Korovkin-Type Theorems for Weakly Nonlinear and Monotone Operators

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Abstract. In this paper we prove analogs of Korovkin’s theorem in the context of weakly nonlinear and monotone operators acting on Banach lattices of functions of several variables. Our results concern the convergence almost everywhere, the convergence in measure and the convergence in $L^p$-norm. Several results illustrating the theory are also included.

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1. Introduction

Korovkin’s theorem [20, 21] provides a very simple test of convergence to the identity for any sequence $(T_n)_n$ of positive linear operators that map $C([0,1])$ into itself: the occurrence of this convergence for the functions $1, x$ and $x^2$. In other words, the fact that

$$\lim_{n \to \infty} T_n(f) = f \quad \text{uniformly on } [0,1]$$

for every $f \in C([0,1])$ reduces to the status of the three aforementioned functions. Due to its simplicity and usefulness, this result has attracted a great deal of attention leading to numerous generalizations. Part of them are included in the authoritative monograph of Altomare and Campiti [2], and the excellent survey of Altomare [3]. For some very recent contributions to this topic, see [4, 5, 27].

Recently, the present authors have extended the Korovkin theorem to the framework of sublinear and monotone operators acting on function spaces endowed with the topology of uniform convergence on compact sets. See [15, 17, 18].
The aim of the present paper is to prove that similar results hold in the context of the three usual modes of convergence used in measure theory: convergence almost everywhere, convergence in probability and the convergence in $L^p$ norm (also known as the convergence in $p$-mean).

The necessary background on our nonlinear framework is summarized in Section 1. We deal with the class of sublinear and monotone operators acting on Banach lattices of functions which verify the property of translatability relative to the multiples of unity. This was introduced in [18], motivated by our interest in Choquet’s theory of integration, but there are many other examples outside that theory, mentioned in this paper. The continuity of sublinear operators can be characterized via the notion of norm as in the linear case. As far as we know, the result of Theorem 1 asserting that every sublinear and monotone operator is Lipschitz continuous is new. It extends a classical result due to Krein [22] who considered the case of linear functionals.

The extension of the Korovkin theorem for the aforementioned modes of convergence makes the object of Theorem 1 (Sect. 3) and, respectively, of Theorem 2 and Theorem 3 (Sect. 4). As a consequence of Theorem 1, we provide in Sect. 5 an alternative proof of the Lebesgue differentiation theorem. The pointwise convergence of the Bernstein–Kantorovich sequences and the Szász–Mirakjan–Kantorovich sequences (as well as of their Choquet counterparts obtained by replacing the Lebesgue integral by Choquet integral with respect to a submodular capacity) is presented in Sect. 6. Our results in this respect extend to a classical theorem of Lorentz [23] (Theorem 2.1.1, p. 30).

The paper ends with a section of further results and comments.

2. Preliminaries on Nonlinear Operators

In what follows, we denote by $X$ a metric measure space, that is, a triple $(X, d, \mu)$ consisting of a space $X$ endowed with the metric $d$ and the measure $\mu$, defined on the sigma algebra $\mathcal{B}(X)$ of Borel subsets of $X$. Notice that every set can be turned into a metric measure space by considering on it the discrete metric and any finite combination (with positive coefficients) of Dirac measures.

Attached to $X$ is the vector lattice $\mathcal{F}(X)$ of all real-valued functions defined on $X$, endowed with the pointwise ordering. Among the vector sub-lattices of $\mathcal{F}(X)$, which play a role in the extension of Korovkin’s results, we mention here

$$\mathcal{F}_b(X) = \{f \in \mathcal{F}(X) : f \text{ bounded}\},$$

$$C(X) = \{f \in \mathcal{F}(X) : f \text{ continuous}\},$$

$$C_b(X) = \{f \in \mathcal{F}(X) : f \text{ continuous and bounded}\},$$

$$UC_b(X) = \{f \in \mathcal{F}(X) : f \text{ uniformly continuous and bounded}\},$$

and

$$C_c(X) = \{f \in \mathcal{F}(X) : f \text{ continuous and having a compact support}\},$$
to which one should add the usual sublattices of measurable functions,
\[ \mathcal{M}(X) = \{ f \in \mathcal{F}(X) : f \text{ Borel measurable} \}, \]
\[ \mathcal{AC}_b(X) = \{ f \in \mathcal{F}(X) : f \text{ bounded and almost everywhere continuous} \} \]
as well as all Banach function spaces (including the spaces \( L^p(\mu) \) with \( p \in [1, \infty] \)). A thorough presentation of Banach function spaces can be found in the book by Bennett and Sharpley [6].

Notice that \( C(X), \mathcal{AC}_b(X) \) and \( L^p(\mu) \) are sublattices of \( \mathcal{M}(X) \). The spaces \( C(X), C_b(X) \) and \( UC_b(X) \) coincide when \( X \) is a compact metric space. \( AC_b(X) \) coincides with the space of Riemann integrable functions when \( X \) is a compact \( N \)-dimensional interval of \( \mathbb{R}^N \) and \( \mu \) is the Lebesgue measure. Notice also that the spaces \( C_b(X), UC_b(X) \) and \( L^p(\mu) \) are Banach lattices with respect to appropriate norms, precisely, the sup norm
\[ \| f \|_{\infty} = \sup \{ |f(x)| : x \in X \}, \]
in the case of the first two spaces, and the \( L^p \)-norm
\[ \| f \|_p = \begin{cases} \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p} & \text{if } p \in [1, \infty) \\ \text{esssup}_{x \in X} |f(x)| & \text{if } p = \infty \end{cases} \]
in the case of spaces \( L^p(\mu) \).

An important family of Lipschitz continuous functions in \( C(X) \) is that associated with the metric \( d \) by the formulas:
\[ d_x : X \to \mathbb{R}, \quad d_x(y) = d(x, y) \quad (x \in X). \]

See [10] and [24] for the necessary background on Banach lattices used in this paper.

As is well known, all norms on the \( N \)-dimensional real vector space \( \mathbb{R}^N \) are equivalent. When endowed with the sup norm and the coordinate-wise ordering, \( \mathbb{R}^N \) can be identified (algebraically, isometrically and in order) with the Banach lattice \( C(\{1, ..., N\}) \), where \( \{1, ..., N\} \) carries the discrete topology.

Suppose that \( X \) and \( Y \) are two metric spaces and \( E \) and \( F \) are, respectively, ordered vector subspaces (or subcones of the positive cones) of \( \mathcal{F}(X) \) and \( \mathcal{F}(Y) \) that contain the unity. An operator \( T : E \to F \) is said to be weakly nonlinear if it satisfies the following two conditions:

(SL) (Sublinearity) \( T \) is subadditive and positively homogeneous, that is,
\[ T(f + g) \leq T(f) + T(g) \quad \text{and} \quad T(\alpha f) = \alpha T(f) \]
for all \( f, g \) in \( E \) and \( \alpha \geq 0 \).

(TR) (Translatability) \( T(f + \alpha \cdot 1) = T(f) + \alpha T(1) \) for all functions \( f \in E \) and all numbers \( \alpha \geq 0 \).

