Line arrangements with the maximal number of triple points

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Abstract

The purpose of this note is to study configurations of lines in projective planes over arbitrary fields having the maximal number of intersection points where three lines meet. We give precise conditions on ground fields $\mathbb{F}$ over which such extremal configurations exist. We show that there does not exist a field admitting a configuration of 11 lines with 17 triple points, even though such a configuration is allowed combinatorially. Finally, we present an infinite series of configurations which have a high number of triple intersection points.

Keywords arrangements of lines, combinatorial arrangements, Sylvester-Gallai problem

Mathematics Subject Classification (2000) 52C30, 05B30, 14Q10

1 Introduction

Configurations of points and lines have been the classical object of study in geometry. They come up constantly in various branches of contemporary mathematics, among others serving as a rich source of interesting examples and counter-examples. By way of example, in algebraic geometry, arrangements of lines have been studied recently by Teitler in [21] in the context of multiplier ideals, in [10] as counter-examples to the containment problem for symbolic powers of ideals of points in the complex projective plane and in [5] in the setup of the Bounded Negativity Conjecture and Harbourne constants.

In combinatorics point line arrangements are subject of classical interest and current research. Notably, in the last year we have witnessed a spectacular proof of a long standing conjecture motivated by the Sylvester-Gallai theorem on the number of ordinary lines. It has been proved by Green and Tao [13] with methods closely related to the real algebraic geometry. Symmetric $(n_k)$ configurations (mostly in the real Euclidean plane) are another classical topic of study in combinatorics. Such configurations with triple points (i.e. $k = 3$) are well understood, see the beautiful monograph by Grünbaum [14]. The classification of $(n_4)$ configurations has witnessed much progress in recent years and is almost completed, mainly due to works of Bokowski and his coauthors, see e.g. [7], [8] for an up to date account on that path of research. In the present note we take up a slightly different point of view and

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investigate what the maximal number of triple points in a configuration of \( s \) lines over an arbitrary ground field is. In Theorem 2.1 which is our main result, we give a full classification for \( s \leq 11 \). The interest in this particular bound is explained by the observation that it is the first value of \( s \) for which a configuration with maximal combinatorially possible number of triple points cannot be realized over any field. To the best of our knowledge this is the first example of this kind. More precisely, non-realizable configurations were known previously, probably the first example was found by Lauffer [17], see also [20] and [12]. However, the known examples concern configurations whose numerical invariants (for example Lauffer’s example is numerically the Desargues (10_3) configuration) allow several combinatorial realizations and some of them are not realizable over any field. In the case studied here, there are two combinatorial realizations possible but none of them has a geometrical realization. It is then natural to ask to what extend the combinatorial upper bound on the number of triple points found by Schönhein [19] (see also equation (3) below) can be improved in general. Our result is rendered by a series of configurations with a high number of triple points presented in Section 3. This series of examples sets some limits to possible improvements in [3].

Given a positive integer \( s \) and a projective plane over a field with sufficiently many elements, it is easy to find \( s \) lines intersecting in exactly \((s^2)\) distinct points. In fact this is the number of intersection points of a general arrangement of \( s \) lines (such arrangements in algebraic geometry are called star configurations, see [11]) and this is also the maximal possible number of points in which at least two out of given \( s \) lines intersect.

Given a configuration of \( s \) mutually distinct lines, let \( t_k \) denote the number of points where exactly \( k \geq 2 \) lines meet. Then there is the obvious combinatorial equality

\[
\binom{s}{2} = \sum_{k \geq 2} t_k \binom{k}{2}.
\]

Let \( T_k(s) \) denote the maximal number of \( k \)-fold points in an arrangement of \( s \) distinct lines in the projective plane over an arbitrary field \( \mathbb{F} \). The discussion above shows that

\[
T_2(s) = \binom{s}{2}.
\]

The aim of this note is to investigate the numbers \( T_3(s) \).

