1}(t) = \dot{j}_0 \Omega A_0 + \dot{J}_+ e^{J_0 \Omega A_0} \Omega A_+ e^{-J_- \Omega A_-} + \dot{J}_- e^{J_0 \Omega A_0} e^{J_+ \Omega A_+} \Omega A_- e^{-J_- \Omega A_-} e^{-J_0 \Omega A_0}. \quad (S1.10)$$

Multiplying both expressions by $\Omega^{-1}$ on the left and using Eq. (S1.4) gives us

$$H(t) = \dot{j}_0 A_0 + \Omega^{-1} \dot{J}_+ e^{J_0 \Omega A_0} \Omega A_+ e^{-J_- \Omega A_-} + \Omega^{-1} \dot{J}_- e^{J_0 \Omega A_0} e^{J_+ \Omega A_+} \Omega A_- e^{-J_- \Omega A_-} e^{-J_0 \Omega A_0}. \quad (S1.11)$$

Then we also know that the symplectic matrices obey $S \Omega S^\dagger = \Omega$. This allows us to rewrite Eq. (S1.11) as

$$H(t) = \dot{j}_0 A_0 + \dot{J}_+ e^{J_0 \Omega A_0} A_+ e^{-J_- \Omega A_-} + \dot{J}_- e^{J_0 \Omega A_0} e^{J_+ \Omega A_+} A_- e^{-J_- \Omega A_-} e^{-J_0 \Omega A_0}, \quad (S1.12)$$

which, after multiplying out the matrices, leaves us with

$$H(t) = \begin{pmatrix} \dot{j}_0 + 2\sinh(4J_+ \dot{J}_-) \quad 2e^{-2iJ_0} [i \cosh(4J_+) \dot{J}_- + \dot{J}_+] \\ e^{2iJ_0} [-i \cosh(4J_+) \dot{J}_- + \dot{J}_+] \quad \dot{j}_0 + 2\sinh(4J_+) \dot{J}_- \end{pmatrix}. \quad (S1.13)$$

However, we also know that the Hamiltonian matrix is given by

$$H(t) = \begin{pmatrix} 1 + 2f(t)/\omega_0 & 2f(t)/\omega_0 \\ 2f(t)/\omega_0 & 1 + 2f(t)/\omega_0 \end{pmatrix}. \quad (S1.14)$$

Equating Eq. (S1.13) and Eq. (S1.14) allows us to identify the following differential equations

$$1 + 2f(t) = \dot{j}_0 + 2\sinh(4J_+) \dot{J}_-, \quad 2f(t) = 2e^{-2iJ_0} [i \cosh(4J_+) \dot{J}_- + \dot{J}_+]. \quad (S1.15)$$
Now, we have three coefficients, yet we also know that the matrix $S(t)$ consists of four independent components, $\alpha(t)$ and $\beta(t)$, which are both complex functions. The two Bogoliubov relations add two additional constraints to $S(t)$, namely $|\alpha(t)|^2 - |\beta(t)|^2 = 1$ and $\alpha(t)\beta'(t) - \alpha'(t)\beta(t) = 0$, which means that the three differential equations can be written as two. In practice, it means that if we know two of the $J$-coefficients, we can always find the third one by using the Bogoliubov coefficients.

By manipulating Eq. (S1.15), it is possible to isolate the three coefficients $J_0$, $J_+$, and $J_-$, into the following three differential equations [41]

\[
\begin{align*}
\dot{J}_0 &= 1 + 2f(t) \left[ 1 - \sin(2J_0) \tanh(4J_+) \right], \\
\dot{J}_+ &= f(t) \cos(2J_0), \\
\dot{J}_- &= f(t) \frac{\sin(2J_0)}{\cosh(4J_+)}.
\end{align*}
\] (S1.16)

We note, however, that $J_-$ does not feature in the first and second equations for $\dot{J}_0$ and $\dot{J}_+$, which means that it can be entirely solved once the other two equations have been solved. This confirms that $S(t)$ is fully determined by only two real parameters.

We now wish to relate $J_0$, $J_+$, and $J_-$ to the functions $P(t)$ and $Q(t)$ in Eq. (9). For the derivation of $P(t)$ and $Q(t)$, see Appendix B in Ref [38]. By rewriting $S(t)$ in Eq. (S1.6) as a single symplectic operator, we find that $\alpha(t)$ and $\beta(t)$ can be written as [41]

\[
\begin{align*}
\alpha(t) &= e^{-iJ_0} \left[ \cosh(2J_+) \cosh(2J_-) - i \sinh(2J_+) \sinh(2J_-) \right], \\
\beta(t) &= e^{-iJ_0} \left[ \cosh(2J_+) \sinh(2J_-) - i \sinh(2J_+) \cosh(2J_-) \right].
\end{align*}
\] (S1.17)

Also from Eq. (9), we identify the following relationships

\[
\begin{align*}
P(t) &= \text{Re}[\alpha] + \text{Re}[\beta], \\
\bar{P}(t) &= \text{Im}[\alpha] + \text{Im}[\beta], \\
Q(t) &= \text{Im}[\beta] - \text{Im}[\alpha], \\
\bar{Q}(t) &= \text{Re}[\alpha] - \text{Re}[\beta].
\end{align*}
\] (S1.18)

