USING FUNCTIONAL METHODS TO COMPUTE
QUANTUM EFFECTS IN THE LIOUVILLE MODEL

T. L. Curtright

Department of Physics, University of Miami
P. O. Box 248046, Coral Gables, FL 33124 USA

G. I. Ghandour

Faculty of Science, Physics Department, Kuwait University
P. O. Box 5969/13060, Safat, KUWAIT

ABSTRACT

We use time-independent canonical transformation methods to discuss the energy
eigenfunctions for the simple linear potential, pedagogically setting the stage for some
field theory calculations to follow. We then discuss the Schrödinger wave-functional
method of calculating correlation functions for Liouville field theory. We compare this
approach to earlier treatments, in particular we check against known weak-coupling
results for the Liouville field defined on a cylinder. Finally, we further set the stage
for future Liouville calculations on curved two-manifolds and briefly discuss simple
quantum mechanical systems with time-dependent Hamiltonians.

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\[^2\] curtright@phyvax.ir.miami.edu
\[^3\] ghassan@kuc01.kuniv.edu.kw
QUANTUM MECHANICAL FOREPLAY

Consider a particle moving in a one-dimensional linear potential as described by the Hamiltonian

$$H = p^2 + q,$$

where $q$ and $p$ are canonically conjugate position and momentum variables. As every well-educated physicist knows, in the coordinate representation the energy eigenfunctions for this problem are Airy functions, easily found by solving the eigenvalue problem in the momentum representation and then executing a Fourier transform. (cf. §24 in [2].)

Nevertheless, let us construct the energy eigenfunctions from scratch using the machinery of canonical transformations. Since the energy spectrum is obviously continuous and unbounded either above or below, $-\infty < E < +\infty$, it is clearly a good idea to map the above Hamiltonian onto a free particle Hamiltonian with the same properties. So we take the free Hamiltonian to be just the momentum itself.

$$H_{\text{free}} = P.$$  

Now, what is a generating function which produces a canonical transformation from the “interacting” variables $(q,p)$ to the “free” variables $(Q,P)$ so that $H = H_{\text{free}}$? A simple answer is

$$F(q,Q) = -\frac{1}{3}Q^3 - qQ.$$  

We may check this statement at the classical level as follows.

$$p_{\text{classical}} = \frac{\partial}{\partial q} F = -Q, \quad P_{\text{classical}} = -\frac{\partial}{\partial Q} F = (Q^2 + q).$$

Hence $p_{\text{classical}}^2 + q = P_{\text{classical}}$ as anticipated.

The significance of the above for the quantum mechanical problem was first appreciated in a general setting by Dirac, in his famous 1932 article [2] which motivated Feynman’s development of the path integral, although Dirac was not inclined to provide any details for such simple examples as that of the case at hand. If we follow Dirac’s advice and exponentiate the above generating function we obtain a transformation functional which produces energy eigenfunctions in coordinate space for the linear potential, $\Phi_E(q)$, from those for the free particle, $\Psi_E(Q)$. Thus

$$\Phi_E(q) = \int_{-\infty}^{+\infty} dQ \ e^{iF(q,Q)} \Psi_E(Q).$$

The free particle eigenfunctions are given here by $\Psi_E(Q) = \frac{1}{\sqrt{2\pi}} e^{iEQ}$.

We easily check that the exponentiated generating function provides an exact solution of the coordinate space partial differential equation

$$H \ e^{iF} = -H_{\text{free}} \ e^{iF},$$

Some other simple examples have been explicitly discussed by G. Ghandour, Phys. Rev. D35 (1987) 1289. Also see T.L. Curtright, Differential Geometric Methods in Theoretical Physics: Physics and Geometry, L.-L. Chau and W. Nahm, ed’s., Plenum Press, 1990, pp 279-289.
where
\[ H = -\frac{\partial^2}{\partial q^2} + q, \quad H_{\text{free}} = -i \frac{\partial}{\partial Q}. \]

(7)

Then if we integrate by parts and discard terms at \( Q = \pm \infty \), we verify the above relation between \( \Phi_E(q) \) and \( \Psi_E(Q) \).

\[
H \Phi_E(q) = \int_{-\infty}^{+\infty} dQ \ (H \ e^{iF(q,Q)}) \Psi_E(Q) = \int_{-\infty}^{+\infty} dQ \ (-H_{\text{free}} \ e^{iF(q,Q)}) \Psi_E(Q)
\]

\[
= \int_{-\infty}^{+\infty} dQ \ e^{iF(q,Q)} H_{\text{free}} \Psi_E(Q) = E \Phi_E(q).
\]

(8)

Of course, we recognize in all this the well-known integral representation for the Airy function.

\[
Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \ e^{-i(z^3/3 + zx)} = \frac{1}{\pi} \int_0^{+\infty} dz \ \cos(z^3/3 + zx).
\]

(9)

That is, \( \Phi_E(q) = Ai(q - E) \). This is precisely the representation for the energy eigenfunctions which follows by the textbook momentum-space/Fourier transform approach mentioned earlier.\(^5\)