### 2 Arrangements with many triple points

Deciding the existence or non-existence of a configuration with certain properties is a problem which can, in principle, be always solved by combinatorial and semi-algebraic methods. The combinatorial part evaluates the collinearity conditions and checks whether the resulting incidence table can be filled in or not. The restrictions are imposed by the number of lines intersecting in configuration points and the condition that two lines cannot intersect in more than one point. Table 2 on page 6 is an example of what we call an incidence table.

If a configuration is combinatorially possible, then we assign coordinates to the equations of configuration lines and check if the system of polynomial equations resulting from evaluating combinatorial data has solutions. Typically this is the case and this is where the semi-algebraic part comes into the picture, as one has
to exclude various degenerations, for example points or lines falling together. This kind of conditions is given by inequalities.

This note, in a sense, is a field case study of the effective applicability of the approach described above. Evaluating all conditions carefully, one can actually study moduli spaces of configurations in the spirit of [2] and [1]. However, since we are interested in the existence of configurations over arbitrary fields, we do not dwell on this aspect of the story.

Some of the computations were supported by the symbolic algebra program Singular [9].

The equality (1) yields the following upper bound on the number $t_3$ of triple points in an arrangement of $s$ lines:

$$ t_3 \leq \left\lfloor \frac{(s)}{3} \right\rfloor. \quad (2) $$

This naive bound has been improved by Kirkman in 1847, [15], with a correction of Schönheim in 1966, [19]. Theorem 3.1 shows that this bound is close to be attained, on the other hand Theorem 2.1 shows that there is place for some further improvements. Let

$$ U_3(s) := \left\lfloor \frac{s - 1}{2} \right\rfloor \cdot \frac{s}{3} - \varepsilon(s), \quad (3) $$

where $\varepsilon(s) = 1$ if $s \equiv 5 \mod (6)$ and $\varepsilon(s) = 0$ otherwise. Then

$$ T_3(s) \leq U_3(s). \quad (4) $$

We refer to Section 5 in [8] for a nice discussion of historical backgrounds. In the next table we present a few first numbers resulting from (4).

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| $U_3(s)$ | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 8 | 12 | 13 | 17 | 20 |

It is natural to ask to which extend the numbers appearing in the above table are sharp. Our main result is the following classification Theorem.

**Theorem 2.1.**  

a) For $1 \leq s \leq 6$, there are configurations of lines with $U_3(s)$ triple points in projective planes $\mathbb{P}(F)$ over arbitrary fields.

b) A configuration of 7 lines with 7 triple points exists only in characteristic 2 (the smallest such configuration is the Fano plane $\mathbb{P}^2(F_2)$).

c) A configuration of 8 lines with 8 triple points exists over any field containing a non-trivial third degree root of 1. Moreover such a configuration always arises by taking out one line from a configuration of 9 lines with 12 triple points.

d) A configuration of 9 lines with 12 triple points exists over any field containing a non-trivial third degree root of 1.

e) A configuration of 10 lines with 13 triple points exists only

e1) over a field $F$ of characteristic 2 containing a non-trivial third root of unity. In this case one of the points has in fact multiplicity 4;

e2) over any field $F$ of characteristic 5.
f) There is no configuration of 11 lines with 17 triple points. There exist configurations of 11 lines with 16 triple points.

Proof. The first part of the Theorem is well known for $s \leq 9$. We go briefly through all the cases for the sake of the completeness and discuss $s = 10$ in more detail as this configuration seems to be new.

For $s = 1, 2$ there is nothing to prove. For $s = 3, 4$ we take 3 lines in a pencil and an arbitrary fourth line. The case $s = 5$ is easy as well, see Figure 1.

![Figure 1: s = 5](image1.png)

![Figure 2: s = 6](image2.png)

We pass to the case $s = 6$ adding the line through both double points, see Figure 2.

For $s = 7$ we obtain the famous Fano plane $\mathbb{F}_2$. It is well known that this configuration is possible only in characteristic 2. The picture below (Figure 3) indicates collinear points as lying on the segments or on the circle.

