Convolution operators via orthogonal polynomials

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Abstract

In this paper we aim to generalize results obtained in the framework of fractional calculus by the way of reformulating them in terms of operator theory. In its own turn, the achieved generalization allows us to spread the obtained technique on practical problems that connected with various physical - chemical processes.

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1 Introduction

The foundation of models describing various physical - chemical processes can be obtained by virtue of fractional calculus methods, the central point of which is a concept of the Riemann-Liouville operator acting in the weighted Lebesgue space. In its own turn, the operator theory methods play an important role in applications and need not any of special advertising. Having forced by these reasons, we deal with mapping theorems for operators acting on Banach spaces in order to obtain afterwards the desired results applicable to integral operators. We also note that our interest was inspired by lots of previously known results related to mapping theorems for fractional integral operators obtained by mathematicians such as Rubin B.S. [12],[13],[14], Vakulov B.G. [19], Samko S.G. [17],[18], Karapetyants N.K. [2],[3].

In this paper we offer one method of studying the Sonin operator [16]. We claim the existence and uniqueness theorem formulated in terms of the Jacoby series coefficients which gives us an opportunity to find and classify a solution of the Sonin-Abel equation due to an asymptotic of the right side. Let us remind that the so called mapping theorem for the Riemann-Liouville operator (the particular case of the Sonin operator) were firstly studied by H. Hardy and Littlewood [1] and nowadays is known as the Hardy-Littlewood theorem with limit index. However there was an attempt to extend this theorem on some class of weighted Lebesgue spaces defined as functional spaces endowed with the following norm

\[ \|f\|_{L^p(I,\beta,\gamma)} := \|f\|_{L^p(I,\mu)}, \mu(x) = \omega^{\beta,\gamma}(x) := (x-a)^\beta(b-x)^\gamma, \beta,\gamma \in \mathbb{R}, I := (a,b). \]
In this direction the mathematicians such as Rubin B.S., Karapetyants N.K. [2] and others had success. All these create the prerequisite to invent another approach for studying the Riemann-Liouville operator mapping properties that was successfully applied in the paper [5].

Assume that the functions

\[ \varrho, \vartheta \in L_1(I'), \ I' = (0, b - a) \subset \mathbb{R}. \]

are such that the so-called Sonin condition holds

\[ \varrho \ast \vartheta = 1. \]

Consider the operators

\[ sI^\varrho_{a+}\varphi(x) := \int_a^x \varrho(x-t)f(t)dt, \ \varphi \in L_1(I, \beta, \gamma); \]

\[ sD^\vartheta_{a+}f(x) := \frac{d}{dx} \int_a^x \vartheta(x-t)f(t)dt, \ f \in sI^\varrho_{a+}(L_1(I, \beta, \gamma)). \]

In an ordinary way (see [15]), we prove that

\[ sD^\vartheta_{a+}sI^\varrho_{a+}\varphi = \varphi, \ \varphi \in L_1(I, \beta, \gamma). \]

Consider the Abel-Sonin equation under most general assumptions on the right part

\[ sI^\varrho_{a+}\varphi(x) := \int_a^x \varrho(x-t)\varphi(t)dt = f \in L_1(I, \beta, \gamma). \tag{1} \]

We use the following notations for Jacobi polynomials and related expressions

\[ p_n^{\beta, \gamma}(x) = \delta_n(x-a)^{-\beta}(b-x)^{-\gamma} \frac{d^n}{dx^n} [(x-a)^{\beta+n}(b-x)^{\gamma+n}], \ \beta, \gamma > -1, \ n \in \mathbb{N}_0, \]

where

\[ \delta_n(\beta, \gamma) = \frac{(-1)^n}{(b-a)^{n+\beta+\gamma+1/2}} \sqrt{\frac{(\beta+\gamma+2n+1)\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}}, \]

\[ \delta_0(\beta - 1, \gamma - 1) = \frac{\Gamma(\beta+\gamma)}{(b-a)^{(\beta+\gamma)-1/2}}, \ \delta_0'(\beta - 1, \gamma - 1) = \Gamma(\beta+\gamma), \]

\[ \delta_0(\beta, \gamma) = \frac{1}{\sqrt{\Gamma(\beta+1)\Gamma(\gamma+1)}}, \ \beta + \gamma + 1 = 0. \]

\[ \hat{C}_k^n(\beta, \gamma) := \sum_{i=0}^{k} C_{n-k}^i (\begin{pmatrix} n+\beta \\ i \end{pmatrix}) C_k^i (\begin{pmatrix} n-i \gamma \\ i \end{pmatrix}) i!. \]

