On strongly almost lacunary statistical $A$-convergence defined by Musielak-Orlicz function

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Abstract: We study some new strongly almost lacunary statistical $A$-convergent sequence space of order $\alpha$ defined by a Musielak-Orlicz function. We also give some inclusion relations between the newly introduced class of sequences with the spaces of strongly almost lacunary $A$-convergent sequence of order $\alpha$. Moreover we have examined some results on Musielak-Orlicz function with respect to these spaces.

Key Words: Almost convergence; Statistical convergence; Lacunary sequence; Musielak-Orlicz function; $A$-convergence.

AMS Classification No: 40A05; 40A25; 40A30; 40C05.

1 Introduction

The concept of statistical convergence was initially introduced by Fast [2], which is closely related to the concept of natural density or asymptotic density of subsets of the set of natural numbers $N$. Later on, it was studied as asummability method by Fridy [4], Fridy and Orhan [6], Freedman and Sember [3], Schoenberg [18], Malafosse and Rakočević [10] and many more mathematicians. Moreover, in recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous

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1The work of the authors was carried under the Post Doctoral Fellow under National Board of Higher Mathematics, DAE, project No. NBHM/PDF.50/2011/64
functions on locally compact spaces. Also, statistical convergence is closely related to the concept of convergence in probability.

By the concept of almost convergence, we have a sequence \( x = (x_k) \in \ell_\infty \) if its Banach limit coincides. The set \( \hat{c} \) denotes set of all almost convergent sequences. Lorentz [8] proved that,

\[
\hat{c} = \{ x \in \ell_\infty : \lim_m t_{mn}(x) \text{ exist uniformly in } n \},
\]

where

\[
t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}.
\]

Similarly, the space of strongly almost convergent sequence was defined as, \( \hat{c} = \{ x \in \ell_\infty : \lim_m t_{mn}(\|x - Le\|) \text{ exists uniformly in } n \text{ for some } L \} \), where, \( e = (1, 1, ...) \). (see Maddox [9])

A lacunary sequence is defined as an increasing integer sequence \( \theta = (k_r) \) such that \( k_0 = 0 \) and \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \).

Note: Throughout this paper, the intervals determined by \( \theta \) will be denoted by \( J_r = (k_{r-1}, k_r] \) and the ratio \( \frac{k_r}{k_{r-1}} \) will be defined by \( \phi_r \).

2 Preliminary concepts

Let \( 0 < \alpha \leq 1 \) be given. The sequence \( (x_k) \) is said to be statistically convergent of order \( \alpha \) if there is a real number \( L \) such that,

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0,
\]

for every \( \varepsilon > 0 \). In this case, we write \( S^\alpha - \lim x_k = L \). The set of all statistically convergent sequences of order \( \alpha \) will be denoted by \( S^\alpha \).

For any lacunary sequence \( \theta = (k_r) \), the space \( N_\theta \) defined as, (Freedman et al.[3])
\[ N_\theta = \left\{ (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\} . \]

The space \( N_\theta \) is a BK space with the norm,

\[ \| (x_k) \|_\theta = \sup_r h_r^{-1} \sum_{k \in I_r} |x_k|. \]

Let \( \theta = (k_r) \) be a lacunary sequence and \( 0 < \alpha \leq 1 \) be given. The sequence \( x = (x_k) \in w \) is said to be \( S_\theta^\alpha \)-statistically convergent (or lacunary statistically convergent sequence of order \( \alpha \)) if there is a real number \( L \) such that

\[ \lim_{r \to \infty} \frac{1}{h_r^\alpha} |\{ k \in I_r : |x_k - L| \geq \epsilon \}| = 0, \]

where \( I_r = (k_{r-1}, k_r] \) and \( h_r^\alpha \) denotes the \( \alpha \)-th power \((h_r)^\alpha \) of \( h_r \), that is, \( h^\alpha = (h_1^\alpha, h_2^\alpha, ... h_r^\alpha, ...) \). We write \( S_\theta^\alpha - \lim x_k = L \). The set of all \( S_\theta^\alpha \)-statistically convergent sequences will be denoted by \( S_\theta^\alpha \).

