Switching of behavior: From hyperchaotic to controlled magnetoconvection model

Cite as: AIP Advances 9, 125235 (2019); https://doi.org/10.1063/1.5129708
Submitted: 01 October 2019 . Accepted: 05 December 2019 . Published Online: 26 December 2019

Javeria Ayub, Muhammad Aqeel, Javeria Nawaz Abbasi, Danish Ali Sunny, and Zainab Rana
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Javeria Ayub, 1,a) Muhammad Aqeel, 1,b) Javeria Nawaz Abbasi, 2,c) Danish Ali Sunny, 1,d) and Zainab Rana 1,e)

AFFILIATIONS
1 Department of Applied Mathematics and Statistics, Institute of Space Technology, Islamabad 44000, Pakistan
2 Department of Mathematics, COMSATS University, Islamabad, Islamabad Campus, Islamabad 45550, Pakistan

a) Electronic mail: jayub8162511@gmail.com
b) Electronic mail: medraqeel@gmail.com
c) Electronic mail: javeria_nawaz@comsats.edu.pk
d) Electronic mail: danish53@gmail.com
e) Electronic mail: zainab.rana16@yahoo.com

ABSTRACT
The switching of behavior, from the hyperchaotic to controlled magnetoconvection model, is studied by a feedback control technique. The magnetoconvection model shows hyperchaotic oscillations for different values of parameters: Rayleigh number \( r \), Chandrasekhar number \( Q \), and diffusivity ratio \( l \). Chaotic responses of the magnetoconvection model are considered through boundedness and Lyapunov exponents to specify the place where the controller needs to be applied. The controller for the magnetoconvection model is calculated by using the concept of the Lie derivative, which is the most significant facet of control analytical techniques. Speed and dislocated feedback techniques are also utilized with the consideration of stability analysis through feedback gains. To show the advantages of the feedback control technique, we give a comparison with other control techniques such as speed and dislocated feedback techniques. Simulation results indicate that the analytical strategy for controlling the oscillation is effective and controlled within a small duration of time.

I. INTRODUCTION
A. Motivation
It was a great breakthrough discovery that irregular motion with strong oscillatory behavior is termed chaotic. Chaos is divided into two major classes: one in which we induce the oscillation and the other class in which we control this oscillatory behavior. The objective of the 2nd class of chaos is to construct a controlled model for the production of optional results that confirm control stability. However, the classical theory of optimal control problems is grouped according to a linear or nonlinear multivariable system of an ordinary differential equation. In the literature, controlling the oscillatory behavior of chaos has a lot of benefits such as magnetomechanical oscillators, oscillating chemical reactions, and cardiac tissues.1-7 The strong oscillatory behavior such as chaotic behavior is induced in most of the systems to be useful in secure communication as encryption of data.5-7 Chaotic models have been controlled by the implementation of control techniques for some beneficial purposes. These techniques provide the circumstances for a system to have controlled output for some specific input variables. Controlling of chaos is divided into two approaches: one is feedback control to keep up the desired results and the other one is nonfeedback control. Literature provides many techniques such as the Ott, Grebogi, and Yorke method, frequency continuous control, state space exact feedback linearization, the adaptive method, bang-bang control, time-delayed, and many others.8-15 In this paper, we have utilized three feedback techniques: state space linearization, speed, and dislocated feedback. State space linearization is an analytical technique that is based on the Lie derivative. In speed and dislocated feedback, the controller can be calculated through linear stability.

The magnetoconvection phenomenon16 represents the coupled stress fluid layers in which a simple, non-Newtonian fluid is...
considered, which has a significant effect on the fluid flow. In this phenomenon, magnetic field, induced current, and electromagnetic interactions are considered, which also have a strong effect on the motion of a fluid. Chertovskih et al. considered the magnetic field generated in three-dimensional space, which reported the hyperchaotic dynamical behavior without the conversion to ordinary differential equations. In addition, Layek and Pati constructed a 5D nonlinear system in convection with heat flux in non-Fourier form and established interesting dynamics. Those models related to the above-described phenomenon have vast applications in polymer flows, lubrication systems, medical sciences, the dynamo theory, and many others.

