Hausdorff dimension of critical fluctuations in abelian gauge theories

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The geometric properties of the critical fluctuations in abelian gauge theories such as the Ginzburg-Landau model are analyzed in zero background field. Using a dual description, we obtain scaling relations between exponents of geometric and thermodynamic nature. In particular we connect the anomalous scaling dimension \( \eta \) of the dual matter field to the Hausdorff dimension \( D_H \) of the critical fluctuations, which are fractal objects. The connection between the values of \( \eta \) and \( D_H \), and the possibility of having a thermodynamic transition in finite background field, is discussed.

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Anderson has proposed the breakdown of a generalized rigidity associated with proliferation of defect structures in an order parameter as a general means of characterizing phase transitions. In the context of three-dimensional superfluids and extreme type-II superconductors, such ideas have recently been put on a quantitative level. It has been explicitly demonstrated that in three spatial dimensions abelian gauge theories such as the Ginzburg-Landau theory describing type-II superconductors, suffer a continuous phase transition driven by a proliferation of topological defects in the order parameter, which are closed loops of quantized vorticity. These loops are induced by transverse phase fluctuations in a complex scalar order parameter. Such fluctuations are prominent in, for instance, doped Mott-Hubbard insulators.

In this paper, we investigate the non-trivial geometric properties of these critical fluctuations, and give a geometric interpretation of the anomalous scaling dimension of the condensate order parameter both for a charged and neutral condensate. In addition, we discuss the connection between the geometric properties of the zero-field critical fluctuations and the possibility of having a thermodynamic finite-field phase transition involving unbinding of loops of quantized vorticity.

We emphasize that the main results to be presented are quite general, and apply to the static critical sector of any theory of a complex scalar matter field coupled to a fluctuating gauge-field in three spatial dimensions, provided the symmetry group of the theory is abelian.

The Hamiltonian for the system is given by

\[
H(q,u_\phi) = m_\phi^2 |\phi|^2 + \frac{u_\phi}{2} |\phi|^4 + |D_\mu \phi|^2 + \frac{1}{4} F^2, \tag{1}
\]

where \( F^2 = F_{\mu \nu} F^{\mu \nu} \), \( F_{\mu \nu} = \partial_\mu h_\nu - \partial_\nu h_\mu \), \( D_\mu = \partial_\mu - iq h_\mu \), and \( \phi = |\phi| \exp(i \theta) \) is a complex matter field coupled to a massless gauge field \( h \) with coupling constant \( q \). The \( |\phi|^4 \) term mediates a short-range repulsion, while the gauge-field \( h \) mediates long-ranged interactions. \( m_\phi \) is the mass parameter for the \( \phi \)-field, and \( u_\phi \) is a self coupling.

Consider Eq. representing a 3D condensate with charge \( q \neq 0 \) sustaining stable topological objects in the form of closed vortex loops. Then the theory with \( q = 0 \) is a field-theoretical description of the ensemble of these stable topological objects, constituting a dual description of the original theory. The theory with \( q \neq 0 \) is also a direct field-theoretical description of a neutral condensate. Thus, in 3D, the gauge-theory \( H(q \neq 0, u_\phi) \) describing a charged condensate has field-theoretical description of its critical fluctuations or topological defects in terms of a theory isomorphic to \( H(q = 0, u_\phi) \) describing a similar but neutral condensate, and vice versa.

In this sense, a charged condensate has neutral vortices with only short-ranged steric interactions, while a neutral condensate has charged vortices with long-ranged interactions. In the former case, the long-ranged interactions between vortex segments are rendered short-ranged by fluctuations of the gauge-field in the original theory, i.e. the dual gauge-field is massive with mass given by the charge of the original problem.

The anomalous dimension \( \eta \) for the \( \phi \) field is defined via the relation

\[
G(x,y) = \langle \phi(x) \phi^\dagger(y) \rangle = \frac{G(|x - y|/\xi)}{|x - y|^{d - 2 + \eta}}, \tag{2}
\]

where \( G(z) \) is some scaling function, \( \xi \) is a correlation length, and \( d \) is the spatial dimension of the system. This correlation function has a geometric interpretation, yielding the probability amplitude of finding any particle-path connecting \( x \) and \( y \). In the present work, the particle trajectories correspond to vortex lines.

