Sine-Gordon solitons in networks: Scattering and transmission at vertices

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Abstract – We consider the sine-Gordon equation on metric graphs with simple topologies and derive vertex boundary conditions from the fundamental conservation laws together with successive space-derivatives of sine-Gordon equation. We analytically obtain traveling-wave solutions in the form of standard sine-Gordon solitons such as kinks and antikinks for star and tree graphs. We show that for this case the sine-Gordon equation becomes completely integrable just as in case of a simple 1D chain. This simple analysis provides a cornerstone for the numerical solution of the general case, including a quantification of the vertex scattering. Applications of the obtained results to Josephson junction networks and DNA double helix are discussed.

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Introduction. – Nonlinear wave dynamics described by the sine-Gordon equation is of importance in a variety of topics in physics, such as as elastic and stress wave propagation in solids, liquids and tectonic plates (see, e.g., [1–7]), transport in Josephson junctions [8], and topological quantum fields [2,6]. Continuous and discrete forms of the sine-Gordon equation have been used so far for the description of wave transport in different media. However, there are structures for which the wave dynamics cannot be described within the traditional continuous or discrete approaches. These are networks and branched structures where the transmission through a branching point (network vertex) should be described by vertex conditions. Early studies of nonlinear evolution equations in branched structures are [9–11], and in the last few years one can observe rapidly growing interest in nonlinear waves and soliton transport in networks described by the nonlinear Schrödinger equation [12–17]. Integrable boundary conditions following from the conservation laws were formulated, and soliton solutions yielding reflectionless transport across the graph vertex were derived in [12], see also [18] for the case of a discrete nonlinear Schrödinger equation. Burioni et al. [19,20] studied the discrete nonlinear Schrödinger equation and computed transport and reflection coefficients as a function of the wave number of a Gaussian wave packet and the length of a graph attached to a defect site.

In this paper we address the wave dynamics in networks described by the sine-Gordon equation on metric graphs, which can be used for modeling of soliton transport in DNA double helix, tectonic plates and Josephson junction networks. The latter has attracted much attention in condensed matter physics [21,22]. Another interesting application of sine-Gordon equations, or, more generally, nonlinear Klein-Gordon equations, on metric graphs can be networks of granular chains [11,23]. Recently, soliton dynamics in networks was studied by considering the 2D sine-Gordon equation on Y and T junctions [24], and the metric graph limit was also studied numerically. See also [25] for similar results for the 2D Nonlinear Schrödinger equation on “fat” graphs.

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and fundamental conservation laws such as energy, charge and momentum conservations together with the asymptotic conditions at infinities: \( \partial_x u_1(x_1, t) \), \( \partial_t u_1(x_1, t) \to 0 \) and \( u_1(x_1, t) \to 2\pi n_1 \) as \( x_1 \to -\infty \), and \( \partial_x u_k(x_k, t) \), \( \partial_t u_k(x_k, t) \to 0 \) and \( u_k(x_k, t) \to 2\pi n_k \) as \( x_k \to \infty \), \( k = 2, 3 \), for some integer \( n_k \), \( k = 1, 2, 3 \).

For the primary star graph in fig. 1, the energy is defined as
\[
E(t) = \sum_{k=1}^{3} \frac{1}{B_k} \int_{B_k} \left[ \frac{1}{2} (u_{kt}^2 + a_k^2 u_{xx}^2) + \beta_k (1 - \cos u_k) \right] dx,
\]
where \( B_1 = (-\infty, 0), B_{2,3} = (0, +\infty) \). Then
\[
\dot{E} = \frac{a_1^2}{\beta_1} u_{1x} u_{1t} \bigg|_{x_1=0} - \frac{a_2^2}{\beta_2} u_{2x} u_{2t} \bigg|_{x_2=0} - \frac{a_3^2}{\beta_3} u_{3x} u_{3t} \bigg|_{x_3=0},
\]
and by (2) the energy conservation reduces to
\[
\frac{a_1^2}{\beta_1} u_{1x} \bigg|_{x_1=0} = \frac{a_2^2}{\beta_2} u_{2x} \bigg|_{x_2=0} + \frac{a_3^2}{\beta_3} u_{3x} \bigg|_{x_3=0}.
\]

