The Steiner $k$-eccentricity on trees

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Abstract

We study the Steiner $k$-eccentricity on trees, which generalizes the previous one in the paper [X. Li, G. Yu, S. Klavžar, On the average Steiner 3-eccentricity of trees, arXiv:2005.10319, 2020]. To support the algorithm, we achieve much stronger properties for the Steiner $k$-ecc tree than that in the previous paper. Based on this, a linear time algorithm is devised to calculate the Steiner $k$-eccentricity of a vertex in a tree. On the other hand, the lower and upper bounds of the average Steiner $k$-eccentricity index of a tree on order $n$ are established based on a novel technique which is quite different from that in the previous paper but much easier to follow.

Keywords: Steiner distance, Steiner tree, Steiner eccentricity, graph algorithms

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1 Introduction

In this paper we consider connected, simple, undirected graphs $G = (V(G), E(G))$. For basic graph notation and terminology we follow the book of West [28], while for algorithmic

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and computational terminology we use [4, 9].

The standard distance $d_G(u, v)$ between vertices $u$ and $v$ in graph $G$ is the length of a shortest path between $u$ and $v$ in $G$. If $S \subseteq V(G)$, $|S| \geq 2$, then the Steiner distance $d_G(S)$ is the minimum size among all connected subgraphs of $G$ containing $S$, that is,

$$d_G(S) = \min\{|E(T)| : T \text{ is a subtree of } G, S \subseteq V(T)\}.$$

Note that if $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$. If $k \geq 1$, then the Steiner $k$-eccentricity of a vertex $v$ in graph $G$ is

$$\text{ecc}_k(v, G) = \max\{d_G(S) : v \in S \subseteq V(G), |S| = k\}.$$ 

Note that, by definition, $\text{ecc}_1(v, G) = 0$. $S \subseteq V(G)$ is a Steiner $k$-ecc $v$-set if $|S| = k$, $v \in S$, and $d_G(S) = \text{ecc}_k(v, G)$. A corresponding minimum Steiner tree $T$ is called a Steiner $k$-ecc $v$-tree (corresponding to the $k$-set $S$). We will also shortly say that $T$ is a $\text{MST}(S, G)$. The average Steiner $k$-eccentricity of a graph $G$ is the mean value of all vertices’ Steiner $k$-eccentricities in $G$, that is,

$$\text{aecc}_k(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ecc}_k(v, G),$$

which is an extension of the average eccentricity of a graph [7, 8].

The Steiner tree problem is NP-hard on general graphs [9, 16], but it can be solved in polynomial time on trees [2]. The Steiner distance on some special graph classes such as trees, joins, Corona products, threshold and product graphs, has been studied in [1, 3, 11, 23, 26]. The average Steiner $k$-distance is closely related to the $k$-th Steiner Wiener index. Both of them were studied on trees, complete graphs, paths, cycles and complete bipartite graphs [6, 12]. The average Steiner distance and the Steiner Wiener index were investigated in [5, 18, 20], while for some work on the Steiner diameter see [23, 26]. The Steiner $k$-diameter was compared with the Steiner $k$-radius in [15, 24]. Closely related invariants were also studied, for instance Steiner Gutman index [21], Steiner degree distance [13], Steiner hyper-Wiener index [25], multi-center Wiener indices [14], and Steiner (revised) Szeged index [10]. We especiall point to the substantial survey [22] on the Steiner distance and related results and to the recent investigation of isometric subgraphs for Steiner distance [27].

