CHARACTERISTIC SETS OF MATROIDS

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Abstract

We investigate possible linear, algebraic, and Frobenius flock characteristic sets of matroids. In particular, we classify possible combinations of linear and algebraic characteristic sets when the algebraic characteristic set is finite or cofinite. We also show that the natural density of an algebraic characteristic set in the set of primes may be arbitrarily close to any real number in the unit interval.

Frobenius flock realizations can be constructed from algebraic realizations, but the converse is not true. We show that the algebraic characteristic set may be an arbitrary cofinite set even for matroids whose Frobenius flock characteristic set is the set of all primes. In addition, we construct Frobenius flock realizations in all positive characteristics from linear realizations in characteristic 0, and also from Frobenius flock realizations of the dual matroid.

1 Introduction

A matroid is a combinatorial structure generalizing the concept of linear independence of vectors in a vector space [Whi35]. Explicitly, it is equivalent to the collection of all linearly independent subsets of a fixed set of vectors in a vector space. However, not all matroids have linear representations, and the existence of a linear representation can depend on the field. The linear characteristic set of a matroid $M$, which we denote $\chi_L(M)$, is the set of characteristics of those fields over which $M$ does have a linear representation. The linear characteristic set of a matroid is either a finite set of positive primes or a cofinite set containing 0 [Rad57, Vam75]. Conversely, any such set occurs as the linear characteristic set of some matroid [Kah82, Rei].

Similar to the linear independence in a vector space, algebraic independence in a field extension also defines a matroid. For a matroid $M$ on a set $E$, an algebraic representation over $K$ is a pair $(L, \phi)$ consisting of a field extension $L$ of $K$ and a map $\phi: E \to L$ such that any $I \subseteq E$ is independent in $M$ if and only if the set $\phi(I)$ is algebraically independent over $K$. If a matroid has a linear representation over a field $K$, then it also has an algebraic representation over $K$. Conversely, an algebraic representation over a field of characteristic 0 can be turned into a linear representation over a field of characteristic 0 by using derivations. However, there are matroids with algebraic representations in positive characteristic, but not linear representations [Lin86].
The algebraic characteristic set of a matroid $M$, denoted by $\chi_A(M)$, is the set of characteristics of the fields in which a matroid $M$ has algebraic representations. A classification analogous to that for linear characteristic sets is not known. Nonetheless, our first result shows that the algebraic characteristic set can be an essentially arbitrary finite or cofinite set, with a restriction only for characteristic 0:

**Theorem 1.** Let $C_L \subseteq C_A \subseteq \mathbb{P} \cup \{0\}$ be finite or cofinite subsets. Suppose either that $0 \in C_L$ and $C_L$ is cofinite, or that $0 \notin C_A$ and $C_L$ is finite. Then there exists a matroid $M$ such that $\chi_L(M) = C_L$ and $\chi_A(M) = C_A$.

Here, and throughout the paper, $\mathbb{P}$ denotes the set of all primes.

Unlike the linear characteristic set, the algebraic characteristic set of a matroid may be neither finite nor cofinite [EH91, Ex. 2]. We extend this example to show that the possible densities of the algebraic characteristic set are dense in the interval $[0, 1]$:

**Theorem 2.** Let $0 \leq \alpha \leq 1$ be a real number and $\epsilon > 0$. Then there exists a matroid $M$ such that $|d(\chi_A(M)) - \alpha| < \epsilon$, where $d(\chi_A(M))$ refers to the natural density of $\chi_A(M)$ in the set of all primes.

Recall that the natural density of a set of primes is defined as:

$$d(S) = \lim_{N \to \infty} \frac{|\{p \in S \mid p < N\}|}{|\{p \in \mathbb{P} \mid p < N\}|}$$

if that limit exists.

While derivations of algebraic representations in positive characteristic do not always give linear representations of the same matroid, Lindström found cases where they did and used that to prove that for $p$ a prime, the so-called Lazarson matroids $M_p$ have algebraic characteristic set consisting of just $p$ [Lin85]. Gordon extended this technique to give examples of matroids with some special non-singleton finite algebraic characteristic sets [Gor88]. He even went so far as to speculate that matroids with non-empty finite linear characteristic set had finite algebraic characteristic set, which is false by Theorem 1.

Inspired by Lindström’s work, Bollen, Draisma, and Pendavingh constructed what they called a Frobenius flock from an algebraic realization of a matroid. The Frobenius flock of an algebraic realization is a collection of compatible linear realizations of different matroids, which collectively agree with the algebraic matroid [BDP18]. The examples of Lindström and Gordon corresponded to the case where the flock was determined by a single one of these linear realizations. Given the tight connection between the Frobenius flock realizations and algebraic realizations in these cases, it is natural to wonder how different they are in general. We can define the flock characteristic set $\chi_F(M) \subseteq \mathbb{P}$, analogously to the linear and algebraic characteristic sets, and the flock characteristic set is an upper bound on the algebraic characteristic set. However, we show that their difference can be an arbitrary finite set of primes:
Theorem 3. In Theorem 1, the matroids constructed with infinite algebraic characteristic set also have $\chi_F(M) = \mathbb{P}$.

It would be interesting to know if the flock characteristic set can be an arbitrary cofinite set, like the linear and algebraic characteristic sets can be. However, the combinations of flock characteristic set and linear characteristic set are constrained by the following:

**Theorem 4.** Let $M$ be a matroid. If $0 \in \chi_L(M)$, then $\chi_F(M) = \mathbb{P}$.

Theorem 4, together with results quoted above, show that Theorem 1 constructs all possible triples of linear, algebraic, and flock characteristic sets, in the case where the linear characteristic set includes $0$.

The method for proving Theorem 4 involves “stretching” linear flocks (which are Frobenius flocks, but with a possible different automorphism than Frobenius). A consequence of this construction, is the following, which disproves [Bol18, Conj. 8.21]:

**Theorem 5.** If $M$ is a matroid, and $M^*$ is its dual matroid, then $\chi_F(M^*) = \chi_F(M)$.

While we don’t know about cofinite flock characteristic sets, any single prime may be a flock characteristic set [Lin85, BDP18]. Moreover, we show that certain finite sets are also possible:

**Theorem 6.** Let $C$ be any Gordon-Brylawski set of primes. Then there exists a matroid $M$ with $\chi_L(M) = \chi_A(M) = \chi_F(M) = C$.

Gordon-Brylawski sets are sets of primes satisfying a certain technical condition, given in Definition 17 below. Although we don’t know if the cardinality of a Gordon-Brylawski set is bounded, Example 18 shows that a Gordon-Brylawski set may have as many as 80 elements.

Recently, matroids over hyperfields and tracts have attracted much attention, as a generalization of both matroids and matroid realizations over a field [BB19, Su23]. In general, hyperfields and tracts don’t seem to have a natural notion of characteristic, and so it’s not clear what would be the analogous definition of a characteristic set. Nonetheless, Frobenius flock realizations are equivalent to realizations over a certain skew hyperfield introduced in [Pen], and so our results do cover characteristic sets for this specific class of hyperfields. Further investigation of the characteristic sets of Frobenius flocks could give some insight into what kind of behavior we should expect for realizations over tracts and skew hyperfields in general.

The remainder of this paper is organized as follows. In Section 2, we construct matroids to prove Theorem 1. In Section 3, we construct matroids whose algebraic characteristic set is neither finite nor cofinite, and prove Theorem 2. In Section 4, we recall the definition of linear flocks from [BDP18] and prove the Theorem 4. Finally in Section 5, we examine examples of matroids with finite flock characteristic sets and prove Theorem 6.
2 Specified characteristic sets

In this section, we introduce a lemma of Evans and Hrushovski and use it to construct matroids with specified linear and algebraic characteristic sets. Evans and Hrushovski constructed algebraic realizations of matroids using matrices of endomorphisms of a fixed one-dimensional group. Moreover, they showed that for certain matroids, all algebraic realizations are equivalent to realizations by such matrices.

