Nucleon-Nucleon scattering from the dispersive N/D method: next-to-leading order study

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Abstract

We consider nucleon-nucleon (NN) interactions from Chiral Effective Field Theory applying the N/D method. We calculate the discontinuity of the NN partial-wave amplitudes across the left-hand cut (LHC) by including one-pion exchange (OPE), once-iterated OPE and leading irreducible two-pion exchange (TPE) calculated in Chiral Perturbation Theory (ChPT). We discuss both uncoupled and coupled partial-waves. Phase shifts and mixing angles are typically rather well reproduced, and a clear improvement of the results obtained previously with only OPE is manifest. We also show that the contributions to the discontinuity across the LHC are amenable to a chiral expansion. Our method also establishes correlations between the S-wave effective ranges and scattering lengths based on unitarity, analyticity and chiral symmetry.
1 Introduction

The application of ChPT, the low-energy effective field theory of QCD, to the problem of nuclear forces was elaborated in Ref. [1] and first put in practice in Ref. [2]. Its application to $NN$ scattering has reached nowadays a sophisticated and phenomenologically successful status [2–8]. See Refs. [9–14] for related reviews. In particular, Refs. [3, 4] take the next-to-next-to-next-to-leading order (N$^3$LO) potential and reproduce $NN$ phase shift data up to $E_{\text{lab}} \sim 200$ MeV accurately, with $E_{\text{lab}}$ the laboratory-frame kinetic energy.

However, the use of the $NN$ potential calculated in ChPT up to some order in a Lippmann-Schwinger equation, as originally proposed in Ref. [1], is known to yield regulator dependent results. That is, the chiral counterterms present in the potential are not able to reabsorb all the ultraviolet divergences that result in the solution of the Lippmann-Schwinger equation [5, 6, 15–22]. Stable results with the $NN$ potential determined from OPE are obtained in Refs. [5, 16] for $\Lambda < 4$ GeV. This is achieved by promoting counterterms from higher to lower orders in the partial waves with attractive $1/r^3$ tensor force generated by OPE [16]. The extension of these ideas to higher orders in the chiral potential is undertaken in Refs. [20, 21] by treating perturbatively subleading contributions to the $NN$ potential beyond OPE. Recently, it has been shown in Ref. [23] that OPE is renormalizable in manifestly Lorentz covariant baryon-ChPT, while this is not the case when the heavy-baryon expansion is used as shown in Ref. [19].

Regulator dependence can also be avoided by employing dispersion relations (DRs), that involve only convergent integrals once enough number of subtractions are taken. This technique was recently applied in Refs. [24, 25] employing the N/D method [26] and OPE. Refs. [24, 25] argued that this method could be applied to higher orders in the chiral expansion by calculating perturbatively in ChPT the discontinuity of a partial wave amplitude along the LHC. Within ChPT, this discontinuity stems from multi-pion exchanges and it constitutes, together with the subtractions constants, the input required to solve the N/D method. We want to investigate explicitly the chiral expansion of the discontinuity along the LHC and extend the calculations in Refs. [24, 25] by including the leading irreducible and reducible TPE, as calculated in ChPT in Ref. [27]. One of the main aims of the work is to show quantitatively that the referred chiral expansion of the discontinuity of a $NN$ partial wave amplitude along the LHC is meaningful. The leading contribution is OPE, $\mathcal{O}(p^0)$, and the subleading ones, once-iterated OPE and irreducible TPE, have both similar size and could be booked in the chiral counting as the latter, which is explicitly $\mathcal{O}(p^2)$. It is also shown that further contributions to the discontinuity along the LHC by increasing the numbers of pion ladders in $NN$ reducible diagrams are more suppressed because they contribute only deeper in the complex plane and move further away from the low-energy physical region. We achieve a reproduction of $NN$ phase shifts and mixing angles that offers a clear improvement compared with that obtained in Refs. [24, 25] with OPE.

The N/D method was used in Refs. [28–31] to study $NN$ scattering. Ref. [28] was restricted to $NN$ S-waves and took only OPE as input along the LHC. Refs. [29, 30] also included other heavier mesons as source for the discontinuity in line with the meson theory of nuclear forces. Ref. [31] modeled the LHC discontinuity by OPE and one or two ad-hoc poles. No attempt was done in these works to offer a systematic procedure to improve the calculation of the discontinuity along the LHC. The main novelty that modern chiral effective field theory of nuclear forces can offer to us in connection with the N/D method consists precisely in calculating systematically such input discontinuity. This is the point that we want to elaborate further in the present research. We also mention Ref. [32] where the N/D method is used in connection with ChPT and $NN$ scattering. It is important to stress that we do not perform any truncation of the LHC and we keep its full extent in all the dispersive integrals considered in the center-of-mass-frame (c.m.) three-momentum squared complex plane, while this is not the case in Ref. [32]. In the latter reference the dispersive integrals along the LHC are cut at the c.m. three-momentum squared
value \(-9M^2_\pi/4\). Additionally, we do not model the way that short-distance contributions enter in the \(NN\) partial waves, while Ref. [32] introduces a specific set up based in a conformal mapping.

The contents of the paper are organized as follows. After this introduction we explain the formalism and deduce the proper integral equations (IEs) for the uncoupled waves in Sec. 2. The expansion of the discontinuity along the LHC in powers of three-momentum and pion masses (the so-called chiral expansion) and in the number of pions exchanged is the object to which Sec. 3 is devoted. The method is applied to the \(^1S_0\) and the uncoupled \(P\)-waves in Secs. 4 and 5, respectively. The constraints that result from requiring the proper threshold behavior for partial waves with orbital angular momentum greater or equal 2 are the contents of Sec. 6. Then, the numerical results for the uncoupled \(D, F, G\) and \(H\) waves are considered in Secs. 7–10. We quantify the different contributions to the discontinuity of a partial wave across the LHC in Sec. 11, where it is shown quantitatively the dominance of OPE and the subleading role of TPE. Section 12 provides the extension of the formalism to the coupled-partial-wave case. This is then applied to the systems \(^3S_1 - ^3D_1\), Sec. 13, \(^3P_2 - ^3F_2\), Sec. 14, \(^3D_3 - ^3G_3\), Sec. 15, \(^3F_4 - ^3H_4\), Sec. 16 and to \(^3G_5 - ^3I_5\) in Sec. 17. Conclusions and outlook are then provided in Sec. 18.

2 The N/D method: uncoupled waves

A \(NN\) partial wave amplitude in the three-momentum squared plane (that we call the \(A\) plane) has two disjoint cuts. The right-hand cut (RHC) is due to the intermediate states in \(NN\) scattering and then it extends from threshold (\(A = 0\)) up to \(A = \infty\). It comprises the elastic cut due to \(NN\) intermediate states, as well as the inelastic cuts, whose lighter thresholds are due to \(n\)-pion production giving contribution for \(A^2 \geq n^2M^2_\pi/4 + nmM_\pi\), with \(M_\pi\) the pion mass and \(m\) the nucleon mass. There is also the LHC which lower energy contributions are due to the exchange of \(n\) pions for \(A \leq -M^2_\pi/n^2/4\), so that OPE extends for \(A \leq -M^2_\pi/4\), TPE for \(A \leq -M^2_\pi\), and so on.

We first start by considering the uncoupled \(NN\) partial waves. Below, in Sec. 12, we present the generalization to coupled waves.

The two cuts present in a given \(NN\) partial wave, \(T_{J\ell S}(A)\), with \(S\) the total spin, \(\ell\) the orbital angular momentum and \(J\) the total angular momentum, can be separated by writing it as the quotient of a numerator \((N_{J\ell S}(A))\) and a denominator \((D_{J\ell S}(A))\) function

\[
T_{J\ell S}(A) = \frac{N_{J\ell S}(A)}{D_{J\ell S}(A)}, \tag{1}
\]

such that the functions \(N_{J\ell S}(A)\) and \(D_{J\ell S}(A)\) have only LHC and RHC, respectively. This is the essential point of the N/D method, first introduced in Ref. [26] to study \(\pi\pi\) scattering.

In the rest of this section we skip the subscripts \(J\ell S\) since we always refer to a definite \(NN\) partial wave. In addition, since all the functions involved in Eq. (1) are real at least in a finite interval along the real axis, they fulfill the Schwartz reflection principle

\[
f(z^*) = f(z)^* . \tag{2}
\]

As a result their discontinuity across a cut along the real axis is given entirely by the knowledge of the imaginary part of the function, because \(f(z + i0^+) - f(z - i0^-) = 2i\text{Im}f(z + i0^+), \) with \(z \in \mathbb{R}\).

Elastic unitarity in our normalization requires

\[
\text{Im}T(A) = \frac{m\sqrt{A}}{4\pi}|T|^2 , \quad A \geq 0 . \tag{3}
\]
In the following we designate

\[ \rho(A) = \frac{m\sqrt{A}}{4\pi}, \quad A \geq 0, \]  

(4)
as the phase-space factor in Eq. (3). Equation (3) has a simpler expression when given as the imaginary part of the inverse of the partial wave along the RHC:

\[ \text{Im} \frac{1}{T} = -\rho(A), \quad A \geq 0. \]  

(5)

Eq. (5), together with Eq. (1), translates in the following equation for \( \text{Im} D(A) \) along the RHC,

\[ \text{Im} D(A) = -\rho(A) N(A), \quad A > 0. \]  

(6)

On the other hand, the discontinuity of a \( NN \) partial wave along the LHC is denoted by

\[ T(A + i0^+) - T(A - i0^+) = 2i\Delta(A), \quad \Delta(A) \equiv \text{Im} T(A + i0^+), \quad A \leq -\frac{M^2}{4}. \]  

(7)

From Eqs. (1) and (7) this in turn implies the following result for \( \text{Im} N(A) \) along the LHC

\[ \text{Im} N(A) = \Delta(A) D(A), \quad A \leq -\frac{M^2}{4}. \]  

(8)

Next, we want to make use of Eqs. (6) and (8) to write down dispersive integrals for \( D(A) \) and \( N(A) \), respectively. For that we need to take into account the high-energy behavior of these functions. The relation in our normalization between the \( S \)- and \( T \)-matrix in partial waves, Eq. (3), is

\[ S(A) = 1 + 2i\rho(A) T(A), \]  

(9)

with \( S(A) \) the \( S \)-matrix element. Inverting the previous equation it follows that \( T(A) = \mathcal{O}(A^{-\frac{7}{2}}) \) at high-energies, \( A \in \mathbb{R} \) and \( A \to \infty \), because \( S(A) = \mathcal{O}(1) \) along the RHC.

Let us assume that \( D(A) = \mathcal{O}(A^{n_0}) \) for \( A \to \infty \) then because

\[ N(A) = T(A) D(A), \]  

(10)
it follows that for real \( A \) and \( A \to +\infty \), \( N(A) = \mathcal{O}(A^{n_0 - \frac{1}{2}}) \). Since \( N(A) \) has only LHC this limit is also valid for any other direction in the \( A \) plane for \( A \to \infty \), according to the the Sugawara and Kanazawa theorem \cite{Sugawara1981, Kanazawa1981}. As a result of the high-energy behavior of \( N(A) \) and \( D(A) \), if we divide simultaneously both functions by \( (A - C)^n \), \( n > n_0 \), we can write down unsubtracted DRs for the new functions \( \hat{D}(A) \) and \( \hat{N}(A) \) defined as

\[ \hat{D}(A) = \frac{D(A)}{(A - C)^n}, \]  

\[ \hat{N}(A) = \frac{N(A)}{(A - C)^n}. \]  

(11)

\footnote{Inelastic channels due to (multi-)pion production are not included in our low-energy analysis.}
To avoid unnecessary complications in the technical derivations we take \(-M_\pi^2/4 < C < 0\), and the following DRs, on account of Eqs. (6) and (8), result

\[
\bar{D}(A) = \sum_{i=1}^{n} \frac{\bar{\delta}_i}{(A - C)^i} - \frac{1}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2) \tilde{N}(q^2)}{q^2 - A},
\]

\[
\tilde{N}(A) = \sum_{i=1}^{n} \frac{\tilde{\nu}_i}{(A - C)^i} + \frac{1}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) \tilde{D}(k^2)}{k^2 - A},
\]

\begin{equation}
\text{with}
\end{equation}

\[
L = -\frac{M_\pi^2}{4}.
\]

Coming back to our original functions \(D(A)\) and \(N(A)\) by multiplying both sides of Eq. (12) by \((A - C)^n\), it results

\[
D(A) = \sum_{i=1}^{n} \frac{\delta_i(A - C)^{n-i}}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2) N(q^2)}{(q^2 - A)(q^2 - C)^n},
\]

\[
N(A) = \sum_{i=1}^{n} \frac{\nu_i(A - C)^{n-i}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2 - A)(k^2 - C)^n}.
\]

It is convenient to relabel the coefficients in the polynomial term of the previous equation and define \(\delta_i \equiv \bar{\delta}_{n-i+1}\) and \(\nu_i \equiv \tilde{\nu}_{n-i+1}\) so that Eq. (14) is rewritten as

\[
D(A) = \sum_{i=1}^{n} \frac{\delta_i(A - C)^{i-1}}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2) N(q^2)}{(q^2 - A)(q^2 - C)^n},
\]

\[
N(A) = \sum_{i=1}^{n} \frac{\nu_i(A - C)^{i-1}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2 - A)(k^2 - C)^n}.
\]

and we recover standard \(n\)-time subtracted DRs. In the previous equation one has to take the limit \((A + i0^+)\) for real values of \(A\) along the integration intervals. Since it is possible to divide simultaneously \(N(A)\) and \(D(A)\) by a constant, because only its ratio is relevant for obtaining \(T(A)\), we normalize the function \(D(A)\) in the following as

\[
D(0) = 1.
\]

In this way, one of the subtraction constants \(\delta_i\) in Eq. (15) is superfluous.

