QUANTUM $K$-THEORY CHEVALLEY FORMULAS
IN THE PARABOLIC CASE

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Abstract. We derive cancellation-free Chevalley-type multiplication formulas for the $T$-equivariant quantum $K$-theory ring of Grassmannians of type $A$ and $C$, and also those of two-step flag manifolds of type $A$. They are obtained based on the uniform Chevalley formula in the $T$-equivariant quantum $K$-theory ring of arbitrary flag manifolds $G/B$, which was derived earlier in terms of the quantum alcove model, by the last three authors.

1. Introduction

Y.-P. Lee defined the (small) quantum $K$-theory of a smooth projective variety $X$, denoted by $QK(X)$ (see [Lee04], and also [Giv00]). This is a deformation of the ordinary $K$-ring of $X$, analogous to the relation between quantum cohomology and ordinary cohomology. The deformed product is defined in terms of certain generalizations of Gromov-Witten invariants (i.e., the structure constants in quantum cohomology), called quantum $K$-invariants of Gromov-Witten type.

Given a simply-connected simple algebraic group $G$ over $\mathbb{C}$, with Borel subgroup $B$, and maximal torus $T \subset B$, we consider the corresponding flag manifold $G/B$, the $T$-equivariant $K$-theory $K_T(G/B)$, and the $T$-equivariant quantum $K$-ring $QK_T(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[K]} \mathbb{Z}[\Lambda][Q]$, where $\mathbb{Z}[\Lambda][Q]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_i = Q^{\alpha_i}$, $i \in I$, with $I$ the index set for the simple roots $\alpha_i$ of $G$; $QK_T(G/B)$ has a $\mathbb{Z}[\Lambda][Q]$-basis given by the classes $[O^w]$ of the structure sheaves of the (opposite) Schubert varieties $X^w \subset G/B$ indexed by the elements $w$ of the Weyl group $W = \langle s_i := s_{\alpha_i} \mid i \in I \rangle$ of $G$. Also, given a (standard) parabolic subgroup $P_J \supset B$ corresponding to a subset $J$, we also consider the partial flag manifold $G/P_J$, the $T$-equivariant $K$-theory $K_T(G/P_J)$, and the $T$-equivariant quantum $K$-ring $QK_T(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[K]} \mathbb{Z}[\Lambda][Q_K]$, where $\mathbb{Z}[\Lambda][Q_K]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_k$, $k \in K := I \setminus J$; $QK_T(G/P_J)$ has a $\mathbb{Z}[\Lambda][Q_K]$-basis given by the (opposite) Schubert classes $[O^w_J]$, for $y \in W^J$, where $W^J$ denotes the set of minimal-length coset representatives for the cosets in $W/W_J$, where $W_J := \langle s_j \mid j \in J \rangle \subset W$. A Chevalley formula (in cohomology, $K$-theory, or their quantum versions) expresses the Schubert basis expansion of the product between an arbitrary Schubert class and the class of a line bundle, or a Schubert class indexed by a simple reflection (i.e., a divisor class). Having an explicit Chevalley formula in the quantum $K$-ring of an arbitrary flag manifold is important because this algebra is uniquely determined by products with divisor classes $[BCMP18]$, together with its $K_T(\text{pt})$-module structure; here, $K_T(\text{pt}) = R(T)$, the representation ring of $T$, is identified with the group algebra $\mathbb{Z}[\Lambda]$ of the weight lattice $\Lambda$ of $G$.

A cancellation-free Chevalley formula in the $T$-equivariant quantum $K$-theory of $G/B$ was recently given in [LNS20] (see also [LNS21]); cf. the related conjecture in [LP07]. This formula is

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expressed in terms of the so-called quantum alcove model, which was introduced in [LL15]. It generalizes the formula in the $T$-equivariant $K$-theory of $G/B$ in [LP07], which can easily be restricted to the partial flag manifold $G/P_J$ for $J \subset I$. However, such a restriction does not work in quantum $K$-theory, because of the lack of functoriality. In contrast, we know (see [Kat19]) that for a subset $J \subset I$, the $(\mathbb{Z}[\Lambda])$-linear push-forward $(\pi_J)_* : K^T_T(G/B) \to K^T_T(G/P_J)$, induced by the natural projection $\pi_J : G/B \to G/P_J$ with $P_J$ the (standard) parabolic subgroup of $G$ corresponding to $J$, yields a surjective $\mathbb{Z}[\Lambda]$-module homomorphism from $QK^\text{poly}_T(G/B) := K^T_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q] \subset QK^T_T(G/B)$ onto $QK^\text{poly}_T(G/P_J) := K^T_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_K]$ such that

$$\Phi_J([O^w_i] : [O_{G/B}(-w_k)]) = [O^{|w_i|}_J] : [O_{G/P_J}(-w_k)]$$

for $w \in W$ and $k \in K := I \setminus J$, where $|w|$ denotes the minimal-length coset representative for the coset $wW_J$ in $W/W_J$; here, $\mathbb{Z}[\Lambda][Q]$ (resp., $\mathbb{Z}[\Lambda][Q_K]$) is the polynomial ring with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_i$, $i \in I$, (resp., $Q_k$, $k \in K = I \setminus J$).

Originally, in [Kat19], the fact above was proved by using the relationship between the $T$-equivariant $K$-group of a (full or partial) semi-infinite flag manifold and the $T$-equivariant quantum $K$-theory of a (full or partial) flag manifold. Here we should mention that the existence of the surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Phi_J$ can also be verified by using the $K$-theoretic analog, conjectured in [LLMS18], of the Peterson homomorphism ($K$-Peterson homomorphism for short), which is a homomorphism of $\mathbb{Z}[\Lambda]$-algebras from the $K$-homology of the affine Grassmannian to (the localization, with respect to the positive part $Q^\vee_+$ of the coroot lattice $Q^\vee$, of) the quantum $K$-ring of $G/P_J$; a (new) proof of the existence of the $K$-Peterson homomorphism has been given recently by [CL22]. Indeed, as stated in the proof of [CL22] Lemma 2.12, under the $K$-Peterson homomorphism (which is a $\mathbb{Z}[\Lambda]$-algebra homomorphism) in the case of the Borel subgroup $B$, the classes of the structure sheaves of Schubert varieties in the affine Grassmannian indexed by the minimal-length coset representatives for $W_{af}/W$, with $W_{af}$ the affine Weyl group and $W$ the finite Weyl group, are sent injectively to the corresponding (opposite) Schubert classes in $QK^T_T(G/B)$ multiplied by explicit monomials in the Novikov variables corresponding to anti-dominant coroots in $-Q^\vee_+$. Hence, by composing the inverse of the $K$-Peterson homomorphism in the case of $B$ with the $K$-Peterson homomorphism (which is also a $\mathbb{Z}[\Lambda]$-algebra homomorphism) in the case of $P_J \supset B$, we obtain the desired surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Phi_J$; here we use the fact that all the (opposite) Schubert classes will lie in the image of the $K$-Peterson homomorphism in the case of $B$ if we multiply them by a monomial in the Novikov variables corresponding to a (fixed) regular anti-dominant coroot in $-Q^\vee_+$. The details of these arguments are explained in Appendix A.

In this paper, on the basis of the fact above, we derive cancellation-free Chevalley formulas in the $T$-equivariant quantum $K$-ring $QK^T_T(G/P_J)$ of the partial flag manifold $G/P_J$, where $P_J \supset B$ is the (standard) parabolic subgroup of $G$ corresponding to $J \subset I$ in the following two cases: (i) $G$ is of type $A$ or $C$ and $J = I \setminus \{k\}$ for $k \in I$; (ii) $G$ is of type $A$ and $J = I \setminus \{k_1, k_2\}$ for $k_1, k_2 \in I$ with $k_1 \neq k_2$. More precisely, the mentioned Chevalley formulas express the quantum multiplication in $QK^T_T(G/P_J)$ with the class of the line bundle associated to the anti-dominant fundamental weight $-\omega_k$ for $k \in I \setminus J$. Our strategy is the following: start with the Chevalley formula for $QK^\text{poly}_T(G/B) \subset QK^T_T(G/B)$ in [LNS20]; apply the $\mathbb{Z}[\Lambda]$-module surjection $\Phi_J : QK^\text{poly}_T(G/B) \to QK^\text{poly}_T(G/P_J)$ (which respects quantum multiplications) above; perform all cancellations, which arise via a sign-reversing involution. In addition, as an application of our Chevalley formulas, we prove the positivity property of certain structure constants of the quantum $K$-ring of a Grassmannian of type $C$ and that of a two-step flag manifold of type $A$, as well as that for an arbitrary full flag manifold.
The resulting Chevalley formulas for Grassmannians of types $A$ and $C$ and also those for two-step flag manifolds of type $A$ are no longer uniform, and they might also involve several cases. This fact validates our approach of deriving them from the uniform formula for $G/B$. Note that, in many cases, the opposite approach works better, namely the formulas for Grassmannians are obtained first, because they are easier.

We now compare our work with two related papers. In [KNS22], a quantum $K$-theory Chevalley formula is given in $QK_T(G/P_J)$, where $J = I \setminus \{k\}$, for the line bundle associated to $-\varpi_k$, assuming that $\varpi_k$ is a minuscule fundamental weight in type $A$, $D$, $E$, or $B$. The formulas are expressed in terms of the quantum Bruhat graph (on which the quantum alcove model is based). The approach in the present paper is simpler, and has the advantage of being easier to be extended to other partial flag manifolds; in fact, we also obtain a quantum $K$-theory Chevalley formula for two-step flag manifolds of type $A$. On another hand, the Chevalley formulas in [BCMP18] for cominuscule varieties are of a different nature than the corresponding cases of the formulas in this paper. Indeed, the role of the quantum Bruhat graph is not transparent in [BCMP18].

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2. Background

Consider a simply-connected simple algebraic group $G$ over $\mathbb{C}$, with Borel subgroup $B$, and maximal torus $T$. Let $\mathfrak{g}$ be the corresponding finite-dimensional simple Lie algebra over $\mathbb{C}$, and $W$ its Weyl group, with length function denoted by $\ell(\cdot)$. Let $\Phi$, $\Phi^+$, and $\Phi^-$ be the set of roots, positive roots, and negative roots of $\mathfrak{g}$, respectively, and let $\Delta$ be the corresponding weight lattice. Let $\alpha_i$, $i \in I$, be the simple roots, $\Delta := \{\alpha_i \mid i \in I\}$ the set of all simple roots, $\theta$ the highest root, and $\alpha_i^\vee$ the coroot associated with the root $\alpha_i$. The reflection corresponding to $\alpha_i$ is denoted, as usual, by $s_{\alpha_i}$, and we let $s_i := s_{\alpha_i}$, $i \in I$, be the simple reflections. Set $\rho := (1/2) \sum_{\alpha \in \Phi^+} \alpha$.

Let $J$ be a subset of $I$. We denote by $W_J := \langle s_i \mid i \in J \rangle$ the parabolic subgroup of $W$ corresponding to $J$, and we identify $W/W_J$ with the corresponding set of minimal coset representatives, denoted by $W^J$; note that if $J = \emptyset$, then $W^J = W$ is identical to $W$. For $w \in W$, we denote by $[w] = [w]^J \in W^J$ the minimal coset representative for the coset $wW_J$ in $W/W_J$.

2.1. The quantum Bruhat graph. We start with the definition of this graph, which plays a fundamental role in our combinatorial model.

**Definition 1.** The quantum Bruhat graph $QB(W)$ is the $\Phi^+$-labeled directed graph whose vertices are the elements of $W$, and whose directed edges are of the form: $w \xrightarrow{\beta} v$ for $w, v \in W$ and $\beta \in \Phi^+$ such that $v = ws_{\beta}$, and such that either of the following holds: (i) $\ell(v) = \ell(w) + 1$; (ii) $\ell(v) = \ell(w) + 1 - 2\langle \rho, \beta^\vee \rangle$. An edge satisfying (i) (resp., (ii)) is called a Bruhat (resp., quantum) edge.

In [BFP99], it is proved that the quantum Bruhat graph $QB(W)$ has the following property (called the shellability): for all $x, y \in W$, there exists a unique directed path from in $QB(W)$ from $x$ to $y$ whose edge labels are increasing with respect to an arbitrarily fixed reflection order on $\Phi^+$. We recall an explicit description of the edges of the quantum Bruhat graphs of types $A$ and $C$. These results generalize the well-known criteria for covers of the Bruhat order in these cases [BB05].

In type $A_{n-1}$, the Weyl group elements (i.e., permutations) $w \in W(A_{n-1}) = S_n$ are written in one-line notation $w = [w(1), \ldots, w(n)]$. For simplicity, we use the same notation $(i, j)$ with
1 \leq i < j \leq n \text{ for the root } \alpha_{ij} \text{ and the reflection } s_{\alpha_{ij}}, \text{ which is the transposition } t_{ij} \text{ of } i \text{ and } j. \text{ We have } \theta = (1, n). \text{ We recall a criterion for the edges of the type } A_{n-1} \text{ quantum Bruhat graph. We need the circular order } \prec_i \text{ on } [n] \text{ starting at } i, \text{ namely } i \prec_i i + 1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i - 1. \text{ It is convenient to think of this order in terms of the numbers } 1, \ldots, n \text{ arranged clockwise on a circle, in this order. We make the convention that, whenever we write } a < b < c < \cdots; \text{ i.e., the leftmost of the chain } a < b < c < \cdots \text{ we are writing is } a, \text{ we refer to the circular order } \preceq a.

**Proposition 2** ([Len12]). For \( w \in S_n \) and \( 1 \leq i < j \leq n \), we have an edge \( w \xrightarrow{(i,j)} w(i,j) \) if and only if there is no \( k \) such that \( i < k < j \) and \( w(i) < w(k) < w(j) \).

If there is a position \( k \) as above, we say that the transposition of \( w(i) \) and \( w(j) \) straddles \( w(k) \). We also let \( w[i,j] := [w(i), w(i+1), \ldots, w(j)] \). We continue to use this terminology and notation for the other classical types.

The Weyl group of type \( C_n \) is the group of signed permutations. These are bijections \( w \) from \([\overline{n}] := \{1 < 2 \cdots n \overline{n} < n \overline{n} - 1 \overline{n} \cdots \overline{n}\} \) to \([\overline{n}] \) satisfying \( w(\overline{i}) = \overline{w(i)} \). Here \( \overline{\tau} \) is viewed as \( \overline{i} \), so \( \overline{\overline{i}} = i \), and \( \text{sign}(\overline{\tau}) = -1 \). We use both the window notation \( w = [w(1), \ldots, w(n)] \) and the full one-line notation \( w = [w(1), \ldots, w(n), w(\overline{1}), \ldots, w(\overline{n})] \) for signed permutations. For simplicity, given \( 1 \leq i < j \leq n \), we denote by \( (i, j) \) the root \( \varepsilon_i - \varepsilon_j \) and the corresponding reflection, which is identified with the composition of transpositions \( t_{ij} t_{\overline{j}} \). Similarly, for \( 1 \leq i < j \leq n \), we denote by \( (i, \overline{j}) = (j, \overline{\tau}) \) the root \( \varepsilon_i + \varepsilon_j \) and the corresponding reflection, which is identified with the composition of transpositions \( t_{ij} t_{\overline{j}} \). Finally, we denote by \( (i, \overline{\tau}) \) the root \( 2\varepsilon_i \) and the corresponding reflection, which is identified with the transposition \( t_{\overline{\tau}} \). We have \( \theta = (1, \overline{\tau}) \).

We now recall the criterion for the edges of the type \( C_n \) quantum Bruhat graph. We need the circular order \( \prec_i \) on \([\overline{n}] \), which is defined similarly to the circular order on \([n] \), by thinking of the numbers \( 1, 2, \ldots, n, \overline{n}, \overline{n} - 1, \ldots, \overline{1} \) arranged clockwise on a circle, in this order. We make the same convention as above that, whenever we write \( a < b < c < \cdots \), we refer to the circular order \( \preceq = a \).

**Proposition 3** ([Len12]). Let \( w \in W(C_n) \) be a signed permutation.

1. Given \( 1 \leq i < j \leq n \), we have an edge \( w \xrightarrow{(i,j)} w(i,j) \) if and only if there is no \( k \) such that \( i < k < j \) and \( w(i) < w(k) < w(j) \).
2. Given \( 1 \leq i < j \leq n \), we have an edge \( w \xrightarrow{(i,\overline{j})} w(i,\overline{j}) \) if and only if \( w(i) < w(\overline{j}) \), \( \text{sign}(w(\overline{j})) = \text{sign}(w(i)) \), and there is no \( k \) such that \( i < k < \overline{j} \) and \( w(i) < w(k) < w(\overline{j}) \).
3. Given \( 1 \leq i \leq n \), we have an edge \( w \xrightarrow{(i,\overline{\tau})} w(i,\overline{\tau}) \) if and only if there is no \( k \) such that \( i < k < \overline{\tau} \) (or, equivalently, \( i < k \leq n \)) and \( w(i) < w(k) < w(\overline{\tau}) \).

### 2.2. The quantum alcove model.

We need basic notions related to the combinatorial model known as the alcove model, which was defined in [LP07]. In particular, we need the notion of a \( \lambda \)-chain of roots, where \( \lambda \) is a weight. In this section, we recall definitions of these notions from [LP07].

Let \( \Lambda \) be the weight lattice of \( G \) and set \( h \overline{R}^\ast := \Lambda \otimes \mathbb{Z} \mathbb{R} \). For \( \alpha \in \Phi \) and \( k \in \mathbb{Z} \), we define a hyperplane \( H_{\alpha,k} \) by \( H_{\alpha,k} := \{ \xi \in h \overline{R}^\ast \mid \langle \xi, \alpha^\vee \rangle = k \} \). We denote by \( s_{\beta,k}, \beta \in \Phi \) and \( k \in \mathbb{Z} \), the reflection with respect to \( H_{\alpha,k} \). Then, an alcove is defined to be a connected component of the space

\[
\bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} H_{\alpha,k}.
\]

If two alcoves \( A \) and \( B \) have a common wall, then \( A \) and \( B \) are said to be adjacent. Let us take adjacent alcoves \( A \) and \( B \). If the common wall of \( A \) and \( B \) is contained in a hyperplane \( H_{\alpha,k} \) for
some $\alpha \in \Phi$ and $k \in \mathbb{Z}$, and the vector $\alpha$ points a direction from $A$ to $B$, then we write $A \xrightarrow{\alpha} B$. We define a specific alcove $A_0$, called the fundamental alcove, by

$$A_0 := \{ \xi \in \mathfrak{h}_R^+ \mid \langle \xi, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \}.$$ 

In addition, for $\lambda \in \Lambda$, we define an alcove $A_\lambda$ by $A_\lambda := A_0 + \lambda = \{ \xi + \lambda \mid \xi \in A_0 \}$.

**Definition 4** ([LP07] Definitions 5.2, 5.4). (1) An alcove path is a sequence $(A_0, A_1, \ldots, A_m)$ of alcoves such that for each $0 \leq k \leq m - 1$, $A_k$ and $A_{k+1}$ are adjacent. If an alcove path $\Pi = (A_0, \ldots, A_m)$ is shortest among all alcove paths from $A_0$ to $A_m$, we say that $\Pi$ is reduced.

(2) Let $\lambda \in \Lambda$. A $\lambda$-chain of roots is a sequence $\Gamma = (\beta_1, \ldots, \beta_m)$ of roots such that there exists an alcove path $\Pi = (A_0 = A_0, \ldots, A_m = A_\lambda)$ such that

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m.$$ 

If $\Pi$ is reduced, then we also say that $\Gamma$ is reduced.

Let $\lambda \in \Lambda$. Take a $\lambda$-chain $\Gamma = (\beta_1, \ldots, \beta_m)$ and corresponding alcove path $(A_0, \ldots, A_m)$. Set $r_i := s_{\beta_i}$, $i = 1, \ldots, m$. Below, we present an explicit description of the chains of roots corresponding to the anti-dominant fundamental weights in the classical types, i.e., $\lambda = -\varpi_k$.