We also use the following notations

\[ f_n(\beta, \gamma) = \int_a^b f(x)p_n^{\beta, \gamma}(x)\omega^{\beta, \gamma}(x)dx, \ S_nf := \sum_{k=0}^n f_n p_n^{\beta, \gamma}, \ A_{mn}^{-1} := \int_a^b p_n^{\beta, \gamma}(x) \left( sD^\vartheta_{a+}p_m^{\beta, \gamma} f(x) \right) \omega^{\beta, \gamma}(x)dx, \]

and short-hand notations \( f_n = f_n(\beta, \gamma), \ p_n = p_n^{\beta, \gamma}, \ (sI^\varrho_{a+}\varphi)_n(\beta, \gamma) := \varphi_n^\vartheta(\beta, \gamma), \) if their meaning is quite clear. We need the following auxiliary lemmas.
Lemma 1. Let \( \vartheta \in L_1(I') \), then for \( \beta, \gamma > 0 \), we have

\[
\int_{a}^{b} p_m^{\beta,\gamma}(x) \left( s D_{a+}^{\vartheta} p_n(x) \right) \omega^{\beta,\gamma}(x) dx = C_m(\beta, \gamma) \int_{a}^{b} p_{m+1}^{\beta-1,\gamma-1}(x) \left( s I_{a+}^{\vartheta} p_n(x) \right) \omega^{\beta-1,\gamma-1}(x) dx,
\]

\[
C_m(\beta, \gamma) := \frac{\delta_m^{\beta,\gamma}(\beta, \gamma)}{\delta_{m+1}^{\beta-1,\gamma-1}(\beta - 1, \gamma - 1)}, \quad m, n \in \mathbb{N}_0.
\]

Proof. Using the formula

\[
p_n(x) = \int_{a}^{x} p_n'(\tau) d\tau + p_n(a),
\]

it is not hard to calculate

\[
\int_{a}^{b} \vartheta(x-t) p_n^{\beta,\gamma}(t) dt = p_n^{\beta,\gamma}(a) \int_{a}^{x} \vartheta(x-t) dt + \int_{a}^{x} \vartheta(x-t) dt \int_{a}^{t} p_n'(\tau) d\tau =
\]

\[
= p_n^{\beta,\gamma}(a) \int_{a}^{x} \vartheta(x-t) dt + \int_{a}^{x} p_n'(\tau) d\tau \int_{a}^{t} \vartheta(x-t) dt =
\]

\[
= p_n^{\beta,\gamma}(a) \int_{a}^{x} \vartheta(x-t) dt + \int_{a}^{x} p_n'(\tau) d\tau \int_{a}^{t} \vartheta(t-\tau) dt =
\]

\[
= p_n^{\beta,\gamma}(a) \int_{a}^{x} \vartheta(x-t) dt + \int_{a}^{x} dt \int_{a}^{t} \vartheta(t-\tau) p_n'(\tau) d\tau. \quad (3)
\]

Hence

\[
\left( s D_{a+}^{\vartheta} p_n \right)(x) = \frac{d}{dx} \int_{a}^{x} \vartheta(x-t) p_n^{\beta,\gamma}(t) dt = p_n^{\beta,\gamma}(a) \vartheta(x-a) + \int_{a}^{x} \vartheta(x-t) p_n'(t) dt. \quad (3)
\]

Note that making change of a variable \( x-t = b-\tau \), we have

\[
\int_{a}^{b} \left| \int_{a}^{x} \vartheta(x-t) p_n'(t) dt \right| dx \leq \int_{a}^{b} |p_n'(t)| dt \int_{a}^{b} |\vartheta(x-t)| dx = \int_{a}^{b} |p_n'(t)| dt \int_{a}^{b} |\vartheta(b-\tau)| d\tau < \infty.
\]

Therefore \( s D_{a+}^{\vartheta} p_n \in L_1(I) \). Now the claimed result can be established by virtue of integration by parts

\[
\int_{a}^{b} p_m^{\beta,\gamma}(x) \left( s D_{a+}^{\vartheta} p_n(x) \right) \omega^{\beta,\gamma}(x) dx := \int_{a}^{b} p_m^{\beta,\gamma}(x) \frac{d}{dx} s I_{a+}^{\vartheta} p_n(x) \omega^{\beta,\gamma}(x) dx =
\]

\[
= \delta_{m}(\beta, \gamma) \int_{a}^{b} \varphi_{m}^{(m)}(x) \frac{d}{dx} s I_{a+}^{\vartheta} p_n(x) dx = -\delta_{m}(\beta, \gamma) \int_{a}^{b} \varphi_{m}^{(m-1)}(x) s I_{a+}^{\vartheta} p_n(x) dx =
\]

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Using the generalized Minkovskii inequality, we get

\[ \omega_1(x) = \frac{\delta_m(\beta, \gamma)}{\delta_{m+1}(\beta - 1, \gamma - 1)} \int_a^b p_m^{\beta-1, \gamma-1}(x) (s f'_{a+n}) (x) \omega^{\beta-1, \gamma-1}(x) dx = \]

\[ = \frac{\delta'_m(\beta, \gamma)}{\delta'_{m+1}(\beta - 1, \gamma - 1)} \int_a^b p_m^{\beta-1, \gamma-1}(x) (s f'_{a+n}) (x) \omega^{\beta-1, \gamma-1}(x) dx, \quad m, n \in \mathbb{N}_0. \]

\[ \square \]

The following lemma that plays the principal role in extension of the previous result.

