A survey of Hirota’s difference equations

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January 13, 2022

Abstract

A review of selected topics in Hirota’s bilinear difference equation (HBDE) is given. This famous 3-dimensional difference equation is known to provide a canonical integrable discretization for most important types of soliton equations. Similarly to the continuous theory, HBDE is a member of an infinite hierarchy. The central point of our exposition is a discrete version of the zero curvature condition explicitly written in the form of discrete Zakharov-Shabat equations for $M$-operators realized as difference or pseudo-difference operators. A unified approach to various types of $M$-operators and zero curvature representations is suggested. Different reductions of HBDE to 2-dimensional equations are considered. Among them discrete counterparts of the KdV, sine-Gordon, Toda chain, relativistic Toda chain and other typical examples are discussed in detail.

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1 Introduction

In 1981 R.Hirota published the paper \[1\] which summarized his earlier studies of discretizing nonlinear integrable equations \[2,3\]. The main result is a compact bilinear equation which can be viewed as an integrable discrete analogue of the 2-dimensional Toda lattice. In Hirota’s original notation it has the form

\[
[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)] \tau \cdot \tau = 0, \tag{1.1}
\]

where \(Z_i\) are arbitrary constants, \(\tau = \tau(x_1, x_2, x_3)\), \(D_i \equiv D_{x_i}\) and Hirota’s \(D\)-operator is defined for a linear differential operator \(F(\partial_y)\) by

\[
F(D_x)f(x) \cdot g(x) = F(\partial_y)f(x + y)g(x - y) \Big|_{y=0}. \tag{1.2}
\]

In the more explicit notation eq. (1.1) looks as follows:

\[
Z_1 \tau(x_1 + 1, x_2, x_3) \tau(x_1 - 1, x_2, x_3) + Z_2 \tau(x_1, x_2 + 1, x_3) \tau(x_1, x_2 - 1, x_3) + Z_3 \tau(x_1, x_2, x_3 + 1) \tau(x_1, x_2, x_3 - 1) = 0. \tag{1.3}
\]

This equation is often called Hirota’s bilinear difference equation (HBDE). Its simplicity is surprising and delusive at the same time: each detail is controlled by integrability and hides meaningful mathematical structures whereas some even simpler looking equations turn out to be untractable by analytical methods.

One of the most impressive outcomes of Hirota’s works is that HBDE is shown to unify many if not all known soliton equations. More precisely, it contains them in an encoded form. Performing a scaling limit for appropriate combinations of parameters and variables, one is able to obtain the Korteweg - de Vries (KdV) equation, Kadomtsev-Petviashvili (KP) equation, modified KdV (MKdV) and modified KP (MKP) equations, two-dimensional Toda lattice (2DTL) equation, sine-Gordon (SG) equation, Benjamin-Ono equation, etc. Their discrete analogues are produced from HBDE by choosing suitable dependent and independent variables. Furthermore, eq. (1.1) was shown to possess soliton solutions and Bäcklund transformations for generic values of parameters. These facts suggest to consider HBDE as a fundamental classical soliton equation, from which the typical examples can be obtained as particular cases.

Recently, bilinear equations of this form emerged \[7,8\] in the context of quantum integrable systems as the model-independent functional relations \[9,10\] for eigenvalues of quantum transfer matrices. This was our motivation for revisiting classical nonlinear difference equations.

These notes aim at reviewing selected topics in HBDE and further clarifying basic elements of the theory. In our exposition, we deal solely with equations themselves saying almost no word about their solutions\[4\]. Likewise their continuous counterparts, completely discretized nonlinear integrable equations are known to possess soliton and finite-gap solutions. However, a systematic treatment of these and other particular classes of solutions could be a separate enterprise which requires much more space. We shall confine ourselves to elaborating discrete versions of commutation representations and auxiliary linear problems on a formal algebraic level. At the same time some important elements of our approach are essentially motivated by the finite-gap theory.

The difference soliton equations are intimately connected with the differential ones. We already mentioned that the latter are obtained from the former by a scaling limit. Better to say, HBDE was just designed to enjoy this property. The fact that such an equation does exist is by no means trivial. A link in the opposite direction was established by T.Miwa \[11\] who noticed that discrete Hirota’s equations can be obtained from the continuous KP hierarchy by choosing the time flows to be certain infinite combinations of the standard flows of the hierarchy. This idea was further developed in the papers \[12,13\] as a method to produce discrete soliton equations from continuous ones. The interrelation between discrete and continuous integrable hierarchies looks like a kind of Fourier duality: they provide complementary descriptions of the same object, namely, of the infinite dimensional grassmannian \[14,15\].

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\[1\] Just because of this we do not draw any distinction between discrete and difference equations. Usually, in the latter case it is implied that solutions are functions of a continuous variable with certain analytical properties.
In this survey we do not give a systematic treatment of the connection between discrete and continuous hierarchies. The problem to describe the limiting procedure that would be compatible with the whole hierarchy is technically involved. However, it is impossible not to refer to continuous hierarchies at all. We agree to a compromise restricting ourselves by a few typical examples.

It is assumed that the reader is familiar with the basic notions of the continuous theory such as Lax and Zakharov-Shabat equations, zero curvature conditions in scalar and matrix forms, commuting flows, infinite hierarchies, $\tau$-function etc.

Let us outline the contents of the paper. (More detailed descriptions are given in short introductions to each section.)

Sect. 2 can be considered as a part of the Introduction. We tried to collect here different forms of 3-dimensional HBDE known in the literature. All of them are equivalent. Simple transformations between them are listed.

Sections 3, 4 and 5 form the main body of the paper. To figure out the key principles underlying the variety of integrable difference equations, we need a number of definitions and axioms. They are given in Sect. 3. All the notions explained in this section are essentially used in the sequel. In Sect. 4, the discrete version of the zero curvature representation is presented. Filling in some gaps in the existing literature, we give explicit forms of the $M$-operators (realized as difference operators) for discrete flows. Sect. 5 is devoted to various types of associated auxiliary linear problems. They provide a "linearization" of the original nonlinear equation. The related notion of Bäcklund transformation is discussed. The Baker-Akhiezer functions are introduced as special formal solutions to the linear problems.

Sections 6 and 7 are more technical and might be interesting mostly for experts. They can be skipped without loss of understanding. In Sect. 6 it is explained how to extend the $M$-operator approach to arbitrary discrete flows defined in Sect. 3. In the general case the $M$-operators contain negative powers of first order difference operators. In Sect. 7 we dwell upon hierarchies of bilinear discrete equations and suggest the notion of "higher" discrete flows with the corresponding zero curvature representation. We conjecture that all "higher" ($N$-term) discrete Hirota’s equations known in the literature are consequences of the 3-term ones. This assertion is proved for the first nontrivial example of the 4-term equations in 4 variables.

Sect. 8 deals with 2-dimensional reductions of HBDE, the corresponding $L$-$M$ pairs and auxiliary linear problems. The list of reductions includes discrete analogues of the KdV equation, the 1D Toda chain, the AKNS system, the relativistic Toda chain, the sine-Gordon equation, the Liouville equation and some others.

In the Appendix, we present main elements of a different approach to HBDE based on Miwa’s transformation. This is the method for generating discrete soliton equations suggested in ref. [12]. We show how it works for the simplest examples and comment on the continuum limit which is in a sense an "inverse" Miwa transformation.

## 2 Equivalent forms of the bilinear equation

Hirota’s difference equation exists in several forms. Historically, they emerged as integrable discretizations of particular continuous hierarchies (e.g. KP, 2DTL). In this section we give a list of most popular forms of Hirota’s difference equation and explicitly demonstrate that they are equivalent. However, it is useful to bear in mind all of them since one or another may be more convenient in a particular problem.

- **a) Hirota’s original form**: 

\[
Z_1 \tau(x_1 + 1)\tau(x_1 - 1) + Z_2 \tau(x_2 + 1)\tau(x_2 - 1) + Z_3 \tau(x_3 + 1)\tau(x_3 - 1) = 0 \tag{2.1}
\]

(here and below we often skip the variables that do not undergo shifts). Note that the 3 variables enter in a symmetric fashion and the equation is invariant under their permutations and a simultaneous permutation of $Z_i$’s. The equation is also invariant under changing the sign of any one of the variables and under the transformation

\[
\tau(x_1, x_2, x_3) \rightarrow \chi_0(x_1 + x_2 + x_3)\chi_1(x_2 + x_3 - x_1)\chi_2(x_1 + x_3 - x_2)\chi_3(x_1 + x_2 - x_3)\tau(x_1, x_2, x_3), \tag{2.2}
\]
where $\chi_i$ are arbitrary functions. The transformation
\[
\tau(x_1, x_2, x_3) \rightarrow Z_1^{-x_1^2/2} Z_2^{-x_2^2/2} Z_3^{-x_3^2/2} \tau(x_1, x_2, x_3)
\] (2.3)
converts eq. (2.13) into the canonical form,
\[
\tau(x_1 + 1)\tau(x_1 - 1) + \tau(x_2 + 1)\tau(x_2 - 1) + \tau(x_3 + 1)\tau(x_3 - 1) = 0 .
\] (2.4)
This equation does not contain any free parameters.

- a') "Gauge invariant" form:
\[
Y(x_1, x_2 + 1, x_3)Y(x_1, x_2 - 1, x_3) = \frac{(1 + Y(x_1, x_2, x_3 + 1))(1 + Y(x_1, x_2, x_3 - 1))}{(1 + Y^{-1}(x_1, x_2, x_3))(1 + Y^{-1}(x_1 - 1, x_2, x_3))},
\] (2.5)
where
\[
Y(x_1, x_2, x_3) \equiv \frac{\tau(x_1, x_2, x_3 + 1)\tau(x_1, x_2, x_3 - 1)}{\tau(x_1 + 1, x_2, x_3)\tau(x_1 - 1, x_2, x_3)}
\] (2.6)
is a gauge invariant quantity: the "gauge" transformation (2.2) does not change it. This form is a discrete counterpart of nonlinear integrable equations written in terms of potentials and fields rather than $\tau$-functions. Some particular cases of this equation emerge naturally in thermodynamic Bethe ansatz [8, 9].

- b) KP-like form:
\[
(\chi_0(2p_1 + 2p_2 + 2p_3)\chi_1(2p_1)\chi_2(2p_2)\chi_3(2p_3))\tau^{p_1,p_2,p_3}
\] (2.8)
is a solution too. Again, the coefficients in (2.7) can be made equal to 1 by means of the transformation $\tau^{p_1,p_2,p_3} \rightarrow \left(\frac{z_1 - z_3}{z_2 - z_3}\right)^{p_1p_2} \left(\frac{z_1 - z_2}{z_2 - z_3}\right)^{p_2p_3} \tau^{p_1,p_2,p_3}$, bringing eq. (2.7) into its canonical form.

- b') MKP-like form:
\[
(\chi_0(2p_1 + 2p_2 + 2p_3)\chi_1(2p_1)\chi_2(2p_2)\chi_3(2p_3))\tau^{p_1,p_2,p_3}
\] (2.10)
Note that the combination of the arguments $p_1 + p_2 + p_3 - p_0$ is the same for all $\tau$-functions in this equation. In other words, the hyperplane $p_1 + p_2 + p_3 - p_0 = \text{const}$ is invariant. Therefore, this equation actually depends on three variables rather than four: say, $p_1, p_2, p_3$. Since sum of the coefficients in eq. (2.10) is zero, like in eq. (2.7), they differ by a reparametrization of $z_i$'s only.

- c) 2DTL-like form:
\[
\nu^{l,i+1}_{n} \tau^{l,i+1}_{n} + (\mu - \nu)^{l,i+1}_{n} \tau^{l,i+1}_{n} = \mu^{l+1,i}_{n+1} \tau^{l+1,i}_{n-1},
\] (2.11)
where \( \tau_{n}^{I,J} \) is a function of the 3 variables and \( \mu, \nu \) are arbitrary constants. The variables \( l, \bar{l} \) are called light cone coordinates. Note that in this form the permutation symmetry is lost. However, an analogue of eq. (2.2) holds true: if \( \tau_{n}^{I,J} \) solves eq. (2.11), then \( \chi_{0}(2n + 2l)\chi_{1}(2l)\chi_{2}(2l)\chi_{3}(2n - 2l)\tau_{n}^{I,J} \) is a solution, too. The transformation

\[
\tau_{n}^{I,J} \rightarrow \left( \frac{\mu}{\nu} - 1 \right)^{-I} \left( -\frac{\mu}{\nu} \right)^{-n^{2}/2} \tau_{n}^{I,J}
\]

allows one to hide the coefficients in eq. (2.11):

\[
\tau_{n+1}^{I,J} + \tau_{n}^{I,J,\bar{l}+1} + \tau_{n}^{I+J,\bar{l}+1} = 0,
\]

(2.13)

which is referred to as its canonical form.

For reader’s convenience we present here the linear substitutions making the canonical forms of equations a), b), c) equivalent.

a)\(\iff\) b): \(\tau(x_{1}, x_{2}, x_{3}) = \tau^{p_{1}, p_{2}, p_{3}}\),

\[
p_{1} = \frac{1}{2}(-x_{1} + x_{2} + x_{3}), \quad p_{2} = \frac{1}{2}(x_{1} - x_{2} + x_{3}), \quad p_{3} = \frac{1}{2}(x_{1} + x_{2} - x_{3}),
\]

(2.14)

\[
x_{1} = p_{2} + p_{3}, \quad x_{2} = p_{1} + p_{3}, \quad x_{3} = p_{1} + p_{2},
\]

(2.15)

b)\(\iff\) c): \(\tau^{p_{1}, p_{2}, p_{3}} = \tau^{n, \bar{l}}\),

\[
n = p_{2} + p_{3}, \quad l = p_{1}, \quad \bar{l} = p_{2},
\]

(2.16)

\[
p_{1} = l, \quad p_{2} = n - \bar{l}, \quad p_{3} = \bar{l},
\]

(2.17)

a)\(\iff\) c): \(\tau(x_{1}, x_{2}, x_{3}) = \tau^{n, \bar{l}}\),

\[
n = x_{1}, \quad l = \frac{1}{2}(-x_{1} + x_{2} + x_{3}), \quad \bar{l} = \frac{1}{2}(x_{1} - x_{2} + x_{3}),
\]

(2.18)

\[
x_{1} = n, \quad x_{2} = n + l - \bar{l}, \quad x_{3} = l + \bar{l}.
\]

(2.19)

Clearly, these linear substitutions are not unique. All other possibilities can be obtained from the given one by applying a transformation of the form \( (x_{1}, x_{2}, x_{3}) \rightarrow (\pm x_{P(1)}, \pm x_{P(2)}, \pm x_{P(3)}) \), where \( P \) is a permutation. Using formulas (2.14)-(2.19) one can easily obtain gauge invariant forms of equations b) and c).

3 Definitions: the nomenclature of flows

Here we introduce a practical set of definitions and axioms which will help us to develop a systematic viewpoint to the zoo of non-linear integrable equations and their commutation representations. This viewpoint is in fact more general than we need for HBDE itself. Differential as well as ”mixed” differential-difference non-linear equations fit the scheme, too. Our approach is motivated by algebro-geometric solutions [2] to soliton equations expressed through Riemann theta-functions. However, since the goal is to clarify formal algebraic structures, we never refer to the solutions explicitly.

3.1 The variables and kinematical constraints

The ”unknown function” entering bilinear equations is always denoted by \( \tau \). This function depends on infinite set of independent variables which are called flows or times. The last two words will be used as synonyms. For each particular equation only a finite number of the time variables take non-zero values.

The flows are labeled by points of the complex plane \( \mathbb{C} \). Call points \( \lambda \in \mathbb{C} \) labels. We make distinction between discrete and continuous flows.

- **Discrete flows**: With each ordered pair of points \( \lambda, \mu \in \mathbb{C}, \lambda \neq \mu \), a discrete flow \( l = l_{\lambda \mu} \) is associated. To put it differently, the flows are attached to vectors \( \lambda \mu \), i.e., each discrete flow has two labels.

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2These are not more than conventional names. In general the both time variables may take complex values.
Continuous flows: With each point $\lambda \in \mathbb{C}$ an infinite sequence of times $\{t_1, t_2, t_3, \ldots\}^{(\lambda)}$ is associated. All the variables $t_j$ have the common label $\lambda$.

In each particular equation which we are going to consider only a finite number of labels are involved. To axiomatize this situation, we say that for all but a finite number of labels $\lambda \in \mathbb{C}$ and for all but a finite number of ordered pairs of labels the corresponding variables are implied to be equal to zero. This condition makes the definition very close to the adelic ideology from the algebraic number theory. The definition in such an abstract form may seem to be overcomplicated and too general. However, this standpoint is useful since it provides an adequate formalization of the simple fact that the number of independent variables in equations of an integrable hierarchy can be arbitrary but finite.

Sometimes it is convenient to say that those variables which are non-zero are switched on while all others are switched off. According to the above definition, the set of labels corresponding to the switched on variables is always finite.

Having this in mind, it is worthwhile to rephrase the definition making it a little bit more concrete\footnote{For each concrete example this unified notation is still not very convenient to work with. In the technical part of the paper this will be changed and simplified. However, for the sake of clarity and definiteness, it is better to introduce general notions and definitions using the unified notation.}.

Let $\{\lambda_{\alpha}\}$, $\alpha \in I$, be a finite set of marked points in $\mathbb{C}$. Here, $I$ is just the finite set of labels corresponding to the variables that are switched on. By $l_{\alpha\beta}$ $(\alpha \neq \beta)$ denote the discrete variable associated with $\lambda_{\alpha}\lambda_{\beta}$. By $t_{(\alpha)}^{(j)}$, $j = 1, 2, \ldots$ denote the continuous times associated with $\lambda_{\alpha}$. The $\tau$-function is a function of these variables:

$$\tau = \tau\left(l_{\alpha\beta}; \{t_{(\alpha)}^{(j)}\}\right).$$

Let $\mathcal{G}$ be the graph whose vertices are the marked points (labels) $\lambda_{\alpha}$, $\alpha \in I$, and whose edges are vectors $\lambda_{\alpha}\lambda_{\beta}$. The edges have orientation that is indicated by an arrow looking from $\alpha$ to $\beta$. This graph will be referred to as the graph of flows. It encodes the kinematic structure of the equation.

We stress that the only essential elements of the graph are vertices and their ordered pairs. All other graphical elements are introduced for convenience of the visualization. In particular, the vectors may intersect on the complex plane but the intersection points should be considered as not belonging to the graph. It is also worth emphasizing that the vectors are just convenient names of flows. They should not be mixed with "directions" of the flows in any sense of this word.

The introduced variables are not independent. There are certain "kinematical" constraints imposed on them.

The first group of constraints involves discrete variables only. The constraints arise when the graph of flows $\mathcal{G}$ has cycles. It is enough to fix the constraints for the following two cases:

i) The 2-cycle:

$$\tau(l_{\alpha\beta} + 1, l_{\beta\alpha} + 1) = \tau(l_{\alpha\beta}, l_{\beta\alpha}). \quad (3.1)$$

![Diagram of 2-cycle]

Informally, this means that $l_{\beta\alpha}$ is identified with $-l_{\alpha\beta}$.

ii) The 3-cycle:

$$\tau(l_{\alpha\beta} + 1, l_{\beta\gamma} + 1, l_{\gamma\alpha} + 1) = \tau(l_{\alpha\beta}, l_{\beta\gamma}, l_{\gamma\alpha}). \quad (3.2)$$

For each concrete example this unified notation is still not very convenient to work with. In the technical part of the paper this will be changed and simplified. However, for the sake of clarity and definiteness, it is better to introduce general notions and definitions using the unified notation.
The corresponding rules for longer cycles follow from these two. According to these rules, one can subsequently remove all the cycles and reduce the graph to a tree. The tree graphs correspond to kinematically independent flows. Formally, it is sufficient to consider graphs without cycles. However, introducing cycles sometimes makes the set of variables more symmetric though non-minimal.

