A 2-ADIC CONTROL THEOREM FOR MODULAR CURVES

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Abstract. We study the behaviour of ordinary parts of the homology modules of modular curves, associated to a decreasing sequence of congruence subgroups $\Gamma_1(N2^r)$ for $r \geq 2$, and prove a control theorem for these homology modules.

1. Introduction

Hida theory studies the modular curves associated to the following congruence subgroups, for primes $p \geq 5$ and $(p, N) = 1$,
\[
\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Np).
\]
Let $Y_r$ denote the Riemann surface associated to the congruence subgroup $\Gamma_1(Np^r)$. One of the important results in Hida theory [3] is that the projective limit of ordinary parts of the homology modules, i.e., $W^{\text{ord}} := \lim_{\leftarrow r} H_1(Y_r, \mathbb{Z}_p)^{\text{ord}}$, is a free $\Lambda$-module of finite rank and
\[
W^{\text{ord}}/a_r W^{\text{ord}} = H_1(Y_r, \mathbb{Z}_p)^{\text{ord}}, \tag{**}
\]
for all $r \geq 1$, where $a_r$ denotes the augmentation ideal of $\mathbb{Z}[1 + p^r \mathbb{Z}_p]$ and $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. In [1], Emerton gave a proof of these results above for primes $p \geq 5$, using algebraic topology of the Riemann surfaces $Y_r$.

Emerton’s proof for $p \geq 5$ holds for $p = 3$ with $N > 1$ verbatim, but for $p = 2$ we show that similar results hold only after passing to smaller congruence subgroups. Moreover, there is no restriction on $N$, i.e., $N$ can be equal to 1 (unlike when $p = 3$) (cf. Theorem 5.2 in the text). As a consequence of these results, we proved control theorems for ordinary 2-adic families of modular forms, see [2]. Some amount of calculations will be omitted and the reader should refer to those in [1] for more details.

2. Preliminaries

Throughout this note, let $p = 2$, $q = 4$, and $N \in \mathbb{N}$ such that $(p, N) = 1$. We look at the modular curves associated to the following congruence subgroups
\[
\cdots \subset \Gamma_1(Np^r) \subset \cdots \subset \Gamma_1(Nq).
\]
If we take the homology with $\mathbb{Z}$-coefficients of the tower of modular curves, we get a tower of finitely generated free abelian groups
\[
\cdots \to \Gamma_1(Np^r)^\text{ab} \to \cdots \to \Gamma_1(Nq)^\text{ab}, \tag{2.1}
\]
because for $r \geq 2$, $H_1(\Gamma_1(Np^r) \backslash \mathbb{H}, \mathbb{Z}) = \Gamma_1(Np^r)^\text{ab}$, where $\mathbb{H}$ denotes the upper half-plane. To understand (2.1), we introduce the congruence subgroups for $r \geq 2$:
\[
\Phi^2_r = \Gamma_1(Nq) \cap \Gamma_0(p^r).
\]
Clearly, we have $\Gamma_1(Np^r) \subset \Phi^2_r \subset \Gamma_1(Nq)$ and $\Gamma_1(Np^r)$ is a normal subgroup of $\Phi^2_r$. For $r \geq 2$, we define $\Gamma_r := \text{Ker } (\mathbb{Z}_p^\times \to (\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times)$, which is a subgroup of $\Gamma_2$ with index $p^{r-2}$. Set $\Gamma := \Gamma_2$.

We define a morphism of groups
\[
\Phi^2_r \xrightarrow{\eta_r} \Gamma/\Gamma_r
\]
via the formula
\[
(\begin{array}{cc} a & b \\ c & d \end{array}) \mapsto d \mod \Gamma_r.
\] (2.2)

**Lemma 2.1.** The map $\eta_r$ is surjective.

**Proof.** Given a $\bar{d} \in \Gamma/\Gamma_r$, we can take a lift $d$ of $\bar{d}$ of the form $1 + kqN$ for some $k \in \mathbb{Z}$, because for any $\alpha, \beta \in \Gamma, \alpha \equiv \beta \pmod{\Gamma_r}$ if and only if $\alpha - \beta \in p^r\mathbb{Z}_p$.

Now, take $c$ to be $Np^r$. Clearly $(c, d) = 1$, and hence there exists $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. We see that $\alpha = (\begin{array}{cc} a & b \\ c & d \end{array}) \in \Phi^2_r$ and $\eta_r(\alpha) = \bar{d}$. \qed

**Remark 2.2.** The restriction of $\eta_r$ to $\Phi^2_r \cap \Gamma^0(p)$, which we denote by $\text{Res}(\eta_r)$, is also surjective onto $\Gamma/\Gamma_r$. Moreover, we have the following commutative diagram
\[
\begin{array}{cccc}
\Phi^2_{r+1} & \xrightarrow{\eta_{r+1}} & \Gamma/\Gamma_{r+1} \\
\downarrow{\text{Res}(\eta_r)} & & \downarrow \\
\Phi^2_r \cap \Gamma^0(p) & \xrightarrow{\text{Res}(\eta_r)} & \Gamma/\Gamma_r,
\end{array}
\]
where the group $\Gamma^0(p) = \{ (\begin{array}{cc} a & b \\ c & d \end{array}) \in \text{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \}$ and $t = (\begin{array}{cc} 1 & 0 \\ 0 & p \end{array})$.

