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Résumé. En construisant des exemples explicites, nous montrons que la méthode de Quebbemann permet d’obtenir de nombreuses classes d’isomorphisme de réseaux extrémaux de dimension 64. Beaucoup de ces exemples n’ont pas d’automorphismes non triviaux.

Abstract. By constructing explicit examples, we show that the method of Quebbemann yields many isomorphism classes of extremal lattices of rank 64. Many of these examples have no non-trivial automorphisms.

1. Introduction

By a lattice, we mean an integral positive-definite lattice. Let \( L \) be an even unimodular lattice with the bilinear form \( \langle \cdot, \cdot \rangle_L : L \times L \rightarrow \mathbb{Z} \). We put
\[
\min(L) := \min\{ \langle x, x \rangle_L \mid x \in L, \ x \neq 0 \}.
\]
It is well-known that the rank \( n \) of \( L \) is divisible by 8, and that \( \min(L) \) satisfies
\[
\min(L) \leq 2 + 2 \left\lfloor \frac{n}{24} \right\rfloor.
\]
We say that an even unimodular lattice \( L \) of rank \( n \) is extremal if the equality holds in (1.1). Extremal lattices are an important research subject, because they give rise to sphere packings of high density.

Not so many explicit examples of extremal lattices are known. Moreover, since the construction of these examples involves very special algebraic objects, each of the known examples has a large automorphism group. For example, the automorphism group \( \text{O}(\Lambda) \) of the Leech lattice \( \Lambda \) is of order \( 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 \).

On the other hand, it was shown in [8] that the number of isomorphism classes of extremal lattices of rank 40 is \( > 8.45 \times 10^{51} \). Since this bound was proved by means of a mass formula, we do not obtain explicit examples of extremal lattices of rank 40 from this result.

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We consider extremal lattices of rank 64. Quebbemann [10] gave a method to construct extremal lattices $Q$ of rank 64 from certain ternary codes $B$. We call an extremal lattice of rank 64 a Quebbemann lattice if it is obtained by (a generalization of) this method. See Section 2 for the precise definition. A remarkable property of Quebbemann’s construction is that the condition on the ternary code $B$ required in order for the lattice $Q$ to be extremal is an open condition. Therefore we expect that, by generating sufficiently general ternary codes $B$, we obtain many extremal lattices of rank 64. Another nice feature of this construction is that we can calculate the set $\text{Min}(Q)$ of non-zero minimal-norm vectors of a Quebbemann lattice $Q$ (that is, the set of vectors $v$ with $\langle v, v \rangle_Q = 6$). It turns out that, by means of $\text{Min}(Q)$, it is possible to compute the automorphism group of $Q$, and to compare the isomorphism class of $Q$ with isomorphism classes of other Quebbemann lattices.

The purpose of this note is to generalize Quebbemann’s construction slightly, and to show that this method indeed yields many mutually non-isomorphic extremal lattices of rank 64, by choosing the ternary code $B$ (pseudo-)randomly, and that their automorphism groups are often very small. Our main result below is proved by producing Quebbemann lattices $Q$ explicitly.

**Theorem 1.1.** Quebbemann’s method yields

1. at least 300 isomorphism classes of extremal lattices $Q$ of rank 64 such that $O(Q) = \{\pm 1\}$, and
2. at least 100 isomorphism classes of extremal lattices $Q$ of rank 64 such that $O(Q) \cong \{\pm 1\} \times \mathbb{Z}/8\mathbb{Z}$.

See Section 2 for the method to produce Quebbemann lattices, Section 3.1 for a method to enumerate minimal-norm vectors, Section 3.2 for a method to distinguish isomorphism classes, and Section 3.3 for the computation of the automorphism groups. We exhibit a few examples in detail in Section 4. The computation data of a part of the lattices in Theorem 1.1 is available from the author’s web-page [12]. (The whole data is too large to be put on a website.) These data are written in the Record format of GAP [3].

In Chapter 8.3(d) of [2], Conway and Sloane constructed a Quebbemann lattice that is different from the one given in Quebbemann’s original paper [10], and suggested that there exist several isomorphism classes of Quebbemann lattices. In [9], Quebbemann showed by means of a mass formula that there exist at least two isomorphism classes. The ease with which we can make non-isomorphic Quebbemann lattices suggests that the number of isomorphism classes is very huge.

Unimodular lattices with no non-trivial automorphisms have been studied by many authors since the work of Bannai [1]. In [6], an even unimodular
Quebbemann’s extremal lattices

Figure 2.1. Dynkin diagram of type $E_8$

lattice of rank 64 without non-trivial automorphisms is constructed. This lattice is, however, not extremal.

In [7], Nebe constructed an extremal lattice of rank 64 by a different method. The order of the automorphism group is at least 587520. In [4] and [5], the existence of extremal Type II $\mathbb{Z}_{2^k}$-codes of length 64 was shown. The isomorphism classes and the automorphism groups of the associated extremal lattices are, however, not clear. In [11], another extremal lattice of rank 64 was constructed by means of a generalized quadratic residue code. Its automorphism group is of order 119040.

