Incorporation particle creation and annihilation into Bohm’s Pilot Wave model

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Abstract.
The purpose of this paper is to come up with a Pilot Wave model of quantum field theory that incorporates particle creation and annihilation without sacrificing determinism; this theory is subsequently coupled with gravity.

1. Introduction
The purpose of this paper is to redo Pilot Wave model for the case of creation and annihilation of particles. It is easy to see, however, that creation and annihilation of particles, being discrete can not be part of a differential equation that defines determinism. Durr and others have proposed that deterministic evolution of quantum states is being ”interrupted” with discrete jumps; the timing of the latter is determined by probabilistic laws (see [1]).

In this paper, however, I will propose a competing view which allows the determinism to hold at all times. I will accomplish this by claiming that there is no true creation/annihilation of particles. The number of particles is always fixed, but some of the particles hide from our view. In particular, there is extra compactified coordinate $x_4 \in \Gamma$, $x_4 + 2\pi = x_4$. Whenever $0 \leq x_4 < \pi$ the particle is visible and when $\pi \leq x_4 < 2\pi$ it is not. A particle undergoes continuous trajectory in all four coordinates, and whenever it happens to intersect either $x_4 = 0$ or $x_4 = \pi$ an appearance of creation/annihilation takes place.

Noteably, this means that the number of visible particles can be smaller than the actual number of particles, but it can not be larger. Therefore, we are working within realm of effective field theory with an upper bound on the total number of particles. This, in particular, means that lack of unitarity is one price we have to pay. For the purposes of this paper, we will pretend that we already have such effective theory, and our goal is to come up with Pilot Wave model within that theory.

Sections 2 and 3 will be devoted to do what we set out above in the context of flat space time. Then, in Section 4 we will couple our model to gravity. It has to be understood, however, that the prescription of adding gravity in Section 4 is meant to be generic for many different Pilot Wave models; therefore, it is independent of details of Chapters 2 and 3.

2. Creation and annihilation of particles
Let us now formulate more precisely the model of creation and annihilation of particles mentioned in the previous section. We will introduce a notation: $\vec{x}^{(4)} = (x_1, x_2, x_3, x_4)$ (where $x_4 + 2\pi = x_4$),
and \( \vec{x}^{(3)} = (x_1, x_2, x_3) \) (thus, \( \vec{x}^{(4)} = (\vec{x}^{(3)}, x_4) \)). In this notation, the phenomenon of particles hiding from our view can be expressed in definition of field operator, \( \phi(\vec{x}^{(4)}) = \phi(\vec{x}^{(3)}, x_4) \) as

\[
\phi(\vec{x}^{(3)}, x_4) = \phi(\vec{x}^{(3)}), \quad 0 \leq x_4 < \pi; \phi(\vec{x}^{(3)}, x_4) = 1, \quad \pi \leq x_4 < 2\pi
\]

(2.1)

This means that if we act with this operator on an empty state, we get

\[
\phi(\vec{x}^{(3)}, x_4)|0\rangle = |\phi(\vec{x}^{(3)})\rangle, \quad 0 \leq x_4 < \pi; \phi(\vec{x}^{(3)}, x_4)|0\rangle = |0\rangle, \quad \pi \leq x_4 < 2\pi
\]

(2.2)

Now, in terms of \( \vec{x}^{(4)} \), the left hand side of the above equations always amounts to ONE particle being present, regardless of the values of \( x_4 \). The right hand side, on the other hand, can give us either one particle state (if \( 0 \leq x_4 < \pi \)) or empty state (if \( \pi \leq x_4 < 2\pi \)). This essentially defines what we meant when we say particles hide.

For simplicity we will assume from now on that there is only one kind of particles, and it happens to be spin 0. Then the configuration space is \((\mathbb{R}^3 \times \Gamma)^N\), where \( \Gamma \) denotes the extra compactified coordinate and \( N \) is the total number of particles (or, in other words, the largest possible number of visible ones). Then we can define the probability amplitude in \((\mathbb{R}^3 \times \Gamma)^N\) as

\[
\psi(x_1^{(4)}, \ldots, x_N^{(4)}) = \langle 0|\phi(x_1^{(4)}) \cdots \phi(x_N^{(4)}) e^{iH(t-t_0)}|\psi(t_0)\rangle
\]

