Symmetry Detection of Rational Space Curves from their Curvature and Torsion

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Abstract

We present a novel, deterministic, and efficient method to detect whether a given rational space curve is symmetric. By using well-known differential invariants of space curves, namely the curvature and torsion, the method is significantly faster, simpler, and more general than an earlier method addressing a similar problem \cite{3}. We present timings from an implementation in \textit{Sage} to support this claim, and a complexity analysis of the algorithm completes the study.

1. Introduction

The problem of detecting the symmetries of curves and surfaces has attracted the attention of many researchers throughout the years, because of its interest in fields like Pattern Recognition \cite{8,11,20,23,39,40,42,44,46}, Computer Graphics \cite{6,7,27,29,30,33,34,37}, and Computer Vision \cite{5,9,21,25,26,28,43,41}. The introduction in \cite{3} contains an extensive account of the variety of approaches used in the above references.

A common characteristic in most of these papers is that the methods focus on computing \textit{approximate} symmetries more than exact symmetries, which is perfectly reasonable in many applications, where curves and surfaces often serve as merely approximate representations of a more complex shape. Some exceptions appear here: If the object to be considered is discrete (e.g. a polyhedron), or is described by a discrete object, like for instance a control polygon or a control polyhedron, then the symmetries can be determined exactly \cite{5,9,21,25}. Examples of the second class are Bézier curves and tensor product surfaces. Furthermore, in these cases the symmetries of the curve or surface follow from those of the underlying discrete object. Another exception appears in \cite{23}, where the
authors provide a deterministic method to detect rotation symmetry of an implicitly defined algebraic plane curve and to find the exact rotation angle and rotation center. The method uses a complex representation of the curve and is generalized in [24] to detect mirror symmetry as well. Recently, the problem of determining whether a rational plane or space curve is symmetric has been addressed in [1] (2, 3] using a different approach. The common denominator in these papers is the following observation: If a rational curve is symmetric, i.e., invariant under a nontrivial isometry f, then this symmetry induces another parametrization of the curve, different from the original parametrization. Assuming that the initial parametrization is proper (definition below), the second parametrization is also proper. Since two proper parametrizations of the same curve are related by a Möbius transformation [36], determining the symmetries is reduced to finding this transformation, therefore translating the problem to the parameter space. This observation leads to algorithms for determining the symmetries of plane curves with polynomial parametrizations [1] and of plane and space curves with rational parametrizations [3], although in the latter case of general space curves only involutions were considered. The more general problem of determining whether two rational plane curves are similar was considered in [3].

In this paper we address the problem of deterministically finding the symmetries of a rational space curve, defined by means of a proper parametrization. Notice that since we deal with a global object, i.e., the set of all the space points defined by a rational parametrization, and not just a piece of it, an approach like that of [2, 3, 9, 21, 25] is not suitable here. Instead we again employ the above observation, but now in addition use well-known differential invariants of space curves, namely the curvature and the torsion. The improvement over the method in [3] is threefold: First of all, we are now able to find all the symmetries of the curve instead of just the involutions. Secondly, the new algorithm is considerably faster and can efficiently handle even curves with high degrees and large coefficients in reasonable timings. Finally, the method is simpler to implement and requires fewer assumptions on the parametrization.

Some general facts on symmetries of rational curves are presented in Section 2. Section 3 provides an algorithm for checking whether a curve is symmetric. The determination of the symmetries themselves is addressed in Section 4. Finally, in Section 5 we report on the performance of the algorithm, by presenting a complexity analysis and providing timings for several examples, including a comparison with the curves tested in [3].

2. Symmetries of rational curves

Throughout the paper, we consider a rational space curve \( C \subset \mathbb{R}^3 \), neither a line nor a circle, parametrized by a rational map

\[
\mathbf{x} : \mathbb{R} \rightarrow C \subset \mathbb{R}^3, \quad \mathbf{x}(t) = (x(t), y(t), z(t)).
\] (1)

The components \( x(t), y(t), z(t) \) of \( \mathbf{x} \) are rational functions of \( t \), and they are defined for all but a finite number of values of \( t \). Note that rational curves are
irreducible. We assume that the parametrization \( I \) is proper, i.e., birational or, equivalently, injective except for perhaps finitely many values of \( t \). This can be assumed without loss of generality, since any rational curve can quickly be properly reparametrized. For these claims and other results on properness, the interested reader can consult [36] for plane curves and [4, §3.1] for space curves.

We recall some facts from Euclidean geometry [15]. An isometry of \( \mathbb{R}^3 \) is a map \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) preserving Euclidean distances. Any isometry \( f \) of \( \mathbb{R}^3 \) is linear affine, taking the form

\[
  f(x) = Qx + b, \quad x \in \mathbb{R}^3,
\]

with \( b \in \mathbb{R}^3 \) and \( Q \in \mathbb{R}^{3 \times 3} \) an orthogonal matrix. In particular \( \det(Q) = \pm 1 \). Under composition, the isometries of \( \mathbb{R}^3 \) form the Euclidean group, which is generated by reflections, i.e., symmetries with respect to a plane, or mirror symmetries. An isometry is called direct when it preserves the orientation, and opposite when it does not. In the former case \( \det(Q) = 1 \), while in the latter case \( \det(Q) = -1 \). The identity map \( \text{id}_{\mathbb{R}^n} \) of \( \mathbb{R}^n \) is called the trivial symmetry.

The classification of the nontrivial isometries of Euclidean space includes reflections (in a plane), rotations (about an axis), and translations, and these combine in commutative pairs to form twists, glide reflections, and rotatory reflections. Composing three reflections in mutually perpendicular planes through a point \( p \), yields a central inversion (also called central symmetry), with center \( p \), i.e., a symmetry with respect to the point \( p \). The special case of rotation by an angle \( \pi \) is of special interest, and it is called a half-turn. Rotation symmetries are direct, while mirror and central symmetries are opposite.

By Bézout’s theorem, the rotations, reflections, and central inversions are the only isometries leaving an irreducible algebraic space curve, different from a line, invariant. We say that an irreducible algebraic space curve is symmetric, if it is invariant under one of these (nontrivial) isometries. In that case, we distinguish between a mirror symmetry, rotation symmetry and central symmetry. If the curve is neither a line nor a circle, it has a finite number of symmetries [3].

