Stability and Robustness Analysis of Nonlinear Systems via Contraction Metrics and SOS Programming

Erin M. Aylward\textsuperscript{1} Pablo A. Parrilo\textsuperscript{1} Jean-Jacques E. Slotine\textsuperscript{2}
\textsuperscript{1}Laboratory for Information and Decision Systems \textsuperscript{2}Nonlinear Systems Laboratory
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

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Abstract

Contraction analysis is a stability theory for nonlinear systems where stability is defined incrementally between two arbitrary trajectories. It provides an alternative framework in which to study uncertain interconnections or systems with external inputs, where it offers several significant advantages when compared with traditional Lyapunov analysis. Contraction-based methods are particularly useful for analyzing systems with uncertain parameters and for proving synchronization properties of nonlinear oscillators. Existence of a contraction metric for a given system is a necessary and sufficient condition for global exponential convergence of system trajectories. For systems with polynomial or rational dynamics, the search for contraction metrics can be made fully algorithmic through the use of convex optimization and sum of squares (SOS) programming. The search process is made computationally tractable by relaxing matrix definiteness constraints, whose feasibility indicate existence of a contraction metric, into SOS constraints on polynomial matrices. We illustrate the results through examples from the literature, emphasizing the advantages and contrasting the differences between the contraction approach and traditional Lyapunov techniques.

1 Introduction

Contraction analysis is a stability theory for nonlinear systems where stability is defined incrementally between two arbitrary trajectories \[13\]. The existence of a contraction metric for a nonlinear system ensures that a suitably defined distance between nearby trajectories is always decreasing, and thus trajectories converge exponentially and globally. One important application of contraction theory is its use in studying the synchronization of nonlinear coupled oscillators \[23\]. These oscillators present themselves in a variety of research fields such as mathematics, biology, neuroscience, electronics, and robotics. The use of coupled oscillators in each of these fields, as well as how contraction theory can be used to analyze networks of coupled identical nonlinear oscillators can be found in \[23\] and the references listed therein.

Contraction theory nicely complements Lyapunov theory, a standard nonlinear stability analysis technique, as it provides an alternative framework in which to study convergence and robustness properties of nonlinear systems. For autonomous systems one can interpret the search for a contraction metric as the search for a Lyapunov function with a certain structure. This statement will be explained further in Section \[5\] There are, however, advantages to searching for a contraction metric instead of searching explicitly for a Lyapunov function. In particular, as we will show, contraction metrics are useful for analyzing uncertain nonlinear systems. In general, nonlinear systems with uncertain parameters can prove quite troublesome for standard Lyapunov methods, since the uncertainty can change the equilibrium point of the system in very complicated ways, thus forcing the use of parameter-dependent Lyapunov functions in order to prove stability for a range of the uncertain parameter values.

Much of the literature on parameter-dependent Lyapunov functions focuses on linear systems with parametric uncertainty \[6\ \[4\ \[3\ \[1\]. However, if a linear model is being used to study a nonlinear system around an equilibrium point, changing the equilibrium of the nonlinear system, necessitates relinearization around the new equilibrium. If the actual position of the equilibrium, in addition to the stability properties of the equilibrium, of
the nonlinear system depends on the uncertainty, it may be impossible to obtain any kind of closed form expression of the equilibrium in terms of the uncertain parameters. Thus, parameterizing the linearization in terms of the uncertainty may not be an option.

A well-studied method of dealing with specific forms of nonlinearities is to model the nonlinear system as a linear system with bounded uncertainty. In particular, in [2] polytopic linear differential inclusions (LDIs), norm-bound LDIs, and diagonal norm-bound LDIs are considered. These techniques are computationally tractable as they reduce to convex optimization problems. Though these methods work for various kinds of uncertainty, it is also desirable to find methods to study the stability of nonlinear systems that do not easily admit linear approximations with the nonlinearities covered with uncertainty bounds.

Contraction theory provides a framework in which to study the stability behavior of more general uncertain nonlinear systems. This framework eliminates many of the restrictions and problems that may be encountered when trying to analyze uncertain nonlinear systems with traditional linearization techniques or Lyapunov methods. This results from the fact that if a nominal system is contracting with respect to a certain contraction metric, it is often the case that the uncertain system with additive or multiplicative uncertainty within a certain range will still be contracting with respect to the same metric, even if the perturbation changes the position of the equilibrium of the system. Thus, it is possible to determine stability of the system for a range of values of the uncertain parameter without explicitly tracking how the uncertainty changes the location of the equilibrium. These ideas will be discussed further in Section 5.

Another interesting feature of the contraction framework is its relative flexibility in incorporating inputs and outputs. For instance, to prove contraction of a class of systems with external inputs, it is sufficient to show the existence of a contraction metric with a certain structure. This feature, which will be discussed in Section 6, is central in using contraction theory to prove synchronization of coupled nonlinear oscillators.

To translate the theoretical discussion above into effective practical tools, it is desirable to have efficient computational methods to numerically obtain contraction metrics. Sum of squares (SOS) programming provides one such method. SOS programming is based on techniques that combine elements of computational algebra and convex optimization, and has been recently used to provide efficient convex relaxations for several computationally hard problems [20]. In this paper we will show how SOS programming enables the search for contraction metrics for the class of nonlinear systems with polynomial dynamics. We discuss how to use SOS methods to find bounds on the maximum amount of uncertainty allowed in a system in order for the system to retain the property of being contracting with respect to the contraction metric of the unperturbed system. We also use SOS methods to optimize the contraction matrix search to obtain a metric that provides the largest symmetric uncertainty interval for which we can prove the system is contracting.

This paper is organized as follows: in Section 2 we give background material on contraction theory. Section 3 discusses sum of squares (SOS) polynomials and matrices. We present next an algorithm which uses SOS programming to computationally search for contraction metrics for nonlinear systems. We discuss why contraction theory is useful for studying systems with uncertain dynamics in Section 4 and external inputs in Section 6. Finally, in Section 7 we present our conclusions, and outline possible directions for future work.
2 Contraction Analysis

Contraction analysis is a relatively recently developed stability theory for nonlinear systems analysis [13]. The theory attempts to answer the question of whether the limiting behavior of a given dynamical system is independent of its initial conditions. More specifically, contraction analysis is a theory in which stability is defined incrementally between two arbitrary trajectories. It is used to determine whether nearby trajectories converge to one another. This section summarizes the main elements of contraction analysis; a much more detailed account can be found in [13].

We consider deterministic dynamical systems of the form

\[ \dot{x} = f(x(t), t), \]  

where \( f \) is a nonlinear vector field and \( x(t) \) is an \( n \)-dimensional state vector. For this analysis it is assumed that all quantities are real and smooth and thus that all required derivatives or partial derivatives exist and are continuous. This existence and continuity assumption clearly holds for polynomial vector fields.

Under the assumption that all quantities are real and smooth, from equation (1) we can obtain the differential relation

\[ \delta \dot{x}(t) = \frac{\partial f}{\partial x}(x(t), t) \delta x(t), \]  

where \( \delta x(t) \) is an infinitesimal displacement at a fixed time. For notational convenience from here on we will write \( x \) for \( x(t) \), but in all calculations it should be noted that \( x \) is a function of time.

