Abstract

Actions of $U(n)$ on $U(n+1)$ coadjoint orbits via embeddings of $U(n)$ into $U(n+1)$ are an important family of examples of multiplicity free spaces. They are related to Gelfand-Zeitlin completely integrable systems and multiplicity free branching rules in representation theory. This paper computes the Hamiltonian local normal forms of all such actions, at arbitrary points, in arbitrary $U(n+1)$ coadjoint orbits. The results are described using combinatorics of interlacing patterns; gadgets that describe the associated Kirwan polytopes.

1 Introduction

A Hamiltonian action of a compact connected Lie group $K$ on compact symplectic manifold $(M, \omega)$ with an equivariant moment map is a multiplicity free space if the ring of $K$-invariant functions $C^\infty(M)^K$ is a commutative Poisson subalgebra [GS84a]. The moment map of a multiplicity free space identifies the orbit space, $M/K$, with a convex polytope called the Kirwan polytope after [Kir84]. Compact multiplicity free spaces are classified by their Kirwan polytope and the principal isotropy subgroup of the action [Kno10]. The local classification of multiplicity free spaces (in a neighbourhood of an orbit) is a crucial step in the proof of the classification theorem for compact multiplicity free spaces. It is equivalent to the classification of smooth affine spherical varieties for $G = K^\mathbb{C}$. Smooth affine spherical varieties are classified by their weight monoids [Los09].

One particularly concrete family of examples of multiplicity free spaces is provided by the action of a unitary group, $U(n)$, on a coadjoint orbit of the unitary group $U(n+1)$ via an embedding of $U(n)$ into $U(n+1)$ (Section 3.1). The Kirwan polytopes of these spaces can be described as the set of points $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ that satisfy the so-called interlacing inequalities,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_n \geq \lambda_{n+1},$$ (1)

where $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$ are fixed parameters determined by the coadjoint orbit. The main result of this paper (Theorem 3.3) is the computation of the local classifying data of these spaces at arbitrary points in arbitrary $U(n+1)$ orbits. This result has two interesting features. First, the classifying data are described in terms of combinatorial gadgets called interlacing patterns that encode the combinatorics of the Kirwan polytopes after [Kir84].

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polytope (see Section 3.2). An example of an interlacing pattern is illustrated below. It corresponds to certain points in $U(8)$ coadjoint orbits diffeomorphic to $U(8)/U(2) \times U(1) \times U(2) \times U(1) \times U(1) \times U(1)$.

![Interlacing pattern diagram]

The second interesting feature is the proof (given in Section 4). Rather than using the classification of smooth affine spherical varieties, the classifying data are computed directly by elementary means. Following several standard reductions, the main step in this proof is the explicit computation of the isotropy representations (Section 4.1). It is shown that they are certain products of standard representations and trivial representations of factors of the isotropy subgroup, which has a block diagonal form. The block diagonal factors of the isotropy subgroup that act by standard representations correspond to “parallelogram shapes” that appear in the interlacing pattern. For example, the isotropy subgroup corresponding to the interlacing pattern above is $U(1) \times U(1) \times U(2) \times U(1)$ and the isotropy representation is $\{0\} \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}$ (see Example 5). The computation of this representation relies on the relationship between the combinatorics of interlacing patterns and divisibility properties of characteristic polynomials of certain Hermitian matrices.

Motivation for this work is provided by the Gelfand-Zeitlin\(^1\) commutative completely integrable systems [GS83]. Although Gelfand-Zeitlin systems have been studied extensively in recent years (see e.g. [ALL18, BMZ18, CKO17, Lan18]), very little is known about their local normal forms as integrable systems near singular fibers (see Example 6). An ongoing program aims to use the results of this paper to prove topological and symplectic local normal forms for Gelfand-Zeitlin systems. The multiplicity free spaces studied in this paper, as well as the associated Gelfand-Zeitlin systems, have analogues for orthogonal groups and orthogonal coadjoint orbits. The local models of those multiplicity free spaces can be computed in a similar fashion.

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## 2 Hamiltonian group actions and local normal forms

This section fixes conventions, notation, and recalls the statement of the Marle-Guillemin-Sternberg local normal form. Standard references are [Aud04, GS84b] modulo conventions.

### 2.1 Hamiltonian group actions

Let $K$ be a connected Lie group. Denote its Lie algebra by $\mathfrak{k}$, the dual vector space by $\mathfrak{k}^*$, and the dual pairing by $\langle -, \cdot \rangle$. Let $\text{Ad}$ and $\text{Ad}^*$ denote the adjoint and coadjoint actions respectively, i.e $\langle \text{Ad}_{k}^* \xi, X \rangle = \langle \xi, \text{Ad}_{k^{-1}} X \rangle$ for $k \in K$, $\xi \in \mathfrak{k}^*$, and $X \in \mathfrak{k}$. Given a left action of $K$ on a manifold $M$, the fundamental vector field of $X \in \mathfrak{k}$ is

$$X_p = \frac{d}{dt} \bigg|_{t=0} \exp(tX) \cdot p,$$

$p \in M$.

\(^1\)Also spelled Gelfand-Cetlin and Gelfand-Tsetlin.
Let \((M, \omega)\) a symplectic manifold. A left action of \(K\) on \(M\) is Hamiltonian if there exists an equivariant map \(\Phi : M \to \mathfrak{t}^*\) such that 
\[
\mathcal{L}_X\omega = d(\Phi, X).
\]
A map \(\Phi\) with this property is called a moment map. The tuple \((M, \omega, \Phi)\) is a Hamiltonian \(K\)-manifold. Hamiltonian \(K\)-manifolds \((M, \omega, \Phi)\) and \((M', \omega', \Phi')\) are isomorphic if there exists a \(K\)-equivariant, symplectic diffeomorphism \(\varphi : (M, \omega) \to (M', \omega')\) such that \(\Phi' \circ \varphi = \Phi\).