By an Orlicz function, we mean a function \( M : [0, \infty) \to [0, \infty) \), which is continuous, non-decreasing and convex with \( M(0) = 0,M(x) > 0 \), for \( x > 0 \) and \( M(x) \to \infty \), as \( x \to \infty \).

The idea of Orlicz function is used to construct the sequence space, (see Lindenstrauss and Tzafriri [7]),

\[ \ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\} . \]

This space \( \ell_M \) with the norm,

\[ \| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} \]

becomes a Banach space which is called an Orlicz sequence space.

Musielak [12] defined the concept of Musielak-Orlicz function as \( \mathcal{M} = (M_k) \). A sequence \( \mathcal{N} = (N_k) \) defined by
\[ N_k(v) = \sup \{ |v|u - M_k(u) : u \geq 0 \}, k = 1, 2, .. \]

is called the complementary function of a Musielak-Orlicz function \( \mathcal{M} \). The Musielak-Orlicz sequence space \( t_{\mathcal{M}} \) and its subspace \( h_{\mathcal{M}} \) are defined as follows:

\[ t_{\mathcal{M}} = \{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \}, \]
\[ h_{\mathcal{M}} = \{ x \in w : I_{\mathcal{M}}(cx) < \infty, \forall c > 0 \}, \]

where \( I_{\mathcal{M}} \) is a convex modular defined by,

\[ I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}. \]

It is considered \( t_{\mathcal{M}} \) equipped with the Luxemburg norm

\[ \| x \| = \inf \left\{ k > 0 : I_{\mathcal{M}} \left( \frac{x}{k} \right) \leq 1 \right\} \]

or equipped with the Orlicz norm

\[ \| x \|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}. \]

A Musielak-Orlicz function \( (M_k) \) is said to satisfy \( \Delta_2 \)-condition if there exist constants \( a, K > 0 \) and a sequence \( c = (c_k)_{k=1}^{\infty} \in \ell^1_+ \) (the positive cone of \( \ell^1 \)) such that the inequality

\[ M_k(2u) \leq KM_k(u) + c_k \]

holds for all \( k \in N \) and \( u \in R_+ \), whenever \( M_k(u) \leq a \).

If \( A = (a_{nk})_{n,k=1}^{\infty} \) is an infinite matrix, then \( Ax \) is the sequence whose nth term is given by \( A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k. \)

We consider a sequence \( x = (x_k) \) which is said to be strongly almost lacunary statistical \( A \)-convergent of order \( a \) (or \( S^a_\alpha(A, \mathcal{M}, (s)) \)-statistically convergent) if,

\[ \lim_{r \to \infty} \frac{1}{h_r^a} \left| \left\{ k \in I_r : \sum_{k \in I_r} \left( M_k \left( \frac{|f_km(A_k(x) - L)|}{\rho(k)} \right) \right)^{(\alpha_k)} \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } m, \]
where \( I_r = (k_{r-1}, k_r] \) and \( h^\alpha_r \) denotes the \( \alpha \)-th power \( (h^\alpha_r) \) of \( h_r \), that is, \( h^\alpha = (h^\alpha_1, h^\alpha_2, \ldots) \) and \( \mathcal{M} = (M_k) \) is a Musielak-Orlicz function.

Also we have introduced the space of strongly almost lacunary \( A \)-convergent sequences with respect to Musielak-Orlicz function \( \mathcal{M} = (M_k) \) as follows:

\[
\hat{N}_\theta^\alpha(A, \mathcal{M}, (s)) = \left\{ (x_k) : \lim_{r \to \infty} \frac{1}{h^\alpha_r} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} = 0, \text{ for some } L \text{ and } \rho(k) > 0 \right\}.
\]

We give some inclusion relations between the sets of \( S_\theta^\alpha(A, \mathcal{M}, (s)) \)-statistically convergent sequences and strongly almost lacunary \( A \)-convergent sequence space \( \hat{N}_\theta^\alpha(A, \mathcal{M}, (s)) \). Also some results defined by Musielak-Orlicz function are studied with respect to these sequence spaces.

