Generalized Clifford-Severi Inequality and
the Volume of Irregular Varieties

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Abstract

We give a sharp lower bound for the selfintersection of a nef line bundle $L$ on an irregular variety $X$ in terms of its continuous global sections and the Albanese dimension of $X$, which we call the Generalized Clifford-Severi inequality. We also extend the result to nef vector bundles and give a slope inequality for fibred irregular varieties. As a byproduct we obtain a lower bound for the volume of irregular varieties; when $X$ is of maximal Albanese dimension the bound is $\text{vol}(X) \geq 2^n! \omega_X$ and it is sharp.

1 Introduction and preliminaries

Geometry of irregular varieties has been deeply developed in the last 30 years. The seminal results of Green and Lazarsfeld on Generic Vanishing theorems, the generalized Castelnuovo-de Franchis theorem by Catanese and Ran, the systematic use of the Fourier-Mukai techniques associated to the Albanese map and further developments by Lazarsfeld, Ein, Hacon, Chen, Pareschi, Popa and many others have provided a rather complete understanding of birational properties of such varieties.

Another fruitful approach to understand the geometry of an irregular variety is the study of their continuous linear series. See for example [5] for one of the first instances of this approach in the case of surfaces. The recent work of Mendes-Lopes, Pardini and Pirola on Brill-Noether theory and continuous families of divisors produces a deep understanding of their geometry in higher dimensions (see [15], [16], and [14]).

In the study of biregular geometry of varieties, inequalities relating the degree and the number of global sections of a line bundle play a special role. Our main result is the following (see Theorem 4.1):

Main Theorem (Generalized Clifford-Severi Inequality)

Let $X$ be a smooth, projective variety of dimension $n$, over an algebraically closed field of characteristic 0. Let $a : X \to A$ be a generating map to an Abelian variety and let $L \in \text{Pic}(X)$ be a nef line bundle.

(i) If $X$ is of maximal $a$-dimension then

$$L^n \geq \delta(L) n! h^0_a(L).$$

In particular, if $L \leq K_X$, then $L^n \geq 2n! h^0_a(L)$.

(ii) Assume $n > \dim a(X) = k \geq 1$, let $G$ be a general fibre of the algebraic fibre space induced by $a$ and let $M$ be the continuous moving part of $L$. If $M|_G$ is big then

$$L^n \geq \delta(L) k! h^0_a(L).$$

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(iii) Assume that $n > \dim a(X) = k \geq 1$ and that $L_{|\mathcal{O}}$ is big. Then

$$L^n \geq k! h^0_a(L).$$

Here $\delta(L)$ is a real number between 2 and 1 which depends on the degree of subcanonicity of $L$ (the minimal $r$ such that $L \geq rK_X$) and $h^0_a(L)$ is the continuous rank of $L$, i.e., the minimal value of $h^0(X, L \otimes \alpha)$ for $\alpha \in \hat{A}$.

We can see this theorem as a wide generalization of the classical Severi inequality for surfaces of maximal Albanese dimension:

$$K_S^2 \geq 4 \chi \omega_S.$$  

This inequality was stated by Severi in the 30's ([24]). Many years later, Catanese ([4]) showed a gap in the proof and proposed the inequality as a conjecture. Manetti gave a proof of the conjecture in the case $K_S$ ample, together with a profound analysis of the positivity properties of $\Omega^1_S$ ([13]). His approach provides further developments and refinements (see [17] and [26]).

The key argument for a complete proof of the Severi inequality without extra hypotheses is given by Pardini ([20], [14]). She deduces the inequality from another well known one: the slope inequality for fibred surfaces ([25], [8]), by using in a quick and clever way the property of being of maximal Albanese dimension. This method, which we call Pardini’s covering trick is completely general, and allows to apply a general philosophy: given an inequality verified by any maximal Albanese dimension variety you can remove all the numerical data involving lower dimensional subvarieties and obtain a new inequality.

In the present paper we generalize the Severi inequality by an induction argument on the dimension of $X$. For this, we combine three different ingredients:

- A suitable version of Xiao’s method for fibrations reducing the problem to étale covers and lower dimensional varieties (induction step). This is done in subsection 5.1.
- The analysis of the behavior of continuous linear series on $X$, i.e., $|L \otimes \alpha|$ for $\alpha \in \hat{A}$. It allows to assume good behavior of the linear system on an étale covering of $X$. This is done in Section 3.
- The use of Pardini’s covering trick ([20]) to remove unnecessary invariants. This is done in subsection 5.2.

The initial step of the induction process (the case of curves) is just a continuous version of Clifford’s Lemma which we can also consider as a 1-dimensional version of Severi inequality. That’s the reason why we add Clifford in the name of the inequality.

Most results can be extended to Weil divisors on normal $\mathbb{Q}$-Gorenstein varieties just understanding the behavior under the desingularization process. Some of the techniques are presented in general for Weil divisors, as Xiao’s method in Subsection 5.1, and in Corollary 4.5 we reinterpret the Generalized Clifford-Severi inequality as a lower bound for the volume of an irregular variety. The (canonical) volume of a general type variety is an interesting invariant of its birational class. In general it can be very small and is not easy to find higher lower bounds (see for example [6]). In the presence of global differential forms it was conjectured by Reid that the value of $\text{vol}(X)$ should have a high lower bound. As far as we know even in the case of irregular 3-folds there are few known results (see for example [7], section 3). The bound we give here confirms Reid’s conjecture for 1-forms. The best bound is obtained in the case of maximal Albanese dimension varieties of general type, for which it is sharp (double covers of abelian varieties):

$$\text{vol}(X) \geq 2n! \chi(\omega_X).$$

During the final preparation of this work, the author has been informed that the content of Corollary B (i), in the case of minimal Gorenstein varieties, was independently proven by Zhang ([27]). There the strategy...
of proof relies on applying Pardini’s trick to the higher dimensional version of the so called Relative Noether inequality, bounding the linear sections of a nef line bundle on an fibred variety.

There are several particular cases of the main result of independent interest. The first one is an extension to vector bundles

Corollary A (Generalized Clifford-Severi inequality for nef Vector Bundles) Let $X$ be a projective, smooth variety of dimension $n$, over an algebraically closed field of characteristic 0. Let $a: X \to A$ be a generating map to an Abelian variety such that $\dim a(X) = k$ and let $G$ be a general fibre of the induced algebraic fibre space. Let $\mathcal{F}$ be a nef vector bundle on $X$ with top Segre class $s(\mathcal{F})$. Assume that $k = n$ or that $\mathcal{F}|_G$ is big. Then

$$s(\mathcal{F}) \geq k! h^0_a(\mathcal{F}).$$

When $L = K_X$ we obtain the sharp generalization of the classical Severi inequality or, equivalently, the aforementioned lower bound for the volume of irregular varieties.

Corollary B (Generalized Severi inequality for $K_X$ or the volume of irregular varieties) Let $X$ be an irregular, minimal, projective variety of dimension $n$ over an algebraically closed field of characteristic 0, with at most canonical singularities. Let $X'$ be any desingularization. Then

(i) If $X$ is of maximal Albanese dimension then

$$\text{vol}(X') = K^n_X \geq 2n! \chi \omega_X$$

and this bound is sharp (double covers of abelian varieties).