In the case when \( T \) is unital (that is, \( T(1) = 1 \)), the condition of translatability takes the form
\[ T(f + \alpha \cdot 1) = T(f) + \alpha 1, \quad (2.1) \]
for all \( f \in E \) and \( \alpha \geq 0 \).
In this paper, we are especially interested in those weakly nonlinear operators which preserve the ordering, that is, which verify the following condition:

(M) (Monotonicity) \( f \leq g \) in \( E \) implies \( T(f) \leq T(g) \) for all \( f, g \) in \( E \).

Some authors prefer the term of isotonicity for monotonicity to avoid any confusion with the monotonicity of subdifferentials in convex analysis. However in our paper, we do not touch the problem of differentiability.

Crandall and Tartar [11] have noticed that if an operator \( T : L^\infty(\mu) \to L^\infty(\mu) \) is translatable and unital, then it is monotone if, and only if, it is Lipschitz with Lipschitz constant at most 1. In particular, this remark holds for \( \mathbb{R}^N \) endowed with the sup-norm and any operator \( T : \mathbb{R}^N \to \mathbb{R}^N \) which verifies the property (2.1).

Real analysis and harmonic analysis offer numerous example of weakly nonlinear and monotone operators, so is the operator \( T \) acting from the Lebesgue space \( L^p(\mathbb{R}^N) \) (\( p > 1 \)) into itself via the formula:

\[
(Tf)(x) = \sup_{r>0} \frac{1}{\operatorname{vol}(B_r(x))} \int_{B_r(x)} f(y) \, dy. \tag{2.2}
\]

The same holds for the operator \( S : L^1(\mu) \to L^1(\mu) \), associated with a Borel probability measure \( \mu \) on \((0,1)\), and defined by the formula

\[
(Sf)(t) = \sup_{\mu(A) \leq t} \frac{1}{\mu(A)} \int_A f(s) \, d\mu.
\]

Notice that \( S \) is also unital.

Many more examples can be found in our paper [18].

Meantime, it is important to notice the existence of sublinear operators which are neither monotone nor translatable, an example being the Hardy–Littlewood maximal operator \( M : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \) (\( p > 1 \)), defined by the formula

\[
Mf(x) = \sup_{r>0} \frac{1}{\operatorname{vol}(B_r(x))} \int_{B_r(x)} |f(y)| \, dy.
\]

However, \( M \) is monotone and translatable on the positive cone of \( L^p(\mathbb{R}^N) \).

A stronger condition than translatability is that of comonotonic additivity,

(CA) \( T(f + g) = T(f) + T(g) \) whenever the functions \( f, g \in E \) are comonotone in the sense that

\[
(f(s) - f(t)) \cdot (g(s) - g(t)) \geq 0 \quad \text{for all } s, t \in X.
\]

The condition of comonotonic additivity occurs naturally in the context of Choquet’s integral (and thus in the case of Choquet-type operators, which are sublinear, comonotonic additive and monotone). See [16,17] as well as the references therein.

Suppose that \( E \) and \( F \) are two Banach lattices and \( T : E \to F \) is an operator (not necessarily linear or continuous).

If \( T \) is a positively homogeneous operator, then

\[
T(0) = 0.
\]
As a consequence, every positively homogeneous and monotone operator $T$ maps positive elements into positive elements,

$$T(x) \geq 0 \quad \text{for all } x \geq 0. \quad (2.3)$$

Consequently, for linear operators the property (2.3) is equivalent to monotonicity.

Every sublinear operator is convex and a convex operator is sublinear if and only if it is positively homogeneous.

The notion of norm can be introduced for every continuous sublinear operator $T : E \rightarrow F$ via the formulas

$$\|T\| = \inf \{ \lambda > 0 : \|T(f)\| \leq \lambda \|f\| \text{ for all } f \in E \}$$

$$= \sup \{ \|T(f)\| : f \in E, \|f\| \leq 1 \}.$$

A sublinear operator may be discontinuous, but when it is continuous, it is Lipschitz continuous:

**Lemma 1.** If $T : E \rightarrow F$ is a continuous sublinear operator, then

$$\|T(f) - T(g)\| \leq 2 \|T\| \|f - g\| \quad \text{for all } f \in E.$$  

**Proof.** Indeed, for all $f, g \in E$, we have $T(f) \leq T(g) + T(f - g) \leq T(g) + |T(f - g)|$ and $T(g) \leq T(f) + |T(g - f)|$, whence

$$|T(f) - T(g)| = \sup \{- (T(f) - T(g)), T(f) - T(g)\} \leq |T(f - g)| + |T(g - f)|.$$  

Therefore, $\|T(f) - T(g)\| \leq \|T(f - g)\| + \|T(g - f)\| \leq 2 \|T\| \|f - g\|$ and the proof is done. \qed

It is a remarkable fact that all sublinear and monotone operators are automatically continuous. This was first proved by Krein for positive linear functionals [22] and later generalized in several linear contexts by various authors (including Klee, Lozanovsky, Namioka and Schaefer). See [1].

**Theorem 1.** Every sublinear and monotone operator $T : E \rightarrow F$ is Lipschitz continuous and the Lipschitz constant of $T$ equals $\|T\|$, that is,

$$\|T(f) - T(g)\| \leq \|T\| \|f - g\| \quad \text{for all } f, g \in E.$$  

**Proof.** Notice first that for all $f, g \in E$ we have $f \leq g + |f - g|$, whence $T(f) - T(g) \leq T(|f - g|)$ due to the monotonicity and subadditivity of $T$. Interchanging the role of $f$ and $g$, we infer that $-(T(f) - T(g)) \leq T(|f - g|)$. Therefore,

$$|T(f) - T(g)| = \sup \{T(f) - T(g), -(T(f) - T(g))\} \leq T(|f - g|). \quad (2.4)$$

Now, the continuity of $T$ can be established by reductio ad absurdum. Indeed, if $T$ were not continuous, then there would exist a sequence $(x_n)_n$ of elements of $E$ such that $\|f_n\| \leq 1/(n2^n)$ and $\|T(f_n)\| \geq n$. Taking into account the inequality $|T(f_n)| \leq T(|f_n|)$, and replacing each $x_n$ by $|f_n|$ if necessary, one may restrict ourselves to the case where all elements $f_n$ belong to $E_+$. Then the series $\sum f_n$ is absolutely converging, with sum $f \geq 0$. Since $T$ is monotone,
\[ T(f) \geq T(f_n), \]

which implies, \( \|T(f)\| \geq \|T(f_n)\| \geq n \) for all positive integers \( n \), a contradiction.

Once the continuity of \( T \) was established, we infer from (2.4) that
\[
\|T(f) - T(g)\| \leq \|T(|f - g|)\| \leq \|T\|\|f - g\|,
\]
which ends the proof. \( \square \)

**Remark 1.** The proof of Theorem 1 shows that every sublinear and monotone operator \( T : E \rightarrow F \) verifies the condition \( |T(f) - T(g)| \leq T(|f - g|) \) for all \( f, g \) in \( E \).

