On the bound states of the Dirac equation in the extreme Kerr metric

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Received 8 August 2008, in final form 10 August 2008
Published 3 November 2008
Online at stacks.iop.org/CQG/25/225022

Abstract
We study the eigenvalues of the angular equation arising after the separation of the Dirac equation in the extreme Kerr metric. For this purpose a self-adjoint holomorphic operator family associated with this eigenvalue problem is considered. We show that the eigenvalues satisfy a first-order nonlinear differential equation with respect to the black hole mass and we solve it. Finally, we prove that there exist no bound states for the Dirac equation in the aforementioned metric.

PACS numbers: 04.62.+v, 04.70.−s

1. Introduction

The last ten years have been characterized by an increasing interest in studying the behavior of Dirac particles in the geometry of an extreme black hole [1–4]. The main picture arising from the aforementioned studies is that under certain conditions on the physical parameters the Dirac equation in the Reissner–Nordström, extreme Kerr and extreme Kerr–Newman metrics admits the existence of bound state solutions. In turn, this leads to the tempting interpretation of such systems made of an extreme black hole plus a fermion as a new kind of atomic system with an extreme charged or uncharged black hole as its nucleus and around an electronic cloud.

A still open problem is to understand if such results could be relevant for astrophysics. The present work represents a contribution in this direction. Despite the common belief that the formation of an extreme Kerr black hole (EKBH) is of only academic interest it is our opinion that we cannot a priori exclude that EKBHs play no role in astrophysics. For instance, relativistic Dyson rings admit a continuous transition to an EKBH [5]. Studies about the existence of such rings based on numerical computations can be found in [6, 7]. Moreover, relativistic Dyson rings could emerge from astrophysical scenarios like stellar core-collapses with high angular momentum [8] or they might simply be present in central regions of galaxies. Finally, it has been recently proved that the only possible candidate for a black hole limit for
stationary and axisymmetric, uniformly rotating perfect fluid bodies with a cold equation of state as well as for isentropic stellar models with a non-zero temperature is the EKBH [9]. Hence, it appears reasonable to study the Dirac equation in the geometry of an extreme black hole.

In what follows we restrict our attention to extreme Kerr black holes. Although it is not the most general model of the exterior region of a black hole we can analyze theoretically, it represents indeed the most realistic model in astrophysics since in general black holes are embedded in environments that are rich in gas and plasma and, consequently any net charge is neutralized by the ambient plasma. The main problem connected with the existence of bound states for the Dirac equation in the extreme Kerr and Kerr–Newman metrics [2, 3] is that an energy eigenvalue \( \omega \) has to satisfy a complicated set of conditions. In fact, according to theorem 1 in [2] \( \omega \in \mathbb{R} \) is an energy eigenvalue of the Dirac equation in the Kerr–Newman metric if and only if there exists an eigenvalue \( \lambda \in \mathbb{R} \) of the angular Dirac equation (arising from the Dirac equation after the separation of variables by means of the Chandrasekhar ansatz [10]) such that

\[
\omega(a^2 + M^2) + ka + eQM = 0, \quad m_e^2 - \omega^2 > 0, \quad \lambda^2 + M^2m_e^2 - (2M\omega + eQ)^2 > \frac{1}{4}
\]

and either

\[
m_e[M[\omega] - \operatorname{sgn}(\omega)(2M\omega + eQ)] \sqrt{m_e^2 - \omega^2} + \operatorname{sgn}(\omega)\lambda = 0,
\]

\[
Mm_e^2 - \omega(2M\omega + eQ) \sqrt{m_e^2 - \omega^2} + \sqrt{\lambda^2 + M^2m_e^2 - (2M\omega + eQ)^2} = 0
\]

or

\[
1 + N + \frac{Mm_e^2 - \omega(2M\omega + eQ)}{\sqrt{m_e^2 - \omega^2}} + \sqrt{\lambda^2 + M^2m_e^2 - (2M\omega + eQ)^2} = 0
\]

for some positive integer \( N \) where \( m_e \) and \( k = \pm 1/2, \pm 3/2, \ldots \) are the mass and the azimuthal quantum number of a spin 1/2 particle with charge \( e \), respectively and \( M, Q \) and \( a = M \) are the mass, the charge and the angular momentum per unit mass of the black hole. This result furnishes necessary and sufficient conditions for the existence of an energy eigenvalue of the Dirac equation in the non-extreme Kerr–Newman metric as well as in the extreme case. However, reference [11] already showed that the Dirac equation in the non-extreme Kerr–Newman metric does not admit any bound state, thus implying that the system (1.1) with (1.2) or (1.3) can never be satisfied. Turning back to [2] lemma 5, under the assumptions (1.1) with (1.2) or (1.3), states that if the Dirac equation in the extreme Kerr metric \( (a = M, Q = 0) \) has a normalizable time-periodic solution with azimuthal quantum number \( k \), then \( \frac{k}{2} < Mm_e < \frac{|k|}{2} \) and the energy of this particle is

\[
\omega = -\frac{kM}{2J}
\]

where \( J \) is the angular momentum of the black hole. Clearly, this result does not state the existence of a bound state but it simply says that if a bound state exists its energy is given by (1.4) provided the system (1.1) with (1.2) or (1.3) is satisfied. Finally, theorem 2 in [2] asserts that for a fixed azimuthal quantum number there exist two sequences \( (a^-_n)_{n \in \mathbb{N}} \) and \( (a^+_n)_{n \in \mathbb{N}} \) with the properties

\[
a^-_n < 0 < a^+_n, \quad |a^\pm_n| \in \left[ \frac{|k|}{2m_e}, \frac{|k|}{\sqrt{2m_e}} \right], \quad \lim_{n \to \infty} |a^\pm_n| = \frac{|k|}{2m_e}
\]
such that the Dirac equation has a normalizable time-periodic solution with azimuthal quantum number \( k \) in the extreme Kerr metric. Unfortunately, the proof of this result relies on the previous lemma which in turn already assumes that \( \omega \) is an energy eigenvalue or in other words that the system (1.1) with (1.2) or (1.3) admits a non-trivial solution. Thus, the question if the system (1.1) with (1.2) or (1.3) admits a solution or not is still open. The main purpose of our work is addressed to answer this question. In particular we shall prove that the system (1.1) with (1.2) or (1.3) admits no solution. This result together with theorem 1 in [2] implies that there exists no bound state for the Dirac equation in the extreme Kerr metric.