It is then possible to write $P(t)$ and $Q(t)$ in terms of $J_0$, $J_+$, and $J_-$ as

\[
\begin{align*}
P(t) &= e^{2J_0} \left[ \cos(J_0) \cosh(2J_0) - \sin(J_0) \sinh(2J_0) \right], \\
Q(t) &= e^{-2J_0} \left[ \sin(J_0) \cosh(2J_0) - \cos(J_0) \sinh(2J_0) \right].
\end{align*}
\] (S1.19)

And similarly, the second derivatives can be found. They are however quite long expressions, so we do not print them here. We then recall that $P(t)$ and $Q(t)$ are determined by the following two differential equations

\[
\begin{align*}
\dot{P}(t) + \left[ 1 + 4f(t)/\omega_0 \right] P(t) &= 0, \\
\dot{Q}(t) + \left[ 1 + 4f(t)/\omega_0 \right] Q(t) &= 0.
\end{align*}
\] (S1.20)

By then inserting the expressions in Eq. (S1.19) and their derivatives into Eq. (S1.20), and using the relations in Eq. (S1.16), it is possible to show that $J_0$, $J_+$ and $J_-$ and their relationship are also solutions to these equations.

Next, we note that it is also possible to define $J_0$, $J_+$, and $J_-$ in terms of $P(t)$ and $Q(t)$. Previously, it was shown that [41]

\[
\begin{align*}
\cosh(4J_+) &= |\alpha^2(t) - \beta^2(t)|, \\
\cosh(4J_-) &= \frac{|\alpha(t)|^2 + |\beta(t)|^2}{|\alpha^2(t) - \beta^2(t)|}, \\
e^{-2iJ_0} &= \frac{\alpha^2(t) - \beta^2(t)}{|\alpha^2(t) - \beta^2(t)|}.
\end{align*}
\] (S1.21)

With the help of the relations in Eq. (S1.21), we can identify

\[
\begin{align*}
\cosh(4J_+) &= \left| \left( i\dot{P}(t) + P(t) \right) \left( \dot{Q}(t) - iQ(t) \right) \right|, \\
\cosh(4J_-) &= \frac{1}{2} \left| \left( i\dot{P}(t) + P(t) \right) \left( \dot{Q}(t) - iQ(t) \right) \right|^2, \\
e^{-2iJ_0} &= \frac{1}{2} \left( i\dot{P}(t) + P(t) \right) \left( \dot{Q}(t) - iQ(t) \right) \left( \dot{Q}(t) - iQ(t) \right)^*.
\end{align*}
\] (S1.22)
Finally, we note that the solutions $P(t)$ and $Q(t)$ are valid for any choice of driving function $f(t)$. The case of parametric modulations explored in the main text leads to Mathieu’s equation, but many other driving patterns can be considered using these methods.

**Supplemental note 2. Approximate solutions to the dynamics**

In this supplemental note, we outline the derivation of the dynamics generated by the Hamiltonian in Eq. (1). We solve the dynamics perturbatively using two separate methods: firstly, we use a two-time perturbative solution of the Mathieu equation, which underpins the dynamics, and secondly, we use a perturbative solution of the unitary time-evolution operator $\hat{U}(t)$ in Eq. (4) to first order. We also examine the stability and error of the solutions, where we show that the approximate solution for $\hat{U}(t)$ produces an error in the Bogoliubov coefficients that grows in time, while the Mathieu equation result in an error that oscillates in time.

1. **Approximate solutions to the Mathieu equation**

We here outline the derivations of the approximate solutions to Mathieu’s equation shown in Eq. (12) in the main text[69]. These solutions are valid for $2\lambda/\omega_0 \ll 1$ and were first presented in Refs [38] and [22]. The method used to obtain the approximate solutions are based on a two-time solution presented in Ref [70].

To simplify the notation in this section, we redefine $\lambda/\omega_0 \rightarrow \lambda$ and $t\omega_0 \rightarrow t$ to be dimensionless parameters. The differential equation that we obtain from solving the dynamics is given by

$$\frac{d^2 y}{dt^2} + [1 + 4\lambda \cos(2t + \phi_p)] y = 0,$$

(S2.1)

where $\lambda$ is the driving strength and $\phi_p$ is the phase of the drive. Defining $\epsilon = -2\lambda$ and $a = 1$, we can write Eq. (S2.1) in the standard form, which we recognize as Mathieu’s equation with an extra phase

$$\frac{d^2 y}{dt^2} + [a - 2\epsilon \cos(2t + \phi_p)] y = 0.$$

(S2.2)

We now wish to find an approximate form of the solutions $y$. The method presented in Ref [70] makes the assumption that the system evolves with two distinct time-scales: a slow time-scale $T = \epsilon t$, and the faster time-scale $t$. We assume that the solutions $y$ depend on both time scales, such that $y(t,T)$. In what follows, we use the slower scale to infer corrections to the zeroth-order behavior of the solution.