For the same star graph as in fig. 1, the charge \( Q \) is given by
\[
2\pi Q = \frac{a_1}{\sqrt{\beta_1}} \int_{-\infty}^{0} u_{1x} dx + \sum_{k=2}^{3} \frac{a_k}{\sqrt{\beta_k}} \int_{0}^{+\infty} u_{kx} dx.
\]
From \( \dot{Q} = 0 \) and (2) we obtain the sum rule
\[
\frac{a_1}{\sqrt{\beta_1}} = \frac{a_2}{\sqrt{\beta_2}} + \frac{a_3}{\sqrt{\beta_3}}.
\]

The initial boundary problem (IBVP) (1), (2) and (4) with appropriate initial conditions and asymptotic conditions at infinities is now well defined. However, to see the infinite number of constants of motion, we must search for other additional conditions for parameters.

**Integrability and traveling-wave solutions.** – We consider the momentum defined by
\[
P = \sum_{k=1}^{3} \frac{a_k}{\beta_k} \int_{B_k} u_{kx} u_{kt} dx,
\]
such that
\[
\dot{P} = \frac{a_1}{\beta_1} \left[ \frac{1}{2} (u_{1x}^2 + a_1^2 u_{1xx}^2) - \beta_1 (1 - \cos u_1) \right] \bigg|_{x_1=0} - \sum_{k=2}^{3} \frac{a_k}{\beta_k} \left[ \frac{1}{2} (u_{kx}^2 + a_k^2 u_{kxx}^2) - \beta_k (1 - \cos u_k) \right] \bigg|_{x_k=0}.
\]

For \( \dot{P} = 0 \), we impose the condition,
\[
\beta_1 = \beta_2 = \beta_3 = \beta(> 0),
\]
which simplifies the sum rule (6) to
\[
a_1 = a_2 + a_3.
\]
Then (8) becomes
\[2\beta P = a_1^n u_1^2(x,0,t) - a_2^n u_2^2(0,t) - a_3^n u_3^2(x,0,t)\]
\[-a_2 a_3}{a_2 + a_3} (a_2 u_2(0,t) - a_3 u_3(0,t))^2,\]
where we used (4), (9) and (10) in obtaining the last expression. Thus, \(P = 0\) yields \(a_2 u_2(0,t) = a_3 u_3(0,t)\), which together with (4), (9) and (10) gives
\[a_1 u_1(0,t) = a_2 u_2(0,t) = a_3 u_3(0,t).\]  
(11)

Conditions on higher-order space-derivatives may be available from higher-order conservations, where the analysis becomes more laborious. However, they can also be obtained directly from (1), (2) via \(a_1^n u_{1xx}(x,t) = (u_{1ht} + \beta \sin u_1)_{xx} = (u_{kht} + \beta \sin u_k)_{xx} = a_2^n u_{3xx}(x,t)\) for \(k = 2, 3\). Thus,
\[a_1^n u_{1xx}(0,t) = a_2^n u_{2xx}(0,t) = a_3^n u_{3xx}(0,t),\]  
(12)

and similarly, taking successive \(x\)-derivatives of (1), we obtain the conditions on higher-order space-derivatives. It should be emphasized that the momentum conservation requires (9)–(11), from which (12) follows. Now we shall prove that eqs. (2), (11) and (12) give a scaling function which guarantees the infinite number of constants of motion.

Let us introduce two functions defined on the bonds from 1 to \(k = 2, 3\) as \(v_{1,k} \equiv u_k \left(\frac{\alpha \sqrt{2}}{\sqrt{4 - k}}\right)\) for \(x < 0\) and \(v_k \left(\frac{\alpha \sqrt{2}}{\sqrt{4 - k}}\right)\) for \(x > 0\). Thanks to the vertex boundary conditions (VBC) in (11) and (12), together with the continuity condition in (2), both of \(v_{1,k}\) with \(k = 2, 3 \in C^2(\infty, \infty)\) and satisfy \(v_{1,2} = v_{1,3} = v(x,t)\), where \(v(x,t)\) is a solution of the dimensionless sine-Gordon equation
\[v_{tt} - v_{xx} + \sin v = 0\]  
(13)
defined on the real line. This fact is identical to the expression of \(u_k(x,t)\) in terms of the function \(v\) as
\[u_k(x,t) = v \left(\frac{\sqrt{2}}{\alpha_k}, \frac{\sqrt{2}}{\alpha_k}\right), x \in B_k \quad (k = 1, 2, 3).\]  
(14)
The scaling function \(v\) in (14) together with the sum rule (10) guarantees the infinite number of constants of motion and thereby the complete integrability of the sine-Gordon equation on the network.

