X-Secure T-Private Federated Submodel Learning

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Abstract

The problem of (information-theoretic) X-secure T-private federated submodel learning represents a setting where a large scale machine learning model is partitioned into K submodels and stored across N distributed servers according to an X-secure threshold secret sharing scheme. Various users wish to successively train (update) the submodel that is most relevant to their local data while keeping the identity of their relevant submodel private from any set of up to T colluding servers. Inspired by the idea of cross-subspace alignment (CSA) for X-secure T-private information retrieval, we propose a novel CSA-RW (read-write) scheme for efficiently (in terms of communication cost) and privately reading from and writing to a distributed database. CSA-RW is shown to be asymptotically/approximately optimal in download/upload communication cost, and improves significantly upon available baselines from prior work. It also answers in the affirmative an open question by Kairouz et al. by exploiting synergistic gains from the joint design of private read and write operations.

1 Introduction

The rise of machine learning is marked by fundamental tradeoffs between competing concerns. Central to this work are 1) the need for abundant training data, 2) the need for privacy, and 3) the need for low communication cost. Federated learning [1–4] is a distributed machine learning paradigm that addresses the first two concerns by allowing distributed users/clients (e.g., mobile phones) to collaboratively train a shared model that is stored in a cluster of databases/servers (cloud) while keeping their training data private. The users retrieve the current model, train the model locally with their own training data, and then aggregate the modifications as focused updates. Thus, federated learning allows utilization of abundant training data while preserving its privacy. However, this incurs higher communication cost as each update involves communication of all model parameters.

Communicating the full model may be unnecessary for large scale machine learning tasks where each user’s local data is primarily relevant to a small part of the overall model, Federated Submodel Learning (FSL) [5] builds on this observation by partitioning the model into multiple submodels and allowing users to selectively train and update the submodels that are most relevant to their local view. What makes this challenging is the privacy constraint. The identity of the submodel that is being retrieved and updated by a user must remain private. Prior works [5–9] that assume centralized storage of all submodels are generally able to provide relatively weaker privacy guarantees such as plausible deniability through differential privacy mechanisms that perturb the data and rely on secure inter-user peer-to-peer communication for secure aggregation.
On the other hand, it is noted recently by Kim and Lee in [10] that if the servers that store the submodels are distributed, then stronger information theoretic guarantees such as ‘perfect privacy’ may be attainable, without the need for user-to-user communication. Indeed, in this work we focus on this setting of distributed servers and perfect privacy. The challenge of federated submodel learning in this setting centers around three key questions.

**Q1 Private Read:** How can a user efficiently retrieve the desired submodel from the distributed servers without revealing which submodel is being retrieved?

**Q2 Private Write:** How can a user efficiently update the desired submodel to the distributed servers without revealing which submodel is being updated?

**Q3 Synergy of Private Read-Write:** Are there synergistic gains in the joint design of retrieval and update operations, and if so, then how to exploit these synergies?

The significance of these fundamental questions goes well beyond federated submodel learning. As recognized by [5] the private read question (Q1) by itself is equivalent to the problem of Private Information Retrieval (PIR) [11,12], which has recently been studied extensively from an information theoretic perspective [13–47]. Much less is known about Q2 and Q3, i.e., the fundamental limits of private-write, and joint read-write solutions from the information theoretic perspective. Notably, Q3 has also been highlighted previously as an open problem by Kairouz et al in [2].

The problem of privately reading and writing data from a distributed memory falls under the larger umbrella of Distributed Oblivious RAM (DORAM) [48] primitives in theoretical computer science and cryptography. With a few limited exceptions (e.g., a specialized 4-server construction in [49] that allows information theoretic privacy), prior studies of DORAM generally take a cryptographic perspective, where privacy is guaranteed subject to computational hardness assumptions, and the number of memory blocks is assumed to be much larger than the size of each block. In contrast, the focus of this work is on Q2 and Q3 under information theoretic privacy. Furthermore, because our motivation comes from federated submodel learning, the size of a submodel is assumed to be significantly larger than the number of submodels (see motivating examples in [5] and Observation 10 in Section 3.1).

**Overview:** We consider the federated submodel learning setting where the global model is partitioned into $K$ submodels, and stored among $N$ distributed servers according to an $X$-secure threshold secret sharing scheme, i.e., any set of up to $X$ colluding servers can learn nothing about the stored models, while the full model can be recovered from the data stored by any $X+1$ servers. One at a time, users update the submodel most relevant to their local training data. The updates must be $T$-private, i.e., any set of up to $T$ colluding servers must not learn anything about which submodel is being updated. The contents of the updates must be $X_\Delta$-secure, i.e., any set of up to $X_\Delta$ colluding servers must learn nothing about the contents of the submodel updates. The size of a submodel is significantly larger than the number of submodels, which is significantly larger than 1, i.e., $L \gg K \gg 1$ where $L$ is the size of a submodel and $K$ is the number of submodels.

Since the private-read problem (Q1) is essentially a form of PIR, our starting point is the $X$-secure $T$-private information retrieval scheme (XSTPIR) of [28]. In particular, we build on the idea of cross-subspace alignment (CSA) from [28], and introduce a new private read-write scheme, called CSA-RW as an answer to Q1 and Q2. To our knowledge CSA-RW is the first communication-efficient private federated submodel learning scheme that achieves information-theoretically perfect

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1By perfect privacy we mean that absolutely no information is leaked about the identity of a user’s desired submodel to any set of colluding servers up to a target threshold.
privacy. CSA-RW also answers Q3 in the affirmative as it exploits query structure from the private-read operation to reduce the communication cost for the private-write operation. The evidence of synergistic gain in CSA-RW from a joint design of submodel retrieval and submodel aggregation addresses the corresponding open problem highlighted in Section 4.4.4 of [2]. In terms of comparisons against available baselines, we note (see Observation 8 in Section 3.1) that CSA-RW improves significantly in both the communication efficiency and the level of privacy compared to [10]. In fact, CSA-RW achieves asymptotically optimal download cost when \( X \geq X_\Delta + T \), and is order-wise optimal in terms of the upload cost. Compared with the 4 server construction of information theoretic DORAM in [49], (where \( X = 1, T = 1, X_\Delta = 0, N = 4 \)) CSA-RW has better communication efficiency (the assumption of \( L \gg K \) is important in this regard). For example, as the ratio \( L/K \) approaches infinity, CSA-RW achieves total communication cost (i.e., the summation of the download cost and the upload cost, normalized by the submodel size) of 6, versus the communication cost of 8 achieved by the construction in [49].

**Notation:** Bold symbols are used to denote vectors and matrices, while calligraphic symbols denote sets. For a positive integer \( N \), \( [N] \) denotes the set \( \{1, 2, \ldots, N\} \). For a real number \( x \), \( \langle x \rangle^+ \) denotes \( \max(x, 0) \). For two tuples of the same length comprised of matrices, \( X = (X_1, X_2, \ldots, X_N) \), \( Y = (Y_1, Y_2, \ldots, Y_N) \), the entry-wise product is defined as
\[
X \odot Y \triangleq (X_1 Y_1, X_2 Y_2, \ldots, X_N Y_N),
\]
the summation is defined as
\[
X + Y \triangleq (X_1 + Y_1, X_2 + Y_2, \ldots, X_N + Y_N).
\]
We also define the transpose of the tuple \( X^T = (X_1^T, X_2^T, \ldots, X_N^T) \). The notation \( \text{diag}(D_1, D_2, \ldots, D_n) \) denotes the block diagonal matrix, i.e., the main-diagonal blocks are square matrices \( (D_1, D_2, \ldots, D_n) \) and all off-diagonal blocks are zero matrices. For a positive integer \( K \), \( \mathbf{I}_K \) denotes the \( K \times K \) identity matrix. For two positive integers \( k, K \) such that \( k \leq K \), \( e_K(k) \) denotes the \( k \)-th column of the \( K \times K \) identity matrix. The notation \( \tilde{O}(a \log^2 b) \) suppresses polylog terms. It may be replaced with \( O(a \log^2 b) \) if the field supports the Fast Fourier Transform (FFT), and with \( O(a \log^2 b \log \log(b)) \) if it does not.

