There is good support for the embedding of the Standard Model fermions in the chiral $16 \, SO(10)$ representation. Such an embedding is provided by the realistic free fermionic heterotic-string models. In this talk we demonstrate the existence of solutions with three generations and $SO(10)$ observable gauge group, in the case of compactification on an torus–fibred Calabi–Yau space over a Hirzebruch base surface. The $SO(10)$ symmetry is broken to $SU(5) \times U(1)$ by a Wilson line. The overlap with the realistic free fermionic heterotic–string models is discussed.

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1. Introduction

There has been considerable interest in the construction realistic particle physics vacua with $N = 1$ supersymmetry, three families of quarks and leptons and a grand unified gauge group $H$ upon compactification of Hořava–Witten theory $^1$ on a complex Calabi–Yau 3–fold $^2$. The latter can be an elliptically–fibred manifold $X$ over a base complex 2–fold $B$ or, more generally, it can be a torus–fibred 3–fold $Z$. While $X$ is assumed to possess a section $\sigma : B \rightarrow X$, $Z$ need not carry a section. The $E_8$ gauge group on the observable sector decomposes as $E_8 \supset G \times H$, where $H$ is the gauge group of the effective field theory and $G$ appears as the structure group of a holomorphic, stable vector bundle over the 3–fold. We are interested in the case when $G = SU(4)$ and $H = SO(10)$. A nonperturbative vacuum state
of the resulting GUT theory on the observable sector is specified by a set of M–theory 5–branes wrapping a holomorphic 2–cycle on the 3–fold. The 5–branes are described by a 4–form cohomology class \( [W] \) satisfying the anomaly–cancellation condition. This class is Poincaré–dual to an effective cohomology class in \( H_2(X, \mathbb{Z}) \) that can be written as
\[
[W] = \sigma_*(\omega) + c(F - N) + dN,
\]
where \( c, d \) are integers, \( \omega \) is a class in \( B \), and \( \sigma_*(\omega) \) is its pushforward to \( X \) under \( \sigma \). The rules to construct these vacua explicitly can be summarised as follows (see refs. 2, 3 for more details):

a) Semistability condition: the spectral data specifying a semistable, holomorphic vector bundle on the 3–fold can be written in terms of an effective class \( \eta \in H^2(B, \mathbb{Z}) \) and coefficients \( \lambda, \kappa_i \) satisfying
\[
\lambda \in \mathbb{Z}, \quad \eta = c_1(B) \mod 2,
\]
with the \( \kappa_i \) either all integer or all half an odd integer, or alternatively
\[
\lambda = \frac{2m + 1}{2}, \quad m \in \mathbb{Z}, \quad c_1(B) \text{ even},
\]
with the same requirements on the \( \kappa_i \). Above, \( c_1 \) denotes the first Chern class.

b) Involution conditions: for a vector bundle \( V_X \) on \( X \) to descend to a vector bundle \( V_Z \) on \( Z \) it is necessary that
\[
\tau_B(\eta) = \eta, \quad \sum_i \kappa_i = \eta \cdot c_1(B).
\]

c) Effectiveness condition: a sufficient condition for \( [W] \) in eqn. (1) to be an effective class is
\[
12c_1(B) \geq \eta, \quad c \geq 0, \quad d \geq 0.
\]
d) Commutant condition: for \( H = SO(10) \) this condition reads
\[
\eta \geq 4c_1(B).
\]
e) Three–family condition:
\[
\lambda \eta (\eta - nc_1(B)) = 6.
\]

2. Vacua over Hirzebruch surfaces \( F_r \).

We take the base manifold \( B \) to be the Hirzebruch surface \( F_r \), \( r \geq 0 \). The latter is a \( \mathbb{C}P^1 \)–fibration over \( \mathbb{C}P^1 \). A basis for \( H_2(F_r, \mathbb{Z}) \) composed of effective classes is given by the class of the base \( \mathbb{C}P^1 \), denoted \( S \), plus the
class of the fibre $\mathbb{CP}^1$, denoted $E$. Their intersections are $S \cdot S = -r$, $S \cdot E = 1$, $E \cdot E = 0$. All effective curves in $F_r$ are linear combinations of $S$ and $E$ with nonnegative coefficients. The Chern classes of $F_r$ are $c_1(F_r) = 2S + (r + 2)E$, $c_2(F_r) = 4$. It is proved in ref. 2 that, over the base $F_r$, one can construct torus–fibred Calabi–Yau 3–folds $Z$ whose fundamental group is $\mathbb{Z}_2$ when $r = 0, 2$. For those allowed values of $r$, any class $\eta \in H_2(F_r, \mathbb{Z})$ is $\tau_B$–invariant. In what follows we will work with an arbitrary allowed value of $r$. Let us write $\eta \in H_2(F_r, \mathbb{Z})$ as $\eta = sS + eE$, for some integers $s, e$ to be determined imposing the conditions summarized in section 1. We can now go to eqn. (1) and write explicit expressions for the homology class $[W]$ that is being wrapped by the fivebranes on the torus–fibred Calabi–Yau 3–fold $Z$. We have $\omega = (24 - s)S + (12r + 24 - e)E$,

\begin{equation}
 c = 112 + \frac{3}{\lambda} - 12\lambda - \sum_i \kappa_i^2,
\end{equation}

\begin{equation}
 d = 16 + \frac{3}{\lambda} - 12\lambda + \sum_i \kappa_i - \sum_i \kappa_i^2.
\end{equation}