In summary, as a result of the discussion in this Section, we can state the following conclusion: If there exists an N/D representation of the on-shell \(N\ N\ partial wave, Eq. (1), then the \(D(A)\) and \(N(A)\) functions must satisfy \(n\)-time subtracted DRs, Eq. (15), for \(n\) large enough.

To solve Eq. (15) it is useful to insert the expression for \(N(A)\) into that of \(D(A)\) and then we end with the following integral equation (IE) for \(D(A)\),

\[
D(A) = \sum_{i=1}^{n} \frac{\delta_i(A - C)^{n-i}}{\pi} \int_{0}^{\infty} dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 - C)^{n-i+1}}
\]

\[
+ \frac{(A - C)^n}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2 - C)^n} \int_{0}^{\infty} dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 - k^2)}. \]

\begin{equation}
\text{(17)}
\end{equation}
Notice that on the right-hand side (r.h.s.) of the previous equation $D(A)$ is only needed along the LHC. We solve numerically this IE by discretization and determine $D(A)$ for $A \leq -\frac{M^2}{4}$. Once this is known, we can then calculate $D(A)$ and $N(A)$ for any other values of $A$ making use of the DRs in Eq. \ref{eq:DRs}.

The integrals along the RHC in Eq. \ref{eq:IE} can be done algebraically. We define the function $g(A, k^2)$ as

$$g(A, k^2) \equiv \frac{1}{\pi} \int_0^\infty dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 - k^2)} = \frac{im/4\pi}{\sqrt{A + i0^+ + k^2 + i0^+}}.$$ \hspace{1cm} (18)

Here, the $+i0^+$ is relevant for calculating this function when needed in the dispersive integrals above. In terms of $g(A, k^2)$ one also has

$$\frac{\partial^{i-1} g(A, C)}{\partial C^{n-1}} = \frac{(i-1)!}{\pi} \int_0^\infty dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 - C)^i}.$$ \hspace{1cm} (19)

It is also clear that once the $D(A)$ and $N(A)$ are expressed in the form of standard DRs, Eq. \ref{eq:DRs}, it is not really necessary to take the subtraction point $C$ with the same value for both functions. In practice we take $C = 0$ for the function $N(A)$. For the function $D(A)$ we always take one subtraction at $C = 0$, because then it is straightforward to impose the normalization condition Eq. \ref{eq:normalization}. Let us stress that DRs are independent of the value taken for the subtraction point since a change in $C$ would be reabsorbed in a change of the values of the subtraction constants $\delta_i$, $\delta_i$ for $D(A)$ and $\nu_i$ for $N(A)$.

It should be noticed that once we find a value of $n$ for which a solution of Eq. \ref{eq:IE} exists, it could occur that if we increase the number of subtractions no solution exists with the subtraction constants fixed (or fitted) from data. If this is the case, there should exist a delicate interplay between the polynomial and the dispersive integral contributions for a solution to exist. We find that this situation is absent for the uncoupled waves analyzed, but it is the case for the coupled $^3S_1 - ^3D_1$ system.

### 3 The input $\Delta(A)$ function

In Refs. \cite{24} and \cite{25} the input function $\Delta(A)$ was calculated from OPE. We now extend this calculation and determine $\Delta(A)$ including as well leading TPE, both irreducible TPE and once-iterated OPE from the results of Ref. \cite{27}.\footnote{For a large enough number of subtractions, typically three or more, it is more advantageous numerically to solve the integral equations in the form corresponding to Eq. \ref{eq:IE}. In this way, one avoids having too large numbers for large values of $A$ that could cause problems to the numerical subroutines for inverting matrices.} The relevant Feynman diagrams are depicted schematically in Fig. \ref{fig:FeynmanDiagrams} where the solid lines are nucleons, the dashed lines are pions and the angular lines indicate how each diagram should be cut to give contribution to $\Delta(A)$. From left to right in Fig. \ref{fig:FeynmanDiagrams} the first diagram is OPE, the second and third ones correspond to irreducible TPE while the last one is once-iterated OPE. This latter diagram contains both irreducible and reducible contributions, explicitly separated in Ref. \cite{27}.

Notice that the calculation of the imaginary part of the diagrams in Fig. \ref{fig:FeynmanDiagrams} along the LHC is finite. When cutting the loop diagrams for TPE, as indicated in Fig. \ref{fig:FeynmanDiagrams} an extra Dirac-delta function originates (beyond those required by energy-momentum conservation) that reduces the momentum integration to a finite domain.

It is known since long \cite{1} that $NN$ irreducible diagrams are amenable to a chiral expansion. This source of $\Delta(A)$ could then be calculated perturbatively and improved order by order in the chiral expansion in a systematic way. In the standard chiral counting \cite{1}, OPE is $O(p^0)$ and leading irreducible TPE is $O(p^2)$.

\footnote{Leading TPE means that the vertices employed in the calculation are the lowest-order ones in the chiral expansion that stem from the $O(p)$ $\pi N$ Lagrangian.}
Figure 1: From left to right OPE and TPE diagrams. The solid lines are nucleons, the dashed ones are pions and the angular lines indicate the way the diagram should be cut to contribute to $\Delta(A)$.

Figure 2: $NN$ reducible diagram with $n$-time iterated OPE. The meaning of the lines is the same as in Fig. 1. The vertical dots indicate extra pion ladders. This diagram only contributes to $\Delta(A)$ for $A \leq -n^2 M_\pi^2 / 4$. 
Regarding the $NN$ reducible diagrams, they give contribution to $\Delta(A)$ by cutting the OPE ladders. Indeed, an $n$-time iterated OPE diagram, see Fig. [2], contributes only to $\Delta(A)$ by putting on-shell all the $n$ pion lines, that is, for $A \leq -n^2M_N^2/4$. This is obvious if we keep in mind the fact that the Schrödinger propagator for each of the $NN$ intermediate states cannot be cut because it is proportional to $1/(A - q^2)$, with $q$ a three-momentum that stems from the linear combination of loop and external three-momenta, and $A < 0$. Then, from Cutkosky rules, the cutting of just one pion line requires to cut the rest of lines because no nucleon line can be cut for $A < 0$ and the angular line in Fig. [2] must go through all the pion ladders, as shown in the figure. This establishes a natural hierarchy of pion ladders at low energies, because by adding one extra ladder we move deeper in the LHC and then further away from the low-energy physical region that has $A > 0$. For a given $NN$ reducible diagram with $n$ pion ladders we can also consider its chiral corrections, which will be relatively suppressed by higher orders in the chiral expansion with respect to the simplest diagram with $n$ pion ladders, depicted in Fig. [2] which is calculated from the lowest-order $\mathcal{L}_{\pi N}$ Lagrangian.

Then, increasing both the number of pions exchanged and the chiral order of the calculation reduce the weight of a diagram to $\Delta(A)$ at low energies. As we discuss in more detail below in Sec. [11] irreducible and reducible TPE diagrams contribute with a similar size, so that we book the relative suppression of increasing the number of pion ladders by one in a reducible $NN$ diagram as $\mathcal{O}(p^2)$, the same amount as an irreducible loop counts in the chiral expansion. In this way, in order to proceed with the calculation of $\Delta(A)$, giving a $NN$ irreducible Feynman diagram we count its chiral order in the standard manner [1] and book its contribution to $\Delta(A)$ according to the latter. For a $NN$ reducible diagram, a leading two-pion ladder (calculated with the lowest-order $\pi N$ vertex) counts as $\mathcal{O}(p^2)$ and each extra leading pion ladder counts additionally as $\mathcal{O}(p^2)$. On top of that, we add the chiral order corresponding to other parts of the diagram that are $NN$ irreducible as well as the increase in the chiral order of the $\pi N$ vertices employed to calculate the pion ladders. The result of this addition is the final chiral order corresponding to the considered $NN$ reducible diagram to $\Delta(A)$.

4 Uncoupled waves: $^1S_0$

In this section we study the $^1S_0$ partial wave. We first take once-subtracted DRs, $n = 1$, and the IE for $D(A)$, Eq. (17), with $C = 0$ reads

$$D(A) = 1 - \nu_1 A g(A, 0) + \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2} g(A, k^2),$$

with $N(A)$, Eq. (15), given by

$$N(A) = \nu_1 + \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2(k^2 - A)}.$$

We have one free parameter $\nu_1$ that can be fixed in terms of the $^1S_0$ scattering length $a_s$ by taking into account the relation in our normalization

$$\frac{4\pi D}{mN} = -\frac{1}{a_s} + \frac{1}{2}r_s A - i\sqrt{A} + \mathcal{O}(A^2),$$

with $r_s$ the $^1S_0$ effective range. Since for $A = 0$ we have $N(0) = \nu_1$ and $D(0) = 1$, it follows that

$\nu_1 = -\frac{4\pi a_s}{m}$.

\footnote{We have explicitly checked this conclusion for twice and three-time iterated OPE.}
The experimental value for the $^1S_0$ scattering length is $a_s = -23.76 \pm 0.01$ fm.

The phase shifts obtained by solving the IE of Eq. (20) are shown in Fig. 3 as a function of the c.m. three-momentum. The (red) solid line corresponds to our results from Eqs. (20) and (23) with $\Delta(A)$ calculated up-to-and-including $O(p^2)$ contributions, and they are compared with the neutron-proton $(np)$ $^1S_0$ phase shifts of the Nijmegen PWA [36] (black dashed line) and with the OPE results of Ref. [24] (blue dotted line). As we see, there is a clear improvement when including TPE.

We can also predict $r_s$ from our results by expanding Eq. (22) up-to-and-including $O(A)$, it results

$$r_s = \frac{m}{2\pi^2a_s} \int_{-\infty}^{\infty} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} \left\{ \sqrt{-k^2} - \frac{1}{a_s} \right\}.$$  \hspace{1cm} (24)

Our calculation of $D(A)$ at $O(p^2)$ gives the numerical result

$$r_s = 2.64 \text{ fm},$$ \hspace{1cm} (25)

quite close already to its experimental value $r_s = 2.75 \pm 0.05$ fm or the value $r_s = 2.670$ fm determined in Ref. [37] for the NijmII potential.

It is important to stress that Eq. (24) exhibits a clear correlation between the effective range and the scattering length for the $^1S_0$ partial wave. This correlation can be written as

$$r_s = a_0 + \frac{\alpha_{-1}}{a_s} + \frac{\alpha_{-2}}{a_s^2},$$ \hspace{1cm} (26)

where the coefficients $\alpha_{-1}, \alpha_{-2}$ are independent of the scattering length $a_s$. This follows because $D(A)$ satisfies the linear integral equation Eq. (20), that we now rewrite as $L[D(A)] = 1 + a_s \frac{A}{m} Ag(A,0)$, with the linear operator $L[D(A)]$ defined as

$$L[D(A)] = D(A) - \frac{A}{\pi} \int_{-\infty}^{\infty} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} g(A,k^2).$$ \hspace{1cm} (27)
The solution $D(A)$ can be split as the sum of two terms $D_0(A) + a_s D_1(A)$, with $D_{0,1}(A)$ independent of $a_s$ and satisfying

$$L[D_0(A)] = 1,$$
$$L[D_1(A)] = \frac{4\pi}{m} Ag(A,0). \quad (28)$$

Substituting $D(A) = D_0(A) + a_s D_1(A)$ into Eq. (24) we then have the following expressions for the coefficients

$$\alpha_0 = \frac{m}{2\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2} \sqrt{-k^2},$$
$$\alpha_{-1} = \frac{m}{2\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2} \left[ D_0(k^2) \sqrt{-k^2} - D_1(k^2) \right],$$
$$\alpha_{-2} = -\frac{m}{2\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D_0(k^2)}{(k^2)^2}. \quad (29)$$

Notice that in our formalism this correlation between $r_s$ and $a_s$ stems from unitarity and analyticity and it makes sense as long as the once-subtracted DR, Eq. (20), exists. Our next-to-leading order (NLO) solution gives the numerical values:

$$\alpha_0 = 2.44 \text{ fm},$$
$$\alpha_{-1} = -4.61 \text{ fm}^2,$$
$$\alpha_{-2} = 5.26 \text{ fm}^3. \quad (30)$$

The $^1S_0$ effective range was predicted by the knowledge of the scattering length and the chiral TPE potential in the first entry of Ref. [6] by renormalizing the Lippmann-Schwinger equation with boundary conditions and imposing the hypothesis of orthogonality of the wave functions determined with different energy.[6] The correlation Eq. (26) was established in this reference with the numerical values for the coefficients: $\alpha_0 = 2.59 \sim 2.67 \text{ fm}$, $\alpha_{-1} = -5.85 \sim (-5.64) \text{ fm}^2$ and $\alpha_{-2} = 5.95 \sim 6.09 \text{ fm}^3$. The intervals of numerical values arise from the values taken for the $c_i$ counterterms of the $O(p^2) \pi N$ Lagrangian [6], because subleading TPE was also considered in that reference. The fact that we can derive the correlation of Eq. (26) from basic properties of $NN$ partial wave amplitudes, namely, analyticity, unitarity and chiral symmetry, is an important result that also reinforces the assumption of orthogonality of wave functions employed in Ref. [6].

