Cross-cavity quantum Rabi model

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Received 13 April 2016, revised 8 August 2016
Accepted for publication 12 August 2016
Published 19 September 2016

Abstract
We introduce the cross-cavity quantum Rabi model describing the interaction of a single two-level system with two orthogonal boson fields and propose its quantum simulation by two-dimensional, bichromatic, first-sideband driving of a single trapped ion. We provide an introductory survey of the model, including its diagonalization in the two-level system basis, numerical spectra and its characteristics in the weak, ultra strong and deep-strong coupling regimes. We also show that the particular case of identical field frequencies and couplings allows us to cast the model as two parity deformed oscillators in any given coupling regime.

Keywords: quantum electrodynamics, trapped ion, Rabi model

(Some figures may appear in colour only in the online journal)

1. Introduction

The Rabi model [1, 2] is an integrable model describing the interaction of atomic angular momentum with an external classical magnetic field. The introduction of a quantum field instead of a classical field produces the quantum Rabi model (QRM); e.g. the interaction of just the single neutral atom with a quantum field, under minimal coupling, the long wave and the two-level approximations, instead of a collection of them [3]. In the weak coupling regime, which happens in standard experiments where the coupling parameter is small compared to the field frequency, a rotating wave approximation (RWA) can be implemented and the QRM becomes the Jaynes–Cummings model (JCM). This was the first version of the QRM to be analytically solved [4]. The validity of the RWA is broken as the coupling strength grows and the full QRM remained unsolvable for any given coupling strength to field frequency ratio until recently [5]. The original and complementary proposals for an analytic solution [6–8] and a
series of proposals to realize the QRM in different experimental platforms, both quantum [9–16] and classical [17–20], have rekindled the interest on the QRM [21–27].

While extensions for the QRM where the number of qubits [28–33] or fields [34–36] are increased have been studied in the literature, here, we want to focus in one configuration that might prove interesting. Let us imagine a two-level atom interacting with the fields of two cavities in an orthogonal configuration, under minimal coupling and the long wavelength approximation, we can arrive to what we will call a cross-cavity QRM

\[
\hat{H} = \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_{j=1}^{2} g_j (\hat{a}_j^\dagger + \hat{a}_j) \hat{\sigma}_j,
\]

where the qubit has an energy gap \(\omega_3\) and is described by the Pauli matrices, \(\hat{\sigma}_j\), with \(j = 1, 2, 3\), the boson fields have frequencies \(\omega_j\) and are described by the annihilation (creation) operators, \(\hat{a}_j\) (\(\hat{a}_j^\dagger\)) with \(j = 1, 2\), and the strength of their couplings is provided by the parameters \(g_j\) with \(j = 1, 2\). Figure 1 shows a sketch of our gedankenexperiment where each of the orthogonal fields interact with its corresponding dipole component of the two-level system. This effective model can be related to the vibrational modes of a single polyatomic molecule interacting with an external magnetic field under linear Jahn–Teller coupling in just two dimensions instead of three [37, 38] and to the Raman adiabatic driving of a single four-level atom coupled to two cavity electromagnetic field modes [39]. Note that time evolution in the restricted case of weak couplings and fields of equal frequencies has been given in the literature [40, 41].

In the following, we will propose a trapped-ion quantum simulation of our cross-cavity QRM. We will diagonalize the Hamiltonian in the qubit basis following a Fulton–Gouterman (FG) approach [42, 43] and study a series of particular cases in order to provide a first stage survey of the model. First, we will consider the case of both fields weakly coupled to the qubit and recover a cross-cavity JCM that conserves the total number of excitations. We will show that this model reduces to that of a single field JCM coupled to a second field through a beam splitter and recover its spectrum and eigenstates analytically, using a perturbation approach up to second order, as well as numerically, computing the diagonalization of the auxiliary FG field Hamiltonians. Next, we will show a case when the total number of excitations is no

![Figure 1. Sketch of the cross-cavity Rabi model, two orthogonal field modes interact under minimal coupling and long wavelength approximation with one two-level system.](image-url)
longer conserved but there exists a different constant of motion given by the difference between the excitation number of the fields plus the excitation number of the qubit. In this case, the field frequencies are degenerate and show identical coupling to the qubit. Interestingly, the model reduces to a Hamiltonian describing one field coupled to the qubit under the JCM and the other under the anti-JCM and diagonalization in the qubit basis yields effective auxiliary FG Hamiltonians corresponding to two parity deformed oscillators for any given coupling parameter. Here, we can only provide numerical results. Then, we will move to numerically study two regimes where we cannot recover information about the constants of motion. In the first of these regimes one field is weakly coupled to the qubit and the other field coupling belongs to the deep-strong coupling regime and, in the second regime, both fields are coupled to the qubit in the deep-strong coupling regimes.

