Chain conditions for rings with enough idempotents with applications to category graded rings

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ABSTRACT
We obtain criteria for when a ring with enough idempotents is left/right artinian or noetherian in terms of local criteria defined by the associated complete set of idempotents for the ring. We apply these criteria to object unital category graded rings in general and, in particular, to the class of skew category algebras. Thereby, we generalize results by Nastasescu-van Oystaeyen, Bell, Park, and Zelmanov from the group graded case to groupoid, and in some cases category, gradings.

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1. Introduction
Throughout this article all rings are associative but not necessarily unital. If a ring $S$ is unital, then we always assume that $S$ is nonzero and that $1_S$ denotes the multiplicative unit of $S$. Recall that a ring is called left/right artinian (noetherian) if it satisfies the descending (ascending) chain condition on its poset of left/right ideals. The main purpose of the present article is to establish criteria for artinianity and noetherianity for rings with enough idempotents (see Theorem 1). The secondary purpose is to apply this result to the setting of category graded rings and thereby obtaining similar results for this class of rings (see Theorems 2–4).

Here is an outline of the article. In Section 2, we analyze chain conditions for the class of rings with enough idempotents, introduced by Fuller in [4] (see Definition 6). To this end, we consider rings with complete sets of idempotents $\{e_i\}_{i \in I}$ which are, what we call, strong (see Definition 10). By this we mean that for all $i, j \in I$ with $e_i Se_j$ nonzero or $e_j Se_i$ nonzero, we have $e_i \in e_i Se_j Se_i$ and $e_j \in e_j Se_i Se_j$. We establish the following:

Theorem 1. Suppose $S$ is a ring with enough idempotents and let $\{e_i\}_{i \in I}$ be a complete set of idempotents for $S$.

1. If $S$ is left/right artinian (noetherian), then $I$ is finite and for every $i \in I$ the ring $e_i Se_j$ is left/right artinian (noetherian).
2. Suppose $\{e_i\}_{i \in I}$ is strong. Then $S$ is left/right artinian (noetherian) if and only if $I$ is finite and for every $i \in I$ the ring $e_i Se_j$ is left/right artinian (noetherian).

In Section 3, we state our conventions on categories and groupoids (see Definition 13). We also introduce the class of hom-set strong categories (see Definition 15). This is a class of categories which strictly contains the class of groupoids (see Example 17).
In Section 4, we consider the class of object unital category (and groupoid) graded rings (for our conventions on graded rings, see Definition 19). We use Theorem 1 and results from the group graded case by Năstăsescu-van Oystaeyen [10] (see Theorem 25) and Bell [1] (see Theorem 27), to establish the following:

**Theorem 2.** Suppose $G$ is a groupoid and let $S$ be a ring which is strongly $G$-graded and object unital.

(1) Let $G$ be torsion-free. Then $S$ is left/right artinian if and only if $G_0$ is finite and for every $a \in G_0$ the ring $S_a$ is left/right artinian and $S_{G(a)}$ is finitely generated as a left/right $S_a$-module.

(2) Let $G$ be polycyclic-by-finite. Then $S$ is left/right noetherian if and only if $G_0$ is finite and for every $a \in G_0$ the ring $S_a$ is left/right noetherian.

For the definitions of the classes of torsion-free and polycyclic-by-finite groupoids, see Definitions 24 and 26.

In Section 5, we analyze chain conditions for skew category algebras which are defined by skew category systems of unital rings (see Definition 28). Using a result by Park [12] (see Theorem 32) from the setting of skew group rings, and the results established in Section 4, we prove the following:

**Theorem 3.** Suppose $G$ is a groupoid and let $\alpha = \{ \alpha_g : R_{d(g)} \to R_{c(g)} \}_{g \in G_1}$ be a skew category system of unital rings. Then the associated skew category algebra $R \ast_\alpha G$ is left/right artinian if and only if $G$ is finite and for every $a \in G_0$ the ring $R_a$ is left/right artinian.

Note that Theorem 3 already was obtained in [11] using a different method. In Section 5, we also show the following theorem, using a result by Zelmanov [14] (see Theorem 33) and the results in Section 4.

**Theorem 4.** Suppose $G$ is a hom-set strong category and let $T$ be a unital ring. Then the associated category ring $T[G]$ is left/right artinian if and only if $T$ is left/right artinian and $G$ is finite.

Theorems 2–4 generalize classical results from the group graded case to the groupoid (and in the case of the last theorem category) graded situation.

