Secure Coded Multi-Party Computation for Massive Matrix Operations

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Abstract

In this paper, we consider a secure multi-party computation problem (MPC), where the goal is to offload the computation of an arbitrary polynomial function of some massive private matrices to a network of workers. The workers are not reliable, some may collude to gain information about the input data. The system is initialized by sharing a (randomized) function of each input matrix to each server. Since the input matrices are massive, the size of each share is assumed to be at most $1/k$ fraction of the input matrix, for some $k \in \mathbb{N}$. The objective is to minimize the number of workers needed to perform the computation task correctly, such that even if an arbitrary subset of $t-1$ workers, for some $t \in \mathbb{N}$, collude, they cannot gain any information about the input matrices. We propose a sharing scheme, called polynomial sharing, and show that it admits basic operations such as adding and multiplication of matrices, and transposing a matrix. By concatenating the procedures for basic operations, we show that any polynomial function of the input matrices can be calculated, subject to the problem constraints. We show that the proposed scheme can offer order-wise gain, in terms of number of workers needed, compared to the approaches formed by concatenation of job splitting and conventional MPC approaches.

Index Terms

multi-party computation, polynomial sharing, secure computation, massive matrix computation

I. INTRODUCTION

With the growing size of datasets in use cases such as machine learning and data sciences, it is inevitable to distribute the computation tasks to some external entities, which are not necessarily trusted. In this set-up, some of the major challenges are protecting the privacy of the data, guaranteeing the correctness of the result, and ensuring the efficiency of the computation.

The problem of processing private information on some external parties has been studied in the context of secure multi-party computation (MPC). Informally, in an MPC problem, there are some private data inputs, available in some source nodes, and the goal is to offload the computation of specific function of those inputs to some parties. These parties are not reliable. Some of them may collude to gain information about the private data or even behave adversarial to make the result incorrect. Thus, the objective is to design a scheme, probably based on coding and randomization techniques, to ensure the privacy of data and correctness of the result. To ensure privacy, some MPC solutions, such as [2], [3], rely on cryptographic hardness assumptions, while others, such as [4]–[6], are protected based on information-theoretic measures (See [7] for a survey on different approaches of MPC). In particular, the BGW scheme, named after its inventors, Ben-Or, Goldwasser, and Wigderson in [4], relies on Shamir secret sharing [8] to develop an information-theoretically private MPC scheme, to calculate any polynomial of private inputs. Shamir secret sharing is an approach that allows us to share a secret, i.e. private input, among some parties, such that if the number of colluding nodes is less than a threshold, they are not able to gain any information about the data. BGW exploits the fact that Shamir secret sharing admits basic operation such as addition and multiplication (at the cost of some communication among nodes). There have been some efforts to improve the efficiency of MPC algorithms, but mainly focusing on the communication loads (see [9]).

This is an extended version of the paper, partially presented in IEEE Communication Theory Workshop (CTW), May 2018, and IEEE International Symposium on Information Theory (ISIT), June 2018 [1].
However, less effort has been dedicated to the cases where the input data is massive. One approach would be to split the computation to some smaller subtasks and dedicate a group of workers to execute each subtask, using conventional MPC approaches. In this paper we argue that the idea of concatenation of job splitting and multi-party computation could be significantly sub-optimum in terms of number of workers needed.

On seemingly irrelevant area, extensive efforts have been dedicated to using coding theory to improve the efficiency of distributed computing, mainly to cope with the stragglers [10]–[20]. The core idea is based on partitioning each input data into some smaller inputs and then encoding smaller inputs. The ultimate goal is to design the code such that the computation per worker node is limited to what it can handle, and also the final result can be derived from the outcome of a subset of worker nodes. This is done for matrix to vector multiplication in [11], and matrix to matrix multiplication in [12]–[16]. In particular, in [13], the code is designed such that the results of different workers form a maximum separable code (MDS), meaning that the final result can be recovered from any subset of servers with the minimum size. That approach has been extended to general matrix partitioning in [14], [15], and to the cases where only an approximate result of the matrix multiplication is needed [17], [20].

In this paper we consider a system including \( \Gamma \) sources, \( N \) workers, and one master. There is a link between each source and worker. All of the workers are connected to each other and also are connected to the master node. Each source will send a function of its data (so called a share) to each worker. We assume that workers have limited computation resources. As a proxy to that limit, we assume that size of each share can be up to a certain fraction of the corresponding input. The workers process their inputs, and in between, they may communicate with each other. After that the workers will send a message to the master node, such that the master node can recover the required result of the function of the sources. The sharing and the computation procedures must be designed such that if any subset of \( t - 1 \) workers collude, for some \( t \in \mathbb{N} \), they can not gain any information about the inputs and the result. Also the master should not gain any additional information, beyond the result, about the inputs. Motivated by recent result in coding for matrix multiplication, as an extension to Shamir secret sharing, in this paper, we propose a new sharing approach called polynomial sharing. We show the proposed sharing approach admits basic operations such as addition, multiplication, and transposing, by developing basic procedure for each of them. Finally we prove that using these procedures, we can compute any polynomial function, while preserving the privacy subject to the storage limit of each worker node. We show that the number of servers needed to compute a function using this approach is order-wise less than what we need compared to approaches based on job splitting and conventional BGW scheme.

The rest of the paper is organized as follows. In Section II we formally state the problem setting. In Section III we review some preliminaries and conventional approaches for MPC. In section IV we state the main result. In Section V we review some motivating examples. In Section VI we present the polynomial sharing scheme. In Section VII we show several procedures to perform basic operations, such as addition, multiplication, and transposing, using the proposed sharing scheme. In Section VIII we present the algorithm to calculate general polynomials. Finally in Section IX we show an extended model of the sharing and algorithm.

A. Concurrent and Follow-up Results

The early version of this paper submitted to IEEE ISIT 2018, in Jan. 2018, which appeared in June 2018 [1]. In addition, it was presented in CTW 2018 in May 2018. We also presented a generalized version of [1] in [21].

In parallel in [19], the authors consider a different set-up, where the goal is to compute an arbitrary polynomial \( G(X_1), G(X_2), \ldots, G(X_K) \) for possibly private inputs \( X_1, X_2, \ldots, X_K \). The idea is then to use coding over this computation to form coded redundancy in order to deal with stragglers and/or guarantee privacy. The major difference between [19] and what we do in this paper is that in [19], the input \( X_k \) in
$G(X_k)$ is not too large for one server to store it. This is in contrast with the set-up we consider in this paper.

In [22], published in June 2018 on arXiv, the authors focus on multiplications of two massive matrices. The major difference is that [22], the function is only multiplication of two servers, and without considering privacy at the master node.

Our work in [1], [21] also has been followed by some works. In [23], the authors improve the communication and computation cost of the scheme of [22]. In [24], the authors rely on arithmetic progressions to improve the sharing scheme to further reduce the number of servers needed.

Some other configurations are also considered in [25], where the master should not know which part of some external data set has been used for computation. In [26], the authors propose a code for private matrix multiplication that is flexible to achieve a trade-off between number of servers needed and communication load.

II. Problem Setting

Consider an MPC system including $\Gamma$ source nodes, $N$ worker nodes, and one master node, for some $\Gamma, N \in \mathbb{N}$ (see Fig. 1).

Each source is connected to every single worker. In addition, every pair of workers are connected to each other. However there is no communication link between the sources. In addition, all of the workers are connected to the master node. All of these links are orthogonal, secure, and error free.

Each source $\gamma \in [\Gamma]$ has access to a matrix $X_\gamma$, chosen independently and uniformly at random from $\mathbb{F}^{m \times m}$ for some finite field $\mathbb{F}$, and $m \in \mathbb{N}$. The master node aims to know the result of a function $Y = G(X_1, X_2, \ldots, X_\Gamma)$, where $G : (\mathbb{F}^{m \times m})^\Gamma \rightarrow \mathbb{F}^{m \times m}$ is an arbitrary polynomial function. The system operates in three phases: (i) sharing, (ii) computation and communication, and (iii) reconstruction. The detailed description of these phases is as follows.

1) Sharing: In this phase, the source $\gamma$ will send $X_\gamma \rightarrow n = F_{\gamma \rightarrow n}(X_\gamma)$ to worker $n$ where $F_{\gamma \rightarrow n} : \mathbb{F}^{m \times m} \rightarrow \mathbb{F}^{m \times \frac{m}{k}}$, for some $k \in \mathbb{N}$, $k|m$, denotes the sharing function, used at source $\gamma \in [\Gamma]$, to share data with worker $n \in [N]$. The number $k$ represents the limit on the storage size at each worker.

2) Computation and Communication: In this phase, the workers will process what they received, and in between, they may send some messages to other workers and continue processing. We define the set $\mathcal{M}_{n \rightarrow n'} \in \mathbb{F}^*$ as the vector of all messages that worker $n$ sends to the worker $n'$ in this phase, for $n, n' \in [N]$.

3) Reconstruction: In this phase, every worker will send a message to the master node. More precisely, worker $n$ will send the message $Y_n \in \mathbb{F}^*$ to the master.

This scheme must satisfy three constraints.

1) Correctness: The master must be able to recover $Y$ from $Y_1, Y_2, \ldots, Y_N$. More precisely

$$H(Y|Y_1, Y_2, \ldots, Y_N) = 0.$$  \hspace{1cm} (1)

2) Privacy for workers: Let $t \in [N]$. Any arbitrary subset including $t - 1$ workers, must not gain any information about the inputs. In particular, for any $S \subset [N], |S| \leq t - 1$

$$H(X_j, j \in [\Gamma] | \bigcup_{n \in S} \{\mathcal{M}_{n \rightarrow n}, n' \in [N]\}, \tilde{X}_\gamma, \gamma \in [\Gamma], n \in S) = H(X_j, j \in [\Gamma]).$$  \hspace{1cm} (2)

$t$ is called the security threshold of the system.

3) Privacy for the master: The master must not gain any additional information about the inputs, beyond the result of the function. In other word, $Y$ is the only new information that is revealed. More precisely

$$H(X_1, X_2, \ldots, X_\Gamma | Y, Y_1, Y_2, \ldots, Y_N) = H(X_1, X_2, \ldots, X_\Gamma | Y).$$  \hspace{1cm} (3)

$\mathbb{F}^*$ means any matrix with any possible size which arrays belong to $\mathbb{F}$.
**Definition 1.** For some $k \in \mathbb{N}$ and $t \in \mathbb{N}$, and polynomial function $G$, we define $N_G^*(t, k)$ as the minimum number of workers needed to calculate $G$, while correctness, privacy of the workers, and privacy at the master are satisfied.

Figure 1. An MPC system including $\Gamma = 3$ private inputs $X_1, X_2$ and $X_3 \in \mathbb{F}^{m \times m}$, $N = 9$ workers, and a master node. All communication links are secure and error free. The size of the shares given to each worker is a fraction of the size of the worker. Input node $\gamma$ sends $\tilde{X}\gamma_n \in \mathbb{F}^{m \times m}$ to worker $n$, for some $k \in \mathbb{N}, k \mid m$. Workers will process their inputs while interacting with each other. Finally worker $n$ sends $Y_n$ to the master node. The master node aims to know the result of a function e.g. $G(X_1, X_2, X_3) = X^T_1 X_2 + X_3$, subject to the correctness (1) and privacy (3), (2).