Next, we consider twice-subtracted DRs, $n = 2$,

$$D(A) = 1 + \delta_2 A - \nu_1 \frac{A(A + M_\pi^2)}{\pi} \int_0^{\infty} dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 + M_\pi^2)q^2} - \nu_2 A(A + M_\pi^2) g(A, -M_\pi^2)$$
$$+ \frac{A(A + M_\pi^2)}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^2} \int_0^{\infty} dq^2 \frac{\rho(q^2)q^2}{(q^2 - A)(q^2 + M_\pi^2)(q^2 - k^2)} \quad (31)$$
$$N(A) = \nu_1 + \nu_2 A + \frac{A^2}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2 - A)(k^2)^2}, \quad (32)$$

$^5$Since the potentials involved are singular this orthogonality condition does not follow like the case of a regular potential but must be imposed, which is a working assumption of the formalism of Ref. [6].
where, the extra subtraction in the function $D(A)$ is taken at $C = -M^2$, while for the function $N(A)$ the two subtractions are taken at $C = 0$, see Eq. (15). The subtraction constant $\nu_1$ is also given by Eq. (23). Next we fix $\delta_2$ in terms of $r_s$. For that, according to Eq. (22), one needs to expand

$$\frac{4\pi D}{mN} + i\sqrt{A}$$

up-to-and-including $O(A)$ terms. In this expansion, one should consider carefully the combination of the first integral on the r.h.s. of Eq. (31) with $im\sqrt{A}/4\pi$

$$\frac{A(A + M^2)}{\pi} \int_0^\infty dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 + M^2)}q^2 + \frac{im\sqrt{A}}{4\pi}$$

$$= \frac{A(A + M^2)}{M^2} \left[ g(A, -M^2) - g(A, 0) \right] + \frac{im\sqrt{A}}{4\pi} = \frac{mA}{4\pi M^2}. \quad (34)$$

For the rest of terms the expansion is straightforward because the limit $A = 0$ can be taken directly inside the integrals. One ends with the following expression for $\delta_2$,

$$\delta_2 = \frac{a_s}{M^2}(1 - \frac{1}{2}r_s M^2) + \frac{\nu_2}{\nu_1} \left[ 1 + \nu_1 M^2 g(0, -M^2) \right] - \frac{M^2}{\pi} \int_{-L}^L dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2}g(k^2, -M^2) , \quad (35)$$

which is then substituted in Eq. (31) and our final expression for the twice-subtracted DR of $D(A)$ results:

$$D(A) = 1 + A \left\{ \frac{a_s}{M^2}(1 - \frac{1}{2}r_s M^2) + \frac{\nu_2}{\nu_1} \left[ 1 + \nu_1 M^2 g(0, -M^2) \right] \right\}$$

$$- A(A + M^2) \left[ \nu_2 g(A, -M^2) - \nu_1 g(A, -M^2) \right]$$

$$+ \frac{A}{\pi} \int_{-L}^L dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} \left\{ \frac{A + M^2}{k^2 + M^2} \left[ k^2 g(A, k^2) + M^2 g(A, -M^2) \right] - M^2 g(k^2, -M^2) \right\} . \quad (36)$$

The subtraction constant $\nu_2$ is fitted to the $np$ Nijmegen PWA phase shifts for $\sqrt{A} \leq 150$ MeV.

$$\nu_2 = 0.17 M^4 , \quad (39)$$

and the resulting curve is shown by the (magenta) dash-dotted line in Fig. 3. We see that this curve follows closely the experimental phase shifts. It is also interesting to remark that the results with $n = 1$, that were able to predict the experimental value for $r_s$ so closely, can be exactly reproduced, as expected, in terms of twice-subtracted dispersion relations with $\nu_2 = -1.346 M^4$.

5 Uncoupled $P$-waves

In this section we discuss the application of the method to the uncoupled $P$-waves.

---

6Since Ref. [36] does not provide errors we always perform least square fit, without weighting.
Theory: OPE
Theory: No free parameters
Theory: $a_V$ fixed
Nijmegen data

Figure 4: (Color online.) Phase shifts of the $^3P_0$ $NN$ partial wave. The (red) solid line corresponds to our results with $n = 1$, Eq. (40), the (magenta) dash-dotted line is our results with $n = 2$, Eq. (41). The (blue) dotted line is the OPE result from Ref. [24] and the (black) dashed line is the Nijmegen PWA phase shifts, which almost coincides with the $n = 2$ result.

5.1 $^3P_0$ wave

For the $^3P_0$ uncoupled wave we also consider first the once-subtracted DR already used for the $^1S_0$ case, i.e. Eqs. (20) and (21). The only important difference is that for $P$- and higher orbital-angular-momentum ($\ell$) partial waves we have the threshold behavior $T_{JLS}(0) = 0$ at $A = 0$, so that $\nu_1 = 0$. Hence, for $\ell \geq 1$ Eqs. (20) and (21) reduce to

\[
D_{JLS}(A) = 1 + \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_{JLS}(k^2)}{k^2} g(A,k^2), \\
N_{JLS}(A) = \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_{JLS}(k^2)}{k^2(k^2 - A)}.
\]

(40)

Notice that there are no free subtraction constants in Eq. (40) and the emerging results are then predictions of our approach. In Fig. 4 we show our results by the (red) solid line. We see a great improvement compared with the OPE result of Ref. [24] given by the (blue) dotted line.

Next, we consider the twice-subtracted DRs of Eqs. (15) and (17) but now with $\nu_1 = 0$ and $C = 0$. They can be written as

\[
D(A) = 1 + \delta_2 A - \nu_2 A^2 g(A,0) + \frac{A^2}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} g(A,k^2), \\
N(A) = \nu_2 A + \frac{A^2}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2 - A)(k^2)^2}.
\]

(41)

In this equation we take $C = 0$ for all the subtractions, because no infrared divergences are generated in the integrals along the RHC for $\nu_1 = 0$. Notice that this was not the case for the $^1S_0$ partial wave because of the first integral on the r.h.s. of Eq. (31). The subtraction constant $\nu_2$ can be fixed straightforwardly
Figure 5: (Color online.) Phase shifts of the \( ^3P_1 \) \( NN \) partial wave. The (red) solid line corresponds to our results with twice-subtracted DRs, with \( a_V \) fixed and \( \delta_2 \) fitted, Eq. (44). The (blue) dotted line is the OPE result from Ref. [24] and the (black) dashed line is the Nijmegen PWA phase shifts.

to the experimental scattering volume\(^7\)

\[ \nu_2 = \frac{4\pi a_V}{m} . \] (42)

For the \( ^3P_0 \) partial wave we have \( a_V = 0.890 \, M^{-3}_\pi \), a value that is derived from the Nijmegen PWA phase shifts \([36]\). Finally, the subtraction constant \( \delta_2 \) is fitted to data with the resulting value

\[ \delta_2 = -0.30 \, M^{-2}_\pi . \] (43)

The resulting curve is shown by the (magenta) dash-dotted line in Fig. 4, that perfectly agrees with the phase shifts of \([36]\) (given by the black dashed line). The reproduction of the data is so good that the fit is completely insensitive to the upper limit of \( \sqrt{A} \) fitted, in the range shown in the figure.

5.2 \( ^3P_1 \) wave

This partial wave illustrates our conclusion in Sec. 2 with respect to the fact that \( n \) should be large enough in order to write down meaningful DRs for \( D(A) \) and \( N(A) \), Eqs. (15). Here, the once-subtracted DR, Eq. (40), does not have solution\(^8\).

However, the twice-subtracted DR, Eq. (41), is meaningful and has a solution. The scattering volume for the \( ^3P_1 \) partial wave is \( a_V = -0.543 \, M^{-3}_\pi \) from the phase shifts of Ref. [36]. The subtraction constant \( \delta_2 \) is fitted to the phase shifts up to \( \sqrt{A} \leq 150 \, \text{MeV} \) with the result

\[ \delta_2 = 3.60 \, M^{-2}_\pi . \] (44)

As we can see in Fig. 5, our results, given by the (red) solid line, closely follow the Nijmegen PWA phase shifts, particularly for \( \sqrt{A} < 200 \, \text{MeV} \).

\(^7\)That we define as \( a_V = \lim_{A \to 0^+} \delta(A)/A^{3/2} \), with \( \delta(A) \) the phase shifts.

\(^8\)The numerical outcome depends on the lower limit of integration when discretizing the integral equation for \( D(A) \).
Figure 6: (Color online.) Phase shifts of the $^1P_1 \, NN$ partial wave. The (red) solid line corresponds to our results with a once-subtracted DR, $n = 1$. The (magenta) dash-dotted line represents the case with twice-subtracted DR, $n = 2$, with $a_V$ fixed and $\delta_2$ fitted. The (blue) dotted line is the OPE result from Ref. [24] and the (black) dashed line is the Nijmegen PWA phase shifts.

5.3 $^1P_1$ wave

Now we consider the singlet uncoupled wave $^1P_1$. As usual, we discuss first the once-subtracted DR and then the twice-subtracted case. The former has no free parameters. For the latter case the scattering volume, $a_V = -0.939 \, M_{\pi}^{-3}$, is used to fix $\nu_2$ and $\delta_2$ is fitted to data. However, now the fit is not very sensitive to this subtraction constant, which is determined only within a large interval of positive values from 0.8 up to 27 $M_{\pi}^{-2}$, depending on the upper limit for $\sqrt{A}$ taken in the fit.

Our results for once- and twice-subtracted DRs are almost identical, as can be seen by comparing the (red) solid and (magenta) dash-dotted lines in Fig. 6 respectively. Both curves are overlapping and reproduce fairly well the data for $\sqrt{A} < 200$ MeV.

6 Uncoupled waves: $\ell \geq 2$

A partial wave amplitude with $\ell \geq 2$ should vanish as $A^\ell$ in the limit $A \to 0$. This behavior is not directly implemented by the DR Eq. (15), unless some constraints are imposed. The right threshold behavior can be achieved by taking the subtraction point $C = 0$ in $N(A)$ and then imposing $\nu_i = 0$ for $i = 1, \ldots, \ell$ in Eq. (15). In this way, since $T = N/D$ and $D(0) = 1$, one has that $T \to A^\ell$ for $A \to 0$, being necessary to consider at least $\ell$-time subtracted DRs. In practice we take the minimum number of subtractions, $n = \ell$, and $C = 0$ in Eq. (15), so that we end with the equations

$$D(A) = 1 + \sum_{i=2}^{\ell} \delta_i A^{i-1} + \frac{A^\ell}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^\ell} g(A, k^2),$$

(45)

$$N(A) = \frac{A^\ell}{\pi} \int_{-\infty}^{\ell} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^\ell(k^2 - A)},$$

(46)
The $\delta_i$ are free parameters that should be fitted data. They are proportional to derivatives of the function $D(A)$ at $A = 0$ namely:

$$
\delta_n = (n - 1)! D^{(n-1)}(0), \ n \geq 2 ,
$$

with

$$
D^{(n)}(0) = \frac{\partial^n D(A)}{\partial A^n} \bigg |_{A=0} .
$$

Nevertheless, as $\ell$ increases, rescattering effects giving rise to the unitarity cut are less important because of the centrifugal barrier and then $D(A) \simeq 1$ for $A$ small and negative. This manifests in the fact that the $\delta_i$ can be taken equal to zero except the one with the largest subscript, $i = \ell$, that is fitted to data providing a good reproduction of the latter in most of the cases. This is the situation that corresponds to the smoothest $D(A)$ for small and negative values of $A$. This rule stems from our study of $NN$ partial wave amplitudes with $\ell \geq 2$, and it holds not only for the uncoupled waves but it is also applicable to the coupled ones. Even if we released all the $\delta_i$ there is no improvement in the reproduction of data with respect to that obtained when only $\delta_{\ell} \neq 0$.