2 Problem Statement: XSTPFSL

As shown in Figure 1, consider \( K \) (initial) submodels \( \left( W_1^{(0)}, W_2^{(0)}, \ldots, W_K^{(0)} \right) \), each of which consists of \( L \) uniformly\(^4\) i.i.d. random symbols from a finite\(^5\) field \( \mathbb{F}_q \). Specifically, for \( k \in [K] \), we have
\[
W_k^{(0)} = \left[ W_k^{(0)}(1), W_k^{(0)}(2), \ldots, W_k^{(0)}(L) \right]^T,
\]
\[
H \left( \left( W_k^{(0)} \right)_{k \in [K]} \right) = KL,
\]
\[\text{\footnote{There is another standard definition of the notation } \tilde{O} \text{ which fully suppresses polylog terms, i.e, } O(a \text{ polylog}(b)) \text{ is represented by } \tilde{O}(a), \text{ regardless of the exact form of } \text{polylog}(b). \text{ The definition used in this paper emphasizes the dominant factor in the polylog term.}}\]
\[\text{\footnote{If the FFT is not supported by the field, Schönhage–Strassen algorithm \cite{50} can be used for fast algorithms that require convolutions, with an extra factor of } \log \log b \text{ in the complexity.}}\]
\[\text{\footnote{It is not necessary for the model data } W \text{ to be i.i.d. uniform for the proposed CSA-RW scheme. The proposed CSA-RW scheme remains valid (correct, secure and private) even if the model data is arbitrarily distributed. The same is true for model updates } \Delta_i \text{ and the desired submodel indices } \theta_i. \text{ i.i.d. uniform assumptions on these entities are needed primarily for converse arguments.}}\]
\[\text{\footnote{The field size does not need to be too large, for example, according to Theorem }1 \text{ it is sufficient to have } q \geq 2N, \text{ where } N \text{ is the number of servers.}}\]
in $q$-ary units. We associate submodel updates with users and time slots, so that the $t^{th}$ update is made by User $t$ at time slot $t$. For all $t \in \mathbb{N}$, the storage at the $n^{th}$ server at time $t$ is denoted as $S_n^{(t)}$, where $t=0$ corresponds to the initial storage. Similarly, $(\mathbf{W}_1^{(t)}, \mathbf{W}_2^{(t)}, \cdots, \mathbf{W}_K^{(t)})$ denotes the submodels at time $t$. A full cycle of XSTPFSL consists of two parts — the retrieval (read) phase and the update (write) phase. At the beginning of the retrieval phase, User $t$ privately generates uniformly distributed desired index $\theta_t$ and the user-side randomness $Z_U^{(t)}$, which is needed to protect the user’s privacy. The user wishes to retrieve the submodel $\mathbf{W}_{\theta_t}^{(t-1)}$ without revealing anything about $(\theta_1, \theta_2, \cdots, \theta_t)$ to any set of up to $T$ colluding servers. To this end, the user generates $N$ read-queries $(Q_1^{(t,\theta_t)}, Q_2^{(t,\theta_t)}, \cdots, Q_N^{(t,\theta_t)})$, where $Q_n^{(t,\theta_t)}$ is intended for the $n^{th}$ server, such that,

$$H\left(Q_n^{(t,\theta_t)} \mid \theta_t, Z_U^{(t)}\right) = 0, \quad \forall n \in [N].$$

Upon receiving the query, each of the $N$ servers responds to the user with an answer $A_n^{(t,\theta_t)}$,
such that,

\[ H \left( A_n^{(t,\theta_t)} \mid S_n^{(t-1)}, Q_n^{(t,\theta_t)} \right) = 0, \quad \forall n \in [N]. \] (4)

The user must be able to recover the desired submodel \( W_{\theta_t}^{(t-1)} \) after collecting the answers from all \( N \) servers.

\[ \text{[Correctness]} \quad H \left( W_{\theta_t}^{(t-1)} \mid \left( A_n^{(t,\theta_t)} \right)_{n \in [N]}, \left( Q_n^{(t,\theta_t)} \right)_{n \in [N]}, \theta_t \right) = 0. \]

This completes the retrieval phase.

At the beginning of the update phase, User \( t \) generates privately an increment for the \( \theta_t^{\text{th}} \) submodel, \( \Delta_t = [\Delta_t(1), \Delta_t(2), \ldots, \Delta_t(L)]^T \), which is comprised of \( L \) i.i.d. uniformly distributed symbols from the finite field \( \mathbb{F}_q \), i.e., \( H(\Delta_t) = L \), in \( q \)-ary units. User \( t \) wishes to update the submodel \( W_{\theta_t}^{(t-1)} \) with the increment \( \Delta_t \), such that User \( t+1 \) is able to retrieve the submodel \( W_{\theta_t}^{(t)} = W_{\theta_t}^{(t-1)} + \Delta_t \) if \( \theta_{t+1} = \theta_t \), and the submodel \( W_{\theta_t'}^{(t)} = W_{\theta_t'}^{(t-1)} + \Delta_t \) if \( \theta_{t+1} = \theta_t' \). To this end, User \( t \) generates \( N \) write-queries \( \left( P_n^{(t,\theta_t)} ; P_2^{(t,\theta_t)} , \ldots, P_n^{(t,\theta_t)} \right) \) for the \( N \) servers, where \( P_n^{(t,\theta_t)} \) is intended for the \( n^{\text{th}} \) server. As before, any \( T \) colluding servers must learn nothing about the desired index \( \theta_t \), and \( \forall n \in [N] \),

\[ H \left( P_n^{(t,\theta_t)} \mid \theta_t, Z_n^{(t)} , \left( A_n^{(t,\theta_t)} \right)_{n \in [N]}, \Delta_t \right) = 0. \] (5)

Upon receiving the query, each of the servers updates its storage according to its current storage \( S_n^{(t-1)} \) and the queries \( \left( P_n^{(t,\theta_t)} , Q_n^{(t,\theta_t)} \right) \) so that \( \forall n \in [N], t \in \mathbb{Z}^+ \),

\[ H \left( S_n^{(t)} \mid S_n^{(t-1)}, P_n^{(t,\theta_t)}, Q_n^{(t,\theta_t)} \right) = 0. \] (6)

To correctly reflect such modification to the \( \theta_t^{\text{th}} \) submodel, for all \( t \in \mathbb{Z}^+, \) \( k \in [K] \), we define \( W_k^{(t)} \), recursively as follows.

\[ W_k^{(t)} = \begin{cases} W_k^{(t-1)} + \Delta_t & k = \theta_t, \\ W_k^{(t-1)} & k \neq \theta_t. \end{cases} \] (7)

Next we formalize the security and privacy constraints. \( T \)-privacy guarantees that at any time, any set of up to \( T \) colluding servers learn nothing about the indices \( (\theta_1, \theta_2, \ldots, \theta_t) \).

\[ \text{[T-Privacy]} \quad I \left( \left( \theta_t \right)_{\tau \in [t]} ; \left( P_n^{(t,\theta_t)} ; Q_n^{(t,\theta_t)} \right)_{n \in T} \bigg| \left( S_n^{(\tau-1)} ; P_n^{(\tau-1,\theta_{t-1})} , Q_n^{(\tau-1,\theta_{t-1})} \right)_{\tau \in [t], n \in T} \right) = 0, \quad \forall T \subseteq [N], |T| = T, t \in \mathbb{Z}^+, \] (8)

where for all \( n \in [N] \), we define \( P_n^{(0,\theta_0)} = Q_n^{(0,\theta_0)} = \emptyset \).

