ANALYTICAL MODELING OF LAMINATED COMPOSITE PLATES USING KIRCHHOFF CIRCUIT AND WAVE DIGITAL FILTERS

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Abstract. A physical-based numerical algorithm using Kirchhoff circuit is detailed for modelling the free vibration of moderate thick symmetrically laminated plates based on the first order shear deformation theory (FSDT). With the help of multidimensional passivity of analog circuit and nonlinear optimization solvers, the philosophy gives rise to a nonlinear programming (NLP) model that can apply further to explore stability characteristics and optimum performance of the resultant multidimensional wave digital filtering network representing the FSDT plate. Various optimization methods exploiting gradient-based and direct search methods are adopted with efficient broad search power to tackle the NLP model. As a result, the necessary Courant-Friedrichs-Levy stability criterion can be fully satisfied at all time with least restriction on the spatially discretized geometry of the scattering problem. With full stability guaranteed, the waveform is analyzed by the power cepstrum for spectra peaks detection, which has led to more accurate estimate of various vibration effects in predicting nature frequencies with different fiber orientations, stacking sequences, stiffness ratios and boundary conditions. These results have shown in excellent agreement with early published works based on the finite element solutions of the high-order shear deformation theory and other well known numerical techniques.

1. Introduction. Laminated composite plates are finding increasingly use in military, aerospace structures, marine vehicles, automotive parts, recreation and building industries, and many other applications. As is consisting of many parallel layers with fibre-reinforced material, the specific geometry can generally achieve more desirable structural properties in mechanics and better performance than conventional materials. Due to their complex mechanical behaviour with the existence of different kinds of couplings between extension, shear, bending and twisting, the laminate is often manifestly represented by sets of linear and/or nonlinear partial differential equations (PDEs) with properly imposed initial and boundary conditions. In most cases of vibration-influenced design involving complex patterns and realistic source configurations, these differential equations are far too complicated to be solved by an explicit analytic formula and thus numerical approaches are sought.

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Many numerical methods have been developed in recent years attempted to deal with the static and dynamic analyses of isotropic and laminated composite plates, e.g., finite difference time domain (FDTD) [19, 37], finite element (FE) [34, 35], differential quadrature (DQ) [26, 27], Rayleigh-Ritz [23], Galerkin [1], pseudo-spectral [9], Runge-Kutta [28], radial basis functions (RBFs) [37, 9], mesh-free [5] etc. These methods, however, have respective drawbacks in that they may be too complex mathematically for routine engineering analysis, be relatively easy to be used but only limited to some special cases, need initial trial functions or require large amounts of computational effort and consequently high cost.

As the composite material usage is experiencing continuous growth, so the need to perform accurate yet efficient analysis will remain high. Therefore, the future calls not only for increased understanding and more accurate modelling but also for incorporating multidisciplinary subjects with fresh ideas and approaches to better predict and compute the dynamic behaviours of the laminate. Moreover, it is equally essential to investigate how the resultant numerical models can provide an efficient, effective and most importantly economic process to speed-up the solution of the composite plate system when is compared with conventional approaches such as the commonly used finite elements.

In this work, we introduce an alternative numerical approach named multidimensional wave digital filtering (MDWDF) method [11, 12, 13], which involves multiple principles of modelling and simulation, lumped circuit theory, MD digital signal processing and mathematical optimization, to achieve solutions of PDE systems efficiently and effectively. A schematic flow diagram towards modelling and implementing a general MDWDF network can be viewed in Fig. 1, which consists of blocks of systems input, multidimensional digital signal processing (MDDSP), and network output. In particular, by adopting a generic tool (i.e. Kirchhoff paradigm) the tasks of MDDSP start with finding a suitable MD lumped electrical network based on the Kirchhoff’s loop rule and MD inductors, which are served as a passage from a well-conditioned mechanical system to a passive electronic equivalent circuit.

The synthesized schematic design with its proper analogy using the standard circuit elements (e.g. resistor, inductor, capacitor, gyrator and ideal transformer, etc.) is a multidimensional Kirchhoff circuit (MDKC) whose behaviors are entirely equivalent to those of the mechanical governing equations described by a set of linear/or nonlinear time-dependent PDEs. From this lumped network, the continuous system can then be equivalently treated by a discrete approximation using a generalized trapezoidal rule, while carrying out the numerical computation by means of passive MD wave digital filters.

Due to its physical-based nature, since inception, the approach can be generally applied to causal and passive physical systems with finite propagation speed, i.e. PDE system of hyperbolic type. As the original system is represented by the MDKC with the validity of the Kirchhoff rules (i.e. MD passivity) that guarantee conservation of power, a vector of Lyapunov functions for which the function itself is an error bound, is formed that guarantees the numerical stability of the PDE system in accordance with the Lyapunov theory [11]. Built on the same class of finite difference schemes (one of classes of numerical integration method), the analysis of numerical stability relies on the time step restriction of the discrete network for which the passivity requirements of the whole analog electrical circuit in terms of the MDKC is satisfied. To achieve the necessary criterion, a classic well-defined
Courant-Friedrichs-Levy (CFL) bound is adopted to guarantee the transformed numerical scheme (i.e. MDWDF network) being fully stable with least restriction on the spatially discretized geometry of the scattering problem. The resultant simply implies that the ratio of time-spatial step is bounded by the phase velocity of a wave [11, 12, 13, 14].

Apart from the type of stability discussed above for the solutions of hyperbolic PDEs, there are various types of stability, which may be concerned with solutions of mixed sets of difference equations and algebraic equations (DAEs) describing dynamical systems [47, 48, 8]. The solution to the DAEs problem can be manipulated to achieve its corresponding variational system with some degrees of transformation from the original problem. As a result, the Lyapunov stability is involved in dealing with such system for which the stability of solutions is near to a point of equilibrium. By looking at the fundamental matrix solution of the variational system, which is of boundedness, one can successfully arrive at the stability guarantee of the dynamic system. Although, two methodologies are different in solving their own problems each involving PDEs or DAEs, the way of achieving and maintaining numerical stability at all time is to limit the solutions within an error bound involving the Lyapunov functions, which are the key factor ultimately contributing to the success of problem solving.

As is possessed of a great variety of outstanding features [11], the technique of MDWDF has proven to be elegant and of high quality in good agreement with other conventional numerical techniques such as finite elements. Some of these features concern not only the conservation of passivity and stability [12, 13, 38, 39], but also a suppression of parasitic oscillations and low round-off noise [15], a great level of computational parallelism [40, 45], and full local inter-connectivity [12, 41]. We note that in a physical system, the passivity is a result of energy conservation, and low round-off noise characteristics together with suppression of parasitic oscillation have essentially led to high accuracy of the algorithm proposed. Furthermore, massive parallelism combined with local inter-connectivity is inherent in the MDWDF models when they are performed with the temporal step updates.

Previous investigation [42, 43, 44] at the free vibration analysis of laminated composite plates with particularly focusing on cross-ply layups have shown demonstrative evidences of computational efficiency and high numerical accuracy. These good performances are directly resulted from an optimum MDWDF network with a specific formulated NLP model being solved by the genetic algorithm (GA) [16]. The NLP model, in essence, arises from the satisfactory MD-passivity of linear and/or nonlinear dynamic elements (e.g. the inductance and capacitance considered in [14, 38, 39]) to the full guarantee of the network optimality. As the network stability is constrained by the necessary CFL condition, numerical instabilities, due to unavoidable operations implied by the need for rounding or truncation and overflow correction, can be fully excluded. As a consequence, the network is convergent at all time leading to uniquely a high degree of robustness and fault tolerance. Meanwhile an insight into the robustness of the network was established together with a qualitative understanding behaviours of the laminar system by justifying numerical accuracy of the network with respect to different scale of CFL numbers.

Standing on our previous successful attempts [42, 43, 44], this study aims to show the robustness and expansion of MDWDF network capacities in analysing more complex mechanical behaviour by involving the bending-twisting anisotropy for free vibration analysis of symmetrically laminated composite FSDT plates on an elastic
foundation. The mechanical anisotropy not only highlights the difference from the previous model with cross-ply layups but also imposes a significant potential on the numerical study of angle-ply material. Due to the increase of network complexity to handle the physical significance of laminate in force and moment resultants, it is important to establish a NLP model with suitable numerical solvers for handling these resultants.

To resolve the NLP model of the newly derived MDWDF network so that the imposed CFL stability criterion can be fully satisfied, we primarily concentrate on evaluating various optimization methods with efficient broad search power that allows algorithmic iterations to violate some or all of the nonlinear inequality constraints during the course of the minimization procedure. The NLP solvers adopted in this study for the performance comparison include the active set algorithm (ASA) \[17\], interior point algorithm (IPA) \[6\], GA \[16\] and pattern search algorithm (PSA) \[22\]. We note that ASA and IPA both pertain to the gradient-based methods, while GA and PSA both are classified into the direct search methods.

Gradient-based methods, which use the first or derivative information, are the preferred methods in some cases when the objective and constraint functions are expensive to evaluate because such methods often require a minimum number of evaluations of these functions. However, usage of the gradient-based methods requires that the objective and constraints have certain smoothness properties. On the other hand, the direct search methods do not use information from the derivatives of optimization function, and can be stochastic in nature, which is proceeded by conducting a series of exploratory moving around the current point that eventually reaches an optimal point. This, however, comes with a price to pay that is encompassed by slow convergence process, but can be more tolerant to the presence of noise in the objective function and constraints. Although these optimizers are popular to a wide range of applications across many industries and functional areas, they are, for the first time, being applied to deal with the NLP model of MDWDF network for numerically modelling and implementation of physical systems.

Comparison study of algorithmic performance in terms of the numerical accuracy, stability and efficiency are made among these optimization techniques to search complex solution spaces within the NLP model so that the MDWDF network with suitable nonlinear solvers embedded can always achieve their best performance in an effective and efficient manner. Having network established with these nonlinear optimizers, the resultant signal of plate vibration is analysed by the power cepstrum defined as “the power spectrum of the logarithm of the power spectrum” \[4\] to detect spectra peaks corresponding to the spacing between peaks/or vibration harmonics i.e. the periodicity. We note that the independent variable of a cepstrum is called quefrency and peaks in the cepstrum are called rahmonics. Since the fundamental frequency is not changing too rapidly, nor too high, the cepstrum analysis can work best and is adopted in this report.

By looking at a specific periodical interval in the quefrequency that corresponds to the highest rahmonic in the cepstrum, one can thus estimate the smallest time period of the waveform with a peak location close/or match to the periodicity in the quefrequency. As a result, an accurate solution of the fundamental frequency can be predicted with various vibration effects for different fibre orientations, stacking sequences, stiffness ratios and boundary conditions. These results, which are then carefully validated by cross referencing, have shown in excellent agreement with
early published works based on the FEM solutions of HSDT and other well known techniques [20, 36, 46].

2. Continuous multidimensional system modeling. Let us consider a rectangular laminated plate being confined in a bounded domain $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$ and symmetrically structured at the mid-plane $\Omega_0$ with $N$ layers perfectly bounded. By approximating time dependence of the in-plane transverse displacement through the thickness of the material in the $z$-direction, the stress equations of motion based on the first order shear deformation theory (FSDT) for free vibration of symmetrically laminated plate can be modelled as a set of PDEs [18, 36]:

$$
\begin{align*}
I_0 \frac{\partial^2 w}{\partial t^2} - \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} &= 0, \\
I_2 \frac{\partial^2 \psi_x}{\partial x^2} - \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x &= 0, \\
I_2 \frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_y &= 0.
\end{align*}
$$

(2.1)

Here $I_i$ are the mass inertia tensor components defined as $I_0 = \rho h, I_2 = \frac{\rho h^3}{12}$ with the mass density $\rho$ [18], $w$ is denoted by the mid-plane displacements on the plane $z = 0$, and $\psi_x$ and $\psi_y$ are the bending rotations of a transverse normal about the $y$– and $x$–axes, respectively. In addition, the shear forces per unit length ($Q_x, Q_y$) and bending moments per unit length ($M_x, M_y, M_{xy}$) are expressed in terms of both the transverse displacement and bending rotations using the plate constitutive equations [18, 36] as

$$
\begin{align*}
\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} &= \kappa_s \bar{A}_Q \begin{bmatrix} \frac{\partial w}{\partial y} + \psi_x \\ \frac{\partial w}{\partial x} + \psi_y \end{bmatrix}, \\
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} &= \bar{D}_M \begin{bmatrix} \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \end{bmatrix}, \\
\bar{A}_Q &= \begin{bmatrix} \bar{A}_{55} & \bar{A}_{45} \\ \bar{A}_{45} & \bar{A}_{44} \end{bmatrix}, \\
\bar{D}_M &= \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{16} \\ \bar{D}_{12} & \bar{D}_{22} & \bar{D}_{26} \\ \bar{D}_{16} & \bar{D}_{26} & \bar{D}_{66} \end{bmatrix}.
\end{align*}
$$

(2.2)

It is of interest to note that the shear forces in Eq. (2.2) adopt a shear correction factor $\kappa_s$ to compensate for the difference between the assumed constant average shear strain and the true parabolic distribution. Furthermore, elements $A_{ij}$ and $D_{ij}$ of matrices $\bar{A}_Q$ and $\bar{D}_M$ represent the transverse shear stiffness and bending stiffness of a composite laminate, respectively. These elements can be defined in terms of the laminate stiffness matrix $[\bar{Q}]_k$ [18] in the $k$th layer of the laminate as:

$$
\begin{align*}
A_{ij} &= \sum_{k=1}^N (Q_{ij})_k (h_k - h_{k-1}), (i, j = 4, 5) \\
D_{ij} &= \frac{1}{3} \sum_{k=1}^N (Q_{ij})_k (h_k^3 - h_{k-1}^3), (i, j = 1, 2, 6).
\end{align*}
$$

(2.4)

(2.5)