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Conventions. Elements of a vector space or a lattice are written as row vectors. For a lattice $L$ or a quadratic space $L$, we denote by $\langle \cdot, \cdot \rangle_L$ the symmetric bilinear form on $L$. The automorphism group $O(L)$ of $(L, \langle \cdot, \cdot \rangle_L)$ acts on $L$ from the right.

2. Quebbemann lattice

2.1. Quebbemann’s construction. We recall Quebbemann’s construction [10] of extremal lattices of rank 64. See also Chapter 8.3(d) of [2]. In fact, our construction below is slightly more general than Quebbemann’s original.

Let $E$ be the root lattice of type $E_8$, that is, $E$ is the lattice of rank 8 generated by vectors $e_1, \ldots, e_8$ with $\langle e_i, e_i \rangle_E = 2$ that form the dual graph in Figure 2.1. It is well-known that $E$ is unimodular.

We consider the $\mathbb{F}_3$-quadratic space $U := E/3E$. A subspace $V$ of $U$ is said to be maximal isotropic if $\dim V = 4$ and $\langle v, v \rangle_U = 0$ holds for all $v \in V$. There exists a direct sum decomposition

$$ (2.1) \quad U = V \oplus W, \quad \text{where } V \text{ and } W \text{ are maximal isotropic subspaces}. $$

Let $\mathcal{D}$ be the set of ordered pairs $(V, W)$ of maximal isotropic subspaces of $U$ satisfying $V \cap W = 0$. Since $\langle \cdot, \cdot \rangle_U$ is non-degenerate, we have a natural isomorphism

$$ (2.2) \quad W \cong \text{Hom}(V, \mathbb{F}_3) \quad \text{for each } (V, W) \in \mathcal{D}. $$
Let $S$ denote the orthogonal direct sum $E^8$ of eight copies of $E$. Then $S$ is unimodular of rank 64. For $i = 1, \ldots, 8$, we denote by $U_i$ the $i^{\text{th}}$-factor of $S/3S = U^8$. We choose and fix an element

$$
\Delta := ((V_1, W_1), \ldots, (V_8, W_8))
$$

of $D^8$. We put $T := \{1, \ldots, 8\}$, and for a subset $J$ of $T$, we put

$$
U_J := \bigoplus_{j \in J} U_j, \quad V_J := \bigoplus_{j \in J} V_j, \quad W_J := \bigoplus_{j \in J} W_j.
$$

Then we have

$$
S/3S = U_T = V_T \oplus W_T.
$$

We consider $U_T$, $V_T$ and $W_T$ as $\mathbb{F}_3$-vector spaces. Let $B$ be a linear subspace of $V_T$ with dim $B = 8$. Note that $W_T$ can be regarded as the dual space of $V_T$ by (2.2). We put

$$
B^\perp := \{z \in W_T \mid \langle z, g \rangle_{U_T} = 0 \text{ for all } g \in B\}.
$$

Then we have dim $B^\perp = 24$. Let $\pi: S \to S/3S = U_T$ denote the natural projection. For any elements $\pi, \pi'$ of $B \oplus B^\perp \subset U_T$, we have $\langle \pi, \pi' \rangle_{U_T} = 0$, and hence, for any elements $x, x'$ of the pre-image $\pi^{-1}(B \oplus B^\perp)$, we have $\langle x, x' \rangle_S \equiv 0 \mod 3$. We denote by $Q(\Delta, B)$ the lattice whose underlying $\mathbb{Z}$-module is $\pi^{-1}(B \oplus B^\perp)$ and whose bilinear form $\langle \cdot, \cdot \rangle_Q$ is given by

$$
\langle \cdot, \cdot \rangle_Q := \frac{1}{3} \langle \cdot, \cdot \rangle_S.
$$

Then $Q := Q(\Delta, B)$ is an even lattice, and we have

$$
\det Q = \left(\frac{1}{3}\right)^{64} \det S \cdot [S : Q]^2 = \left(\frac{1}{3}\right)^{64} \left(\frac{|U_T|}{|B \oplus B^\perp|}\right)^2 = 1.
$$

**Definition 2.1.** For $J = \{i, j\} \subset T$ with $|J| = 2$, we denote by $p_J: B \to V_J$ the projection to the $J$-factor. We say that $B$ satisfies $p_2$-condition if $p_J$ is an isomorphism for all $J \subset T$ with $|J| = 2$.

**Remark 2.2.** If the projection $p_J: B \to V_J$ is an isomorphism, then the projection $p_{T \setminus J}: B^\perp \to W_{T \setminus J}$ to the $(T \setminus J)$-factor is also an isomorphism. Hence, if $B$ satisfies $p_2$-condition, then $B^\perp$ satisfies the following $p_6$-condition: for all $J' \subset T$ with $|J'| = 6$, the projection $p_{J'}: B^\perp \to W_{J'}$ to the $J'$-factor is an isomorphism.

It is obvious that $p_2$-condition imposes an open condition on the Grassmannian variety of 8-dimensional subspaces $B$ of $V_T$.

