From Snakes to Stars, the Statistics of Collapsed Objects - I. Lower–order Clustering Properties.

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ABSTRACT

The highly nonlinear regime of gravitational clustering is characterized by the presence of scale-invariance in the hierarchy of many-body correlation functions. Although the exact nature of this correlation hierarchy can only be obtained by solving the full set of BBGKY equations, useful insights can be obtained by investigating the consequences of a generic scaling ansatz. Extending earlier studies by Bernardeau \& Schaeffer (1992) we calculate the detailed consequences of such scaling for the implied behaviour of a number of statistical descriptors, including some new ones, developed to provide useful diagnostics of scale-invariance. We generalise the two-point cumulant correlators (now familiar to the literature) to a hierarchy of multi-point cumulant correlators (MCC) and introduce the concept of reduced cumulant correlators (RCC) and their related generating functions. The description of these quantities in diagrammatic form is particularly attractive. We show that every new vertex of the tree representation of higher-order correlations has its own reduced cumulant associated with it and, in the limit of large separations, MCCs of arbitrary order can be expressed in terms of RCCs of the same and lower order. Generating functions for these RCCs are related to the generating functions of the underlying tree vertices for matter distribution. Relating the generating functions of RCCs with the statistics of collapsed objects suggests a scaling ansatz of a very general form for the many-body correlation functions which, in turn, induces a similar hierarchy for the correlation functions of overdense regions.

In this vein, we compute the lower-order $S_N$ parameters and two-point cumulant correlators $C_{NM}$ for overdense regions and study how they vary as a function of the initial power spectrum of primordial density fluctuations. We also show that these results match those obtained by the extended Press-Schechter formalism in the limit of large mass.

Key words: Cosmology: theory – large-scale structure of the Universe – Methods: statistical

1 INTRODUCTION

Gravity is scale-free. In the absence of any externally-imposed length scale, such as might be set by initial conditions, it is therefore reasonable to assume that gravitational clustering of particles in the universe should evolve towards a scale-invariant form, at least on small scales where gravitational effects presumably dominate over initial conditions. Observations offer support for such an idea, in that the observed two-point correlation function $\xi(x)$ of galaxies is reasonably well represented by a power law over quite a large range of length scales,

$$\xi(r) = \left(\frac{r}{5h^{-1}\text{Mpc}}\right)^{-1.8}$$

(Groth \& Peebles 1977; Davis \& Peebles 1983) for, say, $r$ between $100h^{-1}\text{kpc}$ and $10h^{-1}\text{ Mpc}$. Higher-order correlation functions of galaxies also appear to satisfy a scale-invariant form, with $\xi_N \propto \xi_2^{N-1}$ as expected from the application of a
general scaling ansatz (Groth & Peebles 1977; Fry & Peebles 1978; Davis & Peebles 1983; Szapudi et al. 1992). For example, the observed three point function, $\xi_3$, is well-established to have a hierarchical form

$$\xi_3(x_a, x_b, x_c) = \xi_{abc} = Q(\xi_{ab} + \xi_{ac} + \xi_{bc}).$$  

where $\xi_{ab} = \xi(x_a, x_b)$, etc. In a similar spirit, the four-point correlation function can be expressed as a combination of four-point graphs with two different topologies – “snake” and “star” – with corresponding amplitudes $R_a$ and $R_b$ respectively.

$$\xi_4(x_a, x_b, x_c, x_d) = \xi_{abcd} = R_a(\xi_{abc} + \xi_{acd} + \ldots (12 \text{ terms})) + R_b(\xi_{abc} + \xi_{acd} + \ldots (4 \text{ terms}))$$

(Fry & Peebles 1978, Fry 1984a, Bonometto et al. 1990, Valdarnini & Borgani 1991). It is natural on the basis of these considerations to infer that all N-point correlation functions can be expressed as a sum over all possible N-tree graphs with (in general) different amplitudes for each tree diagram. It is also generally assumed that there is no shape-dependence in these tree amplitudes in the highly non-linear regime:

$$\xi_N(r_1, \ldots r_N) = \sum_{\alpha, N \text{-trees}} Q_{N,\alpha} \sum_{\text{labellings}} \prod_{i=1}^{N-1} \xi(r_i, r_j).$$

Some numerical experiments have confirmed the validity of such an assumption, at least as far as three-point correlation function is concerned (Scoccimarro et al. 1998).

This tree-level model of hierarchical clustering however is a particular case of a more general scaling ansatz proposed by Balian & Schaeffer (1988), in which the N-point correlation functions can be written in the form

$$\xi_N(\lambda r_1, \ldots \lambda r_N) = \lambda^{-\gamma(N-1)} \xi_N(r_1, \ldots r_N).$$

Rigorous theoretical motivation for these assumptions are, however, yet to be forthcoming. Considerable progress has been made recently in understanding the evolution of gravitational clustering in limit of weak clustering, i.e. in the quasi-linear regime (e.g. Sahni & Coles 1995). In particular, it has been shown that, in the quasi-linear regime, a tree-level hierarchy of correlation functions develops which satisfies $\xi_N \propto \xi_2^{N-1}$ at the limit of vanishing variance. On the other hand, the intermediate regime, where loop corrections to tree-level diagrams become important, and the highly non-linear regime, where perturbative calculations break down completely, remain poorly understood and answers to many questions are still unclear.

Theoretical studies of the evolution of correlation functions in the highly nonlinear regime have mainly concentrated on possible closure schemes for the Born-Bogoliubov-Green-Kirkwood-Yvon (BBGY) equations (Davis & Peebles 1977), stability properties of these closure schemes (Ruamsuwan & Fry 1992) and different phase-space separation schemes for hierarchical solutions of arbitrary order (Fry 1982; Hamilton 1988). Although a general solution of the BBGY equations in nonlinear regime is still lacking, various interesting characteristics of the count probability distribution function (CPDF) and void probability distribution function (VPF) have been established by Balian & Schaeffer (1992). These predictions have also been tested against numerical simulations, in both two and three dimensions, and were found to be in good agreement.

However, most of the analytical results available so far relate to quantities associated only with one-point cumulants. Examples are the so-called $S_N$ parameters, as well as the CPDF and VPF mentioned above. Multi-point cumulant correlators (MCC), which we shall introduce in this paper, are a natural generalization of the $S_N$ parameters and the two-point cumulant correlators (2CC) which are generally used to analyze galaxy catalogs and compare observational data with different models of structure formation. Cumulant correlators were proposed recently by Szapudi & Szalay (1997) and shown to be very efficient for extracting clustering information from galaxy surveys. In particular, they can be used to test gravitational instability scenario and constrain the initial power spectral index. Analytical predictions regarding the 2CCs were made by Bernardardeau (1995) using tree-level perturbation theory in the limit of large separation. In particular, he showed how the bias of overdense cells with respect to the underlying mass distribution is related to the generating function of the 2CCs. These predictions were also successfully tested against numerical simulations. Such calculations were also extended for application to projected (angular) catalogs and have now been shown to be in reasonable agreement with estimates of these quantities extracted from the APM survey (Munshi, Melott & Coles 1998).

Bernardeau & Schaeffer (1992) developed a systematic series expansion for the multi-point void probability distribution function (MVPF) which is essentially the generating function of the multi-point count probability distribution function (MCPF). This expansion again applies only in the highly nonlinear regime. Using such a series expansion, based on powers of $\xi_i/\xi$, where $\xi_i$ is the variance in the ith cell, they were able to evaluate the bias associated with overdense regions. The bias thus calculated, which was analytically shown to depend only on a scaling variable associated with the collapsed object and an intrinsic property of the object, was later found to be in very good agreement with numerical calculations. It is clearly an interesting task to extend such analysis further and check predictions against numerical simulations.

If one can believe that galaxies are in some way identified with overdensities in the matter distribution then observed estimates of the two-point cumulant correlators from a galaxy survey depend upon the bias of such regions relative to the underlying mass distribution. Multipoint cumulant correlators allow more information to be deduced about the properties of bias, because the quantities related to the $S_N$ parameters of overdense cells. Complete knowledge of all $S_N$ parameters (which are related to the generating function of MCCs to arbitrary order) can also help us in constructing the count probability distribution, and related void probability distribution, for collapsed objects.

