Annihilators of local cohomology modules and restricted flat dimensions

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ABSTRACT
Yoshizawa investigated when local cohomology modules have an annihilator that does not depend on the choice of the defining ideal. In this paper we refine his results and investigate the relationship between annihilators of local cohomology modules and restricted flat dimensions.

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1. Introduction
Throughout this paper, let $R$ be a commutative noetherian ring.

The annihilators of local cohomology modules have been widely studied. For example, Faltings’ annihilator theorem [5] states that if $R$ is a homomorphic image of a regular ring and $M$ is a finitely generated $R$-module, then, for ideals $a$ and $b$ of $R$, there exists an integer $n$ such that $b^nH^i_a(M) = 0$ for all integers $i < \lambda^a_b(M)$, where $\lambda^a_b(M) = \inf\{\text{depth } M_p + \text{ht}(a + p)/p \mid p \in \text{Spec } R \setminus V(b)\}$; see also [2, Theorem 9.5.1].

Yoshizawa [8] investigated the following question by using Takahashi’s classification theorem of the dominant resolving subcategories of the category $\text{mod } R$ of finitely generated $R$-modules [7, Theorem 5.4].

Question 1.1. Let $M$ be a finitely generated $R$-module. Let $p$ be a prime ideal of $R$. When does there exist an element $s \in R \setminus p$ such that $sH^i_I(M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R)$?

Note that the annihilator $s$ does not depend on the choice of the ideals $I$.

The purpose of this paper is to investigate the relationship between such annihilators of local cohomology modules and large/small restricted flat dimensions.

First, we deal with [8, Proposition 2.5(4),(5)], which states that if $R$ is a ring of finite Krull dimension or a Cohen-Macaulay ring, then the resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is dominant (see Definition 2.2) for all prime ideals $p$ of $R$, where

$$\mathcal{R}(p) = \left\{ M \in \text{mod } R \right\} \text{ There exists } s \in R \setminus p \text{ such that } sH^i_I(M) = 0 \text{ for all ideals } I \text{ of } R \text{ and all integers } i < \text{grade}(I, R) \right\}.

We prove the dominance of $\mathcal{R}(p)$ without assuming that $R$ has finite Krull dimension or is a Cohen-Macaulay ring.
**Theorem 1.2.** Let $p$ be a prime ideal of $R$. The resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is dominant.

Theorem 1.2 leads to the following corollary, which removes from [8, Corollary 4.3] all the assumptions.

**Corollary 1.3.** Let $M$ be a finitely generated $R$-module, and let $p$ be a prime ideal of $R$. Then there exists an element $s \in R \setminus p$ such that $sH^1_I(M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R) - \text{Rfd}_{R_p} M_p$.

Using Corollary 1.3 and large/small restricted flat dimensions, we obtain the following theorem, the second assertion of which removes from [8, Theorem 4.1] the assumption that the resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is dominant. The first assertion of the following theorem provides a complete answer to Question 1.1.

**Theorem 1.4.** Let $M$ be a finitely generated $R$-module. Let $p$ be a prime ideal of $R$. Consider the following three conditions.

(a) The module $M$ belongs to $\mathcal{R}(p)$, that is, there exists an element $s \in R \setminus p$ such that $sH^1_I(M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R)$.
(b) The large restricted flat dimension $\text{Rfd}_{R_p} M_p$ is at most 0.
(c) The small restricted flat dimension $\text{rfd}_{R_p} M_p$ is at most 0.

Then the following statements hold.

1. The implications (a) $\iff$ (b) $\implies$ (c) always hold.
2. The implication (a) $\implies$ (b) holds if $\text{grade}(q, R) = \text{depth } R_q$ for all $q \in U(p)$.
3. The implication (a) $\iff$ (c) holds if $\text{cmd } R_p \leq 1$ or $\text{CM-dim}_{R_p} M_p < \infty$.
4. The implication (a) $\implies$ (c) holds if $\text{CM-dim}_{R_p} M_p < \infty$ and $\text{grade}(p, R) = \text{depth } R_p$.

Here cmd and CM-dim stand for Cohen-Macaulay defect and Cohen-Macaulay dimension, respectively (see Definition 4.4).

The organization of this paper is as follows. In Section 2, we state our convention, basic notions and their properties for later use. In Section 3, we prove Theorem 1.2. In Section 4, we investigate the relationship between annihilators of local cohomology modules and large/small restricted flat dimensions, and prove Corollary 1.3, and a result part of which is Theorem 1.4.

### 2. Basic definitions and properties

In this section, we give several definitions and their properties. We begin with our convention.