Defining the slow and fast time-scales as $T$ and $t$ allows us to split the absolute derivative $d/dt$ in Eq. (S2.2) into two independent parts:

$$\frac{d}{dt} = \partial_t + \epsilon \partial_T.$$  

(S2.3)

We can then rewrite Eq. (S2.2) as

$$(\partial_t + \epsilon \partial_T)^2 y(t,T) + [a - 2\epsilon \cos(2t + \phi_p)] y(t,T) = 0.$$  

(S2.4)

We then expand the solution $y(t,T)$ for small $\epsilon$ (this is also where the assumption for $\epsilon \ll 1$ comes in) to find

$$y(t,T) = y_0(t,T) + \epsilon y_1(t,T) + O(\epsilon^2).$$

(S2.5)

We then insert Eq. (S2.5) into Eq. (S2.4). The assumption of small $\epsilon$ also allows us to expand the second-order derivative as $(\partial_t + \epsilon \partial_T)^2 \approx \partial_t + 2\epsilon \partial_t \partial_T + O(\epsilon^2)$. Suppressing the dependence on $t$ and $T$ in the solutions for notational clarity, we find

$$\partial_t^2 y_0 + 2\epsilon \partial_t \partial_T y_0 + \epsilon \partial_T^2 y_1 + [a - 2\epsilon \cos(2t + \phi_p)] y_0 + a \epsilon y_1 + O(\epsilon^2) = 0.$$  

(S2.6)

To zeroth order in $\epsilon$, we recover the differential equation for the free harmonic oscillator with solution $y_0(t,T)$, which is also the limiting case of Eq. (S2.2) as $\epsilon \rightarrow 0$. It is given by

$$\partial_T^2 y_0 + a y_0 = 0.$$  

(S2.7)
We use the following trial solution to solve Eq. (S2.7):  
\[ y_0(t, T) = A(T) e^{i \sqrt{\alpha} t} + A^*(T) e^{-i \sqrt{\alpha} t}. \]  
(S2.8)

Here, \( A(T) \) is a yet-to-be-determined oscillation amplitude which only depends on the slow time-scale \( T \). Next, we examine the equation for \( y_1(t, T) \). The remaining terms in Eq. (S2.6) are

\[ \partial^2_T y_1 + 2 \partial_T y_0 + y_1 a - 2 \cos(2t + \phi_p) y_0 = 0, \]  
(S2.9)

where we have divided by \( \epsilon \).

To solve Eq. (S2.9), we insert our zeroth-order solution for \( y_0(t, T) \) shown in Eq. (S2.8). We find

\[ \partial^2_T y_1 + y_1 a + 2i \sqrt{\alpha} \left( \frac{\partial A(T)}{\partial T} e^{i \sqrt{\alpha} t} - \frac{\partial A^*(T)}{\partial T} e^{-i \sqrt{\alpha} t} \right) \]

\[ - 2 \cos(2t + \phi_p) \left( A(T) e^{i \sqrt{\alpha} t} + A^*(T) e^{-i \sqrt{\alpha} t} \right) = 0. \]  
(S2.10)

We now specialize to \( a = 1 \). The expression in Eq. (S2.10) can be written as

\[ \partial^2_T y_1 + y_1 + 2i \left( \frac{\partial A(T)}{\partial T} e^{it} - \frac{\partial A^*(T)}{\partial T} e^{-it} \right) - \left( e^{i(2t + \phi_p)} + e^{-i(2t + \phi_p)} \right) (A(T) e^{it} + A^*(T) e^{-it}) = 0. \]  
(S2.11)

We then multiply out the exponentials in the second term of Eq. (S2.11) to find

\[ \partial^2_T y_1 + y_1 + 2i \left( \frac{\partial A(T)}{\partial T} - A^*(T) e^{i \phi_p} \right) e^{it} + \left( 2i \frac{\partial A^*(T)}{\partial T} + A(T) e^{-i \phi_p} \right) e^{-it} \]

\[ - A(T) e^{3it + i \phi_p} - A^*(T) e^{-3it - i \phi_p} = 0. \]  
(S2.12)

Now, in order for the zeroth-order perturbative solution \( y_0(t, T) \) to be stable, we require that secular terms such as resonant terms \( e^{\text{i} \epsilon t} \) vanish. If this is not the case, the perturbation \( y_1 \) will grow linearly in time and diverge [70]. In addition, we can neglect terms that oscillate much faster, such as \( e^{3\text{i} \epsilon t} \). To make sure the resonant terms do not contribute to the approximate solution, we require that

\[ \left( 2i \frac{\partial A^*(T)}{\partial T} + A(T) e^{-i \phi_p} \right) = 0. \]  
(S2.13)

We then adopt the trial solution \( A(T) = (c_1 - i c_2) e^{(T + i \phi_p)/2} + (c_3 - i c_4) e^{-(T - i \phi_p)/2} \), where the parameters \( c_1, c_2, c_3 \) and \( c_4 \) are to be determined. Inserting this trial solution in Eq. (S2.13) allows us to fix two of the coefficients. We find that in order for Eq. (S2.13) to be zero, we require that \( c_1 = c_2 \) and \( c_3 = -c_4 \). As a result, the trial solution \( A(T) \) becomes

\[ A(T) = c_1 (1 - i) e^{(T + i \phi_p)/2} + c_3 (1 + i) e^{-(T - i \phi_p)/2}. \]  
(S2.14)

Then, inserting Eq. (S2.14) into the expression for \( y_0(t, T) \) shown in Eq. (S2.8), and recalling that \( T = \epsilon t \), we find

\[ y_0(t) = A(\epsilon t) e^{i \epsilon t} + A^*(\epsilon t) e^{-i \epsilon t} \]

\[ = (c_1 (1 - i) e^{i \epsilon t/2} + c_3 (1 + i) e^{-i \epsilon t/2}) e^{i t + i \phi_p/2} + (c_1 (1 + i) e^{i \epsilon t/2} + c_3 (1 - i) e^{-i \epsilon t/2}) e^{-i t - i \phi_p/2}. \]  
(S2.15)