This one-dimensional group construction simultaneously generalizes the realization of linear matroids as algebraic matroids and the realization of rational matroids as algebraic matroids over any field using monomials. The important point for us is that it depends on a choice of one-dimensional connected algebraic group $G$ over an algebraically closed field $K$. Such groups can be classified as either $G_a$, the additive group of $K$, $G_m$, the multiplicative group of $K$, or $E$ an elliptic curve over $K$ [BCD20, Sec. 2.1]. In each of these cases the ring of endomorphisms of the algebraic group is an integral domain $E$, which can be shown to be contained in a (possibly non-commutative) division ring $D$. The one-dimensional group construction turns a linear representation of a matroid over $D$ into an algebraic representation over $K$. The standard translation of linear matroids into algebraic matroids corresponds to the group $G_a$, with $k \in K$ corresponding to the function $x \mapsto kx$, which is an endomorphism of $G_a$. Likewise, the endomorphisms of the multiplicative group $G_m$ are just the integers with $n$ corresponding to the multiplicative endomorphism $x \mapsto x^n$, and then the group construction translates an integer matrix into monomials.

**Lemma 7** (Lem. 3.4.1 in [EH91]). Let $\Phi$ be a collection of equations in the variables $x_0, \ldots, x_n$ including the equations and inequalities:

\[ x_0 = 0, \quad x_1 = 1, \quad \text{and } x_i \neq x_j \text{ (for all } i \neq j) , \]

\[ x_i = x_j + x_k \text{ (where } j, k \neq 0, k \neq i \neq j), \quad x_i = x_j \cdot x_k \text{ (where } i, j, k \neq 0, 1 \text{ and } k \neq i \neq j). \]

Then there exists a matroid $M$ satisfying the following properties:

1. $M$ has a linear realization over an infinite field $K$ if and only if there exist (distinct) values for $x_0, \ldots, x_n$ in $K$ which simultaneously satisfy every equation in $\Phi$.

2. $M$ has an algebraic realization over a field $K$ if and only if there exists a linear representation of $M$ over the division ring generated by the ring of endomorphisms of a 1-dimensional algebraic group $G$ over a field of the same characteristic as $K$.

From now on, we will refer to systems of equations satisfying the conditions of Lemma 7 to mean the form in the first paragraph. Note that, because of the required inequalities, a solution to such a system in a division ring $Q$ will always mean an assignment of distinct values of $Q$ for the variables.
We now recall the classification of the endomorphism rings of a one-dimensional algebraic group. If characteristic of $K$ is 0, then the endomorphism ring of $G_a$, $G_m$, or an elliptic curve is, respectively, $K$, $\mathbb{Z}$, and either $\mathbb{Z}$ or an order in an imaginary quadratic number field [Sil86 Thm. VI.6.1(b)]. If characteristic of $K$ is $p > 0$, then the endomorphism ring of of $G_m$ is again $\mathbb{Z}$, but the endomorphisms of $G_a$ are instead isomorphic to the non-commutative ring of $p$-polynomials, denoted $K[F]$ [Hum75 Sec. 20.3]. Elements of $K[F]$ are written as polynomials in an indeterminate $F$, with coefficients in $K$, but with the multiplication rule defined by $Fa = a^p F$, if $a \in K$. In addition to the same possibilities as characteristic 0, the endomorphism ring of an elliptic curve in positive characteristic may be an order in a quaternion ring [Sil86 Cor. III.9.4].

**Lemma 8.** Let $n > 1$ be an integer. Then there exists a system of equations $\Phi_n$, satisfying the conditions in Lemma 7, whose variables include $y_i$ for $1 \leq i \leq n + 1$, and $w$, with the following properties:

1. For any solution in a division ring $Q$ to the system of equations $\Phi_n$, the variables satisfy $y_i = y_1^i$ for $2 \leq i \leq n + 1$ and $w = y_1^{n+1} + ny_1^{n-1} + (n - 1)y_1^{n-2}$ and the inequality $y_1^{n-1} + y_1^{n-2} \neq 0$.

2. For any field $K$, there exists a finite set $S \subset K$ of elements algebraic over the prime subfield of $K$, such that for any $t \in K \setminus S$, there exists a solution to $\Phi_n$ with $y_1 = t$.

**Proof.** We define the system $\Phi_n$ using variables denoted $x_0, x_1, y_1, \ldots, y_{n+1}, z_1, \ldots, z_{n-1}$, $w_1, \ldots, w_{2n-3}, w$, and satisfying the following equations:

$$
\begin{align*}
    x_0 &= 0 \\
    x_1 &= 1 \\
    y_2 &= y_1 \cdot y_1 \\
    y_3 &= y_2 \cdot y_1 \\
    &\vdots \\
    y_n &= y_{n-1} \cdot y_1 \\
    y_{n+1} &= y_n \cdot y_1 \\
    z_1 &= y_1 + x_1 \\
    z_2 &= z_1 \cdot y_1 \\
    &\vdots \\
    z_{n-1} &= z_{n-2} \cdot y_1 \\
    w_1 &= y_3 + y_1 \\
    w_2 &= w_1 + z_1 \\
    w_3 &= w_2 \cdot y_1 \\
    w_4 &= w_3 + z_2 \\
    w_5 &= w_4 \cdot y_1 \\
    &\vdots \\
    w_{2n-3} &= w_{2n-2} \cdot y_1 \\
    w &= w_{2n-3} + z_{n-1}
\end{align*}
$$

If we let $t$ denote the value of $y_1$, then we can recursively evaluate the variables in terms of $t$: 

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\[
\begin{align*}
x_0 &= 0 & y_1 &= t & z_1 &= t + 1 & w_1 &= t^3 + t \\
x_1 &= 1 & y_2 &= t^2 & z_2 &= t(t + 1) & w_2 &= t^3 + 2t + 1 \\
y_3 &= t^3 & z_3 &= t^2(t + 1) & w_3 &= t^4 + 2t^2 + t \\
\vdots & & \vdots & & \vdots \\
y_n &= t^n & z_{n-1} &= t^{n-2}(t + 1) & w_n &= t^5 + 3t^3 + 2t^2 \\
y_{n+1} &= t^{n+1}
\end{align*}
\]

This proves the first claim.

For the second claim, we need to show that there exists a finite set \( S \) in \( K \), such that any element \( t \) in \( K \setminus S \) provides a solution to \( \Phi_n \). To show that, let us consider the above solution to \( \Phi_n \) as polynomials in \( t \) using the variable assignments as above and let \( P \) be the set of those polynomials. Let \( S \) consist of all roots of equations of the form \( p - q = 0 \), for all distinct \( p, q \in P \). Now, we need to show that for all \( p, q \in P \), \( p - q \) are non-zero polynomials, independent of the characteristic, to know that \( S \) is finite. All the polynomials in \( P \) are monic, and the difference between monic polynomials of different degrees is non-zero, so it sufficient to show that \( p - q \neq 0 \) for polynomials \( p \) and \( q \) of the same degree.

The polynomials with degree 1 are \( y_1 \) and \( z_1 \). The difference between \( y_1 \) and \( z_1 \) is 1, so they are distinct. Similarly, degree 2 elements are \( y_2 \) and \( z_2 \), their difference is \( t \) so they are distinct. For \( 3 \leq i \leq n \), the elements with degree \( i \) are \( t^i, t^i + t^{i-1}, t^i + (i - 2)t^{i-2} + (i - 3)t^{i-3} \) and \( t^i + (i - 1)t^{i-2} + (i - 2)t^{i-3} \). The difference between these terms are either a monic polynomial, \( (i - 2)t^{i-2} + (i - 3)t^{i-3} \) or \( (i - 1)t^{i-2} + (i - 2)t^{i-3} \). The terms \( (i - 2)t^{i-2} + (i - 3)t^{i-3} \) and \( (i - 1)t^{i-2} + (i - 2)t^{i-3} \) are not zero because a prime cannot divide consecutive integers. So degree \( i \) elements are distinct for \( 3 \leq i \leq n \). The elements with degree \( n + 1 \) are \( y_{n+1}, w_{2n-3}, \) and \( w_{2n-2} \). The difference between these terms are either a monic polynomial, \( (n - 1)t^{n-1} + (n - 2)t^{n-2}, \) or \( nt^{n-1} + (n - 1)t^{n-2} \). These are not zero since because, again, a prime cannot divide consecutive integers. Thus, \( S \) is a finite set of elements, all algebraic over the prime subfield of \( F \) and for any \( t \) outside of \( S \), each of the variables in the solution to \( \Phi_n \) with \( y_1 = t \) will be distinct. 

We now use Lemma 8 together with additional equations in order to construct matroids with specified characteristic sets.