The infinitesimal squared distance between two trajectories is \( \delta x^T \delta x \). Using (2), the following equation for the rate of change of the squared distance between two trajectories is obtained:

\[ \frac{d}{dt}(\delta x^T \delta x) = 2 \delta x^T \dot{x} = 2 \delta x^T \frac{\partial f}{\partial x} \delta x. \]  

If \( \lambda_1(x, t) \) is the largest eigenvalue of the symmetric part of the Jacobian \( \frac{\partial f}{\partial x} \) (i.e. the largest eigenvalue of \( \frac{1}{2}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T) \)), then it follows from (3) that

\[ \frac{d}{dt}(\delta x^T \delta x) \leq 2 \lambda_1(x, t) \delta x^T \delta x. \]  

Integrating both sides gives

\[ ||\delta x|| \leq ||0|| e^{\int_0^t \lambda_1(x, t) dt}. \]  

If \( \lambda_1(x, t) \) is uniformly strictly negative (i.e. \( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T < 0 \forall x, t \)), it follows from (5) that any infinitesimal length \( ||\delta x|| \) converges exponentially to zero. By path integration the distance of any finite path also converges exponentially to zero.

A more general definition of length can be given by

\[ \delta z^T \delta z = \delta x^T M(x, t) \delta x \]  

where \( M(x, t) \) is a symmetric, uniformly positive definite and continuously differentiable metric (formally, this defines a Riemannian manifold). This notion of infinitesimal distance
defined with respect to a metric can be used to define a finite distance measure between two trajectories with respect to this metric. Specifically, the distance between two points \(P_1\) and \(P_2\) with respect to the metric \(M(x, t)\) is defined as the shortest path length, in other words the smallest path integral \(\int_{P_1}^{P_2} \sqrt{\delta x^T M(x, t) \delta x}\), between these two points. Accordingly a ball of center \(c\) with radius \(R\) is defined as the set of all points whose distance to \(c\) with respect to \(M(x, t)\) is strictly less than \(R\).

Under the definition of infinitesimal length given in (6), the equation for its rate of change becomes
\[
\frac{d}{dt}(\delta x^T M \delta x) = \delta x^T (\frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M}) \delta x
\]  
where \(M\) is shorthand notation for \(M(x, t)\). Convergence to a single trajectory occurs in regions where \((\frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M})\) is uniformly negative definite. It should be noted that \(\dot{M} = \frac{\partial M}{\partial x} \frac{dx}{dt} + \frac{\partial M}{\partial t}\). The above analysis leads to the following definition and theorem:

**Definition 1 ([13]).** Given the system equations \(\dot{x} = f(x, t)\), a region of the state space is called a contraction region with respect to a uniformly positive definite metric \(M(x, t)\) if \((\frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M})\) is uniformly negative definite in that region.

**Theorem 1 ([13]).** Consider the system equations \(\dot{x} = f(x, t)\). Assume a trajectory starts in a ball of constant radius that is defined with respect to the metric \(M(x, t)\), that is centered at another given trajectory, and that is contained at all times in a contraction region with respect to the metric \(M(x, t)\). Then the first trajectory will remain in that ball and converge exponentially to the center trajectory. Furthermore, global exponential convergence to the center trajectory is guaranteed if the whole state space is a contraction region with respect to the metric \(M(x, t)\).

Definition 1 provides sufficient conditions for a system to be contracting. Namely, the following should be satisfied:

1. The matrix \(M(x, t)\) must be a uniformly positive definite matrix, i.e.,
\[
M(x, t) \succeq \epsilon I \succ 0 \quad \forall x, t.
\]  
2. The metric variation \(\frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M}\) must be a uniformly negative definite matrix, i.e.,
\[
R(x, t) = \frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M} \preceq -\epsilon I \prec 0 \quad \forall x, t.
\]  
An explicit rate of convergence of trajectories \(\beta\) can be found by finding a \(M(x, t)\) that satisfies (8) and
\[
\frac{\partial f}{\partial x} M + M \frac{\partial f}{\partial x} + \dot{M} \preceq -\beta M.
\]  

The notation above is standard; \(\succ, \preceq\), and \(\succeq\) mean positive definite and positive semidefinite respectively, while \(\prec\) and \(\preceq\) mean negative definite and negative semidefinite respectively. If the system dynamics are linear and \(M(x, t)\) is constant (i.e. \(M(x, t) = M\), the conditions
above reduce to those in standard Lyapunov analysis techniques. Lyapunov theory shows that the system $\dot{x}(t) = Ax(t)$ is stable (i.e., all trajectories converge to 0) if and only if there exists a positive definite matrix $M$ (i.e., $M \succ 0$) such that $A^T M + MA \prec 0$.

It should be noted that if a global contraction metric exists for an autonomous system, all trajectories converge to a unique equilibrium point, and we can always produce a Lyapunov function for the system from the contraction metric \[13\]. We assume, without loss of generality, that the equilibrium is at the origin. If the system dynamics are $f(x)$ and $M(x)$ is a time-invariant contraction metric for the system, then $V(x) = f(x)^T M(x) f(x)$ is a Lyapunov function for the system since $V(x) > 0$ and $\dot{V} = f(x)^T (\frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M}) f(x) \leq -\beta V$. This shows that $\dot{x} = f(x)$ tends to 0 exponentially, and thus that $x$ tends towards a finite equilibrium point.

For a constant metric $M(x, t) = M$, this reduces to Krasovskii’s Method \[9\]. We note that for systems with uncertainty there are good reasons to search for a contraction metric to create Lyapunov function of this structure instead of searching for a Lyapunov function directly. These reasons will become clear in Section 5.

The problem of searching for a contraction metric thus reduces to finding a matrix function $M(x, t)$ that satisfies the conditions above. As we will see, SOS methods will provide a computationally convenient approach to this problem.

### 3 Sum of Squares (SOS) Polynomials and Programs

The main computational difficulty of problems involving constraints such as the ones in \[8\] and \[9\] is the lack of efficient numerical methods that can effectively handle multivariate nonnegativity conditions. A convenient approach for this, originally introduced in \[17\], is the use of sum of squares (SOS) relaxations as a suitable replacement for nonnegativity. We present below the basic elements of these techniques.

A multivariate polynomial $p(x_1, x_2, \ldots, x_n) = p(x) \in \mathbb{R}[x]$ is a sum of squares (SOS) if there exist polynomials $f_1(x), \ldots, f_m(x) \in \mathbb{R}[x]$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x). \tag{11}$$

The existence of a SOS representation for a given polynomial is a sufficient condition for its global nonnegativity, i.e., equation (11) implies that $p(x) \geq 0 \forall x \in \mathbb{R}^n$. The SOS condition (11) can be shown to be equivalent to the existence of a positive semidefinite matrix $Q$ such that

$$p(x) = Z^T(x) Q Z(x) \tag{12}$$

where $Z(x)$ is a vector of monomials of degree less than or equal to $\deg(p)/2$. This equivalence of descriptions between (11) and (12) makes finding an SOS decomposition a computationally tractable procedure. Finding a symmetric positive semidefinite $Q$ subject to the affine constraint (12) is a semidefinite programming problem \[17\] \[19\].

