Stable Hollow-lifting Modules and related concepts

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Abstract. In this paper, we introduce and study the concept of stable hollow-lifting modules as a generalization of hollow-lifting modules and stable lifting modules. We call a module is stable hollow-lifting if every stable submodule S such that M/S is hollow there is a direct summand D such that D \subseteq S and S/D \ll M/D. Moreover, we call a module M is strongly stable hollow-lifting if stable submodule S such that M/S is hollow there is a direct summand D such that D \subseteq S and S/D \ll M/D. Several characterizations and properties of (strongly) stable hollow-lifting modules are obtained. Also, the relation among the class stable hollow-lifting modules and other known classes of modules is investigated. Moreover, unlike hollow-lifting modules we prove that a finite direct sum of stable hollow-lifting modules is stable hollow-lifting.

Key words: Lifting Modules, Hollow-lifting Modules, stable hollow-lifting modules, strongly hollow-lifting Modules.

1. Introduction:

Let R be a ring with identity and M be a left R-module. Recall that, a module M is lifting if every submodule is coessential in a direct summand of M [3]. Many authors studied lifting modules and their generalizations. A module M is called hollow-lifting if for every submodule N of M with M/N hollow there is a direct summand D such that D \subseteq N and N/D is small in M/D [6]. Also, a module M is stable lifting if every stable submodule of M is coessential in a direct summand of M [2]. A module M is called FI-hollow-lifting if every fully invariant submodule N of M with M/N hollow is coessential in a direct summand of M [3].

Throughout this paper R will denote arbitrary associative ring with identity and all R-modules are unitary left R-module. N \subseteq M will mean N is a submodule of a module M. A submodule N of a module M is called a small in M (denoted by N \ll M ) if for any X \subseteq M, M=N+X implies X=M. An R-module M is called hollow if every proper submodule is small in M. The module M is called local if has a unique maximal submodule N which contains all proper submodules of M. Let K, N be submodules of M such that K \subseteq N \subseteq M. K is called lie above a N in M if N/K \ll M/K [9]. Recall that a submodule K of M is fully invariant if g(K) \subseteq K for all g \in \text{End} (M ). An R-module M is called duo if every submodule of M is fully invariant [8]. Moreover, a submodule of a module M is a called a stable if f(N) \subseteq N for each homomorphism f:N \rightarrow M. An R-module is called fully stable if every submodule of M is stable [1]. Moreover, every fully invariant direct summand is stable [2].

2. Stable hollow-lifting modules.
Definition (2.1): A module $M$ is called stable hollow-lifting if every stable submodule $S$ of $M$ with $M/S$ is hollow is lies above a direct summand of $M$.

Examples and Remarks (2.2):

1- Every hollow-lifting module is stable hollow-lifting but the converse is not true in general. For example, $Z_2 \oplus Z_6$ as $Z$-module is stable hollow-lifting which it is not hollow-lifting.

2- Every FI-hollow-lifting module is stable hollow-lifting while the converse is not true in general. For example, $Z$ as $Z$-module is stable hollow-lifting which it is not FI-hollow-lifting.

3- Every stable lifting (and then FI-lifting) module is stable hollow-lifting but the converse is not true in general (we have no example yet).

Theorem (2.3): A module $M$ is stable hollow-lifting if and only if for each stable submodule $S$ with $M/S$ is hollow has a supplement submodule $T$ such that $S \cap T$ is direct summand of $M$.

Proof : ($\Rightarrow$). Let $S$ be a stable submodule of stable hollow-lifting module $M$. Then $S$ contains a direct summand $D$ of $M$ such that $S/D \ll M/D$. Let $H$ be a submodule of $M$ such that $M=H \oplus D$. By modular law, $S=H \oplus (S \cap H)$. It is clear that $M=H+T$. Let $(S \cap H)+X=H$ where $X \subseteq H$. So $M=H+T=H+X$. Thus $M=H+X$ and $M/D=(S+X)/D=S/D+(X+D)/D$. But $S/D \ll M/D$, then $M=H+X$. Since $M=H \oplus D$ and $X \subseteq H$ then $X=H$ and so $S/H \ll M/H$. Therefore, $S$ has strong supplement submodule $H$ in $M$.