(2.3)

It should be emphasized that on the right hand side of the above equation, both \( |\psi(t)\rangle = e^{iH(t-t_0)}|\psi(t_0)\rangle \), as well as the Hamiltonian \( H \) are defined in terms of \( \vec{x}^{(3)} \), not \( \vec{x}^{(4)} \). The only part where \( \vec{x}^{(4)} \) comes in is the definition of field operators \( \vec{x}_k^{(4)} \). The reason we can mix \( \vec{x}^{(3)} \) with \( \vec{x}^{(4)} \) is that we have an expression of \( \phi(\vec{x}^{(4)}) \) in terms of \( \phi(\vec{x}^{(3)}) \) that we mentioned earlier.

Now, it is clear that the total number of visible particles can be smaller than the actual number of particles, but it can not be larger. Thus, we are dealing with effective field theory with upper bound on total number of particles. On the other hand, \( e^{iHt} \) can generate states with larger number of particles than allowed. Therefore, we will throw away these states, and, subsequently, include extra coefficient in transition from \( \psi \) to the probability density \( \rho \) in order to make sure that the latter is normalized to 1 among allowed states:

\[
\rho(x_1^{(4)}, \ldots, x_N^{(4)}) = \frac{\langle \psi(x_1^{(4)}, \ldots, x_N^{(4)}) \rangle^2}{\int d^4x_1 \cdots d^4x_N |\psi(x_1^{(4)}, \ldots, x_N^{(4)})|^2}
\]

(2.4)

It is important to point out that \( \rho \) is not a probability that we would postulate in Copenhagen interpretation. We are seeking a deterministic model. The sole source of probability is the ignorance of the observer regarding initial conditions.

As far as the deterministic law goes, this is something we are about to write (see the rest of the chapter). Thus, strictly speaking, the above probabilistic equation should be a consequence of the latter. The reason we have written probability first is that we are ”cheating”: we have first decided what probability we want to get and then we will write the deterministic law that accomplishes this.

However, it would be easier to cheat if we will assume that \( \rho \) has some physical meaning. Thus, we will view \( \rho \) as a physical field on configuration space, and our goal is to come up with dynamics such that the subjective probability happens to coincide with objective \( \rho \). This will allow us to use \( \rho \) as one of the parameters in our deterministic equation.

It should be noticed that, for any given \( \vec{x}^{(3)} \), \( \rho \) is constant throughout \( x_4 \in [0, \pi) \) and \( x_4 \in [\pi, 2\pi) \), respectively, while there is a discontinuity at \( x_4 = 0 \) and \( x_4 = \pi \). Thus, a particle should move towards the boundary of one semicircle if it thinks that eventually crossing that boundary might be favorable. Apart from that, once the particle has reached a boundary, its
trajectory has to remain continuous. Surprisingly enough, both of the above features are common place in electrostatics: step function charge distribution can produce continuous electric field, and, of course, the Coulomb’s law is non local. Therefore, we follow idea mentioned in [3] and introduces electrostatics into our theory, by defining electric field $\vec{E}$ and charge density $\sigma$ as

$$\vec{E} = \rho \vec{v}; \quad \sigma = -\frac{\partial \rho}{\partial t} \quad (2.5)$$

Of course, this is not literal electric field or charge: the theory holds both for charged and neutral fields. We are only using electrostatic notation for mathematical analogy. On any event, it is easy to see that if the ”Coulumb’s law” holds, then the continuity equation holds as well, which guarantees that $\rho$ will be the ”equilibrium probability density” of our deterministic Pilot Wave model.

We now use the method of images in order to compute $\vec{E}$ in compactified geometry. We consider an imaginary situation where coordinates $x_{4k}$ extend to infinity, and charge density is periodic in these coordinates,

$$\sigma(\vec{x}) = \sigma(\vec{x} + 2\pi \hat{x}_{4k}) \quad (2.6)$$

By translational symmetry, we notice that

$$\vec{E}(\vec{x}) = \vec{E}(\vec{x} + 2\pi \hat{x}_{4k}) \quad (2.7)$$

from which it is easy to see that the identical copy of $\vec{E}$ satisfies the same differential equation in the compactified geometry, without having any discontinuities at $x_{4k} = 2\pi n$. Thus, this is a solution we are seeking. Now, from the non-compact case, we can show that the area of the sphere is given by