Unfortunately, there is at present no robust theoretical framework other than the hierarchical ansatz within which such objects can be studied. One-point statistics of collapsed objects have been studied by Mo & White (1996) using the extended
Press-Schechter formalism (Press & Schechter 1974; Bond et al. 1991; Bower 1991; Lacey & Cole 1993; Kauffmann & White 1993) but it remains to be seen whether this formulation can be used to derive multi-point statistics for collapsed objects. The same is largely true of “peaks theory” which is based on an argument used by Kaiser (1984) to explain the strong clustering of Abell clusters using the properties of high-density regions in Gaussian fields. This formalism was subsequently developed by Bardeen et al. (1986) and its use in cosmology is now widespread. However, there is evidence that correspondence between dark haloes and density peaks is not particularly good (Frenk et al. 1998; Katz, Quinn & Gelb 1993) and one needs to take the gravitationally-induced motions of peaks into account to compute the final clustering properties of halos. Progress has recently been made in overcoming these difficulties by Bond & Myers (1996a,b) in the “Peak-Patch Formalism” at the expense of considerable technical complexity. Other efforts to study statistics of collapsed objects include use of the Zel’dovich approximation in nonlinear regime after suitably smoothing the initial potential and using the statistics of singularities of different types (Catelan et al. 1998; Lee & Shandarin 1997, 1998).

It is also important to note that earlier derivation of bias and $S_3$ for collapsed objects in the context of hierarchical ansatz were done within the context of certain specific approximations and it is interesting to check their validity by extending such studies to higher orders, as they are widely used for error estimations of measurements from galaxy catalogs (Szapudi & Colombi 1996), as well as the derivation of mass function of collapsed objects (Valageas & Schaeffer 1997).

In a completely different context it was shown by Colombi et al. (1996) that normalized one point cumulants or $S_N$ parameters for underlying mass distribution do not become constant in highly non-linear regime, in direct contrast to hierarchical ansatz. Which is also connected to the question, if gravitational clustering at highly non-linear regime do satisfy the stable clustering ansatz (Peebles, 1980). Connection between stable clustering and hierarchical ansatz still remains unclear and it is also related to the question, if different initial conditions do lead to similar halo profiles. In a recent study, using stability analysis of nonlinear clustering Yano & Gouda (1998) showed that stable clustering may not be the most natural outcome of gravitational clustering and departure from stable clustering ansatz may mean departure from hierarchical ansatz. However their results depend on specific closure schemes. In any case such issues can only be tested if different prediction regarding hierarchical ansatz are derived analytically and tested against numerical simulations to find out their limitations, which is one of the motivation behind present study.

The paper is organized as follows. In next section we develop a diagrammatic method to represent cumulant correlators of arbitrary order. Using these rules it is also possible to compute these quantities to arbitrary order. In section 3 we use the method of Bernardeau & Schaeffer (1992) to compute generating functions of RCCs to 5th order. In section 4 we use results of section 3 to compute $S_N$ parameters for collapsed objects. We discuss these results in the last section, and place them in the context of other developments in this field.

2 MULTIP-POINT CUMULANT CORRELATORS AND THEIR GENERATING FUNCTIONS

Higher-order reduced correlation functions are typically dominated by contributions from high peaks of the density fields. It is reasonable, therefore, to infer that the generating functions of MCCs are related in some way to the statistics of collapsed objects. According to the hierarchical ansatz, those tree-level diagrams which contribute to reduced correlation function of the form $\langle \delta (x_a)^p \ldots \delta (x_b)^w \rangle$ are of order $p + \ldots + w$. MCCs can be constructed by identifying $p$ position arguments as $x_a$, $q$ arguments as $x_b$, and so on, with the $x_a$ representing the positions of “cells” defined by some implicit smoothing of the density field. Not all contributions constructed in this way are of same order of magnitude. The dominant contribution will always come from those terms which have the minimum number of external lines joining different cells, since every extra line joining different cells contributes an extra factor of $\xi_{ij}/\bar{\xi}_i < 1$ to the overall amplitude of the diagram.

With such a construction, the complete tree diagram can be separated in two parts: (i) external lines joining different cells; and (ii) internal lines joining the different vertices in the same cell in such a way as to keep the external tree topology unchanged. The number of internal degrees of freedom of the diagram corresponds to the number of different ways the internal vertices in a particular node can be arranged without altering the external connectivity of the tree. The external tree configuration connecting different nodes representing different cells on the other hand is very similar to the underlying tree structure of matter correlation function and depends only on the number of nodes in the tree. The number of different topologies for external tree increases with the number of nodes.

The difference between the external tree structure connecting different nodes and the underlying tree structure of the matter correlations connecting different vertices is that nodes in the tree representation of MCCs carry an internal degree of freedom due to the internal tree structure and represent an entire cell, whereas for the underlying matter distribution, each vertex represents a single point in space and does not carry any internal structure. If we denote by $C_p$ the number ways $p$ internal points can be arranged with one external leg remaining to be connected with another cell we can express the 2CCs as

$$\langle \delta^p (x_a) \delta^q (x_b) \rangle = [C_{p1} \xi_{ab} C_{q1}] \bar{\xi}_1^{p+q-2} = C_{pq} \xi_{ab} \bar{\xi}_1^{p+q-2}$$

Similarly, we can define $C_{p11}$ as an internal degree of freedom for a node with $p$ internal vertices and 2 external legs. The 3CCs now can be expressed in terms $C_{p11}$ and $C_{p1}$. In practice $C_{p11}$ can be described as number of different possible ways in which $p$ internal vertices can be rearranged keeping two external legs free:

$$\langle \delta^p (x_a) \delta^q (x_b) \delta^r (x_c) \rangle = [C_{q1} \xi_{ab} C_{p11} \xi_{ac} C_{r1} + C_{p1} \xi_{ab} C_{q11} \xi_{bc} C_{r1} + C_{p1} \xi_{ac} C_{r11} \xi_{bc} C_{q1}] \bar{\xi}_1^{p+q+r-3} = C_{pqr} \xi_{abc} \bar{\xi}_1^{p+q+r-3}$$
A similar decomposition of 4CCs and 5CCs can be used to define $C_{p111}$ and $C_{p1111}$:

$$\langle \delta^p(x_a) \delta^q(x_b) \delta^r(x_c) \delta^s(x_d) \rangle = \left[ C_{p1} \xi_{ab} C_{p1} \xi_{cd} C_{11} + (\text{cyclic permutations}) \right] \xi^{p+q+r+s-4}$$

and

$$\langle \delta^p(x_a) \delta^q(x_b) \delta^r(x_c) \delta^s(x_d) \rangle = \left[ C_{p111} \xi_{ab} C_{p11} \xi_{cd} C_{111} + (\text{cyclic permutations}) \right] \xi^{p+q+r+s+t-5}$$

It is also possible to consider this set of new parameters $C_{p1...1}$ describing the internal degrees of freedom as a set of reduced cumulant correlators (RCCs) which can be used to decompose MCCs $C_{p...}$.. The number of subscripts that appear in the RCCs $C_{p1...1}$ thus defined represents the number of external trees and $p$ represents the number of internal vertices. The total contribution from a cell is equal to product of all vertices in the internal tree structure and of the number of different ways one can rearrange the trees keeping the number of internal vertices and external legs fixed. The highest order vertex that appears in an internal tree of $C_{p1}$ is $p$, in $C_{p11}$ it is $p+1$ and so on. For the purposes of counting different possible arrangements, the external hands are to be considered distinguishable, as they link different external nodes and hence represent one particular topology of the external tree. Internal lines joining different vertices carry an weight of $\xi_i$, where, as above, $\xi_i$ is the variance in the $i^{th}$ cell. External lines carry an weight of $\xi_{ij}$, where $\xi_{ij}$ represents the magnitude of the correlation function joining $i^{th}$ and $j^{th}$ cells. Using these rules one can evaluate RCCs to any order. We list here the contributions from different topologies at lower orders:

$$C_{11} = 1, C_{21} = 2 \nu_2, C_{31}^{(a)} = 6 \nu_2^2, C_{31}^{(b)} = 3 \nu_3$$

