A NOTE ON SIERPİŃSKI PROBLEM RELATED TO TRIANGULAR NUMBERS

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Abstract. In this note we show that the system of equations

\[ tx + ty = tp, \quad ty + tz = tq, \quad tz + tx = tr, \]

where \( t_x = \frac{x(x+1)}{2} \) is a triangular number, has infinitely many solutions in integers. Moreover we show that this system has rational three-parametric solution. Using this result we show that the system

\[ tx + ty = tp, \quad ty + tz = tq, \quad tz + tx = tr, \quad t_x + ty + tz = ts \]

has infinitely many rational two-parametric solutions.

1. Introduction

By a triangular number we call the number of the form

\[ t_n = 1 + 2 + \ldots + n - 1 + n = \frac{n(n+1)}{2}, \]

where \( n \) is a natural number. Triangular number can be interpreted as a number of circles necessary to build equilateral triangle with a side of length \( n \). There are a lot of papers related to the various types of diophantine equations containing triangular numbers and various generalizations of them [3, 4, 5, 6, 7]. One of my favourite is a little book [6] written by W. Sierpiński.

On the page 33. of his book W. Sierpiński stated an interesting question related to the triangular numbers. This question is following:

Question 1.1. Is it possible to find three different triangular numbers with such a property that sum of any two is a triangular number? In other words: is it possible to find solutions of the system of equations

\[ (1) \quad tx + ty = tp, \quad ty + tz = tq, \quad tz + tx = tr, \]

in positive integers \( x, y, z, p, q, r \) satisfying the condition \( x < y < z \)?

In the next section we give all integer solutions of the system (1) satisfying the condition \( x < y < z < 1000 \) and next we construct two one-parameter polynomial solutions of our system (Theorem 2.1).

In section 3 we change perspective a bit and ask about rational parametric solutions of our problem. Using very simple reasoning we are able to construct rational parametric solution with three variable parameters (Theorem 3.2).

Finally, in the last section we consider the system of equations

\[ tx + ty = tp, \quad ty + tz = tq, \quad tz + tx = tr, \quad t_x + ty + tz = ts. \]

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We give some integer solutions of this system. Next, we use parametrization obtained in the Theorem 3.2 to obtain infinitely many rational solutions depending on two parameters. In order to prove this we show that certain elliptic curve defined over the field $\mathbb{Q}(u, v)$ has positive rank.

### 2. Integer solutions of the system $t_x + t_y = t_p$, $t_y + t_z = t_q$, $t_z + t_x = t_r$

In order to find integer solutions of the system (1) we use the computer. We are looking for solutions satisfying the condition $x < y < z < 1000$. We find 44 solutions in this range. Results of our search are presented in Table 1.

| $x$ | $y$ | $z$ | $p$ | $q$ | $r$ |
|-----|-----|-----|-----|-----|-----|
| 9   | 13  | 44  | 16  | 46  | 45  |
| 14  | 51  | 104 | 53  | 116 | 105 |
| 20  | 50  | 209 | 54  | 215 | 210 |
| 23  | 30  | 90  | 38  | 95  | 93  |
| 27  | 124 | 377 | 127 | 397 | 378 |
| 35  | 65  | 86  | 74  | 108 | 93  |
| 35  | 123 | 629 | 128 | 641 | 630 |
| 41  | 119 | 285 | 126 | 309 | 288 |
| 44  | 245 | 989 | 249 | 1019| 990 |
| 51  | 69  | 104 | 86  | 125 | 116 |
| 54  | 143 | 244 | 153 | 283 | 250 |
| 62  | 99  | 322 | 117 | 337 | 328 |
| 65  | 135 | 209 | 150 | 249 | 219 |
| 66  | 195 | 365 | 206 | 414 | 371 |
| 74  | 459 | 923 | 465 | 1031| 926 |
| 76  | 90  | 144 | 118 | 170 | 163 |
| 77  | 125 | 132 | 147 | 182 | 153 |
| 77  | 125 | 207 | 147 | 242 | 221 |
| 83  | 284 | 494 | 296 | 570 | 501 |
| 105 | 170 | 363 | 200 | 401 | 378 |
| 105 | 363 | 390 | 378 | 533 | 404 |
| 105 | 551 | 924 | 561 | 1076| 930 |

