TWISTED VERSION OF STRONG OPENNESS PROPERTY IN $L^p$

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Abstract. In this article, we present a twisted version of strong openness property in $L^p$ with applications.

1. Introduction

The strong openness property is an important feature of multiplier ideal sheaves and used in the study of several complex variables, algebraic geometry and complex geometry (see e.g. [22, 25, 2, 36, 7, 37, 16, 29, 35, 6, 15, 4, 3, 12]).

Recall that the multiplier ideal sheaf $I(\varphi)$ (see [33, 30, 32, 9]) was defined as the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable, where $\varphi$ is a plurisubharmonic function (weight) on a complex manifold. In [22], Guan-Zhou proved strong openness property for multiplier ideal sheaves i.e. $I(\varphi) = I_{+}(\varphi) := \cup_{q>1} I(q\varphi)$, which was a conjecture posed by Demailly (see [8, 9]). Two dimensional case of the conjecture was proved by Jonsson-Mustat˘a [28].

When $I(\varphi) = \mathcal{O}$, Demailly’s strong openness conjecture degenerates to the openness conjecture, which was posed by Demailly-Kollár [11] and proved by Berndtsson [1] (2-dimensional case was proved by Favre-Jonsson [13]).

Using the strong openness property of multiplier ideal sheaves, Guan-Zhou [23] gave a characterization of the multiplier ideal sheaves with weights of Lelong number one. Some generalized versions can be referred to [25].

Recently, Xu [34] completed the algebraic approach to the openness conjecture, which was conjectured by Jonsson-Mustat˘a [28].

1.1. Background. Let $\varphi$ be a plurisubharmonic function on a domain $D \subset \mathbb{C}^n$ containing the origin $o$. Let $I$ be an ideal of $\mathcal{O}_o$, which is generated by $\{f_j\}_{j=1,...,l}$. Denote that

$$\log |I| := \log \max_{1 \leq j \leq l} |f_j|,$$

and $c_{o,p}^l(\varphi) := \sup\{c \geq 0 : |I|^p e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } o\}$. When $p = 2$, $c_{o,2}^l(\varphi)$ is the jumping number $c_{o}^l(\varphi)$ (see [27]).

In [26], Guan-Zhou established the following twisted version of strong openness property by using their solutions ([24]) of two conjectures posed by Demailly-Kollár and Jonsson-Mustat˘a.
Theorem 1.1. (see [26]) Let \( a(t) \) be a positive measurable function on \( (-\infty, +\infty) \) such that \( a(t)e^t \) is strictly increasing and continuous near \( +\infty \). Then the following three statements are equivalent:

(A) \( a(t) \) is not integrable near \( +\infty \);

(B) \( a(-2c_o^1(\varphi)\varphi)\exp(-2c_o^1(\varphi)\varphi + 2\log |I|) \) is not integrable near \( o \) for any \( \varphi \) and \( I \) satisfying \( c_o^1(\varphi) < +\infty \);

(C) \( a(-2c_o^1(\varphi)\varphi + 2\log |I|)\exp(-2c_o^1(\varphi)\varphi + 2\log |I|) \) is not integrable near \( o \) for any \( \varphi \) and \( I \) satisfying \( c_o^1(\varphi) < +\infty \).

Remark 1.3. When \( p = 2 \) and condition (2) holds, Theorem 1.2 is a general version of Theorem 1.1.

Let \( I = \mathcal{O}_o \), and denote \( c_o^1(\varphi) \) by \( c_o(\varphi) \). Theorem 1.2 implies the following.

Corollary 1.4. Let \( \varphi < 0 \) be a plurisubharmonic function on a neighborhood \( U \ni o \) with \( c_o^1(\varphi) \in (0, +\infty) \). Let \( \kappa : \mathbb{R} \to (0, +\infty) \) be a decreasing function satisfying that \( \kappa \) is not integrable near \( +\infty \). Then for any neighborhood \( V \ni o \) one has

\[
|\varphi|^{-1}\kappa(\log(-\varphi))e^{-2c_o(\varphi)\varphi} \notin L^1(V).
\]

When \( \kappa \in C^1(0, +\infty) \) satisfies \( \kappa(t) \geq -5\kappa'(t) \) for \( t \gg 1 \), Corollary 1.4 can be referred to [5].