By Lemma 2.1 we have the following short exact sequence of groups
\[
1 \rightarrow \Gamma_1(Np^r) \rightarrow \Phi^2_r \xrightarrow{\eta_r} \Gamma/\Gamma_r \rightarrow 1.
\]

The action of $\Phi^2_r$ on $\Gamma_1(Np^r)$ by conjugation induces an action of $\Phi^2_r/\Gamma_1(Np^r) = \Gamma/\Gamma_r$ on $\Gamma_1(Np^r)^{ab}$. Thus $\Gamma$ acts naturally on $\Gamma_1(Np^r)^{ab}$. The morphisms in the chain
\[
\cdots \rightarrow \Gamma_1(Np^r)^{ab} \rightarrow \cdots \rightarrow \Gamma_1(Nq)^{ab}
\]
are clearly $\Gamma$-equivariant.

If $r \geq s > 1$, we denote by $\Phi^s_r$ the subgroup of $\Phi^2_r$ containing $\Gamma_1(Np^r)$ whose quotient by $\Gamma_1(Np^r)$ equals $\Gamma_s/\Gamma_r$, i.e., $\Phi^s_r := \Gamma_1(Np^r) \cap \Gamma_0(p^r)$. Moreover, we have
\[
\Gamma_1(Np^r)^{ab} \rightarrow \Phi^s_r \rightarrow \Gamma_s/\Gamma_r \rightarrow 1.
\]

For any $s > 1$, let $\gamma_s$ denote a topological generator of $\Gamma_s$. Then the augmentation ideal $\mathfrak{a}_s$ of $\Lambda = \mathbb{Z}_p[[\Gamma]]$ is a principal ideal generated by $\gamma_s - 1$. Similarly, for $i > 0$, $\Gamma_{s+i} = (\gamma_s^i)$ and $\mathfrak{a}_{s+i} = (\gamma_s^i - 1)$. Clearly, for any $r \geq s > 1$, the augmentation ideal of $\mathbb{Z}[\Gamma_s/\Gamma_r]$ is $\mathfrak{a}_s$, and
\[
\mathfrak{a}_s \Gamma_1(Np^r)^{ab} = [\Phi^s_r, \Gamma_1(Np^r)]/[\Gamma_1(Np^r), \Gamma_1(Np^r)] \subset \Gamma_1(Np^r)^{ab},
\]
and the last inclusion follows since $\Gamma_1(Np^r)$ is a normal subgroup of $\Phi^s_r$. The extension
\[
1 \rightarrow \Gamma_1(Np^r)/[\Phi^s_r, \Gamma_1(Np^r)] \rightarrow \Phi^s_r/[\Phi^s_r, \Gamma_1(Np^r)] \rightarrow \Gamma_s/\Gamma_r \rightarrow 1
\]
is a central extension of a cyclic group, thus the middle group is abelian, implying that
\[
[\Phi^s_r, \Phi^s_r] = [\Phi^s_r, \Gamma_1(Np^r)].
\]
The equality holds because of $\Phi_s^r \supseteq \Gamma_1(Np^r)$ and the fact that the commutator subgroup of the group $\Phi_s^r/[\Phi_s^r, \Gamma_1(Np^r)]$ is trivial.

**Remark 2.3.** The following diagram is commutative

\[
\begin{array}{ccc}
\Phi_s^r \cap \Gamma^0(p) & \xrightarrow{i} & \Phi_s^r \\
\Gamma_1(Np^r) & \sim & \Gamma_1(Np^r) \\
\end{array}
\]

The diagonal map is an isomorphism, by Remark 2.2. Since $\Gamma_s/\Gamma_r$ is finite, we see that the inclusion $i$ is an isomorphism. This remark is useful in proving Lemma 3.6.

To prove Theorems 4.1 and 5.2, we need to understand the images of these morphisms

\[\Gamma_1(Np^r)^{ab} \rightarrow \Gamma_1(Np^s)^{ab}\]

in the chain of homology groups as in (2.1). Unfortunately, we do not have a good characterization these images for $r \geq s > 1$ in general, and so we cannot get a good description of the projective limit. This morphism can be factored as

\[\Gamma_1(Np^r)^{ab} \rightarrow \Gamma_1(Np^r)^{ab}/a_s \hookrightarrow \Phi_s^{ab} \rightarrow \Gamma_1(Np^s)^{ab},\]

and the problem is that the second and third morphisms may not be isomorphisms, in general.

Hida observed that if one applies a certain projection operator arising from the Atkin $U$-operator to all these modules then they become isomorphisms, in which case we have a good control over the images of the morphisms in (2.1). So we now define the Atkin $U$-operator and study their properties.

### 3. Hecke Operators

Suppose $G, H$ are two subgroups of a group $T$, and $t \in T$ such that $[G : t^{-1}Ht \cap G] < \infty$. Then one has

\[G^{ab} \xrightarrow{V} (t^{-1}Ht \cap G)^{ab} \xrightarrow{\sim} (H \cap tGt^{-1})^{ab} \rightarrow H^{ab},\]

where $V$ is the transfer map, the isomorphism is given by conjugating with $t$, and the last morphism is induced by $H \cap tGt^{-1} \hookrightarrow H$. Taking the composition of all these we obtain a morphism

\[[t] : G^{ab} \rightarrow H^{ab},\]

the \textit{“Hecke operator”} corresponding to $t$.

In our case, take $T = \text{GL}_2(\mathbb{Q})$, $G = H$ is a congruence subgroup of $\text{SL}_2$ of level divisible by $p$, and $t = \left(\begin{smallmatrix} 1 & 0 \\ b & p \end{smallmatrix}\right)$. We denote the corresponding Hecke operator by $U_2$.

