Prescaling in a far-from-equilibrium Bose gas

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Non-equilibrium conditions give rise to a class of universally evolving low-energy configurations of fluctuating dilute Bose gases at a non-thermal fixed point. While the fixed point and thus full scaling in space and time is generically only reached at very long evolution times, we here propose that systems can show prescaling much earlier, on experimentally accessible time scales. During the prescaling evolution, some well-measurable short-distance properties of the spatial correlations already scale with the universal exponents of the fixed point while others still show scaling violations. Prescaling is characterized by the evolution obeying already, to a good approximation, the conservation laws which are associated with the asymptotically reached non-thermal fixed point, defining its belonging to a specific universality class. In our simulations, we consider $N = 3$ spatially uniform three-dimensional Bose gases of particles labeled, e.g., by different hyperfine magnetic quantum numbers, with identical inter- and intra-species interactions. In this system, the approach of a non-thermal fixed point is marked by low-energy phase excitations self-similarly redistributing towards smaller wave numbers. During prescaling, the full $U(N)$ symmetry of the model is broken while the conserved transport, reflecting the remaining $U(1)$ symmetries, leads to the buildup of a rescaling quasicondensate distribution.

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Far from equilibrium, comparatively little is known about the possibilities nature reserves for the structure and states of quantum many-body systems. Much progress has been made recently, with new insight gained in the context of prethermalization [1, 2], generalized Gibbs ensembles [3–5], many-body localization [6], critical and prethermal dynamics [7–10], decoherence and revivals [11], and (wave) turbulence [12, 13]. With these examples it has been demonstrated that a quantum system quenched far from equilibrium can show types of relaxation behavior distinctly different from what is known in classical statistics. More recently, the approach of a non-thermal fixed point [14–17] has been observed, exhibiting universal scaling in time and space.

Universal scaling evolution, in classical systems coupled to a temperature bath, has been discussed in the context of dynamical critical phenomena [18, 19], coarsening and phase-ordering kinetics [20], as well as glassy dynamics and ageing [21]. For quantum systems, different types of prethermal dynamics after quenches have been discussed, including scaling phenomena [2, 22–42]. Non-thermal fixed points have been identified [14–17, 43–50], paving the way to a unifying description of universal dynamics. It remains, though, an unresolved question how in general quantum many-body systems evolve from a given initial state to such a fixed point. Here, we propose prescaling as a generic feature of the evolution towards the fixed point.

Universal scaling dynamics associated with a non-thermal fixed point is characterized by scaling evolution of correlation functions. For example, the occupation number $n_a(k, t) = \langle \Phi^\dagger_a(k, t) \Phi_a(k, t) \rangle$ of an $(N$-component) Bose field $\Phi_a(k, t)$, at the fixed point, evolves according to

$$n_a(k, t) = (t/t^{\ref})^{\alpha} f_{S,a}(t/t^{\ref})^{\beta} k^d,$$  

with universal scaling function $f_{S,a}(k) = n_a(k, t^{\ref})$ depending on a single $d$-dimensional variable only, scaling exponents $\alpha, \beta$, and some reference time $t^{\ref}$ within the scaling regime [15]. At the fixed point, all correlation functions of the system are expected to exhibit scaling. Generically, the fixed point is reached only in a certain scaling limit, such as, for $\beta > 0$, at asymptotic times and infinite volume. However, the question arises how the scaling limit is reached and to what extent and when scaling is already seen at finite times.

In equilibrium, fixed points of renormalization-group flows describe correlations at a continuous, e.g. second-order phase transition. They correspond to a pure rescaling of the correlations, in momentum or position space, under the change of the flow parameter such as a scale beyond which fluctuations are averaged over. In the context of critical phenomena as well as fundamental particle physics, more general functional renormalization flows are known to be attracted to partial fixed points where a symmetry is obeyed by the flow equations and thus conserved during the further flow [51]. In such situations, still away from the actual fixed point, scaling violations can occur for some quantities while others already show scaling.

Eq. (1) implies that, at a non-thermal fixed point, the flow parameter is the evolution time itself [15]. A few basic symmetry properties constrain the scaling exponents $\alpha$ and $\beta$, and allow a classification independent of the details of microscopic properties and initial conditions. Conservation laws in the isolated system constrain the particular realization of the scaling. For instance, if $\alpha = \beta d$, the total particle density, $\int d^d k n_a(k, t) = \text{const}$, is conserved at the fixed point (1). Besides this scaling relation also the value of $\beta$ depends on the conservation law [15].