Similarly, we require that any set of up to \( X_\Delta \) colluding servers learn nothing about the increment \( \Delta_t \). To this end, the \( X_\Delta \)-security constraint, \( X_\Delta \geq 0 \), is formalized as follows.

\[ \text{[X_\Delta-Security]} \quad I \left( \Delta_t ; \left( P_n^{(t,\theta_t)} \right)_{n \in X} \bigg| \left( S_n^{(\tau-1)} ; P_n^{(\tau-1,\theta_{t-1})} , Q_n^{(\tau,\theta_t)} \right)_{\tau \in [t], n \in X} \right) = 0, \]
The storage at the servers is structured according to an $X$-secure threshold secret sharing scheme which strives for both maximal security and recoverability. On one hand this means that at any time, the storage at any set of up to $X$ colluding servers must reveal nothing about the submodels. On the other hand, all submodels must be recoverable from the storage of any $(X + 1)$ servers. To this end, storage randomness $Z_S$, which is independent of the submodels, is used to form the initial $X$-secure storage.

\[
H \left( S_n^{(0)} \mid \left( W_k^{(0)} \right)_{k \in [K]}, Z_S \right) = 0, \forall n \in [N],
\]

\[
[X\text{-Security}] \quad I \left( \left( W_k^{(t)} \right)_{k \in [K]} ; \left( S_n^{(t)} \right)_{n \in X} \right) = 0, \forall X \subset [N], |X| = X, t \in \mathbb{N},
\]

\[
[\text{Recoverability}] \quad H \left( \left( W_k^{(t)} \right)_{k \in [K]} \middle| \left( S_n^{(t)} \right)_{n \in X'} \right) = 0, \forall X' \subset [N], |X'| = X + 1, t \in \mathbb{N},
\]

where for all $n \in [N]$, let us define $S_n^{-1} = \emptyset$.

There is a subtle difference in the security constraint that we impose on the storage, and the previously specified security and privacy constraints on updates and queries. To appreciate this difference let us make a distinction between the notions of an internal adversary and an external adversary. We say that a set of colluding servers forms an internal adversary if those colluding servers have access to not only their current storage, but also their entire history of previous stored values and queries. Essentially the internal adversary setting represents a greater security threat because the servers themselves are dishonest and surreptitiously keep records of all their history in an attempt to learn from it. In contrast, we say that a set of colluding servers forms an external adversary if those colluding servers have access to only their current storage, but not to other historical information. Essentially, this represents an external adversary who is able to steal the current information from honest servers who do not keep records of their historical information. Clearly, an external adversary is weaker than an internal adversary. Now let us note that while the $T$-private queries and the $X_\Delta$-secure updates are protected against internal adversaries, the $X$-secure storage is only protected against external adversaries. This is mainly because we will generally allow $X > \max(X_\Delta, T)$, i.e., a higher security threshold for storage, than for updates and queries. Note that once the number of compromised servers exceeds $\max(X_\Delta, T)$, the security of updates and the privacy of queries is no longer guaranteed. In such settings the security of storage is still guaranteed, albeit in a weaker sense. On the other hand if the number of compromised servers is small enough, then indeed secure storage may be guaranteed in a stronger sense, even against internal adversaries. We refer the reader to Observation 4 in Section 3.1 for further insight into this aspect.

The independence of noise, increments, desired indices and submodels is specified so that for all $t \in \mathbb{Z}^+$, we have

\[
H \left( \left( W_k^{(0)} \right)_{k \in [K]} ; (\Delta_{\tau})_{\tau \in [t]} ; (\theta_{\tau})_{\tau \in [t]} ; \left( Z_U^{(\tau)} \right)_{\tau \in [t]} ; Z_S \right) = H \left( \left( W_k^{(0)} \right)_{k \in [K]} \right) + H \left( (\Delta_{\tau})_{\tau \in [t]} \right) + H \left( (\theta_{\tau})_{\tau \in [t]} \right) + H \left( \left( Z_U^{(\tau)} \right)_{\tau \in [t]} \right) + H(Z_S).
\]
normalized by $L$. The upload cost $U_t$ is the expected number of $q$-ary symbols uploaded by User $t$, also normalized by $L$. If the scheme is time-invariant, i.e., the download cost and the upload cost is independent of $t$, we use shorthand notations $(U, D)$ instead of $(U_t, D_t)$. The asymptotic capacity of XSTPFSL, denoted as $C_\infty$, is the supremum of reciprocal of achievable download costs\footnote{The reason for focusing on the download cost is in part that by our achievability scheme, with high levels of data security, i.e., when the value of $X$ is close to the number of servers $N$, the download cost dominates the upload cost, see Theorem\textsuperscript{1} and Figure\textsuperscript{2}. Indeed, high levels of data security are desirable, as discussed above.} over $t \in \mathbb{Z}^+$ of all XSTPFSL schemes, in the limit as the number of submodels $K \to \infty$.

## 3 Main Results

The main contribution of this work is the CSA-RW scheme, whose performance is specified in the following theorem.

**Theorem 1.** Define $\ell \triangleq N - (X + T)$. If $\ell$ divides $L$, $q \geq \ell + N$, and $X_\Delta + T \leq X \leq N - T - 1$, then the following (upload, download) cost pair is achievable by the CSA-RW scheme.

$$
(U, D) = \left( (N - X + X_\Delta + T) + N\ell K / L, N / \ell \right).
$$

The proof of Theorem\textsuperscript{1} is presented in Section\textsuperscript{4}. Note that Theorem\textsuperscript{1} is not limited to asymptotic settings. However, in the limit as $L / K \to \infty$, we have

$$
(U, D) \stackrel{L / K \to \infty}{=} \left( N - X + X_\Delta + T, \frac{N}{N - (X + T)} \right).
$$

The following corollary underscores the download optimality of CSA-RW in terms of the asymptotic (large $K$) capacity.

**Corollary 1.** When $X \geq X_\Delta + T$, the asymptotic capacity is

$$
C_\infty = \left( 1 - \frac{X + T}{N} \right)^+.
$$

Corollary\textsuperscript{1} follows directly from the asymptotic capacity results of XSTPIR [28], because the retrieval phase of XSTPFSL is equivalent to XSTPIR. Note that we must have $X \geq T$, because any $X + 1$ colluding servers reveal everything about the submodels, hence they can figure out the desired indices $\theta_t, t \in \mathbb{Z}^+$ by monitoring which submodels are modified. Therefore, when $X_\Delta = 0$, the asymptotic capacity of XSTPFSL is fully settled. We conjecture that in general, the condition $X \geq X_\Delta + T$ is required for the feasibility (bounded communication cost as $K \to \infty$) of the asymptotic setting.

### 3.1 Observations

1. The upload cost achieved by CSA-RW is bounded ($O(1)$), according to Theorem\textsuperscript{1} which is order-wise optimal.

2. The assumptions that $\ell$ divides $L$ and $q \geq \ell + N$ are mild assumptions: $L$ can be rounded up to the nearest integer multiple of $\ell$ with negligible impact as $L \gg \ell$, and a field size of $\ell + N$ is not too large (does not scale with $K$ or $L$).
3. Theorem 1 contains a trade-off between the level of data security $X$, the upload cost $U$ and the download cost $D$ in the CSA-RW scheme. In particular, it is possible to achieve stricter data security levels and lower upload cost simultaneously, at the cost of increased download cost. To see this tradeoff explicitly, let us consider two examples where $N = 10$, $X_\Delta = T = 1$ and $N = 10$, $X_\Delta = 1$, $T = 2$. For the two examples, with various choices of $X \in \{2, 3, \cdots, 8\}$ and $X \in \{3, 4, \cdots, 7\}$ respectively, we plot the achievable upload costs versus download costs in Figure 2.

4. Recall that the $X$-security constraint only requires protection against an external adversary who can access the current storage but not the past history at any $X$-servers. However, a closer look at the CSA-RW scheme reveals that if the number of compromised servers is no more than $\min(X_\Delta, T)$, then even an internal adversary, i.e., an adversary who has access to the entire history of all previous stored values and queries seen by the compromised servers, can still learn nothing about the stored submodels.