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Phi_s^{a}$, we see that

\[t^{-1}At = \left(\begin{smallmatrix} a & bp \\ cp & d \end{smallmatrix}\right) \text{ and } tAt^{-1} = \left(\begin{smallmatrix} a & bp \\ cp & d \end{smallmatrix}\right).

**Remark 3.1.** Observe that $(1,1)$, $(2,2)$-entries of $A$ and of $t^{\pm 1}At^{\mp 1}$ are the same.
It is easy to see that \( t^{-1}\Phi_r^s t \cap \Phi_r^s = \Phi_r^s \cap \Gamma^0(p), \Phi_r^s \cap t\Phi_r^s t^{-1} = \Phi_r^{s+1} \), where the group \( \Gamma^0(p) \) is as in Remark 2.2. Thus, the Atkin \( U \)-operator (resp. \( U' \)-operator) is by definition the composition

\[
\Phi_r^{s+1} \xrightarrow{V} (\Phi_r^s \cap \Gamma^0(p))^{s+1} \xrightarrow{t^{-1}} \Phi_r^{s+1} \xrightarrow{t^{-1}} \Phi_r^{s+1}, \tag{3.1}
\]

(resp., the composition of the first two of above morphisms).

**Lemma 3.2.** Suppose that \( r \geq s > 1, r' \geq s' > 1, r \geq r', s \geq s' \), so that \( \Phi_r^s \subset \Phi_r^{s'} \).

Then the following diagram commutes

\[
\begin{array}{ccc}
\Phi_r^{s+1} & \xrightarrow{V} & \Phi_r^{s+1} \\
\downarrow & & \downarrow \\
\Phi_r^{s+1} & \xrightarrow{t^{'-1}} & \Phi_r^{s+1}
\end{array}
\]

Thus, the Atkin \( U \)-operator commutes with the morphism \( \Phi_r^{s+1} \rightarrow \Phi_r^{s+1} \).

**Proof.** The proof is similar to the proof of [1, Lem. 3.1]. The final statement follows from (3.1), since the Atkin \( U \)-operator, by definition, is the composition of \( U' \)-operator and the morphism induced by the inclusion of groups \( \Phi_r^{s+1} \subset \Phi_r^s \).

**Corollary 3.3.** For \( r \geq s > 1 \), each \( \Phi_r^{s+1} \) is a \( \mathbb{Z}[U] \)-module via the action of \( U \) and morphisms between these modules (arising from the inclusions) are morphisms of \( \mathbb{Z}[U] \)-modules. Hence, the cokernels of these morphisms acquire a \( \mathbb{Z}[U] \)-module structure.

Suppose \( \pi \) denote the morphism \( \pi : \Phi_r^{s+1} \rightarrow \Phi_r^{s+1} \) and \( \pi' \) for the morphism \( \pi' : \Phi_r^{s+1} \rightarrow \Phi_r^{s+1} \). Then, by Lemma 3.2.2, we have

\[
U' \circ \pi = \pi' \circ U' = U \in \text{End}_\mathbb{Z}(\Phi_r^{s+1}). \tag{3.2}
\]

By the definition of \( U' \), we see that \( \pi \circ U' = U \in \text{End}_\mathbb{Z}(\Phi_r^{s+1}) \).

By Corollary 3.3.3, the cokernel of the morphism \( \Gamma_1(Np^{r})^{s+1} \rightarrow \Phi_r^{s+1} \), for \( r \geq s > 1 \), is a \( \mathbb{Z}[U] \)-module and this cokernel is isomorphic to the group \( \Gamma_s/\Gamma_r \). Hence, the group \( \Gamma_s/\Gamma_r \) is a \( \mathbb{Z}[U] \)-module. Observe that \( \Phi_r^s = \Gamma_1(Np^{r}) \).

**Lemma 3.4.** The operator \( U \) acts on \( \Gamma_s/\Gamma_r \) as multiplication by \( p \).

**Proof.** The operator \( U \) acts on \( \Gamma_s/\Gamma_r \) as a multiplication by \( p \) if and only if it acts on \( \Phi_r^{s+1} \) as \( \bar{A} \mapsto \bar{A}^p \). The operator \( U \) is the composition of the following morphisms:

\[
\begin{array}{ccc}
\Phi_r^s & \xrightarrow{V} & (\Phi_r^s \cap \Gamma^0(p))^{s+1} \\
\downarrow & & \downarrow \\
\Phi_r^s & \xrightarrow{t^{-1}} & \Phi_r^s^{s+1}
\end{array}
\]

Let \( \{\alpha_i = \left( \begin{smallmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right)_{i=0}^{p-1} \} \) be the coset representatives of the group \( \Phi_r^s \cap \Gamma^0(p) \) in \( \Phi_r^s \). If we use these representatives to define the map in (3.3), then the transfer map looks like \( \bar{A} \mapsto \bar{A}^p \). By Remark 3.1, \( t\bar{A}^{p-1} \) and \( \bar{A}^p \) represent the same coset mod \( \Gamma_1(Np^{r}) \) and hence we are done.