Very recently, the Steiner 3-eccentricity of trees was investigated in [19]. A linear-time algorithm was developed to calculate the Steiner 3-eccentricity of a vertex in a tree, and lower and upper bounds for the average Steiner 3-eccentricity index on trees were derived. In this paper we extend these results to arbitrary $k \geq 2$. In the next section we propose a linear algorithm to calculate the Steiner $k$-eccentricity of a vertex in a tree. In Section 3 we establish lower and upper bounds of the average Steiner $k$-eccentricity on trees. We conclude this paper by presenting several possibilities for future work.
2 Steiner $k$-eccentricity of vertices in trees

The techniques from [19] that enabled to calculate the Steiner 3-eccentricity in a tree are not suitable for calculating the Steiner $k$-eccentricity of a vertex in a tree for arbitrary $k \geq 2$. In this section we establish new, stronger structural properties for the Steiner $k$-ecc $v$-tree for a vertex $v$ in a tree, and then apply them to devise a linear time algorithm to calculate the Steiner $k$-eccentricity of a vertex in a tree.

2.1 Two key structural properties

Before stating the two properties, let us introduce some notation and terminology on trees. A vertex of a tree of degree at least 3 is a branching vertex. Let $L(T)$ denote the set of pendent vertices (leaves) of a tree $T$. If $u$ and $v$ are vertices of a tree $T$, then we will denote the (unique) $u.v$-path in $T$ by $P(u,v,T)$. Given a vertex $v \in V(T)$ and a leaf $u \in L(T)$, let $w$ be the nearest branching vertex to $u$ on $P(v,u,T)$. If there is no branching vertex on $P(v,u,T)$, we set $w = v$. Then we say that the sub-path $P(w,u,T)$ of $P(v,u,T)$ is a quasi-pendent path (with respect to $u$ and $v$).

In the rest we will use the following earlier lemma, also without explicitly mentioning it.

**Lemma 2.1** [19, Lemmas 2.4, 2.5] If $T$ is a tree and $v \in V(T)$, then the following holds.

(i) If $k > |L(T)|$, then every $k$-ecc $v$-set contains all the leaves of $T$. The same conclusion holds if $v$ is a leaf and $k = |L(T)|$.

(ii) If $2 \leq k \leq |L(T)|$ and $S$ is a $k$-ecc $v$-set, then every vertex from $S \setminus \{v\}$ is a leaf of $T$.

For our first structural result, we need one more lemma.

**Lemma 2.2** Let $k \geq 2$, let $v$ be a vertex of a tree $T$, let $T_v^k$ be a Steiner $k$-ecc $v$-tree, and let $T_v^{k-1}$ be a Steiner $(k-1)$-ecc $v$-tree. Then there exists a leaf $u \in L(T_v^k) \setminus L(T_v^{k-1})$ such that the quasi-pendent path $P(w,u,T_v^k)$ has no common edge with $T_v^{k-1}$.

**Proof.** If $k = 2$, then $T_v^1$ is a tree on a single vertex $v$, hence the conclusion is clear. Assume in the rest that $k \geq 3$ and suppose on the contrary that every leaf $u \in L(T_v^k) \setminus L(T_v^{k-1})$ satisfies that the quasi-pendant path $P(w,u,T_v^k)$ has common edges with $T_v^{k-1}$. Then to every leaf $u \in L(T_v^k)$ we can associate its private leaf of $L(T_v^{k-1})$. Hence the number of leaves in $T_v^{k-1}$ is not less than that in $T_v^k$. This contradicts the fact (by Lemma 2.1) that the Steiner $(k-1)$-ecc $v$-set corresponding to $T_v^{k-1}$ has one less element than the Steiner $k$-ecc $v$-set corresponding to $T_v^k$.

\[\Box\]
Theorem 2.3 Let $k \geq 2$, and let $v$ be a vertex of a tree $T$. Then every Steiner $k$-ecc $v$-tree contains some Steiner $(k - 1)$-ecc $v$-tree.

Proof. The case $k = 2$ is trivial, hence assume in the rest that $k \geq 3$. Let $T^k_v$ be a Steiner $k$-ecc $v$-tree and suppose on the contrary that it contains no Steiner $(k - 1)$-ecc $v$-tree. If $T^{k-1}_v$ is an arbitrary Steiner $(k - 1)$-ecc $v$-tree, then, by Lemma 2.2 we may select a leaf $u$ from $T^k_v$ such that the quasi-pendant path $P(w, u, T^k_v)$ does not have common edges with $T^{k-1}_v$.

