Motifs in Derived Algebraic Geometry

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Abstract

We formalize the concept of sheaves of sets on a model site by considering variables thereof, or motifs, and we construct functorially defined derived algebraic stacks from them, thereby eliminating the necessity to choose derived extensions as explained in [TV4].

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1 Introduction

It has been observed in the past ([D1], [D2], [TV1]) that one can formalize Algebraic Geometry by writing it in purely categorical terms by simply starting with some symmetric monoidal base category $\mathcal{C}$ and by considering its category $\text{Comm}(\mathcal{C})$ of commutative and unital monoids and letting $\text{Aff} = (\text{Comm}(\mathcal{C}))^{\text{op}}$ be the category of affine schemes over $\mathcal{C}$. On $\text{Aff}$ one then puts some topology $\tau$, be it the Zariski, etale, fffqc or any topology that one so wishes to then develop a notion of stacks and higher stacks on $(\text{Aff}, \tau)$. One would obtain in this manner what is referred to as Relative Algebraic Geometry, classical Algebraic Geometry corresponding to the case of having $\mathcal{C} = (\mathbb{Z} - \text{Mod}, \otimes)$. It is in an attempt to develop Relative Algebraic Geometry over symmetric monoidal $\infty$-categories that Toen observed one could use the fact that model categories give rise to $\infty$-categories via the Dwyer-Kan simplicial localization technique, to start with a symmetric monoidal model category $(\mathcal{C}, \otimes)$ in the sense of [Ho], thereby introducing Homotopical Algebraic Geometry ([T5], [TV1], [TV3], [TV6]), or Algebraic Geometry over model categories. In particular if for some fixed commutative ring $k$ one considers $\mathcal{C} = \text{sk-Mod}$, the category of simplicial objects in $k-\text{Mod}$, one obtains Derived Algebraic Geometry ([TV6]).

The need to introduce stacks, higher stacks and derived stacks can be seen from a classification problem perspective. As recounted in [TV7], one can start with a contravariant functor $F$ from a category $\mathcal{C}$ of geometric objects to $\text{Set}$, where for $X$ a geometric object, $F(X)$ classifies families of objects parametrized by $X$, and $F$ being valued in $\text{Set}$ this classification is really done up to equality. One may relax that condition and ask that classification be done up to isomorphism as well, whence the introduction of contravariant functors into $\text{Gpd}$, which would correspond to considering 1-stacks. Another motivation for making such a generalization as pointed out in [TV4] is that such set-valued moduli functors $F$ may not be representable and only admit a coarse moduli space, or not of the expected dimension, so following [K], a natural approach amounts to seeing such spaces as truncations of higher spaces, or derived spaces, smooth, as opposed to being singular, and of the expected dimension. This would correspond to seeking natural extensions of $F$ to functors $F_1 : \mathcal{C}^{\text{op}} \to \text{Gpd}$ that make the following diagram commutative, where we take $\mathcal{C} = k-C\text{Alg} = \text{Comm}(k-\text{Mod})$, $k$ a commutative ring, since
that will be our main point of interest:

\[
\begin{array}{ccc}
k\text{-Alg} & \xrightarrow{F} & \text{Set} \\
\downarrow^{F_1} & & \\
\text{Gpd} & \xleftarrow{\pi_0} & \\
\end{array}
\]

Next one may want to further relax the classification scheme by also allowing classification up to equivalence. At an elementary level this would mean having a \( \text{Cat} \)-valued functor, but morphisms in \( \text{Cat} \) would have to be inverted, and this is possible only if we have functors valued in \( \infty \)-categories, hence we consider \( \infty \)-stacks, or stacks for short. This would correspond to looking at extensions \( F_\infty \) of \( F_1 \) and \( F \) that make the following diagram commutative:

\[
\begin{array}{ccc}
k\text{-Alg} & \xrightarrow{F} & \text{Set} \\
\downarrow^{F_1} & & \\
\text{Gpd} & \xleftarrow{\pi_0} & \\
\downarrow^{F_\infty} & & \\
\text{sSet} & \xleftarrow{\Pi_1} & \\
\end{array}
\]

Finally, if one considers obstruction theory, as pointed out in [17] Derived Algebraic Geometry is a natural formalism in which such a theory can be written out, and this would correspond to finding derived extensions \( \mathbb{R}F \) of
$F$, $F_1$ and $F_\infty$ that make the following diagram commutative:

Now as discussed in [TV4], not all such extensions will work, there are constraints that have to be met in addition to having the above diagram commutative, for instance having the right derived tangent stack, something that would be known at the onset. Moreover there is no canonical choice of a derived extension. This is our main motivation for introducing a formalism where one would not have to worry about having to pick a derived extension. In addition we would like to functorially construct a derived extension, as exposed in the section that follows.

2 Construction

We reproduce (1) above as it is presented in [T7]:

where for $X \in sSet$, $\Pi_1(X) = G\gamma(X)$, $\gamma(X)$ the path category of $X$, and $G : \text{Cat} \to \text{Gpd}$, $A \mapsto GA = A[\Sigma^{-1}]$, $\Sigma = A_1 (\mathcal{J}, \mathcal{GJ})$. We have
\[ \text{Ob}(\Pi_1(X)) = X_0, \text{ morphisms in } \Pi_1(X) = A_1 \text{ and formal inverses.} \]

We also have \( \pi_0 : \text{Cat} \to \text{Set} \) is the connected component functor on categories (\([\text{McL}]\)).