In order to arrive at a proper circuit interpretation that can fully describe the continuous-time physical behavior of the plate system, the stress-displacement relations from the CLPT must be incorporated into the stress equations of motion in Eq. (2.1). Combining with the partial derivatives resulted from inverse of the stress-displacement in Eqs. (2.2) and (2.3) with respect to the coordinate time, it simply leads to a system of hyperbolic PDEs:
where the self-explanatory notations are defined as

\[ v = \frac{\partial \psi}{\partial t}, w_x = \frac{\partial \psi_x}{\partial t}, w_y = \frac{\partial \psi_y}{\partial t}, Q_x = |\bar{A}Q|, Z_M = |\bar{D}M|. \]

We note that $|\bar{A}Q|$ and $|\bar{D}M|$ are determinants of matrices $\bar{A}Q$ and $\bar{D}M$ in Eqs. (2.2) and (2.3), respectively, while $\bar{A}_{ij}$ and $\bar{D}_{ij}$ represent cofactors of these adjoint matrices:

\[ \begin{align*}
\bar{A}_{45} &= -\bar{A}_{45}, \quad \bar{A}_{54} = \bar{A}_{55} \\
\bar{D}_{11} &= \bar{D}_{22} \bar{D}_{66} - \bar{D}_{26}^2, \quad \bar{D}_{12} = -(\bar{D}_{12} \bar{D}_{66} - \bar{D}_{16} \bar{D}_{26}), \\
\bar{D}_{16} &= \bar{D}_{12} \bar{D}_{26} - \bar{D}_{16} \bar{D}_{22} \\
\bar{D}_{22} &= \bar{D}_{11} \bar{D}_{66} - \bar{D}_{16}^2, \quad \bar{D}_{26} = -(\bar{D}_{11} \bar{D}_{26} - \bar{D}_{12} \bar{D}_{16}), \\
\bar{D}_{66} &= \bar{D}_{11} \bar{D}_{22} - \bar{D}_{12}^2.
\end{align*} \]

Given the first-order PDEs of Eqs. (2.6) and (2.7), the translation from the plate system to a lumped electrical network starts with adopting the scaled dependent variables $i_k, k = 1, \ldots, 8$ denoted by the graphical current together with positive constants $r_j, j = 1, 2, 3$ for the system variables as:

\[ (v, Q_x, Q_y) \triangleq (r_1 i_1, i_2, i_3), (w_x, w_y, M_x, M_y, M_{xy}) \]

\[ \triangleq (r_2 i_4, r_3 i_5, i_6, i_7, i_8). \]

Following by the principle of circuit theory, the differential equations of motion involved in Eqs. (2.6) and (2.7) can be explicitly expressed to achieve the following network operations responsible for each system variable:

\[ \begin{align*}
D_1(t - x)(i_1 + i_2) + D_1(t + x)(i_1 - i_2) + D_2(t + y)(i_1 - i_3) \\
+ D_2(t - y)(i_1 + i_3) + L_1 D_4(i_1) &= 0 \\
-D_1(t + x)(i_1 - i_2) + D_1(t - x)(i_1 + i_2) - R_G L i_4 \\
+ L_2 L D_4(i_2 - i_3) + L_2 D_4(i_2) &= 0 \\
D_2(t - y)(i_1 + i_3) - D_2(t + y)(i_1 - i_3) + L_3 D_4(i_3) \\
- L_2 L D_4(i_2 - i_3) - R_G L i_5 &= 0
\end{align*} \]
that represents behaviors of the laminated composite

\begin{align}
D_3(t-x)(i_4 + i_6) + D_3(t+x)(i_4 - i_6) + R_{GU} i_2 = 0 \\
D_5(t-y)(i_4 + i_8) + D_5(t+y)(i_4 - i_8) + L_4 D_l(i_4) = 0 \\
D_4(t-y)(i_5 + i_7) + D_4(t+y)(i_5 - i_7) + R_{GL} i_3 = 0 \\
D_6(t-x)(i_5 + i_8) + D_6(t+x)(i_5 - i_8) + L_5 D_l(i_5) = 0 \\
- D_3(t+x)(i_4 - i_6) + D_3(t-x)(i_4 + i_6) + L_6 L \partial dx L D_l(i_6 + i_7) = 0 \\
L_{6R} D_l(i_6) + L_{6R} L \partial dx L D_l(i_6 - i_8) = 0 \\
- D_4(t+y)(i_5 - i_7) + D_4(t-y)(i_5 + i_7) + L_{6L} L \partial dx L D_l(i_6 - i_7) = 0 \\
- L_{7R} D_l(i_7) + L_{7R} L \partial dx L D_l(i_7 - i_8) = 0 \\
- D_5(t+y)(i_4 - i_8) + D_5(t-y)(i_4 + i_8) + L_{8U} D_l(i_8) - L_{8R} L \partial dx L D_l(i_8 - i_6) = 0 \\
- D_6(t+x)(i_5 - i_8) + D_6(t-x)(i_5 + i_8) + L_{8D} D_l(i_8) - L_{7R} L \partial dx L D_l(i_7 - i_8) = 0 \\
\end{align}

(2.13)

We particularly note that the plate system (and hence the derived graphical circuit) is formed symmetrically by multiplying sets of factors: \((r_1 Z_M Z_Q, Z_M, Z_M)\) and \((r_2 Z_M Z_Q, r_3 Z_M Z_Q, Z_Q, Z_Q, Z_M Z_Q)\) to Eqs. (2.12) and (2.13).

Since from Eqs. (2.12) and (2.13), we have arrived at the format of MD partial derivative operators \(D_j(\cdot)\) defined below as a result of the direct space-time-domain analysis [12]:

\begin{align}
D_j(t \pm x) = \frac{1}{4} (\delta_j D_l \pm r_1 Z_M Z_Q D_x), (j, \ell) = (1, 1), (3, 2), (6, 3) \\
D_j(t \pm y) = \frac{1}{4} (\delta_j D_l \pm r_1 Z_M Z_Q D_y), (j, \ell) = (2, 1), (5, 2), (4, 3) \ .
\end{align}

(2.14)

We notice that these operators all involve \(D_l\), and comprise all the combinations \(D_l \pm D_x\) and \(D_l \pm D_y\) where \(D_l(\cdot), D_x(\cdot)\) and \(D_y(\cdot)\) are denoted as the partial derivatives with respect to the temporal and spatial coordinates in \(x\) and \(y\) axes, respectively.

Conforming to the principle of Kirchhoff’s voltage law for which the total voltage around a closed loop must be zero, the derived network operations achieve an equivalent MD lumped electrical circuit. The MD circuit is known as the MD Kirchhoff circuit (MDKC) in Fig. 3 that represents behaviors of the laminated composite system in Eqs. (2.6) and (2.7).

In view of the MDKC of Fig. 3, it contains explicitly three main loops placed in the left sub-circuit, while five main loops are routed in the right sub-circuit. In between the two sub-circuits, there are two non-negative gyrators with the gyration resistances \(R_{GU} \triangleq r_2 Z_M Z_Q\), and \(R_{GL} \triangleq r_3 Z_M Z_Q\) coupled physically with pairs of currents \((i_2, i_4)\), and \((i_3, i_5)\) flowing between the left back-end and the right front-end loops, respectively. These eight loops primarily dominated by passive MD inductors that are structured by one- or two-port elements are proceeded in a clockwise circular motion corresponding to Eqs. (2.12) and (2.13).

Given the port elements mainly consisting of resistor, linear inductor, ideal transformer, and gyration, clearly, the MD inductor formed by Eq. (2.14) is the direct result of the well-known transformation from a symmetric lossless two-port in T-junction (otherwise known as the two-port Jaumann structure) [11]. Thus, MD-passivity of the specific structure can be achieved by properly dispatching the auxiliary positive constants \(\delta_j\) awaited to be determined.
On top of that, there are four inductive coupling T-loops [32] used to exchange signals between loops of \((Q_x, Q_y), (M_x, M_y), (M_x, M_{xy})\) and \((M_y, M_{xy})\), correspondingly flowing with pairs of currents \((i_2, i_3), (i_6, i_7), (i_6, i_8)\) and \((i_7, i_8)\) across a symmetric positive definite (SPD) matrix of coupled inductances in the left and right sub-circuits. Notably the inductive T-loops that involve with loops of \((M_x, M_y)\) and \((M_x, M_{xy})\) are resulted from the bending-twisting anisotropy, i.e. \(D_{16} \neq 0, D_{26} \neq 0\). Clearly, this mechanical anisotropy imposes a significant effect on the study of angle-ply material. The physical significance of laminate in force and moment resultants also highlights the difference from the one previously presented in [42, 43] only for cross-ply material. Details of these coupling inductances with coils representing either the self-inductances or the mutual ones can be viewed in Appendix A.

Another important feature of the MDKC is that two extra ideal transformers of ratio 1/1 are applied to reallocate currents within the synthesized loops representing the plate variables \(w_x\) and \(w_y\). The premise of the added ideal transformers is that they make the circuit designers easier to develop the desired circuit with the way where the design is partitioned into its building blocks. As a result, complexity of the routing crossings with currents within these loops can be greatly reduced enabling design flexibility.

3. Development of a multidimensional wave digital filtering network with a NLP core.

3.1. Multidimensional wave digital filtering network. Let us consider a two-port element depicted in the MDKC of Fig. 3 in terms of MD inductor of inductances \(L_{j} \geq 0, j = 1, 2, 3\) corresponding to the temporal coordinate \(t\), and the spatial coordinates \(x\) and \(y\):

\[
\bar{u}(x, y, t) = (L_1D_t \pm L_2D_x \pm L_3D_y)i(x, y, t)
\]

Clearly, the continuous form can be generally approximated by means of an appropriate difference relation using the trapezoidal rule in the spatial domain [12] to arrive at Eq. (3.2) with the port resistance \(\hat{r}\) as:

\[
\bar{u}(x_m, y_n, t_k) + \bar{u}(x_m \pm T_x, y \pm T_y, t_k \pm T_t) = \hat{r}(\bar{i}(x_m, y_n, t_k) - \bar{i}(x_m \pm T_x, y \pm T_y, t_k \pm T_t)).
\]

Here the mesh voltage \(\bar{u}\) and current \(\bar{i}\) involving in \((x, y, t)\) take discrete values as \([x_m, y_n, t_k] \triangleq [mT_x, nT_y, kT_t]\) for all \((k, m, n) \in N\) with the spatial step sizes \(T_x\) and \(T_y\) in \(x\) and \(y\) coordinates, respectively, as well as the temporal step size \(T_t\). Greatly facilitated by the trapezoidal rule, the approximation of one-port self-inductors with inductances \(L_{j}, j \in \{1, 4, 5\}\) involving \(T_i\) in Eq. (A.4) yields positive port resistances \(r_j^i = \frac{2L_i}{T_i^2}\). Similarly, the following port resistances are obtained for two-port MD inductors formed in Eq. (2.14) with SPD matrix of inductances involving \((T_x, T_t)\) or \((T_y, T_t)\) and shown in Eqs. (ids-1) and (A.7):

\[
\begin{align*}
\hat{r}_j &= \frac{r_x Z_{j,2} Z_{3}}{2} = \frac{8}{T_i^2}, (j, \ell) \in \{(1, 1), (3, 2), (6, 3)\} \\
\hat{r}_j &= \frac{r_y Z_{j,3} Z_{2}}{2} = \frac{8}{T_i^2}, (j, \ell) \in \{(2, 1), (5, 2), (4, 3)\}.
\end{align*}
\]

To complete the transformation from the reference circuit of Fig. 3 to the desired WDF arrangement, the incident and reflected wave quantities [11] shall be
introduced for each port using the voltage wave representation as follows:

\[ a \triangleq \bar{u} + R\bar{i}, \quad b \triangleq \bar{u} - R\bar{i} \]  

(3.4)

\[ a \triangleq \bar{u} + R\bar{i}, \quad b \triangleq \bar{u} - R\bar{i} \]  

(3.5)

We note that the wave quantities defined in Eq. (3.4) are primarily for a scalar port with the non-negative port resistance \( R \) connected with an incident wave \( a \) and a reflected wave \( b \), while its vector counterpart is given in Eq. (3.5) with \( \vec{R} \) denoted by a matrix of constant port resistance.

Bring together the various port adapters with elements of WDF representation, and following by the standard WDF procedures \cite{11}, a complete MDWDF network depicted in Fig. 4 can be ready to perform free vibration analysis of the laminated plate with various fiber orientation angles. Readers are suggested to refer to \cite{11} for better explaining the underlying theory and physical facts concerning how elements are translated from the reference domain with proper analogy to the WDF.

In view of the MDWDF network of Fig. 4, it consists of non-reactive and reactive parts responsible for dealing with the system variables and initial-boundary conditions. The reactive part of the network contains self-adapters with waves that play a key role in embedding the first-order derivative initial values, which have been thoroughly investigated in \cite{42, 43}. Meanwhile, the non-reactive part comprises explicitly series adapters mainly conducting activities of determining system quantities. As a result, reflected waves \( b_i, i = 1, \ldots, 12 \) (respectively, incident waves \( a_i, i = 1, \ldots, 12 \)) are interactively pumped in (respectively, pumped out) through adapters marked by \( N'(1) \) \cite{12} (respectively, \( N'(2) \)) with shift operators \( T_i \) \cite{42, 43}. This way, it harmoniously controls the flow-in/flow-out frequency of state quantities \( c_i \) and \( d_i \). Accordingly, relations between the reflected and incident scalar and vector waves within series connection of N-port adapters can be described explicitly as

\[ \sum_j a_j = 2b_j - \frac{2R_i}{\sum_j R_j} \sum_j a_j, \quad j \in \{1, 3, s_1\} \text{ for } v \]

\[ \{G_2, 5, T_1\} \cup \{7, T_2, s_4\} \text{ for } w_x \]

\[ \{9, T_3, s_5\} \cup \{G_4, 11, T_4\} \text{ for } w_y \]

\[ b_j = a_j - 2 \left( \sum_j \frac{R_j}{R_i} \right)^{-1} \frac{R_i}{R_j} \sum_j a_j, \]  

(3.6)

\[ j \in \{JQ, CQ, Gr\} \text{ for } Q_x/Q_y \]

\[ \{JM_{x,y}, CM_{x,y}, TM_{x,y}\} \text{ for } M_x/M_y \]

\[ \{TJM_{x,y}, CM_{x,y}, TM_{x,y}\} \text{ for } M_{x,y} \]  

(3.7)

Here the scalar and matrix forms of the port resistances between the gyrator couplings and ideal transformers can be set as follows

\[ R_{G_2} = R_5 + R_{T_1}, \quad R_{G_4} = R_{11} + R_{T_4}, \quad R_{Gr} = \frac{R_{JQ}}{Q} + \frac{R_{CQ}}{Q} \]  

(3.9)

\[ R_{T_2} = R_7 + R_{s_4}, \quad R_{T_3} = R_9 + R_{s_5}, \quad \frac{R_{TM_{x,y}}}{M_{x,y}} = \frac{R_{JM_{x,y}}}{M_{x,y}} + \frac{R_{CM_{x,y}}}{M_{x,y}}. \]  

(3.10)

This is due to memoryless of the gyrators coupling between two subsystems in the reference circuit where no reflection (feedback of signal, which mainly cause the parasitic oscillations \cite{11}) comes back from the gyrators when the junctions are perfectly terminated.

