Square-Root Algorithms for Maximum Correntropy Estimation of Linear Discrete-Time Systems in Presence of non-Gaussian Noise

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Abstract

Recent developments in the realm of state estimation of stochastic dynamic systems in the presence of non-Gaussian noise have induced a new methodology called the maximum correntropy filtering. The filters designed under the maximum correntropy criterion (MCC) utilize a similarity measure (or correntropy) between two random variables as a cost function. They are shown to improve the estimators’ robustness against outliers or impulsive noises. In this paper we explore the numerical stability of linear filtering technique proposed recently under the MCC approach. The resulted estimator is called the maximum correntropy criterion Kalman filter (MCC-KF). The purpose of this study is two-fold. First, the previously derived MCC-KF equations are revised and the related Kalman-like equality conditions are proved. Based on this theoretical finding, we improve the MCC-KF technique in the sense that the new method possesses a better estimation quality with the reduced computational cost compared with the previously proposed MCC-KF variant. Second, we devise some square-root implementations for the newly-designed improved estimator. The square-root algorithms are well known to be inherently more stable than the conventional Kalman-like implementations, which process the full error covariance matrix in each iteration step of the filter. Additionally, following the latest achievements in the KF community, all square-root algorithms are formulated here in the so-called array form. It implies the use of orthogonal transformations for recursive update of the required filtering quantities and, thereby, no loss of accuracy is incurred. Apart from the numerical stability benefits, the array form also makes the modern Kalman-like filters better suited to parallel implementation and to very large scale integration (VLSI) implementation. All the MCC-KF variants developed in this paper are demonstrated to outperform the previously proposed MCC-KF version in two numerical examples.

Keywords: Maximum correntropy criterion, Kalman filter, square-root filtering, robust estimation.

1. Introduction

In the past few years, the study of filtering techniques under the maximum correntropy criterion (MCC) has become an important aspect of a hidden state estimation of stochastic dynamic systems in the presence of non-Gaussian noise [1, 2, 3, 4]. The MCC methodology implies that a statistical metric of a similarity between two random variables (or correntropy) is used as a cost function (or performance index) for designing the corresponding estimation method. The resulted MCC filters have become the methods of choice in signal processing and machine learning due to its robustness against outliers or impulsive noises compared to the classical Kalman filtering (KF); e.g., see the discussion in [5, 6, 7, 8, 9] and many others.

Being a linear estimator, the KF is an attractive and simple technique that requires only the computation of mean and covariance for constructing the optimal estimate of unknown dynamic state under the minimum mean square (MMS) criterion. For Gaussian systems, this estimate is optimal, i.e. the KF reduces to an MMS estimate rather than a linear MMS estimate. It is clear that in non-Gaussian setting, the classical KF exhibits sub-optimal behavior only. Due to this fact, there was a need for a new estimator that improves the KF robustness against outliers or impulsive noises.

For linear non-Gaussian state-space models, the robust maximum correntropy Kalman filter (MCKF) and the maximum correntropy criterion Kalman filter (MCC-KF) have been recently developed in [10, 11] and [12], respectively. As all Kalman-like filtering algorithms, they compute the first two moments (i.e. the mean and the covariance) for constructing the optimal estimate. However, in contrast to the classical KF, these recent developments utilize the robust MCC as the optimality criterion, instead of using the MMS cost function. As a result, the new filters are shown to outperform the classical KF and several nonlinear Kalman-like filtering techniques in the presence of non-Gaussian uncertainties in the state-space models. Nevertheless, little attention is paid to numerical stability of the Kalman-like filters developed under the MCC strategy, although the classical KF is widely known to suffer from the influence of roundoff errors, severely; see [13, 14]. Our research has tended to focus on the MCC-KF technique and the design of its numerically stable square-root implementations.

The purpose of this paper is two-fold. First, we revise the previously derived MCC-KF equations and prove the related Kalman-like equality conditions. Based on this theoretical find-
ing, we improve the previously proposed MCC-KF algorithm in the sense that the new filter (abbreviated as IMCC-KF) possesses a better estimation quality with the reduced computational cost. Second, we devise some square-root IMCC-KF implementations grounded in numerically robust orthogonal transformations. The square-root strategy is the most popular approach used for enhancing the filter numerical robustness; see [15, 16, 17, 18] etc. It implies the Cholesky decomposition of error covariance matrix and, then, recursive re-calculation of its Cholesky factors instead of using full matrix. Following the latest achievements in the KF community, all square-root algorithms are formulated here in the so-called array form. This means that numerically stable orthogonal transformations are used as far as possible for updating the Cholesky factors in each iteration step. This provides a more reliable estimation procedure as explained in [19, Chapter 12]. Apart from numerical advantages, array Kalman-like algorithms are easier to implement than the explicit filter equations, because all required quantities are simply read off from the corresponding filter post-arrays. As mentioned in [18], this makes the modern KF-like algorithms better suited to parallel implementation and to very large scale integration (VLSI) implementation. Finally, all algorithms developed in this paper are demonstrated to outperform the previously proposed MCC-KF technique in two numerical examples.

