Stochastic Optimal Control with Delay in the Control I: solving the HJB equation through partial smoothing

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Abstract

Stochastic optimal control problems governed by delay equations with delay in the control are usually more difficult to study than the the ones when the delay appears only in the state. This is particularly true when we look at the associated Hamilton-Jacobi-Bellman (HJB) equation. Indeed, even in the simplified setting (introduced first by Vinter and Kwong [40] for the deterministic case) the HJB equation is an infinite dimensional second order semilinear Partial Differential Equation (PDE) that does not satisfy the so-called “structure condition” which substantially means that the control can act on the system modifying its dynamics at most along the same directions along which the noise acts. The absence of such condition, together with the lack of smoothing properties which is a common feature of problems with delay, prevents the use of the known techniques (based on Backward Stochastic Differential Equations (BSDEs) or on the smoothing properties of the linear part) to prove the existence of regular solutions of this HJB equation and so no results on this direction have been proved till now.

In this paper we provide a result on existence of regular solutions of such kind of HJB equations. This opens the road to prove existence of optimal feedback controls, a task that will be accomplished in the companion paper [26]. The main tool used is a partial smoothing property that we prove for the transition semigroup associated to the uncontrolled problem. Such results hold for a specific class of equations and data which arises naturally in many applied problems.

Key words: Optimal control of stochastic delay equations; Delay in the control; Lack of structure condition; Second order Hamilton-Jacobi-Bellman equations in infinite dimension; Smoothing properties of transition semigroups.

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1 Introduction

Optimal control problems governed by delay equations with delay in the control are usually harder to study than the ones when the delay appears only in the state (see e.g. [2, Chapter 4] and [24, 25]). This is true already in the deterministic case but things get worse in the stochastic case. When one tries to apply the dynamic programming method the main difficulty is the fact that, even in the simplified setting introduced first by Vinter and Kwong [40] in the deterministic case (see e.g. [24] for the stochastic case), the associated HJB equation is an infinite dimensional second order semilinear PDE that does not satisfy the so-called “structure condition”, which substantially means that the control can act on the system modifying its dynamics at most along the same directions along which the noise acts.

The absence of such condition, together with the lack of smoothing properties which is a common feature of problems with delay, prevents the use of the known techniques, based on BSDE’s (see e.g. [19]) or on fixed point theorems in spaces of continuous functions (see e.g. [4, 5, 12, 22, 23] or in Gauss-Sobolev spaces (see e.g. [9, 21]), to prove the existence of regular solutions of this HJB equation: hence no results in this direction have been proved till now. The viscosity solution technique can still be used (see e.g. [24]) but to prove existence (and possibly uniqueness) of solutions that are merely continuous. This is an important drawback in this context, since, to prove the existence of optimal feedback control strategies through the dynamic programming approach, one needs at least the differentiability of the solution in the “space-like” variable.

The main aim of this paper is to provide a new result of existence of regular solutions of such HJB equations that holds when the state equation depends linearly on the history of the control and when the cost functional does not depend on such history. Such results will be exploited in the companion paper [26] to solve the corresponding stochastic optimal control problem finding optimal feedback control strategies. This allows to treat satisfactorily a specific class of state equations and data which arise naturally in many applied problems (see e.g. [3, 14, 18, 24, 25, 29]).

The key tool to prove such results is the proof of a “partial” smoothing property for the transition semigroup associated to the uncontrolled equation which we think is interesting in itself and is presented in Section 3.
We believe that such tool may allow to treat also examples where the state equation depends on the history of the state variable, too. To keep things simpler, here we choose to develop and present the result when this does not happen leaving the extension to a subsequent paper.

1.1 Plan of the paper

The plan of the paper is the following:

- in Section 3 we give some notations and we present the problem and the main assumptions;
- in Section 4 we prove the partial smoothing property for the Ornstein-Uhlenbeck transition semi-group, and we explain how to adapt it to an infinite dimensional setting;
- in Section 5 we introduce some spaces of functions where we will perform the fixed point argument and we prove regularity of some convolutions type integrals;
- in Section 6 we solve the HJB equation in mild sense.

2 Preliminaries

2.1 Notation

Let $H$ be a Hilbert space. The norm of an element $x$ in $H$ will be denoted by $|x|_H$ or simply $|x|$, if no confusion is possible, and by $\langle \cdot, \cdot \rangle_H$, or simply by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $H$. We denote by $H^*$ the dual space of $H$. Usually we will identify $H$ with its dual $H^*$. If $K$ is another Hilbert space, $\mathcal{L}(H,K)$ denotes the space of bounded linear operators from $H$ to $K$ endowed with the usual operator norm. All Hilbert spaces are assumed to be real and separable.

In what follows we will often meet inverses of operators which are not one-to-one. Let $Q \in \mathcal{L}(H,K)$. Then $H_0 = \ker Q$ is a closed subspace of $H$. Let $H_1$ be the orthogonal complement of $H_0$ in $H$: $H_1$ is closed, too. Denote by $Q_1$ the restriction of $Q$ to $H_1$: $Q_1$ is one-to-one and $\text{Im} Q_1 = \text{Im} Q$. For $k \in \text{Im} Q$, we define $Q^{-1}(k) := Q_1^{-1}(k)$.

The operator $Q^{-1} : \text{Im} Q \to H$ is called the pseudoinverse of $Q$. $Q^{-1}$ is linear and closed but in general not continuous. Note that if $k \in \text{Im} Q$, then $Q_1^{-1}(k)$ is the unique element of $\{ h : Q(h) = k \}$ with minimal norm (see e.g. [22], p.209).

In the following, by $(\Omega, \mathcal{F}, \mathbb{P})$ we denote a complete probability space, and by $L^2_p(\Omega \times [0,T], H)$ the Hilbert space of all predictable processes $(Z_t)_{t \in [0,T]}$ with values in $H$, normed by $\|Z\|_{L^2_p(\Omega \times [0,T], H)}^2 = \mathbb{E} \int_0^T |Z_t|^2 \, dt$.

Next we introduce some spaces of functions. Let $H$ and $Z$ be real separable Hilbert spaces. By $B_b(H,Z)$ (respectively $C_b(H,Z)$, $UC_b(H,Z)$) we denote the space of all functions $f : H \to Z$ which are Borel measurable and bounded (respectively continuous and bounded, uniformly continuous and bounded).

Given an interval $I \subseteq \mathbb{R}$ we denote by $C(I \times H, Z)$ (respectively $C_b(I \times H, Z)$) the space of all functions $f : I \times H \to Z$ which are continuous (respectively continuous and bounded). $C^{0,1}(I \times H, Z)$ is the space of functions $f \in C(I \times H)$ such that for all $t \in I$ $f(t, \cdot )$ is Fréchet differentiable. By $UC^{1,2}_b(I \times H, Z)$ we denote the linear space of the mappings $f : I \times H \to Z$ which are uniformly continuous and bounded together with their first time derivative $f_t$ and its first and second space derivatives $\nabla f, \nabla^2 f$.

If $Z = \mathbb{R}$ we do not write it in all the above spaces.

2.2 C-derivatives

We first recall the definition of $C$-directional derivatives given in [32], Section 2, and in [20]. Here $H, K, Z$ are Hilbert spaces.

**Definition 2.1** Let $H, K, Z$ be real Hilbert spaces. Let $C : K \to H$ be a bounded linear operator and let $f : H \to Z$. 


\begin{itemize}
  \item The C-directional derivative $\nabla^C$ at a point $x \in H$ in the direction $k \in K$ is defined as:
  \[
  \nabla^C f(x; k) = \lim_{s \to 0} \frac{f(x + sCk) - f(x)}{s}, \quad s \in \mathbb{R},
  \]
  provided that the limit exists.
  
  \item We say that a continuous function $f$ is C-Gâteaux differentiable at a point $x \in H$ if $f$ admits the C-directional derivative in every direction $k \in K$ and there exists a linear operator, called the C-Gâteaux differential, $\nabla^C f(x) \in \mathcal{L}(K, Z)$, such that $\nabla^C f(x; k) = \nabla^C f(x)k$ for $x \in H$, $k \in K$. The function $f$ is C-Gâteaux differentiable on $H$ if it is C-Gâteaux differentiable at every point $x \in H$.
  
  \item We say that $f$ is C-Fréchet differentiable at a point $x \in H$ if it is C-Gâteaux differentiable and if the limit in \((2.1)\) is uniform for $k$ in the unit ball of $K$. In this case we call $\nabla^C f(x)$ the C-Fréchet derivative (or simply the C-derivative) of $f$ at $x$. We say that $f$ is C-Fréchet differentiable on $H$ if it is C-Fréchet differentiable at every point $x \in H$.
\end{itemize}

Note that, in doing the C-derivative, one considers only the directions in $H$ selected in the image of $C$. When $Z = \mathbb{R}$ we have $\nabla^C f(x) \in K^*$. Usually we will identify $K$ with its dual $K^*$ so $\nabla^C f(x)$ will be treated as an element of $K$.

If $f : H \to \mathbb{R}$ is Gâteaux (Fréchet) differentiable on $H$ we have that, given any $C$ as in the definition above, $f$ is C-Gâteaux (Fréchet) differentiable on $H$ and

\[
\langle \nabla^C f(x), k \rangle_K = \langle \nabla f(x), Ck \rangle_H
\]

i.e. the C-directional derivative is just the usual directional derivative at a point $x \in H$ in direction $Ck \in H$. Anyways the C-derivative, as defined above, allows us to deal also with functions that are not Gâteaux differentiable in every direction.

Now we define suitable spaces of C-differentiable functions.

**Definition 2.2** Let $I$ be an interval in $\mathbb{R}$ and let $H$, $K$ and $Z$ be suitable real Hilbert spaces.

\begin{itemize}
  \item We call $C^{1, C}_b(H, Z)$ the space of all functions $f : I \times H \to Z$ which admit continuous and bounded C-Fréchet derivative. Moreover we call $C^{0, 1, C}_b(I \times H, Z)$ the space of functions $f : I \times H \to Z$ belonging to $C_b(I \times H, Z)$ and such that, for every $t \in I$, $f(t, \cdot) \in C^{1, C}_b(H, Z)$. When $Z = \mathbb{R}$ we omit it.
  
  \item We call $C^{2, C}_b(H, Z)$ the space of all functions $f$ in $C^{1}_b(H, Z)$ which admit continuous and bounded directional second order derivative $\nabla^C \nabla f$; by $C^{0, 2, C}_b(I \times H, Z)$ we denote the space of functions $f \in C_b(I \times H, K)$ such that for every $t \in I$, $f(t, \cdot) \in C^{2, C}_b(H, Z)$. When $Z = \mathbb{R}$ we omit it.
  
  \item For any $\alpha \in (0, 1)$ and $T > 0$ (this time $I$ is equal to $[0, T]$) we denote by $C^{\alpha, 1, C}_F([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H, Z) \cap C^{\alpha, 1, C}_b((0, T] \times H)$ such that the map $(t, x) \mapsto t^\alpha \nabla^C f(t, x)$ belongs to $C_b([0, T] \times H, K)$. The space $C^{\alpha, 1, C}_F([0, T] \times H)$ is a Banach space when endowed with the norm

\[
\|f\|_{C^{\alpha, 1, C}_F([0, T] \times H)} = \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in [0, T] \times H} t^\alpha \|\nabla^C f(t, x)\|_K.
\]

When clear from the context we will write simply $\|f\|_{C^{0,1,c}}$.
  
  \item For any $\alpha \in (0, 1)$ and $T > 0$ we denote by $C^{\alpha, 2, C}_F([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H, K) \cap C^{0, 2, C}_b((0, T] \times H)$ such that for all $t \in (0, T]$, $x \in H$ the map $(t, x) \mapsto t^\alpha \nabla^C \nabla f(t, x)$ is bounded and continuous as a map from $(0, T] \times H$ with values in $H \times K$. The space $C^{\alpha, 2, C}_F([0, T] \times H)$ turns out to be a Banach space if it is endowed with the norm

\[
\|f\|_{C^{\alpha, 2, C}_F([0, T] \times H)} = \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in [0, T] \times H} \|\nabla^C f(t, x)\|_H + \sup_{(t, x) \in [0, T] \times H} t^\alpha \|\nabla^C \nabla f(t, x)\|_{H \times K}.
\]
\end{itemize}
3 Setting of the problem and main assumptions

3.1 State equation

In a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we consider the following controlled stochastic differential equation in \(\mathbb{R}^n\) with delay in the control:

\[
\begin{cases}
    dy(t) = a_0 y(t)dt + b_0 u(t)dt + \int_{-d}^0 b_1(\xi) u(t+\xi)d\xi + \sigma dW_t, & t \in [0, T] \\
    y(0) = y_0, \\
    u(\xi) = u_0(\xi), & \xi \in [-d, 0),
\end{cases}
\]

where we assume the following.

**Hypothesis 3.1**

(i) \(W\) is a standard Brownian motion in \(\mathbb{R}^k\), and \((\mathcal{F}_t)_{t \geq 0}\) is the augmented filtration generated by \(W\);

(ii) \(a_0 \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n), \sigma\) is in \(\mathcal{L}(\mathbb{R}^k; \mathbb{R}^n)\);

(iii) the control strategy \(u\) belongs to \(U\) where

\[U := \{ z \in L^2_\mathbb{P}(\Omega \times [0, T], \mathbb{R}^m) : u(t) \in U \text{ a.s.} \}\]

where \(U\) is a closed subset of \(\mathbb{R}^n\);

(iv) \(d > 0\) (the maximum delay the control takes to affect the system);

(v) \(b_0 \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)\);

(vi) \(b_1 \in L^2([-d, 0], \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)).\) \((b_1\) is the density of the time taken by the control to affect the system).