The rest of this paper is organized as follows. In section 2 we shortly derive the Dirac equation in the EKBH. Section 3 is devoted to deriving and solving a nonlinear ODE for the eigenvalues \( \lambda \) with respect to the black hole mass parameter. In section 4 we construct a quasi-linear PDE for the eigenvalues with respect to the energy of the particle and the mass of the black hole. Moreover, we derive a formal power solution for \( \lambda \). Finally, in section 5 we show that the solution set of the system (1.1) with (1.2) or (1.3) is empty. This result implies that there are no bound state solutions for the Dirac equation in the extreme Kerr metric.

2. The Dirac equation in the extreme Kerr metric

In Boyer–Lindquist coordinates \(( t, r, \vartheta, \varphi)\) with \( r > 0, 0 \leq \vartheta \leq \pi, 0 \leq \varphi < 2\pi\) the extreme Kerr metric [4] is given by

\[
\text{ds}^2 = \left(1 - \frac{2Mr}{\Sigma}\right) \text{dt}^2 + \frac{4M^2r}{\Sigma} \text{dr} \text{d\varphi} - \frac{\Sigma}{\Delta} \text{dr}^2 - \Sigma \, \text{d\vartheta}^2 - (r^2 + M^2) \sin^2 \vartheta \frac{\tilde{\Sigma}}{\Sigma} \text{d\varphi}^2
\]

(2.1)

where

\[
\Sigma := \Sigma(r, \vartheta) = r^2 + M^2 \cos^2 \vartheta, \quad \Delta := \Delta(r) = (r - M)^2
\]

\[
\tilde{\Sigma} := \tilde{\Sigma}(r, \vartheta) = 1 - M^2 \gamma^2(r) \sin^2 \vartheta, \quad \gamma(r) := \frac{r - M}{r^2 + M^2}
\]

and \( M \) is the mass of a spinning black hole with angular momentum \( J = M^2 \). Since the equation \( \Delta = 0 \) has a double root at \( r_0 := M \) the Cauchy horizon and the event horizon coincide.

In the following we consider a spin 1/2 particle with mass \( m_e \) and charge \( e \) in the extreme Kerr background. The behavior of such a particle is governed by the Dirac equation, a linear system of four coupled partial differential equations. In the extreme Kerr metric the Dirac equation can be easily obtained from the results in [12] by setting the Kerr parameter \( a \) equal to the mass \( M \) of the black hole and it has the form

\[
(R + A)\Psi = 0
\]

(2.2)

where

\[
R = \begin{pmatrix}
im_e r & 0 & \sqrt{\Delta}D_+ & 0 \\
0 & -im_e r & 0 & \sqrt{\Delta}D_- \\
\sqrt{\Delta}D_- & 0 & -im_e r & 0 \\
0 & \sqrt{\Delta}D_+ & 0 & im_e r
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
-Mm_e \cos \vartheta & 0 & 0 & \mathcal{L}_+ \\
0 & Mm_e \cos \vartheta & -\mathcal{L}_- & 0 \\
0 & \mathcal{L}_+ & -Mm_e \cos \vartheta & 0 \\
-\mathcal{L}_- & 0 & 0 & Mm_e \cos \vartheta
\end{pmatrix}
\]
with $D_\pm$ and $L_\pm$ defined as

$$D_\pm = \frac{\partial}{\partial r} \pm \frac{1}{\Delta} \left[ (r^2 + M^2) \frac{\partial}{\partial t} + M \frac{\partial}{\partial \phi} \right],$$

$$L_\pm = \frac{\partial}{\partial \vartheta} + \frac{1}{2} \cot \vartheta \pm i \left( M \sin \vartheta \frac{\partial}{\partial t} + \csc \vartheta \frac{\partial}{\partial \phi} \right),$$

respectively.

By rearranging (2.2) we can write the Dirac equation in the Hamiltonian form

$$i \frac{\partial}{\partial t} \Psi = H \Psi \quad (2.3)$$

where $H$ is a first-order $4 \times 4$ matrix differential operator acting on spinors $\Psi$ on hypersurfaces $t = \text{const}$. Similarly as in [12] we can construct a positive scalar product

$$\langle \Psi | \Phi \rangle = \int_{-\infty}^{+\infty} dr \int_0^\pi d\vartheta \int_0^{2\pi} d\phi \overline{\Psi(t, r, \vartheta, \phi)} \Phi(t, r, \vartheta, \phi) \frac{r^2 + M^2}{\Delta} \quad (2.4)$$

where $\overline{\Psi}$ denotes the complex conjugated transposed spinor. In the present work we are interested in time-periodic solutions

$$\Psi(t, r, \vartheta, \phi) = e^{-i\omega t} \Psi_0(r, \vartheta, \phi) \quad (\omega \in \mathbb{R})$$

of the Dirac equation (2.2) where $\omega_0$ is normalizable, that is $\langle \Psi | \Phi \rangle = \langle \Psi_0 | \Phi_0 \rangle = 1$. Note that if such a solution exists, then $\omega$ is an eigenvalue of $H$ for the eigenspinor $\Psi_0$ and $\omega$ represents the particle energy of the bound state $\Psi$.