Here, motivated by the concept of partial fixed points [22], we propose the existence of prescaling [52] in an isolated, $(N = 3)$-component dilute Bose gases in $d = 3$ spatial dimensions, quenched far out of equilibrium. Solving the field equations of motion within a semi-classical Truncated-Wigner
FIG. 1. (a) Time evolution exhibiting violations of universal scaling, at larger distances, of the single-component first-order coherence function $g_1^{(1)}(r) = g_1^{(1)}(r, t)$ (at five different times, colored dots). The yellow dotted lines show the scaling form (4), with $k_s(t)$ fitted to match the first zero of the sine. Grey dashed lines show the polynomial approximation of the sinc given in the legend. Up to the first zero in $r$, the numerical results at later times agree well with the single-scale function (4). At the latest time shown finite-size effects cause the first zero to disappear (purple dots). (b) Corresponding second-order coherence function measuring the spatial fluctuations of the relative phases between components 1 and 2, $g_{12}^{(2)}(r, t)$ (colored dots, same times as in (a)). Inset: Rescaled $\bar{g}_{12}^{(2)}(\bar{r}; t_{\text{ref}})$, with $\bar{r} = t/t_{\text{ref}}$, showing that violations of scaling are considerably weaker than for $g_1^{(1)}$.

approach we show that, during the approach of a non-thermal fixed point, the system prescales. This means that certain correlation functions, already at early times and at short distances, scale with the universal exponents predicted for the fixed point which itself can be reached only much later and in a finite-size system may not be reached at all. In our case these functions measure the spatial coherence of the local phase-angle differences between the components. At the same time, weak scaling violations occur in the evolution of other quantities and only slowly vanish, here in the single-component momentum occupations. While the $U(N)$ symmetry of the model is spontaneously broken by the flow, remaining $U(1)$ symmetries, reflecting the conservation laws obeyed during prescaling, are not. We emphasize that the scaling violations affect not only the scaling exponents but in particular also the shape of the scaling functions.

The spatially uniform Bose gases consist of identical particles distinguished only by a single property such as the hyperfine magnetic quantum numbers of the atoms forming the gas. The system in three spatial dimensions is described by an $O(3) \times U(1)$ as well as $U(3)$ symmetric Gross-Pitaevskii (GP) model with quartic contact interaction in the total density

$$H_{O(3)} = \int d^3x \left[ -\Phi_a \frac{\nabla^2}{2m} \Phi_a + \frac{g}{2} \Phi_a^4 + \frac{U}{2} \Phi_a^2 \Phi_b \Phi_b \right],$$

where we use units implying $\hbar = 1$, space-time field arguments are suppressed, $m$ is the particle mass, and it is summed over the Bose fields, $a, b = 1, 2, 3$, obeying standard commutators $[\Phi_a(x, t), \Phi_b^\dagger(y, t)] = i \delta_{ab} \delta(x - y)$. The gases are thus assumed to occupy the same space and be subject to identical inter- and intra-species contact interactions quantified by $g$.

The universal scaling dynamics at the non-thermal fixed point studied here can be described in a perturbative manner in terms of a low-energy effective theory for the phase-angle excitations of the Bose field $\Phi_a(x, t) = [\rho_a^{(0)} + \delta \rho_a(x, t)]^{1/2} \exp(\delta \theta_a(x, t))$, on the background of a constant mean phase $\theta_a^{(0)} = 0$ and density $\rho_a^{(0)}$, defining the breaking of the $U(3)$ symmetry [53]. After integrating out the density fluctuations $\delta \rho_a$, the linear modes of this effective model are given by the total phase $\sum_a \delta \theta_a$, with Bogoliubov dispersion $\omega_\beta(k) = \sqrt{\epsilon_k + 2g\rho^{(0)}}, \epsilon_k = k^2/2m$, and $N - 1$ gapless Goldstone excitations of the relative phases, e.g. $\delta \theta_a - \delta \theta_1$, with free-particle dispersion $\omega_\beta(k) = \epsilon_k$. These modes are interacting non-linearly, as a consequence of the kinetic-energy term in the Hamiltonian (2). The effective model, hence, is similar in character to the non-linear Luttinger-liquid description of low-energy phonons in a single dilute Bose gas, with the marked difference of a universal non-local coupling function appearing in the non-linear terms in $\delta \theta_a$ [53].