Remark 1.5. Corollary 1.4 gives an affirmative answer to a question posed in [5], i.e., the condition \( \kappa(t) \geq -5\kappa'(t) \) for \( t \gg 1 \) (Condition (2) of Theorem 1.2 in [5]) can be removed.
2. Preparation

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin $o$, and let $\varphi$ and $\psi$ be plurisubharmonic functions on $D$. Take $T = -\sup_D \psi$. Let $f$ be a holomorphic function on a neighborhood of the origin $o$.

We call a positive measurable function $c$ on $(T, +\infty)$ in class $Q_T$ if the following two statements hold:
1. $c(t)e^{-t}$ is decreasing with respect to $t$;
2. $\lim \inf_{t \to +\infty} c(t) > 0$.

Denote
$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, o) \in I(\varphi + \psi)_o \& \tilde{f} \in O(\{\psi < -t\}) \right\},$$
by $G(t)$, where $t \in [T, +\infty)$ and $c \in Q_T$.

In §11, we obtain the following concavity of $G(t)$.

**Theorem 2.1. (see [21])** If there exists $t \in [T, +\infty)$ satisfying $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (\int_{T_1}^T c(t)e^{-t}dt, \int_{T_1}^{+\infty} c(t)e^{-t}dt)$, $\lim_{t \to T_+} G(t) = G(T)$ and $\lim_{t \to +\infty} G(t) = 0$, where $h(t) = \int_{T_1}^t c(l)e^{-l}dl$ and $T_1 \in (T, +\infty)$.

**Theorem 2.1** implies the following corollary, which will be used in the proof of Theorem 2.2.

**Corollary 2.2. (see [21])** Let $c \in Q_T$. If $\int_{T_1}^{+\infty} c(t)e^{-t}dt = +\infty$ for some $T_1 > T$, and $(f, o) \notin I(\varphi + \psi)_o$, then $G(t) = +\infty$ for any $t \geq T$, i.e., there is no holomorphic function $f$ on $\{\psi < -t\}$ satisfying $(\tilde{f} - f, o) \in I(\varphi + \psi)_o$ and $\int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty$.

Let $p \in (0, +\infty)$. Let $f$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o$, and let $I$ be an ideal of $O_o$. Denote that $C_{f, I, \Phi}(D) = \inf \{ \int_D |\tilde{f}|^2 e^{-\Phi} : (\tilde{f} - f, o) \in I \& \tilde{f} \in O(D) \}$, where $\Phi$ is a plurisubharmonic function on $D$.

**Theorem 2.1** implies the following proposition (the case $p = 2$ can be referred to [15]).

**Proposition 2.3.** Let $f$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o$, and let $\varphi$ be a negative plurisubharmonic function on $D$. If $c_{\varphi, p}(\varphi) < +\infty$, then

$$\frac{1}{r} \int_{\{2c_{\varphi, p}(\varphi) < r\}} |f|^p \geq C_{f_1, I(\varphi_1 + 2c_{\varphi, p}(\varphi)\varphi_0_1)}(D) > 0$$
holds for any $r \in (0, 1)$, where $f_1 = f^{[\varphi]}$ and $\varphi_1 = (2[\frac{p}{2}] - p) \log |f|$.

**Proof.** It suffices to prove the case $\int_{\{2c_{\varphi, p}(\varphi) < r\}} |f|^p < +\infty$. Theorem 2.1 ($\varphi \sim \varphi_1, \psi \sim 2c_{\varphi_1}(\varphi)\varphi, f \sim f_1$ and $c \sim 1$, here $\sim$ means the former replaced by the latter) implies that

$$\int_{\{2c_{\varphi_1}(\varphi) < -t\}} |f_1|^2 e^{-\varphi_1} \geq G(t)$$

(2.2)

$$\geq e^{-t}G(0)$$

$$= e^{-t}C_{f_1, I(\varphi_1 + 2c_{\varphi_1}(\varphi)\varphi_0_1)}$$

$$\geq C_{f_1, I(\varphi_1 + 2c_{\varphi_1}(\varphi)\varphi_0_1)}(D) > 0.$$

This completes the proof.
holds for any $t \geq 0$. Taking $t = -\log r$, inequality (2.2) becomes

$$\frac{1}{r} \int_{\{2c_{o,p}^f(\varphi)\varphi < \log r\}} |f|^p \geq C_{f_1, I(\varphi_1 + 2c_{o,p}^f(\varphi))o, \varphi_1}(D).$$

