EXISTENCE OF SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS WITH RESONANCE AT HIGHER EIGENVALUES

SHU-ZHI SONG
School of Mathematics and Statistics, Southwest University
Chongqing 400715, China

SHANG-JIE CHEN
School of Mathematics and Statistics, Chongqing Technology and Business University
Chongqing 400067, China

CHUN-LEI TANG*
School of Mathematics and Statistics, Southwest University
Chongqing 400715, China

(Communicated by Andrea Malchiodi)

Abstract. We study the following Kirchhoff type problem:

\[
\begin{cases}
-a - b \int_{\Omega} \left| \nabla u \right|^2 \, dx \nabla u = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Note that \( F(x, t) = \int_0^t f(x, s) \, ds \) is the primitive function of \( f \). In the first result, we prove the existence of solutions by applying the \( G \)-Linking Theorem when the quotient \( \frac{4}{p} \frac{F(x, t)}{t^2} \) stays between \( \mu_k \) and \( \mu_{k+1} \) allowing for resonance with \( \mu_{k+1} \) at infinity. In the second result, for the case that the quotient \( \frac{4}{p} \frac{F(x, t)}{t^2} \) stays between \( \mu_1 \) and \( \mu_2' \) allowing for resonance with \( \mu_2' \) at infinity, we find a nontrivial solution by using the classical Linking Theorem and argument of the characterization of \( \mu_2' \). Meanwhile, similar results are obtained for degenerate problem.

1. Introduction and main results. We consider the following nonlocal Kirchhoff type problem with Dirichlet boundary condition

\[
\begin{cases}
-a - b \int_{\Omega} \left| \nabla u \right|^2 \, dx \nabla u = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \), \( a \geq 0, b > 0 \) are real constants and \( f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) is a Caratheodory function that satisfies the subcritical growth condition:

\[
(f_0) \quad |f(x, t)| \leq C(1 + |t|^{p-1}) \quad \text{for some } 4 < p < 2^*.
\]

2010 Mathematics Subject Classification. 35J60, 35A15.

Key words and phrases. Kirchhoff type, nonlinear eigenvalues, resonance, Cerami condition, link.

Supported by the National Natural Science Foundation of China (No. 11471267;11271388), Chongqing Basis and Frontier Research Project(Grant No. cstc2014jcyjA00035).

* Corresponding author: Chun-Lei Tang.
where $C$ is a positive constant, $2^* = \begin{cases} 6, & N = 3, \\ +\infty, & N = 1,2. \end{cases}$

Let $F(x,t) = \int_0^t f(x,s)ds$ be the primitive function of $f$. The Euler functional corresponding to problem (1) is

$$I_a(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \int_\Omega F(x,u)dx$$

defined on the Hilbert space $H = H^1_0(\Omega)$ equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2dx\right)^{\frac{1}{2}} \forall u \in H.$$ 

In particular, if $a = 0$, then

$$I_0(u) = \frac{b}{4} \|u\|^4 - \int_\Omega F(x,u)dx.$$ 

Since $f$ satisfies the subcritical growth condition ($f_0$), we know at once that $I_a \in C^1(H,\mathbb{R})$ and finding weak solutions of problem (1) is equivalent to finding critical points of functional $I_a$ in $H$. Note that $\Omega$ is a bounded domain in $\mathbb{R}^N$ which means the embedding $H \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1,2^*)$, compact for $s \in [1,2^*)$. Hence, for $s \in [1,2^*)$, there exists $\tau_s > 0$ such that

$$\|u\|_{L^s} \leq \tau_s \|u\| \forall u \in H, \quad (2)$$

where $\|u\|_{L^s}$ denotes the norm of $L^s(\Omega)$.

Problem (1) is related to the stationary analogue of the equation

$$u_{tt} - (a + b \int_\Omega |\nabla u|^2dx)\triangle u = f(x,u),$$

which was first proposed by Kirchhoff [9] in 1883 to describe the transversal oscillations of a stretched string, where $u$ denotes the displacement, $f$ is the external force, $b$ represents the initial tension, and $a$ is related to the intrinsic properties of the string. Some early classical studies of Kirchhoff equations were given by Bernstein [2] and Pohožaev [16]. However, (1) received much attention only after Lions [12] proposed an abstract framework to the problem. It is pointed out in Chipot-Lovat [5] that (1) models several physical and biological systems where $u$ describes a process which depends on the average of itself (for example, the population density).

To state the assumptions, we now recall the definition and some basic properties of eigenvalues about the following two problems:

$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega, \end{cases} \quad (3)$$

and

$$\begin{cases} -\|u\|^2\triangle u = \mu u^3, & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega. \end{cases} \quad (4)$$

Denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \cdots$ the eigenvalues of (3) and by $\varphi_1, \varphi_2, \varphi_3 \cdots$ the normalized eigenfunctions. It is well known that the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $\left\{\frac{\varphi_k}{\sqrt{\lambda_k}}\right\}_{k \in \mathbb{N}}$ is an orthonormal basis of $H$. Moreover, $\lambda_1$ is the smallest, called the principal eigenvalue. It is simple, isolated and it is the only eigenvalue with an eigenfunction of constant sign $\varphi_1 > 0$. So, in terms of the Lagrange multiplier rule, we know that $\lambda_1$ can be characterized as

$$\lambda_1 = \inf \left\{\|u\|^2 : u \in H, \int_\Omega |u|^2dx = 1\right\}.$$
For $u \in H$, one has
\[ \|u\|^2 \geq \lambda_1 \int_{\Omega} |u|^2 dx \quad \text{(Poincare inequality)}. \] (5)

Nevertheless, $H$ can be split with $H = E_1 \bigoplus E_1^\perp$, where $E_1 = \text{span}\{\varphi_1\}$ and $E_1^\perp = \bigoplus_{j \geq 2} \text{ker}(\Delta - \lambda_j)$.

Define
\[ \Sigma = \{ u \in H : \Psi(u) = \int_{\Omega} |u|^4 dx = 1 \} \]
and
\[ \mu'_1 = \inf \{ \|u\|^4 : u \in \Sigma \}. \]

As shown in [10], $\mu'_1 > 0$ is the principal eigenvalue of (4) and $\psi_1 > 0$ is an eigenfunction corresponding to $\mu'_1$ with $\|\psi_1\|_{L^4} = 1$.

Meanwhile, as shown in [17], problem (4) has a sequence of eigenvalues with the variational characterization
\[ \mu_k = \inf_{\Lambda \in \Sigma_k} \sup_{u \in \Lambda} \left( \int_{\Omega} |\nabla u|^2 dx \right) = \inf_{\Lambda \in \Sigma_k} \sup_{u \in \Lambda} \|u\|^4, \] (6)
where $\Sigma_k = \{ \Lambda \subset \Sigma : \text{there exists a continuous, odd and surjective} \ h : S_1^{k-1} \to \Lambda \}$ and $S_1^{k-1}$ denotes the unit sphere in $\mathbb{R}^k$. Besides $\mu_1 = \mu'_1$, it is not known whether the sequence $\{\mu_k\}_{k \in \mathbb{N}}$ contains all eigenvalues of problem (4).

Let
\[ \lim_{|t| \to \infty} \frac{4F(x, t)}{bt^4} = \mu \quad \text{uniformly in } x \in \Omega. \] (7)

We may call $f$ is “3-sublinear” at infinity if $\mu = 0$, $f$ is “3-superlinear” at infinity if $\mu = \infty$ and $f$ is “asymptotically 3-linear” at infinity if $0 < \mu < +\infty$. Especially, we say problem (1) is resonant near infinity at $\mu_k$ if $\mu = \mu_k$ in (7). We mention the works for the case of “3-superlinear” at infinity in [4, 13, 18, 21, 24]; for the case of “3-sublinear” at infinity in [24].

The case of “asymptotically 3-linear” at infinity can be found in [4, 3, 8, 10, 11, 14, 17, 18, 22, 23, 24]. In these papers, we may see, the Euler functional $I_a$ associated with problem (1) is coercive when $\mu < \mu_1$, see [22, 23, 24]. It is still true that the functional $I_a$ is coercive if (7) is satisfied with $\mu = \mu_1$ and the following condition:
\[ (f_1) \quad \lim_{|t| \to \infty} [f(x, t)t - 4F(x, t)] = +\infty \quad \text{uniformly in } x \in \Omega \]

(see [22, 23]). In such case, problem (1) is resonant near infinity at $\mu_1$ from the left side. So, if $\mu \leq \mu_1$, the existence of a global minimum is obtained because $I_a$ is coercive and weakly lower semicontinuous on $H$. Furthermore, if $f$ satisfies some conditions near zero, multiplicity of solutions for problem (1) was obtained by using Local Linking Theorem or invariant sets of descent flow method, see [22, 23, 24]. It is interesting to inspect what happens if we allow $\mu > \mu_1$ in (7). See, for example, in [3], the author handled the case that
\[ \lim_{|t| \to \infty} \frac{4F(x, t)}{bt^4} > \mu_1 \quad \text{uniformly in } x \in \Omega \]
and
\[ \lim_{t \to 0} \frac{2F(x, t)}{at^2} < \lambda_1 \quad \text{uniformly in } x \in \Omega \]
and obtained the existence of at least one nontrivial solution by the Mountain Pass Theorem. We also refer readers to [8, 11] for similar results. If \( \mu \in (\mu_k, \mu_{k+1}) \) in (7) (nonresonance at infinity), nontrivial solutions of problem (1) were obtained by using the Yang index and critical groups in [10, 18]. We refer to [17] for the case of resonance at an arbitrary eigenvalue under a Landesman-Lazer type condition.