**Proposition 2.3** (Quebbemann [10]). Let $U_T = V_T \oplus W_T$ be the decomposition of $U_T = S/3S$ associated with an element $\Delta$ of $D^8$, and let $B$ be an 8-dimensional subspace of $V_T$. Suppose that $B$ satisfies $p_2$-condition. Then $\min(Q(\Delta, B)) = 6$ holds, that is, $Q(\Delta, B)$ is an extremal lattice of rank 64.
Proof. We write $x \in Q(\Delta, B)$ as $x = (x_1, \ldots, x_8)$, where $x_i \in E$ is the $i^{th}$ component by the embedding $Q(\Delta, B) \hookrightarrow S = E^8$. We put

$$x := \pi(x) = (x_1, \ldots, x_8) \in B \oplus B^\perp \subset U_T,$$

where $x_i \in U_i$ is $x_i \mod 3E$. Decomposing each $x_i$ as $y_i + z_i$ with $y_i \in V_i$ and $z_i \in W_i$, we obtain $x = y + z$, where

$$y := (y_1, \ldots, y_8) \in B, \quad z := (z_1, \ldots, z_8) \in B^\perp.$$

Suppose that $\langle x, x \rangle_Q \leq 4$. We show that $x = 0$. Since

$$\langle x, x \rangle_S = \sum \langle x_i, x_i \rangle_E \leq 12,$$

we see that at least two of the components $x_i$ are zero. Since at least two of $y_i$ are zero, the assumption that $B$ satisfy $p_2$-condition implies $y = 0$. Therefore we have $x = z$. In particular, each $x_i$ belongs to $W_i$. Since $W_i$ is isotropic in $U = E/3E$, we have $\langle x_i, x_i \rangle_E \equiv 0 \mod 3$ and hence $\langle x_i, x_i \rangle_E \equiv 0 \mod 6$. Combining this with (2.3), we see that at most two of $x_i$ are nonzero. Since $B^\perp$ satisfies $p_6$-condition by Remark 2.2, we see that $z = 0$. Therefore $x = 0$, and hence $x \in 3S$. If $x$ were non-zero, we would have $\langle x, x \rangle_S \geq 18$, which is a contradiction. □

**Definition 2.4.** An extremal lattice of rank 64 of the form $Q(\Delta, B)$, where $\Delta$ is an element of $D^8$ and $B$ is an 8-dimensional subspace of $V_T$ satisfying $p_2$-condition, is called a Quebbemann lattice.

### 2.2. Maximal isotropic subspaces of $U$.

Recall that the lattice $E$ is equipped with a basis $e_1, \ldots, e_8$ in Figure 2.1. We write elements of $E$ or of $U = E/3E$ as row vectors with respect to $e_1, \ldots, e_8$. The automorphism group $O(E)$ of $E$ is generated by the reflections

$$x \mapsto x - \langle x, e_i \rangle_E e_i$$

with respect to the vectors $e_i$ ($i = 1, \ldots, 8$) of norm 2, and the order of $O(E)$ is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. The natural homomorphism $O(E) \to O(U)$ to the automorphism group of the $F_3$-quadratic space $U$ is injective. Let $\mathcal{V}$ be the set of maximal isotropic subspaces $V$ of $U$. We can prove the following by the standard orbit-stabilizer algorithm using GAP [3].

**Proposition 2.5.** The size of $\mathcal{V}$ is 2240, and $O(E)$ acts transitively on the set $\mathcal{V}$.

Let $V_0 \in \mathcal{V}$ be the maximal isotropic subspace with basis $v_1, \ldots, v_4$ in Table 2.1. The stabilizer subgroup $\text{Stab}(V_0)$ of $V_0$ in $O(E)$ is of order $2^8 \cdot 3^5 \cdot 5$. Let $\mathcal{W}(V_0)$ be the set of all $W \in \mathcal{V}$ such that $V_0 \cap W = 0$. We have $|\mathcal{W}(V_0)| = 729$. 


Table 2.1. Bases of maximal isotropic subspaces $V_0, W^I, W^{II}$

$v_1 = (0, 0, 0, 0, 0, 1, 2), \quad v_2 = (0, 0, 0, 1, 2, 0, 0, 0),$
$v_3 = (0, 1, 0, 0, 1, 0, 0, 1), \quad v_4 = (1, 0, 2, 0, 0, 0, 0, 1).
$v_1^I = (1, 2, 1, 0, 2, 2, 0, 2), \quad v_2^I = (0, 1, 2, 0, 0, 0, 0, 0),$
$v_3^I = (2, 2, 1, 0, 1, 0, 2, 2), \quad v_4^I = (1, 0, 2, 0, 1, 0, 2, 1).
$v_1^{II} = (1, 2, 1, 0, 2, 2, 0, 2), \quad v_2^{II} = (1, 1, 1, 0, 0, 0, 0, 1),$
$v_3^{II} = (0, 2, 0, 0, 1, 0, 2, 0), \quad v_4^{II} = (1, 2, 2, 2, 1, 0, 2, 0).

**Proposition 2.6.** The action of $\text{Stab}(V_0)$ decomposes $W(V_0)$ into two orbits of size 648 and 81. The orbit of size 648 contains $W^I$ with basis $v_1^I, \ldots, v_4^I$ in Table 2.1, and the orbit of size 81 contains $W^{II}$ with basis $v_1^{II}, \ldots, v_4^{II}$ in Table 2.1.

**Remark 2.7.** The basis $v_1^*, \ldots, v_4^*$ of $W$ above is dual to the basis $v_1, \ldots, v_4$ of $V_0$ by the canonical pairing (2.2).