By means of the Chandrasekhar ansatz [13]

$$\Psi_0(r, \vartheta, \phi) = e^{-ik\phi} \begin{pmatrix} f_1(r) g_1(\vartheta) \\ f_2(r) g_2(\vartheta) \\ f_2(r) g_1(\vartheta) \\ f_1(r) g_2(\vartheta) \end{pmatrix}, \quad k \in \{ \pm 1/2, \pm 3/2, \ldots \}$$

the Dirac equation decouples into the equations

$$R \Psi = \lambda \Psi, \quad A \Psi = -\lambda \Psi \quad (2.5)$$

with separation parameter $\lambda \in \mathbb{R}$. Finally, defining

$$f(r) := \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \quad g(\vartheta) := \begin{pmatrix} g_1(\vartheta) \\ g_2(\vartheta) \end{pmatrix}$$

the Dirac equation can be separated into a radial part

$$\begin{pmatrix} (r - M) \frac{d}{dr} + i \frac{V(r)}{r-M} & im_r - \lambda \\ -im_r - \lambda & (r - M) \frac{d}{dr} - i \frac{V(r)}{r-M} \end{pmatrix} f(r) = 0, \quad (2.5)$$

$$\begin{pmatrix} \frac{d}{d\vartheta} + \frac{1}{2} \cot \vartheta - Q(\vartheta) & \lambda - \lambda M \cos \vartheta \\ \lambda + \lambda M \cos \vartheta & \frac{d}{d\vartheta} - \frac{1}{2} \cot \vartheta - Q(\vartheta) \end{pmatrix} g(\vartheta) = 0 \quad (2.6)$$

where

$$V(r) = \omega(r^2 + M^2) + \kappa M, \quad Q(\vartheta) = M \omega \sin \vartheta + k \csc \vartheta.$$ 

As in [2] we introduce the following definition:

**Definition 2.1.** We say that $\omega \in \mathbb{R}$ is an energy eigenvalue of (2.2) if there exists a $\lambda \in \mathbb{R}$ and non-trivial solutions $f$ of (2.5) and $g$ of (2.6) satisfying the normalization conditions

$$\int_{-\infty}^{+\infty} dr \frac{r^2 + M^2}{\Delta} |f(r)|^2 = 1, \quad \int_0^{\pi} d\vartheta \sin \vartheta |g(\vartheta)|^2 = 1. \quad (2.7)$$
3. An ordinary differential equation for the eigenvalues $\lambda$

By means of the transformation
$$\tilde{g}(\vartheta) := \begin{pmatrix} \tilde{g}_1(\vartheta) \\ \tilde{g}_2(\vartheta) \end{pmatrix} = \sqrt{\sin \vartheta} g(\vartheta)$$
the angular equation takes the form
$$(\tilde{\mathcal{L}}g)(\vartheta) := \begin{pmatrix} 0 & \omega \sin \vartheta \\ -\frac{d}{d\vartheta} + i \frac{k}{\sin \vartheta} & 0 \end{pmatrix} \tilde{g} + M \begin{pmatrix} -m_e \cos \vartheta & \omega \sin \vartheta \\ \omega \sin \vartheta & m_e \cos \vartheta \end{pmatrix} \tilde{g} = \lambda \tilde{g}$$
(3.1)
with $\vartheta \in (0, \pi)$. It is straightforward to check that the solutions $\tilde{g}_1$ and $\tilde{g}_2$ of (3.1) have the following useful property:
$$\tilde{g}_1(\pi - \vartheta) = \tilde{g}_2(\vartheta), \quad \tilde{g}_2(\pi - \vartheta) = \tilde{g}_1(\vartheta).$$
(3.2)
We can associate the minimal operator $\mathcal{A}_0$ with the formal differential expression $\mathcal{L}$ acting in the Hilbert space $\mathcal{H} := L_2((0, \pi)^2, C^2)$ of square integrable vector functions with respect to the scalar product
$$(\tilde{g}_1, \tilde{g}_2) = \int_0^\pi \int_0^\pi \tilde{g}_1(\vartheta) \tilde{g}_2(\vartheta) \, d\vartheta.$$ 

The operator $\mathcal{A}_0$ given by $D(\mathcal{A}_0) = C_0^\infty((0, \pi), C^2)^2$ and $\mathcal{A}_0 \tilde{g} := \tilde{\mathcal{L}}g$ for $\tilde{g} \in D(\mathcal{A}_0)$ is densely defined and closable. Moreover, since the formal differential operator $\mathcal{L}$ is in the limit point case at 0 and $\pi$ it follows that $\mathcal{A}_0$ is essentially self-adjoint. In the following we denote the closure of $\mathcal{A}_0$ by $\mathcal{A}$. To indicate the dependence of the angular operator $\mathcal{A}$ and its eigenvalues $\lambda$ on the parameter $M$ we use the notation $\mathcal{A}(M)$ and $\lambda(M)$.