Notice that assumption (vi) on \(b_1\) does not cover the case of pointwise delay since it is technically complicated to deal with: indeed it gives rise, as we are going to see in next subsection, to an unbounded control operator \(B\), for this reason we leave the extension of our approach to this case for further research.

**Remark 3.2** Our results can be generalized to the case when the process \(y\) is infinite dimensional. More precisely, let \(y\) be the solution of the following controlled stochastic differential equation in an infinite dimensional Hilbert space \(H\), with delay in the control:

\[
\begin{cases}
    dy(t) = A_0 y(t)dt + B_0 u(t)dt + \int_{-d}^0 B_1(\xi) u(t+\xi)d\xi + \sigma dW_t, & t \in [0, T] \\
    y(0) = y_0, \\
    u(\xi) = u_0(\xi), & \xi \in [-d, 0).
\end{cases}
\]

Here \(W\) is a cylindrical Wiener process in another Hilbert space \(\Xi\), and \((\mathcal{F}_t)_{t \geq 0}\) is the augmented filtration generated by \(W\). \(A_0\) is the generator of a strongly continuous semigroup in \(H\). The diffusion term \(\sigma\) is in \(\mathcal{L}(\Xi; H)\) and is such that for every \(t > 0\) the covariance operator

\[Q^0_t := \int_0^t e^{sA_0} \sigma \sigma^* e^{sA_0} ds\]

of the stochastic convolution

\[\int_0^t e^{(t-s)A_0} \sigma dW_s\]

is of trace class and, for some \(\gamma \in (0, 1),\)

\[\int_0^t s^{-\gamma} \text{Tr} e^{sA_0} \sigma \sigma^* e^{sA_0} ds < +\infty.\]

The control strategy \(u\) belongs to \(L^2_\mathbb{P}(\Omega \times [0, T], U_1)\), where \(U_1\) is another Hilbert space, and the space of admissible controls \(U\) is built in analogy with the finite dimensional case requiring control strategies to take values in a given closed subset \(U\) of \(U_1\). On the control operators we assume \(B_0 \in \mathcal{L}(U_1; H), B_1 : [-d, 0] \to \mathcal{L}(U_1, H)\) such that \(B_1 u \in L^2([-d, 0], H)\) for all \(u \in U\). In this case, following again \(24, 40\), the problem can be reformulated as an abstract evolution equation in the Hilbert space \(H\) that this time turns out to be \(H \times L^2([-d, 0], H)\). All the results of this paper hold true in this case, under suitable minor changes that will be clarified along the way.
3.2 Infinite dimensional reformulation

Now, using the approach of [40] (see [24] for the stochastic case), we reformulate equation (3.1) as an abstract stochastic differential equation in the Hilbert space \( H = \mathbb{R}^n \times L^2([-d, 0], \mathbb{R}^n) \). To this end we introduce the operator \( A : D(A) \subset H \rightarrow H \) as follows: for \((y_0, y_1) \in H\)

\[
A(y_0, y_1) = (a_0 y_0 + y_1(0), -y'_1), \quad D(A) = \{(y_0, y_1) \in H: y_1 \in W^{1,2}([-d, 0], \mathbb{R}^n), y_1(-d) = 0\}. \tag{3.3}
\]

We denote by \( A^* \) the adjoint operator of \( A \):

\[
A^*_0(y_0, y_1) = (a_0 y_0, y_1)'(0), \quad D(A^*) = \{(y_0, y_1) \in H: y_1 \in W^{1,2}([-d, 0], \mathbb{R}^n), y_1(0) = y_0\}. \tag{3.4}
\]

We denote by \( e^{tA} \) the \( C_0 \)-semigroup generated by \( A \): for \( y = (y_0, y_1) \in H \),

\[
e^{tA} y_1 = \begin{pmatrix} e^{\tau_0} y_0 + \int_{-\tau_0}^0 1_{[-\tau_0, 0]} e^{(s+\tau_0) a_0} y_1(s) s \, ds \\ y_1(\cdot - t) 1_{[-d+t, 0]}(\cdot) \end{pmatrix} \tag{3.5}
\]

Similarly, denoting by \( e^{tA^*} = (e^{tA})^* \) the \( C_0 \)-semigroup generated by \( A^* \), we have for \( z = (a_0 y_0, y_1) \in H \)

\[
e^{tA^*} = \begin{pmatrix} \int_{-a_0, y_0}^{e^{tA} y_0} z_0 \\ e^{(\cdot + t)a_0} y_0 1_{[-a_0, y_0]}(\cdot) + y_1(\cdot + t) 1_{[-d, 0]}(\cdot) \end{pmatrix} \tag{3.6}
\]

The infinite dimensional noise operator is defined as

\[
G : \mathbb{R}^k \rightarrow H, \quad G y = (\sigma y, 0), \quad y \in \mathbb{R}^k. \tag{3.7}
\]

The control operator \( B \) is bounded and defined as

\[
B : \mathbb{R}^m \rightarrow H, \quad B u = (b_0 u, b_1(\cdot) u), \quad u \in \mathbb{R}^m \tag{3.8}
\]

and its adjoint is

\[
B^* : H^* \rightarrow \mathbb{R}^m, \quad B^*(x_0, x_1) = b_0^* x_0 + \int_{-d}^0 b_1^*(\xi) x_1(\xi) d\xi, \quad (x_0, x_1) \in H. \tag{3.9}
\]

Note that, in the case of pointwise delay the last term of the drift in the state equation \( 3.1 \) is \( u(\cdot - d) \), hence \( b_1(\cdot) \) is a measure: the Dirac delta \( 3.6 \). Hence in this case \( B \) is unbounded as it takes values in \( \mathbb{R}^n \times C^*([-d, 0], \mathbb{R}^n) \) (here we denote by \( C^*([-d, 0], \mathbb{R}^n) \) the dual space of \( C([-d, 0], \mathbb{R}^n) \)).

It will be useful to write the explicit expression of the first component of the operator \( e^{tA} B \) as follows

\[
(e^{tA} B)_{\xi} : \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad (e^{tA} B)_{\xi} u = e^{\xi a_0} b_0 u + \int_{-d}^0 1_{[-a_0, u]} e^{(\xi + t) a_0} b_1(\xi) d\xi, \quad u \in \mathbb{R}^m \tag{3.10}
\]

Given any initial datum \((y_0, u_0) \in H \) and any admissible control \( u \in U \) we call \( y(t; y_0, u_0, u) \) (or simply \( y(t) \) when clear from the context) the unique solution (which comes from standard results on SDE’s, see e.g. [24] Chapter 4, Sections 2 and 3) of \( 3.1 \).

Let us now define the process \( Y = (Y_0, Y_1) \in L_+^2(\Omega \times [0, T], H) \) as

\[
Y_0(t) = y(t), \quad Y_1(t)(\xi) = \int_{-d}^t u(\xi + t - \xi) b_1(\xi) d\xi,
\]

where \( y \) is the solution of equation \( 3.1 \), \( u \) is the control process in \( 3.1 \). By Proposition 2 of [24], the process \( Y \) is the unique solution of the abstract evolution equation in \( H \)

\[
\begin{cases}
    dY(t) = AY(t) dt + Bu(t) dt + GdW_t, & t \in [0, T] \\
    Y(0) = y = (y_0, y_1),
\end{cases} \tag{3.11}
\]

where \( y_0 = x_0 \) and \( y_1(\xi) = \int_{-d}^\xi u_0(\xi - \xi) b_1(\xi) d\xi \). Note that we have \( y_1 \in L^2([-d, 0], \mathbb{R}^n) \). Taking the integral (or mild) form of \( 3.11 \) we have

\[
Y(t) = e^{tA} y + \int_0^t e^{(t-s)A} Bu(s) ds + \int_0^t e^{(t-s)A} GdW_s, \quad t \in [0, T]. \tag{3.12}
\]

\[\text{This can be seen, e.g., by a simple application of Jensen inequality and Fubini theorem.}\]
3.3 Optimal Control problem

The objective is to minimize, over all controls in $U$, the following finite horizon cost:

$$J(t, x, u) = \mathbb{E} \int_t^T \left( \bar{\ell}_0(s, y(s)) + \bar{\ell}_1(u(s)) \right) \, ds + \mathbb{E} \bar{\phi}(x(T)).$$

(3.13)

where $\bar{\ell}_0 : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and $\bar{\phi} : \mathbb{R}^n \to \mathbb{R}$ are continuous and bounded while $\bar{\ell}_1 : U \to \mathbb{R}$ is measurable and bounded from below. Referring to the abstract formulation (3.11), the cost in (3.13) can be rewritten also as

$$J(t, x; u) = \mathbb{E} \left( \int_t^T [\ell_0(s, Y(s)) + \ell_1(u(s))] \, ds + \phi(Y(T)) \right),$$

(3.14)

where $\ell_0 : [0, T] \times \mathcal{H} \to \mathbb{R}$, $\ell_1 : U \to \mathbb{R}$ are defined by setting

$$\ell_0(t, x) := \bar{\ell}_0(t, x_0) \quad \forall x = (x_0, x_1) \in \mathcal{H}$$

(3.15)

$$\ell_1 := \bar{\ell}_1$$

(3.16)

(here we cut the bar only to keep the notation homogeneous) while $\phi : \mathcal{H} \to \mathbb{R}$ is defined as

$$\phi(x) := \bar{\phi}(x_0) \quad \forall x = (x_0, x_1) \in \mathcal{H}.$$  

(3.17)

Clearly, under the assumption above, $\ell_0$ and $\phi$ are continuous and bounded while $\ell_1$ is measurable and bounded from below. The value function of the problem is

$$V(t, x) := \inf_{u \in U} J(t, x; u).$$

(3.18)

We define the Hamiltonian in a modified way, indeed, for $p \in \mathcal{H}$, $u \in U$, we define the current value Hamiltonian $H_{CV}$ as

$$H_{CV}(p; u) := \langle p, u \rangle_{\mathbb{R}^m} + \ell_1(u)$$

and the (minimum value) Hamiltonian by

$$H_{\text{min}}(p) = \inf_{u \in U} H_{CV}(p; u).$$

(3.19)

The associated HJB equation with unknown $v$ is then formally written as

$$\begin{cases} 
-\frac{\partial v(t, x)}{\partial t} = \frac{1}{2} Tr GG^* \nabla^2 v(t, x) + \langle Ax, \nabla v(t, x) \rangle_{\mathcal{H}} + \ell_0(t, x) + H_{\text{min}}(\nabla B v(t, x)), \\
v(T, x) = \phi(x). 
\end{cases}$$

(3.20)

To get existence of mild solutions of (3.20) we will need the following assumption.

**Hypothesis 3.3**

(i) $\phi \in C_b(\mathcal{H})$ and it is given by (3.17) for a suitable $\phi \in C_b(\mathbb{R}^n)$;

(ii) $\ell_0 \in C_b([0, T] \times \mathcal{H})$ and it is given by (3.15) for a suitable $\bar{\ell}_0 \in C_b([0, T] \times \mathbb{R}^n)$;

(iii) $\ell_1 : U \to \mathbb{R}$ is measurable and bounded from below;

(iv) the Hamiltonian $H_{\text{min}} : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous so there exists $L > 0$ such that

$$|H_{\text{min}}(p_1) - H_{\text{min}}(p_2)| \leq L|p_1 - p_2| \quad \forall p_1, p_2 \in \mathbb{R}^m;$$

$$|H_{\text{min}}(p)| \leq L(1 + |p|) \quad \forall p \in \mathbb{R}^m.$$  

(3.21)

To get more regular solutions (well defined second derivative $\nabla^2 v$, which will be used to prove existence of optimal feedback controls) we will need the following further assumption.

**Hypothesis 3.4**
(i) $\ell_0$ is continuously differentiable in the variable $x$ with bounded derivative.

(ii) the Hamiltonian $H_{\min} : \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and, for a given $L > 0$, we have, beyond (3.22),

$$|\nabla H_{\min}(p_1) - \nabla H_{\min}(p_2)| \leq L|p_1 - p_2|, \quad \forall p_1, p_2 \in \mathbb{R}^m; \quad (3.22)$$

**Remark 3.5** The assumption (3.22) of Lipschitz continuity of $H_{\min}$ is satisfied e.g. if the set $U$ is compact. Indeed, for every $p_1, p_2 \in \mathbb{R}^m$

$$|H_{\min}(p_1) - H_{\min}(p_2)| \leq |\langle p_1, u \rangle - \langle p_2, u \rangle|, \quad u \in U$$

and in the case of $U$ compact the Lipschitz property immediately follows. The Lipschitz continuity of $H_{\min}$ is satisfied also in the case when $U$ is unbounded, if the current cost has linear growth at infinity.

Moreover the assumption (3.22) of Lipschitz continuity of $\nabla H_{\min}$ is verified e.g. if the function $\ell_1$ is convex, differentiable with invertible derivative and with $(\ell_1')^{-1}$ Lipschitz continuous since in this case $(\ell_1')^{-1}(p) = \nabla H_{\min}(p)$. 

**Remark 3.6** We list here, in order of increasing difficulty, some possible generalization of the above assumptions and of the consequent results.

(i) All our results on the HJB equation and on the control problem could be extended without difficulties to the case when the boundedness assumption on $\phi$ and $\ell_0$ (and consequently on $\phi$ and $\ell_0$) can be replaced by a polynomial growth assumption: namely that, for some $N \in \mathbb{N}$, the functions

$$x \mapsto \frac{\phi(x)}{1 + |x|^N}, \quad (t, x) \mapsto \frac{\ell_0(t, x)}{1 + |x|^N}, \quad (3.23)$$

are bounded. The generalization of Theorem 3.1 to this case can be achieved by straightforward changes in the proof, on the line of what is done, in a different context, in [7] or in [33].