A scaling analysis of the kinetic equation $\partial_t f_a(k, t) = I[f](k, t)$ governing the momentum–space redistribution of the phase-angle excitations $f_a(k, t) = \delta \rho_a(k, t) \delta \theta_a(-k, t)$ at the fixed point gives the exponents $\alpha$ and $\beta$. Here, $I[f]$ is a quantum-Boltzmann-type collision integral involving scattering terms non-linear in the distributions $f_a$, arising from the non-linear couplings of the $\delta \theta_a$. One obtains, for $N \to \infty$ as well as $N = 1$, the values

$$\beta = 1/2, \quad \alpha = \beta d = 3/2$$

by requiring that both sides of the kinetic equation exhibit the same scaling in momentum and time [53], consistent with differently obtained results for $N \to \infty$ [15, 50]. The relation between $\alpha$ and $\beta$ reflects the conservation of the $d$-dimensional integral $\int k f_a(k, t)$. This particular fixed point has Gaussian character, i.e., in the limit $t \to \infty$, correlation functions factorize and the scaling of $f_a(k, t)$ implies the scaling of $n_a(k, t)$ as well as of higher-order correlators of the $\Phi_a$ [53].

Here, we numerically study the evolution of the system to-
FIG. 2. Prescaling of position-space correlations. (a) Scaling exponents $\beta_i$ describing the time evolution of $k_{\parallel}(t) \sim t^{\beta_i}$ with $i = 1, 2, 4$ as obtained from the polynomial fit of the first-order coherence function $g_1^{(1)}(r,t)$, shown in Fig. 1a, at small distances $r$. (b) Corresponding scaling exponents $\beta_i$ obtained from an analogous polynomial fit of the coherence function $g_{12}^{(2)}(r,t)$, shown in Fig. 1b, at small distances $r$. While $g_1^{(1)}(r)$, up to order $O(r^a)$ shows scaling violations, $g_{12}^{(2)}(r)$ already scales, to a good approximation, with the predicted exponent $\beta = 1/2$ for $t_{ref} + \Delta t \geq 300 \tau_\Xi$. In both panels, the $\beta_i(t)$ are averaged over time windows $[t_{ref} + \Delta t]$ with $\Delta t = 146 \tau_\Xi$. Error bars indicate the absolute error on the window-averaged $\beta_i$ arising from the fitted $k_{\parallel,i}$. See also appendix, Fig. 5a.

Towards this fixed point, starting from a far-from-equilibrium initial condition with momentum occupations of the field excitations constant up to some cutoff scale, $n_{\parallel}(k, \theta_0) = n_{\parallel}(k - |k|)$, drawn as the grey ‘box’ distribution in the inset of Fig. 3a. The initial phase angles $\theta_0(k, \theta_0)$ of the Bose fields $\Phi_0(k, \theta_0) = \sqrt{\tau_\Xi} \exp(i\theta_0(k, \theta_0))$ are chosen randomly on the circle and thus uncorrelated. In practice, such an initial condition can be achieved by, e.g., a strong cooling quench or an initial instability [14, 50]. The evolution induced by such an extreme initial condition will include transport of particles from $k \ll k_{{\parallel}i}$ towards the infrared, while their energy is deposited by a few particles at higher momenta, $k > k_{{\parallel}i}$. In this way the system, after a few collision times, shows universal scaling indicating the approach of a non-thermal fixed point [15, 48, 50].

We compute the time evolution of the correlation functions within the truncated Wigner approximation [54] which is expected to be well justified for the high occupancies prevailing throughout the evolution. The initial-state occupancy is $n_0 \approx 2350$, corresponding to a momentum cutoff $k_{{\parallel}i} = 1.4 \kappa_{{\parallel}i}$. Here, $\kappa_{{\parallel}i} = \Xi^{-1} = [2\mu\rho(0)]^{1/2}$ is a momentum scale set by the inverse healing length corresponding to the total density. A spectral split-step algorithm is used to solve the coupled Gross-Pitaevskii equations derived from (2), on a grid with $N_\Xi = 256^3$ points using periodic boundary conditions. The corresponding physical volume of our system is $V = N_\Xi \Xi^3$. The total particle number is $N = \rho(0)V = 6.7 \cdot 10^9$, i.e., we have $N_\Xi = 2.23 \cdot 10^7$ particles in each of the three components. The correlation functions are averaged over 144 trajectories.