2. Trapped-ion quantum simulation

In order to leave the gedankenexperiment behind and experimentally motivate the study of our cross-cavity QRM, let us extend the recent proposal to simulate the standard QRM with trapped ions \[16\]. We will use two orthogonal pairs of bichromatic driving fields instead of just one

\[
\hat{H}_{\text{II}} = \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \left\{ \nu_j \hat{a}^+_j \hat{a}_j + \sum_{k=-1,1} \Omega_{j,k} \cos \{ \eta_{j,k} (\hat{a}^+_j + \hat{a}_j) \} - \omega_{j,k} t + \phi_{j,k} \hat{\sigma}_j \right\}.
\]

(2)

Here, the two-level trapped-ion is described by the energy gap \(\omega_3\) and the Pauli matrices \(\hat{\sigma}_j\) with \(j = 1, 2, 3\), the quantized center of mass vibration modes by the mechanical oscillation frequencies \(\nu_j\) and the annihilation (creation) operators \(\hat{a}_j (\hat{a}^+_j)\) with \(j = 1, 2, 3\), and we have two red- and blue-detuned, \(k = -1\) and \(k = 1\) in that order, classical driving lasers with frequencies tuned to the first sideband transitions plus some small detuning \(\delta_{j,k}\),

\[
\omega_{j,k} = \omega_3 + k \nu_j + \delta_{j,k},
\]

(3)

these classical fields have associated Lamb–Dicke parameters \(\eta_{j,k}\), phases \(\phi_{j,k}\), and couple to their corresponding dipole components with strength \(\Omega_{j,k}\). Expressing the trigonometric functions in exponential form, using the disentangling property \(e^{ij\hat{a}^+\hat{a}} = e^{-\frac{1}{2}(\hat{a}^+ + \hat{a})^2} e^{ij\hat{a}^+} e^{ij\hat{a}}\), the power series expansion of the exponential, and moving into the rotating frame defined by the uncoupled part of the Hamiltonian

\[
\hat{H}_0 = \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \nu_j \hat{a}^+_j \hat{a}_j,
\]

(4)

yields an effective interaction Hamiltonian after some manipulation

\[
\hat{H}_I = \sum_{j,k} \frac{(-i)^{j-1}}{2} \Omega_{j,k} e^{-\frac{i}{2} \eta_{j,k} \hat{\sigma}_j} \times \left[ \sum_{p,q=0}^{\infty} \frac{(-i \eta_{j,k})^p (-i \eta_{j,k})^q}{p! q!} \hat{a}_j^p \hat{a}^+_j \hat{a}^+_k \hat{a}_k \ e^{i (2 \omega_3 + (p - q) \nu_j + \delta_{j,k})} e^{-i \nu_j \hat{\sigma}_j} \right] \hat{\sigma}_j + \text{h.c.} \ 
\]

(5)
Typically, the ion energy gap is in the optical region and large compared to all other parameters, \( \omega_3 \gg \nu_j, \delta_{j,k}, \Omega_{j,k} \), thus the terms oscillating at optical frequencies, \( 2\omega_3 + (p - q + k)\nu_j + \delta_{j,k} - \phi_{j,k} \), will average to zero in any realistic measurement scenario. This allows us to focus on the approximate effective interaction Hamiltonian

\[
\hat{H}_\text{eff} \approx \sum_{j,k,r,s} \frac{(-i)^{j-1}}{2} \Omega_{j,k} e^{-i\frac{\pi}{2} |\eta_{j,r,k}|^2} \langle \hat{a}_j \rangle \langle \hat{a}_k \rangle \hat{a}_j \hat{a}_k e^{i\left(\nu_j - \delta_{j,k} + \phi_{j,k}\right) t} + \text{h.c.}.
\]

(6)

After this optical RWA, we can do a mechanical RWA where those terms rotating at frequencies proportional to \( \nu_j \) average to zero if and only if the mechanical vibration frequency is larger than the first sideband detunings and coupling strengths, \( \nu_j \gg \delta_{j,k}, e^{-\frac{3}{2} |\eta_{j,r,k}|^2} \Omega_{j,k} \). After some manipulation, we can write the following

\[
\hat{H}_\text{eff} = \frac{\eta_{1,-1}}{2} \Omega_{1,-1} \left[ e^{-\frac{1}{2} |\eta_{1,-1}|^2} \left\{ \frac{\hat{a}_1^\dagger \hat{a}_1!}{(\hat{a}_1^\dagger \hat{a}_1 + 1)!} L_{\hat{a}_1}^{|\eta_{1,-1}|^2} \hat{a}_1 \hat{a}_1 e^{-i\delta_{1,-1} t} e^{i\phi_{1,-1} t} + \text{h.c.} \right\} \right] + \text{h.c.}
\]

(7)

where the function \( L_{\hat{a}}^{|\eta|^2}(x) \) is a generalized Laguerre polynomial. At this point, we should note that working in the Lamb–Dicke regime, \( \eta_{j,k} \sqrt{\langle \hat{a}_j^\dagger \hat{a}_j \rangle} \ll 1 \), and an adequate choice of parameters

\[
\phi_{1,-1} = \phi_{1,1} = \phi_{2,1} = -\frac{\pi}{2}, \quad \phi_{2,-1} = \frac{\pi}{2},
\]

\[
\eta_{j,k} \Omega_{j,k} \exp^{-\frac{1}{2} |\eta_{j,k}|^2} = \eta_{j,k} \Omega_{j,k} \exp^{-\frac{1}{2} |\eta_{j,k}|^2},
\]

