Coherent states on the Grassmannian $U(4)/U(2)^2$: oscillator realization and bilayer fractional quantum Hall systems

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Abstract

Bilayer quantum Hall (BLQH) systems, which underlie a $U(4)$ symmetry, display unique quantum coherence effects. We study coherent states (CS) on the complex Grassmannian $G^4_2 = U(4)/U(2)^2$, orthonormal basis, $U(4)$ generators and their matrix elements in the reproducing kernel Hilbert space $H_\lambda(G^4_2)$ of analytic square-integrable holomorphic functions on $G^4_2$, which carries a unitary irreducible representation of $U(4)$ with index $\lambda \in \mathbb{N}$. A many-body representation of the previous construction is introduced through an oscillator realization of the $U(4)$ Lie algebra generators in terms of eight boson operators. This particle picture allows us to make a physical interpretation of our abstract mathematical construction in the BLQH jargon. In particular, the index $\lambda$ is related to the number of flux quanta bound to a bi-fermion in the composite fermion picture of Jain for fractions of the filling factor $\nu = 2$. The simpler, and better known, case of spin-$s$ CS on the Riemann–Bloch sphere $S^2 = U(2)/U(1)^2$ is also treated in parallel, of which Grassmannian $G^4_2$-CS can be regarded as a generalized (matrix) version.

Keywords: coherent states, Grassmannian coset, oscillator realization, bilayer fractional quantum Hall effect, composite fermion

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1. Introduction

Since Schrödinger first introduced in 1926 the notion of (canonical) coherent states (CS) of the harmonic oscillator, the subject of CS has grown and permeates almost all branches of quantum
physics (see e.g. [1] and [2, 3] for old and recent reviews). Besides, some other important topics in applied mathematics, like the theory of wavelets, are also related to the notion of CS [4]. Later in 1972, Gilmore [5, 6] and Perelomov [7, 8] realized that canonical CS were rooted in group theory (the Heisenberg–Weyl group) and generalized the concept for other type of groups. Actually, Gilmore introduced an algorithm [9], which makes use of CS as variational states to approximate the ground state energy, to study the classical, thermodynamic or mean-field, limit of some algebraic quantum models. This algorithm has proved to be specially suitable to analyze the phase diagram of Hamiltonian models undergoing a quantum phase transition.

Among all physical models where CS play a relevant role, we want to highlight the quantum Hall effect (QHE). Several interesting text books on the subject are namely [10–13]. We briefly remind that QHE deals with two-dimensional electron systems subjected to a perpendicular magnetic field $B$. Electrons make cyclotron motions and their energies are quantized into Landau levels. The number density of magnetic flux quanta is $\rho_\phi = B/\phi$, where $\phi = 2\pi h/e$ is the flux unit. One electron occupies an area $2\pi \ell_B$ with $\ell_B = \sqrt{\hbar/\ell B}$ the magnetic length and the filling factor is $v = \rho_0/\rho_\phi$ with $\rho_0$ the electron number density. QHE has attracted renewed attention owing to its peculiar features associated with quantum coherence. In fact, bilayer quantum Hall (BLQH) systems are much more interesting because they exhibit unique effects originating in the interlayer interaction, like the development of spontaneous quantum coherence across the layers. A bilayer system is made by trapping electrons in two thin layers at the interface of semiconductors. Electrons are transferable between the two layers by applying bias voltages. In the BLQH system one Landau site may accommodate four isospin states $|b \uparrow\rangle$, $|b \downarrow\rangle$, $|a \uparrow\rangle$ and $|a \downarrow\rangle$ in the lowest Landau level, where $|b \uparrow\rangle$ (resp. $|a \downarrow\rangle$) means that the electron is in the bottom layer ‘$b$’ (resp. top layer ‘$a$’) and its spin is up (resp. down), and so on. Therefore, the $U(4)$ symmetry underlies the BLQH system provided the cyclotron energy is large enough. The driving force of quantum coherence is the Coulomb exchange interaction, which is described by an anisotropic $SU(4)$ nonlinear $\sigma$-model in BLQH systems [13]. Actually, it is the interlayer exchange interaction which develops the interlayer coherence. The lightest topological charged excitation in the BLQH system is a (complex projective) $\mathbb{CP}^3 = U(4)/[U(1) \times U(3)]$ skyrmion for filling factor $v = 1$ and a (Grassmannian) $G_2^4 = U(4)/[U(2) \times U(2)]$ bi-skyrmion (two $\mathbb{CP}^3$ skyrmions carrying total charge $2e$) for filling factor $v = 2$. The Coulomb exchange interaction for this last case is described by a Grassmannian $G_2^4$ $\sigma$-model and the dynamical field is a Grassmannian field $Z = z^\mu \sigma_\mu [14] (\sigma_\mu$ are the Pauli matrices in (1)) carrying four complex field degrees of freedom $z^\mu \in \mathbb{C}, \mu = 0, 1, 2, 3$. Also, the parameter space characterizing the $SU(4)$-invariant ground state in the BLQH system at $v = 2$ is precisely $G_2^4$ [15].

Just to mention that other construction of CS on the Grassmannian $G_2^N = U(N)/[U(2)U(N-2)]$ (space of complex two planes in $\mathbb{C}^N$) has been recently discussed in [16], but in connection with loop quantum Gravity, where the quantum states of geometry are the so-called spin network states.

In this paper we make a quite thorough (mathematical) study of CS on $G_2^4$, which we are sure that will be of great physical utility as variational states to study the semi-classical (and thermodynamical limit) analysis of the BLQH system and its quantum phase transitions, just like standard spin-$s$ CS are essential for semi-classical studies of quantum phase transitions in boson condensates. Firstly we follow a geometric approach to the construction of CS on $G_2^4$, in part inspired by the method of orbits in geometric quantization due to Kirillov–Kostant–Souriau [17–19] and the Borel–Weil–Bott theorem [20], which relate quantization, geometry and the representation theory for classical groups. In order to connect this abstract construction with the ‘many-body picture’, we introduce an oscillator realization of the $u(4)$
Lie algebra in terms of eight boson creation, $a^\mu_\mu$, $b^\dagger_\mu$, and annihilation, $a_\mu$, $b_\mu$, $\mu = 0, 1, 2, 3,$ operators. This realization differs from the standard Schwinger boson representation of $u(4)$ in terms of four bosons, leading to the totally symmetric representation and related to the Grassmannian $G^4_2 = CP^3$. A similar oscillator realization to ours, but for the (non-compact) pseudo-Grassmannian $U(2, 2)/U(2)$, has been recently considered in [21], in the context of deformation quantization, recovering some old results of Rühl [22, 23] concerning CS on the conformal group (see also [24–26] on this subject). Other boson realizations of the $u(N)$ Lie algebra appear in the literature, namely by Moshinsky [27–30] in the context of nuclear physics, who demonstrated that the irreps of a unitary algebra are characterized by a partition of the number of particles involved and he showed that a basis of the space underlying the irrep can be constructed from the so-called highest-weight polynomial. CS and oscillator realizations for $SU(N)$ have also been discussed in [31], and an identification and state labeling of the class of irreps of $SU(4)$ with respect to $S(U(2) \times U(2))$ have been identified in [32] (see also [33, 34]). However, we do not find a clear connection with our construction, which is specially designed to the study of BLQH systems.

The paper is organized as follows. In section 2 we remind the Lie algebra structure and coordinate systems of $U(4)$ adapted to the fibration $U(2)^2 \to U(4) \to G_2$ (since there is no confusion, from now on we shall use the short-hand $G_2 = G_2^2$). In section 3 we construct a CS system labeled by points of $G_2$ in the (reproducing kernel) Hilbert space $\mathcal{H}_\lambda(G_2)$ of analytic square-integrable holomorphic functions on $G_2$ with a given measure (orthonormality relations are proved in the appendix). This corresponds to a given square-integrable irreducible representation of $U(4)$ with positive integer index $\lambda$, and we identify the Young tableau associated with it, which motivates the ‘particle picture’ construction later in section 5 (those readers more acquainted with the many-body picture might skip section 3 in a first reading and go to section 5). Before, in section 4 we explicitly compute the generators (pseudospin ladder, imbalance, angular momentum, etc, operators) of the representation of $U(4)$ on $\mathcal{H}_\lambda(G_2)$ and their matrix elements in an orthonormal basis. In section 5, we introduce an oscillator realization of the $u(4)$ Lie algebra in terms of eight boson operators, and express the orthonormal basis of $\mathcal{H}_\lambda(G_2)$ in terms of the Fock basis with constraints in the occupancy numbers. An expression of Grassmannian CS as Bose–Einstein-like condensates is also provided. The spin-frozen case, which is described by standard pseudospin-1 CS on the Riemann–Bloch sphere $S^2 = U(2)/U(1)^2$, is treated in parallel all along the paper, to better appreciate the role played by spin in BLQH systems and to stress the similarities and differences between $G_2$ and $S^2$ CS, the first being a generalized (matrix) version of the second ones. Section 6 is devoted to some comments on the (flux quanta) physical meaning of the representation index $\lambda$ and its relation with the composite fermion (CF) picture of Jain [11, 35] in the fractional QHE.