![Figure 3: s = 7](image3.png)

![Figure 4: s = 8](image4.png)

For $s = 8$ there is the Möbius-Kantor ($8_3$) configuration. This configuration cannot be drawn in the real plane. Collinearity is indicated by segments and the circle arch, see Figure 4. This configuration can be obtained from the next configuration by removing one line.

For $s = 9$ there is the dual Hesse configuration. It is easier to describe the original Hesse configuration. It arises taking the nine order 3 torsion points of a smooth complex cubic curve (which carries the structure of an abelian group and the torsion is understood with respect to this group structure). There are 12 lines passing through the nine points in such a way that each line contains exactly 3 torsion points and there are 4 lines passing through each of the points. See [4] for details. This configuration cannot be drawn in the real plane.

Beside the geometrical realization over the complex numbers, the dual Hesse configuration can be also easily obtained in characteristic 3, more precisely, in the plane...
\( \mathbb{P}^2(\mathbb{F}_3) \) taking all 13 lines and removing from this set all 4 lines passing through a fixed point. We leave the details to the reader.

Before moving on, we record for further reference the following simple but useful fact.

**Lemma 2.2.** Let \( \mathcal{L} = \{L_1, \ldots, L_s\} \) be a configuration of lines. Let \( L \in \mathcal{L} \) be a fixed line. Let \( P_1(L), \ldots, P_r(L) \) be intersection points of \( L \) with other configuration lines with corresponding multiplicities \( m_1(L), \ldots, m_r(L) \). Then

\[
s - 1 = \sum_{i=1}^{r} (m_i - 1).
\]

In particular, if there are only triple points on a line \( L \), then \( s \) is an odd number.

### 2.1 10 lines

Now we come to the case \( s = 10 \). We work over an arbitrary field \( \mathbb{F} \). The upper bound (4) implies that in this case there can be at most 13 points of multiplicity at least 3. We first deal with the case when points of higher multiplicity might appear.

#### 2.1.1 Points of excess multiplicity

The combinatorial equality \( (1) \) implies that there are no points with multiplicity \( m \geq 5 \) and only the following cases with 4-fold points need to be considered:

(i) \( t_4 = 2, \ t_3 = 11, \ t_2 = 0 \).

(ii) \( t_4 = 1, \ t_3 = 12, \ t_2 = 3 \).

Case (i) is excluded since there exists a configuration line \( L \) which does not pass through any of the 4-fold points. Hence this line contains only 3-fold points. Since \( s = 10 \), this contradicts Lemma \( 2.2 \).

Passing to (ii) we start with the 4-fold point \( W \). We denote the lines passing through \( W \) by \( M_1, \ldots, M_4 \) as indicated in the picture below.

![Figure 5](image-url)

The remaining 6 lines \( L_1, \ldots, L_6 \) (not visible in the figure above) intersect pairwise in 15 mutually distinct points — 12 of these points are the 12 configuration
triple points and the remaining 3 points (not visible in the figure above) contribute to \( t_2 \). Note that 6 general lines intersect in 15 mutually distinct points, but in our situation there are additional collinearities which are reflected in the picture above and in the table below. Passing to the details let \( P_{ij} = L_i \cap L_j \) for \( 1 \leq i < j \leq 6 \). Up to renumbering of points we may assume that they are distributed in the following way:

| line | points on the line |
|------|--------------------|
| \( M_1 \) | \( P_{12}, P_{34}, P_{56} \) |
| \( M_2 \) | \( P_{13}, P_{25}, P_{46} \) |
| \( M_3 \) | \( P_{14}, P_{26}, P_{35} \) |
| \( M_4 \) | \( P_{15}, P_{24}, P_{36} \) |

Table 1

Moreover, we may assume that the lines \( L_1, \ldots, L_6 \) have the following equations (we omit “\( = 0 \)” in the equations)

\[
\begin{align*}
L_1 &: x, \\
L_2 &: y, \\
L_3 &: z, \\
L_4 &: x + y + z, \\
L_5 &: ax + by + z, \\
L_6 &: cx + dy + z,
\end{align*}
\]

(5)

with

\[
a, b, c, d \in F^* \quad \text{and} \quad \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & 1 \\ c & d & 1 \end{pmatrix} \neq 0.
\]

(6)

Indeed, the first four equations are obvious. The coefficients at \( z \) in the lines \( L_5 \) and \( L_6 \) can be normalized to 1 since otherwise the star configuration condition would fail. Similarly the conditions in (6) are necessary in order to guarantee that \( L_1, \ldots, L_6 \) form a star configuration.

