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Bounds of operators on the Hilbert sequence space

https://doi.org/10.1515/conop-2020-0104
Received July 7, 2020; accepted August 11, 2020

Abstract: The author has computed the bounds of the Hilbert operator on some sequence spaces [18, 19]. Through this study the author has investigated the bounds of operators on the Hilbert sequence space and the present study is a complement of those previous research.

Keywords: Hilbert matrix, Cesàro matrix, Copson matrix, Difference sequence space, Norm

MSC: 26D15, 40C05, 40G05, 47B37

Introduction

Let \( p \geq 1 \) and \( \omega \) denote the set of all real-valued sequences. The space \( \ell_p \) is the set of all real sequences \( x = (x_k) \in \omega \) such that

\[
\|x\|_{\ell_p} = \left( \sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.
\]

The operator \( T \) is called bounded, if the inequality \( \|Tx\|_{\ell_p} \leq K\|x\|_{\ell_p} \) holds for all sequences \( x \in \ell_p \), while the constant \( K \) is not depending on \( x \). The constant \( K \) is called an upper bound for operator \( T \) and the smallest possible value of \( K \) is called the norm of \( T \). We also seek the inequalities of the form \( \|Tx\|_{\ell_p} > L\|x\|_{\ell_p} \), valid for every \( x \in \ell_p \) with \( x_0 > x_1 > \cdots > 0 \). The lower bound of \( T \) is the greatest possible value of \( L \).

Matrix domain. The matrix domain of an infinite matrix \( A \) in a sequence space \( X \) is defined as \( A(X) = \{ x \in \omega : Ax \in X \} \), which is also a sequence space. In especial case \( X = \ell_p \) we use the notation \( A(p) \) which has the definition

\[ A(p) = \{ x \in \omega : Ax \in \ell_p \}. \]

Note that, for the identity matrix \( A = I \), \( A(p) = \ell_p \). By using matrix domains of special triangle matrices in classical spaces, many authors have introduced and studied new Banach spaces.

Hausdorff matrix. The Hausdorff matrix \( H^\mu = (h_{j,k})_{j,k=0}^\infty \) is defined by

\[
h_{j,k} = \begin{cases} 
\int_0^1 \binom{j}{k} \theta^k (1 - \theta)^{j-k} d\mu(\theta) & j \geq k \\
0 & j < k,
\end{cases}
\]

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where $\mu$ is a probability measure on $[0, 1]$. Hardy’s formula ([9], Theorem 216) states that the Hausdorff matrix is a bounded operator on $\ell_p$ if and only if \[ \int_0^1 \theta^p \, d\mu(\theta) < \infty \] and

\[ \|H^\mu\|_{\ell_p} = \int_0^1 \theta^p \, d\mu(\theta) \quad (1 \leq p < \infty). \] (1.1)

Hausdorff operator has also the following formula as its lower bound.

**Theorem 1.1** (\[3\], Theorem 1). Let $1 \leq p < \infty$, and $H^\mu$ is a bounded Hausdorff matrix on $\ell_p$. Then,

\[ \|H^\mu x\|_{\ell_p} \geq L\|x\|_{\ell_p}, \]

for every decreasing sequence $x$ of non-negative terms, where

\[ L^p = \sum_{k=0}^{\infty} \left( \int_0^1 (1 - \theta)^k \, d\mu(\theta) \right)^p. \] (1.2)

The constant in (1.2) is the best possible, and there is equality only when $x = 0$ or $p = 1$ or $d\mu(\theta)$ is the point mass at 1.

**Cesàro matrix of order $n$.** By letting $d\mu(\theta) = n(1 - \theta)^{n-1} \, d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order $n$, $C^n = (c^n_{j,k})$, is defined by

\[ c^n_{j,k} = \begin{cases} \frac{n+1-j}{j!} & 0 \leq k \leq j \\ 0 & \text{otherwise} \end{cases} \] (1.3)

which according to the Hardy’s formula and relation (1.2) has the bounds

\[ \|C^n\|_{\ell_p} = \left( \frac{\Gamma(n+1)\Gamma(1/p')}{\Gamma(n+1/p')} \right) \quad \text{and} \quad L(C^n) = \left\{ \sum_{k=0}^{\infty} \left( \frac{n}{n+k} \right)^p \right\}^{1/p}. \] (1.4)