We recall the following result from [3]. For this purpose, let us recall first that a Möbius transformation (of the affine real line) is a rational function

\[
  \varphi: \mathbb{R} \rightarrow \mathbb{R}, \quad \varphi(t) = \frac{at + b}{ct + d}, \quad \Delta := ad - bc \neq 0.
\]

In particular, we refer to \( \varphi(t) = t \) as the trivial transformation. It is well known that the birational functions on the line are the Möbius transformations [36].

**Theorem 1.** Let \( x: \mathbb{R} \rightarrow C \subseteq \mathbb{R}^3 \) be a proper parametric curve as in \( I \). The curve \( C \) is symmetric if and only if there exists a nontrivial isometry \( f \) and nontrivial Möbius transformation \( \varphi \) for which we have a commutative diagram

\[
\begin{array}{ccc}
C & f & \rightarrow & C \\
\uparrow & & & \uparrow \\
\mathbb{R} & \rightarrow & \mathbb{R} \\
1x & \mapsto & 1x \\
\mathbb{R} & \rightarrow & \mathbb{R} \\
\end{array}
\]
Moreover, for each isometry \( f \) there exists a unique Möbius transformation \( \varphi \) that makes the above diagram commute.

Since the Möbius transformation \( \varphi \) maps the real line to itself, its coefficients can always be assumed to be real by dividing by a common complex number if necessary. Note that \( \varphi(t) \) is the parameter value corresponding to the image under the symmetry \( f \) of the point on \( C \) with parameter \( t \).

Let the \textit{curvature} \( \kappa \) and \textit{torsion} \( \tau \) of a parametric curve \( x \) be the functions

\[
\kappa = \kappa_x := \frac{\|x' \times x''\|}{\|x'\|^3}, \quad \tau = \tau_x := \frac{\langle x' \times x'', x''' \rangle}{\|x' \times x''\|^2}
\]

of the parameter \( t \). Note that \( \kappa \) is non-negative. The functions \( \kappa^2 \) and \( \tau^2 \) are well-known rational \textit{differential invariants} of the parametrization \( x \), in the sense that

\[
\kappa_{f \circ x} = \kappa_x, \quad \tau_{f \circ x} = \det(Q) \cdot \tau_x
\]

for any isometry \( f(x) = Qx + b \). This follows immediately from \( Q \) being orthogonal and the identity

\[
(Ma) \times (Mb) = \det(M)M^{-T}(a \times b),
\]

which holds for any invertible matrix \( M \) and follows from a straightforward calculation. Although \( \tau_x \) and \( \kappa_x^2 \) are rational for any rational parametrization \( x \), the curvature \( \kappa_x \) is in general not rational.

The following lemma describes the behavior of the curvature and torsion under reparametrization, for instance by a Möbius transformation.

\textbf{Lemma 2.} Let \( x \) be the rational parametrization \([1]\) and let \( \phi \in C^3(U) \), with \( U \subset \mathbb{R} \) open. Then

\[
\kappa_{x \circ \phi} = \kappa_x \circ \phi, \quad \tau_{x \circ \phi} = \tau_x \circ \phi,
\]

whenever both sides are defined.

\textit{Proof.} Writing \( \tilde{x} := x \circ \phi \) and using the chain rule, one finds

\[
\begin{align*}
\tilde{x}'(t) &= x'(\phi(t)) \cdot \phi'(t), \\
\tilde{x}''(t) &= x''(\phi(t)) \cdot (\phi'(t))^2 + x'(\phi(t)) \cdot \phi''(t), \\
\tilde{x}'''(t) &= x'''(\phi(t)) \cdot (\phi'(t))^3 + 3x''(\phi(t)) \cdot \phi'(t) \cdot \phi''(t) + x'(\phi(t)) \cdot \phi'''(t),
\end{align*}
\]

whenever \( t \in U \) and \( x \) is defined at \( \phi(t) \). Therefore

\[
\kappa_{x \circ \phi}(t) = \frac{\|\tilde{x}'(t) \times \tilde{x}''(t)\|}{\|\tilde{x}'(t)\|^3} = \frac{\|x'(\phi(t)) \times x''(\phi(t))\| \cdot |\phi'(t)|^3}{\|x'(\phi(t))\|^3 \cdot |\phi'(t)|^3} = (\kappa_x \circ \phi)(t),
\]

and similarly one finds \( \tau_{x \circ \phi} = \tau_x \circ \phi \). \( \square \)
3. Symmetry detection

In this section we derive a criterion for the presence of symmetries \( f(x) = Qx + b \) of curves of type (1), together with an efficient method for checking this criterion. The cases \( \det(Q) = \pm 1 \) need to be checked separately, but are considered simultaneously using linked \( \pm \) and \( \mp \) signs consistently throughout the paper. The resulting method is summarized in Algorithm Symm\( \pm \).

3.1. A criterion for the presence of symmetries

For any parametric curve \( x \) as in (1), write

\[
\kappa^2_x(t) = \frac{A(t)}{B(t)}, \quad \tau_x(t) = \frac{C(t)}{D(t)},
\]

with \((A, B)\) and \((C, D)\) pairs of coprime polynomials. Let

\[
G_x^\pm := \gcd(K_x, T_x^\pm),
\]

with

\[
K_x(t, s) := A(t)B(s) - A(s)B(t), \quad T_x^\pm(t, s) := C(t)D(s) \mp C(s)D(t)
\]

the result of clearing denominators in the equations

\[
\kappa^2_x(t) - \kappa^2_x(s) = 0, \quad \tau_x(t) \mp \tau_x(s) = 0.
\]

Similarly, associate to any M"obius transformation \( \varphi \) the M"obius-like polynomial

\[
F(t, s) := (ct + d)s - (at + b), \quad ad - bc \neq 0,
\]

as the result of clearing denominators in \( s - \varphi(t) = 0 \). We call \( F \) trivial when \( F(t, s) = s - t \), i.e., when the associated M"obius transformation is the identity. Note that \( F \) is irreducible since \( ad - bc \neq 0 \). The zero set of \( F \) is the graph of \( \varphi \), which is either a rectangular hyperbola with horizontal and vertical asymptotes when \( c \neq 0 \), or a line with nonzero and finite slope \( a/d \) when \( c = 0 \) (see Figure 1).