5. With regard to the condition $X \geq X_\Delta + T$ in Theorem 1, let us note that if the recoverability constraint (12) is relaxed, then the CSA-RW scheme naturally extends to lower security levels, where $X < X_\Delta + T$. This is because any scheme that provides stronger data security guarantee automatically also provides lower security level guarantees.

6. An interesting aspect of the CSA-RW scheme (which answers in the affirmative an open question raised in Section 4.4.4 of [2]) is the synergistic exploitation of the structure of the same query vector across both the retrieval phase and the update phase. Intuitively, consider the vector $W = [W_1, W_2, \cdots, W_K]^T$ that consists of symbols from the $K$ submodels. Also, let us consider the standard basis vector $e_K(\theta_t)$ for some $\theta_t \in [K]$. Note that the symbol of the $\theta_t^{th}$ submodel is retrievable from the inner product $W^T e_K(\theta_t) = W_{\theta_t}$. This is the key to the reading operation. On the other hand, to perform a write, we note the following equation.

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Figure 2: Achievable (upload, download) costs pairs $(U, D)$ for the proposed CSA-RW scheme as both $K \to \infty$ and $L/K \to \infty$ for $N = 10$, $X_\Delta = T = 1$ (the blue curve) and $N = 10$, $X_\Delta = 1$, $T = 2$ (the red curve), with various choices of $X \in \{2, 3, \cdots, 8\}$ and $X \in \{3, 4, \cdots, 7\}$, respectively.
\[ W + \Delta_t e_K(\theta_t) = [W_1, \ldots, W_{\theta_t-1}, W_{\theta_t} + \Delta_t, W_{\theta_t+1} \cdots, W_K]. \] Therefore, intuitively we see how the same secret-shared query vector \( e_K(\theta_t) \) is useful for both the retrieval phase and the update phase. Note that this requires the construction of the scheme to be compatible with both the inner product and the operations of the vector space over the finite field \( \mathbb{F}_q \). Consequently, the idea of CSA turns out to be the communication-efficient solution.

7. Another interesting aspect of CSA-RW is that when the data security level (\( X \)) is greater than \( X_\Delta + T \), it is possible to achieve the upload cost of less than \( N \) in the asymptotic setting, as stated in Theorem \[\mathbf{4}\]. This is made possible by the construction of CSA null-shaper, see Definition \[\mathbf{4}\]. By carefully placing nulls of the CSA code polynomial in the update equation, the storage of a total of \( X - (X_\Delta + T) \) servers is left unmodified. Note that the storage structure (i.e., CSA storage, see Definition \[\mathbf{4}\]) is preserved, and the functionality of the retrieval phase is not affected. Hence, the users do not upload the increments to those servers, and the upload cost is improved. Now let us further clarify the idea with a minimal example where \( X = 2, T = 1, X_\Delta = 0, N = 4 \). Continuing from the previous observation, let us define the following functions.

\[
S(\alpha) = W + \alpha Z_1 + \alpha^2 Z_2,
\]
\[
Q(\alpha) = e_K(\theta_t) + \alpha Z',
\]

where \( Z_1, Z_2, Z' \) are uniformly and independently distributed noise vectors that are used to protect data security and user’s privacy, respectively. Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be 4 distinct non-zero elements from a finite field \( \mathbb{F}_q \). The storage at the 4 servers is \( S(\alpha_1), S(\alpha_2), S(\alpha_3), S(\alpha_4) \), respectively. Similarly, the query for the 4 servers is \( Q(\alpha_1), Q(\alpha_2), Q(\alpha_3), Q(\alpha_4) \), respectively. We note that the storage and the query can be viewed as secret sharing with threshold of \( X = 2 \) and \( T = 1 \), respectively. Now it is easy to see that by the following update equation,

\[
S'(\alpha) = S(\alpha) + \Delta_t Q(\alpha)
\]
\[
= (W + \Delta_t e_K(\theta_t)) + \alpha (Z_1 + \Delta_t Z') + \alpha^2 Z_2,
\]

the user is able to update the symbol of the \( \theta_t^{th} \) message with the increment \( \Delta_t \). And we note that after updating, the storage remains a secret sharing of threshold \( X = 2 \). However, let us define the function \( \Omega(\alpha) = (\alpha_1 - \alpha)/\alpha_1 \), which is referred to as CSA null-shaper, and consider the following update equation.

\[
S'(\alpha) = S(\alpha) + \Omega(\alpha) \Delta_t Q(\alpha).
\]

Now let us inspect the second term on the RHS. Note that

\[
\Omega(\alpha) \Delta_t Q(\alpha) = \frac{1}{\alpha_1} (\alpha_1 - \alpha) \Delta_t (e_K(\theta_t) + \alpha Z')
\]
\[
= \frac{1}{\alpha_1} \Delta_t (\alpha_1 e_K(\theta_t) + \alpha (\alpha_1 Z' - e_K(\theta_t)) - \alpha^2 Z')
\]
\[
= \Delta_t (e_K(\theta_t) + \alpha (Z' - \alpha_1^{-1} e_K(\theta_t)) - \alpha^2 \alpha_1^{-1} Z')
\]
\[
= \Delta_t (e_K(\theta_t) + \alpha I_1 - \alpha^2 I_2),
\]

where \( I_1 = Z' - \alpha_1^{-1} e_K(\theta_t) \) and \( I_2 = \alpha_1^{-1} Z' \). It is obvious that by the new update equation, the user is able to update the symbol of the \( \theta_t^{th} \) message with the increment \( \Delta_t \), while
maintaining the storage as a secret sharing of threshold 2, i.e.,

\[ S'(\alpha) = (W + \Delta_t e_k(\theta_t)) + \alpha(Z_1 + \Delta_t I_1) + \alpha^2(Z_2 + \Delta_t I_2). \]

(26)

However, by the definition of CSA null-shaper, we have \( \Omega(\alpha_1) = 0 \). Thus \( S'(\alpha_1) = S(\alpha_1) \), and it is not necessary to upload the increment \( \Delta_t \) to Server 1. In other words, \( I_1 \) and \( I_2 \) are artificially correlated interference symbols such that the codeword \( \Omega(\alpha_1)Q(\alpha_1) \) is zero, and accordingly, the upload cost is improved. Note that CSA null-shaper does not affect the storage structure because \( X = 2 > T = 1 \). The idea illustrated in this minimal example is generalizable for the framework of CSA, see Section 4 for details.

8. Let us compare our CSA-RW solution with that in [10]. The setting in [10] corresponds to \( X = 0, T = 1, X_{\Delta} = 0 \), and the recoverability constraint (12) is relaxed. Since the recoverability constraint is relaxed, our CSA-RW scheme for \( X = 1, T = 1, X_{\Delta} = 0 \) applies to the setting of [10] \( (X = 1 \) security implies \( X = 0 \) security). To make the comparison more concrete, let us briefly review the construction in [10]. At any time \( t, t' \in \mathbb{N} \), each of the \( N \) servers stores the \( K \) submodels in the following coded form.

\[ W_k^{t(t-1)} + z_k(t) \Delta_t, \forall k \in [K], \]

(27)

where for all \( t \in \mathbb{Z}^+, \left( z_k^{(t)} \right)_{k \in [K]} \) are distinct random scalars generated by User \( t \) and we set \( z_{\theta_t} = 1 \). For completeness we define \( \Delta_0 = 0, z_k^{(0)} = 0, W_k^{(t-1)} = W_k^{(0)}, \forall k \in [K] \). In addition, the \( N \) servers store the random scalars \( \left( z_k^{(t)} \right)_{k \in [K]} \) as well as the increment \( \Delta_t \) according to a secret sharing scheme of threshold 1. In the retrieval phase, User \( t \) retrieves the coded desired submodel \( W_{\theta_t}^{t(t-2)} + z_{\theta_t}^{(t-1)} \Delta_{t-1} \) privately according to a capacity-achieving replicated storage based PIR scheme, e.g., [12]. Besides, User \( t \) also downloads the secret shared random scalars \( \left( z_k^{(t-1)} \right)_{k \in [K]} \) and the increment \( \Delta_{t-1} \) to correctly recover the desired submodel \( W_{\theta_t}^{t(t-1)} \). In the update phase, User \( t \) uploads to each of the \( N \) servers the following update vectors \( P_k^{(t)} \) for all \( k \in [K] \).

