Cup Products and Pairings for Abelian Varieties

Klaus Loerke

2009

Abstract

Let $A_K$ be an abelian variety with semistable reduction over a strictly henselian field of positive characteristic with perfect residue class field. We show that there is a close connection between the pairings of Grothendieck, Bester/Bertapelle and Shafarevic. In particular, we show that the pairing of Bester/Bertapelle can be used to describe the $p$-part of Grothendieck’s pairing in the semistable reduction case, thus proving a conjecture of Bertapelle, [Be03].

1 Introduction

Let $R$ be a discrete valuation ring with field of fractions $K$ and residue class field $k$. We consider an abelian variety $A_K$ over $K$ with dual abelian variety $A'_K$ and Néron models $A$ and $A'$, respectively. In [SGA7], exp. IX, 1.2.1, Grothendieck constructed a canonical pairing $\phi \times \phi' \rightarrow \mathbb{Q}/\mathbb{Z}$ of groups of components $\phi$ of $A$ and $\phi'$ of $A'$. Grothendieck’s pairing represents the obstruction to extend the Poincaré bundle on $A_K \times_K A'_K$ to a Poincaré bundle on $A \times_R A'$. It is conjectured that this pairing is perfect. Indeed, it can be shown that this pairing is perfect in the following cases: $A_K$ has semistable reduction, [We97], $R$ is of equal characteristic 0, [SGA7], $R$ has mixed characteristic and a perfect residue class field, [Beg80], the residue class field $k$ is finite, [McC86].

If $k$ is not perfect, counterexamples can be found, [BB02], corollary 2.5. If $R$ is of equal characteristic $p \neq 0$ with infinite, perfect residue class field, there are only partial results: If $A_K$ is of potentially multiplicative reduction, then Grothendieck’s pairing is perfect, [Bo97].

In this article, we focus on the open case of discrete valuations rings of equal characteristic $p \neq 0$ with perfect residue class field $k$. Since the formation of Néron models is compatible with unramified base change, without loss of generality, we may assume that $R$ is complete and strictly henselian.

In this setting, we enlighten the relation of Grothendieck’s pairing to the pairings of Bester and Shafarevic: The latter pairing

$$SP : H^1(K, A_K)_{(p)} \times \pi_1(A')_{(p)} \rightarrow \mathbb{Q}/\mathbb{Z}$$

is constructed in [Sh62] for the prime-to-$p$-parts. For the prime-to-$p$-parts, the relation of the pairings of Grothendieck and Shafarevic is known and can be summarised in the following exact sequence of pairings:

$$0 \rightarrow \left( \text{Grothendieck’s pairing} \right) \rightarrow \left( \text{cup product for } A_{K,n} \text{ and } A'_{K,n} \right) \rightarrow \left( \text{Shafarevic’s pairing} \right) \rightarrow 0.$$
More precisely, there is the following commutative diagram with exact lines. The inductive limits vary over all $n$ which are prime to $p$:

$$
\begin{array}{cccccc}
0 & \rightarrow & (\phi_A)(p) & \rightarrow & \lim\limits_{\rightarrow} H^1(K, A_{K,n}) & \rightarrow & H^1(K, A_K)(p) & \rightarrow & 0 \\
\downarrow^GP & & \downarrow^U & & \downarrow^SP & & \\
0 & \rightarrow & (\phi_{A'})(p) & \rightarrow & \lim\limits_{\rightarrow} H^0(K, A'_{K,n})^* & \rightarrow & \pi_1(A')^*(p) & \rightarrow & 0.
\end{array}
$$

The vertical morphisms are induced by Grothendieck’s pairing, by the perfect cup product ([SGA5], I, 5.1.8) and by Shafarevic’s pairing (cf. [Be03], 4.1, prop. 2 and theorem 2 together with remark 1 on p. 152).

The $p$-parts of the pairings of Grothendieck and Shafarevic are rather mysterious: In order to extend Shafarevic’s pairing to the $p$-parts of $H^1(K, A_K)$ and $\pi_1(A')$ in the case of good reduction, Bester constructed a pairing

$$
H^2_k(R, A_{p^n}) \times \mathscr{F}(A'_{p^n}) \rightarrow \mathbb{Q}/\mathbb{Z}
$$

of the kernels of $p^n$-multiplication of the Néron models of an abelian variety of good reduction and its dual, cf. [Best78]. Here, $H^*_k$ denotes the functor of local cohomology and $\mathscr{F}$ denotes a rather complicated functor constructed by Bester.

In [Be03], Bertapelle generalised Bester’s pairing to the kernels of $p^n$-multiplication of abelian varieties with semistable reduction. Using this, she extended Shafarevic’s pairing to the $p$-parts in the case of semistable reduction.

Surprisingly, if we replace the cup product in the above diagram by Bester’s pairing, there is a similar exact sequence of the $p$-parts of these pairings: (we still have to assume that $A_K$ has semistable reduction)

$$
0 \rightarrow \left( \text{pairing of } p\text{-parts of groups of components} \right) \rightarrow \left( \text{Bester’s pairing} \right) \rightarrow \left( \text{Shafarevic’s pairing} \right) \rightarrow 0
$$

(1)

This sequence, of course, has to be read like the above sequence of pairings.

The first pairing of this sequence is a perfect pairing of the $p$-parts of groups of components of $A$ and $A'$. Bertapelle conjectured that this pairing coincides with Grothendieck’s pairing. We will prove this conjecture.

In a final step, we show that the pairing of Shafarevic can be seen as a higher-dimensional analogue of the pairing of Grothendieck. To make this explicit, we generalise Bester’s functor $\mathscr{F}$ to some kind of “derived functor”. In this view, the pairings of Grothendieck, Bester and Shafarevic induce a duality of the homotopy and local cohomology of $A$.

In order to prove the conjecture of A. Bertapelle, we need to gain a deep understanding of Bester’s pairing. Bester uses cup products in the derived category to construct his pairing. Since there are no references to (derived) cup products on arbitrary sites in the literature, we start by developing such a theory in greater generality as for example in [EC], V, §1, prop. 1.16.

In the third section we use the theory of derived cup products to analyse Bester’s pairing and compare it to the pairing of Grothendieck. For this result we use the technique of rigid uniformisation of abelian varieties: The duality of $A_K$ and $A'_K$ induces a monodromy pairing, which – on the one hand – is compatible with the pairing of Grothendieck and – on the other hand – allows for a connection to cup products and thus, for a connection to the pairing of Bester.
2 Cup Products in the Derived Category

2.1 ∪-Functors

Let $S$ be a category of sheaves, e.g. the category of abelian sheaves on a Grothendieck topology. Under some hypothesis (cf. [EC], V, §1, prop 1.16) one can extend a functorial pairing of left exact functors $f^*A \times g^*B \rightarrow h_*(A \otimes B)$ to a cup product

$$R^rf^*A \times R^sg^*B \rightarrow R^{r+s}h_*(A \otimes B)$$

of the derived functors for every $r, s \in \mathbb{N}$ which is compatible with the connecting morphisms $d$ and such that for $r = s = 0$ this is the pairing we started with. It is known how to construct such cup products in the case of Čech cohomology using Čech cocycles ([EC], V, §1, remark 1.19) and in the case of group cohomology ([Br], V.3) using the Bar resolution. To clarify these various cup products and their relations, we will develop a theory of derived cup products in arbitrary abelian categories $\mathcal{A}$.

When working with derived categories, we will mainly work in the derived category of complexes that are bounded from below, $D^+\mathcal{A}$, that is, we require that for every complex $X \in D^+\mathcal{A}$ there exists $n \in \mathbb{Z}$ such that for all $i < n$ we have $H^i(X) = 0$.

In general, a left exact functor $f_* : \mathcal{A} \rightarrow \mathcal{B}$ does not extend without more ado to the derived category. To remedy this defect, one constructs the derived functor $Rf_*$. This functor turns distinguished triangles into distinguished triangles. Moreover, it is equipped with a functorial, universal morphism $Q \circ f_* \rightarrow Rf_* \circ Q$, for the canonical quotient functor $Q : K^+\mathcal{A} \rightarrow D^+\mathcal{A}$, from the category of complexes up to homotopy, which are bounded from below, to the derived category.

Composition of functors has a simple description in the setting of derived categories. If $f_*$ and $h_*$ are functors whose composition exists and if $f_*$ turns injective objects into $h_*$-acyclic ones, the following fundamental equation holds:

$$R(h_*f_*) = Rh_*Rf_*.$$

(2)

Cup products involve tensor products; thus, we require that our abelian categories are equipped with a tensor product with the usual properties. More precisely, we require that $\mathcal{A}$ and $\mathcal{B}$ are symmetric monoidal categories with respect to a right-exact tensor product, e.g. [MacL], VII, sections 1 and 7.

Since the main application of cup products is sheaf cohomology, we may not use projective resolutions. However, it is sufficient to consider flat objects and flat resolutions to construct the derived tensor product: Let us call an object $P \in \mathcal{A}$ flat, if $P$ is acyclic for the tensor product. In the category of abelian groups or abelian sheaves this is the usual notion of flatness.

It is clear that the derived category is again a symmetric monoidal category with respect to the derived tensor product. It makes things easier, if we require that the category $\mathcal{A}$ has finite Tor-dimension (or finite weak dimension, as in [CE]), i.e. we require that there exists $n \in \mathbb{N}$ such that every object has a flat resolution of length $\leq n$. This ensures that the derived tensor product does not leave the derived category of complexes that are bounded from below, $D^+\mathcal{A}$.

For details, see proposition 2.2.8.

Crucial to our construction is the notion of adjoint functors. Let us recall that a pair of functors $(f^*, f_*)$ is called adjoint, if there exists a functorial
isomorphism
\[ \text{Hom}(f^* A, B) \cong \text{Hom}(A, f_* B) \]
for every pair of objects, \(A, B\). The property of adjointness has many useful consequences: The right adjoint functor \(f_*\) preserves all limits and the left adjoint functor \(f^*\) preserves all colimits (\[\text{MacL}\], V, 5, theorem 1); in particular, \(f_*\) is left exact and \(f^*\) is right exact. If \(f^*\) is even exact, then \(f_*\) preserves injective objects. Moreover, every pair of adjoint functors induces functorial adjunction morphisms \(f^* f_* \rightarrow \text{id}\) and \(\text{id} \rightarrow f_* f^*\).

When we speak of a functor \(f: \mathcal{A} \rightarrow \mathcal{B}\) of monoidal categories, we do not require that \(f\) is compatible with the tensor product.

2.1.1 Definition. (Cup-Pairing of Functors) Let \(f, g\) and \(h\) be three functors of monoidal categories \(\mathcal{A} \rightarrow \mathcal{B}\).

(i) A cup-pairing of functors \(f \cup g \rightarrow h\) is a functorial morphism
\[ fA \otimes gB \rightarrow h(A \otimes B) \]
for every pair of objects \(A, B \in \mathcal{A}\).

(ii) A cap-pairing of functors \(h \rightarrow f \cap g\) is a functorial morphism
\[ h(A \otimes B) \rightarrow fA \otimes gB \]
for every pair of objects \(A, B \in \mathcal{A}\).

If \(f \cup g \rightarrow h\) is a cup-pairing, we call the corresponding functorial morphism \(fA \otimes gB \rightarrow h(A \otimes B)\) the cup-pairing morphism. Accordingly, we call the corresponding morphism of a cap-pairing the cap-pairing morphism. The central lemma in our further argumentation is the following:

2.1.2 Lemma. Let \(f_*, g_*\) and \(h_*\) be right adjoint to \(f^*, g^*\) and \(h^*\). Then every cup-pairing of functors \(f_* \cup g_* \rightarrow h_*\) induces a cap-pairing of functors \(h^* \rightarrow f^* \cap g^*\) and vice versa. These constructions are inverse to each other.

Proof. The adjunctions \(\text{id} \rightarrow f_* f^*\) and \(\text{id} \rightarrow g_* g^*\) induce together with the pairing morphism the morphism \(A \otimes B \rightarrow f_* f^* A \otimes g_* g^* B \rightarrow h_*(f^* A \otimes g^* B)\). Since \(h^*\) and \(h_*\) are adjoint, this composition induces the cap-pairing morphism \(h^* (A \otimes B) \rightarrow f^* A \otimes g^* B\). By “adjoint” arguments, every cap-pairing morphism induces a cup-pairing morphism of functors. A lengthy calculation shows that these constructions are inverse to each other; it uses arguments as in \[\text{MacL}\], IV, 1, proof of theorem 2, part (v). \(\square\)

With these preparations, we can isolate a class of functors that admit cup-products:

2.1.3 Definition. (\(\cup\)-functor) Let \(f_*: \mathcal{A} \rightarrow \mathcal{B}\) be a functor of monoidal abelian categories. It is called a pre-\(\cup\)-functor, if
\begin{itemize}
  \item \((\cup_1)\) \(f_*\) has a left adjoint \(f^*\) with the following properties:
  \item \((\cup_2)\) \(f^*\) is exact and preserves flat objects.
\end{itemize}

\(^1\)Although the cup-pairing or the cup-product is related to the cup-product of algebraic topology, we have chosen the notion of a “cap-pairing” for duality reasons. It has nothing to do with the cap-product of algebraic topology.
A pre-$\bigcup$-functor $f_*$ together with a cup-pairing $f_* \bigcup f_* \rightarrow f_*$ is called a $\bigcup$-functor.

In the case of a $\bigcup$-functor, we will omit the pairing in the notation. In our applications, it will always be clear, which pairing the functor is equipped with.

**Notation.** Whenever $f_*$ is a pre-$\bigcup$-functor, let us denote by $f^*$ its left adjoint.

The $\bigcup$-functors we have defined enjoy the following properties: Due to ($\bigcup_1$), the functor $f_*$ is left exact; due to ($\bigcup_2$), it preserves injective objects. Hence, compositions of $\bigcup$-functors can be described by the Grothendieck spectral sequence. Moreover, since $f^*$ is exact, it coincides with its derived functor $L^f_*$, i.e., by $X \mapsto f^* X$, the functor $f^*$ defines a well-defined functor of derived categories.

Let us continue with some general adjointness properties in the derived category:

**2.1.4 Proposition.** Let $f_*$ be a pre-$\bigcup$-functor with left adjoint $f^*$. Then, there are the following canonical isomorphisms:

(i) $R\text{Hom}(f^* X, Y) \xrightarrow{\sim} R\text{Hom}(X, Rf_* Y)$. 

(ii) $\text{Hom}(f^* X, Y) \xrightarrow{\sim} \text{Hom}(X, Rf_* Y)$.