**Convention 2.1.** All rings are commutative noetherian rings with identity. Let $R$ be a (commutative noetherian) ring. We denote by $\text{mod } R$ the category of (finitely generated) $R$-modules. All subcategories of $\text{mod } R$ are full and closed under isomorphism. The symbol $\mathbb{N}$ denotes the set of non-negative integers. The grade and the depth of the zero module are $\infty$. For the definitions of a grade and a depth, we refer the reader to [3].

First, we recall the notions of a dominant subcategory and a resolving subcategory of $\text{mod } R$. We denote by $\Omega^n_R M$ the $n$-th syzygy module in a projective resolution of a finitely generated $R$-module $M$. Note that $\Omega^n_R M$ is uniquely determined up to projective summands.

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We denote by \( \mathcal{X} \) the subcategory of \( \text{mod } R \) consisting of the direct summands of finite direct sums of modules in \( \mathcal{X} \).

We say that \( \mathcal{X} \) is dominant if, for each \( p \in \text{Spec } R \), there exists a non-negative integer \( n \) such that \( \Omega^n_{R_p} \kappa (p) \in \text{add } \mathcal{X}_p \), where \( \kappa (p) = R_p/pR_p \) and \( \mathcal{X}_p = \{ X_p \in \text{mod } R_p \mid X \in \mathcal{X} \} \).

We say that \( \mathcal{X} \) is resolving if it satisfies the following four conditions.

(i) \( \mathcal{X} \) contains the projective \( R \)-modules.

(ii) \( \mathcal{X} \) is closed under direct summands, that is, if \( M \) is an \( R \)-module belonging to \( \mathcal{X} \) and \( N \) is a direct summand of \( M \), then \( N \) also belongs to \( \mathcal{X} \).

(iii) \( \mathcal{X} \) is closed under extensions, that is, for an exact sequence \( 0 \to L \to M \to N \to 0 \) of \( R \)-modules, if \( L \) and \( N \) belong to \( \mathcal{X} \), then \( M \) also belongs to \( \mathcal{X} \).

(iv) \( \mathcal{X} \) is closed under syzygies, that is, if \( M \) is an \( R \)-module belonging to \( \mathcal{X} \), then \( \Omega^1_R M \) also belongs to \( \mathcal{X} \).

Next, we recall the subcategory \( \mathcal{R}(p) \) of \( \text{mod } R \) and the subset \( U(p) \) of \( \text{Spec } R \) introduced by Yoshizawa [8] and a property of \( \mathcal{R}(p) \).

**Definition 2.3.** Let \( p \) be a prime ideal of \( R \).

(1) We define the subcategory \( \mathcal{R}(p) \) of \( \text{mod } R \) relative to \( p \) by

\[
\mathcal{R}(p) = \left\{ M \in \text{mod } R \mid \text{ There exists } s \in R \setminus p \text{ such that } sH^i_I(M) = 0 \text{ for all ideals } I \text{ of } R \text{ and all integers } i < \text{grade}(I, R) \right\}.
\]

(2) We denote by \( U(p) \) the generalization closed subset \( \{ q \in \text{Spec } R \mid q \subseteq p \} \) of \( \text{Spec } R \).

**Remark 2.4.** Let \( p \) be a prime ideal of \( R \). If \( q \in U(p) \), then \( \mathcal{R}(p) \subseteq \mathcal{R}(q) \).

**Proposition 2.5.** ([8, Proposition 2.5(3)]) Let \( p \) be a prime ideal of \( R \). Then the subcategory \( \mathcal{R}(p) \) is a resolving subcategory of \( \text{mod } R \).

**Remark 2.6.** Let \( M \) be a finitely generated \( R \)-module, and let \( p \) be a prime ideal of \( R \). Since \( 1_R \in R \setminus p \), if \( H^i_I(M) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \), then \( M \) belongs to \( \mathcal{R}(p) \).

Finally, we recall the definition of large and small restricted flat dimensions and some properties.

**Definition 2.7.** Let \( M \) be a finitely generated \( R \)-module.

(1) The large restricted flat dimension of \( M \) is defined by

\[
\text{Rfd}_R M = \sup_{p \in \text{Spec } R} \{ \text{depth } R_p - \text{depth } M_p \}.
\]

Note that \( \text{Rfd}_{R_p} M_p = \sup_{q \in U(p)} \{ \text{depth } R_q - \text{depth } M_q \} \) holds for each \( p \in \text{Spec } R \).

