The Moufang theorem for non-Moufang loops

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Abstract

We introduce a class of loops, that satisfies the Moufang theorem.

1 Introduction

This note has two sources of motivation. First, it is a problem proposed by A. Rajah at the Loops '11 conference. We say that a variety of loops \( V \) satisfies the Moufang theorem if for every loop \( L \) in \( V \) the following implication holds: for every \( x, y, z \) in \( L \), if \( x(yz) = (xy)z \) then the subloop generated by \( x, y, z \) is a group [5]. Is it true that every variety satisfying the Moufang theorem is contained in the variety of Moufang loops? The other source of motivation is the example (the Steiner loop of order 10) showed in the talk of M. L. Merlini Giuliani at the 3rd Mile High Conference on Nonassociative Mathematics on this subject, details of which can be found in [4].

2 Preliminaries and Result

A Steiner triple system \( \mathcal{S} \) is a \( 2 - (n,3,1) \) design, i.e., an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has precisely three points. A finite Steiner triple system with \( n \) points exists if and only if \( n \equiv 1 \) or \( 3 \) \( (\text{mod} \ 6) \).

A Steiner triple system \( \mathcal{S} \) generates a multiplication on pairs of different points \( x, y \) taking as product the third point of the block joining \( x \) and \( y \). Defining \( x \cdot x = x \) we get the Steiner quasigroup associated with \( \mathcal{S} \). Further, adjoining an element \( e \) with \( ex = xe = x, xx = e \), yields the Steiner loop \( S \). Conversely, a Steiner loop \( S \) determines a Steiner triple system whose points are the elements of \( S \setminus \{ e \} \) and the blocks are the triples \( \{ x, y, xy \} \) for all \( x \neq y \in S \setminus \{ e \} \).

We focus upon Steiner loops arising from special Steiner triple systems namely, from Hall triple systems. A Steiner triple system in which every three non-collinear points generate an affine plane of order 3 over the field \( GF(2) \) is called a Hall triple system [3]. These geometries are of interest because they are the only known examples of perfect matroid designs other than classical projective and affine geometries over finite fields and \( t - (n,k,1) \) designs. The class of Hall triple systems is an exceptional case to the Buekenhout's characterization theorem ([1] p. 368); they are the only non-degenerate non-affine "locally affine"
geometries. Furthermore, the "local affinity" of a non-affine geometry cannot be extended only as far as to dimension 3. In addition, Hall triple systems are the only geometries of this type; see Theorem 1.3 in [6] p. 130.

In this work, Steiner loops associated to Hall triple systems are called Hall loops. Hall loops are not groups (c.f. Theorem in [2] p.250) and not Moufang loops. See Figure 1. Multiplication groups of such loops are determined in [7].

**Theorem 1** If \( L \) is a Hall loop and \( x(yz) = (xy)z \) then the subloop \( < x, y, z > \) is a group for all \( x, y, z \in L \).

**Proof.** Let \( x, y, z \in L \setminus \{e\} \) such that \( x \neq y \neq z \). First, assume that none among \( x, y, z \) is the product of the remaining two elements, i.e., \( x, y, z \) are non-collinear points of the corresponding Hall triple system. Thus, \( x, y, z \) generate an affine plane over the field \( GF(3) \). Hence \( x(yz) \) and \( (xy)z \) are different points of the system:

![Diagram](image)

**Figure 1.** Points corresponding to the associativity \( (xy)z = (x(yz)) \) (green); to Moufang identity \( (zx)(yz) = (z(xy))z \) (red).

Consequently, \( x(yz) \neq (xy)z \) in \( L \).

In the opposite case, when \( x, y, z \) form a block, the subloop \( < x, y, z > \) is the Klein four-group, because of diassociativity of Steiner loops.

**Corollary 2** For Hall loops, the assertion of the Moufang Theorem is fulfilled.

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