(8)

allows us to write

\[
\hat{H}_\text{eff} \approx g_1 \left[ \hat{a}_1^\dagger e^{\frac{i}{2} (\delta_{1,-1} - \delta_{1,1}) t} \hat{a}_1 e^{-\frac{i}{2} (\delta_{1,1} - \delta_{1,-1}) t} \right] \hat{a}_1 \hat{a}_1 e^{-i\delta_{1,-1} t} e^{i\phi_{1,-1} t} + \hat{a}_2 e^{-\frac{i}{2} (\delta_{2,1} + \delta_{2,-1}) t} \right] \hat{a}_2 \hat{a}_2 e^{i\delta_{2,1} t} e^{-i\phi_{2,1} t} - \hat{a}_2 \hat{a}_2 e^{-\frac{i}{2} (\delta_{2,-1} + \delta_{2,1}) t} + \hat{a}_2 \hat{a}_2 e^{i\delta_{2,-1} t} e^{-i\phi_{2,-1} t} \right].
\]

(9)

We can make another transformation and move into the uncoupled rotating frame

\[
\hat{H}' = -\frac{1}{4} (\delta_{1,-1} + \delta_{1,1}) \hat{a}_3 + \frac{1}{2} \sum_{j=1}^2 (\delta_{j,1} - \delta_{j,-1}) \hat{a}_j \hat{a}_j,
\]

(10)

to recover the cross-cavity QRM in equation (1) with parameters

\[
\omega_3 = -\frac{1}{2} (\delta_{1,-1} + \delta_{1,1}) = -\frac{1}{2} (\delta_{2,1} + \delta_{2,-1}),
\]

(11)

\[
\omega_j = \frac{1}{2} (\delta_{j,1} - \delta_{j,-1}),
\]

(12)
\begin{equation}
g_j = \eta_{j\pm 1} \Omega_{j\pm 1} e^{\pm i \eta_{j\pm 1} t}.
\end{equation}

Note, the definition of the qubit energy gap \( \omega_3 \) imposes the restriction that the sum of sideband driving field detunings for each mode must be equal, \( \delta_{1,-1} + \delta_{1,1} = \delta_{2,-1} + \delta_{2,1} \). Nevertheless, this gives us enough freedom to realize the cross-cavity QRM in a vast range of parameter sets.

3. Diagonalization in the qubit basis

The feasibility of a quantum simulation of the cross-cavity QRM gives us a reason to explore the solution of this model. In particular, finding the eigenvalues and eigenstates for the closed system may simplify the study of dissipation in the two-level system, which is a necessity in order to compare with experimental measurements.

We will first diagonalize our cross-cavity QRM in the two-level system basis following a FG approach [42, 43]. For this, we rewrite the cross-cavity QRM Hamiltonian in terms of the qubit raising and lowering operators

\begin{equation}
\hat{\mathcal{H}} = \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \omega_j \hat{a}_j \hat{a}_j^\dagger + \left[ g_1 (\hat{a}_1^\dagger + \hat{a}_1) - i g_2 (\hat{a}_2^\dagger + \hat{a}_2) \right] \hat{\sigma}_+ + \text{h.c.},
\end{equation}

and, in order to avoid imaginary couplings, we will perform a \( \pi/2 \) rotation around the second field photon number, \( \hat{a}_2^\dagger \hat{a}_2 \),

\begin{equation}
\hat{\mathcal{H}}_R = \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \omega_j \hat{a}_j \hat{a}_j^\dagger + \left[ g_1 (\hat{a}_1^\dagger + \hat{a}_1) + g_2 (\hat{a}_2^\dagger - \hat{a}_2) \right] \hat{\sigma}_+ + \text{h.c.}
\end{equation}

Now, we can use Schwinger two-boson representation of \( SU(2) \) and effect a rotation around \( \hat{J}_y = -i (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger) / 2 \), \( \hat{D}_y(\theta) = e^{\frac{i \theta}{2} (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger)} \) with \( \tan \theta/2 = g_2/g_1 \), to obtain the following

\begin{align}
\hat{\mathcal{H}}_D &= \frac{1}{2} \omega_3 \hat{\sigma}_3 + \sum_{j=1}^{2} \Omega_j \hat{a}_j \hat{a}_j^\dagger + \gamma (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger)
&\quad + \frac{1}{\sqrt{g_1^2 + g_2^2}} \left[ g_1^2 (\hat{a}_1^\dagger - \hat{a}_1) - g_2 g_2 (\hat{a}_2^\dagger + \hat{a}_2) \right] \hat{\sigma}_1 \\
&\quad + \frac{i}{\sqrt{g_1^2 + g_2^2}} \left[ g_2^2 (\hat{a}_1^\dagger + \hat{a}_1) + g_1 g_2 (\hat{a}_2^\dagger - \hat{a}_2) \right] \hat{\sigma}_2,
\end{align}

where we have used \( \hat{\sigma}_c = (\hat{\sigma}_1 \pm i \hat{\sigma}_2)/2 \) and we have defined the following mixed field frequencies and beam-splitter parameter

\begin{equation}
\Omega_1 = \frac{\omega_1 g_1^2 + \omega_2 g_2^2}{g_1^2 + g_2^2}, \quad \Omega_2 = \frac{\omega_2 g_1^2 + \omega_1 g_2^2}{g_1^2 + g_2^2}, \quad \gamma = \frac{g_1 g_2}{g_1^2 + g_2^2} (\omega_2 - \omega_1).
\end{equation}