(2) The small restricted flat dimension of \( M \) is defined by

\[
\text{rfd}_R M = \sup_{p \in \text{Spec } R} \{ \text{grade}(p, R) - \text{grade}(p, M) \}.
\]

Note that \( \text{rfd}_{R_p} M_p = \sup_{q \in U(p)} \{ \text{grade}(qR_p, R_p) - \text{grade}(qR_p, M_p) \} \) holds for each \( p \in \text{Spec } R \).

We should remark that the original definitions of large/small restricted flat dimensions are different; they are similar to the definition of flat dimension from which those names come, and the equalities in the above definition turn out to hold. One has \( \text{Rfd}_R M, \text{rfd}_R M \in \mathbb{N} \cup \{ -\infty \} \), and \( \text{Rfd}_R M, \text{rfd}_R M = -\infty \) if and only if \( M = 0 \). Also, \( \text{Rfd}_R M \geq \text{rfd}_R M \). For the details of large/small restricted flat dimensions, we refer the reader to [1, Theorem 1.1] and [4, Theorem 2.4, 2.11 and Observation 2.10].
3. Dominance of $\mathcal{R}(p)$

Throughout this section, let $M$ be a finitely generated $R$-module, and let $p$ be a prime ideal of $R$. In this section, we will prove that the resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is always dominant. We begin with the following lemma, which will be used several times later.

**Lemma 3.1.** One has $\text{rfd}_R M = \sup \{ \text{grade}(I, R) - \text{grade}(I, M) \mid I \text{ is a proper ideal of } R \}$.

**Proof.** By the definition of small restricted flat dimension, there is an inequality

$$\text{rfd}_R M \leq \sup \{ \text{grade}(I, R) - \text{grade}(I, M) \mid I \text{ is a proper ideal of } R \}.$$  

We show that the reverse inequality holds. Let $I$ be a proper ideal of $R$ and put $n = \text{grade}(I, M)$. Then there exists an $M$-regular sequence $x = x_1, \ldots, x_n$ in $I$. By [3, Proposition 1.2.10(d)], we have $\text{grade}(I, M/xM) = 0$. Hence there exists $p \in \text{Ass}(M/xM)$ with $p \supseteq I$ such that $\text{grade}(p, M) = n = \text{grade}(I, M)$ by [3, Proposition 1.2.1 and 1.2.10(d)]. Since $p \supseteq I$, we have $\text{grade}(p, R) \geq \text{grade}(I, R)$. Hence we obtain $\text{grade}(p, R) - \text{grade}(p, M) \geq \text{grade}(I, R) - \text{grade}(I, M)$. Thus the assertion follows.

Using the above lemma, we can prove the following proposition.

**Proposition 3.2.** Suppose that the module $M$ is nonzero. Then $\Omega^s_R M \in \mathcal{R}(p)$, where $s := \text{rfd}_R M$.

**Proof.** Note that $s \in \mathbb{N}$. Let $I$ be a proper ideal of $R$. We show that the inequality $\text{grade}(I, \Omega^s_R M) \geq \text{grade}(I, R)$ holds. Let $s = 0$. Then we obtain $\text{grade}(I, \Omega^s_R M) = \text{grade}(I, M) \geq \text{grade}(I, R)$ by Lemma 3.1. Suppose that $s \geq 1$. Since $\text{grade}(I, M) + s \geq \text{grade}(I, R)$ by Lemma 3.1, the grade lemma [3, Proposition 1.2.9] yields

$$\text{grade}(I, \Omega^s_R M) \geq \min\{\text{grade}(I, R), \text{grade}(I, \Omega^{s-1}_R M) + 1\} \geq \ldots$$

$$\geq \min\{\text{grade}(I, R), \text{grade}(I, M) + s\}$$

$$= \text{grade}(I, R).$$

Since $H^i_I(\Omega^s_R M) = 0$ for all integers $i < \text{grade}(I, \Omega^s_R M)$ by [2, Theorem 6.2.7] and the local cohomology functor $H^i_R(\cdot)$ is the zero functor for all integers $i$, we obtain $H^i_I(\Omega^s_R M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R)$. Therefore, $\Omega^s_R M$ belongs to $\mathcal{R}(p)$.

Now we can archive the purpose of this section by showing the following corollary, which is none other than Theorem 1.2. Note that the subcategory $\mathcal{R}(p)$ is a resolving subcategory of $\text{mod } R$ by Proposition 2.5.

**Corollary 3.3.** The resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is dominant.

**Proof.** Suppose that the module $M$ is nonzero. Put $s = \text{rfd}_R M$ and $r = \text{Rfd}_R M$. In view of [7, Corollary 4.6], it is enough to show that $\Omega^s_R M$ belongs to $\mathcal{R}(p)$. By Proposition 3.2, we obtain $\Omega^s_R M \in \mathcal{R}(p)$. Since $r \geq s$, we have $r - s \geq 0$ and $\Omega^s_R M = \Omega^{r-s}_R M$ belongs to $\mathcal{R}(p)$. Thus the assertion follows.

