LOOSE LEGENDRIAN EMBEDDINGS IN HIGH DIMENSIONAL CONTACT MANIFOLDS

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ABSTRACT. We give an h-principle type result for a class of Legendrian embeddings in contact manifolds of dimension at least 5. These Legendrians, referred to as loose, have trivial pseudo-holomorphic invariants. We demonstrate they are classified up to ambient contact isotopy by smooth embedding class equipped with an almost complex framing. This result is inherently high dimensional: analogous results in dimension 3 are false.

1. Introduction

Let $(\mathbb{R}^{2n+1}, \xi)_{std} = \ker (dz - \sum y_i dx_i)$ be a contact manifold. A Legendrian knot is defined to be a closed, connected, embedded submanifold $L^n \rightarrow Y$ so that $TL \subseteq \xi$. Though we abuse notation and say $L \subseteq Y$, we study parametrized embeddings everywhere in this paper. Legendrian knots of particular interest include knots with topology $S^n$, and/or knots embedded in $(\mathbb{R}^{2n+1}, \xi_{std})$.

Definition 1.1. Let $f : L^n \hookrightarrow (Y^{2n+1}, \xi)$ be a smooth embedding, and let $F_s : TL \rightarrow TY|_L$ be a homotopy of bundle monomorphisms, covering $f$ for all $s$, so that $F_0 = df$ and $F_1(TL)$ is a Lagrangian subspace of $\xi$. (Recall $\xi$ has a canonical conformal symplectic structure.) The pair $(f, F_s)$ is called a formal Legendrian knot.

A Legendrian knot can be thought of as a formal Legendrian by letting $F_s = df$ for all $s$. In particular, we say that two Legendrian knots are formally isotopic if there exists a smooth isotopy $f_t : L \rightarrow Y$ between them, and $df_t$ is homotopic through paths of monomorphisms, fixed at the endpoints, to a path of Lagrangian monomorphisms. Notice that classifying formal Legendrian knots up to formal isotopy is a question purely about smooth topology and bundle theory, we do this for the case $(Y, \xi) = (\mathbb{R}^{2n+1}_{st})$ in Appendix A. There are many infinite families of distinct Legendrian knots which are formally isotopic which can be distinguished with pseudo-holomorphic curve invariants [9].

Informally, we call a Legendrian knot with $n > 1$ loose if it contains a sufficiently thick Weinstein neighborhood of a stabilized Legendrian curve; we give a precise definition in the following section. The principal purpose of this paper is a proof of

Theorem 1.2. Suppose $n > 1$ and fix a contact manifold $(Y^{2n+1}, \xi)$. Then for each formal Legendrian isotopy class there is a loose Legendrian knot in that class,
unique up to ambient contact isotopy.

We will assume that the reader is familiar with the general philosophy of the h-principle; theorems from [19], [15], and [16] are cited explicitly in the paper. A brief outline of the paper follows. In Section 2 we cover a number of definitions from contact topology, including a precise definition of loose knots. We demonstrate an h-principle for \(\epsilon\)-Legendrian knots in Section 3; this allows us to set up controllably transverse local charts.

Section 4 is a review of [16]; there we define the concept of wrinkled embeddings and state an h-principle they satisfy. In the following section we adapt this concept to the Legendrian context, and prove an h-principle for Legendrians with prescribed singularities. Section 6 then describes a method to resolve these singularities. The main theorem is proved in Section 7 using the tools from the previous sections.

In the Conclusion 8 we discuss corollaries of Theorem 1.2 and compare the result to other concepts in contact topology. Finally, we classify formal isotopy classes of Legendrian knots in \( (\mathbb{R}^{2n+1}, \xi_{std}) \) in the Appendix A.

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2. Definitions from Contact Topology

In this section we give some definitions and general facts about Legendrian knots. By Darboux neighborhood in \((Y, \xi)\) we mean an open set \(U \subseteq Y\), together with a contactomorphism to a (geometrically) convex subset of \((\mathbb{R}^{2n+1}, \xi_{std})\). Given a Darboux neighborhood we can define two projections, the Lagrangian projection \((x_i, y_i, z) \mapsto (x_i, y_i)\) and the front projection \((x_i, y_i, z) \mapsto (x_i, z)\). For the former a Legendrian will project to an exact Lagrangian immersion in \(\mathbb{R}^{2n+1}_{std}\), and the \(z\) coordinate can be recovered (up to a constant) by integrating \(\sum y_i dx_i\). A self intersection in this projection is called a Reeb chord.

In the front projection a Legendrian projects to a (highly) singular hypersurface, which nevertheless has well defined tangent fibers everywhere. These tangent planes are nowhere vertical and the coordinate slopes recover the \(y_i\) coordinates of the Legendrian. A Legendrian has a Reeb chord wherever its front is self-tangent after a local vertical translation; in particular a Legendrian immersion has a self intersection exactly where its front is self tangent. The kernel of the differential of the front projection is a Legendrian foliation \(F\) whose leaves are the Legendrians \(\{(x, z) = \text{constant}\}\). A Legendrian thus has singularities in the front projection exactly where it intersects \(F\) non-transversely. In this paper we only make use of cusp singularities in the front, locally given by the equation \(z^2 = x_1^3\).
Definition 2.1. In $\mathbb{R}^3_{\text{std}}$, a stabilization is a Legendrian curve which has the properties depicted in Figure 1. Specifically, it is required to have a unique transverse self-intersection and a single cusp in the front projection, and a single Reeb chord. The length of this Reeb chord is called the action of the stabilization. (We do not distinguish an orientation of a stabilization.)

Remark. Inside a contact 3-manifold $(Y, \xi)$, a bypass is defined to be an embedded topological 2-gon $D$ whose characteristic foliation $TD \cap \xi$ has no singularities on the interior, a negative elliptic singularity on one edge, positive elliptic singularities at the two vertices, and a positive hyperbolic singularity on the remaining edge [21]. See Figure 2. Let $L$ be a Legendrian arc in $Y$. Then there is a Darboux chart $U \subseteq Y$ so that $L \subseteq U$ is a stabilization if and only if there is a Legendrian arc $\alpha \subseteq Y$ connecting the endpoints of $L$, so that $L \cup \alpha$ is the boundary of a bypass, where $L$ (respectively $\alpha$) contains the negative elliptic (positive hyperbolic) singularity. In the coordinate system on $U$, the arc $\alpha$ is parallel to the $y$-axis, defining the self-intersection in the in the front projection. The equivalence of these two definitions is a simple application of the Weinstein neighborhood theorem, but we will not use this alternative interpretation in this paper.

Definition 2.2. Suppose $n > 1$. Let $B \subseteq \mathbb{R}^3_{\text{std}}$ be an open ball containing a stabilization of action $a$, and let $V_\rho = \{ |p| \leq \rho, |q| \leq \rho \} \subseteq T^* \mathbb{R}^{n-1}$. Note that $B \times V_\rho$ is an open convex set in $\mathbb{R}^{2n+1}_{\text{std}}$. Let $\Lambda$ be the cartesian product of the stabilization and the zero section, which is Legendrian in $B \times V_\rho$. We call the pair $(B \times V_\rho, \Lambda)$ a Legendrian twist. A Legendrian twist satisfying $\frac{a}{\rho} < 2$ is called a loose chart. Finally, let $L$ be a Legendrian knot in a contact manifold $(Y, \xi)$. If there is a Darboux chart $U \subseteq Y$ so that $(U, U \cap L)$ is a loose chart then $L$ is called loose.

For any constant $c > 0$, we may change coordinates under the contactomorphism $(x_i, y_i, z) \mapsto (cx_i, cy_i, c^2 z)$. This can make either $a$ or $\rho$ any given size, but not simultaneously. The requirement $\frac{a}{\rho} < 2$ is the essential condition in the above definition; we claim that every Legendrian $L$ contains a Legendrian twist. To show
this it suffices to find a contact 3-ball $B \subseteq Y$ so that $B \cap L$ is a stabilization. Let $B^3$ be any small 3-ball intersecting $L$ in a single arc. Since $n > 1$, the h-principle for isocontact embeddings of positive codimension \cite{m} implies that a $C^0$ perturbation of $B^3$ (fixed near $L$) has the necessary properties.

**Proposition 2.3.** Inside a loose chart, there is another Legendrian twist with parameters $a, \rho$ so that $\frac{a}{\rho}$ is arbitrarily small. A loose chart contains two disjointly embedded loose charts.

**Proof:** The first statement implies the second. The proof is essentially contained in Figure 3. We first rescale coordinates on $B \times V_\rho$ so that $\rho$ is normalized to 1. Fix a small $\delta > 0$, and let $\rho' = 1 - \frac{a}{2} - \delta > 0$. Inside $B \times V_\rho$ we are able to isotope $\Lambda$ to $\Lambda'$, so that $\Lambda' \cap B \times V_{\rho'}$ is a Legendrian twist with action $\delta$ (in the given coordinates). Note that $\frac{\delta}{(1-\frac{a}{2}-\delta)^2}$ can be made arbitrarily small by choosing sufficiently small $\delta$. \hfill \Box

We now define an operation that alters any Legendrian knot so that it becomes loose. This construction is unnecessary for the purpose of constructing a Legendrian isotopy between two loose knots, but we will need it to show the existence portion of Theorem 1.2. This operation was first defined in \cite{m}; there it was introduced (without a name) as an operation to alter Legendrian framings in order to construct Stein manifolds. See Proposition A.3. It was later considered in \cite{m} where it was shown that this operation causes pseudo-holomorphic invariants to become trivial.

In the front, consider a small neighborhood of a cusp singularity. After flattening things out, we can say the neighborhood consists of two horizontal open disks $\{z = 0\}$ and $\{z = 1\}$, connected by a strip with a single cusp. By choosing a smaller neighborhood and rescaling coordinates we can assume this model is arbitrarily large in all $x$ and $y$ directions. Of course any point on a Legendrian admits local coordinates so that the given point is on a cusp in the front, thus a small
Figure 3. We can isotope a loose chart in a neighborhood of itself, so that it contains a Legendrian twist with arbitrarily small action as a subset. This picture is in the front projection, note that all coordinates $y_i = \frac{dz}{dx_i}$ are bounded by $\rho$ neighborhood of any point on a Legendrian admits these coordinates.