($\Leftarrow$). Let $S$ be a stable submodule of $M$. By hypothesis, $S$ has a strong supplement submodule $T$ in $M$. Thus $M=\langle S+T, S \cap T \rangle$ and there is a submodule $H$ of $S$ such that $S=(S \cap T) \oplus H$. So $M=\langle S+T \rangle=(S \cap T)+H+T=H+T$. It is obvious that $H \cap T=0$ and hence $M=H \oplus T$. Let $S/H+X/H=M/H$ for a submodule $X$ of $M$ contains $H$. Then, $M=\langle S+X=(S \cap T)+H+X \rangle$. But $S \cap T \ll S$ so $S \cap T \ll M$, then $M=H+X$. Since $H \subseteq X$, then $M=X$ and so $S/H \ll M/H$. Hence, $M$ is stable hollow lifting module.

Proposition (2.4): The following statements are equivalent for a module $M$:

1- $M$ is stable hollow-lifting.

2- For every stable submodule $S$ such that $M/S$ is hollow there is a decomposition $M=M_1 \oplus M_2$ such that $M_1 \subseteq S$ and $S \cap M_2 \ll M_2$.

3- For every stable submodule $S$ such that $M/S$ is hollow there is a decomposition $M=M_1 \oplus M_2$ such that $M_1 \subseteq S$ and $S \cap M_2 \ll M_2$.

Proof : (1$\Rightarrow$2) Let $S$ be a stable submodule of $M$ such that $M/N$ is hollow. By stable hollow-lifting property of $M$, there is a direct summand $M_1$ of $M$ which contained in $S$ such that $S/M_1 \ll M/M_1$. Let $M_2$ be a submodule of $M$ such that $M=M_1 \oplus M_2$. Let $(S \cap M_2)+Y=M_2$ where $Y \subseteq M$. Then $M=M_1+M_2=M_1+(S \cap M_2)+Y$. Also, $M/M_1=M_1+(S \cap M_2)/M_1+(Y/M_1)/M_1$. Since $S/M_1 \ll M/M_1$, then $M_1+(S \cap M_2)/M_1 \ll M/M_1$ (by [9]). Hence $Y+M_1=M$ and since $M=M_1 \oplus M_2$ and $Y \subseteq M_2$, then $Y=M_2$. This means $S \cap M_2 \ll M_2$.

(2$\Rightarrow$3) It is clear.

(3$\Rightarrow$1) Let $S$ be a stable submodule of $M$ such that $M/S$ is hollow. By (3), there is a decomposition $M=M_1 \oplus M_2$ such that $M_1 \subseteq S$ and $S \cap M_2 \ll M$. We claim that $S/M_1 \ll M/M_1$. Let $S/M_1+X/M_1=M/M_1$ where $X/M_1 \subseteq M/M_1$. Then $S+X=M$ and so by using the Modular law $S=M_1 \oplus (S \cap M_2)$. Thus $M=S+X=M_1 \oplus (S \cap M_2)+X$. But $S \cap M_2 \ll M$, then $M=M_1+X=M$. Therefore, $S/M_1 \ll M/M_1$ (i.e.) $S$ with $M/S$ hollow is lies above a direct summand of $M$. Then $M$ is stable hollow-lifting.
Proposition (2.5): A module $M$ is stable hollow-lifting if and only if every stable submodule $S$ of $M$ decomposes as $S = D \oplus H$ where $D$ is a direct summand of $M$ and $H \subseteq M$.

Proof: The proof essentially follows the proof of [6, Proposition 2.5.] so it is omitted.

It is well-known that classes of lifting modules and hollow-lifting modules are not closed under a (finite) direct sum. In fact, $Z_2 \oplus Z_3$ as $Z$-module is not lifting (see [4]) as well as is not hollow-lifting (see [7, Example 6.1]) while $Z_2$ and $Z_3$ are lifting (and then hollow-lifting) $Z$-modules. Here, we assert that a finite direct sum of stable hollow-lifting modules is stable hollow-lifting.

Proposition (2.6): A finite direct sum of stable hollow-lifting modules is stable hollow-lifting.