**Theorem 3.** Consider the curve \( C \) defined by \( x \) in (1) and let \( G_x^\pm \) be as above. Then \( C \) has a nontrivial symmetry \( f(x) = Qx + b \), with \( \det(Q) = \pm 1 \), if and only if there exists a nontrivial polynomial \( F \) of type (10), associated with a M"obius transformation \( \varphi \), such that \( F \) divides \( G_x^\pm \) and the parametrizations \( x \) and \( x \circ \varphi \) have identical speed,

\[
\|x'\| = \|(x \circ \varphi)'\|.
\]

**Proof.** \( \Rightarrow \): If \( C \) is invariant under an isometry \( f(x) = Qx + b \), with \( \det(Q) = \pm 1 \), by Theorem 1 there exists a M"obius transformation \( \varphi \) such that \( f \circ x = x \circ \varphi \). Let \( F \) be the M"obius-like polynomial associated with \( \varphi \). The points \((t, s)\) for
which \( K_x(t,s) = T^h_x(t,s) = 0 \) are the points satisfying \( \kappa_x(s) = \kappa_x(t) \) and \( \tau_x(s) = \pm \tau_x(t) \). This includes the zeroset \( \{ (t, s) : s = \varphi(t) \} \) of \( F(t,s) \), since
\[
\kappa_x \circ \varphi = \kappa_{x_{0} \circ \varphi} = \kappa_{f_{0} \circ \varphi} = \kappa_x, \quad \tau_x \circ \varphi = \tau_{x_{0} \circ \varphi} = \tau_{f_{0} \circ \varphi} = \det(Q) \tau_x = \pm \tau_x
\]
by Lemma 2 and 5. Since \( F \) is irreducible, Bézout’s theorem implies that \( F \) divides \( K_x \) and \( T^h_x \), and therefore \( G^\pm_x \) as well. Furthermore, since \( Q \) is orthogonal, the parametrizations have equal speed,
\[
\| (x \circ \varphi)' \| = \| (f \circ x)' \| = \| (Qx + b)' \| = \| Qx' \| = \| x' \|.
\]

"\( \Longleftrightarrow \)": Let \( \varphi \) be the nontrivial transformation associated to \( F \). Let \( t_0 \in \mathbb{R} \) be such that \( x(t_0) \) is a regular point on \( C \), and consider the arc length function
\[
s = s(t) := \int_{t_0}^{t} ||x'(t)|| dt,
\]
which (locally) has an infinitely differentiable inverse \( t = t(s) \). By (11),
\[
\left\| \frac{d}{ds} (x \circ t) \right\| = \left\| \frac{dx}{dt} \frac{dt}{ds} \right\| = 1 = \left\| \frac{d}{dt} (x \circ \varphi) \frac{dt}{ds} \right\| = \left\| \frac{d}{ds} (x \circ \varphi \circ t) \right\|,
\]
so that \( x \circ t \) and \( x \circ \varphi \circ t \) are parametrized by arc length. Since \( F \) divides \( G^\pm_x \), any zero \( (t, \varphi(t)) \) of \( F \) is also a zero of \( K_x \) and \( T^h_x \), implying that \( \kappa_x = \kappa_x \circ \varphi \) and \( \tau_x = \pm \tau_x \circ \varphi \). Then, by repeatedly applying Lemma 2
\[
\kappa_{x(t_0)} = \kappa_x \circ t = \kappa_{x_0 \circ t} = \kappa_{x_0 \circ \varphi \circ t}, \quad \tau_{x(t_0)} = \tau_x \circ t = \pm \tau_{x_0 \circ \varphi \circ t}, \quad (12)
\]
and the Fundamental Theorem of Space Curves [17, p. 19] implies that \( x \circ t \) and \( x \circ \varphi \circ t \) coincide up to an isometry \( f(x) = Qx + b \) with \( \det(Q) = \pm 1 \). In other words, locally \( f \circ x \circ t = x \circ \varphi \circ t \) and therefore, since these functions are algebraic, globally \( f \circ x = x \circ \varphi \) and \( f \) is a symmetry of \( C \).

Note that the polynomial \( G^\pm_x \) cannot be identically 0. Indeed, \( G^\pm_x \) is identically 0 if and only if \( K_x \) and \( T^h_x \) are both identically 0, which happens precisely when \( \kappa_x \) and \( \tau_x \) are both constant. If \( \kappa_x = 0 \) then \( C \) is a line, if \( \tau_x = 0 \) and \( \kappa_x \) is a nonzero constant then \( C \) is a circle, and if \( \kappa_x, \tau_x \) are both constant but nonzero then \( C \) is a circular helix, which is non-algebraic. All of these cases are excluded by hypothesis.

3.2. Finding the Möbius-like factors \( F \) of \( G^\pm_x \)
The criterion in Theorem 3 requires to check if a bivariate polynomial \( G = G^\pm_x \) has real factors of the form \( F(t,s) = (ct + d)s - (at + b) \), with \( ad - bc \neq 0 \). Here \( a, b, c, d \) need not be rational numbers, so that we need to factor over the algebraic or real numbers. This problem has been studied by several authors [12, 13, 14, 18]. However, since in our case we are looking for factors of a specific form, we develop an ad hoc method to check the condition.
Let $G$ be the curve in the $(t, s)$-plane defined by $G(t, s)$. Let $t_0$ be such that the vertical line $L$ at $t = t_0$ does not contain any zero of $G$ where the partial derivative $G_s := \frac{\partial G}{\partial s}$ vanishes. These are the points $t_0$ for which the discriminant of $g(s) := G(t_0, s)$ does not vanish, which is up to a factor equal to the Sylvester resultant $\text{Res}_s(G, G_s)$ and has degree at most $(2m_s - 1)m_t$ in $t_0$, with $(m_t, m_s)$ the bidegree of $G$. Therefore one can always find an integer abscissa $t_0$ with this property by checking for at most $(2m_s - 1)m_t + 1$ points $t_0$ whether the gcd of $g(s)$ and $G_s(t_0, s)$ is trivial.

