Rational solutions to the ABS list: Degenerating approach

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Abstract

In the paper we first construct rational solutions for the Nijhoff-Quispel-Capel (NQC) equation by means of bilinear method. These solutions can be transferred to those of $Q_3^\delta$ equation in the Adler-Bobenko-Suris (ABS) list. Then making use of degeneration relation we obtain rational solutions for $Q_2$, $Q_1^\delta$, $H_3^\delta$, $H_2$ and $H_1$. These rational solutions are in Casoratian form and the basic column vector satisfies an extended condition equation set.

Keywords: lattice KdV-type equations, ABS list, Casoratian, rational solutions
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1 Introduction

As one of interpretations of integrability of lattice equations, multidimensional consistency [1] has become increasingly popular in the recent years. With this property and two mild additional requirements on the equations: symmetry and the so-called ‘tetrahedron property’, Adler, Bobenko and Suris classified the integrable models defined on an elementary quadrilateral [2]. The corresponding result is named as ABS list, which consists of nine lattice equations: $Q_4$, $Q_3^\delta$, $Q_2$, $Q_1^\delta$, $A_2$, $A_1^\delta$, $H_3^\delta$, $H_2$, $H_1$. These equations are of form

\begin{align}
Q_4: & \quad p'(u\tilde{u} + \tilde{u}\tilde{u}) - q'(\tilde{u}\tilde{u} + u\tilde{u}) \\
& = \frac{p'q' - q'p'}{1 - p'^2q'^2} (\tilde{u}\tilde{u} + u\tilde{u}) - p'q'(1 + u\tilde{u}\tilde{u}), \quad (1.1a) \\
Q_3^\delta: & \quad p'(1 - q'^2)(u\tilde{u} + \tilde{u}\tilde{u}) - q'(1 - p'^2)(u\tilde{u} + \tilde{u}\tilde{u}) \\
& = (p'^2 - q'^2) (\tilde{u}\tilde{u} + u\tilde{u}) + \delta^2 \left(1 - p'^2\right) \left(1 - q'^2\right), \quad (1.1b) \\
Q_2: & \quad p'(u - \tilde{u})(\tilde{u} - \tilde{u}) - q'(u - \tilde{u})(\tilde{u} - \tilde{u}) + p'q'(p' - q')(u + \tilde{u} + \tilde{u} + \tilde{u}) \\
& = p'q'(p'^2 - q'^2) + q'^2(p' - q'), \quad (1.1c) \\
Q_1^\delta: & \quad p'(u - \tilde{u})(\tilde{u} - \tilde{u}) - q'(u - \tilde{u})(\tilde{u} - \tilde{u}) = \delta^2 p'q'(q' - p'), \quad (1.1d)
\end{align}

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Some of these equations have been known before, for example, $H_1$ is the lattice potential Korteweg-de-Vries (lpKdV) equation\cite{3}, $H_3$ is the lattice Schwarzian KdV (lSKdV) equation\cite{4}, $Q_4$ is known as the Adler’s equation\cite{22}. In equations (1.1), $\delta$ is a constant; $u_{n,m} := u(n,m)$ denotes the dependent variable of the lattice points labeled by $(n,m) \in \mathbb{Z}^2$; $p'$ and $q'$ are the continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables $n$ and $m$, respectively; notations with elementary lattice shifts are denoted by $\tilde{u} := u_{n+1,m}$, $\hat{u} := u_{n,m+1}$, $\tilde{\hat{u}} := u_{n+1,m+1}$.

Various of approaches have been shown to be effective in deriving soliton solutions for the ABS list (1.1) as evidenced by series of papers. Atkinson et al. constructed $N$-soliton solutions to $Q_3$ in terms of the $\tau$-function of the Hirota-Miwa equation. The corresponding solutions were expressed by the usual Hirota’s polynomial of exponentials\cite{8}. By developing Hirota’s direct method, Hietarinta and Zhang derived $N$-soliton solutions to $H$-series of equations and $Q_1$\cite{9}. This method is algorithmic and based on multidimensional consistency, progressing in each case from background solution to $1$-soliton solution to $N$-soliton solutions, where many Casoratian shift formulae were established. Meanwhile, Nijhoff and his collaborators proposed Cauchy matrix approach\cite{7} to catch the $N$-soliton solutions for the ABS list except for the elliptic case of $Q_4$. The authors of the present paper extended Cauchy matrix approach to a generalized case\cite{10}, which can be used to construct more kinds of exact solutions beyond soliton solutions for integrable systems (see also Ref.\cite{11}), such as, multiple-pole solutions. By setting initial value problem, Inverse Scattering Transform was also established to solve $H_1$\cite{12} and the ABS list\cite{13}. As the ‘master’ and the most complicate equation in this list, $Q_4$ was solved by using the Bäcklund transformation\cite{22}.

Different from soliton solution, rational solution is usually expressed by fraction of polynomials. Generally speaking, such type of solutions can be derived from soliton solutions through a special limit procedure (see Refs.\cite{15,16} as examples). Compared with the case in continuous integrable system, it is more difficult to get the rational solutions of lattice equations. In spite of this, until now much progress has been got. Algebraic solutions and lump-like solutions for the Hirota-Miwa equation were, respectively, given in Refs.\cite{17,18}. With the help of bilinear method\cite{9}, rational solutions for $H_3$ as well as lattice Boussinesq equation were shown in recent papers\cite{19,20}. Besides, by imposing reduction conditions on rational solutions for the Hirota-Miwa equation, rational solutions for lpKdV equation and two semi-discrete lpKdV equations were obtained\cite{21}. Recently, transformation approach was proposed to construct the rational solutions for the whole ABS list except for $Q_3$ and $Q_4$\cite{22}.

The present paper is devoted to investigating rational solutions for the ABS list with a different methodology, where $Q_4$ is excluded. The paper is organised as follows. In Sec.2, some necessary
materials are displayed as preliminary, including the re-parameterized ABS lattices and Casoratian. In Sec.3, we derive the rational solutions to lattice KdV-type equations involving lpmKdV equation, lpmKdV equation, NQC equation, together with two Miura transformations. In Sec.4, rational solutions for $Q_3$ are presented. Furthermore, degenerations will be considered to derive the rational solutions for “lower equations” $Q_2, Q_1, H_3, H_2$ and $H_1$. Sec.5 is for conclusions. In addition, an appendix is given as a complement to the paper.

2 Preliminary

Some new parameters are usually introduced such that the list (1.1) can be handled easily. For example, in Ref. [7] the ABS lattice equations (1.1) except for $Q_4$ were re-parameterized so that their solutions can be expressed through Cauchy matrices. These re-parametrisations are given by [7]

\[
\begin{align*}
Q_{3\delta} : & \quad p' = \frac{P}{P+\sqrt{a}} = \frac{p^2-b^2}{p}, \quad q' = \frac{Q}{Q+\sqrt{a}} = \frac{q^2-b^2}{q}, \\
Q_{2,1\delta} : & \quad p' = \frac{P}{P+\sqrt{a}}, \quad q' = \frac{q}{q+\sqrt{a}}, \\
H_{3\delta} : & \quad p' = \frac{P}{a^2-P^2} = \frac{1}{p}, \quad q' = \frac{Q}{a^2-Q^2} = \frac{1}{q}, \\
H_{2,1} : & \quad p' = -p^2, \quad q' = -q^2.
\end{align*}
\]

(2.1)

And the re-parameterized lattice equations are

\[
\begin{align*}
Q_{3\delta} : & \quad P(u\bar{u} + \bar{u}\bar{u}) - Q(u\bar{u} + \bar{u}\bar{u}) = (p^2 - q^2)((\bar{u}\bar{u} + u\bar{u}) + \frac{\delta^2}{4PQ}), \\
Q_{2} : & \quad (q^2 - a^2)(u - \bar{u})(\bar{u} - \bar{u}) - (p^2 - a^2)(u - \bar{u})(\bar{u} - \bar{u}) \\
& \quad + (p^2 - a^2)(q^2 - a^2)(q^2 - p^2)(u + \bar{u} + \bar{u} + \bar{u}) \\
& \quad = (p^2 - a^2)(q^2 - a^2)(q^2 - p^2)((p^2 - a^2)^2 + (q^2 - a^2)^2 - (p^2 - a^2)(q^2 - a^2)), \\
Q_{1\delta} : & \quad (q^2 - a^2)(u - \bar{u})(\bar{u} - \bar{u}) - (p^2 - a^2)(u - \bar{u})(\bar{u} - \bar{u}) = \frac{\delta^2a^4(p^2 - q^2)}{(p^2 - a^2)(q^2 - a^2)}, \\
H_{3\delta} : & \quad P(a^2 - p^2)(u\bar{u} + \bar{u}\bar{u}) - Q(a^2 - p^2)(u\bar{u} + \bar{u}\bar{u}) = \delta(p^2 - q^2), \\
H_{2} : & \quad (u - \bar{u})(\bar{u} - \bar{u}) + (p^2 - q^2)(u + \bar{u} + \bar{u} + \bar{u}) = p^4 - q^4, \\
H_{1} : & \quad (u - \bar{u})(\bar{u} - \bar{u}) = p^2 - q^2,
\end{align*}
\]

