COHERENT STATES AND GEOMETRY ON THE SIEGEL-JACOBI DISK

STEFAN BERCEANU

ABSTRACT. The coherent state representation of the Jacobi group $G_1^J$ is indexed with two parameters, $\mu(=1/\hbar)$, describing the part coming from the Heisenberg group, and $k$, characterizing the positive discrete series representation of SU(1,1). The Ricci form, the scalar curvature and the geodesics of the Siegel-Jacobi disk $D_1^J$ are investigated. The significance in the language of coherent states of the transform which realizes the fundamental conjecture on the Siegel-Jacobi disk is emphasized. The Berezin kernel, Calabi’s diastasis, the Kobayashi embedding, and the Cauchy formula for the Siegel-Jacobi disk are presented.

CONTENTS

1. Introduction
2. Coherent states - an introduction
3. The coherent states - a bridge between quantum mechanics and geometry
4. Coherent states on the Siegel-Jacobi disk
   4.1. The generators of the Lie algebra $g^J_1$
   4.2. Formulas for the Heisenberg group $H_1$
   4.3. Formulas for SU(1,1)
   4.4. The differential action
   4.5. The Jacobi group $G_1^J$
5. Geometry on the Siegel-Jacobi disk
   5.1. The symmetric Fock space
   5.2. Two-forms
   5.3. Geodesics
   5.4. Embeddings
6. Appendix
   Acknowledgments
   References

1991 Mathematics Subject Classification. 81R30,32Q15,53C22,81V80,81S10.

Key words and phrases. Coherent states, representations of coherent state Lie algebras, Jacobi group, geodesics, embeddings.
1. Introduction

The Jacobi group of index \( n \), \( G^J_n \), is the semidirect product of the symplectic group \( \text{Sp}(n, \mathbb{R}) \) with the real \((2n+1)\)-dimensional Heisenberg group \( H_n \). The points of the homogeneous domain \( \mathcal{D}^J_n \) associated to the Jacobi group \( G^J_n \), called the Siegel-Jacobi domain \([67, 68]\), are in \( \mathcal{D}_n \times \mathbb{C}^n \), where \( \mathcal{D}_n \) denotes the Siegel ball. The Jacobi groups are unimodular, nonreductive, algebraic groups, and the Siegel-Jacobi spaces are reductive, nonsymmetric domains associated to the Jacobi groups \([67, 68, 15]\) by the generalized Harish-Chandra embedding \([60]\). The holomorphic irreducible unitary representations of the Jacobi groups based on Siegel-Jacobi domains have been constructed \([23, 24, 63, 64, 65, 15]\). Among many other applications \([12, 13, 14, 15, 17, 18]\), the Jacobi group is responsible for the squeezed states \([42, 62, 49, 70, 41]\) in quantum optics \([52, 1, 61, 36]\).

We have associated to the Jacobi group \( G^J_n \) coherent states in the meaning of Perelomov \([57]\), based on the Siegel-Jacobi space \( \mathcal{D}^J_n \). Similar constructions have been considered in \([17, 58]\). In \([12]\) we have defined coherent states based on the Siegel-Jacobi disk \( \mathcal{D}^J_1 \). In the present paper we revisit our previous construction in \([12]\), introducing a parameter \( \mu \in \mathbb{R} \) to describe the holomorphic representation associated with the Heisenberg group \( H_1 \), while the integer \( k \) indexes the holomorphic positive discrete series of \( \text{SU}(1,1) \) \([2]\). The standard realization of the position-momentum operators \( \hat{q} = q \), \( \hat{p} = -i \hbar \frac{\partial}{\partial q} \) in quantum mechanics corresponds to \( \mu = \frac{1}{\hbar} \) \([32]\).

In this paper we look at several results obtained in our previous publications devoted to the Jacobi group \([12]-[18]\) from the point of view of a program of investigation of differential geometry using the coherent states defined on symmetric domains \([4]-[8]\), this time considering coherent states based on the Siegel-Jacobi spaces.

In order to make the paper self-contained, the notation adopted for coherent states is briefly recalled in \( \S 2 \). More details are given in \([11, 12]\). Our point of view on the deep relationship between coherent states and geometry is summarized in \( \S 3 \). In \( \S 4 \) we collect basic facts on the Jacobi algebra \( \mathfrak{g}^J_1 \) and the Jacobi group \( G^J_1 \). Using the Bargmann transform, Remark \( 4 \) determines \( \mu = \frac{1}{\hbar} \) in order to have a representation of the Heisenberg group compatible with the standard quantum mechanics. The FC-transform, relating the normalized and the unnormalized Perelomov coherent state vectors based on the Siegel-Jacobi disk \( \mathcal{D}^J_1 \), which has an important significance below, is written down explicitly in Lemma \( 2 \). Most of the result in \( \S 5 \) devoted to the geometry of the Siegel-Jacobi disk, are new. From the homogeneous metric \( ds^2(z, w)_{k\mu} \), we determine in Proposition \( 2 \) the Ricci form, which depends only on \( w \in \mathcal{D}_1 \), and the Siegel-Jacobi disk is not an Einstein manifold. The scalar curvature of \( \mathcal{D}^J_1 \) is constant and negative. Similar results where obtained also in \([69]\), working on the Siegel-Jacobi upper half plane. In Proposition \( 8 \) it is observed that the change of coordinates that realizes the fundamental conjecture \([66, 38, 34]\) for the Siegel-Jacobi disk \([16, 18, 17]\) has a significance in the context of coherent states. A discussion of the geodesics of the Siegel-Jacobi disk is presented in \( \S 5.3 \). The Berezin kernel, Calabi’s diastasis, the Kobayashi embedding, and the Cauchy formula for the Siegel-Jacobi disk are calculated in \( \S 5.4 \). The paper ends with an appendix containing the definition of the reductive homogeneous spaces and a few other notions of differential geometry which appear in the body of the paper. In \([16, 17, 18]\) we have presented some physical systems based
COHERENT STATES AND GEOMETRY ON THE SIEGEL-JACOBI DISK 3

In this paper the Hilbert space $H$ is endowed with a scalar product antilinear in the first argument, $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$, $x, y \in H, \lambda \in \mathbb{C} \setminus 0$. We denote the imaginary unit $\sqrt{-1}$ by $i$, and the Real and Imaginary part of a complex number by $\Re$ and respectively $\Im$, i.e. we have for $z \in \mathbb{C}$, $z = \Re z + i \Im z$.

2. Coherent states - an introduction

In order to fix the notation on coherent states [57], let us consider the triplet $(G, \pi, \mathfrak{h})$, where $\pi$ is a continuous, unitary, irreducible representation of the Lie group $G$ on the separable complex Hilbert space $H$.

For $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, let us define the (unbounded) operator $d\pi(X)$ on $H$ by $d\pi(X).v := \frac{d}{dt}\big|_{t=0} \pi(\exp tX).v$, whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra $\mathfrak{g}$ on the smooth vectors $H^\infty$ of $H$, the derived representation, and we denote $X.v := d\pi(X).v$ for $X \in \mathfrak{g}, v \in H^\infty$.

Let us now denote by $H$ the isotropy group with Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$. We consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$ [57]. We consider manifolds $M$ which are CS-orbits, i.e. which admit a holomorphic embedding $\iota_M : M \hookrightarrow \mathbb{P}(\bar{H}^\infty)$ [18, 55, 11].

It can be shown [10] that the CS-manifolds are reductive spaces. Let us denote by $m$ the vector space orthogonal to the Lie algebra $\mathfrak{h}$ of $H$ in $\mathfrak{g}$, i.e. we have the vector space decomposition $\mathfrak{g} = \mathfrak{h} + m$. The tangent space to $M$ at $o$ can be identified with $m$ and if $\exp : \mathfrak{g} \rightarrow G$ is the exponential mapping, then $G/H = \exp(m)$, see definition at p. 104 in [40] and the Appendix.

We can introduce the normalized (unnormalized) vectors $e_x$ (respectively, $e_z$) defined on $G/H$

$$e_x = \exp\left(\sum_{\phi \in \Delta_+} x_\phi X^+_\phi - \bar{x}_\phi X^-_\phi\right)e_0, \quad e_z = \exp\left(\sum_{\phi \in \Delta_+} z_\phi X^+_\phi\right)e_0,$$

where $e_0$ is the extremal weight vector of the representation $\pi$, $\Delta_+$ are the positive roots of the Lie algebra $\mathfrak{g}$ of $G$, and $X_\phi, \phi \in \Delta$, are the generators. $X^+_\phi$ ($X^-_\phi$) corresponds to the positive (respectively, negative) generators. See details in [57, 11].

Let us denote by FC the change of variables $x \rightarrow z$ in formula (2.1) such that

$$e_x = \tilde{e}_z, \quad \tilde{e}_z := (e_z, e_z)^{-\frac{1}{2}} e_z, \quad z = FC(x).$$

The reason for calling the transform (2.2) FC (fundamental conjecture) is explained later, see Proposition 3.

The coherent vector mapping $\varphi$ is defined locally, on a coordinate neighborhood $V_0$ (cf. [10, 11]):

$$\varphi : M \rightarrow \mathfrak{h}, \quad \varphi(z) = e_z,$$

where $\mathfrak{h}$ denotes the Hilbert space conjugate to $H$. The vectors $e_z \in \mathfrak{h}$ indexed by the points $z \in M$ are called Perelomov’s coherent state vectors. Using Perelomov’s coherent
vectors, we consider Berezin’s approach to quantization of Kähler manifolds with the supercomplete sets of vectors [19]-[22] in the formulation of Rawnsley [59, 26, 27].