III. Preliminaries: $k = 1$

As described in Section II, we have three constraints: limited storage size at each worker node, correctness, and privacy. If there is no storage limit at the workers, i.e., $k = 1$, the problem is reduced to a version of secure multi-party computation, which has been extensively studied in the literature. In particular, in [4], Ben-Or, Goldwasser, and Wigderson propose a scheme referred as BGW for $k = 1$. As we will see having the limit of $k > 1$ will drastically change the problem. To be self contained, here we briefly describe BGW scheme in several Examples.

**Example 1 (BGW scheme for addition).** Assume that we have two sources 1 and 2 with private inputs $X_1 = A \in \mathbb{F}^{m \times m}$ and $X_2 = B \in \mathbb{F}^{m \times m}$, respectively. These sources will share their inputs with the workers. There are at most $t - 1$ semi-honest workers among the workers, where $t \in [N]$.

BGW protocol for this problem is as follows:

1) **Phase 1 - Sharing:**

First phase of BGW scheme is based on Shamir secret sharing [8]. Source 1 develops the polynomial

$$F_A(x) = A + A_1 x + \cdots + A_{t-1} x^{t-1},$$

where $F_A(0) = A$ is the private input at source 1 and the other coefficients, $A_i$, $i \in \{1, 2, \ldots t - 1\}$, are chosen from $\mathbb{F}^{m \times m}$ independently and uniformly at random. Source 1 uses $F_A(x)$ to share $A$ with the workers. This means it will send $F_A(\alpha_n)$ to worker $n$, for some distinct $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F}$. We know from the Lagrange interpolation rule that if we have $t$ points from $F_A(x)$ we can determine this polynomial [27]. Therefore, any subset of including $t$ workers collaboratively can reconstruct $F_A(0) = A$. Intuitively, the reason is that we need $t$ unknown variables for determining $F_A(x)$ and so we need at least $t$ equations. Since the coefficients of the $F_A(x)$, i.e., $A_i, i \in \{1, 2, \ldots t - 1\}$, are chosen in $\mathbb{F}^{m \times m}$ independently and uniformly at random, so any subset of including less than $t$ parties cannot reconstruct $A$ and indeed gain no information about it. For formal proof see [8].
This approach is called \((N, t)\) sharing of \(A\) (or interchangeably \((N, t)\) Shamir secret sharing of \(A\)) and \(F_A(\alpha_n)\) is called the share of \(A\) to worker \(n\).

Similarly source 2 develops \(F_B(x)\) according to the following equation and share its secrets among the workers.

\[
F_B(x) = B + B_1x + \cdots + B_{t-1}x^{t-1},
\]

where \(F_B(0) = B\) is private input at source 2 and \(B_i, \ i \in \{1, 2, \ldots t - 1\}\), are chosen in \(\mathbb{F}^{m \times m}\) independently and uniformly at random. Source 2 shares \(F_B(\alpha_n)\) to worker \(n\).

2) Phase 2 - Computation and Communication:

Shamir secret sharing scheme has the linearity property, which is very important. It means that if we have the shared secrets \(A\) and \(B\) using \(F_A(x)\) and \(F_B(x)\) among \(N\) workers, in order to share the secret \(pA + qB\) among the parties, for some constants \(p, q \in \mathbb{F}\), we just need that worker \(n\) locally calculates \(pF_A(\alpha_n) + qF_B(\alpha_n)\). This implies that the share of \(pA + qB\) are available at the workers using random polynomial \(pF_A(x) + qF_B(x)\). For this problem its sufficient to choose \(p = q = 1\). Note that in this particular example, no communication among nodes are needed.

3) Phase 3 - Reconstruction:

In this phase worker \(n\) will send \(Y_n = F_A(\alpha_n) + F_B(\alpha_n)\), which has been calculated in phase two, to the master node. If the master has \(F_A(\alpha_n) + F_B(\alpha_n)\) for \(t\) or more distinct \(\alpha_n\)’s it can recover all coefficients of degree \(t - 1\) polynomial \(F_A(x) + F_B(x)\). In particular, it will recover \(Y = F_A(0) + F_B(0) = A + B\). It is also easy to verify that both privacy constraints are satisfied.

Example 2 (BGW scheme for multiplication). Assume that we have two sources 1 and 2 with private inputs \(X_1 = A \in \mathbb{F}^{m \times m}\) and \(X_2 = B \in \mathbb{F}^{m \times m}\), respectively. In addition, the master aims to know \(Y = A^T B\). There are \(t - 1\) semi-honest adversaries among the \(N\) workers.

BGW protocol for this problem operates as follows:

1) Phase 1 - Sharing:

This phase is exactly the same as the first phase of Example 1. Therefore, at the end of this phase worker \(n\) has \(F_A(\alpha_n)\) and \(F_B(\alpha_n)\) for some distinct \(\alpha_n\), where \(F_A(x)\) and \(F_B(x)\) are defined in (4) and (5), respectively.

2) Phase 2 - Computation and Communication:

In this phase, worker \(n\) calculates \(F_A^T(\alpha_n)F_B(\alpha_n)\), simply by multiplying his shares of \(A\) and \(B\). Let us define the polynomial \(H(x) = F_A^T(x)F_B(x)\). We note that \(H(0) = A^T B\) and \(\text{deg}(H(x)) = 2t - 2\). Therefore, by having at least \(2t - 1\) samples of this polynomial, \(A^T B\) can be solved for. Therefore, if \(N \geq 2t - 1\), and these workers send their result to the master, it can calculate \(A^T B\). However, this will violate privacy at the master [5]. The reason is that by having \(H(\alpha_n) = F_A^T(\alpha_n)F_B(\alpha_n)\) for \(2t - 1\) distinct \(\alpha_n\)’s, the master can recover \(H(x)\), and then by factorizing it, it can recover \(F_A(x)\) and \(F_B(x)\), and thus \(A\) and \(B\).

The idea of BGW protocol to solve this challenge is as follows. We know if \(N \geq 2t - 1\), there exists a vector \(r = (r_1, r_2, \ldots, r_N)\) such that

\[
H(0) = A^T B = \sum_{n=1}^{N} r_n H(\alpha_n).
\]

Now worker \(n\) will share \(H(\alpha_n) = F_A^T(\alpha_n)F_B(\alpha_n)\) with other workers using Shamir secret sharing. In other words, worker \(n\) forms a polynomial of degree \(t - 1\),

\[
F_n(x) = H(\alpha_n) + H^{(1)}_1 x + H^{(2)}_2 x^2 + \cdots + H^{(t-1)}_{t-1} x^{t-1},
\]
where $H_i^{(n)}$, $i \in [t-1]$, $n \in [N]$, are chosen independently and uniformly at random in $\mathbb{F}^{m \times m}$. Worker $n$ will send to worker $n'$ the value of $F_n(\alpha_n')$, for $n, n' \in [N]$. Then each worker $n$ calculates $\sum_{n'=1}^N r_{n'} F_{n'}(\alpha_n)$. Let us define $F(x)$ as

$$F(x) \triangleq \sum_{n'=1}^N r_{n'} F_{n'}(x) = \sum_{n'=1}^N r_{n'} H(\alpha_{n'}) + x \sum_{n'=1}^N r_{n'} H_1^{(n')} + \cdots + x^{t-1} \sum_{n'=1}^N r_{n'} H_{t-1}^{(n')}.$$

(7)

Then, we have the following observations:

(i) Due to (6) $F(0) = \sum_{n'=1}^N r_{n'} H(\alpha_{n'}) = A^T B$.

(ii) Worker $n$ has access to $F(\alpha_n) = \sum_{n'=1}^N r_{n'} F_{n'}(\alpha_n)$.

(iii) $\sum_{n'=1}^N r_{n'} H_i^{(n')}, i \in [t-1]$, are independent with uniform distribution in $\mathbb{F}^{m \times m}$.

Thus $F(\alpha_n)$ is indeed a Shamir share of $A^T B$. Thus if $t$ of the workers send $F(\alpha_n)$ to the master, it can recover $F(0) = A^T B$.

3) Phase 3 - Reconstruction:

Each worker $n$ sends $F(\alpha_n)$ to the master. It can recover $F(0) = A^T B$, if it has $F(\alpha_n)$ from $t$ workers.

We note that the master can also recover $\sum_{n'=1}^N r_{n'} H_i^{(n')}, i \in [t-1]$, which reveal no information about $A$ and $B$. Therefore, the privacy at the master is guaranteed. In addition, whatever is shared with a worker is based on Shamir secret sharing with new random coefficients. This can be used to prove that the privacy at the workers is guaranteed.

This example shows that if the number of workers is $N \geq 2t-1$, the system can calculate multiplication.

We note that after Phase 2 of Example 2 each worker has the share of $A^T B$. Now let us assume that the target is to calculate $A^T B + C$, for some $A, B, C \in \mathbb{F}^{m \times m}$. Then, it is enough that each worker adds share of $A^T B$ with share of $C$. Following this idea, we can use BGW algorithm to calculate any polynomials of the inputs, with at least $2t-1$ workers. For formal description see [4].

Consider the condition where there is a storage limit for each worker node, i.e., $k > 1$. One approach to deal with this case is to split the job into smaller job and use BGW scheme for each sub-job, as we see in the following example.

Example 3 (Concatenation of Job Splitting and BGW, $k = 2$). Here, we revisit Example 2 but here, we assume that $k = 2$. We partition each matrix into two sub-matrices as follows:

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

(8)

(9)

where $A_i, B_i \in \mathbb{F}^{m \times \frac{m}{2}}$, for $i \in \{1, 2\}$.

We note that

$$A^T B = \begin{bmatrix} A_1^T B_1 & A_1^T B_2 \\ A_2^T B_1 & A_2^T B_2 \end{bmatrix}.$$

Therefore, to calculate $A^T B$, we can use BGW scheme with four groups of workers to calculate $A_i^T B_j, i, j \in \{1, 2\}$, each of them with at least $2t-1$ workers, following Example 2. Therefore, using this scheme, we need $N \geq 4(2t-1)$ workers.

One can see that with concatenation of job-splitting plus BGW described in Example 3 the minimum number of workers needed to calculate addition and multiplication of two matrices is $N = kt$ and $N = k^2(2t-1)$, respectively. In this paper, we propose an algorithm which reduces the number of workers significantly, as stated in the next sections.
IV. MAIN RESULT

The main result of this paper is as follows:

**Theorem 1.** For any $k, t \in \mathbb{N}$ and any polynomial function $G$, 

$$N_G^*(t, k) \leq \min\{2k^2 + 2t - 3, k^2 + kt + t - 2\}.$$ 

*Remark 1.* To prove the Theorem 1 we propose a scheme which is based on the novel approach for sharing called polynomial sharing and some procedures to calculate any basic function such as addition and multiplication.

*Remark 2.* Recall that concatenation of job splitting and MPC for multiplication needs $k^2(2t - 1)$ workers. However in the proposed scheme we can do multiplication with at most $\min\{2k^2 + 2t - 3, k^2 + kt + t - 2\}$ workers, which is an orderwise improvement. For Example for $t = 200$ and $k = 16$, the proposed scheme needs $N = 909$ workers, while the job splitting approach needs $N = 102144$ workers.

V. MOTIVATING EXAMPLE

Here, we revisit Example 3 with $k = 2$ and $t = 4$ and propose a solution that needs 13 workers, as compared to $4 \times 7 = 28$ workers in solution of Example 3.

**Example 4 (Proposed Approach for $k = 2, t = 4$).** In this Example we aim to show how to use a new approach to securely calculate $L = A^T B$.