Evaluating collinearity conditions in Table 1 above we obtain the following system of linear and quadratic equations:

\[
\begin{align*}
a - b - c + d &= 0 \\
-ad + a - c + d &= 0 \\
a - bc &= 0 \\
bc - d &= 0
\end{align*}
\]

(7)

Additionally, the condition that the lines \( M_1, \ldots, M_4 \) belong to the same pencil gives

\[
-ab + a + bc - 1 = 0.
\]

(8)

A solution to the above system of equations (7) and (8) satisfying additionally the non-equality condition (6) exists only in characteristic 2. This has been verified with the aid of Singular. Moreover, in that case \( a \) satisfies

\[
a^2 + a + 1 = 0.
\]

and then, consequently, \( b = a^2, c = a^2 \) and \( d = a \).

It follows that the configuration (ii) exists in \( \mathbb{P}^2(F_{2^q}) \), for all \( q \geq 2 \) (the case \( q = 1 \) is excluded as there are evidently not enough points in the Fano plane. The
configuration lines are then given by equations

\[ L_1: x, \quad L_2: y, \quad L_3: z, \]
\[ L_4: x + y + z, \quad L_5: ax + a^2y + z, \quad L_6: a^2x + ay + z, \]
\[ M_1: x + y, \quad M_2: ax + z, \quad M_3: a^2x + y + z, \quad M_4: x + a^2y + z \]

Then the configuration points have coordinates:

\[ W = (1 : 1 : a), \quad P_{12} = (0 : 0 : 1), \quad P_{13} = (0 : 1 : 0), \quad P_{14} = (0 : 1 : 1), \]
\[ P_{15} = (0 : 1 : a^2), \quad P_{24} = (1 : 0 : 1), \quad P_{25} = (1 : 0 : a), \quad P_{26} = (1 : 0 : a^2), \]
\[ P_{34} = (1 : 1 : 0), \quad P_{35} = (a : 1 : 0), \quad P_{36} = (1 : a : 0), \quad P_{46} = (a^2 : a : 1), \]
\[ P_{56} = (1 : 1 : 1). \]

The incidence table in this case reads:

\[ \begin{array}{cccccccccccc}
W & P_{12} & P_{13} & P_{14} & P_{15} & P_{24} & P_{25} & P_{34} & P_{35} & P_{36} & P_{46} & P_{56} \\
L_1 & + & + & + & + & & & & & & & \\
L_2 & + & & & & + & + & & & & & \\
L_3 & & + & & & & + & + & + & & & \\
L_4 & & & + & & + & & + & + & & & \\
L_5 & & & & + & & + & & + & & & \\
L_6 & & & & & + & & + & & + & & \\
M_1 & & & & & & + & & & + & & \\
M_2 & & & & & & & + & & & & \\
M_3 & & & & & & & & + & & & \\
M_4 & & & & & & & & & + & & \\
\end{array} \]

Table 2

2.1.2 Points of multiplicity 3

Now we pass to the case that there are no quadruple points, hence \( t_3 = 13 \) and consequently \( t_2 = 6 \). There is an odd number of 2-fold points on each configuration line. This implies that there is a configuration line \( M_1 \) containing exactly three 2-fold points \( D_1, D_2, D_3 \). We have again two cases:

(A) The lines \( M_2, M_3, M_4 \) passing through the points \( D_1, D_2, D_3 \) meet in a single point \( W \).
(B) The lines $M_2$, $M_3$, $M_4$ form a triangle with vertices $Z_1$, $Z_2$, $Z_3$.