2. Rings with enough idempotents

In this section, we analyze chain conditions for the class of rings with enough idempotents. To this end, we introduce the class of rings with complete sets of idempotents which are, what we call, strong. At the end of this section, we prove Theorem 1. Throughout this article, we put $\mathbb{N} := \{1, 2, 3, \ldots\}$. For the rest of the section, $R$ and $S$ denote rings. For the convenience of the reader, we first gather some well known results that we will use later.

**Proposition 5.** The following assertions hold.

(1) Suppose $M$ is a left/right $S$-module and let $R \subseteq S$. If $M$ is artinian (noetherian) as a left/right $R$-module, then $M$ is artinian (noetherian) as a left/right $S$-module.

(2) Suppose $R$ is a direct summand of the left/right $R$-module $S$. Then, for any right/left ideal $I$ of $R$, the equality $IS \cap R = IR$ (or $SI \cap R = RI$) holds. In particular, if $S$ is right/left artinian (noetherian) and $R$ is unital, then $R$ is right/left artinian (noetherian).

(3) Suppose $n \in \mathbb{N}$ and let $R_1, \ldots, R_n$ be rings. Then $R_1 \times \cdots \times R_n$ is left/right artinian (noetherian) if and only if all of the rings $R_1, \ldots, R_n$ are left/right artinian (noetherian).

(4) Suppose $R$ is unital and let $M$ be a left (right) $R$-module. Let $n \in \mathbb{N}$ and suppose $M_1, \ldots, M_n$ are left/right $R$-submodules of $M$ such that $M = M_1 \oplus \cdots \oplus M_n$. Then $M$ is left/right artinian (noetherian) if and only if all of the modules $M_1, \ldots, M_n$ are left/right artinian (noetherian).
**Proof.** See a standard book on the theory of rings and modules e.g. [13].

**Definition 6.** Recall from [4] that $S$ is said to have enough idempotents if there exists a set $\{e_i\}_{i \in I}$ of nonzero orthogonal idempotents in $S$, called a complete set of idempotents for $S$, such that $S = \bigoplus_{i \in I} e_i S = \bigoplus_{i \in I} e_i S$. For the rest of this section, $S$ denotes a ring with enough idempotents where $\{e_i\}_{i \in I}$ is a fixed complete set of idempotents for $S$. Given $i, j \in I$, we use the notation $S_{ij} := e_i S e_j$ and $S_i := S_i$. We also put $S_0 := \bigoplus_{i \in I} S_i$.

**Proposition 7.** If $S$ is left/right artinian (noetherian), then $I$ is finite.

**Proof.** Suppose $I$ is infinite. Then we may assume that $\mathbb{N} \subseteq I$.

Define a set of left ideals $\{I_i\}_{i \in \mathbb{N}}$ of $S$ by $I_i = Se_i + Se_{i+1} + Se_{i+2} + \cdots$ for $i \in \mathbb{N}$. Then $e_i \in I_i \setminus I_{i+1}$ and thus $I_i \supseteq I_{i+1}$ for all $i \in \mathbb{N}$. Therefore $S$ is not left artinian.

Define a set of right ideals $\{I_i\}_{i \in \mathbb{N}}$ of $S$ by $I_i = e_i S + e_{i+1} S + e_{i+2} S + \cdots$ for $i \in \mathbb{N}$. Then $e_i \in I_i \setminus I_{i+1}$ and thus $I_i \supseteq I_{i+1}$ for all $i \in \mathbb{N}$. Therefore $S$ is not right artinian.

Define a set of left ideals $\{K_i\}_{i \in \mathbb{N}}$ of $S$ by $K_i = Se_i + Se_{i+1} + \cdots + Se_i$ for $i \in \mathbb{N}$. Then $e_i \in K_i \cap K_{i+1}$ and thus $K_i \subseteq K_{i+1}$ for all $i \in \mathbb{N}$. Therefore $S$ is not left noetherian.

Define a set of right ideals $\{L_i\}_{i \in \mathbb{N}}$ of $S$ by $L_i = e_i S + e_{i+1} S + \cdots + e_i S$ for $i \in \mathbb{N}$. Then $e_{i+1} \in L_{i+1} \setminus L_i$ and thus $L_i \supseteq L_{i+1}$ for all $i \in \mathbb{N}$. Therefore $S$ is not right noetherian.

