Rogue waves in discrete-time quantum walks

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Rogue waves are rapid and unpredictable events of exceptional amplitude reported in various fields, such as oceanography and optics, with much of the interest being targeted towards their physical origins and likelihood of occurrence. Here, we use the all-round framework of discrete-time quantum walks to study the onset of those events due to a random phase modulation, unveiling its long-tailed statistics, distribution profile, and dependence upon the degree of randomness. We find that those rogue waves belong the Gumbel family of extreme value distributions.

Introduction. — Rogue or freak waves, unpredictable and rare huge walls of water appearing from nowhere and vanishing without a trace, have been known and feared for centuries by seafarers. The first solid account of the phenomenon took place in 1995 when data collected on the Draupner oil platform in the North Sea revealed a 26-meter wave rising out of a background with about half significant wave height [1]. Years later, analogies between such ocean wave phenomena and light propagation in optical fibers surged in the framework of the nonlinear Schroedinger equation [2]. Since then, interest in ubiquitous wave phenomena displaying long-tailed statistics, when outliers occur more often than expected from Gaussian statistics, has skyrocketed in various fields (for a recent review, see [3]). Optics, particularly, has been a powerful testbed for investigating rogue waves thanks to the spatial and timescales involved and, in addition, optical rogue waves include a bunch of novel phenomena, not necessarily featuring a hydrodynamics counterpart [4].

One of the key challenges in the field is to find out precisely how those events emerge so as to be able to predict and control them. There is a long-standing debate on whether rogue waves emerge as linear or nonlinear processes [3] and what is the role of noise and randomness [5]. It is natural to assume that nonlinearity plays an important role due to modulational instability [6, 7], collisions between solitons [8], and so forth. On the other hand, some studies suggest that linear interference of random fields are crucial [9–20], with nonlinear effects responsible for extra wave focusing [21–23]. Indeed, linear models can display rogue waves on their own when augmented with the right ingredients as shown in [10]. This has been shown experimentally in microwave transport in randomly distributed scatterers [9], 2D photonic crystal resonators [13], and very recently by measuring linear light diffraction patterns in the presence of long-range spatial memory effects in the random input [13].

Interest on linear rogue waves has been raising considerably over the past few years. Yet, it is surprising that quantum mechanics has barely been taken into consideration. Even though the dynamics of a single quantum particle can be mapped into linear optics, investigating the onset of rogue-like events in the very domain of quantum mechanics has its own appeal. It could, for instance, shed new light on the dynamics of disordered systems and related features such as Anderson localization. With that in mind, we set about to explore the occurrence of rogue quantum amplitudes using the discrete-time quantum walk (DTQW) approach [24]. It is basically a cellular automaton [25] whose updating rules are run by a preset sequence of quantum gates. Given recent experimental advances in the field [26, 27] as well as their wide range of applications, from quantum algorithms [28] to simulation of involved phenomena in condensed matter physics [15, 30–34], DTQWs make for a suitable starting point.

We report the manifestation of rogue waves in the Hadamard one-dimensional DTQW induced by random phase fluctuations. We do so by unveiling the long-tailed statistics of the occupation probability amplitudes (which is analogous to light intensity in optics) over the space-time set of events. We show that an intermediate level of disorder scaling as $N^{-r}$ maximizes the likelihood of rogue events. That has to do with a fair balance between localization and mobility, for which the localization length $\propto N^{2r}$, $N$ being the number of sites. Furthermore, extreme-value analysis is carried out for the amplitude block maximum over time and we find that the resulting distribution falls into the Gumbel class.

Quantum walk model. — We consider a single-particle DTQW in one dimension [24] defined by a two-level (coin) space $H_C \equiv \{ |\uparrow\rangle, |\downarrow\rangle \}$ and a position space $H_P \equiv \{|n\rangle\}$, such that the full Hilbert space reads $H = H_C \otimes H_P$. An arbitrary state at a given instant $t$ can be written as $|\psi_n(t)\rangle = \sum_{n \geq 1} a_n(t) |\uparrow\rangle + b_n(t) |\downarrow\rangle$, satisfying the normalization condition $\sum_n |a_n(t)|^2 + |b_n(t)|^2 = 1$.