**Example 1** (Coadjoint orbits). Let \(\Theta \subset \mathfrak{t}^*\) an orbit of the coadjoint action of \(K\). Given \(\xi \in \Theta\), the tangent space \(T_\xi \Theta \subset \mathfrak{t}^*\) is the set of elements of the form \(\text{ad}_{\hat{X}}^* \xi, \ X \in \mathfrak{t}\). The Kostant-Kirillov-Souriau symplectic form \(\omega_{\text{KKS}}\) on \(\Theta\) is defined pointwise by the formula 
\[
(\omega_{\text{KKS}})(\text{ad}_{\hat{X}}^* \xi, \text{ad}_{\hat{Y}}^* \xi) = \langle \xi, [X, Y] \rangle.
\]
The inclusion map \(\iota : \Theta \to \mathfrak{t}^*\) is a moment map for the coadjoint action of \(K\) on \((\Theta, \omega_{\text{KKS}})\). △

**Example 2** (Homomorphisms). Let \((M, \omega, \Phi)\) a Hamiltonian \(K\)-manifold, \(H\) a Lie group, and \(\varphi : H \to K\) a Lie group homomorphism. Let \((d\varphi)^* : \mathfrak{t}^* \to \mathfrak{h}^*\) denote the linear map dual to \(d\varphi : \mathfrak{h} \to \mathfrak{t}\). Then the action of \(H\) on \(M\) defined via the action of \(K\) and the homomorphism \(\varphi\) is Hamiltonian and \((d\varphi)^* \circ \Phi\) is a moment map. △

Let \(U(n)\) denote the group of \(n \times n\) unitary matrices, with Lie algebra \(\mathfrak{u}(n)\), and let \(\mathcal{H}_n\) denote the set of \(n \times n\) Hermitian matrices, \(X = X^\dagger\), where \(X \mapsto X^\dagger\) denotes conjugate transpose. Fix the isomorphism \n\[
\mathcal{H}_n \to \mathfrak{u}(n)^*, \quad X \mapsto \left(A \mapsto \frac{1}{\sqrt{-1}} \text{Tr}(XA)\right).
\]
It is equivariant with respect to the action of \(U(n)\) on \(\mathcal{H}_n\) by conjugation, \(k \cdot X = kXk^\dagger\).

**Example 3** (Representations). Identify \(\mathbb{C}^n \cong M_{n \times 1}(\mathbb{C})\). The standard symplectic form on \(\mathbb{C}^n\) is 
\[
\omega_{\text{std}}(x, y) = \frac{1}{2\sqrt{-1}}(x^\dagger y - y^\dagger x), \quad x, y \in M_{n \times 1}(\mathbb{C}).
\]
The action of \(U(n)\) on \(\mathbb{C}^n\) by the standard representation is Hamiltonian with moment map 
\[
\Phi(x) = -\frac{1}{2}xx^\dagger.
\]
More generally, suppose that \(V\) is a real vector space equipped with a linear symplectic form \(\omega_V\). Let \(\rho : K \to Sp(V, \omega_V)\) be a representation of \(K\) on \(V\) by symplectic transformations. Then the action of \(K\) on \((V, \omega_V)\) defined by \(\rho\) is Hamiltonian with moment map \(\Phi_V\) defined by the condition 
\[
\frac{1}{2} \omega_V(d\rho(X)v, v) = \langle \Phi_V(v), X \rangle, \quad \forall v \in V.
\]

**Example 4** (Isotropy representations). Let \((M, \omega, \Phi)\) a Hamiltonian \(K\)-manifold. Given \(p \in M\), let \(K \cdot p\) denote the orbit of the action of \(K\) through \(p\) and let \(K_p \leq K\) denote the isotropy subgroup: the subgroup of elements that fix \(p\). Let \(K_{\Phi(p)}\) denote the isotropy subgroup of \(\Phi(p)\). Then \(K_p \leq K_{\Phi(p)}\). The symplectic slice at \(p \in M\) is the vector space 
\[
W_p = T_p(K \cdot p)/\{T_p(K \cdot p) \cap T_p(K \cdot p)\}^{\omega},
\]
where \(T_p(K \cdot p)^{\omega}\) denotes the subspace of elements \(X \in T_pM\) such that \(\omega_p(X, Y) = 0\) for all \(Y \in T_p(K \cdot p)\). The restriction of \(\omega_p\) to \(T_p(K \cdot p)^{\omega}\) descends to a symplectic form on \(W_p\) denoted \(\omega_p\). The linearization of the action of \(K_p\), a.k.a. the isotropy representation, preserves the subspaces \(T_p(K \cdot p)^{\omega}\) and \(T_p(K \cdot p) \cap T_p(K \cdot p)^{\omega}\), so it descends to an action of \(K_p\) on \((W_p, \omega_p, \Phi_W)\) by symplectic transformations. Thus \((W_p, \omega_p, \Phi_W)\) is a Hamiltonian \(K_p\)-manifold, where \(\Phi_W\) is defined as in Example 3. △
2.2 Marle-Guillemin-Sternberg local normal forms

Given a connected Lie group $K$, Marle-Guillemin-Sternberg data (MGS data) is a tuple $(\xi, L, W, \omega_W)$ where $\xi \in \mathfrak{t}^*$, $L$ is a Lie subgroup of $K_{\xi}$, and $(W, \omega_W)$ is a symplectic vector space equipped with a representation of $L$ by symplectic transformations.

Given MGS data $(\xi, L, W, \omega_W)$, [GS84b, Mar85] construct a Hamiltonian $K$-manifold, denoted $M(\xi, L, W, \omega_W)$, with the following properties. Let $m = \xi/\Gamma$ and identify $m^\ast$ with a $L$-invariant complement of $\Gamma^\ast$ in $\mathfrak{t}^\ast$. As a manifold, $M(\xi, L, W, \omega_W)$ is the total space of the vector bundle

$$K \times_L (m^\ast \times W) \to K/L$$

(6)

associated to the principal bundle $L \to K \to K/L$ and the representation $m^\ast \times W$. The symplectic structure on $M(\xi, L, W, \omega_W)$ is determined by the data $(\xi, L, W, \omega_W)$ (see [GS84b, GS84c, Mar85] for more details).

With respect to this diffeomorphic description of $M(\xi, L, W, \omega_W)$, the Hamiltonian action of $K$ and the corresponding moment map are

$$k' \cdot [k, \eta, w] = [k' k, \eta, w],$$

$$\Phi([k, \eta, w]) = \text{Ad}^*_k(\eta + \Phi_W(w) + \xi).$$

(7)

Let $(M, \omega, \Phi)$ be a Hamiltonian $K$-manifold. The Marle-Guillemin-Sternberg data of a point $p \in M$ is $(\Phi(p), K_p, W_p, \tau_p)$, where $K_p$ is the isotropy subgroup of $p$ and $(W_p, \tau_p)$ is the symplectic slice at $p$ equipped with the isotropy representation of $K_p$ as described in Example 4.

**Theorem 2.1** (Marle-Guillemin-Sternberg local normal forms). [GS84c, Mar85] Let $(M, \omega, \Phi)$ a Hamiltonian $K$-manifold. For all $p \in M$ there exists $K$-invariant neighbourhoods $U \subset M$ of the orbit $K \cdot p$ and $U' \subset M(\Phi(p), K_p, W_p, \tau_p)$ of the orbit $K \cdot [e, 0, 0]$ and an isomorphism of Hamiltonian $K$-manifolds $\varphi: U \to U'$ such that $\varphi(p) = [e, 0, 0]$.