Taking equal port resistances of ideal transformer into account, the status of reflection-free at the junctions simply yields

\[ a_j = 0, \quad j \in \{G_1, G_2, G_3, G_4, T_2, T_3, TM_1, TM_4\} \]
in Eq. (3.6) being simplified as

\[ b_{c_2} = -(a_5 + a_{T_1}), \quad b_{c_3} = -(a_{11} + a_{T_1}), \]  

(3.11)

\[ b_{T_2} = -(a_7 + a_{s_4}), \quad b_{T_3} = -(a_9 + a_{s_5}) \]  

(3.12)

while, the reflected vector waves in Eq. (3.8) yields

\[ k_{Gr} = -\left(\mathbf{a}_{JQ} + \mathbf{a}_{Q} + \mathbf{a}_{Gr}\right), \quad k_{TM_{s,y}} = -\left(\mathbf{a}_{JM_{s,y}} + \mathbf{a}_{CM_{s,y}}\right) \]  

(3.13)

Referring back to the wave flow diagrams of the ideal transformer, the missing incoming waves of these reflection-free ports/junctions \( a_j \) and \( \mathbf{a}_j \) can be immediately realized from other outgoing waves defined in Eqs. (3.6) and (3.8) as

\[ a_{T_2} = -b_{T_1}, \quad a_{T_3} = -b_{T_4}, \quad a_{T_{M_1}} = -b_{T_{M_2}}, \quad a_{T_{M_4}} = -b_{T_{M_5}}. \]  

(3.14)

Likewise, the incoming waves can also be derived on the basis of the gyrator coupling as in [42, 43]. In addition, the 2-port adapter marked by \( N'(1) \) and \( N''(1) \) can be readily determined in terms of the state inputs denoted by \( c_j \) and the state outputs denoted by \( d_j \) by using standard results [12]. The reader is referred to Appendix B.1 for details of these elements, while Appendix B.2 is provided for the port resistances \( r_j \) and \( \mathbf{R}_j \) at each series port adapter with operations described above.

Now combining with the series N-port adapters in Eqs. (3.6) and (3.8) and the reflection-free ports/junctions in Eqs. (3.12)-(3.13), solution functions to the PDEs of Eqs. (2.6) and (2.7) can now be described at the discrete points \( (x_m, y_n, t_k) \) in the form of incident wave variables \( a_j \) and \( a_{s_j} \) (indices are omitted) as:

\[ v = \frac{R}{\sum R_c}(a_{s_1} + a_1 + a_3) \]  

(3.15)

\[ [Q_x, Q_y]' = (2\mathbf{R}_{Gr})^{-1}(\mathbf{a}_{JQ} + \mathbf{a}_{CQ} + \mathbf{a}_{Gr}) \]  

(3.16)

\[ w_x = \frac{R}{\sum R_{w_y}}(a_7 + a_{T_2} + a_{s_4}) \]  

(3.17)

\[ w_y = \frac{R}{\sum R_{w_y}}(a_9 + a_{T_3} + a_{s_5}) \]  

(3.18)

\[ [M_x, M_y]' = \sum R^{-1}_{M_{s,y}}(a_{JM_{s,y}} + a_{CM_{s,y}} + a_{TM_{s,y}}) \]  

(3.19)

\[ [', M_{xy}] = \sum R^{-1}_{M_{xy}}(a_{JT_{M_{s,y}}} + a_{CT_{M_{s,y}}} + a_{JT_{M_{s,y}}}) \]  

(3.20)

The summation of port resistance defined in each junction is given as the following:

\[ \sum R_v \triangleq R_{s_1} + R_1 + R_3; \quad \sum R_{w,xy} \triangleq R_7 + R_{T_2} + R_{s_4}; \]  

\[ \sum R_{w,y} \triangleq R_9 + R_{T_3} + R_{s_5}; \]  

\[ \sum R_{M_{s,y}} \triangleq R_{JM_{s,y}} + R_{CM_{s,y}} + R_{TM_{s,y}}; \]  

\[ \sum R_{M_{xy}} \triangleq R_{JT_{M_{s,y}}} + R_{CT_{M_{s,y}}} + R_{JT_{M_{s,y}}}. \]  

Since the inception of of Eqs. (3.15) and (3.20), the WDF interpolation for each boundary node can be easily applied, just like the building block, to the MDWDF network. This can be done by replacing state inputs of the plate variables in the non-reactive part of the network by their corresponding state outputs. Details of the network implementation involving the accommodation of initial and boundary conditions into the open-field network can be referred to [42, 43].

3.2. MDWDF network stability and NLP problem. We note from previous section that the MDKC is represented by Eqs. (2.6) and (2.7), a well-conditioned plate model. The resultant MDWDF network used to solve that continuum model
must be numerically stable. Since the MDKC is accomplished with full MD-passivity, the passivity of the overall circuit gives rise to the following theorem to provide the MDWDF network a full stability with least restriction, which must be constrained by the well known CFL criterion.

**Theorem 3.1.** The MDWDF network of Fig. 4 used to perform the free vibration of symmetrical FSDT laminate governed by Eqs. (2.6) and (2.7) is numerically stable as long as the time step restriction subject to the CFL necessary condition is constrained by:

\[ T_t \leq T_z / \sqrt{2C_g}. \]  

(3.21)

Here \( T_z \in W = \{ z : \sqrt{x^2 + y^2} \leq z \leq \sqrt{1-x^2 - y^2} \} \) is defined as an isotropic spatial step, \( h \triangleq T_z/T_y \) and \( \chi \triangleq T_2/T_t \) are set as the ratio of the spatial step and that of the spatial-to-time step, respectively. In addition, the self-explanatory notations \( C_g \) with \( C_j \triangleq I_jZ_MZ_Q, j = 0, 2 \) and \( \ell_j, j = 1, \ldots, 6 \) are all defined as constants by

\[
C_g \triangleq \min \left\{ \frac{C_0}{I_0 \sqrt{h} \min \{\ell_1, \ell_2/h\}}, \frac{C_2}{I_2 \sqrt{h} \min \{\ell_3, \ell_5/h\}}, \frac{C_2 \sqrt{h}}{I_2 \min \{\ell_4, h\ell_6\}} \right\}
\]

(3.22)

\[
\ell_1 \triangleq Z_M(A_{15}^e + A_{55}^e)/\kappa_s, \ell_2 \triangleq Z_M(A_{44}^e + A_{45}^e)/\kappa_s
\]

(3.23)

\[
\ell_3 \triangleq Z_Q(D_{11}^c + D_{12}^c + D_{16}^c), \ell_4 \triangleq Z_Q(D_{12}^c + D_{22}^c + D_{26}^c)
\]

(3.24)

\[
\ell_5 \triangleq Z_Q(D_{16}^c + \frac{1}{2}D_{66}^c), \ell_6 \triangleq Z_Q(D_{26}^c + \frac{1}{2}D_{66}^c)
\]

(3.25)

**Proof.** See Appendix C. for detail.

Results obtained from Theorem 3.1 specifically give rise to the following interesting remarks:

**Remark 1.**

1. Provided Eq. (3.3) together with \( \ell_j \) in Eqs. (3.23) and (3.25), the possible choice of \( \delta_j > 0 \) awaiting to be determined in Eqs. (2.14) can be set straightforward by

\[
\delta_1 = \min\{\ell_1, \ell_2/h\}, \delta_3 = \min\{\ell_3, \ell_5/h\}, \delta_4 = \min\{\ell_4, h\ell_6\}
\]

(3.26)

with \( \delta_2 = h\delta_1, \delta_5 = h\delta_3 \) and \( \delta_6 = \delta_4/h \). Thus, the MD-passivity of individual elements (and hence the MD-passivity of the entire circuit) can be fully guaranteed.

2. The positive constants \( r_j, j \in \{1, 2, 3\} \) associated with the graphical currents \( i_k, k \in \{1, 4, 5\} \) defined in Eq. (2.12) are bounded as \( 0 < r_j \leq \bar{r}_j, j = 1, 2, 3 \) where \( \bar{r}_j \) are given by

\[
\bar{r}_1 = \frac{\chi}{Z_MZ_Q} \sqrt{\frac{2}{1+1/h^2}} \min\{\ell_1, \ell_2/h\}
\]

\[
\bar{r}_2 = \frac{\chi}{Z_MZ_Q} \sqrt{\frac{2}{1+1/h^2}} \min\{\ell_3, \ell_5/h\}
\]

\[
\bar{r}_3 = \frac{\chi}{Z_MZ_Q} \sqrt{\frac{2}{1+1/h^2}} \min\{\ell_4, h\ell_6\}
\]

(3.27)
3. When the scenario of a single isotropic layer is considered with material properties $E_{ij} = E, G_{ij} = G$ and $\varepsilon_{ij} = \varepsilon$, the phase velocity in Eq. (3.22) can be realized more simply as $C_g = \sqrt{\kappa G/\rho}$ [41].

Following from Theorem 3.1 and making use of $\delta_j, j = 1, 3, 4$ in Eq. (3.26) and $L_j, j = 1, 4, 5$ in Eqs. (C.2) and (C.3) with $\bar{r}_j, j = 1, 2, 3$ in Eq. (3.27), the core of an optimum MDWDF network to achieve its best performance while still keeping numerical stability is formulated by solving a simple NLP with nonlinear constraints as:

$$\max_{\chi \geq \chi^0} CFL(\chi) \triangleq \frac{C_g}{\chi}$$

subject to

$$\begin{bmatrix}
    \bar{L}_1(\chi) & \triangleq & C_0 \bar{r}_1^2(\chi) - \delta_1(1 + h) \\
    \bar{L}_4(\chi) & \triangleq & C_2 \bar{r}_2^2(\chi) - \delta_3(1 + h) \\
    \bar{L}_5(\chi) & \triangleq & C_2 \bar{r}_3^2(\chi) - \delta_4(1 + 1/h)
\end{bmatrix} \geq 0. \quad (3.29)$$

Remark 2.

1. The variable $\chi$ in Eqs. (3.28) and (3.29) is subjected to a lower bound $\chi^0 = \sqrt{2}C_g$ according to Eq. (3.21) leading to the cost function of the NLP, which is defined as the CFL number and is bounded by $[0, 1/\sqrt{2}]$. Any model with a given CFL number outside the bound will cause the network unstable and thus incur a divergent scheme according to Theorem 3.1.

2. Provided the lower bound $\chi^0$, the NLP model with three nonlinear constraints can be solved iteratively by global optimization techniques with a boost in terms of initial guess given by the golden section search method [33].

3. Greatly benefited from the optimum or sub-optimum of the NLP model, one, thus, can completely assure the performance of the MDWDF network in terms of:
   - full guarantee of achieving the optimality or sub-optimality of the network by effectively and accurately predicting the desired fundamental frequency.
   - side effects mitigation such as vibratory displacement, and balancing overflow corrections and roundoff errors, which are most attributed to the use of large time steps by the network, with the shortest simulation time, i.e. fast convergence.

3.3. Solvers of the nonlinear programming problem. As stated in Introduction, the NLP model of Eqs. (3.28) and (3.29) can be possibly solved by NLP solvers, e.g. gradient-based and direct search methods. Generally speaking, gradient methods are preferable to tackle NLP models as they use information from the derivatives to guide the search and find optimum solutions much quicker. On the other hand, direct search methods explore the parameter space in a systematic manner without the need for derivatives of the optimization function. A natural question then is what the trade-off is between computational efficiency and effectiveness. In this subsection, we briefly state regarding their concepts and stopping criteria as the difference between these two types of method is significant.

- Active set algorithm (ASA) [17]: The ASA solver attempts to solve NLP models by focusing on solutions of the Karush-Kuhn-Tucker (KKT) equations [33] with active constraints at the solution point being implicitly included. The KKT equations are analogous to the condition where the gradient must be
zero at a minimum, and are served as necessary conditions for the optimality of a constrained optimization problem where a sequence of subproblems that are used to compute the Lagrange multipliers is solved by the SQP algorithm (an inner algorithm). As the algorithm is a two-phase iterative methods (first phase focuses on feasibility, while the second one focuses on optimality), termination for the solver can be occurred when both the maximum constraint violations and the infinity norm of the scaled vector of first-order optimality [33] are met.

- Interior point algorithm (IPA) [6]: Analogous to the ASA solver, the IPA solver is also a two-phase derivative-based optimization method, which utilizes the KKT equations and combines with the projected conjugate gradient algorithm (an inner algorithm) to compute the Lagrange multipliers for a sequence of approximate problems. Unlike the ASA solver trying to learn the true active set, the IPA solver uses the barrier terms to ensure iterates lying in the interior of the feasible region. Since the IPA server is a two-phase iterative methods, termination tolerance for solver of this type is set at the same conditions as the ASA solver does.