**Proposition 8.** Suppose $S$ is left/right artinian (noetherian) and $i \in I$. Then the ring $S_i$ is left/right artinian (noetherian).

**Proof.** From Proposition 7 it follows that $S_0$ is unital with $1_{S_0} = \sum_{j \in I} e_j$. Since $S_0$ is a direct summand of $S$ as a right/left $S_0$-module it follows from Proposition 5(2) that $S_0$ is left/right artinian (noetherian). By Proposition 5(3) the ring $S_i$ is left/right artinian (noetherian).

**Proposition 9.** Given $S$ and $\{e_i\}_{i \in I}$ the following properties are equivalent:

1. $\forall (i, j, k) \in I \times I \times I$ if two of the additive groups $S_{ij}, S_{jk}$ and $S_{ik}$ are nonzero, then the third one is also nonzero and $S_{ij} S_{jk} = S_{ik}$;
2. $\forall (p, q) \in I \times I$ if one of the additive groups $S_{pq}$ and $S_{qp}$ is nonzero, then the other one is also nonzero and $S_{pq} S_{qp} = S_p$;
3. $\forall (p, q) \in I \times I$ if one of the additive groups $S_{pq}$ and $S_{qp}$ is nonzero, then the other one is also nonzero and $e_p \in S_{pq} S_{qp}$.

**Proof.** (1)$\Rightarrow$(2): Suppose (1) holds and take $(p, q) \in I \times I$.

- Case 1: $S_{pq} \neq \{0\}$. Since $S_p \neq \{0\}$ it follows from (1), with $i = p, j = q$ and $k = p$, that $S_{qp} \neq \{0\}$ and $S_{pq} S_{qp} = S_p$.
- Case 2: $S_{qp} \neq \{0\}$. Since $S_q \neq \{0\}$ it follows from (1), with $i = q, j = p$ and $k = q$, that $S_{pq} \neq \{0\}$ and $S_{qp} S_{pq} = S_q$.

(2)$\Rightarrow$(1): Suppose (2) holds and take $(i, j, k) \in I \times I \times I$.

- Case 1: $S_{ij} \neq \{0\}$ and $S_{jk} \neq \{0\}$. By (2) we get that $\{0\} \neq S_j = S_j S_j = S_j S_j S_{jk} S_{jk} \subseteq S_j S_{jk} S_{jk}$. Therefore $S_{jk} \neq \{0\}$. By (2) again we get that $S_{jk} = S_{jk} S_{jk} = S_{jk} S_{jk} S_{jk} \subseteq S_{jk} S_{jk} \subseteq S_{jk}$. Thus $S_{jk} S_{jk} = S_{jk}$.
- Case 2: $S_{ij} \neq \{0\}$ and $S_{ik} \neq \{0\}$. By (2) we get that $\{0\} \neq S_i = S_i S_i = S_i S_i S_{ik} S_{ik} \subseteq S_i S_{ik} S_{ik}$. Therefore $S_{ik} \neq \{0\}$. The same calculation as in Case 1 shows that $S_i S_{ik} = S_{ik}$.
- Case 3: $S_{ik} \neq \{0\}$ and $S_{jk} \neq \{0\}$. By (2) we get that $\{0\} \neq S_k = S_k S_k = S_k S_k S_{jk} S_{jk} \subseteq S_k S_{jk} S_{jk}$. Therefore $S_{jk} \neq \{0\}$. The same calculation as in Case 1 shows that $S_j S_{jk} = S_{jk}$.

(2)$\iff$(3): This is clear since $e_p \in S_p$ for all $p \in I$.\[QED\]
Definition 10. If $S$ and $\{e_i\}_{i \in I}$ satisfy any of the three equivalent properties in Proposition 9, then we say that $\{e_i\}_{i \in I}$ is a strong complete set of idempotents for $S$.

Proposition 11. Let $\{e_i\}_{i \in I}$ be a strong complete set of idempotents for $S$. Suppose $S_{ij} \neq \{0\}$ for some $i, j \in I$.

1. The poset of left (right) ideals of $S_i$, $S_j$ is isomorphic to the poset of left (right) $S_i$-submodules of $S_{ij}$.
2. The ring $S_i$, $S_j$ is left (right) artinian/noetherian if and only if the left $S_j$-module (right $S_i$-module) $S_{ji}$ is artinian/noetherian.

