THE EQUIVARIANT EULER CHARACTERISTIC OF $\mathcal{A}_3[2]$

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Abstract. We compute the weighted Euler characteristic, equivariant with respect to the action of the symplectic group of degree six over the field of two elements, of the moduli space of principally polarized abelian threefolds together with a level two structure.

1. Introduction

Let $\mathcal{A}_g[2]$ denote the moduli space of principally polarized abelian varieties of dimension $g$ together with a full level 2 structure. Similarly, let $\mathcal{M}_g[2]$ denote the moduli space of smooth curves of genus $g$ together with a full level 2 structure. Note that we are considering these as coarse moduli spaces. The two types of moduli spaces are connected through the Torelli morphism $t_g: \mathcal{M}_g[2] \to \mathcal{A}_g[2]$ sending a smooth curve to its Jacobian. There is an action of $\text{Sp}_{2g}(2)$, the symplectic group of degree $2g$ over the field of two elements, on $\mathcal{A}_g[2]$ and $\mathcal{M}_g[2]$ via its action on the level 2 structure (for some more details see for instance [2]).

For $g \leq 2$, $\mathcal{M}_g[2]$ is isomorphic to the locus $\mathcal{H}_g[2]$ consisting of hyperelliptic curves. The space $\mathcal{H}_g[2]$ can in turn be described as a disjoint union of copies of $\mathcal{M}_{0,2g+2}$ (see [6], [10] and [11]), the moduli space of smooth genus 0 curves together with $2g + 2$ marked points. The cohomology of $\mathcal{M}_{0,2g+2}$, together with the action of the symmetric group $S_{2g+2}$, can (because of purity) be computed using counts of points over finite fields (see for instance [4] and [9]). In Section 5.1 respectively Section 5.2 below, we compute in this way the $\text{Sp}_{2g}(2)$-action and Hodge structure of the cohomology of $\mathcal{M}_1[2] \cong \mathcal{A}_1[2]$ respectively the $\text{Sp}_2(4)$-action and Hodge structure of the cohomology of $\mathcal{M}_2[2]$. By adding the complement of $t_2(\mathcal{M}_2[2])$ inside $\mathcal{A}_2[2]$ consisting of products of elliptic curves we also compute the $\text{Sp}_2(4)$-equivariant weighted Euler characteristic of $\mathcal{A}_2[2]$. For a definition of this type of Euler characteristic see Section 2.

The main result of this article is Table 1 which contains the $\text{Sp}_2(6)$-equivariant weighted Euler characteristic of $\mathcal{A}_3[2]$. This is based upon the work of the second author in [2] in which the $\text{Sp}_2(6)$-equivariant cohomology of $\mathcal{M}_3[2]$ is computed, see Section 3 below. There are two other loci consisting of either products of an indecomposable abelian surface and an elliptic curve, or products of three elliptic curves. The cohomology of these loci are computed in Section 5 and Section 6 respectively.

We note in Section 4 that the weighted Euler characteristic of $\mathcal{A}_3[2]$ contains much fewer classes than the weighted Euler characteristic of its different loci. This cancellation property was noted also in [1] for the integer valued Euler characteristic of local systems upon the corresponding strata inside $\mathcal{A}_3$, the moduli space of principally polarized abelian threefolds with no level structure. The Hodge structure of the cohomology of $\mathcal{A}_3$ was previously known, see [7].
2. Euler characteristics

For a quasi-projective variety $X$ defined over $\mathbb{C}$, let $W_k H^i(X)$ denote the weight $k$-part of $H^i(X)$, the $i$th Betti cohomology group with complex coefficients. For an action of a finite group $G$ on $X$, let the $G$-equivariant weighted Euler characteristic of $X$ be the virtual representation of $G$ defined as

$$e_X(v) = \sum_{i,k \geq 0} (-1)^i W_k H^i(X) v^k.$$ 

This Euler characteristic is additive in the sense that if $X = Y \sqcup Z$, where $Y$ and $Z$ are preserved by $G$, then $e_X(v) = v^{2\text{codim}X(Y)} e_Y(v) + v^{2\text{codim}X(Z)} e_Z(v)$. Note that if $X$ fulfills purity, in the sense of Dimca and Lehrer in [4], then one can from this Euler characteristic determine the individual cohomology groups as representations of $G$.