**Proof.** Since $f^*$ and $f_*$ are adjoint and $f_*$ preserves injective objects, the first assertion is (2). Moreover, we have $\text{Hom}(X, Y) = H^0(\text{RHom}(X, Y))$, hence (ii). \hfill $\square$

We need the following simple lemma to prove our main theorem:

**2.1.5 Lemma.** Let $f^*$, $g^*$ and $h^*$ be exact functors $\mathcal{A} \rightarrow \mathcal{B}$ with the property that $g^*$ respects flat objects. Then every cap-pairing $h^* \rightarrow f^* \cap g^*$ extends to a cap-pairing in the derived category, that is, the cap-pairing induces a functorial morphism

$$h^*(X \otimes^L Y) \rightarrow f^* X \otimes^L g^* Y$$

for complexes $X, Y \in D^+ \mathcal{A}$.

**Proof.** Let $P \rightarrow Y$ be a quasi-isomorphism with a complex $P$ consisting of flat objects. Since $g^*$ respects flat objects, we have morphisms $h^*(X \otimes^L Y) = h^*(X \otimes P) \rightarrow f^* X \otimes g^* P = f^* X \otimes^L g^* Y$. \hfill $\square$

**2.1.6 Theorem. (Cup product)** Let $f_* \cup g_* \rightarrow h_*$ be a cup-pairing of pre-$\bigcup$-functors $\mathcal{A} \rightarrow \mathcal{B}$.

(i) There exists a cup-pairing $Rf_* \cup Rg_* \rightarrow Rh_*$, i.e. there exists a functorial morphism, the derived cup product,

$$Rf_* X \otimes^L Rg_* Y \rightarrow Rh_*(X \otimes^L Y),$$

for complexes $X$ and $Y$ in the derived category.

(ii) This morphism induces the cup product

$$\bigcup: R^r f_* A \otimes R^s g_* B \rightarrow R^{r+s} h_*(A \otimes B)$$

for objects $A, B \in \mathcal{A}$ and every pair of integers $r, s \in \mathbb{N}$, with the following properties:
(iii) For \( r = s = 0 \), this is the pairing of functors we started with and it is compatible with exact sequences in the following sense:

(iv) Let \( 0 \to A' \to A \to A'' \to 0 \) be an exact sequence in \( \mathcal{A} \) let \( B \in \mathcal{A} \) be another object. There is a commutative diagram

\[
\begin{array}{ccc}
R^r f_* A'' \otimes R^s g_* B & \longrightarrow & R^{r+s} h_*(A'' \otimes B) \\
\downarrow & & \downarrow \\
R^{r+1} f_* A' \otimes R^s g_* B & \longrightarrow & R^{r+s+1} h_*(A' \otimes B)
\end{array}
\]

and a diagram, which commutes up to sign, with interchanged roles of the exact sequence and \( B \).

By the properties (iii) and (iv), the cup product is uniquely determined.

Proof. The cup-pairing of \( f_* \cup g_* \to h_* \) induces a cap-pairing \( h^* \to f^* \cap g^* \) by lemma \([2.1.2]\). Using lemma \([2.1.3]\), this cap-pairing extends to the derived category. By lemma \([2.1.2]\), the derived cup-pairing induces the cup-pairing \( Rf_* \cup Rg_* \to Rh_* \). This is the derived cup product. Let us now take the \((r+s)\)th cohomology of the derived cup product. Together with the canonical morphism \( A \otimes B \to A \otimes B \) this induces

\[
H^{r+s}(Rf_* A \otimes^L Rg_* B) \longrightarrow R^{r+s} h_*(A \otimes^L B) \longrightarrow R^{r+s} h_*(A \otimes B).
\]

There exists a canonical morphism \( \alpha, [CE], IV, \S 6, \) proposition 6.1:

\[
\alpha : R^r f_* A \otimes R^s g_* B \longrightarrow H^{r+s}(Rf_* A \otimes^L Rg_* B).
\]

The composition of those morphisms is the cup product. This is the construction of the cup product. Now, let us prove its properties: For \( r = s = 0 \), we show that this morphism coincides with the cup-pairing we started with: Let \( A, B \in \mathcal{A} \) and consider the following commutative diagram in the derived category

\[
\begin{array}{ccc}
h^*(f_* A \otimes g_* B) & \longrightarrow & f^* f_* A \otimes^L g^* g_* B \longrightarrow A \otimes B \\
\uparrow & & \uparrow \\
h^*(f_* A \otimes^L g_* B) & \longrightarrow & f^* f_* A \otimes^L g^* g_* B \longrightarrow A \otimes^L B \\
\downarrow & & \downarrow \\
h^*(Rf_* A \otimes^L Rg_* B) & \longrightarrow & f^* Rf_* A \otimes^L g^* Rg_* B \longrightarrow A \otimes^L B.
\end{array}
\]

It is induced by the canonical morphisms \( \otimes^L \to \otimes \) and \( f_* \to Rf_* \) (and for \( g_* \) accordingly). Again, we use the fact that the exact functors \( f^* \), \( g^* \) and \( h^* \) define functors on the level of derived categories. Since \( Rh_* \) is right adjoint to \( h^* \), this diagram induces

\[
\begin{array}{ccc}
f_* A \otimes g_* B & \longrightarrow & Rh_*(A \otimes B) \\
\downarrow \varphi & & \downarrow \\
f_* A \otimes^L g_* B & \longrightarrow & Rh_*(A \otimes^L B) \\
\downarrow \quad & & \downarrow \\
Rf_* A \otimes^L Rg_* B & \longrightarrow & Rh_*(A \otimes^L B).
\end{array}
\]

If we take the 0-th cohomology, the morphism \( \varphi \) induces an isomorphism, and we see that for \( r = s = 0 \) the cup product morphism of (ii) coincides with the cup-pairing we started with. Finally, we show that the cup product is compatible
with exact sequences. Every exact sequence induces a triangle \( A' \to A \to A'' \to A'[1] \). This triangle induces the following morphism of triangles:

\[
\begin{array}{ccc}
Rf_*A' \otimes^L Rg_*B & \longrightarrow & Rh_*(A' \otimes^L B) \\
\downarrow & & \downarrow \\
Rf_*A \otimes^L Rg_*B & \longrightarrow & Rh_*(A \otimes^L B) \\
\downarrow & & \downarrow \\
Rf_*A'' \otimes^L Rg_*B & \longrightarrow & Rh_*(A'' \otimes^L B) \\
\downarrow & & \downarrow \\
Rf_*A'[1] \otimes^L Rg_*B & \longrightarrow & Rh_*(A'[1] \otimes^L B).
\end{array}
\]

By means of the morphism \( \alpha \) and the canonical morphism \( \otimes^L \to \otimes \), the last square induces a diagram

\[
\begin{array}{ccc}
R^r f_*A'' \otimes R^s g_*B & \longrightarrow & R^{r+s}h_*(A'' \otimes B) \\
\downarrow & & \downarrow \\
R^{r+1} f_*A' \otimes R^s g_*B & \longrightarrow & R^{r+s+1}h_*(A' \otimes B).
\end{array}
\]

To prove that the cup product is uniquely determined by these properties, consider a sequence \( 0 \to A \to I \to Z \to 0 \) with an \( f_* \)-acyclic object \( I \). Since \( R^r f_* I = 0 \) for \( r \geq 1 \), the left vertical morphism is bijective for \( r \geq 1 \) and surjective for \( r = 0 \). Inductively, we can infer that the cup product morphisms are uniquely determined by property (iii) and (iv).

**Remark.** In the category of sheaves, we will study another possibility to construct the morphism \( \alpha \). That is why we do not study the morphism \( \alpha \) in detail, here. In fact, the existence proof for \( \alpha \), which is given in [CE], is categorial.

The proof of the theorem shows that the cup product is a natural morphism. This fact is not visible if one constructs cup products as in [EC], V, §1, proposition 1.16. Moreover, the fundamental properties of cup products as in (iii) and (iv) turn out to be formal properties of a morphism in the derived category. Finally, we show that composition of \( \cup \)-functors is compatible with composition of cup products:

**2.1.7 Proposition.** Let \( h_* \), \( f_* \) and \( g_* \) be \( \cup \)-functors with \( g_* = h_* f_* \). If their cup-pairing morphisms are compatible, i.e., if the canonical diagram

\[
\begin{array}{ccc}
g_* A \otimes g_* B & \longrightarrow & g_*(A \otimes B) \\
\| & & \| \\
(h_* f_* A) \otimes (h_* f_* B) & \longrightarrow & h_*(f_* A \otimes f_* B) \longrightarrow h_* f_*(A \otimes B)
\end{array}
\]

(5)

commutes, then the cup-products of \( Rh_* \), \( Rf_* \) and \( Rg_* \) are compatible, that is the following diagram commutes:

\[
\begin{array}{ccc}
Rg_* X \otimes^L Rg_* Y & \longrightarrow & Rg_*(X \otimes^L Y) \\
\| & & \| \\
(Rh_* Rf_* X) \otimes^L (Rh_* Rf_* Y) & \longrightarrow & Rh_*(Rf_* X \otimes^L Rf_* Y) \longrightarrow Rh_* Rf_*(X \otimes^L Y).
\end{array}
\]

\( \square \)
2.1.8 Corollary. Let \( h_* \) and \( f_* \) be \( \cup \)-functors. If the composition \( g_* := h_* f_* \) exists, then this composition is a \( \cup \)-functor and it is compatible with cup products, i.e., the conclusion of the above proposition holds.

Proof. Since \( h_* \) and \( f_* \) have exact left adjoints \( f^* \) and \( h^* \), the composition \( g_* \) has an exact left adjoint \( g^* := f^* h^* \), which clearly respects flat objects. The cup pairings for \( h_* \) and \( f_* \) induce a cup pairing for \( g_* \) by diagram (5). \( \square \)

In the case that the cup-pairing-morphism of the composition \( g_* = h_* f_* \) is compatible with the cup-pairing morphisms of the functors \( h_* \) and \( g_* \), we will say that the cup-pairings of \( h_* \), \( f_* \) and \( g_* \) are compatible. As an important computational tool we can construct the following pairing of spectral sequences:

2.1.9 Corollary. Let \( h_* \), \( f_* \) and \( g_* = h_* f_* \) be three \( \cup \)-functors with compatible cup-pairings. Then there is the following commutative diagram of pairings of spectral sequences:

\[
\begin{align*}
R^r h_* R^s f_* A \times R^r h_* R^s f_* B & \longrightarrow R^{r+s} h_* R^{r+s} f_* (A \otimes B) \\
\downarrow & \downarrow \\
R^{r+s} g_* A \times R^{r+s} g_* B & \longrightarrow R^{r+s} g_* (A \otimes B)
\end{align*}
\]

for objects \( A \) and \( B \), where the horizontal morphisms are induced by cup products. \( \square \)

2.2 Flat Sheaves

In this section, we give a brief survey on flat sheaves and flat resolutions for the category of abelian sheaves on an arbitrary Grothendieck topology. In contrast to the exposition of flat sheaves e.g. in [RD] or [KS] we will not use stalks.

In the following, we will use the tensor product both in the category \( \mathcal{S} \) of sheaves and in the category \( \mathcal{P} \) of presheaves. Although we will use the same notation, it will be clear in which category we are working and, therefore, which tensor product we refer to. Remember that the sheaf tensor product is obtained from the presheaf tensor product by sheafification: If \( s: \mathcal{P} \to \mathcal{S} \) denotes the sheafification functor and if \( i: \mathcal{S} \to \mathcal{P} \) denotes the inclusion functor, we have \( A \otimes B = s(iA \otimes iB) \) for sheaves \( A \) and \( B \).

To begin with, recall that an abelian group \( P \) is flat if and only it is torsion free ([CA], I, §2.5, prop. 3). This fact generalises to the category of abelian sheaves and presheaves.

2.2.1 Definition. (Torsion Free Sheaf) Let \( A \) be a sheaf (or presheaf) of abelian groups. It is called torsion free if the multiplication morphism \( n: A \to A \) is a monomorphism for every integer \( n \geq 1 \).

Obviously, a (pre)sheaf \( A \) is torsion free if and only if the groups \( A(U) \) are torsion free for every \( U \). This is the key to the following proposition:

2.2.2 Proposition. Let \( P \) be an abelian sheaf or a presheaf. Then \( P \) is flat if and only if it is torsion free.
Proof. Let $P$ be a flat sheaf or a flat presheaf. Tensoring it with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ shows that it is torsion free. To prove the other implication, recall that the group of sections over $U$ of the tensor product of presheaves $A$ and $B$ is defined to be $A(U) \otimes B(U)$. Therefore, the characterisation of flat groups as torsion free groups implies the converse implication in the category of presheaves. It remains to show the implication “torsion free” $\Rightarrow$ “flat” for sheaves: let $P$ be a torsion free sheaf. The functor $- \otimes P$ is equal to $s(i- \otimes iP)$. This functor preserves monomorphisms. Hence $P$ is flat. □

This simple characterisation of flatness is the key to the following assertions about flat sheaves of abelian groups.

2.2.3 Corollary. Let $P$ be a flat sheaf (or presheaf) of abelian groups.

(i) Every subsheaf of $P$ is flat,

(ii) the sheafification of $P$ is flat, and

(iii) $H^0(X, P)$ is a flat group for every object $X$ of the site.

Proof. Using the characterisation of flat sheaves, it is clear that a subsheaf of a flat sheaf is flat. Let $P$ be a flat presheaf and let $T$ be any sheaf of abelian groups. The calculation

$$\text{Hom}(sP \otimes F, T) = \text{Hom}(sP, \text{Hom}(F, T))$$
$$= \text{Hom}(P, \text{Hom}(iF, iT))$$
$$= \text{Hom}(P \otimes iF, iT)$$
$$= \text{Hom}(s(P \otimes iF), T)$$

shows that the functor $(sP \otimes -) = s(P \otimes i-)$ respects monomorphisms. Hence, $sP$ is flat. Finally, $(iii)$ is trivial. □

Let $f$ be an arbitrary morphisms of sites. In our applications we are interested in the case of the morphism “change of topology” (e. g. $R_{fl} \to R_{et}$) and in the case of “restriction of site” (e. g. the morphism induced by $\text{Spec } K \hookrightarrow \text{Spec } R$) These morphisms induce functors $f^*$ and $f_*$. We want to study their flatness preserving properties.

2.2.4 Lemma. Let $(P_i)_{i \in I}$ be a directed system of flat groups. The group $\varinjlim P_i$ is flat.

Proof. Since the inductive limit is exact and commutes with tensor products, the functor $- \otimes \varinjlim P_i$ is isomorphic to the exact functor $\varinjlim (- \otimes P_i)$. □

Of course, the same is true for a directed limit of flat sheaves.

