Topological Vortex Lines in Two-Gap Superconductor

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Based on the $U(1)$ gauge potential decomposition theory and the $\phi$-mapping method, we study the vortex lines in two-gap superconductor and obtain the condition, under which the vortices can carry an arbitrary fraction of magnetic flux. It has been pointed out that the Chern-Simon action is a topological invariant, which is just the total sum of all the self-linking numbers and all the linking numbers of the knot family.

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I. INTRODUCTION

Superconductors allow for a rich variety of topological defects and phase transitions [1]. The best known topological object is the Abrikosov vortex in one-gap superconductors. But the advent of two-gap superconductors has opened a new possibility for us to have far more interesting topological properties [2]. The equivalence between the Ginzburg-Landau-Gross-Pitaevskii (GLGP) functional and the nonlinear $O(3)$ $\sigma$ model makes knotted solitons exist in the two-gap system [3]; Monopoles different from those in one-gap superconductors have been also discussed in Ref. [4]. Here we shall particularly interest in the topological vortex lines in two-gap superconductor. The purpose of the paper is twofold. First we find out the condition, under which the vortices can carry an arbitrary fraction of magnetic flux quantum. The second purpose is to research the knotted configurations in the two-gap superconductor.

The intriguing possibilities of topological defects carrying a fraction of flux quantum have long attracted interest, and several nontrivial realizations were identified [1]. Simultaneously, it is well known that knots as string structure of finite energy appear in a variety of physical scenarios, including the structure of elementary particles [5, 6], the early cosmology [7, 8], Bose-Einstein condensation [9]. In geometry, a knot is an embedding map $\gamma : S^1 \to R^3$. Two or more knots together are called a link, i.e., a family knots, which possesses many important topological numbers, including self-linking number and Gauss linking number [10]. Here we research the vortex lines and the knotted vortex lines in two-gap superconductor from the point of view of topology and reveal the inner relationship between the Chern-Simon action and the topological characteristic numbers of the knot family.

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II. FRACTIONAL MAGNETIC FLUX

Two-gap superconductor can be described by a two–flavor (denoted by \( \alpha = 1, 2 \)) GLGP functional, whose free energy density is given by

\[
F = \frac{1}{2m_1} \left| \left( \hbar \partial_\lambda + i \frac{2e}{c} A_\lambda \right) \Psi_1 \right|^2 + \frac{1}{2m_2} \left| \left( \hbar \partial_\lambda - i \frac{2e}{c} A_\lambda \right) \Psi_2 \right|^2 + V + \frac{B^2}{8\pi},
\]

where \( V = -b_\alpha |\Psi_\alpha|^2 + \frac{\kappa_2}{m_\alpha} |\Psi_\alpha|^4 \). The two condensates are characterized by the different effective masses \( m_\alpha \), the different coherence lengths \( \xi_\alpha = h/\sqrt{2m_\alpha b_\alpha} \) and the different concentrations \( N_\alpha = \langle |\Psi_\alpha|^2 \rangle = b_\alpha/c_\alpha \). Then it is known that the contribution of the condensate \( \Psi_1 \) is \( \kappa_1 = (\frac{N_1}{m_1})/(\frac{N_1}{m_1} + \frac{N_2}{m_2}) \) and the condensate \( \Psi_2 \) contributes \( \kappa_2 = (\frac{N_2}{m_2})/(\frac{N_1}{m_1} + \frac{N_2}{m_2}) \) to the two-gap system. What is the importance in the present GLGP model is that the two charged fields are not independent but nontrivial coupled through the electromagnetic field. This kind of nontrivial coupling indicates that in this system there should be a nontrivial, hidden topology. In order to find out the topological structure and to investigate it conveniently, we introduce the decomposition of \( U(1) \) gauge potential theory and the \( \phi \)-mapping method \[1\]–[2]. In the theory of two-gap superconductor, the condensate wave function \( \Psi_\alpha \) are the order parameter of the charged continuum, which are sections of the complex line bundle: \( \Psi_\alpha(x) = \phi_\alpha + i\theta_\alpha \). The \( U(1) \) covariant derivative \( D_\mu \Psi_\alpha \) are introduced to describe the interaction between \( \Psi_\alpha \) and the electromagnetic field: \( D_\mu \Psi_\alpha(x) = \partial_\mu \Psi_\alpha - igA_\mu \Psi_\alpha \), where \( \mu = 0, 1, 2, 3 \) denote the four-dimensional space-time, \( g = 2e/c\hbar \), and \( A_\mu \) is the \( U(1) \) gauge potential. Then the \( U(1) \) gauge field tensor is given by: \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Defining two-dimensional unit vectors in terms of \( \phi_\alpha^a \) by: \( m_\alpha^a = \frac{\phi_\alpha^a}{\|\phi_\alpha\|} \), \( (a = 1, 2) \), \( \|\phi_\alpha\|^2 = \phi_\alpha^a \phi_\alpha^a = \Psi_\alpha^* \Psi_\alpha \), one can prove \[13\] that \( A_\mu \) can be decomposed in terms of \( m_\alpha^a \),