Another method to guarantee the right behavior at threshold was developed in Ref. [24]. We refer to this reference for further details. The neat result is that the $D(A)$ function should fulfill the set of constraints

$$
\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^\lambda} = 0 , \ (\lambda = 2, \ldots, \ell)
$$

with $\ell \geq 2$. In order to fulfill them a set of $\ell - 1$ CDD poles [38] are included in the $D(A)$ function, whose residues are adjusted by imposing Eq. (49). The final expressions are Eq. (46), that is the same as here, and a different equation for $D(A)$

$$
D(A) = 1 + A \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D^{(2)}(k^2)}{k^2} g(A, k^2) + \frac{A \sum_{n=0}^{\ell-2} c_n A^n}{(A - B)^{\ell-1}} ,
$$

with $B$ corresponding to the position of the CDD poles that is finally sent to infinity. Notice that at low energies ($A \ll B$) the addition of the CDD poles reduce to change the function $D(A)$ by a polynomial of degree $\ell - 1$. In this sense, this method based on the constraints Eq. (49) and the addition of the CDD poles Eq. (50) is a particular case at low energies of the most general solution with $\ell$ subtractions, Eq. (45). We do not further use the method of Ref. [24] because unless $\Delta(A)$ vanishes fast enough in the infinite, e.g. like $1/A$ in OPE, it implies to use integrals along the LHC that grow with powers of $B \to \infty$ which makes very hard its numerical manipulation. In particular, standard numerical subroutines used to invert a matrix and find the numerical solution of the IE, do not provide the right answer for large $B$. This is the situation at NLO because $\Delta(A) \to A$ for $A \to \infty$, and it would be even worse if higher orders in the chiral expansion of $\Delta(A)$ were implemented.

7 Uncoupled waves: $D$-waves

For the $D$-waves one has to solve Eq. (45) with $\ell = 2$. For the case of the singlet $^1D_2$ partial wave a fit to data is not appropriate here because it produces negative values of $\delta_2$, that in turn gives rise to a

---

9Strictly speaking, they correspond to the derivatives from the left of $D(A)$ at $A = 0$, that is, the limit $A \to 0^-$ is the proper one in order to avoid the branch cut singularity in $D(A)$ due to the onset of the unitarity cut for $A > 0$. 

10
resonance in the low-energy region, just a bit above the energy range fitted. The resulting curve fitting data up to $\sqrt{A} \leq 150$ MeV is shown by the (magenta) dash-dotted line with $\delta_2 = -0.22 M^{-2}_\pi$ in the left panel of Fig. 7. To avoid the resonance behavior we then impose that $\delta_2 \geq 0$, with the (red) solid curve in the same figure corresponding to $\delta_2$ fitted, Eq. (51). Though there is a clear improvement compared with the OPE results of Ref. [24] (blue dotted line), higher order corrections are needed to provide an accurate reproduction of data.

We follow the same steps for the $3D_2$ partial wave. In this case we observe a numerical behavior not seen before when solving Eq. (45). There is a dependence on the lower limit of integration along the LHC that can be reabsorbed, however, in the value of the free parameter $\delta_2$. In this way, the resulting phase shifts below $\sqrt{A} = 300$ MeV are stable under changes in the lower limit of integration. The phase shifts with $\sqrt{A} < 200$ MeV are fitted with

$$\delta_2 = -0.18^{+0.02}_{-0.01} M^{-2}_\pi,$$

where the errors show the variation in this parameter when the lower limit of integration varies from $-4^2$ to $-187^2$ GeV$^2$. The phase shifts obtained are the (red) solid line in the right panel of Fig. 7 (the other lines obtained with the different values mentioned for the lower limit of integration overlap each other and cannot be distinguished in the scale of the figure.) We also see a clear improvement in the reproduction of data when moving from OPE to TPE, specially for $\sqrt{A} < 200$ MeV.

### 8 Uncoupled waves: $F$-waves

Here we study the uncoupled $F$ waves, namely, $1F_3$ and $3F_3$. For these waves Eq. (45) is applied with $\ell = 3$ and it requires three subtractions, with two free parameters $\delta_2$ and $\delta_3$, proportional to $D'(0)$ and $D''(0)$, in that order, according to Eq. (47). In the following we use the derivatives $D^{(n)}(0)$ as free parameters, which we consider more natural parameters for the polynomial in front of the integral in Eq. (45).

The partial wave $1F_3$ is quite insensitive to $D'(0)$ and slightly dependent on $D''(0)$, which is required to be positive for a better reproduction of data. In the left panel of Fig. 8 we show by the (red) solid line...
Figure 8: (Color online.) Phase shifts for $^1F_3$ (left panel) and $^3F_3$ (right panel). $^1F_3$: The (red) solid and (magenta) dash-dotted lines correspond to the NLO results with $D''(0) = 1.7$ and $10 M_\pi^{-2}$. $^3F_3$: The (red) solid line is for $D''(0) = 20 M_\pi^{-2}$. In both cases, the once-subtracted DR phase shifts, from Eq. (40), are given by the (cyan) double-dotted lines. The OPE result from Ref. [24] is the (blue) dotted line. The Nijmegen PWA analysis is the (black) dashed line.

the outcome with $D''(0) = 1.7 M_\pi^{-2}$, the resulting value in a fit to data up to $\sqrt{A} = 150$ MeV, while the (magenta) dash-dotted line corresponds to take $D''(0) = 10 M_\pi^{-2}$. Despite the large variation in the value of $D''(0)$ the two lines run very close to each other, which clearly shows how little the results depend on the actual values of $D''(0)$. In all these curves $D'(0)$ is fixed to 0. The OPE results are quite similar as the NLO ones.

The $^3F_3$ partial wave is also insensitive to $D'(0)$, but the fit clearly prefers a value for $D''(0)$ around $20 M_\pi^{-2}$. The outcome at NLO is shown by the (red) solid line. One observes a clear improvement in the reproduction of data from OPE to NLO.

One can check whether the $F$-waves could be already treated in perturbation theory. For that we propose to use the once-subtracted DR, Eq. (40), that has no the right threshold behavior which requires a partial wave to vanish as $A^3$ when $A \to 0$. The origin for the failure to reproduce the proper threshold behavior rests in the resummation of the right-hand-cut undertaken by the $D(A)$ function. For a perturbative wave (Born approximation) unitarity requirements should be of little importance. The outcome from Eq. (40) is shown by the (cyan) double-dotted lines in Fig. 8, which run very close to the (red) solid lines, our NLO results that implement by construction the correct threshold behavior. The $^3F_3$ wave seems less perturbative than the $^1F_3$, because the once-subtracted DR provides results that are more different compared with the full results. Had we applied the once-subtracted DR for the $D$-waves the outcome would have been very different to the results discussed in Sec. 7 and shown in Fig. 7 (particularly for the $^3D_2$ that would not even match the correct sign). This indicates that the $F$-waves can be treated in good approximation in perturbation theory, while this is not the case for the $D$ waves yet.

9 Uncoupled waves: $G$-waves

We now proceed to discuss the $G$-waves and solve Eq. (45) with $\ell = 4$. The situation here follows the general rule discussed in Sec. 6, so that it is enough to release only $D^{(3)}(0)$, which acts then as the active degree of freedom. From the best fits obtained with one free parameter $D^{(3)}(0)$, if we release further the other two parameters, $D'(0)$ and $D''(0)$, no improvement is obtained. For the partial wave $^3G_4$ we obtain $D^{(3)}(0) = 1.21 M_\pi^{-6}$ and for the $^3G_4$ a large value of $D^{(3)}(0)$ gives a perfect reproduction of data,
Figure 9: (Color online.) Phase shifts for $^1G_4$ (left panel) and $^3G_4$ (right panel). Full results are the (red) solid lines. The once-subtracted DR phase shifts, from Eq. (40), are given by the (cyan) double-dotted lines. The OPE result from Ref. [24] is the (blue) dotted line. The Nijmegen PWA analysis is the (black) dashed line.

Figure 10: (Color online.) Phase shifts for $^1H_5$ (left panel) and $^3H_5$ (right panel). Full results are the (red) solid lines. The once-subtracted DR phase shifts, from Eq. (40), are given by the (cyan) double-dotted lines. The OPE result from Ref. [24] is the (blue) dotted line. The Nijmegen PWA analysis is the (black) dashed line.

$D^{(3)}(0) = 117 \, M_\pi^{-6}$. In both cases the reproduction of data is very good, particularly for the latter case, as shown in Fig. 9 by the (red) solid line. The $^3G_4$ wave is the most perturbative one, as one can see by the fact that the once-subtracted DR results (shown by the cyan dashed lines in Fig. 9) are much closer to the full results than for the $^1G_4$ case.

10 Uncoupled waves: $H$-waves

The same rule of Sec. 6 regarding the number of active free parameters is observed as in the case of the $F$- and $G$-waves, so that we only release $D^{(4)}(0)$. For the $^1H_5$ the best results are obtained with $D^{(4)}(0) \simeq -1 \, M_\pi^{-8}$, corresponding to the (red) solid line in the left panel of Fig. 10. The results reproduce data fairly well. The lines obtained from OPE [24] and by employing a once-subtracted DR run close to our full ones at NLO. For the $^3H_5$ wave we obtain the best value $D^{(4)}(0) \simeq 2.5 \, M_\pi^{-8}$, and the results are shown by the (red) solid line of the right panel of Fig. 10. The reproduction of data is almost perfect in the range shown. The once-subtracted DR and OPE results are very similar between them and run
rather close to the full results, indicating the perturbative nature of the $H$-waves.

11 Quantifying contributions to $\Delta(A)$

For any partial wave there is always a term, corresponding to the last line in Eq. (17), that gives the nested contribution of the LHC to the function $D(A)$. This type of integration along the LHC is the proper one to ascertain the relative size of the different contributions to $\Delta(A)$, because once the $D(A)$ function is known along the LHC then any scattering quantity can be calculated. It is then not illuminating to look directly to the relative sizes of the different contributions to $\Delta(A)$, but better one should look at the amount that they contribute to the integral along the LHC. Since this integration involves the very same function that we want to calculate, we evaluate it by substituting $D(k^2) \rightarrow 1$, although any other constant value would be equally valid to ascertain relative differences. In this way, we can then perform an a priori quantitative study about the importance of the different contributions in $\Delta(A)$ when solving Eq. (17).

At the practical level we have used Eq. (17) with changes in its form because of different selection of the subtraction point $C$, as explained above. We display in Eq. (52) the integrals used for each wave to quantify the weight in our results of the different contributions to $\Delta(A)$. All the integrals require two or more subtractions so that they are convergent. Indeed twice- or more subtracted DRs have been used in all the partial waves in Secs. 4–10.

\[
\ell \leq 1 : \frac{A(A + M^2)}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{q^2 \rho(q^2)}{(q^2 - A)(q^2 - k^2)(q^2 + M^2)} ,
\]

\[
\ell \geq 2 : \frac{A^\ell}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^\ell} \int_{0}^{\infty} dq^2 \frac{\rho(q^2)}{(q^2 - A)(q^2 - k^2)} .
\]

Let us analyze first the case of the $^{1}S_0$. For that we show in the left panel of Fig. 11 the corresponding integral in Eq. (52), while in the right panel we plot directly $\Delta(A)$. In both cases we distinguish between OPE (green dash-dotted line), irreducible TPE (blue dotted line) and reducible TPE (magenta dashed line). The total result is given for the integral (left panel) and it corresponds to the (red) solid line. We see that the integral is clearly dominated by the OPE contribution, despite the irreducible TPE contribution overpasses OPE in $\Delta(A)$ at around $-2M^2$. The next contribution in importance is irreducible TPE and the least important by far is reducible TPE. The latter contribution is so much suppressed because for the $^{1}S_0$ it is proportional to $m^4$. 

Figure 11: Left panel: different contributions to the integral in Eq. (52) with $\ell = 0$. Right panel: contributions to $\Delta(A)$. These contributions comprise irreducible TPE (blue dotted line), reducible TPE (magenta dashed line) and OPE (green dash-dotted line). The total result, only shown for the left panel, is the (red) solid line.
The dominance of OPE in the integral at low energies along the RHC is because: i) it starts to contribute the soonest in all of them; ii) the integrand in Eq. (52) is enhanced at low three-momenta by the factor $1/(k^2)^2$ for $\ell \leq 1$. Because of these reasons every contribution to $\Delta(A)$ that involves the exchange of a larger number of pions should be increasingly suppressed. Let us recall that precisely the threshold for each contribution to $\Delta(A)$ controls its exponential suppression for large radial distances in the $NN$ potential, as $\exp(-nm\pi r)$ for an $n$-pion exchange contribution. Notice also that one can see clearly in the right panel of Fig. 11 that OPE increases in absolute value very fast towards its threshold, at $-M^2_{\pi}/4$. This is because OPE at low energies has a typical value for its derivative proportional to $1/A^2$, which implies a large relative change between the onset of OPE and that of TPE. We can say from the left panel of Fig. 11 that it is justified to calculate perturbatively the different contributions to $\Delta(A)$ for the $^1S_0$.

The case of the $P$-waves is shown in Fig. 12. From top to bottom we show the partial waves $^3P_0$, $^3P_1$ and $^1P_1$, in that order. The left panels show the integral in Eq. (52) and the right ones the different contributions to $\Delta(A)$. The notation is the same as used in Fig. 11. By comparing the (green) dash-dotted and (red) solid lines in the left panels of Fig. 12 one clearly observes the dominance of the OPE contribution. For the $^3P_0$ wave both irreducible and reducible TPE are sizeable but tend to cancel mutually. The actual extent of this cancellation could be sensitive to the exact values of the function.
$D(k^2)$ (substituted by 1 in the integral along the LHC in Eq. (52)). We also observe that the irreducible and reducible TPE contributions are typically of similar size as a global picture for the $P$-waves. The pattern of results shown for the integral then again suggests that a perturbative treatment for the different contributions to $\Delta(A)$, in the form discussed in Sec. 3, is meaningful.