This integral expression for the linear potential energy eigenfunctions, as provided by the generating function of a canonical transformation, is quite useful. It allows us to compute correlation functions involving the interacting variables \( q \) and \( p \) through a reduction to correlation functions involving the free variables \( Q \) and \( P \). As an illustration consider the propagator

\[
\Delta(q_1, q_2; t) = \int_{-\infty}^{+\infty} dE \ e^{-iEt} \ \Phi_E(q_1) \ \Phi^*_E(q_2).
\]

(10)

For the linear potential problem, a closed-form result for \( \Delta \) follows immediately from the above integral representation. Although the generating function is cubic in \( Q \), it is still only necessary to evaluate Gaussian integrals as evident in the following steps.

\[
\Delta(q_1, q_2; t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} dE \ \int_{-\infty}^{+\infty} dQ_1 \ \int_{-\infty}^{+\infty} dQ_2 \ e^{-iE(t-Q_1+Q_2)} e^{-i(Q_1^3/3+q_1Q_1)} e^{i(Q_2^3/3+q_2Q_2)}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dQ_1 \ \int_{-\infty}^{+\infty} dQ_2 \ \delta(t - Q_1 + Q_2) \ e^{-i(Q_1^3/3+q_1Q_1)} e^{i(Q_2^3/3+q_2Q_2)}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dQ_1 \ e^{-i(Q_1^3/3+q_1Q_1)} e^{i((Q_1-t)^3/3+q_2(Q_1-t))}.
\]

(11)

\(^5\)Interestingly, as a well-informed reader will further recognize, \( F(q, Q) \) is the “canonical unfolding” of the elementary catastrophe germ \( G = Q^3 \). (For example, see R. Gilmore, *Catastrophe Theory for Scientists and Engineers*, John Wiley & Sons, 1981.) We note in passing that the canonical unfoldings of all seven elementary catastrophes (with co-dimension \( \leq 4 \)) provide generating functions for canonical transformations from interacting systems involving linear potentials to free systems, in complete parallel with the development which we have given here for \( F(q, Q) \).
So the terms cubic in $Q_1$ cancel in the exponent, which leaves just a Gaussian in $Q_1$ and hence the result.

$$\Delta(q_1,q_2;t) = \frac{1}{2\pi} \sqrt{\frac{i}{it}} \exp \left( \frac{i(q_1 - q_2)^2}{4t} - \frac{i(q_1 + q_2)t}{2} - \frac{i t^3}{12} \right).$$ \hspace{1cm} (12)

The reader is encouraged to compare this method of computing the propagator with the more familiar path integral method [5].

There are two special features of the linear potential problem which are not generally true [3]. Firstly, the expression for the classical generating function $F$ works to solve exactly the quantum mechanical linear potential problem. In general, this is not the case as there are usually quantum corrections (i.e. terms $O(\hbar)$) which must be added to $F$. Secondly, both the normalization and overall phase of the energy eigenfunction are unchanged by the canonical transformation for the linear potential. In general, this is again not the case as it is usually necessary to introduce an energy dependent normalization and phase factor $N(E)$, so

$$\Phi_E(q) = N(E) \int_{-\infty}^{+\infty} dQ \, e^{iF(q,Q)} \Psi_E(Q).$$ \hspace{1cm} (13)

We shall next discuss an example, Liouville field theory, where both of these more general features arise. Fortunately, the resulting modifications are so simple in the Liouville case that most if not all of the essential information is still contained in the classical generating function.

**FUNCTIONAL APPROACH TO LIOUVILLE THEORY**

Liouville field theory consists of a scalar field $\phi$ in exponential self-interaction. In this section, we will consider Liouville theory on a two-dimensional spacetime cylinder, $(0 \leq \sigma \leq 2\pi, -\infty < \tau < +\infty)$, subject to periodic boundary conditions in $\sigma$. As Liouville himself demonstrated for the classical theory, this interacting field is canonically equivalent to a free pseudoscalar field $\psi$. The equivalence for fixed time, $\tau$, is most readily apparent from the existence of a generating functional

$$F[\phi,\psi] = \int d\sigma \left( \phi \partial_\sigma \psi - \frac{2m^2}{g^2} e^{g\phi} \sinh(g\psi) \right).$$ \hspace{1cm} (14)

For the classical theory, canonical momenta are obtained from $F$ in the usual way through functional derivation.

$$\pi_{\phi,\text{classical}} = \frac{\delta F}{\delta \phi}, \quad \pi_{\psi,\text{classical}} = -\frac{\delta F}{\delta \psi}. \hspace{1cm} (15)$$

Evaluation of these derivatives leads to a pair of first-order equations, the so-called Bäcklund transformations for the classical theory.

$$\partial_\tau \phi = \pi_{\phi,\text{classical}} = \partial_\sigma \psi - \frac{2m}{g} e^{g\phi} \sinh(g\psi).$$
\[ \partial_\tau \psi = \pi_{\psi,\text{classical}} = \partial_\sigma \phi + \frac{2m}{g} e^{g\phi} \cosh(g\psi). \quad (16) \]

Integrability conditions for these equations follow from alternately differentiating with respect to \(\sigma\) and \(\tau\). Consistency requires that the classical \(\phi\) and \(\psi\) fields must obey the Liouville and free wave equations.

\[ (\partial_\tau^2 - \partial_\sigma^2) \phi + \frac{4m^2}{g} e^{2g\phi} = 0, \quad (\partial_\tau^2 - \partial_\sigma^2) \psi = 0. \quad (17) \]

For the classical Liouville field theory, the Bäcklund equations may be integrated to obtain all solutions for the interacting field \(\phi\) in terms of the free field \(\psi\). Hence, all interacting field functionals \(g[\phi, \pi_\phi]\) may be expressed in terms of free field functionals, \(G[\psi, \pi_\psi]\), and the classical theory is completely solved. However, there remains a long way to go to arrive at a quantum Liouville theory, even with a complete set of classical solutions.

