Accurate inference for a one parameter distribution based on the mean of a transformed sample

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Abstract: A great deal of inference in statistics is based on making the approximation that a statistic is normally distributed. The error in doing so is generally \(O(n^{-1/2})\) and can be very considerable when the distribution is heavily biased or skew. This note shows how one may reduce this error to \(O(n^{-1/(j+1)})\), where \(j\) is a given integer. The case considered is when the statistic is the mean of the sample values from a continuous one-parameter distribution, after the sample has undergone an initial transformation.

Keywords: Accurate inference; Confidence interval; Cornish-Fisher transformations; Edgeworth expansion; Lehmann alternative; Quantile.

1 Introduction and summary

Given a random sample of size \(n\), the usual confidence interval for the population mean is of the order \(O(n^{-1/2})\). The aim of this note is to show how a more accurate confidence interval of the order \(O(n^{-1/(j+1)})\), where \(j\) is a given integer, can be obtained. This is an important problem because finding accurate confidence intervals for the population mean is an everyday problem faced by many scientists and engineers.

Suppose for some statistic \(\psi_n\), \(Y_n(\theta) = n^{1/2}\{\psi_n - g(\theta)\}/\sigma(\theta) \to N(0,1)\) as \(n \to \infty\), where \(\theta \in \mathbb{R}\). Cornish and Fisher (1937) and Fisher and Cornish (1960) obtained an Edgeworth type expansion for the distribution of \(Y_n(\theta)\), and an asymptotic expansion for its percentile points when this distribution is parameter-free. Withers (1984) gave a simplified version of their results which reduced the labor of their application. Withers (1983) considered the more general case, where the distribution of \(Y_n(\theta)\) does depend on \(\theta\). In this case a parameter-free transformation \(V_{nx}(\cdot)\) was given such that

\[ V_{nx_1}(\psi_n) < \theta < V_{nx_2}(\psi_n) \text{ with probability } 1 - \alpha \tag{1.1} \]

provided

\[ \Phi(x_1) - \Phi(x_2) = 1 - \alpha, \text{ e.g. } x_1 = -x_2 = \Phi^{-1}(1 - \alpha/2), \tag{1.2} \]
assuming \( g(\cdot) \) is an increasing function, where \( \Phi(\cdot) \) denotes the distribution of a standard normal random variable. If \( g(\cdot) \) is decreasing the inequalities in (1.1) are reversed.

In Section 2, we show how this theory applies to \( \psi_n = \bar{X}_n \), the mean of \( X_1, \ldots, X_n \) i.i.d. \( F(x, \theta) \) on \( R \), where \( F(x, \theta) \) is of known parametric form.

In applications \( \{X_i\} \) will generally not be the original observations \( \{Y_i\} \), say, but will be given by \( X_i = h(Y_i) \), where \( h(\cdot) \) is a transformation chosen from considerations of efficiency, robustness or ease of computation of the first few cumulants of \( F(x, \theta) \) as functions of \( \theta \). So, if \( Y_1 \sim R(x, \theta) \)

\[
F(x, \theta) = R(h^{-1}(x), \theta) \tag{1.3}
\]

for \( h(\cdot) \) one to one increasing.

However, \( \{Y_i\} \) need not lie in \( R \). Their distribution may, in fact, depend on parameters, \( \lambda \), other than \( \theta \) provided \( F(x, \theta) \) does not depend on \( \lambda \). Note that \( \theta \) itself may be a reparameterisation of an original parameter.

In Section 3 the efficiency and robustness of this class of procedures is considered.

In Section 4 this theory is applied to the ‘Lehmann alternative’: \( F(x, \theta) = R(x^\theta) \), where by suitable choice of \( h(\cdot) \), \( R(\cdot) \) may be any continuous distribution.

For many parameter inference problems see Withers (1989).

2 The general case

Let \( X_1, \ldots, X_n \) be a random sample for a distribution \( F(x, \theta) \) on \( R \) such that \( g(\theta) = EX_1 \) is a known one-to-one function from the parameter space, assumed to be some subset of \( R \). Set

\[
\sigma(\theta)^2 = \text{var}X_1, \quad T_n = \bar{X}_n - g(\theta), \quad Y_n(\theta) = n^{1/2}T_n/\sigma(\theta).
\]