\[
A_\mu = A_{\alpha\mu} = \frac{1}{g} \epsilon_{ab} m_\alpha^a \partial_\mu m_\alpha^b - \partial_\mu \theta,
\]

where \( \theta \) is a phase factor. Since \( \partial_\mu \theta \) does not contribute to the gauge field tensor \( F_{\mu\nu} \), we obtain the two kinds of decomposition expression:

\[
F_{\mu\nu} = F_{\alpha\mu\nu} = \frac{2}{g} \epsilon_{ab} \partial_\mu m_\alpha^a \partial_\nu m_\alpha^b.
\]

As follows, we can see that the two-gap superconductor acquires properties which are qualitatively very different from those of a one-gap superconductor. Introducing a two-dimensional topological tensor current \( K_{\alpha}^{\mu\nu} = (1/4\pi)\epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_\lambda m_\alpha^a \partial_\rho m_\alpha^b \), using \( \partial_\mu (\phi_\alpha^a/\|\phi_\alpha\|) = (\partial_\mu \phi_\alpha^a) /\|\phi_\alpha\| + \phi_\alpha^a \partial_\mu (1/\|\phi_\alpha\|) \) and considering the Green’s function relation in \( \phi \) space: \( \partial_\alpha \partial_\mu \|\phi_\alpha\| = 2\pi \delta^2(\hat{\phi}_\alpha) \) \( (\partial_\alpha = \partial/\partial \phi_\alpha^a) \), one can prove that \[11\]: \( K_{\alpha}^{\mu\nu} = \delta^2(\phi_\alpha) D^{\mu\nu}(\phi_\alpha/\|\phi_\alpha\|) \). Denoting the spacial components of \( K_{\alpha}^{\mu\nu} \) by \( K_{\alpha}^{\mu} \), we have

\[
K_{\alpha}^{\mu} = \delta^2(\phi_\alpha) D^\mu(\phi_\alpha /\|\phi_\alpha\|),
\]

in which \( D^\mu(\phi_\alpha /\|\phi_\alpha\|) = D^{\mu\alpha}(\phi_\alpha /\|\phi_\alpha\|) \) is the Jacobian vector. The expression of Eq. \[3\] provides an important conclusion: \( K_{\alpha}^{\mu} = 0 \), if and only if \( \phi_\alpha \neq 0; \) \( K_{\alpha}^{\mu} \neq 0 \), if and only if \( \phi_\alpha = 0 \). So it is necessary to study the zero points of \( \phi_\alpha \) to determine the nonzero solutions of \( K_{\alpha}^{\mu} \). The implicit function theory \[14\] shows that under the regular condition \( D^\mu(\phi_\alpha /\|\phi_\alpha\|) \neq 0 \), the general solutions of

\[
\phi_1^\alpha(t, x^1, x^2, x^3) = 0, \quad \phi_2^\alpha(t, x^1, x^2, x^3) = 0
\]
can be expressed as $\tilde{x}_\alpha = \tilde{x}_{\alpha k}(s, t)$, which represent the world surfaces of $N$ moving isolated singular strings $L_k$ with string parameter $s$ ($k = 1, 2 \cdots N$). This indicates that there are vortex lines located at the zero points of the $\tilde{\phi}_\alpha$ field.