**Hamiltonian $K$-manifolds $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ are equivalent** if there exists an automorphism $\psi$ of $K$, a symplectomorphism $F': (M, \omega) \to (M', \omega')$, and an $\text{Ad}^*_K$-fixed element $\xi \in \mathfrak{t}^*$ such that:

1. $\psi(k) \cdot F'(m) = F(k \cdot m)$, and
2. $\Phi + \xi = (d\psi)^* \circ \Phi' \circ F$.

Marle-Guillemin-Sternberg data $(\xi, L, W, \omega_W)$ and $(\xi', L', W', \omega_{W'})$ for $K$ are equivalent if the corresponding model spaces are equivalent as Hamiltonian $K$-manifolds. For instance, if $p$ and $p'$ are in the same $K$-orbit, then the MGS data of $p$ and $p'$ are equivalent.

3 Statement of the main theorem

The following notation will be useful in the remainder of the paper. Given a sequence of real numbers $\tau = (\tau_1, \ldots, \tau_n)$, let $[\tau]$ denote the set of elements in $\tau$. Let $\tau_i$ denote the $i$th element of $[\tau]$ in decreasing order. Let $m(\tau)$ denote the size of $[\tau]$. Let $n_+(\tau)$ denote the number of times $\tau$ occurs in $\tau$. Let $n_i(\tau)$ denote the number of times $\tau_i$ occurs in $\tau$. 


3.1 Multiplicity free $U(n)$ actions on $U(n+1)$ coadjoint orbits

Given a non-increasing sequence of real numbers $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$, let $\Theta_\Lambda$ denote the set of matrices in $\mathcal{H}_{n+1}$ with eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$. Then $\Theta_\Lambda$ is the orbit of

$$\Lambda := \begin{pmatrix}
\lambda_1 \\
\ddots \\
\lambda_{n+1}
\end{pmatrix} \quad (8)$$

under the action of $U(n+1)$ by conjugation and the map $k \mapsto k\lambda k^\dagger$ descends to a $U(n+1)$-equivariant diffeomorphism

$$U(n+1)/U(n_1(\underline{\lambda})) \times \cdots \times U(n_{m(\underline{\lambda})}(\underline{\lambda})) \to \Theta_\Lambda. \quad (9)$$

The map (2) defines a $U(n)$-equivariant diffeomorphism of $\Theta_\Lambda$ with a coadjoint orbit of $U(n+1)$. Let $\omega_\Lambda$ denote the symplectic form on $\Theta_\Lambda$ defined by this identification and the Kostant-Kirillov-Souriau symplectic form defined in Example 1. For all $p \in \Theta_\Lambda$,

$$(\omega_\Lambda)_p([X, p], [Y, p]) = \frac{1}{\sqrt{-1}} \Tr (p[Y, X]) \quad \forall X, Y \in u(n+1). \quad (10)$$

With respect to (2), $(\Theta_\Lambda, \omega_\Lambda, \iota: \Theta_\Lambda \to \mathcal{H}_{n+1})$ is a Hamiltonian $U(n+1)$-manifold, where $\iota$ denotes inclusion. Let $K = U(n)$ and let $\varphi: K \to U(n+1)$ be an embedding of $K$ as a Lie subgroup of $U(n+1)$. With respect to the identification (2), $(d\varphi)^*$ is a linear projection $\mathcal{H}_{n+1} \to \mathcal{H}_n$. By Example 2, $(\Theta_\Lambda, \omega_\Lambda, \Phi)$ is a Hamiltonian $K$-manifold with moment map

$$\Phi = (d\varphi)^* \circ \iota: \Theta_\Lambda \to \mathcal{H}_n. \quad (11)$$

It is well-known that $(\Theta_\Lambda, \omega_\Lambda, \Phi)$ are multiplicity free spaces for all possible choices of $\underline{\lambda}$ and $\varphi$ (this follows from Lemma 4.1 below).

3.2 Interlacing patterns

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$ and $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ be non-increasing sequences of numbers that satisfy the interlacing inequalities (1). The inequalities (1) are represented by attaching labels to a fixed set of $2n+1$ vertices arranged on a triangular grid as illustrated by the following example:

$$\begin{array}{cccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7
\end{array} \quad (12)$$

If a vertex labelled $x$ appears to the left of a vertex labelled $y$, then $x \geq y$. The labels on the top row correspond to $\underline{\lambda}$ and the labels on the bottom row correspond to $\underline{\mu}$.

The (labelled) interlacing pattern of a pair of sequences $(\underline{\lambda}, \underline{\mu})$ that satisfy (1) is the labelled undirected plane graph obtained by adding straight edges to the diagram above according to the following rule: two
vertices are connected by an edge iff they are nearest neighbours and their labels are equal. For example, the following is the interlacing pattern of \((\lambda, \mu)\) where \(\lambda = (6, 6, 5, 3, 2, 1, 0)\) and \(\mu = (6, 5, 4, 3, 3, 1, 1)\).

Three types of connected components can occur in interlacing patterns: \(\nabla\)-shapes, \(\Delta\)-shapes, and \(\square\)-shapes. In the example (13): the components labelled 6 and 2 are \(\nabla\)-shapes, the components labelled 4 and 1 are \(\Delta\)-shapes, and the components labelled 5 and 3 are \(\square\)-shapes. By convention, an isolated vertex on the top row is a \(\nabla\)-shape and an isolated vertex on the bottom row is a \(\Delta\)-shape.

If \(\lambda = (\lambda_1, \ldots, \lambda_{n+1})\) is fixed, then the set of pairs \((\lambda, \mu)\) that satisfy (1) (equivalently, the set of labelled interlacing patterns whose labels on the top row are given by \(\lambda\)) is in bijection with elements of the polytope

\[
\Delta_\lambda := \{ \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n | (\lambda, \mu) \text{ satisfies (1)} \}.
\]

Given \((\Theta_\Lambda, \omega_\Lambda, \Phi)\) as in the previous section, a point \(p \in \Theta_\Lambda\) determines a pair \((\lambda, \mu)\) that satisfies (1), where \(\mu = (\mu_1, \ldots, \mu_n)\) denotes the eigenvalues of \(\Phi(p)\) arranged in non-increasing order. Thus, every \(p \in \Theta_\Lambda\) has an associated labelled interlacing pattern. As observed in [GS83], the polytope \(\Delta_\lambda\) defined above is the Kirwan polytope of \((\Theta_\Lambda, \omega_\Lambda, \Phi)\), i.e.

