Matching coefficients for improved staggered bilinears *

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We calculate one-loop matching factors for bilinear operators composed of improved staggered fermions. We compare the results for different improvement schemes used in the recent literature, including the HYP action and an action close to the Asqtad action. We find that all improvement schemes substantially reduce the size of the one-loop contributions to matching factors. The resulting corrections are comparable to, or smaller than, those found with Wilson and domain-wall fermions.

Staggered fermions provide an attractive option for calculating weak matrix elements. The main advantages are (i) that unquenched simulations are significantly faster than with other fermion discretizations, and (ii) that the axial symmetry allows the calculation of weak matrix elements relevant to kaon mixing and decays. Progress has been slowed, however, by the large $O(a^2)$ errors observed with unimproved staggered fermions—in particular large staggered-flavor (hereafter “taste”) symmetry breaking—and the large size of some 1-loop contributions to matching factors.

Both problems are alleviated with improved staggered fermions. The key ingredient is the use of “smeared” links, which reduce the taste-breaking couplings of high momentum gluons to quarks. This is one part of the Symanzik improvement program applied at tree level. Smeared links are expected to substantially reduce the non-perturbative taste-symmetry breaking, and this is observed in the pion spectrum. It turns out that the largest reduction is for multiple APE smearing or HYP links. The smeared links are also expected to reduce the 1-loop contributions to matching factors, since the taste-changing couplings are responsible for large tadpole-like contributions. The purpose of our present work is to test the latter expectation with a variety of smearing choices.

We carry out this test using “hypercube bilinears”, i.e. bilinear operators in which the quark and antiquark fields reside on the same hypercube. These are the most local bilinears which match onto the complete set of continuum operators, having all spins and tastes. They are also the building blocks for the four-fermion operators we use in calculations of weak matrix elements.

We compare the following choices of improved actions and operators (and use the associated numbers to refer to them below):\footnote{For details of the definitions of the various smeared links we refer to the original references.}

1. Unimproved staggered action and operators made gauge invariant using the original links.

2. Unimproved action and operators except that all links are replaced by the Fat-7 smeared links introduced in Ref. 3.

3. Unimproved action and operators except that all links are replaced by the $O(a^2)$ improved links introduced in Ref. 3 (these involve Fat-7 smearing and the additional double-staple “Lepage” term).

4. Unimproved action and operators except that all links are replaced by HYP links 4A and 4B below.

5. Asqtad-like action (includes the Naik term but uses unimproved Wilson glue instead of the improved gluon action used by the MILC collaboration) 5 and operators containing $O(a^2)$ improved links.
We tadpole-improve all links (using the fourth-root of the plaquette to define $u_0$) except for the HYP links, for which we discuss mean-field improvement below.

The new feature introduced into perturbative calculations by smearing is that a link in, say, the $\mu$-th direction couples to gluons polarized in all directions $\nu$, not only to $\nu = \mu$ as for the original links. This introduces a vertex factor

$$\delta_{\nu,\mu} D_\mu(k) + (1 - \delta_{\nu,\mu}) G_{\nu,\mu}(k)$$

(1)

where

$$D_\mu(k) = 1 - d_1 \sum_{\nu \neq \mu} \tilde{s}_\nu^2 + d_2 \sum_{\nu, \rho \neq \mu} \tilde{s}_\nu^2 \tilde{s}_\rho^2$$

$$- d_3 \tilde{s}_\nu^2 \tilde{s}_\rho^2 \tilde{s}_\sigma^2 - d_4 \sum_{\nu \neq \mu} \tilde{s}_\nu^4,$$  

(2)

with $\tilde{s}_\nu = \sin(k_\nu/2)$, etc., and

$$G_{\nu,\mu}(k) = \tilde{s}_\mu \tilde{s}_\nu G_{\nu,\mu}(k)$$

(3)

$$\tilde{G}_{\nu,\mu}(k) = d_1 - d_2 \frac{(\tilde{s}_\rho^2 + \tilde{s}_\sigma^2)}{2} + d_3 \frac{\tilde{s}_\rho^2 \tilde{s}_\sigma^2}{3} + d_4 \tilde{s}_\nu^2,$$  

(4)

with all indices $(\mu, \nu, \rho, \sigma)$ different. This general form encompasses all our smearing choices and agrees with previous results.