$$A = \frac{2\pi^{2N}}{(2N-1)!} r^{4N-1} \quad (2.8)$$

This tells us that the electric field is

$$\vec{E}(\vec{x}) = \frac{(2N-1)!}{2\pi^{2N}} \sum_{a_1, \ldots, a_N} \int d^{4N} x' \sigma(\vec{x}') (\vec{x}' - \vec{x} + \vec{R}_{a_1, \ldots, a_N}) \frac{1}{|\vec{x}' - \vec{x} + \vec{R}_{a_1, \ldots, a_N}|^{4N}} \quad (2.9)$$

where

$$\vec{R}_{a_1, \ldots, a_N} = \sum_{i} 2\pi a_i \hat{x}_{4i} \quad (2.10)$$

is a displacement of an image charge. Now by substituting back

$$\vec{E} \rightarrow \rho^{(4)} \vec{v}; \quad \sigma \rightarrow \frac{\partial |\psi|^2}{\partial \tau} \quad (2.11)$$

we obtain a guidance equation

$$\vec{v}(\vec{x}) = \frac{(2N-1)!}{2\pi^{2N} \rho^{(4)}(\vec{x})} \sum_{a_1, \ldots, a_N} \int d^{4N} x' \rho^{(4)}(\vec{x}') (\vec{x}' - \vec{x} + \vec{R}_{a_1, \ldots, a_N}) \frac{1}{|\vec{x}' - \vec{x} + \vec{R}_{a_1, \ldots, a_N}|^{4N}} \quad (2.12)$$

1 While in [3] they mentioned electrostatic idea, the discussion seemed to be brief, so I am not sure whether or not they had the same motivation. On any event, as far as I can tell after brief look at the paper, they were not explicitly talking about particle creation/annihilation, the extra compactified coordinates or other things I am focusing on.
3. Decoherence

So far we have successfully came up with a model that reproduces desired probability density. However, there is one more ingredient to Pilot Wave model: effective collapse. It is well known that sometimes wave function in a configuration space splits into several non-overlapping branches that correspond to different outcomes of the measurement (such as, for example, arrow of measuring apparatus pointing in different directions). If a particle happened to be in one of these branches, it can not go to the other branch, since it is prevented from being in a region between branches due to low value of $|\psi|^2$ in that region.

This phenomenon, however, is strictly a consequence of locality. In particular, a particle can not enter the region between the branches because of the low value of $|\psi|^2$ in the immediate vicinity; it does not care that $|\psi|^2$ will become high again if it travels far enough to get into another branch. Our electrostatic model, however, is non-local. Thus, a possibility of unwanted travel between parallel universes is a natural concern that arises in the context of our model.

More specific source of the problem is that the electric charge density, $\sigma = -\partial \rho / \partial t$ is nearly zero between the branches, so nothing keeps the particle from that region. The only reason the probability of finding a particle between the branches is small is that $\vec{v}$ is inversely proportional to $\rho$, which simply means that once it entered the region in question it will fly away very fast. But nothing prevents it from flying away into a different branch! In fact, since the velocity of the particle is parallel to electric field, we know it would be the case if electric field lines of one branch intersect the other one (this issue was also mentioned at the end of chapter 3 in [3])!

At the same time, it should be understood that the non-locality in our model was introduced on purpose. After all, in light of the fact that $\rho$ is constant along each of the semicircles of $\Gamma$, a particle needs to have non-local ways of seeing whether or not it should move towards the edge of $\Gamma$ in order to cross on the other side. Thus, we would like to have it both ways: we want a particle to see the other side of $\Gamma$, but, at the same time, we do not want it to see the other branches of splitted wave function.

One idea that comes to mind is replacing Coulomb’s interaction with Yukawa, since this would give us the short-range interaction that we desire. This, however, will create non-zero divergence in the region of zero charge, which would violate continuity equation. Therefore, this proposal is not appropriate. The failure of Yukawa’s approach to address the issue also has deeper physical meaning. A sequence of short range non-localities will, inevitably, lead to long range one. Thus, we have a choice of either unwanted long-range non-locality (Coulomb’s case) or a failure of desired short-range ones (Yukawa’s case) in a form of violation of continuity equation.