**Corollary 2.8.** The action of $O(E)$ decomposes $D$ into two orbits. One orbit contains $(V_0, W^I)$ with the stabilizer subgroup $G^I$ of order 480, and the other orbit contains $(V_0, W^{II})$ with the stabilizer subgroup $G^{II}$ of order 3840.

2.3. Construction of $Q(\Delta, B)$ with an automorphism of order 8.
We fix a pair $(V, W) \in D$, and consider the 8-tuple

$$\Delta_0 := ((V, W), \ldots, (V, W)) \in D^8.$$  

Let $G$ be the stabilizer subgroup of $(V, W)$ in $O(E)$, and let $\gamma$ be an element of order 8 in $G$. (The stabilizer subgroup $G^I$ (resp. $G^{II}$) in Corollary 2.8 contains 120 elements (resp. 1360 elements) of order 8.) We define an automorphism $\tilde{\gamma}$ of $S = E^8$ by

$$x = (x_1, \ldots, x_8) \mapsto x^{\tilde{\gamma}} = (x_2^{\gamma}, \ldots, x_8^{\gamma}, x_1^{\gamma}).$$

The action of $\tilde{\gamma}$ on $S/3S = U_T$ preserves the decomposition $U_T = V_T + W_T$ associated with $\Delta_0$ above. For $v \in V_T = V^8$, we denote by $B(\gamma, v)$ the linear subspace of $V_T$ generated by the orbit of $v$ under the action of $\langle \tilde{\gamma} \rangle \cong \mathbb{Z}/8\mathbb{Z}$. If $B(\gamma, v)$ is of dimension 8 and satisfies $p_2$-condition, then $Q(\Delta_0, B(\gamma, v))$ is a Quebbemann lattice invariant under the action of $\langle \tilde{\gamma} \rangle$ on $S$. In particular, the automorphism group $O(Q)$ of $Q := Q(\Delta_0, B(\gamma, v))$ contains an element $\tilde{\gamma}(Q) := \tilde{\gamma}|Q$ of order 8.
3. Computations on Quebbemann lattices

We fix an 8-tuple $\Delta = ((V_1, W_1), \ldots, (V_8, W_8)) \in \mathcal{D}^8$. Let $B$ be an 8-dimensional linear subspace of $V_T = V_1 \oplus \cdots \oplus V_8$ satisfying p2-condition, and we consider the extremal lattice $Q(\Delta, B)$ of rank 64.

3.1. Enumeration of minimal-norm vectors. In this section, we explain a method to calculate the set $\text{Min}(Q(\Delta, B)) := \{ x \in Q \mid \langle x, x \rangle_Q = 6 \}$, which is the set of all minimal-norm vectors of $Q := Q(\Delta, B)$. A norm-type is an 8-tuple $[n_1, \ldots, n_8]$ of non-negative even integers $n_i$ such that $\sum n_i = 18$. For $x = (x_1, \ldots, x_8) \in \text{Min}(Q(\Delta, B))$, we put

$$\nu(x) := [\langle x_1, x_1 \rangle_E, \ldots, \langle x_8, x_8 \rangle_E],$$

and call it the norm-type of $x$. For a non-negative even integer $a$, we put $N(a, E) := \{ v \in E \mid \langle v, v \rangle_E = a \}$, and let $N(a, U) \subset U$ be the image of $N(a, E)$ by the natural projection $E \to U$. (See Table 3.1.) A minimal-norm vector $x \in \text{Min}(Q(\Delta, B))$ is said to be of divisible type if $x \in 3S$ holds, or equivalently, if only one of $x_1, \ldots, x_8$ (say $x_i$) is non-zero and $x_i$ is written as $3x'_i$ by some $x'_i \in N(2, E)$, or equivalently, if the norm-type $\nu(x)$ of $x$ is obtained by a permutation of components from $[0, \ldots, 0, 18]$. Since $|N(2, E)| = 240$, there exist exactly $240 \times 8$ minimal-norm vectors of divisible type.

**Proposition 3.1.** Let $x \in Q(\Delta, B)$ be a minimal-norm vector that is not of divisible type. Then the norm-type $\nu(x)$ of $x$ is obtained by a permutation of components from one of the following:

$$[0, 0, 0, 0, 6, 6, 6] \quad \text{(of type } 0^56^3),$$

$$[0, 2, 2, 2, 2, 2, 4, 4] \quad \text{(of type } 0^12^54^2),$$

$$[0, 2, 2, 2, 2, 2, 6] \quad \text{(of type } 0^12^66^1),$$

$$[2, 2, 2, 2, 2, 2, 2, 4] \quad \text{(of type } 2^74^1).$$

**Proof.** As in the proof of Proposition 2.3, we see that $\bar{x} := \pi(x) \in B \oplus B^\perp$ is decomposed uniquely as $\bar{y} + \bar{z}$, where $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_8) \in B$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_8) \in B^\perp$. Suppose that $\bar{y} = 0$. Since $x$ is not of divisible type, we
see that $\mathbf{z} = \mathbf{z}$ is not zero. Since $B^\perp$ satisfies $p_6$-condition, at most five of $z_1, \ldots, z_8$ are zero. Since $\mathbf{x}_i = z_i \in W_i$, we have $\langle x_i, x_i \rangle_E \equiv 0 \mod 3$ and hence $\langle x_i, x_i \rangle_E \in \{0, 6, 12, 18\}$. Combining these, we see that $\nu(x)$ is of type $0^56^3$. Suppose that $y \neq 0$. Since $B$ satisfies $p_2$-condition, at most one of $y_1, \ldots, y_8$ is zero. Hence at most one of $x_1, \ldots, x_8$ is zero. Therefore $\nu(x)$ is either of type $0^12^54^2$ or $0^12^66^1$ or $2^74^1$.