**Lemma 2.** Let \( \vartheta \in L_2(I', -k, 0) \), \( k > 0 \), \( \beta, \gamma > 0 \), then the following estimate holds

\[ \left| \int_a^b p_m^{\beta-1, \gamma-1}(x) s f'_{a+n} f(x) \omega_1(x) dx \right| \leq C \| f \|_{L_2(I', \beta, \gamma)}, \quad m = 0, 1, 2, \ldots, \]

where \( \omega_1(x) = (x - a)^{\beta-1}(b - x)^{\gamma-1}. \)

**Proof.** Let us consider the following reasonings

\[ \left| \int_a^b \omega_1(x) p_m^{\beta-1, \gamma-1}(x) s f'_{a+n} f(x) dx \right| = \left| \int_a^b f(t) dt \int_a^b \vartheta(x-t) \omega_1(x) p_m(x) dx \right| \leq \]

\[ \leq \| f \|_{L_2(I', \beta, \gamma)} \left( \int_a^b \omega_1^{-1}(t) \left| \int_t^b \vartheta(x-t) \omega_1(x) p_m(x) dx \right|^2 dt \right)^{1/2} = I_1. \]

Using the generalized Minkovskii inequality, we get

\[ I_1 \leq \| f \|_{L_2(I', \beta, \gamma)} \int_a^b p_m(x) \omega_1(x) \left( \int_a^b |\vartheta(x-t)|^2 \omega_1^{-1}(t) dt \right)^{1/2} dx \leq \]

\[ \leq \| f \|_{L_2(I', \beta, \gamma)} \left( \int_a^b \omega_1(x) dx \int_a^b |\vartheta(x-t)|^2 \omega_1^{-1}(t) dt \right)^{1/2} . \]

Making the change of the variable twice, we have

\[ \int_a^b \omega_1(x) dx \int_a^x |\vartheta(x-t)|^2 \omega_1^{-1}(t) dt = \int_a^b \omega_1(x) dx \int_0^x |\vartheta(t)|^2 \omega_1^{-1}(x-t) dt = \]

\[ = \int_0^{b-a} \omega_1(x+a) dx \int_0^x |\vartheta(t)|^2 \omega_1^{-1}(x + a - t) dt = \int_0^{b-a} |\vartheta(t)|^2 dt \int_0^{b-a} \omega_1^{-1}(x + a - t) \omega_1(x+a) dx = \]

\[ \int_0^{b-a} \omega_1(x+a) dx \int_0^x |\vartheta(t)|^2 \omega_1^{-1}(x + a - t) dt = \int_0^{b-a} |\vartheta(t)|^2 dt \int_0^{b-a} \omega_1^{-1}(x + a - t) \omega_1(x+a) dx = \]

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Lemma 3. Let
\[
|\vartheta(t)|^2 t^{-k} k! \int_0^{b-a} \omega^{-1}(x + a - t) \omega_1(x + a) dx.
\]

Note that, without lose of generality, we may make the following representation \((\delta_1 + \delta_2 = k)\)
\[
t^k \int_t^{b-a} \omega_1(x + a) \omega^{-1}(x + a - t) dx = t^{-k} \int_t^{b-a} x^{\beta - 1} (b - a - x) (x - t)^{-\beta} (b - a - x + t)^{-\gamma} dx =
\]
\[
t^{\delta_1 + \delta_2} \int_t^{b-a} x^{\beta - 1} (b - a - x) (x - t)^{-\beta} (b - a - x + t)^{-\gamma + \delta_2} dx \leq
\]
\[
C t^{\delta_1 + \delta_2} \int_t^{b-a} x^{\beta - 1} (b - a - x) (x - t)^{-\beta} (b - a - x + t)^{-\delta_2} dx \leq
\]
\[
C \int_t^{b-a} x^{\beta - 1 + \delta_1} (b - a - x) (x - t)^{-\beta} t^{\delta_2} (b - a - x + t)^{-\delta_2} dx \leq
\]
\[
C \int_t^{b-a} (b - a - x)^{-1 + \delta_2} (x - t)^{-\beta} t^{\delta_2} (b - a - x + t)^{-\delta_2} dx \leq
\]
\[
C \int_t^{b-a} (b - a - x)^{-1 + \delta_2} (x - t)^{-\beta} t^{\delta_2} (b - a - x + t)^{-\delta_2} dx = B(\delta_1, \delta_2) C.
\]

Combining these estimates, we obtain
\[
I_1 \leq B(\delta_1, \delta_2) \| f \|_{L_2(I, \beta, \gamma)} \int_0^{b-a} |\vartheta(t)|^2 t^k dt.
\]

The last estimate proves the desired result. \(\square\)

**Lemma 3.** Let \(\vartheta \in L_2(I'), \beta, \gamma > 0\), then the following estimate holds
\[
\| s I_a^\vartheta f \|_{L_2(I, \beta, \gamma)} \leq C \| f \|_{L_2(I, \beta, \gamma)}.
\]