In this work we will stick with the first option: long range Coulomb’s law and undesired long range non-locality. In order to avoid undesired effects from other branches, we will set up a dynamics in such a way that these branches disappear! In other words, yes, a particle can see other branches if they are out there. But, they are not! Of course, we now have to modify deterministic dynamics of a wave function in such a way that the latter, in fact, diminishes in the regions where we want it to. This can be formally done by adding to the standard quantum field theory Hamiltonian ($H_Q$) an extra one ($H_B$):

$$H = H_Q + H_B$$

(3.1)

where the $H_B$ is newly introduced influence of a beable particle on a wave function. It is designed in such a way that the wave function within any of the branches not occupied by a beable dies out, while the wave function within the one that is occupied is left unchanged.

Technically, $H_B$ diminishes the value of wave function in any given branch as long as a particle is not in that branch. Thus, if it is in the gap between the branches, then all of them die off; but, in light of the normalization, the uniform impact on $\psi$ has no bearing on $\rho$. Now, suppose there are only two branches: 1 and 2. If branch number 1 happened to be bigger than branch number 2, then the particle spends more time within the former than it does within the latter.
Thus, branch 1 dies out at a slower rate. In light of normalization of $\rho$ this means that branch 1 will actually increase, until its total probability becomes 1, while branch 2 will decrease to 0.

Now, the above paragraph, by itself, is not what we want: it might imply that if the total probability of branch 1 is 0.7 while the one of branch 2 is 0.3, then the branch 1 will end up a survivor with probability much higher than 0.7. This is where more familiar effective collapse phenomenon comes into picture to help us. If, at the first moment the particle happened to be within branch 2, then it would take some small portion of time for a particle to leave the branch 2. If the rate of disassociation of a branch is high enough, then during this small time portion the size of the branch 1 decreased from 0.7 to 0.07.

The unbiased picture starts when the particle is in the region between branches. However, since the value of $\rho$ is very small in that region, and velocity of a particle is inversely proportional to $\rho$, the particle spends very little time there. In particular, the travel from the boundary of branch 2 to the neutral point takes much shorter time than the time spent within branch 2. During this short travel time, the probabilities within either of the two branches only underwent very small change: the probability within branch 2 changed from 0 to $\delta$, while in the branch 1 it changed from 0.3 to 0.2 and within branch 1 it changed from 0.07 to 0.06. Then, the neutral competition starts between 0.06 and 0.2; and, predictably, 0.2 wins.

Thus, just like more standard models tell us, the deciding factor is which branch did a particle happen to occupy immediately after the split, not which branch is bigger. Just like in more standard Pilot Wave models, locality is a key element of the argument (here, as always, by locality we mean the one in multidimensional configuration space, not the one in ordinary space). This time, however, for the reasons stated earlier we could not use the locality of the influence of wave on a particle; instead, we have used the locality in the trajectory of the particle, itself (namely, the continuity of the latter). Thus, while the particle can non-locally see the other branch, it is not allowed to jump there. During the short time it is forced to take in order to leave the original branch, the other branch dies off which is the ultimate reason of collapse of a wave function.

Let us now go ahead and come up with a precise definition of $H_B$. In order to do this, we have to define the notion of being in the same branch as our particle. We will do that by postulating the field $\chi$ that obeys the following equation:

$$\frac{\partial \chi}{\partial t} = a\nabla^2 \chi + be^{k|x-\vec{x}|^2} - \frac{\lambda \chi(\vec{x})}{\rho(\vec{x})},$$

(3.2)

where $\lambda$ is assumed to be very small. From similarity with diffusion equation, $\rho \ll \lambda$ serves as a barrier of penetration of $\chi$, while in the $\rho \gg \lambda$ region $\chi$ fills out the whole connected region, regardless of specifics of $\rho$. Thus, if a particle is inside one of the branches, $\chi$ will be nearly constant inside that one branch and nearly zero everywhere else.

Now, the simplest thing we can now do is define $H_B$ as an imaginary expression $-i\delta/\chi$, where $\delta$ is some constant. From the similarity with diffusion equation, we know that $\chi > 0$. Thus, if $\chi$ is reasonably large, $e^{-iH_B t} \approx 1$. At the same time, if $\chi \ll \delta$, we get $e^{-iH_B t} \approx 0$ which, of course, would diminish the value of $\psi$ within the wrong branches.

