DIOPHANTINE APPROXIMATION BY ALMOST EQUILATERAL TRIANGLES

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Abstract. A two-dimensional continued fraction expansion is a map $\mu$ assigning to every $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ a sequence $\mu(x) = T_0, T_1, \ldots$ of triangles $T_n$ with vertices $x_{ni} = (p_{ni}/d_{ni}, q_{ni}/d_{ni}) \in \mathbb{Q}^2, d_{ni} > 0, p_{ni}, q_{ni}, d_{ni} \in \mathbb{Z}, i = 1, 2, 3,$ such that

$$\det \begin{pmatrix} p_{n1} & q_{n1} & d_{n1} \\ p_{n2} & q_{n2} & d_{n2} \\ p_{n3} & q_{n3} & d_{n3} \end{pmatrix} = \pm 1 \quad \text{and} \quad \bigcap_n T_n = \{x\}.$$ 

We construct a two-dimensional continued fraction expansion $\mu^*$ such that for densely many (Turing computable) points $x$ the vertices of the triangles of $\mu(x)$ strongly converge to $x$. Strong convergence depends on the value of

$$\lim_{n \to \infty} \frac{\sum_{i=1}^3 \text{dist}(x, x_{ni})}{(d_{n1} + d_{n2} + d_{n3})},$$

(“dist” denoting euclidean distance) which in turn depends on the smallest angle of $T_n$. Our proofs combine a classical theorem of Davenport Mahler in diophantine approximation, with the algorithmic resolution of toric singularities in the equivalent framework of regular fans and their stellar operations.

1. Introduction and statement of the main results

Following Lagarias [10], by a two-dimensional continued fraction expansion we mean a map $\mu$ assigning to every $(\alpha, \beta) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ a sequence $\mu(\alpha, \beta)$ of triangles $T_n \subseteq \mathbb{R}^2$ with rational vertices $(p_{ni}/d_{ni}, q_{ni}/d_{ni}), d_{ni} > 0, p_{ni}, q_{ni}, d_{ni} \in \mathbb{Z}, i = 1, 2, 3,$ $n = 0, 1, \ldots$ such that $\bigcap_n T_n = \{(\alpha, \beta)\}$ and

$$\det \begin{pmatrix} p_{n1} & q_{n1} & d_{n1} \\ p_{n2} & q_{n2} & d_{n2} \\ p_{n3} & q_{n3} & d_{n3} \end{pmatrix} = \pm 1. \quad (1)$$

Triangles $T \subseteq \mathbb{R}^2$ with vertices $(p_i/d_i, q_i/d_i) \in \mathbb{Q}, \ d_i > 0, i = 1, 2, 3,$ having the unimodularity property (1) are said to be regular, being affine counterparts of regular three-dimensional cones in $\mathbb{R}^3$, [5, 1.2.16], [7, p.146]. Regular cones, in turn, are the basic constituents of regular fans, i.e., complexes of regular cones, [5, 3.1.2], [7, V, 4.11]. Regular triangles, cones and fans will find use throughout this paper.

For any point $x = (x_1, x_2) \in \mathbb{R}^2$ we let $G_x = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}$ be the subgroup of the additive group $\mathbb{R}$ generated by $x_1, x_2, 1$. Thus $\text{rank}(G_x) = 1$ iff $x \in \mathbb{Q}^2$. Points $x$ with $\text{rank}(G_x) \in \{2\}$ are Lebesgue-negligible, and their continued fraction approximation is a routine variant of the classical one. We will mostly consider points $y = (\alpha, \beta)$ with $\text{rank}(G_y) = 3$, called rank 3 points. Any such point automatically lies in the interior of every regular triangle containing it.

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Suppose a rank 3 point \( x \) is approximated by a sequence of triangles \( R_0, R_1, \ldots \). Good approximations are in conflict with the tendency of the \( R_n \) to degenerate into needle-like triangles like those of \([2, 8, 15]\). One may naturally wonder whether the many negative results on strong convergence of two-dimensional continued fraction expansions, \([3, p.22]\) and generalized Farey sequences, \([8, 4.1]\), have a geometric counterpart in the fact that certain scalene triangles are inevitable. For any triangle \( T \) let \( \text{diam}(T) \) be the length of the longest side of \( T \). From a theorem of Davenport–Mahler \([6]\) we have a first partial answer:

**Theorem 1.1.** Infinitely many rank 3 points \( x \in \mathbb{R}^2 \) have the property that for every two-dimensional continued fraction expansion \( \mu \), no subsequence \( E_0 \supseteq E_1 \supseteq \cdots \) of \( \mu(x) = T_0, T_1, \ldots \) such that the angles of every \( E_n \) are \( > \arcsin(23^{1/2}/6) \approx \pi/(3.3921424) \approx 53^\circ \), satisfies \( \bigcap_n E_n = \{x\} \). Thus, letting for \( i = 1, 2, 3 \), \( x_{ni} = (p_{ni}/d_{ni}, q_{ni}/d_{ni}) \) denote the \( i \)th vertex of \( T_n \), we have

\[
\lim_{n \to \infty} \inf_{i=1,2,3} \frac{\text{dist}(x, x_{ni})}{(2d_{ni}d_{n2}d_{n3})^{-1/2}} \geq \lim_{n \to \infty} \inf_{i=1,2,3} \frac{\text{diam}(T_n)}{\text{area}(T_n)^{1/2}} \geq 2 \left( \frac{13}{23} \right)^{1/4} \approx 1.734138878. \tag{2}
\]

And yet, toric resolution of singularities \([5, 7]\)—in the equivalent algorithmic-combinatorial framework \([1]\) of desingularization of fans in \( \mathbb{R}^3 \)—yields:

**Theorem 1.2.** For all \( \epsilon > 0 \) there is a two-dimensional continued fraction expansion \( \mu^* \) having the following property: for densely many rank 3 points \( x \in \mathbb{R}^2 \), the sequence \( \mu^*(x) = E_{i0}, E_{i1}, \ldots \) satisfies

\[
\lim_{n \to \infty} \frac{\max_{i=1,2,3} \text{dist}(x, v_{ni})}{(2d_{n1}d_{n2}d_{n3})^{-1/2}} \leq \lim_{n \to \infty} \frac{\text{diam}(E_{n})}{\text{area}(E_{n})^{1/2}} < 2 \left( \frac{1}{3} \right)^{1/4} + \epsilon \approx 1.52 + \epsilon, \tag{3}
\]

with \( v_{ni} = (p_{ni}/d_{ni}, q_{ni}/d_{ni}) \) the \( i \)th vertex of the triangle \( E_n \), \( i = 1, 2, 3 \).