The different smearing choices are distinguished by the values of the coefficients $d_{1-4}$:

1. Unimproved: $d_1 = d_2 = d_3 = d_4 = 0$;
2. Fat-7 links: $d_1 = d_2 = d_3 = 1, d_4 = 0$;
3 & 5. $O(a^2)$ improved: $d_1 = 0, d_2 = d_3 = d_4 = 1$;
4. HYP links: We consider two choices for the smearing parameters which define the HYP link.

4A. Those from Ref. [3], which were determined using a non-perturbative optimization procedure: $d_1 = 0.89, d_2 = 0.96, d_3 = 1.08, d_4 = 0$.

4B. The second choice leads to the same coefficients as for Fat-7 links (case 2 above) and thus removes $O(a^2)$ taste-symmetry breaking. Note, however, that the two-gluon vertices are not the same as for the Fat-7 links, so that one-loop tadpole diagrams differ.

Using these new vertices, we calculate the one-loop matching to continuum bilinears regularized in the NDR scheme:

$$O^\text{cont}_i = O^\text{lat}_i + \frac{C_F g^2}{16\pi^2} \sum_j (\delta_{ij} 2d_i \ln(\mu a) + c_{ij}) O^\text{lat}_j$$

(5)

| Operator-i | (1) | (2) | (3) | (4A) | (4B) | (5) |
|------------|-----|-----|-----|------|------|-----|
| (1 ⊗ 1)   | -29.4 | 1.9 | -4.4 | -0.6 | -0.1 | -2.2 |
| (1 ⊗ \xi_\mu) | -8.6 | 2.5 | -2.6 | 1.8 | 2.5 | -0.3 |
| (1 ⊗ \xi_{\mu\nu}) | 0.6 | 2.9 | -2.8 | 4.0 | 4.9 | -0.8 |
| (1 ⊗ \xi_{\mu\nu}) | 5.2 | 3.3 | -4.0 | 6.0 | 7.3 | -2.1 |
| (1 ⊗ \xi_{\nu}) | 8.7 | 3.8 | -5.6 | 8.0 | 9.7 | -3.8 |
| (\gamma_{\mu} ⊗ 1) | 0 | 0 | 0 | 0 | 0 | 1.4 |
| (\gamma_{\mu} ⊗ \xi_\mu) | -4.9 | 0.8 | 2.9 | -0.9 | -1.2 | 4.3 |
| (\gamma_{\mu} ⊗ \xi_\nu) | 0.2 | -0.1 | 3.0 | 1.3 | 1.8 | -1.5 |
| (\gamma_{\mu} ⊗ \xi_{\mu\nu}) | -3.4 | 0.4 | -0.1 | 0.3 | 0.4 | 1.4 |
| (\gamma_{\mu} ⊗ \xi_{\mu\nu}) | 2.5 | -0.2 | -5.5 | 2.7 | 3.7 | -4.0 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 0.2 | 0.1 | -2.5 | 1.6 | 2.1 | -1.0 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 4.9 | -0.2 | -7.9 | 4.2 | 5.7 | -6.5 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 2.8 | 0.0 | -5.0 | 3.0 | 4.0 | -3.6 |
| (\gamma_{\mu} ⊗ 1) | 1.6 | 0.4 | -1.3 | 1.9 | 2.3 | -0.0 |
| (\gamma_{\mu} ⊗ \xi_\mu) | 0.8 | 0.9 | 2.1 | 1.0 | 0.9 | 3.3 |
| (\gamma_{\mu} ⊗ \xi_\nu) | 3.0 | 0.0 | -4.5 | 3.0 | 4.0 | -3.2 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 4.6 | 1.8 | 6.7 | 0.3 | -0.2 | 7.8 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 1.3 | 0.4 | -1.4 | 1.9 | 2.3 | -0.1 |
| (\gamma_{\mu} ⊗ \xi_{\nu\rho}) | 4.9 | -0.2 | -7.4 | 4.2 | 5.7 | -6.1 |