For any real random variable \( X \) set

\[
K_r(X) = \text{rth cumulant of } X,
\]

\[
\ell_r(X) = \kappa_2(X)^{-r/2}\kappa_r(X) - \delta_{r,2}, \tag{2.1}
\]

where \( \delta_{r,s} = 1 \) if \( r = s \) and \( \delta_{r,s} = 0 \) if \( r \neq s \). Suppose that for some \( j \geq 0 \), \( \kappa_r(X_1) \) exists for \( 1 \leq r \leq j + 2 \) and

\[
\limsup_{t \to \pm \infty} \left| \int \exp(itx)dF(x, \theta) \right| < 1. \tag{2.2}
\]

This condition rules out many discrete lattice distributions. Then by Theorem 3, page 541 of Feller (1971),

\[
P(Y_n(\theta) \leq x) = \Phi(x) - \phi(x) \sum_{r=1}^j n^{-r/2}U_r(x) + o\left(n^{-j/2}\right) \quad \text{as } n \to \infty
\]

uniformly in \( x \), where \( U_r(x) \) is a polynomial in \( x \) defined in terms of \( \ell_1, \ldots, \ell_{r+2} \), where \( \ell_r = \ell_r(X_1) \). Note that \( \ell_1 = \ell_2 = 0 \). \( U_r = R_{r+2} \) is defined by Feller. In particular, by equation (6.50) of Stuart
and Ord (1987),

\[
U_1 = \ell_3 H_2 / 6, \\
U_2 = \ell_4 H_3 / 24 + \ell_3^2 H_5 / 72, \\
U_3 = \ell_5 H_4 / 120 + \ell_4 \ell_4 H_6 / 144 + \ell_3^3 H_8 / 1296, \\
U_4 = \ell_6 H_5 / 720 + \ell_5^2 H_7 / 1152 + \ell_5 \ell_5 H_7 / 720 + \ell_3^4 H_9 / 1728 + \ell_4^2 H_9 / 31104,
\]

where \( H_r \) is the Hermite polynomial: \( H_r(x) = \exp(x^2/2)\left(-\partial/\partial x\right)^r \exp(-x^2/2) \). For example, \( H_1, \ldots, H_{10} \) are given by equation (6.23) of Stuart and Ord (1987). Cornwall and Fisher (1937) used this to show (for a more general situation but assuming all cumulants exist) that

\[
P(Y_n(\theta) \leq x) \equiv \Phi (\xi_n(x)) = \Phi (\xi_{n_j}(x)) + o\left(n^{-j/2}\right),
\]

where

\[
\xi_n(x) = x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x), \quad \xi_{n_j}(x) = x - \sum_{r=1}^{j} n^{-r/2} f_r(x)
\]

and \( f_r(x) \) is a polynomial in \( x \) depending on \( \ell_1, \ldots, \ell_{r+2} \):

\[
f_r(x) = \beta'_r a_r(x),
\]

where

\[
\beta'_1 = \ell_3, \quad \beta'_2 = (\ell_4 - \ell_3^2), \quad \beta'_3 = (\ell_5 - \ell_3 \ell_4, \ell_3^3), \\
\beta'_4 = (\ell_6, \ell_3 \ell_5, \ell_3^2 \ell_4, -\ell_3^3)
\]

and

\[
a_1(x) = H_2 / 6, \quad a_2(x) = \left(H_3(x)/24, (4x^3 - 7x)/36\right)', \\
a_3(x) = \left(H_4(x)/120, (11x^4 - 42x^2 + 15) / 144, (69x^4 - 187x^2 + 52) / 648\right)', \\
a_4(x) = \left(H_5(x)/720, (5x^5 - 32x^3 + 35x) / 384, (7x^5 - 48x^3 + 51x) / 360, \\
(111x^5 - 547x^3 + 456x) / 864, (948x^5 - 3628x^3 + 2473x) / 7776\right)'.
\]

They further showed that if \( z_\alpha = \Phi^{-1}(1 - \alpha) \) then

\[
Y_n(\theta) \leq \eta_n(z_\alpha) \quad \text{with probability } 1 - \alpha,
\]

where

\[
\eta_n(y) = y + \sum_{r=1}^{\infty} n^{-r/2} g_r(y)
\]

and \( g_r(y) \) is a polynomial in \( x \) depending on \( \ell_1, \ldots, \ell_{r+2} \):

\[
g_r(y) = \beta'_r b_r(y),
\]