Let $S^k_v$ be the Steiner $k$-ecc $v$-set corresponding to $T^k_v$ and set $S_1 = S^k_v \setminus \{u\}$. Then $S_1$ is a $(k - 1)$-set containing the vertex $v$. Moreover, the tree $T_1 = T^k_v \setminus (P(w, u, T^k_v) \setminus \{w\})$ is a MST$(S_1, T)$. By the assumption, the size of $T_1$ is strictly less than that of $T^{k-1}_v$, that is,

$$|E(T_1)| < |E(T^{k-1}_v)|.$$ (1)

Let $S_2 = S^{k-1}_v \cup \{u\}$, where $S^{k-1}_v$ is the Steiner $(k - 1)$-ecc $v$-set corresponding to the tree $T^{k-1}_v$. Then $S_2$ is a $k$-set which contains the vertex $v$. Let $T_2$ be a MST$(S_2, T)$. In the following we are going to show that the size of $T_2$ is larger than that of $T^k_v$.

Since the quasi-pendant path $P(w, u, T^k_v)$ does not share any edge with $T^{k-1}_v$ and must be a sub-path of the quasi-pendant path $P(w', u, T_2)$, the size of $T_2$ satisfies

$$|E(T_2)| \geq |E(T^{k-1}_v)| + |E(P(w, u, T^k_v))|.$$ (2)

Combining (1) and (2) we obtain that

$$|E(T_2)| \geq |E(T^{k-1}_v)| + |E(P(w, u, T^k_v))|$$
$$> |E(T_1)| + |E(P(w, u, T^k_v))|$$
$$= |E(T^k_v)|.$$ (3)

Hence $|E(T_2)| > |E(T^{k}_v)|$. Since $T_2$ is a minimum Steiner tree on a $k$-set containing $v$, (3) contradicts the fact that $T^k_v$ is a Steiner $k$-ecc $v$-tree.

Theorem 2.3 thus asserts that a Steiner $k$-ecc $v$-tree contains some Steiner $(k - 1)$-ecc $v$-tree. The question now is, how to determine such a Steiner $(k - 1)$-ecc $v$-tree. The message of the next result is that for our purposes, any Steiner $(k - 1)$-ecc $v$-tree will do. Before stating the theorem, we need some more notation. If $H$ is a subgraph of a graph $G$, and $v \in V(G)$, then the distance from $v$ to $H$ is $d_G(v, H) = \min\{d_G(v, u) : u \in V(H)\}$. The eccentricity of $H$ in $G$ is $ecc_G(H) = \max\{d_G(v, H) : v \in V(G)\}$.

Theorem 2.4 Let $k \geq 1$, and let $v$ be a vertex of a tree $T$. If $T_1$ and $T_2$ are Steiner $k$-ecc $v$-trees of $T$, then $ecc_T(T_1) = ecc_T(T_2)$. 


Proof. There is nothing to be proved if \( T_1 = T_2 \). Hence assume in the rest that \( T_1 \) and \( T_2 \) are different Steiner \( k \)-ecc \( v \)-trees of \( T \). If \( k = 1 \), then a (unique) Steiner 1-ecc\( v \)-tree is induced by the vertex \( v \) itself. Since all longest paths starting from \( v \) have the same length, the assertion of the theorem is clear for \( k = 1 \). Hence we may also assume in the rest of the proof that \( k \geq 2 \).

Let \( P_1 \) and \( P_2 \) be longest paths from vertices of \( V(T) \) to trees \( T_1 \) and \( T_2 \), respectively. Let \( u_1 \) and \( u_2 \) be the two endpoints of \( P_1 \) with \( u_1 \in V(T_1) \), and let \( w_1 \) and \( w_2 \) be the two endpoints of \( P_2 \) with \( w_1 \in V(T_2) \). Set \( T_0 = T_1 \cap T_2 \). To prove the theorem it suffices to prove that \( u_1 \in V(T_0) \) and \( w_1 \in V(T_0) \). By symmetry, it suffices to prove the first assertion, that is, \( u_1 \in V(T_0) \).