For a random walk of length \( N \) in \( d = 3 \), the probability of going from \( x \) to \( y \) is given by

\[
P(x,y;N) = \left( \frac{3}{2 \pi N} \right)^{3/2} \exp \left[ -\frac{(x-y)^2}{2N} \right]. \tag{3}
\]

The correlation function \( G(x,y) \) of the corresponding gaussian field theory is found by summing up \( P(x,y;N) \) for all \( N \)

\[
G(x,y) = \sum_N P(x,y;N) \propto \frac{1}{|x - y|}. \tag{4}
\]
Comparing this with Eq. 3, we find \( \eta = 0 \), as expected. The random walker traces out a fractal path with Hausdorff dimension \( D_H \). Moreover, in general the distance between two points \( x \) and \( y \) \( N \) walks apart is given by

\[
\langle |x - y|^2 \rangle \propto N^{2\Delta},
\]

where \( \Delta \) is the wandering exponent which for the gaussian 3D case is \( \Delta = 1/2 \). Inverting Eq. 3 we find that the total length of the random walker with linear size as \( L^{1/\Delta} \), hence the Hausdorff dimension of the random walker is given by \( D_H = 1/\Delta \). If we set \( x = y \) in Eq. 3 we find that the unnormalized distribution \( D(N) \) of loops of perimeter \( N \), at the critical point, is given by

\[
D(N) \propto \frac{1}{N} \sum_x P(0; N) \propto N^{-\alpha},
\]

with \( \alpha = 5/2 \) for purely random walkers. The extra factor \( N^{-1} \) in Eq. 3 comes from the arbitrariness in defining the starting position along the loop. Hence, for the case of strict random walkers in 3D, described by a gaussian field theory \( H = m^2 \phi^2 + (\nabla \phi)^2 \), the corresponding set of values for the two geometric and one thermodynamic exponents is given by \( (\Delta, \alpha, \eta) = (1/2, 5/2, 0) \).

Beyond the gaussian case, exact exponents cannot be obtained analytically; however we will derive scaling relations for them. When Eq. 3 is invoked, a generalized probability function \( P(x, y; N) \) may be written on the form

\[
P(x, y; N) \propto \frac{1}{N^\rho} F\left(\frac{|x-y|}{N^{\Delta}}\right),
\]

where \( F(x) \) is a scaling function, and normalizability of \( P \) implies \( \rho = d \Delta \). From Eq. 3, we find that \( P(x, y; N) \) should scale with \( N \) as \( N^{1-\alpha} \), which yields the scaling relation \( \rho = \alpha - 1 \). Conversely, summing over all \( N \) in Eq. 3 to find the correlation function \( G(x, y) \), we obtain

\[
G(x, y) = \sum_N P(x, y; N) \propto \frac{1}{|x-y|^{d-\Delta}},
\]

giving the scaling relation \( \eta = \frac{d-1}{\alpha} + 2 - d \). Combining the above, we find

\[
\eta + D_H = 2, \quad D_H = \frac{d}{\alpha - 1}.
\]

A computation of the geometric exponent \( \alpha \) yields the thermodynamic exponent \( \eta \) and the Hausdorff dimension \( D_H \). Note that both \( \eta \) and \( D_H \) are sensitive functions of \( \alpha \), \( \partial \eta/\partial \alpha = -\partial D_H/\partial \alpha = d/\alpha \), such that a precise determination of \( \eta \) and \( D_H \) requires great precision in the determination of \( \alpha \). The above reinforces the statement that a geometric transition of the vortex tangle at criticality of the gauge theory Eq. 3 can be assigned a genuine thermodynamic order parameter via a dual formulation of the original theory in three spatial dimensions. The random walker is represented by a gaussian theory, Eq. 3 with \( (u_0 = 0, q = 0) \), for which \( \eta = 0 \). This corresponds to \( D_H = 2 \), such that the random walk in three dimensions traces out a path that precisely fills a cross-sectional area of the system. Note that \( D_H < 2 \iff \eta > 0 \), while \( D_H > 2 \iff \eta < 0 \).

The Hamiltonian Eq. 3 with \( (u_0 \neq 0, q \neq 0) \) has a dual field theory corresponding to Eq. 3 with \( q = 0 \) describing the neutral vortex tangle of a charged superconductor. The \( |\phi|^4 \)-term in Eq. 3 represents a steric repulsion, i.e. the vortex loops can not overlap, leading to a random walk problem with self-avoiding links (but not necessarily self-avoiding sites), in the sense that parallel vortex segments repel, perpendicular vortex segments can cut, while antiparallel vortex segments can annihilate. Hence, this is not a standard self-avoiding path problem. However, we expect \( \Delta > 1/2 \) or equivalently \( D_H < 2 \), since steric repulsion should result in a vortex-loop tangle packing space less densely than for the non-interacting case, so that \( \eta > 0 \). The repulsive interaction between parallel vortex segments also leads to a more efficient suppression of long loops than for the non-interacting case, so that \( \alpha > 5/2 \).