Proof. We show the “left” part of the proof and leave the “right” part to the reader. Since (2) obviously follows from (1) it is enough to show (1). Define maps $\alpha : \{\text{left ideals of } S_i\} \rightarrow \{\text{left } S_j\text{-submodules of } S_{ij}\}$ and $\beta : \{\text{left } S_j\text{-submodules of } S_{ij}\} \rightarrow \{\text{left ideals of } S_j\}$ by $\alpha(I) = e_jSI$ and $\beta(M) = e_iSM$ for left ideals $I$ of $S_i$ and left $S_j$-submodules $M$ of $S_{ij}$. Then, clearly, $\alpha$ and $\beta$ are well defined inclusion preserving maps. Take a left ideal $I$ of $S_i$ and a left $S_j$-submodule $M$ of $S_{ij}$. By Proposition 9 we get that $\beta(\alpha(I)) = \beta(e_jSI) = S_jS_jI = S_jI = I$ and $\alpha(\beta(M)) = \alpha(e_iSM) = S_jS_iM = S_jM = M$. This proves (1).

Proposition 12. Let $\{e_i\}_{i \in I}$ be a finite strong complete set of idempotents for $S$. Suppose that for every $i \in I$ the ring $S_i$ is left/right artinian (noetherian). Then $S$ is left/right artinian (noetherian).

Proof. We show the “left” part of the proof and leave the “right” part to the reader. Take $i, j \in I$. By Proposition 11(2) it follows that the left $S_j$-module $S_{ij}$ is artinian (noetherian). By Proposition 5(1) it follows that $S_{ij}$ is artinian (noetherian) as a left $S_0$-module. Since $I$ is finite, it follows from Proposition 5(4) that $S = \bigoplus_{k \in I} S_{kl}$ is artinian (noetherian) as a left $S_0$-module. By Proposition 5(1), $S$ is left artinian (noetherian).

Proof of Theorem 1. The “only if” statements follow from Propositions 7 and 8. The ”if” statement follows from Proposition 12.

3. Hom-set strong categories

In this section, we state our conventions on categories and groupoids. We introduce the class of hom-set strong categories and we show that this class strictly contains the class of groupoids. At the end of the section, we show a finiteness result for hom-set strong categories.

Definition 13. For the rest of this article, unless otherwise stated, we let $G$ denote a small category. Recall that this means that the collection of objects in $G$, denoted by $G_0$, and the collection of morphisms in $G$, denoted by $G_1$, are sets. The domain and codomain of $g \in G_1$ is denoted by $\text{dom}(g)$ and $\text{cod}(g)$, respectively; we indicate this by writing $g : \text{dom}(g) \rightarrow \text{cod}(g)$. Given $a, b \in G_0$ we let $G(a, b)$ denote the set of morphisms $b \rightarrow a$. We put $G(a) := G(a, a)$ for $a \in G_0$. We always regard $G_0$ as a subset of $G_1$. Therefore, we denote the identity morphism $a \rightarrow a$ by $a$. We put $G_2 := \{(g, h) \in G_1 \times G_1 \mid \text{dom}(g) = \text{cod}(h)\}$. If $(g, h) \in G_2$, then the composition of $g$ and $h$ is written as $gh$. We say that $G$ is finite if $G_1$, and hence also $G_0$, is finite. Recall that a category $G$ is said to be a groupoid if all morphisms in $G$ are isomorphisms. In that case, the inverse of a morphism $g : a \rightarrow b$ in $G$ is denoted by $g^{-1} : b \rightarrow a$.

Proposition 14. For a category $G$, the following properties are equivalent:

1. $\forall(a, b, c) \in G_0 \times G_0 \times G_0$ if two of the sets $G(a, b)$, $G(b, c)$ and $G(a, c)$ are nonempty, then the third set is nonempty and $G(a, b)G(b, c) = G(a, c)$;
2. $\forall(x, y) \in G_0 \times G_0$ if one of the sets $G(x, y)$ and $G(y, x)$ is nonempty, then the other set is nonempty and $G(x, y)G(y, x) = G(x)$;
3. $\forall(x, y) \in G_0 \times G_0$ if one of the sets $G(x, y)$ and $G(y, x)$ is nonempty, then the other set is nonempty and $x \in G(x, y)G(y, x)$.
This follows from Proposition 14(3) and the fact that for every morphism \( \alpha \):

\[
\text{Proof. } (1) \Rightarrow (2): \text{Suppose (1) holds and take } (x, y) \in G_0 \times G_0.
\]

- Case 1: \( G(x, y) \neq \emptyset \). Since \( G(x, x) \neq \emptyset \) it follows from (1), with \( a = x, b = y \) and \( c = a \), that \( G(y, x) \neq \emptyset \) and \( G(x, y)G(y, x) = G(x) \).
- Case 2: \( G(y, x) \neq \emptyset \). Since \( G(y, y) \neq \emptyset \) it follows from (1), with \( a = y, b = x \) and \( c = y \), that \( G(x, y) \neq \emptyset \) and \( G(y, x)G(y, y) = G(y) \).