(ii) Since our results on the HJB equation are based on smoothing properties (proved in Section 4) which holds also for measurable functions, we could consider current cost and final cost only measurable instead of continuous. The proofs would be very similar but using different underlying spaces.

(iii) Using the approach of [23] it seems possible to relax the Lipschitz assumptions on the Hamiltonian function asking only local Lipschitz continuity of the Hamiltonian function, but paying the price of requiring differentiability of the data.

In this paper we do not perform all such generalizations since we want to concentrate on the main point: the possibility of solving the HJB equation and the control problem without requiring the so-called structure condition.

### 4 Partial smoothing for the Ornstein-Uhlenbeck semigroup

This section is devoted to what we call the “partial” smoothing property of Ornstein-Uhlenbeck transition semigroups. First, in Subsection 4.1 we give two results (Theorem 4.1 for the first $C$-derivative and Proposition 4.5 for the second derivative) for a general Ornstein-Uhlenbeck transition semigroup in real separable Hilbert space $H$. Then in Subsection 4.2 we prove two specific results for our problem (Propositions 4.9 and 4.11).

#### 4.1 Partial smoothing in a general setting

Let $H, \Xi$ be two real and separable Hilbert spaces and let us consider the Ornstein-Uhlenbeck process $X^x(\cdot)$ in $H$ which solves the following SDE in $H$:

$$\begin{align*}
\left\{ \begin{array}{l}
dX(t) = AX(t)dt + GdW_t, \quad t \geq 0 \\
X(0) = x,
\end{array} \right.
\end{align*}
\tag{4.1}$$

In this paper we do not perform all such generalizations since we want to concentrate on the main point: the possibility of solving the HJB equation and the control problem without requiring the so-called structure condition.
where $A$ is the generator of a strongly continuous semigroup in $H$, $(W_t)_{t \geq 0}$ is a cylindrical Wiener process in $\Xi$ and $G : \Xi \to K$. In mild form, the Ornstein-Uhlenbeck process $X^x$ is given by

$$X^x(t) = e^{tA}x + \int_0^t e^{(t-s)A}GdW_s, \quad t \geq 0. \quad (4.2)$$

$X$ is a Gaussian process, namely for every $t > 0$, the law of $X(t)$ is $\mathcal{N}(e^{tA}x, Q_t)$, the Gaussian measure with mean $e^{tA}x$ and covariance operator $Q_t$, where

$$Q_t = \int_0^t e^{sA}GG^*e^{sA^*}ds.$$  

The associated Ornstein-Uhlenbeck transition semigroup $R_t$, is defined by setting, for every $f \in B_b(H)$ and $x \in H$,

$$R_t[f](x) = \mathbb{E}f(X^x(t)) = \int_K f(z + e^{tA}x)\mathcal{N}(0, Q_t)(dz). \quad (4.3)$$

where by $X^x$ we denote the Ornstein-Uhlenbeck process above with initial datum given by $x \in H$.

It is well known (see e.g. [10, Section 9.4]), that $R_t$ has the strong Feller property (i.e. it transforms bounded measurable functions into continuous ones) if and only if

$$\text{Im } e^{tA} \subseteq \text{Im } Q_t^{1/2}, \quad (4.4)$$

and that such property is equivalent to the so-called null-controllability of the linear control system identified by the couple of operators $(A, G)$ (here $z(\cdot)$ is the state and $a(\cdot)$ is the control):

$$z'(t) = Az(t) + Ga(t), \quad z(0) = x.$$ 

(see again [10, Appendix B]). Under (4.4) $R_t$ also transforms any bounded measurable function $f$ into a Fréchet differentiable one, the so-called “smoothing” property, and

$$\|\nabla R_t[f]\|_\infty \leq \Gamma(t)\|\xi(H)\|\|f\|_\infty$$

where $\Gamma(t) := Q_t^{-1/2}e^{tA}$.

Here we take another Hilbert space $K$, a bounded operator $C : K \to H$ and extend the smoothing property in two directions: searching for $C$-derivatives and applying $R_t$ to a specific class of bounded measurable functions (see [31] for results in this direction in finite dimension).

Let $P : H \to H$ be a bounded linear operator; given any $\phi : \text{Im}(P) \to \mathbb{R}$ measurable and bounded we define a function $\tilde{\phi} \in B_b(H)$, by setting

$$\tilde{\phi}(x) = \phi(Px) \quad \forall x \in H. \quad (4.5)$$

We prove that, under further assumptions on the operators $A$, $G$, $C$ and $P$, the semigroup $R_t$ maps functions $\phi$, defined as in (4.5), into $C$-Fréchet differentiable functions.

**Theorem 4.1** Let $A$ be the generator of a strongly continuous semigroup in $H$. Let $G : \Xi \to H$. Let $K$ be another real and separable Hilbert space and let $C : K \to H$ be a linear bounded operator. Let $\tilde{\phi} : \text{Im}(P) \to \mathbb{R}$ be measurable and bounded and define $\phi : H \to \mathbb{R}$ as in (4.5). Fix $t > 0$ and assume that $P e^{tAC} : K \to H$ is well defined.

Then, $R_t[\phi]$ is $C$-Fréchet differentiable if

$$\text{Im } (Pe^{tAC}) \subseteq \text{Im } Q_t^{1/2}. \quad (4.6)$$

In this case we have, for every $k \in K$,

$$\langle \nabla^C(R_t[\phi])(x), k \rangle_K = \int_H \tilde{\phi}(Pz + Pe^{tA}x) \left< Q_t^{-1/2}P e^{tAC}k, Q_t^{-1/2}z \right>_H N(0, Q_t)(dz). \quad (4.7)$$

Moreover for every $k \in K$ we have the estimate

$$\| \langle \nabla^C(R_t[\phi])(x), k \rangle_K \| \leq \|\tilde{\phi}\|_\infty \left\| Q_t^{-1/2}Pe^{tAC} \right\|_{\mathcal{L}(K; H)} \| k \|_K,$$
Remark 4.2

In [10], Remark 9.29, it is showed that the analogous of condition 4.6 is also a necessary condition for the Fréchet differentiability of $\langle \cdot \rangle$. Consequently, we have, arguing exactly as in [10], proof of Theorem 9.26,

$$
d(t, y, z) = \frac{dN(y, Q_t)}{dN(0, Q_t)}(z) = \exp \left\{ \langle Q_t^{-1/2}y, Q_t^{-1/2}z \rangle_H - \frac{1}{2} \| Q_t^{-1/2}y \|^2_H \right\},
$$

which gives (4.7). Consequently

$$
|\langle \nabla^C(\beta_t)[\phi](x), k \rangle_K| \leq \| \nabla^C(\beta_t)[\phi](x) \|_K \leq \| \phi \|_\infty \left( \int_H \langle Q_t^{-1/2}P_t^zCk, Q_t^{-1/2}z \rangle_H^2 N(0, Q_t)(dz) \right)^{1/2} = \| \phi \|_\infty \| Q_t^{-1/2}P_t^zCk \|_{\mathcal{L}(K;H)}.
$$

This gives the claim. □

Remark 4.3

We consider two special cases of the previous Theorem 4.1 that will be useful in next section.

(i) Let $K = H$ and $C = I$. In this case Theorem 4.1 gives Fréchet differentiability: for $t > 0 \beta_t[\phi]$ is Fréchet differentiable if

$$
\mathrm{Im} \left( P_t^z \right) \in \mathrm{Im} Q_t^{-1/2}
$$

and we have, for every $h \in H$,

$$
\langle \nabla(\beta_t[\phi])(x), h \rangle_H = \int_H \phi \left( P_t^z + P_t^z \right) \langle Q_t^{-1/2}P_t^z h, Q_t^{-1/2}z \rangle_H^2 N(0, Q_t)(dz).
$$

Moreover for every $h \in H$ we have the estimate

$$
|\langle \nabla(\beta_t[\phi])(x), h \rangle_H | \leq \| \phi \|_\infty \| Q_t^{-1/2}P_t^z \|_{\mathcal{L}(H;H)} |h|_H.
$$

(ii) Let $K_0$ and $K_1$ be two real and separable Hilbert spaces and let $K = K_0 \times K_1$ be the product space. Now, given any $\phi \in B_0(K_0)$, we define, in the same way as in [3.7], a function $\phi \in B_0(K)$, by setting

$$
\phi(k) = \phi(k_0) \quad \forall k = (k_0, k_1) \in K.
$$

Let $P: K \to K_0$ be the projection on the first component of $K$: for every $k = (k_0, k_1) \in K$, $P_k = k_0$. Theorem 4.1 says that $\beta_t[\phi]$ is C-Fréchet differentiable for every $t > 0$ if

$$
\mathrm{Im} \left( (e^{tAC}_0)^\circ \right) \in \mathrm{Im} Q_t^{-1/2}, \quad \forall t > 0
$$

and we have, for every $h \in H$,

$$
\langle \nabla(\beta_t[\phi])(x), h \rangle_H = \int_H \phi \left( P_t^z + P_t^z \right) \langle Q_t^{-1/2}P_t^z h, Q_t^{-1/2}z \rangle_H^2 N(0, Q_t)(dz).
$$

Moreover for every $h \in H$ we have the estimate

$$
|\langle \nabla(\beta_t[\phi])(x), h \rangle_H | \leq \| \phi \|_\infty \| Q_t^{-1/2}P_t^z \|_{\mathcal{L}(H;H)} |h|_H.
$$
and we have, for every $k \in K$,
\[
(\nabla^C (R_t [\phi]) (x), k)_K = \int_K \tilde{\phi} (x_0 + (e^{tA} x_0) \langle Q_t^{-1/2} \left( (e^{tA} C k) \right)_0, Q_t^{-1/2} z \rangle_K, N(0, Q_t) (dz)).
\]  
(4.13)

Moreover for every $k \in K$ we have the estimate
\[
| (\nabla^C (R_t [\phi]) (x), k)_K | \leq \| \phi \|_{\infty} \| Q_t^{-1/2} \left( (e^{tA} C) \right)_0, 0 \|_{\mathcal{L}(K; K)} | k |_K.
\]  
(4.14)

Remark 4.4 In Theorem 4.1 we prove the partial smoothing for functions $\phi$ defined as in (4.5) for functions $\tilde{\phi}$ bounded and measurable. The boundedness assumption on $\phi$ (and consequently on $\tilde{\phi}$) can be replaced by a polynomial growth assumption: namely that, for some $N \in \mathbb{N}$,
\[
x \mapsto \frac{\tilde{\phi}(x)}{1 + |x|^N}
\]
is bounded. The generalization of Theorem 4.1 to this case can be achieved by straightforward changes in the proof, on the line of what is done in [7] or in [33].

4.1.1 Second derivatives

We now prove that, if $\phi$ is more regular, also $\nabla^C R_t [\phi]$ and $\nabla R_t [\phi]$ have more regularity. This fact, in the context of our model (see Subsection 4.2.4), will be used in Section 5 to prove $C^2$ regularity of the solution of the HJB equation.

Proposition 4.5 Let $A$ be the generator of a strongly continuous semigroup in $H$. Let $G : \Xi \to H$. Let $K$ be another real and separable Hilbert space and let $C : K \to H$ be a linear bounded operator. Let $\tilde{\phi} : \text{Im}(P) \to \mathbb{R}$ be measurable and bounded and define $\phi : H \to \mathbb{R}$ as in (4.5). Fix $t > 0$ and assume that $P e^{tA} C : K \to H$ is well defined. Assume that (4.7) holds true. If $\tilde{\phi}$ is such that $\phi \in C_b^1 (H)$, then for every $t > 0$ the first order derivatives $\nabla^C R_t [\phi]$ and $\nabla R_t [\phi]$ exist and are bounded, with the second one given by
\[
(\nabla (R_t [\phi]) (x), h)_H = R_t \left[ (\nabla \phi, e^{tA} h)_H \right] (x), \quad \forall h \in \mathcal{H}.
\]  
(4.15)

Moreover the second order derivatives $\nabla \nabla^C R_t [\phi], \nabla^C \nabla R_t [\phi]$ exist, coincide, and we have
\[
\langle (\nabla \nabla^C (R_t [\phi]) (x)) k, h \rangle_H = \int_H \langle \nabla \partial (P z + P e^{tA} x), P e^{tA} h \rangle_H \langle (Q_t)^{-1/2} (P e^{tA} C k), (Q_t)^{-1/2} z \rangle_H N(0, Q_t) (dz).
\]  
(4.16)

Finally for every $k \in K, h \in H$ we have the estimate
\[
| \langle (\nabla \nabla^C (R_t [\phi]) (x)) h, k \rangle_H | \leq \| \nabla \partial \|_{\infty} \| Q_t^{-1/2} P e^{tA} C \|_{\mathcal{L}(K; H)} | k |_K | h |_H.
\]  
(4.17)