B. Hyperchaotic magnetoconvection model

The theoretical study of the chaotic convection in coupled-stress electrically conducted fluid layers is important technically as well as quite challenging. So, the system defined by equations in Ref. 16 helps in the deep dynamical and bifurcation analysis. Layek and Pati considered the infinite horizontal coupled-stress fluid layer confined between two stress free layers in the Cartesian coordinate system \((X, Y, Z)\). In this system, the \(X\)-axis positions the horizontal plane and the \(Z\)-axis represents the vertical plane. Here, additional considerations of Rayleigh-Benard convection are applied in the electrically conducting system. \(h\) describes the distances between the isothermal parallel layers, and \(D\) represents the large separation between two vertical sides. An externally applied magnetic field is denoted by \(B_0\), in the direction opposite to gravity. The author utilized the traditional hydromagnetic motion and heat transfer mathematical phenomena with induction equations to construct the model equation with initial and boundary conditions in (1). These conductive state perturbation equations to the nondimensional form has been calculated by the authors in Ref. 16. Finally, the four-dimensional nonlinear magnetoconvection system is stated with the tedious calculation,

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - cx_1 - Qx_4), \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - bx_3, \\
\dot{x}_4 &= l(mx_1 - x_4),
\end{align*}
\]

where \(a = \mu/\varrho_0\kappa\) is the Prandtl number and is a dimensionless parameter, \(c = 1 + \Lambda c_z\) is the modified coupled stress parameter, \(Q = B_0^2 h^2/\mu_0\eta\mu\) is the Chandrasekhar number, \(r = Ra/Re\), is the normalized Rayleigh number, \(l = \eta/\kappa\) is the diffusivity ratio, and the other parameters are \(b = 4\pi^2/\Lambda\), which is the geometrical parameter, and \(m = \pi^2/\Lambda^2\), which is the thermal expansion coefficient, and the remaining variables are listed in Table I. The variables defined in the model equations \(x_1\) to \(x_4\) are dimensionless and interaction variables, which define the interaction between the \(X\)-coordinate, \(Z\)-coordinate, \(T\)-temperature, and \(V\)-fluid velocity. Especially, the variable “\(x_4\)” is associated with the magnetic field and a linear interaction with “\(x_1\)” Model equations (2) show hyperchaotic oscillations in the presence of a magnetic field. The system bifurcates for two parameters: normalized Rayleigh number “\(r\)” and Chandrasekhar number “\(Q\)” which is defined in terms of magnetic field.

The total dynamics of the magnetoconvection fluid flow model (2) is dependent on two parameters such as the Chandrasekhar number \(Q\) that induced the magnetic field and the Rayleigh number \(r\). The equilibria switch their behavior about these two parameters and bifurcation occurs. However, the model is very sensitive to the

| TABLE I. Magnetoconvection parameters. |
|----------------------------------------|
| Symbols | Names |
|---------|-------|
| \(\mu\) | Newtonian viscosity |
| \(\varrho_0\) | Fluid density at reference temperature \(T_0\) |
| \(\kappa\) | Thermal diffusivity |
| \(\eta\) | Magnetic diffusivity |
| \(\mu_0\) | Magnetic permeability |
| \(B_0\) | Magnetic field |
| \(h\) | Distances between the two isothermal horizontal planes |
| \(\Lambda = \pi^2 + k^2\) | \(k\) is the horizontal wave number |

\(\Lambda\) is the Chandrasekhar number.
diffusivity ratio parameter. Before we transform the model defined in (2) into the controlled system, we have to estimate the dynamics with Lyapunov theories.

### C. State space exact linearization

State space feedback linearization is utilized to control chaotic behavior with a suitable controller. In this control technique, we transform the nonlinear system into the equivalent but in linear form,

\[
\dot{X} = f(X). \tag{3}
\]

Suppose (3) is a nonlinear system where \( X \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) are the state vector and vector field, respectively. For a nonlinear single input controller, the controlled nonlinear dynamical system is represented as

\[
\dot{X} = f(X) + g(X)u. \tag{4}
\]