(2.2)

where in (2.2a) $(p, P) = p$ and $(q, Q) = q$ are the points on the elliptic curve

\[
\{(x, X)|X^2 = (x^2 - a^2)(x^2 - b^2)\},
\]

(2.3)

in (2.2d)

\[
P^2 = a^2 - p^2, \quad Q^2 = a^2 - q^2,
\]

(2.4)

and in $Q_{3\delta}$ and $Q_2$ the dependent variable $u$ has been scaled by

\[
u \to u(b^2 - a^2), \quad u \to \frac{a^4u}{(p^2 - a^2)^2(q^2 - a^2)^2},
\]

respectively. In the present paper, we focus on constructing rational solutions for the ABS list (2.2), where some Casoratian techniques developed in recent literatures [9, 23] will be adopted.
Casoratian can be viewed as the discrete version of Wronskian. In general, for a given basic column vector 
\[ \phi(\alpha, \beta, l) = (\phi_1(\alpha, \beta, l), \phi_2(\alpha, \beta, l), \ldots, \phi_N(\alpha, \beta, l))^T, \]  
with \( \{\phi_j(\alpha, \beta, l) = \phi_j(n, m, \alpha, \beta, l)\} \), the corresponding Casoratian can be written as 
\[ f = |\phi(\alpha, \beta, 0), \phi(\alpha, \beta, 1), \ldots, \phi(\alpha, \beta, N - 1)| \]
\[ = |0_{\alpha, \beta}, 1_{\alpha, \beta}, \ldots, N - 1_{\alpha, \beta}| \]
\[ = |(0, 1, \ldots, N - 1)_{\alpha, \beta}|. \]  
(2.5)

Here and hereafter, we employ the short-hand notations [24], such as 
\[ |(0, 1, \ldots, N - 1)_{\alpha, \beta}| = |(N-1)_{\alpha, \beta}|, \]  
(2.7a)
\[ |(0, 1, \ldots, N - 2, N)_{\alpha, \beta}| = |(N-2, N)_{\alpha, \beta}|. \]  
(2.7b)

In addition to the above notations, we need the following Laplace expansion identity for Casoratian verification.

**Lemma 1.** [24] Suppose that \( G \) is a \( N \times (N - 2) \) matrix, and \( a, b, c, d \) are \( N \)-th-order column vectors, then 
\[ |G, a, b| |G, c, d| - |G, a, c| |G, b, d| + |G, a, d| |G, b, c| = 0. \]  
(2.8)

### 3 Rational solutions for lattice KdV-type equations

In this section, we first review the usual Casoratian solutions for the lattice KdV-type equations, including lpKdV equation, lpmKdV equation, NQC equation and two Miura transformations. Then by introducing an extended condition equation set, we derive rational solutions for the lattice KdV-type equations. The results play key roles in the construction of the rational solutions for Q3δ.

#### 3.1 Casoratian solutions for lpKdV equation

The lpKdV equation reads

\[ (p - q + \tilde{w} - \tilde{w})(p + q + w - \tilde{w}) = p^2 - q^2, \]  
(3.1)

which is equivalent to equation (2.2f) by a change of dependent variable \( w = u + np + mq + u_0 \) (\( u_0 \) is a constant) and reduces to the pKdV equation after a double continuum limit. The lpKdV equation (3.1) admits bilinear form [9]

\[ \mathcal{H}_1 \equiv \tilde{g}\tilde{f} - \tilde{g}\tilde{f} + (p - q)(\tilde{f}\tilde{f} - f\tilde{f}) = 0, \]  
(3.2a)
\[ \mathcal{H}_2 \equiv \tilde{g}\tilde{f} - \tilde{g}\tilde{f} + (p + q)(f\tilde{f} - \tilde{f}\tilde{f}) = 0 \]  
(3.2b)

under dependent transformation

\[ w = \frac{g}{f}. \]  
(3.3)

Casoratian solutions to equation (3.2) can be summarized by the following result [25].
Proposition 1. The bilinear equation (3.2) possesses Casoratians
\[ f = |(N-1)_{0,0}|, \quad g = |(N-2, N)_{0,0}|, \quad \text{in which the column vector } \phi(0,0,l) \text{ satisfies} \]
\[ p\phi(\alpha, \beta, l) = \tilde{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \]  
\[ q\phi(\alpha, \beta, l) = \hat{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \]  
\[ p\psi(\alpha, \beta, l) = \tilde{\psi}(\alpha, \beta, l) - \psi(\alpha, \beta, l+1), \]  
\[ q\hat{\psi}(\alpha, \beta, l) = \psi(\alpha, \beta, l) + \hat{\psi}(\alpha, \beta, l+1), \]  
\[ \phi(\alpha, \beta, l) = A^m \psi(\alpha, \beta, l), \]
where \( \psi(\alpha, \beta, l) = (\psi_1(\alpha, \beta, l), \psi_2(\alpha, \beta, l), \ldots, \psi_N(\alpha, \beta, l))^T \) is an auxiliary vector and \( A^m \) is a \( N \times N \) matrix that only depends on \( m \) but is independent of \( n, \alpha, \beta \) and \( l \).

For the detailed proof, one can see Ref. [25].

3.2 Casoratian solutions for lpmKdV equation

The lpmKdV equation can be described as
\[ v_a((p-a)\tilde{v}_a - (q-a)\hat{v}_a) = \tilde{v}_a((p+a)\tilde{v}_a - (q+a)\hat{v}_a), \]  
where \( a \) is a non-zero constant. Under transformation
\[ v_a = \frac{h}{f}, \]
equation (3.6) is bilinearized into
\[ H_{11} \equiv (p-a)\tilde{f}h + (q+a)f\tilde{h} - (p+q)\tilde{f}\tilde{h} = 0, \]  
\[ H_{12} \equiv (p+a)f\tilde{h} + (q-a)\tilde{f}h - (p+q)\tilde{f}\tilde{h} = 0, \]

or
\[ H_{21} \equiv (p-a)\tilde{f}h - (q-a)f\tilde{h} - (p-q)\tilde{f}\tilde{h} = 0, \]  
\[ H_{22} \equiv (p+a)f\tilde{h} - (q+a)\tilde{f}h - (p-q)\tilde{f}\tilde{h} = 0. \]

The Casoratian solutions to these two bilinear equations are shown as follows.

Proposition 2. The bilinear equations (3.8) and (3.9) possess Casoratians
\[ f = |(N-1)_{0,0}|, \quad h = a^N |(N-1)_{-1,0}|, \quad \text{in which the column vectors } \phi(0,0,l) \text{ and } \phi(-1,0,l) \text{ satisfy system (3.5) together with} \]
\[ a\phi(\alpha, \beta, l) = \hat{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \]
In (3.15), for
\[ a\psi(\alpha, \beta, l) = \tilde{\psi}(\alpha, \beta, l) - \psi(\alpha, \beta, l + 1), \quad (3.11b) \]
\[ p\tilde{\varphi}(\alpha, \beta, l) = \varphi(\alpha, \beta, l) + \tilde{\varphi}(\alpha, \beta, l + 1), \quad (3.11c) \]
\[ q\varphi(\alpha, \beta, l) = \tilde{\varphi}(\alpha, \beta, l) - \varphi(\alpha, \beta, l + 1), \quad (3.11d) \]
\[ a\varphi(\alpha, \beta, l) = \tilde{\varphi}(\alpha, \beta, l) - \varphi(\alpha, \beta, l + 1), \quad (3.11e) \]
\[ \phi(\alpha, \beta, l) = B_{[n]}\varphi(\alpha, \beta, l), \quad (3.11f) \]

where \( \varphi(\alpha, \beta, l) = (\varphi_1(\alpha, \beta, l), \varphi_2(\alpha, \beta, l), \ldots, \varphi_N(\alpha, \beta, l))^T \) is an auxiliary vector and \( \hat{\circ} \) denotes shift with respect to \( \alpha \), i.e., \( \hat{\phi}(\alpha, \beta, l) = \phi(\alpha + 1, \beta, l) \); \( B_{[n]} \) is a \( N \times N \) matrix that only depends on \( n \) but is independent of \( m, \alpha \) and \( l \).