Say now that $X$ is a variety defined over $\mathcal{O}[\frac{1}{N}]$, where $\mathcal{O}$ is a ring of integers of an algebraic number field, together with an action of a finite group $G$. Say furthermore that there is a polynomial $P(t)$, with complex coefficients and of degree $2 \dim X$, such that $P(q) = |X(\mathbb{F}_q)|$ for almost all prime powers $q$. We can, using the Lefschetz fixed point theorem, from this information determine the weighted Euler characteristic of $X(\mathbb{C})$. The set $X(\mathbb{F}_q)$ consists of the fixed points of Frobenius. By counting the fixed points of Frobenius composed with elements of $G$ we can in the same way determine the $G$-equivariant weighted Euler characteristic of $X(\mathbb{C})$. This will be called a twisted point count. For a reference, see [5, Appendix A].

3. Decomposable and indecomposable abelian threefolds

We say that an abelian threefold is indecomposable if it is not isomorphic to a product of abelian varieties of lower dimension. We denote the corresponding locus in $A_3[2]$ by $A_{3}^{\text{in}}[2]$.

The Torelli morphism $t_3$ gives an isomorphism $M_3[2] \cong A_3^t[2]$ (on the level of coarse moduli spaces). The moduli space $M_3[2]$ can be decomposed as a disjoint union

$$M_3[2] = Q[2] \sqcup H_3[2]$$

where $Q[2]$ denotes the locus consisting of curves whose canonical model is a plane quartic curve and where $H_3[2]$ denotes the hyperelliptic locus.

The cohomology groups of $Q[2]$ and $H_3[2]$ were determined as representations of $\text{Sp}_2(6)$ by the second author in [2]. For completeness, we repeat the results in Table 3 and Table 4.

There are two types of decomposable abelian threefolds. The threefold can either be isomorphic to a product of an indecomposable abelian surface and an elliptic curve or to a product of three elliptic curves. We denote the corresponding loci in $A_3[2]$ by $A_{3,1}[2]$ and $A_{1,1,1}[2]$ respectively.

4. The main result

We have the decomposition

$$A_3[2] = t_3(Q[2]) \sqcup t_3(H_3[2]) \sqcup A_{2,1}[2] \sqcup A_{1,1,1}[2]$$

and below we compute the cohomology groups of each of the spaces on the right hand side as representations of $\text{Sp}_2(6)$. Moreover, we will see that each cohomology group $H^i$ of a space on the right hand side is pure of weight $2i$ and Tate type $(i, i)$. 
By the additivity of the weighted Euler characteristic,
\[ e_{\mathcal{A}_3[2]}(v) = c_{t_5(\mathbb{Q}[2])}(v) + v^2 e_{t_5(\mathbb{Q}[2])}(v) + v^4 e_{\mathcal{A}_2[2]}(v) + v^6 e_{\mathcal{A}_1[1]}(v). \]

Putting the results together for the different strata we get the Sp(2)-equivariant weighted Euler characteristic of \( \mathcal{A}_3[2] \), see Table 1. Each column in this table corresponds to an irreducible representation of Sp(2). The irreducible representations are denoted \( \phi_{dn} \) where \( d \) is the dimension of the representation and \( n \) is letter used to distinguish different representations of the same dimension, see [3].

| \( e_{\mathcal{A}_3[2]}(v) \) | \( \phi_{1a} \) | \( \phi_{2a} \) | \( \phi_{3a} \) | \( \phi_{4a} \) | \( \phi_{5a} \) | \( \phi_{6a} \) | \( \phi_{7a} \) | \( \phi_{8a} \) | \( \phi_{9a} \) |
|--------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                          | 1 + \( v^4 + v^6 + v^{12} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( e_{\mathcal{A}_3[2]}(v) \) | \( v^{12} \) | 0 | \( v^4 \) | 0 | \( v^6 \) | 0 | 0 | 0 | 0 |
|                          | \( -v^{10} + v^8 + v^{14} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1. The Sp(2)-equivariant weighted Euler characteristic of \( \mathcal{A}_3[2] \).