### 3. The Case of Almost Everywhere Convergence

We start with an extension of Korovkin’s theorem in the context of sublinear and monotone operators acting on the vector lattice \( \mathcal{AC}_b(X) \) of all bounded and almost everywhere continuous \( f : X \rightarrow \mathbb{R} \).

**Theorem 2.** Suppose that \( X \) is a locally compact subset of the Euclidean space \( \mathbb{R}^N \) endowed with a positive Borel measure \( \mu \) and \( E \) is a vector sublattice of \( \mathcal{F}(X) \) that contains the following set of test functions: 1, \( \pm \text{pr}_1, \ldots, \pm \text{pr}_N \) and \( \sum_{k=1}^N \text{pr}_k^2 \).

(i) If \( (T_n)_n \) is a sequence of sublinear and monotone operators from \( E \) into \( E \) such that
\[ T_n(f)(x) \rightarrow f(x) \quad \text{a.e.} \]
for each of the \( 2N + 2 \) aforementioned test functions, then this property extends to all nonnegative functions \( f \) in \( E \cap \mathcal{AC}_b(X) \).

(ii) If, in addition, each operator \( T_n \) is translatable, then \( T_n(f)(x) \rightarrow f(x) \) a.e. for every \( f \in E \cap \mathcal{AC}_b(X) \).

Moreover, in both cases (i) and (ii), the family of test functions can be reduced to 1, \( -\text{pr}_1, \ldots, -\text{pr}_N \) and \( \sum_{k=1}^N \text{pr}_k^2 \) provided that \( X \) is included in the positive cone of \( \mathbb{R}^N \).

**Proof.** Let \( f \in E \cap \mathcal{AC}_b(\Omega) \) and let \( \omega \) be a point of continuity of \( f \), which is also a point where
\[ T_n(h)(\omega) \rightarrow h(\omega) \]
for each of the functions \( h \in \{1, \pm \text{pr}_1, \ldots, \pm \text{pr}_N \text{ and } \sum_{k=1}^N \text{pr}_k^2\} \).

Then for \( \varepsilon > 0 \) arbitrarily fixed, there is \( \delta > 0 \) such that
\[
|f(x) - f(\omega)| \leq \varepsilon \quad \text{for every } x \in X \text{ with } \|x - \omega\| \leq \delta.
\]

If \( \|x - \omega\| \geq \delta \), then
\[
|f(x) - f(\omega)| \leq \frac{2\|f\|_{\infty}}{\delta^2} \cdot \|x - \omega\|^2,
\]
so that
\[
|f(x) - f(\omega)| \leq \varepsilon + \frac{2\|f\|_{\infty}}{\delta^2} \cdot \|x - \omega\|^2 \quad \text{for all } x \in X. \quad (3.1)
\]
Denoting
\[ M = \max \{ \text{pr}_1(\omega), ..., \text{pr}_N(\omega), 0 \}, \]
once can restate (3.1) as
\[
|f(x) - f(\omega)| \leq \varepsilon + \frac{2\|f\|_\infty}{\delta^2} \left[ \sum_{k=1}^N \text{pr}_k^2(x) + 2 \sum_{k=1}^N \text{pr}_k(x)(M - \text{pr}_k(\omega)) \right] + 2M \sum_{k=1}^N (-\text{pr}_k(x)) + \|\omega\|^2 .
\]

Taking into account Remark 1 and the properties of sublinearity and monotonicity of the operators \( T_n \), we infer in the case where \( f \geq 0 \) that
\[
|T_n(f) - f(\omega)| \leq |T_n(f) - T_n(f(\omega) \cdot 1) + f(\omega)T_n(1) - f(\omega)|
\leq T_n(|f - f(\omega)|) + f(\omega)|T_n(1) - 1|
\leq \varepsilon T_n(1) + \frac{2\|f\|_\infty}{\delta^2} \left[ T_n \left( \sum_{k=1}^N \text{pr}_k^2 \right) + 2 \sum_{k=1}^N (M - \text{pr}_k(\omega))T_n(\text{pr}_k) \right]
+ 2M \sum_{k=1}^N T_n(-\text{pr}_k) + \|\omega\|^2 T_n(1) + f(\omega)|T_n(1) - 1|.
\]

By our choice of \( \omega \),
\[
\lim_{n \to \infty} \sup_{\omega} |T_n(f)(\omega) - f(\omega)| \leq \varepsilon ,
\]
whence we conclude that \( T_n(f)(\omega) \to f(\omega) \), since \( \varepsilon > 0 \) was arbitrarily fixed.

(ii) Suppose in addition that each operator \( T_n \) is also translatable. According to assertion (i),
\[ T_n(f + \|f\|_\infty) \to f + \|f\|_\infty \quad \text{a.e.} \]
Since the operators \( T_n \) are translatable, \( T_n(f + \|f\|_\infty) = T_n(f) + \|f\|_\infty T_n(1) \)
and by our hypotheses \( T_n(1) \to 1 \) a.e. Therefore, \( T_n(f) \to f \) a.e.

As concerns the last assertion of Theorem 2, notice that when \( X \) is included in the positive cone of \( \mathbb{R}^N \) one can restate the estimate (3.1) as
\[
|f(x) - f(\omega)| \leq \varepsilon
+ \frac{2\|f\|_\infty}{\delta^2} \left[ \sum_{k=1}^N \text{pr}_k^2(x) + 2 \sum_{k=1}^N (-\text{pr}_k(x))\text{pr}_k(\omega) + \|\omega\|^2 \right] ,
\]
which leads to
\[
|T_n(f) - f(\omega)| \leq T_n(|f - f(\omega)|) + f(\omega)|T_n(1) - 1|
\leq \varepsilon + \frac{2\|f\|_\infty}{\delta^2} \left[ T_n \left( \sum_{k=1}^N \text{pr}_k^2(x) \right) + 2\text{pr}_k(\omega) \sum_{k=1}^N T_n(-\text{pr}_k(x)) + \|\omega\|^2 T_n(1) \right]
+ f(\omega)|T_n(1) - 1| .
\]
in the case where \( f \geq 0 \). Then the proof continues verbatim as in the cases (i) and (ii).

**Corollary 1.** Let \( \mathcal{R}(K) \) be the vector lattice of all Riemann integrable functions defined on an \( N \)-dimensional compact interval \( K = \prod_{k=1}^{N} [a_k, b_k] \) and let \( (T_n)_n \) be a sequence of sublinear and monotone operators from \( \mathcal{R}(K) \) into itself such that

\[
T_n(f) \to f \quad \text{a.e.}
\]

for each of the functions \( 1, \pm pr_1, \ldots, \pm pr_N \) and \( \sum_{k=1}^{N} pr_k^2 \). Then this convergence also occurs for all nonnegative functions \( f \in \mathcal{R}(K) \). It occurs for all Riemann integrable functions defined on \( K \) when the operators \( T_n \) are weakly nonlinear and monotone.

