Polynomial solutions of $q$-Heun equation and ultradiscrete limit

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ABSTRACT
We study polynomial-type solutions of the $q$-Heun equation, which is related with quasi-exact solvability. The condition that the $q$-Heun equation has a non-zero polynomial-type solution is described by the roots of the spectral polynomial, whose variable is the accessory parameter $E$. We obtain sufficient conditions that the roots of the spectral polynomial are all real and distinct. We consider the ultradiscrete limit to clarify the roots of the spectral polynomial and the zeros of the polynomial-type solution of the $q$-Heun equation.

1. Introduction

It is widely known that classical orthogonal polynomials play important roles in mathematics and physics. Among them, the Legendre polynomial, the Chebyshev polynomial, the Gegenbauer polynomial and the Jacobi polynomial are essentially described by the hypergeometric function $\binom{2}{1}(\alpha, \beta; \gamma; z)$, which satisfies the hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{dy}{dz} - \alpha\beta y = 0. \quad (1)$$

It is a standard form of second-order Fuchsian differential equation with three regular singularities $\{0, 1, \infty\}$.

A $q$-difference analogue of the hypergeometric function is written as

$$\binom{2}{1}(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} x^n, \quad (\lambda, q)_n = \prod_{i=0}^{n-1} (1 - \lambda q^i), \quad (2)$$

and it is called the basis hypergeometric function or $q$-hypergeometric function. The basis hypergeometric function satisfies the basic (or $q$-difference) hypergeometric equation

$$(x - q)f(x/q) - ((a + b)x - q - c)f(x) + (abx - c)f(qx) = 0. \quad (3)$$

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Note that every coefficient of \( f(x/q), f(x) \) and \( f(qx) \) is linear in \( x \).

A standard form of second-order Fuchsian differential equation with four regular singularities \( \{0, 1, t, \infty\} \) is given by

\[
\frac{d^2y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-t} \right) \frac{dy}{dx} + \frac{\alpha \beta z - B}{z(z-1)(z-t)} y = 0 \tag{4}
\]

with the condition \( \gamma + \delta + \epsilon = \alpha + \beta + 1 \), and it is called Heun’s differential equation. Note that the condition \( \gamma + \delta + \epsilon = \alpha + \beta + 1 \) implies that the local exponents about \( z = \infty \) are \( \alpha \) and \( \beta \) (see [8]). The parameter \( B \) is called an accessory parameter, which is independent from the local exponents. As we shall explain later, polynomial solutions of the Heun equation have different features from the hypergeometric polynomials (the Jacobi polynomial and so on).

A \( q \)-difference analogue of Heun’s differential equation was given by Hahn [3] in 1971 as the form

\[
a(x)qg(x/q) + b(x)g(x) + c(x)g(qx) = 0 \tag{5}
\]

such that \( a(x), b(x), c(x) \) are polynomials which satisfy \( \text{deg}_x a(x) = \text{deg}_x c(x) = 2, a(0) \neq 0 \neq c(0) \) and \( \text{deg}_x b(x) \leq 2 \). Recently the \( q \)-Heun equation was recovered by two methods [14], one is by degeneration of Ruijsenaars–van Diejen operator [10,17], and the other is by specialization of the linear \( q \)-difference equation related with \( q \)-Painlevé VI equations [6]. We adopt the expression of the \( q \)-Heun equation as

\[
(x - q^{h_1 + 1/2} t_1)(x - q^{h_2 + 1/2} t_2)g(x/q) + q^{\alpha_1 + \alpha_2} (x - q^{l_1 - 1/2} t_1)(x - q^{l_2 - 1/2} t_2)g(qx)
- \{(q^{\alpha_1} + q^{\alpha_2})x^2 + Ex + q^{(h_1 + h_2 + l_1 + l_2 + \alpha_1 + \alpha_2)/2}(q^{\beta/2} + q^{-\beta/2})t_1 t_2\}g(x) = 0. \tag{6}
\]

Note that the parameter \( E \) in Equation (6) may be regarded as an accessory parameter, which was already pointed out by Hahn. By the limit \( q \to 1 \), we recover Heun’s differential equation (see [3,14]).

In this paper we investigate polynomial-type solutions of the \( q \)-Heun equation. Here we recall polynomial solutions of Heun’s differential equation (4) by following [2,13] (see also [8,18]). Set

\[
y = \sum_{n=0}^{\infty} \tilde{c}_n x^n, \quad (\tilde{c}_0 = 1), \tag{7}
\]

and substitute it to the differential equation (4). Then the coefficients satisfy \( t(n+1)(n+\gamma)\tilde{c}_{n+1} = [n \{ (n-1+\gamma) (1+t) + t\delta + \epsilon \} + B] \tilde{c}_n \)

\[
- (n-1+\alpha)(n-1+\beta)\tilde{c}_{n-1}, \quad (n = 1, 2, \ldots). \tag{8}
\]

If \( t \neq 0,1 \) and \( \gamma \notin \mathbb{Z}_{<0} \), then \( \tilde{c}_n \) is a polynomial in the accessory parameter \( B \) of degree \( n \) and we denote it by \( \tilde{c}_n(B) \). Moreover we assume that \( \alpha = -N \) or \( \beta = -N \) for some \( N \in \mathbb{Z}_{\geq 0} \). Let \( B_0 \) be a solution to the equation \( \tilde{c}_{N+1}(B) = 0 \). Then it follows from (8) for \( n = N + 1 \) that \( \tilde{c}_{N+1}(B_0) = 0 \). By applying (8) for \( n = N+2, N+3, \ldots \), we have \( \tilde{c}_n(B_0) = 0 \) for \( n \geq N + 3 \). Hence, if \( \tilde{c}_{N+1}(B_0) = 0 \), then the differential equation (4) has a non-zero polynomial solution. More precisely, we have the following proposition.
Proposition 1.1 ([8,18]): Assume that \( t \not\in \{0,1\} \), \( \gamma \not\in \mathbb{Z}_{\geq 0} \), \((\alpha + N)(\beta + N) = 0\) and \( N \in \mathbb{Z}_{\geq 0} \). If \( B \) is a solution to the equation \( \tilde{c}_{N+1}(B) = 0 \), then the differential equation (4) has a non-zero polynomial solution of degree no more than \( N \).

We call \( \tilde{c}_{N+1}(B) \) the spectral polynomial, and it does not appear in the setting of the hypergeometric polynomials because the hypergeometric equation does not have accessory parameters. It is fundamental to study the zeros of the spectral polynomial. Although the spectral polynomial \( \tilde{c}_{N+1}(B) \) may have non-real roots or multiple roots in general, we obtained a sufficient condition that the spectral polynomial has only distinct real roots in [2]. Namely, if \((\alpha + N)(\beta + N) = 0\), \( N \in \mathbb{Z}_{\geq 0} \), \( \delta, \epsilon \) and \( \gamma \) are real, \( \gamma > 0 \), \( \delta + \epsilon + \gamma + N > 1 \) and \( t < 0 \), then the equation \( \tilde{c}_{N+1}(B) = 0 \) has all its roots real and distinct. Note that it can be proved by applying the argument of Sturm sequence (see [2] for details).

On the \( q \)-deformed case, quasi-exact solvability of the \( q \)-Heun equation (i.e. invariance of some finite-dimensional subspace with respect to the \( q \)-Heun operator defined in Equation (10)) was shown in [15] and it indicates polynomial-type solutions of the \( q \)-Heun equation by diagonalizing the \( q \)-Heun operator on the finite-dimensional subspace. In this paper, we investigate the polynomial-type solutions of the \( q \)-Heun equation by using the recursive relations like Equation (8), and we also apply the argument of Sturm sequence. Recall that the \( q \)-Heun equation was introduced in Equation (6). Set \( \lambda_1 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 - \beta + 2)/2 \). Under the assumption that \(-\lambda_1 - \alpha_1(=N)\) is a non-negative integer and \( \beta \not\in \{1,2,\ldots,N\} \), we obtain the algebraic equation \( c_{N+1}(E) = 0 \), which is an analogue of \( \tilde{c}_{N+1}(B) = 0 \), such that the \( q \)-Heun equation has a solution of the form

\[
f(x) = x^{\lambda_1} \sum_{n=0}^{N} c_n(E_0)x^n,
\]

if \( E = E_0 \) is a solution to \( c_{N+1}(E) = 0 \) (see Proposition 2.2). We call \( c_{N+1}(E) \) the spectral polynomial of the \( q \)-Heun equation. We consider real-root property for the spectral polynomial \( c_{N+1}(E) \) in Theorem 3.2.