Note that the results for \( \phi_{1a} \) agree with the computation of the cohomology groups of \( \mathcal{A}_3 \) together with their Hodge structure in [7]. Note also that only 13 of the 30 irreducible representations of Sp(2) occur in \( e_{\mathcal{A}_3[2]}(v) \) and that for each irreducible representation the coefficients of \( v^i \) are all either zero or \( \pm 1 \). This is in sharp contrast to the cohomology of the individual pieces - all irreducible representations except \( \phi_{7a} \) occur in some cohomology group of some piece and they occur with multiplicities up to 14.

5. AN INDECOMPOSABLE ABELIAN SURFACE AND AN ELLIPTIC CURVE

As in the genus 3 case, \( t_2 \) gives an isomorphism \( \mathcal{M}_2[2] \cong \mathcal{A}_2^\text{in}[2] \), where \( \mathcal{A}_2^\text{in}[2] \) denotes the indecomposable locus inside \( \mathcal{A}_2[2] \).

There is a close relationship between \( \mathcal{A}_2[2] \) and the product space \( \mathcal{M}_2[2] \times \mathcal{A}_1[2] \). Let \( C \) be a genus 2 curve with level 2 structure represented by the symplectic basis \( (e_1, e_2, f_1, f_2) \) of Jac(C)[2] and let \( E \) be an elliptic curve with level 2 structure \( (e_3, f_3) \). Then \( t_2(C) \times E \) is an abelian threefold and \( t_2(C)[2] \times E[2] \) is a six dimensional vector space over \( \mathbb{F}_2 \) with a symplectic pairing given by

\[ e_i \cdot e_j = f_i \cdot f_j = 0 \]

and

\[ e_i \cdot f_j = \delta_{i,j} \]

for all \( i \) and \( j \), where we identify \( e_i \) with \( t_2(e_i) \) and \( f_i \) with \( t_2(f_i) \). Clearly, not all level 2 structures on \( t_2(C) \times E \) arise in this way but those that do are permuted by the group Sp(4) \times Sp(2). Let \( \mathcal{C} \) be the quotient set Sp(6)/(Sp(4) \times Sp(2)).

We may then describe the locus \( \mathcal{A}_2[2] \) as

\[ \mathcal{A}_2[2] = \prod_{c \in \mathcal{C}} (\mathcal{M}_2[2] \times \mathcal{A}_1[2]), \]

where \( (\mathcal{M}_2[2] \times \mathcal{A}_1[2])_c \) is an isomorphic copy of \( \mathcal{M}_2[2] \times \mathcal{A}_1[2] \) indexed by \( c \) and the components are permuted as

\[ g(\mathcal{M}_2[2] \times \mathcal{A}_1[2])_c = (\mathcal{M}_2[2] \times \mathcal{A}_1[2])_{gc} \]
for $g \in \text{Sp}_2(6)$. In terms of cohomology groups this means that
\[ H^i(A_2[2]) = \text{Ind}_{\text{Sp}_2(4) \times \text{Sp}_2(2)} H^i(\mathcal{M}_2[2] \times A_1[2]). \]
By the K"unneth theorem we have that
\[ H^i(\mathcal{M}_2[2] \times A_1[2]) \cong \bigoplus_{p+q=i} H^p(\mathcal{M}_2[2]) \otimes H^q(A_1[2]). \]
Thus, in order to understand the action of $\text{Sp}_2(4) \times \text{Sp}_2(2)$ on $H^i(\mathcal{M}_2[2] \times A_1[2])$ it is enough to understand the action of $\text{Sp}_2(4)$ on $H^i(\mathcal{M}_2[2])$ and the action of $\text{Sp}_2(2)$ on $H^i(A_1[2])$ for all $i$.