This diagram is made commutative by selecting derived extensions to \( \text{sk-CAlg} \) of sheaves, 1-stacks and stacks, which depends very much on the context, so one may inquire whether working with motifs, or variables, instead of specializations thereof, would provide something that is less of an ad hoc construction. This is motivated in particular by the fact that mentioning “sheaves” and “derived stacks” in the above diagram can be made precise using the formalism of motifs as defined presently. We briefly remind the reader of the definition of motivic frame as introduced in \([\text{G}]\): given a mathematical construct \( X \), one can formalize its construction by using what is called a motivic frame \( x = \{ x^{(n)} \to x^{(n+1)} \}_{n \geq 0} \) with \( z^{(n)} \) gluing maps and \( x^{(n)} \) variables, or motifs, the idea being that \( X \) would be constructed from objects of a different nature, that can in a first approximation be collected into classes, and that the \( x^{(i)} \)'s would provide variables for objects of each class. The gluing maps would indicate in what manner are the different objects put together to form \( X \). The choice of the word motif is mainly one of semantics; “variable” is a rather crude way to refer to the \( x^{(i)} \)'s, relative to “motif”, which is essentially a pattern, originally meant to describe a unifying, elementary model, which captures the nature of the objects within each class in addition to being a simple variable. A motif depending on other motifs will simply be referred to as a higher motif.

We let \( \text{Sh} \) be a motif for sheaves of sets on \( \text{k-CAlg} \). Hence we are looking for a commutative diagram of motifs. Further we regard the move from sheaves to derived stacks as a functorial operation, thus we would like to replace \( \pi_0 \circ \Pi_1 \) on the right by an adjoint \( \text{Set} \to \text{sSet} \) denoted \( i \) that we would like to argue is identical to \( j \) on the left. This is made possible by the following observation: the connected component functor can also be defined from \( \text{sSet} \) and this is what we will use, we define it as the coequalizer (\([\text{I}]\)):

\[
\begin{align*}
X_1 & \xrightarrow{d_1} X_0 \\
& \xrightarrow{d_0} X_0 \to \pi_0(X)
\end{align*}
\]

for \( X \in \text{sSet} \). This would give \( \pi_0 \) as \( \pi_0 \dashv cs_* \), \( cs_* \) the constant simplicial
functor:

\[ cs_* : \text{Set} \rightarrow \text{sSet} \]
\[ S \mapsto cs_*(S) = \underline{S} \]

where \( \underline{S} \) is such that \( \underline{S}_n = S \) for all \( n \geq 0 \) and all face and degeneracy maps are \( \text{id}_S \). By definition, \( j = cs_* \), where \( j : k\text{-CAlg} \rightarrow \text{sk-CAlg} \) is the natural inclusion functor that sends a \( k \)-algebra \( A \) to the constant simplicial object \( \underline{A} \) in \( \text{sk-CAlg} \) \((\ref{eq:17})\). Hence we take \( j = i = cs_* \).

One would then promote such a functor to the status of motif, denoted \( s \), or more precisely a variable functor \( s : \text{Set} \rightarrow \text{sSet} \) a specialization of which would be \( cs_* \). The aim would be to find a motif for a derived stack, denoted \( dSt \), depending on \( s \) and \( Sh \), hence a higher motif, that would make the following diagram commute:

\[
\begin{array}{ccc}
k\text{-CAlg} & \xrightarrow{Sh} & \text{Set} \\
\downarrow{s} & & \downarrow{s} \\
\text{sk-CAlg} & \xrightarrow{dSt} & \text{sSet}
\end{array}
\]  \tag{2}

The interest of having such a motif is that it would comprise all possible derived extensions given \( s \) and \( Sh \). The aim of the present paper is to prove that such a higher motif is a derived stack, i.e. that one can functorially construct derived stacks, and that such a construction is choice-free if written in the language of motifs.

We now define \( s \) to be a functorial simplicial frame on \( k\text{-CAlg} \), which in addition preserves finite limits. Recall from [H] that for \( A \) an object of \( k\text{-CAlg} \), we define a simplicial frame on \( A \) to be a simplicial object \( \hat{A} \in (k\text{-CAlg})^{\Delta^{op}} = \text{sk-CAlg} \), together with an equivalence \( cs_*A \rightarrow \hat{A} \) in the Reedy model category structure on \( \text{sk-CAlg} \) such that the induced map \( A \rightarrow A_0 \) is an isomorphism and if \( A \) is fibrant in \( k\text{-CAlg} \), so is \( \hat{A} \) in \( \text{sk-CAlg} \). We then define a functorial simplicial frame on \( k\text{-CAlg} \) to be given by a pair \( (G, j) \), where \( G : k\text{-CAlg} \rightarrow \text{sk-CAlg} \) is a functor, \( j : \text{id} \Rightarrow G \) is such that \( j_A : cs_*A \rightarrow G(A) \) is a simplicial frame for any \( A \in k\text{-CAlg} \). We will frequently abuse notation and refer to \( G \) from \( s = (G, j) \) as \( s \) itself.
Given a motif \( Sh \), and a functorial simplicial frame \( s \) on \( k\text{-}CAlg \), we have a corresponding higher motif \( dSt[Sh, s] = dSt \) which we aim to prove is a derived stack. Recall from \([T7]\) that in the definition of a derived stack we have to use hypercovers, which necessitate the introduction of coaugmented, cosimplicial objects \( A \to B_* \) in \( sk\text{-}CAlg \). Here we would take \( A = s(A) \), and \( B_* \) is a functor:

\[
B : \Delta \to \text{sk-CAlg} \\
\quad n \mapsto B_n = s(B_n)
\]

Note that \( B \in \text{cs} \cdot \text{sk-CAlg} \) where \( \text{cs} \) stands for cosimplicial. Since each \( B_n \) is in \( \text{sk-CAlg} \), we take it to be some \( s(B_n) \) for \( B_n \in \text{k-CAlg} \). Moreover \( s \) being a functor, if \( B_* \in \text{csk-CAlg} \), then \( B_* = s(B_*) \) is a cosimplicial object in \( \text{sk-CAlg} \). We are now ready to give the conditions \( dSt \) have to satisfy to be a derived stack. According to \([T7]\) applied to our setting, the following conditions must be met:

- For any equivalence \( s(A) \to s(B) \) in \( \text{sk-CAlg} \), \( A, B \in \text{k-CAlg} \), the induced morphism \( dSt(sA) \to dSt(sB) \) is an equivalence in \( sSet \).