The last constraint describes the interrelation between a discrete flow $\lambda_\alpha \lambda_\beta$ and the ”adjacent” continuous ones (i.e., corresponding to the endpoints $\lambda_\alpha$ and $\lambda_\beta$):

iii) Miwa’s rule [11], [12]:

$$
\begin{align*}
\tau(l_{\alpha\beta} + 1; t^{(\alpha)}, t^{(\beta)}) &= \tau(l_{\alpha\beta}; t^{(\alpha)} - [\lambda_{\beta} - \lambda_{\alpha}], t^{(\beta)}), \\
\tau(l_{\alpha\beta} - 1; t^{(\alpha)}, t^{(\beta)}) &= \tau(l_{\alpha\beta}; t^{(\alpha)} - [\lambda_{\alpha} - \lambda_{\beta}]),
\end{align*}
$$

Here $\tau(t) \equiv \tau(\{t_j\})$ and the short-hand notation

$$f(t \pm [z]) \equiv f(t_1 \pm z, t_2 \pm \frac{1}{2}z^2, t_3 \pm \frac{1}{3}z^3, \ldots)$$

is used for a function $f$ of the infinite sequence of variables $t = \{t_1, t_2, \ldots\}$. The second relation (3.4) follows from the rule (i) and the first one.

The relations (3.3), (3.4) should be understood as formal rules which allow one to translate the infinite sequence of continuous time shifts into the shift of a single discrete variable and vice versa. We do not care about convergency of the infinite substitutions, i.e., the $\tau$-function is considered to be a formal series in $\lambda$’s (in the left hand sides, $\lambda$’s are implicitly present in the definition of discrete flows). In known examples of algebro-geometric solutions, the $\tau$-function is considered to be a true function, not merely a formal series. In this case there are some additional restrictions to the domains of all variables and labels. They ensure the convergency of the infinite substitutions. Meanwhile, for algebro-geometric solutions elements of the graph $\mathcal{G}$ acquire a transparent interpretation on a Riemann surface as punctures and cuts. Furthermore, the discrete time variables discribe discontinuities of the Baker-Akhiezer function on the cuts.

We refer to the discrete flows $\lambda_\alpha \lambda_\beta$ as elementary discrete flows. One may introduce more complicated flows which can be thought of as ”superpositions” of the elementary ones. Specifically, fix several elementary flows, say, $l_1, l_2, \ldots, l_M$ (here $l_i \equiv l_{\alpha_i\beta_i}$ for some $\alpha_i, \beta_i$) and consider the $\tau$-function as a function of a new variable $y$ as follows: $\tau[y] \equiv \tau(l_1 + y, l_2 + y, \ldots, l_M + y)$. In the time evolution with respect to the new variable $y$ the ”elementary” variables $l_i$ simultaneously get shifted by $y$ while the others are constants. Let us call flows of this type composite discrete flows.

To put it differently, let $\partial_i$ be the vector field corresponding to the flow $l_i$. Then the vector field corresponding to the composite flow $y$ is $\partial_y := \sum_{i=1}^{M} \partial_i$, so

$$\exp(\partial_y) \tau(\{l_{\alpha_i\beta_i}\}) = \tau(\{l_{\alpha_i\beta_i} + 1\}) \exp(\partial_y).$$

However, one should be careful since due to (3.2) the simultaneous shift of $l_{\alpha\beta}$ and $l_{\beta\gamma}$ is equivalent to an elementary flow.

The precise definition is as follows:
• Composite discrete flows are labeled by finite sets of vectors \( \{\lambda_\alpha, \lambda_\beta\} \), \( i = 1, 2, \ldots, M \) such that \( \beta_i \neq \alpha_j \) for any \( i, j \). Let \( y \) be the corresponding time variable, then the evolution is defined by

\[
\tau[y] = \tau(\{(l_{\alpha_i} + y\})
\]

where \( l_{\alpha_i} \beta_i \) and other elementary variables are supposed to be constants.

The distinction between elementary and composite flows can be extended to the continuous flows, too. For the reason which will be more clear later, it is natural to consider the continuous times \( t^{(\alpha)}_1 \) as elementary flows. At this stage we motivate this definition by the fact that due to Miwa’s rule \( (3.3) \) they can be obtained as the result of a scaling limit from discrete elementary flows. Similarly, higher continuous times \( t^{(\alpha)}_j \) with \( j \geq 2 \) are limits of composite discrete flows. Therefore, we call them composite.

Let us summarize. We introduced several notions and definitions which are extensively used throughout the paper. First of all, a partial classification of flows and time variables has been suggested. We have defined discrete and continuous flows and distinguished between elementary and composite flows. To any particular equation one may assign a graph which explicitly shows the kinematical structure of the equation and possible constraints imposed on the flows. In order to make this clear, we give some examples.

### 3.2 Examples

Here we illustrate the above notions by familiar examples. To make the graphs of flows more informative, let us add a new graphical element: fat dots mean that the corresponding continuous times are non-zero.

In the KP hierarchy the graph of flows consists of one ”fat” point with corresponding continuous ”times” \( \{t_j\} \). The set of discrete flows is empty. The \( \tau \)-function is \( \tau(t) \equiv \tau(t_1, t_2, \ldots) \).

In the 2DTL hierarchy the graph of flows consists of two ”fat” points with the corresponding times \( \{t_j\} \) and \( \{\bar{t}_j\} \). The discrete flow associated to the vector connecting the two points is the discrete ”time” \( n \) of the 2DTL \( \tau \)-function \( \tau_n(t; \bar{t}) \).

The graphs of flows for the discrete KP and 2DTL equations are as follows:

All continuous times are switched off. In both cases only three independent discrete flows are switched on. This agrees with the continuous case where first non-trivial equations of the KP and 2DTL hierarchies

\[\text{Taking into account Miwa’s rule (iii), it would be more precise to say that the continuous times can not be made equal to zero by a transformation of the form (3.3), (3.4).}\]
have three independent variables $t_1, t_2, t_3$ and $t_1, \bar{t}_1, n$, respectively. In the continuum limit, all the lines except the vertical one in the 2DTL figure shrink up to fat points.

The discretized KP and 2DTL hierarchies are illustrated by the following graphs:

Higher equations of the hierarchies involve more than three elementary flows. Labels of these flows are analogous to the number of a higher flow in the continuous hierarchies. The labels are complex numbers. This looks like a kind of Fourier duality between a parameter marking equations of the hierarchy and the time variable corresponding to a particular flow: continuous flows are marked by a discrete "label" whereas discrete flows are marked by a continuous label.

4 Discrete Zakharov-Shabat representation of Hirota’s equation

The reformulation of classical nonlinear integrable equations as flatness conditions for a two-dimensional connection is the basic constituent of the theory. The flatness means that subsequent shifts along any pair of the time flows commute. These conditions are known as Zakharov-Shabat equations or zero curvature representation. In the paper [1] R.Hirota gave an example of the discretized zero curvature representation for eq. (1.1). In the physical language, the discrete connection is a lattice gauge field. The approach emphasizing the relation to gauge field theories on the lattice was developed by S.Saito and N.Saitoh [21]. We present these results in a modified form which makes the theory completely parallel to the 2DTL theory [22].

The discrete zero curvature condition is equivalent to commutativity of certain multivariable difference operators. The existence of such a "commutation representation" is a hallmark of integrability. At the same time, if a commutation representation exists, it is not unique. In particular, there are different (in fact infinitely many) ways to represent HBDE as a zero curvature condition.

The general scheme is as follows. Choose any time flow to be the "reference" one, i.e., the one in which all the $M$-operators are going to act as differential or difference operators. Commutativity of the flows means that any pair of such $M$-operators obeys a compatibility condition which is just one of the Zakharov-Shabat equations. This fact allows one to relate different hierarchies to each other. In general $M$-operators are pseudo-difference or difference operators with matrix coefficients. Here we consider only the case of difference operators. More general examples are given later in Sect.6. When the reference flow is taken to be an elementary one, the coefficients are scalar functions. Sect. 4.3 contains an example of the zero curvature condition for HBDE realized by $2 \times 2$-matrix difference operators.

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We use this shorter name for what is usually called "quantum pseudo-differential operator".
4.1 Basic $M$-operators

So far all elementary discrete flows enjoyed equal rights. No one of them was better than any other one. We are going to break this equality of rights and distinguish a reference flow. It may be any flow including composite and continuous ones. For simplicity, we start with the case when the reference flow is discrete and elementary. Other cases are discussed later.

The idea is to assign difference operators to flows. These operators act to functions of the reference flow variable. We call them $M$-operators. In this section we consider the simplest $M$-operators which are basic blocks of more general ones.

Let us specify the notation and take the reference flow to be $\lambda_0 \rightarrow \lambda_1 \rightarrow \lambda_2$. The double index notation is inconvenient for practical purposes. Dealing with a limited number of flows, it is worthwhile to give them simpler though less systematic names. Unless otherwise stated, the letter $u$ will be reserved for the reference variable corresponding to an elementary discrete flow. So we set $u = l_{01}$.

Let $\lambda_2$ be any label different from $\lambda_0$, $\lambda_1$. In this situation, the graph of flows is a triangle:

Its sides $\lambda_0 \lambda_2$ and $\lambda_1 \lambda_2$ define flows which we call adjacent to the reference flow $u$ in the obvious sense. In general, a flow $\lambda_\alpha \lambda_\gamma$ (resp., $\lambda_\beta \lambda_\gamma$) is said to be left adjacent (resp., right adjacent) to the flow $\lambda_\alpha \lambda_\beta$.

Coming back to the triangle graph of flows, we set $l_{02} = l$, $l_{12} = l'$. The $\tau$-function will be denoted by $\tau_{u,l'}$. There are only two independent flows. According to (ii), we have

$$\tau_{u,l'}^{l,l'+1} = \tau_{u}^{l+1,l'}.$$  \hfill (4.1)

Coefficients of $M$-operators are expressed through $\tau$. Let us take $u, l$ as independent variables. By definition, the $M$-operator assigned to the left adjacent flow $l$ is

$$M_u^l = e^{\partial_u} - \lambda_0^{01} \frac{\tau_{u,l}^{l,l'+1}}{\tau_{u}^{l+1,l'+1}},$$  \hfill (4.2)

where the coefficient $\lambda_2^{01}$ is expressed through the three labels:

$$\lambda_2^{01} = \frac{1}{\lambda_2 - \lambda_0} - \frac{1}{\lambda_1 - \lambda_0}, \quad \lambda_1^{02} = -\lambda_2^{01}.$$  \hfill (4.3)

The shift operator $e^{\partial_u}$ has standard commutation relations with functions of $u$: $e^{\pm \partial_u} f(u) = f(u \pm 1)e^{\pm \partial_u}$. Note that $M_u^u = e^{\partial_u}$. It is implied that the $\tau$-function in (4.2) might depend on all the other variables which are switched off in this particular case. When they are switched on, they enter eq. (4.2) as parameters. Their values are the same for each of the four $\tau$-functions in the ratio. As a rule, we do not indicate them explicitly if this does not lead to a confusion.

Once the $M$-operator for the left adjacent flow is written, it can be translated into the one for the right adjacent flow $l'$ by passing to independent variables $u, l'$. In eq. (4.2), the (implicit) argument $l'$ is the same in each $\tau$-function. Using (4.1), we rewrite them in such a way that $l$ is the same and implicit. The rule (ii) tells us that the shift $l' \rightarrow l' + 1$ is equivalent the simultaneous shifts $u \rightarrow u + 1$ and $l \rightarrow l + 1$. Thinking about $M$-operators as generating shifts of the discrete variables by 1, it is natural to define the $M$-operator for the right adjacent flow $l'$ as follows:

$$\tilde{M}_{l'}^u = e^{-\partial_u} M_u^l,$$
or, more explicitly,

\[ \tilde{M}'_u = 1 - \lambda^{01}_2 e^{-\frac{\lambda^{01}_2 + 1}{\tau_{u+1}} - 1} e^{-\partial u}. \] (4.4)

It is also useful to introduce

\[ M'_u = e^{-\partial l} M_u, \quad \tilde{M}'_u = e^{-\partial l'} \tilde{M}_u \] (4.5)

which are difference operators in two variables. It trivially follows from the construction that they commute:

\[ [M'_u, \tilde{M}'_u] = 0. \] (4.6)

We have defined \( M \)-operators for elementary discrete flows adjacent to the reference one. In this case they have especially simple form. They are first order difference operators in \( u \). \( M \)-operators corresponding to composite and non-adjacent flows have more complicated structure.

Let us comment on continuous reference flows. According to (3.3), the continuum limit in \( u \) means \( e^{\partial u} \to 1 - \lambda^{-1} \partial t_1 + O(\lambda^{-2}) \), where \( t_1 \) is the first continuous flow with labels \( \lambda_0 \) and \( \lambda = (\lambda_1 - \lambda_0)^{-1} \). The limiting form of the \( M \)-operator (4.2) as \( \lambda \to \infty \) is

\[ M^{(t)} = \partial t_1 - \partial t_1 \log \frac{\tau_{l+1}}{\tau_l} - (\lambda_2 - \lambda_0)^{-1}. \]

This is a first order differential operator in the reference continuous time variable \( t_1 \). It generates shifts in the discrete variable \( l \). The fact that it is an operator of the first order suggests to call the continuous flow \( t_1 \) elementary (see the end of Sect. 3.1).

### 4.2 Discrete Zakharov-Shabat equations

Two independent flows are not enough for deriving bilinear equations. Non-trivial bilinear equations for \( \tau \) arise starting from 3 independent discrete flows, in which case the graph of flows should contain at least 4 vertices. So let us fix 4 labels \( \lambda_\alpha, \alpha = 0, 1, 2, 3 \) and consider the general graph with 4 vertices:

```
\begin{verbatim}
\begin{tikzpicture}
  \node (l) at (2,0) {$l$};
  \node (t1) at (3,3) {$\lambda_1$};
  \node (t2) at (5,3) {$\lambda_2$};
  \node (t3) at (5,-3) {$\lambda_3$};
  \node (t4) at (3,-3) {$\lambda_0$};
  \node (u) at (1,-1) {$u$};
  \node (m) at (2,1) {$m$};
  \node (mbar) at (1,-3) {$\bar{m}$};

  \draw (l) -- (t1);
  \draw (l) -- (t2);
  \draw (l) -- (t3);
  \draw (l) -- (t4);
  \draw (u) -- (m);
  \draw (u) -- (mbar);

  \node (lprime) at (2,-3) {$l'$};
  \node (t1prime) at (3,6) {$\bar{\lambda}_1$};
  \node (t2prime) at (5,6) {$\bar{\lambda}_2$};
  \node (t3prime) at (5,-6) {$\bar{\lambda}_3$};
  \node (t4prime) at (3,-6) {$\bar{\lambda}_0$};
  \node (ubar) at (1,1) {$\bar{u}$};

  \draw (lprime) -- (t1prime);
  \draw (lprime) -- (t2prime);
  \draw (lprime) -- (t3prime);
  \draw (lprime) -- (t4prime);
  \draw (ubar) -- (m);
  \draw (ubar) -- (mbar);

  \draw (lprime) -- (l);
  \draw (t1prime) -- (t1);
  \draw (t2prime) -- (t2);
  \draw (t3prime) -- (t3);
  \draw (t4prime) -- (t4);
  \draw (m) -- (mbar);
\end{tikzpicture}
\end{verbatim}
```

The simplified notation for the flows is clear from the picture. Like in (4.3), we set

\[ \lambda_{\alpha\beta}^\gamma = \frac{1}{\lambda_{\gamma} - \lambda_{\alpha}} - \frac{1}{\lambda_{\beta} - \lambda_{\alpha}}, \quad \lambda_{\beta\gamma}^\alpha = -\lambda_{\alpha\beta}^\gamma \] (4.7)

for all possible values of the pairwise distinct indices.

Let \( \lambda_0 \lambda_1 \) be the reference flow, as before. The left (resp., right) adjacent flows are \( \lambda_0 \lambda_2, \lambda_0 \lambda_3 \) (resp., \( \lambda_1 \lambda_2, \lambda_1 \lambda_3 \)). Each of them has its own \( M \)-operator of the form (4.2) (resp., (4.4)).
The key point is to extend the trivial commutation \((4.6)\) to all the flows in the graph adjacent to \(u\):

\[
[M_0^m, M_0^n] = [M_l^m, M_l^n] = [\bar{M}_0^m, \bar{M}_0^n] = 0. 
\] (4.8)

In contrast to eq. \((4.6)\), these are non-trivial requirements which give bilinear equations for \(\tau\). Written in terms of \(M\)-operators, commutation relations \((4.8)\) are discrete Zakharov-Shabat equations.

In the following we proposes we use the notation like \(M^l_0 = M^l_0(u, l, \bar{l}, \ldots)\) to indicate the dependence of \(M\)-operators on the variables.

**Proposition 4.1** The discrete Zakharov-Shabat equations

\[
M^m_0(m, l + 1)M^l_0(m, l) = M^l_0(m + 1, l)M^m_0(m, l), 
\] (4.9)

\[
\bar{M}^m_0(\bar{m}, \bar{l} + 1)\bar{M}^l_0(\bar{m}, \bar{l}) = \bar{M}^l_0(\bar{m} + 1, \bar{l})\bar{M}^m_0(\bar{m}, \bar{l}), 
\] (4.10)

\[
\bar{M}^l_0(l + 1, \bar{l})M^l_0(l, \bar{l}) = M^l_0(l, \bar{l} + 1)\bar{M}^l_0(l, \bar{l}), 
\] (4.11)

are equivalent to the following bilinear relations for \(\tau\):

\[
\lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l, \lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l + H_1(l, m; u)\bar{r}_{u+1}^l = 0, 
\] (4.12)

\[
\lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l, \lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l + H_2(l, \bar{m}; u)\bar{r}_{u+1}^\bar{l} = 0, 
\] (4.13)

\[
\lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l, \lambda_2^{0,1}r_{u+1}^l \bar{r}_{u+1}^l + H_3(l, \bar{l}; \bar{u})\bar{r}_{u+1}^\bar{l} = 0, 
\] (4.14)

respectively, where \(H_1\) are arbitrary functions such that \(H_1(l, m; u + 1) = H_1(l, m; u)\).

**Proof.** The proof consists in the straightforward commutation of \(M\)-operators. The \(M\)-operators read:

\[
M^l_0 = e^{\partial_\bar{u}} - \lambda_3^{0,1}\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l}, 
\] (4.15)

\[
\bar{M}^m_0 = 1 - \lambda_3^{0,1}\frac{r_{u-1}^m \bar{r}_{u-1}^m}{r_{u-1}^m \bar{r}_{u-1}^m}e^{-\partial_\bar{u}}, 
\] (4.16)

\(M^m_0\) is given by \((4.13)\) with the changes \(l \rightarrow m\) and \(3 \rightarrow 2\), \(M^l_0\) is given by \((4.16)\) with the changes \(\bar{m} \rightarrow \bar{l}\) and \(3 \rightarrow 2\). The details for eq. \((4.9)\) are given below.

Eq. \((4.9)\) reads:

\[
(e^{\partial_\bar{u}} - \lambda_2^{0,1}\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l})(e^{\partial_\bar{u}} - \lambda_3^{0,1}\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m}) =
\]

\[
= (e^{\partial_\bar{u}} - \lambda_3^{0,1}\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m})(e^{\partial_\bar{u}} - \lambda_2^{0,1}\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l}). 
\] (4.17)

The terms \(e^{2\partial_\bar{u}}\) and those which do not contain shift operators cancel automatically. The comparison of the coefficients in front of \(e^{\partial_\bar{u}}\) yields

\[
\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l} = \frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m}, 
\]

\[
\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m} = \frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l}, 
\]

\[
\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m} = \frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l}, 
\]

or

\[
\lambda_2^{0,1}\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l} - \lambda_3^{0,1}\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m} = \frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l}, 
\] (4.18)

\[
\lambda_3^{0,1}\frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m} - \lambda_2^{0,1}\frac{r_{u+1}^l \bar{r}_{u+1}^l}{r_{u+1}^l \bar{r}_{u+1}^l} = \frac{r_{u+1}^m \bar{r}_{u+1}^m}{r_{u+1}^m \bar{r}_{u+1}^m}. 
\] (4.19)

The denominators in both sides differ from the numerators by the shift \(u \rightarrow u + 1\). Therefore, their ratio is a "quasiconstant" in \(u\), so the equation is equivalent to eq. \((4.12)\). This completes the proof.
Now we restore the "equality of rights" of elementary flows by imposing the requirement that Zakharov-Shabat equations should hold for any choice of the reference flow. For example, let \( l \) be the reference flow. Construct \( \mathcal{M} \)-operators for the flows \( u, \bar{u} \) adjacent to \( l \) (see the figure). Then we require that the operators \( \mathcal{M}^{u}_l, \mathcal{M}^{\bar{u}}_l, \mathcal{M}^{\bar{m}}_l \) commute with each other (of course some of them commute automatically due to (4.6)). Note, however, that \( \mathcal{M} \)-operators constructed with respect to different reference flows are not required to be commuting, e.g. \([\mathcal{M}^{\bar{m}}_u, \mathcal{M}^{\bar{m}}_l] \neq 0\).