Using the notion of a SOS polynomial as a primitive, we can now introduce a convenient class of optimization problems. A sum of squares program is a convex optimization problem.
of the form:

\[
\min \sum_{j=1}^{J} w_j c_j \\
\text{subject to } a_{i,0} + \sum_{j=1}^{J} a_{i,j}(x) c_j \text{ is SOS for } i = 1, \ldots, I,
\]

where the \(c_j\)'s are the scalar real decision variables, the \(w_j\)'s are given real numbers that define the objective function, and the \(a_{i,j}(x)\) are given multivariate polynomials. There has recently been much interest in SOS programming and SOS optimization as these techniques provide convex relaxations for various computationally hard optimization and control problems; see e.g. [17, 18, 12, 20] and the volume [7].

A SOS decomposition provides an explicit certificate of the nonnegativity of a scalar polynomial for all values of the indeterminates. In order to design an algorithmic procedure to search for contraction metrics, we need to introduce a similar idea to ensure that a polynomial matrix is positive definite for every value of the indeterminates. A natural definition is as follows:

**Definition 2 ([5]).** Consider a symmetric matrix with polynomial entries \(S(x) \in \mathbb{R}[x]^{m \times m}\), and let \(y = [y_1, \ldots, y_m]^T\) be a vector of new indeterminates. Then \(S(x)\) is a sum of squares matrix if the scalar polynomial \(y^T S(x) y\) is a sum of squares in \(\mathbb{R}[x, y]\).

For notational convenience, we also define a stricter notion:

**Definition 3.** A matrix \(S(x)\) is strictly SOS if \(S(x) - \epsilon I\) is a SOS matrix for some \(\epsilon > 0\).

Thus, a strictly SOS matrix is a matrix with polynomial entries that is positive definite for every value of the indeterminates. An equivalent definition of an SOS matrix can be given in terms of the existence of a polynomial factorization: \(S(x)\) is a SOS matrix if and only if it can be decomposed as \(S(x) = T(x)^T T(x)\) where \(T(x) \in \mathbb{R}[x]^{p \times m}\). For example,

\[
M(x) = \begin{bmatrix}
\omega^2 + \alpha^2(x^2 + k)^2 & \alpha(x^2 + k) \\
\alpha(x^2 + k) & 1
\end{bmatrix}
\]

is a SOS matrix for all values of \(\alpha\) and \(k\). Indeed, this follows from the decomposition \(M(x) = T(x)^T T(x)\), where

\[
T(x) = \begin{bmatrix}
\omega & 0 \\
\alpha(x^2 + k) & 1
\end{bmatrix}.
\]

SOS matrices have also been used recently by Hol and Scherer [8] and Kojima [10] to produce relaxations of polynomial optimization problems with matrix positivity definiteness constraints.

### 4 Computational Search for Contraction Metrics via SOS Programming

As explained in Section 2, given a dynamical system, the conditions for a contraction metric to exist in regions of the state-space are given by a pair of matrix inequalities. In the case
of metrics $M(x)$ that do not depend explicitly on time, relaxing the matrix definiteness conditions in (8) and (9) to SOS matrix based tests makes the search for contracting metrics a computationally tractable procedure. More specifically, the matrix definiteness constraints on $M(x)$ (and $R(x)$) can be relaxed to SOS matrix constraints by changing the inequality $M(x) - \epsilon I \succeq 0$ in (8) (where $\epsilon$ is an arbitrarily small constant) to the weaker condition that $M(x)$ be a strictly SOS matrix. With these manipulations we see that existence of SOS matrices $M(x)$ and $R(x)$ is a sufficient condition for contraction.

**Lemma 1.** Existence of a strictly SOS matrix $M(x)$ and a strictly SOS matrix $-R(x) = -(\partial f/\partial x)^T M + M \partial f/\partial x + \dot{M}$ is a sufficient condition for global contraction of an autonomous system $\dot{x} = f(x)$ with polynomial dynamics.

**Proof.** By Theorem 1, a sufficient condition for contraction of any nonlinear system is the existence of uniformly positive definite $M(x)$ and $-R(x)$. A sufficient condition for uniform positive definiteness of $M(x)$ and $-R(x)$ is the existence of strictly SOS matrices $M(x)$ and $-R(x)$.

This lemma can easily be extended to existence of certain SOS matrices implying contraction with a convergence rate $\beta$ by redefining $R(x)$ as $R(x) = \partial f/\partial x^T M + M \partial f/\partial x + \dot{M} + \beta M$. At this point, we do not know if the full converse of Lemma 1 holds. If a system is exponentially contracting, it is known that a contraction matrix always exists [13]. Nevertheless, a system with polynomial dynamics may certainly be contracting under non-polynomial metrics. Furthermore, even if a positive definite contraction matrix with polynomial entries $M$ exists, it may not be the case that it is a SOS matrix. We notice, however, that some of these issues, such as the gap between “true” contracting metrics and SOS-based ones, can be bridged by using the more advanced techniques explained in [18].

### 4.1 Search Algorithm

One main contribution of this work is to show how sum of squares (SOS) techniques can be used to algorithmically search for a time-invariant contraction metric for nonlinear systems with polynomial dynamics. Existence of a contraction metric for nonlinear systems certifies contraction (or convergence) of system trajectories. For systems with polynomial dynamics, we can obtain a computationally tractable search procedure by restricting ourselves to a large class of SOS-based metrics.

As suggested by Lemma 1, the main idea is to relax the search for matrices that satisfy matrix definiteness constraints $M(x) \succ 0$ and $-R(x) \succ 0$ into SOS-matrix sufficient conditions. Equivalently, we want to find a polynomial matrix $M(x)$ that satisfies SOS matrix constraints on $M(x)$ and $R(x)$. The SOS feasibility problem can then be formulated as finding $M(x)$ and $R(x)$ such that $y^T M(x) y$ is SOS and $-y^T R(x) y$ is SOS.

More specifically, the detailed steps in the algorithmic search of contraction metrics for systems with polynomial dynamics are as follows:

1. Choose the degree of the polynomials in the contraction metric, and write an affine parametrization of the symmetric matrices of that degree. For instance, if the degree is equal to two, the general form of $M(x)$ is...
2. Calculate \( \frac{\partial f}{\partial x} \) and define \( R(x) := \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M} \). Thus, \( R(x) \) will also be a symmetric matrix with entries that depend affinely on the same unknown coefficients \( a_i, b_i, \) and \( c_i \).

3. Change matrix constraints \( M(x) \succ 0 \ \forall x \), and \( R(x) = \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M} \prec 0 \ \forall x \) into scalar constraints on quadratic functions \( p(x, y) = y^T M(x) y > 0 \ \forall x, y \), and \( r(x, y) = y^T R(x) y = y^T \left( \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M} \right) y < 0 \ \forall x, y \), where \( y \) is an \( n \times 1 \) vector of new indeterminates.

4. Impose SOS constraints on \( p(x, y) \), and \( -r(x, y) \), and solve the associated SOS feasibility problem. If a solution exists, the SOS solver will find values for the unknown coefficients, such that the constraints are satisfied.