If $G$ has a Möbius-like factor $F$ as in \textbf{[10]}, then the zero set of $F$ intersects $\mathcal{L}$ in a single point $p_0 = (t_0, \xi)$ satisfying

$$(ct_0 + d)\xi - (at_0 + b) = 0.$$  \hspace{1cm} (13)

Since $G_s(p_0) \neq 0$, the equation $F(t, s) = 0$ implicitly defines a function $s = s(t)$ in a neighborhood of $p_0$. Moreover, by differentiating the identity $F(t, s(t)) = 0$ once and twice with respect to $t$, and evaluating at $p_0$, we find the relationships

$$-a + ds_0' + c \cdot (\xi + t_0s_0') = 0, \hspace{1cm} (14)$$

$$ds_0'' + c \cdot (2s_0' + t_0s_0'') = 0, \hspace{1cm} (15)$$

where $\xi = s(t_0)$, $s_0' := s'(t_0)$, and $s_0'' := s''(t_0)$. In order to find expressions for $s_0', s_0''$, we now use that the function $s(t)$ is also implicitly defined by $G(t, s) = 0$, because $F$ is a factor of $G$ and $G_s(p_0) \neq 0$. Differentiating once and twice the identity $G(t, s(t)) = 0$ with respect to $t$ gives

$$s' = -\frac{G_t(t, s)}{G_s(t, s)}, \hspace{1cm} s'' = -\frac{G_{tt}(t, s) + 2G_{ts}(t, s)s' + G_{ss}(t, s)(s')^2}{G_s(t, s)}.$$  \hspace{1cm} (16)

Evaluating these expressions at $p_0$ yields expressions $s_0' = s_0'(\xi)$ and $s_0'' = s_0''(\xi)$.

Now we distinguish the cases $d \neq 0$ and $d = 0$. In the first case, we may assume $d = 1$ by dividing all coefficients in the Möbius transformation by $d$. In that case $2s_0' + t_0s_0'' = 2(ad - bc)d/(ct_0 + d)^2 \neq 0$ and, using \textbf{[13]–[15]},

$$c(\xi) = \frac{-s_0''}{2s_0' + t_0s_0'}, \hspace{0.5cm} a(\xi) = s_0' + c(\xi)(\xi + t_0s_0'), \hspace{0.5cm} b(\xi) = -a(\xi)t_0 + \xi + c(\xi)t_0\xi.$$  \hspace{1cm} (17)

The polynomial $F$ is a factor of $G$ if and only if the resultant $\text{Res}_s(F, G)$ is identically 0. Substituting $a(\xi), b(\xi), c(\xi)$, and $d = 1$ into this resultant yields a polynomial $P(t)$, whose coefficients are rational functions of $\xi$. Let $R(\xi)$ be the gcd of the numerators of these coefficients and of $g(\xi)$. The real roots $\xi$ of $R(\xi)$ for which $a(\xi), b(\xi), c(\xi)$ are well defined and $\Delta(\xi) := a(\xi) - b(\xi)c(\xi) \neq 0$ correspond to the Möbius-like factors $F$ of $G$ as in \textbf{[10]} with $d = 1$.

On the other hand, when $d = 0$ we may assume $c = 1$, and \textbf{[13]–[15]} yield rational expressions

$$a_0(\xi) = \xi + t_0s_0', \hspace{1cm} b_0(\xi) = -a(\xi)t_0 + t_0\xi.$$  \hspace{1cm} (18)

Substituting $a_0(\xi), b_0(\xi), c = 1$, and $d = 0$ into the resultant $\text{Res}_s(F, G)$ yields a polynomial $P_0(t)$, whose coefficients are rational functions of $\xi$. Let $R_0(\xi)$ be
Figure 1: The zeroset (solid) of the polynomial $G$ intersects the vertical line $L$ (dashed) in the points $(2, \pm \sqrt{8})$ and $(2, \pm 1/\sqrt{12})$ in Example 1.

Theorem 4. The polynomial $G$ has a real Möbius-like factor $F$ as in (10) with $d \neq 0$ (resp. $d = 0$) if and only if $R(\xi)$ (resp. $R_0(\xi)$) has a real root. Furthermore, every such real root provides a factor of this form.

Note that the cases $d = 0$ and $d \neq 0$ can be computed in parallel.

Example 1. Consider the bivariate polynomial

$$G(t, s) = 3s^4t^4 - 6s^4t^3 + 3s^4t^2 - 6s^2t^4 - s^2t^2 + 2s^2t - s^2 + 2t^2.$$ 

The vertical line $L := \{t = t_0 := 2\}$ does not intersect the zeroset of $G$ in a point where $G_s$ vanishes, since the discriminant of $g(\xi) := G(t_0, \xi) = 12\xi^4 - 97\xi^2 + 8$ is nonzero (see Figure 1). Evaluating (16) at $p_0 = (2, \xi)$ yields

$$s'_0 = -\frac{18\xi^4 - 97\xi^2 + 4}{\xi(24\xi^2 - 97)},$$

$$s''_0 = \frac{16416\xi^{10} - 206316\xi^8 + 879669\xi^6 - 1387682\xi^4 + 55302\xi^2 + 1552}{\xi^3(24\xi^2 - 97)^3}.$$
When \( d \neq 0 \), we may assume \( d = 1 \) and Equations (17) yield

\[
\begin{align*}
c(\xi) &= -\frac{1}{2} \left( 116416\xi^{10} - 206316\xi^8 + 879669\xi^6 - 1387682\xi^4 + 55302\xi^2 + 1552 \right), \\
a(\xi) &= -\frac{1}{2} \left( 72\xi^8 + 2019\xi^6 - 21192\xi^4 + 40138\xi^2 - 4656 \right), \\
b(\xi) &= -\frac{(9504\xi^{10} - 163836\xi^8 + 879621\xi^6 - 1434145\xi^4 + 133646\xi^2 - 4656)\xi}{(504\xi^8 - 5511\xi^6 + 20905\xi^4 - 36290\xi^2 - 1552)(12\xi^2 - 1)}.
\end{align*}
\]

Substituting these expressions into the resultant \( \text{Res}_x(F, G) \) and taking the gcd of the numerators of its coefficients and \( g \) yields a polynomial \( R(\xi) = \xi^2 - 8 \).

We find \( F_1(t, s) = -st + \sqrt{2}t + s \) for \( \xi = \sqrt{8} \) and \( F_2(t, s) = -st - \sqrt{2}t + s \) for \( \xi = -\sqrt{8} \) as factors of \( G \). In the case \( d = 0 \), we may assume \( c = 1 \) and we get

\[
\begin{align*}
a_0(\xi) &= -\frac{(\xi^2 - 8)(12\xi^2 - 1)}{\xi(24\xi^2 - 97)}, \\
b_0(\xi) &= \frac{18\xi^4 - 97\xi^2 + 4}{\xi(24\xi^2 - 97)}.
\end{align*}
\]

Here \( R_0(\xi) = 12\xi^2 - 1 \) and we obtain \( F_3(t, s) = st - \frac{1}{3}\sqrt{3} \) for \( \xi = \frac{1}{\sqrt{12}} \) and \( F_4(t, s) = st + \frac{1}{3}\sqrt{3} \) for \( \xi = -1/\sqrt{12} \). The entire computation takes a fraction of a second when implemented in Sage on a modern laptop. For more details we refer to the worksheet accompanying this paper [31].