Then, we recall that \( \epsilon = -2 \lambda \). With a small rearrangement, the final expression for the perturbative solution is

\[ y_0(t) = 2 \left( c_1 e^{-\lambda t} + c_3 e^{\lambda t} \right) \cos(t + \phi_p/2) + 2 \left( c_3 e^{-\lambda t} - c_1 e^{\lambda t} \right) \sin(t + \phi_p/2). \]  
(S2.16)

The remaining coefficients \( c_1 \) and \( c_3 \) can then be fixed by examining the initial conditions for \( y_0(t) \), which in our case are the initial conditions for \( P(t) \) and \( Q(t) \). We recall from the main text that they are \( P(t=0) = 1 \) and \( \dot{P}(t=0) = 0 \), as well as \( Q(t=0) = 0 \) and \( \dot{Q}(t=0) = 1 \). Given these initial conditions, we find in the case of \( P(t) \) that

\[ c_1 = \frac{(\lambda - 1) \cos(\phi_p/2) - (\lambda + 1) \sin(\phi_p/2)}{4 [\lambda \cos(\phi_p) - 1]}, \quad c_3 = \frac{(\lambda - 1) \cos(\phi_p/2) + (\lambda + 1) \sin(\phi_p/2)}{4 [\lambda \cos(\phi_p) - 1]}. \]  
(S2.17)
For $Q(t)$, we find that the coefficients become

$$c_1 = \frac{\sin(\phi_p/2) - \cos(\phi_p/2)}{4 \left[ \lambda \cos(\phi_p) - 1 \right]}, \quad c_3 = \frac{\sin(\phi_p/2) + \cos(\phi_p/2)}{4 \left[ \lambda \cos(\phi_p) - 1 \right]}.$$  (S2.18)

Inserting these into the solution $y_0(t)$ in Eq. (S2.16), we obtain the approximate solutions for $P(t)$ and $Q(t)$ [22, 38]:

$$P(t) = \frac{\left[ \lambda \cos(t + \phi_p) - \cos(t) \right] \cosh(\lambda t) + \left[ \sin(t + \phi_p) - \lambda \sin(t) \right] \sinh(\lambda t)}{\lambda \cos(\phi_p) - 1},$$

$$Q(t) = \frac{\cos(t + \phi_p) \sinh(\lambda t) - \sin(t) \cosh(\lambda t)}{\lambda \cos(\phi_p) - 1},$$  (S2.19)

which are equivalent to those shown in Eq. (12) in the main text after expanding in $\lambda$ to first order, and with factors of $\omega_0$ restored.

2. Time-evolution perturbation theory

We now present an alternative method by which the dynamics can be solved. To treat the dynamics of the Hamiltonian in Eq. (1) in the main text, we make use of time-dependent perturbation theory. We start by writing down the time-evolution operator $\hat{U}(t)$

$$\hat{U}(t) = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^t dt' \hat{H}(t') \right].$$  (S2.20)

We now divide the Hamiltonian in Eq. (1) into two parts: One that contains a modified free evolution term with $\hat{a}^\dagger \hat{a}$, and the other part that contains the interaction term

$$\hat{H}_0(t) = \hbar [\omega_0 + 2 f(t)] \hat{a}^\dagger \hat{a},$$

$$\hat{H}_I(t) = \hbar f(t) (\hat{a}^2 + \hat{a}^\dagger),$$

where we have ignored a scalar term since it results in a global phase. We then consider the frame that rotates with $\hat{H}_0(t)$. For the choice of $f(t) = \lambda \cos(2\omega_0 t + \phi_p)$ in this paper, the evolution generated by $\hat{H}_0(t)$ is given by

$$\hat{U}_0(t) = \exp \left[ -i \int_0^t dt' (\omega_0 + 2 f(t')) \hat{a}^\dagger \hat{a} \right] = e^{-i\theta(t) \hat{a}^\dagger \hat{a}}.$$  (S2.21)

FIG. 3. Plot showing the difference between a numerically obtained solution of Mathieu’s equation and the approximate expressions in Eq. (S2.19). Here $\Delta P$ and $\Delta Q$ are the differences between the numerically obtained solution and the approximate solutions plotted for various values of $\lambda/\omega_0$. As the driving strength increases, the errors start diverging. For the values used in this paper $\lambda/\omega_0 = 0.01$, the solutions stay stable and the errors small.
where we have defined \( \theta(t) = \left[ \omega_0 t + \frac{2}{\omega_0} \cos(\omega_0 t + \phi) \sin(\omega_0 t) \right] \). The interaction Hamiltonian in this frame evolves with \( \hat{U}_0(t) \) such that

\[
\hat{H}_I(t) = \hat{U}_0^\dagger(t) \hat{H}_I(t) \hat{U}_0(t) = h f(t) \left( e^{2i\theta(t)} \hat{a}^\dagger \hat{a} + e^{-2i\theta(t)} \hat{a}^2 \right),
\]

where we have used the fact that \( e^{ix} \hat{a} e^{-ix} \hat{a}^\dagger = e^{-ix} \hat{a} \). The evolution operator in the interaction frame is therefore

\[
\hat{U}_I(t) = \mathcal{T}\exp \left[ -i \int_0^t dt' f(t') \left( e^{2i\theta(t')} \hat{a}^\dagger \hat{a} + e^{-2i\theta(t')} \hat{a}^2 \right) \right].
\]