Proof: Let $M = \bigoplus_{i=1}^{n} M_i$ where $M_i$ is stable hollow-lifting module for all $i = 1 \ldots n$ and let $S$ be a stable submodule of $M$ with $M/S$ hollow. Following [4, proposition (4.5)], $S = \bigoplus_{i=1}^{n} (M_i \cap S)$ and $M_i \cap S$ is a stable submodule of $M_i$ for each $i = 1 \ldots n$. From the known fact that epimorphic image of hollow module is hollow [4], since $M/S$ is hollow so $M/(M \cap S)$. Now Since $M_i$ is stable hollow-lifting for each $i$, there is a direct summand $D_i$ of $X_i$ such that $D_i \subseteq M_i \cap S$ and $(M \cap S)/D_i \subset M/D_i$ for every $i$. It is clear that $D = \bigoplus_{i=1}^{n} U_i$ is direct summand of $M$ and since a finite sum of small submodules is small, $\bigoplus_{i=1}^{n} (M_i \cap S)/D_i \subset \bigoplus_{i=1}^{n} U_i = M/D$. Thus, one can conclude that $M = \bigoplus_{i=1}^{n} M_i$ is stable hollow-lifting.

Corollary (2.7): A finite direct sum of hollow-lifting modules is stable hollow-lifting.

Proposition (2.8): A module $M$ is stable hollow-lifting if and only if for any stable submodule $S$ of $M$ such that $M/S$ is hollow there is an idempotent endomorphism $\alpha$ of $M$ such that $\alpha(M) \subseteq S$ and $(I - \alpha)(S)$ is small in $(I - \alpha)(M)$.

Proof: Let $S$ be a stable submodule of $M$ with $M/S$ hollow. By Theorem (2.3), since $M$ is stable hollow-lifting module, then $S$ has a strong supplement submodule $D$ in $M$. Thus $M = S + D$, $S \cap D < D$ and $S = (S \cap D) \oplus X$ where $X$ is a submodule of $H$. Hence $M = S + D = (S \cap D) + X + D$. Let $\alpha : M \rightarrow X$ be the natural projection mapping. One can easily see that $\alpha$ is idempotent endomorphism of $M$ and also $(I - \alpha)(S) = S \cap (I - \alpha)(M) = S \cap D < D$. So $(I - \alpha)(S) \subset (I - \alpha)(M)$. Conversely, let $S$ be a stable submodule of $M$ with $M/S$ hollow. By hypothesis, there is an idempotent endomorphism $\alpha$ with $\alpha(M) \subseteq S$ and $(I - \alpha)(S)$ is small in $(I - \alpha)(M)$. From [10], $M = \alpha(M) \oplus (I - \alpha)(M)$. Also, $(I - \alpha)(S) \subset S \cap (I - \alpha)(M)$, then $S \cap (I - \alpha)(M) \subset (I - \alpha)(M)$. Therefore, $M$ is stable hollow-lifting.

Following [7] (3.1), If $M$ is hollow-lifting (fl-hollow-lifting), then $M/U$ is hollow-lifting (fl-hollow-lifting) for every fully invariant submodule $U$ of $M$. For stable hollow-lifting modules this result is not still valid, for example $Z$ as $Z$-module is stable hollow-lifting while $Z/12Z$ is not stable hollow-lifting $Z$-module and $12Z$ is fully invariant submodule of $Z$.

We call a module $M$ have the property $S$ if every nonzero submodule of $M$ contains a stable submodule of $M$ is stable.

Proposition (2.9): Let $M$ be a stable hollow-lifting module which have the property $S$. Then, $M/S$ is stable hollow-lifting for every submodule $S$ of $M$.