| $x$ | $y$ | $z$ | $p$ | $q$ | $r$ |
|-----|-----|-----|-----|-----|-----|
| 114 | 429 | 650 | 444 | 779 | 660 |
| 131 | 174 | 714 | 218 | 735 | 726 |
| 131 | 245 | 714 | 278 | 755 | 726 |
| 135 | 154 | 531 | 205 | 553 | 548 |
| 161 | 260 | 924 | 306 | 960 | 938 |
| 170 | 469 | 755 | 499 | 889 | 774 |
| 189 | 305 | 406 | 359 | 508 | 448 |
| 216 | 390 | 854 | 446 | 939 | 881 |
| 230 | 741 | 870 | 776 | 1143| 900 |
| 237 | 527 | 650 | 578 | 837 | 692 |
| 245 | 714 | 989 | 755 | 1220| 1019|
| 252 | 272 | 702 | 371 | 753 | 746 |
| 278 | 370 | 594 | 463 | 700 | 656 |
| 286 | 405 | 494 | 496 | 639 | 571 |
| 293 | 390 | 854 | 488 | 939 | 903 |
| 299 | 441 | 560 | 533 | 713 | 635 |
| 350 | 629 | 781 | 720 | 1003| 856 |
| 476 | 634 | 665 | 793 | 919 | 818 |
| 581 | 774 | 935 | 968 | 1214| 1101|
| 588 | 645 | 689 | 873 | 944 | 906 |
| 609 | 779 | 923 | 989 | 1208| 1106|
| 714 | 798 | 989 | 1071| 1271| 1220|

**Table 1**

The above solutions show that the answer on Sierpiński’s question is easy. However, due to abundance of solutions it is natural to ask if we can find infinitely many solutions of the system (1).

**Theorem 2.1.** The system of equations (1) has infinitely many solutions in integers.

**Proof.** In order to find infinitely many solutions of our problem we examined the Table 1. We find that for any $u \in \mathbb{N}$ the values of polynomials given by

\[
\begin{align*}
    x &= (u+1)(2u+5), \\
    y &= (u+2)(2u^2 + 8u + 7), \\
    z &= (2u^2 + 7u + 4)(2u^2 + 7u + 7)/2, \\
    p &= 2u^3 + 12u^2 + 24u + 15, \\
    q &= (4u^4 + 28u^3 + 73u^2 + 87u + 40)/2, \\
    r &= (4u^4 + 28u^3 + 71u^2 + 77u + 30)/2.
\end{align*}
\]
or

\[x = (u + 3)(2u + 3), \quad p = 2u^3 + 12u^2 + 24u + 16,
\]

\[y = (u + 1)(2u^2 + 10u + 13), \quad q = (4u^4 + 36u^3 + 121u^2 + 177u + 92)/2,
\]

\[z = (2u^2 + 9u + 8)(2u^2 + 9u + 11)/2, \quad r = (u + 2)(u + 3)(2u + 3)(2u + 5)/2,
\]

are integers and are solutions of the system (1).

Using the parametric solutions we have obtained above we get

**Corollary 2.2.** Let \( f(X) = X(X + a) \), where \( a \in \mathbb{Z} \setminus \{0\} \). Then there is infinitely many positive integer solutions of the system

\[(*) \quad f(x) + f(y) = f(p), \quad f(y) + f(z) = f(q), \quad f(z) + f(x) = f(r).
\]

**Proof.** It is clear that we can assume \( a > 0 \). Now, note that if sextuple \( x, y, z, p, q, r \) is a solution of system (1), then the sextuple \( ax, ay, az, ap, aq, ar \) is a solution of the system (*).

In the light of the above corollary we end this section with the following

**Question 2.3.** Let \( f \in \mathbb{Z}[x] \) be a polynomial of degree two with two distinct roots in \( \mathbb{C} \). Is the system of equations

\[f(x) + f(y) = f(p), \quad f(y) + f(z) = f(q), \quad f(z) + f(x) = f(r),
\]

solvable in different positive integers ?

3. **Rational solutions of the system**

In the view of the Theorem 2.1 it is natural to state the following

**Question 2.3.** Is the set of one-parameter polynomial solutions of the system (1) infinite?

We suppose that the answer for this question is YES. Unfortunately, we are unable to prove this. So, it is natural to ask if instead polynomials we can find rational parametric solutions.