### 3 Main Results

#### Theorem 3.1

Let \( \alpha, \beta \in (0, 1] \) be real numbers such that \( \alpha \leq \beta \), \( \mathcal{M} \) be a Musielak-Orlicz function and \( \theta = (k_r) \) be a lacunary sequence, then \( \hat{N}_\theta^\alpha(A, \mathcal{M}, (s)) \subset S_\theta^\beta \).

Proof: Let \( x \in \hat{N}_\theta^\alpha(A, \mathcal{M}, (s)) \).

For \( \varepsilon > 0 \) given, let us denote \( \Sigma_1 \) as the sum over \( k \in I_r, |t_{km}(A_k(x) - L)| \geq \varepsilon \) and \( \Sigma_2 \) denote the sum over \( k \in I_r, |t_{km}(A_k(x) - L)| < \varepsilon \) respectively.

As \( h^\alpha_r \leq h^\beta_r \) for each \( r \), we may write,

\[
\frac{1}{h^\alpha_r} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} = \frac{1}{h^\alpha_r} \left[ \Sigma_1 \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} + \Sigma_2 \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} \right] \geq \frac{1}{h^\alpha_r} \left[ \Sigma_1 \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} + \Sigma_2 \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k} \right] \geq \frac{1}{h^\alpha_r} \Sigma_1 \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho(k)} \right) \right]^{s_k}
\]
\[ \geq \frac{1}{h_r} \sum_{k \in I_r} \min([M_k(\varepsilon_1)]^h, [M_k(\varepsilon_1)]^H), \varepsilon_1 = \frac{\rho}{\rho^3} \]

As \( x \in \widetilde{N}_\alpha^\alpha(A, M, (s)) \), the left hand side of the above inequality tends to zero as \( r \to \infty \). Therefore, the right hand side of the above inequality tends to zero as \( r \to \infty \), hence \( x \in \hat{S}_\beta^\beta \).

**Corollary 3.2.** Let \( 0 < \alpha \leq 1 \), \( M \) be a Musielak-Orlicz function and \( \theta = (k_r) \) be a lacunary sequence, then

\[ \hat{N}_\alpha^\alpha(A, M, (s)) \subset \hat{S}_\alpha^\alpha. \]

**Theorem 3.3.** Let \( M \) be a Musielak-Orlicz function, \( x = (x_k) \) be a bounded sequence and \( \theta = (k_r) \) be a lacunary sequence. If \( \lim_{r \to \infty} \frac{h_r}{h_r^3} = 1 \), then \( x \in \hat{S}_\beta^\beta \Rightarrow x \in \hat{N}_\alpha^\alpha(A, M, (s)). \)

**Proof:** Suppose that \( x = (x_k) \) be a bounded sequence that is \( x \in \ell_\infty \) and \( \hat{S}_\beta^\beta - \lim x_k = L \).

As \( x \in \ell_\infty \), then there is a constant \( T > 0 \) such that \( |x_k| \leq T \). Given \( \varepsilon > 0 \), we have,

\[ \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right]^{\xi_k} \]

\[ \geq \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right]^{\xi_k} + \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \right]^{\xi_k} \]

\[ \leq \frac{1}{h_r} \sum_{k \in I_r} \max \left\{ [M_k \left( \frac{r}{\rho^{(k)}} \right)]^h, [M_k \left( \frac{r}{\rho^{(k)}} \right)]^H \right\} + \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{r}{\rho^{(k)}} \right) \right]^{\xi_k} \]

\[ \leq \max \{ [M_k(K)]^h, (M_k(K)]^H \} + \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{r}{\rho^{(k)}} \right) \right]^{\xi_k} \]

\[ \leq K, \frac{r}{\rho^{(k)}} = \varepsilon \]

Hence, \( x \in \widetilde{N}_\alpha^\alpha(A, M, (s)). \)
Theorem 3.4. If \( \lim s_k > 0 \) and \( x = (x_k) \) is strongly \( \hat{N}^\alpha_\theta(A, \mathcal{M}, (s)) \)-summable to \( L \) with respect to the Musielak-Orlicz function \( \mathcal{M} \), then \( \hat{N}^\alpha_\theta(A, \mathcal{M}, (s)) - \lim x_k \) is unique.