The first case is impossible. This follows similarly to the case (e1) with a 4-fold point. Indeed, the six lines $L_1, \ldots, L_6$ not visible in the Figure form a star configuration, i.e. there are 15 mutually distinct intersection points $P_{ij} = L_i \cap L_j$. Up to renumbering incidences between the lines $M_i$ and the points $P_{jk}$ are as in the Table.

Moreover the condition that $M_2$, $M_3$ and $M_4$ meet at one point $W$ is the same as in equation (8). This gives the same solution as in the previous case (e1). It is easy to check that this implies that the line $M_1$ goes through $W$, a contradiction.

Now we consider the remaining case (B). There are three 3-fold points on the line $M_1$. The points $Z_1$, $Z_2$, $Z_3$ must be also 3-fold points of the configuration and on each of the lines $M_2$, $M_3$, $M_4$ there are two more 3-fold points. This gives altogether twelve 3-fold points. Hence the six remaining lines $L_1, \ldots, L_6$ have a 3-fold intersection point. We call this point $D$, and assume that $L_4$, $L_5$ and $L_6$ pass through $D$. We can assume that $D = (1 : 1 : 1)$ and then the equations of the lines $L_i$ are

$$
L_1 : x,
L_2 : y,
L_3 : z,
L_4 : ax - (a + 1)y + z,
L_5 : bx - (b + 1)y + z,
L_6 : cx - (c + 1)y + z,
$$

with $a, b, c$ mutually distinct and different from zero. The combinatorics implies that each of $Z_1$, $Z_2$, $Z_3$ lie on one of the lines $L_4$, $L_5$, $L_6$. Up to renumbering we can assume $Z_1 \in L_4$, $Z_2 \in L_5$, $Z_3 \in L_6$. Then the incidence table is determined as follows:

| line | points on the line |
|------|-------------------|
| $M_1$ | $P_{14}$, $P_{25}$, $P_{36}$ |
| $M_2$ | $Z_2$, $Z_3$, $P_{12}$, $P_{34}$ |
| $M_3$ | $Z_1$, $Z_3$, $P_{15}$, $P_{23}$ |
| $M_4$ | $Z_1$, $Z_2$, $P_{13}$, $P_{26}$ |

Table 3

See the text after Table 3 for hints how to fill in such a table.
Using equations as in equation (9) and evaluating incidences we obtain the following conditions

\[
\begin{align*}
ab + ac + a - bc &= 0 \\
ac + a - b + c &= 0 \\
ab + c &= 0 \\
a + bc &= 0
\end{align*}
\]

This implies that \(b^2 = 1\). If \(b = -1\) then \(a = -1\), a contradiction. If \(b = 1\) then \(a = 3\) and \(a^2 + 1 = 0\). It follows that \(\text{char } F = 2\) or \(\text{char } F = 5\). In the first case \(a = c\), a contradiction. In the second case we obtain \(a = 3\), \(b = 1\), \(c = 2\), thus the lines are

\[
\begin{align*}
L_1 &: x, \\
L_2 &: y, \\
L_3 &: z, \\
L_4 &: 3x + y + z, \\
L_5 &: x + 3y + z, \\
L_6 &: 2x + 2y + z, \\
M_1 &: x + y + z, \\
M_2 &: 2x + 4y, \\
M_3 &: 3y + z, \\
M_4 &: 2x + z
\end{align*}
\]

The points have coordinates

\[
\begin{align*}
D &= (1 : 1 : 1), \\
Z_1 &= (2 : 3 : 1), \\
Z_2 &= (4 : 3 : 2), \\
Z_3 &= (4 : 3 : 1), \\
P_{12} &= (0 : 0 : 1), \\
P_{13} &= (0 : 1 : 0), \\
P_{14} &= (0 : 4 : 1), \\
P_{15} &= (0 : 4 : 3), \\
P_{23} &= (1 : 0 : 0), \\
P_{25} &= (1 : 0 : 4), \\
P_{26} &= (1 : 0 : 3), \\
P_{34} &= (4 : 3 : 0), \\
P_{36} &= (3 : 2 : 0).
\end{align*}
\]