$$C_{111} = \nu_2, C_{211}^{(a)} = 2 \nu_3, C_{411}^{(b)} = 2 \nu_2^2, C_{411}^{(a)} = 3 \nu_4, C_{311}^{(b)} = 6 \nu_4 \nu_2, C_{311}^{(a)} = 12 \nu_4 \nu_2, C_{311}^{(d)} = 6 \nu_3^2$$

$$C_{411}^{(a)} = 4 \nu_5, C_{411}^{(b)} = 6 \nu_4 \nu_2^2, C_{411}^{(c)} = 24 \nu_4 \nu_2^2, C_{411}^{(d)} = 24 \nu_4 \nu_2^2, C_{411}^{(e)} = 24 \nu_4 \nu_2^2, C_{411}^{(f)} = 24 \nu_4 \nu_2^2, C_{411}^{(g)} = 96 \nu_3 \nu_5 \nu_2$$

$$C_{1111} = \nu_3, C_{2111}^{(a)} = 2 \nu_4, C_{2111}^{(b)} = 6 \nu_3 \nu_2$$

$$C_{3111}^{(a)} = 6 \nu_2 \nu_4, C_{3111}^{(b)} = 18 \nu_2 \nu_4, C_{3111}^{(c)} = 18 \nu_3 \nu_2^2, C_{3111}^{(d)} = 18 \nu_3 \nu_2^2, C_{3111}^{(e)} = 3 \nu_5, C_{3111}^{(f)} = 18 \nu_3^2$$

The superscripts $a, b, \ldots$ denote different configurations of internal vertices with fixed external legs and the coefficients in each term represent the number of possible internal rearrangements. We denote the amplitudes associated with $n^{th}$ order vertex of matter correlation function by $\nu_n$.

The sum of all internal vertices appearing in the tree-representation of MCCs $\langle \delta(x_a)^p \cdots \delta(x_b)^m \rangle$ will satisfy the following equation, which guarantees that only the leading-order diagrams with the minimum numbers of external legs are considered:

$$\sum_{i}^p i + \sum_{j}^q j + \ldots = 2(p + q + \ldots) - 2$$

This means that, for one point cumulants, $\sum_{i}^p i = 2p - 2$ and for 2CCs $\sum_{i}^p i + \sum_{j}^q j = 2(p + q) - 2$. These are the same as the conditions met by the leading-order tree-level contributions of cumulants and cumulant correlators (Bernardeau 1995). This can also be understood by noticing, for example, that the expression for the 3CC contains factors $\xi_0^{-1} \xi_0^{-1} \xi_0^{-1}$ for each individual cells and two correlation functions connecting three different cells i.e. $\xi_{ab} \xi_{bc}$ (or any cyclic permutation), so finally we get the order to be equal to $2(p + q + r - 3) + 4 = 2(p + q + r) - 2$.

For reasons we shall discuss in the next section, we now define generating function of the RCCs by following equations:

$$\mu_1(y) = \beta(y) = \sum_{n=1}^{\infty} C_{n!} y^n = y - 6 y^2/2! + (6 y^2 + 3 y^3) y^3/3! - \ldots$$

$$\mu_2(y) = \sum_{n=1}^{\infty} C_{n!}^{(1)} y^n = -\nu_2 y + (2 \nu_2^2 + 2 \nu_3) y^2/2! - (6 \nu_3 + 18 \nu_2 \nu_3 + 3 \nu_4) y^3/3! + (4 \nu_3 + 48 \nu_2 \nu_4 + 36 \nu_3^2 + 144 \nu_2^2 \nu_3 + 24 \nu_4) y^4/4! + \ldots$$

$$\mu_3(y) = \sum_{n=1}^{\infty} C_{n!}^{(1)} y^n = \nu_3 y - (6 \nu_2 \nu_3 + 2 \nu_4) y^2/2! + (36 \nu_3^2 + 18 \nu_3^2 + 24 \nu_2 \nu_4 + 3 \nu_5) y^3/3! - \ldots$$

The total number of trees appearing in the representation of the $S_p$ parameters (related to $C_p$) is $p^{p-2}$. Similarly, for $C_{p1}$, the total number of terms is $p^{p-1}$; for $C_{p11}$ it is $p^p$, for $C_{p111}$ it is $p^{p+1}$ and $C_{p1111}$ in general will contain $p^{p+r-2}$ terms, where $r$
is the number of external lines joining different cells in the tree representation of the RCCs. For one point cumulants, or $S_N$ parameters, $r = 0$, for 2CCs $r = 1$, and so on.

### 3 TREE REPRESENTATION OF MULTI-POINT CUMULANTS

Let us denote the joint probability distribution function of $q$ different cells having cell occupation of $N_1, \ldots, N_q$ by $P(N_1, \ldots, N_q)$. The generating function of this distribution is $P(\lambda_1, \ldots, \lambda_q) = \exp[\chi(\lambda_1, \ldots, \lambda_q)]$.

\[
\exp[\chi(\lambda_1, \ldots, \lambda_q)] = P(\lambda_1, \ldots, \lambda_q) = \sum_{N_1, \ldots, N_q} P(N_1, \ldots, N_q) \lambda_1^{N_1} \ldots \lambda_q^{N_q}.
\]  

Extending the known relation for one-point statistics (Balian & Schaeffer 1989) and employing the vertex generating function

\[
G(\tau) = \sum_{n=1}^{\infty} \frac{\nu_n^q}{n!} (\tau)^n,
\]
Figure 2. The tree representation for multi-point cumulants can be understood by considering the external tree which represents how different cells are linked with each other and the internal trees in each cell, i.e. how internal vertices are arranged in a tree structure with fixed number of external legs. In this figure we represent different internal trees which appear in the multi-point cumulant correlators.

Bernardeau & Schaeffer (1992) showed that it is possible to relate \( \exp(\chi(\lambda_1, \ldots, \lambda_q)) \) to \( G(\tau) \) using

\[
\chi(\lambda_1, \ldots, \lambda_q) = -\sum_{i=1}^{q} -\frac{1}{\xi_i}(y_i G(\tau_i) - \frac{1}{2} y_i \tau_i G'(\tau_i))
\]

\[
\tau_i = -y_i G'(\tau_i) - \sum_{i \neq j} \frac{\xi_{ij}}{\xi_i} y_j G'(\tau_j)
\]

The correlation of different cells enters through \( \xi_{ij}/\xi_i \) in eq. (15). If we ignore this term in the second equation, which governs the behaviour of \( \tau_i \), then the statistics of different cells are independent of each other. This is a kind of zeroth approximation when only one-point statistics are used in the description. If we expand \( \chi(\lambda_1, \ldots, \lambda_q) \) in a series of powers of \( \xi_{ij}/\xi_i \), then we will recover the effect of correlations between different cells of progressively higher order. Expanding eq. (16) we can write:

\[
\tau_i^{(0)} + \tau_i^{(1)} + \tau_i^{(2)} + \ldots + y_i G'(\tau_i^{(0)}) + \tau_i^{(1)} + \tau_i^{(2)} = -\sum_{i \neq j} \frac{\xi_{ij}}{\xi_i} y_j G'(\tau_j^{(0)}) + \tau_j^{(1)} + \tau_j^{(2)}.
\]

Corrections up to second order in \( (\xi_{ij}/\xi_i) \) were computed by Bernardeau & Schaeffer (1992). These and the next higher-order terms are as follows:

\[
\tau_i^{(0)} = -y_i G'(\tau_i^{(0)})
\]

\[
\tau_i^{(1)} = \frac{1}{[1 + y_i G''(\tau_i^{(0)})]} \sum_{i \neq j} \frac{\xi_{ij}}{\xi_i} y_j G'(\tau_j^{(0)}) = \frac{1}{[1 + y_i G''(\tau_i^{(0)})]} \sum_{i \neq j} \frac{\xi_{ij}}{\xi_i} \tau_j^{(0)}
\]
\( \tau_i^{(2)} = -\frac{1}{1 + y_i G''(\tau_i^{(0)})} \left[ y_i \tau_i^{(1)2} G''(\tau_i^{(0)}) + \sum_{i \neq j} \frac{\xi_{ij} y_j G''(\tau_j^{(0)}) \tau_j^{(1)}}{\xi_i} \right] \)