**Proof.** Using the generalized Minkovskii inequality, we get
\[
\| s I_a^\vartheta f \|_{L_2(I, \beta, \gamma)} \leq \int_a^b \| f(t) \| \left( \int_t^b \| \vartheta(x - t) \|^2 \omega(x) dx \right)^{1/2} dt \leq
\]
\[
\leq \| f \|_{L_2(I, \beta, \gamma)} \left( \int_a^b \omega(x) dx \int_a^x \| \vartheta(x - t) \|^2 \omega^{-1}(t) dt \right)^{1/2} = \| f \|_{L_2(I, \beta, \gamma)} \times I_1,
\]

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here $\omega(x) := \omega^{\beta,\gamma}(x)$. Repeating the reasonings of Lemma 2 and making the change of variable twice, we have

$$I_1 = \int_0^{b-a} \omega(x+a)dx \int_0^x |\vartheta(t)|^2 \omega^{-1}(x+a-t)dt = \int_0^{b-a} |\vartheta(t)|^2 dt \int_t^{b-a} \omega^{-1}(x+a-t)\omega(x+a)dx.$$  

Note that

$$\int_t^{b-a} \omega^{-1}(x+a-t)\omega(x+a)dx = \int_t^{b-a} x^{\beta}(b-a-x)^{\gamma}(x-t)^{-\beta}(b-a-x+t)^{-\gamma}dx \leq$$

$$\leq C \int_t^{b-a} x^{\beta}(b-a-x)^{\gamma}(x-t)^{-\beta}(b-a-x)^{-\gamma}dx \leq C \int_t^{b-a} (x-t)^{-\beta}dx, \quad t \in (0, b-a).$$

The last estimate proves the desired result.  

2 The main theorem

Before formulating the main theorem, let us make the following notations

$$B^{\beta,\gamma}_p(f, \xi) := \sum_{n=1}^{\infty} |f_n|^p \xi^n, \quad f_n := \int_a^b f(x)p^{\beta,\gamma}_n(x)\omega(x)dx.$$  

Consider the Abel-Sonin equation under most general assumptions on the right-hand side

$$s I^\varrho_a \varphi = f \in L^2(I, \beta, \gamma).$$

We have the following theorem

**Theorem 1.** Assume that $2 \leq p < \infty$, $0 < \beta, \gamma < 1$, the following conditions hold

$$B^{\beta-1,\gamma-1}_p(s I^\varrho_a f, \xi) < \infty, \quad \sum_{m=0}^{\infty} f_m^{\varrho} p^{\beta-1,\gamma-1}_m(a) = 0, \quad (4)$$

where $\xi = (5/2 + \max\{\beta, \gamma\})(p-2) + 2$, then there exists a unique solution of the Abel-Sonin equation in $L_p(I, \beta, \gamma)$ represented by its series. Moreover, in the case $p = 2$, $0 < \beta, \gamma < 1$, we claim that conditions (4) are necessary, so we have a criterion.

**Proof.** The sufficiency part of existence: Using formula (1), we obtain

$$\delta'_{m+1}(\beta-1, \gamma-1) \int_a^b p^{\beta,\gamma}_m(x) \left(s D^\varrho_{a+k} f\right)(x)\omega^{\beta,\gamma}(x)dx =$$

$$= \delta'_m(\beta, \gamma) \int_a^b p^{\beta-1,\gamma-1}_m(x) \left(s I^\varrho_a S_k f\right)(x)\omega^{\beta-1,\gamma-1}(x)dx, \quad k, m, n \in \mathbb{N}_0.$$  

\(\square\)
Combining Lemma 2, Jacoby series expansion, that is given by virtue of the fact \( f \in L_2(I, \beta, \gamma) \), we can easily extend the last relation as follows

\[
\int_a^b p_m^{\beta, \gamma}(x) \left( sD_{a+}^\theta S_k f \right) (x) \omega^{\beta, \gamma}(x) dx \to \delta'_m(\beta, \gamma) \int_a^b p_m^{\beta-1, \gamma-1}(x) \left( sI_{a+}^\theta \right) (x) \omega^{\beta-1, \gamma-1}(x) dx, \quad k \to \infty, \quad m, n \in \mathbb{N}_0. \tag{6}
\]

It is clear that we can rewrite last relation in the following form

\[
\left| \sum_{n=0}^\infty A_{mn}^{-1} f_n \right| = C_m \left| \int_a^b p_m^{\beta-1, \gamma-1}(x) \left( sI_{a+}^\theta \right) (x) \omega^{\beta-1, \gamma-1}(x) dx \right|,
\]

where

\[
C_m = \frac{\delta'_m(\beta, \gamma)}{\delta'_{m+1}(\beta - 1, \gamma - 1)} = \sqrt{(m + 1)(\beta + \gamma + m)}.
\]

Thus, due to the theorem conditions we have

\[
\sum_{m=1}^\infty \left| \sum_{n=0}^\infty A_{mn}^{-1} f_n \right|^p m^{\xi-p} \leq C \mathfrak{B}_{p}^{\beta-1, \gamma-1}(sI_{a+}^\theta f, \xi) < \infty.
\]