\[ P_k^{(t)} = \begin{cases} z_k^{(t)} \Delta_t - z_k^{(t-1)} \Delta_{t-1}, & k \neq \theta_{t-1}, \\ z_k^{(t)} \Delta_t, & k = \theta_{t-1}. \end{cases} \]

(28)

Also, User \( t \) uploads the secret shared random scalars \( \left( z_k^{(t)} \right)_{k \in [K]} \) and the increment \( \Delta_t \) to the \( N \) servers. To perform an update, each of the \( N \) servers updates all of the submodels \( k \in [K] \) according to the following equation.

\[ \left( W_k^{(t-2)} + z_k^{(t-1)} \Delta_{t-1} \right) + P_k^{(t)} = W_k^{(t-1)} + z_k^{(t)} \Delta_t. \]

(29)

Perhaps the most significant difference between our CSA-RW scheme and the construction in [10] is that the latter does not guarantee the privacy of successive updates, i.e., by monitoring the storage at multiple time slots, the servers are eventually able to learn about the
submodel indices from past updates.\footnote{This is because the update vectors for the \(K\) submodels at any time \(t\) can be viewed as \(\mathcal{P}_t = \text{span}\{\Delta_1, \Delta_{t-1}\}\). For any two consecutive time slots \(t\) and \(t + 1\), it is possible to determine \(\text{span}\{\Delta_t\} = \text{span}\{\Delta_1, \Delta_{t-1}\} \cap \text{span}\{\Delta_{t+1}, \Delta_t\} = \mathcal{P}_t \cap \mathcal{P}_{t+1}\). Due to the fact that for User \(t\), the update vector for the \(\theta_{t+1}^{\ell_{t+1}}\) submodel (6) only lies in \(\text{span}\{\Delta_t\}\), any curious server is able to obtain information about \(\theta_{t-1}\) from \(\mathcal{P}_t\) if \(\Delta_t\) is linearly independent of \(\Delta_{t-1}\).}

On the other hand, our construction guarantees information theoretic privacy for an unlimited number of updates, without extra storage overhead. Furthermore, we note that the normalized download cost achieved by the construction in \cite{10} cannot be less than 2, whereas CSA-RW achieves download cost of less than 2 with large enough \(N\). For the asymptotic setting \(L/K \to \infty\), the upload cost achieved by \cite{10} is at least \(2N + 1\), while CSA-RW achieves the upload cost of at most \(N\). The lower bound of upload cost of XSTPFSL is characterized in \cite{10} as \(NK\). However, our construction of CSA-RW shows that it is possible to do better.\footnote{It is claimed in \cite{11} that the achieved upload cost is \(NLK + L + K\). However, with compression, e.g., entropy encoding, it is possible to achieve lower upload cost. For example, in the asymptotic setting \(L/K \to \infty\), the upload cost of \((2N + 1 + 1/(N - 1))\) may be achievable.}

In particular, for the asymptotic setting \(K \to \infty\), \(L/K \to \infty\), the upload cost of less than \(N\) is achievable by the CSA-RW scheme.

9. Let us also briefly review the 4-server information-theoretic DORAM construction in \cite{49} to see how our CSA-RW scheme improves upon it. Note that the setting considered in \cite{49} is a special case of our problem where \(X = 1, T = 1, X_\Delta = 0\), and the recoverability constraint is also relaxed. First, we note that in the asymptotic setting \(L/K \to \infty\), the upload cost achieved by the CSA-RW is the same as that in \cite{49}. Therefore, for this comparison we focus on the retrieval phase and the download cost. Specifically, the information-theoretic DORAM construction in \cite{49} partitions the four servers into two groups, each of which consists of 2 servers. For the retrieval phase, the first group emulates a 2-server PIR, storing the \(K\) submodels secured with additive random noise, i.e., \(W + Z\). The second group emulates another 2-server PIR storing the random noise \(Z\). To retrieve the desired submodel privately, the user exploits a PIR scheme to retrieve the desired secured submodel, as well as the corresponding random noise. Therefore, with capacity-achieving PIR schemes, the download cost is 4 (for large \(K\)). On the other hand, our CSA-RW scheme avoids the partitioning of the servers and improves the download cost by jointly exploiting all 4 servers. Remarkably, with the idea of cross-subspace alignment, out of the 4 downloaded symbols, the interference symbols align within 2 dimensions, leaving 2 dimensions interference-free for the desired symbols, and consequently, the asymptotically optimal download cost of 2 is achievable. Lastly, the CSA-RW scheme also generalizes efficiently to arbitrary numbers of servers.

10. Finally, let us explore the practical significance of the asymptotic limits \(K \to \infty\), \(L/K \to \infty\), with an example. Suppose we have \(N = 6\) distributed servers, we require security and privacy levels of \(X_\Delta = T = 1, X = 3\), so that \(\ell = 2\), and we operate over \(\mathbb{F}_8\). Consider an e-commerce recommendation application similar to what is studied in \cite{5}, where a global model with a total of 3,500,000 symbols (from \(\mathbb{F}_8\)) is partitioned into \(K = 50\) submodels. Each of the submodels is comprised of \(L = 70,000\) symbols. Note that \(L \gg K \gg 1\). Now according to Theorem \ref{1} the normalized upload cost achieved by the CSA-RW scheme is
$U = (6 - 3 + 1 + 1) + \frac{5 \times 2 \times 50}{20000} \approx 5.00857$. On the other hand, the normalized download cost achieved is $D = 6/2 = 3$. Evidently, the asymptotic limits $K \to \infty, L/K \to \infty$ are fairly accurate for this non-asymptotic setting. For this particular example, the upload cost is increased by only 0.17% compared to the asymptotic limit. On the other hand, the download cost is increased by only $1.4 \times 10^{-22}$% compared to the lower bound (evaluated for $K = 50$) from [28].

4 Proof of Theorem 1

Let $X_{\Delta} + T \leq X \leq N - T - 1$. Set $\ell = N - (X + T), L = \mu \ell$. We need a total of $(\ell + N)$ distinct constants from the finite field $F_q, q \geq \ell + N$, denoted as $(f_1, f_2, \cdots, f_\ell)$ and $(\alpha_1, \alpha_2, \cdots, \alpha_N)$. Furthermore, $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ are non-zero elements. \forall k \in [K], t \in \mathbb{N}, u \in [\mu], l \in [\ell], define

$$W_k^{(t)}(u, l) = W_k^{(t)}(\ell(u - 1) + l),$$

i.e., the $L = \mu \ell$ symbols of each of the $K$ submodels are partitioned into $\mu$ groups, each of which consists of $\ell$ symbols. Similarly, for the increments, we define

$$\Delta_t(u, l) = \Delta_t(\ell(u - 1) + l).$$

For all $t \in \mathbb{N}, u \in [\mu], l \in [\ell]$, let us define,

$$\dot{W}_{u,l}^{(t)} = \left[ W_1^{(t)}(u, l), W_2^{(t)}(u, l), \cdots, W_K^{(t)}(u, l) \right]^T.$$  

Further, let us set $Z_s = \left\{ Z_{u,l}^{(0,x)} \right\}_{u \in [\mu], l \in [\ell], x \in [X]}$ where $Z_{u,l}^{(0,x)}$ are i.i.d. uniform column vectors from $F_q^{K}$, \forall $u \in [\mu], l \in [\ell], x \in [X]$. For all $t \in \mathbb{Z}^+$, we also set $Z_{u,l}^{(t)} = \left\{ Z_{u,l}^{(t,s)} \right\}_{l \in [\ell], s \in [T]} \cup \left\{ Z_{u,l}^{(t,x)} \right\}_{u \in [\mu], l \in [\ell], x \in [X_{\Delta}]}$ where for all $t \in \mathbb{Z}^+, u \in [\mu], l \in [\ell], s \in [T], Z_{u,l}^{(t,s)}$ are i.i.d. uniform column vectors from $F_q^K$ and for all $t \in \mathbb{Z}^+, u \in [\mu], l \in [\ell], x \in [X_{\Delta}], Z_{u,l}^{(t,x)}$ are i.i.d. uniform scalars from $F_q$. Furthermore, for all $t \in \mathbb{Z}^+, u \in [\mu], l \in [\ell], x \in [X]$, let $Z_{u,l}^{(t,x)}$ be $K \times 1$ column vectors from the finite field $F_q$. Let us start from the following definitions.