Returning to the lab-frame, the full evolution can be written as \( \hat{U}(t) = \hat{U}_0(t) \hat{U}_I(t) \). When \( \lambda t \ll 1 \), we can expand the exponential in Eq. (S2.24) to first order in \( \lambda \) to find

\[
\hat{U}_I(t) = 1 - i \lambda \int_0^t dt' \cos(2\omega_0 t' + \phi_p) \left( e^{2i\theta(t')} \hat{a}^\dagger \hat{a} + e^{-2i\theta(t')} \hat{a}^2 \right) + O(\lambda^2).
\]

We then examine the evolution of \( \hat{a} \). We find

\[
\hat{a}(t) = \hat{U}^\dagger(t) \hat{a} \hat{U}(t) = \hat{U}_I^\dagger(t) \hat{U}_0(t) \hat{U}_0(t) \hat{U}_I(t) = e^{-i\theta(t)} \hat{U}_I^\dagger(t) \hat{a} \hat{U}_I(t).
\]

Then, we insert the approximate form of \( \hat{U}_I(t) \) shown in Eq. (S2.25) into Eq. (S2.26) to find

\[
\hat{a}(t) = e^{-i\theta(t)} \left\{ \hat{a} + i \lambda \int_0^t dt' \cos(2\omega_0 t' + \phi_p) \left[ (e^{2i\theta(t')} \hat{a}^\dagger \hat{a} + e^{-2i\theta(t')} \hat{a}^2) \hat{a} \right] + O(\lambda^2) \right\} + O(\lambda^2).
\]

Expanding and evaluating the integral, we may identify the Bogoliubov coefficients \( \alpha(t) \) and \( \beta(t) \) as per Eq. (8).

\[
\alpha(t) \approx e^{-i\omega_0 t} - 2i \frac{\lambda}{\omega_0} \cos(\omega_0 t + \phi) \sin(\omega_0 t),
\]

\[
\beta(t) \approx -\frac{\lambda}{4\omega_0} e^{-i(\omega_0 t + \phi)} \left[ (e^{4i\omega_0 t} - 1) e^{2i\phi} + 4i\omega_0 t \right].
\]

As can be seen, both coefficients contain linear corrections of \( \lambda \), but they are a bit different from those derived in the previous section.

### 3. Error analysis of the perturbative solutions

The first step we perform in order to determine the error of the solutions is to plot the perturbative solutions in Eq. (S2.19) against numerically obtained solutions of Mathieu’s equation. We do so in Fig. 3, where we have defined \( \Delta P(t) \) and \( \Delta Q(t) \) as the deviations away from the numerical result. The phase is \( \phi_p = \pi/2 \). As can be seen, for short times the error remains similar in magnitude to the driving amplitude \( \lambda/\omega_0 \).

Another way in which we can quantify the errors of our approximate solutions is by considering the Bogoliubov normalization relation \( |\alpha|^2 - |\beta|^2 = 1 \). Starting with the solutions obtained by expanding \( \hat{U}(t) \), shown in Eq. (S2.28), we find to second order in \( \lambda \) that

\[
|\alpha(t)|^2 - |\beta(t)|^2 = 1 - (\omega_0 t)^2 + \frac{7}{8}(\lambda/\omega_0)^2 + \text{oscillating terms}.
\]

Here, we note from Eq. (S2.29) that the error grows with \( t \), which means that the solutions derived in Section 2.2 will become increasingly inaccurate.
FIG. 4. Plot showing the errors in the (a) Bogoliubov condition and (b) number of quanta given the numerically obtained solutions (black), the perturbative solutions of Mathieu’s equation in Eq. (13) (purple dotted), and the solutions from the expanded \( \hat{U}(t) \) in Eq. (S2.28) (blue dashed). The parameters are \( \lambda/\omega_0 = 0.02 \), \( \phi = 0 \), \( \phi_p = \pi/2 \), and \( r = \sqrt{10} \). The approximate solution to Mathieu’s equation follows the outline of the exact solution, but it fails to replicate some of the faster oscillations. As shown in Eq. (S2.30), the approximate solution is accurate whenever \( \omega_0 t \) is a multiple of \( \pi \), given that \( \phi_p = \pi/2 \).

In contrast, using the expressions for the perturbative solutions to the Mathieu equation in Eq. (13), we find that the error is given by

\[
|\alpha(t)|^2 - |\beta(t)|^2 \approx \frac{1 - \lambda \cos(2t\omega_0 + \phi_p)/\omega_0}{1 - \lambda \cos(\phi_p)/\omega_0}.
\]  

(S2.30)

We note that our solution is exact whenever \( \phi_p = n\pi/2 \) and \( 2\omega_0 t + \phi_p = n\pi/2 \), for integer \( n \). For example, when \( \phi_p = \pi/2 \), the solution is exact at \( \omega_0 t = \pi/2 \).