**Proposition 9.** Let \( C \) be a finite set of primes. Then there exists a matroid \( M \) such that \( \chi_L(M) = \chi_A(M) = C \).
Proof. Let \( n \) be the product of the primes in \( C \). We use the system \( \Phi_n \) from Lemma 8 and add the equation \( y_{n+1} = w + y_{n-2} \). Now, use Lemma 7 to construct a matroid \( M \). If \( \Phi_n \) has a solution in a division ring \( Q \), then by Lemma 8, the two sides of our added equation evaluate to \( y_{n+1} = y_1^{n+1} \) and \( w + y_{n-2} = y_1^{n+1} + ny_1^{n-1} + ny^{n-2} \), with \( y_1^{n-1} + y_1^{n-2} \) non-zero in \( Q \). Therefore, for the equation to hold, \( n \) must be 0, which means the characteristic of \( Q \) is contained in \( C \). In other words, \( \chi_L(M) \subseteq C \). Also, since the endomorphism ring of a 1-dimensional group can only have positive characteristic if the field of definition has the same characteristic, then \( \chi_A(M) \subseteq C \).

On the other hand, for any infinite field \( K \) whose characteristic is contained in \( C \), we can choose \( t \in K \) outside a finite set and have a solution to \( \Phi_n \) with \( y_i = t^i \). Furthermore, because \( n = 0 \) in \( K \), this will also be a solution with the additional equation, showing that \( \chi_L(M) \supset C \) and completing the proof of the proposition. \( \square \)

Lemma 10. Let \( C \) be the union of \( \{0\} \) and a cofinite set of primes. Then there exists a set of equations \( \Phi_C \), satisfying the set of constraints in Lemma 8 such that if \( \Phi_C \) has a solution over a division ring \( Q \), then the characteristic of \( Q \) is contained in \( C \). Conversely if \( K \) is any infinite field whose characteristic is contained in \( C \), then \( \Phi_C \) has a solution in \( K \).

Proof. Let \( n \) be the product of the finite set of primes not in \( C \) and consider \( \Phi_n \) from Lemma 8. We will construct a system of equations \( \Phi_C \), by adding a variable \( v \) and the equation \( v = w + y_{n-2} \) to \( \Phi_n \).

If \( Q \) is a division ring of characteristic not in \( C \), then by Lemma 8 for any solution in \( Q \), \( y_{n-2} = y_1^{n-2} \) and \( w = y_1^{n+1} - y_1^{n-2} \), where we’re using the fact that \( n = 0 \) in \( Q \). By the added equation, then, \( v = y_1^{n+1} \) and therefore \( v = y_{n+1} \), so the variables are distinct. We conclude that \( \Phi_C \) does not have a solution with distinct values over a division ring with characteristic not in \( C \).

Conversely, if \( K \) is an infinite field whose characteristic is in \( C \), then we can choose \( t \) outside of the finite set \( S \) from Lemma 8 to get a solution with \( y_i = t^i \) and \( v = t^{n+1} + nt^{n-1} + nt^{n-2} \). We only need to justify that \( v \) does not coincide with any of the variables used in \( \Phi_n \). First, as a polynomial in \( t \), it has a different degree than all except \( y_{n+1} \), \( w_{2n-3} \), and \( w_{2n-2} \). The differences between \( v \) and each of these are polynomials with leading coefficients \( n \), 1, and 1 respectively, so they are distinct elements of \( K \), because \( n \) is non-zero in \( K \). We conclude that \( \Phi_C \) has a solution in \( K \). \( \square \)

Proposition 11. Let \( C \) be the union of \( \{0\} \) and a cofinite set of primes. Then there exists a matroid \( M \) such that \( \chi_L(M) = C \) and \( \chi_A(M) = \{0\} \cup \mathbb{P} \).

Proof. Let \( \Phi_C \) be the system of equations from Lemma 10. Then, use Lemma 7 to construct a matroid \( M \). By these two lemmas, \( M \) is realizable over any infinite field of characteristic contained in \( C \) and not realizable over any field of characteristic not contained in \( C \). Therefore, \( \chi_L(M) = C \). In particular, \( M \) is realizable over \( Q \), which is the field of fractions of the endomorphism ring of \( G_m \), so \( M \) is algebraically realizable over any field. \( \square \)
Proposition 12. Let \( C \) be the union of \( \{0\} \) and a cofinite set of primes. Then there exists a matroid \( M \) with \( \chi_L(M) = \chi_A(M) = C \).

Proof. We start with the system \( \Phi_C \) as in Lemma 10, to which we add the variables \( u_1, u_2, \) and \( u_3 \) and the equations \( u_2 = u_1 \cdot u_1, u_3 = u_2 \cdot u_1, \) and \( 1 = x_1 = u_3 + u_1 \) to get \( \Phi \).

Let \( M \) be a matroid constructed from this system according to Lemma 7. Any solution to \( \Phi \) in a division ring of characteristic 0 must satisfy \( u_3^3 + u_1 - 1 = 0 \). This polynomial is irreducible in \( \mathbb{Q} \), so the value \( u_1 \) takes must be degree three over \( \mathbb{Q} \). However, the ring of endomorphisms of \( G_m \) or an elliptic curve is contained in either the rationals, a quadratic number field, or a quaternion algebra over \( \mathbb{Q} \), and all elements of these rings have degree at most 2 over \( \mathbb{Q} \). Therefore, any algebraic realization of \( M \) must come from the algebraic group \( G_a \), whose endomorphism ring has the same characteristic as the field of definition.

Then, by Lemma 10, the characteristic of any division ring having solutions to \( \Phi \), and thus to \( \Phi_C \) must be contained in \( C \), and thus \( \chi_A(M) \subset C \).

On the other hand, we want to show that \( \chi_L(M) \supset C \). Let \( K \) be a transcendental extension of an algebraically closed field whose characteristic is contained in \( C \). Let \( u_1 \) be any root of the polynomial \( u_3^3 + u_1 - 1 \) so long as \( u_1 \neq -1 \) (which is only possible in characteristic 3, and in characteristic 3 there are also other roots). Then, set \( u_2 = u_1^2 \), and \( u_3 = u_1^3 \), and we claim that 0, 1, \( u_1 \), \( u_2 \), and \( u_3 \) are distinct. We consider the possible equalities: First, if \( u_1, u_2, \) or \( u_3 \) is zero, then \( u_1 = 0 \), which is not possible because the polynomial has a non-zero constant term. Second, if \( u_1 = 1, u_2 = u_1, \) or \( u_3 = u_2 \), then that implies \( u_1 = 1 \), but the defining polynomial for \( u_1 \) evaluates to 1 \( \neq 0 \) in all characteristics at \( u_1 = 1 \). Third, if \( u_2 = 1 \) or \( u_3 = u_1 \), then that implies \( u_1 = \pm 1 \), and we’ve assumed that \( u_1 \neq -1 \) and shown that \( u_1 = 1 \) is not possible. Fourth, if \( u_3 = 1 \), then substituting \( u_3 = u_1^3 \) into the defining polynomial yields \( u_1 = 0 \), which is a contradiction.

Now choose \( t \) to be transcendental over the prime field of \( K \). Then it is not a root of \( u_1^3 + u_1 - 1 \) and not equal to \( u_2 \) or \( u_3 \), either. Furthermore, all of the variables defined in \( \Phi_C \) are polynomials of \( t \), and thus transcendental over the prime subfield of \( K \), and so distinct from the \( u_i \)'s. This shows that all variables are distinct in this solution, and therefore \( C \subset \chi_L(M) \), which completes the proof of the proposition. \( \Box \)

Proposition 13. Let \( C \) be a finite set of primes. Then there exists a matroid \( M \) with \( \chi_L(M) = C \) and \( \chi_A(M) = \chi_F(M_P) = \mathbb{P} \).