Proof. We first prove (4.15). Let $\tilde{\phi} : \text{Im} P \to \mathbb{R}$ be such that, defining $\phi$ as in (4.5), $\phi \in C_b^1 (H)$. For any $h \in H$ we have, applying the dominated convergence theorem,
\[
(\nabla R_t [\phi] (x), h)_H = \lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \int_H \phi (z + e^{tA} (x + \alpha h)) N(0, Q_t) (dz) - \int_H \phi (z + e^{tA} x) N(0, Q_t) (dz) \right]
\]
\[
= \int_H \lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \phi (z + e^{tA} (x + \alpha h)) - \phi (z + e^{tA} x) \right] N(0, Q_t) (dz)
\]
\[
= \int_H \langle \nabla \phi (z + e^{tA} x), e^{tA} h \rangle_H N(0, Q_t) (dz) = R_t \left[ (\nabla \phi, e^{tA} h)_H \right] (x).
\]
The boundedness of \( \langle \nabla (R_t [\phi]) (x), h \rangle \) easily follows. We compute the second order derivatives starting from \( \nabla \nabla^G R_t [\phi] \). Using (4.7) and the Dominated Convergence Theorem we get, for \( h \in H, k \in K \),
\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \langle \nabla^G (R_t [\phi]) (x + \alpha h), k \rangle - \langle \nabla^G (R_t [\phi]) (x), k \rangle \right] = \lim_{\alpha \to 0} \frac{1}{\alpha} \int_H \langle \tilde{\phi} (P z + Pe^{t A} x + \alpha h) - \tilde{\phi} (P z + Pe^{t A} x) \rangle \left\langle (Q_t)^{-1/2} (Pe^{t A} C_k), (Q_t)^{-1/2} z \right\rangle_H N(0, Q_t) (dz)
\]
\[
= \int_H \langle \nabla \tilde{\phi} (P z + Pe^{t A} x), Pe^{t A} h \rangle_H \left\langle (Q_t)^{-1/2} (Pe^{t A} C_k), (Q_t)^{-1/2} z \right\rangle_H N(0, Q_t) (dz).
\]

Similarly, using (4.15) and (4.5), we get, for \( h \in H, k \in K \),
\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \langle \nabla (R_t [\phi]) (x + \alpha C k), h \rangle_H - \langle \nabla (R_t [\phi]) (x), h \rangle_H \right] = \lim_{\alpha \to 0} \frac{1}{\alpha} \int_H \langle \nabla \tilde{\phi} (P z + Pe^{t A} x + \alpha C k), Pe^{t A} h \rangle_H - \langle \nabla \tilde{\phi} (P z + Pe^{t A} x), Pe^{t A} h \rangle_H N(0, Q_t) (dz)
\]
\[
= \int_H \langle \nabla \tilde{\phi} (P z + Pe^{t A} x), Pe^{t A} h \rangle_H \left\langle (Q_t)^{-1/2} (Pe^{t A} C_k), (Q_t)^{-1/2} z \right\rangle_H N(0, Q_t) (dz).
\]

The above immediately implies (4.10) and the estimate (4.17).

\[\blacksquare\]

### 4.2 Partial smoothing in our model

In the setting of Section 3 we assume that Hypothesis 3.1 holds true. We take \( H = \mathbb{R}^n \times L^2(-d, 0; \mathbb{R}^n) \), \( \Xi = \mathbb{R}^k \), \( (\Omega, \mathcal{F}, \mathbb{P}) \) a complete probability space, \( W \) a standard Wiener process in \( \Xi \), \( A \) and \( G \) as in (3.3) and (3.7). Then, for \( x \in H \), we take the Ornstein-Uhlenbeck process \( X^z(\cdot) \) given by (4.2). The associated Ornstein-Uhlenbeck transition semigroup \( R_t \) is defined as in (4.3) for all \( f \in B_0(H) \).

The operator \( P \) of the previous subsection here is the projection \( \Pi_0 \) on the first component of the space \( H \), similarly to Remark 1.3 (ii). Hence, given any \( \phi \in B_0(\mathbb{R}^n) \), we define, as in (3.17) a function \( \phi \in B_0(H) \), by setting
\[
\phi(x) = \tilde{\phi}(\Pi_0 x) = \tilde{\phi}(x_0) \quad \forall x = (x_0, x_1) \in H.
\]

For such functions, the Ornstein-Uhlenbeck semigroup \( R_t \) is written as
\[
R_t [\phi](x) = \mathbb{E} \phi(X^x(t)) = \mathbb{E} \tilde{\phi}(X^x(t)_{x_0}) = \int_H \tilde{\phi}(z + e^{t A} x_0) \mathcal{N}(0, Q_t) (dz).
\]

Concerning the covariance operator \( Q_t \) we have the following.

**Lemma 4.6** Let \( A \) be defined in (3.3), let \( G \) be defined by (3.7) and let \( t \geq 0 \). Let \( Q_t^0 \) be the selfadjoint operator in \( \mathbb{R}^n \) defined as
\[
Q_t^0 := \int_0^t e^{s A_0} \sigma^s \sigma^* e^{s A_0^*} ds.
\]

Then for every \( (x_0, x_1) \in H \) we have
\[
Q_t^0(x_0, x_1) = (Q_t^0 x_0, 0)
\]

and so
\[
\text{Im} \ Q_t = \text{Im} \ Q_t^0 \times \{0\} \subseteq \mathbb{R}^n \times \{0\}
\]

Hence, for every \( \tilde{\phi} \in B_0(\mathbb{R}^n) \) and for the corresponding \( \phi : H \to \mathbb{R} \) defined in (4.18) we have
\[
R_t [\phi](x) = \int_{\mathbb{R}^n} \tilde{\phi}(z_0 + (e^{t A} x_0) \mathcal{N}(0, Q_t^0) (dz_0).
\]
Proof. Let \((x_0, x_1) \in \mathcal{H}\) and \(t \geq 0\). By direct computation we have
\[
Q_t \left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) = \int_0^t e^{sA}GG^*e^{sA^*} \left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) ds
\]
\[
= \int_0^t e^{sA} \left( \begin{array}{c} \sigma \sigma^* \\ 0 \\ 0 \end{array} \right) e^{sA^*} \left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) ds
\]
\[
= \int_0^t e^{sA} \left( \begin{array}{c} \sigma \sigma^* e^{sA}x_0 \\ 0 \end{array} \right) ds = \int_0^t \left( e^{sa_0} \sigma \sigma^* e^{sa_0}x_0 \right) ds
\]
from which the first claim \((4.21)\) follows. The second claim \((4.22)\) is immediate. \(\square\)

Remark 4.7. The statement of the above lemma holds true (substituting \(\mathbb{R}^n\) with the Hilbert space \(K_0\) introduced below) also in the following more general setting. Let \(\mathcal{H} = K_0 \times K_1\) where \(K_0\) and \(K_1\) are both real separable Hilbert spaces. Let \(\Xi\) be another separable Hilbert space (the noise space) and consider the Ornstein-Uhlenbeck process
\[
X(t) = e^{tA}x + \int_0^t e^{(t-s)A}GdW_s, \quad t \geq 0,
\]
where \(A\) generates a strongly continuous semigroup on \(\mathcal{H}\), and \(G \in \mathcal{L}(\Xi, \mathcal{H})\). Assume that
\begin{itemize}
  \item \(G = (\sigma, 0)\) for \(\sigma \in \mathcal{L}(\Xi, K_0)\) so \(GG^* = \left( \begin{array}{c} \sigma \sigma^* \\ 0 \\ 0 \end{array} \right) \) with \(\sigma \sigma^* \in \mathcal{L}(K_0)\);
  \item for every \(k_0 \in K_0, t \geq 0\),
    \[
    e^{tA}(k_0, 0) = (e^{tA_0}k_0, 0),
    \]
    where \(A_0\) generates a strongly continuous semigroup in \(K_0\);
\end{itemize}
then the claim still hold. Indeed in such case we have, for \(t \geq 0, k_0 \in K_0, k_1 \in K_1\),
\[
\left( e^{tA^*}(k_0, k_1) \right)_0 = e^{tA_0^*}k_0,
\]
where \(A_0^*\) is the adjoint of \(A_0\). So, for \(t \geq 0\),
\[
Q_t^0 k_0 = \int_0^t e^{sa_0} \sigma \sigma^* e^{sa_0}k_0 ds
\]
and
\[
Q_t(k_0, k_1) = (Q_t^0 k_0, 0).
\]
This works, in particular, in the case described in Remark 3.2. \(\square\)

We now analyze when Theorem 4.1 can be applied in the cases \(C = I\) or \(C = B\) concentrating on the cases when the singularity at \(t = 0^+\) of \(\|Q_t^{-1/2}P_0e^{tAC}\|\) is integrable, as this is needed to solve the HJB equation \((6.2)\).

4.2.1 \(C = I\)

By Theorem 4.1 we have our partial smoothing (namely \((4.13)\) and \((4.14)\)) for \(C = I\) if
\[
\text{Im} P_0e^{tA} \subseteq \text{Im}\ Q_t^{1/2}.
\]
By Lemma 4.6 and 3.5 this implies
\[
\text{Im} e^{tA} \subseteq \text{Im}(Q_t^0)^{1/2}.
\]
Since, clearly, \(e^{tA}\) is invertible and \(\text{Im}(Q_t^0)^{1/2} = \text{Im} Q_t^0\), then \((4.26)\) is true if and only if the operator \(Q_t^0\) is invertible. On this we have the following result, taken from [42] Theorem 1.2, p.17 and 35.

\[\text{Indeed once we know that } e^{tA}(k_0, 0)_1 = 0 \text{ then } (4.21) \text{ is equivalent to } (4.25).\]
Lemma 4.8 The operator $Q_t^0$ defined in \[(4.20)\] is invertible for all $t > 0$ if and only if
\[
\text{Im}(\sigma, a_0 \sigma, \ldots, a_0^{n-1} \sigma) = \mathbb{R}^n.
\]
This happens if and only if the linear control system identified by the couple $(a_0, \sigma)$ is null controllable. In this case, for $t \to 0^+$,
\[
\|(Q_t^0)^{-1/2}\| \sim t^{-r-1/2}
\]
where $r$ is the Kalman exponent, i.e. the minimum $r$ such that
\[
\text{Im}(\sigma, a_0 \sigma, \ldots, a_0^{n} \sigma) = \mathbb{R}^n.
\]
Hence $r = 0$ if and only if $\sigma$ is onto.

We now pass to the smoothing property.

Proposition 4.9 Let $A$ and $G$ be defined respectively by \[(4.18)\] and \[(4.19)\]. Let $\bar{\phi} : \mathbb{R}^n \to \mathbb{R}$ be measurable and bounded and define, as in \[(4.15)\], $\phi : \mathcal{H} \to \mathbb{R}$, by setting $\phi(x) = \bar{\phi}(x_0)$ for every $x = (x_0, x_1) \in \mathcal{H}$. Then, if $Q_t^0$ is invertible, we have the following:
(i) the function $(t, x) \mapsto R_t[\phi](x)$ belongs to $C_b((0, +\infty) \times \mathcal{H})$. Moreover it is Lipschitz continuous in $x$ uniformly in $t \in [0, t_1]$ for all $0 < t_0 < t_1 < +\infty$.

(ii) Fix any $t > 0$. $R_t[\phi]$ is Fréchet differentiable and we have, for every $h \in \mathcal{H}$,
\[
\left\langle \nabla (R_t[\phi]) (x), h \right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^n} \bar{\phi} (z_0 + (e^{tA}x)_0) \left\langle (Q_t^0)^{-1/2} (e^{tA}h)_0, (Q_t^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} N(0, Q_t^0)(dz_0).
\]
(4.27)

where $(e^{tA}x)_0$, $(e^{tA}h)_0$ are given by \[(4.25)\]. Moreover for every $h \in \mathcal{H}$ we have the estimate
\[
|\left\langle \nabla (R_t[\phi]) (x), h \right\rangle_{\mathcal{H}}| \leq \|\bar{\phi}\|_{\infty} \left\|(Q_t^0)^{-1/2} (e^{tA})_0 \right\|_{L(\mathcal{H};\mathbb{R}^n)} |h|_{\mathcal{H}}.
\]
Hence for all $T > 0$ there exists $C_T$ such that
\[
|\left\langle \nabla R_t[\phi] (x), h \right\rangle_{\mathcal{H}}| \leq C_T t^{-r-1/2} \|\bar{\phi}\|_{\infty} |h|_{\mathcal{H}}, \quad t \in [0, T],
\]
(4.28)

where $r$ is the Kalman exponent which is 0 if and only if $\sigma$ is onto.

(iii) Fix any $t > 0$. $R_t[\phi]$ is $B$-Fréchet differentiable and we have, for every $k \in \mathbb{R}^m$,
\[
\left\langle \nabla^B (R_t[\phi]) (x), k \right\rangle_{\mathbb{R}^m} = \int_{\mathbb{R}^n} \bar{\phi} (z_0 + (e^{tA}x)_0) \left\langle (Q_t^0)^{-1/2} (e^{tA}Bk)_0, (Q_t^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} N(0, Q_t^0)(dz_0).
\]
(4.29)

Moreover, for every $k \in \mathbb{R}^m$,
\[
|\left\langle \nabla^B (R_t[\phi]) (x), k \right\rangle_{\mathbb{R}^m}| \leq \|\bar{\phi}\|_{\infty} \left\|(Q_t^0)^{-1/2} (e^{tA}B)_0 \right\|_{L(\mathbb{R}^m;\mathbb{R}^n)} |k|_{\mathbb{R}^m}.
\]
(4.30)

Hence for all $T > 0$ there exists $C_T$ such that
\[
|\left\langle \nabla^B R_t[\phi] (x), k \right\rangle_{\mathbb{R}^m}| \leq C_T t^{-r-1/2} \|\bar{\phi}\|_{\infty} |k|_{\mathbb{R}^m}, \quad t \in [0, T],
\]
(4.31)

where $r$ is the Kalman exponent which is 0 if and only if $\sigma$ is onto.

