Geometry and Physics on $w_\infty$ Orbits.

K. G. Selivanov

Physikalisches Institut der Universität Bonn
Nußallee 12, D-5300 Bonn 1
Germany

Abstract

We apply the coadjoint orbit technique to the group of area preserving diffeomorphisms (APD) of a 2D manifold, particularly to the APD of the semi-infinite cylinder which is identified with $w_\infty$. The geometrical action obtained is relevant to both $w$ gravity and 2D turbulence. For the latter we describe the hamiltonian, which appears to be given by the Schwinger mass term, and discuss some possible developments within our approach. Next we show that the set of highest weight orbits of $w_\infty$ splits into subsets, each of which consists of highest weight orbits of $\tilde{w}_N$ for a given $N$. We specify the general APD geometric action to an orbit of $\tilde{w}_N$ and describe an appropriate set of observables, thus getting an action and observables for $\tilde{w}_N$ gravity. We compute also the Ricci form on the $\tilde{w}_N$ orbits, what gives us the critical central charge of the $\tilde{w}_N$ string, which appears to be the same as the one of the $W_N$ string.

*This work was supported in part by the Deutsche Forschungsgemeinschaft
†on leave of absence from ITEP, Moscow
1 Introduction

$w_\infty$ - or, more generally, the algebra of area preserving diffeomorphisms (APD) of a 2D manifold - has appeared in mathematical physics in very different problems. First, as it was explained by V.I. Arnold [Arn], motion of an inviscid incompressible fluid (in a space of any dimension) can be viewed as a hamiltonian flow on a coadjoint orbit of APD. Then, $w_\infty$ appeared to be an $N\to\infty$ limit of Zamolodchikov’s $W_N$ algebra [Ba]. Another remarkable appearance of $w_\infty$ - as a symmetry of the $c=1$ string - was discovered in [KlePo] and [Wit]. In [Ko] APD was discussed in the context of 3D Cern-Simons theory. After all, APD is apparently a symmetry of the Nambu-Goto string action (and therefore, of the Polyakov’s one), also it is a symmetry of the 2D QCD action, and we will not even try to list all appearances of $w_\infty$ in the recent enormous developments of the theory of integrable equations.

In our consideration of $w_\infty$ here we adopt the coadjoint orbit ideology.

In section 2 we obtain the geometrical action (that is the kinetic term corresponding to the Kirillov symplectic form on the coadjoint orbit) for APD without specifying the underlying 2D manifold and also without the central extension term. As known from [AlSha] and [Wie], the geometrical actions for the group of the diffeomorphisms of the circle and for the loop group (with the central extensions) are nothing but the Liouville and WZW actions correspondingly. That is why it was interesting to look at the APD case. The action (2.18) (possibly with some contributions from lower dimensional cycles and from the boundary of the manifold, see discussions of section 3 ) has all the rights to serve as an action for $w_\infty$ gravity in its 3D formulation. Apart from the kinetic term we also discuss possible Hamiltonians, particularly, the Hamiltonian for the motion of the ideal fluid. It was funny to understand that the latter was given by the so called Schwinger mass term (we have to say in advance that the dual space to the Lie algebra of APD is essentially the space of abelian gauge potentials). We present some speculations about a possible utilization of the APD action in the recently proposed theory of
2D turbulence [Po].

In section 3 we specify our 2D manifold to be the semi-infinite cylinder, the area preserving vector fields on which form the standard \( w_\infty \) (on the cylinder it would be rather \( w_{1+\infty} \)). We then discuss the structure of the coadjoint orbits. Actually, due to presence of huge ideals in \( w_\infty \), the set of orbits splits into subsets consisting of coadjoint orbits of \( \bar{w}_N \)'s (not to be confused with \( w_N \)'s and \( W_N \)'s !). After describing the appropriate representatives of the orbits, we specify the geometric action of the section 2 (including in addition a central extension term) thus getting geometrical actions for \( \bar{w}_N \). Apart from the action we describe the set of observables. The action and the observables should be considered as the ones for \( \bar{w}_N \) gravity.

In section 4 we discuss the possibility of geometrical formulation of the critical \( \bar{w}_N \) string, following the ideology of [BoRa] developed for the case of the bosonic (“Virasoro”) string. Computing the curvature of the ghost bundle over the \( \bar{w}_N \) orbit we get an (upper) critical central charge which appeared to be the same as the one for the \( W_N \) string [Pop].

In section 5 we give some prospects and open questions.

We will not try to make our definitions and arguments rigorous; our consideration will rather be of a heuristic character.

## 2 Geometrical action and Hamiltonians

The Kirillov bracket on a coadjoint orbits as well as the entire coadjoint orbit machinery is present now in physicist’s folklore so we shall be short in general definitions following basically notations of [AlSha].