**Definition 2.4.** Let $L^n$ be a Legendrian knot in $(Y, \xi)$. Let $M \subseteq D^n$ be a compact, codimension 0 manifold, so that $M \cap \partial D^n = \emptyset$. Choose a Morse function $h : M \to [0, 2]$, which is identically zero near $\partial M$ and has all critical values larger than 1. Choose a point on $L$ and local coordinates as above, suitably large to accommodate $h$. On the compactly supported set they disagree, replace the disk $\{z = 0\}$ with the set $\{z = h(x)\}$. This Legendrian knot is called the $M$-stabilization of $L$, denoted $s_M(L)$. See Figure 4.

This construction does not depend on the choice of neighborhood, since any small disk in $L$ can be taken to any other by ambient contactomorphism. A priori, $s_M(L)$ may depend on the isotopy class of embeddings $M \subseteq D^n$; we assume this data is included in order to define $s_M(L)$. In fact, Theorem 1.2 implies $s_M(L)$ is determined up to Legendrian isotopy by only $\chi(M)$ and the formal Legendrian isotopy class of $L$ when $n > 1$. For the case $n = 1$ the reader can check that $D^1$-stabilizing a knot is equivalent to stabilizing a curve twice, once with each orientation.

**Proposition 2.5.** For any Legendrian knot of dimension $n > 1$ and any $M \subseteq D^n$, $s_M(L)$ is loose.

Proof: In the coordinates defined above, there is visibly a Legendrian twist with action 1, see Figure 4. The radius of the neighborhood in the $x$ directions is determined by the topology of the embedding $M \subseteq D^n$, but the radius in the $y$ directions
Figure 4. An $M$-stabilization of a small neighborhood. Here $M$ is the annulus. Any $M$-stabilized Legendrian contains a loose chart, shown here as the region between the thin curves.

may be taken to be arbitrarily large, as discussed above. By rescaling the $x$ and $y$ coordinates in inverse proportion (keeping the contact form fixed), we exhibit a loose chart.

$$\square$$

**Proposition 2.6.** Let $L$ be a Legendrian knot, and suppose $\chi(M) = 0$. Then $s_M(L)$ is formally isotopic to $L$.

**Proof:** Identify $M \subseteq L$ as the set $\{h(x) > 1\}$, as in Definition 2.4. We first describe a smooth isotopy, undoing the $M$-stabilization. The $y$ coordinates of our knot are given by the gradient of $h$. Fixing this near $\partial M$, we can homotope the gradient to a nonzero vector field, since $\chi(M) = 0$. We interpret this as an isotopy which alters the $y$ coordinates but has a fixed front projection. We can then push $M$ down through the $\{z = 1\}$ plane without the knot self-intersecting.

It remains to show that this smooth isotopy, $f_t$, is actually the base of a formal Legendrian isotopy, that is, we need to homotope $df_t$ through bundle monomorphisms to a Lagrangian monomorphism. Since we avoid the singular set, the obvious straight line path through bundle maps projects non-singularly to the $x$ coordinate plane. It follows that this path is in fact through monomorphisms. $\square$

3. $\epsilon$-Legendrian Knots

We demonstrate an h-principle for $\epsilon$-Legendrian knots in this section. The advantage of working with $\epsilon$-knots rather than formal knots is that it gives us a set
of Darboux coordinates around every point, so that \( L \) has a smooth front projection. For the purposes of this paper \( \epsilon = \frac{\pi}{7} \) is sufficiently small. First, we define a \textit{Legendrian plane field} to be a Lagrangian subfield of the distribution \( \xi \).

**Definition 3.1.** An embedded submanifold \( L^n \subseteq (Y,\xi) \) is called \( \epsilon \)-Legendrian if there is a Legendrian plane field along \( L \), \( \lambda \), which is \( \epsilon \)-close to \( TL \). Here, two \( n \)-planes are said to be \( \epsilon \)-close if the projection from one plane to the other is an isomorphism and the angle between any vector and its projection is less than \( \epsilon \) (in some fixed metric).

We use this opportunity to discuss the general problem of \( A \)-directed embeddings, which we will discuss in other contexts throughout the paper. Let \( L \) be an \( n \)-manifold, and \( Y \) a manifold of larger dimension. Let \( A \subseteq Gr_n(Y) \), where \( Gr_n(Y) \) denotes the bundle of \( n \)-planes in \( TY \), with fiber \( \operatorname{Gr}_{\dim(Y),n} \). An \( A \)-directed embedding is an embedding \( L \rightarrow Y \) so that \( TL \subseteq A \). A \textit{formal} \( A \)-directed embedding is a smooth embedding \( f: L \rightarrow Y \), together with a path of bundle monomorphisms \( F_s: TL \rightarrow TY \) covering \( f \), so that \( F_0 = df \) and \( \operatorname{Im}(F_1) \subseteq A \). To say an \( h \)-principle holds for \( A \)-directed embeddings is to say the inclusion of \( A \)-directed embeddings into formal \( A \)-directed embeddings is a weak homotopy equivalence (with the \( C^\infty \) topologies). In particular, it induces a bijection on \( \pi_0 \) of these spaces: for every formal \( A \)-directed isotopy class, there is exactly one \( A \)-directed embedding up to \( A \)-directed isotopy.

Even under the assumption that \( A \) is open, an \( h \)-principle for \( A \)-directed embeddings is not generally true. For example if \( L = S^2 \) and \( Y = \mathbb{R}^3 \), the \( h \)-principle for \( A \)-directed embeddings fails for any proper subset \( A \subseteq Gr_{3,2} \). In [19], it is shown that an \( h \)-principle holds for all open \( A \), if \( L \) is an open manifold. Furthermore, the concept of convex integration is used there to prove an \( h \)-principle holds for \( A \)-directed embeddings of closed manifolds, under the assumption \( A \) is open and \textit{ample}. Rather than stating the original definition, we give the ampltenseness criterion 19.1.1 from [15].

**Proposition 3.2.** Let \( A \subseteq Gr_n(Y) \), fix \( p \in Y \), and let \( S \in Gr_{n-1}(Y)_p \) be a \((n-1)\)-plane contained inside an element of \( A \). Let \( \Omega_{p,S} = \{ v \in T_pY; \operatorname{Span}\{S,v\} \in A_p \} \). Assume for every choice of \( p \) and \( S \), the convex hull of each connected component of \( \Omega_{p,S} \) is equal to \( T_pY \). Then \( A \) is ample.

Let \((Y,\xi)\) be contact, and let \( A \subseteq Gr_n(Y) \) be the subset of \( n \)-planes which deviate from a Lagrangian plane in \( \xi \) by angle less than \( \epsilon \). In these terms an embedding \( L \rightarrow Y \) is \( \epsilon \)-Legendrian if and only if it is \( A \)-directed. Assume that \( S \) is an \((n-1)\)-plane which makes an angle less than \( \epsilon \) with some Legendrian plane. Then \( \Omega_{p,S} \) is connected, open, and scalar invariant. This implies the convex hull of \( \Omega_{p,S} \) is all of \( T_pY \), and thus \( A \) is ample by Proposition 3.2.

Convex integration implies an \( h \)-principle for \( \epsilon \)-Legendrian knots; this means the space of \( \epsilon \)-Legendrian knots is weakly homotopy equivalent to formal \( \epsilon \)-Legendrian knots. If furthermore \( \epsilon < \frac{\pi}{2} \) then the space of formal \( \epsilon \)-Legendrian knots is weakly
homotopy equivalent to the space of formal Legendrian knots, simply because this is true in each fiber.

**Proposition 3.3.** Let $\epsilon < \frac{\pi}{2}$. Then the natural inclusion of $\epsilon$-Legendrian knots into formal Legendrians knots is a weak homotopy equivalence. In particular, every formal Legendrian is formally homotopic to an $\epsilon$-Legendrian, and any formal isotopy between two $\epsilon$-Legendrians can be $C^0$ perturbed (rel endpoints) to an $\epsilon$-Legendrian isotopy.

### 4. Review of Wrinkled Embeddings

In this section, we review concepts from [16] needed for the proof of Theorem 1.2. While attempting to be minimally complete, it would be to the reader’s advantage to understand the constructions there more thoroughly. Theorem 1.2 can be thought of as an application of Eliashberg/Mishachev’s ideas to contact topology.

As discussed in the previous section, an h-principle for $A$-directed embeddings of a closed manifold $L$ is not generally true, even if we assume $A$ is open. The motivation of the definitions in [16] is to prove an h-principle for all open $A$, by relaxing the notion of embedding. Specifically, a wrinkled embedding is a smooth map which is a topological embedding, but is allowed to have prescribed singularities. These singularities have well defined tangent fibers, allowing us to define $A$-directed wrinkled embeddings. The main theorem from [16] is an h-principle for $A$-directed wrinkled embeddings, for any open $A$. We now make these statements precise, which we will adapt to a local, codimension 1 situation.

**Definition 4.1.** Let $W : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a smooth, proper map, which is a topological embedding. Suppose $W$ is a smooth embedding away from a finite collection of spheres, $\{S_j^{n-1}\}$. Suppose, in some coordinates near these spheres, $W$ can be parametrized by $W(u, \vec{v}) = (\vec{v}, u^3 - 3u(1-|\vec{v}|^2), \frac{1}{3}u^5 - \frac{2}{3}u^3(1-|\vec{v}|^2) + u(1-|\vec{v}|^2)^2)$, where our domain coordinates lie in a small neighborhood of the sphere $\{|\vec{v}|^2 + u^2 = 1\} \subseteq \mathbb{R}^n$. Then $W$ is called a wrinkled embedding, and the spheres $S_j^{n-1}$ are called the wrinkles.