\( \square \)
We would like to define an action of $\Gamma$ on $\Phi^s_{r+1}$ and call it the nebentypus action. This can be done as follows: For $r \geq 2$, if $d \in \Gamma_r$, then choose an element $\alpha = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ of $\text{SL}_2(\mathbb{Z})$ such that $p^{r+1} | c$ and $p | b$, i.e., $\alpha \in \Phi^2_{r+1} \cap \Gamma^0(p)$. Such an $\alpha$ exists, because

$$\Phi^s_{r+1} \cap \Gamma^0(p) \to \Gamma/\Gamma_{r+1} \to \Gamma/\Gamma_r.$$ 

The nebentypus action of $d$ on $\Phi^s_{r+1}$ is given by conjugation by $\alpha$. This action is well-defined because if $\alpha_1$ and $\alpha_2$ denote two lifts of $\tilde{d}$, then $\alpha_1^{-1}\alpha_2 \in \Gamma_1(Np^{r+1})\cap \Gamma^0(p) \subseteq \Phi^s_r$ and hence for any element $x \in \Phi^s_r$, $\alpha_1^{-1}\alpha_2 x \alpha_2^{-1}\alpha_1 = x$ in $\Phi^s_{r+1}$. Now we shall show that the actions of $U$ and $\Gamma$ commutes.

**Lemma 3.5.** If $r \geq s > 1$, the actions of $U$ and $\Gamma$ commutes on $\Phi^s_{r+1}$.

**Proof.** Though the proof of this lemma is similar to the proof of [1, Lem. 3.5.], here we make some remarks in between, hence we briefly recall its proof. It is easy to see that $\alpha(\Phi^s_r \cap \Gamma^0(p))\alpha^{-1} = \Phi^s_r \cap \Gamma^0(p)$ for any $\alpha \in \Phi^1_{r+1} \cap \Gamma^0(p)$, since $\alpha\Phi^s_r \alpha^{-1} \subseteq \Phi^s_r$. Look at the following commutative diagram

\[
\begin{array}{ccc}
\Phi^s_{r+1} & \xrightarrow{\alpha-\alpha^{-1}} & \Phi^s_r \\
V \downarrow & & V \\
(\Phi^s_r \cap \Gamma^0(p))_{ab} & \xrightarrow{\alpha-\alpha^{-1}} & (\alpha(\Phi^s_r \cap \Gamma^0(p))\alpha^{-1})_{ab} = (\Phi^s_r \cap \Gamma^0(p))_{ab} \\
\downarrow {t-t^{-1}} & & \downarrow {\alpha t^{-1}(\alpha t^{-1})^{-1}} \\
\Phi^s_{r+1} & \xrightarrow{\alpha-\alpha^{-1}} & (\alpha\Phi^s_{r+1}\alpha^{-1})_{ab} = \Phi^s_{r+1} \\
\downarrow {\alpha-\alpha^{-1}} & & \downarrow {\alpha(\alpha^{-1})_{ab}} \\
\Phi^s_{r+1} & \xrightarrow{\alpha-\alpha^{-1}} & (\alpha\Phi^s_{r+1}\alpha^{-1})_{ab} = \Phi^s_{r+1}.
\end{array}
\]

The top square in the diagram above commutes because if $\{\gamma_1, \ldots, \gamma_q\}$ form a set coset representatives for the group $\Phi^s_r \cap \Gamma^0(p)$ in $\Phi^s_r$, so is the set $\{\alpha\gamma_1\alpha^{-1}, \ldots, \alpha\gamma_q\alpha^{-1}\}$. Observe that, this diagram commutes even if $\alpha \in \Phi^1_{r+1} \cap \Gamma^0(p)$. The last square commutes by the functoriality of the transfer map.

We now prove the commutativity of the middle square, i.e., the map

$$\alpha t^{-1}(\alpha t^{-1})^{-1} : (\Phi^s_r \cap \Gamma^0(p))_{ab} \to \Phi^s_{r+1}$$

is $t - t^{-1}$. If $g \in \Phi^s_r \cap \Gamma^0(p)$, then

$$\alpha t^{-1}(\alpha t^{-1})^{-1} \alpha^{-1} = (\alpha t^{-1})\alpha^{-1} = (\alpha t^{-1}t)\alpha^{-1}.$$ 

Since $\alpha t^{-1}t \in \Gamma_1(Np^{r+1})$ for $\alpha \in \Phi^1_{r+1} \cap \Gamma^0(p)$, we see that the conjugation by $\alpha t^{-1}t$ induces identity on $\Phi^s_{r+1}$ (because elements of $\Phi^s_{r+1}$ do commute in $\Phi^s_{r+1}$).

In the above diagram composition of the vertical morphisms on either side are the operator $U$ and it commutes with the automorphism of $\Phi^s_r$ induced by conjugation by $\alpha$, but we know $\Gamma$ acts on $\Phi^s_r$ by conjugation by such elements $\alpha$. \qed

Observe that the inclusion $\Gamma_1(Np^r) \subseteq \Phi^s_r$ gives rise to the another transfer map

$$\Phi^s_{r+1} \xrightarrow{V} \Gamma_1(Np^r)_{ab}$$

**Lemma 3.6.** The transfer morphism $V : \Phi^s_{r+1} \to \Gamma_1(Np^r)_{ab}$ commutes with the action of $U$ on its source and target.
Proof. It suffices to prove that the following diagram (in which $V$ denotes the transfer maps between various abelianizations) commutes:

$$
\begin{array}{ccc}
\Phi_{r}^{s \text{ab}} & \xrightarrow{V} & \Phi_{r}^{r \text{ab}} \\
\downarrow & & \downarrow \\
(\Phi_{r}^{s} \cap \Gamma^{0}(p))^{\text{ab}} & \xrightarrow{V} & (\Phi_{r}^{r} \cap \Gamma^{0}(p))^{\text{ab}} \\
\downarrow t-t^{-1} & & \downarrow t-t^{-1} \\
\Phi_{r+1}^{s \text{ab}} & \xrightarrow{V} & \Phi_{r+1}^{r \text{ab}}
\end{array}
$$

The top square in the diagram above commutes because of functoriality of the transfer map. The commutativity of the bottom square follows by the following calculation.