Prescaling is most easily detected in position-space correlations. In Fig. 1a, we show the first-order spatial coherence function $g_{12}^{(1)}(r,t) = \langle \Phi_0^\dagger(x + r,t)\Phi_0(x,t) \rangle$. In the scaling regime, $r \gg \Xi$ and $t \geq 200 \tau_\Xi$ ($\tau_\Xi = 2\pi\rho(0)^{-1}$), it is found to approach the exponential $\times$ cardinal-sine form

$$g_{12}^{(1)}(r,t) \approx \rho(0)^{-1} r^{-k_{\parallel,i}|r|} \sin(k_{\parallel,i}(t)|r|), \tag{4}$$

$$(\sin(x) = \sin(x)/x)$$

where the particle density $\rho(0)$ is uniform while the phase oscillates and fluctuates on a scale given by the inverse coherence length $k_{\parallel,i}$. At the fixed point, this length is rescaling in time as $k_{\parallel,i}(t) \sim t^{\beta_i}$, with universal exponent $\beta_i$. Our numerical data confirms this scaling form, for $t = 245 \tau_\Xi$ up to the first zero of the sine, see Fig. 1a.

We stress that, as the non-linear term in (2) couples the total densities, it suppresses total density fluctuations but not fluctuations of the local density differences between the components. Hence, Goldstone excitations of the relative phases can become relevant. To demonstrate this we study the evolution of the second-order coherence function $g_{ab}^{(2)}(r,t) = \langle \Phi_0^\dagger(x + r,t)\Phi_0(x,t) \rangle$ sensitive to the relative phases $\theta_a - \theta_b$, see Fig. 1b for $(a,b) = (1,2)$.

The temporal scaling analysis of the numerically determined functions $g_{1}^{(1)}(r,t), g_{12}^{(2)}(r,t)$ provides a direct way to extract the scaling exponent $\beta_i$. A polynomial fit of $g_{ab}^{(1)}(r,t) = c_0(1 - k_{\parallel,i}(t)r + c_2[k_{\parallel,i}(t)]^2 - c_4[k_{\parallel,i}(t)]^4 + O(r^5))$ at $r \gg \Xi$, avoiding the short-distance thermal peak, allows studying scaling violations. Note that, for $c_0 = 1$, $c_2 = 1/3$, $c_4 = 1/30$, this corresponds to the Taylor-series of the form (4). Scaling is realized to order $m$ when $\beta_i \equiv \beta$ in $k_{\parallel,i}(t) \sim t^{\beta_i}$ for all $i \leq m$ with non-vanishing $c_i$. Taking the logarithmic derivative of $k_{\parallel,i}(t)$ with respect to $t$ and averaging it over a fixed time window gives the $\beta_i$ shown in Fig. 2, for $i = 1, 2, 4$, for both, $g_{1}^{(1)}$ and $g_{12}^{(2)}$. See the appendix for details.

The value $\beta_1 = 0.5$ found, at late times, for the scaling of $k_{\parallel,1}$, for $i = 1, 2$, parameterizing $g_{1}^{(1)}$, and for $i = 1, 2, 4$ in the case of $g_{12}^{(2)}$, confirms, to a very good approximation, the analytically predicted value $\beta = 1/2$ [15, 50]. While initially, the $g_{1}^{(1)}$ does not scale, scaling is approached at later times, when the $\beta_i$ are close to each other, corroborating that the fixed point is reached at asymptotically long evolution times. Depending on the observable, the system, at any finite time, appears close to but is away from the fixed point. This can also be expected from the oscillating long-distance form of $g_{1}^{(1)}$ in Fig. 1a which violates scaling. In contrast, the coherence function $g_{12}^{(2)}$ scales earlier over much larger distances (see inset of Fig. 1b).

As the system approaches the fixed point we hence observe prescaling. This is quantitatively seen in the scaling exponents $\beta_i$ shown in Fig. 2, see also the appendix. Furthermore, while the scaling exponents settle in to stationary values for lower orders, scaling in the higher orders is not yet fully developed within the time window considered. Note that the finite size of the system does not lead to continued scaling far beyond $t = 400 \tau_\Xi$. Comparing Figs. 2a and b we find that the time scales for establishing the scaling functions and the associated scaling exponents depend on the observable. This can also be intuitively concluded from comparing Figs. 1a and b.