- Genetic algorithm (GA) [16]: Different from the above classical gradient-based algorithms, the GA solver is based on the mechanics of natural selection process by trying to mimic biological evolution for NLP models. The core of GA consisting of reproduction, crossover and mutation processes repeatedly modifies a population of individual solutions by randomly selecting individuals from the current population, and uses them as parents to produce the children for the next generation. Over successive generations, the population evolves itself and the best point in the population approaches an optimal solution. In this study, the augmented Lagrange genetic algorithm (ALGA) is employed, which combines the Lagrangian and penalty parameters to generate a sequence of improving approximate solutions for NLP. Termination tolerances are various including the generation number, function tolerance, and constraint tolerance.

- Pattern search algorithm (PSA) [22]: The PSA solver is a class of direct search methods, which do not require any gradient information of the objective function, and is proceeded by conducting a series of exploratory moves around the current point within a set of points (or meshes) before a new iterate is identified. Moving with the core operators of search direction, mesh, and poll method, the PSA minimizes a sequence of subproblems looking for one whose value of the objective function is lower than that at the current point. In particular, the augmented Lagrange pattern search algorithm (ALPSA) is adopted to solve our NLP model. When the subproblem formulated by combining the objective function and nonlinear constraints is minimized to a required accuracy and satisfies feasibility conditions, the Lagrangian estimates are updated. Otherwise, the penalty parameter is increased to yield a new subproblem formulation.

4. Numerical results. Taking the full advantage of the NLP modeling to explore the robustness of MDWDF network, in this section, extensive study for numerical performances of various NLP solvers within the MDWDF network is presented in terms of accuracy and stability for symmetrically laminated composite FSDT plates. More specifically, we focus on the vibration effects of various square plates with cross-ply layups combining with various fiber orientations, stacking sequences, stiffness ratios and boundary conditions. Supported by Theorem 3.1, we concentrated,
for the first time, on exploiting various NLP solvers with suitable broad search power including the gradient-based methods (ASA and IPA) and direct search methods (ALGA and ALPSA).

Optimum solution obtained by the MDWDF network primarily involves two major phases of calculation: phase-I optimal or sub-optimal solution of the NLP model following by phase-II the mechanical operations of the MDWDF network. In the phase-I process, comprehensive comparison of algorithmic performance tackling the NLP model is present based on the these NLP methods. Simulation results have led to optimum CFL numbers as one of key inputs to the phase-II calculation that yields the best normalized fundamental frequencies. These results are shown to be in excellent agreement with FEM solutions based on the HSDT \cite{20,36} and other well known numerical technique \cite{46}.

Unless otherwise specified, computational geometries of the square laminates are set by $a \times b = 1 \times 1$ along the $x$- and $y$-direction, the span-to-thickness ratio $a/h = 10$ with grids = $21 \times 21$. In addition to the geometry setting, the plates are assumed to be of the same thickness, density and made of the same linearly elastic composite material with fibers being uniformly distributed throughout the 2-3 plane. As a result, properties of individual layer are specified below:

\[
\begin{align*}
E_1/E_2 & = \text{open}, E_2 = 7 \times 10^9 (N/m^2) \\
G_{12} & = G_{13} = 0.6E_2(N/m^2), G_{23} = 0.5E_2(N/m^2), \\
\nu_{12} & = 0.25, \rho = 1.0(\text{kg/m}^3). 
\end{align*}
\]

Here $E_i, i = 1, 2$ stand for the Young’s modulus in the $i$ principal coordinate, $\nu_{ij}$ are denoted as the Poissons ratio in the $i - j$ principal coordinates and is related to the Young’s modulus by $\nu_{12}/\nu_{21} = E_1/E_2$, and the engineering constants $G_{ij}$ represent the shear moduli in the $i - j$ principal coordinates.

On top of that, each board of the plate is subjected to the hard type simply supported (SS2) boundary condition. The reader is referred to literatures \cite{36,21,10,29} for detail of the displacement boundary condition and \cite{42,43} for the state outputs derivation on boundaries. Furthermore, in order to analyze effects of the free vibration, a periodic equation is explicitly expressed involving with the spatial variation of lateral displacement $W$ and the radial frequency of natural plate vibration $\omega$ as

\[
w_0(x, y, t) = W(x, y)(\cos(\omega t) + \sin(\omega t)), W(x, y) = \sin(\alpha \pi x/a) \sin(\beta \pi y/b) \quad (4.2)
\]

where $\omega$ corresponds to different mode shapes based on different values of $\alpha$ and $\beta$ in Eq. (4.2), accordingly different shapes of $w$.

To study the convergence of frequency parameters, stacking layers of the laminate are arranged as $[0^\circ/90^\circ/0^\circ]$, $[0^\circ/90^\circ/0^\circ/90^\circ]$ and $[0^\circ/90^\circ/90^\circ/90^\circ/0^\circ]$ for three-layer, four-layers and five-layer cross-ply layups, respectively. Geometries of these square laminated plates are illustrated in Fig. 2. We note that for these symmetric schemes, the laminate stiffness has been greatly simplified as $\bar{A}_{44} = \bar{A}_{55} \neq 0, \bar{A}_{45} = 0$ and $\bar{D}_{16} = \bar{D}_{26} = 0$. Other types of laminate scheme, e.g. angle-ply or quasi-isotropic laminates \cite{24,36,31,30}, can also be studied. Unlike the cross-ply laminates, these special laminates, however, impose a non-orthotropic bending behaviors. Thus, numerical examples considered in this paper are limited to thick symmetric cross-ply laminates with layers of equal thickness.

To estimate waveforms of free vibration, the trapezoidal rule integration is carried out for the transverse velocity $v(x, y, t) = \frac{dw(x,y,t)}{dt}$ and its acceleration term
\[
\frac{d v(x,y,t)}{d t} = -\omega^2 w(x,y,t) \quad \text{at a period of time } [t_{k-1}, t_k] \text{ leading to the vibration frequency}
\]

\[
\omega = \sqrt{\frac{2}{T_k} \left\| v(x,y,t_{k-1}) - v(x,y,t_k) \right\|} \quad \text{at a period of time } [t_{k-1}, t_k]
\]

where \( \| \cdot \| \) denotes the Euclidean norm. For \( t \in [0, t_k] \), Eq. (4.3) shows peaks of signal \( \omega \) repeated itself every half-cycle regularly with every other receiving an overall \( \pi \) phase shift where the fundamental radian frequency can, in principle, be obtained for various ratios of span-to-thickness. However, there are difficulties and challenges arising from (i) that not all waveforms are periodic, (ii) those that are periodically oscillated may be changing in fundamental frequency over the time of interest, (iii) signals which are periodic with fundamental interval \( T \) could be also periodic with intervals \( 2T, 3T, \) etc.

By looking at the characteristic of the signal, a reliable and commonly used approach for obtaining an estimate of the dominant fundamental frequency (\( \omega_f \)) is to employ the frequency analysis of the signal. In particular, as the frequency spectra maintain many regularly spaced vibration harmonics, then the Fourier analysis of the logarithm of the power spectrum (i.e. spectrum of spectrum, which is called the (power) ‘cepstrum’ [4]) will reveal peaks corresponding to the spacing between the vibration harmonics i.e. the periodicities. Note that the new Fourier frequency domain is called ‘quefrequency’ and peaks in the cepstrum are called ‘rähmonics’.

As the fundamental frequency is not changing too rapidly, nor too high, the cepstrum analysis of the vibration frequency works best and is adopted in this report. Thus, one can obtain an estimate of the fundamental frequency by measuring the periodical interval \( (T_C) \) of the first harmonic in the quefrequency with the highest rähmonic in the cepstrum. It is worth noting that since these periodic signals may be changing in fundamental frequency over the time of interest, one can ideally check the first harmonic by matching the highest rähmonic with a corresponding peak of vibration frequency.

To ensure that the network stability based on the necessary CFL criterion is guaranteed at all time, the NLP model derived by Eqs. (3.28) and (3.29) must be securely constrained by the global optimization methods that have been briefly discussed in Section 3.3. These algorithms are performed by MATLAB optimization toolbox with option parameters set in Table 1 and Table 2 for gradient-based methods and direct search methods, respectively. From these, significant vibration characteristics of the plate can all be studied with various stacking sequences and stiffness ratios. Here the vibration effects include the normalized fundamental frequencies, strain energy and total kinetic energy with respect to different scales of CFL number. Note that the normalized fundamental frequency \( \bar{\omega}_f \) and its corresponding shear correction \( \kappa_s \) is defined by

\[
\bar{\omega}_f = \omega_f a^2 \sqrt{\frac{\rho}{E_2 h}}, \text{ with } \kappa_s = 5/6 \quad (4.4)
\]

\[
\bar{\omega}_f = \omega_f a^2 \sqrt{\frac{\rho h}{D_0}}, \text{ with } \kappa = \pi^2/12 \quad (4.5)
\]

where \( D_0 = E_2 h^3/12(1 - \mu_{12}\mu_{21}) \) is the flexural rigidity. In addition, the strain energy (prefix \( SE \)) and total kinetic energy (prefix \( KE \)) [18] are defined by integrating by parts for the stress, strain and displacements at the \( k \)th layer through the thickness of the plate as follows:
shows details of the optimization process performed by the

3.1

with full details.

χ

instability. We note that the magnitudes

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and

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the space-to-temporal ratio

CFL

Robustness of the network.

4.1.2.

different models with the following scales
to fast convergence of the network. To explore robustness of t he network, eight more
are further tested on the network to show their corresponding p late energies

Table

In addition, convergence of this NLP optimization is presented in the first-half of

maximum constraint violation depicted in Fig.

χ

following by the locally optimal solution found. Staring with the lower bo und of

objective function value at every iteration, while the tolerance crit eria are checked

χ

of the space-to-temporal ratio (\( \chi \)) depicted in Fig.

stiffness ratio involving at most 7 iterations and 16 functions counte d when the max-

ASA solvers. The optimization process includes principally the evaluat ion of the

4.1.1. NLP model optimization. Provided by the algorithm termination criteria
listed in Table 1, Fig. 5 shows details of the optimization process performed by the
ASA solvers. The optimization process includes principally the evaluation of the
objective function value at every iteration, while the tolerance criteria are checked
following by the locally optimal solution found. Staring with the lower bound of
\( \chi_0 = \sqrt{2}\sigma_y \) as is suggested by Theorem 3.1, the algorithm is terminated for each
stiffness ratio involving at most 7 iterations and 16 functions counted when the maximum
constraint violation depicted in Fig. 5(a2)-5(d2) and the first-order optimality
depicted in Fig. 5(a3)-5(d3) both are met. As a result, optimums or suboptimums of the space-to-temporal ratio (\( \chi \)) are obtained and illustrated in Fig. 5(a1)-5(d1). In addition, convergence of this NLP optimization is presented in the first-half of
Table 3 with full details.

4.1.2. Robustness of the network. Supported by the optimum or sub-optimum of the space-to-temporal ratio \( \chi_{\text{min}} \), and hence that of the CFL criterium number \( \text{CFL}_{\text{max}} \), the necessary MDWDF network stability takes effect immediately leading to fast convergence of the network. To explore robustness of the network, eight more
different models with the following scales

\[
\begin{align*}
\chi_1 &= \chi_{\text{min}} - 3\%,
\chi_2 &= \chi_{\text{min}} - 1.5\%,
\chi_3 &= \chi_{\text{min}} - 1.0\%,
\chi_4 &= \chi_{\text{min}} + 1.0\%,
\chi_5 &= \chi_{\text{min}} + 1.5\%,
\chi_6 &= \chi_{\text{min}} + 3.0\%.
\end{align*}
\]

(4.6)

are further tested on the network to show their corresponding plate energies \( L_{KE} \)
and \( L_{SE} \), and thus to give an insight into the issue of network robustness against
instability. We note that the magnitudes \( \chi_{\text{min}} - \epsilon_1 \% \) (or \( +\epsilon_1 \% \)) and \( \chi_{\text{min}} + \epsilon_2 \%

indicate their values are $\epsilon_1\%$ below (or above) and $\epsilon_2x$ (times) above $\chi_{\min}$, respectively.

Provided the model 4 containing $\chi_4 = \chi_{\min}$ as the reference one, the percentage error distribution of $L_{SE}$ measured by the first 800 temporal steps for the scheduled scales $\chi_i$ in Eq. (4.6) are depicted in the accompanying Fig. 6(a)-6(d) for $E_1/E_2 = 10, 20, 30, 40$, respectively, within the first 250 temporal steps. Clearly, models 5-9 with scales $\chi_i > \chi_{\min}, i = 5,\ldots,9$ are well stable at all time as a result of Theorem 3.1. In contrast, models 1-3 with scales $\chi_i < \chi_{\min}, i = 1, 2, 3$ are unstable and have eventually led to ill-posed systems. Similar results can be also observed by Fig. 7(a)-7(d) representing the percentage error distribution of $L_{KE}$ for $E_1/E_2 = 10, 20, 30, 40$, respectively. Reasons for the network instability may be caused initially by $L_4 < 0$ (as can be seen clearly in Tables 4 and 5 for $E_1/E_2 = 10, 20$ and $E_1/E_2 = 30, 40$, respectively) and hence not satisfying the MD passivity of the reference circuit resulting in the original system ill-conditioned.

Further to the issue involving the ill-posed problem with $\chi < \chi_{\min}$, we implement the network with a bigger time step ($T_t \triangleq \frac{T_z}{\chi}$), which may also directly link to instability of the network due to excessive round off error measured. In particular, for $E_1/E_2 = 10$, Fig. 7(a) shows model 1 with $\chi_1$ contains error starting to build up at the 108 temporal step and eventually causing the system diverged quicker than models 2 and 3 with scales $\chi_2$ and $\chi_3$, which start the divergence at the 182 and 331 temporal steps, respectively. Fig. 7(b)-(d) also shows similar effect of divergence for $E_1/E_2 = 20, 30, 40$, respectively.