\( \tau_i^{(3)} = -\frac{1}{1 + y_i G''(\tau_i^{(0)})} \left[ y_i \tau_i^{(1)3} G'''(\tau_i^{(0)}) + y_i \frac{\tau_i^{(1)3}}{3!} G^{IIV}(\tau_i^{(0)}) - \sum_{i \neq j} \frac{\xi_{ij} y_j G''(\tau_j^{(0)}) \tau_j^{(1)2} + \xi_{ij} y_j G''(\tau_j^{(0)}) \tau_j^{(1)2} \tau_j^{(1)}}{\xi_i} \right] \)

\( \tau_i^{(4)} = -\frac{1}{1 + y_i G''(\tau_i^{(0)})} \left[ \frac{\tau_i^{(1)2}}{2} G''(\tau_i^{(0)}) + \frac{\tau_i^{(1)2}}{4!} G^{IV}(\tau_i^{(0)}) + \frac{\tau_i^{(1)4}}{4!} G^{IV}(\tau_i^{(0)}) \right] + \sum_{i \neq j} \frac{\xi_{ij} y_j \left( G''(\tau_j^{(0)}) \tau_j^{(3)} + G''(\tau_j^{(0)}) \tau_j^{(1)2} \tau_j^{(1)} + G^{IV}(\tau_j^{(0)}) \tau_j^{(1)2} \right)}{\xi_i} \right] \]

 Certain predictions of such a series expansion have already been tested against numerical simulations, at least to linear order. Such predictions from the tree-level hierarchy relate to the two-point cumulant correlators (2CCs) which satisfy the relation \( \langle \delta^p(x_1) \delta^q(x_2) = [C_{pq} C_{q1} \xi_{ab}] \rangle \). This property of factorization is related to the fact that the bias associated with an overdense object is an intrinsic property of the object. The bias associated with a pair comprising two different objects can be factorised into two separate quantities which each represent the intrinsic bias of one constituent. In recent work Munshi & Melott (1998) were able to show that predictions such as \( C_{pq} = C_{q1} C_{qj} \) are satisfied very accurately in highly non-linear regime when two over-dense cells are not separated by a large distance and \((\xi_{ij} / \bar{E}) < 1\) is satisfied. Other related predictions regarding bias have also been tested successfully by Munshi et al. (1998). The Hierarchical ansatz implies natural predictions for other higher order quantities, such as one point cumulants for collapsed objects which we will also compute here. A comparison of such predictions against numerical simulations is left for future work.

In a similar vein, we can consider a series expansion of \( \chi(\lambda_1, \ldots, \lambda_q) \) in \( (\xi_{ij} / \bar{E}) \) we get higher order correction terms that involve contribution from the \( \tau_i \) to the same order. In leading order it can be shown that all over-dense cells are uncorrelated. Each higher-order term encodes the effect of one more extra neighbouring cells on occupancy of a given single cell.

\[ \chi^{(0)}(\lambda_i) = -\sum_{i} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} \tau_i^{(0)} G''(\tau_i^{(0)}) \]

\[ \chi^{(1)}(\lambda_i) = -\sum_{i} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) \tau_i^{(1)} + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G''(\tau_i^{(0)}) \tau_i^{(1)} + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G'''(\tau_i^{(0)}) \tau_i^{(1)2} \tau_i^{(1)} \]

\[ \chi^{(2)}(\lambda_i) = -\sum_{i} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) \tau_i^{(2)} + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G''(\tau_i^{(0)}) \tau_i^{(2)} + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G'''(\tau_i^{(0)}) \tau_i^{(2)2} \tau_i^{(1)} \]

\[ \chi^{(3)}(\lambda_i) = -\sum_{i} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) \tau_i^{(3)} - \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G''(\tau_i^{(0)}) \tau_i^{(3)} - \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G'''(\tau_i^{(0)}) \tau_i^{(3)} + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} G''''(\tau_i^{(0)}) \tau_i^{(3)} \]

\[ \chi^{(4)}(\lambda_i) = -\sum_{i} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) \tau_i^{(4)} - \sum_{i} \frac{y_i}{\xi_i} G''(\tau_i^{(0)}) \tau_i^{(4)} \]

After a tedious but straightforward exercise in algebra it is possible to express every order of \( \chi(\lambda_1, \ldots, \lambda_q) \) in terms of \( \tau_i^{(0)} \). It is interesting to note that every new power of \( (\xi_{ij} / \bar{E}) \) in the expansion of \( \chi(\lambda_1, \ldots, \lambda_q) \) adds a new “star” vertex of same order. What is most remarkable however is that such an expansion is also able to reproduce other diagrams such as the “snake” and hybrid of “stars” and “snakes” of lower orders.

\[ \chi^{(0)}(\lambda_i) = -\sum_{i=1}^{q} \frac{y_i}{\xi_i} G'(\tau_i^{(0)}) + \frac{1}{2} \sum_{i} \frac{y_i}{\xi_i} \tau_i^{(0)} G'(\tau_i^{(0)}) \]

\[ \chi^{(1)}(\lambda_i) = \frac{1}{2} \sum_{i} \sum_{i \neq j} \frac{\tau_i^{(0)} \xi_j \tau_j^{(0)} \xi_j \tau_j^{(0)}}{\xi_i} \]

\[ \chi^{(2)}(\lambda_i) = \frac{1}{2} \sum_{i} \sum_{i \neq j} \sum_{i \neq k} \frac{\tau_i^{(0)} \xi_j \xi_j \tau_j^{(0)} \xi_j \tau_j^{(0)}}{\xi_i} \]
\[ \chi^{(3)}(\lambda_i) = \frac{1}{3!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \frac{\mu_3(\tau^{(0)}_i)}{\xi_i} \frac{\tau^{(0)}_j}{\xi_j} \frac{\tau^{(0)}_k}{\xi_k} \frac{\tau^{(0)}_l}{\xi_l} \text{ Star Topology (4)} \]

\[ + \frac{1}{2} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \frac{\tau^{(0)}_i}{\xi_i} \frac{\tau^{(0)}_j}{\xi_j} \frac{\mu_2(\tau^{(0)}_k)}{\xi_k} \frac{\tau^{(0)}_l}{\xi_l} \text{ Snake Topology (12)} \]

\[ \chi^{(4)}(\lambda_i) = \frac{1}{4!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \mu_4(\tau^{(0)}_i) \frac{\tau^{(0)}_j}{\xi_j} \frac{\tau^{(0)}_k}{\xi_k} \frac{\tau^{(0)}_l}{\xi_l} \text{ Star Topology (5)} \]

\[ + \frac{1}{2!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \frac{\tau^{(0)}_i}{\xi_i} \frac{\tau^{(0)}_j}{\xi_j} \frac{\mu_2(\tau^{(0)}_k)}{\xi_k} \frac{\mu_2(\tau^{(0)}_l)}{\xi_l} \text{ Hybrid Topology (60)} \]

\[ + \frac{1}{2!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \frac{\tau^{(0)}_i}{\xi_i} \frac{\tau^{(0)}_j}{\xi_j} \frac{\mu_2(\tau^{(0)}_k)}{\xi_k} \frac{\mu_2(\tau^{(0)}_l)}{\xi_l} \text{ Snake Topology (60). (20)} \]

We have reintroduced quantities \( \mu_n \) that represent vertices in the tree-level diagrams of MCCs, \( n \) representing the number of external legs associated with such vertices. In terms involving ordered sums we can replace \( \frac{1}{2} \sum_{i \neq j} \) by \( \sum_{\text{pairs}} \) (i.e. sum over all pairs) and similarly \( \frac{3}{3!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \) by \( \sum_{\text{triplets}} \) (i.e. sum over all triplets) and additional terms which contribute to loop corrections to lower order terms. Similarly \( \frac{4}{4!} \sum_{i \neq j} \sum_{j \neq k} \sum_{k \neq l} \sum_{l \neq m} \) can be replaced by \( \sum_{\text{quadruplets}} \) and loop contribution to lower order terms.