2. Maximum Correntropy Criterion Kalman Filter

Consider the state-space equations

\[ x_k = F_{k-1} x_{k-1} + G_{k-1} w_{k-1}, \quad k \geq 1, \] (1)
\[ z_k = H_k x_k + v_k \] (2)

where \( x_k \in \mathbb{R}^n \) and \( z_k \in \mathbb{R}^m \) are the unknown dynamic state and the observable measurement vector, respectively. The processes \( \{w_k\} \) and \( \{v_k\} \) are zero-mean, white, uncorrelated, and have known covariance matrices \( Q_k \) and \( R_k \), respectively. They are also uncorrelated with the initial state \( x_0 \), which has the mean \( \bar{x}_0 \) and the covariance matrix \( \Sigma_0 \).

The KF associated with state-space model (1), (2) yields the linear MMS estimate, \( \hat{x}_{0\mid 0} \), of the unknown dynamic state, given the available measurements \( \{z_1, \ldots, z_k\} \). To improve the filter estimation quality in the presence of non-Gaussian noise, the MCC optimality criterion can be used instead of the MMS cost function for deriving the corresponding filtering equations. The performance index to be optimized under the MCC (with Gaussian kernel) approach is found as follows [4, 12]:

\[ J_m(x_k) = G_x (\|z_k - H_k x_k\|) + G_x (\|x_k - F_{k-1} x_{k-1}\|) \]

where \( G_x (\|x_k - y_k\|) = \exp \left\{ -\|x_k - y_k\|^2 / (2\sigma^2) \right\} \), and \( \sigma > 0 \) is the kernel size or bandwidth.

Minimization of the objective function \( J_m \) with respect to \( x_k \) implies \( \partial J_m / \partial x_k = 0 \) and yields the equation [4]:

\[ (x_k - F_{k-1} x_{k-1}) = \frac{G_x (\|z_k - H_k x_k\|)}{G_x (\|x_k - F_{k-1} x_{k-1}\|)} H_k^T (z_k - H_k x_k). \] (3)

We note that the best estimate for state vector \( x_{k-1} \) at time point \( k - 1 \) is a posteriori estimate \( \hat{x}_{k-1\mid k-1} \). Hence, from (3) one obtains the following nonlinear equation, which needs to be solved with respect to \( x_k \):

\[ x_k = F_{k-1} \hat{x}_{k-1\mid k-1} + \frac{G_x (\|z_k - H_k x_k\|)}{G_x (\|x_k - F_{k-1} \hat{x}_{k-1\mid k-1}\|)} H_k^T (z_k - H_k x_k). \] (4)

The fixed point correntropy filter developed in [4] and the MCC-KF method proposed in [12] suggest to use a fixed point rule for solving the mentioned nonlinear equation with initial approximation \( x_0^{(0)} = \hat{x}_{0\mid 0} \) at the right-hand side of (4). Besides, both techniques imply only one iteration of the fixed point rule and, hence, by substituting \( x_k \approx \hat{x}_{k\mid k-1} \) into the right-hand side of formula (4) we obtain the following recursion

\[ \hat{x}_{k\mid k-1} = F_{k-1} \hat{x}_{k-1\mid k-1} + \frac{G_x (\|z_k - H_k \hat{x}_{k\mid k-1}\|)}{G_x (\| \hat{x}_{k\mid k-1\mid k-1} - F_{k-1} \hat{x}_{k-1\mid k-1}\|)} H_k^T (z_k - H_k \hat{x}_{k\mid k-1}). \]

Next, the MCC-KF method designed in [12] integrates the KF minimum-variance estimation with the maximum correntropy filtering. In particular, the cited paper utilizes the norm \( \| \cdot \|_{p_{k-1}} \) induced by the inverse measurement covariance matrix \( R_k^{-1} \) in the numerator and the norm \( \| \cdot \|_{p_{k-1}} \) induced by the inverse predicted process covariance matrix \( F_{k-1} P_{k-1|k-1} F_{k-1}^T + Q_{k-1} \) in the denominator of the recursion above. Thus, the MCC-KF is given as follows; see Algorithm 2 in [12]:

 Initialization:

\[ \hat{x}_{0\mid 0} = \mathbf{E} \{ x_0 \}, \quad P_{0\mid 0} = \mathbf{E} \{ (x_0 - \hat{x}_{0\mid 0})(x_0 - \hat{x}_{0\mid 0})^T \}. \] (5)

Prior estimation:

\[ \hat{x}_{k\mid k-1} = F_{k-1} \hat{x}_{k-1\mid k-1}, \quad P_{k\mid k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T. \] (6)

Posterior estimation:

\[ L_k = \frac{G_x (\|z_k - H_k \hat{x}_{k\mid k-1\mid k-1}\|)}{G_x (\| \hat{x}_{k\mid k-1\mid k-1} - F_{k-1} \hat{x}_{k-1\mid k-1}\|)}. \] (7)

\[ K_k = (P_{k\mid k-1}^{-1} + L_k H_k^T R_k^{-1} H_k)^{-1} L_k H_k^T R_k^{-1}, \]
\[ \hat{x}_{k\mid k} = \hat{x}_{k\mid k-1} + K_k^T (z_k - H_k \hat{x}_{k\mid k-1}), \]
\[ P_{k\mid k} = (I - K_k^T H_k) P_{k\mid k-1} (I - K_k^T H_k)^T + K_k^T R_k (K_k^T)^T. \] (8)

In the equations above, we use the new notation \( K_k^T \) for the gain matrix \( (P_{k\mid k-1}^{-1} + L_k H_k^T R_k^{-1} H_k)^{-1} L_k H_k^T R_k^{-1} \) appeared in Algorithm 2 in [12], emphasizing the dependence of this quantity on the scalar \( L_k \). This also helps us to distinguish this matrix from the classical KF feedback gain in the rest of our paper.