In fact, from (14) and the sum rule (10) we find that all the conservation laws \([28, 29]\)
\[\int_{-\infty}^{\infty} \left(\frac{\beta}{a_k}\right)^{\frac{1}{2}} u_{k} \left(\frac{\alpha \sqrt{2}}{\sqrt{4 - k}}\right) dx = \text{const}\]  
(15)
of the sine-Gordon eq. (13) on the real line also hold on the star graph, because
\[\sum_{k=1}^{3} \left(\frac{\beta}{a_k}\right)^{\frac{1}{2}} u_{k} \left(\frac{\alpha \sqrt{2}}{\sqrt{4 - k}}\right) dx = \text{const}\]
where we used (4), (9) and (10) in obtaining the last expression. Thus, \(P = 0\) yields \(a_2 u_2(0,t) = a_3 u_3(0,t)\), which together with (4), (9) and (10) gives
\[a_1 u_1(0,t) = a_2 u_2(0,t) = a_3 u_3(0,t).\]  
(11)

The conservation of energy \(E\), charge \(Q\) and momentum \(P\) are just special cases.

From now on, we shall prescribe \(\beta = 1\) without loss of generality. Equation (13) has a number of explicit soliton solutions, for instance: kink “+” and anti-kink “−” solutions which can be written as \([3, 4]\)
\[v(x,t) = 4 \arctan \left(\frac{\exp \left(\frac{x - x_0 - vt}{\sqrt{1 - \nu^2}}\right)}{\sqrt{1 - \nu^2}}\right),\]  
(17)
where \(|v| < 1\) is the velocity of the kink. Other soliton solutions include breathers, kink-kink collisions and kink-antikink collisions \([4]\), to name just a few, see also \([30]\) for further multisoliton-type solutions. If the sum rule in (10) holds, then all these solutions, or, more generally, all solutions of (13), transfer via (14) to solutions on the metric graph. For instance, the kinks then provide reflectionless transmission of energy through the graph vertex, where the speed and energy of a kink traveling in positive direction (\(\nu > 0\)) split according to the ratios \(a_2/a_1\) and \(a_3/a_1\), respectively. On the other hand, launching two suitably fine tuned kinks on bonds 2 and 3 in negative direction, their joint energy is transmitted to bond 1.

Before entering into the numerical analysis of kink dynamics, we comment on other VBCs originating in the local scattering properties at each vertex. The VBC (2) of continuity and (4) of local flux conservation with \(\beta_k = 1(\alpha = 1, 2, 3)\) are also called \(\delta VBC\). They naturally appear \([24]\), see also \([25]\) for a similar construction for the case of the NLS, and \([27]\), Chapt. 8) for an overview of related linear results) by considering the 2D sine-Gordon equation \(\partial_t^2 u - \Delta u + \sin u = 0\) on a “fat” graph, i.e., a 2D branched domain with Neumann boundary conditions, where \(w_2/w_1 = a_2^2/a_1^2\) and \(w_3/w_1 = a_3^2/a_1^2\) are the relative widths of the (fat) bonds.