3.3. The complete algorithm

Once each M"obius-like factor \( F \) of \( G^\pm \) and the corresponding M"obius transformation \( \varphi \) have been determined, Condition (11) can be checked quickly. Squaring and clearing denominators yields a polynomial \( W(t) = 0 \) of degree at most \( 24m - 4 \), where \( m \) is the maximum degree of the numerators and denominators of the components of \( x \).

To check whether \( W(t) \) is identically zero one can directly compute it, for instance if \( m \) is small. Alternatively, one can evaluate \( W(t) \) at any \( 24m - 4 + 1 \) points and use that \( W(t) \) is identically zero precisely when it is zero at these points. When the coefficients of \( \varphi \) are written in terms of an algebraic number \( \xi \) with minimal polynomial \( M(\xi) = 0 \), checking whether \( W \) is zero at \( t_0 \) amounts to checking whether \( M(\xi) \) divides \( W(t_0) \) as a polynomial in \( \xi \). We thus arrive at Algorithm Symm for determining the number of symmetries of the curve \( C \).

4. Determining the symmetries

Algorithm Symm detects whether the parametric curve \( \mathbf{x} \) from [1] has nontrivial symmetries. In the affirmative case we would like to determine these symmetries. By Theorem (1) every such symmetry corresponds to a M"obius transformation \( \varphi = (at+b)/(ct+d) \), which corresponds to a M"obius-like factor \( F \) of \( G \) computed by Algorithm Symm. In this section we shall see how the symmetry \( f(\mathbf{x}) = Q\mathbf{x} + b \) can be computed from \( \varphi \).
Algorithm Symm±

Require: A proper parametrization \( x \) of a space curve \( C \), not a line or a circle.
Ensure: The number of symmetries \( f(x) = Qx + b \), with \( \det(Q) = \pm 1 \), of \( C \).

1: Find the bivariate polynomials \( K, T \) and \( G \) from (7) and (8).
2: Find the resultant \( \text{Res}_s(F, G^\pm) \), with \( F \) as in (10).
3: Let \( t_0 \) be such that the discriminant of \( g^\pm(\xi) := G^\pm(t_0, \xi) \) does not vanish.
4: Find the gcd \( R(\xi) \) of \( g^\pm \) and the numerators of the coefficients of the polynomial \( P(t) \) obtained by substituting \( d = 1 \) and (17) into \( \text{Res}_s(F, G^\pm) \).
5: Find the real roots of \( R(\xi) \) for which (9) is well defined, each defining a Möbius transformation by substituting (17) and \( d = 1 \) in (3).
6: Let \( n \) be the number of these Möbius transformations satisfying (11).
7: Find the gcd \( R_0(\xi) \) of \( g^\pm \) and the numerators of the coefficients of the polynomial \( P_0(t) \) obtained by substituting \( c = 1, d = 0, (18) \) into \( \text{Res}_s(F, G^\pm) \).
8: Find the real roots of \( R_0(\xi) \) for which (9) is well defined, each defining a Möbius transformation by substituting (18), \( c = 1 \) and \( d = 0 \) in (3).
9: Let \( n_0 \) be the number of these Möbius transformations satisfying (11).
10: Return “The curve has \( n + n_0 \) symmetries with \( \det(Q) = \pm 1 \).”

Theorem 1 implies the identity

\[
Qx(t) + b = x(\varphi(t)).
\]

Let us distinguish the cases \( d \neq 0 \) and \( d = 0 \). In the latter case, (19) becomes

\[
Qx(t) + b = x(\varphi(t)) = x(\tilde{a}/t + \tilde{b}), \quad \tilde{a} := b/c, \quad \tilde{b} := a/c.
\]

Applying the change of variables \( t \rightarrow 1/t \) and writing \( \tilde{x}(t) := x(1/t) \), we obtain

\[
Qx(t) + b = \tilde{x}(\tilde{a}t + \tilde{b}).
\]

Without loss of generality, we assume that \( \tilde{x}(t) \) is well defined at \( t = \tilde{b} \) and that \( x'(0), x''(0) \) are well defined, nonzero, and not parallel. The latter statement is equivalent to requiring that the curvature \( \kappa_x(t) \) at \( t = 0 \) is well defined and distinct from 0. This can always be achieved by applying a change of parameter of the type \( t \rightarrow t + \alpha \). Observe that \( \varphi(t) \) can be determined before applying this change, because afterwards the new Möbius transformation is just \( \varphi(t + \alpha) \).

Evaluating (20) at \( t = 0 \) yields

\[
Qx(0) + b = \tilde{x}(\tilde{b}),
\]

while differentiating once and twice and evaluating at \( t = 0 \) yields

\[
Qx'(0) = \tilde{a} \cdot \tilde{x}'(\tilde{b}), \quad Qx''(0) = \tilde{a}^2 \cdot \tilde{x}''(\tilde{b}).
\]

Using (6) and that \( Q \) is orthogonal, taking the cross product in (22) yields

\[
Q(x'(0) \times x''(0)) = \det(Q) \cdot \tilde{a}^3 \cdot \tilde{x}'(\tilde{b}) \times \tilde{x}''(\tilde{b}).
\]
Multiplying $Q$ by the matrix $B := [x'(0), x''(0), x'(0) \times x''(0)]$ therefore gives

$$C := [\dot{a} \cdot \dot{x}'(\hat{b}), \dot{a}^2 \cdot \ddot{x}''(\hat{b}), \det(Q) \cdot \dot{a}^3 \cdot \dot{x}'(\hat{b}) \times \ddot{x}''(\hat{b})]$$

and $Q = CB^{-1}$. One sets $\det(Q) = 1$ to find the orientation-preserving symmetries, and $\det(Q) = -1$ to find the orientation-reversing symmetries. One finds $b$ from \[21\].