There are at least three major routes one can take to arrive at a quantized Liouville theory. These involve either operator methods [7], Schrödinger functional techniques [8], or path integrals [9]. Among these three major routes, there are also several minor variations, and while all these approaches may indeed be equivalent in principle, they are not necessarily equivalent in practice.

For example, one operator approach to quantize the theory is to convert the classical relations between \(\phi\) and \(\psi\) into well-defined operator expressions in such a way that the locality and conformal transformation properties expected of the expressions do indeed hold. This is a laborious procedure, but it has been carried out for many of the classical relations between Liouville and free fields [10]. Unfortunately, even when valid operator relations have been obtained, there still remains the difficult task of using those operator results to evaluate correlation functions.

Here we shall follow the canonical functional methods pedagogically discussed in the previous section for the linear potential. We will use the generating functional \(F[\phi, \psi]\) within the Schrödinger equal-time functional formalism to construct energy eigenfunctionals \(\Phi_E[\phi]\) for the quantum Liouville theory from well-known free field energy eigenfunctionals \(\Psi_E[\psi]\). We then approach the task of evaluating correlation functions in a fashion completely analogous to the evaluation of the propagator for the linear potential. As in that simple case, the end result for the Liouville theory, at least in principle, is to reduce the problem to the evaluation of free field functional integrals. Indeed, in the weak-coupling limit, we immediately obtain a series of Gaussian functional integrals which evaluates to reproduce results obtained using operator methods [11]. Again, as in the case of the propagator for the linear potential, this functional approach to correlation functions should be compared to that using path integrals [9]. Unfortunately, here we will be forced to leave a detailed comparison with path integral results as an exercise for the interested student. Suffice it to say that the functional methods seem to be easier to implement than operator methods, and they may provide a useful bridge between operator methods and path integral techniques.
Let us return to the Bäcklund transformations, Eqn. (14). These first-order functional derivative relations are most conveniently written for the quantum Liouville theory as

$$D_\pm(\sigma) \ e^{iF} = 0,$$

where

$$D_\pm(\sigma) \equiv -i \left( \frac{\delta}{\delta \phi(\sigma)} \pm \frac{\delta}{\delta \psi(\sigma)} \right) - (\partial_\sigma \psi(\sigma) \mp \partial_\sigma \phi(\sigma)) + \frac{2m}{g} e^{g\phi(\sigma) \pm g\psi(\sigma)}.$$  

Considering two such functional derivatives, in the form

$$D_\pm(\sigma_1)D_\pm(\sigma_2)e^{iF}$$

as \(\sigma_1 \rightarrow \sigma_2\), leads to the conclusion that the Liouville and free field Hamiltonians have equivalent effects on \(e^{iF}\), just as in the case of the linear potential, Eqn. (6). Consequently, the Liouville energy eigenfunctionals are functional transforms of the free field eigenfunctionals.

$$\Phi_E[\phi] = N(E) \int d\psi \ e^{iF[\phi,\psi]} \ \Psi_E[\psi].$$  

As in the previous quantum mechanics example, the classical form for the generating functional serves to provide an exact transformation between interacting and free theories. However, there is one important difference between the simple quantum mechanics example and the Liouville theory. The parameters in \(F\) are renormalized, as explained below.

In any case, correlation functions are now given in terms of functional integrals. For example, the expectation of an exponentiated Liouville field is

$$\langle E_1 | e^{\alpha g\phi} | E_2 \rangle = \int d\phi \ \Phi^*_{E_1}[\phi] \ e^{\alpha g\phi} \ \Phi_{E_2}[\phi]$$

$$= N(E_1)^* N(E_2) \int d\psi_1 \int d\psi_2 \ \Psi^*_{E_1}[\psi_1] \ O[\psi_1, \psi_2] \ \Psi_{E_2}[\psi_2]$$

where the effective operator on the space of \(\psi\) functionals is

$$O[\psi_1, \psi_2] = \int d\phi \ \exp (-iF^*[\phi, \psi_1] + iF[\phi, \psi_2] + \alpha g\phi).$$

In principle, this should be equivalent to the operator results of Braaten et al. [10], but in practice, we believe that this functional form for the expectation value can sometimes lead to more immediate results. For example, perturbation theory in the non-zero mode effects for expectations between low energy states is immediately developed from the functional expression.