**Theorem 3.2.** There is three-parameter rational solution of the system (1).

**Proof.** Let \( u, v, w \) be a parameters. Let us note that the system (1) is equivalent to the system

\[
\begin{aligned}
y &= u(p - x), \quad u(y + 1) = p + x + 1, \\
z &= v(q - y), \quad v(z + 1) = q + y + 1, \\
x &= w(r - z), \quad w(x + 1) = r + z + 1.
\end{aligned}
\]

Because we are interested in rational solutions, we can look on the system of equations (2) as on the system of linear equations with unknowns \( x, y, z, p, q, r \). This system has a solution given by

\[
\begin{aligned}
x(u, v, w) &= \frac{(u - 1)w(1 + u - 2v + 2uv + v^2 + uw^2 + (-1 - u + v^2 + uv^2)w)}{(1 - u^2)(v^2 - 1)(w^2 - 1) + 8uvw}, \\
y(u, v, w) &= \frac{u(v - 1)(1 - u + v - uv + (-2 + 2v)w + (1 + u + v + uv)w^2)}{(1 - u^2)(v^2 - 1)(w^2 - 1) + 8uvw}, \\
z(u, v, w) &= \frac{v(w - 1)(1 - 2u + u^2 - v + u^2v + (1 + 2u + u^2 - v + u^2v)w)}{(1 - u^2)(v^2 - 1)(w^2 - 1) + 8uvw}.
\end{aligned}
\]
and the quantities p, q, r can be calculated from the identities

\begin{align*}
p(u, v, w) &= \frac{ux(u, v, w) + y(u, v, w)}{u}, \\
q(u, v, w) &= \frac{vy(u, v, w) + z(u, v, w)}{v}, \\
r(u, v, w) &= \frac{wz(u, v, w) + x(u, v, w)}{w}.
\end{align*}

Remark 3.3. It is clear that the same reasoning can be used in order to prove that the system

\begin{align*}
f(x) + f(y) &= f(p), \\
f(y) + f(z) &= f(q), \\
f(z) + f(x) &= f(r),
\end{align*}

where \( f \in \mathbb{Z}[x] \) is a polynomial of degree two with two different roots, has rational parametric solution depending on three parameters.

4. Rational solutions of the system

\begin{align*}
t_x + ty = t_p, \\
t_y + tz = t_q, \\
t_z + tx = t_r, \\
t_x + ty + tz = t_s.
\end{align*}

We start this section with quite natural Question 4.1. Is the system of equations

\begin{align*}
t_x + ty = t_p, \\
t_y + tz = t_q, \\
t_z + tx = t_r, \\
t_x + ty + tz = t_s.
\end{align*}

solvable in integers?

This question is mentioned in the very interesting book [1, page 292] and it is attributed to K. R. S. Sastry. In this book we can find triple of integers \( x = 11, y = z = 14 \) (which was find by Ch. Ashbacher). These number with \( p = r = 18, q = 20, s = 23 \) satisfy the system (3). It is clear that the question about different numbers \( x, y, z, p, q, r, s \) which satisfy (3) is more interesting. Using the computer search we find some 7-tuples of different integers which are the solutions of (3). Our results are contained in Table 2.

| x  | y  | z  | p   | q   | r   | s   |
|----|----|----|-----|-----|-----|-----|
| 230| 741| 870| 776 | 1143| 900 | 1166|
| 609| 779| 923| 989 | 1208| 1106| 1353|
| 714| 798| 989| 1071| 1271| 1220| 1458|
| 1224|1716|3219|2108|3648|3444|3848|

Table 2

It is quite likely that there is a polynomial solution of the system (3). However we are unable to find such a polynomials.

Now we use the parametric solutions obtained in theorem 3.2 to deduce the following

Theorem 4.2. The system of diophantine equations (3) has infinitely many rational solutions depending on two parameters.