2.2.5 Proposition. Let $P$ be a flat sheaf, then $f^*P$ and $f_*P$ are flat.

Proof. Due to proposition 2.2.2 the assertion for the case of the functor $f_*$ is trivial. Things are more complicated for the case of the functor $f^*$ induced by a morphism $f : S \to S'$ of sites. Let $P$ be a flat sheaf on $S'$. In a first step to obtain $f^*P$ from $P$, we have to extend the sheaf $P$ to a presheaf on $S$. Then we sheafify this presheaf to get the sheaf $f^*P$: The presheaf-extension to $S$ is
given by the presheaf \( T \mapsto \lim_{\to} P(T) \) for some inductive system, cf. [EC], II, §2, prop. 2.2 and the following remarks. According to the above lemma, this limit is a flat group, thus, the presheaf is flat, thus, the associated sheaf is flat. □

The above proof gets easier, if \( j: Y \to X \) is a restriction map, i.e. if \( j \) belongs to the category of the Grothendieck topology. In this case, \( j^*P \) is the restriction of \( P \) to \( Y \); this sheaf is clearly flat.

In “classical” sheaf theory we can argue with stalks to prove various properties of flat sheaves, e.g. [KS], proposition 2.4.12. In the general setting, the functor \( j_! \) is of great help (cf. [EC], II, §3, remark 3.18). Let \( j: Y \to X \) be a morphism of schemes that belongs to the category of the Grothendieck topology. It induces a functor \( j_! \), extension by 0, between presheaf categories \( \mathcal{P}_Y \to \mathcal{P}_X \) as follows: Let \( F \) be a presheaf over \( Y \). We set

\[
(j_! F)(T) = \bigoplus_{\varphi \in \text{Hom}_X(T,Y)} F(T_{\varphi})
\]

for every \( X \)-scheme \( T \). Here \( T_{\varphi} \) is the scheme \( T \), regarded as an \( Y \)-scheme by means of \( \varphi \). By sheafification, this functor induces a functor of the appropriate categories of sheaves. We will denote this functor by \( j_! \), too. It has the following fundamental properties.

2.2.6 Lemma.
(i) The functor \( j_! \) is left adjoint to \( j^* \),
(ii) \( j_! \) is exact (in particular, the functor \( j^* \) preserves injective objects) and
(iii) we have \( j_! \mathcal{Z} \otimes F = j! j^* F \) for every sheaf \( F \). In particular, \( j_! \mathcal{Z} \) is flat.

Proof. (i) and (ii) are [EC], II, §3, remark 3.18. A calculation, similar to the proof of 2.2.3, shows that the functor \( j_! \mathcal{Z} \otimes - \) is isomorphic to the exact functor \( j_! j^* \). □

We have now collected all technical facts about flat sheaves that we will need in the following. Let us continue with flat resolutions and their properties:

2.2.7 Proposition. Let \( \mathcal{S} \) be a category of sheaves of abelian groups and let \( F \) be a sheaf. Then there exists an exact sequence

\[
0 \to P_1 \to P_0 \xrightarrow{\varepsilon} F \to 0
\]

with flat sheaves \( P_0, P_1 \).

This lemma generalises the appropriate property for the category of abelian groups: The Tor-dimensions (or weak dimension as in [CE]) of the categories of abelian groups, abelian presheaves and abelian sheaves are \( \leq 1 \).

Proof. For every object \( U \) of the Grothendieck topology with structure morphism \( j: U \to X \), the sheaf \( j_! \mathcal{Z} \) is flat. We define an epimorphism of presheaves and thus of sheaves

\[
\varepsilon: P_0 := \bigoplus_{\substack{j: U \to R \in F(U) \atop s \in F(U)}} j_! \mathcal{Z} \to F
\]

by \( 1 \mapsto s \in F(U) \) for the component \( (j: U \to R, s) \). As the sheaf \( \bigoplus j_! \mathcal{Z} \) is flat, the kernel \( P_1 := \ker \varepsilon \) it is flat, too. □
Remark. For later use, let us remark that this proposition and lemma 2.2.6 imply that the category $\mathcal{P}$ of abelian presheaves has enough projectives in form of the presheaves $j_! \mathbb{Z}$. Since the presheaf functor $j^*$ is exact, the presheaf functor $j_!$ preserves projectives. In fact, we have $\text{Hom}(j_! \mathbb{Z}, F) = F(U)$; this functor is exact on the category of presheaves. Since flat sheaves need not be projective, we cannot expect that every sheaf has a projective resolution of length 1.

2.2.8 Proposition. Let $* \in \{b, +, -, \emptyset\}$ and let $X \in D^* \mathcal{I}$ be a complex. Then, there exists a quasi-isomorphism $P \to X$ with a complex $P \in D^* \mathcal{I}$ consisting of flat sheaves. In particular, for any complex, bounded from below, there exists a quasi-isomorphism $P \to X$ with a complex $P$ of flat sheaves that is bounded from below.

Proof. This is a special case of a more general result. Cf. [KS], chapter I, exc. I.23. □

Using this result, it is easy to work with the derived tensor product. Remember that $A \otimes^L B$ is obtained by choosing a quasi-isomorphism $P \to B$ with a complex $P$ consisting of flat sheaves and setting $A \otimes^L B := A \otimes P$. The above proposition implies that in this process we do not leave the category of complexes bounded from below. Thus, we have a convergent spectral sequence to compute the hypercohomology of double complexes obtained from the derived tensor product.

Finally, we want to construct the Künneth spectral sequence. Unfortunately, proofs for the existence of the Künneth spectral sequence can only be found for categories with enough projectives, e.g. for module categories, [EC], XVII, §3, (iv) or [G], I, §5, theorem 5.5.1. These proofs do not generalise without more ado to the category of sheaves.

2.2.9 Proposition. Let $X$ and $Y$ be complexes of sheaves. There exists the homological Künneth spectral sequence

$$E^2_{r,s} = \bigoplus_{s_1 + s_2 = s} \text{Tor}_r(H_{s_1}(X), H_{s_2}(Y)) \implies E^r+s = H_{r+s}(X \otimes^L Y).$$

Proof. In a first step, we prove the existence of the Künneth spectral sequence in the category $\mathcal{P}$ of presheaves. The proof given in [G], I, §5, theorem 5.5.1 (or 5.4.1, respectively) is purely categorial and rests on the existence of (projective) Cartan-Eilenberg resolutions. Since $\mathcal{P}$ has enough projectives (remark on page 11), we can construct a projective Cartan-Eilenberg resolution (in $\mathcal{P}$ those resolutions are called “résolution injective” or “résolution projective”, respectively) in $\mathcal{P}$ and construct the Künneth spectral sequence for the complexes of presheaves $iX$ and $iY$ as it is shown there.

Since the functor of sheafification is exact, and since tensor products and flat sheaves are compatible with sheafification, we sheafify the Künneth spectral sequence in the category $\mathcal{P}$ to obtain a spectral sequence in the category $\mathcal{S}$ of sheaves. Routine verifications show that this spectral sequence has the right $E_2$ and limit terms and thus is the Künneth spectral sequence in the category of abelian sheaves. □
Remarks.

(i) Since the Tor-dimension of the categories of abelian sheaves and presheaves is 1 by proposition 2.2.7, the Künneth spectral sequence collapses to a family of exact sequences

\[ 0 \to \bigoplus_{s_1 + s_2 = s} H_{s_1}(X) \otimes H_{s_2}(Y) \to H_s(X \otimes^L Y) \to \bigoplus_{s_1 + s_2 = s - 1} \text{Tor}_1(H_{s_1}(X), H_{s_2}(Y)) \to 0. \]

(ii) The first edge morphism induces the morphism \( \alpha \) of (4). For this equality cf. [CE], XVII, §3, proposition 3.1.

2.3 Cup Products in the Category of Abelian Sheaves

In this section, we study some fundamental functors of sheaf cohomology. It is an important observation that they all come along as adjoint functors. Using the theory of \( \cup \)-functors, we will see that they all admit cup products. In our applications, we are mainly interested in sheaves on the étale and the flat site of a discrete valuation ring \( R \), so let us adopt our notation to this situation. If nothing else is specified, we are working with a fixed site over \( R \) with its abelian category of abelian sheaves. Of course, all these results are true for the category of abelian sheaves on any site or topological space.

The following facts are well-known, e.g. [EC].

2.3.1 Proposition. The following pairs of functors are adjoint:

(i) Sheafification \( s : \mathcal{P} \to \mathcal{I} \) and inclusion \( i : \mathcal{I} \to \mathcal{P} \) for the category of abelian sheaves on any fixed site.

(ii) \( f^* \) and \( f_* \) associated to the canonical morphism \( f : R_{\text{fl}} \to R_{\text{ét}} \).

(iii) \( j^* \) and \( j_* \) for a morphism \( j : Y \to X \) of schemes.

(iv) The functors “constant \( R \)-sheaf” \( G \mapsto \tilde{G} \) and \( H^0(R, -) \).

(v) The functor \( G \mapsto i_* \tilde{G} \) and \( H^0_k(R, -) := \ker(H^0(R, -) \to H^0(K, -)) \), the functor “sections with support over \( k \)” of local cohomology with respect to the closed immersion \( i : \text{Spec} k \hookrightarrow \text{Spec} R \).

It is common to denote the functor \( H^0(R, -) \) by \( \Gamma \) and to denote its derived functor in the sense of derived categories by \( R\Gamma \). Moreover, let us write \( \Gamma_k \) for the functor \( H^0_k(R, -) \) and \( R\Gamma_k \) for its derived functor.

In fact, even the functors of (i) and (iv) can be seen as an incarnation of a functor associated to a morphism of sites: Let \( \mathcal{C} \) be the category of the Grothendieck topology \( \mathfrak{T} \). Then the category \( \mathcal{C} \) with trivial coverings defines a Grothendieck topology. Let us denote this topology by \( \mathcal{C} \) again. There is a canonical morphism \( \mathcal{C} \to \mathfrak{T} \). This morphism induces the functors \( s \) and \( i \).

To study the functor of (iv), consider the morphism \( h : \{\ast\} \to (\text{Schemes}/R) \), \( \ast \mapsto R \). Then \( h^* \) is the functor “constant sheaf” and \( h_* \) is the functor \( H^0(R, -) \).

There are the following extended adjointness properties for the functors (i) – (iv) and for the tensor product with its adjoint \( \mathcal{H}\mathcal{O}\mathcal{M} \).

2.3.2 Proposition. Let \( \varphi_* \) be one of the functors of proposition 2.3.1 (i) – (iv). There are canonical isomorphisms

\[ \varphi_* \mathcal{H}\mathcal{O}\mathcal{M}(\varphi^* X, Y) \to \mathcal{H}\mathcal{O}\mathcal{M}(X, \varphi_* Y). \]

\(^2\)For a brief introduction to local cohomology, see e.g. [LC].
(ii) \( \text{Hom}(X \otimes Y, Z) \xrightarrow{\sim} \text{Hom}(X, \text{Hom}(Y, Z)) \).

(iii) \( \text{Hom}(X \otimes Y, Z) \xrightarrow{\sim} \text{Hom}(X, \text{Hom}(Y, Z)) \),

for suitable sheaves or complexes of sheaves. These isomorphisms extend to the derived category:

(iv) \( R\varphi_* \text{RHom}(\varphi^*X, Y) \xrightarrow{\sim} \text{RHom}(X, R\varphi_* Y) \).

(v) \( \text{RHom}(X \otimes^L Y, Z) \xrightarrow{\sim} \text{RHom}(X, \text{RHom}(Y, Z)) \).

(vi) \( \text{RHom}(X \otimes^L Y, Z) \xrightarrow{\sim} \text{RHom}(X, \text{RHom}(Y, Z)) \),

for suitable complexes \( X, Y, Z \) in the derived category. Moreover, the functors \( \otimes^L \) and \( \text{RHom} \) are adjoint:

(vii) \( \text{Hom}(X \otimes^L Y, Z) \xrightarrow{\sim} \text{Hom}(X, \text{RHom}(Y, Z)) \).

To prove this proposition, we need the following well-known lemma

2.3.3 Lemma. Let \( P \) be a flat sheaf and let \( I \) be an injective sheaf. Then, the sheaf \( \text{Hom}(P, I) \) is injective.

Proof. Indeed, by the adjunction formula, the functor \( \text{Hom}(-, \text{Hom}(P, I)) \) is isomorphic to the exact functor \( \text{Hom}(- \otimes P, I) \); hence, \( \text{Hom}(P, I) \) is injective. \( \square \)

Proof of proposition 2.3.2. The isomorphisms of (i) – (iii) are well-known: [EC], II, 3.22 for (i), ibid. 3.19 for (iii). Let us continue with the “derived” isomorphisms: To show (iv), we can assume that \( X \) consists of flat, and \( Y \) of injective objects. By lemma 2.3.3 the sheaf \( \text{Hom}(f^*X, Y) \) is injective and the formula is (i). In the same way, we can show (v) and (vi). Let \( Z \) be a complex of injective objects and let \( Y \) be a complex of flat objects. Then, the formulas of (v) and (vi) are the formulas of (ii) and (iii). Finally, by taking the 0-th cohomology of (vi) we get (vii). \( \square \)

2.3.4 Proposition. The functors of proposition 2.3.1 are \( \cup \)-functors.

Proof. It is known that the corresponding left adjoint functors are exact, [EC], II, introduction to §3 with II, §2, 2.6. Using the characterisation of flat groups and flat sheaves being torsion free, it is clear that the functor \( G \mapsto \hat{G} \) preserves flats objects. Using proposition 2.2.5 it follows that the functors \( G \mapsto i_* \hat{G}, f^* \) and \( j^* \) preserve flat objects, too. Finally, we have to construct canonical cup-pairings. In the case of the functors of (i) – (iv) of proposition 2.3.1 the functor \( \varphi^* \) even commutes with tensor products, i.e., there exists a canonical cap-pairing morphism \( \varphi^*(A \otimes B) \xrightarrow{\sim} \varphi^* A \otimes \varphi^* B \). By lemma 2.1.2 this isomorphism gives rise to the desired cup-pairing.

Finally, let us show that the functor \( H^0_k(R, -) \) is equipped with a canonical cup-pairing. Consider the commutative diagram with exact second line

\[
\begin{array}{cccc}
H^0_k(R, A) \otimes H^0_k(R, B) & \longrightarrow & H^0(R, A) \otimes H^0(R, B) & \longrightarrow & H^0(K, A) \otimes H^0(K, B) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0_k(R, A \otimes B) & \longrightarrow & H^0(R, A \otimes B) \longrightarrow H^0(K, A \otimes B)
\end{array}
\]

Keep in mind that the first line needs not to be exact. However, it is a complex. The second and third vertical morphisms are the cup-pairings of \( H^0(R, -) \) and \( H^0(K, -) \). \( \square \)
Since the functors of proposition 2.3.1 are $\cup$-functors, they admit cup products by theorem 2.1.6.