Hence, \( u \in \mathbb{R} \) is a control parameter, \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) are vector fields, and both \( f \) and \( g \) are continuous on \( \mathbb{R} \). Let

\[
ad_t g(X) = [f, ad_t^{-1}g](X) \quad (l \geq 1), \quad ad_0^l g(X) = g(X), \tag{5}
\]

where \([,]\) defines the Lie brackets between two vector fields. The verification of two necessary conditions implies that the dynamical system possesses a linear transformation. They are controllability and involutivity, which imply that \( L \) is a full rank matrix and \( M \) is involutive about \( X_0 \), respectively, where

\[
L = [g(X_0), ad_t^1g(X_0), ad_t^2g(X_0), \ldots, ad_t^{24}g(X_0)] \tag{6}
\]

and

\[
M = \text{span}\{g, ad_t g, ad_t^2 g, \ldots, ad_t^{24} g\}. \tag{7}
\]

Then, there exists a real-valued output function \( \lambda(X) \), which is defined inside the neighborhood \( \Omega(X_0) \) of \( X_0 \) and satisfies the following equations:

\[
Lg\lambda(X) = Lad_t^1g\lambda(X) = \ldots = Lad_t^{24}g\lambda(X) = 0, \quad x \in \Omega(X_0), \tag{8}
\]

\[
Lad_t^{24}\lambda(X_0) \neq 0. \tag{9}
\]

However, \( L_t\beta(x) \) represents the Lie derivative of a function \( \beta(x) \) with respect to \( \alpha \) vector field. Furthermore, there exists a linear transformation \( Z = T(X) \) of (4) in the neighborhood \( \Omega(X_0) \) as

\[
Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \\ \vdots \\ z_{24} \end{bmatrix} = \begin{bmatrix} T_1(X) \\ T_2(X) \\ \vdots \\ T_{24}(X) \end{bmatrix} = \begin{bmatrix} \lambda(X) \\ L_2\lambda(X) \\ L_2^2\lambda(X) \\ \ldots \\ L_2^{24}\lambda(X) \end{bmatrix}, \tag{10}
\]

and the linear control law is

\[
v = \xi_1(X) + \xi_2(X)u, \quad \text{where} \quad \xi_1 = L_2\lambda(X) \quad \text{and} \quad \xi_2 = L_2L_2^{-1}\lambda(X). \tag{11}
\]

Hence, a linear controllable transformed system is depicted in Fig. 2, developed for (4) through (10), (11) and expressed as

\[
Z_1 = z_2, \quad Z_2 = z_3, \quad \ldots \quad Z_{24} = z_l, \quad Z_l = v. \tag{12}
\]

### II. Chaotic Responses in Magnetoconvection Model

System (2) is dissipative for \( a, b, c, l > 0 \) and satisfies the following equation:

\[
\nabla \cdot F = -ac - 1 - b - l < 0. \tag{13}
\]

![FIG. 2. State space exact linearization.](image1)

![FIG. 3. Phase portraits of system model (2): (a) \( x_1x_2x_3 \)-phase space and (b) \( x_2x_3x_4 \)-phase space.](image2)
AIP Advances 9, 125235 (2019); doi: 10.1063/1.5129708
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System model (2) produces hyperchaotic behavior for \((a, Q, r, l, m, b, c) = (10, 1573, 616, 0.1, (4/9\pi^2), 8/3, 8.8463)\), which depicts the random behavior in phase portraits shown in Figs. 3(a), 3(b), 4(a), and 4(b).

A. Lyapunov exponents

A bifurcation parameter, Rayleigh number \(r\), changes the behavior of system model (2) from periodic to upgraded chaotic stream lines. Another bifurcation parameter that induces a magnetic field in system model (2) is represented as \(Q\), which also upgraded the chaos. The magnetoconvection model presents the chaotic behavior for \((a, Q, r, l, m, b, c) = (10, 1573, 616, 0.1, (4/9\pi^2), 8/3, 8.8463)\) and \((a, Q, r, l, m, b, c) = (10, 1573, 619, 0.1, (4/9\pi^2), 8/3, 8.8463)\) in Ref. 16. Lyapunov exponents define the convergence and divergence of two nearby trajectories, and the method is presented in Ref. 25. For system (2), the numerical values are calculated and presented in Figs. 5(a) and 5(b).