**Corollary 1.3.** There is a two-dimensional continued fraction expansion \( \mu^* \) such that for a dense set \( \mathcal{D} \) of rank 3 points \( x = (\alpha, \beta) \in \mathbb{R}^2 \), all angles of every triangle of the sequence \( \mu^*(x) = E_{0}^*, E_{1}^*, \ldots \) are \( > \arcsin(23^{1/2}/6) \). Thus,

\[
\lim_{n \to \infty} \frac{\max_{i=1,2,3} \text{dist}(x, v_{ni}^*)}{(2d_{n1}^*d_{n2}^*d_{n3}^*)^{-1/2}} \leq \lim_{n \to \infty} \frac{\text{diam}(E_{n}^*)}{\text{area}(E_{n}^*)^{1/2}} < 2 \cdot \left( \frac{13}{23} \right)^{1/4}, \tag{4}
\]

with \( v_{ni}^* = (p_{ni}^*/d_{ni}, q_{ni}^*/d_{ni}) \) denoting \( i \)th vertex of the triangle \( E_n^* \), \( i = 1, 2, 3 \).

**Corollary 1.4.** With the notation of Corollary 1.3, let \( \rho \subseteq \mathbb{R}^3 \) be the half-line originating at \((0, 0, 0)\) and passing through \((\alpha, \beta, 1)\). For each \( n = 0, 1, \ldots \), pick a vertex \( v_n^* = (p_n^*/d_n^*, q_n^*/d_n^*) \) of \( E_n^* \) having smallest denominator. Then the sequence \( v_0^*, v_1^*, \ldots \) strongly converges to \( x \), in the sense that \( \lim_{n \to \infty} \text{dist}(\rho, (p_n^*, q_n^*, d_n^*)) = 0 \).

Our continued fraction expansions strongly converging over the dense set \( \mathcal{D} \) of Corollaries 1.3-1.4 inherit from the Farey expansion the following properties:

**Approximation steps by Farey sums (=Farey mediants)** Each triangle \( E_{n+1}^* \) in \( \mu^*(x) \) is obtained from \( E_n^* \) via finitely many computations of mediants of pairs of vertices of consecutive triangles. Thus, passing to homogeneous integer coordinates of the vertices of the \( E_n^* \), \( \mu^*(x) \) is an “expansion” along the ray \( \rho \) in the more restrictive sense of Brentjes \([3, 2.3]\) and \([4, p.21]\).

**Turing computability** \( \mathcal{D} \) contains a subset \( \mathcal{D}' \) of points, also dense in \( \mathbb{R}^2 \), which are the output of an enumerating Turing machine. (See Remark 3.2.) Thus for any \( y \in \mathcal{D}' \) we have a two-dimensional continued fraction algorithm in the sense of \([4, p.21]\)

Farey sums, unimodularity, computability issues, angles, expansions, and the estimates (2)-(4) are not needed for the proof of the main result of \([12]\), stating that the set of points for which strong convergence fails is Lebesgue-negligible.
Finally, in Remark 4.1, our Corollary 1.4 is comparatively discussed with Grabiner’s [8, Theorem 4.1] stating that for all \( n \geq 2, 3, \ldots \), no \( n \)-dimensional Farey continued fraction algorithm is strongly convergent.

2. Proof of Theorem 1.1

Following [5, p.29], a ray \( \rho \) in \( \mathbb{R}^3 \) is a half-line having the origin \( 0 = (0, 0, 0) \) as its extremal point. Thus for some nonzero vector \( w = (x, y, z) \in \mathbb{R}^3 \) we may write

\[
\rho = (w) = \{ \lambda w \in \mathbb{R}^3 \mid 0 \leq \lambda \in \mathbb{R} \}.
\]

A nonzero integer vector \( v = (a, b, c) \in \mathbb{Z}^3 \subseteq \mathbb{R}^3 \) is said to be primitive if \( \gcd(a, b, c) = 1 \). In other words, moving from \( 0 \) along the ray \( (v) \), \( v \) is the first integer point \( \neq 0 \). Following [5, 1.2.14, 1.2.16] by a rational, three-dimensional, simplicial cone ("simplex cone", or "simple cone" in [7, V, 1.8]) in \( \mathbb{R}^3 \) we mean a set \( \sigma \subseteq \mathbb{R}^3 \) of the form

\[
\sigma = \langle v_1, v_2, v_3 \rangle = \left\{ \sum_{i=1}^{3} \lambda_i v_i \left| 0 \leq \lambda_i \in \mathbb{R} \right. \right\},
\]

for primitive vectors \( v_1, v_2, v_3 \in \mathbb{Z}^3 \) whose linear span coincides with \( \mathbb{R}^3 \). The \( v_i \) are said to be the primitive generating vectors of \( \sigma \) ("minimal" generators, in [5, p.30]). They are uniquely determined by \( \sigma \). (See [7, p.146], where the notation \( \text{pos}[v_1, v_2, v_3] \) is used instead of \( \langle v_1, v_2, v_3 \rangle \). In [5, 1.2.1] one finds the notation \( \text{Cone}(\{v_1, v_2, v_3\}) \).