\((2) \Rightarrow (1): \text{Suppose (2) holds and take } (a, b, c) \in G_0 \times G_0 \times G_0.
\]

- Case 1: \( G(a, b) \neq \emptyset \) and \( G(b, c) \neq \emptyset \). Then \( G(a, c) \supseteq G(a, b)G(b, c) \neq \emptyset \) so that \( G(a, c) \neq \emptyset \). From (2), with \( x = c \) and \( y = b \), we get that \( G(c, b) \neq \emptyset \) and \( G(c, b)G(b, c) = G(c) \). Thus \( G(a, c) = G(a, c)G(c) = G(a, c)G(c, b)G(b, c) \subseteq G(a, b)G(b, c) \subseteq G(a, c) \Rightarrow G(a, b)G(b, c) = G(a, c) \).
- Case 2: \( G(a, b) \neq \emptyset \) and \( G(a, c) \neq \emptyset \). From (2), with \( x = a \) and \( y = b \), it follows that \( G(b, a) \neq \emptyset \). Thus \( G(b, c) \supseteq G(b, a)G(a, c) \neq \emptyset \) and hence \( G(b, c) \neq \emptyset \). A calculation similar to Case 1 gives \( G(a, b)G(b, c) = G(a, c) \).
- Case 3: \( G(a, c) \neq \emptyset \) and \( G(b, c) \neq \emptyset \). From (2), with \( x = c \) and \( y = b \), it follows that \( G(b, c) \neq \emptyset \). Thus \( G(a, b) \supseteq G(a, c)G(c, b) \neq \emptyset \) and hence \( G(a, b) \neq \emptyset \). A calculation similar to Case 1 gives \( G(a, b)G(b, c) = G(a, c) \).

\((2) \iff (3): \text{This is clear since } x \in G(x) \text{ for all } x \in G_0. \)

**Definition 15.** If a category \( G \) satisfies any of the equivalent properties in Proposition 14, then we say that \( G \) is a hom-set strong category.

**Proposition 16.** If \( G \) is a groupoid, then \( G \) is a hom-set strong category.

**Proof.** This follows from Proposition 14(3) and the fact that for every morphism \( g : b \rightarrow a \) in \( G \) the relation \( a = gg^{-1} \in G(a, b)G(b, a) \) holds.

**Example 17.** Not all hom-set strong categories are groupoids. Indeed, let \( M \) be a monoid with identity element 1. Take a nonempty set \( X \). We now define the category \( MX \) in the following way. As set of objects in \( MX \) we take \( X \). As morphisms in \( MX \) we take all triples \( g = (m, x, y) \), for \( x, y \in X \), where we put \( d(g) = y \) and \( c(g) = x \). The composition of \( g = (m, x, y) \) with \( h = (n, y, z) \), for \( m, n \in M \) and \( x, y, z \in X \), is defined as \( gh = (mn, x, z) \). Then, for all \( x, y \in X \), the set \( MX(x, y) \) is nonempty and \( (1, x, x) = (1, x, y)(1, y, x) \in MX(x, y)MX(y, x) \). Thus, by Proposition 14(3), it follows that \( MX \) is a hom-set strong category. On the other hand, it is easily checked that \( MX \) is a groupoid if and only if \( M \) is a group.

**Proposition 18.** Suppose \( G \) is a hom-set strong category. Then \( G \) is finite if and only if \( G_0 \) is finite and for every \( a \in G_0 \) the monoid \( G(a) \) is finite.