The corresponding curves for the $D$-waves, $\ell = 2$ in Eq. (52), are shown in Fig. 13. Again we observe a clear dominance of OPE in the integral of Eq. (52). For the $^1D_2$ wave the irreducible TPE is larger than the reducible contribution, but for the $^3D_2$ the situation is reversed. So we conclude that typically they should be considered of similar size, as argued in Sec. 3.

The $F$-waves show an overwhelming dominance of the OPE contribution to the integral in Eq. (52) with $\ell = 3$, see the left panels of Fig. 14. This is in agreement with our discussion in Sec. 3, where we argue that these waves could be treated perturbatively. In addition these waves present small corrections to the phase shifts from higher orders, as shown in Fig. 8. We also see that irreducible and reducible TPE have similar sizes (see e.g. the right panels in Fig. 14). A similar situation occurs for the $G$- and $H$-waves, shown in Figs. 15 and 16, respectively. The fact that OPE and the total result for the integration in Eq. (52) coincide for the $F$- and higher partial waves clearly indicates their perturbative character. Notice that this is not the case for lower values of $\ell \leq 2$.

The expressions for $\Delta(A)$ can be algebraically obtained for the different partial waves from the expressions given in Ref. [27]. A closer look at them would be appropriate in order to disentangle the origin of the somewhat surprising result that irreducible and reducible TPE contributions to $\Delta(A)$ have typically a similar size. To illustrate this point let us consider the $^3P_0$ wave for which, as shown in the top panel on the right of Fig. 12, both reducible and irreducible TPE have opposite sign but similar magnitude.
Figure 14: Left panels: different contributions to the integral in Eq. (52) with $\ell = 3$. Right panels: Contributions to $\Delta(A)$. From top to bottom we show the $^1F_3$ and $^3F_3$ partial waves, respectively. The meaning of the lines is the same as in Fig. 11.

Figure 15: Left panels: different contributions to the integral in Eq. (52) with $\ell = 4$. Right panels: Contributions to $\Delta(A)$. From top to bottom we show the $^1G_4$ and $^3G_4$ partial waves, respectively. The meaning of the lines is the same as in Fig. 11.
The different contributions to $\Delta(A)$ are:

$$
\Delta_{OPE} = -\frac{g_A^2 \pi M_A^2}{16 f^2 A}, \quad A < -\frac{M_A^2}{4},
$$

$$
\Delta_{IRR} = \frac{1}{4608 f^4 A^2 \pi} \left\{ -2 \left[ A(M_A^2 + A) \left[ 3M_A^4 + A(-M_A^2 + 2A) + 2g_A^2 (-3M_A^4 + 5A(-M_A^2 + 2A) + g_A^4 \left\{ -87M_A^4 + A(59M_A^2 + 98A) \right\} + 6M_A^2 \left[ -M_A^2 + g_A^2 (2M_A^2 - 6A) - 3A + g_A^4 (29M_A^2 + 21A) \right] \times \log \left( \frac{(-A)^{\frac{1}{2}}}{M_A} + \left(1 - \frac{A}{M_A^2}\right)^{\frac{1}{2}} \right) \right] \right] , \quad A < -M_A^2 ,
$$

$$
\Delta_{VGV} = \frac{g_A^4 m}{3840 f^4 A^2} \left\{ -4M_A^5 - 20M_A^3 (-A)^{\frac{3}{2}} + 24(-A)^{\frac{5}{2}} - 15M_A^1 (-A)^{\frac{3}{2}} \log \left( -1 + 2 \left(\frac{(-A)^{\frac{1}{2}}}{M_A}\right) \right) \right\} , \quad A < -M_A^2 ,
$$

where we have, from top to bottom, the OPE, irreducible TPE and reducible TPE (labelled with the subscript $VGV$) contributions, respectively. We see in $\Delta_{VGV} the presence in the numerator of the nucleon mass and an extra factor of $\pi$ compared with $\Delta_{IRR}$, as expected for a reducible diagram. However, we observe the presence of much bigger numerical factors in the numerator of $\Delta_{IRR}$, which in the end make that both contributions have similar size. In order to see this effect more clearly let us separate from $\Delta_{IRR}$ and $\Delta_{VGV}$ the terms proportional to $g_A^4$ and with the largest power of $A$, the ones that dominate for $|A|$ considerable larger than $M_A^2$. Their ratio, in this order, is

$$
\frac{\delta_{VGV}}{\delta_{IRR}} = \frac{-\pi m}{36 \left(\frac{(-A)^{\frac{1}{2}}}{245}\right)}.
$$
Again this equation exhibits clearly the large ratio of scales $\frac{\pi m}{(-A)^{1/2}}$, as expected, but at the same time it has a large numerical enhancement from the irreducible contribution by the factor $245/36 \approx 7$. This is large enough to make both contributions similarly sized because the previous ratio becomes

$$\frac{\delta_{\text{VG}}}{\delta_{\text{IRR}}} \simeq \frac{3M_\pi}{(-A)^{1/2}} \sim \frac{M_\pi}{(-A)^{1/2}} = \mathcal{O}(1) \ , \ A < -M_\pi^2 .$$

(55)

The presence of numerical factors enhancing $\Delta_{\text{IRR}}$ are in part due to combinatorial reasons, by putting on-shell the two pions when cutting diagrams in order to evaluate their imaginary part along the LHC, see Fig. 1. As an example, let us take proton-proton (pp) scattering. Then, the reducible part of diagram 1.d) only contributes by exchanging two $\pi^0$, which contains a factor $1/2$ because of the indistinguishability of them. But the diagram 1.c), in addition to $\pi^0\pi^0$, it also contains $\pi^+\pi^-$ as intermediate state. As a result the diagram 1.c) at low energies is enhanced by a factor 3 compared with diagram 1.d).

12 Coupled partial waves

The spin triplet $NN$ partial waves with total angular momentum $J$ mix the orbital angular momenta $\ell = J - 1$ and $\ell' = J + 1$ (except the $^3P_0$ wave that is uncoupled.) Each coupled partial wave is determined by the quantum numbers $S, J, \ell$ and $\ell'$. In the following for simplifying the notation we omit them and indicate, for given $J$ and $S$, the different partial waves by $t_{ij}$, with $i = 1$ corresponding to $\ell = J - 1$ and $i = 2$ to $\ell' = J + 1$. In matrix notation, one has a symmetric $2 \times 2$ $T$-matrix. In our normalization, the relation between the $T$- and $S$-matrix reads

$$S(A) = I + i2\rho(A)T(A)$$

$$= \begin{pmatrix}
\cos 2\epsilon_J e^{i2\delta_1} & i\sin 2\epsilon_J e^{i(\delta_1 + \delta_2)} \\
 i\sin 2\epsilon_J e^{i(\delta_1 + \delta_2)} & \cos 2\epsilon_J e^{i2\delta_2}
\end{pmatrix} ,$$

(56)

where $I$ is the $2 \times 2$ unit matrix, $\epsilon_J$ is the mixing angle, and $\delta_1$ and $\delta_2$ are the phase shifts for the channels with orbital angular momentum $J - 1$ and $J + 1$, in this order.

Above threshold ($A > 0$), and below pion production, the unitarity character of the $S$-matrix, $SS^\dagger = S^\dagger S = I$, can be expressed in terms of the (symmetric) $T$-matrix as

$$\text{Im}T^{-1}(A) = -\rho(A)I ,$$

(57)

where $\rho(A)$ was already defined in Eq. (4). In the following, the imaginary parts above threshold of the inverse of the $T$-matrix elements, $t_{ij}(A)$, play an important role,

$$\text{Im} \frac{1}{t_{ij}(A)} \equiv -\nu_{ij}(A) , A > 0 .$$

(58)

From Eq. (56), one can easily express the different $\nu_{ij}$ in terms of phase shifts and the mixing angle along the physical region. It implies that we can write the diagonal partial waves as $t_{ii}$ and the mixing amplitude $t_{12}$ as $t_{ii} = (e^{2i\delta_i} \cos 2\epsilon_J - 1)/2i\rho$ and $t_{12} = e^{i(\delta_1 + \delta_2)} \sin 2\epsilon_J/2\rho$, respectively. With these equalities it is
straightforward to obtain for \( A > 0 \):

\[
\nu_{11}(A) = \rho(A) \left[ 1 - \frac{\frac{1}{2} \sin^2 2\epsilon_J}{1 - \cos 2\epsilon_J \cos 2\delta_1} \right]^{-1},
\]

\[
\nu_{22}(A) = \rho(A) \left[ 1 - \frac{\frac{1}{2} \sin^2 2\epsilon_J}{1 - \cos 2\epsilon_J \cos 2\delta_2} \right]^{-1},
\]

\[
\nu_{12}(A) = 2\rho(A) \frac{\sin(\delta_1 + \delta_2)}{\sin 2\epsilon_J}.
\]

Eq. (58) generalizes Eq. (3), valid for an uncoupled partial wave. Indeed, if we set \( \epsilon_J = 0 \) in \( \nu_{11}(A) \) and \( \nu_{22}(A) \), the uncoupled case is recovered. Note also that \( \nu_{ij}(A)/\rho(A) \geq 1 \) and for \( A \to \infty \) one expects that \( \nu_{ij}(A) = O(A^2) \) as \( \rho(A) \) itself, because the absolute value of the trigonometric functions in Eqs. (59)-(61) is bounded by 1.

We apply the N/D method, discussed in Sec. 2, to each partial wave \( t_{ij} \) separately,

\[
t_{ij}(A) = \frac{N_{ij}(A)}{D_{ij}(A)}.
\]

We define \( \ell_{ij} \) as \( \ell_{11} = \ell, \ell_{22} = \ell' = \ell + 2 \) and \( \ell_{12} = (\ell + \ell')/2 = \ell + 1 \). From the previous equation and Eq. (58) it follows that

\[
\text{Im}D_{ij}(A) = -N_{ij}(A)\nu_{ij}(A), \quad A > 0,
\]

\[
\text{Im}N_{ij}(A) = D_{ij}(A)\Delta_{ij}(A), \quad A < L,
\]

where \( \text{Im}t_{ij}(A) \equiv \Delta_{ij}(A) \) along the LHC. The only formal difference with respect to Eqs. (6) and (8) is that now instead of \( \rho(A) \) we have \( \nu_{ij}(A) \) in Eq. (63). We can then follow the same line of reasoning as given in Sec. 2 and write down unsubtracted dispersion relations for \( D_{ij}/(A-C)^n \) and \( N_{ij}/(A-C)^n \) for large enough \( n \). Multiplying them by \( (A-C)^n \) we derive the proper dispersion relations valid for \( D_{ij}(A) \) and \( N_{ij}(A) \), as done in Sec. 2. In this way, our general equations for the coupled channel case arise:

\[
D_{ij}(A) = \sum_{p=1}^{n} \delta_{ij}^{(p)} (A-C)^{p-1} - \sum_{p=1}^{n} \nu_{ij}^{(p)} \frac{(A-C)^n}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2 - A)(q^2 - C)^{n-p+1}} \frac{\nu_{ij}(q^2)}{(q^2 - k^2)}
\]

\[
N_{ij}(A) = \sum_{p=1}^{n} \nu_{ij}^{(p)} (A-C)^{p-1} + \frac{(A-C)^n}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2 - A)(k^2 - C)^n}.
\]

Of course, as in the uncoupled partial wave case, we rewrite conveniently the previous equations whenever we take the subtractions at different subtraction points, that is, not all of the them taken at the same \( C \). In particular we impose the normalization condition

\[
D_{ij}(0) = 1,
\]

so that one subtraction for \( D_{ij}(A) \) is always taken at \( C = 0 \), and this gives

\[
\delta_{ij}^{(1)} = 1.
\]
We will indicate below case by case where the subtractions are taken.