Note that, $C^1 = C$ is the well-known Cesàro matrix

\[ c_{j,k} = \begin{cases} \frac{1}{n+1} & 0 \leq k \leq j \\ 0 & \text{otherwise} \end{cases} \]

for all $j, k \in \mathbb{N}$. That is,

\[ C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]

with the $\ell_p$-norm $\|C\|_{\ell_p} = p^*$ (where $p^*$ is the conjugate of $p$ i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$) and lower bound $L(C) = \zeta(p)^{1/p}$, where $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$. For more examples

\[ C^2 = \begin{pmatrix} 2/3 & 0 & 0 & \cdots \\ 3/6 & 1/3 & 0 & \cdots \\ 3/6 & 2/6 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 3/4 & 1/4 & 0 & \cdots \\ 6/10 & 3/10 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

The Cesàro matrix domain $ces(n, p)$ is the set of all sequences whose $C^n$-transforms are in the space $\ell_p$, that is

\[ ces(n, p) = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{n+j} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\}, \]
which is a Banach space with the norm \( \|x\|_{\text{ces}(n, p)} = \left( \sum_{j=0}^{\infty} \left| \frac{1}{j!} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p \right)^{1/p} \).

Note that, for special case \( n = 1 \), we use the notation \( \text{ces}(p) \) instead of \( \text{ces}(1, p) \). For more information about the Cesàro sequence space \( \text{ces}(n, p) \), the reader can refer to [21].

**Gamma matrix.** By letting \( d\mu(\theta) = n\theta^{n-1} d\theta \) in the definition of the Hausdorff matrix, the Gamma matrix of order \( n \), \( \Gamma^n = (\gamma^n_{j,k}) \), is

\[
\gamma^n_{j,k} = \begin{cases} \binom{n+k-1}{k} / (j!)^{-1} & 0 \leq k \leq j \\ 0 & \text{otherwise}, \end{cases}
\]

which according to the Hardy’s formula and relation (1.2) has the bounds

\[
\|\Gamma^n\|_{\ell_p} = \frac{np}{np - 1} \quad \text{and} \quad L(\Gamma^n) = \left\{ \sum_{k=0}^{\infty} \left( \frac{n+k}{k} \right)^{-p} \right\}^{1/p}.
\]

Note that, \( \Gamma^1 \) is the well-known Cesàro matrix. For more examples:

\[
\Gamma^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/3 & 2/3 & 0 & \cdots \\ 1/6 & 2/6 & 3/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \Gamma^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/4 & 3/4 & 0 & \cdots \\ 1/10 & 3/10 & 6/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\]

The sequence space associated with the matrix \( \Gamma^n \), is the set \( \{ x = (x_k) \in \omega : \Gamma^n x \in \ell_p \} \) or

\[
gam(n, p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left| \frac{1}{j!} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\},
\]

which is called the Gamma space of order \( n \). In special case \( n = 1 \), we show the Gamma sequence space \( \text{gam}(1, p) \) by the notation \( \text{ces}(p) \).

**Hilbert matrix.** The Hilbert matrix \( H = (h_{j,k}) \) is defined by

\[
h_{j,k} = \frac{1}{j+k+1},
\]

for all integers \( j, k \). That is,

\[
H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\]

By theorem 323 of [7], \( H \) is a bounded operator and

\[
\|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi \text{csc}(\pi/p) \quad \text{and} \quad L(H) = \zeta(p)^{1/p}.
\]

The sequence space associated with the Hilbert matrix, \( h\text{hil}(p) \), is defined by

\[
h\text{hil}(p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{x_k}{j+k+1} \left| x_k \right|^p < \infty \right\},
\]

which has the norm

\[
\|x\|_{h\text{hil}(p)} = \left( \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{x_k}{j+k+1} \left| x_k \right|^p \right)^{1/p}.
\]
According to the inequality which has proved in Theorem 11.5 of [2]
\[ \|x\|_{ces(p)} \leq \|x\|_{hil(p)} \leq \frac{\pi}{p} \csc(\pi/p) \|x\|_{ces(p)}, \]
we obtain that \( hil(p) = ces(p) \). On the other hand, since \( ces(p) \) and \( \ell_p \) are isomorphic, hence the isomorphism \( hil(p) \cong \ell_p \) will be resulted.