Proof. Point (ii) immediately follows from the invertibility of $Q_t^0$, the discussion just before Lemma \[(4.16)\] and Theorem \[(4.17)\]. Point (i) follows from point (ii) and from the continuity of trajectories of the Ornstein-Uhlenbeck process \[(4.11)\] with $A$ and $G$ given by \[(4.33)\] and \[(4.37)\]. Point (iii) follows observing that the operator $\Pi_0 e^{tA}B : \mathbb{R}^m \to \mathbb{R}^n$, given in \[(4.10)\] is well defined and hence, thanks to the invertibility of $Q_t^0$, Theorem \[(4.11)\] can be applied.
4.2.2 $C = B$

By Theorem 4.1 we have the partial smoothing (4.13 and 4.14) for $C = B$ if

$$\text{Im} \Pi_0 e^{tA} B \subset \text{Im} Q_t^{1/2} = \text{Im} Q_t^0$$

(4.32)

Since, as proved e.g. in [22] (Lemma 1.1, p. 18 and formula (2.11) p. 210),

$$\text{Im}(Q_t^{1/2}) = \text{Im}(\sigma, a_0 \sigma, \ldots, a_0^{n-1} \sigma),$$

then, using (3.10), 4.32 is verified if and only if

$$\text{Im} \left( e^{ta_0 b_0} + \int_{-d}^0 1_{[-t,0]} e^{(t+r)a_0 b_1} (dr) \right) \subseteq \text{Im}(\sigma, a_0 \sigma, \ldots, a_0^{n-1} \sigma).$$

(4.33)

We now provide conditions, possibly weaker than the invertibility of $Q_t$, under which (4.33) is verified and the singularity at $t = 0^+$ of $\|Q_t^{-1/2} \Pi_0 e^{tA} B\|$ is integrable. We first recall the following result (see [22], Proposition 2.1, p. 211).

**Proposition 4.10** If $F_1$ and $F_2$ are linear bounded operators acting between separable Hilbert spaces $X$, $Y$, $Z$ such that $\|F_1^* f\| = \|F_2^* f\|$ for any $f \in Z^*$, then $\text{Im} F_1 = \text{Im} F_2$ and $\|F_1^{-1} z\| = \|F_2^{-1} z\|$ for all $z \in \text{Im} F_1$.

**Proposition 4.11** Assume that Hypothesis 3.1 holds. Assume moreover that, either

$$\text{Im}(e^{ta_0 b_0}) \subseteq \text{Im} \sigma, \forall t > 0; \quad \text{Im} b_1(s) \in \text{Im} \sigma, \text{ a.e. } \forall s \in [-d,0]$$

(4.34)

or

$$\text{Im} \left( e^{ta_0 b_0} + \int_{-d}^0 1_{[-t,0]} e^{(t+r)a_0 b_1} (dr) \right) \subseteq \text{Im} \sigma, \forall t > 0.$$  

(4.35)

Then, for any bounded measurable $\phi$ as in (4.18), $R_t [\phi]$ is $B$-Fréchet differentiable for every $t > 0$, and, for every $h \in \mathbb{R}^m$, $\langle \nabla^B(R_t [\phi](x), k) \rangle_{\mathbb{R}^m}$ is given by (4.29) and satisfies the estimate (4.30). Moreover for all $T > 0$ there exists $C_T$ such that

$$| \langle \nabla^B(R_t [\phi])(x), k \rangle_{\mathbb{R}^m} | \leq C_T t^{-1/2} \| \phi \|_{\infty} |k|_{\mathbb{R}^m}.$$  

(4.36)

**Proof.** Consider the following linear deterministic controlled system in $\mathcal{H}$:

$$\begin{cases}
  dX(t) = AX(t)dt + Gu_1(t)dt \\
  X(0) = Bh,
\end{cases}$$

(4.37)

where the state space is $\mathcal{H}$, the control space is $U_1 = \mathbb{R}^k$, the control strategy is $u_1 \in L^2_{\text{loc}}([0, +\infty); U_1)$, the initial point is $Bh$ with $h \in \mathbb{R}^m$. Define the linear operator

$$\mathcal{L}^0_t : L^2([0,t]; U_1) \to \mathbb{R}^n, \quad u_1(\cdot) \mapsto \int_0^t e^{a_0(t-s)} \sigma u_1(s) ds.$$  

Then the first component of the state trajectory is

$$X^0(t) = \Pi_0 e^{tA} Bk + \mathcal{L}^0_t u_1$$

(4.38)

Hence $X^0$ can be driven to 0 in time $t$ if and only if

$$\Pi_0 e^{tA} Bk \in \text{Im} \mathcal{L}^0_t$$

In such case, by the definition of pseudoinverse (see Subsection 2.1), we have that the control which brings $X^0$ to 0 in time $t$ with minimal $L^2$ norm is $(\mathcal{L}^0_t)^{-1} \Pi_0 e^{tA} Bk$ and the corresponding minimal square norm is

$$\mathcal{E}(t, Bk) := \min \left\{ \int_0^t |u_1(s)|^2 ds : X(0) = Bk, \ X^0(t) = 0 \right\} = \| (\mathcal{L}^0_t)^{-1} \Pi_0 e^{tA} Bk \|^2_{L^2(0,t; U_1)}.$$  

(4.39)
Since for all \( z \in \mathbb{R}^n \) we have
\[
\| (Q_t^0)^{1/2} z \|_{\mathbb{R}^n}^2 = | \langle Q_t^0 z, z \rangle_{\mathbb{R}^n} | = \| (L_t^0)^* z \|_{L^2(0,t;U_1)}^2
\]
then, by Proposition \ref{prop:invertibility}, we get
\[
\text{Im} \left( (Q_t^0)^{1/2} \right) = \text{Im} L_t^0.
\]
and
\[
\| (L_t^0)^{-1} \Pi_0 e^{tA} Bk \|_{L^2(0,t;U_1)} = \| (Q_t^0)^{-1/2} \Pi_0 e^{tA} Bk \|_{\mathbb{R}^n}.
\]
Hence, by \ref{lem:invertibility}, to estimate \( \| (Q_t^0)^{-1/2} \Pi_0 e^{tA} Bk \|_{\mathbb{R}^n} \) it is enough to estimate the minimal energy to steer \( X^0 \) to 0 in time \( t \). When \ref{lem:invertibility} holds we see, by simple computations, that the control
\[
\bar{u}_{1}(s) = -\frac{1}{t} \sigma^{-1} e^{s a_0 b_0 k} - \sigma^{-1} b_1(\cdot) k_{1_{[-d,0]}(-t)}, \quad s \in [0,t],
\]
where \( \sigma^{-1} \) is the pseudoinverse of \( \sigma \), brings \( X^0 \) to 0 in time \( t \). Hence, for a suitable \( C > 0 \) we get
\[
E(t, Bk) \leq \int_0^t \bar{u}_{1}^2(s) ds \leq C \left( \frac{1}{t} + \| b_1 \|_{L^2([-d,0];L(\mathbb{R}^m;\mathbb{R}^n))} \right) \| k \|_{\mathbb{R}^n}^2.
\]
So, for a, possibly different constant \( C \), we get
\[
\| (Q_t^0)^{-1/2} (e^{tA} Bk) \|_{L^2(\mathbb{R}^m;\mathbb{R}^n)} \leq C t^{-\frac{1}{2}} \| k \|_{\mathbb{R}^n} \text{ and the estimate is proved. If we assume } \ref{lem:invertibility} \text{ we can take as a control, on the line of } \ref{thm:invertibility}, \text{ Theorem 2.3-(iii), p.210},
\]
\[
\tilde{u}_{1}(s) = -\sigma^{-1} e^{(t-s) a_0} (Q_t^0)^{-1} \left( e^{t a_0 b_0 k} + \int_0^1 1_{[-t,0]} e^{(t+r) a_0 b_1}(dr) k \right), \quad s \in [0,t],
\]
and use that the singularity of the second term as \( t \to 0^+ \) is still of order \( \frac{1}{t} \) since \ref{lem:invertibility} holds (see e.g. \ref{thm:invertibility}, Theorem 1). Once this estimate is proved, the proof of the B-Fréchet differentiability is the same as the one of Proposition \ref{prop:invertibility} (iii).

\begin{remark} \label{rem:generalization}

The above results can be generalized to the case, introduced in Remark \ref{rem:invertibility} above, when the first component of the space \( \mathcal{H} \) is infinite dimensional.

\begin{itemize}
  \item For the case \( C = I \) the required partial smoothing holds if we ask, in place of the invertibility of \( Q_t^0 \), that, for every \( t > 0 \),
  \[
  \text{Im} \left( e^{tA} \right)_0 \subseteq \text{Im} (Q_t^0)^{1/2}
  \]
  which would imply that the linear operator \( (Q_t^0)^{-1/2} (e^{tA})_0 \) is continuous from \( K_1 \) into itself.

  \item For the case \( C = B \), the required partial smoothing holds if we ask that, for every \( t > 0 \),
  \[
  \text{Im} \left( e^{tA} B \right)_0 \subseteq \text{Im} (Q_t^0)^{1/2}
  \]
  which would imply that the operator \( (Q_t^0)^{-1/2} (e^{tA} B)_0 \) is continuous from \( U \) to \( K_1 \).
\end{itemize}

Clearly, in this generalized setting the estimates \ref{lem:invertibility} and \ref{lem:invertibility} do not hold any more and they depend on the specific operators \( A, B, \sigma \).
\end{remark}

5 Smoothing properties of the convolution

By Proposition \ref{prop:invertibility} we know that if \ref{lem:invertibility} or \ref{lem:invertibility} hold and \( \phi \) is as in \ref{lem:invertibility} with \( \tilde{\phi} \) measurable and bounded then \( \nabla^B(R_t[\phi]\langle x \rangle) \) exists and its norm blows up like \( t^{-1/2} \) at \( 0^+ \). Moreover if \( \phi \in C_0(\mathbb{R}^n) \), then \( R_t[\phi] \in C_0([0,T] \times \mathcal{H}) \), see e.g. \ref{thm:invertibility} Proposition 6.5.1 (or the discussion at the end of \ref{thm:invertibility}).

We now prove that, given \( T > 0 \), for any element \( f \) of a suitable family of functions in \( C_b([0,T] \times \mathcal{H}) \), a similar smoothing property for the convolution integral \( \int_0^T R_{t-s} [f(s,\cdot)](x) ds \) holds. This will be a crucial step to prove the existence and uniqueness of the solution of our HJB equation in next section.

For given \( \alpha \in (0,1) \) we define now a space designed for our purposes.
Definition 5.1 Let $T > 0$, $\alpha \in (0, 1)$. A function $g \in C_b([0,T] \times \mathcal{H})$ belongs to $\Sigma^1_{T,\alpha}$ if there exists a function $f \in C^0_{\alpha}([0,T] \times \mathbb{R}^n)$ such that

$$g(t,x) = f\left(t, (e^{tA}x)_0\right), \quad \forall (t,x) \in [0,T] \times \mathcal{H}.$$

If $g \in \Sigma^1_{T,\alpha}$, for any $t \in (0,T]$ the function $g(t, \cdot)$ is both Fréchet differentiable and $B$-Fréchet differentiable. Moreover, for $(t,x) \in [0,T] \times \mathcal{H}$, $h \in \mathcal{H}$, $k \in \mathbb{R}^m$,

$$\langle \nabla g(t,x), h \rangle_{\mathcal{H}} = \langle \nabla f\left(t, (e^{tA}x)_0\right), (e^{tA}h)_0 \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \langle \nabla^B g(t,x), k \rangle_{\mathbb{R}^m} = \langle \nabla f\left(t, (e^{tA}x)_0\right), (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n}.$$

This in particular imply that, for all $k \in \mathbb{R}^m$

$$\langle \nabla^B g(t,x), k \rangle_{\mathbb{R}^m} = \langle \nabla g(t,x), Bk \rangle_{\mathbb{R}^n \times L^2([-d,0];\mathbb{R}^n)}, \quad (5.1)$$

which also means $B^* \nabla g = \nabla^B g$. For later notational use we call $\bar{f} \in C_b((0,T] \times \mathbb{R}^n; \mathbb{R}^m)$ the function defined by

$$\langle \bar{f}(t,y), k \rangle_{\mathbb{R}^m} = t^\alpha \langle \nabla f\left(t, (e^{tA}Bk)_0 \right), k \rangle_{\mathbb{R}^n}, \quad (t,y) \in (0,T] \times \mathbb{R}^n, \quad k \in \mathbb{R}^m,$$

which is such that

$$t^\alpha \nabla^B g(t,x) = \bar{f}\left(t, (e^{tA}x)_0\right).$$

We also notice that if $g \in \Sigma^1_{T,\alpha}$, then in order to have $g$ $B$-Fréchet differentiable it suffices to require $(e^{tA}B)_0$ bounded and continuous.

When (4.1) or (4.3) hold we know, by Proposition 4.11 that the function $g(t,x) = R_t[\phi](x)$ for $\phi$ given by (4.18) with $\phi$ bounded and continuous, belongs to $\Sigma^1_{T,1/2}$.

Lemma 5.2 The set $\Sigma^1_{T,\alpha}$ is a closed subspace of $C^0_{\alpha}([0,T] \times \mathcal{H})$.