Next we address the case $d \neq 0$. After dividing the coefficients of $\varphi$ by $d$ if necessary, we may assume $d = 1$. As before, we assume that $x(t)$, and therefore any of its derivatives, is well defined at $t = 0$, and we again assume that the curvature $\kappa_x(0)$ is well defined and nonzero. Differentiating (19) once and twice, we get

$$Qx'(t) = x'(\varphi(t)) \cdot \varphi'(t) = x'\left(\frac{at + b}{ct + 1}\right) \frac{\Delta}{(ct + 1)^2}, \quad (24)$$

$$Qx''(t) = x''(\varphi(t)) \varphi'(t)^2 + x'(\varphi(t)) \varphi''(t)$$

$$= x''\left(\frac{at + b}{ct + 1}\right) \frac{\Delta^2}{(ct + 1)^4} - 2x'\left(\frac{at + b}{ct + 1}\right) \frac{c\Delta}{(ct + 1)^3}. \quad (25)$$

Evaluating (24) and (25) at $t = 0$ yields

$$Qx'(0) = x'(b) \cdot \Delta, \quad Qx''(0) = x''(b) \cdot \Delta^2 - 2x'(b) \cdot c\Delta. \quad (26)$$

Using (0) and that $Q$ is orthogonal, taking the cross product in (26) yields

$$Q(x'(0) \times x''(0)) = \det(Q) \cdot \Delta^3 \cdot x'(b) \times x''(b). \quad (27)$$

Since $\varphi$ is known, the matrix $Q$ can again be determined from its action on $x'(0), x''(0), \text{and } x'(0) \times x''(0)$, which is given by Equations (26) and (27). One finds $b$ by evaluating (19) at $t = 0$.

Once $Q$ and $b$ are found, one can compute the set of fixed points of $f(x) = Qx + b$ to determine the elements of the symmetry, i.e., the symmetry center, axis, or plane. For rotations, the rotation angle is the angle between any (non-fixed) point and its image under $f$.

**Example 2.** Let $C \subset \mathbb{R}^3$ be the crunode space curve parametrized by

$$x : t \mapsto \left(\frac{t}{t^4 + 1}, \frac{t^2}{t^4 + 1}, \frac{t^3}{t^4 + 1}\right).$$

Applying Algorithm Symm⁺ we get $G^+(t, s) = (t - s)(t + s)$. The first factor corresponds to the identity map $\varphi_1(t) = t$ and the trivial symmetry $f_1(x) = x$. The second factor corresponds to the Möbius transformation $\varphi_2(t) = -t$. Clearly $\varphi_2$ satisfies Condition (11), so that Theorem 3 implies that $C$ has a nontrivial, direct symmetry $f_2(x) = Q_2x + b_2$. With $a = -1, b = 0, c = 0, d = 1, \text{and } \det(Q) = 1,$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad Q_2 := CB^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
Evaluating (19) at $t = 0$ gives $b_2 = (I - Q_2)x(0) = 0$, so that $C$ is invariant under $f_2(x) = Q_2x$, which is a half-turn about the $y$-axis. Since there are no other factors in $G^+$, there are no direct symmetries corresponding to a Möbius transformation with $d = 0$.

As for the opposite symmetries, applying Algorithm Symm yields $G^-(t,s) = (st-1)(st+1)$, whose factors correspond to the Möbius transformations $\phi_3(t) = 1/t$ and $\phi_4(t) = -1/t$. A direct computation shows that $\phi_3(t)$ and $\phi_4(t)$ satisfy Condition (11), so that they correspond to symmetries $f_3(x) = Q_3x$ and $f_4(x) = Q_4x$, with

$$Q_3 = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}.$$

The sets of fixed points of these isometries are the planes $\Pi_3 : z - 2x = 0$ and $\Pi_4 : z + 2x = 0$, which intersect in the symmetry axis of the half-turn; see Figure 2.

5. Performance

5.1. Complexity

Let us determine the complexity of Algorithm Symm. In addition to using the standard Big O notation $O$ for the space- and time-complexity analysis, we use
the Soft $O$ notation $\tilde{O}$ to ignore any logarithmic factors in the time-complexity analysis.

**Theorem 5.** Algorithm Symm$^\pm$ finishes in $\tilde{O}(m^6)$ integer operations, with $m$ be the maximal degree of the numerators and denominators of the components of the parametric curve $x$ in $[1]$.

**Proof.** Step 1. Using the Schönhage-Strassen algorithm, two polynomials of degree $m$ with integer coefficients can be multiplied in $\tilde{O}(m)$ (integer) operations [15], Table 8.7. Therefore the computation of $\kappa^2$ and $\tau$ can be carried out in $\tilde{O}(m)$ operations as well, resulting in rational functions whose numerators and denominators have degree $O(m)$. As a consequence, $K$ and $T = T^\pm$ can also be computed in $\tilde{O}(m)$ operations, and have degrees $O(m)$ in $t$ and $s$. The bivariate gcd $G = G^\pm$ can be computed in $\tilde{O}(m^3)$ operations using Brown’s modular gcd algorithm [10, Eq. (95)], and has degree $O(m)$ in both variables. Step 1 therefore takes $O(m^3)$ operations.

Step 2. Since $F(t, s) = (ct + d)s - (at + b)$, the resultant $\text{Res}_s(F, G)$ is the polynomial in $t$ obtained by replacing $s$ by $(at + b)/(ct + d)$ and clearing denominators. Writing $G(t, s) = \sum_{k=0}^{m_0} G_k(t)s^k$ as a sum of $m_0 + 1 = O(m)$ terms, with each $G_k(t)$ a polynomial of degree $O(m)$, gives

$$\text{Res}_s(F, G) = \sum_{k=0}^{m_0} G_k(t)(at + b)^k(ct + 1)^{m_0 - k}, \quad (28)$$

which is a polynomial of degree $O(m)$ in $a, b, c,$ and $t$.

Step 3. For any integer $t_0$, the two polynomials $G(t_0, s)$, $G_s(t_0, s)$ have degree $O(m)$, so that their (univariate) gcd can be computed in $\tilde{O}(m)$ operations. Since we need to consider at most $O(m^2)$ values of $t_0$, Step 3 takes $\tilde{O}(m^3)$ operations.