The above examples have clearly pointed out a key criterion for establishing a robust MDWDF network to integrate the plate system, i.e. proper selection of $\chi$ and hence the CFL number so that the process of transition from the passive circuit system to a fully-stable algorithm can be carried out in an effective and efficient manner. In particular, having the optimal solution ($\chi_{\min}$ and hence $CFL_{\max}$) of the NLP model secured in the first place, the MDWDF network is clearly maintained to be well-conditioned for the plate system with faster convergence achieved. Likewise, for any $\chi > \chi_{\min}$ it amounts to smaller CFL numbers, while keeping numerical stability at all time. We note that the smaller CFL number of the model, sometimes, is interpreted as being very safe, but making little progress towards the steady states, due to a smaller temporal step. Notably, these results are consistent with those appeared in $[42, 43]$ with material properties $E_1/E_2 = 10$ and $a/h = 5$.

4.2. Optimality of fundamental frequency. Having the network stability limited by the CFL bound $[0, C_g/\chi_{\min}]$ for the four-layer cross-ply layup $[0^\circ/90^\circ/90^\circ/0^\circ]$ square laminates with $E_1/E_2 = 10, 20, 30, 40$, the desired normalized fundamental frequency $\bar{\omega}_f(\chi_{\min})$ can then be obtained by the cepstrum process $[4]$. More specifically, by matching the highest harmonic with periodical interval ($T_{CP}$) in the quefrequency domain of Figs 8.1(a)-9.2(h), one can observed, without any difficulty, the corresponding first harmonic with periodicity ($T_{FP}$) in time domain of Figs. 8.1(a)-9.2(c), respectively. The resultant cepstrum process is depicted in Table 6, while the minimum value $\bar{\omega}_f$ is of very much agreement with those obtained by TSDDT-FEM $[36]$. Taking Eq. (4.2) into account, Table 7 depicts the minimum $\bar{\omega}_f$ for the first six modes of the plate with $E_1/E_2 = 10, 20, 30, 40$, while a stable isometric view of the first six well defined modes is only shown for $E_1/E_2 = 10$ in Fig. 10.1-10.6.

To validate that the obtained minimum value $\bar{\omega}_f$ is actually an optimum one (which means $\chi_{\min} = \chi_{opt}$), let us consider again those models listed in Tables 4
and 5 for $E_1/E_2 = 10, 20, 30, 40$. Having the time step determined by $\chi$ and following by the network stability constrained by the CFL bound $[0, C_g/\chi_{\text{min}}]$, the normalized fundamental frequency $\bar{\omega}_f(\chi)$ for every model of $\chi$ is obtained and depicted in Fig. 11(a)-11(d). Clearly, $\bar{\omega}_f(\chi_{\text{min}})$ shown in Fig. 11(a), 11(b) and 11(d), are optimum for $E_1/E_2 = 10, 20, 40$, respectively. Meanwhile Fig. 11(c) indicates the sub-optimum $\bar{\omega}_f$ obtained by $\chi_{\text{min}}$ for $E_1/E_2 = 30$. These results have numerically verified the 3rd claim of Remark 2 that the optimum of NLP directly leads to the optimality or sub-optimality of the network in terms of predicting the optimal normalized fundamental frequency.

To further validate the effectiveness of the network, the study of network stability and optimality is also carried out for the four-layer cross-ply layup $[0^\circ/90^\circ/90^\circ/0^\circ]$ plate with a broader range of stiffness ratios, which are set as $E_1/E_2 = 2n, n = 1, \ldots, 20$. Results depicted in Fig. 12(a) and 12(b) for the optimal CFL number and optimum $\bar{\omega}_f$ corresponding to each $E_1/E_2$ ratio clearly suggest that for fixed thickness of plates, the network stability and frequency optimality for higher stiffness ratios can be secured only by smaller CFL numbers. The exceptional presentation in terms of achieving smoothed results at a sustainable level as shown in Fig. 12 has also led to a strong indication that the MDWDF network is very robust and competitive with standard numerical approaches, e.g. FEMs.

4.3. Performance comparison w.r.t nonlinear optimization solvers. The previous section has suggested that solutions of the NLP model significantly play a key role, which takes effects directly on the network stability, convergence and accuracy, and it only yields local solutions, in this section we continue to explore optimum solutions of the NLP model using other three global optimization techniques mentioned in Section 3.3. Performance comparison of these four NLP solvers can fall into two categories: accuracy and efficiency.

4.3.1. Numerical effectiveness and efficiency. Adopting the MATLAB optimization toolbox with option parameters listed in Tables 1 and 2 for gradient-based and direct-search methods, respectively, these solvers are implemented for four-layer cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ plates with $E_1/E_2 = 10, 20, 30, 40$, and terminated smoothly when the tolerance criteria are completely met. As can be expected further that solving the NLP model of Eqs. (3.28)-(3.29) by the four different nonlinear solvers ASA, IPA, ALGA, and ALPSA, the network stability for every stiffness ratio of the plate can be all guaranteed by measuring the global (or near the global) solution $\chi_{\text{min}}$.

More specifically, optimization processes performed by IPA solvers within the phase-I calculation are depicted in Fig. 13 and 14 illustrating the convergence behavior with the cost function evaluation at every iteration w.r.t the tolerance criteria (the maximum constraint violation and the first-order optimality) for which details are provided in Table 3. Similar results of algorithm convergence involving the ALGA and ALPSA solvers are portrayed in Figs. 15 and 16 containing the score histograms at every iteration w.r.t. the tolerance criteria (number of functions evaluated and their corresponding function values). Table 9 details the processes performed by these direct-search solvers.

In view of Figs. 5, 13, 14, 15, and 16, it is strongly suggested that numerical effectiveness in terms of algorithm convergence is fully demonstrated by these solvers. For the numerical efficiency in terms of CPU runtime, as can be seen from Fig. 17(b), clearly the deterministic IPA is the most efficient solver to achieve an
optimum solution; conversely, the stochastic ALGA naturally takes the longest time to identify an solution.

4.3.2. Numerical accuracy. With these solvers employed to tackle the phase-I NLP model optimization, Table 8 presents $E_\omega(\%)$ as a result of the phase-II MDWDF network operation for the four-layer cross-ply laminated plate $[0^\circ/90^\circ/90^\circ/0^\circ]$ with $E_1/E_2 = 10, 20, 30, 40$. These results use the TSDT-FEM [36] as a reference. Clearly from Table 8 (also from Fig. 17(a)), the percentage errors of the normalized fundamental frequency has evidently shown that the ALPSA solver can help identify solutions more accurately than the other three NLP solvers for $E_1/E_2 = 10, 20$, while the ASA solver can perform better solutions than the other three solvers for $E_1/E_2 = 30, 40$.

Overall results as portrayed in Fig. 17(c) (also from Table 8) outperform FSDT-EFG [7] and FSDT-GRBF [46] regardless stiffness ratio. In addition, the results obtained are in very good agreement with the FEM solution based on the TSDT-FEM [36] where the percentage errors measured are 0.19%, 0.82%, 0.59%, 0.46% corresponding to the stiffness ratios $E_1/E_2 = 10, 20, 30, 40$.

Computationally demonstrated by the above accurate predictions of the normalized fundamental frequency for the four-layer laminate with cross-ply layups, the design of an optimum network is further strengthened by carrying out more tests on the plate with three-layer and five-layer cross-ply stacking sequences $[0^\circ/90^\circ/0^\circ]$ and $[0^\circ/90^\circ/90^\circ/90^\circ/0^\circ]$. Detail of optimum results based on various NLP solvers are listed in Tables 10 and 11 for three-layer and five-layer cross-ply sequences, respectively. In particular, Tables 10 reports $E_\omega(\%)$ using the FEM-based HSDT results [20] as the reference. In both layup cases, numerical performances of these solvers are very much the same similar to the four-layer layup case, and reader can verify that the predicted normalized fundamental frequencies are in good coincidence with those reported in [20].

Taking the algorithmic accuracy and efficiency into consideration, we may conclude that the ALGA and ALPSA solvers pertaining to the class of direct search method can assist the MDWDF network to achieve its best performance in terms of accuracy for integrating the plate system. On the other hand, the gradient-based methods of ASA and IPA solvers are the best choices for the network to model the plate system effectively in terms of algorithm convergence. We note that these results are validated only for specifically applying to the MDWDF network modeling of composite laminated plate.

5. Conclusions. A comprehensive development of a novel MDWDF networks based on a NLP model has been carried out with the stability analysis and performance evaluation for the free vibration responses of a 2D symmetrically laminated composite FSDT plate. Provided optimum solutions of the NLP model as a passage to gain smooth transition from the passive circuit system to a fully-stable algorithm in an effective and efficient manner, the well-developed MDWDF network was convergent adequately at all time. Meanwhile, numerical performances in terms of the measurement accuracy and efficiency were discussed fully, for the first time, involving various NLP solvers to examine the plate effects in predicting nature frequencies based on different stiffness ratios and stacking sequences. As far as computational performance of these solvers was concerned, surely the ALGA and ALPSA solvers could identify optimal solution that helps the specific MDWDF network to achieve its best performance in terms of accuracy for integrating the
plate system with $E_1/E_2 = 10, 20$. Likewise, the ASA and IPA solvers were the best choices for the network towards effectively and efficiently modeling the plate system with $E_1/E_2 = 30, 40$. The resultant normalized fundamental frequencies were carefully validated by cross referencing with early published works, which explicitly illustrate that performances of optimum MDWDF networks embedded with a suitable NLP solver were in very good agreement with that of the FEM solutions based on the high-order shear deformation theory.

**Appendix A.** Coupled inductances with coils representing self-inductances and the mutual ones in the MDKC. Having the MDKC established for the laminated system of Eqs. (2.6) and (2.7), there are four inductive coupling T-loops [32] following from Eqs. (2.12) and (2.13) being constructed with the SPD form in subsystems of the MDKC as follows:

$$\mathbf{L}_{\text{CQ}} = \begin{bmatrix} L_2 + L_{2,3} & -L_{2,3} \\ -L_{2,3} & L_3 + L_{2,3} \end{bmatrix}, \mathbf{L}_{\text{CMJ}} = \begin{bmatrix} L_{6L} + L_{6L,7L} & -L_{6L,7L} \\ -L_{6L,7L} & L_{7L} + L_{6L,7L} \end{bmatrix}$$

(A.1)

$$\mathbf{L}_{\text{CMJ}_{xy}} = \begin{bmatrix} L_{6R} + L_{6R,8U} & -L_{6R,8U} \\ -L_{6R,8U} & L_{8U} + L_{6R,8U} \end{bmatrix}, \quad \mathbf{L}_{\text{CMJ}_{xy}} = \begin{bmatrix} L_{7R} + L_{7R,8D} & -L_{7R,8D} \\ -L_{7R,8D} & L_{8D} + L_{7R,8D} \end{bmatrix}$$

(A.2)

(A.3)

Here the inductances and their corresponding mutual ones of the coils can be read in detail with constants $C_j = L_j Z_M Z_Q, j = 0, 2$ by

$$L_1 = \frac{1}{2} C_0 - \delta_1 - \delta_2, L_2 = Z_M \frac{\bar{A}_2}{\kappa}, L_3 = Z_M \frac{\bar{A}_3}{\kappa}, L_{2,3} = \delta_1, L_{3} = Z_M \frac{\bar{A}_3}{\kappa}, L_{2,3} = \delta_2$$

(A.4)

$$L_{6L} = k_1 Z_Q \bar{D}_{11} - L_{6L,7L} - \frac{1}{2} \delta_3, \quad L_{6R} = (1 - k_1) Z_Q \bar{D}_{11} - L_{6R,8U} - \frac{1}{2} \delta_3, \quad \frac{\partial \bar{D}_{11}}{\partial t_1} \leq k_1 \leq 1 + \frac{\partial \bar{D}_{11}}{\partial t_1}$$

(A.5)

$$L_{7L} = k_2 Z_Q \bar{D}_{22} - L_{6L,7L} - \frac{1}{2} \delta_4, \quad L_{7R} = (1 - k_2) Z_Q \bar{D}_{22} - L_{7R,8D} - \frac{1}{2} \delta_4, \quad \frac{\partial \bar{D}_{22}}{\partial t_2} \leq k_2 \leq 1 + \frac{\partial \bar{D}_{22}}{\partial t_2}$$

(A.6)

$$L_{8U} = k_3 Z_Q \bar{D}_{66} - L_{6R,8U} - \delta_5, \quad L_{8D} = (1 - k_3) Z_Q \bar{D}_{66} - L_{7R,8D} - \delta_6, \quad \frac{\partial \bar{D}_{66}}{\partial t_6} \leq k_3 \leq 1 + \frac{\partial \bar{D}_{66}}{\partial t_6}$$

(A.7)

$$L_{2,3} = -Z_M \frac{\bar{A}_3}{\kappa}, L_{6L,7L} = -Z_Q \bar{D}_{12}, L_{6R,8U} = -Z_Q \bar{D}_{16}, L_{7R,8D} = -Z_Q \bar{D}_{26}$$

(A.8)

**Appendix B.** Wave quantities and port resistances within port adapters.

**B.1. State and wave elements.** The 2-port adapter marked by $N'(1)$ and $N''(1)$ [12] yields the state inputs $c_j$ and the incident waves $a_j$ in terms of the reflected waves $b_j$ and state outputs $d_j$, respectively, by using the standard results [12], i.e. for $k = 1, \ldots, 6$,

$$\begin{cases} c_j = -\frac{1}{2}(b_{j-1} + b_j), j = 2k - 1 \\ c_j = -\frac{1}{2}(b_{j-1} + b_j), j = 2k \end{cases}, \begin{cases} a_j = d_j + d_{j+1}, j = 2k - 1 \\ a_j = d_j - d_{j-1}, j = 2k \end{cases}$$

(B.1)

Other state inputs are defined by $c_{s_j} = b_{s_j}, j \in \{1, 4, 5, 9, 10\}$, $\mathbf{L}_{\text{CQ}} = \mathbf{L}_{\text{CM}} = \mathbf{b}_{\text{CM}}, j \in \{x, y, x'y, y'x\}$, while incident waves are set by $a_{s_j} = -d_{s_j}, j = \ldots$
1, 4, 5; \( a_{cmj} = -d_{cmj}, j = 1, \ldots, 6 \). In addition, the state input-output relations are given below to show some level of parallelism, i.e. each \( d_j \) at the \( k \)-th time grid can be updated simultaneously from \( c_j \) at the \( (k-1) \)-th time grid:

\[
d_j(m, n, k) = c_j(m, n, k-1), j \in \{1, 4, 5\}
\]

\[
c_j(m \pm 1, n, k - 1), j \in EB \triangleq \{1, 5, 9\} \cup WB \triangleq \{2, 6, 10\}
\]

\[
c_j(m, n \pm 1, k - 1), j \in NB \triangleq \{3, 7, 11\} \cup SB \triangleq \{4, 8, 12\}
\]

\[
\hat{Z}_{CMj}(m, n, k) = \hat{Z}_{CM}(m, n, k-1), j \in \{x, y, xy, yxy\}
\]

We note from Eq. (B.2) that the plus sign ‘+’ occurs when \( j \in \{EB \cup NB\} \).