2. The group $U(4)$: coordinate systems and generators

Let us firstly describe very briefly the structure of the group $U(4)$ of unitary $4 \times 4$ matrices, reminding its Lie algebra basis and putting coordinates on it. In this paper we are interested in the Lie algebra basis adapted to the noncanonical chain of subgroups

$$U(4) \supset U(2) \times U(2) \supset U(1) \times U(1).$$

The corresponding matrix representation is useful, for instance, when studying isospin $SU(4)$ symmetry in bilayer spin (namely, quantum Hall) systems, to emphasize the spin $SU(2)$ symmetry in the, let us say, bottom ($b$ or pseudospin $-1/2$) and top ($a$ or pseudospin $1/2$) layers. The pseudospin rotates when particles are transferred from one layer to the
other. More precisely, we denote the $U(4)$ generators in the fundamental representation by $\tau_{\mu\nu} \equiv \sigma_2 \otimes \sigma_\nu$, $\mu, \nu = 0, 1, 2, 3$ where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(1)
denote the Pauli matrices (plus $\sigma_0$). We shall introduce, for convenience, the interlayer ladder matrices

$$\tau_{+\mu} \equiv \frac{1}{2}(\tau_{1\mu} + i\tau_{2\mu}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \tau_{-\mu} \equiv \frac{1}{2}(\tau_{1\mu} - i\tau_{2\mu}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(2)
and the Lorentz-like generators

$$m_{\mu\nu} = \frac{1}{4} \left( \sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu \right),$$

(3)
where $\sigma_\nu \equiv \sigma^\nu = \eta^{\nu\alpha} \sigma_\alpha$ and we shall use the metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ to rise and lower indices. The Einstein summation convention will also be used unless otherwise stated. Note that $m_{\mu\nu}$ can be expressed in terms of $\tau_{\pm\mu}$, $m_{\mu\nu}$, and the ‘pseudospin third component’ $\tau_{00}$ as (we denote $\tilde{\tau}_{\pm\mu} = \tau_{\pm\mu} = \eta^{\mu\nu} \tau_{0\nu}$):

$$[m_{\mu\nu}, m_{\alpha\beta}] = \eta_{\nu\alpha} m_{\mu\beta} - \eta_{\nu\beta} m_{\mu\alpha},$$

$$[\tilde{\tau}_{-\mu}, m_{\alpha\beta}] = \eta_{\mu\alpha} \tilde{\tau}_{-\beta} - \eta_{\mu\beta} \tilde{\tau}_{-\alpha}, \quad [\tilde{\tau}_{+\mu}, m_{\alpha\beta}] = \delta_{\mu\alpha} \tilde{\tau}_{+\beta} - \delta_{\mu\beta} \tilde{\tau}_{+\alpha},$$

$$[\tau_{+\mu}, \tilde{\tau}_{-\nu}] = \eta_{\mu\nu} \tau_{00}, \quad [\tau_{00}, \tau_{\pm\mu}] = \pm 2 \tau_{\pm\mu},$$

$$[\tau_{00}, m_{\mu\nu}] = 0.$$

(4)
The linear Casimir operator is $C_1 = \tau_{00}$. The quadratic Casimir operator can be written in several forms as

$$C_2 = \frac{1}{2} \delta^{\mu\nu} \eta^{\alpha\beta} m_{\mu\alpha} \tau_{\beta\nu} - \frac{1}{2} \tau_{00}^2,$$

$$= \frac{1}{2}(\tau_{00} \tilde{\tau}_0^\mu + \tau_{00} \tilde{\tau}_3^\mu) + \frac{1}{2}(\tilde{\tau}_{-\mu} \tau_3^\mu + \tilde{\tau}_{+\mu} \tau_0^\mu) - \frac{1}{2} \tau_{00}^2$$

$$= \frac{1}{2} \tau_{00}^2 + 2s_0^2 + s_3^2 + \frac{1}{2}(\tau_{-\mu} \tau_3^\mu + \tau_{+\mu} \tau_0^\mu),$$

(5)
which, for the current fundamental (four-dimensional) representation, is simply $\frac{1}{2} {\tau}_{00}^2$. In the last equality we have also introduced the angular momentum

$$s_{aj} = \frac{1}{4} (\tilde{\tau}_{0j} + \tilde{\tau}_{3j}) = \begin{pmatrix} -\frac{1}{2} \sigma_j & 0 \\ 0 & 0 \end{pmatrix}, \quad s_{bj} = \frac{1}{4} (\tau_{0j} - \tau_{3j}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \sigma_j \end{pmatrix}, \quad j = 1, 2, 3, (6)$$
of the top ($a$) and bottom ($b$) layers. The relative sign between $s_a$ and $s_b$ has a sense that will be explained later (it could be assimilated to the space-fixed and body-fixed rigid-rotor angular momentum operators). Note that $s_a^2 + s_b^2 = -\frac{1}{2} m_{\mu\nu} m^{\mu\nu}$. In the BLQH literature [13] it is customary to define the spin $\tau_j^{\text{spin}} = \tau_{0j}$ and pseudospin $\tau_j^{\text{pseudospin}} = \tau_{0j}$ matrices, together with the remaining nine isospin matrices $\tau_{jk}$. Note that $\tau_j^{\text{spin}} = 2(s_{bj} - s_{aj})$.

The fundamental representation of the group $U(4)$ is defined as usual

$$U(4) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{4 \times 4}(C) : g^\dagger g = 1 = gg^\dagger \right\},$$

(7)
where, in terms of the $2 \times 2$ complex matrices $A, B, C, D$ in (7), the restrictions are explicitly written as

$^1$ Although we are in principle in a non-relativistic setting, relativistic notation turns out to be quite convenient.
\[ g^i g = 1 \iff \begin{cases} 
 D^i D + B^i B = \sigma_0 \\
 A^i A + C^i C = \sigma_0 \\
 A^i B + C^i D = 0, 
\end{cases} \]

(8)

together with those restrictions of \( gg^i = 1 \). In this paper we shall use a set of complex coordinates to parametrize \( U(4) \). This parametrization will be adapted to the complex Grassmannian \( \mathbb{G}_2 = U(4)/U(2)^2 \). It can be obtained through a block-orthonormalization process of the matrix columns of:

\[ \begin{pmatrix} \sigma_0 & 0 \\
-Z^i & \sigma_0 \end{pmatrix} \rightarrow g = \begin{pmatrix} \sigma_0 & Z \\
-Z^i & \sigma_0 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 \\
0 & \Delta_2 \end{pmatrix}, \quad \begin{cases} 
\Delta_1 = (\sigma_0 + ZZ^i)^{-1/2} \\
\Delta_2 = (\sigma_0 + Z^i Z)^{-1/2}. 
\end{cases} \]

Actually, we can identify

\[ Z = Z(g) = BD^{-1} = -A^{-1} C^i, \quad Z^i = Z^i(g) = -CA^{-1} = D^{-1} B^i, \]

\[ \Delta_1 = (AA^i)^{1/2}, \quad \Delta_2 = (DD^i)^{1/2}. \]

(9)

The positive-matrix conditions \( AA^i > 0 \) and \( DD^i > 0 \) are then equivalent to:

\[ \sigma_0 + ZZ^i > 0, \quad \sigma_0 + Z^i Z > 0. \]

(10)

Let us conclude this section by giving a complete local parametrization of \( U(4) \) adapted to the fibration \( U(2)^2 \rightarrow U(4) \rightarrow \mathbb{G}_2 \). Any element \( g \in U(4) \) (in the present patch, containing the identity element) admits the Iwasawa decomposition

\[ g = \begin{pmatrix} A & B \\
C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\
-Z^i \Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\
0 & U_2 \end{pmatrix}, \]

(11)

where the matrices

\[ U_1 = \Delta_1^{-1} A, \quad U_2 = \Delta_2^{-1} D \]

belong to \( U(2) \) and represent spin rotations in the top and bottom layers, respectively. Likewise, a parametrization of any \( U \in U(2) \) (in a patch containing the identity), adapted to the quotient \( \mathbb{C}P^1 = S^2 = U(2)/U(1)^2 \) (the Hopf fibration) is

\[ U = \begin{pmatrix} a & b \\
c & d \end{pmatrix} = \begin{pmatrix} \delta & z\delta \\
-z\delta & \delta \end{pmatrix} \begin{pmatrix} u_1 & 0 \\
0 & u_2 \end{pmatrix}, \]

(12)

where \( z = b/d \in \mathbb{C} \simeq S^2 \) (the one-point compactification of \( \mathbb{C} \) by inverse stereographic projection), \( \delta = (1 + z\bar{z})^{-1/2} \) and the phases \( u_1 = a/|a|, u_2 = d/|d| \).

3. Coherent states, closure relations and orthonormal basis

Firstly, let us consider the Hilbert space \( L^2(U(4)) \), \( d\mu \) of square-integrable complex functions \( \psi(g) \) on \( U(4) \) with invariant scalar product

\[ \langle \psi | \psi' \rangle = \int_{U(4)} d\mu(g) \psi(g) \overline{\psi'(g)} \]

(13)

given through the invariant Haar measure \( d\mu(g) \), which can be decomposed as:

\[ d\mu(g) = d\mu(g)|_{G_2} d\mu(g)|_{U(2)^2}, \]

\[ d\mu(g)|_{G_2} = \det(\sigma_0 + Z^i Z)^{-1} |dZ|, \]

\[ d\mu(g)|_{U(2)^2} = dv(U_1) dv(U_2), \]

(14)

where we are denoting by \( dv(U) \) the Haar measure on \( U(2) \), which can be in turn decomposed as:

\[ dv(U) = dv(U)|_{G_2} dv(U)|_{U(1)^2}, \]

\[ dv(U)|_{G_2} = (1 + z\bar{z})^{-1} |dz|, \]

\[ dv(U)|_{U(1)^2} = -\mu_1 d\mu_1 d\mu_2 d\mu_2. \]