**Motivation.** Parallel to the several research on the finite Hilbert operator, see [1, 4, 8, 23], there is some information about the infinite version of Hilbert matrix [24–27]. Recently the author [15, 16] has introduced some factorizations for the infinite Hilbert matrix based on the generalized Cesàro matrix and Cesàro and Gamma matrices of order \( n \). The author has also computed the norm and the lower bound of Hilbert operator on some sequence spaces [18, 19]. Through this study the author has tried to compute the norm and lower bound of several operators on the Hilbert sequence space that have not been done before.

## 2 Bounds of Cesàro and Copson operators on the Hilbert sequence space

In this study, we intend to find the bounds of some well-known operators on the Hilbert sequence space. In so doing we need the following lemma which is the combination of Lemmas 2.1 in [18] and [19].

**Lemma 2.1.** Let \( U \) be a bounded operator on \( \ell_p \) and \( A_p \) and \( B_p \) be two matrix domains such that \( A_p \sim \ell_p \). Then, the following statements hold:

1. If \( BT \) is a bounded operator on \( \ell_p \), then \( T \) is a bounded operator from \( \ell_p \) into \( B_p \) and
   \[ \|T\|_{\ell_p,B_p} = \|T\|_{\ell_p} \quad \text{and} \quad L(T)_{\ell_p,B_p} = L(BT). \]
2. If \( T \) has a factorization of the form \( T = UA \), then \( T \) is a bounded operator from the matrix domain \( A_p \) into \( \ell_p \) and
   \[ \|T\|_{A_p,\ell_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p,\ell_p} = L(U). \]
3. If \( BT = UA \), then \( T \) is a bounded operator from the matrix domain \( A_p \) into \( B_p \) and
   \[ \|T\|_{A_p,B_p} = \|U\|_{\ell_p} \quad \text{and} \quad L(T)_{A_p,B_p} = L(U). \]

In particular, if \( AT = UA \), then \( T \) is a bounded operator from the matrix domain \( A_p \) into itself and \( \|T\|_{A_p} = \|U\|_{\ell_p} \) and \( L(T)_{A_p} = L(U) \). Also, if \( T \) and \( A \) commute then \( \|T\|_{A_p} = \|T\|_{\ell_p} \) and \( L(T)_{A_p} = L(T) \).

More recently, several mathematicians have investigated the problem of finding the norm of operators on some matrix domains [5, 6, 10–14, 17, 20, 21]. Throughout this research, we use the notations \( \|\cdot\|_{\ell_p} \) and \( L(\cdot) \), for the norm and lower bound of operators on sequence space \( \ell_p \) and \( \|\cdot\|_{X,Y} \) and \( L(\cdot)_{X,Y} \) for the norm and lower bound of operators from the matrix domain \( X \) into the sequence space \( Y \).

**Hilbert matrix of order** \( n \). For a non-negative integer \( n \), we define the Hilbert matrix of order \( n \), \( H^n = (h^n_{j,k}) \), by
\[
h^n_{j,k} = \frac{1}{j + k + n + 1} \quad (j, k = 0, 1, \ldots).
\]
Note that for \( n = 0 \), \( H^0 = H \) is the well-known Hilbert matrix. For more examples:
\[
H^1 = \begin{pmatrix}
1/2 & 1/3 & 1/4 & \cdots \\
1/3 & 1/4 & 1/5 & \cdots \\
1/4 & 1/5 & 1/6 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad H^2 = \begin{pmatrix}
1/3 & 1/4 & 1/5 & \cdots \\
1/4 & 1/5 & 1/6 & \cdots \\
1/5 & 1/6 & 1/7 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
Similarly, the sequence space associated with the Hilbert matrix of order \( n \), \( \text{hil}(n, p) \), is defined by

\[
\text{hil}(n, p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p < \infty \right\},
\]

which has the norm \( \|x\|_{\text{hil}(n,p)} = \left( \sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p \right)^{\frac{1}{p}} \).

For non-negative integers \( n, j \) and \( k \), let us define the matrix \( B^n = (b^n_{j,k}) \) by

\[
b^n_{j,k} = \frac{(k+1) \cdots (k+n)}{(j+k+1) \cdots (j+k+n+1)} = \binom{n+k}{k} \beta(j+k+1, n+1) \quad (j, k = 0, 1, \ldots),
\]

where the \( \beta \) function is

\[
\beta(m, n) = \int_0^1 z^{m-1} (1-z)^{n-1} \, dz \quad (m, n = 1, 2, \ldots).
\]

Consider that for \( n = 0 \), \( B^0 = H \), where \( H \) is the Hilbert matrix.