The incidence table is

|   | \(D\) | \(Z_1\) | \(Z_2\) | \(Z_3\) | \(P_{12}\) | \(P_{13}\) | \(P_{14}\) | \(P_{15}\) | \(P_{23}\) | \(P_{25}\) | \(P_{26}\) | \(P_{34}\) | \(P_{36}\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(L_1\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(L_2\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(L_3\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(L_4\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(L_5\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(L_6\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(M_1\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(M_2\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(M_3\) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \(M_4\) | + | + | + | + | + | + | + | + | + | + | + | + | + |

Table 4

Passing to the last assertion f) of Theorem 2.1 we assume that a configuration of 11 lines with 17 triple points exists. Then (11) implies that there are 4 double points in the configuration. Hence each line meets 10 other lines, there is an even number of double points on each line. If there are 4 double points on a line, then this condition fails on the 4 lines meeting the given one in the double points. Hence, there must be 2 pairs of lines in the configuration with double points situated in the intersection points of lines from different pairs as indicated in the figure below.
Now, there are two cases

(I) the line $W_1W_2$ belongs to the configuration,

(II) the line $W_1W_2$ is not a configuration line.

We begin with the case (I)

Let $L$ be the line $W_1W_2$. In the figure above there are 5 configuration lines. The remaining lines $L_1, \ldots, L_6$ must form a star configuration and their intersection points $P_{ij} = L_i \cap L_j$ have to distribute in five collinear triples lying on the lines $L, M_1, M_2, N_1$ and $N_2$. Up to renumbering the points the collinear triples are

$$
\begin{align*}
P_{12}, P_{34}, P_{56} \\
P_{13}, P_{25}, P_{46} \\
P_{14}, P_{26}, P_{35} \\
P_{15}, P_{24}, P_{36} \\
P_{16}, P_{23}, P_{45}
\end{align*}
$$

Indeed, the first column contains the points lying on the line $L_1$. The first row is then completed just by assigning numbers. The index 2 must appear somewhere in the second row. Since the pairs 3, 4 and 5, 6 cannot be distinguished in this stage (similarly as the particular points within these pairs), we have the freedom to label that point $P_{25}$. These labeling determines the rest of the table.

Without loss of generality as in (5) and (6) we may assume that

$$
\begin{align*}
L_1 : x, & & L_2 : y, & & L_3 : z, \\
L_4 : x + y + z, & & L_5 : ax + by + z, & & L_6 : cx + dy + z,
\end{align*}
$$
with
\[ a, b, c, d \in \mathbb{F}^* \quad \text{and} \quad \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & 1 \\ c & d & 1 \end{pmatrix} \neq 0. \]

We can now compute coordinates of all points \( P_{ij} \) for \( 1 \leq i < j \leq 6 \) and evaluate collinearity conditions (11). This leads to the following system of equations:

\[
\begin{align*}
    a - b - c + d &= 0 \\
    -ad + a - c + d &= 0 \\
    a - bc &= 0 \\
    bc - d &= 0 \\
    ad - a + b - d &= 0
\end{align*}
\] (12)

This system has a solution satisfying conditions (11) only if the ground field \( \mathbb{F} \) has characteristic 2. In that case we have

\[ a = d = \varepsilon, \quad b = c = \varepsilon^2, \]

with \( \varepsilon \) a solution of the equation \( x^2 + x + 1 = 0 \), i.e. a primitive root of unity of order 3. This implies that the equations of the lines \( L, N_1, N_2, M_1, M_2 \) are (up to ordering)

\[ x + y, \quad \varepsilon x + z, \quad \varepsilon^2 x + y + z, \quad x + \varepsilon^2 y + z, \quad \varepsilon y + z. \]

These lines belong all to the pencil of lines passing through the point \( (1 : 1 : \varepsilon) \), a contradiction.