To describe the Lie algebra of APD of the manifold \( M \), pick a measure \( \omega \) on \( M \) - which in 2D case we are concerned with here is also a symplectic form. The element of LieAPD is a vector field \( \epsilon \) on \( M \) preserving \( \omega \), that is \( \mathcal{L}_\epsilon \omega = 0 \) with \( \mathcal{L}_\epsilon \) being a Lie derivative, which implies that the one-form \( i_\epsilon \omega \)
is closed,
\[ di_\epsilon \omega = O \]  \hspace{1cm} (2.1)
Equivalently one can say that \( \epsilon \) is divergentless.

Further, for a given one-form \( A \) on \( M \) one can construct a linear function on \( \text{LieAPD} \) as
\[ \langle A, \epsilon \rangle = \int i_\epsilon A \omega \]  \hspace{1cm} (2.2)
From (2.1) it follows that \( A \) and \( A + d\varphi \) define the same function \( \langle A, \epsilon \rangle \), so the dual space to \( \text{LieAPD} \), \( (\text{LieAPD})^* \) is identified with the space of one-forms on \( M \) modulo exact forms (with the proper definition of the space of the forms, particularly in the case of noncompact \( M \)).

To get an explicit parametrization of the group APD, pick (local) coordinates \((\sigma, \tau)\) of \( M \) such that (locally)
\[ \omega = d\tau \wedge d\sigma \]  \hspace{1cm} (2.3)
Then an \((X, Y)\) element of APD acts as
\[ (\sigma, \tau) \rightarrow (X(\sigma, \tau), Y(\sigma, \tau)) \]  \hspace{1cm} (2.4)
with the obvious constraint
\[ \partial(X, Y)/\partial(\sigma, \tau) = X_\sigma Y_\tau - X_\tau Y_\sigma = 1 \]  \hspace{1cm} (2.5)
\((x, y)\) will always stand for the inverse of \((X, Y)\).

The coadjoint action of APD can be defined by saying that \( A \) transforms as a one-form:
\[ (A_\sigma(\sigma, \tau), A_\tau(\sigma, \tau))^{(X,Y)} = (AX(X,Y)X_\sigma + AY(X,Y)Y_\sigma, AX(X,Y)X_\tau + AY(X,Y)Y_\tau) \]  \hspace{1cm} (2.6)
The infinitesimal coadjoint action by a vector field \( \epsilon \) is, of course, given by
\[ \delta_\epsilon A = \mathcal{L}_\epsilon A \]  \hspace{1cm} (2.7)
Some \((A_\sigma, A_\tau)\) can be chosen as a representative of an orbit \(O\). Then (2.6) gives a convenient parametrization of \(O\).

Now the Kirillov’s form \(\Omega\) at the point \((A_\sigma, A_\tau)^{(X,Y)}\) of \(O\) on the tangent-to-\(O\) vectors \(\delta_{\epsilon_1}A^{(X,Y)}\) and \(\delta_{\epsilon_2}A^{(X,Y)}\) takes the value

\[
\Omega_{A^{(X,Y)}}(\delta_{\epsilon_1}A^{(X,Y)}, \delta_{\epsilon_2}A^{(X,Y)}) = \langle A^{(X,Y)}, [\epsilon_1, \epsilon_2] \rangle
\]

(2.8)

Identifying \(\delta A^{(X,Y)}\) in the explicit parametrization (2.6) with (2.7), one easily derives that the corresponding vector field \(\epsilon_1\) is given by

\[
\epsilon_1^\mu \partial_\mu = (Y, \tau \delta_1 X - X, \tau \delta_1 Y) \partial_\sigma + (-Y, \sigma \delta_1 X + X, \sigma \delta_1 Y) \partial_\tau
\]

(2.9)

To write out the symplectic form \(\Omega\) (2.8) one needs to compute the commutator of \(\epsilon_1\) (2.9) and \(\epsilon_2\) (similar formula). One could do it straightforwardly however more useful is to note first that the commutator of two APD vector fields (in 2D case) is a hamiltonian one and then to see that the corresponding Hamiltonian \(h_{[\epsilon_1, \epsilon_2]}\) reads

\[
h_{[\epsilon_1, \epsilon_2]} = \delta_1 X \delta_2 Y - \delta_1 Y \delta_2 X
\]

(2.10)

Indeed, from (2.1) follows that (locally)

\[
i_{\epsilon_1} \omega = dh_1
\]

(2.11)

where, of course, the function \(h_1\) is not defined \(a\ priory\) on the whole \(M\). Hence the commutator is (locally) given by

\[
i_{[\epsilon_1, \epsilon_2]} \omega = d\{h_1, h_2\}
\]

where \(\{, \}\) is the Poisson bracket, corresponding to \(\omega\). However, by definition,

\[
\{h_1, h_2\} = dh_2(\epsilon_1)
\]

(2.12)

the right hand side of which is defined globally on \(M\) and can be also computed as \(i_{\epsilon_2} \omega(\epsilon_1)\), easily giving (2.10).
We are in a position now to write out the symplectic form:

$$\Omega_{A(X,Y)}(\delta_1 A^{(X,Y)}, \delta_2 A^{(X,Y)}) = \int F^{(X,Y)}(\delta_1 X \delta_2 Y - \delta_1 Y \delta_2 X)$$  \((2.13)\)

where \(F^{(X,Y)}\) is the “curvature” for a “gauge potential” \(A^{(X,Y)}, F^{(X,Y)} = dA^{(X,Y)}\).