(2.4)
where
\[
\begin{align*}
    b_1(y) &= H_2(y)/6, \\
    b_2(y) &= \left(\frac{H_3(y)}{24}, \frac{2y^3 - 5y}{36}\right)^{\prime}, \\
    b_3(y) &= \left(\frac{H_4(y)}{120}, \frac{y^4 - 5y^2 + 2}{24}, \frac{12y^4 - 53y^2 + 17}{324}\right)^{\prime}, \\
    b_4(y) &= \left(\frac{H_5(y)}{720}, \frac{3y^5 - 24y^3 + 29yz}{384}, \frac{2y^5 - 17y^3 + 21y}{180}, \frac{14y^5 - 103y^3 + 107y}{288}, \frac{252y^5 - 1688y^3 + 1511y}{7776}\right)^{\prime},
\end{align*}
\]

and \(g_5, g_6\) may be obtained from V, VI, pages 214, 215 of Fisher and Cornish (1960), by setting \(a = b = 0, c = \ell_3, d = \ell_4, e = \ell_5\), etc.

So, under \((2.2)\), if
\[
\sigma = \sigma(\theta), \kappa_3(X_1), \ldots, \kappa_{j+2}(X_1)
\]
exist and do not depend on \(\theta\), then
\[
G_n^{-1}(1 - \alpha) = G_{nj}^{-1}(1 - \alpha) + o\left(n^{-(j+1)/2}\right),
\]
where
\[
G_n(x) = P\left(\bar{X}_n - g(\theta) \leq x\right),
\]
\[
G_{nj}^{-1}(1 - \alpha) = n^{-1/2} \sigma \eta_{nj}(z_\alpha), \quad \eta_{nj}(y) = y + \sum_{r=1}^{j} n^{-r/2} g_r(y).
\]

In particular, the confidence interval
\[
g^{-1}\left(\bar{X}_n - G_{nj}^{-1}(\alpha/2)\right) \leq \theta \leq g^{-1}\left(\bar{X}_n - G_{nj}^{-1}(1 - \alpha/2)\right)
\]
has level \(1 - \alpha + o(n^{-j/2})\), in fact \(1 - \alpha + O(n^{-(j+1)/2})\) if \(\kappa_{j+3}(X_1)\) is finite. More generally, if \((2.5)\) is weakened to allow \(\sigma(\theta)\) to vary with \(\theta\) and \(Y_n(\cdot)\) is one to one increasing (or decreasing), and \(x_1, x_2\) satisfy \((1.2)\), then a confidence interval with level \(1 - \alpha + O(n^{-(j+1)/2})\) is
\[
Y_n^{-1}(\eta_{nj}(x_2)) \leq \theta \leq Y_n^{-1}(\eta_{nj}(x_1))
\]
with the inequalities reversed if \(Y_n(\cdot)\) is decreasing.

These formulae have been shown to be extremely accurate. One can judge the number of significant places when approximating \(G_n^{-1}\) by \(G_{nj}^{-1}\) by the size of the successive terms \(n^{-r/2} g_r(z_\alpha)\), which generally alternate in sign. See, for example, Fisher and Cornish (1960).

Withers (1983) – for a more general situation – showed how to obtain a confidence interval for \(\theta\) in the more usual situations, where the cumulants depend on \(\theta\). This dependency is expressed by writing \(g_r(x) = g_r(x, \theta)\), etc.

The main purpose of the present note is to apply these results to the case of the sample mean under the assumptions \((2.1), (2.2)\). When the initial transformation \(h(\cdot)\) is independent of \(\theta\) then applying Withers (1983) to \(\bar{X}_n\), one obtains that a confidence interval of level \(1 - \alpha + o(n^{-j/2})\) is
\[
V_{nx_{x1j}}(\bar{X}_n) < \theta < V_{nx_{x2j}}(\bar{X}_n),
\]

\[\frac{\sigma}{\sqrt{n}}, \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.\]
where \( V_{nxj}(t) = g^{-1}(S_{nxj}(t)) \), \( x_1 \), \( x_2 \) satisfy (12), and

$$ S_{nxj}(t) = t + \sum_{i=1}^{j+1} n^{-i/2}Q_i(t) $$

(2.9)

for \( \{Q_i(t)\} \) given by Withers (1983) in terms of \( P_i(t) = \sigma(g^{-1}(t))g_{i-1}(x, g^{-1}(t)) \) for \( \{g_i\} \) as above, where \( g_0(x, \theta) = x \). Here, we have assumed \( g(\cdot) \) to be increasing. If \( g(\cdot) \) is decreasing the inequalities in (2.8) are reversed.

In particular, \( Q_1 = -P_1 \), \( Q_2 = -P_2 - \hat{P}_1Q_1 \) and \( Q_3 = -P_3 - \hat{P}_2Q_1 - \hat{P}_1Q_1^2/2 \).