Now we pass to the second case (II). In this situation we start with the following figure

There are now 5 remaining configuration lines, which we call as usual \( L_1, \ldots, L_5 \). They form a star configuration, i.e. there are 10 mutually distinct intersection points \( P_{ij} = L_i \cap L_j \), for \( 1 \leq i < j \leq 5 \). Up to renumbering these points are distributed as follows
| line | points on the line |
|------|-------------------|
| $M_1$ | $P_{12}, P_{34}$ |
| $M_2$ | $P_{15}, P_{23}$ |
| $N_3$ | $P_{33}$ |
| $N_1$ | $P_{14}, P_{25}$ |
| $N_2$ | $P_{24}, P_{35}$ |
| $M_3$ | $P_{45}$ |

Table 5

The Table 5 is filled as follows. The first two rows are filled just by assigning labels. They imply immediately that $Z_4 \in L_4$ and $Z_5 \in L_5$. There is another triple point of the configuration not depicted in Figure 9. The line $L_2$ cannot pass through this point so that it must be $P_{13}$ (note that $L_4$ and $L_5$ intersect $N_3$ already in $Z_4$ and $Z_5$ respectively). The points $P_{24}$ and $P_{25}$ must then lie one on $N_1$ and the other on $N_2$. We have selected the labeling in such a way, that $Z_i \in L_i$ holds for all $i = 1, \ldots, 5$.

Similarly as in case (I), without loss of generality, we may assume that the equations of the lines $L_i$ are

$$
\begin{align*}
L_1 & : x, \\
L_2 & : y, \\
L_3 & : z, \\
L_4 & : x + y + z, \\
L_5 & : ax + by + z.
\end{align*}
$$

Evaluating the conditions $Z_i \in L_i$ for $i = 1, \ldots, 5$ we obtain the following system of equations (one condition, $Z_2 \in L_2$, is satisfied automatically)

$$
\begin{align*}
-a^2 + ab^2 + ab - b^2 &= 0 \\
-2a^2 + ab^2 + ab - a &= 0 \\
-2ab^2 + ab - a + b^2 &= 0 \\
-a^2 - ab - a + b^2 &= 0
\end{align*}
\tag{13}
$$

This system is equivalent to the system

$$
\begin{align*}
3(a - b^2) &= 0 \\
ab - 2b^2 + a &= 0 \\
a^2 - ab + b^2 - a &= 0
\end{align*}
\tag{14}
$$

The latter system has to be treated differently in case of characteristic of $F$ equal either 2 or 3 but all cases lead to the same solution $a = b = 1$, we omit the details. This solution means $L_4 = L_5$, a contradiction.

We conclude the proof of Theorem 2.1 with an example of a configuration of 11 lines with 16 triple points. Our example is constructed over an arbitrary field $F$ which contains the golden section ratio. Our example is dual to the example in [8, page 398, figure (i)].

Turning to details, let $b \in F$ satisfy $b^2 + b - 1 = 0$ and let the configuration lines be given by the following equations:

$$
\begin{align*}
L_1 & : x, \\
L_2 & : y, \\
L_3 & : z, \\
L_4 & : x + y + z, \\
L_5 & : -bx + z, \\
L_6 & : bx + y + bz, \\
L_7 & : y + z, \\
L_8 & : b^2x + by + z, \\
L_9 & : bx - by - z, \\
L_{10} & : -b^3x - y - bz, \\
L_{11} & : -b^2x + (1 - b)y.
\end{align*}
$$
The configuration points are then easily computed to be:

\[ P_1 = (0 : -1 : 1), \quad P_2 = (1 : 0 : 0), \quad P_3 = (0 : 1 : 0), \]
\[ P_4 = (1 : 0 : -1), \quad P_5 = (-1 : b + 1 : -b), \quad P_6 = (-1 : b : 0), \]
\[ P_7 = (1 : 0 : b), \quad P_8 = (0 : -b + 1), \quad P_9 = (0 : -1 : b), \]
\[ P_{10} = (1 : 0 : -b^2), \quad P_{11} = (1 - b : -b : b), \quad P_{12} = (-1 : b : -b), \]
\[ P_{13} = (1 : 1 : -1) \quad P_{14} = (0 : 0 : 1), \quad P_{15} = (1 : 1 : 0), \]
\[ P_{16} = (1 : 1 : -2). \]

