Unlocked-relative-phase states in arrays of Bose-Einstein condensates

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Phase engineering techniques are used to control the dynamics of long-bosonic-Josephson-junction arrays built by linearly coupling Bose-Einstein condensates. Just at the middle point of the underlying discrete energy band of the system, unlocked-relative-phase states with uniform particle density across the array are shown to be stationary along with the locked-relative-phase Bloch waves. In finite, experimentally-feasible systems, such states find ranges of dynamical stability that depend on the ratio of coupling to interaction energy. This ratio determines also different decay regimes, which include the recurrence of staggered soliton trains in the condensates that are formed around Josephson loop currents at the junctions. Additionally, the preparation of maximally out-of-relative-phase (or splay) states is demonstrated to evolve into the quasiperiodic oscillation of the uniform density of the condensates that keeps constant the total density of the system. Such behavior is altered by the presence of noise, which gives rise to the appearance of localized Josephson currents and eventually breaks down the axial uniformity of the system.

I. INTRODUCTION

The arrays of Josephson junctions, either in superconducting or nonlinear-optical systems, have been very successful in the development of technical applications. Three types of time periodic states have been studied in series arrays of Josephson Junctions: in-phase states (or locked-phase junctions), splay states (with the phases of the junctions evenly distributed), and incoherence states (with nonuniform distribution of junction phases) [1]. Equivalent states have also been found in the related systems of globally coupled (discrete) Ginzburg-Landau equations [2]. As far as we know, only the first type of such stationary states, having locked-phase junctions, have been explored within the scope of bosonic Josephson junctions made by arrays of coupled Bose-Einstein condensates (BECs). This situation may derive from the fact that these systems lack in general the global-coupling arrangement of junctions used in superconductors or optics. Instead, BECs are usually connected by the next-neighbor coupling established through the barriers of optical-lattice potentials operating in a tight binding regime [3]. Ultimately, both configurations, next-neighbor and global coupling, can be considered as limit cases of linear coupling with different spatial ranges [4].

The Josephson effect was soon realized in ultracold-gas experiments [5–7], addressing mainly phenomena associated to single and short Josephson junctions. Regarding extended junctions, special theoretical attention has been paid to systems of two linearly coupled one-dimensional (1D) BECs, which configure a single long bosonic Josephson junction. Beyond the symmetric and anti-symmetric uniform states typical of the point-like junction in a double-well potential, many works have focused on the stationary nonlinear waves known as Josephson vortices [8–14], which have been recently observed in experiments [15].

Concerning the study of junction arrays, up to date only the arrays of point-like Josephson junctions have been experimentally realized [16]. Theoretically, particular features of the arrays of long Josephson junctions have been explored, ranging from the superfluid-insulator transition [17], the motion of bright solitons [18], the exotic phases in the presence of gauge fields [19], the stabilization of sets of localized dark solitons and Josephson vortices [20], or the generation of transverse Josephson vortices [21].

In this work, we consider arrays of long-bosonic Josephson junctions that are brought about by the stack of linearly-coupled elongated BECs. The junctions are described through the relative phases of next-neighbor condensates, and their dynamics is studied within the
Gross-Pitaevskii theory. Our center of interest is the existence and stability of array states whose junctions have not locked relative phases. On the one hand, we show that there exist a set of stationary states, living just at the middle of the discrete energy band, that, in spite of sharing energy and density profile with uniform Bloch waves, break their locked relative phase. Dynamically stable states of this type can be found at high coupling, and hence they are relevant for experimental realization. On the other hand, we address the dynamics of maximally out-of-relative-phase states that mimic the splay states in globally coupled junctions. Although stationary states of this type cannot be found in next-neighbor coupling arrangements, we show how these states evolve through oscillations of the uniform density of the condensates that preserve the total density in the absence of noise. Different decay scenarios of unlocked-relative-phase states that involve the emergence of solitons and localized Josephson currents are discussed.

II. ARRAY OF COUPLED ELONGATED BECS

Figure 1 shows the prototypical arrangement of the considered arrays. In this example, the system is made of $M = 5$ elongated BECs linearly coupled along their common axial $x$-direction, forming a ring-shaped array. A linear coupling of energy $\hbar \Omega/2$ connects next-neighbor components and determines an effective transverse distance $\delta_y = \sqrt{\hbar/m \Omega}$ between them. Along the axial direction, the interparticle interaction defines a healing length $\xi = \hbar/\sqrt{m \xi y}$, where $n$ is the axial atomic density of the BEC, $g > 0$ is the contact-interaction strength, and $m$ is the atomic mass. The ratio $\nu = (\xi/\delta_y)^2 = \hbar \Omega/g n$ regulates the amount of particle tunneling across the condensate junctions. As it has been recently proposed \cite{19, 20}, such a system is feasible to experimental realization with ultra-cold gases loaded in optical lattices.