In Fig. 4, we compare the error of the approximate solutions shown in Eq. (S2.28) and Eq. (13) with a numerically obtained solution of Mathieu’s equation for an initial coherent state \( |re^{i\phi}\rangle \). The parameters are set to \( \phi = 0 \), \( \phi_p = \pi/2 \), \( \lambda/\omega_0 = 0.01 \) and \( r = \sqrt{10} \). We note a few things from this figure. Firstly, we note that the exact solutions (black lines) have a periodicity that is about twice that of the approximate solutions to Mathieu’s equation (purple dotted lines). The missed oscillations can also be observed as errors in Fig. 3. It might be possible to further improve the accuracy of the two-time scale solutions by adding a third time-scale, which is stretched by \( \sqrt{\epsilon} \). We leave such an analysis to future work. Secondly, we note that the error of the solutions from expanding \( \hat{U}(t) \) (blue dashed lines) grow in time, and thereby diverge from the numerical solutions to a greater extent than those obtained from Mathieu’s equation. They are however more accurate for shorter time-scales, since they reproduce the shorter oscillations of the numerically obtained solutions.

4. Mathieu equation stability analysis

The Mathieu equation is numerically unstable, which means that certain parameter combinations result in diverging solutions. When \( a = 1 \), there are in fact no stable solutions that can be obtained. However, since we measure the state at the beginning of each cooling cycle, we effectively reset the instabilities that would have been introduced for the full running time of the protocol. In this way, the inclusion of measurements also prevents the buildup of instability from the modulations of the potential.

We may examine the stability of Mathieu’s equation by plotting the solution to Eq. (11) in the long-time limit \( \omega_0 t = 10^3 \) as a function of the generic parameters \( a \) and the perturbation \( \epsilon \). We do so in Fig. 5. The purple regions indicate where the solutions reach a value greater than \( 10^3 \). The black dashes line indicates where \( a = 1 \), which, as expected, does not contain stable solutions.

The extension to non-unitary dynamics is likely to change the stability of the equations of motion, however such a stability analysis would require a full analytical solution of the open systems dynamics with the time-dependent frequency modulation. We leave this to future work.
Supplemental note 3. Protocol with strong homodyne measurements

Instead of projecting into the coherent-state basis, we may also consider homodyne measurements of the state. For completeness, we show here that the quanta of position eigenstates can also be reduced using the parametric modulations, however we find that the phase relation differs from that identified in the main text for coherent states.

We start again with the fact that \( \hat{a}(t) = \alpha(t) \hat{a} + \beta(t) \hat{a}^\dagger \). We then let the initial state \( |x_0\rangle \) be an eigenstate of the generalised quadrature operator \( \hat{x}_0 = (e^{i\theta} \hat{a}^\dagger + e^{-i\theta} \hat{a}) / \sqrt{2} \), where \( \theta \) is a phase. We can then define eigenstates of \( \hat{x}_0 \) such that \( \hat{x}_0 |x_0\rangle = x_0 |x_0\rangle \), where \( x_0 \) is the eigenvalue. Note that \( |x_0\rangle \) is not a proper normalised eigenstate because it is not square-integrable. The following overlap with the Fock state \( |n\rangle \) is however well-defined

\[
\langle \hat{x}_0 | n \rangle = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} e^{-x^2/2} H_n(x) e^{-in\theta},
\]

where \( H_n(x) \) is the Hermite polynomial. This means that we can expand the position eigenstates in the Fock basis as

\[
|x_0\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n| x_0 \rangle = \sum_{n=0}^{\infty} \frac{1}{\pi^{1/4} \sqrt{2^n n!}} e^{-x^2/2} H_n(x) e^{in\theta} |n\rangle.
\]

The number of quanta for a single position eigenstate is then given by

\[
\langle \hat{a}^\dagger \hat{a}(t) | x_0 \rangle = |\alpha(t)|^2 \langle x_0 | \hat{a}^\dagger \hat{a} | x_0 \rangle + \alpha(t) \beta(t) \langle x_0 | \hat{a}^2 | x_0 \rangle + \beta^*(t) \alpha(t) \langle x_0 | \hat{a}^\dagger \hat{a} | x_0 \rangle + |\beta(t)|^2 \langle x_0 | \hat{a}^\dagger \hat{a} | x_0 \rangle + 1.
\]

We then use the Fock basis expansion in Eq. (S3.2) to find

\[
\begin{align*}
\langle x_0 | \hat{a}^\dagger \hat{a} | x_0 \rangle &= \sum_{n=0}^{\infty} \frac{n}{\pi^{1/2} 2^n n!} e^{-x^2} H_n^2(x), \\
\langle x_0 | \hat{a}^2 | x_0 \rangle &= \sum_{n=0}^{\infty} \frac{1}{\pi^{1/2} 2^{n+1} n!} e^{-x^2} H_{n+2}(x) H_n(x) e^{-2in\theta}, \\
\langle x_0 | \hat{a}^\dagger^2 | x_0 \rangle &= \sum_{n=0}^{\infty} \frac{1}{\pi^{1/2} 2^{n+1} n!} e^{-x^2} H_n(x) H_{n+2}(x) e^{2in\theta}.
\end{align*}
\]