- For any coaugmented, cosimplicial object \( s(A) \to s(B_*) \), corresponding to a ffqc-hypercovering in \( \text{dk-Aff} = \text{sk-CAlg}^{op} \), the induced morphism \( dSt(sA) \to \text{holim}_{n \in \Delta} dSt(sB_n) \) is an equivalence in \( sSet \).

- For any finite family \( \{ sA_i \} \) in \( \text{sk-CAlg} \), the natural morphism \( dSt(\prod sA_i) \to \prod dSt(sA_i) \) is an equivalence in \( sSet \).

Since the construction of derived stacks is intimately linked to that of sheaves of sets on \( \text{k-CAlg} \), we also give their definition as given in \([TV6]\). We regard sets as constant simplicial sets. A functor \( F : \text{k-CAlg} \to \text{Set} \) is a sheaf if:

- For any equivalence \( A \to B \) in \( \text{k-CAlg} \), the induced morphism \( F(A) \to F(B) \) is an equivalence in \( sSet \).
• For any finite family \( \{A_i\}_{i \in I} \) in \( k\text{-CAlg} \), the natural morphism
\[
F(\prod_{i \in I} A_i) \to \prod_{i \in I} F(A_i)
\]
is an isomorphism in \( \text{Ho}(sSet) \).

• For any cosimplicial object \( A \to B \) in \( k\text{-CAlg} \) corresponding to a ffqc-hypercover \( \text{Spec } B \to \text{Spec } A \) in \( k\text{-Aff} \), the induced morphism
\[
F(A) \to \text{holim}_{n \in \Delta} F(B_n)
\]
is an isomorphism in \( \text{Ho}(sSet) \).

### 3 Statement of the theorem and proof

**Theorem 3.1.** A higher motif \( dSt \) as defined by the commutative diagram (2) is a derived stack.

There are three points to be checked, and each will be the subject of a subsection.

#### 3.1 \( dSt \) preserves equivalences

Suppose \( s(A) \to s(B) \) is an equivalence in \( \text{sk-CAlg} \), for \( A, B \in k\text{-CAlg} \). If we denote equivalences by \( \sim \), we have:
\[
\begin{array}{ccc}
\text{cs}_*(A) & \xrightarrow{\sim} & s(A) \\
\downarrow & & \\
\text{cs}_*(B) & \xrightarrow{\sim} & s(B)
\end{array}
\]
each of those morphisms maps to an isomorphism in \( \text{Ho}(\text{sk-CAlg}) \), giving an isomorphism \( \text{cs}_*(A) \to \text{cs}_*(B) \) by composition. Then by [TV6] the constant simplicial functor \( \text{cs}_* : k\text{-CAlg} \to \text{Ho}(\text{sk-CAlg}) \), that is \( \text{Hom}_{\text{Ho}(\text{sk-CAlg})}(\text{cs}_* A, \text{cs}_* B) \cong \text{Hom}_{k\text{-CAlg}}(A, B) \), so this isomorphism gives an equivalence \( A \to B \) in \( k\text{-CAlg} \). Now \( dSt \) is built from a motif of sheaves \( \text{Sh} \) which is a sheaf, and by the sheaf condition \( A \to B \) implies \( \text{Sh}(A) \to \text{Sh}(B) \) in \( sSet \). Since each of \( \text{Sh}(A) \) and \( \text{Sh}(B) \) are sets and are regarded as constant simplicial sets in \( sSet \), this reads
$cs_*Sh(A) \sim \rightarrow cs_*Sh(B)$. By definition of $s = (G, j)$, since we have a map $Sh(A) \rightarrow Sh(B)$ the natural transformation $j$ gives us a commutative diagram:

$$
\begin{array}{ccc}
cs_*Sh(A) & \sim & s(Sh(A)) \\
\downarrow & & \downarrow p \\
\cdots & \cdots & \cdots \\
\end{array}
$$

and by using the 2 out of 3 property twice in this diagram we have that $p$ is a weak equivalence. Finally by commutativity of (2) $dSt(sA) = s(Sh(A)) \sim \rightarrow s(Sh(B)) = dSt(sB)$ so $dSt$ does preserve equivalences.

### 3.2 $dSt$ satisfies hyperdescent

Consider a coaugmented cosimplicial object $s(A) \rightarrow s(B_*)$ in sk-$CAlg$ corresponding to a ffqc-hypercover $Spec(s(B_*)) \rightarrow Spec(s(A))$ in dk-$Aff = sk-CAlg^{op}$. We have to show $dSt(sA) \sim \rightarrow \text{holim}_{n \in \Delta} dSt(sB_n)$ in $sSet$. Recall from [TV1] that the ffqc topology on dk-$Aff$ induces ffqc hypercoverings: if $Spec s(A)$ is an object of the site (dk-$Aff, ffqc$), a homotopy ffqc-hypercover of $Spec s(A)$ is a $Spec s(B_*)$ in $Ho(sdk-Aff)$ along with a morphism $Spec s(B_*) \rightarrow Spec s(A)$ in $Ho(sdk-Aff)$ such that for all $n \geq 0$:

$$
\text{Spec } s(B_*)^{\mathbb{R} \Delta^n} \rightarrow \text{Spec } s(B_*)^{\mathbb{R} \partial \Delta^n} \times_{\text{Spec } s(A)^{\mathbb{R} \partial \Delta^n}} \text{Spec } s(A)^{\mathbb{R} \Delta^n}
$$

is a ffqc-covering in dk-$Aff$. From [T7] a finite family $\{f_i : sA \rightarrow sB_i\}$ in sk-$CAlg$ is a ffqc-covering if each $f_i$ is flat, and if the induced morphism of affine schemes:

$$
\prod \text{Spec } \pi_0(sB_i) \rightarrow \text{Spec } \pi_0(sA)
$$

is surjective, where $f_i$ flat means the induced morphism $\text{Spec } \pi_0(sB_i) \rightarrow \text{Spec } \pi_0(sA)$ is flat and for all $i > 0$, the natural morphism:

$$
\pi_i(sA) \otimes_{\pi_0(sA)} \pi_0(sB_i) \rightarrow \pi_i(sB_i)
$$

is an isomorphism.

We first have to show $A \rightarrow B_*$ is a coaugmented cosimplicial object in $k-CAlg$ corresponding to a ffqc-hypercovering $Spec B_* \rightarrow Spec A$ in $k-Aff$.  

9
3.2.1 Spec $B_* \rightarrow$ Spec $A$ is a ffqc-hypercovering in $k$-$Aff$

The aim of this subsection is to show that for all $n \geq 0$:

$$\text{Spec } B_*^{R\Delta^n} \rightarrow \text{Spec } B_*^{R\partial \Delta^n} \times_{h_{\text{Spec } A^{R\partial \Delta^n} \text{Spec } A^{R\Delta^n}}}^h \text{Spec } A^{R\Delta^n}$$

(3)

is a ffqc-covering in $k$-$Aff$. This will be done in two steps. We will first show

$$(cs_*(B_*))^{op,R\Delta^n} \rightarrow (cs_*(B_*))^{op,R\partial \Delta^n} \times_{(cs_*(A))^{op,R\partial \Delta^n}}^h (cs_*(A))^{op,R\Delta^n}$$

and then we will show (3) is a ffqc-covering in $k$-$Aff$.

We first rewrite:

$$\text{Spec } s(B_*)^{R\Delta^n} \rightarrow \text{Spec } s(B_*)^{R\partial \Delta^n} \times_{h_{\text{Spec } s(A)^{R\partial \Delta^n}}}^h s(A)^{R\Delta^n}$$

as:

$$(s(B_*))^{op,R\Delta^n} \rightarrow (s(B_*))^{op,R\partial \Delta^n} \times_{(s(A))^{op,R\partial \Delta^n}}^h (s(A))^{op,R\Delta^n}$$

From [TV3] sdk-$Aff$ being a simplicial model category it is tensored and cotensored over $sSet$, hence for $F_* \in$ sdk-$Aff$ we have by adjunction:

$$\text{Hom}(\Delta^n \otimes F_*, (s(B_*))^{op}) \cong \text{Hom}(F_*, (s(B_*))^{op,\Delta^n})$$

as well as:

$$\text{Hom}(\Delta^n \otimes F_*, (cs_*(B_*))^{op}) \cong \text{Hom}(F_*, (cs_*(B_*))^{op,\Delta^n})$$

Since the exponential map is natural in both arguments, having a map $(s(B_*))^{op} \rightarrow (cs_*(B_*))^{op}$, we have maps:

$$\begin{array}{ccc}
(s(B_*))^{op,\Delta^n} & \xrightarrow{\phi} & (cs_*(B_*))^{op,\Delta^n} \\
\downarrow \partial_0 & & \downarrow \partial_0 \\
(s(B_*))^{op,\Delta^n} & \xrightarrow{\phi_0} & (cs_*(B_*))^{op,\Delta^n} \\
\downarrow \gamma & & \downarrow \gamma \\
(s(B_*))^{op,R\Delta^n} & \xrightarrow{\gamma \phi_0} & (cs_*(B_*))^{op,R\Delta^n}
\end{array}$$

(4)

where $\partial_0$ is the degree zero map with $\phi_0$ the map on degree zero elements induced by $\phi$, and $\gamma$ is the canonical functor from a given category to its homotopy category. In the same manner we have maps:

$$(s(B_*))^{op,R\partial \Delta^n} \rightarrow (cs_*(B_*))^{op,R\partial \Delta^n}$$

(5)
\[(sA)^{\text{op}, \mathbb{R} \Delta^n} \rightarrow (cs_* A)^{\text{op}, \mathbb{R} \Delta^n} \quad (6)\]

and:

\[(sA)^{\text{op}, \mathbb{R} \partial \Delta^n} \rightarrow (cs_* A)^{\text{op}, \mathbb{R} \partial \Delta^n} \quad (7)\]

which induce a map of homotopy fiber products:

\[(s(B_*))^{\text{op}, \mathbb{R} \partial \Delta^n} \times^{h}_{(sA)^{\text{op}, \mathbb{R} \partial \Delta^n}} (sA)^{\text{op}, \mathbb{R} \Delta^n} \quad (8)\]

which when combined with the bottom horizontal map of (4) yields:

\[(s(B_*))^{\text{op}, \mathbb{R} \Delta^n} \rightarrow (s(B_*))^{\text{op}, \mathbb{R} \partial \Delta^n} \times^{h}_{(sA)^{\text{op}, \mathbb{R} \partial \Delta^n}} (sA)^{\text{op}, \mathbb{R} \Delta^n} \quad (9)\]

The top horizontal map is a ffqc-covering in dk-Aff. We would like to fill the bottom map and show that it is a ffqc-covering as well. In order to do this we will show that both vertical maps are equivalences, which will give us a bottom horizontal map since we work in Ho(dk-Aff). This will also tell us this map gives us a ffqc-covering. In a first time to show the vertical maps above are weak equivalences, we will need all maps in (4), (5), (6) and (7) to be equivalences so we prove the more general fact:

**Lemma 3.2.1.1.** For \( K = \Delta^n \) or \( K = \partial \Delta^n \), \( A \in \text{csk-CAlg} \), the map \((sA)^{\text{op}, \mathbb{R}K} \rightarrow (cs_* A)^{\text{op}, \mathbb{R}K}\) is a weak equivalence.