It is remarkable that the prescaling exponents $\beta_i$ found for $g_{1}^{(1)}$ and $g_{ab}^{(2)}$ as shown in Fig. 2 agree with the (for $N = 1$ and $N \to \infty$) analytically predicted value $\beta = 1/2$ to a very
good accuracy, effectively leaving little space for anomalous deviations. This furthermore suggests that the universality class does not depend on $N$, reflecting that the $U(N)$ symmetry is broken during prescaling while the $U(1)$ symmetries are still intact, as no condensate is present yet, in particular in the relative-phase degrees of freedom. Similar values for $\alpha$ and $\beta$ have been found in recent experiments on a quasi one-dimensional three-component spinor Bose gas in which additional spin-changing interactions and Zeeman shifts break the $U(3)$ symmetry of the model considered here [16], freezing out also some of the relative-phase degrees of freedom. In the experiment, the scaling evolution is seen for relatively short distances and times.

In momentum space, the scaling evolution is seen as a self-similar transport to lower momenta. Fig. 3a shows that, for times $t \gtrsim t_{\text{ref}} = 31 \xi$, and within a limited range of low momenta, the evolution of the angle-averaged momentum distribution $n_1(k, t)$ exhibits approximate scaling in time $t$ and radial momentum $k = |k|$ according to (1). Rescaling the distributions at different times they appear to collapse to a single universal scaling function (main frame). Note that the transport of energy to the UV induces a thermal tail to appear which will prevent the system from reaching the fixed point, mathematically at $t \to \infty$, even in an infinite volume as the microscopic interaction properties gradually increase the thermal fraction of the gas.

The Fourier transform of (4) gives the scaling form

$$n_a(k, t) = C k_\Lambda(t) \left[4k_\Lambda^4(t) + |k|^4 \right]^{-1},$$

with some constant $C$ and $k_\Lambda(t) \sim t^\beta$, for $k \lesssim \xi$, see Fig. 3a as well as further details given in the appendix. The observed power-law fall-off beyond the inverse coherence-length scale, $n_a(k) \sim k^{-\zeta}$ for $k \gg k_\Lambda(t)$, with $\zeta \approx 4$, confirms predictions obtained by means of a large-$N$ non-perturbative kinetic theory [50, 55]. Hence, as the momentum occupancies approach the scaling behavior (1), we find the relation $\alpha \approx 3\beta$, which reflects the temporal conservation of the total particle number in each component and thus a conserved $U(1)$ symmetry.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{(a) Universal scaling dynamics of one of the single-component occupation numbers, $n_1(k) \equiv n_1(k, t)$ according to (1). The inset shows the evolution starting from the initial distribution $n_1(k, t_0) = n_0 \Theta(k_0 - |k|)$, identical in all three components $a$ (grey line), with $n_0 = (4\pi k_0^3)^{-1} \rho(0)$, $k_0 = 1.4 \xi$, at five different times (colored dots). The collapse of the data to the universal scaling function $f_{\text{sc}}(k) = n_1(k, t_{\text{ref}})$, with reference time $t_{\text{ref}} \xi = 31$ shows the scaling (1) in space and time. For the window between $t_{\text{ref}} \xi \lesssim t \lesssim 350 \xi$, we extract exponents $\alpha = 1.62 \pm 0.37$, $\beta = 0.53 \pm 0.09$, see also Fig. 4a. (b) Universal scaling dynamics of the correlator measuring the spatial fluctuations of the relative phases, $C_{12}(k, t) = \langle |\Phi^0 \Phi^0(k, t)|^2 \rangle$ for the same system. The universal scaling of $C_{12}(k, t)$ confirms our hypothesis and analytic prediction that the relative phases between the components scale similarly as $n_a$. Note that $C_{12}(k, t)$ does not show a plateau in the IR but follows a scaling function with similar fall-off at higher momenta as the one for $n_a(k, t)$. The power-law $C_{12} \sim k^{-4}$ in the scaling regime is only slightly modified as compared to that of $n_a$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Prescaling of momentum-space correlations. (a) Scaling exponents $\alpha/3$ and $\beta$ obtained from least-square rescaling fits of the occupancy spectra $n_1(k) \equiv n_1(k, t)$ shown in Fig. 3a. The exponents correspond to the mean required to collapse the spectra within the time window $[t_{\text{ref}} \xi, t_{\text{ref}} \xi + \Delta t]$ with $\Delta t = 146 \xi$, and the momentum window $[k_{\text{min}}, k_{\text{max}}]$, with $k_{\text{min}}$ set by the lowest non-zero radial momentum at $t = t_{\text{ref}} \xi$ and $k_{\text{max}} = 0.45 \xi^{-1}$, such that the slight bend of $n_1(k)$ to a steeper power law is excluded. Error bars denote the least-square fit error. (b) The same exponents extracted from the collapse of the momentum space correlations $C_{12}(k) \equiv C_{12}(k, t)$ shown in Fig. 3b, for the same time window. While $n_1(k)$ still shows scaling violations, prescaling is indicated by the scaling of $C_{12}(k)$ at $t_{\text{ref}} \xi + \Delta t \gtrsim 300 \xi$.}
\end{figure}
We stress that also here scaling violations prevail up to the maximum time $t \approx 400 \text{zs}$, see Fig. 4a, when finite-size effects gradually become relevant. During the late period, $t_{\text{ref}} = 200 \text{zs} \leq t \leq 350 \text{zs}$, one obtains the scaling exponents $\alpha = 1.62 \pm 0.37, \beta = 0.53 \pm 0.09$, with a trend towards a smaller $\beta$, cf. similar results found in [15]. The scaling violations are seen as a gradual change of the form of the distribution at low momenta, violating the scaling (1), cf. Fig. 5b in the appendix.