**Theorem 4.1** Let \( x \) be any one of the elementary flows shown in the picture and let \( v, \bar{v} \) be the corresponding left and right adjacent flows such that \( x, v, \bar{v} \) are independent. Then the commutativity conditions

\[
[\mathcal{M}^{x}_v, \mathcal{M}^{\bar{v}}_x] = 0
\]

imposed simultaneously for any three independent reference flows \( x \) are equivalent to the equations

\[
\lambda_0^{03} r_{u} l^{,m+1,m+1} + \lambda_0^{01} r_{u} l^{,m+1,m} + \lambda_0^{02} r_{u} l^{,m+1,m} = 0 , \tag{4.20}
\]

\[
\lambda_1^{02} r_{u} l^{,i+1,m+1} + \lambda_0^{03} r_{u} l^{,i+1,m+1} + \lambda_0^{01} r_{u} l^{,i+1,m+1} = 0 , \tag{4.21}
\]

\[
\lambda_3^{01} r_{u} l^{,i+1,i+1} - \lambda_3^{02} r_{u} l^{,i+1,i+1} = \lambda_0^{01} r_{u} l^{,i+1,i+1} . \tag{4.22}
\]

**Sketch of proof.** By virtue of the previous proposition, it is enough to show that the functions \( H_i \) are constants: \( H_1 = -H_2 = -H_3 = \lambda_0^{02} \). This can be done straightforwardly by writing down bilinear equations arising from Zakharov-Shabat equations for \( \mathcal{M} \)-operators corresponding to each choice of the reference flow and demanding their consistency with each other.

We see that equations (4.20), (4.22) coincide with the KP and Toda-like forms of HBDE (2.7), (2.11), respectively. (Eq. (4.21) coincides with the KP-like form after the change \( u \to -u \).) The three equations differ by the choice of the independent variables only that agrees with substitutions (2.14)-(2.19). The transition from one triple of independent variables to another should be done according to the rules (i) and (ii) (see (3.1), (3.2)). Using these rules, it is easy to see that the three equations (4.20)-(4.21) are equivalent to each other.

The 4-variable MKP-like form (2.10) of the Hirota equation follows from (4.20) by applying the rule (3.2). Namely, fix an extra label \( \mu_0 \), and consider the flows \( \mu_0 \lambda_0^{\alpha} \), \( \alpha = 0, \ldots, 3 \) with time variables \( p_\alpha \). From (3.2) we have, for instance, \( \tau^{m+1,p_2}_{u,p_0+1} = \tau^{m,p_2+1}_{u,p_0} \) and so on. This change of variables converts equation (4.20) into eq. (2.10).

### 4.3 Matrix realization of the zero curvature condition

We restrict ourselves by giving an example which illustrate the general scheme outlined in the introduction to this section.

Consider the graph of flows which is a reduced version of the one from Sect. 4.2:
The variables which we do not need here are switched off. The simplified ad hoc notation is clear from the figure. This choice of independent variables corresponds to the discretized 2DTL equation.

Our goal here is to write the zero curvature condition with another choice of the reference flow. Specifically, let it be the composite flow labeled by the pair of vectors $\lambda_0 \lambda_3, \lambda_1 \lambda_2$. Let $y$ be the corresponding "composite" time variable. According to the definition given in Sect. 3.1, the $\tau$-function depends on $y$ as follows:

$$\tau_{u}[y] \equiv \tau_{u}[y + \bar{y}].$$

In other words, we set, by definition, $\partial_y := \partial_l + \partial_{\bar{l}}$, so the shift operator $e^{\partial_y}$ acts to the $\tau$-function by shifting $l, \bar{l}$ simultaneously:

$$e^{\partial_y} \tau_{u} = \tau_{u+1, \bar{l}+1} e^{\partial_y}.$$

Introduce the following difference operators with $2 \times 2$-matrix coefficients:

$$L_n(l, \bar{l}) = \begin{pmatrix} e^{\partial_y} + \nu & \tau_{n+1} \bar{l} \
-\mu \tau_{n+1} l & -\mu \tau_{n+1} \bar{l} \end{pmatrix},$$

$$M_n(l, \bar{l}) = \begin{pmatrix} 1 & -\mu \tau_{n+1} \bar{l} \\
\tau_{n+1} \bar{l} & -e^{\partial_y} - \mu \tau_{n+1} \bar{l} \end{pmatrix},$$

where we put $\mu \equiv \lambda_1^{[1]}$, $\nu \equiv \lambda_3^{[1]}$ for the sake of brevity.

**Proposition 4.2** The matrix discrete Zakharov-Shabat equation

$$L_n(l, \bar{l}) M_n(l, \bar{l}) = M_{n+1}(l, \bar{l}) L_n(l, \bar{l})$$

is equivalent to the bilinear relation

$$\tau_{n+1} \bar{l} + \tau_n l = H_n(l, \bar{l}) \tau_{n+1} \bar{l} + \tau_n l,$$

where $H_n(l, \bar{l})$ is periodic in $n$ with period 1: $H_{n+1}(l, \bar{l}) = H_n(l, \bar{l})$.

This bilinear equation coincides with eq. (4.14). We omit the proof since it is absolutely straightforward after the $L$-$M$ pair is given. A way to derive matrix $M$-operators from the scalar ones is discussed in Sect. 5.

Like in Theorem 4.1, the validity of the zero curvature condition for $M$-operators constructed with respect to all possible independent reference flows implies the bilinear equation with a fixed constant function $H$. It has the form (2.11).

**Remark 4.1** In the 2DTL interpretation, the operator $M_n$ generates evolution in the chiral discrete "space-time" variable $\bar{l}$ whereas $L_n$ generates shifts along the $n$-lattice. In our scheme, both $M_n$ and $L_n$ are "$M$-operators" rather than "$L$-operators". We write $L_n$ according to the tradition which is justified by the case when an additional reduction of the 2DTL is implied.

It is instructive to look at the continuous version of this zero curvature condition. It provides the zero curvature representation of the 2DTL with the composite continuous reference flow defined by the vector field $\partial_y := \partial_l + \partial_{\bar{l}}$ (see Sect. 3). This representation naturally arises when one embeds the 2DTL into the 2-component KP hierarchy. The Zakharov-Shabat equation

$$\partial_{\bar{l}} L_n = M_{n+1} L_n - L_n M_n,$$
where

\[
L_n = \begin{pmatrix}
\frac{\partial_y - \partial_{t_1}(\log \tau_{n+1})}{\tau_n} & -\frac{\tau_{n+1}}{\tau_n} \\
\frac{\tau_n}{\tau_{n+1}} & 0
\end{pmatrix},
\]

\[
M_n = \begin{pmatrix}
0 & \frac{\tau_{n+1}}{\tau_n} \\
-\frac{\tau_{n-1}}{\tau_n} & \partial_y
\end{pmatrix}
\]

is equivalent to

\[
\partial_{t_1} \partial_{\bar{t}_1} \log \frac{\tau_{n+1}}{\tau_n} = -\frac{\tau_{n+1}}{\tau_n} \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} - \frac{\tau_{n+2} \tau_n}{\tau_{n+1}^2},
\]

which is the 2DTL equation in the bilinear form.

**Remark 4.2** The 1D Toda chain (1DTC) is a reduction of the 2DTL such that \(\tau_n\) does not depend on \(t_1 + \bar{t}_1\), i.e., \(\partial_y\) commutes with \(\tau_n\). Therefore, in this case \(\partial_y\) can be considered as a c-number. Identifying it with the spectral parameter, one recognizes eqs. (4.27) as the standard \(L-M\)-pair for the 1DTC realized by \(2 \times 2\)-matrices depending on the spectral parameter (see e.g. [23]).

## 5 Linearization of the Hirota equations

Zero curvature conditions studied in the previous section are equivalent to compatibility of an overdetermined system of linear difference equations for a "wave function" \(\psi\). These linear equations are called **auxiliary linear problems** (ALP). They play a very important role in the theory. Common solutions to ALP carry the complete information about solutions to the nonlinear equations. All the properties of the latter can be translated into the language of the ALP. This is what we mean by the linearization of HBDE.

In accordance with the diversity of zero curvature representations there are many types of the ALP. This section deals with the most important examples.

We begin with the scalar linear problems associated with the \(M\)-operators (4.15), (4.16) for elementary discrete flows adjacent to the reference one. They are simple first order linear difference equations with coefficients expressed through the \(\tau\)-function. The formal solution of a special form is called (formal) Baker-Akhiezer function. It depends on a spectral parameter. Baker-Akhiezer functions are formal analogues of Bloch solutions. The formula for Baker-Akhiezer functions in terms of the \(\tau\)-function was suggested for the first time in [24]. General solutions to the ALP are linear combinations of Baker-Akhiezer functions with different spectral parameters. In a similar way, one may define dual Baker-Akhiezer functions as formal solutions to the linear problems for adjoint operators.

Given a solution to the ALP, one may consider Bäcklund transformations. Furthermore, a "duality" between coefficient functions and solutions of the ALP allows one to define a chain of successive Bäcklund transformations described by the Bäcklund flow. We consider two types of the Bäcklund flows. It is shown that in the particular case when the solutions to the ALP are Baker-Akhiezer functions the Bäcklund flows can be identified with elementary discrete flows adjacent to the reference flow.

There is a "gauge freedom" in the ALP which can be fixed by certain normalization of \(\psi\). Mostly we use the gauge which leads to the simplest possible form of the linear equations. Another choice – \(z_0\)-gauge – is briefly discussed in Sect. 5.4. This gauge makes equations more symmetric for the price of introducing an auxiliary point \(z_0 \in \mathbb{C}\) and complication of coefficient functions.

The ALP associated with the matrix \(M\)-operators are also discussed. In fact matrix \(M\)-operators can be most conveniently derived using the scalar ALP. The matrix linear problems are obtained by rearranging the scalar ones. More precisely, in order to rearrange the scalar ALP in such a way that the reference flow is taken to be composite, one has to pass to difference operators with matrix coefficients.
5.1 Scalar linear problems

The commutativity of $M$-operators [4.8] implies that they have a common set of eigenfunctions. Equivalently, the discrete Zakharov-Shabat equations (4.9)-(4.11) for $M$-operators imply compatibility of the linear problems

$$M_{u}^{l} \tilde{\psi}^{l,i}(u) = \psi^{l+1,i}(u), \quad (5.1)$$
$$\bar{M}^{l} \tilde{\psi}^{l,i}(u) = \psi^{l,i+1}(u) \quad (5.2)$$

for any elementary discrete flows $l, \bar{l}$ adjacent to $u$. Note that the "eigenvalues" put equal to 1 in the r.h.s. can be made arbitrary by changing normalization of $\psi$. Our choice in (5.1), (5.2) is most convenient in the purely discrete case though it does not lead to a smooth continuum limit.

More explicitly, eqs. (5.1), (5.2) read (see (4.13), (4.16)):

$$\psi^{l,i}(u + 1) - \lambda_{3}^{01} V^{l,i}(u) \psi^{l,i}(u) = \psi^{l+1,i}(u), \quad (5.3)$$
$$\psi^{l,i}(u) - \lambda_{2}^{01} C^{l,i}(u) \psi^{l,i}(u - 1) = \psi^{l,i+1}(u), \quad (5.4)$$

where

$$V^{l,i}(u) := \frac{\tau_{u}^{l,i} \tau_{u+1}^{l+1,i}}{\tau_{u+1}^{l,i} \tau_{u}^{l+1,i}}, \quad (5.5)$$
$$C^{l,i}(u) := \frac{\tau_{u}^{l,i} \tau_{u-1}^{l+1,i}}{\tau_{u-1}^{l,i} \tau_{u}^{l+1,i}}, \quad (5.6)$$

These formulas become more symmetric in terms of the "unnormalized" wave function

$$\rho_{u}^{l,i} = \psi^{l,i}(u) t_{u}^{l,i}. \quad (5.7)$$

Plugging this into (5.3), (5.4), we get:

$$\tau_{u}^{l+1,i} \rho_{u+1}^{l,i} - \lambda_{3}^{01} \tau_{u+1}^{l,i} \rho_{u}^{l+1,i} = \tau_{u+1}^{l,i} \rho_{u}^{l+1,i}; \quad (5.8)$$
$$\tau_{u}^{l+1,i} \rho_{u-1}^{l,i} - \lambda_{2}^{01} \tau_{u-1}^{l,i} \rho_{u}^{l+1,i} = \tau_{u}^{l,i} \rho_{u}^{l+1,i}, \quad (5.9)$$

Let us show that a carefully performed continuum limit of these equations yields the familiar linear problems for the 2DTL. Let

$$\lambda_{3} = \lambda_{0} + \epsilon, \quad \lambda_{2} = \lambda_{1} + \bar{\epsilon}, \quad \epsilon, \bar{\epsilon} \to 0,$$

so

$$e^{\partial_{i}} \to 1 - \epsilon \partial_{i}, \quad e^{\partial_{i}} \to 1 - \bar{\epsilon} \partial_{i}$$

by virtue of Miwa’s rule (3.3) applied to the discrete flows $\lambda_{0} \lambda_{3}, \lambda_{1} \lambda_{2}$, respectively. Let us change the normalization of the $\psi$-function introducing the $\varphi$-function as follows:

$$\varphi^{l,i}(u) = (\lambda_{0} - \lambda_{3})^{i} \psi^{l,i}(u) \quad (5.10)$$

so the linear problems read

$$(\lambda_{0} - \lambda_{3}) \varphi^{l,i}(u + 1) - \lambda (\lambda_{3} - \lambda_{1}) V^{l,i}(u) \varphi^{l,i}(u) = \varphi^{l+1,i}(u),$$
$$(\lambda_{0} - \lambda_{2}) \varphi^{l,i}(u) - \lambda (\lambda_{2} - \lambda_{1}) C^{l,i}(u) \varphi^{l,i}(u - 1) = (\lambda_{0} - \lambda_{2}) \varphi^{l,i+1}(u), \quad (5.11)$$

where $\lambda \equiv (\lambda_{1} - \lambda_{0})^{-1}$. We get as $\epsilon, \bar{\epsilon} \to 0$:

$$\begin{cases}
\partial_{t_{1}} \varphi(u) = \varphi(u + 1) + \left( \lambda + \partial_{t_{1}} \log \frac{\tau_{u+1}}{\tau_{u}} \right) \varphi(u) \\
\partial_{t_{1}} \varphi(u) = \lambda \frac{\tau_{u+1} \tau_{u-1}}{\tau_{u}^{2}} \varphi(u - 1).
\end{cases} \quad (5.12)$$

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The transformation
\[ \tau_u \to \lambda^{-u^2} e^{-\lambda u_1 \tau_u} \]
eliminates the constant \( \lambda \) turning the linear problems into the familiar form
\[
\begin{cases}
\partial_{\tau_1} \varphi(u) = \varphi(u + 1) + v(u) \varphi(u) \\
\partial_{\tau_1} \varphi(u) = c(u) \varphi(u - 1).
\end{cases}
\]  
(5.13)

Here
\[ v(u) = \partial_{\tau_1} \log \frac{\tau_{u+1}}{\tau_u}, \quad c(u) = \frac{\tau_{u+1} \tau_{u-1}}{\tau_u^2} \]
(for the sake of notational simplicity we use here the same letter for the transformed function). The Zakharov-Shabat equation
\[
\left[ \partial_{\tau_1} - e^{\partial_u} - v(u), \partial_{\tau_1} - c(u) e^{-\partial_u} \right] = 0
\]
yields the 2DTL equation in the form
\[
\partial_{\tau_1} \partial_{\tau_1} \log \frac{\tau_{u+1}}{\tau_u} = \frac{\tau_{u+1} \tau_{u-1}}{\tau_u^2} - \frac{\tau_u + 2 \tau_u}{\tau_{u+1}}.
\]  
(5.14)

These are continuous analogues of eqs. (4.11), (4.19), respectively.

5.2 Bäcklund transformations

The ALP in the form \( \text{(5.8), (5.9)} \) have a remarkable property emphasized in ref. \( \text{[21]} \): they are symmetric under interchanging \( \tau \) and \( \rho \). Furthermore, one may treat them as linear problems for the function \( \tau \), the compatibility condition being a bilinear equation for \( \rho \). This equation is again HBDE of the same form. In ref. \( \text{[21]} \), this fact was referred to as the "duality" between "potentials" \( \tau \) and "wave functions" \( \rho \). This "duality" shows up most transparently in the fully discretized case.

More precisely, rewriting eqs. \( \text{(5.3), (5.4)} \) as linear equations for
\[
\tilde{\psi}^{l,j}(u) = \frac{\tau_{l,j+1}}{\tau_{l,j+1}^2} \left( \psi_{l,j+1}(u + 1) \right)^{-1} = \left( \psi_{l,j+1}(u + 1) \right)^{-1},
\]
(see \( \text{(5.7)} \)), we get
\[
\left( e^{-\partial_u} - \lambda_3^{01} \tilde{V}^{l,j}(u) \right) \tilde{\psi}^{l,j}(u) = \tilde{\psi}^{l-1,j}(u),
\]
\[
\left( 1 - \lambda_2^{01} \tilde{C}^{l,j}(u + 1) e^{\partial_u} \right) \tilde{\psi}^{l,j}(u) = \tilde{\psi}^{l,j-1}(u),
\]
(5.15) (5.16)

where \( \tilde{V} \) and \( \tilde{C} \) are given by the same formulas \( \text{(5.3), (5.4)} \) with \( \rho \) in place of \( \tau \). The difference operators in the l.h.s. are adjoint to the operators \( \text{(4.13), (4.14)} \) with \( \tau \to \rho \). The formal adjoint operator is defined by the rule \( (f(u) e^{k \partial_u})^l = e^{-k \partial_u} f(u) \). It then follows that the compatibility conditions are described by Theorem 4.1 with \( \tau \) replaced by \( \rho \).

Therefore, passing from a given solution \( \tau \) to \( \rho \) we have got a new solution to HBDE. This is a Bäcklund-type transformation also known under the names "Darboux" or "Bäcklund-Darboux" transformation. For a comprehensive discussion of transformations of this kind see \( \text{[23]} \). The bilinear form of Bäcklund transformations was suggested by R.Hirota \( \text{[3]} \).

One may repeat the procedure once again starting from \( \rho \) and, moreover, consider a chain of successive transformations of this type. It is natural to introduce an additional discrete variable \( b \) to mark steps of the "flow" along this chain and let \( \tau_{u,b}^{l,j}, \rho_{u,b}^{l,j} \) be \( \tau \) and \( \rho \) at \( b \)-th step, respectively. The first Bäcklund flow is defined by
\[
\tau_{u,b+1}^{l,j} = \rho_{u,b}^{l,j}.
\]  
(5.17)

This means that \( \tau \) at the next step of the "Bäcklund time" \( b \) is put equal to a solution \( \rho \) of the linear equations \( \text{(5.8), (5.9)} \). Then these linear problems become bilinear equations for \( \tau_b \):
\[
\tau_{u,b}^{l+1,j} - \lambda_3^{01} \tau_{u+1,b}^{l+1,j} \tau_{u,b+1}^{l,j} - \tau_{u+1,b}^{l,j} \tau_{u,b+1}^{l+1,j} = 0,
\]  
(5.18)
\[ \tau_{u,b}^{l+1} - \tau_{u,b}^{l}(u) = \lambda_{01}^{l} \tau_{u,b}^{l+1}(u) - \lambda_{12}^{l} \tau_{u,b}^{l} u_{-1,b+1}, \]  
(5.19)
where \( \bar{l} \) (resp., \( l \)) in eq. (5.18) (resp., (5.19)) is skipped.