It is also pertinent to note that, at every order, the hierarchical ansatz reproduces the underlying tree structure of matter correlation, but with different amplitudes for the vertices. It also produces different loop corrections to lower order diagrams. Since we are only interested in the leading order contributions to the statistics of collapsed objects (important in large separation limit) we will only focus on tree level diagrams. If we however focus on statistics of the initial Gaussian field it is important to note that since the reduced correlation functions of odd orders are zero, one needs to consider these corrective terms which are neglected here as in this case they will provide the dominant contributions.

The expression of \( \mu_n \) to arbitrary order can be used to construct statistics of collapsed objects, such as the probability distribution function or the void probability function, quantities which are generally easier to extract from numerical simulations than the higher order moments themselves. We begin with:

\[
\begin{align*}
\mu_1(y) & = -yg'(\tau(y)) \\
\mu_2(y) & = -yg''(\tau(y)) \\
\mu_3(y) & = -yg'''(\tau(y)) \\
\mu_4(y) & = \frac{-yg^{(IV)}(\tau(y))}{1 + yg''(\tau(y))^2} + 3y^2g'''(\tau(y))^2 \\
\mu_5(y) & = \frac{-yg^{(V)}(\tau(y))}{1 + yg''(\tau(y))^3} + 10y^2g^{(IV)}(\tau(y))g''(\tau(y)) - 15y^3g'''(\tau(y))^3 \\
\end{align*}
\]

The series expansion of these functions will reproduce terms which we have earlier obtained from counting internal degrees of freedoms associated with different nodes in tree representation of the RCCs. One can then construct the successive approximations to \( \mathcal{P} \) as follows:

\[
\begin{align*}
\mathcal{P}^{(0)}(\lambda_1, \ldots, \lambda_q) & = \prod_i \mathcal{P}(\lambda_i) \\
\mathcal{P}^{(1)}(\lambda_1, \ldots, \lambda_q) & = \sum_{i,j} \frac{\tau^{(0)}_{ij}}{\xi_i} \mathcal{P}(\lambda_i) \mathcal{P}(\lambda_j) \prod_{k \neq i,j} \mathcal{P}(\lambda_k) \\
\mathcal{P}^{(2)}(\lambda_1, \ldots, \lambda_q) & = \sum_{i,j,k} \frac{\mu_2(y)}{\xi_i} \mathcal{P}(\lambda_i) \mathcal{P}(\lambda_j) \mathcal{P}(\lambda_k) \prod_{l \neq i,j,k} \mathcal{P}(\lambda_l) \\
\mathcal{P}^{(3)}(\lambda_1, \ldots, \lambda_q) & = \sum_{i,j,k,l} \frac{\mu_3(y)}{\xi_i} \mathcal{P}(\lambda_i) \mathcal{P}(\lambda_j) \mathcal{P}(\lambda_k) \mathcal{P}(\lambda_l) \prod_{m \neq i,j,k,l} \mathcal{P}(\lambda_m) \\
\end{align*}
\]
Using the generating functions we have obtained (21), we can then express the count probability distribution function to successive order:

\[ P^{(0)}(N) = \prod_i P(N_i); \]

\[ P^{(1)}(N) = \sum_{k \neq i,j} P(N_k) \prod_{i,j \neq k} b(N_j) P(N_j) \xi_{ij} P(N_i) b(N_i); \]

\[ P^{(2)}(N) = \sum_{i \neq j,k} P(N_i) \prod_{i,j \neq k} \nu_2(N_i) P(N_i) \xi_{ij} P(N_j) b(N_i) \xi_{jk} P(N_k) b(N_k); \]

\[ P^{(3)}(N) = \sum_{m \neq i,j,k,l} P(N_m) \prod_{i,j,k,l \neq m} \nu_3(N_m) P(N_m) \xi_{ij} P(N_i) b(N_i) \xi_{jk} P(N_j) b(N_j) \xi_{kl} P(N_k) b(N_k); \]

\[ P^{(4)}(N) = \sum_{r \neq i,j,k,l,m} P(N_r) \prod_{i,j,k,l,m \neq r} \nu_4(N_r) P(N_r) \xi_{ij} P(N_i) b(N_i) \xi_{jk} P(N_j) b(N_j) \xi_{kl} P(N_k) b(N_k) \xi_{lm} P(N_l) b(N_l) \xi_{mn} P(N_m) b(N_m). \]

The quantity \( P(\lambda) \) plays the role of the generating function for the one-point probability distribution \( P(N) \). If one incorporates the "bias" for a cell of occupation number \( N \) by \( b(N) \) then the generating function for \( P(N)b(N) \) can be seen to be \( P(\lambda) \tau(y)/\xi \). Similarly, \( P(\lambda)\mu_s(y)/\xi \) is the generating function for \( P(N)\nu_s(N) \). To summarise:

\[ \sum \lambda^N P(N) = P(\lambda); \]

\[ \sum \lambda^N P(N)b(N) = P(\lambda) \frac{\tau(\lambda)}{\xi_2}; \]

\[ \sum \lambda^N P(N)\nu_2(N) = P(\lambda) \frac{\mu(\lambda)}{\xi_2}; \]

\[ \sum \lambda^N P(N)\nu_s(N) = P(\lambda) \frac{\mu_s(\lambda)}{\xi_2}. \]

Using the generating functions we have obtained (21), we can then express the count probability distribution function to successive order:

\[ P^{(0)}(N) = \prod_i P(N_i); \]

\[ P^{(1)}(N) = \sum_{k \neq i,j} P(N_k) \prod_{i,j \neq k} b(N_j) P(N_j) \xi_{ij} P(N_i) b(N_i); \]

\[ P^{(2)}(N) = \sum_{i \neq j,k} P(N_i) \prod_{i,j \neq k} \nu_2(N_i) P(N_i) \xi_{ij} P(N_j) b(N_i) \xi_{jk} P(N_k) b(N_k); \]

\[ P^{(3)}(N) = \sum_{m \neq i,j,k,l} P(N_m) \prod_{i,j,k,l \neq m} \nu_3(N_m) P(N_m) \xi_{ij} P(N_i) b(N_i) \xi_{jk} P(N_j) b(N_j) \xi_{kl} P(N_k) b(N_k); \]

\[ P^{(4)}(N) = \sum_{r \neq i,j,k,l,m} P(N_r) \prod_{i,j,k,l,m \neq r} \nu_4(N_r) P(N_r) \xi_{ij} P(N_i) b(N_i) \xi_{jk} P(N_j) b(N_j) \xi_{kl} P(N_k) b(N_k) \xi_{lm} P(N_l) b(N_l) \xi_{mn} P(N_m) b(N_m). \]

where we have used the notation \( P^{(i)}(N) = P^{(i)}(N_1, \ldots, N_q) \) for arbitrary number of cells \( q \). The product terms involve pairs for \( i = 1 \), triplets for \( i = 2 \), quadruplets for \( i = 3 \) and pentuplets for \( i = 4 \). The functions \( b(N) = \mu_2(N) \) and \( \mu_3(N) \), which describe the effect of correlations between different cells, will each satisfy two normalization conditions satisfied by the multi-point count probability distribution function (CPDF) \( P(N_1, \ldots, N_k) \). The first set of these conditions is obtained by taking the first order moment of \( P(N_1, \ldots, N_k) \) with respect to all but one of the \( N_k \), and the second set is obtained by taking moments with respect to all \( N_k \) and using the definition of the underlying matter correlation function. Hence we impose

\[ \sum P(N)\nu_s(N) = 0; \quad \sum N P(N)\nu_s(N) = \bar{N}\nu_s, \]

where \( \nu_s \) are the amplitudes associated with vertices of underlying mass distribution in the highly nonlinear regime.