(15)
We have used the Iwasawa decomposition of an element $g$ given in (11), (12) and denoted by $|dz|$ and $|dZ|$ the Lebesgue measures on $\mathbb{C}$ and $\mathbb{C}^4$, respectively (see appendix for more explicit expressions of this measure). The group $U(4)$ is represented in $L^2(U(4), \mu)$ as (left-action) $|U(g')\psi\rangle(g) = \psi(g'^{-1}g)$. This representation is reducible and we shall restrict it to an irreducible subspace. As we want to restrict ourselves to the quotient $U(4)/U(2)^2$, we chose as fiducial (ground state, lowest weight) vector $\psi_0^g(g) = \det(D)^h$ for $g$ given in (11) and $\lambda$ an integer number that will eventually label the corresponding irreducible representation. In fact, $\psi_0^g(g)$ is invariant (up to a phase) under $U(2)^2 \subset U(4)$ since, for $g' = (0\quad 0 \quad 0 \quad 1) \in U(2)^2$, we have
\begin{equation}
\psi_0^g(g'^{-1}g) = \det(U_2^\dagger D)^h = \det(U_2^\dagger)^h \psi_0^g(g).
\end{equation}
Under a general element $g' = (E\quad C)^{\dagger} \in U(4)$, the vector $\psi_0^g$ transforms as
\begin{equation}
\psi_0^g(g) \equiv \psi_0^g(g'^{-1}g) = \det(B'^\dagger B + D'^\dagger D)^h = \det(B'^\dagger Z + D'^\dagger)^h \psi_0^g(g),
\end{equation}
where we have used the relations (9) to write $Z = BD^{-1}$. The set of functions in the orbit of $\psi_0^g$ under $U(4)$
\begin{equation}
S_g \equiv \{\psi_0^g = U(g)\psi_0^g, \ g \in U(4)\}
\end{equation}
defines a system of CS. Note that $\psi_0^g$ and $\psi_0^{g'}$ are equivalent (up to a phase) if $g'g \in U(2)^2 \subset U(4)$. We shall prove that this CS system fulfills the resolution of the identity
\begin{equation}
1 = c_s \int_{G_2} d\mu(g)|_{G_2} |\psi_0^g|^2 |\psi_0^g|,
\end{equation}
with a suitable normalization constant $c_s$. Before, let us obtain some auxiliary results. Note that, introducing $Z'^\dagger = D'^{-1}B'^\dagger$ as in (9), the state (17) can be written as
\begin{equation}
\psi_0^g(g) = \det(\sigma_0 + Z'^\dagger Z)^{\dagger} \psi_0^g(g).
\end{equation}
We also realize that $|\psi_0^g(g)|^2 = \det(DD'^\dagger)^{\dagger} = \det(\sigma_0 + Z'^\dagger Z)^{-\dagger}$. To prove (19), we would like to have an expansion of $\det(\sigma_0 + Z'^\dagger Z)^k$ in terms of orthogonal polynomials. For this purpose, let us prove an interesting identity that will be useful in the sequel.

**Lemma 3.1. Let us denote by**
\begin{equation}
D_{q_s,q_b}^j(X) = \frac{(j+q_s)!(j-q_b)!}{(j+q_b)!(j-q_s)!} \sum_{k = \text{max}(0,q_s+q_b)}^{\text{min}(j+q_s,j+q_b)} \binom{j+q_s}{k} \binom{j-q_b}{k-q_s+q_b} \times x_1^{j+q_s-k+q_b} x_2^k x_6^{k-q_s+q_b},
\end{equation}
the usual Wigner’s $D$-matrices for $SU(2)$ (see e.g. [36]), where $j \in \mathbb{N}/2$ (the spin) runs on all non-negative half-integers and $q_s,q_b = -j,-j+1,\ldots,-j+1, j$, and $X$ represents here an arbitrary $2 \times 2$ complex matrix with entries $x_{\mu\nu}$. For every $\lambda \in \mathbb{N}$ the following identity holds:
\begin{equation}
\det(\sigma_0 + X)^\lambda = \sum_{m=0}^{\lambda} \sum_{j=0;\frac{1}{2}}^{\lambda - m/2} \frac{2j+1}{\lambda+1} \binom{\lambda+1}{j+m+1} D_{q_b}^j(X)^m \sum_{q=-j}^{j} D_{q_b}^q(X),
\end{equation}
where the sum on $j$ runs over half-non-negative integers: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, (\lambda - m)/2$.

**Proof.** We shall proceed by induction on $\lambda$. For $\lambda = 1$ we have
\begin{equation}
\det(\sigma_0 + X) = 1 + \text{tr}(X) + \det(X)
\end{equation}
with \( \text{tr}(X) \) and \( \text{det}(X) \) homogeneous polynomials of degree 1 and 2 in \( \lambda \), respectively. Wigner matrices \( D^{j}_{qq}(X) \) are homogeneous polynomials of degree \( 2j \) in \( \lambda \). For the spin-0 singlet representation of \( U(2) \) we have \( D^{0}_{00}(X) = 1 \) and for the spin-1/2 fundamental representation of \( U(2) \) we have \( \sum_{q=-1/2}^{1/2} D^{1/2}_{qq}(X) = \text{tr}(X) \), and therefore

\[
\frac{1}{2} \sum_{m=0}^{1} \sum_{j=0}^{\lambda - m/2} \frac{2j + 1}{2} \left( \begin{array}{c} \lambda + 1 \\ 2j + m + 1 \end{array} \right) \left( \begin{array}{c} \lambda + 1 \\ m \end{array} \right) \text{det}(X)^m \sum_{q=-j}^{j} D^{1/2}_{qq}(X) \\
= 1 + \text{tr}(X) + \text{det}(X) = \text{det}(\sigma_0 + X).
\]

Thus we proved the identity (22) for \( \lambda = 1 \). Let us assume that (22) holds for some natural \( \lambda \). Inspired by Euler’s theorem, we shall define the following differential operator:

\[
D_{\lambda} \equiv -\lambda + t \frac{\partial}{\partial t},
\]

which will be useful in the sequel. Applying \( D_{\lambda+1} \) to \( \text{det}(\sigma_0 + tX)^{\lambda+1} \) gives

\[
D_{\lambda+1} \text{det}(\sigma_0 + tX)^{\lambda+1} = - (\lambda + 1) \text{det}(\sigma_0 + tX)^{\lambda} (1 - \text{det}(tX)),
\]

where we have used that \( \text{tr}(tX) \) and \( \text{det}(tX) \) homogeneous polynomials of degree 1 and 2 in the parameter \( t \). Assuming now that (22) holds for some natural \( \lambda > 1 \) and inserting it in the rhs of (25), after some algebraic manipulations we arrive to

\[
\sum_{m=0}^{\lambda} \sum_{j=0}^{\lambda - m/2} (2j + 1) \left( \begin{array}{c} \lambda + 1 \\ 2j + m + 1 \end{array} \right) \left( \begin{array}{c} \lambda + 1 \\ m \end{array} \right) \text{det}(tX)^m \sum_{q=-j}^{j} D^{1}_{qq}(tX) \\
= \sum_{m=0}^{\lambda + 1} \sum_{j=0}^{\lambda + 1 - m/2} (2j + 2m - (\lambda + 1)) \left( \begin{array}{c} \lambda + 2 \\ 2j + m + 1 \end{array} \right) \text{det}(tX)^m \sum_{q=-j}^{j} D^{1}_{qq}(tX).
\]

Taking into account that \( \text{det}(tX)^m \sum_{q=-j}^{j} D^{1}_{qq}(tX) = t^{2m+2j} \text{det}(tX)^m \sum_{q=-j}^{j} D^{1}_{qq}(X) \) (that is, a homogeneous polynomial of degree \( 2m+2j \) in the X entries), we recognize \( (2j + 2m - (\lambda + 1)) \) in the rhs of (26) as the eigenvalue of \( D_{\lambda+1} \). Thus we proved that

\[
D_{\lambda+1} \text{det}(\sigma_0 + tX)^{\lambda+1} = D_{\lambda+1} \sum_{m=0}^{\lambda + 1} \sum_{j=0}^{\lambda + 1 - m/2} (2j + 1) \left( \begin{array}{c} \lambda + 2 \\ 2j + m + 1 \end{array} \right) \left( \begin{array}{c} \lambda + 1 \\ m \end{array} \right) \text{det}(tX)^m \sum_{q=-j}^{j} D^{1}_{qq}(tX),
\]

which coincides with the result of applying \( D_{\lambda+1} \) to both sides of (22) with \( \lambda \) replaced by \( \lambda + 1 \). The fact that \( D_{\lambda}(f(t) + k) = D_{\lambda}f(t) - \lambda k \), for any constant \( k \), eliminates any arbitrariness in \( f(t) \). Therefore, for \( t = 1 \), we conclude that the equality (22) is also true for \( \lambda + 1 \), thus achieving the proof by induction.

Now we are in condition to prove the following interesting result

**Theorem 3.2.** The set of homogeneous polynomials

\[
\phi^{j,m}_{q_a,q_b}(Z) = \sqrt{\frac{2j + 1}{\lambda + 1} \left( \begin{array}{c} \lambda + 1 \\ 2j + m + 1 \end{array} \right) \left( \begin{array}{c} \lambda + 1 \\ m \end{array} \right) \text{det}(Z)^m D^{j}_{q_a,q_b}(Z)}, \quad 2j + m \leq \lambda, \quad q_a, q_b = -j, \ldots, j.
\]
of degree \(2j + 2m\) verifies the following closure relation (the reproducing Bergman kernel):

\[
\sum_{\lambda} \left( \frac{\lambda - m}{2} \right)^{-j} \sum_{m=0}^{\lambda - j} \sum_{j=0}^{\lambda-m} \phi_{q_0, q_0=0}^{j, m}(Z) \phi_{q_0, q_0=0}^{j, m}(Z) = \det(\sigma_0 + Z^*Z)^{-j}
\]

and constitutes an orthonormal basis of the \(\mathcal{H}_\lambda\) dimensional Hilbert space \(\mathcal{H}_\lambda(\mathbb{G}_2) = L^2(\mathbb{G}_2, d\mu_\lambda)\) of analytic square-integrable holomorphic functions on \(\mathbb{G}_2\) with measure

\[
d\mu_\lambda(Z, Z') = c_\lambda |\phi_0^\lambda(g)|^2 d\mu(g)|_{\mathbb{G}_2} = c_\lambda \det(\sigma_0 + Z^*Z)^{-\lambda} |dZ|,
\]

where \(c_\lambda = 12d_\lambda/\pi^4\) is a normalization constant.