Throughout this paper we will use the following notation:

\[
A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1 \} \quad \text{and} \quad V = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0 \} \cup \{0\}.
\]

Let \( \rho \subseteq V \) be a ray and \( v_0, v_1, \ldots \) a sequence of primitive integer vectors in \( V \). We then say that \( v_0, v_1, \ldots \) strongly converge to \( \rho \) if \( \lim_{n \to \infty} \text{dist}(\rho, v_n) = 0 \). Letting \( v_n \) be the orthogonal projection of \( \langle v_n \rangle \cap A \) into \( z = 0 \), and \( r \in \mathbb{R}^2 \) the orthogonal projection of \( \rho \cap A \) into \( z = 0 \), we equivalently say that the sequence \( v_0, v_1, \ldots \) strongly converges to \( r \), [8, (4), p.37].

Following [5, 1.2.16] and [7, V, 1.10], a rational cone \( \sigma = \langle v_1, v_2, v_3 \rangle \subseteq \mathbb{R}^3 \) is said to be regular if \( \{v_1, v_2, v_3\} \) is a basis of the free abelian group \( \mathbb{Z}^3 \). Equivalently, the \( 3 \times 3 \) integer matrix with row vectors \( v_1, v_2, v_3 \) has determinant \( \pm 1 \). Thus for \( p_i/d_i, q_i/d_i, d_i > 0 \), \( p_i, q_i, d_i \in \mathbb{Z} \), \( \gcd(p_i, q_i, d_i) = 1 \), \( i = 1, 2, 3 \), the triangle

\[
T = \text{conv}((p_1/d_1, q_1/d_1), (p_2/d_2, q_2/d_2), (p_3/d_3, q_3/d_3))
\]

is regular iff the primitive integer vectors \( v_i = (p_i, q_i, d_i) \) generate a regular cone contained in \( V \). If \( T \) is regular, it is easy to see ([2, Corollary 11]) that its area depends only on the least common denominators (henceforth, denominators, [10, p.466]) \( d_i \) of the vertices of \( T \),

\[
\text{area}(T) = \frac{1}{2d_1 d_2 d_3}.
\]

A ray \( \rho = ((x, y, z)) \subseteq \mathbb{R}^3 \) is said to be irrational if \( x, y, z \) are linearly independent over \( \mathbb{Q} \). Thus the ray \( ((\alpha, \beta, 1)) \) is irrational iff \( (\alpha, \beta) \) is a rank 3 point. Every irrational \( \rho \) has no integer points except the origin. The converse is not true, e.g., for the ray \( ((x, y, 1)) \) whenever \( (x, y) \) is a rank 2 point. For any \( 0 \leq \theta < \pi/3 \), an irrational ray \( \rho \subseteq V \) is said to be a \( \theta \)-accessible if there is a sequence \( \sigma_0 \not\supseteq \sigma_1 \not\supseteq \ldots \) of regular three-dimensional cones \( \sigma_i \subseteq V \) such that \( \bigcap_i \sigma_i = \{\rho\} \), and for every \( n = 0, 1, \ldots \), the angles of the triangle \( \sigma_n \cap A \) are all \( > \theta \). If \( \rho \) is not \( \theta \)-accessible we say that \( \rho \) is \( \theta \)-inaccessible.

As usual, "conv" denotes convex hull, and "diam" is short for "diameter."
Lemma 2.1. For every $\theta$ satisfying $\arcsin(23^{1/2}/6) < \theta < \pi/3$ there is an irrational $\theta$-inaccessible ray $\rho \subseteq V$.

Proof. Fix an arbitrary $\theta$ satisfying $\arcsin(23^{1/2}/6) < \theta < \pi/3$, and let

$$c_{\theta} = \frac{1}{3\sin \theta}.$$  \hfill (9)

Thus $\frac{2}{\sqrt{3}} < c_{\theta} < \frac{2}{\sqrt{2}}$. For any such $c_{\theta}$, Davenport and Mahler [6, Proof of Theorem 2(b)] exhibit a pair $(\alpha, \beta)$ of real numbers satisfying the following condition:

$(\dagger)$ $\alpha, \beta, 1$ are linearly independent over $\mathbb{Q}$ and there are only finitely many triplets $(a, b, c) \in \mathbb{Z}^3$ such that $0 < a^2 + b^2$ and $|a \alpha + b \beta + c| < c_{\theta}/(a^2 + b^2)$.

Let $\rho = \rho(\theta) = ((\alpha, \beta, 1))$. By $(\dagger)$, $\rho \subseteq V$ is an irrational ray. Arguing by way of contradiction we will show that $\rho$ is $\theta$-inaccessible.

Indeed, suppose there exists a sequence $\sigma_0 \supsetneq \sigma_1 \supsetneq \ldots$ of regular cones in $V$ such that $\bigcap_i \sigma_i = \{ \rho \}$ and the angles of each triangle $\sigma_i \cap A$ are all $> \theta$ (absurdum hypothesis). For fixed but otherwise arbitrary $m = 0, 1, \ldots$ let us write for simplicity

$$\sigma = \sigma_m = \langle u_1, u_2, u_3 \rangle,$$

with $u_i = (p_i, q_i, r_i)$ for suitable integers $p_i, q_i, r_i$ with $r_i > 0$, $i = 1, 2, 3$. Let the matrix $M$ be defined by

$$M = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}.$$  \hfill (10)

We can safely assume $\det M = 1$. The cone $\sigma$ determines the triangle $S = \sigma \cap A$ with vertices $(p_i/r_i, q_i/r_i, 1)$, $i = 1, 2, 3$. Let $S_\perp^i$ be the orthogonal projection of $S$ to the plane $z = 0$,

$$S_\perp^i = \text{conv}(u_1, u_2, u_3) = \text{conv}((p_1/r_1, q_1/r_1), (p_2/r_2, q_2/r_2), (p_3/r_3, q_3/r_3)).$$