**Proof.** We use the fact that \( X^{\mathbb{R}K} = (X^{\mathbb{R}K})_0 \) as shown in [TV6], obtained by first taking a fibrant replacement of \( X_* \), followed by taking the exponential by \( K \), and then taking the degree zero component. We will use the following fact from [HI]: that dk-Aff being a simplicial model category, for any fibrant objects \( X \) and \( Y \) in dk-Aff, \( g : X \rightarrow Y \) is an equivalence if and only if for any cofibrant \( Z \) in dk-Aff we have \( \text{Hom}(Z, X) \simeq_{\text{sSet}} \text{Hom}(Z, Y) \). Thus we fix some cofibrant object \( F_* \) in dk-Aff and consider

\[
\text{Hom}(F_*, (sA)^{\text{op}, \mathbb{R}K}) = \text{Hom}(QF_*, (sA)^{\text{op}, \mathbb{R}K}) \\
\simeq \text{Hom}(K \otimes F_*, R((sA)^{\text{op}})) \\
= \text{Hom}(Q(K \otimes F_*), R((sA)^{\text{op}})) \quad (10)
\]
Now again we can invoke that same result from [Hi] for $R((sA)_{\text{op}}) \to R((cs_*A)_{\text{op}})$ since in the diagram below, vertical maps are trivial cofibrations, the top horizontal map is a weak equivalence by definition of $s$, so by the 2 out of 3 property applied twice, the bottom horizontal map is a weak equivalence as well:

\[
\begin{array}{ccc}
(sA)_{\text{op}} & \longrightarrow & (cs_*A)_{\text{op}} \\
\downarrow & & \downarrow \\
R((sA)_{\text{op}}) & \longrightarrow & R((cs_*A)_{\text{op}})
\end{array}
\]

Being an equivalence from (10) we have:

\[
\text{Hom}(F_*, (sA)_{\text{op,RK}}) \cong \text{Hom}(Q(K \otimes F_*), R((sA)_{\text{op}})) \\
\cong \text{Hom}(Q(K \otimes F_*), R((cs_*A)_{\text{op}})) \\
\cong \text{Hom}(F_*, (cs_*A)_{\text{op,RK}})
\]

hence $(sA)_{\text{op,RK}} \to (cs_*A)_{\text{op,RK}}$ is a weak equivalence, more precisely, a Reedy equivalence in the Reedy model structure for $\text{sdk-} Aff$, and by 15.3.11 of [Hi] this implies its degree zero component $(sA)_{\text{op,RK}} \to (cs_*A)_{\text{op,RK}}$ is an equivalence as well.

**Corollary 3.2.1.2.** The maps (4), (5), (6), (7) are all weak equivalences.

**Proof.** For the first two equations it’s immediate, we just take $A = B_*$. For the other two $A \in k\text{-CAlg}$ gives $sA \in \text{sk-CAlg}$, hence $(sA)_{\text{op}} \in \text{dk-Aff}$, regarded as a constant simplicial object $cs_*((sA)_{\text{op}}) \in \text{sdk-Aff}$. Now for $K = \Delta^n$ or $K = \partial \Delta^n$, we have:

\[
\begin{array}{ccc}
(sA)_{\text{op,RK}} & \longrightarrow & (cs_*A)_{\text{op,RK}} \\
\| & & \| \\
(cs_*((sA)_{\text{op}}))_{\text{RK}} & \longrightarrow & (cs_*((cs_*A)_{\text{op}}))_{\text{RK}}
\end{array}
\]

Since $(sA)_{\text{op}} \simto (cs_*A)_{\text{op}}$, $cs_*$ being fully faithful, $cs_*((sA)_{\text{op}}) \simto cs_*((cs_*A)_{\text{op}})$ and following the same reasoning as in the proof of the previous lemma we would find that the bottom horizontal map above is a weak equivalence, hence so is the top horizontal map.

As a consequence of having those equivalences we prove the fiber products in (8) are equivalent:
Lemma 3.2.1.3. The map of homotopy fiber products:

\[
\begin{aligned}
(s(B_*))^{\text{op}, \mathbb{R}\partial \Delta^n} \times^h_{(sA)^{\text{op}, \mathbb{R}\partial \Delta^n}} (sA)^{\text{op}, \mathbb{R}\Delta^n} \\
(c_{s_*}(B_*))^{\text{op}, \mathbb{R}\partial \Delta^n} \times^h_{(c_{s_*}A)^{\text{op}, \mathbb{R}\partial \Delta^n}} (c_{s_*}A)^{\text{op}, \mathbb{R}\Delta^n}
\end{aligned}
\]  

(11)

is a weak equivalence.