Let us denote by $\mathfrak{F}_\delta = L^2_{hol}(M, d\nu_M)$ the space of holomorphic, square integrable functions with respect to the scalar product on $M$

$$(2.4) \quad (f, g)_{\mathfrak{F}_\delta} = \int_M \bar{f}(z)g(z) \, d\nu_M(z, \bar{z}), \quad d\nu_M(z, \bar{z}) = \frac{\Omega_M(z, \bar{z})}{K_M}, \quad K_M = (e_z, e_{\bar{z}}).$$

The positive real function $K_M = K_M(z, \bar{z})$ in (2.4), also denoted $K(z)$, is the Bergman kernel, expressed as the scalar product of coherent states based on the Kähler homogeneous manifold $M = G/H$, $\Omega_M$ is the normalized $G$-invariant volume form

$$(2.5) \quad \Omega_M := (-1)^\binom{n}{\frac{n}{2}} \frac{1}{n!} \omega \wedge \ldots \wedge \omega, $$

and the $G$-invariant Kähler two-form $\omega$ on the $2n$-dimensional manifold $M = G/H$ with Kähler potential

$$(2.6) \quad f(z, \bar{z}) = \ln K(z, \bar{z})$$

is given by

$$(2.7) \quad \omega_M(z) = i \sum_{\alpha, \beta \in \Delta_+} h_{\alpha\bar{\beta}}(z) \, dz_\alpha \wedge d\bar{z}_\beta,$$

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln(e_z, e_{\bar{z}}), \quad h_{\alpha\bar{\beta}} = \bar{h}_{\beta\alpha}. $$

The hermitian (Bergman) metric of $M$ in local coordinates is

$$(2.9) \quad ds^2_M(z, \bar{z}) = \sum_{\alpha, \beta} h_{\alpha\bar{\beta}} \, dz_\alpha \otimes d\bar{z}_\beta = \sum_{\alpha, \beta} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln(K_M(z, \bar{z})) \, dz_\alpha \otimes d\bar{z}_\beta.$$

and the condition of the metric to be a Kählerian one is (cf. (6) p. 156 in [45])

$$(2.10) \quad \frac{\partial h_{\alpha\beta}}{\partial z_\gamma} = \frac{\partial h_{\gamma\beta}}{\partial z_\alpha}, \quad \alpha, \beta, \gamma = 1, \ldots, n.$$ 

If $\{\varphi_n(z)\}_{n=1}^\infty$ is an orthonormal base of functions of $\mathfrak{F}_\delta$ and

$$(2.11) \quad K_M(z, \bar{w}) := (e_z, e_{\bar{w}}),$$

then the Bergman kernel admits the series expansion

$$(2.12) \quad K_M(z, \bar{w}) = \sum_{n}^\infty \varphi_n(z)\bar{\varphi}_n(w).$$

For compact manifolds $M$, $\mathfrak{F}_\delta$ is finite dimensional and also the sum (2.12) is finite.

Let us introduce the map $\Phi : \mathfrak{F}^* \to \mathfrak{F}_\delta$,

$$(2.13) \quad \Phi(\psi) := f_\psi, \quad f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\delta} = (e_z, \psi)_{\delta}, \quad z \in V_0 \subset M,$$
where we have identified the space $\mathcal{H}$ complex conjugate to $\mathcal{H}$ with the dual space $\mathcal{H}^\star$ of $\mathcal{H}$. (2.4) is nothing else but Parseval overcompleteness identity [22].

\[
(\psi_1, \psi_2) = \int_{M=G/H} (\psi_1, e_z)(e_{\bar{z}}, \psi_2) \, d\nu_M(z, \bar{z}).
\]

The relation (2.4) can be interpreted in the language of geometric quantization [46]. Together with the Kähler manifold $(M, \omega)$, we also consider the triple $\sigma = (L, h, \nabla)$, where $L$ is a holomorphic line bundle on $M$, $h$ is the hermitian metric on $L$ and $\nabla$ is a connection compatible with metric and the Kähler structure [9]. With respect to a local holomorphic frame for the line bundle, the metric can be given as

\[
h(s_1, s_2)(z) = \hat{h}(z) \bar{\hat{s}}_1(z) \hat{s}_2(z),
\]

where $\hat{s}_i$ is a local representing function for the section $s_i$, $i = 1, 2$, and $\hat{h}(z) = (e_z, e_{\bar{z}})^{-1}$. (2.4) is the local representation of the scalar product on the line bundle $\sigma$. The connection $\nabla$ has the expression $\nabla = \partial + \partial \ln \hat{h} + \bar{\partial}$. The curvature of $L$ is defined as $F(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$, and locally $F = \bar{\partial} \partial \ln \hat{h}$ [31]. The Kähler manifold $(M, \omega)$ is quantizable if there exists a triple $\sigma$ such that $F(X, Y) = -i\omega(X, Y)$ and we have (2.8).

The square of the length of a vector $X \in \mathbb{C}^n$, measured in this metric at the point $z \in M$, is [33]

\[
\tau_M^2(z, X) := \sum_{\alpha, \beta} h_{\alpha\beta}(z) X_\alpha \bar{X}_\beta.
\]

If the length $l$ of a piecewise $C^1$-curve $\gamma : [0, 1] \ni t \mapsto \gamma(t) \in M$ is defined as

\[
l(\gamma) := \int_0^1 \tau_M(\gamma(t), \gamma'(t)) \, dt,
\]

then the Bergman distance between two points $z_1, z_2 \in M$ is

\[
d_B(z_1, z_2) = \inf \{ l(\gamma) : \gamma \text{ is a piecewise curve s.t. } \gamma(0) = z_1, \gamma(1) = z_2 \}.
\]

We denote the “normalized” Bergman kernel by

\[
\kappa_M(z, z') := \frac{K_M(z, z')}{\sqrt{K_M(z)K_M(z')}} = \left(\bar{e}_z, e_{\bar{z}'}\right) = \frac{(e_z, e_{\bar{z}'})}{||e_z|| ||e_{\bar{z}'}||},
\]

Introducing in the above definition the series expansion (2.12), with the Cauchy-Schwartz inequality, we have that

\[
|\kappa_M(z, z')| \leq 1.
\]

In this paper by the Berezin kernel $b_M : M \times M \to [0, 1] \in \mathbb{R}$ we mean:

\[
b_M(z, z') = |\kappa_M(z, z')|^2.
\]
3. The coherent states - a bridge between quantum mechanics and geometry

In [4] we have advanced the proposal of a program of investigation of the deep relationship between coherent states and the geometry of the manifolds on which the coherent states are defined. We have investigated how the coherent states permit to find: 1) the geodesics; 2) the conjugate locus; 3) the cut locus; 4) the divisors; 5) the Calabi’s diastasis. Also, we wanted to find a geometric meaning of transition amplitudes and of different distances and angles in quantum mechanics for coherent states. The obtained results are established for very particular manifolds, mostly on hermitian symmetric spaces [4]-[8]. Our favorite example was the complex Grassmann manifold and its noncompact dual [6].

In [4, 5, 7, 8] we have proved that

Remark 1. For symmetric spaces the dependence \( z(t) = FC(tx) \) from (2.2) gives geodesics in \( M \) with the property that \( z(0) = p \) and \( \dot{z}(0) = x \).

Let \( \xi : \mathfrak{g} \setminus 0 \to \mathbb{P}(\mathfrak{g}) \) be the the canonical projection \( \xi(z) = [z] \). The Fubini-Study metric in the nonhomogeneous coordinates \([z]\) is the hermitian metric on \( \mathbb{C}P^\infty \) [13]

\[
\|ds^2\|_{FS}([z]) = \frac{(d\ z, d\ z)(z, z) - (d\ z, z)(z, d\ z)}{(z, z)^2}.
\]

The elliptic Cayley distance [30] between two points in the projective Hilbert space \( \mathbb{P}(\mathbb{H}) \) is defined as

\[
d_C([z_1], [z_2]) = \arccos \frac{|\kappa_M(z_1, z_2)|}{||z_1|| ||z_2||}.
\]

The Fubini-Study metric (3.1) and the Cayley distance (3.2) are independent of the homogeneous coordinates \( z \) representing \([z] = \xi(z)\).

Calabi’s diastasis [28], in the context of coherent states as used by Cahen, Gutt and Rawnsley [27], reads:

\[
D_M(z, z') = -\ln b_M(z, z') = -2 \ln |\bar{e}_z, \bar{e}_{z'}|.
\]

Let \( M \) be a homogeneous Kähler manifold \( M = G/H \) to which we associate the Hilbert space of functions \( \mathfrak{g}_M \) with respect to the scalar product (2.4). We can make the following assertions

Remark 2. Let us suppose that the Kähler manifold \( M \) admits a holomorphic embedding

\[
\iota_M : M \hookrightarrow \mathbb{C}P^\infty, \iota_M(z) = [\varphi_0(z) : \varphi_1(z) : \ldots].
\]

The Hermitian metric (2.9) on \( M \) is the pullback of the Fubini-Study metric (3.1) via the embedding (3.4), i.e.

\[
d s^2_M(z) = \iota_M^* d s^2_{FS}(\iota_M(z)).
\]

The angle defined by the normalized Bergman kernel (2.14) can be expressed via the embedding (3.4) as function of the Cayley distance (3.2)

\[
\theta_M(z_1, z_2) = \arccos |\kappa_M(z_1, \bar{z}_2)| = \arccos |\bar{e}_{z_1}, \bar{e}_{z_2}|_M = d_C(\iota_M(z_1), \iota_M(z_2)).
\]
We have also the relation
\begin{equation}
\Delta_H(z_1, z_2) \geq \theta_M(z_1, z_2).
\end{equation}

The following (Cauchy) formula is true
\begin{equation}
(\bar{e}_{z_1}, \bar{e}_{z_2})_M = (\iota_M(z_1), \iota_M(z_2))_{\mathbb{C}P^\infty}.
\end{equation}

The Berezin kernel (2.16) admits the geometric interpretation via the Cayley distance as
\begin{equation}
b_M(z_1, z_2) = \cos^2 d_C(\iota_M(z_1), \iota_M(z_2)) = 1 + \cos(2d_C(\iota_M(z_1), \iota_M(z_2)))^2.
\end{equation}

**Proof.** The assertion (3.5) is known from [43]. We introduce in the expression (3.1) of the Fubini-Study hermitian metric on $\mathbb{C}P^\infty$ the change of coordinates realized by the embedding $\iota_M$ (3.4) and we get
\begin{equation}
ds^2_{FS}(z) = \left(\sum d\bar{\varphi}_i d\varphi_i - \left| \sum d\bar{\varphi}_i \varphi_i \right|^2 \right) / \left(\sum \bar{\varphi}_i \varphi_i \right)^2.
\end{equation}

But the Bergman kernel $K$ admits the expansion (2.12)
\begin{equation}
K(z, \bar{z}) = \sum \bar{\varphi}_i(z) \varphi_i(z),
\end{equation}
and we write down (3.10) in the coordinates $z$ of the manifold $M$ as
\begin{equation}
ds^2_{FS}(z) = \frac{K(\bar{z})}{K} ds^2(z) = K \frac{d\bar{z}}{\bar{z}} (d\bar{z}) K - (d\bar{z}) K (d\bar{z}) K.
\end{equation}

In the above expression we take into account (2.6), i.e. $K = e^f$, which implies
\begin{equation}
d\bar{z} K = K \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,
\end{equation}
\begin{equation}
dz K = K \sum \frac{\partial f}{\partial z_k} dz_k d\bar{z}_j + K \sum \frac{\partial^2 f}{\partial z_k \partial \bar{z}_j} dz_k d\bar{z}_j.
\end{equation}

Now we introduce the expressions (3.12) into (3.11) and we get
\begin{equation}
ds^2_{FS}(z) = \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j,
\end{equation}

i.e. (3.5). The formulas (3.6), (3.8) are proved in [4], [5], [8]. The inequality (3.7) is proved in [25], [51].