2.3.5 Theorem. Let $\varphi_*$ be any of the left adjoint functors of proposition 2.3.1. There is a canonical morphism $R\varphi_*X \otimes^L R\varphi_*Y \to R\varphi_*(X \otimes^L Y)$ for complexes $X, Y$ in the derived category $D^+_{\text{fl}}$. It induces morphisms

$$R^r\varphi_*A \otimes R^s\varphi_*B \to R^{r+s}\varphi_*(A \otimes B)$$

for sheaves $A, B$ with the properties of theorem 2.1.6. □

Remark. By the adjunction of $R\mathbb{H}om$ and $\otimes^L$, the cup product morphism is the same as a morphism $R\varphi_*A \to R\mathbb{H}om(R\varphi_*B, R\varphi_*(A \otimes^L B))$, which, again, we will refer to as the cup product.

It is an easy calculation to show that this cup product coincides with the product that is constructed in [EC], V, §1, proposition 1.16 using Godement resolutions. In fact, we have already proven this fact implicitly in theorem 2.1.6: Both products share the same properties that characterise a cup product.

2.4 The Cup Product of Čech Cohomology

In some cases, Čech cohomology can be used to compute the “true” derived functor cohomology groups $H^*(X, -)$. In this section, we show that in these cases the cup product, which is defined on Čech cocycles, can be used to compute the cup product of derived functor cohomology. Remember that the Čech functors $\check{H}^*(\mathcal{U}, -)$ for a covering $\mathcal{U}$ of $U$ and $\check{H}^*(U, -)$ are universal, cohomological $\delta$-functors on the category $\mathcal{P}$ of presheaves, [A62], theorem 3.1. For a sheaf $F$, both groups $\check{H}^0(\mathcal{U}, iF)$ and $\check{H}^0(U, iF)$ coincide with the group $H^0(U, F)$. Thus, the relation between Čech cohomology and derived functor cohomology is described by the Čech spectral sequences

$$\check{H}^r(\mathcal{U}, R^s iF) \Longrightarrow H^{r+s}(U, F) \quad \text{and} \quad \check{H}^r(U, R^s iF) \Longrightarrow H^{r+s}(U, F) \quad (6)$$

for the inclusion functor $i: \mathcal{P} \to \mathcal{S}$. Before we can state the first lemma, we need some notations: Let $\mathcal{U} = (U_i \xrightarrow{j_i} U)_{i \in I}$ be a covering. By $j_{i_0, \ldots, i_n}$, we denote the morphism $U_{i_0} \times_U \ldots \times_U U_{i_n} \to U$. For an abelian group $G$, we denote by $G$ the constant presheaf associated to $G$.

2.4.1 Lemma. Let $\mathcal{U}$ be a covering of $U$. The canonical sequence of presheaves

$$C_\mathcal{U}(\mathbb{Z}) : \cdots \to \bigoplus_{i_0, i_1, i_2 \in I} \mathbb{Z} \to \bigoplus_{i_0, i_1 \in I} (j_{i_0, i_1})_! \mathbb{Z} \to \bigoplus_{i \in I} (j_i)_! \mathbb{Z}$$

is a projective resolution of the cokernel of the latter pair of morphisms in the category $\mathcal{P}$ of presheaves. We will denote this cokernel by $C_\mathcal{U}\mathbb{Z}$. This presheaf is flat.
Proof. This is [A62], proof of theorem 3.1 for the proof that the sequence is acyclic and the remark on page 11 that it consists of projectives. In [A62], ibid, it is shown that for \( V \to U \) this sequence induces a canonical, exact sequence

\[
\cdots \to \bigoplus_{s \times s \times s} \to \bigoplus_{s \times s} \mathbb{Z} \to \bigoplus_{s} \mathbb{Z}
\]

for a set \( S \), depending on \( V \). The cokernel of the last map, \( c_{U} \mathbb{Z} (V) \), is trivial if \( S \) is empty and it is isomorphic to \( \mathbb{Z} \) otherwise. Hence, this presheaf is flat. □

In [A62], proof of theorem 3.1 it is shown that this complex induces the Čech complex via \( \check{C}(U, F) = \text{Hom}(C_{U}(\mathbb{Z}), F) \). This resolution will play an important role in the following. Moreover, since the functor \( \text{Hom}(\cdot, F) \) is left exact, we have

\[
\check{H}^{0}(U, F) = H^{0}(\check{C}(U, F)) = \text{Hom}(c_{U} \mathbb{Z}, F).
\]

(7)

This means that the Čech functor is represented by the presheaf \( c_{U} \mathbb{Z} \).

2.4.2 Proposition. Let \( \mathfrak{U} = (U, U)_{i \in I} \) be a covering.

(i) The Čech functor \( \check{H}^{0}(\mathfrak{U}, -) \) is a ∪-functor.

(ii) In particular, there exists a cup product for \( \check{\Gamma} := \check{H}^{0}(\mathfrak{U}, -) \).

(iii) This cup product induces a cup product for the Čech functor \( \check{\Gamma} := \check{H}^{0}(U, -) \).

Proof. Let us prove that \( \check{H}^{0}(\mathfrak{U}, -) \) has a left adjoint. We define the functor “constant presheaf with respect to the covering \( \mathfrak{U} \)”, \( c_{U} G \), for an abelian group \( G \) by

\[
c_{U} G := \check{G} \otimes c_{U} \mathbb{Z},
\]

where \( \check{G} \) is the constant presheaf associated to \( G \). An easy calculation using equation (7) shows that the functor \( c_{U} \) is left adjoint to the Čech functor.

Since \( c_{U} \mathbb{Z} \) is flat, the functor \( c_{U} \) is exact. Since the presheaf \( c_{U} \mathbb{Z} \) consists of the group \( \mathbb{Z} \) or 0 at every level, the tensor product \( c_{U} \mathbb{Z} \otimes c_{U} \mathbb{Z} \) is isomorphic to \( c_{U} \mathbb{Z} \). In particular, there is a canonical isomorphism

\[
c_{U} \otimes (\check{F} \otimes \check{G}) \cong (c_{U} \mathbb{Z} \otimes \check{F}) \otimes (c_{U} \mathbb{Z} \otimes \check{G}),
\]

(8)

This can be regarded as a cap-pairing \( c_{U} \to c_{U} \cap c_{U} \) and, thus, there is a cup-pairing \( \check{\Gamma}_{U} \cup \check{\Gamma}_{U} \to \check{\Gamma}_{U} \).

These facts imply that \( \check{\Gamma}_{U} \) admits a cup product. Now, \( \check{\Gamma} \) is defined to be \( \lim_{\rightarrow \mathfrak{U}} \check{\Gamma}_{U} \), where the limit varies over all coverings \( \mathfrak{U} \) of \( U \). By this limit we define the cup product for the Čech-functor \( \check{\Gamma} \). Remember that \( \lim_{\rightarrow} \) turns flat sheaves into flat sheaves, remark after proposition 2.2.4. Consequently, taking the direct limit is compatible with the derived tensor product. This way, we obtain the cup product for \( \check{\Gamma} \). □

We want to describe the cup product of the Čech functor: The cap-pairing morphism of \( c_{U} \mathbb{Z} \) for \( G = F = \mathbb{Z} \) can be extended to a canonical pairing of the resolutions:

\[
C_{U}^{r+s}(\mathbb{Z}) \to C_{U}^{r}(\mathbb{Z}) \otimes C_{U}^{s}(\mathbb{Z})
\]

\[
e_{i_{0},...,i_{r+s}} \mapsto e_{i_{0},...,i_{r}} \otimes e_{i_{r},...,i_{r+s}}.
\]

In particular, for \( r = s = 0 \) this pairing induces the isomorphism \( c_{U} \mathbb{Z} \to \sim c_{U} \mathbb{Z} \otimes c_{U} \mathbb{Z} \) of the above proof. Moreover, this pairing induces a pairing of Čech
complexes as follows: Let \( a = (a_0, \ldots, a_r) \in C^r(U, A) \) and let \( b = (b_0, \ldots, b_s) \in C^s(U, B) \). Their image in \( C^{r+s}(U, F \otimes G) \) is given by the Alexander-Whitney formula:

\[
(a \cup b)_{i_0, \ldots, i_r + i_s} = a_{i_0, \ldots, i_r} \otimes b_{i_r, \ldots, i_s},
\]

(9)

cf. [EC], V, §1, remark 1.19. This suggests that the cup product of Čech cohomology is given by the Alexander-Whitney formula and, in the case that sheaf cohomology is given by Čech cohomology, even the cup product is given by (9). We are going to prove these facts in the following.

2.4.3 Proposition. The cup product defined on Čech cocycles by the Alexander-Whitney-formula agrees with the cup product of the \( \cup \)-functor \( \check{H}^0(U, -) \).

Proof. The complex \( C_U(Z) \) is a projective resolution of \( c_U(Z) \). Since \( \check{H}^0(U, F) = \text{Hom}(c_U(Z), F) \), there are canonical isomorphisms

\[
R\check{\Gamma}_UF \xrightarrow{\sim} R\text{Hom}(c_U(Z), F) \xrightarrow{\sim} \text{Hom}(C_U(Z), F) \xrightarrow{\sim} C(U, F).
\]

(10)

The derived cup product of \( \check{\Gamma}_U \) can therefore be written as the following morphism in the derived category:

\[
\text{Hom}(C_U(Z), F) \otimes^L \text{Hom}(C_U(Z), G) \longrightarrow \text{Hom}(C_U(Z), F \otimes^L G).
\]

(11)

Let us replace \( F \) and \( G \) by flat, i.e. torsion free, resolutions. Then, the complex \( \text{Hom}(C_U(Z), F) \) consists of direct sums of flat summands \( F(U_i) \) for morphisms \( U_i \to U \). Hence, it consists of flat groups. Consequently, we can replace the derived tensor product \( \otimes^L \) by the ordinary tensor product \( \otimes \). Consequently, the morphism is given by the Alexander-Whitney formula. \( \square \)

Let us emphasise that equation (10) states that the derived functor \( R\check{\Gamma}_U \) of the Čech functor actually is the Čech complex relative to the covering \( \mathfrak{U} \).

Remark. Using the same arguments, we can show that cup products of group cohomology can equivalently be defined by \( \cup \)-functors and by the Alexander-Whitney formula on the Bar resolution, cf. [Br], V.3.

By Čech cohomology, we refer to the cohomology with respect to the functors \( \check{H}^*(U, -) \) and \( \check{H}^*(U, -) \). We say that Čech cohomology and derived functor cohomology agree in dimension \( r \) if the edge morphism

\[
\check{H}^r(U, F) \longrightarrow H^r(U, F) \quad \text{or} \quad \check{H}^r(U, F) \longrightarrow H^r(U, F)
\]

of the Čech spectral sequence \( \mathcal{B} \) is an isomorphism. For example, this is the case for the latter edge morphism for \( r = 0, 1 \).

2.4.4 Theorem. If Čech cohomology agrees with derived functor cohomology, the cup product of Čech cohomology agrees with the cup product of derived functor cohomology. In particular, it can be calculated by the Alexander-Whitney-formula.

Proof. For the case of \( \check{H}^*(U, -) \), this is corollary 2.1.9 and proposition 2.4.3. In the case of \( \check{H}^*(U, -) \), we take the limit over all coverings \( \mathfrak{U} \) over the diagram of proposition 2.4.7 and continue as in corollary 2.1.9. \( \square \)
2.4.5 Corollary. The cup product $H^1(U,A) \times H^0(U,B) \to H^1(U,A \otimes B)$ can be computed by Čech cohomology.

We will see an explicit example for such a computation in the proof of proposition 3.3.11.

3 The $p$-Part of Grothendieck’s Pairing in the Case of Semistable Reduction

3.1 Rigid Uniformisation

In the following, let $R$ be a complete, discrete valuation ring of equal characteristic $p \neq 0$ with algebraically closed residue class field $k$. Let $K := \text{quot } R$ be its field of fractions and let $A_K$ be an abelian variety over $K$ with dual abelian variety $A'_K$.

In this section, we summarise some results on rigid uniformisation of abelian varieties as we need them in the following. These results are due to [R70], [BL91] for rigid uniformisation and [BX96] for formal Néron models. For the following proposition, it is crucial that $K$ is complete:

3.1.1 Proposition. (Rigid Uniformisation) Let $R$ and $K$ be as above and let $A_K$ be an abelian variety over $K$ with semistable reduction.

(i) There exists a $K$-group scheme $E_K$, which is an extension of an abelian variety $B_K$ with good reduction by a split torus $T_K$ of dimension $d$, such that in the category of rigid $K$-groups the abelian variety $A_K$ is isomorphic to a quotient of $E_K$ by a split lattice $M_K$ of rank $d$. This means, $M_K$ is a closed analytic subgroup of $E_K$, which is isomorphic to the constant $K$-group scheme $\mathbb{Z}^d$. This data can be summarised by the following exact sequences:

$$0 \to M_K \to E_K \to A_K \to 0$$

$$0 \to T_K \to E_K \to B_K \to 0$$

where the morphism $E_K \to A_K$ exists in the category of rigid $K$-groups.

(ii) Let $A'_K$ be the dual abelian variety whose uniformisation is denoted by $M'_K$, $E'_K$, $T'_K$. Then, $M'_K$ is the character group of $T_K$ (and vice versa). The abelian varieties $B_K$ and $B'_K$ are dual to each other. □

The exact sequence of rigid uniformisation extends to the level of formal Néron models as follows:

3.1.2 Proposition.

(i) The rigid uniformisation above gives rise to an exact sequence of formal Néron models:

$$0 \to M \to E \to A \to 0.$$  

The induced morphism $E^0 \to A^0$ is an isomorphism.
This sequence induces the following exact sequence of groups of components
\[ 0 \rightarrow M \rightarrow \phi_E \rightarrow \phi_A \rightarrow 0. \]

Proof. See [BX96], theorem 2.3 and proposition 5.4. \qed

To simplify notation, let us write \( M \) for both the Néron model of \( M_K \) and the (abstract) group which \( M_K \) is associated to.

We will need the following notions of duality: Let \( G \) be any (abstract) abelian group. We define \( G^\vee \) to be the group \( \text{Hom}(G, \mathbb{Z}) \), and \( G^* \) to be the Pontrjagin dual \( \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \). If \( G \) is finite, there is a canonical isomorphism \( G^* \rightarrow \sim \text{Ext}^1(G, \mathbb{Z}) \), induced by the Ext-sequence of \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0 \).