We also need to recall the more general quantum alcove model [LL15]. We refer to [LNS20, Section 3.2] for more details. In the next definition, we use the following notation: for $\beta \in \Phi$,

$$|\beta| := \begin{cases} \beta & \text{if } \beta \in \Phi^+, \\ -\beta & \text{if } \beta \in \Phi^- . \end{cases}$$

**Definition 5** ([LL15]). A subset $A = \{j_1 < j_2 < \cdots < j_s\}$ of $[m] := \{1, \ldots, m\}$ (possibly empty) is a $w$-admissible subset if we have the following directed path in the quantum Bruhat graph $\text{QB}(W)$:

$$\Pi(w, A) : \quad w \xrightarrow{|\beta_1|} wr_{j_1} \xrightarrow{|\beta_2|} wr_{j_1}r_{j_2} \xrightarrow{|\beta_3|} \cdots \xrightarrow{|\beta_s|} wr_{j_1}r_{j_2} \cdots r_{j_s} \xrightarrow{\text{end}(w, A)} .$$

We denote by $A^-$ the subset of $A$ corresponding to quantum steps in $\Pi(w, A)$. Let $\mathcal{A}(w, \Gamma)$ be the collection of all $w$-admissible subsets corresponding to the $\lambda$-chain $\Gamma$, and $\mathcal{A}_\lambda(w, \Gamma)$ its subset consisting of all those $A$ with $A^- = \emptyset$ (i.e., $\Pi(w, A)$ is a saturated chain in Bruhat order). For convenience, we identify an admissible subset $J = \{j_1 < \cdots < j_s\}$ with the corresponding sequence of roots $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$ in the $\lambda$-chain $\Gamma$ (in case of multiple occurrences of a root in $\Gamma$, we specify which one is considered). Also, we define statistics $\text{down}(w, A)$ for $A \in \mathcal{A}(w, \Gamma)$ as follows:

$$\text{down}(w, A) := \sum_{j \in A^-} |\beta_j|.$$ 

In addition, let $H_{\beta_j, -t_j}$, $j = 1 \ldots m$, be the hyperplane containing the common wall of $A_{j-1}$ and $A_j$. Then we define $\text{wt}(w, A)$ by

$$\text{wt}(w, A) := -ws_{\beta_{j_1}, -t_{j_1}} \cdots s_{j_s, -t_{j_s}}(-\lambda).$$

We use the same notation as in Section 2.1, and we start with type $A_{n-1}$. It is proved in [LP07, Corollary 15.4] that, for any $k = 1, \ldots, n - 1$, we have the following reduced $(-\varpi_k)$-chain of roots, denoted by $\Gamma(k)$ (note that all the roots in this $(-\varpi_k)$-chain are negated for simplicity of notation, and hence they are all positive roots):

$$\begin{align*}
(1, n), & \quad (1, n-1), \quad \ldots, \quad (1, k+1), \\
(2, n), & \quad (2, n-1), \quad \ldots, \quad (2, k+1), \\
& \quad \ldots \\
(k, n), & \quad (k, n-1), \quad \ldots, \quad (k, k+1).
\end{align*}$$

(1)
In type $A_{n-1}$, we have the (Dynkin) diagram automorphism

$$\omega : [n-1] \to [n-1], \quad l \mapsto n-l.$$  

By applying the diagram automorphism $\omega$ to $\Gamma(n-k)$, we obtain another reduced $(-\pi_k)$-chain (with all the roots negated), denoted by $\Gamma^*(k)$:

$$(1, n), (2, n), \ldots, (k, n),
(1, n-1), (2, n-1), \ldots, (k, n-1), \ldots
(1, k+1), (2, k+1), \ldots, (k, k+1).$$

In type $C_n$, let

$$\Gamma(k) := \Gamma'_2 \cdots \Gamma'_k \Gamma_1(k) \cdots \Gamma(k),$$

where

$$\Gamma'_j := ((1, j), (2, j), \ldots, (j-1, j)), \quad \Gamma_j(k) := ((1, j), (2, j), \ldots, (j-1, j),
(j, k+1), (j, k+2), \ldots, (j, n)), \quad \Gamma_j(n, k) := ((j, n), (j, n-1), \ldots, (j, k+1)).$$

It is proved in [Len10] Lemma 4.1 that $\Gamma(k)$ is a reduced $(-\varpi_k)$-chain (with all the roots negated), for $1 \leq k \leq n$.

### 2.3. The quantum $K$-theory of flag manifolds.

In order to describe the (small) $T$-equivariant quantum $K$-ring $\mathbb{Q}K_T(G/B)$, for the finite-dimensional flag manifold $G/B$, we associate a variable $Q_i$ to each simple coroot $\alpha_i^\vee$, and set $\mathbb{Z}[Q] := \mathbb{Z}[Q_i \mid i \in I], \mathbb{Z}[\mathbb{Q}] := \mathbb{Z}[Q_i \mid i \in I]$; for each $\xi = \sum_{i \in I} d_i\alpha_i^\vee$ in $Q^{\vee,+}$, we set $Q^\xi := \prod_{i \in I} Q_i^{d_i}$. Also, we set $\mathbb{Z}[\Lambda][Q] := \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Z}[\Lambda][\mathbb{Q}] := \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}} \mathbb{Z}[Q]$, where $\mathbb{Z}[\Lambda]$ is the group algebra of the weight lattice $\Lambda$ of $G$, and is identified with the representation ring $R(T) = K_T(pt)$. Following [Lee04] (and also [Giv00]), we define the quantum $K$-ring $\mathbb{Q}K_T(G/B)$ to be the $\mathbb{Z}[\Lambda][Q]$-module $K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\mathbb{Q}]$, equipped with the quantum product $\star$ given in terms of quantum $K$-invariants of Gromov-Witten type. The quantum $K$-ring $\mathbb{Q}K_T(G/B)$ has a $\mathbb{Z}[\Lambda][\mathbb{Q}]$-basis given by the classes $[O^w]$ of the structure sheaves of the (opposite) Schubert varieties $X^w \subseteq G/B$ of codimension $\ell(w)$, for $w \in W$.

We consider the maximal (standard) parabolic subgroup of $P_I \supset B$ of $G$ corresponding to the subset $J := I \setminus \{k\}$, for some $k \in I$. The $T$-equivariant quantum $K$-ring $\mathbb{Q}K_T(G/P_J)$ of the partial flag manifold $G/P_J$ is defined as $K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_k]$, where $K_T(G/P_J)$ is the $T$-equivariant $K$-theory of $G/P_J$, and $\mathbb{Z}[\Lambda][Q_k]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the single (Novikov) variable $Q_k = Q^{\alpha_k^\vee}$ corresponding to the simple coroot $\alpha_k^\vee$. The (opposite) Schubert classes $[O^y_J]$, for $y \in W^J$, form a $\mathbb{Z}[\Lambda][Q_k]$-basis.

We also consider the (standard) parabolic subgroup $P_J \supset B$ of $G$ corresponding to the subset $J := I \setminus \{k_1, k_2\}$, for some $k_1, k_2 \in I$ with $k_1 \neq k_2$. In this case, the $T$-equivariant quantum $K$-ring $\mathbb{Q}K_T(G/P_J)$ is defined as $K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_{k_1}, Q_{k_2}]$, where $\mathbb{Z}[\Lambda][Q_{k_1}, Q_{k_2}]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the two (Novikov) variables $Q_{k_1}, Q_{k_2}$. As in the maximal parabolic case, the (opposite) Schubert classes $[O^y_J]$, for $y \in W^J$, form a $\mathbb{Z}[\Lambda][Q_{k_1}, Q_{k_2}]$-basis.

For an arbitrary subset $J \subseteq I$, let $\pi_J : G/B \to G/P_J$ be the natural projection, and let $(\pi_J)_* : K_T(G/B) \to K_T(G/P_J)$ denote the induced push-forward, which is $\mathbb{Z}[\Lambda]$-linear. Also, it is well-known that $\pi_J([O^w]) = [O^w_J]$ for each $w \in W$, where $[w]$ denotes the minimal-length coset representative for the coset $wP_J$ in $W/W_J$, and that $\pi_J([O_{G/B}(-\varpi_k)]) = [O_{G/P_J}(-\varpi)]$ for $k \in K = I \setminus J$ (see, for example, [MNS22] Section 9.2]). Now, we set $\mathbb{Q}K_T^{\text{poly}}(G/B) :=$
$K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q] \subset QK_T(G/B)$, and $QK_T^{\text{poly}}(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_K]$, where $\mathbb{Z}[\Lambda][Q_K]$ is the ring of polynomials with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_k = Q^{\alpha_k}_k$, $k \in K := I \setminus J$. Based on the finiteness result on the quantum multiplication in $QK_T(G/P_J)$ with the line bundle classes $[\mathcal{O}_{G/P_J}(-\varpi_k)]$ for $k \in K = I \setminus J$ (see also [ACT22]), Kato proved (see [Kat19]) that the $(\mathbb{Z}[\Lambda]-linear)$ push-forward $(\pi_I)_*: K_T(G/B) \to K_T(G/P_J)$ induces a surjective $\mathbb{Z}[\Lambda]-module$ homomorphism $\Phi_J : QK_T^{\text{poly}}(G/B) \to QK_T^{\text{poly}}(G/P_J)$ such that for $w \in W$ and $k \in K = I \setminus J$, the following equality holds:

$$\Phi_J([\mathcal{O}_w^G] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_{J}^G] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)],$$

by defining $\Phi_J(Q^\xi) := Q^{[\xi]}_J$ for each $\xi \in Q^{\vee,+}$, where $[\xi]^J := \sum_{k \in I \setminus J} c_k^J \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$. Namely, Kato proved the following.

**Theorem 6 ([Kat19]).** Let $J$ be an arbitrary subset of $I$. Then, the surjective $\mathbb{Z}[\Lambda]-module$ homomorphism

$$\Phi_J : QK_T^{\text{poly}}(G/B) \to QK_T^{\text{poly}}(G/P_J)$$

defined by $\Phi_J(Q^\xi[\mathcal{O}_w^G]) = Q^{[\xi]}_J[\mathcal{O}_{J}^G]$ for $w \in W$ and $\xi \in Q^{\vee,+}$, where $[\xi]^J = \sum_{k \in I \setminus J} c_k^J \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$, has the following multiplicativity:

$$\Phi_J([\mathcal{O}_w^G] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_{J}^G] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)]$$

for $w \in W$ and $k \in K = I \setminus J$.

In Appendix [A] we give another proof of the existence of the multiplicative $\mathbb{Z}[\Lambda]-module$ surjection $\Phi_J$ above by using the $K$-Peterson homomorphism, which is a homomorphism of $\mathbb{Z}[\Lambda]$-algebras from the $K$-homology of the affine Grassmannian associated to $G$ to (the localization, with respect to $Q^{\vee,+}$, of) the quantum $K$-ring $QK_T(G/P_J)$; a (new) proof of the existence of the $K$-Peterson homomorphism has been given by [CL22].

We now recall the (cancellation-free) quantum $K$-theory Chevalley formula in [LNS20, Theorem 47] (see also [LNS21, Theorem 12]) for $G/B$, which is based on the quantum alcove model; in fact, we use the slight modification corresponding to the multiplication by the class $[\mathcal{O}(\varpi_k)] := [\mathcal{O}_{G/B}(\varpi_k)]$ of the line bundle associated to $-\varpi_k$. Throughout this paper, we denote by $|S|$ for a set $S$ the cardinality of $S$. This formula is expressed in terms of a $(-\varpi_k)$-chain of roots, cf. Section 2.2.

**Theorem 7.** Let $k \in I$, and fix a reduced $(-\varpi_k)$-chain $\Gamma(k)$. Then, in $QK_T^{\text{poly}}(G/B) \subset QK_T(G/B)$, we have for $w \in W$,

$$[\mathcal{O}(\varpi_k)] \cdot [\mathcal{O}_w^G] = \sum_{A \in A(w, \Gamma(k))} (-1)^{|A|} Q_\text{down}(w, A) e^{-\text{wt}(w, A)} [\mathcal{O}_{\text{end}(w, A)}].$$

(4)

**Remark 8.** The right-hand side of equation (4) is cancellation-free. Indeed, suppose, for a contradiction, that there exist two admissible subsets $A, A' \in A(w, \Gamma(k))$ satisfying $\text{end}(w, A) = \text{end}(w, A')$ and $(-1)^{|A|} = (-1)^{|A'|}$ (together with $\text{down}(w, A) = \text{down}(w, A')$ and $\text{wt}(w, A) = \text{wt}(w, A')$). Here we know (see [BF90], and also [Pos15]) that for directed paths $p_1, p_2$ in $\text{QB}(W)$ starting from the same element $v \in W$ and ending at the same element $u \in W$, the equality $(-1)^{l(p_1)} = (-1)^{l(p_2)}$ holds, where $l(\cdot)$ denotes the length of a directed path. This contradicts the equality $(-1)^{|A|} = (-1)^{|A'|}$, as desired.
Let \( N_{w,u}^{u,\xi} \in \mathbb{Z}[P] \), with \( v, w, u \in W^J \), \( \xi \in Q_{I,J}^{\vee,+} := \sum_{i \in I \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee \), denote the structure constants of \( QK_T(G/P_J) \) defined by:
\[
[O^u] \cdot [O^w] = \sum_{u \in W^J, \xi \in Q_{I,J}^{\vee,+}} N_{w,u}^{u,\xi} Q^\xi [O^u].
\]

Let \( \rho_J \) be a half of the sum of all positive roots of \( P_J \), and set \( \deg(Q^\xi) := 2(\rho - \rho_J, \xi) \) for \( \xi \in Q_{I,J}^{\vee,+} \). It is expected that the structure constants of \( QK_T(G/P_J) \) have the following positivity property.

**Conjecture 9 (BCMP18 Conjecture 2.2).** For \( v, w, u \in W^J \) and \( \xi \in Q_{I,J}^{\vee,+} \), we have
\[
(-1)^{\ell(v)+\ell(w)+\ell(u)+\deg(Q^\xi)} N_{v,w}^{u,\xi} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 | \gamma \in -\Delta].
\]

The positivity property of the structure constants \( N_{s_k,w}^{u,\xi} \), with \( k \in K = I \setminus J \), is proved for cominuscule varieties \( G/P_J \), which include Grassmannians of type \( A \), by Buch-Chaput-Mihalcea-Perrin in [BCMP18] by writing explicitly the structure constants. Also, the positivity property of the structure constants \( N_{w,u}^{u,\xi} \), with \( \xi = 0 \), is proved by Anderson-Griffeth-Miller [AGM11] since these are the structure constants of the ordinary \( T \)-equivariant \( K \)-theory \( K_T(G/P_J) \). In this paper, we prove the positivity property of the structure constants \( N_{s_k,w}^{u,\xi} \), with \( k \in K = I \setminus J \), for full flag manifolds of arbitrary types, two-step flag manifolds of type \( A \), and Grassmannians of type \( C \).

Let us define \( C_w^{u,\xi} \in \mathbb{Z}[P] \), with \( w, u \in W, \xi \in Q_{I,J}^{\vee,+} \), by:
\[
[O(-w_k)] \cdot [O^w] = \sum_{u \in W^J, \xi \in Q_{I,J}^{\vee,+}} C_w^{u,\xi} Q^\xi [O^w].
\]

Since it is well-known that \([O^{sk}] = 1 - e^{-w_k}[O(-w_k)] \) for \( k \in I \setminus J \), we see that
\[
[O^{sk}] \cdot [O^w] = (1 - e^{-w_k}[O(-w_k)]) \cdot [O^w] = [O^w] - e^{-w_k}[O(-w_k)] \cdot [O^w] = (1 - e^{-w_k} C_w^{u,0}) [O^w] + \sum_{\xi \in Q_{I,J}^{\vee,+} \setminus \{0\}} (-e^{-w_k} C_w^{u,\xi}) Q^\xi [O^w] + \sum_{u \in W \setminus \{w\}, \xi \in Q_{I,J}^{\vee,+}} (-e^{-w_k} C_w^{u,\xi}) Q^\xi [O^u].
\]

Hence it follows that for \( w, u \in W^J \) and \( \xi \in Q_{I,J}^{\vee,+} \),
\[
N_{s_k,w}^{u,\xi} = \begin{cases} 1 - e^{-w_k} C_w^{u,0} & \text{if } u = w \text{ and } \xi = 0, \\ -e^{-w_k} C_w^{u,\xi} & \text{otherwise.} \end{cases}
\]

For the proof of the positivity property, we need the following lemma.

**Lemma 10.** Let \( w \in W \). Let \( \lambda \in \Lambda \) be a dominant weight, and take a reduced \((-\lambda)\)-chain \( \Gamma \). For \( A \in \mathcal{A}(w, \Gamma) \), we have \( \text{wt}(w, A) \in -\lambda + Q^+ \).

**Proof.** Let \( A \in \mathcal{A}(w, \Gamma) \). We denote by \( \Lambda_0^A \) the set of all level-zero weights of the (untwisted) affine Lie algebra \( g_{af} = (g \otimes \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d \) associated to \( g \); in the following, we regard \( \lambda \) as an element of \( \Lambda_0^A \).

We use quantum Lakshmibai-Seshadri (QLS) paths of shape \( \lambda \), which are defined in [LNSSS17, Definition 3.1]. We first assume that \( \Gamma \) is the lex \((-\lambda)\)-chain, defined in [LNS21, Section 4.2]. In this case, we know from [LNS20, Proposition 3.1] that there exists a QLS path \( \eta \) of shape \( \lambda \) such
that \(\text{wt}(w, A) = -\text{wt}(\eta)\), where \(\text{wt}(\eta) := \eta(1)\). Let us write \(\eta\) in the form \(\eta = (\nu_1, \ldots, \nu_s; 0 = a_0 < a_1 < \cdots < a_s = 1)\), with \(\nu_1, \ldots, \nu_s \in W\lambda\) and \(a_0, \ldots, a_s \in \mathbb{Q}\). Then we see that \(\nu_k \in \lambda - Q^+, k = 1, \ldots, s\), since \(\lambda \in \Lambda\) is dominant and \(W\) is the finite Weyl group. Hence we have

\[
\text{wt}(\eta) = \eta(1) = \sum_{k=1}^{s} (a_k - a_{k-1})\nu_k \in \lambda - \sum_{j \in I} a_j.
\]

Also, we have

\[
\text{wt}(\eta) = \eta(1) = \nu_s + \sum_{k=1}^{s-1} a_k(\nu_k - \nu_{k+1}).
\]

Since \((\nu_k, \nu_{k+1})\) is an \(a_k\)-chain (see [Lit95 Section 4]), it follows that \(a_k(\nu_k - \nu_{k+1}) \in Q\) for \(k = 1, \ldots, s - 1\). In addition, we have that \(\nu_s \in \lambda - Q^+\). Hence we see that \(\text{wt}(\eta) \in \lambda + Q\). Therefore, we deduce that \(\text{wt}(\eta) \in \lambda - Q^+\), as desired.

We next assume that \(\Gamma\) is an arbitrary reduced \((-\lambda)\)-chain. Then we know that \(\Gamma\) can be deformed to the lex \((-\lambda)\)-chain \(\Gamma'\) by repeated application of Yang-Baxter transformations in [KLN21 Section 3.1] (see also [LNS21 Remark 40]). In this situation, [KLN21 Theorems 3.2 and 3.4] implies that there exists a bijection \(Y : \mathcal{A}(w, \Gamma) \rightarrow \mathcal{A}(w, \Gamma')\), given by quantum Yang-Baxter moves, such that \(\text{wt}(w, Y(A)) = \text{wt}(w, A)\) for all \(A \in \mathcal{A}(w, \Gamma)\). Here we note that [KLN21 Theorem 3.2] states that \(Y\) is a surjection ([FK20 Section 2]), i.e., a “signed bijection”, where \(\mathcal{A}(w, \Gamma)\) and \(\mathcal{A}(w, \Gamma')\) are regarded as signed sets equipped with sign functions. However, since \(-\lambda\) is anti-dominant, we have no sign-reversing involution on any non-empty subset of \(\mathcal{A}(w, \Gamma)\) or \(\mathcal{A}(w, \Gamma')\). Therefore, \(Y\) is, in fact, a bijection. Since \(\Gamma'\) is the lex \((-\lambda)\)-chain and \(Y(A) \in \mathcal{A}(w, \Gamma')\), we deduce that \(\text{wt}(w, A) = \text{wt}(w, Y(A)) \in -\lambda + Q^+\). This proves the lemma. 

Note that if there exists an edge \(x \rightarrow y\) in \(QB(W)\) for \(x, y \in W\), then we have \(\ell(y) \equiv \ell(x) + 1\) mod 2 by the definition of \(QB(W)\). This implies that for \(w \in W\) and \(A \in \mathcal{A}(w, \Gamma(k))\), we have

\[
(-1)^{|A|} = (-1)^{\ell(\text{end}(w, A)) - \ell(w)}.
\]

In this section, we prove the positivity property of structure constants for full flag manifolds as a corollary of the Chevalley formula (Theorem 7). We will give a proof of the positivity property for Grassmannians of type \(C\) (resp., two-step flag manifolds of type \(A\)) in Section 3.2 (resp., Section 12).

