Final state interaction in kaons decays.

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Abstract

The kaons decays to the pairs of charged and neutral pions are considered in the framework of the non-relativistic quantum mechanics. The general expressions for the decay amplitudes to the two different channels accounting for the strong interaction between pions are obtained. The developed approach allows one to estimate the contribution of terms of any order in strong interaction and correctly takes into account the electromagnetic interaction between the pions in the final state.

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1 Introduction

It has long been known [1, 2] that the K-mesons decays with a pions in the final state can give unique information on the pions s-wave scattering lengths $a_0, a_2$, whose values are predicted by Chiral Perturbation Theory (ChPT) with high accuracy [3].

Recently the high quality data on $K^\pm \rightarrow \pi^\mp \pi^0 \pi^0$ decays have been obtained by NA48/2 collaboration at CERN [4]. The dependence of the decay rate

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on the invariant mass of neutral pions \( M^2 = (p_1 + p_2)^2 \) reveals a prominent anomaly (cusp) at the charged pions production threshold \( M_c^2 = 4m_c^2 \).

As was explained in \cite{5, 6} this anomaly is due to the possibility for the kaon to decay to three charged pions, which after charge exchange reaction \( \pi^+\pi^- \to \pi^0\pi^0 \) gives the observed neutral pions. This possibility is provided by mass difference of charged and neutral pions. The detail consideration of this decay using the technique of non-relativistic field theory \cite{7} or ChPT \cite{8} supports the proposed picture.

Nevertheless there are two challenges crucial in scattering lengths extraction from kaons decays. One needs a reliable way to estimate the contribution of higher order terms in strong interaction and calculates the electromagnetic interaction among the charged pions.

These issues are very close connected with each other. Calculation of the electromagnetic interaction in every order of strong interaction \cite{9} doesn’t solve the problem of bound states (pionium atoms), as to take into account electromagnetic interaction leading to unstable bound states one needs expressions for decay amplitudes including the strong interaction between pions in all orders \cite{10}. The problem of correct accounting of the electromagnetic effects are also necessary in a wide class of decays with two pions in the final state as for instance \( K_{e4} \) decay \cite{11, 12}.

The phenomenon of cusp in elastic scattering at the threshold relevant to inelastic channel is known for many years and was widely discussed in the framework of non-relativistic quantum mechanics \cite{13, 14, 15}. For the elastic process \( \pi^0\pi^0 \to \pi^0\pi^0 \) this anomaly at the \( \pi^+\pi^- \) threshold was firstly discussed in the framework of ChPT in \cite{16}. In the present work we consider the kaon decay to pion pairs with pions of different masses. Using the well known results of quantum mechanics we obtain the matrix elements for decay \( K \to \pi\pi \) where the final pions consist from pions of different masses \( (\pi^+\pi^-, \pi^0\pi^0) \).

## 2 Two channel decay

We are interested in two channel decay of kaon to the pion pairs in the final state, where the pions in the pair can be neutral or charged. The well examples are \( K_L \to \pi\pi \) as well as \( K^\pm \to \pi^+\pi^- e^\pm \mu \) \( (K_{e4} \) decay). In what follows all quantities relevant to the neutral pions pair \( (\pi^0\pi^0) \) are labeled by index "n", whereas the charged pions pairs \( (\pi^+\pi^-) \) are labeled by index "c".

We do not consider here the electromagnetic interaction in the pair, the effect discussed in our previous work \cite{12}. Our main goal is to obtain the matrix elements of the kaon decay to the pion pair accounting for different masses of
neutral and charged pions and the possibility of charged exchange reaction
\( \pi^+\pi^- \rightarrow \pi^0\pi^0 \) and the elastic scattering of pions in the final state.
The general form of matrix element for kaon decay to the final state with
two charged or neutral pions can be written in the operator form:

\[
M_c = \int \Psi_c^+(r)M_0(r)d^3r; \quad M_n = \int \Psi_n^+(r)M_0(r)d^3r
\]  

(1)

The two component operator \( M_0 = \begin{pmatrix} M_c^{(0)}(r) \\ M_n^{(0)}(r) \end{pmatrix} \), where \( M_c^{(0)}(r), M_n^{(0)}(r) \)
are the matrix elements of kaon decay to noninteracting charged and neutral
pions pairs, while \( \Psi_c(r), \Psi_n(r) \) are the appropriate two component wave
functions.