For computing the norm of Cesàro and Copson operators on the Hilbert matrix domain we need the following lemma.

**Lemma 2.2** ([18], Lemma 2.3). The Hilbert matrix, \( H \), and the Hilbert matrix of order \( n \), \( H^n \), have the following factorizations based on the Cesàro matrix of order \( n \) of the forms:

(i) \( H = B^n C^n \),

(ii) \( H^n = C^n B^n \),

(iii) \( C^n H = H^n C^n \),

(iv) \( B^n \) is a bounded operator on \( \ell_p \) and \( \|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p') \Gamma(1/p)}{\Gamma(n+1)} \).

**Theorem 2.3.** The identity operator, \( I \), is a bounded operator from the Cesàro space \( \text{ces}(n, p) \) into the Hilbert space \( \text{hil}(p) \) and

\[
\|I\|_{\text{ces}(n,p), \text{hil}(p)} = \frac{\Gamma(n+1/p') \Gamma(1/p)}{\Gamma(n+1)}.
\]

In particular, the identity operator is a bounded operator from \( \text{ces}(p) \) into \( \text{hil}(p) \) and \( \|I\|_{\text{ces}(p), \text{hil}(p)} = \frac{n}{p'} \csc\left(\frac{\pi}{p}\right) \).

**Proof.** According to Lemma 2.2, the Hilbert operator has a factorization of the form \( H = B^n C^n \). Since the map \( \text{ces}(n, p) \to \ell_p, x \mapsto C^n x \) is an isomorphism between these two spaces, hence

\[
\|I\|_{\text{ces}(n,p), \text{hil}(p)} = \sup_{x \in \text{ces}(n, p)} \frac{\|Ix\|_{\text{hil}(p)}}{\|x\|_{\text{ces}(n,p)}} = \sup_{x \in \text{ces}(n, p)} \frac{\|Hx\|_{\ell_p}}{\|C^n x\|_{\ell_p}} = \sup_{x \in \text{ces}(n, p)} \frac{\|B^n C^n x\|_{\ell_p}}{\|C^n x\|_{\ell_p}} = \sup_{y \in \ell_p} \frac{\|B^n y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p') \Gamma(1/p)}{\Gamma(n+1)}.
\]

The author has recently introduced a factorization for the Hilbert matrix based on the Gamma matrix of order \( n \) of the form:
Theorem 2.4 ([16], Theorem 2.8). The Hilbert matrix has a factorization of the form $H = S^n I^n$, where the matrix $S^n = (s_{i,j}^n)$ has the entries

$$s_{i,j}^n = \frac{(1 - 1/n)(j + 1) + (k + 1)}{(j + k + 1)(j + k + 2)} \quad (j, k = 0, 1, \ldots),$$

is a bounded operator on $\ell_p$ and

$$\|S^n\|_{\ell_p} = \pi \left(1 - \frac{1}{np}\right) \csc(\pi/p).$$

In particular, for $n = 1$, $H = BC$, where $C$ is the Cesàro matrix and $B$ is a bounded operator with $\|B\|_{\ell_p} = \frac{\pi}{p} \csc(\pi/p)$.

Theorem 2.5. The identity operator, $I$, is a bounded operator from the Gamma space $\Gamma(n, p)$ into the Hilbert space $\mathcal{H}(p)$ and

$$\|I\|_{\Gamma(n, p), \mathcal{H}(p)} = \pi \left(1 - \frac{1}{np}\right) \csc(\pi/p).$$

In particular, the identity operator is a bounded operator from $\Gamma(n, p)$ into $\mathcal{H}(p)$ and $\|I\|_{\Gamma(n, p), \mathcal{H}(p)} = \frac{\pi}{p} \csc(\pi/p)$.

Proof. The map $\Gamma(n, p) \to \ell_p$, $x \mapsto \Gamma^n x$ is an isomorphism between these two spaces. According to Theorem 2.4 we have

$$\|I\|_{\Gamma(n, p), \mathcal{H}(p)} = \sup_{x \in \Gamma(n, p)} \frac{\|x\|_{\mathcal{H}(p)}}{\|x\|_{\Gamma(n, p)}} = \sup_{x \in \Gamma(n, p)} \frac{\|\Gamma^n x\|_{\ell_p}}{\|x\|_{\ell_p}} = \frac{\|\Gamma^n\|_{\ell_p}}{\|\ell_p\|_{\ell_p}} = \frac{\pi}{n+1} \csc(\pi/p) = \pi \left(1 - \frac{1}{np}\right) \csc(\pi/p).$$

In particular, for $n = 1$, $\Gamma^1 = C$, hence we obtain the result.