Here, we note that the expressions on the last two lines in Eq. (S3.4) can be negative, since the Hermite polynomials contain odd powers of \( x \) for odd \( n \). Thus it is possible to contain cooling with a strong homodyne measurement as well.
We now wish to derive an expression for the number of quanta akin to that in Eq. (16) in the main text, which shows the photon number for coherent states. Such an expression tells us what phase relationship we need between the phase of the generalised quadrature eigenstate \( \theta \) and the phase of the parametric modulation \( \phi_p \) for the system to be cooled. Since only the second two lines in Eq. (S3.4) have phases, we write them as

\[
\langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_{n+2}(x).
\] (S3.5)

Inserting this into Eq. (S3.3) and expanding to first order in \( \lambda \) (which requires us to assume that \( x \) is small), we find

\[
\langle \hat{a}^\dagger \hat{a}(t) \rangle \approx \langle x_0 \hat{a}^\dagger \hat{a} \rangle + \frac{\lambda}{\omega_0} \left\{ \langle x_0 \hat{a}^\dagger \hat{a} \rangle [\cos(\phi_p) - \cos(2\omega_0 t + \phi_p)] - 2t\omega_0 x_0 \sin(2\theta + \phi_p) \right\} + O(\lambda^2).
\] (S3.6)

The number of quanta can only decrease if the last term of Eq. (S3.6) is negative. To find out whether that is the case, we must first examine the sign of \( \bar{x}_0 \). Around \( x \sim 0 \), the following recurrence relation for the Hermite polynomials holds: \( H_{n+2}(0) = -2(n+2)H_n(0) \). Using this, we find that

\[
\bar{x}_0(x \sim 0) = -2 \sum_{n=0}^{\infty} \frac{(n+1)H_n^2(0)}{\pi^{1/2} 2^{n+1} n!}.
\] (S3.7)

Since all terms inside the sum in Eq. (S3.7) are positive, we deduce that \( \bar{x}_0 < 0 \) for small \( x \). This means that we require the term with \( \sin(2\theta + \phi_p) \) to be maximally negative, which is true when \( 2\theta + \phi_p = 3\pi/2 \). We note that this phase relationship is different from the coherent states, which required \( 2\theta + \phi_p = \pi/2 \). We plot \( \langle \hat{a}(t) \rangle \) as a function of time in Fig. 6 for various choices of the phase \( \phi_p \). The parameters are \( x = 0.5, \theta = 0, \) and \( \lambda/\omega_0 = 0.02 \).

For larger \( x \), however, \( \bar{x}_0 \) in Eq. (S3.7) is positive (this can be checked numerically), which means that the system is instead cooled when \( 2\theta + \phi_p = \pi/2 \). Since we cannot deterministically prepare the system in the eigenstate \( |x_0 \rangle \) where \( x_0 \) is small, we conclude that while homodyne measurements are an option for feedback cooling, heterodyne measurements are more reliable since the phase relation for coherent states remains the same regardless of the measurement outcome.

**Supplemental note 4. Applying parametric modulations to a thermal state**

Driving alone is insufficient to cool down arbitrary quantum states lacking a phase reference such as thermal states. To see this, we now consider the effect of applying the cycle to a thermal state of the quantum oscillator at inverse temperature \( \beta' = 1/k_B T \), given by,

\[
\hat{\rho}_{th} = \frac{e^{-\beta' \hat{H}_0}}{Z}, \quad \text{where} \quad Z = \text{tr} \{ e^{-\beta' \hat{H}_0} \}.
\] (S4.1)
We can compute \( \langle \hat{n}(t) \rangle_{\text{th}} \) to find,

\[
\langle \hat{n}(t) \rangle_{\text{th}} = (|\alpha(t)|^2 + |\beta(t)|^2) \hat{n} + |\beta(t)|^2 \\
= \hat{n} + (1 + 2\hat{n})|\beta(t)|^2,
\]

where we used the Bogoliubov identity \( |\alpha(t)|^2 = |\beta(t)|^2 + 1 \) and the fact that for thermal states, \( \langle \hat{a}^2(t) \rangle_{\text{th}} = \langle \hat{a}^{\dagger 2}(t) \rangle_{\text{th}} = 0 \). Here \( \hat{n} = \text{Tr} \{ \hat{a}^{\dagger} \hat{n} \} = (e^{\frac{\Delta}{k_B T}} - 1)^{-1} \cdot e^{\frac{\Delta}{k_B T}} \). Since all quantities in Eq. (S4.2) are positive, the mean quanta cannot decrease by the driving alone. We note that this is true regardless of what dynamics we are considering, since this expression is completely general in terms of the Bogoliubov coefficients. In other contexts, the limits of algorithmic cooling with Gaussian resources have been considered [71].

**Supplemental note 5. Limits to the cooling protocol**

In Fig. 2, one of the measurements around \( \omega_0 t = 600 \) results in a coherent state with a very low number of quanta \( |\xi|^2 < 0.1 \). In this instance, the parametric modulations do not cool the state further, instead they heat the system. Here, we explore why this occurs. The same feature can be observed for the case of single-mode squeezing. We can therefore study the limits to a single cycle of the cooling protocol presented in this work by comparing the single-mode squeezing that results from the parametric modulation with a generic single-mode squeezing operation.