If $\sigma_d = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, where $d$ runs through coset representatives of $\Gamma_r$ in $\Gamma_s$, forms a set of coset representatives for the group $\Gamma_{1}(Np^{r}) \cap \Gamma^{0}(p)$ in $\Phi_{r}^{s \text{ab}}$, then so are $t\sigma_d t^{-1} = \left( \begin{smallmatrix} a & b/p \\ c & d \end{smallmatrix} \right)$ for the group $\Gamma_{1}(Np^{r})$ in $\Phi_{r}^{r \text{ab}}$ (by Remark 2.3).

In this section, we have defined the $U$-operators for the congruence subgroups $\{ \Phi_{r}^{s} \}$ and proved that morphisms between these congruence subgroups respects the action of $U$ and this action commutes with the action of $\Gamma$.

4. Ordinary parts

Let $A$ be a commutative finite $\mathbb{Z}_p$-algebra and $U$ be a non-zero element of $A$. It well-known that $A$ factors as a product of local rings. Let $A^{\text{ord}}$ denote the product of all those local rings of $A$ in which the projection of $U$ is a unit. This is a flat $A$-algebra.

Let $M$ be any module in the abelian category of $\mathbb{Z}_p[X]$-modules which are finitely generated as $\mathbb{Z}_p$-modules. In this case, we take $A$ to be the image of $\mathbb{Z}_p[X]$ in $\text{End}_{\mathbb{Z}_p}(M)$, which is a finite $\mathbb{Z}_p$-algebra, and $U$ to be the image of $X$. We define

$$M^{\text{ord}} := M \otimes_A A^{\text{ord}}$$

and call this the ordinary part of $M$. Observe that taking ordinary parts is an exact functor on our abelian category.

If we consider $X$ to be the $U$-operator corresponding to the prime $p$, we may consider the ordinary part of the $\mathbb{Z}_p$-homology of the curve $Y_r$, i.e., the module $(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}}$, which is a $\Gamma$-module by Lemma 3.5.

We have the following theorem for the prime $p = 2$, which is similar to Theorem 3.1 in [3] for $p \geq 5$ and for the congruence subgroups $\Gamma_{1}(Np^{r})$ for $r \geq 1$.

Theorem 4.1. If $r \geq s > 1$, then the morphism of abelian groups

$$(\Gamma_1(Np^r) \otimes \mathbb{Z}_p)^{\text{ord}}/a_s \to (\Gamma_1(Np^s) \otimes \mathbb{Z}_p)^{\text{ord}}$$

is an isomorphism.

Proof. We shall show that

$$(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}}/a_s \xrightarrow{\sim} (\Phi_{r}^{s \text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}} \xrightarrow{\sim} (\Gamma_{1}(Np^{s})_{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}}. \quad (4.1)$$

If $\pi : \Phi_{r}^{s \text{ab}} \to \Phi_{r-1}^{s \text{ab}}$ is the morphism induced by the inclusion $\Phi_{r}^{s} \subset \Phi_{r-1}^{s}$, then $U' \circ \pi = U \in \text{End}(\Phi_{r}^{s \text{ab}})$, $\pi \circ U' = U \in \text{End}(\Phi_{r-1}^{s \text{ab}})$.
By Lemma 3.2, we have the following diagram

\[
\begin{array}{ccccccc}
\Phi_{s-1}^{ab} & \xrightarrow{\pi} & \Phi_{s-2}^{ab} \\
U' & \downarrow & U' \\
\Phi_{s}^{ab} & \xrightarrow{\pi} & \Phi_{s-1}^{ab} \\
U' & \downarrow & U' \\
\Phi_{s+1}^{ab} & \xrightarrow{\pi} & \Phi_{s}^{ab} \\
\end{array}
\]

The existence of \(U'\) implies that upon tensoring over \(\mathbb{Z}_p\) and taking the ordinary parts \(\pi\) induces an isomorphism (and \(U^{-1} \circ U'\) provides an inverse to \(\pi\))

\[(\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}} = (\Phi_{r-1}^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}.\]

By induction on \(r\), we obtain the second isomorphism in (4.1), i.e.,

\[(\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}} = (\Phi_s^{ab} \otimes \mathbb{Z}_p)^{\text{ord}} = (\Gamma(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}.\]

To prove the first isomorphism consider the short exact sequence

\[1 \rightarrow \Gamma_1(Np^r)^{ab}/a_s \rightarrow \Phi_r^{ab} \rightarrow (\Gamma_s/\Gamma_r) \rightarrow 1.\]

By tensoring this sequence with \(\mathbb{Z}_p\) and then taking the ordinary parts to obtain

\[1 \rightarrow (\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}/a_s \rightarrow (\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}} \rightarrow (\Gamma_s/\Gamma_r)^{\text{ord}} \rightarrow 1,
\]

because \(\mathbb{Z}_p\) is flat as a \(\mathbb{Z}\)-module and ordinary parts preserves exactness. By Lemma 3.4, the operator \(U\) acts on \(\Gamma_s/\Gamma_r\) as multiplication by \(p\) and so is a nilpotent operator, as \(\Gamma_s/\Gamma_r\) is a \(p\)-torsion group. Thus \((\Gamma_s/\Gamma_r)^{\text{ord}} = 0\), and hence the Theorem follows. \(\Box\)