**4. Coherent states on the Siegel-Jacobi disk**

4.1. **The generators of the Lie algebra $g_1^\prime$.** The Heisenberg group is the group with the 3-dimensional real Lie algebra isomorphic to the Heisenberg algebra
\begin{equation}
h_1 \equiv \langle is1 + \alpha a^\dagger - \bar{\alpha}a >_{s, \alpha, \bar{\alpha} \in \mathbb{R}},
\end{equation}
where $a^\dagger (a)$ are the boson creation (respectively, annihilation) operators which verify the canonical commutation relations (1.3a).

Let us also consider the Lie algebra of the group SU(1, 1):
\begin{equation}
su(1, 1) = \langle 2i \theta K_0 + yK_+ - \bar{y}K_- >_{\theta, y \in \mathbb{R}},
\end{equation}
where the generators $K_{0,+,−}$ verify the standard commutation relations (4.4b).

Now let us define the Jacobi algebra as the the semi-direct sum \[ [12]\]
\begin{align}
\mathfrak{g}_1^J := \mathfrak{h}_1 \times \mathfrak{su}(1, 1),
\end{align}
where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}_J^1$, i.e. $[\mathfrak{h}_1, \mathfrak{g}_J^1] = \mathfrak{h}_1$, determined by the commutation relations (4.4a), (4.4d):
\begin{align}
(4.4a) & \quad [a, a^\dagger] = 1, \\
(4.4b) & \quad [K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_−, K_+] = 2K_0, \\
(4.4c) & \quad [a, K_+] = a^\dagger, \quad [K_−, a^\dagger] = a, \quad [K_+, a^\dagger] = [K_−, a] = 0, \\
(4.4d) & \quad [K_0, a^\dagger] = \frac{1}{2} a^\dagger, [K_0, a] = -\frac{1}{2} a.
\end{align}

**Remark 3.** The Jacobi groups are unimodular, non-reductive, algebraic groups of Harish-Chandra type. The Siegel-Jacobi domains are reductive, non-symmetric manifolds associated to the Jacobi groups by the generalized Harish-Chandra embedding.

**Proof.** From the algebraic relations (4.4), we can conclude that the Siegel-Jacobi disk $D_1^J$ is a non-symmetric space. Indeed, $D_1^J = G_1^J / H$, where $H = U(1) \times \mathbb{C}$ is connected and the Jacobi algebra admits the decomposition of the type \[ [60]\], i.e. $\mathfrak{g}_J^1 = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is generated by $K_0$ and $1$. We have $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and the Siegel-Jacobi domains are reductive homogeneous spaces, in sense of Nomizu \[56\], but because $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ is not true, the Siegel-Jacobi disk is not a symmetric space. Also, because of the presence of the Heisenberg subalgebra in the Jacobi algebra, the Jacobi group is a non-reductive one.

From the definition of the Jacobi algebra $\mathfrak{g}_1^J$, we see that the Jacobi group is an algebraic group of Harish-Chandra type \[60\], as was underlined in \[15\] for $\mathfrak{g}_J^1$. Indeed, in the direct sum of vector spaces, $\mathfrak{g}_J^1 = \mathfrak{p}_+ + \mathfrak{t}_\mathbb{C} + \mathfrak{p}_-$, with properties $[\mathfrak{t}_\mathbb{C}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm$, and $\mathfrak{p}_+ = \mathfrak{p}_-$, we have the identification $\mathfrak{p}_+ \equiv \langle K_+, a^\dagger \rangle$, $\mathfrak{p}_- \equiv \langle K_0, 1 \rangle$, and $\mathfrak{t}_\mathbb{C} \equiv \langle K_0, 1 \rangle$, see also \[12\]. The generalized Harish-Chandra embedding of the homogeneous space $D_1^J$ into $\mathfrak{p}_+$ is determined by $gK \mapsto \zeta = (z, w)$, where $\exp \zeta = (g)_+$, as explained in detail \[13\], and the $G_1^J$-invariant structure of $D_1^J$ is determined by the natural inclusion $D_1^J \hookrightarrow P_+ = \exp \mathfrak{p}_+$.

### 4.2. Formulas for the Heisenberg group $H_1$

The displacement operator $D_\mu(\alpha)$ (or simply, $D(\alpha)$), $\mu \in \mathbb{R}_+$:
\begin{align}
D_\mu(\alpha) = \exp \sqrt{\mu} (\alpha a^\dagger - \bar{\alpha} a) = \exp(-\frac{1}{2} \mu |\alpha|^2) \exp(\sqrt{\mu} a^\dagger) \exp(-\sqrt{\mu} \bar{\alpha} a),
\end{align}
has the composition property
\begin{align}
D_\mu(\alpha_2)D_\mu(\alpha_1) = e^{i \theta_\mu(\alpha_2, \alpha_1)} D_\mu(\alpha_2 + \alpha_1), \quad \theta_\mu(\alpha_2, \alpha_1) := \mu \Im(\alpha_2 \bar{\alpha}_1).
\end{align}
The coherent states associated to the Heisenberg group $H_1$ are defined on the homogeneous space $\mathbb{C} = H_1 / \mathbb{R}$ taking in (2.2) as $e_0$ the vacuum $e_0^H$, with $\mathbf{a} e_0^H = 0$ and then (2.1) has the form
\begin{align}
\mathbf{e}_\alpha := D_\mu(\alpha) e_0 = e^{-\frac{1}{2} \mu |\alpha|^2} e_\alpha, \quad e_\alpha := e^{\sqrt{\mu} \alpha} a^\dagger e_0.
\end{align}
a particular case of the relation

\[
D_\mu(\alpha)e_z = e^{-\mu\bar{\alpha}(z+\frac{\alpha}{\bar{\alpha}})}e_{\alpha+z}.
\]

Equation (4.7) corresponds to the projective representation of the parallel transport in the Hilbert space \( F_\mu \) with scalar product (4.12); see equation (4.17) in [20], or equation (107) in [29]; see p. 47 in [37]. For the Fock representation

\[
\beta_\mu(\alpha, t) := e^{i\mu t}D_\mu(\alpha)
\]

of the Heisenberg algebra (1.1) we have to take \( d\beta_\mu(1) = i\mu \), and

\[
\beta_\mu(\alpha, t)e_z = e^{i\mu t}e^{-\mu\bar{\alpha}(z+\frac{\alpha}{\bar{\alpha}})}e_{\alpha+z},
\]

where \( \mu = 2\pi m, \ m \in \mathbb{R} \). In [12] we have taken \( \mu = 1 \).

The orthonormal base of \( n \)-vectors consists of the vectors

\[
|n> := (n!)^{-\frac{1}{2}}(a^\dagger)^n|0>; \quad <n', n> = \delta_{nn'}.
\]

Perelomov’s CS-vectors associated to the Heisenberg group, defined on \( M := H_1/\mathbb{R} = \mathbb{C} \), are

\[
e_z := e^{\sqrt{\mu}z}a^\dagger e_0 = \sum \frac{(\sqrt{\mu}z)^n}{(n!)^{1/2}}|n>.
\]

The corresponding holomorphic functions associated to (4.9) are (see e.g. formula (4.3) in [20], or Theorem 3.4 in [39], or equation (112) in [29]; Bargmann took \( \mu = 1 \) in equation (1.6) in [3] as we took in our previous publication [12]):

\[
f_{|n>}(z) := <e_z, |n> = \frac{(\sqrt{\mu}z)^n}{(n!)^{1/2}}.
\]

The reproducing kernel \( K_\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is

\[
K_\mu(z, z') := <e_z, e_{z'}> = \sum f_{|n>}(z)f_{|n>}(z') = e^{\mu z z'},
\]

where the vector \( e_z \) is given by (4.9), while the function \( f_{|n>}(z) \) is given by (4.10).