Remark. Taking the uniformisation of the dual abelian variety \( A'_K \) into consideration, there is the bijective map evaluation of characters
\[ \bar{\phi}_{T'} \rightarrow \sim \text{Hom(Hom}(T'_K, \mathbb{G}_m), \mathbb{Z}), \]
for an element \( \bar{t} \in \phi_{T'} = T'(R)/T'^0(R) = T'_K(K)/T'^0(R) \) and the discrete valuation \( \nu \) on \( K \). This map will play a central role in the following; we will denote it by \( \nu^* \).

### 3.2 The Monodromy Pairing

In [SGA7], Grothendieck suggested comparing the perfect monodromy pairing to Grothendieck’s pairing of groups of components in order to show that Grothendieck’s pairing is perfect in the case of semistable reduction. This is done in [We97]. We will sketch the main ideas without giving proofs. In the following, let \( A_K \) be an abelian variety with semistable reduction. Let \( M, E \) etc. be the objects of rigid uniformisation we obtained in proposition 3.1.2.

We have two exact sequences of component groups
\[ 0 \rightarrow M' \rightarrow \phi_{E'} \rightarrow \phi_{A'} \rightarrow 0 \quad \text{and} \]
\[ 0 \rightarrow M \rightarrow \phi_E \rightarrow \phi_A \rightarrow 0. \]

Again, we write \( M \) for the abstract group \( M_K \) is associated to. There is a canonical isomorphism \( \phi_T \rightarrow \sim \phi_E \), [BX96], theorem 4.11, and, thus, there are isomorphisms
\[ \phi_{E'} \rightarrow \sim \phi_{T'} \rightarrow \sim M'^\vee. \]

The last map is the map “evaluation of characters”, \( \nu^* \), as in (12).

#### 3.2.1 Proposition. The isomorphisms obtained from (14) induce a commutative diagram
\[ M \xrightarrow{\phi_E} \phi_A \xrightarrow{\phi_{A'}} \widehat{\phi}_{E'} \xrightarrow{\widehat{\phi}} M'^\vee \xrightarrow{\sigma} \text{Ext}^1(\phi_{A'}, \mathbb{Z}) \rightarrow 0. \]

In particular, the induced map \( \sigma \) is bijective. Hence, it induces a perfect pairing \( \phi_A \times \phi_{A'} \rightarrow \mathbb{Q}/\mathbb{Z} \).
We will refer to this pairing of component groups as the monodromy pairing.

**Proof.** The exact sequence in the first line is clear. The second one is obtained by applying $\text{Hom}(-, \mathbb{Z})$ to the first sequence of (13); it yields the exact sequence

$$
\text{Hom}(\phi_{E'}, \mathbb{Z}) \rightarrow \text{Hom}(M', \mathbb{Z}) \rightarrow \text{Ext}^1(\phi_{A'}, \mathbb{Z}) \rightarrow \text{Ext}^1(\phi_{E'}, \mathbb{Z})
$$

Since $\phi_{E'}$ is free, the latter group is trivial. Hence, we have the exact sequence of the second line. The commutativity of the first square is shown in [We97], section 3 and 4. It induces the morphism $\sigma$ that is clearly bijective. Since the group $\text{Ext}^1(\phi_{A'}, \mathbb{Z})$ is isomorphic to $\text{Hom}(\phi_{A'}, \mathbb{Q}/\mathbb{Z})$, the isomorphism $\sigma$ induces a perfect pairing of groups of components. \hfill $\Box$

**3.2.2 Proposition.** In the situation of semistable reduction, the monodromy pairing and Grothendieck’s pairing of groups of components coincide up to sign. That is, the following diagram commutes up to sign

$$
\begin{array}{ccc}
\phi_T & \longrightarrow & \phi_A \\
\downarrow \phi^* & & \downarrow \text{GP} \\
M' & \longrightarrow & \phi_{A'}^*
\end{array}
$$

In particular, in the case of an abelian variety with semistable reduction, Grothendieck’s pairing is perfect.

**Proof.** See [We97], proposition 5.1. \hfill $\Box$

**3.3 Bester’s Pairing**

In [Bes78], M. Bester constructed Bester’s pairing using local cohomology. Let us review the theory of local cohomology in the framework of étale cohomology. (cf. [EC] pages 73–78)

The topological space $\text{Spec} R$ consists of two points and there are two immersions

$$
\text{Spec} k \xleftarrow{i} \text{Spec} R \xrightarrow{j} \text{Spec} K,
$$

the first being a closed immersion, the second being an open immersion. We consider the functor $H^i_k(R, -)$ on the category of abelian sheaves on $\text{Spec} R$ which is defined by the exact sequence

$$
0 \longrightarrow H^i_k(R, F) \longrightarrow H^0(R, F) \longrightarrow H^0(K, F)
$$

(15)

for a sheaf $F$ on the flat or the étale site of $\text{Spec} R$. Obviously, it is left exact so we can define its derived functors $H^i_k(R, -) := R^i H^0_k(R, -)$. Of course, there is a long exact cohomology sequence for every exact sequence, but there is another important sequence for every sheaf $F$:

**3.3.1 Proposition. (Sequence of Local Cohomology)** Let $F$ be a sheaf. Then, sequence (13) is part of the long exact sequence of local cohomology:

$$
\ldots \longrightarrow H^i_k(R, F) \longrightarrow H^i(R, F) \longrightarrow H^i(K, F) \longrightarrow H^{i+1}_k(R, F) \longrightarrow \ldots
$$

for every $i \geq 0$. 

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Of course, this works for both the étale and the flat topology.

\textit{Proof.} Lemma 3.3.8 or [Bes78], Appendix, p. 172. □

As counterpart to local cohomology, Bester constructs a functor $\mathcal{F}$. We will sketch its construction: As always, let $R$ be a complete, discrete valuation ring of equal characteristic $p \neq 0$ with algebraically closed residue class field $k$. By the Cohen structure theorem for complete, local rings, \( [N] \), theorem 31.1 and theorem 31.10, the ring $R$ has a uniquely determined field of coefficients, which is isomorphic to $k$, and $R$ is isomorphic to $k[[x]]$.

Bester constructs the functor $\mathcal{F}$ for finite, flat $R$-group schemes $N$, whose order is a power of $p$, as follows: Choose a smooth, connected, formal resolution of $N$, i.e. a family of resolutions

$$
\zeta_i : 0 \longrightarrow N_{R_i} \longrightarrow A_i \longrightarrow B_i \longrightarrow 0
$$

of smooth and connected schemes $A_i$ and $B_i$ over $R_i := R/m^i$ such that $\zeta_i = \zeta_{i+1} \otimes_R R_i$ (cf. [Bes78], 1, lemma 3.1). We define $\mathcal{F}(N)$ to be the pro-sheaf over Spec $k$ associated to

$$
\mathcal{F}(N) := (\text{coker}(\pi_1(\alpha_i A_i) \rightarrow \pi_1(\alpha_i B_i)))_{i \in \mathbb{N}}.
$$

Here, $\pi_1$ denotes the fundamental group in the sense of proalgebraic groups, [O66], II.7-4. Due to the Cohen structure theorem, there exist canonical morphisms $\alpha_i : \text{Spec } R_i \rightarrow \text{Spec } k$. They induce the Weil restriction functors $\alpha_i^*$, that coincide with the Greenberg functor, [BLR], p. 276. By means of the Greenberg functor, we regard an $R$-group as a proalgebraic group over $k$. Bester shows that $\mathcal{F}$ is independent of the resolution chosen and, indeed, defines a functor.\(^3\)

The Greenberg functor $(\alpha_i^*)_{i \in \mathbb{N}}$ has some nice properties: it is exact on smooth group schemes; more precisely, its first derived functor $R^1 \alpha_i^* G$ is trivial for smooth groups $G$ and it respects identity components of smooth group schemes ([Bes78], 1, lemma 1.1). In the following, we will omit the functor $\alpha_i^*$ in the notation and write simply $N$ for the proalgebraic group $(\alpha_i^* N_{R_i})_{i \in \mathbb{N}}$.

Bester’s pairing is built upon Cartier duality, $N \times N^D \rightarrow \mathbb{G}_m$. Since we are interested in groups $N$ whose order is a power of $p$, the image of this morphism is contained in some $\mu_{p^n,R}$. In this case, the Cartier pairing can be equivalently written as

$$
N \times N^D \longrightarrow \lim_{\longrightarrow} \mu_{p^n,R} =: \mu_{p^{\infty}}, \quad (16)
$$

for any finite, flat group $N$ whose order is a power of $p$.

The main part of [Bes78] is devoted to the proof of the following theorem:

\textbf{3.3.2 Theorem. (Bester’s Pairing)} Let $N$ be a finite, flat $R$-group scheme whose order is a power of $p$. Then, there exists a perfect pairing

$$
H^2_{\text{et}}(R, N) \times \mathcal{F}(N^D) \longrightarrow \mathbb{Q}/\mathbb{Z}.
$$

We will call this pairing Bester’s pairing. □

\(^3\)Later, we will discuss another approach to Bester’s functor $\mathcal{F}$. 

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Remarks.

(i) Let \( \tilde{G} \) be the constant group scheme which is associated to an abstract group \( G \). Then, we have \( \mathcal{F}(\tilde{G}) = G \), \cite{Bes78}, 1, lemma 3.7 and remark 3.8.

(ii) Let \( N \) be any finite, flat \( R \)-group scheme. If the order of \( N \) is prime to \( p \), then \( N \) is étale. Since \( R \) is strictly henselian, \( N \) is constant. Thus, \( \mathcal{F}(N) \) is isomorphic to \( N(R) = N(K) \), regarded as a constant group scheme, and \( H^2_k(R, N^D) \) is isomorphic to \( H^1(K, N^D) \) by proposition \ref{H2kRND} and lemma \ref{H2kRND} (i). We can define a perfect pairing
\[ H^2_k(R, N) \times \mathcal{F}(N^D) \longrightarrow \mathbb{Q}/\mathbb{Z} \]
for all finite, flat \( R \)-group schemes by means of the cup product for the prime-to-\( p \)-part and Bester’s pairing for the \( p \)-part of \( N \). Later, we will see that Bester’s pairing is the well suited extension of the cup product to the \( p \)-part.

In the following, we want to study the pairing for tori; more precisely, we want to study Bester’s pairing for the group \( \mu_{p^n,R} \) and its Cartier dual \( \mathbb{Z}/p^n \).

Let \( \mathcal{F} \) denote the Néron model of \( G_{m,K} \). It is isomorphic to \( \mathcal{F} \cong \bigcup_{r \in \mathbb{Z}} \pi^r G_{m,R} \) for a uniformising element \( \pi \in R \). We will encounter the exact sequence
\[ 0 \longrightarrow \mu_{p^n,R} \longrightarrow \mathcal{F} \longrightarrow n\mathcal{F} \longrightarrow 0 \] (17)
for \( n \in \mathbb{N}_{\geq 1} \). Since the \( n \)-power map of \( G_{m,R} \) is epimorphic in the flat topology, the subgroup \( n\mathcal{F} \) of \( \mathcal{F} \) is given by \( n\mathcal{F} = \bigcup_{r \in n\mathbb{Z}} \pi^r G_{m,R} \). It is clear that this group scheme is smooth, too.

We need some cohomological facts:

3.3.3 Lemma. Let \( G \) be a smooth \( R \)-group scheme.

(i) \( H^i(R, G) = 0 \) for all \( i \geq 1 \).

(ii) \( H^1_k(R, G) = G(K)/G(R) \). In particular we have

(iii) \( H^1_k(R, G_{m,R}) = \mathbb{Z} \), \( H^1_k(R, \mathcal{F}) = 0 \).

(iv) \( H^2_k(R, \mu_{n,R}) = H^1_k(R, n\mathcal{F}) = \mathbb{Z}/n \) for all \( n \in \mathbb{N} \).

(v) \( H^2_k(R, A^0) = \phi_A \).

Proof. (i) is shown in \cite{DixExp}, theorem 11.7. Using this, we can show (ii): Let \( G \) be a smooth \( R \)-group scheme. We have the following part of the sequence of local cohomology:
\[ H^0(R, G) \longrightarrow H^0(K, G) \longrightarrow H^1_k(R, G) \longrightarrow H^1(R, G). \]

Since the latter term is trivial by (i), this implies (ii). Since both \( G_{m,R} \) and \( \mathcal{F} \) are smooth, we get (iii) by (ii). Let us prove (iv). Consider the exact sequence of \ref{17}. It induces the following exact cohomology sequence
\[ H^1_k(R, \mathcal{F}) \longrightarrow H^1_k(R, n\mathcal{F}) \longrightarrow H^2_k(R, \mu_{n,R}) \longrightarrow H^2_k(R, \mathcal{F}). \]

The group \( H^1_k(R, \mathcal{F}) \) is trivial by (iii). Furthermore, the last term is trivial: The sequence of local cohomology induces the exact sequence
\[ H^1(K, G_{m,K}) \longrightarrow H^2_k(R, \mathcal{F}) \longrightarrow H^2(R, \mathcal{F}). \]
We have used $\mathcal{G} \otimes_R K = \mathcal{G}_{m,K}$. The outer terms of this sequence are trivial due to Hilbert 90 in the case of $H^1(K, \mathcal{G}_{m,K})$ and due to (i) in the case $H^2(R, \mathcal{G})$.

Thus, we have the equality $H^n_k(R, \mu_{p^n,R}) = H^n_k(R, n\mathcal{G})$ of (iv). Furthermore, since $n\mathcal{G}$ is smooth, we can conclude by means of (ii) that $H^n_k(R, n\mathcal{G})$ is isomorphic to $(n\mathcal{G})(K)/(n\mathcal{G})(R) = K^*/(n\mathcal{G})(R)$.

Now, consider the diagram

$$
\begin{array}{c}
0 \longrightarrow \mathcal{G}_{m,R} \longrightarrow n\mathcal{G} \longrightarrow i_\ast n\mathbb{Z} \longrightarrow 0 \\
\| \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{G}_{m,R} \longrightarrow \mathcal{G} \longrightarrow i_\ast \mathbb{Z} \longrightarrow 0.
\end{array}
$$

Since $\mathcal{G}_{m,R}$ is smooth, this diagram induces a diagram with exact lines of $R$-valued points. In particular, the cokernel of the second vertical morphism is $K^*/(n\mathcal{G})(R) = H^1_k(R, n\mathcal{G})$. Obviously, it is isomorphic to $\mathbb{Z}/n$.

Finally, (v) is a direct consequence of (ii).

\[ \square \]

3.3.4 Lemma. Let $f : R_{\text{fl}} \rightarrow R_{\text{et}}$ denote the canonical morphism of sites induced by the identity map. Then we have:

(i) $R^r f_* G = 0$ for a smooth group $G$ and every $r \geq 1$.