The readers are referred to [12] for a detailed derivation and properties of the MCC-KF estimator under consideration. In the cited paper, the MCC-KF is shown to outperform the classical KF, the fixed point correntropy filter from [4] and several nonlinear filtering techniques when the non-Gaussian uncertainties arise in stochastic system (1), (2).
It is worth noting here that because of utilizing only one iteration of a fixed point rule for solving the underlying nonlinear equation (4), we have $G_r ([\tilde{\xi}_{kk-1} - F_k \tilde{\xi}_{kk-1}]) = G_r ([0]) = 1$ since $\tilde{\xi}_{kk-1} = F_{k-1} \tilde{\xi}_{k-1}$. Hence, both methods in [4, 12] can be simplified since the denominator in (8) is equal to 1. For further iterates, this is not the case and the difference might be considerable. For this reason, the general form of (8) is used in this paper.

The kernel size $\sigma$ plays a significant role in the behavior of any correntropy filter. For instance, the MCKF developed in [10] was shown to be reduced to the standard KF as $\sigma \to \infty$. Here, we follow the adaptive strategy suggested in [12] and implemented in [20] for choosing $\sigma$ (i.e. $\sigma = \|z_k - H_k \tilde{\xi}_{kk-1}\|_R_1$ in each iteration step) in order to provide a fair comparative study with the earlier published MCC-KF method. This strategy is also motivated by a case study presented in [4]. We stress that the problem of optimal kernel size selection is beyond the scope of this paper.

In this paper, we explain how the MCC-KF estimation quality can be further enhanced. The new improved filter is based on the Kalman-like equations proved in Section 3. Additionally, we derive two numerically stable square-root implementations, which are the main purpose in the present study.

3. New Improved Maximum Correntropy Criterion KF

The previously proposed MCC-KF algorithm was shown to be coincident with the classical KF when $L_k = 1$; see [12]. To begin designing a new estimator, we first note that for the classical KF the following formulas hold [21, p. 128-129]:

$$K_k = P_{kk} H_k^T R_k^{-1}$$

$$= (I - K_k H_k) P_{kk-1} (\text{12})$$

$$P_{kk} = P_{kk-1} + L_k H_k^T R_k^{-1}$$

$$= (I - K_k H_k) P_{kk-1}$$

$$= (I - K_k H_k) P_{kk-1} (I - K_k H_k)^T + K_k R_k K_k^T$$

(13)  

We stress the following theoretical result.

Lemma 1. Consider state-space model (1), (2) where non-Gaussian uncertainties might arise. Similarly to the classical KF, the following formulas can be proved when the filter feedback gain obeys (9):

$$K_k^f = \left( P_{kk-1}^{-1} + L_k H_k^T R_k^{-1} \right)^{-1} L_k H_k^T R_k^{-1}$$

$$= (I - K_k^f H_k) P_{kk-1} (I - K_k^f H_k)^T + K_k^f R_k K_k^T$$

(14) (15) (16)

where $L_k$ is computed by formula (8).

Proof: First, we note that formula (9) of the original MCC-KF, i.e. $K_k^f = (P_{kk-1}^{-1} + L_k H_k^T R_k^{-1} H_k)^{-1} L_k H_k^T R_k^{-1}$, is obtained by a simple substitution of (19) into (17). Next, we prove the equivalence of formulas (17) and (18) used for the feedback gain computation $K_k^f$. Having substituted (20) into (17), we arrive at

$$K_k^f = P_{kk-1} L_k H_k^T R_k^{-1} - K_k^f H_k P_{kk-1} L_k H_k^T R_k^{-1}$$

(17)

and, finally, we get

$$K_k^f = P_{kk-1} L_k H_k^T \left( R_k + H_k P_{kk-1} L_k H_k^T \right)^{-1}$$

Hence,

$$K_k^f (I + H_k P_{kk-1} L_k H_k^T R_k^{-1}) = P_{kk-1} L_k H_k^T R_k^{-1}$$

$K_k^f (R_k + H_k P_{kk-1} L_k H_k^T) R_k^{-1} = P_{kk-1} L_k H_k^T R_k^{-1}$$

(18)

For the classical KF we have $K_k = (P_{kk-1}^{-1} + L_k H_k^T R_k^{-1})^{-1}$ where $R_{k,k} = H_k P_{kk-1} L_k H_k^T + R_k$. Similarly we define $R_e^f = H_k^T P_{kk-1} L_k H_k^T + R_k$ and the formula above can be written in the following form:

$$K_k^f = P_{kk-1} L_k H_k^T \left( R_e^f \right)^{-1}$$

(19)

where $R_e^f = H_k P_{kk-1} L_k H_k^T + R_k$. This is exactly equation (18).