Theorem 2.6. The Cesàro operator of order $n$, $C^n$, is a bounded operator from the sequence space $\mathcal{H}(p)$ into $\ell_p$ and

$$L(C^n)_{\ell_p} = \frac{\Gamma(n + 1)}{\Gamma(n + 1/p)\Gamma(1/p)}.$$

In particular, the Cesàro operator has the lower bound $L(C^n)_{\ell_p} = \frac{\pi}{\pi} \sin(\pi/p)$. 

Proof. According to Lemma 2.2, the Hilbert operator $H$ has a factorization of the form $H = B^n C^n$, which results in the inequality

$$\|Hx\|_{\ell_p} \leq \frac{\Gamma(n + 1/p)}{\Gamma(n + 1)} \|C^n x\|_{\ell_p},$$

where the constant is best possible. Now, by taking infimum from both sides of the inequality

$$\frac{\|C^n x\|_{\ell_p}}{\|Hx\|_{\ell_p}} \geq \frac{\Gamma(n + 1)}{\Gamma(n + 1/p)\Gamma(1/p)},$$

the proof is complete.

Corollary 2.7. The Cesàro operator of order $n - 1$, $C^{n-1}$, is a bounded operator from the Cesàro space $\Gamma(n, p)$ into the Hilbert space $\mathcal{H}(p)$ and

$$\|C^{n-1}\|_{\Gamma(n, p), \mathcal{H}(p)} = \pi \left(1 - \frac{1}{np}\right) \csc(\pi/p).$$

In particular, the identity operator is a bounded operator from $\Gamma(n, p)$ into $\mathcal{H}(p)$ and $\|I\|_{\Gamma(n, p), \mathcal{H}(p)} = \frac{\pi}{p} \csc(\pi/p).$
Proof. The sequence spaces $ces(n, p)$ and $\ell_p$ are isomorphic. According to the theorem 2.4 and the identity $C^n = I^n C^{n-1}$ we have $H C^{n-1} = S^n C^n$ that results in

$$
\|C^{n-1}\|_{ces(n,p), hil(p)} = \sup_{x \in ces(n,p)} \|C^{n-1} x\|_{hil(p)} = \sup_{x \in ces(n,p)} \|H C^{n-1} x\|_{\ell_p}
$$

$$
= \sup_{x \in ces(n,p)} \|S^n C^n x\|_{\ell_p} = \sup_{y \in \ell_p} \|S^n y\|_{\ell_p}
$$

$$
= \|S^n\|_{\ell_p} = \pi \left(1 - \frac{1}{np}\right) \csc(\pi/np).
$$

In particular, for $n = 1$, $C^0 = I$ and $C^1 = C$, where $I$ is the identity matrix and $C$ is the Cesàro matrix. Therefore we have the desired result.

\[\square\]

**Theorem 2.8.** The Cesàro operator of order $n$, $C^n$, is a bounded operator from $hil(p)$ into $hil(n, p)$ and

\[(i)\quad \|C^n\|_{hil(p), hil(n,p)} = \frac{\Gamma(n+1) \Gamma(1/p)}{\Gamma(n+1/p)} , \]

\[(ii)\quad L(C^n)_{hil(p), hil(n,p)} = \left\{ \sum_{k=0}^{\infty} \left( \frac{n}{n+k} \right)^p \right\}^{1/p} .
\]

In particular, Cesàro operator is a bounded operator from $hil(p)$ into $hil(1, p)$ and $\|C\|_{hil(p), hil(1, p)} = p^*$ and $L(C)_{hil(p), hil(1, p)} = \zeta(p)^{1/p}$.