Proof. Let \( n \) be the product of the primes in \( C \). Consider the system of equations \( \Phi \) consist-
ing of $\Phi_n$ from Lemma 8 together with additional variables $u_1, \ldots, u_8$ and the equations:

- $u_3 = u_2 + x_1$
- $u_4 = u_1 \cdot u_3$
- $u_5 = u_2 \cdot u_1$
- $u_6 = u_5 + w$
- $u_7 = u_6 + y_{n-2}$
- $u_8 = u_4 + y_{n+1}$
- $u_8 = u_7 + u_1$

Let $M$ be the matroid defined from $\Phi$ by Lemma 7. If we have any solution to $\Phi$ in a division ring $Q$, then there exists $t \in Q$ such that $y_i = t^i$ and $w = t^{n+1} + nt^{n-2} + (n - 1)t^{n-2}$ by Lemma 8. If we let $a$ and $b$ be the values of $u_1$ and $u_2$, respectively. Then, the other variables satisfy:

- $u_3 = b + 1$
- $u_4 = ab + a$
- $u_5 = ba$
- $u_6 = ba + t^{n+1} + nt^{n-1} + (n - 1)t^{n-2}$
- $u_7 = ba + t^{n+1} + nt^{n-1} + nt^{n-2}$
- $u_8 = ab + a + t^{n+1}$
- $= ba + a + t^{n+1} + nt^{n-1} + nt^{n-2}$

If $Q$ is commutative, then $ab = ba$ and so the last equation implies that $nt^{n-1} + nt^{n-2} = 0$. Since $t^{n-1} + t^{n-2}$ is non-zero by Lemma 8 then $n = 0$, which means that the characteristic of a commutative field which has solutions to $\Phi$ must be contained in $C$. In particular, $\chi_L(M) \subset C$.

Conversely, let $K = \mathbb{F}_p(a, b, t)$, where $p \in C$ and consider the solution formed by setting $y_i = t^i$, $u_1 = a$, $u_2 = b$, and assigning the other variables as above. Then the variables $u_1, \ldots, u_8$ are distinct polynomials. Moreover, the variables $u_i$ are not contained in $\mathbb{F}_p(t)$, whereas all the variables used by the system $\Phi_n$ are contained in $\mathbb{F}_p(t)$, so these are also distinct. This shows that $\chi_L(M) = C$.

Finally, we want to show that $M$ is algebraically realizable over the field $\overline{\mathbb{F}_p}$ for any prime $p$. Since $M$ is linearly realizable when $p \in C$, it is sufficient to consider the case when $p \notin C$, so $n$ is non-zero. We construct an algebraic realization by finding a solution to $\Phi$ over the ring $\overline{\mathbb{F}_p}[F]$, which is the endomorphism ring of $G_a$. We first choose $\alpha \in \overline{\mathbb{F}_p} \setminus \mathbb{F}_p^{n-1} \setminus \mathbb{F}_p^{n-2}$. Thus, $\alpha^{p^{n-1}} - \alpha$ and $\alpha^{p^{n-2}} - \alpha$ are non-zero, so we set $\beta = (\alpha^{p^{n-1}} - \alpha)^{-1}$ and $\gamma = (\alpha^{p^{n-2}} - \alpha)^{-1}$. Then, let $y_1 = F$, $u_1 = \beta F^{n-1} + \gamma F^{n-2}$, $u_2 = n\alpha$, and the other
variables as:

\[
\begin{align*}
  u_3 &= n\alpha + 1 \\
  u_4 &= (n\beta\alpha^{p^n-1} + \beta)F^{n-1} + (n\gamma\alpha^{p^n-2} + \gamma)F^{n-2} \\
  &= (n\alpha\beta + n + \beta)F^{n-1} + (n\alpha\gamma + n + \gamma)F^{n-2} \\
  u_5 &= n\alpha\beta F^{n-1} + n\alpha\gamma F^{n-2} \\
  u_6 &= F^{n+1} + (n\alpha\beta + n)F^{n-1} + (n\alpha\gamma + n - 1)F^{n-2} \\
  u_7 &= F^{n+1} + (n\alpha\beta + n)F^{n-1} + (n\alpha\gamma + n)F^{n-2} \\
  u_8 &= F^{n+1} + (n\alpha\beta + n + \beta)F^{n-1} + (n\alpha\gamma + n + \gamma)F^{n-2}
\end{align*}
\]

All of these are distinct values in \( \mathbb{F}_p(F) \) and satisfy the equations in \( \Phi \). Moreover, they are distinct from the variables used in \( \Phi_n \), because those all lie in the subfield \( \mathbb{F}_p(F) \). We conclude that \( \chi_A(M) = \mathbb{P} \).

**Proof of Theorems 7 and 9.** We consider the different combinations of whether \( C_A \) and \( C_L \) are finite or cofinite. First, suppose that \( C_A \) is finite, which implies that \( C_L \subset C_A \) is also finite and that neither \( C_A \) nor \( C_L \) contains \( 0 \). By Proposition 13, there exists a matroid \( M_1 \) such that \( \chi_L(M_1) = C_L \) and \( \chi_A(M_1) = \mathbb{P} \). By Proposition 9, there exists another matroid \( M_2 \) such that \( \chi_L(M_2) = \chi_A(M_2) = C_A \). Since the characteristic set of a direct sum is the intersection of the characteristic sets, \( \chi_L(M_1 \oplus M_2) = C_L \) and \( \chi_A(M_1 \oplus M_2) = C_A \).

Second, suppose that \( C_A \) is cofinite, but \( C_L \) is finite, which again implies that \( 0 \) is not in \( C_L \) and \( C_A \). Then by Proposition 12, there exists a matroid \( M_1 \) such that \( \chi_L(M_1) = \chi_A(M_1) = C_A \cup \{0\} \). By Proposition 13, there exists a matroid \( M_2 \) such that \( \chi_L(M_2) = C_L \) and \( \chi_A(M_2) = \mathbb{P} \). Again, the characteristic sets of a direct sum are the intersections of the characteristic sets, so \( \chi_L(M_1 \oplus M_2) = C_L \) and \( \chi_A(M_1 \oplus M_2) = C_A \).

Third, suppose that \( C_A \) and \( C_L \) are both cofinite, which implies that \( 0 \in C_L \subset C_A \). Similarly, we use Proposition 12 to construct a matroid \( M_1 \) such that \( \chi_L(M_1) = \chi_A(M_1) = C_A \). Moreover, by Theorem 4 whose proof doesn’t use anything in this section, \( \chi_F(M_1) = \mathbb{P} \). By Proposition 11, there exists a matroid \( M_2 \) such that \( \chi_L(M_2) = C_L \) and \( \chi_A(M_2) = \mathbb{P} \cup \{0\} \), and consequently \( \chi_F(M_2) = \mathbb{P} \). Because characteristic sets of direct sums are the intersections of characteristic sets, \( \chi_L(M_1 \oplus M_2) = C_L \) and \( \chi_A(M_1 \oplus M_2) = C_A \). The same is true for Frobenius flock characteristic sets, by Theorems 4.11, 4.13, and 4.18 in [Bol18], so \( \chi_F(M_1 \oplus M_2) = \mathbb{P} \).

\section{Infinite algebraic characteristic sets}

The following proposition gives explicit examples of algebraic characteristic set which are neither finite nor cofinite. Our construction works similarly to Example 2 in [EH91].
Proposition 14. Let $n$ be a positive integer. Then there exists a matroid $M_n$ such that

$$\chi_A(M_n) = \{ p \in \mathbb{P} : p \not\equiv 1 \mod n \}.$$ 

Proof. Let $k$ be the least integer such that $m = kn$ is greater than 6. We define a new system of equations $\Phi_n$ satisfying the conditions in Lemma 7, in terms of the variables $x_0, x_1, y_1, \cdots, y_{m-1}, z_1, z_2, z_3$ by the following equations:

\[
\begin{align*}
x_0 &= 0 & y_2 &= y_1 \cdot y_1 & z_2 &= y_k \cdot z_1 \\
x_1 &= 1 & y_3 &= y_2 \cdot y_1 & z_3 &= z_1 \cdot y_k \\
&\quad \vdots \\
y_{m-1} &= y_{m-2} \cdot y_1 & y_{m-1} &= y_{m-2} \cdot y_1 \\
x_1 &= y_{m-1} \cdot y_1
\end{align*}
\]

Now, use Lemma 7 to construct a matroid $M_n$ from the equations $\Phi_n$. Any algebraic realization of $M_n$ will yield a solution to these equations in the division ring of the endomorphism ring of a 1-dimensional group over $K$. This solution must satisfy:

\[
\begin{align*}
x_0 &= 0 & y_2 &= y_1^2 & z_2 &= y_k z_1 \\
x_1 &= 1 \cdot y_1^m & y_3 &= y_1^3 & z_3 &= z_1 y_k \\
&\quad \vdots \\
y_{m-1} &= y_1^{m-1}
\end{align*}
\]

Because the $y_i$’s are distinct from 1 in this solution, $y_1$ must be a primitive $m$th root of unity. Also, in order for $z_2 = y_k z_1$ and $z_3 = z_1 y_k$ to be distinct, the division ring must be non-commutative, which implies that $0 \not\in \chi_A(M_n)$ and in positive characteristic, a solution must come from a non-commutative endomorphism ring of an elliptic curve or $G_m$.