The main goal of this paper is to establish the existence of weak solutions for problem (1) when the quotient \( \frac{4F(x,t)}{bt^4} \) exhibits between \( \mu_k \) and \( \mu_{k+1} \) at infinity. We introduce the main results as follows:

**Theorem 1.1.** Assume that \( a > 0 \) and \( \mu_k < \mu_{k+1} \) are two consecutive eigenvalues of problem (4). If the nonlinearity \( f \) satisfies \((f_0)\) and the following conditions:

\[
(f_2) \quad \mu_k < \liminf_{|t| \to \infty} \frac{4F(x,t)}{bt^4} \leq \limsup_{|t| \to \infty} \frac{4F(x,t)}{bt^4} \leq \mu_{k+1} \quad \text{uniformly in} \ x \in \Omega;
\]

\[
(f_3) \quad \lim_{|t| \to \infty} |f(x,t)t - 4F(x,t) + a\lambda t^2| = +\infty \quad \text{uniformly in} \ x \in \Omega;
\]

then problem (1) has at least a weak solution in \( H \).

**Remark 1.** We do not assume in \((f_2)\) that the limit as \(|t| \to \infty\) of the quotient \( \frac{4F(x,t)}{bt^4} \) necessarily exists and moreover, we assume that \( \liminf_{|t| \to \infty} \) is bigger that \( \mu_k \), allowing for resonance with respect to \( \mu_{k+1} \) and nonresonance with respect to \( \mu_k \) which implies the Euler functional \( I_k \) is not coercive at infinity. The possibility of resonance with respect to \( \mu_{k+1} \) necessitates an additional asymptotic condition, and this is \((f_3)\). We point out the characterization of variational eigenvalues \( \mu_k \) defined by \((6)\) plays an essential role in the geometry of the Euler functional associated to problem (1) under \((f_2)\). But it becomes invalid in such case that the quotient \( \frac{4F(x,t)}{bt^4} \) stays between \( \mu_k \) and \( \mu_{k+1} \), allowing for resonance with respect to \( \mu_k \) and nonresonance with respect to \( \mu_{k+1} \), that is,

\[
\mu_k \leq \liminf_{|t| \to \infty} \frac{4F(x,t)}{bt^4} \leq \limsup_{|t| \to \infty} \frac{4F(x,t)}{bt^4} < \mu_{k+1} \quad \text{uniformly in} \ x \in \Omega.
\]

The next result is related to a second eigenvalue of problem (4). For our purpose, the characterization of \( \mu_2 \) defined by \((6)\) is not convenient. Instead, we will use an alternative one defined as follows which will be proved in the next section. Let \( \mu'_2 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \|\gamma(t)\|^4 \),

where

\[ \Gamma_0 = \{ \gamma \in C([-1,1],\Sigma) : \gamma(-1) = -\psi_1, \gamma(1) = \psi_1 \}. \]

We are now ready to state our results.

**Theorem 1.2.** Problem (1) has at least a weak solution in \( H \) under hypotheses \((f_0),(f_3)\) and the following conditions:

\[
(f_4) \quad \mu_1 \leq \liminf_{|t| \to \infty} \frac{4F(x,t)}{bt^4} \leq \limsup_{|t| \to \infty} \frac{4F(x,t)}{bt^4} \leq \mu'_2 \quad \text{uniformly in} \ x \in \Omega;
\]

\[
(f_5) \quad \lim_{|t| \to \infty} \int_{\Omega} F(x,t\psi_1)dx - \frac{bt^4}{4} \leq \frac{a\sqrt{\mu_1}}{2} t^2 \to +\infty \quad \text{uniformly in} \ x \in \Omega.
\]

In addition, suppose that \( f \) satisfies \( f(x,0) = 0 \) and
(\(f_6\)) there exist \(\delta_1 > 0\), \(q \in (1,2)\) and \(k_1 > 0\), such that
\[
F(x,t) \geq k_1 |t|^q \quad \text{for all } |t| \leq \delta_1;
\]

(\(f_7\)) \[
\left[ \int_\Omega f(x,t\psi_1)t\psi_1 dx - b\mu_1 t^4 - a\sqrt{\mu_1} t^2 \right] > 0 \quad \text{for all } t \neq 0;
\]

problem (1) has a nontrivial solution in \(H\).

Remark 2. If the left inequality of (\(f_4\)) is strict, we see, (\(f_5\)) is redundant. Moreover, from (\(f_4\)), the following two conditions
\[
\liminf_{|t| \to \infty} \frac{4F(x,t)}{bt^4} = \mu_1 \quad \text{uniformly in } x \in \Omega
\]
and
\[
\limsup_{|t| \to \infty} \frac{4F(x,t)}{bt^4} = \mu'_2 \quad \text{uniformly in } x \in \Omega
\]
are permitted at the same time which means problem (1) may be doubly resonant at infinity.

Remark 3. There are functions \(f(x,t)\) satisfying the assumptions in Theorem 1.2. For example, let
\[
f(t) = b\mu t^3 + K t^3, \quad \text{where } \mu \in (\mu_1,\mu'_2], \quad K > 0 \text{ large enough.}
\]

In the following, we consider problem (1) with \(a = 0\), that is,
\[
\begin{cases}
-b \int_\Omega |\nabla u|^2 dx \Delta u = f(x,u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Problem (8) is called degenerate elliptic Kirchhoff equation. Such degenerate partial differential equations arise naturally because nonlinear elliptic equations often may have vanishing coefficients or small parameters in higher order partial derivatives. We state the results as follows:

Theorem 1.3. Assume \(\mu_k < \mu_{k+1}\) are two consecutive eigenvalues of problem (4). Problem (8) has at least a weak solution in \(H\) if the nonlinearity \(f\) satisfies (\(f_0\)), (\(f_1\)) and (\(f_2\)).

Theorem 1.4. Let (\(f_0\)), (\(f_1\)), and (\(f_4\)) hold. Besides, assume that \(f\) satisfies
\[
(f_8) \quad \lim_{|t| \to \infty} \left[ \int_\Omega F(x,t\psi_1) dx - \frac{b\mu_1}{4} t^4 \right] \to +\infty \quad \text{as } n \to \infty,
\]

problem (8) has at least a weak solution in \(H\). Furthermore, if \(f\) satisfies \(f(x,0) = 0\) and the following conditions:

(\(f_9\)) there exist \(\delta_2 > 0\), \(r \in (1,4)\) and \(k_2 > 0\) such that
\[
F(x,t) \geq k_2 |t|^r \quad \text{for all } |t| \leq \delta_2;
\]

(\(f_{10}\)) \[
\int_\Omega f(x,t\psi_1)t\psi_1 dx - b\mu_1 t^4 > 0 \quad \text{for all } t \neq 0;
\]
then problem (8) has at least a nontrivial solution in \(H\).

Remark 4. It is clear that the assumption (\(f_4\)) implies problem (8) may be doubly resonant at infinity. If the left inequality of (\(f_4\)) is strict, then (\(f_8\)) is redundant.
2. Construction of a second eigenvalue. This section is devoted to the construction of a second eigenvalue $\mu'_2$ which has been introduced ahead.

**Proposition 1.** Let

$$
\mu'_2 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \|\gamma(t)\|^4,
$$

where

$$
\Gamma_0 = \{ \gamma \in C([-1,1], \Sigma) : \gamma(-1) = -\psi_1, \gamma(1) = \psi_1 \}.
$$

Then, $\mu'_2$ is an eigenvalue of problem (4).

**Proof.** Let $\Phi(u) = \|u\|^4$ and $\Psi(u) = \int_{\Omega} |u|^4 \, dx$ $u \in H$. We show first that $\Phi(u)$ satisfies the $(PS)$ condition restricted to $\Sigma$, that is, if $\{u_n\} \subset \Sigma$ such that

$$
\Phi(u_n) \text{ is bounded and } (\Phi|_{\Sigma})' \to 0 \text{ as } n \to \infty,
$$

then $\{u_n\}$ possesses a convergent subsequence. Note that the norm of the derivative at $u \in \Sigma$ of $\Phi|_{\Sigma}$ is defined as

$$
\|\Phi'^*\|_{H^*} = \inf \|\Phi'(u) - t\Psi'(u)\|_{H^*}, \quad t \in \mathbb{R}
$$

where $\|\cdot\|_{H^*}$ denotes the norm on the dual space $H^*$. Let $\{u_n\} \subset \Sigma$ and $\{t_n\} \subset \mathbb{R}$ be sequences such that, for some constant $c$,

$$
\|\Phi(u_n)\| \leq c
$$

and

$$
\left|\|u_n\|^2 \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx - t_n \int_{\Omega} |u_n|^2 u_n v \, dx \right| \leq \varepsilon_n \|v\|
$$

for all $v \in H$ and $\varepsilon_n \to 0$. From (11), we see that $\{u_n\}$ is bounded in $H$. Consequently, for a subsequence,

$$
u_n \to u \text{ weakly in } H \quad \text{and} \quad u_n \to u \text{ strongly in } L^4(\Omega).
$$