In general it would be impossible to solve the roots of the spectral polynomial \( c_{N+1}(E) \) explicitly as well as the spectral polynomial \( \tilde{c}_{N+1}(B) \) of the Heun equation. Then we adopt an idea from the ultradiscrete limit \( q \to +0 \), as the case of the \( q \)-Painlevé equations [4,5,16], in order to understand the roots of the spectral polynomial. In Section 4, we investigate the asymptotics of the roots of the spectral polynomial \( c_{N+1}(E) \) as \( q \to +0 \). In some cases the spectral polynomial \( c_{N+1}(E) \) tends to the \( q \)-Pochhammer function \((-c^{-1}q^{-d}x;q)_{N+1}(:=\prod_{k=0}^{N}(1+c^{-1}q^{-d+k}x))\) as \( q \to +0 \) for some \( c \in \mathbb{R}_{>0} \) and \( d \in \mathbb{R} \) and the asymptotics of roots of the spectral polynomial is described as \(-cq^d,-cq^{d-1},\ldots,-cq^{d-N}\). We can also obtain the behaviour of the polynomial-type solution of the \( q \)-Heun equation as \( q \to +0 \) for each root of the spectral polynomial, and it is described by a certain product of two \( q \)-Pochhammer functions (see Theorems 4.3 and 4.6), where similar functions appear in the literature [9,19]. Note that in some cases the roots of the spectral polynomial \( \tilde{c}_{N+1}(E) \) as \( q \to +0 \) may not be written as \(-cq^d,-cq^{d-1},\ldots,-cq^{d-N}\).

Recently, the \( q \)-Heun equation also appears in the study of degenerations of the Askey–Wilson algebra [1] from a more algebraic perspective, and it might be applicable...
for the study of solutions of the $q$-Heun equation in the form of the summation of the $q$-hypergeometric functions.

This paper is organized as follows. In Section 2, we consider the polynomial-type solution of the $q$-Heun equation and introduce the spectral polynomial $c_{N+1}(E)$. In Section 3, we show the real root property of the spectral polynomial. In Section 4, we analyse the solutions to $c_{N+1}(E) = 0$ by applying the ultradeer limit $q \to 0$. In Section 5, we give concluding remarks. In the Appendix, we introduce theorems on the ultradeer limit of an algebraic equation and continuation of the solutions as $q \to 0$. Throughout this paper, we assume $0 < q < 1$.

2. Polynomial-type solutions of the $q$-Heun equation

Let $A^{(4)}$ be the operator defined by

$$
A^{(4)} = x^{-1}(x - q^{h_1+1/2}t_1)(x - q^{h_2+1/2}t_2)T_q^{-1} + q^{\alpha_1+\alpha_2}x^{-1}(x - q^{h_1-1/2}t_1)(x - q^{l_2-1/2}t_2)T_q^{-1}
$$

$$
- \{q^{\alpha_1} + q^{\alpha_2}x + q^{(h_1+h_2+1/2+\alpha_1+\alpha_2)/2}(q^{\beta/2} + q^{-\beta/2})t_1t_2x^{-1}\},
$$

where $T_q^{-1}g(x) = g(x/q)$ and $T_qg(x) = g(qx)$. Then the $q$-Heun equation is written as

$$
(A^{(4)} - E)g(x) = 0,
$$

where $E$ is a constant. The action of $A^{(4)}$ to $x^\mu$ is written as

$$
A^{(4)}x^\mu = d^{(4),+}(\mu)x^{\mu+1} + d^{(4),0}(\mu)x^\mu + d^{(4),-}(\mu)x^{\mu-1},
$$

where

$$
d^{(4),+}(\mu) = q^{-\mu}(1 - q^{\alpha_1+\mu})(1 - q^{\alpha_2+\mu}),
$$

$$
d^{(4),0}(\mu) = -q^{-\mu}(q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + (q^{l_1}t_1 + q^{l_2}t_2)q^{\alpha_1+\alpha_2+2\mu-1/2}),
$$

$$
d^{(4),-}(\mu) = t_1t_2q^{h_1+h_2+1}q^{-\mu}(1 - q^{\mu-\lambda_1})(1 - q^{\mu-\lambda_2})
$$

and

$$
\lambda_1 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 - \beta + 2)/2,
$$

$$
\lambda_2 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 + \beta + 2)/2.
$$

In a special case, it is shown in [15] that the operator $A^{(4)}$ preserves a finite-dimensional space (see [12] for the Heun equation).

Proposition 2.1 ([15]): Let $\lambda \in \{\lambda_1, \lambda_2\}, \alpha \in \{\alpha_1, \alpha_2\}$ and assume that $-\lambda - \alpha (:= N)$ is a non-negative integer. Let $V^{(4)}$ be the space spanned by the monomials $x^{\lambda+k}$ ($k = 0, \ldots, N$), i.e.

$$
V^{(4)} = \{c_0x^{\lambda} + c_1x^{\lambda+1} + \cdots + c_Nx^{\lambda+N} \mid c_0, c_1, \ldots, c_N \in \mathbb{C}\}.
$$

Then the operator $A^{(4)}$ preserves the space $V^{(4)}$. 

**Proof:** The proposition follows from $d^{(4),-}(\lambda) = 0$ and $d^{(4),+}(\lambda + N) = 0$. See [15] for details.

Note that the values $\lambda_1$ and $\lambda_2$ in Equation (14) are exponents of $q$-Heun equation (Equation (11)) about $x=0$, and the values $\alpha_1$ and $\alpha_2$ are exponents about $x=\infty$ (see [15]).

Set

$$g(x) = x^\lambda \sum_{n=0}^{N} c_n x^n \quad (c_0 \neq 0),$$

and substitute it into Equation (11). Under the assumption of Proposition 2.1, we have

$$d^{(4),-}(\lambda) = 0, \quad d^{(4),+}(\lambda + N) = 0$$

and

$$c_1 d^{(4),-}(\lambda + 1) + c_0(d^{(4),0}(\lambda) - E) = 0,$$

$$c_n d^{(4),-}(\lambda + n) + c_{n-1}(d^{(4),0}(\lambda + n - 1) - E)$$

$$+ c_{n-2} d^{(4),+}(\lambda + n - 2) = 0, \quad (2 \leq n \leq N),$$

$$c_N(d^{(4),0}(\lambda + N) - E) + c_{N-1}d^{(4),+}(\lambda + N - 1) = 0.$$  \hspace{1cm} (17)

From now on, we investigate the case $\lambda = \lambda_1$. If $(\beta =)\lambda_2 - \lambda_1 \notin \{1, 2, \ldots, N\}$, then we have $d^{(4),-}(\lambda_1 + n) \neq 0$ for $n = 1, 2, \ldots, N$ and the coefficients $c_n$ ($n = 1, 2, \ldots, N$) are determined recursively. If we regard $E$ as an indeterminate and set $c_0 = 1$, then $c_n$ is a polynomial of $E$ of degree $n$ and we denote it by $c_n(E)$. Then we have

$$c_n(E)t_1t_2q^{h_1+h_2}(1 - q^n)(1 - q^{n-\beta})$$

$$- c_{n-1}(E)[Eq^{n-1+\lambda_1} + q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + (q^{l_1}t_1 + q^{l_2}t_2)q^{2(n-1+\lambda_1)+\alpha_1+\alpha_2-1/2}]$$

$$+ c_{n-2}(E)q(1 - q^{n-2+\lambda_1+\alpha_1})(1 - q^{n-2+\lambda_1+\alpha_2}) = 0,$$ \hspace{1cm} (18)

for $n = 1, 2, \ldots, N$, where $c_{-1}(E) = 0$ and $c_0(E) = 1$. We define the polynomial $c_{N+1}(E)$ by

$$c_{N+1}(E)t_1t_2q^{h_1+h_2}(1 - q^{N+1})(1 - q^{N+1-\beta})$$

$$- c_N(E)[Eq^{N+\lambda_1} + q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + (q^{l_1}t_1 + q^{l_2}t_2)q^{2(N+\lambda_1)+\alpha_1+\alpha_2-1/2}]$$

$$+ c_{N-1}(E)q(1 - q^{N-1+\lambda_1+\alpha_1})(1 - q^{N-1+\lambda_1+\alpha_2}) = 0$$ \hspace{1cm} (19)

in the case $N + 1 - \beta \neq 0$ and

$$c_{N+1}(E) = c_N(E)[Eq^{N+\lambda_1} + q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + (q^{l_1}t_1 + q^{l_2}t_2)q^{2(N+\lambda_1)+\alpha_1+\alpha_2-1/2}]$$

$$- c_{N-1}(E)q(1 - q^{N-1+\lambda_1+\alpha_1})(1 - q^{N-1+\lambda_1+\alpha_2})$$ \hspace{1cm} (20)

in the case $N + 1 - \beta = 0$.