**Proof.** Because any three bilinear equations in (3.8) and (3.9) can lead to the remainder, here we just study equations \( H_{11}, H_{12} \) and \( H_{21} \).

We use shifted \( H_{11} \) in the following form
\[ H_{11} \equiv (p - a)\tilde{f}h + (q + a)\tilde{f}h - (p + q)\tilde{f}h = 0. \quad (3.12) \]

By virtue of equation (3.11a), it is easy to know that \( h \) in (3.10) can be rewritten as
\[ h = a^N[0_{-1,0}, (N - 2)_{0,0}], \quad (3.13) \]

In (3.12), for \( f, h, \tilde{f}, \tilde{h} \) and \( \tilde{f} \), we make use of (A.1a), (A.1c), (A.2a), (A.2c) and (A.2g), respectively, and get
\begin{align*}
(pq)^N & \left[ (p - a)\tilde{f}h + (q + a)\tilde{f}h - (p + q)\tilde{f}h \right] \\
& = a^N\frac{A_{[m]}}{A_{[n]}}[(0_{-1,0}, (N - 3)_{0,0}, \phi(0, 0, N - 2))|(N - 2)_{0,0}, A_{[m]}A^{-1}_{[n]}\phi(0, 0, N - 2)] \\
& - |(N - 2)_{0,0}, \phi(0, 0, N - 2))|0_{-1,0}, (N - 3)_{0,0}, A_{[m]}A^{-1}_{[n]}\phi(0, 0, N - 2)] \\
& - |(N - 3)_{0,0}, \phi(0, 0, N - 2), A_{[m]}A^{-1}_{[n]}\phi(0, 0, N - 2)|0_{-1,0}, (N - 2)_{0,0}), \quad (3.14) \\
\end{align*}

which vanishes in the light of Lemma (11) where \( G = (N - 3)_{0,0}, (a, b, c, d) = (0_{-1,0}, \phi(0, 0, N - 2), N - 2_{0,0}, A_{[m]}A^{-1}_{[n]}\phi(0, 0, N - 2)). \)

Similarly, for proving \( H_{12} \), we consider its shifted form
\[ H_{12} \equiv (p + a)\tilde{f}h + (q - a)\tilde{f}h - (p + q)\tilde{f}h = 0. \quad (3.15) \]

In (3.15), for \( f, h, \tilde{f}, \tilde{h} \) and \( \tilde{f} \), we use (A.1b), (A.1d), (A.2a), (A.2c) and (A.2d), respectively. Then we have
\begin{align*}
H_{12} & \equiv (p + a)\tilde{f}h + (q - a)\tilde{f}h - (p + q)\tilde{f}h \\
& = a^N(pq)^{N+2}\frac{B_{[n]}}{B_{[n]}}[(0_{-1,0}, (N - 3)_{0,0}, B_{[n]}B^{-1}_{[n]}\phi(0, 0, N - 2))|(N - 2)_{0,0}, \phi(0, 0, N - 2)] \\
& + |(N - 2)_{0,0}, B_{[n]}B^{-1}_{[n]}\phi(0, 0, N - 2))|0_{-1,0}, (N - 3)_{0,0}, \phi(0, 0, N - 2)] \\
& - |(N - 3)_{0,0}, \phi(0, 0, N - 2), B_{[n]}B^{-1}_{[n]}\phi(0, 0, N - 2)|0_{-1,0}, (N - 2)_{0,0}) \\
\end{align*}
by using Lemma 1, in which $G(\hat{f}, \hat{h})$. Thus we complete the proof.

Next, we adopt the down-tilde-hat version of $\mathcal{H}_{21}$, i.e.,

$$
\mathcal{H}_{21} \equiv (p-a)f\hat{h} - (q-a)f\hat{h} - (p-q)f\hat{h} = 0. 
$$

(3.17)

For (3.17), $f$, $f$, $h$, $h$, $f$ are provided by (A.1a), (A.1b), (A.1c), (A.1d) and (A.1m), respectively. Now we obtain

$$
\mathcal{H}_{21} \equiv (p-a)f\hat{h} - (q-a)f\hat{h} - (p-q)f\hat{h} = 0. 
$$

(3.18)

by using Lemma 1 in which $G = (\hat{N}-3)_{0,0}, (a, b, c, d) = (N-2)_{0,0}, (\hat{f}, \hat{h})$. Thus we complete the proof.

The lqKdV equation (3.11) is related to the lpmKdV equation (3.6) by

$$
p-q + \tilde{w} - \tilde{w} = \frac{1}{v_a}((p-a)v_a - (q-a)v_a) 
$$

(3.19a)

$$
= \frac{1}{v_a}((p+a)v_{a,b} - (q+a)v_a), 
$$

(3.19b)

$$
p + q + w - \tilde{w} = \frac{1}{v_a}((p-a)v_a + (q+a)v_a) 
$$

(3.19c)

$$
= \frac{1}{v_a}((p+a)v_a + (q-a)v_a), 
$$

(3.19d)

which serve as the Miura transformation. Substituting dependent transformations (3.3) and (3.7) into (3.19), one can easily find that (3.3), (3.8) and (3.9) compose the bilinear forms for the system (3.19). Therefore, system (3.19) have Casoratian solutions (3.3) and (3.7), in which $f, g$ and $h$ are given by (3.11) and (3.10), where the basic column vector $\phi(\alpha, \beta, l)$ satisfies systems (3.5) and (3.11).

### 3.3 Casoratian solutions for NQC equation

The NQC equation was firstly introduced in Ref. 3 by direct linearization method. This equation has the form

$$
\frac{1 + (p-a)S(a, b) - (p+b)\tilde{S}(a, b)}{1 + (q-a)S(a, b) - (q+b)\tilde{S}(a, b)} = \frac{1 - (q+a)\tilde{S}(a, b) + (q-b)\tilde{S}(a, b)}{1 - (p+a)\tilde{S}(a, b) + (p-b)\tilde{S}(a, b)} 
$$

(3.20)
where $a$ and $b$ are non-zero constants. Through different parameter choices, (3.20) can yield lpmKdV equation (3.1) and lpmKdV equation (3.6). In Ref. [3], it was revealed that there is a Miura transformation between lpmKdV equation (3.6) and NQC equation (3.20), which is given by equation (3.1) and lpmKdV equation (3.6). In Ref. [3], it was revealed that there is a Miura transformation between lpmKdV equation (3.6) and NQC equation (3.20), which is given by

\begin{align}
1 + (p - a)S(a, b) - (p + b)\bar{S}(a, b) &= \overline{v}_av_b, \\
1 + (q - a)S(a, b) - (q + b)\bar{S}(a, b) &= \overline{v}_av_b, \\
1 + (p - b)S(a, b) - (p + a)\bar{S}(a, b) &= v_av_b, \\
1 + (q - b)S(a, b) - (q + a)\bar{S}(a, b) &= v_av_b,
\end{align}

(3.21)

where $v_a$ satisfies the lpmKdV equation (3.6). Equation (3.20) can be derived from (3.21) in the light of equality $\frac{\overline{v}_av_b}{v_av_b} = \frac{(v_av_b)^{-1}}{(v_av_b)}$. Now rather than discussing equation (3.20), we turn to consider system (3.21). Through transformations

\[ v_a = \frac{h}{f}, \quad v_b = \frac{s}{f}, \quad S(a, b) = \frac{\theta}{f}, \]

(3.22)

system (3.21) can be bilinearized into

\begin{align}
\mathcal{H}_{31} &\equiv f\hat{f} + (p - a)\hat{\theta}\hat{f} - (p + b)\hat{\theta}f - \hat{h}s = 0, \\
\mathcal{H}_{32} &\equiv f\hat{f} + (q - a)\hat{\theta}\hat{f} - (q + b)\hat{\theta}f - \hat{h}s = 0, \\
\mathcal{H}_{33} &\equiv f\hat{f} + (p - b)\hat{\theta}\hat{f} - (p + a)\hat{\theta}f - h\hat{s} = 0, \\
\mathcal{H}_{34} &\equiv f\hat{f} + (q - b)\hat{\theta}\hat{f} - (q + a)\hat{\theta}f - h\hat{s} = 0.
\end{align}