**Proof.** One applies Theorem 2, by taking into account Lebesgue’s characterization of Riemann integrable functions: a function \( f : K \to \mathbb{R} \) belongs to \( \mathcal{R}(K) \) if and only if \( f \) is bounded and the set points where it is not continuous has Lebesgue measure zero. See [10], p. 323, for the case \( N = 1 \) and [31], pp. 110-113, Sec.11.1, for an arbitrary \( N \geq 1 \).

**Remark 2.** (The necessity of hypotheses in Theorem 2) Though being sufficient for the fulfillment of Theorem 2, none of the three conditions imposed on the operators \( T_n \) (sublinearity, monotonicity and translatability) is necessary. See the case of the sequence of operators,

\[
S_n : \mathcal{R}([0, 1]) \to \mathcal{R}([0, 1]), \quad S_n(f) = f + f^2/n,
\]

which fails all these assumptions despite its converges to the identity of \( \mathcal{R}([0, 1]) \).

**Remark 3.** The proof of Theorem 2 still works in the variant where the space \( \mathcal{AC}_b(X) \) is replaced by \( C_b(X) \) and the convergence a.e. is replaced, respectively, by pointwise convergence (or even by uniform convergence on compact subsets).

**Remark 4.** Suppose that \( (T_n)_n \) is a sequence of weakly linear and monotone operators from \( E \) into \( E \) and let \( \Omega_f \) be a set consisting of points of continuity of a function \( f \in E \cap \mathcal{F}_b(X) \). An inspection of the argument of Theorem 2 shows that \( T_n(f)(x) \to f(x) \) for every \( x \in \Omega_f \) provided that

\[
T_n(h)(x) \to h(x)
\]

for every \( x \in \Omega_f \) and every test function \( h \in \{1, \pm pr_1, \ldots, \pm pr_N, \sum_{k=1}^{N} pr_k^2 \} \).

This remark also works in the case of Corollary 1.

In connection with Remark 4, let us mention the following nonlinear generalization of a result due to Altomare. See [4], Section 2.

**Theorem 3.** Suppose that \( X \) is a subset of the Euclidean space \( \mathbb{R}^N \), \( \omega \) is point in \( X \) and \( E \) is a sublattice of \( \mathcal{F}(X) \) that contains the unit 1 and also the following set of test functions: \( 1, \pm pr_1, \ldots, \pm pr_N \) and \( \sum_{k=1}^{N} pr_k^2 \).
If \((T_n)_n\) is a sequence of sublinear and monotone functionals defined on \(E\) such that
\[
T_n(f)(x) \to f(\omega) \quad \text{for } x \in X,
\]
whenever \(f\) is one of the test functions \(1, \pm pr_1, \ldots, \pm pr_N\) and \(\sum_{k=1}^{N} pr_k^2\), then
\[
\lim_{n \to \infty} T_n(f) = f(\omega) \quad \text{for } x \in X
\]
for all nonnegative functions \(f \in \mathcal{F}(X)\), which are continuous at \(\omega\). The conclusion occurs for all functions \(f \in \mathcal{F}_b(X)\) continuous at \(\omega\) when the functionals \(T_n\) are weakly nonlinear and monotone.

The proof is similar to that of Theorem 2 and the details are left to the reader as an exercise.

4. The Case of Convergence in Measure and of Convergence in \(L^p\)-Norm

The convergence almost everywhere is related to other modes of convergence. Indeed, the convergence almost everywhere implies local convergence in measure, that is, \(f_n \to f\) a.e. implies
\[
\lim_{n \to \infty} \mu(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon \}) = 0
\]
for every \(\varepsilon > 0\) and every Borel set \(A\) with \(\mu(A) < \infty\). The converse fails, but if \(\mu\) is \(\sigma\)-finite, then \((f_n)_n\) converges to \(f\) locally in measure if and only if every subsequence has in turn a subsequence that converges to \(f\) almost everywhere.

In the next section, we will discuss the connection between the global convergence in measure and the convergence in \(p\)-mean. Recall that \(f_n \to f\) globally in measure if
\[
\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon \}) = 0
\]
for every \(\varepsilon > 0\). When \(\mu(X) < \infty\), the two types of convergence in measure coincide and we will refer to each of them as convergence in measure. In the same context, every sequence of measurable functions \(f_n\) that converges almost everywhere to a function \(f\) also converges to \(f\) in measure. The converse assertion is false: there exists a sequence of measurable functions on \([0, 1]\] that converges to zero in Lebesgue measure, but does not converge at any point at all. See [9], Theorem 2.2.3, p. 111 and Example 2.2.4, p.112.

The analog of Theorem 2 in the case of convergence in measure is as follows.

**Theorem 4.** Suppose that \(X\) is a compact subset of the Euclidean space \(\mathbb{R}^N\) endowed with a positive Borel measure \(\mu\) and let \((T_n)_n\) be a sequence of sublinear and monotone operators from \(C(X)\) into itself such that
\[
\lim_{n \to \infty} \mu(\{x \in X : |T_n(f) - f| \geq \varepsilon \}) = 0
\]
for each of the functions \(1, \pm pr_1, \ldots, \pm pr_N\) and \(\sum_{k=1}^{N} pr_k^2\) and each \(\varepsilon > 0\).
Then this convergence occurs for all nonnegative functions $f \in C(X)$. It occurs for all functions in $C(X)$ provided that the operators $T_n$ are weakly nonlinear and monotone.

Proof. Let $f \in C(X)$ be a nonnegative function and let $\varepsilon > 0$ arbitrarily fixed. Due to the uniform continuity of the function $f$, for every $\alpha \in (0, \varepsilon/2)$, there is $\delta > 0$ such that

$$|f(t) - f(x)| \leq \alpha + \delta \|t - x\|^2 \quad \text{for all } t, x \in X,$$

the reason by reductio ad absurdum. Proceeding as in the proof of Theorem 2, we infer that for all $x \in X$ and $n \in \mathbb{N}$, we have

$$|T_n(f) - f(x)| = |T_n(f) - T_n(f(x) \cdot 1) + f(x)T_n(1) - f(x)|$$

$$\leq T_n(|f - f(x)|) + f(x)|T_n(1) - 1|$$

$$\leq \alpha + \delta \left[ T_n \left( \sum_{k=1}^{N} \text{pr}_k^2 \right) + 2 \sum_{k=1}^{N} (M - \text{pr}_k(x))T_n(\text{pr}_k) \right]$$

$$+ 2M \sum_{k=1}^{N} T_n(-\text{pr}_k) + \|x\|^2 T_n(1) \right] + f(x)|T_n(1) - 1|,$$

where $M = \max \{\text{pr}_1(\omega), ..., \text{pr}_N(\omega), 0 : x \in X\}$.

