Upper bounds for critical probabilities in Bernoulli Percolation models

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Abstract

We consider bond and site Bernoulli Percolation in both the oriented and the non-oriented cases on $\mathbb{Z}^d$ and obtain rigorous upper bounds for the critical points in those models for every dimension $d \geq 3$.

Keywords: upper bounds for percolation; critical thresholds; dynamical coupling.

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1 Introduction

The study of Bernoulli percolation on $\mathbb{Z}^d$ is more than 60 years old and the existence of a non-trivial phase transition for $d \geq 2$ is well established for the model and several of its variants, but the exact value of the critical parameter $p_c$ is seldom known. A celebrated result of Kesten (see [8]) proved that the critical probability in Bernoulli bond percolation on $\mathbb{Z}^2$ is $1/2$. Beyond that, a handful of planar lattices had their critical probability established and planarity was always the key factor. On the other hand, for dimensions $d \geq 3$, there is not much hope of finding exact values for the critical probability and the best we can expect are numerical results via Monte Carlo Methods (see the very efficient algorithm in [10]), statistical estimates based on a comparison with dependent percolation (see Section 6.2 of [1] for an overview) or rigorous bounds (see for instance [11, 13]). In this paper, we focus on finding rigorous upper bounds for site and bond Bernoulli Percolation on $\mathbb{Z}^d$ for every $d \geq 3$, in both the oriented and the non-oriented cases.

The primary tools we use are couplings between the models we seek to understand and models where bounds or precise values for the critical probabilities are known. Although those bounds are still far from the values obtained from Monte Carlo simulations, we believe that most of them are the best rigorous upper bounds in the literature.

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The remainder of the text is organized as follows: in Section 2 we define more precisely the models and state the main results, in Section 3 we establish some dynamical couplings, in Section 4 we prove the theorems and in Section 5 we give a numerical table with upper bounds for the critical probabilities in homogeneous Bernoulli percolation models for dimensions up to $d = 9$.

2 The models and main results

For $d \geq 1$, the underlying graph for all the models will be the $d$-dimensional hypercubic lattice in which the set of vertices is $\mathbb{Z}^d$ and the set of edges is the set of non ordered pairs $E(\mathbb{Z}^d) := \{ \langle v, u \rangle : v, u \in \mathbb{Z}^d \text{ and } |v - u| = 1 \}$. We abuse notation and denote this graph simply by $\mathbb{Z}^d$.

2.1 Bond percolation

Given $p \in [0, 1]$, consider a family of independent random variables $\{X_e\}_{e \in E(\mathbb{Z}^d)}$, where, for each $e \in E(\mathbb{Z}^d)$, $X_e$ has Ber$(p)$ distribution. Let $\mu_e$ be the law of $X_e$, and let $\mathbb{P}_p := \prod_{e \in E(\mathbb{Z}^d)} \mu_e$ be the resulting product measure. We declare an edge $e$ to be open if $X_e = 1$ and closed otherwise.

We first consider the non-oriented case and denote by $\{x \leftrightarrow y\}$ the event where $x, y \in \mathbb{Z}^d$ are connected by an open path, i.e., there exist $x_0, \ldots, x_n$ such that $x_0 = x, x_n = y$ and each $\langle x_{j-1}, x_j \rangle$ belongs to $E(\mathbb{Z}^d)$ and is open for $j = 1, \ldots, n$. Let $C^b_0 := \{ x \in \mathbb{Z}^d : 0 \leftrightarrow x \}$ be the open cluster of the origin, and $|C^b_0|$ its size.

We define the percolation probability by $\theta^b_d(p) := \mathbb{P}_p(|C^b_0| = \infty)$. The critical point for the non-oriented bond Bernoulli percolation model will be denoted by

$$p^b_c(d) = \sup\{ p \geq 0 : \theta^b_d(p) = 0 \}.$$

We now consider the oriented case. Let $\{e_1, \ldots, e_d\}$ be the set of positive unit vectors of $\mathbb{Z}^d$. We denote by $\{x \rightarrow y\}$ the event where $x, y \in \mathbb{Z}^d$ are connected by an oriented open path, i.e., there exist $x_0, \ldots, x_n$ such that $x_0 = x, x_n = y$ and for each $j = 1, \ldots, n$, we have $x_j = x_{j-1} + e$, for some $e \in \{e_1, \ldots, e_d\}$ and $\langle x_{j-1}, x_j \rangle$ is open. Let $\vec{C}^b_0 := \{ x \in \mathbb{Z}^d : 0 \rightarrow x \}$ be the oriented open cluster of the origin, and $|\vec{C}^b_0|$ its size.

Analogously, we define $\vec{\theta}^b_d(p) := \mathbb{P}_p(|\vec{C}^b_0| = \infty)$ the corresponding oriented percolation probability, and we denote the critical point for the oriented bond Bernoulli percolation model by

$$\vec{p}^b_c(d) = \sup\{ p \geq 0 : \vec{\theta}^b_d(p) = 0 \}.$$

Our results are the following:
**Theorem 1.** Consider non-oriented bond Bernoulli percolation and let $p^*(d)$ be the unique solution in $(0,1)$ of
\[
2 \prod_{i=0}^{2} \left( 1 - (1 - p)^{d+1} \right) = 2 - \sum_{i=0}^{2} (1 - p)^{d+1}.
\]

Then, for every $d \geq 3$, we have $p_c(d) \leq p^*(d)$.