Similarly, defining the second Bäcklund flow (the Bäcklund ”time” is now denoted by \( \bar{b} \)),

\[ \tau_{u,\bar{b}}^{l+1} - \tau_{u,\bar{b}}^{l}(u) = \lambda_{01}^{l} \tau_{u,\bar{b}}^{l+1} + \lambda_{12}^{l} \tau_{u,\bar{b}}^{l}, \]  
(5.20)
we get from (5.8), (5.9):

\[ \lambda_{01}^{l} \tau_{u,\bar{b}}^{l+1} + \lambda_{12}^{l} \tau_{u,\bar{b}}^{l} = \tau_{u,b}^{l} \tau_{u,b}^{l+1}, \]  
(5.21)
In these equations one immediately recognizes different forms of HBDE. A time discretization of the Toda chain by means of Darboux transformations was considered in [26].

The Bäcklund flows can be defined by a zero curvature condition. Given any solution \( \psi \) to the ALP (5.3), (5.4), introduce the operator

\[ B_{u}^{l} = e^{-\partial_{u}} \left( e^{\partial_{b}} - \frac{\psi(u+1)}{\psi(u)} \right). \]  
(5.23)
Then eq. (5.18) is represented as the commutativity condition \([B_{u}^{l}, M_{u}^{l}] = 0\). A similar \( B \)-operator exists for the second Bäcklund flow.

### 5.3 Baker-Akhiezer functions

Each of the ALP (5.3), (5.4) is a first order linear difference equation in two variables. Assuming HBDE (4.20)-(4.22) hold, we are going to construct a 1-parametric family of their common solutions of a special form. These solutions \( \psi(u) = \psi(u; z) \) are called Baker-Akhiezer functions. They depend on the spectral parameter \( z \in \mathbb{C} \).

Let us switch on an extra elementary flow shown by the dotted line:

The corresponding time variable is \( p_{z} \). Let

\[ \lambda_{z}^{\alpha \beta} = \frac{1}{z - \lambda_{\alpha}} - \frac{1}{\lambda_{\beta} - \lambda_{\alpha}}. \]  
(5.24)
Note the identity

\[ \lambda_{z}^{\alpha \beta} \lambda_{z}^{\beta \gamma} = \lambda_{\alpha}^{\beta \gamma} \lambda_{z}^{\alpha \gamma}. \]  
(5.25)
Then, assuming the 3-term Hirota equations hold for for the triples \( (u, l, p_{z}) \) and \( (u, \bar{l}, p_{z}) \) of independent variables,

\[ \psi^{l+1}(u; z) = (\lambda_{z}^{01})^{u} (\lambda_{z}^{03})^{-l} \left( \frac{\lambda_{z}^{02}}{\lambda_{z}^{01}} \right)^{l+1} \tau_{u,p_{z}+1}^{l} \tau_{u,p_{z}}^{l} \bigg|_{p_{z}=0} \]  
(5.26)
is a formal common solution to eqs. (5.3), (5.4) for any \( z \). Indeed, under the substitution (5.26) eq. (5.3) turns into eq. (4.20) for the triple \( (u, l, p_{z}) \) while eq. (5.4) turns into eq. (4.22) for the triple \( (u, \bar{l}, p_{z}) \).
Therefore, the new label \( z \) is identified with the spectral parameter. Formula (5.26) for the \( \psi \)-function coincides with the Japanese formula [15], [24] because due to (3.3) we have

\[
\frac{\tau_{u,l}^{(l)}(-z - \lambda_0)}{\tau_{u,l}^{(l)}(0)}.
\]

The general solution to the ALP can be represented in the form

\[
\psi(u) = \int d^2z \mu(z) \psi(u; z)
\]  

with arbitrary ”measure” \( \mu(z) \) on the complex plane. In other words, this is a linear combination of Baker-Akhiezer functions with different spectral parameters.

Note that the ”\( B \)-operator” (5.23) in which \( \psi \) is taken to be the Baker-Akhiezer function coincides with an \( M \)-operator. Indeed, we have the following formula for the \( M \)-operator (4.15) in terms of the Baker-Akhiezer function:

\[
M_u^i = \lim_{z \to \lambda_3} \left( e^{\partial_z} - \frac{\psi(u + 1; z)}{\psi(u; z)} \right).
\]  

The dual Baker-Akhiezer function \( \psi^* \) is defined by

\[
\psi^{*l,i}(u; z) = (\lambda_3^{01})^{-u} (\lambda_3^{03})^{-l} \left( \frac{\lambda_3^{02}}{\lambda_3^{01}} \right)^{-l} \frac{\tau_{u,pz}^{(l)}(-z - \lambda_0)}{\tau_{u,pz}^{(l)}(0)} |_{p_z = 0}.
\]  

It satisfies the equations

\[
(M_u^l(u - l - 1))^i \psi^{*l,i}(u; z) = \psi^{*l-1}(u; z),
\]

\[
(M_u^l(u - 1, l - 1))^i \psi^{*l,i}(u; z) = \psi^{*l-1}(u; z),
\]  

where the difference operators in the l.h.s. are formally adjoint to the operators (4.13), (4.16).

### 5.4 The \( z_0 \)-gauge

Equations (5.23), (5.24) imply a specific choice of normalization of the \( \psi \)-function. Indeed, multiplying \( \psi \) by any function, one can change the form of the equations. This is a kind of a ”gauge freedom”. There is no canonical way to fix the gauge. The gauge that we systematically use throughout the paper is the most economic one in the sense that the ALP for discrete flows have the simplest possible form. Here we are going to discuss another choice which has its own advantages.

This more general gauge requires to fix an additional point \( z_0 \in \mathbb{C} \) different from the vertices of the graph of flows. The gauge is defined by the following normalization condition for the Baker-Akhiezer function \( \Psi(u; z) \):

\[
\Psi(u; z_0) = 1.
\]  

We call it \( z_0 \)-gauge. Given this condition, it is natural to represent \( \Psi \) in the form

\[
\Psi(u; z) = \frac{\psi(u; z)}{\psi(u; z_0)}
\]

and rewrite the ALP (5.3), (5.4) for \( \psi \) in terms of \( \Psi \). In this way, we get

\[
(\lambda_3^{0l} U(u, l)e^{\partial_z} - \lambda_3^{03} W(u, l)e^{\partial_z}) \Psi(u; z) = \lambda_3^{01} \Psi(u; z),
\]  

where

\[
U(u, l) = \frac{\tau_{u,p+1}^{(l)} \tau_{u + 1, p}^{(l)}}{\tau_{u,p}^{(l)} \tau_{u + 1, p}^{(l)}}, \quad W(u, l) = \frac{\tau_{u,p+1}^{(l)} \tau_{u + 1, p}^{(l)}}{\tau_{u,p}^{(l)} \tau_{u + 1, p}^{(l)}}.
\]  

A general prescription for writing down the equation should be clear from comparison with the picture.
With this prescription at hand, a similar equation can be written for any pair of flows such that one of them is reference and another one is left adjacent.

A nice feature of the \( z_0 \)-gauge is that there is no need to care about equations for right adjacent flows. They are automatically produced by the same prescription if one changes the ”orientation” of the reference flow (i.e., consider \( \lambda_1 \lambda_0^\ast \) as the reference flow). In order to express everything in the same variables, one should apply the rules (3.1) \((u \rightarrow -u)\) and (3.2) \((\bar{p}_0 \rightarrow p_0)\). We stress that eq. (5.4) cannot be obtained from eq. (5.3) in this way. In that gauge we need two types of equations separately for left and right adjacent flows.

The Baker-Akhiezer function that solves all these equations in the \( z_0 \)-gauge has the form

\[
\Psi(u; z) = \prod_{\alpha \beta} \left( \frac{\lambda_0^{\alpha \beta}}{\lambda_{\alpha \beta} \nu_{\alpha \beta}} \right) \frac{\nu_{p_0, p_{\ast}}}{\nu_{p_0 + 1, p_{\ast}}} \bigg|_{p_0 = p_{\ast} = 0}.
\]

As usual, the \( \tau \)-functions depend on all the skipped variables as parameters. Due to (5.25) the form of the prefactor is consistent with (3.1), (3.2).

Our previous gauge is a limiting case of the \( z_0 \)-gauge as \( z_0 \rightarrow \lambda_0 \). However, the limit is singular and, therefore, it needs a regularization. As a result, the symmetry of the \( z_0 \)-gauge becomes broken.

## 5.5 Two-component formalism

Linear equations (5.3), (5.4) can be brought into another form in which they become first order partial difference equations for a 2-component vector function. Their compatibility yields matrix Zakharov-Shabat equations presented in Sect. 4.3.

Let us use the notation of Sect. 4.3 and, accordingly, denote \( \psi_n \equiv \psi(n) \), \( V_n^{\gamma \bar{\gamma}} \equiv V^{\gamma \bar{\gamma}}(n) \), \( C_n^{\kappa \bar{\kappa}} \equiv C^{\kappa \bar{\kappa}}(n) \). Then eqs. (5.3), (5.4) read:

\[
\psi_{n+1}^{\gamma \bar{\gamma}} = \psi_n^{\gamma \bar{\gamma}} - \nu V_n^{\gamma \bar{\gamma}} \psi_n^{\gamma \bar{\gamma}},
\]

\[
\psi_{n+1}^{\gamma \bar{\gamma}} = \psi_n^{\gamma \bar{gamma}} - \mu C_n^{\gamma \bar{\gamma}} \psi_n^{\gamma \bar{\gamma}}.
\]

These equations allow us to find out how the vector

\[
\begin{pmatrix}
\psi_n^{\gamma \bar{\gamma}} \\
\psi_{n-1}^{\gamma \bar{\gamma}}
\end{pmatrix}
\]

transforms under shifts in \( n \) and \( \bar{\gamma} \).

Combining eqs. (5.3), (5.4) we have, for instance (here \( \partial_y \equiv \partial_l + \partial_{\bar{\gamma}} \)):

\[
\psi_{n+1}^{\gamma \bar{\gamma}} = \psi_n^{\gamma \bar{\gamma}} + \nu V_n^{\gamma \bar{\gamma}} \psi_n^{\gamma \bar{\gamma}}
\]

\[
= \psi_n^{\gamma \bar{\gamma}} + \mu C_n^{\gamma \bar{\gamma}} \psi_n^{\gamma \bar{\gamma}} - \nu V_n^{\gamma \bar{\gamma}} \psi_{n-1}^{\gamma \bar{\gamma}}
\]

\[
= (\nu V_n^{\gamma \bar{\gamma}} \psi_{n-1}^{\gamma \bar{\gamma}} - \nu V_n^{\gamma \bar{\gamma}} \psi_{n-1}^{\gamma \bar{\gamma}})
\]

\[
= (\nu V_{n+1}^{\gamma \bar{\gamma}} + \mu C_{n+1}^{\gamma \bar{\gamma}}) \psi_{n+1}^{\gamma \bar{\gamma}} - \nu V_{n+1}^{\gamma \bar{\gamma}} \psi_{n-1}^{\gamma \bar{\gamma}} - \mu C_{n+1}^{\gamma \bar{\gamma}} \psi_{n-1}^{\gamma \bar{\gamma}}.
\]

(5.38)
Proceeding in the same way, we get:

\[
\begin{pmatrix}
\psi_{n+1}^{l,i} \\
\psi_n^{l,i}
\end{pmatrix} = 
\begin{pmatrix}
e^{\partial_y} + \nu \lambda_{n,\ell}^{l,i} + \mu \lambda_{n+l}^{l,i} & -\mu \nu \lambda_{n+1,l}^{l,i} \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_n^{l,i} \\
\psi_{n-1}^{l,i}
\end{pmatrix},
\] (5.44)

\[
\begin{pmatrix}
\psi_{n+1}^{l,i} \\
\psi_n^{l,i+1}
\end{pmatrix} = 
\begin{pmatrix}
1 & -\mu \lambda_n^{l,i}
\end{pmatrix} \begin{pmatrix}
\psi_n^{l,i} \\
\psi_{n-1}^{l,i}
\end{pmatrix},
\] (5.39)

The operators in the r.h.s. provide a matrix \( L-M \)-pair for HBDE which differs from \( [4, 23] \) by a "gauge" transformation. Recall that the Baker-Akhiezer function \( 5.26 \) has \( \tau_n \) in the denominator, so the two components of the vector \( 5.37 \) have different denominators. In the 2-component formalism, it is natural to require the denominators to be the same. This condition partially fixes the gauge.

Thus, introducing the vector \( (\psi_n, \chi_n) \) with the second component

\[
\chi_n^{l,i} = \frac{\tau_{n-1}^{l,i}}{\tau_n^{l,i}} \psi_n^{l,i},
\] (5.40)

we rewrite eqs. \( 5.39 \) in the form

\[
L_n(l, \bar{l})(\begin{pmatrix}
\psi_n^{l,i} \\
\chi_n^{l,i}
\end{pmatrix}) = \begin{pmatrix}
\psi_{n+1}^{l,i} \\
\chi_{n+1}^{l,i}
\end{pmatrix},
\] (5.41)

\[
M_n(l, \bar{l})(\begin{pmatrix}
\psi_n^{l,i} \\
\chi_n^{l,i}
\end{pmatrix}) = \nu \begin{pmatrix}
\psi_{n+1}^{l,i+1} \\
\chi_{n+1}^{l,i+1}
\end{pmatrix},
\] (5.42)

with the \( L \) and \( M \)-operators given by eqs. \( 4.23 \).

Equations \( 5.41 \), \( 5.42 \) imply some useful difference equations for \( \psi_n^{l,i} \). Excluding \( \chi_n \), we get from \( 5.41 \):

\[
\psi_{n+1}^{l,i+1} - \psi_n^{l,i} = \left( \nu \frac{\tau_{n+1}^{l,i+1}}{\tau_n^{l,i+1}} + \mu \frac{\tau_{n-l}^{l,i+1}}{\tau_n^{l,i+1}} \right) \psi_n^{l,i} + \mu \nu \frac{\tau_{n+1}^{l,i+1}}{\tau_n^{l,i+1}} \psi_{n-1}^{l,i}.
\] (5.43)

Similarly, excluding \( \chi_n \) from eq. \( 5.42 \) and using HBDE \( 2.11 \), we get a 4-term equation for \( \psi_n \):

\[
\psi_{n+1}^{l,i+1} - \psi_n^{l,i+1} + \psi_n^{l,i} = -\nu \frac{\tau_{n+2}^{l,i+1}}{\tau_n^{l,i+1}} \psi_n^{l,i+1} + (\nu - \mu) \frac{\tau_{n+1}^{l,i+1}}{\tau_n^{l,i+1}} \psi_{n-1}^{l,i+1} \psi_{n-1}^{l,i}.
\] (5.44)

Here it is implied that \( \tau_n^{l,i} \) satisfies HBDE \( 4.22 \) in the corresponding notation. For more information on eq. \( 5.44 \) see the next section.

The continuous analogue of eq. \( 5.43 \) is

\[
(\partial_t + \partial_x) \varphi = \varphi_{n+1} + \left( \partial_x \log \frac{\tau_n}{\tau_{n+1}} \right) \varphi_n + \frac{\tau_n - \tau_{n+1}}{\tau_n^2} \varphi_{n-1}.
\] (5.45)

This is obvious from \( 5.12 \) (we write \( \varphi_n \) instead of \( \varphi(u) \) according to the present notation). It is a discrete non-stationary Schrödinger equation. Eq. \( 5.44 \) in the continuum limit turns into

\[
\partial_t \partial_x \varphi_n - \left( \partial_x \log \frac{\tau_n}{\tau_{n+1}} \right) \partial_t \varphi_n - \frac{\tau_n - \tau_{n+1}}{\tau_n^2} \varphi_n = 0.
\] (5.46)

It is a continuous two-dimensional Schrödinger equation in magnetic field. Its quasiperiodic solutions were studied in the paper \( 27 \) by means of the algebro-geometric approach. The corresponding theory for discrete two-dimensional equations similar to \( 5.44 \) was proposed in \( 28 \).
6 Pseudo-difference $M$-operators

In this section we study the general form of $M$-operators which satisfy conditions of the zero curvature form with $M$-operators for elementary discrete flows adjacent to the reference flow.

Starting from the scalar ALP for a pair of left and right adjacent flows it is not difficult to find the $M$-operators for non-adjacent flows. Indeed, it is possible to exclude the reference flow from the pair of linear equations. Then the right adjacent flow turns out to be written in terms of the left adjacent one. Considering the latter as a new reference flow, one obtains a general $M$-operator for any elementary discrete flow in terms of any (elementary discrete) reference flow. In general these are pseudo-difference operators, i.e., they contain negative powers of first order difference operators.

This construction can be extended to more general operators which generate some new flows commuting with the elementary discrete flows. We call them adjoint flows. The corresponding pseudo-difference operators are constructed in Sect. 6.2 with the help of two arbitrary independent solutions $\psi, \psi^*$ to the ALP and the adjoint ALP.

6.1 $M$-operators for arbitrary elementary discrete flows and the corresponding linear problems

Rearranging eqs. (6.3), (6.4), it is possible to find out $M$-operators for arbitrary elementary discrete flows, not only for adjacent to the reference flow. The idea is to exclude shifts in $u$ and then consider $l$ as a new reference flow.

From (6.3) we have

$$\psi^{l,l}(u+1) = \psi^{l+1,l}(u) + \lambda_1^{01} V^{l,l}(u) \psi^{l,l}(u).$$

Plugging this into (6.4) written in the form

$$\psi^{l,l}(u+1) - \psi^{l,l+1}(u+1) = \lambda_2^{01} C^{l,l}(u+1) \psi^{l,l}(u),$$

in place of $\psi^{l,l}(u+1)$ and $\psi^{l,l+1}(u+1)$, we get

$$\psi^{l+1,l+1}(u) + \lambda_3^{01} V^{l,l+1}(u) \psi^{l,l+1}(u) = \psi^{l+1,l}(u) + (\lambda_2^{01} C^{l,l}(u) - \lambda_2^{01} C^{l,l+1}(u+1)) \psi^{l,l}(u).$$

Using (4.22), we find

$$\lambda_3^{01} V^{l,l}(u) - \lambda_2^{01} C^{l,l}(u+1) = \lambda_3^{01} V^{l,l+1}(u) - \lambda_2^{01} C^{l+1,l}(u) = \lambda_3^{01} \frac{V^{l+1,l+1} - V^{l,l+1}}{C^{l+1,l+1} - C^{l,l+1}}. (6.1)$$

Note that the first equality follows already from eq. (1.11) which is a weaker condition than (1.22).

Therefore, $\psi$ obeys the following 4-term linear equation:

$$\psi^{l+1,l+1}(u) - \psi^{l+1,l}(u) = \lambda_3^{01} \frac{V^{l+1,l+1} - V^{l,l+1}}{C^{l+1,l+1} - C^{l,l+1}} \psi^{l,l+1}(u) + \lambda_2^{01} \frac{V^{l+1,l+1} - V^{l+1,l}}{C^{l+1,l+1} - C^{l+1,l}} \psi^{l,l}(u). (6.2)$$

in which we recognize eq. (5.43) from Sect. 5.5.