Choose a rank $N$ such that $\|f\|_\infty |T_n(1) - 1| \leq \alpha$ for every $n \geq N$. Then for $n \geq N$, the set

$$\{x \in X : |T_n(f)(x) - f(x)| \geq \varepsilon\}$$

is included in the set of points $x \in X$, where

$$T_n \left( \sum_{k=1}^{N} \text{pr}_k^2 \right)(x) + 2 \sum_{k=1}^{N} (M - \text{pr}_k(x))T_n(\text{pr}_k)(x)$$

$$+ 2M \sum_{k=1}^{N} T_n(-\text{pr}_k)(x) + \|x\|^2 T_n(1)(x) \geq \frac{\varepsilon - 2\alpha}{\delta}.$$

This implies that

$$\mu(\{x \in X : |T_n(f) - f| \geq \varepsilon\})$$

$$\leq \mu \left( \left\{ x \in X : T_n(\sum_{k=1}^{N} \text{pr}_k^2) + \cdots \geq (\varepsilon - 2\alpha)/\delta \right\} \right)$$

for every $n \geq N$. Taking into account our hypothesis and the fact that the sum of sequences convergent in measure is also a sequence convergent in measure (see, Bogachev [9], Corollary 2.2.6, p. 113), we conclude the proof of the first part of Theorem 4.

For the second part, if $f \in C(X)$ is an arbitrary real-valued function, we will apply the preceding reasoning to $f + \|f\|_\infty \geq 0$ to infer that $T_n(f + \|f\|_\infty) \to f + \|f\|_\infty$ in measure. We have

$$T_n(f + \|f\|_\infty) = T_n(f) + \|f\|_\infty T_n(1),$$
because the operators $T_n$ are assumed to be weakly nonlinear and $T_n(1) \to 1$ by our hypotheses. The proof ends by using again the algebraic operations with sequences convergent in measure.

**Remark 5.** When $X$ is locally compact, then the statement of Theorem 4 still works if global convergence in measure is replaced by the following condition of local convergence,

$$\lim_{n \to \infty} \mu (\{ x \in K : |T_n(f) \to f| \geq \varepsilon \}) = 0$$

for every $\varepsilon > 0$ and every compact subset $K$ of $X$.

**Remark 6.** As in the case of Theorem 2, none of the three conditions imposed on the operators $T_n$ (sublinearity, monotonicity and translatability) is necessary for the fulfillment of Theorem 4. See the example offered by Remark 2.

We continue this section by considering the case of convergence in $L^p$-norm. The existing literature includes many papers containing various generalization of Korovkin’s theorem in the context of operators acting on $L^p$-spaces. See Altomare [3], Altomare and Campiti [2], Berens and DeVore [7, 8], Donner [13, 14], Swetits and Wood [28], and Wulbert [30] to cite just a few. However, all of them refer to the case of linear and positive operators. The next theorem provides nonlinear generalization bases on sequences of weakly nonlinear and monotone operators.

**Theorem 5.** Let $\mu$ be a positive Borel measure on $\mathbb{R}^N$ with compact support and let $(T_n)_n$ be a sequence of sublinear and monotone operators from the Banach lattice $L^p(\mu)$ into itself, where $p \in [1, \infty)$. If $M = \sup_n \|T_n\| < \infty$ and

$$T_n(f) \to f \quad \text{in } p\text{-mean}$$

for each of the test functions $1, \pm pr_1, \ldots, \pm pr_N$ and $\sum_{k=1}^N pr_k^2$, then this convergence occurs for all nonnegative functions $f \in L^p(\mu)$. It occurs for all functions in $L^p(\mu)$ provided that the operators $T_n$ are weakly nonlinear and monotone.

**Proof.** Let $f \in L^p(\mu)$, $f \geq 0$. Since $C_c(\mathbb{R}^N)$ is dense into $L^p(\mu)$ in the $L^p$-norm (see [9], Corollary 4.2.2, p. 252) and the lattice operations in a Banach lattice are continuous ([24], Proposition 1.1.6, p. 6), it follows that for every $\varepsilon > 0$ there exists a nonnegative continuous function $g$ with compact support $\text{supp } g$ such that

$$\|f - g\|_{L^p} < \varepsilon.$$

Then,

$$\|T_n(f) - f\|_{L^p} \leq \|T_n(f) - T_n(g)\|_{L^p} + \|T_n(g) - gT_n(1)\|_{L^p}$$

$$+ \|gT_n(1) - g\|_{L^p} + \|g - f\|_{L^p}$$

$$\leq \|T_n\| \|f - g\|_{L^p} + \|T_n(g) - gT_n(1)\|_{L^p}$$

$$+ \|g\|_{\infty} \|T_n(1) - 1\|_{L^p} + \|f - g\|_{L^p}$$

$$\leq \varepsilon(1 + M) + \|T_n(|g - g(x)|)(x)\|_{L^p} + \|g\|_{\infty} \|T_n(1) - 1\|_{L^p}.$$
Using the uniform continuity of $g$, one can infer (by reductio ad absurdum) that there exists a number $\delta \geq \|g\|_{\infty} + 1$ such that

$$|g(y) - g(x)| \leq \varepsilon + \delta \cdot \|y - x\|^2$$

for all $x$ and $y$ in an open and bounded neighborhood of the support of $g$, say \{ $z : d(z, \text{supp} g < 1$} and therefore for all $x$ and $y$ in $\mathbb{R}^N$. Here, $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^N$.

According to our hypotheses, there exists a rank $n_0$ such that

$$\|T_n(h) - h\|_{L^p} < \varepsilon/\delta$$

for all $n \geq n_0$ and all $h \in \{1, \pm p_1, \ldots, \pm p_N, \sum_{k=1}^N p_k^2\}$. Put

$$\alpha = \max_{x \in K} \{p_1(x), \ldots, p_N(x), 0\},$$

where $K$ denotes the support of $\mu$. We have

$$\|T_n(|g - g(x)|)(y)\|_{L^p} \leq \|T_n(\varepsilon)(x) + \delta T_n(\|y - x\|^2)(x)\|_{L^p} \leq \varepsilon \|T_n(1)\|_{L^p} + \delta \|T_n(\|y - x\|^2)(x)\|_{L^p} \leq \varepsilon (\varepsilon + \mu(K)) + \delta \|T_n(\|y - x\|^2)(x)\|_{L^p} \leq \varepsilon (\varepsilon + \mu(K)) + \delta \|T_n(\|y - x\|^2)(x)\|_{L^p}.$$ 