Proof. We are looking at the following diagram:

\[
\begin{aligned}
(s(B_*))^{\text{op}, \mathbb{R}\partial \Delta^n} \times^h_{(sA)^{\text{op}, \mathbb{R}\partial \Delta^n}} (sA)^{\text{op}, \mathbb{R}\Delta^n} \\
\downarrow & \quad (c_{s_*}(B_*))^{\text{op}, \mathbb{R}\partial \Delta^n} \times^h_{(c_{s_*}A)^{\text{op}, \mathbb{R}\partial \Delta^n}} (c_{s_*}A)^{\text{op}, \mathbb{R}\Delta^n} \\
\leftarrow & \quad (sA)^{\text{op}, \mathbb{R}\partial \Delta^n} \leftarrow (sA)^{\text{op}, \mathbb{R}\Delta^n} \\
\downarrow & \quad (c_{s_*}(B_*))^{\text{op}, \mathbb{R}\partial \Delta^n} \\
\leftarrow & \quad (c_{s_*}A)^{\text{op}, \mathbb{R}\Delta^n} \leftarrow (c_{s_*}A)^{\text{op}, \mathbb{R}\Delta^n}
\end{aligned}
\]

To prove that the back vertical map is an equivalence we will use 15.10.10 from [Hi]: in a model category in which we have a diagram such as the one...
in which all objects are fibrant, the squares in the front and in the back are pullbacks, $p$ and $p'$ are fibrations, $r_B$, $r_C$ and $r_D$ are equivalences, then so is $r_A$. We apply this to our setting, where our initial model category is $sdk$-$Aff$, in which we took fibrant replacements of $sA$, $sB_*$, $cs_*A$ and $cs_*B_*$ already. By 9.3.9 of [Hi] since $sdk$-$Aff$ is a simplicial model category the exponentials of such objects are fibrant as well, and so are their zero components by 15.3.11 of [Hi], so all objects are fibrant as needed. The pullback squares are given by the homotopy fiber products of the previous diagram, the maps $r_B$, $r_C$ and $r_D$ are the vertical equivalences in that diagram, so $r_A$ would be the map of homotopy fiber products. We take for $p$ and $p'$ the following maps:

$$(sA)^{op,\Delta^n} \rightarrow (sA)^{op,\partial\Delta^n}$$

and:

$$(cs_*A)^{op,\Delta^n} \rightarrow (cs_*A)^{op,\partial\Delta^n}$$

We will use the following fact 9.3.9 2b) from [Hi]: if $X$ is fibrant in a simplicial model category and $j : K \rightarrow L$ is an inclusion in $sSet$, then $X^L \rightarrow X^K$ is a fibration. Applying this to the first map above for example, one gets that:

$$(R(sA)^{op})^{\Delta^n} \rightarrow (R(sA)^{op})^{\partial\Delta^n}$$

$$
\begin{array}{ccc}
(sA)^{op,\Delta^n} & \rightarrow & (sA)^{op,\partial\Delta^n} \\
\| & & \| \\
(sA)^{op,\Delta^n} & & (sA)^{op,\partial\Delta^n}
\end{array}
$$

is a fibration, more precisely, a Reedy fibration, hence by 15.3.11 of [Hi] the
degree zero component is a fibration as well:

\[
[(sA)^{\text{op}}, R\Delta^n]_0 \longrightarrow [(sA)^{\text{op}}, R\partial\Delta^n]_0
\]

One would show in the same manner that \((cs_\ast A)^{\text{op}}, R\Delta^n \rightarrow (cs_\ast A)^{\text{op}}, R\partial\Delta^n\) is a fibration. This completes the proof that the map of homotopy fiber products is a weak equivalence.

**Lemma 3.2.1.4.** The map \((cs_\ast (B_\ast))^{\text{op}}, R\Delta^n \rightarrow (cs_\ast (B_\ast))^{\text{op}}, R\partial\Delta^n \times^h_{(cs_\ast A)^{\text{op}}, R\partial\Delta^n} (cs_\ast A)^{\text{op}}, R\Delta^n\) gives a ffqc-covering in \(dk\text{-}Aff\).

**Proof.** The previous lemma shows that the right vertical map in the diagram:

\[
\begin{array}{ccc}
(s(B_\ast))^{\text{op}}, R\Delta^n & \longrightarrow & (s(B_\ast))^{\text{op}}, R\partial\Delta^n \times^h_{(sA)^{\text{op}}, R\partial\Delta^n} (sA)^{\text{op}}, R\Delta^n \\
\downarrow & & \downarrow \\
(cs_\ast (B_\ast))^{\text{op}}, R\Delta^n & \longrightarrow & (cs_\ast (B_\ast))^{\text{op}}, R\partial\Delta^n \times^h_{(cs_\ast A)^{\text{op}}, R\partial\Delta^n} (cs_\ast A)^{\text{op}}, R\Delta^n
\end{array}
\]

is an equivalence. The vertical map on the left has already been shown to be an equivalence as well. Hence in \(\text{Ho}(dk\text{-}Aff)\) we have a bottom horizontal map as shown in the diagram above. Further from [TV6] the definition of hypercovers uses the higher homotopy groups \(\pi_i\) for \(i \geq 0\). Recall that for \(C\) a simplicial model category, \(A \in C, |A| = \text{Map}_C(1, A) \in \text{Ho}(sSet)\). If \(C\) is pointed \(|A|\) has a natural basepoint and we can define \(\pi_i(A) = \pi_i(|A|, *)\). Here \(\text{Map}_C\) is the simplicial hom in the simplicial model category \(C\) which for us we take to be \(sk\text{-}CAlg\). By 9.3.3 of [HI] since 1 is cofibrant, if \(A \simto B\) is an equivalence of fibrant objects, then \(\text{Map}_C(1, A) \rightarrow \text{Map}_C(1, B)\) is an equivalence in \(sSet\), i.e. \(|A| \simto |B|\), and by definition of higher homotopy groups on \(sk\text{-}CAlg\) we conclude that being a ffqc-covering is an invariant on weak equivalence classes, whence the result.