The relation (6.1) allows us to rewrite it in the form

$$\frac{t_{l+1,l}}{t_{l,l+1}} \Delta_{l} \frac{t_{l+1,l}}{t_{l,l+1}} \Delta_{l} \psi^{l,l}(u) + \lambda_2^{01} \psi^{l,l}(u) = 0, (6.3)$$

where

$$\Delta_{l} \equiv e^{2i} - 1.$$

(6.4)
This equation looks like a discrete 2-dimensional Laplace equation in a curved space. It can be formally rewritten as

\[
\left( \bar{\Delta}_l + \lambda_1^{01} \frac{\tau_{u+1}^t \psi_{l+1,t}^u}{\tau_{u+1}^{t+1}} + \lambda_2^{01} \frac{\tau_{l+1}^t \psi_{l,t+1}^u}{\tau_{l+1}^{t+1}} \right) \psi_{l,t}^u(u) = 0, \quad (6.5)
\]

or, finally,

\[
\psi_{l,t+1}^u(u) = \left( 1 - \frac{\tau_{u+1}^{l+1} \lambda_1^{01} \tau_{u+1}^{l+1}}{\tau_{u+1}^{t+1}} \lambda_2^{01} \frac{\tau_{u+1}^{l+1} \tau_{u+1}^{l+1}}{\tau_{u+1}^{t+1}} \right) \psi_{l,t}^u(u). \quad (6.6)
\]

To avoid a confusion, we stress that the operator inside the brackets acts to the variable \( l \) whereas \( u \) enters as a parameter. This operator should be identified with the \( M \)-operator generating the flow \( \bar{l} \) realized as a pseudo-difference operator in \( l \).

\[
\begin{array}{c}
\lambda_1 \\
\bar{l} \\
\lambda_2 \\
 \downarrow \\
\lambda_0 \\
\bar{u} \\
\lambda_3 \\
\end{array}
\]

In other words, letting \( l \) to be the reference flow, we, therefore, have got an \( M \)-operator for the flow \( \bar{l} \) which is not adjacent to \( l \).

In the limit when the points \( \lambda_0, \lambda_1 \) merge, the flow \( \bar{l} \) becomes left adjacent to \( l \). Let us demonstrate how the corresponding difference \( M \)-operator is reproduced from the r.h.s. of eq. (6.6). Let \( \lambda_1 - \lambda_0 = \epsilon, \epsilon \to 0 \). We have

\[
\frac{\lambda_1^{01}}{\epsilon} + \lambda_2^{01} \to 1 + \epsilon \left( \epsilon^2 + \lambda_3^{02} \right) + O(\epsilon^2),
\]

\[
\frac{\tau_{u+1}^{l+1}}{\tau_{u}^{l+1}} \to 1 + O(\epsilon),
\]

\[
\frac{\tau_{u+1}^{l+1} \tau_{u+1}^{l+1}}{\tau_{u+1}^{t+1} \tau_{u}^{l+1} \tau_{u+1}^{l+1}} \to 1 - \epsilon \lambda_3^{02} \left( 1 - \frac{\tau_{u+1}^{l+1} \tau_{u+1}^{l+1}}{\tau_{u+1}^{t+1} \tau_{u+1}^{l+1}} \right) + O(\epsilon^2)
\]

as \( \epsilon \to 0 \) (in the last line eq. (4.22) is used), so the naive limit of the r.h.s. (6.6) is zero. However, we should take into account the change of normalization which is implied when the former flow \( \bar{l} \) turns into a left adjacent flow to \( l \). This is achieved by \( \psi_{l+1}^u \to -\epsilon \psi_{l}^u \), so in the limit the right \( M \)-operator

\[
M_l = e_{\bar{u}} - \lambda_2^{03} \frac{\tau_{u+1}^{l+1} \tau_{u+1}^{l+1}}{\tau_{u+1}^{t+1} \tau_{u+1}^{l+1}}
\]

is reproduced.

For illustrative purposes, let us give continuous analogues of the above formulas. Rather than to perform the limit directly, it is much easier to use the continuous version (5.13) of the linear problems from the very beginning. Making use of eq. (5.14), we find the analogue of eq. (6.2):

\[
\partial_1 \partial_{\bar{t}_1} \varphi(u) - v(u) \partial_{\bar{t}_1} \varphi(u) - c(u) \varphi(u) = 0. \quad (6.7)
\]

The analogues of eqs. (5.3), (5.6) read

\[
\frac{\tau_{u+1}^{t+1}}{\tau_{u+1}^{t+1}} \partial_{\bar{t}_1} \varphi(u) = \varphi(u), \quad (6.8)
\]

\[
\left( \partial_{\bar{t}_1} - \frac{\tau_{u+1}^{t+1} \tau_{u+1}^{t+1}}{\tau_{u+1}^{t+1} \tau_{u+1}^{t+1}} \right) \varphi(u) = 0, \quad (6.9)
\]

respectively.
6.2 Adjoint flows

Finally, we extend the above scheme to incorporate more general flows which we call *adjoint*. Let

\[ A_l^a = 1 + w \Delta_l^{-1} w^* , \quad \Delta_l = e^{\partial_a} - 1 \]  \hspace{1cm} (6.10)

be a pseudo-difference operator, where \( w, w^* \) are as yet arbitrary functions of all the time variables. By \( a \) we denote the time variable corresponding to the adjoint flow that we are going to define. In this section, the reference flow is \( l \). The \( M \)-operator for an elementary discrete flow \( p \) (see the figure) is

\[ M_p^l = e^{\partial_l} - \lambda_p V_p^l \frac{\tau_p^{l+1}}{\tau_p^{l+1} \tau_p^{l+1}} , \quad \lambda_p \equiv \lambda_p^4. \]  \hspace{1cm} (6.11)

**Proposition 6.1** The commutativity condition

\[ [e^{-\partial_a} A_l^a, e^{-\partial_p} M_p^l] = 0 \]  \hspace{1cm} (6.12)

holds only if \( w \) and \( w^* \) satisfy the linear equations

\[
\begin{cases}
(e^{\partial_l} + \lambda_p V_p^l, a) w_p^l = \omega w_p^{l+1} \\
(e^{-\partial_l} + \lambda_p V_p^l, a) w_p^{l+1} = \omega w_p^l ,
\end{cases}
\]  \hspace{1cm} (6.13)

where

\[ V_p^l, a \equiv \frac{\tau_p^{l+1, a}}{\tau_p^{l+1} \tau_p^{l+1}} \]

and \( \omega \) is an arbitrary constant.

The proof is by straightforward computation. Equations (6.13) are necessary conditions for vanishing of the pseudo-difference part of the commutator. Here are main steps of the proof. Eq. (6.12),

\[
\left( e^{\partial_l} - \lambda_p V_p^l, a + 1 \right) \left( 1 + w_p^l \Delta_l^{-1} w_p^l \right) = \left( 1 + w_p^{l+1} \Delta_l^{-1} w_p^{l+1} \right) \left( e^{\partial_l} - \lambda_p V_p^l, a \right),
\]  \hspace{1cm} (6.14)

can be rewritten as

\[
\lambda_p \left( V_p^{l+1, a} - V_p^l, a \right) + w_p^{l+1} w_p^{l-1} - w_p^{l+1} w_p^l
= \left( w_p^l - \lambda_p V_p^l, a - \omega w_p^{l+1} \right) \Delta_l^{-1} w_p^l
- w_p^{l+1} \Delta_l^{-1} \left( w_p^{l+1} - \lambda_p V_p^l, a - \omega w_p^{l+1} \right).\]  \hspace{1cm} (6.15)

Since the l.h.s. does not contain negative powers of \( \Delta_l \), the r.h.s. should be zero. This condition implies eqs. (6.13).

The ALP for the left adjacent \( p \) to the reference flow \( l \) and its adjoint read (cf. (5.3))

\[
\begin{cases}
\psi^p(l + 1) - \lambda_p V_p^p \psi^p(l) = \psi^{p+1}(l) \\
\psi^p(l - 1) - \lambda_p V_p^{l-1} \psi^p(l) = \psi^{p-1}(l),
\end{cases}
\]  \hspace{1cm} (6.15)
Whence we identify

\[ w^p,a = \psi^p,a(l + 1), \quad w^*p,a = \psi^{*p,a}(l + 1), \]

where \( \psi, \psi^* \) are arbitrary solutions to the linear problems (6.15).

The operator (6.10) acquires the form

\[ A^q_i = 1 + \psi^{a+1}(l) \Delta_i^{-1} \psi^{*a}(l + 1). \]  \hspace{1cm} (6.16)

The commutativity condition (6.12) is equivalent to a nonlinear equation for \( \tau^p,a \). The adjoint flow is defined by two arbitrary solutions \( \psi, \psi^* \) to the linear problems (6.15). For continuous hierarchies, pseudo-differential analogues of the operators (6.16) and corresponding adjoint flows were studied in [29].

As an example, let us show that taking \( \psi, \psi^* \) to be the Baker-Akhiezer function \( \psi(l; z) \) and its dual, one reproduces eq. (6.6). According to the prescription of Sect. 5.3, the Baker-Akhiezer function and its dual read

\[ \psi^a(l; z) = \zeta^a(\lambda^0 z) \left. \frac{t_{p,a}}{t_{p,a}^{(l)}} \right|_{p_z=0}, \quad \psi^{*a}(l; z) = \zeta^{-a}(\lambda^0 z) \left. \frac{t_{p,a}^{(l)}}{t_{p,a}(p_z)} \right|_{p_z=0}, \]

where \( \zeta^a \) is a normalization factor specified below. Here \( p_z \) is the time variable corresponding to the flow \( \lambda^0 \rightarrow \lambda^0 \left. \right|_{p_z=0} \) left adjacent to \( l \). Let \( z \rightarrow \lambda_1 \). In the limit \( p_z \) coincides with \( u \). Plugging (6.17) with \( \zeta = \lambda_z^{02} \) into (6.16), we do reproduce the operator in the r.h.s. of eq. (6.6).

7 On the hierarchy of bilinear difference equations

Integrable partial differential equations always can be included in an \textit{infinite hierarchy}. Infinite number of commuting flows generates infinite number of evolution equations.

The hierarchies of discrete integrable equations are less studied. First of all, it is not quite clear what are "higher discrete flows" on the space of pseudo-difference operators. An understanding of this matter is necessary if one is going to extend the Zakharov-Shabat formalism to the higher Hirota equations known in the literature [12], [30].

There are two "complimentary" points of view on this matter. First, one might consider 3-term HBDE (4.20) as a counterpart of the whole infinite hierarchy itself. In this case it should be understood as an infinite set of equations (continuously numbered by labels \( \lambda_\alpha \)'s) for a function of infinitely many variables \( l_{\alpha\beta} \) associated with \( \lambda_\alpha \lambda_\beta \). Second, one might expect that composite discrete flows are good candidates for true analogues of the higher continuous flows. This is justified by analysing the continuum limit. Indeed, to get a higher continuous flow as a limiting case, one should start from a composite discrete flow with specially adjusted labels.

Our goal in this section is to show how these two approaches could be consistent with each other. A natural conjecture is that the \( N \)-term "higher" Hirota equations for a function of \( N \) variables are consequences of the basic 3-term equation (4.20) treated as a \textit{hierarchy}. This means that the 3-term equation is supposed to hold for each triple of the \( N \) variables with the corresponding \( \lambda_\alpha \)'s. To support this conjecture, the case of 4-term HBDE is considered in detail. Besides, an extension of the Zakharov-Shabat scheme to this case is suggested.
7.1 Higher equations of the hierarchy

Higher Hirota difference equations known in the literature [30] are written for a function \( \tau(l_1, \ldots, l_N) \) of \( N \) variables. They have the form

\[
\begin{vmatrix}
1 & z_1 & z_1^2 & \ldots & z_1^{N-2} & \tau_1 \hat{\tau}_1 \\
1 & z_2 & z_2^2 & \ldots & z_2^{N-2} & \tau_2 \hat{\tau}_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & z_N & z_N^2 & \ldots & z_N^{N-2} & \tau_N \hat{\tau}_N \\
\end{vmatrix} = 0,
\]

(7.1)

where \( z_i \) are arbitrary constants and

\[
\tau_i \equiv \tau(l_1, l_2, \ldots, l_{i-1}, l_i + 1, l_{i+1}, \ldots, l_N) ,
\]

\[
\hat{\tau}_i \equiv \tau(l_1 + 1, l_2 + 1, \ldots, l_{i-1} + 1, l_i, l_{i+1} + 1, \ldots, l_N + 1).
\]

(7.2)

In the more compact form they read

\[
\sum_{j=1}^{N} \Lambda_j \tau_j \hat{\tau}_j = 0.
\]

(7.3)

The constants \( \Lambda_j \) are subject to the only condition \( \sum_{j=1}^{N} \Lambda_j = 0 \). For \( N = 3 \) one gets usual Hirota’s bilinear difference equation, where \( z_i \) are certain rational functions of \( \lambda_\alpha \)‘s. Like in the 3-term case, the variables \( l_i \) will be identified with elementary discrete flows.

The transformation

\[
\tau(l_1, \ldots, l_N) \rightarrow \exp \left[ \frac{1}{2N-4} \sum_{k=1}^{N} \log \Lambda_k \left( \sum_{j=1}^{N} l_j \right)^2 \right] \tau(l_1, \ldots, l_N)
\]

(7.4)

sends eq. (7.9) to the canonical form,

\[
\sum_{j=1}^{N} \tau_j \hat{\tau}_j = 0,
\]

(7.5)

which does not contain any free parameters.

An analogue of Hirota’s original form,

\[
\left( \sum_{j=1}^{N} \Lambda_j \exp(D_{x_j}) \right) \tau \cdot \tau = 0,
\]

(7.6)

is obtained from eq. (7.4) by the linear change of variables

\[
x_j = -l_j + \frac{1}{N-2} \sum_{i=1}^{N} l_i
\]

(7.7)

which generalizes (2.13).

7.2 Zero curvature conditions for composite flows

We are going to show that the zero curvature condition written for the composite discrete flows introduced in Sect. 3.1 lead to the "higher" bilinear equations of the form (7.1). The "higher" \( M \)-operators are obtained as products of the elementary ones.

In this section we deal with the following graph of flows:
The reference flow is $u$. Other notation is clear from the picture. For simplicity, we consider left adjacent flows only. All what follows can be easily reformulated in terms of the right adjacent flows.

According to the definition of composite flows (Sect. 3.1), we introduce a "higher" $M$-operator $M_u^{qr}$ generating evolution in the composite flow labeled by the pair of vectors $\lambda_0 \lambda_2$, $\lambda_0 \lambda_3$ as the product of elementary $M$-operators of the form \[ M_u^r = e^{\partial_u} M_u^0 e^{\partial_r} M_u^r, \] or, in more details, \[ M_u^{qr}(q, r) = M_u^0(q, r + 1) M_u^r(q, r). \] (7.8)

Due to the zero curvature condition for elementary flows $M_u^r = M_u^{rq}$. The compatibility of this composite flow with an elementary flow $\lambda_0 \lambda_1$ reads \[ M_u^p(p, q + 1, r + 1) M_u^{qr}(p, q, r) = M_u^{qr}(p + 1, q, r) M_u^p(p, q, r). \] (7.10)

Clearly, this zero curvature condition is an immediate consequence of eq. (4.9) and definition (7.8) provided eq. (4.9) holds for any pair of the flows from the triple $p, q, r$.

A few words about the notation. For the notational simplicity, we put \[ \lambda_{21} = \lambda_r, \quad \lambda_{01} = \lambda_q, \quad \lambda_{41} = \lambda_p. \]

Since in this section we deal with too many variables, it is also convenient to change slightly the notation for the $\tau$-function writing the argument $u$ in brackets: $\tau_u \rightarrow \tau(u)$. Other variables are indices, as before. This will remind us that $u$ is the reference variable. Needless to say that unshifted variables will be skipped when possible. Throughout this section, the notation \[ V_u^{(p)}(p, q, r) = \frac{\tau_{p,q,r}(u) \tau_{p+1,q,r}(u+1)}{\tau_{p+1,q,r}(u) \tau_{p,q,r}(u+1)} \] is used.

Now we are ready to elaborate eq. (7.10) explicitly. Eq. (7.10) reads \[ \left( e^{\partial_u} - \lambda_p V_u^{(p)}(p, q + 1, r + 1) \right) \left( e^{\partial_u} - \lambda_q V_u^{(q)}(p, q, r + 1) \right) \left( e^{\partial_u} - \lambda_r V_u^{(r)}(p, q, r) \right) = \left( e^{\partial_u} - \lambda_q V_u^{(q)}(p + 1, q, r + 1) \right) \left( e^{\partial_u} - \lambda_r V_u^{(r)}(p + 1, q, r) \right) \left( e^{\partial_u} - \lambda_p V_u^{(p)}(p, q, r) \right). \] (7.12)

The coefficients in front of $e^{2\partial_u}$ give \[ \lambda_p V_u^{(p)}(p, q + 1, r + 1) + \lambda_q V_u^{(q)}(p, q, r + 1) + \lambda_r V_u^{(r)}(p, q, r) = \lambda_p V_{u+1}^{(p)}(p, q, r) + \lambda_q V_{u+1}^{(q)}(p + 1, q, r + 1) + \lambda_r V_{u+1}^{(r)}(p + 1, q, r). \] (7.13)

This is a direct corollary of 3-term HBDE. To see this, recall eq. (4.18) from the proof of Theorem 4.1. In the present notation it reads \[ \lambda_p V_u^{(p)}(p, q + 1, r + 1) + \lambda_q V_{u+1}^{(q)}(p, q, r + 1) = \lambda_p V_{u+1}^{(p)}(p, q, r + 1) + \lambda_q V_u^{(q)}(p + 1, q, r + 1). \] (7.14)
Here \( r + 1 \) enters as a parameter. A similar equation can be written down for the pair \( p, r \), where now \( q \) enters as a parameter:
\[
\lambda_p V_{u+1}^{(p)}(p, q, r + 1) + \lambda_r V_{u+2}^{(r)}(p, q, r) = \lambda_p V_{u+2}^{(p)}(p, q, r) + \lambda_r V_{u+1}^{(r)}(p + 1, q, r). \tag{7.15}
\]

Taking their sum, we obtain eq. (7.13):

Comparing coefficients in front of \( e^{\theta_n} \), we get:
\[
\lambda_p \lambda_q V_u^{(p)}(p, q, r + 1) V_u^{(p)}(p, q + 1, r + 1) + \lambda_q \lambda_r V_{u+1}^{(r)}(p, q, r) V_u^{(q)}(p, q, r + 1)
+ \lambda_p \lambda_r V_{u+1}^{(r)}(p, q, r + 1) V_{u+1}^{(r)}(p, q, r)
= \lambda_p \lambda_q V_u^{(p)}(p + 1, q, r + 1) V_{u+1}^{(p)}(p, q, r) + \lambda_q \lambda_r V_{u+1}^{(r)}(p + 1, q, r) V_u^{(q)}(p + 1, q, r + 1)
+ \lambda_p \lambda_r V_{u+1}^{(r)}(p, q + 1, r + 1) V_{u+1}^{(r)}(p, q, r). \tag{7.16}
\]

In a similar way, it is easy to show that this equality also follows from HBDE. Of course this is a trivial result provided the zero curvature conditions (4.9) hold for both elementary \( M \)-operators in the product (7.9).