(ii) $R^1 f_* \mu_{p^n,R} = f_* \mathcal{G}_{m,R}/(f_* \mathcal{G}_{m,R}) \cdot p^n$. The sheaves $R^r f_* \mu_{p^n,R}$ are trivial for $r \neq 1$.

(iii) $H^r_{\text{fl}}(R, R^1 f_* \mu_{p^n,R}) = H^{r+1}_{\text{fl}}(R, \mu_{p^n,R})$. These groups are trivial for $r \neq 0$.

(iv) $H^r_{\text{et}}(R, R^1 f_* \mu_{p^n,R}) = H^{r+1}_{\text{et}}(R, \mu_{p^n,R})$. For $r = 1$, this group is isomorphic to $\mathbb{Z}/p^n$ and it is trivial for $r \neq 1$.

Proof. (i) is [DixExp], lemma 11.1 with theorem 11.7. Statement (ii) is an easy consequence of the Kummer sequence $0 \rightarrow \mu_{p^n,R} \rightarrow \mathcal{G}_{m,R} \rightarrow \mathcal{G}_{m,R} \rightarrow 0$ on the étale site of $\text{Spec } R$. It yields the following long exact sequence on the étale site of $\text{Spec } R$:

$$
0 \longrightarrow f_* \mathcal{G}_{m,R} \xrightarrow{p^n} f_* \mathcal{G}_{m,R} \longrightarrow R^1 f_* \mu_{p^n,R} \longrightarrow R^1 f_* \mathcal{G}_{m,R}.
$$

(18)

It is exact on the left, since $f_* \mu_{p^n,R} = 0$. Now, (i) implies (ii).

Statements (iii) and (iv) follow with the Leray spectral sequences

$$
H^r_{\text{fl}}(R, R^s f_* \mu_{p^n,R}) \Rightarrow H^{r+s}_{\text{fl}}(R, \mu_{p^n,R}),
$$

$$
H^r_{\text{et}}(R, R^s f_* \mu_{p^n,R}) \Rightarrow H^{r+s}_{\text{et}}(R, \mu_{p^n,R})
$$

and with (ii): The spectral sequences are degenerate with $E^{r,s}_{2} = 0$ for $s \neq 1$, hence $H^r_{\text{fl}}(R, R^1 f_* \mu_{p^n,R}) = H^{r+1}_{\text{fl}}(R, \mu_{p^n,R})$. Again, the Kummer sequence induces the long exact cohomology sequence

$$
H^r(R, \mathcal{G}_{m,R}) \xrightarrow{p^n} H^r(R, \mathcal{G}_{m,R}) \longrightarrow H^{r+1}(R, \mu_{p^n,R}) \longrightarrow H^{r+1}(R, \mathcal{G}_{m,R}).
$$

Since $H^r(R, \mathcal{G}_{m,R}) = 0$ for $r \geq 1$, we have $H^r(R, \mu_{p^n,R}) = 0$ for $r \geq 2$.

By the same arguments applied to the sequence of local cohomology, it follows (iv) in the case $r \neq 1$. The case $r = 1$ is lemma 3.3.3

\[ \square \]

In the following, we will often drop the indices $\text{et}$ and $\text{fl}$. However, it will be unambiguous relative to which site we compute the cohomology groups: When
working relative to the étale site, we will use the functor $f_*$, to indicate that $f_*N$ refers to the sheaf, defined by the group scheme $N$ on the small étale site. When working on the flat site, we identify the scheme $N$ with the appropriate sheaf by means of the Yoneda lemma.

Using these cohomological facts, we want to study Bester’s pairing for the torus $T_K$ and its Néron model $T$ of the rigid uniformisation of abelian varieties. Using lemma 3.3.3, the group $H^1_k(R,T^0)$ is isomorphic to the group of components of $T$.

Since $T_K$ is split, we can reduce to the case of $T_K \cong G_{m,K}^*$; thus its Néron model $T$ is isomorphic to $G_{m}^*$ with its identity component $T_0 \cong G_{m,R}^*$. The exact sequence

$$0 \to G_{m,R}^* \to p^n \to i_*p^n \to 0$$

induces the following exact sequence

$$0 \to H^1_k(R,G_{m,R})^i \to H^1_k(R,p^n) \to H^1_k(R,i_*p^n) \to 0.$$

The latter term is trivial: it fits into the sequence of local cohomology (proposition 3.3.1):

$$0 = H^0(K,i_*p^nZ) \to H^1_k(R,i_*p^nZ) \to H^1(R,i_*p^nZ) = H^1(k,p^nZ) = 0.$$

Due to lemma 3.3.3, there is an epimorphism

$$H^1_k(R,G_{m,R})^i \to H^1_k(R,p^n) = H^2_k(R,\mu_{p^n,R}).$$

Let $M' = \text{Hom}(G_{m,K},G_{m,K}) \cong \mathbb{Z}$ denote the character group of the torus $T = G_{m,K}$. Evaluation of characters $\bar{x} \mapsto (\phi \mapsto \nu(\phi(x)))$ yields a bijective map

$$H^1_k(R,G_{m,R}) = K^*/R^* \xrightarrow{\nu^*} M'\vee.$$

Finally, the exact sequence $0 \to \mathbb{Z}^p \to \mathbb{Z} \to \mathbb{Z}/p^n \to 0$ induces the connecting morphism of the long Ext-sequence

$$\text{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\delta} \text{Ext}^1(\mathbb{Z}/p^n,\mathbb{Z}).$$

The last group is isomorphic to the Pontrjagin dual of $\mathbb{Z}/p^n$, hence, we can write this map as $M'\vee \to (\mathbb{Z}/p^n)^\vee$. Since $\mathcal{F}(\mathbb{Z}/p^n) = \mathbb{Z}/p^n$, these maps fit into the following diagram:

$$\begin{array}{ccc}
H^1_k(R,G_{m,R}) & \xrightarrow{\nu^*} & H^2_k(R,\mu_{p^n,R}) \\
\downarrow_{\nu^*} & & \downarrow_{\text{BP}} \\
M'\vee & \xrightarrow{\delta} & \mathcal{F}(\mathbb{Z}/p^n)^\vee.
\end{array}$$

(19)

To simplify notation, let us write $q := p^n$. The most important result of this section is the following proposition. It will be crucial to the proof that Grothendieck’s pairing can be described by Bester’s pairing.

3.3.5 Proposition. Diagram (19) commutes, i.e. Bester’s pairing is compatible with evaluation of characters.
We will prove this proposition in the setting of derived categories. To do so, let us recall some basic facts on the derived category. Moreover, we will translate some previous results into the setting of derived categories. We start with the following basic “two of three” properties for triangulated categories. They can be seen as a generalisation of the 5-lemma of homological algebra.

3.3.6 Lemma. Let

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & Y'
\end{array}
\]

be two distinguished triangles with morphisms \(X \rightarrow X'\) and \(Y \rightarrow Y'\) such that the resulting square commutes. Then there exists a morphism \(Z \rightarrow Z'\) such that the resulting diagram is a morphism of triangles. If two of these three morphisms are isomorphisms, so is the third.

Proof. The first assertion is the axiom \([\text{TR3}]\) of \([\text{RD}]\), \S 1, the second assertion is \([\text{ibid}]\) proposition 1.1.

We use the following notations:

\[
\Gamma_{k,R} := H^0_k(R, -), \quad \Gamma_R := H^0(R, -), \quad \Gamma_K := H^0(K, -),
\]

with the corresponding derived functors \(R\Gamma_{k,R}\), \(R\Gamma_R\) and \(R\Gamma_K\). Since these functors are \(\bigcup\)-functors by proposition 2.3.4, they preserve injective sheaves. This ensures the existence of various spectral sequences involving those functors.

Let us prove some cohomological facts we need in order to prove proposition 3.3.5. There is a “derived version” of lemma 3.3.4:

3.3.7 Lemma.\(^{(i)}\) \(Rf_*\mu_q,R = (R^1f_*\mu_q,R)[-1]\).

\(^{(ii)}\) \(R\Gamma_{k,R,\text{ét}} \circ Rf_*\mu_q,R = R\Gamma_{k,R,\text{ét}}\mu_q,R = H^2(R, \mu_q,R)[-2]\).

\(^{(iii)}\) \(R\Gamma_{R,\text{ét}} \circ Rf_*\mu_q,R = R\Gamma_{R,\text{ét}}\mu_q,R = H^1(R, \mu_q,R)[-1]\) and analogous over \(K\).

\(^{(iv)}\) \(Rf_*\mathbb{Z}/q = \mathbb{Z}/q\).

We regard the cohomology objects as a complex concentrated in degrees 1, 2, 1 and 0 respectively.

Proof. In the cases of \(^{(ii)}\) and \(^{(iii)}\) keep in mind that \(f_*\) respects injective sheaves, i.e. we can use the fundamental equation \(R\Gamma \circ Rf_* = R(\Gamma \circ f_*)\), for the appropriate functor \(\Gamma_R\) etc.

In all four cases, we have a complex \(X\) which has non trivial cohomology in one level, only (namely \(Rf_*\mu_q,R\) in level 1, \(R\Gamma_{k,R}\mu_q,R\) in level 2, etc., see lemma 3.3.4). By standard arguments in the derived category, the claimed equality follows.

There is the following counterpart of the sequence of local cohomology, lemma 3.3.1 in the derived world:

3.3.8 Lemma. There is a distinguished triangle

\[
R\Gamma_{k,R}(X) \rightarrow R\Gamma_R(X) \rightarrow R\Gamma_K(X) \rightarrow R\Gamma_{k,R}(X)[1]
\]

for every complex \(X\).
Proof. As in [RD], IV, §1, “Motif B” (p. 218).

If we take cohomology of this triangle, we obtain the exact sequence of local cohomology, proposition 3.3.1.

3.3.9 Lemma. There are canonical isomorphisms
(i) $H^2_\ell(R, \mathcal{F}(\mu_{q,R})) = \mathbb{Z}/q$.
(ii) $\pi_1(R^1f_*\mu_{q,R}) = \mathcal{F}(\mu_{q,R})$.

Proof. (i) is shown in the proof of [Bes78], 2, lemma 6.2. Statement (ii) is shown for a group of height 1 in [Bes78], 2, lemma 5.13, with lemma 4.7. For the general case, use the isomorphism $\mathcal{F}(\mu_{q,R}) = \pi_1(R^1\alpha_*\mu_{q,R})$, [Bes78], 1, lemma 3.7, where $\alpha_*$ denotes the Greenberg functor, see the introduction to this section. Using [ADT], remark after theorem III, 10.4, the pro-algebraic group scheme $R^1\alpha_*\mu_{q,R}$ can be identified with $H^1(R, \mu_{q,R})$, regarded as a pro-algebraic group scheme. Due to the Leray spectral sequence, this object can be identified with $R^1f_*\mu_{q,R}$.

Finally, we need to evaluate the cup product induced by Cartier duality. Let $S$ be $R$ or $K$. It is easy to see that the Cartier pairing is given by

$$\mu_{n,S} \times \mathbb{Z}/n \to \mu_{n,S} \subseteq \mathbb{G}_{m,S} \quad \text{(20)} \quad (\bar{\zeta}, \bar{i}) \mapsto \zeta^i.$$

The Cartier pairing induces the following cup product:

3.3.10 Lemma. Let $N$ be a finite, flat group of order $n$ over $S$. Then, Cartier duality $N \times N^D \to \mu_{n,S}$ induces cup products

$$R^r f_* N \times R^s f_* N^D \to R^{r+s} f_* \mu_{n,S}$$

$$H^r(S, N) \times H^s(S, N^D) \to H^{r+s}(S, \mu_{n,S})$$

for the morphism $f: S_{\text{fl}} \to S_{\text{ét}}$.

Proof. The Cartier pairing $N \times N^D \to \mathbb{G}_{m,S}$ induces a morphism $N \otimes N^D \to \mu_{n,S}$. By composition with the cup product, theorem 2.3.5, this induces the desired cup product.

The description of (20) extends to the cup product in the following sense:

3.3.11 Lemma. Let $S$ be $R$ or $K$.

(i) $H^1(S, \mu_{n,S}) = S^*/(S^*)^n$.

(ii) The cup product for $r = 1, s = 0$ is given by

$$H^1(S, \mu_{n,S}) \times H^0(S, \mathbb{Z}/n) \to H^1(S, \mu_{n,S}) \quad (\bar{\zeta}, \bar{i}) \mapsto \zeta^i$$

for an element $\bar{\zeta} \in H^1(S, \mu_n) = S^*/(S^*)^n$ and $\bar{i} \in \mathbb{Z}/n$. 

25
Proof. (i) follows from the Kummer sequence and the triviality of $H^1(K, G_m)$ and $H^1(R, G_m)$ due to Hilbert 90 and lemma 3.3.3.

Due to corollary [2.4.5] we can describe the cup product in terms of Čech cohomology: An element of $\hat{H}^1(S, F)$ can be represented by some element of $H^1(U, F)$ for some covering $U = (U_i \to S)_{i \in I}$. Without loss of generality, we can assume that every $U_i$ is connected such that $(\mathbb{Z}/n)(U_i) = \mathbb{Z}/n$. The cup product of (ii) on Čech cohomology is induced by the Alexander-Whitney formula on Čech cochains, cf. (9)

$$C^1(U, \mu_{n,S}) \times C^0(U, \mathbb{Z}/n) \to C^1(U, \mu_{n,S} \otimes \mathbb{Z}/n)$$

with $(\zeta \cup i)_{j_0,j_1} = \zeta_{j_0,j_1} \otimes i_{j_1}$. The canonical map $\hat{H}^1(U, \mu_{n,S} \otimes \mathbb{Z}/n) \to \hat{H}^1(U, \mu_{n,S})$ is given by $\zeta \cup i \mapsto \zeta^i$ with $(\zeta^i)_{j_0,j_1} = \zeta^{j_1}_{j_0,j_1}$. The composition of both maps yields the description of (ii). □

Using these results, we can prove the commutativity of (19) in the setting of derived categories:

Proof of proposition 3.3.6 In a first step, let us elaborate on the construction of Bester’s pairing as it is sketched in [Bes78], 2, lemma 6.3. Doing so, we see in which way Bester’s pairing is built upon the cup product for $f : R_{\text{fl}} \to R_{\text{et}}$. In a second step, we use this description to evaluate it for $\mu_{q,R}$ and its Cartier dual $\mathbb{Z}/q$.