We finally prove:

**Lemma 3.2.1.5.** The map \((B_\ast)^{\text{op}}, R\Delta^n \rightarrow (B_\ast)^{\text{op}}, R\partial\Delta^n \times^h_{A^{\text{op}}, R\partial\Delta^n} A^{\text{op}}, R\Delta^n\) gives a ffqc-covering in \(k\text{-}Aff\).
We start from:

\[(cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} \longrightarrow (cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} \times_{(cs_*A)^{\text{op},\mathbb{R}^\Delta^n}} (cs_*A)^{\text{op},\mathbb{R}^\Delta^n} \]

\[\downarrow \downarrow \downarrow \downarrow \]

\[(B_*)^{\text{op},\mathbb{R}^\Delta^n} \quad (B_*)^{\text{op},\mathbb{R}^\Delta^n} \times_{A^{\text{op},\mathbb{R}^\Delta^n}} A^{\text{op},\mathbb{R}^\Delta^n} \]

we will show there is a bottom horizontal map making this diagram commutative, which in addition would provide a ffqc-covering in \(k\text{-Aff}\). First we will show that \((cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} = cc_*(B_*^{\text{op},\mathbb{R}^\Delta^n})\) and that we have a same result with \(\Delta^n\) interchanged with \(\partial\Delta^n\), and that the same would hold if we considered \(A\) instead of \(B_*\). Then we will argue that:

\[cc_*(B_*^{\text{op},\mathbb{R}^\Delta^n}) \times_{cc_*(A^{\text{op},\mathbb{R}^\Delta^n})} cc_*(A^{\text{op},\mathbb{R}^\Delta^n}) = cc_*(B_*^{\text{op},\mathbb{R}^\Delta^n} \times_{A^{\text{op},\mathbb{R}^\Delta^n}} A^{\text{op},\mathbb{R}^\Delta^n})\]

and finally using the constant nature of the constant cosimplicial functor \(cc_*\) that we do have a ffqc-covering in \(k\text{-Aff}\) as desired. First:

**Lemma 3.2.1.6.** \((cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} = cc_*(B_*^{\text{op},\mathbb{R}^\Delta^n})\)

**Proof.** By definition:

\[(cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} = \gamma(([cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n}]_0)\]

We first observe that \((cs_*(B_*))^{\text{op}} = cc_*(B_*^{\text{op}})\). Upon exponentiating by \(\Delta^n\) we get:

\[(cs_*(B_*))^{\text{op},\Delta^n} \longrightarrow (cc_*(B_*^{\text{op}}))^{\Delta^n} \]

\[\gamma\]

(12)

where the last equality follows since the exponentiation is taken relative to the simplicial model category structure on \(sk\text{-}CAlg\). For the same reason upon taking the degree zero part:

\[(((cs_*(B_*))^{\text{op},\Delta^n})_0 = (cc_*(B_*^{\text{op},\Delta^n}))_0\]

\[= cc_*(B_*^{\text{op},\Delta^n})_0\]

\[= cc_*(B_*^{\text{op},\Delta^n})\]

and upon applying the canonical functor \(\gamma\) we get:

\[(cs_*(B_*))^{\text{op},\mathbb{R}^\Delta^n} = \gamma(((cs_*(B_*))^{\text{op},\Delta^n})_0 = \gamma(cc_*(B_*^{\text{op},\Delta^n}))\]

\[= cc_*(B_*^{\text{op},\Delta^n}) = cc_*(B_*^{\text{op},\mathbb{R}^\Delta^n}).\]

\[\Box\]
By construction we have the same result if we replace $\Delta^n$ by $\partial\Delta^n$, or use $A$ instead of $B_*$. We now show:

**Lemma 3.2.1.7.**

\[ cc_*( (B_*)^{op, R\partial\Delta^n} ) \times_{cc_*( A^{op, R\partial\Delta^n} )}^{h} cc_*( A^{op, R\Delta^n} ) = cc_*( (B_*)^{op, R\partial\Delta^n} ) \times_{A^{op, R\partial\Delta^n}}^{h} E \]

**Proof.** We first focus on the homotopy fiber product on the left above. Following \[\text{[H]}\] it is defined by using a functorial factorization which we will denote by $E$ as in the following diagram:

\[ E(cc_*( (B_*)^{op, R\partial\Delta^n} )) \times_{cc_*( A^{op, R\partial\Delta^n} )}^{h} E(cc_*( A^{op, R\Delta^n} )) = E(cc_*( B_*)^{op, R\partial\Delta^n} ) \times_{cc_*( A^{op, R\partial\Delta^n} )}^{h} cc_*( A^{op, R\Delta^n} ) \]

By definition of the constant cosimplicial functor $cc_*$, a map $cc_*( A ) \rightarrow cc_*( B )$ is given by a map $A \rightarrow B$ which is the same in all degrees, hence can be denoted by $cc_*( A \rightarrow B )$. By definition of the functorial factorization, $E(cc_*( X )) = cc_*( EX )$, hence a map $cc_*( A ) \leftarrow E(cc_*( X )) = cc_*( EX )$ can be denoted $cc_*( A \leftarrow E( X ) )$. It follows:

\[ cc_*( (B_*)^{op, R\partial\Delta^n} ) \times_{cc_*( A^{op, R\partial\Delta^n} )}^{h} cc_*( A^{op, R\Delta^n} ) \]

\[ = E(cc_*( (B_*)^{op, R\partial\Delta^n} )) \times_{cc_*( A^{op, R\partial\Delta^n} )}^{h} cc_*( A^{op, R\Delta^n} ) \]

\[ = \lim \bigg( E(cc_*( (B_*)^{op, R\partial\Delta^n} )) \rightarrow cc_*( A^{op, R\partial\Delta^n} ) \leftarrow E(cc_*( A^{op, R\Delta^n} )) \bigg) \]

\[ = \lim cc_*( E((B_*)^{op, R\partial\Delta^n} )) \rightarrow A^{op, R\partial\Delta^n} \leftarrow E(A^{op, R\Delta^n}) \]

\[ = cc_*( \lim E((B_*)^{op, R\partial\Delta^n} )) \rightarrow A^{op, R\partial\Delta^n} \leftarrow E(A^{op, R\Delta^n}) \]

17
since $cc_*$ preserves limits as a right adjoint, and this equals $cc_*((B_*)^{op,R\partial\Delta^n} \times_{A^{op,R\partial\Delta^n}} A^{op,R\Delta^n})$ as claimed.