According to [14] (theorem 5.8) the domain of $\mathcal{A}(0)$ is given by
$$D(\mathcal{A}) = \{ g \in \mathcal{H} : \tilde{g} \text{ is absolutely continuous and } \mathcal{A}(0)\tilde{g} \in \mathcal{H} \}.$$ 

Since $\mathcal{A}(M) = \mathcal{A}(0) + T(M)$ with the bounded multiplication operator
$$T(M) = M \begin{pmatrix} -m_e \cos \vartheta & \omega \sin \vartheta \\ \omega \sin \vartheta & m_e \cos \vartheta \end{pmatrix},$$
its domain of the definition $D(\mathcal{A})$ is independent of $M \in \mathbb{C}$ (see [15] chapter IV, section 1, theorem 1.1). Moreover, for $M \in \mathbb{R}$ it results that $T(M)$ is a symmetric perturbation of $\mathcal{A}(0)$ and theorem 4.10, chapter V, section 4 in [15] implies that $\mathcal{A}(M)$ is self-adjoint. According to the classification in [15] (chapter VII, section 3) $\mathcal{A}(M)$ forms a self-adjoint holomorphic operator family of type (A) in the variable $M \in \mathbb{C}$. Further, the spectrum of $\mathcal{A}(0)$ is discrete and consists of simple eigenvalues given by lemma 3.3, chapter 3, section 1.2 in [16]
$$\lambda_{n,k}(M) = \text{sign}(n) (|k| - \frac{1}{2} + |n|), \quad n \in \mathbb{Z}\setminus\{0\}. \quad (3.3)$$

This means that $\mathcal{A}(0)$ has compact resolvent and theorem 2.4, chapter V, section 2 in [15] yields that $\mathcal{A}(M)$ has compact resolvent for all $M \in \mathbb{C}$. This implies that the eigenvalues $\lambda_{n,k} = \lambda_{n,k}(M), n \in \mathbb{Z}\setminus\{0\}$ of $\mathcal{A}(M)$ are simple and depend holomorphically on $M$. Moreover, the first derivative of $\mathcal{A}$ with respect to $M$ is given by
$$\frac{d\mathcal{A}}{dM} = \begin{pmatrix} -m_e \cos \vartheta & \omega \sin \vartheta \\ \omega \sin \vartheta & m_e \cos \vartheta \end{pmatrix},$$
which yields the following estimates for the growth rate of the eigenvalues (see [15] chapter VII, section 3.4)
$$\left\| \frac{d\lambda_{n,k}}{dM} \right\| \leq \max \{|m_e|, |\omega|\}.$$
Here, \( \| \cdot \| \) denotes the operator norm of a \( 2 \times 2 \) matrix. In addition, theorem 4.10, chapter V, section 3 in [15] implies that
\[
\min_{n \in \mathbb{Z} \setminus \{0\}} |\lambda_{n,k} - \lambda_{n,k}(0)| \leq \| T(M) \| \leq \max \{|m_c|, |\omega|\}
\]
for each eigenvalue \( \lambda_{n,k} \) of \( A(M) \). Finally, by interchanging the components of \( \tilde{g}(\vartheta) \) it is easy to check that
\[
\lambda_{n,k} \text{ is an eigenvalue of } A \text{ if and only if } -\lambda_{n,k} \text{ is an eigenvalue of } A \text{ with } k, M \text{ and } m_c \text{ replaced by } -k, -M \text{ and } -m_c, \text{ respectively.}
\]
Since the eigenvalues depend holomorphically on \( M \) the following identity holds:
\[
\lambda_{n,k}(\omega, m_c; M) = -\lambda_{-n,-k}(\omega, -m_c; -M)
\]
for all \( M \in \mathbb{R} \). Therefore, without loss of generality we can always restrict our attention to the case \( k \geq \frac{1}{2} \).

**Theorem 3.1.** For fixed \( k, \omega \) and \( m_c \), the eigenvalue \( \lambda_{n,k} \) of \( A \) satisfies the first-order nonlinear separable differential equation
\[
\frac{d\lambda_{n,k}}{dM} = 2(M\omega + k)\frac{2\omega \lambda_{n,k} - m_c}{4\lambda_{n,k}^2 - 1} \quad (3.4)
\]
where \( \lambda_{n,k}(0) \) is given by (3.3).

**Proof.** For simplicity in notation we omit in the following the indices \( n \) and \( k \) of \( \lambda \). Let \( \tilde{g} \) be an eigenfunction of \( A \) for the eigenvalue \( \lambda \) which is normalized by the condition \( (\tilde{g}, \tilde{g}) = 1 \).

Introducing the functions
\[
U(\vartheta) := \tilde{g}_1^2(\vartheta) + \tilde{g}_2^2(\vartheta), \quad V(\vartheta) := \tilde{g}_2^2(\vartheta) - \tilde{g}_1^2(\vartheta), \quad W(\vartheta) := 2\tilde{g}_1(\vartheta)\tilde{g}_2(\vartheta)
\]
and employing (3.1) it can be easily checked that \( U, V \) and \( W \) satisfy the following system of ODEs:
\[
U'(\vartheta) = -2f(\vartheta)V(\vartheta) + Mm_c \cos \vartheta W(\vartheta), \quad f(\vartheta) = M\omega \sin \vartheta + \frac{k}{\sin \vartheta}, \quad (3.5)
\]
\[
V'(\vartheta) = -2f(\vartheta)U(\vartheta) + 2\lambda W(\vartheta), \quad (3.6)
\]
\[
W'(\vartheta) = 2Mm_c \cos \vartheta U(\vartheta) - 2\lambda V(\vartheta). \quad (3.7)
\]

From analytic perturbation theory (see [15], chapter VII, section 3.4) we have
\[
\frac{d\lambda}{dM} = \left( \frac{dA}{dM} \tilde{g}, \tilde{g} \right) = \int_0^\pi d\vartheta \tilde{g}'(\vartheta) \begin{pmatrix} -m_c \cos \vartheta & \omega \sin \vartheta \\ \omega \sin \vartheta & m_c \cos \vartheta \end{pmatrix} \tilde{g}(\vartheta) = m_c I_1 + \omega I_2 \quad (3.8)
\]
with
\[
I_1 = \int_0^\pi d\vartheta \cos \vartheta V(\vartheta), \quad I_2 = \int_0^\pi d\vartheta \sin \vartheta W(\vartheta).
\]