1) **Phase 1 - Sharing:**

Consider the following polynomial matrices

$$F_A(x) = A_1 + A_2x + A_3x^4 + A_4x^5 + A_5x^6,$$

$$F_B(x) = B_1 + B_2x^2 + B_3x^4 + B_4x^5 + B_5x^6,$$

where in the above equations $A_1, A_2, B_1, 2$ and $B_2$ are defined in (8) and (9) in Example 3. In addition, $A_i, B_i$, for $i \in [3, 5]$, are matrices chosen independently and uniformly at random in $\mathbb{F}^{m \times \frac{m}{2}}$. As one can see these polynomials follow certain patterns. More precisely in these polynomials the coefficients of some powers of $x$ are zero. In particular in $F_A(x)$ the coefficients of $x^2$ and $x^3$ are zero and in the $F_B(x)$ the coefficients of $x$ and $x^3$ are zero. We choose $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F}$, independently and uniformly at random. Source nodes 1 and 2 will share $F_A(\alpha_n)$ and $F_B(\alpha_n)$ with worker $n$, respectively. We call this form of sharing as polynomial sharing.

2) **Phase 2 - Computation and Communication:**

Worker $n$ calculates $F_A^T(\alpha_n)F_B(\alpha_n)$. Consider the polynomial matrix $H(x)$ of degree 12, defined as,

$$H(x) = \sum_{n=0}^{12} H_n x^n \triangleq F_A^T(x)F_B(x).$$

(10)

We note that

$$H_0 = A_1^TB_1,$$

$$H_1 = A_2^TB_1,$$

$$H_2 = A_3^TB_2,$$

$$H_3 = A_4^TB_2.$$ 

(11)

If the master has $H(\alpha_n)$ for at least 13 distinct $\alpha_i$, then with probability approaching to one, as $|\mathbb{F}| \to \infty$, we can calculate all the coefficients of $H(x)$, including $H_0 = A_1^TB_1$, $H_1 = A_2^TB_1$, $H_2 = A_3^TB_2$, and $H_3 = A_4^TB_2$. 

In particular, there are some \( r_n^{(i,j)} \), \( i, j \in \{1, 2\} \) and \( n \in [N] \), such that
\[
A_i^T B_j = \sum_{n=1}^{N} r_n^{(i,j)} H(\alpha_n). \tag{12}
\]

**Remark 3:** Note that \( r_n^{(i,j)} \), \( i, j \in \{1, 2\} \) and \( n \in [N] \) are only function of \( \alpha_n \), \( n \in [N] \), and is available everywhere.

We note that we can not share samples of \( F_A^T(x) F_B(x) \) with the master directly, otherwise the privacy constraint at the master will be violated. This is because the master can factorize \( H(x) \), and from there, it can solve for \( A \) and \( B \).

To ensure privacy at the master, workers do some communication in order to prepare the scene for the next stage and cast the result of the local computation in the form of polynomial sharing. Worker \( n \) forms \( Q^{(n)}(x) \), defined as
\[
Q^{(n)}(x) \triangleq \left[ \begin{array}{c} r_n^{(1,1)} H(\alpha_n) \\ r_n^{(2,1)} H(\alpha_n) \end{array} \right] + x^2 \left[ \begin{array}{c} r_n^{(1,2)} H(\alpha_n) \\ r_n^{(2,2)} H(\alpha_n) \end{array} \right] + R_0^{(n)} x^4 + R_1^{(n)} x^5 + R_2^{(n)} x^6, \tag{13}
\]
where \( R_i^{(n)} \), \( i \in \{0, 1, 2\} \), are chosen independently and uniformly at random in \( \mathbb{F}^{m \times m} \). Recall that \( H(\alpha_n), r_n^{(1,1)}, r_n^{(1,2)}, r_n^{(2,1)} \), and \( r_n^{(2,2)} \) are available at worker \( n \). Thus worker \( n \) has all the information to form \( Q^{(n)}(x) \). Then worker \( n \) sends \( Q^{(n)}(\alpha_{n'}) \) to worker \( n', n' \in [N] \).

In the end, worker \( n' \) has access to the matrices \( \{ Q^{(1)}(\alpha_{n'}), Q^{(2)}(\alpha_{n'}), \ldots, Q^{(N)}(\alpha_{n'}) \} \). Now worker \( n' \) will calculate \( \sum_{n=1}^{13} Q^{(n)}(\alpha_{n'}) \). Consider the polynomial matrix
\[
Q(x) \triangleq \sum_{n=1}^{N} Q^{(n)}(x). \tag{14}
\]
Thus worker \( n' \) has access to \( Q(\alpha_{n'}) \).

We note that
\[
Q(x) \triangleq \sum_{n=1}^{N} Q^{(n)}(x) = \sum_{n=1}^{N} \left[ \begin{array}{c} r_n^{(1,1)} H(\alpha_n) \\ r_n^{(2,1)} H(\alpha_n) \end{array} \right] + x^2 \sum_{n=1}^{N} \left[ \begin{array}{c} r_n^{(1,2)} H(\alpha_n) \\ r_n^{(2,2)} H(\alpha_n) \end{array} \right] + x^4 \sum_{n=1}^{N} R_0^{(n)} + x^5 \sum_{n=1}^{N} R_1^{(n)} + x^6 \sum_{n=1}^{N} R_2^{(n)}
\]
\[
= \left[ A_1^T B_1 \right] + x^2 \left[ A_1^T B_2 \right] + x^4 \sum_{n=1}^{N} R_0^{(n)} + x^5 \sum_{n=1}^{N} R_1^{(n)} + x^6 \sum_{n=1}^{N} R_2^{(n)}
\]
\[
= L_1 + L_2 x^2 + L_4 x^4 + L_5 x^5 + L_6 x^6, \tag{15}
\]
where (a) follows from (12), and \( L_1 \triangleq \left[ A_1^T B_1 \right], L_2 \triangleq \left[ A_1^T B_2 \right], L_4 \triangleq \sum_{n=1}^{N} R_0^{(n)}, L_5 \triangleq \sum_{n=1}^{N} R_1^{(n)}, \)
and \( L_6 \triangleq \sum_{n=1}^{N} R_2^{(n)} \). Note that, \( L_i, i \in \{4, 5, 6\} \) have independent and uniform distribution in \( \mathbb{F}^{m \times m} \). Because of the randomness injected in \( R(x) \) and its degree, one can easily verify that the \( Q(x) \) is in the form of our desired polynomial matrix and worker \( n \) has access to \( Q(\alpha_n) \).

3) **Phase 3 - Reconstruction:**
Worker \( n \) will send \( Q(\alpha_n) \) to the master, \( n \in [N] \). The master then calculates \( A^T B \) from what it receives by polynomial interpolation.

\[\diamondsuit\]
This algorithm guarantees the privacy at the workers. An intuitive explanation is that in each interaction with the workers, the algorithm uses some independent random matrices, as the coefficients of the sharing polynomials, such that any subset, including 3 workers cannot gain any additional information. This will be proven formally later in Appendix B for the general case. Also there is no information leakage at the master.

Now let us consider the case, where the objective is to calculate $C^T A^T B$. We note that at the end of above algorithm, the share of $A^T B$ is available at each worker, and thus we can follow the same algorithm to multiply $C^T$ to $A^T B$. In general, we can calculate any polynomial function of the inputs.

VI. POLYNOMIAL SHARING

This scheme is motivated by [13], which is a coding technique for matrix multiplication in the distributed system with stragglers.

Definition 2. Let $A \in \mathbb{F}^{m \times m}$, partitioned as

$$ A \triangleq [A_1 \ A_2 \ldots \ A_k], \quad (16) $$

where $A_i \in \mathbb{F}^{m \times \frac{m}{k}}$, for some $k \in \mathbb{N}$ and $k|m$. The polynomial matrix $F_{A,b,t,k}(x)$, for some $b \in [k]$, is defined as

$$ F_{A,b,t,k}(x) \triangleq \sum_{n=1}^{k} A_n x^{b(n-1)} + \sum_{n=1}^{t-1} A_{k+n} x^{k^n+n-1}, \quad (17) $$

where $A_i, i = 1, 2, \ldots, k$ are defined in (16) and $A_i, i = k+1, k+2, \ldots, k+t-1$, are chosen independently and uniformly at random from $\mathbb{F}^{m \times \frac{m}{k}}$. We say that matrix $A$ is $(b, t, k)$ polynomial-shared with workers in $[N]$, if $F_{A,b,t,k}(\alpha_n)$ is sent to worker $n, n \in [N]$, where $\alpha_n \in \mathbb{F}$ are distinct constants assigned to worker $n, n \in [N]$.

Theorem 2. Let $t, k, m \in \mathbb{N}$, $N \geq \min\{2k^2 + 2t - 3, k^2 + tk + t - 2\}$, $k|m$, and $A, B \in \mathbb{F}^{m \times m}$. Define

$$ H(x) \triangleq F_{A,1,t,k}(x)F_{B,k,t,k}(x), $$

then for some large enough $|\mathbb{F}|$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F}$, such that for any $i_1, i_2, \ldots, i_{k+t-1} \in [N]$, we have

$$ H(A|F_{A,1,t,k}(\alpha_{i_1}), F_{A,1,t,k}(\alpha_{i_2}), \ldots, F_{A,1,t,k}(\alpha_{i_{k+t-1}})) = 0, \quad (18) $$

$$ H(B|F_{B,k,t,k}(\alpha_{i_1}), F_{B,k,t,k}(\alpha_{i_2}), \ldots, F_{B,k,t,k}(\alpha_{i_{k+t-1}})) = 0, \quad (19) $$

and

$$ H(A^T B|H(\alpha_1), H(\alpha_2), \ldots, H(\alpha_N)) = 0. \quad (20) $$

In addition, if we choose $\alpha_1, \alpha_2, \ldots, \alpha_N$, independently and uniformly at random in $\mathbb{F}$, the probability of (18), (19), and (20) approaches to one, as $|\mathbb{F}| \to \infty$.

Proof. See Appendix A

VII. PROCEDURES

In this section, we explain several procedures to do basic operations such as addition, multiplication, and transposing, using polynomial sharing without leaking any information. This procedures will allow us to calculate any polynomial function of the input, subject to the constraints (1), (2), and (3).
A. Addition

Let \( A, B \in \mathbb{F}^{m \times m} \)

\[
A = \begin{bmatrix} A_1 & A_2 & \ldots & A_k \end{bmatrix},
\]

\[
B = \begin{bmatrix} B_1 & B_2 & \ldots & B_k \end{bmatrix},
\]

where \( A_i, B_i \in \mathbb{F}^{m \times \frac{m}{k}} \), for \( i \in [k] \) and \( k|m \). In addition assume \( A \) and \( B \) are \((b, t, k)\) polynomial-shared with workers in \([N]\) for some \( k \in \mathbb{N} \). The objective is to \((b, t, k)\) polynomial-share \( L \triangleq A + B \) with workers in \([N]\). We follow a one-phase procedure.

1) Computation phase:

In this phase, Worker \( n \) calculates \( F_{A,b,t,k}(\alpha_n) + F_{B,b,t,k}(\alpha_n) \).