\[\Delta_\lambda = \{ (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n | \mu_1 \geq \cdots \geq \mu_n, \exists p \in \Theta_\Lambda \text{ with eigenvalues } \mu_1, \ldots, \mu_n \} \]

The notation \(\lambda \in [\lambda]\)-shape denotes the set of all \(\lambda \in [\lambda]\) such that the connected component of the interlacing pattern of \((\lambda, \mu)\) labelled by \(\lambda\) is a \(\nabla\)-shape. Similar notation is used for other sets. For example, any pair \((\lambda, \mu)\) satisfying (1) satisfies the identity

\[
\sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^n \mu_i = \sum_{\lambda \in [\lambda]} \sum_{\nabla\text{-shape}} \lambda - \sum_{\mu \in [\mu]} \sum_{\Delta\text{-shape}} \mu.
\]

Remark 3.1. An unlabelled interlacing pattern is an undirected plane graph that can be obtained from a labelled interlacing pattern by erasing the labels. In other words, the edges in an unlabelled interlacing pattern must correspond to a configuration of equalities and strict inequalities that is allowed by (1). For instance, the following is an unlabelled interlacing pattern.

\[
\nabla / \nabla / \Delta \cdot \Delta \cdot
\]

On the other hand, the following is not an unlabelled interlacing pattern.

\[
\nabla / \nabla / \nabla \cdot \Delta \cdot
\]
If $\mu$ and $\mu'$ are contained in the relative interior of the same face of $\Delta_{\lambda}$, then the unlabelled interlacing patterns of $(\lambda, \mu)$ and $(\lambda, \mu')$ are the same. Thus the set of unlabelled interlacing patterns obtained by erasing labels from labelled interlacing patterns of pairs $(\lambda, \mu)$, $\lambda$ fixed, is in natural bijection with the set of faces of $\Delta_{\lambda}$. The partial order on faces of $\Delta_{\lambda}$ corresponds to an obvious partial order on the set of all such unlabelled interlacing patterns. Thus, they encode $\Delta_{\lambda}$ as an abstract polytope. It is also straightforward to read the local moment cone of a point $\underline{\mu} \in \Delta_{\lambda}$ from the unlabelled interlacing pattern of $(\lambda, \underline{\mu})$. The intersection of this local moment cone with the standard lattice in $\mathbb{R}^n$ is the weight monoid of the corresponding smooth affine spherical variety that appears in the classification of [Kno10].

**Remark 3.2.** The interlacing patterns described here occur as rows in larger diagrams, also called interlacing patterns, that describe points and faces of Gelfand-Zeitlin polytopes as well as fibers of Gelfand-Zeitlin systems (see e.g. [ACK18, CKO17, Pab14, BMZ18]). Some authors use an equivalent combinatorial gadget called *ladder diagrams* and introduce terminology such as $W$-blocks, $M$-blocks, and $N$-blocks that is used here.

### 3.3 Statement of the main theorem

Let $K = U(n)$ and let $(\lambda, \mu)$ be a pair of non-increasing sequences $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ and $\mu = (\mu_1, \ldots, \mu_n)$ that satisfy the interlacing inequalities (1). Let $M := \text{diag}(\mu_1, \ldots, \mu_n)$. The stabilizer subgroup $K_M$ for the conjugation action of $K$ is a block diagonal subgroup isomorphic to $U(n_1(\underline{\mu})) \times \cdots \times U(n_m(\underline{\mu}))$. Define

$$W_{(\lambda, \mu)} := \bigoplus_{\mu \in [\underline{\mu}]} \mathbb{C}^{n(\mu)},$$

and the block-diagonal subgroup

$$L_{(\lambda, \mu)} := L_1 \times \cdots \times L_{m(\underline{\mu})} \leq U(n_1(\underline{\mu})) \times \cdots \times U(n_m(\underline{\mu})) = K_M$$

where

$$L_i = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \mid k \in U(n_i(\underline{\mu}) - 1) \right\} \leq U(n_i(\underline{\mu}))$$

if the component of the interlacing pattern of $(\lambda, \mu)$ labelled $\underline{\mu}$ is a $\square$-shape, and $L_i = U(n_i(\underline{\mu}))$ otherwise. Equip $W_{(\lambda, \mu)}$ with the representation of $L_{(\lambda, \mu)}$ where the factor $L_i$ acts by the standard representation on the corresponding factor $\mathbb{C}^{n(\mu)}$ if the component of the interlacing pattern of $(\lambda, \mu)$ labelled $\underline{\mu}$ is a $\square$-shape, and it acts trivially otherwise.

**Example 5.** Consider the interlacing pattern in Figure 13. Then $M = \text{diag}(6, 5, 4, 3, 3, 1, 1)$,

$$L_{(\lambda, \mu)} = \left\{ \begin{pmatrix} k_6 \\ k_5 \\ 1 \\ k_3 \\ 1 \\ k_1 \end{pmatrix} \mid k_6, k_5, k_1 \in U(1), k_3 \in U(2) \right\}$$

$$W_{(\lambda, \mu)} = \{0\} \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}.$$

The representation of $L_{(\lambda, \mu)}$ on $W_{(\lambda, \mu)}$ is $(k_6, k_5, k_3, k_1) \cdot (z_5, z_3) = (k_5z_5, k_3z_3)$. △
For $\mu \in \underline{\mu}$, define $r_\mu \geq 0$ such that

\[
    r_\mu^2 = - \left( \prod_{\lambda \in [\lambda]} (\mu - \lambda) \right) \left( \prod_{\tau \in [\mu], \Delta\text{-shape}} \frac{1}{(\mu - \tau)} \right)
\]

(21)

if the connected component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\mu$ is a $\Delta$-shape, and $r_\mu = 0$ otherwise.

If the connected component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\mu$ is a $\Delta$-shape, then $r_\mu^2 > 0$.

Provided that the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\mu = \mu_i$ is not a $\Delta$-shape, define

\[
    C_i := C_\mu := \frac{n + 1}{n} \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i - \mu + \sum_{\tau \in [\mu], \Delta\text{-shape}} \frac{r_\tau^2}{\mu - \tau}.
\]

(22)

Finally, define a linear symplectic form on $W(\underline{\lambda}, \underline{\mu})$ by the formula

\[
    \omega(\underline{\lambda}, \underline{\mu})(u, w) := \frac{1}{\sqrt{-1}} \sum_{\mu \in [\mu], \Delta\text{-shape}} \frac{-u_\mu^\dagger w_\mu + w_\mu^\dagger u_\mu}{C_\mu},
\]

(23)

for all $u, w \in W(\underline{\lambda}, \underline{\mu})$, where $u_\mu$ denotes the projection of $u$ to the factor $\mathbb{C}^{n_\mu(\mu)}$.