Within a mean field approximation at zero temperature, the dynamics of an $M$-condensate array follows the Gross-Pitaevskii (GP) equations

$$i \hbar \frac{\partial \Psi_j}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g |\Psi_j|^2 \right) \Psi_j - \frac{\hbar \Omega}{2} \left( \Psi_{j-1} + \Psi_{j+1} \right),$$

for the complex order parameters $\Psi_j(x,t) = \sqrt{n_j(x,t)} \exp[|\Theta_j(x,t)|]$, with density $n_j(x,t)$ and phase $\Theta_j(x,t)$, of the elongated BECs $j = 0, 1, \ldots, M-1$.

The transverse dynamics inside each BEC is assumed to be frozen by means of a tight transverse confinement. For the sake of analytical treatment, we assume infinite length along the $x$-direction, hence no external potential enters Eq. (1). The Josephson dynamics in the elongated junctions separating the BECs can be described through the relative phases $\varphi_j(x,t) = \Theta_{j+1}(x,t) - \Theta_j(x,t)$ and relative densities $\delta_j(x,t) = n_{j+1}(x,t) - n_j(x,t)$.

The periodic configuration along $y$ admits stationary states in the form of transverse Bloch waves with transverse quasimomentum $\hbar K_x$, which we additionally set to have also a definite axial momentum $\hbar K_z$ as

$$\Psi_{j,k}(x,t) = \sqrt{n} \exp \left[ i (k \cdot r_j - \mu_k t/\hbar) \right],$$

where $k \cdot r_j = K_x x + K_y y_j \equiv \Theta_j(x)$ are the time-independent phases, and $\mu_k = gn + \hbar^2 K_z^2/2m - \hbar \Omega \cos(K_k \delta y)$ is the chemical potential, which takes values in the discrete energy band $\mu_k \in gn + \hbar^2 K_z^2/2m + [-\hbar \Omega, \hbar \Omega]$ of width $2\hbar \Omega$. The momentum and spatial vectors are denoted by $k = (K_x, K_y)$ and $r_j = (x,y_j)$, respectively, and $y_j = j \delta y$ represents the discrete transverse coordinate. The quasimomentum can take only $M$ different integer values within the first Brillouin zone $K_k = 2\pi k/M \delta y$ with $k \in \{0, \pm 1, \pm 2, \ldots, \lfloor M/2 \rfloor \}$ where $\lfloor M/2 \rfloor$ is the greatest integer less than or equal to $M/2$. All the Bloch states with constant density are dynamically stable if $k \leq M/4$, irrespective of the coupling $\Omega$ \cite{21}.

In a Bloch state, for a given time and a fixed axial position, the phases of the BEC components of quasimo-
mentum $K_k$ are uniformly separated $\Theta_j(x = 0, t = 0) = 2\pi kj/M$, so that the relative phases $\varphi_j = 2\pi k/M, \forall j$, are locked everywhere and for every time. The top panel of Fig. [3] shows schematically this fact for a $M = 4$ array in a dynamically stable Bloch wave with $k = 1$, just at the middle of the discrete energy band. The BEC phases of the array $\Theta_j(x = 0, t = 0) = [0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}]$ (circles in the left chart), and the relative phases $\varphi_j = [\pi, \frac{\pi}{2}, \frac{3\pi}{2}, \pi]$ (triangle in the right chart) are represented on the unit circle, in a phasor diagram.

Interestingly, along with the Bloch waves, there exist alternative stationary states that have constant density everywhere. Such states, which break the monotonic variation of the phase across the array, exist by virtue of the discrete nature of the system, since equivalent states sharing energy and density profile with the Bloch waves can not exist in continuous periodic potentials. The middle and bottom panels of Fig. [3] depict stationary configurations of the array phases belonging to this set of states. We will generically refer to the states of this set as unlocked-relative-phase states, in contrast to the locked relative phases of the Bloch waves.