**Corollary 11.** Let \(G\) be of an arbitrary type, \(J = \emptyset\) (hence \(P_J = B\)), and \(k \in I\). Then, for \(w, u \in W^J\) and \(\xi \in Q^{\vee, +}\), we have

\[
(-1)^{1 + \ell(u) + \ell(\xi) + \text{deg}(Q^\xi)} N_{s_k, w}^{\xi, u} \in \mathbb{Z}_{>0}[e_\gamma - 1 | \gamma \in -\Delta].
\]

**Proof.** Let \(w \in W\). Take \(A \in \mathcal{A}(w, \Gamma(k))\) such that \(A^- = \emptyset\). If \(A = \emptyset\), then we have

\[
(-1)^{|A|} Q^\downbow(w, A) e^{-\text{wt}(w, A)[\mathcal{O}_{\text{end}(w, A)}]} = e^w\varpi_k[\mathcal{O}^w].
\]

Since there exists no \(A \in \mathcal{A}(w, \Gamma(k))\) such that \(\text{end}(w, A) = w\) and \(\text{down}(w, A) = 0\) except for \(A = \emptyset\), we have \(C_w^{u, 0} = e^{w\varpi_k}\). In addition, we have \(\text{deg}(Q^0) = 0\). Hence it follows that

\[
N_{s_k, w}^{u, 0} = 1 - e^{w\varpi_k - \varpi_k} = (-1)^{1 + \ell(u) + \ell(w) + \text{deg}(Q^0)} (e^{w\varpi_k - \varpi_k} - 1);
\]
note that \(w \varpi_k - \varpi_k \in -Q^+\). Since
\[
e^{-\mu} = \prod_{i \in I} (e^{-\alpha_i})^{c_i} = \prod_{i \in I} ((e^{-\alpha_i} - 1) + 1)^{c_i} = \prod_{i \in I} \left( \sum_{k=0}^{c_i} \binom{c_i}{k} (e^{-\alpha_i} - 1)^k \right) \in \mathbb{Z}_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta]
\]
for \(\mu = \sum_{i \in I} c_i \alpha_i \in Q^+\), we deduce that
\[
(-1)^{1+\ell(w)+\ell(u)+\deg(Q^0)} N_{s_k,w}^{u,\xi,\lambda} \in \mathbb{Z}_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta],
\]
as desired.

Next, take \(A \in \mathcal{A}(w, \Gamma(k)) \setminus \{\emptyset\}\). Then we have
\[
(-1)^{|A|} Q^{\text{down}(w,A)} e^{-\text{wt}(w,A)\mathcal{O}_{\text{end}(w,A)}} = (-1)^{\ell(\text{end}(w,A)) - \ell(w)} Q^{\text{down}(w,A)} e^{-\text{wt}(w,A)\mathcal{O}_{\text{end}(w,A)}}.
\]
Also, by Lemma 10 we have \(\text{wt}(w, A) \in -\varpi_k + Q^+\) for \(A \in \mathcal{A}(w, \Gamma(k))\). Here we set
\[
\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda} := \{A \in \mathcal{A}(w, \Gamma(k)) \mid \text{end}(w, A) = u, \text{down}(w, A) = \xi, \text{wt}(w, A) = \lambda\},
\]
for \(u \in W, \xi \in Q^{\vee,+}\), and \(\lambda \in -\varpi_k + Q^+\). Then by Theorem 7 we have
\[
C_{w,\xi}^{u,\lambda} = \sum_{\lambda \in -\varpi_k + Q^+} \sum_{A \in \mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}} (-1)^{|A|} e^{-\text{wt}(w,A)} = (-1)^{\ell(u) - \ell(w)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| e^{-\lambda}.
\]
Since \(\deg(Q_j) = 2(\rho, \alpha_j^\vee) = 2\) for all \(j \in I\), we have \(\deg(Q^\xi) \in 2\mathbb{Z}\). Therefore, we see that
\[
N_{s_k,w}^{u,\xi,\lambda} = -e^{-\varpi_k} \cdot (-1)^{\ell(u) - \ell(w)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| e^{-\lambda}.
\]
= \((-1)^{1+\ell(u) + \ell(w) + \deg(Q^\xi)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| e^{-\varpi_k - \lambda}.
\]
This implies that
\[
(-1)^{1+\ell(u) + \ell(w) + \deg(Q^\xi)} N_{s_k,w}^{u,\xi,\lambda} \in \mathbb{Z}_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta],
\]
as desired. This proves the corollary. \(\square\)

3. Quantum K-theory Chevalley formulas in the maximal parabolic case

Given a maximal parabolic subgroup \(P_J\) for \(J = I \setminus \{k\}\), we will derive cancellation-free parabolic Chevalley formulas for the quantum multiplication in \(QK_T(G/P_J)\) with \(\mathcal{O}(-\varpi_k) := [\mathcal{O}_{G/P_J}(-\varpi_k)]\). Based on Theorem 6 explained in Section 2.3, we obtain certain formulas from equation 11 in Theorem 7 for \(QK_T^{\text{poly}}(G/B) \subset QK_T(G/B)\) by applying \(\Phi_J\); this argument works for an arbitrary fundamental weight \(\varpi_k\) of \(G\) of any type. However, upon applying \(\Phi_J\), there are many terms to be canceled in the corresponding formula in \(QK_T^{\text{poly}}(G/P_J) \subset QK_T(G/P_J)\). For any fundamental weight \(\varpi_k\) in types \(A\) and \(C\), we cancel out all these terms via a sign-reversing involution, and obtain a cancellation-free formula. We rely on the structure of the corresponding \((-\varpi_k)\)-chain of roots \(\Gamma(k)\) in Section 2.2 as well as the quantum Bruhat graph criteria in Section 2.1.
Remark 12. Upon applying the above procedure, there are no cancellations among the terms corresponding to \( w \)-admissible subsets \( A \) with \( A^- = \emptyset \), by Remark \([\text{5}]\).

Remark 13. If \( G \) is of type \( A_{n-1} \), then the partial flag manifold \( G/P_J \) for \( J = I \setminus \{k\} \) is isomorphic to the Grassmannian \( \text{Gr}(k, n) \) defined as:

\[
\text{Gr}(k, n) := \{ V \mid V \text{ is a subspace of } \mathbb{C}^n \text{ such that } \dim V = k \}.
\]

Also, if \( G \) is of type \( C_n \), then the partial flag manifold \( G/P_J \) for \( J = I \setminus \{k\} \) is isomorphic to the isotropic Grassmannian \( \text{IG}(k, 2n) \) defined as:

\[
\text{IG}(k, 2n) := \left\{ V \mid V \text{ is a subspace of } \mathbb{C}^{2n} \text{ such that } \dim V = k, \text{ and } V \text{ is isotropic with respect to } (-,-) \right\};
\]

where \((-,-)\) denotes a non-degenerate skew symmetric bilinear form on \( \mathbb{C}^{2n} \).

3.1. Type \( A_{n-1} \). We start with type \( A_{n-1} \), and we fix the anti-dominant fundamental weight \(-w_k\). Note that \( w \in W^J \) is equivalent to \( w[1, k] \) and \( w[k+1, n] \) being increasing sequences.

Lemma 14. Consider \( w \in W^J \). We have an edge \( w \stackrel{(i,j)}{\rightarrow} w(i,j) \) in the quantum Bruhat graph on \( S_n \), with \( i \leq k < j \), if and only if one of the following two conditions holds:

1. the edge is a Bruhat cover, with \( w(i) = a, w(j) = a + 1 \), and \( w(i,j) \in W^J \);
2. the edge is a quantum one, and \( (i,j) = \alpha_k \).

Proof. We implicitly use several times the quantum Bruhat graph criterion in Proposition \([\text{2}]\) as well as the fact that \( w[1, k] \) and \( w[k+1, n] \) are increasing sequences. Letting \( a := w(i) \), and assuming that the edge is a Bruhat cover, we cannot have \( w(j) > a + 1 \) because the value \( a + 1 \) would be straddled by the transposition \((i, j)\). Indeed, this would happen irrespective of \( a + 1 \) being in \( w[1, k] \) or \( w[k+1, n] \). So we must have \( w(j) = a + 1 \). Now assume that \( w(i) > w(j) \). If \( i < k \), then the value \( w(k) \) would be straddled, while if \( j > k + 1 \), then the value \( w(k+1) \) would be straddled. So we must have \( i = k \) and \( j = k + 1 \). \( \square \)

We can now give a short proof of [KNS22, Theorem I] in type \( A_{n-1} \), which is restated below in terms of the quantum alcove model.

Theorem 15. In type \( A_{n-1} \), consider \( 1 \leq k \leq n - 1 \) and \( w \in W^J \).

1. If \( w \geq [s_\theta] \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(-w_k)] \cdot [\mathcal{O}^w] = e^{w_{\mathcal{O}^w}} \sum_{A \in \mathcal{A}(w, \Gamma(k))} (-1)^{|A|} \left( [\mathcal{O}^{\text{end}(w,A)}] - Q_k [\mathcal{O}^{\text{end}(w,A)s_k}] \right).
\]

2. If \( w \not\geq [s_\theta] \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(-w_k)] \cdot [\mathcal{O}^w] = e^{w_{\mathcal{O}^w}} \sum_{A \in \mathcal{A}(w, \Gamma(k))} (-1)^{|A|} [\mathcal{O}^{\text{end}(w,A)}].
\]

Remark 16. As will be seen in the proof below, for \( w \in W^J \), the condition \( w \geq [s_\theta] \) is equivalent to the condition \( w(k) = n \) and \( w(k+1) = 1 \).

Example 17. We give some examples of the Chevalley formula in the case that \( n = 4 \) and \( k = 2 \). Note that \( [s_\theta] = s_3s_1s_2 \). Also, we have \( \Gamma(2) = ((1, 4), (1, 3), (2, 4), (2, 3)) \) (with all roots negated).

1. Let \( w = s_3s_1s_2 = [s_\theta] \). Table \([\text{1}]\) is the list of all admissible subsets \( A \in \mathcal{A}(w, \Gamma(2)) \) and their statistics \( \text{end}(w,A) \), \( \text{down}(w,A) \), together with \( [\text{end}(w,A)] \); note that \( \text{wt}(w,A) = -s_2w_2 \) for all \( A \in \mathcal{A}(w, \Gamma(2)) \).
Thus Theorem 15 (1) holds in this case.

By applying the surjection $\Phi : \mathcal{K}^\text{poly}(G/B) \to \mathcal{K}^\text{poly}(G/P_J)$, explained in Theorem 10, to equation (7), we obtain the following cancellation-free formula in $\mathcal{K}^\text{poly}(G/P_J) \subset \mathcal{K}^\text{poly}(G/B)$:

$$[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2 s_3 s_1 s_2}] = e^{s_2 s_3 s_1 s_2} [\mathcal{O}^{s_3 s_1 s_2} - [\mathcal{O}^{s_2 s_3 s_1 s_2} - Q_2(\mathcal{O}^{s_3 s_1}] + Q_2(\mathcal{O}^{s_2 s_3 s_1})]$$

(7)

By applying the surjection $\Phi : \mathcal{K}^\text{poly}(G/B) \to \mathcal{K}^\text{poly}(G/P_J)$, explained in Theorem 10, to equation (7), we obtain the following cancellation-free formula in $\mathcal{K}^\text{poly}(G/P_J) \subset \mathcal{K}^\text{poly}(G/B)$:

$$[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2 s_3 s_1 s_2}] = e^{s_2 s_3 s_1 s_2} [\mathcal{O}^{s_3 s_1 s_2} - [\mathcal{O}^{s_2 s_3 s_1 s_2} - Q_2(\mathcal{O}^{s_3 s_1} + Q_2(\mathcal{O}^{s_2}])$$

(8)

By applying the surjection $\Phi : \mathcal{K}^\text{poly}(G/B) \to \mathcal{K}^\text{poly}(G/P_J)$ to equation (8), we obtain the following cancellation-free formula in $\mathcal{K}^\text{poly}(G/P_J) \subset \mathcal{K}^\text{poly}(G/B)$; here, the

| $A$ | $\text{end}(w, A)$ | $\text{down}(w, A)$ |
|-----|-------------------|---------------------|
| $\emptyset$ | $s_3 s_1 s_2$ | $s_3 s_1 s_2$ |
| $\{1\}$ | $s_2 s_3 s_1 s_2$ | $s_2 s_3 s_1 s_2$ |
| $\{4\}$ | $s_3 s_1$ | $e$ |
| $\{1, 4\}$ | $s_2 s_3 s_1$ | $s_2$ |

Table 1. The list of all admissible subsets $A \in \mathcal{A}(s_3 s_1 s_2, \Gamma(2))$

| $A$ | $\text{end}(w, A)$ | $\text{down}(w, A)$ |
|-----|-------------------|---------------------|
| $\emptyset$ | $s_2$ | $s_2$ |
| $\{2\}$ | $s_1 s_2$ | $s_3 s_2$ |
| $\{3\}$ | $s_3 s_2$ | $s_3 s_2$ |
| $\{4\}$ | $e$ | $e$ |
| $\{2, 3\}$ | $s_3 s_1 s_2$ | $s_3 s_1 s_2$ |
| $\{2, 4\}$ | $s_1$ | $e$ |
| $\{3, 4\}$ | $s_3$ | $e$ |
| $\{2, 3, 4\}$ | $s_3 s_1$ | $e$ |

Table 2. The list of all admissible subsets $A \in \mathcal{A}(s_2, \Gamma(2))$
underlined terms in the first equality are canceled out:

\[
[O(-w_2)] \cdot [O^{s_2}] = e^{s_2 \omega_2}([O^{s_2}] - [O^{s_1 s_2}] - [O^{s_3 s_2}] - Q_2 [O^e] \\
+ [O^{s_3 s_1 s_2}] + Q_2 [O^e] + Q_3 [O^e] - Q_2 [O^e]) \\
= e^{s_2 \omega_2}([O^{s_2}] - [O^{s_1 s_2}] - [O^{s_3 s_2}] + [O^{s_3 s_1 s_2}]).
\]

Also, we deduce that \( A_c(w, \Gamma(2)) = \emptyset, \{2\}, \{3\}, \{2, 3\} \). Therefore, we see that

\[
(\text{RHS of equation } (3)) = e^{s_2 \omega_2} ([O^{s_2}] - [O^{s_1 s_2}] - [O^{s_3 s_2}] + [O^{s_3 s_1 s_2}]). \\
= [O(-w_2)] \cdot [O^{s_2}].
\]

Thus Theorem \ref{thm:main}(2) holds in this case.

**Proof of Theorem \ref{thm:main}**. The result is clear when \( w \) is the identity (indeed, a \( w \)-admissible subset is either empty or consists only of the transposition \((k, k + 1)\)); so we can assume that \( w(k) > w(k + 1) \).

Let \( A \) be a generic \( w \)-admissible subset in \( A(w, \Gamma(k)) \). Given the structure of the \((\cdot, \omega_i)\)-chain \( \Gamma(k) \) in \ref{fig:Gamma} and Lemma \ref{lem:sign-reversing}, we can see that a quantum step in a path \( \Pi(w, A) \) must correspond to the transposition \( \alpha_k = (k, k + 1) \), which is the last one in \( \Gamma(k) \). All other steps are Bruhat covers of the form specified in Lemma \ref{lem:sign-reversing}(1). Moreover, the structure of \( \Gamma(k) \) combined with the fact that \( w \in W^J \) imply that \( A \) contains at most one root labeling a Bruhat cover in \( \Pi(w, A) \) of the following forms: \((i, \cdot)\) for each \( i \leq k \), and \((\cdot, j)\) for each \( j > k \). All these facts will be used implicitly.

By Deodhar’s criterion for the Bruhat order on the symmetric group \[BB05\] Theorem 2.6.3, we can see that \( w \geq [s_q] = [2, 3, \ldots, k, n, 1, k + 1, \ldots, n - 1] \) (in one-line notation) if and only if \( w(k) = n \) and \( w(k + 1) = 1 \). Thus, we consider the following cases; whenever there are terms to be canceled, we describe the sign-reversing involution mentioned above.

**Case 1**: \( w(k) < n \). Let \( q > k + 1 \) be such that \( w(q) = w(k) + 1 \leq n \). We pair every \( A \) containing \((k, k + 1)\), but not \((k, j)\) with \( j > k + 1 \), with \( A' := A \cup \{(k, q)\} \). It is clear that \( A' \) is also \( w \)-admissible, and in fact the root \((k, q)\) is the predecessor of \((k, k + 1)\) in \( A' \). Moreover, we have

\[
[\text{end}(w, A')] = [\cdots (k, q)(k, k + 1)] = [\cdots (k, k + 1)(k + 1, q)] = [\text{end}(w, A)],
\]

as well as \( \text{down}(w, A) = \text{down}(w, A') \) and \( \text{wt}(w, A) = \text{wt}(w, A') \). The latter property is a consequence of the fact that all the affine reflections in the definition of \( \text{wt}(\cdot, \cdot) \) in \[LNS20\] Equation (12) fix \( \omega_k \); for more details, see \[LP07\] Corollary 8.2 and the discussion preceding it. Finally, as the cardinalities of \( A \) and \( A' \) differ by 1, their contributions to the parabolic Chevalley formula for \( G/P_J \) have opposite signs. We have thus proved that the involution \( A \leftrightarrow A' \) is sign-reversing.

**Case 2**: \( w(k) = n \) and \( w(k + 1) > 1 \). Let \( p < k \) be such that \( w(p) = w(k + 1) - 1 \geq 1 \). We pair every \( A \) containing \((k, k + 1)\), but not \((i, k + 1)\) with \( i < k \), with \( A' := A \cup \{(p, k + 1)\} \). We continue the reasoning like in Case 1.

**Case 3**: \( w(k) = n \) and \( w(k + 1) = 1 \). It is clear that no \( w \)-admissible subset \( A \) can contain transpositions of the form \((i, k + 1)\) with \( i < k \), and \((k, j)\) with \( j > k + 1 \). Furthermore, there is a 2-to-1 correspondence between \( A(w, \Gamma(k)) \) and \( A_c(w, \Gamma(k)) \): every \( A \in A_c(w, \Gamma(k)) \) corresponds to itself and \( A \cup \{(k, k + 1)\} \). Like above, we can check that \( \text{wt}(w, A') = \text{wt}(w, A \cup \{(k, k + 1)\}) \). Finally, based on the above facts and Remark \ref{rem:sign-reversing} we can see that there are no cancellations of terms corresponding to the elements of either \( A_c(w, \Gamma(k)) \) or \( A(w, \Gamma(k)) \setminus A_c(w, \Gamma(k)) \).

It is now easy to see that the uncanceled terms in the resulting combinatorial formula are precisely those in \ref{eq:uncanceled} in Case 3, and those in \ref{eq:uncanceled prime} in Cases 1 and 2. \( \square \)

### 3.2. Type \( C_n \)

As we move beyond type \( A \), we note that the following analogue of Lemma \ref{lem:sign-reversing} exists: \[KNS22\] Lemma 5.1] for any simply laced type and \( \omega_k \) minuscule. Below we present the
corresponding result in type $C_n$, which works for any $w_k$; this result is easily proved based on the quantum Bruhat graph criterion in Section 2.1.

**Lemma 18.** Consider $1 \leq k \leq n$ and $w \in W^J$ in type $C_n$. We have a quantum edge $w \xrightarrow{\alpha} ws_\alpha$ in $QB(W)$, with $\alpha \in \Phi^+ \setminus \Phi^+_J$, if and only if $w \neq e$ and one of the following two conditions holds:

1. $\alpha = \alpha_k$;
2. $\alpha = (k, \overline{k})$, $w(k) = \overline{a}$ for $1 \leq a \leq n$, and $w[k + 1, n] \subseteq \{a + 1, \ldots, n\}$ if $k < n$.

Let us now turn to a short proof in the case of $w_k$ in type $C_n$, where $1 \leq k \leq n$. Note that $w \in W^J$ is equivalent to $w[1, k]$ and $w[k + 1, n]$ being increasing sequences (with respect to the total order on $[n]$), as well as $w[k + 1, n]$ consisting of positive entries. We also need to introduce more notation. The $(-w[2])$-chain $\Gamma(k)$ in (2) has an obvious splitting $\Gamma(k) = \Gamma_1(k)\Gamma_2(k)$, where $\Gamma_1(k) := \Gamma_2 \cdots \Gamma'_k$ and $\Gamma_2(k) := \Gamma_1(k) \cdots \Gamma_k(k)$. This induces a splitting $A = A^1 \sqcup A^2$ of any $w$-admissible subset $A$, where $A_i = A \cap \Gamma^i(k)$, for $i = 1, 2$.