(3.23)

For the Casoratian solutions to (3.22), one has

**Proposition 3.** Bilinear system (3.23) admits the solutions

\[ f = |(\overline{N} - 1)_{0,0}|, \quad h = a^N|(\overline{N} - 1)_{-1,0}|, \quad s = b^N|(\overline{N} - 1)_{0,-1}|, \quad \theta = -\frac{1}{a + b}((ab)^N|(\overline{N} - 1)_{-1,-1}| - |(\overline{N} - 1)_{0,0}|), \]

(3.24)

in which the basic column vector $\phi(\alpha, \beta, l)$ satisfies (3.5a), (3.5b), (3.11a) and and

\begin{align}
b\phi(\alpha, \beta, l) &= \dot{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l + 1), \\
p\chi(\alpha, \beta, l) &= \dot{\chi}(\alpha, \beta, l) - \chi(\alpha, \beta, l + 1), \\
q\chi(\alpha, \beta, l) &= \dot{\chi}(\alpha, \beta, l) - \chi(\alpha, \beta, l + 1), \\
a\chi(\alpha, \beta, l) &= \dot{\chi}(\alpha, \beta, l) - \chi(\alpha, \beta, l + 1), \\
b\dot{\chi}(\alpha, \beta, l) &= \chi(\alpha, \beta, l) + \dot{\chi}(\alpha, \beta, l + 1), \\
\phi(\alpha, \beta, l) &= C_{[\beta]}\chi(\alpha, \beta, l), \\
p\varpi(\alpha, \beta, l) &= \ddot{\varpi}(\alpha, \beta, l) - \varpi(\alpha, \beta, l + 1), \\
q\varpi(\alpha, \beta, l) &= \ddot{\varpi}(\alpha, \beta, l) - \varpi(\alpha, \beta, l + 1), \\
a\ddot{\varpi}(\alpha, \beta, l) &= \ddot{\varpi}(\alpha, \beta, l) + \ddot{\varpi}(\alpha, \beta, l + 1), \\
b\ddot{\varpi}(\alpha, \beta, l) &= \ddot{\varpi}(\alpha, \beta, l) - \varpi(\alpha, \beta, l + 1).
\end{align}

(3.25)
\[ \phi(\alpha, \beta, l) = D_{[\alpha]} \varpi(\alpha, \beta, l), \]  

where \( \chi(\alpha, \beta, l) = (\chi_1(\alpha, \beta, l), \chi_2(\alpha, \beta, l), \ldots, \chi_N(\alpha, \beta, l))^T \) and \( \varpi(\alpha, \beta, l) = (\varpi_1(\alpha, \beta, l), \varpi_2(\alpha, \beta, l), \ldots, \varpi_N(\alpha, \beta, l))^T \) are two auxiliary vectors and \( \cdot \) denotes shift with respect to \( \beta \), i.e., \( \hat{\phi}(\alpha, \beta, l) = \phi(\alpha, \beta + 1, l) \); \( C_{[\beta]} \) is a \( N \times N \) matrix that only depends on \( \beta \) but is independent of \( n, m, \alpha \) and \( l \); \( D_{[\alpha]} \) is a \( N \times N \) matrix only depends on \( \alpha \) but is independent of \( n, m, \beta \) and \( l \). In \( [3.24] \) we add factor \((ab)^N\) in \( \theta \) to formally avoid the constraint \( b \neq -a \). One can see the explicit expressions for the two simplest rational solutions listed in Sec.\( 3.4 \).

**Proof.** Noting that the structure of \( \theta \), we rewrite \( (3.23) \) to

\[ \mathcal{H}_{41} \equiv (p - a) \hat{\theta} - (p + b) \hat{\theta} f + (a + b) \hat{h} s = 0, \]  
\[ \mathcal{H}_{42} \equiv (q - a) \hat{\theta} - (q + b) \hat{\theta} f + (a + b) \hat{h} s = 0, \]  
\[ \mathcal{H}_{43} \equiv (p - b) \hat{\theta} f - (p + a) \hat{\theta} f + (a + b) h \hat{s} = 0, \]  
\[ \mathcal{H}_{44} \equiv (q - b) \hat{\theta} f - (q + a) \hat{\theta} f + (a + b) h \hat{s} = 0, \]

where

\[ \theta = (ab)^N |(\tilde{N} - 1)_{-1,-1}|. \]

For proving \( (3.26a) \), we make use of the replacements \( f = \frac{1}{bN} \hat{\theta} \) and \( h = \frac{a^N}{bN} b \) and consider equation

\[ \mathcal{H}_{41} \equiv (p - a) \hat{\theta} - (p + b) \hat{\theta} s + a^N (a + b) \hat{s} s = 0, \]  

where \( \hat{\theta} = (ab)^N |0_{-1,-1}, (\tilde{N} - 2)_{0,-1}|. \) In \( (3.27) \), for \( \hat{\theta}, \hat{s}, \hat{\theta}, \hat{s} \) and \( s \), we use \( A.2c, A.1g, A.2i, A.2k \) and \( A.1e \), respectively. Then we have

\[ \mathcal{H}_{41} \equiv (p - a) \hat{\theta} - (p + b) \hat{\theta} s + a^N (a + b) \hat{s} s \]

\[ = (ab^2)^N (pb)^{-N+2} \left[ C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2) \right] \]

\[ - |(\tilde{N} - 2)_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2)| |(\tilde{N} - 3)_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2)\]

\[ - |(\tilde{N} - 2)_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2)| |(\tilde{N} - 2)_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2)\]

\[ = 0, \]

where Lemma \( 11 \) was considered, in which \( G = (\tilde{N} - 3)_{0,-1}, (a, b, c, d) = (0_{-1,-1}, \phi(0, -1, N - 2), N - 2_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0, -1, N - 2)) \). Similarly, one can prove bilinear equations \( (3.26b)-(3.26d) \). Thus we finish the verification.

\[ \square \]

Let us conclude Propositions 1-3 as follows.

**Proposition 4. The Casoratians**

\[ f = |(\tilde{N} - 1)_{0,0}|, \quad g = |(\tilde{N} - 2, N)_{0,0}|, \quad h = a^N |(\tilde{N} - 1)_{-1,0}|, \quad s = b^N |(\tilde{N} - 1)_{0,-1}|, \]
\[ \theta = -\frac{1}{a+b}((ab)^N|\lambda - 1| - |\lambda - 1|) \tag{3.29} \]

solve the bilinear equations (3.32), (3.33) and (3.34), where the basic column vector \( \dot{\phi}(\alpha, \beta, l) \) satisfies the condition equation set (3.34), (3.35), (3.36) together with

\[ b\psi(\alpha, \beta, l) = \psi(\alpha, \beta, l) - \psi(\alpha, \beta, l+1), \tag{3.30a} \]
\[ b\varphi(\alpha, \beta, l) = \varphi(\alpha, \beta, l) - \varphi(\alpha, \beta, l+1). \tag{3.30b} \]

The additional equations (3.30) implies that \( v_b \) given by (3.22) also satisfies the lpmKdV equation (3.6) with \( a \rightarrow b \). According to the different forms of \( A_{[m]} \), \( B_{[n]} \), \( C_{[\beta]} \) and \( D_{[\alpha]} \), at least two types of solutions can be obtained (cf. Ref. [25]). For example, when \( A_{[m]} \), \( B_{[n]} \), \( C_{[\beta]} \) and \( D_{[\alpha]} \) are, respectively, diagonal matrices defined as

\[ A_{[m]} = \text{Diag}((q^2 - k_j^2)^m)_{N \times N}, \quad B_{[n]} = \text{Diag}((p^2 - k_j^2)^n)_{N \times N}, \tag{3.31a} \]
\[ C_{[\beta]} = \text{Diag}((b^2 - k_j^2)^\beta)_{N \times N}, \quad D_{[\alpha]} = \text{Diag}((a^2 - k_j^2)^\alpha)_{N \times N}. \tag{3.31b} \]

Then (3.29) together with

\[ \phi_j(\alpha, \beta, l) = \rho_j^{(0)+}(p+k_j)^n(q+k_j)^m(a+k_j)^\alpha(b+k_j)^\beta k_j^l \]
\[ \quad + \rho_j^{(0)-}(p-k_j)^n(q-k_j)^m(a-k_j)^\alpha(b-k_j)^\beta(-k_j)^l, \tag{3.32a} \]
\[ \psi_j(\alpha, \beta, l) = (q^2 - k_j^2)^m \phi_j(\alpha, \beta, l), \quad \varphi_j(\alpha, \beta, l) = (p^2 - k_j^2)^n \phi_j(\alpha, \beta, l), \tag{3.32b} \]
\[ \chi_j(\alpha, \beta, l) = (b^2 - k_j^2)^\beta \phi_j(\alpha, \beta, l), \quad \omega_j(\alpha, \beta, l) = (a^2 - k_j^2)^\alpha \phi_j(\alpha, \beta, l) \tag{3.32c} \]

for \( j = 1, 2, \ldots, N \) provides the usual multi-soliton solutions for the lattice KdV-type equations, where \( \{ \rho_j^{(0)+} \} \) are constants.