4. Relationships of $\mathcal{R}(p)$ with restricted flat dimensions

In this section, we investigate the relationship between annihilators of local cohomology modules and large/small restricted flat dimensions.

First, we prove the following result, which is none other than Corollary 1.3. This result removes from [8, Corollary 4.3] all the assumptions, that is, the assumption that the resolving subcategory $\mathcal{R}(p)$ of $\text{mod } R$ is dominant, that $\sup_{q \in U(p)} \left\{ \text{depth } R_q - \text{depth } M_q \right\} \geq 0$, and that depth $R_q = \text{grade}(q, R)$ for all $q \in U(p)$. Note that, in the case $\text{Rfd}_{R_p} M_p = -\infty$, we get $-(-\infty)$, which is interpreted as $\infty$.
**Proposition 4.1.** (cf. [8, Corollary 4.3]) Let $M$ be a finitely generated $R$-module, and let $p$ be a prime ideal of $R$. Then there exists an element $s \in R \setminus p$ such that $sH^i_I(M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R) - \text{Rfd}_p(M)$.

**Proof.** Put $r = \text{Rfd}_R M$. Note that $r \in \mathbb{N} \cup \{-\infty\}$. If $r = -\infty$, then $M_p = 0$. Hence there exists an element $s \in R \setminus p$ such that $sM = 0$. Since local cohomology functors are $R$-linear by [2, Properties 1.2.2], we obtain $sH^i_I(M) = 0$ for all ideals $I$ of $R$ and all integers $i$. Let $r \geq 0$. We use induction on $r$. Let $r = 0$. Then we have $\text{depth} R - \text{depth} M_p = 0$ for all $q \in U(p)$. Since the resolving subcategory $\mathcal{R}(p)$ of mod $R$ is dominant by Corollary 3.3, it follows from [7, Theorem 1.1] that $M$ belongs to $\mathcal{R}(p)$. Suppose that $r \geq 1$. Then the depth lemma [3, Proposition 1.2.9] yields

$$\text{Rfd}_p(\Omega^1_R M)_p = \sup_{q \in U(p)} \{\text{depth} R_q - \text{depth} (\Omega^1_R M)_q\} = \sup_{q \in U(p)} \{\text{depth} R_q - \text{inf} \{\text{depth} M_q + 1, \text{depth} R_q\}\} = \sup\{\sup_{q \in U(p)} (\text{depth} R_q - \text{depth} M_q) - 1, 0\} = \sup\{\text{Rfd}_p M_p - 1, 0\} = r - 1.$$ 

Using the induction hypothesis, we find an element $s \in R \setminus p$ such that $sH^i_I(\Omega^1_R M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R) - r + 1$. We take an exact sequence $0 \to \Omega^1_I M \to P \to M \to 0$ with a finitely generated projective $R$-module $P$. This provides an exact sequence $H^1_I(P) \to H^1_I(M) \to H^1_I(\Omega^1_R M) \to H^1_I(P)$ for all ideals $I$ of $R$ and all integers $i$. Since $P$ is a direct summand of a free module, [2, Theorem 6.2.7] yields $H^1_I(P) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R)$. Thus, from the above exact sequence, we get an isomorphism $H^1_I(M) \cong H^1_I(\Omega^1_R M)$ for all $i < \text{grade}(I, R) - 1$. Hence $sH^1_I(M) \cong sH^1_I(\Omega^1_R M) = 0$ for all ideals $I$ of $R$ and all integers $i < \text{grade}(I, R) - r$, as $r \geq 1$. The proof of the corollary is completed. \hfill $\square$

Next, we introduce two invariants defined similarly to large/small restricted flat dimensions.

**Definition 4.2.** Let $M$ be a finitely generated $R$-module.

1. We set $\text{Rfd}_R^d M := \sup_{p \in \text{Spec } R} (\text{depth} p - \text{grade}(p, M))$. Note that $\text{Rfd}_R^d M \geq \text{Rfd}_R M$, and that $\text{Rfd}_R^d M_p = \sup_{q \in U(p)} (\text{depth} q - \text{grade}(q, M))$ for each $p \in \text{Spec } R$.
2. We set $\xi(p, M) := \sup_{q \in U(p)} (\text{grade}(q, R) - \text{grade}(q, M))$ for each $p \in \text{Spec } R$. Note that $\text{Rfd}_R^d M_p \geq \xi(p, M)$ for each $p \in \text{Spec } R$.