**Theorem 19.** In type $C_n$, given $w \in W^J$, we have the following cancellation-free formula in $QK_T^{\ast}(G/P_J) \subset QK_T(G/P_J)$:

$$\mathcal{O}(w, A) = \sum_{A \in A_k(w, \Gamma(k))} (-1)^{|A|}e^{-\text{wt}(w, A)[\mathcal{O}^{\text{end}(w, A)}]} - Q_k \sum_{A \in A_k(w, \Gamma(k))} (-1)^{|A|}e^{-\text{wt}(w, A)[\mathcal{O}^{\text{end}(w, A)s_{2i_k}}]}.$$  

**Remark 20.** As will be seen in the proof below, for $w \in W^J$, the condition $w \geq |s_\theta|$ is equivalent to the condition $w(k) = \overline{a}$.

**Example 21.** In this example, we consider the case that $n = 3$ and $k = 2$. Note that $|s_\theta| = s_1s_2s_3s_2$. Recall that $\Gamma(2) = ((1, \overline{2}), (1, 3), (1, \overline{3}), (1, 2), (2, \overline{3}), (2, 3))$ (with all roots negated). Let $w = s_2s_3s_2$. Then the list of all admissible subsets $A \in \mathcal{A}(w, \Gamma(2))$ and their statistics $\text{end}(w, A)$, down$(w, A)$, together with $\text{end}(w, A^1)$, $\text{down}(w, A)$, is given in Table 3. Note that $\text{wt}(w, A) = -s_2s_3s_2w_2$ for all $A \in \mathcal{A}(w, \Gamma(2))$.

By Theorem 7, in $QK_T^{\ast}(G/B)$, we have:

$$\mathcal{O}(w, A) = e^{s_2s_3s_2}((O^s_{2, 3, 8, 2}) - [O^s_{1, 2, 8, 3}] - [O^s_{2, 8, 3, 1}] - [O^s_{8, 1, 2, 8, 3}] - Q_2Q_3[O^e] - Q_2[O^s_{2, 8, 3}]$$
$$+ [O^s_{1, 2, 8, 3}] + Q_2Q_3[O^s_{1, 1}] + Q_2[O^s_{1, 2, 8, 3}] + [O^s_{8, 1, 2, 8, 3}] + Q_2[O^s_{2, 8, 3}] + Q_2Q_3[O^s_{1, 8, 2, 3}]$$
$$- Q_2Q_3[O^s_{1, 8, 2, 3}] - Q_2[O^s_{1, 8, 2, 3}] - Q_2Q_3[O^s_{1, 8, 2, 3}]$$

By applying the surjection $\Phi_J : QK_T^{\ast}(G/B) \to QK_T^{\ast}(G/P_J)$ to equation (10), we obtain the following cancellation-free formula in $QK_T^{\ast}(G/P_J) \subset QK_T(G/P_J)$; here, the underlined terms in the first equality are canceled out:

$$\mathcal{O}(w, A) = e^{s_2s_3s_2}((O^s_{2, 3, 8, 2}) - [O^s_{1, 2, 8, 3}] - [O^s_{2, 8, 3, 1}] - [O^s_{8, 1, 2, 8, 3}] - Q_2[O^e] - Q_2[O^s_{2, 8, 3}]$$
$$+ [O^s_{1, 2, 8, 3}] + Q_2[O^e] + Q_2[O^s_{1, 8, 2, 3}] + [O^s_{8, 1, 2, 8, 3}] + Q_2[O^s_{2, 8, 3}] + Q_2[O^e]$$
$$- Q_2[O^e] - Q_2[O^s_{1, 8, 2, 3}] - Q_2[O^s_{1, 8, 2, 3}] + Q_2[O^s_{1, 8, 2, 3}] + Q_2[O^s_{1, 8, 2, 3}]$$

$$= e^{s_2s_3s_2}((O^s_{2, 3, 8, 2}) - 2[O^s_{1, 2, 8, 3}] - [O^s_{2, 8, 3, 1}] + 2[O^s_{8, 1, 2, 8, 3}] + Q_2[O^e] - Q_2[O^s_{2, 8, 3}]).
Table 3. The list of all admissible subsets $A \in \mathcal{A}(s_2s_3s_2, \Gamma(2))$

| A            | $\text{end}(w, A^1)$ | $\text{end}(w, A)$ | $\text{end}(w, A)$ | $\text{down}(w, A)$ |
|--------------|------------------------|---------------------|---------------------|---------------------|
| $\emptyset$  | $s_2s_3s_2$            | $s_2s_3s_2$         | $s_2s_3s_2$         | $0$                 |
| $\{1\}$      | $s_1s_2s_3s_2$         | $s_1s_2s_3s_2$      | $s_1s_2s_3s_2$      | $0$                 |
| $\{4\}$      | $s_2s_3s_2$            | $s_2s_3s_1s_2$      | $s_2s_3s_1s_2$      | $0$                 |
| $\{5\}$      | $s_2s_3s_2$            | $s_1s_2s_3s_2$      | $s_1s_2s_3s_2$      | $0$                 |
| $\{7\}$      | $s_2s_3s_2$            | $\emptyset$         | $\emptyset$         | $\alpha_2^\vee + \alpha_3$ |
| $\{8\}$      | $s_2s_3s_2$            | $s_2s_3$             | $s_2$               | $\alpha_2^\vee$    |
| $\{1,4\}$    | $s_1s_2s_3s_2$         | $s_1s_2s_3s_1s_2$   | $s_1s_2s_3s_1s_2$   | $0$                 |
| $\{1,7\}$    | $s_1s_2s_3s_2$         | $s_2$               | $\alpha_2^\vee$    |
| $\{1,8\}$    | $s_1s_2s_3s_2$         | $s_1s_2s_3$          | $s_1s_2$            | $\alpha_2$        |
| $\{4,6\}$    | $s_2s_3s_2$            | $s_1s_2s_3s_1s_2$   | $s_1s_2s_3s_1s_2$   | $0$                 |
| $\{4,8\}$    | $s_2s_3s_2$            | $s_2s_3s_1$          | $s_2$               | $\alpha_2^\vee$    |
| $\{5,7\}$    | $s_2s_3s_2$            | $s_1$               | $\alpha_2 + \alpha_3$ |
| $\{5,8\}$    | $s_2s_3s_2$            | $s_1s_2s_3$          | $s_1s_2$            | $\alpha_2$        |
| $\{7,8\}$    | $s_2s_3s_2$            | $s_2$               | $\alpha_2 + \alpha_3$ |
| $\{1,4,7\}$  | $s_1s_2s_3s_2$         | $s_2s_1$            | $s_2$               | $\alpha_2 + \alpha_3$ |
| $\{1,4,8\}$  | $s_1s_2s_3s_2$         | $s_1s_2s_3s_1$      | $s_1s_2$            | $\alpha_2$        |
| $\{1,7,8\}$  | $s_1s_2s_3s_2$         | $s_1s_2$            | $s_1s_2$            | $\alpha_2 + \alpha_3$ |
| $\{4,6,7\}$  | $s_2s_3s_2$            | $s_2s_1$            | $s_2$               | $\alpha_2 + \alpha_3$ |
| $\{4,6,8\}$  | $s_2s_3s_2$            | $s_1s_2s_3s_1$      | $s_1s_2$            | $\alpha_2$        |
| $\{5,7,8\}$  | $s_2s_3s_2$            | $s_1s_2$            | $s_1s_2$            | $\alpha_2 + \alpha_3$ |
| $\{1,4,7,8\}$| $s_1s_2s_3s_2$         | $s_1s_2s_1$         | $s_1s_2$            | $\alpha_2 + \alpha_3$ |
| $\{4,6,7,8\}$| $s_2s_3s_2$            | $s_1s_2s_1$         | $s_1s_2$            | $\alpha_2 + \alpha_3$ |

Also, we deduce that $\mathcal{A}_{w}(w, \Gamma(2)) = \{\emptyset, \{1\}, \{4\}, \{5\}, \{1,4\}, \{4,6\}\}$; note that only two elements $A = \{1\}, \{1,4\}$ of $\mathcal{A}_{w}(w, \Gamma(2))$ satisfy $\text{end}(w, A^1) \geq [s_9]$. Therefore, we see that

(RHS of equation [21])

$$
e^{s_2s_3s_2[O]} (O^{s_2s_3s_2} - O^{s_1s_2s_3s_2} - O^{s_2s_3s_1s_2} - O^{s_1s_2s_3s_1s_2} + [O^{s_1s_2s_3s_1s_2}])
\quad - Q_2 e^{s_2s_3s_2[O]} (-O^{[s_1]} + [O^{s_2[O]}])
\quad = e^{s_2s_3s_2[O]} ([O^{s_2s_3s_2}] - 2[O^{s_1s_2s_3s_2}] - [O^{s_2s_3s_1s_2}] + 2[O^{s_1s_2s_3s_1s_2}] + Q_2[O^2] - Q_2[O^{s_2}])
= [O^{(\emptyset)}] \cdot [O^{s_2s_3s_2}];$$

here, we have used $s_{2s_2} = s_2s_3s_2$ for the first equality. Thus Theorem [19] holds in this case.

**Proof of Theorem [19]** We assume that $w$ is not the identity, as this case is trivial. We follow the same procedure outlined above, and describe the sign-reversing involution canceling terms obtained from the Chevalley formula for $G/B$.

We carry out the proof in the case $k < n$, and refer to $k = n$ at the end. Consider a generic $w$-admissible subset $A$ in $\mathcal{A}(w, \Gamma(2))$, corresponding to a term in the Chevalley formula for $G/B$. Like in type $A$, the structure of $\Gamma(k)$ combined with the fact that $w \in W^J$ imply that $A$ contains at most one root labeling a Bruhat cover in $\Pi(w, A)$ from each row in the display of $\Gamma_j(k)$ in [B].

We focus on those $A$ with $A^{-} \neq \emptyset$. By Lemma [18] we have $A^{-} \subseteq \{\alpha_k = (k, k+1), 2\varepsilon_k = (k, \overline{k})\}$. Note that both of these roots appear only once in the $(-\pi_k)$-chain $\Gamma(k)$, with $(k, k + 1)$ being the last one, while $(k, \overline{k})$ appears in the last segment $\Gamma_1(k)$. In fact, we have either $A^{-} = \{(k, k + 1)\}$ or $A^{-} = \{(k, \overline{k})\}$. Indeed, assuming that $(k, \overline{k}) \in A^{-}$, and considering the signed permutation $u$ in $\Pi(w, A)$ to which $(k, \overline{k})$ is applied, we have $u \in W^J$ and $u[k+1, n] \subseteq \{a+1, \ldots, n\}$, where
a := |u(k)| = u(\overline{k})$; therefore, it is impossible for $(k, k + 1)$ to correspond to a quantum step in $\Pi(w, A)$.

Now assume that $A^- = \{(k, k + 1)\}$, and let $v := \text{end}(w, A \setminus \{(k, k + 1)\})$. We clearly have $v \in W^J$. We will pair $A$ with another $w$-admissible subset $A'$, such that their contributions to the parabolic Chevalley formula for $G/P_J$ cancel out. We must have one of the following cases, where $1 \leq a < b \leq n$.

Case 1: $v(k) = b$, $v(k + 1) = a$, and $A$ does not contain $(k, j)$ with $j > k + 1$.

Subcase 1.1: $b < n$. This case is completely similar to Case 1 in the type $A$ proof. Indeed, there clearly exists $q > k + 1$ such that $v(q) = b + 1 \leq n$. We let $A' := A \cup \{(k, q)\}$, so $(A')^- = \{(k, k + 1)\}$, and continue the reasoning as above.

Subcase 1.2: $b = n$. We let $A' := A \cup \{(k, \overline{k})\}$, and we have $(A')^- = \{(k, k + 1)\}$.

Case 2: $v(k) = \overline{b}$, $v(k + 1) = b$. We let $A' := A \cup \{(k, \overline{k})\}$, and we have $(A')^- = \{(k, \overline{k})\}$.

Case 3: $v(k) = \overline{b}$, $v(k + 1) = a$, and $A$ contains neither $(k, \overline{k})$, nor $(k, j)$ for $j > k + 1$.

Subcase 3.1: $k + 2 \leq n$ and $v(k + 2) < b$. Consider $q > k + 1$ largest such that $v(q) < b$. We let $A' := A \cup \{(k, \overline{k})\}$, and we have $(A')^- = \{(k, k + 1)\}$.

Subcase 3.2: $k + 2 > n$ or $v(k + 2) > b$. We let $A' := (A \setminus \{(k, k + 1)\}) \cup \{(k, \overline{k + 1}), (k, \overline{k})\}$, and we have $(A')^- = \{(k, \overline{k})\}$.

We claim that in all cases,

$$A' \in \mathcal{A}(w, \Gamma(k)), \quad [\text{end}(w, A)] = [\text{end}(w, A')], \quad \text{and} \quad \text{wt}(w, A) = \text{wt}(w, A').$$

Furthermore, it is not hard to check that these cases completely pair up all $w$-admissible subsets $A$ with $A^- = \{(k, k + 1)\}$, either among themselves (in Cases 1.1, 1.2, and 3.1), or with $A$ satisfying $A^- = \{(k, \overline{k})\}$ (in Cases 2 and 3.2); see below for a discussion of the latter $A$ which are not paired up above.

Indeed, let us consider, for instance, Case 2. We cannot have $(k, \overline{k}) \in A$, because the corresponding up step in Bruhat order would not be a cover (by the classical part of the criterion in Proposition $\mathbb{B}(3)$). Moreover, $A$ cannot contain any root of the form $(k, j)$ with $j > k + 1$, as the corresponding reflection would bring a positive entry to position $k$, whereas $v(k)$ is negative. Therefore, the roots $(k, \overline{k})$ and $(k, k + 1)$ are the last two in $A'$, while the step corresponding to $(k, \overline{k})$ is a quantum one (by the criterion in Proposition $\mathbb{B}(3)$). Moreover, we have

$$[\text{end}(w, A')] = [v(k, \overline{k})(k, k + 1)] = [v(k, k + 1)(k + 1, \overline{k + 1})] = [\text{end}(w, A)].$$

The weight preservation is verified by noting that all affine hyperplanes corresponding to the roots in $\Gamma^2(k)$ contain $\varpi_k$; so the corresponding affine reflections fix $\varpi_k$, and are thus irrelevant for the weight computation.

On the other hand, in the Chevalley formula for $G/B$, the quantum steps corresponding to both roots $(k, k + 1)$ and $(k, \overline{k})$ contribute the variable $Q_k$. Indeed, as indicated above, we have the following coroot splitting: $(2\epsilon_k)^{\gamma} = \alpha_k^{\vee} + (\alpha_{k+1}^{\vee} + \cdots + \alpha_n^{\vee})$. Finally, since the cardinalities of $A$ and $A'$ differ by 1, we conclude that the involution $A \leftrightarrow A'$ is indeed sign-reversing. In this way, the contributions to the parabolic Chevalley formula for $G/P_J$ of all $A$ with $A^- = \{(k, k + 1)\}$ are canceled.

We have now exhausted all $w$-admissible sets $A$ with $A^- = \{(k, k + 1)\}$. Thus, it remains to discuss the contributions of the remaining $A$ with $A^- \neq \emptyset$, i.e., $A^- = \{(k, \overline{k})\}$ and $A$ is not among the $A'$ in Cases 2 and 3.2. So from now on we work under this assumption. We previously considered the signed permutation $u \in W^J$ in $\Pi(w, A)$ to which $(k, \overline{k})$ is applied, and observed that $u[k + 1, n] \subseteq \{a + 1, \ldots, n\}$, where $a := |u(k)|$. If $(k, \overline{k})$ is followed by another root in $A$, then this
can only be \((k, k+1)\); but this situation was considered in Case 2 above, which means that \((k, \overline{k})\) must be the last root in \(A\). Moreover, \(A\) cannot contain any root of the form \((k, j)\) with \(j > k\), because we would be in Case 3.2. The following two cases cover all remaining possibilities, and we continue to use the above notation.

Case 4: \(u(k) \neq \overline{1}\) (i.e., \(a \neq 1\)), and \(A^2\) contains no root \((i, \overline{k})\) with \(i < k\). There clearly exists \(p < k\) such that \(u(p) = a - 1\). We let \(A' := A \cup \{(p, \overline{k})\}\), where the root \((p, \overline{k})\) is taken from \(\Gamma^2(k)\).

We have \(\langle A' \rangle = \{(k, \overline{k})\}\). Like above, we verify that the terms corresponding to \(A\) and \(A'\) cancel out, so we can extend the sign-reversing involution above by pairing \(A\) with \(A'\).

Now recall that, in general, \(A^2\) contains at most one root \((i, \overline{k})\) with \(i < k\). Whenever it contains one, the values in positions \(i\) and \(k\) of the signed permutation to which this reflection is applied are of the form \(b-1\) and \(\overline{b}\), respectively. Thus, the remaining case consists of the following \(w\)-admissible subsets \(A\).

Case 5: \(u(k) = \overline{1}\) (i.e., \(a = 1\)), and \(A^2\) contains no root \((i, \overline{k})\) with \(i < k\). We clearly have \(A \setminus \{(k, \overline{k})\} \in \mathcal{A}_{\xi}(w, \Gamma(k))\), where we recall that \(\langle k, \overline{k} \rangle\) is the last root in \(A\). Now let \(u' := \text{end}(w, A^1)\).

Based on the structures of \(\Gamma(k)\) and \(A\), we have \(u'(k) = u(k) = \overline{1}\). But this is equivalent to \(u' \geq \lfloor s_g \rfloor = [2, 3, \ldots, k, \overline{1}, k+1, \ldots, n]\) (in the window notation), by Deodhar’s criterion for the type \(C\) Bruhat order [BB05, Chapter 8, Exercise 6]. In the same way as above, we can see that \(\text{wt}(w, A) = \text{wt}(A \setminus \{(k, \overline{k})\})\). The above facts imply that the terms corresponding to this case make up the second sum in \(\mathcal{I}\). By Remark 12 there are no cancellations between these terms.

We conclude by considering \(k = n\), and noting that the proof reduces to Cases 4 and 5 above. \(\square\)

We now prove the positivity property of structure constants for isotropic Grassmannians as a corollary of Theorem 19.