Let us calculate

\[
\xi - p = (5/2 + \max\{\beta, \gamma\})(p - 2) + 2 - p = (1/2 + \max\{\beta, \gamma\})(p - 2) + p - 2.
\]

It implies that

\[
\sum_{m=1}^\infty \left| \sum_{n=0}^\infty A_{mn}^{-1} f_n \right|^p M_p = m^{1/2 + \max\{\beta, \gamma\}}.
\]

Having applied the Zigmund-Marczincevich theorem, we get that there exists a function \( \psi \in L_p(I, \beta, \gamma) \), such that

\[
\sum_{n=0}^\infty A_{mn}^{-1} f_n = \psi_m, \quad m \in \mathbb{N}_0.
\]

As the consequences, we have

\[
\int_a^b p_m^{\beta, \gamma}(x) \left( sD_{a+}^\theta S_k f \right) (x) \omega^{\beta, \gamma}(x) dx \to \int_a^b p_m^{\beta, \gamma}(x) \psi(x) \omega^{\beta, \gamma}(x) dx, \quad m \in \mathbb{N}_0, \quad k \to \infty; \tag{7}
\]

\[
\int_a^b p_m^{\beta, \gamma}(x) \psi(x) \omega^{\beta, \gamma}(x) dx = C_m \int_a^b p_{m+1}^{\beta-1, \gamma-1}(x) \left( sI_{a+}^\theta \right) (x) \omega^{\beta-1, \gamma-1}(x) dx. \tag{8}
\]
Using simple reasonings, it is not hard to calculate the following formula
\[
\int_a^b p_m^{\beta,\gamma}(x) S_1(x) \omega^{\beta,\gamma}(x) dx = C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) \omega^{\beta-1,\gamma-1}(x) dx \int_a^x S_1(x) dt =
\]
\[
= C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) \left( s I_{a+}^\theta s I_{a+}^\theta + S_1(x) \right) \omega^{\beta-1,\gamma-1}(x) dx, \quad t = 0, 1, \ldots .
\]

Now, using the Jacobi series expansion for the function \( \psi \) (it is possible by virtue of the Zigmund-Marczincevich theorem) and Lemmas 23 we can extend the previous relation as follows
\[
C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) \left( s I_{a+}^\theta s I_{a+}^\theta + \psi \right) \omega^{\beta-1,\gamma-1}(x) dx = \int_a^b p_m^{\beta,\gamma}(x) \psi(x) \omega^{\beta,\gamma}(x) dx.
\]

Combining this relation with (8), we get
\[
\int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) \left( s I_{a+}^\theta s I_{a+}^\theta + \psi \right) \omega^{\beta-1,\gamma-1}(x) dx = \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) \left( s I_{a+}^\theta f \right) \omega^{\beta-1,\gamma-1}(x) dx,
\]
\[
m = 0, 1, \ldots .
\]

It implies that
\[
s I_{a+}^\theta s I_{a+}^\theta = s I_{a+}^\theta f + \tilde{C} \text{ a.e.}, \quad (9)
\]
where
\[
\tilde{C} = \int_a^b p_0^{\beta-1,\gamma-1} \left\{ s I_{a+}^\theta f - s I_{a+}^\theta s I_{a+}^\theta \right\} \omega^{\beta-1,\gamma-1}(x) dx.
\]

Analogously to the reasonings given above, having applied Lemmas 23 it is not hard to establish the following equality
\[
\int_a^b p_m^{\beta,\gamma}(x) s I_{a+}^\theta \omega^{\beta,\gamma}(x) dx = C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) s I_{a+}^\theta s I_{a+}^\theta \omega^{\beta-1,\gamma-1}(x) dx.
\]

Hence, using (9) we obtain
\[
\int_a^b p_m^{\beta,\gamma}(x) s I_{a+}^\theta \omega^{\beta,\gamma}(x) dx =
\]
\[
= C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) s I_{a+}^\theta s I_{a+}^\theta f \omega^{\beta-1,\gamma-1}(x) dx + C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) s I_{a+}^\theta \tilde{C} \omega^{\beta-1,\gamma-1}(x) dx.
\]