**Proof.** The "only if" statement is clear. Now we show the "if" statement. Suppose \( G_0 \) is finite and let all monoids \( G(a) \), for \( a \in G_0 \), be finite. Take different \( c, d \in G_0 \) with \( G(c, d) \neq \emptyset \). By Proposition 14(3) there are \( p : c \rightarrow d \) and \( q : d \rightarrow c \) such that \( pq = c \). Define maps \( \alpha : G(c, d) \rightarrow G(c) \) and \( \beta : G(c) \rightarrow G(c, d) \) by \( \alpha(g) = qg \), for \( g \in G(c, d) \), and \( \beta(h) = ph \), for \( h \in G(c) \). Then \( \beta \alpha = \id_{G(c, d)} \), which, in particular, implies that \( \alpha \) is injective. Since \( G(c) \) is finite this implies that \( G(c, d) \) is finite. Thus, since \( G_0 \) is finite, we get that \( G_1 = \bigcup_{a,b \in G_0} G(a, b) \) is finite.

4. Hom-set-strongly category graded rings

In this section, we study the class of object unital category (and groupoid) graded rings. At the end of the section, we prove Theorem 2.
**Definition 19.** For the rest of this section, we let $S$ denote a ring which is $G$-graded. Recall from [6, 7] that this means that there for every $g \in G_1$ is an additive subgroup $S_g$ of $S$ such that $S = \bigoplus_{g \in G_1} S_g$ and for all $g, h \in G_1$, the inclusion $S_g S_h \subseteq S_{gh}$ holds, if $(g, h) \in G_2$, and $S_g S_h = \{0\}$, otherwise. Note that if $H$ is a subcategory of $G$ then $S_H := \bigoplus_{h \in H} S_h$ is a subring of $S$. Following [2, 3] (see also [8]) we say that the $G$-grading on $S$ is object unital if for all $a \in G_0$ the ring $S_a$ is unital and for all $g \in G_1$ and all $s \in S_g$ the equalities $1_S S_g S = 1_S S_{g(e)} = s$ hold. In that case, $S$ is a ring with enough idempotents with $\{1_{S_a} \}_{a \in G_0}$ as a complete set of idempotents and the following equality holds for all $a, b \in G_0$:

$$1_{S_a} S_{a b} = S_{G(a,b)}.$$  \hspace{1cm} (1)

Following [6] we say that $S$ is strongly $G$-graded if for all $(g, h) \in G_2$, the equality $S_g S_h = S_{gh}$ holds.

**Proposition 20.** Suppose $G$ is a hom-set strong category and let $S$ be an object unital $G$-graded ring. Then the following properties are equivalent:

1. $\forall (a, b, c) \in G_0 \times G_0 \times G_0$ if two of the groups $S_{G(a,b)}$, $S_{G(b,c)}$ and $S_{G(a,c)}$ are nonzero, then the third is also nonzero and $S_{G(a,b)} S_{G(b,c)} = S_{G(a,c)}$;
2. $\forall (p, q) \in G_0 \times G_0$ if one of the groups $S_{G(p,q)}$ and $S_{G(q,p)}$ is nonzero, then the other one is also nonzero and $S_{G(p,q)} S_{G(q,p)} = S_{G(p)}$;
3. $\forall (p, q) \in I \times I$ if one of the groups $S_{G(p,q)}$ and $S_{G(q,p)}$ is nonzero, then the other one is also nonzero and $1_{G(p,q)} S_{G(p,q)}$;
4. The set $\{1_{S_a} \}_{a \in G_0}$ is a strong complete set of idempotents for $S$.

**Proof.** This follows from Proposition 9, Proposition 14 and equation (1).

**Definition 21.** Suppose $G$ is a hom-set strong category and let $S$ be an object unital $G$-graded ring. If $S$ satisfies any of the equivalent properties in Proposition 20, then we say that $S$ is hom-set-strongly $G$-graded.

**Theorem 22.** Suppose $S$ is an object unital $G$-graded ring.

1. If $S$ is left/right artinian (noetherian), then $G_0$ is finite and, for every $a \in G_0$, the ring $S_{G(a)}$ is left/right artinian (noetherian).
2. Suppose $G$ is a hom-set strong category and let $S$ be hom-set-strongly $G$-graded. Then $S$ is left/right artinian (noetherian) if and only if $G_0$ is finite and $S_{G(a)}$ is left/right artinian (noetherian) for all $a \in G_0$.

**Proof.** This follows from Theorem 1 and Proposition 20.

**Proposition 23.** Suppose $S$ is an object unital $G$-graded ring. If $G$ is a groupoid, then $S$ is hom-set-strongly $G$-graded.