The traditional separation of the fields into zero (\(q\) and \(Q\)) and non-zero (\(\hat{\phi}\) and \(\hat{\psi}\)) modes at fixed \(\tau\) is given by [10]

$$\phi(\sigma) = q + \hat{\phi}(\sigma), \quad \hat{\phi}(\sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \left( a_n e^{-i\sigma} + b_n e^{i\sigma} \right),$$

\(\dagger\)

N.B. The field integrations in this result, and in the expectation of \(\exp(\alpha g\phi)\), are over all field configurations at fixed time. That is, \(\int d\phi\) and \(\int d\psi\) here, and elsewhere in this paper, are not path integrals, but rather Schrödinger functional integrals.
\[ \psi(\sigma) = Q + \hat{\psi}(\sigma), \quad \hat{\psi}(\sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} (A_n e^{-\imath n\sigma} + B_n e^{\imath n\sigma}). \]  

(24)

However, one should keep in mind that the expansion coefficients in the functional formalism used here are not operator valued.

For the free field, we may explicitly display the time dependence of the modes to obtain right-movers

\[ \hat{\psi}_+ (\tau, \sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} B_n e^{-\imath n(\tau - \sigma)}, \]  

(25)

and left-movers

\[ \hat{\psi}_- (\tau, \sigma) = \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} A_n e^{-\imath n(\tau + \sigma)}. \]  

(26)

For \( \hat{\phi} \), the time-dependence is not so simple. Rather, it is convenient to combine the left- and right-moving modes for fixed time using projection operators. We first define these projections for the free field.

\[ \hat{\psi}(\sigma) = \sum_{n \neq 0} \psi_n e^{\imath n\sigma} = \hat{\psi}_+(\sigma) + \hat{\psi}_-(\sigma), \quad \hat{\psi}_\pm(\sigma) = P_\pm \hat{\psi}(\sigma) = \sum_{n > 0} \psi_{\pm n} e^{\pm \imath n\sigma}, \]  

(27)

\[ P_\pm \equiv \frac{1}{2} \left( 1 \mp \imath \frac{\partial_\sigma}{|\partial_\sigma|} \right), \quad |\partial_\sigma| \equiv \sqrt{-\frac{\partial^2}{\partial \sigma^2}}. \]  

(28)

We then note that this separation of the modes is a well-defined procedure even for the interacting field.

\[ \hat{\phi}(\sigma) = \sum_{n \neq 0} \phi_n e^{\imath n\sigma} = \hat{\phi}_+(\sigma) + \hat{\phi}_-(\sigma), \quad \hat{\phi}_\pm(\sigma) = P_\pm \hat{\phi}(\sigma) = \sum_{n > 0} \phi_{\pm n} e^{\pm \imath n\sigma}. \]  

(29)

Using this separation into zero and non-zero modes, we may now parallel the operator approach in Braaten et al. and perform a perturbative analysis of the effects of the non-zero modes for low energy expectation values of exponentiated Liouville fields.

The non-zero mode free field vacuum functional is

\[ \Psi_{\text{vacuum}}[\hat{\psi}] = \exp \left( -\frac{1}{2} \int_0^{2\pi} d\sigma \, |\partial_\sigma| \hat{\psi}(\sigma) \, |\partial_\sigma| \hat{\psi}(\sigma) \right), \]  

(30)

where \( |\partial_\sigma| \hat{\psi}(\sigma) = |\partial_\sigma| \hat{\psi}(\sigma) = \sum_{n > 0} |n| (\psi_{+n} e^{\imath n\sigma} + \psi_{-n} e^{-\imath n\sigma}) \). From this we construct low energy free field wave functionals, \( \Psi_k[\hat{\psi}] = \Psi_k[Q] \Psi_{\text{vacuum}}[\hat{\psi}] \), with \( E \equiv g^2 k^2 / 4\pi \) and \( \Psi_k[Q] = N_k \exp(i g k Q) \), where it is understood that \( g \) is small and \( k \) is of order unity. The functional transform (20) then yields low energy Liouville wave functionals.

\[ \Phi_k[\phi] = N_k \int dQ \, \exp(i g k Q) \int d\hat{\psi} \, e^{i F[\phi, \hat{\psi}]} \Psi_{\text{vacuum}}[\hat{\psi}]. \]  

(31)

We now split-off the non-zero mode contributions by writing

\[ F[\phi, \psi] = \int_0^{2\pi} d\sigma \left( \phi \, \partial_\sigma \psi - \frac{2m}{g^2} e^{gq} \sin(gQ) \right) + F_{\text{int}}[\phi, \psi], \]  

(32)
where the interaction between zero and non-zero modes is contained in

\[ F_{\text{int}} = -\frac{m}{g^2} e^{g(q+Q)} \int_0^{2\pi} d\sigma \left( e^{g(\hat{\phi} + \hat{\psi}) - 1} - \frac{m}{g^2} e^{g(q-Q)} \int_0^{2\pi} d\sigma \left( e^{g(\hat{\phi} - \hat{\psi}) - 1} \right) \right). \] (33)

We shift \( \hat{\psi} \to \hat{\psi} + \hat{\phi}_+ - \hat{\phi}_- \), complete the square, and eliminate the \( \int d\sigma \hat{\phi} \partial_\sigma \hat{\psi} \) term to obtain

\[ \Phi_k[\phi] = N_k \exp \left( -\frac{1}{2} \int_0^{2\pi} d\sigma \hat{\phi}(\sigma) |\partial_\sigma| \hat{\phi}(\sigma) \right) \int dQ \exp \left( igkQ - \frac{4\pi mi}{g^2} e^{gq} \sinh(gQ) \right) \times \int d\hat{\psi} \exp \left( i F_{\text{int}}[\phi, \psi + \hat{\phi}_+ - \hat{\phi}_-] - \frac{1}{2} \int_0^{2\pi} d\sigma \hat{\psi}(\sigma) |\partial_\sigma| \hat{\psi}(\sigma) \right). \] (34)