In the following, we prove $C_{f_1, I(\varphi_1 + 2c_{o,p}^f(\varphi))o, \varphi_1}(D) > 0$ by contradiction: if $C_{f_1, I(\varphi_1 + 2c_{o,p}^f(\varphi))o, \varphi_1}(D) = 0$, there exist holomorphic function $\{F_j\}_{j=1,2,...}$ on $D$, such that $(F_j - f, o) \in I(\varphi_1 + 2c_{o,p}^f(\varphi))o$ and $\lim_{j \to +\infty} \int_D |F_j|^2e^{-\varphi_1} = 0$. As $e^{-\varphi_1}$ has locally positive lower bound on $D$, there exists a subsequence of $\{F_j\}$ compactly convergent to 0. It follows from the closedness of $I(\varphi_1 + 2c_{o,p}^f(\varphi))o$ under the topology of compact convergence (see [17]) that $(f_1, o) \in I(\varphi_1 + 2c_{o,p}^f(\varphi))o$, which contradicts that $|f|^pe^{-2c_{o,p}^f(\varphi)\varphi}$ is not integrable near $o$. Hence, we obtain $C_{f_1, I(\varphi_1 + 2c_{o,p}^f(\varphi))o, \varphi_1}(D) > 0$. \hfill $\square$

Choosing $(f, o) \in I \subseteq O_o$ such that $c_{o,p}^f(\varphi) = c_{o,p}^f(\varphi)$, Proposition 2.3 implies the following result.

**Corollary 2.4.** Let $p \in (0, +\infty)$, and let $I$ be an ideal of $O_o$. Let $\varphi$ be a negative plurisubharmonic function on $D$. If $c_{o,p}^f(\varphi) < +\infty$, then

$$\frac{1}{r} \int_{\{2c_{o,p}^f(\varphi)\varphi < \log r\}} |I|^p$$

has positive lower bounds independent of $r \in (0, 1)$.

The following Lemma (see [24]) implies a solution of a conjecture posed by Jonsson and Mustaţă.

**Lemma 2.5.** (see [24]) Let $B \in (0, 1]$. Let $\varphi$ be a negative plurisubharmonic function on pseudoconvex domain $D \subset \mathbb{C}^n$. Let $f$ be a bounded holomorphic function on $D$. Assume that $|f|^pe^{-\varphi}$ is not locally integrable near $z_0 \in D$. Then we obtain that

$$\liminf_{R \to +\infty} R \frac{1}{B} \mu(\{-R - B < -\log |f| < -R\}) \geq \frac{C}{2eB},$$

where $C > 0$ is a constant independent of $B$ and $\mu$ is the Lebesgue measure on $\mathbb{C}^n$.

Taking $R = kB$ in inequality (2.4), for any given $\epsilon > 0$, there exists $k_o$ depending on $B$, such that for any $k \geq k_o$, one can obtain that

$$e^{(k+1)B} \frac{1}{B} \mu(\{-(k+1)B - \varphi - 2\log |f| < -(kB)\}) \geq \frac{C - \epsilon}{2},$$

i.e.

$$\mu(\{-(k+1)B < \varphi - 2\log |f| < -(kB)\}) \geq e^{-(k+1)B} B \frac{C - \epsilon}{2}.$$  

Take sum $k \geq k_o$ in inequality (2.5), and let $B$ go to 0, one can obtain that

$$\liminf_{R \to +\infty} R \mu(\{\varphi - 2\log |f| < -R\}) \geq \frac{C}{2}.$$  

Inequality (2.6) implies the following Proposition, which will be used in the proof of Theorem 1.2.
Proposition 2.6. (see [24]) Let $p \in (0, +\infty)$, and let $I$ be an ideal of $O_o$. Let $\varphi$ be a negative plurisubharmonic function on pseudoconvex domain $D$ (the origin $o \in D$). If $c^p_{o,p}(\varphi) < +\infty$, then

\begin{equation}
\frac{1}{r} \mu\{|2c^p_{o,p}(\varphi)\varphi - p \log |I| < \log r\}
\end{equation}

has positive lower bounds independent of $r \in (0, 1)$.