**Theorem 2.** Consider oriented bond Bernoulli percolation on $\mathbb{Z}^d$.

1) If $d$ is even, then $\overrightarrow{p}_c^b(d) \leq 1 - (1/3)^{2/d}$;

2) For $d \geq 4$, we have that
\[
\overrightarrow{p}_c^b(d) \leq \frac{1}{d} + \frac{C_d}{d^2},
\]
where
\[
C_d = 1 + \frac{8}{d} + \frac{d^{5/2}}{(\sqrt{2\pi})^{d-1}} \left[ \frac{d-1}{d-3} \right] e^{\frac{1}{d}}.
\]

3) For any dimension $d \geq 2$, we have that $\overrightarrow{p}_c^b(d+1) \leq f(d)$, where $f(d)$ is the unique solution in $(0,1)$ of
\[
p = \overrightarrow{p}_c^b(d) \left[ p + (1-p)^{(d+1)/d} \right].
\]

**Remark:** It is known (see [3]) that $\overrightarrow{p}_c^b(d) \sim 1/d$, hence the third upper bound above is asymptotically sharp.

### 2.2 Site percolation

Given a parameter $p \in [0,1]$, we consider a family $\{X_v\}_{v \in \mathbb{Z}^d}$ of independent Bernoulli random variables with parameter $p$. As before, $\mathbb{P}_p$ will denote the resulting product measure. A vertex $v \in \mathbb{Z}^d$ is declared to be open if $X_v = 1$ and closed otherwise.

Now, the sequence of vertices $(x_0, \ldots, x_n)$ is said to be an open path if all the vertices are open and for each $j = 1, \ldots, n$, $\langle x_{j-1}, x_j \rangle \in E(\mathbb{Z}^d)$. If in addition to these conditions, for each $j = 1, \ldots, n$, we have $x_j = x_{j-1} + e$ for some $e \in \{e_1, \ldots, e_d\}$, the sequence is said to be an oriented open path. Having these definitions we define $C_0^s$, $\overrightarrow{C}_0^s$, $\theta_0^s(p)$ and $\overrightarrow{\theta}_0^s(p)$ accordingly.

Finally, the critical points for non-oriented and oriented site Bernoulli percolation are respectively given by
\[
p_c^s(d) = \sup\{p \geq 0 : \theta_0^s(p) = 0\} \quad \text{and} \quad \overrightarrow{p}_c^s(d) = \sup\{p \geq 0 : \overrightarrow{\theta}_0^s(p) = 0\}.
\]

For site percolation models, our results are the following.

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Theorem 3. Consider non-oriented site Bernoulli percolation on \( \mathbb{Z}^d \).

1) If \( d \) is even, then \( p^c_\ast(d) \leq 1 - (0,32)^{2/d} \);

2) If \( d \) is divisible by 3, then \( p^c_\ast(d) \leq 1 - (0,5)^{3/d} \);

3) For any dimension \( d \geq 2 \), we have that \( p^c_\ast(d+1) \leq g(d) \), where \( g(d) \) is the unique solution in \( (0,1) \) of

\[
p = p^c_\ast(d) \left[ p + (1 - p)^{2d/(2d-1)} \right].
\]

Theorem 4. Consider oriented site Bernoulli percolation on \( \mathbb{Z}^d \).

1) If \( d \) is even, then \( p^c_\ast(d) \leq 1 - (0,25)^{2/d} \);

2) For any dimension \( d \geq 2 \), we have that \( p^c_\ast(d) \leq h(d) \), where \( h(d) \) is the unique solution in \( (0,1) \) of

\[
p = p^c_\ast(d) \left[ p + (1 - p)^{(d+1)/d} \right].
\]

3 The Dynamical Couplings

Although we are only interested in homogeneous percolation, some of the tools we will use are related to couplings between anisotropic bond percolation models.

We now define the anisotropic (or inhomogeneous) non-oriented and oriented bond percolation models. For each \( i = 1, \ldots, d \), let \( E_i = \{(x, x + e_i) : x \in \mathbb{Z}^d\} \) be the set of edges parallel to \( e_i \). Given \( p_1, \ldots, p_d \in [0,1] \), we consider a family of independent random variables \( \{X_e\}_{e \in E(\mathbb{Z}^d)} \), but now, for each \( e \in E_i \), \( X_e \) has \( \text{Ber}(p_i) \) distribution, \( i = 1, \ldots, n \). The open cluster of the origin \( C^b_0 \) and the oriented open cluster of the origin \( \vec{C}^b_0 \) are defined analogously. The probabilities that \( |C^b_0|, |\vec{C}^b_0| \) are infinite, will be denoted by \( \theta(d_1, \ldots, p_d) \) and \( \vec{\theta}(p_1, \ldots, p_d) \) respectively.