Since

$$\|y - x\|^2 = \sum_{k=1}^N p_k^2(y) + 2 \sum_{k=1}^N \sum_{k=1}^N (\alpha - p_k(x))p_k(y) + \sum_{k=1}^N p_k^2(x) + 2\alpha \sum_{k=1}^N (-p_k(y)),$$

it follows that

$$T_n(\|y - x\|^2)(x) \leq T_n \left( \sum_{k=1}^N p_k^2 \right)(x) + 2 \sum_{k=1}^N (\alpha - p_k(x))T_n(p_k)(x) + 2\alpha \sum_{k=1}^N T_n(-p_k)(x) + \sum_{k=1}^N p_k^2(x)T_n(1)(x)$$

$$= T_n \left( \sum_{k=1}^N p_k^2 \right)(x) - \sum_{k=1}^N p_k^2 + 2 \sum_{k=1}^N (\alpha - p_k(x))[T_n(p_k)(x) + T_n(-p_k)(x)]$$

$$+ 2 \sum_{k=1}^N p_k(x)[T_n(-p_k)(x) + p_k(x)] - \sum_{k=1}^N p_k^2(x) + \sum_{k=1}^N p_k^2(x)T_n(1)(x),$$
whence
\[
\delta \| T_n(\|y - x\|^2(x))\|_{L^p} \leq \delta \| T_n \left( \sum_{k=1}^{N} p r_k^2 \right)(x) - \sum_{k=1}^{N} pr_k^2(x)\|_{L^p}
\]
\[
+ 4\alpha \delta \sum_{k=1}^{N} \| T_n(-pr_k)(x) + pr_k(x)\|_{L^p}
\]
\[
+ 4\alpha \delta \sum_{k=1}^{N} \| T_n(pr_k)(x) - pr_k(x)\|_{L^p}
\]
\[
+ 2\alpha \delta \sum_{k=1}^{N} \| T_n(-pr_k)(x) + pr_k(x)\|_p
\]
\[
+ \alpha^2 \delta \sum_{k=1}^{N} \| T_n(1)(x) - 1\|_{L^p}
\]
\[
\leq \varepsilon (1 + 10\alpha + \alpha^2)
\]
for \(n\) sufficiently large. This ends the proof in the case where \(f \geq 0\).

In the general case, notice first that \(T_n(f) \to f\) in \(p\)-mean for every \(f \in L^p(\mu)\) with \(\|f\|_\infty < \infty\). Indeed, \(T_n(f + \|f\|_\infty) \to f + \|f\|_\infty\) by the discussion above and \(T_n(f + \|f\|_\infty) = T_n(f) + \|f\|_\infty T_n(1)\) due to the fact that the operators \(T_n\) are weakly nonlinear.

If \(f\) is an arbitrary function in \(L^p(\mu)\), then it can be approximated by step functions. Every step function \(h\) is bounded in the sup-norm, so that \(\|T_n(h) - h\|_{L^p} \to 0\). Finally, the inequality
\[
\|T_n(f) - f\|_{L^p} \leq \|T_n(f) - T_n(h)\|_{L^p} + \|T_n(h) - h\|_{L^p} + \|h - f\|_{L^p}
\]
\[
\leq M \|f - h\|_{L^p} + \|T_n(h) - h\|_{L^p} + \|h - f\|_{L^p}
\]
allows us to conclude that \(\|T_n(f) - f\|_{L^p} \to 0\). \(\square\)

It is conceivable that Theorem 5 still works in the case of an arbitrary finite Borel measure on \(\mathbb{R}^N\), but at the moment we lack a valid argument.

In the setting of linear and continuous operators acting on \(L^1[0, 1]\), Wulbert [30] has proved a Korovkin-type theorem that avoids the hypothesis of monotonicity. It is an open question whether his result admits an analog within the framework of continuous sublinear operators \(T_n : L^1[0, 1] \to L^1[0, 1]\).

5. An Application to Lebesgue Differentiation Theorem

The Lebesgue differentiation theorem is an important result in real analysis that can be stated as follows.

**Theorem 6.** Let \(f \in L^1(\Omega)\) be the Lebesgue space associated with an open subset \(\Omega\) of \(\mathbb{R}^N\). Consider any collection \(\mathcal{R}\) of closed \(N\)-dimensional intervals with sides parallel to the axes, with non-empty interior, centered at the origin 0 and containing sequences \((R_n)_n\) contracting to 0 as \(\text{diam}R_n \to 0\). Assume furthermore that the intervals in \(\mathcal{R}\) are comparable, i.e., for any two \(R_i, R_j\)
$R_j \in \mathcal{R}$ either $R_i \subset R_j$ or $R_j \subset R_i$. (Example: the collection of all $N$-dimensional closed cubic intervals centered at $0$.)

Then, for almost every $x$ and for every $(R_n)_n \subset \mathcal{R}$ with $\text{diam} R_n \to 0$ one has

$$
\lim_{n \to \infty} \frac{1}{\text{vol}(R_n)} \int_{R_n} f(x-y)dy = f(x) \quad \text{a.e.}
$$

A simple proof of this theorem can be found in a paper by de Guzman and Rubio [12]. In what follows, we present an alternative argument based on our Theorem 2.

Indeed, we deal here with the sequence of linear and positive operators $T_n : L^1(\Omega) \to L^1(\Omega)$ defined by the formula:

$$
T_n f(x) = \frac{1}{\text{vol}(R_n)} \int_{R_n} f(x-y)dy.
$$

Clearly, these operators are weakly nonlinear. Theorem 2 applies to those functions $f$ which are bounded in the sup-norm, but this can be easily arranged by approximating an arbitrary $f \in L^1(\Omega)$ by step functions $f_\varepsilon$ and noticing that

$$
\int_{R_n} \left| \frac{1}{\text{vol}(R_n)} \int_{R_n} f(x-y)dy - \frac{1}{\text{vol}(R_n)} \int_{R_n} f_\varepsilon(x-y)dy \right| dx \\
\leq \int_{R_n} |f - f_\varepsilon| dx < \varepsilon
$$

for all $n$. Taking into account that every sequence converging in mean has a subsequence converging almost everywhere, we may reduce the proof of Theorem 6 to the case where $f$ vanishes outside a compact set and $\sup_x |f(x)| < \infty$. Under these circumstances, the verification of the property of a.e. convergence for the test functions $1, \text{pr}_1, \ldots, \text{pr}_N$ and $\sum_{k=1}^{N} \text{pr}_{2k}$ is a simple exercise. For example, in the one-dimensional case, one has to observe that

$$
\frac{1}{R_n - r_n} \int_{r_n}^{R_n} dy = 1
$$

$$
\frac{1}{R_n - r_n} \int_{r_n}^{R_n} (x-y)dy = \frac{(x-r_n)^2 - (x-R_n)^2}{2(R_n - r_n)} = x - \frac{R_n + r_n}{2} \to x
$$

$$
\frac{1}{R_n - r_n} \int_{r_n}^{R_n} (x-y)^2dy = \frac{(x-r_n)^3 - (x-R_n)^3}{3(R_n - r_n)} \\
= x^2 - x(R_n + r_n) + \frac{R_n^2 + R_nr_n + r_n^2}{3} \to x^2
$$

for every $x \in \Omega$ and every sequence of intervals $[r_n, R_n]$ with $r_n < 0 < R_n$ that contracts to the origin.

Remark 7. The last computations allow us to construct counterexamples showing that the convergences asserted by Theorems 2, 4 and 5 fail for functions of variable sign in the absence of the condition of translatability. See the
case of sublinear and monotone operators $T_n : L^1(0, 1) \to L^1(0, 1)$ defined by

$$T_n f(x) = \frac{1}{R_n - r_n} \max \left\{ \int_{r_n}^{R_n} f(x - y)dy, 0 \right\}.$$ 

6. The Convergence of Some Sequences of Bernstein-Type Operators

In this section, we present some concrete examples illustrating the above results in the context of Choquet’s nonlinear integral. The necessary background on this integral is covered by the Appendix at the end of our paper [18].