Proof. It is clear that $\Sigma^1_{T,\alpha}$ is a vector subspace of $C^0_{\alpha}([0,T] \times \mathcal{H})$. We prove now that it is closed. Take any sequence $g_n \to g$ in $C^0_{\alpha}(\{0,T\} \times \mathcal{H})$. Then to every $g_n$ we associate the corresponding function $f_n$ and $\bar{f}_n$. The sequence $\{f_n\}$ is a Cauchy sequence in $C_b((0,T] \times \mathbb{R}^n)$. Indeed for any $\epsilon > 0$ take $(t_\epsilon, y_\epsilon)$ such that

$$\sup_{(t,y) \in [0,T] \times \mathbb{R}^n} |f_n(t,y) - f_m(t,y)| < \epsilon + |f_n(t_\epsilon, y_\epsilon) - f_m(t_\epsilon, y_\epsilon)|$$

Then choose $x_\epsilon \in \mathcal{H}$ such that $y_\epsilon = (e^{t_\epsilon A}x_\epsilon)_0$ (this can always be done choosing e.g. $x_\epsilon = (e^{-t_\epsilon A}y_\epsilon, 0)$). Hence we get

$$\sup_{(t,y) \in [0,T] \times \mathbb{R}^n} |f_n(t,y) - f_m(t,y)| < \epsilon + |g_n(t_\epsilon, x_\epsilon) - g_m(t_\epsilon, x_\epsilon)| \leq \epsilon + \sup_{(t,x) \in [0,T] \times \mathcal{H}} |g_n(t,x) - g_m(t,x)|.$$

Since $\{g_n\}$ is Cauchy, then $\{f_n\}$ is Cauchy, too. So there exists a function $f \in C_b([0,T] \times \mathbb{R}^n)$ such that $f_n \to f$ in $C_b((0,T] \times \mathbb{R}^n)$. This implies that $g(t,x) = f(t, (e^{tA}x)_0)$ on $[0,T] \times \mathcal{H}$. With the same argument we get that there exists a function $\bar{f} \in C_b((0,T] \times \mathbb{R}^n; \mathbb{R}^m)$ such that $\bar{f}_n \to \bar{f}$ in $C_b((0,T] \times \mathbb{R}^n; \mathbb{R}^m)$. This implies that $t^\alpha \nabla^B g(t,x) = \bar{f}(t, (e^{tA}x)_0)$ on $[0,T] \times \mathcal{H}$. \hfill \Box

Next, in analogy to what we have done defining $\Sigma^1_{T,\alpha}$, we introduce a subspace $\Sigma^2_{T,\alpha}$ of functions $g \in C^0_{\alpha,2}(\{0,T\} \times \mathcal{H})$ that depends in a special way on the variable $x \in \mathcal{H}$.

Definition 5.3 A function $g \in C_b([0,T] \times \mathcal{H})$ belongs to $\Sigma^2_{T,\alpha}$ if there exists a function $f \in C^0_{\alpha,2}([0,T] \times \mathbb{R}^n)$ such that for all $(t,x) \in [0,T] \times \mathcal{H}$,

$$g(t,x) = f\left(t, (e^{tA}x)_0\right).$$

If $g \in \Sigma^2_{T,\alpha}$ then for any $t \in (0,T]$ the function $g(t, \cdot)$ is Fréchet differentiable and

$$\langle \nabla g(t,x), h \rangle_{\mathcal{H}} = \langle \nabla f\left(t, (e^{tA}x)_0\right), (e^{tA}h)_0 \rangle_{\mathbb{R}^n}, \quad \forall (t,x) \in [0,T] \times \mathcal{H}, h \in \mathcal{H}.$$

Moreover also $\nabla^B g(t, \cdot)$ is $B$-Fréchet differentiable and

$$\langle \nabla^B (\nabla g(t,x)h), k \rangle_{\mathbb{R}^m} = \langle \nabla^2 f\left(t, (e^{tA}x)_0\right) (e^{tA}h)_0, (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n}, \quad \forall (t,x) \in [0,T] \times \mathcal{H}, h \in \mathcal{H}, k \in \mathbb{R}^m.$$
We also notice that, since the function $f$ is twice continuously Fréchet differentiable the second order derivatives $\nabla^2 \nabla g$ and $\nabla^2 \nabla g$ both exist and coincide:

$$\left\langle \nabla^2 \nabla g(t,x), h \right\rangle_{\mathcal{H}} = \left\langle \nabla^2 \nabla g(t,x), k \right\rangle_{\mathcal{H}}$$

Again for later notational use we call \( f_1 \in C_b([0,T] \times \mathbb{R}^n; \mathbb{R}^m) \) the function defined by

$$\left\langle f_1(t,y), h \right\rangle_{\mathcal{H}} = \left\langle \nabla f(t,y), (e^{tA} Bh) h \right\rangle_{\mathbb{R}^n}$$

which is such that

$$\nabla^B g(t,x) = f_1(t, (e^{tA} x)_0)$$

Similarly, we call \( f \in C_b((0,T] \times \mathbb{R}^n; \mathcal{L}(\mathcal{H}, \mathbb{R}^m)) \) the function defined by

$$\left\langle f(t,y), h \right\rangle_{\mathcal{H}} = t^\alpha \left\langle \nabla^2 f(t,y), (e^{tA} h)_0, (e^{tA} Bh)_0 \right\rangle_{\mathbb{R}^n}$$

which is such that

$$t^\alpha \nabla^B \nabla g(t,x) = t^\alpha \nabla^B g(t,x) = f(t, (e^{tA} x)_0).$$

We now pass to the announced smoothing result.

**Lemma 5.4** Let (4.34) or (4.35) hold true. Let \( T > 0 \) and let \( \psi : \mathbb{R}^m \to \mathbb{R} \) be a continuous function satisfying Hypothesis (5.3), estimates (5.2). Then

i) for every \( g \in \Sigma^1_{T,1/2} \), the function \( \hat{g} : [0,T] \times \mathcal{H} \to \mathbb{R} \) belongs to \( \Sigma^1_{T,1/2} \) where

$$\hat{g}(t,x) = \int_0^t R_{t-s} \psi(\nabla^B g(s, \cdot))(x) ds.$$  \hspace{5mm} (5.2)

Hence, in particular, \( \hat{g}(t, \cdot) \) is B-Fréchet differentiable for every \( t \in (0,T] \) and, for all \( x \in \mathcal{H} \),

$$\left| \nabla^B (\hat{g}(t, \cdot))(x) \right|_{(\mathbb{R}^m)_*} \leq C \left( t^{1/2} + \| g \|_{C_{1/2}^{0,1}} \right).$$  \hspace{5mm} (5.3)

If \( \sigma \) is onto, then \( \hat{g}(t, \cdot) \) is Fréchet differentiable for every \( t \in (0,T] \) and, for all \( h \in \mathcal{H}, x \in \mathcal{H} \),

$$\left| \nabla (\hat{g}(t, \cdot))(x) \right|_{\mathcal{H}^*} \leq C \left( t^{1/2} + \| g \|_{C_{1/2}^{0,1}} \right).$$  \hspace{5mm} (5.4)

ii) Assume moreover that \( \psi \in C^1(\mathbb{R}^m) \). For every \( g \in \Sigma^2_{T,1/2} \), the function \( \hat{g} \) defined in (5.2) belongs to \( \Sigma^2_{T,1/2} \). Hence, in particular, the second order derivatives \( \nabla^2 \nabla \hat{g}(t, \cdot) \) and \( \nabla^2 \nabla \hat{g}(t, \cdot) \) exist, coincide and for every \( t \in (0,T] \) and, for all \( x \in \mathcal{H} \),

$$\left| \nabla^2 \nabla \hat{g}(t, \cdot)(x) \right|_{(\mathbb{R}^m)_*^*} \leq C \| g \|_{C_{1/2}^{0,2}}$$  \hspace{5mm} (5.5)

If \( \sigma \) is onto, then \( \hat{g}(t, \cdot) \) is twice Fréchet differentiable and for every \( t \in (0,T] \), for all \( h \in \mathcal{H} \) and \( x \in \mathcal{H} \),

$$\left| \nabla^2 (\hat{g}(t, \cdot))(x) \right|_{\mathcal{H}^* \times \mathcal{H}^*} \leq C \| g \|_{C_{1/2}^{0,2}}$$  \hspace{5mm} (5.6)

**Proof.** We start by proving that (5.2) is B-Fréchet differentiable and we exhibit its B-Fréchet derivative. Recalling (4.3), we have

$$\int_0^t R_{t-s} \left[ \psi \left( \nabla^B \left( g(s, \cdot) \right) \right) \right](x) ds = \int_0^t \int_\mathcal{H} \psi \left( \nabla^B \left( g(s, \cdot) \right) \left( z + e^{(t-s)A} x \right) \right) \mathcal{N}(0, Q_{t-s})(dz)$$

By the definition of \( \Sigma^2_{T,1/2} \), we see that

$$s^{1/2} \nabla^B g(s, z + e^{(t-s)A} x) = \hat{f} \left( s, (e^{sA} x)_0 + (e^{tA} x)_0 \right) \quad \forall t \geq s > 0, \forall x, z \in \mathcal{H}.$$  \hspace{5mm} (5.7)
Hence the function $\hat{f}$ associated to $\hat{g}$ is
\[
\hat{f}(t, y) = \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \hat{f} \left( s, (e^{sA}z)_0 + y \right) \right) \mathcal{N}(0, Q_{t-s})(dz)
\]
with, by our assumptions on $\psi$,
\[
\|\hat{f}\|_\infty \leq C \int_0^t \left( 1 + s^{-1/2}\|\hat{f}\|_\infty \right) ds
\]
To compute the $B$-directional derivative we look at the limit
\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) (x + \alpha Bk) ds - \int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) \right] (x) ds \right].
\]
From what is given above we get
\[
\int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) \right] (x + \alpha Bk) ds =
\]
\[
= \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \bar{f} \left( s, (e^{sA}z)_0 + (e^{tA}(x + \alpha Bk))_0 \right) \right) \mathcal{N}(0, Q_{t-s})(dz) ds =
\]
\[
= \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \bar{f} \left( s, (e^{sA}z)_0 + (e^{tA}x)_0 \right) \right) d(t, t - s, \alpha Bk, z) \mathcal{N}(0, Q_{t-s})(dz) ds,
\]
where
\[
d(t_1, t_2, y, z) = \frac{d\mathcal{N} ( ((e^{t_1A}y)_0, 0 ), Q_{t_2} ) (z)}{d\mathcal{N} ( (0, Q_{t_2}) ) (z)}
\]
\[
= \exp \left\{ \frac{1}{2} \left< \mathcal{N}^{1/2} \left( (e^{t_1A}y)_0, 0 \right), Q_{t_2}^{-1/2} \right>_{\mathcal{H}} - \frac{1}{2} \left< Q_{t_2}^{-1/2} \left( (e^{t_1A}y)_0, 0 \right), Q_{t_2}^{-1/2} \right>_{\mathcal{H}} \right\}. \quad (5.8)
\]
Hence
\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) \right] (x + \alpha Bk) ds - \int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) \right] (x) ds \right] =
\]
\[
= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \bar{f} \left( s, (e^{sA}z)_0 + (e^{tA}x)_0 \right) \right) \frac{d(t, t - s, \alpha Bk, z) - 1}{\alpha} \mathcal{N}(0, Q_{t-s})(dz) ds
\]
\[
= \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \bar{f} \left( s, (e^{sA}z)_0 + (e^{tA}x)_0 \right) \right) \left< Q_{t-s}^{-1/2} \left( (e^{tA}Bk)_0, 0 \right), Q_{t-s}^{-1/2} \right>_{\mathcal{H}} \mathcal{N}(0, Q_{t-s})(dz) ds.
\]
Since the above limit is uniform for $k$ in the unit sphere, then we get the required $B$-Fréchet differentiability and
\[
\left< \nabla^B \left( \int_0^t R_{t-s} \left[ \psi \left( \nabla^B(g(s, \cdot)) \right) ds \right] (x), k \right>_{\mathbb{R}^m} =
\]
\[
= \int_0^t \int_{\mathcal{H}} \psi \left( s^{-1/2} \bar{f} \left( s, (e^{sA}z)_0 + (e^{tA}x)_0 \right) \right) \left< (Q_{t-s})^{-1/2} \left( e^{tA}Bk \right)_0, Q_{t-s}^{-1/2} \right>_{\mathbb{R}^m} \mathcal{N}(0, Q_{t-s})(dz) ds.
\]
Finally we prove the estimate \( 5.3 \). Using the above representation and the Hölder inequality we have
\[
\left| \left\langle \nabla^B \left( \int_0^t R_{t-s} \left[ \psi \left( \nabla^B (g(s, \cdot)) \right) \right] ds \right) (x), k \right\rangle \right| \leq \nabla \left\langle \int_0^t \left( 1 + s^{-1/2} \tilde{f} (s, (e^{A}z)_0 + (e^{A}x)_0) \right) \left| \left( Q^0_{t-s} \right)^{-1/2} \left( e^{tA}Bk \right)_0 \right| (Q^0_{t-s})^{-1/2}z_0 \right| ds \right| \leq C \int_0^t \left( 1 + s^{-1/2} \|g\|_{C^{0,1}([1,\infty])} \right) \left| \left( Q^0_{t-s} \right)^{-1/2} \left( e^{tA}Bk \right)_0 \right| \left\| \left( Q^0_{t-s} \right)^{-1/2} \left( e^{tA}Bk \right)_0 \right\| ds \leq C \int_0^t \left( 1 + s^{-1/2} \|g\|_{C^{0,1}([1,\infty])} \right) (t-s)^{-1/2} |k|_{\mathbb{R}^m} ds \leq C \left( t^{1/2} + \|g\|_{C^{0,1}([1,\infty])} \right) |h|_{\mathbb{R}^m}.
\]
Observe that in the last step we have used the estimate
\[
\left\| \left( Q^0_{t-s} \right)^{-1/2} \left( e^{tA}Bh \right)_0 \right\|_{\mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)} \leq C (t-s)^{-1/2}
\]
which follows from the proof of Proposition 4.10 (or Proposition 4.11). Moreover we have also used that
\[
\int_0^t (t-s)^{-1/2}s^{-1} \frac{1}{2} ds = \int_0^1 (1-x)^{-1/2} x^{-1/2} dx = \beta(1/2, 1/2),
\]
where by \( \beta(\cdot, \cdot) \) we mean the Euler beta function.