Using Lemmas 23 in an absolutely analogous way, we obtain
\[
C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) s I_{a+}^\theta s I_{a+}^\theta f \omega^{\beta-1,\gamma-1}(x) dx = \int_a^b p_m^{\beta,\gamma}(x) f \omega^{\beta,\gamma}(x) dx.
\]
In addition, using a trivial equality
\[
\int_a^x \theta(x-t) dt = \int_a^x \theta(t-a) dt,
\]
we get
\[
\hat{C} \int_a^b p_{m}^{\beta,\gamma}(x) \theta(x-a) \omega^{\beta,\gamma}(x) dx = C_m \int_a^b p_{m+1}^{\beta-1,\gamma-1}(x) s I_{a+}^{\hat{e}} \omega^{\beta-1,\gamma-1}(x) dx.
\]
Therefore, combining the above results, we get
\[
\int_a^b p_{m}^{\beta,\gamma}(x) s I_{a+}^{\hat{e}} \omega^{\beta,\gamma}(x) dx = \int_a^b p_{m}^{\beta,\gamma}(x) f \omega^{\beta,\gamma}(x) dx + \hat{C} \int_a^b p_{m}^{\beta,\gamma}(x) \theta(x-a) \omega^{\beta,\gamma}(x) dx.
\]
Thus, due to the basis property of Jacoby polynomials in \( L_2(I, \beta, \gamma) \), we have
\[
I_{a+}^{\hat{e}} \omega = f + \hat{C} \omega(x-a) \ a.e.
\]
Now let us express the constant \( \hat{C} \) in terms of the theorem conditions. Applying Lemmas 2, 3 we can easily prove
\[
\int_a^b \omega^{\beta-1,\gamma-1}(x) I_{a+}^1 S_t \psi(x) dx =
\]
\[
= \int_a^b \omega^{\beta-1,\gamma-1}(x) s I_{a+}^{\hat{e}} I_{a+}^0 \psi(x) dx \rightarrow \int_a^b \omega^{\beta-1,\gamma-1}(x) s I_{a+}^{\hat{e}} I_{a+}^0 \psi(x) dx, \ t \rightarrow \infty.
\]
Therefore, we have
\[
\int_a^b \omega^{\beta-1,\gamma-1}(x) s I_{a+}^{\hat{e}} \{ s t_a f - I_{a+}^1 S_t \psi \} \omega^{\beta-1,\gamma-1}(x) dx \rightarrow \hat{C}, \ t \rightarrow \infty.
\]
(10)
Consider more precisely
\[
\int_a^x S_t \psi(t) dt = \sum_{m=0}^t \psi_m \int_a^x p_m^{\beta,\gamma}(t) dt = \sum_{m=0}^t \psi_m \frac{p_m^{\beta-1,\gamma-1}(x) - p_m^{\beta-1,\gamma-1}(a)}{\sqrt{(m+1)(\beta + \gamma + m)}}.
\]
It follows that
\[
\int_a^b \omega^{\beta-1,\gamma-1}(x) I_{a+}^1 S_t \psi(x) dx = \int_a^b \sum_{m=0}^t \psi_m \frac{p_m^{\beta-1,\gamma-1}(x) - p_m^{\beta-1,\gamma-1}(a)}{\sqrt{(m+1)(\beta + \gamma + m)}} \omega^{\beta-1,\gamma-1}(x) dx =
\]
\[
= - \sum_{m=0}^t \frac{\psi_m p_m^{\beta-1,\gamma-1}(a)}{\sqrt{(m+1)(\beta + \gamma + m)}} \int_a^b \omega^{\beta-1,\gamma-1}(x) dx = -B(\beta, \gamma)(b-a)^{\beta+\gamma-1} \sum_{m=0}^t \frac{\psi_m p_m^{\beta-1,\gamma-1}(a)}{\sqrt{(m+1)(\beta + \gamma + m)}} =
\]
9
\[ -B(\beta, \gamma)(b - a)^{\beta+\gamma-1} \sum_{m=0}^{\ell} \frac{C_m f^\theta_m p^\beta_{m+1, \gamma-1}(a)}{\sqrt{(m + 1)(\beta + \gamma + m)}} = -B(\beta, \gamma)(b - a)^{\beta+\gamma-1} \sum_{m=0}^{\ell} f^\theta_m p^\beta_{m+1, \gamma-1}(a). \]

Hence

\[
\int_a^b p^\beta_{-1, \gamma-1} \left\{ s I^\theta_{a+f} - I^1_{a+S I^\theta_{\psi}} \right\} \omega^\beta_{-1, \gamma-1}(x) dx = B(\beta, \gamma)(b - a)^{\beta+\gamma-1} \sum_{m=0}^{\ell} f^\theta_m p^\beta_{m+1, \gamma-1}(a) + f^\theta_0 =
\]

\[
= \sqrt{B(\beta, \gamma)}(b - a)^{\beta+\gamma-1/2} \sum_{m=0}^{\ell} f^\theta_m p^\beta_{m+1, \gamma-1}(a) + f^\theta_0 =
\]

\[
= \sqrt{B(\beta, \gamma)}(b - a)^{\beta+\gamma-1/2} \sum_{m=0}^{\ell} f^\theta_m p^\beta_{m-1, \gamma-1}(a) \to C, \quad t \to \infty.
\]

The necessity part of existence: Now assume that there exists a solution of the Abel-Sonin equation in \( L_2(I, \beta, \gamma) \), \( 0 < \beta, \gamma < 1 \). Then using Lemmas \( 23 \) we can easily establish (analogously to the above) the following equality

\[
\int_a^b p^\beta_{m, \gamma}(x) \psi(x) \omega^\beta_{\gamma}(x) dx = C_m \int_a^b p^\beta_{m+1, \gamma-1}(x) \left( s I^\theta_{a+f} + s I^\theta_{a+\psi} \right)(x) \omega^\beta_{-1, \gamma-1}(x) dx =
\]

\[
= C_m \int_a^b p^\beta_{m+1, \gamma-1}(x) \left( s I^\theta_{a+f} \right)(x) \omega^\beta_{-1, \gamma-1}(x) dx, \quad m \in \mathbb{N}_0.
\]