We have to be a bit careful, since $H$ was defined on $(\mathbb{R}^3 \times \Gamma)^N$ we are working on. In order to draw a link between the two spaces, for every state $|s\rangle$ in our Fock space we will define a set of points $S(|s\rangle)$ in our configuration space that correspond to $|s\rangle$. It is easy to see from our construction that the above set is an uncountable union of semicircles in $\vec{x}_i$, each corresponding to different value of $\vec{x}^{(2)}$. We will now define $H_B$ on the Fock space as follows:

$$H_B|s\rangle = -\left( \int_{S(|s\rangle)} d^{4N} \vec{x} \frac{i\delta}{\chi(\vec{x})} \right) |s\rangle$$

(3.3)
In light of the structure of diffusion equation, it is possible that \( \chi \) might have different values on the same semicircle which might lead to some ambiguity. It is, however, reasonable to expect that \( \chi \) will either be consistently large or consistently small throughout each semicircle; but that should be investigated more closely in further research.

4. Adding gravity to Pilot Wave model

In order to avoid the issue of quantum gravity altogether, I propose to introduce to view gravity as a classical field, outside of the quantum fields (such as \( \phi \)) which are subjects to above model. Furthermore, in order to be able to use the results from previous sections, all of which are based on flat space, I will not view gravity as geometry either. Our space is fundamentally flat, and the presence of curvature is merely an illusion produced due to the coincidence that gravity happened to couple to all other fields (including itself) equally. This implies that we can unambiguously define Hilbert space, and also we can continue to use flat-space-based Coulomb’s Law results from previous sections.

The coupling of gravity to other fields is expressed in explicit dependence of Hamiltonian on the metric. Thus, the evolution in \( \vec{x}^{(3)} \) will change from

\[
\frac{d}{dt} |\psi(t)\rangle = e^{iH(t-t_0)}|\psi(t_0)\rangle
\]

(4.1)

where \( H(g_{\mu \nu}(\mathbb{R}^3)) \) tells us that \( H \) depends on the value of metric on \( t = \text{const} \) hypersurface \( \mathbb{R}^3 \); but, at the same time, it depends on all components of a metric, including the timelike ones. In light of flat geometry, after the above modification is made, the rest of the equations from previous section remain unchanged.

The hard part, however, is writing down how gravitational field is produced. In particular, since gravitational field is classical, we would like it to be an exact solution either to Einstein’s equation or to some modification of it. Now, from Noether’s Theorem, we know that the exact solution to Einstein’s equation only exists if the energy-momentum tensor is conserved. Now, we have three options of sources (all of which, as we will see, will have some conceptual issues to adress, and the lack of conservation of energy-momentum is one of the problems):

1) Bohmian particle. In order to avoid black hole configuration, we can simply replace delta function with Gaussian. This will produce a continuous energy-momentum distribution:

\[
T_{\mu \nu}(x^{(3)}) = \sqrt{\frac{\pi}{a}} N \sum_{k=1}^{N} \sum_{0 \leq \mu, \nu \leq 3} \frac{dx^\mu_B}{d\tau_k} \frac{dx^\nu_B}{d\tau_k} m_k f(x^{5k\pm4}) e^{-a|\vec{x}^{(3)} - \vec{r}_k(x^{(3)})|^2}
\]

(4.2)

where \( \vec{r}_k(x_B) = (x^{5k+1}_B, x^{5k+2}_B, x^{5k+3}_B) \), \( d\tau_k = \sum_{0 \leq \mu, \nu \leq 3} g_{\mu \nu} dx^\mu dx^\nu \),

(4.3)

\( m_k \) is a mass of particle number \( k \) (which is introduced for a more general case where more than one kind of field is considered), and

\[
f(x_4) = \frac{1}{2} (1 + \tan^{-1}(n \sin x_4))
\]

(4.4)

is a differential approximation to step function that is needed in order to assure that the hidden particles do not contribute to gravitational field. The differential approximation to step function is needed in place of actual step function in order to avoid singular effects on gravity produced by the latter. Finally, in the above expressions the Einstein’s convention was ignored in order to avoid confusion between \( N \)-term sums and 4-term ones.
2) The source of energy momentum tensor is a wave function. Thus, the energy momentum tensor is given by

\[ T_{\mu\nu}(\vec{x}(3)) = \sum_{k=1}^{N} \frac{\partial \psi}{\partial x^{5k+\mu}} \frac{\partial \psi}{\partial x^{5k+\nu}} \]  