The fact which is not obvious from the very beginning is that eqs. (7.14), (7.15), (7.16) imply one of the higher Hirota equations (namely, (7.1) at \( N = 4 \)). Proceeding like in the proof of Proposition 4.1, we rewrite (7.16) in the form
\[
\tau^{(u + 1)} \left( \frac{\lambda_q \lambda_r \tau^{p+1,q+1}(u + 2)}{\tau^{p+1,q+1}(u + 1)} - \frac{\lambda_p \lambda_r \tau^{p+1,r+1}(u + 2)}{\tau^{p+1,r+1}(u + 1)} \right)
- \lambda_p \lambda_q \frac{\tau^{p+1,q+1}(u + 1)}{\tau^{p+1,q+1}(u + 1)} \left( \frac{\tau^{p+1}(u + 1)}{\tau^{p+1}(u + 1)} - \frac{\tau^{q+1}(u)}{\tau^{r+1}(u + 1)} \right)
- \lambda_p \lambda_r \frac{\tau^{q+1,r+1}(u + 1)}{\tau^{r+1}(u + 1)} \left( \frac{\tau^{u + 1}}{\tau^{u + 1}} - \frac{\tau^{r+1}(u + 2)}{\tau^{r+1}(u + 1)} \right) \tag{7.17}
\]
(cf. (7.18)). Multiplying both sides by \((\lambda_q - \lambda_r) \tau^{p+1,r+1}(u + 1)\tau^{r+1}(u + 1)\) and using HBDE in the form
\[(\lambda_q - \lambda_r) \tau^{p+1,r+1}(u) \tau^{q+1}(u + 1) = \lambda_q \tau^{r+1}(u) \tau^{q+1}(u + 1) - \lambda_r \tau^{q+1}(u) \tau^{r+1}(u + 1), \tag{7.18}
\]
we get
\[
\frac{\tau^{(u + 1)} \tau^{p+1,q+1,r+1}(u)}{\tau^{(u + 2)} \tau^{p+1,q+1,r+1}(u + 1)} A^{p,q,r}(u) = B^{p,q,r}(u), \tag{7.19}
\]
where
\[
A^{p,q,r}(u) = (\lambda_q - \lambda_r) \lambda_q \lambda_r \tau^{p+1,r+1}(u + 2) \tau^{p+1}(u + 1)
- \lambda_p \lambda_q \frac{\tau^{p+1,q+1}(u + 2)}{\tau^{p+1,q+1}(u)} (\lambda_q \tau^{p+1,r+1}(u) \tau^{p+1,q+1}(u + 1))
+ \lambda_p \lambda_r \frac{\tau^{p+1,r+1}(u + 2)}{\tau^{p+1,q+1}(u)} (\lambda_r \tau^{p+1,q+1}(u) \tau^{p+1,r+1}(u + 1)),
\]
\[
B^{p,q,r}(u) = (\lambda_q - \lambda_r) \lambda_q \lambda_r \tau^{p+1}(u) \tau^{r+1}(u + 1)
- \lambda_p \lambda_q \frac{\tau^{r+1}(u + 1)}{\tau^{r+1}(u + 2)} (\lambda_q \tau^{p+1}(u + 1) \tau^{q+1}(u + 2))
+ \lambda_p \lambda_r \frac{\tau^{q+1}(u)}{\tau^{q+1}(u + 2)} (\lambda_r \tau^{p+1}(u + 1) \tau^{r+1}(u + 2)).
\]
The last two terms can be transformed further using eq. (7.18). The result is

\[ A^{p,q,r}(u) = h^{p,q,r}(u + 1) - \lambda_p^2(\lambda_q - \lambda_r) \frac{\tau_{q+1,r+1}(u)\tau_{p+1,q,r+1}(u + 1)}{\tau_{p+1,q+1,r+1}(u)}, \]

\[ B^{p,q,r}(u) = h^{p,q,r}(u) - \lambda_p^2(\lambda_q - \lambda_r) \frac{\tau_{q+1,r+1}(u)\tau_{u+1}\tau_{p+1}(u + 2)}{\tau(u + 2)}, \]

where

\[ h^{p,q,r}(u) = \lambda_q\lambda_r(\lambda_q - \lambda_r)\tau_{p+1}(u)\tau_{q+1,r+1}(u + 1) + \lambda_r\lambda_p(\lambda_r - \lambda_p)\tau_{q+1}(u)\tau_{p+1,r+1}(u + 1) + \lambda_p\lambda_q(\lambda_p - \lambda_q)\tau_{r+1}(u)\tau_{p+1,q+1}(u + 1). \] (7.20)

Finally, eq. (7.19) turns into

\[ \frac{\tau(u+1)\tau_{p+1,q+1,r+1}(u)}{\tau(u+2)\tau_{p+1,q+1,r+1}(u+1)} = \frac{h^{p,q,r}(u)}{h^{p,q,r}(u + 1)}, \] (7.21)

which leads to

\[ h^{p,q,r}(u) + (\lambda_q - \lambda_q)(\lambda_q - \lambda_r)(\lambda_r - \lambda_p)\tau(u + 1)\tau_{p+1,q+1,r+1}(u) = 0. \] (7.22)

Eq. (7.22) has the form (7.1) at \( N = 4 \). This completes the calculation.

\( M \)-operators for arbitrary composite flows can be defined as a straightforward generalization of (7.8):

\[ M_{p_1\cdots p_N}^u = \exp \left( \sum_{j=1}^{N} \partial_{p_j} \right) \prod_{i=1}^{N} (e^{-\partial_{p_i}} M_{p_i}^u). \] (7.23)

Note that the order of the operators in the product is not essential since the operators \( e^{-\partial_{p_i}} M_{p_i}^u \) commute due to the zero curvature condition (4.9). For simplicity it is assumed that all the flows \( p_i \) are left adjacent to the reference flow \( u \). Operators (7.23) generate discrete analogues of higher flows of the KP hierarchy (see the corresponding graph of flows in Sect. 3.2). There is now straightforward to write down similar operators for right adjacent flows which would generate higher flows of the discrete 2DTL hierarchy.

**Conjecture.** All the higher HBDE (7.1) follow from the compatibility of the composite flows generated by the \( M \)-operators (7.23) and elementary discrete flows.

The calculation given above shows that the conjecture is true for the 4-term bilinear equation. Unfortunately, we are not aware of any other proof than this sophisticated calculation that is hard to perform in the general case. The conjecture claims that all the higher bilinear equations are corollaries of eq. (4.20) considered as a hierarchy, i.e., valid for all triples of adjacent flows.

### 8 Reductions of Hirota’s equations

The hierarchy of discrete Hirota’s equations admits several important reductions. **Reduction** means imposing a constraint compatible with the hierarchy, so that the number of independent variables gets reduced. In this way one is able to construct discrete analogues of KdV, SG and other interesting equations.

The simplest way to impose a constraint is to require the \( \tau \)-function to be **stationary** with respect to a particular flow \( s \) (possibly, up to a "gauge" transformation (2.2)). Non-trivial examples emerge when the stationary flow is composite. As for the commutation representation, there are two options.

First, the stationary flow can be reference. Then \( M \)-operators become free of differentiation because the symbol \( \partial_s \) commutes with all the coefficients. In other words, \( \partial_s \) can be considered as a \( c \)-number and identified with a **spectral parameter**. This is a natural origin of \( M \)-operators depending on (rational) spectral parameter.
Alternatively, one may take as the reference flow any flow \( y \) other than \( s \). Then any \( M \)-operator \( M^{(f)} \) generating a flow \( f \) contains operators \( \partial_y \). Since coefficients of the operator \( M^{(f)} \) do not depend on \( s \), the compatibility condition for the flows \( s \) and \( f \) acquires the Lax-type form

\[
M^{(s)}(f + 1)M^{(f)} = M^{(f)}M^{(s)}(f),
\]

where \( M^{(f)} \) plays the role of Lax operator. Unlike the Zakharov-Shabat scheme, where each zero curvature condition involves two different time flows (apart from the reference flow), Lax equations are written for each flow separately. The Lax equation represents the time flow as a similarity (i.e., isospectral) transformation of the Lax operator. It is natural to call \( M^{(s)} \) the L-operator of the reduced hierarchy. This is a natural origin of \( L \)-operators which are difference (or differential) rather than pseudo-difference (or pseudo-differential) operators. In the light of this general scheme, we pass to considering examples.

8.1 KdV-like reductions

0). Discrete d’Alembert equation (a trivial example). Let \( u \) be an elementary discrete flow dealt with in Sects. 3, 4. The stationarity condition with respect to this flow, \( \tau(u + 1) = \tau(u) \), immediately leads (see (4.20)) to the relation

\[
\tau_{l+1,m}^{l,m} = \tau_{l,m}^{l+1,m+1},
\]

(8.1)

where \( l, m \) are any other two elementary flows. This is the discrete 2D d’Alembert equation written in ”light cone” coordinates. The general solution is \( \tau_{l,m} = \chi_+(l)\chi_-(m) \) with arbitrary \( \chi_\pm \); this is just the allowed ”gauge” freedom (2.8) of the \( \tau \)-function. So we see that this reduction is too strong – it respects trivial solutions only. In this sense it is incompatible with the hierarchy. To get nontrivial examples, one should either impose stationarity conditions with respect to the composite (”higher”) flows or impose periodic conditions in \( u \) with period \( N > 1 \) (e.g. \( \tau(u + 2) = \tau(u) \)).

1). Discrete KdV equation. Consider the graph of flows

and put \( \lambda_2^{01} = \lambda_q, \lambda_3^{01} = \lambda_p \) for brevity. In this notation, eq. (4.20) takes the form

\[
\lambda_q \tau^{p+1,q}_{u+1} - \lambda_p \tau^{p,q+1}_{u+1} + (\lambda_p - \lambda_q)\tau^{p+1,q+1}_{u+1} = 0.
\]

(8.2)

To get the discrete KdV reduction, we impose the constraint

\[
\tau^{p+1,q+1}_{u+1} = \tau^{p,q}_{u+1},
\]

(8.3)

i.e., the \( \tau \)-function is stationary with respect to the composite flow labeled by the pair of vectors \( \lambda_0\lambda_2, \lambda_0\lambda_3 \). This condition converts the 3-dimensional equation (8.2) into the following 2-dimensional one:

\[
\lambda_q \tau^{p+1,q}_{u+1} - \lambda_p \tau^{p,q+1}_{u+1} + (\lambda_p - \lambda_q)\tau^{p,q}_{u+1} = 0.
\]

(8.4)

This is the discrete KdV equation in the bilinear form \([2],[3]\). The discrete KdV equation is also known in the form \([4]\)

\[
V(u,p) - V(u-1,p-1) = \kappa(V^{-1}(u,p) - V^{-1}(u-1,p)).
\]

(8.5)

\footnote{The Lax operator for the hierarchy without any reduction is in general an infinite series in negative powers of first order difference operators. The theory based on the Lax representation with Lax operators of this kind is not considered here.}
The equivalence to eq. (8.4) follows from the identification

\[ V(u, p) = \frac{\tau_p u^{p+1}}{\tau_u^{p+1} u^{p+1}}, \quad \kappa = \frac{\lambda_q}{\lambda_p}. \]

Let us turn to \( M \) and \( L \) operators. Following the history of the KdV equation, we begin with difference operators with scalar coefficients. Let \( u \) be the reference variable. Then the (composite) stationary flow is generated by the \( M \)-operator of the type (7.4):

\[ M_u = M_u(p, q + 1)M_u(p, q) = L^{(KdV)} = e^{2\partial_x} - \left( \lambda_p \frac{\tau_u^{p-1} u^{p+1}}{\tau_u^{p+1} u^{p+1}} + \lambda_q \frac{\tau_u^{p-1} u^{p+1}}{\tau_u^{p+1} u^{p+1}} \right) e^{\partial_x} + \lambda_p \lambda_q. \quad (8.6) \]

We call this 2-nd order difference operator the Lax operator of the discrete KdV equation. The spectral problem \( L^{(KdV)} \psi = E \psi \) is a discrete analogue of the stationary Schrödinger equation that is an auxiliary linear problem for KdV. The \( p \)-evolution generated by the \( M \)-operator of the type (4.13) is isospectral: \( L^{(KdV)}(p + 1) = M_u L^{(KdV)}(p) (M_u)^{-1} \).

If the reference flow coincides with the stationary one, we get \( 2 \times 2 \) matrix \( M \)-operators depending on a spectral parameter \( z \). The spectral parameter is an eigenvalue of the shift operator along the stationary flow acting to the \( \psi \)-function: \( \exp(\partial_x + \partial_y) \psi = z^2 \psi \). Consider the vector

\[ \begin{pmatrix} \psi^u_p \\ \chi^u_p \end{pmatrix} = \begin{pmatrix} \psi^{(u)} \\ \chi^{p} \end{pmatrix}. \]

Repeating the argument of Sect. 5.5, we obtain the following linear problems for shifts in \( u \) and \( p \) respectively:

\[ \begin{pmatrix} \psi^u_{p+1} \\ \chi^u_{p+1} \end{pmatrix} = \begin{pmatrix} \lambda_p \frac{\tau_p \tau_u^{p+1} u^{p+1}}{\tau_u^{p+1} u^{p+1}} & \frac{\tau_p}{\tau_u^{p+1}} \\ z^2 \frac{\tau_p}{\tau_u^{p+1}} & \lambda_q \end{pmatrix} \begin{pmatrix} \psi^u_p \\ \chi^u_p \end{pmatrix}, \quad (8.7) \]

\[ \begin{pmatrix} \psi^p_{u+1} \\ \chi^p_{u+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\tau_u}{\tau_u^{p+1}} \\ z^2 \frac{\tau_u}{\tau_u^{p+1}} & \lambda_q - \lambda_p \end{pmatrix} \begin{pmatrix} \psi^p_u \\ \chi^p_u \end{pmatrix}. \quad (8.8) \]

Compatibility of these linear problems yields the discrete matrix Zakharov-Shabat equations with the spectral parameter \( z \).

Equations for the first component \( \psi^u_p \) read:

\[ \psi^p_{u+1} + \frac{1}{\lambda_p - \lambda_q} \left( \lambda_q \frac{\tau_p \tau_u^{p+1} u^{p+1}}{\tau_u^{p+1} u^{p+1}} - \lambda_p \frac{\tau_p \tau_u^{p+1} u^{p+1}}{\tau_u^{p+1} u^{p+1}} \right) \psi^p_u = (z^2 - \lambda_p \lambda_q) \psi^p_{u-1}, \quad (8.9) \]

\[ \psi^p_{u+1} + (\lambda_p - \lambda_q) \left( \frac{\tau_u^2}{\tau_u^{p+1}} \right) \psi^p_u = z^2 \psi^p_{u-1}. \quad (8.10) \]

Eq. (8.9) coincides with the spectral problem for the Lax operator (8.6) provided the \( \tau \)-function obeys the bilinear relation (8.4).

2). Discrete Boussinesq equation [73]. In this example the \( \tau \)-function \( \tau^{p,q} \) satisfies \( \tau^{p+1,q+1} = \tau^{p,q} \). Eq. (8.2) reduces to

\[ \lambda_q \tau^{p+1,q,p+1} - \lambda_p \tau^{p,q+1} \tau^{p,q-1} + (\lambda_p - \lambda_q) \tau^{p+1,q+1} \tau^{p-1,q-1} = 0, \quad (8.11) \]

where \( u \) enters implicitly as a parameter. Different kinds of Lax and Zakharov-Shabat representations for this equation can be written down straightforwardly.

In a similar way, it is possible to define \( A_n \)-type reductions. In this case the \( \tau \)-function obeys the condition \( \exp(\sum_{\alpha=1}^{n} \partial_{\alpha}^u) \tau = \tau \). This becomes an actual reduction for higher Hirota equations with the number of variables greater than \( n \).
8.2 Discrete time 1D Toda chain and its relatives

This group of examples includes the discrete time 1D Toda chain (1DTC), discrete AKNS system (in particular, discrete nonlinear Schrödinger equation), discrete time relativistic Toda chain and discrete Heisenberg ferromagnet (HF). These models differ by choosing dependent and independent variables while the type of reduction is essentially the same.

Let the graph of flows and the notation be the same as in Sect. 4.3. Now the τ-function is required to be stationary with respect to the composite flow labeled by the pair of vectors $\lambda_0\lambda_1$, $\lambda_1\lambda_2$, i.e.,

$$\tau_{n+1,l} = \tau_{n,l}.$$  \hspace{1cm} (8.12)

The stationary flow is generated by the composite $M$-operator $e^{\partial_l} M_n^l e^{-\partial_l} M_n^l$ with the reference flow $n$. This operator should be identified with the Lax operator of the discrete time 1DTC:

$$L^{(TC)} = e^{\partial_l} - \left( \nu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} + \mu \frac{\tau_{n-1,l}^{l+1}}{\tau_{n,l}^{l+1}} \right) + \nu \mu \frac{\tau_{n-1,l}^{l+1}}{(\tau_{n,l}^{l+1})^2} e^{-\partial_l}. \hspace{1cm} (8.13)$$

This is a 2-d order difference operator in $n$ with scalar coefficients. The spectral problem $L^{(TC)} \psi = E \psi$ is a 1D discrete stationary Schrödinger equation (cf. (5.43)). The $l$-dynamics does not change the spectrum of $L^{(TC)}$.

Replacing $e^{\partial_l} = e^{\partial_l + \partial_l} \to z^2$, we get from (2.23) an $L$-$M$ pair for the discrete time 1DTC realized by $2 \times 2$ matrices depending on the spectral parameter $z$:

$$L_n^{(TC)}(z) = \begin{pmatrix} z^2 + \nu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} + \mu \frac{\tau_{n-1,l}^{l+1}}{\tau_{n,l}^{l+1}} & -\mu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} \\ \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} & 0 \end{pmatrix}, \hspace{1cm} (8.14)$$

$$M_n^{(TC)}(z) = \begin{pmatrix} 1 & -\mu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} \\ \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} & z^2 - \mu \frac{\tau_{n-1,l}^{l+1}}{\tau_{n,l}^{l+1}} \end{pmatrix}. \hspace{1cm} (8.15)$$

Correspondingly, the 2D discrete Schrödinger equation (5.44) becomes a 1D spectral problem in the variable $l$,

$$\psi_{n+1}^{l+1} + (\nu - \mu) \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} \psi_{n+1}^{l} = z^2 \left( \psi_{n}^{l} + \nu \frac{\tau_{n-1,l}^{l+1}}{\tau_{n,l}^{l+1}} \psi_{n+1}^{l} \right), \hspace{1cm} (8.16)$$

with the non-standard dependence on the spectral parameter. Another useful form of this equation is

$$\bar{\psi}_{n+1}^{l-1} - \nu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} \bar{\psi}_{n}^{l-1} = \left( z + (\nu - \mu) z^{-1} \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} \right) \bar{\psi}_{n+1}^{l-1}, \hspace{1cm} (8.17)$$

where $\bar{\psi}_{n}^{l} = z^{-l} \psi_{n}^{l}$.

The equation of motion in the bilinear form is the 2-dimensional reduction of eq. (2.11):

$$\nu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} - \mu \frac{\tau_{n+1,l}^{l+1}}{\tau_{n,l}^{l+1}} = (\nu - \mu)(\tau_{n+1,l}^{l})^2, \hspace{1cm} (8.18)$$

which after the linear change $l \to m = l + n$, $\tau_{n}^{l} \to \tau_{n}(m)$ coincides with Hirota’s original form of this equation [3]:

$$(1 + g^{-2})\tau_{n}(m + 1)\tau_{n}(m - 1) - \tau_{n-1}(m)\tau_{n-1}(m) = g^{-2}(\tau_{n}(m))^2, \hspace{1cm} (8.19)$$

where

$$g^{-2} = \frac{\nu}{\mu} - 1.$$
In terms of the new dependent variable

$$\phi_n(m) = \log \frac{\tau_{n+1}(m)}{\tau_n(m)}$$

(8.20)
eq(8.19) acquires the form

$$\exp \left( \phi_n(m + 1) + \phi_n(m - 1) - 2\phi_n(m) \right) = \frac{1 + g^2 \exp(\phi_{n+1}(m) - \phi_n(m))}{1 + g^2 \exp(\phi_n(m) - \phi_{n-1}(m))}$$

(8.21)

that is the discrete time 1DTC equation studied by Yu.Suris [31].

The continuum limit in $m$ is straightforward: set $m \to m/\epsilon$, $g^2 = -\epsilon^2$, then $\phi_n(m \pm 1) \to \phi_n \pm \epsilon \phi'_n + \frac{1}{2} \epsilon^2 \phi''_n$ as $\epsilon \to 0$. Expanding eq. (8.21) in $\epsilon$, we get the familiar 1DTC equation:

$$\phi_n'' = e^{\phi_n - \phi_{n-1}} - e^{\phi_{n+1} - \phi_n}.$$  

(8.22)

It is interesting to note that eq. (8.21) has another continuum limit yielding the sine-Gordon equation. To see this, let us redefine the field $\phi$ before taking the limit:

$$\phi_n(m) = i(-1)^{n-m} \varphi_n(m),$$

(8.23)

so the equation reads

$$\exp \left( i\varphi_{n+1}(m) + i\varphi_{n-1}(m) - i\varphi_n(m + 1) - i\varphi_n(m - 1) \right) = \frac{1 + g^{-2} \exp(i\varphi_{n+1}(m) + i\varphi_n(m))}{1 + g^{-2} \exp(-i\varphi_{n-1}(m) - i\varphi_n(m))}.$$  

(8.24)

It is the field $\varphi_n(m)$ that is now assumed to have a smooth continuum limit in $n, m$. Setting $m \to m/\epsilon$, $n \to n/\epsilon$, $g^{-2} = -i\epsilon^2$, $\epsilon \to 0$ and expanding in $\epsilon$ as before, we get the SG equation

$$\left( \partial_n^2 - \partial_m^2 \right) \varphi = 2 \sin(2\varphi).$$

(8.25)

(The same limit for the field $\phi$ in eq. (8.21) would give the d’Alembert equation $(\partial_n^2 - \partial_m^2) \varphi = 0$.) A discrete analogue of the SG equation of a different kind is given below in Sect. 8.3.