We can analyze that system model (2) represents the hyperchaotic behavior for interval \(587 \leq r < 619\) with two positive Lyapunov exponents. Within the interval \(643 \leq r \leq 655\) and value \(r = 619\), system (2) possesses one positive \(L_1 > 0\), and for \(600 < r < 620\), system (2) possesses \(L_1 > L_2 > 0\), which represents the chaotic and hyperchaotic behavior, respectively. Moreover, by Kaplan and Yorke, the Lyapunov exponent dimension can be calculated as \(D_L = j + \sum_{i=1}^{\lfloor j \rfloor} L_i\), where “\(j\)” is the largest integer such that \(L_1 + \cdots + L_j > 0\).

System model (2) exhibits hyperchaos as presented in Fig. 6 due to the Lyapunov dimension \(D_L > 3\) for \(570 \leq r \leq 640\) and chaos with the Lyapunov dimension \(3 > D_L > 2\) for \(670 \leq r\). For the existence of hyperchaotic and chaotic behavior, the Lyapunov dimension must lie within the interval \(3 < D_L < 4\) and \(2 < D_L < 3\), respectively.

B. Lyapunov stability defines ultimate bound

Chaotic behavior is unpredictable and trajectories are dense in the phase space. That is why a system that possesses chaos must be bounded for some particular initial condition. Lyapunov functions are the energy functions and are positive definite. For the time derivative of the energy function less than zero, all trajectories must be confined within a boundary of the Lyapunov function. Here,
we utilized the Lyapunov function theory to compute the ultimate boundedness of system model (2).

**Theorem II.1.** If \(l, m, a, Q > 0\), then system (2) has the following bound defined by

\[
\Psi = \{ (x_1, x_2, x_3, x_4) | l m x_1^2 + x_2^2 + x_3^2 + a Q x_4^2 \leq K^2 \} \quad (14)
\]

with the following constraint:

\[
\Omega = \{ (x_1, x_2, x_3, x_4) | (x_2 - \omega x_1)^2 + (l m a x_1^2 + b x_3^2 + l a Q x_4^2) \geq (\omega x_1)^2 \} \quad (15)
\]

**Proof.** Suppose a Lyapunov function \(V(x_1, x_2, x_3, x_4)\) is positive definite,

\[
V(x_1, x_2, x_3, x_4) = l m x_1^2 + x_2^2 + x_3^2 + a Q x_4^2, \quad (16)
\]

and its respective time derivative is

\[
\dot{V}/2 = l m x_1 x_3 + x_2 x_3 + x_3 x_4 + a Q x_4 x_1, \quad (19)
\]

\[
= -l m a x_1^2 - b x_3 - l a Q x_4^2 + (l m a + r) x_1 x_2, \quad (17)
\]

\[
= -(x_2 - \omega x_1)^2 - (l m a x_1^2 + b x_3^2 + l a Q x_4^2) + (\omega x_1)^2, \quad (15)
\]

where \(\omega\) is represented as

\[
\omega = \frac{l m a + r}{2}.
\]

Here, \(\dot{V}/2 \leq 0\) only for

\[
(x_2 - \omega x_1)^2 + (l m a x_1^2 + b x_3^2 + l a Q x_4^2) \geq (\omega x_1)^2. \quad (17)
\]

Hence, \(\Psi\) is an ultimate bound with \(l, m, a, Q > 0\) and with the above-mentioned constraint.

The ultimate boundedness of the magnetoconvection model is presented in Figs. 7(a) and 7(b) for \(K = 10\).
where
\[
g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\quad \text{and} \quad u \in \mathcal{R}.
\] (20)

Now, we have implemented the method described in Sec. I C. We have stated Lemma III.1 with the verification of the necessary conditions.

**Lemma III.1.** For any \( x_0 \in \mathcal{R}^3 \), \( \exists \) is an open set \( P(x_0) \), such that \( x_0 \in P(x_0) \), where \( L = [g(x_0), ad_1 g(x_0), ad_2^2 g(x_0), ad_1^2 g(x_0)] \) matrix has full rank and \( M \) is involutive.

\[
M = [g(x), \ad_1 g(x), \ad_2^2 g(x), \ad_1^2 g(x)],
\]
(22)

det(L) = \begin{vmatrix} 0 & aQ & a'cQ + aQ \ & aQ(ac + l) + (l - amQ) + a(r - x_3) \ & -aQ(r - x_3) \ & -aQx_2 \ & l(l - amQ) \end{vmatrix},
\]
(23)

and
\[
= a^2 Q^2 x_2 (2r - br + 2(b - 1)x_3).
\]
(24)

Hence, \( \det(L) \neq 0 \) for all \( x_2 \neq 0 \); therefore, \( L \) is a full rank matrix. The involutiveness of \( M = \text{span}\{g, \ad_1 g, \ad_2^2 g\} \) is obvious, \([g, \ad_1 g] = 0\), \([g, \ad_2^2 g] = 0\), and \([\ad_1 g, \ad_2^2 g] = 0\).