**Proposition 2.2:** Let $\lambda_1$ be the value in Equation (14), $\alpha \in [\alpha_1, \alpha_2]$ and assume that $-\lambda_1 - \alpha (:= N)$ is a non-negative integer and $\beta \notin \{1, 2, \ldots, N\}$. Set $c_{-1}(E) = 0$, $c_0(E) = 1$ and we
determine the polynomials $c_n(E) \ (n = 1, \ldots, N)$ recursively by Equation (18). Assume that $E = E_0$ is a solution of the algebraic equation
\[
c_{N+1}(E) = 0, \tag{21}
\]
(see Equations (19), (20)). Then the q-Heun equation has a non-zero solution of the form
\[
f(x) = x^{λ_1} \sum_{n=0}^{N} c_n(E_0)x^n. \tag{22}
\]

**Proof:** The condition that the function $f(x)$ in Equation (22) satisfies the q-Heun equation is equivalent to Equation (18) substituted by $E = E_0$ for $n = 1, 2, \ldots, N$ and
\[
c_N(E_0)[q^{1/2}(q^{h_1}t_1 + q^{h_2}t_2) + Eq^{N+\lambda_1} + (q^{l_1}t_1 + q^{l_2}t_2)q^{2(N+\lambda_1)+\alpha_1+\alpha_2-1/2}
- c_{N-1}(E_0)q(1-q^{N-1+\lambda_1+\alpha_1})(1-q^{N-1+\lambda_1+\alpha_2}) = 0. \tag{23}
\]
Then Equation (18) substituted by $E = E_0$ for $n = 1, 2, \ldots, N$ follows from the definition of the polynomials $c_n(E) \ (n = 1, \ldots, N)$, and Equation (23) follows from $c_{N+1}(E_0) = 0$ by Equations (19) and (20). □

We call $f(x)$ in Equation (22) a polynomial-type solution, which is a product of $x^{λ_1}$ and a polynomial. If the accessory parameter $E$ of the q-Heun equation satisfies the equation $c_{N+1}(E) = 0$, then the q-Heun equation has a polynomial-type solution. We call $c_{N+1}(E)$ the spectral polynomial of the q-Heun equation.

### 3. Real root property of the spectral polynomial of the q-Heun equation

We may use the theory of Sturm sequence from the three term relations for $c_n$ and we obtain real root property of the spectral polynomial $c_{N+1}(E)$. The following lemma is obtained by the argument of Sturm sequence, which was essentially applied to Lamé equation and Heun equation in [2,13,18].

**Lemma 3.1:** Let $N$ be a non-negative integer. Assume that $d_n > 0$ and $d'_{n+1} > 0$ for $n = 1, \ldots, N$ and $p_n > 0$ and $q_n \in \mathbb{R}$ for $n = 1, \ldots, N + 1$. Set $c_{-1}(E) = 0$, $c_0(E) = 1$, and determine the polynomial $c_n(E) \ (n = 1, 2, \ldots, N + 1)$ recursively by
\[
d_n c_n(E) = (p_n E + q_n) c_{n-1}(E) - d'_n c_{n-2}(E). \tag{24}
\]
In the case $d_{N+1} = 0$, we set $c_{N+1}(E) = (p_{N+1} E + q_{N+1}) c_N(E) - d'_{N+1} c_{N-1}(E)$. Then the polynomial $c_n(E) \ (n = 1, 2, \ldots, N + 1)$ has $n$ real distinct zeros $s_j^{(n)} \ (j = 1, \ldots, n)$ such that
\[
\begin{align*}
s_1^{(n)} &< s_1^{(n-1)} < s_2^{(n)} < s_2^{(n-1)} < \cdots < s_{n-1}^{(n-1)} < s_{n-1}^{(n)} < s_n^{(n)}
\end{align*}
\]
for $n = 2, \ldots, N + 1$.

**Proof:** It follows from the assumption that $c_n(E)$ is a polynomial of degree $n$ such that the coefficient of $E^n$ is positive. The polynomial $c_1(E) = (p_1 E + q_1)/d_1$ has one real zero. We
show that if \( r \in \{1, 2, \ldots, N - 1\} \) and the polynomials \( c_{r-1}(E) \) and \( c_r(E) \) has real distinct zeros such that
\[
\sigma_1^{(r)} < \sigma_1^{(r-1)} < \sigma_2^{(r)} < \sigma_2^{(r-1)} < \cdots < \sigma_{r-1}^{(r)} < \sigma_{r-1}^{(r-1)} < \sigma_r^{(r)},
\] (26)
then the polynomial \( c_{r+1}(E) \) has \( r+1 \) real distinct zeros such that
\[
\sigma_1^{(r+1)} < \sigma_1^{(r)} < \sigma_2^{(r)} < \sigma_2^{(r+1)} < \cdots < \sigma_{r+1}^{(r)} < \sigma_{r+1}^{(r+1)}.
\] (27)
Since \( c_{r-1}(\sigma_{r-1}^{(r-1)}) = 0, \sigma_{r-1}^{(r-1)} < \sigma_r^{(r)} \) and \( c_r(E) \to +\infty \) as \( E \to +\infty \), we have \( c_{r-1}(\sigma_r^{(r)}) > 0 \). Moreover it follows from Equation (26) and \( c_{r-1}^{(r-1)}(s_{r-j}^{(r)}) = 0 \) (\( j = 1, 2, \ldots, r - 1 \)) that \( c_{r-1}(\sigma_{r-1}^{(r-1)}) < 0, c_{r-1}(s_{r-2}^{(r)}) > 0, \ldots, \) i.e. \( (-1)^j c_{r-1}(s_{r-j}^{(r)}) > 0 (j = 1, 2, \ldots, r - 1) \). By \( c_{r-1}(s_{r-j}^{(r)}) = 0 \) and Equation (24), we obtain \( d_{r+1} c_{r+1}(s_{r-j}^{(r)}) = -d_{r+1} c_{r-1}(s_{r-j}^{(r)}) \). Therefore \( (-1)^j c_{r+1}(s_{r-j}^{(r)}) > 0 \) for \( j = 0, 1, \ldots, r - 1 \), which follows from the assumption \( d_{r+1} > 0 \) and \( d_{r+1} > 0 \). Since \( c_{r+1}(s_r^{(r)}) < 0 \) and \( c_{r+1}(E) \to +\infty \) as \( E \to +\infty \), there exists a real number \( s_{r+1}^{(r+1)} \) such that \( s_r^{(r)} < s_{r+1}^{(r+1)} \) and \( c_{r+1}(s_{r+1}^{(r+1)}) = 0 \). It follows from \( c_{r+1}(s_{r-j}^{(r)}) c_{r+1}(s_{r-j+1}^{(r)}) < 0 \) and the intermediate value theorem that there exists a real number \( s_{r-j+1}^{(r+1)} \) such that \( s_r^{(r)} < s_{r-j}^{(r+1)} < s_{r-j+1}^{(r+1)} \) and \( c_{r+1}(s_{r-j+1}^{(r+1)}) = 0 \) for \( j = 1, 2, \ldots, r - 1 \). It also follows from \( (-1)^j c_{r+1}(s_r^{(r)}) > 0 \) and \( c_{r+1}(E) \to (-1)^r + \infty \) as \( E \to -\infty \) that there exists a real number \( s_1^{(r+1)} \) such that \( s_1^{(r+1)} < s_1^{(r)} \) and \( c_{r+1}(s_1^{(r+1)}) = 0 \). Therefore the polynomial \( c_{r+1}(E) \) has \( r+1 \) real distinct zeros \( s_j^{(r+1)} (j = 1, 2, \ldots, r + 1) \) which satisfy Equation (27).