**Theorem 3.3.** Let $K = U(n)$ and let $(\Theta_\Lambda, \omega_\Lambda, \Phi)$ be the Hamiltonian $K$-manifold associated to a non-increasing sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$ and an embedding $\varphi: K \to U(n + 1)$ as in Section 3.1. Then, the Marle-Guillemin-Sernberg local normal form data of $p \in \Theta_\Lambda$ is equivalent to

\[
    (M, L(\underline{\lambda}, \underline{\mu}), W(\underline{\lambda}, \underline{\mu}), \omega(\underline{\lambda}, \underline{\mu}))
\]

(24)

where $(\underline{\lambda}, \underline{\mu})$ is determined by $p$ as in Section 3.2 and $M, L(\underline{\lambda}, \underline{\mu}), W(\underline{\lambda}, \underline{\mu})$, and $\omega(\underline{\lambda}, \underline{\mu})$ are as defined above.

The proof of Theorem 3.3, given in Section 4, describes an explicit linear isomorphism between the isotropy representation at $p$ and the symplectic representation $(W(\underline{\lambda}, \underline{\mu}), \omega(\underline{\lambda}, \underline{\mu}))$.

**Remark 3.4.** It is straightforward to check that as $L(\underline{\lambda}, \underline{\mu})$-representations,

\[
    \mathfrak{m}^* \cong \bigoplus_{\mu \in [\mu], \Delta\text{-shape}} (\mathbb{R} \times \mathbb{C}^{n_\mu(\mu)-1})
\]

(25)

where if the component of the interlacing pattern labelled $\mu_i$ is a $\Delta$-shape, then the factor $L_i \cong U(n_i(\mu) - 1)$ acts on the corresponding factor $\mathbb{R} \times \mathbb{C}^{n_i(\mu)-1}$ as the product of the trivial representation and the standard representation. Otherwise the factor $L_i$ acts trivially. The moment map of the local normal form $M(M, L(\underline{\lambda}, \underline{\mu}), W(\underline{\lambda}, \underline{\mu}), \omega(\underline{\lambda}, \underline{\mu}))$ is easily computed by combining Example 3 and (7).

**Example 6.** Let $\lambda_1 > \lambda_2 > \lambda_3$ and let $p \in \Theta_\Lambda$ such that the eigenvalues of $\Phi(p)$ are $\mu_1 = \mu_2 = \lambda_2$. The interlacing pattern of $p$ is
It follows from Theorem 3.3 that the orbit through \( p \) is a Lagrangian \( U(2)/U(1) \cong S^3 \) and a neighbourhood of this orbit is isomorphic to a neighbourhood of the zero section in \( T^*S^3 \), equipped with the Hamiltonian action of \( U(2) \) by cotangent lift of the action of \( U(2) \) on \( S^3 \). This particular example was derived by [Ala09] who used it to show that the Gelfand-Zeitlin systems on regular \( U(3) \) coadjoint orbits are isomorphic, in a neighbourhood of this Lagrangian \( S^3 \) fiber, to an integrable system for the normalized geodesic flow on \( T^*S^3 \). △

4 Proof of Theorem 3.3

Let \( K = U(n) \) and fix an arbitrary non-increasing sequence \( \Delta = (\lambda_1, \ldots, \lambda_{n+1}) \). Several standard reductions are in order.

First, any two embeddings \( K \to U(n + 1) \) are related by an inner automorphism of \( U(n + 1) \) and such an inner automorphism determines an equivalence of the associated Hamiltonian \( K \)-manifolds. Thus, it is sufficient to compute the MGS data with respect to the embedding

\[
\varphi: K \to U(n + 1), \quad k \mapsto \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.
\]

(26)

With respect to (2),

\[
(d\varphi)^*: \mathcal{H}_{n+1} \to \mathcal{H}_n, \quad (d\varphi)^*(X) = X^{(n)},
\]

(27)

where \( X^{(n)} \) denotes the bottom right principal \( n \times n \) submatrix of \( X \). Thus \( \Phi(X) = X^{(n)} \).

Second, it is sufficient to compute the MGS data for points of the form

\[
p = \left( \begin{array}{c|c|c|c|c}
c & z_1 & z_2 & \cdots & z_{n-1} \\
\hline
z_1 & \mu_1 & & & \\
 z_2 & & \mu_2 & & \\
 \vdots & & & \ddots & \\
 z_{n-1} & & & & \mu_{n-1} \\
 z_n & & & & \mu_n \\
\end{array} \right), \quad z_i \in \mathbb{C} \text{ and } c = \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^n \mu_i,
\]

(28)

where \( \mu_1 \geq \cdots \geq \mu_n \). Indeed, every point in \( O_\Delta \) can be brought to this form by the action \( U(n) \), so its MGS data is equivalent to the MGS data of a point of this form. Note that \( p \in \Phi^{-1}(M) \) if and only if \( p \) is of the form (28).

Before giving the final reduction, recall from [GS83] that the condition \( p \in O_\Delta \), for \( p \) of the form (28), is equivalent to the following equality of characteristic polynomials,

\[
\prod_{i=1}^{n+1} (x - \lambda_i) = (x - c) \prod_{i=1}^n (x - \mu_i) - \sum_{i=1}^n z_i \prod_{j=1}^n (x - \mu_j).
\]

(29)
Re-write \( p \) in block form

\[
p = \begin{pmatrix}
  c & z_1^\dagger & z_2^\dagger & \cdots & z_m^\dagger \\
  z_1 & \mu_1 I_{n_1(\mu)} & 0 & \cdots & 0 \\
  z_2 & 0 & \mu_2 I_{n_2(\mu)} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  z_m & 0 & 0 & \cdots & \mu_m I_{n_m(\mu)} \\
\end{pmatrix}, \quad z_i \in M_{n_i(\mu) \times 1}(\mathbb{C}).
\]

where \( m = m(\underline{\mu}) \). If \( \mu = \underline{\mu} \), let \( z_\mu = z_i \) denote the corresponding block. Then (29) becomes

\[
\prod_{\lambda \in [\underline{\lambda}]} (x - \lambda)^{n_\lambda(\underline{\lambda})} = (x - c) \prod_{\mu \in [\underline{\mu}]} (x - \mu)^{n_\mu(\underline{\mu})} - \sum_{\mu \in [\underline{\mu}]} ||z_\mu||^2 (x - \mu)^{n_\mu(\underline{\mu})-1} \prod_{r \in [\underline{\mu}] \setminus \tau \neq \mu} (x - \tau)^{n_r(\underline{\tau})}. \quad (31)
\]

The following lemma is well-known. It’s proof is left as an exercise using the fact that \( p \in \mathcal{O}_\Lambda \) iff \( p \) satisfies (31).