(4.5)

Furthermore, here are two options of defining \( \psi \):

a) By using the evolution based on \( H_Q \)

b) By using the evolution based on \( H_Q + H_B \)

It is easy to see that option "2a" predicts gravitational interaction between different branches of wave function, which means that the gravitational field produced by objects in our universe will be lost among gravitational fields produced in all of the other parallel ones. On the other hand, in cases "1" and "2b", the influence of acceleration of particle on a wave leads to energy momentum non-conservation. By the way, even in 2a, non-conservation occurs in a much milder form, namely it is a consequence of non-unitarity of the theory (due to the upper bound on particle number). Thus, in order to avoid additional problem, namely the interaction between universes, we will stick to options 1 and 2b, even though it means we have to face non-conservation of energy in a more severe form. (For this reason, it is important to be able to modify gravity in such a way that it has exact solutions for non-conserved sources. We can attempt to find a solution by evolving our metric along timelike curve. However, if we select timelike geodesics, they might intersect which would lead to possible contradiction between the results produced through the evolution along each of the two geodesics unless the sources meet some condition. That condition would be of a form of "curl equals zero" which is just as bad as energy conservation that we were trying to avoid.

Therefore, we instead introduce a flat set of coordinates that are guaranteed not to intersect, while gravity is viewed as merely a field in these coordinates (which, by the way, gives us a good excuse not to generalize our electrostatics from previous chapter to curved space). Since Einstein's equation for arbitrary metric is too complicated to analyze, we will consider a linear approximation. Within this approximation we will first convince ourselves that the equation is not invertible, and then find a way to modify it to make it such. The linear approximation to Einstein's equation is

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{2} \partial^\alpha \partial_\alpha h_{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\alpha h^\alpha_{\nu} + \frac{1}{2} \partial_\nu \partial_\alpha h^\alpha_{\mu} - \frac{1}{2} \eta_{\mu\nu} \partial_\alpha h^{\alpha\beta} - \frac{1}{2} \partial_\mu \partial_\nu h + \frac{1}{2} \eta_{\mu\nu} \partial^\alpha \partial_\alpha h \]  

(4.6)

We will now compute the above expression for all choices of \((\mu\nu)\). However, we will leave out all terms except for the second time derivatives. We will denote the expressions not containing second time derivatives by dots. Thus, we get

\[ R_{00} - \frac{1}{2} R g_{00} = \cdots ; \quad R_{11} - \frac{1}{2} R g_{11} = \frac{1}{2} (\partial_0 \partial_0 h^{22} + \partial_0 \partial_0 h^{33}) + \cdots \]

\[ R_{10} - \frac{1}{2} R g_{10} = \cdots ; \quad R_{11} - \frac{1}{2} R g_{12} = -\frac{1}{2} \partial_0 \partial_0 h^{12} + \cdots \]  

(4.7)

The rest of the components are obvious from the permutting indices. Now, in order to find a dynamics, we have to solve the above equation for \( \partial_0 \partial_0 h^{\mu\nu} \). Thus, in order to see whether or not the equation is solvable, we have to compute a determinant of corresponding 10 \( \times \) 10 tensor. In order to write that tensor in the most transparent form, we will define coordinates of vectors it acts upon as follows:

\[ (v_1, \cdots, v_{10}) = \partial_0 \partial_0 (h^{00}, h^{11}, h^{22}, h^{33}, h^{01}, h^{02}, h^{03}, h^{12}, h^{13}, h^{23}) \]  

(4.8)
In these coordinates, our equation takes the form \( A\vec{v} = \vec{w} \), where
\[
A = \frac{1}{2} \text{diag} (0_{1\times1}, B_{3\times3}, 0_{3\times3}, -I_{3\times3}) \tag{4.9}
\]
and
\[
B = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix} \tag{4.10}
\]
As a result of \( 0_{1\times1} \) and \( 0_{3\times3} \) components of a diagonal, the determinant is zero. This confirms our earlier prediction that Einstein’s equation, as it stands, is not invertible. However, if we add a diagonal \( \epsilon I_{10\times10} \), then the \( 0\)-s will become \( \epsilon I\)-s (where \( I \) is an identity matrix). This, together with non-zero determinant of \( B \) will prove the non-zero determinant of \( A + \epsilon I \).