The first coupling is the content of Proposition 1 in [3]:

Proposition 1. Consider inhomogeneous non-oriented Bernoulli bond percolation in \( \mathbb{Z}^d \). Let \( p_1, \ldots, p_{d+1} \in [0,1] \) and let \( \tilde{p}_d \in [0,1] \) be such that

\[
(1 - \tilde{p}_d) = (1 - p_d)(1 - p_{d+1}).
\]

Then, \( \theta_{d+1}(p_1, \ldots, p_d, p_{d+1}) \geq \theta_d(p_1, \ldots, p_{d-1}, \tilde{p}_d) \).

Now we define bond and site percolation models in the triangular lattice \( T \). This lattice is simply \( \mathbb{Z}^2 \) with extra edges of the form \( \langle v, v + (1,1) \rangle \). That is, \( T = (V_T, E_T) \) where \( V_T = \mathbb{Z}^2 \) and \( E_T \) is the set of non ordered pairs \( \{(v, u) : v - u = (1,0), (0,1) \text{ or } (1,1)\} \). We will denote by \( \theta_T^b \) and \( \theta_T^s \) the corresponding percolation probability for bond and site models, respectively.
We will consider inhomogeneous bond percolation on $T$, where the corresponding parameters $(p_1, p_2, p_3)$ will refer to edges of the form $\langle v, v + (1, 0) \rangle$, $\langle v, v + (0, 1) \rangle$ and $\langle v, v + (1, 1) \rangle$, respectively.

In the next proposition, we construct a monotonic coupling between inhomogeneous bond percolation on the triangular lattice $T$ with parameters $(p_1, p_2, p_3)$ and on $\mathbb{Z}^3$ with the same parameters.

**Proposition 2.** Let $(p_1, p_2, p_3) \in [0, 1]$ and consider two inhomogeneous bond Bernoulli percolation processes on the triangular lattice and on $\mathbb{Z}^3$, both with parameters $(p_1, p_2, p_3)$. Then

$$\theta_T^b(p_1, p_2, p_3) \leq \theta_3^b(p_1, p_2, p_3).$$

**Proof.** We will construct a dynamic coupling between the percolation process on $\mathbb{Z}^3$ with parameters $(p_1, p_2, p_3)$ and an infection process over $T$. We will do it in such a way that the law of infected sites in $T$ is the same as the law of the open cluster of the origin for anisotropic percolation on $\mathbb{Z}^3$ and also that, if the infection process survives, the open cluster of the origin of the process in $\mathbb{Z}^3$ must be infinite.

The coupling will be built based on a susceptible-infected strategy described as follows. First, we declare the origin of $T$ as the *initial* infected component. Next, at each time-step, we possibly grow the infected component and associate each new vertex $v$ of the infected component in $T$ to a vertex $x(v)$ in the open cluster of the origin in $\mathbb{Z}^3$. More precisely, consider a vertex $v$ in the infected component of $T$ and a neighbor $v + u$ out of the infected component. The vertex $v$, in the infected component in $T$, will be associated with some vertex $x(v)$ in the open cluster of the origin in $\mathbb{Z}^3$. According to $u = \pm(1, 0), \pm(0, 1)$ or $\pm(1, 1)$ we denote $\tau(u) = \pm(1, 0, 0), \tau(u) = \pm(0, 1, 0)$ or $\tau(u) = \pm(0, 0, 1)$ respectively. If $\langle x(v), x(v) + \tau(u) \rangle$, is open, we infect $v + u$. (and write $x(v + u) = x(v) + \tau(u)$).

Let’s define the sequence of sets $(I_n, x(I_n), R_n, S_n)_{n \geq 0}$. Here, $I_n$ represents the infected vertices in $T$ and $x(I_n)$ represents the vertices in $\mathbb{Z}^3$ associated with the infected vertices. $R_n$ represents the removed edges of $T$. Finally, given $I_n, x(I_n)$ and $R_n$, the susceptible edges set is given by

$$S_n := \{ \langle v, u \rangle : v \in I_n \text{ and } u \notin I_n \} \cap R_n^C.$$

At time $n = 0$, we set

- $I_0 = \{ 0 \} \subset V_T$;
- $R_0 = \emptyset \subset E_T$;
- $x(0) = 0 \in \mathbb{Z}^3$;
- $S_0 := \{ \langle v, u \rangle : v \in I_0 \text{ and } u \notin I_0 \} \cap R_0^C = \{ \langle 0, \pm(1, 0) \rangle, \langle 0, \pm(1, 0) \rangle, \langle 0, \pm(1, 1) \rangle \}$.