Every allowed choice of $r$, plus every choice of the rational coefficients $\kappa_i$ subject to the conditions indicated in each case, gives rise to a different vacuum $[W]$:

- $s = 9$: $\sum_i \kappa_i = 30$ and $\sum_i \kappa_i^2 \leq 46$,

\begin{equation}
 [W] = \alpha_s \left( 15S + \left( \frac{15}{2}r + 18 \right)E \right) + \left( 112 - \sum_i \kappa_i^2 \right)(F - N) + \left( 46 - \sum_i \kappa_i^2 \right)N.
\end{equation}

- $s = 10$: $\sum_i \kappa_i = 34$ and $\sum_i \kappa_i^2 \leq 34$,

\begin{equation}
 [W] = \alpha_s \left( 14S + (7r + 17)E \right) + \left( 96 - \sum_i \kappa_i^2 \right)(F - N) + \left( 34 - \sum_i \kappa_i^2 \right)N.
\end{equation}

- $s = 11$: $\sum_i \kappa_i = 34$ and $\sum_i \kappa_i^2 \leq 66$,

\begin{equation}
 [W] = \alpha_s \left( 13S + \left( \frac{13}{2}r + 18 \right)E \right) + \left( 128 - \sum_i \kappa_i^2 \right)(F - N) + \left( 66 - \sum_i \kappa_i^2 \right)N.
\end{equation}

- $s = 13$: $\sum_i \kappa_i = 38$ and $\sum_i \kappa_i^2 \leq 38$,

\begin{equation}
 [W] = \alpha_s \left( 11S + \left( \frac{11}{2}r + 18 \right)E \right) + \left( 96 - \sum_i \kappa_i^2 \right)(F - N) + \left( 38 - \sum_i \kappa_i^2 \right)N.
\end{equation}

- $s = 14$: $\sum_i \kappa_i = 38$ and $\sum_i \kappa_i^2 \leq 54$,

\begin{equation}
 [W] = \alpha_s \left( 10S + (5r + 19)E \right) + \left( 112 - \sum_i \kappa_i^2 \right)(F - N) + \left( 54 - \sum_i \kappa_i^2 \right)N.
\end{equation}
\[ s = 15: \sum_i \kappa_i = 42 \text{ and } \sum_i \kappa_i^2 \leq 58, \]
\[ [W] = \sigma_+ \left( 9S + \left( \frac{9}{2}r + 18 \right) E \right) + (112 - \sum_i \kappa_i^2) (F - N) + (58 - \sum_i \kappa_i^2) N. \tag{15} \]

\[ s = 18: \sum_i \kappa_i = 46 \text{ and } \sum_i \kappa_i^2 \leq 78, \]
\[ [W] = \sigma_+ \left( 6S + (3r + 19) E \right) + (128 - \sum_i \kappa_i^2) (F - N) + (78 - \sum_i \kappa_i^2) N. \tag{16} \]

\[ s = 22: \sum_i \kappa_i = 54 \text{ and } \sum_i \kappa_i^2 \leq 54, \]
\[ [W] = \sigma_+ \left( 2S + (r + 19) E \right) + (96 - \sum_i \kappa_i^2) (F - N) + (54 - \sum_i \kappa_i^2) N. \tag{17} \]

3. Overlap with the free fermionic models

In this section we elaborate briefly on the overlap with the free fermionic models. Amazingly enough, the structure of the manifolds constructed in ref. 2, up to the imposition of the three generation condition, precisely coincides with the structure of the manifold that underlies the free fermionic models.