In addition, from lemma 1 in [17] the following estimates hold:
\[
|U(\vartheta)|, |V(\vartheta)|, |W(\vartheta)| \leq C \sin^{2k} \vartheta \quad (3.9)
\]
with some constant \( C > 0 \). Since without loss of generality \( k \) can be assumed positive, it results that \( U, V \) and \( W \) vanish at \( \vartheta = 0 \) and \( \vartheta = \pi \). If we multiply (3.6) by \( \sin \vartheta \), integrate by parts and take into account that \( \int_0^\pi d\vartheta U(\vartheta) = 1 \) we obtain
\[
2(M\omega + k) = I_1 + 2\lambda I_2 + 2M\omega I_3, \quad I_3 := \int_0^\pi d\vartheta \cos^2 \vartheta U(\vartheta). \quad (3.10)
\]
If we multiply (3.7) by \( \cos \vartheta \) and integrate by parts we get
\[
2 \lambda I_1 + I_2 = 2 M m_e I_3. \tag{3.11}
\]

The next step consists in computing the integral entering on the lhs in (3.10) and (3.11). By means of (3.6) we can rewrite (3.5) and (3.7) in terms of the functions \( U \) and \( V \) and their first derivatives as follows:
\[
U'(\vartheta) + 2 f(\vartheta) V(\vartheta) = \frac{M m_e}{2 \lambda} \cos \vartheta (V'(\vartheta) + 2 f(\vartheta) U(\vartheta)), \tag{3.12}
\]
\[
(V'(\vartheta) + 2 f(\vartheta) U(\vartheta))' = 4 M m_e \lambda \cos \vartheta U(\vartheta) - 4 \lambda^2 V(\vartheta). \tag{3.13}
\]

If we derivate (3.12) once with respect to \( \vartheta \) and make use of (3.13) we obtain
\[
(U'(\vartheta) + 2 f(\vartheta) V(\vartheta))' = -\frac{M m_e}{2 \lambda} \sin \vartheta (V'(\vartheta) + 2 f(\vartheta) U(\vartheta)) + 2 M m_e \cos \vartheta (M m_e \cos \vartheta U(\vartheta) - \lambda V(\vartheta)). \tag{3.14}
\]

Insertion of (3.16) into (3.8) gives (3.4). This completes the proof. \( \square \)

**Theorem 3.2.** For fixed \( k, \omega \neq 0 \) and \( m_e \) there exists a unique solution of (3.4) subjected to the initial condition (3.3). The solution in the implicit form is given by
\[
\omega \left( \lambda_{n,k}^2(M) - \lambda_{n,k}^2(0) \right) + m_e (\lambda_{n,k}(M) - \lambda_{n,k}(0)) = \frac{\omega^2 - m_e^2}{2 \omega} \ln \left( \frac{2 \omega \lambda_{n,k}(M) - m_e}{2 \omega \lambda_{n,k}(0) - m_e} \right) = M \omega^2 (M \omega + 2k). \tag{3.17}
\]

**Proof.** For simplicity in notation we omit in the following the indices \( n \) and \( k \) of \( \lambda \). Since (3.4) is separable the solution (3.17) satisfying the initial condition (3.3) can be obtained by computing the integrals entering in the following expression:
\[
\int_{\lambda_{(0)}}^{\lambda_{(M)}} d\lambda \frac{4 \lambda^2 - 1}{2 \omega \lambda - m_e} = 2 \int_0^M dM (M \omega + k). \tag{3.18}
\]

**Corollary 3.3.** For fixed \( k, m_e \) and \( \omega = 0 \) the solution of (3.4) subjected to the initial condition (3.3) is given by
\[
\lambda_{n,k}(M) = 1 + \left( -3 c_{n,k}(M) + \sqrt{9 c_{n,k}^2(M) - 1} \right)^{2/3},
\]
\[
c_{n,k}(M) = 2 m_e k M + \lambda_{n,k}(0) - \frac{4}{3} \lambda_{n,k}^3(0)
\]
for \( |c_{n,k}(M)| \geq 1/3. \)


Proof. For simplicity in notation we omit in the following the indices \( n \) and \( k \) of \( \lambda \). When \( \omega = 0 \), (3.4) reduces to
\[
\frac{d\lambda_{n,k}}{M} = \frac{2m_k}{4\lambda^2 - 1}.
\]
Taking into account that the above ODE is separable, the computation of
\[
\int_{\lambda(0)}^{\lambda(M)} d\lambda(4\lambda^2 - 1) = -2m_k \int_0^M dM
\]
gives rise to the following cubic equation for \( \lambda \):
\[
\frac{4}{3} \lambda^3(M) - \lambda(M) + c = 0, \quad c := 2kMm_c + \lambda(0) - \frac{2}{3} \lambda^3(0).
\]
Since \( c \in \mathbb{R} \) the only real root of the above equation is
\[
\lambda = 1 + \left(-3c + \sqrt{9c^2 - 1}\right)^{2/3}/2\sqrt{-3c + \sqrt{9c^2 - 1}}
\]
for \( 9c^2 \geq 1 \). \( \Box \)

4. A quasi-linear PDE for the eigenvalues \( \lambda \)

Analogous to [17] we can study the eigenvalues of the angular problem as a function of the parameters \( \mu := Mm_c \) and \( \nu := M\omega \). Since the procedure is the same as in [17] with the only exception that now \( a = M \) where \( a \) is the angular momentum per unit mass of the black hole we limit ourselves to present the main result.

Theorem 4.1. For fixed \( k \) and \( n \) the \( \text{n}^{th} \) eigenvalue \( \lambda = \lambda_n(k; \mu, \nu) \) is an analytical function in \( (\mu, \nu) \in \mathbb{R}^2 \) satisfying the first-order quasi-linear partial differential equation
\[
(\mu + 2\nu\lambda) \frac{\partial \lambda}{\partial \mu} + (\nu + 2\mu\lambda) \frac{\partial \lambda}{\partial \nu} + 2k\mu - 2\mu\nu = 0 \quad (4.1)
\]
with \( \lambda_n(k, 0, 0) \) given by (3.3).