It remains to be shown that \( c_{N+1}(E) \) has \( N+1 \) real distinct zeros \( s_j^{(N+1)} (j = 1, 2, \ldots, N+1) \) which satisfy Equation (27) for \( r = N \). If \( d_{N+1} > 0 \), then it is shown by applying the previous discussion. If \( d_{N+1} = 0 \), then \( c_{N+1}(E) \) is defined separately and it is reduced to the case \( d_{N+1} = 1 \). If \( d_{N+1} < 0 \), then we set \( \tilde{d}_{N+1} = -d_{N+1} \) and \( \tilde{c}_{N+1}(E) = -c_{N+1}(E) \). Since \( \tilde{d}_{N+1} > 0 \), it is shown that \( \tilde{c}_{N+1}(E) \) has \( N+1 \) real distinct zeros \( s_j^{(N+1)} (j = 1, 2, \ldots, N+1) \) which satisfy Equation (27) for \( r = N \). Obviously the zeros of \( c_{N+1}(E) \) coincide with those of \( \tilde{c}_{N+1}(E) \).

In Lemma 3.1 we obtained that if \( d_n > 0 \) and \( d_n' > 0 \) for \( n = 1, \ldots, N \) and \( p_n > 0 \) for \( n = 1, \ldots, N+1 \), then the polynomial \( c_{N+1}(E) \) has \( N+1 \) real distinct zeros. We apply the lemma for the three term relation in Equation (18).

**Theorem 3.2:** Assume that \( N = -\lambda - \alpha_1 \) is a non-negative integer, \( 0 < q < 1 \) and \( t_1, t_2, h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta \) are all real. If one of the following conditions is satisfied, then the equation \( c_{N+1}(E) = 0 \) in Equation (21) has all its roots real and distinct.

(i) \( t_1 t_2 > 0, \alpha_2 - \alpha_1 < 1 \) and \( \beta < 1 \).
(ii) \( t_1 t_2 > 0, \alpha_2 - \alpha_1 > N \) and \( \beta > N \).
(iii) \( t_1 t_2 < 0, \alpha_2 - \alpha_1 > N \) and \( \beta < 1 \).
(iv) \( t_1 t_2 < 0, \alpha_2 - \alpha_1 < 1 \) and \( \beta > N \).
Proof: The theorem is shown by applying Lemma 3.1 to Equations (18)–(20). As for (i), it follows from the condition of (i) that \( n + \lambda_1 + \alpha_2 < 1 \) for \( n = 1, 2, \ldots, N \) and \( n - \beta > 0 \) for \( n = 1, 2, \ldots \). Hence the assumption of Lemma 3.1 is confirmed. (iv) is shown similarly. (ii) and (iii) follows from the lemma by multiplying \(-1\).

4. Analysis of the spectral polynomial by the ultradiscrete limit

In the previous sections, it was shown that the condition for the eigenvalue \( E \) such that \( q \)-Heun equation admits a non-zero polynomial solution is described by the roots of the spectral polynomial \( c_{N+1}(E) \). However, it is not possible to solve the algebraic equation \( c_{N+1}(E) = 0 \) explicitly. In this section, we investigate the solution of \( c_{N+1}(E) = 0 \) by the ultradiscrete limit \( q \to +0 \). Detailed properties on convergence will be discussed in the Appendix.

We define the equivalence of functions of the variable \( q \) by

\[
a(q) \sim b(q) \iff \lim_{q \to +0} \frac{a(q)}{b(q)} = 1.
\]

(28)

We also define the equivalence \( \sum_{j=0}^{M} a_j(q)E^j \sim \sum_{j=0}^{M} b_j(q)E^j \) of the polynomials of the variable \( E \) by \( a_j(q) \sim b_j(q) \) for \( j = 0, \ldots, M \).

We investigate solutions of \( q \)-Heun equation in the form

\[
f(x) = x^{-1} \sum_{n=0}^{N} c_n(E)x^n,
\]

(29)

where \( \lambda_1 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 - \beta + 2)/2 \) under the condition of Theorem 3.2(i). For simplicity, we assume that \( t_1 > 0, t_2 > 0, h_1 < h_2 \) and \( l_1 < l_2 \). From now on, we assume that

\[
N = -\lambda_1 - \alpha_1 \in \mathbb{Z}_{\geq 0}, \beta < 1, \alpha_2 - \alpha_1 < 1, t_1 > 0, t_2 > 0, h_1 < h_2, l_1 < l_2.
\]

(30)

Recall that the polynomials \( c_n(E) \ (n = 1, 2, \ldots, N) \) are determined recursively by Equation (18), i.e.

\[
c_n(E) = c_{n-1}(E)[E^{n-1+\lambda_1} + q^{1/2}(q^h t_1 + q^{l_2}) + (q^h t_1 + q^{l_2})^{2(n-1+\lambda_1)+\alpha_1+\alpha_2-1/2}] - c_{n-2}(E)[q(1 - q^{n-2+\lambda_1+\alpha_1})(1 - q^{n-2+\lambda_1+\alpha_2})],
\]

(31)

with the initial condition \( c_0(E) = 1 \) and \( c_{-1}(E) = 0 \). The spectral polynomial \( c_{N+1}(E) \) is determined by setting \( n = N+1 \) in Equation (31). As \( q \to +0 \),

\[
(1 - q^n)(1 - q^{n-\beta}) \sim 1, \quad (n = 1, 2, \ldots)
\]

and

\[
q^{-h_1-h_2+1}(1 - q^{n-2+\lambda_1+\alpha_1})(1 - q^{n-2+\lambda_1+\alpha_2}) \sim q^{2n-1-h_1-h_2-\beta}, \quad (n = 2, 3, \ldots, N + 1).
\]

(32)
Proposition 4.1: for $n = 1, 2, \ldots, N + 1$ under the assumption that there are no cancellation of the leading terms of the coefficients of $E_j$ ($j = 0, 1, \ldots, n - 1$) in the right-hand side with respect to the limit $q \to +0$.

If $1 + h_2 - l_2 - \beta > 0$ (resp. $2N + 1 + h_2 - l_2 - \beta < 0$), then we have $q^{1/2-h_2} + q^{2n-1/2-l_2-\beta} \sim q^{1/2-h_2}$. (resp. $q^{1/2-h_2} + q^{2n-1/2-l_2-\beta} \sim q^{2n-1/2-l_2-\beta}$) for $n = 1, 2, \ldots, N + 1$. We investigate the behaviour of the spectral polynomial $c_{N+1}(E)$ and the associated solutions of the $q$-Heun equation as $q \to +0$ for the case $1 + h_2 - l_2 - \beta > 0$ or $2N + 1 + h_2 - l_2 - \beta < 0$.