**Proof.** Suppose $G$ is a groupoid. By Proposition 16 $G$ is hom-set strong. Take $p, q \in G_0$. We consider two cases. Case 1: $S_{G(p,q)} \neq \{0\}$. Then there is $g : q \rightarrow p$ with $S_g \neq \{0\}$. Since $S$ is strongly $G$-graded it follows that $1_{S_p} \in S_p = S_g S_{g^{-1}} \subseteq S_{G(p,q)} S_{G(q,p)}$. Thus $S_{G(q,p)} \neq \{0\}$. Case 2: $S_{G(q,p)} \neq \{0\}$. By a calculation similar to the one in Case 1 it follows that $S_{G(p,q)} \neq \{0\}$ and that $1_{S_p} \in S_{G(p,q)} S_{G(q,p)}$. By Proposition 20(3) it follows that $S$ is hom-set-strongly $G$-graded.

**Definition 24.** Recall that a group is called torsion-free if the only element in the group of finite order is the identity. More generally, we say that a groupoid $G$ is torsion-free if for every $a \in G_0$ the group $G(a)$ is torsion-free.
Theorem 25. Suppose $H$ is a torsion-free group with identity element 1 and let $T$ be a unital and $H$-graded ring. Then $T$ is left/right artinian if and only if $T_1$ is left/right artinian and $T$ is finitely generated as a left/right $T_1$-module.

Proof. This is [10, Corollary 9.6.2] (see also [5, Theorem 1.2]). \hfill \square

Definition 26. Recall that a group is called polycyclic-by-finite if it has a finite length subnormal series with each factor a finite group or an infinite cyclic group. More generally, we say that a groupoid $G$ is polycyclic-by-finite if for every $a \in G_0$ the group $G(a)$ is polycyclic-by-finite.

Theorem 27. Suppose $H$ is a polycyclic-by-finite group with identity element 1 and let $T$ be a unital and strongly $H$-graded ring. Then $T$ is left/right noetherian if and only if $T_1$ is left/right noetherian.

Proof. See [1, Proposition 2.5] and in a slightly more general case [5, Theorem. 1.1] \hfill \square

Proof of Theorem 2. This follows from Theorem 22, Proposition 23, Theorems 25 and 27. \hfill \square

5. Skew category algebras

In this section, we apply the previous results to analyse chain conditions for skew category algebras which are defined by skew category systems of unital rings. At the end of the section, we prove Theorems 3 and 4.

Definition 28. For the rest of the article, let $R = \{R_a\}_{a \in G_0}$ be a collection of unital rings and let $\alpha = (\alpha_a : R_a \to R_{\alpha a})_{a \in G_1}$ be a collection of ring isomorphisms. Following [9] we say that $\alpha$ is a skew category system if $\alpha$ is a functor from $G$ to the category of unital rings, that is, if $\alpha(gh) = \alpha(g)\alpha(h)$ for all $(g, h) \in G_2$. Again following [9], we say that the associated skew category algebra of $G$ over $R$, denoted by $R *_{\alpha} G$, is the set of formal finite sums of elements of the form $rg$ for $r \in R_{\alpha g}$ and $g \in G_1$. The addition in $R *_{\alpha} G$ is defined by the relations $rg + r'g = (r + r')g$ for $r, r' \in R_{\alpha g}$ and $g \in G_1$. The multiplication in $R *_{\alpha} G$ is defined by the additive extensions of the relations $rg \cdot r'h = r\alpha_g(r')gh$, for $r \in R_{\alpha g}, r' \in R_{\alpha h}$ and $(g, h) \in G_2$, and $rg \cdot r'h = 0$, when $g, h \in G_1$ but $(g, h) \notin G_2$. The ring $R *_{\alpha} G$ is $G$-graded if we put $(R *_{\alpha} G)_g = R_{\alpha g}g$ for $g \in G_1$. In fact, with this grading, $R *_{\alpha} G$ is strongly $G$-graded. Also $R *_{\alpha} G$ is object unital since for each $a \in G_0$, the ring $(R *_{\alpha} G)_a = R_a$ is unital with $1_{R_a} = 1_{R_{\alpha a}}$. If $G$ is a groupoid (group, monoid), then $R *_{\alpha} G$ is called a skew groupoid (group, monoid) algebra of $G$ over $R$. If all the rings in $R$ coincide with a ring $T$ and the ring isomorphisms in $\alpha$ are identity maps, then $R *_{\alpha} G$ is called a category algebra of $G$ over $T$ and is denoted by $T[G]$. In that case, if $G$ is groupoid (group, monoid), then $T[G]$ is called a groupoid (group, monoid) algebra of $G$ over $T$. If $a \in G_0$, then we put $\alpha(a) = (\alpha_g : R_a \to R_{\alpha a})_{g \in G(a)}$.