The corresponding geometrical action reads

$$S = \int dt \omega(A_X(X,Y) \dot{X} + A_Y(X,Y) \dot{Y})$$  \((2.14)\)

where the diffeomorphism \((X,Y)\) parametrizing a point of \(\mathcal{O}\) is assumed to be time \((t, not \tau!)\) -dependent and \(\dot{X} (\dot{Y})\) is the time-derivative of \(X (Y)\).

Let us compute the variation of \(S\) \((2.14)\) - to observe the correspondence to \(\Omega\) \((2.13)\) and for future use. Straightforwardly varying one gets

$$\delta S = \int dt \omega F_{XY}(X,Y)(\delta X \dot{Y} - \delta Y \dot{X})$$  \((2.15)\)

Taking the vector field \(\epsilon\) defined as in \((2.9)\) and the vector field \(\zeta\) defined as in \((2.9)\) with \(\dot{X} (\dot{Y})\) instead of \(\delta X (\delta Y)\) (of course, \(\epsilon\) and \(\zeta\) are divergent free, for variations of \(X\) and \(Y\) as well as their time-dependence are restricted by the constraint \((2.5)\)) and using \((2.10)\) one sees that \(\delta S\) \((2.15)\) rewrites as

$$\delta S = \int dt \Omega_{A(X,Y)}(\delta_\epsilon A^{(X,Y)}, \delta_\zeta A^{(X,Y)})$$  \((2.16)\)

as it should according to the canons of hamiltonian mechanics (apparently, \(\delta_\zeta A^{(X,Y)} = d/dt A^{(X,Y)}\)).

It is useful to rewrite the action \((2.14)\) in a little different form. Making change of variables \((\sigma, \tau) \rightarrow (x(\sigma, \tau), y(\sigma, \tau))\) in the integration over \(M\) in \((2.14)\) (recall that \((x, y)\) is the inverse of \((X,Y)\)) one gets

$$S = \int dt \omega(A_\tau(\sigma, \tau)(\dot{x}_\tau - x_\tau \dot{y}) + A_\sigma(\sigma, \tau)(-\dot{x}_\sigma + x_\sigma \dot{y}))$$  \((2.17)\)

The action \((2.17)\) can be rewritten in an even nicer form after elevating the constraint \((2.5)\) (which is, of course, the same for both \((X,Y)\) and \((x, y)\)) to
the action and combining $A$ with a lagrange multiplier $A_t$ into a one-form $\hat{A}$ on the 3D manifold $R^1 \times M$ ($S^1 \times M$)

$$S = \int \hat{A} \wedge \omega - \hat{A} \wedge dx \wedge dy \quad (2.18)$$

We feel a need to recall that from the triple $(A_t, A_\tau, A_\sigma)$ only $A_t$ should be integrated over in the path integral quantization of the theory.

The equations of motion of a geometrical action say that a system rests at one point of the phase space. If one wishes to have more interesting dynamics - for example dynamics of the ideal fluid - then one needs to include some hamiltonian which could a priori be an arbitrary function on a phase space. In the case at hand such functions are given by arbitrary "gauge invariant" functionals of $A^{(X,Y)}$. One obvious sample is the "curvature" $F^{(X,Y)}$.

Consider a function $f$ defined by

$$F^{(X,Y)} = f \omega \quad (2.19)$$

An arbitrary functional of $f$, in particular, any power of $f$ also gives a function on $\mathcal{O}$. Taking a local function of $f$ and integrating it over $M$ with the measure $\omega$, one obviously gets an invariant of the coadjoint action of APD and hence an invariant of any hamiltonian flow on $\mathcal{O}$. However one should keep in mind that on some orbits powers of $f$ are ill-defined (before any quantization!), see section 3.

To describe the ideal fluid Hamiltonian we for simplicity take $M$ to be a disc $D$ and in some parametrization $(\sigma, \tau)$ (consistent with (2.2)) pick the metric

$$g_{\mu, \nu} dz^\mu \otimes dz^\nu = d\sigma \otimes d\sigma + d\tau \otimes d\tau$$

The metric allows us to construct another function the $\mathcal{O}$,

$$\mathcal{H} = \frac{1}{2} \langle A^{(X,Y)}, \rho_{A^{(X,Y)}} \rangle \quad (2.20)$$

where $\rho_A$ is a divergenless vector field given by

$$\rho_A^\mu \partial_\mu = (g^{\mu, \nu} - \frac{\partial^\mu \partial^\nu}{\partial^2}) A_\nu \partial_\mu \quad (2.21)$$
with $\partial^2$ being laplacian.