**Remark 8.1** The discrete KdV reduction discussed in Sect. 8.1 is formally a particular case of the present reduction when $\lambda_0$ and $\lambda_1$ merge. However, it is more convenient to consider them separately. Note also that there exists a continuum limit leading to the KdV equation from eq. (8.18), too. In this sense the discrete time 1DTC is sometimes considered as a discretization of the KdV equation.

The discrete version of the decoupled nonlinear Schrödinger equation (called also the AKNS system) is essentially the same reduction, i.e., the stationary flow is the same. The difference is in picking out other independent flows to be involved in the equations. Specifically, instead of the $n$-flow one may consider any other elementary discrete flow $p$ left adjacent to $l$.

We would like to show how to derive the discrete AKNS system directly from the bilinear Hirota equations. The basic bilinear equations in question are

$$\lambda_p \tau_n^{p+1,1} \tau_{n+p+1,1} + (\mu - \lambda_p) \tau_n^{p,1} \tau_{n+p+1,1} = \mu \tau_{n+1}^{p,1} \tau_{n+p+1,1},$$

(8.26)

$$\nu \tau_n^{p+1,1} \tau_{n+p+1,1} - \lambda_p \tau_n^{p,1} \tau_{n+p+1,1} = \nu - \lambda_p \tau_{n+p+1,1}.$$  

(8.27)

The first one is eq. (4.22) written for the triple $(p, n, \bar{l})$. The second one is obtained from eq. (4.21) written for the triple $(l, p, n)$ by taking into account the invariance $\tau^{l+1,1} = \tau^{l,1}$. It is easy to see that in terms of the quantities

$$Q^{p,1} = \frac{\tau_{n+p,1}}{\tau_{n,1}}, \quad R^{p,1} = \frac{\tau_{n-1,1}}{\tau_{n+p-1,1}}$$

(8.28)

$(n$ is fixed) these equations can be rewritten as follows:

$$(\nu - \lambda_p)(\lambda_p - \mu)Q^{p,1} + (\lambda_p - \mu)Q^{p,l+1} + (\nu - \lambda_p)Q^{p+l,1} = \lambda_p - \mu Q^{p+1,l+1}.$$  

(8.29)

$$(\nu - \lambda_p)(\lambda_p - \mu)R^{p+1,l} + (\lambda_p - \mu)R^{0,l+1} + (\nu - \lambda_p)R^{p+1,l+1} = \lambda_p - \mu R^{p+1,l+1}.$$  

(8.30)
This system is equivalent to the discrete AKNS system given in ref. [33].

There exists another choice of dependent variables which converts eqs. (8.18), (8.26), (8.27) into a discrete analogue of the relativistic Toda chain (RTC) [32]. Passing again to the variable \( m = l + n \), like in eq. (8.24), we put

\[
x_m(p) = -\log \frac{\tau_n^p(m + 1)}{\tau_n^p(m)}.
\]

(8.31)

The claim is that the equation of motion for \( x_m(p) \) has the form

\[
\frac{(1-\alpha \exp(x_{m+1}(p)-x_m(p)))(1-\beta \exp(x_m(p)-x_{m-1}(p-1)))(1-\gamma \exp(x_m(p)-x_{m}(p+1))}{(1-\alpha \exp(x_m(p)-x_{m-1}(p)))(1-\beta \exp(x_{m+1}(p+1)-x_m(p)))(1-\gamma \exp(x_{m}(p-1)-x_{m}(p)))} = 1,
\]

(8.32)

where

\[
\alpha = \frac{\nu}{\mu - \mu}, \quad \beta = \frac{\nu (\mu - \lambda_p)}{\mu \lambda_p (\mu - \nu)}, \quad \gamma = \frac{\lambda_p}{\lambda_p - \mu},
\]

(8.33)

which is the discrete time RTC equation suggested in ref. [33].

Let us outline the way to derive eq. (8.32). The basic bilinear relations (8.18), (8.26), (8.27) read

\[
(\nu - \mu)(\tau_n^p(m))^2 - \nu\tau_n^p(m + 1)\tau_n^p(m - 1) = -\mu \tau^p_{n+1}(m)\tau^p_{n-1}(m),
\]

(8.34)

\[
(\mu - \lambda_p)\tau_n^p(m + 1)\tau_n^{p+1}(m) + \lambda_p \tau_n^p(m + 1)\tau_n^p(m) = \mu \tau^p_{n+1}(m + 1)\tau^p_{n-1}(m),
\]

(8.35)

\[
\nu \tau^p_{n+1}(m + 1)\tau_n^{p+1}(m - 1) - \lambda_p \tau^p_{n+1}(m)\tau_n^{p+1}(m) = (\nu - \lambda_p)\tau^p_{n+1}(m)\tau_n^{p+1}(m),
\]

(8.36)

respectively. It is straightforward to show that the following two bilinear relations are direct corollaries of the basic ones:

\[
(\nu - \lambda_p)\tau_n^{p+1}(m + 1) - (\mu - \lambda_p)\tau_n^{p+1}(m + 1)\tau_n^p(m + 1) = (\nu - \mu)\tau_n^{p+1}(m + 1)\tau_n^p(m + 1),
\]

(8.37)

\[
\lambda_p(\nu - \mu)\tau_n^p(m)\tau_n^{p+1}(m + 1) + (\mu - \lambda_p)\tau_n^p(m + 1)\tau_n^{p+1}(m - 1) = \mu(\nu - \lambda_p)\tau_n^{p+1}(m + 1)\tau_n^{p+1}(m).
\]

(8.38)

Eq. (8.37) is obtained by eliminating \( \tau_n^p(m) \) from (8.34), (8.36) (i.e. by dividing them by \( \tau_n^{p+1}(m + 1), \tau_n^{p+1}(m) \) respectively and taking their sum) and making use of eq. (8.34). Eq. (8.38) is obtained in a similar way by eliminating \( \tau_n^{p+1}(m) \) from (8.35), (8.36) and making use of eq. (8.37). Now eq. (8.32) easily follows from (8.34), (8.35) and (8.38).

**Remark 8.2** The bilinearization of the usual (continuous time) RTC was suggested in ref. [34]. The equivalence of the RTC and the "semi-discretized" AKNS system (with discrete "space" and continuous time variables) was recently proved in ref. [33].

We conclude this subsection with a few words about the discrete Heisenberg ferromagnet [12]. This equation fits the scheme in the following way. The reduction is the same. In addition to the flow \( p \) from the previous example, one should introduce yet another elementary discrete flow \( q \) left adjacent to \( l \). The \( \tau \)-function now depends on four variables: \( \tau = \tau_n^q(p, q) \). Fix \( n, l \) and consider the following four functions of \( p, q \): \( \tau_n^{l+1}(p, q), \tau_n^{l-1}(p, q), \tau_n^{l+1}(p, q), \tau_n^{l-1}(p, q) \). It can be shown (using, in particular, one of the higher Hirota equations in 4 variables) that certain combinations of these functions satisfy a system of nonlinear difference equations in the variables \( p, q \). This system is equivalent to the discrete HF model discussed in detail in ref. [12], where it was treated in a slightly different way as a part of the reduced 2-component 2DTL hierarchy. As in the case of the discrete AKNS system, the aforementioned embedding into the 1-component 2DTL hierarchy leads to equivalent equations of motion. We omit the details.

### 8.3 Periodic reductions

Periodic reductions of the continuous 2DTL hierarchy give rise to a number of very important equations. For example, the 2-periodic reduction \( \tau_{n+2} = \tau_n \) contains the sine-Gordon (SG) equation. The same periodic constraint can be imposed in the discretized set-up, thus providing us with a discrete analogue of the SG equation.

\[\text{Footnote}\quad \text{The idea of this derivation is due to S.Kharchev (unpublished).}\]
We remark that periodic reductions can be treated on equal footing with stationary reductions. Indeed, the flow \( p \to p + 2 \) is, formally, a degenerate case of a composite flow when the corresponding labels pairwise merge on the complex plane. The periodicity \( \tau^{p+2} = \tau^p \) means stationarity with respect to this degenerate "composite" flow. However, this point of view does not seem to be useful in practice. Usually it is more convenient to treat periodic reductions separately.

Let us consider the 2DTL-like form of HBDE (2.11) with the constraint

\[
\tau_n^{l+2} = \tau_n^l. \tag{8.39}
\]

It becomes the following system of two-dimensional equations:

\[
\begin{align*}
\nu \tau_0^{l,l+1,l+1,l+1} - (\nu - \mu) \tau_0^{l,l,l+1,l+1} &= \mu \tau_1^{l,l+1,l+1,l}, \\
\nu \tau_1^{l,l+1,l+1,l} - (\nu - \mu) \tau_1^{l,l,l+1,l+1} &= \mu \tau_0^{l,l+1,l+1,l}. \tag{8.40}
\end{align*}
\]

The SG field \( \Phi^{l\bar{l}} \) on the square lattice \((l,\bar{l})\) is given by

\[
\Phi^{l\bar{l}} = \frac{1}{2} \log \frac{\tau_0^{l\bar{l}}}{\tau_1^{l\bar{l}}}. \tag{8.41}
\]

Rearranging eqs. (8.40), one gets a closed equation for \( \Phi^{l\bar{l}} \),

\[

\nu \sinh(\Phi^{l\bar{l}} + \Phi^{l+1\bar{l}+1} - \Phi^{l\bar{l}+1} - \Phi^{l+1\bar{l}}) = \mu \sinh(\Phi^{l\bar{l}} + \Phi^{l+1\bar{l}+1} + \Phi^{l\bar{l}+1} + \Phi^{l+1\bar{l}}), \tag{8.42}
\]

which is known as discrete SG equation \([4]\) written in the light cone coordinates.

Let us mention another useful form of the discrete SG equation \([36, 37]\). Set

\[
S^{l\bar{l}} = \exp\left(-2\Phi^{l+1\bar{l}+1} - 2\Phi^{l\bar{l}+1}\right) = \frac{\tau_1^{l,l+1,l+1}}{\tau_0^{l,l,l+1,l+1}}, \tag{8.43}
\]

\[
\tilde{S}^{l\bar{l}} = \exp\left(-2\Phi^{l\bar{l}} - 2\Phi^{l+1\bar{l}+1}\right) = \frac{\tau_1^{l,l+1,l+1}}{\tau_0^{l,l,l+1,l+1}}, \tag{8.44}
\]

then, evidently,

\[
\tilde{S}^{l\bar{l}+1} \tilde{S}^{l\bar{l}+1} = S^{l\bar{l}} S^{l\bar{l}+1\bar{l}+1}. \tag{8.45}
\]

On the other hand, the discrete SG equation implies that

\[
\tilde{S}^{l\bar{l}} = \frac{\mu - \nu S^{l\bar{l}}}{\mu S^{l\bar{l}} - \nu}, \tag{8.45}
\]

so eq. (8.42) converts into

\[
S^{l\bar{l}} S^{l+1\bar{l}+1} = \frac{(\mu - \nu S^{l\bar{l}})(\mu - \nu S^{l+1\bar{l}})}{(\mu S^{l\bar{l}+1\bar{l}+1} - \nu)(\mu S^{l+1\bar{l}+1\bar{l}+1} - \nu)} \tag{8.46}
\]

(cf. \([23]\)).

We now turn to the zero curvature representation. Let \( l \) be the reference flow. The shift \( n \to n + 2 \) is generated by the scalar \( L \)-operator

\[
L^{(SG)} = e^{i \theta} + \nu \left( \frac{\tau_1^{l,l+1\bar{l}}}{\tau_0^{l,l+1\bar{l}}} + \frac{\tau_1^{l,l+1\bar{l}+2}}{\tau_0^{l,l+1\bar{l}+2}} \right) e^{i \theta} + \nu^2, \tag{8.47}
\]

However, this representation is not convenient for describing evolution in \( \bar{l} \).

The matrix \( L-M \) pair with spectral parameter is more appropriate here. To derive it, one should take the "stationary" flow \( 2n \) to be the reference one and repeat the arguments given in Sect. 5.5 with necessary modifications. The operator \( e^{i \theta} \) should be substituted by a spectral parameter \( z^2 \). Omitting the details, we present the result.
The discrete Liouville equation (DLE) reads

\[ -\nu^{-1} \begin{pmatrix} \psi^{l+1,\bar{l}} \\ \chi^{l+1,\bar{l}} \end{pmatrix} = \begin{pmatrix} -\frac{z\tau_0^{l,\bar{l}}}{\nu\tau_1^{l+1,\bar{l}}} & z \frac{\tau_0^{l,\bar{l}}}{\nu\tau_1^{l+1,\bar{l}}} \\ z \frac{\tau_0^{l+1,\bar{l}}}{\nu\tau_1^{l,\bar{l}}} & z \frac{\tau_0^{l+1,\bar{l}}}{\nu\tau_1^{l,\bar{l}}} \end{pmatrix} \begin{pmatrix} \psi^{l,\bar{l}} \\ \chi^{l,\bar{l}} \end{pmatrix}, \]  

(8.48)

which is akin to (8.7). Denoting the matrices in the r.h.s. of eqs. (8.48), (8.49) by \( M^{(+)}, M^{(-)} \) respectively, we can write the compatibility condition

\[ M^{(+)}(l,\bar{l}+1)M^{(-)}(l,\bar{l}) = M^{(-)}(l+1,\bar{l})M^{(+)}(l,\bar{l}) \]  

(8.50)

which yields the discrete SG equation.

\( N \)-periodic reductions (\( \tau_{n+N} = \tau_n \)) can be treated in a similar way. They correspond to \( N \)-periodic Toda lattices in discrete time. It is also possible to impose periodicity with respect to any one of the composite flows. In the remaining part of this subsection we briefly comment on an important class of such reductions which are discrete analogues of Intermediate Long Wave (ILW) equations.

The universal form of reductions from the 2DTL to the family of continuous ILW equations is most transparently written in terms of the \( \tau \)-function of the 2DTL hierarchy. The reduction to the ILW\(_k\) equation reads \[ \tau_{n+k}(t_1, t_2, \ldots; \bar{t}_1, \bar{t}_2, \ldots) = \tau_n(t_1, t_2, \ldots; \bar{t}_1, \bar{t}_2, \ldots), \]  

(8.51)

where \( h \) is a fixed parameter. This parameter interpolates between the \( k \)-periodic reduction \( (h = 0) \) and the Benjamin-Ono equation \( (h \to \infty) \). In words, the \( \tau \)-function should not depend on a particular combination of \( n \) and \( t_1 \). This suggests a discretization of the ILW\(_k\) equation. According to our general rules of discretization, one should substitute \( t_1 \) by an elementary discrete flow \( p \). Then it is natural to substitute eq. (8.51) by

\[ \tau_{n+k}^p = \tau_n^p, \]  

(8.52)

where \( l \) and \( k \) are integer parameters. The particular cases are the discrete KdV equation \( (l = k = 1) \) and the \( k \)-periodic reduction \( (l = 0) \). In the continuum limit we get the continuous ILW\(_k\) equation.

### 8.4 Discrete Liouville equation

The discrete Liouville equation (DLE) and its \( A_n \)-generalizations (discrete time 2DTL with open boundaries) form a very important special class of discrete integrable systems which in general does not fit the reduction scheme discussed in this section. We include it here for the only reason that the DLE is, formally, a degenerate case of the discrete SG equation. The relationship between these two integrable systems deserves further study.

The DLE can be obtained from the discrete SG equation as a result of a certain scaling limit. Let us rescale \( S^{l,\bar{l}} \to \mu S^{l,\bar{l}} \) in eq. (8.40). Clearly, this rescaling means a constant shift of the field: \( \Phi^{l,\bar{l}} \to \Phi^{l,\bar{l}} - \frac{1}{4} \log \mu \). Then, taking the limit \( \mu \to 0 \) in eq. (8.40) (keeping shifts in \( \bar{l} \) alive!), one arrives at the DLE

\[ S_{L}^{l+1,\bar{l}+1} = (\nu^{-1} - S_{L}^{l+1,\bar{l}})(\nu^{-1} - S_{L}^{l+1,\bar{l}}), \]  

(8.53)

where, formally,

\[ S_{L}^{l,\bar{l}} = \lim_{\mu \to 0} (\mu^{-1}S^{l,\bar{l}}). \]  

(8.54)
Setting
\[ S^{\ell,\bar{\ell}}_L = \exp \left( -2\Phi^{\ell+1,\bar{\ell}}_L - 2\Phi^{\ell,\bar{\ell}+1}_L \right), \] (8.55)
we get, in place of eq. (8.42), the DLE written in terms of the discrete Liouville field \[ T \]
\[ 2\nu \sinh (\Phi^{\ell,\bar{\ell}}_L + \Phi^{\ell+1,\bar{\ell}+1}_L - \Phi^{\ell,\bar{\ell}+1}_L - \Phi^{\ell+1,\bar{\ell}}_L) = \exp \left( \Phi^{\ell,\bar{\ell}}_L + \Phi^{\ell+1,\bar{\ell}+1}_L + \Phi^{\ell+1,\bar{\ell}}_L + \Phi^{\ell,\bar{\ell}+1}_L \right), \] (8.56)
or, in a simpler form,
\[ \exp \left( -2\Phi^{\ell+1,\bar{\ell}}_L - 2\Phi^{\ell,\bar{\ell}+1}_L \right) - \exp \left( -2\Phi^{\ell,\bar{\ell}}_L - 2\Phi^{\ell+1,\bar{\ell}+1}_L \right) = \nu^{-1}. \] (8.57)

In the continuum limit one should put \( l \to \nu^{1/2}x_+, \bar{l} \to \nu^{1/2}x_- \), \( S^{\ell,\bar{\ell}}_L \to \exp \left( -4\Phi(x_+,x_-) \right) \). Expanding in \( \nu^{-1} \to 0 \), we get the continuous Liouville equation
\[ 2\partial_{x_+} \partial_{x_-} \Phi(x_+,x_-) = e^{4\Phi(x_+,x_-)}. \] (8.58)

The bilinear form of eq. (8.53) is available via the substitution
\[ S^{\ell,\bar{\ell}}_L = \frac{T^1(l + 1,\bar{l}) T^1(l,\bar{l} + 1)}{T^0(l + 1,\bar{l}) T^2(l,\bar{l} + 1)} \] (8.59)
after which the DLE becomes equivalent (up to a "gauge freedom", see below) to the bilinear relation
\[ T^a(l + 1,\bar{l}) T^{\bar{a}}(l,\bar{l} + 1) - T^a(l,\bar{l}) T^{\bar{a}}(l + 1,\bar{l} + 1) = \nu^{-1} T^{a-1}(l + 1,\bar{l}) T^{a+1}(l,\bar{l} + 1) \] (8.60)
with the condition
\[ T^a(l,\bar{l}) = 0 \quad \text{unless} \quad a = 0, 1, 2. \] (8.61)

This condition implies the discrete d’Alembert equation (8.4) for \( T^0 \) and \( T^2 \), so they have to have a factorized form \( T^0(l,\bar{l}) = \chi^0(l) \tilde{\chi}^0(\bar{l}) \), \( T^2(l,\bar{l}) = \chi^2(l) \tilde{\chi}^2(\bar{l}) \) with arbitrary and independent functions \( \chi^{0,2}, \tilde{\chi}^{0,2} \). This is just the aforementioned gauge freedom.

