THE BAUM-CONNES ASSEMBLY MAP AND
THE GENERALIZED BASS CONJECTURE

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May 2007

INTRODUCTION

In the early 1980’s, P. Baum and A. Connes defined an assembly map

(0.1) \[ A^a_G : KK^G(C(EG), \mathbb{C}) \to K^G_t(C^*(G)) \]

where \( G \) denotes a locally compact group, \( EG \) the classifying space for proper \( G \)-actions, \( C(EG) \) the \( G \)-algebra of complex-valued functions on \( EG \) vanishing at infinity, and \( KK^G(C(EG), \mathbb{C}) \) the \( G \)-equivariant \( KK \)-groups of \( (EG) \) with coefficients in \( \mathbb{C} \), while \( K^G_t(C^*(G)) \) represents the topological \( K \)-groups of the reduced \( C^* \)-algebra of \( G \). The original details of this map appeared (a few years later) in [BC1] and [BC2], with further elaborations in [BCH]. As shown in [BC3], when \( G \) is discrete the left-hand side admits a Chern character which may be represented as

\[ ch^BC_*^G(G) : KK^G(C(EG), \mathbb{C}) \to \bigoplus_{x \in fin(<G>)} H_*^{BG_x; \mathbb{C}} \otimes HP_{er}^*(\mathbb{C}) \]

where \( fin(<G>) \) is the set of conjugacy classes of \( G \) corresponding to elements of finite order, \( G_x \) the centralizer of \( g \) in \( G \) where \( x = <g> \), and \( HP_{er}^*(\mathbb{C}) \) the periodic cyclic homology of \( \mathbb{C} \). Note that \( H_*^{BH; \mathbb{C}} \otimes HP_{er}^*(\mathbb{C}) \) are simply the 2-periodized complex homology groups of \( BH \), and (via the classical Atiyah-Hirzebruch Chern character) can be alternatively viewed as the complexified \( K \)-homology groups of \( BH \). Upon complexification, the map \( ch^BC_*^G(G) \) is an isomorphism. The original construction of Baum and Connes \( A^a_G \) was analytical. Motivated by the need to construct a homotopical analogue to their map, we constructed an assembly map in [O1] which we will denote here as

\[ A^h^G_*^G \otimes \mathbb{C} : H_*^{BG_x; \mathbb{K}(\mathbb{C})} \otimes \mathbb{C} \to K^G_t(C^*(G)) \otimes \mathbb{C} \]

where \( \mathbb{K}(\mathbb{C}) \) denotes the 2-periodic topological \( K \)-theory spectrum of \( \mathbb{C} \). The construction of this map amounted to an extension of the classical assembly map constructed in [L] which was designed to take into account the contribution coming from the conjugacy classes of finite order. The two

Key words and phrases. Baum-Connes Assembly map, Baum-Connes Conjecture, Bass Conjecture.
essential features of $A_{*,h}^G \otimes \mathbb{C}$, shown in [O1], were (i) it factors through $K_*^t(\mathbb{C}[G]) \otimes \mathbb{C}$ (where $K_*^t(\mathbb{C}[G])$ denotes the Bott-periodized topological $K$-theory of the complex group algebra, topologized with the fine topology), and (ii) the composition of $A_{*,h}^G \otimes \mathbb{C}$ with the complexified Chern-Connes-Karoubi-Tillmann character $c_h^G : K_*^e(\mathbb{C}[G]) \otimes \mathbb{C} \to HC_*^e(\mathbb{C}[G])$ was effectively computable (see below). What we did not do in [O1] was show that $A_{*,a}^G \otimes \mathbb{C}$ and $A_{*,h}^G \otimes \mathbb{C}$ agree. Since this initial work, there have been numerous extensions and reformulations of the Baum-Connes assembly map, as well as of the original Baum-Connes conjecture, which states that the map in (0.1) is an isomorphism. These extensions typically are included under the umbrella term “Isomorphism Conjecture”, (formulated for both algebraic and topological $K$-theory; cf. [DL], [FJ], [LR]). Thanks to [HP], we now know that the different formulations of these assembly maps (e.g., homotopy-theoretic vs. analytical) agree.