### B.2. Vector wave variables and their corresponding port resistances.

The following defines the vector waves used within the MDWDF network:

\[
\begin{align*}
\mathcal{A}_{CM_w} & \triangleq [a_{CM1}, a_{CM2}]', \mathcal{B}_{CM_w} \triangleq [b_{CM1}, b_{CM2}]' \\
\mathcal{A}_{ JM_w} & \triangleq [a_{JM1}, a_{JM2}]', \mathcal{B}_{ JM_w} \triangleq [b_{JM1}, b_{JM2}]' \\
\mathcal{A}_{CM_v} & \triangleq [a_{CM1}, a_{CM3}]', \mathcal{B}_{CM_v} \triangleq [b_{CM1}, b_{CM3}]' \\
\mathcal{A}_{JM_v} & \triangleq [a_{JM1}, a_{JM3}]', \mathcal{B}_{JM_v} \triangleq [b_{JM1}, b_{JM3}]'
\end{align*}
\]  

The matrices of port resistance corresponding to the above vector waves are given by:

\[
\begin{align*}
R_{CM_w} & \triangleq \frac{2L_{CM_w}}{T_l}, R_{JM_w} \triangleq \begin{bmatrix} R_6 & 0 \\ 0 & R_{12} \end{bmatrix}, R_{CM_v} \triangleq \begin{bmatrix} R_{TM1} & 0 \\ 0 & R_{TM4} \end{bmatrix} \\
R_{CM_v} & \triangleq \frac{2L_{CM_v}}{T_l}, R_{JM_v} \triangleq \begin{bmatrix} R_{TM2} & 0 \\ 0 & R_{TM8} \end{bmatrix}, R_{CM_v} \triangleq \begin{bmatrix} R_{TM3} & 0 \\ 0 & R_{TM10} \end{bmatrix}
\end{align*}
\]  

The above scalar resistances defined for each port adapter in Fig. 5 can be determined from Eqs. (3.3) using WDFs as follows:

\[
\begin{align*}
R_{ ej} & = \hat{r}_j, j \in \{1, 4, 5\} \\
R_{kj} & = \hat{r}_j, (k, j) \in \{(1, 1), (2, 1), (5, 3), (6, 3), (9, 6), (10, 6)\} \\
R_{kj} & = \hat{r}_j, (k, j) \in \{(3, 2), (4, 2), (7, 5), (8, 5), (11, 4), (12, 4)\}
\end{align*}
\]  

### Appendix C. Proof of Theorem 3.1.

**Proof.** From the region of \( W \), clearly the maximal range of \( x \) and \( y \) occurs at which the two surfaces meet, i.e., \( \sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2} \). In other words, for any point \((x, y, z)\) within the region satisfies the inequality \( \sqrt{x^2 + y^2} \leq 1/\sqrt{2} \). For unitary spatial steps \( T_x \) and \( T_y \) along the \( x \) and \( y \) axes, respectively, the inequality generally implies

\[
T_z \triangleq \sqrt{T_x^2 + T_y^2}/\sqrt{2} \leq 1.
\]
The satisfaction of MDKC being passive can immediately imply non-negativity of every inductance i.e. $L_j \geq 0$, $j = 1, \ldots, 8$, which suggest that the auxiliary constants $\delta_j > 0$ in Eq. (3.3) are bounded by

\[ \begin{align*}
L_1 &\geq 0 : \delta_1 + \delta_2 \leq r_1^2 C_0, L_2 &\geq 0 : \delta_1 \leq \ell_1, L_3 &\geq 0 : \delta_2 \leq \ell_2, \\
L_4 &\geq 0 : \delta_3 + \delta_5 \leq r_2^2 C_2, L_5 &\geq 0 : \delta_4 + \delta_6 \leq r_3^2 C_6 \\
L_{6L} &\geq 0 : \delta_3 \leq 2Z_Q(k_1 \bar{D}_{11} + \bar{D}_{12}), L_{6R} &\geq 0 : \delta_3 \leq 2Z_Q((1 - k_1)\bar{D}_{11} + \bar{D}_{16}) \\
L_{7L} &\geq 0 : \delta_4 \leq 2Z_Q(\bar{D}_{12}^r + k_2 \bar{D}_{22}), L_{7R} &\geq 0 : \delta_4 \leq 2Z_Q((1 - k_2)\bar{D}_{22} + \bar{D}_{26}) \\
L_{8U} &\geq 0 : \delta_5 \leq \ell_5, L_{8D} &\geq 0 : \delta_6 \leq \ell_6.
\end{align*} \tag{C.2, C.3, C.4, C.5} \]

In particular, Eqs. (C.4) and (C.5) simply imply $L_6 \triangleq L_{6L} + L_{6R} \geq 0$ and $L_7 \triangleq L_{7L} + L_{7R} \geq 0$ leading to

\[ \begin{align*}
L_6 &\geq 0 : \delta_3 \leq \delta_3 \triangleq Z_Q(\bar{D}_{11} + \bar{D}_{12} + \bar{D}_{16}), \\
L_7 &\geq 0 : \delta_4 \leq \delta_4 \triangleq Z_Q(\bar{D}_{12} + \bar{D}_{22} + \bar{D}_{26}).
\end{align*} \tag{C.7, C.8} \]

Now combining $T_z$ defined in Eq. (C.1) with $\delta_2 = \hbar \delta_1$ and $r_1 \geq \sqrt{(\delta_1 + \delta_2)/C_0}$ as a result of Eqs. (3.3) and (C.2), respectively, we obtain a lower bound of the ratio of space-time step $T_z/T_t$ from the requirements of $\delta_j, j = 1, 2$ in Eq. (C.2) as:

\[ r_1 Z_M Z_Q \sqrt{\frac{\delta_1^2 + \delta_2^2}{2\delta_1 \delta_2}} \geq \sqrt{\frac{Z_Q}{I_2^2 \sqrt{\delta_1 \delta_2}}} \geq \sqrt{2} \sqrt{\frac{C_0}{I_2^2 \sqrt{\min\{\ell_1, \frac{\delta_1}{\hbar}\}}}} \tag{C.9} \]

with help from the following

\[ \delta_1 \leq \min\{\ell_1, \ell_2/\hbar\}, \sqrt{(x^2 + y^2)(x + y)/x^2y^2} \geq \sqrt{8/\sqrt{x y}}, \forall x, y \geq 0. \tag{C.10} \]

Similarly, the passivity of $L_j$ involving with $\delta_j, j = 3, \ldots, 6$ satisfies the following inequalities for $T_z/T_t$:

\[ \begin{align*}
r_2 Z_M Z_Q \sqrt{\frac{\delta_3^2 + \delta_4^2}{2\delta_3 \delta_4}} &\geq \sqrt{2} \sqrt{\frac{C_2}{I_2^2 \sqrt{\min\{\ell_3, \frac{\delta_3}{\hbar}\}}}} \tag{C.11} \\
r_3 Z_M Z_Q \sqrt{\frac{\delta_5^2 + \delta_6^2}{2\delta_5 \delta_6}} &\geq \sqrt{2} \sqrt{\frac{C_6}{I_2^2 \sqrt{\min\{\ell_4, \frac{\delta_5}{\hbar}\}}}} \tag{C.12}
\end{align*} \]

Notably, given $\delta_5 = \hbar \delta_3$ and $\delta_6 = \hbar \delta_6$ admitted again from Eq. (3.3), Eq. (C.7) together with Eq. (C.6) imply immediately $\delta_3 \leq \min\{\ell_3, \ell_5/\hbar\}$ and $\delta_4 \leq \min\{\ell_4, \hbar \ell_6\}$, respectively. In addition, inequalities in Eqs. (C.11) and (C.12) are realized by combining with the lower bound of $r_j, j = 2, 3$ given in Eqs. (C.3) as $r_2 \geq \sqrt{(\delta_3 + \delta_5)/C_2}$ and $r_3 \geq \sqrt{(\delta_4 + \delta_6)/C_2}$.

Let us apply the well-defined CFL number as $C_g T_t/T_z$ where $C_g$ can be realized as the phase velocity of a wave [41] traveling freely across the whole plate according to the system of Eqs. (2.6) and (2.7). Now gathering together Eqs. (C.9)-(C.12), it amounts that for a maximal isotropic spatial step $T_z$ the phase velocity is thus arrived as Eq. (3.22) leading to the time step being constrained for the stability of the MDWDF network with least restriction as

\[ T_t \leq T_z/(\sqrt{2} C_g). \]

The above time step restriction explicitly leads to the stability of the MDWDF network, which is limited by the necessary CFL condition as

\[ C_g T_t/T_z \leq 1/\sqrt{2} \iff \chi \geq \sqrt{2} C_g. \tag{C.13} \]
Referring back to the leftmost equalities of Eqs. (C.9)-(C.11) and making use of the relationships \( \delta_2 = h\delta_1 \), \( \delta_5 = h\delta_3 \) and \( \delta_6 = \delta_4/h \), clearly ranges of the scaled constants \( r_j, j = 1, 2, 3 \) can be immediately located by

\[
0 < r_1 = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2 \delta_2 + \delta_5}{\delta_1 + \delta_5}} = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \delta_1 \quad (C.14)
\]

\[
\leq \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \min \{ \ell_1, \ell_2/h \} = \bar{r}_1 \quad (C.15)
\]

\[
0 < r_2 = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2 \delta_3 + \delta_5}{\delta_3 + \delta_5}} = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \delta_3 \quad (C.16)
\]

\[
\leq \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \min \{ \ell_3, \ell_5/h \} = \bar{r}_2 \quad (C.17)
\]

\[
0 < r_3 = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2 \delta_4 + \delta_6}{\delta_4 + \delta_6}} = \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \delta_4 \quad (C.18)
\]

\[
\leq \frac{\chi}{Z_M Z_Q} \sqrt{\frac{2}{1+1/h^2}} \min \{ \ell_4, h\ell_6 \} = \bar{r}_3. \quad (C.19)
\]

This completes the proof.

\[
\square
\]

### Table 1. Option parameters set for the gradient-based NLP solvers.

| Options for ASA selection | Value | Options for ASA termination |
|---------------------------|-------|-----------------------------|
| Number of objective function evaluations | 150   | Max. number of iterations | 400 |
| Max. change for finite-difference gradients | 0.1   | Tolerance on objective function | 10^{-8} |
| Min. change for finite-difference gradients | 10^{-8} | Max. change for finite-difference gradients | 0.1 |
| Finite difference type | forward | Tolerance of maxi. constraint | 10^{-6} |
|                         |       | Max. number of SQP iterations | 40 |
|                         |       | Tolerance of SQP constraint violation | 10^{-6} |

| Options for IPA selection | Value | Options for IPA termination |
|---------------------------|-------|-----------------------------|
| Number of objective function evaluations | 150   | Max. number of iterations | 1000 |
| Initial barrier value | 0.1   | Max. change for finite-difference gradients | 0.1 |
| Max. change for finite-difference gradients | 10^{-8} | Tolerance of objective function | 10^{-8} |
| Min. change for finite-difference gradients | 0.1   | Max. change for finite-difference gradients | 10^{-8} |
| Finite difference type | forward | Tolerance of maxi. constraint | 10^{-6} |
|                         |       | Max. number of PCG iterations | 2 |
|                         |       | Tolerance of PCG algorithm | 10^{-10} |

### Table 2. Option parameters set for the direct search NLP solvers.

| Options for ALGA selection | Value | Options for ALGA termination |
|---------------------------|-------|-----------------------------|
| Size of population | 30    | Max. number of iterations | 100 |
| Probability of crossover | 0.85  | Tolerance on objective function | 10^{-8} |
| Probability of mutation | 1    | Tolerance on function value | 10^{-8} |
| Size of elitism | 2     | Max. stall generations | 10^{-6} |
| Initial penalty parameter | 30   | Min. tolerance for mesh size | 10^{-6} |
| Penalty update parameter | 50   | Tolerance of nonlinear constraint | 10^{-6} |

| Options for ALPSA selection | Value | Options for ALPSA termination |
|---------------------------|-------|-----------------------------|
| Number of objective function evaluations | 500   | Max. number of iterations | 100 |
| Initial penalty parameter | 30    | Tolerance on objective function | 10^{-8} |
| Penalty update parameter | 100   | Tolerance on function value | 10^{-8} |
| Poll method | b^{*}PM | Min. tolerance for mesh size | 10^{-6} |
| Search method/Iteration limit | b^{*}SM/30 | Tolerance of nonlinear constraint | 10^{-6} |

\( a^{*}PM \): GPS positive basis 2N; \( b^{*}SM \): Latin hypercube search
Table 3. The optimum value of $\chi_{\text{min}}$ and its corresponding optimization processes performed by the gradient-based methods (ASA and IPA) with the algorithm termination options listed in Table 1 for the four-layer cross-ply \([0^\circ/90^\circ/90^\circ/0^\circ]\) laminated square plates with $E_1/E_2 = 10, 20, 30, 40$. 