Next, we need to prove the equivalence between formulas (19), (20) and (21) for computing a posteriori error covariance matrix, $P_{kk}$. First, taking into account the matrix inversion lemma, i.e. Sherman-Morrison-Woodbury formula [22]:

$$(A + UCV)^{-1} = A^{-1} - A^{-1} \left(U^{-1} + V A^{-1} U^{-1}\right)A^{-1}$$

and by substituting (18) into equation (19) we obtain

$$P_{kk} = (P_{kk-1}^{-1} + L_k H_k^T R_k^{-1})^{-1} = P_{kk-1}$$

$$= P_{kk-1} L_k H_k^T R_k^{-1}$$

(20)

The last expression in the formula above is exactly equation (20). Hence, the Kalman-like formulas (19), (20) hold when the feedback gain $K_k^f$ obeys (9).

Finally, we wish to prove (21). With trivial manipulations, formula (20) is transformed to the form

$$P_{kk} = (I - K_k^f H_k) P_{kk-1} + K_k^f R_k \left(K_k^f\right)^T - K_k^f R_k \left(K_k^f\right)^T$$

(21)

Next, by substituting (20) into (17), we have

$$K_k^f = (I - K_k^f H_k) P_{kk-1} L_k H_k^T R_k^{-1}$$

(22)

Taking into account the fact that $P_{kk-1}$ is symmetric, and by substituting (23) into (22), we derive equation (21) as follows:

$$P_{kk} = (I - K_k^f H_k) P_{kk-1} + K_k^f R_k \left(K_k^f\right)^T$$

$$= (I - K_k^f H_k) P_{kk-1} L_k H_k^T R_k^{-1} \left(K_k^f\right)^T$$

$$= (I - K_k^f H_k) (P_{kk-1} L_k H_k^T R_k^{-1} + K_k^f R_k \left(K_k^f\right)^T - R_k) \left( K_k^f \right)^T$$

(23)

Hence, the algebraic equivalence between expressions (20), (21) is proved. This means that the Kalman-like formulas (19) – (21) hold for a posteriori error covariance matrix when the feedback gain $K_k^f$ obeys (9) or equivalently (17), (18). This completes the proof.
Remark 1. It is interesting to note that there is a method-invariant form for calculating a posteriori error covariance $P_{k|k}$ in the classical KF and the MCC-KF approach. More precisely, equations (15) and (20) are the same in their forms, except the way of computing the feedback gain; see the term $K_k$ in (15) and the term $K_k^T$ in (20). We stress that the other formulas for covariance calculation differ in their forms.

Remark 2. The classical KF equation (16) is used in the original MCC-KF algorithm proposed in [12] for calculating the error covariance matrix $P_{k|k}$; see equation (11). In contrast to the previously-proposed MCC-KF version, Lemma 1 suggests to use formula (21) instead. It involves the covariance inflation parameter $L_k$ in computing $P_{k|k}$ (which is not the case for the original MCC-KF). It is worth noting here that this inflation parameter is computed based on the MCC cost function and can serve as a scale to control information inflation of $P_{k|k}$.

The theoretical result obtained in Lemma 1 suggests that equation (21) should be used instead of (11) in the MCC-KF computational scheme. More precisely, formula (9) for the feedback gain computation seems to be inconsistent with the MCC-KF error covariance calculation by (11) because of the missing multiplier $L_k$ in (11); see formula (21). Hence, the accuracy of the MCC-KF filter might be improved with the above-proven theoretical result. To begin constructing the new improved MCC-KF technique (IMCC-KF), we first replace equation (11) by (21). The results of numerical experiments presented in Section 5 confirm that this simple amendment improves the state estimation accuracy of the original MCC-KF.

Next, we note that the MCC-KF feedback gain $K_k^T$ calculation in (9) requires two $n \times n$ and one $m \times m$ matrices’ inversions, because $P_{k|k-1} \in \mathbb{R}^{n \times n}$, $(P_{k|k-1}^{-1} + L_k L_k^T K_k^{-1} H_k) \in \mathbb{R}^{m \times m}$ and $R_k \in \mathbb{R}^{m \times m}$. Therefore, such MCC-KF implementation becomes impractical when the dimensions of the dynamic state and the measurement vector increase. Apart from the computation complexity issue, it is also preferable to avoid the matrix inversion operation. The latter is particularly advantageous from the numerical stability viewpoint. In our novel IMCC-KF technique, we use formula (18) instead of (9) in the feedback gain computation $K_k^T$. This modification avoids two $n \times n$ matrices’ inversions and requires only one inversion of the matrix $R_k^{-1} = H_k H_k^T + R_k$. The described amendment can be performed because of the algebraic equivalence of equations (9) and (17), (18) proved in Lemma 1. Additionally, following the discussion in [21, p. 129] we suggest to use computationally simpler expression (20) than (21) for calculating $P_{k|k}$. As proved in Lemma 1, these are mathematically equivalent and, hence, can be both utilized in the $P_{k|k}$ calculation. In summary, all the mentioned improvements yield the new IMCC-KF estimator summarized in the form of Algorithm 1 below.