The homogeneous space \( \mathbb{C} = H_1/R \) has the \( H_1 \)-invariant Kähler two-form \( \omega_\mu \)

\[-i\omega_\mu(z) = \mu dz \wedge d\bar{z}.\]

The scalar product (2.4) on the Segal-Bargmann-Fock space \( \mathfrak{F}_\mu = L^2_{hol}(\mathbb{C}, \rho_\mu) \) (in [3] Bargmann works with \( \mu = 1 \)) is

\[
(f, g)_\mu = \int_\mathbb{C} \bar{f}g\rho_\mu d\Re z d\Im z, \quad \rho_\mu = \frac{\mu}{\pi} \exp \left( -\mu|z|^2 \right).
\]

Note that

**Remark 4.** Under the differential realization on \( \mathfrak{F}_\mu \) of the creation and annihilation operators

\[
a^\dagger = \sqrt{\mu} z, \quad a = \frac{1}{\sqrt{\mu}} \frac{\partial}{\partial z},
\]

\[
(f, g)_\mu = \int_\mathbb{C} \bar{f}g\rho_\mu d\Re z d\Im z, \quad \rho_\mu = \frac{\mu}{\pi} \exp \left( -\mu|z|^2 \right).
\]
the operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ are hermitian conjugate with respect to the scalar product (4.12). The standard realization

\begin{equation}
\hat{q} = q, \quad \hat{p} = -i\hbar \frac{\partial}{\partial q}
\end{equation}

in $\mathcal{H} = L^2(\mathbb{R}, d x)$ of the position and momentum operators, where

\begin{equation}
\mathbf{a} = \lambda(\hat{q} + i \hat{p}), \quad \mathbf{a}^\dagger = \lambda(\hat{q} - i \hat{p}),
\end{equation}

corresponds to the choice of $\mu$ in (4.11), (4.12) and $\lambda$ in (4.15) as

\begin{equation}
\mu \hbar = 1, \quad 2\hbar \lambda^2 = 1.
\end{equation}

Proof. It easy to verify that in the realization (4.13), we have

\begin{equation}
(a f, g)_{\mu} = (f, a^\dagger g)_{\mu}.
\end{equation}

Following Bargmann [3], who considered the case $\mu = 1$, we shall find the kernel $B$ which determines the (Bargmann) transform $\mathcal{B} : \mathcal{H} = L^2(\mathbb{R}, d x) \rightarrow \mathcal{F}_\mu$,

\begin{equation}
f(z) = \int B(z, q)\psi(q) \, dq,
\end{equation}

such that if $\mathcal{B} : \mathcal{H} \ni \psi \mapsto f \in \mathcal{F}_\mu$, then, in accord with the representation (4.13), $a^\dagger \psi \mapsto \sqrt{\mu} z f$ and $a \psi \mapsto \frac{1}{\sqrt{\mu}} \frac{\partial}{\partial z} f$. But

\begin{align}
\int \mathcal{B}(a^\dagger \psi) \, dq &= \int (a^\dagger \mathcal{B})\psi \, dq = \sqrt{\mu} z f = \sqrt{\mu} \int z B \psi \, dq; \\
\int \mathcal{B}(a \psi) \, dq &= \int (a \mathcal{B})\psi \, dq = \frac{1}{\sqrt{\mu}} \frac{\partial}{\partial z} f = \frac{1}{\sqrt{\mu}} \int \frac{\partial B}{\partial z} \psi \, dq.
\end{align}

If in (4.18) we introduce (4.15), (4.14), we get for the kernel $B$ of the Bargmann transform (4.17) the system of differential equations

\begin{align}
\sqrt{\mu} z B &= \lambda(q + \hbar \frac{\partial}{\partial q}) B; \\
\frac{1}{\sqrt{\mu}} \frac{\partial B}{\partial z} &= \lambda(q - \hbar \frac{\partial}{\partial q}) B.
\end{align}

Solving the system (4.19), we get (4.16) and the kernel $B$ of the Bargmann transform (4.17) is determined as

\begin{equation}
B(z, q) = B_0 e^{\frac{i}{\hbar}(\sqrt{2z^2} - \frac{q^2}{2})}, \quad B_0 = (\pi \hbar)^{-\frac{1}{2}}.
\end{equation}

The value of $B_0$ in (4.20) was fixed choosing the normalization constant such that

\begin{equation}
\int B(z, q)B(\bar{w}, q) \, dq = K_{\frac{1}{h}}(z, \bar{w}).
\end{equation}

Note also that the normalized solution in $\mathcal{H}$ of the equations $a \varphi_0 = 0$ is $\varphi_0(q) = (2\pi \hbar)^{-\frac{1}{4}} e^{-\frac{q^2}{2\hbar}}$ from where we obtain the values of the n-particle vectors (4.8) in $\mathcal{H}$. □
4.3. Formulas for $\text{SU}(1, 1)$. Let us denote by $S_k$ the unitary irreducible positive discrete series representation $D^+_k$ of the group $\text{SU}(1, 1)$ with Casimir operator $C = K_0^2 - K_1^2 - K_2^2 = k(k - 1)$, where $k$ is the Bargmann index for $D^+_k$ [2]. Let us introduce the notation $S_k(z) = S_k(w)$, where $w \in \mathbb{C}$, $|w| < 1$ and $z \in \mathbb{C}\setminus0$, are related by (4.21c). We have the relations:

\begin{align}
(4.21a) \quad S_k(z) &:= \exp(z K_+ - \bar{z} K_-), \\
(4.21b) \quad S_k(w) &= \exp(w K_+) \exp(\xi K_0) \exp(-\bar{w} K_-); \\
(4.21c) \quad w &= \frac{z}{|z|} \tanh(|z|), \quad \xi = \ln(1 - w\bar{w}), \quad z \neq 0,
\end{align}

and $w = 0$ for $z = 0$ in (4.21c).

Here we underline that, in accord with the Remark [1]

**Remark 5.** The change of variables in (4.21c) is of the type $FC$ in the meaning of Remark [1] as in (2.2), i.e. the dependence $w(t) = w(tz) = \frac{z}{|z|} \tanh(t|z|)$ gives geodesics for the symmetric space $\mathcal{D}_1$ with the property that $w(0) = z$ and $w(0) = 0$.

**Proof.** Indeed, we have to verify that $w(t) = \frac{z}{|z|} \tanh(t|z|)$ verifies the equation of geodesics on Siegel disk $\mathcal{D}_1$, $G_2 = 0$, where $G_2$ is given in (5.33b) below. The fact that $\mathcal{D}_1$ is a symmetric space is evident from the structure of the algebra $\mathfrak{su}(1, 1)$ given in (4.4b) which admits a decomposition of the type (6.1a) with property (6.2).

In order to construct coherent states associated to $\text{SU}(1, 1)$ defined on the Siegel disk $\mathcal{D}_1 = \text{SU}(1, 1)/\text{U}(1)$, in formula (2.2) we take $e_0 = e^K_o \equiv e_{kk}$ as the extreme vector of the $D^+_k$ representation such that $K_0 e_{kk} = k e_{kk}, k > 0$.

In the orthonormal base (4.22),

\begin{equation}
(4.22) \quad e_{k,k+m} := a_{km}(K_+)^m e_{kk}, \quad a_{km}^2 = \frac{\Gamma(2k)}{m!\Gamma(m + 2k)},
\end{equation}

Perelomov’s CS-vectors for $\text{SU}(1, 1)$, based on the unit disk $\mathcal{D}_1 = \text{SU}(1, 1)/\text{U}(1)$, are

\begin{equation}
(4.23) \quad e_z := e^{z K_+} e_0 = \sum z^n K_+^n e_0 = \sum z^n e_{k,k+n} n!a_{kn},
\end{equation}

and the corresponding holomorphic functions are (see e.g. equation 9.14 in [2])

\begin{equation}
(4.24) \quad f_{e_{kk+n}}(z) := (e_z, e_{k,k+n}) = \sqrt{\frac{\Gamma(n + 2k)}{n!\Gamma(2k)}} z^n.
\end{equation}

The reproducing kernel $K_k : \mathcal{D}_1 \times \bar{\mathcal{D}_1} \rightarrow \mathbb{C}$ is

\begin{equation}
(4.25) \quad K_k(z, z') := (e_z, e_{z'}) = \sum f_{e_{kk+m}}(z)\bar{f}_{e_{kk+m}}(z') = (1 - z\bar{z})^{-2k},
\end{equation}

where the vector $e_z$ is given by (4.23), while the function $f_{e_{kk+m}}(z)$ is given by (4.24).

The Siegel disk $\mathcal{D}_1$ has the Kähler two-form $\omega_k$

\begin{equation}
- i \omega_k(w) = \frac{2k}{(1 - w\bar{w})^2} \, dw \wedge d\bar{w},
\end{equation}

where $w \in \mathcal{D}_1$.
SU(1, 1)-invariant to the linear fractional transformation

\[ (4.26) \quad g \cdot w = \frac{aw + b}{\delta}, \quad \delta = \bar{b}w + \bar{a}, \quad SU(1, 1) \ni g = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix}, \quad \text{where } |a|^2 - |b|^2 = 1. \]

The action \((4.26)\) is transitive, and the Siegel disk \(\mathcal{D}_1\) is a homogeneous manifold. As was already mentioned in the proof of Remark 5, \(\mathcal{D}_1\) is symmetric: the solution \(g \in SU(1, 1)\) of the equation \(\sigma : \mathcal{D}_1 \ni w \mapsto -w \in \mathcal{D}_1\) is given in \((4.26)\) with \(a = \pm 1, \ b = 0\). Then \(\sigma^2 = 1\), and 0 is an isolated fixed point of \(\sigma\). By homogeneity, it can be shown that for \(\forall w \in \mathcal{D}_1\), there exists a \(\sigma_w\) such that \(\sigma_w^2 = 1\), and \(w\) is an isolated fixed point of \(\sigma_w\).

The scalar product \((2.4)\) on \(\mathcal{D}_1 = SU(1, 1)/U(1)\) in \(\mathfrak{g}_k = L_{hol}^2(\mathcal{D}_1, \rho_k)\) is (see e.g. equation 9.9 in [2])

\[ (4.27) \quad (\phi, \psi)_k = \int_{|z|<1} \bar{f}_\phi(z)f_\psi(z)\rho_k \ d\Re z \ d\Im z, \quad \rho_k = \frac{2^{k-1}-1}{\pi}(1 - |z|^2)^{2k-2}, \quad 2k = 2, 3, \ldots \]

In order to prove Proposition 1, we recall [12]

**Remark 6.** Let \(g \in SU(1, 1)\) has the form \((4.26)\).

The following relations hold:

\[ (4.28) \quad S_k(z)e_0 = (1 - |w|^2)^ke_w. \]

\[ (4.29) \quad S_k(g)e_w = (\delta)^{-2k}e_{g,w}. \]

### 4.4. The differential action.

We shall suppose that we know the derived representation \(d\pi\) of the Lie algebra \(\mathfrak{g}_1^J\) of the Jacobi group \(G_1^J\). We associate to the generators \(a, a^\dagger\) of the Heisenberg group and to the generators \(K_{0,+,+}^\dagger\) of the group SU(1, 1) the operators \(a, a^\dagger\), respectively \(K_{0,+,+}^\dagger\), where \((a^\dagger)^\dagger = a\), \(K_0^\dagger = K_0, K_{\pm}^\dagger = K_{\mp}\). We take in the definition \((2.2)\) \(e_0 = e_0^H \otimes e_0^K\), i.e. we impose to the cyclic vector \(e_0\) to verify simultaneously the conditions

\[ (4.30a) \quad ae_0 = 0, \]
\[ (4.30b) \quad K_-e_0 = 0, \]
\[ (4.30c) \quad K_0e_0 = ke_0; \quad k > 0, 2k = 2, 3, \ldots. \]

We consider in \((4.30c)\) the positive discrete series representations \(D_k^+\) of SU(1, 1) (cf. §9 in [2]).