Suppose on the contrary that \( u_1 \in V(T_1) \setminus V(T_0) \). Let \( s \) be a leaf of \( T_1 \) such that \( u_1 \) is on the path \( P(v, s, T_1) \). Then there must be a vertex \( w_0 \in V(T_0) \) and a leaf \( t \) of \( T_2 \) such that \( E(P(w_0, s, T_1)) \cap E(P(w_0, t, T_2)) = \emptyset \), see Fig. 1. Note that \( w_0 \) may be the vertex \( v \).

![Figure 1: The configuration of the vertices \( w_0, u_1, u_2, w_1, w_2, s \) and \( t \).](image)

We claim that \( V(P_1) \cap V(T_2) = \emptyset \). Otherwise, let \( x \in V(T_2) \cap V(P_1) \). Then the path \( E(P(x, v, T_1)) \setminus E(P(x, v, T_2)) \neq \emptyset \), since \( E(P(w_0, u_1, T_1)) \neq \emptyset \). So the two paths \( P(x, v, T_1) \) and \( P(x, v, T_2) \) form a cycle in the original graph \( T \). This contradicts to the fact that \( T \) is a tree. In the same way, we obtain that \( V(P_2) \cap V(T_2) = \emptyset \).

Since \( |E(P_1)| = d_T(u_2, T_1) = \text{ecc}_T(T_1) \) and \( |E(P(w_0, t, T_2))| = d_T(t, T_1) \), we have

\[
|E(P_1)| \geq |E(P(w_0, t, T_2))|.
\] (4)

Moreover, since we have assumed that \( u_1 \in V(T_1) \setminus V(T_0) \), we infer that \( |E(P(u_1, w_0, T))| > 0 \). Together with \( (4) \) this yields

\[
|E(P(u_2, w_0, T))| = |E(P_1)| + |E(P(u_1, w_0, T))| \\
\geq |E(P(w_0, t, T_2))| + |E(P(u_1, w_0, T))| \\
> |E(P(w_0, t, T_2))|.
\] (5)
Now we pay attention to the tree \( T_2 \). Let \( S \) be the Steiner \( k \)-ecc \( v \)-set corresponding to the tree \( T_2 \). Let \( S' = S \setminus \{ t \} \cup \{ u_2 \} \). Then \( S' \) is a \( k \)-set containing the vertex \( v \). In the following, we will establish a contradiction that the tree \( T'_2 = MST(S', T) \) has more edges than the tree \( T_2 \). Recall that \( T_2 \) is a Steiner \( k \)-ecc \( v \)-tree.

Let \( P(w, t, T_2) \) be the quasi-pendant path with respect to \( v \) in \( T_2 \) and distinguish the following cases.

**Case 1**: \( w \in V(P(w_0, t, T_2)) \setminus \{ w_0 \} \).
In this case the tree \( T'_2 = MST(S', T) \) can be represented as \( T'_2 = T_2 \setminus P(w, t, T_2) \cup P(w_0, u_2, T) \). Since the path \( P(w, t, T_2) \) is a sub-path of \( P(w_0, t, T_2) \), \( |E(P(w_0, t, T_2))| \geq |E(P(w, t, T_2))| \) holds. Combining this fact with (5) we have:

\[
|E(T'_2)| = |E(T_2)| - |E(P(w, t, T_2))| + |E(P(w_0, u_2, T))| \\
\geq |E(T_2)| - |E(P(w_0, t, T_2))| + |E(P(w, t, T_2))| \\
> |E(T_2)|.
\]