The relation in (12) (with $v = u_n$) implies that $\{t_n\}$ remains bounded. Thus, from (2) and letting $v = u_n - u$ in (12), one has

$$
\left|\|u_n\|^2 \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx \right| \leq \left| t_n \int_{\Omega} |u_n|^2 u_n (u_n - u) \, dx \right| + \varepsilon_n \|u_n - u\|
$$

$$
\leq |t_n| \|u_n\|^2 \|u_n - u\|_{L^4} + c\varepsilon_n
$$

$$
\leq \tau_3^2 |t_n| \|u_n\|^3 \|u_n - u\|_{L^4} + c\varepsilon_n
$$

$$
\leq c\|u_n - u\|_{L^4} + c\varepsilon_n
$$

$$
\to 0 \quad \text{as } n \to \infty.
$$

So, we deduce from (14) that

$$
\int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx \to 0 \quad \text{as } n \to \infty.
$$

In addition, from (13), we know

$$
\int_{\Omega} \nabla u \cdot \nabla (u_n - u) \, dx \to 0 \quad \text{as } n \to \infty.
$$
Then, we have
\[
\|u_n - u\|^2 = \int_\Omega \nabla (u_n - u) \cdot \nabla (u_n - u) \, dx
= \int_\Omega \nabla u_n \cdot \nabla (u_n - u) \, dx - \int_\Omega \nabla u \cdot \nabla (u_n - u) \, dx
\rightarrow 0 \quad \text{as } n \rightarrow \infty
\]
by (15) and (16).

Next, we turn to the study of the geometry of \(\Phi|_\Sigma\). It is clear that \(\Gamma_0\) defined by (10) is nonempty (take, e.g., \(\psi \in H\) with \(\psi \notin \text{span}\{\psi_1\}\), consider the path \(t\psi_1 + (1 - |t|)\psi\)). From Remark 2.6 in [6], we need to deduce
\[
\inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \Phi(\gamma(t)) \geq \max\{\Phi(\psi_1), \Phi(-\psi_1)\}.
\]
(17)

For this purpose, we introduce the continuous map \(\vartheta : H \rightarrow \mathbb{R}\) defined by
\[
\vartheta(u) = \int_\Omega \psi_3^1 \, u \, dx
\]
and
\[V = \{u \in H : \vartheta(u) = 0\},\]
which is a closed linear subspace of \(H\). We see
\[H = \text{span}\{\psi_1\} \oplus V.\]

Indeed, set \(k = \vartheta(u)\) for \(u \in H\), then
\[\vartheta(u - k\psi_1) = \vartheta(u) - k\vartheta(\psi_1) = \vartheta(u) - k = 0,
\]
which implies \(v = u - k\psi_1 \in V\), that is, \(u = k\psi_1 + v\). We define the following quantity
\[
\mu_V = \inf_{u \in V \cap \Sigma} \Phi(u) = \inf_{u \in V \cap \Sigma} \|u\|^4.
\]

We claim
\[\mu_V > \mu_1.
\]
It is evident that \(\mu_1 \leq \mu_V\). Suppose that \(\mu_1 = \mu_V\). Then we can find a sequence \(\{u_n\} \subseteq V\), such that
\[
\|u_n\|_{L^4} = 1 \quad \forall n \geq 1
\]
and
\[
\|u_n\|^4 \rightarrow \mu_1 = \mu_V.
\]
(19)

It follows from (19) that the sequence \(\{u_n\}\) is bounded and so we may assume
\[u_n \rightharpoonup u \quad \text{weakly in } H, \quad u_n \rightarrow u \quad \text{strongly in } L^4(\Omega).
\]
(20)

Thus, one has \(\|u\|_{L^4} = \lim_{n \rightarrow +\infty} \|u_n\|_{L^4} = 1\) by (18) and
\[
\|u\|^4 \leq \lim\inf_{n \rightarrow \infty} \|u_n\|^4 = \lim_{n \rightarrow \infty} \|u_n\|^4 = \mu_1
\]
by (19). So, \(\|u\|^4 = \mu_1, \|u\|_{L^4} = 1\), which immediately implies
\[u = \pm \psi_1.
\]
(21)
On the other hand, the fact \( \{ u_n \} \subseteq V \) shows \( \vartheta(u_n) = 0 \) \( \forall n \geq 1 \). Combining with (20), we have
\[
\left| \int_{\Omega} \psi_1^3 u dx \right| = \left| \int_{\Omega} \psi_1^3 u dx - \int_{\Omega} \psi_1^3 u_n dx \right| = \left| \int_{\Omega} \psi_1^3 (u - u_n) dx \right|
\leq \int_{\Omega} |\psi_1^3 (u - u_n)| dx \leq \|\psi_1\|_{L^4}^3 \| u_n - u \|_{L^4} \to 0 \quad \text{as} \quad n \to \infty,
\]
that is,
\[
0 = \int_{\Omega} \psi_1^3 u dx,
\]
a contradiction to (21). Hence \( \mu_\nu > \mu_1 \).

In addition, we may deduce
\[
\gamma(t) \cap V \setminus \{0\} \neq \emptyset \quad \forall \gamma \in \Gamma_0.
\]
(22)

Let \( \gamma \in \Gamma_0 \) and consider \( \vartheta \circ \gamma : [-1, 1] \to \mathbb{R} \). Evidently this is a continuous function such that
\[
-1 = \vartheta(\gamma(-1)) < 0 < \vartheta(\gamma(1)) = 1.
\]
So, by Bolzano’s theorem, we know that there exists a constant \( t_0 \in (-1, 1) \) such that
\[
\vartheta(\gamma(t_0)) = 0,
\]
which implies \( \gamma(t_0) \in V \). Note that \( \gamma(t_0) \neq 0 \) is ensured by \( \gamma(t) \in \Sigma \) and then (22) is satisfied by the arbitrariness of \( \gamma \). So,
\[
\inf_{\gamma \in \Gamma_0} \max_{t \in [-1, 1]} \Phi(\gamma(t)) \geq \inf_{V \cap \Sigma} \Phi(u) = \mu_\nu > \mu_1 = \Phi(\psi_1) = \Phi(-\psi_1),
\]
that is, (17) is satisfied. Thus, we have
\[
\mu'_2 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1, 1]} \Phi(\gamma(t)) = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1, 1]} \|\gamma(t)\|_4^4
\]
is a critical value of \( \Phi|_{\Sigma} \), which means \( \mu'_2 \) is the other definition of second eigenvalue of problem (4).

\[\square\]

3. Proof of the Theorems. We introduce the symmetric cone
\[
\mathbb{C}_{k+1} = \left\{ u \in H : \int_{\Omega} |\nabla u|^4 dx \geq \mu_{k+1} \int_{\Omega} |u|^4 dx \right\}
\]
and
\[
\mathbb{D} = \left\{ u \in H : \int_{\Omega} |\nabla u|^4 dx \geq \mu'_2 \int_{\Omega} |u|^4 dx \right\}.
\]

**Definition 3.1.** (See [20]) Let \( Q \) be a submanifold of a Banach space \( X \) with relative boundary \( \partial Q \), \( S \) be a closed subset of a Banach space \( Y \) and \( G \) be a subset of \( C^0(\partial Q, Y \setminus S) \). We say \( S \) and \( \partial Q \) are \( G \)-linking if for any map \( h \in C^0(Q,Y) \) such that \( h \mid_{\partial Q} \in G \) there holds \( h(Q) \cap S \neq \emptyset \).

**Theorem 3.2.** (\( G \)-Linking Theorem) Suppose \( X, Y \) are real Banach spaces. Consider a closed subset \( S \subseteq Y \) and a submanifold \( Q \subseteq X \) with relative boundary \( \partial Q \). \( G \) is a subset of \( C^0(\partial Q, Y \setminus S) \). Set \( \Gamma = \{ h \in C^0(Q,Y) : h \mid_{\partial Q} \in G \} \). Suppose \( S \) and \( \partial Q \) are \( G \)-linking and \( I \in C^1(Y, \mathbb{R}) \) satisfies
\[
(a) \text{ there exists } h_0 \in \Gamma \text{ such that } \sup_{x \in Q} I(h_0(x)) < +\infty;
\]

(b) there exist two constants $\alpha, \beta$ with $\beta > \alpha$ such that
\[ \inf_{y \in S} I(y) \geq \beta \]
and
\[ \sup_{x \in \partial Q} I(h(x)) \leq \alpha \quad \forall h \in \Gamma; \]

(c) $I$ satisfies the Cerami condition which is stated in [1]. Then, the number
\[ c := \inf_{h \in \Gamma} \sup_{x \in Q} I(h(x)) \]
defines a critical values $c \geq \beta$ of $I$.

Remark 5. Similar conclusions can be found in [20] if we replace the Cerami condition by the Palais-Smale condition. The conclusion in [20] is based on a linking structure and on a deformation lemma which can be ensured by the Palais-Smale condition. It was shown in [1] that the Cerami condition suffices to get the deformation lemma. Thus, we deduce immediately that Theorem 3.2 is still correct. With the same reasoning, the classical Linking Theorem (see [15] Theorem 8.4) holds true under the Cerami condition.