For such calculations it is convenient to write \( P_i \) in the form

$$ P_i(t) = M_i(t)b_{i-1}(x), \quad i \geq 1, $$

where \( M_i(t) = m_i(g^{-1}(t)) \), \( m_i(\theta) = \sigma(\theta)\beta_{i-1} \), \( \beta_0 = 1 \) and \( b_0(x) = x \) with \( \{\beta_r, b_r\} \) given by (2.3), (2.4). Setting \( K_r(\theta) = \kappa_r(X_1) \), one obtains

$$ m_1(\theta) = \sigma(\theta), \quad m_2(\theta) = \sigma(\theta)^{-2}K_3(\theta), \quad m_3(\theta) = (\sigma(\theta)^{-3}K_4(\theta), \sigma(\theta)^{-5}K_3(\theta)^2), $$

eq \text{etc, and so}

$$ Q_1(t) = -M_1(t)x, $$

$$ Q_2(t) = -M_2(t)b_1(x) + x^2D_tM_1(t)^2/2, $$

$$ Q_3(t) = -M_3(t)b_2(x) + xb_1(x)D_tM_1(t) - x^3D_tM_1(t)^3/6, $$

(2.10) (2.11) (2.12)

and so forth, where \( D_t = \partial/\partial t \).

### 3 Efficiency and robustness

So far our concern has been to obtain accurate inference on the parameter of the original distribution \( R(x, \theta) \) from the size of \( \overline{X}_n \), where \( X_i \equiv h(Y_i) \) and \( h(\cdot) \) is a given transformation. We now consider the efficiency of the procedure, and its robustness to outliers, as these factors are important in the choice of \( h(\cdot) \).

Let \( F_n \) be the empirical distribution of \( \{Y_i\} \), which we shall suppose lie in \( R^a \). Corresponding to (2.8) is the point estimate

$$ \hat{\theta}_n = \theta(F_n), $$

where \( \theta(F) = g^{-1}(\int h \, df) \) and \( g(\cdot) \) is fixed by the choice of \( h(\cdot) \):

$$ g(\theta) = \int x \, dF(x, \theta) = \int h(y) \, dR(y, \theta). $$

The influence function of \( \hat{\theta}_n \) is

$$ I_{\theta}(x, F) = \left( h(x) - \int h \, df \right) \cdot \hat{g} \left( g^{-1}(\int h \, df) \right) $$
which evaluated at \( F(\cdot) = R(\cdot, \theta) \) gives \( I_\theta(x) = (h(x) - g(\theta))/\hat{g}(\theta) \). So, to reduce the effect of outliers it is desirable that \( h(\cdot) \) be bounded. Also

\[
n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, V(\theta, h))
\]
as \( n \to \infty \), where

\[
V(\theta, h) = \int I_\theta(x)^2 \, dR(x, \theta) = \left( \int h(x)^2 \, dR(x, \theta) - g(\theta)^2 \right)/\hat{g}(\theta)^2 = \sigma(\theta)^2/\hat{g}(\theta)^2. \tag{3.1}
\]

The asymptotic efficiency of \( \hat{\theta}_n \) or of the confidence interval \( [\bar{y}, \bar{\eta}] \) is inversely proportional to \( V(\theta, h) \). Note that \( V(\theta, h) \) is minimized by \( h = q_\theta \), where \( q_\theta \) is Fisher’s score function

\[
q_\theta(x) = \partial/\partial \theta \log dR(x, \theta)/dR(x, 0).
\]

The maximum likelihood estimate \( \theta_n^* \) is asymptotically equivalent to this choice in the sense that \( n^{1/2}(\theta_n^* - \hat{\theta}_n) \xrightarrow{d} 0 \) as \( n \to \infty \). However, the results of Section 2 have assumed \( h(\cdot) \) is independent of \( \theta \), so can only be applied to \( \theta_n^* \) when \( q_\theta(x) \) has the form \( a(\theta)b(x) \).

4 Lehmann’s alternative

In this section, we illustrate the results of Sections 2 and 3 when the original sample \( \{Y_i\} \) has distribution \( R(x, \theta) = F_0(x)^\theta \), where \( \theta > 0 \), and \( F_0(\cdot) \) is a continuous distribution. This is sometimes known as ‘Lehmann’s alternative’. By (1.3), \( \{X_i = h(Y_i)\} \) have distribution \( F(x, \theta) = R(x)^\theta \), where \( R(x) = F_0(h^{-1}(x)) \). So, by suitable choice of \( h(\cdot) \), \( R(\cdot) \) may be chosen to be any continuous distribution on \( R \). The cumulative generating function for \( X_1 \)

\[
K_R(t) = \log \int \exp(tx) \, dR(x)^\theta.
\]

However, it is sometimes easier to calculate the necessary cumulants directly.