Let us define \( Q(x) \) as

\[
Q(x) \triangleq F_{A,b,t,k}(x) + F_{B,b,t,k}(x).
\]

Thus worker \( n' \) has access to \( Q(\alpha_{n'}) \). One can see that

\[
Q(x) = F_{A,b,t,k}(x) + F_{B,b,t,k}(x)
\]

\[
= \sum_{j=1}^{k} A_j x^{b(j-1)} + \sum_{j=1}^{t-1} A_{k+j} x^{k^2+j-1} + \sum_{j=1}^{k} B_j x^{b(j-1)} + \sum_{j=1}^{t-1} B_{k+j} x^{k^2+j-1}
\]

\[
= \sum_{j=1}^{k} (A_j + B_j) x^{b(j-1)} + \sum_{j=1}^{t-1} (A_{k+j} + B_{k+j}) x^{k^2+j-1}
\]

\[
= \sum_{j=1}^{k} L_j x^{b(j-1)} + \sum_{j=1}^{t-1} L_{k+j} x^{k^2+j-1},
\]

where \( L_j \) defined as \( L_j \triangleq A_j + B_j \) for \( j \in [k] \), and \( L_{j+k} \triangleq A_{j+k} + B_{j+k} \), for \( j \in [t-1] \). We note that \( L = [L_1 \ L_2 \ \ldots \ L_k] \). In addition \( L_{k+j}, j \in [t-1] \) have independent and uniform distribution in \( \mathbb{F}^{m \times \frac{m}{k}} \). Thus, one can easily verify that the \( Q(x) \) is in the form of our desired polynomial matrix required for \((b, t, k)\) polynomial-sharing of \( A + B \) with workers in \([N]\) and worker \( n \) has access to \( Q(\alpha_n) \). These phases are detailed in Algorithm 1.

Algorithm 1 Addition

\begin{enumerate}
\item Inputs: Matrices \( A, B \in \mathbb{F}^{m \times m} \) which are \((b, t, k)\) polynomial-shared with workers in \([N]\).
\item Worker \( n \) adds his shares of \( A \) and \( B \).
\item End.
\end{enumerate}

B. Multiplication by Constant

Let \( A \in \mathbb{F}^{m \times m} \), and

\[
A = \begin{bmatrix} A_1 & A_2 & \ldots & A_k \end{bmatrix},
\]

for \( A_i \in \mathbb{F}^{m \times \frac{m}{k}} \), for \( i \in [k] \) and \( k|m \), be \((b, t, k)\) polynomial-shared with workers in \([N]\). The objective is to \((b, t, k)\) polynomial-share \( L \triangleq qA \) with workers in \([N]\), where \( q \) is a constant in \( \mathbb{F} \).

One can see that if each worker locally multiplies his shares of \( A \) by \( q \), matrix \( L \) is \((b, t, k)\) polynomial-shared with workers in \([N]\). More precisely Consider the polynomial matrix

\[
Q(x) \triangleq qF_{A,b,t,k}(x).
\]

(23)
Thus worker \( n' \) has access to \( Q(\alpha_{n'}) \). One can see that
\[
Q(x) = qF_{A,b,t,k}(x)
\]
\[
= q \sum_{j=1}^{k} A_j x^{b(j-1)} + q \sum_{j=1}^{t-1} A_{k+j} x^{k^2+j-1}
\]
\[
= \sum_{j=1}^{k} qA_j x^{b(j-1)} + \sum_{j=1}^{t-1} qA_{k+j} x^{k^2+j-1}
\]
\[
= (a) \sum_{j=1}^{k} L_j x^{b(j-1)} + \sum_{j=1}^{t-1} L_{k+j} x^{k^2+j-1},
\]
where (a) follows from \( L = qA \), and \( L_j \triangleq qA_j \) for \( j \in [k] \), and \( L_{j+k} \triangleq qA_{j+k} \), for \( j \in [t-1] \). In addition, \( L_{k+j}, j \in [t-1] \) have independent and uniform distribution in \( \mathbb{F}_q^{m \times m} \). Thus, one can easily verify that \( Q(x) \) is in the form of our desired polynomial matrix and worker \( n \) has access to \( Q(\alpha_n) \). This step is detailed in Algorithm 2.

**Algorithm 2 Multiplication by Constant**

**Inputs:** Matrix \( A \in \mathbb{F}^{m \times m} \) which is \((b, t, k)\) polynomial-shared with workers in \([N]\), and \( q \in \mathbb{F} \).

1. Worker \( n \) multiplies its share of \( A \) by \( q \).

2. End.

**Remark 4:** Polynomial sharing scheme has the linearity property. More precisely, if matrices \( A \) and \( B \) are \((b, t, k)\) polynomial-shared with workers in \([N]\), in order to \((b, t, k)\) polynomial-share \( L = qA + pB \) with workers in \([N]\), it’s enough that each worker just locally calculates the same computation on its shares.

**C. Multiplication of Two Matrices**

Similar to the Shamir secret sharing scheme, calculating the shares of the multiplication of two matrices is not as simple as calculating the addition. We need to do some communication. In what follows, we explain the procedure.

Let us assume that \( A, B \in \mathbb{F}^{m \times m} \)
\[
A = \begin{bmatrix} A_1 & A_2 & \ldots & A_k \end{bmatrix},
\]
\[
B = \begin{bmatrix} B_1 & B_2 & \ldots & B_k \end{bmatrix},
\]
where \( A_i, B_i \in \mathbb{F}^{m \times m} \), for \( i \in [k] \) and \( k|m \), are \((1, t, k)\) and \((k, t, k)\) polynomial-shared with workers in \([N]\), respectively. The goal is to \((b, t, k)\) polynomial-share \( L \triangleq A^T B \) with workers in \([N]\). We follow three phases including computation, communication and aggregation.

1) **Phase I - Computation:**

Worker \( n \) calculates \( F_{A,1,t,k}(\alpha_n)F_{B,k,t,k}(\alpha_n) \).

Consider the polynomial matrix \( H(x) \) of degree \( 2(k^2 + t - 2) \), defined as,
\[
H(x) = \sum_{n=0}^{2(k^2+t-2)} H_n x^n \triangleq F_{A,1,t,k}^T(x)F_{B,k,t,k}(x).
\]

We note that
\[
H_{i-1+k(j-1)} = A_j^T B_j,
\]
for \( i, j \in [k] \).

According to (20), if \( N \geq \min\{2k^2 + 2t - 3, k^2 + tk + t - 2\} \), then with probability approaching to one, as \( |\mathbb{F}| \to \infty \), we can calculate all the coefficients of \( H(x) \), including \( H_{i-1+k(j-1)} = A_i^T B_j \), for \( i, j \in [k] \), from \( H(\alpha_n), n \in [N] \). In particular, there are some \( r_n^{(i,j)} \), \( i, j \in [k] \) and \( n \in [N] \), such that

\[
A_i^T B_j = \sum_{n=1}^{N} r_n^{(i,j)} H(\alpha_n). \tag{27}
\]

**Remark 5:** Note that \( r_n^{(i,j)} \), \( i, j \in [k] \) and \( n \in [N] \) are only function of \( \alpha_n, n \in [N] \), and is available everywhere.

Up to now worker \( n \) has access to \( H(\alpha_n) \). The challenge is to find a way to change the local knowledge of the \( H(\alpha_n) \) to \((b, t, k)\) polynomial-share of \( L \), for each worker \( n \).

2) **Phase 2 - Communication:**

Worker \( n \) forms the matrix \( H^{(n)} \), defined as

\[
H^{(n)} \triangleq \begin{bmatrix}
H(\alpha_n)r_n^{(1,1)} & H(\alpha_n)r_n^{(1,2)} & \ldots & H(\alpha_n)r_n^{(1,k)} \\
H(\alpha_n)r_n^{(2,1)} & H(\alpha_n)r_n^{(2,2)} & \ldots & H(\alpha_n)r_n^{(2,k)} \\
\vdots & \vdots & \ddots & \vdots \\
H(\alpha_n)r_n^{(k,1)} & H(\alpha_n)r_n^{(k,2)} & \ldots & H(\alpha_n)r_n^{(k,k)}
\end{bmatrix}. \tag{28}
\]

Then worker \( n \), \((b, t, k)\) polynomial-shares \( H^{(n)} \) with workers in \([N]\). Worker \( n \) forms the following polynomial,

\[
F_{H^{(n)}, b, t, k}(x) = \sum_{j=1}^{k} H_j^{(n)} x^{b(j-1)} + \sum_{j=1}^{t-1} H_{k+j}^{(n)} x^{2j+1},
\]

where \( H_j^{(n)} \triangleq \begin{bmatrix}
H(\alpha_n)r_n^{(1,j)} \\
H(\alpha_n)r_n^{(2,j)} \\
\vdots \\
H(\alpha_n)r_n^{(k,j)}
\end{bmatrix} \), for \( j \in [k] \) and \( H_{k+j}^{(n)} \), \( j = 1, 2, \ldots, t - 1 \), are chosen independently and uniformly at random in \( \mathbb{F}^{m \times \frac{m}{k}} \). All workers follow the same algorithm. Thus, in the end worker \( n' \) has access to the matrices \( \{F_{H^{(1)}, b, t, k}(\alpha_{n'}), F_{H^{(2)}, b, t, k}(\alpha_{n'}), \ldots, F_{H^{(N)}, b, t, k}(\alpha_{n'})\} \).

3) **Phase 3 - Aggregation:**

Now worker \( n' \) calculates \( \sum_{n=1}^{N} F_{H^{(n)}, b, t, k}(\alpha_{n'}) \).

Consider the polynomial matrix

\[
Q(x) \triangleq \sum_{n=1}^{N} F_{H^{(n)}, b, t, k}(x). \tag{30}
\]

Thus worker \( n' \) has access to \( Q(\alpha_{n'}) \).
We note that

\[
Q(x) = \sum_{n=1}^{N} F_{H(n),b,t,k}(x)
\]

\[
= \sum_{n=1}^{N} \sum_{j=1}^{k} H_j^{(n)} x^{b(j-1)} + \sum_{n=1}^{N} \sum_{j=1}^{t-1} H_{k+j}^{(n)} x^{k+j-1}
\]

\[
= \sum_{n=1}^{N} \sum_{j=1}^{k} \left[ \begin{array}{c}
H(\alpha_n)r_n^{(1,j)} \\
H(\alpha_n)r_n^{(2,j)} \\
\vdots \\
H(\alpha_n)r_n^{(k,j)}
\end{array} \right] x^{b(j-1)} + \sum_{n=1}^{N} \sum_{j=1}^{t-1} H_{k+j}^{(n)} x^{k+j-1}
\]

\[
= 1 + \sum_{n=1}^{N} \sum_{j=1}^{t-1} H_{k+j}^{(n)} x^{k+j-1}
\]

(31)

where (a) follows from (27), and (b) follow from definition \( L_j \triangleq \left[ \begin{array}{c}
A_1^T B_j \\
A_2^T B_j \\
\vdots \\
A_k^T B_j
\end{array} \right] \) for \( j \in [k] \), and \( L_{j+k} \triangleq \sum_{n=1}^{N} H_{k+j}^{(n)} x^{k+j-1} \).

\[\sum_{n=1}^{N} H_{k+j}^{(n)} \text{ for } j \in [t-1]. \]

Note that, \( L_{k+j} \) for \( j \in [t-1] \) have independent and uniform distribution in \( \mathbb{R}^{m \times \frac{m}{k}} \). Thus, one can easily verify that the \( Q(x) \) is in the form of the desired polynomial matrix required to \((b, t, k)\) polynomial-sharing of \( L = A^T B \) with workers in \([N]\) and worker \( n \) has access to \( Q(\alpha_n) \). These phases are detailed in Algorithm 3.

D. Transposing

Let \( A \in \mathbb{R}^{m \times m} \)

\[
A = \left[ \begin{array}{ccccc}
A_1 & A_2 & \ldots & A_k
\end{array} \right],
\]

where \( A_i \in \mathbb{R}^{m \times \frac{m}{k}} \), for \( i \in [k] \) and \( k|m \), is \((b, t, k)\) polynomial-shared with workers in \([N]\). The goal is to \((b, t, k)\) polynomial-share \( L \triangleq A^T \) with workers in \([N]\). We follow three phases including splitting, communication and aggregation.