5. Use the obtained coefficients \( a_i, b_i, c_i \) to construct the contraction metric \( M(x) \) and the corresponding \( R(x) \).

6. Optionally, for graphical presentation, independent verification, or if the convex optimization procedure runs into numerical error, further testing can be done to verify the validity of the computed solution. To do this, we can check if the matrix constraints \( M(x) \succ 0 \), and \( R(x) \prec 0 \) hold over a range of the state space by finding and plotting the eigenvalues over this range. If a true feasible solution does not exist, the minimum eigenvalue of \( M(x) \) will be negative or the maximum eigenvalue of \( R(x) \) will be positive. Either one of these cases violates the matrix constraints which certify contraction. In most semidefinite programming solvers, the matrix \( Q \) in [12] is computed with floating point arithmetic. If \( Q \) is near the boundary of the set of positive semidefinite matrices, it is possible for the sign of eigenvalues that are zero or close to zero to be computed incorrectly from numerical roundoff and for the semidefinite program solver to encounter numerical difficulties. Numerical issues are further discussed in Section 6.3.1.

7. An explicit lower bound on the rate of convergence can be found by using bisection to compute the largest \( \beta \) for which there exist matrices \( M(x) \succ 0 \) and \( R_\beta(x) = \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M} + \beta M \prec 0 \).

For the specific examples presented later in the paper, we have used SOSTOOLS, a SOS toolbox for MATLAB developed for the specification and solution of sums of squares programs [19]. The specific structure of SOS matrices, or equivalently, the bipartite form of the polynomials \( p(x, y) \) and \( r(x, y) \) is exploited through the option \texttt{sparsemultipartite} of the command \texttt{sosineq} that defines the SOS inequalities. Future versions of SOSTOOLS will allow for the direct specification of matrix SOS constraints.
We present next are two examples of using this procedure to search for contraction metrics for nonlinear systems with polynomial dynamics. The systems studied are a model of a jet engine with controller, and a Van der Pol oscillator.

4.2 Example: Moore-Greitzer Jet Engine Model

The algorithm described was tested on the following dynamics, corresponding to a Moore-Greitzer model of a jet engine, with stabilizing feedback operating in the no-stall mode [11]. In this model, the origin is translated to a desired no-stall equilibrium. The state variables correspond to \( \phi = \Phi - 1, \psi = \Psi - \Psi_{co} - 2 \), where \( \Phi \) is the mass flow, \( \Psi \) is the pressure rise and \( \Psi_{co} \) is a constant [11]. The dynamic equations take the form:

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\
3\phi - \psi
\end{bmatrix}
\]  

(13)

The only real-valued equilibrium of the system is \( \phi = 0, \psi = 0 \). This equilibrium is stable.

The results of the algorithmic search for SOS matrices \( M(x) \) and \(-R(x)\) of various orders are given in Table 1. Values in the table, except the final row, are output values from SeDuMi [22], the semidefinite program solver used as the optimization engine in solving the SOS program. CPU time is the number of seconds it took for SeDuMi’s interior point algorithm to find a solution. As expected, the computation time increases with the degree of the polynomial entries of \( M(x) \). Feasibility ratio is the final value of the feasibility indicator. This indicator converges to 1 for problems with a complementary solution, and to –1 for strongly infeasible problems. If the feasibility ratio is somewhere in between, this is usually an indication of numerical problems. The values \( \text{pinf} \) and \( \text{dinf} \) detect the feasibility of the problem. If \( \text{pinf} = 1 \), then the primal problem is infeasible. If \( \text{dinf} = 1 \), the dual problem is infeasible. If \( \text{numerr} \) is positive, the optimization algorithm (i.e., the semidefinite program solver) terminated without achieving the desired accuracy. The value \( \text{numerr} = 1 \) gives a warning of numerical problems, while \( \text{numerr} = 2 \) indicates a complete failure due to numerical problems.

As shown in Table 1 for this system no contraction metric with polynomial entries of degree 0 or 2 could be found. This can be certified from the solution of the dual optimization problem. Since SeDuMi is a primal-dual solver, this infeasibility certificates are computed as a byproduct of the search for contraction metrics.

An explicit lower bound for the rate of convergence of the trajectories of the jet engine model, i.e., the largest value \( \beta \) for which matrices \( M(x) \succ 0 \) and \( R_\beta(x) = \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + M + \beta M \prec 0 \) were found, was \( \beta = 0.818 \).

We remark that for this system, it is also possible to prove stability using standard Lyapunov analysis techniques. However, we illustrate stability of this example from a contraction viewpoint because contraction theory offers a good approach to study this system when there is parametric uncertainty in the plant dynamics or feedback equations. For example, in the no-stall mode, the jet dynamics equations are

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\
-\frac{1}{2}\phi^3 - u
\end{bmatrix}
\]  

(14)
where \( u \) is a control variable. If a nominal stabilizing feedback control \( u \) can be found (e.g., using backstepping [11] or some other design method), the SOS techniques described in Section 5.1 provide a way to find other stabilizing feedback controls which are centered around the nominal control. For example, if a stabilizing linear feedback control \( u = k_1 \phi + k_2 \psi \) can be found, we can interpret \( k_1 \) and \( k_2 \) as uncertain parameters and use the methods described in Section 5.1 to search for ranges of gain values centered around the nominal values \( k_1 \) and \( k_2 \) that will also stabilize the system.

### 4.3 Example: Van der Pol Oscillator

A classic example that has played a central role in the development of nonlinear dynamics is given by the Van der Pol equation

\[
\ddot{x} + \alpha (x^2 + k) \dot{x} + \omega^2 x = 0,
\]

with \( \alpha \geq 0, k, \) and \( \omega \) as parameters. Historically this equation arose from studying nonlinear electric circuits used in the first radios [21]. When \( k < 0 \), the solutions of (15) behave like a harmonic oscillator with a nonlinear damping term \( \alpha (x^2 + k) \dot{x} \). The term provides positive damping when \( |x| > k \) and negative damping when \( |x| < k \). Thus, large amplitude oscillations will decay, but if they become too small they will grow larger again [21]. If \( k > 0 \) all trajectories converge to the origin.

In Table 2 we present the results of running the contraction matrix search algorithm for the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-x_1 \alpha (x_1^2 + k) + \omega^2 x_1
\end{bmatrix},
\]

with \( \alpha = 1, \omega = 1 \), which is the state-space version of the Van der Pol oscillator (15). We present solution for various values of \( k \), with a contraction matrix with entries that are quartic polynomials.