**Proof.** Replacing \(X = Z^*Z\) in (22) we have

\[
\sum_{\lambda} \left( \frac{\lambda - m}{2} \right)^{-j} \sum_{m=0}^{\lambda - j} \sum_{j=0}^{\lambda-m} \frac{2j + 1}{\lambda + 1} \left( \frac{\lambda + 1}{m} \right) \det(Z^*Z)^{-m} \sum_{q=-j}^{j} D^j_{qq}(Z^*Z)
\]

\[
= \det(\sigma_0 + Z^*Z)^{\lambda}.
\]

Using determinant and Wigner’s \(D\)-matrix properties [36]

\[
det(Z^*Z)^{\lambda} \sum_{q=-j}^{j} D^j_{qq}(Z^*Z) = det(Z^*Z)^{\lambda} \sum_{q=-j}^{j} D^j_{qq}(Z^*Z)
\]

and the definition of the functions (28), we see that (32) reproduces (29). On the other hand, the number of linearly independent polynomials \(\prod_{i=1}^{n} z_i^{n_i}\) of fixed degree of homogeneity

\[
n = \sum_{j=1}^{n} n_j (n + 1)(n + 2)(n + 3)/6 = \binom{n+3}{3}
\]

(\(\text{the number of ways of distributing } n \text{ quanta among four levels},\) which coincides with the number of linearly independent polynomials (28) with degree of homogeneity \(n = 2m + 2j\) for \(n \leq \lambda\). For \(\lambda < n\), \(\lambda < 2j + 2m < 2\lambda\), the degeneracy is \(\binom{2\lambda - n + 3}{3}\) (the number of ways of distributing \(2\lambda - n\) quanta among four levels). The total number of linearly independent polynomials is

\[
\sum_{n=0}^{\lambda} \left( \begin{array}{c} n + 3 \\ 3 \end{array} \right) + \sum_{n=\lambda+1}^{2\lambda} \left( \begin{array}{c} 2\lambda - n + 3 \\ 3 \end{array} \right) = (\lambda + 1)(\lambda + 2)(\lambda + 3)/12,
\]

which coincides with the dimension (30). This proves that the set of polynomials (28) is a basis for analytic functions \(\phi \in \mathcal{H}_\lambda(\mathbb{G}_2)\). Moreover, this basis turns out to be orthonormal under the projected integration measure (31). We address the interested reader to the appendix for details.

Let us introduce bracket notation and put

\[
\langle j, m | q_0, q_0 = 0 \rangle Z \equiv \phi_{q_0, q_0=0}^{j, m}(Z) \det(\sigma_0 + Z^*Z)^{-\lambda/2}.
\]

(We remove the label \(\lambda\) from the definition of \(\langle j, m | q_0, q_0 = 0 \rangle\) for the sake of brevity.) This makes \(\mathcal{H}_\lambda(\mathbb{G}_2)\) a reproducing kernel Hilbert space, that is, a Hilbert space of functions \(\phi\) in which pointwise evaluation \(\phi(Z)\) is a continuous linear functional. The resolution of the identity for an orthonormal basis in \(\mathcal{H}_\lambda(\mathbb{G}_2)\) then adopts the form

\[
1 = \sum_{m=0}^{\lambda} \left( \frac{\lambda - m}{2} \right)^{-j} \sum_{j=0}^{\lambda-m} \sum_{q_0, q_0=-j}^{j} |j, m \rangle \langle j, m | q_0, q_0 = 0 \rangle Z.
\]

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and the formal ket \(|Z\rangle\) is

\[ |Z\rangle = \det(\sigma_0 + Z^4 Z)^{-1/2} \sum_{m=0}^{\lambda} \sum_{j=0}^{(\lambda-m)/2} \sum_{q_{a}, q_{b} = -j}^{j} \psi_{q_{a}, q_{b}}^{|m|} (Z) |j_{q_{a}, q_{b}}\rangle. \] (36)

Actually, we can identify \(|Z\rangle\) with the CS \(|\psi_{q_{a}}^{|a}\rangle\) up to a phase. From the CS overlap

\[ \langle Z'|Z\rangle = \frac{\det(\sigma_0 + Z^4 Z)^{1/2}}{\det(\sigma_0 + Z^4 Z)^{1/2} \det(\sigma_0 + Z^4 Z)^{1/2}} \] (37)

we see that \(|Z\rangle\) is normalized. Moreover, using the orthogonality properties of the homogeneous polynomials \(\psi_{q_{a}, q_{b}}^{|m|} (Z)\), it is direct to prove the announced resolution of unity (19), now written as:

\[ 1 = c_\lambda \int_{G_2} |Z\rangle \langle Z|d\mu(g)|G_2|. \] (38)

It is interesting to compare the \(U(4)/U(2)^2\) CS (36) with the well known \(U(2)/U(1)^2\) or spin-s CS

\[ |z\rangle = (1 + |z|^2)^{-s/2} \sum_{q=-s}^{s} \psi_{q}(z) |s, q\rangle, \quad \psi_{q}(z) = \left( \frac{2s}{s + q} \right)^{1/2} z^q, \] (39)

with \(z \in \mathbb{C}\) (the stereographic projection of the sphere \(S^2 = U(2)/U(1)^2\) onto the complex plane), for which the CS overlap and the resolution of the identity acquire the form

\[ \langle z'|z\rangle = \frac{(1 + \bar{z}' z)^{2s}}{(1 + |z|^2)^{s}(1 + |z'||^2)^{s}}, \quad 1 = \frac{2s + 1}{\pi} \int_{\mathbb{C}} |z|^2 \frac{d^2z}{(1 + |z|^2)^2}. \] (40)

We perceive a similar structure between \(U(4)/U(2)^2\) and \(U(2)/U(1)^2\) CS, although the case \(U(4)/U(2)^3\) is more involved and can be regarded as a generalized (matrix \(Z\)) version of the standard (scalar \(z\)) case.

We finish this section with an explicit form of the unirep of \(U(4)\) on \(\mathcal{H}_s(G_2)\) in the form of a Corollary.

**Corollary 3.3.** For any holomorphic function \(\phi \in \mathcal{H}_s(G_2)\) and any \(g' \in U(4)\), the following action

\[ [U_{\phi}^s g] (Z) \equiv \det(D^{g^4} + B^{g^4} Z^{g^4}) \phi(Z'), \quad Z' = (A^{g^4} Z - C^{g^4}) (D^{g^4} - B^{g^4} Z)^{-1} \] (41)

defines a square-integrable unitary irreducible representation of \(U(4)\) on \(\mathcal{H}_s(G_2)\).

Note that if we define \(\psi(g) = \psi_{q_{a}}^{|a}|g\rangle \phi(Z), \ Z = Z(g), \) then

\[ [U_{\phi}^s g] (Z) = \left( \psi_{q_{a}}^{|a}(g) \right)^{-1} [U^{s} \phi] (g). \] (42)

The unitarity of \(U\) in \(L^2(U(4), \ d\mu)\) directly implies the unitarity of \(U^{s}\) in \(\mathcal{H}(G_2)\). Irreducibility follows from the fact that, for example, for \(\phi(Z) = 1\), the transformed function

\[ [U_{\phi}^s g] (Z) \equiv \det(D^{g^4} + B^{g^4} Z)^{g^4} \sum_{m=0}^{\lambda} \sum_{j=0}^{(\lambda-m)/2} \sum_{q_{a}, q_{b} = -j}^{j} c_{q_{a}, q_{b}}^{|m|} (g') \psi_{q_{a}, q_{b}}^{|m|} (Z) \] (43)

is expanded in terms of all basis functions \(\psi_{q_{a}, q_{b}}^{|m|} (Z)\) with non-zero coefficients \(c_{q_{a}, q_{b}}^{|m|} (g') = \det(D^{g^4} y_{q_{a}, q_{b}}^{|m|} (B'D^{-1})),\) as follows from (29).

Our irrep turns out to correspond to the one denoted by the Young Tableau of shape \([\lambda, \lambda]\) with two rows of \(\lambda\) boxes each (we use the ‘English notation’). This irrep arises in the Clebsch–Gordan decomposition of a tensor product of \(N = 2\lambda\) four-dimensional (fundamental,
elementary) representations of $U(4)$. The dimension of the tableau $[\lambda, \lambda]$ can be obtained from the so-called hook-length formula (which is a special case of the Weyl’s character formula, see e.g. [37]) and turns out to coincide with the dimension $d_\lambda$ of $\mathcal{H}_\lambda(G_{22})$ in (30). For example, for $\lambda = 1$ ($N = 2$ ‘particles or quanta’) we have $[1] \otimes [1] = [2] \otimes [1, 1]$ or

$$
\begin{array}{c}
\begin{array}{c}
\otimes \quad \otimes \\
\otimes \quad \otimes \\
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\otimes \quad \otimes \\
\otimes \quad \otimes \\
\end{array}
\end{array}
\Rightarrow
4 \times 4 = 10 + 6
$$

(44)

so that $[1, 1]$ has dimension $d_1 = 6$. For $\lambda = 2$ ($N = 4$ ‘particles or quanta’) we have

$$
\begin{array}{c}
\begin{array}{c}
\otimes \quad \otimes \quad \otimes \quad \otimes \\
\otimes \quad \otimes \quad \otimes \quad \otimes \\
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\otimes \quad \otimes \quad \otimes \quad \otimes \\
\otimes \quad \otimes \quad \otimes \quad \otimes \\
\end{array}
\end{array}
\Rightarrow
4 \times 4 = 10 + 6
$$

(45)

and the dimension of $[2, 2]$ (the last young tableau) is precisely $d_2 = 20$. After discussing an oscillator realization of the previous construction later in section 5, we will provide in section 6 a ‘CF’ picture (a term imported from the QHE jargon [11]) to physically interpret the $[\lambda, \lambda]$ configurations as two fermions bound to $\lambda$ flux quanta each. Before, let us state some interesting results concerning the basic operators and their matrix elements.