Let $\psi : \mathbb{R} \to \mathbb{R}^2$ be the plane curve, defined by $\psi(u) = (\psi^1(u), \psi^2(u)) = (u^3 - 3u, \frac{1}{5}u^5 - \frac{2}{3}u^3 + u)$, shown in Figure 3 (we assume $\psi$ is horizontal outside a compact subset). Let $\psi_\delta$ be a rescaling of this function, defined by $\psi_\delta(u) = (\delta^{3/2}\psi^1(\frac{u}{\sqrt{\delta}}), \delta^{5/2}\psi^2(\frac{u}{\sqrt{\delta}}))$. This is well defined even when $\delta < 0$, in this case $\psi_\delta$ is smooth and graphical. We define $\psi_0(u) = (u^3, \frac{1}{5}u^5)$, which makes $\psi_\delta$ a continuous family of plane curves.

In these terms, wrinkled embeddings are locally modeled by $W(u, \vec{v}) = (\vec{v}, \psi_{1-|\vec{v}|^2}(u))$. Therefore wrinkles have two kinds of singularities: on the singular sphere $\{|\vec{v}|^2 +
$u^2 = 1 \}$, there are cusp singularities everywhere on the lower and upper hemisphere. Along the equator \( \{ u = 0 \} \), we see “unfurled swallowtail” singularities. See Figure 6.

An embryo of a wrinkle is defined to be the isolated singularity with a local model given by

\[
(u, \vec{v}) \mapsto (u^3 + 3u|\vec{v}|^2, \vec{v}, \frac{1}{5}u^5 + \frac{3}{2}u^3|\vec{v}|^2 + u|\vec{v}|^4)
\]

with \((u, \vec{v})\) in a neighborhood of the origin. For \( t \in (-\epsilon, \epsilon) \), let \( W_t(\vec{v}, u) = (\vec{v}, \psi_{t - |\vec{v}|^2}(u)) \). Then \( W_t \) is smooth for \( t < 0 \) and has a single wrinkle when \( t > 0 \). At \( t = 0 \), there is an embryo singularity at \((u, \vec{v}) = 0\). We allow embryo singularities whenever we discuss parametric families of wrinkled embeddings. Generically, these occur with codimension 1 in parameter space, and are isolated points in the embedding. We do not distinguish a time orientation, so an embryo can either create a wrinkle in forward time, or allow one to disappear.

Even though a wrinkled embedding is singular, it does have well defined tangent fibers of dimension \( n \) everywhere. For example, let \( p \) be a cusp singularity point.
given in coordinates by \( f(u) = (u^2, u^3) \). Even though \( df \) is trivial at the point \( u = 0 \), small neighborhoods of this point are \( C^1 \) close to uniformly horizontal. Therefore we define the tangent fiber to be horizontal at that point. One can similarly check that the tangent fibers near an unfurled swallowtail or embryo singularity uniformly approach horizontal in the coordinates given above.

Therefore, given a wrinkled embedding \( W : \mathbb{R}^n \to \mathbb{R}^{n+1} \), we have defined the hyperplane bundle \( TW \subseteq T^{\mathbb{R}^{n+1}}|_{\text{Im}(W)} \). This allows us to define \( A \)-directed wrinkled embeddings: wrinkled embeddings with \( TW \subseteq A \). We quote the h-principle from [10], again specialized for our purposes:

**Theorem 4.2.** Let \( f : \mathbb{R}^n \to \mathbb{R}^{n+1} \) be a graphical smooth map, and let \( \nu^0 \subset T^{\mathbb{R}^{n+1}}|_{\text{Im}(f)} \) be a nowhere vertical hyperplane distribution so that \( \nu = \text{Im}(df) \) outside of a compact set, \( C \). Then there is a wrinkled embedding \( W : \mathbb{R}^n \to \mathbb{R}^{n+1} \) so that \( TW \) is \( C^0 \) close to \( \nu \) and \( W \) is \( C^0 \) close to \( f \), equal outside \( C \). This also holds parametrically: for families \( f_t \) and \( \nu_t \) with \( t \in D^m \) and \( \nu_t = \text{Im}(df_t) \) at \( \partial D^m \), we can find a family of wrinkled embeddings \( W_t \), \( C^0 \) close to \( f_t \), so that \( TW_t \) is \( C^0 \) close to \( \nu_t \) and \( W_t = f_t \) at \( \partial D^m \) and outside \( C \).

5. WRINKLED LEGENDRIANS

Say we wanted to prove that embedded Legendrian knots satisfy an h-principle, despite knowing this is false. We will see this reduces simply to solving the local extension problem: given a formal Legendrian \( f : \mathbb{R}^n \to \mathbb{R}^{2n+1}_{\text{std}} \) which is Legendrian outside a compact set, we need to show we can find a Legendrian embedding \( C^0 \) close to \( f \), and equal to it outside the compact set. The \( C^0 \) close condition is essential: we have no lower bounds on the size of our local charts and we need to avoid self intersections.

The set of Legendrian planes in \( Gr_n(\mathbb{R}^{2n+1}_{\text{std}}) \) is not open, so none of our theorems about directed embeddings apply immediately. The advantage of the local picture is it allows us to re-interpret the geometry in the front projection. Any smooth embedding \( W : \mathbb{R}^n \to \mathbb{R}^{n+1} \) which is never vertical defines a Legendrian \( L \subseteq \mathbb{R}^{2n+1}_{\text{std}} \). Assume \( f \) projects to a smooth hypersurface \( H \subseteq \mathbb{R}^{n+1} \). The \( y \) coordinates of \( f \) define a hyperplane field \( \nu = \ker(dz - \sum_i y_i dx_i) \subseteq T^{\mathbb{R}^{n+1}}|_H \). Then \( L \) is \( C^0 \) close to \( f \) if \( W \) is \( C^0 \) close to \( H \) and \( TW \) is \( C^0 \) close to \( \nu \).

In fact \( W \) need not be smooth, since a smooth Legendrian need not have a smooth front projection. At this point we would like to use Theorem 4.2, but first we need to study wrinkled singularities to determine if they have smooth Legendrian lifts. Wrinkled embeddings have a natural tangent bundle. More precisely, given a non-vertical wrinkled embedding \( W : \mathbb{R}^n \to \mathbb{R}^{n+1} \) there are unique smooth functions \( y_i(v, u) \) so that \( dz = y_i dx_i \) everywhere. At a cusp singularity \( (x, z) = (u^2, u^3) \), this function is given by \( y = \frac{2}{3}u \). In this case, the triple \( (x, y, z) \) is a smooth embedding, therefore the cusp singularity is the front projection of a smooth Legendrian curve.

For unfurled swallowtails (as well as embryos) the functions \( y_i \) are uniquely defined, but the induced map \( L : \mathbb{R}^n \to \mathbb{R}^{2n+1}_{\text{std}} \) is not an embedding: \( dL \) has rank
n − 1 at these singularities. For now we “define away” this problem.

**Definition 5.1.** Let \( L \) be closed and connected. A *wrinkled Legendrian* is a smooth map \( f : L \to (Y, \xi) \), which is a topological embedding, satisfying the following properties. The image of \( df \) is contained in \( \xi \) everywhere, and \( df \) is full rank outside a subset of codimension 2. This singular set is required to be diffeomorphic to a disjoint union of \((n − 2)\)-spheres \( \{S_j^{n-2}\} \), called *Legendrian wrinkles*. We assume each \( S_j^{n-2} \) is contained in a Darboux chart \( U_j \), so that the front projection of \( L \cap U_j \) is a wrinkled embedding, smooth outside of a compact set. (In particular, the front projection of each \( S_j^{n-2} \) is the unfurled swallowtail singularities of a single wrinkle in the front.)

A wrinkle Legendrian is therefore a smooth Legendrian embedding outside a set of codimension 2, however it is permitted to contain the singularity defined as the Legendrian lift of the unfurled swallowtail. Our definition is slightly stronger than this: we also require a “global trivialization” of each Legendrian wrinkle given by the Darboux charts \( U_j \). We emphasize that \( \{U_j\} \) is considered part of the data of a wrinkled Legendrian: for a given map \( f : L \to (Y, \xi) \), different choices of \( \{U_j\} \) are considered to be different as wrinkled Legendrians. However notice there is no requirement for these Darboux charts to be disjoint, and in fact we often take them to be equal when multiple Legendrian wrinkles are contained in a single Darboux chart.