Consider next Eq. 3 with \( (u_0 \neq 0, q = 0) \) for \( d = 3 \), which has a dual field theory corresponding to Eq. 3 with \( q \neq 0 \) describing the charged vortex tangle of a neutral condensate. A long-ranged (anti) Biot-Savart interaction is mediated by the gauge-field. This is a relevant perturbation, in renormalization group sense, to a steric contact repulsion. The geometric properties of the charged vortex tangle are a result of a balance between attractive forces mediated by the gauge field, and the steric repulsion. As the numerical simulations show, we find \( \Delta < 1/2 \), corresponding to \( D_H > 2 \) which means that the vortex tangle is more compact than the ensemble of pure random walkers, due to the fact that an attractive long-ranged Biot-Savart interaction between oppositely oriented vortex segments overcompensates the steric repulsion so as to contract the vortex-loop tangle not only compared to the pure \( |\phi|^4 \)-case, but even compared to the noninteracting case. The tangle thus packs space so that it more than fills a cross-sectional area of the system.

The fluctuation-dissipation theorem provides a bound on \( \eta \) via the susceptibility \( \chi_\phi = \int d^d x G(x) \sim \xi^{2-\eta} \), which is bounded by the volume \( L^d \) of the system, \( L^{2-\eta} = L^d \cdot L^{2-d-\eta} < L^d \), so \( \eta > 2 - d \). Eq. 3 gives a geometric interpretation of this bound. Specializing to \( d = 3 \), \( \eta = -1 \) corresponds to topological excitations with \( D_H = 3 \), an upper limit.

For \( d = 3 \), the continuous phase transition in a superfluid or extreme type-II superconductor has recently been demonstrated to be driven by a proliferation of vortex loops. From the above, \( \eta = -1 \) means that a single vortex loop at \( T_c \) packs space completely, i.e. its perimeter \( N \) scales as \( N \propto L^3 \), implying that the vortex-tangle collapses on itself, rendering the transition discontinuous. This may be been seen from the standard scaling rela-
tion $\beta = \nu (d - 2 + \eta) / 2$ for critical exponents. Formally, this implies that the limit $\eta \to (2 - d)^+$ corresponds to the limit $\beta \to 0^+$, characteristic of a discontinuous transition. More informally, a collapse of a vortex tangle may be viewed as mediated by an effective attractive vortex interaction, a situation akin to what is known in type-I superconductors. Deep in the type-I regime, it is known that superconductors suffer a weakly discontinuous transition.

Monte Carlo simulations have been performed on the lattice version of Eq. (3) in the phase-only approximation, to determine precise values of $\alpha$, both for $q = 0$ and $q \neq 0$. We have also performed simulations on pure random walkers described by the theory $H(q = 0, u, \phi = 0)$. They reveal that a determination of $\alpha$ is less fraught with finite-size effects than a determination of $D_H$. Thus, we focus on determining $\alpha$. The model we consider is

$$H = -J \sum_{<i,j>} \cos(\theta_i - \theta_j - qC h_{ij}) + \frac{1}{2} (\nabla \times \mathbf{h})^2, \quad (10)$$

where the site-variable $\theta_i$ is the phase of the complex matter field $\phi = |\phi| \exp(i\theta)$ of Eq. (3) when the system is discretized, $J$ is essentially a bare phase-stiffness, and the link-variable $h_{ij} = \int_0^1 d\mathbf{f} \cdot \mathbf{h}$. The charge $qC$ is the (original) charge entering in the simulations. Up to this point we have considered a general charge $q$ irrespective of whether it couples to the original condensate or the resulting vortex tangle. The numerical simulations are performed on the phase of the condensate, hence the concept of original and dual are fixed in terms of the numerical simulations. Consequently we introduce the charges $q_C$ for the condensate and $q_V$ for the vortices.