Also the Fourier transform of $C_{12}(\mathbf{r}, t)$ giving the coherence function $C_{12}(\mathbf{k}, t) = \langle |\langle \Phi | \Phi_0(\mathbf{k}, t) \rangle |^2 \rangle$ shows scaling earlier than $n_1$, cf. Figs. 3b and 4b. $C_{12}$ does not show the same IR plateau as $n_1$ but the same power-law fall-off $\sim k^{-4}$ at larger momenta. Thus, $g^{(2)}_{12}$ deviates from the exp $\times$ sinc-form (4), cf. the yellow dotted line in Fig. 1b.

Prescaling, observable in the relatively early evolution after a quench far from equilibrium, has the potential to play a decisive role for universal scaling evolution phenomena and their accessibility in experiments with ultracold atomic gases.

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APPENDIX

In the following we provide further details of the numerical analysis of prescaling in momentum and position space. To support our observation of prescaling further, we show, in Fig. 5a, the time evolution of characteristic scales $k_{\Lambda_i}(t)$, $i = 1, 2, 4$, extracted, as described in the main text, by fitting a scaling function of the form

$$S_{\text{fit}}^{(1)}(r) = c_0 \left( 1 - k_{\Lambda_1}(t)r + \frac{[k_{\Lambda_2}(t)r]^2}{3} - \frac{[k_{\Lambda_4}(t)r]^4}{30} \right)$$

(AI)

to the low-distance behavior of the numerically computed coherence function $g^{(1)}_{12}(r)$, see Fig. 1a. The three resulting scales change slowly in time. $k_{\Lambda_2}$ settles approximately to a constant at early times around $t \approx 100 \text{zs}$, reflecting scaling with $\beta_2 \approx 0.5$. At later times, also the first- and higher-order terms approach this scaling behavior, indicating that a scaling form is reached at late times.

In the main text we show that our numerical data confirms, at late evolution times, the proposed scaling form (4) to a very good accuracy, and thus the absence of a quadratic term $\propto |\mathbf{k}|^2$.
in the denominator of (5). This is seen in Fig. 5b where we compare the momentum distribution, at different times, with the two different scaling functions (5) as well as

\[ n^G_a(k, t) = \frac{C_d \delta^2 k \lambda(t)}{|k^2 + \{|k|^2|^2/2}. \tag{A2} \]

We find that, at early times, the data is better described by the scaling function (A2). In contrast, at late times, our data rather confirms the scaling function (5). The inset clearly shows that, in the late-time scaling regime, a single power law prevails in the inverse of the momentum distribution after subtracting a constant, while at early times, the quadratic term violates scaling.