### III. LOCKED- AND UNLOCKED-RELATIVE-PHASE STATES

#### A. Stationary states

The Bloch waves situated just at the middle of the energy band, that is with $k = M/4$ (as the case shown in Fig. [2]), present $\frac{\pi}{2}$ relative phases that add to $\pi$ phase jumps between second-neighbor condensates. This fact cancels the coupling dependence in the GP Eq. [1], since $\Psi_{j+1} + \Psi_{j-1} = 0$, and thus in the chemical potential $\mu_{M/4} = gn + \hbar^2 K^2/2m$. The resulting configuration resembles the features of splay states in globally coupled oscillators [2], which in turn model Josephson-junction arrays [1]. There, the splay states are characterized by oscillator phases that add to neutralize the coupling. To do so, the phases are maximally out of phase, evenly distributed around the unit circle in a phasor diagram, and their existence is accompanied by a high degeneracy induced by the all-to-all coupling. Although in the setup considered here the nearest-neighbor connection imposes a more restricted scenario, the neutralization of the coupling term in GP equation [1] leads also to new degenerate stationary states.

In canceling the coupling, the condensate phases and the relative phases in a Bloch wave with $k = M/4$ fulfill

$$ \Theta_{j+1}(x) - \Theta_{j-1}(x) = \varphi_j + \varphi_{j-1} = \pi . \quad (3) $$

This precise configuration allows us to introduce an extra degree of freedom, a phase $\Delta \varphi$ that modifies neither the energy nor the chemical potential of the system when added to every second component, so that

$$ \varphi_j = \frac{\pi}{2} + \Delta \varphi, $$

$$ \varphi_{j-1} = \frac{\pi}{2} - \Delta \varphi. \quad (4) $$

The arbitrary phase $\Delta \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ can be added to and subtracted from the relative phases of consecutive junctions to get a new, degenerate stationary state. In this way the relative phases of the array, which are locked for the Bloch wave, become unlocked without energy cost. This degeneracy reflects the symmetry of the short range coupling between condensates. Simultaneously, the Josephson current $J_j = \Omega n \sin(\varphi_j)$ flowing between components $j$ and $j+1$, which measures the particle tunneling through the junctions, is reduced by a factor $\cos(\Delta \varphi)$ with respect to the phase-locked configuration. Note that the condensate phases resulting from Eq. [4] are in general not uniformly distributed in a phasor diagram, which corresponds to a diagram of the so-called incoherent states [1].

The set of stationary states generated by the operation given in Eq. [4] can only be found in arrays whose number of components $M$ is a multiple of four, where the Bloch wave with $k = M/4$ exists. The simplest state of this type appears in an system with $M = 4$. From its Bloch wave with $k = 1$, new steady configurations can be chosen as $\Theta_j(x) = K_\pi x + [0, \frac{\pi}{2} - \Delta \varphi, \pi, \frac{3\pi}{2} - \Delta \varphi]$, and the relative phases read $\varphi_j = [\frac{\pi}{2} - \Delta \varphi, \pi - \Delta \varphi, \frac{3\pi}{2} - \Delta \varphi]$. In particular, for $\Delta \varphi = \pi/2$ one gets a stationary configuration with $\Theta_j(x) = K_\pi x + [0, 0, \pi, \pi]$ and $\varphi_j = [0, 0, \pi, 0, \pi$, which is represented in the bottom panel of Fig. [2] and that we will refer to hereafter as out-of-phase state. In this limit case the Josephson current in the array cancels. This is the maximally out of phase, stationary pattern achievable in the relative phase diagram of a system with nearest-neighbor coupling. For higher-$M$ arrays, the same phase pattern can be found repeated in out-of-phase stationary states that are degenerate with the Bloch wave of $k = M/4$, that is $\Theta_j(x) = K_\pi x + [0, 0, \pi, 0, \pi, 0, \pi, \ldots]$, and $\varphi_j = [0, 0, \pi, 0, \pi, 0, \pi, \ldots]$.

These phase patterns reflect the configurations associated with the boundary of the Brillouin zone in a supercell structure with double period. They can also be seen as the phases induced by a train of transverse dark solitons whose nodes are located at every other junction of the array.

#### 1. Linear stability

The linear stability of unlocked-relative-phase states, with generic stationary phases $\Theta_j(x) = K_\pi x + [0, 0, \pi, 0, \pi, 0, \pi, \ldots]$, can be found from the Bogoliubov equations for the linear excitations $[u_j(x), v_j(x)]$ with en-
ergy $\mu + h\omega$ \[22\]:

$$H u_j + g n e^{i2\Theta} v_j - \frac{h\Omega}{2} (u_{j-1} + u_{j+1}) = h\omega u_j,$$

$$-H v_j - g n e^{-i2\Theta} u_j + \frac{h\Omega}{2} (v_{j-1} + v_{j+1}) = h\omega v_j,$$

where $H = -(h^2/2m)\partial_x^2 + 2gn - \mu$, and $\mu = \mu_M/4 = gn + h^2K_x^2/2m$.