### 3.4 Rational solutions for lattice KdV-type equations

For H3_δ and Q1_δ in the ABS list, the existence of \( \delta \neq 0 \) plays a crucial role in the procedure of obtaining rational solutions from their soliton solutions [19]. While for H1 and H2, not involving \( \delta \), it does not work to derive their rational solution from the soliton solutions (3.29) with (3.32) by taking limit. To avoid this shortcoming, we consider the following system

\[ (p-c)\phi(\alpha, \beta, l) = \tilde{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \tag{3.33a} \]
\[ (q-c)\phi(\alpha, \beta, l) = \tilde{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \tag{3.33b} \]
\[ (a-c)\phi(\alpha, \beta, l) = \tilde{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \tag{3.33c} \]
\[ (b-c)\phi(\alpha, \beta, l) = \tilde{\phi}(\alpha, \beta, l) - \phi(\alpha, \beta, l+1), \tag{3.33d} \]
\[ (p-c)\psi(\alpha, \beta, l) = \tilde{\psi}(\alpha, \beta, l) - \psi(\alpha, \beta, l+1), \tag{3.33e} \]
\[ (q+c)\tilde{\psi}(\alpha, \beta, l) = \psi(\alpha, \beta, l) + \tilde{\psi}(\alpha, \beta, l+1), \tag{3.33f} \]
\[ (a-c)\psi(\alpha, \beta, l) = \tilde{\psi}(\alpha, \beta, l) - \psi(\alpha, \beta, l+1), \tag{3.33g} \]
\[ (b+c)\tilde{\psi}(\alpha, \beta, l) = \psi(\alpha, \beta, l) + \tilde{\psi}(\alpha, \beta, l+1), \tag{3.33h} \]
\[ \phi(\alpha, \beta, l) = A_{[m]} \psi(\alpha, \beta, l), \tag{3.33i} \]
\[ (p+c)\tilde{\varphi}(\alpha, \beta, l) = \varphi(\alpha, \beta, l) + \tilde{\varphi}(\alpha, \beta, l+1), \tag{3.33j} \]
satisfies the extended condition equation set bilinear equations (3.2), (3.8), (3.9) and (3.23), where the basic column vector $f$. For (3.31), solutions to (3.33) can be described as

\[ \phi_j(\alpha, \beta, l) = \rho_j^{(0)}(p + k_j)^n(q + k_j)^m(a + k_j)^{\alpha}b - k_j)^{\beta}(c - k_j)^l, \]

\[ \psi_j(\alpha, \beta, l) = (q^2 - k_j^2)^{-m}\phi_j(\alpha, \beta, l), \quad \varphi_j(\alpha, \beta, l) = (p^2 - k_j^2)^{-n}\phi_j(\alpha, \beta, l), \]

\[ \chi_j(\alpha, \beta, l) = (b^2 - k_j^2)^{-\beta}\phi_j(\alpha, \beta, l), \quad \varpi_j(\alpha, \beta, l) = (a^2 - k_j^2)^{-\alpha}\phi_j(\alpha, \beta, l). \]

with \( j = 1, 2, \ldots, N \).

From the previous subsections, one knows that \( f_0 = |(N - 1)_{0, 0}|, g_0 = |(N - 2)_{0, 0}|, h_0 = a^N|(N - 1)_{-1, 0}|, s_0 = b^N|(N - 1)_{0, -1}| \) and \( \theta_0 = -\frac{1}{a + b}((ab)^N|(N - 1)_{-1, -1}| - |(N - 1)_{0, 0}|) \) solve the bilinear equations (3.2), (3.3), (3.9) and (3.23), where the basic column vector $\phi(\alpha, \beta, l)$ satisfies system (3.33) with $c = 0$. In the case of (3.31), by direct calculation one knows that (cf. Ref. [26])

\[ f_c = f_0, \quad g_c = g_0 + N\gamma f_0, \quad h_c = h_0, \quad \theta_c = \theta_0, \]

where in $f_c, g_c, h_c, s_c$ and $\theta_c$, the basic column vector $\phi(\alpha, \beta, l)$ satisfies the extended condition equation set (3.33). Substituting (3.35) into bilinear forms (3.2), (3.3), (3.9) and (3.23), one can easily find that $f_c, g_c, h_c, s_c$ and $\theta_c$ also satisfy these bilinear equations. Therefore, (3.23) together with (3.34) also yields multi-soliton solutions for the lattice KdV-type equations. Furthermore, we can get the following result.

**Proposition 5.** The Casoratians

\[ f = |(N - 1)_{0, 0}|, \quad g = |(N - 2)_{0, 0}|, \quad h = a^N|(N - 1)_{-1, 0}|, \quad s = b^N|(N - 1)_{0, -1}| \]

\[ \theta = -\frac{1}{a + b}((ab)^N|(N - 1)_{-1, -1}| - |(N - 1)_{0, 0}|) \]

solve the bilinear equations (3.2), (3.3), (3.9) and (3.23), where the basic column vector $\phi(\alpha, \beta, l)$ satisfies the extended condition equation set (3.33).
Analogous to the earlier proofs of Propositions 1-3, we can deduce the verification of Proposition 5. Constant $c$ in (3.33) guarantees that one can get rational solutions for the lattice KdV-type equations. In fact, $A_{[m]}$, $B_{[n]}$, $C_{[\beta]}$ and $D_{[\alpha]}$ are taken, respectively, as lower triangular Toeplitz matrices (For more properties of this type matrices, one can refer to Ref. [16])

\[ A_{[m]} = (\gamma_{s,j}(q,m))_{N \times N}, \quad \gamma_{s,j}(q,m) = \left\{ \begin{array}{ll} \frac{1}{(2s-2)!} \partial_k^{2(s-j)}(q^2 - k^2)^m |_{k=0}, & s \geq j, \\ 0, & s < j, \end{array} \right. \quad (3.37a) \]

\[ B_{[n]} = (\gamma_{s,j}(p,n))_{N \times N}, \quad \gamma_{s,j}(p,n) = \left\{ \begin{array}{ll} \frac{1}{(2s-2)!} \partial_k^{2(s-j)}(p^2 - k^2)^n |_{k=0}, & s \geq j, \\ 0, & s < j, \end{array} \right. \quad (3.37b) \]

\[ C_{[\beta]} = (\gamma_{s,j}(b,\beta))_{N \times N}, \quad \gamma_{s,j}(b,\beta) = \left\{ \begin{array}{ll} \frac{1}{(2s-2)!} \partial_k^{2(s-j)}(b^2 - k^2)^\beta |_{k=0}, & s \geq j, \\ 0, & s < j, \end{array} \right. \quad (3.37c) \]

\[ D_{[\alpha]} = (\gamma_{s,j}(a,\alpha))_{N \times N}, \quad \gamma_{s,j}(a,\alpha) = \left\{ \begin{array}{ll} \frac{1}{(2s-2)!} \partial_k^{2(s-j)}(a^2 - k^2)^\alpha |_{k=0}, & s \geq j, \\ 0, & s < j. \end{array} \right. \quad (3.37d) \]

The generic basic Casoratian column vector $\phi(\alpha, \beta, l)$ for (3.33) can then be taken as

\[ \phi(\alpha, \beta, l) = A_+ \phi^+(\alpha, \beta, l) + A_- \phi^-(\alpha, \beta, l) \quad (3.38a) \]

with

\[ \phi^\pm(\alpha, \beta, l) = (\phi_0^\pm(\alpha, \beta, l), \phi_1^\pm(\alpha, \beta, l), \ldots, \phi_{N-1}^\pm(\alpha, \beta, l))^T, \quad (3.38b) \]

\[ \phi_s^\pm(\alpha, \beta, l) = \frac{1}{(2s)!} \partial_k^{2s}[(p \pm k)^n(q \pm k)^m(a \pm k)^\alpha(b \pm k)^\beta(c \pm k)^l+\frac{1}{2}]|_{k=0}, \quad (3.38c) \]

where $A_\pm$ are two arbitrary non-singular lower triangular Toeplitz matrices. In (3.35c) we added $\frac{1}{2}$ in the factor $(c \pm k)^l+\frac{1}{2}$ to avoid zero derivative.