Proof: Let $S$ be a stable submodule of $M$ and let $K/S$ be a stable submodule of $M/S$. By the property $S$ of $M$ thus $K$ is stable submodule of $M$. Since $M$ is stable hollow-lifting, then there is a direct summand $D$ of $M$ such that $D \subseteq K$ such that $K/D < M/D$. Let $M = D \oplus H$ where $H$ is a submodule of $M$. So $S + D \subseteq K$ and then $S + D/S \subseteq K/S$. Let $\alpha : M/D \rightarrow M/S + D$ as follows $\alpha(m + D) = m + (S + D)$ for all $m \in M$. It is easy to check that $\alpha$ it is epimorphism. Now, since $K/D < M/D$ so $\alpha(K/D) \subseteq M/S + D$ and hence $K/S = D < M/S + D$. So by Third Isomorphism Theorem, $(K/S)/(S + D/S) \subset (M/(S + D)/S)$ in $M/S$. Since every stable submodule is fully invariant [8] so by [6], $M/S = D \oplus H/S = (D \oplus H/S) \oplus (H \oplus S/S)$. In other words, $(D \oplus S/S)$ is direct summand of $M/S$. Therefore, $M/S$ is stable hollow-lifting.
Corollary (2.10): Let $M$ be a fully stable module. Then, if $M$ is stable hollow-lifting, then $M/S$ is stable hollow-lifting for each submodule $S$ of $M$.

Proposition (2.11): Every stable coclosed submodule $N$ of stable hollow-lifting module $M$ with $M/N$ hollow is direct summand.

Proof: Let $N$ be a stable coclosed submodule of $M$ with $M/N$ is hollow. Since $M$ is stable hollow-lifting, then $N$ contains a direct summand $D$ such that $N/D mM/D$. So by coclosed of $N$, $N = D$ and so $N$ is direct summand of $M$.

Proposition (2.12): Let $M$ be an indecomposable stable hollow-lifting module. Then, every maximal submodule $S$ of $M$ is unique.

Proof: Let $S$ be a maximal submodule of $M$ which is stable and let $K$ be another maximal submodule of $M$. Thus $S + K = M$ and also since $S$ is maximal so $M/S$ is simple and hence is hollow. By hypothesis, $M$ is stable hollow-lifting, then there is a direct summand $D$ of $M$ such that $S/D mM/D$. But $M$ is indecomposable, thus $D = (0)$ and so $S$ is hollow. Thus $K = M$ which is contradiction. Therefore, $S$ is unique maximal.

Proposition (2.13): Let $M$ be stable hollow-lifting module with $Rad(M) = 0$. Then every maximal submodule $S$ of $M$ with $M/S$ is direct summand.

Proof: Let $S$ be a stable of $M$ with $M/S$ is hollow. By hypothesis, $M$ is stable hollow-lifting, then $S = D + K$ where $D$ is a direct summand of $M$ and $K \ll M$. But $Rad(M) = 0$ so $D = (0)$. Thus $S = D$ which is direct summand.

3. Strongly stable hollow-lifting Modules

Definitions (3.1): A module $M$ is called strongly stable hollow-lifting if every stable submodule $S$ of $M$ with $M/S$ is hollow is lies above a stable direct summand of $M$.

Directly, from above definition, we have every strongly stable hollow-lifting module strongly hollow-lifting. The converse is not true; by Proposition (2.6) the $Z \oplus Z_s$ as $Z$-module is stable hollow-lifting while it is not strongly stable hollow-lifting.

Remark (3.2):

i. Every strongly FI-hollow-lifting module is strongly stable hollow-lifting. Note that $Z$ as $Z$-module is strongly stable hollow-lifting which it is not strongly FI-hollow-lifting.

ii. In general there is no directly relation between hollow-lifting modules and strongly stable hollow-lifting modules. Indeed, $Z$ as $Z$-module is strongly stable hollow-lifting while it is not hollow-lifting. On other direction, $Z \oplus Z_4$ is lifting $Z$-module [4] and hence is hollow-lifting while it is not strongly stable hollow-lifting.

By using the same technique of [10, (4.11.11)], we have the next result:

Proposition (3.3): The following are equivalent for a module $M$:

i. $M$ is strongly stable lifting.

ii. For each stable submodule $S$ of $M$ with $M/S$ hollow there is a stable direct summand $H_1$ of $M$ which contains $S$ and $S \cap H_2$ is small in $M$ where $M = H_1 \oplus H_2$.

iii. Each stable submodule $S$ of $M$ with $M/S$ hollow can be decomposes as $S = K_1 \oplus K_2$ where $K_1$ is stable direct summand of $M$ and $K_2$ is small submodule of $M$.