We investigate the magnetic flux carried by the vortices: $\Phi = \oint_{\partial \Sigma} A \cdot d\ell$. It is due to the different contributions of the two gaps that the magnetic flux can be expressed as: $\Phi = \kappa_1 \int_{\partial \Sigma} A_1 \cdot d\ell + \kappa_2 \int_{\partial \Sigma} A_2 \cdot d\ell$. Using the Stokes' theorem and the $\phi$-mapping method, we get $\Phi = \kappa_1 \Phi_0 \int_{\Sigma} \delta^2(\tilde{\phi}_1) D(\tilde{\phi}_1) d\Sigma + \kappa_2 \Phi_0 \int_{\Sigma} \delta^2(\tilde{\phi}_2) D(\tilde{\phi}_2) d\Sigma$, in which $\Phi_0 = \frac{hc}{2} \mu_0$ stands for the standard flux quantum in two-gap superconductor. In $\delta$-function theory [15], we can obtain a decomposition expression on the 2-dimensional surface $\Sigma$: $\delta^2(\phi_\alpha) = \sum_{l=1}^{N} \frac{\delta^2(\tilde{x} - \tilde{x}_l)}{|D(\phi_\alpha/\tilde{x})|}$, where $\tilde{x}_l$ are the crossing points of vortices and the surface $\Sigma$, and the positive integer $\beta_l$ is the Hopf index of the $\phi$-mapping, which means that when $\tilde{x}$ covers the neighborhood of the point $\tilde{x}_l$ once, the vector field $\tilde{\phi}$ covers the corresponding region in $\phi$ space $\beta_l$ times. Then the magnetic flux in two-gap system can be reexpressed as

$$\Phi = \kappa_1 \Phi_0 \sum_{l=1}^{N} \beta_l^1 \eta_l^1 + \kappa_2 \Phi_0 \sum_{l=1}^{N} \beta_l^2 \eta_l^2,$$  

(5)

in which $\eta_l^\alpha = \text{sgn}D(\phi_\alpha/\tilde{x}) = \pm 1$ is the Brouwer degree of the $\phi$-mapping. When $\beta_l^1 = \beta_l^2$ and $\eta_l^1 = -\eta_l^2$, it is easy to see that $\Phi = (\kappa_1 - \kappa_2) W^1 \Phi_0$, where $W^\alpha = \sum_{l=1}^{N} \beta_l^\alpha \eta_l^\alpha$ is the winding number of the $\tilde{\phi}_\alpha$ around $\tilde{x}_l$. Since there exists the relation $\kappa_1 - \kappa_2 = \cos \tilde{\theta}$, one can arrive at the result that in the case of $W^1 = -W^2$, the vortices in two-gap superconductor can carry an arbitrary fraction of magnetic flux quantum; When $\beta_l^1 = \beta_l^2$ and $\eta_l^1 = \eta_l^2$, one can get $\Phi = \sum_{l=1}^{N} (\kappa_1 + \kappa_2) \beta_l^1 \eta_l^1 \Phi_0 = W^1 \Phi_0$. Thus in this case the vortices can carry only an integer number of magnetic flux quanta; When $\beta_l^1 \neq \beta_l^2$, we obtain the general form of the magnetic flux in two-gap superconductor: $\Phi = \kappa_1 W^1 + \kappa_2 W^2$.

### III. THE KNOTTED VORTEX LINES

We will discuss the knotted vortex lines in the two-gap system. In $\delta$-function theory [15], one can prove that in three-dimensional space

$$\delta^2(\phi_\alpha) = \sum_{k=1}^{N} \beta_k^\alpha \int_{L_k} \delta^3(\tilde{x} - \tilde{x}_{\alpha k}(s)) ds,$$  

(6)

where $D(\phi_\alpha/\tilde{x})_{\Sigma_k} = \frac{1}{2} \epsilon_{u v} \epsilon_{m n} \partial \phi_\alpha^m \partial u^v (\partial \phi_\alpha^n \partial u^v)$, and $\Sigma_k$ is the $k$th planar element transverse to $L_k$ with local coordinates $(u^1, u^2)$. The positive integer $\beta_k$ is the Hopf index of the $\phi$-mapping. Meanwhile taking notice of the definition of Jacobian, the direction vector of $L_k$ is given by

$$\frac{dx^i}{ds} \bigg|_{\tilde{x}_{\alpha k}} = \frac{D^i(\phi_\alpha/\tilde{x})}{D(\phi_\alpha/\tilde{x})} \bigg|_{\tilde{x}_{\alpha k}}. $$  