**Proof.** (i) Since $hil(p)$ and $\ell_p$ are isomorphic, hence by applying Lemmas 2.1 and 2.2 we have

$$
\|C^n\|_{hil(p), hil(n,p)} = \sup_{x \in hil(p)} \|C^n x\|_{hil(n,p)} = \sup_{x \in hil(p)} \|H^n C^n x\|_{\ell_p}
$$

$$
= \sup_{x \in hil(p)} \|C^n H x\|_{\ell_p} = \sup_{y \in \ell_p} \|C^n y\|_{\ell_p}
$$

$$
= \|C^n\|_{\ell_p} = \frac{\Gamma(n+1) \Gamma(1/p)}{\Gamma(n+1/p)} ,
$$

which gives the desired result. (ii) The proof is similar to the previous part. Hence

$$
L(C^n)_{hil(p), hil(n,p)} = L(C^n) = \left\{ \sum_{k=0}^{\infty} \left( \frac{n}{n+k} \right)^p \right\}^{1/p} .
$$

\[\square\]

**Copson matrix of order** $n$. Transposing the Cesàro matrix of order $n$ results the Copson matrix of order $n$, $C^T$ that has the $\ell_p$-norm

$$
\|C^T\|_{\ell_p} = \frac{\Gamma(n+1) \Gamma(1/p)}{\Gamma(n+1/p)} ,
$$

by Helinger-Toeplitz theorem, which states

**Theorem 2.9** ([2], Proposition 7.2). Suppose that $1 < p, q < \infty$. A matrix $A$ maps $\ell_p$ into $\ell_q$ if and only if the transposed matrix, $A^T$, maps $\ell_q$ into $\ell_p$. We then have

$$
\|A\|_{\ell_p, \ell_q} = \|A^T\|_{\ell_q, \ell_p} .
$$

We say that $Q = (q_{n,k})$ is a quasi-summability matrix if it is an upper-triangular matrix, i.e. $q_{n,k} = 0$ for $n < k$, and $q_{n,k} = 1$ for all $k$. The product of two quasi-summability matrices is also a quasi-summability matrix and all these matrices have the lower bound 1 on $\ell_p$, according to the following theorem.

**Theorem 2.10** ([3], Theorem 2). Let $p$ be fixed, $1 < p < \infty$, and let $T$ be a quasi-summability matrix. If $x \in \ell_p$ satisfies $x_0 > x_1 > \cdots > 0$, then

$$
\|Tx\|_q \geq \|x\|_{\ell_p} .
$$
Now, according to the above theorem we have
\[ L(C^{nt}) = 1. \]

Similar to the Cesàro matrix, we have the following results for the Copson operator of order \( n \).

**Theorem 2.11.** The Copson operator of order \( n \), \( C^{nt} \), is a bounded operator from \( \text{hil}(n, p) \) into \( \text{hil}(p) \) and
(i) \[ \|C^{nt}\|_{\text{hil}(n, p),\text{hil}(p)} = \frac{\Gamma(n+1)}{\Gamma(n+1/p)} \cdot \]
(ii) \[ L(C^{nt})_{\text{hil}(n, p),\text{hil}(p)} = 1. \]

In particular, Copson operator is a bounded operator from \( \text{hil}(1, p) \) into \( \text{hil}(p) \) and \( \|C^t\|_{\text{hil}(1, p),\text{hil}(p)} = p \) and \( L(C^t)_{\text{hil}(1, p),\text{hil}(p)} = 1. \)

**Theorem 2.12.** The Copson operator of order \( n \), \( C^{nt} \), is a bounded operator from the sequence space \( \text{hil}(n, p) \) into \( \ell_p \) and
\[ L(C^{nt})_{\text{hil}(n, p),\ell_p} = \frac{\Gamma(n+1)}{\Gamma(n+1/p)} \Gamma(1/p^*). \]

In particular, the Copson operator has the lower bound \( L(C^t)_{\text{hil}(1, p),\ell_p} = \frac{p}{\pi} \sin(\pi/p) \).

## 3 Bounds of Gamma operator on the Hilbert sequence space

In this section we intend to compute the bounds of Gamma operator on the Hilbert matrix domain, but we need the following lemma first.