**Corollary 22.** Let \(G\) be of type \(C_n\), and \(J = I \setminus \{k\}\) for an arbitrary fixed \(1 \leq k \leq n\). Then, for \(w, u \in W_J\) and \(\xi \in Q_{I \setminus J}^{\vee+}\), we have

\[
(-1)^{1 + \ell(w) + \ell(u) + \text{deg}(\xi)} N_{s_k, w}^{u, \xi} \in \mathbb{Z}_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta].
\]

**Proof.** Take \(A \in \mathcal{A}_\xi(w, \Gamma(k))\) such that \(\text{end}(w, A^1) \geq \lfloor s_g \rfloor\), and set \(v := \text{end}(w, A)\). Recall from the proof of Theorem 19 that there exists a quantum edge \(v \xrightarrow{\frac{2\nu}{s_k}} v s_{2s_k} \in \mathcal{Q}(W)\). Also, by Case 5 in the proof of Theorem 19, we have \(v(k) = \overline{1}\). Note that \(v \in W_J\), and hence that \(v(1) < \cdots < v(k), v(k+1) < \cdots < v(n)\). It follows that \(1 = v s_{2s_k}(k) < v s_{2s_k}(1) < \cdots < v s_{2s_k}(k-1)\) and \(v s_{2s_k}(k+1) < \cdots < v s_{2s_k}(n)\). Therefore, if we take a cyclic permutation \(\sigma := (1, k, k-1, \ldots, 2) \in W\), then we have \(|v s_{2s_k}| = v s_{2s_k} \sigma\). Hence we see that

\[
|A| + 1 = |A \cup \{(k, \overline{k})\}|
\equiv \ell(v s_{2s_k}) - \ell(w)
\equiv \ell(v s_{2s_k} \sigma) - \ell(\sigma) - \ell(w)
\equiv \ell([v s_{2s_k}]) - (k - 1) - \ell(w)
\equiv \ell(w) + \ell([v s_{2s_k}]) + k - 1
\]

modulo 2. Thus, we obtain

\[
(-1)^{|A|+1} = (-1)^{\ell(w) + \ell([v s_{2s_k}]) + k - 1}.
\]
It is easy to check (see, for example, [GW09, Section 3.1.5, Exercise 4]) that

\[
2\rho_J = \begin{cases} 
\sum_{i=1}^{k-1} i(k-1)\alpha_i + \sum_{i=1}^{n-k-1} i(2(n-k) - i + 1)\alpha_{k+1} + \frac{(n-k)(n-k+1)}{2}\alpha_n & \text{if } k \neq n, \\
\sum_{i=1}^{n-1} i(n-i)\alpha_i & \text{if } k = n.
\end{cases}
\]

Since

\[
\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } |i-j| = 1 \text{ and } i \neq n, \\
-2 & \text{if } i = n \text{ and } j = n-1, \\
0 & \text{otherwise},
\end{cases}
\]

we have

\[
2\langle \rho_J, \alpha_k^\vee \rangle = \begin{cases} 
-(k-1) - 2(n-k) & \text{if } k \neq n-1, n, \\
-(n-2) - 2 \cdot \frac{1 \cdot 2}{2} & \text{if } k = n-1, \\
-(n-1) & \text{if } k = n
\end{cases}
\]

\[\equiv k - 1 \mod 2.\]

In addition, we have \(2\langle \rho, \alpha_k^\vee \rangle = 2\). Therefore, we see that

\[
\deg(Q_k) = 2\langle \rho, \alpha_k^\vee \rangle - 2\langle \rho_J, \alpha_k^\vee \rangle \equiv 2 - (k-1) \equiv k - 1 \mod 2.
\]

We set

\[
\mathcal{A}(w, \Gamma(k))_{u,\alpha_k^\vee,\lambda} := \{ A \in \mathcal{A}_d(w, \Gamma(k)) \mid \text{end}(w, A^{(1)}) \geq |s_\theta|, \text{end}(w, A)s_2e_k = u, \text{wt}(A \cup \{(k, \overline{k})\}) = \lambda \}. \]

Then, since \(\text{wt}(w, A) \in -\varpi_k + Q^+\) by Lemma 11, we deduce from Theorem 19 that

\[
C_w^{u,\alpha_k^\vee} = \sum_{\lambda \in -\varpi_k + Q^+} \sum_{A \in \mathcal{A}(w, \Gamma)^0_{u,\alpha_k^\vee,\lambda}} (-1)^{|A|+1}Q_k e^{-\text{wt}(w, A \cup \{(k, \overline{k})\})} = (-1)^{\ell(w) + \ell(u) + k - 1}Q_k \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\alpha_k^\vee,\lambda}| e^{-\lambda}.
\]

Therefore, we obtain

\[
N_{u,w}^{v,\gamma} = (-1)^{1+\ell(w) + \ell(u) + k - 1}Q_k \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\alpha_k^\vee,\lambda}| e^{-\varpi_k - \lambda}.
\]

This implies that

\[
(-1)^{1+\ell(w) + \ell(u) + \deg(Q_k)} N_{u,w}^{v,\gamma} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 \mid \gamma \in -\Delta], \tag{11}
\]

as desired. Equation (11), together with the positivity property of \(N_{u,v}^{w,0}\) for \(u, v, w \in W^J\), proves the corollary.

\[\square\]

4. Quantum \(K\)-theory Chevalley formulas for two-step flag manifolds

In this section, we concentrate on the case of type \(A_{n-1}\); note that \(I = [n-1]\) in this case. Let us consider the (standard) parabolic subgroup \(P_J \supset B\) corresponding to \(J = I \setminus \{k_1, k_2\}\) for some
$k_1, k_2 \in I$ with $k_1 < k_2$; the resulting partial flag manifold $G/P_J$ is isomorphic to a two-step flag manifold $\text{Fl}(k_1, k_2; n)$ defined as:

$$\text{Fl}(k_1, k_2; n) := \left\{(V_1, V_2) \mid V_1 \text{ and } V_2 \text{ are subspaces of } \mathbb{C}^n \text{ such that } V_1 \subsetneq V_2, \dim V_1 = k_1, \text{ and } \dim V_2 = k_2 \right\}.$$  

The purpose of this section is to derive cancellation-free parabolic Chevalley formulas for the quantum multiplication in $QK_T(G/P_J)$ with $|O(-\omega_k)|$, for $k = k_1$ and $k = k_2$. For this purpose, as in Section 3, we examine all the terms to be canceled in certain formulas obtained from equation (3) in Theorem 4 in $QK_T^{\text{poly}}(G/B) \subset QK_T(G/B)$, by applying the map $\Phi_J : QK_T^{\text{poly}}(G/B) \to QK_T^{\text{poly}}(G/P_J)$.

4.1. Some lemmas on admissible subsets. Note that for $w \in W = W(A_{n-1}) = S_n, w \in W^J$ is equivalent to $w[1, k_1], w[k_1 + 1, k_2], \text{ and } w[k_2 + 1, n]$ being increasing sequences (see \text{[BB05, Lemma 2.4.7]}).

We first consider the case $k = k_1$. We will make repeated use of the following.

Lemma 23. Consider $w \in W^J$. We have an edge $w \xrightarrow{(i,j)} w(i, j)$ in the quantum Bruhat graph $\text{QB}(W)$, with $i \leq k_1 < j$, if and only if one of the following two conditions holds:

1. the edge above is a Bruhat cover, and $w(i, j) \in W^J$;
2. the edge above is a quantum one, and $(i, j) = (k_1, k_2 + 1) \text{ or } (i, j) = (k_1, 1)$.

Proof. As in the proof of Lemma 14, we implicitly use Proposition 2 as well as the fact that $w[1, k_1], w[k_1 + 1, k_2], \text{ and } w[k_2 + 1, n]$ are increasing sequences. Assume first that the edge above is a Bruhat cover. Then, since $(i, j) \notin W_J$, \text{[BB05, Corollary 2.5.2]} implies that $w(i, j) \in W^J$, as desired. Assume next that the edge above is a quantum one; note that $w(i) > w(j)$ in this case. If $i < k_1$, then the value $w(k_1)$ would be straddled between $w(i)$ and $w(j)$. Hence we must have $i = k_1$. Also, if $k_1 + 1 < j \leq k_2$, then the value $w(k_1 + 1)$ would be straddled between $w(k_1)$ and $w(j)$; if $j > k_2$, then the value $w(k_2 + 1)$ would be straddled between $w(k_1) \text{ and } w(j)$. Hence we must have $j = k_1 + 1 \text{ or } j = k_2 + 1$. This proves the lemma.

Lemma 24. Consider $w \in W^J$, and assume that we have a quantum edge $w \xrightarrow{(k_1, k_2 + 1)} w(k_1, k_2 + 1)$ in $\text{QB}(W)$. Then, for $k_1 + 1 \leq j \leq k_2$, we have an edge $w(k_1, k_2) \xrightarrow{(k_1, j)} w(k_1, k_2 + 1)w(k_1, j)$ in $\text{QB}(W)$ if and only if $j = k_1 + 1$. In this case, the edge $w(k_1, k_2) \xrightarrow{(k_1, j)} w(k_1, k_2 + 1)w(k_1, j)$ is a Bruhat cover.

Proof. Set $v := w(k_1, k_2 + 1)$. Since we have a quantum edge $w \xrightarrow{(k_1, k_2 + 1)} v$ in $\text{QB}(W)$, Proposition 2 implies the following:

$$v(k_1) < v(k_1 + 1) < v(k_1 + 2) < \cdots < v(k_2) < v(k_2 + 1).$$

If $k_1 + 1 < j \leq k_2$, then the value $v(k_1 + 1)$ would be straddled between $v(k_1)$ and $v(j)$. Hence we must have $j = k_1 + 1$. In this case, by Proposition 2 we have a Bruhat edge $v \xrightarrow{(k_1, k_1 + 1)} v(k_1, k_1 + 1)$. This proves the lemma.

As a corollary of Lemmas 23 and 24, we immediately obtain the following.

Lemma 25. Let $w \in W^J$, and take $A = \{j_1 < \cdots < j_s\} \in A(w, \Gamma(k_1))$. If the directed path $\Pi(w, A)$ contains a quantum edge, then $\Pi(w, A)$ is one of the following forms; here $\rightarrow_B$ indicates a Bruhat edge, while $\rightarrow_Q$ indicates a quantum edge:
(1) \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-1} \frac{(k_1,k_1+1)}{Q} \to w_s \); 

(2) \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-1} \frac{(k_1,k_2+1)}{Q} \to w_s \); 

(3) \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-2} \frac{(k_1,k_2+1)}{Q} \to w_{s-1} \frac{(k_3,k_1+1)}{B} \to w_s \).

In view of this lemma, we divide the set \( \mathcal{A}(w, \Gamma(k_1)) \) into the disjoint union of the following four subsets:

1. \( \mathcal{A}_0(w, \Gamma(k_1)) \) (defined in Section 21); 
2. \( \mathcal{A}_1(w, \Gamma(k_1)) := \{ A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A) \) is of the form (1) in Lemma 25\}; 
3. \( \mathcal{A}_2(w, \Gamma(k_1)) := \{ A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A) \) is of the form (2) in Lemma 25\}; 
4. \( \mathcal{A}_3(w, \Gamma(k_1)) := \{ A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A) \) is of the form (3) in Lemma 25\}.

Then it follows that 
\[ \mathcal{A}(w, \Gamma(k_1)) = \mathcal{A}_0(w, \Gamma(k_1)) \sqcup \mathcal{A}_1(w, \Gamma(k_1)) \sqcup \mathcal{A}_2(w, \Gamma(k_1)) \sqcup \mathcal{A}_3(w, \Gamma(k_1)). \]

Also, we can verify the following:

- if \( A \in \mathcal{A}_0(w, \Gamma(k_1)) \), then \( \down(w, A) = 0 \), and hence \( Q^{\down(w,A)} = 0 \);
- if \( A \in \mathcal{A}_1(w, \Gamma(k_1)) \), then \( \down(w, A) = \alpha^\vee_{k_1} \), and hence \( Q^{\down(w,A)} = Q_{k_1} \);
- if \( A \in \mathcal{A}_2(w, \Gamma(k_1)) \) or \( A \in \mathcal{A}_3(w, \Gamma(k_1)) \), then \( \down(w, A) = \alpha^\vee_{k_1} + \cdots + \alpha^\vee_{k_2} \), and hence \( Q^{\down(w,A)} = Q_{k_1}Q_{k_2} \).

Therefore, by equation (4), we deduce that 
\[
[O(-\varpi_{k_1})] \cdot [O^w] = e^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_0(w, \Gamma(k_1))} (-1)^{|A|} [O^{\text{end}(w,A)}] 
+ e^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [O^{\text{end}(w,A)}] 
+ e^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\text{end}(w,A)}] 
+ e^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\text{end}(w,A)}].
\]

Next, we consider the case \( k = k_2 \). In this case, we use \( \Gamma^*(k_2) \) instead of \( \Gamma(k_2) \). From Lemma 25 by applying the diagram automorphism \( \omega \), we obtain the following.

**Lemma 26.** Let \( w \in W^J \), and take \( A = \{ j_1 < \cdots < j_s \} \in \mathcal{A}(w, \Gamma^*(k_2)) \). If the directed path \( \Pi(w, A) \) contains a quantum edge, then \( \Pi(w, A) \) is one of the following forms; here, \( \rightarrow \) indicates a 
Bruhat edge, while \( \overleftarrow{\cdots} \) indicates a quantum edge:

1. \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-1} \frac{(k_2,k_2+1)}{Q} \to w_s \); 
2. \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-1} \frac{(k_1,k_2+1)}{Q} \to w_s \); 
3. \( \Pi(w, A) : w = w_0 \to \cdots \to w_{s-2} \frac{(k_1,k_2+1)}{Q} \to w_{s-1} \frac{(k_2,k_2+1)}{B} \to w_s \).

In view of this lemma, we divide the set \( \mathcal{A}(w, \Gamma^*(k_2)) \) into the disjoint union of the following four subsets:
Theorem 31. \( A_{\omega}(w, \Gamma^*(k_2)) \) (already defined);

(2) \( A_1(w, \Gamma^*(k_2)) := \{ A \in A(w, \Gamma^*(k_2)) \mid \Pi(w, A) \text{ is of the form (1) in Lemma 26} \};

(3) \( A_2(w, \Gamma^*(k_2)) := \{ A \in A(w, \Gamma^*(k_2)) \mid \Pi(w, A) \text{ is of the form (2) in Lemma 26} \};

(4) \( A_3(w, \Gamma^*(k_2)) := \{ A \in A(w, \Gamma^*(k_2)) \mid \Pi(w, A) \text{ is of the form (3) in Lemma 26} \}.

4.2. Parabolic Chevalley formulas for two-step flag manifolds. We state cancellation-free parabolic Chevalley formulas for the equivariant quantum \( K \)-theory \( W/P_J \); the proofs of these results will be given in Sections 4.3 and 4.4. First, we assume that \( k = k_1 \). Take and fix \( w \in W^J \).

Theorem 27. If \( w(k_1) < w(k_1 + 1) \), then we have the following cancellation-free formula:

\[
[O(-\omega_{k_1})] \cdot [O^w] = e^{w\omega_{k_1}} \sum_{A \in A_\omega(w, \Gamma(k_1))} (-1)^{|A|} [Q^{end(w,A)}].
\]

We consider the following condition:

(Q) \( w(k_1) > w(k_2) \) and \( w(k_1 + 1) > w(k_2 + 1) \).

Remark 28. As mentioned at the beginning of Section 4.1, \( w[k_1 + 1, k_2] \) is an increasing sequence for \( w \in W^J \). Hence condition (Q) implies that \( w(k_1) > w(k_2) \geq w(k_1 + 1) > w(k_2 + 1) \).

Theorem 29. Assume that \( w(k_1) > w(k_1 + 1) \), and assume that condition (Q) does not hold.

(1) If \( w(1) < w(k_1 + 1) \) or \( w(k_1) < w(k_2) \), then we have the following cancellation-free formula:

\[
[O(-\omega_{k_1})] \cdot [O^w] = e^{w\omega_{k_1}} \sum_{A \in A_\omega(w, \Gamma(k_1))} (-1)^{|A|} [Q^{end(w,A)}].
\]

(2) If \( w(1) > w(k_1 + 1) \) and \( w(k_1) > w(k_2) \), then we have the following cancellation-free formula:

\[
[O(-\omega_{k_1})] \cdot [O^w] = e^{w\omega_{k_1}} \sum_{A \in A_\omega(w, \Gamma(k_1))} (-1)^{|A|} \left( [Q^{end(w,A)}] - Q_{k_1} [Q^{end(w,A)s_{k_1}}] \right).
\]

Also, we consider the following condition:

(Full) both of the following hold:

(1) \( w(k_1) = n \) and \( w(k_2 + 1) = 1 \); and

(2) \( w(k_1 + 1) \) is the minimum element in the sequence \( w[1, k_2] \).

Remark 30. Condition (Full) holds if and only if condition (Q) holds and \( w(1) > w(k_1 + 1) \), \( w(k_1) > w(n) \); note that the inequality \( w(1) > w(k_1 + 1) \), together with condition (Q), implies that \( w(1) > w(k_2 + 1) \).

Theorem 31. Assume condition (Q).

(1) Assume that \( w(k_1) < w(n) \).

(a) If \( w(1) < w(k_1 + 1) \), then we have the following cancellation-free formula:

\[
[O(-\omega_{k_1})] \cdot [O^w] = e^{w\omega_{k_1}} \sum_{A \in A_\omega(w, \Gamma(k_1))} (-1)^{|A|} [Q^{end(w,A)}].
\]

(b) If \( w(1) > w(k_1 + 1) \), then we have the following cancellation-free formula:

\[
[O(-\omega_{k_1})] \cdot [O^w] = e^{w\omega_{k_1}} \sum_{A \in A_\omega(w, \Gamma(k_1))} (-1)^{|A|} \left( [Q^{end(w,A)}] - Q_{k_1} [Q^{end(w,A)s_{k_1}}] \right).
\]

(2) Assume that \( w(k_1) > w(n) \).
Remark 34. Condition (Full)

We consider the following analog of condition (Full):

Next, we assume that \( k = k_2 \).

**Theorem 32.** If \( w(k_2) < w(k_2 + 1) \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(w_{k_2})] \cdot [\mathcal{O}^w] = e^{w_{k_2}} \sum_{A \in A_c(w, \Gamma(k_2))} (-1)^{|A|} [\mathcal{O}^{\text{end}(w,A)}].
\]

Recall condition (Q) above.

**Theorem 33.** Assume that \( w(k_2) > w(k_2 + 1) \), and assume that condition (Q) does not hold.

1. If \( w(k_2) < w(n) \) or \( w(k_1 + 1) < w(k_2 + 1) \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(w_{k_2})] \cdot [\mathcal{O}^w] = e^{w_{k_2}} \sum_{A \in A_c(w, \Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\text{end}(w,A)}].
\]

2. If \( w(k_2) > w(n) \) and \( w(k_1 + 1) > w(k_2 + 1) \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(w_{k_2})] \cdot [\mathcal{O}^w] = e^{w_{k_2}} \sum_{A \in A_c(w, \Gamma^*(k_2))} (-1)^{|A|} \left( [\mathcal{O}^{\text{end}(w,A)}] - Q_{k_2} [\mathcal{O}^{\text{end}(w,A) s_{k_2}}] \right).
\]

We consider the following analog of condition (Full):

(Full)* both of the following hold:

1. \( w(k_1) = n \) and \( w(k_2 + 1) = 1 \); and
2. \( w(k_2) \) is the maximum element in the sequence \( w[k_1 + 1, n] \).

**Remark 34.** Condition (Full)* holds if and only if condition (Q) holds and \( w(n) < w(k_2) \), \( w(k_2 + 1) < w(1) \); note that the inequality \( w(n) < w(k_2) \), together with condition (Q), implies that \( w(n) < w(k_1) \).

**Theorem 35.** Assume condition (Q).

1. Assume that \( w(1) < w(k_2 + 1) \).
   a. If \( w(k_2) < w(n) \), then we have the following cancellation-free formula:

\[
[\mathcal{O}(w_{k_2})] \cdot [\mathcal{O}^w] = e^{w_{k_2}} \sum_{A \in A_c(w, \Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\text{end}(w,A)}].
\]
(b) If \( w(k_2) > w(n) \), then we have the following cancellation-free formula:

\[
[O(-\varpi_{k_2})] \cdot [O^w] = e^{w \varpi_{k_2}} \sum_{A \in A_{<}(w, \Gamma^*(k_2))} (-1)^{|A|^2} \left( [O^{|\text{end}(w,A)\cdot k_2|}] - Q_{k_2}[O^{|\text{end}(w,A)\cdot k_2|}] \right).
\]

(2) Assume that \( w(1) > w(k_2 + 1) \).

(a) If \( w(k_1) < w(n) \), then we have the following cancellation-free formula:

\[
[O(-\varpi_{k_2})] \cdot [O^w] = e^{w \varpi_{k_2}} \sum_{A \in A_{<}(w, \Gamma^*(k_2))} (-1)^{|A|} [O^{|\text{end}(w,A)|}] .
\]

(b) If \( w(k_2) < w(n) < w(k_1) \), then we have the following cancellation-free formula:

\[
[O(-\varpi_{k_2})] \cdot [O^w] = e^{w \varpi_{k_2}} \sum_{A \in A_{<}(w, \Gamma^*(k_2))} (-1)^{|A|} [O^{|\text{end}(w,A)|}] - Q_{k_1} Q_{k_2} [O^{|\text{end}(w,A)(k_1,k_2+1)|}] .
\]

(3) If condition (Full)* holds, then we have the following cancellation-free formula:

\[
[O(-\varpi_{k_2})] \cdot [O^w] = e^{w \varpi_{k_2}} \sum_{A \in A_{<}(w, \Gamma^*(k_2))} (-1)^{|A|} [O^{|\text{end}(w,A)|}] - Q_{k_1} Q_{k_2} [O^{|\text{end}(w,A)(k_1,k_2+1)|}] .
\]

**Example 36.** In this example, we consider the case that \( n = 6 \) and \( (k_1, k_2) = (2, 4) \). Let \( w = s_4 s_1 s_2 s_3 s_5 s_4 s_3 s_2 \). Then, \( w \) satisfies condition (Q), and we see that \( w(k_2 + 1)(= w(5)) < w(1) < w(k_1 + 1)(= w(3)) \). Recall that \( \Gamma(2) = ((1, 6), (1, 5), (1, 4), (1, 3), (2, 6), (2, 5), (2, 4), (2, 3)) \). Then Table 3 is the list of all admissible subset \( A \in \mathcal{A}(w, \Gamma(5)) \) and their statistics \( \text{end}(w, A), \text{down}(w, A) \), together with \( |\text{end}(w, A)| \).