$\mathcal{H}$ (2.20) is the sought hamiltonian. To see this, notice that the variation of the canonical action $S - \int dt \mathcal{H}$ is equal to

$$\int dt (\Omega_{A(x,y)}(\delta_A A^{(x,y)}, d/dt A^{(x,y)}) - \langle \delta_A A^{(x,y)}, \rho_{A(x,y)} \rangle)$$

(2.22) (we have utilized (2.16)). (2.22) is equivalent to the statement that

$$d/dt F^{(x,y)} + \delta_{\rho_{A(x,y)}} F^{(x,y)} = 0$$

what, taking into account (2.7), (2.19) and (2.21), gives the equation of 2D flow of the ideal fluid (with $f$ from (2.19) being nothing but the vorticity).

The Hamiltonian (2.20) might have something to do with the recently proposed theory of 2D turbulence [Po]. Canonically, the time-independent distribution is given by $\exp(-\beta \mathcal{H})$ while the measure of integration (to compute correlators) is defined from the symplectic form $\Omega$ (2.13). Naively, there is no place for turbulence. For example, enstrophy, being one of the APD invariants discussed above, can be just moved away from any correlator. However, the necessity of the ultraviolet cut off in the integration over $\mathcal{O}$ (UV-cut off in a "physical" fluid is provided by viscosity, as argued in [Po]; note also that on some orbits of $w_\infty$ the Hamiltonian has to be regularized even in classical case, see next section) changes the situation. The flows are not hamiltonian any more and hence the orbits (and their invariants!) are not invariant. In this case something has to be changed in the original definition of the correlators because the regularized system, instead of walking along the phase space $\mathcal{O}$, moves somehow transversally to it. Polyakov points out in his paper an heuristic similarity of the enstrophy flux in turbulence theory with the axial anomaly in field theory. In our approach there is a very natural place for this analogy. A coadjoint orbit can be viewed as a factor of the group by an isotropy subgroup, in group variables the latter appearing as a gauge symmetry of the action. Moving transversally to the orbit looks very much like an anomaly in this gauge symmetry. We will return to this problem somewhere else.
3 Action and Observables for $\bar{w}_N$ case

The standard definition of $w_{\infty}$ [Ba] is given in terms of generators

$$w^j_m, \ m = 0, \pm 1, \pm 2, \ldots \ j = 0, 1, 2, \ldots$$

with the Lie bracket

$$[w^j_m, w^k_n] = (m(k + 1) - n(j + 1))w^{j+k}_{m+n}$$

This algebra can be viewed as a Poisson algebra formed by Hamiltonians

$$h^j_m = -i\tau^{j+1} \exp(im\sigma)$$

with the Poisson bracket defined by the symplectic form (2.3) or, equivalently, as the algebra of (analytical) hamiltonian vector fields, related to (3.3) by the rule (2.11). The underlying manifold is then identified naturally with the semi-infinite cylinder $\tau \geq 0, \sigma + 2\pi \sim \sigma$. On the semi-infinite cylinder one need not make a difference between area preserving and hamiltonian vector fields, furthermore on the semi-infinite cylinder $j = -1$ is forbidden by the requirement that vector fields from LieAPD must be tangent to the boundary ($\tau = 0$).

To describe (LieAPD)$^*$ take the dual basis $f^j_m$ defined by

$$\langle f^j_m, w^k_n \rangle = i\delta^{j,k}\delta_{m+n,0}$$

We will consider only orbits, corresponding to the highest weight representations, that is orbits with representative $f$ of the type

$$f = \sum_j b^j f^j_0$$

where $b^j$ are numbers.

The coadjoint action is defined as usual by

$$\langle \delta_{w^j_m} f, w^k_n \rangle + \langle f, [w^j_m, w^k_n] \rangle = 0$$

8
Now consider orbits whose representative (3.4) consists only of a finite number of $f^j_0$, that is there exists such an $N$ that

$$b^N \neq 0 \text{ and } b^j = 0 \text{ for } j > N \tag{3.7}$$

From the definitions (3.3)-(3.6) it is easy to see that an isotropy subalgebra $I_N$ for an orbit of the type (3.5), (3.7) is generated by

$$w^j_0, \, j \leq N \text{ and } w^j_m, \, j > N, \text{ every } m \tag{3.8}$$

So the orbits (3.5), (3.7) are in fact orbits of $\bar{\omega}_{N+2}$, particularly at $N = 0$ - Virasoro orbits. One can say that these orbits possess a hidden $w_\infty$ symmetry but it is very well hidden (the algebra of constraints forms an ideal in the algebra of observables). Note that for the algebra (3.1), (3.2) with the central extension (which could be nontrivial only in the Virasoro subalgebra [Ba] and is given by Gel’fand-Fuks cocycle) there exists a richer zoo the types of isotropy subalgebras being in direct correspondence to ones of the Virasoro coadjoint orbits discussed e.g. in [Wit2].