5.1. The moduli space of elliptic curves with level two structure. In order to understand the action of $\text{Sp}_2(4)$ on $H^i(A_1[2])$ we note that $\text{Sp}_2(2)$ is isomorphic to the symmetric group $S_3$ and that $A_1[2]$ is isomorphic to $\mathcal{M}_{0,4}$, the moduli space of four ordered points on $\mathbb{P}^1$. Under these identifications, the action of $\text{Sp}_2(2)$ is given by permuting the first three points.

Since $\mathcal{M}_{0,4}$ is pure in the sense of Dimca and Lehrer we can deduce the action of $\text{Sp}_2(2)$ on the cohomology groups by a twisted point count, see Section 2. Simple computations give, where $F$ is the Frobenius,
\[
\begin{align*}
|\mathcal{M}^{F^{\text{id}}}_{0,4}| &= \frac{(q+1)(q-1)(q-2)}{|\text{PGL}_2(F_q)|} = q - 2\\
|\mathcal{M}^{F^{(12)}}_{0,4}| &= \frac{(q+1)(q^2-q)q}{|\text{PGL}_2(F_q)|} = q \\
|\mathcal{M}^{F^{(123)}}_{0,4}| &= \frac{(q+1)(q^2-q)(q^2-q)}{|\text{PGL}_2(F_q)|} = q + 1.
\end{align*}
\]
Thus, the traces of $(\text{id}, (12), (123))$ on $H^0(A_1[2])$ and $H^1(A_1[2])$ are $(1,1,1)$ and $(2,0,-1)$, respectively. In other words, $H^i(A_1[2])$ is the trivial representation of $\text{Sp}_2(2)$ while $H^1(A_1[2])$ is the standard representation.

5.2. The moduli space of genus two curves with level two structure. In order to understand the action of $\text{Sp}_2(4)$ on $H^i(\mathcal{M}_2[2])$ we note that $\text{Sp}_2(4)$ is isomorphic to the symmetric group $S_6$ and that $\mathcal{M}_2[2]$ is isomorphic to $\mathcal{M}_{0,6}$, the moduli space of six ordered points on $\mathbb{P}^1$. Under these identifications, the action of $\text{Sp}_2(4)$ on $\mathcal{M}_2[2]$ is given by permuting the points. Also, $\mathcal{M}_{0,6}$ is pure so we can again deduce the action of $\text{Sp}_2(4)$ on the cohomology via twisted point counts. Simple computations give, where $F$ is the Frobenius,
\[
\begin{align*}
|\mathcal{M}^{F^{\text{id}}}_{0,6}| &= \frac{(q+1)(q-1)(q-2)(q-3)(q-4)}{|\text{PGL}_2(F_q)|} = q^3 - 9q^2 + 26q - 24 \\
|\mathcal{M}^{F^{(12)}}_{0,6}| &= \frac{(q+1)(q-1)(q-2)(q-3)(q-4)}{|\text{PGL}_2(F_q)|} = q^3 - 3q^2 + 2q \\
|\mathcal{M}^{F^{(123)}}_{0,6}| &= \frac{(q+1)(q^3-q^2)(q^2-q)}{|\text{PGL}_2(F_q)|} = q^3 - q^2 - 2q \\
|\mathcal{M}^{F^{(1234)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 - 3q^2 - 2q + 8 \\
|\mathcal{M}^{F^{(12345)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 - q^2 \\
|\mathcal{M}^{F^{(123456)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 - q - 3 \\
|\mathcal{M}^{F^{(123456)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 + q^2 \\
|\mathcal{M}^{F^{(123456)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 + q^2 + q + 1 \\
|\mathcal{M}^{F^{(123456)}}_{0,6}| &= \frac{(q^2-q)(q^3-q^2)(q^2-q^3)}{|\text{PGL}_2(F_q)|} = q^3 + q - 1.
\end{align*}
\]
By the Künneth theorem we have that
\[
\text{for } g \in \text{Sp}_2(4) \text{ we therefore know } H^i(M_2[2] \times A_1[2]) \text{ as a representation of Sp}_2(4) \times \text{Sp}_2(2). \]
Inducing this representation to Sp_2(6) gives us \( H^i(A_{2,1}[2]) \)
and \( \text{Sp}_2(6) \). We give the result in Table 5. As an aside we note that the complement of \( t_2(M_2[2]) \) inside \( A_2[2] \) consists of products of elliptic curves. The cohomology of this locus can be computed using the same techniques as in Section 6, but in this case we will omit the details. Let us denote the irreducible representations of \( \text{Sp}_2(4) \cong S_6 \) by \( \lambda \), which are indexed in the standard way by \( \lambda \), a partition of 6. Adding the contributions from the two loci we find that,
\[
e_{A_2[2]}(v) = (1 + v^2)s_6 - v^4(s_{5,1} + s_{4,2}) + v^6s_{3,2,1}.
\]