As a natural first step we searched for a constant contraction metric. None could be found algorithmically. This was expected as it is easily shown analytically that a constant contraction matrix for this system does not exist. If \( M \) is constant, then

\[
M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad \frac{\partial f}{\partial x} = \begin{bmatrix} 0 \\ -1 - 2x_1 x_2 \end{bmatrix}, \quad \frac{1}{x_1^2 - k},
\]

\[
R = \begin{bmatrix}
-2b - 4bx_1 x_2 & a - bx_1^2 - kb - c - 2cx_1 x_2 \\
2b - 2cx_1^2 - 2kc & a - bx_1^2 - kb - c - 2cx_1 x_2
\end{bmatrix}.
\]

| Degree of polynomials in \( M(x) \) | 0 | 2 | 4 | 6 |
|-------------------------------------|---|---|---|---|
| CPU time (sec)                      | 0.140 | 0.230 | 0.481 | 0.671 |
| Feasibility ratio                   | -1.000 | -0.979 | 1.003 | 0.990 |
| \( p_{\text{inf}} \)                | 1 | 1 | 0 | 0 |
| \( d_{\text{inf}} \)                | 0 | 0 | 0 | 0 |
| \( \text{numerr} \)                 | 0 | 1 | 0 | 0 |
| \( M > 0, R < 0 \) conditions met?  | no | no | yes | yes |

Table 1: Contraction matrix search results for closed-loop jet engine dynamics.
Degree of polynomials in $M(x)$

| $k$ | 4   | 4   | 4   | 4   | 0.001 | 0.01 | 0.1  | 1    | 4   | 4   | 4   | 4   |
|-----|-----|-----|-----|-----|-------|------|------|------|-----|-----|-----|-----|
| pinf | 1    | 1   | 1   | 0   | 0     | 0    | 0    | 0    | 0   | 0   | 0   | 0   |
| dinf | 0    | 0   | 0   | 0   | 0     | 0    | 0    | 0    | 0   | 0   | 0   | 0   |
| numerr | 1   | 1   | 1   | 1   | 1     | 0    | 0    | 0    | 0   | 0   | 0   | 0   |

$M \succ 0, R \prec 0$ conditions met?

| no | no | no | no | no | yes | yes | yes | yes | yes |

Table 2: Contraction matrix search results for oscillator dynamics.

Figure 1: Phase plots of Van der pol Oscillator.

(a) $k = 0.5$  
(b) $k = -0.5$

For $R$ to be negative definite $R_{11}$ must be negative for all values of $x_1, x_2$. In other words $-2b - 4bx_1x_2 \leq 0$ or $-1 \leq 2x_1x_2$. This clearly does not hold for all values of $x_1$, and $x_2$. A more complicated analysis (or a duality argument) also shows why there is no contraction matrix with quadratic entries for this system.

The algorithm finds a contraction function for the system $\ddot{x} + (x^2 + k)\dot{x} + x = 0$ when $k > 0$ but not when $k < 0$. As shown in Figure 1 the trajectories of the oscillator converge to zero when $k > 0$, and converge to a limit-cycle when $k < 0$. Thus, the results of the contraction metric search is as expected. Since all trajectories converge to the origin when $k > 0$ we expect that a contraction metric exists for the system. In the case where $k < 0$ the origin is an unstable fixed point and thus the system is not contracting.

Since for $k < 0$ the system is not contracting, we should not be able to find a contraction function. It should be noted that the converse does not hold. The fact that we cannot find a contraction function does not necessarily mean that the system is not contracting. This is because finding an SOS representation of the constrained quadratic functions is a sufficient condition for their positivity, not a necessary one.

It should be noted that for the example above, we can prove stability through Lyapunov analysis, and SOS programming can also be used to find Lyapunov functions. However, we illustrate this example here as contraction theory applied to a slightly modified version of this system provides a nice way to prove synchronization of coupled Van der Pol oscillators. This will be discussed in Section 6. This synchronization property is much more difficult to prove with standard Lyapunov methods.
5 Contraction Metrics and Systems with Uncertain Dynamics

5.1 Uncertainty Analysis with Contraction Metrics and SOS Programming

From the robust control perspective, one of the most appealing features of contraction theory is the fact that it provides a natural framework in which to study uncertain nonlinear systems where the parametric uncertainty changes the location of the equilibrium points. In general, standard Lyapunov analysis does not handle this situation particularly well, since the Lyapunov function must track the changes in the location of the steady-state solutions, thus forcing the use of parameter-dependent Lyapunov functions. However, in general it may be impossible to obtain any kind of closed form expression of the equilibria in terms of the parameters, thus complicating the direct parametrization of possible Lyapunov functions.

Much attention has been given to robust stability analysis of linear systems (e.g., [6, 4, 3, 2, 24]). Less attention, however, has been paid to nonlinear systems with moving equilibria. Two papers addressing this issue are [14, 1]. The approach in [14] is to consider systems described by the equations

\[ \dot{x} = f(x) + h(x), \]  

where \( x \) is a real \( n \)-vector, \( f \) and \( h \) are continuously differentiable functions, and \( h(x) \) represents the uncertainties or perturbation terms. Given an exponentially stable equilibrium \( x_e \), [14] establishes sufficient conditions by using the linearization of the system to produce Lyapunov functions which prove existence and local exponential stability of an equilibrium \( \tilde{x}_e \) for (17) with the property \( |x_e - \tilde{x}_e| < \varepsilon \) where \( \varepsilon \) is sufficiently small.

Since the approach in [14] is essentially based on a fixed Lyapunov function, it is more limited than our approach using contraction theory and SOS programming, and can prove stability only under quite conservative ranges of allowable uncertainty. By “allowable” we mean that if the uncertainty is in this range, the equilibrium remains exponentially stable under the uncertainty. A quantitative measure of this conservativeness will be given in Section 5.2.1 where we discuss the results of the method applied to an uncertain model of a Moore-Greitzer jet engine and compare them to an approach via contraction theory and SOS programming.

The approach in [1] is to linearize the dynamics around an equilibrium which is a function of the uncertain parameter \( x_0 = g(\delta) \), \( \delta \in \Omega \) and then use structured singular values to determine the eigenvalues of the linearized system \( \frac{d\tilde{x}}{dt} = A(\delta)\tilde{x} \) if \( A(\delta) \) is rational in \( \delta \). If \( A(\delta) \) is marginally stable, no conclusions can be made about the stability of the nonlinear system.

The contraction theory framework eliminates the need for linearization, and even the need to know the exact position of the equilibrium, in order to analyze stability robustness in uncertain nonlinear systems. In contrast to the Lyapunov situation, when certain classes of parametric uncertainty are added to the system, a contraction metric for the nominal system will often remain a contraction metric for the system with uncertainty, even if the perturbation has changed the equilibrium of the nonlinear system.
As noted in Section 2 if a global time-invariant contraction metric exists for an autonomous system, and an equilibrium point exists for the system, all trajectories converge to a unique equilibrium point, and we can always produce a Lyapunov function of the form $V(x) = f(x)^T M(x) f(x)$. When a system contains parametric uncertainty, this formula yields the parameter-dependent Lyapunov function $V(x, \delta) = f(x, \delta)^T M(x) f(x, \delta)$ for ranges of the parametric uncertainty $\delta$ where the contraction metric for the nominal system is still a contraction metric for the system with perturbed dynamics. Thus, if a contraction metric can be found for the system, we can easily construct a Lyapunov function which tracks the uncertainty for a certain range.

5.1.1 Case 1: Bounds on the uncertainty range for which the system remains contractive with respect to the nominal metric.