4.1. The case $1 + h_2 - l_2 - \beta > 0$

If $1 + h_2 - l_2 - \beta > 0$, then we have

$$c_n(E) \sim t_1^{-1} t_2^{-1} (E^{n-1-h_2-l_2+1} + t_1 q^{1/2-h_2}) c_{n-1}(E) - t_1^{-1} t_2^{-1} q^{2n-1-h_2-l_2-\beta} c_{n-2}(E)$$

for $n = 1, 2, \ldots, N + 1$ under the assumption that there are no cancellation of the leading terms of the coefficients of $E_j$ ($j = 0, 1, \ldots, n - 1$) in the right-hand side with respect to the limit $q \to +0$. Since $c_0(E) = 1$ and $c_{-1}(E) = 0$, we have $c_1(E) \sim t_1^{-1} t_2^{-1} (E^{h_2-l_2+1} + t_1 q^{1/2-h_2})$ and

$$c_2(E) \sim t_1^{-2} t_2^{-2} (E^{h_2-l_2+1} + t_1 q^{1/2-h_2}) (E^{h_2-l_2+1} + t_1 q^{1/2-h_2})$$

$$- t_1^{-1} t_2^{-1} q^{3-l_2-\beta}.$$  

If $1 - 2h_2 \neq 3 - l_1 - l_2 - \beta$, i.e. $2 + 2h_2 - l_1 - l_2 - \beta \neq 0$, then there are no cancellation of the leading terms. If $2 + 2h_2 - l_1 - l_2 - \beta > 0$, then we may ignore the term $t_1^{-1} t_2^{-1} q^{3-l_2-\beta}$ and we have

$$c_2(E) \sim t_1^{-2} t_2^{-2} (E^{h_2-l_2+1} + t_1 q^{1/2-h_2}) (E^{h_2-l_2+1} + t_1 q^{1/2-h_2}).$$

Moreover we can obtain the following proposition as $q \to +0$.

**Proposition 4.1:** If $1 + h_2 - l_2 - \beta > 0$ and $2 + 2h_2 - l_1 - l_2 - \beta > 0$ then we have

$$c_n(E) \sim t_1^{-1} t_2^{-1} (E^{n-1-h_2-l_2+1} + t_1 q^{1/2-h_2}) c_{n-1}(E)$$

for $n = 1, 2, \ldots, N + 1$.

**Proof:** Let $k \in \{1, 2, \ldots, N\}$ and assume that Equation (37) holds for $n \leq k$. Then we have

$$t_1^{-1} t_2^{-1} (E^{k-h_2-l_2+1} + t_1 q^{1/2-h_2}) c_k(E) - t_1^{-1} t_2^{-1} q^{2k+1-l_1-l_2-\beta} c_{k-1}(E)$$

$$\sim \{t_1^{-2} t_2^{-2} (E^{k-h_2-l_2+1} + t_1 q^{1/2-h_2}) (E^{k-h_2-l_2+1} + t_1 q^{1/2-h_2})$$

$$- t_1^{-1} t_2^{-1} q^{2k+1-l_1-l_2-\beta}\} c_{k-1}(E).$$

(38)

Since $1 - 2h_2 < 3 - l_1 - l_2 - \beta \leq 2k + 1 - l_1 - l_2 - \beta$, we may neglect the term $t_1^{-1} t_2^{-1} q^{2k+1-l_1-l_2-\beta} c_{k-1}(E)$ and we have Equation (37) for $n = k+1$. □
Theorem 4.2: We assume Equation (30), $1 + h_2 - l_2 - \beta > 0$ and $2 + 2h_2 - l_1 - l_2 - \beta > 0$.

(i) The spectral polynomial $c_{N+1}(E)$ satisfies

$$c_{N+1}(E) \sim (t_1 t_2)^{-N-1} q^{(N/2 + \lambda_1 - h_1 - h_2)(N+1)} \times (E + q^{1/2 - N + h_1 - \lambda_1} t_1)(E + q^{3/2 - N + h_1 - \lambda_1} t_1) \cdots (E + q^{1/2 - h_1 - \lambda_1} t_1)(E + q^{1/2 + h_1 - \lambda_1} t_1).$$

(ii) There exist solutions $E_k(q)$ $(k = 1, 2, \ldots, N + 1)$ to the equation $c_{N+1}(E) = 0$ for sufficiently small $q$ such that

$$E_k(q) \sim -q^{-k + 3/2 + h_1 - \lambda_1} t_1.$$

Proof: We obtain (i) by applying Proposition 4.1 repeatedly. (ii) follows from Corollary A.4.

Remark: If $2 + 2h_2 - l_1 - l_2 - \beta < 0$, then the zeros of the polynomials $c_n(E)$ have a different feature. It follows from Equation (35) and the assumption $2h_2 - l_1 - l_2 - \beta < -2$ that

$$c_2(E) \sim t_1^{-2} t_2^{-2} (q^{1 - 2h_1 - 2h_2 + 2\lambda_1} E^2 + t_1 q^{1/2 - h_2} q^{h_1 - h_2 + \lambda_1} E - t_1 t_2 q^{3 - h_1 - l_2 - \beta}).$$

The spectral polynomial $c_{N+1}(E)$ would be different from Equation (39). See [7,11] for details.

We investigate the polynomial solution of $q$-Heun equation for the case $1 + h_2 - l_2 - \beta > 0$ and $2 + 2h_2 - l_1 - l_2 - \beta > 0$ as $q \to +0$. Then the spectral polynomial $c_{N+1}(E) = 0$ has solutions which satisfy Equation (40). Write the normalized polynomial solution of the $q$-Heun equation as

$$x^{\lambda_1} \sum_{n=0}^{N} c_n(E_k) x^n,$$

where $\lambda_1 = (h_1 + h_2 - l_1 - l_2 - \alpha_1 - \alpha_2 + \beta + 2)/2$, $c_0(E_k) = 1$ and $E_k$ is an abbreviation of $E_k(q)$.

Theorem 4.3: Let $k \in \{1, 2, \ldots, N + 1\}$. Assume Equation (30), $1 + h_2 - l_2 - \beta > 0$, $2 + 2h_2 - l_1 - l_2 - \beta > 0$ and the value $E = E_k$ is a solution of the characteristic equation $c_{N+1}(E) = 0$ such that $E_k \sim -q^{3/2 - k + h_1 - \lambda_1} t_1$ as $q \to +0$.

(i) The coefficients of the normalized polynomial solution in Equation (42) satisfy

$$c_n(E_k) \sim -q^{n-k+1/2-h_2} t_2^{-1} c_{n-1}(E_k), \quad 1 \leq n \leq k - 1,$$

$$c_n(E_k) \sim q^{2n+1/2+h_2-l_1-l_2-\beta} t_1^{-1} c_{n-1}(E_k), \quad k \leq n \leq N.$$
We have
\[ \sum_{n=0}^{N} c_n(E_k)x^n \sim \prod_{j=1}^{k-1} (1 - q^{j-k+1/2-h_2}t_2^{-1}x) \prod_{j=k}^{N} (1 + q^{2j+1/2+h_2-l_1-l_2-\beta}t_1^{-1}x). \] (44)

There exists \( q_j \in \mathbb{R}_{>0} \) for \( j = 1, 2, \ldots, M \) such that the polynomial \( \sum_{n=0}^{N} c_n(E_k)x^n \) has a zero \( x = x_j(q) \) for \( 0 < q < q_j \) which is continuous on \( q \) and satisfies
\[ \lim_{q \to +0} \frac{x_j(q)}{q^{j-1/2+h_2}t_2} = 1, \quad (j = 1, \ldots, k-1), \]
\[ \lim_{q \to +0} \frac{x_j(q)}{-q^{-2j-1/2-h_2+l_1+l_2+\beta}t_1} = 1, \quad (j = k, \ldots, N). \] (45)

**Proof:** Since Equation (42) is a solution of \( q \)-Heun equation, the coefficients \( c_n(E_k) \) satisfies
\[ c_n(E_k) \sim t_1^{-1}t_2^{-1}(-t_1q^{3/2-k+h_1-\lambda_1}q^{n-1-h_1-h_2+\lambda_1} + t_1q^{1/2-h_2})c_{n-1}(E_k), \]
\[ -t_2^{-1}t_1^{-1}q^{2n-1-l_1-l_2-\beta}c_{n-2}(E_k). \] (46)

for \( n = 1, \ldots, N + 1 \), if the leading terms on the right-hand side are not cancelled. Note that \( c_{-1}(E_k) = 0 \), \( c_{N+1}(E_k) = 0 \) and \( c_0(E_k) = 1 \). If \( 1 \leq n \leq k-1 \), then the term \( t_2^{-1}q^{1/2-h_2}c_{n-1}(E_k) \) does not affect the leading term and the right-hand side of Equation (46) is equivalent to
\[ -t_2^{-1}q^{1/2+n-k-h_2}c_{n-1}(E_k) - t_1^{-1}t_2^{-1}q^{2n-1-l_1-l_2-\beta}c_{n-2}(E_k). \] (47)

We now show that \( c_n(E_k) \sim -t_2^{-1}q^{1/2-h_2+n-k}c_{n-1}(E_k) \) for \( 1 \leq n \leq k-1 \) and \( k \geq 2 \). It follows from Equation (46) for the case \( n = 1 \) that \( c_1(E_k) \sim -t_2^{-1}q^{3/2-k-h_2} \). If \( c_{n-1}(E_k) \sim -t_2^{-1}q^{1/2-h_2+n-1-k}c_{n-2}(E_k) \), then it follows from \( 2 + 2h_2 - l_1 - l_2 - \beta > 0 \) that the term \( q^{1/2+n-k}c_{n-1}(E_k) \) is stronger than \( q^{2n-1-l_1-l_2-\beta} \). Hence we may ignore the term \( t_1^{-1}t_2^{-1}q^{n-1-l_1-l_2-\beta}c_{n-2}(E_k) \), the leading term of the right-hand side is contained in \( -t_2^{-1}q^{1/2-h_2+n-k}c_{n-1}(E_k) \) and we have \( c_n(E_k) \sim -t_2^{-1}q^{1/2-h_2+n-k}c_{n-1}(E_k) \) for \( 1 \leq n \leq k-1 \).