In the former case, the endomorphism ring is contained in a quaternion algebra over $\mathbb{Q}$, all of whose elements have degree at most 2 over $\mathbb{Q}$. On the other hand, a primitive $m$th root of unity for $m > 6$ has degree at least 3 over $\mathbb{Q}$, so the elliptic curve case is not possible.

Therefore, an algebraic realization corresponds to a solution to $\Phi_n$ in the division ring $Q$ of the ring of $p$-polynomials. The element $y_k = y_1^k$ is a primitive $n$th root of unit. If $p \equiv 1 \mod n$, then the polynomial $t^n - 1$ splits in $\mathbb{F}_p$, and so $y_k$ is contained in $\mathbb{F}_p$. However, any element of $\mathbb{F}_p$ is in center of $Q$, which contradicts the inequality between $z_2 = y_k z_1$ and $z_3 = z_1 y_k$. Therefore, if $M_p$ is algebraically realizable over a field, the field must have positive characteristic $p \not\equiv 1 \mod n$.

Conversely, suppose that $p \not\equiv 1 \mod n$, and we will construct a solution to $\Phi_n$ in $\mathbb{F}_p[F]$. We choose $y_1$ to be a primitive $m$th root of unit in $\mathbb{F}_p$ and set $y_i = y_1^i$. In particular, $y_k$ is a primitive $n$th root of unity, which is not contained in $\mathbb{F}_p$ because $p \not\equiv 1 \mod n$. We set $z_1 = F$, so that $z_2 = y_k F$ and $z_3 = F y_k = y_k F$ are distinct because $y_k \not\in \mathbb{F}_p$. Thus, $M_p$ is algebraically realizable over $\mathbb{F}_p$. \qed
The proof of Theorem 2 uses the following elementary lemma from analysis, whose proof we include for the convenience of the reader.

**Lemma 15.** Let \((x_n)\) be a sequence of positive numbers such that \(x_n \to 0\) as \(n \to \infty\) but \(\sum_{n=1}^{\infty} x_n = \infty\). Then for any \(a, \delta > 0\), there exists a finite set of integers \(A\) such that \(a - \delta < \sum_{n \in A} x_n < a + \delta\).

**Proof.** Let \(N\) be such that \(x_n < 2\delta\) for all \(n \geq N\). Let \(M \geq N\) be the minimal index such that \(\sum_{n=N}^{M} x_n > a - \delta\), which exists since \(\sum_{n=1}^{\infty} x_n = \infty\). Then, by minimality, \(\sum_{n=N}^{M-1} x_n \leq a - \delta\), so

\[
\sum_{n=N}^{M} x_n = \sum_{n=N}^{M-1} x_n + x_M < (a - \delta) + 2\delta < a + \delta.
\]

Thus \(A = \{N, N + 1, \ldots, M\}\) is a set as in the lemma statement. \(\square\)

**Proof of Theorem 2.** Let \(q\) be a fixed prime and \(M_q\) be the matroid obtained from the Proposition [14] with \(\chi_A(M_q) = \{p \text{ prime} : p \not\equiv 1 \text{ mod } q\}\). By Dirichlet’s theorem on arithmetic progressions, the set of primes \(p\) such that \(p \equiv 1 \text{ mod } q\) has natural density \(1/(q - 1)\) and therefore, \(\chi_A(M_q)\) has natural density \((q - 2)/(q - 1)\).

More generally, for any finite set \(S\) of primes, the algebraic characteristic set of the direct sum \(\bigoplus_{q \in S} M_q\) is the set of primes \(p\) such that \(p \not\equiv 1 \text{ mod } q\) for all \(q \in S\). By the Chinese Remainder Theorem, there are \(\prod_{q \in S}(q - 2)\) non-zero congruence classes modulo \(\prod_{q \in S} q\) which satisfy these congruence inequalities for all \(q \in S\). Therefore, by Dirichlet’s theorem on arithmetic progression, the natural density of \(\chi_A(\bigoplus_{q \in S} M_q)\) is \(\prod_{q \in S}(q - 2)/(q - 1)\).

Now, we proceed to find a suitable set \(S\). We let \(q_n\) denote the \(n\)th prime, and set

\[
x_n = -\log \left( \frac{q_n - 2}{q_n - 1} \right) = -\log \left( 1 - \frac{1}{q_n - 1} \right) \geq \frac{1}{q_n - 1} \geq \frac{1}{q_n}.
\]

Then \(x_n \to 0\) since \(q_n \to \infty\), and since \(\sum_{n=1}^{\infty} 1/q_n\) diverges, so does \(\sum_{n=1}^{\infty} x_n\). Therefore, Lemma [15] with \(a = -\log \alpha\) and \(\delta = \log(\alpha + \epsilon) - \log(\alpha)\) gives us a finite set \(A\) such that

\[
\left| a - \sum_{n \in A} \log \frac{q_n - 2}{q_n - 1} \right| < \delta.
\]

Because \(\log\) is a concave function, \(\delta < \log(\alpha) - \log(\alpha - \epsilon)\), which implies

\[
\left| \alpha - \prod_{n \in A} \frac{q_n - 2}{q_n - 1} \right| < \epsilon.
\]

Then, consider \(M = \bigoplus_{n \in A} M_{q_n}\), and we have shown that the density of \(\chi_A(M)\) is \(\prod_{n \in A}(q_n - 2)/(q_n - 1)\), and so \(|d(\chi_A(M)) - \alpha| < \epsilon.\) \(\square\)
4 Stretching Frobenius flocks

In this section, we prove Theorem 4 establishing the existence of Frobenius flocks for any matroid which is linear over a field of characteristic 0 and Theorem 5 proving that the Frobenius flock representability of a matroid is closed under duality. Both results use the same technical tool, which is a way of stretching linear flocks, which are a more general object than Frobenius flocks.

For the definition of the linear flock, we need notations and definitions of deletion and contraction for vector spaces. Let $E$ be a finite set and $K$ a field. For $v \in K^E$, and $I \subseteq E$, define $v_I \in K^I$ be the restriction of $v$ to the coordinates indexed by $I$ and for a linear subspace $V \subseteq K^E$ and $I \subseteq E$ define deletion and contraction to be

$$V \setminus I = \{ v_{E \setminus I} \mid v \in V \} \quad \text{and} \quad V/I = \{ v_{E \setminus I} \mid v \in V, v_I = 0 \},$$

respectively, both of which are subspaces of $K^{E-I}$. Since $V \setminus I$ is the projection of $V$ to $K^I$, and $V/(E-I)$ is the kernel of that projection, the rank-nullity theorem implies that $\dim V \setminus I + \dim V/(E-I) = \dim V$. It is also easy to see that when applied to disjoint sets, deletion, and contraction commute with each other, and also that multiple deletions or contractions can be combined.

Each vector space $V \subseteq K^E$ defines a matroid whose bases are the sets $B$ such that $V \setminus (E-B) = K^B$. We denote it by $M(V)$. The deletion and contraction of vector spaces are closely related to deletion and contraction of matroids. For instance, for any $I \subseteq E$, $M(V/I) = M/I$ and $M(V \setminus I) = M \setminus I$.

Now suppose that $\phi$ is an automorphism of $K$. Then for any $v \in K^E$ we can define an action of $\phi$ coordinate-wise:

$$\phi v = (\phi(v_i))_{i \in E}$$

and for a vector space $V \subseteq K^E$, we have $\phi V = \{ \phi v \mid v \in V \}$, which is also a vector space.

Following [Bol18, Def. 4.1], a $\phi$-linear flock of $E$ over $K$ is defined to be a map $V_\bullet: \alpha \mapsto V_\alpha$ which assigns a $d$-dimensional linear subspace $V_\alpha \subseteq K^E$ to each $\alpha \in \mathbb{Z}^E$, such that:

(LF1) $V_\alpha/i = V_{\alpha+e_i} \setminus i$ for all $\alpha \in \mathbb{Z}^E$ and $i \in E$; and

(LF2) $V_{\alpha+1} = \phi V_\alpha$ for all $\alpha \in \mathbb{Z}^E$.