6. Products of three elliptic curves

There is a close relationship between the product space \((A_1[2])^3 \) and the locus \( A_{1,1,1}[2] \). Let \( E_1, E_2 \) and \( E_3 \) be three elliptic curves with level 2 structures \((e_1, f_1), (e_2, f_2) \) and \((e_3, f_3) \), respectively. Then \( E_1 \times E_2 \times E_3 \) is an abelian threefold and \( E_1[2] \times E_2[2] \times E_3[2] \) is a six dimensional vector space over \( \mathbb{F}_2 \) with a symplectic pairing given by
\[
e_i \cdot e_j = f_i \cdot f_j = 0
\]
for all \( i \) and \( j \)
\[
e_i \cdot f_j = \delta_{i,j}.
\]
Clearly, not all level 2 structures on \( E_1 \times E_2 \times E_3 \) arise in this way but those that do are permuted by the group \((\text{Sp}_2(2))^3 \) while the three curves themselves are permuted by the symmetric group \( S_3 \). Let \( \mathcal{C} \) be the quotient set \( \text{Sp}_2(6)/(S_3 \ltimes (\text{Sp}_2(2))^3) \). We may describe the locus \( A_{1,1,1}[2] \) as
\[
A_{1,1,1}[2] \cong \prod_{c \in \mathcal{C}} (A_1[2])_c^3,
\]
where \((A_1[2])_c^3 \) is an isomorphic copy of \((A_1[2])^3 \) indexed by \( c \) and the components are permuted as
\[
g(A_1[2])_c^3 = (A_1[2])_{gc}^3
\]
for \( g \in \text{Sp}_2(6) \). In terms of cohomology groups this means that
\[
H^i(A_{1,1,1}[2]) = \text{Ind}_{S_3 \ltimes (\text{Sp}_2(2))}^{\text{Sp}_2(6)} H^i((A_1[2])^3).
\]
By the Künneth theorem we have that
\[
H^i((A_1[2])^3) \cong \bigoplus_{p+q+r=i} H^p(A_1[2]) \otimes H^q(A_1[2]) \otimes H^r(A_1[2]).
\]
Thus, in order to understand the action of \( S_3 \ltimes (\text{Sp}_2(2))^3 \) on \( H^i((A_1[2])^3) \) it is enough to understand the action of \( \text{Sp}_2(2) \) on \( H^i(A_1[2]) \) for all \( i \) and the action of \( S_3 \) on the factors. Since the action of \( \text{Sp}_2(2) \) was described in Section 5.4.4 we only consider the action of \( S_3 \) on the factors. Let \( \alpha \otimes \beta \otimes \gamma \in H^p(A_1[2]) \otimes H^q(A_1[2]) \otimes H^r(A_1[2]) \subseteq
where the signs are a consequence of the Künneth isomorphism. Since $S_3$ is generated by transpositions and $H^{p+q+r}((A_1[2])^3)$ is generated by elements of the form $\alpha \otimes \beta \otimes \gamma$ for all possible choices of $p$, $q$ and $r$, this determines the action of $S_3$ on $H^{p+q+r}((A_1[2])^3)$.

We now have all the information we need in order to understand the action of $S_3 \ltimes (Sp_2(2))^3$ on $H^i((A_1[2])^3)$.