This means that, at time $n = 0$, only the vertex 0 is infected, and it can potentially infect any of its neighbours, so all edges containing the origin are susceptible. After that, in each
step, an infected vertex tries to infect a non-infected vertex through a susceptible edge (if the latter exists). From now on, we choose an arbitrary, but fixed, ordering of the edges in \( T \). Suppose that \( I_n, x(I_n), R_n \) and \( S_n \) are already defined. If there is no susceptible edge then the process stops. More specifically, if \( S_n = \emptyset \), then for all \( k \geq 1 \),

\[
\begin{align*}
&\bullet I_{n+k} = I_n; \\
&\bullet R_{n+k} = R_n; \\
&\bullet S_{n+k} = S_n.
\end{align*}
\]

Otherwise, if there exists some susceptible edge, then the infected vertex in the smallest (in the previously fixed ordering) such edge tries to infect its non-infected neighbour. Let us write it in symbols. Suppose \( S_n \neq \emptyset \) and let \( g_n \) be the smallest edge in \( S_n \). Since \( g_n \in S_n \), it has to be equal to some \( \langle v, v + u_n \rangle \), where \( v \in I_n \), \( v + u_n \notin I_n \), and \( u_n \in \{ \pm (0,1), \pm (1,0), \pm (1,1) \} \).

We set \( \tau(\pm (1,0)) = \pm (1,0,0) \), \( \tau(\pm (0,1)) = \pm (0,1,0) \) and \( \tau(\pm (1,1)) = \pm (0,0,1) \). Then, \( v \) infects \( v + u_n \) if \( \langle x(v), x(v) + \tau(u_n) \rangle \) is open in \( \mathbb{Z}^3 \). More precisely, if \( \langle x(v), x(v) + \tau(u_n) \rangle \) is open in \( \mathbb{Z}^3 \) then we write

\[
I_{n+1} := I_n \cup \{ v + u_n \},
\]

and define

\[
x(v + u_n) := x(v) + \tau(u_n).
\]

Otherwise, if \( \langle x(v), x(v) + \tau(u_n) \rangle \) is closed in \( \mathbb{Z}^3 \), we set \( I_{n+1} := I_n \).

Now that we have explored \( g_n \), we remove it and write

\[
R_{n+1} := R_n \cup \{ g_n \}.
\]

Next, to conclude our induction step, we set

\[
S_{n+1} := \{ \langle v, u \rangle : v \in I_{n+1} \text{ and } u \notin I_{n+1} \} \cap R_{n+1}^C.
\]

Observe that the function \( x : \cup_n I_n \to \mathbb{Z}^3 \) is injective. In fact, if \( v = (v_1, v_2) \in I_n \), then by construction, we have that \( x(v) = (x_1, x_2, x_3) \) satisfies

\[
v_1 = x_1 + x_3 \text{ and } v_2 = x_2 + x_3.
\]

Now, observe that the image of \( x \) is contained in the open cluster of the origin \( C_0^k \) of \( \mathbb{Z}^3 \). Since \( x \) is injective, \( | \cup_n I_n | \leq | C_0^k | \). Also, note that \( \cup_n I_n \) has the same law as \( C_0^T \), where \( C_0^T \) is the open cluster of the origin in \( T \) with parameters \( p_1, p_2, p_3 \). Therefore,

\[
\theta_T^k(p_1, p_2, p_3) \leq \theta_3^k(p_1, p_2, p_3),
\]

and the proof of Proposition follows. \( \square \)
To conclude this section, we construct two couplings that will be used to prove Item 3 of Theorem 2 along with its site version, which will be used in the proofs of Item 3 of Theorem 3 and Item 2 of Theorem 4. These couplings are reminiscent of the coupling in [6] and give an upper bound for the critical probability for any percolation model in $\mathbb{Z}^{d+1}$ as a function of the corresponding critical probability in $\mathbb{Z}^d$. Since our goal is to obtain better upper bounds, we need to improve the coupling.

**Proposition 3.** Consider oriented bond Bernoulli percolation on $\mathbb{Z}^{d+1}$ and suppose that

$$p > \overline{\theta}^b_{d+1}(d) \left[ p + (1 - p)^{(d+1)/d} \right].$$

Then, $\overline{\theta}^b_{d+1}(p) > 0$.

**Proof.** Let $\mathcal{E} = \{e_1, \ldots, e_d\}$ denote the set of positive unit vectors of $\mathbb{Z}^d$. To avoid ambiguities, we denote the set of positive unit vectors of $\mathbb{Z}^{d+1}$ by $\{u_1, \ldots, u_{d+1}\}$. Consider the multigraph $\mathbb{Z}^{d+1}_\mathcal{E}$ defined as follows: the set of vertices is $\mathbb{Z}^{d+1}$ and the set of edges is given by $E(\mathbb{Z}^{d+1}_\mathcal{E}) := (\bigcup_{i=1}^d E_i) \cup E_\mathcal{E}$ where $E_i = \{(x, x + u_i) : x \in \mathbb{Z}^{d+1}\}$, $i = 1, \ldots, d$, and we define $E_\mathcal{E} := \{(v, v + u_{d+1}) : v \in \mathbb{Z}^{d+1}, e \in \mathcal{E}\}$. In words, each edge of $\mathbb{Z}^{d+1}$ parallel to $u_{d+1}$ is partitioned into another $|\mathcal{E}|$ edges indexed by $\mathcal{E}$ in $\mathbb{Z}^{d+1}_\mathcal{E}$, while edges parallel to all other directions remain unmodified.