Before defining a topology on the space of wrinkled Legendrians, we first define an additional singularity that allows Legendrian wrinkles to appear and disappear in families. Again this is completely analogous to wrinkled embeddings. We define a *Legendrian embryo* to be the singularity given by the Legendrian lift of the embryo singularity defined in the previous section for maps \( \mathbb{R}^n \to \mathbb{R}^{n+1} \). Again these singularities are generically codimension 1 in parameter space and isolated points in the domain \( L \). For reference, a Legendrian embryo is given in Darboux coordinates by

\[
\begin{align*}
x_1 &= u^3 + 3u|\vec{v}|^2 \\
(x_2, \ldots, x_n) &= \vec{v} \\
y_1 &= \frac{1}{3}(u^2 + |\vec{v}|^2) \\
(y_2, \ldots, y_n) &= 2u(|\vec{v}|^2 - \frac{1}{3}u^2)\vec{v} \\
z &= \frac{1}{5}u^5 + \frac{2}{3}u^3|\vec{v}|^2 + u|\vec{v}|^4
\end{align*}
\]

with domain coordinates \((u, \vec{v})\) in a neighborhood of the origin in \( \mathbb{R}^n \). As before, we allow Legendrian embryos whenever we discuss parametric families of wrinkled Legendrians. When a Legendrian wrinkle is born we add a new \( U_j \) to the collection of Darboux charts which contains the Legendrian embryo, and it is required to contain the created wrinkle throughout its entire “lifetime”. To topologize the space of wrinkled Legendrians we use the \( C^\infty \) topology on the space of maps, together with independent \( C^\infty \) topologies for the Darboux charts \( U_j \).
We claim that Legendrian wrinkles have a canonical coorientation in $L$. A point on a Legendrian wrinkle is given in Darboux coordinates by

\[
\begin{align*}
x_1 &= u^3 + 3ux_2 \\
y_1 &= \frac{1}{3}(u^2 - x_2) \\
y_2 &= ux_2 + \frac{1}{3}u^3 \\
y_i &= 0 \text{ for } i > 2 \\
z &= \frac{1}{5}u^5 - \frac{2}{3}u^3x_2 + ux_2^2.
\end{align*}
\]

We see $\{u = 0, x_2 = 0\}$ is the singular set. $df$ has rank $n - 1$ on this set, and its kernel is spanned by $\partial_u$. Let $\beta : (-\epsilon, \epsilon) \to L$ be a path with $\beta(0) = 0$, so that $\dot{\beta}$ is in the kernel of $df$ at that point. Then the second derivative of $f \circ \beta$ at $s = 0$ defines a nonzero vector $v \in TY$. For another choice of path $\beta_1$, $v$ is scaled by $\left(\frac{||\dot{\beta}_1(0)||}{||\dot{\beta}(0)||}\right)^2$ and added to $df(\dot{\beta}_1(0))$. Thus at the wrinkle, there is a canonical $n$-plane containing the $\text{Im}(df)$, and it is canonically cooriented.

Also notice that $\text{Im}(df)$ contains the tangent space of the singular set, and thus a Legendrian wrinkle has a canonical normal framing in $L$.

If our goal is to prove an h-principle for wrinkled Legendrian knots, first we must define a map from wrinkled Legendrians to formal Legendrian knots. Given a wrinkled Legendrian $f : L \to (Y, \xi)$, we can $C^\infty$ perturb the map $f$ near the Legendrian wrinkles so that the perturbation $\tilde{f}$ is a smooth embedding. This is a contractable choice and can be made completely canonical by choosing a fixed perturbation of the model in Definition 5.1. To define the structure map $F_s$, we need to find a homotopy through bundle monomorphisms from $df$ to a Lagrangian map $F_1 : TL \to \xi$; we use the Darboux charts $\{U_j\}$ to define this. Consider the $(n + 1)$-plane bundle given by $P = \text{span}(\partial_{y_i}, \partial_z)$. Because $f$ is a perturbation of a wrinkled embedding which is nowhere vertical, $df$ is homotopic to a map which is everywhere transverse to $P$. The set of bundle maps $TL \to TU_j$ which are transverse to $P$ forms a contractable space, since they can be identified with the graphs of all $(n + 1) \times n$ matrices. Similarly the set of Lagrangian planes transverse to $P$ is contractable, being equivalent to the space of skew symmetric $n \times n$ matrices. Therefore we can homotope $df$ through monomorphisms to a Lagrangian map $F_1$ transverse to $P$ everywhere, and this homotopy is canonical up to a contractable choice.

**Theorem 5.2.** The map defined above from the space of wrinkled Legendrians to the space of formal Legendrian knots is a weak homotopy equivalence. In particular if two Legendrian knots are formally isotopic, then there is an isotopy through wrinkled Legendrians between them.

This theorem is essentially a combination of the h-principles discussed so far. We first prove a lemma to reduce the problem to Darboux charts, which is a citation of the Holonomic Approximation Theorem (Theorem 3.1.2 in [15]). As explained in Section 3, $\epsilon$ should be thought of as being roughly $\frac{\pi}{5}$; we reserve the words “close”
and “small” for an arbitrarily small size, which may depend on $\epsilon$.

**Lemma 5.3.** Let $L_t : L \rightarrow (Y, \xi)$ be a family of $\epsilon$-Legendrian knots with $t \in D^m$, which are Legendrian at $\partial D^m$. Then, we can perturb $L_t$ through $\epsilon$-Legendrians ($C^0$ small, fixed at $\partial D^m$) to $\bar{L}_t$, and find a finite collection of (continuous families of) Darboux balls $\{B^k_t\}$ which are disjoint for all $t$, so that $\bar{L}_t$ is Legendrian outside $\bigcup_{k} B^k_t$. Furthermore, we can arrange that $\bar{L}_t \cap B^k_t$ has a graphical front projection for all $t$ and $k$.

**Proof:** Choose a fixed point $t_0 \in D^m$, and let $\theta_t : Y \rightarrow Y$ be a family of ambient diffeomorphisms extending $L_t$, so that $\theta_{t_0}$ is the identity. For each $p \in L_{t_0}$, let $U \ni p \in D^m$, let $U_{p,t} : B_{std}^{2n+1} \rightarrow Y$ be a small Darboux neighborhood around $\theta_t(p)$, so that $U_{p,t}(0) = p$ and $(U_{p,t})_*(\langle \partial x_i \rangle) = T_p L_t$ at this point. We choose $U_{p,t}$ small enough so that $\lambda_t$ and $(U_{p,t})_*(\langle \partial x_i + y_i \partial z \rangle)$ are $\epsilon$-close, and also $(U_{p,t})_*(\langle \partial z \rangle) \notin \lambda_t$. The set $\bigcup_{k} (U_{p,t}^{-1}(B_{std}^{2n+1})), p \in L_{t_0}$) is an open cover of $L_{t_0}$: we choose a finite subcover, indexed by the points $p_k$. The cover defines a triangulation of $L_{t_0}$, so that each $(n-i)$-simplex is contained in $i + 1$ of these charts, let $K$ be the codimension 1 skeleton. Note that for all $t$, $\{U_{p_k,t}\}$ is a finite covering of $L_t$ by Darboux balls.

In coordinates, $L_t \cap U_k = \{z = z(x), y_i = y_i(x)\}$, and $\lambda = \\text{span}_i \{\partial x_i + y_i(x) \partial z + \sum_j g_j(x) \partial y_i \}$, for some functions $g_j(x)$. The Lagrangian condition on $\lambda$ is equivalent to $g'_j = g_j$, therefore the functions $z(x), y_i(x), g'_j(x)$ together define a section of the second jet space of the $x$ coordinates, $J^2(B^n)$.

To construct $\bar{L}_t$, we perturb $L_t$ inductively on each Darboux chart. Let $L^k_t = U_{p_k,t}(L_t) \subseteq B_{std}^{2n+1}$. Then the family of $\epsilon$-Legendrians $(L^k_t, \lambda_t)$ defines a family of non-holonomic sections of $J^2(B^n)$. We cite the $m$-parametric Holonomic Approximation Theorem [15]. It states that, on some open neighborhood $V$ of some $C^0$ perturbation of $K \cap L^k_t$, we may $C^0$ approximate the family of sections $(z, y_i, g'_j)_{t \in K}$ with sections $(\tilde{z}, \tilde{y}_i, \tilde{g}'_j)_{t \in K}$ which are holonomic on $V$. Over the intersection with previous charts, we fix our section where it is already holonomic and use the extension form of Holonomic Approximation. This defines a $\epsilon$-Legendrian family $\bar{L}_t$ which is $C^0$ close to $L_t$. $\bar{L}_t$ is Legendrian on $V$, since the section defining it is holonomic there. Furthermore $\partial \bar{L}_t$ is $C^0$ close to $\lambda_t$ over $V$, thus $\bar{L}_t$ remains graphical on $V$, in every chart $U_{p_k,t}$. To find $B^k_t$, take an open subset of $U_{p_k,t}$ which does not intersect $K$, but whose boundary is contained in $V$.

**Proof of Theorem 5.2** Suppose $L_t$ is a family of formal Legendrian knots, Legendrian at $t \in \partial D^m$. First, Proposition 5.3 states we can first make $L_t$ $\epsilon$-Legendrian for all $t$. The previous lemma then constructs a moving collection of Darboux balls, so that $L_t$ is Legendrian outside. Therefore in each Darboux ball we have an $\epsilon$-Legendrian family, which is Legendrian outside of a compact region. Furthermore the front projection of $L_t$ is a smooth graphical isotopy of a hypersurface, the $y$-coordinates determine a nowhere vertical hyperplane field along this front. We then apply Theorem 4.2 to find a family of wrinkled embeddings $W_{s,t}$ ($s \in [0,1]$), so that $W_{0,t}$ is the front of $L_t$, $W_{s,t}$ is always $C^0$ close to the front of $L_t$, and $TW_{1,t}$ is $C^0$ close to the given hyperplane field. This ensures the wrinkled Legendrian lift
of \( W_{1,t} \) is \( C^0 \) close to \( L_t \), and therefore it remains embedded.

This theorem holds in all dimensions. In the case \( n = 1 \), a generic wrinkled Legendrian will be smooth, since Legendrian wrinkles are codimension 2 submanifolds. However, an isotopy will contain Legendrian embryos, so we cannot conclude two Legendrian embeddings which are wrinkled isotopic are in fact Legendrian isotopic. In fact, crossing through a Legendrian embryo in this dimension is equivalent to a Legendrian curve stabilization (or destabilization). This implies a theorem originally proved in [18]: if two Legendrian 1-knots are formally isotopic, they are Legendrian isotopic after a finite number of stabilizations. It’s also true that two smoothly isotopic Legendrian 1-knots are formally isotopic after some number of stabilizations, this is a simple calculation in the algebraic topology of frame bundles (see Appendix A).

6. Twist Markings

Theorem 5.2 is a major portion of the work in proving Theorem 1.2. In this section we systemitize a method to resolve the singularities of wrinkled Legendrians.