Perelomov’s coherent state vectors associated to the group \(G_1^J\) with Lie algebra the Jacobi algebra \((4.3)\), based on the Siegel-Jacobi disk

\[ (4.31) \quad \mathcal{D}_1^J := H_1/\mathbb{R} \times SU(1, 1)/U(1) = \mathbb{C} \times \mathcal{D}_1, \]

are defined as

\[ (4.32) \quad e_{z,w} := e^{\sqrt{-1}z}a^\dagger wK^+e_0, \quad z \in \mathbb{C}, \quad |w| < 1. \]

The general scheme \([11]\) associates to elements of the Lie algebra \(\mathfrak{g}\) holomorphic first order differential operators: \(X \in \mathfrak{g} \rightarrow X \in \mathcal{D}_1\). The space of functions on which these operators act in the case of the Jacobi group will be made precise later, see \((5.1)\).
The following lemma expresses the differential action of the generators of the Jacobi algebra as operators of the type \( A_1 \), i.e. first order holomorphic differential operators with polynomial coefficients in the variables \((w, z)\) on \( D_1^J \), see [12]:

**Lemma 1.** The differential action of the generators (4.3) of the Jacobi algebra \( A_1^J \) is given by the formulas:

\[
\begin{align*}
(4.33a) & \quad \alpha = \frac{\partial}{\sqrt{\mu \partial z}}, \quad \alpha^\dagger = \sqrt{\mu} z + w \frac{\partial}{\sqrt{\mu \partial z}}, \\
(4.33b) & \quad \mathbb{K}_- = \frac{\partial}{\partial w}; \quad \mathbb{K}_0 = k + \frac{1}{2} w \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}; \\
(4.33c) & \quad \mathbb{K}_+ = \frac{1}{2} \mu z^2 + 2kw + zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w},
\end{align*}
\]

where \( z \in \mathbb{C}, |w| < 1 \).

4.5. The Jacobi group \( G_1^J \). Following [12], let us introduce the normalized coherent state vectors of the type (2.1), based on the Siegel-Jacobi disk

\[
(4.34) \quad \psi_{\eta,w} := D_\mu(\alpha)S_k(w)e_0; \quad \alpha \in \mathbb{C}, \quad w \in \mathbb{C}, |w| < 1.
\]

We find [12] a relation between the normalized vector (4.34) and the unnormalized Perelomov’s CS-vector (4.32), which play an important role in the proof of Proposition

**Lemma 2.** The vectors (4.34), (4.32), i.e.

\[
\psi_{\eta,w} := D_\mu(\eta)S_k(w)e_0; \quad e_{z,w} := \exp(\sqrt{\mu}z\alpha^\dagger + w^2K_+)e_0.
\]

are related by the relation

\[
\begin{align*}
(4.35a) & \quad \psi_{\eta,w} = (1 - w\bar{w}^k \exp(-\frac{1}{2} \mu \bar{\eta} z)e_{z,w}, \quad z = \eta - w\bar{\eta}, \\
(4.35b) & \quad e_{z,w} = (1 - w\bar{w})^{-2k} \exp(\frac{1}{2} \mu \bar{\eta} z)e_{\eta,w}, \quad \eta = \frac{z + \bar{z}w}{1 - \bar{w}w},
\end{align*}
\]

where the change of variables \((\eta, w) \mapsto (z, w)\) in (4.35)

\[
(4.36) \quad (z, w) = FC(\eta, w), \quad z = \eta - w\bar{\eta}
\]

is a FC-transform in the sense of (2.2) for coherent states defined on the Siegel-Jacobi disk \( D_1^J \).

Let us introduce the notation \( D_1^J \ni \zeta := (z, w) \in (\mathbb{C} \times D_1) \). Following the methods of [12], we get the reproducing kernel \( K : D_1^J \times D_1^J \to \mathbb{C}, \) \( K_{k\mu}(\zeta, \zeta') := (e_{z,\bar{w}}, e_{z',\bar{w}'})\):

\[
(4.37) \quad K_{k\mu}(\zeta, \zeta') = (1 - w\bar{w})^{-2k} \exp(\mu F(\zeta, \zeta'), \quad F(\zeta, \zeta') = \frac{2z^2 z + z^2 \bar{w}' + \bar{z}' \bar{w}}{2(1 - w\bar{w})}.\]

In particular, the kernel on diagonal, \( K_{k\mu}(z, \bar{w}) = (e_{z,\bar{w}}, e_{z,\bar{w}}) \), reads

\[
(4.38) \quad K_{k\mu}(z, \bar{w}) = (1 - w\bar{w})^{-2k} \exp(\mu F(z, w), \quad F(z, w) = \frac{2z^2 z + z^2 \bar{w}' + \bar{z}' \bar{w}}{2(1 - w\bar{w})},
\]

and evidently \( K_{k\mu}(z, \bar{w}) > 0, \forall (z, w) \in D_1^J \).
The holomorphic, transitive and effective action of the Jacobi group $G^J_1$ on the manifold $D^J_1$ (4.31) can be seen from the Proposition 1 proved in [12] in the case $\mu = 1$:

**Proposition 1.** Let us consider the action $S_k(g)D_\mu(\alpha)e_{z,w},$ where $g \in SU(1, 1)$ has the form (4.26), $D_\mu(\alpha)$ is given by (4.5), and the coherent state vector $e_{z,w}$ is defined in (4.32). Then we have the formula (4.39) and the relations (4.40), (4.41):

\[
S_k(g)D_\mu(\alpha)e_{z,w} = \lambda e_{z_1,w_1}, \quad \lambda = \lambda(g, \alpha; z, w),
\]

\[
z_1 = \frac{\gamma}{\delta}, \quad \gamma = z + \alpha - \bar{\alpha}w; \quad \delta = \bar{b}w + \bar{a}, \quad w_1 = g \cdot w = \frac{aw + b}{\delta},
\]

\[
\lambda = (\delta)^{-2k} \exp\left(-\frac{1}{2} \mu \lambda_1\right), \quad \lambda_1 = \bar{a}(z + \gamma) + \frac{\bar{b} \gamma^2}{\delta}.
\]

If we consider the representation $T_{k\mu}(g, \alpha, t) = S_k(g)\beta_\mu(\alpha, t)$ of the Jacobi group $G^J_1$, then $T_{k\mu}(g, \alpha, t)e_{z,w} = \lambda e^{i\mu}e_{z_1,w_1}$.

5. **Geometry on the Siegel-Jacobi disk**

5.1. **The symmetric Fock space.** In accord with the general scheme of [2] the scalar product (2.14) of functions from the space $\mathfrak{F}_{k\mu} = L^2_{hol}(D^J_1, \rho_{k\mu})$ corresponding to the kernel $K_{k\mu}$ defined by (4.38) on the manifold (4.31) is:

\[
(\phi, \psi)_{k\mu} = \int_{D^J_1} f_\phi(z, w)f_\psi(z, w)\rho_{k\mu} \, d\nu, \quad \rho_{k\mu} = \Lambda(1 - \bar{w}w)^{2k}e^{-\mu - \frac{\mu^2 + 1}{2(1 - \bar{w}w)}}.
\]

where the value of the $G^J_1$-invariant measure $d\nu$

\[
d\nu = \mu \frac{d\Re w \, d\Im w}{(1 - \bar{w}w)^3} \, d\Re z \, d\Im z
\]

is obtained in (5.19). Note that the value of the normalization constant in (5.1) is the same as in the case $\mu = 1$, i.e.

\[
\Lambda = \frac{4k - 3}{2\pi^2}.
\]

The base of orthonormal functions associated to the CS-vectors attached to the Jacobi group $G^J_1$, defined on the manifold $D^J_1$ (4.31) [12], [14]

\[
f_{n > e_{k', k' + m}}(z, w) := (e_{z, \bar{w}}|n > e_{k', k' + m}), \quad z \in \mathbb{C}, \quad |w| < 1, k = k' + \frac{1}{4}, 2k' \in \mathbb{Z}_+
\]

consists of the holomorphic polynomials

\[
f_{n > e_{k', k' + m}}(z, w) = f_{e_{k', k' + m}}(w) \frac{P_n(\sqrt{\mu}z, w)}{\sqrt{n!}},
\]

where the monomials $f_{e_{k', k' + m}}$ are defined in (4.24), while

\[
P_n(z, w) = n! \sum_{p=0}^{[\frac{n}{2}]} \frac{w^p}{p!} \frac{z^{n-2p}}{(n-2p)!},
\]
The series expansion (2.12) of the reproducing kernel (1.37) reads
\[ K_{k\mu}(z, w; \bar{z}', \bar{w}') = \sum_{n,m} f_{|n>,e_{\nu'},k'+m}(z, w) \tilde{f}_{|n>,e_{\nu'},k'+m}(z', w'). \]

5.2. Two-forms. We follow further the general prescription of §2. We calculate the Kähler potential on \( D_J^1 \) as the logarithm of the reproducing kernel \( f := \ln K_{k\mu} \), i.e.,
\[ f = \frac{\mu \bar{z} \bar{z} + z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})} - 2k \ln(1 - w\bar{w}). \]

The Kähler two-form \( \omega \) is obtained with formulas (2.7), (2.8), i.e.,
\[ -i \omega = h_{z\bar{z}} dz \wedge d\bar{z} + h_{z\bar{w}} dz \wedge d\bar{w} - h_{\bar{z}\bar{w}} d\bar{z} \wedge dw + h_{\bar{w}\bar{w}} dw \wedge d\bar{w}. \]

The volume form is:
\[ -\omega \wedge \omega = 2 \left| \begin{array}{cc} h_{z\bar{z}} & h_{z\bar{w}} \\ h_{\bar{z}w} & h_{\bar{w}\bar{w}} \end{array} \right| dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}. \]

With the Kähler potential (5.8), we get the metric coefficients in (5.9)
\[ h_{z\bar{z}} = \frac{\mu}{P}, \quad h_{z\bar{w}} = \frac{\eta}{P}, \quad h_{\bar{w}w} = \frac{\mu |\eta|^2}{P^2} + \frac{2k}{P^2}, \quad P = (1 - w\bar{w}), \]
where \( \eta \) was defined previously in (4.35).

