NEUMANN HEAT CONTENT ASYMPTOTICS WITH
SINGULAR INITIAL TEMPERATURE AND SINGULAR
SPECIFIC HEAT

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Abstract. We study the asymptotic behavior of the heat content on a compact Riemannian manifold with boundary and with singular specific heat and singular initial temperature distributions imposing Robin boundary conditions. Assuming the existence of a complete asymptotic series we determine the first three terms in that series. In addition to the general setting, the interval is studied in detail as are recursion relations among the coefficients and the relationship between the Dirichlet and Robin settings.

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1. Introduction

1.1. Historical framework. Let $M$ be a compact $m$-dimensional manifold with boundary $\partial M$ and Riemannian metric $g$. Let $D$ be an operator of Laplace type which drives the evolution process (see Section 1.2 for details). Let $\phi$ represent the initial temperature of $M$ and let $T(x;t)$ be subsequent temperature of the manifold. A good general reference to the heat equation is provided by [16, 18]. We impose suitable boundary conditions $\mathcal{B}$ to ensure the process is well defined. Let $\rho$ be the specific heat. We suppose for the moment $\phi$ and $\rho$ are smooth and let $\beta(t)$ be the heat content of the manifold. There is a complete asymptotic series for $\beta$ as $t \to 0$ with locally computable coefficients. We postpone for the moment a precise definition to avoid complicating the exposition at this stage.

The problem was first studied by [2, 4] in the context of domains in Euclidean space with smooth boundaries. We note that regions with polygonal boundaries were examined in [5]. Subsequent results for regions with fractal boundaries were obtained in [1, 3]. Apart from these papers, most of the work on the heat content asymptotics has been in the smooth category and we shall always assume the boundary of $M$ to be smooth. The “functorial method” has been used to express the asymptotic coefficients in terms of geometrical quantities for Dirichlet and Robin boundary conditions by [6, 7, 8, 9, 17]. Other authors [26, 27, 28, 29, 30] have examined these coefficients using other methods. Various other boundary conditions also have been considered [13, 20, 21] as have variable geometries [19]. Motivated by work of [15] for the growth of the heat trace asymptotics, the growth of the heat content asymptotics has been examined [12, 22, 51].

The case where the initial temperature and the specific heat are singular in a controlled fashion at the boundary will form the centerpiece of this paper – we refer to Section 1.3 for precise definitions. A rigorous treatment was provided in [14] where the initial temperature is singular on the boundary but the specific heat is smooth. Subsequently, the case where the specific heat can be singular as well was studied for Dirichlet boundary conditions [10, 13]. The present paper examines the situation for Neumann boundary conditions.

1.2. Operators of Laplace type. Let $(M, g)$ be a compact Riemannian manifold of dimension $m$ with smooth boundary $\partial M$. Let $V$ be a smooth vector bundle over
M and let $D : C^\infty(V) \to C^\infty(V)$ be a second order partial differential operator on the space of smooth sections to $V$. We adopt the *Einstein convention* and sum over repeated indices. We say that $D$ is of *Laplace type* if the leading symbol of $D$ is given by the metric tensor, i.e. locally $D$ has the form:

$$D = -\{g^{\mu\nu}\partial_{\partial \epsilon_\mu}\partial_{\partial \epsilon_\nu} + A^\rho\partial_{\partial \epsilon_\rho} + B\}.$$

The following *Bochner formalism* [18] permits us to work tensorially:

**Lemma 1.1.** There exists a unique connection $\nabla$ on $V$ and a unique endomorphism $E$ of $V$ so that $D = D(g, \nabla, E) = -(g^{\mu\nu}\nabla_{\partial \epsilon_\mu}\nabla_{\partial \epsilon_\nu} + E)$. The connection 1-form $\omega$ of $\nabla$ and the endomorphism $E$ are given by

$$\omega_\mu = \frac{1}{2}(g_{\mu\nu}A^\nu + g^\sigma\Gamma_{\sigma\epsilon\mu}Id) \text{ and } E = B - g^{\mu\nu}(\partial_{\partial \epsilon_\nu}\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\Gamma_{\mu\nu}^\sigma).$$

If $D = d^*d + dd^*$ is the Laplacian on the space of smooth $p$-forms, then $\nabla$ is the Levi-Civita connection and $E$ is given in terms of curvature. If $D$ is the spin Laplacian, then $\nabla$ is the spinor connection and $E$ is a multiple of the scalar curvature. We shall use the Levi-Civita connection and $\nabla$ to covariantly differentiate tensors of all types.

We must impose suitable boundary conditions. Let $e_m$ be the inward unit normal vector field on the boundary. Let $S_R$ be an auxiliary endomorphism of $V|_{\partial M}$ and let $\nabla$ be the connection on $V$ given in Lemma 1.1. Let $B_D$ and $B_R$ be the Dirichlet and the Robin boundary operators which are defined, respectively, by setting:

$$B_D\phi = \phi|_{\partial M} \quad \text{and} \quad B_R\phi = (\nabla e_m\phi + S_R\phi)|_{\partial M} \quad \text{for } f \in C^\infty(V).$$

The Neumann boundary operator is defined by taking $S_R = 0$. We let $D_{D/R}$ denote either the Dirichlet or the Robin realization of $D$ henceforth. It is convenient to have a common notation.

### 1.3. The initial temperature and the specific heat

Let $r$ be the geodesic distance to the boundary. This is a continuous function on all of $M$ which is smooth near $\partial M$. If $\bar{y} = (y^1, ..., y^{m-1})$ is a system of local coordinates near a point $P \in \partial M$, then $\bar{x} = (y, r)$ for $r \in [0, \epsilon]$ is an adapted system of local coordinates near $P$ in $M$ for some $\epsilon > 0$; the curves $\gamma_\mu(s) := (y, s)$ are unit speed geodesics perpendicular to the boundary and $\partial_r$ is the inward pointing unit geodesic normal.

To have a common notation, let $W = V$ or let $W = V^*$. If $\alpha \in C$, let $K_\alpha(W)$ be the vector space of all sections $\Phi$ to $W$ which are smooth on the interior of $M$ such that $r^\alpha \Phi$ is smooth near $\partial M$. The parameter $\alpha$ controls the blow up (resp. decay) of $\Phi$ near the boundary if $\Re(\alpha) > 0$ (resp. $\Re(\alpha) < 0$). Let $\phi \in K_\alpha(V)$ represent the initial temperature of $M$ and let $\rho \in K_{\alpha_2}(V^*)$ give the specific heat. We shall always assume that

$$\Re(\alpha_1) < 1, \quad \Re(\alpha_2) < 1, \quad \text{and} \quad \alpha_1 + \alpha_2 \notin \mathbb{Z}. \quad (1.1)$$

The first two conditions ensure that $\phi$ and $\rho$ are in $L^1$. The final condition is imposed to avoid interactions between the interior and boundary terms in the asymptotic series for the heat content. We will return to this point subsequently.

Expand $\phi \in K_{\alpha_1}(V)$ near $\partial M$ in a modified Taylor series:

$$\phi(y, r) \sim r^{-\alpha_1} \sum_{j=0}^{\infty} (\phi^j(y)) r^j \quad \text{as} \quad r \downarrow 0^+ \text{ where } \phi^j := \frac{1}{j!}(\nabla_{\partial \epsilon_j})^j(r^\alpha_1 \phi)|_{\partial M}. \quad (1.2)$$

We shall usually be working with scalar operators and can choose a local section $s$ so that $\nabla e_m s = 0$. We may then regard $\phi$ as a function and the above expansion as a Taylor series. A similar expansion, of course, is valid for the specific heat $\rho$ where we regard $\rho$ as a section to the dual bundle and covariantly differentiate with respect to the dual connection on $V^*$ which has connection 1-form $-\omega^*_\nu$. 

1.4. The heat equation and the heat content. If \( \phi \in K_{\alpha_1}(V) \), then the subsequent temperature \( T := e^{-tD_B} \phi \) is characterized by the relations:

\[
(\partial_t + D)T = 0 \quad \text{(evolution equation)},
\]

\[
\lim_{t \to 0} T(\cdot; t) = \phi \quad \text{(initial condition)},
\]

\[
\mathcal{B}T(\cdot; t) = 0 \text{ for } t > 0 \quad \text{(boundary condition)}.
\]

Let \( \langle \cdot, \cdot \rangle \) be the natural pairing between \( V \) and the dual bundle \( V^* \), let \( dx \) be the Riemannian measure on \( M \), let \( dy \) be the Riemannian measure on \( \partial M \), and let \( \rho \in K_{\alpha_2}(V^*) \) be the specific heat of the manifold. The total heat content of the manifold is defined by setting:

\[
\beta(\phi, \rho, D, \mathcal{B})(t) := \int_M \langle e^{-tD_B} \phi, \rho \rangle dx.
\]

There is a smooth heat kernel \( K = K_{D, \mathcal{B}} \) so that \( T(x; t) = \int_M K(x, \tilde{x}; t) \phi(\tilde{x}) d\tilde{x}, \) and

\[
\beta(\phi, \rho, D, \mathcal{B})(t) = \int_{M \times M} \langle K(x, \tilde{x}; t) \phi(\tilde{x}), \rho(x) \rangle d\tilde{x} dx.
\]

This is well defined for \( \phi \in L^1(V) \) and \( \rho \in L^1(V^*) \); it was for this reason that we assumed \( \Re(\alpha_1) < 1 \) and \( \Re(\alpha_2) < 1 \) in Equation 1.4.

If \( D_B \) is self-adjoint, as will be the case for either the Dirichlet or the Robin realization of the scalar Laplacian, then we can take a spectral resolution \( \{ \phi_n, \lambda_n \} \) for \( D_B \). Here the \( \{ \phi_n \} \) is an orthonormal basis for \( L^2(V) \) such that \( D\phi_n = \lambda_n \phi_n \) and \( \mathcal{B}\phi_n = 0 \). We then have

\[
K(x, \tilde{x}; t) = \sum_n e^{-t\lambda_n} \phi_n(x) \otimes \phi_n(\tilde{x}).
\]

This series converges in the \( C^\infty \) topology for \( t > 0 \). The temperature \( T \) is smooth for \( t > 0 \). However, the convergence to \( \phi \) as \( t \downarrow 0 \) in Equation 1.4 is not pointwise but in \( L^1(V) \).