Proof. By the strong openness property, we have $|I|^p e^{-2c^p_{o,p}(\varphi)\varphi}$ is not integrable on any neighborhood of $o$. There exists $(f, o) \in I$ such that $|f|^p e^{-2c^p_{o,p}(\varphi)\varphi}$ is not integrable on any neighborhood of $o$.

Denote that $f_1 = f^{[\frac{p}{2}]}$ and $\varphi_1 = (2[\frac{p}{2}] - p) \log |f|$, then $|f_1|^2 e^{-\varphi_1 - 2c^p_{o,p}(\varphi)\varphi}$ is not integrable on any neighborhood of $o$. Replacing $\varphi$ by $2c^p_{o,p}(\varphi)\varphi + \varphi_1$, and $f$ by $f_1$ and $R$ by $-\log r$ in inequality (2.6), we obtain that

\[
\liminf_{r \to 0} \frac{1}{r} \mu\{|2c^p_{o,p}(\varphi)\varphi - p \log |I| < \log r\}
\]

\[
= \liminf_{r \to 0} \frac{1}{r} \mu\{|2c^p_{o,p}(\varphi)\varphi + \varphi_1 - 2 \log |f_1| < \log r\}
\]

\[> 0,\]

which implies that $\mu\{|2c^p_{o,p}(\varphi)\varphi - p \log |I| < \log r\}$ has positive lower bounds independent of $r \in (0, 1)$. $\square$

The following two lemmas will be used to prove Theorem 1.2.

**Lemma 2.7.** Let $a(t)$ be a positive measurable function on $(-\infty, +\infty)$, such that $a(t)e^t$ is increasing near $+\infty$, and $a(t)$ is not integrable near $+\infty$. Then there exists a positive measurable function $\tilde{a}(t)$ on $(-\infty, +\infty)$ satisfying the following statements:

1. there exists $T < +\infty$ such that $\tilde{a}(t) \leq a(t)$ for any $t > T$;
2. $\tilde{a}(t)e^t$ is strictly increasing and continuous near $+\infty$;
3. $\tilde{a}(t)$ is not integrable near $+\infty$.

Proof. There exists $T \in \mathbb{Z}$ such that $a(t)e^t$ is increasing on $(T - 1, +\infty)$. As $a(t)$ is not integrable near $+\infty$ and $a(t)e^t$ is increasing on $(T - 1, +\infty)$, we have $\sum_{n=T-1}^{+\infty} a(n) = +\infty$ and $a(n) \leq a(n+1)e$. Then there exists a sequence of real numbers $\{b_n\}_{n=T}^{+\infty}$ satisfying that: (1) $\sum_{n=T}^{+\infty} b_n = +\infty$; (2) $b_{n+1} \leq a(n)$ for any $n \geq T - 1$; $b_n < b_{n+1}e$ for any $n \geq T$.

Take

\[
\tilde{a}(t) = \begin{cases} \frac{b_n}{e^t} \left(\frac{b_{n+1}}{b_n}\right)^{t-n} & \text{if } t \in [n, n+1) \subset [T, +\infty), \\ 1 & \text{if } t \in (-\infty, T). \end{cases}
\]

Now, we prove that $\tilde{a}$ satisfies the three statements in Lemma 2.7. Following from $b_n < b_{n+1}e$, $b_{n+1} \leq a(n)$ for any $n \geq T - 1$ and $a(t)e^t$ is increasing on $(T - 1, +\infty)$, we obtain that $\tilde{a}(t) \leq \frac{1}{e^t} \max \{b_n, b_{n+1}\} \leq \frac{b_{n+1}}{e^t} \leq \frac{a(n)}{e^t} \leq a(t)$ for any $t \in [n, n+1) \subset [T, +\infty)$. As $b_n < b_{n+1}e$, then $\tilde{a}(t)e^t$ is strictly increasing near $+\infty$. The continuity of $\tilde{a}(t)$ on $(T, +\infty)$ is just from the construction of $\tilde{a}(t)$. Note that

\[
\int_{T}^{+\infty} \tilde{a}(t)dt \geq \sum_{n=N}^{+\infty} \frac{1}{e^t} \min \{b_n, b_{n+1}\} \geq \sum_{n=N}^{+\infty} \frac{b_n}{e^{T+1}} = +\infty
\]

holds for any integer $N \geq T$. 