**Table 4.** The list of all admissible subsets \( A \in \mathcal{A}(s_4 s_1 s_2 s_3 s_5 s_4 s_3 s_2, \Gamma(2)) \)

| \( A \) | \( \text{end}(w, A) \) | \( \text{end}(w, A) \) | \( \text{down}(w, A) \) |
|---|---|---|---|
| \( \emptyset \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | 0 |
| \{4\} | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | 0 |
| \{6\} | \( s_4 s_5 s_1 \) | \( s_4 \) | \( \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee \) |
| \{8\} | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee \) |
| \{4, 6\} | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee \) |
| \{4, 8\} | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( \alpha_2 + \alpha_3 + \alpha_4 \) |
| \{4, 6, 8\} | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2 \) | \( \alpha_2 + \alpha_3 + \alpha_4 \) |

By Theorem 7 in \( \mathcal{Q}K_T^{\text{poly}}(G/B) \), we have:

\[
[O(-\varpi_2)] \cdot [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2}] = e^{s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2} [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_2}] - Q_2 Q_3 Q_4 [O^{s_4 s_5 s_1}] - Q_2 [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_3}] + Q_2 Q_3 Q_4 [O^{s_4 s_5 s_2 s_1}] + Q_2 [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_3}] + Q_2 Q_3 Q_4 [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_3}] - Q_2 Q_3 Q_4 [O^{s_4 s_5 s_1 s_2 s_3 s_4 s_3}] .
\]

(12)

By applying the surjection \( \Phi_T : \mathcal{Q}K_T^{\text{poly}}(G/B) \to \mathcal{Q}K_T^{\text{poly}}(G/P_J) \) to equation [12], we obtain the following cancellation-free formula in \( \mathcal{Q}K_T^{\text{poly}}(G/P_J) \subset \mathcal{Q}K_T(G/P_J) \); here, the underlined terms in
the first equality are canceled out:

\[
[\mathcal{O}(-w_2)] \cdot [\mathcal{O}^{s_4 s_5 s_2 s_3 s_4 s_3 s_2} - [\mathcal{O}^{s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2}] - Q_2 Q_3 Q_4 [\mathcal{O}^{s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2}] + Q_2 Q_3 Q_4 [\mathcal{O}^{s_4 s_5 s_2 s_3 s_4 s_3 s_2}].
\]

Also, we deduce that \( \mathcal{A}_{\varphi}(w, \Gamma(2)) = \{\emptyset, \{4\}\} \). Therefore, we see that

(RHS of the equation in Theorem 31(2)(b))

\[
= [\mathcal{O}(-w_2)] \cdot [\mathcal{O}^{s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2} - [\mathcal{O}^{s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2}] - Q_2 Q_4 [\mathcal{O}^{s_4 s_5 s_1}] + Q_2 Q_4 [\mathcal{O}^{s_4 s_5 s_2}].
\]

Thus Theorem 31(2)(b) holds in this case.

**Remark 37.** In [Xu21] Theorem 4.5, Xu obtained a Chevalley formula for incidence varieties, that is, for the two-step flag manifold \( G/P \) in the case that \( J = I \setminus \{1, n - 1\} \), by a completely different method of proof than ours of Theorems 27, 29, 31, 32, 33, and 35. We can verify that in this case, our Chevalley formula coincides with the one in [Xu21] for incidence varieties. As an example, we compare Theorem 20 (2) with [Xu21, Equation (9) of Theorem 4.5]; this is a most complicated case. As for Theorems 27, 29, 31, we can also compare our formulas and Xu’s ones by the same argument as below. As for Theorem 31, \( w \) should be the unique element of \( W^J \) such that \( w(1) = n \) and \( w(n) = 1 \), and hence we can compare the formulas by direct calculation. As for Theorems 32, 33, and 35, we can show the coincidence of the formulas from that of the formulas in Theorems 27, 29, and 31 by applying the diagram automorphism \( \omega \) (see Section 4.3).

Throughout this remark, we assume that \( k_1 = 1, k_2 = n - 1 \). Note that under this assumption, for \( 1 \leq i, j \leq n \) with \( i \neq j \), there exists a unique \( w \in W^J \) such that \( w(1) = i \) and \( w(n) = j \); in such a case, we write \( w = [i, j] \), as in [Xu21].

We assume that \( w \in W^J \) satisfies the following:

- \( w(k_1) > w(k_1 + 1) \),
- condition (Q) does not hold,
- \( w(1) > w(k_1 + 1) \),
- \( w(k_1) > w(k_2) \),

and set \( i := w(1), j := w(n) \) (i.e., \( w = [i, j] \)). Under these assumptions, we see that \( i + 1 \equiv j \) mod \( n \) if and only if \( i = n - 1 \) and \( j = n \) (i.e., \( w = [n - 1, n] \)).

Let us compute the product \([\mathcal{O}^{s_1}] \cdot [\mathcal{O}^w]\) by our Chevalley formula. Recall that

\[
\Gamma(1) = ((1, n), (1, n - 1), \ldots, (1, 2))
\]
Theorems 27, 29, and 31, respectively, by applying the diagram automorphism proof of the results stated in the previous subsection. Since Theorems 32, 33, and 35 follow from 4.3.

Proofs of parabolic Chevalley formulas: Part 1. This result agrees with the first equation of [Xu21, Equation (9) of Theorem 4.5].

By the well-known formula $[O^s] = 1 - e^{-\omega_1} [O(-\omega_1)]$ (cf., [BCMP18 Section 4.1]), we see that $[O^s] : [O^{n-1,n}]$

Next, we consider the case $w = [i, j] \neq [n-1, n]$. In this case, we see that $i+1 \neq j$ mod $n$. Since condition (Q) does not hold, we have $w(n) \neq 1$. Also, we have $w(n) \neq n$; this is because if $w(n) = n$, then $w$ must be $[n-1, n]$ under our assumptions. These facts imply that $w(1) = i = n$ and $w(2) = 1$. By Proposition 2, we deduce that $A_{\omega}(w, \Gamma(1)) = \{\emptyset\}$. Therefore, we compute:

Again, since $[O^s] = 1 - e^{-\omega_1} [O(-\omega_1)]$, we see that $[O^s] : [O^{n-1,n}]$

This result agrees with the first equation of [Xu21 Equation (9) of Theorem 4.5].

4.3. Proofs of parabolic Chevalley formulas: Part 1. In this and the next subsection, we give proofs of the results stated in the previous subsection. Since Theorems 32, 33 and 35 follow from Theorems 27, 29 and 31 respectively, by applying the diagram automorphism $\omega : [n-1] \rightarrow [n-1]$, it suffices to prove Theorems 27, 29 and 31. Note that the diagram automorphism $\omega$ induces a group automorphism $\omega : W \rightarrow W, s_i \mapsto s_{\omega(i)}$, together with a linear automorphism $\omega : h^{\ast}_R \rightarrow h^{\ast}_R$,.
By using the involution $\iota$, and also an isomorphism $\omega : G/P_J \xrightarrow{\sim} G/P_{\omega(J)}$ of varieties; recall that $G$ is simply-connected. Hence, as mentioned in Sections 8.1 and 8.3 of [MNS22], we see that there exists a $\mathbb{Z}$-module isomorphism $\omega : QK_T(G/P_J) \xrightarrow{\sim} QK_T(G/P_{\omega(J)})$ such that

$$e^\mu[O^\omega] \mapsto e^{\omega(\mu)}[O^\omega]$$

for $\mu \in \Lambda$, $w \in W_J$, and such that $\omega(Q_l) = Q_{\omega(l)}$ for $l \in I \setminus J$. In this subsection, we give proofs of Theorems 27 and 29.

By Remark 8, we obtain the following.

**Lemma 38.** The sum

$$e^w \sum_{A \in \mathcal{A}_e(w, \Gamma(k_1))} (-1)^{|A|} [O^\text{end}(w, A)]$$

is cancellation-free.

Also, by making use of Proposition 2, we can verify the following.

**Lemma 39.** The following hold.

1. We have $\mathcal{A}_3(w, \Gamma(k_1)) \neq \emptyset$ if and only if $w(k_1) > w(k_1 + 1)$.
2. We have $\mathcal{A}_2(w, \Gamma(k_1)) \neq \emptyset$ if and only if condition (Q) holds.

**Remark 40.** It is obvious that $\mathcal{A}_2(w, \Gamma(k_1)) \neq \emptyset$ if and only if $\mathcal{A}_3(w, \Gamma(k_1)) \neq \emptyset$.

**Remark 41.** If $w(k_1) > w(k_1 + 1)$, then we have

$$\mathcal{A}_1(w, \Gamma(k_1)) = \{ A \uplus \{(k_1, k_1 + 1)\} \mid A \in \mathcal{A}_e(w, \Gamma(k_1)) \}. \quad (13)$$

Also, if condition (Q) holds, then we have

$$\mathcal{A}_2(w, \Gamma(k_1)) = \{ A \uplus \{(k_1, k_1 + 1)\} \mid A \in \mathcal{A}_e(w, \Gamma(k_1)) \}, \quad (14)$$

$$\mathcal{A}_3(w, \Gamma(k_1)) = \{ A \uplus \{(k_1, k_1 + 1), (k_1, k_1 + 1)\} \mid A \in \mathcal{A}_e(w, \Gamma(k_1)) \}. \quad (15)$$

**Proof of Theorem 27.** By Lemma 39, we have $\mathcal{A}(w, \Gamma(k_1)) = \mathcal{A}_e(w, \Gamma(k_1))$. Therefore, the theorem follows from Lemma 38. \hfill \square

In the rest of this subsection, we assume that $w(k_1) > w(k_1 + 1)$, and assume that condition (Q) does not hold. Hence we have $\mathcal{A}_2(w, \Gamma(k_1)) = \mathcal{A}_3(w, \Gamma(k_1)) = \emptyset$.

First, assume that $w(1) < w(k_1 + 1)$. Take the maximal $1 \leq p \leq k_1$ such that $w(p) < w(k_1 + 1)$. Then, we can define an involution $\iota$ on $\mathcal{A}_1(w, \Gamma(k_1))$ as follows: set

$$\mathcal{A}_1^1(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_1(w, \Gamma(k_1)) \mid (p, k_1 + 1) \in A \},$$

$$\mathcal{A}_1^2(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_1(w, \Gamma(k_1)) \mid (p, k_1 + 1) \notin A \},$$

and define $\iota$ by

$$A \in \mathcal{A}_1^2(w, \Gamma(k_1)) \mapsto \iota(A) := A \uplus \{(p, k_1 + 1)\} \in \mathcal{A}_1^1(w, \Gamma(k_1)),$$

$$A \in \mathcal{A}_1^1(w, \Gamma(k_1)) \mapsto \iota(A) := A \setminus \{(p, k_1 + 1)\} \in \mathcal{A}_1^2(w, \Gamma(k_1)).$$

This $\iota$ has the following properties:

- $\text{end}(w, \iota(A)) = \text{end}(w, A)(p, k_1)$ (and hence $|\text{end}(w, \iota(A))| = |\text{end}(w, A)|$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution $\iota$, we obtain the following.
Lemma 42. Assume that $w(k_1) > w(k_1 + 1)$, and assume that condition (Q) does not hold. If $w(1) < w(k_1 + 1)$, then

$$e^{w \circ k_1} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1}[O^{\text{end}(w,A)}] = 0.$$ 

Remark 43. Even if we assume condition (Q), the identity in Lemma 42 is still valid, if all the conditions of this lemma other than the negation of condition (Q) hold. This is because the involution $\iota$ above is well-defined whether or not condition (Q) holds.

Next, assume that $w(k_1) < w(k_2)$. Take the minimal $k_1 + 1 \leq q \leq k_2$ such that $w(k_1) < w(q)$. Then, we can define an involution $\iota$ on $A_1(w, \Gamma(k_1))$ as follows: set

$$A_1^1(w, \Gamma(k_1)) := \{ A \in A_1(w, \Gamma(k_1)) \mid (k_1, q) \in A \},$$

$$A_1^2(w, \Gamma(k_1)) := \{ A \in A_1(w, \Gamma(k_1)) \mid (k_1, q) \notin A \},$$

and define $\iota$ by

$$A \in A_1^1(w, \Gamma(k_1)) \mapsto \iota(A) := A \cup \{(k_1, q)\} \in A_1^1(w, \Gamma(k_1)),$$

$$A \in A_1^1(w, \Gamma(k_1)) \mapsto \iota(A) := A \setminus \{(k_1, q)\} \in A_1^2(w, \Gamma(k_1)).$$

This $\iota$ has the following properties:

- $\text{end}(w, \iota(A)) = \text{end}(w, A)(k_1 + 1, q)$ (and hence $|\text{end}(w, \iota(A))| = |\text{end}(w, A)|$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution $\iota$, we obtain the following.

Lemma 44. Assume that $w(k_1) > w(k_1 + 1)$, and assume that condition (Q) does not hold. If $w(k_1) < w(k_2)$, then

$$e^{w \circ k_1} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1}[O^{\text{end}(w,A)}] = 0.$$ 

Proof of Theorem 2.4 (1). By Lemmas 42 and 44 we deduce that

$$e^{w \circ k_1} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1}[O^{\text{end}(w,A)}] = 0.$$ 

Therefore, we obtain the desired cancellation-free formula from Lemma 38 together with the fact that $A_2(w, \Gamma(k_1)) = A_3(w, \Gamma(k_1)) = \emptyset$. \hfill $\Box$

We assume that $w(1) > w(k_1 + 1)$ and $w(k_1) > w(k_2)$ until the end of this subsection. Let $A \in A_1(w, \Gamma(k_1))$, and set $y := \text{end}(w, A \setminus \{(k_1, k_1 + 1)\})$. Since $A \setminus \{(k_1, k_1 + 1)\}$ contains only Bruhat steps, we see that $y(k_1 + 1) < y(1)$ and $y(k_2) < y(k_1)$. Therefore, if we set $z := y s_{k_1} = \text{end}(w, A)$, then we have

- $z(k_1) < z(1) < z(2) < \cdots < z(k_1 - 1),$
- $z(k_1 + 2) < z(k_1 + 3) < \cdots < z(k_2) < z(k_1 + 1)$, and
- $z(k_2 + 1) < z(k_2 + 2) < \cdots < z(n);$

hence, if we take cyclic permutations $\sigma_1 := (1, k_1, k_1 - 1, \ldots, 2)$ (if $k_1 = 1$, then we take $\sigma_1 := e$, the identity permutation) and $\sigma_2 := (k_1 + 1, k_1 + 2, \ldots, k_2)$ (if $k_1 + 1 = k_2$, then we take $\sigma_2 := e$), then we deduce that $|z| = \sigma_1 \sigma_2$. Note that the definitions of $\sigma_1$ and $\sigma_2$ do not depend on the choice of $A$. Thus, for $A, B \in A_1(w, \Gamma(k_1))$ with $A \neq B$, it follows that

$$|\text{end}(w, A)| = \text{end}(w, A) \sigma_1 \sigma_2 \neq \text{end}(w, B) \sigma_1 \sigma_2 = |\text{end}(w, B)|.$$
Lemma 45. Assume that \( w(k_1) > w(k_1 + 1) \), and assume that condition \((Q)\) does not hold. If \( w(1) > w(k_1 + 1) \) and \( w(k_1) > w(k_2) \), then the sum

\[
\sum_{A \in \mathcal{A}_1(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{(\text{end}(w, A))}]
\]

is cancellation-free.

Remark 46. Note that we do not use the negation of condition \((Q)\) in the proof of Lemma 45. Hence the sum (16) is cancellation-free whether or not we assume condition \((Q)\), if all the conditions of Lemma 45 other than the negation of \((Q)\) hold.

Remark 47. If \( \mathcal{A}_1(w, \Gamma(k_1)) \neq \emptyset \) (or equivalently, \( w(k_1) > w(k_1 + 1) \)), then equation (13) shows that

\[
\sum_{A \in \mathcal{A}_1(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{(\text{end}(w, A))}]
\]

is cancellation-free.

Proof of Theorem 29(2). The desired identity follows from Lemmas 38, 45, and Remark 47 together with the fact that \( \mathcal{A}_2(w, \Gamma(k_1)) = \mathcal{A}_3(w, \Gamma(k_1)) = \emptyset \).

4.4. Proofs of parabolic Chevalley formulas: Part 2. In this subsection, we give a proof of Theorem 31 since we assume condition \((Q)\), we have \( w(k_1) > w(k_1 + 1) \); see Remark 28.

First, assume that \( w(k_1) < w(n) \). Then, we can take the minimal \( k_2 + 1 \leq q \leq n \) such that \( w(k_1) < w(q) \), and define an involution \( \iota \) on \( \mathcal{A}_l(w, \Gamma(k_1)) \), \( l = 2, 3 \), as follows: for each \( l = 2, 3 \), we set

\[
\mathcal{A}_l^1(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_l(w, \Gamma(k_1)) \mid (k_1, q) \in A \},
\]

\[
\mathcal{A}_l^2(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_l(w, \Gamma(k_1)) \mid (k_1, q) \notin A \},
\]

and define \( \iota \) by

\[
A \in \mathcal{A}_l^1(w, \Gamma(k_1)) \mapsto \iota(A) := A \cup \{(k_1, q)\} \in \mathcal{A}_l^1(w, \Gamma(k_1)),
\]

\[
A \in \mathcal{A}_l^2(w, \Gamma(k_1)) \mapsto \iota(A) := A \setminus \{(k_1, q)\} \in \mathcal{A}_l^2(w, \Gamma(k_1)).
\]

This \( \iota \) has the following properties:

- \( \text{end}(w, \iota(A)) = \text{end}(w, A)(k_2 + 1, q) \) (and hence \( |\text{end}(w, \iota(A))| = |\text{end}(w, A)| \));
- \( |\iota(A)| = |A| \pm 1 \).

By using the involution \( \iota \), we obtain the following.

Lemma 48. Assume condition \((Q)\). If \( w(k_1) < w(n) \), then for \( l = 2, 3 \),

\[
\sum_{A \in \mathcal{A}_l(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{(\text{end}(w, A))}] = 0.
\]
Lemma 50. Assume condition (Q-A). By using the involution $\iota$ is not well-defined; we will explain this situation later.

Remark 49. $\iota$ and define $\iota$ and $\iota$ and imply that the sum $\iota$ and $\iota$ is cancellation-free. Therefore, Theorem 31 (1) (b) follows from Lemma 38.

Proof of Theorem 31 (1). By Lemma 48, we deduce that $\mathcal{O}(w, A) = 0$ by Remark 43. Therefore, Theorem 31 (1) (a) follows from Lemma 38.

Assume now that $w(1) > w(k_1 + 1)$. Note that $w(k_1) > w(k_2)$ by condition (Q). Hence Remark 46 implies that the sum $\iota$ and $\iota$ is cancellation-free. Therefore, Theorem 31 (1) (b) follows from Lemma 38 and Remark 47.

Next, assume that $w(k_1) > w(n)$. We consider the following auxiliary condition:

(Q-A) there exists $1 \leq l \leq k_1$ such that $w(k_2 + 1) < w(l) < w(k_1 + 1)$.

Assume condition (Q-A), and that $w(1) < w(k_2 + 1)$. We take the maximal $1 \leq p_{A_2} \leq k_1$ such that $w(p_{A_2}) < w(k_2 + 1)$. Then, we can define an involution $\iota_{A_2}$ on $\mathcal{A}_2(w, \Gamma(k_1))$ as follows: set $\mathcal{A}_2^1(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_2(w, \Gamma(k_1)) \mid (p_{A_2}, k_2 + 1) \in A \}$, $\mathcal{A}_2^2(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_2(w, \Gamma(k_1)) \mid (p_{A_2}, k_2 + 1) \notin A \}$, and define $\iota_{A_2}$ by

$A \in \mathcal{A}_2^1(w, \Gamma(k_1)) \mapsto \iota_{A_2}(A) := A \cup \{(p_{A_2}, k_2 + 1) \} \in \mathcal{A}_2^1(w, \Gamma(k_1)),$

$A \in \mathcal{A}_2^2(w, \Gamma(k_1)) \mapsto \iota_{A_2}(A) := A \setminus \{(p_{A_2}, k_2 + 1) \} \in \mathcal{A}_2^2(w, \Gamma(k_1)).$

Remark 49. If condition (Q-A) does not hold, then the above $\iota_{A_2} : \mathcal{A}_2^1(w, \Gamma(k_1)) \rightarrow \mathcal{A}_2^2(w, \Gamma(k_1))$ is not well-defined; we will explain this situation later.

This $\iota_{A_2}$ has the following properties:

- $\iota(w, \iota_{A_2}(A)) = \iota(w, A)(p_{A_2}, k_1)$ (and hence $\iota(\iota_{A_2}(A)) = \iota(A))$;
- $|\iota_{A_2}(A)| = |A| \pm 1$.

By using the involution $\iota_{A_2}$, we obtain the following.

Lemma 50. Assume condition (Q). If $w(1) < w(k_2 + 1)$ and condition (Q-A) hold, then $\mathcal{O}(w, A) = 0$.