The following example is instructive. Let \( M = [0, \pi] \) and let \( D = -\partial_x^2 \). The spectral resolution of the Dirichlet Laplacian is given by \( \{ n^2, \sqrt{2/\pi} \sin(nx) \}_{n \geq 1} \). Thus if \( \phi = 1 \), we have

\[
T(x; t) = \frac{2}{\pi} \sum_{n=1, n\text{ odd}}^\infty \frac{2}{n^2} e^{-tn^2} \sin(nx).
\]

This is smooth for \( t > 0 \). However, since \( T(0; t) = 0 \) for \( t > 0 \) and \( \phi = 1 \), the convergence is not pointwise. The associated heat content is given by:

\[
\beta(1, 1, \Delta, \mathcal{B}_\mathcal{P})(t) = \frac{8}{\pi} \sum_{n=1, n\text{ odd}}^\infty \frac{1}{n^2} e^{-tn^2}
= \pi - \frac{4}{\sqrt{\pi}} t^{1/2} + O(t^k) \text{ as } t \downarrow 0 \forall k \geq 1.
\]

1.5. The form of the asymptotic series. We shall need to consider certain integrals which are divergent and which need to be regularized. For example, the integral \( \int_M \langle \phi, \rho \rangle dx \) is divergent if \( 1 < \Re(\alpha_1 + \alpha_2) < 2 \). The Riemannian measure is not in general a product near the boundary. Since, however, \( dx = dydr \) on the boundary of \( M \), we may decompose

\[
\langle \phi, \rho \rangle dx = \langle \phi^0, \rho^0 \rangle e^{-\alpha_1 - \alpha_2} dydr + O(e^{1-\alpha_1-\alpha_2}) \text{.}
\]
Let $C_{\varepsilon} := \{ x \in M : r(x) \leq \varepsilon \}$ be a small collared neighborhood of the boundary. For $\Re(\alpha_1 + \alpha_2) < 2$ and $\alpha_1 + \alpha_2 \neq 1$, define:

$$T^\text{Reg}_{\Re}(\phi, \rho) := \int_{M-C_{\varepsilon}} (\phi, \rho) dx + \int_{C_{\varepsilon}} \{ (\phi, \rho) dx - (\phi^0, \rho^0) r^{-\alpha_1 - \alpha_2} dy dr \} + \int_{\partial M} (\phi^0, \rho^0) dy \times \varepsilon^{1-\alpha_1 - \alpha_2}(1 - \alpha_1 - \alpha_2)^{-1}.$$

This is clearly independent of $\varepsilon$ and agrees with $\int_M (\phi, \rho) dx$ if $\Re(\alpha_1 + \alpha_2) < 1$. The regularization $T^\text{Reg}(\phi, \rho)$ is a meromorphic function of $\alpha_1 + \alpha_2$ with a simple pole at $\alpha_1 + \alpha_2 = 1$. More generally, the integrals $\langle D^n \phi, \rho \rangle$ which appear in Conjecture 1.2 below need regularization if $\Re(\alpha_1 + \alpha_2) > 1 - 2n$. Poles can appear whenever $\Re(\alpha_1 + \alpha_2) = 1 - k$. These poles are evident in the formulas given subsequently; we expect the local formulas for the asymptotic expansion of the heat content may involve log terms when $\alpha_1 + \alpha_2 \in {\mathbb Z}$ and for that reason excluded these values in Equation (1.a).

To simplify the notation, we set:

$$\beta^M_n(\phi, \rho, D) := (-1)^n / n! \cdot T^\text{Reg}_{\Re}\{ \langle D^n \phi, \rho \rangle \}.$$

If $M$ is a closed manifold, these are the invariants which would appear in the heat content expansion. We assume that the following conjecture (which extends the discussion of [10]) holds henceforth. We will justify the powers $t^{(1+j-\alpha_1-\alpha_2)/2}$ subsequently in Section 4 using dimensional analysis. We refer to [14] where a related result was established when the specific heat is smooth.

**Conjecture 1.2.** If $(\alpha_1, \alpha_2)$ satisfy Equation (1.a), then there is a complete asymptotic series as $t \downarrow 0$ of the form:

$$\beta(\phi, \rho, D, B_{\Re})(t) \sim \sum_{n=0}^{\infty} t^n \beta^M_n(\phi, \rho, D) + \sum_{j=0}^{\infty} t^{(1+j-\alpha_1-\alpha_2)/2} \beta^M_{j, \alpha_1, \alpha_2}(\phi, \rho, D, B_{\Re}).$$

The coefficients $\beta^M_{j, \alpha_1, \alpha_2}(\phi, \rho, D, B_{\Re})$ are given by integrals of local invariants over the boundary.

**Remark 1.3.** This conjecture has been established in [14] using the calculus of pseudo-differential operators in the special case that $\alpha_2 \in {\mathbb N}$ or that $\alpha_1 \in {\mathbb N}$. The extension to the present setting is motivated by that work.

Let $R_{ijkl}$ denote the Riemann curvature tensor; with our sign convention, we have that $R_{1221} = +1$ on the unit sphere $S^2$ in $\mathbb{R}^3$. Let $\text{Ric}$ denote the Ricci tensor, let $\tau$ denote the scalar curvature, and let $L_{ab}$ denote the second fundamental form. We let indices $\{i, j, k, l\}$ range from 1 to $m$ and index a local orthonormal frame for $TM$; we let indices $\{a, b, c\}$ range from 1 to $m - 1$ and index a local orthonormal frame for $T\partial M$. On the boundary, $\epsilon_m$ will always denote the inward unit geodesic normal and $\cdot \cdot$ will denote the components of the covariant derivative. In Section 2 we will perform a careful analysis of the local formulas involved and show:

**Lemma 1.4.** There exist universal constants $\varepsilon^\nu_{\Re, \alpha_1, \alpha_2}$ so that:

$$\beta^M_{0, \alpha_1, \alpha_2}(\phi, \rho, D, B_{\Re}) = \int_{\partial M} \varepsilon_{\Re, \alpha_1, \alpha_2}(\phi^0, \rho^0) dy,$$

$$\beta^M_{1, \alpha_1, \alpha_2}(\phi, \rho, D, B_{\Re}) = \int_{\partial M} \{ \varepsilon_{\Re, \alpha_1, \alpha_2}(\phi^1, \rho^0) + \varepsilon^1_{\Re, \alpha_1, \alpha_2}(L_{a00} \phi^0, \rho^0) \}.$$
Conjecture 1.5. A related result was established when the specific heat is smooth: $\alpha$ is connected as this would fail if we only considered real variables. The invariant $\beta_{2,\alpha,\alpha_2}$ but we postponed it until we could introduce the appropriate notation. Note that $\epsilon$ on the half-line to identify the constants $S_\alpha$.

Lemma 1.6. We shall assume the following henceforth; it is properly part of Conjecture 1.2 henceforth; it is properly part of Conjecture 1.2, although the functions appearing in Lemma 1.4 have an analytic extension to the connected open set defined by Equation (1.a).

1.6. The invariant $\beta_{2,\alpha,\alpha_2}$. In Section 3, we will make a special case computation on the half-line to identify the constants $\epsilon_{D/R,\alpha,\alpha_2}$ of Lemma 1.4. Set

$$\epsilon_{D/R,\alpha,\alpha_2} := \begin{cases} -1 & \text{if } B = B_D \\ +1 & \text{if } B = B_R \end{cases}.$$  

Lemma 1.6. If $(\alpha_1, \alpha_2)$ satisfies Equation (1.a), then

$$\epsilon_{D/R,\alpha,\alpha_2} := \epsilon_{D/R} \cdot 2^{-\alpha_1-\alpha_2} \pi^{-1/2} \Gamma \left( \frac{2-\alpha_1-\alpha_2}{2} \right) \cdot \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}{\Gamma(2-\alpha_1-\alpha_2)} + 2^{-\alpha_1-\alpha_2} \pi^{-1/2} \Gamma \left( \frac{2-\alpha_1-\alpha_2}{2} \right) \Gamma(1 + \alpha_2 - 1) \cdot \left( \frac{\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_2)} + \frac{\Gamma(1-\alpha_2)}{\Gamma(1-\alpha_1)} \right).$$

The following result is an immediate consequence of Lemma 1.6 using the usual properties of the $\Gamma$ function. We will give another proof in Section 4 that arises from certain functional properties of the heat content asymptotics.

Lemma 1.7. We have the recursion relations:

1. $\epsilon_{D/R,\alpha,2,\alpha_2} = \frac{2(\alpha_1 - 2)(\alpha_2 - 1)}{3 - \alpha_1 - \alpha_2} \epsilon_{D/R,\alpha_1,\alpha_2}$.

2. $\epsilon_{D/R,\alpha,1,2,\alpha_2} = \frac{2(\alpha_2 - 2)(\alpha_1 - 1)}{3 - \alpha_1 - \alpha_2} \epsilon_{D/R,\alpha_1,\alpha_2}$.

3. $\epsilon_{D/R,\alpha,1,2,\alpha_2} = -\frac{2(\alpha_1 - 1)(\alpha_2 - 1)}{3 - \alpha_1 - \alpha_2} \epsilon_{D/R,\alpha_1,\alpha_2}$.

Note that the roles of Neumann and Dirichlet boundary conditions are interchanged in Assertion (3).

1.7. Heat content asymptotics for Dirichlet boundary conditions. Dirichlet boundary conditions have been treated previously in [10, 14]; we summarize those results as follows:

Theorem 1.8. If $(\alpha_1, \alpha_2)$ satisfy Equation (1.a), then:

$$\beta_{\alpha,\alpha_2}^M(\phi, \rho, D, B_D) = \int_{\partial M} \epsilon_{D/R,\alpha_1,\alpha_2}(\phi, \rho) dy,$$
β^M_{0,1,02}(ϕ, ρ, D, B_D) = \int_{BM} \left\{ -\frac{1}{2} (\varepsilon_{D,01-1,02} + \varepsilon_{D,01,02-1}) L_{aa}(ϕ^0, ρ^0) \\
+ \varepsilon_{D,01-1,02} (ϕ^1, ρ^0) + \varepsilon_{D,01,02-1} (ϕ^1, ρ^1) \right\} dy.

β^M_{2,01,02}(ϕ, ρ, D, B_D) = \int_{BM} \left\{ -\frac{1}{2} (\varepsilon_{D,01-2,02} + \varepsilon_{D,01,02-1}) L_{aa}(ϕ^0, ρ^0) \\
+ \varepsilon_{D,01,02} (E ϕ^0, ρ^0) + \varepsilon_{D,01-2,02} (ϕ^0, ρ^0) + \varepsilon_{D,01,02-2} (ϕ^0, ρ^2) \\
- \frac{1}{2} (\varepsilon_{D,01-1,02-1} + \varepsilon_{D,01,02-2}) L_{aa}(ϕ^1, ρ^1) \\
+ (\frac{1}{2} \varepsilon_{D,01-2,02 - \frac{1}{2} \varepsilon_{D,01,02-1} + \frac{1}{2} \varepsilon_{D,01,02-2}} (L_{ab} L_{ab} + R_{mm})(ϕ^0, ρ^0) \\
- \varepsilon_{D,01,02} (ϕ_{01,01}^0, ρ_{01}^0) + 0 r ϕ^0, ρ^0 + \varepsilon_{D,01-1,02-1} (ϕ^1, ρ^1) \\
+ (\frac{1}{2} \varepsilon_{D,01-2,02} + \frac{1}{2} \varepsilon_{D,01,02-2} + \frac{1}{2} \varepsilon_{D,01,02-1} - \frac{1}{2} \varepsilon_{D,01,02}) L_{aa} L_{ab}(ϕ^0, ρ^0) \right\} dy.

1.8. Heat content asymptotics for Robin boundary conditions. The following is the main result of this paper; it will be established in Section 4 using various functorial properties of these invariants:

**Theorem 1.9.** If (ρ_1, ρ_2) satisfy Equation (L, α), then:

β^M_{0,1,02}(ϕ, ρ, D, B_R) = \int_{BM} \varepsilon_{R,01,02}(ϕ^0, ρ^0) dy.