Using inverse Laplace transforms, it is now possible to relate the distributions \( P(N) \) or \( P(N)\nu_s(N) \) in terms of their generating functions \( P(\lambda) \) or in terms of \( \mu_s(y) \). In turns out that the results can be expressed in terms of functions of a scaling
variable \( x = N/N_c \), where \( N_c = \bar{N}_2 \) and \( \bar{N} \) is the typical occupation number of cells under consideration.

\[
P(N) = \frac{1}{2\pi i} \int dy \frac{\mu_2(y)}{y} \exp(yx); \\
\mu_2(x) = \frac{1}{2\pi i} \int dy \frac{\mu_2(y)}{y} \exp(yx); \\
\nu_2(x) = \frac{1}{2\pi i} \int dy \frac{\mu_2(y)}{y} \exp(yx); \\
\nu_3(x) = \frac{1}{2\pi i} \int dy \frac{\mu_3(y)}{y} \exp(yx); \\
\nu_4(x) = \frac{1}{2\pi i} \int dy \frac{\mu_4(y)}{y} \exp(yx),
\]

In turn the scaling functions \( h(x) \) and \( b(x) \), which describe the behaviour of \( P(N) \), \( b(N) \) and \( \nu_s(N) \) for different length scales and for different levels of non-linearity, are expressed in terms of the vertex-generating functions \( \mu_s(y) \) we introduced earlier in equation (21):

\[
h(x) = -\frac{1}{2\pi i} \int \frac{d\lambda}{\lambda^{N+1}} \frac{\tau(\lambda)}{\xi} \frac{1}{\xi_2 N_c} h(x) \]

\[
P(N)b(N) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda^{N+1}} \frac{\tau(\lambda)}{\xi} \frac{1}{\xi_2 N_c} b(x)\]

\[
P(N)\nu_2(N) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda^{N+1}} \frac{\mu_2(\lambda)}{\xi} \frac{1}{\xi_2 N_c} \nu_2(x) \]

\[
P(N)\nu_3(N) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda^{N+1}} \frac{\mu_3(\lambda)}{\xi} \frac{1}{\xi_2 N_c} \nu_3(x).
\]

(26)

In this way are much more stable than differential count statistics and easier to test against numerical simulations, as they depend less on specific models for underlying mass distribution characterized by \( G(\tau) \).

4 MULTI-POINT CUMULANTS AND STATISTICS OF COLLAPSED OBJECTS

We can now discuss the statistics of overdense cells in terms of their vertex representation. Using the results of the previous section, we can assign a weight \( M_s \) to vertices of the trees representing the distribution of overdense cells, such that with \( M_s \) plays the same role as \( \nu_s \) in the tree expansion for the underlying distribution. A complete knowledge of all the nodes for overdense cells (or their generating function) guarantees a full statistical description for collapsed objects, including the
one-point cumulants $S^c_N$ and cumulant correlators $C^c_{NM}$, the subscripting referring to the collapsed objects which we assume can be selected by applying a density threshold to the cells. In the following calculations give expressions for orders $\leq 5$. The quantities $M_n$ can be written

$$M_n(N) = \frac{\nu_4(N)}{b^2(N)}; \quad M_n(> N) = \frac{\nu_4(N)}{b^2(N)}.$$

We can write the one-point cumulants in terms of the amplitudes $M_n$ as follows, which are similar to the case of underlying mass distribution:

$$S^c_0(> x) = 3\frac{\nu_2(> x)}{b^2(> x)} = 3M_4(> x)$$
$$S^c_2(> x) = \frac{4\nu_2(> x)}{b^2(> x)} + 12\left(\frac{\nu_2(> x)}{b^2(> x)}\right)^2$$
$$= 4M_3(> x) + 12M_2(> x)^2$$
$$S^c_4(> x) = \frac{5\nu_2(> x)}{b^2(> x)} + 60\frac{\nu_3(> x)}{b^3(> x)} \left(\frac{\nu_2(> x)}{b^2(> x)}\right)^3$$
$$+ 60\left(\frac{\nu_2(> x)}{b^2(> x)}\right)^3$$
$$= 5M_4(> x) + 60M_3(> x)M_2(> x) + 60M_2(> x)^3.$$

The two-point cumulants are given by

$$C^c_{21}(> x) = 2\frac{\mu_2(> x)}{b(> x)^2} = 2M_2(> x)$$
$$C^c_{31}(> x) = 3\frac{\mu_3(> x)}{b(> x)^3} + 6\left(\frac{\mu_2(> x)}{b(> x)}\right)^2$$
$$= 3M_4(> x) + 6M_2(> x)^2$$
$$C^c_{41}(> x) = 4\frac{\mu_4(> x)}{b(> x)^3} + 36\frac{\mu_3(> x)}{b^2(> x)^3} \left(\frac{\mu_2(> x)}{b(> x)}\right)^2$$
$$+ 24\left(\frac{\mu_2(> x)}{b(> x)}\right)^3$$
$$= 4M_4(> x) + 36M_3(> x)M_2(> x) + 24M_2(> x)^3.$$

In the following two subsections we obtain approximate forms for the amplitudes $M_n$ in two different limiting cases. For the determination of $S_N$ and $C_{NM}$ parameters for all values of mass it is necessary to evaluate contour integrals relating $\nu_2(x)$ and $\mu_2(y)$ numerically, an exercise we do not attempt here.

### 4.1 The case of moderately over-dense objects ($x << 1$)

The asymptotic forms of the different scaling functions in the limit $x << 1$ are depend upon the limiting behaviour of the scaling functions when $y >> 1$. In this case, the scaling function $\sigma(y)$ is known to exhibit a power law profile with amplitude $a$ and power-law index $\omega$ both depending on initial power spectral index (for a detail derivation see Balian & Schaeffer 1989, and Bernardeau & Schaeffer 1992):

$$\sigma(y) = ay^{-\omega}; \quad h(x) = a(1 - \omega)\Gamma(\omega)x^{-\omega - 2},$$

where $\Gamma(\cdot)$ is the Gamma function, and we use the specific form of $G(\tau)$ given by Bernardeau & Schaeffer (1992):

$$G(\tau) = \left(1 + \frac{\tau}{\kappa}\right)^{-\kappa}.$$  

The asymptotic behaviour of $G(\tau)$ can then be expressed as

$$G(\tau) \propto c\tau^{-\kappa},$$

and this can be used to compute the asymptotic form for $\sigma(y)$:

$$\tau(y) = (y\omega\kappa)^{1/(1+\kappa)}$$
$$\sigma(y) = (1 + \frac{\kappa}{2}\kappa(\omega, 2 + \kappa))^{-\kappa/2}\kappa^\kappa y^{\kappa(\omega, 2 + \kappa)}.$$

Using these expressions we can relate $c$ and $\kappa$ with $a$ and $\omega$:

$$\kappa = \frac{2\omega}{(1 - \omega)}(kc)^{1/(k+2)} = (2\omega)^{1/2},(1 - \omega)\kappa = k^{-\omega}c^{1 - \omega}.$$  

It is now possible to express the asymptotic form of $\mu_n$ as a function of $y$ and in terms of parameters describing the asymptotic behaviour of $\sigma(N_c)$, i.e. $a$ and $\omega$:

$$\mu_1(y) = \tau(y) = (2\omega)^{1/2}y^{(1-\omega)/2}.$$
Figure 3. Lower-order cumulants and cumulant correlators are plotted as a function of $\omega$ for the limiting case of $x << 1$. There are no analytical results connecting $\omega$ with the initial spectral index, $n$, but simulations suggest that $\omega$ increases with the amount of small-scale power. Determinations of $\omega$ for different scale free initial conditions in 2D and 3D were made by Munshi et al. (1997). Our results show that both cumulants and cumulant correlators for overdense regions increase with the amount of small-scale power, which reflects similar a behaviour for matter cumulants and cumulant correlators.

\[
\begin{align*}
\mu_2(y) & = \frac{1 + \omega}{2} \\
\mu_3(y) & = \frac{1 - \omega^2}{4} (2\omega a)^{-1/2} y^{-(1-\omega)/2} \\
\mu_4(y) & = \frac{(1 - \omega^2)(6 + 2\omega)}{8} (2\omega a)^{-1} y^{-(1-\omega)}. 
\end{align*}
\]

Using these asymptotic forms and the definitions of $h(>x)\mu_n(>x)$ introduced earlier in equations (28), we can can write

\[
\begin{align*}
\nu_1(>x)h(>x) & = h(>x)\mu_1(>x) = (2\omega a)^{1/2} x^{(\omega-1)/2} \\
\nu_2(>x)h(>x) & = \frac{1 + \omega}{2} \\
\nu_3(>x)h(>x) & = \left(\frac{1 - \omega^2}{4}\right) (2\omega a)^{-1/2} x^{(1-\omega)/2} \\
\nu_4(>x)h(>x) & = \frac{(1 - \omega^2)(6 + 2\omega)}{64\omega^3} (2\omega a)^{-1} x^{2(1-\omega)} \\
\nu_5(>x)h(>x) & = \frac{(1 - \omega^2)(6 + 2\omega)}{64\omega^3} (2\omega a)^{-1} x^{2(1-\omega)} \\
\end{align*}
\]