We prove this proposition in two steps. First, we will define a multigraph $\mathbb{Z}^{d+1}_\mathcal{E}$ and show that, with certain parameters, the model on $\mathbb{Z}^{d+1}_\mathcal{E}$ is equivalent to the homogeneous model on $\mathbb{Z}^{d+1}$ with parameter $p$. To conclude, we will show that if $p$ satisfies Inequality (1), then this new model on $\mathbb{Z}^{d+1}_\mathcal{E}$ dominates a supercritical model on $\mathbb{Z}^d$.

Consider now inhomogeneous oriented bond Bernoulli percolation on $\mathbb{Z}^{d+1}_\mathcal{E}$, where edges in $\bigcup_{i=1}^d E_i$ are open with probability $p$ and edges in $E_\mathcal{E}$ are open with probability $q$, where $q$ is such that $(1 - p) = (1 - q)|\mathcal{E}|$. Clearly, the distribution of the open cluster in this model is the same as in the homogeneous model on $\mathbb{Z}^{d+1}$ with parameter $p$. We will construct a coupling showing that the model on $\mathbb{Z}^{d+1}_\mathcal{E}$ with parameters $p$ and $q$ as above, dominates the homogeneous model on $\mathbb{Z}^d$ with parameter $p/[1 - (1 - p)q]$.

First, for each $e_i \in \mathcal{E}$, let $\sigma(e_i) := u_i \in \mathbb{Z}^{d+1}$. Then, for each $v \in \mathbb{Z}^{d+1}$ and each $e \in \mathcal{E}$, let $A(v, e)$ be the event where either the edge $\langle v, v + \sigma(e) \rangle$ is open or, for some $k \geq 1$,

- the edges $\langle v + iu_{d+1}, v + (i + 1)u_{d+1} \rangle$, $i = 0, \ldots, k - 1$, are open and
- the edges $\langle v + iu_{d+1}, v + iu_{d+1} + \sigma(e) \rangle$, $i = 0, \ldots, k - 1$ are closed, and
- $\langle v + ku_{d+1}, v + ku_{d+1} + \sigma(e) \rangle$ is open.

In the event $A(v, e)$, we define $u(v, e) = v + \sigma(e)$ if $\langle v, v + \sigma(e) \rangle$ is open, or $u(v, e) =
\(v + k u_{d+1} + \sigma(e)\) if \(k \geq 1\) is such that the above three conditions are met. Observe that

\[
P(A(v, e)) = p \sum_{i=0}^{\infty} [q(1-p)]^i = \frac{p}{1 - (1-p)q}
\]

\[
= \frac{p}{1 - (1-p)(1-(1-p)^{1/d})}
\]

\[
= \frac{p}{p + (1-p)^{(d+1)/d}}.
\]

Similarly to what was done in Proposition 2, we now build the sequence of sets \((I_n, x(I_n), R_n, S_n)_{n \geq 0}\). For \(n = 0\), we set

- \(I_0 = \{0\} \subset \mathbb{Z}^d\);
- \(R_0 = \emptyset \subset E(\mathbb{Z}^d)\);
- \(x(0) = 0 \in \mathbb{Z}^{d+1} \).

Suppose that, for some \(n \geq 0\), \(I_n, x(I_n)\) and \(R_n\) are already defined. Then \(S_n \subset E(\mathbb{Z}^d)\) is given by

\[
S_n := \{\langle v, u \rangle : v \in I_n \text{ and } u \not\in I_n\} \cap R_n^C.
\]

If \(S_n = \emptyset\), our sequence becomes constant, that is,

\[
(I_{n+k}, x(I_{n+k}), R_{n+k}, S_{n+k}) = (I_n, x(I_n), R_n, S_n), \quad \forall k \geq 1.
\]

(2)

Otherwise, let \(g_n = \langle v, v + e \rangle\) be the smallest (according to a prefixed ordering, as in Proposition 2) edge of \(S_n\), where \(v \in I_n, e \in \mathcal{E}\), and \(v + e \not\in I_n\). We set \(R_{n+1} = R_n \cup \{g_n\}\). We also set

\[
I_{n+1} = \begin{cases} 
I_n \cup \{v + e\}, & \text{if } A(x(v), e) \text{ occurs;} \\
I_n, & \text{otherwise.}
\end{cases}
\]

In case \(A(x(v), e)\) occurs, we set \(x(v + e) = u(x(v), e)\).

Once our sequence \((I_n, x(I_n), R_n, S_n)_{n \geq 0}\) is built, note that by construction, the function \(x : \cup_n I_n \longrightarrow \mathbb{Z}^{d+1}_\mathcal{E}\) is injective. In fact, for each \(v \in \cup_n I_n\), the projection of \(x(v)\) into \(\mathbb{Z}^d\) is equal to \(v\). The conclusion of the proof follows in a similar way to that of Proposition 2.

Proposition 4. Consider non-oriented site Bernoulli percolation on \(\mathbb{Z}^{d+1}\) and suppose that

\[
p > p_c^*(d) \left[p + (1-p)^{2d/(2d-1)}\right].
\]

(3)

Then \(\theta_{d+1}^*(p) > 0\).