A rotation of \( \pi/4 \) around \( \hat{\sigma}_2 \) yields a Hamiltonian of the FG type [42, 43]

\begin{equation}
\hat{\mathcal{H}}_{FG} = \hat{A} \hat{\sigma}_3 + \hat{B} \hat{\sigma}_1 + \hat{C} \hat{\sigma}_2 + \hat{D} \hat{\sigma}_3,
\end{equation}
with auxiliary operators for our rotated cross-cavity QRM

\[
\begin{align*}
\hat{A}_D &= \Omega_1 \hat{a}_1^\dagger \hat{a}_1 + \Omega_2 \hat{a}_2^\dagger \hat{a}_2 + \gamma (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger), \\
\hat{B}_D &= -\frac{1}{2} \omega_3, \\
\hat{C}_D &= \frac{i}{\sqrt{g_1^2 + g_2^2}} [g_1^2 (\hat{a}_1^\dagger \hat{a}_1) - g_2^2 (\hat{a}_2^\dagger \hat{a}_2) + g_1 g_2 (\hat{a}_1 \hat{a}_2 - \hat{a}_2 \hat{a}_1)], \\
\hat{D}_D &= \frac{i}{\sqrt{g_1^2 + g_2^2}} [g_1^2 (\hat{a}_1^\dagger \hat{a}_1) - g_2^2 (\hat{a}_2^\dagger \hat{a}_2) - g_1 g_2 (\hat{a}_1 \hat{a}_2 + \hat{a}_2 \hat{a}_1)].
\end{align*}
\]

(19)

In order to diagonalize this in the two-level basis, we need an operator \( \hat{R} \) such that

\[
[\hat{R}, \hat{A}] = [\hat{R}, \hat{B}] = [\hat{R}, \hat{C}] = [\hat{R}, \hat{D}] = 0.
\]

(20)

We can choose the boson field parity

\[
\hat{R} = \hat{U}_{12} = e^{i\pi \sum_n \hat{a}_n^{\dagger} \hat{a}_n},
\]

(21)

to write a FG unitary transformation

\[
\hat{U}_{FG} = \frac{1}{2\sqrt{2}} [(1 + \hat{R})(\hat{\sigma}_1 + \hat{\sigma}_3) + (1 - \hat{R})(\hat{\sigma}_1 - i\hat{\sigma}_3)],
\]

(22)

that diagonalizes our cross-cavity QRM Hamiltonian in the qubit basis

\[
\hat{H}_{FG}^{(D)} = \hat{U}_{FG} \hat{H}_{FG} \hat{U}_{FG}^{\dagger},
\]

(23)

\[
= (\hat{A} + \hat{D}) \hat{1} + (\hat{B} - i\hat{C}) \hat{R} \hat{\sigma}_3.
\]

(24)

Here, the diagonal form of our Hamiltonian in the qubit basis is

\[
\hat{H}_{D}^{(D)} = \hat{H}_{D}^{(+)} |e\rangle \langle e| + \hat{H}_{D}^{(-)} |g\rangle \langle g|,
\]

(25)

with the auxiliary Hamiltonians in terms of just the two field modes

\[
\begin{align*}
\hat{H}_{D}^{(+)} &= \Omega_2 \hat{a}_2^\dagger \hat{a}_2 + \gamma (\hat{a}_1 \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1^\dagger) + \frac{1}{2} \omega_3 \hat{N}_{12} \\
&\quad + \frac{1}{\sqrt{g_1^2 + g_2^2}} [\hat{a}_1^\dagger (g_1^2 \hat{N}_{12} + g_2^2 \hat{N}_{12}) + \hat{a}_2^\dagger (g_1^2 \hat{N}_{12} + g_2^2 \hat{N}_{12})] \\
&\quad - \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} [\hat{a}_2^\dagger (1 \mp \hat{N}_{12}) + (1 \mp \hat{N}_{12}) \hat{a}_2].
\end{align*}
\]

(26)

These field Hamiltonians describe two driven boson fields interacting through a beam-splitter. The driving function depends on the parity of both fields. These coupled and driven oscillators may be feasible of diagonalization using Bargmann representation and we will address this in a future manuscript. Here, we are concerned with just a survey of the possibilities provided by our cross-cavity QRM.