These wave functions would satisfy to couple Shrödinger equations
\[
-\Delta \Psi_c(r) + U_{cc}\Psi_c(r) + U_{cn}\Psi_n(r) = k_c^2 \Psi_c(r) \\
-\Delta \Psi_n(r) + U_{nn}\Psi_n(r) + U_{nc}\Psi_c(r) = k_n^2 \Psi_n(r)
\]  

(2)

where \( U_{ij} \) are the strong potentials describing elastic \( cc \rightarrow cc; nn \rightarrow nn \) scattering
and charge exchange reaction \( cn \rightarrow cn \). \( k_c, k_n \) are the charge and neutral pions momenta in the appropriate center of mass system.

According to the general principles of scattering theory the asymptotic behavior
of the wave functions \( \Psi_c(r), \Psi_n(r) \) can be written through the s-wave amplitudes
\( f_{cc}, f_{nn}, f_{cn}, f_{nc} \) in the following form:

\[
\Psi_c(r) = \begin{pmatrix} \frac{\text{sink}_cr}{k_c r} & 0 \\ 0 & \frac{\text{sink}_nr}{k_n r} \end{pmatrix} + \begin{pmatrix} e^{-ik_c r}f^*_{cc} \\ e^{-ik_n r}f^*_{nc} \end{pmatrix} \\
\Psi_n(r) = \begin{pmatrix} 0 & \frac{\text{sink}_cr}{k_c r} \\ \frac{\text{sink}_nr}{k_n r} & 0 \end{pmatrix} + \begin{pmatrix} e^{-ik_c r}f^*_{cn} \\ e^{-ik_n r}f^*_{nn} \end{pmatrix}
\]  

(3)

The first columns in these expressions describe the noninteracting s-waves
pions pairs, whereas the second columns correspond to the interacting charge
and neutral pions pair in the far asymptotic of corresponding wave function.

One can rewritten these equations through the elements of appropriate S-
matrix \[17\] :

\[
S_{cc} = 1 + 2ik_c f_{cc}; \quad S_{nn} = 1 + 2ik_n f_{nn}; \quad S_x = 2i \sqrt{k_n k_c} f_x
\]  

(4)

Substituting these relations in the expressions (3) one immediately obtains:

\[
\Psi_c^*(r) = \begin{pmatrix} i e^{-ik_c r} - S_{cc} e^{ik_c r} \\ -i e^{-ik_n r} S_{nn} e^{ik_n r} \end{pmatrix} + \begin{pmatrix} e^{-ik_c r}f^*_{cc} \\ e^{-ik_n r}f^*_{nn} \end{pmatrix} \\
\Psi_n^*(r) = \begin{pmatrix} i e^{-ik_n r} - S_{nn} e^{ik_n r} \\ -i e^{-ik_c r} S_{cc} e^{ik_c r} \end{pmatrix} + \begin{pmatrix} e^{-ik_n r}f^*_{nn} \\ e^{-ik_c r}f^*_{cc} \end{pmatrix}
\]  

(5)

\[1\]Throughout this paper we restricted by s-wave \( \pi\pi \) scattering in the final state.
From the other hand the wave functions $\Psi_c(r)$ and $\Psi_n(r)$ can be constructed as the linear combination of two real solutions of equations (2)

$$
\Psi^{(1)} = \begin{pmatrix} \Psi_c^{(1)} \\ \Psi_n^{(1)} \end{pmatrix} \quad \Psi^{(2)} = \begin{pmatrix} \Psi_c^{(2)} \\ \Psi_n^{(2)} \end{pmatrix}
$$

with the standard boundary conditions $\Psi^{(1)}(0) = \Psi^{(2)}(0) = 0$. Keeping this in mind we will look for the desired wave functions in the form:

$$
\Psi^*_c(r) = A_c^{(1)} \Psi^{(1)} + A_c^{(2)} \Psi^{(2)}; \quad \Psi^*_n(r) = A_n^{(1)} \Psi^{(1)} + A_n^{(2)} \Psi^{(2)}
$$

where $A_c^{(1)}, A_c^{(2)}, A_n^{(1)}, A_n^{(2)}$ are arbitrary complex numbers.