4. Infinitesimal generators and matrix elements

Let us denote by $T_{\mu \nu}$ and $M_{\mu \nu}$ the infinitesimal (differential) generators of the finite action (41) fulfilling the same commutation relations as the matrix generators $t_{\mu \nu}$ and $m_{\mu \nu}$ in (4). Writing $Z = z^{\mu} \sigma_{\mu}, z^{\mu} \in \mathbb{C}, z^2 = \bar{z} \mu, \bar{\partial}_\mu = \partial/\partial z^\mu$ and $\bar{\partial}_\mu = \partial/\partial \bar{z}_\mu = \partial/\partial z^\mu$, these generators have the following expression:

$$
\begin{align*}
M_{\mu \nu} &= z^{\mu} \partial_{\nu} - z^{\nu} \partial_{\mu}, \\
T_{\mu 0} &= 2(z^{\mu} \partial_{\mu} - \lambda), \\
T_{\mu \nu} &= \bar{\partial}_{\mu}, \\
T_{\nu \mu} &= \bar{\partial}_{\nu} - z^{\mu} T_{\mu 0}.
\end{align*}
$$

(46)

where we are using the notation $T_{\pm \mu} = (T_{1 \mu} \pm i T_{3 \mu})/2$ and $T_{\pm 0} = T_{30} = m^{\mu \nu} T_{\pm \mu \nu}$, as in (2) and (3). For example, from the general expression (41), we can compute the infinitesimal action of $g' = e^{i \gamma T_{\pm}}$ ($B' = 0 = C$ and $A' = e^{i \gamma m_0} = D'$) on wave functions as $[U_{g'}(Z) = e^{i \gamma m_0} \phi(e^{i \gamma Z}) = \phi(Z) + i T_{30} \phi(Z) + O(\gamma^2)$). The other generators are calculated in a similar way. Let us compute their action on the orthonormal basis functions (28). Firstly we see that the homogeneous polynomials in (28) are eigenfunctions of the (pseudospin third component) operator $T_{30} = T_3^0$ since

$$
T_{30}^{0} \phi_{q_0, q_0}^{j, m} = (2j + 2m - \lambda) \phi_{q_0, q_0}^{j, m},
$$

(47)

where the eigenvalue $2(2j + 2m - \lambda)$ could be related to an ‘imbalance’ or particle difference between layers $a$ and $b$ (see next section). Similarly, we can compute the action of the lowering interlayer ladder operators ($T_{30}^{\pm} = m^{\mu \nu} T_{3 \mu \nu}$)

$$
\begin{align*}
T_{30}^{+} \phi_{q_0, q_0}^{j, m} &= C_{j, m+2j+1}^{j, m+2j+1} + C_{j, m+2j+1}^{j, m+2j+1} \phi_{q_0, q_0}^{j, m+1} + C_{j, m+1}^{j, m+1} \phi_{q_0, q_0}^{j, m+1} + C_{j, m+1}^{j, m+1} \phi_{q_0, q_0}^{j, m+1}, \\
T_{30}^{-} \phi_{q_0, q_0}^{j, m} &= C_{j, m+2j+1}^{j, m+2j+1} \phi_{q_0, q_0}^{j, m+1} - C_{j, m+1}^{j, m+1} \phi_{q_0, q_0}^{j, m+1} + C_{j, m+1}^{j, m+1} \phi_{q_0, q_0}^{j, m+1} + C_{j, m+1}^{j, m+1} \phi_{q_0, q_0}^{j, m+1}.
\end{align*}
$$

10
\[ T^2 \varphi_{q_0,q_0}^m = iC_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} - iC_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} + iC_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} + iC_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} \]

and the raising interlayer ladder operators

\[ T^3 \varphi_{q_0,q_0}^m = C_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} - C_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} + C_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} - C_{j,m+2j+1}^{j_{-1/2}} \varphi_{q_0,q_0}^{j_{-1/2}} \]

with

\[ C_{j,m} = \sqrt{\left( j + q_0 \right)(j + q_0 - m) \eta(\lambda - (m - 2))}. \]

The differential representation of the top and bottom layer angular momentum in (6) is

\[ S_{\alpha j} = \frac{1}{2}(T_{0j} + q_{0j}) = \frac{1}{2}(M_{0j} - i\epsilon_{jk}M_{k\ell}) \text{ and } S_{\beta j} = \frac{1}{2}(T_{0j} - T_{3j}) = \frac{1}{2}(M_{0j} + i\epsilon_{jk}M_{k\ell}). \]

The action of the spin third component is

\[ S_{3 \varphi_{q_0,q_0}^m} = \varphi_{q_0,q_0}^{m}, \quad \epsilon = a, b \]

and the action of the ladder spin operators is

\[ S_{L \varphi_{q_0,q_0}^m} = \sqrt{\left( j + q_0 \right)(j + q_0 + 1)} \varphi_{q_0,q_0}^{m}, \quad \epsilon = a, b \]

where \( S_{\alpha \beta} = S_{\alpha 1} \mp S_{\alpha 2} \) and \( S_{\alpha \pm} = S_{\alpha 1} \pm iS_{\alpha 2}. \) Note that \( S_{\alpha \pm} \) and \( S_{\alpha \pm} \) have conjugated definitions (\( \pm \leftrightarrow \mp \)). This fact is related to the transformation property of wave functions in (41) which, for pure rotations \( (C' = 0 = B', A' = V_a, D' = V_b, V \in SU(2), \epsilon = a, b) \) gives \( [U_c^\ell \varphi](Z) = \varphi(V_a^\ell Z V_b), \) so that rotations \( V_0 \) on the layer \( a \) are represented by the inverse \( V_0^\ell. \) This fact resembles the difference between space-fixed and body-fixed rigid-rotor angular momentum operators, as commented after equation (6).

For completeness, we also give the action of \( U(2)^2 \)-invariant (i.e., commuting with \( M_{\mu \nu} \)) quadratic operators: \( M^2 = M_{0\mu} M_{0\mu}, T_{0\pm} = T_{0\mu} T_{\mu \pm} = T_{\mu \pm} T_{0\mu} = T_{+\mu} T_{-\mu}, \) \( T_{-\mu} = T_{+\mu}, \) which results in

\[ M^2 \varphi_{q_0,q_0}^m = -8j(j + 1) \varphi_{q_0,q_0}^m, \]

\[ T_{\lambda} \varphi_{q_0,q_0}^m = 4\sqrt{m(2j + m + 1)(\lambda - m - 1)(\lambda - m + 1)} \varphi_{q_0,q_0}^{m-1}, \]

\[ \tilde{T}_{\lambda} \varphi_{q_0,q_0}^m = -4\sqrt{2j + m + 1}(\lambda - m + 1) \varphi_{q_0,q_0}^{m+1}, \]

\[ \tilde{T}_{\lambda} \varphi_{q_0,q_0}^m = -4(2j^2 + m(m - \lambda - 2) + j(2m - \lambda - 1)) \varphi_{q_0,q_0}^m. \]
With these ingredients, the value of the quadratic Casimir operator (5) (written in terms of \( T_{\mu} \)) in the Hilbert space \( \mathcal{H}_G(\mathbb{G}_2) \) is easily computed and gives:
\[
\mathcal{C}_2 \varphi_{i\mu}^{j\nu} = \lambda (\lambda + 4) \varphi_{i\mu}^{j\nu}, \quad \forall j, m, q, q_b.
\]
(54)