(ii) If $\dim \text{alb}_X(X) = k < n$, $X$ is of general type and the continuous base locus $W$ of $\omega_X$ verifies $\text{kod}(W|_G) \leq 0$, for a general fibre $G$ of the induced Albanese fibre space, then

$$\text{vol}(X') = K^n_X \geq 2k! h^0_{\text{alb}_X}(\omega_X)$$

(iii) If $\dim \text{alb}_X(X) = k < n$ and $X$ is of general type, then

$$\text{vol}(X') = K^n_X \geq k! h^0_{\text{alb}_X}(\omega_X)$$

As another instance of the Main Theorem we have

Corollary C (Clifford-Severi inequality for adjoint divisors) Assume that $X$ is a $n$-dimensional variety of maximal Albanese dimension.

(i) If $L$ is a GV-sheaf (for example, if $L = K_X + D$, with $D$ nef), then $L^n \geq \delta(L) n! \chi(X, L)$.

(ii) If $L = K_X + D$, with $D$ big and nef, then $L^n \geq \delta(L) n! h^0(X, L)$.

For the classification of irregular surfaces with small invariants the following is an interesting result

Corollary D (Decomposable $K_S$ on surfaces) Let $S$ be a smooth surface of maximal Albanese dimension and such that $K_S \equiv L_1 + L_2$ (numerically), with $L_i$ nef line bundles. Then

$$K^2_S \geq 4\chi \omega_S + 4h^1_\omega(L_1).$$
From the existence of a higher dimensional slope inequality-type we could deduce a Severi inequality for the canonical sheaf (see [2], section 5). It turns out that from the Generalized Clifford-Severi inequality we obtain a slope inequality just considering $L = \omega_f$ in the main theorem:

**Corollary E (Slope inequality)** Let $f : X \longrightarrow B$ be a relatively minimal fibration onto a smooth curve $B$ of genus $b$ with general fibre $F$. Assume that $X$ is of maximal Albanese dimension. Then

(i) If $b = 0$ then $K_f^b \geq 2n! \chi_f$.

(ii) If $b \geq 1$, then $K_f^b \geq 2n! [\chi(\omega_f) + h^1(\omega_f)] \geq 2n! (\chi_X - 2\chi_B \chi_F)$.

The paper is divided as follows. The general set-up is in section 2. In section 3 we obtain a slope inequality just considering $\omega_f$ (see Definition 2.9 and Remark 2.10), in such a way that $\delta(L) = 2$ if and only if $L$ is numerically subcanonical ($r = 1$). In section 4 we study the behavior of linear systems under étale Galois covers. Here, the concepts of continuous fixed part and continuous moving part of $L$ play an special role. This analysis is a cornerstone in the proof of main theorem and provides results of independent interest on linear series on irregular varieties. It turns out that (see Theorems 3.2 and 3.5 for more complete results):

**Theorem F** Up to composing a blow-up with an étale Galois covering, $\lambda : \tilde{X} \longrightarrow X$

(i) For any $\alpha \in \tilde{A}$ we have a decomposition: $\lambda^*(L \otimes \alpha) = \tilde{W} + \tilde{N}_\alpha$ where the divisor $\tilde{W}$ is the fixed component (and does not depend on $\alpha$) and the moving part $\tilde{N}_\alpha$ is base point free.

(ii) The map $a \circ \lambda$ factorizes through the algebraic fibre space induced by the linear system $|\lambda^*L|$. In particular $\dim \phi_{\lambda^*L}(\tilde{X}) \geq \dim a(X)$ and so $|\lambda^*L|$ is generically finite provided $X$ is of maximal $a$-dimension.

Section 4 is devoted to state the main result, and to prove several versions and corollaries. In Section 5 we prove the Main Theorem. There, an account of Xiao’s method especially adapted to étale Galois covers of fibrations onto $\mathbb{P}^1$ is given.

**Notations and conventions** Varieties are assumed to be smooth, projective, defined over an algebraically closed field $k$ of characteristic 0, except otherwise stated. We use the notation $L$ for a (Cartier) divisor or its associated line bundle interchangeably, except for the canonical sheaf and divisor which will be denoted by $\omega_X$ and $K_X$ respectively. We use additive or multiplicative notation interchangeably. Given an abelian variety $A$ we denote by $\hat{A} = \text{Pic}^0(A)$ its dual abelian variety, by $\hat{A}_d$ the subgroup of its $d$-torsion elements and by $\hat{A}_\text{tors} = \bigcup_{d \in \mathbb{N}} \hat{A}_d$ the set of all its torsion elements. Given an irregular variety $X$ we set $\text{Pic}^r(X)$ for the set of numerically torsion line bundles on $X$, i.e., the set of $M$ such that $M^{\otimes r} \in \text{Pic}^0(X)$ for some $r \in \mathbb{N}$. Maps $a : X \longrightarrow A$, where $A$ is an Abelian variety are assumed to be generating, i.e., that $a(X)$ generates $A$ as a group. Given a map $f : X \longrightarrow Y$ (not necessarily surjective) we say that it is generically finite if $X \longrightarrow f(X)$ is. We say that $f$ factorizes through an algebraic fiber space of dimension $k$ with general fibre $G$ if $\dim f(X) = k$ and so its Stein factorization decomposes $f = g \circ h : X \longrightarrow Z \longrightarrow Y$ where $Z$ is normal of dimension $k$, $g$ is an algebraic fiber space and $G$ is a general fibre of $g$. As usual, an algebraic fiber space of
dimension 1 will be called a \textit{fibration}. We will use \( \equiv \) for numerical equivalence and given two divisors \( D_1, D_2 \) we denote \( D_1 \preceq D_2 \) if \( D_2 - D_1 \) is pseudoeffective, i.e., it is a limit of (real) effective divisors or, equivalently, its product with arbitrary nef line bundles is nonnegative.

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## 2 The continuous rank and the subcanonicity index

In this section we introduce two numbers associated to line bundles which will appear in the Main Theorem. The first one, the \textit{continuous rank} can be defined for any coherent sheaf on irregular varieties, and the second one (the \textit{subcanonicity degree}) for a line bundle on any variety. The fundamental properties of these two numbers are that both behave well under algebraic equivalence, blow-ups, étale Galois coverings and hyperplane sections.

Given a line bundle on an irregular variety \( X \), the behavior of the continuous system given by \( |L \otimes \alpha| \) is studied in [15] and in [16]. Our interest relies on general elements in the continuous family.

**Definition 2.1.** Let \( X \) be a smooth irregular variety and \( F \) a coherent sheaf on \( X \). Let \( A \) be an abelian variety and \( \alpha : X \longrightarrow A \) a generating map. Define

\[
h^0_a(F) := \min \{ h^i(X, F \otimes a^* \alpha) \mid \alpha \in \text{Pic}^0(A) \}
\]

\( h^0_a(F) \) will be called the \textit{continuous rank} of \( F \).