Definition 1. (CSA Storage) For any $t \in \mathbb{N}$, the storage at the $N$ servers is said to form the CSA storage if for all $n \in [N], S_n^{(t)}$ have the following form.

$$S_n^{(t)} = \left( S_{n,1}^{(t)}, S_{n,2}^{(t)}, \cdots, S_{n,\mu}^{(t)} \right),$$

where for all $u \in [\mu]$

$$S_{n,u}^{(t)} = \left[ \frac{1}{f_1 - \alpha_n} \dot{W}_{u,1}^{(t)} + \sum_{x \in [X]} \alpha_n^{x-1} \dot{Z}_{u,1}^{(t,x)} \right] \left[ \frac{1}{f_2 - \alpha_n} \dot{W}_{u,2}^{(t)} + \sum_{x \in [X]} \alpha_n^{x-1} \dot{Z}_{u,2}^{(t,x)} \right] \cdots \left[ \frac{1}{f_{\ell} - \alpha_n} \dot{W}_{u,\ell}^{(t)} + \sum_{x \in [X]} \alpha_n^{x-1} \dot{Z}_{u,\ell}^{(t,x)} \right].$$
Definition 2. (CSA Query) For any \( t \in \mathbb{Z}^+ \), the retrieve-queries \( (Q_n^{(t,\theta_t)})_{n \in [N]} \) from User \( t \) for the \( N \) servers are said to form a CSA query if for all \( n \in [N] \), we have
\[
Q_n^{(t,\theta_t)} = \left( Q_{n,1}^{(t,\theta_t)}, Q_{n,2}^{(t,\theta_t)}, \ldots, Q_{n,\mu}^{(t,\theta_t)} \right),
\]
where for all \( n \in [N] \),
\[
Q_{n,1}^{(t,\theta_t)} = \cdots = Q_{n,\mu}^{(t,\theta_t)} = \begin{bmatrix} e_K(\theta_t) + (f_1 - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \mathbf{Z}_1^{(t,s)} \\ e_K(\theta_t) + (f_2 - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \mathbf{Z}_2^{(t,s)} \\ \vdots \\ e_K(\theta_t) + (f_\ell - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \mathbf{Z}_\ell^{(t,s)} \end{bmatrix}.
\]

Definition 3. (CSA Increment) For any \( t \in \mathbb{Z}^+ \), the write-queries \( (P_n^{(t,\theta_t)})_{n \in [N]} \) from User \( t \) for the \( N \) servers are said to form a CSA increment if for all \( n \in [N] \), we have
\[
P_n^{(t,\theta_t)} = \left( P_{n,1}^{(t,\theta_t)}, P_{n,2}^{(t,\theta_t)}, \ldots, P_{n,\mu}^{(t,\theta_t)} \right),
\]
where for all \( n \in [N] \), \( u \in [\mu] \),
\[
P_{n,u}^{(t,\theta_t)} = \text{diag} \left( \left( \frac{1}{f_1 - \alpha_n} \Delta_t(u,1) + \sum_{x \in [X_\Delta]} \alpha_n^{x-1} \mathbf{Z}_u^{(t,x)} \right) \mathbf{I}_K, \ldots, \left( \frac{1}{f_\ell - \alpha_n} \Delta_t(u,\ell) + \sum_{x \in [X_\Delta]} \alpha_n^{x-1} \mathbf{Z}_u^{(t,x)} \right) \mathbf{I}_K \right).
\]

Definition 4. (CSA Null-shaper) Define the set \( F = [X - (X_\Delta + T)] \). The CSA null-shaper is defined as follows.
\[
\Omega_n = (\Omega_{n,1}, \Omega_{n,2}, \ldots, \Omega_{n,\mu}),
\]
where for all \( n \in [N] \), \( u \in [\mu] \),
\[
\Omega_{n,1} = \cdots = \Omega_{n,\mu} = \text{diag} \left( \left( \prod_{f \in F} (\alpha_f - \alpha_n) \right) \mathbf{I}_K, \ldots, \left( \prod_{f \in F} (\alpha_f - \alpha_n) \right) \mathbf{I}_K \right).
\]

Note that the CSA null-shaper is a constant, and most importantly, for all \( n \in F, u \in [\mu] \), we have \( \Omega_{n,u} = 0 \).

Definition 5. (CSA-RW Scheme) The initial storage at the \( N \) servers is the CSA storage at time 0, i.e., \( (S_n^{(0)})_{n \in [N]} \). At time \( t, t \in \mathbb{Z}^+ \), in the retrieval phase, the user uploads the CSA query for the \( N \) servers and retrieves the desired submodel \( W_{\theta_t}^{(t-1)} \) from the answers returned by the \( N \) servers, which are constructed as follows.
\[
A_n^{(t,\theta_t)} = \left( S_n^{(t-1)} \right)^T \circ Q_n^{(t,\theta_t)}, n \in [N].
\]

In the update phase, the user uploads the CSA increment for the \( N \) servers and each of the servers updates its storage according to the following equation.
\[
S_n^{(t)} = S_n^{(t-1)} + \Omega_n \circ P_n^{(t,\theta_t)} \circ Q_n^{(t,\theta_t)}, n \in [N].
\]
Now let us prove the correctness of the presented construction. We start from the following lemmas.

**Lemma 1.** The CSA-RW scheme allows User $t$ to retrieve the desired submodel $W_{\theta_t}^{(t-1)}$ while satisfying the $T$-privacy constraint.

**Proof.** By definition, for all $n \in [N]$, we have

$$A_n^{(t,\theta_t)} = \left( S_n^{(t-1)} \right)^T Q_{n,u}^{(t,\theta_t)} \bigg|_{u \in [\mu]}. \quad (43)$$

For all $u \in [\mu]$, we have

$$\left( S_n^{(t-1)} \right)^T Q_{n,u}^{(t,\theta_t)} = \sum_{l \in [\ell]} \left( \frac{1}{f_l - \alpha_n} W_{u,l}^{(t-1)} + \sum_{x \in [X]} \alpha_n x^{t-2} Z_{u,l}^{(t-1,x)} \right) \left( e_K(\theta_t) + (f_l - \alpha_n) \sum_{s \in [T]} \alpha_n s^{t-1} Z_{l,s}^{(t,s)} \right)$$

$$= \sum_{l \in [\ell]} \left( \frac{1}{f_l - \alpha_n} W_{u,l}^{(t-1)} e_K(\theta_t) + \sum_{i \in [X+T]} \alpha_n i^{t-1} I_{u,l,i}^{(t)} \right)$$

$$= \sum_{l \in [\ell]} \left( \frac{1}{f_l - \alpha_n} W_{\theta_t}^{(t-1)} (u, l) + \sum_{i \in [X+T]} \alpha_n i^{t-1} I_{u,i}^{(t)} \right), \quad (46)$$

where for all $l \in [\ell], i \in [X+T], I_{u,l,i}^{(t)}$ are various linear combinations of inner products of $W_{u,l}^{(t-1)}$, $e_K(\theta_t)$, $(Z_{u,l}^{(t-1,x)})_{x \in [X]}$ and $(Z_{l,s}^{(t,s)})_{s \in [T]}$, whose exact forms are irrelevant, and $I_{u,i}^{(t)} = \sum_{l \in [\ell]} I_{u,l,i}^{(t)}$.