5. Oscillator realization

It is well known the oscillator (Schwinger) realization of the \( SU(2) \) angular momentum operators \( \hat{S}_+ \), \( \hat{S}_- \) in terms of two bosonic modes \( a \) and \( b \) as
\[
\hat{S}_+ = \frac{1}{2} (a^\dagger - b^\dagger b), \quad \hat{S}_- = a^\dagger b, \quad \hat{S}_z = b^\dagger a,
\]
(55)
and the expression of spin-\( s \) basis states \(| s, q \rangle \), \( q = -s, \ldots, s \), in terms of Fock states \(| 0 \rangle \) denotes the Fock vacuum
\[
|n_a\rangle \otimes |n_b\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}}{\sqrt{n_a!n_b!}} | 0 \rangle
\]
(56)
as
\[
|s, q\rangle = \frac{(a^\dagger)^{s+q}(b^\dagger)^{s-q}}{\sqrt{(s+q)!(s-q)!}} | 0 \rangle = \frac{\varphi_j(a^\dagger) \varphi_q(b^\dagger)}{\sqrt{(2s)!}} | 0 \rangle = |s+q\rangle_a \otimes |s-q\rangle_b,
\]
(57)
where we have used the monomials \( \varphi_q \) in (39) as operator functions, since this notation will be generalized in a natural way later in equation (75) for a Fock representation of the basis functions \(|j_{i\mu}^{m}\rangle \) of \( \mathcal{H}_G(\mathbb{G}_2) \). Note that the total number of quanta is fixed to \( n_a + n_b = (s + q) + (s - q) = 2s \). The lowest weight state \(|s, -s\rangle = (b^\dagger)^s|0\rangle \) is often regarded as a boson condensate and the rest of states \(|s, q\rangle \) as excitations above this condensate. The \( SU(2) \) spin-\( s \) CS (39) can also be written as
\[
|z\rangle = \frac{1}{\sqrt{2s+1}} \left( \frac{b^\dagger + za^\dagger}{\sqrt{1 + |z|^2}} \right)^{2s} | 0 \rangle = \frac{e^{zs}}{(1 + |z|^2)^{s}} |s, -s\rangle.
\]
(58)
The natural generalization to \( U(4) \) requires four bosonic modes \( a, b, c \) and \( d \), for which the basis states
\[
|n_a\rangle \otimes |n_b\rangle \otimes |n_c\rangle \otimes |n_d\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}(c^\dagger)^{n_c}(d^\dagger)^{n_d}}{\sqrt{n_a!n_b!n_c!n_d!}} | 0 \rangle,
\]
(59)
with \( n_a + n_b + n_c + n_d = N \) the total (fixed, linear Casimir) number of ‘particles or quanta’, all belong to the totally symmetric irreducible representation of \( U(4) \). This representation is related to the quotient \( \mathbb{CP}^3 = U(4)/U(3) \times U(1) \) (the complex projective space) whose points \( z_a, z_b, z_c, z_d \in \mathbb{C} \) (in a certain patch) label the CS
\[
|z_a, z_b, z_c\rangle = \frac{1}{\sqrt{N!}} \left( \frac{d^\dagger + z_a c^\dagger + z_b b^\dagger + z_d a^\dagger}{\sqrt{1 + |z_a|^2 + |z_b|^2 + |z_d|^2}} \right)^N | 0 \rangle.
\]
(60)
These CS also verify a resolution of the identity similar to the one in (40) but replacing the \( \mathbb{CP}^1 \) integration measure by the corresponding \( \mathbb{CP}^3 \) integration measure. Fields taking values in the target manifold \( \mathbb{CP}^3 \) describe Goldstone bosons, \( SU(4) \)-skyrmions and small fluctuations around the ground state in the BLQH system at filling factor \( \nu = 1 \) [13].

However, these are not the CS (36) we are dealing with in this paper. Actually, the CS (36) will be related to the filling factor \( \nu = 2 \) in the BLQH system. The question is: Is there a boson realization like (60) but for the CS (36) labeled by points \( Z \) in the complex Grassmannian \( \mathbb{G}_2 \)? The answer is positive and it will be given later in proposition 5.1.
The most popular oscillator realization of the Lie algebra \( u(n) \) is that in terms of bilinear products of \( n \) creation and annihilation operators (Schwinger representation) leading to the totally symmetric representation (for example, the Bose–Einstein–Fock basis \((59)\) for \( n = 4 \)). Although perhaps less known, other realizations of \( u(n) \) in terms of more than \( n \) bosonic modes have also been used in the literature \([27–30]\), which describe more general representations than the symmetric one. Let us provide an oscillator realization for the (non-symmetric) \( U(4) \) representation given in the previous sections.

Note that, defining \( \mathcal{Z} = \sum_{j=0}^{n} j |j\rangle \) and \( \mathcal{Z}^\dagger = (a^\dagger \ b^\dagger) \), the angular momentum operators \((55)\) can be compactly written as

\[
S_\mu = \frac{i}{2} \text{tr}(\mathcal{Z}^\dagger \sigma_\mu \mathcal{Z}),
\]

with \( S_3 = S_1 + i S_2, S_z = S_3 \) and \( \mathcal{Z}^\dagger \mathcal{Z} = 2 S_0 = a^\dagger a + b^\dagger b \) the total number of quanta, which is fixed to \( N = 2s \). This construction can be straightforwardly extended to \( u(4) \) by defining now

\[
\mathcal{Z} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \\ b_0 & b_1 \\ b_2 & b_3 \end{pmatrix}.
\]

The oscillator realization of the \( u(4) \) generators \( \tau_{\mu\nu} \) is given by

\[
\mathcal{T}_{\mu\nu} = \text{tr}(\mathcal{Z}^\dagger \tau_{\mu\nu} \mathcal{Z}).
\]

Indeed, one can easily verify that \([\mathcal{T}_{\mu\nu}, \mathcal{T}_{\mu'\nu'}] = \text{tr}(\mathcal{Z}^\dagger [\tau_{\mu\nu}, \tau_{\mu'\nu'}] \mathcal{Z})\), and therefore \((63)\) defines a (unitary) representation of \( u(4) \) in the Fock space

\[
|n_\lambda \rangle \otimes |n_\mu \rangle = \left( n_\lambda^a \right)^{n_\lambda^b} |n_\lambda^a \rangle | n_\mu^b \rangle = \prod_{\mu=0}^{3} \frac{(a_\mu^+)^{n_\mu^a} (b_\mu^+)^{n_\mu^b}}{\sqrt{n_\mu^a! n_\mu^b!}} |0\rangle.
\]

Let us look for the expression of the basis states \( |j_{\lambda\mu}^{l,m}\rangle \) in \((34)\) in terms of the Fock basis \((64)\). It is clear that some constraints must be imposed to the occupancy numbers \( n_\mu^a \) and \( n_\mu^b \) in order to obtain a \( d_\lambda \)-dimensional Hilbert space. In particular, we shall see that the constraint \( \mathcal{Z}^\dagger \mathcal{Z} = a^\dagger a + b^\dagger b = \lambda I_{2x2} \) is fulfilled on the basis states \( |j_{\lambda\mu}^{l,m}\rangle \), where \( I_{2x2} \) denotes the \( 2 \times 2 \) identity operator. Firstly we have to fix the total number of quanta \( \sum_{\mu=0}^{3} n_\mu^a + n_\mu^b = 2\lambda \), that is, the linear Casimir operator \( \mathcal{T}_{00} = \sum_{\mu=0}^{3} a_\mu^+ a_\mu + b_\mu^+ b_\mu \) is fixed to \( 2\lambda \). From \((47)\), we also see that the interlayer imbalance operator \( \mathcal{T}_{30} = \sum_{\mu=0}^{3} a_\mu^+ a_\mu - b_\mu^+ b_\mu \) provides the relation \( \sum_{\mu=0}^{3} (n_\mu^a - n_\mu^b) = 2(j + 2m - \lambda) \), so that, when the homogeneity degree \((2j + 2m)\) of \( \psi_{j_{\lambda\mu}^{l,m}} \) equals \( \lambda \) (half the total number of quanta), the configuration \( |j_{\lambda\mu}^{l,m}\rangle \) is balanced (same number of quanta in both layers \(a\) and \(b\)). Therefore, the lowest-weight (zero homogeneity degree) state \( |\varphi_0 \rangle \equiv |j_{0,0}^{0,0}\rangle \) is made of \( 2\lambda \) quanta in the bottom layer \(b\) and can expressed in terms of Fock states as:

\[
|\varphi_0 \rangle = \frac{\text{det}(b^\dagger)^3}{\lambda! \sqrt{\lambda + 1} |0\rangle} = \prod_{k=0}^{\lambda} \frac{(-1)^k}{\sqrt{\lambda + 1}} |k \rangle \bigg| \lambda - k \bigg|_k.
\]

Indeed, one can easily check that \( |\varphi_0 \rangle \) fulfills the constraint \( \mathcal{Z}^\dagger \mathcal{Z} = \lambda I_{2x2} \).

Applying ladder operators \((48), (49), (52)\) and \((53)\) to the lowest-weight state \((65)\) we have been able to obtain the expression of the basis states \( |j_{\lambda\mu}^{l,m}\rangle \) in terms of Fock states \((64)\).
step by step. In the process we find extra restrictions to the number \( n_a^\mu \) and \( n_b^\mu \) of quanta in layers \( a \) and \( b \):

\[
n_0^a + n_1^a + n_2^a + n_3^a = 2(j + m),
\]

(66)

which says that the homogeneity degree \( 2(j + m) \) of \( \phi_{q_a,q_b}^{j,m} \) represents the total number of quanta in the top layer \( a \). Other restriction is

\[
n_0^a + n_1^a + n_0^b + n_3^b = \lambda = n_1^a + n_3^a + n_1^b + n_3^b,
\]

(67)

which states that the total number of ‘even’ (\( \mu = 0, 2 \)) and ‘odd’ (\( \mu = 1, 3 \)) quanta in both layers must be balanced. In the ‘composite bi-fermion’ picture (82) of the next section, ‘even and odd’ (flux) quanta are attached to the ‘first and second’ fermions, respectively. Another interesting restriction is

\[
n_0^a + n_2^a - n_0^b - n_2^b = -2q_a,
\]

(68)

\[
n_1^a + n_1^b - n_1^b - n_1^b = 2q_b.
\]

which says that the ‘magnetic quantum numbers’ \( q_a \) and \( q_b \), measure the imbalance between \( \mu = \{0, 1\} \) (spin up) and \( \mu = \{2, 3\} \) (spin down) type ‘flux’ quanta (see next section for a physical interpretation) inside layers \( a \) and \( b \), respectively. Note the difference of sign in the definition of \( q_a \) and \( q_b \).