We show that \( c_n(E_k) \sim t_1^{-1}q^{1/2+2n+h_2-l_1-l_2-\beta}c_{n-1}(E_k) \) for \( k \leq n \leq N \). It follows from the three term relation that
\[ c_{n-2}(E_k) \sim t_1q^{3/2-2n-h_2+l_1+l_2+\beta}c_{n-1}(E_k) - t_1t_2q^{-2n+1+l_1+l_2+\beta}c_n(E_k) \] (48)

for \( k + 1 \leq n \leq N + 1 \), if the leading terms on the right-hand side are not cancelled. In the case \( n = N+1 \), we have \( c_N(E_k) \sim t_1^{-1}q^{1/2+2N+h_2-l_1-l_2-\beta}c_{N-1}(E_k) \) by Equation (48). We assume that \( c_n(E_k) \sim t_1^{-1}q^{1/2+2n+h_2-l_1-l_2-\beta}c_{n-1}(E_k) \) for some \( n \) such that \( k + 1 \leq n \leq N \). Since \( q^{3/2-2n-h_2+l_1+l_2+\beta}q^{1/2+2n+h_2-l_1-l_2-\beta} \) is stronger than \( q^{-2n+1+l_1+l_2+\beta} \), the term \( t_1t_2q^{-2n+1+l_1+l_2+\beta}c_n(E_k) \) in the right-hand side of Equation (48) is negligible and we have \( c_{n-1}(E_k) \sim t_1^{-1}q^{-3/2+2n+h_2-l_1-l_2-\beta}c_{n-2}(E_k) \). Thus we have shown \( c_n(E_k) \sim t_1^{-1}q^{1/2+2n+h_2-l_1-l_2-\beta}c_{n-1}(E_k) \) for \( k \leq n \leq N \). Therefore we obtain (i).
We show (ii). Write
\[
\prod_{j=1}^{N} (1 + s_j x) = \sum_{n=0}^{N} d_n x^n, \quad s_j = \begin{cases} t_2^{-1} q^{j-k+1/2-h_2}, & j = 1, \ldots, k-1, \\ -t_2^{-1} q^{2j+1/2+h_2-h_1-h_2-\beta}, & j = k, \ldots, N. \end{cases}
\] (49)

Then it follows from \(-1/2 - h_2 < 3/2 + h_2 - l_1 - l_2 - \beta < 2k + 1/2 + h_2 - l_1 - l_2 - \beta\) that \(s_j\) is stronger than \(s_{j+1}\) for \(j = 1, \ldots, N - 1\). Hence the assumption of Theorem A.3 is confirmed and we have \(d_n \sim \prod_{j=1}^{n} s_j\) for \(n = 1, \ldots, N\). On the other hand, it follows from (i) that \(c_{n+1}(E_k) \sim s_{n+1} c_n(E_k)\). Combining with \(d_0 = 1 = c_0(E_k)\), we obtain (ii).

(iii) follows from Theorem A.3. \(\blacksquare\)

Note that the polynomial solution of \(q\)-Heun equation in Equation (44) takes the form of a product of two \(q\)-Pochhammer functions as \(q \to +0\). It is known that certain products of two \(q\)-Pochhammer functions for arbitrary \(q\) arise in the context of quantum \(6j\)-symbols (or \(q\)-Racah polynomials) in [9, Equation (2.5)] and also in the analysis of difference operators related to \(U_q(s l_2)\) in [19, §3.3.2], but we do not have direct connection between these results and Equation (44).

### 4.2. The case \(2N + 1 + h_2 - l_2 - \beta < 0\)

If \(2N + 1 + h_2 - l_2 - \beta < 0\), then

\[
c_n(E) \sim t_1^{-1} t_2^{-1} (Eq^{-1-h_1-h_2+\lambda_1} + q^{2n-1/2-h_2-\beta} t_1) c_{n-1}(E)
\]

\[
- q^{2n-1-l_1-l_2-\beta} t_2^{-1} t_1^{-1} c_{n-2}(E)
\] (50)

for \(n = 1, 2, \ldots, N + 1\) under the assumption of no cancellation as in the case \(1 + h_2 - l_2 - \beta > 0\) with respect to the limit \(q \to +0\). Then we have \(c_1(E) \sim t_1^{-1} t_2^{-1} (Eq^{-h_1-h_2+\lambda_1} + q^{3/2-h_2-\beta} t_1)\) and

\[
c_2(E) \sim t_1^{-2} t_2^{-2} (Eq^{-h_1-h_2+\lambda_1} + q^{7/2-h_2-\beta} t_1) (Eq^{-h_1-h_2+\lambda_1} + q^{3/2-h_2-\beta} t_1)
\]

\[
- q^{3-3l_1-l_2-\beta} t_1^{-1} t_2^{-1}.
\] (51)

If \(2 + l_1 - l_2 - \beta < 0\), then we may ignore the term \(t_1^{-1} t_2^{-1} q^{3-3l_1-l_2-\beta}\) and we have

\[
c_2(E) \sim t_1^{-2} t_2^{-2} (Eq^{-h_1-h_2+\lambda_1} + q^{7/2-h_2-\beta} t_1) (Eq^{-h_1-h_2+\lambda_1} + q^{3/2-h_2-\beta} t_1).
\] (52)

Moreover we can obtain the following proposition as \(q \to +0\).

**Proposition 4.4**: Let \(k \in \{1, 2, \ldots, N\}\). If \(2N + 1 + h_2 - l_2 - \beta < 0\) and \(2k + l_1 - l_2 - \beta < 0\) then \(c_n(E)\) satisfies

\[
c_n(E) \sim t_1^{-1} t_2^{-1} (Eq^{-1-h_1-h_2+\lambda_1} + q^{2n-1/2-h_2-\beta} t_1) c_{n-1}(E)
\] (53)

for \(n = 1, 2, \ldots, k + 1\).
There exist solutions $E_k$ for $n = 1, 2, \ldots, N + 1$. As Theorem 4.2 in the previous subsection, we obtain the following theorem:

**Theorem 4.5:** We assume Equation (30), $2N + 1 + h_2 - l_2 - \beta < 0$ and $2N + l_1 - l_2 - \beta < 0$, then $c_n(E)$ satisfies

$$c_n(E) \sim q^{n-1-h_2+h_1}t_1^{-1}(-E + t_1 q^{n-3/2+l_1+\lambda_2} - 1)c_{n-1}(E)$$

for $n = 1, 2, \ldots, N + 1$. We investigate the polynomial solution of $q$-Heun equation for the value $E = E_k$ such that $E_k \sim -q^{k-3/2+l_1+\lambda_2}t_1$ ($k \in \{1, 2, \ldots, N + 1\}$) in the case $2N + 1 + h_2 - l_2 - \beta < 0$ and $2N + l_1 - l_2 - \beta < 0$ as $q \to +0$. Write the normalized polynomial solution as

$$x^{l_2} \sum_{n=0}^{\infty} c_n(E_k)x^n,$$

where $c_0(E_k) = 1$. Then we have the following theorem which can be proved similarly to Theorem 4.3.