**Definition 6.1.** Let \( L \subseteq (Y, \xi) \) be a wrinkled Legendrian with \( k \) wrinkles, and let \( \Phi \subseteq L \) be an embedded codimension 1 smooth compact submanifold with boundary. Assume \( \Phi \) has the topology of a sphere with \( k \) open disks removed. Then \( \Phi \) is called a twist marking if the singular set of \( L \) is equal to \( \partial \Phi \), and there is a small collar neighborhood of the singular set so that \( \Phi = \{ u = 0, x_2 \leq 0 \} \subseteq L \) in terms of the coordinates following Definition 5.1. If \( L \) has a Legendrian embryo, we require that it is contained in the interior of \( \Phi \) with local model given by \( \Phi = \{ u = 0 \} \).

**Definition 6.2.** We use the following topology on the space of wrinkled Legendrians with twist markings. We put the \( C^\infty \) topology on both the space of wrinkled Legendrians as well as the space of embedded submanifolds. We also specify a relation to accomodate discrete changes in the topology of \( \Phi \): if \( L_t \) is a path of wrinkled Legendrians containing a Legendrian embryo, and \( \Phi_t \) is a path of twist markings on \( L_t \) so that \( \Phi_t \) acquires another puncture at the embryo, the path \((L_t, \Phi_t)\) is defined to be continuous.

Recall Legendrian wrinkles have neighborhoods with front projection given by \( \{(x_1, \ldots, x_n, z); (x_1, z) = \psi_{x_2}(u), u \in (-\epsilon, \epsilon)\} \), where \( \{(x_2, u) = (0, 0)\} \) is the singular set. Let \( \delta > 0 \) be some small number. \( \Phi \) should be thought of as a formal representation of the neighborhood \( \{(x_1, z) = \psi_{x_3}(u); u \in (-\epsilon, \epsilon)\} \), where the \( x_1 \) direction is transverse to \( \Phi \). This interpretation allows us to resolve all singularities at \( \partial \Phi \).

**Proposition 6.3.** In \((Y, \xi)\), let \((L_t, \Phi_t)\) be a family of wrinkled Legendrians with twist markings. Then we can construct an isotopy of Legendrian knots \( \tilde{L}_t : L \to (Y, \xi) \), so that \( \tilde{L}_t \) identical to \( L_t \) outside any small neighborhood of \( \Phi_t \).
Proof: We only need to check things for our given models; it suffices to work in the front projection. Note that two (possibly wrinkled) Legendrian embeddings are $C^0$ close if their fronts are $C^1$ close. Check that when $\delta > 0$ is small, $\psi_\delta(u)$ is $C^1$ close to the horizontal axis, and identical to it outside a small neighborhood of the origin.

On the interior of $\Phi$, we find coordinates so that $(L, \Phi) = ((z = 0), \{(x, z) = (0, 0)\})$, and replace this by $\tilde{L} = \{(x, z) = \psi_\delta(u); u \in \mathbb{R}\}$. This alteration is $C^1$ small in the front projection and is contained in a neighborhood of $\Phi$; it remains to describe the behavior near $\partial \Phi$.

Let $m_\delta : \mathbb{R} \to \mathbb{R}$ be a smoothing of the function $\max(\delta, \cdot)$. Near points on $\partial \Phi$, we have coordinates so that $(L, \Phi) = \{(x, z) = \psi_{2z}(u); u \in \mathbb{R}\}$, which we replace with $\tilde{L} = \{(x, z) = \psi_{m_\delta(2z)}(u)\}$. This is nonsingular and compatible with our definition of the interior of $\Phi$. Since both $L$ and $\tilde{L}$ are $C^1$ close to horizontal near the singular set this alteration is $C^1$ small. See Figure 7.

Finally, we check this construction near a Legendrian embryo singularity. Let $L_t$ be a path with a unique embryo, we may choose coordinates so that the front of $L_t$ is given by $\{(x, z) = \psi_{1-r^2}(u); r^2 = x_2^2 + \ldots + x_n^2, u \in \mathbb{R}\}$, and $\Phi_t = \{(x_1 = 0, x_2^2 + \ldots + x_n^2 \geq t)\}$, here $t \in (-\epsilon, \epsilon)$. We then replace this path of wrinkled Legendrians with the path $\tilde{L}_t$, with front $\{(x, z) = \psi_{m_\delta(t-r^2)}(u)\}$. $\square$

We now prove a proposition that relates resolutions of Legendrian wrinkle singularities with loose charts. We say two properly embedded Legendrians $(U_0, L_0)$ and $(U_1, L_1)$ in the Darboux charts $U_i$ are equivalent if we can find contact inclusions $\iota_0 : U_0 \to U_1$ and $\iota_1 : U_1 \to U_0$, so that $L_0 = \iota_0^{-1}(L_1)$ and $L_1 = \iota_1^{-1}(L_0)$. As our main example, notice the Legendrian in $\mathbb{R}^3_{std}$ with front projection $\{(x, z) = \psi(\mathbb{R})\}$ (Figure 5) is equivalent to any stabilization (Definition 2.1, Figure 4).

Consider the properly embedded wrinkled Legendrian $\Lambda : B^n \to B_{std}^{2n+1}$ with front given by $\{(x, z) = \psi_{1-r^2}(u); r^2 = x_2^2 + \ldots + x_n^2, u \in \mathbb{R}\}$; we refer to this as an inside-out wrinkle. Recall that Legendrian wrinkles are given by $\{(x, z) = \psi_{1-r^2}(u)\}$. An inside-out wrinkle contains a single Legendrian wrinkle at $\{r^2 = 1, x_1 = 0\}$, however the embedding is not compactly supported. Notice the front
projection is singular on the set \( \{ \| \vec{v} \|^2 - \frac{\vec{x}^2}{4} = 1 \} \).

An inside-out wrinkle is not technically a wrinkled Legendrian as described in Definition 5.1 because there is no Darboux chart containing the Legendrian wrinkle so that the front is graphical outside of a compact subset. Let \( D_0^{n-1} = \{ r \leq 1, u = 0 \} \). Then \( D_0^{n-1} \) is easily checked to be a twist marking on \( \Lambda \); we now show that resolving \( \Lambda \) along this twist marking is equivalent to a loose chart.

**Proposition 6.4.** Let \((\Lambda, D_0^{n-1})\) be an inside out wrinkle with the twist marking defined above. Let \( \tilde{\Lambda} \) be the resolution of \( D_0^{n-1} \) at any scale, and let \( B \) be any Darboux ball containing \( D_0^{n-1} \). Let \((U, L)\) be any loose chart. Then \((B, \tilde{\Lambda})\) and \((U, L)\) are equivalent models, in terms of mutual pairwise inclusion by contactomorphism.

**Proof:** Let \( \delta \) be the scale of the resolution along \( D_0^{n-1} \), as in of Proposition 6.3. \( \tilde{\Lambda} \) is a product neighborhood of the curve \( \psi_\delta \), and smaller scale resolution can be realized by a compactly supported isotopy. Therefore \( \tilde{\Lambda} \) contains a subset which is an arbitrarily wide product neighborhood of the curve \( \psi \). Since a loose chart is an arbitrarily wide product neighborhood of a stabilization (Proposition 2.3), the proposition follows from the observation that \( \psi \) is equivalent to a stabilization. \( \square \)

### 7. Completing the Main Proof

From now on, we will be interested in wrinkled Legendrians which are extended from an inside-out wrinkle. That is, we consider singular Legendrians \( L \) with only Legendrian wrinkle singularities, so that on a fixed set \( V \), \( V \cap L \) is an inside-out wrinkle; and outside \( V \), \( L \) is a wrinkled Legendrian (meaning the Legendrian wrinkles outside \( V \) are required to be contained in Darboux charts so that \( L \) is graphical outside a compact subset). We call such an object a *prepared wrinkled Legendrian*.

Notice that the definition of a twist marking is equally valid for prepared wrinkled Legendrians: a twist marking is required to have one boundary component on each Legendrian wrinkle, including the inside out wrinkle. Given a twist marking on a prepared Legendrian wrinkle, we may associate to it an element of \( \pi_{n-1} L \) as follows. A twist marking is an embedded \( S^{n-1} \) with a number of punctures. Besides the inside out wrinkle, each boundary component is contained in a Darboux neighborhood \( U_j \) (as in Definition 5.1). Because \( U_j \cap L \) is contractable, we can cap off this boundary component in a unique canonical way by choosing a cap in \( U_j \cap L \). After repeating this process we are left with an \((n-1)\)-disk whose boundary is contained in \( V \). This gives a well defined element of \( \pi_{n-1}(L, V) \) which is isomorphic to \( \pi_{n-1} L \) since \( V \) is contractable.

Given a prepared wrinkled Legendrian, we say a twist marking is *nulhomotopic* if the element described above is \( 0 \in \pi_{n-1} L \). Note that Proposition 6.3 only relies on the local model near the Legendrian wrinkle singularities, thus this proposition continues to hold for prepared wrinkled Legendrians.
Figure 8. Building a Legendrian twist inside $L$. Pictured here is the case $n = 3$.

**Proposition 7.1.** Let $L$ be a prepared wrinkled Legendrian with a nulhomotopic twist marking $\Phi$. Then the smooth Legendrian obtained from resolving $L$ along $\Phi$ is canonically formally isotopic to the wrinkled Legendrian obtained by resolving only the inside-out wrinkle along $D_0^{n-1}$.

**Proof:** Let $\tilde{L}$ be the smooth Legendrian resolved along $\Phi$, and let $L_0$ be the wrinkled Legendrian obtained from resolving the inside-out wrinkle along $D_0^{n-1}$. Clearly they are equal outside of $\bigcup U_j \cup D_0^{n-1} \cup \Phi$. By our definition of how a wrinkled Legendrian is interpreted as a formal Legendrian, we simply ignore the Legendrian wrinkles and treat the chart $U_j$ as if it were graphical. Therefore, compared to $\tilde{L}$, $L_0$ lacks a Legendrian twist along $\Phi$, and also along disks in each $U_j$. Instead it has an additional Legendrian twist along $D_0^{n-1}$. But these two sets can be homotoped from one to the other, using the assumption that $\Phi$ is nulhomotopic. □

The next proposition is a topological observation without much depth. However, it is the only place the assumption $n > 1$ is essentially used. It is also the only step in the proof that does not obviously extend to families $L_t$ with $t \in D^m$ for $m > 1$.