Proof. We know that the functions \( x, y, z, p, q, r, s \in \mathbb{Q}(u, v, w) \) we have obtained in the proof of the Theorem 3.2 satisfied the system (3). So in order to find solutions of the system (3) it is enough to consider the last equation \( t_x + ty + tz = t_s \). If we
put calculated quantities \(x, y, z\) into the equation \(t_x + t_y + t_z = t_s\), use the identity \(8t_s + 1 = (2s + 1)^2\) and get rid of denominators we get the equation of quartic curve \(C\) defined over the field \(\mathbb{Q}(u, v)\) given by

\[ C : h^2 = a_4(u, v)w^4 + a_3(u, v)w^3 + a_2(u, v)w^2 + a_1(u, v)w + a_0(u, v) =: h(w), \]

where

\[ a_0(u, v) = a_4(-u, v) = (u - 1)^2(-1 + u + 2v + 2uv - v^2 + w^2)^2, \]

\[ a_1(u, v) = a_3(-u, v) = 4(u - 1)(v^2 - 1)(\frac{u^4 - 1}{u - 1}(v^2 + 1) + 2(u - 1)(u^2 + 4u + 1)v), \]

\[ a_2(u, v) = 4(1 - 10u^2 + u^4)v^2 + 8(u^4 - 1)v(1 + v^2) + 2(3 + 2u^2 + 3u^4)(1 + v^4). \]

Because the polynomial \(h \in \mathbb{Q}(u, v)[w]\) has not multiple roots the curve \(C\) is smooth. Moreover, we have \(\mathbb{Q}(u, v)\)-rational point on \(C\) given by

\[ Q = (0, (u - 1)(-1 + u + 2v + 2uv - v^2 + w^2)). \]

If we treat \(Q\) as a point at infinity on the curve \(C\) and use the method described in [2] page 77] we conclude that \(C\) is birationally equivalent over \(\mathbb{Q}(u, v)\) to the elliptic curve with the Weierstrass equation

\[ E : Y^2 = X^3 - 27f(u, v)X - 27g(u, v), \]

where

\[ f(u, v) = u^4(v^8 + 1) + 4u^2(u^4 - 1)v(v^6 + 1) + (1 + 8u^2 - 22u^4 + 8u^6 + u^8)v(1 + v^4) + 4(u^4 - 1)(u^4 - 3u^2 - 1)v(1 + v^2) + 2(3 - 16u^2 + 29u^4 - 16u^6 + 3u^8)v^4, \]

\[ g(u, v) = (u^2(v^4 + 1) + 2(u^4 - 1)v(v^2 + 1) + 2(2 - 5u^2 + 2u^4)v^2) \times (-2f(u, v) + 3(u^2 - 1)^2v^2(1 + u^2 - 2v + 2u^2v + v^2 + u^2v^2)^2). \]

The mapping \(\varphi : C \ni (w, h) \mapsto (X, Y) \in E\) is given by

\[ w = a_4(u, v)^{-1}\left(\frac{16a_4(u, v)^2Y - 27d(u, v)}{24a_4(u, v)X - 54c(u, v)}\right) - \frac{a_3(u, v)}{4} \]

\[ h = a_4(u, v)^{-2}\left(\frac{16a_4(u, v)^2Y - 27d(u, v)}{24a_4(u, v)X - 54c(u, v)}\right)^2 + \frac{8a_4(u, v)X}{9} + c(u, v). \]

We should note that the quantity \(a_4(u, v)^{\frac{2}{3}} = ((u + 1)(1 + u + \ldots))^3\) is a polynomial in \(\mathbb{Z}[u, v]\), so our mapping is clearly rational. The quantities \(c, d \in \mathbb{Z}[u, v]\) are given below:

\[ c(u, v) = -4(u + 1)^2 \times \left( -u^2(u + 1)^2(v^8 + 1) + (u^2 - 1)(1 - 10u - 2u^2 - 10u^3 + u^4)v(v^6 + 1) + 2(u - 1)^2(3 - 4u^2 - 6u^3 + 4u^5 + 3u^4)v^2(v^4 + 1) + (u - 1)^2(15 + 10u + 2u^2 + 10u^3 + 10u^4)v^3(v^2 + 1) + 2(10 + 20u - 5u^2 - 46u^3 - 5u^4 + 20u^5 + 10u^6)v^4 \right) / 3, \]

\[ d(u, v) = 16(u - 1)u(u + 1)^4v(v^2 - 1) \times (-1 + u^2 + 2u^2v + v^2 + u^2v^2)(1 + u^2 - 2v + 2u^2v + v^2 + u^2v^2) \times (-1 + u^2 - 4v - 4u^2v + 10v^2 - 10u^2v^2 - 4v^3 - 4u^2v^3 - v^4 + u^2v^4). \]
In order to finish the proof of our theorem we must show that the set of $\mathbb{Q}(u, v)$-rational points on the elliptic curve $E$ is infinite. This will be proved if we find a point with infinite order in the group $E(\mathbb{Q}(u, v))$ of all $\mathbb{Q}(u, v)$-rational points on the curve $E$. In general this is not an easy task. First of all note that there is torsion point $T$ of order 2 on the curve $E$ given by