We can estimate the range of uncertainty under which the contraction metric for the nominal system is still a contraction metric for the perturbed system. To calculate this range, a SOS program can be written to minimize or maximize the amount of uncertainty allowed subject to the constraint $R_\delta(x) = \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M}(f_\delta(x)) < 0$, where $f_\delta(x)$ are the dynamics for the system with parametric uncertainty. The uncertainty bound is a decision variable in the SOS program and enters the constraint above in the $\frac{\partial f}{\partial x}$ and $f_\delta(x)$ terms.

If we have more than one uncertain parameter in the system, we can find a polytopic inner approximation of the set of allowable uncertainties with SOS Programming. For example, if we have two uncertain parameters, we can algorithmically find a polytope in parameter space for which the original metric is still a contraction metric. The convex hull of four points, each of which can be found by entering one of the four combinations, $(\delta_1, \delta_2) = (\gamma, \gamma)$, $(\delta_1, \delta_2) = (\gamma, -\gamma)$, $(\delta_1, \delta_2) = (-\gamma, \gamma)$, or $(\delta_1, \delta_2) = (-\gamma, -\gamma)$, into the uncertainty values in $f_\delta = [\delta_1 \delta_2]^T(x)$ and then maximizing $\gamma$ subject to the constraint $R_\gamma(x) = \frac{\partial f}{\partial x}^T M + M \frac{\partial f}{\partial x} + \dot{M}(f_\gamma(x)) < 0$, defines a polytope over which stability is guaranteed.

5.1.2 Case 2: Search for a contraction metric that guarantees the largest symmetric uncertainty interval for which the system is contractive.

Alternatively, we can instead optimize the search for a metric $M(x)$ that provides the largest symmetric uncertainty interval for which we can prove the system is contracting. If the scalar uncertainty $\delta$ enters the system dynamics affinely, in other words if $f(x) = f_1(x) + \delta f_2(x)$, we can perform this optimization as follows. First write $R(x, \delta) = R_0(x) + \delta R_1(x)$. To find the largest interval $(-\gamma, \gamma)$ such that for all $\delta$ that satisfy $-\gamma < \delta < \gamma$ the system is contracting, introduce the following constraints into an SOS program:

$$M(x) > 0, \quad R_0(x) + \gamma R_1(x) < 0, \quad R_0(x) - \gamma R_1(x) < 0.$$

We note that $\gamma$ multiplies the scalar decision coefficients $a_i$, $b_i$, and $c_i$ in $R_1(x)$ and thus we must use a bisection procedure to find the maximum value of $\gamma$ for which there exists SOS matrices $M(x)$, $R_0(x)$ and $R_1(x)$ that satisfy the constraints above.

If there are two uncertain parameters that enter the system dynamics affinely, we can extend the procedure above as follows: To find the largest uncertainty square with width...
Table 3: Range of perturbation where closed-loop uncertain jet engine dynamics given in (19) are contracting with respect to the nominal metric.

and height $\gamma$ such that for all $\delta_1$ and $\delta_2$ that satisfy $-\gamma < \delta_1 < \gamma$ and $-\gamma < \delta_2 < \gamma$ the system is contracting, first write $R(x, \delta_1, \delta_2) = R_0(x) + \delta_1 R_1(x) + \delta_2 R_2(x)$, Then introduce the following constraints into and SOS program:

\[
M(x) > 0, \quad R_0(x) + \gamma R_1(x) + \gamma R_2(x) < 0, \quad R_0(x) + \gamma R_1(x) - \gamma R_2(x) < 0
\]

\[
R_0(x) - \gamma R_1(x) + \gamma R_2(x) < 0, \quad R_0(x) - \gamma R_1(x) - \gamma R_2(x) < 0. \quad (18)
\]

Next, as in the scalar uncertainty case, use a bisection procedure to find the maximum value of $\gamma$ for which there exists SOS matrices $M(x)$, $R_0(x)$, $R_1(x)$ and $R_2(x)$ that satisfy the constraints above. In the case of a large number of uncertain parameters, standard relaxation and robust control techniques can be used to avoid an exponential number of constraints.

5.2 Example: Moore-Greitzer Jet Engine Model with Uncertainty

5.2.1 Scalar Additive Uncertainty

As described above, SOS programming can be used to find ranges of uncertainty under which a system with uncertain perturbations is still contracting with the original contraction metric. The contraction metric found for the deterministic system continues to be a metric for the perturbed system over a range of uncertainty even if the uncertainty shifts the equilibrium point and trajectories of the system. For the Moore-Greitzer jet engine model, the dynamics in (13) were perturbed by adding a constant term $\delta$ to the first equation.

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} = 
\begin{bmatrix}
-\psi - \frac{3}{2} \phi^2 - \frac{1}{2} \phi^3 + \delta \\
3 \phi - \psi
\end{bmatrix} \quad (19)
\]

In Table 3 we display the ranges of $\delta$ where the system was still contracting with the original contraction metric for 4th and 6th degree contraction metrics. Note the range of allowable uncertainty is not symmetric.

When instead we optimized the contraction metric search to get the largest symmetric $\delta$ interval we obtained the results listed in Table 4. A 6th degree contraction function finds the uncertainty range $|\delta| \leq 1.023$. Because a Hopf bifurcation occurs in this system at $\delta \approx 1.023$, making the system unstable for $\delta > 1.023$, we can conclude that the 6th degree contraction metric is the highest degree necessary to find the maximum range of uncertainty for which the system is contracting. The Hopf bifurcation is shown in Figure 2.

Using the techniques in [14] we computed the allowable uncertainty range for the system given in (19) as $|\delta| \leq 5.1 \times 10^{-3}$. In the notation of [14], we calculated the other parameters in Assumption 1 of [14] as: $h = [\delta, \ 0]^T$, $|A^{-1}|_{\infty} = 1$, $|Dh(x_e)|_{\infty} = 0$, $a = \frac{1}{30}$, and $|h(x_e)|_{\infty} = |\delta|$, where $\delta$ is the perturbation term in (19). The allowable range $|\delta| \leq 1.023$ computed via contraction theory and SOS programming is much larger than the allowable uncertainty range $|\delta| \leq 5.1 \times 10^{-3}$ computed with the techniques in [14].
5.2.2 Scalar Multiplicative Uncertainty

The approaches in Section 5.1 also apply to multiplicative uncertainty, since the multiplicative coefficients enter affinely in the constraints in the SOS program. Tables 5 and 6 present the results of the described uncertainty analysis on the following system, which is equation (13) with multiplicative uncertainty.

\[
\dot{\phi} \quad \dot{\psi} = \begin{bmatrix}
-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\delta\phi^3 \\
3\phi - \psi
\end{bmatrix}.
\]

(20)

5.2.3 Multiple Uncertainties

We consider next the system that results from introducing two additive uncertainties to the jet dynamics in equation (13). We computed an uncertainty polytope (shown in Figure 3) for which the system

\[
\dot{\phi} \quad \dot{\psi} = \begin{bmatrix}
-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\delta_1\phi^3 + \delta_1 \\
3\phi - \psi + \delta_2
\end{bmatrix}
\]

(21)

is guaranteed to be contracting with respect to the original metric. Table 7 shows the results of optimizing the contraction metric to find the largest uncertainty square with width and height \(\gamma\) such that for all \(\delta_1\) and \(\delta_2\) that satisfy \(-\gamma < \delta_1 < \gamma\) and \(-\gamma < \delta_2 < \gamma\) the system is contracting.

| Degree of polynomials in \(M(x)\) | 4       | 6             |
|----------------------------------|---------|---------------|
| \(\delta\) range                | \(|\delta| \leq 0.938\) | \(|\delta| \leq 1.023\) |

Table 5: Range of perturbation for which the uncertain system given in (20) is contracting with respect to the nominal metric.
| Degree of polynomials in $M(x)$ | 4   | 6   | 8   |
|--------------------------------|-----|-----|-----|
| $\delta$ range                | $(1 - 0.247, 1 + 0.247)$ | $(1 - 0.356, 1 + 0.356)$ | $(1 - 0.364, 1 + 0.364)$ |

Table 6: Symmetric range of perturbation where uncertain closed-loop jet engine dynamics given in (20) are contracting.