Proof. The strategy of the proof is similar to the previous proposition. As before, let \(\{e_1, \ldots, e_d\}\) denote the set of positive unit vectors of \(\mathbb{Z}^d\) and let \(\{u_1, \ldots, u_{d+1}\}\) denote the
set of positive unit vectors of \( \mathbb{Z}^{d+1} \). We will consider a graph where each vertex of \( \mathbb{Z}^{d+1} \) will be partitioned into another \( 2d - 1 \) vertices.

For each \( v \in \mathbb{Z}^{d+1} \), we define the set of split vertices \( V_v := \{ v^{(1)}, \ldots, v^{(2d-1)} \} \). Let \( \mathbb{Z}^{d+1}_v \) be the graph with vertex set \( \bigcup_{v \in \mathbb{Z}^{d+1}} V_v \) and edge set

\[
E(\mathbb{Z}^{d+1}_v) := \{ (x, y) : x \in V_v \text{ and } y \in V_u, \text{ for some } \langle v, u \rangle \in E(\mathbb{Z}^{d+1}) \}.
\]

Given \( p \) satisfying (3), let \( q \) be such that \((1 - p) = (1 - q)^{2d-1}\). We will consider the non-oriented site Bernoulli percolation model on \( \mathbb{Z}^{d+1}_v \) with parameter \( q \). Note that the distribution of the open cluster of the origin in this new model is the same as in the model on \( \mathbb{Z}^{d+1} \) with parameter \( p \).

To build the coupling between \( \mathbb{Z}^{d+1}_v \) with parameter \( q \) and \( \mathbb{Z}^d \) with parameter \( p/(1-(1-p)q) \), we again construct a sequence of sets \( (I_n, x(I_n), R_n, S_n)_{n \geq 0} \). Since we are considering a site model, for each \( n \geq 0 \), the sets \( R_n \) and \( S_n \) will be, respectively, the sets of removed and susceptible vertices (instead of edges) at step \( n \). We start with two infected vertices (otherwise the origin should have been split into \( 2d \) vertices, instead of \( 2d - 1 \)). For \( n = 0 \), we set

- \( I_0 = \{0, e_1\} \subset \mathbb{Z}^d \);
- \( R_0 = \emptyset \subset \mathbb{Z}^d \);
- \( x(0) = 0^{(1)} \in \mathbb{Z}^{d+1}_v \) and \( x(e_1) = u_1^{(1)} \in \mathbb{Z}^{d+1}_v \).

Inductively, if for some \( n \geq 0 \), the sets \( I_n, x(I_n) \) and \( R_n \) are already defined, we set

\[
S_n = \{ u \in \mathbb{Z}^d : \langle v, u \rangle \in E(\mathbb{Z}^d) \text{ for some } v \in I_n \} \cap (I_n \cup R_n)^C. \tag{4}
\]

If \( S_n = \emptyset \), the sequence becomes constant, as in (2). Otherwise, let \( a_n \) be the smallest (in a preset ordering) vertex of \( S_n \). We can write \( a_n = v + e \), such that \( v \in I_n \) and \( e \in \{ \pm e_1, \ldots, \pm e_d \} \). In this case, let \( j_n \) denote the number of susceptible neighbors of \( v \) (including \( a_n \)), that is

\[
j_n := |\{ u \in S_n : \langle v, u \rangle \in E(\mathbb{Z}^d) \}|.
\]

Since we start with two infected vertices, necessarily \( |S_n| \neq 2d \), and then \( 1 \leq j_n \leq 2d - 1 \).

For each \( i \in \{1, \ldots, d\} \), we set the notation \( \sigma(\pm e_i) = \pm u_i \).

Consider the event \( A_n \) where, either, for some \( \ell \in \{1, \ldots, 2d - 1\} \), the vertex \( (x(v) + \sigma(e))^{(\ell)} \) is open or, for some \( k \geq 1 \) and \( \ell \in \{1, \ldots, 2d - 1\} \),

- the vertices \( (x(v) + i u_{d+1})^{(j_n)} \), \( i = 1, \ldots, k \), are open, and
- the vertices \( (x(v) + i u_{d+1} + \sigma(e))^{(j)} \), are closed \( \forall j \in \{1, \ldots, 2d - 1\}, i = 0, \ldots k - 1 \), and
- the vertex \( (x(v) + k u_{d+1} + \sigma(e))^{(\ell)} \) is open.
Note that $A_n$ has probability

$$
P(A_n) = (1 - (1 - q)^{2d-1}) \sum_{i=0}^{\infty} \left[ q(1 - q)^{2d-1} \right]^i
$$

$$
= \frac{p}{1 - (1 - p)q}
$$

$$
= \frac{p}{1 - (1 - p)(1 - (1 - p)^{1/(2d-1)})}
$$

$$
= \frac{p}{p + (1 - p)^{2d/2d-1}}.
$$

To conclude our induction step, we set

- $R_{n+1} = \begin{cases} R_n \cup \{v + e\}, & \text{if } A_n \text{ does not occur;} \\ R_n, & \text{otherwise; } \end{cases}$

- $I_{n+1} = \begin{cases} I_n \cup \{v + e\}, & \text{if } A_n \text{ occurs;} \\ I_n, & \text{otherwise. } \end{cases}$

In the event $A_n$, if for some $\ell \in \{1, \ldots, 2d - 1\}$ the vertex $(x(v) + \sigma(e))^{(\ell)}$ is open, we set $x(v + e) = (x(v) + \sigma(e))^{(\ell)}$. Otherwise, let $k \geq 1$ and $\ell \in \{1, \ldots, 2d - 1\}$ be such that the above three conditions are satisfied, in this case, we set $x(v + e) = (x(v) + ku_{d+1} + \sigma(e))^{(\ell)}$.