The assumed regularity of $\sigma$ means that $S_\perp^i$ is regular. By (8), the area of $S$ equals $(2r_1 r_2 r_3)^{-1}$. Let us display the transpose inverse of $M$ by writing

$$L = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$  \hfill (10)

By definition, the rows of $L$ are the primitive generating vectors of the dual cone of $\sigma$, [5, 1.2.3], [7, 1, 4.1]. For each $i = 1, 2, 3$, let the function $f_i: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f_i(x, y) = a_i x + b_i y + c_i$. Then $f_i(u_i) = 1/r_i$ and $f_i$ constantly vanishes over the segment $\text{conv}(u_j, u_k)$, where $(i \neq j, j \neq k, k \neq i)$. Writing for short

$$\delta_i = (a_i^2 + b_i^2)^{1/2},$$

it follows that

$$\text{dist}(u_j, u_k) = \frac{\delta_i}{r_j r_k} \quad \text{and} \quad \text{dist}(u_i, \text{line}(u_j, u_k)) = \frac{1}{r_i \delta_i}.$$  \hfill (11)

By our standing absurdum hypothesis, for each $i = 1, 2, 3$ the angle $\theta_i$ of the triangle $S_\perp^i$ satisfies the inequalities $\theta < \theta_i < \pi/2$. Therefore,

$$\sin \theta < \sin \theta_i = \frac{r_i}{\delta_i \delta_k}.$$  \hfill (12)

Claim: In the notation of $(\dagger)$, (9) and (10), for at least one $i \in \{1, 2, 3\}$ we have $|a_i \alpha + b_i \beta + c_i| < c_{\theta}/(a_i^2 + b_i^2)$. 

For otherwise, for all $i$ we have
\[ |a_i\alpha + b_i\beta + c_i| = \delta_i \cdot \text{dist}((\alpha, \beta), \text{line}(u_j, u_k)) \geq \delta_i^2, \]
whence by (11),
\[ \text{dist}(u_j, u_k) \cdot \text{dist}((\alpha, \beta), \text{line}(u_j, u_k)) \geq \frac{c_0}{\delta_i^2 r_j r_k}. \]
From (12) we obtain
\[ 2 \cdot \text{area}(T_i) \geq c_0 \cdot \sum \frac{1}{\delta_i^2 r_j r_k}. \]
As a consequence,
\[ \frac{1}{c_0} \geq r_1 r_2 r_3 \cdot \left( \sum \frac{1}{\sum \delta_i^2 r_j r_k} = \sum \frac{r_i}{\delta_i^2} = \sum \frac{\delta_i \delta_j \sin \theta_i}{\delta_i^2} > \sin \theta \cdot \sum \frac{\delta_i \delta_j}{\delta_i^2} \geq 3 \sin \theta, \right. \]
which is impossible. This settles our claim.

Let us now turn back to the sequence $\sigma_0 \supseteq \sigma_1 \supseteq \ldots$ of regular cones introduced at the outset of this proof. For each $n = 0, 1, \ldots$ let us now write $\sigma_n = (v_{1n}, v_{2n}, v_{3n})$, where, for each $i = 1, 2, 3$, $v_{in} = (p_{in}, q_{in}, r_{in}) \in \mathbb{Z}^3$ and $(p_{in}/r_{in}, q_{in}/r_{in}, 1)$ is the $i$th vertex of the triangle $S_n = \sigma_n \cap A$. Let
\[ M_n = \begin{pmatrix} p_{1n} & q_{1n} & r_{1n} \\ p_{2n} & q_{2n} & r_{2n} \\ p_{3n} & q_{3n} & r_{3n} \end{pmatrix} \]
and $L_n$ be the transpose inverse of $M_n$. Since $\rho$ is an irrational ray, the vector $(\alpha, \beta, 1)$ lies in the (relative) interior of each triangle $S_n$. Thus from $\bigcap \sigma_i = \{ \rho \}$ it follows that for every $n$ there exists $m$ such that for all $l > m$ no row of $L_n$ is a row of $L_l$. Our claim then yields infinitely many triplets $(a, b, c)$ of integers such that $0 < a^2 + b^2$ and $|a\alpha + b\beta + c| < c_0/(a^2 + b^2)$. This contradicts the Davenport-Mahler result (4). \hfill $\Box$

The proof of Theorem 1.1 immediately follows from Lemma 2.1 upon setting $x = (\alpha, \beta)$. The first inequality of (2) is a consequence of (8).

3. Proof of Theorem 1.2 and Corollary 1.3

Further details will be needed about the set of $\theta$-accessible rays in $\mathbb{R}^3$. First of all, recalling the notational stipulations (7), let us equip the set
\[ \mathcal{R} = \{ \rho \subseteq \mathcal{V} \mid \rho \text{ is a ray} \} \]
with the topology inherited from the real projective plane. Thus a subset of $\mathcal{R}$ is open if it coincides with the set of all rays intersecting $U$, for some relatively open set $U \subseteq A \subseteq \mathbb{R}^3$.

Next we prepare the two-dimensional counterpart of the classical operation of taking Farey medians of segments in $[0, 1]$ with rational vertices. This is frequently found in diophantine approximation (sometimes called “Farey sums”) [2, 3.1], [8, 2.1], [15, p.441], and is also a main tool for the resolution of singularities of fans in the theory of toric varieties, [5, §11.1], [7, III, 2.1]. It will find pervasive use in this section.