In the more explicit notations of section 2, the basic element $f^j_m$ can be represented as $f^j_m(\sigma, \tau)$ (the function $f(\sigma, \tau)$ from (2.19))

$$f^j_m(\sigma, \tau) = (-1)^j \partial^{j+1} \delta(\tau) \exp(im\sigma) \tag{3.9}$$

Specifying $M$ of section 2 and substituting representative $A$ corresponding to $f$ (3.5) with $f^j_m$ from (3.9) into the action (2.17) one gets

$$S = -\sum_j b_j \sum_{k=0}^{j+1} \binom{j+1}{k} \int dtd\sigma y_k \dot{x}_{j+1-k} \tag{3.10}$$

where $y_k$ ($x_\tau$) stands for $d^k/d\tau^k \, y_{/\tau=0}$ ($d^k/d\tau^k \, x_{/\tau=0}$). The derivatives of $x$ and $y$ are not independent, the derivatives of $y$ being expressed in terms of $x$ due to the constraint (2.5) (henceforth we say (2.5) meaning its analog for $(x,y)$) and boundary condition

$$y(\sigma,0) = 0 \tag{3.11}$$

9
For example, restricting eq.(2.5) to the boundary $\tau = 0$ gives

$$y_1 = \frac{1}{x_{0,\sigma}}$$  \hspace{1cm} (3.12)

Hitting (2.5) by $\partial/\partial \tau$ and putting $\tau = 0$ gives

$$y_2 = -\frac{\partial_\sigma(x_1x_{0,\sigma})}{(x_{0,\sigma})^3}$$  \hspace{1cm} (3.13)

and so on.

Consider for example the case $N = 1$ ($\tilde{w}_3$ - case). Plugging (3.12), (3.13) into (3.10) gives

$$S = \int dt d\sigma \left( -b^0 \frac{\dot{x}_0}{x_{0,\sigma}} - 2b^1 \frac{\dot{x}_1}{x_{0,\sigma}} + b^1 \frac{\dot{x}_1(x_1x_{0,\sigma})_{\sigma}}{(x_{0,\sigma})^3} \right)$$  \hspace{1cm} (3.14)

As we already said, $w_\infty$ admits a central extension [Ba] given by the Gel’fand-Fuks cocycle defined on the boundary $\tau = 0$:

$$\alpha(\epsilon, \zeta) = \frac{1}{48\pi} \int d\sigma (\partial_\sigma^3 \epsilon \zeta - \epsilon \partial_\sigma^3 \zeta)_{\tau=0}$$  \hspace{1cm} (3.15)

where $\epsilon$ and $\zeta$ are vector fields from LieAPD (recall that by definition they are tangent to the boundary).

Switching the central extension results in c-term contributions to the symplectic form $\Omega$ (2.13) and to the action (2.17) (or (3.10)), the contributions to the action being the 2D-gravity Wess-Zumino term [Po2]:

$$\Delta S = -\frac{c}{24\pi} \int dt d\sigma \left( \frac{\dot{x}_{0,\sigma}x_{0,\sigma}}{(x_{0,\sigma})^2} - \frac{\dot{x}_0(x_{0,\sigma})^2}{(x_{0,\sigma})^3} \right)$$  \hspace{1cm} (3.16)

where $c$ is dual to the central element of the Lie algebra. Due to the fact that switching the central extension affects only the Virasoro subalgebra, all the formulae on the way to (3.16) are parallel to those in [AlSha] and it is not fun to reproduce them here.

Thus we claim that the action given by the sum of the terms (3.10) and (3.16) is the action for $\tilde{w}_{N+2}$ gravity ((3.14) and (3.16) for $\tilde{w}_3$-case).
The observables for $\bar{w}_{N+2}$ gravity in the present formalism read

$$w^j_m = \int f^{(X,Y)} h^j_m \omega$$

(3.17)

with $h^j_m$ from (3.3) and $f^{(X,Y)}$ is $f$ from (3.5),(3.9) moved by the element

$(X,Y)$ of (centrally extended) APD:

$$f^{(X,Y)}(\sigma, \tau) = f(X(\sigma, \tau), Y(\sigma, \tau)) - c/24\pi \partial_\tau \delta(\tau) s(X_0, \sigma)$$

(3.18)

where $\delta(\tau)$ is the Dirac $\delta$-function and $s(X, \sigma)$ is the Schwartzian derivative,

$$s(X, \sigma) = \frac{\partial^3 X}{\partial \sigma X} - \frac{3}{2} \left( \frac{\partial^2 X}{\partial \sigma X} \right)^2$$

Making in (3.17) a change of variables $(\sigma, \tau) \rightarrow (x(\sigma, \tau), y(\sigma, \tau))$ one gets

$$w^j_m = i \sum_k b^k \int d\sigma \partial_x^{k+1} (y^{j+1} \exp(imx))/\tau=0$$

$$+ i \frac{c}{24\pi} \delta^{j,0} \int d\sigma \frac{s(x_0, \sigma)}{x_0, \sigma} \exp(imx_0)$$

(3.19)

Note that due to the boundary condition (3.11) $w^j_m$ automatically equals zero for $j > N$.