Our approach is variational based on the critical point theory for $C^1$–functionals. Specifically, we shall look for some weak solutions by applying the classical Linking Theorem and Theorem 3.2 ($G$–Linking theorem). Furthermore, in order to show the weak solutions are nonzero, we shall make more accurate estimate on the energy functional in the proofs of Theorem 1.2 and Theorem 1.4. We divide the proofs into two cases.

Notation: Throughout the paper, we denote by $c$ and $c_i$ various positive constants which may vary from place to place.

Case I: $a > 0$.
In order to prove our theorems, we first verify that $I_a$ satisfies the compactness condition.

Lemma 3.3. Any bounded Cerami sequence of $I_a$ has a converging subsequence in $H$ if hypotheses $(f_0)$ is satisfied.

Proof. Let $\{u_n\} \subset H$ be a bounded Cerami sequence of $I_a$. By the reflexivity of $H$, we can assume that there exists a function $u \in H$ such that
\[ u_n \rightharpoonup u \text{ weakly in } H, \quad u_n \to u \text{ strongly in } L^p(\Omega), \quad u_n(x) \to u(x) \text{ a.e. } x \in \Omega. \quad (23) \]
By $(f_0)$ and (2), we see
\[
\left| \int_{\Omega} f(x, u_n)(u - u_n) dx \right| \leq C \int_{\Omega} \left( 1 + |u_n|^{p-1} \right) |u - u_n| dx \\
\leq C \left( |\Omega| \|u\|_p^{p-1} + \|u_n\|_{L_p}^{p-1} \right) \|u - u_n\|_{L_p} \\
\leq C \left( |\Omega| \|u\|_p^{p-1} + \tau^{p-1} \|u_n\|_{L_p}^{p-1} \right) \|u - u_n\|_{L_p} \\
\to 0 \quad \text{as } n \to \infty.
\]
Since $\{u_n\}$ is a bounded Cerami sequence of $I_a$, we obtain
\[
|\langle I_a'(u_n), u - u_n \rangle| \leq \|I_a'(u_n)\|_{H^*} \|u - u_n\| \\
\leq \|I_a'(u_n)\|_{H^*} \|u\| + \|I_a'(u_n)\|_{H^*} \|u_n\| \to 0 \quad \text{as } n \to \infty.
\]
So, we have
\[ (a + b\| u_n \|^2) \int_{\Omega} \nabla u_n \cdot \nabla (u - u_n) \, dx \]
\[ = (I'_a(u_n), u - u_n) + \int_{\Omega} f(x, u_n)(u - u_n) \, dx \to 0 \text{ as } n \to \infty, \]
which implies
\[ \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx \to 0 \text{ as } n \to \infty. \]  \hspace{1cm} (24)
In addition, from (23), we know
\[ \int_{\Omega} \nabla u \cdot \nabla (u_n - u) \, dx \to 0 \text{ as } n \to \infty. \]  \hspace{1cm} (25)
Then, we have
\[ \| u_n - u \|^2 = \int_{\Omega} \nabla (u_n - u) \cdot \nabla (u_n - u) \, dx \]
\[ = \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx - \int_{\Omega} \nabla u \cdot \nabla (u_n - u) \, dx \to 0 \text{ as } n \to \infty \]
by (24) and (25). Hence, \( u_n \to u \) in \( H \) due to the uniform convexity of \( H \).

Remark 6. It is easy to see Lemma 3.3 is still correct for \( a = 0 \).

Lemma 3.4. \( I_a \) (\( a > 0 \)) satisfies the Cerami condition if \( (f_0) \) and \( (f_3) \) hold.

Proof. Let \( \{ u_n \} \subset H \) be a Cerami sequence, i.e.,
\[ I_a(u_n) \to c \text{ and } (1 + \| u_n \|) I'_a(u_n) \to 0 \text{ as } n \to \infty. \]  \hspace{1cm} (26)
Due to Lemma 1, it suffices to show that \( \{ u_n \} \) is bounded in \( H \). For this we suppose by contradiction that, passing to subsequence, \( \| u_n \| \to \infty \) as \( n \to \infty \).

Letting \( v_n = \frac{u_n}{\| u_n \|} \), there is a function \( v \in H \) such that
\[ v_n \rightharpoonup v \text{ weakly in } H, \]
\[ v_n \to v \text{ strongly in } L^2(\Omega), \]
\[ v_n(x) \to v(x) \text{ a.e. } x \in \Omega. \]  \hspace{1cm} (27)
On the other hand, set \( u_n = \phi_n + w_n \) where \( \phi_n \in E_1 \) and \( w_n \in E_1^\perp \). From \( (f_0) \) and \( (f_3) \), there exists a constant \( c_1 > 0 \) such that
\[ f(x, t)t - 4F(x, t) + a\lambda_1 t^2 \geq -c_1 \]  \hspace{1cm} (28)
for all \( t \in \mathbb{R} \) and \( x \in \Omega \). By the definition of \( E_1 \) and \( E_1^\perp \), one has
\[ \| u_n \|^2 = \| \phi_n \|^2 + \| w_n \|^2, \quad \| u_n \|^2_{L^2} = \| \phi_n \|^2_{L^2} + \| w_n \|^2_{L^2}. \]
Then, combining (26) and (28), we see
\[ 4(c + 1) \geq 4I_a(u_n) - \langle I'_a(u_n), u_n \rangle \]
\[ = a\| u_n \|^2 + \int_{\Omega} (f(x, u_n)u_n - 4F(x, u_n)) \, dx \]
\[ = a\| u_n \|^2 - a\lambda_1 \| u_n \|^2 + \int_{\Omega} (f(x, u_n)u_n - 4F(x, u_n) + a\lambda_1 |u_n|^2) \, dx \]
\[ \geq a(1 - \frac{\lambda_1}{\lambda_2})\| w_n \|^2 \geq -c_1|\Omega|, \]
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. The above inequality shows $\|w_n\| \leq c$. So, up to subsequence if necessary, $\frac{w_n}{\|w_n\|} \to 0$ weakly in $H$ and then

$$
\phi_n \to v \text{ weakly in } E_1, \quad \frac{\phi_n}{\|\phi_n\|} \to v \text{ strongly in } L^2(\Omega).
$$

Notice that $\phi_n \in E_1$ means $\int_{\Omega} \|
abla \phi_n\|^2 \, dx = \lambda_1 \int_{\Omega} |\phi_n|^2 \, dx$. So, by weak lower semi-continuity of the norm and (5), one has

$$
\lambda_1 \int_{\Omega} |v|^2 \, dx \leq \int_{\Omega} |\nabla v|^2 \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla v_n|^2 \, dx \leq \limsup_{n \to \infty} \int_{\Omega} |\nabla v_n|^2 \, dx
$$

$$
= \limsup_{n \to \infty} \left[ \int_{\Omega} \left( \frac{\nabla \phi_n}{\|\phi_n\|} \right)^2 \, dx + \int_{\Omega} \left( \frac{\nabla w_n}{\|w_n\|} \right)^2 \, dx \right]
$$

$$
\leq \lim_{n \to \infty} \int_{\Omega} \left( \frac{\nabla \phi_n}{\|\phi_n\|} \right)^2 \, dx + \lim_{n \to \infty} \int_{\Omega} \left( \frac{\nabla w_n}{\|w_n\|} \right)^2 \, dx
$$

$$
= \lim_{n \to \infty} \lambda_1 \int_{\Omega} \left( \frac{\phi_n}{\|\phi_n\|} \right)^2 \, dx = \lambda_1 \int_{\Omega} |v|^2 \, dx.
$$

So, $\int_{\Omega} |\nabla v|^2 \, dx = \lim_{n \to \infty} \int_{\Omega} |\nabla v_n|^2 \, dx = 1$ and then $v \in E_1 \setminus \{0\}$. We deduce immediately that $v(x) > 0$ or $v(x) < 0$. We only consider the case $v(x) > 0$, $v(x) < 0$ is similar. It follows from (27) that $u_n(x) \to +\infty$ as $n \to \infty$. By Fatou’s lemma, (f3) and (5), we obtain

$$
4(c + 1) \geq 4I_a(u_n) - \langle I'_a(u_n), u_n \rangle
$$

$$
= a\|u_n\|^2 - a\lambda_1 \int_{\Omega} |u_n|^2 \, dx + \int_{\Omega} \left( f(x, u_n)u_n - 4F(x, u_n) + a\lambda_1 |u_n|^2 \right) \, dx
$$

$$
\geq \int_{\Omega} \left( f(x, u_n)u_n - 4F(x, u_n) + a\lambda_1 |u_n|^2 \right) \, dx \to +\infty \text{ as } n \to \infty.
$$

This is a contradiction. Therefore, $\{u_n\}$ is bounded in $H$. \hfill \Box

**Proof of Theorem 1.1.** By (f0) and the left inequality of (f2), there exist an $\varepsilon_1 > 0$ with $\mu_k + \varepsilon_1 < \mu_{k+1}$ and a positive constant $c = c(\varepsilon_1) > 0$ such that

$$
|F(x, t)| \geq \frac{b}{4} \left( \mu_k + 2\varepsilon_1 \right)t^4 - c \quad \forall (x, t) \in \Omega \times \mathbb{R}.
$$