In the limit cases $\Delta \varphi = \pm \frac{\pi}{2}$, Eqs. \[6\] can be readily solved by making use of the Fourier expansions $u_j(x) = \sum_q u_q \exp\{i[K_x x + q_x x + q_y y_j]\}$ and $v_j(x) = \sum_q v_q \exp\{-i[K_x x - q_x x - q_y y_j]\}$, where $q_p = 2\pi p/M\delta y$ is the transverse momentum of the excitation for integer $p = 0, \pm 1, \pm 2, \ldots [M/2]$. The Bogoliubov equations get decoupled for each two-dimensional wave number $q = (q_x, q_p)$, and the resulting dispersion is

$$h\omega = \frac{h^2K_x q_x}{m} +$$

$$\pm \sqrt{\left(\xi_x - h\Omega \cos\left(\frac{2\pi p}{M}\right)\right)\left(\xi_x - h\Omega \cos\left(\frac{2\pi p}{M}\right) + 2gn\right)},$$

where $\xi_x = h^2q_x^2/2m$. The energy branches with $p < M/2$ produce imaginary frequencies $\omega$ that are associated to unstable modes. Due to their plane wave character, these modes are not localized. The maximum imaginary frequency leading the decay of these out-of-phase states depends on the ratio $h\Omega \cos(2\pi p/M)/gn = \nu \cos(2\pi p/M)$. The analysis is simpler for $K_x = 0$. In this case, if such ratio is less than one the maximum imaginary frequency has $\text{Im}[h\omega] < gn$ and corresponds to $q_x = 0$, otherwise it reaches $\text{Im}[h\omega] = gn$ and corresponds to quasimomenta $q_x = \pm \sqrt{2(\nu \cos(2\pi p/M) - 1)}/\xi_x$; for high $\nu \gg 1$ the range of unstable modes becomes localized around this maxima (see Fig. 3). As we show next, different decay dynamics result from each case, and the mentioned localization of the unstable modes in momentum space suggests a way to find stable steady configurations of out-of-phase states in finite systems, where the momentum can only take discrete values. Stability is found when the set of these discrete momenta do not sample the small ranges of unstable modes.

2. Dynamics

We report on the typical dynamics of out-of-phase states by numerically solving the GP Eqs. \[1\] for an array of $M = 4$ condensates. As has been shown in the linear analysis, the ratio $\nu = h\Omega/\sqrt{gn}$ determines the conditions for the stability of the system, and only the branch $p = 0$ of Eq. \[6\] produces unstable modes. To demonstrate this fact, we have selected three case examples with $\nu = 0.5, 10$, and 22, that represent respectively different dynami-
FIG. 5. Staggered solitons during the emergence of the stripe phase in the evolution of the out-of-phase state presented in Fig. 4. At the junction between condensates $j = 1$ and $j = 2$, consecutive, counter-rotating loop currents are centered at the zeros of the Josephson current $J_1$. The loops are closed by condensate axial currents, and their direction is reversed in the next soliton recurrence. Time, length and densities are given in the same units of Fig. 4 whereas current is given in arbitrary units.

cal regimes. In all of them, the system is constrained to evolve in an axially finite domain of length $L = 17.9 \xi$ and periodic boundary conditions. A white noise perturbation has been added to the initial stationary state in order to simulate a more realistic scenario.

The out-of-phase state is dynamically stable for $\nu = 22$ since the unstable frequencies predicted by Eq. (1) occur for axial momenta (around $q_x = \pm 6.48/\xi$) that are not sampled by the momenta $k = 2\pi/L \times n = 0.35/\xi \times n$, for integers $n$, determined by the finite system. For the given axial length, there are in fact many other instances of coupling above $\nu \approx 17.8$ that provide stability, e.g., systems with $\nu = 19.9, 22, 24.5$ or 27 are equally stable. Our results of the nonlinear time evolution of the array in the presence of initial noise (not shown here because of their flat, uniform density and phase profiles) confirm the linear analysis. In this way, the dynamical stability of these systems makes it possible their experimental realization.

However, for smaller coupling values, at $\nu = 10$, and $\nu = 0.5$, the instability cannot be prevented, and the out-of-phase configurations decay during the time evolution. An observable common feature, as can be seen in Figs. 4 and 6, is the synchronization between the components with initial equal phases. Nevertheless, the system dynamics presents notable differences in both cases. For $\nu = 10$, Fig. 4 shows a quasi-periodic time evolution during which trains of solitons emerge in the axial direction of each component, breaking temporarily the uniformity of the $\pi$ relative phases and creating localized Josephson loop currents, and vanish, returning the array to its initial configuration. The number of solitons is given by the standing waves created with the wavenumbers $q_x = \pm 4.2/\xi$ of the only two unstable frequencies (for this coupling strength). The solitons are staggered in the condensates with $\pi$ relative phase, which form trains of dark-bright solitons when the condensates are combined (see Fig. 5). As a result, the total density profile of the system shows stripes of high contrast. The dark-bright sequence and also the Josephson loop currents are reversed with each new recurrence. The time recurrences of the solitons, thus of the striped state, suggest that this configuration is also dynamically unstable for the parameters considered here, as it also happens in similar types of modulation instability [23].