β^M_{1,01,02}(ϕ, ρ, D, B_R) = \int_{BM} \left\{ \varepsilon_{R,01-1,02}(ϕ^0, ρ^0) + \varepsilon_{R,01,02-1}(ϕ^0, ρ^1) \\
- \frac{1}{2} \varepsilon_{R,01,02}(ϕ^1, ρ^0) + \frac{1}{2} \varepsilon_{R,01-1,02-1}(S_R φ^0, ρ^0) \\
+ \left\{ \frac{1}{2} \varepsilon_{R,01,02} (ϕ^0, ρ_{01}) + 0 \cdot r φ^0, ρ^0 \\
+ \varepsilon_{R,01,02}(E ϕ^0, ρ^0) + \varepsilon_{R,01-1,02}(ϕ^1, ρ^1) \\
+ \left\{ \frac{1}{2} \varepsilon_{R,01,02} (ϕ^0, ρ_{01}) + \varepsilon_{R,01-1,02} (ϕ^1, ρ^1) \\
+ \left\{ \varepsilon_{R,01-1,02-1} + \frac{1}{2} \varepsilon_{R,01,02-2} (S_R φ^0, ρ^0) \\
+ \left\{ \frac{1}{2} \varepsilon_{R,01-1,02} (ϕ^0, ρ^0) + \frac{1}{2} \varepsilon_{R,01,02-1} (S_R φ^0, ρ^0) \right\} \right\} \right\} dy.$
2. Local invariants

By assumption, we may express the asymptotic coefficients \( \beta_{j,\alpha_1,\alpha_2}^{BM} \) as the integrals of local invariants over the boundary:

\[
\beta_{j,\alpha_1,\alpha_2}^{BM}(\phi, \rho, D, B_D/R) = \int_{\partial M} \beta_{j,\alpha_1,\alpha_2}^{BM}(\phi, \rho, D, B_D/R)(y) dy.
\]  
(2.a)

Fix a system of local coordinates \( x = (x^1, \ldots, x^m) \). Set \( \bar{\partial}_x^2 = (\partial_{x^i})^{\alpha_1} \ldots (\partial_{x^m})^{\alpha_m} \). We have

\[
\beta_{j,\alpha_1,\alpha_2}^{BM}(\phi, \rho, D, B_D/R)(y) = \beta_{j,\alpha_1,\alpha_2}^{BM}(\phi^\ell, \rho^\ell, \partial_{x}^{\ell} A^K, \partial_{x}^{\ell} B, \partial_{x}^{\ell} S_R) .
\]  
(2.b)

Here we need only consider a finite number of jets; we omit the \( S_R \) variables for Dirichlet boundary conditions and for Robin boundary conditions only differentiate \( S_R \) tangentially. The local invariants are polynomial in the derivatives of the structures involved with coefficients that depend smoothly on the metric.

2.1. A coordinate formulation. We summarize the arguments briefly - this also serves to motivate the power \( (1 + j - \alpha_1 - \alpha_2)/2 \) of \( t \) in Conjecture 1.2. Fix a point \( P \in \partial M \) and choose local coordinates \( \bar{y} = (y^1, \ldots, y^{m-1}) \) on the boundary which are centered at \( P \). Let \( \vec{x} := (\bar{y}, r) \) be the associated adapted coordinates. Let indices \( a, b, \ldots \) range from 1 to \( m - 1 \). We then have

\[
d s^2 = g_{ab} dy^a \circ dy^b + dr \circ dr \quad \text{where} \quad g_{ab} = g_{ab}(y, r).
\]

We further normalize the choice of coordinates so that \( g_{ab}(P) = \delta_{ab} \). This eliminates the smooth dependence on the metric tensor and ensures that the invariants of Equation (2.1.a) are polynomial in the jets of the structures. For a smooth function \( f \) and a multi-index \( \vec{a} = (a_1, \ldots, a_m) \) we let

\[
f_{\vec{a}} = \bar{\partial}^{a_1}_{x^1} \ldots \bar{\partial}^{a_m}_{x^m} f.
\]

We define

\[
\text{ord}(g_{\vec{a}, \vec{b}}) := |\vec{a}|, \quad \text{ord}(A^\ell_{\vec{b}}) := |\vec{b}| + 1, \quad \text{ord}(B_\ell) := |\vec{b}| + 2, \quad \text{ord}(\rho^\ell) := \ell, \quad \text{ord}(S_{R, \vec{a}}) := |\vec{a}| + 1.
\]

Here, since \( S_R \) is only defined on the boundary, \( \vec{a} \) only reflects tangential indices; we delete the \( S_{R, \vec{a}} \) variables from consideration for Dirichlet boundary conditions as they play no role.

**Lemma 2.1.** \( \beta_{j,\alpha_1,\alpha_2}^{BM}(\phi, \rho, D, B)(y) \) is homogeneous of weighted order \( j \).

**Proof.** Fix \( P \in \partial M \) and choose adapted coordinates so \( g_{ab}(P) = \delta_{ab} \). Then the local formula for \( \beta_{j,\alpha_1,\alpha_2}^{BM} \) is polynomial in the variables \( \{g_{ij/\vec{a}}, A^K_{\vec{b}}, B_\ell, \phi^\ell, \rho^\ell, S_{R, \vec{a}}\}_{|\vec{a}| > 0} \).

We use dimensional analysis. Let \( c > 0 \), and consider the operator \( D_c := e^{-cD} \). It is then clear in a purely formal sense that \( e^{-t(c^{-2}D)} = e^{-(c^{-2}t)D} \) and thus if \( T_c \) is the temperature distribution defined by \( D_c \), then

\[
T_c(x; t) = T(x; c^{-2}t).
\]  
(2.2)

We justify this formal computation by verifying that the relations of Equation (1.7) are satisfied. Suppose that \( T_c \) is defined by Equation (2.2). Then:

\[
(\partial_t + D_c)T_c(x; t) = c^{-2} \{\partial_t T + DT\} (x; c^{-2}t) = 0 \quad \text{and} \quad \lim_{t \downarrow 0} T(\cdot; c^{-2}t) = \phi(\cdot).
\]

We must check the boundary conditions are satisfied by \( T_c \); this is immediate with Dirichlet boundary conditions so we set \( B_{D_c} := B_D \) and ignore the subscript. The situation concerning Robin boundary conditions requires more work. We use
Lemma 1.1 to see that $D_c$ and $D$ determine the same connection. However the normal rescales; $e_m(c^2y) = c^{-1}e_m$. Set:

$$B_{R,c} := \nabla e^{-1}e_m + S_{R,c} \text{ where } S_{R,c} := c^{-1}S_R.$$  \hspace{1cm}  \text{(2.d)}

We complete the proof that $T_c = e^{-TD_c}\phi$ by verifying $T_c$ satisfies the rescaled Robin boundary conditions:

$$B_{R,c}T_c = e^{-1}\left((\nabla e^{-1}e_m + S_{R})T(.,c^{-2}t)\right)|_{\partial M} = c^{-1}B_RT(.,c^{-2}t) = 0.$$  \hspace{1cm}  \text{(2.g)}

Since $g_c = c^2g$, $dx_c = c^m dx$. We verify that the interior invariants $\beta_n^M$ rescale properly:

$$\beta_n^M(\phi, \rho, D_c) = (-1)^n/n! \cdot T^\phi_{\text{Reg}}\{D^n \phi, \rho\} = \frac{(-1)^n}{m} \cdot c^{m-2n} T^\phi_{\text{Reg}}\{D^n \phi, \rho\}.$$  \hspace{1cm}  \text{(2.e)}

We apply Conjecture 1.2 and use Equation (2.e) to expand:

$$\beta(\phi, \rho, D_c, B_{D/R,c}))(t) \sim \sum_{n=0}^{\infty} t^n c^{m-2n} \beta_n^M(\phi, \rho, D) + \sum_{j=0}^{\infty} t^{(1+j-a_1-a_2)/2} \beta_j^{M+1}(\phi, \rho, D_c, B_{D/R,c})).$$  \hspace{1cm}  \text{(2.f)}

We apply Equation (2.e) to see:

$$\beta(\phi, \rho, D_c, B_{D/R,c})(t) = \int_M \langle T_n(x; t, \rho(x))dx_c = c^m \int_M \langle T(x; c^{-2}t, \rho(x))dx

= c^m \beta(\phi, \rho, D, B_{D/R,c})(c^{-2}t) \sim c^m \sum_{n=0}^{\infty} (c^{-2}t)^n \beta_n^M(\phi, \rho, D)

+ c^m \sum_{j=0}^{\infty} (c^{-2}t)^{(1+j-a_1-a_2)/2} \beta_j^{M+1}(\phi, \rho, D, B_{D/R,c}).$$  \hspace{1cm}  \text{(2.g)}

We equate powers of $t$ in the asymptotic expansions of Equation (2.f) and Equation (2.g) to see:

$$\beta^{M+1}_{j,a_1,a_2}(\phi, \rho, D_c, B_{D/R,c}) = c^{m+1} \int_{a_1+a_2-1}^{a_1+a_2} \beta_j^{M+1}(\phi, \rho, D, B_{D/R,c})(y)dy.$$  \hspace{1cm}  \text{(2.h)}

We use Equation (2.h), since $dy_c = c^{-1}dy$, Equation (2.h) implies:

$$\beta_j^{M+1}(\phi, \rho, D_c, B_{D/R,c}) = c^{m-1-j+a_1+a_2} \int_{a_1+a_2} \beta_j^{M+1}(\phi, \rho, D, B_{D/R,c})(y)dy.$$

Consequently (ignoring divergence terms), we have the following relation for the local invariants:

$$\beta_j^{M+1}(\phi, \rho, D_c, B_{D/R,c})(y) = c^{j+a_1+a_2} \beta_j^{M+1}(\phi, \rho, D, B_{D/R,c})(y).$$  \hspace{1cm}  \text{(2.i)}

We wish to apply the formalism of Equation (2.1). We may expand:

$$\phi \sim r_{c-1}(\phi^0 + r_{c}\phi_{c} + ...) \sim r^{-a_1}(\phi^0 + r_{c}\phi_{c} + ...)$$

Equating terms in the asymptotic series, using the fact that $r_c = c \cdot r$, and arguing similarly with $\rho$ yields:

$$\phi^c = c^{a_1-t} \phi^t \text{ and } \rho^c = c^{a_2-t} \rho^t.$$  \hspace{1cm}  \text{(2.j)}
We renormalize the coordinate system and set \( \bar{x}_c = c \cdot \bar{x} \); \( \partial_{x_c} = c^{-1} \partial_x \). We introduce the coordinate systems \( \bar{x} \) and \( \bar{x}_c \) into the notation as a computational aid; the local invariants \( \beta^{M \alpha_j,\alpha_k}(y) \) do not, of course, depend on the choice of the coordinate system. We compute:

\[
g_{c,ij}(\bar{x}_c,P) = c^2 g_{ij}(c^{-1} \partial_x, c^{-1} \partial_x)(\bar{x},P) = c^{-2} g_{ij}(\bar{x},P) = \delta_{ij}.
\]

Consequently, \( g_{c,ij}(\bar{x}_c) = g_{ij}(\bar{x}) \) and we have that:

\[
g_{c,ij,\bar{z}}(\bar{x}_c,P) = c^{-|\bar{z}|} g_{ij,\bar{z}}(\bar{x},P).
\]

We expand:

\[
D_c = c^{-2}(g^{ij} \partial_{x_i} \partial_{x_j} + A^k \partial_{x_k} + B)
\]

\[
= -(g^{ij} \partial_{x_i} \partial_{x_j} + c^{-1} A^k \partial_{x_{c,k}} + c^{-2} B)
\]

to see that \( A^k_c(\bar{x}_c,P) = c^{-1} A^k(\bar{x},P) \) and \( B_c(\bar{x}_c,P) = c^{-2} B(\bar{x},P) \). Consequently

\[
A^k_c(\bar{x}_c,P) = c^{-1-|\bar{z}|} A^k_b(\bar{x},P) \quad \text{and} \quad B_{c,\bar{z}}(\bar{x}_c,P) = c^{-2-|\bar{z}|} B_{c,\bar{z}}(\bar{x},P). \tag{2.k}
\]

Similarly, by Equation (2.d) we have:

\[
S_{R,c,\bar{d}}(\bar{x}_c,P) = c^{1-|\bar{d}|} S_{R,\bar{d}}(\bar{x},P). \tag{2.l}
\]