1) Phase 1 - Splitting:

Define \( F_i(x) \triangleq F_{A,b,t,k}(\frac{m}{k}(i-1) : \frac{m}{k}i : \cdot) \).

(32)

\[\]
**Algorithm 3** Multiplication of Two Matrices

**Inputs:** Number $N$ of the workers. Matrices $A$, $B$ which are $(1, t, k)$ and $(k, t, k)$ polynomial-shared with workers in $[N]$, respectively. $r_n^{(i,j)}$, $i, j \in [k]$ and $n \in [N]$ are only function of $\alpha_n$, $n \in [N]$, and are available everywhere.

**Phase 1 - Computation:**
1. Worker $n$ calculates $H(\alpha_n) \triangleq F_{A,1,t,k}^T(\alpha_n)F_{B,k,t,k}(\alpha_n)$.

**Phase 2 - Communication:**
2. Worker $n$ will $(b, t, k)$ polynomial-shares the matrix

$$H^{(n)} \triangleq \begin{bmatrix} H(\alpha_n)r_n^{(1,1)} & H(\alpha_n)r_n^{(1,2)} & \ldots & H(\alpha_n)r_n^{(1,k)} \\ H(\alpha_n)r_n^{(2,1)} & H(\alpha_n)r_n^{(2,2)} & \ldots & H(\alpha_n)r_n^{(2,k)} \\ \vdots & \vdots & \ddots & \vdots \\ H(\alpha_n)r_n^{(k,1)} & H(\alpha_n)r_n^{(k,2)} & \ldots & H(\alpha_n)r_n^{(k,k)} \end{bmatrix},$$

with workers in $[N]$.

**Phase 3 - Aggregation:**
3. Worker $n$ calculates the sum of the messages received in the last phase.

4. End.

In other words, $F_i(x)$, $i \in [k]$, is a sub-matrix of $F_{A,b,t,k}(x)$, including rows form $\frac{m}{k}(i-1)$ to $\frac{m}{k}i$. One can see that we have

$$F_i(x) = \sum_{n=1}^{k} A_{i,n}x^{b(n-1)} + \sum_{n=1}^{t-1} A_{i,k+n}x^{k^2+n-1}, \quad (33)$$

where

$$A \triangleq \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ A_{21} & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \ldots & A_{kk} \end{bmatrix},$$

and $A_{i,j} \in \mathbb{F}^{m \times m}$, for $i, j \in [k + t - 1]$. Since each worker $n$ has access to $F_{A,b,t,k}(\alpha_n)$, then it has access to $F_i(\alpha_n)$, too. According to (19), if $N \geq (k^2 + t - 1)$, then by having $F_i(\alpha_n)$, for $n = 1, 2, \ldots, N$, with probability approaching to one, as $|\mathbb{F}| \rightarrow \infty$, we can calculate all the coefficients of $F_i(x)$, including $A_{i,n}$, for $i \in [k]$. In particular there are some $r_n^{(j)}$, such that for $i, j \in [k]$, and $n \in [N],$

$$A_{i,j} = \sum_{n=1}^{N} r_n^{(j)}F_i(\alpha_n). \quad (34)$$

**Remark 6:** Note that $r_n^{(j)}$, $j \in [k]$ and $n \in [N]$, are only function of $\alpha_n$, $n \in [N]$, and is available everywhere.

2) **Phase 2 - Communication:**
Worker $n$ forms $H^{(n)}$ defined as

$$H^{(n)} \triangleq \begin{bmatrix} F_1(\alpha_n)r_n^{(1)} & F_2(\alpha_n)r_n^{(1)} & \ldots & F_k(\alpha_n)r_n^{(1)} \\ F_1(\alpha_n)r_n^{(2)} & F_2(\alpha_n)r_n^{(2)} & \ldots & F_k(\alpha_n)r_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ F_1(\alpha_n)r_n^{(k)} & F_2(\alpha_n)r_n^{(k)} & \ldots & F_k(\alpha_n)r_n^{(k)} \end{bmatrix}. \quad (35)$$
3) Phase 3 - Aggregation:

Then worker \( n \), \((b, t, k)\) polynomial-shares \( H^{(n)} \) with workers in \([N]\). According to (17) one can see that

\[
F_{H^{(n)}, b, t, k}(x) = \sum_{j=1}^{k} H^{(n)}_j x^{b(j-1)} + \sum_{j=1}^{t-1} H^{(n)}_{k+j} x^{k^2+j-1},
\]

(36)

where \( H^{(n)}_j \triangleq \begin{bmatrix} F_j(\alpha_n) r_{n}^{(1)} \\ F_j(\alpha_n) r_{n}^{(2)} \\ \vdots \\ F_j(\alpha_n) r_{n}^{(k)} \end{bmatrix} \) for \( j \in [k] \), and \( H^{(n)}_{k+j}, j = 1, 2, \ldots, t - 1 \), are chosen independently and uniformly at random in \( \mathbb{F}^{m \times \frac{m}{t}} \). Thus, in the end worker \( n' \) has access to the matrices \( \{F_{H^{(1)}, b, t, k}(\alpha_{n'}), F_{H^{(2)}, b, t, k}(\alpha_{n'}), \ldots, F_{H^{(N)}, b, t, k}(\alpha_{n'})\} \).

3) Phase 3 - Aggregation:

Now worker \( n' \) will calculate \( \sum_{n=1}^{N} F_{H^{(n)}, b, t, k}(\alpha_{n'}) \) as follows. Consider the polynomial matrix

\[
Q(x) \triangleq \sum_{n=1}^{N} F_{H^{(n)}, b, t, k}(x).
\]

(37)

Thus worker \( n' \) can calculate \( Q(\alpha_{n'}) \).

We note that

\[
Q(x) = \sum_{n=1}^{N} F_{H^{(n)}, b, t, k}(x)
\]

\[
= \sum_{n=1}^{N} \sum_{i=1}^{k} A^{i}_i, \ldots, \sum_{i=1}^{k} A^{i}_k \\
= \sum_{i=1}^{k} \sum_{j=1}^{N} A^{i}_{j} x^{b(i-1)} + \sum_{j=1}^{N} H^{(n)}_{k+j} x^{k^2+j-1}
\]

(38)

where (a) follows from (34), and \( L_i \triangleq \begin{bmatrix} A_{i,1} \\ A_{i,2} \\ \vdots \\ A_{i,k} \end{bmatrix} \) for \( i \in [k] \), and \( L_{j+k} \triangleq \sum_{n=1}^{N} H^{(n)}_{k+j} x^{k^2+j-1} \), for \( j \in [t - 1] \). In addition, \( L_{k+j}, j \in [t - 1] \), have independent and uniform distribution in \( \mathbb{F}^{m \times \frac{m}{t}} \). Thus,
one can easily verify that the $Q(x)$ is in the form of our desired polynomial matrix and worker $n$ has access to $Q(\alpha_n)$.

These phases are detailed in Algorithm 4.

**Algorithm 4** Transposing

**Inputs:** Number $N$ of the workers. Matrix $A$ which is $(b, t, k)$ polynomial-shared with workers in $[N]$, $r_{n}^{(j)}$ for $j \in [k]$, and $n \in [N]$ are only function of $\alpha_n$, $n \in [N]$, and is available everywhere.

**Phase 1 - Splitting:**
1: worker $n$ calculates $F_{i}(\alpha_n) \triangleq F_{A,b,t,k}(\alpha_n)(\frac{m}{E} (i - 1) : \frac{m}{E} i:,)$, for $i \in [m]$.

**Phase 2 - Communication:**
2: Worker $(b, t, k)$ polynomial-shares the matrix

$$
H^{(n)} \triangleq \begin{bmatrix}
F_{1}(\alpha_n)r_{n}^{(1)} & F_{2}(\alpha_n)r_{n}^{(1)} & \cdots & F_{k}(\alpha_n)r_{n}^{(1)} \\
F_{1}(\alpha_n)r_{n}^{(2)} & F_{2}(\alpha_n)r_{n}^{(2)} & \cdots & F_{k}(\alpha_n)r_{n}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
F_{1}(\alpha_n)r_{n}^{(k)} & F_{2}(\alpha_n)r_{n}^{(k)} & \cdots & F_{k}(\alpha_n)r_{n}^{(k)}
\end{bmatrix},
$$

with workers in $[N]$.

**Phase 3 - Aggregation:**
3: Worker $n$ calculates the sum of the messages received in the last phase.
4: End.

**E. Changing the jump**

Let $A \in F^{m \times m}$

$$
A = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix},
$$

where $A_i \in F^{m \times m}$, for $i \in [k]$ and $k|m$, is $(b, t, k)$ polynomial-shared with workers in $[N]$. The goal is to $(b', t, k)$ polynomial-share $A$ with workers in $[N]$, where $b' \neq b$.

According to (19), if $N \geq (k + t - 1)$, then with probability approaching to one, as $|F| \to \infty$, we can calculate all the coefficients of $F_{A,b,t,k}(x)$, including $A_j$, for $j \in [k]$. In particular, there are some $r_{n}^{(j)}$, $j \in [k]$ and $n \in [N]$, such that

$$
A_j = \sum_{n=1}^{N} r_{n}^{(j)} F_{A,b,t,k}(\alpha_n). \quad (39)
$$

**Remark 7:** Note that $r_{n}^{(j)}$, $j \in [k]$ and $n \in [N]$, are only function of $\alpha_n$, $n \in [N]$, and is available everywhere.

1) **Phase 1 - Communication:**

Worker $n$ forms $H^{(n)}$ defined as

$$
H^{(n)} \triangleq \begin{bmatrix} r_{n}^{(1)} F_{A,b,t,k}(\alpha_n) & r_{n}^{(2)} F_{A,b,t,k}(\alpha_n) & \cdots & r_{n}^{(k)} F_{A,b,t,k}(\alpha_n) \end{bmatrix}. \quad (40)
$$

Then worker $n$, $(b', t, k)$ polynomial-shares $H^{(n)}$ with workers in $[N]$. According to (17) one can see that

$$
F_{H^{(n)},b',t,k}(x) = \sum_{j=1}^{k} H_{j}^{(n)} x^{b'(j-1)} + \sum_{j=1}^{t-1} H_{k+j}^{(n)} x^{k^2 + j - 1}, \quad (41)
$$
Algorithm 5 Changing the jump

Inputs: Number $N$ of the workers. Matrix $A$ which is $(b, t, k)$ polynomial-shared with workers in $[N]$, $r_n^{(j)}, j \in [k]$, and $n \in [N]$ are only function of $\alpha_n$, $n \in [N]$, and is available everywhere.

Phase 1 - Communication:
1: Worker $n$, $(b', t, k)$ polynomial-shares the matrix

$$H^{(n)}(x) \triangleq \begin{bmatrix} r_n^{(1)}F_{A,b,t,k}(\alpha_n) & r_n^{(2)}F_{A,b,t,k}(\alpha_n) & \ldots & r_n^{(k)}F_{A,b,t,k}(\alpha_n) \end{bmatrix},$$

with workers in $[N]$.  

Phase 2 - Aggregation:
2: Worker $n$ calculates the sum of the messages received in the last phase.
3: End.
VIII. PROPOSED ALGORITHM

In Section VII at the end of Procedures 1-5 the output is in the form of polynomial sharing. This allows us to calculate any polynomial by concatenating these procedures accordingly. In this section we detail how to do that. In this proposed algorithm we use the arithmetic representation of the function described in Appendix C. The details of the Algorithm is described in Algorithm 6. In the following theorem we claim that this algorithm satisfies constraints (1), (2), and (3).