Now, to construct a linear transformation \( Z = [\lambda(x), L_T \lambda(x), L_T^2 \lambda(x), L_T^3 \lambda(x)]^T \), the output function \( \lambda(x) \) must be evaluated first. Hence, for a real-valued output function, we have considered Eqs. (8) and (9) for all \( x_2 \neq 0 \)

\[
L_T \lambda(x) = aQ(-r + x_3) \lambda_{x_1} - aQ x_2 \lambda_{x_1} = 0.
\]
(27)

**Proof.** From Eqs. (18) and (20), the following can easily be calculated as
\[
ad_1 g(x) = \begin{bmatrix} aQ \\ 0 \\ 0 \\ 1 \end{bmatrix},
ad_2^2 g(x) = \begin{bmatrix} a'Q + aQ \\ -aQ(r - x_3) \\ -aQx_2 \\ l^2 - almQ \end{bmatrix},
\]
and \( \ad_1^2 g(x) = \begin{bmatrix} aQ(ac + l) + (l - amQ) + a(r - x_3) \\ aQ(-1 + ac + l) + (1 - b + ac + l)x_3 \\ -a(-1 + b + ac + l)x_3 \\ l^2 - al(ac + 2)Qx_2 \end{bmatrix} \) (21)

So, from Eq. (21),

\[
L_T \lambda(x) = aQ(-r + x_3) \lambda_{x_1} + x_1^2 + \xi,
\]
(28)

which defines the nonlinear output function. Lie derivatives for an output function \( \lambda(x) \) are

\[
L_T^2 \lambda(x) = aQ(b(r - 2x_1) x_1 x_2 - bx_3) + 2x_2(-r x_3 + x_2 + x_1 x_3),
\]
(29)
\[
L_T^3 \lambda(x) = aQ(-2x_1 x_2 x_3 + 2b x_1 x_2 - 2r x_1 + 4x_2 - 2(-1 + b) x_1 x_3 + x_2(-2 - b)r - 2(-1 + b) x_1(-c x_1 - x_2 - Q x_3)),
\]
(30)
\[
L_T^4 \lambda(x) = aQ(-2x_1 x_2 x_3 + 2b x_1 x_2 x_3 + 2x_2(-r x_1 + 4x_2 - 2(-1 + b) x_1 x_3 + x_2(-2 - b)r - 2(-1 + b) x_1(-c x_1 - x_2 - Q x_3)),
\]
(31)

\[
L_T^1 \lambda(x) = aQ\left( -2 + b \right) r - 2\left( -1 + b \right) x_1(3 + b + ac) x_1 x_3 + aQ\left( -2 - 2b \right) r^2 x_1 + \left( -2 + b \right) x_1(3 + b + ac) x_1 x_3
\]
+ \left( -2 + 3b \right) r x_1 x_3 - 2\left( -1 + b \right)(3 + b + ac)x_2 x_3 + x_1(2x_1^2 - x_2^2)\right)\left( c x_1 - x_2 + Q x_3 \right) - \left( r x_1 - x_2 - x_3 \right)
\]
\times \left( 8x_2 + (2 + 2b) r(3 + b + ac)x_2 - 2a(x_2 + aQ x_2) - 2(-1 + b) \left( -2x_1 x_2 + (3 + b + ac)x_1 x_3
\right)
\right) + \left( x_2(x_1 - x_2) - b x_3 \right) \left( 6b^2 x_1 x_2 + b^2 (r - 8x_1) - 2 \left( -2 r x_1^2 + (3 + ac)x_1 x_2 + 2x_1^2 x_3
\right)
\right.
\]
+ \left( x_2(-2 + Q x_3) \right) + \left. b(-3r x_1^2 + 2(2x_1 x_3 + ax_2(c x_1 - x_2 + Q x_3)), \right)
\]
(32)
The new transformation with inverse transformation becomes