Thus, Lemma 2.7 holds. □

Lemma 2.8. (see [20]) For any two measurable spaces \((X_i, \mu_i)\) and two measurable functions \(g_i\) on \(X_i\), respectively \((i \in \{1, 2\})\), if \(\mu_1(\{g_1 \geq r^{-1}\}) \geq \mu_2(\{g_2 \geq r^{-1}\})\) for any \(r \in (0, r_0]\), then \(\int_{\{g_1 \geq r_0^{-1}\}} g_1 d\mu_1 \geq \int_{\{g_2 \geq r_0^{-1}\}} g_2 d\mu_2\).

3. Proofs of Theorem 1.2 and corollary 1.4

In this section, we prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. We prove Theorem 1.2 in two cases, that \(a(t)\) satisfies condition (1) or condition (2).

Case (1). \(a(t)\) is decreasing near \(+\infty\).

Firstly, we prove (B) \(\Rightarrow\) (A) and (C) \(\Rightarrow\) (A). Consider \(I = 1\) and \(\varphi = \log |z_1|\) on the unit polydisc \(\Delta^n \subset \mathbb{C}^n\). Note that \(c_{o,p}(\log |z_1|) = 1\) and

\[
\int_{\Delta^n} a(-2 \log |z_1|) \frac{1}{|z_1|^2} = (\pi r_0^2)^n \int_{\Delta^n} a(-2 \log |z_1|) \frac{1}{|z_1|^2} = (\pi r_0^2)^n \int_{0}^{r_0} a(-2 \log r) r^{-1} dr = (\pi r_0^2)^n \int_{-2 \log r_0}^{+\infty} a(t) dt,
\]

then we obtain (B) \(\Rightarrow\) (A) and (C) \(\Rightarrow\) (A).

Then, we prove (A) \(\Rightarrow\) (B). The strong openness property shows that there exists \((f, o) \in I\) such that \(|f|^p e^{-2c_{o,p}(\varphi)}\varphi\) is not integrable near \(o\). It suffices to prove that \(|f|^p e^{-2c_{o,p}(\varphi)}\varphi a(-2c_{o,p}(\varphi)\varphi)\neq (n)\) is not integrable near \(o\).

Denote that \(f_1 = f^{\frac{2}{p}}\) and \(\varphi_1 = (\frac{2}{p} - p) \log |f|\), where \(m := \min\{n \in \mathbb{Z} : n \geq m\}\). Note that \(|f|^p e^{-2c_{o,p}(\varphi)}\varphi = |f_1|^2 e^{-\varphi_1 - 2c_{o,p}(\varphi)\varphi}\) and it’s not integrable near \(o\). Then we assume that \(|f_1|^2 e^{-\varphi_1 - 2c_{o,p}(\varphi)\varphi} a(-2c_{o,p}(\varphi)\varphi)\) is integrable near \(o\) to get a contradiction. Set \(c(t) = a(t) e^t + 1\). There exists a pseudoconvex domain \(D \subset \mathbb{C}^n\) containing the origin \(o\) such that \(\int_{D} |f_1|^2 e^{-\varphi_1 - 2c_{o,p}(\varphi)\varphi} a(-2c_{o,p}(\varphi)\varphi) < +\infty\), \(\int_{D} |f_1|^2 e^{-\varphi_1} = \int_{D} |f|^p < +\infty\) and \(c(t)e^{-t} = a(t) + e^{-t}\) is decreasing on \((T, +\infty)\), where \(T = -\sup_{D} 2c_{o,p}(\varphi)\varphi\). Note that \(\lim_{t \to +\infty} c(t) > 0\), then \(c \in \mathbb{Q}\). As \(a(t)\) is not integrable near \(+\infty\), so is \(c(t)e^{-t}\). Using Corollary 2.2, \(f, \varphi\) and \(\psi\) are replaced by \(f_1, \varphi_1\) and \(2c_{o,p}(\varphi)\varphi\) respectively, as \((f_1, o) \notin I(\varphi_1 + 2c_{o,p}(\varphi)\varphi)\), then we have \(G(T) = +\infty\). By the definition of \(G(T)\), we obtain that

\[
G(T) \leq \int_{D} |f_1|^2 e^{-\varphi_1} c(-2c_{o,p}(\varphi)\varphi) < +\infty
\]

which contradicts to \(G(T) = +\infty\). Thus, we obtain (A) \(\Rightarrow\) (B).