Therefore, in the matrix form, we have

$$\begin{bmatrix}
S_{1,u}^{(t-1)} & Q_{1,u}^{(t,\theta_t)} \\
S_{2,u}^{(t-1)} & Q_{2,u}^{(t,\theta_t)} \\
\vdots & \vdots \\
S_{N,u}^{(t-1)} & Q_{N,u}^{(t,\theta_t)}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{f_1 - \alpha_1} \cdots \frac{1}{f_1 - \alpha_N} & 1 \cdots \alpha_{X+T-1} \\
\frac{1}{f_2 - \alpha_1} \cdots \frac{1}{f_2 - \alpha_N} & 1 \cdots \alpha_{2X+T-1} \\
\vdots & \vdots & \vdots \\
\frac{1}{f_{\ell} - \alpha_1} \cdots \frac{1}{f_{\ell} - \alpha_N} & 1 \cdots \alpha_{\ell X+T-1}
\end{bmatrix}
\begin{bmatrix}
W_{\theta_t}^{(t-1)}(u, 1) \\
W_{\theta_t}^{(t-1)}(u, 2) \\
\vdots \\
W_{\theta_t}^{(t-1)}(u, \ell) \\
I_{u,1}^{(t)} \\
\vdots \\
I_{u,X+T}^{(t)}
\end{bmatrix}, \quad (47)$$

where $C$ is an $N \times N$ Cauchy-Vandermonde matrix, which is invertible, see, e.g., [51]. Thus for all $u \in [\mu]$, the user is able to retrieve the desired symbols $(W_{\theta_t}^{(t-1)}(u, l))_{l \in [\ell]}$ by inverting the matrix.

Therefore, the desired submodel $W_{\theta_t}^{(t-1)}$ is retrieved. To verify the $T$-privacy, we note that by the construction of the query, the vector $e_K(\theta_t)$, which carries the information of desired index $\theta_t$, is protected by the MDS($N, T$) coded uniform i.i.d. random noise vectors. Thus the queries for the $N$ servers form a secret sharing of threshold $T$, and are independent of the queries and the increments of all prior users $\tau, \tau \in [t - 1]$. Thus $T$-privacy is guaranteed.

**Lemma 2.** The CSA-RW scheme correctly updates the desired submodel and achieves CSA storage at each timeslot $t$ while guaranteeing $T$-privacy and $X_\Delta$-security.
Proof. Let us first inspect the second term on the RHS of (42). For all \( n \in [N] \), we have

\[
\Omega_n \circ P_n^{(t, \theta_t)} \circ Q_n^{(t, \theta_t)} = \left( \Omega_{n,u} P_{n,u}^{(t, \theta_t)} Q_{n,u}^{(t, \theta_t)} \right)_{u \in [\mu]},
\]

where for all \( u \in [\mu] \), we have

\[
\Omega_{n,u} P_{n,u}^{(t, \theta_t)} Q_{n,u}^{(t, \theta_t)} = \left[ \prod_{f \in F} (\alpha_f - \alpha_{n-a}) \right] \cdot \left[ \prod_{f \in (\alpha_f - f_1)} \left( \frac{1}{f_1 - \alpha_{n-a}} \Delta_t(u, 1) + \sum_{x \in [X]} \alpha_n^{-1} Z_{u,x}^{(t, x)} \right) \right] \cdot \left( e_K(\theta_t) + (f_1 - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \tilde{Z}_1^{(t,s)} \right)
\]

\[
= \left[ \prod_{f \in (\alpha_f - f_1)} \left( \frac{1}{f_1 - \alpha_{n-a}} \Delta_t(u, 1) e_K(\theta_t) + \sum_{i \in [X]} \alpha_n^{i-1} I_{u,i}^{(t)} \right) \right] \cdot \left( e_K(\theta_t) + (f_1 - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \tilde{Z}_1^{(t,s)} \right)
\]

\[
= \left[ \prod_{f \in (\alpha_f - f_1)} \left( \frac{1}{f_1 - \alpha_{n-a}} \Delta_t(u, 1) e_K(\theta_t) + \sum_{i \in [X]} \alpha_n^{i-1} I_{u,i}^{(t)} \right) \right] \cdot \left( e_K(\theta_t) + (f_1 - \alpha_n) \sum_{s \in [T]} \alpha_n^{s-1} \tilde{Z}_1^{(t,s)} \right)
\]

For all \( l \in [\ell] \), \( i \in [X] \), \( I_{u,i}^{(t)} \) and \( \tilde{Z}_1^{(t,s)} \) are various linear combinations of the vectors \( e_K(\theta_t) \) and \( \tilde{Z}_1^{(t,s)} \), whose exact forms are not relevant. Note that in (50), we multiply the last two terms in each row by the second term in each row with the second term. Note that for all \( l \in [\ell] \), \( \prod_{f \in F} (\alpha_f - f_1) \) is the remainder of the polynomial division \( (\prod_{f \in F} (\alpha_f - \alpha_{n-a})) / (f_1 - \alpha_n) \), and they are non-zero due to the fact that \( (f_1, f_2, \cdots, f_\ell), \alpha_1, \alpha_2, \cdots, \alpha_N \) are distinct. Now by the definition of the update equation in (42), we have

\[
S_{n}^{(t)} = S_{n}^{(t-1)} + P_{n}^{(t, \theta_t)} \circ Q_{n}^{(t, \theta_t)}
\]

\[
= \left( S_{n,1}^{(t-1)}, \cdots, S_{n,\mu}^{(t-1)} \right) + \left( \Omega_{n,1} P_{n,1}^{(t, \theta_t)} Q_{n,1}^{(t, \theta_t)}, \cdots, \Omega_{n,\mu} P_{n,\mu}^{(t, \theta_t)} Q_{n,\mu}^{(t, \theta_t)} \right)
\]

\[
= \left( S_{n,1}^{(t-1)} + \Omega_{n,1} P_{n,1}^{(t, \theta_t)} Q_{n,1}^{(t, \theta_t)}, \cdots, S_{n,\mu}^{(t-1)} + \Omega_{n,\mu} P_{n,\mu}^{(t, \theta_t)} Q_{n,\mu}^{(t, \theta_t)} \right)
\]
We note that for all \( u \in [\mu] \), we have
\[
S_{n,u}^{(t-1)} + \Omega_{n,u} \mathbf{P}^{(t,\theta_t)} \mathbf{Q}_{n,u}^{(t,\theta_t)} = \begin{bmatrix}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,1}^{(t-1)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,1}^{(t-1,x)}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,2}^{(t-1)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,2}^{(t-1,x)}
\vdots
\frac{1}{1-\alpha_n} \mathbf{W}_{u,\ell}^{(t-1)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,\ell}^{(t-1,x)}
\end{bmatrix} + \begin{bmatrix}
\frac{1}{1-\alpha_n} \Delta_t(u,1) e_K(\theta_t) & + \sum_{i \in [X]} \alpha_n^{i-1} \mathbf{I}_{u,1,i}^{(t)}
\frac{1}{1-\alpha_n} \Delta_t(u,2) e_K(\theta_t) & + \sum_{i \in [X]} \alpha_n^{i-1} \mathbf{I}_{u,2,i}^{(t)}
\vdots
\frac{1}{1-\alpha_n} \Delta_t(u,\ell) e_K(\theta_t) & + \sum_{i \in [X]} \alpha_n^{i-1} \mathbf{I}_{u,\ell,i}^{(t)}
\end{bmatrix}
\] (56)
\[
= \begin{bmatrix}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,1}^{(t)} & + \Delta_t(u,1) e_K(\theta_t) + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,1}^{(t,x)}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,2}^{(t)} & + \Delta_t(u,2) e_K(\theta_t) + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,2}^{(t,x)}
\vdots
\frac{1}{1-\alpha_n} \mathbf{W}_{u,\ell}^{(t)} & + \Delta_t(u,\ell) e_K(\theta_t) + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,\ell}^{(t,x)}
\end{bmatrix}
\] (57)
\[
= \begin{bmatrix}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,1}^{(t)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,1}^{(t,x)}
\frac{1}{1-\alpha_n} \mathbf{W}_{u,2}^{(t)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,2}^{(t,x)}
\vdots
\frac{1}{1-\alpha_n} \mathbf{W}_{u,\ell}^{(t)} & + \sum_{x \in [X]} \alpha_n^{x-1} \mathbf{Z}_{u,\ell}^{(t,x)}
\end{bmatrix}
\] (58)