By construction, for each $v \in \cup_n I_n$, the projection of $x(v)$ onto $\mathbb{Z}^d$ is equal to $v$. Therefore $x : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d+1}_V$ is also injective. It follows that the site percolation model in $\mathbb{Z}^{d+1}_V$ (with $e^{(1)}_1$ declared open), dominates the infection process $(I_n)_{n \geq 0}$. Since $|\cup_n I_n|$ has the same distribution as the size of the open cluster of $\{0, e_1\}$ in a supercritical percolation model on $\mathbb{Z}^d$, the proof follows.

\[ \square \]

### 4 Proof of Theorems

In this section, we prove the theorems using the couplings of the last section.

#### 4.1 Proof of Theorem \[ \text{T1} \]

**Proof.** Given Propositions \[ \text{T1} \] and \[ \text{T2} \], the proof of Theorem \[ \text{T1} \] is quite straightforward. Recall that for bond percolation on the triangular lattice with parameters $(p_1, p_2, p_3)$, assuming $p_1, p_2, p_3 < 1$, we have (see Theorem 11.116 in [7])

$$
\theta_\text{T}(p_1, p_2, p_3) > 0 \iff p_1 + p_2 + p_3 - p_1p_2p_3 > 1.
$$

10
Consider now the partition
\[ d = \left\lfloor \frac{d}{3} \right\rfloor + \left\lfloor \frac{d+1}{3} \right\rfloor + \left\lfloor \frac{d+2}{3} \right\rfloor. \]

Let \( p > p^*(d) \) and set, for \( i = 1, 2, 3, \)
\[ p_i = 1 - (1 - p)^{\frac{d+1-i}{3}}. \]

Then, iteratively applying Proposition 1 and finally Proposition 2, we have
\[ \theta_d(p) \geq \theta_3(p_1, p_2, p_3) \geq \theta_T(p_1, p_2, p_3) > 0, \]
which concludes the proof.

4.2 Proof of Theorem 2

Proof. First, we remark that Proposition 1 holds mutatis mutandis for inhomogeneous oriented bond Bernoulli percolation. As a consequence, we have the following corollary which we state without proof.

Lemma 1. Let \( k \in \mathbb{N} \) be such that \( d \) is divisible by \( k \). Given \( p \in [0, 1] \), let \( \bar{p} \) be such that
\[ (1 - \bar{p}) = (1 - p)^{d/k}. \]
Then, \( \theta_d^k(p) \geq \theta_d^k(\bar{p}). \)

Taking \( k = 2 \) in Lemma 1 and using Liggett’s upper bound \( p_c^d(2) \leq 2/3 \) (see [9]), the first item of Theorem 2 follows. The third item is equivalent to Proposition 3.

Finally, the second item is implicitly proved in [1]. There, the authors defined a quantity \( \lambda(1/d, \ldots, 1/d) \) and showed that (see Equation (3.2)), if \( p \in [0, 1] \) satisfies
\[ \phi(d) := \lambda(1/d, \ldots, 1/d) \leq dp - 1, \]
then \( \theta^c_d(p) > 0. \) In particular,
\[ p_c^d(d) \leq \frac{1 + \phi(d)}{d}. \]
The conclusion follows from the estimate given in Subsection 3.1 of [1] (see the equation above (3.5)), that is
\[ \phi(d) \leq \frac{1}{d} + \frac{8}{d^2} + \frac{d^{3/2}}{(\sqrt{2\pi})^{d-1}} \left[ \frac{d-1}{d-3} \right] \left[ \frac{1}{d^3} \right] e^{1/2}. \]
Inhomogeneous percolation is not well defined for site models, and therefore, we are not able to formulate a version of Proposition\textsuperscript{1} for site percolation. But note that Lemma\textsuperscript{1} only involves homogeneous models. In the following, we give its site version.

**Lemma 2.** Consider non-oriented site Bernoulli percolation on $\mathbb{Z}^d$. Let $k \in \mathbb{N}$ be such that $d$ is divisible by $k$. Given $p \in [0, 1]$, let $\tilde{p}$ be such that

$$(1 - \tilde{p}) = (1 - p)^{d/k}.$$  

Then, we have $\theta_{s,d}(p) \geq \theta_{s,k}(\tilde{p})$.

**Proof.** The lemma follows by a coupling between non-oriented site Bernoulli percolation on $\mathbb{Z}^d$, with parameter $p$, and on $\mathbb{Z}^k$, with parameter $\tilde{p}$. To establish this coupling, we again construct a sequence of vertex sets $(I_n, x(I_n), R_n, S_n)_{n \geq 0}$.