We can also define similar invariants

$$rfd_R^d M = \sup_{p \in \text{Spec } R} (\text{grade}(p, R) - \text{depth} M_p)$$

for a finitely generated $R$-module $M$, and

$$\xi^d(p, M) = \sup_{q \in U(p)} (\text{grade}(q, R) - \text{depth} M_q)$$

for a prime ideal $p$ of $R$ and a finitely generated $R$-module $M$. These turn out to coincide with $rfd_R M$ and $\xi(p, M)$, respectively.

**Proposition 4.3.** Let $M$ be a finitely generated $R$-module.

1. One has $rfd_R M = \sup_{p \in \text{Spec } R} (\text{grade}(p, R) - \text{depth} M_p)$.
2. One has $\xi(p, M) = \sup_{q \in U(p)} (\text{grade}(q, R) - \text{depth} M_q)$ for each $p \in \text{Spec } R$. 
Proof. (1) It is enough to show that \( \text{rfd}_R M \leq \sup_{p \in \text{Spec } R} \{ \text{grade}(p, R) - \text{depth } M_p \} \). We take \( p \in \text{Spec } R \) such that \( \text{rfd}_R M = \text{grade}(p, R) - \text{grade}(p, M) \). By [3, Proposition 1.2.10(a)], we find \( u \in V(p) \) such that \( \text{grade}(p, M) = \text{depth } M_u \). Since \( p \subseteq u \), we get \( \text{grade}(p, R) \leq \text{grade}(u, R) \). Hence we obtain \( \text{rfd}_R M = \text{grade}(p, R) - \text{grade}(p, M) \leq \text{grade}(u, R) - \text{depth } M_u \). Thus the assertion follows.

(2) Let \( p \in \text{Spec } R \). It is enough to show that \( \xi(p, M) \leq \sup_{q \in U(p)} \{ \text{grade}(q, R) - \text{depth } M_q \} \). We take \( q \in U(p) \) such that \( \xi(p, M) = \text{grade}(q, R) - \text{grade}(qR_p, M_p) \). By [3, Proposition 1.2.10(a)], we find \( u \in U(p) \) with \( q \subseteq u \) such that \( \text{grade}(qR_p, M_p) = \text{depth } (M_p)_{uR_p} = \text{depth } M_u \). Since \( q \subseteq u \), we get \( \text{grade}(q, R) \leq \text{grade}(u, R) \). Hence we obtain \( \xi(p, M) = \text{grade}(q, R) - \text{grade}(qR_p, M_p) \leq \text{grade}(u, R) - \text{depth } M_u \). Thus the assertion follows. \( \square \)

To state the theorem below, we introduce Cohen-Macaulay defect and Cohen-Macaulay dimension.

**Definition 4.4.** Let \( R \) be a local ring.

(1) The **Cohen-Macaulay defect** is defined by \( \text{cmd } R = \dim R - \text{depth } R \).

(2) We denote by \( \text{CM-dim}_R M \) the **Cohen-Macaulay dimension** of a finitely generated \( R \)-module \( M \). The ring \( R \) is Cohen-Macaulay if and only if \( \text{CM-dim}_R M < \infty \) for all finitely generated \( R \)-modules. For the definition and their basic properties, we refer the reader to [6, §3].

The following theorem is one of the main results of this paper, which is a detailed version of Theorem 1.4.

**Theorem 4.5.** Let \( M \) be a finitely generated \( R \)-module, and let \( p \) be a prime ideal of \( R \). Consider the following five conditions.

\[
(L)' \quad \text{There is an inequality } \text{Rfd}'_{R_p} M_p \leq 0.
(L) \quad \text{There is an inequality } \text{Rfd}_{R_p} M_p \leq 0.
(S) \quad \text{There is an inequality } \text{rfd}_{R_p} M_p \leq 0.
(G) \quad \text{There is an inequality } \xi(p, M) \leq 0.
(Y) \quad \text{The module } M \text{ belongs to } \mathcal{R}(p).
\]

Then the following statements hold.

(1) (a) Condition (S) is satisfied if and only if \( H^i_{IR_p} (M_p) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(IR_p, R) \).

(b) Condition (G) is satisfied if and only if \( H^i_{IR_p} (M_p) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \).

(2) The following implications always hold true.

\[
(L)' \quad \Longrightarrow \quad (L) \quad \Longrightarrow \quad (S) \quad \Longrightarrow \quad (G) \quad \Longrightarrow \quad (Y)
\]

(3) (a) If \( p \in \text{Min } R \), then condition \( (L)' \) is satisfied.