Proposition 29. If $R *_{\alpha} G$ is left/right artinian (noetherian), then $G_0$ is finite and, for every $a \in G_0$, the skew monoid ring $R_a *_{\alpha(a)} G(a)$ is left/right artinian (noetherian).

Proof. This follows from Theorem 22(1). \hfill \square

Proposition 30. The set $\{1_{R_a}a\}_{a \in G_0}$ is a strong complete set of idempotents for $R *_{\alpha} G$ if and only if the category $G$ is hom-set strong. In that case, the ring $R *_{\alpha} G$ is hom-set-strongly $G$-graded.

Proof. Put $S = R *_{\alpha} G$. Suppose $\{1_{R_a}a\}_{a \in G_0}$ is a strong complete set of idempotents for $S$ and take $(x, y) \in G_0 \times G_0$ such that one of the sets $G(x, y)$ and $G(y, x)$ is nonempty.
• Case 1: \( G(x, y) \neq \emptyset \). Then \( 1_{R_x} x S_1 R_y y = S_{G(x, y)} \neq \emptyset \). By Proposition 9(3) it follows that \( S_{G(y, x)} = 1_{R_x} y S_1 R_y x \neq \emptyset \) and \( 1_{R_x} x \in S_{G(x, y)} S_{G(y, x)} \). Thus, in particular, \( x \in G(x, y) G(y, x) \).

• Case 2: \( G(y, x) \neq \emptyset \). By an argument similar to the one used in Case 1 it follows that \( G(x, y) \neq \emptyset \) and \( y \in G(y, x) G(x, y) \).

By Proposition 14(3) it follows that \( G \) is hom-set strong.

Suppose that \( G \) is hom-set strong and take \( (p, q) \in G_0 \) such that one of the additive groups \( 1_{R_p} p S_1 R_q q = S_{G(p, q)} \) and \( 1_{R_q} q S_1 R_p p = S_{G(q, p)} \) is nonzero.

• Case 1: \( S_{G(p, q)} \neq \emptyset \). Then \( G(p, q) \neq \emptyset \). Proposition 14(3) implies that \( G(q, p) \neq \emptyset \) and \( p \in G(p, q) G(q, p) \). Thus \( 1_{R_p} p \in S_{G(p, q)} S_{G(q, p)} \).

• Case 2: \( S_{G(p, q)} \neq \emptyset \). By an argument similar to the one used in Case 1 it follows that \( G(p, q) \neq \emptyset \) and \( q \in G(q, p) G(p, q) \) and \( 1_{R_q} q \in S_{G(q, p)} S_{G(p, q)} \).

By Proposition 9(3) it follows that \( \{ 1_{R_a} a \}_{a \in G_0} \) is a strong complete set of idempotents for \( S \). The last statement follows from Proposition 20.

Proposition 31. Suppose \( G \) is hom-set strong. Then \( R \star_{\alpha} G \) is left/right artinian (noetherian) if and only if \( G_0 \) is finite and, for every \( a \in G_0 \), the skew monoid ring \( R_a \star_{\alpha(a)} G(a) \) is left/right artinian (noetherian).

Proof. This follows from Theorem 22(2) and Proposition 30.

For the rest of the article, \( T \) denotes a fixed unital ring.

Theorem 32. Suppose \( H \) is a group and let \( \alpha : H \to \text{Aut}(T) \) be a group homomorphism. Then the associated skew group ring \( T \star_{\alpha} H \) is left/right artinian if and only if \( H \) is finite and \( T \) is left/right artinian.

Proof. This is a result by Park [12, Theorem 3.3]

Theorem 33. Suppose \( M \) is a monoid. Then the monoid algebra \( T[M] \) is left/right artinian if and only if \( M \) is finite and \( T \) is left/right artinian.

Proof. This is a result by Zelmanov [14, Corollary at p. 562].

Proposition 34. If the category algebra \( T[G] \) is left/right artinian, then \( T \) is left/right artinian and \( G \) is finite.

Proof. This follows from Theorem 33 and Propositions 18 and 29.

Proof of Theorems 3 and 4. This follows from Proposition 31, Theorems 32, 33, and Proposition 34.

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