Proof: Let \( \lim s_k = s > 0 \). Suppose that \( \hat{N}^\alpha_\theta(A, \mathcal{M}, (s)) - \lim x_k = L \), and \( \hat{N}^\alpha_\theta(A, \mathcal{M}, (s)) - \lim x_k = L_1 \). Then,

\[
\lim_{r \to \infty} \frac{1}{h^r} \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} = 0, \text{ for some } \rho_1^{(k)} > 0
\]

and

\[
\lim_{r \to \infty} \frac{1}{h^r} \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} = 0, \text{ for some } \rho_2^{(k)} > 0.
\]

Define \( \rho^{(k)} = \max(2\rho_1^{(k)}, 2\rho_2^{(k)}) \). As \( \mathcal{M} \) is nondecreasing and convex, we have,

\[
\frac{1}{h^r} \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k} \leq \frac{D}{h^r} \sum_{k \in \mathcal{I}} \frac{1}{2^{s_k}} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} + \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} \leq \frac{D}{h^r} \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_1^{(k)}} \right) \right]^{s_k} + D \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho_2^{(k)}} \right) \right]^{s_k} \to 0, (r \to \infty),
\]

where \( \sup_{k} s_k = H \) and \( D = \max(1, 2^{H-1}) \). Hence,

\[
\lim_{r \to \infty} \frac{1}{h^r} \sum_{k \in \mathcal{I}} \left[ M_k \left( \frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k} = 0.
\]

As \( \lim s_k = s \), we have,

\[
\lim_{k \to \infty} \left[ M_k \left( \frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s_k} = \left[ M_k \left( \frac{|L - L_1|}{\rho^{(k)}} \right) \right]^{s}
\]

and so \( L = L_1 \). Thus the limit is unique.

Theorem 3.5. Let \( A = (a_{mk}) \) be an infinite matrix of complex numbers and let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function satisfying \( \Delta_2 \)-condition. If \( x \)
is strongly almost lacunary $A$-convergent sequences with respect to $\mathcal{M}$, then $\hat{N}_\theta^a(A) = \hat{N}_\theta^a(A, \mathcal{M})$.

Proof: Let $x \in \hat{N}_\theta^a(A)$.

Then, \( \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |t_{km}(A(x) - L)| = 0 \), uniformly in $m$.

Let us define two sequences $y$ and $z$ such that,

$$\left( |t_{km}(A_k(y) - L)| \right) = \begin{cases} 
|t_{km}(A_k(x) - L)| & \text{if } |t_{km}(A_k(x) - L)| > 1; \\
\theta & \text{if } |t_{km}(A_k(x) - L)| \leq 1.
\end{cases}$$

$$\left( |t_{km}(A_k(z) - L)| \right) = \begin{cases} 
\theta & \text{if } |t_{km}(A_k(x) - L)| > 1; \\
|t_{km}(A_k(x) - L)| & \text{if } |t_{km}(A_k(x) - L)| \leq 1.
\end{cases}$$

Hence, \( |t_{km}(A_k(x) - L)| = |t_{km}(A_k(y) - L)| + |t_{km}(A_k(z) - L)| \).

Also, \( |t_{km}(A_k(y) - L)| \leq |t_{km}(A_k(x) - L)| \) and \( |t_{km}(A_k(z) - L)| \leq |t_{km}(A_k(x) - L)| \).

Since, $\hat{N}_\theta^a(A)$ is normal, so we have $y, z \in \hat{N}_\theta^a(A)$.

Let $\sup_k M_k(2) = T$

Then,

$$\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right)$$

$$= \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(y) - L)| + |t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right)$$

$$\leq \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2} M_k \left( \frac{2|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) + \frac{1}{2} M_k \left( \frac{2|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) \right]$$

$$< \frac{1}{2} \frac{1}{h_r} \sum_{k \in I_r} K_1 \left( \frac{|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) M_k(2) + \frac{1}{2} \frac{1}{h_r^2} \sum_{k \in I_r} K_2 \left( \frac{|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) M_k(2)$$
\[
\leq \frac{1}{2} h_r \sum_{k \in I_r} K_1 \left( \frac{|t_{km}(A_k(y) - L)|}{\rho^{(k)}} \right) \sup M_k(2) + \frac{1}{2} h_r \sum_{k \in I_r} K_2 \left( \frac{|t_{km}(A_k(z) - L)|}{\rho^{(k)}} \right) \sup M_k(2)
\]

\[\to 0 \text{ as } r \to \infty.\]

Hence \( x \in \hat{N}_\theta^a(A, \mathcal{M}). \) This completes the proof.