We can now complete the proof of lemma 3.2.1.5 with the previous results the ffqc-covering in $dk\text{-}Aff$ given by Lemma 3.2.1.4 now reads:

$$cc_*((B_*)^{op,R\Delta^n} \to cc_*((B_*)^{op,R\partial\Delta^n} \times_{A^{op,R\partial\Delta^n}} A^{op,R\Delta^n}))$$

and again by the constant nature of $cc_*$ this map being a ffqc-covering implies:

$$(B_*)^{op,R\Delta^n} \to (B_*)^{op,R\partial\Delta^n} \times_{A^{op,R\partial\Delta^n}} A^{op,R\Delta^n}$$  \hspace{1cm} (13)

is a ffqc-covering in $k\text{-}Aff$, i.e. $A \to B_*$ corresponds to a homotopy ffqc-hypercovering in $k\text{-}Aff$.

3.2.2 From $Sh$ being a sheaf to $dSt$ being a derived stack

Now that $A \to B_*$ corresponds to a ffqc-hypercovering in $k\text{-}Aff$, $Sh$ being a sheaf we obtain an isomorphism in $Ho(sSet)$: $Sh(A) \to \text{holim}_{n\in\Delta}Sh(B_n)$, or equivalently an equivalence in $sSet$. Now because in the definition $s = (G,j)$ of $s$ we have a natural transformation $j$ we obtain the following commutative diagram:

$$
\begin{array}{ccc}
cc_*Sh(A) & \sim & cc_*\text{holim}_{n\in\Delta}Sh(B_n) \\
\downarrow & & \downarrow \\
\sim & & \sim \\
\text{s}(Sh(A)) & \sim & \text{s(\text{holim}_{n\in\Delta}Sh(B_n)))} \\
\| & & \| \\
dSt(sA) & & \text{holim}_{n\in\Delta}s(Sh(B_n))) \\
\| & & \| \\
& & \text{holim} \text{ dSt(sB_n)}
\end{array}
$$

where $dSt(sA) = sSh(A)$ by commutativity of (2), $s(\text{holim} \ Sh(B_n))) = \text{holim} \ sSh(B_n)$ since as shown in [Hi], homotopy limits are equalizers, hence limits, and $s$ preserves finite limits by definition, and finally this last object is equal to $\text{holim} \ dSt(sB_n)$ again by commutativity of (2). Further $cc_*$ being fully faithful, $Sh(A) \xrightarrow{\sim} \text{holim}_{n\in\Delta}Sh(B_n)$ implies $cc_*Sh(A) \xrightarrow{\sim} cc_*\text{holim}_{n\in\Delta}Sh(B_n)$, hence $sSh(A) \to s(\text{holim} \ Sh(B_n)))$ is an equivalence by the 2 out of 3 property, and it follows that $dSt(sA) \xrightarrow{\sim} \text{holim} dSt(sB_n)$ i.e. $dSt$ satisfies descent.
3.3 \(dSt\) preserves finite products

We consider a finite family \(\{sA_i\}\) in \(skCA\). We need to show the natural morphism:

\[
dSt(\prod sA_i) \rightarrow \prod dSt(sA_i)
\]

is an equivalence in \(sSet\). We have \(dSt(\prod sA_i) = dSt(\prod A_i)\) since \(s\) preserves finite products. This latter object is equal to \(s\text{Sh}(\prod A_i)\) by commutativity of (2). From \(\text{Sh}(\prod A_i)\) since \(\text{Sh}\) is a sheaf, we have \(\text{Sh}(\prod^h A_i) \cong \prod \text{Sh}(A_i)\) an isomorphism in \(\text{Ho}(sSet)\), hence an equivalence in \(sSet\). Keeping in mind that we really consider sets as constant simplicial sets in \(sSet\), this reads \(cs_*\text{Sh}(\prod^h A_i) \cong cs_* \prod \text{Sh}(A_i)\), so we have the following commutative diagram by virtue of the existence of the natural transformation \(j\):

\[
\begin{array}{c}
cs_*\text{Sh}(\prod^h A_i) \cong \rightarrow cs_* \prod \text{Sh}(A_i) \\
\cong \downarrow \cong \downarrow \\
\text{Sh}(\prod^h A_i) \quad \cong \rightarrow \quad \text{Sh}(A_i) \\
\cong \downarrow \cong \downarrow \\
dSt(\prod A_i) \quad \cong \rightarrow \quad \prod s(\text{Sh}(A_i)) \\
\cong \downarrow \cong \downarrow \\
dSt(\prod sA_i) \quad \cong \rightarrow \quad \prod dSt(sA_i)
\end{array}
\]

where we used the 2 out of 3 property to have an equivalence \(s(\text{Sh}(\prod A_i)) \rightarrow \prod s(\text{Sh}(A_i))\), hence \(dSt(\prod sA_i) \cong \rightarrow \prod dSt(sA_i)\). This completes the proof that \(dSt\) is a derived stack, the claim of Theorem 3.1.

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