## 5. Iwasawa modules

We have the following inverse system indexed by natural numbers \(r \geq 2,\)

\[\cdots \rightarrow \Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p \rightarrow \cdots \rightarrow \Gamma_1(Np^2)^{ab} \otimes \mathbb{Z}_p.\]

Define the Iwasawa module by

\[W := \lim_{r \geq 2} \Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p.\]

The profinite group \(\Gamma\) acts on the \(\mathbb{Z}_p\)-module \(\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p\) through its finite quotient \(\Gamma/\Gamma_r\). Thus the Iwasawa module \(W\) becomes a module over the completed group algebra

\[\Lambda := \mathbb{Z}_p[[\Gamma]] = \lim_{r \geq 2} \mathbb{Z}_p[\Gamma/\Gamma_r].\]

Though the Iwasawa module \(W\) is difficult to understand, by Theorem 4.1, we can understand the ordinary part of \(W\) very well. To make the statement clear, let us slightly abstract the situation.

Let \(\{M_r\}_{r \geq 2}\) be a system of \(\Lambda\)-modules. Further, assume that each \(M_r\) is pointwise fixed by \(\Gamma_r\) and hence a module over \(\Lambda/\mathfrak{a}_s\Lambda = \mathbb{Z}_p[\Gamma/\Gamma_r]\). For each \(r \geq s \geq 2\), we have a map \(M_r \rightarrow M_s\) such that it factors via

\[M_r/\mathfrak{a}_s M_r \rightarrow M_s.\]
Define \( W := \lim_{s \to 2} M_r \). We have a collection of maps \( W \to M_r \) for each \( r \geq 2 \) and they factor as \( W/a_r W \to M_r \).

**Proposition 5.1.** Assume that each \( M_r \) is \( p \)-adically complete and for each \( r \geq s \geq 2 \), \( M_r/a_r M_r \to M_s \) is an isomorphism. Then \( W/a_r W \to M_s \) is an isomorphism.

**Proof.** For \( r \geq s \geq 2 \), the maps \( M_r \to M_s \) are surjective, and hence the canonical map from \( W \to M_s \) is also surjective. We shall show that the kernel is \( a_r W \).

Since each \( M_r \) is \( p \)-adically complete and is point-wise fixed by \( \Gamma_r \), we have \( M_r = \lim_{s \to 2} M_r/n^s M_r \), where \( \Gamma_r = (\gamma_r) \) and \( n = (\gamma_r - 1, p) \), i.e., each \( M_r \) is \( n \)-adically complete.

By induction on \( i \), we get that \( \gamma_s^i - 1/\gamma_s - 1 \in (\gamma_s - 1, p)^i \). In particular, we have \( \gamma_2^{p^r - 2} - 1/\gamma_2 - 1 \in m = (\gamma_2 - 1, p) \). Hence, \( m^{p^r - 2} \subseteq ((\gamma_2 - 1)^{p^r - 2}, p) \subseteq n \subseteq m = (a_2, p) \).

As a result, we see that each \( M_r \) is \( m \)-adically complete, since they are \( n \)-adically complete. Once we have that each \( M_r \) is \( m \)-adically complete, then proving the injectivity of the above map is quite similar to the proof of [1, Prop. 5.1]. □

The following Theorem is an immediate consequence of the Proposition above.

**Theorem 5.2.** For any \( r \geq 2 \), we have

\[
W^{\text{ord}}/a_r W^{\text{ord}} \cong (\Gamma_1(Np^r)^{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}}
\]

is the \( \Gamma_r \)-co-invariants of \( W^{\text{ord}} \).

**Proof.** This follows from Proposition 5.1 together with Theorem 4.1 □

The above Theorem is a key ingredient for the proof of the Theorem 5.3. The \( \Lambda \)-module \( W^{\text{ord}} \) is a compact \( \Lambda \)-module (under the projective limit of the \( p \)-adic topologies on each module \( \Gamma_1(Np^r)^{\text{ab}} \otimes \mathbb{Z}_p \), which are free of finite rank over \( \mathbb{Z}_p \), and also since \( W^{\text{ord}} \) is a direct factor of \( W \).

Furthermore, Theorem 5.2 implies that the projective limit topology on \( W^{\text{ord}} \) coincides with its \( m \)-adic topology (where \( m = (a_2, p) \subset \Lambda \) denotes the maximal ideal of \( \Lambda \)), because the kernels of the projections \( \Lambda \to \mathbb{Z}_p/p^n \mathbb{Z}_p[\Gamma/\Gamma_r] \) are co-final with the sequence of ideals \( m^r \) in \( \Lambda \). Thus \( W^{\text{ord}} \) is a \( \Lambda \)-module, compact in its \( m \)-adic topology such that

\[
W^{\text{ord}}/m = W^{\text{ord}}/(a_2, p) = (\Gamma_1(Nq)^{\text{ab}} \otimes \mathbb{Z}_p/p)^{\text{ord}}
\]

is a finite dimensional \( \mathbb{Z}_p/p\mathbb{Z}_p \)-module, of dimension \( d \) (say). By Nakayama’s lemma, we have that \( W^{\text{ord}} \) is a finitely generated \( \Lambda \)-module with a minimal generating set has cardinality \( d \). We have the following theorem for the prime \( p = 2 \), which is similar to the main theorem in [2] for \( p \geq 5 \).