![Figure 3: Polytopic region of uncertainty where closed-loop jet engine dynamics given in (21) are contracting with respect to nominal metric.](image)

Table 7: Symmetric range of perturbation where uncertain closed-loop dynamics given in (21) are contracting.
6 Contraction Metrics for Systems with External Inputs

6.1 Stability Analysis of Systems with External Inputs

Another interesting feature of the contraction framework is the relative flexibility in incorporating inputs and outputs. For instance, to prove contraction of a class of systems with external inputs, it is sufficient to show the existence of a polynomial contraction metric with a certain structure. This is described in the following theorem.

**Theorem 2.** Let

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\
\vdots &= \\
\dot{x}_k &= f_k(x_1, x_2, \ldots, x_n) \\
\dot{x}_{k+1} &= f_{k+1}(x_1, x_2, \ldots, x_n) + v_{k+1}(u) \\
\vdots &= \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + v_n(u)
\end{align*}
\]  

be a set of nonlinear coupled differential equations where only the last \(n-k\) depend explicitly on \(u(t)\). If there exists a \(n \times n\) matrix \(M(x_1, \ldots, x_k)\) such that \(M > 0\) and \(\dot{M} + \partial f^T/\partial x M + M \partial f/\partial x < 0\) then the system is contracting for all possible choices of \(u(t)\).

**Proof.** For notational convenience, let \(\dot{x}_1 = [\dot{x}_1 \ldots \dot{x}_k]^T = f_1(x_1, x_2)\) and \(\dot{x}_2 = [\dot{x}_{k+1} \ldots \dot{x}_n]^T = f_2(x_1, x_2, u)\). The metric \(M(x_1, x_2, \ldots, x_k) = M(x_1)\) is independent of \(x_2\), and thus \(\partial M_{ij}/\partial x_2 = 0\) \(\forall i, j\). Since \(\partial M_{ij}/\partial t\) also vanishes, it follows that \(\forall i, j, \dot{M}_{ij} = \partial M_{ij}/\partial x_1 \dot{x}_1 + \partial M_{ij}/\partial x_2 \dot{x}_2 + \partial M_{ij}/\partial t = \partial M_{ij}/\partial x_1 \dot{x}_1\). Thus \(\dot{M}(x_1)\) is not a function of \(u(t)\). In addition, \(\partial f/\partial x\) has no dependence on \(u(t)\) because \(f(x, u) = h(x) + v(u)\). Thus, if there exists a \(n \times n\) matrix \(M(x_1, \ldots, x_k)\) such that \(M > 0\) and \(\dot{M} + \partial f^T/\partial x M + M \partial f/\partial x < 0\), then the system in (22) is contracting for any value of \(u(t)\). \(\square\)

Through the example considered in the following section, we will illustrate how Theorem 2 is particularly useful in proving synchronization of nonlinear oscillators, an issue explored in more detail in [23]. Theorem 2 can be easily extended to the case where \(u(t)\) is a vector (i.e. \(u(t) = [u_1(t), \ldots, u_m(t)]^T\)).

6.2 Coupled Oscillators

Contraction Theory is a useful tool to study synchronization behaviors of various configurations of coupled oscillators. For simplicity, we only consider here the case a pair of unidirectionally coupled oscillators; more complicated and general couplings are discussed in [23].
A state-space model of two unidirectionally coupled oscillators (only one oscillator influences the other) is
\[
\begin{align*}
\dot{x} &= f(x, t) \\
\dot{y} &= f(y, t) + u(x) - u(y),
\end{align*}
\] (23)
where \(x, y \in \mathbb{R}^m\), are the state vectors, \(f(x, t)\) and \(f(y, t)\) are the dynamics of the uncoupled oscillators, and \(u(x) - u(y)\) is the coupling force\(^1\). The following theorem is a slightly modified version of Theorem 2 in [23].

**Theorem 3.** If \(\dot{y} = f(y) + u(y) - u(x)\) in (23) is contracting with respect to \(y\) over the entire state space for arbitrary \(u(x)\), the two systems will reach synchrony (i.e. \(y(t)\) and \(x(t)\) will tend toward the same trajectory) regardless of initial conditions.

**Proof.** The system \(\dot{y} = f(y) - u(y) + u(x)\) with input \(u(x)\) is contracting with respect to \(y\) over the entire state space and \(y(t) = x(t)\) is a particular solution. Thus, by the properties of contraction, all solutions converge exponentially to \(y(t) = x(t)\). \(\square\)

Theorem [23] becomes especially powerful when the vector field appearing in the second subsystem of (23) has the structure described in equation (22)\(^3\). We illustrate this in the next example.

### 6.3 Example: Coupled Van der Pol Oscillators

Consider two identical Van der Pol oscillators coupled as
\[
\begin{align*}
\dot{x} + \alpha(x^2 + k)x + \omega^2 x &= 0 \\
\dot{y} + \alpha(y^2 + k)y + \omega^2 y &= \alpha \eta (\dot{x} - \dot{y})
\end{align*}
\] (24)
where \(\alpha > 0, \omega > 0, k\) are arbitrary constants. We note that if \(k < 0\), trajectories of the individual oscillator dynamics converge to a limit cycle. See Figure [1(b)] We first write these coupled systems in state-space form to get the equations in the form of (23). Their state-space form is
\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} &=
\begin{bmatrix}
x_2 \\
-\alpha(x_1^2 + k)x_2 - \omega^2 x_1 \\
-\alpha(y_1^2 + k + \eta)y_2 - \omega^2 y_1 + \alpha \eta x_2
\end{bmatrix}
\end{align*}
\] (25)

\(^1\)An example of coupled oscillators whose state-space representation is in this form is
\[
\begin{align*}
\dot{x} + \alpha(x^2 + k)x + \omega^2 x &= 0 \\
\dot{y} + \alpha(y^2 + k)y + \omega^2 y &= \alpha \eta (\dot{x} - \dot{y})
\end{align*}
\] where \(\alpha > 0, \omega > 0, k\) are arbitrary constants.

\(^2\)By contracting with respect to \(y\) for arbitrary \(u(x)\) we mean that the system \(\dot{y} = f(y) - u(y) + u(x)\), where \(y\) is the state vector and \(u(x)\) is an arbitrary driving function, is contracting for all inputs \(u(x)\).