We use Equation (2.a), Equation (2.k), and Equation (2.l) together with the observation that \( \beta^{M \alpha_j,\alpha_k}(y) \) is bilinear in \((\phi, r)\) to see:

\[
\beta^{M \alpha_j,\alpha_k}(\phi, r, D_c, B_D/\partial) = c^{\alpha_1 - |\bar{d}|} g_{ij,\bar{z}}(e^{-1/2}) A^\ell_b, c^{-2-|\bar{z}|} B_{c,\bar{z}}, \tag{2.m}
\]

\[
c^{-1-|\bar{d}|} S_{R,\bar{d}}(\bar{x},P).
\]

We use Equation (2.k) and Equation (2.m) to conclude therefore:

\[
\beta^{M \alpha_j,\alpha_k}(e^{-1} r^\phi, e^{-1} \rho^\ell, c^{-|\bar{d}|} g_{ij,\bar{z}}, c^{-1-|\bar{d}|} e^{-2-|\bar{z}|} B_{c,\bar{z}}) = c^{-1-|\bar{d}|} S_{R,\bar{d}}(\bar{x},P).
\]

It now follows that \( \beta^{M \alpha_j,\alpha_k}(y) \) is homogeneous of weighted order \( j \). There can be divergence terms which integrate to zero on the boundary and which are not controlled by this analysis. These terms play no role in the asymptotic coefficients and may therefore be ignored.

\[\square\]

2.2. A tensorial formulation. We have just discussed the homogeneity property of the invariants \( \beta_j \) from a coordinate point of view. We now consider the same property from a more invariant point of view. Fix a point \( P \in \partial M \). Choose geodesic coordinates on the boundary centered at \( P \). Form an adapted coordinate system for \( P \) in \( M \). Then the only non-zero first derivatives of the metric are \( g_{ab,m}(P) = \partial_{x_m} g_{ab}(P) \). The second fundamental form is given by setting:

\[
L_{ab} := g(\nabla_{\bar{y}_a} \partial_{\bar{m}}, \partial_r) = \Gamma_{abm} = -\frac{1}{2} g_{ab,m}.
\]

Keeping in mind the necessity to preserve the condition \( g_{ab}(P) = \delta_{ab} \) and observing that the Levi-Civita connection is unchanged by rescaling, we see that:

\[
L_{ab}(g_c) = c^2 g(\nabla_{e^{-1} \partial_{x_m}} c^{-1} \partial_{\bar{y}_a}, c^{-1} \partial_{\bar{y}_b}) = c^{-1} L_{ab}(g).
\]
Thus $L$ is homogeneous of degree 1 under rescaling. Similarly we have:
\[
R_{ijkl}(g_c)(P) = e^{2c}g((\nabla_c^{-1} \partial_{x_i} \nabla_c^{-1} \partial_{x_j} - \nabla_c^{-1} \partial_{x_j} \nabla_c^{-1} \partial_{x_i})c^{-1} \partial_{x_k}c^{-1} \partial_{x_l})
\]
so that $R_{ijkl}$ has order 2. Arbitrary partial derivatives of the metric at $P$ may now be expressed in terms of these tensors and their covariant derivatives; there are universal curvature relations (see, for example, the discussion in [24]) but these play no role. Rather than considering tangential derivatives of $S$ are universal curvature relations (see, for example, the discussion in [24]) but these play no role. Rather than considering tangential derivatives of $S_R$ with respect to the Levi-Civita connection of $M$, we consider covariant derivatives of $S_R$ with respect to the connection defined by $D$ and the Levi-Civita connection on the boundary. Let $\Omega_{ij}$ be the curvature of the connection defined by $D$. Let $\prime$ denote the components of multiple covariant differentiation with respect to the Levi-Civita connection of $M$ and the connection $\nabla$ defined by $D$ and let $\prime$ defined similarly using the Levi-Civita connection of $\partial M$ and $\nabla$; we can only differentiate $L$ and $S_R$ tangentially. Each covariant derivative adds 1 to the order. The derivatives of the total symbol of $D$ may then be expressed in terms of these variables so we may regard $\beta_{j,\alpha_1,\alpha_2}(\phi,\rho,D,B)(y)$ as a polynomial in the variables
\[
\{\phi^{\ell},\rho^{\ell},D_{\alpha_1}\alpha_2..._{\alpha_k},R_{\alpha_1\beta_1\alpha_2..._{\alpha_k}},E_{\alpha_1..._{\alpha_k}},S_{\Omega_1\Omega_2..._{\Omega_k}},R_{\alpha_1\alpha_2..._{\alpha_k}}\}
\]
which is invariantly defined and which is homogeneous of total order $j$. H. Weyl’s theorem [32] then lets us write this in terms of contractions of indices; the structure group is the orthogonal group $O(m - 1)$ so the index $m$ plays a distinguished role.

2.3. The proof of Lemma 1.4. Lemma 1.4 now follows from Lemma 2.1 and the discussion in Section 2.2 once a suitable basis of Weyl invariants is written down. We integrate by parts and ignore divergence terms to replace the invariants $\int_{\partial M} \langle \phi^{\alpha a}, \rho^{\beta b} \rangle dy$ and $\int_{\partial M} \langle \phi^{\alpha a}, \rho^{\beta a} \rangle dy$ by $-\int_{\partial M} \langle \phi^{\alpha a}, \rho^{\beta a} \rangle dy$. □

3. The proof of Lemma 1.6

We must determine the coefficient $\varepsilon_{D/R,\alpha_1,\alpha_2}$ of Lemma 1.4 which describes $\beta_{0,\alpha_1,\alpha_2}(\phi,\rho,D,B)$. We begin with a computation on the strip:

$$S := \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_1 < 1, \alpha_2 < 1, \alpha_1 + \alpha_2 > 1\}.$$

The Dirichlet setting was discussed in [10],[11] so we concentrate on Robin boundary conditions; we may set $S_R = 0$ as this plays no role in $\beta_{0}^{\partial M}$ and simplify matters by considering Neumann boundary conditions. Let $M = [0,\infty)$, let $D = -\partial^2_x$, and let $K_{D/\mathcal{N}}(x,y;t)$ be the Dirichlet or Neumann heat kernel on $M$ for $t > 0$. Let $\chi$ be a plateau function which is identically equal to 1 near $x = 0$ and which has compact support in $[0,\epsilon)$ for $\epsilon$ small. Set

$$\phi_{\alpha_1}(x) := x^{-\alpha_1}\chi(x) \quad \text{and} \quad \rho_{\alpha_2}(x) := x^{-\alpha_2}\chi(x). \quad (3.1)$$

We shall use this notation subsequently as well. We have:

$$\phi^\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0 \end{cases} \quad \text{and} \quad \rho^\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0 \end{cases}.$$  

The heat content on $[0,\infty)$ is:

$$\beta(\phi_{\alpha_1}, \rho_{\alpha_2}, D, B_{D/\mathcal{N}})(t) = \int_0^\infty \int_0^\infty K_{B_{D/\mathcal{N}}}(x_1, x_2; t)\phi_{\alpha_1}(x_1)\rho_{\alpha_2}(x_2)dx_1dx_2.$$  

Then $\beta(\phi_{\alpha_1}, \rho_{\alpha_2}, D, B_{D})(t)$ is finite for all $t > 0$ if and only if $\alpha_1 < 1$ and $\alpha_2 < 1$. We also suppose that $\alpha_1 + \alpha_2 > 1$. In [10] we have shown that

$$\beta(\phi_{\alpha_1}, \rho_{\alpha_2}, D, B_{D})(t) = \varepsilon_{D,\alpha_1,\alpha_2}(1-\alpha_1-\alpha_2)/2 + O(1).$$  

In the special case of the half-line we have that
\[ K_{B_n}(x_1, x_2; t) = K_{B_D}(x_1, x_2; t) + 2(4\pi)^{-1/2}e^{-(x_1 + x_2)^2/(4t)}. \]

We compute:
\[ \beta(\phi_{\alpha}, \rho_{\alpha}, D, B_N)(t) = \beta(\phi_{\alpha}, \rho_{\alpha}, D, B_D)(t) \]
\[ + 2(4\pi t)^{-1/2} \int_0^\infty \int_0^\infty e^{-(x_1 + x_2)^2/(4t)}x_1^{-\alpha_1}x_2^{-\alpha_2}dx_1dx_2 \]
\[ + 2(4\pi)\int_0^\infty \int_0^\infty e^{-(x_1 + x_2)^2/(4t)}x_1^{-\alpha_1}x_2^{-\alpha_2}(\chi(x_1)\chi(x_2) - 1)dx_1dx_2. \]

The second term in the right hand side above is evaluated by a change of variable \( x_2 = x_1\sigma. \) By Tonelli’s Theorem we have that it equals
\[ 2(4\pi t)^{-1/2} \int_0^\infty \int_0^\infty e^{-(x_1 + x_2)^2/(4t)}x_1^{-\alpha_1}x_2^{-\alpha_2}dx_1dx_2. \]

Note that if \( \alpha_1 < 1 \) and \( \alpha_2 < 1 \) then
\[ \int_0^\infty \sigma^{-\alpha_2}(1 + \sigma)^{\alpha_1+\alpha_2-2}d\sigma = \int_0^1 (\sigma^{-\alpha_1} + \sigma^{-\alpha_2})(1 + \sigma)^{\alpha_1+\alpha_2-2}d\sigma. \]

We will show that the third term in the right hand side of Equation (5.1) vanishes up to all orders. First note that there exists \( \epsilon > 0 \) so \( \chi(x) = 1, 0 \leq x \leq \epsilon. \) Secondly note that there exists a constant \( C < \infty \) so \( |\chi(x)| \leq C, x \geq 0. \) Hence
\[ |\chi(x_1)\chi(x_2) - 1| \leq (C^2 + 1)1_{(\epsilon, \infty)}(x_1)1_{(\epsilon, \infty)}(x_2). \]

It follows that the third term in the right hand side of Equation (5.1) is bounded in absolute value by
\[ 2(C^2 + 1)(4\pi t)^{-1/2} \int_0^\infty \int_0^\infty e^{-(x_1 + x_2)^2/(4t)}x_1^{-\alpha_1}x_2^{-\alpha_2}dx_1dx_2 \]
\[ \leq (C^2 + 1)\int_0^\infty \int_0^\infty e^{-(x_1^2 + x_2^2)/(16t)}x_1^{-\alpha_1}x_2^{-\alpha_2}dx_1dx_2 \]
\[ = O(e^{-x^2/(5t)}). \]

By Lemma 3.4 we have \( \beta_{\partial M}^{\alpha_1, \alpha_2}(\phi, \rho, D, B) = \int_{\partial M} \varepsilon_{\partial M, \alpha_1, \alpha_2}(\phi^0, \rho^0)dy. \) The above discussion permits us to decompose:
\[ \varepsilon_{\partial M, \alpha_1, \alpha_2} = \varepsilon_{\partial M, \alpha_1, \alpha_2} + \tilde{\kappa}_{\alpha_1, \alpha_2} \]
\[ \kappa_{\alpha_1, \alpha_2} := 2^{-\alpha_1-\alpha_2-1/2}(\frac{2-\alpha_1-\alpha_2}{2}) \int_0^1 (\sigma^{-\alpha_1} + \sigma^{-\alpha_2})(1 + \sigma)^{\alpha_1+\alpha_2-2}d\sigma \]
\[ \tilde{\kappa}_{\alpha_1, \alpha_2} := 2^{-\alpha_1-\alpha_2-1/2}(\frac{2-\alpha_1-\alpha_2}{2}) \int_0^1 (\sigma^{-\alpha_1} + \sigma^{-\alpha_2})(1 - \sigma)^{\alpha_1+\alpha_2-2}d\sigma. \]
The integral defining \( \kappa_{\alpha_1, \alpha_2} \) clearly is well defined for \( \alpha_1 < 1 \) and \( \alpha_2 < 1 \). By replacing \( \sigma \) by \( \frac{1}{T} \), we see that the integral with respect to \( \sigma \) over \([0, 1] \) defining \( \kappa_{\alpha_1, \alpha_2} \) is equal to the integral with respect to \( \sigma \) over \([1, \infty) \). Thus the integral with respect to \( \sigma \) over \([0, 1] \) is \( \frac{1}{T} \) the integral over \([0, \infty) \). We may now use Gradshteyn and Ryzhik [25] (see 3.251.11) and Gradshteyn and Ryzhik [25] (see 8.3801 and 8.384.1) to evaluate these integrals and establish Lemma 1.6 on the strip. By Conjecture 1.5, we may then use analytic continuation to establish Lemma 1.6 for the full parameter range given in Equation (1.9). □

4. The proof of Theorem 1.9

Functional properties

4.1. Direct sum and product formulas.

Lemma 4.1. The coefficients of Lemma 1.4 are independent of the fiber dimension of the underlying vector bundle.