Proof. Same proof as in theorem 1, section III in [17] with \( \nu \) replaced now by \( -\nu \). \( \Box \)

In order to derive formal power series solutions of (4.1) it is convenient to introduce a new function \( \Lambda(\mu, \nu) \) defined by the relation \( \lambda(\mu, \nu) = \Lambda(\mu, \nu) + \lambda_n(0, 0) \) and new independent variables \( \bar{\mu} := \mu - \nu \) and \( \bar{\nu} := \mu + \nu \). Hence, (4.1) becomes
\[
a_1(\bar{\mu}, \bar{\nu}, \Lambda) \frac{\partial \Lambda}{\partial \bar{\mu}} + a_2(\bar{\mu}, \bar{\nu}, \Lambda) \frac{\partial \Lambda}{\partial \bar{\nu}} = f(\bar{\mu}, \bar{\nu}) \quad (4.2)
\]
with
\[
a_1(\bar{\mu}, \bar{\nu}, \Lambda) := \bar{\mu} [1 - 2(\Lambda + \lambda_n(0, 0))], \quad a_2(\bar{\mu}, \bar{\nu}, \Lambda) := \bar{\nu} [1 + 2(\Lambda + \lambda_n(0, 0))]
\]
(4.3)
and
\[
f(\bar{\mu}, \bar{\nu}) := \frac{1}{2} (\bar{\nu}^2 - \bar{\mu}^2) - k(\bar{\nu} + \bar{\mu}). \quad (4.4)
\]
Since \( \Lambda(0, 0) = 0, a_i(0, 0, 0) = 0 \) for all \( i = 1, 2 \) and \( f(0, 0) = 0 \) we can apply a method similar to that developed in [18] to study formal power series solutions of (4.2). In what
follows we are interested in the existence and uniqueness of the formal solution
\[ \Lambda(x) = \sum_{|\alpha| \geq 1} \Lambda_\alpha x^\alpha, \quad \alpha = (m, n) \in \mathbb{N}^2, \quad |\alpha| = m + n, \quad x = (\tilde{\mu}, \tilde{v}) \in \mathbb{R}^2 \] (4.5)
centered at the origin for the equation (4.2). Moreover, we will investigate the convergence of the formal power series solutions (4.5) by computing its Gevrey order. We recall that a function \( F(x) \) with \( x = (x_1, x_2) \in \mathbb{R}^2 \) is said to be of Gevrey-\( \{s\} \) class with \( (s_1, s_2) \in \mathbb{R}^2 \) if the power series
\[ B_s[F](x) = \sum_{|\alpha| \in \mathbb{N}^2} F_\alpha x^\alpha \quad 1^{(2)} := (1, 1), \quad (\alpha!)^{(2)} := (m!)^{s_1 - 1}(n!)^{s_2 - 1} \]
converges in a neighborhood of \( x = 0 \). By \( G^{(s)} \) we denote the set of all formal power series of Gevrey-\( \{s\} \) class. Furthermore, \( F(x) \in G^{(1,1)} \) if and only if \( F(x) \) is a convergent power series near \( x = 0 \). For further details we refer to [19].

Let \( \mathcal{J} \) be the Jacobi matrix of the vector field
\[ x \mapsto (a_1(x, 0), a_2(x, 0)) = (\tilde{\mu}(1 - 2\lambda_n(k, 0, 0)), \tilde{v}(1 + 2\lambda_n(k, 0, 0))) \]
with \( x \) defined by (4.5). Then, we have
\[ \mathcal{J} = \left( \frac{\partial a_i(x, 0)}{\partial x_j} \right)_{x=0, i,j=1,2} = \left( \begin{array}{cc} 1 - 2\lambda_n(k, 0, 0) & 0 \\ 0 & 1 + 2\lambda_n(k, 0, 0) \end{array} \right). \] (4.6)

**Lemma 4.2.** For all \( k = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \) and \( n \in \mathbb{Z} \setminus \{0\} \) it results \( \det \mathcal{J} \neq 0 \).

**Proof.** An elementary computation involving (3.3) gives
\[ \det \mathcal{J} = 1 - 4\lambda_n^2(k, 0, 0) = 1 - 4(k - 1/2 + \frac{1}{2} + |n|)^2. \]
Since \( |k| \geq 1/2 \) and \( |n| \geq 1 \) it follows that \( |k| - \frac{1}{2} + |n| \geq 1 \). Hence, \( \det \mathcal{J} \leq -3 \). \( \square \)

In the next lemma we show that the so-called Poincaré condition is satisfied by (4.2).

**Lemma 4.3.** For all \( \alpha \in \mathbb{N}^2 \) it results
\[ |\lambda_1 m + \lambda_2 n - f_\alpha(0)| > c|\alpha|, \quad f_\alpha(0) := \frac{\partial f}{\partial \Lambda} \bigg|_{x=0} \] (4.7)
where \( \lambda_1 \) and \( \lambda_2 \) denote the eigenvalues of the Jacobi matrix \( \mathcal{J} \), \( f \) is given by (4.4) and \( c \) is a positive constant independent of \( \alpha \in \mathbb{N}^2 \).