It is interesting to note the power of the bias $b(>x)$ that appears in each of these expressions is precisely that required that, when divided by the appropriate quantity to evaluate $M_s$, the parameters for collapsed objects in the limit $x << 1$ all become independent of $x$:

\[
\begin{align*}
M_3(x) & = \frac{(1 + \omega) \Gamma\left[\frac{1+\omega}{2}\right]}{4 \Gamma[1 + \omega];} \\
M_4(x) & = \frac{(1 - \omega^2) \Gamma\left[\frac{1+\omega}{2}\right]^3}{16\omega^2 \Gamma[\omega^2 \Gamma[\frac{2-\omega}{2}];} \\
M_5(x) & = \frac{(1 - \omega^2)(6 + 2\omega)}{64\omega^3} \frac{\Gamma\left[\frac{1+\omega}{2}\right]^4}{\Gamma[\omega^2 \Gamma[2 - \omega].} 
\end{align*}
\]

The implications for cumulants and cumulant correlators of overdense regions are illustrated in Figure 3.
4.2 The case of highly over-dense objects ($x >> 1$)

It is also possible to evaluate the $S_N$ parameters in the limit when $x >> 1$. The dominant contribution to this case comes from the singular nature of $\sigma(N_x)$ for small negative values of $y$. The value of $y = y_s$ for which $\sigma(y)$ becomes singular is given by the solution of the equation

$$y_s = -\frac{\tau_s}{G'(\tau_s)},$$

(40)

where $\tau_s$ can be evaluated by solving the implicit equation

$$\tau_s = \frac{G'(\tau_s)}{G''(\tau_s)}$$

(41)

which really depends on the exact modelling of $G'(\tau)$ in the nonlinear regime. However, as we will show, in the limit of rare events where $x$ takes very large values, the hierarchical amplitudes become independent of the initial conditions and this considerably simplifies things. The quantity $\sigma(y)$ plays a very important role in determining various scaling properties and is known to have a singularity for small but negative values of $y$ which determines the asymptotic values of $S_N$ for large values of $x$. We can express $\sigma(y)$ near the singularity as

$$\sigma - \sigma_s = -a_s \Gamma\left( -\frac{3}{2} \right)(y - y_s)^{3/2};$$

(42)

with

$$a_s = \frac{1}{\Gamma\left( -\frac{3}{2} \right)} G'(\tau_s) G''(\tau_s) \left( \frac{2G'(\tau_s) G'''(\tau_s)}{G''(\tau_s)} \right).$$

(43)

Such a singularity produces an exponential cut-off in the power law profile of $h(x)$ above, and which we can express as

$$h(x) \approx -a_s y_s x^{-5/2} \exp(y_s x).$$

(44)

Differentiating $y$ twice with respect to $x$ and using the fact that the first order derivative of $\tau$ with respect to $y$ vanishes at the singular point $y_s$ it is possible to show that, near the singular point,

$$(\tau - \tau_s) = \left( \frac{2G'(\tau_s) G'''(\tau_s)}{G''(\tau_s)} \right)^{1/2} (y - y_s)^{1/2}.$$

(45)

Similarly, the other $\mu_n(y)$ functions can also be expressed as a function of $y$ near the singularity $y_s$. We list only those terms which will dominate in determining the $S_N$ parameters for large values of $x$.

$$\mu_2(y) = \frac{-y G''(\tau)}{1 + y G''(\tau)} - \frac{G''(\tau)}{G''(\tau)} \left( \frac{2G'(\tau) G'''(\tau)}{G''(\tau)} \right)^{-1/2} (y - y_s)^{-1/2},$$

$$\mu_3(y) = \frac{-y G'''(\tau)}{(1 + y G''(\tau))^3} \left( \frac{2G'(\tau) G'''(\tau)}{G''(\tau)} \right)^{-3/2} (y - y_s)^{-3/2}$$

$$\mu_4(y) = \frac{-y G^{(IV)}(\tau)}{(1 + y G''(\tau))^4} + \frac{3y^2 G''''(\tau)}{(1 + y G''(\tau))^5} = \frac{3G'(\tau)^3}{G''(\tau)^2 G'(\tau)} \left( \frac{2G'(\tau) G'''(\tau)}{G''(\tau)} \right)^{-5/2} (y - y_s)^{-5/2}.$$

(46)

Using these relations in the expressions for $\mu_n(> x)$, it is possible to show that,

$$b(> x) = \mu_1(> x) = -\left( \frac{x}{G'(\tau_s)} \right)$$

$$\mu_2(> x) = \left( \frac{x}{G'(\tau_s)} \right)^2$$

(47)

$$\mu_3(> x) = -\left( \frac{x}{G'(\tau_s)} \right)^3$$

$$\mu_4(> x) = \left( \frac{x}{G'(\tau_s)} \right)^4.$$

(48)

Using the appropriate definitions of the $S_N$ and $C_{NM}$ parameters, it is now straightforward to obtain

$$S_N = N^{N-2}, \quad C_{N1} = N^{N-1}, \quad C_{N11} = N^N.$$  

(49)

It is interesting that all the $\mu_n(> x)$ become equal to unity in this limit, so that the $S_N$ parameters are simply equal to the number of tree diagrams of that order, i.e. $N^{N-2}$. Exactly the same result has also been derived from the extended Press-Schechter formalism (Press & Schechter 1974; Mo et al. 96). It is remarkable and encouraging that the hierarchical ansatz, which is known to be valid only in the highly non-linear regime, predicts the same results as this alternative formalism, which is based on smoothing of the initial density field.
5 DISCUSSION

Our starting point for this study has been the most general form of hierarchical model for the matter correlation functions in the highly non-linear regime. This is equivalent to the assumption that the correlation hierarchy can be built up as a tree structure where each vertex is assigned a different weight (the vertex amplitude), but which is independent of geometrical form factors. The amplitudes associated with vertices are left arbitrary. We have adopted the method of generating functions introduced by Bernardeau & Schaeffer (1992) to show that such a hierarchy induces a similar tree structure for overdense cells, which we take to represent collapsed objects.

We then expanded the generating functions for the joint probability distributions of several cells in terms of a series in powers of \( \xi_{ij}/\bar{\xi}_i \). At zeroth order it reproduces one-point PDF with no correlation between different cells. At linear order of \( \langle \xi_{ij}/\bar{\xi}_i \rangle \) two-point correlations between different cells can be used to compute the bias of overdense cells with respect to the underlying matter distribution. At second order of \( \langle \xi_{ij}/\bar{\xi}_i \rangle \) a hierarchical form for the three-point correlation of overdense cells is reproduced and this can be used to compute the hierarchical amplitude \( M_3 \) associated with collapsed objects. Similarly, each higher-order term reproduces the amplitude of a new vertex and the associated multipoint cumulant correlators (MCCs) for collapsed objects. Our analysis is also able to reproduce the exact number of all the higher-order hybrid and snake diagrams with their associated tree amplitudes. We found that the one-point and the multi-point statistics of collapsed objects depend physically on properties intrinsic to the objects, and thus depend mathematically only on the scaling variable \( x = N/N_c \).

Assuming a very general model for the generating function \( G(\tau) \), for the underlying mass we have computed cumulants \( S_N \) and cumulant correlators \( C_{ij} \) for overdense cells. We have carried out this analysis in two limiting cases, of moderately overdense cells \( x << 1 \) and extremely overdense cells \( x >> 1 \), respectively. We find that for extremely overdense cells the results are independent of initial conditions, the amplitudes associated with vertices of arbitrary order tend asymptotically to unity, and the generating functions can be expressed in a closed form \( G(\tau) = \exp(-\tau) \). This closed-form solution can be used to compute all other statistical quantities for over-dense cells such as the count probability distribution function (CPDF) or the void probability function (VPF) for overdense cells. For moderately overdense cells, the results depend on the initial power spectrum and we find that the \( S_N \) parameters for overdense cells decrease with increasing small-scale power in this case.