As in (5) and (6), given a cone $\psi = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $i \neq j \in \{1, 2, 3\}$ we use the notation
\[ \langle u_i, u_j \rangle = \{ \alpha u_i + \beta u_j \in \mathbb{R}^3 \mid 0 \leq \alpha, \beta \in \mathbb{R} \}. \]
The two-dimensional cones \( \langle u_1, u_2 \rangle \), \( \langle u_3, u_1 \rangle \), and \( \langle u_2, u_3 \rangle \) are the two-dimensional faces of \( \psi \). The vector \( u_1 + u_2 \in \mathbb{Z}^3 \) is called the (Farey) \textit{midpoint} of \( u_1 \) and \( u_2 \). In case \( \psi \) is regular, \( u_1 + u_2 \) is primitive; the three-dimensional simplicial cones \( \langle u_1 + u_2, u_2, u_3 \rangle \) and \( \langle u_1 + u_2, u_1, u_3 \rangle \) are said to be obtained by the \textit{binary starring} of \( \psi \) at \( u_1 + u_2 \). We write, respectively,

\[
\langle u_1, u_2, u_3 \rangle \mapsto^* \langle u_1 + u_2, u_2, u_3 \rangle \text{ and } \langle u_1, u_2, u_3 \rangle \mapsto^* \langle u_1 + u_2, u_1, u_3 \rangle.
\]

Both cones \( \langle u_1 + u_2, u_2, u_3 \rangle \) and \( \langle u_1 + u_2, u_3, u_1 \rangle \) are regular. As usual, a \textit{simplicial complex} in \( \mathbb{R}^n \) is a finite set \( \mathcal{K} \) of simplexes in \( \mathbb{R}^n \), closed under taking faces, and having the further property that any two elements of \( \mathcal{K} \) intersect in a common face, [7, I, p.66]. For every simplicial complex \( \Sigma \), the point set union of the simplexes of \( \Sigma \) is called the \textit{support} of \( \Sigma \), denoted \( |\Sigma| \), [7, I, 1.9]. As a special case of a general definition, when \( |\Sigma| \subseteq \mathbb{R}^2 \) coincides with the point set union of the triangles in \( \Sigma \), we say that the simplicial complex \( \Sigma \) is \textit{regular} ("unimodular" in [14]) if so are all its triangles. We also say that \( \Sigma \) is a regular \textit{triangulation} of its support. Regular triangulations are affine counterparts of regular fans, [5, 3.1.18], [7, V, 4.11]. If \( \Sigma_1 \) and \( \Sigma_2 \) have the same support and every simplex of \( \Sigma_1 \) is contained in some simplex of \( \Sigma_2 \), we say that \( \Sigma_1 \) is a \textit{subdivision} of \( \Sigma_2 \).

For any triangle \( \text{conv}(P, Q, R) \), let \( PQR \) denote the angle with vertex \( Q \). By a traditional abuse of notation, we will use the same notation for angles and their measure.

For the proof of the next lemma we let

\[
n_0 = (p_0, q_0, r_0), \quad n_1 = (p_1, q_1, r_1), \ldots
\]

enumerate (in some prescribed lexicographic order) the totality of primitive vectors \( n = (p,q,r) \in \mathbb{Z}^3 \) satisfying the inequality \(|p| + |q| > 0\). For each \( i = 0, 1, \ldots \) let the plane \( n^i_A \subseteq \mathbb{R}^3 \) be defined by \( p_ix + q_iy + r_iz = 0 \). It follows that \( n_i \cap A \) is a line, denoted \( \Lambda_i \). Moreover, the sequence

\[
\Lambda_0, \Lambda_1, \ldots
\]

(14) gives all possible rational lines lying on \( A \), (i.e., all lines containing at least two distinct rational points of \( A \)).

**Lemma 3.1.** For any \( \theta < \pi/3 \) the set of irrational \( \theta \)-accessible rays is dense in \( \mathcal{R} \).

**Proof.** It suffices to prove the lemma under the more restrictive condition

\[
\pi/4 < \theta < \pi/3.
\]

(15)

The set \( \mathcal{U} \) of all relatively open right triangles with rational vertices lying on the plane \( A \) is a basis of the natural topology of \( A \) inherited by restriction from the usual topology of \( \mathbb{R}^3 \). So for any nonempty \( \mathcal{O} \in \mathcal{U} \) we must show that some \( \theta \)-accessible ray has a nonempty intersection with \( \mathcal{O} \).

Let \( \mathcal{O} \) denote the closure of \( \mathcal{O} \). There exists a uniquely determined (necessarily rational, simplicial, three-dimensional) cone \( \tau \subseteq V \) such that the triangle \( \tau \cap A \) coincides with \( \mathcal{O} \).

Toric resolution of singularities, [5, §11.1, p.113], [7, VI, proof of 8.5, p.165], (whose fan-theoretic reformulation for \( \tau \) amounts to starring at primitive integer vectors arising from iterated applications of Blichfeldt’s theorem in the Geometry of Numbers, [11, 9, p.35], [13, 1.2], [14, p.544]) yields a \textit{regular fan over} \( \tau \), i.e., a complex \( \Delta \) of regular cones and their faces, such that \( \tau \) coincides with the point set union of the cones of \( \Delta \). Interestingly enough, from the input data consisting of the vertices of \( \mathcal{O} \), a regular fan \( \Delta \) can be effectively computed having special minimality properties, [1], which are characteristic of desingularizations of fans in
R³, and are reminiscent of the Hirzebruch-Jung continued fraction algorithm for the smallest resolution of singularities of fans in R², [5]. Let τ = (a, b, c) ∈ Δ be a three-dimensional cone with τ ∩ A = conv(A, B, C). Then without loss of generality we can write

\[\frac{\pi}{3} < A_0C, \quad A_0B \geq C_0A \geq A_0B, \quad A_0B < \frac{\pi}{3}.\]

Preamble. In case conv(A, B) ⊆ A₀, via one binary starring we replace τ by τ₀ = (a, b + c, c), then give new names a₀, b₀, c₀ to the primitive generating vectors of τ₀ and write A₀, B₀, C₀ for the corresponding vertices of the triangle τ₀ ∩ A in such a way that

\[\frac{\pi}{3} < A_0C, \quad A_0C_0 \geq C_0B_0 \geq A_0B_0C_0, \quad A_0B_0C_0 < \frac{\pi}{3}.\]