**Theorem 3.6.** Let \( A = (a_{mk}) \) be an infinite matrix of complex numbers and let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function satisfying \( \Delta_2 \)-condition. If

\[
\liminf_{\nu \to \infty} k \frac{M_k \left( \frac{\nu}{\rho^{(k)}} \right)}{\nu} > 0, \text{ for some } \rho^{(k)} > 0,
\]

then, \( \hat{N}_\theta^a(A) = \hat{N}_\theta^a(A, \mathcal{M}). \)

Proof: If \( \hat{N}_\theta^a(A) = \hat{N}_\theta^a(A, \mathcal{M}) \) for some \( \rho^{(k)} > 0 \), then there exists a number \( \gamma > 0 \) such that

\[
M_k \left( \frac{\nu}{\rho^{(k)}} \right) \geq \gamma \left( \frac{\nu}{\rho^{(k)}} \right), \forall \nu > 0, \text{ and some } \rho^{(k)} > 0.
\]

Let \( x \in \hat{N}_\theta^a(A, \mathcal{M}) \). Then,

\[
\frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \geq \frac{1}{h_r} \sum_{k \in I_r} \nu \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right) \geq \gamma \frac{1}{h_r} \sum_{k \in I_r} \left( \frac{|t_{km}(A_k(x) - L)|}{\rho^{(k)}} \right)
\]

Hence, \( x \in \hat{N}_\theta^a(A) \). This completes the proof.

**Theorem 3.7.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function where \( (M_k) \) is pointwise convergent. Then, \( \hat{N}_\theta^a(A, \mathcal{M}, (s)) \subset \hat{S}_\theta^a(A, \mathcal{M}, (s)) \) if and only if

\[
\lim_k M_k \left( \frac{\nu}{\rho^{(k)}} \right) > 0 \text{ for some } \nu > 0, \rho^{(k)} > 0.
\]

Proof: Let \( \epsilon > 0 \) and \( x \in \hat{N}_\theta^a(A, \mathcal{M}, (s)). \)

Also, if \( \lim_k M_k \left( \frac{\nu}{\rho^{(k)}} \right) > 0 \), then there exists a number \( c > 0 \) such that
\[ M_k \left( \frac{\nu}{\rho^{(k)}} \right) \geq c, \text{ for } \nu > \varepsilon. \]

Let us consider, \( I_r^1 = \{ i \in I_r : \left[ M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho^{(k)}} \right) \right] \geq \varepsilon \} \).

Then,
\[
\frac{1}{h_r^s} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho^{(k)}} \right) \right]^{s_k} \geq \frac{1}{h_r^s} \sum_{k \in I_r^1} \left[ M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho^{(k)}} \right) \right]^{s_k} \geq c \frac{1}{h_r^s} |t_{km}(A_0(\varepsilon))|
\]

Hence, it follows that \( x \in \hat{S}_\theta^a(A, \mathcal{M}, (s)) \).

Conversely, let us assume that the condition does not hold good. For a number \( \nu > 0 \), let \( \lim_{k} M_k \left( \frac{\nu}{\rho^{(k)}} \right) = 0 \) for some \( \rho > 0 \). Now, we select a lacunary sequence \( \theta = (n_r) \) such that \( M_k \left( \frac{1}{\rho^{(k)}} \right) < 2^{-r} \) for any \( k > n_r \).