For $k = 1, \ldots, 8$, let $e_1^{(k)}, \ldots, e_8^{(k)}$ be the copy of the basis $e_1, \ldots, e_8$ of $E$ in the $k$th factor of $S = E^8$. We use the ordered set

\[(3.2) \quad e_1^{(1)}, \ldots, e_8^{(1)}, e_1^{(2)}, \ldots, e_8^{(2)}, \ldots, e_1^{(8)}, \ldots, e_8^{(8)}\]

of vectors as a basis of $S$ and of $S/3S = U^8 = U_T$.

**Proposition 3.2.** The ternary code $B \oplus B^\perp \subset U_T$ is generated by row vectors of a $32 \times 64$ matrix of the echelon form as in Figure 3.1, where $I_8$ is the identity matrix of size 8, $A_i$ are $4 \times 8$ matrices whose row vectors form a basis of $W_i \subset U$ for $i = 3, \ldots, 6$, $C_{\mu \nu}$ are some $8 \times 8$ matrices, and the blank blocks are zero matrices.

**Proof.** Since the projection $p_{12}: B \to V_1 \oplus V_2$ to the $(12)$-factor and the projection $p_{\overline{78}}: B^\perp \to W_1 \oplus \cdots \oplus W_6$ to the $(123456)$-factor are both isomorphisms, the projection

\[P_{12}: B \oplus B^\perp \to U \oplus U\]

to the $(12)$-factor is surjective, and its kernel $\text{Ker} P_{12}$ is mapped isomorphically to $W_3 \oplus \cdots \oplus W_6$ by the projection

\[P_{3456}: \text{Ker} P_{12} \to U \oplus U \oplus U \oplus U\]

to the $(3456)$-factor. □
We make the symmetric group $\mathcal{S}_8$ act on $S = E^8$ by
\[(x_1, \ldots, x_8)^\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(8)}) \quad \text{for } \sigma \in \mathcal{S}_8.\]

For $\Delta = ((V_1, W_1), \ldots, (V_8, W_8)) \in \mathcal{D}^8$, we put
\[\Delta^\sigma := ((V_{\sigma(1)}, W_{\sigma(1)}), \ldots, (V_{\sigma(8)}, W_{\sigma(8)})).\]

Then we have $Q(\Delta, B)^\sigma = Q(\Delta^\sigma, B^\sigma)$ in $S$. If $x \in \text{Min}(Q(\Delta, B))$ is of norm-type $[n_1, \ldots, n_8]$, then $x^\sigma \in \text{Min}(Q(\Delta^\sigma, B^\sigma))$ is of norm-type $[n_{\sigma(1)}, \ldots, n_{\sigma(8)}]$.

Let $n = [n_1, \ldots, n_8]$ be a norm-type obtained by a permutation of components from one of the norm-types in (3.1). We calculate the set $\mathcal{M}(n)$ of codewords $\bar{x} = \pi(x) \in B \oplus B^\perp$ corresponding $x \in \text{Min}(Q(\Delta, B))$ with $\nu(x) = n$ by the following method.

First we choose a permutation $\tau \in \mathcal{S}_8$ such that $n^\tau = [n_{\tau(1)}, \ldots, n_{\tau(8)}]$ satisfies
\[n_{\tau(1)} \leq n_{\tau(2)} \leq \cdots \leq n_{\tau(8)}.\]

We then transform a generator matrix of $(B \oplus B^\perp)^\tau = B^\tau \oplus B^\tau^\perp$ into the echelon form in Figure 3.1, and search for $\bar{x}_1, \ldots, \bar{x}_8 \in U$ satisfying conditions (3.3) below by back track search; that is, if we find $(\bar{x}_1, \ldots, \bar{x}_i)$ satisfying the first $i$ conditions of (3.3), then we search for $\bar{x}_{i+1}$ satisfying the $(i+1)^{st}$ condition of (3.3).

Recall that $N(n_i, U)$ is the image of $N(n_i, E)$ by the natural map $E \to U$.