(29)

It follows from the definition of $\mu_k$ in (6) that there exists some $\Lambda_1 \in \Sigma_k$ such that

$$
\sup_{u \in \Lambda_1} \|u\|_4^4 \leq \mu_k + \varepsilon_1.
$$

(30)

From (29) and (30), for any $u \in \Lambda_1$ and $t > 0$, we have

$$
I(tu) = \frac{a}{2} \|tu\|^2 + \frac{b}{4} \|tu\|^4 - \int_{\Omega} F(x, tu) \, dx
$$

$$
\leq \frac{a}{2} \|u\|^2 + \frac{bt^4}{4} - \frac{b}{4} \left( \mu_k + 2\varepsilon_1 \right)t^4 \|u\|_4^4 + c|\Omega|
$$

$$
\leq \frac{a}{2} \sqrt{\mu_k + \varepsilon_1} t^2 - \frac{b}{4} \varepsilon_1 t^4 + c|\Omega|.
$$

(31)

On the other hand, defining

$$
G_1(x, s) = F(x, s) - \frac{b}{4} \mu_{k+1} s^4 - \frac{a\lambda_1}{2} s^2 \quad \forall (x, s) \in \Omega \times \mathbb{R},
$$


we get
\[ G_1'(x, s)s - 4G_1(x, s) = f(x, s)s - 4F(x, s) + a\lambda_1 s^2. \]
From (f_3), for any \( M > 0 \), there is a constant \( C_M > 0 \) such that
\[ G_1'(x, s)s - 4G_1(x, s) \geq M \]
for all \( x \in \Omega \) and \( |s| > C_M \). Hence, for \( s > C_M \), from the above inequality, we have
\[ \frac{d}{ds} \left( \frac{G_1(x, s)}{s^4} \right) = \frac{G_1'(x, s)s - 4G_1(x, s)}{s^5} \geq \frac{M}{s^5}. \]
Integrating the above expression over the interval \([t, T] \subset (C_M, +\infty)\), one has
\[ \frac{G_1(x, t)}{t^4} \leq \frac{G_1(x, T)}{T^4} + \frac{M}{4} \left( \frac{1}{T^4} - \frac{1}{t^4} \right). \]
Since \( \limsup_{T \to +\infty} \frac{G_1(x, T)}{T^4} \leq 0 \), letting \( T \to +\infty \), we obtain \( G_1(x, t) \leq -\frac{M}{4} \) for \( t \geq C_M \) and \( x \in \Omega \). Similarly, \( G_1(x, t) \leq -\frac{M}{4} \) for \( t \leq -C_M \) and \( x \in \Omega \). Hence, we have by the arbitrariness of \( M \), that
\[ \lim_{|t| \to +\infty} G_1(x, t) = -\infty \text{ uniformly for } x \in \Omega. \]
Combining (f_0) with the above equality, we find a positive constant \( c \) such that
\[ G_1(x, t) < c \quad \forall (x, t) \in \Omega \times \mathbb{R}. \]
Then, for any \( u \in C_{k+1} \), one has
\[
I_a(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x, u) dx
\]
\[
= \frac{a}{2} \|u\|^2 - \frac{a\lambda_1}{2} \int_\Omega |u|^2 dx + \frac{b}{4} \left( \|u\|^4 - \mu_{k+1} \|u\|_{L^4}^4 \right)
\]
\[
- \int_\Omega \left( F(x, u) - \frac{b\mu_{k+1}}{4} |u|^4 - \frac{a\lambda_1}{2} |u|^2 \right) dx
\]
\[
\geq -\int_\Omega G_1(x, u) dx \geq -c|\Omega|. \quad (32)
\]
Notice that (31) implies
\[ I_a(tu) \to -\infty \text{ as } t \to +\infty \text{ uniformly for } u \in \Lambda_1, \]
which, combining (32), shows immediately that there exists a constant \( \rho_1 > 0 \) large enough such that
\[ \alpha = \max_{u \in \Lambda_1, t \geq \rho_1} I_a(tu) < -c|\Omega| = \beta \leq \inf_{u \in C_{k+1}} I_a(u). \quad (33) \]
Now set \( \rho_1 \Lambda_1 = \{ tu : u \in \Lambda_1, t \geq \rho_1 \} \) and
\[ G = \{ h \in C(S^{k-1}, \rho_1 \Lambda_1) : h \text{ is odd} \}, \]
where \( S^{k-1} \) is the boundary of the closed unit ball \( B^k \) in \( R^k \), that is \( S^{k-1} = \partial B^k \).
We claim that \( C_{k+1} \) and \( S^{k-1} \) are \( G \)-Linking. In fact, for any \( h \in G \), by (33), we get \( h(S^{k-1}) \cap C_{k+1} = \emptyset \), which shows that \( G \) is a subset of \( C(S^{k-1}, H \setminus C_{k+1}) \).
On the other hand, letting
\[ \Gamma_1 = \{ h \in C(B^k, H) : h|_{S^{k-1}} \in G \}, \]
we have the following conclusion: \( \Gamma_1 \) is nonempty and \( h(B^k) \cap C_{k+1} \neq \emptyset \) for all \( h \in \Gamma_1 \). The proof may be found in [7]. We include it for completeness.
Note that $\Lambda_1 \in \Sigma_k$. By the definition of $\Sigma_k$, there exists a continuous odd surjection $h : S^{k-1} \to \Lambda_1$. Define $\bar{h} : B_k \to H$ by $\bar{h}(ts) = tp_1h(s)$ for any $s \in S^{k-1}$ and any $t \in [0, 1]$. Thus, $\bar{h} \in \Gamma_1$, i.e., $\Gamma_1$ is nonempty.

Moreover, for any $h \in \Gamma_1$, if $0 \in h(B_k)$, one has $h(B_k) \cap C_{k+1} \neq \emptyset$. Otherwise, let $\pi : H \setminus \{0\} \to \Sigma$ be defined by
\[
\pi(u) = \frac{u}{\|u\|_L^2} \quad \forall u \in H
\]
(the radial retraction of $H \setminus \{0\}$ onto $\Sigma$). Consider the map $\tilde{h} : S^k \to H$,
\[
\tilde{h}(x_1, \cdots, x_{k+1}) = \begin{cases} 
\pi \circ h(x_1, \cdots, x_{k+1}), & x_{k+1} \geq 0, \\
-\pi \circ h(-x_1, \cdots, -x_{k+1}), & x_{k+1} < 0.
\end{cases}
\]
It is straightforward to verify that $\tilde{h}(S^k) \in \Sigma_{k+1}$. Thus, $\|u\|^4 \geq \mu_{k+1} \int_\Omega u^4 \, dx$ for any $u \in \tilde{h}(S^k)$, i.e. $u \in C_{k+1}$. Indeed $(\pi \circ h)(x) \in C_{k+1}$ implies $h(x) \in C_{k+1}$. Then $h(B_k) \cap C_{k+1} \neq \emptyset$.

Furthermore, (b) of the $G$–Linking Theorem is satisfied from (33), and (c) holds by Lemma 3.4. As to (a) of the $G$–Linking Theorem, it is easy to verify from the compactness of $B^k$. Therefore, Theorem 1.1 holds with the critical value
\[
c : = \inf_{h \in \Gamma_1} \sup_{x \in B^k} I_a(h(x)).
\]

\[\square\]

Proof of Theorem 1.2. From Lemma 3.4, we need only to study the “geometry” of $I_a$. Define
\[
G_2(x, s) = F(x, s) - \frac{b}{4} \mu_2 s^4 - \frac{a\lambda_1}{2} s^2 \quad \forall (x, s) \in \Omega \times \mathbb{R},
\]
one has
\[
\lim_{|t| \to \infty} G_2(x, t) = -\infty \quad \text{uniformly for } x \in \Omega
\]
and then $G_2(x, t) < c$ for some $c > 0$. For any $u \in \mathcal{D}$, we deduce easily that
\[
I_a(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \int_\Omega F(x, u) \, dx
\]
\[
= \frac{a}{2} \|u\|^2 - \frac{a\lambda_1}{2} \int_\Omega |u|^2 \, dx + \frac{b}{4} \|u\|^4 - \mu_2 \|u\|_4^4
\]
\[
- \int_\Omega \left( F(x, u) - \frac{b\mu_2}{4} |u|^4 - \frac{a\lambda_1}{2} |u|^2 \right) \, dx
\]
\[
\geq -\int_\Omega G_2(x, u) \, dx \geq -c|\Omega|.
\]
On the other hand, we deduce from (f3) that
\[
I_a(t \psi_1) \to -\infty \quad \text{as } t \to +\infty, \quad \forall u \in \text{span}\{\psi_1\}.
\]
So, there exits a constant $\rho_2 > 0$ large enough such that
\[
\max \{I_a(\rho_2 \psi_1), I_a(-\rho_2 \psi_1)\} < -c|\Omega| \leq \inf_{\mathcal{D}} I_a.
\]