**Theorem 5.3 (Main Result).** The module \( W^{\text{ord}} \) is free of finite rank over \( \Lambda \), and its \( \Lambda \)-rank is equal to \( d \).

As a corollary, we see that, for \( r \geq 2 \), the \( \mathbb{Z}_p \)-rank of the free \( \mathbb{Z}_p \)-module \( (\Gamma_1(Np^r)^{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}} \) is \( d \). In particular, these \( \mathbb{Z}_p \)-ranks are independent of \( p^r \) in the level. Using this result, we have proved control theorems for ordinary 2-adic families of modular forms, see [2]. The classical versions of this theorem for \( p = 2, 3 \) do not seem to be explicitly available in the literature, though an ad`elic version of it can be found in [4].
6. Reflexivity results

To prove Theorem 5.3 it enough to show that $\mathbf{W}^{\text{ord}}$ is a reflexive $\Lambda$-module [5]. We show this by considering the duality theory of the modules $(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p)^{\text{ord}}$ and showing that they are reflexive as $\mathbb{Z}_p[\Gamma/\Gamma_r]$-modules. Now, we briefly recall the notion of reflexivity, and the necessary results. For more details, see [1] [6].

Suppose that $R$ is a commutative ring, $G$ is a finite group, and $M$ is a left $R[G]$-module. Let $N$ be any $R$-module. Then $\text{Hom}_R(M, N)$ is a right $R[G]$-module, via

$$(f * g)(x) := f(g^{-1}x).$$

Since the ring $R[G]$ is naturally a bi-module over itself, via the ring multiplication, $R[G] \otimes_R N$ is an $\text{R}[G]$-bi-module, making $\text{Hom}_{R[G]}(M, R[G] \otimes_R N)$ a right $R[G]$-module.

**Lemma 6.1** ([1]). *There is a canonical isomorphism of right $R[G]$-modules*

$$\text{Hom}_R(M, N) = \text{Hom}_{R[G]}(M, R[G] \otimes_R N).$$

In particular, when $N = R$, we see that $M^\ast$ and $\text{Hom}_{R[G]}(M, R[G])$ are canonically isomorphic as right $R[G]$-modules, where $M^\ast := \text{Hom}_R(M, R)$, the $R$-dual of $M$. The analogue of the above lemma for right $R[G]$-modules is also true. Hence,

$$\text{Hom}_R(M^\ast, R) = \text{Hom}_{R[G]}(M^\ast, R[G]).$$

are canonically isomorphic as left $R[G]$-modules.

By definition of $M^\ast$, there is a natural morphism of $R$-modules $M \to \text{Hom}_R(M^\ast, R)$, which is also a morphism of left $R[G]$-modules. If this natural morphism of $R$-modules is an isomorphism, then we say that $M$ is a reflexive $R$-module. Thus we have proved:

**Lemma 6.2.** *If $M$ is a left $R[G]$-module which is reflexive as an $R$-module, then $M$ is reflexive as an $R[G]$-module.*

The crux of this Lemma is that to check the reflexivity of $R[G]$-module $M$ over $R[G]$, it is enough to check it over $R$. Now we need to understand how to use the reflexivity results for modules over $\mathbb{Z}_p[\Gamma/\Gamma_r]$ to show the reflexivity of $\mathbf{W}^{\text{ord}}$ as a $\Lambda$-module.

7. Proof of Theorem 5.3

For $r \geq 2$, and $N \in \mathbb{N}$ such that $(p, N) = 1$. We define the cohomology of $Y_r$ as

$$H^1(Y_r, \mathbb{Z}_p) := \text{Hom}_\mathbb{Z}(\Gamma_1(Np^r)_{\text{ab}}, \mathbb{Z}_p) = \text{Hom}_{\mathbb{Z}_p}(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p).$$

The ring $\Lambda$ acts on $\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p$ through its quotient $\Lambda_r := \Lambda/\mathfrak{a}_r = \mathbb{Z}_p[\Gamma/\Gamma_r]$. More generally, if $r \geq s > 1$ then the ring $\Lambda_s$ is equal to $\Lambda_r/\mathfrak{a}_s$, hence $\Lambda_r \hookrightarrow \Lambda_s$. Thus we get the following sequence of morphisms of $\Lambda_r$-modules

$$\text{Hom}_{\Lambda_r}(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p, \Lambda_r) \to \text{Hom}_{\Lambda_s}(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p, \Lambda_r)/\mathfrak{a}_s$$

$$\to \text{Hom}_{\Lambda_s}(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p, \Lambda_s) = \text{Hom}_{\Lambda_s}(\Gamma_1(Np^r)_{\text{ab}} \otimes \mathbb{Z}_p/\mathfrak{a}_s, \Lambda_s).$$

If $M$ is any $\mathbb{Z}_p[U]$-module, which is finitely generated as a $\mathbb{Z}_p$-module, then so is the $\mathbb{Z}_p$-dual $M^\ast := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$. Here $M^\ast$ is a $\mathbb{Z}_p[U]$-module via the dual action of $U$. Clearly $(M^\ast)^{\text{ord}} = (M^{\text{ord}})^\ast$, i.e., taking ordinary parts commutes with duals. Thus
we may take ordinary parts of the above diagram of homomorphisms to obtain a diagram
\[ \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r) \rightarrow \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r)/a_s \rightarrow \text{Hom}_{\Lambda_r}((\Gamma_1(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_s) \]

By Theorem 4.1, we have
\[ \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r) \rightarrow \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r)/a_s \rightarrow \text{Hom}_{\Lambda_r}((\Gamma_1(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_s) \]

**Lemma 7.1.** The morphism
\[ \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r)/a_s \rightarrow \text{Hom}_{\Lambda_r}((\Gamma_1(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_s) \]
is an isomorphism.