After the shift, the interaction becomes

\[ F_{\text{int}}[\phi, \psi + \hat{\phi}_+ - \hat{\phi}_-] = -\frac{m}{g^2} e^{g(q+Q)} \int_0^{2\pi} d\sigma \left( e^{g(2\hat{\phi}_+ + \hat{\psi}) - 1} - \frac{m}{g^2} e^{g(q-Q)} \int_0^{2\pi} d\sigma \left( e^{g(2\hat{\phi}_- - \hat{\psi}) - 1} \right) \right). \] (35)

Now expand \( \exp(i F_{\text{int}}) \) in powers of \( F_{\text{int}} \) to obtain a series of Gaussian integrals

\[ \int d\hat{\psi} \exp \left( i F_{\text{int}}[\phi, \psi + \hat{\phi}_+ - \hat{\phi}_-] - \frac{1}{2} \int_0^{2\pi} d\sigma \hat{\psi}(\sigma) |\partial_\sigma| \hat{\psi}(\sigma) \right) \]

\[ = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d\hat{\psi} e^{-\frac{1}{2} \int_0^{2\pi} d\sigma \hat{\psi}(\sigma) |\partial_\sigma| \hat{\psi}(\sigma)} \left( F_{\text{int}}[\phi, \psi + \hat{\phi}_+ - \hat{\phi}_-] \right)^n \] (36)

Thus \( \Phi_k[\phi] \) reduces to a series of \( \hat{\psi} \) Gaussian integrals with coefficients which depend on the zero modes, \( q \) and \( Q \). Subsequent integration over \( Q \) yields a \( q \) and \( \phi \) dependent series.

\[ \Phi_k[\phi] = \sum_{n=0}^{\infty} \Phi_k^{(n)}[q, \phi] \] (37)

In this series, \( \Phi^{(n)} \) is obtained by keeping only the term involving \( (F_{\text{int}})^n \) in the previous expansion.

The lowest order non-trivial results for the expectation of \( e^{\alpha g \phi} \) are obtained by keeping terms up to and including \( (F_{\text{int}})^2 \) in this series. The evaluation of the resulting Gaussian integrals is straightforward. To \( O(g^6) \) we obtain

\[ \langle \Phi_{k_1} | e^{\alpha g \phi} | \Phi_{k_2} \rangle = \int d\phi \Phi_{k_1}^*[\phi] e^{\alpha g \phi} \Phi_{k_2}[\phi] \]

\[ = N_{k_1} N_{k_2} Z_{k_1 k_2}(\alpha, m, g) e^{\alpha^2 g^2 \zeta(N)/2\pi} \left( 1 + 8 \left( \alpha - (k_1^2 + k_2^2) \right) \left( \frac{g^2}{4\pi} \right)^2 \zeta_2 \right. \]

\[ \left. + \left( \alpha(2 - \alpha) \left( \frac{4}{3} + \frac{2}{3} \alpha + \alpha^2 \right) + 2 \alpha(2 - \alpha)(k_1^2 + k_2^2) - (k_1^2 - k_2^2)^2 \right) \left( \frac{g^2}{4\pi} \right)^3 \zeta_3 + O(g^8) \right), \] (38)
where the zero-mode matrix element is defined by

\[ Z_{k_1 k_2}(\alpha, m, g) = \frac{2e^{\pi(k_2 - k_1)}}{g^2 \Gamma(\alpha)} \left( \frac{g}{2\pi m} \right)^{\alpha} \left| \Gamma \left( \frac{\alpha + i(k_1 + k_2)}{2} \right) \Gamma \left( \frac{\alpha + i(k_1 - k_2)}{2} \right) \right|^2, \]

and where \( N_k \) is a normalization factor required by the zero-mode wave function. The exact form of \( N_k \) is not important for the present discussion, but the possibility of such normalization factors is important, as we shall see. The usual ultraviolet divergence in the vacuum expectation value of an “un-normal-ordered” exponential is present here and is given by

\[ e^{\alpha^2 \frac{\zeta_1(N)}{2\pi}} = \int d\psi \ e^{\alpha \hat{g} \hat{\phi}} \left| \Psi_{\text{vacuum}}[\hat{\psi}] \right|^2 \]

where we have imposed a mode cutoff, \( N \), to write \( \zeta_1(N) = \sum_{n=1}^{N} \frac{1}{n} \).