**Lemma 4.1.** Let \( p \) of the form (30). Then \( p \in \mathcal{O}_\Lambda \) if and only if for all \( \mu \in [\underline{\mu}] \), \( ||z_\mu||^2 = r_\mu^2 \). Moreover, the action of \( K_M \) on \( \Phi^{-1}(M) \) is transitive.

The final reduction concerns the isotropy subgroup. Given \((\underline{\lambda}, \underline{\mu})\), define \( \tilde{p} \in \mathcal{O}_\Lambda \) of the form (30) such that for all \( \mu \in [\underline{\mu}] \),

\[
   z_\mu = \begin{pmatrix}
   r_\mu \\
   0 \\
   \vdots \\
   0
\end{pmatrix}.
\]

By construction, \( K_{\tilde{p}} = L_{(\underline{\lambda}, \underline{\mu})} \). The MGS data of every other point \( p \in \Phi^{-1}(M) \) is equivalent to that of \( \tilde{p} \) by Lemma 4.1.

**Remark 4.2.** Many of the facts mentioned in this section are also useful for studying Gelfand-Zeitlin systems [GS83, CKO17].

### 4.1 The isotropy representation

Continuing from the previous section, this section computes the isotropy representations at the points \( \tilde{p} \in \Phi^{-1}(M) \) as described in (30), (32) and Lemma 4.1.

**Lemma 4.3.** Let \( p \in \Phi^{-1}(M) \) and let \( c, z \) be defined as in (30). The subspace \( T_p(K \cdot p)^\omega \) consists of all matrices of the form

\[
   \begin{pmatrix}
   0 & (c - M)x^\dagger + z^\dagger X^\dagger \\
   (c - M)x + Xz & 0
\end{pmatrix}, \quad X \in \mathfrak{k}, \ x \in M_{n \times 1}(\mathbb{C})
\]

such that

\[
   0 = x^\dagger z + z^\dagger x \\
   0 = xz^\dagger + z x^\dagger + [X, M].
\]

The subspace \( T_p(K \cdot p) \cap T_p(K \cdot p)^\omega \) consists of all matrices of the form

\[
   \begin{pmatrix}
   0 & z^\dagger Y^\dagger \\
   Yz & 0
\end{pmatrix}, \quad Y \in \mathfrak{k}. \]


Proof. Denote
\[
\eta := \begin{pmatrix} 0 \\ 0 \\ Y \end{pmatrix}, \quad \xi := \begin{pmatrix} 0 \\ -x^T \end{pmatrix}, \quad X, Y \in \mathfrak{k}, x_0 \in \sqrt{-1}\mathbb{R}, x \in M_{n \times 1}(\mathbb{C}).
\]
The tangent space \( T_p \mathcal{O}_\Lambda \) consists of elements of the form \([\xi, p]\). Since diagonal elements of \( u(n + 1) \) act trivially, set \( x_0 = 0 \). Then elements of \( T_p \mathcal{O}_\Lambda \) have block form
\[
[\xi, p] = \begin{pmatrix} -x^1z - z^1x \\ (c - M)x + Xz \\ xz^T + zx^T + [X, M] \end{pmatrix}, \quad X \in \mathfrak{k}, x \in M_{n \times 1}(\mathbb{C}).
\]
Elements of \( T_p(K \cdot p) \) have block form
\[
[\eta, p] = \begin{pmatrix} 0 \\ x^T \end{pmatrix}, \quad Y \in \mathfrak{k}.
\]
Recall,
\[
T_p(K \cdot p)^\omega = \{[\xi, p] \in T_p \mathcal{O}_\Lambda \mid (\omega_\Lambda)_p([\xi, p], [\eta, p]) = 0 \forall Y \in \mathfrak{k} \}.
\]
By (10),
\[
\sqrt{-1}(\omega_\Lambda)_p([\xi, p], [\eta, p]) = \text{Tr}(p[\xi, \eta])
\]
\[
= -\text{Tr}(z^1Yx) - \text{Tr}(zx^TY) + \text{Tr}(M[X, Y])
\]
\[
= \text{Tr}([([M, X] - xz^T - zx^T)Y]).
\]
Let \( \sqrt{-1}E_{i,i}, E_{i,j} - E_{j,i} \) and \( \sqrt{-1}(E_{i,j} + E_{j,i}) \) be standard basis elements for \( \mathfrak{k} \) (where \( E_{i,j} \) denotes the matrix whose \( i, j \)-entry is 1 and all other entries are 0). Plugging these elements in for \( Y \) yields a system of equations,
\[
0 = x_i\bar{x}_i + z_i\bar{z}_i \quad \forall i
\]
\[
0 = (\mu_j - \mu_i)(X_{j,i} + X_{i,j}) - (x_j\bar{x}_i + z_j\bar{z}_i - x_i\bar{x}_j - z_i\bar{z}_j) \quad \forall i \neq j
\]
\[
0 = (\mu_j - \mu_i)(X_{j,i} - X_{i,j}) - (x_j\bar{x}_i + z_j\bar{z}_i + x_i\bar{x}_j + z_i\bar{z}_j) \quad \forall i \neq j,
\]
(36)
(37)
where \( X_{i,j} \) denotes the \( i, j \) entry of \( X \) which in turn is equivalent to the system of equations
\[
0 = x_i\bar{x}_i + z_i\bar{z}_i \quad \forall i
\]
\[
0 = (\mu_j - \mu_i)X_{j,i} - (x_j\bar{x}_i + z_j\bar{z}_i) \quad \forall i \neq j.
\]
This system of equations is equivalent to the system of matrix equations (34). It follows from (34) that the block diagonal parts of \([\xi, p] \in T_p(K \cdot p)^\omega \) are zero, so \([\xi, p]\) has the form (33) subject to the equations (34).
By properties of equivariant moment maps, \( T_p(K \cdot p) \cap T_p(K \cdot p)^\omega = T_p(K_M \cdot p) \) [GS84b]. Elements of \( T_p(K_M \cdot p) \) have block form of (35), which completes the proof. \( \square \)

Equations (34) dictate the form of the vectors \( (c - M)x + Xz \), as the next two lemmas demonstrate.