**Proof.** Taking into account that \( \frac{\partial f}{\partial \Lambda} = 0 \) and employing (4.6) we obtain
\[ |\lambda_1 m + \lambda_2 n - f_\alpha(0)|^2 = (1 - 2\lambda_n(0))^2 m^2 + (1 + 2\lambda_n(0))^2 n^2 + 2(1 - 4\lambda_n^2(0))mn \] (4.8)
where \( \lambda_n(k, 0, 0) \) if \( \lambda_n(k, 0) \). To prove (4.7) we have to distinguish between the cases \( n > 0 \) and \( n < 0 \). In what follows we give the proof for \( n > 0 \) since the case \( n < 0 \) can be treated analogously. Let us rewrite (3.3) as \( \lambda_{n,k}(0) = \text{sign}(n)\lambda_{n,k}(0) \) with \( \lambda_{n,k}(0) = |k| - \frac{1}{2} + |n| \). For \( n > 0 \) (4.8) becomes
\[ |\lambda_1 m + \lambda_2 n - f_\alpha(0)|^2 = (1 - 2|\lambda_n(k, 0)|)^2 m^2 + (1 + 2|\lambda_n(k, 0)|)^2 n^2 + 2(1 - 4\lambda_n^2(k, 0))mn. \]
Since \( 1 + 2|\lambda_n(k, 0)| > 1 - 2|\lambda_n(k, 0)|) \) for all \( n \in \mathbb{N} \setminus \{0\} \) and \( k = \pm 1/2, \pm 3/2, \ldots \) the above expression can be majorized as
\[ |\lambda_1 m + \lambda_2 n - f_\alpha(0)|^2 \geq |1 - 2|\lambda_n(k, 0)||^2(m + n)^2. \]
Hence, (4.7) is satisfied for some positive constant \( c := |1 - 2|\lambda_n(k, 0)||. \) \( \square \)
Finally, note that
\[
\left. \frac{\partial a_i}{\partial \Lambda} (\tilde{\mu}, \tilde{\nu}, \Lambda) \right|_{(0,0,0)} = 0 \quad \text{for all} \quad i = 1, 2. \tag{4.9}
\]
Since the Poincaré condition (4.7) and (4.9) are satisfied, theorems 1.1–2, section 1.2 in [18] imply that the equation (4.2) has a unique formal power solution
\[
\Lambda(\tilde{\mu}, \tilde{\nu}) = \sum_{|\alpha| \geq 1} \Lambda_\alpha \tilde{\mu}^m \tilde{\nu}^n, \quad \alpha = (m, n) \in \mathbb{N}_0^2, \quad |\alpha| = m + n. \tag{4.10}
\]
Furthermore, since \( \Lambda \in G^{(1,1)} \) it results that (4.10) converges. The last task is to compute a recurrence relation for the coefficients \( \Lambda_\alpha \). To this purpose we rewrite (4.10) as follows:
\[
\Lambda(\tilde{\mu}, \tilde{\nu}) = \sum_{m,n=0}^\infty \Lambda_{m,n} \tilde{\mu}^m \tilde{\nu}^n, \quad m + n \geq 1. \tag{4.11}
\]
Since
\[
\Lambda(\tilde{\mu}, \tilde{\nu})^2 = \sum_{m,n=0}^\infty \left( \sum_{r=0}^m \sum_{s=0}^n \Lambda_{r,s} \Lambda_{m-r,n-s} \right) \tilde{\mu}^m \tilde{\nu}^n, \quad m + n \geq 1, \quad r + s \geq 1
\]
from (4.2) we obtain the identity
\[
\sum_{m,n=0}^\infty \left( (m \pm n + nc_+) \Lambda_{m,n} + (n - m) \sum_{r=0}^m \sum_{s=0}^n \Lambda_{r,s} \Lambda_{m-r,n-s} \right) \tilde{\mu}^m \tilde{\nu}^n = \frac{1}{2} (\tilde{\mu}^2 - \tilde{\nu}^2) - k(\tilde{\nu}^2 + \tilde{\mu})
\]
with \( m + n \geq 1 \) and \( c_\pm := 1 \pm 2 \lambda_\alpha(k, 0, 0) \). If we compare the terms of equal order in \( \tilde{\mu} \) and \( \tilde{\nu} \) it follows that
\[
\begin{align*}
\Lambda_{1,0} &= \frac{k}{2 \lambda_\alpha(k, 0, 0) - 1}, & \Lambda_{0,1} &= -\frac{k}{2 \lambda_\alpha(k, 0, 0) + 1}, & \Lambda_{11} &= 0, \\
\Lambda_{2,0} &= \frac{(2 \lambda_\alpha(k, 0, 0) - 1)^2 - 4 k^2}{4(2 \lambda_\alpha(k, 0, 0) - 1)^3}, & \Lambda_{0,2} &= \frac{(2 \lambda_\alpha(k, 0, 0) + 1)^2 - 4 k^2}{4(2 \lambda_\alpha(k, 0, 0) + 1)^3},
\end{align*}
\]
and for \( m + n \geq 3 \) the coefficients \( \Lambda_{m,n} \) are given by the relation
\[
\Lambda_{m,n} = \frac{m - n}{m \pm n + nc_+} \sum_{r=0}^m \sum_{s=0}^n \Lambda_{r,s} \Lambda_{m-r,n-s}, \quad r + s \geq 1. \tag{4.12}
\]
Finally, note that for \( m = n \) we have \( \Lambda_{n,n} = 0 \) for all \( n \in \mathbb{N} \).

5. Analysis of the radial system (2.5)

In this section we show that there exists no bound state for the Dirac equation in the extreme Kerr metric. The main idea behind the proof is that after a suitable transformation the deficiency indices of the transformed radial operator are zero. In fact, the deficiency index of a differential operator simply counts the number of square integrable solutions.