Hence

\[ \mathfrak{B}^{\beta-1, \gamma-1}(s I^\theta_{a+f}, 2) < \infty. \]

Since \( \psi \) is a solution, then \( \mathfrak{B} \) is fulfilled, where \( \hat{C} = 0 \). We can also establish \( \Omega \), in the way that was used above. Having repeated the above reasonings, we come to the relation

\[
\sum_{m=0}^{\infty} f^\theta_m p^\beta_{m-1, \gamma-1}(a) = 0.
\]

The proof of uniqueness: Assume that there exists a solution \( \psi \) and another solution \( \phi \) in \( L_p(I, \beta, \gamma) \) of the Sonin-Abel equation, and let us denote \( \xi := \psi - \phi \). Denote

\[ I_n := \left( a + \frac{1}{n}, b - \frac{1}{n} \right), \]

then the following assumptions are fulfilled

\[ \bigcup_{n=1}^{\infty} I_n = I, \quad I_n \subset I_{n+1}, \quad \mu(I \setminus I_n) \to 0, \quad n \to \infty, \quad L_p(I, \beta, \gamma) \subset L_p(I_n). \]

The verification is left to a reader. In terms of these denotations, it is also clear that

\[ \bigcup_{n=1}^{\infty} C^\infty_0(I_n) = C^\infty_0(I), \quad C^\infty_0(I_n) \subset C^\infty_0(I_{n+1}). \]
Let us show that
\[
\forall \eta \in C_0^\infty(\Omega), \forall \xi \in L_p(I, \beta, \gamma), \exists h \in L_p(I, \beta, \gamma) : \\
\int_a^b \xi(x) \eta(x) dx = \int_a^b \xi(x) s I^e_{b-} \omega h(x) dx.
\] (11)

It is not hard to prove that \( s D^\theta_{b-} \eta(x) \in C(\bar{I}) \), the proof is left to a reader. Moreover, we have the following. Consider
\[
\omega^{-1}(x) s D^\theta_{b-} \eta(x) = (x-a)^{-\beta} (b-x)^{1-\gamma} (b-x)^{-1} s D^\theta_{b-} \eta(x).
\]

Let us show that \( s D^\theta_{b-} \eta(b) = 0 \). In accordance with the reasonings applied to obtain formula (3), we analogously get
\[
(s D^\theta_{b+} \eta)(x) = \eta(b) \vartheta(b-x) - \int_x^b \vartheta(t-x) \eta'(t) dt = - \int_x^b \vartheta(t-x) \eta'(t) dt.
\]

It is clear that
\[
\left| \int_x^b \vartheta(t-x) \eta'(t) dt \right| \leq C \int_x^b |\vartheta(t-x)| dt = \int_0^{b-x} |\vartheta(t)| dt.
\]

Since \( \vartheta \in L_2(I', -\varepsilon) \), then
\[
\int_0^{b-x} |\vartheta(t)| dt = 0, \ x = b.
\]

Hence we obtain the desired result i.e. \( s D^\theta_{b-} \eta(b) = 0 \). We can also get without any difficulties, by using the previous results, the following relation
\[
\frac{d}{dx} s D^\theta_{b+} \eta(x) = \int_x^b \vartheta(t-x) \eta''(t) dt,
\]

and it is clear that
\[
\frac{d}{dx} s D^\theta_{b+} \eta(x) = 0, \ x = b.
\]

Now it gives us the following
\[
(b-x)^{-1} s D^\theta_{b-} \eta(x) = (b-x)^{-1} \left\{ s D^\theta_{b-} \eta(x) - s D^\theta_{b-} \eta(b) \right\} \to 0, \ x \to b.
\]

Hence the function \( \omega^{-1} D^\theta_{b-} \eta \) belongs to \( L_{p'}(I, \beta, \gamma) \), if \( \beta < 1/(p'-1) \) (in particularly it is fulfilled if \( 1 < p' \leq 2 \)). It implies that we have a representation \( D^\theta_{b-} \eta = \omega h \), where \( h \) belongs to \( L_{p'}(I, \beta, \gamma) \). By virtue of the fact \( \eta \in C_0^\infty(I) \), we can easily prove a relation \( s I^e_{b-} s D^\theta_{b-} \eta = \eta \) (a reader only ought to repeat the proof given above corresponding to the polynomial case). Hence
\[
\eta = s I^e_{b-} \omega h, \ h \in L_{p'}(I, \beta, \gamma).
\]
Taking into account the reasonings given above, we get formula (2) which, on account of the Fubini theorem, can be rewritten as follows

\[ \int_a^b \xi(x) \eta(x) dx = \int_a^b \xi(x) \cdot I^{\mu} \omega h(x) dx = \int_a^b I^{\mu + 1} \xi(x) h(x) \omega(x) dx = 0. \]