The Fréchet differentiability and the estimate \( 5.4 \) is proved exactly in the same way using the fact that \( \sigma \) is onto and Proposition 4.10.

Now we consider the case of \( g \in \Sigma^2_{T,1/2} \). We start by proving that \( 5.2 \) is Fréchet differentiable and, in order to compute the Fréchet derivative, we use \( 4.11 \) (which is true for every \( \phi \in C^1_0 (H) \), see its proof) looking at the limit, for \( h \in \mathcal{H} \),
\[
\left\langle \nabla \tilde{g}(t, x), h \right\rangle_{\mathcal{H}} = \lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \int_0^t R_{t-s} \left[ \psi \left( \nabla^B (g(s, \cdot)) \right) \right] (x + \alpha h) ds - \int_0^t R_{t-s} \left[ \psi \left( \nabla^B (g(s, \cdot)) \right) \right] (x) ds \right] = \int_0^t \left[ \left\langle \nabla \left( \psi \left( \nabla^B (g(s, \cdot)) \right) \right), e^{(t-s)A}h \right\rangle_{\mathcal{H}} \right] (x) ds = \int_0^t \left[ \left\langle \nabla \psi \left( \nabla^B (g(s, \cdot)) \right), \nabla \nabla^B (g(s, \cdot)) e^{(t-s)A}h \right\rangle_{\mathbb{R}^m} \right] (x) ds = \int_0^t \left[ \left\langle \nabla \psi \left( \nabla^B (g(s, z) + e^{(t-s)A}x) \right), \nabla \nabla^B (g(s, z) + e^{(t-s)A}x) e^{(t-s)A}h \right\rangle_{\mathbb{R}^m} \right] N(0, Q_{t-s}) (dz) ds = \int_0^t \left[ \left\langle \nabla \psi \left( \tilde{f}_1 \left( s, (e^{sA}z)_0 + (e^{sA}x)_0 \right) \right), s^{-1/2} \tilde{f} (s, (e^{sA}z)_0 + (e^{sA}x)_0) e^{(t-s)A}h \right\rangle_{\mathbb{R}^m} \right] N(0, Q_{t-s}) (dz) ds.
\]
Now, from calculations similar to the ones performed in the first part we arrive at
\[
\left\langle \nabla^B \left\langle \nabla \tilde{g}(t, x), h \right\rangle_{\mathcal{H}}, k \right\rangle_{\mathbb{R}^m} = \int_0^t \left[ \left\langle \nabla \psi \left( \tilde{f}_1 \left( s, (e^{sA}z)_0 + (e^{sA}x)_0 \right) \right), s^{-1/2} \tilde{f} (s, (e^{sA}z)_0 + (e^{sA}x)_0) e^{(t-s)A}h \right\rangle_{\mathbb{R}^m} \right] \left( \left( Q^0_{t-s} \right)^{-1/2} \left( e^{tA}Bk \right)_0 \right) (Q^0_{t-s})^{-1/2}z_0 ds \right].
\]
Since \( \nabla \psi \) is bounded (as it satisfies \( 5.21 \)) then \( 5.10 \) easily follows by the definition of \( \tilde{f} \) and from \( 5.10 \). Second order differentiability and estimate \( 5.6 \), when \( \sigma \) is onto, follow in the same way.

### 6 Regular solutions of the HJB equation

We show first, in Subsection 6.1 that the HJB equation \( 6.2 \) admits a unique mild solution \( v \) which is \( B- \) Fréchet differentiable. Then (Subsection 6.2) we prove a further regularity result whose proof is more complicated than the previous one and that will be useful to solve completely the control problem in the forthcoming companion paper [20].
6.1 Existence and uniqueness of mild solutions

We start showing how to rewrite \((6.3)\) in its integral (or “mild”) form as anticipated in the introduction, formula \((6.3)\). Denoting by \(\mathcal{L}\) the generator of the Ornstein-Uhlenbeck semigroup \(R_t\), we know that, for all \(f \in C_b^2(\mathcal{H})\) such that \(\nabla f \in D(A^*)\) (see e.g. \[8\] Section 5 or also \[11\] Theorem 2.7):

\[
\mathcal{L}[f](x) = \frac{1}{2} \text{Tr} \, GG^* \nabla^2 f(x) + (x, A^* \nabla f(x)).
\]

The HJB equation \((6.2)\) can then be formally rewritten as

\[
\begin{align*}
\frac{\partial v(t,x)}{\partial t} &= \mathcal{L}[v(t,\cdot)](x) + \ell_0(t,x) + H_{\text{min}}(Bv(t,x)), & t \in [0,T], \ x \in \mathcal{H}, \\
v(T,x) &= \tilde{\phi}(x_0).
\end{align*}
\]

By applying formally the variation of constants formula we then have

\[
v(t,x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t}[H_{\text{min}}(Bv(s,\cdot)) + \ell_0(s,\cdot)](x) \, ds, \quad t \in [0,T], \ x \in H,
\]

We use this formula to give the notion of mild solution for the HJB equation \((6.2)\).

**Definition 6.1** We say that a function \(v : [0,T] \times \mathcal{H} \rightarrow \mathbb{R}\) is a mild solution of the HJB equation \((6.2)\) if the following are satisfied:

1. \(v(T,\cdot, \cdot) \in C_{1/2}([0,T] \times \mathcal{H})\);
2. equality \((6.3)\) holds on \([0,T] \times \mathcal{H}\).

**Remark 6.2** Since \(C_{1/2}([0,T] \times \mathcal{H}) \subset C_b([0,T] \times \mathcal{H})\) (see Definition \(6.1\)) the above Definition \(6.1\) requires, among other properties, that a mild solution is continuous and bounded up to \(T\). This constrains the assumptions on the data, e.g. it implies that the final datum \(\phi\) must be continuous and bounded. As recalled in Remark \((4.1)\) (i) and (ii) we may change this requirement in the above definition asking only polynomial growth in \(x\) and/or measurability of \(\phi\). Most of our main results will remain true with straightforward modifications.

Since the transition semigroup \(R_t\) is not even strongly Feller we cannot study the existence and uniqueness of a mild solution of equation \((6.2)\) as it is done e.g. in \[22\]. We then use the partial smoothing property studied in Sections \(4\) and \(5\). Due to Lemma \(5.4\) the right space where to seek a mild solution seems to be \(\Sigma_{T,1/2}\); indeed our existence and uniqueness result will be proved by a fixed point argument in such space.

**Theorem 6.3** Let Hypotheses \((6.4)\) and \((6.5)\) hold and let \((4.3)\) or \((4.7)\) hold. Then the HJB equation \((6.2)\) admits a mild solution \(v\) according to Definition \(6.1\). Moreover \(v\) is unique among the functions \(w\) such that \(w(T,\cdot, \cdot) \in \Sigma_{T,1/2}\) and it satisfies, for suitable \(C_T > 0\), the estimate

\[
||v(T,\cdot, \cdot)||_{C_{1/2}} \leq C_T \left(||\phi||_{\infty} + ||\ell_0||_{\infty}\right).
\]

Finally if the initial datum \(\phi\) is also continuously \(B\)-Fréchet (or Fréchet) differentiable, then \(v \in C_{b}^{0,1,B}([0,T] \times \mathcal{H})\) and, for suitable \(C_T > 0\),

\[
||v||_{C_{b}^{0,1,B}} \leq C_T \left(||\phi||_{\infty} + ||\nabla B \phi||_{\infty} + ||\ell_0||_{\infty}\right)
\]

(substituting \(\nabla B \phi\) with \(\nabla \phi\) if \(\phi\) is Fréchet differentiable).

**Proof.** We use a fixed point argument in \(\Sigma_{T,1/2}\). To this aim, first we rewrite \((6.3)\) in a forward way. Namely if \(v\) satisfies \((6.3)\), then, setting \(w(t,x) := v(T-t,x)\) for any \((t,x) \in [0,T] \times \mathcal{H}\), we get that \(w\) satisfies

\[
w(t,x) = R_t[\phi](x) + \int_0^t R_{t-s}[H_{\text{min}}(B w(s,\cdot)) + \ell_0(s,\cdot)](x) \, ds, \quad t \in [0,T], \ x \in H,
\]

where \(H_{\text{min}}(Bv(t,x))\) is meant in right-derivative sense if the initial datum \(\phi\) is differentiable.
which is the mild form of the forward HJB equation

\[
\begin{cases}
\frac{\partial v(t,x)}{\partial t} = \mathcal{L}[w(t,\cdot)](x) + \ell_0(t,x) + H_{\min}(\nabla^B w(t,x)), & t \in [0, T], \, x \in \mathcal{H}, \\
w(0,x) = \phi(x).
\end{cases}
\]  

(6.7)

Define the map \(\mathcal{C}\) on \(\Sigma_{T,1/2}\) by setting, for \(g \in \Sigma_{T,1/2}\),

\[
\mathcal{C}(g)(t,x) := R_t[g](x) + \int_0^t R_{t-s} [H_{\min}(\nabla^B g(s,\cdot)) + \ell_0(s,\cdot))(x) \, ds, \quad t \in [0, T],
\]

(6.8)

By Proposition 5.11 and Lemma 5.4 (i) we deduce that \(\mathcal{C}\) is well defined in \(\Sigma_{T,1/2}\) and takes its values in \(\Sigma_{T,1/2}\). Since in Lemma 5.2 we have proved that \(\Sigma_{T,1/2}\) is a closed subspace of \(C^{0,1,B}_{1/2}([0,T] \times \mathcal{H})\), once we have proved that \(\mathcal{C}\) is a contraction, by the Contraction Mapping Principle there exists a unique (in \(\Sigma_{T,1/2}\)) fixed point of the map \(\mathcal{C}\), which gives a mild solution of (6.2).

Let \(g_1, g_2 \in \Sigma_{T,1/2}\). We evaluate \(\|\mathcal{C}(g_1) - \mathcal{C}(g_2)\|_{\Sigma_{T,1/2}} = \|\mathcal{C}(g_1) - \mathcal{C}(g_2)\|_{C^{0,1,B}_{1/2}}\). First of all, arguing as in the proof of Lemma 5.3 we have, for every \((t,x) \in [0,T] \times \mathcal{H}\),

\[
\begin{align*}
|\mathcal{C}(g_1)(t,x) - \mathcal{C}(g_2)(t,x)| &= \left| \int_0^t R_{t-s} \left[ H_{\min}(\nabla^B g_1(s,\cdot)) - H_{\min}(\nabla^B g_2(s,\cdot)) \right] (x) \, ds \right| \\
&\leq \int_0^t s^{-1/2} L \sup_{y \in \mathcal{H}} |s^{1/2}\nabla^B (g_1(s,y) - g_2(s,y))| \, ds \leq 2Lt^{1/2}\|g_1 - g_2\|_{C^{0,1,B}_{1/2}}.
\end{align*}
\]

Similarly, arguing exactly as in the proof of (6.3), we get

\[
\begin{align*}
t^{1/2}|\nabla^B \mathcal{C}(g_1)(t,x) - \nabla^B \mathcal{C}(g_2)(t,x)| &= t^{1/2} \left| \nabla^B \left[ \int_0^t R_{t-s} \left[ H_{\min}(\nabla^B g_1(s,\cdot)) - H_{\min}(\nabla^B g_2(s,\cdot)) \right] (x) \, ds \right] 
\right| \\
&\leq t^{1/2} L \|g_1 - g_2\|_{C^{0,1,B}_{1/2}} \int_0^t (t-s)^{-1/2}s^{-1/2} \, ds \leq t^{1/2} L \beta (1/2,1/2) \|g_1 - g_2\|_{C^{0,1,B}_{1/2}}.
\end{align*}
\]

Hence, if \(T\) is sufficiently small, we get

\[
\|\mathcal{C}(g_1) - \mathcal{C}(g_2)\|_{C^{0,1,B}_{1/2}} \leq C \|g_1 - g_2\|_{C^{0,1,B}_{1/2}}
\]

(6.9)

with \(C < 1\). So the map \(\mathcal{C}\) is a contraction in \(\Sigma_{T,1/2}\) and, if we denote by \(v\) its unique fixed point, then \(v := w(T - \cdot, \cdot)\) turns out to be a mild solution of the HJB equation (6.2), according to Definition 6.3.

Since the constant \(L\) is independent of \(t\), the case of generic \(T > 0\) follows by dividing the interval \([0,T]\) into a finite number of subintervals of length \(\delta\) sufficiently small, or equivalently, as done in [32], by taking an equivalent norm with an adequate exponential weight, such as

\[
\|f\|_{\mathcal{H}_{C^{0,1,B}_{1/2}}} = \sup_{(t,x) \in [0,T] \times \mathcal{H}} \left| e^\eta t f(t,x) \right| + \sup_{(t,x) \in [0,T] \times \mathcal{H}} e^\eta t^{1/2} \left\| \nabla^B f(t,x) \right\|_{(\mathcal{H})^*},
\]

The estimate (6.3) follows from Proposition 4.11 and Lemma 5.4.