\[
\begin{align*}
\bar{x}_1 &\in N(n_{\tau(1)}, U), \\
\bar{x}_2 &\in N(n_{\tau(2)}, U), \\
\bar{x}_3 &:= \bar{x}_1 C_{13} + \bar{x}_2 C_{23} + \bar{u}_3 A_3 \in N(n_{\tau(3)}, U), \text{ where } \bar{u}_3 \in F_3^4, \\
\bar{x}_4 &:= \bar{x}_1 C_{14} + \bar{x}_2 C_{24} + \bar{u}_4 A_4 \in N(n_{\tau(4)}, U), \text{ where } \bar{u}_4 \in F_3^4, \\
\bar{x}_5 &:= \bar{x}_1 C_{15} + \bar{x}_2 C_{25} + \bar{u}_5 A_5 \in N(n_{\tau(5)}, U), \text{ where } \bar{u}_5 \in F_3^4, \\
\bar{x}_6 &:= \bar{x}_1 C_{16} + \bar{x}_2 C_{26} + \bar{u}_6 A_6 \in N(n_{\tau(6)}, U), \text{ where } \bar{u}_6 \in F_3^4, \\
\bar{x}_7 &:= \bar{x}_1 C_{17} + \bar{x}_2 C_{27} + (\bar{u}_3, \bar{u}_4) C_{37} + (\bar{u}_5, \bar{u}_6) C_{47} \in N(n_{\tau(7)}, U), \\
\bar{x}_8 &:= \bar{x}_1 C_{18} + \bar{x}_2 C_{28} + (\bar{u}_3, \bar{u}_4) C_{38} + (\bar{u}_5, \bar{u}_6) C_{48} \in N(n_{\tau(8)}, U).
\end{align*}
\]

If we find $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_8)$ satisfying all conditions in (3.3), then $\bar{x}$ belongs to $\mathcal{M}(n^\tau)$ and hence
\[\bar{x}^{\tau^{-1}} = (\bar{x}_{\tau^{-1}(1)}, \ldots, \bar{x}_{\tau^{-1}(8)})\]
is an element of $\mathcal{M}(n)$. All elements of $\mathcal{M}(n)$ are obtained in this way.

Using the maps $N(a, E) \to N(a, U)$, we can make from $\mathcal{M}(n)$ the set $M(n)$ of vectors $x \in \text{Min}(Q(\Delta, B))$ with $\nu(x) = n$. Taking the union of these sets $M(n)$ together with the set of minimal-norm vectors of divisible type, we obtain the set $\text{Min}(Q(\Delta, B))$ of all minimal-norm vectors of $Q(\Delta, B)$. 
Remark 3.3. Thanks to the permutation $\tau$, we have $|N(n_{\tau(i)},U)| \leq |N(n_{\tau(j)},U)|$ for $i < j$, and hence, in the back track search above, there exist few possibilities of $x_i$ for small indexes $i$. By this trick, the enumeration of $\text{Min}(Q(\Delta, B))$ becomes tractable.

Remark 3.4. We know that $|\text{Min}(L)| = 2611200$ for an extremal lattice $L$ of rank 64 by the theory of theta series and modular forms. (See, for example, [2, Chapter 7.7].) Hence we can confirm easily that we left no minimal-norm vectors uncounted.

3.2. Isomorphism classes. In order to distinguish isomorphism classes of two extremal lattices $L$ and $L'$ of rank 64, we use the distribution of intersection patterns of minimal-norm vectors. Let $\text{Min}(L)$ be the set of vectors $v \in L$ with $\langle v, v \rangle_L = 6$. For $v, v' \in \text{Min}(L)$, we have $\langle v, v' \rangle_L \in \{0, \pm 1, \pm 2, \pm 3, \pm 6\}$. For $k = 0, 1, 2, 3, 6$, we put

$$a_k(v) := \frac{1}{2} \left| \{v' \in \text{Min}(L) \mid \langle v, v' \rangle_L = k \text{ or } -k \} \right|.$$ 

We have $a_6(v) = 1$ and $\sum a_k(v) = 1305600$. The triple

$$a(v) := [a_1(v), a_2(v), a_3(v)]$$

is called the intersection pattern of $v$. For a triple $a = [a_1, a_2, a_3]$ of non-negative integers with $a_1 + a_2 + a_3 + 1 \leq 1305600$, we put

$$\mathcal{A}_L(a) := \{v \in \text{Min}(L) \mid a(v) = a\}, \quad A_L(a) := \frac{1}{2} |\mathcal{A}_L(a)|,$$

and call the function $A_L$ the distribution of intersection patterns. It is obvious that, if $A_L \neq A_{L'}$, then $L$ and $L'$ are not isomorphic.

Remark 3.5. In fact, the calculation of intersection patterns $a(v)$ of all elements $v$ of $\text{Min}(Q(\Delta, B))/\{\pm 1\}$ takes most of the computation time in the proof of Theorem 1.1.

3.3. Automorphism group. Let $L$ be an extremal lattice of rank 64, and let $\Gamma$ be a subgroup of $O(L)$. We will apply Proposition 3.7 below to the cases $\Gamma = \{\pm 1\}$ and $\Gamma = \{\pm 1\} \times (\tilde{\gamma}Q)$, where $\tilde{\gamma}Q$ is the automorphism of $Q(\Delta_0, B(\gamma, v))$ introduced in Section 2.3.

Definition 3.6. An ordered list $(v_1, \ldots, v_{64})$ of vectors in $\text{Min}(L)$ is said to be a $\Gamma$-rigidifying basis if the following hold:

(a) The vectors $v_1, \ldots, v_{64}$ form a basis of $L \otimes \mathbb{Q}$.

(b) The group $\Gamma$ acts on the set $\mathcal{A}_L(a(v_1))$ transitively.

(c) Suppose that $i > 1$. Then the set

$$\{v' \in \mathcal{A}_L(a(v_i)) \mid \langle v', v_j \rangle_L = \langle v_i, v_j \rangle_L \text{ for all } j < i \}$$

consists of a single element $v_i$.