$$L_g L_x^2 \lambda(x) = -a^2 Q^2 x_2((-2 + b)r - 2(-1 + b)x_3).$$

The system has been controlled for the initial condition $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.4$, and $x_4(0) = 0.1$. The controller is activated for the interval $G = (0, 55)$. In Figs. 9(a), 9(b), 10(a), and 10(b) for $t_1 = -0.2, t_2 = -0.4, t_3 = -0.25$, and $t_4 = -0.2$, we have calculated the time history for different values of $G = 10, 20, 30$, and 50. We can easily observe in Figs. 9(a), 9(b), 10(a), and 10(b) that the system is stabilized in the defined interval and successfully achieves the desired goals.

B. Speed and dislocated feedback

We have already discussed an effective technique state-space linearization, and here we compare it with other feedback techniques, i.e., dislocated and speed feedback techniques. Recently, Yan controlled a new hyperchaotic Chen system by using the feedback control method, and another new hyperchaotic system was controlled to its unstable equilibrium by Dou et al. Similarly, Zhu and Aziz and AL-Azzawi controlled an upgraded chaos in the hyperchaotic Lorenz system and modified hyperchaotic pan system to the unstable equilibrium points by using the feedback control method. An article presented some problems of their strategies and treatment.

This section discusses the stabilization of the hyperchaotic system to the equilibrium point with the help of two simple feedback techniques, i.e., speed and dislocated feedback. The speed and dislocated feedback control techniques are used to suppress chaos to their respective equilibria. In these techniques, we add a controller in the form of feedback gain parameter. We choose the controller in such a way that we can drag the trajectories $(x_1, x_2, x_3, x_4)$ of the magnetocovention model to equilibrium point, i.e., $E: (0, 0, 0, 0)$. The Routh-Hurwitz criterion is utilized to analyze the conditions where the controlled system is asymptotically stable. Also, theoretical interpretation and numerical simulation are given for the suggested controller gains. Consider the nonlinear system

$$\dot{x} = bx,$$  

one can control $x_3$ to the goal $x_G$. As $x_3 \to x_G$, the state vector goals become

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \to \begin{bmatrix} \pm \sqrt{\frac{b_0}{c + Qm}} \\ \pm \sqrt{\frac{b_0}{c + Qm}} \\ x_G \\ \pm m \sqrt{\frac{b_0}{c + Qm}} \end{bmatrix}$$

A linear controllable system from Eq. (12) is

$$Z_1 = z_1, Z_2 = z_2, Z_3 = z_3, Z_4 = z_4, Z_4 = v,$$  

and let $v = \hat{v} + \tau Z, \tau = [\tau_1, \tau_2, \tau_3, \tau_4]$, where $v$ is a linear control law represented as

$$v = \tau_1 z_1 + \tau_2 z_2 + \tau_3 z_3 + \tau_4 z_4.$$  

However, $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathbb{R}$ according to the Routh-Hurwitz and the Hopf bifurcation existence criteria and always less than and equal to zero. We can write an input nonlinear control law using Eq. (11) as

$$u = \frac{v - \xi_1(x)}{\xi_2(x)},$$

where

$$\xi_1 = L_x^2 \lambda(x),$$

$$\xi_2 = L_g L_x^2 \lambda(x).$$

The feedback control design for the controlled magnetocovention model is depicted in Fig. 8.

Utilizing the following equation and varying the values of $x_3$ in Eq. (41),

$$\zeta = -a Q x_3 \left( \frac{x_0}{2} - r + \frac{x_0}{2} \right).$$

\[ \text{FIG. 8. Controlled design.} \]
FIG. 9. The control of hyperchaos in magnetoconvection for (a) \( x_G = 10 \) and (b) \( x_G = 20 \).

and after applying controllers, we get

\[
\dot{x} = bx + u. \tag{44}
\]

The characteristic equation of system (44) is

\[
\lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4 = 0, \tag{45}
\]

where \( c_1, c_2, c_3, \) and \( c_4 \) are the coefficients. According to the Routh-Hurwitz criterion,\textsuperscript{32} if the characteristic polynomial satisfies the following conditions:

- \( \lambda^2 + c_1 \lambda + c_2 = 0; \ c_1 > 0, c_2 > 0, \)
- \( \lambda^2 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0; \ c_1 > 0, c_3 > 0, c_1 c_2 - c_3 > 0, \)
- \( \lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4 = 0; \ c_1 > 0, c_4 > 0, c_1 c_2 - c_3 > 0, \)
- \( (c_1 c_2 - c_3) c_3 - c_1 c_4 > 0, \)

then all roots must have negative real parts, which implies stability. In this section, we will use speed and dislocated feedback control strategies to control the magnetoconvection system and stabilize it to an unstable equilibrium point \( E: (0, 0, 0, 0) \) with \( (a, Q, r, l, m, b, c) = (10, 1573, 616, 0.1, (4/9 \pi^2), 8/3, 8.8463) \).

1. Speed feedback control method

In speed feedback, the controller is added as a combination of multiplication between the velocity function with the feedback gain.\textsuperscript{32}

**Theorem III.2.** Let the controlled hyperchaotic magnetoconvection be

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - cx_1 - Qx_4) + u_1, \\
\dot{x}_2 &= rx_1 - x_2 - x_1 x_3 + u_2, \\
\dot{x}_3 &= x_1 x_2 - bx_3 + u_3, \\
\dot{x}_4 &= l(mx_1 - x_4) + u_4,
\end{align*}
\tag{46}
\]
where \( u_j (j = 1, 2, 3, 4) \) are the control inputs. One can drag the chaotic trajectories \((x_1, x_2, x_3, x_4)\) to the equilibrium point \(E: (0, 0, 0, 0)\) by choosing suitable control inputs. Here, we took \( u_1 = u_2 = u_3 = 0 \) and \( u_4 = k_1 x_4 \), where \( k_1 \) is the feedback coefficient and \( k_1 > 0.35 \). For the above-mentioned values, system (46) gradually converges to an unstable equilibrium point \(E: (0, 0, 0, 0)\).

**Proof.** After applying controllers, system (2) becomes
\[
\begin{align*}
\dot{x}_1 &= a(x_2 - cx_1 - Qx_4) + u_1, \\
\dot{x}_2 &= rx_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - bx_3, \\
\dot{x}_4 &= l(mx_1 - x_4) + k_1 x_4.
\end{align*}
\]
(47)

The Jacobian matrix calculated at the origin is
\[
J|_{(0, 0, 0, 0)} = \begin{pmatrix}
-ac & a & 0 & -aQ \\
r & -1 & 0 & 0 \\
0 & 0 & -b & 0 \\
lm + k_1 lm & 0 & 0 & -l - k_1 l
\end{pmatrix}
\]
with the characteristic equation
\[
(-b - \lambda) \left( lm + k_1 lm \right) \left( -aQ - aQ \lambda \right) + (-l - k_1 l - \lambda) \left( ac - ar + \lambda + acl + \lambda^2 \right) = 0,
\]
(48)

\( \lambda_1 = -b < 0 \), and according to the Routh-Hurwitz criterion, if \( k_1 > 0.35 \), then all the roots must have negative real parts. Thus, system (47) will gradually converge to \(E: (0, 0, 0, 0)\).

Figures 11(a)–11(d) show the behavior of trajectories \((x_1, x_2, x_3, x_4)\) as \( k_1 \) varies from 0.35 to 0.42. One can see that the controller reduces the large oscillations and converges to the origin.

### 2. Dislocated feedback control method

If a linear variable of the system is multiplied by a feedback gain and is added on the right side to the opposite variable, then this method is called a dislocated feedback method. This type of controller dislocated the variable to another variable.

**Theorem III.3.** Let the controlled hyperchaotic magnetoconvection be
\[
\begin{align*}
\dot{x}_1 &= a(x_2 - cx_1 - Qx_4) + u_1, \\
\dot{x}_2 &= rx_1 - x_2 - x_1 x_3 + u_2, \\
\dot{x}_3 &= x_1 x_2 - bx_3 + u_3, \\
\dot{x}_4 &= l(mx_1 - x_4) + u_4,
\end{align*}
\]
(49)

where \( u_j (j = 1, 2, 3, 4) \) are the control inputs. One can drag the chaotic trajectories \((x_1, x_2, x_3, x_4)\) to an equilibrium point \(E: (0, 0, 0, 0)\) by choosing suitable control inputs. Here, we took \( u_2 = u_3 = u_4 = 0 \) and \( u_1 = k_2 x_2 \), where \( k_2 < -2.61 \). For the above-mentioned values, system (49) gradually converges to an equilibrium point \(E: (0, 0, 0, 0)\).