By abuse of notation, if the map is clear by the context, we will usually write \( \alpha \) instead of \( a^* \alpha \).

**Remark 2.2.** If \( RS_a \) is the Fourier-Mukai functor associated to the map \( a \) (see [21]), we have that \( h^0_a(F) = \text{rank} RS^a_0(F) \).

**Definition 2.3.** If \( a : X \longrightarrow A \) is a map from \( X \) to an abelian variety, we will say that \( X \) is of maximal \( a \)-dimension if \( \dim X = \dim a(X) \). When \( A = \text{Alb}(X) \) and \( a = \text{alb}_X \) the definition corresponds to the classical notion of a maximal Albanese dimension variety.

**Example 2.4.** As simple examples for line bundles \( L \in \text{Pic}(X) \), we have

- If \( L = K_X + D \) with \( D \) nef, and \( X \) is of maximal \( a \)-dimension, then \( h^0_a(L) = \chi(X, L) \) by Generic Vanishing theorem. If \( \dim \text{alb}(X) = \dim X - 1 \) then we still have \( h^0_a(F) \geq \chi(X,F) \).

- The same holds for any \( GV \)-sheaf, as higher direct images of relative dualizing sheaves (see [22]).

- If \( L = K_X + D \) with \( D \) big and nef, then \( h^0_a(L) = \chi(X, L) = h^0(X, L) \) by Kawamata-Viehweg vanishing theorem.

**Remark 2.5.** The author was informed by Pirola that using the arguments of [15] and [16] and bounding the obstructions to deform a global section of \( L \), the following inequality holds: \( h^0_a(L) \geq h^1(X, L) - h^1(X, L) \) (unpublished).

**Remark 2.6.** The Severi inequality can be restated as \( K_S^2 \geq 4h^0_{\text{alb}_S}(\omega_S) \).

Let’s see now a non trivial example
Proposition 2.7. Let $f : X \rightarrow B$ be a fibration onto a smooth, projective curve of genus $b$, with general fibre $F$. Assume that $X$ is of general type and of maximal Albanese dimension. Let $a = \text{alb}_X$. Then
\[ h^0_{\alpha}(\omega_f) = h^0_{\alpha}(1) + \chi(X, \omega_f) = h^0_{\alpha}(f) + \chi(X, \omega_X) - 2 \chi(B, \omega_B) \chi(F, \omega_F). \]
If $b = 0, 1$ we have in fact that $h^0_{\alpha}(\omega_f) = \chi(X, \omega_X) - (2b - 2) \chi(F, \omega_F) \geq \chi(X, \omega_X)$.

Proof. Let $A = \text{Alb}(X)$. The subset of $\text{Pic}^0(A)$ where $h^0(X, \omega_f \otimes a^* \alpha)$ takes its minimum value is an open set, so it is enough to compute this value for a general $\alpha \in \hat{A}_{\text{tors}}$. In this case $R^i f_*(\omega_f \otimes \alpha)$ is locally free on $B$ ([10]), of rank $h^i(F, \omega_F \otimes \alpha)$, which is zero for $i \geq 1$ by generic vanishing applied to the map $\nu : F \hookrightarrow X \rightarrow A$ (clearly $F$ is of maximal $\pi$-dimension). Hence $H^i(X, \omega_f \otimes \alpha) = H^i(B, f_*(\omega_f \otimes \alpha)) = 0$ for $i \geq 2$, and the result follows.

When $b = 0, 1$, consider the étale covering given by $\alpha, g : \overline{X} \rightarrow X$. We can choose $\alpha$ of prime order $p$ such that $\pi^* \alpha \neq O_F$, and so $\pi^* \alpha^{\otimes i} \neq O_F$ for all $1 \leq i \leq p - 1$. Hence, the induced map $\overline{f} = f \circ g : \overline{X} \rightarrow B$ is a fibration (the fibres are the connected étale cover of $F$ given by $\pi^* \alpha \neq O_F$). Then $f_*(\omega_f \otimes \alpha)$ is a direct summand of $\overline{f}_*(\omega_{\overline{f}})$, and so it is a nef vector bundle on $B$. If $b = 0$ this proves $H^1(X, \omega_f \otimes \alpha) = H^1(B, f_*(\omega_f \otimes \alpha)) = 0$. If $b = 1$, we also obtain that $H^1(X, \omega_f \otimes \alpha) = H^1(B, \omega_X \otimes \alpha)) = 0$ by generic vanishing.

Finally let us see how $h^0_{\alpha}$ behaves under étale covers, blow-ups and hyperplane sections.

Proposition 2.8. Let $a : X \rightarrow A$ be a a generating map to an abelian variety $A$, and let $L \in \text{Pic}(X)$.