Proposition 3.7. If a $\Gamma$-rigidifying basis exists, then $O(L)$ is equal to $\Gamma$. 

Table 4.1. Matrix $G'_0$

\[
\begin{bmatrix}
0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & \end{bmatrix}
\]

Proof. Note that $O(L)$ preserves each subset set $A_L(a)$ of $\text{Min}(L)$ for any $a$. In particular, we have $v^g \in A_L(a(v))$ for any $g \in O(L)$ and any $v \in L$. Let $g$ be an arbitrary element of $O(L)$. By (b), there exists an element $g' \in \Gamma$ such that $v^g = v^{g'}$. By (c), we can prove that $v^g_i = v^{g'}_i$ holds for all $i = 1, \ldots, 64$ by induction on $i$. Then (a) implies that $g = g'$.

4. Examples

4.1. Examples without non-trivial automorphisms. Let $V_0$ be the maximal isotropic subspace of $U$ with basis $v_1, \ldots, v_4$ in Table 2.1. By this basis, an element of $V_0$ is expressed by a vector in $F_3^4$, and hence an element of $V_0$ is expressed by a vector in $F_3^{32}$. Let $G_0$ be the $8 \times 32$ matrix with components in $F_3$ of the form $[I_8 | G'_0]$, where $I_8$ is the identity matrix of size 8 and $G'_0$ is given in Table 4.1. (This matrix $G'_0$ was produced by choosing components pseudo-randomly.) Let $B_0$ be the linear subspace of $V_0$ generated by the row vectors of $G_0$. Then $B_0$ satisfies $p_2$-condition. Recall that we have given maximal isotropic subspaces $W_I$ and $W_{II}$ in Table 2.1. Let $Q_I$ (resp. $Q_{II}$) be the Quebbemann lattice $Q(\Delta_I, B_0)$ (resp. $Q(\Delta_{II}, B_0)$), where

$\Delta_I = ((V_0, W_I), \ldots, (V_0, W_I)) \in D^8$, $\Delta_{II} = ((V_0, W_{II}), \ldots, (V_0, W_{II})) \in D^8$.

Then the distributions of intersection patterns of $Q_I$ and $Q_{II}$ are as in Table 4.2. (The left table is of $Q_I$ and the right is of $Q_{II}$.) In these tables, the intersection patterns $a = [a_1, a_2, a_3]$ are sorted by the lexicographic order. We can readily see that $Q_I$ and $Q_{II}$ are not isomorphic. Both of $Q_I$ and $Q_{II}$ have $\{\pm 1\}$-rigidifying basis, and hence $O(Q_I)$ and $O(Q_{II})$ are equal to $\{\pm 1\}$.

4.2. Examples with an automorphism of order 8. Let $\gamma$ be an element of $O(E)$ represented by the matrix in Table 4.3 with respect to the basis $e_1, \ldots, e_8$ of $E$. Then $\gamma$ is of order 8 and belongs to the stabilizer subgroup $G^I$ of $(V_0, W^I) \in D$. Let $v = (v_1, \ldots, v_8) \in V_0^8$ be

\[
\begin{pmatrix}
2210 & 0120 & 0201 & 1001 & 0201 & 0222 & 0122 & 0122
\end{pmatrix}
\]
Table 4.2. Distributions of intersection patterns of $Q^I$ and $Q^{II}$

| no. | $a_1$ | $a_2$ | $a_3$ | $A_{Q^I}(a)$ | no. | $a_1$ | $a_2$ | $a_3$ | $A_{Q^{II}}(a)$ |
|-----|-------|-------|-------|-------------|-----|-------|-------|-------|--------------|
| 1   | 568092| 40191 | 612   | 1           | 1   | 568422| 40131 | 602   | 1           |
| 2   | 568290| 40155 | 606   | 1           | 2   | 568488| 40119 | 600   | 2           |
| 3   | 568356| 40143 | 604   | 3           | 3   | 568554| 40107 | 598   | 1           |
| 4   | 568488| 40119 | 606   | 2           | 4   | 568620| 40095 | 596   | 3           |
| 5   | 568554| 40107 | 598   | 2           | 5   | 568686| 40083 | 594   | 7           |
| 6   | 568620| 40095 | 596   | 3           | 6   | 568752| 40071 | 592   | 5           |
| 7   | 568686| 40083 | 594   | 2           | 7   | 568818| 40059 | 590   | 8           |
| 8   | 568752| 40071 | 588   | 6           | 8   | 568844| 40047 | 588   | 8           |
| 9   | 568818| 40035 | 586   | 8           | 9   | 568950| 40023 | 584   | 11          |
| 10  | 568884| 40047 | 588   | 9           | 10  | 569016| 40023 | 584   | 11          |