Algorithm 6 Proposed Algorithm

Assume that the values of the $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F}$ are chosen uniformly at random in $\mathbb{F}$, and are known at all workers.

**Phase 1- Secret Sharing:** This phase will be done by every source node. Source node $\gamma \in [\Gamma]$ takes the following steps.
1: calculates $F_{X_{\gamma}, 1, t, k}(x)$.
2: sends $F_{X_{\gamma}, 1, t, k}(\alpha_n)$ to worker $n \in [N]$.

**Phase 2- Computation and communication:** This phase will be done by every worker node. All of the workers consider the arithmetic representation of the function according to rules in Appendix C. Worker $n \in [N]$ takes the following steps.
3: If it assess all of the gates, it goes to the next phase, otherwise consider the first non-assessed gate. Call that gate $g$.
4: If $g$ is a addition gate, it follows Procedure 1 and go to step 3.
5: If $g$ is a multiplication by constant gate, it follows Procedure 2 and go to step 3.
6: Assume that up and down inputs of the $g$ are called $B$, and $A$, respectively. Also call the output of the gate $C$.
7: If it has access to the $F_{B^T, 1, t, k}(\alpha_n)$, it follows Procedure 4 to access $F_{B, 1, t, k}(\alpha_n)$, and goes to step 9.
8: If it has access to the $F_{B, 1, t, k}(\alpha_n)$, it follows Procedure 5 to access $F_{B, 1, t, k}(\alpha_n)$, and goes to step 9.
9: If it has access to the $F_{A, 1, t, k}(\alpha_n)$, it follows Procedure 4 to access $F_{A^T, 1, t, k}(\alpha_n)$.
10: Follows Procedure 3 to access $F_{C, 1, t, k}(\alpha_n)$ and it goes to step 3.

**Phase 3- Reconstruction:** This phase will be done by every worker node. Worker $n \in [N]$ takes the following steps.
11: Stores the output of the arithmetic representation of the function in $Y_n$.
12: Sends $Y_n$ to the master node.
13: End.

**Theorem 3.** Algorithm 6 satisfies constraints (1), (2), and (3), for any $X_1, X_2, \ldots, X_\Gamma \in \mathbb{F}^{m \times m}$, and polynomial function $G : (\mathbb{F}^{m \times m})^\Gamma \rightarrow \mathbb{F}^{m \times m}$.

**Proof.** For Constraints (2), and (3) see Appendix B. For Constraint (1), note that the master will receive the polynomial sharing of the result, from all of the workers, and according to Theorem 2 these information is enough to reconstruct the result.

IX. EXTENSION

In the proposed scheme in Section VIII in order to share the matrix according to the polynomial sharing scheme, we partition it column-wise. This model of partitioning and sharing can be extended. In general, we can partition the matrix into some blocks and share the matrix according to this configuration. This approach is inspired and motivated by entangled polynomial code [14], or MatDot code [15].
Definition 3. Let $A \in \mathbb{F}^{z \times v}$, for some $z, v \in \mathbb{N}$, be

$$A = \begin{bmatrix}
A_{0,0} & A_{0,1} & A_{0,2} & \ldots & A_{0,m-1} \\
A_{1,0} & A_{1,1} & A_{1,2} & \ldots & A_{1,m-1} \\
A_{2,0} & A_{2,1} & A_{2,2} & \ldots & A_{2,m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{p-1,0} & A_{p-1,1} & A_{p-1,2} & \ldots & A_{p-1,m-1}
\end{bmatrix},$$

where $A_{i,j} \in \mathbb{F}^{k \times k'}$, for some $k, k' \in \mathbb{N}$, $k|z$, and $k'|v$.

We define entangled polynomial matrix $F_{A,b,p,m,n,t,s}(x)$, for some $p, m, n, t \in \mathbb{N}$ and symbol $s \in \{+,-\}$ as

$$F_{A,b,p,m,n,t,+(x)} \triangleq \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} A_{j,k} x^{j+bp} + \sum_{q=0}^{t-2} A'_{q} x^{npm+q},$$

$$F_{A,b,p,m,n,t,-(x)} \triangleq \sum_{j=0}^{p-1} \sum_{k=0}^{m-1} A_{j,k} x^{(p-1-j)+bp} + \sum_{q=0}^{t-2} A'_{q} x^{npm+q},$$

where $A'_{q}, q \in \{0,1,\ldots,t-2\}$ are chosen independently and uniformly at random from $\mathbb{F}^{k \times k'}$.

Now with this method of sharing we can do more general models of polynomial calculation. For example assume that $A$ and $B$ be matrices divided into $pm$ and $pn$ sub-matrices of equal size $k \times k'$, respectively. Let $C \triangleq A^{T}B$. Assume that $C$ is divided to $m \times n$ sub-matrices of equal size of $k' \times k'$ as follows:

$$C = \begin{bmatrix}
C_{0,0} & C_{0,1} & C_{0,2} & \ldots & C_{0,n-1} \\
C_{1,0} & C_{1,1} & C_{1,2} & \ldots & C_{1,n-1} \\
C_{2,0} & C_{2,1} & C_{2,2} & \ldots & C_{2,n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{m-1,0} & C_{m-1,1} & C_{m-1,2} & \ldots & C_{m-1,n-1}
\end{bmatrix},$$

Similar to procedure [3] we can calculate $C$ by first sharing $A$ and $B$ using $F_{A,1,p,m,n,t,+(x)}$ and $F_{B,m,p,n,m,t,-(x)}$ respectively. The each node $n$ calculates $F_{A,1,p,m,n,t,+(\alpha_{n})}F_{B,m,p,n,m,t,-(\alpha_{n})}$. One can see that in $F_{A,1,p,m,n,t,+(\alpha_{n})}F_{B,m,p,n,m,t,-(\alpha_{n})}$, is a degree $2npm + 2t - 4$ polynomial, where $C_{i,j}$ is equal to $H_{p-1+pi+jpm}$. In addition, overall, the $\min\{2pmn + 2t - 3, mpn + 2mp + np + nt - 2n - 2p + 1\}$ coefficients in this polynomial is not zero. Thus if the number of workers is at least $\min\{2pmn + 2t - 3, mpn + 2mp + np + nt - 2n - 2p + 1\}$, similar to Theorem [3] we can calculate $C$. Similar procedures in Section VII we can calculate any polynomial function $G$, satisfying constraints (1), (2), and (3) (see [21]).

X. CONCLUSION AND DISCUSSION

In this paper, we developed a new secure multiparty computation for massive inputs. The proposed solution offers significant gains compared to schemes based on splitting the data into smaller pieces, and applying conventional multiparty computation. In this work, we assumed that some of the nodes are semi-honest, meaning they are curious, but follow the protocol. The next step is to consider the case where nodes are adversarial, which has been addressed in [28]. There are many open problems in this direction. This includes exploring communication efficiency, the tradeoff between number of servers and communication load, having a network of heterogeneous servers, various network topologies, and the cases where some communication links are eavesdropped. In addition, investigating the case where the sources are collocated and can be encode together would be interesting. Here we assume that we want to calculate $G(X_{1}, X_{2}, \ldots, X_{r})$, where inputs are massive. One interesting direction is to consider the case, where the goal is to calculate $G(X_{1}^{(i)}, X_{2}^{(i)}, \ldots, X_{r}^{(i)})$, for $i = 1, \ldots, K$, for some integer $K$. This would be in the intersection of this work and [19]. Exploiting the sparsity of the input data in this calculation would be of great interest (see [20]).
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APPENDIX A
PROOF OF THE THEOREM 2

In order to prove the Theorem 2 first we prove the following lemma.

Lemma 4. Let \( A, B \in \mathbb{F}^{m \times m} \)

\[
A = \begin{bmatrix} A_1 & A_2 & \ldots & A_k \end{bmatrix}, \\
B = \begin{bmatrix} B_1 & B_2 & \ldots & B_k \end{bmatrix},
\]

where \( A_i, B_i \in \mathbb{F}^{m \times m} \), for \( i \in [k] \) and \( k \mid m \). Assume that these matrices are shared among \( N \) workers using polynomial matrices \( F_{A,1,t,k}(x) \) and \( F_{B,k,t,k}(x) \) respectively. Define

\[
H(x) = \sum_{n=0}^{2(k^2-t-2)} h_n x^n \triangleq F_{A,1,t,k}^T(x) F_{B,k,t,k}(x). \tag{45}
\]

If \( t < k + 1 \) then the number of zero coefficients of \( H(x) \) is \((k - t + 1)(k - 1)\).

Proof. We have

\[
F_{A,1,t,k}^T(x) = \sum_{n=1}^{k} A_n^T x^{n-1} + x^{k^2} \sum_{n=1}^{t-1} A_{k+n}^T x^{n-1}, \\
F_{B,k,t,k}(x) = \sum_{n=1}^{k} B_n x^{k(n-1)} + x^{k^2} \sum_{n=1}^{t-1} B_{k+n} x^{n-1}.
\]

The power of \( x \) with nonzero coefficients in \( \sum_{n=1}^{k} A_n^T x^{n-1} \) are \( \{0, 1, 2, \ldots, k^2 - 1\} \).

The power of \( x \) with nonzero coefficients in \( x^{k^2} \sum_{n=1}^{t-1} A_{k+n}^T x^{n-1} \) are \( \{k^2, k^2 + 1, k^2 + 2, \ldots, k^2 + k - 1 + t - 2\} \).

The power of \( x \) with nonzero coefficients in \( x^{k^2} \sum_{n=1}^{t-1} B_{k+n} x^{n-1} \) are \( \{k^2 + ik + j | i \in [0, k - 1], j \in [0, t - 2]\} \).

The power of \( x \) with nonzero coefficients in \( x^{2k^2} \sum_{n=1}^{t-1} A_{k+n}^T x^{n-1} \) are \( \{2k^2, 2k^2 + 1, 2k^2 + 2, \ldots, 2k^2 + 2t - 4\} \).

If \( k - 1 \leq t - 2 \), then \( k + 1 \leq t \). Thus for all \( i = [0, 2k^2 + 2t - 4], \) the coefficients of \( x^i \) is nonzero. Assume that \( t \leq k \), then \( k - t + 1 \) numbers of zero coefficients in every interval \([k^2 + ik, k^2 + (i + 1)k - 1]\) for \( 1 \leq i \leq k - 1 \). Thus the number of total zero coefficients is

\[(k - t + 1)(k - 1)\].