Finally the proof of the last statement follows observing that, if \(\phi\) is continuously \(B\)-Fréchet (or Fréchet) differentiable, then \(R_t[\phi]\) is continuously \(B\)-Fréchet differentiable with \(\nabla^B R_t[\phi]\) bounded in \([0,T] \times \mathcal{H}\), see Lemma 4.13 formula (4.15). This allows to perform the fixed point, exactly as done in the first part of the proof, in \(C^{0,1,B}_{1/2}([0,T] \times \mathcal{H})\) and to prove estimate (6.5). \(\square\)

**Corollary 6.4** Let Hypotheses 5.1 and 5.3 hold and let \(\sigma\) be onto. Then the mild solution of equation (6.2) found in the previous theorem is also Fréchet differentiable, and the following estimate holds true

\[
\|v(T - \cdot, \cdot)\|_{C^{0,1}_{1/2}} \leq C_T \left( \|\tilde{\phi}\|_{\infty} + \|\tilde{\ell}_0\|_{\infty} \right)
\]

(6.10)

for a suitable \(C_T > 0\).
Proof. Let \( v \) be the mild solution of equation (6.2), and \( \forall t \in [0, T] \), \( x \in \mathcal{H} \) define \( w(t, x) := v(T - t, x) \), so that \( w \) satisfies (6.6), so that by applying the last statement of Lemma 3.4 it is immediate to see that \( w \in C_{1/2}([0, T] \times \mathcal{H}) \). By differentiating (6.6) we get

\[
\nabla w(t, x) = \nabla R_t[\phi](x) + \frac{1}{t} \int_0^t R_{t-s}[H_{\min}(\nabla^B w(s, \cdot) + \ell_0(s, \cdot))](x) \, ds, \quad t \in [0, T], \ x \in \mathcal{H},
\]

By Lemma 4.5 the above recalled variation of Lemma 5.4 and estimate (5.4), we get that

\[
|\nabla w(t, x)| \leq Ct^{-1/2}\|\phi\|_{\infty} + Ct^{1/2} \left( 1 + \|w\|_{C_{1/2}} + \|\ell_0\|_{\infty} \right), \quad t \in [0, T], \ x \in \mathcal{H},
\]

which gives the claim using the estimate for \( \|w\|_{C_{1/2}} \) given in (6.4).

\[ \square \]

6.2 Second derivative of mild solutions

The further regularity result we are going to prove is interesting in itself, but is also crucial to solve the control problem, as will be seen in the forthcoming companion paper [26]. A similar result can be found in [22], Section 4.2. Here we use the same line of proof but we need to argue in a different way to get the apriori estimates.

Theorem 6.5 Let Hypotheses 3.7, 3.9 and 3.4 hold. Let also 4.3, or 4.33 hold. Let \( v \) be the mild solution of the HJB equation (6.2) as from Theorem 6.3. Then we have the following.

(i) If \( \phi \) is continuously differentiable then we have \( v \in \Sigma^2_{2,T,1/2} \), hence the second order derivatives \( \nabla^B \nabla v \) and \( \nabla^B v \) exist and are equal. Moreover there exists a constant \( C > 0 \) such that

\[
|\nabla^2 v(t, x)| \leq C \left( \|\nabla \phi\|_{\infty} + \|\nabla \ell_0\|_{\infty} \right),
\]

(6.11)

Finally, if \( \sigma \) is onto, then also \( \nabla^2 v \) exists and is continuous and, for suitable \( C > 0 \),

\[
|\nabla^2 v(t, x)| \leq C \left( (T - t)^{-1/2}\|\nabla \phi\|_{\infty} + (T - t)^{1/2}\|\nabla \ell_0\|_{\infty} \right).
\]

(6.13)

(ii) If \( \phi \) is only continuous then the function \( (t, x) \mapsto (T - t)^{1/2}v(t, x) \) belongs to \( \Sigma^2_{2,T,1/2} \). Moreover there exists a constant \( C > 0 \) such that

\[
|\nabla v(t, x)| \leq C \left( (T - t)^{-1/2}\|\phi\|_{\infty} + \|\nabla \ell_0\|_{\infty} \right),
\]

(6.14)

Finally, if \( \sigma \) is onto, then also \( \nabla^2 v \) exists and is continuous in \( [0, T) \times \mathcal{H} \) and, for suitable \( C > 0 \),

\[
|\nabla^2 v(t, x)| \leq C \left( (T - t)^{-1}\|\nabla \phi\|_{\infty} + (T - t)^{1/2}\|\nabla \ell_0\|_{\infty} \right).
\]

(6.16)

Proof. We start proving (i) by applying the Contraction Mapping Theorem in a closed ball \( B_T(0, R) \) \( (R \) to be chosen later) of the space \( \Sigma^2_{2,T,1/2} \). By Proposition 4.3 and Lemma 5.4 we deduce that the map \( C \) defined in (3.18) brings \( \Sigma^2_{2,T,1/2} \) into \( \Sigma^2_{2,T,1/2} \). Moreover, for every \( g \in \Sigma^2_{2,T,1/2} \), we get, first using (3.21),

\[
|C(g)(t, x)| \leq |R_t[\phi](x)| + \int_0^t R_{t-s}[H_{\min}(\nabla^B g(s, \cdot) + \ell_0(s, \cdot))](x) ds \leq \|\phi\|_{\infty} + tL \left( 1 + \|g\|_{C_{1/2}} \right) + t\|\ell_0\|_{\infty};
\]

second by (4.14), (5.11), (3.21) (calling \( M := \sup_{[0, T]} \|e^{tA}\| \))

\[
|\nabla C(g)(t, x)| \leq |\nabla R_t[\phi](x)| + \left| \nabla \int_0^t R_{t-s}[H_{\min}(\nabla^B g(s, \cdot) + \ell_0(s, \cdot))](x) ds \right|
\]

\[
\leq M\|\nabla \phi\|_{\infty} + M\int_0^t \left( \|H_{\min}(\nabla^B g(s, \cdot) + \ell_0(s, \cdot))\|_{\infty} + \|\ell_0\|_{\infty} \right) ds
\]

\[
\leq M \left[ \|\nabla \phi\|_{\infty} + t\|\ell_0\|_{\infty} + Lt^{1/2}\|g\|_{C_{1/2}} \right];
\]

(6.17)
third by (4.17) (with (4.34) or (4.35)), and (5.5)

\[
t^{1/2} |\nabla^0 B \nabla C(g)(t, x)| \leq t^{1/2} |\nabla^0 B \nabla R_0[\phi](x)| + t^{1/2} |\nabla^1 B \nabla \int_0^t R_{t-s} \left[ H_{\text{min}} \left( \nabla^0 B g(s, \cdot) \right) + \ell_0(s, \cdot) \right](x)ds |
\]

\[
\leq C |\nabla \phi|_\infty + Ct^{1/2} \int_0^t (t - s)^{-1/2} |\nabla \ell_0(s, \cdot)|_\infty ds + Ct^{1/2} |g|_{C^0_{1/2} B}^{0, a, b}
\]

\[
\leq C \left[ |\nabla \phi|_\infty + 2t |\nabla \ell_0|_\infty + t^{1/2} |g|_{C^0_{1/2} B}^{0, a, b} \right],
\]

with the constant C (that may change from line to line) given by the quoted estimates. Hence, for \( g \in B_T(0, R) \), we get, for given \( C_1 > 0 \),

\[
||C(g)||_{C^0_{1/2} B}^{0, a, b} \leq C_1 \left[ ||\phi||_{C^0_T} + T + T ||\ell_0||_{C^0_T} \right] + TL ||g||_{C^0_{1/2} B}^{0, a, b} + (ML + C) T^{1/2} ||g||_{C^0_{1/2} B}^{0, a, b}
\]

\[
\leq C_1 \left[ ||\phi||_{C^0_T} + T + T ||\ell_0||_{C^0_T} \right] + \rho(T) R
\]

where we define

\[
\rho(T) := TL + (ML + C) T^{1/2}.
\]

Now take \( g_1, g_2 \in \Sigma^2_{T, 1/2} \). Arguing as in the above estimates we have, for every \((t, x) \in [0, T] \times \mathcal{H}\),

\[
|C(g_1)(t, x) - C(g_2)(t, x)| = \left| \int_0^t R_{t-s} \left[ H_{\text{min}} \left( \nabla^0 B g_1(s, \cdot) \right) - H_{\text{min}} \left( \nabla^0 B g_2(s, \cdot) \right) \right](x)ds \right|
\]

\[
\leq tL ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b}
\]

\[
|\nabla C(g_1)(t, x) - \nabla C(g_2)(t, x)| = \left| \nabla \int_0^t R_{t-s} \left[ H_{\text{min}} \left( \nabla^0 B g_1(s, \cdot) \right) - H_{\text{min}} \left( \nabla^0 B g_2(s, \cdot) \right) \right](x)ds \right|
\]

\[
\leq M \int_0^t \left| \nabla H_{\text{min}} \left( \nabla^0 B g_1(s, \cdot) \right) \nabla \nabla^0 B g_1(s, \cdot) - \nabla H_{\text{min}} \left( \nabla^0 B g_2(s, \cdot) \right) \nabla \nabla^0 B g_2(s, \cdot) \right|_{\infty} ds
\]

\[
\leq 2ML^{1/2} \left[ ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b} ||g_1||_{C^0_{1/2} B}^{0, a, b} + ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b} \right]
\]

and, using (5.12),

\[
t^{1/2} |\nabla^0 B \nabla C(g_1)(t, x) - \nabla^0 B \nabla C(g_2)(t, x)| = t^{1/2} |\nabla^1 B \nabla \int_0^t R_{t-s} \left[ H_{\text{min}} \left( \nabla^0 B g_1(s, \cdot) \right) - H_{\text{min}} \left( \nabla^0 B g_2(s, \cdot) \right) \right](x)ds |
\]

\[
\leq t^{1/2} ML \beta (1/2, 1/2) \left[ ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b} ||g_1||_{C^0_{1/2} B}^{0, a, b} + ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b} \right].
\]

Hence, for \( g_1, g_2 \in B_T(0, R) \), we have, recalling (6.21) and the way C is found in (6.19),

\[
||C(g_1) - C(g_2)||_{C^0_{1/2} B}^{0, a, b} \leq L \left( T + MT^{1/2} (2 + \beta (1/2, 1/2)) (1 + R) \right) ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b}
\]

\[
\leq \rho(T)(1 + R) ||g_1 - g_2||_{C^0_{1/2} B}^{0, a, b},
\]

(6.22)

Now, by (6.20) and (6.22), choosing any \( R > C_1 \left[ ||\phi||_{C^0_T} + T + T ||\ell_0||_{C^0_T} \right] \) we can find \( T_0 \) sufficiently small so that \( \rho(T_0) < 1/2 \) and so, thanks to (6.20) and (6.22), C is a contraction in \( B_{T_0}(0, R) \). Let then \( w \) be the unique fixed point of \( C \) in \( B_{T_0}(0, R) \); it must coincide with \( v(T - \cdot, \cdot) \) for \( t \in [0, T_0] \). This procedure can be iterated arriving to cover the whole interval \([0, T]\) if we give an apriori estimate for the norm \( ||w||_{C^0_{1/2} B}^{0, a, b} \).

By the last statement of Theorem 6.3 we already have an apriori estimate for \( ||w||_{\infty} + ||\nabla^0 B w||_{\infty} \). To get the estimate for \( \nabla w \) and \( \nabla^0 B \nabla w \) we use the (6.17) and (6.19) where we put \( w \) in place of \( Cg \) and \( g \). From the first line of (6.19) and (5.12) we get

\[
||\nabla^0 B \nabla w(t, \cdot)||_{\infty} \leq C \left( ||\nabla \phi||_{\infty} + 2T ||\nabla \ell_0||_{\infty} \right) + L \int_0^t (t - s)^{-1/2} ||\nabla^0 B \nabla w(s, \cdot)||_{\infty} ds
\]
which, thanks to the Gronwall Lemma (see [27], Subsection 1.2.1, p.6) give the apriori estimate for $\nabla^B \nabla w$. Then from the second line of (6.17) we get

$$\|\nabla w(t, \cdot)\|_\infty \leq M \|\nabla \phi\|_\infty + T \|\nabla \ell_0\|_\infty + ML \int_0^t \|\nabla^B \nabla w(s, \cdot)\|_\infty ds$$

which gives the apriori estimate for $\nabla w$ using the previous one for $\nabla^B \nabla w$. Estimate (6.13) follows by repeating the same arguments above but replacing $\nabla^B \nabla$ with $\nabla^2$.

We now prove (ii). Let $v$ be the mild solution of (6.2) and, for all $\varepsilon \in [0, T]$, $x \in \mathcal{H}$, call $\phi^\varepsilon(x) = v(T - \varepsilon, x)$. Then $v$ is the unique mild solution, on $[0, T - \varepsilon] \times \mathcal{H}$, of the equation (for $t \in [0, T]$, $x \in H$)

$$v(t, x) = R_{T-\varepsilon-t} \phi^\varepsilon(x) + \int_t^{T-\varepsilon} R_{s-t} \left[ H_{\min}(\nabla^B v(s, \cdot)) + \ell_0(s, \cdot) \right] (x)ds \quad (6.23)$$

This fact can be easily seen by applying the semigroup property of $R_t$ (see e.g. [22] Lemma 4.10 for a completely similar result).

Now, by Theorem 6.3 $\phi^\varepsilon$ is continuously differentiable, so we can apply part (i) of this theorem to (6.23) getting the required $C^2$ regularity. Estimates (6.14)-(6.15)-(6.16) follows using estimates (6.11)-(6.12)-(6.13) with $\phi^\varepsilon$ in place of $\phi$ and then using the arbitrariness of $\varepsilon$ and applying (6.4) to estimate $\phi^\varepsilon$ in term of $\phi$.

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