(b) If \( p \in \text{Ass } R \), then condition (S) is satisfied.

(c) If \( \text{grade}(p, R) = 0 \), then condition (G) is satisfied.

(4) Suppose that \( \text{CM-dim}_{R_p} M_p < \infty \). Then the equivalence \( (L) \Leftrightarrow (S) \) holds true. Hence the implication \( (S) \Rightarrow (Y) \) holds true.

(5) Suppose that \( \text{cmd } R_p \leq 1 \). Then the equivalences \( (L)' \Leftrightarrow (L) \Leftrightarrow (S) \) hold true. Hence the implication \( (S) \Rightarrow (Y) \) holds true.

(6) Suppose that \( \text{CM-dim}_{R_p} M_p < \infty \) and \( \text{grade}(p, R) = \text{depth } R_p \). Then the implication \( (G) \Rightarrow (L) \) holds true. Hence the equivalences \( (L) \Leftrightarrow (S) \Leftrightarrow (G) \Leftrightarrow (Y) \) hold true.
Proof. The latter assertions of statements (4)–(7) follow from the former ones and (2).

(1a) There are equivalences

\[
(S) \iff \text{grade}(IR_p, M_p) \geq \text{grade}(IR_p, R_p) \quad \text{for all ideals } I \text{ of } R
\]

\[
\Rightarrow H_{IR_p}^i(M_p) = 0 \quad \text{for all ideals } I \text{ of } R \quad \text{and all integers } i < \text{grade}(IR_p, R_p),
\]

where the first equivalence follows from Lemma 3.1 and the second follows from [2, Theorem 6.2.7].

(1b) Suppose that \( \text{grade}(qR_p, M_p) \geq \text{grade}(q, R) \) for all \( q \in U(p) \). Let \( I \) be an ideal of \( R \). If \( IR_p = R_p \), then \( \text{grade}(IR_p, M_p) = \infty \geq \text{grade}(I, R) \). Suppose that \( IR_p \) is a proper ideal of \( R_p \). By the proof of Lemma 3.1, we obtain \( q \in U(p) \) with \( q \geq I \) such that \( \text{grade}(qR_p, M_p) = \text{grade}(IR_p, M_p) \). Hence \( \text{grade}(IR_p, M_p) = \text{grade}(qR_p, M_p) \geq \text{grade}(q, R) \geq \text{grade}(I, R) \). Thus there are equivalences

\[
(G) \iff \text{grade}(IR_p, M_p) \geq \text{grade}(I, R) \quad \text{for all ideals } I \text{ of } R
\]

\[
\Rightarrow H_{IR_p}^i(M_p) = 0 \quad \text{for all ideals } I \text{ of } R \quad \text{and all integers } i < \text{grade}(I, R),
\]

where the last equivalence follows from [2, Theorem 6.2.7].

(2) Since \( \text{Rfd}_{IR_p}^i M_p \geq \text{Rfd}_{R_p}^i M_p \geq \text{Rfd}_{R_p}^i M_p \geq \xi(p, M), (L') \Rightarrow (L) \Rightarrow (S) \Rightarrow (G) \) hold true. It follows from Proposition 4.1 that the implication \( (L) \Rightarrow (Y) \) holds true. Finally, we show that \( (Y) \Rightarrow (G) \) holds true. Suppose that there exists an element \( s \in R \setminus p \) such that \( sH^i_{IR_p}(M_p) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \). Then [2, Corollary 4.3.3] yields \((s/1)H^i_{IR_p}(M_p) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \). Since \( s \in R \setminus p \), the element \( s/1 \) is a unit of \( R_p \). Hence we get \( H^i_{IR_p}(M_p) = 0 \) for all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \). Hence \( (G) \) follows from (1b). Therefore, \( (Y) \Rightarrow (G) \) holds true.

(3a) Suppose that \( p \in \text{Min } R \). Then we have \( U(p) = \{p\} \) and \( \text{depth } R_p = 0 \). Hence we obtain \( \text{Rfd}_{IR_p}^i M_p = \text{depth } R_p - \text{grade}(pR_p, M_p) \leq \text{depth } R_p = 0 \). Thus \( (L') \) holds.

(3b) Suppose that \( p \in \text{Ass } R \). Then we have \( \text{depth } R_p = 0 \). Hence we obtain \( \text{Rfd}_{IR_p}^i M_p \leq \sup_{q \in U(p)} \text{grade}(qR_p, M_p) = \text{depth } R_p = 0 \). Thus \( (S) \) holds.