(i) Let $\mu : \hat{A} \rightarrow A$ a degree $m$ isogeny and consider the base change diagram

```
\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\mu}} & X \\
\downarrow & & \downarrow \\
\hat{A} & \xrightarrow{\mu} & A
\end{array}
\]
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Then $h^i_{\alpha}(\hat{\mu}^*(L)) = h^i_{\mu^*\alpha}(\mu^*L) = mh^i_{\alpha}(L)$.

(ii) If $\sigma : X' \rightarrow X$ is a blow-up, then $h^0_{\alpha}(\sigma^*L) = h^0_{\alpha}(L)$.

(iii) If $H$ is a smooth divisor divisor on $X$ such that $H - L$ is nef and $a(H)$ generates $A$, then $h^0_{\alpha}(L|_H) \geq h^0_{\alpha}(L)$.

Proof. (i) Let $N = \text{Ker} \mu \subseteq \hat{A}$, which is of order $m$. We have $\hat{\mu}_*(\mathcal{O}_\hat{X}) = \bigoplus_{\gamma \in N} a^*(\gamma)$. Let $Z_i \subseteq \hat{A}$ be the jumping locus of the value $h^i(X, a^*(\alpha))$, which is a proper closed set. For any $\beta = a^*(\alpha) \notin a^*(Z_i)$ we have
\[ h^i(X, \hat{\mu}^*L \otimes \alpha^* \beta) = \bigoplus_{\gamma \in N} h^i(X, L \otimes a^*(\alpha \otimes \gamma)) = mh^i_{\alpha}(L) \]
since $\alpha \otimes \gamma \notin Z_i$ for all $\gamma \in N$.

(ii) Obvious.

(iii) For $\alpha \in \hat{A}$ we can consider the exact sequence
\[ 0 \rightarrow \mathcal{O}_X((L - H) \otimes \alpha) \rightarrow \mathcal{O}_X(L \otimes \alpha) \rightarrow \mathcal{O}_H(L_H \otimes \alpha) \rightarrow 0 \]
Taking cohomology we have $h^0(X, (L - H) \otimes \alpha) = 0$ for $\alpha$ general. 

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Let us consider now the second invariant we need. As in the case of curves, the smaller is the degree of a nef line bundle, the bigger is the ratio between its degree and its global sections. The best behavior, given by Clifford's lemma, holds for subcanonical line bundles. We introduce now an invariant which measures exactly this relation in terms of $r$-subcanonicity. More concretely we have

**Definition 2.9.** Given a nef line bundle $L$ we define

(i) $r(L) = \inf \{ r \geq 1 \mid L \preceq rK_X \}$ (degree of subcanonicity of $L$). Note that if this set is empty, then $r(L) = \infty$.

(ii) $\delta(L) = \frac{2r(L)}{2r(L) - 1}$.

**Remark 2.10.** The following are easy properties of $\delta$:

1. $\delta(L)$ is a decreasing function of $r$, varying between 2 and 1. $\delta(L) = 2$ if and only if $L$ is (numerically) subcanonical and $\delta(L) = 1$ if and only if $r(L) = \infty$.
2. $\delta(L)$ is a decreasing function of $L$, i.e., if $L_1 \preceq L_2$ then $\delta(L_1) \geq \delta(L_2)$.
3. $\delta(L)$ increases by hyperplane section, i.e., if $M$ is a smooth section of an ample line bundle then $\delta(L|M) \geq \delta(L)$.
4. If $\mu : \tilde{X} \to X$ is an étale Galois covering and $\tilde{L} = \mu^*L$, then $\delta(\tilde{L}) = \delta(L)$.
5. $\delta(L)$ increases by blow-up, i.e., if $\sigma : X' \to X$ is any blow-up, then $\delta(\sigma^*(L)) \geq \delta(L)$.
6. If $X$ is a variety of general type, then $K_X$ is big and hence for all $L$ there exists an $r$ such that $rK_X - L$ is effective. Then we always have $\delta(L) > 1$. The case $X = A$ an abelian variety is just the opposite: $\delta(O_A) = 2$ and for all nef $L \neq O_A$ we have $\delta(L) = 1$.

## 3 Some properties of continuous linear systems

Let $a : X \to A$ be a generating map to an abelian variety and let $L \in \text{Pic}(X)$. We are going to study the geometry of the continuous linear systems $|L \otimes \alpha|$ for $\alpha$ general. A good presentation and an analysis of the generic base loci of the main continuous system associated to $L$ is developed in [15] and [16]. Here we consider two related problems: the continuous resolution of base points of a continuous linear system (Theorem 3.2), and the behavior of a general $|L \otimes \alpha|$ up to an étale covering (Theorem 3.5).

It is well known that there exists a nonempty open set

$$U \subseteq U_{a,L} = \{ \alpha \in \hat{A} \mid h^0(X, L \otimes \alpha) = h^0_0(L) \}.$$  

such that for $\alpha \in U$, if we consider the decomposition $L \otimes \alpha = W_\alpha + N_\alpha$ into its fixed and moving part respectively, then the divisors $W_\alpha$ belong to the same algebraic class, and the same occurs with the divisors $N_\alpha$.

We define now the **continuous moving part** and the **continuous fixed part** of $L$ as follows. Consider first the evaluation map

$$ev_U := \oplus ev_\alpha : \bigoplus_{\alpha \in U} H^0(X, L \otimes \alpha) \otimes \alpha^{-1} \to L.$$  

We have that $\text{Im} ev_U = \mathcal{I}_U \otimes L$, where $\mathcal{I}_U$ is an ideal sheaf. In the proof of Theorem 3.2 we will see that this sheaf does not depend on the chosen open set $U$ verifying the conditions above. Consider its decomposition

$$\mathcal{I}_U = \mathcal{O}_X(-W) \otimes \mathcal{I}_B$$  

with $\text{codim}_X B \geq 2$. 

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Definition 3.1.  
(i) $W$ is the continuous fixed part of $L$.

(ii) $M = L - W$ is the continuous moving part of $L$.

Set $\text{Im}ev_\alpha = \mathcal{I}_\alpha \otimes L \otimes \alpha$ where $\mathcal{I}_\alpha = \mathcal{O}(-W_\alpha) \otimes \mathcal{I}_{B_\alpha}$ with $\text{codim}_X(B_\alpha) \geq 2$. Observe that, by construction we have

$$\mathcal{I}_\alpha \otimes L \otimes \alpha \subseteq M \otimes \alpha \subseteq L \otimes \alpha.$$ 

Hence we have that

$$h^0_\alpha(L) = h^0_\alpha(M).$$

Now we are going to see that the continuous fixed and moving parts of $L$ behave as the linear ones up to a suitable étale covering. Following Pareschi and Popa ([23]), recall that a line bundle $L$ on $X$ is continuously globally generated with respect to the map $a$ if the continuous evaluations maps $ev_V$ defined above are surjective for all non-empty open sets $V \subseteq \hat{A}$.

Theorem 3.2. With the previous notation

(i) The sheaf $\mathcal{F} = \text{Im}(ev_V)$ is continuously globally generated with respect to $a$.

(ii) Up to a blow-up $\sigma$ and an étale Galois covering (more concretely, a multiplication map) $\mu$

$$\begin{array}{ccc}
\tilde{X} \xrightarrow{\mu} X' & \xrightarrow{\sigma} & X \\
\downarrow \tilde{a} & & \downarrow a \\
\hat{A} & \xrightarrow{a} & \hat{A}
\end{array}$$