**Lemma 3.1.** Let \( n \) be a positive integer number. The Following identities hold between the Hilbert and Gamma matrices of order \( n \).
(i) \[ B^n I^n = B^{n-1}, \]
(ii) \[ H I^n = B^{n-1} C^n, \]
(iii) \[ H^n I^n = I^n H^{n-1}. \]

**Proof.** (i) By applying the identity \( \sum_{j=0}^{\infty} z^j = (1 - z)^{-1} \) for \( |z| < 1 \), we deduce that
\[
(B^n I^n)_{i,k} = \sum_{j=0}^{\infty} \binom{n+j}{j} \beta(i+j+1, n+1) \frac{(n+k-1)}{(n+j)}
\]
\[
= \binom{n+k-1}{k} \sum_{j=0}^{\infty} \beta(i+j+k+1, n+1)
\]
\[
= \binom{n+k-1}{k} \int_{0}^{1} z^{i+k} (1-z)^{n-1} dz
\]
\[
= \binom{n+k-1}{k} \beta(i+k+1, n) = b_{i,k}^{n-1}.
\]

(ii) By applying the identity \( H = B^n C^n \), commutative property of Hausdorff matrices and part (i) we have
\[
HI^n = B^n C^n I^n = B^n I^n C^n = B^{n-1} C^n.
\]
(iii) By applying lemma 2.2, part (i) and the identity \( C^n = I^n C^{n-1} \) we have
\[
H^n \Gamma^n = C^n B^n \Gamma^n = C^n B^{n-1} = I^n H^{n-1}.
\]

\[\square\]

**Corollary 3.2.** The Gamma operator of order \( n \), \( \Gamma^n \), is a bounded operator from the Cesàro space \( ces(n, p) \) into the Hilbert space \( hil(p) \) and
\[
\|\Gamma^n\|_{ces(n, p), hil(p)} = \frac{\Gamma(n - 1/p)I(1/p)}{I(n)}.
\]

**Proof.** The sequence spaces \( ces(n, p) \) and \( \ell_p \) are isomorphic. According to Lemma 3.1 part (ii)
\[
\|\Gamma^n\|_{ces(n, p), hil(p)} = \sup_{x \in ces(n, p)} \|\Gamma^n x\|_{hil(p)} = \sup_{x \in ces(n, p)} \frac{\|\Gamma^n x\|_{\ell_p}}{\|x\|_{ces(n, p)}} = \sup_{y \in \ell_p} \|\Gamma^n y\|_{\ell_p} = \frac{\|\Gamma^n\|_{\ell_p}}{\|\Gamma^n\|_{\ell_p}} = \frac{\Gamma(n - 1/p)I(1/p)}{I(n)}.
\]

So the proof is complete. \[\square\]

**Theorem 3.3.** The Gamma operator of order \( n \), \( \Gamma^n \), is a bounded operator from the sequence space \( hil(p) \) into \( \ell_p \) and
\[
L(\Gamma^n)_{hil(p), \ell_p} = \frac{1}{\pi} \left( \frac{np}{np - 1} \right) \sin(\pi/p).
\]

In particular, the Cesàro operator has the lower bound \( L(C)_{hil(p), \ell_p} = \frac{p}{\pi} \sin(\pi/p) \).

**Proof.** From Theorem 2.4 we obtain the inequality
\[
\|Hx\|_{\ell_p} \leq \frac{n - 1}{np} \csc(\pi/p) \|\Gamma^n x\|_{\ell_p},
\]
where the constant is best possible. Now, by taking infimum from the both sides of the inequality
\[
\frac{\|\Gamma^n x\|_{\ell_p}}{\|Hx\|_{\ell_p}} \geq \frac{1}{\pi} \left( \frac{np}{np - 1} \right) \sin(\pi/p),
\]
we gain the desired result. \[\square\]

**Theorem 3.4.** Let \( n \) be a positive integer number. The Gamma operator of order \( n \), \( \Gamma^n \), is a bounded operator from the Hilbert space \( hil(n - 1, p) \) into the Hilbert space \( hil(n, p) \) and
(i) \( \|\Gamma^n\|_{hil(n-1, p), hil(n, p)} = \frac{np}{np - 1} \)
(ii) \( L(\Gamma^n)_{hil(n-1, p), hil(n, p)} = \left\{ \sum_{k=0}^{\infty} \binom{n+k}{k}^{-p} \right\}^{1/p} \).

In particular, Cesàro operator is a bounded operator from \( hil(p) \) into \( hil(1, p) \) and \( \|C\|_{hil(p), hil(1, p)} = p^* \) and \( L(C)_{hil(p), hil(1, p)} = \zeta(p)^{1/p} \).