**Case 2**: \( w \in V(T_0) \).
Now the tree \( T'_2 = MST(S', T) \) can be represented as \( T'_2 = T_2 \setminus P(w, t, T_2) \cup P(w, u_2, T) \). Recall that the path \( P(w, t, T_2) \) is composed of two sub-paths which are \( P(w, w_0, T_2) \) and \( P(w_0, t, T_2) \) respectively. And \( P(w, u_2, T) \) is also composed of two sub-paths which are \( P(w, w_0, T_2) \) and \( P(u_2, w_0, T) \). By (5) we can estimate as follows:

\[
|E(T'_2)| = |E(T_2)| - |E(P(w, t, T_2))| + |E(P(w, u_2, T))| \\
= |E(T_2)| - (|E(P(w, w_0, T_2))| + |E(P(w_0, t, T_2))|) \\
+ (|E(P(w, w_0, T_2))| + |E(P(u_2, w_0, T))|) \\
= |E(T_2)| - |E(P(w_0, t, T_2))| + |E(P(u_2, w_0, T))| \\
> |E(T_2)|.
\]

In both cases we have thus proved that \( |E(T'_2)| > |E(T_2)| \), a contradiction to the fact that \( T_2 \) is a Steiner \( k \)-ecc \( v \)-tree.

\[ \qed \]

### 2.2 A linear time algorithm

By Theorems 2.3 and 2.4, the problem to calculate the Steiner \( k \)-eccentricity of a given vertex of a tree can be reduced to recursively finding a longest path starting at a given
A vertex. This is formally done in Algorithm 1.

**Algorithm 1:** k-ECC(v, T, k)

**Input:** A vertex v, a tree T, and an integer k ≥ 2

**Output:** The Steiner k-eccentricity of v in T

1. if the number of leaves is less than k then
   2. return |V(T)| − 1;

3. end

4. else

5. \( ecc = 0; \)

6. for \( i = 1 \) to \( k - 1 \) do

7. Longest Path(v, T, path);

8. \( ecc = ecc + |E(P)|; \)

9. Path Shrinking(v, T, path);

10. end

11. return ecc

12. end

To explain Steps 1-3 of Algorithm 1, we state the following lemma.

**Lemma 2.5** Let \( k \geq 3 \) and let v be a vertex of a tree T. If \( |L(T)| < k \), then the Steiner \( k\)-ecc v-tree is the entire tree T.

**Proof.** The cardinality of the set \( S = \{v\} \cup L(T) \) is at most k, since \( |L(T)| < k \). Moreover, the \( MST(S, T) \) is the entire tree T. Hence the Steiner \( k\)-ecc v-tree is the entire T.

Steps 1-12 form the recursive reduction which consists of finding \( k - 1 \) times a longest path starting at a vertex. In Step 7 we use the depth-first search (DFS) algorithm to find a longest path starting at a given vertex, the details are present in Algorithm 2. Step 9 shrinks the path obtained in Step 7 into a single vertex for the purpose of the next loop, the details are presented in Algorithm 3. Algorithms 2 and 3 are borrowed from 14 where one can find additional details on them. For the statement of these algorithms we
recall that if $v$ is a vertex of a graph $G$, then the set of its neighbours is denoted by $N_G(v)$. 

**Algorithm 2: Longest Path**

Input: A vertex $v$, a tree $T$ rooted at $v$, and an array named `path` to store a longest path starting at $v$

Output: the length of a longest path starting at $v$

1. $max = 0$; $temp = max$
2. for each vertex $u \in N_T(v)$ which has not been visited till now do
3.   $temp = \text{Longest Path}(u, T, \text{path})$
4.   if $temp > max$ then
5.     $\text{path}[v] = u$
6.     $max = temp$
7. end
8. end
9. return $max + 1$

**Algorithm 3: Path Shrinking**

Input: A tree $T$, a vertex $v$, and an array named `path` to store a longest path starting at $v$

Output: A new tree obtained by shrinking the longest path into the single vertex $v$

1. $w = v$
2. while $\text{path}[w] \neq \emptyset$ do
3.   for each vertex $x \in N_T(w)$ do
4.     remove the edge $(w, x)$ from $T$
5.     add a new edge between $x$ and $v$ in $T$
6. end
7. $w = \text{path}[w]$
8. end

**Theorem 2.6** Algorithm 1 computes the Steiner $k$-eccentricity of a vertex in a tree and can be implemented to run in $O(k(n + m))$ time, where $n$ and $m$ are the order and the size of the tree, respectively.