**Theorem 4.6:** Let $k \in \{1, 2, \ldots, N + 1\}$. Assume Equation (30), $2N + 1 + h_2 - l_2 - \beta < 0$ and $2N + l_1 - l_2 - \beta < 0$ and the value $E = E_k$ is a solution of the characteristic equation $c_{N+1}(E) = 0$ such that $E_k \sim -q^{k-3/2+l_1+\lambda_2}t_1$ as $q \to +0$.

(i) The coefficients of the normalized polynomial solution in Equation (58) satisfy

$$c_n(E_k) \sim t_2^{-1}q^{2n-1/2-l_2-\beta}c_{n-1}(E_k), \quad 1 \leq n \leq k - 1,$$

$$c_n(E_k) \sim -t_1^{-1}q^{n-k/2-l_1}c_{n-1}(E_k), \quad k \leq n \leq N.$$
(ii) We have

\[ \sum_{n=0}^{N} c_n(E_k)x^n \sim \prod_{j=1}^{k-1} (1 + t_2^{-1} q^{2j-1/2-l_2-\beta} x) \prod_{j=k}^{N} (1 - t_1^{-1} q^{j-k+1/2-l_1} x). \]  

(60)

(iii) There exists \( q_j \in \mathbb{R}_{>0} \) for \( j = 1, 2, \ldots, M \) such that the polynomial \( \sum_{n=0}^{N} c_n(E_k)x^n \) has a zero \( x = x_j(q) \) for \( 0 < q < q_j \) which is continuous on \( q \) and satisfies

\[
\lim_{q \to +0} \frac{x_j(q)}{-t_2 q^{-2j+1/2+l_2+\beta}} = 1, \quad (j = 1, \ldots, k-1),
\]

\[
\lim_{q \to +0} \frac{x_j(q)}{t_1 q^{k-j-1/2+l_1}} = 1, \quad (j = k, \ldots, N). \quad (61)
\]

Note that the polynomial solution of \( q \)-Heun equation in Equation (60) takes the form of a product of two \( q \)-Pochhammer functions as \( q \to +0 \).

5. Concluding remarks

In this paper, we investigated polynomial-type solutions of the \( q \)-Heun equation. We defined the spectral polynomial of the accessory parameter \( E \) in the case that the parameters of the \( q \)-Heun equation satisfies the assumption of Proposition 2.2. The polynomial-type solution of the \( q \)-Heun equation exists, if the accessory parameter is a root of the spectral polynomial. Then we obtained sufficient conditions that all the roots of the spectral polynomial is real and distinct in Section 3. To find the behaviour of the roots of the spectral polynomial and the associated polynomial-type solution of the \( q \)-Heun equation, we considered the ultradiscrete limit \( q \to +0 \) and we obtained the behaviour of them in the case \( 1 + h_2 - l_2 - \beta > 0 \) and \( 2 + 2h_2 - l_1 - l_2 - \beta > 0 \) and the case \( 2N + 1 + h_2 - l_2 - \beta < 0 \) and \( 2N + l_1 - l_2 - \beta < 0 \).

We point out problems which should be clarified in the near future.

One problem is to consider the ultradiscrete limit of the spectral polynomial and the associated polynomial-type solution of the \( q \)-Heun equation for the case \( -2N - 1 < h_2 - l_2 - \beta < -1 \).

In [15], the variants of the \( q \)-Heun equation, i.e. \((A^{(3)} - E)g(x) = 0\) and \((A^{(2)} - E)g(x) = 0\), were discussed. It was also shown that the two equations has quasi-exact solvability, and they have polynomial-type solutions for some special cases. Then it would be possible to discuss the spectral polynomial, real root property of them, and analysis of the roots of the spectral polynomial by the ultradiscrete limit.

As a different direction to the variants of the \( q \)-Heun equation, we may consider degenerations of the \( q \)-Heun equation as Heun’s differential equation admits degenerations such as singly confluent Heun equation, doubly confluent Heun equation, bi-confluent Heun equation and tri-confluent Heun equation by confluence of the singularities (see [8]). Then it would be possible to consider polynomial-type solutions to degenerations of the \( q \)-Heun equation.

Acknowledgments

The authors are grateful to Simon Ruijsenaars for valuable comments and fruitful discussions.
Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The third author was supported by Japan Society for the Promotion of Science (JSPS) KAKENHI [grant numbers JP26400122, JP18K03378].

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Appendix. Ultradiscrete limit of the algebraic equation

We consider the ultrasymmetric limit (i.e. $q \to +0$) of the algebraic equation

$$
\sum_{j=0}^{M} \tilde{c}_j(q) x^j = 0.
$$

(A1)

We assume that $\tilde{c}_j(q)/((-1)^{p_j}q^{k_j}) \to 1$ as $q \to +0$ ($j = 0, 1, \ldots, M$), where $p_j \in \{0, 1\}$, $c_j > 0$ and $\lambda_j \in \mathbb{R}$.

Let $P$ and $N$ be the subsets of $\{0, 1, \ldots, M\}$ such that $P = \{j \mid p_j = 0\}$ and $N = \{j \mid p_j = 1\}$. Then the algebraic equation is written as

$$
\sum_{j \in P} c_j(q) x^j = \sum_{j \in N} c_j(q) x^j,
$$

(A2)

where $c_j(q) = (-1)^{p_j} \tilde{c}_j(q)$. Set $x = q^t$. Then $c_j(q)x^j/(c_j(q)x^j) \to 1$ as $q \to +0$. Since $q^\alpha + q^\beta$ can be approximated as $q^{\min(\alpha, \beta)}$ by the limit $q \to +0$, we obtain the following ultrasymmetric equation;

$$
\min(\lambda_j + jt)_{j \in P} = \min(\lambda_j + jt)_{j \in N}.
$$

(A3)

On the solution of the ultrasymmetric equation, we immediately obtain the following proposition.

**Proposition A.1:** Let $t_0$ be a solution to Equation (A3) such that the minimum is attained at $k \in P$ and $k' \in N$. Then we have $t_0 = (\lambda_{k'} - \lambda_k)/(k - k')$ and $\lambda_j + jt_0 \geq \lambda_k + kt_0 = \lambda_{k'} + k't_0$ for all $j$.

Here we assume that

$$
\lambda_j + jt_0 > \lambda_k + kt_0 = \lambda_{k'} + k't_0, \quad j \in \{0, 1, \ldots, M\} \setminus \{k, k'\}, \quad k \in P, \quad k' \in N.
$$

(A4)

By substituting $x = c q^{t_0}$ into the algebraic equation and observing the coefficient of the leading term $q^{\lambda_k} q^{kt_0} (= q^{\lambda_{k'}} q^{k't_0})$, the constant $c$ should satisfy $c = (c_k/c_{k'})^{1/(k' - k)}$.

We are going to justify that the value $x = (c_k/c_{k'})^{1/(k' - k)} q^{t_0}$ is asymptotic to a solution of the algebraic equation as $q \to +0$.

**Theorem A.2:** Let $P$ and $N$ be sets such that $P \cap N = \phi$ and $P \cup N = \{0, 1, \ldots, M\}$, $c_j(q)$ ($j \in \{0, 1, \ldots, M\}$) be a function such that $c_j(q)/q^j \to c_j$ as $q \to +0$ for some $c_j \in \mathbb{R}_{>0}$. We assume Equation (A4). Then the algebraic equation

$$
\sum_{j \in P} c_j(q) x^j = \sum_{j \in N} c_j(q) x^j
$$

(A5)

has a solution $x = x(q)$ ($0 < q < q_0$) such that

$$
\lim_{q \to +0} \frac{x(q)}{(c_k/c_{k'})^{1/(k' - k)} q^{\lambda_{k'} - \lambda_k}/(k - k')} = 1
$$

(A6)

for some $q_0 \in \mathbb{R}_{>0}$.