**Proof.** After applying controllers, system (2) becomes
\[
\begin{align*}
\dot{x}_1 &= a(x_2 - cx_1 - Qx_4) + k_2 x_2, \\
\dot{x}_2 &= rx_1 - x_2 - x_1 x_3, \\
\dot{x}_3 &= x_1 x_2 - bx_3, \\
\dot{x}_4 &= l(mx_1 - x_4).
\end{align*}
\]
(50)

**FIG. 11.** Controlled hyperchaotic magnetoconvection system for different values of gain coefficient: (a) \( k_1 = 0.36 \), (b) \( k_1 = 0.38 \), (c) \( k_1 = 0.40 \), and (d) \( k_1 = 0.42 \).
FIG. 12. Controlled hyperchaotic magnetoconvection system for different values of gain coefficient: (a) $k_2 = -2.66$, (b) $k_2 = -2.68$, (c) $k_2 = -2.70$, and (d) $k_2 = -2.80$.

FIG. 13. Comparison of control results for $x_0 = 10$, $k_1 = 0.42$, and $k_2 = 2.8$ where controlled values are presented as (a) $x_1$, (b) $x_2$, (c) $x_3$, and (d) $x_4$. 
The Jacobian matrix and the characteristic equation at the origin are

$$
J|_{(0,0,0)} = \begin{pmatrix}
-ac & a + k_2 & 0 & -aQ \\
r & -1 & 0 & 0 \\
0 & 0 & -b & 0 \\
im & 0 & 0 & -l
\end{pmatrix}
$$

and

$$
(-b - \lambda)(lm(-aQ - aQ\lambda) + (-l - \lambda)((ac - ar - k_2r + \lambda + acl + \lambda^2)) = 0, 
$$

(51)

where $\lambda_1 = -b < 0$. According to the Routh-Hurwitz criterion in system (49), if $k_2 < -2.61$, then all the roots must have negative real parts. Thus, system (49) will gradually converge to $E: (0, 0, 0, 0)$. □

Figures 12(a)–12(d) show the behavior of trajectories ($x_1$, $x_2$, $x_3$, $x_4$) as $k_2$ varies from $-2.61$ to $-2.8$ and gradually converges to the equilibria at the origin.

IV. COMPARISON OF RESULTS AND DISCUSSION

In Figs. 13(a)–13(d), the performance of three control strategies, i.e., state space, speed, and dislocated feedback, is presented. These control strategies effectively control the chaos, but according to the comparison, one can see that the state space linearization feedback controls the trajectories efficiently and more effectively as compared to the other two control strategies. Also for $x_3$ and $x_4$ trajectories, the controller of state space linearization and speed feedback controls effectively.

According to the accuracy point of view, state space linearization is recommended as it is an analytical technique, whereas with some series of steps, we can design an exact form of controller that controls the system to the required goals. On the other hand, in the speed and dislocated feedback, there is no need to design a controller as compared to state space linearization. Through stability analysis, we have defined the interval for control gain and the effective control values on an ad hoc basis.

V. CONCLUSION

In this article, we have analyzed the controlled results of the magnetoconvection model, constructed from the state space linearization, speed, and dislocated feedback techniques. The magnetoconvection model produces the chaotic and hyperchaotic behavior for some specific values of parameters, such as Rayleigh number $r$, Chandrasekhar number $Q$, and diffusivity ratio $l$. In order to control these chaotic and hyperchaotic trajectories, a controller is designed by the state space linearization, which is independent of parametric values and is more effective. It can also control the trajectories of the magnetoconvection model to the required goals and takes less time as compared to the other two techniques such as speed and dislocated feedback.

ACKNOWLEDGMENTS

We are thankful to Institute of Space Technology, Islamabad, and the Higher Education Commission (HEC) of Pakistan for providing the research environment and sufficient resources in order to conduct this study.

The authors declare that they have no conflict of interest.

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