Also, we take the maximal $1 \leq p_{A_3} \leq k_1$ such that $w(p_{A_3}) < w(k_1 + 1)$. We can define an involution $\iota_{A_3}$ on $\mathcal{A}_3(w, \Gamma(k_1))$ as follows: set $\mathcal{A}_3^1(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p_{A_3}, k_1 + 1) \in A \}$, $\mathcal{A}_3^2(w, \Gamma(k_1)) := \{ A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p_{A_3}, k_1 + 1) \notin A \}$.
and define \( \iota_{A_3} \) by

\[
A \in \mathcal{A}_3^2(w, \Gamma(k_1)) \mapsto \iota_{A_3}(A) := A \cup \{(p_{A_3}, k_1 + 1)\} \in \mathcal{A}_3^2(w, \Gamma(k_1)), \\
A \in \mathcal{A}_3^1(w, \Gamma(k_1)) \mapsto \iota_{A_3}(A) := A \setminus \{(p_{A_3}, k_1 + 1)\} \in \mathcal{A}_3^1(w, \Gamma(k_1)).
\]

**Remark 51.** If condition (Q-A) does not hold, then the above \( \iota_{A_3} : \mathcal{A}_3^2(w, \Gamma(k_1)) \to \mathcal{A}_3^1(w, \Gamma(k_1)) \) is not well-defined for the same reason as \( \iota_{A_2} \).

This \( \iota_{A_3} \) has the following properties:

- \( \text{end}(w, \iota_{A_3}(A)) = \text{end}(w, A)(p_{A_3}, k_1) \) (and hence \( |\text{end}(w, \iota_{A_3}(A))| = |\text{end}(w, A)| \));
- \( |\iota_{A_3}(A)| = |A| \pm 1 \).

By using the involution \( \iota_{A_3} \), we obtain the following.

**Lemma 52.** Assume condition (Q). If \( w(1) < w(k_2 + 1) \) and condition (Q-A) holds, then

\[
e^{w \omega_{k_1}} \sum_{A \in \mathcal{A}_3(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} O^{[\text{end}(w, A)]} = 0.
\]

Next, assume that condition (Q-A) does not hold, but assume that \( w(1) < w(k_2 + 1) \). Take the maximal \( 1 \leq p \leq k_1 \) such that \( w(p) < w(k_2 + 1) \). Set

\[
\mathcal{A}_2^C(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2(w, \Gamma(k_1)) \mid (p, k_2 + 1) \in A\}, \\
\mathcal{A}_2^{C,1}(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2^C(w, \Gamma(k_1)) \mid \rho \in A\}, \\
\mathcal{A}_2^{C,2}(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2^C(w, \Gamma(k_1)) \mid \rho \notin A\}.
\]

Observe that if \( A \in \mathcal{A}_2^C(w, \Gamma(k_1)) \), then we must have \((p, k_2 + 1) \in A \); if not, then \( A \) cannot contain a quantum step \((k_1, k_2 + 1)\), which contradicts the definition of \( \mathcal{A}_2^C(w, \Gamma(k_1)) \). Thus, the above \( \iota_{A_2} : \mathcal{A}_2^C(w, \Gamma(k_1)) \to \mathcal{A}_2^{C,1}(w, \Gamma(k_1)) \) is not well-defined. Hence we need another involution.

In fact, we can define an involution on \( \mathcal{A}_2^{C,1}(w, \Gamma(k_1)) \) similar to \( \iota_{A_2} \) as follows. We set

\[
\mathcal{A}_2^{C,1}(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2^C(w, \Gamma(k_1)) \mid (p, k_2 + 1) \in A\}, \\
\mathcal{A}_2^{C,2}(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2^C(w, \Gamma(k_1)) \mid (p, k_2 + 1) \notin A\}.
\]

Then we can define an involution \( \iota \) on \( \mathcal{A}_2^{C,2}(w, \Gamma(k_1)) \) by

\[
A \in \mathcal{A}_2^{C,2}(w, \Gamma(k_1)) \mapsto \iota(A) := A \cup \{(p, k_2 + 1)\} \in \mathcal{A}_2^{C,1}(w, \Gamma(k_1)), \\
A \in \mathcal{A}_2^{C,1}(w, \Gamma(k_1)) \mapsto \iota(A) := A \setminus \{(p, k_2 + 1)\} \in \mathcal{A}_2^{C,2}(w, \Gamma(k_1)).
\]

This \( \iota \) has the following properties:

- \( \text{end}(w, \iota(A)) = \text{end}(w, A)(p, k_1) \) (and hence \( |\text{end}(w, \iota(A))| = |\text{end}(w, A)| \));
- \( |\iota(A)| = |A| \pm 1 \).

By using the involution \( \iota \), we obtain the following.

**Lemma 53.** Assume condition (Q). If \( w(1) < w(k_2 + 1) \), and if condition (Q-A) does not hold, then

\[
e^{w \omega_{k_1}} \sum_{A \in \mathcal{A}_2^{C,2}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} O^{[\text{end}(w, A)]} = 0.
\]

Similarly, we set

\[
\mathcal{A}_3^C(w, \Gamma(k_1)) := \{A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p, k_2 + 1) \notin A\}, \\
\mathcal{A}_3^{C,1}(w, \Gamma(k_1)) := \{A \in \mathcal{A}_3^C(w, \Gamma(k_1)) \mid (p, k_2 + 1) \in A\}.
\]
Observe that if \( A \in \mathcal{A}_3^1(w, \Gamma(k_1)) \), then we must have \((p, k_1 + 1) \not\in A\); if not, then \( A \) cannot contain a quantum step \((k_1, k_2 + 1)\). However, we can define an involution on \( \mathcal{A}_3^1(w, \Gamma(k_1)) \) similar to \( \iota_{\mathcal{A}_3^1} \) as follows. We set

\[
\begin{align*}
\mathcal{A}_3^{C,1}(w, \Gamma(k_1)) & := \{ A \in \mathcal{A}_3^C(w, \Gamma(k_1)) \mid (p, k_1 + 1) \in A \}, \\
\mathcal{A}_3^{C,2}(w, \Gamma(k_1)) & := \{ A \in \mathcal{A}_3^C(w, \Gamma(k_1)) \mid (p, k_1 + 1) \not\in A \}.
\end{align*}
\]

Then we can define an involution \( \iota \) on \( \mathcal{A}_3^C(w, \Gamma(k_1)) \) by

\[
\begin{align*}
A \in \mathcal{A}_3^{C,2}(w, \Gamma(k_1)) & \mapsto \iota(A) := A \sqcup \{(p, k_1 + 1)\} \in \mathcal{A}_3^{C,1}(w, \Gamma(k_1)), \\
A \in \mathcal{A}_3^{C,1}(w, \Gamma(k_1)) & \mapsto \iota(A) := A \setminus \{(p, k_1 + 1)\} \in \mathcal{A}_3^{C,2}(w, \Gamma(k_1)).
\end{align*}
\]

This \( \iota \) has the following properties:

- \( \end(w, \iota(A)) = \end(w, A)(p, k_1) \) (and hence \( |\end(w, \iota(A))| = |\end(w, A)| \));
- \( |\iota(A)| = |A| \pm 1 \).

By using the involution \( \iota \), we obtain the following.

**Lemma 54.** Assume condition (Q). If \( w(1) < w(k_2 + 1) \), and if condition (Q-A) does not hold, then

\[
e^{w \otimes k_1} \sum_{A \in \mathcal{A}_3^{C}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\end(w, A)}] = 0.
\]

It remains to examine cancellations for the set

\[
\mathcal{A}_3^{2}(w, \Gamma(k_1)) := \mathcal{A}_2^1(w, \Gamma(k_1)) \cup \mathcal{A}_3^1(w, \Gamma(k_1)).
\]

The desired involution on \( \mathcal{A}_3^{2}(w, \Gamma(k_1)) \) is given as follows:

\[
\begin{align*}
A \in \mathcal{A}_2^1(w, \Gamma(k_1)) & \mapsto \iota(A) := (A \setminus \{(p, k_1 + 1)\}) \cup \{(k_1, k_1 + 1)\} \in \mathcal{A}_3^1(w, \Gamma(k_1)), \\
A \in \mathcal{A}_3^1(w, \Gamma(k_1)) & \mapsto \iota(A) := (A \setminus \{(k_1, k_1 + 1)\}) \cup \{(p, k_2 + 1), (p, k_1 + 1)\} \in \mathcal{A}_2^1(w, \Gamma(k_1)).
\end{align*}
\]

This \( \iota \) has the following properties:

- \( \end(w, \iota(A)) = \end(w, A)(p, k_1) \) (and hence \( |\end(w, \iota(A))| = |\end(w, A)| \));
- \( |\iota(A)| = |A| \pm 1 \).

By using the involution \( \iota \), we obtain the following.

**Lemma 55.** Assume condition (Q). If \( w(1) < w(k_2 + 1) \), and if condition (Q-A) does not hold, then

\[
e^{w \otimes k_1} \sum_{A \in \mathcal{A}_3^{C}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\end(w, A)}] + e^{w \otimes k_1} \sum_{A \in \mathcal{A}_3^{C}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\end(w, A)}] = 0.
\]

**Proof of Theorem 31 (2) (a).** By Lemmas 50, 52, 53, 54 and 55, we have

\[
e^{w \otimes k_1} \sum_{A \in \mathcal{A}_3^{C}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [O^{\end(w, A)}] = 0.
\]

Also, since \( w(1) < w(k_2 + 1) \), \( w(k_1 + 1) \), Remark 43 implies that

\[
e^{w \otimes k_1} \sum_{A \in \mathcal{A}_3(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [O^{\end(w, A)}] = 0.
\]

These observations, together with Lemma 38, prove the desired cancellation-free identity. \( \square \)

**Remark 56.** In the proof of Theorem 31 (2) (a), we do not use the assumption that \( w(k_1) > w(n) \).
Now, we assume that \( w(k_1) > w(n) \) and \( w(k_2 + 1) < w(1) < w(k_1 + 1) \), which are the assumptions of Theorem 31 (2) (b); note that \( w(1) < w(k_1 + 1) < w(k_1) \) by condition (Q), and hence \( k_1 \neq 1 \). In this case, the same proof as that of Lemma 52 yields the following.

**Lemma 57.** Assume condition (Q). If \( w(k_2 + 1) < w(1) < w(k_1 + 1) \), then $$e^{w_{\sigma k_1}} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\text{end}(w, A)}] = 0.$$ 

**Remark 58.** We do not need the assumption that \( w(k_1) > w(n) \) for Lemma 57.

In contrast, the sum over \( A_2(w, \Gamma(k_1)) \) is cancellation-free. Indeed, let \( A \in A_2(w, \Gamma(k_1)) \), and set \( y := \text{end}(w, A \setminus \{(k_1, k_2 + 1)\}) \). Note that \( A \setminus \{(k_1, k_2 + 1)\} \) contains only Bruhat steps. Hence we see that \( y(k_2 + 1) < y(1) \) and \( y(n) < y(k_1) \). Therefore, if we set \( z := y(k_1, k_2 + 1) = \text{end}(w, A) \), then

- \( z(k_1) < z(1) < z(2) < \cdots < z(k_1 - 1) \),
- \( z(k_1 + 1) < z(k_1 + 2) < z(k_2) \), and
- \( z(k_2 + 2) < z(k_2 + 3) < \cdots < z(n) < z(k_2 + 1) \);

hence, if we take cyclic permutations \( \sigma_1 := (1, k_1, k_1 - 1, \ldots, 2) \) and \( \sigma_2 := (k_2 + 1, k_2 + 2, \ldots, n) \) (if \( k_2 + 1 = n \), then we take \( \sigma_2 := e \) ), then we have \( |\text{end}(w, A)| = |\text{end}(w, A)\sigma_1 \sigma_2| \). Note that these \( \sigma_1 \) and \( \sigma_2 \) does not depends on the choice of \( A \). Thus, for \( A, B \in A_2(w, \Gamma(k_1)) \) with \( A \neq B \), it follows that

$$|\text{end}(w, A)| = |\text{end}(w, A)\sigma_1\sigma_2 = |\text{end}(w, B)|.$$

This, together with Remark 38 proves the following.

**Lemma 59.** Assume condition (Q). If \( w(k_1) > w(n) \) and \( w(k_2 + 1) < w(1) < w(k_1 + 1) \), then the sum $$e^{w_{\sigma k_1}} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\text{end}(w, A)}]$$

is cancellation-free.

**Remark 60.** If \( A_2(w, \Gamma(k_1)) \neq \emptyset \) (or equivalently, if condition (Q) holds), then equation (14) shows that

$$e^{w_{\sigma k_1}} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\text{end}(w, A)}]$$

$$= -e^{w_{\sigma k_1}} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\text{end}(w, A)(k_1, k_2 + 1)}].$$

**Proof of Theorem 31 (2) (b).** Since \( w(1) < w(k_1 + 1) \), Remark 43 implies that $$e^{w_{\sigma k_1}} \sum_{A \in A_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\text{end}(w, A)}] = 0.$$ 

Hence, by Lemmas 38, 57, 59, and Remark 60, we obtain the desired cancellation-free formula. □

It only remains to prove Theorem 31 (3). To do so, we assume condition (Full). By the same argument as in the proof of Lemma 59, we see that for \( A \in A_2(w, \Gamma(k_1)) \), \( |\text{end}(w, A)| = |\text{end}(w, A)\sigma_1 \sigma_2| \), where \( \sigma_1 \) and \( \sigma_2 \) are the cyclic permutations defined above (if \( k_1 = 1 \), then we take \( \sigma_1 := e \) ). In addition, since \( w(k_1 + 1) < w(1) \) by condition (Full) (2), it follows that \( \text{end}(w, A)(k_1) < \text{end}(w, A)(k_1 + 1) \) (if \( k_1 = 1 \), then we need only the inequality \( \text{end}(w, A)(k_1 + 1) < \text{end}(w, A)(1) \)). Therefore, for \( A \in A_2(w, \Gamma(k_1)) \), note that \( A \setminus \{(k_1, k_1 + 1)\} \in A_2(w, \Gamma(k_1)) \), the following hold:

- \( \text{end}(w, A)(k_1) < \text{end}(w, A)(1) < \text{end}(w, A)(2) < \cdots < \text{end}(w, A)(k_1 - 1) \),
Exercise 4) that $N_{\text{positivity}}$ property under the assumptions of Theorems 32, 33, and 35 follows by the same arguments.

Proof. We give a proof of Corollary 63 under the assumptions of Theorems 27, 29, and 31. The positivity property.

Corollary 63. Let $G$ be of type $A_{n-1}$, $J = I \setminus \{k_1, k_2\}$ for arbitrarily fixed $1 \leq k_1 < k_2 \leq n - 1$, and $k = k_1$ or $k = k_2$. Then, for $w, u \in W^J$ and $\xi \in Q^+_A$, we have

$$(-1)^{1 + \ell(w) + \ell(u) + \deg(\xi)} N_{w, u}^{\xi}_{k_1, k_2} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 \mid \gamma \in -\Delta].$$

Proof. We give a proof of Corollary 63 under the assumptions of Theorems 27, 29, and 31. The positivity property under the assumptions of Theorems 27, 29, and 31 follows by the same arguments as those for Theorems 27, 29, and 31. Note that the positivity property under the assumptions of Theorems 27, 29, and 31 (a) and (2) (a) has already been known because of the positivity property of $N_{w, v}^{(0)}$ for $u, v, w \in W^J$. First, it is easy to check (see, for example, [GW09, Section 3.1.5, Exercise 4]) that

$$2\rho_J = \sum_{i=1}^{k_1-1} i(k_1 - i)\alpha_i + \sum_{i=1}^{k_2-1} i(k_2 - k_1 - i)\alpha_{k_1 + i} + \sum_{i=1}^{n-k_2-1} i(n - k_2 - i)\alpha_{k_2 + i}.$$
Since
\[ \langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases} \]
we have
\[ 2\langle \rho_J, \alpha_{k_1}^\vee \rangle = -(k_1 - 1) - (k_2 - k_1 - 1) = 2 - k_2, \]
\[ 2\langle \rho_J, \alpha_{k_2}^\vee \rangle = -(k_2 - k_1 - 1) - (n - k_2 - 1) = 2 - n + k_1. \]
In addition, we know that \( 2\langle \rho, \alpha_{k_1}^\vee \rangle = 2 \). Therefore,
\[ \deg(Q_{k_1}) = 2\langle \rho - \rho_J, \alpha_{k_1}^\vee \rangle = -k_2 \]
\[ \deg(Q_{k_2}) = 2\langle \rho - \rho_J, \alpha_{k_2}^\vee \rangle = n - k_1, \]
and hence
\[ \deg(Q_{k_1}Q_{k_2}) = 2\langle \rho - \rho_J, \alpha_{k_1}^\vee + \alpha_{k_2}^\vee \rangle = n - k_1 + k_2. \]

Let us consider the structure constants \( N_{k_1,k_2}^{\nu,\xi} \) with \( \xi \neq 0 \) under the assumptions of Theorems 29(2) and 31(1)(b). We maintain the setting of Lemma 35 except for the negation of condition (Q) (see Remark 46). Take \( A \in A_1(w, \Gamma(k_1)) \), and set \( u := \text{end}(w, A) \), \( u_0 := \text{end}(w, A) \). Then, by the proof of Lemma 35 we have \( u = u_0\sigma_1\sigma_2 \), where \( \sigma_1 = (1, k_1, k_1 - 1, \ldots, 2) \) and \( \sigma_2 = (k_1 + 1, k_1 + 2, \ldots, k_2) \). Therefore, we see that
\[ (-1)^{|A|}e^{w\varpi_{k_1}}Q_{k_1}[O^{\text{end}(w, A)}] = (-1)^{\ell(u_0) - \ell(w)}e^{w\varpi_{k_1}}Q_{k_1}[O^u], \]
\[ = (-1)^{\ell(u_0) - \ell(w)}e^{w\varpi_{k_1}}Q_{k_1}[O^u]. \]
We set
\[ A_1(w, \Gamma(k_1))_u := \{ A \in A_1(w, \Gamma(k_1)) \mid \text{end}(w, A) = u \} \]
for \( u \in W^J \). Then, for \( u \in W^J \), we deduce from Theorems 29(2) and 31(1)(b) that
\[ C_w^{u,\alpha_{k_1}^\vee} = e^{w\varpi_{k_1}} \sum_{A \in A_1(w, \Gamma(k_1))_u} (-1)^{|A|} = (-1)^{\ell(u) + \ell(w) + \deg(Q_{k_1})}A_1(w, \Gamma(k_1))_u|e^{w\varpi_{k_1}}, \]
and hence that
\[ N_{k_1,k_2}^{u,\alpha_{k_1}^\vee} = (-1)^{1 + \ell(w) + \ell(u) + \deg(Q_{k_1})}A_1(w, \Gamma(k_1))_u|e^{w\varpi_{k_1} - \varpi_{k_2}}. \]

Since \( w\varpi_{k_1} - \varpi_{k_2} \in -Q^+ \) and hence \( e^{w\varpi_{k_1} - \varpi_{k_2}} \in Z_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta] \), we conclude that
\[ (-1)^{1 + \ell(w) + \ell(u) + \deg(Q_{k_1})}N_{k_1,k_2}^{u,\alpha_{k_1}^\vee} \in Z_{\geq 0}[e^{\gamma} - 1 \mid \gamma \in -\Delta], \quad (17) \]
as desired. Equation (17), together with the positivity property of \( N_{u,v}^{w,0} \) for \( u, v, w \in W^J \), implies Corollary 63 under the assumptions of Theorems 29(2) and 31(1)(b).

Next, we consider the structure constants \( N_{k_1,k_2}^{u,\xi} \) with \( \xi \neq 0 \) under the assumption of Theorem 31(2)(b). We maintain the setting of Lemma 59 Take \( A \in A_2(w, \Gamma(k_1)) \), and set \( u :=
We set $u$ and hence that $e^u$. Again, since $e^u$, we consider the structure constants $N_{k_1,k_2}$ as desired. Equations (17), (20), together with the positivity property of $\xi$, in the case $\xi = 0$. The positivity property (Novikov) variables $\sigma$. Another proof of the existence of the multiplicative surjection $A_2(w, \Gamma(k_1))_u := \{ A \in A_2(w, \Gamma(k_1)) \mid \text{end}(w, A) = u \}$ for $u \in W^J$. Then, for $u \in W^J$, we deduce from Theorem 31 (2) (b) that

$$C_w^{u_1,k_1+\alpha_{k_2}^v} = e^{w_{\alpha_{k_1}}_1} \sum_{A \in A_2(w, \Gamma(k_1))_u} (-1)^{|A|} = (-1)^{(\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2}) | A_2(w, \Gamma(k_1))_u | e^{w_{\alpha_{k_1}}_1}},$$

and hence that

$$N_{s,k_1,w}^{u,v} = (-1)^{(1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2}) | A_2(w, \Gamma(k_1))_u | e^{w_{\alpha_{k_1}}_1}}.$$  \hspace{1cm} (18) Again, since $e^{w_{\alpha_{k_1}}_1} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 \mid \gamma \in -\Delta]$, we conclude that

$$(-1)^{(1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2}) | N_{s,k_1,w}^{u,v} | e^{w_{\alpha_{k_1}}_1} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 \mid \gamma \in -\Delta]}, \hspace{1cm} (19)$$
as desired. Equation (19), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, implies Corollary 63 under the assumption of Theorem 31 (2) (b).