Abbreviating $KK_*^e(C(EG), \mathbb{C})$ as $K_*^G(EG)$ (read: the equivariant $K$-homology of the proper $G$-space $EG$), our main result is

**Theorem 1.** There is a commuting diagram

$$
\begin{array}{ccc}
K_*^G(EG) & \xrightarrow{A_{*,DL}^G} & K_*^t(\mathbb{C}[G]) \\
\downarrow \text{id} & & \downarrow \text{id} \\
HC_*^{fin}(\mathbb{C}[G]) & \xrightarrow{HC_*^e} & HC_*^e(\mathbb{C}[G])
\end{array}
$$

where $A_{*,DL}^G$ is the homotopically defined assembly map of [DL], $HC_*^e(\mathbb{C}[G]) := \bigoplus_{x \in fin(<G>)} HC_*^{e}(\mathbb{C}[G])_x \cong \bigoplus_{x \in fin(<G>)} H(BG_x; \mathbb{C}) \otimes HC_*^e(\mathbb{C})$ is the elliptic summand of $HC_*^e(\mathbb{C}[G])$ [JOR], the lower horizontal map is the obvious inclusion, and the Chern character $c_h^G$ becomes an isomorphism upon complexification for $* \geq 0$.

Let $\beta$ denote a bounding class, $(G, L)$ a discrete group equipped with a word-length, and $H_{\beta,L}(G)$ the rapid decay algebra associated with this data [JOR]. We write $K_*^t(H_{\beta,L}(G))$ for the Bott-periodic topological $K$-theory of the topological algebra $H_{\beta,L}(G)$. The Baum-Connes assembly map for $H_{\beta,L}(G)$ is defined to be the composition

$$(BC) \quad A_{*,DL}^G : K_*^G(EG) \xrightarrow{A_{*,DL}^G} K_*^t(\mathbb{C}[G]) \xrightarrow{ch_*} K_*^t(H_{\beta,L}(G))$$

where the second map is induced by the natural inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$. In [JOR], we conjectured that the image of $ch_* : K_*^t(H_{\beta,L}(G)) \to HC_*^e(H_{\beta,L}(G))$ lies in the elliptic summand $f_{\text{in}}HC_*^e(H_{\beta,L}(G))$ (conjecture $\beta$-SrBC). As the inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$ sends $f_{\text{in}}HC_*^e(\mathbb{C}[G])$ to $f_{\text{in}}HC_*^t(H_{\beta,L}(G))$, naturality of the Chern character $c_h^G$ and Theorem 1 implies

**Corollary 2.** If $A_{*,DL}^G$ is rationally surjective, then $\beta$-SrBC is true.

Since going down and then across is rationally injective, we also have (compare [O1])

**Corollary 3.** The assembly map $A_{*,DL}^G \otimes \mathbb{Q}$ is injective for all discrete groups $G$.

We do not claim any great originality in this paper. In fact, Theorem 1, although not officially appearing in print before this time, has been a “folk-theorem” known to experts for many years. The connection between the Baum-Connes Conjecture (more precisely a then-hypothetical Baum-Connes-type Conjecture for $\mathbb{C}[G]$) and the stronger Bass Conjecture for $\mathbb{C}[G]$ discussed in [JOR] was noted by the author in [O2].
There is some overlap of this paper with the results presented in [Ji]. A special case of Theorem 1 (for \( * = 0 \) and \( \mathbb{C}[G] \) replaced by the \( \ell^1 \)-algebra \( \ell^1(G) \)) appeared as the main result of [BCM].

**Proof of Theorem 1**

We use the notation \( F_*^{fin}(\mathbb{C}[G]) \) to denote the elliptic summand \( \bigoplus_{x \in fin(<G)>} F_*(\mathbb{C}[G]) \) where \( F_*(-) = HH_*(-), HN_*(-), HC_*(-) \) or \( HPer_*(-) \). To maximize consistency with [LR], we write \( S \) for the (unreduced) suspension spectrum of the zero-sphere \( S^0 \), \( HN(R) \) resp. \( HH(R) \) the Eilenberg-MacLane spectrum whose homotopy groups are the negative cyclic resp. Hochschild homology groups of the discrete ring \( R \), and \( K^a(R) \) the non-connective algebraic \( K \)-theory spectrum of \( R \), with \( K^a_*(R) \) representing its homotopy groups. By [LR], diag. 1.6] there is a commuting diagram

\[
\begin{align*}
H_*^G(EG; S) & \rightarrow K_*^a(\mathbb{Z}[G]) \\
H_*^G(EG; HN(\mathbb{Z})) & \rightarrow HN_*^{fin}(\mathbb{Z}[G]) \rightarrow HN_*^a(\mathbb{Z}[G]) \\
H_*^G(EG; HH(\mathbb{Z})) & \rightarrow HH_*^{fin}(\mathbb{Z}[G]) \rightarrow HH_*^a(\mathbb{Z}[G])
\end{align*}
\]

(1.1)

where the top horizontal map is the composition

\[
H_*^G(EG; S) \rightarrow H_*^G(EG; K^a(\mathbb{Z})) \rightarrow K_*^a(\mathbb{Z}[G])
\]

referred to as the the restricted assembly map for the algegraic \( K \)-groups of \( \mathbb{Z}[G] \). The other two horizontal maps are the assembly maps for negative cyclic and Hochschild homology respectively. The upper left-hand map is induced by the map from the sphere spectrum to the Eilenberg-MacLane spectrum \( HN \), which may be expressed as the composition of spectra \( S \rightarrow K^a(\mathbb{Z}) \rightarrow HN \). By [LR], the composition on the left is a rational equivalence.