### 4. Weak coupling regime: cross-cavity JCM

When both fields are weakly coupled to the qubit, we can start from the cross-cavity QRM and move into the uncoupled rotating frame, \( \hat{H}_0 \), implement a RWA to neglect terms with frequency \( \Delta_j = \omega_j + \omega_3 \), and keep the terms with frequencies
Then, we move into a frame given by the free boson fields
\[
\hat{H}_n = \sum_j \delta_j \hat{a}_j^\dagger \hat{a}_j,
\]
and, after implementing a rotation of \(\pi/2\) around the frame defined by the number of bosons in the second mode, we obtain an effective cross-cavity JC model
\[
\hat{H}_{cJC} = \delta_1 \hat{a}_1^\dagger \hat{a}_1 + \delta_2 \hat{a}_2^\dagger \hat{a}_2 + g_1 (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + g_2 (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1^\dagger).
\]
Either the original cross-cavity simulation with weak couplings or this effective JC model for the two fields can be implemented in our trapped ion simulation. Note that the cross-cavity JC model conserves the total excitation number and, thus, parity
\[
\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} (\sigma_1 + 1), \quad \hat{\Pi} = e^{i\eta \hat{N}},
\]
such that \([\hat{N}, \hat{H}_{cJC}] = [\hat{\Pi}, \hat{H}_{cJC}] = 0\). As a result, in order to find the spectrum, we could stop here and partition the corresponding Hilbert space into subspaces of dimension 2\(\langle \hat{N} \rangle + 1\) for each and every average value of the excitation number \(\langle \hat{N} \rangle = 0, 1, 2, \ldots\). Then, the spectra should be brought together and arranged with utmost care not to skip elements. But let us keep with the FG diagonalization and make the rotation around \(\hat{J}_y\) mentioned before to obtain an effective Hamiltonian
\[
\hat{H}_D = \sum_{j=1}^{2} \Omega_j \hat{a}_j^\dagger \hat{a}_j + \gamma (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \sqrt{g_1^2 + g_2^2} (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1^\dagger),
\]
describing a standard JC model interacting with a second boson field through a beam splitter with modified parameters
\[
\Omega_1 = \frac{\delta_1 g_1^2 + \delta_2 g_2^2}{g_1^2 + g_2^2}, \quad \Omega_2 = \frac{\delta_2 g_1^2 + \delta_1 g_2^2}{g_1^2 + g_2^2}, \quad \gamma = \frac{g_1 g_2}{g_1^2 + g_2^2} (\omega_2 - \omega_1).
\]
Let us make a second stop here and notice that, for near resonance frequencies, we could treat the beam splitter term as a perturbation for the JC model and approximate the spectrum and eigenstates. For the sake of space, we will only write the unperturbed energy in the original frame
\[
E^{(0)}_{\pm} = \left( N + \frac{1}{2} \right) \omega_3 + \left( n + \frac{1}{2} \right) \Omega_1 + \left( N - n \right) \Omega_2 \pm \frac{1}{2} \Omega_R(n),
\]
\[
E^{(0)}_{0} = \left( N - \frac{1}{2} \right) \omega_3.
\]
Note that it involves the total excitation number of the subspace, \(N\), the ladder label of the dressed state for the JC model, \(n\), and the Rabi frequency \(\Omega_R(n)\) that goes as the square root of the dressed state label for resonance and near-resonance conditions.

Let us continue and move this Hamiltonian into a FG form by defining the auxiliary operators,
that lead to choosing the parity of just the boson fields as the auxiliary operator.

Finally, we get a weak-coupling Hamiltonian diagonalized in the qubit basis

\[ \hat{H}_{JC} = \Omega_1 \hat{a}_1 + \Omega_2 \hat{a}_2 + \gamma (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger), \quad \hat{B}_{JC} = 0, \]
\[ \hat{C}_{JC} = \frac{1}{2} \sqrt{g_1^2 + g_2^2} (\hat{a}_1 - \hat{a}_1^\dagger), \quad \hat{D}_{JC} = \frac{1}{2} \sqrt{g_1^2 + g_2^2} (\hat{a}_1 + \hat{a}_1^\dagger), \] (36)

Figure 2. (a) A thousand scaled energies of the spectra and (b) logarithm of the absolute value of the relative error between the numeric results, \( \Delta \epsilon_j = E_j^{\text{cQRM}} - E_j^{\text{FG}} \), for the cross-cavity QRM in the weak coupling regime, \( \omega_1 = \omega_2 = \omega_3 = \omega, \ g_1 = 0.001 \omega \) and \( g_2 = 0.002 \omega \).

describe two fields coupled by a beam splitter, one of them nonlinearly driven by a function proportional to the two field parity. The second conserved quantity and a closed form for the spectrum escapes our efforts at the moment but figure 2(a) shows a thousand eigenenergies for the cross-cavity QRM with parameters in the weak coupling regime for both fields and figure 2(b) shows the logarithm of the absolute value of the relative error between the numeric and analytic spectra

\[ \Delta \epsilon_j = E_j^{\text{cQRM}} - E_j^{\text{FG}}. \] (39)

We also calculated the spectra via perturbation theory using second order correction. The error for these results is not shown but yields values similar to those presented. We want to make emphasis, again, that care must be exerted when ordering the analytic eigenvalues.