Actually, to obtain (38), it is necessary to remove a similar divergence due to the exponentials in the generating functional. To this end we have replaced \( e^{\hat{g} \hat{\phi}} \) appearing in (33) by \( e^{-\frac{\zeta_1(N)}{2\pi}} e^{\hat{g} \hat{\phi}} \). We may think of this as a renormalization of the “mass” appearing in \( F_{int} \). This is not an unexpected renormalization. However, it is not the whole story for mass renormalization, as is evident from the \( O(g^4) \) terms in (38). The \( \alpha \zeta_2 \) term on the RHS is a problem (the quantum version of the equations of motion would fail) and must be eliminated. This may be achieved by making an additional finite renormalization of the mass in \( F \), which acquires an \( \alpha \) dependence and cancels the \( \alpha \zeta_2 \) term through its appearance in the zero-mode expression (39). The structure of certain terms in the perturbation series suggests that this finite mass renormalization is of the form \( \frac{2}{g^2} \sin(g^2/2) \), although we have only checked this fully to \( O(g^4) \). Nonetheless, this form is supported to all orders by comparison with the operator results in Braaten et al. [10]. In summary, the mass appearing in \( F \) is renormalized according to

\[ m \rightarrow e^{-\frac{\zeta_1(N)}{2\pi}} \xi m, \quad \xi = \left( \frac{2}{g^2} \sin(g^2/2) \right). \]

Based on exact results within the operator formalism and for the effective potential of the Liouville theory, we also anticipate another quantum correction to the exponentials in \( F \), in the form of a finite renormalization of \( g \).

\[ g \rightarrow \frac{g}{1 + \frac{g^2}{2\pi}}. \]

However, we have not checked this correction within the functional formalism. To do so to lowest order would require perturbative results beyond \( O(g^6) \).

Finally, other anomalous \( O(g^4) \) non-zero mode contributions in (38) can be eliminated through changes in normalizations. The terms \( (k_1^2 + k_2^2) \left( \frac{g^2}{2\pi} \right)^2 \zeta_2 \) may be removed by changing the normalizations of the states \( \Phi_{k_1}^{*}[\phi] \) and \( \Phi_{k_2}[\phi] \). At this time, we have no firm conjectures about the form for such normalization changes in higher orders.

After these renormalizations, the results in (38) agree to \( O(g^6) \) with those of Braaten et al. [11] obtained through the use of operator techniques. Perhaps it will be
possible to extend these perturbative results to obtain correct closed-form expressions to all orders in $g$ within the functional framework. Or, perhaps it will be possible to numerically evaluate the functional expressions to obtain non-perturbative results. We leave these as open problems for the interested reader.

LIOUVILLE THEORY ON CURVED SURFACES

Perhaps functional methods are also useful when the $(\tau, \sigma) = (z^0, z^1)$ manifold is not intrinsically flat. The classical relations between $\phi$ and $\psi$ have been discussed for this case by Preitschopf and Thorn [12]. Classically, these fields now satisfy the covariant equations

$$D^\mu D_\mu \phi = \frac{1}{2g} R - \frac{4m^2}{g} e^{2g\phi}, \quad D^\mu (\partial_\mu \psi - \frac{1}{g} \omega_\mu) = 0,$$

where the connection and scalar curvature,

$$\omega_\mu = \eta_{ab} e^a_\mu e^b_\lambda \partial_\nu e^\nu_\lambda, \quad R = -2\epsilon^{\mu\nu} \partial_\mu \omega_\nu,$$

and the zweibein $e^a_\mu$ are given functions that depend on $(z^0, z^1)$.

Canonical equivalence of the $\phi$ and $\psi$ fields in this curved surface situation again follows from a generating functional. In this case, as opposed to the cylinder, that generating functional depends explicitly on $z^0$ since the zweibein and connection do.

$$F[\phi, \psi; z^0] = \int dz^1 \left( \phi \left( \partial_1 \psi - \frac{1}{g} \omega_1(z) \right) - \frac{2m}{g^2} e^{g\phi} e^a_1(z) V_a(\psi) \right)$$

where the tangent space vector $V$ is given by $(V_0, V_1) = (\cosh(g\psi), \sinh(g\psi))$. Therefore, to understand the quantum theory in this situation, we require an extension of the previous formalism to the case where the generating functional has explicit time dependence. In principle, this extension is known, but in practice, this is still an open problem for the Liouville quantum theory, which we intend to discuss further elsewhere. Here it will suffice to return to the simple setting of quantum mechanics to discuss the case of generating functionals with explicit time dependence, again in the context of the linear potential.

TIME-DEPENDENT QUANTUM MECHANICS

It is not difficult to extend the above functional methods for solving quantum mechanical models to the case of systems with explicitly time-dependent generating functionals. In fact, the free particle is probably the most familiar example of the method which involves just such an extension. Let us briefly review this situation. As is well-known, the general solution for the time-dependent wave function of a free particle moving in one dimension can be written as

$$\Psi(q, t) = \int_{-\infty}^{+\infty} dQ \frac{e^{i(Q-Q_0)^2/2\hbar}}{\sqrt{2\pi\hbar}} \Psi_0(Q),$$