Proof. Let \( V = V_1 \oplus V_2 \) be a direct sum vector bundle over \( M \), let \( D = D_1 \oplus D_2 \), let \( \phi = \phi_1 \oplus \phi_2 \), and let \( \rho = \rho_1 \oplus \rho_2 \). With Robin boundary conditions, we take \( S_R = S_{R,1} \oplus S_{R,2} \). The problem decouples and consequently

\[
\beta(\phi, \rho, D, B)(t) = \beta(\phi_1, \rho_1, D_1, B)(t) + \beta(\phi_2, \rho_2, D_2, B)(t).
\]

This implies that:

\[
\beta^{\alpha M}_{j,\alpha_1,\alpha_2}(\phi, \rho, D, B) = \beta^{\alpha M}_{j,\alpha_1,\alpha_2}(\phi_1, \rho_1, D_1, B) + \beta^{\alpha M}_{j,\alpha_1,\alpha_2}(\phi_2, \rho_2, D_2, B).
\]

The desired result now follows; again, we ignore divergence terms when passing to the local formulas. □

Suppose that \((M_1, g_1)\) is a closed Riemannian manifold of dimension \( m_1 \) and that \((M_2, g_2)\) is a Riemannian manifold with boundary of dimension \( m_2 \). We form \((M, g) := (M_1 \times M_2, g_1 \oplus g_2)\). If \( D_i \) are operators of Laplace type on \( M_i \), we form \( D = D_1 \oplus 1 \oplus D_2 \) on \( V := V_1 \otimes V_2 \).

Lemma 4.2. Adopt the notation established above. We have:

1. \( \beta(\phi, \rho, D, B_{D/R})(t) = \beta(\phi_1, \rho_1, D_1)(t) \cdot \beta(\phi_2, \rho_2, D_2, B_{D/R})(t) \).

2. \( \beta^{\alpha M}_{j,\alpha_1,\alpha_2}(\phi, \rho, D, B_{D/R})(x_1, y_2) = \sum_{k+2\ell=j} (-1)^{\ell} \frac{1}{k!}(D^k \phi_1)(x_1) \cdot \beta^{\alpha M}_{k,\alpha_1,\alpha_2}(\phi_2, \rho_2, D_2, B_{D/R})(y_2) \).

3. The coefficients of Lemma 1.4 are independent of the dimension \( m \).

4. \( \varepsilon_{D/R, \alpha_1, \alpha_2} = 0, \varepsilon_{D/R, \alpha_1, \alpha_2} = 0, \varepsilon_{D/R, \alpha_1, \alpha_2} = -\varepsilon_{D/R, \alpha_1, \alpha_2} \).

Proof. One has purely formally that

\[
e^{-tD\varepsilon_{D/R, \alpha}} \phi = e^{-tD_1 \phi_1} \otimes e^{-tD_2 \varepsilon_{D/R, \alpha} \phi_2}.
\]

We verify Equation (4.4) using Equation (1.9) as follows. Let \( T_t \) be the temperature distributions defined by \( \phi \) on \( M_i \). We compute:

\[
(\partial_t + D)(T_1 \otimes T_2) = \{ (\partial_t + D_1)T_1 \} \otimes T_2 + T_1 \otimes \{ (\partial_t + D_2)T_2 \} = 0,
\]

\[
\lim_{t \downarrow 0} \{ T_1 \otimes T_2 \}(t) = \lim_{t \downarrow 0} \{ T_1(t) \} \otimes \lim_{t \downarrow 0} \{ T_2(t) \} = \phi_1 \otimes \phi_2,
\]

\[
B_{D/R}(\phi_1 \otimes \phi_2) = \phi_1 \otimes B_{D/R} \phi_2 = 0.
\]
Assertion (1) is now immediate. Since the boundary conditions play no role on the closed manifold \( M_1 \), we have
\[
\beta(\phi_1, \rho_1, D_1)(t) \sim \sum_{n=0}^{\infty} t^n (-1)^n \int_{M_1} \langle D^n \phi_1, \rho_1 \rangle dx_1.
\]
Assertion (2) now follows from Assertion (1) by equating terms in the asymptotic expansions.

We argue as follows to prove Assertion (3). Let \( M_1 = S^1 \), let \( D_1 = -\partial_x^2 \), and let \( \phi_1 = \rho_1 = 1 \). Since \( T_1(x_1; t) = 1 \),
\[
\beta(\phi_1, \rho_1, D_1)(t) = 2\pi.
\]
We use Assertion (2) to see that:
\[
\rho(1) = 1.
\]

Consequently Assertion (2) yields:
\[
\beta(\phi_1, \rho_1, D_1)(t) = 2\pi.
\]

We complete the proof by considering Assertion (4). With Robin boundary conditions, we take \( S_R = 0 \) as it plays no role in the evaluation of the coefficients we are considering. Let \( (M_1, g_1, \phi_1, \rho_1, D_1) \) be arbitrary. Let \( M_2 = [0, \pi] \) with the usual flat metric, let \( D_2 = \partial_x^2 \), and let \( \phi_{a_1} \) and \( \rho_{a_2} \) be as given in Equation (3.a).

Since the structures on \( M_2 \) are flat,
\[
\beta_{j, a_1, a_2}(\phi_2, \rho_2, D_2, \mathcal{B}_{D/R}) = \begin{cases} \varepsilon_{D/R, a_1, a_2} & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}.
\]

Consequently Assertion (2) yields:
\[
\beta_{2, a_1, a_2}(\phi, \rho, D, \mathcal{B}_{D/R}) = -\varepsilon_{D/R, a_1, a_2} \int_{M_1} \langle D_1 \phi_1, \rho_1 \rangle dx_1
\]
\[
= \varepsilon_{D/R, a_1, a_2} \int_{M_1} \langle \phi_{1, a_2} + E \phi_1 + 0 \cdot \tau_1 \phi_1, \rho_1 \rangle dx_1
\]
\[
= \varepsilon_{D/R, a_1, a_2} \int_{M_1} \left\{ \langle \phi_{1, a_1, 1}, 0 \rangle + \langle E \phi_1, \rho_1 \rangle + \langle 0, \rho_1 \rangle \right\} dx_1. \tag{4.b}
\]

On the other hand, when we use the formulas of Lemma 1.4 directly, most of the terms vanish and we obtain:
\[
\beta^{\mathcal{B}_{D/R}}_{2, a_1, a_2}(\phi, \rho, D, \mathcal{B}_R)
\]
= \[
\frac{1}{2} \int_{M_1} \left\{ \varepsilon_{D/R, a_1, a_2}^2 \phi_{1, a_1, 0} + \varepsilon_6 D_{D/R, a_1, a_2} E \phi_1 + \varepsilon_{13} D_{D/R, a_1, a_2} \tau_1 \phi_1, \rho_1 \right\} dx_1. \tag{4.c}
\]
The desired result follows by comparing Equation (4.b) with Equation (4.c).

4.2. Index shifting. The following observation is simply change of notation:

Lemma 4.3.

1. Suppose that \( \phi^0 = 0 \). Then \( \phi \) can also be regarded as being an element \( \tilde{\phi} \in K_{a_1 - 1}(V) \) where \( \tilde{\phi} = \phi^{a_1 + 1} \). We have:
\[
\beta^{\mathcal{B}_{D/R}}_{j-1, a_1, a_2}(\phi, \rho, D, \mathcal{B}_{D/R}) = \beta^{\mathcal{B}_{D/R}}_{j, a_1, a_2}(\phi, \rho, D, \mathcal{B}_{D/R}).
\]

2. Suppose that \( \rho^0 = 0 \). Then \( \rho \) can also be regarded as being an element \( \tilde{\rho} \in K_{a_2 - 1}(V^*) \) where \( \tilde{\rho} = \rho^{a_2 + 1} \). We have:
\[
\beta^{\mathcal{B}_{D/R}}_{j-1, a_1, a_2-1}(\phi, \tilde{\rho}, D, \mathcal{B}_{D/R}) = \beta^{\mathcal{B}_{D/R}}_{j, a_1, a_2}(\phi, \rho, D, \mathcal{B}_{D/R}).
\]
We have
\[ \frac{1}{4} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1 - 1, \alpha_2}, \quad \frac{3}{4} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1, \alpha_2 - 1}, \]
\[ \frac{7}{4} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1 - 2, \alpha_2}, \quad \frac{5}{4} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1, \alpha_2 - 2}, \]
\[ \frac{1}{2} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1 - 1, \alpha_2}, \quad \frac{6}{4} \frac{\varepsilon}{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1, \alpha_2 - 1}, \]
\[ \frac{9}{4} \gamma_{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1 - 1, \alpha_2}, \quad \frac{11}{4} \gamma_{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon}{D/R, \alpha_1, \alpha_2 - 1}. \]

Proof. Suppose \( \phi^0 = 0. \) Then

\[ \phi \sim r^{-(\alpha_1 - 1)}(r_1^1 + r^2 \phi^2 + ...) \sim r^{-(\alpha_1 - 1)}(\phi^1 + r \phi^2 + ...). \]

Consequently, \( \phi \in K_{\alpha_1 - 1}. \) We expand:

\[ \beta(\phi, \rho, D, B_{D/R})(t) \]
\[ \sim \sum_{n=0}^{\infty} t \beta^M_n(\phi, \rho, D) + \sum_{j=0}^{\infty} t^{(1+j-\alpha_1-\alpha_2)/2} \beta^{BM}_{j, \alpha_1, \alpha_2}(\phi, \rho, D, B_{D/R})(t) \]
\[ = \beta(\phi, \rho, D, B_{D/R})(t) \]
\[ \sim \sum_{n=0}^{\infty} t \beta^M_n(\phi, \rho, D) + \sum_{k=0}^{\infty} t^{(1+k-(\alpha_1-1)-\alpha_2)/2} \beta^{BM}_{k, \alpha_1 - 1, \alpha_2}(\phi, \rho, D, B_{D/R})(t). \]

We set \( k + 1 = j \) and omit terms in the boundary asymptotic expansions to prove Assertion (1). The proof of Assertion (2) is similar and therefore omitted.