From the phase distributions of the matter field we can extract vortex loops. These loops have charge $q_V$ and are described by the field theory $H(q_V, u, \phi, \mu)$. Hence, we can study the critical properties of the charged field theory $H(q_V, u, \phi)$ by considering the geometric properties of the thermally excited vortex-loop tangle at the critical temperature in the 3DXY model. Conversely, the geometric properties of the vortex tangle with $qC \neq 0$ yield the critical properties of the neutral field theory $H(q_V = 0, u, \phi)$ of Eq. (3).

The simulations with $q_C = 0$ are described elsewhere, while for $qC \neq 0$ the simulations proceed as follows. For every site on the lattice a phase change $\theta_i \to \theta'_i$ is attempted, and accepted or rejected according to the Metropolis algorithm. Then a change in $h_{ij} \to h_{ij} + \delta h$ is attempted, and accepted or rejected according to the Metropolis algorithm. When updating $h_{ij}$ we update all the link variables on a randomly oriented elementary plaquette containing $h_{ij}$ as one of its four edges. Updating of $\mathbf{h}$ in this fashion guarantees that the gauge-fixing condition $\nabla \cdot \mathbf{h} = 0$ is enforced at all times. For $qC = 0$, the simulations were performed for a system of size $L \times L \times L$ with $L = 100$, while those for $qC \neq 0$ were performed with $L = 64$.

During the simulations we have sampled the distribution function $D(N)$, Eq. (7), obtaining $\alpha$. The results are shown in Fig. 3 and listed in Table I. The value of $\alpha$ obtained for $qC \neq 0$ (dual neutral), which is the hardest system to simulate, gives a value for $\eta$ in good agreement with high-precision results for $\eta$ of the pure $\phi^4$-theory. This serves as a useful benchmark on our method of extracting $\eta$. For $qC = 0$ we have simulated much larger systems than for $qC \neq 0$. The deviation from the gaussian value $\alpha = 5/2$ is substantial, and of opposite sign compared to $qC \neq 0$. Given the size of the system we consider for $qC = 0$, it is unlikely that this is a finite-size artifact. An $\alpha < 5/2$ guarantees $\eta < 0$ for the $qC = 0$ (dual charged) case, contrary to the value of $\eta > 0$ for $qC \neq 0$. In particular, the inset of Fig. 1 lends strong support to the proposition that $\eta(qC \neq 0) > 0$, while $\eta(qC = 0) < 0$.

The value $\eta < 0$ obtained for the original neutral, dual charged case, is significant: It implies that $D_H > 2$ for this case. Whether $D_H > 2$ or $D_H < 2$ is of great import to the possibility of having a genuine phase-transition driven by a vortex-loop unbinding even in the presence of a finite background field such as magnetic induction in type-II superconductors. A vortex system tackles configurational entropy more easily if it is compressible than if it is incompressible. For the charged case the gauge-field fluctuations render the system compressible. In the neutral case, the system expands screening strings of closed vortex loops to a larger extent than for the charged case, as substitutes for the gauge-field fluctuations. This is why $D_H(qC = 0) > D_H(qC \neq 0)$. There is an infinitely larger amount of screening vortex-strings in the neutral case (dual charged) than for the charged case (dual neutral), which is the true significance of the fact that $\eta$ is smaller for $qC = 0$ than for $qC \neq 0$. The possibility of the zero-field vortex-loop blowout transition surviving the presence of a finite field is much greater in a neutral superfluid or an extreme type-II superconductor, than in a charged condensate with a priori good screening.

Given the significance of $D_H > 2$, we elaborate on the fact that for the neutral (dual charged) case, we find $\eta < 0$. The Lehmann-representation of the Fourier transform $G(p)$ of Eq. (2) is sometimes used to argue that $\eta$ obeys the strict inequality $\eta > 0$. The Lehmann-representation of $G(p)$ is given by

$$\tilde{G}(k) = \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{k^2 + \mu^2}, \quad (11)$$

where $1 = \int_0^\infty d\mu^2 \rho(\mu^2)$, and $\rho(\mu^2) = Z\delta(\mu^2 - m_{sc}^2) + \sigma(\mu^2)$. The propagator for the gaussian case would be $G(k) = 1/(k^2 + m_{sc}^2)$, where $m_{sc}^2$ refers to the bare massparameter in Eq. (2). Thus, $\eta > 0$ follows if $0 < Z < 1$, which holds for a uniformly positive $\rho(\mu^2)$. However, in theories with a local gauge symmetry, the two-point correlation function is a gauge-dependent quantity. Thus, $\rho(\mu^2)$ may in principle be made negative for certain values of $\mu$ by a gauge-transformation. This invalidates the reason-
ing leading to the strict inequalities \( Z < 1 \) and \( \eta > 0 \).