Now we prove Theorem 2. First note that based on Lemma 4, the number of nonzero coefficients of \( H(x) \) is \( \min\{2k^2 + 2t - 3, k^2 + kt + t - 2\} \). Assume that \( N = \min\{2k^2 + 2t - 3, k^2 + kt + t - 2\} \), and \( \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F} \). Define

\[
H \triangleq \begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \ldots & \alpha_1^N \\
\alpha_2^1 & \alpha_2^2 & \ldots & \alpha_2^N \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_N^1 & \alpha_N^2 & \ldots & \alpha_N^N \end{bmatrix},
\]
where \( i_1, i_2, \ldots, i_N \in \{0, 1, \ldots, 2(k^2 + t - 2)\} \) are the indexes of the nonzero coefficients of the \( H(x) \).
Also, for any \( i_1, i_2, \ldots, i_{k+t-1} \in [N] \), define
\[
A_{i_1, i_2, \ldots, i_{k+t-1}} \triangleq \begin{bmatrix}
\alpha_{i_1}^0 & \alpha_{i_1}^1 & \alpha_{i_1}^{k-1} & \alpha_{i_1}^{k^2} & \alpha_{i_1}^{k^2+1} & \alpha_{i_1}^{k^2+t-2} \\
\alpha_{i_2}^0 & \alpha_{i_2}^1 & \alpha_{i_2}^{k-1} & \alpha_{i_2}^{k^2} & \alpha_{i_2}^{k^2+1} & \alpha_{i_2}^{k^2+t-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{i_{k+t-1}}^0 & \alpha_{i_{k+t-1}}^1 & \alpha_{i_{k+t-1}}^{k-1} & \alpha_{i_{k+t-1}}^{k^2} & \alpha_{i_{k+t-1}}^{k^2+1} & \alpha_{i_{k+t-1}}^{k^2+t-2} \\
\end{bmatrix},
\]
\[
B_{i_1, i_2, \ldots, i_{k+t-1}} \triangleq \begin{bmatrix}
\alpha_{i_1}^0 & \alpha_{i_1}^k & \alpha_{i_1}^{(k-1)k} & \alpha_{i_1}^{k^2} & \alpha_{i_1}^{k^2+1} & \alpha_{i_1}^{k^2+t-2} \\
\alpha_{i_2}^0 & \alpha_{i_2}^k & \alpha_{i_2}^{(k-1)k} & \alpha_{i_2}^{k^2} & \alpha_{i_2}^{k^2+1} & \alpha_{i_2}^{k^2+t-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{i_{k+t-1}}^0 & \alpha_{i_{k+t-1}}^k & \alpha_{i_{k+t-1}}^{(k-1)k} & \alpha_{i_{k+t-1}}^{k^2} & \alpha_{i_{k+t-1}}^{k^2+1} & \alpha_{i_{k+t-1}}^{k^2+t-2} \\
\end{bmatrix}.
\]

To prove Theorem 2, it’s enough to show that for large enough \(|\mathbb{F}|\), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F} \), such that for any \( i_1, i_2, \ldots, i_{k+t-1} \in [N] \), we have
\[
det A_{i_1, i_2, \ldots, i_{k+t-1}} \neq 0, \tag{46}
\]
\[
det B_{i_1, i_2, \ldots, i_{k+t-1}} \neq 0, \tag{47}
\]
\[
det H \neq 0, \tag{48}
\]
and if we choose \( \alpha_1, \alpha_2, \ldots, \alpha_N \), independently and uniformly at random in \( \mathbb{F} \), the probability of (46), (47), and (48) approaches to one, as \(|\mathbb{F}| \to \infty\).

Define
\[
f(\alpha_1, \alpha_2, \ldots, \alpha_N) \triangleq \left( \prod_{i_1, i_2, \ldots, i_{k+t-1} \in [N]} \det A_{i_1, i_2, \ldots, i_{k+t-1}} \det B_{i_1, i_2, \ldots, i_{k+t-1}} \right) \det H.
\]

As proved in [29], for large enough \(|\mathbb{F}|\), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F} \), such that for any \( i_1, i_2, \ldots, i_{k+t-1} \in [N] \), we have \( f(\alpha_1, \alpha_2, \ldots, \alpha_N) \neq 0 \). Also, based on Schwartz-Zippel Lemma [30], [31], if we choose \( \alpha_1, \alpha_2, \ldots, \alpha_N \), independently and uniformly at random in \( \mathbb{F} \), the probability of (46), (47), and (48) approaches to one, as \(|\mathbb{F}| \to \infty\).
APPENDIX B

PROOF OF PRIVACY IN THEOREM 3

Recall that in this algorithm, to share any information, we always add some random matrices to it. We claim that this protocol satisfies privacy constraints (2) and (3). In order to formally prove that, we use the following two lemmas.

**Lemma 5.** Assume that \( r(x) \) is a polynomial of degree \( t - 1 \), where coefficients are chosen uniformly at random from \( \mathbb{F} \). Define

\[
\tilde{r} \triangleq (r(\alpha_1), r(\alpha_2), \ldots, r(\alpha_t))^T,
\]

for some distinct \( \alpha_1, \alpha_2, \ldots, \alpha_t \in \mathbb{F} \). Then \( \tilde{r} \) has a uniform distribution over \( \mathbb{F}^t \).

**Proof.** Assume that

\[
r(x) = \sum_{n=0}^{t-1} a_n x^n,
\]

where \( a_i \)'s are chosen independently and uniformly at random from \( \mathbb{F} \). Also assume that \( r_1, r_2, \ldots, r_t \) are \( t \) random elements chosen uniformly at random in \( \mathbb{F} \). According to Lagrange interpolation rule (see [27]) we know that the following set of equations has a unique answer.

\[
\begin{align*}
    r(1) &= r_1, \\
    r(2) &= r_2, \\
    & \vdots \\
    r(t) &= r_t.
\end{align*}
\]

It means that if we know the values of \( r_1, r_2, \ldots, r_t \), we can uniquely determine the values of \( a_i \), for \( i \in [0, t - 1] \). Also its obvious that if we know the values of \( a_0, a_1, \ldots, a_{t-1} \), we can uniquely determine the values of \( r(1), r(2), \ldots, r(t) \). Therefore there is a one to one mapping between the vectors \( a = (a_0, a_1, \ldots, a_{t-1})^T \) and \( \tilde{r} = (r(1), r(2), \ldots, r(t))^T \). Note that \( a_i, i = 0, 1, \ldots, t - 1 \), are chosen independently and uniformly at random in \( \mathbb{F} \). Therefore \( a \) has a uniform distribution over \( \mathbb{F}^t \). Because of the one-to-one mapping between \( a \) and \( \tilde{r} \), the vector \( \tilde{r} \) has a uniform distribution over \( \mathbb{F}^t \), too. \( \square \)

**Corollary 6.** Assume that \( \mathbf{R}(x) \) is a polynomial matrix of degree \( t - 1 \) where coefficients are chosen uniformly at random in \( \mathbb{F}^{p \times q} \). Define \( \tilde{\mathbf{R}} \) as

\[
\tilde{\mathbf{R}} \triangleq (\mathbf{R}(\alpha_1), \mathbf{R}(\alpha_2), \ldots, \mathbf{R}(\alpha_t))^T,
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_t \in \mathbb{F} \). Then \( \tilde{\mathbf{R}} \) has a uniform distribution over \( (\mathbb{F}^{p \times q})^t \).

**Proof.** The proof directly follows from Lemma 5. \( \square \)

**Lemma 7.** Assume that we have \( \mathcal{A} = \{ \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n \} \) where \( \mathbf{A}_i \in \mathbb{F}^{p \times q}, i = 1, 2, \ldots, n \). Assume that we have \( m \) polynomials \( \mathbf{I}_1(x), \mathbf{I}_2(x), \ldots, \mathbf{I}_m(x) \) of degree at most \( n - 1 \) where their coefficients are in \( \mathcal{A} \). Consider the polynomials

\[
\begin{align*}
    \mathbf{T}_1(x) &= \mathbf{I}_1(x) + x^n \mathbf{R}_1(x), \\
    \mathbf{T}_2(x) &= \mathbf{I}_2(x) + x^n \mathbf{R}_2(x), \\
    & \vdots \\
    \mathbf{T}_m(x) &= \mathbf{I}_m(x) + x^n \mathbf{R}_m(x),
\end{align*}
\]

where for \( 1 \leq i \leq m \), \( \mathbf{R}_i(x) \) is a polynomial of degree \( t \) where coefficients are chosen uniformly at random from \( \mathbb{F}^{p \times q} \). In addition, assume that there are \( N \) workers and worker \( j \) has access to the value of \( \mathbf{T}_i(\alpha_j) \), for \( 1 \leq i \leq m \), and some distinct values of \( \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F} \). If there are \( t \) semi-honest
workers, they can not gain any additional information about the elements of $A$, i.e., $I(A; \tilde{T}) = 0$, where $T$ defined as

$$\tilde{T} \triangleq \begin{bmatrix} T_1(\alpha_{i_1}) & T_1(\alpha_{i_2}) & \ldots & T_1(\alpha_{i_t}) \\ T_2(\alpha_{i_1}) & T_2(\alpha_{i_2}) & \ldots & T_2(\alpha_{i_t}) \\ \vdots & \vdots & \ddots & \vdots \\ T_m(\alpha_{i_1}) & T_m(\alpha_{i_2}) & \ldots & T_m(\alpha_{i_t}) \end{bmatrix}. \tag{50}$$

$$\tilde{\mathbf{I}} \triangleq \begin{bmatrix} I_1(\alpha_{i_1}) & I_1(\alpha_{i_2}) & \ldots & I_1(\alpha_{i_t}) \\ I_2(\alpha_{i_1}) & I_2(\alpha_{i_2}) & \ldots & I_2(\alpha_{i_t}) \\ \vdots & \vdots & \ddots & \vdots \\ I_m(\alpha_{i_1}) & I_m(\alpha_{i_2}) & \ldots & I_m(\alpha_{i_t}) \end{bmatrix}, \quad \tilde{\mathbf{R}} \triangleq \begin{bmatrix} R_1(\alpha_{i_1}) & R_1(\alpha_{i_2}) & \ldots & R_1(\alpha_{i_t}) \\ R_2(\alpha_{i_1}) & R_2(\alpha_{i_2}) & \ldots & R_2(\alpha_{i_t}) \\ \vdots & \vdots & \ddots & \vdots \\ R_m(\alpha_{i_1}) & R_m(\alpha_{i_2}) & \ldots & R_m(\alpha_{i_t}) \end{bmatrix}. \tag{51}$$

Proof. Assume that the semi-honest workers are $i_1, i_2, \ldots, i_t$. Define

$$\tilde{\mathbf{I}} \triangleq \begin{bmatrix} I_1(\alpha_{i_1}) & I_1(\alpha_{i_2}) & \ldots & I_1(\alpha_{i_t}) \\ I_2(\alpha_{i_1}) & I_2(\alpha_{i_2}) & \ldots & I_2(\alpha_{i_t}) \\ \vdots & \vdots & \ddots & \vdots \\ I_m(\alpha_{i_1}) & I_m(\alpha_{i_2}) & \ldots & I_m(\alpha_{i_t}) \end{bmatrix}, \quad \tilde{\mathbf{R}} \triangleq \begin{bmatrix} R_1(\alpha_{i_1}) & R_1(\alpha_{i_2}) & \ldots & R_1(\alpha_{i_t}) \\ R_2(\alpha_{i_1}) & R_2(\alpha_{i_2}) & \ldots & R_2(\alpha_{i_t}) \\ \vdots & \vdots & \ddots & \vdots \\ R_m(\alpha_{i_1}) & R_m(\alpha_{i_2}) & \ldots & R_m(\alpha_{i_t}) \end{bmatrix}.$$ 

The elements of the three matrices $\tilde{T}, \tilde{\mathbf{I}}, \tilde{\mathbf{R}}$ are random variables chosen in $F$. The adversaries jointly know the elements of the matrix $\tilde{T}$. We claim that in this situation the adversaries do not gain any additional information about the elements of $A$. For any $T, I, \tilde{I} \in \mathbb{F}^{mp \times tq}$ we have

$$\Pr(\tilde{I} = I|\tilde{T} = T) = \frac{\Pr(\tilde{T} = T|\tilde{I} = I) \Pr(\tilde{I} = I)}{\sum_{I_j \in \mathbb{F}^{mp \times tq}} \Pr(\tilde{T} = T|\tilde{I} = I_j) \Pr(\tilde{I} = I_j)} = \frac{\Pr(\tilde{R} = T - I|\tilde{I} = I) \Pr(\tilde{I} = I)}{\sum_{I_j \in \mathbb{F}^{mp \times tq}} \Pr(\tilde{R} = T - I|\tilde{I} = I_j) \Pr(\tilde{I} = I_j)} = \frac{\Pr(\tilde{I} = I)}{\sum_{I_j \in \mathbb{F}^{mp \times tq}} \Pr(\tilde{I} = I_j)} = \Pr(\tilde{I} = I),$$

where (a) follows form Bayesian Rule, (b) follows form that according to Corollary 6 matrix $\tilde{R}$ has a uniform distribution over $\mathbb{F}^{mp \times tq}$, and $\Pr(\tilde{R} = T - I) = \frac{1}{|\mathbb{F}|^{|mp \times tq|}}$. Thus we have $H(\tilde{I}|\tilde{T}) = H(\tilde{I})$. Therefore $I(\tilde{I}, \tilde{T}) = 0$.