Similarly, the so-called \(\delta^\prime\) VBCs \([27]\), Chapt. 8) consist of (11) and
\[a_1 u_1(0, t) - a_2 u_2(0, t) - a_3 u_3(0, t) = 0,\]  
(18)
which conserve charge and energy for all values of the \(a_k\). A simple calculation shows both \(\delta\) and \(\delta^\prime\) VBCs can be derived from (10) and (14), but the inverse derivation is not possible. We note that (11) and (18) conserve \(E\) and \(Q\), but if conservation of \(P\) is enforced, then eq. (18) reduces to (2). Most importantly, (10) and (14) give the existence of the infinite number of constants of motion (as shown in (15), (16), which is equivalent to the complete integrability of the sine-Gordon equation on the graph.
**Vertex transmission.** — An important issue for wave dynamics in networks is the scattering at vertices. The sum rule in (10) allows the tuning of the vertex scattering to achieve reflectionless transmission. We now give numerical solutions of (1) with β=1 (k=1,2,3), using 2nd order in space and time finite differences, where we first focus on (2) and (4) as VBC, i.e., the δ case. Figure 2 shows the reflectionless propagation of a kink in the special case that the sum rule (10) holds.

In fig. 3 we numerically treat the transmission of solitons through the graph vertex when the sum rule (10) is violated. In (a)–(c) we consider the “natural” case a_k = 1, k=1,2,3, and the same kink initial condition as in fig. 2. The total energy is still conserved (by (4)), but in contrast to the reflectionless case from fig. 2, there now is significant reflection at the vertex. To demonstrate and quantify the dependence of the vertex transmission on the a_k in some more detail, in fig. 3(d) we essentially return to the situation of fig. 2. That is, we set a_1 = 1, a_2 = 0.7, but let a_3 vary and plot the reflection coefficient R, defined as the ratio of the energies in the incoming bond at initial time t = 0 and at t = 15. At a_3 = 0.3, corresponding to fig. 2, we have R = 0, i.e., zero reflection.

Additionally, the red line in fig. 3(d) shows the analogous simulation for the case of δ’ vertex conditions (11) and (18). Again we have zero reflection at a_3 = 0.3, while violating the sum rule gives qualitatively similar but slightly stronger reflections than the δ case. Two points should be noted: the simulations in figs. 2 and 3 have also confirmed the conservation (up to numerical discretization effects) of Q and P so long as the sum rule (10) holds (see fig. 4); the simulations do not use the soliton properties of the kinks, but only the fact that they are travelling-wave solutions for which we have formulas for the initial conditions. Thus, these numerical results can be transferred to general nonlinear Klein–Gordon equations that admit travelling-wave solutions.

**Other graph topologies.** — Our results can be extended to other simple topologies such as general star graphs, tree graphs, loop graphs and their combinations. Exact traveling-wave solutions of sine-Gordon models on such graphs with one incoming semi-infinite bond can be obtained similarly to the above case of a star graph with three bonds, leading to generalizations of the sum rule. We illustrate this for the tree graph from fig. 5, consisting of three “layers” b_1, b_{i1}, b_{ij}, where i, j run over the given bonds.

On each bond b_1, b_{i1}, b_{ij} we have a sine-Gordon equation given by (1). Setting β_i = β_{i1} = β_{ij} = 1 for all i, j, the a_{i1} and a_{ij} have to be determined from the sum rule like (10) at each vertex. For instance, at the three nodes
Finally, it can be shown that the above approach can be applied to obtain exact traveling-wave solutions of sine-Gordon models on other (than above) graphs consisting of at least two semi-infinite bonds and any subgraph between them. In this case one needs to impose the pertinent vertex conditions like (20) or (22) at the vertices connecting the semi-infinite bonds with the subgraph.

### Conclusions.

In this work we studied sine-Gordon equations on simple metric graphs, and derived vertex boundary conditions for charge, energy and momentum conservation, and additionally conditions on parameters, for which the problem has explicit analytical soliton solutions. We find the sum rule (10) for bond-dependent coefficients at each vertex of the graph, which makes the sine-Gordon equation on the graph completely integrable. It is shown that the obtained solutions provide the reflectionless transmission of solitons at the graph vertex. This is also illustrated numerically by quantifying the reflections for a case where these conditions are violated, and we discussed how to generalize the results to other graph topologies. The results can be directly applied to several important problems such as Josephson junction network and DNA double helix. In such approach our model corresponds to continuous version of the system considered in [21,22]. Finally, a very important application can be DNA double-helix models where the energy transport is described in terms of sine-Gordon equations [31,32]. Base pairs of the DNA double helix can be considered as a branched system and modeled by a star graph [32]. Then the H-bond energy between two base pairs in such system can be characterized by the parameter, \( \beta \).

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