we have that for all $\alpha \in \hat{A}$

$$\lambda^*(L \otimes \alpha) = \tilde{W}_\alpha + \tilde{N}_\alpha$$

is the decomposition in the fixed and moving divisor, $\lambda = \mu \circ \sigma$, the linear system $|\tilde{N}_\alpha|$ is base point free and the divisor $\tilde{W}$ does not depend on $\alpha$.

Proof. (i) This is basically the content of Remark 4.2 in [1]. To sketch a proof observe that given a point $p \in X$, if there exists a nonempty open subset $V \subseteq U$ such that $p$ is a base point of the linear systems $|L \otimes \alpha|$ for all $\alpha \in V$, then $p$ is also a base point of these linear systems for all $\alpha \in U$. Indeed, all the sections in $H^0(X, L \otimes \alpha)$, $\alpha \in U$, are limits of those with $\alpha \in V$, since on $U$ the dimensions $h^0(X,L\otimes\alpha)$ are constant.

Hence, for any open set $V \subseteq \hat{A}$ we have

$$\text{Im}(ev_V) \supseteq \text{Im}(ev_{V\cap U}) = \text{Im}(ev_U) = \mathcal{F}.$$ 

Finally, observe that by construction we have for any $\alpha \in U$

$$\mathcal{I}_\alpha \otimes (L \otimes \alpha) \subseteq \mathcal{F} \otimes \alpha \subseteq L \otimes \alpha$$

and hence $H^0(X, \mathcal{F} \otimes \alpha) = H^0(X, L \otimes \alpha)$ and so we have $\text{Im}(ev_{U,L}) = \text{Im}(ev_{U,\mathcal{F}})$.

(ii) Consider a blow-up $\sigma : X' \rightarrow X$ such that $\sigma^*(\mathcal{O}_X(-W) \otimes \mathcal{I}_B) = \mathcal{O}_{X'}(-W')$. If we set $L' = \sigma^*L$, then we have that $\sigma^*\mathcal{F} = L'(-W')$ is continuously globally generated with respect to $a' = a \circ \sigma$. Then we can apply a result of Debarre (cf. [9] Proposition 3.1): there exists an étale Galois covering $\mu : \tilde{X} \rightarrow X'$ which we can assume is induced by a multiplication map on $\hat{A}$, such that for any $\alpha \in \hat{A}$, the line bundles $\tilde{L}(-\mu^*W') \otimes \alpha = \mu^*L'(-W') \otimes \alpha$ are globally generated. Hence, if we set $\tilde{W} = \mu^*W'$ we have the statement. 

\hfill \square
Remark 3.4. Observe that in the previous construction we have the following properties

- For all $\alpha \in \tilde{A}$ the line bundles $\tilde{N}_\alpha$ on $\tilde{X}$ are algebraically equivalent.
- $h^0(\tilde{N}_\alpha) = h^0(\tilde{L}) = (\deg \mu) h^0(L') = (\deg \mu) h^0(M) = (\deg \mu) h^0(L)$.
- For all $l \geq 1$ we have $h^0(\tilde{X}, \tilde{N}_\alpha^{\otimes l}) = (\deg \mu) h^0(X, (M \otimes \alpha)^{\otimes l})$.
- Since $\tilde{N}_\alpha$ are base point free, if $L$ is nef then $(\deg \mu) L^n = (\tilde{L})^n \geq (\tilde{N}_\alpha)^n$.

To finish the section we are going to see that given a nef line bundle with continuous sections, the image of the map induced by its linear sections, up to étale base change, factorizes the map $a$. Hence the dimension of the image of $X$ through the linear system $|L|$ is bounded by the $a$-dimension of $X$. In particular, it is maximal when $X$ is of maximal $a$-dimension. This will be a crucial point in the proof of Main Theorem.

In order to do that, consider the following notation. Given a line bundle $L$ consider the morphism $\psi_L : X' \to \mathbb{P}^m$ it induces on a suitable blow-up $X'$ of $X$. We denote by

$$\phi_L : X' \to Z_L$$

the algebraic fibre space induced by $\psi_L$.

Theorem 3.5. Let $X$ be a smooth $n$-dimensional variety and $a : X \to A$ a generating map to an abelian variety with $k = \dim a(X)$. Let $L \in \text{Pic} X$, such that $h^0_a(L) \neq 0$. Let $M$ be its continuous moving part. Consider all the previous notation of this section. Then

(i) The map $\lambda$ in Theorem 3.2 (ii) verifies that $\tilde{a}$ factorizes through $\phi_L$. In particular $\dim \phi_L(\tilde{X}) \geq k$. Moreover, for all $\alpha \in \tilde{A}$ we have that $N_\alpha \in \pi^*_L \text{Pic} Z_L$.

(ii) The linear system $|\tilde{L}|$ induces a generically finite map provided one of the following conditions hold

- $|L|$ induces a generically finite map.
- $X$ is of maximal $a$-dimension.
- $M_G$ is big, where $G$ is the generic fibre of the map $a$.

Proof. (i) Let $T$ be a general (connected) fibre of $\phi_L$, and let $R \in \text{Pic} (Z_L)$ such that $\tilde{N}_0 = \phi_L^* (R)$. By Remark 3.4, if $h^0_a(L) \neq 0$ then $h^0(\tilde{N}_0) \neq 0$ and hence for all $\alpha \in \tilde{A}$ we have

$$h^0(\tilde{X}, \tilde{N}_0 \otimes \tilde{a}^* \alpha) \neq 0.$$

By projection formula we have that $h^0(Z_L, R \otimes (\phi_L)_* (\tilde{a}^* \alpha)) \neq 0$ for all $\alpha \in \tilde{A}$. The sheaf $(\phi_L)_* (\tilde{a}^* \alpha)$ is torsion free of rank $h^0(T, (\tilde{a}^* \alpha)|_T)$ and so it must be non zero for all $\alpha$. Since $(\tilde{a}^* \alpha) \in \text{Pic}^0(T)$ this can only happen if $(\tilde{a}^* \alpha)|_T = \mathcal{O}_T$. Hence the natural composition map

$$\tilde{A} \to \text{Pic}^0(\tilde{X}) \to \text{Pic}^0(T)$$

is zero. Dualizing we obtain that for general $T$ the map $\tilde{a}$ contracts $T$ to a point.

The rest of the statement follows immediately from this factorization.

(ii) If $|L|$ induces a generically finite map clearly so does $|\tilde{L}|$. By (i) the same holds if $X$ is of maximal $a$-dimension ($k = n$). For the rest, observe that the fibres $G'$ of the map $a \circ \lambda$ are just disconnected copies of $G$. On the other hand, by construction $\tilde{N}_0|_{G'}$ is big if and only if $M_{G'}$ is (see Remark 3.5 (iii)), hence we can assume that $\tilde{N}_0$ is $\tilde{a}$-big. But if $r < n$ this is not possible since the fibres $G'$ are covered by those of $\phi_L$. 

\[
\]
Remark 3.6. Bigness of $M|_G$ follows from bigness of $M$ itself, but in general it can be difficult to check. Here we have two sufficient conditions.

- If $L|_G$ is big (for example, if $L$ itself is big) and $\text{kod} W|_G \leq 0$, then $M|_G$ is big.
- If $L$ is continuously globally generated in codimension 2 (i.e. $W = \emptyset$), then bigness of $M$ is equivalent to bigness of $L$, i.e., $L^n > 0$.
- Continuous global generation of $L$ outside of the ramification locus of the Albanese map of $X$, is implied by $M$-regularity of $L$ (see [23]).