The striking similarity between eqs. (8.43) and (8.53) is transparent after the replacing \( T^a(l,\bar{l}) \to \tau^a_{\ell,\bar{l}} \).

Furthermore, taking into account the periodicity \( \tau^a_{\ell,\bar{l}} = \tau^a_{\ell,\bar{l}+1} \), they become formally identical. (Equivalently, using the gauge freedom, one can set \( T^2(l,\bar{l}) = T^0(l,\bar{l}) \) in eq. (8.56).) It would be interesting to link them directly on the level of solutions.

Acknowledgements

I am grateful to A.Gorsky, D.Lebedev, O.Lipan, A.Marshakov, A.Mironov, A.Orlov, P.Wiegmann and, especially, to S.Kharchev and I.Krichever for many interesting and helpful discussions. It is a pleasure to thank CMAT de l’Ecole Polytechnique, where this work was initiated, for the hospitality. This work was supported in part by RFBR grant 97-02-19085 and by ISTC grant 015.

Appendix: bilinear difference equations from continuous hierarchies

In the Appendix we give an alternative point of view to the difference Hirota equations. It relies on the famous Miwa transformation (3.3) which so far was obscure in our exposition. Given a continuous integrable hierarchy (such as KP or 2DTL), this relation can be used as a definition of the elementary discrete flows. This definition leads to the same discrete flows as in Sect. 4.2. This approach has as many advantages as disadvantages. The main advantage is much more direct and instructive connection with the Grassmannian approach to continuous hierarchies and their \( \tau \)-functions. The main disadvantage is misleadingly less invariant formulation which is inconvenient in some cases.
The Miwa transformation

Let $\tau(t_1, t_2, t_3, \ldots) \equiv \tau(t)$ be the $\tau$-function of the continuous KP hierarchy. It is a function of infinite number of "times" $t_i$ and satisfies infinitely many bilinear equations. The $\tau$-function solves all equations of the hierarchy simultaneously.

In general the $\tau$-function can be represented as an infinite dimensional determinant \[13\]-\[17\]. It turns out that there exists a choice of independent variables such that the determinant reduces to a finite dimensional one. This choice is provided by the Miwa transformation \[11\]:

$$ t_k = t_k^{(0)} - \frac{1}{k} \sum_{\alpha \in \mathcal{I}} p_{\alpha} \mu_{\alpha}^{-k}, \quad k = 1, 2, \ldots \quad (A1) $$

Here, the summation runs over a finite set $I$, $t_k^{(0)}$ are "background values" of the times, $\mu_{\alpha}$ are arbitrary complex numbers (called Miwa’s variables) and $p_{\alpha}$ are integers (sometimes called multiplicities of $\mu_{\alpha}$’s).

**Remark** The Miwa transformation plays an important role in revealing integrable structures of matrix models of 2D gravity. In particular, the easiest proof of the fact that the partition functions of the hierarchy simultaneously.

In what follows we use the short hand notation \[3\].

**Important fact:** The $\tau$-function of the KP hierarchy obeys the identity

$$ \tau \left( t^{(0)} + \sum_{\alpha = 1}^{N} \left[ \mu_{\alpha}^{-1} \right] - \left[ \mu_{\alpha}^{-1} \right] \right) = \frac{\tau(t^{(0)}) \prod_{\alpha > \beta}^{N} (\nu_{\alpha} - \nu_{\beta})}{\prod_{\alpha > \beta}^{N} (\nu_{\alpha} - \nu_{\beta}) \prod_{\alpha < \beta}^{N} (\mu_{\alpha} - \mu_{\beta})} \det_{1 \leq \alpha, \beta \leq N} K(\nu, \mu), \quad (A2) $$

where

$$ K(\nu, \mu) = \frac{\tau(t^{(0)}) + [\nu^{-1} - [\mu^{-1}]]}{(\nu - \mu) \tau(t^{(0)})}. \quad (A3) $$

Here $N \geq 1$ and $\mu_{\alpha}, \nu_{\alpha}$ are arbitrary complex numbers. A useful particular case of this formula is

$$ \tau \left( t^{(0)} - \sum_{\alpha = 1}^{N} [\mu_{\alpha}^{-1}] \right) = \frac{\det_{1 \leq \alpha, \beta \leq N} (\varphi_\alpha(\mu_{\beta}))}{\prod_{\alpha < \beta}^{N} (\mu_{\alpha} - \mu_{\beta})}, \quad (A4) $$

where

$$ \varphi_m(\mu) = \frac{1}{(m - 1)!} \lim_{\nu \to \infty} \nu^{2m-1} \frac{\partial^{m-1}}{\partial \nu^{m-1}} K(\nu, \mu). \quad (A5) $$

When one translates the KP theory into the language of free fermions \[13\] formula \[A2\] becomes nothing else than Wick’s theorem, $K(\nu, \mu)$ being the fermionic propagator on a Riemann surface.

Instead of treating eq. \[A2\] as an identity, one may follow another way. Given a function $K(\nu, \mu)$ with a simple pole at $\nu = \mu$, this equation can be used as a definition of the l.h.s. This simply means that we disregard the dependence on background times $t_k^{(0)}$ assuming they are fixed. The $\tau$-function in Miwa’s variables satisfy certain bilinear relations to which formula \[A2\] gives a solution in the form of a finite dimensional determinant.

In the case of the 2DTL hierarchy the Miwa transformation goes in a similar way. The $\tau$-function $\tau_\alpha(t_1, t_2, \ldots; \bar{t}_1, \bar{t}_2, \ldots) \equiv \tau_\alpha(t; \bar{t})$ depends on the discrete time $n$ and two infinite sets of continuous times $t_i$ and $\bar{t}_i$. We set

$$ t_k = t_k^{(0)} - \frac{1}{k} \sum_{\alpha \in \mathcal{I}} p_{\alpha} \mu_{\alpha}^{-k}, $$

$$ \bar{t}_k = \bar{t}_k^{(0)} - \frac{1}{k} \sum_{\alpha \in \mathcal{I}} \bar{p}_{\alpha} \bar{\mu}_{\alpha}^k, \quad k = 1, 2, \ldots \quad (A6) $$

where $\bar{\mu}_{\alpha}$ is an independent set of Miwa’s variables with multiplicities $\bar{p}_{\alpha}$.
The following analogue of eq. (A2) holds:

$$
\tau_{n-N} \left( t^{(0)} - \sum_{\alpha=1}^{N} [\mu_{\alpha}^{-1}]; \bar{t}^{(0)} + \sum_{\alpha=1}^{N} [\bar{\mu}_{\alpha}] \right) = \frac{\tau_{n}(t^{(0)}; \bar{t}^{(0)}) \prod_{\alpha=1}^{N} \mu_{\alpha}^{-1}}{\prod_{\alpha<\beta}(\mu_{\alpha} - \mu_{\beta})(\bar{\mu}_{\beta} - \bar{\mu}_{\alpha})} \det_{1 \leq \alpha, \beta \leq N} J_{n}(\mu_{\alpha}, \bar{\mu}_{\beta}),
$$

(A7)

where

$$
J_{n}(\mu, \bar{\mu}) = \frac{\tau_{n-1}(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}])}{\tau_{n}(t^{(0)}; \bar{t}^{(0)})}.
$$

(A8)

Note that in this case the function $J_{n}(\mu, \bar{\mu})$ does not necessarily have first order pole at $\mu = \bar{\mu}$.

### Discrete flows

Discrete equations for the $\tau$-function listed in Sect. 2 are obtained if one fixes Miwa’s variables $\mu_{\alpha}$ and consider flows in the multiplicities $p_{\alpha}$. We give a few examples.

**Example 1.** Set

$$
\tau^{p_{1}, p_{2}, p_{3}} = \tau \left( t^{(0)} - \sum_{\alpha=1}^{3} p_{\alpha} [\mu_{\alpha}^{-1}] \right)
$$

(A9)

and consider $\bar{t}^{(0)} = t^{(0)} - \sum_{\alpha=1}^{3} p_{\alpha} [\mu_{\alpha}^{-1}]$ as a new ”background”. According to eq. (A7),

$$
\tau^{p_{1}+1} = \tilde{\varphi}_{1}(\mu_{1}), \quad \tau^{p_{1}+1, p_{2}+1} = \frac{\tilde{\varphi}_{1}(\mu_{1}) \tilde{\varphi}_{1}(\mu_{2})}{\mu_{1} - \mu_{2}}
$$

(A10)

with some functions $\tilde{\varphi}_{1}, \tilde{\varphi}_{2}$. Combining the zero determinant with two identical lines,

$$
0 = \begin{vmatrix}
\tilde{\varphi}_{1}(\mu_{1}) & \tilde{\varphi}_{1}(\mu_{2}) & \tilde{\varphi}_{1}(\mu_{3}) \\
\tilde{\varphi}_{1}(\mu_{1}) & \tilde{\varphi}_{1}(\mu_{2}) & \tilde{\varphi}_{1}(\mu_{3}) \\
\tilde{\varphi}_{2}(\mu_{1}) & \tilde{\varphi}_{2}(\mu_{2}) & \tilde{\varphi}_{2}(\mu_{3})
\end{vmatrix},
$$

(A11)

and expanding it in the first line, we get an equation of the form (2.7). Since its coefficients do not depend on the chosen background, the equation holds for all values of $p_{1}, p_{2}, p_{3}$.

**Example 2.** Repeating the previous argument for

$$
\tau^{p_{1}, p_{2}, p_{3}} = \tau \left( t^{(0)} + p_{0} ([\mu_{1}^{-1}] - [\nu^{-1}]) + \sum_{\alpha=1}^{3} p_{\alpha} ([\nu^{-1}] - [\mu_{\alpha}^{-1}]) \right)
$$

and making use of eq. (A2), we get eq. (2.10).

**Example 3.** In eq. (A7), let $N = 2$, $\mu_{1} = \mu$, $\mu_{1} = \bar{\mu}$, $\mu_{2} \to \infty$, $\bar{\mu}_{2} \to 0$:

$$
\tau_{n-2}(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}]) = \begin{vmatrix}
\tau_{n-1}(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}]) & \tau_{n-1}(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)}) \\
\tau_{n-1}(t^{(0)}; \bar{t}^{(0)} + [\bar{\mu}]) & \tau_{n-1}(t^{(0)}; \bar{t}^{(0)})
\end{vmatrix}
$$

(A12)

Denoting

$$
\tau_{n}^{l \bar{l}} = \tau_{n}(t^{(0)} - l[\mu^{-1}]; \bar{t}^{(0)} - l[\bar{\mu}]),
$$

(A13)

we get the equation

$$
\tau_{n}^{l \bar{l} + 1} = \tau_{n}^{l \bar{l}} + \tau_{n-1}^{l \bar{l}+1} = (\bar{\mu}/\mu) \tau_{n+1}^{l \bar{l}+1} \tau_{n-1}^{l \bar{l}+1}
$$

(A14)

**Example 4.** Example 1 can be generalized in the following way. Consider a $N \times N$-matrix with the lines $\varphi_{1}(\mu_{1}), \varphi_{1}(\mu_{1}), \varphi_{2}(\mu_{1}), \varphi_{3}(\mu_{1}), \ldots, \varphi_{N-1}(\mu_{i}), i = 1, 2, \ldots, N$, so that the first two lines coincide.
and determinant of this matrix is zero. Then, expanding in the first row, like in Example 1, we get the "higher" bilinear difference equation of the form \( (A1) \).

**Example 5.** At last, we show how to derive HBDE in the KP-like form from eq. \( (A4) \) in a direct way.\footnote{This argument is taken from ref. \[13\].} When two or more \( \mu_\alpha \)'s coincide, both the numerator and denominator in the r.h.s. of eq. \( (A4) \) equal zero. Resolving the indeterminacy, we have

\[
\tau \left( \mu^{(0)} - \sum_{\alpha=1}^{N} p_\alpha [\mu_\alpha^{-1}] \right) = \frac{\det \left( M_{ij}^{(N)} \right)}{\prod_{\alpha<\beta}(\mu_\alpha - \mu_\beta)^{p_\alpha p_\beta}}, \quad 1 \leq i, j \leq N, \quad (A15)
\]

and all \( \mu_\alpha \)'s are now distinct. Here \( N \equiv \sum_{\alpha=1}^{N} p_\alpha \), \( M_{ij}^{(N)} \) is the \( N \times N \)-matrix having the rows

\[
\varphi_i(\mu_1), \varphi'_i(\mu_1), \varphi''_i(\mu_1), \ldots, \varphi_i^{(p_1-1)}(\mu_1), \\
\varphi_i(\mu_2), \varphi'_i(\mu_2), \varphi''_i(\mu_2), \ldots, \varphi_i^{(p_2-1)}(\mu_2), \\
\varphi_i(\mu_N), \varphi'_i(\mu_N), \varphi''_i(\mu_N), \ldots, \varphi_i^{(p_N-1)}(\mu_N), \quad 1 \leq i \leq N. \quad (A16)
\]

We need the well known Jacobi identity for determinants:

\[
D[i_1,j_1]D[i_2,j_2] - D[i_1,j_2]D[i_2,j_1] = D[i_1,i_2]D[j_1,j_2], \quad i_1 < i_2, \ j_1 < j_2, \quad (A17)
\]

where \( D \) is determinant of a square matrix and \( D[i_1,i_2,j_1,j_2] \) denotes minors of this matrix with \( i_1, i_2, j_1, j_2 \) rows and \( j_1, j_2 \)-th columns removed. Applying this identity to the matrix \( M_{ij}^{(N)} \) in \( (A15) \) for \( i_1 = N - 1, i_2 = N, j_1 = \sum_{\alpha=1}^{a} p_\alpha, j_2 = j_1 + \sum_{\alpha=a+1}^{b} p_\alpha, 1 \leq a < b \leq N \), we get, in the short hand notation,

\[
(\mu_\alpha - \mu_\beta)\tau^p_{\alpha\beta} - \tau^p_{\alpha\beta} = \tau^p_{\alpha\beta} - \tau^p_{\alpha\beta}, \quad (A18)
\]

where \( \tau^p_{\alpha\beta} \) is defined by the same formula \( (A12) \) with the matrix \( M_{ij}^{(N-1)} = M_{ij}^{(N-1)} \) for \( 1 \leq i \leq N - 2, M_{N-1,j}^{(N-1)} = M_{N,j}^{(N)} \).

Let \( \mu_\epsilon \) be a third Miwa variable (different from \( \mu_\alpha, \mu_\beta \)) with the multiplicity \( p_\epsilon \) not shown explicitly in eq. \( (A18) \). Multiplying this equality by \( \tau^{p_\epsilon-1}/\tau \) and then writing down a couple of similar equations obtained by cyclic permutations of the indices \( a, b, c \), we see that the sum of these three equations coincides with eq. \( (2.7) \).

**Remark** The discrete flows discussed here coincide with those introduced in the main body of the paper if one fixes the following choice of the labels \( \lambda_0 \) and \( \lambda_1 \): \( \lambda_0 = \infty, \lambda_1 = 0 \). (To remove a label to infinity, one should use a different normalization.)

**Continuum limit**

As it is clear from eq. \( (A1) \), inverse Miwa's variables \( \mu_\alpha^{-1} \) play the role of lattice spacings for the discrete flows. So, to perform the limit to continuous equations, it is necessary for \( \mu_\alpha \) to tend to infinity with a simultaneous rescaling of \( p_\alpha \)'s.

Here is the typical example (the KP hierarchy). In this example we follow ref. \[12\]. Introduce three (a priori independent) lattice spacings \( \varepsilon_i = \mu_\alpha^{-1}, i = 1, 2, 3 \), and rescale \( p_i \rightarrow p_i/\varepsilon_i \). It is then convenient to rewrite the KP-like form \( (2.7) \) of HBDE in terms of Hirota's \( D \)-operator \( (1.2) \):

\[
(\varepsilon_1(\varepsilon_2 - \varepsilon_3)e^{-\varepsilon_1/2}D_{\varepsilon_1} + (\varepsilon_2/2)D_{\varepsilon_2} + (\varepsilon_3/2)D_{\varepsilon_3} + \\
+ \varepsilon_2(\varepsilon_3 - \varepsilon_1)e^{\varepsilon_2/2}D_{\varepsilon_2} + (\varepsilon_1/2)D_{\varepsilon_1} + (\varepsilon_3/2)D_{\varepsilon_3} + \\
+ \varepsilon_3(\varepsilon_1 - \varepsilon_2)e^{\varepsilon_3/2}D_{\varepsilon_3} + (\varepsilon_1/2)D_{\varepsilon_1} + (\varepsilon_2/2)D_{\varepsilon_2} - (\varepsilon_3/2)D_{\varepsilon_3}) \tau \cdot \tau = 0. \quad (A19)
\]
This equation serves as a "generating function" for a part of the continuous KP hierarchy. To see this, we express operators $D_{pi}$ through Hirota’s derivatives with respect to the continuous flows $t_k$,

$$D_{pi} = -\sum_{k=1}^{\infty} \frac{1}{\rho^i} \varepsilon_{j} \bar{D} \cdot D_{t_k}, \quad i = 1, 2, 3,$$

in accordance with (A1). Substituting this into eq. (A19) and expanding it in a power series in $\varepsilon_i$, we have:

$$\begin{align*}
\varepsilon_1 & \left( \varepsilon_2 - \varepsilon_3 \right) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l P_j \left( \frac{1}{2} \tilde{D} \right) P_k \left( -\frac{1}{2} \tilde{D} \right) P_l \left( -\frac{1}{2} \tilde{D} \right) + \\
& + \varepsilon_2 \left( \varepsilon_3 - \varepsilon_1 \right) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l P_j \left( -\frac{1}{2} \tilde{D} \right) P_k \left( \frac{1}{2} \tilde{D} \right) P_l \left( -\frac{1}{2} \tilde{D} \right) + \\
& + \varepsilon_3 \left( \varepsilon_1 - \varepsilon_2 \right) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l P_j \left( -\frac{1}{2} \tilde{D} \right) P_k \left( -\frac{1}{2} \tilde{D} \right) P_l \left( \frac{1}{2} \tilde{D} \right) \tau \cdot \tau = 0, \quad (A20)
\end{align*}$$

where $\tilde{D} \equiv \left( D_{t_1}, D_{t_2}/2, \ldots, D_{t_k}/k, \ldots \right)$ and $P_j(t)$ are Schur polynomials defined by

$$\exp \left( \sum_{k=1}^{\infty} \frac{t_k}{k} z^k \right) = \sum_{m=0}^{\infty} P_m(t) z^m. \quad (A21)$$

Extracting the coefficients in front of $\varepsilon_1^j \varepsilon_2^k \varepsilon_3^l$, we obtain the infinite set of bilinear equations,

$$\begin{align*}
P_{j-1} \left( \frac{1}{2} \tilde{D} \right) & \quad P_{j-1} \left( -\frac{1}{2} \tilde{D} \right) & \quad P_j \left( -\frac{1}{2} \tilde{D} \right) \quad \tau \cdot \tau = 0, \\
P_{k-1} \left( \frac{1}{2} \tilde{D} \right) & \quad P_{k-1} \left( -\frac{1}{2} \tilde{D} \right) & \quad P_k \left( -\frac{1}{2} \tilde{D} \right) \quad \tau \cdot \tau = 0, \\
P_{l-1} \left( \frac{1}{2} \tilde{D} \right) & \quad P_{l-1} \left( -\frac{1}{2} \tilde{D} \right) & \quad P_l \left( -\frac{1}{2} \tilde{D} \right) \quad \tau \cdot \tau = 0,
\end{align*} \quad (A22)$$

which for $1 \leq j < k < l$ form a subset of the whole KP hierarchy in the bilinear form.

The leading term as $\varepsilon_i \to 0$ in (A20) corresponds to $(j,k,l) = (1,2,3)$ in (A22). In this case eq. (A22) gives (the bilinear form of) the KP equation itself:

$$(D_{t_1}^4 - 4D_{t_1}D_{t_2} + 3D_{t_2}^2) \tau \cdot \tau = 0. \quad (A23)$$

This example shows once again that the discrete hierarchy has a more transparent structure than the continuous one. The continuum limit brings artificial complications.

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