At low $\nu$ the decay of out-of-phase states is characterized by the presence of several unstable modes whereby a quasiperiodic behavior cannot not reached. The interaction of unstable modes produces complicated scenarios that can soon give rise to chaotic dynamics. An example is shown in Fig. 6 for $\nu = 0.5$, where a few moving solitons and unsteady localized Josephson currents can be seen to emerge and interact within the array components. After this, the axial and transverse dynamics of the array are strongly coupled and the evolution increases progressively in complexity.

The dynamical regimes of generic unlocked-phase states with $\Delta \varphi < |\pi/2|$ do not present significant differ-
FIG. 7. Same as Fig. 4 for an unlocked-phase state with $\Delta \varphi = \pi/4$ and $\nu = 10$. Differences with respect to those shown for out-of-phase states. For detailed comparison, Fig. 7 shows the time evolution of an unlocked-relative-phase state with $\Delta \varphi = \pi/4$ and $\nu = 10$, sharing the rest of parameters with the out-of-phase state of Fig. 4. In this case, the recurrences of the soliton trains present lower contrast and occur at higher rate. Curiously, the synchronization is only clearly observable between second-neighbor condensates with initial $\pi$ relative phases. In addition, consecutive solitons in the soliton trains of component $j = 0$ and $j = 2$ evolve through merging, or alternatively splitting, in order to produce new, reversed staggered configurations.

B. Non-stationary maximally out-of-phase junctions

In the absence of all-to-all coupling, there is no stationary state presenting junctions with relative phases that were uniformly distributed on the phasor diagram. Such a system would be the equivalent of the splay states in the array of globally coupled Josephson junctions [1]. However, it is still possible to prepare a splay state in arrays with an odd number of components, and to use it as initial state in order to monitor its subsequent time evolution. Although this configuration leads to the generation of non-uniform Josephson currents in the array, our numerical results show that, in the absence of perturbative noise, the dynamics preserves the total constant density, and each condensate follows a quasi-periodic behavior of oscillating density without axial variations. However, the presence of noise breaks the axial uniformity within the components of the array and eventually produces the decay of the quasiperiodic behavior.

As a case example, we have prepared a maximally-out-of-relative-phase state in an array with $M = 5$ components and $\nu = 1$. The relative phases $\varphi_j = [0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5]$, are determined from setting the condensate phases $\Theta_j = [0, 0, 2\pi/5, 6\pi/5, 2\pi/5]$. Figure 8 shows the subsequent time evolution after adding perturbative noise to the initial configuration. During the initial stage, the evolution follows the same described behavior as in the absence of noise. Beyond $t \approx 60$, as can be seen, the noise induces local variations of the Josephson currents between components that modify the flat density profiles. In spite of this fact, the
initially in-phase condensates keep their synchronization during the whole time evolution.

IV. CONCLUSIONS

The present work contributes to the characterization of the arrays of long-bosonic Josephson junctions built with linearly coupled Bose-Einstein condensates. In these systems, we have demonstrated the existence and stability conditions of extended, uniform density states with unlocked relative phases. These states appear from the effective cancellation of the coupling in the equations of motion of the array, which allows for a new energy degeneracy in the system (other than the usual of the Bloch waves with equal absolute value of the quasimomentum) that is conditioned by the next-neighbor coupling of the discrete array. Regimes of stability, of quasiperiodic recurrence of striped density, and of complex, chaotic dynamics have been found depending on the ratio of coupling to interaction energy; and higher ratios favor stability. The typical decay dynamics of these states shares features with other modulation instabilities, proceeding through the condensate density variation according to the growth of standing waves created by unstable modes. Simultaneously, trains of counter-rotating Josephson loop currents centered at the junctions play equivalent role to that of regular vortex dipoles created at the nodal lines of dark solitons in continuous systems.

Alternatively, we have explored the preparation and dynamics of maximally out-of-relative-phase states, which are well known in globally coupled junctions. Although the next-neighbor connection of the BECs do not allow for such stationary configuration, the prepared state evolves, in the absence of noise, into a quasiperiodic oscillation of the uniform density of the condensates that keeps constant the total density of the system.

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