Quantization of the theory thus obtained will for sure be considered somewhere else.

4 Ricci Form on $\bar{w}_N$ Orbits

In this section we adopt the ideology developed in [BoRa] for the case of the bosonic (“Virasoro”) string. They suggested to consider its Hilbert space as some holomorphic homogeneous bundle over general coadjoint Virasoro orbit (an orbit of the type $diff S^1/S^1$ which proved to be a Kähler manifold). The bundle is proposed to be a tensor product of a matter (bosonic) bundle and of a ghost vacuum bundle over $diff S^1/S^1$, the former being tuned in such a way that its curvature cancels the curvature of the latter. As explained in
[BoRa] the curvature of the ghost bundle is given by the Ricci form of the base Kähler manifold \( \text{diff} S^1 / S^1 \) in their case. They computed the Ricci form of \( \text{diff} S^1 / S^1 \) and it appeared to be equal

\[
Ric(L_m, L_n) = \left( -\frac{26}{12} m^3 + \frac{1}{6} m \right) \delta_{m+n,0} \quad (4.1)
\]

(the generators of the Virasoro algebra \( L_m \) are viewed in (4.1) as vector fields on the coadjoint orbit and (4.1) gives a value of the Ricci form on them). The number 26 in (4.1) gives the value of the critical central charge for the bosonic \( (w_0) \) string.

We transfer all the ideology and a lot of technical details from [BoRa] to \( w_\infty \)-case (the technique of Ricci form computation for an infinite-dimensional homogeneous Kähler manifold has been developed in [Fr]).

First, one easily sees that \( \bar{w}_N \) orbit (3.5), (3.7) has a homogeneous complex structure consistent with the symplectic form, thus being a homogeneous Kähler manifold. Indeed, the complex structure \( J \) on the orbit \( \mathcal{O} \) is built in a canonical way from the following decomposition of the algebra \( w_\infty \):

\[
w_\infty = I_N \oplus w_N^+ \oplus w_N^- \quad (4.2)
\]

where \( I_N \) is the isotropy subalgebra (3.8), \( w_N^+ \) (\( w_N^- \)) is spanned by \( w_m^j \), \( j \leq N, m > 0 \) (\( w_m^j \), \( j \leq N, m < 0 \)).

Define a vector field \( \zeta_{w_m^k} \) on \( \mathcal{O} \) in such a way that its action on functions on \( \mathcal{O} \) is just the coadjoint action. Then on two such vectors at point \( f \) the symplectic form \( \Omega \) (2.8) takes the value

\[
\Omega_f(\zeta_{w_m^k}, \zeta_{w_m^j}) = ib^{k+j} \delta_{m+l,0}(k + j + 2) \quad (4.3)
\]

with \( b^k \) in (4.3) and (3.5), (3.7) being the same. The form (4.3) is apparently consistent with the complex structure (4.2), hence \( \mathcal{O} \) is a Kähler manifold.

The Kähler metric \( g^K \) on vectors \( \zeta_{w_m^k} \) and \( \zeta_{w_m^j} \) at a point \( f \) takes the value

\[
g_f^K(\zeta_{w_m^k}, \zeta_{w_m^j}) = \Omega_f(\zeta_{w_m^k}, J\zeta_{w_m^j}) = b^{k+j}(k + j + 2) \quad (4.4)
\]
We will need some more general Hermitian metric on $O$

$$g_f(\zeta w^k, \zeta w^l) = g^{k+l}_i$$

with $g^k_i = 0$ at $k > N$. Consider now the metric-compatible-(with the metric (4.5))-Hermitian connection $\nabla \zeta w^k$ on $O$ and define the operator $\phi$ on $TO_f$ as

$$\phi \zeta w^k V = \nabla \zeta w^k V - L \zeta w^k V$$

(4.6)

Obviously, $\phi$ is an ultralocal (tensorial) operator preserving the complex structure and the metric. Due to these properties it is easily computable in general [Fr], [BoRa], and in our case it reads (one needs only the restriction of $\phi$ to $TO_f^+$, so in the formulae below $j \leq N$, $n > 0$)

(i \leq N)

$$\phi \zeta w^i \zeta w^j = (i + 1)n\zeta w^{i+j}$$

(4.7a)

(i > N)

$$\phi \zeta w^i \zeta w^j = -(j + 1)m + (i + 1)n\zeta w^{i+j}$$

(4.7b)

$m > 0, i \leq N$

$$\phi \zeta w^i \zeta w^j = ((j + 1)m + (i + 1)n)\zeta w^{i+j} \theta(N - i - j + 1)\theta(n - m)$$

(4.7c)

$m > 0, i \leq N$

$$\phi \zeta w^i \zeta w^j = -\sum_{i,j}((j + 1)m + (i + 1)(p + m))\theta(N - i - j + 1)$$

(4.7d)

$$(g^{-1}_{p+i,j}g_{p+i,j}^k \zeta w^p)$$

where $(g^{-1}_{p,i,j})$ is the matrix inverse of $(g_{p})^{i,j}$ and $\theta(j)$ is $1$ when $j > 0$ and $0$ otherwise.