Let $f : R_{\text{fl}} \to R_{\text{et}}$ be the canonical morphism of sites. In the derived category, the cup product for $f_\ast$ and $\mu_{q,R}$, $\mu^{\text{Der}}_{q,R} = \mathbb{Z}/q$ is a morphism

$$Rf_\ast \mu_{q,R} \to R\mathcal{H}om(Rf_\ast \mathbb{Z}/q, Rf_\ast \mu_{q,R}).$$

Bester shows that there is a morphism

$$d : R\mathcal{H}om(Rf_\ast \mathbb{Z}/q, Rf_\ast \mu_{q,R}) \to R\mathcal{H}om(R\mathcal{F} \circ Rf_\ast \mathbb{Z}/q, R\mathcal{F} \circ Rf_\ast \mu_{q,R})$$

where $R\mathcal{F}$ is defined to be $\pi_1[1] \circ R\alpha_* = L\pi_0 \circ R\alpha_*$ for the Greenberg functor $\alpha_*$, cf. introduction on page 20. Again, we will omit the functor $R\alpha_*$ in the notation. By composition with the derived cup product morphism, this gives rise to a morphism

$$Rf_\ast \mu_{q,R} \to R\mathcal{H}om(R\mathcal{F} \circ Rf_\ast \mathbb{Z}/q, R\mathcal{F} \circ Rf_\ast \mu_{q,R}).$$

As $Rf_\ast \mathbb{Z}/q = \mathbb{Z}/q$ and $\pi_1(\mathbb{Z}/q) = 0$ and $\pi_0(\mathbb{Z}/q) = \mathbb{Z}/q = \mathcal{F}(\mathbb{Z}/q)$, we can conclude that $R\mathcal{F} \circ Rf_\ast \mathbb{Z}/q = \mathbb{Z}/q$. To evaluate $R\mathcal{F} \circ Rf_\ast \mu_{q,R}$ we need more preparations: We know the following:

(i) $Rf_\ast \mu_{q,R} = R^1f_\ast \mu_{q,R}[-1]$, cf. Lemma 3.3.7.

(ii) $\pi_1(R^1f_\ast N) = \mathcal{F}(N)$, by lemma 3.3.9 and

(iii) $\pi_0(R^1f_\ast \mu_{q,R}) = 0$. This group is trivial since the sheaf $R^1f_\ast \mu_{q,R}$ fits into an exact sequence

$$0 \to f_\ast G_{m,R} \xrightarrow{q} f_\ast G_{m,R} \to R^1f_\ast \mu_{q,R} \to 0,$$

cf. (18). As the (pro-)scheme $f_\ast G_{m,R}$ is connected, so is the pro-scheme $R^1f_\ast \mu_{q,R}$.

4 Despite of the notation, $R\mathcal{F}$ is not the derived functor of $\mathcal{F}$. In fact, $\mathcal{F}$ defines an exact functor. Bester chooses this notation because $\pi_1(R^1f_\ast N) = \mathcal{F}(N)$. Cf. lemma 3.3.9.
Using arguments similar to those in the proof of lemma 3.3.7, we can conclude that $R \mathcal{F} \circ R f_* \mu_{q,R} = \mathcal{F} (\mu_{q,R})$.

Applying $R \Gamma_{k,R}$, we get

$$R \Gamma_{k,R} \circ R f_* \mu_{q,R} \to R \Gamma_{k,R} \circ R \hom (\mathcal{F} (\mathbb{Z}/q), \mathcal{F} (\mu_{q,R})).$$

Again, with lemma 3.3.7, this morphism can be written as

$$H^2_k (R, \mu_{q,R}) [-2] \to R \Gamma_{k,R} \circ R \hom (\mathcal{F} (\mathbb{Z}/q), \mathcal{F} (\mu_{q,R})). \quad (22)$$

Next, we want to show that in this very situation the functor $R \Gamma_{k,R}$ commutes with $R \hom$. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/q \to 0$ induces a triangle

$$R \Gamma_{k,R} \circ R \hom (\mathbb{Z}, F (\mu_{q,R})) \to R \Gamma_{k,R} \circ R \hom (\mathbb{Z}/q, F (\mu_{q,R})) \to R \Gamma_{k,R} \circ R \hom (\mathbb{Z}, F (\mu_{q,R}))[1].$$

Since the functors $\hom (\mathbb{Z}, -)$ (for sheaves) and $\hom (\mathbb{Z}, -)$ (for abstract groups) are isomorphic to the identity, we have

$$R \Gamma_{k,R} \circ R \hom (\mathbb{Z}, -) = R \hom (\mathbb{Z}, R \Gamma_{k,R} -).$$

Therefore, this triangle is isomorphic to the triangle

$$R \hom (\mathbb{Z}, R \Gamma_{k,R} \mathcal{F} (\mu_{q,R})) \to R \hom (\mathbb{Z}, R \Gamma_{k,R} \mathcal{F} (\mu_{q,R})) \to R \Gamma_{k,R} \circ R \hom (\mathbb{Z}/q, \mathcal{F} (\mu_{q,R}))[1].$$

Due to the “two of three” properties of triangulated categories, the first term is isomorphic to $R \hom (\mathbb{Z}/q, R \Gamma_{k,R} \mathcal{F} (\mu_{q,R}))$. Since $\mathcal{F} (\mathbb{Z}/q) = \mathbb{Z}/q$, we can write the cup product morphism (22) as

$$\beta: H^2_k (R, \mu_{q,R}) [-2] \to R \hom (\mathcal{F} (\mathbb{Z}/q), R \Gamma_{k,R} \mathcal{F} (\mu_{q,R})). \quad (23)$$

Let us take hypercohomology of the second term. The corresponding spectral sequence is

$$\text{Ext}^r (\mathcal{F} (\mathbb{Z}/q), H^2_k (R, \mathcal{F} (\mu_{q,R}))) \Rightarrow H^{r+s} (R \hom (\mathcal{F} (\mathbb{Z}/q), R \Gamma_{k,R} \mathcal{F} (\mu_{q,R}))),$$

and it induces the edge morphism $E^2 \to E^0_2$:

$$\eta: H^2 (R \hom (\mathcal{F} (\mathbb{Z}/q), R \Gamma_{k,R} \mathcal{F} (\mu_{q,R}))) \to \hom (\mathcal{F} (\mathbb{Z}/q), H^2_k (R, \mathcal{F} (\mu_{q,R}))).$$

Since $H^2_k (R, \mathcal{F} (\mu_{q,R})) = \mathbb{Z}/q$ (lemma 3.3.9), we can compose these morphisms to obtain Bester’s pairing

$$\text{BP} = \eta \circ H^2 (\beta): H^2_k (R, \mu_{q,R}) \to \hom (\mathcal{F} (\mathbb{Z}/q), \mathbb{Z}/q) = \mathcal{F} (\mathbb{Z}/q)^{*}.$$

Next, we want to show that Bester’s pairing is compatible with ordinary cup products. Therefore, consider the triangle of lemma 3.3.8. Together with the cup product morphism, it induces the following morphism of distinguished
triangles. The lower right morphism is the morphism $d$ of (21):

\[
\begin{align*}
\Gamma k \circ & R f_\ast \mu q, R \rightarrow \Gamma k \circ R Hiom(Z/q, R f_\ast \mu q, R) \\
\Gamma R \circ & R f_\ast \mu q, R \rightarrow \Gamma R \circ R Hiom(Z/q, R f_\ast \mu q, R) \\
\Gamma K \circ & R f_\ast \mu q, R \rightarrow \Gamma K \circ R Hiom(Z/q, R f_\ast \mu q, R) \\
\Gamma k \circ & R f_\ast \mu q, R[1] \rightarrow \Gamma k \circ R Hiom(Z/q, R f_\ast \mu q, R)[1] \\
\end{align*}
\]

By the the same “two of three” argument as above, the functors \( \Gamma R \), \( \Gamma K \) and \( \Gamma k, R \) commute with \( Hiom \). Using lemma 3.3.7, we can write this diagram as

\[
\begin{align*}
H^2_k(R, \mu q, R)[−2] \rightarrow & R Hom(Z/q, H^2_k(R, \mu q, R)[−2]) \\
H^1(R, \mu q, R)[−1] \rightarrow & R Hom(Z/q, H^1(R, \mu q, R)[−1]) \\
H^1(K, \mu q, R)[−1] \rightarrow & R Hom(Z/q, H^1(K, \mu q, R)[−1]) \\
H^2_k(R, \mu q, R)[−1] \rightarrow & R Hom(Z/q, H^2_k(R, \mu q, R)[−2])[1] \\
R Hom(\mathcal{F}(Z/q), R \Gamma k (\mu q, R))[1].
\end{align*}
\]

Due to corollary 2.1.9 the functors \( R \Gamma \) and \( R \Gamma k \) transform the cup product of \( f_\ast \) into the cup product for \( H^0 \) or \( H^0_k \) on the small étale site. Now, we take hypercohomology. Using lemma 3.3.9 and the edge morphisms \( E^1 \rightarrow E^2 \), this induces the following diagram:

\[
\begin{align*}
0 & \rightarrow R^\ast / (R^\ast)^q = H^1(R, \mu q, R) \times \mathbb{Z}/q \rightarrow H^1(R, \mu q, R) \\
K^\ast / (K^\ast)^q & = H^1(K, \mu q, R) \times \mathbb{Z}/q \rightarrow H^1(K, \mu q, R) \\
\mathbb{Z}/q & \rightarrow H^2_k(R, \mu q, R) \times \mathbb{Z}/q \rightarrow H^2_k(R, \mu q, R) \\
0 & \rightarrow H^2_k(R, \mu q, R) \times \mathcal{F}(\mathbb{Z}/q) \rightarrow H^2_k(\mu q, R) \rightarrow \mathbb{Z}/q.
\end{align*}
\]

This diagram shows that Bester’s pairing is compatible with the cup product. To analyse the cup products, we can use the cup product defined on Čech cocycles. Lemma 3.3.11 shows that it is given by \((\bar{x}, \bar{i}) \mapsto \bar{x}^i\), for \( \bar{x} \in K^\ast / (K^\ast)^q \) and \( \bar{i} \in \mathbb{Z}/q \).

Consequently, Bester’s pairing can be described as follows: Lift an element \( \zeta \in H^2_k(R, \mu q, R) \) to an element \( \bar{x} \in K^\ast / (K^\ast)^q \). Then, \( \langle \zeta, i \rangle_{BP} = \nu(x^i) \in \mathbb{Z}/q \).
Consequently, the diagram of pairings
\[ \begin{array}{ccc}
H^1(R, G_m) \times M' & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
H^2(R, \mu_q) \times \mathbb{Z}/q & \longrightarrow & \mathbb{Z}/q \subseteq \mathbb{Q}/\mathbb{Z}
\end{array} \tag{24} \]
commutes. The upper pairing is given by \((\bar{x}, n) \mapsto \nu(x^n)\), for \(\bar{x} \in K^*/R^*\), \(n \in M' = \mathbb{Z}\). The lower pairing is given by \((\bar{x}, \bar{n}) \mapsto \nu(x^n)\) for \(\bar{x} \in K^*/(K^*)^q\), \(\bar{n} \in \mathbb{Z}/q\). Thus, diagram (19) commutes. \(\square\)

**Remark.** If we replace the integer \(q\) in diagram (24) by an integer \(n\) which is prime to \(p\) then we can infer that the cup product \(H^2_k(R, \mu_p) = H^1_1(K, \mu_p) \times H^0(K, \mathbb{Z}/n) \rightarrow \mathbb{Z}/n\) is compatible with the map evaluation of characters, \(\nu^*\). See proposition 3.3.11. We will use this observation to combine Bester’s pairing for the \(p\)-part of a groups and the cup product for the prime-to-\(p\)-part of a group to a new pairing that is compatible with the map \(\nu^*\).

In \cite{Bes78}, Bester gives a description of Bester’s pairing for groups of height 1. We can find the result of proposition 3.3.5 in this description as follows: Crucial for Bester’s description is the exact sequence
\[ 0 \longrightarrow R^1 f_* N \longrightarrow \text{Lie} N \otimes \Omega^1_{R/k} \longrightarrow \text{Lie} N \otimes \Omega^1_{R/k} \longrightarrow 0 \]
(cf. \cite{Bes78}, 2, lemma 4.3). In the case of \(N = \mu_p,\) it reads
\[ 0 \longrightarrow f_* G_m/(f_* G_m)^p \mu_{p,R} \xrightarrow{\text{dlog}} \Omega^1_{R/k} \longrightarrow \Omega^1_{R/k} \longrightarrow 0 \]
(cf. \cite{ADT}, III, example 5.9). The morphism \(\text{dlog}: x \mapsto \frac{dx}{x}\) of this sequence is part of the diagram (cf. \cite{Bes78}, bottom of p. 164)
\[ \begin{array}{ccc}
0 & \longrightarrow & \mathcal{H}om(F(\mathbb{Z}/p), \mathbb{Z}/p) \\
\downarrow & & \downarrow \text{BP} \\
0 & \longrightarrow & \mathcal{H}om(\mathcal{O}_k, \mathcal{O}_k) \\
\downarrow & & \downarrow i_1=\text{res}^* \\
0 & \longrightarrow & H^2_k(R, \mu_{p,R}) \\
\downarrow & & \downarrow \text{dlog} \\
0 & \longrightarrow & H^1_k(R, \Omega^1_{R/k})
\end{array} \]
where \(i_1\) is induced by the residue map (\cite{Bes78}, section 2.5). Since \(\text{res} \circ \text{dlog}\) equals the valuation on \(R\) (\cite{Se59}, II, no. 12, proposition 5'), we can conclude that Bester’s pairing for \(\mu_{p,R}\) and \(\mathbb{Z}/p\) is given by the evaluation map.

### 3.4 Bester’s Pairing vs. Grothendieck’s Pairing

From now on, let \(A_K\) be an abelian variety with semistable reduction and Néron model \(A\). Let \(M_K, E_K\) etc. denote the data of rigid uniformisation as in proposition 3.1.1 and 3.1.2.

In the last section, we gave an overview of Bester’s pairing for finite, flat \(R\)-group schemes. Unfortunately, the kernel of \(n\)-multiplication \(A_n\) of the Néron model of an abelian variety with semistable reduction is known to be only a quasi-finite, flat group scheme. In \cite{Be03}, lemma 14, it is shown that it is sufficient to consider only the finite part \(A^f_n\) of \(A_n\) to obtain a suitable duality result for \(A_n\) and \(A^f_n\). Recall that every quasi-finite \(R\)-group scheme \(N\) is the disjoint union of open and closed subschemes \(N^f\) and \(N'\) where \(N^f\) is finite and \(N'\) has an empty special fibre (\cite{BLR}, 2.3, proposition 4).
Since we are interested in a comparison between Bester’s pairing and Grothendieck’s pairing, we assume \( n \in \mathbb{N} \) to be big enough to kill \( \phi_A \) and \( \phi_{A'} \). To simplify notation, we will just write \( \phi \) instead of \( i_* \phi \).