Here $e_i$ is the $i$th unit vector in $\mathbb{Z}^n$ and $1 \in \mathbb{Z}^n$ is the vector whose entries are all 1. If $\phi = F^{-1}$, where $F: x \rightarrow x^p$ is the Frobenius map, then we call a $F^{-1}$-linear flock a Frobenius flock [BDP18, Sec. 4].

For each $\alpha \in \mathbb{Z}^n$, the vector space $V_\alpha$ defines a matroid $M(V_\alpha)$ whose bases are the $d$-element sets $B$ such that $V \setminus (E-B) = K^B$. The union of these sets of bases for all $\alpha \in \mathbb{Z}^n$, is also a matroid, which we call the support matroid of $V_\alpha$ [BDP18, Lem. 17]. Let $M$ be a matroid. If there exists a Frobenius flock $V_\bullet$ with support matroid $M$, then $V_\bullet$ is a Frobenius flock representation of $M$.

We now establish a lemma allowing us to stretch Frobenius flock representations:
Lemma 16. Let $V_\bullet$ be a $\phi$-linear flock over a field $K$. Suppose that $\psi$ is an automorphism of $K$ such that $\psi^m = \phi$. Then, there exists a $\psi$-linear flock $V'_\bullet$ where $V'_{\alpha m} = V_\alpha$ for all $\alpha \in \mathbb{Z}^n$, and whose support matroid is the same as the support matroid of $V_\bullet$.

Proof. Let $\beta \in \mathbb{Z}^n$, and write $\beta = m\alpha + (r_1, \ldots, r_n)$ where $0 \leq r_i < m$ and $\alpha \in \mathbb{Z}^n$. For any $0 \leq k < m$, we define the sets $I_{\leq k} = \{i : r_i < k\}$, $I_{> k} = \{i : r_i > k\}$ and $I_k = \{i : r_i = k\}$.

Now let us define the $K$-vector space

$$V'_\beta = \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{> k} \setminus I_{\leq k},$$

and we claim that as $\beta$ ranges over all elements of $\mathbb{Z}^n$, $V'_\bullet$ defines a $\psi$-linear flock. Note that a term $\phi^k V_\alpha / I_{> k} \setminus I_{\leq k}$ in the definition of $V'_\beta$ is a subspace of $K^{I_k}$ and so the direct sum gives a vector subspace of $K^E$ via the isomorphism $K^E \cong \bigoplus_{k=0}^{m-1} K^{I_k}$. Also, if $\beta = m\alpha$, meaning that $r_i = 0$ for all $i$, then only the $k = 0$ summand of the definition of $V'_\beta$ is non-trivial, and this shows that $V'_{m\alpha} = V_\alpha$.

As noted above, the rank-nullity theorem implies that

$$d = \dim V_\alpha = \dim V / I_{> 0} + \dim V \setminus I_0.$$

By induction, and because the sets $I_k$ partition $E$, $d = \sum_{k=0}^{m-1} \dim V_\alpha / I_{> k} \setminus I_{\leq k}$, which implies that $\dim V'_\beta = d$.

We check the axiom (LF2) of a linear flock first. Consider

$$\beta + 1 = m\alpha' + (r'_1, \ldots, r'_n)$$

and if we define $\alpha' = \alpha + e_{I_{m-1}}$. $I_j' = \{i : r'_i = j\} = I_j - 1$ for $1 \leq j \leq m - 1$ and $I_0' = I_{m-1}$, then similarly to the decomposition $\beta' = m\alpha' + (r'_1, \ldots, r'_n)$, where $r_i = j$ if and only if $i \in I_j'$. In addition, we also define $I'_{< k} = \bigcup_{j<k} I_j'$ and $I'_{> k} = \bigcup_{j>k} I_j'$, which means that $I'_{< k} = I_{< k-1} \cup I_{m-1}$ and $I'_{> k} = I_{> k-1} - I_{m-1}$, where $-$ denotes the set difference, to distinguish it from matroid deletion.

For $I \subseteq E$, the following generalization holds in analogy with Lemma 9 of [BDP18],

(LF1') \quad $V_{\alpha} / I = V_{\alpha + e_I} \setminus I$ for all $\alpha \in \mathbb{Z}^n$ and $I \subseteq \{1, 2, \ldots, n\}$ where $e_I = \sum_{i \in I} e_i$. 

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Then, we have

\[ V'_{\beta+1} = \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/I_{>k} \setminus I_{<k} \]

by definition of \( V' \).

\[ = \left( \bigoplus_{k=1}^{m-1} \psi^k V_{\alpha+e_{I_{m-1}}} / (I_{>k-1} - I_{m-1}) \setminus (I_{<k-1} \cup I_{m-1}) \right) \oplus V_{\alpha+e_{I_{m-1}}} / I_{<m-1} \]

by the above identities

\[ = \left( \bigoplus_{k=1}^{m-1} \psi^k V_{\alpha}/I_{>k-1} \setminus I_{<k-1} \right) \oplus V_{\alpha+1} \setminus I_{<m-1} \]

by (LF1)

\[ = \left( \bigoplus_{k=1}^{m-1} \psi^k V_{\alpha}/I_{>k-1} \setminus I_{<k-1} \right) \oplus \phi V_{\alpha} \setminus I_{<m-1} \]

by (LF2)

\[ = \left( \bigoplus_{k=1}^{m-1} \psi^k V_{\alpha}/I_{>k-1} \setminus I_{<k-1} \right) \oplus \psi \cdot \psi^{m-1} V_{\alpha} \setminus I_{<m-1} \]

because \( \phi = \psi^m \)

This completes the proof of (LF2).

Now we consider the axiom (LF1), which says that \( V_\beta/i = V_{\beta+e_i} \setminus i \). We first consider the case when \( i \notin I_{m-1} \) and let \( j = r_i \), so that \( i \in I_j \). Therefore, the vector \( \beta + e_i \) can be written as \( m \alpha + (r_1', \ldots, r_n') \), where \( r_1', \ldots, r_n' < m \) and \( r_k' = r_k \) unless \( k = i \) in which case \( r_i' = r_i + 1 \). Then, if \( I'_{<k} = \{ i : r_i' < k \} \) and \( I'_{>k} = \{ i : r_i' > k \} \), as usual, then \( I'_{<j+1} = I_{<j+1} \setminus \{ i \} \) and \( I'_{>j} = I_{>j} \cup \{ i \} \), but other than these two exceptions, \( I'_{<k} = I_{<k} \) and \( I'_{>k} = I_{>k} \). Therefore, the definition of \( V' \) gives us:

\[ V'_{\beta+e_i} = \left( \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/I_{>k} \setminus I_{<k} \right) \oplus (\psi^j V_{\alpha}/(I_{>j} \cup \{ i \}) \setminus I_{<j}) \]

\[ \oplus (\psi^{j+1} V_{\alpha}/I_{>j+1} \setminus (I_{<j+1} - \{ i \})) \]

The deletion of the \( i \)th component only affects the summand contained in \( K^{E_j} \), which
is the last summand, so by combining the deletions:

\[ V'_{\beta+e_i} \setminus i = \left( \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha+e_i} / (I_{>k} \setminus I_{<k}) \right) \oplus \left( \psi^j V_{\alpha} / (I_{>j} \cup \{i\}) \setminus I_{<j} \right) \]

\[ \oplus \left( \psi^{j+1} V_{\alpha} / (I_{>j+1} \setminus I_{<j+1}) \right) \]

\[ = \left( \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha} / (I_{>k} \setminus I_{<k}) \right) \oplus \left( \psi_j V_{\alpha} / (I_{>j} \cup \{i\}) \setminus I_{<j} \right) \]

\[ = \left( \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha} / (I_{>k} \setminus I_{<k}) \right) / \{i\} = V'_\beta / \{i\}, \]

because the contraction of \{i\} only affects the \( k = j \) summand. This completes the proof of (LF1) when \( i \not\in I_{m-1} \).