Table 1
Diagonal matching coefficients $c_{ii}$ for the improvement choices listed in the text. The components $\mu, \nu, \rho, \sigma$ are all different.

where $d_i = (3, 0, -1)$ for spins ($S/P, V/A, T$), and the labels $i, j$ run over spin-tastes. For example, the continuum operators are

$$O^\text{cont}_{i=(\gamma_\sigma ⊗ \xi_\rho)} = \frac{1}{a^2} Q_{a,a}^{\gamma_\sigma} \xi_\rho Q_{\gamma_{\rho\sigma}}^{\gamma_\rho}. \ \ \ \ \ \text{\xi_\rho} = \gamma_\rho^*.$$  

The superscripts on $Q$ indicate the continuum flavor—we consider only flavor non-singlets.

The details of the calculation are presented in Ref. [10]. To check our results, we do two independent calculations using different methods. We focus here on the diagonal matching coefficients $c_{ii}$—four of the off-diagonal coefficients are non-vanishing but all are small, irrespective of smearing. Results for the $c_{ii}$ are given in Table [4].

These are the matching coefficients if the NDR scale is taken to be $\mu = 1/a$. This is expected to be within a factor of 2 of the optimum value (“$q^*$”), and in any case the coefficients depend only weakly on this scale. It is sufficient here to quote only one decimal place—more accurate results can be found in Ref. [11].

We see from the table that all the choices of smeared link that we consider significantly reduce the size of the largest matching coefficients. Note
that, in present simulations with $1/a \approx 2$ GeV, $\alpha_{MS} \approx 0.19$ so $c_{ii} = 5$ corresponds to a 10% correction. Thus the matching coefficient for the unimproved staggered fermion scalar bilinear ($c = -29.4$) leads to $\sim 60\%$ corrections, so that perturbation theory is not reliable. For all smearing choices $|c| < 10$, which is a considerable improvement, and makes the corrections comparable to those for Wilson ($c_S = -0.1$, $c_P = -9.7$, $c_V = -7.8$, $c_A = -2.9$, $c_T = -4.3$ in the tadpole improvement scheme of Ref. [11]) and domain-wall fermions ($c_{S/P} = -11.2$, $c_{V/A} = -5.3$, $c_T = -2.0$, with the domain-wall mass parameter $M = 1.7$ [12]). The “best” choice is case 2, i.e. Fat-7 links.

The corrections for HYP fermions can be further reduced using mean-field improvement. This is essentially a copy of the tadpole improvement scheme applied to the smeared links. We define $u_0^{\text{SM}}$ to be the fourth-root of the plaquette built out of smeared links. Each smeared link in the action and operators is then divided by $u_0^{\text{SM}}$. This is straightforward to implement in simulations and in perturbation theory. We expect this to reduce the matching factors by largely removing the residual fluctuations in the smeared links, but that the reduction should be by a smaller factor than for the original tadpole improvement applied to unimproved, Fat-7 or $O(a^2)$ links. This is borne out by our results, which are quoted in Table 2. We have also applied this “second-level” of mean-field improvement to the Fat-7 and $O(a^2)$ improved links, for which the effect is to slightly increase the corrections.3

After mean-field improvement, the HYP matching factors are nearly as small as those for Fat-7 links in Table 1. Both have corrections at the 10% level. We expect the corrections for four-fermion operators to be roughly twice this size, and this is borne out by our preliminary results for four-fermion operators with HYP links [13]. Given that HYP links lead to much greater reduction in non-perturbative taste-symmetry breaking in the pion spectrum, we are encouraged to pursue our calculations of weak matrix elements using HYP fermions.

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3It turns out that, after mean-field improvement, Fat-7 and HYP links with “Fat-7” coefficients (case 4B) have identical 1-loop matching factors.