For the partial waves with \( \ell_{ij} \geq 2 \) we have to guarantee the right threshold behavior such that \( t_{ij}(A) \rightarrow A^{\ell_{ij}} \) for \( A \rightarrow 0^+ \). This is done as in Sec. 6 by considering \( \ell_{ij} \)-time DRs with all the subtraction constants in \( N_{ij}(A) \) taken at \( C = 0 \) and with vanishing value. For the function \( D_{ij}(A) \), apart of the subtraction taken at \( C = 0 \), the rest of them are taken at \( C \neq 0 \). The resulting IEs are

\[
D_{ij}(A) = 1 + \sum_{p=2}^{\ell_{ij}} \delta^{(ij)}_p A(C)^{p-2} + \frac{A(C)^{\ell_{ij}-1}}{\pi^2} \int_{-\infty}^{\infty} dk^2 \Delta_{ij}(k^2) D_{ij}(k^2) \]

\[
\times \int_{0}^{\infty} dq^2 \frac{\nu_{ij}(q^2)(q^2)^{\ell_{ij}-1}}{(q^2 - A)(q^2 - k^2)(q^2 - C)^{\ell_{ij}-1}},
\]

\[
N_{ij}(A) = \frac{A^{\ell_{ij}}}{\pi} \int_{-\infty}^{\infty} dk^2 \Delta_{ij}(k^2) D_{ij}(k^2) \frac{(k^2)^{\ell_{ij}}}{(k^2)^{\ell_{ij}}(k^2 - A)}.
\]

Notice that we have rewritten the \((\ell_{ij} - 1)\)th degree polynomial in \( D_{ij}(A) \) so that the coefficients \( \delta^{(ij)}_p \) have a simpler relation with \( D_{ij}(A) \). Indeed, one can deduce straightforwardly

\[
\delta^{(ij)}_p = \frac{(-1)^p}{C^{p-1}} \left[ \sum_{i=0}^{p-2} \frac{(-1)^i}{i!} C^i D^{(i)}(C) - 1 \right].
\]

That is, \( \delta^{(ij)}_p \) is proportional to the Taylor expansion of \( D_{ij}(A) \) at around \( A = C \) up to order \( p - 2 \) evaluated at \( A = 0 \) minus \( D_{ij}(0) = 1 \). In the practical applications that follow we always take \( C = -M_\pi^2 \).

The situation with all the \( \delta^{(ij)}_p \) equal to zero corresponds to \( D_{ij}(0) = 1 \) and \( D_{ij}^{(n)}(0) = 0 \) (this is the so-called pure perturbative case for a high orbital-angular-momentum wave). On the other hand, the rule given in Sec. 6 for a \( n \)-time subtracted DR corresponds to having \( D_{ij}(0) = 1 \), \( D_{ij}^{(p)} = 0 \) for \( 1 \leq p < n - 2 \) and \( D_{ij}^{(n-2)}(0) \neq 0 \).

As shown explicitly in Ref. 25, the \( \nu_{22}(A) \) function diverges as \( A^{-\frac{3}{2}} \) for \( A \rightarrow 0 \). This requires some care in order to avoid infrared divergent integrals, a problem already noticed in Ref. 39. This issue is cured in Eq. (69) because \( C \neq 0 \). Then, the factor \((q^2)^{\ell_{22} - 1}\) cancels, at least partially, the threshold divergence in \( \nu_{22}(A) \) so that the integral is convergent. Notice that \( \ell_{22} \geq 2 \), with its smallest value for the \( 3D_1 \) wave. The function \( \nu_{12}(A) \) also diverges at threshold but only as \( A^{-\frac{1}{2}} \), so that it does not give rise to any infrared divergent integral. For completion, we recall that the \( \nu_{11}(A) \) vanishes for \( A \rightarrow 0 \) as \( A^{\frac{1}{2}} \). In the following we define the function \( g_{ij}(A, k^2, C; m) \) as

\[
g_{ij}(A, k^2, C; m) = \int_{0}^{\infty} dq^2 \frac{\nu_{ij}(q^2)(q^2)^m}{(q^2 - A)(q^2 - k^2)(q^2 - C)^m}.
\]

The main difference with respect to the uncoupled case is that now one has to solve simultaneously three N/D equations for \( ij = 11, 12 \) and 22, which are linked between each other because of the \( \nu_{ij}(A) \) functions. They depend on the phase shifts \( \delta_1, \delta_2 \) and on the mixing angle \( \epsilon_I \), defined in Eq. (63), which constitute also the final output of our approach. Thus, we follow an iterative approach, as already done in Ref. 25, as follows. Given an input for \( \delta_1, \delta_2 \) and \( \epsilon_I \), one solves the three integral equations for \( D_{ij}(A) \) along the LHC. Then, the scattering amplitudes on the RHC can be calculated. In terms of them, the phase shifts \( \delta_1 \) and \( \delta_2 \) are obtained from the phase of the \( S \)-matrix elements \( S_{11} \) and \( S_{22} \), while \( \sin 2\epsilon_I = \frac{2p_{t12}n_{12}}{n_{12}} \), according to Eq. (68). In this way a new input set of \( \nu_{ij} \) functions, Eqs. (69)-(71), is provided. These are used again in the integral equations, and the iterative procedure...
is finished when convergence is found (typically, the difference between two consecutive iterations in the three independent functions $D_{ij}$ along the LHC is required to be less than one per mil.)

It can be shown straightforwardly that unitarity is fulfilled in our coupled channel equations, solved in the way just explained, if $|S_{11}(A)|^2 = |S_{22}(A)|^2 = \cos^2 2\epsilon_J$ for $A > 0$. From the fact that $\text{Im} t_{12} = \nu_{12}|t_{12}|^2$, according to Eq. (63), and $\sin 2\epsilon_J = 2\rho|t_{12}||n_{12}|$ (the latter equality is valid only when convergence is reached), it results that the phase of $t_{12}$ is $\delta_1 + \delta_2$, as required by unitarity, Eq. (56). By construction the phase shifts are equal to one-half of the phase of the $S$-matrix diagonal elements when convergence is achieved, so that Eq. (56) is satisfied if $|S_{11}| = |S_{22}| = \cos 2\epsilon_J$.

For the initial input one can use e.g. the results given by Unitarity ChPT [40], the LO results obtained from Ref. [25] or some put-by-hand phase shifts and mixing angle. For the latter case a good choice is to take as initial input for $\delta_1$ and $\delta_2$ the resulting phase shifts obtained by treating $t_{11}$ and $t_{22}$ as uncoupled waves. We find no dependence in our final unitary results regarding the input taken.

### 13 Coupled waves: $^{3}S_{1} - ^{3}D_{1}$

For the $^{3}S_{1} - ^{3}D_{1}$ system, we write down a once-subtracted DR for the partial wave $^{3}S_{1}$ and twice-subtracted DRs for the $^{3}D_{1}$ and mixing partial wave, in order to guarantee that the position of the deuteron pole is the same in all of the tree partial waves. The explicit expressions for the $^{3}S_{1}$ partial wave are:

\[
D_{11}(A) = 1 - \nu_{1} A g_{11}(A,0) + \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{k^2} g_{ij}(A,k^2),
\]

\[
N_{11}(A) = \nu_{1} + \frac{A}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{k^2(k^2 - A)},
\] (73)

where the function $g_{ij}(A,k^2)$ is defined as

\[
g_{ij}(A,k^2) = \frac{1}{\pi} \int_{0}^{\infty} dq^2 \frac{\nu_{ij}(q^2)}{(q^2 - A)(q^2 - k^2)}.
\] (74)

The subtraction constant $\nu_{1}$ is fixed in terms of the experimental $^{3}S_{1}$ scattering length, $a_{t} = 5.424 \pm 0.004$ fm, analogously as we did already for the $^{1}S_{0}$ in Sec. 4

\[
\nu_{1} = \frac{4\pi a_{t}}{m}.
\] (75)

For the mixing partial wave, $\ell_{12} = 1$, and $^{3}D_{2}$ with $\ell_{22} = 2$, we have

\[
N_{ij}(A) = \frac{A_{\ell_{ij}}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2 - A)(k^2)_{\ell_{ij}}},
\]

\[
D_{ij}(A) = 1 - \frac{A}{k_{d}^{2}} + \frac{A(A - k_{d}^{2})}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2)_{\ell_{ij}}^{2}} g_{ij}^{(d)}(A,k^2),
\] (76)

with the new integration along the RHC

\[
g_{ij}^{(d)}(A,k^2) = \frac{1}{\pi} \int_{0}^{\infty} dq^2 \frac{\nu_{ij}(q^2)(q^2)_{\ell_{ij}}^{-1}}{(q^2 - A)(q^2 - k^2)(q^2 - k_{d}^{2})}.
\] (77)
Figure 17: (Color online.) From top to bottom and left to right: Phase shifts for $^3S_1$, $^3D_1$ and the mixing angle $\epsilon_1$, respectively. The (red) solid line corresponds to the results obtained from Eqs. (73) and (76) with the $^3S_1$ scattering length as experimental input. The OPE result from Ref. [25] is the (blue) dotted line. The Nijmegen PWA analysis is the (black) dashed line.

The function $g_{ij}(A, k^2)$ and $g_{ij}^{(d)}(A, k^2)$ were already introduced in Ref. [25]. Notice that these functions have to be evaluated numerically. In Eq. (76) one extra subtraction is taken at $k^2_d$, which is the three-momentum squared of the deuteron pole position obtained for the $^3S_1$ wave from Eq. (73). In other words, $k^2_d$ is the value of $A$ at which $D_{11}(k^2_d) = 0$ in each step in the iterative process for solving Eqs. (73) and (76). No extra subtraction constants are introduced because we require $D_{12}(k^2_d) = D_{22}(k^2_d) = 0$, so that all three coupled partial waves have the deuteron at the same position, $A = k^2_d$.

We solve Eqs. (73) and (76) with different input which is provided by the results of Ref. [40] by varying the parameter $g_0$ in that reference. We observe some dependence in the outcome solutions so that we require a criterion of maximum stability under changes in $g_0$. E.g. let us take the slope at threshold of the mixing angle $\epsilon_1$, denoted by $a_\epsilon$ and defined by

$$a_\epsilon = \lim_{A \to 0^+} \sin \frac{2\epsilon_1}{A^2} = 1.128 \ M_\pi^{-3}, \quad (78)$$

with the value obtained from the Nijmegen PWA phase shifts. This quantity has a minimum as a function of the input used that indeed gives the closest value to the experimental one in Eq. (78). We obtain $a_\epsilon = 1.10 \sim 1.14 \ M_\pi^{-3}$. Precisely the mixing angle is by far the most sensitive quantity to the input data for obtaining the final solution by iteration. Then, it is certainly a welcome fact that the best results are obtained for the input that generates most stable results under changes of itself. The results obtained by solving Eqs. (73) and (76), with $\nu_1$ fixed from the experimental $^3S_1$ scattering length, Eq. (75), are
shown by the (red) solid line in Fig. 17. We see that these curves tend to follow data quite closely already, specially below $\sqrt{A} \simeq 100$ MeV. Let us notice as well the clear and noticeable improvement in the reproduction of data compared with the OPE results of Ref. [25].

This improvement is also clear in the value obtained for the deuteron binding energy, $E_d = -k_d^2/m$. At NLO we obtain $E_d = 2.35-2.38$ MeV, a value much closer to experiment $E_d = 2.22$ MeV than the one obtained at LO in Ref. [25], $E_d = 1.7$ MeV. A similar situation also occurs for the $^3S_1$ effective range, $r_t$.

Proceeding similarly as done in Sec. 4 for $r_s$, we derive an integral expression for calculating $r_t$:

$$ r_t = -\frac{m}{2\pi^2 a_t} \int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2) D_{11}(k^2)}{(k^2)^2} \left\{ \frac{1}{a_t} + \frac{4\pi k^2}{m} g_{11}(0, k^2) \right\} - \frac{8}{m} \int_0^\infty dq^2 \frac{\nu_{11}(q^2) - \rho(q^2)}{(q^2)^2} . $$

The last integral on the r.h.s. of the previous equation was not present in Eq. (24) because it is a coupled-wave effect, due to the mixing between the $^3S_1$ and $^3D_1$. This equation also exhibits the correlation between $a_t$ and $r_t$, although in a more complicated manner than for the $^1S_0$ partial wave, Eq. (26), because $\nu_{11}(A)$ depends nonlinearly on $D_{11}(A)$. We obtain the value

$$ r_t = 1.36 - 1.39 \text{ fm} , $$

Figure 18: (Color online.) Left panels: Different contributions to the integrals in Eq. (87). Right panels: Contributions to $\Delta(A)$. From top to bottom we show the results for $^3S_1$, $^3D_1$ and mixing wave, respectively. OPE is given by the (green) dash-dotted line, the (magenta) dashed line is the reducible TPE contribution and the (blue) dotted line is the irreducible TPE contribution. In the left panels we show the total result by the (red) solid line.
to be compared with its experimental value, \( r_t = 1.759 \pm 0.005 \) \( \text{fm} \). At LO Ref. [25] obtained the much lower result \( r_t = 0.46 \) \( \text{fm} \) when only \( a_t \) was taken as experimental input.