**Proposition 7.2.** Let $n > 1$. Suppose $L_t : L \to (Y, \xi)$ is an isotopy of prepared wrinkled Legendrian. Then there is a nulhomotopic twist marking $\Phi_t \subseteq L_t$.

**Proof:** For each $t$ and $j$, we may choose $D_j^{n-1} \subseteq U_j$ so that it approaches the Legendrian wrinkle as specified in Definition 6.1.

For all $j$ pick points $p_j^t \in D_j^{n-1}$, including a point $p_0^t \in D_0^{n-1}$ for the inside-out wrinkle. Because each wrinkle $S_j^{n-2}$ is canonically cooriented, this induces a coorientation on $D_j^{n-1}$. First we describe $\Phi_0$. If there are no Legendrian wrinkles besides the inside-out wrinkle, we simply let $\Phi_0 = D_0^{n-1}$. Otherwise, for each $j$, we
find a curve $\alpha_j$, connecting $p_0$ to $p^0_j$. We require that the $\alpha_j$ are mutually disjoint, and do not intersect any $D^{n-1}_j$ on their interior. Furthermore we ask that $\alpha_j$ is transverse to $D^{n-1}_j$ at $p^0_j$, and the outward tangent to $\alpha_j$ matches the coorientation on $D^{n-1}_j$. Let $S$ be the boundary of a small neighborhood of $\alpha = \bigcup_j \alpha_j$. For any $j$, $S \cap D^{n-1}_j$ is a small $(n-2)$-sphere which bounds a small disk in both $S$ and $D^{n-1}_j$. Discard these disks, and smooth corners to get a connected smooth manifold (we also do this for $D^{n-1}_0$). After doing this for all $j$ we obtain a manifold $\Phi_0$, satisfying all the conditions in Definition 6.1.

We now construct $\Phi_t$. An isotopy of wrinkled Legendrian embeddings has embryo singularities at points isolated in both space and time. On any subinterval of time not containing an embryo, the isotopy is induced by an ambient contact isotopy of $(Y, \xi)$. On such intervals we can simply let the ambient isotopy act on $\alpha$, which naturally gives us an isotopy of twist markings. Thus it suffices to describe $\Phi_t$ in a small time interval around an embryo singularity at time $t_0$.

We first consider an embryo singularity where a wrinkle $S^{n-2}_j$ disappears in forward time. At the embryo $T\Phi_{t_0} = \text{Im}(dL_{t_0})$, since this equation is satisfied for all points on $\partial\Phi_t$ when $t < t_0$. When $t > t_0$, $\Phi_t$ has one less puncture, and has a long “tentacle” with no boundary components. This can be retracted inside a neighborhood of $p_0$, and the isotopy can be continued. For wrinkle creation, notice this process can be reversed. Immediately before an embryo occurs we can extend a tentacle out from $p_0$ to contain it. Furthermore while keeping everything embedded, we can do this so that $T\Phi_{t_0}$ is tangent to $\text{Im}(dL_{t_0})$, with given orientation.

We now complete the proof of Theorem 1.2. We start by proving the existence portion.

Proposition 7.3. Let $n > 1$, and suppose $(f,F_s)$ is a formal Legendrian knot in $(Y,\xi)$. Then there is a Legendrian knot which is formally isotopic to $(f,F_s)$.

For any Legendrian $L$, $s_{S^1 \times D^{n-1}}(L)$ is a loose knot in the same formal isotopy class as $L$, so this proposition implies the existence theorem for loose Legendrian knots. This proposition is essentially proved in [12], let us first outline their proof here. Since immersed Legendrians satisfy an h-principle, we focus on the set of formal Legendrian isotopy classes in a fixed regular Legendrian homotopy class. A simple calculation shows this set of formal isotopy classes admits a transitive $\mathbb{Z}$ action, and we then show that $M$-stabilization generates this action by $\chi(M)$. Thus given any formal Legendrian isotopy class we can first find a Legendrian immersion $L$ in the correct regular homotopy class, which will generically be embedded. Then we just pick an $M$ so that $s_M(L)$ is in the correct formal isotopy class; notice we can realize any integer by $\chi(M)$ since $n > 1$. Much of this proof is explained in the Appendix [A] though there are a number of gaps. To the author’s knowledge a complete proof does not exist in the literature, though it has been a “known theorem” since [12].
Here we give a different proof of the statement using our h-principle method. It is a distinct proof in that it does not require any knowledge about the set of all formal isotopy classes. In particular, the case for general $Y$ and $L$ is no more difficult that the case of spheres in $\mathbb{R}^{2n+1}_{std}$, the above proof is more difficult to extend to cases where $\pi_1 Y \neq 0$ or when $L$ is not nullhomologous.

**Proof:** Let $L_0 \subseteq (Y, \xi)$ be a formal Legendrian knot. By Theorem 5.2 there is a wrinkled Legendrian $L_1$ which is formally isotopic to $L_0$. Choose a small neighborhood of $L_1$ disjoint from all Legendrian wrinkles, and let $L_2$ be the $S^1 \times D^{n-1}$-stabilization of $L_1$. Then $L_2$ is formally Legendrian isotopic to $L_1$ by Proposition 2.6. Proposition 2.5 implies $L_2$ is loose, thus Proposition 6.3 implies there is a Darboux chart $V$, so that $(V, V \cap L_2)$ is contactomorphic to the standard resolution of an inside-out wrinkle.

In $L_2$, replace $V \cap L_2$ with an unresolved inside-out wrinkle; this defines a prepared wrinkled Legendrian, $L_3$. Proposition 7.2 implies there is a nulhomotopic twist marking $\Phi$ on $L_3$. By resolving $L_3$ along $\Phi$ as in Proposition 6.3 we get a smooth Legendrian $\tilde{L}$, and Proposition 7.1 says that $\tilde{L}$ is formal Legendrian isotopic to $L_2$. □

It remains to prove that any two loose Legendrians which are formally isotopic are Legendrian isotopic. The proof is nearly identical to the proof of their existence. We will in fact prove a slightly stronger statement:

**Theorem 7.4.** Let $n > 1$. Suppose $L_0$, $L_1 \subseteq (Y, \xi)$ are two loose Legendrian knots, with a formal Legendrian isotopy between them. Then they are Legendrian isotopic, and further the Legendrian isotopy can be chosen to be formally isotopic (rel endpoints) to the given formal isotopy.

**Proof:** Suppose $L_0$, $L_1$ are loose Legendrian knots with a formal isotopy $L_t$ between them. Let $B$ be a Darboux ball so that $B \cap L_0$ is a loose chart. We then can pick an ambient smooth isotopy $\zeta_t$ so that $B \cap \zeta_t^1(L_t) = B \cap L_0$, which is the identity on $\{t = 0\} \cup \{s = 0\}$ and $\zeta_t$ is a contact isotopy. Though $\zeta_t$ cannot be made into a contact isotopy, $\zeta_t^1(L_t)$ is a formal Legendrian isotopy with bundle homtoopy $d\zeta_t^1 \circ F_t$, where $F_t : TL \to TY$ is the bundle homtoopy for $L_t$. Thus $\zeta_t^1(L_t)$ is a formal isotopy between $L_0$ and $L_1$, we relabel it $L_t$.

Now we apply Theorem 5.2 to find a wrinkled Legendrian isotopy between $L_0$ and $L_1$, which we denote $L_t'$. We apply the theorem as an extension from the closed set $\bar{B}$, so $L_t'$ retains the property that $B \cap L_t'$ is a fixed loose chart. By Proposition 6.4 we can find a smaller ball $\tilde{B} \subseteq B$ so that $\tilde{B} \cap L_t$ is isotopic to the standard resolution of an inside-out wrinkle. Let $\Lambda_t$ be the prepared wrinkled Legendrian isotopy which is equal to $L_t$ outside $\tilde{B}$, and $\tilde{B} \cap \Lambda_t$ is that inside-out wrinkle, unresolved. We then apply Proposition 7.2 to get a path of nulhomotopic twist markings $\Phi_t \subseteq \Lambda_t$. Since $\Lambda_0$ is smooth outside $\tilde{B}$, $\Phi_0$ is a disk with boundary in $\tilde{B}$; since $\Phi_0$ is nulhomotopic we can assume $\Phi_0 \subseteq B$, and similarly for $\Phi_t$ in $\Lambda_t$. Resolve the prepared wrinkled isotopy $\Lambda_t$ along $\Phi_t$ using Proposition 6.3 this defines a genuine Legendrian isotopy $\tilde{\Lambda}_t$. By definition $\tilde{\Lambda}_0 = L_0$ and $\tilde{\Lambda}_1 = L_1$. 

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Furthermore, Proposition 7.1 implies the path $\tilde{A}_t$ is formally Legendrian isotopic to $L_t$.

8. Conclusion

We take some time to discuss how Theorem 1.2 relates to other results in the field. The term “loose Legendrian knot” comes from 3-dimensional contact topology [11], there it means a Legendrian knot whose complement is overtwisted. Our concept of loose knots is significantly different: loose knots exist in high dimensional Darboux charts. In both cases looseness is a hypothesis about the global contact topology of the knot complement which allows us to apply h-principle results to Legendrians. Unlike the overtwisted complement case, the geometric model defined here is not contained in any compact region of the complement; instead it is required to intersect the (standard) tubular neighborhood of the knot in a prescribed way.