Let \( A = I \) and define a sequence \( x \) by putting,
\[
A_k(x) = \begin{cases} \nu & \text{if } n_{r-1} < k \leq \frac{n_r + n_{r-1}}{2}; \\ \theta & \text{if } \frac{n_r + n_{r-1}}{2} < k \leq n_r. \end{cases}
\]

Therefore,
\[
\frac{1}{h_r^s} \sum_{k \in I_r} \left[ M_k \left( \frac{|A_k(x)|}{\rho^{(k)}} \right) \right]^{s_k} = \frac{1}{h_r^s} \sum_{n_{r-1} < k \leq \frac{n_r + n_{r-1}}{2}} M_k \left( \frac{\nu}{\rho^{(k)}} \right) < \frac{1}{h_r^s} \frac{1}{2^{r-1}} \left[ \frac{n_r + n_{r-1}}{2} - n_{r-1} \right] = \frac{1}{2^r} \rightarrow 0 \text{ as } r \rightarrow \infty.
\]

Thus we have \( x \in \hat{N}_\theta^a(A, \mathcal{M}, (s)) \).
But,
\[
\lim_{r \to \infty} \frac{1}{h_r^n} \left\{ \{ k \in I_r : \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A(x))|}{\rho(k)} \right) \right] \} \right\} = \lim_{r \to \infty} \frac{1}{h_r^n} \left\{ \{ k \in (n_r - n_{r-1}) : \sum_{k \in I_r} \left[ M_k \left( \frac{v}{\rho(k)} \right) \right] \} \right\}
\]
\[
= \lim_{r \to \infty} \frac{1}{h_r^n} n_r = \frac{1}{2}.
\]

So, \( x \notin \hat{S}_0(A, \mathcal{M}, (s)) \).

**Theorem 3.8.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function. Then \( \hat{S}_0^\alpha(A, \mathcal{M}, (s)) \subset N_0^\alpha(A, \mathcal{M}, (s)) \) if and only if \( \sup_v \sup_k M_k \left( \frac{v}{\rho(k)} \right) < \infty \).

Proof: Let \( x \in \hat{S}_0^\alpha(A, \mathcal{M}, (s)) \). Suppose \( h(v) = \sup_k M_k \left( \frac{v}{\rho(k)} \right) \) and \( h = \sup_v h(v) \). Let
\[
I_r^2 = \left\{ k \in I_r : M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho(k)} \right) < \varepsilon \right\}.
\]

Now, \( M_k(v) \leq h \) for all \( k, v > 0 \). So,
\[
\frac{1}{h_r^n} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho(k)} \right) \right] = \frac{1}{h_r^n} \sum_{k \in I_r} \left[ M_k \left( \frac{|t_{km}(A(x) - L)|}{\rho(k)} \right) \right] \leq h \frac{1}{h_r^n} |t_{km}(A_0(\varepsilon))| + h(\varepsilon).
\]

Hence, as \( \varepsilon \to 0 \), it follows that \( x \in N_0^\alpha(A, \mathcal{M}, (s)) \).

Conversely, suppose that
\[
\sup_v \sup_k M_k \left( \frac{v}{\rho(k)} \right) = \infty.
\]

Then, we have
\[
0 < v_1 < v_2 < ... < v_{r-1} < v_r < ...
\]
so that $M_n \left( \frac{v_r}{p_k} \right) \geq h_r^\alpha$ for $r \geq 1$. Let $A = I$. We set a sequence $x = (x_k)$ by,

$$A_k(x) = \begin{cases} v_r & \text{if } k = n_r \text{ for some } r = 1, 2, ..; \\ \theta & \text{otherwise.} \end{cases}$$

Then,

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \left\{ \left\lfloor \frac{\sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(x))|}{\rho(k)} \right)^{s_k}}{\varepsilon} \right\rfloor \right\} = \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(x))|}{\rho(k)} \right)^{s_k} \geq \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} M_k \left( \frac{|v_r - L|}{\rho(k)} \right) \geq \lim_{r \to \infty} \frac{1}{h_r^\alpha} = 1$$

Hence, $x \in \hat{S}_0^\alpha(A, \mathcal{M}, (s))$.

But,

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} M_k \left( \frac{|t_{km}(A_k(x)) - L|}{\rho(k)} \right) = \lim_{r \to \infty} \frac{1}{h_r^\alpha} \left[ M_n \left( \frac{|v_r - L|}{\rho(k)} \right) \right] \geq \lim_{r \to \infty} \frac{1}{h_r^\alpha} = 1$$

So, $x \in \hat{N}_0^\alpha(A, \mathcal{M}, (s))$.

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