(3c) Suppose that \( \text{grade}(p, R) = 0 \). Let \( I \) be an ideal of \( R \) and \( i \) be an integer with \( i < \text{grade}(I, R) \).

(3d) Suppose that \( \text{grade}(p, R) = 0 \). Let \( I \) be an ideal of \( R \) and \( i \) be an integer with \( i < \text{grade}(I, R) \).

(4) By [4, Theorem 2.8] and [6, Theorem 3.8], we have \( \text{Rfd}_{IR_p}^i M_p = \text{CM-dim } R_p M_p = \text{depth } R_p - \text{depth } M_p \). Then there are inequalities and equalities

\[
\text{CM-dim } R_p M_p = \text{Rfd}_{IR_p}^i M_p \\
\geq \text{Rfd}_{IR_p}^i M_p \\
\geq \text{depth } R_p - \text{depth } M_p \\
= \text{CM-dim } R_p M_p,
\]

where the second inequality follows from [4, Observation 2.12]. Hence we obtain the equalities

\[
\text{Rfd}_{IR_p}^i M_p = \text{Rfd}_{IR_p}^i M_p = \text{CM-dim } R_p M_p = \text{depth } R_p - \text{depth } M_p. \quad \text{Thus } (L') \Rightarrow (S) \text{ holds true.}
\]

(5) By [4, Lemma 3.1(ii)], we have \( \text{grade}(qR_p, R_p) = \text{depth } R_q \) for all \( q \in U(p) \). Then we obtain \( \text{Rfd}_{IR_p}^i M_p = \text{Rfd}_{IR_p}^i M_p = \text{Rfd}_{IR_p}^i M_p. \quad \text{Thus } (L') \Rightarrow (L) \Rightarrow (S) \text{ hold true.}
(6) If \( \xi(p, M) \leq 0 \), then depth \( R_p = \text{grade}(p, R) \leq \text{grade}(pR_p, M_p) = \text{depth} M_p \). Hence we have \( \text{Rfd}_{R_p} M_p = \text{CM-dim}_{R_p} M_p = \text{depth} R_p - \text{depth} M_p \leq 0 \), where the first equality follows from [4, Theorem 2.8] and the second follows from [6, Theorem 3.8]. Thus \((G) \Rightarrow (L)\) holds true.

(7) We have \( \text{Rfd}_{R_p} M_p = \xi(p, M) \). Thus \((S) \Leftrightarrow (G)\) holds true.

(8) We have \( \text{Rfd}_{R_p} M_p = \text{Rfd}_{R_p} M_p = \xi(p, M) \). Thus \((L) \Leftrightarrow (S) \Leftrightarrow (G) \Leftrightarrow (Y)\) hold true.

(9) Note that \( R = R_m = R_p \). We see from (7) that \((S) \Leftrightarrow (G)\) holds, and there are equivalences

\[
(Y) \Leftrightarrow H^i_I(M) = 0 \text{ for all ideals } I \text{ of } R \text{ and all integers } i < \text{grade}(I, R)
\]

\[
\Leftrightarrow H^i_{I_{R_m}}(M_m) = 0 \text{ for all ideals } I \text{ of } R \text{ and all integers } i < \text{grade}(I, R)
\]

\[
\Leftrightarrow (G),
\]

where the first equivalence holds since any element of \( R \setminus m \) is a unit of \( R \) and the third follows from (1b).

(10) If \( \text{ht } p \leq 1 \), then \( \text{cmd } R_p = \dim R_p - \text{depth } R_p \leq \text{ht } p \leq 1 \). Then \((S) \Rightarrow (Y)\) holds true by (5). Let \( \text{ht } p = 2 \). Since \( R \) is local and \( \dim R \leq 2 \), we have \( p = m \). It follows from (9) that \((S) \Rightarrow (Y)\) holds true. \(\square\)

Here, we give a remark on when the assumptions posed in the statements of the above theorem are satisfied.

**Remark 4.6.** Let \( M \) be a finitely generated \( R \)-module, and let \( p \) be a prime ideal of \( R \).

(1) Suppose that the local ring \( R_p \) is Cohen-Macaulay. Then \( \text{CM-dim}_{R_p} M_p < \infty \) and \( \text{grade}(q, R) = \text{grade}(qR_p, R_p) = \text{depth} R_q \) for all \( q \in \text{U}(p) \) by [6, Theorem 3.9] and [3, Theorem 2.1.3].

(2) Suppose that the \( R_p \)-module \( M_p \) has finite projective dimension or more generally finite Gorenstein dimension. Then \( \text{CM-dim}_{R_p} M_p < \infty \) by [6, Theorem 3.7].