The final expression of the basis states \( |\phi_{q_a,q_b}^{j,m}\rangle \) in terms of Fock states (64) is

\[
|\phi_{q_a,q_b}^{j,m}\rangle = \frac{1}{\sqrt{2j+1}} \sum_{q=-j}^{j} (-1)^{q-j-q} |\phi_{q_a,q_b}^{j,m}\rangle \otimes |\phi_{q_a,q_b}^{j,m}\rangle,
\]

(69)

where

\[
|\phi_{q_a,q_b}^{j,m}\rangle = \sum_{k=0}^{\text{max}(0,q+q')} G_{q,q'}^{j,m}(k) \begin{vmatrix} k & j+m+q-k \\ j+m+q' & k-q-q' \end{vmatrix},
\]

(70)

(either for layers \( a \) and \( b \)) with

\[
G_{q,q'}^{j,m}(k) = \sqrt{\frac{2j+1}{(2j+1)!m! (j+q')!(j-q')!}} \\
\times \sqrt{(j+m+q-k)!(j+m+q'-k)!(k-q-q')!} \\
\times \sum_{p=0}^{m} (-1)^p \binom{j+q'}{k-m+p} \binom{j-q'}{k-m+p-q-q'} \binom{m}{p}.
\]

(71)

As the simplest example, let us provide the explicit expression of the basis states \( |\phi_{q_a,q_b}^{j,m}\rangle \) for two quanta (\( \lambda = 1 \)):

\[
\begin{align*}
|0,0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_b, \\
|\frac{1}{2},0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_b, \\
|\frac{1}{2},1\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_b, \\
|\frac{1}{2},\frac{1}{2}\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_b, \\
|\frac{1}{2},-\frac{1}{2}\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_b.
\end{align*}
\]

We do not present here the (rather cumbersome) steps to get this result. We must acknowledge the benefits of Mathematica add-on packages like ‘Quantum Algebra’ to check this and some other expressions along this section. These packages are available at [38].
\[
\begin{bmatrix}
\frac{1}{2}, 0 \\
-\frac{1}{2}, \frac{1}{2}
\end{bmatrix} = \frac{1}{\sqrt{2}} \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).
\]
\[
\begin{bmatrix}
\frac{1}{2}, 0 \\
\frac{1}{2}, -\frac{1}{2}
\end{bmatrix} = \frac{1}{\sqrt{2}} \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).
\]
\[
0, 0 = \frac{1}{\sqrt{2}} \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).
\] (72)

One can prove that the set of vectors \( |\psi_{q,d}^{j,m}\rangle\) constitutes an orthonormal set for each layer, that is
\[
\{ |\psi_{q,d}^{j,m}\rangle, |\psi_{q',d'}^{j,m}\rangle \} = \delta_{j,j'} \delta_{m,m'} \delta_{q,q'} \delta_{d,d'}.
\] (73)

After some algebra, one can realize that the states (70) can be obtained as
\[
|\psi_{q,d}^{j,m}\rangle = \sqrt{\frac{(\lambda - 2j - m)! (\lambda + 1 - m)!}{\lambda! (\lambda + 1)!}} \psi_{q,d}^{j,m}(a^\dagger) |0\rangle,
\] (74)

and an equivalent expression for the layer \( b \), where we are treating now the homogeneous polynomials \( \psi_{q,d}^{j,m} \) in (28) as operator functions, since there is not ordering problem (all \( a^\dagger \) and \( b^\dagger \) commute). Therefore, the basis states (69) can be obtained from the Fock vacuum \( |0\rangle \) as
\[
|\psi_{q,d}^{j,m}\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{q=-j}^{j} (-1)^{q-j} \lambda_q \psi_{q,d}^{j,m}(a^\dagger) \psi_{q,d}^{j,-2j-m}(b^\dagger) |0\rangle.
\] (75)

This is the \( SU(4) \) version of equation (57) for the spin-\( s \) basis states \( |s, q\rangle \) of \( SU(2) \), with the role of the spin \( s \) played now by \( \lambda \) and the role of the monomials \( \phi_q(z) \) played now by the homogeneous polynomials \( \psi_q(z) \).

At this point, we are in condition to provide a boson realization like (58) and (60) but for the CS (36) labeled by points \( Z \) in complex Grassmannian \( \mathbb{G}_2 \).

**Proposition 5.1.** Let us denote by \( \tilde{a} = \frac{1}{2} \eta^{\mu\nu} \text{tr}(\sigma_\mu a^\dagger) \sigma_\nu \) and \( \tilde{b} = \frac{1}{2} \eta^{\mu\nu} \text{tr}(\sigma_\mu b) \sigma_\nu \). The CS \( |Z\rangle \) in (36) can be written as a boson condensate
\[
|Z\rangle = \frac{1}{\lambda! \sqrt{\lambda + 1}} \left( \frac{\det(\tilde{b}^\dagger + Z\tilde{a}^\dagger)}{\sqrt{\det(\sigma^0 + Z\sigma^3)}} \right) \lambda^\lambda |0\rangle.
\] (76)

**Proof.** Using similar steps as in the proof of lemma 3.1 and theorem 3.2, we can also proof that, for any \( 2 \times 2 \) matrices \( A, B \) and \( C \) with \( A \) invertible, the following identity holds
\[
\frac{\det(A + BC)^\lambda}{\lambda! \sqrt{\lambda + 1}} = \sum_{m=0}^{\lambda} \sum_{j=0}^{\lambda-m/2} \sum_{q=d=0}^{j} V_{d-q}^{j,m}(A, B) \psi_{d-q}^{j,m}(C),
\] (77)

with
\[
V_{d-q}^{j,m}(A, B) = \frac{1}{\sqrt{2j + 1}} \sum_{q=-j}^{j} (-1)^{q-j} \lambda_q \psi_{q,d}^{j,m}(B) \psi_{q,-d}^{j,-2j-m}(A).
\] (78)

Taking into account the following properties
\[
A^{-1} = \frac{\tilde{A}}{\det(A)}, \quad \det(\tilde{A}) = \det(A'), \quad \mathcal{D}_{d-q}^{j} (\tilde{X}) = (-1)^{j+q_d+q_B} \mathcal{D}_{q,d}^{j} (X')
\] (79)
and identifying \( A' \to \hat{b}'^\dagger, B' \to \hat{b}' \) and \( C \to Z \), the expression (76) reduces to (36) through the identification (75).

For \( Z = 0 \) we recover the lowest-weight state \( |\psi_0\rangle \) in equation (65) since \( \det(\hat{b}') = \det(\hat{b}) \).

To finish, let us provide another expression of the CS \( |Z\rangle \) in (36), now as an exponential of creation operators.

**Proposition 5.2.** Let us denote by \( T_+^\mu = T_+^\mu \sigma_\mu = 2\hat{a}'^\dagger \). The CS \( |Z\rangle \) in (36) and (76) can be written as the exponential action on the lowest-weight state

\[
|Z\rangle = \frac{e^{i\mu(\hat{Z}T_+^\mu)}}{\det(\sigma_0 + \hat{Z}^\dagger \hat{Z})^{1/2}} |\psi_0\rangle.
\]

(80)

Proving (80) is equivalent to prove that

\[
e^{i\mu(\hat{Z}T_+^\mu)} |\psi_0\rangle = \sum_{m=0}^\lambda \sum_{j_0}^{(\lambda-m)/2} \sum_{q_0}^{j_0} \varphi_{q_0}^{j_0, \mu} (Z) |j_0, \mu\rangle,
\]

which can be done by induction on the homogeneity degree in \( Z \). We shall not give here the (rather cumbersome) details and only shall point out that the equivalence of the expressions (36), (76) and (80) for CS on \( U(4)/U(2) \) is the counterpart of the equivalence of (39) and (58) for CS on \( U(2)/U(1) \).

### 6. Physical interpretation and some comments

Let us propose a physical interpretation of the previous abstract mathematical construction by making use of the fractional QH effect notion of CF [11]. The CF theory maps the strongly interacting system of electrons in a partially filled Landau level to a system of weakly interacting particles called CFs, which are bound states of an electron and a certain number of flux quanta (quantized vortices). The hierarchy of fractional QH states is understood by the use of CFs. Bilayer CF states have also been studied [11]. Here we shall try to make compatible our construction with the CF picture of the BLQH system at filling factor \( \nu = 2 \) and its fractions.

In the BLQH system at filling factor \( \nu = 2 \), there are two electrons in one Landau site. Charged excitations are bi-Skyrmions in the \( \nu = 2 \) BLQH system [14]. The \( G_2 \)-Skyrmion has the general expression

\[
\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2) = \left( \begin{array}{cc} \mathcal{Z}_1^1 & \mathcal{Z}_1^2 \\ \mathcal{Z}_2^1 & \mathcal{Z}_2^2 \end{array} \right),
\]

(82)

where \( \mathcal{Z}_1 \) and \( \mathcal{Z}_2 \) are two \( CP^3 \) fields orthogonal one to another \( \mathcal{Z}_1 \cdot \mathcal{Z}_2 = 0 \). The reader can note the similarity between the bi-Skyrmion (82) and the bosonic matrix (62). Though there are two fields, \( (\mathcal{Z}_1, \mathcal{Z}_2) \), we cannot distinguish them quantum mechanically since they describe two electrons in the same Landau site. Thus, \( \mathcal{Z} \) is not exactly a set of two independent \( CP^3 \) fields.

In fact, two fields \( \mathcal{Z} \) and \( \mathcal{Z}' \) are indistinguishable when they are related by a local \( V \in U(2) \) transformation \( \mathcal{Z}' = \mathcal{Z}V \). The identification \( \mathcal{Z}' \sim \mathcal{Z} \) leaves only four complex field degrees of freedom \( Z = z^\mu \sigma_\mu, z^\mu \in \mathbb{C}, \mu = 0, 1, 2, 3 \). Here we have restricted to one Landau site of the Lowest Landau Level. Hence the parameter space characterizing the \( U(4)/U(2) \)-invariant ground state in the BLHQ system at \( \nu = 2 \) contains four complex independent variables. They are
also the four complex Goldstone modes associated with a spontaneous breakdown of the $U(4)$ symmetry.