where

7 Again, the $\frac{1}{2g}$ coefficients of $R$ and $\omega_\mu$ are expected to be modified by quantum effects to $\frac{1}{2} g + \frac{g}{2\pi}$. 
the initial data is specified by $\Psi_0(Q)$. We may view this as a canonical transformation, from the variable $q$, whose time development is given by the Hamiltonian $H = \frac{1}{2} p^2$, to the variable $Q$, whose time development is nil (i.e. $H_0 \equiv 0$). As such, we may write the wave function at time $t$ as $\Psi(q,t) = \int_{-\infty}^{+\infty} dQ \, e^{iF(q,Q;t)} \Psi_0(Q)$. The quantum generator of the transformation is $F(q,Q;t) = (q - q_c(t))^2/2t + i/2 \ln(2\pi it)$, and it provides a solution of the evolution equation $\left(H - i \frac{\partial}{\partial t}\right) e^{iF(q,Q;t)} = H_0 \, e^{iF(q,Q;t)} \equiv 0$, where the Hamiltonian $H$ is written in the coordinate basis, $H = -\frac{1}{2} \partial^2/\partial q^2$. Note that the $\ln(2\pi it)$ term in $F$ is a quantum correction to the classical generating function, which is just $F_{\text{classical}} = (q - Q)^2/2t$. This quantum correction serves to cancel the second $q$ derivative term in the quantum equation for $F$, $\frac{\partial}{\partial t} F - \frac{i}{2} \frac{\partial^2}{\partial q^2} F + \frac{1}{2} \left(\frac{\partial}{\partial q} F\right)^2 = 0$. Immediately, we see the generalization to any theory for which the propagator is known. The logarithm of the exact propagator is ($i$ times) an exact time dependent quantum generating function for the theory. Thus we may obtain $F$ for a one-dimensional system defined by any quadratic Hamiltonian, or for the Coulomb problem, and transform the dynamics into $H_0 = 0$.

However, for the Liouville theory we do not have the exact propagator. Rather, we are given \cite{15} which tells us how to transform directly from the exponentially interacting theory to a “free” theory on a curved manifold. As a very simple analogy, let us consider a particle in a linear potential with a time-dependent coefficient, $f(t)$. The time-dependent Hamiltonian is

$$H(t) = \frac{1}{2} p^2 + f(t) \, q. \quad (46)$$

Time-dependent wave functions for this system can be obtained from free-particle wave functions governed by the Hamiltonian

$$H_{\text{free}} = \frac{1}{2} P^2. \quad (47)$$

A time-dependent generating function $F_2(q,P;t)$ which provides the connection between the two Hamiltonians is

$$F_2(q,P;t) = (q - q_c(t)) (P + p_c(t)) + S_c(t), \quad (48)$$

where $q_c(t)$ and $p_c(t)$ represent any particular classical solution to the “specified acceleration” problem,

$$dq_c/dt = p_c, \quad dp_c/dt = -f(t), \quad (49)$$

and where $S_c$ is the corresponding classical action.

$$S_c(t) = \frac{1}{2} q_c(t) p_c(t) - \frac{1}{2} \int_0^t d\tau \, f(\tau) \, q_c(\tau). \quad (50)$$

Again, we check the classical requirements on $F_2(q,P;t)$ for it to implement the necessary canonical transformation. Thus

$$p = \frac{\partial}{\partial q} F_2(q,P;t) = P + p_c, \quad Q = \frac{\partial}{\partial P} F_2(q,P;t) = q - q_c, \quad (51)$$
and by directly substituting these
\[ H(t) = \frac{1}{2} p^2 + p_c P + fQ + \frac{1}{2} p_c^2 + f q_c = H_{\text{free}} - \frac{\partial}{\partial t} F_2. \] (52)

We may now clarify how one arrives at the above form for \( F_2(q, P; t) \). We simply observe that any solution of the classical specified acceleration problem differs from a particular solution only by a free particle solution. Hence, \( P = p - p_c(t) \) and \( Q = q - q_c(t) \) are indeed free particle variables.

We use \( F_2 \) in the quantum theory as before. It connects a general time-dependent wave function \( \Phi(q, t) \) for a particle governed by (46) to a free particle wave function \( \Psi(P, t) \) through the integral relation
\[ \Phi(q, t) = \frac{1}{2\pi} \int dP \, e^{iF_2(q, P; t)} \Psi(P, t). \] (53)

To convert this to the previous type of generating functional, \( F(q, Q; t) \), we carry out a Fourier transform. So
\[ e^{iF(q, Q; t)} = \frac{1}{2\pi} \int dP \, e^{-iQP} \, e^{iF_2(q, P; t)} = e^{i(q-q_c(t))p_c(t)+iS_c(t)} \delta(Q-q+q_c(t)). \] (54)

The presence of a Dirac delta function in this result is perhaps surprising. Nonetheless, it is straightforward to check that \( e^{iF(q, Q; t)} \) obeys the time-dependent partial differential equation
\[ \left( H(t) - i \frac{\partial}{\partial t} \right) e^{iF(q, Q; t)} = H_{\text{free}} e^{iF(q, Q; t)}. \] (55)

Thus the relation between \( \Phi(q, t) \) and a free particle wave function \( \Psi(Q, t) \) is
\[ \Phi(q, t) = \int dQ \, e^{iF(q, Q; t)} \Psi(Q, t) = e^{i(q-q_c(t))p_c(t)+iS_c(t)} \Psi(q-q_c(t), t). \] (56)

A straightforward check shows that \( \Phi(q, t) \) as given by this expression does indeed obey the time-dependent Schrödinger equation, \((-\frac{1}{2} \frac{\partial^2}{\partial q^2} + f(t)q) \Phi(q, t) = i \frac{\partial}{\partial t} \Phi(q, t)\), provided that \( \Psi(Q, t) \) is a solution of the free wave equation, \((-\frac{1}{2} \frac{\partial^2}{\partial q^2}) \Psi(q, t) = i \frac{\partial}{\partial t} \Psi(q, t)\).