Now suppose that \( i \in I_{m-1} \). In this case \( \beta + e_i = m\alpha' + (r'_1, \ldots, r'_n) \) where \( \alpha' = \alpha + e_i \), \( I'_k = \{i : r'_i = k\} = I_k \) for \( k \neq 0, m-1 \), \( I'_{m-1} = I_{m-1} \setminus \{i\} \) and \( I'_0 = I_0 \cup \{i\} \). Then,

\[ V'_{\beta+e_i} = \left( \bigoplus_{k=1}^{m-2} \psi^k V_{\alpha+e_i} / (I_{>k} - \{i\}) \setminus (I_{<k} \cup \{i\}) \right) \oplus (V_{\alpha+e_i} / (I_{>0} - \{i\})) \]

\[ \oplus \left( \psi^{m-1} V_{\alpha+e_i} / (I_{<m-1} \cup \{i\}) \right) \]

\[ V'_{\beta+e_i} \setminus i = \left( \bigoplus_{k=1}^{m-2} \psi^k V_{\alpha+e_i} / (I_{>k} \setminus \{i\}) \setminus (I_{<k} \cup \{i\}) \right) \oplus (V_{\alpha+e_i} / (I_{>0} - \{i\}) \setminus \{i\})) \]

\[ \oplus \left( \psi^{m-1} V_{\alpha+e_i} / (I_{<m-1} \cup \{i\}) \right) \]

\[ V'_{\beta+e_i} \setminus i = \left( \bigoplus_{k=1}^{m-2} \psi^k V_{\alpha} / (I_{>k} \setminus I_{<k}) \right) \oplus (V_{\alpha} / I_{>0}) \oplus \left( \psi^{m-1} V_{\alpha} \setminus I_{<m-1} / \{i\} \right) \]

(by (LF1) in each summand)

\[ = \left( \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha} / (I_{>k} \setminus I_{<k}) \right) / \{i\} \]

\[ = V'_\beta / \{i\}, \]

which completes the proof of (LF1) and thus that \( V'_\beta \) is a matroid flock.

Finally, we want to show that the support matroids of \( V_\bullet \) and \( V'_\bullet \) are the same. Since \( V_{m\alpha} = V_\alpha \), any basis of the support matroid of \( V_\bullet \) will also be a basis of the support matroid of \( V'_\bullet \). For the converse, we suppose that \( \beta \) is any coordinate in \( \mathbb{Z}^n \) and \( B \) is any subset
of $E$. Then, with $\alpha$, $I_{>k}$, and $I_{<k}$ as before,
\[
V'_\beta = \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/I_{>k} \setminus I_{<k}
\]
and
\[
V'_\beta/(E - B) = \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/I_{>k}/(I_k - B) \setminus I_{<k}.
\]

The deletion of a vector space always contains the contraction of the same set, and thus,
\[
V'_\beta/(E - B) \subset \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/I_{>k}/(I_k - B)/(I_{<k} \cap B)
\]
\[
= \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha}/(E - B)/(I_{>k} \cap B) \setminus (I_{<k} \cap B).
\]

However, this last expression is the same construction that was used to make $V'_\beta$, but applied to $V_{\alpha}/(E - B)$. Therefore, its vector space dimension is the same as that of $V_{\alpha}/(E - B)$, which, by the containment, implies that $\dim V'_\beta/(E - B) \leq \dim V_{\alpha}/(E - B)$. If $B$ is a basis of the support matroid of $V_\bullet$, then $\dim V'_\beta = |B|$, which means that $B$ is also a basis of the support matroid of $V_\bullet$. This concludes the proof that $V_\bullet$ and $V_\bullet'$ have the same support matroids.

Using Lemma 16, we now prove Theorem 4 and Theorem 5.

**Proof of Theorem 4.** By [Ing71], if $0 \in \chi_L(M)$, then $M$ has a representation over a finite extension of the rationals, i.e. a number field $K$. Let $O_K$ be the ring of integers in the number field $K$. Then, using going-up theorem [Mar18, Thm. 20], for any prime $p$, there exists a prime ideal $\mathfrak{p} \subset O_K$ such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. By [Mar18, Thm. 14], $O_K$ is a Dedekind domain and if $J$ is any non-zero ideal in $O_K$, then $O_K/J$ is finite. So $\mathfrak{p}$ is a maximal ideal and $O_K/\mathfrak{p}$ is a finite field. The containment of $\mathbb{Z}$ in $O_K$ induces a ring-homomorphism $\mathbb{Z} \to O_K/\mathfrak{p}$, and the kernel is $\mathfrak{p} \cap \mathbb{Z} = (p)$. So, we obtain an embedding $\mathbb{F}_p \to O_K/\mathfrak{p}$. Then $O_K/\mathfrak{p}$ is an extension of finite degree over $\mathbb{F}_p$. Thus, $O_K/\mathfrak{p} \cong \mathbb{F}_{p^n}$ for some $n$. Also, any localization of a Dedekind domain at a non-zero prime ideal is a discrete valuation ring [DF99 Thm. 15, Ch. 16]. So, there exists a valuation $\nu : K^* \to \mathbb{Z}$ whose valuation ring has residue field isomorphic to $\mathbb{F}_{p^n}$. Using this valuation with [BCL20, Lem. 3.5], we can construct a linear flock with trivial automorphism over a finite field $\mathbb{F}_{p^n}$.

Now consider the inverse Frobenius automorphism $F^{-1} : x \mapsto x^{-p}$ of $\mathbb{F}_{p^n}$, whose iteration $F^{-m}$ is the trivial automorphism. Then, using Lemma 16 with $\psi = F^{-1}$, we have $M$ has a Frobenius flock representation over a field of characteristic $p$. Therefore, $\chi_F(M) = \mathfrak{p}$. 

\[\square\]
**Proof of Theorem** Let \( V_\bullet : \alpha \mapsto V_\alpha \) be a Frobenius flock representation over \( K \) of \( M \). Then the dual of \( V_\bullet \) is defined as \( V^*_\bullet : \alpha \mapsto V^\perp_\alpha \) [Bol18, Def. 4.15], which form a \( F \)-linear flock over \( K \) with support matroid \( M^* \) [Bol18, Thm. 4.16]. By [Bol18, Thm. 4.30], the flock \( V^*_\bullet \) is determined by its skeleton, which is a finite number of matroid realizations, together with a finite number of compatibility conditions, all of which are algebraic. Therefore, we can assume that the skeleton is defined over a finite extension of \( \mathbb{F}_p \), and thus that \( V^*_\bullet \) is a \( F \)-linear flock over a finite extension of \( \mathbb{F}_p \).

Since the Frobenius endomorphism \( F \) has finite order in a finite extension of \( \mathbb{F}_p \), then \( F^{-m} = F \) for some \( m \). Then, using Lemma [16] we get a \( F^{-1} \)-linear flock with the support matroid \( M^* \). Therefore, \( M^* \) has a Frobenius flock representation over \( K \) which implies \( \chi_F(M) \subseteq \chi_F(M^*) \). Furthermore, since \((M^*)^* = M\), we have the inclusion in the other direction, and so \( \chi_F(M^*) = \chi_F(M) \).

\[ \square \]

5 Finite Frobenius flock characteristic sets

In this section, we give an examples of matroids with finite, non-singleton Frobenius flock characteristic set, based on the construction in [Gor88].

**Definition 17.** Consider a set of primes \( \{p_1, p_2, \ldots, p_k\} \) and let \( n = p_1 \cdots p_k + 1 \) and \( s = \lceil \log_2 n \rceil \). For \( 0 \leq i \leq s \), set \( b_i = \lfloor n/2^{(s-i+1)} \rfloor \). Then \( b_0 = 0, b_1 = 1, b_2 = 2 \) or \( 3 \) and in general, \( b_i = 2b_{i-1} \) or \( 2b_{i-1} + 1 \). The Brylawski matrix \( N_n \), introduced in a slightly different form in [Bry82], is the \( 3 \times (2s + 6) \) matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & b_i & b_i & b_s & b_s \\
\end{pmatrix}
\]

We call the set of primes \( \{p_1, p_2, \ldots, p_k\} \) a Gordon-Brylawski set, if for each pair of indices \( 0 \leq i < j \leq s \) such that \((i, j) \not\in \{(0, 1), (1, 2)\}\), and each prime \( p_i \), the difference \( b_j - b_i \) is not 0 or \( \pm 1 \) modulo \( p_i \) [Gor88].