**Proof.** The correctness of Algorithm 1 is ensured by Theorems 2.3 and 2.4. By Lemma 2.5, the Steiner $k$-eccentricity of a vertex in a tree is equal to the size of the tree if its number of leaves is less than $k$. There is a linear-time algorithm to find all leaves of a tree by the depth-first search (DFS) algorithm [4]. Hence Steps 1-3 can be implemented in $O(n + m)$ time. Similarly, each loop in Steps 6-9 can be implemented in $O(n + m)$ time, thus all loops require $O(k(n + m))$ time.

To conclude the section we again point out that the structural properties to support
the algorithm(s) from [19] only ensure calculation of the Steiner 3-eccentricity. Hence we need to develop a new approach that works for general $k$.

3 Upper and lower bounds

In this section we establish an upper and a lower bound on the average Steiner $k$-eccentricity index of a tree for $k \geq 3$. These bounds were earlier proved in [19] in the special case $k = 3$. It is appealing that to obtain the bound for the general case, the proof idea is quite different and significantly simpler that the one in [19]. For the new approach, the following construction is essential.

$\pi$-transformation: Let $T$ be a tree and let $P = P(u, v, T)$ be a path with at least one edge, such that every internal vertex of $P$ is of degree 2 in $T$. Let $X$ be the maximal subtree containing $u$ in the tree $T \setminus E(P)$, and $Y$ be the maximal subtree containing $v$ in the graph $T \setminus E(P)$. We may without loss of generality assume that $\text{ecc}_T(u, X) \leq \text{ecc}_T(v, Y)$. Then the $\pi$-transformation $\pi(T)$ of $T$ is defined as

$$
T' = \pi(T) = T \setminus \{(u, w) : w \in N_X(u)\} \cup \{(v, w) : w \in N_X(v)\}.
$$

The inverse transformation is is $T = \pi^{-1}(T') = T' \setminus \{(v, w) : w \in N_X(v)\} \cup \{(u, w) : w \in N_X(u)\}$. See Fig. 2

$T' = \pi(T)$ and $T = \pi^{-1}(T')$

**Lemma 3.1** Let $T$, $P$, $u$, $v$, $X$, $Y$, and $T'$ be as in the definition of the $\pi$-transformation. If $w \in V(P) \cup V(X)$, then in $T'$ there exists a Steiner $k$-ecc $w$-set $S$ such that $S \cap (V(Y) \setminus \{v\}) \neq \emptyset$.

**Proof.** Let $S'$ be a Steiner $k$-ecc $w$-set in $T'$ such that $S \cap (V(Y) \setminus \{v\}) = \emptyset$, and set $Q = S' \setminus \{w\}$. Since $k \geq 3$, the cardinality of $Q$ is at least two. Let $v' \in V(Y)$ such that the distance between $v$ and $v'$ is $\text{ecc}_{T'}(v, Y)$. Consider the following two cases.

**Case 1:** $Q \cap V(X) = \emptyset$.

In this case the vertices of $Q$ are all in $P$. Let $w' \in Q$ be the nearest vertex to $v$. Construct a new vertex set $S'' = (S' \setminus \{w'\}) \cup \{v'\}$. 
3 UPPER AND LOWER BOUNDS

Case 2: $Q \cap V(X) \neq \emptyset$.

Let $w' \in Q \cap V(X)$. Construct a new vertex set $S'' = (S' \setminus \{w'\}) \cup \{v'\}$.

In each of the two cases, the size of $MST(S'', T')$ is not less than the size of $MST(S', T')$, hence the assertion.

Lemma 3.2 Under the notation of Lemma 3.1, $aecc(T) \geq aecc(T')$.