$$T = (3u^2(v^4 + 1) + 6(u^4 - 1)v(v^2 + 1) + 6(u^2 - 2)(2u^2 - 1)v^2, 0).$$

It is clear that this point is not suitable for our purposes. Fortunately in our case we can find another point

$$P = \left( \frac{3}{4}((3 - 2u^2 + 3u^4)(v^4 + 1) + 8(u^4 - 1)v(v^2 + 1) + 2(5 - 14u^2 + 5u^4)v^2), \frac{27}{8}(u^2 - 1)^2(v^2 - 1)((u^2 + 1)(v^4 + 1) + 4(u^2 - 1)v(v^2 + 1) + 6(u^2 + 1)v^2) \right).$$

Now, if we specialize the curve $E$ for $u = 2, v = 3$, we obtain the elliptic curve

$$E_{2, 3} : Y^2 = X^3 - 28802736X + 40355763840$$

with the point $P_{2,3} = (5736, 252720)$, which is the specialization of the point $P$. As we know, the points of finite order on the elliptic curve $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$ have integer coordinates [8] page 177], while

$$2P_{2,3} = (765489/100, -518102487/1000) ;$$

therefore, $P_{2,3}$ is not a point of finite order on $E_{2,3}$, which means that $P$ cannot be a point of finite order on $E$. Therefore, $E$ is a curve of positive rank. Hence, its set of $\mathbb{Q}(u, v)$-rational points is infinite and our theorem is proved.

Let us note the obvious

**Corollary 4.3.** Let $f \in \mathbb{Z}[x]$ be a polynomial of degree two with two distinct rational roots. Then the system of equations

$$f(x)+f(y) = f(p), \quad f(y)+f(z) = f(q), \quad f(z)+f(x) = f(r), \quad f(x)+f(y)+f(z) = f(s),$$

has infinitely many rational parametric solutions depending on two parameters.

**Example 4.4.** Using the method of proof of the above theorem we produce now an example of rational functions $x, y, z \in \mathbb{Q}(u)$ which satisfy system [3] for some $p, q, r, s \in \mathbb{Q}(u)$ which can be easily find (with the computer of course). Because the considered quantities are rather huge we put here $v = 2$. Then we have that

$$P_2 + T_2 = (1404u^4 + 219u^2 + 4, 8(9u^2 + 1)^2(81u^2 + 1)),$$

where $P_2, T_2$ are specializations of points $P, T$ in $v = 2$ respectively. Now, we have that

$$(w, h) = \varphi^{-1}(P_2 + T_2) =$$

$$\left( \frac{(9u - 1)^2(9u + 1)(63u^2 + 17)}{3(u + 1)(6561u^4 + 1134u^3 + 306u - 1)}, \frac{8(9u - 1)(9u + 1)F(u)}{9(u + 1)^2(6561u^4 + 1134u^3 + 306u - 1)^2} \right)$$

where

$$F(u) = 43046721u^8 + 11573604u^7 + 6388956u^5 + 1285956u^6 + 680886u^4 + 919836u^3 + 93636u^2 + 10404u + 1.$$
Using the calculated values and the definition of $x, y, z$ given in the proof of Theorem 3.2, we find that the functions

\[
x(u) = \frac{2(u - 1)(9u - 1)^2(9u + 1)^2(63u^2 + 17)(81u^2 + 1)}{G(u)},
\]
\[
y(u) = \frac{3u(11 + 42u^2 + 2187u^4)(1 + 2754u^2 + 3645u^4)}{G(u)},
\]
\[
z(u) = \frac{2(27u^2 - 18u - 5)(81u^2 - 48u - 1)(135u^2 + 18u + 7)(243u^3 - 99u^2 + 57u - 1)}{3G(u)},
\]

where

\[
G(u) = -23914845u^9 + 110008287u^8 - 18528264u^7 + 15956352u^6 + 473850u^5 - 940410u^4 - 91008u^3 - 96264u^2 - 33u + 35
\]

satisfied the system of equations (3).

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