It is also interesting to diagonalize the \( ^3S_1 - ^3D_1 \) S-matrix around the deuteron pole position. This allows us to obtain two interesting quantities [41], apart from the deuteron binding energy. One of them is the asymptotic \( D/S \) ratio \( \eta \) of the deuteron. To evaluate this quantity we diagonalize the \( ^3S_1 - ^3D_1 \) S-matrix by an orthogonal matrix \( O \),

\[
O = \begin{pmatrix}
\cos \varepsilon & -\sin \varepsilon \\
\sin \varepsilon & \cos \varepsilon 
\end{pmatrix},
\]

Such that

\[
S = O \begin{pmatrix} S_0 & 0 \\ 0 & S_2 \end{pmatrix} O^T,
\]

with \( S_0 \) and \( S_2 \) the S-matrix eigenvalues. The parameter \( \eta \) can be expressed in terms of the mixing angle \( \varepsilon \) as [41, 42]

\[
\eta = -\tan \varepsilon.
\]

We also evaluate the residue of the eigenvalue \( S_0 \) at the deuteron pole position

\[
S_0 = \frac{N_p^2}{\sqrt{-k^2_d + i\sqrt{A}}} + \text{regular terms}.
\]

We obtain the following numerical values:

\[
\eta = 0.029, \quad N_p^2 = 0.73 \, \text{fm}^{-1},
\]

that are close to the calculations \( \eta = 0.0271(4) \) [43], \( \eta = 0.0263(13) \) [44] and \( \eta = 0.0268(7) \) [45], as well as to the Nijmegen PWA results [46]

\[
\eta = 0.02543(7), \quad N_p^2 = 0.7830(7) \, \text{fm}^{-1}. \tag{86}
\]

Apart from the IE on Eqs. (73) and (76) we also tried other ones by including more subtractions, so that more experimental input could be fixed. Namely, fixing simultaneously: i) \( a_t \) and \( a_c \) or ii) \( a_t, r_t \) and \( E_d \) or iii) \( a_t, r_t, E_d \) and \( a_c \). However, either the coupled-channel iterative process does not converge or we end with the solution corresponding to the uncoupled-wave case.

We also consider here analogous integrals along the LHC to those used in Sec. [11] in order to quantify the different contributions to \( \Delta(A) \),

\[
\ell_{11} = 0 : \frac{A^2}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{\nu_{11}(q^2)}{(q^2 - A)(q^2 - k^2)} ,
\]

\[
\ell_{12} = 1 : \frac{A(A - k^2_d)}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{12}(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{\rho(q^2)q^2}{(q^2 - A)(q^2 - k^2)(q^2 - k^2_d)} ,
\]

\[
\ell_{22} = 2 : \frac{A(A - k^2_d)}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{\nu_{22}(q^2)q^2}{(q^2 - A)(q^2 - k^2)(q^2 - k^2_d)} , \tag{87}
\]

where two subtractions are required in order to have convergent integrals in Eq. (87), as already pointed out in the uncoupled-wave case. For the mixing partial wave, we have taken the integration along the
RHC as it were elastic, using $\rho(q^2)$ instead of $\nu_{12}(q^2)$, because the latter would require the actual function $D_{12}(k^2)$ as it is very sensitive to coupled-channel effects. From the left panels of Fig. 18 we see that the total integral is dominated by OPE in all cases. Nevertheless, for $^3S_1$ the individual contributions of the reducible and irreducible TPE are sizeable but of different sign, so that they cancel to a large extent and the dominance of the OPE contribution results. We see that, as a whole, the reducible and irreducible contributions are of similar absolute size but with opposite signs.

14 Coupled waves: $^3P_2 - ^3F_2$

![Phase shift plots](image)

Figure 19: (Color online.) From top to bottom and left to right: Phase shifts for $^3P_2$, $^3F_2$ and the mixing angle $\epsilon_2$, in order. The (red) solid line corresponds to the results obtained with once-subtracted DRs for $^3P_2$, while twice-subtracted DRs are used for the latter partial wave to obtain the (green) dash-dotted line. The (blue) dotted line is the results with only OPE from Ref. [25]. The Nijmegen PWA phase shifts are given by (black) dashed line.

In this section we consider the coupled wave system $^3P_2 - ^3F_2$ making use of Eqs. (69) and (70) with $\ell_{11} = 1$, $\ell_{12} = 2$ and $\ell_{22} = 3$. In the following we always take $C = -M^2_\pi$ in Eq. (69) and instead of the coefficients $\delta^p_{ij}$ we directly use $D_{ij}^{(n)}(C)$, $n = 0, \ldots, \ell_{ij} - 2$, as the free parameters. As discussed in Sec. 6 it is enough to $D_{ij}^{(\ell_{ij} - 2)}(C)$ as the only active free parameter for every partial wave.

We find that the results are all quite insensitive to $D_{22}(-M^2_\pi)$ and $D'_{22}(-M^2_\pi)$, as one would expect because $F$-waves are expected to be perturbative, as already discussed in Sec. 8. This is another confirmation of this conclusion. The fitted parameter $D'_{22}(-M^2_\pi)$ turns to be negative and of several units of size, but essentially the same results are obtained as long as $D'_{22}(-M^2_\pi) \lesssim -1 M^{-2}_\pi$. Regarding $D_{22}(-M^2_\pi)$ we
Figure 20: (Color online.) Left panels: Different contributions to the integrals in Eq. (91). Right panels: Contributions to $\Delta(A)$. From top to bottom we show the results for $^3P_2$, $^3F_2$ and mixing wave, respectively. The meaning of the lines is the same as in Fig. 15.
fix it to 1. Our results are then only sensitive to \( D_{12}(-M_π^2) \) with the best fitted value

\[
D_{12}(-M_π^2) = 1.1 .
\] (88)

From these results we can calculate the \( P \)-wave scattering volume, which is just given by the first derivative at \( A = 0 \) of the function \( N_{11}(A) \). This is straightforwardly worked out from Eq. (70), with the result, \( a_V = 0.12 \ M_π^{-3} \), that is a 20\% off its phenomenological value \( a_V = 0.0964 \ M_π^{-3} \) obtained from Ref. [36]. To improve this situation we employ a twice-subtracted DR by taking \( N(66) \) for the two subtractions in the function \( N(A) \) at \( C = 0 \) with \( \nu_1^{(11)} = 0 \) and

\[
\nu_2^{(11)} = \frac{4\pi a_V}{m} ,
\] (89)

in terms of the experimental value of \( a_V \). Now \( D_{11}(-M_π^2) \) is also a free parameter fitted to data,

\[
D_{11}(-M_π^2) = 0.1 ,
\] (90)

while for \( D_{12}(-M_π^2) \) and \( D_{22}(M_π^2) \) the same values as in the case of the once-subtracted DR for \( 3P_2 \) are employed, since no improvement in the reproduction of data results by varying them. The resulting phase shifts and mixing angle are shown by the (red) solid line, than when it is not imposed (green dashed line). In all the partial waves we observe a noticeable improvement of the OPE results of Ref. [25].

At the practical numerical level it is interesting to remark that for the coupled waves the mixing angle is small. Then, as a first approximation, one can study separately the waves with orbital angular momentum \( J - 1 \) and \( J + 1 \) as if they were uncoupled. In this way, it is more efficient numerically to fit the free parameters present in them than if the full iterative process of coupled waves were taken. Once this is done, the mixing is included but we first keep the values obtained in the uncoupled-wave limit for the free parameters fitted then, so that it only remains to determine those present in the mixing partial wave. We then vary around the parameters fixed by the uncoupled-wave case until the full results are stable.

With regard to the integrals along the LHC in order to quantify the different contributions to \( \Delta(A) \), we have now, according to the number of subtractions taken in the DRs for each partial wave, the following expressions:

\[
\ell_{11} = 1 : \quad \frac{A(A + M_π^2)}{π^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{\nu_{11}(q^2)q^2}{(q^2 - A)(q^2 - k^2)(q^2 + M_π^2)} ,
\]

\[
\ell_{12} = 2 : \quad \frac{A(A + M_π^2)}{π^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{12}(k^2)}{(k^2)^2} \int_{0}^{\infty} dq^2 \frac{\rho(q^2)q^2}{(q^2 - A)(q^2 - k^2)(q^2 + M_π^2)} ,
\]

\[
\ell_{22} = 3 : \quad \frac{A(A + M_π^2)^2}{π^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2)}{(k^2)^3} \int_{0}^{\infty} dq^2 \frac{\nu_{22}(q^2)(q^2)^2}{(q^2 - A)(q^2 - k^2)(q^2 + M_π^2)^2} .
\] (91)

For \( 3F_2 \) and the mixing partial wave the situation is as usual, so that the OPE contribution dominates the respective integral along the LHC. However, for the \( 3P_2 \) the reducible TPE contribution is much larger than the OPE one. We consider that this situation is very specific for this partial wave. This is manifest by the fact that the OPE contribution in this wave is in absolute value more than one order of magnitude smaller than in the other \( P \)-waves, namely, \( 1P_1, 3P_0, 3P_1 \) and the mixing wave in the \( 3S_1 - 3D_1 \).
system. This can be easily checked by comparing the two panels in the first row of Fig. 20 with Fig. 12 and the two panels in the last row of Fig. 18. On the other hand, we also observe that the reducible and irreducible TPE contributions have typically similar size in absolute value, taking a whole picture of all the partial waves involved in the $^3P_2 - ^3F_2$ system.

15 Coupled waves: $^3D_3 - ^3G_3$

The orbital momenta attached to the the $^3D_3 - ^3G_3$ system are $\ell = 2$, 3 and 4 for the $^3D_3$, mixing wave and $^3G_5$ coupled waves, in this order. These values are used in Eqs. (69) and (70) to provide the appropriate IEs.

![Graphs showing phase shifts and mixing angles for $^3D_3$, $^3G_3$ and the mixing angle $\epsilon_3$.](image)

Figure 21: (Color online.) From top to bottom and left to right: Phase shifts for $^3D_3$, $^3G_3$ and the mixing angle $\epsilon_3$, respectively. The (red) solid line corresponds to our NLO results, the (blue) dotted line is the results with only OPE from Ref. [25] and the Nijmegen PWA phase shifts are given by (black) dashed line.

The fit is not able to fix a definite value for $D_{11}(-M_c^2)$, which is always given with large uncertainties and very much dependent on the upper limit of the energy taken in the fit. Then, we fix it to 1 and the curves are basically the same. For the mixing wave we also have $D_{12}(C) = 1$. Regarding the first derivative $D'_1(C)$ a slightly negative value, e.g. $-0.1 \, M_c^{-2}$, offers the best results. This corresponds basically to the situation with the perturbative values for the mixing wave. For the $^3G_3$ wave the fit is also consistent with a smooth behavior for the $D_{22}(A)$ function for $A < 0$. In this case, $D_{22}(C) = 1$, $D'_{22}(C) = 0$ and $D''_{22}(C) > 1 \, M_c^{-4}$, that is, only the highest order derivative is different from zero with the value of the function at $C$ equal to 1, according to the rule given in Sec. 6. The resulting phase shifts and mixing angle are shown in Fig. 21 by the (red) solid line. We already see that the phase shifts for...
\[ \Delta(A) = \int_{-\infty}^{\infty} dk \frac{\Delta_{ij}(k^2)}{(k^2)^{\ell_{ij}} - 1} \int_{0}^{\infty} dq \frac{\mu_{ij}(q^2)(q^2)^{\ell_{ij}} - 1}{(q^2 - A)(q^2 - k^2)(q^2 + M_\pi^2)^{\ell_{ij}} - 1}, \] (92)

where \( \ell_{ij} = 2, 3 \) and 4, \( \mu_{11} = \nu_{11}, \mu_{22} = \nu_{22} \) and \( \mu_{12} = \rho \). The results are shown in Fig. 22. We see that for \( ^3G_3 \) and the mixing wave the integral is dominated by OPE. However, for \( ^3D_3 \) the irreducible and reducible TPE contributions are large, indeed each of them is larger than OPE, though they have opposite signs and then cancel mutually to a large extent. This is why OPE is still the most important contribution to the total result, but we then expect for this wave that the higher order contributions will play a more prominent role. Indeed, \( ^3D_3 \) is the wave for which the reproduction of data is still poor in Fig. 21.
16 Coupled waves: $^3F_4 - ^3H_4$

In this case the direct use of Eqs. (69) and (70) does not provide a stable solution for the $^3H_4$ wave. We have to perform an extra subtraction in the $^3H_4$ partial wave in order to end with meaningful (convergent) results. The resulting IEs to be solved are:

\[
\begin{align*}
ij = 11, 12 : D_{ij} &= 1 + \sum_{p=2}^{\ell_{ij}} \delta_p^{(ij)} A(A - C)^{p-2} + \frac{A(A - C)^{\ell_{ij} - 1}}{\pi^2} \int_{-\infty}^{L} \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2)^{\ell_{ij}}(k^2 - A)} g_{ij}(A, k^2, C; \ell_{ij} - 1), \\
ij = 22 : D_{22} &= 1 + \sum_{p=2}^{6} \delta_p^{(22)} A(A - C)^{p-2} + \frac{A(A - C)^5}{\pi^2} \int_{-\infty}^{L} \frac{\Delta_{22}(k^2)D_{22}(k^2)}{(k^2)^6} g_{22}(A, k^2, C; 5),
\end{align*}
\]

with the $N_{ij}(A)$ functions given by:

\[
\begin{align*}
ij = 11, 12 : N_{ij}(A) &= \frac{A^{\ell_{ij} - 1}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)D_{ij}(k^2)}{(k^2)^{\ell_{ij}}(k^2 - A)}, \\
ij = 22 : N_{22}(A) &= \nu_{6}^{(22)} A^5 + \frac{A^6}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2)D_{22}(k^2)}{(k^2)^6(k^2 - A)}.
\end{align*}
\]