In case conv(A, B) ∉ A₀, we just set τ₀ = (a, b, c) = (a₀, b₀, c₀) and let

\[A_0 = \langle a_0 \rangle \cap A, \quad B_0 = \langle b_0 \rangle \cap A, \quad C_0 = \langle c_0 \rangle \cap A.\]

In any case, the regular cone τ₀ satisfies (16) as well as

\[\text{conv}(A_0, B_0) \not\in A_0.\]  

Next we proceed with the following two steps:

Step 1. Let the point J be defined by J ∈ conv(A₀, B₀), A₀J = τ₀ = π/3. The existence of J is ensured by (16). By (15) and (17), there exists a point I satisfying the following conditions:

\[I \in \text{conv}(A_0, B_0), \quad I \not\in \mathbb{Q}^2, \quad I \not\in A_0, \quad \frac{\pi - 3\theta}{6} < A_0IC_0 \leq \frac{\pi}{3}.\]  

Since I is not a rational point, the ray ρ through I is contained in exactly one of the two three-dimensional cones obtained by binary starring τ₀ at a + b. Letting τ₁ denote such cone, we may write in more detail

\[\tau_1 = \langle a_1, b_1, c_0 \rangle \in \{\langle a_0, a_0 + b_0, c_0 \rangle, \langle b_0, a_0 + b_0, c_0 \rangle\}.\]

Keeping c₀ fixed and proceeding as in the classical slow continued fraction algorithm, we have a sequence of regular three-dimensional cones containing ρ ⊃ I,

\[\tau_0 \supsetneq \tau_1 \supsetneq \ldots,\]

where τ_n = (a_n, b_n, c_n) ⊃ ρ, and τ_{n+1} is the result of a binary starring of τ_n. The sequence does not terminate, because I ∉ Q². Let us write

\[B_nC_0A_n \text{ shrinks to zero as } n \text{ tends to } \infty, \text{ then } \bigcap_{\{a_i, b_i\}} = \{\rho\}.\]

Further, \(\lim_{n} A_nB_nC_0 = A_0IC_0\). Since by (18), I ∉ A₀, then for some ζ ∈ R and 0 < m ∈ Z the triangle conv(A_m, B_m, C₀) satisfies

\[B_m \not\in A_0, \quad \theta < \zeta = A_mB_mC_0 < \pi/3, \quad \{A_0, B_0\} \cap \{A_m, B_m, C_0\} = \emptyset.\]  

Step 1 is completed.

Step 2. Next we keep fixed the primitive generator b_m of τ_m = (a_m, b_m, c_m), and proceed as in Brentjes’ [3, 2.3(2), p.19]. For all 0 ≤ p, q ∈ Z, the cone \(\sigma_{p,q} = \langle a_m + pb_m, c_m + qb_m, b_m \rangle\) is regular and is contained in τ_m. Further, \(\sigma_{p,q}\) is obtainable from τ_m by a sequence of p + q binary starings. (See [8, Fig 3, p.58] for an illustration of these binary starings, with the caveat that all vertices therein have different names from those of our triangles here.) Let

\[A_{p,q} = \langle a_m + pb_m \rangle \cap A, \quad C_{p,q} = \langle c_m + qb_m \rangle \cap A.\]

For all p, q \(A_{p,q}B_mC_{p,q}\) is equal to ζ. Further, \(\bigcap_{p,q} \sigma_{p,q} = \{b_m\}\). Next let
\( \Pi_{p,q} \) be the plane in \( \mathbb{R}^3 \) determined by the three points \( \mathbf{0}, A_{p,q}, C_{p,q} \).

\( \Lambda_{p,q} \) be the rational line \( \Pi_{p,q} \cap \mathbb{A} \).

\( \Omega_{p,q} \subseteq \mathbb{A} \) be the bisector of the angle \( A_{p,q} \overrightarrow{B_m} C_{p,q} \) of the triangle \( \sigma_{p,q} \cap \mathbb{A} \).

For infinitely many pairs \((p',q')\) of integers \(\sigma_{p',q'} \cap \mathbb{A}\) of the two intersecting lines \(\Lambda_{p',q'}\) and \(\Omega_{p',q'}\) can be made arbitrarily close to \(\pi/2\). Correspondingly,

\[ B_m \overrightarrow{C_{p',q'}} A_{p',q'} \] and \( B_m A_{p',q'} C_{p',q'} \) get arbitrarily close to \((\pi - \zeta)/2 > \pi/3\). (20)

Thus by (19), for (infinitely many pairs of) integers \(p', q' > 0\), the regular cone \(\sigma_{p',q'} = \langle a_m + p'b_m, c_m + q'b_m, b_m \rangle \) intersects \(\mathbb{A}\) in a triangle having all angles \(\theta < \zeta, \zeta < \pi/3\), and also satisfying

\[ \sigma_{p',q'} \cap \Lambda_0 = \emptyset \] and \(\{A_0, B_0, C_0\} \cap \{A_{p',q'}, B_m, C_{p',q'}\} = \emptyset\). (21)

Letting “relint” denote relative interior, we automatically have

\[ \text{relint} (A_{p',q'}, B_m, C_{p',q'}) \subseteq \text{relint} (A_0, B_0, C_0) \subseteq \mathcal{O}. \] (22)

We finally introduce the following notational abbreviations:

\[ \tau_n = \sigma_{p',q'} = \langle a_m + p'b_m, b_m, c_m + q'b_m \rangle = \langle a_n, b_n, c_n \rangle \]

and

\[ \tau_n \cap \mathbb{A} = \text{conv}(A_n, B_n, C_n), \]

where

\[ A_n = \langle a_n \rangle \cap \mathbb{A}, \quad B_n = \langle b_n \rangle \cap \mathbb{A}, \quad C_n = \langle c_n \rangle \cap \mathbb{A}. \]

Now Step 2 is completed.