Let \( \mathbb{C}^\delta \) denote the complex numbers \( \mathbb{C} \) equipped with the discrete topology. Tensoring with \( \mathbb{C} \) and combined with the inclusion of group algebras \( \mathbb{Z}[G] \hookrightarrow \mathbb{C}^\delta[G] \), (1.1) yields the commuting diagram

\[
\begin{align*}
H_*^G(EG; \mathbb{Q}) \otimes \mathbb{C} & \rightarrow K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} \\
\cong & \\
HN_*^{fin}(\mathbb{C}[G]) & \rightarrow HN_*^a(\mathbb{C}[G])
\end{align*}
\]

(1.2)

Next, we consider the transformation from algebraic to topological \( K \)-theory, induced by the map of group algebras \( \mathbb{C}^\delta[G] \rightarrow \mathbb{C}[G] \) which is the identity on elements. By the results of [CK], [W] and [T], there is a commuting diagram

\[
\begin{align*}
K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} & \rightarrow K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\
\cong & \\
HN_*^a(\mathbb{C}[G]) & \rightarrow HPer_*^a(\mathbb{C}[G])
\end{align*}
\]

(1.3)
where \( ch_*(\mathbb{C}[G]) \) is the Connes-Karoubi Chern character for the fine topological algebra \( \mathbb{C}[G] \), and the bottom map is the transformation from negative cyclic to periodic cyclic homology.

We can now consider our main diagram

\[
\begin{array}{cccccc}
H^G_c(EG; \mathbb{C}) \otimes K_*(\mathbb{C}) & \rightarrow & K^*_c(\mathbb{C}[G] \otimes \mathbb{C}) & \rightarrow & K^*_c(\mathbb{C}[G] \otimes \mathbb{C} K_*(\mathbb{C}) & \rightarrow & K^*_c(\mathbb{C}[G] \otimes \mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) \\
\end{array}
\]

(1.4) \[ H^t_{Per}(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) \rightarrow H^t_{Per}(\mathbb{C}[G]) \]

The top left square commutes by (1.2), and the middle top square commutes by (1.3). The upper right square commutes by virtue of the fact that the Connes-Karoubi-Chern character is a homomorphism of graded modules, which maps the \( K^*_c(\mathbb{C}) \)-module \( K^*_c(\mathbb{C}[G]) \) to the \( H^t_{Per}(\mathbb{C}[G]) \)-module \( H^t_{Per}(\mathbb{C}[G]) \), with the map of base rings induced by isomorphism \( ch_*(\mathbb{C}[\{id\}]) : K^*_c(\mathbb{C}) \otimes \mathbb{C} \longrightarrow H^t_{Per}(\mathbb{C}). \) The lower left square commutes trivially, while the lower right commutes by the naturality of the inclusion \( H^t_{Per}(\mathbb{C}[G]) \hookrightarrow H^t_{Per}(\mathbb{C}[G]) \) with respect to the module structure over \( H^t_{Per}(\mathbb{C}) \). Summarizing, we get a commuting diagram

\[
\begin{array}{cccccc}
H^G_c(EG; \mathbb{C}) \otimes K_*(\mathbb{C}) & \rightarrow & K^*_c(\mathbb{C}[G] \otimes \mathbb{C}) & \rightarrow & K^*_c(\mathbb{C}[G]) & \rightarrow & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & H^t_{Per}(\mathbb{C}[G]) & \rightarrow & HC^*_c(\mathbb{C}[G]) \\
\end{array}
\]

(1.5)

where the bottom square is induced by the transformation \( H^t_{Per}(\_ \rightarrow HC^*_c(\_ \_)) \), which respects the summand decomposition indexed on conjugacy classes. Restricted the elliptic summand yields the map \( H^t_{Per}(\mathbb{C}[G]) \rightarrow HC^*_c(\mathbb{C}[G]) \) which is an isomorphism for \( * \geq 0 \), implying the result stated in Theorem 1.

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