We apply Assertion (1) to prove Assertion (3). Suppose \( \phi^0 = 0. \) By Lemma 1.8

\[ \beta^M_{0, \alpha_1, \alpha_2}(\phi, \rho, D, B_{D/R}) = \int_{\partial M} \frac{1}{\varepsilon_{D/R, \alpha_1, \alpha_2}}(\phi^1, \rho^0)dy, \]
\[ \beta^{BM}_{0, \alpha_1 - 1, \alpha_2}(\phi, \rho, D, B_{D/R}) = \int_{\partial M} \frac{1}{\varepsilon_{D/R, \alpha_1 - 1, \alpha_2}}(\phi^0, \rho^0)dy. \]

Consequently, \( \frac{1}{4} \varepsilon_{D/R, \alpha_1, \alpha_2} = \frac{\varepsilon_{D/R, \alpha_1, \alpha_2 - 1}}{4}. \) The remaining assertions of the Lemma are established similarly. \( \square \)

4.3. Relating Robin and Dirichlet Boundary Conditions. We work on the interval \([0, \pi]. \) Let \( 0 \neq c \in \mathbb{R}. \) Set

\[ A := \partial_x + c, \quad A^* := -\partial_x + c, \quad D = A^* A = AA^* = (-\partial_x^2 - c^2). \]

Let \( B_D \) be the Dirichlet boundary operator and let \( B_R \phi = 0 \) define the boundary condition \( A^* \phi|_{\partial M} = 0; \) this is the associated Robin boundary operator:

\[ B_D \phi := \phi(0) \oplus \phi(\pi), \quad B_R \phi := (\phi(0) - c \phi(0)) \oplus (-\phi(\pi) + c \phi(\pi)), \]
\[ S_R(0) = -c, \quad S_R(\pi) = +c, \quad E = -c^2. \]

Lemma 4.4. If \( \Re(\alpha_1) < -1 \) and \( \Re(\alpha_2) < -1, \) then:

1. \( \partial_t \beta(\phi, \rho, D, B_{D/R})(t) = -\beta(D \phi, \rho, D, B_{D/R})(t). \)
2. \( \partial_t \beta(\phi, \rho, D, B_{D/R})(t) = -\beta(\phi, D \rho, D, B_{D/R})(t). \)
3. \( \partial_t \beta(\phi, \rho, D, B_R)(t) = -\beta(A^* \phi, A^* \rho, D, B_D)(t). \)
4. \( \partial_t \beta(\phi, \rho, D, B_D)(t) = -\beta(A \phi, A \rho, D, B_R)(t). \)
5. \( \beta^{BM}_{j, \alpha_1 + 1, \alpha_2}(D \phi, \rho, D, B_{D/R}) = -\frac{1}{2}(1 + j - \alpha_1 - \alpha_2) \beta^{BM}_{j, \alpha_1, \alpha_2} \phi, \rho, D, B_{D/R}). \)
6. \( \beta^{BM}_{j, \alpha_1, \alpha_2 + 1}(\phi, D \rho, D, B_{D/R}) = -\frac{1}{2}(1 + j - \alpha_1 - \alpha_2) \beta^{BM}_{j, \alpha_1, \alpha_2} \phi, \rho, D, B_{D/R}). \)
7. \( \beta^{BM}_{j, \alpha_1 + 1, \alpha_2 + 1}(A^* \phi, A^* \rho, D, B_D) = -\frac{1}{2}(1 + j - \alpha_1 - \alpha_2) \beta^{BM}_{j, \alpha_1, \alpha_2} \phi, \rho, D, B_R). \)
Thus we may establish Assertion (3) by computing:

\begin{equation}
\beta_{\alpha_1, \alpha_2}^{\beta M} + (A\phi, A\rho, D, B_D) = -\frac{1}{2} (1 + j - \alpha_1 - \alpha_2) \beta_{\alpha_1, \alpha_2}^{\beta M} (\phi, \rho, D, B_D).
\end{equation}

(9) $\varepsilon_{\alpha_1, \alpha_2}^R = \left\{-\frac{1}{\alpha_1 - 2} \varepsilon_{\alpha_1 - 2, \alpha_2} - \frac{1}{\alpha_2 - 2} \varepsilon_{\alpha_1, \alpha_2 - 1}\right\}$. 

(10) $\varepsilon_{\alpha_1, \alpha_2}^R = \left\{-\frac{1}{\alpha_1 - 2} \varepsilon_{\alpha_1 - 2, \alpha_2} - \frac{1}{\alpha_2 - 2} \varepsilon_{\alpha_1, \alpha_2 - 1}\right\}$. 

(11) $\varepsilon_{\alpha_1, \alpha_2}^R = \left\{-\frac{1}{\alpha_1 - 1} \varepsilon_{\alpha_1 - 1, \alpha_2 - 1} - \frac{1}{\alpha_2 - 2} \varepsilon_{\alpha_1, \alpha_2 - 2}\right\}$. 

(12) $\varepsilon_{\alpha_1, \alpha_2}^R = \left\{-\frac{1}{\alpha_1 - 1} \varepsilon_{\alpha_1 - 1, \alpha_2 - 1} + \frac{2}{\alpha_2 - 2} \varepsilon_{\alpha_1, \alpha_2 - 2}\right\}$. 

Proof. We begin by establishing the spectral resolutions of the Dirichlet realization $D_D$ and of the Robin realization $D_R$. For $n = 1, 2, \ldots$, set:

\[ \phi_n^D := \sqrt{\frac{2}{\pi}} \sin(n\pi) \] 

and \[ \lambda_n := n^2 + c^2. \] 

Because \( \{\phi_n^D\} \) is a complete orthonormal basis for $L^2$, because $D\phi_n^D = \lambda_n \phi_n^D$, and because $\phi_n^D(0) = \phi_n^D(\pi) = 0$, we may conclude that \( \{\phi_n^D, \lambda_n\}_{n \in \mathbb{N}} \) is the Dirichlet spectral resolution of $D$. Similarly, set:

\[ \phi_n^R := \lambda_n^{-\frac{1}{2}} A\phi_n^D; \]

\( \{\phi_n^R\} \) is a complete orthonormal basis for $L^2$ with $D\phi_n^R = \lambda_n \phi_n^R$. Since

\[ A^*\phi_n^R|_{\partial M} = \lambda_n^{-1/2} A^* A\phi_n^D|_{\partial M} = \lambda_n^{-1/2} D\phi_n^D|_{\partial M} = \lambda_n^{1/2} \phi_n^D|_{\partial M} = 0, \]

$B_R A\phi_n^R = 0$ so the eigenfunctions $\phi_n^R$ satisfy Robin boundary conditions. Consequently \( \{\phi_n^R, \lambda_n\}_{n \in \mathbb{N}} \) is a Robin spectral resolution of $D$. If $\phi \in L^1$, let

\[ \gamma_n^{D/R}(\phi) := \int_0^\pi \phi(x) \phi_n^{D/R}(x) dx \]

be the associated Fourier coefficients. We then have

\[ \beta(\phi, \rho, D, B_D)(t) = \sum_{n=1}^{\infty} e^{-t \lambda_n} \gamma_n^{D/R}(\phi) \gamma_n^{D/R}(\rho), \]

\[ \partial_t \beta(\phi, \rho, D, B_D)(t) = \sum_{n=1}^{\infty} -\lambda_n e^{-t \lambda_n} \gamma_n^{D/R}(\phi) \gamma_n^{D/R}(\rho). \]

Since \( \Re(\alpha_1) < -1 \) and \( \Re(\alpha_2) < -1 \), we may integrate by parts to see

\[ \gamma_n^{D/R}(D\phi) = \int_M D\phi \cdot \phi_n^{D/R} dx = \int_M \phi \cdot D\phi_n^{D/R} dx = \lambda_n \int_M \phi \cdot \phi_n^{D/R} dx = \lambda_n \gamma_n^{D/R}(\phi). \]

Assertions (1) and (2) now follow. To prove Assertion (3), we note:

\[ \gamma_n^{D}(A^*\phi) = \int_M A^* \phi \cdot \phi_n^{D} dx = \int_M \phi \cdot A\phi_n^D dx = \sqrt{\lambda_n} \int_M \phi \cdot \phi_n^{R} dx = \sqrt{\lambda_n} \gamma_n^{R}(\phi). \]

Thus we may establish Assertion (3) by computing:

\[ \partial_t \beta(\phi, \rho, D, B_R)(t) = \sum_{n=1}^{\infty} -\lambda_n e^{-t \lambda_n} \gamma_n^{R}(\phi) \cdot \gamma_n^{R}(\rho) \]

\[ = -\sum_{n=1}^{\infty} e^{-t \lambda_n} \gamma_n^{D}(A^*\phi) \cdot \gamma_n^{D}(A^*\rho) = -\beta(A^*\phi, A^*\rho, D, B_D)(t). \]

Assertion (4) follows similarly once we observe that \( \{A^*\phi_n^R/\sqrt{\lambda_n}, \lambda_n\}_{n \in \mathbb{Z}} \) is dually a spectral resolution of the Dirichlet Laplacian; the roles of $A$ and $A^*$ being interchanged.
We now establish Assertions (5)-(8). Since $D\phi \in K_{\alpha_1+2}(V)$, we have:

$$
\partial \beta(\phi, \rho, D, B_D) = \sum_{n=0}^{\infty} n \cdot (-1)^n t^{n-1} / n! I_{\text{Reg}}^{\alpha}(D^n \phi, \rho)
$$

$$
+ \frac{1}{2} \sum_{j=0}^{\infty} \left( j + 1 - \alpha_1 - \alpha_2 \right) t^{(j-1-\alpha_1-\alpha_2)/2} \beta_{\alpha_1, \alpha_2}^{\alpha}(\phi, \rho, D, B_D) / \alpha_1 \alpha_2
$$

$$
= -\beta(D\phi, D, B_D)(t) \sim - \sum_{\ell=0}^\ell (-1)^\ell t^\ell / \ell! I_{\text{Reg}}(D^{\ell+1}\phi, \rho)
$$

$$
- \sum_{k=0}^\infty t^{(k+1-(\alpha_1+2)-\alpha_2)/2} \beta_{\alpha_1, \alpha_2}^{\alpha}(D\phi, \rho, D, B_D).
$$

The term with $n = 0$ plays no role. Setting $n = \ell + 1$ equates the summation involving the regularized integrals. Setting $k = j$ and equating terms in the asymptotic series for the boundary terms establishes Assertion (5); the proof of Assertion (6) is similar. To prove Assertions (7) and (8), we may integrate by parts to express series for the boundary terms establishes Assertion (5); the proof of Assertion (6) is similar. To prove Assertions (7) and (8), we may integrate by parts to express:

$$
I_{\text{Reg}}(D^n A^* \phi, A^* \rho) = I_{\text{Reg}}(AD^n A^* \phi, \rho) = I_{\text{Reg}}(D^n A^* \phi, \rho)
$$

$$
I_{\text{Reg}}(D^{n+1} \phi, \rho) = I_{\text{Reg}}(D^{n+1} \phi, \rho) = I_{\text{Reg}}(D^{n+1} \phi, \rho).
$$

The remainder of the argument now follows similarly.