Note that the scaling form (A2) corresponds to the angle-averaged spatial first-order coherence function \( g^{\langle x \rangle}(x - y, t) = (\Phi(x, t)\Phi(y, t)) \) of the Bose field \( \Phi(x, t) \) having the form of a pure exponential,

\[ g^{\langle x \rangle}_a(r, t) = \rho^{(0)}_a \exp \left\{ -\frac{k \lambda(t)|r|}{|r|} \right\}, \tag{A3} \]

with uniform particle density \( \rho^{(0)}_a \) and background phase \( \vartheta^{(0)}_a \). Its Fourier transform yields the occupation number distribution of the bosons, with normalization constant \( C_d = \rho^{(0)}_a \Gamma(d + 1/2)/\pi^{(d+1)/2} \) and momentum exponent \( \zeta = d + 1 \).

[1] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Science 337, 1318 (2012).
[2] J. Berges, S. Borsanyi, and C. Wetterich, Phys. Rev. Lett. 93, 142002 (2004), arXiv:hep-ph/0403234 [hep-ph].
[3] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Science 348, 207 (2015).
[4] E. T. Jaynes, Phys. Rev. 106, 620 (1957).
[5] E. T. Jaynes, Phys. Rev. 108, 171 (1957).
[6] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science 349, 842 (2015).
[7] S. Braun, M. Friesdorf, S. S. Hodgman, S. Schreiber, J. P. Romzheimer, A. Riera, M. del Rey, I. Bloch, J. Eisert, and U. Schneider, PNAS 112, 3641 (2015).
[8] E. Nicklas, M. Kuhl, M. Höfer, A. Johnson, W. Müssel, H. Strobel, J. Tomkovic, T. Gasenzer, and M. K. Oberthaler, Phys. Rev. Lett. 115, 245301 (2015), arXiv:1509.02173 [cond-mat.quant-gas].
[9] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Science 347, 167 (2015), arXiv:1410.8487 [cond-mat.quant-gas].
[10] C. Eigen, J. A. P. Liddle, R. Lopes, A. E. Cornell, R. P. Smith, and Z. Hadzibabic (2018), arXiv:1805.09802 [cond-mat.quant-gas].
[11] B. Rauer, S. Erne, T. Schweigler, F. Cataldini, M. Tavora, and J. Schmiedmayer, Science (2018), 10.1126/science.aan7938.
[12] V. E. Zakharov, V. S. L’vov, and G. Falkovich, Kolmogorov Spectra of Turbulence I: Wave Turbulence (Springer, Berlin, 1992).
[13] N. Navon, A. L. Gaunt, R. P. Smith, and Z. Hadzibabic, Nature (London) 539, 72 (2016), arXiv:1609.01271 [cond-mat.quant-gas].
[14] J. Berges, A. Rothkopf, and J. Schmidt, Phys. Rev. Lett. 101, 041603 (2008), arXiv:0803.0131 [hep-th].
[15] A. Pinheiro Orioli, K. Boguslavski, and J. Berges, Phys. Rev. D 92, 025041 (2015), arXiv:1503.02498 [hep-ph].
[16] M. Prüfer, P. Kunkel, H. Strobel, S. Lannig, D. Linnemann, C.-M. Schmied, J. Berges, T. Gasenzer, and M. K. Oberthaler (2018), arXiv:1805.11881 [cond-mat.quant-gas].
[17] S. Erne, R. Buecker, T. Gasenzer, J. Berges, and J. Schmiedmayer (2018), arXiv:1805.12310 [cond-mat.quant-gas].
[18] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
[19] H. Janssen, in Dynamical critical phenomena and related topics, Lecture Notes in Physics, vol. 104 (Springer, Heidelberg 1979, 1979) p. 26.
[20] A. J. Bray, Adv. Phys. 43, 357 (1994).
[21] P. Calabrese and A. Gambassi, J. Phys. A: Math. Gen. 38, R133 (2005).
[22] G. Aarts, G. F. Bonini, and C. Wetterich, Phys. Rev. D 63, 025012 (2000).
[23] A. Lamacraft, Phys. Rev. Lett. 98, 160404 (2007).