Note that \((0, 1)\) and possibly \((1, 2)\) are the only pairs of indices \((i, j)\) such that \( b_j - b_i \) can equal 1. Since these differences will therefore be 1 uniformly modulo all primes \( p_i \), this does not cause a problem for the construction.

**Example 18.** A computation shows that the 80 consecutive primes beginning with 12811987 form a Gordon-Brylawski set.

The following proposition proves Theorem [6]

**Proposition 19.** Let \( N_n \) be the Brylawski matrix where \( n = p_1 \cdots p_k + 1 \) and \( M_n \) be the matroid which is linearly represented over \( \mathbb{F}_{p_1} \) by the matrix \( N_n \). Then, \( \chi_F(M_n) \subseteq \{p_1, \ldots, p_k\} \). If \( \{p_1, \ldots, p_k\} \) is a Gordon-Brylawski set, then \( \chi_F(M_n) = \{p_1, \ldots, p_k\} \).
Proof. Assume that $M_n$ has a Frobenius flock representation over a field $K$ of characteristic $p$. Let $A$ be the restriction of $M_n$ to its first 4 elements, corresponding to first for columns of $N_n$. Then $A$ is isomorphic to $U_{3,4}$, which is rigid [BDPT8 Lem. 53], which means that any valuations on the matroid $U_{3,4}$ is projectively equivalent to the valuation which is constant 0. Then by [BDPT8 Lem. 46], there exist $\alpha \in \mathbb{Z}^K$ such that $M_\alpha = M(V_\alpha)$ contains the elements of $A$ as a circuit. Now we want to show that $V_\alpha$ equals the row space of the Brylawski matrix $N_n$ over $\mathbb{F}_p$. Let $B$ be the matrix representing $V_\alpha$ and $v'_1, \ldots, v'_6, w'_1, u'_1, \ldots, w'_s, u'_s$ denote the columns of $B$, analogous to the labeling of the columns of the Brylawski matrix $N_n$ in Definition 17.

As the first four elements form a circuit, we may use row operations on the matrix $B$ in such a way that the columns corresponding to this circuit are as in the matrix $N_n$. Since $\{v_1, v_2, v_3\}$ is a circuit, the third entry in $v'_3$ is 0 and since $\{v_3, v_4, v_5\}$ is a circuit, then the two non-zero entries of $v'_5$ are the same. Then by the column scaling we get that the $v'_3$ is the fifth column of the matrix $N_n$. We can similarly conclude that, after scaling the columns $v'_3$ and $v'_4$ are the same as the corresponding columns of $N_n$.

The fact that $\{v_2, v_6, w_1\}$ is a circuit forces the first and last entries of $w_1$ to be the same, and the circuit $\{v_5, u_1, w_1\}$ is a circuit means the middle entry is the sum of the other two. Therefore, after scaling, we can assume that $w'_1$ is the same as $w_1$. We now use induction to show that for $i \geq 2$, $w'_i$ and $u'_i$ are the same as $w_i$ and $u_i$, after scaling. Here, the circuit $\{v_3, w_1, w_1\}$ forces the first two entries of $w_1$ to be 1 and 2 respectively, after scaling. Then, the one of the minors

\[
\begin{vmatrix}
 v_1 & u_{i-1} & w_i \\
 1 & 0 & 1 \\
 0 & 1 & 2 \\
 0 & b_{i-1} & b_i
\end{vmatrix} = b_i - 2b_{i-1} \quad \text{or} \quad
\begin{vmatrix}
 v_6 & u_{i-1} & w_i \\
 1 & 0 & 1 \\
 0 & 1 & 2 \\
 1 & b_{i-1} & b_i
\end{vmatrix} = b_i - 2b_{i-1} - 1
\]

is zero, depending on whether $b_i = 2b_{i-1}$ or $b_i = 2b_{i-1} + 1$ and then this forces the last entry of $w'_i$ to be $b_i$. Finally, $u'_i$ is forced to agree with $u_i$, after scaling, because of the circuits $\{v_5, u_1, w_1\}$ and $\{v_2, v_3, u_i\}$. Therefore, we conclude that $V_\alpha$ is represented by the Brylawski matrix $N_n$. We also have the minor

\[
\begin{vmatrix}
 v_1 & w_1 & u_s \\
 1 & 1 & 0 \\
 0 & 2 & 1 \\
 0 & 1 & b_s
\end{vmatrix} = 2b_n - 1 = n - 1 = p_1 \ldots p_k
\]

which corresponds to a circuit of $M_\alpha$, and therefore forces the characteristic $p$ to be one of $p_1, \ldots, p_k$.

We have shown that $p = p_i$ for some $i$, so $\chi_F(M_n) \subseteq \{p_1, \ldots, p_k\}$. Since the set of primes $\{p_1, p_2, \ldots, p_k\}$ is a Gordon-Brylawski set, then by [Gor88 Thm. 5], we have $\chi_L(M_n) = \{p_1, \ldots, p_k\}$, therefore $\chi_F(M_n) = \{p_1, \ldots, p_k\}$. \qed
References

[BB19] Matthew Baker and Nathan Bowler, *Matroids over partial hyperstructures*, Adv. Math. 343 (2019), 821–863.

[BCD20] Guus P. Bollen, Dustin Cartwright, and Jan Draisma, *Matroids over one-dimensional groups*, Int. Math. Res. Not. IMRN (2020), rnaa175.

[BDP18] Guus P. Bollen, Jan Draisma, and Rudi Pendavingh, *Algebraic matroids and Frobenius flocks*, Adv. Math. 323 (2018), 688–719.

[BK80] Tom Brylawski and D. Kelly, *Matroids and combinatorial geometries*, Carolina Lecture Series, University of North Carolina, Department of Mathematics, Chapel Hill, NC, 1980.

[Bol18] Guus Pieter Bollen, *Frobenius flocks and algebraicity of matroids*, Ph.D. thesis, Eindhoven University of Technology, 2018.

[Bry82] Tom Brylawski, *Finite prime-field characteristic sets for planar configurations*, Linear Algebra Appl. 46 (1982), 155–176.

[DF99] David Dummit and Richard M. Foote, *Abstract Algebra*, Prentice Hall, Upper Saddle River, N.J, 1999.

[EH91] David M. Evans and Ehud Hrushovski, *Projective planes in algebraically closed fields*, Proc. Lond. Math. Soc. s3-62 (1991), no. 1, 1–24.

[Gor88] Gary Gordon, *Algebraic characteristic sets of matroids*, J. Combin. Theory Ser. B 44 (1988), no. 1, 64–74.

[Hum75] James E. Humphreys, *Linear Algebraic Groups*, Springer, New York, NY, 1975.

[Ing71] Aubrey W. Ingleton, *Representation of matroids*, Combinatorial mathematics and its applications, vol. 23, London, 1971, pp. 149–167.

[Kah82] Jeff Kahn, *Characteristic sets of matroids*, J. Lond. Math. Soc. s2-26 (1982), no. 2, 207–217.

[Lin85] Bernt Lindström, *On the algebraic characteristic set for a class of matroids*, Proc. Amer. Math. Soc. 95 (1985), no. 1, 147.

[Lin86] Bernt Lindström, *A non-linear algebraic matroid with infinite characteristic set*, Discrete Math. 59 (1986), no. 3, 319–320.

[Mar18] Daniel Marcus, *Number Fields*, Springer, New York, 2018.
[Pen] Rudi Pendavingh, *Field extensions, derivations, and matroids over skew hyperfields*, preprint, available at https://arxiv.org/abs/1802.02447.

[Rad57] R. Rado, *Note on independence functions*, Proc. Lond. Math. Soc. s3-7 (1957), no. 1, 300–320.

[Rei] R. Reid, *Obstructions to representations of combinatorial geometries*, (unpublished, results appear as appendix to Chapter 24 of [BK80]).

[Sil86] Joseph H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, New York, NY, 1986.

[Su23] Ting Su, *Matroids over skew tracts*, European J. Combin. 109 (2023), 103643.

[Vam75] P. Vamos, *A necessary and sufficient condition for a matroid to be linear*, Möbius Algebras (Proc. Conf. Univ. Waterloo, 1971) (1975), 166–173.

[Whi35] Hassler Whitney, *On the abstract properties of linear dependence*, Amer. J. Math. 57 (1935), no. 3, 509.