To prove Assertion (9), we assume $\Re(\alpha_1) < -1$ and $\Re(\alpha_2) < -1$. We then use analytic continuation to obtain the result on the full parameter range. Adopt the notation of Equation (5.3) and set $\phi = \phi_{\alpha_1}$, and $\rho = \rho_{\alpha_2}$. Only the value at $r = 0$ is relevant; integration over the boundary is just evaluation at 0. Since $D = -(\partial_x^2 - c^2)$, the metric and associated connection are flat. We have:

$$
A^* \phi = (-\partial_x + c)(r^{-\alpha_1}) = r^{-\alpha_1-1}\{\alpha_1 + cr\},
$$

$$
A^* \rho = (-\partial_x + c)(r^{-\alpha_2}) = r^{-\alpha_2-1}\{\alpha_2 + cr\},
$$

$$
\phi^0 = 1, \quad \phi^1 = 0, \quad \rho^0 = 1, \quad \rho^1 = 0,
$$

$$
(A^* \phi)^0 = \alpha_1, \quad (A^* \phi)^1 = c, \quad (A^* \rho)^0 = \alpha_2, \quad (A^* \rho)^1 = c,
$$

$$
\phi^0 = 0, \quad \rho^0 = 0, \quad (A^* \rho)^2 = 0, \quad (A^* \phi)^2 = 0,
$$

$$
S_0(0) = -c, \quad E = -c^2.
$$

We use Assertion (7), Lemma 1.4 and Lemma 1.3 (3) to see:

$$
\beta_{\alpha_1+1, \alpha_2+1}^{\alpha}(A^* \phi, A^* \rho, D, B_D)
$$

$$
= \epsilon_{\alpha_1, \alpha_2+1}^1 (A^* \phi)^1 (A^* \rho)^1 + \epsilon_{\alpha_1, \alpha_2+1}^3 (A^* \phi)^0 (A^* \rho)^1
$$

$$
= c \cdot \{ \alpha_2 \epsilon_{\alpha_1, \alpha_2+1}^2 + \alpha_1 \epsilon_{\alpha_1, \alpha_2+1}^2 \}
$$

$$
= -\frac{1}{2} (2 - \alpha_1 - \alpha_2) \cdot \epsilon_{\alpha_1, \alpha_2+1}^1 (A^* \phi, \rho, D, B_D)
$$

$$
= \frac{1}{2} (2 - \alpha_1 - \alpha_2) \cdot c \epsilon_{\alpha_1, \alpha_2},
$$

We solve these equations to establish see:

$$
\epsilon_{\alpha_1, \alpha_2} = \frac{2}{2 - \alpha_1 - \alpha_2} (\alpha_2 \epsilon_{\alpha_1, \alpha_2+1}^2 + \alpha_1 \epsilon_{\alpha_1, \alpha_2+1}^2)
$$

Applying Lemma 1.4 (3) then yields:

$$
\epsilon_{\alpha_1, \alpha_2} = \{ -\frac{1}{2} \epsilon_{\alpha_1, \alpha_2-1} - \frac{1}{2} \epsilon_{\alpha_1, \alpha_2-1} \}
$$

Assertion (10) and Assertion (11) follow from Assertion (9) by using Lemma 1.3 to index shift. We take $j = 2$ and examine the coefficient of $c^2$ to complete the proof of Assertion (12) by computing:
\[
\beta^{BM}_{2,\alpha_1+1,\alpha_2+1}(A^\ast \phi, A^\ast \rho, D, B_D) \\
= (A^\ast \phi)^1(A^\ast \rho)^1 \xi_{D,\alpha_1+1,\alpha_2+1} - c^2(A^\ast \phi)^0(A^\ast \rho)^0 \xi_{D,\alpha_1+1,\alpha_2+1} \\
= c^2 \cdot \xi_{D,\alpha_1,\alpha_2} - c^2 \alpha_1 \alpha_2 \xi_{D,\alpha_1+1,\alpha_2+1} \\
= -\frac{1}{2}(3 - \alpha_1 - \alpha_2)\beta^{BM}_{2,\alpha_1+1}(\phi, \rho, D, B_R) \\
= -\frac{1}{2}(3 - \alpha_1 - \alpha_2)c^2\{\varepsilon_{R,\alpha_1,\alpha_2} - \varepsilon_{R,\alpha_1,\alpha_2}\}.
\]

We solve these equations and use Lemma 1.7 to see:
\[
\varepsilon_{16}^{\alpha,\alpha_1,\alpha_2} = -\frac{2}{3 - \alpha_1 - \alpha_2} \xi_{D,\alpha_1,\alpha_2} + \frac{2\alpha_1 \alpha_2}{3 - \alpha_1 - \alpha_2} \xi_{D,\alpha_1+1,\alpha_2+1} + \varepsilon_{R,\alpha_1,\alpha_2} \\
= -\frac{1}{3 - \alpha_1 - \alpha_2} \xi_{R,\alpha_1,\alpha_2} - \xi_{D,\alpha_1+1,\alpha_2+1} + \frac{2\alpha_1 \alpha_2}{3 - \alpha_1 - \alpha_2} \varepsilon_{R,\alpha_1,\alpha_2}.
\]

\[\square\]

4.4. The proof of Lemma 1.7
Lemma 1.7 (3) focuses attention on the shifted invariants \(\varepsilon_{R,\alpha_1,\alpha_2}\) and \(\varepsilon_{R,\alpha_1,\alpha_2-1}\). Lemma 1.4 expresses these invariants in terms of the fundamental invariants \(\varepsilon_{B,\alpha_1,\alpha_2}\). These relations follow from the standard properties of the \(\Gamma\) function. It is instructive, however, to use Lemma 1.4 to establish these properties as an independent check on our work.

We adopt the notation of Lemma 1.3 with \(c = 0\). We use Equation (3.3) to define \(\phi_{\alpha_1}\) and \(\rho_{\alpha_2}\) on \([0, \pi]\). We take \(j = 0\) and apply Lemma 1.4 and Lemma 1.4 to see:
\[
\beta^{BM}_{0,\alpha_1+1,\alpha_2}(D\phi_{\alpha_1}, \rho_{\alpha_2}, D, B_D/R) \\
= -\beta^{BM}_{0,\alpha_1+1,\alpha_2}(\hat{\alpha}_1(\hat{\alpha}_1 + 1)\phi_{\alpha_1+2}, \rho_{\alpha_2}, D, B_D/R) = -\hat{\alpha}_1(\hat{\alpha}_1 + 1)\varepsilon_{D/R,\alpha_1+2,\alpha_2} \\
= -\frac{1}{2}(\hat{\alpha}_1 - \hat{\alpha}_2) \beta^{BM}_{0,\alpha_1+1,\alpha_2}(\phi, \rho, D, B_D/R) = -\frac{1}{2}(\hat{\alpha}_1 - \hat{\alpha}_2)\varepsilon_{D/R,\alpha_1,\alpha_2}.
\]

Consequently:
\[
-\hat{\alpha}_1(\hat{\alpha}_1 + 1)\varepsilon_{D/R,\alpha_1+2,\alpha_2} = -\frac{1}{2}(\hat{\alpha}_1 - \hat{\alpha}_2)\varepsilon_{D/R,\alpha_1,\alpha_2}.
\]

Lemma 1.7 (1) now follows after replacing setting \(\hat{\alpha}_1 = \alpha_1 - 2\) and \(\hat{\alpha}_2 = \alpha_2\). The proof of Lemma 1.7 (2) is similar. To prove Lemma 1.7 (3), we set \(c = 0\) to obtain \(A = -A^*\); the roles of \(D\) and \(R\) are then symmetric in Lemma 1.4 so we may compute:
\[
\beta^{BM}_{0,\alpha_1+1,\alpha_2+1}(D\phi_{\alpha_1}, \rho_{\alpha_2}, D, B_D/R) \\
= \hat{\alpha}_1\hat{\alpha}_2\beta^{BM}_{0,\alpha_1+1,\alpha_2+1}(\phi_{\alpha_1+1}, \rho_{\alpha_2+1}, D, B_D/R) = \hat{\alpha}_1\hat{\alpha}_2\varepsilon_{D/R,\alpha_1+1,\alpha_2+1} \\
= -\frac{1}{2}(\hat{\alpha}_1 - \hat{\alpha}_2)\beta^{BM}_{0,\alpha_1+1,\alpha_2+1}(\phi, \rho, D, B_D/R) = -\frac{1}{2}(\hat{\alpha}_1 - \hat{\alpha}_2)\varepsilon_{D/R,\alpha_1,\alpha_2}.
\]

Lemma 1.7 (3) now follows after setting \(\hat{\alpha}_1 = k_1 - 1\).

4.5. Warped products. Let \(\mathbb{T}^{m-1} = [0, 2\pi]^{m-1}\) denote the torus where we identify \(0 \sim 2\pi\) to form a closed manifold. Let \((y_1, ..., y_{m-1})\) be the usual periodic parameters where \(0 \leq y_i \leq 2\pi\), and let
\[
M := \mathbb{T}^{m-1} \times [0, \pi].
\]

Let \(f_a \in C^\infty([0, \pi])\) be a collection of smooth functions which have compact support near \(r = 0\) with \(f_a(0) = 0\). Let \(\chi(r) \in C^\infty([0, \pi])\) be the mesa function described
previously; $\chi$ is identically 1 near $r = 0$ and identically 0 near $r = 1$. Set

$$ds^2_M = \sum_a e^{2f_a(r)}dy_a \circ dy_a + dr \circ dr, \quad \phi_2 := \chi(r)r^{-\alpha_1},$$

$$D_M := -\sum_a e^{-2f_a(r)}\partial^2_{y_a} - \partial_r^2, \quad \rho_2 := \chi(r)r^{-\alpha_2},$$

$$D_2 := -\partial_r^2, \quad \phi_M(y, r) := \phi_2(r), \quad \rho_M(y, r) = e^{-\sum_a f_a(r)}\rho_2(r).$$

Let $B = B_T$ or let $B_R = \partial_r + S_{R,0}$. 

**Lemma 4.5.** Adopt the notation established above. Then:

$$\beta_{j,\alpha_1,\alpha_2}(\phi_M, \rho_M, D, B)(y, 0) = \beta_{j,\alpha_1,\alpha_2}(\phi_2, \rho_2, D_2, B)(0).$$

**Proof.** Let $T_2 := e^{-tD_2} \phi_2$ and let $T_M := e^{-tD_M} \phi_M$. We show that

$$T_M(y, r; t) = T_2(r; t)$$

by verifying that $T$ satisfies the defining relations of Equation (1.11)

$$(\partial_t + D_M)T_2 = (\partial_t + D_2)T_2 = 0, \quad \lim_{t \downarrow 0} T_2(\cdot; t) = \phi_2 = \phi_M, \quad BT_2(\cdot; t) = 0.$$

Since the volume element on $M$ is given by $e^{\sum_a f_a(r)}dydr$, we complete the proof by computing:

$$\beta(\phi_M, \rho_M, D_M, B)(t) = \int_{\mathbb{T}^{n-1} \times [0, 1]} T_2(r; t) \{e^{-\sum_a f_a(r)}\rho_2(r)\} e^{\sum_a f_a(r)}dydr$$

$$= \int_{\mathbb{T}^{n-1} \times [0, 1]} T_2(r; t)\rho_2(r)dydr = (2\pi)^{m-1} \int_{[0, 1]} T_2(r; t)\rho_2(r)dr$$

$$= (2\pi)^{m-1} \beta(\phi_2, \rho_2, D_2, B)(t). \quad \square$$

**4.6. The proof of Theorem 1.9.** We have computed most of the unknown coefficients in Lemma 4.6. We use the notation of Section 4.5 and apply Lemma 4.5 to compute the remaining coefficients; the recursion relations of Lemma 4.7 also play a crucial role as do our previous computations. We adopt the notation of Equation (3.10) and consider the following example. Let $\phi_2 := r^{-\alpha_1}\chi$ and $\rho_2 := r^{-\alpha_2}\chi$ where $\chi$ is identically 1 near $r = 0$ and vanishes for $r \geq \frac{1}{2}$. We then have:

$$\phi_2 = \rho_2 = 1, \quad \phi_2^\ell = \rho_2^\ell = 0 \text{ for } \ell \geq 1.$$ 