Lemma 4.4. Let \( p \in \Phi^{-1}(M) \) and let \( z \) be defined as in (30). Let \( X \in \mathfrak{k} \) and \( x \in M_{n \times 1}(\mathbb{C}) \) such that
\[
0 = xz^T + zx^T + [X, M].
\]
If the the component of the interlacing pattern of \((\Delta, \mu)\) labelled \( \mu \) is not a \( \Delta \)-shape, then
\[
(Xz)_\mu = \left( \sum_{\tau \in [\mu] \setminus \Delta_{\text{shape}}} \frac{\tau^2}{\mu - \tau} \right) x_\mu.
\]
(38)
Proof. Let \( \mu \neq \nu \) distinct elements of \( \mu \). Let \( X_{\mu,\nu}, x_\mu, z_\mu, \) etc. denote the corresponding blocks of \( X, x, \) and \( z \). By (38), the \( \mu,\nu \) block of \( X \) is given by the formula

\[
X_{\mu,\nu} = \frac{1}{\mu - \nu} (x_\mu z_\nu^\dagger + z_\mu x_\nu^\dagger), \quad \forall \mu \neq \nu.
\]

By Lemma 4.1, if the component of the interlacing pattern of \((\lambda, \mu)\) labelled \( \mu \) is not a \( \bigtriangleup \)-shape, then \( z_\mu = 0 \). Thus

\[
(Xz)_\mu = \sum_{\tau \in [\mu]} X_{\mu,\tau} z_\tau = \sum_{\substack{\tau \in [\mu] \\ \tau \neq \mu}} \frac{1}{\mu - \tau} x_\mu z_\tau^\dagger = \left( \sum_{\substack{\tau \in [\mu] \\ \bigtriangleup\text{-shape}}} \frac{|z_\tau|^2}{\mu - \tau} \right) x_\mu = \left( \sum_{\substack{\tau \in [\mu] \\ \bigtriangleup\text{-shape}}} \frac{r_\tau^2}{\mu - \tau} \right) x_\mu. \quad \square
\]

Recall the definition of \( C_\mu \) from (22).

**Lemma 4.5.** Let \( p, X, \) and \( x \) as in Lemma 4.4 such that (38) holds. Assume that the component of the interlacing pattern of \((\lambda, \mu)\) labelled \( \mu \) is not a \( \bigtriangleup \)-shape. Then, \( C_\mu = 0 \) if and only if the component of the interlacing pattern of \((\lambda, \mu)\) labelled \( \mu \) is a \( \bigtriangleup \)-shape.

**Proof.** First, note that it is sufficient to prove

\[
\prod_{\lambda \in [\lambda]} (x - \lambda) = (x - c) \prod_{\mu \in [\mu]} (x - \mu) - \sum_{\mu \in [\mu]} r_\mu^2 \prod_{\tau \in [\mu]} (x - \tau). \quad (39)
\]

Indeed, since the component of the interlacing pattern labelled \( \mu \) is not a \( \bigtriangleup \)-shape, plugging in \( x = \mu \) yields

\[
\prod_{\lambda \in [\lambda]} (\mu - \lambda) = \left( \mu - c - \sum_{\tau \in [\mu]} \frac{r_\tau^2}{\mu - \tau} \right) \prod_{\tau \in [\mu]} (\mu - \tau) = -C_\mu \prod_{\tau \in [\mu]} (\mu - \tau) \quad (40)
\]

and the factor

\[
\prod_{\tau \in [\mu]} (\mu - \tau) \quad (41)
\]

is non-zero.

Second, applying Lemma 4.1 \( r_\mu = 0 \) when the component labelled \( \mu \) is not a \( \bigtriangleup \)-shape) and rearranging,
observe that
\[
(x - c) \prod_{\mu \in [\underline{\mu}]} (x - \mu)^{n_{\mu}(\underline{\mu})} - \sum_{\mu \in [\underline{\mu}]} r_{\mu}^2 (x - \mu)^{n_{\mu}(\underline{\mu}) - 1} \prod_{\tau \in [\underline{\mu}]} (x - \tau)^{n_{\tau}(\underline{\mu})}
\]
\[
= (x - c) \prod_{\mu \in [\underline{\mu}]} (x - \mu)^{n_{\mu}(\underline{\mu})} \prod_{\tau \in [\underline{\mu}]} (x - \tau)^{n_{\tau}(\underline{\mu})} - \sum_{\mu \in [\underline{\mu}]} r_{\mu}^2 (x - \mu)^{n_{\mu}(\underline{\mu}) - 1} \prod_{\tau \in [\underline{\mu}]} (x - \tau)^{n_{\tau}(\underline{\mu})} \prod_{\tau \neq \mu} (x - \tau)^{n_{\tau}(\underline{\mu})}
\]
\[
= \left( (x - c) \prod_{\mu \in [\underline{\mu}]} (x - \mu) - \sum_{\mu \in [\underline{\mu}]} r_{\mu}^2 \prod_{\tau \in [\underline{\mu}]} (x - \tau) \right) \cdot \prod_{\tau \in [\underline{\mu}]} (x - \tau)^{n_{\tau}(\underline{\mu}) - 1} \prod_{\tau \neq \mu} (x - \tau)^{n_{\tau}(\underline{\mu})}
\]
(42)

Then (39) follows by combining (42) and (31), which completes the proof.

For \( p \in \Phi^{-1}(M) \), let \( V_p \subseteq \mathbb{C}^n \) denote the image of injective linear map
\[
T: T_p(K \cdot p)^\omega \to \mathbb{C}^n, \quad \begin{pmatrix} 0 & (c - M)X + Xz \end{pmatrix} \mapsto (c - M)x + Xz
\]
(43)
and let \( U_p \subseteq V_p \) denote the image of \( T_p(K \cdot p) \cap T_p(K \cdot p)^\omega \). Specialize to the case of \( \tilde{p} \) and recall that \( K_{\tilde{p}} = L(\Delta \underline{\mu}) \). The map \( T \) is \( K_{\tilde{p}} \)-equivariant with respect to the action of \( K_{\tilde{p}} \) on \( \mathbb{C}^n \) as a block-diagonal subgroup of \( K = U(n) \) acting by the standard representation. Decompose \( \mathbb{C}^n = \bigoplus_{i=1}^m \mathbb{C}^{n_i(\underline{\mu})} \), \( m = m(\underline{\mu}) \). The subspaces \( V_{\tilde{p}} \) and \( U_{\tilde{p}} \) have the form \( \bigoplus_{i=1}^m V_i \) (respectively \( \bigoplus_{i=1}^m U_i \)) for some subspaces \( U_i \subseteq V_i \subseteq \mathbb{C}^{n_i(\underline{\mu})} \). The map \( T \) descends to an isomorphism of \( K_{\tilde{p}} \)-representations,
\[
W_{\tilde{p}} = T_{\tilde{p}}(K \cdot \tilde{p})^\omega / (T_{\tilde{p}}(K \cdot \tilde{p}) \cap T_{\tilde{p}}(K \cdot \tilde{p})^\omega) \cong \bigoplus_{i=1}^m V_i / U_i.
\]
(44)
The representation of \( K_{\tilde{p}} = L_1 \times \cdots \times L_m \) on the right is given in each component by the inclusion \( L_i \subseteq U(n_i(\underline{\mu})) \) and the standard representation of \( U(n_i(\mu)) \) on \( \mathbb{C}^{n_i(\underline{\mu})} \). This representation of \( L_i \) preserves the subspaces \( U_i \subseteq V_i \) so it induces a representation on \( V_i / U_i \).