Step 4. For any integer $t_0$ and unknown $\xi$, evaluating [10] at $(t_0, \xi)$ takes $O(m^2)$ operations, yielding rational functions $s'_0(\xi)$ and $s''_0(\xi)$ whose numerator and denominator have degrees $O(m)$. Substituting these rational functions into [17] takes $\tilde{O}(m)$ operations and yields rational functions

$$a(\xi) = \frac{a_1(\xi)}{a_2(\xi)}, \quad b(\xi) = \frac{b_1(\xi)}{b_2(\xi)}, \quad c(\xi) = \frac{c_1(\xi)}{c_2(\xi)}, \quad (29)$$

whose numerator and denominator have degree $O(m)$. Substituting these rational functions into (28) followed by binomial expansion, i.e., computing

$$\sum_{n=0}^{k} \binom{k}{n} a_1^n b_2^{m_0 - k + n} b_1^{k-n} c_1^{m_0 - n} t^n a_2^{m_0} b_2^{m_0}, \quad \sum_{n=0}^{m_0 - k} \binom{m_0 - k}{n} c_1^n c_2^{m_0 - n} t^n c_2^{m_0}, \quad (30)$$

involves raising polynomials of degree $O(m)$ to the power $O(m)$, which can be computed in $O(m^3)$ operations using binary exponentiation and FFT-based multiplication [43, §8.2]. All powers $a_i^l, b_i^l, c_i^l, t^l$, with $l = 0, \ldots, m_0$, in the above expression can therefore be computed in $O(m^4)$ operations, resulting
in polynomials of degree $O(m^2)$ in $\xi$. All remaining products can be computed in $O(m^3)$ operations, resulting in polynomials of degree $O(m^2)$.

Now the rational functions in $(30)$ are determined, the product of their numerators can be carried out in $O(m)$ ring operations, the ring now being the polynomials in $\xi$. Since these polynomials have degree $O(m^2)$, the product of $G_k(t)$ and the rational functions in $(30)$ takes $O(m^2)$ integer operations, and yields the terms in the sum $(28)$. After factoring out the common denominator $(a_2b_2c_2)^m$, this sum involves $O(m)$ polynomials of degree $O(m)$ in $t$, which requires $O(m^2)$ additions of polynomials of degree $O(m^2)$ in $\xi$. This involves $O(m^2)$ integer operations and yields a polynomial of degree $O(m)$ in $t$, whose coefficients $P_l(\xi)$ are polynomials of degree $O(m^2)$. The gcd of $g(\xi)$ with the $P_l(\xi)$ can be computed in $O(m^3)$ operations, resulting in a polynomial $R(\xi)$ of degree $O(m)$, since $g(\xi)$ has degree $O(m)$. Step 4 therefore takes $O(m^4)$ operations.

Step 5. One determines whether $R(\xi)$ has real roots using root isolation, which takes $O(m^3)$ operations [22 Theorem 17].

Step 6. Writing $x = (x_1/x_2, y_1/y_2, z_1/z_2)$, we find that

$$\|x'\|^2 = \frac{(x_2x'_1 - x_1x'_2)^2y_2^4z_2^4 + (y_2y'_1 - y_1y'_2)^2x_2^4z_2^4 + (z_2z'_1 - z_1z'_2)^2x_2^4y_2^4}{x_2^4y_2^4z_2^4}$$

can be computed in $O(m)$ operations. From Step 3 we already know the expansions of the powers $(at + b)^l$, $(ct + d)^l$ and their products, thus determining $x \circ \varphi$. Taking the derivative of $x \circ \varphi$ and then squaring involves multiplying and adding polynomials of degree $O(m)$ in $t$ and $O(m^2)$ in $\xi$, which requires $O(m^2)$ operations. Similarly we determine $[(y \circ \varphi)'^2$ and $[(z \circ \varphi)'^2$ in $O(m^2)$ operations. The resulting rational functions have numerator and denominator of degree $O(m)$ in $t$ and $O(m^2)$ in $\xi$, and can be added in $O(m^2)$ operations. Clearing denominators again takes $O(m^2)$ operations, and results in the polynomial $W = W(t; \xi)$ of degree $O(m)$ in $t$ and of degree $O(m^2)$ in $\xi$.

The most expensive case takes place when the Möbius transformation $\varphi$ is defined in terms of the minimal polynomial $M(\xi)$ of a root $\xi$ of $R(\xi)$. Univariate polynomials can be factored over the rationals in $O(m^6)$ operations [35], but there are algorithms that perform better in practice whose complexity has not been thoroughly analyzed yet [19,32]. In particular $R(\xi)$ can be factored over the rationals in $O(m^6)$ operations, yielding as irreducible factors these minimal polynomials of degree $O(m)$. For evaluating $W$ at a point $t_0$, we need to add $O(m)$ polynomials of degree $O(m^3)$ in $\xi$, which requires $O(m^4)$ operations. As explained in Section 3.3 we need to evaluate $W$ at $O(m)$ distinct values $t_0$, resulting in a total number of $O(m^4)$ operations. For each polynomial $W(t_0; \xi)$ and each minimal polynomial $M(\xi)$, we check whether their gcd is equal to $M(\xi)$ in $O(m^2)$ operations [45 Corollary 11.6], requiring $O(m^4)$ operations in total. Step 6 therefore requires $O(m^6)$ operations.

Steps 7–10. These steps have the same complexity as Steps 4–6.

\[ \square \]
Table 1: Average CPU time (seconds) of the algorithm in [3] \( t_{\text{old}} \) and Algorithm \( \text{Symm}^\pm \) \( t_{\text{new}} \) for parametric curves given in [4].

| curve            | degree | \( t_{\text{old}} \) | \( t_{\text{new}} \) |
|------------------|--------|-----------------------|-----------------------|
| twisted cubic    | 3      | 0.26                  | 0.15                  |
| cusp             | 4      | 0.52                  | 0.16                  |
| half-turn 1      | 4      | 2.22                  | 0.14                  |
| crunode          | 4      | 39.60                 | 0.42                  |
| inversion 1      | 7      | 6.80                  | 0.19                  |
| space rose       | 8      | 57.60                 | 0.25                  |
| inversion 2      | 11     | 75.60                 | 0.22                  |

To test Algorithm \( \text{Symm}^\pm \) for symmetric curves with higher degree, Table 2 lists the timings for a family of space rose curves of increasing degree \( m = 4j + 4 \),

| degree | \( t_{\text{new}} \) |
|--------|-----------------------|
|        | 0.66                  |
| 12     | 0.92                  |
| 16     | 1.47                  |
| 20     | 2.30                  |
| 24     | 4.38                  |

| degree | \( t_{\text{new}} \) |
|--------|-----------------------|
| 28     | 5.33                  |
| 32     | 6.53                  |
| 36     | 8.77                  |
| 40     | 15.88                 |
| 44     | 18.11                 |

5.2. Experimentation

Algorithm \( \text{Symm}^\pm \) was implemented in the computer algebra system \textbf{Sage} [38], using \textbf{Singular} [16] as a back-end, and was tested on a Dell XPS 15 laptop, with 2.4 GHz i5-2430M processor and 6 GB RAM. Additional technical details are provided in the Sage worksheet [31].