Now, adding this extra term is equivalent to adding \( \epsilon \partial_0 \partial_0 h^{\mu\nu} \) to our equations. Since we would like to have a covariant theory, we instead add a term \( \epsilon \partial_\alpha \partial_\alpha h^{\mu\nu} \), which for a general metric generalizes to \( \epsilon \partial_\alpha \partial_\alpha g^{\mu\nu} \):
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \epsilon \partial_\alpha \partial_\alpha g_{\mu\nu} = T_{\mu\nu} \tag{4.11}
\]
which corresponds to a Lagrangian
\[
S = \int d^4x \sqrt{-g} \left( R + kT + \frac{\epsilon}{2} \partial_\alpha g^{\mu\nu} \partial_\alpha g_{\mu\nu} \right) \tag{4.12}
\]
where \( \epsilon \)-term likewise breaks the general relativistic covariance.

It is important to notice that, in light of \( \partial_\alpha \) being used in place of \( \nabla_\alpha \), the \( \epsilon \)-term in the above equation is not covariant. In fact, in light of the fact that \( \nabla_\alpha g_{\mu\nu} = 0 \), we couldn’t have possibly made it covariant even if we wanted to. However, it can be argued based on Noether’s theorem that violation of covariance is necessary if we desire to violate a conservation law.

At the same time, however, not all violations of covariance would have served our purpose. For example, adding mass term would have had no impact on above matrix and, therefore, determinant would have continued to be zero. This can be explained by saying that, while mass term does violate the covariance we are familiar with, the Lagrangian might still have been symmetric with respect to something else that we don’t easily recognize. This would also lead to a conservation law that we would not recognize either, but the latter would have been just as restrictive as the energy momentum one mass term would have gotten rid of. On the other hand, the kinetic term above does not produce any unrecognized symmetries/conservation laws, since the determinant of coefficient is non-zero.

It is important to admit, however, that in the above analysis we were working exclusively in the linear approximation. For the general metric, it is quite possible for determinant to accidentally become zero. In fact, if the determinant changes sign between two points, then its continuity implies a presence of hypersurface in which it is 0. The surface, itself, is infinitely smaller than any three dimensional region and, therefore, insignificant. We do, however, have to avoid the impact of that surface onto the future region.

In order to do this, we have to tame the behavior of time derivatives somehow. For now, we will settle on violating relativity (our excuse is that we have already done that, anyway, with Pilot Wave model), although for a future it might be worthwhile trying to find more covariant ways of doing that. As we said before, we can write the second time derivatives in a form of a single vector \( \vec{v} \in \mathbb{R}^{10} \), which allows us to write the dynamics as \( \dot{\vec{v}} = A^{-1} \vec{w} \), for the tensor \( A \) specified earlier. We will now replace it with
\[
\vec{v} = \left( \frac{\tan^{-1} (\eta \det A^{-1})}{\eta \det A^{-1}} \right)^{1/10} A^{-1} \vec{w} \tag{4.13}
\]
where $\eta$ is a very small number. It is easy to see that the determinant of modified version of $A^{-1}$ coincides with the determinant of $A^{-1}$ itself as long as the latter does not blow up. When it does, however, the former approaches $\pi/2\eta$.

To satisfy more mathematically minded people, it might be an interesting project for future to modify principle of least action in such a way that the above modification of dynamics is dictated to us by substituting un-modified Lagrangian into modified Lagrange's equation (that would probably require us to single out the preferred variables in order for modification to be applied to $\partial_t \partial_t g_{\mu\nu}$ instead of, say, $(\partial_t \partial_t g_{\mu\nu})^2$, since the covariant problem of finding extreme point of an action does not give us the dynamics we want). Either that, or else we can search for the modifications of Lagrangian itself in order for principle of least action to dictate us the above modified dynamics. I will leave both of these questions for the future research.

As far as the above equation of motion, itself, is concerned, we can do even better by removing $1/10$ from the power of the above coefficient. In this case, near the singular region the second time derivatives will approach zero, which is just the opposite of the issue we were trying to avoid! We have to be careful if we want to do that, because this might potentially freeze the singularity once it is reached. What comes to our rescue is that the first time derivatives continue to be non-zero. The space variation of the latter will eventually allow us to get out of the singular region. But, of course, this needs to be investigated more closely.