In the free fermionic formalism 5 a model is specified in terms of a set of boundary condition basis vectors and one-loop GSO projection coefficients. These fully determine the partition function of the models, the spectrum and the vacuum structure. The three generation models of interest here are constructed in two stages. The first corresponds to the NAHE set of boundary basis vectors \( \{1, S, b_1, b_2, b_3\} \). The second consists of adding to the NAHE set three additional boundary condition basis vectors, typically denoted \( \{\alpha, \beta, \gamma\} \). The gauge group at the level of the NAHE set is \( SO(6)^3 \times SO(10) \times E_8 \), which is broken to \( SO(4)^3 \times U(1)^3 \times SO(10) \times SO(16) \) by the vector \( 2\gamma \). Alternatively, we can start with an extended NAHE set \( \{1, S, \xi_1, \xi_2, b_1, b_2\} \), with \( \xi_1 = 1 + b_1 + b_2 + b_3 \). The set \( \{1, S, \xi_1, \xi_2\} \) produces a toroidal Narain model with \( SO(12) \times E_6 \times E_8 \) or \( SO(12) \times SO(16) \times SO(16) \) gauge group depending on the GSO phase \( c(\xi_1, \xi_2) \). The basis vectors \( b_1 \) and \( b_2 \) then break \( SO(12) \to SO(4)^3 \), and either \( E_6 \times E_8 \to E_6 \times U(1)^2 \times E_8 \) or \( SO(16) \times SO(16) \to SO(10) \times U(1)^3 \times SO(16) \). The vectors \( b_1 \) and \( b_2 \) correspond to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold modding. The three sectors \( b_1, b_2 \) and \( b_3 \) correspond to the three twisted sector of the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold, with each producing eight generations in the \( 27 \), or \( 16 \), representations of \( E_6 \), or \( SO(10) \), respectively. In the case of \( E_6 \) the untwisted sector produces an additional \( 3 \times (27 + \bar{27}) \), whereas in the \( SO(10) \) model it produces \( 3 \times (10 + \bar{10}) \). Therefore, the Calabi–Yau manifold that corresponds to the
$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at the free fermionic point in the Narain moduli space has $(h_{21}, h_{11}) = (27, 3)$.

To note the overlap with the construction of ref. 2 we construct the $\mathbb{Z}_2 \times \mathbb{Z}_2$ at a generic point in the moduli space. For this purpose, let us start with the compactified torus $T_1^2 \times T_2^2 \times T_3^2$ parameterized by three complex coordinates $z_1, z_2$ and $z_3$, with the identification

$$z_i = z_i + 1; \quad z_i = z_i + \tau_i,$$

where $\tau$ is the complex parameter of each torus $T^2$. We consider $\mathbb{Z}_2$ twists and possible shifts of order two:

$$z_i \rightarrow (-1)^{\epsilon_i} z_i + \frac{1}{2} \delta_i,$$

subject to the condition that $\prod_i (-1)^{\epsilon_i} = 1$. This condition insures that the holomorphic three–form $\omega = dz_1 \wedge dz_2 \wedge dz_3$ is invariant under the $\mathbb{Z}_2$ twist. Under the identification $z_i \rightarrow -z_i$, a single torus has four fixed points at

$$z_i = \{0, 1/2, \tau/2, (1 + \tau)/2\}.$$  \hfill (20)

The first model that we consider is produced by the two $\mathbb{Z}_2$ twists

$$\alpha : (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3)$$

$$\beta : (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3).$$  \hfill (21)

There are three twisted sectors in this model, $\alpha, \beta$ and $\alpha \beta = \alpha \cdot \beta$, each producing 16 fixed tori, for a total of 48. The untwisted sector adds three additional Kähler and complex deformation parameters producing in total a manifold with $(h_{21}, h_{11}) = (51, 3)$. We refer to this model as $X_1$. This manifold admits an elliptic fibration over a base $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. This can be seen from the Borceia–Voisin classification of elliptically fibered Calabi–Yau manifolds 7 and from ref. 8.

Next we consider the model generated by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twists in (21), with the additional shift

$$\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + \frac{1}{2}).$$  \hfill (22)

This model has fixed tori from the three twisted sectors $\alpha, \beta$ and $\alpha \beta$. The product of the $\gamma$ shift in (22) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the $\gamma$ shift, half the fixed tori from each twisted sector are paired. Therefore, the action of this shift is to reduce the total number of fixed tori from the twisted sectors by a factor of 1/2. Consequently, in this model $(h_{21}, h_{11}) = (27, 3)$. This model therefore reproduces the data of the
$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at the free-fermion point in the Narain moduli space. We refer to this model as $X_2$.

The manifold $X_1$ therefore coincides with the manifold $X$ of ref. 2, the manifold $X_2$ coincides with the manifold $Z$, and the $\gamma$-shift in eq. (22) coincides with the freely acting involution $\tau_X$ of ref. 2. Thus, the free fermionic models admit precisely the structure of the Calabi-Yau manifolds considered in ref. 2.

4. Conclusions

We discussed in this paper the construction of nonperturbative flipped $SU(5)$ vacua in Hořava–Witten theory. The flipped $SU(5)$ model 9 played a pivotal role in the development of the realistic free fermionic heterotic string models 10. Hořava–Witten theory provides the framework to extend the study of these models to the nonperturbative domain. Details of these investigations will be presented in forthcoming publications.

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