Before moving on to different regimes, we want to bring forward Peres lattices [44] as a tool that has been used to visually search for regular or irregular behavior of the spectrum in Dicke models [45, 46]. They work in the following way, if a Hamiltonian is integrable and we plot the average of conserved quantities for a given eigenstate versus the corresponding eigenvalue, it will form a regular lattice of points because each eigenstate can be labeled by the quantum numbers provided by the constants of motion. If we add a perturbation to the
system, rendering it nonintegrable, the plot of the former constants of motion mean value will deform according to the perturbation; a small perturbation does not completely destroy the regularity of the lattice but a large one does. Thus, a Peres lattice may work as a visual tool to qualitatively address spectral regularity, or lack thereof. For example, most probably, the cross-cavity JC model is integrable due to the form casting it as a single field JC model where the field is coupled through a beam splitter to another field. We found that the total excitation is a constant of motion, thus, we obtain a regular lattice if we plot the mean value of the excitation number versus the corresponding energy as expected, figure 3. Still, we need find a second constant of motion to confirm if the model is integrable or not.

5. Fields with identical frequencies and couplings

We can provide a small amount of intuition for all coupling regimes if we consider two boson fields with the same frequency, $\omega_1 = \omega_2 = \omega$, and identical coupling strengths, $g_1 = g_2 = g$. In this exceptional configuration, we can follow the FG diagonalization shown in section 3 step by step, with the slight deviation of introducing a rotation of $\omega_3$ around the operator

$$\hat{\eta} = -\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} (\hat{\delta}_3 + 1),$$

composed from the conserved operators from JC and anti-JC dynamics [47]. This slight change allows us to recover an effective Hamiltonian where the first field mode is coupled to the qubit under anti-JC dynamics and the second under JC dynamics

$$\hat{H}_{\text{exc}} = \sum_j \delta_j \hat{a}_j^\dagger \hat{a}_j + \sqrt{2} g \{ (\hat{a}_1^\dagger - \hat{a}_2) \hat{\sigma}_z + (\hat{a}_1 - \hat{a}_2^\dagger) \hat{\sigma}_z \},$$

with field detunings provided by the following expressions

$$\delta_1 = \omega + \omega_3, \quad \delta_2 = \omega - \omega_3.$$  

This Hamiltonian will conserve the operator $\hat{\eta}$, $[\hat{\eta}, \hat{H}_{\text{exc}}] = 0$, which relates to Schwinger's two-boson SU(2) representation as $\hat{J}_z = (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)/2$; in other words, it is the total excitation number, or population inversion, of the combined SU(2) representations. If we choose the mean value of operator $\hat{\eta}$ to partition the corresponding Hilbert space, we will

![Figure 3. Mean value of the excitation number, $\langle \hat{N} \rangle$, for the eigenstates of the cross-cavity QRM in the weak coupling regime, $\omega_1 = \omega_2 = \omega_3 = \omega$, $g_1 = 0.001\omega$ and $g_2 = 0.002\omega$. Note the ordered lattice form due to the fact that the total excitation number is conserved by the cross-cavity JC model.](image-url)
finish with infinite dimensional subspaces for each $\langle \hat{n} \rangle = 0, \pm 1, \pm 2, \ldots$ Note that either the original cross-cavity QRM simulation with degenerate field frequencies and balanced couplings or this effective $aJC$–$JC$ Hamiltonian can be implemented in our trapped ion simulation. Following the rest of the procedure, we produce the auxiliary operators

$$
\begin{align*}
\hat{A}_{\text{exc}} &= \delta_1 \hat{a}_1^\dagger \hat{a}_1 + \delta_2 \hat{a}_2^\dagger \hat{a}_2, \quad \hat{B}_{\text{exc}} = 0, \\
\hat{C}_{\text{exc}} &= \frac{1}{\sqrt{2}} g(\hat{a}_1^\dagger - \hat{a}_1 + \hat{a}_2^\dagger - \hat{a}_2), \quad \hat{D}_{\text{exc}} = \frac{1}{\sqrt{2}} g(\hat{a}_1^\dagger + \hat{a}_1 - \hat{a}_2^\dagger - \hat{a}_2).
\end{align*}
$$

(43)

Here, the diagonal form of our exceptional Hamiltonian in the qubit basis is

$$
\hat{H}^{(D)}_{\text{exc}} = \hat{H}^{(+)}_{\text{exc}} |e\rangle \langle e| + \hat{H}^{(-)}_{\text{exc}} |g\rangle \langle g|,
$$

(44)

with the auxiliary field Hamiltonians

$$
\hat{H}^{(+)}_{\text{exc}} = \sum_{j=1}^2 \delta_j \hat{a}_j^\dagger \hat{a}_j + g(\hat{A}_{\pm j}^\dagger + \hat{A}_{\pm j}),
$$

(45)

where we have defined nonlinear parity deformed operators

$$
\begin{align*}
\hat{A}_{\pm j} &= -(-1)^j \frac{1}{\sqrt{2}} \hat{a}_j(1 \pm (-1)^j)\hat{1}_{12}, \\
\hat{A}_{\pm j}^\dagger &= -(-1)^j \frac{1}{\sqrt{2}} (1 \pm (-1)^j)\hat{1}_{12}\hat{a}_j^\dagger.
\end{align*}
$$

(46)