We can obtain $\nu_{6}^{(22)}$ by making use of a once-subtracted DR for the $^3H_4$ partial wave, which has a large orbital angular momentum, so that this DR provides accurate results. Recall our results for the once-subtracted DR in the uncoupled partial waves with $\ell \geq 3$ presented by the (cyan) double-dotted lines in Figs. [8][10]. For $A \to 0$ one has that $T(A) \to N(A) \to \nu_{6}^{(22)} A^5$, so that this counterterm is directly related with the behavior of the phase shifts at threshold, as it could be foreseen from our previous experience with the $P$-waves. In this way, we obtain

\[
\nu_{6}^{(22)} = 0.079 \ M_x^{-12}.
\]

The coefficients $\delta_p^{(ij)}$ are expressed in terms of the functions $D_{ij}(A)$ and their derivatives at $A = C$, according to Eq. (71), with $C = -M_x^2$ as we always take. For $^3F_4$ we use $D_{11}(C) = 1$ and $D_{11}'(C) = 0$, because other values different from the pure perturbative ones do not improve the reproduction of data. For the $^3H_4$ one can also think of the pure perturbative values $D_{22}(C) = 1$ and $D_{22}^{(n)}(C) = 0$, $n = 1, 2, 3$. However, we have realized that a little change in $\nu_{6}^{(22)}$ requires a change of $O(1)$ in $\delta_{6}^{(22)}$, keeping only negative values. In this way, we have fixed the latter coefficient to a negative value of $O(1)$ and then adjust slightly $\nu_{6}^{(22)}$ with respect to the value calculated in Eq. (95). Typically we find just a slightly smaller value for $\nu_{6}^{(22)}$ than that in Eq. (95), $\nu_{6}^{(22)} \simeq 0.078 \ M_x^{-12}$. Regarding the mixing wave we find that no improvement in the reproduction of data is accomplished for $D_{12}^{(n)}(C), 0 \leq n \leq 2$, with values different from the pure perturbative ones, which are the ones taken. We show our NLO results in Fig. [23] by the (red) solid line. One observes that still an improvement (higher orders) is needed to reproduce the $^3F_4$ phase shifts, such deviation is also observed in ChPT potential approaches, see e.g. [17]. For the $^3H_4$ and $\epsilon_4$ the reproduction is much better. The (blue) dotted line corresponds to the OPE results that run close to the NLO ones.\footnote{No OPE results for the $^3F_4 - ^3H_4$ and $^3G_5 - ^3F_5$ are worked out in Ref. [25]. We obtain them by employing the same IEs as in NLO but keeping only in $\Delta_{ij}(A)$ the OPE contribution.}
As usual we also study the size of the different contributions to $\Delta(A)$ by evaluating the pertinent integrals along the LHC:

$$
\ell_{ij} = 3, 4 : \frac{A(A + M_\pi^2)^{\ell_{ij} - 1}}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2)}{(k^2)^{\ell_{ij}}} \int_{0}^{\infty} dq^2 \frac{\mu_{ij}(q^2)(q^2)^{\ell_{ij} - 1}}{(q^2 - A)(q^2 - k^2)(q^2 + M_\pi^2)^{\ell_{ij} - 1}},
$$

$$
\ell_{22} = 5 : \frac{A(A + M_\pi^2)^5}{\pi^2} \int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2)}{(k^2)^6} \int_{0}^{\infty} dq^2 \frac{\nu_{22}(q^2)(q^2)^5}{(q^2 - A)(q^2 - k^2)(q^2 + M_\pi^2)^5},
$$

with $\mu_{ij}$ defined after Eq. (92). The results are shown in Fig. 24. We see that for all the waves the total result of the integrals is dominated by OPE. Though for the $^3F_4$ the independent contributions of reducible and irreducible TPE are not small, they cancel each other almost exactly.

### 17 Coupled waves: $^3G_5 - ^3I_5$

In the $^3G_5 - ^3I_5$ system we have $\ell_{11} = 4$, $\ell_{12} = 5$ and $\ell_{22} = 6$. However, the resulting IEs from Eqs. (69) and (70) do not provide convergent results because the $^3I_5$ partial wave requires an extra subtraction, so
Figure 24: (Color online.) Left panels: Different contributions to the integrals in Eq. (96). Right panels: Contributions to $\Delta(A)$. From top to bottom we show the results for $^3F_4$, $^3H_4$ and mixing wave, respectively. The meaning of the lines is the same as in Fig. 18.
that we can finally obtain results independent of the limits of integration. We then have:

\[
ij = 11, 12 : D_{ij} = 1 + \sum_{p=2}^{7} \delta_p^{(ij)} A (A - C)^{p-2} + \frac{A (A - C)^{\ell_{ij} - 1}}{\pi^2} \int_{-\infty}^{L} \frac{\Delta_{ij}(k^2) D_{ij}(k^2)}{(k^2)^{\ell_{ij}}} g_{ij}(A, k^2, C; \ell_{ij} - 1),
\]

\[
ij = 22 : D_{22} = 1 + \sum_{p=2}^{7} \delta_p^{(22)} A (A - C)^{p-2} + \frac{A (A - C)^6}{\pi^2} \int_{-\infty}^{L} \frac{\Delta_{22}(k^2) D_{22}(k^2)}{(k^2)^7} g_{22}(A, k^2, C; 6),
\]

with the \(N_{ij}(A)\) functions given by

\[
ij = 11, 12 : N_{ij}(A) = \frac{A^{\ell_{ij}-1}}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{ij}(k^2) D_{ij}(k^2)}{(k^2)^{\ell_{ij}}(k^2 - A)},
\]

\[
ij = 22 : N_{22}(A) = \nu^{(22)}_7 A^6 + \frac{A^7}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta_{22}(k^2) D_{22}(k^2)}{(k^2)^7(k^2 - A)}. \tag{98}
\]

We can predict \(\nu^{(22)}_7\) by employing a free-parameter once-subtracted DR for an uncoupled \(3I_5\), as we did in the previous section to calculate \(\nu^{(22)}_6\) for the \(3H_4\) wave. In this way we obtain the number

\[
\nu^{(22)}_7 = -0.178 \ M_{\pi}^{-14}. \tag{99}
\]

We have also tried fits to data by releasing this number and the results obtained confirm this prediction. Regarding the coefficients \(\delta_p^{(ij)}\) the same quality in the reproduction of data is obtained by taking \(\delta_p^{(ij)} = 0\) except for the coefficient with the highest \(p\) for every \(ij\), namely, \(p = 4\) for \(ij = 11\), \(p = 5\) for \(ij = 12\) and \(p = 7\) for \(ij = 22\), which are fitted to data. Then, the coupled-wave system \(3G_5 - 3I_5\) illustrates again the rule of Sec. 6. For the fitted coefficients we have \(D^{(2)}_{11}(C) < -0.5\), \(D^{(3)}_{12}(C) < -0.5\) and \(D_{22}(C)^{(5)} \neq 0\), in appropriate powers of \(M_{\pi}^{-2}\). For the last constant, one has to take into account that a change in \(D_{22}(C)^{(5)}\) of \(\mathcal{O}(1)\) can be reabsorbed in slight changes of \(\nu^{(22)}_7\) around the value given in Eq. (99), similarly as in Sec. 17 for the \(3F_4 - 3H_4\) system.

The resulting phase shifts are shown in Fig. 25 in which, for definiteness, we take the values \(D^{(2)}_{11}(C) = -1 \ M_{\pi}^{-4}\), \(D^{(3)}_{12}(C) = -1 \ M_{\pi}^{-6}\) and \(D_{22}^{(5)} = -10 \ M_{\pi}^{-10}\). The NLO phase shifts are shown by the (red) solid line. We see that they follow closely the \(NN\) phase shifts of Ref. [36]. For the \(3I_5\) partial-wave phase shifts the reproduction is perfect. The LO results, given by the (blue) dotted line, are also obtained with the same values for the \(\delta_p^{(ij)}\). We observe that the reproduction of the \(3G_5\) phase shifts is worse than in the NLO case, and only slightly worse for the \(3I_5\) phase shifts. For \(\epsilon_5\) the LO result is similar as in NLO. We have also varied the \(\delta_p^{(11)}\) (\(p = 2, 3, 4\)) for the LO calculation in order to improve the reproduction of the \(3G_5\) phase shifts but no gain is obtained.

It has been already noticed in Refs. [3,17] that the \(3G_5\) phase shifts, even with a chiral N\(^3\)LO potential, are not well reproduced after solving the corresponding Lippmann-Schwinger equation, either with finite [4] or infinite three-momentum cutoff [37], as well as by calculating in perturbation theory [3,27]. Our results in Fig. 25 for the \(3G_5\) are closer to data than the ones in those references, despite our calculation is only a NLO one. However, our nonperturbative approach already includes one free parameter exclusively for the \(3G_5\), which is not the case in Refs. [3,4,17].

As usual we also study the size of the different contributions to \(\Delta(A)\) by evaluating the appropriate
The results are shown in Fig. 26. We see that for all the waves the total result of the integrals is dominated by OPE. However, for the $^3G_5$ the iterated and irreducible TPE contributions are not small. Nevertheless, they cancel almost exactly so that the net contribution is mostly given by OPE.

Finally, we show in Table I the minimum number of free parameters needed for every partial wave in our present study at NLO. When the free parameter is only determined within broad intervals (its order of magnitude is not even fixed) then we do not consider it as a free parameter, but better as having a constraint. We do not consider either as free parameters those subtraction constants that take their expected perturbative values. We give in the box next to the right of the one with the name of the partial wave, the minimum number of free parameters for this partial wave, in the explained sense. We have in total 14 free parameters. One should be aware that the number of free parameters does not necessarily increase with the accuracy up to which $\Delta(A)$ is calculated in ChPT. There is no such a close connection between subtraction constants and the chiral order in which the $\Delta(A)$ as the situation between the number of chiral counterterms and the chiral order in which the NN potential is calculated. E.g.
Figure 26: (Color online.) Left panels: Different contributions to the integrals in Eq. (100). Right panels: Contributions to $\Delta(A)$. From top to bottom we show the results for $^3G_5$, $^3I_5$ and mixing wave, respectively. The meaning of the lines is the same as in Fig. 18.
we have one free parameter for the $^1S_0$ and $^3S_1 - ^3D_1$ waves both at LO \cite{24,25} and now at NLO.

\section*{Conclusions}

We have applied the N/D method to study $NN$ scattering within ChPT. The basic input in this method is the imaginary part along the left-hand cut of a given $NN$ partial wave, that we denote by $\Delta(A)$. This is calculated within ChPT up to some order in the chiral expansion. Here we have included OPE and leading TPE contributions, extending the results of Refs. \cite{24,25}, which only considered OPE. The standard ChPT counting clearly establishes that OPE is $\mathcal{O}(p^0)$, while irreducible TPE is $\mathcal{O}(p^2)$. We have also discussed that increasing the pion ladders in $NN$ reducible diagrams is suppressed because it gives rise to contributions to $\Delta(A)$ for $A$ deeper in the LHC and further away from the low-energy physical region. We have employed suitable integrals along the LHC to properly quantify the different contributions to $\Delta(A)$, better than just to compare numerical values directly from this quantity. It follows that OPE is indeed the dominant contribution while irreducible and reducible TPE are subleading. We have shown by explicit evaluation that the reducible TPE contribution to $\Delta(A)$ is typically of the same size in absolute value as irreducible TPE, because the latter is enhanced by numerical factors. We then count both of them in the chiral expansion for $\Delta(A)$ as $\mathcal{O}(p^2)$, as the irreducible TPE part does.

Our reproduction of phase shifts and mixing angles is already quite good for most of the partial waves. Typically is as good or better than the one achieved with an NLO calculation of the $NN$ potential, which is then employed to solve a Lippmann-Schwinger equation (either exactly or performing a distorted wave approximation) \cite{11,14,21}. Giving this promising results, $N^2$LO and $N^3$LO calculations of $\Delta(A)$ should be pursuit in the future to fully ascertain the power of the method in the study of $NN$ scattering, here applied up to NLO.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$^1S_0$ & $^1P_1$ & $^3P_0$ & $^3P_1$ & $^3S_1 - ^3D_1$ & $^3P_2 - ^3F_2$ & $^3D_1$ \\
\hline
$^3P_1$ & $^1P_1$ & $^3P_2$ & $^3P_2 - ^3F_2$ & $^3D_1$ & $^3P_3$ & $^3D_3 - ^3G_3$ \\
\hline
$^1D_2$ & $^3D_2$ & $^3P_3$ & $^3P_3$ & $^3F_3 - ^3H_4$ & $^3D_3$ & $^3G_4$ \\
\hline
$^1F_3$ & $^3F_3$ & $^3P_3$ & $^3F_3 - ^3H_4$ & $^3F_4$ & $^3G_4$ & $^3G_5 - ^3H_5$ \\
\hline
$^1G_4$ & $^3G_4$ & $^3P_3$ & $^3F_4 - ^3H_4$ & $^3F_4$ & $^3G_5$ & $^3G_5 - ^3H_5$ \\
\hline
$^1H_5$ & $^3H_5$ & $^3P_3$ & $^3F_5$ & $^3F_5$ & $^3G_5$ & $^3G_5 - ^3H_5$ \\
\hline
\end{tabular}
\caption{The minimum number of free parameters for each partial wave in our study at NLO is given in the box to the right of the wave.}
\end{table}
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