It is easy to understand that $S(a,b)$ given by (3.22) with (3.36) and (3.38) satisfies symmetric property $S(a,b) = S(b,a)$. When $N = 2$, solutions $w$, $v_a$ and $S(a,b)$ are, respectively, given by

\[ w = \frac{2cpq}{pq + 2c(mp + nq)} + 2c, \quad (3.39a) \]

\[ v_a = \frac{-2cpq}{a(pq + 2c(mp + nq))} + 1, \quad (3.39b) \]

\[ S(a,b) = \frac{2cpq}{ab(pq + 2c(mp + nq))}, \quad (3.39c) \]

and when $N = 3$, solutions $w$, $v_a$ and $S(a,b)$ read, respectively,

\[ w = \frac{6cpq(pq + 2c(mp + nq))^2}{( -3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2) 
+ 8c^3(3nmpq(mp + nq) + (n^3 - n)q^3 + (m^3 - m)p^3)) + 3c), \quad (3.40a) \]

\[ v_a = \frac{6cpq}{a^2}(pq + 2c(mp + nq))\left(2cpq - a(pq + 2c(mp + nq))\right) 
/ ( -3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2) \],

\[ S(a,b) = \frac{6cpq}{a^2}(pq + 2c(mp + nq))\left(2cpq - a(pq + 2c(mp + nq))\right) 
/ ( -3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2) \]
\[ S(a, b) = \frac{6cpq}{a^2b^2} \left( -2cpq + a(pq + 2c(mp + nq)) \right) \left( -2cpq + b(pq + 2c(mp + nq)) \right) \]
\[
/ \left( -3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2) + 8c^3(3nmpq(mp + nq) + (n^3 - n)q^3 + (m^3 - m)p^3)) \right).
\]

It is noteworthy that (3.39c) and (3.40c) also hold for \( b = -a \).

4 Rational solutions for the ABS list \( Q_3^\delta \)

4.1 Rational solutions for the \( Q_3^\delta \)

In Ref. [7], soliton solutions for \( Q_3^\delta \) were written as a linear combination of four terms each of which contains as an essential ingredient the soliton solution of NQC equation with different values of the branch point parameters which enter in that equation. In the following, we still adopt this result to present the rational solution for the \( Q_3^\delta \).

**Theorem 1.** The rational solution of \( Q_3^\delta \) is formulated by

\[
\begin{align*}
\frac{u}{A} &= F(a, b) \left[ 1 - (a + b)S(a, b) \right] + B \left[ 1 - (a - b)S(a, -b) \right] \\
&\quad + C \left[ 1 + (a + b)S(-a, -b) \right] + D \left[ 1 + (a + b)S(-a, -b) \right],
\end{align*}
\]

in which \( S(\pm a, \pm b) \) are the rational solutions of the NQC equation with parameters \( \pm a, \pm b \); the function \( F(a, b) \) is defined as

\[
F(a, b) = \left( \frac{P}{(p-a)(p-b)} \right)^n \left( \frac{Q}{(q-a)(q-b)} \right)^m,
\]

and \( P, Q \) are defined by (2.3); \( A, B, C \) and \( D \) are constants subject to the single constraint

\[
AD(a + b)^2 - BC(a - b)^2 = -\delta^2/16ab.
\]

The proof of soliton solutions to the \( Q_3^\delta \) presented in Ref. [7] is based on \( lpKdV \) equation (3.1), Miura transformations (3.19) and (3.21), as well as symmetric property \( S(a, b) = S(b, a) \). Since rational solutions for these equations have been shown in previous section, naturally we achieve the verification of Theorem 1. We omit it here.

4.2 Degeneration

We now consider the problem of degeneration of rational solutions into the remaining “lower” equations \( Q_2, Q_1^\delta, H_3^\delta, H_2 \) and \( H_1 \) in the ABS list (2.2). To do so we follow the degenerations given in Ref. [7] which are limits on the parameters \( a \) and \( b \) and the dependent variable \( u \), where a small parameter \( \epsilon \) is introduced, and all degenerations are obtained in the limit \( \epsilon \to 0 \). The degeneration relations between \( Q_3^\delta \) and “lower equations” \( Q_2, Q_1^\delta, H_3^\delta, H_2 \) and \( H_1 \) can be depicted by Fig.1.

![Degeneration relation](Image)
4.2.1 Q₃ᵢ → Q₂

The degeneration from Q₃ᵢ to Q₂ is

\[ b = a(1 - 2\epsilon), \quad u \rightarrow \frac{\delta}{4a^2} \left( \frac{1}{\epsilon} + 1 + (1 + 2u)\epsilon \right). \]  \hspace{1cm} (4.4)

Making the following replacements of constants in (4.1)

\[ A \rightarrow \frac{\delta}{4a^2}A\epsilon, \quad B \rightarrow \frac{\delta}{8a^2}\left( \frac{1}{\epsilon} + 1 - \xi_0 + ((3 + \xi_0^2)/2 + 2AD)\epsilon \right), \]
\[ C \rightarrow \frac{\delta}{8a^2}\left( \frac{1}{\epsilon} + 1 + \xi_0 + ((3 + \xi_0^2)/2 + 2AD)\epsilon \right), \quad D \rightarrow \frac{\delta}{4a^2}D\epsilon, \]  \hspace{1cm} (4.5)

we find the rational solution for Q₂:

\[ u = \frac{1}{4}((\xi + \xi_0)^2 + 1) + a(\xi + \xi_0)S(-a, a) + a^2(Z(a, -a) + Z(-a, a)) + \]
\[ AD + \frac{1}{2}A\rho(a)(1 - 2aS(a, a)) + \frac{1}{2}D\rho(-a)(1 + 2aS(-a, -a)), \]  \hspace{1cm} (4.6)

in which

\[ \xi = 2a\left( \frac{p}{a^2 - p^2}n + \frac{q}{a^2 - q^2}m \right), \quad \rho(a) = \left( \frac{p+a}{p-a} \right)^n \left( \frac{q+a}{q-a} \right)^m, \]
\[ Z(a, -a) = \left. -\frac{1}{2a}\partial_\epsilon S(a, 2ae - a) \right|_{\epsilon=0}, \quad Z(-a, a) = \left. \frac{1}{2a}\partial_\epsilon S(-a, a - 2ae) \right|_{\epsilon=0}, \]  \hspace{1cm} (4.7)

and \( \xi_0, A \) and \( D \) are the constants which may be chosen arbitrarily.

4.2.2 Q₂ → Q₁ᵢ

To achieve the rational solution for Q₁ᵢ we degenerate from (4.6) by taking

\[ u \rightarrow \frac{\delta^2}{4\epsilon^2} + \frac{1}{\epsilon}u. \]

Meanwhile, we replace the constants appearing in solution (4.6) by

\[ A \rightarrow \frac{2A}{\epsilon}, \quad D \rightarrow \frac{2D}{\epsilon}, \quad \xi_0 \rightarrow \xi_0 + \frac{2B}{\epsilon}. \]  \hspace{1cm} (4.8)

Then the rational solutions for Q₁ᵢ can be described as

\[ u = A\rho(a)(1 - 2aS(a, a)) + B(\xi + \xi_0 + 2aS(-a, a)) + D\rho(-a)(1 + 2aS(-a, -a)), \]  \hspace{1cm} (4.9)

where constants \( A, B, D \) and \( \xi_0 \) are chosen to satisfy the single constraint

\[ AD + \frac{1}{4}B^2 = \frac{\delta^2}{16}. \]  \hspace{1cm} (4.10)
4.2.3 Q3δ → H3δ

By setting

\[ b = \frac{1}{\epsilon^2}, \quad u \rightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} u, \]  

(4.11)

and

\[ A \rightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} A, \quad B \rightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} B, \quad C \rightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} C, \quad D \rightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} D, \]  