We consider the sets
\[
\Gamma_2 = \{ \gamma \in C([-1, 1]; H) : \gamma(-1) = -\rho_2 \psi_1, \gamma(1) = \rho_2 \psi_1 \};
\]
\[
\partial Q_1 = \{ -\rho_2 \psi_1, \rho_2 \psi_1 \};
\]
and
\[
\partial Q_1 = \{ -\rho_2 \psi_1, \rho_2 \psi_1 \};
\]
We claim that the pair \((Q_1, \partial Q_1)\) is linking with \(\mathbb{D}\) in \(H\) (see [15] Definition 8.1). It is obvious that \(\partial Q_1 \cap \mathbb{D} = \emptyset\). Next, we show that

\[
\gamma([-1,1]) \cap \mathbb{D} \neq \emptyset \quad \forall \gamma \in \Gamma_2,
\]

where \(\Gamma_2\) is defined by (36). If \(0 \in \gamma([-1,1])\), then we are done since \(0 \in \mathbb{D}\). So, suppose that \(0 \notin \gamma([-1,1])\). Note the definition of \(\pi\) in (34). Since

\[
(\pi \circ \gamma)(-1) = \pi(\gamma(-1)) = \pi(-\rho_2 \psi_1) = -\psi_1
\]

and

\[
(\pi \circ \gamma)(1) = \pi(1) = \pi(\rho_2 \psi_1) = \psi_1,
\]

we deduce \(\pi \circ \gamma \in \Gamma_0\), where \(\Gamma_0\) is defined by (10). From the definition of \(\mu'_2\) in (9), we may find a constant \(t_0 \in [-1,1]\) such that

\[
\|\nabla \pi(\gamma(t_0))\|^4 \geq \mu'_2,
\]

which shows \(\pi(\gamma(t_0)) \in \mathbb{D}\). Since \(\gamma \in \Gamma_2\) is arbitrary, we conclude that the pair \((Q_1, \partial Q_1)\) is linking with \(\mathbb{D}\). Thus, from (35) and Lemma 3.4, the Linking Theorem (see [15] Theorem 8.4) produces a critical point \(u_1 \in H\) of \(I_a\) such that

\[
c = I_a(u_1) = \inf_{\gamma \in \Gamma_2} \max_{-1 \leq t \leq 1} I_a(\gamma(t)). \tag{37}
\]

Now, we show that \(u_1 \neq 0\). According to (37), it suffices to produce a \(\gamma_* \in \Gamma_2\) such that

\[
I_a(\gamma_*(t)) < 0 \quad \forall t \in [-1,1].
\]

For a fixed \(\mu > \mu'_2\), according to the definition of \(\mu'_2\), we may find a \(\tilde{\gamma}_0 \in \Gamma_0\) such that

\[
\mu'_2 \leq \max_{-1 \leq t \leq 1} \|\tilde{\gamma}_0(t)\|^4 < \mu. \tag{38}
\]

It is clear from assumptions \((f_0), (f_4)\) and \((f_6)\) that

\[
F(x, t) \geq k_1|t|^q - \frac{c}{4}|t|^4 \quad \forall (x, t) \in \Omega \times \mathbb{R} \tag{39}
\]

and some \(c > 0\). Then, using (38) and (39), for every \(t \in [-1,1]\) and \(s > 0\), one has

\[
I_a(s\tilde{\gamma}_0(t)) = \frac{as^2}{2}\|\tilde{\gamma}_0(t)\|^2 + \frac{bs^4}{4}\|\tilde{\gamma}_0(t)\|^4 - \int_\Omega F(x, s\tilde{\gamma}_0(t))
\]

\[
\leq \frac{as^2}{2}\|\tilde{\gamma}_0(t)\|^2 + \frac{bs^4}{4}k_1s^q - k_1s^q \min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|^q_\Sigma + \frac{c}{4}s^4
\]

\[
= \frac{as^2}{2}\|\tilde{\gamma}_0(t)\|^2 + \frac{bs^4}{4}k_1s^q - k_1s^q \min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|^q_\Sigma.
\]

Note that \(\tilde{\gamma}_0(t) \in \Sigma\) means \(\tilde{\gamma}_0(t) \neq 0\) for every \(t \in [-1,1]\). Thus, we have \(\min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|^q_\Sigma > 0\). Combining the fact that \(q < 2\), we deduce by the above inequality that there exists a constant \(s_0 \in (0, \rho_2)\) such that

\[
\max_{t \in [-1,1]} I_a(s_0\tilde{\gamma}_0(t)) < 0, \tag{40}
\]

which, immediately, implies

\[
I_a(s_0\psi_1) = I_a(s_0\tilde{\gamma}_0(1)) < 0 \quad \text{and} \quad I_a(-s_0\psi_1) = I_a(s_0\tilde{\gamma}_0(-1)) < 0.
\]
Meanwhile, using the assumption \((f_7)\), for \(t \in [s_0, \rho_2]\), we have
\[
\frac{d}{dt} I_a(t\psi_1) = \frac{1}{t} \langle I_a'(t\psi_1), t\psi_1 \rangle
= \frac{1}{t} \left[ a\|t\psi_1\|^2 + b\|t\psi_1\|^4 - \int_{\Omega} f(x, t\psi_1) t\psi_1 \, dx \right]
= \frac{1}{t} \left[ a\sqrt{\mu_1} t^2 + b\mu_1 t^4 - \int_{\Omega} f(x, t\psi_1) t\psi_1 \, dx \right]
= -\frac{1}{t} \left[ \int_{\Omega} f(x, t\psi_1) t\psi_1 \, dx - b\mu_1 t^4 - a\sqrt{\mu_1} t^2 \right]
< 0.
\]
So,
\[
I_a(t\psi_1) \leq I_a(s_0\psi_1) < 0 \quad \forall t \in [s_0, \rho_2].
\tag{41}
\]
Similarly, we have
\[
I_a(-t\psi_1) \leq I_a(-s_0\psi_1) < 0 \quad \forall t \in [s_0, \rho_2].
\tag{42}
\]
Define the linear path \(\gamma_1 : [-1, -\frac{1}{2}] \mapsto H\) as
\[
\gamma_1(t) = (-1 - 2t)(-\rho_2\psi_1) + 2(t + 1)(-s_0\psi_1) \quad \forall t \in \left[-1, -\frac{1}{2}\right].
\]
Evidently \(\gamma_1\) is continuous and
\[
\gamma_1(-1) = -\rho_2\psi_1 \quad \gamma_1\left(-\frac{1}{2}\right) = -s_0\psi_1.
\]
Moreover, from \((42)\), it follows that \(I_a|_{\gamma_1} < 0\). Set
\[
\gamma_2(t) = s_0\tilde{\gamma}_0(2t) \quad \forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right],
\]
where \(\tilde{\gamma}_0 \in \Gamma_0\) is introduced earlier in the proof. We see that \(\gamma_2\) is continuous, \(I_a|_{\gamma_2} < 0\) (see \((40)\)) and
\[
\gamma_2\left(-\frac{1}{2}\right) = -s_0\psi_1, \quad \gamma_2\left(\frac{1}{2}\right) = s_0\psi_1.
\]
Define another linear path \(\gamma_3 : \left[\frac{1}{2}, 1\right] \mapsto H\) as
\[
\gamma_3(t) = (2t - 1)(\rho_2\psi_1) + 2(1 - t)(s_0\psi_1) \quad \forall t \in \left[\frac{1}{2}, 1\right].
\]
We see immediately that \(\gamma_3\) is continuous, \(I_a|_{\gamma_3} < 0\) (see \((41)\)) and
\[
\gamma_3\left(\frac{1}{2}\right) = s_0\psi_1, \quad \gamma_3(1) = \rho_2\psi_1.
\]
Now, by concatenating the paths \(\gamma_1, \gamma_2, \gamma_3\), we generate a continuous path \(\gamma_*\) joining \(-\rho_2\psi_1\) with \(\rho_2\psi_1\). Thus, \(\gamma_* \in \Gamma_2\) with \(I_a(\gamma_*(t)) < 0\) and so \(u_1 \neq 0\).

**Case II: \(a=0\).**

**Lemma 3.5.** When \(a = 0\), \(I_0\) satisfies the Cerami condition under the condition \((f_0), (f_1)\) and \((f_2)\).
Proof. Note that
\[ I_0(u) = \frac{b}{4}||u||^4 - \int_{\Omega} F(x,u)dx \quad \text{if} \ a = 0. \]

Let \( u_n \subset H \) be a Cerami sequence of \( I_0 \). It needs only to show that \( u_n \) is bounded in \( H \) due to Remark 6. We may assume by contraction that
\[ I_0(u_n) \to c, \quad (1 + ||u_n||)I_0'(u_n) \to 0 \] and \( ||u_n|| \to \infty \) as \( n \to \infty \).