It remains to consider the structure constants $N_{s,k_1,w}^{u,\xi}$ with $\xi \neq 0$ under the assumption of Theorem 31 (3) and consider the structure constants $N_{s,k_1,w}^{u,\xi}$ for $\xi \neq 0$. The positivity property in the case $\xi = \alpha_{k_1}^v$ has already been proved by equation (17). Hence it suffices to consider the case $\xi = \alpha_{k_1}^v + \alpha_{k_2}^v$. We maintain the setting of Lemma 61. We set

$$A_3(w, \Gamma(k_1))_u := \{ A \in A_2(w, \Gamma(k_1)) \cup A_3(w, \Gamma(k_1)) \mid \text{end}(w, A) = u \}$$

for $u \in W^J$. Then, by the same argument as that for equation (18), we deduce from Theorem 31 (3) that

$$N_{s,k_1,w}^{u,v} = (-1)^{(1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2}) | A_3(w, \Gamma(k_1))_u | e^{w_{\alpha_{k_1}}_1}},$$

and hence conclude that

$$(-1)^{(1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2}) | N_{s,k_1,w}^{u,v} | e^{w_{\alpha_{k_1}}_1} \in \mathbb{Z}_{\geq 0}[e^\gamma - 1 \mid \gamma \in -\Delta]}, \hspace{1cm} (20)$$
as desired. Equations (17), (20), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, implies Corollary 63 under the assumption of Theorem 31 (3). This completes the proof of the corollary.

\textbf{APPENDIX A. ANOTHER PROOF OF THE EXISTENCE OF THE MULTIPlicative SURJECTION $\Phi_J$}

In this appendix, we mainly use the notation of Section 2.3. In addition, we set $QK^\text{poly}_J(G/B) := K_T(G/B) \otimes Z[\Lambda] Z[\Lambda][Q]$, where $Z[\Lambda][Q]$ is the polynomial ring with coefficients in $Z[\Lambda]$ in the (Novikov) variables $Q_i = Q_i^{\alpha_i^v}$, $i \in I$; also, for an arbitrary subset $J \subseteq I$, we set $QK^\text{poly}_J(G/P_J) := K_T(G/P_J) \otimes Z[\Lambda] Z[\Lambda][Q_K]$, with $K := I \setminus J$, where $Z[\Lambda][Q_K]$ is the polynomial ring with coefficients in $Z[\Lambda]$ in the variables $Q_k$, $k \in K$. It is known (see [Kat19]) that there exists a surjective
such that $\Phi_J$ from $QK_T^{\text{poly}}(G/B)$ onto $QK_T^{\text{poly}}(G/P_J)$ such that $\Phi_J(Q^J[O^w]) = Q^{[J]}[O^w]$ for $w \in W$ and $\xi \in Q^{\nu,+}$, where $[\xi]^J := \sum_{k \in I \cup J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\nu,+}$. In this appendix, based on results in [CL22], we give another (short) proof of the existence of such a $\mathbb{Z}[\Lambda]$-algebra homomorphism. First of all, we note that $QK_T^{\text{poly}}(G/B)$ is a $\mathbb{Z}[\Lambda]$-subalgebra of $QK_T(G/B) = K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q]$ by [CL22, Corollary 1.2].

Let us briefly recall the main result of [CL22]. Following [CL22], let $Gr_G$ denote Pressley-Segal’s model of the affine Grassmannian associated to a simple and simply-connected complex Lie group $G$; more precisely, let $Gr_G$ be the space of polynomial based loops in a (fixed) maximal compact subgroup of $G$, equipped with an ind-variety structure (see [PSS02, Chapter 8] for details). We denote by $K^T(Gr_G)$ the $T$-equivariant $K$-homology (in the topological sense) of the affine Grassmannian $Gr_G$, equipped with the Pontryagin product $\odot$ coming from the group product on the topological group $Gr_G$. Then, we have two bases. One is a basis (called the localization basis) $O_\xi := [O_{x_\xi}]$, $\xi \in Q^\vee$, of $K^T(Gr_G)$ over Frac($\mathbb{Z}[\Lambda]$), where $x_\xi$ is the $T$-fixed point of $Gr_G$ corresponding to the cocharacter of $T$ associated to $\xi \in Q^\vee$. More precisely, if we consider the $\mathbb{Z}[\Lambda]$-algebra $O_\xi$ equipped with the product $\odot$ defined by $O_{\xi_1} \odot O_{\xi_2} := O_{\xi_1 + \xi_2}, \xi_1, \xi_2 \in Q^\vee$, then we have an injective $\mathbb{Z}[\Lambda]$-algebra homomorphism $K^T(Gr_G) \rightarrow \bigoplus_{\xi \in Q^\vee} \text{Frac}(\mathbb{Z}[\Lambda])O_\xi$ which fixes every $O_\xi$. Another is indeed a basis of $K^T(Gr_G)$ over $\mathbb{Z}[\Lambda]$ given as follows. Let $W_\af = W \ltimes Q^\vee$ be the affine Weyl group of $G$, and let $W_\af^\vee$ denote the set of minimal-length coset representatives for $W_\af/W$. We know from [LS10, Section 3] that an element $w_\xi \in W_\af$, with $w \in W$ and $\xi \in Q^\vee$, lies in $W_\af^\vee$ if and only if $\xi \in Q^\vee$ is anti-dominant and $w$ is of minimal length in its coset $wW_\xi$ in $W/W_\xi$, where $W_\xi \subset W$ is the stabilizer of $\xi$ in $W$; note that if $\xi \in Q^\vee$ is anti-dominant, then $\xi \in -Q^{\nu,+}$. In particular, if $\xi \in Q^\vee$ is regular anti-dominant, then $w_\xi \in W_\af^\vee$ for all $w \in W$. For each $w_\xi \in W_\af^\vee$, there exists a complex cell (called an affine Schubert cell) in $Gr_G$ containing the $T$-fixed point $w_\xi \in Gr_G$ of finite dimension; the class of the structure sheaf of the Zariski closure of this cell is denoted by $O_{w_\xi}$, and is called the affine Schubert class associated to $w_\xi \in W_\af^\vee$. Then we know that the classes $O_{w_\xi}, w_\xi \in W_\af^\vee$, form a $\mathbb{Z}[\Lambda]$-basis of $K^T(Gr_G)$.

Now the main result of [CL22] is stated as follows.

**Theorem 64 ([CL22 Theorem 1.1])**. Let $J$ be an arbitrary subset of $I$. Then, there exist a $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Psi_J : K^T(Gr_G) \rightarrow QK_T(G/P_J)[(Q^{\nu,+})^{-1}]$, where $QK_T(G/P_J)[(Q^{\nu,+})^{-1}]$ denotes the localization of $QK_T(G/P_J)$ with respect to the monomials in the Novikov variables $Q_i = Q_i^\vee, i \in I$. Moreover, $\Psi_J(O_{w_\xi}) = Q^{[J]}[O_{w_\xi}^w]$ for each $w_\xi \in W_\af^\vee$, where $[\xi]^J := \sum_{k \in I \cup J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^\vee$, and $[O_{w_\xi}^w]$ denotes the (opposite) Schubert class in $K_T(G/P_J)$ associated to the minimal-length coset representative $[w] \in W^J$ for the coset $wW_\xi$ in $W/W_J$.

Note that in the case $J = \emptyset$, i.e., $P_J = B$, the $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Psi := \Psi_\emptyset$ is injective since the affine Schubert classes $O_{w_\xi}, w_\xi \in W_\af^\vee$, form a $\mathbb{Z}[\Lambda]$-basis of $K^T(Gr_G)$ and $\Psi([O_{w_\xi}^w]) = Q^\xi[O^w]$. We will construct a surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Phi_J$ from $QK_T^{\text{poly}}(G/B)$ to $QK_T^{\text{poly}}(G/P_J)$ such that $\Phi_J(Q^\xi[O^w]) = Q^{[J]}[O_{w_\xi}^w]$ for $w \in W$ and $\xi \in Q^{\nu,+}$, where $[\xi]^J := \sum_{k \in I \cup J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\nu,+}$. We first note that for each element $v \in QK_T^{\text{poly}}(G/B) = K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q]$, there exists a sufficiently regular anti-dominant coroot $\eta \in -Q^{\nu,+}$ such that $Q^\eta v \in QK_T(G/B)[(Q^{\nu,+})^{-1}]$ lies in the image of the map $\Psi, i.e., Q^\eta v \in \Psi(K^T(Gr_G))$; by the injectivity of $\Psi$, there exists a unique $u \in K^T(Gr_G)$ such that $\Psi(u) = Q^\eta v$. Indeed, we may assume that $v = Q^\xi[O^w]$ for some $w \in W$ and $\xi \in Q^{\nu,+}$ since each $v \in QK_T^{\text{poly}}(G/B)$ is, by its definition, a finite
linear combination with coefficients in \( \mathbb{Z}[\Lambda] \) of such elements. Hence we can take a sufficiently regular anti-dominant coroot \( \eta \in Q^V \) such that \( \xi + \eta \in Q^V \) is also regular anti-dominant; note that we have \( \eta \in -Q^{V,+} \) since \( \eta \in Q^V \) is anti-dominant. Then it follows that \( \Psi(u) = Q^\eta u \) by Theorem 6.4. Now we define \( \Phi_J(v) := Q^{|\xi|^J} \cdot \Psi_J(u) \in QK^\text{poly}_T(G/P_J) \). We can easily verify that the element \( Q^{|\xi|^J} \cdot \Psi_J(u) \) does not depend on the choice of (a sufficiently regular anti-dominant coroot) \( \eta \in -Q^{V,+} \), and hence that \( \Phi_J \) is a well-defined surjective \( \mathbb{Z}[\Lambda] \)-module homomorphism from \( QK^\text{poly}_T(G/B) \) onto \( QK^\text{poly}_T(G/P_J) \). Indeed, if \( v = Q^\xi [O^w] \) with \( w \in W \) and \( \xi \in Q^{V,+} \), then \( \Phi_J(Q^\xi [O^w]) = Q^{\xi^J} [O^{|w|}] \).

Also, for \( v_1, v_2 \in QK^\text{poly}_T(G/B) \), we can take sufficiently regular anti-dominant coroots \( \eta_1, \eta_2 \in -Q^{V,+} \) such that \( Q^\eta_1 v_1, Q^\eta_2 v_2 \in \Psi(K^T(G_G)) \); hence there exist uniquely \( u_1, u_2 \in K^T(G_G) \) such that \( \Psi(u_1) = Q^\eta_1 v_1 \) and \( \Psi(u_2) = Q^\eta_2 v_2 \). Since \( \Psi = \Psi_0 \) is a \( \mathbb{Z}[\Lambda] \)-algebra homomorphism, we have \( Q^{\eta_1 + \eta_2} v_1 \cdot v_2 = Q^{|\eta_1|} v_1 \cdot Q^{|\eta_2|} v_2 = \Psi(u_1 \circ u_2) \) in \( QK^T(G/B)[[Q^{V,+}]] \), where \( u_1 \circ u_2 \in K^T(G_G) \). Therefore, we see that \( \Phi_J(v_1 \cdot v_2) = Q^{\eta_1 + \eta_2} \cdot \Phi_J(u_1 \circ u_2) = Q^{\eta_1 + \eta_2} \cdot \Phi_J(u_1) \cdot \Phi_J(v_1) \cdot \Phi_J(v_2) \) in \( QK^T(G/P_J)[[Q^{V,+}]] \) since \( \Psi_J \) is a \( \mathbb{Z}[\Lambda] \)-algebra homomorphism. This proves that the map \( \Phi_J \) is a \( \mathbb{Z}[\Lambda] \)-algebra homomorphism from \( QK^\text{poly}_T(G/B) \) to \( QK^\text{poly}_T(G/P_J) \), as desired.

Finally, since \( [O^s_i] = 1 - e^{|\omega_i|} [O_{G/B}(\omega_i)] \) in \( K^T(G/B) \) for all \( i \in I \) and \( [O^s_k] = 1 - e^{|\omega_k|} [O_{G/P}(\omega_k)] \) in \( K^T(G/P) \) for all \( k \in K = I \setminus J \), it follows that \( \Phi_J([O_{G/B}(\omega_k)]) = [O_{G/P_J}(\omega_k)] \), and hence that \( \Phi_J([O^w] \cdot [O_{G/B}(\omega_k)]) = \Phi_J([O^w]) \cdot \Phi_J([O_{G/B}(\omega_k)]) = [O^w] \cdot [O_{G/P_J}(\omega_k)] \) for all \( k \in K = I \setminus J \).

Thus we have given a proof of the following fact; cf. Theorem 6, due to Kato (\cite{Kat19}).

**Corollary 65.** Let \( J \) be an arbitrary subset of \( I \). Then, there exist a surjective \( \mathbb{Z}[\Lambda] \)-algebra homomorphism

\[
\Phi_J : QK^\text{poly}_T(G/B) \to QK^\text{poly}_T(G/P_J)
\]

such that \( \Phi_J(Q^\xi [O^w]) = Q^{\xi^J} [O^{|w|}] \) for \( w \in W \) and \( \xi \in Q^{V,+} \), where \( [\xi]^J := \sum_{k \in I \setminus J} c_k \alpha^\vee_k \) for \( \xi = \sum_{i \in I} c_i \alpha^\vee_i \in Q^{V,+} \), and \( [w] \in W^J \) denotes the minimal-length coset representative for the coset \( wW_J \) in \( W/W_J \). Also, for each \( k \in K = I \setminus J \), the following equality holds for all \( w \in W \):

\[
\Phi_J([O^w] \cdot [O_{G/B}(\omega_k)]) = [O^{|w|}] \cdot [O_{G/P_J}(\omega_k)].
\]

**APPENDIX B. WEIHONG XU’S CONJECTURE ABOUT A CANCELLATION-FREE PARABOLIC CHEVALLEY FORMULA IN TYPE A (WITH WEIHONG XU)**

In this appendix, we mention the relation between our results and a conjecture due to Weihong Xu, which is expected to be a cancellation-free Chevalley formula in type A for an arbitrary subset \( J \subset I \).

Let \( G \) be of type \( A_{n-1} \). Take \( 1 \leq k_1 < k_2 < \cdots < k_m \leq n - 1 \), and set \( J := I \setminus \{k_1, \ldots, k_m\} \). In this case, the partial flag manifold \( G/P_J \) is isomorphic to the \( m \)-step flag manifold \( \text{Fl}(k_1, \ldots, k_m; n) \), defined as:

\[
\text{Fl}(k_1, \ldots, k_m; n) := \left\{ (V_1, \ldots, V_m) \mid V_i, i = 1, \ldots, m, \text{ is a subspace of } \mathbb{C}^n \text{ such that } \dim V_i = k_i, \text{ and } V_1 \subset V_2 \subset \cdots \subset V_m \right\}.
\]

For a directed path

\[
p : w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} w_r,
\]
in $QB(W)$, we define $\ell(p) \geq 0$, $\text{end}(p) \in W$, and $\text{wt}(p) \in Q^{\vee,+}$ by

$$
\ell(p) := r, \\
\text{end}(p) := w_r, \\
\text{wt}(p) := \sum_{1 \leq k \leq r} \gamma_k' \\
\quad \text{if } w_{k-1} \rightarrow w_k \text{ is a quantum edge}.
$$

Also, for $1 \leq a \leq n-1$, the quantum $a$-Bruhat graph $QB_a(W)$ is defined to be the subgraph of $QB(W)$ having only those edges whose labels are of the form $(i,j)$ such that $i \leq a < j$. In addition, we define a total order $\triangleleft$ on $\Phi^+$ as follows: for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, we define $(i,j) \triangleleft (k,l)$ if $(j > l)$ or $(j = l$ and $i < k)$.

Xu formulated the following conjecture on a cancellation-free Chevalley formula for $QK(G/P_J)$, the non-equivariant quantum $K$-theory of $G/P_J$, and checked it for all partial flag manifolds with $n < 8$ and $m < 4$ using a computer program.

**Conjecture 66** (Weihong Xu). In $QK(G/P_J)$, for $w \in W^J$, the following cancellation-free formula holds:

$$
[O^{w_k}] \cdot [O^w] = \sum_p (-1)^{\ell(p)-1}Q^{[\text{wt}(p)]/j} [O^{[\text{end}(p)]]},
$$

where the sum on the right-hand side is over all non-empty paths $p$ in $QB_{k_1}(W)$ of the form

$$
p : w = w_0 \overset{(i_1,j_1)}{\rightarrow} w_1 \overset{(i_2,j_2)}{\rightarrow} \cdots \overset{(i_r,j_r)}{\rightarrow} w_r,
$$

such that

1. $(i_1,j_1) \triangleleft (i_2,j_2) \triangleleft \cdots \triangleleft (i_r,j_r)$,
2. for each $0 \leq t \leq r$ (regarding as $k_0 = 0$ and $k_{n+1} = n$) and an edge $(i,j)$
   - there does not exist any paths of the form $v \overset{(i,j)}{\rightarrow} w'$ in $QB_{k_t}(W)$ such that $k_t + 1 \leq j < j' \leq k_{t+1}$,
   - there does not exist any paths of the form $v \overset{(i',j)}{\rightarrow} w'$ in $QB_{k_t}(W)$ such that $k_t + 1 \leq i' < i \leq k_{t+1}$,
3. if there are two edges $(i,j) \overset{(i,j)}{\rightarrow}$ and $(i,j') \overset{(i,j')}{\rightarrow}$ in $p$ such that $(i,j) \triangleleft (i,j')$, then there exists $1 \leq t \leq n-1$ such that $j' \leq k_t < j$,
4. if there are two edges $(i,j) \overset{(i,j)}{\rightarrow}$ and $(i',j) \overset{(i',j)}{\rightarrow}$ in $p$ such that $(i,j) \triangleleft (i',j)$, then there exists $1 \leq t \leq n-1$ such that $i \leq k_t < i'$.

We now compare Xu’s conjectural formula in the case $m = 2$ with our cancellation-free Chevalley formula for two-step flag manifolds. For $w \in W^J$, we obtain the following formula in $QK(G/P_J)$ by applying the surjection $\Phi_J$ to equation (4) and specializing at $e_{11} = 1$ for $\mu \in \Lambda$:

$$
[O(-w_{k_1})] \cdot [O^w] = \sum_{A \in A(w,\Gamma(k_1))} (-1)^{|A|}Q^{[\text{down}(w,A)]/j} [O^{[\text{end}(w,A)]}],
$$

where the sum $\sum_p$ is over all (possibly empty) directed paths in $QB_{k_1}(W)$ satisfying (1) in Conjecture 66. By the formula $[O^{w_{k_1}}] = 1 - [O(-w_{k_1})]$ in $QK(G/P_J)$, we deduce that

$$
[O^{w_{k_1}}] \cdot [O^w] = \sum_p (-1)^{\ell(p)-1}Q^{[\text{wt}(p)]/j} [O^{[\text{end}(p)]}],
$$

for $w \in W^J$, and checked it for all partial flag manifolds with $n < 8$ and $m < 4$ using a computer program.
where the sum $\sum_p$ is over all non-empty directed paths in $QB_{k_1}(W)$ satisfying (1) in Conjecture 66. Here, we can construct certain involutions among non-empty directed paths satisfying (1) but not (2), and those satisfying (1) but not (3) or (4). Furthermore, we can verify that such involutions agree with those constructed in Section 4 by direct calculation. Hence we that equation (21) coincides with our cancellation-free Chevalley formula (Theorems 27, 29, and 31). We can also consider the product $[O^{s_{k_2}}] \cdot [O^w]$ by using the diagram automorphism $\omega$ and the result above for the product $[O^{s_{k_1}}] \cdot [O^w]$. In addition, we can verify that Xu’s conjectural formula (21) coincides with our Chevalley formula for Grassmannians of type $A$ (Theorem 15) in the same way as above.

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