If we introduce the notation
\[ G(z) := \det(h_{\alpha\bar{\beta}})_{\alpha,\beta=1,\ldots,n}, \]
then the Ricci form of the Bergman metric is (see p. 90 in [54])
\[ \rho_M(z) := i \sum_{\alpha,\beta=1}^n \text{Ric}_{\alpha\bar{\beta}}(z) \, dz_\alpha \wedge d\bar{z}_\beta, \quad \text{Ric}_{\alpha\bar{\beta}}(z) = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \ln G(z). \]

For the determinant \( G \) (5.12) for the Siegel-Jacobi disk we find
\[ G(z, w) = \frac{2k\mu}{(1 - w\bar{w})^3}. \]

The scalar curvature at a point \( p \in M \) of coordinates \( z \) is (see p. 294 in [44])
\[ s_M(p) = \sum_{\alpha,\beta=1}^n (h_{\alpha\bar{\beta}})^{-1} \text{Ric}_{\alpha\bar{\beta}}(z). \]

Following [50], let us introduce also the positive definite (1,1)-form on \( M = G/H \)
\[ \tilde{\omega}_M(z) := i \sum_{\alpha,\beta \in \Delta_+} \tilde{h}_{\alpha\bar{\beta}}(z) \, d z_\alpha \wedge d\bar{z}_\beta, \quad \tilde{h}_{\alpha\bar{\beta}}(z) := (n + 1)h_{\alpha\bar{\beta}}(z) - \text{Ric}_{\alpha\bar{\beta}}(z), \]
which is Kähler, with Kähler potential \( \tilde{f} = \ln(K(z)^{n+1}G(z)) \).

With (5.8)-(5.16), we obtain
Proposition 2. The Kähler two-form $\omega_{k\mu}$ on $D_1^J$, $G_1^J$-invariant to the action (1.40) is:

\begin{equation}(5.17)\end{equation}

\[-i \omega_{k\mu}(z, w) = 2k \frac{dw \wedge d\bar{w}}{P^2} + \mu \frac{A \wedge \bar{A}}{P}, \quad A = d\eta + \bar{\eta} dw, \quad \eta = \frac{z + \bar{z}w}{P}.\]

The hermitian metric on $G_1^J$ corresponding to the Kähler two-form (5.17) is:

\begin{equation}(5.18)\end{equation}

\[ds^2_k(z, w) = 2k \frac{dw \otimes d\bar{w}}{P^2} + \mu \frac{A \otimes \bar{A}}{P}.\]

The volume form (5.10) is:

\begin{equation}(5.19)\end{equation}

\[\omega_k \wedge \omega_k = 4k\mu(P)^{-3/4} d\Re z \wedge d\Im z \wedge d\Re w \wedge d\Im w.\]

The Kähler two-form (5.16) for $D_1^J$, corresponding to the Kähler potential $\tilde{f}(z, w) = 3[\mu f(z, w) - (2k + 1) \ln(1 - w\bar{w})]$ reads:

\begin{equation}(5.20)\end{equation}

\[-i \tilde{\omega}_{D_1^J} = 3 \left[(2k + 1) \frac{d w \wedge d\bar{w}}{P^2} + \frac{A \wedge \bar{A}}{P}\right].\]

The Ricci form (5.13) is:

\begin{equation}(5.21)\end{equation}

\[\rho_{D_1^J}(z, w) = -3i \frac{d w \wedge d\bar{w}}{(1 - w\bar{w})^2},\]

and $D_1^J$ is not an Einstein manifold with respect to the metric (5.18).

The scalar curvature (5.15) has the value:

\begin{equation}(5.22)\end{equation}

\[s_{D_1^J}(p) = -\frac{3}{2k}, \quad p \in D_1^J.\]

We emphasize that result (5.22) was previously obtained in [69].

Now we introduce in the expression of the one-form $A$ in (5.17) the FC-transform (4.36), $z = \eta - w\bar{\eta}$, so $A = d\eta - w \frac{d\bar{\eta}}{P}$, and then $A \wedge \bar{A} = P d\eta \wedge d\bar{\eta}$. We underline the significance of the change of coordinates FC which realizes the fundamental conjecture for the Siegel-Jacobi disk, proved in Proposition 3.1 in [18]:

Proposition 3. The FC-transform (4.36), $FC(\eta, w) = (z, w)$, which appears in Lemma 2, $z = \eta - w\bar{\eta}$, is a homogeneous Kähler diffeomorphism, i.e. $FC^*\omega_{k\mu}(z, w) = \omega_{k\mu}(\eta, w)$, where

\begin{equation}(5.23)\end{equation}

\[\omega_{k\mu}(\eta, w) = \omega_k(w) + \omega_{\mu}(\eta).\]

The Kähler two-form (5.23) is invariant to the action of $G_1^J$ on $\mathbb{C} \times D_1$, $((g, \alpha), (\eta, w)) \rightarrow (\eta_1, w_1)$, where

\begin{equation}(5.24)\end{equation}

\[\eta_1 = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}), \quad w_1 = \frac{aw + b}{bw + a}, \quad g = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \in SU(1, 1).\]
5.3. **Geodesics.** The equations of geodesics on a manifold $M$ with linear connection
with components of the linear connections $\Gamma$ are (see e.g. Proposition 7.8 p. 145 in
[44])

$$\frac{d^2 x_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{dt} \frac{dx_k}{dt} = 0, \quad i = 1, \ldots, n.$$  

$d s^2(z, \bar{z})$ from (2.9) gives the hermitian metric of $M$ in local coordinates, while
the fact the metric is Kählerian imposes the restrictions (2.10). The non-zero Christoffel’s
symbols $\Gamma$ are determined by the equations (cf. (12) at p. 156 in [45])

$$\sum_{\alpha} h_{\alpha\bar{\beta}} \Gamma^\alpha_{\beta\gamma} = \frac{\partial h_{\beta\bar{\gamma}}}{\partial z^\alpha}.$$  

In the variables $(z, w) \in (\mathbb{C}, \mathcal{D}_1)$ the equations of geodesics (5.25) read

$$\frac{d^2 z}{dt^2} + \Gamma^z_{zz} \left( \frac{dz}{dt} \right)^2 + 2 \Gamma^z_{zw} \frac{dz}{dt} \frac{dw}{dt} + \Gamma^w_{ww} \left( \frac{dw}{dt} \right)^2 = 0;$$

$$\frac{d^2 w}{dt^2} + \Gamma^w_{zw} \frac{dz}{dt} \frac{dw}{dt} + \Gamma^w_{ww} \left( \frac{dw}{dt} \right)^2 = 0.$$  

The equations (5.26) which determine the $\Gamma$-symbols for the Siegel-Jacobi disk are

$$\left\{ \begin{array}{l}
h_{zz} \Gamma^z_{zz} + h_{zw} \Gamma^w_{zz} = \frac{\partial h_{ww}}{\partial z}; \\
h_{zz} \Gamma^z_{zw} + h_{ww} \Gamma^w_{zw} = \frac{\partial h_{ww}}{\partial z}.
\end{array} \right.$$  

$$\left\{ \begin{array}{l}
h_{zw} \Gamma^z_{zw} + h_{ww} \Gamma^w_{zw} = \frac{\partial h_{ww}}{\partial z}; \\
h_{zw} \Gamma^z_{ww} + h_{ww} \Gamma^w_{ww} = \frac{\partial h_{ww}}{\partial w}.
\end{array} \right.$$  

With (5.11), we calculate easily the derivatives

$$\frac{\partial h_{zz}}{\partial z} = 0; \quad \frac{\partial h_{zw}}{\partial z} = \frac{\mu}{P^2}; \quad \frac{\partial h_{ww}}{\partial z} = \frac{\bar{\eta}}{P^2}; \quad \frac{\partial h_{ww}}{\partial w} = \frac{\mu \bar{\eta} + \eta \bar{\eta}}{P^2};$$

$$\frac{\partial h_{ww}}{\partial w} = 2 \mu \bar{\eta} \bar{\eta} \bar{w} \frac{\partial h_{ww}}{\partial w} = \frac{\mu \bar{\eta}}{P^2} + 3 \mu \bar{\eta} \bar{\eta} \bar{w} + 4 k \bar{w} \bar{w} \bar{w}.$$  

Introducing (5.31) into (5.28) (5.30), we find for the Christoffel’s symbols $\Gamma$-s the expressions

$$\Gamma^z_{zz} = - \lambda \bar{\eta}, \quad \Gamma^w_{zw} = \lambda; \quad \Gamma^z_{zw} = - \lambda \bar{\eta}^2 \frac{\bar{w}}{P};$$

$$\Gamma^w_{ww} = \lambda \bar{\eta}; \quad \Gamma^z_{ww} = - \lambda \bar{\eta}^3; \quad \Gamma^w_{ww} = \lambda \bar{\eta}^2 + 2 \frac{\bar{w}}{P}, \quad \lambda = \frac{\mu}{2k},$$

and we can formulate
Remark 7. The equations of geodesics on the Siegel-Jacobi corresponding to metric defined by \( \omega_{k\mu} \) are

\[
\begin{align*}
(5.33a) \quad \mu\eta G^2 &= 2kG_3, \quad G_1 = \frac{dz}{dt} + \frac{\eta}{P} \frac{dw}{dt}, \quad G_3 = \frac{d^2 z}{dt^2} + 2\frac{\bar{w}}{P} \frac{d z}{dt} \frac{d w}{dt}, \quad P = 1 - \bar{w}w; \\
(5.33b) \quad \mu G_2^2 &= -2kG_2, \quad G_2 = \frac{d^2 w}{dt^2} + 2\frac{\bar{w}}{P} \left( \frac{d w}{dt} \right)^2.
\end{align*}
\]

The above equations were written in the case \( \mu = 1 \) in [12].