The finite path of binary starlings

\[ \tau_0 \mapsto^* \tau_1 \mapsto^* \cdots \mapsto^* \tau_m \mapsto^* \tau_{m+1} \mapsto^* \cdots \mapsto^* \tau_n \]

results in a new regular cone \(\tau_n\) having the following properties:

(i) \(\text{conv}(A_n, B_n, C_n) \cap \Lambda_0 = \emptyset\), by (21).

(ii) \(\text{conv}(A_n, B_n, C_n) \subseteq \text{relint} (A_0, B_0, C_0)\), by (22).

(iii) \(\theta < A_n, B_n, C_n < \pi/3\), by (19).

(iv) \(B_n A_n, C_n, B_n C_n A_n > \pi/3 > \theta\), by (20).

Claim 1. Suppose we are given a regular cone \(\tau_n\) such that the triangle \(\tau_n \cap \mathbb{A}\) is disjoint from \(\Lambda_1 \cup \cdots \cup \Lambda_k\), has an angle \(\zeta_n\) with \(\theta < \zeta_n < \pi/3\), and has the other two angles \(\pi/3\). Then a finite path of binary starlings produces from \(\tau_n\) a regular cone \(\tau_{n+1}\) such that the triangle \(\tau_{n+1} \cap \mathbb{A}\) is disjoint from \(\Lambda_1 \cup \cdots \cup \Lambda_k\), is strictly contained in the relative interior of \(\tau_n \cap \mathbb{A}\), has an angle \(\zeta_{n+1}\) with \(\theta < \zeta_{n+1} < \pi/3\), and has the other two angles \(\pi/3\).

The proof is by induction on \(k = 0, 1, \ldots\), following Steps 1 and 2 with \(\tau_{n+k}\) in place of \(\tau_n\). Since \(\tau_{n+k+1} \subseteq \tau_{n+k}\), to ensure that \(\tau_{n+k+1} \cap \mathbb{A}\) is disjoint from \(\Lambda_1 \cup \cdots \cup \Lambda_{k+1}\), it is sufficient to guarantee that it is disjoint from \(\Lambda_{k+1}\). (In case \(\Lambda_{k+1}\) contains the largest side of the triangle \(\langle \tau_{n+k+1} \rangle \cap \mathbb{A}\) we perform a preliminary binary starring of \(\tau_{n+k+1}\) as in the preamble above, before taking Steps 1 and 2.)

Having proved our claim, let us fix the following notation, for all \(n, k = 0, 1, \ldots\):

\[ T_n = \text{orthogonal projection of } \tau_n \cap \mathbb{A} \text{ into the plane } z = 0. \] (23)

\[ \eta_k = \tau_n \cap \mathbb{A}, \quad D_k = \eta_k \cap \mathbb{A}, \quad E_n = \text{orthogonal projection of } D_n \text{ into } z = 0. \] (24)

For notational simplicity, the dependence on \(\theta\) and \(\mathcal{O}\) of \(\tau_n, \eta_k, T_n, D_n, E_n\) is tacitly understood.
Claim 2. $\bigcap_l \eta_l$ is a singleton consisting of a $\theta$-accessible ray $\rho = \rho_{O, \theta}$ such that the point $\rho \cap A$ lies in $O$, and the orthogonal projection of $\rho \cap A$ into the plane $z = 0$ is a rank 3 point.

As a matter of fact, the compactness of each triangle $\tau_n \cap A$ ensures that $\bigcap_l \eta_l$ is nonempty. For $i = 1, 2, 3$ let $(p_{ni}/d_{ni}, q_{ni}/d_{ni})$ be the vertices of $T_n$. Recalling (8), from the regularity of $\tau_n$ it follows that area$(T_n) = 1/(2d_{n1}d_{n2}d_{n3})$. After each binary starring $\tau_j \mapsto^* \tau_{j+1}$ as in (13), two vertices of $T_j$ are also vertices of $T_{j+1}$. The denominator of the third vertex of $T_{j+1}$ is strictly greater than the denominator of the third vertex of $T_j$. Thus area$(T_j) \downarrow 0$. Since the $E_i$ are a subsequence of the $T_i$, it follows that area$(E_i) = \text{area}(D_1) \downarrow 0$. Since by (15) for every $n$ all angles of $D_n$, are $> \theta > \pi/4$, elementary geometry shows that $\bigcap_l D_l$ is a singleton point.

(With reference to our remarks in the Introduction, no tendency here is possible for the $D_n$ to degenerate into needle-like triangles—because the area of $D_n$ controls its diameter.) Thus $\bigcap_l \eta_l$ is a singleton ray, which has the desired properties, by Claim 1 and (22). In particular, the orthogonal projection of $\rho \cap A$ into the plane $z = 0$ is a rank 3 point because $\rho \cap A$ lies in no rational line $\Lambda_n$.

Having thus settled Claim 2, the proof of Lemma 3.1 is complete. □

Remark 3.2. Perusal of the proof of Lemma 3.1 shows that, once a fixed finite alphabet $A$ is chosen, and strings over $A$ representing rational points, triangles, and cones are equipped with some prescribed lexicographic order, then for every $\theta < \pi/3$ there exists a Turing machine $M_\theta$ having the following property:

Over any input integer $n \geq 0$ together with the vertices of a triangle $O$, for $\emptyset \not\in O \in \mathcal{U}$, $M_\theta$ outputs the $n$th term $\eta_n$ of a sequence $\eta_0 \supseteq \eta_1 \supseteq \ldots$ of regular cones in $V$ closing down to an irrational ray which intersects $A$ at a point $y$ of $O$, and has the additional property that all angles of every triangle $\eta_n \cap A$ are $> \theta$.

Thus the projection of $y$ into the plane $z = 0$ is a pair of recursively enumerable real numbers. In particular, if $\sin(\theta)$ is rational, the instructions/quintuples of $M_\theta$ can be effectively written down, following the constructive proof of Lemma 3.1.