**Example 6.1.** Let $u$ be a basis vector for the trivial representation of $Sp_2(2)$ and let $v_1$ and $v_2$ be basis vectors for the standard representation of $Sp_2(2)$. Let $\sigma \in Sp_2(2)$ be an element of order 3 acting as

$$\begin{align*}
\sigma.u &= u, \\
\sigma.v_1 &= v_2, \\
\sigma.v_2 &= -v_1 - v_2,
\end{align*}$$

and let $g \in S_3 \ltimes (Sp_2(2))^3$ be the element $g = ((23), (\sigma, \sigma, id))$. We of course have $g.u \otimes u \otimes u = u \otimes u \otimes u$, so $\text{Tr}(g, H^0((A_1[2])^3)) = 1$. In cohomological degree 1 we have

$$g.v_2 \otimes u \otimes u = -(v_1 + v_2) \otimes u \otimes u$$

while $g.\alpha$ has no component in the direction of $\alpha$ for all other choices of $\alpha \in H^1((A_1[2])^3)$. Thus, $\text{Tr}(g, H^1((A_1[2])^3)) = -1$. In degree 2 we have

$$g.u \otimes v_2 \otimes v_2 = u \otimes v_2 \otimes (v_1 + v_2)$$

while $g.\alpha$ has no component in the direction of $\alpha$ for all other choices of $\alpha \in H^2((A_1[2])^3)$. We conclude that $\text{Tr}(g, H^2((A_1[2])^3)) = 1$. Finally, in degree 3 we have

$$g.v_2 \otimes v_2 \otimes v_2 = -(v_1 + v_2) \otimes v_2 \otimes (v_1 + v_2)$$

while $g.\alpha$ has no component in the direction of $\alpha$ for all other choices of $\alpha \in H^3((A_1[2])^3)$. Hence, $\text{Tr}(g, H^3((A_1[2])^3)) = -1$. We thus have

$$\sum_{i=0}^{3} \text{Tr}(g, H^i((A_1[2])^3)) t^i = 1 - t + t^2 - t^3.$$

Similar computations for the other conjugacy classes of $S_3 \ltimes (Sp_2(2))^3$ give the results in Table 2 where

$$P_g((A_1[2])^3, t) := \sum_{i=0}^{3} \text{Tr}(g, H^i((A_1[2])^3)) t^i,$$

is called the equivariant Poincaré polynomial. See Chapter 4 of [8] for a beautiful description of how to compute representatives of $S_3 \ltimes (Sp_2(2))^3$. In Table 2, $\sigma$ is the element of $Sp_2(2)$ described in Example 6.1 while $\tau$ is the element of order 2 acting as

$$\tau.v_1 = -v_1, \quad \tau.v_2 = v_1 + v_2.$$
where $v_1$ and $v_2$ are the same basis vectors of the standard representation considered in Example 6.1.

By inducing the corresponding representations from $S_3 \ltimes (\text{Sp}_2(2))^3$ to $\text{Sp}_2(6)$ we obtain the cohomology of $A_{1,1,1}[2]$ as a representation of $\text{Sp}_2(6)$. We give the result in Table 6.

7. Cohomology groups of strata

In this section we give the cohomology groups of $Q[2]$, $H_3[2]$, $A_{2,1}[2]$ and $A_{1,1,1}[2]$ as representations of $\text{Sp}_2(6)$. The results are presented in Table 6. Each column in these tables corresponds to an irreducible representation of $\text{Sp}_2(6)$. The irreducible representations are denoted $\phi_{d,n}$ where $d$ is the dimension of the representation and $n$ is letter used to distinguish different representations of the same dimension, see 3.