(7)

Then from Eqs. (6) and (7), we obtain the inner structure of $K^i_{\alpha}$

$$K^i_{\alpha} = \sum_{k=1}^{N} W_k^\alpha \int_{L_k} \frac{dx^i}{ds} \delta^3(\tilde{x} - \tilde{x}_{\alpha k}(s)) ds,$$  

(8)

where $W_k^\alpha = \beta_k^\alpha \eta_k^\alpha$ is the winding number of the field $\tilde{\phi}_\alpha$ around $L_k$, with $\eta_k^\alpha = \text{sgn}D(\phi_\alpha/\tilde{x}) = \pm 1$ being the Brouwer degree of $\phi$-mapping. We can see that vortex lines exist at the zeros of the two order parameter field $\Psi_1$ and $\Psi_2$,
between. To explore the topological property of the knotted vortices, introduce the Chern-Simon action, which is defined by

\[ S = \frac{g}{4\pi} \oint_{M^3} A \wedge F = \frac{g}{8\pi} \oint_{M^3} \epsilon^{ijk} A_i F_{jk} d^3x, \tag{9} \]

where \( i; j; k \) denote the three-dimensional space. It can be seen that considering Eq. (2) and substituting Eq. (3) into Eq. (9), one can obtain \( S = \sum_{k=1}^{N} W_k \int_{k} A_idx^i \). Since \( A_i \) satisfies the \( U(1) \) gauge transformation: \( A'_i = A_i + \partial_i \phi \), where \( \phi \in \mathbb{R} \) is a phase factor denoting the \( U(1) \) transformation, when these vortex lines are \( N \) closed curves, i.e., a family of \( N \) knots \( \gamma_k \) \( (k = 1, \cdots, N) \), the terms \( \partial_i \phi \) contributes nothing to the integral

\[ S = \sum_{k=1}^{N} W_k \oint_{\gamma_k} A_idx^i. \tag{10} \]

Then the expression (10) is invariant under the gauge transformation. Meanwhile we know that \( S \) independent of the metric. Therefore one can conclude that \( S \) are topological invariant for the knotted vortex lines in two-gap superconductors.

In the following, we investigate the relations between the Chern-Simon action \( S \) and the topological numbers of a knot family. Let \( \bar{x} \) and \( \bar{y} \) be two points respectively on the knots \( \gamma_k \) and \( \gamma_l \). Noticing the symmetry between the two points, Eq. (10) should be reexpressed as

\[ S = \sum_{k=1, l=1}^{N} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \partial_j A_j dx^j \wedge dy^j. \tag{11} \]

Considering that \( \gamma_k \) and \( \gamma_l \) can be the same one knot or two different knots, we should write Eq. (11) in two parts \((k = l \text{ and } k \neq l)\); furthermore the \( k = l \) part includes both the \( \bar{x} = \bar{y} \) and the \( \bar{x} \neq \bar{y} \) cases. To explore the relation between \( S \) and the topological numbers of a family knot, we should first express \( A_i \) in terms of the vector field which carries the geometric information of the knot family. Define the Gauss mapping \( \bar{m} : S^1 \times S^1 \to S^2, \bar{m} = \frac{\bar{x} - \bar{y}}{\|\bar{x} - \bar{y}\|} \), where \( \bar{m} \) is a three-dimensional unit vector, i.e., a section of sphere bundle \( S^2 \). Let \( \bar{c}(x, y) \) be a two-dimensional unit vector on the sphere \( S^2 \) and satisfy: \( \bar{e} \cdot \bar{e}' = 1, \bar{e} \perp \bar{m} \). Then, using the decomposition of \( U(1) \) gauge potential theory, one can obtain the inner structure of \( A_i \) in terms of \( \bar{e} \) as \( A_i = \epsilon_{ab} \bar{c}^a \partial_i \bar{c}^b \) \( (a, b = 1, 2) \). Then the Chern-Simon action Eq. (11) is expressed as

\[ S = \sum_{k=1, l=1}^{N} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \epsilon_{ab} \partial_i \bar{c}^a \partial_j \bar{c}^b dx^j \wedge dy^j. \tag{12} \]