The curvature $R$ of the connection $\nabla \zeta$ in terms of $\phi$ reads

$$R(\zeta w^m, \zeta w^n)\zeta w^k = \left(\left[\phi \zeta w^m, \phi \zeta w^n\right] - \phi \zeta w^m \zeta w^n \right) \zeta w^k$$

(4.8)
$R$ (4.8) is considered as an operator on $TO_f^+$ and the Ricci form is, by definition, the trace of it. Thus, to compute the Ricci form we need only the diagonal elements of $R$ and they are easily seen to be

$$
\text{diag}_p^k R(\zeta_{w^i_m}, \zeta_{w^j_m})
$$

$\theta(N - i - j + 1)\theta(p - m)\theta(N - i - s + 1)(g^{k,s}_{p - m}g^{s,k}_{p})$  

$\theta(N - i - s + 1)\theta(N - k + 1)(g^{k,s}_{p+m})$  

$\theta(N - i - j + 1)\theta(N - i - s + 1)(g^{k,s}_{p - m}g^{s,k}_{p})$  

$- 2mp$

To obtain the Ricci form one needs to sum over $k$ (up to $N$) and over positive $p$. From the fact that $g^{k,s}_{p}$ is indeed $g^{k,s}_{p}$ it follows that the diagonal element (4.9) is nonzero only at $i = j = 0$. Hence the Ricci form is expressed as

$$
\text{Ric}(\zeta_{w^i_m}, \zeta_{w^j_m}) = \delta^{i+j,0} \delta_{m+n,0} \sum_p
$$

$$
(-\theta(p - m)\sum_{s,k}((k + 1)m + p)((s + 1)m + p)(g^{k,s}_{p - m}g^{s,k}_{p})$  

$+ \sum_{s,k}((k + 1)m + p + m)((s + 1) + p + m)(g^{k,s}_{p+m})g^{s,k}_{p})$  

$- 2mp(N + 1))$

Arbitrary $g^{k+s}_{p}$ can be expanded as

$$
g^{k+s}_{p} = \sum_{q=0}^{N} a^{N}_{q,p} \delta^{k+s,N-q}
$$
and plugging (4.11) into (4.10) gives

\[ \text{Ric}(\zeta_{w^i_m}, \zeta_{w^j_n}) = \delta^{i+j,0} \delta_{m+n,0} \sum_p \]

\[ (-\theta(p - m) \sum_{k,s} ((k + 1)m + p)((s + 1)m + p) \frac{a_{0,p-m}^N}{a_{0,p}^N} \]

\[ + \sum_{k,s} ((k + 1)m + p + m)((s + 1)m + p + m) \frac{a_{0,p}^N}{a_{0,p+m}^N} \]

\[ - 2mp(N + 1)) \]  

(4.12)

We come to a subtle point. The sum over \( p \) in (4.12) is divergent for the Kähler metric (4.4) (that is for \( a_{0,p} = b^p(N + 2) \)). The Ricci form on the orbit \( \mathcal{O} \) of \( w_\infty \) is ill-defined. To define it somehow, note that there exists a class of Hermitian metrics on \( \mathcal{O} \) for which a sum over \( p \) in (4.12) is absolutely convergent. Indeed, one sees that

\[ a_{0,p} = A_1^N p^{N+3} + A_2^N p^{N+1} + ... \]  

(4.13)

\( A_k^N \) are parameters) and arbitrary \( a_{q,p}^N \) for \( q > 0 \) define such a class of metrics and with (4.13) one can proceed with the computation of the Ricci form, getting finally

\[ \text{Ric}(\zeta_{w^i_m}, \zeta_{w^j_n}) = \left( -\frac{26 + 24N^3 + 46N^2 + 46N}{12} \right) m^3 + \frac{N + 1}{6} m \]  

(4.14)

Remarkably, the Ricci form is apparently independent on the parameters \( A_k^N \) from (4.13). To explain this somehow, note that on a finite-dimensional manifold the cohomology class defined by the Ricci form is independent of the metric and the proof can be transfered to the infinite-dimensional case provided the Ricci form is well defined (the sum is convergent). The term \( m^3 \) defines the nontrivial cohomology class on \( \mathcal{O} \) and its independence on the metric is very satisfactory. We don’t have any explanation why the \( m^1 \) term in (4.14) is metric independent.

Note that within the class (4.13) one can come as close as necessary to the Kähler metric while keeping the Ricci form convergent. This and the
independence of the Ricci form of the parameters $A^N_k$ is a justification of our regularization.