(3) Let \( R = k[[X, Y]]/(X^2, XY) \), where \( k \) is a field, and let \( p = (X, Y) R \). Then \( R_p = R \) is not a Cohen-Macaulay ring. However, since \( \text{cmd } R = 1 \), we have \( \text{grade}(q, R) = \text{grade}(qR_p, R_p) = \text{depth} R_q \) for all \( q \in \text{U}(p) \) by [4, Lemma 3.1].

Finally, we investigate which implication does not necessarily hold.

**Proposition 4.7.** Let \( R \) be a local ring with depth \( R = 0 \). Then the following assertions hold.

(1) One has \( \mathcal{R}(p) = \text{mod } R \).

(2) If \( \dim R \geq 2 \), then there exists \( p \in \text{Spec } R \) such that \( \text{depth } R_p > 0 \).

(3) Suppose \( \dim R \geq 2 \), and let \( M = R/p \) for such a prime ideal \( p \) as in (2).

(a) One has \( \text{Rfd}_{R_p} M_p > 0 \). Hence the implication \((Y) \Rightarrow (S)\) does not hold for \( M \) and \( p \). Thus none of the implications \((Y) \Rightarrow (L'), (Y) \Rightarrow (L), (G) \Rightarrow (L)\), \((G) \Rightarrow (L)\), \((G) \Rightarrow (L)\) and \((G) \Rightarrow (S)\) holds for \( M \) and \( p \).

(b) One has \( \text{Rfd}_R M > 0 = \text{Rfd}_R M \). Hence the implication \((S) \Rightarrow (L)\) does not hold for \( M \) and \( m \). Thus none of the implications \((S) \Rightarrow (L), (G) \Rightarrow (L)\) and \((G) \Rightarrow (L)\) holds for \( M \) and \( m \).

(c) One has \( \text{Rfd}'_R R > 0 = \text{Rfd}_R R \). Hence the implication \((L) \Rightarrow (L)'\) does not hold for \( R \) and \( m \).

**Proof.** (1) Since depth \( R = 0 \), we have \( \text{grade}(I, R) = 0 \) for all proper ideals \( I \) of \( R \). Since the local cohomology functor \( H^i_I(\cdot) \) is the zero functor for all ideals \( I \) of \( R \) and all integers \( i < 0 \), we obtain \( H^i_I(M) = 0 \) for all finitely generated \( R \)-module \( M \), all ideals \( I \) of \( R \) and all integers \( i < \text{grade}(I, R) \). Thus \( M \) belongs to \( \mathcal{R}(p) \).

(2) By [4, Lemma 1.4], there exists \( p \in \text{Spec } R \) such that \( \text{depth } R_p = \dim R - 1 > 0 \).
(3a) We have $\text{rfd}_R M_p \geq \text{depth} R_p - \text{depth} M_p = \text{depth} R_p > 0$ by [4, Observation 2.12] and (2). Since $M$ belongs to $R(p)$ by (1), $(Y) \Rightarrow (S)$ does not hold for these $M$ and $p$. Hence the last assertion follows from Theorem 4.5(2).

(3b) We have $\text{rfd}_R M \geq \text{depth} R_p - \text{depth} M_p = \text{depth} R_p > 0$ by (2). Also, $\text{rfd}_R M \leq \text{depth} R = 0$ by [4, (2.12.1)]. Hence $(S) \Rightarrow (L)$ does not hold for these $M$ and $m$. Thus the last assertion follows from Theorem 4.5(2).

(3c) We have $\text{rfd}_R' R \geq \text{depth} R_p - \text{grade}(p, R) = \text{depth} R_p > 0 = \text{rfd}_R R$ by (2).

Remark 4.8. The table below forming an $6 \times 6$ matrix describes the relationships among those five conditions which we have discussed so far. Here, the symbol “○” (resp. “×”) in the $(i,j)$ entry means that the implication from the condition placed in the $(i,1)$ entry to the condition placed in the $(1,j)$ entry always holds (resp. does not always hold).

| (L)' | (L) | (S) | (G) | (Y) |
|------|-----|-----|-----|-----|
| ○    | ×   | ×   | ×   | ○   |
| ×    | ○   | ○   | ○   | ?   |
| ×    | ×   | ○   | ?   | •   |

Our main interest is whether the implication corresponding to the symbol “?” in the $(4,6)$ entry holds or not, which is the implication $(S) \Rightarrow (Y)$. Note that if we find a counterexample to this implication, then it is also a counterexample to the implication $(G) \Rightarrow (Y)$, which corresponds to the symbol “•” in the $(5,6)$ entry.

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