**Proof:** Set $t_0 = (\lambda_{k'} - \lambda_k)/(k - k')$, $x = q^{t_0} u$ and

$$
f(u, q) = q^{-(\lambda_k + kt_0)} \left[ \sum_{j \in P} c_j(q) [q^{t_0} u]^j - \sum_{j \in N} c_j(q) [q^{t_0} u]^j \right].
$$

(A7)

Then the equation $f(q^{-t_0} x, q) = 0$ is equivalent to Equation (A5) for $q > 0$. As $q \to +0$, we have

$$
f(u, q) \sim q^{-(\lambda_k + kt_0)} \left[ \sum_{j \in P} c_j(q) q^j q^{t_0} u^j - \sum_{j \in N} c_j(q) q^j q^{t_0} u^j \right].
$$

(A8)
By the assumption of Equation (A4), we obtain
\[ \lim_{q \to +0} f(u, q) = c_k u^k - c_{k'} u^{k'}, \tag{A9} \]
and define \( f(u, 0) \) by the limit \( q \to +0 \). Then we have \( f((c_k/c_{k'})^{1/(k' - k)}, 0) = 0 \). On the other hand, we have
\[ \lim_{q \to +0} \frac{\partial}{\partial u} f(u, q) = \lim_{q \to +0} q^{-(\lambda_k + k\alpha_0)} \left[ \sum_{j \in P} c_j q^{\lambda_j + j\alpha_0} u^{j-1} - \sum_{j \in N} c_j q^{\lambda_j + j\alpha_0} u^{j-1} \right] \]
\[ = k c_k u^{k-1} - k' c_{k'} u^{k'-1} = \frac{\partial}{\partial u} f(u, 0), \]
\[ \frac{\partial}{\partial u} f(u, 0) \bigg|_{u=(c_k/c_{k'})^{1/(k' - k)}} = (k - k') c_k^{(k'-1)/(k' - k)} c_{k'}^{(k-1)/(k - k')} \neq 0. \tag{A10} \]
By the implicit function theorem, there exists \( q_0 > 0 \) and a continuous function \( u(q) \) on \( 0 \leq q < q_0 \) such that \( f(u(q), q) = 0 \) and \( \lim_{q \to +0} u(q) = (c_k/c_{k'})^{1/(k' - k)} \). Hence Equation (A5) has a solution \( x = x(q) \) \((0 < q < q_0)\) such that
\[ \lim_{q \to +0} \frac{x(q)}{(c_k/c_{k'})^{1/(k' - k)} q^{\lambda_k/(k - k')}} = 1. \tag{A11} \]
\[ \square \]

**Theorem A.3:** Let
\[ \sum_{j=0}^{M} \tilde{c}_j(q) x^j = 0 \tag{A12} \]
be the algebraic equation such that \( \tilde{c}_{M-j}(q)/\tilde{c}_{M-j+1}(q) \to r_j q^{\mu_j} \) as \( q \to +0 \) \((j = 1, 2, \ldots, M)\), where \( r_j \in \mathbb{R}_{\geq 0}, \mu_j \in \mathbb{R} \) and \( \mu_1 < \mu_2 < \cdots < \mu_M \). Then there exists \( q_k \in \mathbb{R}_{>0} \) for \( k = 1, 2, \ldots, M \) such that the algebraic equation has a solution \( x = x_k(q) \) for \( 0 < q < q_k \) which is continuous on \( q \) and satisfies
\[ \lim_{q \to +0} \frac{x_k(q)}{r_k q^{\mu_k}} = 1. \tag{A13} \]

**Proof:** Let \( P \) (resp. \( N \)) be the subsets of \([0, 1, \ldots, M]\) such that \( P = \{0\} \cup \{j \mid r_1 \ldots r_j > 0\} \) and \( N = \{j \mid r_1 \ldots r_j < 0\} \). Without loss of generality, we may assume that the algebraic equation is monic, i.e. \( \tilde{c}_M(q) = 1 \). Then we have \( \tilde{c}_{M-j}(q) \sim r_1 \ldots r_j q^{\mu_1 + \cdots + \mu_j} \) and the algebraic equation is written as
\[ \sum_{j \in P} |\tilde{c}_j(q)| x^j = \sum_{j \in N} |\tilde{c}_j(q)| x^j. \tag{A14} \]
We investigate positive solutions to Equation (A14). By setting \( x = q^t \), the corresponding ultradiscrete equation is written as
\[ \min\{\mu_1 + \cdots + \mu_j + (M - j) t\}_{j \in P} = \min\{\mu_1 + \cdots + \mu_j + (M - j) t\}_{j \in N}. \tag{A15} \]
Let \( k \in \{1, 2, \ldots, M\} \) such that \( r_k < 0 \). Then we have \((k - 1 \in P \text{ and } k \in N)\) or \((k - 1 \in N \text{ and } k \in P)\). It follows from \( \mu_1 < \mu_2 < \cdots < \mu_M \) that \( \mu_1 + \cdots + \mu_{k-1} + (M - k + 1) \mu_k < \mu_1 + \cdots + \mu_j + (M - j) \mu_k \) for \( j \neq k - 1, k \) and the value \( t_k = \mu_k \) satisfies Equation (A15). Hence we can apply Theorem A.2 and there exists \( q_k \in \mathbb{R}_{>0} \) such that the algebraic equation has a solution \( x = x_k(q) \) for \( 0 < q < q_k \) which is continuous on \( q \) and satisfies \( x_k(q)/(|r_k| q^{\mu_k}) \to 1 \) as \( q \to +0 \).
We investigate negative solutions to Equation (A14). Let \( P' \) (resp. \( N' \)) be the subsets of \([0, 1, \ldots, M]\) such that \( P' = \{0\} \cup \{j \mid r_1 \ldots r_j(-1)^j < 0\} \) and \( N' = \{j \mid r_1 \ldots r_j(-1)^j > 0\} \). By setting
\( x = -q' \), the corresponding ultradiscrete equation is written as
\[
\min\{\mu_1 + \cdots + \mu_j + (M - j)t\}_{j \in P} = \min\{\mu_1 + \cdots + \mu_j + (M - j)t\}_{j \in N'}.
\]  
(A16)

Let \( k \in \{1, 2, \ldots, M\} \) such that \( r_k > 0 \). Then we have \((k - 1 \in P' \text{ and } k \in N') \) or \((k - 1 \in N' \text{ and } k \in P') \). It follows from \( \mu_1 < \mu_2 < \cdots < \mu_M \) that \( \mu_1 + \cdots + \mu_{k-1} + (M - k + 1)\mu_k < \mu_1 + \cdots + \mu_{j} + (M - j)\mu_k \) for \( j \neq k - 1, k \) and the value \( t_k = \mu_k \) satisfies Equation (A15). Hence we can apply Theorem A.2 and there exists \( q_k \in \mathbb{R}_{>0} \) such that the algebraic equation has a solution \( x = x_k(q) \) for \( 0 < q < q_k \) which is continuous on \( q \) and satisfies \(-x_k(q)/(r_k q^{\mu_k}) \to 1 \) as \( q \to +0 \).

\( \blacksquare \)

**Corollary A.4:** Assume that
\[
\sum_{j=0}^{M} \tilde{c}_j(q)x^j \sim cq^{\mu_1}\prod_{j=1}^{M}(x + r_j q^{\mu_j}),
\]  
(A17)

where \( r_j \in \mathbb{R}_{\neq 0}, \mu_j \in \mathbb{R} \) and \( \mu_1 < \mu_2 < \cdots < \mu_M \). Then there exists \( q_k \in \mathbb{R}_{>0} \) for \( k = 1, 2, \ldots, M \) such that the algebraic equation \( \sum_{j=0}^{M} \tilde{c}_j(q)x^j = 0 \) has a solution \( x = x_k(q) \) for \( 0 < q < q_k \) which is continuous on \( q \) and satisfies
\[
\lim_{q \to +0} \frac{x_k(q)}{-r_k q^{\mu_k}} = 1.
\]
(A18)

**Proof:** It follows from \( \mu_1 < \mu_2 < \cdots < \mu_M \) that
\[
\prod_{j=1}^{M}(x + r_j q^{\mu_j}) \sim \sum_{j=0}^{M} r_1 \cdots r_j q^{\mu_1 + \cdots + \mu_j}x^{M-j}.
\]
(A19)

Therefore the corollary follows from the theorem.  

\( \blacksquare \)