Table 4.3. Elements of $O(E)$ of order 8

\[
\gamma := \begin{bmatrix}
2 & 1 & 2 & 4 & 3 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -2 & -2 & -2 & -2 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\
-3 & -2 & -4 & -6 & -5 & -4 & -2 & -1 \\
2 & 1 & 3 & 4 & 3 & 2 & 1 & 1
\end{bmatrix}, \quad \gamma' := \begin{bmatrix}
1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 \\
-2 & -1 & -2 & -4 & -3 & -2 & -2 & -1 \\
3 & 2 & 4 & 6 & 5 & 3 & 2 & 1 \\
-2 & -2 & -4 & -5 & -4 & -3 & -2 & -1 \\
1 & 1 & 2 & 3 & 3 & 3 & 2 & 1 \\
-1 & -1 & -1 & -2 & -2 & -2 & -1 & 0 \\
-1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\
2 & 1 & 2 & 3 & 2 & 2 & 1 & 1
\end{bmatrix}
\]

where each component $v_i$ is written with respect to the basis of $V_0$ in Table 2.1. Then the subspace $B(\gamma, v)$ of $V_0^8$ satisfies $p_2$-condition, and we obtain a Quebbemann lattice $Q := Q(\Delta^I, B(\gamma, v))$ with an automorphism
Table 4.4. Distributions of intersection patterns of $Q$ and $Q'$

| no. | $a_1$ | $a_2$ | $a_3$ | $A_Q(a)$ | no. | $a_1$ | $a_2$ | $a_3$ | $A_{Q'}(a)$ |
|-----|-------|-------|-------|----------|-----|-------|-------|-------|------------|
| 1   | 568026| 40203 | 614   | 8        | 1   | 568092| 40191 | 608   | 8          |
| 2   | 568092| 40191 | 612   | 16       | 2   | 568224| 40167 | 606   | 24         |
| 3   | 568290| 40155 | 606   | 24       | 3   | 568290| 40155 | 606   | 8          |
| 4   | 568356| 40143 | 604   | 16       | 4   | 568488| 40119 | 600   | 8          |
| 5   | 568422| 40131 | 602   | 16       | 5   | 568686| 40083 | 594   | 16         |
|     | ...   |       |       |          |     | ...   |       |       |            |
| 100 | 580104| 38007 | 248   | 11240    | 100 | 580104| 38007 | 248   | 10688      |
| 101 | 580170| 37995 | 246   | 12984    | 101 | 580170| 37995 | 246   | 12344      |
| 102 | 580236| 37983 | 244   | 14840    | 102 | 580236| 37983 | 244   | 14656      |
| 103 | 580302| 37971 | 242   | 16712    | 103 | 580302| 37971 | 242   | 16064      |
| 104 | 580368| 37959 | 240   | 18800    | 104 | 580368| 37959 | 240   | 19240      |
| 105 | 580434| 37947 | 238   | 20808    | 105 | 580434| 37947 | 238   | 20104      |
| 106 | 580500| 37935 | 236   | 23184    | 106 | 580500| 37935 | 236   | 22984      |
| 107 | 580566| 37923 | 234   | 25304    | 107 | 580566| 37923 | 234   | 25128      |
| 108 | 580632| 37911 | 232   | 27416    | 108 | 580632| 37911 | 232   | 28064      |
| 109 | 580698| 37899 | 230   | 29720    | 109 | 580698| 37899 | 230   | 29128      |
| 110 | 580764| 37887 | 228   | 32472    | 110 | 580764| 37887 | 228   | 32304      |
|     | ...   |       |       |          |     | ...   |       |       |            |
| 155 | 583734| 37347 | 138   | 40       | 155 | 583734| 37347 | 138   | 32         |
| 156 | 583800| 37335 | 136   | 24       | 156 | 583800| 37335 | 136   | 48         |
| 157 | 583866| 37323 | 134   | 16       | 157 | 583866| 37323 | 134   | 24         |
| 158 | 583932| 37311 | 132   | 8        | 158 | 583932| 37311 | 132   | 24         |
|     |       |       |       |          |     |       |       |       |            |
| total: 1305600 |       |       |       |          | total: 1305600 |       |       |       |            |

$\tilde{\gamma}_Q$ of order 8, where $\Delta^I \in D^8$ is given in the previous subsection. By the method of $\Gamma$-rigidifying basis, we see that $O(Q) = \{\pm 1\} \times \langle \tilde{\gamma}_Q \rangle$. The action of $O(Q)$ decomposes $\text{Min}(Q)$ into $2611200 / 16 = 163200$ orbits. The distribution of intersection patterns is given in Table 4.4 (left).

Let $\gamma'$ be an element of $O(E)$ given in Table 4.3, which is of order 8 and belongs to the stabilizer subgroup $G^{II}$ of $(V_0, W^{II}) \in D$. Let $v' \in V_0^8$ be

$$
(2220 \mid 0102 \mid 2120 \mid 2220 \mid 2202 \mid 1202 \mid 2220 \mid 2112)
$$

Then $B(\gamma', \gamma')$ satisfies $p_2$-condition, and we obtain a Quebbemann lattice $Q' := Q(\Delta^{II}, B(\gamma', v'))$. We see that $O(Q') = \{\pm 1\} \times \langle \tilde{\gamma}_{Q'} \rangle$, and its action decomposes $\text{Min}(Q')$ into $163200$ orbits. The distribution of intersection patterns is given in Table 4.4 (right).

References

[1] E. Bannai, Positive definite unimodular lattices with trivial automorphism groups, Memoirs of the American Mathematical Society, vol. 429, American Mathematical Society, 1990, 70 pages.
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