4. Proofs of Theorem 1.2, Corollary 1.3 and Corollary 1.4

Proof of Theorem 1.2. Recall the definition of the basis $\mathcal{U}$ given at the beginning of the proof of Lemma 3.1. For every nonempty $O \in \mathcal{U}$ and angle $\theta < \pi/3$, the proof constructs the irrational $\theta$-accessible ray $\rho_{O, \theta} \in O$. Let the rank 3 point $x_{O, \theta} \in \mathbb{R}^2$ be defined by

$$x_{O, \theta} = \text{orthogonal projection of } \rho_{O, \theta} \cap A \text{ into the plane } z = 0.$$ 

Let further

$$D_\theta = \{x_{O, \theta} \in \mathbb{R}^2 \mid \text{for some nonempty } O \in \mathcal{U}\}.$$ 

By Lemma 3.1, $D_\theta$ is a dense set of rank 3 points in $\mathbb{R}^2$. Let the sequence $E_{0, O, \theta}, E_{1, O, \theta}, \ldots$ be as in (24)—the dependence on $\theta$ and $O$ being now made explicit. Let the two-dimensional continued fraction expansion $u_\theta$ be defined as follows:

(I) For each $x = x_{O, \theta} \in D_\theta$, $u_\theta(x)$ is the sequence $E_{0, O, \theta}, E_{1, O, \theta}, \ldots$.

(II) For all other pairs $y \in \mathbb{R}^2$ of irrationals, $u_\theta(y)$ is, e.g., the two-dimensional continued fraction expansion in [4].

For every $\varepsilon > 0$ there is $\theta = \theta(\varepsilon)$ so close to $\pi/3$ that the two-dimensional continued fraction expansion $u_\varepsilon = u_{\theta(\varepsilon)}$ satisfies condition (3) in Theorem 1.2. As a matter of fact, on the one hand, for all equilateral triangles, the ratio between diameter (=side length) and square root of area is $2 \cdot 3^{-1/4}$. On the other hand, since all
angles of $E_{n,\gamma}$ are $> \theta$, an elementary geometric argument ensures that the ratio between the diameter of each $E_{n,\gamma}$ and the square root of the area of $E_{n,\gamma}$ is $\leq 2 \cdot 3^{-1/4} + k\theta$, where the constant $k\theta$ is independent of $n$ and $\gamma$, and tends to zero as $\theta$ tends to $\pi/3$ from below.

The proof of Theorem 1.2 is complete.

**Proof of Corollary 1.3.** The proof of the first statement in Corollary 1.3 is a routine variant of the proof of Theorem 1.2, arguing for the special case $\theta^* = \arcsin(23^{1/2}/6)$ in Lemma 3.1, and defining $\mu^* = \mu_{\theta^*}$ as in (I)-(II) above. To verify (4), let $E$ range over the totality $E$ of triangles whose angles are $\geq \arcsin(23^{1/2}/6)$. Then elementary geometry shows that the ratio between the diameter of $E$ and the square root of the area of $E$ attains the maximum value $2 \cdot (13/23)^{1/4}$ when $E$ is the isosceles triangle with two equal angles of $\arcsin(23^{1/2}/6)$ radians; for all other triangles in $E$ this ratio is $< 2 \cdot (13/23)^{1/4}$. The first identity in (4) follows from (8).

The proof of Corollary 1.3 is complete.

**Proof of Corollary 1.4.** Let $k = 2 \cdot (13/23)^{1/4}$. For all $n = 0, 1, \ldots$ we can write without loss of generality

$$v^*_n = v^*_{n,1} = (p^*_n/d^*_n, q^*_n/d^*_n), \quad d^*_n,1 \leq d^*_n,2 \leq d^*_n,3.$$ 

By (4) in Corollary 1.3 we have

$$\text{dist}(\rho, v^*_{n,1}) < d^*_n,1 \cdot \text{diam}(E^*_n) < k \cdot d^*_n,1 (2d^*_n,1 d^*_n,2 d^*_n,3)^{-1/2} = \frac{k(d^*_n,1)^{1/2}}{(2d^*_n,2 d^*_n,3)^{1/2}} \leq \frac{k}{(2d^*_n,3)^{1/2}}.$$ 

By construction, $\lim_{n \to \infty} d^*_n,3 \to \infty$, whence $\lim_{n \to \infty} \text{dist}(\Lambda, v^*_{n,1}) = 0$, as desired to prove that the sequence of primitive integer vectors $(p^*_n, q^*_n, d^*_n,1)$ strongly converges to the ray $\rho$ through $(\alpha, \beta, 1)$. Correspondingly, the vertices $v^*_0, v^*_1, \ldots$ strongly converge to the rank 3 point $(\alpha, \beta) = x$.

**Remark 4.1.** It is instructive enough to compare Corollary 1.4 with [8, Theorem 4.1] stating that for all $n = 2, 3, \ldots$, no $n$-dimensional Farey continued fraction algorithm is strongly convergent.

Both in our approach here and in [8] the unimodular property, the effective computability of approximating sequences of points, the rank of points, their denominators, Farey mediants, stellar operations and two-dimensional continued fraction expansions have a basic role.

However, our algorithmic approach is local; we study the geometric properties of sequences of triangles closing down to a single point $x \in \mathbb{R}^2$, just as the classical continued fraction algorithm does by providing a sequence of rational segments whose vertices converge to a single point $y \in [0, 1]$.

By contrast, following the time-honored tradition of [9, §8] and many other papers (see [15] and references therein), the subject matter of [8] is the generalization of the Farey sequence and its variants, [16]. So these papers deal with two-dimensional continued fraction expansions arising from sequences of finer and finer triangulations $\nabla_n$ of a given fixed domain $D$, in such a way that the denominator of each vertex of each triangle of $\nabla_n$ is $\leq n$, and for every point $x \in D$...
there is a sequence of triangles $T_{x,0} \in \nabla_0$, $T_{x,1} \in \nabla_1$, ... whose intersection is the singleton $\{x\}$.

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