**Algorithm 1. IMCC-KF (Improved conventional version)**

1. **Initialization:** $(k=0) \hat{x}_{0|0} = \bar{x}_0$ and $P_{0|0} = \Pi_0$.
2. **Time Update:** $(k=1, \ldots, N) \triangleright$ Posteriori estimation
   
   \[ \hat{x}_{k|k-1} = F_{k-1} \hat{x}_{k-1|k-1}. \]
   \[ P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T. \]
3. **Measurement Update:** $(k=1, \ldots, N) \triangleright$ Posteriori estimation
   
   \[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k^T (z_k - H_k \hat{x}_{k|k-1}). \]
   \[ P_{k|k} = (I - K_k^T H_k) P_{k|k-1}. \]

The above-presented IMCC-KF is formulated in the so-called conventional form, i.e. Algorithm 1 recursively updates the entire matrices $P_{k|k-1}$ and $P_{k|k}$ in each iteration step of the filter. The method can be improved further by noting that any covariance matrix is symmetric and, hence, only its upper-triangular (or lower-triangular) part is to be re-calculated in each iteration step of the filter, only. To implement the mentioned modification, the square-root (SR) approach is widely used in the KF community. The resulted SR filters are inherently more stable (with respect to roundoff errors) than any conventional implementation and, hence, they are preferable for practical use; see the numerical results of ill-conditioned tests in [23, Chapter 6].

**4. New Square-Root IMCC-KF Implementations**

The most popular approach for designing factored-form KF implementations (square root filters) is grounded in the covariance Cholesky decomposition; see the detailed explanation in [23, p. 18]. The important fact to be taken into account is that the Cholesky decomposition exists and is unique when the symmetric matrix to be decomposed is positive definite [24]. If the matrix is a positive semi-definite, then the Cholesky decomposition still exists, however, it is not unique [25]. More precisely, the Cholesky decomposition implies the factorization of a symmetric positive definite matrix $A$ in the following form: $A = (A^{1/2})^T (A^{1/2})$. Such factors can be made unique by insisting, for instance, that the factors have a triangular form (with positive diagonal elements) or to be symmetric [26]. In most applications, the triangular form is preferred. However, we may remark that sometimes it is not required and, hence, other SR filtering variants might be considered for various reasons; e.g., it saves computations in [26].

In this paper, we use the Cholesky decomposition of a symmetric positive definite matrix $A$ in the following form: $A = A^{1/2} A^{1/2}$ where $A^{1/2}$ is an upper triangular matrix with positive diagonal elements. For convenience, we also write $A^{-1/2} \equiv (A^{1/2})^{-1}$, $A^{-T/2} \equiv (A^{-1/2})^T$. The key idea of the SR filtering strategy is a replacement of the state error covariance matrix, $P$, by its Cholesky factors and, then, re-formulation of the filtering equations in terms of these factors $P^{T/2}$ and $P^{1/2}$ only. Undoubtedly, the SR approach is not free of roundoff errors, however, it is motivated by two considerations [15]: 1) the product $P^{T/2} P^{1/2}$ can never be indefinite, even in the presence of roundoff errors, while roundoff errors sometimes cause the computed value of $P$ to be indefinite; 2) the numerical conditioning of $P^{1/2}$ is generally much better than that of $P$.

More precisely, the condition number $K(P) = K(P^{T/2} P^{1/2}) = [K(P^{1/2})]^T$. This means that while numerical operation with $P$ may encounter difficulties when $K(P) = 10^p$, the SR filter should function until $K(P) = 10^{2p}$, i.e. with double precision.”
Furthermore, modern SR methods imply QR factorization in each iteration step of the filter for updating the corresponding Cholesky factors as follows: first, the pre-array $A$ is built from the filter quantities that are available at the current step. Next, an orthogonal operator $\Psi$ is applied to the pre-array in order to get an upper triangular (or lower triangular) form of the post-array $R$ such that $\Psi A = R$. Finally, the updated filter quantities are simply read off from the post-array $R$, see [19, Chapter 12].

Taking into account that $L_k$ in (8) is a scalar value, two SR-based IMCC-KF implementations are designed, below.

**Algorithm 2. SR-based IMCC-KF (Square-root algorithm)**

**Initialization:** (k=0)
1. Apply Cholesky decomposition: $\Pi_0 = \Pi_0^{T/2} \Pi_0^{1/2}$
2. Set the initial values: $\hat{x}_0 = \tilde{x}_0$ and $\hat{P}_{00} = \Pi_0^{1/2}$.

**Time update:** (k=1, ..., N) $\triangleright$ Priori estimation
3. Repeat line 2 of the IMCC-KF to find $\hat{x}_{k|k-1}$.
4. Compute $L_k$ by MCC-KF formula (8).
5. Finally, taking into account that the error covariance matrix is symmetric and $L_k$ is a scalar, from (26), (27), we obtain $\hat{P}_{k|k} = L_k^2 \hat{K}_k^T (L_k^T)^{-T/2}$ (27).

The last expression in the formula above is exactly the equation in line 7 of the conventional IMCC-KF (Algorithm 1). This completes the proof of algebraic equivalence between the IMCC-KF and its SR-based variant (Algorithm 2).

The analysis of Algorithms 1 and 2 suggests that their implementations demand one $m \times m$ matrix inversion; see lines 5 and 7, respectively. However, in contrast to the conventional implementation (Algorithm 1), the SR-based variant (Algorithm 2) requires the inversion of only upper triangular matrix $(R_k^0)^{1/2}$ instead of $R_k^0$. The latter can be done by the computationally cheap backward substitution.

Eventually, Algorithm 2 can be improved further such that the new method avoids $R_k^0$ (or its Cholesky factor) inversion. Following [18], we develop the extended SR-based version.