The quantum walker evolves as $|\Psi(t + 1)\rangle = \hat{S}|\Psi(t)\rangle$, where the conditional shift operator $\hat{S}$ is responsible for the nearest-neighbor transitions $\hat{S}|\uparrow, n\rangle = |\uparrow, n + 1\rangle$ and $\hat{S}|\downarrow, n\rangle = |\downarrow, n - 1\rangle$ (assuming periodic boundary conditions). $\hat{C} = (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)\sqrt{2}$ is the standard Hadamard coin, $I_P$ is the identity operator acting on the $N$–dimensional position space, and

$$\hat{D} = \sum_c \sum_n e^{iF(c,n,t)} |c, n\rangle \langle c, n|$$

is the phase-gain operator, with $F(c, n, t)$ being a real-valued arbitrary function [35] and $c = \uparrow, \downarrow$. Given the flexibility in choosing $F(c, n, t)$, one is able to produce vari-
ous dynamical regimes. Setting $F = 0$ renders the standard Hadamard quantum walk in which walker spreads out ballistically [24]. Here, instead, we set a static random phase modulation such that $F(c,n,t) = F(c,n) = 2\pi \nu$, where $\nu$ is a random number uniformly distributed within $[-W,W]$, with $W$ being the disorder width. As this setting can lead to Anderson localization [15, 32], we ought to inquire whether rogue waves can be supported given proper initial conditions and amount of noise embedded in $F(c,n)$.

Results. — In order to avoid ambiguity between an actual rogue event (a rare one) and the inevitable Anderson localization in the statistics, we initialize the system in a coin-unbiased [36], fully delocalized state $|\psi(0)\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (|\uparrow, n\rangle + |\downarrow, n\rangle)$. Random phase modulation is introduced at the very first gate operation $D$ [see Eq. (1)] so as to foster inhomogeneity and, as a result, fragmentation of the walker wavefunction. These two ingredients have been proved to be crucial for the development of linear rogue waves [10].

Let us now establish the criteria to identify the rogue waves. A standard approach in oceanography and optics [3] sets that the amplitude of a rogue wave must exceed at least twice the significant wave height, defined as the mean level of disorder, $W$, that maximizes the chances of observing a rogue event somewhere along the system in a coin-unbiased [36], fully delocalized state $|\psi(0)\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (|\uparrow, n\rangle + |\downarrow, n\rangle)$. Random phase modulation is introduced at the very first gate operation $D$ [see Eq. (1)] so as to foster inhomogeneity and, as a result, fragmentation of the walker wavefunction. These two ingredients have been proved to be crucial for the development of linear rogue waves [10].

In order to analyze those distributions in a more quantitative level for the whole range of $W$, Fig. 2(a) shows the ensemble-averaged percentage of events fulfilling $P_n > 2P_{th}$. Fig. 2(b) shows the size dependence of the disorder level $W_c$ above which extreme events have a finite occurrence probability. Irrespective of the threshold level, the minimal disorder strength leading to the occurrence of rogue waves $W_c \propto N^{-1/2}$. It is interesting to stress that the typical localization length of the eigenstates of quantum walks under random phase shifts scales as $\chi \propto 1/W^2$ [15]. Therefore, $\chi \propto N$ at $W_c$. The above result unveils that rogue waves emerge whenever disorder is strong enough to produce effectively localized states in a finite chain with $N$ sites. Curiously, there is an optimal level of disorder, $W_{\text{max}}$, that maximizes the chances of observing a rogue event somewhere along the $N$-site cycle. This suggests that rogue events are more likely to develop when one properly balances localization and mobility. The inset of Fig. 2(a) shows that $W_{\text{max}} \propto N^{-\nu}$, with $\nu \approx 0.19$ over the range of chain sizes considered.

![FIG. 1. (a) Snapshot of the space-time evolution of the occupation amplitude $P_n$ in the Hadamard DTQW on a ring with $N = 100$ sites and disorder strength $W = 0.1$ (single realization). (b,c) Time series and spatial profile extracted from (a). The rogue event is seen at $t = 7174$.](image1)

![FIG. 2. Normalized PDFs for (a) $W = 0.01$ and (b) $W = 0.3$ in semilog scale for an ensemble of 5000 independent realizations of disorder and $10^4$ steps on a cycle with $N = 100$ sites.](image2)
FIG. 3. (a) Number of rogue wave events versus disorder strength $W$ for $N = 50, 100, 200, 400$ and 800 sites (black line to the orange, respectively), averaged over 5000 independent realizations of disorder, each running through $10N$ time steps. Inset shows the scaling of the disorder degree that maximizes the chances of measuring a rogue event, $W_{\text{max}}$, with $N$. (b) Disorder strength $W_c$ above which rogue waves have a finite occurrence probability for distinct threshold levels. The scaling $W_c \propto N^{-1/2}$ unveils that at $W_c$ the localization length $\chi \propto 1/W^2$ is of the order of the chain size.