**Theorem 5.1.** The solution set of the system (1.1) with (1.2) or (1.3) is empty.

**Proof.** In order to apply some results of [20] we bring the radial system (2.5) in a more amenable form by transforming the dependent variable as \( f(r) = (f_1(r), f_2(r))' := (F(r) - iG(r), F(r) + iG(r))' \). Moreover, we introduce a new independent variable \( x \) defined

\[
\frac{\partial a_i}{\partial \Lambda} (\tilde{\mu}, \tilde{\nu}, \Lambda) |_{(0,0,0)} = 0 \quad \text{for all} \quad i = 1, 2. \tag{4.9}
\]
by the relation
\[ \frac{dx}{dr} = \frac{r^2 + M^2}{(r - M)^2}. \]

The solution of the above equation is
\[ x(r) = r - \frac{2M^2}{r - M} + 2M \ln (r - M) \]
and it can be easily seen that \( x \in \mathbb{R} \) since \( x \to +\infty \) for \( r \to +\infty \) and \( x \to -\infty \) for \( r \to M^+ \).

Let \( \mathbb{E} = (F, G)^T \). Hence, (2.5) becomes
\[(\mathbb{E}, \mathbb{E}) = J \frac{d\mathbb{E}}{dx} + B(x) \mathbb{E} = \omega \mathbb{E} \tag{5.1}\]
with
\[ J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \quad B(x) = \begin{pmatrix} \frac{m_x(r(x) - M) + kM}{r^2(x) + M^2} & \frac{\lambda_x(r(x) - M)}{r^2(x) + M^2} \\ \frac{\lambda_x(r(x) - M)}{r^2(x) + M^2} & \frac{m_x(r(x) - M) + kM}{r^2(x) + M^2} \end{pmatrix}. \]

According to the above transformations the integrability condition (2.7) for the radial spinors simplifies to
\[(\mathbb{E}, \mathbb{E}) = \int_{-\infty}^{+\infty} dx (F^2(x) + G^2(x)) < \infty.\]

Note that the formal differential expression \( \mathbb{E} \) is formally symmetric since \( J = -J^* \) and \( B = B^* \). Let \( S_{\text{min}} \) be the minimal operator associated with \( \mathbb{E} \) such that \( S_{\text{min}} \) acts in the Hilbert space \( L^2(\mathbb{R}, dx)^2 \) with respect to the scalar product \((\cdot, \cdot)\). The operator \( S_{\text{min}} \) with the domain of the definition \( D(S_{\text{min}}) = C_0^\infty(\mathbb{R})^2 \) such that \( S_{\text{min}} \mathbb{E} = \mathbb{E} \mathbb{E} \) for \( \mathbb{E} \in D(S_{\text{min}}) \) is densely defined and closable. Let \( S \) denote the closure of \( S_{\text{min}} \). We apply the so-called decomposition method due to Neumark [21]. For this purpose, let \( S_{\text{min}, \pm} \) be the minimal operators associated with \( \mathbb{E} \) when restricted on the half-lines \([0, \infty)\) and \((-\infty, 0)\), respectively. We consider \( S_{\text{min}, \pm} \) acting in the Hilbert spaces \( L^2(\mathbb{R}_\pm, dx)^2 \) with respect to the scalar product \((\cdot, \cdot)\). The operators \( S_{\text{min}, \pm} \) given by \( D(S_{\text{min}, \pm}) = C_0^\infty(\mathbb{R}_\pm)^2 \) with \( S_{\text{min}, \pm} \mathbb{E}_\pm := \mathbb{E} \mathbb{E}_\pm \) for \( \mathbb{E}_\pm \in D(S_{\text{min}, \pm}) \) are densely defined and closable. Note that since the formal differential operator \( \mathbb{E} \) is in the limit point case at \( \pm \infty \) the operators \( S_{\text{min}, \pm} \) are even essentially self-adjoint. In the following we denote the closure of \( S_{\text{min}, \pm} \) by \( S_\pm \). Let \( N_\pm(S_\pm) \) be the deficiency indices of the system (5.1) and let us denote by \( \kappa_\pm \) the number of positive and negative eigenvalues of the matrix \( iJ \). Clearly, we have \( \kappa_+ = 1 = \kappa_- \). Theorem 5.2 (see section 5.1 in [20]) implies that \( N_+(S_+) = 1 = N_-(S_-) \).

According to definition 2.14 (section 2.5 in [20]) the system (5.1) is definite on \( \mathbb{R}_+ \) and on \( \mathbb{R}_- \) (and proposition 5.4 (section 5.1 in [20]) implies that the deficiency indices for \( S \) are
\[ N_+(S) = N_+(S_+) + N_-(S_-) - 2 = 0. \]
Therefore, the system (5.1) does not admit any square integrable solution on the whole real line and this completes the proof. \( \Box \)

6. Conclusions

To conclude, reference [2] showed the existence of eigenvalues in the Kerr–Newman metric under the assumptions that conditions (1.1) with (1.2) or (1.3) are satisfied. On the other hand, in the non-extreme Kerr metric the results of reference [11] (i.e. the non-existence of bound states) imply that these conditions cannot be compatible. This is the bottom line for the combined results of reference [2, 11]. In the present paper, using different methods we
demonstrated a similar result for the extreme Kerr metric. Relying on the results by Lesch and Malamud [20] we proved the absence of eigenvalues in this case. Hence, as in the case of non-extreme Kerr metric, conditions (1.1) with (1.2) or (1.3) cannot be satisfied and therefore the (discrete) eigenvalues do not exist.

We also mention that in [3] a set of necessary and sufficient conditions for the existence of an energy eigenvalue for the Dirac equation in the extreme Kerr–Newman metric have been derived. However, also in that case the conditions are so complicated that it is not clear at all if they admit at least one non-trivial solution. We believe that the approach we used to show the absence of bound state solutions for the Dirac equation in the extreme Kerr metric should work as well in the case of the extreme Kerr–Newman metric since in the latter case the radial system can be written again in a form similar to equation (5.1).

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