Now we discuss the solution of the system (5.33).

a) If in the system of differential equations (5.33), we take \( \mu = 0 \), then from (5.33b) we get \( G_2 = 0 \), i.e. the equation of geodesics on \( D_1 \). The solution of the equation \( G_2 = 0 \), with \( w(0) = 0, \dot{w}(0) = B \), is given by the FC transform (4.21a), i.e.

\[
(5.34) \quad w(t) = \frac{B}{|B|} \tanh(t|B|).
\]

If we introduce the solution (5.34) of the equation \( G_2 = 0 \) of geodesics on \( D_1 \), then the solution of the differential equation \( G_3 = 0 \) with the initial condition \( z_0 = \frac{d z}{dt} \big|_{t=0}, z_1 = z(0) \) is \( z(t) = \frac{2}{B} w(t) + z_1 \), and (5.33a) is satisfied.

b) If \( \mu \neq 0 \), a particular solution \((z, w)\) of (5.33) of the system of geodesics on the Siegel-Jacobi disk \( D_1^J \) is given by \( z = \eta_0 - \bar{\eta}_0 w(t) \), and \( w = w(t) \) with \( w(0) \) given by (5.33a), and \( \eta = \eta_0 \) independent of \( t \). This particular solution has been noted already in [18] in the case \( \mu = 1 \). This is a particular case of the solution \( \eta = \eta_0 + t\eta_1 \) of equation of geodesics \( \frac{d^2 \eta}{dt^2} = 0 \) on the flat manifold \( C \) corresponding to the separation of variables as in (5.23) of \( \omega_{k\mu}(\eta, w) \). We can formulate this observation:

Remark 8. The FC transform given by (4.36) in Lemma 2 is not a geodesic mapping, but it gives geodesics \((z(t), w) = FC(\eta, w)\) on the non-symmetric space \( D_1^J \) with \( w = w(t) \), given by (5.33a), and \( \eta = \eta_0 \).

5.4. Embeddings. We recall that the homogeneous Kähler manifolds \( M = G/H \) which admit an embedding of the type given by Remark 2 are called coherent type manifolds, and the groups \( G \) are called coherent-state type groups [18 55]. We particularize Remark 2 in the case of the Siegel-Jacobi disk and we have

Remark 9. The Jacobi group \( G_1^J \) is a coherent-state type group and the Siegel-Jacobi disk \( D_1^J \) is a quantizable Kähler coherent state manifold. The Hilbert space of functions \( \mathcal{F}_k \) is the space \( \mathcal{F}_{k\mu} = L^2_{hol}(D_1^{J}, P_{k\mu}) \) with the scalar product (5.1). The Kählerian embedding \( \nu_{D_1^{J}} : D_1^{J} \hookrightarrow \mathbb{CP}^\infty \) \( \nu_{D_1^{J}} = [\Phi] = [\phi_0 : \phi_1 : \ldots : \phi_N : \ldots] \) is realized with an ordered version of the base functions \( \Phi = \left\{ f_{n,m}(z,w) \right\} \) given by (5.5), and the Kähler two-form (5.17) is the pullback of the Fubini-Study Kähler two-form (3.1) on \( \mathbb{CP}^\infty \),

\[
\omega_{k\mu} = \nu_{D_1^{J}}^* \omega_{FS}|_{\mathbb{CP}^\infty}, \quad \omega_{k\mu}(z,w) = \omega_{FS}(\left[ \phi_N(z,w) \right]).
\]

The normalized Bergman kernel (2.15) of the Siegel-Jacobi disk \( \kappa_{k\mu} \) expressed in the variables \( \zeta = (z,w), \zeta' = (z',w') \) reads

\[
(5.35) \quad \kappa_{k\mu}(\zeta, \zeta') = \kappa_k(w, \bar{w}) \exp[\mu(F(\zeta, \zeta') - \frac{1}{2}(F(\zeta) + F(\zeta'))],
\]
where \( \kappa_k(w, \bar{w}') \) is the normalized Bergman kernel for the Siegel disk \( D_1 \)

\[
(5.36) \quad \kappa_k(w, \bar{w}') = \left( \frac{(1 - |w|^2)(1 - |w'|^2)}{(1 - \bar{w}'w)^2} \right)^k,
\]

\( F(\zeta, \bar{\zeta}') \) is defined in (4.37), and \( F(\zeta) \) is defined in (4.38). The Berezin kernel of \( D_{J_1} \) is

\[
(5.37) \quad b_{k\mu}(\zeta, \zeta') = b_k(w, w') \exp \mu [2\Re F(\zeta, \bar{\zeta}') - F(\zeta) - F(\zeta')],
\]

where \( b_k(w, w') = |\kappa_k(w, \bar{w}')|^2 \).

With formula (5.7), we get for the diastasis function on the Siegel-Jacobi disk the expression:

\[
(5.37) \quad \frac{D_{k\mu}(\zeta, \zeta')}{2} = k \ln \frac{|1 - \bar{w}'w|^2}{(1 - |w|^2)(1 - |w'|^2)} + \mu \left[ \frac{F(\zeta) + F(\zeta')}{2} - \Re F(\zeta, \bar{\zeta}') \right].
\]

6. Appendix

We recall some definitions from differential geometry of notions used in the paper.

**Definition 1.** (Reductive homogeneous spaces, cf. Nomizu [56]) A homogeneous space \( M = G/H \) is reductive if the Lie algebra \( g \) of \( G \) may be decomposed into a vector space direct sum of the Lie algebra \( h \) of \( H \) and an \( \text{Ad}(H) \)-invariant subspace \( m \), that is

\[
(6.1a) \quad g = h + m, \quad h \cap m = 0,
\]

\[
(6.1b) \quad \text{Ad}(H)m \subset m.
\]

Condition (6.1b) implies

\[
(6.1c) \quad [h, m] \subset m
\]

and, conversely, if \( H \) is connected, then (6.1c) implies (6.1b). Note that \( H \) is always connected if \( M \) is simply connected. The decomposition (6.1a) verifying (6.1b) is called a \( H \)-stable decomposition.

We recall that an analytic Riemannian manifold is called *Riemannian globally symmetric* if each \( p \in M \) is an isolated fixed point of an involutive isometry \( s_p \) of \( M \), see p. 205 in [40]. We recall that if the Lie algebra \( g \) and its subalgebra \( h \) associated with the homogeneous manifold \( M = G/H \) satisfy (6.1), then a necessary and sufficient condition for \( M \) to be a locally symmetric space is

\[
(6.2) \quad [h, h] \subset h, \quad [h, m] \subset m, \quad [m, m] \subset h.
\]

If \( M \) is a complete, simply connected Riemannian locally symmetric space, then \( M \) is a Riemannian globally symmetric space (cf. Theorem 5.6 p 232 in [40]).

Let \( \text{Exp}_p : M_p \to M \) be the geodesic exponential map from the tangent space \( M_p \) at \( p \in M \) to \( M \) (cf. definition at p. 33 in [40]), and also let \( \exp : g \to G \) be the exponential map from the Lie algebra \( g \) to the Lie group \( G \). We also recall that the symmetric spaces have the property that \( \text{Exp}_p(m) = \exp_p(m) \), i.e. the exponential \( \exp_p \) from the Lie algebra to the Lie group at \( p \) gives geodesics such that lines in the tangent space are geodesics in the manifold emerging from the point \( p \). More exactly, if \( \sigma \) is the projection \( \sigma : G \to G/H \), such that \( o = \sigma(e) \), where \( e \) is the unit element in the group.
We also have been used in Remark 8 the notion of geodesic mapping (cf. Definition 5.1 p. 127 in [53])

**Definition 2.** If $f : M \to N$ is a diffeomorphism between manifolds with affine connections, then $f$ is called a geodesic mapping if it maps geodesic curves on $M$ into geodesic curves on $N$.

**Acknowledgments.** This research was supported by the ANCS project program PN 09 37 01 02/2009 and by the UEFISCDI - Romania program PN-II Contract No. 55/05.10.2011. I am grateful to Mircea Bundaru for the suggestions to improve the manuscript. The author would like to express his thanks to Professors Jae-Suk Park, Jae-Hyun Yang and Kisik Kim for the possibility to present a preliminary version of this paper in seminars at the Center for Geometry and Physics, Institute for Basic Science, Pohang, Korea, Department of Mathematics and, respectively, Department of Physics, Inha University Incheon, Korea. The author is indebted to the unknown referees for the suggestions and criticism.