Substituting the expressions (6),(7) in (1) one gets:

$$
M_c = A_c^{(1)} \int \left( (\Psi_c^{(1)} M_c^{(0)} + \Psi_n^{(1)} M_n^{(0)}) \right) d^3r \\
+ A_c^{(2)} \int \left( (\Psi_c^{(2)} M_c^{(0)} + \Psi_n^{(2)} M_n^{(0)}) \right) d^3r = A_c^{(1)} I_1 + A_c^{(2)} I_2
$$

$$
M_n = A_n^{(1)} \int \left( (\Psi_c^{(1)} M_c^{(0)} + \Psi_n^{(1)} M_n^{(0)}) \right) d^3r \\
+ A_n^{(2)} \int \left( (\Psi_c^{(2)} M_c^{(0)} + \Psi_n^{(2)} M_n^{(0)}) \right) d^3r = A_n^{(1)} I_1 + A_n^{(2)} I_2
$$

Making use that any real solution of equations (2) out of the potential range ($U_{ij} = 0$) can be taken in the form:

$$
\Psi(r) = \frac{g \sin(kr + \delta(k))}{kr} = \frac{g}{2ikr} \left( e^{ikr + i\delta(k)} - e^{-ikr - i\delta(k)} \right)
$$

we will look for the real solutions out of potential range as:

$$
\Psi^{(1)}_c(r) = g_1 e^{ik_c r} - g^*_1 e^{-ik_c r} \cdot \frac{2ik_c r}{2ik_c r}; \quad \Psi^{(1)}_n(r) = h_1 e^{ik_n r} - h^*_1 e^{-ik_n r} \cdot \frac{2ik_n r}{2ik_n r}
$$

$$
\Psi^{(2)}_c(r) = g_2 e^{ik_c r} - g^*_2 e^{-ik_c r} \cdot \frac{2ik_c r}{2ik_c r}; \quad \Psi^{(2)}_n(r) = h_2 e^{ik_n r} - h^*_2 e^{-ik_n r} \cdot \frac{2ik_n r}{2ik_n r}
$$

In order to obtain the relations between the unknown coefficients in the above expressions let us at first compare the asymptotic behavior of the

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2In terms of the wave function $\Phi(r) = \frac{\Psi(r)}{r}$ this condition requires the regularity at $r=0$.

3 We consider only the class of strong potentials with the sharp boundary.

4 These factors are functions of pions momenta $k_c, k_n$. 
initial wave functions $\Psi_c(r)$ in (5) with the first raw in the parametrization (10):

$$\begin{align*}
\frac{A_c^{(1)}}{2ik_c} \left( g_1 e^{ik_c r} - g_1^* e^{-ik_c r} \right) &+ \frac{A_c^{(2)}}{2ik_c} \left( g_2 e^{ik_c r} - g_2^* e^{-ik_c r} \right) = \frac{i}{2k_c} \left( e^{-ik_c r} - S_{cc} e^{ik_c r} \right) \\
\frac{A_c^{(1)}}{2ik_n} \left( h_1 e^{ik_n r} - h_1^* e^{-ik_n r} \right) &+ \frac{A_c^{(2)}}{2ik_n} \left( h_2 e^{ik_n r} - h_2^* e^{-ik_n r} \right) = -\frac{i}{2\sqrt{k_n k_c}} S_{cn} e^{ik_n r}
\end{align*}$$

Gathering the structures in front of the appropriate exponents and solving the system of obtained equations after a bit algebra we get:

$$A_c^{(1)} = \frac{h_2^*}{H}; \quad A_c^{(2)} = -\frac{h_1^*}{H}; \quad H = g_1^* h_2 - h_1^* g_2^*;$$

$$S_{cc} = \frac{h_2^* g_1 - h_1^* g_2}{H}; \quad S_{cn} = \sqrt{\frac{k_n}{k_c}} \frac{h_2^* h_1 - h_1^* h_2}{H}$$

(11)

Carry out the same procedure for $\Psi(r)$ we obtain the relevant relations for the case of kaon decay to pair of neutral pions:

$$A_n^{(1)} = -\frac{g_2^*}{H}; \quad A_n^{(2)} = -\frac{g_1^*}{H};$$

$$S_{nn} = -\frac{g_2^* h_1 - g_1^* h_2}{H}; \quad S_{nc} = \sqrt{\frac{k_n}{k_c}} \frac{g_2^* g_2 - g_1^* g_2}{H}$$

(12)

In respect that due to T-invariance $S_{cn} = S_{nc}$ it can be checked that obtained relations satisfied the unitarity constraints:

$$|S_{nn}|^2 + |S_{nc}|^2 = 1; \quad |S_{cc}|^2 + |S_{cn}|^2 = 1$$

(13)

As has been seen from expression (9) the imaginary parts of functions $g_{1(2)}$, $h_{1(2)}$ are determined by appropriate phases.\footnote{The phases are odd functions of relevant momenta $\delta(-k) = -\delta(k)$. For instance, from first equation in (10):

$$g_1 = g e^{i\delta(k_c)} = g \cos \delta(k_c) + ig \sin \delta(k_c) = g \cos \delta(k_c) + ik_c g \frac{\sin \delta(k_c)}{k_c}$$