**Algorithm 3. ESR-based IMCC-KF (extended SR algorithm)**

**Initialization:** (k=0)
1. Apply Cholesky decomposition: $\Pi_0 = \Pi_0^{T/2} \Pi_0^{1/2}$
2. Set $P_{0|0} = \Pi_0^{1/2}$ and $P_{0|0}^{-1} \tilde{x}_0 = \Pi_0^{1/2} \tilde{x}_0$.

**Time update:** (k=1, ..., N) $\triangleright$ Priori estimation
3. Pre-array
4. Read off $\hat{P}_{k|k}^{-1} \tilde{x}_{k|k-1}$ from the post-array.
5. Compute $\hat{P}_{k|k}^{-1} \tilde{x}_{k|k-1}$. which is exactly the equation in line 3 of the conventional IMCC-KF.
**Measurement Update:** \(k=1, \ldots, N\) \(\Rightarrow\) Posteriori estimation

6. Compute \(L_k\) by MCC-KF formula (8).

7. \[ \begin{bmatrix} R^{1/2}_{k} & L^{1/2}_{k} \\ L^{1/2}_{k} & P^{1/2}_{k} \end{bmatrix} \begin{bmatrix} H_k & 0 \\ 0 & P^{1/2}_{k} \end{bmatrix} \begin{bmatrix} -R^{1/2}_{k} \end{bmatrix} \begin{bmatrix} \xi_{k-1}^\star \end{bmatrix} \] Pre-array

8. \[ \begin{bmatrix} \xi_{k}^\star \end{bmatrix} \begin{bmatrix} \xi_{k-1}^\star \end{bmatrix} \] Post-array

where \(\Psi\) is any orthogonal operator such that the first two block columns of the post-array is upper triangular.

9. Read off \(\left( P^{1/2}_{k}\right) \) and \(\left( P^{1/2}_{k} \right) \xi_{k}^\star \) from the post-array.

As can be seen, the state vector estimates \(\hat{x}_{k-1}\) and \(\hat{x}_{k}\) are computed by a simple multiplication of the blocks that are directly read off from the corresponding post-arrays; see equations in lines 5, 9 of Algorithm 3. The block (*) in the post-array of the eSR-based IMCC-KF means that these entries are of no interest in the presented filter implementation.

The novel eSR-based IMCC-KF (Algorithm 3) is algebraically equivalent to the SR-based IMCC-KF (Algorithm 2) and, hence, to the conventional implementation in Algorithm 1. Indeed, from line 3 of Algorithm 3, we have

\[ F_{k-1}P^{T/2}_{k-1}P^{T/2}_{k-1}\hat{x}_{k-1} = F_{k-1}\hat{x}_{k-1} \]

which is exactly the equation in line 2 of the IMCC-KF (Algorithm 1). Its SR-based version (Algorithm 2) utilizes the same formula for the \(\hat{x}_{k-1}\) computation, i.e. \(\hat{x}_{k-1} = F_{k-1}\hat{x}_{k-1}\).

Next, line 7 in Algorithm 3 yields (24) – (26) and (28) – (29).

Taking into account that \(L_k\) is a scalar, from (28), we have

\[ \hat{e}_{k} = \left(R^{L}_{e,k}\right)^{-T/2}L^{1/2}_{k}(z_k - H \hat{x}_{k-1}) = \left(R^{L}_{e,k}\right)^{-T/2}L^{1/2}_{k}e_{k} \]

where \(e_{k} = z_k - H \hat{x}_{k-1}\) is the filter residual and, hence, the quantity \(\hat{e}_{k}\) is its "normalized" version.

From formula (29), we get \(\hat{x}_{k} = \hat{x}_{k-1} + \hat{K}^{e}_{k}\hat{e}_{k}\). By substituting the expressions for the "normalized" feedback gain (27) and the "normalized" residual (30), we get

\[ \hat{x}_{k} = \dot{x}_{k}^\star + \hat{K}^{e}_{k}\hat{e}_{k} \]

The second expression in the formula above is the equation in line 7 of Algorithm 2, meanwhile the last expression coincides with line 6 of Algorithm 1. This completes the proof of algebraic equivalence of all Algorithms 1-3.

5. Numerical Experiments

To fulfill a fair comparative study of the newly-developed methods and the original MCC-KF proposed in [12], we consider the test problem in the cited paper and the accompanying MATLAB codes, which are freely available in [20]. To provide the same experimental conditions for all methods under examination, we incorporate our IMCC-KF algorithms into the cited codes, where the original MCC-KF technique has been already implemented.

**Example 1.** Consider a land vehicle dynamic measured in time intervals of 3 seconds, i.e. \(\Delta t = 3\) sec., with the heading angle \(\psi\) (which is set to \(\psi = 60\) deg.), as follows:

\[ x_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 0 \Delta t \sin \psi \\ \Delta t \cos \psi \end{bmatrix} u_{k-1} + w_{k-1}, \]

\[ z_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k + v_k \]

with the initial conditions being \(\hat{x}_0 = [1, 1, 0, 0]^T\) and \(\Pi_0 = \text{diag}([1, 4, 3, 3])\). Additionally, two cases of the process and measurement noises are employed, below.

**Case 1.** All entries of \(w_k\) and \(v_k\) are comprised of Gaussian noise plus shot noise, i.e. \(w_k = N(\mu_w, Q) + \text{Shot noise}\), and \(v_k = N(\mu_v, R) + \text{Shot noise}\). The means and covariances are taken to be \(\mu_w = 0, \mu_v = 0\) and \(Q = \text{diag}([0.1, 0.1, 0.1, 0.1])\), \(R = \text{diag}([0.1, 0.1])\), respectively.