We may also use \( e^{iF} \) to transform the complete set of time-dependent free particle wave functions \( \Psi_k(Q, t) = e^{ikQ-\frac{1}{2}k^2t} \). The result is to obtain a complete set of time-dependent interacting particle wave functions \( \Phi_k(q, t) = e^{iS_c(t)+i(p_c(t)+k)(q-q_c(t))-\frac{1}{2}k^2t} \). Of course, when \( f(t) \to f \), a constant independent of \( t \), it is possible to obtain the usual stationary state wave functions from \( \Phi_k(q, t) \) by projection \( \Phi_k(q, E) = \int dt \, e^{iEt} \Phi_k(q, t) \).

The result given by Eqn.(56) is quite amusing, and not readily found in the literature. It is related to an old result of Schrödinger, however. To see this, it is useful to generalize a bit to include a quadratic term in the potential, again with a time-dependent coefficient.
\[ H_{\text{oscillator}}(t) = \frac{1}{2} p^2 + \frac{1}{2} k(t) q^2 + f(t) q. \] (57)
Now, any solution for this shifted oscillator system is related to a solution of the unshifted (i.e. $f = 0$) oscillator by the same canonical transformation generated by (48), only now the particular classical solution used in $F_2$ must be a solution of the classical equations

$$
dq_c/dt = pc, \quad dp_c/dt = -k(t)q_c - f(t).
$$

(58)

The action has the same form as in (50). The same relation (56) between time-dependent wave packets also applies, only now $\Phi$ and $\Psi$ are solutions of

$$
\left(-\frac{1}{2} \frac{\partial^2}{\partial q^2} + \frac{1}{2} k(t) q^2 + f(t) q\right) \Phi(q,t) = i \frac{\partial}{\partial t} \Phi(q,t)
$$

(59)

and

$$
\left(-\frac{1}{2} \frac{\partial^2}{\partial q^2} + \frac{1}{2} k(t) q^2\right) \Psi(q,t) = i \frac{\partial}{\partial t} \Psi(q,t).
$$

(60)

This relation between wave packets, Eqn.(56), is the generalization to the time-dependent case of the well-known shift of variable that relates energy eigenstates for the potential $kq^2/2$ to those for the potential $fq + kq^2/2$, with $k$ and $f$ constant, namely $q \to q - f/k$. Accompanying this shift of variable, there is also a shift of energy eigenvalue: $E \to E - f^2/2k$. The corresponding ‘energy’ effect in the general time-dependent case is the phase that appears on the RHS of Eqn.(56). At this point perhaps we should clarify an issue which should have occurred to the reader. What happens if a different classical solution is used to implement the canonical transformation? Well, clearly, the difference between any two solutions of Eqn.(58) is a solution of

$$
\frac{d^2q_d(t)}{dt^2} = -k(t)q_d(t),
$$

the unshifted oscillator equation. Now the reader may easily check that any solution of the unshifted oscillator Schrödinger equation remains a solution of that equation if its spatial argument is translated by $q_d(t)$, and it is multiplied by the phase $e^{iS_d(t)+ip_d(t)(q-q_d(t))}$ (cf. Eqn.(56) with $f = 0$). This is just a generalization to the case of an arbitrary time-dependent $k(t)$, and an arbitrary wave-packet, of the well-known feature for the time development of the “minimal Gaussian packet” of the harmonic oscillator [13].

It remains to further develop these simple ideas involving time-dependent canonical transformations in quantum mechanics and apply them to analyze the Liouville theory on a curved surface. We hope to pursue these matters in the future.

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References

[1] L.D. Landau and E.M. Lifschitz, Quantum Mechanics (Non-relativistic Theory), Third Edition, Pergamon Press, 1977.

[2] P.A.M. Dirac, Physikalische Zeitschrift der Sowjetunion, Vol. 3, pp 64-72, 1932.

[3] G. Ghandour, Phys. Rev. D35 (1987) 1289.

[4] T.L. Curtright, Differential Geometric Methods in Theoretical Physics: Physics and Geometry, L.-L. Chau and W. Nahm, ed’s., Plenum Press, 1990, pp 279-289.

[5] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1965.

[6] R. Gilmore, Catastrophe Theory for Scientists and Engineers, John Wiley & Sons, 1981.

[7] T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309.

[8] T.L. Curtright and G.I. Ghandour, unpublished (1984); T. McCarty, University of Florida Ph.D. Thesis, unpublished (1990).

[9] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.

[10] E. Braaten, T.L. Curtright, and C.B. Thorn, Phys. Lett. 118B (1982) 115; Ann. Phys. (NY) 147 (1983) 365.

[11] E. Braaten, T.L. Curtright, G.I. Ghandour, and C.B. Thorn, Phys. Rev. Lett. 51 (1983) 19; Ann. Phys. (NY) 153 (1984) 147.

[12] C. Preitschopf and C.B. Thorn, Phys. Lett. 250B (1991) 79. Also see, E. D’Hoker, Phys. Lett. 264B (1991) 101.

[13] E. Schrödinger, Naturwiss. 14 (1926) 664.