In this situation, we have several important exact sequences:

3.4.1 Proposition. (i) There are exact sequences

\[
\begin{align*}
H_k^2(R,T_n) \xrightarrow{\varphi} H_k^2(R,A_n) \rightarrow H_k^2(R,(E_n')^D) \rightarrow 0 \\
0 \rightarrow \mathcal{F}(E_n') \rightarrow \mathcal{F}(A_n') \rightarrow \mathcal{F}(M'/nM').
\end{align*}
\]

where \( \mathcal{F}(A_n) \) is defined to be \( \mathcal{F}(A_{f_n}^I) \).

(ii) The image of \( \varphi \) is \( \phi_{A_n} \), and

(iii) the image of \( \psi \) is \( \phi_{A_{n}'} \).

Proof. [Be03], lemmas 11, 12 and 13.

As \( n \) does not have to be a power of \( p \), the groups \( A_n, E_n \) etc. can have non-trivial prime-to-\( p \)-parts. As indicated before, we use the perfect cup product to extend Bester’s pairing from the \( p \)-part of the groups to the whole group and call this combined pairing Bester’s pairing again. In this spirit, you have to read the following:

Since \( E_n \) is a finite, flat \( R \)-group scheme, we can apply Bester duality as in [Best78] and we get the following diagram

\[
\begin{array}{ccc}
H_k^2(R,T_n) & \xrightarrow{\varphi} & H_k^2(R,A_n) \\
\downarrow & & \downarrow \phi_A \\
\mathcal{F}(M'/nM')^* & \xrightarrow{\psi} & \mathcal{F}(A_{n}^I)^*
\end{array}
\]

for the morphism \( \Gamma \) as defined in [Be03], Lemma 14. Essentially, it is induced by the closed immersion \( A_{f_n}^I \hookrightarrow A_n \) and Bester’s pairing for the finite, flat \( R \)-group \( (A_{f_n}^I)^D \), cf. [Be03], proof of lemma 14.

Since the images of the first horizontal arrows are isomorphic to \( \phi_A \) and \( \phi_{A_n}' \), the first square of this diagram induces the following diagram:

\[
\begin{array}{ccc}
H_k^2(R,T_n) & \xrightarrow{\varphi} & H_k^2(R,A_n) \\
\downarrow \phi_A & & \downarrow \phi_A \\
\mathcal{F}(M'/nM')^* & \xrightarrow{\phi_{A_n}'} & \mathcal{F}(A_{n}^I)^*
\end{array}
\]

We call the morphism between component groups to be induced by Bester’s pairing (as it is the kernel of the right square of (25)).

3.4.2 Proposition. The morphisms \( \Gamma \) and \( BP : \phi_A \rightarrow \phi_{A_n}' \) are bijective.

Proof. The images of the left horizontal arrows in diagram (25) are \( \phi_{A_n}' \), and \( \phi_A \) respectively. The first isomorphism in (26) induces a surjection between these groups. Due to duality reasons, both groups must have the same cardinality, thus the surjection is indeed bijective. Using the snake lemma (or a diagram chase in (25)) we see that \( \Gamma \) is bijective, too. \( \square \)
3.4.3 Theorem. The pairing of component groups, induced by Bester’s pairing coincides with Grothendieck’s pairing up to sign.

Proof. Let \( n \) be big enough with \( n\phi_A = 0 = n\phi_{A'} \). Due to the remark on page 29, the cup product, i.e. Bester’s pairing at its prime-to-\( p \)-part, is compatible with the map “evaluation of characters”, \( \nu^* \). Thus, we can argue for Bester’s pairing and the cup product simultaneously.

The composition \( T \to E \to A \) that induces the morphism \( \phi_T \to \phi_A \) has a factorisation \( T^0 \to nT \to A^0 \). Thus, there is a factorisation

\[
H^1_k(R, T^0) \to H^1_k(R, nT) \to H^1_k(R, A^0).
\]

Using lemma 3.3.3, it corresponds to the first line in the following diagram (obtained from the right square of diagram (26)):

\[
\begin{array}{ccc}
\phi_T & \to & H^2_k(R, T_n) \\
\downarrow \nu^* & & \downarrow \mathrm{BP} \\
M^{\nu^*} & \to & M^{\nu^*}/nM^{\nu^*} \\
\end{array} \quad \begin{array}{c}
\phi_A \\
\mathrm{GP} \downarrow \mathrm{BP} \\
\phi'A' \\
\end{array}
\]

Keep in mind that \( M^{\nu^*}/nM^{\nu^*} \cong (M'/nM')^* \), \( T^D_n \cong M'/nM' \) and \( \mathcal{F}(\hat{G}) = G \).

In this situation, we know the following:

(i) The left square commutes by proposition 3.3.5.

(ii) The outer diagram – consisting of \( \nu^* \) and \( \mathrm{GP} \) – commutes up to sign by corollary 3.2.2.

(iii) The right square – consisting of Bester’s pairings – commutes.

Since the morphism \( \phi_T \to \phi_A \) is surjective and since both compositions with \( \mathrm{BP} \) and \( \mathrm{GP} \) coincide up to sign, both \( \mathrm{BP} \) and \( \mathrm{GP} \) must coincide up to sign. □

3.5 A new view on Bester’s pairing

Let \( N \) be a finite, flat group scheme over \( R \). Bester’s functor \( \mathcal{F} \) is defined to be the cokernel of \( \pi_1 \tilde{\alpha}_*G \to \pi_1 \tilde{\alpha}_*H \), where

\[
0 \to N \to G \to H \to 0
\]

is a smooth, connected formal resolution of \( N \) and where \( \tilde{\alpha}_* \) is the Greenberg functor. Since smooth group schemes are acyclic for the Greenberg functor, and \( \pi_1 \) is exact on connected proalgebraic group schemes, we want to view this resolution as an acyclic resolution for the functor \( \pi_1 \circ \tilde{\alpha}_* \). In this view, we want to regard Bester’s functor \( \mathcal{F} \) as the first derived functor of \( \pi_1 \circ \tilde{\alpha}_* \). Unfortunately, this cannot be made more explicit, since the category of \( R \)-group schemes (or formal \( R \)-group schemes) does not have enough injective objects. However, let us assume we had the machinery of (hyper)cohomology to handle this composed functor.

In this case, it would be clear that \( \mathcal{F} \) is independent of the smooth, connected resolution. Moreover, if we assume that the first right-derived functor of \( \pi_1 \) is the functor \( \pi_0 \), the Grothendieck spectral sequence would provide the exact sequence of \( [\text{Bes78}], \text{I, Lemma 3.7} \), easily.

5Another proof that the cup product is compatible with Grothendieck’s pairing is given in \( [\text{Bes03}], \text{4.1, proposition 2} \).
In the following we will construct a spectral sequence that serves as a surrogate for the Grothendieck spectral sequence and converges to Bester’s functor.

Let us write $R^0\pi_1$ for the functor $\pi_1$ and $R^1\pi_1$ for the functor $\pi_0$ on proalgebraic group schemes. For $i \neq 0, 1$ set $R^i\pi_1 := 0$

3.5.1 Proposition. Let $G$ be a (formal) $R$-group scheme with a (formal) smooth resolution. Then there exists a spectral sequence

$$R^r\pi_1R^s\tilde{\alpha}_*G \Rightarrow \mathcal{F}^{r+s}G$$

for a family of objects $\mathcal{F}^n(G)$.

Remarks.

(i) A priori, this spectral sequence is only functorial in (formal) $R$-groups together with a smooth (formal) resolution.

(ii) More general, we can use (formal) $R$-groups, which have a resolution of group schemes that are $\tilde{\alpha}_*$-acyclic.

(iii) If $G$ is a smooth $R$-group scheme, we will omit the functor $\tilde{\alpha}_*$ in the notation and write simply $\pi_1G$ and $\pi_0G$, respectively.

Proof. Let us begin with the following observation: If $G$ is a pro-algebraic group, let $\bar{G}$ denote its universal covering. (cf. [GP], §6.2) Let us view the canonical map $G^* : \bar{G} \to G$ as a complex in degrees 0 and 1. Then we have $H^i(G^*) = R^i\pi_1G$.

Combining this observation with hyper cohomology leads to the result:

Since $G$ has a smooth, hence $\alpha_{i,+}$-acyclic resolution $G \to X^*$, consider the following double complex $K$.

$$
\begin{array}{c}
\alpha_*X^0 \xrightarrow{d'} \alpha_*X^1 \xrightarrow{} \alpha_*X^2 \xrightarrow{} \ldots \\
\alpha_*X^0 \xrightarrow{d''} \alpha_*X^1 \xrightarrow{} \alpha_*X^2 \xrightarrow{} \ldots \\
\end{array}
$$

The cohomology groups $^\prime H^*(\alpha_*X^*)$ are the objects $R^*\alpha_*N$. Furthermore, since “universal covering”, is an exact functor, it commutes with cohomology and we have $^\prime H^*(\alpha_*X^*) = R^*\alpha_*G$

The spectral sequence of hypercohomology

$$^\prime H^r(\mathcal{F}^s(K)) \Rightarrow H^{r+s}(\text{Tot } K) =: \mathcal{F}^{r+s}G$$

leads to the result. □

Notation. For later use let us denote the double complex used in the proof by $K(G)$.

We have seen that the above spectral sequence is functorial in (formal) $R$-groups together with a resolution of (formal) smooth $R$-groups. We can weaken this hypothesis:

Since every quotient of a smooth group scheme over $R_i$ is smooth again, [SGA3], exp. VI A, prop 1.3.1, the existence of a smooth, formal resolution is equivalent to the existence of a monomorphism $N \hookrightarrow G$ with a smooth, formal $R$-group $G$. Moreover:
3.5.2 Proposition. 

(i) Let $M \rightarrow N$ be a morphism of (formal) $R$-group schemes such that there exist monomorphisms into smooth (formal) $R$-groups $M \hookrightarrow G$ and $N \rightarrow H$ (We do not demand that these monomorphisms are compatible). Then there is a morphism of smooth resolutions of $M$ and $N$. In particular, the above spectral sequence is functorial, once there are monomorphisms into smooth (formal) $R$-groups.

(ii) Let 
\[ 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \]
be a short exact sequence of (formal) $R$-group schemes such that there exist monomorphisms into smooth (formal) $R$-groups (Again, we do not demand that these monomorphisms are compatible). Then there exist a short exact sequence of smooth (formal) resolutions of $N'$, $N$ and $N''$. In particular, there is a long exact sequence
\[ 0 \rightarrow F^0(N') \rightarrow F^0(N) \rightarrow F^0(N'') \rightarrow F^1(N') \rightarrow F^1(N) \rightarrow F^1(N'') \rightarrow 0 \]

Proof. The assertion of (i) can be proven as the first part of the proof of [Bes78], I, lemma 3.5. To proof (ii) we need to construct in a first step compatible resolutions of $N'$, $N$ and $N''$. This can be done as in the first part of the proof of ibid. lemma 3.9. Once we have these resolutions, (ii) is clear, since the functor “universal covering” is exact. □

Let us collect some first properties of the functors $F^i$. They can be obtained easily by the above spectral sequence.

3.5.3 Proposition. Let $G$ be an $R$-group scheme.

(i) $F^0(G) = \pi_1 \alpha_* G$.

(ii) $F^1(G)$ is trivial for connected, smooth groups $G$, more general

(iii) $F^0(G) = \pi_1 G$ and $F^1(G) = \pi_0 G$ for smooth $R$-group schemes $G$. □

3.5.4 Proposition. On the category of (quasi) finite, flat $R$-group schemes, the functors $F^1$ and the functor $F$ of Bester coincide.

Proof. Let $0 \rightarrow N \rightarrow G_1 \rightarrow G_2 \rightarrow 0$ be a smooth, connected formal resolution of $N$. Due to [3.5.2] it induces the following exact sequence
\[ \pi_1 G_1 \rightarrow \pi_1 G_2 \rightarrow F^1(N) \rightarrow \pi_0(G_1) = 0. \]
Hence $F^1(N)$ is the same quotient as $F(N)$. □

Let $A_K$ be an abelian variety with semistable reduction and let $n \in \mathbb{N}$ be big enough to kill $\phi$ and $\phi'$. Then there is an exact sequence
\[ 0 \rightarrow A_n \rightarrow A \rightarrow A^0 \rightarrow 0 \quad (27) \]
Since $n$-multiplication on $A^0$ is epimorphic, we have the equality $i_* \phi = A_n / A^0_n$. Remember the following equalities of lemma 3.3.3.
3.5.5 Lemma. Let $A_K$ be an abelian variety with semistable reduction and let $A$ be its Néron model.

(i) $H^1_k(R, A) = 0$ and $H^1_k(R, A^0) = \phi$.
(ii) $H^2_k(R, A) = H^3(K, A_K)$.
(iii) $H^1_k(R, (A^0)_n) = H^1_k(R, A_n)$ and $H^1_k(R, A^0) = H^1_k(R, A)$ for $i \geq 1$.

This allows us to reveal the relationship between the pairings of Grothendieck, Bester and Shafarevic:

3.5.6 Proposition. Let $A_K$ be an abelian variety with semistable reduction and let $n \in \mathbb{N}$ big enough to kill $\phi$ and $\phi'$. There is a commutative diagram

\[
\begin{array}{ccccccccc}
H^1_k(R, A) & n & H^1_k(R, A^0) & n & H^2_k(R, A_n) & n & H^2_k(R, A) & n & H^2_k(R, A_n) & 0 \\
\downarrow & \downarrow \text{GP} & \downarrow \text{BP} & \downarrow \text{SP} & \downarrow \text{GP} & \downarrow \text{BP} & \downarrow \text{SP} & \downarrow \text{GP} & \downarrow & \downarrow \\
(\pi_0 A^0)^* & n & (\pi_0 A')^* & \mathcal{F}(A'_n)^* & n & (\pi_1 A^0)^* & \pi_1 A^* & (\pi_1 A')^* & (\pi_1 A_n)^* & 0
\end{array}
\]

where all vertical morphisms are induced by the perfect pairings of Shafarevic, Bester/Bertapelle and Grothendieck.

Proof. The existence of the horizontal sequences is clear. The squares involving BP and SP commute by [Be03], the square involving BP and GP commute by theorem 3.4.3.

Remark. This diagram, of course, induces the sequence of pairings as in (i).

Acknowledgements. I would like to thank Alessandra Bertapelle and Siegfried Bosch for many helpful remarks on previous versions of this article and for valuable discussions on the arithmetic of abelian varieties.

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