(47)

that realize a Wigner–Heisenberg algebra [48, 49]

$$
\begin{align*}
[\hat{a}_j^\dagger, \hat{a}_j, \hat{A}_{\pm j}] &= -\hat{A}_{\pm j}, \\
[\hat{a}_j^\dagger, \hat{a}_j, \hat{A}_{\pm j}^\dagger] &= \hat{A}_{\pm j}^\dagger, \\
[\hat{A}_{\pm j}, \hat{A}_{\pm k}^\dagger] &= -(-1)^{(j+k)/2}\hat{1}_{12}\hat{a}_j^\dagger, \\
[\hat{A}_{\pm j}, \hat{A}_{\pm k}] &= 2\hat{1}_{12}\hat{a}_j^\dagger \hat{a}_k^\dagger, \\
[\hat{A}_{\pm 1}, \hat{A}_{\pm 2}] &= 2\hat{1}_{12}\hat{a}_1^\dagger \hat{a}_2^\dagger, \\
[\hat{A}_{\pm 1}^\dagger, \hat{A}_{\pm 2}^\dagger] &= 0.
\end{align*}
$$

(48)

(49)
At this point, we can diagonalize these two auxiliary nonlinear parity deformed oscillators to provide a numerical spectrum for degenerate field frequencies and balanced couplings. Figure 4(a) shows a thousand scaled energies, \( E_j/\omega \), from the spectra of the cross-cavity QRM with equal field frequencies and balanced couplings in the ultra strong coupling (USC) regime, \( \omega_1 = \omega_2 = g_1 = g_2 = \omega \). For the sake of comparison, we also implemented a spectral solver for the cross-cavity QRM using up to a hundred photons in each of the boson fields. Figure 4(b) shows the logarithm of the absolute value of the relative error between both numerical results. One must be careful while rotating results to the same reference frame and ordering the combined spectra provided by the two auxiliary nonlinear parity deformed oscillators. Note, our spectral solver using up to a hundred photons on the fields delivers about five thousand converged eigensates that we can trust under a convergence criterion involving the information on the tail of the eigenstate that we will discuss later on.

As mentioned before, the effective JC–aJC Hamiltonian share eigenbasis with the operator \( \hat{\eta} \). Thus, the mean value of this operator for the numerical eigenstates of the nonlinear parity deformed oscillators, once transformed back into the frame of \( \hat{H}_{\text{exc}} \), will form a so-called Peres lattice [44], figure 5, where, for a given subspace defined by a constant value of \( \langle \hat{\eta} \rangle = 0, \pm 1, \pm 2, \ldots \) in the frame of \( \hat{H}_{\text{exc}} \), there will be an infinite number of eigenstates as shown in the figure.

6. Combined weak and deep-strong coupling regimes

Another regime that allows for a perturbative approach is the case when one of the boson fields is weakly coupled to the qubit and the other field is in the deep-strong coupling regime. For example, let us take the first field in the deep-strong coupling regime \( g_j \gg \omega_j \) with \( j = 1, 2, 3 \), and the second in the weak coupling regime, \( g_2 \ll \omega_j \) with \( j = 1, 2, 3 \). In this regime, implementing a rotation of \( \pi/4 \) around \( \hat{\sigma}_2 \) and moving intro a frame defined by displacement operator defined as \( \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \) and auxiliary parameter \( \alpha = g_j/\omega_j \) yields a Hamiltonian of the FG type with auxiliary operators.
Here, the diagonal form of our combined weak and deep-strong coupling regimes Hamiltonian in the qubit basis is

\[ \hat{H}_D = \omega_1 (\hat{a}_1 \dagger \hat{a}_1 - \alpha^2) + \omega_2 (\hat{a}_2 \dagger \hat{a}_2 - \alpha^2), \]

\[ \hat{B}_D = \frac{i}{2} \left\{ -\frac{\pi}{4} [\hat{D}(2\alpha) + \hat{D}^\dagger(2\alpha)] - i g_2 (\hat{a}_1 \dagger + \hat{a}_2)(\hat{D}(2\alpha) - \hat{D}^\dagger(2\alpha)) \right\}, \]

\[ \hat{C}_D = \frac{i}{2} \left\{ -\frac{\pi}{4} [\hat{D}(2\alpha) - \hat{D}^\dagger(2\alpha)] - i g_2 (\hat{a}_1 \dagger + \hat{a}_2)(\hat{D}(2\alpha) + \hat{D}^\dagger(2\alpha)) \right\}, \]

\[ \hat{D}_D = 0. \]  

(50)

Although the contribution related to \( \omega_3 \) is of the order of the diagonal part, we can explore the use of perturbation theory in order to obtain some spectral information from the system.