Recall that if the component of the interlacing pattern labelled \( \underline{\mu}_i \) is a \( \square \)-shape, then \( L_i = U(n_i(\underline{\mu})) \).

**Proposition 4.6.** For all \( i = 1, \ldots, m, m = m(\underline{\mu}) \), there is an isomorphism of \( L_i \) representations
\[
V_i / U_i \cong \begin{cases} \mathbb{C}^{n_i(\underline{\mu})} & \text{if the component of the interlacing pattern of } (\lambda, \underline{\mu}) \text{ labelled } \underline{\mu}_i \text{ is a } \square \text{-shape,} \\ \{0\} & \text{else,} \end{cases}
\]
where \( \mathbb{C}^{n_i(\underline{\mu})} \) denotes the standard representation of \( U(n_i(\underline{\mu})) \).
Proof. In general,

\[ U_i = \{ (Yz)_i \mid Y \in \mathfrak{t}_{\text{M}} \} = \{ Y_i z_i \mid Y_i, z_i \in u(n_i(\mu_i)) \} \]

If the component of the interlacing pattern of \( (\lambda, \mu) \) labelled \( \mu_i \) is a \( \Delta \)-shape, then, by Lemma 4.1, \( z_i \neq 0 \), so \( U_i = \mathbb{C}^{n_i(\mu)} \) and \( V_i / U_i \cong \{ 0 \} \). If the component of the interlacing pattern of \( (\lambda, \mu) \) labelled \( \mu_i \) is not a \( \Delta \)-shape, then, \( z_i = 0 \), so \( U_i = \{ 0 \} \).

It remains to determine the subspace \( V_i \) when the component of the interlacing pattern of \( (\lambda, \mu) \) labelled \( \mu_i \) is not a \( \Delta \)-shape. In this case, it follows by Lemma 4.4 that the block

\[
(c - M)x + Xz)_i = (c - M)x_i + (Xz)_i = C_i x_i,
\]

where \( C_i = C_{\mu_i} \) as defined in (22). By Lemma 4.3,

\[
V_i = \{ ((c - M)x + Xz)_i \mid X \in \mathfrak{t}, x \in \mathbb{C}^n, xz^\dagger + zx^\dagger + [X, M] \} = \{ C_i x_i \mid x_i \in \mathbb{C}^{n_i(\mu)} \}.
\]

(45)

By Lemma 4.5, \( C_i = 0 \) if and only if the component of the interlacing pattern of \( (\lambda, \mu) \) labelled \( \mu_i \) is a \( \mathfrak{d} \)-shape. This completes the proof. \( \Box \)

Thus \( \bigoplus_{i=1}^m V_i / U_i \) is isomorphic to the \( L(\Delta, \mu) \)-representation \( W(\Delta, \mu) \).

Proposition 4.7. The linear symplectic structure on \( W(\Delta, \mu) \) defined via the symplectic form \( \varpi_{\tilde{p}} \) and the isomorphism (44) equals the linear symplectic form \( \omega(\Delta, \mu) \) defined in (23).

Proof. Denote

\[
\eta := \left( \begin{array}{c} 0 \\ -y^\dagger \\ Y \end{array} \right), \quad \xi := \left( \begin{array}{c} 0 \\ -x^\dagger \\ X \end{array} \right), \quad X, Y \in \mathfrak{t}, x, y \in M_{n \times 1}(\mathbb{C}).
\]

Then, using Lemma 4.4,

\[
\sqrt{-1}(\omega_{\lambda})_{\tilde{p}}([\xi, \tilde{p}], [\eta, \tilde{p}]) = \text{Tr} (\tilde{p} [\xi, \eta]) = \text{Tr} ([\tilde{p}, \xi] \eta)
\]

\[
= -\text{Tr} \left( \left( \begin{array}{c|c} 0 & (c - M)x^\dagger + z^\dagger X^\dagger \\ \hline (c - M)x + Xz & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ y^\dagger \\ Y \end{array} \right) \right)
\]

\[
= -((c - M)x^\dagger + z^\dagger X^\dagger) y + \text{Tr}((c - M)x + Xz)y^\dagger
\]

\[
= -((c - M)x^\dagger + z^\dagger X^\dagger) y + \text{Tr}(y^\dagger((c - M)x + Xz))
\]

\[
= -(c - M)(x^\dagger y - y^\dagger x) - z^\dagger X^\dagger y + y^\dagger Xz
\]

\[
= -(c - M)(x^\dagger y - y^\dagger x) - (Xz)^\dagger y + y^\dagger Xz
\]

\[
= -(c - M)(x^\dagger y - y^\dagger x) + \sum_{i=1}^m \left( \sum_{j \neq i} \frac{r_j^2}{\mu_i - \mu_j} \right) (-x^\dagger_i y_i + y^\dagger_i x_i).
\]

(46)

Viewing \([\xi, \tilde{p}]\) and \([\eta, \tilde{p}]\) as representatives of vectors in the isotropy representation,

\[
(\varpi_{\lambda})_{\tilde{p}}([\xi, \tilde{p}], [\eta, \tilde{p}]) = \frac{1}{\sqrt{-1}} \sum_{i=1}^m \left( \sum_{j \neq i} \frac{r_j^2}{\mu_i - \mu_j} \right) (-x^\dagger_i y_i + y^\dagger_i x_i)
\]

\[
= \sum_{i=1}^m C_i (-x^\dagger_i y_i + y^\dagger_i x_i).
\]

(47)

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Applying the isomorphism $T: W_p \rightarrow W_{(\lambda, \mu)}$, $[\xi, p] \mapsto u = (C_i x_i)_i$, $[\eta, p] \mapsto v = (C_i y_i)_i$ yields

$$\omega_{(\lambda, \mu)}(u, w) = \frac{1}{\sqrt{-1}} \sum_{\text{shape}}^m \frac{u_i^\dagger w_i + w_i^\dagger u_i}{C_i}.$$

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DEPARTMENT OF MATHEMATICS & STATISTICS, McMASTER UNIVERSITY, HAMILTON HALL, 1280 
MAIN STREET W, HAMILTON, ON, L8S 4K1, CANADA

E-mail address: lanej5@math.mcmaster.ca