Consequently there exist suitably chosen constants $\{c_j\}$ (which play no role in our development) so that:

$$\beta_{j,\alpha_1,\alpha_2}(\phi_M, \rho_M, D_M, B)(y, 0) = \beta_{j,\alpha_1,\alpha_2}(\phi_2, \rho_2, D_2, B)(0)$$

$$= c_j \cdot S_{R,0}^1 \quad \text{for } j > 0.$$ 

We shall ignore this term and concentrate on other terms. Let $\omega$ be the connection 1-form of the connection on $V$ defined by $D$ and let $\tilde{\omega}$ be the dual connection 1-form of the dual connection on $V^\ast$: $\tilde{\omega} + \omega = 0$. We let $e_a := \partial_{y_a}$ for $1 \leq a \leq m$ and $e_m := \partial_r$; this is an orthonormal frame on $\partial M$. We use Lemma 4.11 to compute:

$$\Gamma_{abm} = -f_a^s \delta_{ab} e^{2f_a} f_s, \quad \Gamma_{ab}^m = -f_a^s e^{2f_a} \delta_{ab},$$

$$\Gamma_{amb} = f_a^s \delta_{am} e^{2f_a}, \quad \Gamma_{am}^b = f_a^s \delta_{am},$$

$$L_{ab}|_{\partial M} = \Gamma_{ab}^m|_{\partial M} = -f_a^s \delta_{ab}|_{\partial M},$$

$$\omega_a = 0, \quad \tilde{\omega}_a = 0,$$

$$\omega_m = -\frac{1}{2} \sum_a f'_a, \quad \tilde{\omega}_m = -\omega_m = \frac{1}{2} \sum_a f'_a.$$ 

Consequently:

$$R_{abm} = g((\nabla_a \nabla_m - \nabla_m \nabla_a)e_b, e_m)|_{\partial M} = \{\Gamma_{ac}^m \Gamma_{mb}^c - \partial_m \Gamma_{ac}^b\}|_{\partial M}$$

$$= \{-(f_a^s)^2 + f'_a + 2(f'_a)^2 \delta_{ab}\}|_{\partial M},$$
We use Equation (1.b) to compute:

\[ \text{Ric}_{mm}|_{\partial M} = -\sum_a \left\{ f''_a + (f'_a)^2 \right\}|_{\partial M}, \]
\[ E|_{\partial M} = \left\{ -\partial_m \omega_m - \omega_m^2 + \omega_m \Gamma^{m}_{\alpha \alpha} \right\}|_{\partial M} \]
\[ = \left\{ \frac{1}{2} \sum_a f''_a - \frac{1}{2} \sum_{a,b} f'_a f'_b + \frac{1}{2} \sum_{a,b} f'_a f'_b \right\}|_{\partial M} \]
\[ = \left\{ \frac{1}{2} \sum_a f''_a - \frac{1}{2} \sum_{a,b} f'_a f'_b \right\}|_{\partial M}. \]

At this stage, the connection 1-form enters in a crucial fashion as we must use the dual connection and the dual connection 1-form \( \tilde{\omega} \) to covariantly differentiate \( \rho \). Thus \( \nabla_{\partial \rho} \psi = (\partial_{\rho} \psi + \omega_{\rho} \psi) \) if \( \psi \in C^\infty(V) \) while \( \nabla_{\partial \rho} \psi = (\partial_{\rho} \psi - \omega_{\rho} \psi) \) if \( \psi \in V^\ast \).

We use Equation (1.13) to compute:

\[ \phi_1^0 = 1, \]
\[ \phi_3^0 = \left\{ \nabla_{\partial \rho} (e^{\rho} \phi_M) \right\}|_{\partial M} = \left\{ (\partial_{\rho} - \frac{1}{2} \sum_a f'_a) \right\}|_{\partial M} = -\frac{1}{2} \sum_a f''_a|_{\partial M}, \]
\[ \phi_3^0 = \frac{1}{2} \left\{ (\nabla_{\partial \rho})^2 (e^{\rho} \phi_M) \right\}|_{\partial M} = \frac{1}{2} \left\{ (\partial_{\rho} - \frac{1}{2} \sum_a f'_a)^2 \right\}|_{\partial M} \]
\[ = \frac{1}{2} \left\{ \sum_{a,b} f'_a f'_b - \frac{1}{2} \sum_a f''_a \right\}|_{\partial M}, \]
\[ \phi_1^0 = e^{-\sum_a f_a}|_{\partial M} = 1, \]
\[ \phi_3^0 = \left\{ \nabla_{\partial \rho} (f_M) \right\}|_{\partial M} = \left\{ (\partial_{\rho} + \frac{1}{2} \sum_a f'_a) (e^{-\sum_a f_a}) \right\}|_{\partial M} = -\frac{1}{2} \sum_a f'a|_{\partial M}, \]
\[ \phi_3^0 = \frac{1}{2} \left\{ (\nabla_{\partial \rho})^2 (f_M) \right\}|_{\partial M} = \frac{1}{2} \left\{ (\partial_{\rho} + \frac{1}{2} \sum_a f'_a)^2 (e^{-\sum_a f_a}) \right\}|_{\partial M} \]
\[ = \frac{1}{2} \left\{ \sum_{a,b} f'_a f'_b - \frac{1}{2} \sum_a f''_a \right\}|_{\partial M}. \]

To ensure that the Robin boundary on \( M \) takes the form \( \mathcal{B} = \partial_r + S_{R,0} \), we set

\[ S_R = S_{R,0} - \omega_{\rho}|_{\partial M} = S_{R,0} + \left\{ \frac{1}{2} \sum_a f'_a \right\}|_{\partial M}. \]

4.6.1. The coefficient of \( \sum_a f'_a \) in \( \beta_0^M \).

\[ 0 = -\frac{1}{2} \varepsilon_{R,0,1,0,2} + \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} + \frac{1}{2} \varepsilon_{R,0,1,0,2} \]

Consequently

\[ \varepsilon_{R,0,1,0,2} = -\frac{1}{2} \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} + \frac{1}{2} \varepsilon_{R,0,1,0,2} \]
\[ = \frac{1}{2} \left\{ \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} \right\} \]
\[ = \frac{1}{2} \left\{ \frac{1}{2} \varepsilon_{R,0,1,0,2} + \frac{1}{2} \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} \right\} \]

We use Lemma (4.13) to shift indices and compute:

\[ \varepsilon_{R,0,1,0,2} = \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} + \frac{1}{2} \varepsilon_{R,0,1,0,2} \]
\[ = \frac{1}{2} \left\{ \varepsilon_{R,0,1,0,2} + \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} \right\} \]
\[ = \frac{1}{2} \left\{ \varepsilon_{R,0,1,0,2} + \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} \right\} \]

4.6.2. The coefficient of \( S_{R,0} \sum_a f'_a \) in \( \beta_2^M \).

\[ 0 = \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} = \varepsilon_{R,0,1,0,2} \]

Consequently

\[ \varepsilon_{R,0,1,0,2} = \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} - \frac{1}{2} \varepsilon_{R,0,1,0,2} \]
\[ = \frac{1}{2} \left\{ \varepsilon_{R,0,1,0,2} + \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} \right\} \]
\[ = \frac{1}{2} \left\{ \varepsilon_{R,0,1,0,2} + \varepsilon_{R,0,1,0,2} - \varepsilon_{R,0,1,0,2} \right\} \]
We have determined all the unknown coefficients in Lemma 1.4. This completes

\[ \frac{1}{2} \frac{2 - 3}{2 - a} \left\{ 2 + \frac{(a_1 - 2)(a_1 - 1)}{a_1 - 2} + \frac{(a_2 - 2)(a_2 - 1)}{a_2 - 2} \right\} \varepsilon R, a_1, a_2 \\
+ \frac{1}{2} \frac{1}{(a_1 - 2)(a_2 - 1)} \left[ 2 + (a_1 - 1) + (a_2 - 1) \right] \varepsilon R, a_1, a_2 - 1 \\
= \frac{a_1 + a_2}{2 - a} \varepsilon R, a_1, a_2 + \frac{1}{2} \frac{a_1 + a_2}{(a_1 - 2)(a_2 - 1)} \varepsilon R, a_1, a_2 - 1. \]

4.6.3. The coefficients of \( \sum f_j^M \) and \( \sum (f'_a)^2 \) in \( \beta^M_2 \). We obtain the relations:

\[ 0 = -\frac{1}{4} \varepsilon R, a_1, a_2 + \frac{1}{2} \varepsilon R, a_1, a_2 - 1 \]

This shows

\[ \varepsilon^9_{R, a_1, a_2} = \frac{11}{4} \varepsilon R, a_1, a_2 = -\frac{1}{4} \varepsilon R, a_1, a_2 + \frac{1}{2} \varepsilon R, a_1, a_2 - \frac{1}{4} \varepsilon R, a_1, a_2 \\
= -\frac{1}{4} \varepsilon R, a_1, a_2 - \frac{1}{4} \varepsilon R, a_1, a_2 - 1 + \frac{1}{2} \varepsilon R, a_1, a_2 \\
= \left\{ -\frac{1}{4} \left( a_1 - 2 \right)(a_1 - 1) \right\} \varepsilon R, a_1, a_2 - 1 \\
= -\frac{1}{2} a_1^2 + a_1 + 2a_2 + 1 \varepsilon R, a_1, a_2. \]

4.6.4. The coefficient of \( \sum_{a,b} f_j^M f'_a \) in \( \beta^M_2 \).

\[ 0 = -\frac{1}{8} \varepsilon R, a_1, a_2 + \frac{1}{7} \varepsilon R, a_1, a_2 + \frac{1}{6} \varepsilon R, a_1, a_2 + \frac{1}{8} \varepsilon R, a_1, a_2 + \frac{1}{2} \varepsilon R, a_1, a_2 + \frac{1}{4} \varepsilon R, a_1, a_2 + \frac{1}{8} \varepsilon R, a_1, a_2 + \frac{1}{16} \varepsilon R, a_1, a_2 + \frac{1}{16} \varepsilon R, a_1, a_2 + \frac{1}{4} \varepsilon R, a_1, a_2 - 1 \\
= -\frac{1}{4} \varepsilon R, a_1, a_2 - 1 + \frac{1}{2} \varepsilon R, a_1, a_2 - 1 \]

This implies

\[ \varepsilon^9_{R, a_1, a_2} = -\frac{1}{2} \varepsilon R, a_1, a_2 - 1 \]

\[ \left\{ \frac{1}{8} \frac{a_1 - 2}{a_1 - 2} + \frac{a_2}{a_2 - 2} \varepsilon R, a_1, a_2 - 1 \right\} \varepsilon R, a_1, a_2 - 1 \\
= \frac{1}{2} \frac{a_1 + a_2}{a_1 - 2} \varepsilon R, a_1, a_2 - 1 \\
= \left\{ -\frac{1}{4} \left( a_1 - 2 \right)(a_1 - 1) \right\} \varepsilon R, a_1, a_2 - 1 \\
= \left\{ -\frac{1}{4} \left( a_1 - 2 \right)(a_1 - 1) \right\} \varepsilon R, a_1, a_2 - 1 \\
= \frac{1}{2} \frac{a_1 + a_2}{a_1 - 2} \varepsilon R, a_1, a_2 - 1 \\
\]

We have determined all the unknown coefficients in Lemma 1.4. This completes the proof of Theorem 1.4.
HEAT CONTENT ASYMPTOTICS

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