**References**

[1] S. T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent states, wavelets, and their generalizations*, Springer-Verlag, New York (2000)

[2] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947) 568–640

[3] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform part I*, Commun. Pure Appl. Math. **14** (1961) 187–214

[4] S. Berceanu, *Coherent states: old geometric methods in new quantum clothes*, 1994, [hep-th/9408008](https://arxiv.org/abs/hep-th/9408008) 4 p; *The coherent states: old geometrical objects in new quantum clothes*, Romanian J. Phys. **47** (2002) 353–358

[5] S. Berceanu, Coherent states and global differential geometry, in *Quantization, Coherent states and complex structures*, Plenum, New York, Edited by J.-P. Antoine, S. T. Ali, W. Lisiecki, I. M. Mladenov and A. Odzijewicz, 131–140, 1995

[6] S. Berceanu, *On the Geometry of complex Grassmann manifold, its noncompact dual and coherent states*, Bull. Belg. Math. Soc. Simon Stevin **4** (1997) 205–243

[7] S. Berceanu, *Coherent states and geodesics: cut locus and conjugate locus*, J. Geom. Phys. **21** (1997) 149–168

[8] S. Berceanu, Coherent states, transition amplitudes and embeddings, in *Quantization, Coherent States and Poisson Structures*, Polish Scientific Publishers PWN (Warsaw), Edited by A. Strasburger, S. T. Ali, J.-P. Antoine and A. Odzijewicz, 133–142, 1998; [dg-ga/9707026](https://arxiv.org/abs/dg-ga/9707026)

[9] S. Berceanu and M. Schlichenmaier, *Coherent state embeddings, polar divisors and Cauchy formulas*, J. Geom. Phys. **34** (2000) 336–358

[10] S. Berceanu and A. Gheorghe, *Differential operators on orbits of coherent states*, Romanian J. Phys. **48** (2003) 545–556; arXiv: 0211054 [math.DG]

[11] S. Berceanu, Realization of coherent state algebras by differential operators, in *Advances in Operator Algebras and Mathematical Physics*, Edited by F. Boca, O. Bratteli, R. Longo and H. Siedentop, The Theta Foundation, Bucharest, 1–24, 2005

[12] S. Berceanu, *A holomorphic representation of the Jacobi algebra*, Rev. Math. Phys. **18** (2006) 163-199; *Errata*, Rev. Math. Phys. **24** (2012) 1292001 (2 pages)

[13] S. Berceanu, A holomorphic representation of Jacobi algebra in several dimensions, in *Perspectives in Operator Algebra and Mathematical Physics*, Edited by F.-P. Boca, R. Purice and S. Stratila, The Theta Foundation, Bucharest 1-25, 2008; arXiv: 0604381 [math.DG]
[14] S. Berceanu and A. Gheorghe, Applications of the Jacobi group to Quantum Mechanics, Romanian J. Phys. 53 (2008) 1013-1021; arXiv: 0812.0448 [math.DG]

[15] S. Berceanu and A. Gheorghe, On the geometry of Siegel-Jacobi domains, Int. J. Geom. Methods Mod. Phys. 8 (2011) 1783–1798; arXiv: 1011.3317 [math.DG]

[16] S. Berceanu, Classical and quantum evolution on the Siegel-Jacobi manifolds, in Proceedings of the XXX Workshop on Geometric Methods in Physics, Edited by P. Kielanowski, S. T. Ali, A. Odzijewicz, M. Schlichenmaier and T. Voronov, Trends in Mathematics, Springer Basel AG, 43–52, 2013

[17] S. Berceanu, A convenient coordinatization of Siegel-Jacobi domains, Rev. Math. Phys. 24 (2012) 1250024 (38 pages); arXiv: 1204.5610 [math.DG]

[18] S. Berceanu, Consequences of the fundamental conjecture for the motion on the Siegel-Jacobi disk, Int. J. Geom. Methods Mod. Phys. 10 (2013) 1250076 (18 pages); arXiv: 1110.5469v2 [math.DG]

[19] F. A. Berezin, Quantization in complex bounded domains (Russian), Dokladi Akad. Nauk SSSR, Ser. Mat. 211 (1973) 1263–1266

[20] F. A. Berezin, Quantization (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974) 1116–1175

[21] F. A. Berezin, Quantization in complex symmetric spaces (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975) 363–402

[22] F. A. Berezin, The general concept of quantization, Commun. Math. Phys. 40 (1975) 153–174

[23] R. Berndt and S. Böcherer, Jacobi forms and discrete series representations of the Jacobi group, Math. Z. 204 (1990) 13–44

[24] R. Berndt and R. Schmidt, Elements of the representation theory of the Jacobi group, Progress in Mathematics 163, Basel, Birkhäuser, 1998

[25] Z. B/suppress locki, The Bergman metric and the pluricomplex Green functions, Trans. Amer. Math. Soc. 124 (1996) 2021–2027

[26] M. Cahen, S. Gutt and J. Rawnsley, Quantization of Kähler manifolds I: geometric interpretation of Berezin’s quantization, J. Geom. Phys. 7 (1990) 45–62

[27] M. Cahen, S. Gutt and J. Rawnsley, Quantization of Kähler manifolds. II, Trans. Math. Soc. 337 (1993) 73–98

[28] E. Calabi, Isometric imbedding of complex manifolds, Ann. Math. 58 (1953) 1–23

[29] P. Cartier, Quantum mechanical commutation relations and the ta functions, in Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., Boulder, Colo., 1965, Amer. Math. Soc., Providence, R. I., 361–383, 1966

[30] A. Cayley, A six memoir upon quantics, Phyl. Trans. R. Soc. London 149 (1859), 61–90; pp 561–592 in Collected mathematical papers, Vol II, Cambridge University Press, Cambridge, 1989

[31] S. S. Chern, Complex manifolds without potential theory, Springer-Verlag, Berlin, 1979

[32] M. Combescure and D. Robert, Coherent states and applications in mathematical physics, Springer, Dordecht, 2012

[33] Z. Dinev, On the Bergman metric representative coordinates, Sci. China Math. 54 (2011) 1357–1374

[34] J. Dorfmeister and K. Nakajima, The fundamental conjecture for homogeneous Kähler manifolds, Acta Math. 161 (1988) 23–70

[35] M. Eichler and D. Zagier, The theory of Jacobi forms, Progress in Mathematics 55, Boston, Birkhäuser, 1985

[36] P. D. Drummond and Z. Ficek, Editors, Quantum Squeezing, Berlin, Springer, 2004

[37] G. B. Folland, Harmonic analysis in phase space, Princeton University Press, Princeton, New Jersey, 1989

[38] S. G. Ginikin, I.I Pjateckiĭ-Šapiro and E. B. Vinberg, Homogenous Kähler manifolds, in Geometry of homogenous bounded domains, E. Vesentini (Ed.), Springer-Verlag, Berlin, Heidelberg, 2011, Lectures given at the Summer School of the C.I.M.E. held at Urbino, Italy, July 3–13, 1967

[39] B. C. Hall, Holomorphic methods in analysis and mathematical physics, in First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998), Contemporary Mathematics 260, Amer. Math. Soc., Providence, RI, (2000) 1–59
[40] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic, New York, 1978
[41] J. N. Hollenhors, *Quantum limits on resonant-mass gravitational-wave detectors*, Phys. Rev. D 19 (1979) 1669–1679
[42] E. H. Kennard, *Zur Quantenmechanik einfacher Bewegungstypen*, Zeit. Phys. 44 (1927) 326–352
[43] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc 92 (1959) 267–290
[44] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol I, Interscience publishers, New York, 1963
[45] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol II, Interscience publishers, New York, 1969
[46] B. Kostant, Quantization and unitary representations I. Prequantization, in *Lecture Notes in Mathematics* 170, edited by C. T. Tam, Springer-Verlag, Berlin, 1970
[47] P. Kramer and M. Saraceno, *Semicoherent states and the group ISp(2, R)*, Physica 114A (1982) 448–453
[48] W. Lisiecki, *Coherent state representations. A survey*, Rep. Math. Phys. 35 (1995) 327–358
[49] E. H. Kennard, *Zur Quantenmechanik einfacher Bewegungstypen*, Zeit. Phys. 44 (1927) 326–352
[50] Q. K. Lu, *Holomorphic invariant forms on a bounded domain*, Sci. China. Math. 51A (2008) 1945–1964
[51] Q. K. Lu, *The conjugate points of CP^∞ and the zeroes of Bergman kernel*, Acta. Math. Sci. 29B (2009) 480–492
[52] L. Mandel L and E. Wolf, *Optical coherence and quantum optics*, Cambridge: University Sci. Press, 1995
[53] J. Mikeš, A. Vanžurová and I. Hinterleitner, *Geodesic mappings and some generalizations*, Palacý University Olomouc, Faculty of Science, Olomouc, 2009
[54] A. Moroianu, *Lectures on Kähler Geometry*, Cambridge University Press, London Mathematical Society Student Texts 69, Cambridge University Press, Cambridge, 2007
[55] K.-H. Neeb, *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics 28, Walter de Gruyter, Berlin, New York, 2000
[56] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. 76 (1954) 33–65
[57] A. M. Perelomov, *Generalized Coherent States and their Applications*, Springer, Berlin, 1986
[58] C. Quesne *Vector coherent state theory of the semidirect sum Lie algebras wsp(2N, R)*, J. Phys. A: Gen. 23 (1990) 847–862
[59] J. H. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford Ser. 28 (1977) 403–415
[60] I. Satake, *Algebraic structures of symmetric domains*, Publ. Math. Soc. Japan 14, Princeton Univ. Press, 1980
[61] S. Sivakumar, *Studies on nonlinear quantum optics*, J. Opt. B Quantum Semiclass. Opt. 2 (2000) R61–R75
[62] P. Stoler, *Equivalence classes of minimum uncertainty packets*, Phys. Rev. D 1 (1970) 3217–3219
[63] K. Takase, *On unitary representations of Jacobi groups*, J. Reine Angew. Math. 430 (1992) 130–149
[64] K. Takase, *On Siegel modular forms of half-integral weights and Jacobi forms*, Trans. Amer. Math. Soc. 351 (1999) 735–780
[65] E. B. Vinberg and S. G. Gindikin, *Kählerian manifolds admitting a transitive solvable automorphism group* (Russian), Math. Sb. 74 (116) (1967) 333–351
[66] J.-H. Yang, *Invariant metrics and Laplacians on the Siegel-Jacobi spaces*, J. Number Theory 127 (2007) 83–102
[67] J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk*, J. Korean Math. Soc. 45 (2008) 781–794
[68] J.-H. Yang, Y.-H. Yong, S.-N. Huh, J.-H. Shin, H.-G. Min, *Sectional curvatures of the Siegel-Jacobi space*, Bull. Korean Math. Soc. 50 (2013) 787–799
[70] H. P. Yuen, *Two-photon coherent states of the radiation field*, Phys. Rev. A 13 (1976) 2226–2243

May 8, 2014

(Stefan Berceanu) National Institute for Physics and Nuclear Engineering, Department of Theoretical Physics, PO BOX MG-6, Bucharest-Magurele, Romania

E-mail address: Berceanu@theory.nipne.ro