Thus, one has for \( n \) large enough,
\[ 4(c+1) \geq 4I_0(u_n) - \langle I_0'(u_n), u_n \rangle \]
\[ = \int_{\Omega} (f(x,u_n)u_n - 4F(x,u_n))dx. \quad (43) \]

From the right inequality of (f2), fixed an \( \varepsilon_2 > 0 \), there exists a \( C_{\varepsilon_2} > 0 \) such that
\[ |F(x,t)| \leq \frac{b(\mu_{k+1} + \varepsilon_2)}{4}t^4 \quad \forall x \in \Omega, |t| \geq C_{\varepsilon_2}. \]

Hence, by (f0) and \( F(x,t) = \int_{0}^{1} f(x,st)ds \), there exists a \( c_1 = c_1(\varepsilon_2) > 0 \) such that
\[ |F(x,t)| \leq \frac{b(\mu_{k+1} + \varepsilon_2)}{4}t^4 + c_1 \quad \forall (x,t) \in \Omega \times \mathbb{R}. \]

Thus, for \( n \) large enough, we have
\[ b||u_n||^4 = 4I_0(u_n) + 4\int_{\Omega} F(x,u_n)dx \]
\[ \leq 4(c+1) + b(\mu_{k+1} + \varepsilon_2)\int_{\Omega} |u_n|^4dx + 4c_1|\Omega| \]
\[ = c_2 + b(\mu_{k+1} + \varepsilon_2)||u_n||^4_{L^4}, \quad (44) \]

where \( c_2 = 4(c+1) + 4c_1|\Omega| \). Let \( v_n = \frac{u_n}{||u_n||} \), then there is \( v \in H \) such that \( u_n \rightharpoonup v \) weakly in \( H \) and \( v_n \to v \) strongly in \( L^4(\Omega) \). Dividing (44) by \( ||u_n||^4 \) and letting \( n \to \infty \), we have
\[ 1 \leq (\mu_{k+1} + \varepsilon_2)||v||^4_{L^4}, \]

which shows that \( \{ x \in \Omega : v(x) \neq 0 \} \) has a positive measure. Now, by (f0) and (f1), there exists a constant \( c > 0 \) such that
\[ f(x,t)t - 4F(x,t) > -c \quad \forall (x,t) \in \Omega \times \mathbb{R}. \]

Hence,
\[ \int_{\Omega} [f(x,u_n)u_n - 4F(x,u_n)]dx \]
\[ = \int_{\{x \in \Omega : v(x) \neq 0\}} [f(x,u_n)u_n - 4F(x,u_n)]dx \]
\[ + \int_{\{x \in \Omega : v(x) = 0\}} [f(x,u_n)u_n - 4F(x,u_n)]dx \]
\[ \geq \int_{\{x \in \Omega : v(x) \neq 0\}} [f(x,u_n)u_n - 4F(x,u_n)]dx - c|\{ x \in \Omega : v(x) = 0 \}|. \]

An application of Fatou’s lemma yields
\[ \int_{\Omega} [f(x,u_n)u_n - 4F(x,u_n)]dx \to +\infty \quad \text{as} \quad n \to \infty, \]
which is a contradiction to (43). Thus, \( u_n \) is bounded in \( H \).

\[ \square \]

**Proof of Theorem 1.3.** Defining

\[ \tilde{G}_1(x, s) = F(x, s) - \frac{b}{4} \mu_{k+1} s^4 \quad \forall (x, s) \in \Omega \times \mathbb{R}, \]
we get

\[ \tilde{G}_1(x, s) s - 4\tilde{G}_1(x, s) = f(x, s) s - 4F(x, s). \]

From \( f_1 \), for any \( M' > 0 \), there exists a constant \( C_{M'} > 0 \) such that

\[ \tilde{G}_1(x, s) s - 4\tilde{G}_1(x, s) \geq M' \]
for all \( x \in \Omega \) and \( |s| > C_{M'} \). Hence, for \( s > C_{M'} \), from the above inequality, we have

\[ \frac{d}{ds} \left( \frac{\tilde{G}_1(x, s)}{s^4} \right) = \frac{\tilde{G}_1(x, s) s - 4\tilde{G}_1(x, s)}{s^5} \geq \frac{M'}{s^5}. \]

Integrating the above expression over the interval \([t, T] \subset (C_{M'}, +\infty)\), one has

\[ \frac{\tilde{G}_1(x, t)}{t^4} \leq \frac{\tilde{G}_1(x, T)}{T^4} + \frac{M'}{4} \left( \frac{1}{T^4} - \frac{1}{t^4} \right). \]

Since \( \lim \sup_{T \to +\infty} \frac{\tilde{G}_1(x, T)}{T^4} \leq 0 \), letting \( T \to +\infty \), we obtain \( \tilde{G}_1(x, t) \leq -\frac{M'}{4} \) for all \( t \geq C_{M'} \) and \( x \in \Omega \). Similarly, \( \tilde{G}_1(x, t) \leq -\frac{M'}{4} \) for all \( t \leq -C_{M'} \) and \( x \in \Omega \). Hence, we have by the arbitrariness of \( M' \), that

\[ \lim_{|t| \to \infty} \tilde{G}_1(x, t) = -\infty \text{ uniformly for } x \in \Omega. \]

(45)

Combining \( f_0 \) with the above equality, we find a positive constant \( c \) such that

\[ \tilde{G}_1(x, t) < c \quad \forall (x, t) \in \Omega \times \mathbb{R}. \]

For any \( u \in \mathcal{C}_{k+1} \), from the above inequality, one has

\[ I_0(u) = \frac{b}{4} \| u \|^4 - \int_\Omega F(x, u) \, dx \]
\[ = \frac{b}{4} \left( \| u \|^4 - \mu_{k+1} \| u \|^4_{L^4} \right) - \int_\Omega \left( F(x, u) - \frac{b\mu_{k+1}}{4} \| u \|^4 \right) \, dx \]
\[ \geq - \int_\Omega \tilde{G}_1(x, u) \, dx \geq -c|\Omega|. \]

(46)

From the left inequality of \( (f_0) \) and \( (f_2) \), one has

\[ |F(x, t)| \geq \frac{b}{4} \left( \mu_k + 2\varepsilon_3 \right) t^4 - c_1 \quad \forall (x, t) \in \Omega \times \mathbb{R}, \]

(47)

for some \( c_1 > 0 \) and \( \varepsilon_3 \in (0, \frac{\mu_{k+1} - \mu_k}{2}) \). Nevertheless, from the definition of the variational eigenvalue \( \mu_k \), it follows that there exists some \( \Lambda_2 \in \Sigma_k \) such that

\[ \sup_{u \in \Lambda_2} \| u \|^4 \leq \mu_k + \varepsilon_3. \]

(48)

Combining (47) with (48), we see

\[ I_0(tu) = \frac{b}{4} \| tu \|^4 - \int_\Omega F(x, tu) \, dx \]
\[ \leq - \frac{b\varepsilon_3}{4} t^4 + c_1|\Omega|, \quad \text{for } u \in \Lambda_2 \text{ and } t > 0. \]

(49)
Proof of Theorem 1.4. We just give a brief proof for completeness. Similar to (45),

\[
\alpha' = \max_{u \in \Lambda_2} I_0(tu) < -c|\Omega| = \beta' \leq \inf_{u \in C_{k+1}} I_0(u).
\]

(50)

Set \( \rho_3 \Lambda_2 = \{tu : u \in \Lambda_2, t \geq \rho_3\} \) and

\[ G' = \{h \in C(S^{k-1}, \rho_3 \Lambda_2) : h \text{ is odd}\}. \]

For any \( h \in G' \), by (50), we get \( h(S^{k-1}) \cap C_{k+1} = \emptyset \), which shows that \( G' \) is a subset of \( C(S^{k-1}, H \setminus C_{k+1}) \). Let

\[ \Gamma_1' = \{h \in C(B^k, H) : h|_{S^{k-1}} \in G'\}. \]

Then, proceeding exactly as in the previous Theorem 1.1, we may verify (50) and Lemma 3.5, the conclusion in Theorem 1.3 is correct.

\[ \square \]

Proof of Theorem 1.4. We just give a brief proof for completeness. Similar to (45), we may deduce from \((f_1), (f_4)\) that

\[ \lim_{|t| \to \infty} \left[ F(x, t) - \frac{bt_2}{4}|t|^4 \right] = -\infty \text{ uniformly for } x \in \Omega, \]

and so, for any \( u \in \mathbb{D} \), one has

\[
I_0(u) = \frac{b}{4} \|u\|^4 - \int_{\Omega} F(x, u)dx = \frac{b}{4} \left( \|u\|^4 - \mu_2 \int_{\Omega} |u|^4 \right) - \int_{\Omega} \left[ F(x, u) - \frac{bt_2}{4}|u|^4 \right] dx \\
\geq -c|\Omega|
\]

(51)

for some \( c > 0 \). The condition \((f_5)\) immediately implies that

\[
I_0(t\psi_1) = \frac{b}{4} \|t\psi_1\|^4 - \int_{\Omega} F(x, t\psi_1)dx \to -\infty \text{ as } |t| \to \infty.
\]

(52)

We see, from (51) and (52), that there exists a constant \( \rho_4 > 0 \) large enough such that

\[
\max \{I_0(\rho_4\psi_1), I_0(-\rho_4\psi_1)\} < -c|\Omega| \leq \inf_{\mathbb{D}} I_0.
\]

Let

\[
\Gamma_2' = \{\gamma \in C([-1, 1]; H) : \gamma(-1) = -\rho_4\psi_1, \gamma(1) = \rho_4\psi_1\};
\]

\[ \partial Q_2 = \{-\rho_4\psi_1, \rho_4\psi_1\}; \]

and

\[ Q_2 = [-\rho_4\psi_1, \rho_4\psi_1]. \]