From (4.14) one can read off the critical central charge $c_{N+2}$ and the intercept $\alpha_0^{N+2}$ (for the Virasoro mode) for the $\bar{w}_{N+2}$ string:

$$c_{N+2} = 26 + 4N^3 + 24N^2 + 46N, \quad \alpha_0^{N+2} = 1 + \frac{1}{6}N^3 + N^2 + \frac{11}{6}N$$  

(4.15)

Amusingly, they are the same as for $W_{N+2}$ string.

5 Conclusion

The appearance of the $\bar{w}_N$ algebra may be considered a little more abstractly than in section 3. First, the space of vector fields on the boundary $\Gamma$ of some 2D manifold $M$ is easily seen to be isomorphic to the factor of the algebra of APD vector fields on $M$ with respect to the subalgebra of vector fields disappearing on $\Gamma$. The latter is in fact an ideal, so that the factor has a structure of a Lie algebra and is isomorphic to the algebra of vector fields on $S^1$. There are smaller ideals in APD in which vector fields disappear on $\Gamma$ with their $N$-th derivatives. The factor of APD with respect to such $N$-th ideal is isomorphic to the $\bar{w}_{N+2}$ algebra.

Physically speaking, higher spin fields are remnants of the extra-dimension. 2D gravity ($w_0$ gravity) can be considered as having $w_\infty$ symmetry, the latter being very well hidden in a sense that the constraints form an ideal in the algebra of observables. Then the constants $b^j$ in the lagrangean (3.10) look like condensates, spontaneously breaking the very well hidden $w_\infty$ symmetry further and further.

The present analysis of $\bar{w}_N$ doesn’t help a lot in the understanding of $W_N$. It would be interesting to understand what are the first principles ruling the deformation of $\bar{w}_N$ gravity to the $W_N$ one. A natural guess is that the deformation follows from a quantization of the former, work on which is in progress.
Also very interesting would be to understand how to get a standard two-dimensional formulation of $w_\infty$ gravity [BHPRSShSt] from our 3D one (see action (2.17) or (2.18)).

Concerning 2D turbulence, our approach seems to be very promising. The Schwinger mass term is intimately related to the axial anomaly which is field theory analog of the turbulence fluxes [Po]. It suggests to make a fermionization of the turbulence problem which is interesting by itself as well as in view of the conformal perspective in 2D turbulence theory. We will be back to all these stuffs somewhere else.

Acknowledgments.

I am grateful to my colleagues from the University of Bonn for their kind hospitality, especially to Niels Obers whose information store and whose knowledge of computer tricks helped me a lot.

Note added in proof.

After the work was completed, A.S.Gorsky informed me that some of the results of section 2 intersected with the ones of [Khe]. And a few days later R.Mkrtchyan came to the University of Bonn with a talk [MM] devoted to the geometrical action for APD of 2D manifold (see first part of the section 2).

References

[AlSha] A. Alekseev and S. Shatashvili, Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2D Gravity, Nuclear Phys. B323 (1989) 719-733

[Arn] V.I. Arnold, Mathematical Methods of Classical Mechanics, (Springer-Verlag, New-York, 1978)

[Ba] I. Bakas, The large-$N$ limit of extended conformal systems, Phys. Lett. 228B (1989) 57

[BHPRSShSt] E. Bergshoeff et al, $w_\infty$ gravity, Phys. Lett. 243B (1990) 350, Quantization deforms $w_\infty$ to $W_\infty$ gravity, Nuclear Physics B363 (1991) 163
[BoRa] M.J. Bowick and S.G. Rajeev, The Holomorphic Geometry of Closed Bosonic String Theory and $DiffS^1/S^1$, Nuclear Physics B293 (1987) 348-384

[Fr] D.S. Freed, in Infinite dimensional groups with applications, ed. V. Kac, (Springer, Berlin, 1985)

[Khe] B. Khesin, Zapiski Nauchnykh Seminarov LOMI, 1991

[KlePo] I. Klebanov and A.M. Polyakov, Mod. Phys. Lett., A6 (1991) 3273

[Ko] I.I. Kogan, Mod. Phys. Lett. A7 (1992) 3717

[MM] R. Manvelyan and R. Mkrtchyan, Geometrical action for $w_\infty$ algebra, to appear

[Po] A.M. Polyakov, The Theory of Turbulence in 2D, Preprint PUPT-1369 (1992), Preprint PUTP-1341 (1992)

[Po2] A.M. Polyakov, Mod. Phys. Lett., A11 (1987) 893

[Pop] C.N. Pope, A Review of $W$ String, Preprint CTP TAMU-30/92 (1992)

[Wie] P.B. Wiegman, Multivalued functionals and geometric approach to quantization of relativistic particles and strings, preprint MIT (1988)

[Wit] E. Witten, Nucl. Phys. B373 (1992) 187

[Wit2] E. Witten, Coadjoint Orbits of the Virasoro Group, Comm. Math. Phys. 114, 1-53 (1988)