The incidence table in this case reads:

|    | \( P_1 \) | \( P_2 \) | \( P_3 \) | \( P_4 \) | \( P_5 \) | \( P_6 \) | \( P_7 \) | \( P_8 \) | \( P_9 \) | \( P_{10} \) | \( P_{11} \) | \( P_{12} \) | \( P_{13} \) | \( P_{14} \) | \( P_{15} \) | \( P_{16} \) |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \( L_1 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_2 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_3 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_4 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_5 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_6 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_7 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_8 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_9 \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_{10} \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
| \( L_{11} \) | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |
|    | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        | +        |

And finally, the configuration is visualized in Figure 10. In this figure the dashed circle indicates the line at the infinity on which parallel lines intersect. So that for example parallel lines \( L_9 \) and \( L_{11} \) intersect in point \( P_{15} \). The line at infinity is a configuration line.
3 Configurations with many triple points

In complex algebraic geometry a point where exactly two lines meet is a node. This is the simplest singularity one encounters and is denoted by $A_1$ in the A-D-E-classification of simple singularities of curves, see for example [3]. Plane curves (not necessarily splitting in lines) with $A_1$ singularities are well understood, see for example [13]. When exactly three lines meet in a point, then there is a $D_4$ singularity in that point. Apart from $A_1$, this is the only simple singularity which can appear in an arrangement of lines. Plane curves containing $D_4$ singularities are way less understood, see for example [15, Section 11]. Results of this note can be considered as a step towards completing this picture.

The construction we present here is directly motivated by the passage from the Hesse configuration to its dual.

Let $E$ be an elliptic curve embedded as a smooth plane cubic. The group law on $E$ is related to the embedding by the following equivalent conditions

\begin{enumerate}
\item the points $P$, $Q$ and $R$ on the curve $E$ are collinear;
\item $P + Q + R = 0$ in the group $E$.
\end{enumerate}

Let $p$ be a prime number $\geq 3$. There are exactly $p^2$ mutually distinct solutions to the equation $pX = 0$ on $E$. These solutions form a subgroup $E(p)$ of $p$-torsion points. Since they form a subgroup and by the above equivalence, any line joining two distinct points in $E(p)$ intersects $E$ in another point which is also an element of $E(p)$. The tangent line to $E$ at 0 is tangent there to order 3, in particular the equation $2X = 0$ on $E$ has no non-trivial solution in $E(p)$. The tangent lines to $E$ at every other point $X \in E(p) \setminus \{0\}$ intersects $E(p)$ in some other point $Y$. This is because the equation $2X + Y = 0$ has a unique solution in $E(p)$ for all $Y \neq 0$. In particular the point $X$ also lies on a line tangent to $E$ at some point $Z \in E(p)$.

Hence, there are altogether $(p^2+4)(p^2-1)/6$ lines determined by pairs of points in $E(p)$. There are $p^2+1$ configuration lines passing through 0 and $p^2+1$ lines passing through every other point in $E(p)$.

Passing to the dual configuration, we obtain thus $p^2$ lines with $t_3(p) := \frac{(p^2-1)(p^2-2)}{6}$ triple points (and $p^2 - 1$ double points corresponding to the tangents at points $X \in E(p) \setminus \{0\}$). The equality in [11] guarantees that there are no other intersection points between the lines. Since $p$ is a prime, there is no rounding in [11] and it cannot be $p^2 \equiv 5 \mod (6)$, thus the difference $U_3(p^2) - t_3(p) = \frac{p^2-1}{3}$.

Hence we have proved our final result.

**Theorem 3.1.** For any prime number $p \geq 3$, there exists a configuration of $p^2$ lines intersecting in $\frac{(p^2-1)(p^2-2)}{6}$ triple points (and $p^2 - 1$ double points).

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