We present tables with timings corresponding to different groups of examples. Table 1 corresponds to the set of examples in [3], making it possible to compare the timing \( t_{\text{new}} \) of Algorithm \( \text{Symm}^\pm \) to the timing \( t_{\text{old}} \) of the algorithm in [3]. It is clear from the table that the algorithm introduced in this paper is considerably faster for each curve.

To test Algorithm \( \text{Symm}^\pm \) for symmetric curves with higher degree, Table 2 lists the timings for a family of space rose curves of increasing degree \( m = 4j + 4 \).
Table 3: CPU times $t_{\text{new}}$ (seconds) for random dense rational parametrizations of various degrees $m$ and coefficients with bitsize bounded by $\tau$.

| $t_{\text{new}}$ | $\tau = 4$ | $\tau = 8$ | $\tau = 16$ | $\tau = 32$ | $\tau = 64$ | $\tau = 128$ | $\tau = 256$ |
|------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $m = 4$          | 0.61      | 0.62      | 0.66      | 0.73      | 0.83      | 1.14      | 1.88      |
| $m = 6$          | 1.65      | 1.76      | 1.72      | 1.89      | 2.13      | 2.80      | 4.55      |
| $m = 8$          | 3.50      | 3.54      | 3.55      | 3.84      | 4.27      | 5.36      | 8.59      |
| $m = 10$         | 7.53      | 7.47      | 7.30      | 7.98      | 8.42      | 9.76      | 15.35     |
| $m = 12$         | 14.46     | 14.35     | 14.30     | 14.84     | 15.98     | 18.35     | 25.87     |
| $m = 14$         | 22.31     | 23.24     | 22.39     | 24.93     | 27.35     | 38.36     |           |
| $m = 16$         | 34.86     | 35.60     | 35.38     | 35.27     | 38.14     | 41.91     | 55.74     |
| $m = 18$         | 53.03     | 52.78     | 52.78     | 51.16     | 54.49     | 60.44     | 78.27     |

which are given parametrically by

$$x(t) = \left( \sum_{i=0}^{j} (-1)^i \frac{2j}{2i} u^{2j-2i} v^{2i}, \sum_{i=0}^{j} (-1)^i \frac{2j}{2i} u^{2j-2i} v^{2i} \right),$$

with $u = (1 - t^2)/(1 + t^2)$ and $v = 2t/(1 + t^2)$. The Algorithm Symm± quickly finds the symmetries of these symmetric curves, also for high degree.

Finally, write $\tau = \lceil \log_2 k \rceil + 1$ for the bitsize of an integer $k$. Table 3 shows average timings for random dense rational parametrizations with various degrees $m$ and coefficients with bitsize at most $\tau$. For very large coefficient bitsizes (> 256, i.e., coefficients with more than 77 digits) and high degrees (> 20) the machine runs out of memory. We have therefore analyzed separately the regime with high degree and the regime with large coefficient bitsize.

Figure 3 presents log-log plots of the CPU times against the degree (left) and against the coefficient bitsizes (right). The (eventually) linear nature of this data suggests the existence of an underlying power law. Least squares approximation yields that the average CPU time $t_{\text{new}}$ satisfies

$$t_{\text{new}} \sim \alpha m^\beta, \quad \alpha \approx 6.7 \cdot 10^{-3}, \quad \beta \approx 3.2$$

in case of random dense rational parametrizations with coefficient bitsize at most $\tau = 4$, and

$$t_{\text{new}} \sim \alpha m^\beta, \quad \alpha \approx 7.9 \cdot 10^{-5}, \quad \beta \approx 3.2$$

for the space rose curves with various degrees $m$. Note that both timings are close to the $\tilde{O}(m^3)$ operations needed by the modular bivariate gcd algorithm in Step 1 in Section 5.1. The reason is that in practice almost all time is spent computing the bivariate gcd in Step 1, which then typically has low degree, so that the remaining calculations take relatively little time.

5.3. An observation on plane curves

If $C$ is planar, then $\tau$ and $T^\pm$ are identically zero, so that $G^\pm = K$. Although Algorithm Symm± is still valid for such curves, we have observed a very poor
Figure 3: The average CPU time $t_{\text{new}}$ (seconds) versus the degree $m$ (left) for random dense rational parametrizations with bitsize $\tau = 4$ and space rose curves, fitted by the power laws (31) and (32), and versus the coefficient bitsizes (right). The symbols indicate the average observed CPU times, and the error bars show the interval of observed CPU times.

 performance in this case. The reason is that, for non-planar curves, the degree of $G^\pm$ is typically small compared to the degrees of $K$ and $T^\pm$. However, for plane curves the degree of $G^\pm$ is equal to the degree of $K$, and then the computation takes a very long time. Therefore, for plane curves, the algorithms in [2] and [3] are preferable.

6. Conclusions

We have presented a new, deterministic, and efficient method for detecting whether a rational space curve is symmetric. The method combines ideas in [1] with the use of the curvature and torsion as differential invariants of space curves. The complexity analysis and experiments show a good theoretical and practical performance, clearly beating the performance obtained in [3]. The algorithm also improves in scope on the algorithm of [3], which can only be applied to find the symmetries for Pythagorean-hodograph curves and involutions of other curves. Finally Algorithm Symm$^\pm$ is simpler than the algorithm in [3], which imposes certain conditions on the parametrization that often lead to a reparametrized, non-sparse curve whose coefficients have a large bitsize. By contrast, the algorithm in this paper has fewer requirements and is efficient even with high degrees.
As a final remark, note that Algorithm Symm+ is based on two conditions in Theorem 3: one involving the curvature and torsion of the curve and the other one involving the arc length. One might wonder whether these two conditions really are independent for the case of rational curves. We included both conditions because we did not succeed in proving that they are dependent, but neither did we find an example of a tentative Möbius transformation not satisfying Condition (11). The relation between the conditions is therefore undetermined, and we pose the question here as an open problem. In any case, in the complexity analysis and experiments we observed that the cost of checking (11) is small compared to the rest of the algorithm.

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