We start by considering the Bernstein–Kantorovich–Choquet polynomial operators for functions of one real variable, $K_{n,\mu}^{(1)} : \mathcal{R}([0, 1]) \to \mathcal{R}([0, 1])$, defined by the formula

$$K_{n,\mu}^{(1)}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot (C) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{[k/(n+1), (k+1)/(n+1)]},$$

where the symbol $(C)$ in front of the integral means that we deal with a Choquet integral (with respect to the capacity $\mu$). As usually in approximation theory,

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad \text{for } t \in [0, 1] \text{ and } n \in \mathbb{N}.$$ 

Due to the properties of Choquet’s integral, when $\mu$ is a submodular capacity, it follows that each operator $K_{n,\mu}^{(2)}$ is weakly nonlinear, monotone and unital from $\mathcal{R}([0, 1])$ into itself. This happens in particular when $\mu$ is the Lebesgue measure $\mathcal{L}$, in which case the Choquet integral reduces to Lebesgue integral and the Bernstein–Kantorovich–Choquet polynomial operators coincide with the Bernstein–Kantorovich polynomial operators:

$$K_{n}^{(1)}(f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \cdot \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt,$$

which act also on $\mathcal{R}([0, 1])$.

It is known that $K_{n}^{(1)}(x^k) \to x^k$ uniformly on $[0, 1]$ for $k \in \{0, 1, 2\}$, so from Remark 4 we infer the following result previously noticed by Lorentz [23], Theorem 2.1.1, p. 30:

**Theorem 7.** $K_{n}^{(1)}(f)(x) \to f(x)$ at each point of continuity of $f \in \mathcal{R}([0, 1])$ (and thus $K_{n}^{(1)}(f)(x) \to f(x)$ a.e. on $[0, 1]$).

Theorem 7 extends verbatim to the case of tensor product multivariate Bernstein–Kantorovich polynomial operators, by using the general Theorem in Haussmann–Pottinger [19], page 213.
A similar result works for the Bernstein–Kantorovich–Choquet polynomial operators $K_{n,\sqrt{\mathcal{L}}}(1)$ associated with the submodular capacity $\mu = \sqrt{\mathcal{L}} : A \rightarrow \sqrt{\mathcal{L}(A)}$ for $A \in \mathcal{B}([0,1])$. Indeed, as we noticed in [15], Section 3, $K_{n,\sqrt{\mathcal{L}}}(1)(-x) \rightarrow -x$ and $K_{n,\sqrt{\mathcal{L}}}(x^k) \rightarrow x^k$ uniformly on $[0,1]$ for $k \in \{0,1,2\}$.

Remark 8. According to Remark 4, it follows that $K_{n,\sqrt{\mathcal{L}}}(f)(x) \rightarrow f(x)$ at each point $x$ of continuity of $f \in \mathcal{R}([0,1])$.

The Bernstein–Kantorovich–Choquet polynomial operators for functions of two real variables are defined by the formula:

$$K_{n,\sqrt{\mathcal{L}}}(2)(f)(x_1,x_2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} p_{n,k_1}(x_1)p_{n,k_2}(x_2) \cdot \left( (C) \int_{k_1/(n+1)}^{(k_1+1)/(n+1)} (C) \int_{k_2/(n+1)}^{(k_2+1)/(n+1)} f(t_1,t_2)d\mu(t_2) \right) d\mu(t_1)$$

and they are weakly linear, monotone and unital operators from $\mathcal{R}([0,1]^2)$ provided that the capacity $\mu$ is submodular.

Remark 9. Since $K_{n,\sqrt{\mathcal{L}}}(2)(f)(x_1,x_2) \rightarrow f(x_1,x_2)$ uniformly on $[0,1]^2$, for each of the test functions $1$, $\pm pr_1$, $\pm pr_2$, $pr_1^2+pr_2^2$, it follows from Remark 4 that $K_{n,\sqrt{\mathcal{L}},\mu}(2)(f)(x_1,x_2) \rightarrow f(x_1,x_2)$ at each point of continuity of the function $f \in \mathcal{R}([0,1]^2)$. Here, $\mathcal{L}$ is the planar Lebesgue measure.

Consider now the case of the Szász–Mirakjan–Kantorovich operators, acting on the space $\mathcal{R}_{loc,b}([0,\infty))$ of all functions bounded on $[0,+,\infty)$ and Riemann integrable on each compact subinterval, by the formula

$$S_n(f)(x) = (n+1)e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(t)dx.$$  

It is known that $S_n(x^k) \rightarrow x^k$ pointwise for $x \in [0,\infty)$, whenever $k \in \{0,1,2\}$. See Walczak [29]. Taking into account Remark 4, we obtain the following result that seems to be new:

**Theorem 8.** If $f \in \mathcal{R}_{loc,b}([0,\infty))$, then $S_n(f)(x) \rightarrow f(x)$ at each point of continuity of $f$ (and thus it converges everywhere on $[0,+,\infty)$).
Remark 10. A Choquet companion to the Szász–Mirakjan–Kantorovich operators is provided by the following sequence of operators:

\[ S_n,\sqrt{L}(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(C) \int_{k/n}^{(k+1)/n} f(t) d\sqrt{L}}{\mu([k/n, (k+1)/n])} \cdot \frac{(nx)^k}{k!}, \quad f \in \mathcal{R}_{loc,b}([0,\infty)), \]

where the integration is performed with respect to the capacity \( \sqrt{L} \). According to [15],

\[ S_n,\sqrt{L}(h)(x) \rightarrow h(x), \]

pointwise for all points \( x \in [0, +\infty) \) and test functions \( h \in \{1, x, -x, x^2\} \). Therefore, \( S_n,\sqrt{L}(f)(x) \rightarrow h(x), \) pointwise at each point of continuity of \( f \in \mathcal{R}_{loc,b}([0,\infty)) \).

7. Further Results and Comments

As was noticed by Korovkin [20,21], his theorem mentioned in the Introduction also works when the unit interval is replaced by the unit circle \( S^1 = \{(\cos \theta, \sin \theta) : \theta \in [0, 2\pi)\} \subset \mathbb{R}^2 \).

Our results show that more is true:

**Theorem 9.** Let \( E \) be a linear subspace of continuous real-valued functions defined on the unit circle that contains the test functions \( 1, \cos, -\cos, \sin \) and \( -\sin \). If \( (T_n)_n \) is a sequence of weakly nonlinear and monotone operators which carry \( E \) into the space \( C(S^1) \) such that \( T_n(f) \rightarrow f \) converges almost everywhere (respectively, in measure or in \( p \)-mean), for each test function, then the same mode of convergence holds for every \( f \in E \).

The proof follows from Theorems 1–3, by remarking that the cosine function can be seen as the restriction of \( pr_1 \) to \( S^1 \), and the sine function as the restriction of \( pr_2 \) to \( S^1 \).