While modification of gravity might look artificial, it is important to point out that it has some benefits beyond the Pilot Wave model. For example, it might adress the problem with singularities that arise in theories involving massive gravitons. As can be seen from the equation 2.151 of [4], the singularities are linked to the equation

$$-3m_g^2 T = \frac{k}{2} T$$

where $m_g$ is a graviton mass. Now, by looking at the above equation, one can immediately see that it does not involve second time derivatives! In our language this means that the above equation puts a restrictions on initial conditions on $t =$ const hypersurface, and if these initial conditions are not met the equation for second time derivatives has no solution. In other words, the equation is not invertible.

Thus, by making it invertible through the kinetic $\epsilon$-term, we automatically get rid of the above static equation, and thus allow for non-singular theories involving massive gravitons. In fact, if the mass of the graviton is singificantly smaller than some function of $\epsilon$ then the the existence of solution around $m_g = 0$, toghether with continuity in $m_g$, implies a nice behavior of the solution. If, on the other hand, $m_g$ is much larger than the above function of $\epsilon$, then the $\epsilon \approx 0$ approximation will imply nearly-sigular behavior. This means that there might be some correlation between $\epsilon$ and the upper bound on graviton mass.

At the same time, we can use Pilot Wave model at hand to obtain a lower bound on $\epsilon$ (after all, if the one didn’t exist, nothing would have stopped us from setting $\epsilon$ to 0). This would then imply a lower bound on the upper bound on $m_g$. It would be interesting to compare this to the graviton masses currently proposed by other arguments.

5. Conclusions

In this paper we have accomplished two things. First of all, we have introduced creation and annihilation of particles into Pilot Wave model. We have done that through introducing extra compactified dimension and postulating that the particles on the wrong side of the semicircle are invisible. This allowed us to view a discrete creation and annihilation events as merely manifestations of continuous trajectories in our extended space. We then introduced Pilot Wave model in order to describe the latter deterministically.
Then, in the second part of the paper, we have added gravity to the theory by associating energy momentum tensor with our beable particles (which is smeared out in space in order to avoid black hole configuration). The latter serves as a source of classical gravitational field. However, as a result of acceleration of beable particles, energy-momentum tensor is not conserved, which means that Einstein's equation, in its original form, has no exact solution.

Thus, we were forced to add extra kinetic term to the gravitational action with very small coefficient. That coefficient breaks general relativistic covariance and, at the same time, makes equation invertible and, therefore, guarantees the existence of a solution within a region bounded by some hypersurface. Nevertheless, we still had to introduce further modification in order to tame the behavior of a solution in vicinity of that surface, which ultimately allows it to extend beyond that surface. Doing so required us to violate relativity.

Finally, we have pointed out that the modification of gravity had a side benefit: massive graviton can be introduced without resulting singularities! More precisely, the standard prediction that the graviton mass should be exactly zero (see [4]) is transformed into a conclusion that the mass is much smaller than some function of $\epsilon$, where $\epsilon$ is a coefficient of our newly introduced kinetic term.

While massive gravitons are not needed for our Pilot Wave model, I believe that the prediction of the upper bound to their mass is important since it refutes a potential aesthetic objection that I went through the hoops of modifying gravity only for one purpose (that is, Pilot Wave model) which might, in some people's eyes, indicate dishonesty of the theory. For that same reason, it is important to explore other potential uses of modification of gravity.

One thing that naturally comes to mind is dark matter problem. There have been a lot of claims that dark matter does not exist and the phenomena that are typically interpreted as the effects of dark matter are, in fact, due to the modifications of gravity that are not felt on our everyday scales but become increasingly important cosmologically. It might be interesting to replace the modifications of gravity that has been proposed before with the one proposed in the current work and compare the differences in cosmological predictions.

Interestingly enough, while the above cosmological problem does not have a direct link to Pilot Wave model, it will have indirect one. In particular, we will use Pilot Wave model in order to judge the value of the constant $\epsilon$ to accomodate it. We will then use this same $\epsilon$ for the dark matter problem! This means that Pilot Wave model will begin to have predictive power, which will hopefully bring it into a mainstream interest.

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