The maximum likelihood estimate is given by \( \theta_n^* = -\overline{X}_n^{-1} \), where \( h(x) = \log F_0(x) \). This yields

\[
\text{Example 4.1 Suppose } R(x) = \exp(x) \text{ on } (-\infty, 0]. \text{ Then } K_R(t) = -\log(1 + t/\theta), \kappa_r(X_1) = (-\theta)^{-r}(r - 1)! , g(\theta) = -\theta^{-1}, \sigma(\theta)^2 = -\theta^2 \text{ and } \ell_r = (-)^r(r - 1)! \text{ for } r > 2. \text{ So, } \xi_n, \eta_n, g_r, f_r, \text{ are independent of } \theta. \text{ Also } Y_n(\theta) = (\theta \overline{X}_n + 1)n^{1/2}. \text{ So, by equation (2.7) a confidence interval with level } 1 - \alpha + O(n^{-1/2}) \text{ is}
\]

\[
N_{nj}(x_2)/|\overline{X}_n| \geq \theta \geq N_{nj}(x_1)/|\overline{X}_n|,
\]

where \( N_{nj}(x) = 1 - n^{-1/2} \eta_{nj}(x) = 1 - \sum_{i=1}^{n} \eta_{i,j} g_{i-1}(x) \), where \( \{g_i\} \) are given by (2.3), (2.4). In this particular example, one may use \( n|\overline{X}_n| \sim \Gamma(x, n) \) and hence \( 2n|\overline{X}_n| \sim \chi^2_{2n} \) to obtain a confidence interval directly. The expansion \( L_n(x) = n + \sqrt{2} \sum_{i=1}^{7} n^{-i/2} g_{i-1}(x) \) for \( \chi^2_{2n} \) is given by equation (3a) of Fisher and Cornish (1960). So, \( \chi^2_{2n} \leq L_n(-x) \) with probability \( 1 - \Phi(x) + O(n^{-7/2}) \). It follows that for this example in terms of (3a), \( N_{nj}(x) = L_n(-x)/(2n) \). □
Example 4.2 Suppose $R(x) = x^\nu$ on $[0, 1]$, where $\nu > 0$ is a given. This corresponds to $h(x) = F_0(x)^{1/\nu}$.

So, it will be both less efficient and less robust than the choice of Example 4.1. However, it serves to illustrate the method when $\eta_n(\cdot)$ depends on $\theta$. In this case the cumulants are best calculated from $EX_i^1 = (1 + r\psi)^{-1}$, where $\psi = (\nu\theta)^{-1}$. So, $g(\theta) = (1 + \psi)^{-1}$ lies in $[0, 1]$. Set $t = g(\theta)$. Then

\[
\sigma(\theta)^2 = (1 + 2\psi)^{-1} - t^2 = t(1 - t)^2(2 - t)^{-1},
\]

\[
K_3(\theta) = (1 + 3\psi)^{-1} - 3(1 + 2\psi)^{-1}t + 2t^3 = t(3 - 2t)^{-1} - 3t^2(2 - t)^{-1} + 2t^3,
\]

and so forth. By (2.10)–(2.12), $Q_1$ is given in terms of

\[
M_1(t) = (1 - t)(2t^{-1} - 1)^{-1/2}
\]

and $Q_2$ is given in terms of

\[
M_2(t) = 2(1 - t)(1 - 2t)/(3 - 2t)
\]

and

\[
D_t M_1(t)^2/2 = (2 - t)^{-2} - t = (2 - t^{-2})(1 - t)(1 - 3t + t^2).
\]

Finally, by (2.8), a confidence interval of level $1 - \alpha + O(n^{-1})$ is given by

\[
(S_{nx1}(X_n)^{-1} - 1)/\nu < \theta < (S_{nx2}(X_n)^{-1} - 1)/\nu,
\]

where $S_{nx1}(t)$ is given by (2.4), (2.9), (2.10) and (4.1)–(4.3). By (3.1), the asymptotic efficiency of this choice is $\{V(\theta, h) for the maximum likelihood estimate $/V(\theta, h) = \theta^2 g'(\theta)^2/\sigma(\theta)^2 = 1 - (\nu\theta + 1)^{-2}$.

\[
\square
\]

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