We leave to the reader the extension of Theorem 9 in some other cases of interest such as the torus \( S^1 \times S^1 \) and the 2-dimensional sphere \( S^2 \).

**Remark 11.** (Mixing metric spaces and spherical domains) Suppose that \( X \) is a compact subset of \( \mathbb{R}^N \). Then the product space \( X \times S^1 \) is a compact subset of \( \mathbb{R}^{N+1} \) and the space \( C(X \times S^1) \) can be identified with the Banach space \( C_{2\pi}(X \times \mathbb{R}) \), of all continuous functions \( f : X \times \mathbb{R} \rightarrow \mathbb{R}, 2\pi \)-periodic in the second variable, endowed with the sup-norm.

The reader can easily check that our Theorems 1-3 extend to the case of sequences of weakly nonlinear operators and monotone operators \( T : C(X \times S^1) \rightarrow C(X \times S^1) \) and the following set of test functions \( f(x) = u(x)v(\varphi) \), where

\[ u \in \left\{ 1, \pm pr_1, \ldots, \pm pr_N \text{ and } \sum_{k=1}^{N} pr_k^2 \right\} \quad \text{and} \quad v \in \{1, \pm \cos \varphi, \pm \sin \varphi \}. \]

The case of uniform convergence has been noticed in [18] (extending the case of positive linear operators settled in [27]).
All our theorems remain valid in the context of Cesàro convergence. If \((T_n)_n\) is a sequence of weakly nonlinear and monotone operators (from a Banach lattice of functions \(E\) into itself), then so is the sequence \(\left(\frac{1}{n} \sum_{k=1}^{n} T_k\right)_n\). As a consequence, adding the conditions imposed by Theorems 1-3, we infer that
\[
\frac{1}{n} \sum_{k=1}^{n} T_k(f) \rightarrow f
\]
almost everywhere, (respectively, in measure or in \(p\)-mean) for every \(f \in E\) whenever it happens for the functions \(1, \pm pr_1, \ldots, \pm pr_N, \sum_{k=1}^{N} pr_k^2\).

Last but not the least, one can extend our results (following the model of Theorem 2 in [18]) using instead of the classical families of test functions on \(\mathbb{R}^N\) the separating functions, which allow us to replace the critical inequalities of the form (3.1) by inequalities that work in the general context of metric spaces. For details concerning these functions, see [25].

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References
[1] Abramovich, Y.A., Aliprantis, C.D.: Positive Operators. Handbook of the Geometry of Banach spaces, I, pp. 85–122. North-Holland, Amsterdam (2001)
[2] Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications. de Gruyter Studies in Mathematics, 17, Berlin (1994, reprinted 2011)
[3] Altomare, F.: Korovkin-type theorems and positive operators. Surv. Approx. Theory 6, 92–164 (2010)
[4] Altomare, F.: On positive linear functionals and operators associated with generalized means. J. Math. Anal. Appl. 502, 125278 (2021)
[5] Altomare, F.: On the convergence of sequences of positive linear operators and functionals on bounded function spaces. Proc. Am. Math. Soc. 149, 3837–3848 (2021)
[6] Bennett, C., Sharpley, R.: Interpolation of operators, Pure and Applied Mathematics, vol. 129. Academic Press Inc, Boston (1988)
[7] Berens, H., DeVore, R.A.: Quantitative Korovkin theorems for \(L^p\)-spaces. In: Lorentz, G.G. (ed.) Approximation Theory II, pp. 289–298. Academic Press, New York (1976)
[8] Berens, H., DeVore, R.A.: Quantitative Korovkin theorems for positive linear operators in $L^p$ space. Trans. Am. Math. Soc. 245, 349–361 (1978)
[9] Bogachev, V.: Measure Theory, vol. I. Springer, Berlin (2007)
[10] Choudary, A.D.R., Niculescu, C.P.: Real Analysis on Intervals. Springer, New Delhi (2014)
[11] Crandall, M.G., Tartar, L.: Some relations between nonexpansive and order preserving maps. Proc. Am. Math. Soc. 78, 385–390 (1980)
[12] de Guzmán, M., Rubio, B.: Remarks on the Lebesgue differentiation theorem, the Vitali lemma and the Lebesgue–Radon–Nikodym theorem. Am. Math. Mon. 79, 341–348 (1972)
[13] Donner, K.: Korovkin’s theorem in $L^p$-spaces. J. Funct. Anal. 42, 12–28 (1981)
[14] Donner, K.: Extension of positive operators and Korovkin theorems. Springer, Berlin (2006)
[15] Gal, S.G., Niculescu, C.P.: A nonlinear extension of Korovkin’s theorem. Mediterr. J. Math. 17, 145 (2020)
[16] Gal, S.G., Niculescu, C.P.: Choquet operators associated to vector capacities. J. Math. Anal. Appl. 500, 125153 (2021)
[17] Gal, S.G., Niculescu, C.P.: A note on the Choquet type operators. Aequationes Math. 95, 433–447 (2021)
[18] Gal, S.G., Niculescu, C.P.: Nonlinear versions of Korovkin’s abstract theorems. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. 116, Article no. 68 (2022)
[19] Haussmann, W., Pottinger, P.: On the construction and convergence of multivariate interpolation operators. J. Approx. Theory. 19, 105–221 (1977)
[20] Korovkin, P.P.: On convergence of linear positive operators in the space of continuous functions (Russian). Doklady Akad. Nauk. SSSR. (NS) 90, 961–964 (1953)
[21] Korovkin, P.P.: Linear Operators and Approximation Theory. Fitzmatgiz, Moscow (1959) (in Russian) [English translation, Hindustan Publ. Corp., Delhi (1960)]
[22] Krein, M.G.: Fundamental properties of normal conical sets in a Banach space. Dokl. Akad. Nauk USSR 28, 13–17 (1940)
[23] Lorentz, G.G.: Bernstein Polynomials. The Chelsea Publishing Company, New York (1986)
[24] Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin (1991)
[25] Niculescu, C.P.: An overview of absolute continuity and its applications. Internat. Ser. Numer. Math. 157, 201–214. Birkhäuser, Basel (2009)
[26] Niculescu, C.P., Olteanu, O.: From the Hahn–Banach extension theorem to the isonoticity of convex functions and the majorization theory. Revista de la Real Academia de Ciencias Exactas Físicas y Naturales. Serie A Matemáticas. 114, 1–19 (2020)
[27] Popa, D.: Korovkin-type results for multivariate functions which are periodic with respect to one variable. Math. Nachr. 291, 1563–1573 (2018)
[28] Swetits, J.J., Wood, B.: Quantitative estimates for $L^p$ approximation with positive linear operators. J. Approx. Theory 38, 81–89 (1983)
[29] Walczak, Z.: On approximation by modified Szasz–Mirakyan operators. Glasnik Matematicki. 37, 303–319 (2002)
[30] Wulbert, D.E.: Convergence of operators and Korovkin’s theorem. J. Approx. Theory 1, 381–390 (1968)
[31] Zorich, V.A.: Mathematical Analysis II. Universitext. Springer, Berlin (2004)

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