Unlike stable hollow-lifting property, strongly stable hollow-lifting property need not be closed under finite direct sum (see the example (ii) in Remark (3.2)).

Proposition (3.4): Let $N = N_1 \oplus N_2$ where $N_1$ and $N_2$ are stable submodules of $N$. If $N_1$ and $N_2$ are strongly hollow-lifting, then so $N$ is strongly hollow-lifting.
Proof: Let $S$ be a stable submodule of $N$ with $N/S$ hollow. Since $N_i+S/S \neq N/S$, then $N/S=N_i+S/S$. But $N_i+S/S \cong N_i/S \cap N_i$ hence $N_i/S \cap N_i$ is hollow. By [1], $S=(S \cap N_j) \oplus (S \cap N_j)$ and $S \cap N_i$ and $S \cap N_j$ are stable submodules of $N_i$ and $N_j$ respectively. Since $N_i$ is strongly stable hollow-lifting, then $S \cap N_i= D_i \oplus K_i$ where $D_i$ is direct summand of $N_i$ and $K_i$ is small in $N_i$. By the same argument, $S \cap N_j= D_j \oplus K_j$ where $D_j$ is direct summand of $N_j$ and $K_j$ is small in $N_j$. Thus $D= D_i \oplus D_j$ is stable direct summand of $N=N_i \oplus N_j$ and $K= K_i \oplus K_j$ is small in $N=N_i \oplus N_j$. But $S=(S \cap N_i) \oplus (S \cap N_j)=(D_i \oplus K_i) \oplus (D_j \oplus K_j)=(D_i \oplus D_j) \oplus (K_i \oplus K_j)$. So $N$ is strongly stable hollow-lifting.

Recall that a module $M$ is weak duo if every direct summand of $M$ is fully invariant [8], equivalently, every direct summand is stable [2].

**Proposition (3.5):** Every weak duo stable hollow-lifting module is strongly stable hollow-lifting.

Proof: It is clear.

**Corollary (3.6):** Every indecomposable stable hollow-lifting module is strongly hollow-lifting.

**Proposition (3.7):** Let $M$ be a fully stable module. Then the following statements are equivalent:

1. $M$ is hollow-lifting.
2. $M$ is stable hollow-lifting.
3. $M$ is strongly stable hollow-lifting.

A module $P$ is said to be Projective if for every epimorphism $\alpha:A \to B$ (where $A$ and $B$ are modules) and for every homomorphism $\beta:P \to B$ there exists a homomorphism $\gamma:P \to A$ such that $\beta=\alpha \gamma$. Moreover, a projective module $P$ is said to be cover of a module $N$ if there is an epimorphism $\alpha:P \to N$ such that $\ker(\alpha)$ is small in $P$. Unlike injective envelope, not each module has a projective cover. [5]

The next result discusses when a factor module of strongly stable hollow-lifting module has projective cover.

**Proposition (3.8):** If $P$ is projective (strongly) stable hollow-lifting module, then for any stable submodule $S$ where $P/S$ hollow, $P/S$ has a projective cover.

Proof: Let $S$ be a stable submodule of $P$ where $P/S$ hollow. Because $P$ is stable hollow-lifting, it implies that $P=P_i \oplus P_j$ where $S$ contains $P_i$ and $P_j \cap S$ is small in $P_j$. Thus, we have the short exact sequence $0 \to P_j \cap S \to P_j/P_j \cap S \to 0$. Since projective property is closed under direct summands [5], so $P_j$ is projective. Therefore, $\ker(\pi)= P_j \cap S \lhd P_j$. So $\pi: P_j \cap S \to P_j/P_j \cap S$ is epimorphism with $P_j$ is projective. So, $P_j \cap S$ have a projective cover. But $P/S=P_i+P_j/S \approx P_i/P_j \cap S$. That is, $P/S$ has a projective cover.

**Remark (3.9):** Note that stable condition in above result is necessary. In fact, $Z$ is stable hollow-lifting $Z$-module and also it is projective but $Z/Z$ has no projective cover and also $2Z$ is not stable submodule of $Z_2$.

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