**Proof.** The matrix domains \( hil(n, p) \) and \( \ell_p \) are isomorphic spaces. By applying Lemma 3.1 we have
\[
\|\Gamma^n\|_{hil(n-1, p), hil(n, p)} = \sup_{x \in hil(n-1, p)} \|\Gamma^n x\|_{hil(n, p)} = \sup_{x \in hil(n-1, p)} \frac{\|\Gamma^n x\|_{\ell_p}}{\|x\|_{hil(n-1, p)}} = \sup_{y \in \ell_p} \|\Gamma^n y\|_{\ell_p} = \frac{\|\Gamma^n\|_{\ell_p}}{\|\Gamma^n\|_{\ell_p}} = \frac{np}{np - 1}.
\]

\[\square\]
(ii) Similar to the proof of previous part

\[ L(I^n)_{hil(n-1),hil(n,p)} = L(I^n) = \left\{ \frac{\sum_{k=0}^{\infty} \binom{n+k}{k}^{-p}}{n+k} \right\}^{1/p}. \]

In special case \( n = 1 \), since \( H^0 = H \) and \( I^1 = C \) we have the particular result. \( \square \)

### 4 Norm of difference operator on the Hilbert sequence space

Let \( n \in \mathbb{N} \) and \( \Delta^n = (\delta_{i,k}^n) \) be the backward difference operator of order \( n \) with entries

\[ \delta_{i,k}^n = \begin{cases} (-1)^{i-k} \binom{n}{j-k} & k \leq j \leq n \n+1 \times k \\ 0 & \text{otherwise} \end{cases} \]

The author has recently computed the norm of backward difference operator on some sequence spaces and this part of study is a complement for those results obtained in [22].

**Theorem 4.1** ([7], Theorem 275). Let \( p > 1 \) and \( T = (t_{m,k}) \) be a matrix operator with \( t_{m,k} \geq 0 \) for all \( m, k \). Suppose that \( K, R \) are two strictly positive numbers such that

\[ \sum_{m=0}^{\infty} t_{m,k} \leq K \text{ for all } k, \quad \sum_{k=0}^{\infty} t_{m,k} \leq R \text{ for all } m, \]

(bounds for column and row sums respectively). Then

\[ \|T\|_{\ell_p} \leq R^{(p-1)/p} K^{1/p}. \]

The above theorem also known as Schur’s theorem.

**Theorem 4.2.** Let \( \Delta^n = (\delta_{i,k}^n) \) be the backward difference operator of order \( n \). Then

(i) \( \|\Delta^n\|_{hil(p),hil(p)} = \|\Delta^n\|_{\ell_p} \)

(ii) \( \|\Delta^n\|_{\ell_p,hil(p)} \leq \frac{1}{n+1} \)

**Proof.** (i) Let \( A = HA^n \). By using the identity \( \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^j = (1 - z)^n \), we have

\[
A_{i,k} = \sum_{j=k}^{n} \delta_{i,j}^n = \sum_{j=k}^{n} \frac{1}{i+j+1} \binom{n}{j-k} \int_0^1 z^{i+k} \, dz = \frac{1}{i+k+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_0^1 z^{i+k} \, dz = \frac{1}{i+k+1} \beta(i+k+1, n+1).
\]

Obviously, \( A \) is a symmetric matrix which implies that \( HA^n = \Delta^n H \). Now,

\[
\|\Delta^n\|_{hil(p),hil(p)} = \sup_{x \in hil(p)} \frac{\|HA^n x\|_{\ell_p}}{\|x\|_{hil(p)}} = \sup_{x \in hil(p)} \frac{\|H x\|_{\ell_p}}{\|x\|_{hil(p)}},
\]

\[
\|\Delta^n\|_{\ell_p,hil(p)} = \sup_{y \in \ell_p} \frac{\|\Delta^n y\|_{\ell_p}}{\|y\|_{\ell_p}} = \|\Delta^n\|_{\ell_p}.
\]
(ii) Let $A$ be the matrix defined in part (i). According to Lemma 2.1 part (i)

$$\|A^n\|_{\ell_\infty, hil(p)} = \|H A^n\|_{\ell_\infty} = \|A\|_{\ell_\infty}.$$ 

By a simple calculation

$$u_k = \sum_{j=0}^{\infty} a_{j,k} = \frac{(n-1)!}{(k+1)\cdots(k+n)},$$

where $u_k$ is the $k^{th}$ column sum of $A$. Since $\frac{1}{n} = u_0 > u_1 > \cdots$ and $A$ is symmetric, hence $R$ and $K$ are both $\frac{1}{n}$ in Schur's theorem. Therefore $\|A\|_{\ell_\infty} \leq \frac{1}{n}$. \hfill $\square$

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