| $E_1/E_2$ | Iter. | F-count | $\chi$ | Max. constr. | $1^{st}$ opt. | Iter. | F-count | $\chi$ | Max. constr. | $1^{st}$ opt. |
|-----------|-------|---------|-------|-------------|--------------|-------|---------|-------|-------------|--------------|
| 10        | 0     | 2       | 2.53311 | 0.0001187  | Infeasible   | 0     | 2       | 3.523114 | 0.5000       | Infeasible   |
|           | 1     | 4       | 29.1438  | -0.0007862 | 3.86         | 1     | 5       | 3.239829 | 0.4950       | 0.2833      |
|           | 2     | 6       | 16.995   | -0.0002634 | 2.22         | 5     | 16      | 2.559054 | 2.486e-2    | 0.1092      |
|           | 3     | 8       | 12.6526  | -4.356e-5  | 0.531        | 9     | 25      | 5.851129 | 5.661e-3    | 3.331e-2    |
|           | 4     | 10      | 11.9075  | -1.283e-6  | 6.50e-2     | 13    | 33      | 12.04055 | 1.342e-1    | 1.866       |
|           | 5     | 12      | 11.8842  | -1.256e-9  | 2.19e-3     | 17    | 41      | 11.88414 | 3.321e-6    | 3.957e-5    |
|           | 6     | 14      | 11.8841* | -1.225e-15 | 2.37e-6     | 20    | 52      | 11.88414*| 2.196e-13   | 0           |
| 20        | 0     | 2       | 2.53311  | 0.0002611  | Infeasible   | 0     | 2       | 3.523114 | 0.5000       | Infeasible   |
|           | 1     | 4       | 53.9773  | -0.005372  | 27.4         | 1     | 5       | 3.24753  | 0.4950       | 0.2794      |
|           | 2     | 6       | 29.4623  | -0.00122   | 4.2          | 5     | 16      | 2.574543 | 4.123e-2    | 9.694e-2    |
|           | 3     | 8       | 19.2631  | -2.112e-4  | 1.59         | 9     | 25      | 9.093263 | 2.175e-3    | 1.320e-2    |
|           | 4     | 10      | 16.5631  | -1.48e-5   | 0.184        | 13    | 33      | 16.41998 | 8.452e-2    | 1.516       |
|           | 5     | 12      | 16.343   | -9.832e-8  | 1.55e-2     | 17    | 41      | 16.34150 | 2.417e-6    | 3.959e-5    |
|           | 6     | 14      | 16.3415* | -4.458e-12 | 1.05e-4     | 20    | 52      | 16.34150*| 2.654e-13   | 0           |
| 30        | 0     | 2       | 2.53311  | 0.0004084  | Infeasible   | 0     | 2       | 3.523114 | 0.5000       | Infeasible   |
|           | 1     | 4       | 78.4373  | -0.01117   | 8.75         | 1     | 5       | 3.247713 | 0.4950       | 0.2754      |
|           | 2     | 6       | 41.7109  | -0.002616  | 5.44         | 5     | 16      | 2.589383 | 5.592e-2    | 8.980e-2    |
|           | 3     | 8       | 25.542   | -5.07e-4   | 2.63         | 9     | 26      | 11.41252 | 3.059       | 8.636       |
|           | 4     | 10      | 20.4244  | -5.079e-5  | 0.486        | 13    | 36      | 21.52822 | 1.733       | 8.5         |
|           | 5     | 12      | 19.7832  | -7.972e-7  | 3.76e-2     | 17    | 44      | 19.77675 | 4.03e-3     | 1.526e-2    |
|           | 6     | 14      | 19.7728  | -2.093e-10 | 6.29e-4     | 20    | 50      | 19.77282 | 2.0e-6      | 3.96e-5     |
|           | 7     | 16      | 19.7728* | -1.789e-17 | 1.92e-7     | 23    | 58      | 19.77282*| 7.136e-9    | 2.069e-14   |
| 40        | 0     | 2       | 2.53311  | 0.0005573  | Infeasible   | 0     | 2       | 3.523114 | 0.5000       | Infeasible   |
|           | 1     | 4       | 102.773  | -0.01904   | 20.1         | 1     | 5       | 3.251616 | 0.4950       | 0.2715      |
|           | 2     | 6       | 53.8883  | -0.004527  | 5.94         | 5     | 16      | 2.603146 | 6.948e-2    | 8.390e-2    |
|           | 3     | 8       | 31.7156  | -9.314e-4  | 3.55         | 9     | 26      | 13.44820 | 3.121       | 10.49       |
|           | 4     | 10      | 23.9651  | -1.183e-4  | 0.874        | 13    | 36      | 24.51400 | 1.666       | 9.295       |
|           | 5     | 12      | 22.7118  | -2.976e-6  | 6.42e-2     | 17    | 44      | 22.68110 | 4.026e-3    | 1.542e-2    |
|           | 6     | 14      | 22.6772  | -2.266e-9  | 1.87e-3     | 20    | 50      | 22.67716 | 1.754e-6    | 3.960e-5    |
|           | 7     | 16      | 22.6772* | -1.351e-15 | 1.28e-6     | 22    | 54      | 22.67716*| 8.846e-9    | 3.960e-7    |
Table 4. Key results measured by the first 800 temporal steps for the study of network robustness and optimality based on the ASA solver w.r.t the four-layer cross-ply layup \([0^\circ/90^\circ/90^\circ/0^\circ]\) laminated square plate with \(E_1/E_2 = 10, 20\).

| \(E_1/E_2\) | Model(\(\chi\)) | \(T_i(\text{ms})\) | CFL no. | \(L_1\) | \(L_4\) | \(L_5\) | \(E_{L_{\text{avg}}} (%)\) | \(E_{L_{\text{avg}}} (%)\) | Div. step |
|--------------|----------------|-----------------|---------|---------|---------|---------|----------------|----------------|-----------|
| \(1(\chi_{\text{min}} - 3.0\%)\) | 4.33 | 0.1532 | 1.18 | -0.19 | 49.95 | [-45.2471] | [-22.2565] | 108 |
| \(2(\chi_{\text{min}} - 5.0\%)\) | 4.27 | 0.1530 | 1.22 | -0.10 | 51.90 | [-63.1224] | [-54.1263] | 192 |
| \(3(\chi_{\text{min}} - 1.0\%)\) | 4.24 | 0.1522 | 1.23 | -0.06 | 52.67 | [-54.454] | [-37.499] | 311 |
| \(4(\chi_{\text{min}}\) | 4.21 | 0.1507 | 1.26 | 0 | 54.03 | reference | X | 96 |
| \(5(\chi_{\text{min}} + 1.0\%)\) | 4.16 | 0.1492 | 1.28 | 0.07 | 55.43 | [-47.12] | [-36.22] | X |
| \(6(\chi_{\text{min}} + 1.5\%)\) | 4.14 | 0.1485 | 1.30 | 0.10 | 56.15 | [-54.15] | [-39.44] | X |
| \(7(\chi_{\text{min}} + 3.0\%)\) | 4.08 | 0.1462 | 1.34 | 0.20 | 58.37 | [-42.19] | [-40.89] | X |
| \(8(\chi_{\text{min}} + 1.5\%)\) | 4.02 | 0.1402 | 1.17 | 0.13 | 46.5 | [-30.32] | [-30.23] | X |
| \(9(\chi_{\text{min}} + 3.0\%)\) | 1.05 | 0.0376 | 21.0 | 48.94 | 1089 | [-30.32] | [-30.24] | X |

\[
L_j = L_j \times 10^4, j = 1, 4, 5. E_1/E_2 = 10 : \chi_{\text{min}} = 11.8841, \quad E_1/E_2 = 20 : \chi_{\text{min}} = 16.3415.
\]

Table 5. Key results measured by the first 800 temporal steps for the study of network robustness and optimality based on the ASA solver w.r.t the four-layer cross-ply layup \([0^\circ/90^\circ/90^\circ/0^\circ]\) laminated square plate with \(E_1/E_2 = 30, 40\).

| \(E_1/E_2\) | Model(\(\chi\)) | \(T_i(\text{ms})\) | CFL no. | \(L_1\) | \(L_4\) | \(L_5\) | \(E_{L_{\text{avg}}} (%)\) | \(E_{L_{\text{avg}}} (%)\) | Div. step |
|--------------|----------------|-----------------|---------|---------|---------|---------|----------------|----------------|-----------|
| \(1(\chi_{\text{min}} - 3.0\%)\) | 2.61 | 0.0933 | 21.68 | -0.45 | 299.1 | [-46.2761] | [-20.2810] | 101 |
| \(2(\chi_{\text{min}} - 5.0\%)\) | 2.57 | 0.0919 | 22.37 | -0.23 | 217.0 | [-49.1062] | [-47.1743] | 147 |
| \(3(\chi_{\text{min}} - 1.0\%)\) | 2.55 | 0.0915 | 22.61 | -0.15 | 219.7 | [-51.1777] | [-47.1231] | 100 |
| \(4(\chi_{\text{min}}\) | 2.53 | 0.0905 | 23.07 | 0 | 225.0 | reference | X | 72 |
| \(5(\chi_{\text{min}} + 1.0\%)\) | 2.50 | 0.0897 | 23.55 | 0.15 | 230.5 | [-51.12] | [-47.51] | X |
| \(6(\chi_{\text{min}} + 1.5\%)\) | 2.49 | 0.0892 | 23.79 | 0.23 | 233.3 | [-49.15] | [-51.76] | X |
| \(7(\chi_{\text{min}} + 3.0\%)\) | 2.46 | 0.0879 | 24.49 | 0.36 | 241.5 | [-47.22] | [-30.11] | X |
| \(8(\chi_{\text{min}} + 1.5\%)\) | 2.41 | 0.0862 | 146.2 | 39.8 | 1644 | [-37.36] | [-30.24] | X |
| \(9(\chi_{\text{min}} + 3.0\%)\) | 0.63 | 0.0226 | 374.9 | 113.7 | 4280 | [-34.38] | [-33.24] | X |

\[
L_j = L_j \times 10^4, j = 1, 4, 5. E_1/E_2 = 30 : \chi_{\text{min}} = 19.7728, \quad E_1/E_2 = 40 : \chi_{\text{min}} = 22.6772.
\]
Table 6. The cepstrum analysis for four-layer cross-ply layups $[0^\circ/90^\circ/90^\circ/0^\circ]$ square laminates.

| $E_1/E_2$ | $T_C$(ms) | $T_{FP}$(ms) | $\omega_f$(rad/s) | $\bar{\omega}_f$(rad/s) | $E_\omega$(%) |
|------------|-----------|--------------|-------------------|------------------------|-------------|
| 10         | 1211.6    | 2414.9       | 2.60175           | 9.83369172            | 0.195963    |
| 20         | 985.2     | 1964.3       | 3.24926           | 12.28105278           | 0.823283    |
| 30         | 852.1     | 1699.3       | 3.69751           | 13.97527101           | 0.599417    |
| 40         | 782.7     | 1561.0       | 4.02499           | 15.21304757           | 0.462573    |

$E_\omega$(%) uses the TSDT-FEM results [36] as the reference.

Table 7. Nondimensionalized nature frequencies based on the ASA solver for the first six modes of the four-layer cross-ply layup $[0^\circ/90^\circ/90^\circ/0^\circ]$ square laminates.

| $E_1/E_2$ | Mode($\alpha,\beta$) | 1(1.1) | 2(1.2) | 3(2.1) | 4(2.2) | 5(1.3) | 6(2.3) |
|------------|-----------------------|--------|--------|--------|--------|--------|--------|
| 10         | 9.83369172            | 18.81513016 | 27.66930905 | 33.264721086 | 34.20327106 | 44.7972894 |
| 20         | 12.28105278           | 22.82830988 | 32.74947498 | 38.428878791 | 40.0837813 | 51.06332472 |
| 30         | 13.97527101           | 25.11064738 | 35.17371580 | 41.92581303 | 45.15087598 | 53.36012568 |
| 40         | 15.21304757           | 26.79312856 | 36.38796514 | 44.14277738 | 48.08409679 | 59.83798712 |
Table 8. Numerical accuracy involving four different NLP solvers for the four-layer cross-ply laminated SS2 square plates $[0^\circ/90^\circ/90^\circ/0^\circ]$.