For fractional filling factors $\nu = \frac{2}{2}$ we can think of the following ‘composite bi-fermion’ picture. We have two electrons attached to $\lambda$ flux quanta each. The first electron can occupy any of the four isospin states $|b \uparrow\rangle, |b \downarrow\rangle, |a \uparrow\rangle$ and $|a \downarrow\rangle$ in the lowest Landau level. Therefore, there are $(\lambda+1)^2$ ways of distributing $\lambda$ quanta among these two states. Due to the Pauli exclusion principle, there are only three states left for the second electron and $(\lambda+2)^2$ ways of distributing $\lambda$ quanta among these three states. However, some of the previous configurations must be identified since both electrons are indistinguishable and $\lambda$ pairs of quanta adopt $(\lambda+1)^2$ equivalent configurations. In total, there are

$$\frac{(\lambda+3)(\lambda+2)}{(\lambda+1)} = \frac{1}{12}(\lambda + 3)(\lambda + 2)(\lambda + 1)$$

ways to distribute $2\lambda$ flux quanta among two identical electrons in four states, which turns out to coincide with the dimension $d_{\lambda}$ in (30) of the Hilbert space $\mathcal{H}\lambda(\mathbb{G}_2)$ of analytic square-integrable holomorphic functions on $\mathbb{G}_2$ introduced in theorem 3.2. Using Haldane’s sphere picture [39] for the fractional QH effect, $\lambda$ is also related to the ‘monopole strength’ in $\mathbb{G}_2$. Like Haldane’s sphere for monolayer systems, we believe that our construction on $\mathbb{G}_2$ will be very convenient for analytical studies of BLQH systems at fractions of $\nu = 2$. In particular, we think that our construction of CS on $\mathbb{G}_2$ will be relevant to study the interlayer macroscopic coherence in the BLQH system and a semi-classical study of quantum phase transitions, which is usually discussed in the simpler spin-frozen limit. Before, an interconnection between our CS and the usual variational wave functions of Laugling, Halperin and Jain [11, 40–42] for correlated electrons in the lowest Landau level would be in order. This is work in progress.

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Appendix A. Orthonormality of homogeneous polynomials

In order to prove the orthonormality relations

$$\langle j', m' | j, m \rangle = \int_{\mathbb{G}_2} d\mu_j(Z, Z') \chi_{j', m'}^{j, m}(Z) \chi_{m', m}^{j', j}(Z) = \delta_{j,j'} \delta_{m,m'} \delta_{q, q'} \delta_{q', q}$$

we shall adopt the following decomposition for a matrix $Z \in \mathbb{G}_2$

$$Z = V_1 Z V_2^\dagger,$$

where

$$V_u = \frac{1}{\sqrt{1 + r_u^2}} \begin{pmatrix} 1 & r_u e^{iu\theta_u} \\ -r_u e^{-iu\theta_u} & 1 \end{pmatrix}, \quad 0 \leq r_u < \infty, \quad 0 \leq \theta_u < 2\pi, \quad u = 1, 2,$$

are unitary matrices and

$$\Xi = \begin{pmatrix} \rho_1 e^{i\theta_1} & 0 \\ 0 & \rho_2 e^{i\theta_2} \end{pmatrix}, \quad 0 \leq \rho_u < \infty, \quad 0 \leq \theta_u < 2\pi, \quad u = 1, 2.$$

Let us perform this change of variables to the invariant measure (31). On the one hand, the Lebesgue measure on $\mathbb{C}^4$ can be written as:

$$|dZ| = J(\rho_1, \rho_2) d\rho_1 d\theta_1 d\rho_2 d\theta_2 ds(V_1) ds(V_2),$$
Let us start evaluating the first integral. For the diagonal matrix

\[ \int d\mu_{\lambda}(Z, Z') = c_{\lambda} J(\rho_1, \rho_2) \Omega(\rho_1, \rho_2) \prod_{u=1}^{2} \rho_u \, d\rho_u \, d\theta_u \] 

so that the invariant measure reads:

\[ d\mu_{\lambda}(Z, Z') = c_{\lambda} J(\rho_1, \rho_2) \Omega(\rho_1, \rho_2) \prod_{u=1}^{2} \rho_u \, d\rho_u \, d\theta_u \] 

where \( c_{\lambda} = \pi^{-4}(\lambda + 1)(\lambda + 2)^2(\lambda + 3) \).

Let us denote by

\[ N'_{j,m} = \sqrt{\frac{2j + 1 + \left( \lambda + 1 \right) \left( \lambda + 1 \right)}{\lambda + 1}} \left( \frac{\lambda + 1}{m} \right) \]

the normalization constants of the basis functions (28). We want to evaluate:

\[ \langle q_j a_0 | q_{a}^0 \rangle = \left| \langle q_j a_0 | q_{a}^0 \rangle \right| \]

Using determinant properties, the Wigner’s \( D \)-matrix multiplication property

\[ D_{q'q}^j(X) D_{q'q'}^j(Y) = D_{q'''q''}^j(XY) \]

the transpositional symmetry

\[ D_{q'q}^j(Y) = D_{q'q}^j(Y^T), \]

and the fact that \( \det(V_{1,2}) = 1 \) and that \( \Xi \) is diagonal, the previous expression can be restated as:

\[ \frac{[\mathcal{J}', \mathcal{J}'][j,m]}{[\mathcal{J}', \mathcal{J}'][q_1, q_0]} = \sum_{q=q'}^{j} \sum_{q'=q'}^{j} c_{\lambda} \prod_{u=1}^{2} \rho_u \, d\rho_u J(\rho_1, \rho_2) \Omega(\rho_1, \rho_2) D_{q'q}^j(\Xi) \]

\[ \times D_{q'q'}^j(\Xi) \det(\Xi)^m \det(\Xi)^m \prod_{u=1}^{2} \int_{G_2} ds(V_u) D_{q'q}^j(V_u) \]

Let us start evaluating the first integral. For the diagonal matrix \( \Xi \) we have that

\[ D_{q'q}^j(\Xi) = \delta_{q_1, q_0}(\rho_1 e^{i\theta}) \delta_{q_2, q_0}(\rho_2 e^{i\theta}) \]

so that

\[ D_{q'q}^j(\Xi) D_{q'q'}^j(\Xi) \det(\Xi)^m \det(\Xi)^m = \rho_1^{(j+q'+q+m+m')} \rho_2^{(j-q-q'-m-m')} \]

\[ \times e^{i(j'q'-q-q'+m+m)} \]

\[ \times e^{i(j'q'-q-q'+m+m)} \]

Integrating out angular variables gives the restrictions

\[ \int_0^{2\pi} \int_0^{2\pi} D_{q'q}^j(\Xi) D_{q'q'}^j(\Xi) \det(\Xi)^m \det(\Xi)^m \, d\theta_1 \, d\theta_2 \]

\[ = 4\pi^2 \delta_{q, q'} \delta_{j+m, j'+m'} \rho_1^{2(j+q+m)} \rho_2^{2(j-q+m)} \]

Integrating the radial part:

\[ 4\pi^2 \sum_{j} \int_0^{\infty} \frac{J(\rho_1, \rho_2) \Omega(\rho_1, \rho_2) \rho_1^{2(j+q+m)} \rho_2^{2(j-q+m)}}{\rho_1 \rho_2} \, d\rho_1 \, d\rho_2 \]

\[ = \frac{1 + 5q^2 - (j + m)^2 + (j + m + 2q^2 + 1)\lambda}{\pi^2(\lambda + 1)} \]

\[ \sum_{j} R_{j+m}^q \]
and putting all together in (A.7) we have:
\[
\frac{\langle j',m' | j,m \rangle_{q_0,q_0}}{\mathcal{N}_{j,m}^{q_0,q_0}} = \delta_{j+m,j'+m'} \sum_{q=-\min(j,j')}^{\min(j,j')} R_{j+m}^{q} \prod_{u=1}^{2} \int_{\mathbb{S}^2} ds(V_u) D_{q_u,q_u}^j(V_u) D_{q_u,q_u}^{j'}(V_u).
\] (A.9)

The last two integrals are easily computable. Actually they are a particular case of the orthogonality properties of Wigner’s $D$-matrices. More explicitly:
\[
\int_{\mathbb{S}^2} ds(V) D_{q_0,q_0}^{j'}(V) D_{q_0,q_0}^j(V) = \int_0^{2\pi} \int_0^{2\pi} r dr d\phi \frac{r^2}{(1 + r^2)^2} D_{q_0,q_0}^{j'}(\mathbf{r}) D_{q_0,q_0}^j(\mathbf{r}) = \delta_{j,j'} \frac{\pi}{2j' + 1}.
\]

Going back to (A.9) it results:
\[
\langle j',m' | j,m \rangle_{q_0,q_0} = \delta_{j,j'} \delta_{m,m'} \delta_{q_0,q_0} \delta_{q_0,q_0} \left( \frac{N_{j,m}}{2j+1} \right)^2 \sum_{q=-j}^{j} \pi^2 R_{j+m}^{q}.
\]

Finally, taking into account the combinatorial identity:
\[
\sum_{q=-j}^{j} \pi^2 R_{j+m}^{q} = \frac{(2j + 1)(3j + 1)}{(2j + 1 + 1)(3j + 1)} R_{j+m}^{j+m},
\]
and the explicit expression of the normalization constants $\mathcal{N}_{j,m}$, we arrive at the orthonormality relations (A.1).

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