When \( \gamma_k \) and \( \gamma_l \) are the same one knot but \( \bar{x} \) and \( \bar{y} \) are different points, we get

\[ S = 2\pi \times \left[ \sum_{k=1}^{N} \frac{1}{4\pi} W_k^2 \oint_{\gamma_k} \oint_{\gamma_k} \bar{m}^*(dS) \right], \tag{13} \]

where \( \bar{m}^*(dS) = \bar{m} \cdot (d\bar{m} \times d\bar{m}) = 2\epsilon_{ab} \partial^a \partial^b dS \) is the pull-back of \( S^2 \) surface element. The expression (13) is just related to the writhing number \( Wr(\gamma_k) \) of \( \gamma_k \) \( (13): \ W_r(\gamma_k) = \frac{1}{2\pi} \oint_{\gamma_k} \oint_{\gamma_k} \bar{m}^*(dS) \). Furthermore when \( k = l \) and \( \bar{x} = \bar{y} \), Eq. (11) should be written as

\[ S = 2\pi \times \left[ \frac{1}{4\pi} \sum_{k=1}^{N} W_k^2 \oint_{\gamma_k} \epsilon_{ab} \bar{c}^a \partial_i \bar{c}^b dx^i \right]. \tag{14} \]
Let $\vec{T}$ be the unit tangent vector of knot $\gamma_k$ at $\vec{x}$ ($\vec{m} = \vec{T}$ when $\vec{x} = \vec{y}$), and $\vec{V}$ is defined as $e_a = e^{ab}V_b$ $(a, b = 1, 2; \vec{V} \perp \vec{T}, \vec{e} = \vec{T} \times \vec{V})$. One can prove that $\frac{1}{4\pi} \oint_{\gamma_k} e_{ab} \partial_i e^b_i dx^i = \frac{1}{4\pi} \oint_{\gamma_k} (\vec{T} \times \vec{V}) \cdot d\vec{V} = Tw(\gamma_k)$, where $Tw(\gamma_k)$ is the twisting number of $\gamma_k$. From the Calugareanu formula [18]: $SL(\gamma_k) = Wr(\gamma_k) + Tw(\gamma_k)$, (where $SL(\gamma_k)$ is the self-linking number of $\gamma_k$), we can see that Eqs. (13) and (14) just compose the self-linking numbers of the $N_k$ knots.

Let us discuss the third case: $\gamma_l$ and $\gamma_k$ are the different knots. It is easy to obtain

$$S = 2\pi \times \left[ \sum_{k,l=1}^{N} \frac{1}{4\pi} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \vec{m}^\ast(dS) \right]. \quad (15)$$

Comparing the expression Eq. (15) with the definition of the Gauss linking number: $Lk(\gamma_k, \gamma_l) = \frac{1}{4\pi} e^{ijk} \oint_{\gamma_k} dx^i \oint_{\gamma_l} dy^j (\vec{x}^k - \vec{y}^k) / ||\vec{x} - \vec{y}||^3 - \frac{1}{4\pi} \oint_{\gamma_k} \oint_{\gamma_l} \vec{m}^\ast(dS)$, we know that in this case the action $S$ is related to the Gauss linking number $Lk(\gamma_k, \gamma_l)$ between $\gamma_k$ and $\gamma_l$ ($k \neq l$). Therefore, we arrive at the important result

$$S = 2\pi \left[ \sum_{k=1}^{N} W_k^2 SL(\gamma_k) + \sum_{k=1, l=1}^{N} W_k W_l Lk(\gamma_k, \gamma_l) \right]. \quad (16)$$

This precise expression just reveals the relationship between the Chern-Simon quantum number $S$ and the linking numbers of the knots family. Since the self-linking and Gauss linking are both the intrinsic characteristic numbers of knotlike configurations in geometry, expression Eq. (16) directly relates $S$ to the topology of the knots family itself, and therefore $S$ can be regarded as an important invariant required to describe the topology of knotted vortex lines in two-gap superconductor. This is just the significance of the introduction and research of topological invariant $S$.

IV. CONCLUSION

In summary, we investigate the topological vortex lines and the knotted vortex lines in the two-gap superconductor. It is due to the different contribution of the two gaps that the vortex lines can carry an arbitrary fraction of magnetic flux. Furthermore, we find out that the Chern-Simon action is a topological invariant for the knot family and reveal that it is just the total sum of all the self-linking numbers and all the linking numbers of the knot family in two-gap superconductor.

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