| Method         | NLP Algorithm | Para. | $E_{1}/E_{2}$ | 10   | 20   | 30   | 40   |
|----------------|---------------|-------|---------------|------|------|------|------|
| FSDT-MDWDF ASA| $\chi_{\text{min}}$ | 11.88413667 | 16.34149675 | 19.77282289 | 22.67715857 |
|               | $CFL_{\text{max}}$ | 0.15072042 | 0.10960942 | 0.09058808 | 0.07898617 |
|               | CPU time(ms)   | 2411.1 | 2425.98 | 2461.76 | 2502.44 |
|               | $\bar{\omega}_{f}$ | 9.83369172 | 12.28105278 | 13.97527101 | 15.21304757 |
|               | $E_{\bar{\omega}}$ (%) | 0.19596347 | 0.82328368 | 0.59941701 | 0.46257394 |
| IPA           | $\chi_{\text{min}}$ | 11.88413707 | 16.34149709 | 19.77282289 | 22.67715898 |
|               | $CFL_{\text{max}}$ | 0.15072041 | 0.10960942 | 0.09058808 | 0.07898617 |
|               | CPU time(ms)   | 2265.58 | 1745.8 | 1824.4 | 1972.86 |
|               | $\bar{\omega}_{f}$ | 9.83369205 | 12.28105303 | 13.97527129 | 15.21304784 |
|               | $E_{\bar{\omega}}$ (%) | 0.19596011 | 0.82328166 | 0.59941904 | 0.46257571 |
| ALGA          | $\chi_{\text{min}}$ | 11.88413869 | 16.34149995 | 19.77282581 | 22.67716160 |
|               | $CFL_{\text{max}}$ | 0.15072040 | 0.10960941 | 0.09058807 | 0.07898617 |
|               | CPU time(ms)   | 6262.2 | 6137.32 | 4223.16 | 4194.09 |
|               | $\bar{\omega}_{f}$ | 9.83370380 | 12.28105800 | 13.97527405 | 15.21304960 |
|               | $E_{\bar{\omega}}$ (%) | 0.19584080 | 0.82324156 | 0.59943894 | 0.46259008 |
| ALPSA         | $\chi_{\text{min}}$ | 11.88415178 | 16.34150453 | 19.77282756 | 22.67716233 |
|               | $CFL_{\text{max}}$ | 0.15072013 | 0.10960929 | 0.09058806 | 0.07898616 |
|               | CPU time(ms)   | 2924.57 | 2781.15 | 2593.6 | 2857.36 |
|               | $\bar{\omega}_{f}$ | 9.83370423 | 12.28105862 | 13.97527458 | 15.21305002 |
|               | $E_{\bar{\omega}}$ (%) | 0.19583654 | 0.82324156 | 0.59944269 | 0.46259008 |
| FSDT-GRBF [36] | $\bar{\omega}_{f}$ | 9.539 | 11.977 | 13.716 | 15.059 |
|               | $E_{\bar{\omega}}$ (%) | 3.18684664 | 3.27868852 | 1.26691621 | 0.55471175 |
| FSDT-EFG [7]  | $\bar{\omega}_{f}$ | 9.670 | 12.115 | 13.799 | 15.068 |
|               | $E_{\bar{\omega}}$ (%) | 1.85730234 | 1.00506618 | 0.66945004 | 0.49527834 |
| FSDT-FEM [36] | $\bar{\omega}_{f}$ | 9.841 | 12.218 | 13.864 | 15.107 |
|               | $E_{\bar{\omega}}$ (%) | 0.12179031 | 0.16342539 | 0.20155485 | 0.2373360 |
| TSDT-EFG [7]  | $\bar{\omega}_{f}$ | 9.842 | 12.218 | 14.154 | 15.145 |
|               | $E_{\bar{\omega}}$ (%) | 0.11164112 | 0.16342530 | 1.90757270 | 0.01320742 |
| TSDT-FEM [36] | $\bar{\omega}_{f}$ | 9.853 | 12.238 | 13.892 | 15.143 |

$E_{\bar{\omega}}$ (%) uses the TSDT-FEM results [36] as the reference.
Table 9. Optimization process performed by the direct search methods (ALGA and ALPSA) with the algorithm termination options for the four-layer cross-ply scheme $[0^\circ/90^\circ/90^\circ/0^\circ]$ and various types of stiffness ratio $E_1/E_2 = 10, 20, 30, 40$.

| $E_1/E_2$ | ALGA | | | ALPSA | | |
|-----------|-------|------------------|---|-------|------------------|---|
|           | Gen. | F-count | $\chi$ | Max. constr. | Iter. | F-count | $\chi$ | Max. constr. | Mesh size |
| 10        | 1    | 1066    | 8.28321 | 1.577e-4 | 0     | 1      | 3.79967 | 2.088e-4 | 1 |
| 3         | 3146 | 2.53318 | 1.187e-3 | 1     | 94    | 5.37203 | 2.596e-4 | 1.074e-3 | 3.333e-4 |
| 5         | 5226 | 2.53311 | 1.187e-3 | 2     | 343   | 14.091  | 0       | 3.333e-4 |
| 7         | 7306 | 2.53311 | 1.187e-3 | 3     | 451   | 11.8842 | 0       | 3.333e-6 |
| 10        | 10516| 11.8842* | |          | 4     | 501    | 11.8842* | 0       | 3.333e-8 |
| 20        | 1    | 530     | 2.53311 | 2.611e-3 | 0     | 1      | 3.79967 | 2.108e-3 | 1 |
| 3         | 1570 | 2.53311 | 2.611e-3 | 1     | 61    | 24.7257 | 0       | 3.333e-4 |
| 5         | 2610 | 2.53311 | 2.611e-3 | 2     | 262   | 16.3415 | 0       | 3.333e-6 |
| 7         | 3668 | 16.4907 | 0       | 3     | 433   | 16.3415* | 0       | 3.333e-8 |
| 10        | 5269 | 16.3415* | |          | 4     | 501    | 11.8842* | 0       | 3.333e-8 |
| 30        | 0    | 0       | 2.53311 | Infeasible | 0     | 1      | 2.53311 | 4.084e-3 | 1 |
| 1         | 1072 | 24.6611 | 0       | 1     | 25    | 43.6032 | 0       | 3.333e-4 |
| 2         | 2138 | 19.7728 | 0       | 2     | 266   | 19.7728 | 0       | 3.333e-6 |
| 3         | 3190 | 19.7728* | |          | 3     | 437    | 19.7728* | 0       | 3.333e-8 |
| 40        | 0    | 0       | 2.53311 | Infeasible | 0     | 1      | 2.53311 | 5.573e-3 | 1 |
| 1         | 1072 | 28.3387 | 0       | 1     | 29    | 48.9484 | 0       | 3.333e-4 |
| 2         | 2138 | 22.6772 | 0       | 2     | 258   | 22.6772 | 0       | 3.333e-6 |
| 3         | 3190 | 22.6772* | |          | 3     | 441    | 22.6772* | 0       | 3.333e-8 |

Table 10. Numerical accuracy of MDWDF networks for the three-layer cross-ply $[0^\circ/90^\circ/0^\circ]$ laminated plate.

| Method | NLP | Algorithm | Para. | $E_1/E_2$ | 10 | 20 | 30 | 40 |
|--------|-----|-----------|-------|-----------|----|----|----|----|
| FSDT-MDWDF ASA $\chi_{\text{min}}$ | 12.9150498 | 17.80436624 | 21.46965556 | 24.53187819 |
| $\omega_f$ | 9.75843249 | 12.09708139 | 13.65712101 | 14.70474902 |
| $E(\%)$ | 0.35793863 | 0.37405733 | 0.59012307 | 0.41481091 |
| IPA $\chi_{\text{min}}$ | 12.91506538 | 17.80436635 | 21.46965596 | 24.53187859 |
| $CFL_{\text{max}}$ | 0.14077503 | 0.10211645 | 0.08468317 | 0.07411249 |
| $\omega_f$ | 9.7584379 | 12.09708146 | 13.65712126 | 14.70474926 |
| $E(\%)$ | 0.35793557 | 0.37406920 | 0.59014307 | 0.41480929 |
| ALGA $\chi_{\text{min}}$ | 12.91506688 | 17.80436836 | 21.46965818 | 24.53190153 |
| $CFL_{\text{max}}$ | 0.14077501 | 0.10211643 | 0.08468316 | 0.07411243 |
| $\omega_f$ | 9.75844939 | 12.09708282 | 13.65712267 | 14.70476301 |
| $E(\%)$ | 0.35792393 | 0.37406920 | 0.59013530 | 0.41471617 |
| ALPSA $\chi_{\text{min}}$ | 12.91825286 | 17.80436836 | 21.46965842 | 24.53190155 |
| $CFL_{\text{max}}$ | 0.14074029 | 0.10208235 | 0.08468306 | 0.07411238 |
| $\omega_f$ | 9.76135232 | 12.10112200 | 13.65713962 | 14.70476302 |
| $E(\%)$ | 0.33334367 | 0.00407583 | 0.00590260 | 0.41471610 |

$HSDT$-FEM [20] \( \omega \) 9.794 12.052 13.577 14.766

$E_{\text{res}}(\%)$ uses the HSDT-FEM results [20] as the reference.
| NLP Algorithm | \( \frac{E_1}{E_2} \) | 10    | 20    | 30    | 40    |
|---------------|----------------|-------|-------|-------|-------|
| ASA \( \chi_{\text{min}} \) | 11.12198459 | 15.31530671 | 18.56722410 | 21.43934366 |
| ASA \( \text{CFL}_{\text{max}} \) | 0.15957798 | 0.11588562 | 0.09558908 | 0.08375115 |
| ASA \( \bar{\omega}_f \) | 9.53036311 | 11.97168174 | 13.53757952 | 5.46066961 |
| IPA \( \chi_{\text{min}} \) | 11.12198460 | 15.31530709 | 18.56722348 | 21.32430073 |
| IPA \( \text{CFL}_{\text{max}} \) | 0.15957798 | 0.11588562 | 0.09558908 | 0.08323011 |
| IPA \( \bar{\omega}_f \) | 9.53036311 | 11.97168174 | 13.53757952 | 5.46066961 |
| ALGA \( \chi_{\text{min}} \) | 11.12198680 | 15.31531111 | 18.56722310 | 21.32430035 |
| ALGA \( \text{CFL}_{\text{max}} \) | 0.15957795 | 0.11588559 | 0.09558908 | 0.08323011 |
| ALGA \( \bar{\omega}_f \) | 9.53036500 | 11.97168518 | 13.69778682 | 15.02075202 |
| ALPSA \( \chi_{\text{min}} \) | 11.12199160 | 15.31530987 | 18.56722509 | 21.32430176 |
| ALPSA \( \text{CFL}_{\text{max}} \) | 0.15957788 | 0.11588560 | 0.09558907 | 0.08323010 |
| ALPSA \( \bar{\omega}_f \) | 9.53036911 | 11.97168421 | 13.69778829 | 15.02075301 |
Figure 1. A schematic flow diagram towards modeling a general MDWDF network.

Figure 2. Geometry of square laminated plates with (a) three-layer cross-ply stacking sequence \([0^\circ/90^\circ/0^\circ]\) (b) four-layer cross-ply stacking sequence \([0^\circ/90^\circ/90^\circ/0^\circ]\), and (c) five-layer cross-ply stacking sequence \([0^\circ/90^\circ/90^\circ/90^\circ/0^\circ]\).

Figure 3. A MDKC representation for a symmetrically laminated composite FSDT plate with free vibration.
Figure 4. A MDWDF network for numerical simulation of the laminated plate system (2.6)-(2.7).

Figure 5. Key parameters obtained by the ASA solver for the four-layer cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ square plate with stiffness ratios $E_1/E_2 = 10, 20, 30, 40$. (a1)-(d1): Objective function value ($\chi$). (a2)-(d2): The maximum constraint violation corresponding to the objective value. (a3)-(d3): The first-order optimality.
Figure 6. MDWDF network robustness indicated by the percentage error distribution ($E_{LSE}(\%)$) w.r.t. various scales of CFL number for four stiffness ratios $E_1/E_2 = 10, 20, 30, 40$ where model 4 of $\chi_{min}$ is used as a reference.

Figure 7. MDWDF network stability indicated by the percentage error distribution ($E_{KE}(\%)$) w.r.t. various scales of CFL number for four stiffness ratios $E_1/E_2 = 10, 20, 30, 40$. 
Figure 8. Vibration waveform and its corresponding power cepstrum for four-layer cross-ply layup \([0^\circ/90^\circ/90^\circ/0^\circ]\) square laminates with the stiffness ratios \(E_1/E_2 = 10, 20\).

Figure 9. Vibration waveform and its corresponding power cepstrum for four-layer cross-ply layup \([0^\circ/90^\circ/90^\circ/0^\circ]\) square laminates with the stiffness ratios \(E_1/E_2 = 30, 40\).
Figure 10. The first six modes vibration for the four-layer cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ SS2 square laminate.

Figure 11. MDWDF network optimality w.r.t. various scales of $\chi$ (model) and different stiffness ratios. (a) $E_1/E_2 = 10 : \bar{\omega}_f(\chi_{min}) = 9.8336$. (b) $E_1/E_2 = 20 : \bar{\omega}_f(\chi_{min}) = 12.1655$. (c) $E_1/E_2 = 30 : \bar{\omega}_f(\chi_{min}) = 13.7703$. (d) $E_1/E_2 = 40 : \bar{\omega}_f(\chi_{min}) = 15.0852$. 
Figure 12. Optimal parameters obtained for the study of network stability and optimality w.r.t. the SS2 simply supported cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ laminated plate with various stiffness ratios and $a/h = 10$: (a) Optimal CFL number. (b) Optimal $\bar{\omega}_f$.

Figure 13. Key parameters obtained by the IPA for the SS2 simply supported four-layer cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ square plate with stiffness ratios $E_1/E_2 = 10, 20$. (a1)-(d1) Objective function value ($\chi$) at every iteration. (a2)-(d2) The maximum constraint violation corresponding to the objective value. (a3)-(d3) The first-order optimality.
Figure 14. Key parameters obtained by the IPA for the SS2 simply supported four-layer cross-ply \([0^\circ/90^\circ/90^\circ/0^\circ]\) square plate with stiffness ratios \(E_1/E_2 = 30, 40\). (a1)-(d1) Objective function value (\(\chi\)) at every iteration. (a2)-(d2) The maximum constraint violation corresponding to the objective value. (a3)-(d3) The first-order optimality.

Figure 15. Key parameters obtained by the ALGA for the SS2 simply supported four cross-ply \([0^\circ/90^\circ/90^\circ/0^\circ]\) square plate with various stiffness ratios \(E_1/E_2 = 10, 20, 30, 40\). (a1)-(d1) Score histograms at every iteration w.r.t. number of individuals. (a2)-(d2) The corresponding fitness values.
Figure 16. Key parameters obtained by the ALPSA for the SS2 simply supported four-layer cross-ply \([0^\circ/90^\circ/90^\circ/0^\circ]\) square plate with stiffness ratios \(E_1/E_2 = 10, 20, 30, 40\). (a1)-(d1) Score histograms at every iteration w.r.t. number of functions evaluated. (a2)-(d2) The corresponding function values.

Figure 17. Feasible comparisons in terms of performance measured by \(E_\omega(\%)\) among (a) nonlinear optimization solvers within the MDWDF network w.r.t. stiffness ratios \(E_1/E_2 = 10, 20, 30, 40\), (b) the CPU runtime of NLP solvers w.r.t stiffness ratios, and (c) MDWDF networks and GRBF method.
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