Remark 3.7. In general bigness of $L|_G$ does not imply bigness of its continuous moving part $M|_G$. Take for example $X = S \times Y$ where $S$ is a general type surface with $p_g = 1$, $q = 0$ and $Y$ is a general type and albanese general type variety of dimension $n-2$, with base point free paracanonical system $N$. Then clearly $L = \omega_X$ is big, its continuous fixed part is $Z = \pi_1^*(D)$ ($D$ the only canonical section of $S$) and $M = \pi_2^*N$, the Albanese map of $X$ is just the Albanese map of $Y$ and $M|_G$ is not big.

4 The generalized Clifford-Severi Inequality

In the previous sections we have introduced all the ingredients needed to state our main results. All of them are particular instances or corollaries of the following main theorem which establishes a sharp lower bound for $L^n$ ($L$ a nef line bundle on an irregular variety) in terms of its continuous global sections and the Albanese dimension of $X$.

Theorem 4.1. (Generalized Clifford-Severi Inequality)

Let $X$ be a smooth, projective variety of dimension $n$, over an algebraically closed field of characteristic 0. Let $\alpha : X \rightarrow A$ be a generating map to an Abelian variety and let $L \in \text{Pic}(X)$ be a nef line bundle.

(i) If $X$ is of maximal $\alpha$-dimension then

$$L^n \geq \delta(L) n! h^0_\alpha(L).$$

In particular, if $L \preceq K_X$, then $L^n \geq 2n! h^0_\alpha(L)$.

(ii) Assume $n > \text{dim}(X) = k \geq 1$, let $G$ be a general fibre of the algebraic fibre space induced by $\alpha$ and let $M$ be the continuous moving part of $L$. If $M|_G$ is big then

$$L^n \geq \delta(L) k! h^0_\alpha(L).$$

(iii) Assume that $n > \text{dim}(X) = k \geq 1$ and that $L|_G$ is big. Then

$$L^n \geq k! h^0_\alpha(L).$$

The proof of this theorem relies on a suitable use of Xiao’s method on étale Galois coverings of $X$ and it is postponed to the next Section.

Remark 4.2. Since $\delta$ is a decreasing function, we do not need to know the exact value of $r(L)$ to obtain an inequality. Hence, part (i) of the theorem can be rephrased as (analogously for (ii)): when $X$ is of maximal $\alpha$-dimension

(i) For any nef $L$ we have $L^n \geq n! h^0_\alpha(L)$.

(ii) If $L \preceq rK_X$ ($r \geq 1$), then $L^n \geq \frac{2r}{r-1} n! h^0_\alpha(L)$. 

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Remark 4.3. (i) The bound given in 4.1 (i) is sharp for the lowest value of $\delta(L)$: take $X = A$ an abelian variety and $L$ an ample line bundle on $A$. Also, in general it is asymptotically sharp for general type varieties $X$ and sufficiently ample line bundles $L = mL$, as shown by asymptotic Riemann-Roch theorem when $m \to \infty$.

(ii) For varieties of non maximal Albanese dimension Theorem 4.1 can not hold without extra hypothesis as those in (ii) and (iii). Indeed, take $L$ to be the pullback of a line bundle on the Albanese image of $X$. Then $L^n = 0$ and in general $h^0_{alb X}(L) \neq 0$.

Since the inequality holds for irregular varieties of any Albanese dimension, we can extend the result of Theorem 4.1 to nef vector bundles just looking to its projectivization.

Corollary 4.4. Let $X$ be a projective, smooth variety of dimension $n$, over an algebraically closed field of characteristic 0. Let $a : X \to A$ be a map to an Abelian variety such that $\dim a(X) = k$ and let $G$ be a general fibre of the induced algebraic fibre space. Let $F$ be a nef vector bundle on $X$ with top Segre class $s(F)$. Assume that $k = n$ or that $F|_G$ is big. Then

$$s(F) \geq k! h^0_{alb}(F).$$

Proof. Let $Y = \mathbb{P}(F)$, $\pi : Y \to X$ the natural projection and $L = \mathcal{O}_Y(1)$ the tautological line bundle. Consider the induced map $a_Y := a \circ \pi : Y \to A$; we have $\dim a_Y(Y) = k$. Observe that $s(F) = \mathcal{O}_Y(1)^{n+l-1}$ whenever $l = \text{rank } F$.

Let $G'$ be the fibre of the algebraic fibre space induced by the Albanese map of $Y$. If $k = n$ then $G' = \mathbb{P}^{l-1}$, $L_{|G'} = \mathcal{O}_{\mathbb{P}^{l-1}}(1)$ and hypothesis of theorem 4.1 (ii) hold. If $k < n$ then $G'$ is a projective bundle on $G$. If $F|_G$ is big, then the hypothesis of (iii) hold. In both cases we use $\delta(L) \geq 1$. \hfill \Box

As a particular case of the Theorem 4.1 we obtain the sharp generalization of the classical Severi inequality when $L = K_X$ is nef, hence when the variety $X$ is minimal. Observe that considering $Y$ any irregular variety of general type this exactly gives a very high (and sharp in the maximal Albanese dimension case) lower bound of the canonical volume of $Y$ just considering a minimal model $X$ of $Y$. To obtain this result we must consider nef Weil divisors on normal Q-Gorenstein varieties. This behavior for irregular varieties was conjectured by Miles Reid but till now very few was known (see [6] and [7], for example).

Corollary 4.5. (Generalized Severi Inequality / Volume of irregular varieties) Let $X$ be an irregular, minimal, projective variety of dimension $n$ over an algebraically closed field of characteristic 0, with at most canonical singularities. Let $X'$ be any desingularization of $X$. Then

(i) If $X$ is of maximal Albanese dimension then

$$\text{vol}(X') = K^n_X \geq 2n! \chi_{\omega X}$$

and this bound is sharp (double covers of abelian varieties).

(ii) If $\dim a(X) = k < n$, $X$ is of general type and the continuous base locus $W$ of $\omega X$ verifies $\text{kod}(W|_G) \leq 0$, for a general fibre $G$ of the induced Albanese fibre space, then

$$\text{vol}(X') = K^n_X \geq 2k! h^0_{alb X}(\omega X)$$

(iii) If $\dim a(X) = k < n$ and $X$ is of general type, then

$$\text{vol}(X') = K^n_X \geq k! h^0_{alb X}(\omega X)$$
Proof. Consider any desingularization \( \sigma : X' \rightarrow X \). The volume of \( X' \) is a birational invariant which coincides with \( K_X^2 \). Since \( X \) has canonical singularities (hence rational), we have that \( \text{Alb} X = \text{Alb} X' =: A \) and \( \text{alb}_X = \text{alb}_{X'} \circ \sigma \) ([3], Ch.2.4). Hence

\[
h^0_{\text{alb}_X}(\omega_X) = h^0_{\text{alb}_{X'}}(\omega_{X'})
\]

and it coincides with \( \chi(\omega_X) = \chi(\omega_{X'}) \) when \( X \) is of maximal Albanese dimension. Fix \( a_0 \in \hat{A} \) such that

\[
h^0(X, \omega_X \otimes a_0) = h^0_{\text{alb}_X}(\omega_X).
\]