| $g$ | $P_6((A_1[2]^3,t)$ | $g$ | $P_6((A_1[2]^3,t)$ | $g$ | $P_6((A_1[2]^3,t)$ |
|-----|----------------------|-----|----------------------|-----|----------------------|
| $(\text{id}, \text{id}, \text{id})$ | $1 + 6t + 12t^2 + 8t^3$ | $(\text{id}, \text{id}, \text{id})$ | $1 + 2t - 2t^2 - 4t^3$ | $(\text{id}, \text{id}, \text{id})$ | $1 + 2t^4$ |
| $(\text{id}, \text{id}, \tau)$ | $1 + 4t + 4t^2$ | $(\text{id}, \text{id}, \tau, \text{id})$ | $1 + 2t$ | $(\text{id}, \text{id}, \tau, \text{id})$ | $1$ |
| $(\text{id}, \text{id}, \sigma)$ | $1 + 3t - 4t^3$ | $(\text{id}, \text{id}, \sigma, \text{id})$ | $1 + 2t + t^2 + 2t^3$ | $(\text{id}, \text{id}, \sigma, \text{id})$ | $1 - t^3$ |
| $(\text{id}, \tau, \tau)$ | $1 + 2t$ | $(\text{id}, \tau, \tau, \text{id})$ | $1 - 2t^2$ | $(\text{id}, \tau, \tau, \text{id})$ | $1$ |
| $(\text{id}, \tau, \sigma)$ | $1 + t - 2t^2$ | $(\text{id}, \tau, \sigma, \text{id})$ | $1$ | $(\text{id}, \tau, \sigma, \text{id})$ | $1$ |
| $(\text{id}, \sigma, \sigma)$ | $1 - 3t^2 + 2t^3$ | $(\text{id}, \sigma, \sigma, \text{id})$ | $1 + t^2$ | $(\text{id}, \sigma, \sigma, \text{id})$ | $1$ |
| $(\tau, \tau, \tau)$ | $1$ | $(\tau, \tau, \tau, \text{id})$ | $1 - t - 2t^2 + 2t^3$ | $(\tau, \tau, \tau, \text{id})$ | $1$ |
| $(\tau, \tau, \sigma)$ | $1 - t$ | $(\tau, \tau, \sigma, \text{id})$ | $1$ | $(\tau, \tau, \sigma, \text{id})$ | $1$ |
| $(\tau, \sigma, \sigma)$ | $1 - 2t + t^2$ | $(\tau, \sigma, \sigma, \text{id})$ | $1 - t + t^2 + t^3$ | $(\tau, \sigma, \sigma, \text{id})$ | $1$ |
| $(\sigma, \sigma, \sigma)$ | $1 - 3t + 3t^2 - t^3$ | $(\sigma, \sigma, \sigma, \text{id})$ | $1 - t + t^2 + t^3$ | $(\sigma, \sigma, \sigma, \text{id})$ | $1$ |

Table 2. Equivariant Poincaré polynomials of $(A_1[2]^3$, for a representative $g$ of every conjugacy class of $S_3 \ltimes (\text{Sp}_2(2))^3$.

| $\phi_{d,n}$ | $\phi_{1,n}$ | $\phi_{2,n}$ | $\phi_{3,n}$ | $\phi_{4,n}$ | $\phi_{5,n}$ | $\phi_{6,n}$ | $\phi_{7,n}$ | $\phi_{8,n}$ | $\phi_{9,n}$ |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $H^0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $H^1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $H^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $H^3$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $H^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $H^5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $H^6$ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| $H^7$ | 1 | 0 | 2 | 1 | 1 | 2 | 1 | 0 | 3 |
| $H^8$ | 5 | 1 | 1 | 4 | 0 | 3 | 2 | 2 | 5 |

Table 3. The cohomology groups of $Q[2]$ as a representation of $\text{Sp}_2(6)$. 
Table 4. The cohomology groups of $H_3[2]$ as a representation of $\text{Sp}_2(6)$.

| $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ |
|-------|-------|-------|-------|-------|
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ | $\phi_{6}$ |
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ | $\phi_{6}$ |
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ | $\phi_{6}$ |

Table 5. The cohomology groups of $A_{2,1}[2]$ as representations of $\text{Sp}_2(6)$.

| $H^0$ | $H^1$ | $H^2$ | $H^3$ |
|-------|-------|-------|-------|
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ |
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ |
| $\phi_{14}$ | $\phi_{12}$ | $\phi_{10}$ | $\phi_{8}$ |

Table 6. The cohomology groups of $A_{1,1,1}[2]$ as representations of $\text{Sp}_2(6)$.
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