Proof. If $v$ is a vertex in $V(Y) \setminus \{v\}$, then for any Steiner $k$-ecc $w$-set $S'$ in $T'$, the size of a minimum Steiner tree on $S'$ in graph $T$ is not less than that in $T'$. So the Steiner $k$-eccentricity of every vertex $w \in V(Y)$ in $T$ is not less than that in $T'$.

If $w$ is a vertex in $V(P) \cup V(X)$, then by Lemma 3.1 there exists a Steiner $k$-ecc $w$-set $S'$ in $T'$, such that $S' \cap (V(Y) \setminus \{v\}) \neq \emptyset$. The size of a minimum Steiner tree on $S'$ in $T$ is not less than that in $T'$. Therefore the Steiner $k$-eccentricity of every vertex $w \in V(P) \cup V(X)$ in $T$ is not less than that in $T'$.

In any case, the Steiner $k$-eccentricity of every vertex $v \in V(T')$ is not larger than that in $T$. As the average Steiner $k$-eccentricity index is the mean value of all vertices' Steiner $k$-eccentricities, the average Steiner $k$-eccentricity of $T'$ is not larger than that of $T$.

If the order of a tree $T$ is not larger than $k$, then a Steiner $k$-ecc $v$-set contains all vertices of $T$ for every $v \in V(T)$. Then every Steiner $k$-ecc $v$-tree is the entire tree $T$ for every vertex $v$. So for a given $k \geq 3$, we just consider the trees where the order of each is more than $k$.

Theorem 3.3 If $k \geq 3$ is an integer, and $T$ a tree on order $n > k$, then

$$k - \frac{1}{n} \leq aecc_k(T) \leq n - 1.$$  

Moreover, the star $S_n$ attains the lower bound, and the path $P_n$ attains the upper bound.

Proof. Repeatedly applying the $\pi$-transformation on $T$ until it is possible, we obtain the star $S_n$. On the other hand, repeatedly applying the $\pi^{-1}$ transformation on $T$ until it is possible, we obtain the path $P_n$. By Lemma 3.2 the $\pi$-transformation does not increase the average Steiner $k$-eccentricity of $T$. Hence the star $S_n$ attains the minimum Steiner $k$-eccentricity, and the path $P_n$ attains the maximum Steiner $k$-eccentricity. Finally, we obtain $aecc_k(S_n) = k - \frac{1}{n}$ and $aecc_k(P_n) = n - 1$ by straightforward computation.

In Fig. 3 an example is given in which the process of constructing extremal graphs, that is, a start and a path, by means of the $\pi$-transformation and the $\pi^{-1}$-transformation.

In [17] the average Steiner 2-eccentricity of trees was investigated. For the sake of our final result, we recall the following result.
4 Conclusions

Figure 3: Constructing extremal graphs using the $\pi$ transformation and the $\pi^{-1}$ transformation. Bold edges denote the paths defined in the transformations.

Lemma 3.4 ([17]) Let $T$ be a tree of order $n$. Then $aecc_2(S_n) \leq aecc_2(T) \leq aecc_2(P_n)$. The left equality holds if and only if $T \cong S_n$, while the right equality holds if and only if $T \cong P_n$.

Combining Theorem 3.3 with Lemma 3.4 we have the following result.

Corollary 3.5 If $k \geq 2$ is an integer, then $S_n$ (resp. $P_n$) attains the minimum (resp. the maximum) average Steiner $k$-eccentricity in the class of trees.

4 Conclusion

In this paper we have derived a linear-time algorithm to calculate the Steiner $k$-eccentricity of a vertex in a tree, and established lower and upper bounds for the average Steiner $k$-eccentricity of a tree. These results extend those from [19] for the case $k = 3$. It remains open to determine the extremal graphs for the average Steiner $k$-eccentricity index on trees for $k \geq 2$. Moreover, the general problem to compute the Steiner $k$-eccentricity of a general graph is widely open, in particular, it is not known whether it is NP-hard.

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