\(^3\)If it does not have such a structure and \(u(x)\) drives each component of \(y\) the only possible contraction metric is a constant.
By Theorem 2, this pair of unidirectional oscillators will reach synchrony regardless of initial conditions if

\[
\dot{y} = f(y) - u(y) + u(x) = \left[ -\alpha(y_1^2 + k + \eta)y_2 - w^2y_1 + \alpha\eta x_2 \right]
\]

(26)
is contracting with respect to y for arbitrary values of \(u(x) = x_2\). We see by Theorem 2 that for this to occur, we must find a contraction metric \(M(y)\) that is only a function of \(y_1\) (i.e. \(M(y) = M(y_1)\)).

When the search algorithm described in Section 4.1 was applied to find a metric that satisfied \(M(y) = M(y_1)\) as well as \(M(y) \succ 0\) and \(R(y) \preceq 0\), none were found. However, it is shown in the appendix, which is a modified version of the appendix of [23], that a metric that satisfies \(M(y) \succ 0\) and \(R(y) \preceq 0\) implies asymptotic convergence of trajectories of system (26). A system with this metric that satisfies \(M(y) \succ 0\) and \(R(y) \preceq 0\) is called semi-contracting [13, 23].

The metric

\[
M(y) = \begin{bmatrix} \omega^2 + \alpha^2(y_1^2 + k + \eta)^2 & \alpha(y_1^2 + k + \eta) \\ \alpha(y_1^2 + k + \eta) & 1 \end{bmatrix}
\]

(27)

that appears in [23] is only a function of \(y_1\) and satisfies \(M(y) \succ 0\) and \(R(y) \preceq 0\) for the system dynamics (26) if \(\alpha > 0\) and \((k + \eta) \geq 0\). For this \(M\) and the system equation (26), we have

\[
R = \dot{M} + \frac{\partial f}{\partial y}^T M + M \frac{\partial f}{\partial y} = \begin{bmatrix} -2\alpha\omega^2y_1^2 - 2\alpha\omega^2(k + \eta) & 0 \\ 0 & 0 \end{bmatrix}.
\]

(28)

For \(\alpha > 0\), \((k + \eta) > 0\), \(M(y) \succ 0\) and \(R(y) \preceq 0\). Since (27) and (28) show analytically that the system (26) is semi-contracting we used our search algorithm to search for a metric with \(M(y) \succ 0\) and \(R(y) \preceq 0\).

6.3.1 Search for a Semidefinite R Matrix: Numerical Problems and Solutions

A minor problem that one may encounter when searching for contraction metrics, depending on the structure of polynomial constraints, is that the resulting optimization problem may be feasible, but not strictly feasible. This can cause numerical difficulties in the algorithms used in the solution procedure. In many cases, however, this can be remedied by introducing a presolving stage in which redundant variables are eliminated. When we ran the search algorithm based on Theorem 2 and only searched for \(M\) as a function of \(y_1\), no valid solution was found even if we only constrained \(R\) to be negative semidefinite and not strictly negative definite. Since the analytic solution (28) was feasible but not strictly feasible, we hypothesized there was numerical error in the algorithm. Based on knowledge of the analytic solution (28), we thus constrained \(R_{22} = 0\) and \(R_{12} = 0\), eliminated redundant variables, and then searched for a solution in the resulting lower dimensional space4. With these constraints in place, a solution was found with the search algorithm.

4Setting \(R_{22} = 0\), and \(R_{12} = 0\) leads to redundant decision coefficients in the polynomial entries of \(M\) and \(R\). If these redundant variables are eliminated through a presolving stage, the search algorithm finds \(M \succ 0\) and \(R \preceq 0\).
7 Conclusions

In this paper we have described how SOS programming enables an algorithmic search for contraction metrics for the class of nonlinear systems with polynomial dynamics. We also have illustrated the results through several examples.

These examples illustrate how contraction analysis offers several significant advantages when compared with traditional Lyapunov analysis. Contraction analysis provides relative flexibility in incorporating inputs and outputs. It is also particularly useful in the analysis of nonlinear systems with uncertain parameters where the uncertainty changes the equilibrium points of the system. It is often the case that if the nominal system is contracting with respect to a metric, the uncertain system with additive or multiplicative uncertainty will still be contracting with respect to the original metric, even if the perturbation changes the equilibrium of the system. In addition, a slightly modified version of the standard algorithmic search allows us to optimize the search to obtain a contraction metric that provides the largest uncertainty interval for which we can prove the system is contracting.

Subjects of future research include a careful evaluation of how the computational resources needed by the algorithm scale with system size, as well as the benefits and limitations of this approach in the context of other nonlinear system analysis techniques.

A Proving Asymptotic Convergence of Coupled Van der Pol Oscillators With a Negative Semidefinite R Matrix.

This appendix is a modified version of the appendix in [23]. Consider the system given in (26). Consider a $2 \times 2$ matrix $M(y)$ that is uniformly positive definite, and a corresponding $R(y)$ matrix that is uniformly negative semidefinite, but not uniformly negative definite. Since (26) is a two-dimensional system, we can assume without loss of generality that $R(y)$ is of the form

$$R(y) = \begin{pmatrix} -K(y) & 0 \\ 0 & 0 \end{pmatrix}$$

where $K(y) > 0 \ \forall \ y$. Let $\delta y = (\delta y_1, \delta y_2)^T = (\delta y, \delta \dot{y})^T$. where $y_1$ and $y_2$ are the variables in equation (26). With this $R(y)$ and $M(y)$ matrices, the general definition of differential length given in (6) and associated equation for rate of change of length (7) are

$$\delta z^T \delta z = \delta y^T M(y) \delta y$$

and

$$\frac{d}{dt} \delta z^T \delta z = \frac{d}{dt} (\delta y^T M(y) \delta y)$$

$$= \delta y^T R(y) \delta y$$

$$= -K(y) \delta y_1^2.$$  \ (29)
Since \( \frac{d}{dt} (\delta y^T M(y) \delta y) \leq 0 \) and \( \delta z^T \delta z \geq 0 \), \( \delta z^T \delta z \) has a limit at \( t \) goes to infinity. We will prove through a Taylor series argument that all trajectories of this system converge asymptotically. If \( \delta y = \delta y_1 \neq 0 \), then
\[
\delta z^T \delta z(t + dt) - \delta z^T \delta z(t) = -K(y)(\delta y_1)^2 dt + O((dt)^2)
\]
while if \( \delta y_1 = 0 \),
\[
\delta z^T \delta z(t + dt) - \delta z^T \delta z(t) = -2K(y)(\delta y_2)^2 \frac{dt^3}{3!} + O((dt)^4).
\]

Since \( \delta z^T \delta z \) converges, \( \delta z^T \delta z(t + dt) - \delta z^T \delta z(t) \) approaches zero asymptotically and hence \( \delta y_1 \) and \( \delta y_2 \) or equivalently \( \delta y \) and \( \delta \dot{y} \) both tend to zero. Thus, for any input \( u(x) \) all solutions of system (26) converge asymptotically to a single trajectory independent of initial conditions, and the unidirectional oscillators given in (25) will reach synchrony asymptotically regardless of initial conditions.
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