First, we recall that $\mathcal{E} = \{e_1, \ldots, e_d\}$ is the set of positive unit vectors of $\mathbb{Z}^d$ and let $(D_{u_1}, \ldots, D_{u_k})$ be a uniform partition of $\mathcal{E}$ into $k$ subsets indexed by the set $\{u_1, \ldots, u_k\} \subset \mathbb{Z}^k$ of positive unit vectors of $\mathbb{Z}^k$. For each $u \in \{u_1, \ldots, u_k\}$, let $D_{-u} = -D_u$.

Consider the non-oriented site Bernoulli percolation model on $\mathbb{Z}^d$, with parameter $p$. For each $v \in \mathbb{Z}^d$ and each $u \in \{\pm u_1, \ldots, \pm u_k\}$, we define the following event $B(v, u) := \{v + e \text{ is open, for some } e \in D_u\}$.

If the event $B(v, u)$ occurs, we set $e(v, u) = v + e$, where $e$ is an open vertex that guarantees the occurrence of $B(v, u)$. Note that $\mathbb{P}(B(v, u)) = \tilde{p}$.

For $n = 0$, we define

- $I_0 = \{0\} \subset \mathbb{Z}^k$;
- $R_0 = \emptyset \subset \mathbb{Z}^k$;
- $x(0) = 0 \in \mathbb{Z}^d$.

If for some $n \geq 0$, the sets $I_n, x(I_n)$ and $R_n$ are already defined, let $S_n$ be as given in \textsuperscript{1}. If $S_n = \emptyset$, the sequence becomes constant as in \textsuperscript{2}. Otherwise, let $v_n$ be the smallest vertex of $S_n$ (according to a preset ordering of $\mathbb{Z}^k$). In this case, we write $v_n = v + u \notin I_n$, where $v \in I_n$ and $u \in \{\pm u_1, \ldots, \pm u_k\}$. To conclude the induction step, we define

- $R_{n+1} = \begin{cases} R_n \cup \{v + u\}, & \text{if } B(x(v), u) \text{ does not occur;} \\ R_n, & \text{otherwise;} \end{cases}$
I_{n+1} = \begin{cases} I_n \cup \{v + u\}, & \text{if } B(x(v), u) \text{ occurs;} \\ I_n, & \text{otherwise.} \end{cases}

If \( B(x(v), u) \) occurs, we define \( x(v + u) = e(x(v), u) \).

Note that, by construction, the function \( x : \bigcup_n I_n \rightarrow \mathbb{Z}^d \) is such that, for each \( v = v_{u_1}u_1 + \cdots + v_{u_k}u_k \in \bigcup_n I_n \), writing \( x(v) = x_{e_1}e_1 + \cdots + x_{e_d}e_d \), we have

\[
\sum_{e \in D_u} x_e = v_u, \quad \forall u \in \{u_1, \ldots, u_k\}.
\]

Therefore, \( x : \bigcup_n I_n \rightarrow \mathbb{Z}^d \) is injective. The conclusion of the lemma follows as in previous couplings.

With Lemma 2 we are now able to conclude the goal of this section.

Proof of Theorem 3. The first item of Theorem 3 follows by taking \( k = 2 \) in Lemma 2 together with Wierman’s upper bound \( p_s^*(2) \leq 0.68 \) (see [12]). Taking \( k = 3 \) in Lemma 2 and using the upper bound \( p_s^*(3) < 1/2 \) of Campanino-Russo (see [2]), the second item follows. The third item follows directly by Proposition 4.

4.4 Proof of Theorem 4

Proof. First, we remark that Lemma 2 and Proposition 4 hold mutatis mutandis for oriented site Bernoulli percolation. Therefore to conclude the proof of Theorem 4, the last ingredient is Liggett’s upper bound \( \overline{p}^s_r(2) \leq 3/4 \) (see [3]).

5 Explicit bounds

In this section, we present a table with upper bounds for bond and site Bernoulli Percolation in \( \mathbb{Z}^d \), up to \( d = 9 \), in both the oriented and the non-oriented cases.

In each column we give a numerical upper bound rounded to four decimals for the critical probability in the head of the column.
To obtain each bound, we used the following:

**Non-oriented bond percolation.** All bounds were obtained from Theorem 1.

**Oriented bond percolation** All bounds came from Theorem 2. For $d = 4$ we used Item 1), for $d = 5$ we used the upper bound for $d = 4$, and Item 3). For $d \geq 6$ we used Item 2).

**Non-oriented site percolation.** The bound for $d = 3$ follows from Campanino-Russo (see [2]). The others follow from Theorem 3. For $d = 4$ and $d = 8$ we used Item 1). For $d = 6$ and $d = 9$ we used Item 2). For $d = 5$ and $d = 7$ we used Item 3) along with the upper bounds obtained for $d = 4$ and $d = 6$.

**Oriented site percolation.** All the bounds came from Theorem 4. For even dimensions the bounds follow from Item 1), and for odd dimensions the bounds follow from Item 2) using the upper bounds for even dimensions.

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