Hence

\[ \int_{I_n} \xi(x) \eta(x) dx = 0, \forall \eta \in C^\infty_0 (I_n). \]

We claim that \( \xi \neq 0 \). Therefore in accordance with the consequence of the Hahn-Banach theorem there exists the element \( \varpi \in L_{p'}(I_n) \), such that

\[ (\varpi, \xi)_{L_2(I_n)} = \| \psi - \phi \|_{L_p(I_n)} > 0. \]

On the other hand, there exists the sequence \( \{ \eta_k \}_{k=1}^\infty \subset C^\infty_0 (I_n) \), such that \( \eta_k \to \varpi \) with respect to the norm \( L_{p'}(I_n) \). Hence

\[ 0 = (\eta_k, \xi)_{L_2(I_n)} \to (\varpi, \xi)_{L_2(I_n)}. \]

Hence \( \psi = \phi \) almost everywhere on the set \( I_n, n = 1, 2, ... \) (the explanation of this reasoning is too simple and can be found in any book devoted to Functional Analysis). In its own turn, it implies that \( \psi = \phi \) almost everywhere on the set \( I \). The uniqueness has been proved. Thus the proof of sufficiency has been completed.

\[ \square \]

**Remark 1.** In terms of Theorem 1 it seems to be easy to formulate necessary conditions of solvability of the Abel-Sonin equation in \( L_p(I, \beta, \gamma) \), \( 0 < \beta, \gamma < 1, 1 < p < 2 \).

For this purpose we need impose the additional conditions, first of them is \( \varrho \in L_{p'}(I') \). Due to this condition we can repeat the reasonings of Lemma 3 and prove

\[ \| sI^{\mu + 1} f \|_{L_2 (I, \beta, \gamma)} \leq C \| f \|_{L_p (I, \beta, \gamma)}. \]  

(12)

The second condition, we need impose, is the Polard condition

\[ 4 \max \left\{ \frac{\beta + 1}{2 \beta + 3}, \frac{\gamma + 1}{2 \gamma + 3} \right\} < p < 4 \min \left\{ \frac{\beta + 1}{2 \beta + 1}, \frac{\gamma + 1}{2 \gamma + 1} \right\}. \]

After that our attention ought to be concentrate upon the following equality

\[ \int_a^b P_m^{\beta, \gamma} (x) \psi(x) \omega^\beta \gamma(x) dx = C_m \int_a^b P_{m+1}^{\beta-1, \gamma-1} (x) (sI_{a+}^{\varrho} sI_a^{\varrho}) (x) \omega^{\beta-1, \gamma-1}(x) dx. \]

It can be proved by applying Lemma 2 estimate (12) and the basis property of Jacoby polynomials. Hence, having taken into account that \( \psi \) is a solution, we get

\[ \int_a^b P_m^{\beta, \gamma} (x) \psi(x) \omega^\beta \gamma(x) dx = C_m \int_a^b P_{m+1}^{\beta-1, \gamma-1} (x) (sI_{a+}^{\varrho} f) (x) \omega^{\beta-1, \gamma-1}(x) dx, m \in \mathbb{N}_0. \]
Having applied the Zigmund-Maczincevich theorem, we have
\[ \sum_{m=1}^{\infty} |\psi_n|^p M_m^{p-2} m^{p-2} \leq C \|\psi\|_{L_p(I,\beta,\gamma)}, \quad M_m = m^{1/2 + \max(\beta,\gamma)}. \]

Hence, by direct calculation, we obtain
\[ \mathfrak{M}^{\beta-1,\gamma-1}(s_I^{\vartheta} f, \xi) < C \sum_{n=1}^{\infty} |n|^{(\beta/2 + \max(\beta,\gamma))(p-2)+2} = \]
\[ = C \sum_{n=1}^{\infty} |\psi_n|^p n^{(\beta/2 + \max(\beta,\gamma))(p-2)} < \|\psi\|_{L_p(I,\beta,\gamma)}^p < \infty. \]

The prove of the fact
\[ \sum_{m=0}^{\infty} f_m = 0 \]

is absolutely analogous to the case \( p = 2 \). We should only repeat the scheme of the reasonings having taken into account the additional conditions. These reasonings can be formed in the following theorem.

**Theorem 2.** Assume that there exists a solution \( \psi \) in \( L_p(I,\beta,\gamma) \) with \( 0 < \beta,\gamma < 1 \) and \( 1 < p < 2 \) of the Abel-Sonin equation, the following additional conditions hold
\[ \rho \in L_p(I'), 4 \max \left\{ \frac{\beta + 1}{2\beta + 3}, \frac{\gamma + 1}{2\gamma + 3} \right\} < p < 4 \min \left\{ \frac{\beta + 1}{2\beta + 1}, \frac{\gamma + 1}{2\gamma + 1} \right\}. \]

Then relations (4) hold.

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