Presently, overtwistedness in high dimensions is not understood, if such a concept exists at all. To justify our use of terminology, we argue that generally there can only be one flexible class of Legendrian knots. Let $(Y, \xi)$ be a high dimensional contact manifold which is “overtwisted”. Let $L$ be a loose Legendrian knot, and suppose $L_{OT}$ is in the same formal isotopy class, and has overtwisted complement. Since an $M$-stabilization takes place in a small neighborhood of a point, $L' = s_{S^1 \times D^{n-1}}(L_{OT})$ is a loose knot, whose complement remains overtwisted. By Theorem 1.2 $L$ is Legendrian isotopic to $L'$. If knots with overtwisted complement have flexibility properties (as in [13] or [7] in the $n = 1$ case), $L_{OT}$ should be ambient isotopic (or at least contactomorphic) to $L'$. Thus in a hypothetical high dimensional overtwisted manifold, we expect a Legendrian knot is loose exactly when it has overtwisted complement.

We can extend our result to a h-principle for loose Legendrian links, where the definition requires each component of the Legendrian to have a loose chart, mutually disjoint and also disjoint from the other components of the Legendrian. Note that a union of loose knots is not necessarily a loose link, a fact also true of loose links in overtwisted 3-manifolds.

In [9], Legendrian contact homology is defined (for a certain class of $(Y, \xi)$), which is a pseudo-holomorphic curve invariant of Legendrian knots. There it is shown that $LCH(s_{D^n}(L)) = 0$ for any knot $L$. Theorem 1.2 implies every loose knot is the $D^n$-stabilization of another knot, thus every loose Legendrian has trivial $LCH$. Suppose $X$ is any exact symplectic filling of $Y$, and $\Gamma$ is an exact Lagrangian with Legendrian boundary, $L$. Then $\Gamma$ induces an augmentation of $LCH(L)$ [8]. The trivial algebra admits no augmentation, thus it follows that loose knots are not fillable by exact Lagrangians. The converse is false: a non-loose Legendrian $T^2$ is described in [10] which admits no exact Lagrangian filling (though it admits a totally real filling).
If a Weinstein manifold admits a handle decomposition so that the top dimensional handles are attached along loose knots, it is expected to inherit flexible properties. This topic is explored in much more depth in [5].

For tight contact structures on 3-dimensional manifolds, one might hope to find a similar flexible class of Legendrians. However, it is known that no suitable one exists: the standard unknot is the unique Legendrian knot in its formal isotopy class [13]. It is further shown there that the only topologically local structure of Legendrian curves is the number of stabilizations, and a result from [17] states that for any fixed \( k \), there are formally isotopic knots which are distinct even after \( k \) stabilizations of any type. Together these results imply there is no flexible class of Legendrian 1-knots, which can be defined independent of the global topology of the knot.

Theorem 1.2 is for parametrized Legendrians, so it implies that loose knots have maximal \( \pi_0 \text{Diff}(L) \) symmetry: any diffeomorphism of \( L \) fixing the classical invariants can be realized by an ambient contact isotopy. Of course any exotic diffeomorphism homotopic to the identity is an example of such a symmetry. In contrast, there exists (in particular) an exotic diffeomorphism of \( S^8 \) which cannot be realized by any contact isotopy of the standard Legendrian unknot [1].

Given a Legendrian knot \( L \subseteq (Y, \xi) \), we define the **twist capacity** of \( L \) to be the non-negative real number \( c(L) = \sup \{ \frac{\rho^2}{a^2} ; \text{there exists a Legendrian twist } (U, U \cap L) \subseteq (Y, L) \text{ with parameters } a, \rho \} \). The paragraph preceding Proposition 2.3 explains why \( c(L) \) is always strictly positive; the proposition itself states that \( c(L) = \infty \) whenever \( c(L) > \frac{1}{2} \) (which occurs exactly when \( L \) is loose). For spheres in \( \mathbb{R}^{2n+1}_{std} \) notice that \( c(L_1 \# L_2) \geq \max(c(L_1), c(L_2)) \), where \( L_1 \# L_2 \) denotes the connect sum of the knots \( L_1 \) and \( L_2 \). In particular, the standard Legendrian unknot has smallest capacity among all Legendrian spheres.

Looking to results unique to high dimensional contact topology, we see that a “sufficiently thick” condition is often crucial. In [22] it is shown that any contact manifold containing a sufficiently thick Weinstein neighborhood of an overtwisted contact submanifold is not fillable by a (semi-positive) symplectic manifold. Note that all contact manifolds contain overtwisted submanifolds. In [14] it is shown that whenever \( r < R < 1 \), there is a contact isotopy of \((\mathbb{R}^{2n} \times S^1, \ker(d\theta - \sum_i y_i dx_i))\) which squeezes \( B^{2n}_R \times S^1 \) inside \( B^{2n}_r \times S^1 \). However, this is shown to be false when \( r < 1 < R \). Though these results and Theorem 1.2 all seem intuitively similar, no concrete connections between these phenomena are presently understood.

**Appendix A. Formal Legendrian Isotopy Classes in \( \mathbb{R}^{2n+1}_{std} \)**

In order for the main result of this paper to useful in practice, we would like to have an explicit way to tell when two knots are formally isotopic. This is purely an issue about bundle theory and algebraic topology. The calculations are not particularly deep, but they are somewhat involved. First we define two invariants of formal Legendrian knots. Some of the details in calculation are left to the reader,
they can also be found in [9].

**Definition A.1.** Let $L$ be a formal Legendrian knot in $(Y,\xi)$. $F_1$ is a bundle map $TL \to \xi|_L$, so every fiber has Lagrangian image. The homotopy class of this map in the space of Lagrangian bundle monomorphisms is called the **rotation class** of $L$. We denote this class $r(L)$.

Immersed Legendrian knots satisfy an h-principle [19], and the rotation class classifies them up to regular Legendrian homotopy. If we have two Legendrian knots which are smoothly homotopic, we can compare their rotation classes as follows. A formal Legendrian defines an isomorphism $\xi|_L \cong TL \otimes \mathbb{C}$, therefore two formal Legendrians together define a difference element in $\text{Aut}_\mathbb{C}(TL \otimes \mathbb{C})$, also known as the gauge group of $\xi|_L$. Two Legendrians have the same rotation class if and only if this difference element is in the component of the identity. If $\xi|_L$ is trivial (which is always the case if $\xi|_Y$ is trivial) then $\text{Aut}_\mathbb{C}(\xi|_L) \cong \text{Map}(L,U_n)$, thus the difference class $r(L_0) - r(L_1)$ is an element of $K^1(L)$ in this case.

**Definition A.2.** Suppose $n$ is odd, and let $L$ be a formal Legendrian knot in $(Y,\xi)$. Assume $L$ is orientable and nullhomologous. Extend $F_s$ to a path $\tilde{F}_s$ in $\text{Aut}_\mathbb{R}(TY|_L)$. Let $R$ be a vector field in $TY|_L$, positively transverse to $\xi$. Then $\tilde{F}_s^{-1}(R)$ is nowhere tangent to $L$, and the linking number of the knot with the vector field does not depend on the choice of lifting $\tilde{F}_s$. This integer is called the **Thurston-Bennequin number** of $L$, denoted $tb(L)$.

*Remark.* When $n$ is even, the definition makes sense but the invariant is uninteresting. In the example $\mathbb{R}^{2n+1}_{\text{std}}$, we can equivalently consider the signed count of self intersections in the Lagrangian projection (regardless of dimension). If $n$ is odd, the intersection product is skew, and the order of the inputs are given by height. For even $n$ the intersection product is commutative, so all the data necessary to calculate $tb(L)$ is contained in the Lagrangian projection. Together with the Lagrangian neighborhood theorem, it follows that $tb(L) = \frac{1}{2}(-1)^{n/2+1}\chi(L)$ in this case.

**Proposition A.3.** Let $L$ be a Legendrian knot, and $M \subseteq D^n$. Then $r(s_M(L)) = r(L)$ always. When $n$ is odd, $tb(s_M(L)) = tb(L) - 2\chi(M)$.

*Proof:* $L$ and $s_M(L)$ are Legendrian regular homotopic by the homotopy $\{z = t \cdot h(x)\}$ so the statement about rotation class is clear (see Definition 2.4 for notation). We calculate $tb(L)$ by taking the signed count of self intersections in the Lagrangian projection. In the course of the homotopy $L$ will intersects itself once for each Morse critical point of $h$. This corresponds to a a sign change for the associated intersection in the Lagrangian projection, so $tb$ changes by $\pm 2$ for each Morse critical point. By explicit calculation we see that the sign corresponds to the parity of the Morse index, so the total change is $2\chi(M)$.

\[\square\]
We now state a classification of formal Legendrian isotopy classes, assuming our ambient manifold is $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$. This tells us that all embeddings of $L$ are smoothly isotopic when $n > 1$ [20]. Similar calculations can be done for any $(Y, \xi)$, but there are smooth obstructions and the bundle theory becomes more difficult.

**Theorem A.4.** We describe formal Legendrian knots up to formal isotopy in $\mathbb{R}^{2n+1}_{\text{std}}$.

(a) Suppose $n$ is odd. If two formal Legendrian knots have the same Thurston-Bennequin number and rotation class, then they are formally Legendrian isotopic.

(b) If two formal Legendrian surfaces in $\mathbb{R}^5_{\text{std}}$ have the same rotation class, they are formally Legendrian isotopic.

(c) Suppose $n > 2$ is even. Then for each rotation class there are at most two formal Legendrian isotopy classes. If $L$ is simply connected, there are exactly two.

**Remarks.** Every set of invariants is realized by a formal Legendrian knot, with the additional note in case (a) that the parity of $tb(L)$ is determined by $r(L)$. However note that Proposition 7.3 is false if $n = 1$: there is no Legendrian realizing a formal Legendrian unknot with $tb = 0$. For $n > 3$ the parity of $tb(L)$ is determined only by the topology of $L$, for example $tb(S^n)$ is odd for any Legendrian sphere in $\mathbb{R}^{2n+1}_{\text{std}}$. To show this, first take the Lagrangian projection of $L$, which is an exact Lagrangian immersion in $\mathbb{R}^{2n}_{\text{std}}$. Notice the parity of $tb(L)$ is equal to the mod 2 count of self interesections of this Lagrangian immersion, in fact this is an invariant of smooth immersions in $\mathbb{R}^{2n}$ up to regular homotopy. Both smooth and Lagrangian immersions satisfy h-principles [19], thus the existence of Lagrangian immersions of a given smooth regular homotopy class is governed by the inclusion map $\pi_n U_n \to \pi_n V_{2n,n}$. For $n$ odd this is a map $\mathbb{Z} \to \mathbb{Z}_2$, and (a stable shift of) Lemma A.6 implies this is the zero map except when $n = 1, 3$.