**Proof.** By Lemma 3.6, we may restrict \( V \) to the ordinary parts to obtain a morphism
\[ (\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}} \rightarrow (\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}. \]

Look at the following commutative diagram
\[ \begin{array}{ccc}
\text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) & \stackrel{V^*}{\longrightarrow} & \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathbb{Z}_p}((\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) & & \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_r)/a_s \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) & \stackrel{\sim}{\longrightarrow} & \text{Hom}_{\Lambda_r}((\Gamma_1(Np^s)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \Lambda_s) \\
\end{array} \]
in which the two horizontal isomorphisms are those provided by Lemma 6.2, because \( \Lambda_r = \mathbb{Z}_p[\Gamma/\Gamma_r] \). The first vertical map \( V^* \) is the dual morphism of \( V \) and the two vertical isomorphisms are a part of Theorem 4.1 and its proof.

Now to prove the Lemma, it suffices to prove that
\[ \text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) \rightarrow \text{Hom}_{\mathbb{Z}_p}((\Phi_r^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) \] (7.1)
is surjective and \( \ker(V^*) = a_s \text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p)^{\text{ord}}, \mathbb{Z}_p) \).

Since \( V \) commutes with \( U \) and taking ordinary parts commutes with taking \( \mathbb{Z}_p \)-duals, the morphism in (7.1) is the ordinary part of the morphism
\[ \text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p), \mathbb{Z}_p) \rightarrow \text{Hom}_{\mathbb{Z}_p}((\Phi_r^{ab} \otimes \mathbb{Z}_p), \mathbb{Z}_p) \] (7.2)

Now, it suffices to show that the morphism \( V^* \) in (7.2) is surjective with kernel equal to \( a_s \text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^{ab} \otimes \mathbb{Z}_p), \mathbb{Z}_p) \), since taking ordinary parts is also exact and commutes with the action of \( \Gamma \). But, this claim was proved in [11, 8.8] for torsion-free groups \( H \) and \( G \) such that \( H \subseteq G \), instead of \( \Gamma \). Observe that, when \( p = 2 \) and \( r \geq s \geq 2 \), the groups \( \Gamma_1(Nq) \) and \( \Phi_s^r \) are torsion-free, since \( \Gamma_1(M) \) is torsion free for all \( M \geq 3 \).
We now have all the information needed to prove Theorem 5.3. Consider the chain of $\Lambda$-modules
$$\cdots \longrightarrow \text{Hom}_{\Lambda}((\Phi^\text{ab}_r \otimes \mathbb{Z}_p)^\text{ord}, \Lambda_r) \longrightarrow \cdots \longrightarrow \text{Hom}_{\mathbb{Z}_p}((\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord}, \mathbb{Z}_p).$$

**Lemma 7.2.** There is a canonical isomorphism
$$\text{Hom}_{\Lambda}(\mathbb{W}^\text{ord}, \Lambda) = \lim_{\leftarrow r} \text{Hom}_{\Lambda}((\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord}, \Lambda_r).$$

**Proof.** We have the following canonical isomorphisms
$$\text{Hom}_{\Lambda}(\mathbb{W}^\text{ord}, \Lambda) = \lim_{\leftarrow r} \text{Hom}_{\Lambda_r}((\mathbb{W}^\text{ord}/a_r, \Lambda_r) = \lim_{\leftarrow r} \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord}, \Lambda_r),$$
where the last isomorphism follows from the Theorem 5.2. □

**Proposition 7.3.** For $r > 1$, there is a canonical isomorphism
$$\text{Hom}_{\Lambda}(\mathbb{W}^\text{ord}, \Lambda)/a_r = \text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord}, \Lambda_r).$$

**Proof.** The claim follows from Lemma 7.1, Lemma 7.2, and Lemma 5.1. □

**Theorem 7.4.** The module $\mathbb{W}^\text{ord}$ is $\Lambda$-free.

**Proof.** Since any finitely generated reflexive $\Lambda$-module is free, it suffices to show that $\mathbb{W}^\text{ord}$ is a reflexive $\Lambda$-module. By Proposition 7.3 and Lemma 6.2 we have:
$$\text{Hom}_{\Lambda}(\text{Hom}_{\Lambda}(\mathbb{W}^\text{ord}, \Lambda), \Lambda) = \lim_{\leftarrow r} \text{Hom}_{\Lambda_r}(\text{Hom}_{\Lambda_r}(\mathbb{W}^\text{ord}, \Lambda)/a_r, \Lambda_r)$$
$$= \lim_{\leftarrow r} \text{Hom}_{\Lambda_r}(\text{Hom}_{\Lambda_r}((\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord}, \Lambda_r), \Lambda_r)$$
$$= \lim_{\leftarrow r} (\Gamma_1(Np^r)^\text{ab} \otimes \mathbb{Z}_p)^\text{ord} = \mathbb{W}^\text{ord}. $$

□

8. Acknowledgements

The author thanks Prof. M. Emerton for his encouragement to work out the details in [1] for the prime $p = 2$. These results turned out be quite useful in [2].

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