On the other hand, form the definition of $A$, we see that $H(\tilde{T}|\tilde{I}, A) = H(\tilde{T}|\tilde{I})$. Therefore, $A \rightarrow \tilde{I} \rightarrow \tilde{T}$ is a Markov chain. According to data processing inequality we have

$$I(A; \tilde{T}) \leq I(\tilde{I}; \tilde{T}) = 0.$$

Therefore if the adversaries know the elements of $\tilde{T}$, they are not able to gain any information about the elements of $A$. 

\textbf{Corollary 8. Assume that}

$$Y \triangleq [Y_1, Y_2, \ldots, Y_k],$$
where $Y_1, Y_2, \ldots, Y_k \in \mathbb{F}^{m \times m}$. Assume that $Y$ is shared using polynomial matrix

$$F_{Y,b,t,k}(x) = \sum_{n=0}^{t-2} Y_n x^{b(n-1)} + \sum_{n=1}^{k} R_n x^{k^2+n},$$

where $R_0, R_1, \ldots, R_{t-2}$ are chosen uniformly at random in $\mathbb{F}^{m \times m}$, among $N$ workers. More precisely, $F_{Y,b,t,k}(\alpha_n)$ is delivered to worker $n$, for some distinct $\alpha_n$ in $\mathbb{F}$. For any subset of $S \subset [N]$, $|S| = t-1$, we have

$$H(Y|F_{Y,b,t,k}(\alpha_i), i \in S) = H(Y).$$

**Proof.** The proof follows directly from Lemma 7. \qed

Intuitively, we can explain the privacy constraints as follows. If we consider any subset of $t-1$ workers, each share that they receive from the sources or any other workers, includes $t-1$ random matrices, excluding the original data itself. So, if we ignore the original data, the number of equations and the number of variables are exactly the same. However, because we have data also, the number of equations is always less than the number of the variables, no matter how data is involved in these equations.

Here we prove the privacy constraints (2), and (3), for the Algorithm (6). Assume that the semi-honest workers are $i_1, i_2, \ldots, i_{t-1} \in [N]$. In order to prove the constraint (2) for Algorithm 6 we must show that for any $X_1, X_2, \ldots, X_R \in \mathbb{F}^{m \times m}$, and polynomial function $G : (\mathbb{F}^{m \times m})^r \rightarrow \mathbb{F}^{m \times m}$ we have

$$H(X_j, j \in [\Gamma]) \bigcup_{n \in S}\{M_{n \rightarrow n}, n' \in [N]\}, \tilde{\gamma}_n, \gamma \in [\Gamma], n \in S = H(X_j, j \in [\Gamma]),$$

where $S = \{i_1, i_2, \ldots, i_{t-1}\}$. Assume that the calculations is done through $R$ rounds. Let us define $M_{n' \rightarrow n}$ as the messages from the worker $n'$ to worker $n$, in the round $r \in R$. One can see that we have

$$M_{n' \rightarrow n} = \bigcup_{r=1}^{R} M_{n' \rightarrow n}^r.$$

Also define $R_{n' \rightarrow n}$ as the set of all random matrices that worker $n'$, use for sending a message to worker $n$, in the round $r$. Let us define

$$M_{S}^{(r)} \triangleq \bigcup_{n' \in [\Gamma], n \in S} M_{n' \rightarrow n},$$

$$R_{S}^{(r)} \triangleq \bigcup_{n' \in [\Gamma], n \in S} R_{n' \rightarrow n},$$

$$X \triangleq \{X_1, X_2, \ldots, X_R\}$$

$$\tilde{X}_S \triangleq \{\tilde{\gamma}_n, \gamma \in [\Gamma], n \in S\}$$

From the definition, we have

$$H(X_j, j \in [\Gamma]) \bigcup_{n \in S}\{M_{n' \rightarrow n}, n' \in [N]\}, \tilde{\gamma}_n, \gamma \in [\Gamma], n \in S =$$

$$H(X|\tilde{X}_S, M_{S}^{(r)}, r \in [R]).$$

Therefore, to prove the privacy constraint (2), it is sufficient to show that

$$I(\tilde{X}_S, M_{S}^{(1)}, M_{S}^{(2)}, \ldots, M_{S}^{(R)}; X) = 0.$$
One can see that
\[ H(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, \ldots, M^{(R)}_S | X) = H(\tilde{X}_S | X) + H(M^{(1)}_S | \tilde{X}_S, X) \]
+ \(H(M^{(2)}_S | \tilde{X}_S, X)\)
\vdots
+ \(H(M^{(R)}_S | \tilde{X}_S, X)\).
\]

(52)

According to Corollary 8, we have \(H(\tilde{X}_S | X) = H(\tilde{X}_S)\). In addition, since in each round \(k \in [R], M^{(k)}_S\) is a function of \(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, M^{(k-1)}_S\), and \(R^{(k)}_S\), then
\[ H(M^{(k)}_S | M^{(k-1)}_S, \ldots, M^{(1)}_S, \tilde{X}_S, X) = H(R^{(k)}_S | M^{(k-1)}_S, \ldots, M^{(1)}_S, \tilde{X}_S, X) \]
\[ = H(R^{(k)}_S) \geq H(M^{(k)}_S), \]
where (a) follows from the fact that \(H(R^{(k)}_S)\) and \(H(M^{(k)}_S)\) have the same size and \(R^{(k)}_S\) has a uniform and independent distribution. Therefore, according to (52) and (53) we have
\[ H(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, \ldots, M^{(R)}_S | X) \geq H(\tilde{X}_S) + H(M^{(1)}_S) + H(M^{(2)}_S) + \ldots + H(M^{(R)}_S) \]
\[ \geq H(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, \ldots, M^{(R)}_S). \]

Therefore,
\[ I(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, \ldots, M^{(R)}_S ; X) \leq 0. \]

Thus
\[ I(\tilde{X}_S, M^{(1)}_S, M^{(2)}_S, \ldots, M^{(R)}_S ; X) = 0. \]

Now we prove constraint (3). Assume that
\[ Y = [Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}], \]
where \(Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)} \in \mathbb{F}^{m \times \frac{t}{2}}\). There exist a polynomial function \(F_{Y,b,t,k}(x)\), where \(F_{Y,b,t,k}(\alpha_i) = Y_i, i \in [N]\), for some distinct \(\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{F}\). According to (17), there exists \(R_0, R_1, \ldots, R_{t-2}\) chosen uniformly at random in \(\mathbb{F}^{m \times \frac{t}{2}}\), where
\[ F_{Y,b,t,k}(x) = \sum_{n=1}^{k} Y^{(n)}x^{b(n-1)} + \sum_{n=0}^{t-2} R_n x^{k^2+n}. \]

One can see that we have
\[ H(Y | Y_1, Y_2, \ldots, Y_N) = 0, \]
\[ H(Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}, R_0, R_1, \ldots, R_{t-2} | Y_1, Y_2, \ldots, Y_N) = 0 \]
\[ H(Y_1, Y_2, \ldots, Y_N | Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}, R_0, R_1, \ldots, R_{t-2}) = 0. \]

(54)

(55)

(56)

We have
\[ H(\mathcal{X} | Y, Y_1, Y_2, \ldots, Y_N) \overset{(a)}{=} H(\mathcal{X} | Y_1, Y_2, \ldots, Y_N) \]
\[ \overset{(b)}{=} H(\mathcal{X} | Y_1, Y_2, \ldots, Y_N, Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}, R_0, R_1, \ldots, R_{t-2}) \]
\[ \overset{(c)}{=} H(\mathcal{X} | Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}, R_0, R_1, \ldots, R_{t-2}) \]
\[ \overset{(d)}{=} H(\mathcal{X} | Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}) \]
\[ = H(\mathcal{X} | Y) \]

where (a) follows from (54), (b) and (c) follow from (56) and (55), and (d) follows from the fact that \(R_0, R_1, \ldots, R_{t-2}\) were chosen uniformly at random in \(\mathbb{F}^{m \times \frac{t}{2}}\).
APPENDIX C
ARITHMETIC CIRCUITS

In this part we want to describe some rules to create the arithmetic circuits corresponding to a function 
\( G(X_{i_1}, X_{i_2}, \ldots, X_{i_r}) \). We know that each arithmetic function can be written as a sum of production terms

\[
G(X_{i_1}, X_{i_2}, \ldots, X_{i_r}) = \sum_{j=1}^{M} G_j(X_{i_1}, X_{i_2}, \ldots, X_{i_r}),
\]

(57)

where \( M \) is the number of monomial terms and all of the \( G_j \) function are monomials, for \( j \in [M] \).

Example 5. Assume that the desired function \( G \) is

\[
G(X_1, X_2, X_3, X_4) = X_2^T X_1^2 X_3 + X_2 X_4 X_3^T.
\]

According to the notations we have

\[
G_1(X_1, X_2, X_3, X_4) = X_2^T X_1^2 X_3,
\]

\[
G_2(X_1, X_2, X_3, X_4) = X_2 X_4 X_3^T.
\]

There are some rules to represent \( G \) by multiplication and addition gates. The rules are described in the following. For simplification we show a multiplication gate by an AND gate, and an addition gate by an OR gate.

1) **Rule 1:**

Assume that the function is in the form of \( \prod_{j=1}^{r} X_j \). In order to represent this function, first we multiply the last two matrices \( (X_{r-1}, X_r) \), then we multiply \( X_{r-2} \) to the result of the previous operation and so on. See Fig. 2.

![Fig. 2](image1.png)

Fig. 2. The circuit representing the order of operations in calculating the function \( G = \prod_{j=1}^{r} X_j \).

2) **Rule 2:**

Assume that the function is in the form of \( \sum_{j=1}^{r} X_j \). In order to represent this function, first we add the last two matrices \( (X_{r-1} + X_r) \), then we add \( X_{r-2} \) to the result of the previous operation, and so on. See Fig. 3.

![Fig. 3](image2.png)

Fig. 3. The circuit representing the order of operations in calculating the function \( G = \sum_{j=1}^{r} X_j \).

3) **Rule 3:**
In general for representing a function, first we apply rule 1 for creating the arithmetic representation of \( G_1, G_2, \ldots, G_M \), and then we apply rule 2. For example the arithmetic representation of example 5 is as Fig. 4.

\[
G(X_1, X_2, X_3, X_4) = X_2^T X_1^2 X_3 + X_2 X_3 X_3^T.
\]

Fig. 4. The circuit corresponding to the function \( G(X_1, X_2, X_3, X_4) = X_2^T X_1^2 X_3 + X_2 X_3 X_3^T \).