where (58) holds from the definitions of \((\mathbf{W}_k^{(t)})_{k \in [K]}\) and \((\hat{\mathbf{W}}_{u,l}^{(t)})_{u \in [\mu], l \in [\ell]}\). Note that for all \( l \in [\ell], x \in [X], \mathbf{Z}_{u,l}^{(t,x)} = \hat{\mathbf{Z}}_{u,l}^{(t-1,x)} + \mathbf{I}_{u,l,x}^{(t)} \). The form shown in (58) is exactly the form defined by CSA storage. \( T \)-privacy is proved by Lemma 1. Besides, it is easy to see that the symbols of the increment vector \( \Delta_t \) are protected by the MDS(\( N, X_\Delta \)) coded uniform i.i.d. random noise symbols, i.e., the CSA increment forms a secret sharing of threshold \( X_\Delta \), and it is independent of the increments by the users \( \tau, \tau \in [t-1] \) and the queries by the users \( \tau, \tau \in [t] \). Thus \( X_\Delta \)-security is guaranteed. This completes the proof of Lemma 2.

Based on Lemma 1, Lemma 2 which hold for all \( t \), the CSA-RW scheme satisfies the correctness, \( T \)-privacy, and \( X_\Delta \)-security constraints for each update \( t, t \in \mathbb{Z}^+ \). Now let us consider the \( X \)-security. By the definition of CSA storage at any time \( t \), it forms a secret sharing of threshold \( X \), i.e., \((\hat{\mathbf{W}}_{u,l}^{(t)})_{u \in [\mu], l \in [\ell]}\) are protected by the MDS(\( N, X \)) coded uniform i.i.d. random noise symbols \((\hat{\mathbf{Z}}_{u,l}^{(t,x)})_{u \in [\mu], l \in [\ell], x \in [X]}\). Thus \( X \)-security is guaranteed. Besides we note that CSA storage at any time \( t \), \((S_n^{(t)})_{n \in [N]}\) can be viewed as an MDS(\( N, X + 1 \)) code of \((\hat{\mathbf{W}}_{u,l}^{(t)})_{u \in [\mu], l \in [\ell]}\) and \((\hat{\mathbf{Z}}_{u,l}^{(t,x)})_{u \in [\mu], l \in [\ell], x \in [X]}\), thus we can recover all of the \( K \) submodels from the storage of any \( (X + 1) \) servers.

Finally, let us calculate the upload and download costs. Note that the costs are time-invariant, so we use the compact notations \((U, D)\). In terms of the upload cost \( U \), by the definition of CSA query, a total of \( N \ell K \) symbols from \( \mathbb{F}_q \) must be uploaded. Furthermore, by the definition of CSA increment, there are a total of \( L = \mu \ell \) independent symbols from \( \mathbb{F}_q \) uploaded for each of the servers. However, according to the definition of CSA null-shaper, for all \( n \in \mathcal{F}, u \in [\mu] \), we have \( \Omega_{n,u} = 0 \). Therefore, it is not necessary to upload the CSA increment for the servers \( n, n \in \mathcal{F} \). In other words, the storage for the servers \( n, n \in \mathcal{F} \) is not changed in the update phase. Therefore,
we have $U = ((N - X + X_\Delta + T)\mu \ell + N\ell K) / (\mu \ell) = (N - X + X_\Delta + T) + N\ell K / L$. On the other hand, the user downloads a total of $N\mu$ symbols from the finite field $\mathbb{F}_q$ in the retrieval phase, thus we have $D = (N\mu) / (\mu \ell) = N / \ell$. The asymptotic setting follows by applying the limit $L/K \to \infty$. This completes the proof of Theorem 1.

Remark 1. (Symmetrization and Randomization) As described, the upload cost is not symmetric across servers, because the CSA increment is not uploaded to the servers $n, n \in F$. However, we claim that symmetry can be achieved by randomizing over the $N$ permutations of the servers that correspond to cyclic shifts. The choice of the set $F$ can be any subset of $[N]$ such that $|F| = X - (X_\Delta + T)$. Therefore, it is possible for each user to uniformly and randomly select the set $F$ from the cyclic permutations to achieve a symmetric upload cost in the average sense.

4.1 Complexity Analysis

To further analyze the CSA-RW scheme, we conduct a brief complexity analysis in this subsection.

(Encoding and Decoding Complexity) First, let us consider the complexity of the encoding and decoding algorithms of our construction. It is worth noting that the computations for producing the CSA storage, CSA query and CSA increment can be regarded as multiplications of (scaled) Cauchy-Vandermonde matrices with various vectors. The computation for recovering the desired submodel by the user from the answers of the $N$ servers can be viewed as solving linear systems defined by Cauchy-Vandermonde matrices. Cauchy-Vandermonde matrices are one important class of structured matrix, for which superfast algorithms have been studied extensively \[52,53\]. Therefore, by these superfast algorithms, the complexity of producing the CSA storage, CSA query and CSA increment is at most $\tilde{O}(\mu K N \log^2 N)$, $\tilde{O}(\ell K N \log^2 N)$ and $\tilde{O}(\mu N \log^2 N)$, respectively.

On the other hand, the complexity of decoding the desired submodel from the answers of the servers is at most $\tilde{O}(\mu N \log^2 N) = \tilde{O}(\mu \ell \log^2 N)$. It is obvious that the encoding/decoding algorithms have a complexity that is almost linear in their output/input sizes.

(Access Complexity) Another complexity of interest is the access complexity. It is clear from the CSA-RW scheme that in the retrieval phase, the access complexity for each of the servers is $\mathcal{O}(KL)$, i.e., all of the $K$ submodels must be read. On the other hand, in the update phase, for each of the servers, a symmetrized (or randomized, see Remark 1) CSA-RW scheme has (average) access complexity of $\mathcal{O}\left(\frac{(N - X + X_\Delta + T)K L}{N}\right)$, i.e., (in the average sense) it is sufficient to update only $1 - \frac{X - (X_\Delta + T)}{N}$ fraction of the symbols of each of the $K$ submodels.

5 Conclusion

We explored the problem of $X$-secure $T$-private federated submodel learning from an information-theoretic perspective. We proposed a novel construction of a cross-subspace alignment based private read and write scheme, the CSA-RW scheme. The CSA-RW scheme answers an existing open question by taking advantage of synergistic gains from the joint design of read and write operations. CSA-RW improves upon baseline schemes available from prior works; it is shown to be order-wise optimal in terms of the upload cost and asymptotically optimal in terms of download cost when $X \geq X_\Delta + T$. Note that the baseline schemes \[10,49\] have relaxed recoverability constraints, i.e., they may require access to more than $X + 1$ servers in order to recover all the stored model data. On the other hand, the CSA-RW scheme presented in this work provides more efficient recoverability, i.e., access to any $X + 1$ servers is enough to recover all stored data. Even with
the burden of this additional constraint, CSA-RW still improves upon the baselines. Conversely, a relaxed recoverability constraint may allow CSA-RW to use other storage formats such as MDS-coded $X$-secure storage [25], which have the potential to reveal richer trade-offs between storage, recoverability, and communication costs for given security and privacy levels. This is a promising direction for future work. Another interesting question for future work has to do with the degree of submodel partitioning. Partitioning the model into a greater number of smaller submodels (increasing $K$, decreasing $L$) has the potential to improve communication costs by making the training more localized, but also carries the risk of increasing communication cost because multiple smaller submodels may need to be trained by each user. This direction leads to multi-message private read [39] and write operations, which are also of fundamental interest.

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