We start by considering a generic single-mode squeezing operator. It is given by \( \hat{S}(z) = e^{(\hat{a}^2 - \hat{a}^{\dagger 2})/2} \), where we define \( z = qe^{i\theta} \) for which \( q, \theta \in \mathbb{R} \). The number of quanta for a squeezed coherent state \( |\xi\rangle \) with \( \xi = e^{i\phi}r \) therefore becomes

\[
\langle \hat{S}^\dagger(z) \hat{a}^\dagger \hat{a} \hat{S}(z) \rangle_{|\xi\rangle} = r^2 \cosh^2(q) - 2r^2 \cos(2\phi - \theta) \cosh(q) \sinh(q) + (r^2 + 1) \sinh^2(q).
\]

Here, we note that squeezing can reduce the number of quanta provided that \( 2\phi - \theta = 2\pi n \), where \( n \) is an integer. Note that the mapping from the evolution operator in Eq. (S1.7) to this single-mode squeezing operator is non-trivial, which is why the phase relationships are different. That is, the phase \( \theta \) presented here is related to \( \phi_p \) in the main-text in a non-trivial manner. However, if \( q \) is too large, the system will instead gain quanta. In fact, there is always an ideal squeezing value \( q \) that minimizes \( \langle \hat{S}^\dagger(z) \hat{a}^\dagger \hat{a} \hat{S}(z) \rangle_{|\xi\rangle} \) for a particular \( r \). To find this \( q \), we differentiate Eq. (S5.1) with respect to \( q \) to find

\[
\frac{d}{dq} \langle \hat{S}^\dagger(z) \hat{a}^\dagger \hat{a} \hat{S}(z) \rangle_{|\xi\rangle} = \sinh(2q) + 2r^2 [\sinh(2q) - \cos(\Delta) \cosh(2q)].
\]

Setting Eq. (S5.2) to zero and solving for \( q \), we find \( q = \log(1 + 4r^2)/4 \). This results in \( \langle \hat{S}^\dagger(z) \hat{a}^\dagger \hat{a} \hat{S}(z) \rangle_{|\xi\rangle} = [\sqrt{1 + 4r^2} - 1]/2 \), which is the lowest number of quanta the state can be cooled to. If we squeeze beyond this value, quanta will instead be added to the system rather than subtracted. For example, given a coherent state with \( r = 1 \), the optimal squeezing value is \( q = \log(5)/4 \), which results in \( \langle \hat{S}^\dagger(z) \hat{a}^\dagger \hat{a} \hat{S}(z) \rangle_{|\xi\rangle} \approx 0.6 \). It is not possible to reduce the number of quanta beyond this value by squeezing alone.

To connect with the dynamics considered in this work, we begin by studying the decomposed evolution operator in Eq. (S1.7). Ignoring the rotating \( e^{-iJ_0 \hat{a}^\dagger \hat{a}} \) with \( \hat{a}^\dagger \hat{a} \), which only adds a phase to the final state, we focus on the two squeezing operators \( e^{-iJ_+ (\hat{a}^{\dagger 2} - \hat{a}^2)} \) and \( e^{-iJ_- (\hat{a}^{\dagger 2} - \hat{a}^2)} \). To compare this product with the single squeezing operator, we must first combine the two exponentials into a single squeezing operator. This has been done in Ref [41], where it was found that

\[
e^{-iJ_+ (\hat{a}^{\dagger 2} + \hat{a}^2)} e^{-iJ_- (\hat{a}^{\dagger 2} - \hat{a}^2)} = e^{i\phi_{sq}(\hat{a}^\dagger \hat{a} + 1)} S(z_{sq}),
\]

where

\[
\phi_{sq} = \text{Arg} \left( \frac{1 - i \tanh(2J_+) \tanh(2J_-)}{1 + i \tanh(2J_+) \tanh(2J_-)} \right), \quad z_{sq} = \frac{i \tanh(2J_+) - \tanh(2J_-)}{1 - i \tanh(2J_+) \tanh(2J_-)}.
\]

Decomposing \( z_{sq} \) in Eq. (S5.4) into real and imaginary parts, we find the squeezing magnitude

\[
r_{sq} = \sqrt{1 - \frac{2}{\cosh(4J_+) \cosh(4J_-) + 1}}.
\]
Then, using Eq. (S1.22) which relate $J_{\pm}$ to $P(t)$ and $Q(t)$, we can use our approximate solutions for parametric modulations shown in Eq. (12) to write

$$ r_{sq} = \sqrt{1 - \frac{4}{2 + P^2(t) + Q^2(t) + \dot{P}^2(t) + \dot{Q}^2(t)}}. $$

(S5.6)

Here we note that the initial conditions for $P(t = 0) = 1$ and $Q(t = 0) = 1$ imply that at $t = 0$, $r_{sq} = 0$. The maximum value that can be achieved is $r_{sq} = 1$ in the limit of large $t$. This comparison demonstrates that the squeezing magnitude $r_{sq}$ is a function of time $t$ and the modulation strength $\lambda$. It is then possible to determine the ideal cycle length for each measurement outcome if the initial coherent state magnitude is known. From our comparison with the single-mode squeezing, we also note that as $t$ increases, some cycles are likely to heat the system rather than cool it, especially if $r_{sq}$ is larger than the ideal squeezing value needed to cool the system.

As a final note, we observe that the error in our approximate solutions for $P(t)$ and $Q(t)$ sometimes implies an imaginary $r_{sq}$ when $\omega_0 t$ is small. This could however be mitigated by developing a more accurate solution to Mathieu’s equation and merely shows a limitation in treating the dynamics as a single squeezing operation.