**Case 2.** The \(w_k\) and \(v_k\) are Gaussian mixture noises. The \(w_k\) is a mixture of \(N(\mu_{w1}, Q_1)\) and \(N(\mu_{w2}, Q_2)\). The \(v_k\) is a mixture of \(N(\mu_{v1}, R_1)\) and \(N(\mu_{v2}, R_2)\). The means and covariances are taken to be \(\mu_{w1} = [-3, -3, -3, -3]^T, \mu_{w2} = [2, 2, 2, 2]^T, \mu_{v1} = [2, 2]^T, \mu_{v2} = [-2, -2]^T, Q_1 = Q_2 = Q\) and \(R_1 = R_2 = R\).

In our comparative study, the following methods are tested: 1) the original MCC-KF scheme given by equations (6) – (11); 2) the original MCC-KF where only one equation is corrected, i.e. formula (11) is replaced by (21); 3) the improved method, i.e. the IMCC-KF (Algorithm 1); 4) the SR-based IMCC-KF (Algorithm 2); and 5) the eSR-based IMCC-KF (Algorithm 3).

We repeat the set of numerical experiments from [12]. More precisely, the stochastic model in Example 1 is simulated for \(k = 1, \ldots, 300\) to generate the measurement history. For that, the shot noise (case 1) and Gaussian mixture noise (case 2) are treated separately. They are implemented exactly as in [20]. Next, the inverse problem is solved, i.e. the optimal dynamic state estimate is computed by each filtering technique under examination. We repeat the experiment \(M = 100\) times. To judge the quality of the estimators, the root mean square error (RMSE) in each component of the state vector averaged over 100 Monte Carlo runs is computed as follows:

\[ \text{RMSE}_{x_k} = \sqrt{\frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} \left( x_{k, \text{exact}} - \hat{x}_{k, j,k} \right)^2} \]

where \(M = 100\) is the number of Monte-Carlo trials, \(N = 300\) is the number of discrete time points, the \(x_{k, \text{exact}}(t_k)\) and \(\hat{x}_{k, j,k}^\star(t_k)\)
are the $i$-th entry of the “true” state vector (simulated) and its estimated value obtained in the $j$-th Monte-Carlo trial, respectively. The results of this set of numerical experiments are summarized in Table 1, where we report the RMSE$_{x_i}$, $i = 1, \ldots, 4$, the $\|\text{RMSE}_e\|_2$ and the CPU time (s) averaged over $M = 100$ Monte-Carlo simulations. All methods were implemented in the same precision (64-bit floating point) in MATLAB running on a conventional PC with processor Intel(R) Core(TM) i5-2410M CPU 2.30 GHz and with 4 GB of installed memory (RAM).

Table 1 allows for the following conclusions. First, having compared the first two rows, we observe that a simple replacement of formula (11) in the previously derived MCC-KF by equation (21) in accordance with Lemma 1 enhances the filter estimation accuracy. Recall that the difference between these two versions is only in the error covariance $P_{kk}$ computation. Although the previously proposed MCC-KF (6)-(11) is formulated in the symmetric Joseph stabilized form (11), the results of numerical experiments suggest that appearance of multiplier $L_k$ in (11) according to Lemma 1 improves the estimation accuracy in all entries of the vector $\hat{x}_{kk}$ in the shot noise (case 1) and in half of the state vector components in the Gaussian mixture noise (case 2). The aggregated $\|\text{RMSE}_e\|_2$ quantity is less for both noise cases when the MCC-KF methodology is implemented via recursion (6)-(10), (21). In summary, the previously developed MCC-KF given by equation (6)-(11) is less accurate compared to its novel implementation presented in Section 3, where $L_k$ is taking into account. Concerning the computational complexity, the difference between these two implementations is imperceptible and, hence, their average CPU time is the same.

Second, Table 1 says that Algorithms 1-3 maintain the same estimation accuracy in both noise case scenarios. This conclusion holds for all entries of the state vector to be estimated and for the aggregated $\|\text{RMSE}_e\|_2$ values. Hence, our theoretical expectations are realized. Indeed, the mentioned finding was anticipated, since Algorithms 1-3 are proved to be algebraically equivalent in Section 4. It is also in line with the main theoretical result proved in Lemma 1. In other words, the outcome of our numerical experiments substantiates the theoretical derivations of Lemma 1 in practice.

Concerning the computational complexity of the original MCC-KF and the novel IMCC-KF algorithm, we observe the following: the CPU time is two times lower for the IMCC-KF than for the MCC-KF. It was expected since the IMCC-KF does not require two $n \times n$ matrices’ inversion for calculating the gain in contrast to the original MCC-KF. Besides, the new IMCC-KF uses a computationally simpler expression for $P_{kk}$ computation than the original MCC-KF.

Next, consider the SR algorithms. In general, they are more computationally expensive, but inherently more stable and reliable than any conventional implementation, because of the use of numerically stable orthogonal transformations in each iteration step of the filter. We observe that for our low-dimensional problem in Example 1 ($n = 4, m = 2$), the average CPU time of the SR IMCC-KF is almost the same as in the corresponding conventional implementation, i.e. in the IMCC-KF.