There exists a desingularization \( \sigma \), an effective \( \mathbb{Q} \)-Cartier divisor \( E_{a_0} \) on \( X' \) which is \( \sigma \)-exceptional, and an effective Cartier divisor \( M_{a_0} \) on \( X' \) such that

\[
\sigma^*(K_X \otimes a_0) - E_{a_0} = M_{a_0}.
\]

Observe that the linear system \( |M_{a_0}| \) may have a base component. On the other hand, since \( X \) has canonical singularities, there exists \( E \), an effective \( \mathbb{Q} \)-Cartier divisor on \( X' \), \( \sigma \)-exceptional such that \( K_{X'} = \sigma^*(K_X) + E \). Hence we have that

\[
K_{X'} \otimes a_0 = M_{a_0} + (E + E_{a_0})
\]

where by construction \( (E + E_{a_0}) \) is a \( \sigma \)-exceptional, integral Cartier divisor. We also have

\[
h^0(X, M_{a_0}) = h^0(X', K_{X'} \otimes a_0) = h^0_{\text{alb}_{X'}}(\omega_{X'}).
\]

Observe that we can apply exactly the same construction as in Theorem 3.2 (ii), just applying Theorem 3.2 (i) to the line bundle \( M_{a_0} \). Hence we obtain that (we keep the notations of Theorem 3.2)

\[
\lambda^*(K_X \otimes a_0) = \tilde{W} + N
\]

where now \( \tilde{W} \) is a \( \mathbb{Q} \)-Cartier divisor whose fractional part is \( \lambda \)-exceptional. Since \( K_X \otimes a_0 \) and \( N \) are nef and \( \tilde{W} \) effective we have that \( (\deg \mu)K_X^n \geq N^n \). By Remark 3.4 we also obtain that (\( \deg \mu \)) \( h^0(\mu, K_X) = h^0(N) \) and bigness of \( N \) on the fibres of the albanese map is equivalent to that of moving part of \( M_{a_0} \). Hence we reduce the problem to the nef Cartier divisor \( N \) on the smooth variety \( \hat{X} \) and observe that by construction we have that \( \delta(N) = 2 \), since \( N \) is subcanonical. Then apply Theorem 4.1.

\[ \square \]

Remark 4.6. Irregular, general type varieties of Albanese dimension \( k \), with \( \text{vol}(X) < 2k! \chi(\omega_X) \) seem to have strong restrictions, if they exist. At least when the minimal model \( X \) is Gorenstein, the albanese map of \( X \) must factorize through a fibration with general fibre \( G \) of dimension \( l \geq 1 \) such that \( K_G^l = 1 \) (hence regular or with \( \chi(\omega_G) \leq 1 \) as follows as a corollary of proof of Theorem 4.1 (see Remark 5.8). In particular, for minimal, Gorenstein \( X \) of Albanese dimension \( n-1 \), the inequality \( K_X^n \geq 2(n-1)! \chi(\omega_X) \) also holds.

For adjoint line bundles we have

Corollary 4.7. When \( X \) is of maximal Albanese dimension

(i) If \( L \) is a GV-sheaf (for example, if \( L = K_X + D \), with \( D \) nef), then \( L^n \geq \delta(L) n! \chi(X, L) \).

(ii) If \( L = K_X + D \), with \( D \) big and nef, then \( L^n \geq \delta(L) n! h^0(X, L) \).

Proof. Just apply (2.4).

\[ \square \]

Remark 4.8. Similar results in the previous corollary can be deduced in the case of non maximal Albanese dimension imposing positivity properties of \( L \) or \( M \) on the fibres of the Albanese map as in Theorem 4.1.

For surfaces we can give a useful result when the canonical sheaf is numerically decomposable.
Corollary 4.9. Let $S$ be a smooth surface of maximal Albanese dimension such that $K_S \equiv L_1 + L_2$ with $L_i$ nef line bundles. Then

$$K_S^2 \geq 4\chi_S + 4h^1_0(L_1)$$

Proof. Let $\rho \in \text{Pic}^n(S)$ such that $L_1 + (L_2 + \rho) = K_S$ and redefine $L_2 = L_2 + \rho$. Observe that $h^1_0(L_1) = h^2_1(L_2)$ and both are (numerically) sub-canonical. Applying Theorem (4.1) to both sheaves and adding up, we obtain

$$L_1^2 + L_2^2 \geq 4(h^0_0(L_1) + h^0_0(L_2)) = 4\chi(S, L_1) + 4h^1_0(L_1) = 2L_1(-L_2) + 4\chi_S + 4h^1_0(L_1)$$

and hence

$$K_S^2 = L_1^2 + 2L_1L_2 + L_2^2 \geq 4\chi_S + 4h^1_0(L_1)$$

Our last application shows as the Clifford-Severi inequality implies a slope inequality for fibred varieties.

Remark 4.10. Given a fibred variety over a smooth curve, $f : X \to B$ with general fibre $F$, and a line bundle $L$ on $X$, slope inequalities relate the invariants of $(X, L)$, $(F, L|_F)$ and $B$. The best slope inequalities hold when some stability properties hold for $L$ (see [2] for a survey on this topic), and then a slope inequality, which we call $f$-positivity of $L$ follows

$$L^n \geq n \frac{h^1_0(F, L|_F)}{\deg f}$$

The case $X$ of general type and $L = \omega_f$ encodes important numerical and geographical properties of the fibration, but $f$-positivity of $\omega_f$ is only known for fibred surfaces. In fact, $f$-positivity of $\omega_f$ for dimensions less or equal to $n$ implies a weaker inequality for the slope

$$K_F^n \geq 2n! \chi_f$$

from which it can be deduced the Severi inequality for $L = \omega_X$ (see [2] Proposition 5.8, for a proof of this statement). Now we prove a vice-versa, namely, that the Severi inequality for $L = \omega_f$ implies the above slope inequality when $b = g(B) = 0$ and a slightly weaker result when $b \geq 1$.

Corollary 4.11. (Slope inequality) If we have a relatively minimal fibration $f : X \to B$ onto a smooth curve $B$ of genus $b$ with general fibre $F$ and $X$ is of general type and of maximal Albanese dimension, then

(i) If $b = 0$ then $K^n_f \geq 2n! \chi_f$.

(ii) If $b \geq 1$, then $K^n_f \geq 2n! [\chi(\omega_f) + h^1_0(\omega_f)] \geq 2n! [\chi_{\omega_X} - 2\chi_{\omega_B} \chi_{\omega_F}]$

Proof. (i) If $b = 0$ just add up the Severi inequalities for $K_X$ and for $K_F$

$$K^n_F = K^n_F + 2nK^n_F \geq 2n! \chi_{\omega_X} + 4n! \chi_{\omega_F} = 2n! (\chi_{\omega_X} - \chi_{\omega_B} \chi_{\omega_F}) = 2n! \chi_f$$

(ii) If $b \geq 1$ then $\omega_f$ is subcanonical and so $\delta(\omega_f) = 2$. Then apply Proposition 2.7 and Theorem 4.1.

5 Proof of Theorem 4.1

As pointed out in the introduction, the proof relies in three basic tools: Xiao’s method, the behavior of linear systems on suitable étale coverings (studied in Section 3) and finally Pardini’s covering trick. We consider them separately.
5.1 Xiao’s method

We will use a simplified version of Xiao’s method for fibrations, in the case where the base curve is \( \mathbb{P}^1 \), and specially adapted to an ulterior process of étale covers. We remind briefly the method in this simplified version and refer to [2], [12], [19] and [25] for details.