Finally, we prove (A) \(\Rightarrow\) (C). If \(I = 1\), following from the above discussion, we have (C) holds. Thus, it is suffices to consider the case \(I \neq 1\).
If \( a(t) \) is decreasing near \(+\infty\), we have \( a(-2c_{o,p}(\varphi)\varphi) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \) for any plurisubharmonic function \( \varphi \) on \( D \) and \( I \) satisfying \( c_{o,p}(\varphi) < +\infty \). Note that there exists a small neighborhood \( D' \) of \( o \) such that \( \log |I| < 0 \) on \( D' \), then \( a(t) \) is decreasing near \(+\infty\) implies that

\[
a(-2c_{o,p}(\varphi)\varphi + p \log |I|) \geq a(-2c_{o,p}(\varphi)\varphi).
\]

Thus we obtain that \( a(-2c_{o,p}(\varphi)\varphi + p \log |I|) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \).

Thus, we prove Theorem 2.2 for the case that \( a(t) \) satisfies condition (1).

Case (2). \( a(t)e^{t} \) is increasing near \(+\infty\).

In this case, the proofs of \((B) \Rightarrow (A)\) and \((C) \Rightarrow (A)\) are the same as the case (1), therefore it suffices to prove \((A) \Rightarrow (B)\) and \((A) \Rightarrow (C)\).

Assume that statement (A) holds. It follows from Lemma 2.8 that there exists a positive function \( \tilde{a}(t) \) on \((-\infty, +\infty)\) satisfying that: \( \tilde{a}(t) \leq a(t) \) near \(+\infty\); \( \tilde{a}(t)e^{t} \) is strictly increasing and continuous near \(+\infty\); \( \tilde{a}(t) \) is not integrable near \(+\infty\). Thus, it suffices to prove that \( \tilde{a}(-2c_{o,p}(\varphi)\varphi) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) and \( \tilde{a}(-2c_{o,p}(\varphi)\varphi + p \log |I|) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) are not integrable near \( o \) for any \( \varphi \) and \( I \) satisfying \( c_{o,p}(\varphi) < +\infty \).

Firstly, we prove \( \tilde{a}(-2c_{o,p}(\varphi)\varphi) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \) for any \( \varphi \) and \( I \) satisfying \( c_{o,p}(\varphi) < +\infty \) by using Corollary 2.4 and Lemma 2.8.

Let \( X_{1} \) be a small neighborhood of \( o \), and let \( X_{2} = (0, 1] \). \( \mu_{1}() = \int |I|^{2} \), and let \( \mu_{2} \) be the Lebesgue measure on \( X_{2} \). Denote that \( Y_{r} = \{-2c_{o,p}(\varphi)\varphi \geq -\log r\} \). Corollary 2.4 shows that there exists a positive constant \( C \) such that \( \mu_{1}(Y_{r}) \geq Cr \) holds for any \( r \in (0, 1] \).

Let \( g_{1} = \tilde{a}(-2c_{o,p}(\varphi)\varphi) \exp(-2c_{o,p}(\varphi)\varphi) \) and \( g_{2}(x) = \tilde{a}(-\log x + \log C)Cx^{-1} \).

As \( \tilde{a}(t)e^{-t} \) is increasing near \(+\infty\), then \( g_{1} \geq \tilde{a}(-\log r)r^{-1} \) on \( Y_{r} \) implies that

\[
\mu_{1}\{g_{1} \geq \tilde{a}(-\log r)r^{-1}\} \geq \mu_{1}(Y_{r}) \geq Cr
\]

holds for any \( r > 0 \) small enough.

As \( \tilde{a}(t)e^{t} \) is strictly increasing near \(+\infty\), then there exists \( r_{0} \in (0, 1) \) such that

\[
\mu_{2}\{x \in (0, r_{0}]: g_{2}(x) \geq \tilde{a}(-\log r)r^{-1}\} = \mu_{2}\{0 < x \leq Cr\} = Cr
\]

for any \( r \in (0, r_{0}] \).