Finally, recall that the SR-based variants (Algorithms 2, 3) are numerically more robust (with respect to roundoff errors) than the conventional implementation (Algorithm 1). However, Example 1 does not suite for examination of numerical insights of the proposed computational schemes. Further, we equip Example 1 with an ill-conditioned measurement model, as it is often done in the KF literature while investigating the filter numerical instability issues; e.g., see [23].

Example 2. Consider a land vehicle dynamic from Example 1

$$z_k = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 + \delta \end{bmatrix} x_k + v_k, \; v_k \sim N(0, R)$$

where $R = \text{diag}([\delta^2, 0^2])$. To simulate roundoff we assume that $\delta^2 < \epsilon_{\text{roundoff}}$, but $\delta > \epsilon_{\text{roundoff}}$ where $\epsilon_{\text{roundoff}}$ denotes the unit roundoff error\(^1\). Again, two cases of the process and measurement noises are employed.

Case 1. All entries of $w_k$ are comprised of Gaussian noise plus shot noise, i.e. $w_k = N(\mu_k, Q) + \text{Shot noise with } \mu_k = 0$ and $Q = \text{diag}[0.1, 0.1, 0.1, 0.1]$.\(^2\)

Case 2. It is the same as in Example 1.

We repeat the experiments described above for 100 Monte Carlo runs and various ill-conditioning parameter values $\delta$. The source of the difficulty is in the matrix $R_{2,k}$ inversion. More precisely, we remark that although rank $H = 2$, the matrix $R_{2,k}$ becomes severely ill-conditioned as $\delta \to \epsilon_{\text{roundoff}}$, i.e. to machine precision limit. In summary, we provide the following set of numerical experiments. For each value of the parameter $\delta$, we solve the state estimation problem as it is done above in Example 1. Then, we summarize the aggregated values $\|\text{RMSE}_e\|_2$ in Table 2 for each filter implementation under examination.

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\(^1\)Computer roundoff for floating-point arithmetic is often characterized by a single parameter $\epsilon_{\text{roundoff}}$ defined in different sources as the largest number such that either $1 + \epsilon_{\text{roundoff}} = 1$ or $1 + \epsilon_{\text{roundoff}}/2 = 1$ in machine precision.

\(^2\)In the context of Example 2, $\epsilon_{\text{roundoff}}$ is a computer roundoff term for floating-point arithmetic.
should be also stressed that the shot noise (case 1) is added only in the process equation in this example. Besides, following the MCC-KF implementation in [20], the noise covariances should be adapted to the sample values when the noises are generated. However, in our numerical experiments we do not adjust the covariance $R$ to the corresponding sample value.

Having analyzed the obtained results presented in Table 2, we can explore the numerical behaviour of each algorithm while growing problem ill-conditioning. We again conclude that for both noise scenarios the previously developed MCC-KF given by equations (6)-(11) is less accurate compared with the implementation via formulas (6)-(10), (21), although equation (11) is formulated in the symmetric stabilized form. Both mentioned algorithms belong to the class of conventional implementations. From the second panel of Table 2, we observe their fast degradation while $\delta \to \epsilon_{\text{roundoff}}$, the machine precision limit. Indeed, already for $\delta \approx 10^{-6}$ they provide essentially no correct digits in the computed state estimate for Gaussian mixture noise, since the $\|\text{RMSE}_2\|$ value is ‘NaN’; see the first two rows in Table 2. In MatLab, the term ‘NaN’ stands for ‘Not a Number’ that actually means the failure of numerical method. The interesting fact is that for shot noise (case 1) the symmetric Joseph stabilized form (11) seems to be more robust to roundoff errors compared to the implementation via equation (21), which is not symmetric because of the multiplier $L_i$. Hence, we can conclude that the original MCC-KF (6)-(11) is less accurate compared with the implementation via formulas (6)-(10), (21), but seems to be more robust to roundoff errors because of the symmetric Joseph stabilized form (11).

Next, we study the new IMCC-KF (Algorithm 1) that is also of conventional (non-square-root) type. We observe that the original MCC-KF based on the Joseph stabilized implementation (11) is less accurate and less robust than Algorithm 1 in the both noise cases under consideration. This finding is reasonable if we recall that the new improved method (Algorithm 1) avoids two $n \times n$ matrices’ inversions compared to the original MCC-KF via (6)-(11) and the algorithm via recursion (6)-(10), (21); see the discussion in Section 3.

Finally, the last two rows of Table 2 are considered. It can be seen that the SR-based algorithms (Algorithms 2, 3) outperform any conventional implementation in the estimation accuracy in the both noise case scenarios. Indeed, they degrade more slowly than any other examined counterpart as $\delta \to \epsilon_{\text{roundoff}}$. Thus, the outcome of these numerical experiments substantiates their inherent numerical stability with respect to roundoff errors.

### 6. Concluding Remarks

In this paper, the recently proposed MCC-KF technique is revised. As a result, the new improved estimator and two its SR-based variants are developed. Although all new algorithms are algebraically equivalent, the results of the numerical experiments suggest that the SR-based filters provide the best estimation quality when solving ill-conditioned state estimation problem in the presence of non-Gaussian noise. The elaborated state estimation techniques are planned to be extended to state estimation of continuous-time nonlinear stochastic systems via the accurate extended continuous-discrete KF approach presented recently in [27, 28, 29, 30].

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