For such disorder level, the typical localization length scales as $\chi \propto N^{2\nu}$.

An increased likelihood of the occurrence of rogue waves between weak and intermediate disorder strengths has been seen in recent experiments carried out on 1D photonic lattices featuring both on-site and coupling disorder [17]. That also suggests that the interplay between localization and delocalization is a key ingredient for the for the generation of extreme events in linear systems. Furthermore, correlated fluctuations – known to yield rich transport properties [38] – have been exploited to enhance the likelihood of occurrence of rogue waves [15,18], some of these largely exceeding the amplitude threshold (referred to as super rogue waves) [18].

Large fluctuations in $F(c,n)$ [cf. Eq. (1)] tend to make localization effects sharper but it does not necessarily mean that the occurrence of rogue waves will follow that up. We shall always keep in mind that a rogue wave is a rare and sudden event whose amplitude should exceed some threshold based on the average amplitude background. In order to produce such abnormal constructive interference at some location via linear dynamics, we need proper synchronization of random waves undergoing different paths and thereby some degree of mobility. Figure 4 shows the evolution of branching patterns highlighting the distribution profile of the rogue events (red spots). In the case of weak disorder, we note that whenever synchronization conditions are met to form a rogue wave, it usually covers a few sites in the neighborhood before disappearing [4(a)]. For intermediate disorder, the rogue events become sparse but more frequent, as a more complex branching profile emerges [4(b)]. If we keep on increasing the disorder width $W$, there will be a stage above which mobility, if any, is restricted to shorter spatial domains given the onset of local resonances. This is seen in Fig. 4(c) in the form of well defined amplitude domains, with few of them giving rise to rogue waves now and then. That is why the rogue-wave likelihood saturates for large $W$ and barely responds to the system size.
for (c) \( W = 0.50 \)

FIG. 5. Extreme-value PDFs in semilog scale for \( N = 100 \) and \( 10^4 \) independent realizations of disorder. At each time step, the maximum probability amplitude \( P_{\text{max}} \) is recorded. The red line is a Gumbel-type fitting given by \( y(x) \propto \exp[-ax - b \exp(-ax)] \), with \( a, b \) depending on \( W \) and \( N \). For intermediate degree of disorder, as in Fig. 5(b), the range of \( P_{\text{max}} \) is visibly more stretched, what again indicates a pronounced likelihood of observing a rogue event.

Final remarks. — We have reported the occurrence of rogue wave events in disordered DTQWs and showed that those indeed belong a class of extreme value phenomena. Using the peak-over-threshold approach borrowed from oceanography, we have also uncovered the long-tailed profile of the distributions. We found that an intermediate degree of disorder \( W_{\text{max}} \propto N^{-\nu} \) yields maximum occurrence of rogue waves due to a proper balance between trapping mechanisms and mobility for which \( \chi \propto N^{2\nu} \). This calls for further investigation in order to assess the intrinsic relationship between localization length and rogue wave generation, specially in the case of correlated phases which has been shown to enhance the occurrence of extreme events [16, 18].

The DTQW studied here also offers the possibility of embedding nonlinearity into \( F(c, n, t) \). In [33], for instance, the authors considered a Kerr-type self-phase modulation and reported the formation of solitonlike pulses. Also, in [33], it was shown that self-trapping can occur for certain coin angles. The stage is thus set for assessing the competition between linear and nonlinear mechanics in the generation of rogue waves in quantum walks.

We hope that our work seeds interest in quantum-mechanical extreme events in general, specially in the context of condensed-matter theory, for the sake of facing Anderson localization phenomena under different light, as well as in the field of quantum information processing, where unexpected events of that nature could lead to potential hazards in the evaluation of some protocol given the unavoidable presence of manufacturing imperfections of the physical components.

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