The diagonal part of the auxiliary field Hamiltonian is provided by the states \( \{m, n, x\} \) where \( m \) and \( n \) are the number of photons in the Fock states for the fields and the qubit state is given by \( x = g, e \). The spectrum for this part does not depend on the qubit state

\[ E_{m,n,x}^{\omega_1} = \omega_1 (m - \alpha^2) - \omega_2 n, \]

and, thus, is two-fold degenerate but the degeneracy is lifted due to the fact that the each qubit state defines two mutually orthogonal subspaces. First and second order corrections due to the perturbation are given in the following

Figure 6. (a) A thousand scaled energies of the spectra and (b) logarithm of the absolute value of the squared root sum of the corresponding eigenstate tail for the cross-cavity QRM with field one in the deep-strong coupling and field two in the weak coupling regimes, \( \omega_1 = \omega_2 = \omega_3 = \omega, g_1 = 2\omega \) and \( g_2 = 0.0001\omega \).
where the $\hat{m}$th element of the displacement operator can be calculated in terms of Laguerre polynomials [50] or Tricomi hypergeometric function [51]. Perturbation theory for two-fold degenerate systems where the subspaces are orthogonal also provides us with viable corrections for the eigenstates but, again, the expression are not amenable for printed form. These corrections allow us to write the mean values for the total excitation number and $\hat{N}$-operator in the original frame. At the moment, our computation power does not allow us to explore true DSC regimes, and we restrict our analysis to coupling parameters $w_1 = g_{21}$ and $w_2 = g_{102}$ for a system on-resonance, $w_3 = \omega_3 = \omega_2 = \omega$. Figure 6(a) shows a thousand components of the spectra and figure 6(b) the logarithm of the squared root sum of the corresponding eigenstate tail. Here, neither the total excitation nor $\hat{N}$ are conserved and their Peres lattices are disordered, figures 7(a) and (b), respectively, but the behavior from previous cases can still be discerned through the disorder.

### 7. Deep-strong coupling regime

In order to complement our spectral survey, we implement a brute force solver in a subspace allowing up to a hundred bosons in each field mode. We will take as convergence measure the information content of the eigenstate tail [45]. In our case of two truncated Hilbert spaces, we can define it as

$$|	au_j| = \sqrt{\sum_{m,n} |c_{m,n,j}^{(j)}|^2},$$

where the complex number $c_{m,n,j}^{(j)}$ is the amplitude corresponding to the state with $m$ photons in the first field, $n$ photons in the second field and the qubit in the state $x = g, e$ of the $j$th
eigenstate of the Hamiltonian expressed in the standard basis of the truncated subspaces, either $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_q$ for the cross-cavity QRM or $\mathcal{H} = \mathcal{H}_d \otimes \mathcal{H}_2$ for the two nonlinear parity deformed driven oscillators, and we take the tail of the eigenstate as the last quarter of the truncated amplitudes, $m_{\text{min}} = \lceil 3 \dim \mathcal{H}_d/4 \rceil$ and $n_{\text{min}} = \lceil 3 \dim \mathcal{H}_2/4 \rceil$, where the operation $\lceil x \rceil$ rounds up $x$ to the next integer. Figure 8(a) shows a thousand scaled energies and figure 8(b) shows the logarithm of the information content within the eigenstate tail for the corresponding eigenstate. We used the following parameter values, $\omega_1 = \omega_2 = \omega_3 = \omega$, $g_1 = 2\omega$ and $g_2 = 2.3\omega$. In the DSC regime, we lack any knowledge about the constants of motion, thus we will get disordered Peres lattices of the excitation number, figure 9(a), or the operator $\hat{J}$, figure 9(b), instead of the ordered lattices we obtained in the exceptional case of equal field frequencies and couplings or the weak-coupling regime, in that order. Again, we
want to emphasize that care must be exerted in the ordering of eigenvalues because the inclusion of a second truncated Hilbert space most probably induces gaps in the spectrum.

8. Conclusion

We have proposed a cross-cavity QRM, where a single two-level system interacts with two orthogonal boson fields under minimal coupling and the long-wavelength approximation, and shown it is feasible of experimental quantum simulation in the trapped ion quantum electrodynamics platform.

We diagonalized our model in the two-level basis following a FG approach and showed that the model reduces to a Hamiltonian where one boson field is coupled to the qubit following the Jaynes–Cummings model and the other boson field follows anti-Jaynes–Cummings coupling for all given coupling parameters in the particular case of fields with same frequencies and couplings. In this peculiar regime, our model is equivalent to two parity deformed driven oscillators coupled via a beam splitter and conserves an operator that provides the total excitation number of the qubit and one field minus the excitation number of the other field. The latter is proportional to the $z$-component of Schwinger two-boson representation of $SU(2)$.

For the sake of completeness, we calculated the spectra in the weak coupling regime and a combination of one field in the weak coupling and the other in the deep-strong coupling regimes through perturbation theory, and provide a numerical spectra in the deep-strong coupling regime but leave a detailed study for future correspondence due to the complexity of the model. We plan to address the integrability of the model in each particular regime in future communications.

We want to emphasize that our cross-cavity Rabi model opens different research avenues to answer questions like the existence of a underlying closed algebra for nonlinear parity driven dual fields and its implications for the existence of entire functions to represent such systems within Bargmann formalism, in mathematical physics. It also relate to the problem of molecular vibration in the presence of external potentials that might allow for engineering of reaction rates in a goal closer to applied physics or physical chemistry where the Longuet-Higgins model was born, to mention just a couple of them.

Acknowledgments

C Huerta Alderete acknowledges financial support through CONACYT master studies grant #331166.

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