Then, \((Q_2, \partial Q_2)\) is linking with \( \mathbb{D} \). (The proof is similar to the corresponding part of Theorem 1.2.) Thus, there exists a critical point \( \widehat{u}_1 \in H \) of \( I_0 \) such that

\[
c = I_0(\widehat{u}_1) = \inf_{\gamma \in \Gamma_2'} \max_{-1 \leq t \leq 1} I_0(\gamma(t)).
\]

Indeed, \( \widehat{u}_1 \) is nonzero under the conditions \((f_9)\) and \((f_{10})\). For a fixed \( \mu > \mu_2' \), we may find \( \widehat{\gamma}_0 \in \Gamma_2' \) such that

\[
\mu_2' \leq \max_{-1 \leq t \leq 1} \|\widehat{\gamma}_0(t)\|^4 < \mu.
\]
It is clear from assumptions \((f_0), (f_1)\) and \((f_2)\) that
\[
F(x,t) \geq k_2 |t|^r - \frac{c}{4} t^4 \quad \forall (x,t) \in \Omega \times \mathbb{R}
\]
and some \(c > 0\). So,
\[
I_0(s\tilde{\gamma}_0(t)) = \frac{b s^4}{4} \|\tilde{\gamma}_0(t)\|^4 - \int_{\Omega} F(x,s\tilde{\gamma}_0(t))
\]
\[
\leq \frac{b\mu}{4} s^4 - k_2 s^r \min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|_{L^r} + \frac{c}{4} s^4
\]
\[
= \frac{b\mu + c}{4} s^4 - k_2 s^r \min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|_{L^r}.
\]

It is easy to see that \(\min_{t \in [-1,1]} \|\tilde{\gamma}_0(t)\|_{L^r} > 0\). Note that \(1 < r < 4\) and then we find some \(s_1 \in (0, \rho_4)\) such that
\[
\max_{t \in [-1,1]} I_0(s_1\tilde{\gamma}_0(t)) < 0.
\]
Thus, we conclude
\[
I_0(s_1\psi_1) = I_0(s_1\tilde{\gamma}_0(1)) < 0 \quad I_0(-s_1\psi_1) = I_0(s_1\tilde{\gamma}_0(-1)) < 0.
\]
According to the assumption \((f_{10})\), for \(t \in [s_1, \rho_4]\), we have
\[
\frac{d}{dt} I_0(t\psi_1) = \frac{1}{t} \left( I'_0(t\psi_1), t\psi_1 \right)
\]
\[
= \frac{1}{t} \left[ b\mu_1 |t|^4 - \int_{\Omega} f(x,t\psi_1)t\psi_1 dx \right]
\]
\[
= - \frac{1}{t} \left[ \int_{\Omega} f(x,t\psi_1)t\psi_1 dx - b\mu_1 |t|^4 \right]
\]
\[
< 0,
\]
which means
\[
I_0(t\psi_1) \leq I_0(s_1\psi_1) < 0 \quad \forall t \in [s_1, \rho_4].
\]
It is also true that
\[
I_0(-t\psi_1) \leq I_0(-s_1\psi_1) < 0 \quad \forall t \in [s_1, \rho_4].
\]
Define the linear path \(\tilde{\gamma}_1 : [-1, -\frac{1}{2}] \to H\) as
\[
\tilde{\gamma}_1(t) = (-1 - 2t)(-\rho_4 \psi_1) + 2(t + 1)(-s_1 \psi_1) \quad \forall t \in \left[ -1, -\frac{1}{2} \right].
\]
Evidently \(\tilde{\gamma}_1\) is continuous, \(I_0|\tilde{\gamma}_1 < 0\), and
\[
\tilde{\gamma}_1(-1) = -\rho_4 \psi_1 \quad \tilde{\gamma}_1(-\frac{1}{2}) = -s_1 \psi_1.
\]
Set
\[
\tilde{\gamma}_2(t) = s_1 \tilde{\gamma}_0(2t) \quad \forall t \in \left[ -\frac{1}{2}, \frac{1}{2} \right],
\]
where \(\tilde{\gamma}_0 \in \Gamma_0\) is introduced earlier in the proof. We see that \(\tilde{\gamma}_2\) is continuous, \(I_0|\tilde{\gamma}_2 < 0\) and
\[
\tilde{\gamma}_2(-\frac{1}{2}) = -s_1 \psi_1, \quad \tilde{\gamma}_2(\frac{1}{2}) = s_1 \psi_1.
\]
Define another linear path \( \hat{\gamma}_3 : [\frac{1}{2}, 1] \rightarrow H \) as
\[
\hat{\gamma}_3(t) = (2t-1)(\rho_4 \psi_1) + 2(1-t)(s_1 \psi_1), \quad \forall t \in \left[ \frac{1}{2}, 1 \right].
\]
We see immediately that \( \hat{\gamma}_3 \) is continuous, \( I_0|_{\hat{\gamma}_3} < 0 \) and
\[
\hat{\gamma}_3\left( \frac{1}{2} \right) = s_1 \psi_1 \quad \hat{\gamma}_3(1) = \rho_4 \psi_1.
\]
Now, by concatenating the paths \( \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \), we generate a continuous path \( \hat{\gamma}_* \) joining \(-\rho_4 \psi_1 \) with \( \rho_4 \psi_1 \). Thus, \( \hat{\gamma}_* \in \Gamma_1^* \) with \( I_0|_{\hat{\gamma}_*} < 0 \). This implies \( I_0(\hat{u}_1) \leq I_0(\hat{\gamma}_*(t)) < 0 \) and so \( \hat{u}_1 \neq 0 \).

Acknowledgements. The authors would like to express sincere thank to the anonymous referees for careful reading of the manuscript and valuable comments.

REFERENCES

[1] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with “strong” resonance at infinity, *Nonlinear Anal.*, 7 (1983), 981–1012.
[2] S. Bernstein, Sur une classe d’´equations fonctionnelles aux d´eriv´ees partielles (Russian) Bull. Acad. Sci. URSS, *Sér. Math. [Izvestia Akad. Nauk SSSR],* 4 (1940), 17–26.
[3] B. T. Cheng, New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems, *J. Math. Anal. Appl.*, 394 (2012), 488–495.
[4] B. T. Cheng and X. Wu, Existence results of positive solutions of Kirchhoff type problems, *Nonlinear Anal.*, 71 (2009), 4883–4892.
[5] M. Chipot and B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.*, 30 (1997), 4619–4627.
[6] M. Cuesta, D. de Figueiredo and J.-P. Gossez, The beginning of the Fučik spectrum for the \( p \)-Laplacian, *J. Differential Equations*, 159 (1999), 212–238.
[7] P. Drábek and S. B. Robinson, Resonance problems for the \( p \)-Laplacian, *J. Funct. Anal.*, 169 (1999), 189–200.
[8] L. Ding, L. Li and J. L. Zhang, Solutions to Kirchhoff equations with combined nonlinearities, *Electron. J. Differential Equations*, 2014, No. 10, 10 pp.
[9] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
[10] P. Kanishka and Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations*, 221 (2006), 246–255.
[11] Z. P. Liang, F. Y. Li and J. P. Shi, Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31 (2014), 155–167.
[12] J.-L. Lions, On some questions in boundary value problems of mathematical physics, *Contemporary developments in continuum mechanics and partial differential equations*, 30, 284–346, North-Holland Math. Stud., North-Holland, Amsterdam-New York, 1978.
[13] A. M. Mao and Z. T. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, *Nonlinear Anal.*, 70 (2009), 1275–1287.
[14] A. M. Mao and S. X. Luan, Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems, *J. Math. Anal. Appl.*, 383 (2011), 239–243.
[15] S. Michael, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Second edition, Springer-Verlag, Berlin, 1996.
[16] S. I. Pohožaev, A certain class of quasilinear hyperbolic equations (Russian), *Mat. Sb. (N.S.)*, 96 (1975), 152–166, 168.
[17] J. J. Sun and C. L. Tang, Resonance problems for Kirchhoff type equations, *Discrete Contin. Dyn. Syst.*, 33 (2013), 2139–2154.
[18] J. Sun and S. B. Liu, Nontrivial solutions of Kirchhoff type problems, *Appl. Math. Lett.*, 25 (2012), 500–504.
[19] J. J. Sun and C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type equations, *Nonlinear Anal.*, 74 (2011), 1212–1222.
[20] S. Z. Song and C. L. Tang, Resonance problems for the $p$-Laplacian with a nonlinear boundary condition, *Nonlinear Anal.*, 64 (2006), 2007–2021.

[21] Y. W. Ye, Infinitely many solutions for Kirchhoff type problems, *Differ. Equ. Appl.*, 5 (2013), 83–92.

[22] Y. Yang and J. H. Zhang, Positive and negative solutions of a class of nonlocal problems, *Nonlinear Anal.*, 73 (2010), 25–30.

[23] Y. Yang and J. H. Zhang, Nontrivial solutions of a class of nonlocal problems via local linking theory, *Appl. Math. Lett.*, 23 (2010), 377–380.

[24] Z. T. Zhang and P. Kanishka, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.*, 317 (2006), 456–463.

Received November 2014; revised May 2016.

E-mail address: sjrdj@ctbu.edu.cn
E-mail address: 11183356@qq.com
E-mail address: tangcl@swu.edu.cn