We have also developed a diagrammatic method for counting different ways in which a general correlation function \( \xi_N \) with \( N \) arguments can be grouped in such a way that \( p \) of its nodes are associated with one particular cell, \( q \) are in a second cell, and so on. In leading order such a grouping of vertices in different cells generates a new level of the tree hierarchy which can be related to the correlation hierarchy for overdense cells. The only difference between the trees for the underlying mass distribution and those for collapsed objects lies in the internal structure of their constituent nodes. Trees associated with collapsed objects have internal degrees of freedom for different vertices that appear in different ways. As we have shown, however, it is possible to group nodes associated with underlying mass trees in such a way as to replicate the tree structure for collapsed objects with same external topology but different internal weights. Our method of counting internal degrees of freedom associated with the vertices matches with analytical results obtained by more laborious means.

In an exactly similar manner it is possible to use the trees associated with collapsed objects to study how clusters or groups of collapsed objects are correlated with each other. In this case, the nodes associated with groups of collapsed objects will have many more internal degrees of freedom as they will consist of nodes of tree structures of collapsed objects which themselves are made of point-like vertices of the underlying mass distribution. Amplitudes associated with nodes appearing in the tree structure of groups of collapsed objects will definitely depend on the mass of objects that constitute the group, and will have scaling properties associated with them. We shall present a complete analysis of this problem in near future.

From an observational point of view, the results contained in this paper are important for estimating finite volume corrections. It is well known that the finite size of galaxy catalogs introduce errors in estimations of cumulants and cumulant correlators. Error estimation for values of the \( S_N \) parameters – normalised moments of the CPDF – extracted from finite catalogs involve volume integrals of the two-point joint probability distribution function. Such an analysis, carried out by Szapudi & Colombi (1996), involves the generating function \( \mu_1(y) = \tau(y) \). Similar estimation procedures for multi-point cumulant correlators will involve higher-order generating functions describing correlation between more than two cells. Our analytical expressions for \( \mu_N \) will be useful in such analyses. In a slightly different approach, it is also possible to talk about the probability distribution in the context of error estimation. Estimated values of different statistical quantities are affected by the presence or absence of rare objects in a finite size catalog. The higher the order of cumulant, the higher the contribution from the high-\( N \) tail of the CPDF, which is highly sensitive to presence or absence of rare objects. It is natural, therefore, to expect that our analytical results determining the generating function for rare objects can be related to the probability distribution of estimated errors. This will extend earlier studies in this direction where only the variance of errors were considered. This is important because we know that finite volume correction not only introduces large error but measurements are more likely to underestimate the true values of cumulants or cumulant correlators.

Cumulants for collapsed objects have also been computed using alternative procedures, chiefly by extension of the Press-Schechter formalism (Bond et al. 1991; Bower 1992). Our results complement these studies, because they are based on a fully nonlinear approach and depend only on the assumption that there is a hierarchical structure of the correlation functions. We think this ansatz is based in relatively firm ground, and can, at least in principle, be directly related to the microscopic physics of gravitational clustering through the BBGKY equations. The analysis can also be applied the MCCs of arbitrary order which have not yet been investigated using any other formalism. Our results for computation of \( S_N \) parameters for collapsed objects match with such an analysis done using extended Press-Schechter formalism. However similar comparison
for moderately over-dense objects are difficult as different prescriptions are used to define collapsed objects in these two formalisms. We plan to compare our theoretical predictions against numerical simulations both in 3D as well as in 2D, and will present these results in due course.

It is also important to note that, owing to our incomplete knowledge of gravitational clustering in the highly non-linear regime, at least two different ansätze are often used to study nonlinear clustering, namely the hierarchical ansatz (which we discuss here) and the stable clustering ansatz and these should not be confused. While the hierarchical ansatz relates higher-order correlation functions with the two-point correlation function, the stable clustering ansatz on the other hand relates the power-law index of the two-point correlation with the initial power spectral index. Thus these two ansätze are not contradictory but complement each other. Our analysis is based completely on the hierarchical ansatz, and we have never used stable clustering to derive our results.

It may also be possible to extend our analysis to the joint probability distribution function of densities of concentric cells, with their centres coinciding with the centre of a dark halo. In such a context it will be interesting to check if our analysis can also predict the inner structure of haloes, i.e. their density profiles. This will also clarify the issue of whether there exists a universal halo profile as proposed by Navarro et al. (1996, 1997) and how it relates to stable clustering and to the hierarchical ansatz. We will report these results elsewhere.

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REFERENCES

Balian R., Schaeffer R., 1989, A & A, 220, 1
Bardeen J.M., Bond J.R., Kaiser N., Szalay A.S., 1986, ApJ, 304, 15
Bernardeau F., 1992, ApJ, 392, 1
Bernardeau F., 1994, ApJ, 433, 1
Bernardeau F., 1995, A&A, 301, 309
Bernardeau F., Schaeffer R., 1992, A&A, 255, 1
Bond J.R., Cole S., Efstathiou G., Kaiser N., 1991, ApJ, 379, 440
Bond J.R., Myers S.T., 1996a, ApJS, 103, 1
Bond J.R., Myers S.T., 1996b, ApJS, 103, 41
Bonometto S.A., Borgani S., Parsch M., Salucci P., 1990, ApJ, 356, 350
Bosch P., Szapudi I., Szalay A.S., 1994, ApJS, 93, 65
Bower R.J., 1991, MNRAS, 248, 332
Baugh C.M., Gaztanaga E., 1996, MNRAS, 280, L37
Catelan P., Lucchin F., Matarrese S., Porciani C., 1998, MNRAS, 297, 692
Colombi S., Bouchet F.R., Hernquist L., 1996, ApJ, 465, 14 [astro-ph/9610253]
Davis M., Peebles P.J.E., 1977, ApJS, 34, 425
Frenk C.S., White S.D.M., Davis M., Efstathiou G., 1988, ApJ, 327, 507
Fry J.N., 1982, ApJ, 262, 424
Fry J.N., 1984, ApJ, 279, 499
Fry J.N., Peebles P.J.E., 1978, ApJ, 211, 19
Groth E., Peebles P.J.E., 1977, ApJ, 217, 385
Hamilton A.J.S., 1988, ApJ, 332, 67
Kaiser N., 1984, ApJL, 284, L9
Katz N., Quinn T., Gelb J.M., 1993, MNRAS, 265, 689
Kauffmann G.A.M., White S.D.M., 1993, MNRAS, 261, 921
Lacey C., Cole S., 1993, MNRAS, 262, 627
Lee J., Shandarin S.F., 1998 ApJ 500, 14
Lee J., Shandarin S.F., astro-ph/9803222
Munshi D., Bernardaeau F., Melott A.L., Schaeffer R., 1998 MNRAS, in press astro-ph/9707009
Munshi D., Melott A.L., 1998, ApJ, submitted, astro-ph/9801011
Munshi D., Coles P., Melott A.L., 1998, MNRAS, submitted, astro-ph/9812271
Mo H.J., Jing Y.P., White S.D.M., 1997, MNRAS, 284, 189
Mo H.J., White S.D.M., 1997, MNRAS, 282, 347
Navarro J.F., Frenk C.S., White S.D.M., 1996, ApJ, 462, 563
Navarro J.F., Frenk C.S., White S.D.M., 1997, ApJ, 490, 49
Peebles, P.J.E., 1980, The Large Scale Structure of the Universe. Princeton University Press, Princeton.
Press W.H., Schechter P.L., 1974, ApJ, 187, 425
Sahni V., Coles P., 1995, Physics Reports, 262, 1
Scoccimarro R., Colombi S., Fry J.N., Frieman J.A., Hivon E., Melott A.L., 1998, ApJ, 496, 586
Szapudi I., Colombi S., 1996, ApJ, 470, 131
Szapudi I., Szalay A.S., 1993, ApJ, 408, 43
Szapudi I., Szalay A.S., 1997, ApJ, 481, L1
Valdarini R., Borgani S., 1991, MNRAS, 251, 575
White S.D.M., 1979, MNRAS, 186, 145
Yano T., Gouda N., 1998, ApJ, 495, 533