[24] D. Rossini, A. Silva, G. Mussardo, and G. E. Santoro, Phys. Rev. Lett. 102, 127204 (2009).
[25] E. G. Dalla Torre, E. Demler, and A. Polkovnikov, Phys. Rev. Lett. 110, 090404 (2013).
[26] A. Gambassi and P. Calabrese, Europhys. Lett. 95, 66007 (2011), arXiv:1012.5294 [cond-mat.stat-mech].
[27] B. Sciolla and G. Biroli, Phys. Rev. B 88, 201110 (2013), arXiv:1211.2572 [cond-mat.stat-mech].
[28] P. Smacchia, M. Knap, E. Demler, and A. Silva, Phys. Rev. B 91, 205136 (2015).
[29] A. Maraga, A. Chiochetta, A. Mitra, and A. Gambassi, Phys. Rev. E 92, 042151 (2015).
[30] A. Maraga, P. Smacchia, and A. Silva, Phys. Rev. B 94, 245122 (2016).
[31] A. Chiochetta, M. Tavora, A. Gambassi, and A. Mitra, Phys. Rev. B 91, 220302 (2015).
[32] A. Chiochetta, M. Tavora, A. Gambassi, and A. Mitra, Phys. Rev. B 94, 134311 (2016).
[33] A. Chiochetta, A. Gambassi, S. Diehl, and J. Marino, Phys. Rev. B 94, 174301 (2016).
[34] A. Chiochetta, A. Gambassi, S. Diehl, and J. Marino, Phys. Rev. Lett. 118, 135701 (2017).
[35] J. Marino and S. Diehl, Phys. Rev. Lett. 116, 070407 (2016).
[36] J. Marino and S. Diehl, Phys. Rev. B 94, 085150 (2016), arXiv:1606.00452 [cond-mat.quant-gas].
[37] K. Damle, S. N. Majumdar, and S. Sachdev, Phys. Rev. A 54, 5037 (1996).
[38] S. Mukerjee, C. Xu, and J. E. Moore, Phys. Rev. B 76, 104519 (2007).
[39] L. A. Williamson and P. B. Blakie, Phys. Rev. Lett. 116, 025301 (2016).
[40] J. Hofmann, S. S. Natu, and S. Das Sarma, Phys. Rev. Lett. 113, 095702 (2014), arXiv:1403.1284 [cond-mat.quant-gas].
[41] L. A. Williamson and P. B. Blakie, Phys. Rev. A 94, 023608 (2016).
[42] A. Bourges and P. B. Blakie, *Phys. Rev. A* **95**, 023616 (2017).
[43] C. Scheppach, J. Berges, and T. Gasenzer, *Phys. Rev. A* **81**, 033611 (2010), arXiv:0912.4183 [cond-mat.quant-gas].
[44] J. Berges and D. Sexty, *Phys. Rev. D* **83**, 085004 (2011), arXiv:1012.5944 [hep-ph].
[45] B. Nowak, D. Sexty, and T. Gasenzer, *Phys. Rev. B* **84**, 020506(R) (2011), arXiv:1012.4437v2 [cond-mat.quant-gas].
[46] B. Nowak, J. Schole, D. Sexty, and T. Gasenzer, *Phys. Rev. A* **85**, 043627 (2012), arXiv:1111.6127 [cond-mat.quant-gas].
[47] J. Schole, B. Nowak, and T. Gasenzer, *Phys. Rev. A* **86**, 013624 (2012), arXiv:1204.2487 [cond-mat.quant-gas].
[48] J. Berges, in *Proc. Int. School on Strongly Interacting Quantum Systems Out of Equilibrium, Les Houches*, edited by T. Giamarchi et al. (OUP, Oxford, 2016) arXiv:1503.02907 [hep-ph].
[49] M. Karl and T. Gasenzer, *New J. Phys.* **19**, 093014 (2017).
[50] I. Chantesana, A. Piñeiro Orioli, and T. Gasenzer (2018), arXiv:1801.09490 [cond-mat.quant-gas].
[51] C. Wetterich, *Phys. Lett. B* **104**, 269 (1981).
[52] C. Wetterich, private communication.
[53] A. N. Mikheev, C.-M. Schmied, and T. Gasenzer, ArXiv e-prints (2018), arXiv:1807.10228 [cond-mat.quant-gas].
[54] P. B. Blakie, A. S. Bradley, M. J. Davis, R. J. Ballagh, and C. W. Gardiner, *Adv. Phys.* **57**, 363 (2008), arXiv:0809.1487v2 [cond-mat.stat-mech].
[55] R. Walz, K. Boguslavski, and J. Berges, *Phys. Rev. D* **97**, 116011 (2018).