(4.12)

rational solution to H3δ can be degenerated from (4.1), which is of form

\[ u = (A + (-1)^{n+m} B) gu_a + ((-1)^{n+m} C + D) \rho^{-1} v_{-a}, \]  

(4.13)
in which \( v_a \) is defined by (3.7) and

\[ \rho = \left( \frac{P}{a-p} \right)^n \left( \frac{Q}{a-q} \right)^m, \]  

(4.14)

where parameters \( P \) and \( Q \) are related to \( p \) and \( q \) by (2.4) and the constants \( A, B, C \) and \( D \) are subject to the constraint

\[ AD - BC = \delta \frac{1}{4a}. \]

4.2.4 Q2 → H2

The degeneration from Q2 to H2 can be arrived at by setting

\[ a = \frac{1}{\epsilon}, \quad u \rightarrow \frac{1}{4} + \epsilon^2 u. \]  

(4.15)

Substituting (4.15) into (4.6) combined with

\[ aS(-a, a) \rightarrow -\epsilon S^{(0)} + O(\epsilon^2), \]

\[ aS(a, a) \rightarrow \epsilon S^{(0)} - 2\epsilon^2 S^{(1)} + O(\epsilon^3), \]

\[ aS(-a, -a) \rightarrow \epsilon S^{(0)} + 2\epsilon^2 S^{(1)} + O(\epsilon^3), \]

\[ a^2(Z(-a, a) + Z(a, -a)) \rightarrow 2\epsilon^2 S^{(1)} + O(\epsilon^3), \]  

(4.16)

and the following choice for the constants

\[ A \rightarrow A(\epsilon + \zeta_1 \epsilon^2 /2), \quad D \rightarrow A(-\epsilon + \zeta_1 \epsilon^2 /2), \quad \xi_0 \rightarrow \epsilon \xi_0 \]  

(4.17)

with unconstrained constants \( \xi_0, \zeta_1 \), the rational solution for H2 reads

\[ u = \frac{1}{4} (\zeta + \zeta_0)^2 - (\zeta + \zeta_0)S^{(0)} + 2S^{(1)} - A^2 + (-1)^{n+m} A(\zeta + \zeta_1 /2 - 2S^{(0)}), \]  

(4.18)

where

\[ \zeta = 2(np + mq). \]  

(4.19)

From (3.39) and (3.40) one know that when \( N = 2 \), \( S^{(0)} \) and \( S^{(1)} \) are taken as

\[ S^{(0)} = \frac{2cpq}{pq + 2c(mp + nq)}, \quad S^{(1)} = 0, \]  

(4.20)
and when $N = 3$, $S^{(0)}$ and $S^{(1)}$ are described as

$$
S^{(0)} = 6cpq(pq + 2c(mp + nq))^2 / (-3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2)) \\
+ 8c^3(3nmpq(mp + nq) + (n^3 - n)q^3 + (m^3 - m)p^3)), \\
S^{(1)} = 12(cpq)^2(pq + 2c(mp + nq))/ (-3pq((pq)^2 - 2cpq(mp + nq) - 4c^2(mp + nq)^2)) \\
+ 8c^3(3nmpq(mp + nq) + (n^3 - n)q^3 + (m^3 - m)p^3)).
$$

4.2.5 $Q_{1 \delta} \rightarrow H_1$

The rational solution to $H_1$ can be obtained from (4.9) through degeneration. Substituting $a = 1 \epsilon, u \rightarrow \epsilon \delta u$ into (4.9) and using

$$
A \rightarrow \frac{\delta}{2} A(1 + \zeta_1 \epsilon), \quad D \rightarrow \frac{\delta}{2} A(-1 + \zeta_1 \epsilon), \quad B \rightarrow \delta B, \quad \xi_0 \rightarrow \epsilon \zeta_0
$$

with constants $\zeta_0, \zeta_1$, we have

$$
u = B(\zeta + \zeta_0 - 2S^{(0)}) + (-1)^{n+m}A(\zeta + \zeta_1 - 2S^{(0)}),
$$

which presents rational solution of $H_1$, where $\zeta_0, \zeta_1, A$ and $B$ satisfies $A^2 - B^2 = -\frac{1}{4}$. Here $S^{(0)}$ is same as the one in solution (4.18).

5 Conclusions

In the present paper, we investigate the rational solutions for the whole ABS list except for Q4. The procedure is different from the one given in Ref. [22]. We make use of Hirota’s bilinear method together with the key results given in Ref. [7]. In order to express the rational solutions for the lattice KdV-type equations uniformly, we consider an extended condition equation set (3.33), where a constant $c$ is introduced. This constant is indispensable and allows one to take spectral parameters’ limit $k \rightarrow 0$. On basis of rational solutions for the lpKdV equation and the Miura transformations (3.19) and (3.21), together with symmetric property $S(a, b) = S(b, a)$, one can finish the proof of Theorem 1. Theorem 1 reveals a fact that solutions to $Q_3 \delta$ can be expressed as a linear combination of four different solutions of the NQC equation with the parameters $a, b$ changing signs holds not only for solitons [7] but also for rational solutions. Besides, from the relation (4.1) one may consider (4.23) as a bilinear form of $Q_3 \delta$. Based on the degeneration relations between $Q_3 \delta$ and “lower equations” $Q_2, Q_1 \delta, H_3 \delta, H_2$ and $H_1$, the rational solutions for the latter equations are also derived.

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A  Casoratian shift formulae

We list shift formulae for the Casoratians

\[ f = |(N - 1)_{0,0} |, \quad g = |(N - 2, N)_{0,0} |, \quad h = |(N - 1)_{-1,0} |, \quad s = |(N - 1)_{0,-1} |, \quad \vartheta = |(N - 1)_{-1,-1} |, \]

where the basic column vector \( \phi(\alpha, \beta, l) \) satisfies the relations \([3.5],[3.11]\) and \([3.25]\).

\[
\begin{align*}
p^{N-2}f &= -|(N-2)_{0,0}, \phi(0, 0, N-2)|, & (A.1a) \\
q^{N-2}f &= -|(N-2)_{0,0}, \phi(0, 0, N-2)|, & (A.1b) \\
(p - a)p^{N-2}h &= a^{N}|0_{-1,0}, (N-3)_{0,0}, \phi(0, 0, N-2)|, & (A.1c) \\
(q - a)q^{N-2}h &= a^{N}|0_{-1,0}, (N-3)_{0,0}, \phi(0, 0, N-2)|, & (A.1d) \\
p^{N-2}s &= -b^{N}|(N-2)_{0,-1}, \phi(0, 0, N-2)|, & (A.1e) \\
q^{N-2}s &= -b^{N}|(N-2)_{0,-1}, \phi(0, 0, N-2)|, & (A.1f) \\
(p - a)p^{N-2}\vartheta &= (ab)^{N}|0_{-1,-1}, (N-3)_{0,-1}, \phi(0, 0, N-2)|, & (A.1g) \\
(q - a)q^{N-2}\vartheta &= (ab)^{N}|0_{-1,-1}, (N-3)_{0,-1}, \phi(0, 0, N-2)|, & (A.1h) \\
(p - b)p^{N-2}\vartheta &= (ab)^{N}|0_{-1,-1}, (N-3)_{-1,0}, \phi(0, 0, N-2)|, & (A.1i) \\
(q - b)q^{N-2}\vartheta &= (ab)^{N}|0_{-1,-1}, (N-3)_{-1,0}, \phi(0, 0, N-2)|, & (A.1j) \\
p^{N-2}(g + pf) &= -|(N-3, N-1)_{0,0}, \phi(0, 0, N-2)|, & (A.1k) \\
q^{N-2}(g + qf) &= -|(N-3, N-1)_{0,0}, \phi(0, 0, N-2)|, & (A.1l) \\
(p - q)p^{N-2}q^{N-2}f &= |(N-3)_{0,0}, \phi(0, 0, N-2), \phi(0, 0, N-2)|, & (A.1m)
\end{align*}
\]