Using the continuity of \( \tilde{a}(-\log r)r^{-1} \) and \( \tilde{a}(-\log r)r^{-1} \) converges to \(+\infty\) ( when \( r \to 0 + 0 \) ), we obtain that

\[
\mu_{1}\{g_{1} \geq r^{-1}\} \geq \mu_{2}\{x \in (0, r_{0}]: g_{2}(x) \geq r^{-1}\}
\]

holds for any \( r > 0 \) small enough. Following from Lemma 2.8 and \( \tilde{a}(t) \) is not integrable near \(+\infty\), we obtain \( \tilde{a}(-2c_{o,p}(\varphi)\varphi) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \).

Then, we prove \( \tilde{a}(-2c_{o,p}(\varphi)\varphi + p \log |I|) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \) for any \( \varphi \) and \( I \) satisfying \( c_{o,p}(\varphi) < +\infty \) by using Proposition 2.0 and Lemma 2.8.

Let \( X_{1} \) be a small neighborhood of \( o \), and let \( X_{2} = (0, 1] \). \( \mu_{1} \) and \( \mu_{2} \) be the Lebesgue measure on \( X_{1} \) and \( X_{2} \), respectively. Let \( Y_{r} = \{-2c_{o,p}(\varphi)\varphi + p \log |I| \geq -\log r\} \). Proposition 2.0 shows that there exists a positive constant \( C \) such that \( \mu_{1}(Y_{r}) \geq Cr \) holds for any \( r \in (0, 1] \).
Let \( g_1 = \tilde{a}(-2c_{o,p}(\varphi)\varphi + p \log |I|) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) and \( g_2(x) = \tilde{a}(-\log x + \log C)x^{-1} \). As \( \tilde{a}(t)e^{-t} \) is increasing near \(+\infty\), then \( g_1 \geq \tilde{a}(-\log r)r^{-1} \) on \( Y_r \) implies that
\[
\mu_1(\{ g_1 \geq \tilde{a}(-\log r)r^{-1} \}) \geq \mu_1(Y_r) \geq Cr
\]
holds for any \( r > 0 \) small enough.

As \( \tilde{a}(t)e^{t} \) is strictly increasing near \(+\infty\), then there exists \( r_0 \in (0,1) \) such that
\[
\mu_2(\{ x \in (0, r_0]: g_2(x) \geq \tilde{a}(-\log r)r^{-1} \}) = \mu_2(\{ 0 < x \leq Cr \}) = Cr
\]
for any \( r \in (0, r_0] \).

Using the continuity of \( \tilde{a}(-\log r)r^{-1} \) and \( \tilde{a}(-\log r)r^{-1} \) converges to \(+\infty\) (when \( r \to 0 + 0 \)), we obtain that
\[
\mu_1(\{ g_1 \geq r^{-1} \}) \geq \mu_2(\{ x \in (0, r_0]: g_2(x) \geq r^{-1} \})
\]
holds for any \( r > 0 \) small enough. Following from Lemma 2.8 and \( \tilde{a}(t) \) is not integrable near \(+\infty\), we obtain \( \tilde{a}(-2c_{o,p}(\varphi)\varphi + p \log |I|) \exp(-2c_{o,p}(\varphi)\varphi + p \log |I|) \) is not integrable near \( o \).

Thus, we prove Theorem 1.2 for the case that \( a(t) \) satisfies condition (2). \( \square \)

**Proof of Corollary 1.4.** Let \( a(t) = \frac{1}{t} \kappa(\log t - \log(2c_o(\varphi))) \). Note that
\[
\int_1^{+\infty} a(t)dt = \int_N^{+\infty} \frac{1}{t} \kappa(\log t - \log(2c_o(\varphi)))dt = \int_{\log N}^{+\infty} \kappa(t - \log(2c_o(\varphi)))dt = +\infty
\]
for \( N \gg 1 \). Then the case \( p = 2, I = 1 \) and \( a(t) \) satisfying condition (1) of Theorem 1.2 implies that Corollary 1.4 holds. \( \square \)

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TWISTED VERSION OF STRONG OPENNESS PROPERTY IN $L^p$

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