At considered low energy one can safely confined by linear term in phases dependence on momenta:

$$g_1 = d_c^{(1)} + ik_c a_c^{(1)}; \quad g_2 = d_c^{(2)} + ik_c a_c^{(2)}$$

$$h_1 = d_n^{(1)} + ik_n a_n^{(1)}; \quad h_2 = d_n^{(2)} + ik_n a_n^{(2)}$$

(15)}
Substituting these relations in expressions (12), (13) after cumbersome, but simple algebra we obtain the energy dependence of S-matrix elements in the two channel case:

\[
S_{cc} = \frac{(1 + i k_c a_{cc})(1 - i k_c a_{nn}) - k_n k_c a_x^2}{(1 - i k_c a_{cc})(1 - i k_c a_{nn}) + k_n k_c a_x^2}
\]

\[
S_{nn} = \frac{(1 + i k_n a_{nn})(1 - i k_c a_{cc}) - k_n k_c a_x^2}{(1 - i k_c a_{cc})(1 - i k_n a_{nn}) + k_n k_c a_x^2}
\]

\[
S_{cn} = S_{nc} = \frac{2i \sqrt{k_c k_n a_x}}{(1 - i k_c a_{cc})(1 - i k_n a_{nn}) + k_n k_c a_x^2}
\]

where:

\[
a_{nn} = \frac{a_n^{(2)} d_c^{(1)} - a_n^{(1)} d_c^{(2)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}}, \quad a_{cc} = \frac{a_c^{(1)} d_c^{(2)} - a_c^{(2)} d_c^{(1)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}},
\]

\[
a_x = \frac{a_n^{(2)} d_c^{(1)} - a_n^{(1)} d_c^{(2)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}}, \quad a_{cc} = \frac{a_c^{(1)} d_c^{(2)} - a_c^{(2)} d_c^{(1)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}}
\]

(16)

Now we are in the position to get the dependence of matrix elements (1) on pairs momenta \(k_c, k_n\). Introducing the real combinations:

\[
M_{0c} = \frac{I_1 d_n^{(2)} - I_2 d_n^{(1)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}}, \quad M_{0n} = \frac{-I_1 d_c^{(2)} + I_2 d_c^{(1)}}{d_n^{(2)} d_c^{(1)} - d_n^{(1)} d_c^{(2)}}
\]

(17)

and making use the expressions (8),(12), (13),(15) we obtain our final result:

\[
M_c = M_{0c} \frac{1 - i k_n a_{nn}}{D} + i k_n M_{0n} \frac{a_x}{D} \quad M_n = M_{0n} \frac{1 - i k_c a_{cc}}{D} + i k_c M_{0c} \frac{a_x}{D}
\]

\[
D = (1 - i k_c a_{cc})(1 - i k_n a_{nn}) + k_n k_c a_x^2
\]

(18)

(19)

For applications it is more convenient to rewritten these relations through the amplitudes of elastic pion-pion scattering \(f_{cc}, f_{nn}\) and charge exchange \(f_x\):

\[
M_c = M_{0c}(1 + i k_c f_{cc}) + i k_n M_{0n} f_x; \quad M_n = M_{0n}(1 + i k_c f_{nn}) + i k_c M_{0c} f_x
\]

\[
f_{cc} = \frac{a_{cc}(1 - i k_n a_{nn}) + i k_n a_x^2}{D}; \quad f_{nn} = \frac{a_{nn}(1 - i k_c a_{cc}) + i k_c a_x^2}{D}; \quad f_x = \frac{a_x}{D};
\]

(20)

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\footnote{The similar expressions are cited in the textbook \cite{15}, but with wrong numerator in the inelastic case.}

\footnote{The integrals \(I_{1,2}\) are real quantities.}

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These relations expressing the decay matrix elements (1) through the amplitudes of pion-pion scattering are the main result of present work. Their application to $K \rightarrow 3\pi$ and $K^\pm \rightarrow \pi^\pm \pi^\mp e^\pm \nu$ decays allow us \cite{10, 12} to take into account the electromagnetic interaction among the charged pions in the final state for any invariant mass of the pion pair.

The first terms in the expansion (20) coincide with appropriate expressions in \cite{5, 6}, i.e. the $M_{0c}, M_{0n}$ introduced above (see eq. (18)) can be interpreted as so called "unperturbed" amplitudes introduced in \cite{5}.

The two channel task considered in the present work permits to estimate the accuracy of the scattering lengths values extracting from experimental data on kaons decays. Moreover obtained expressions allows one to correctly take into account the electromagnetic effects in the final state not only above the charged pions production threshold, but also for bound states. \cite{10, 12}

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