and

\[
\begin{align*}
p^{N-2}\tilde{f} &= \frac{\tilde{B}_{[m]}}{B_{[m]}}|(N-2)_{0,0}, B_{[m]}B^{-1}_{[m]}\phi(0, 0, N-2)|, & (A.2a) \\
q^{N-2}\tilde{f} &= \frac{\tilde{A}_{[m]}}{A_{[m]}}|(N-2)_{0,0}, A_{[m]}A^{-1}_{[m]}\phi(0, 0, N-2)|, & (A.2b) \\
b^{N-2}\tilde{s} &= b^{N}\tilde{C}_{[\beta]}|\tilde{C}_{[\beta]}|(N-2)_{0,-1}, \phi(0, 0, N-2)|, & (A.2c) \\
a^{N-2}\tilde{h} &= a^{N}|\tilde{D}_{[\alpha]}|D_{[\alpha]}|\tilde{D}_{[\alpha]}^{-1}\phi(-1, 0, N-2)|, & (A.2d) \\
(p + a)p^{N-2}\tilde{h} &= a^{N}|\tilde{B}_{[m]}|B_{[m]}B^{-1}_{[m]}\phi(0, 0, N-2)|, & (A.2e) \\
(q + a)q^{N-2}\tilde{h} &= a^{N}|\tilde{A}_{[m]}|A_{[m]}A^{-1}_{[m]}\phi(0, 0, N-2)|, & (A.2f) \\
(p + q)p^{N-2}q^{N-2}\tilde{f} &= \frac{\tilde{A}_{[m]}}{A_{[m]}}|(N-3)_{0,0}, \phi(0, 0, N-2), \phi(0, 0, N-2)|, & (A.2g)
\end{align*}
\]
\[(p + q) p^{N-2} q^{N-2} = \frac{\widehat{B}_{[n]}}{B_{[n]}} \{(N-3)_{0,0}, \phi(0,0,N-2), B_{[n]} \widehat{B}_{[n]}^{-1} \phi(0,0,N-2)\}, \quad (A.2h)\]
\[(p + b) p^{N-2} b^{N-2} = b \frac{\hat{C}_{[\beta]}}{C_{[\beta]}} \{(N-3)_{0,-1}, \phi(0,-1,N-2), C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0,-1,N-2)\}, \quad (A.2i)\]
\[(q + b) q^{N-2} b^{N-2} = b \frac{\hat{C}_{[\beta]}}{C_{[\beta]}} \{(N-3)_{0,-1}, \phi(0,-1,N-2), C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0,-1,N-2)\}, \quad (A.2j)\]
\[(a + b) b^{N-2} = b \frac{\hat{C}_{[\beta]}}{C_{[\beta]}} \{(0,-1,1), (N-3)_{0,-1}, C_{[\beta]} \hat{C}_{[\beta]}^{-1} \phi(0,-1,N-2)\}, \quad (A.2k)\]
\[(p + a) p^{N-2} a^{N-2} = a \frac{\hat{D}_{[\alpha]}}{D_{[\alpha]}} \{(N-3)_{1,0}, \phi(-1,0,N-2), D_{[\alpha]} \hat{D}_{[\alpha]}^{-1} \phi(-1,0,N-2)\} \quad (A.2l)\]
\[(q + a) q^{N-2} a^{N-2} = a \frac{\hat{D}_{[\alpha]}}{D_{[\alpha]}} \{(N-3)_{1,0}, \phi(-1,0,N-2), D_{[\alpha]} \hat{D}_{[\alpha]}^{-1} \phi(-1,0,N-2)\} \quad (A.2m)\]
\[(a + b) a^{N-2} = a \frac{\hat{D}_{[\alpha]}}{D_{[\alpha]}} \{(0,-1,1), (N-3)_{1,0}, D_{[\alpha]} \hat{D}_{[\alpha]}^{-1} \phi(-1,0,N-2)\}, \quad (A.2n)\]
\[q^{N-2} \{\hat{g} - q \hat{f} \} = \frac{\hat{A}_{[m]}}{A_{[m]}} \{(N-3, N-1)_{0,0}, A_{[m]} \hat{A}_{[m]}^{-1} \phi(0,0,N-2)\}. \quad (A.2o)\]

References

[1] F.W. Nijhoff, A.J. Walker, The discrete and continuous Painlevé VI hierarchy and the Garnier systems, *Glasgow Math. J*. 43A (2001) 109–123.

[2] V.E. Adler, A.I. Bobenko, Y.B. Suris, Classification of integrable equations on quad-graphs. The consistency approach, *Commun. Math. Phys*. 233 (2003) 513–543.

[3] F.W. Nijhoff, G.R.W. Quispel, H.W. Capel, Direct linearization of nonlinear difference-difference equations, *Phys. Lett. A* 97 (1983) 125–128.

[4] F.W. Nijhoff, H.W. Capel, The discrete Korteweg-de Vries equation, *Acta Appl. Math*. 39 (1995) 133–158.

[5] V.E. Adler, Bäcklund transformation for the Krichever-Novikov equation, *Int. Math. Res. Not.* 1 (1998) 1–4.

[6] J. Atkinson, Bäcklund transformations for integrable lattice equations, *J. Phys. A: Math. Theor*. 41 (2008) 135202(8pp).

[7] F.W. Nijhoff, J. Atkinson, J. Hietarinta, Soliton solutions for ABS lattice equations: I: Cauchy matrix approach, *J. Phys. A: Math. Theor*. 42 (2009) 404005(34pp).

[8] J. Atkinson, J. Hietarinta, F.W. Nijhoff, Soliton solutions for Q3, *J. Phys. A: Math. Theor*. 41 (2008) 142001(11pp).

[9] J. Hietarinta, D.J. Zhang, Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization, *J. Phys. A: Math. Theor*. 42 (2009) 404006(30pp).

[10] D.J. Zhang, S.L. Zhao, Solutions to the ABS lattice equations via generalized Cauchy matrix approach, *Stud. Appl. Math.* 131 (2013) 72–103.

[11] D.D. Xu, D.J. Zhang, S.L. Zhao, The Sylvester equation and integrable equations: I. The Korteweg-de Vries system and sine-Gordon equation, *J. Nonlin. Math. Phys*. 21(3) (2014) 382–406.

[12] S. Butler, N. Joshi, An inverse scattering transform for the lattice potential KdV equation, *Inver. Prob*. 26 (2010) 115012(28pp).

[13] S. Butler, Multidimensional inverse scattering of integrable lattice equations, *Nonlinearity* 25 (2012) 1613–1634.
[14] J. Atkinson, F.W. Nijhoff, A constructive approach to the soliton solutions of integrable quadrilateral lattice equations, Commun. Math. Phys. 299 (2010) 283–304.

[15] M.J. Ablowitz, J. Satsuma, Solitons and rational solutions of nonlinear evolution equations, J. Math. Phys. 19 (1978) 2180–2187.

[16] D.J. Zhang, Notes on solutions in Wronskian form to soliton equations: KdV-type, arXiv:nlin.SI/0603008 (2006) 45pp.

[17] K. Maruno, K. Kajiwara, S. Nakao, M. Oikawa, Bilinearization of discrete soliton equations and singularity confinement, Phys. Lett. A 229 (1997) 173–182.

[18] B. Grammaticos, A. Ramani, V. Papageorgiou, J. Satsuma, R. Willox, Constructing lump-like solutions of the Hirota-Miwa equation, J. Phys. A: Math. Theor. 40(42) (2007) 12619–12627.

[19] Y. Shi, D.J. Zhang, Rational solutions of the H3 and Q1 models in the ABS lattice list, SIGMA 7 (2011) 046(11pp).

[20] L.J. Nong, D.J. Zhang, Y. Shi, W.Y. Zhang, Parameter extension and the quasi-rational solution of a lattice Boussinesq equation, Chin. Phys. Lett. 30 (2013) 040201(4pp).

[21] W. Feng, S.L. Zhao, Y. Shi, Rational solutions for lattice potential KdV equation and two semi-discrete lattice potential KdV equations, Z. Naturforsch. 71(2a) (2016) 121–128.

[22] D.D. Zhang, D.J. Zhang, Rational solutions to the ABS list: Transformation approach, arXiv: 1702.01266v2 (2017).

[23] K. Kajiwara, Y. Ohta, Bilinearization and Casorati determinant solution to the non-autonomous discrete KdV equation, J. Phys. Soc. Japan 77 (2008) 054004(9pp).

[24] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg-de Vries and the Kadomtsev-Petviashvili equations: the Wronskian technique, Phys. Lett. A 95 (1983) 1–3.

[25] D.J. Zhang, J. Hietarinta, Generalized solutions for the H1 model in ABS list of lattice equations, Nonl. Mod. Math. Phys: Proceedings of the First International Workshop, AIP Conference Proceedings 1212 (2010) 154–161.

[26] J. Hietarinta, D.J. Zhang, Multisoliton solutions to the lattice Boussinesq equation, J. Math. Phys. 51 (2010) 033505(12pp).