It is unknown to the author if there exists a calculable invariant in $\mathbb{Z}_2$ which distinguishes the formal isotopy classes in case (c). Below it is defined as an invariant associated to a smooth isotopy between two Legendrian knots, which is why the $\pi_1 L = 0$ assumption is needed. The invariant in question should be a “Thurston-Bennequin-Kervaire semicharacteristic”, see [2].

**Proof:** We assume some basic facts about frame bundles, see [3, 23]. Given two Legendrian knots construct a smooth isotopy $L_t$ between them, this defines a path $\beta_t : L \to V_{2n+1,n}$ so that $\beta_0$ is a constant map (here $V_{2n+1,n}$ is the Stiefel manifold of $n$-frames in $\mathbb{R}^{2n+1}$). $L$ need not admit a global parallelization, since here $\beta_t$ compares $dL_t$ to $dL_0$ at each point of $L$, and this difference does not depend on a choice of framing at that point. Said differently, maps $L \to Gr_n(\mathbb{R}^{2n+1}_{\text{std}})$ lifting the isotopy $L_t$ can be identified with $\text{Map}(L, V_{2n+1,n})$ by choosing a connection on the tautological bundle over $Gr_n(\mathbb{R}^{2n+1}_{\text{std}})$. Inside $V_{2n+1,n}$, identify $U_n$ as the subset of Legendrian frames. (Though “which frames are Legendrian” depends on the point in $\mathbb{R}^{2n+1}$, these inclusions are all homotopy equivalent to the inclusion $U_n \subseteq O_{2n} \subseteq O_{2n+1} \to V_{2n+1,n}$.) $\beta_1$ has image inside of $U_n$ since $L_1$ is Legendrian, and so $\beta_t$ defines an element $\beta \in \pi_1(\text{Map}(L, V_{2n+1,n}), \text{Map}(L, U_n))$. Notice that
Map$(L, V_{2n+1,n})$ is connected since $V_{2n+1,n}$ is $n$-connected.

Our smooth isotopy can be made into a formal Legendrian isotopy exactly when $\beta = 0$. Conversely, given any $\beta \in \pi_1(\text{Map}(L, V_{2n+1,n}), \text{Map}(L, U_n))$ and a Legendrian knot $L_0$, we can define a formal Legendrian knot $(f, F_s) = (L_0, \beta_s)$. If $L_1$ is a Legendrian realizing this formal Legendrian (which exists by Proposition 7.3), then the obstruction associated to the smooth isotopy between $L_0$ and $L_1$ is $\beta$.

In the long exact sequence for the pair, notice $\partial_s \beta = r(L_0) - r(L_1) \in \pi_0(\text{Map}(L, U_n))$. Thus under the assumption $r(L_0) = r(L_1)$ we can lift $\beta$ to $\beta \in \pi_1(\text{Map}(L, V_{2n+1,n}))$. We pause to prove some lemmas concerning the homotopy groups of frame bundles.

**Lemma A.5.** Consider the fibration $O_{n+1} \rightarrow O_{2n+1} \rightarrow V_{2n+1,n}$. In the homotopy long exact sequence, the map $\pi_{n+1}V_{2n+1,n} \rightarrow \pi_n O_{n+1}$ is injective, except for $n = 2, 6$. For these two values, $\pi_n O_{n+1}$ is trivial.

**Proof:** First, consider the case where $n$ is odd. The kernel of our map is the image of the group $\pi_{n+1}O_{2n+1}$. By Bott periodicity, this group is finite. But $\pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}$, so the image must be trivial.

Next, consider the case where $n$ is even, and not equal to 2 or 6. Consider the map $\pi_n O_{n+1} \rightarrow \pi_n O_{2n+1}$. The first group classifies $(n + 1)$-vector bundles on $S^{n+1}$, whereas the second group classifies stable bundles. Since $TS^{n+1}$ is non-trivial, but stably trivial [3], we know this map must have non-zero kernel. So $\pi_{n+1}V_{2n+1,n} \rightarrow \pi_n O_{n+1}$ has non-zero image. Since $\pi_{n+1} V_{2n+1,n} \cong \mathbb{Z}_2$, this implies the map is injective.

**Lemma A.6.** For all $n > 2$, $\pi_{n+1} U_n \rightarrow \pi_{n+1} V_{2n+1,n}$ is the zero map. For $n = 2$, it is a surjection.

**Proof:** Let $n \neq 2, 6$. Notice that the inclusion $U_n \subseteq V_{2n+1,n}$ factors through $U_n \subseteq O_{2n+1} \rightarrow V_{2n+1,n}$. By the previous lemma, the second map is trivial on $\pi_{n+1}$.

Case $n = 6$: Consider the map $\pi_{n+1} U_n \rightarrow \pi_{n+1} O_{2n+1}$. This is in the stable range, so we can look at the exact sequence

$$\pi_{n+1} U \rightarrow \pi_{n+1} O \rightarrow \pi_{n+1} (O/U) \rightarrow \pi_n U.$$

By Bott periodicity, $\pi_n U \cong 0$, and $\pi_{n+1} (O/U) \cong \pi_{n+1} (\Omega O) \cong \mathbb{Z}_2$. It follows that the map $\pi_{n+1} U_n \rightarrow \pi_{n+1} O_{2n+1}$ is multiplication by 2, as a map $\mathbb{Z} \rightarrow \mathbb{Z}$. Therefore, the map $\pi_{n+1} U_n \rightarrow \pi_{n+1} V_{2n+1,n} \cong \mathbb{Z}_2$ is zero.

Case $n = 2$: Since $\pi_n O_{n+1} \cong 0$, we know $\pi_{n+1} O_{2n+1}$ surjects onto $\pi_{n+1} V_{2n+1,n}$. This, together with the fact that $\pi_{n+1} U_n \rightarrow \pi_{n+1} O_{2n+1}$ is an isomorphism, implies the result. □
Lemma A.7. Let $n$ be odd. From the fibrations $O_{n+1} \to O_{2n+1} \to V_{2n+1,n}$ and $O_n \to O_{n+1} \to S^n$, form the composition map $\tilde{t}b : \pi_{n+1}V_{2n+1,n} \to \pi_nO_{n+1} \to \pi_nS^n$. Then $\tilde{t}b$ is an injection, in fact, it is the map $\mathbb{Z} \mapsto 2\mathbb{Z}$.

Proof: We know from Lemma A.5 that the first map is an injection, so the lemma is equivalent to “Is $\text{Im}(\pi_{n+1}V_{2n+1,n}) \cap \ker(\pi_nS^n)$ trivial in $\pi_nO_{n+1}$?” By the exact sequences, this is equivalent to the intersection $\ker(\pi_nO_{2n+1}) \cap \text{Im}(\pi_nO_n) \subseteq \pi_nO_{n+1}$. This statement is then equivalent to “Suppose $\nu$ is an $(n+1)$-plane bundle on $S^{n+1}$ which is both stably trivial and of zero euler class. Is $\nu$ trivial?” But $\nu$ must be trivial; the tangent bundle of the sphere generates the group of stably trivial vector bundles over $S^{n+1}$, and it has nonzero euler class. The second statement follows since the euler class of this generator is 2.

Returning to the proof of the theorem, recall our isotopy is unobstructed if $\tilde{\beta} \in \pi_1\text{Map}(L,V_{2n+1,n})$ is in the image of $\pi_1\text{Map}(L,U_n)$. Take any degree one map $L \to S^n$. Since $V_{2n+1,n}$ is $n$-connected this map induces an isomorphism $\pi_1\text{Map}(L,V_{2n+1,n}) \cong \pi_{n+1}V_{2n+1,n}$, identifying the image of $\pi_1\text{Map}(L,U_n)$ with that of $\pi_{n+1}U_n$.

For part (b), $n = 2$: Lemma A.6 implies that that $\tilde{\beta}$ is in the image of $\pi_{n+1}U_n$, thus $\beta = 0$.

In part (a), $n$ is odd. We claim $\tilde{t}b(\beta) = \tilde{t}b(L_0) - \tilde{t}b(L_1)$. Since Lemma A.7 says $\tilde{t}b : \pi_{n+1}V_{2n+1,n} \to \pi_nS^n$ is an injection and $\tilde{t}b(L_0) = \tilde{t}b(L_1)$ by hypothesis, this implies $\beta = 0$. Consider the geometric meaning of the maps in Lemma A.7. The first map to $\pi_nO_{n+1}$ can be interpreted as a difference class of the Legendrian framings of the normal bundle induced by the isotopy. The second map, induced by $O_{n+1} \to S^n$, is simply “pick one vector in the frame”, here we think of it as choosing the Reeb vector field. Thus $\tilde{t}b(\beta)$ represents the difference class of the Reeb framings, which equals $\tilde{t}b(L_0) - \tilde{t}b(L_1)$.

For part (c), $n > 2$ is even. $\tilde{\beta} \in \pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}_2$, which implies there are at most two formal Legendrian isotopy classes for the given rotation class. However $\beta$ is an invariant of a smooth isotopy: one can imagine a isotopy from a Legendrian to itself so that $\beta \neq 0$. If such a case exists there will only be one formal isotopy class for the given rotation class. Under the assumption $\pi_1L = 0$, the space of smooth embeddings $L \hookrightarrow \mathbb{R}^{2n+1}_{std}$ is simply connected [6] and thus this cannot occur. □

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