A negative \( \eta \), as found here and in other simulations, all representing basically exact results, agrees with a recent non-perturbative RG calculation\(^2\), which also gives \( \nu = \nu_{ADXY} \) at the "charged" critical point.

At the critical point, the relevant fluctuations are \textit{transverse} phase-fluctuations, or vortices\(^3\). Ignoring amplitude fluctuations yields an effective Hamiltonian governing the transverse \( \theta \)-fluctuations, whose Fourier-transform \( F \) we denote by \( S_k = F((\nabla \theta) \cdot \tau) = -2\pi i (k \times n_k)/k^2 \), where \( n_k \) is the Fourier-transform of the local vorticity. We find, after integrating out the transverse gauge-field, that \( H = \Xi^2(k) \cdot S_k \cdot S_{-k} \), where \( \Xi^2(k) = k^2/(k^2 + 2q^2) \). For \( q = 0 \), we have \( \Xi^2(k) = 1 \), while \( \lim_{k \to 0} \Xi^2(k) \sim k^2 \) for \( q \neq 0 \). The coupling to a fluctuating gauge field softens the transverse phase-fluctuations, providing the effective phase-stiffness with an extra power \( k^2 \) compared to the \( q = 0 \)-case. Thus, \( G^{-1}(k, q = 0) = k^2 + \Sigma(k) \) and \( G^{-1}(k, q \neq 0) = k^4 + \Sigma(k) \). In both cases, the \( k \to 0 \)-limit of the self-energy \( \Sigma(k) \) is given by \( \Sigma(k) \sim k^2 - \eta \). We thus have \( \lim_{k \to 0} G^{-1}(k) \sim k^2 - \eta \) provided \( \eta > 0 \) for \( q = 0 \) and, when invoking the absolute lower bound, \( \eta > -1 \) for \( q \neq 0 \). For a pure \( |\phi|^4 \)-theory, the Lehmann-representation coupled with positive-definiteness of \( \rho(\mu^2) \), holds.

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\[ FIG. 1. \] The vortex-loop distribution function \( D(N) \sim N^{-\alpha} \) as a function of loop-perimeter \( N \), at the critical point for the charged (\( q_c \neq 0, q_v = 0 \)) and neutral (\( q_c = 0, q_v \neq 0 \)) cases. Numerical results for the exponents (\( \alpha, D_H, \Delta, \eta \)) are given in Table I. The system size is \( L \times L \times L \), with \( L = 180 \) for \( q_c = 0 \), and \( L = 64 \) for \( q_c \neq 0 \). Inset shows simulation results for \( N^{5/2}D(N) \sim N^{5/2 - \alpha} \) on a double-logarithmic scale. Top: \( q_c = 0, L = 180 \) (dual charged). Middle: Noninteracting vortex loops (gaussian case), \( L = 64 \). Bottom: \( q_c \neq 0, L = 64 \) (dual neutral). The results demonstrate that \( 5/2 - \alpha > 0 \) for \( q_c = 0 \) (dual charged), while \( 5/2 - \alpha < 0 \) for \( q_c \neq 0 \) (dual neutral). Hence, by Eq. \( \Delta \), \( D_H > 2, \Delta < 0 \) for \( q_c = 0 \), while \( D_H < 2, \Delta > 0 \) for \( q_c \neq 0 \). The latter agrees with other high-precision results for \( \eta \), see Ref. 11. Note that the gaussian result \( \alpha = 5/2 \) is obtained to high precision, for \( L = 64 \).

\[ TABLE I. \] The loop distribution exponent \( \alpha \), as determined from Monte-Carlo simulations. The remaining exponents have been determined from Eq. \( \Delta \). Symbols are explained in the text.

\[ \begin{array}{|c|c|c|c|}
\hline
\text{Exponent} & \text{Gaussian} & q_c = 0, q_v \neq 0 & q_c \neq 0, q_v = 0 \\
\hline
\alpha & 5/2 & 2.312 \pm 0.003 & 2.56 \pm 0.03 \\
\hline
D_H & 2 & 2.287 \pm 0.004 & 1.92 \pm 0.04 \\
\hline
\Delta & 1/2 & 0.437 \pm 0.001 & 0.52 \pm 0.01 \\
\hline
\eta & 0 & -0.287 \pm 0.004 & 0.08 \pm 0.04 \\
\hline
\end{array} \]

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