Let \( X \) be a normal projective variety of dimension \( n \) and let \( D \) be a nef Weil \( \mathbb{Q} \)-Cartier divisor. Let \( L = \mathcal{O}_X(D) \) be its associated rank 1 reflexive sheaf. Assume we have a fibration \( f : X \to \mathbb{P}^1 \) with \( F \) a general fibre of \( f \) and let \( \mathcal{E} = f_*L \). It is a vector bundle since it is torsion free on a smooth curve. Consider its decomposition

\[
\mathcal{E} = f_*L = \bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(a_i)
\]

with \( a_1 \geq a_2 \geq \ldots \geq a_m \geq 0 > a_{m+1} \geq \ldots \geq a_l, l \geq m \geq 0 \), and \( l = h^0(F,L|_F) \). Observe that we have

\[
h^0(X,L) = a_1 + \ldots + a_m + m \quad \text{and so}
\]

\[
a_1 + \ldots + a_m \geq h^0(X,L) - h^0(F,L|_F).
\]

(5.1)

For \( i = 1, \ldots, m \), define \( \mathcal{E}_i = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_i) \). When the \( a_i \)’s are different, these are the pieces of the Harder-Narashiman filtration of \( \mathcal{E}_m \), of associated slopes \( \mu_i = a_i \).

For each \( i = 1, \ldots, m \) such that \( a_i > a_{i+1} \) the composite of the natural sheaf homomorphisms

\[
f^*\mathcal{E}_i(-a_i) \to f^*(f_*L)(-a_i) \to L(-a_iF)
\]

surjects onto a sheaf of ideals of type \( \mathcal{I}_{Z_i} \otimes L(-a_iF) \). Following [19] Lemma 1.1 and Remark therein, up to a suitable desingularization \( \epsilon : \hat{X} \to X \), if we set \( \hat{L} = \epsilon^*L \), we have a decomposition

\[
\hat{L} \sim_{\mathbb{Q}} N_i + \hat{Z}_i + a_i\hat{F}
\]

where:

- \( N_i \) is a nef Cartier divisor on \( \hat{X} \) inducing a base point free linear system.
- \( \hat{Z}_i \) is an effective and fixed \( \mathbb{Q} \)-Cartier divisor.

If \( a_i = a_{i+1} \) we define \( N_i = N_{i+1} \), \( Z_i = Z_{i+1} \). Observe that \( N_1 = \mathcal{O}_{\hat{X}} \) if and only if \( a_1 > a_2 \). We redefine \( a_{m+1} = 0 \) and extend coherently the definition to \( N_{m+1} \) and \( \hat{Z}_{m+1} \).

Moreover, we have that

\[
N_1 \leq N_2 \leq \ldots \leq N_m \leq N_{m+1} \leq L \\
\hat{Z}_1 \geq \hat{Z}_2 \geq \ldots \geq \hat{Z}_m \geq \hat{Z}_{m+1} \geq 0 \\
a_1 \geq a_2 \geq \ldots \geq a_m \geq a_{m+1} = 0
\]

In fact, by construction we have that

\[
N_i + (\hat{Z}_i - \hat{Z}_{m+1}) = N_{m+1}(-a_i\hat{F})
\]

is the decomposition of \( N_{m+1}(-a_i\hat{F}) \) in its moving and fixed part, respectively.

Under these assumptions we can apply Xiao’s Lemma (see [12]) and [19] Lemma 1.2. We define the linear systems \( P_i := N_i|_{\hat{F}} \) which are free from base points and induce maps \( \phi_i : \hat{F} \to \mathbb{P}^{r_i} \). Observe that for \( i = 1, \ldots, m \) we have \( r_i \geq i \).

Define now
Remark 5.4. The same construction can be applied to any selection of indexes \( f \) for a generating map to an abelian variety and continuous Xiao’s method.

Proposition 5.5. Let \( \beta \) be a countable union of proper closed sets. An element \( \beta \) in its complementary set verifies the statement.

\[
I_s = \{ k = 1, \ldots, m \mid \dim \phi_k(\hat{F}) = s \}
\]

and we obtain a partition of the set \( \{ 1, \ldots, m \} \).

Let \( r \) be the maximum index such that \( I_{r-1} \neq \emptyset \) and define decreasingly, for \( s = 1, \ldots, r - 1 \)

\[
b_s = \begin{cases} 
\min I_s & \text{if } I_s \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

Then we have that, for any \( A_1, \ldots, A_{n-r} \) nef \( \mathbb{Q} \)-Cartier divisors the following inequality holds:

\[
A_{t+1} \cdots A_{n-r} \left[ N^r_{m+1} - \left( \sum_{s=r-1}^{r} (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} \left( \sum_{l=0}^{s} P_i^{s-l} P_{l+1}^{l+1} (a_i - a_{i+1}) \right) \right) \right] \geq 0.
\]

In particular, taking \( A_1 = \ldots = A_{n-r} = \hat{L} \) we obtain

\[
L^n = (\hat{L})^n \geq \hat{L}^{n-r} N^r_{m+1} \geq \hat{L}^{n-r} \left[ \sum_{s=r-1}^{r} (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} \left( \sum_{l=0}^{s} P_i^{s-l} P_{l+1}^{l+1} (a_i - a_{i+1}) \right) \right]
\]

Since \( P_{t+1} \geq P_t \) and they are nef, we have that

\[
\sum_{l=0}^{s} (P_i^{s-l} P_{l+1}^{l+1}) \geq (s + 1) P_t^n
\]

and so

\[
L^n \geq \hat{L}^{n-r} \left[ \sum_{s=r-1}^{r} (s + 1) (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} P_i^n (a_i - a_{i+1}) \right] \tag{5.2}
\]

For later reference we need to consider the following special case. Assume that all the induced maps \( \phi_i \) have image of dimension \( n - 1 \), i.e., they are generically finite. Then

\[
L^n \geq n[P_m^{n-1}(a_m - a_{m+1}) + P_{m-1}^{n-1}(a_{m-1} - a_{m}) + \ldots + P_1^{n-1}(a_1 - a_2)] \tag{5.3}
\]

Remark 5.4. The same construction can be applied to any selection of indexes \( I \subseteq \{ 1, \ldots, m \} \).

We are going to see now how the method behaves under a suitable étale Galois covering of \( X \). It turns out that a continuous Xiao’s method hold.

Proposition 5.5. Let \( X \) be a normal projective variety and \( L \) a Weil \( \mathbb{Q} \)-Cartier divisor. Let \( \alpha : X \rightarrow A \) be a generating map to an abelian variety and \( f : X \rightarrow \mathbb{P}^1 \) a fibration. Then, for a very general element \( \beta \in \hat{A} \), the line bundle \( L' := L \otimes \beta \) verifies that the vector bundles \( \mathcal{E}_\alpha = f_*(L' \otimes \alpha) \) are all equal for \( \alpha \in \hat{A}_{\text{tors}} \).

Proof. The continuous family of vector bundles \( \{ f_*(L \otimes \beta) \} \) on \( \mathbb{P}^1 \) for \( \beta \in \hat{A} \) must be constant on a nonempty open set \( U \). Let \( D = \hat{A} \setminus U \). Then

\[
\bigcup_{\alpha \in \hat{A}_{\text{tors}}} (\alpha + D) \neq \hat{A}
\]

being a countable union of proper closed sets. An element \( \beta \) in its complementary set verifies the statement. \( \square \)
Consider a fibration $f : X \rightarrow \mathbb{P}^1$ as in the proposition. Let $L$ be a nef line bundle such that $h^0_a(L) \neq 0$ and such that verifies the conclusion of Proposition 5.5, i.e., for all $\alpha \in \hat{A}_{\text{tors}}$, the sheaves $f_*(L \otimes \alpha)$ are all equal. Keeping all the previous notations, for $i = 1, \ldots, m$ consider the linear systems $L(-a_iF)$. Apply now Theorem 5.2 to all of them and get

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\lambda} & X \\
\downarrow & & \downarrow f \\
\mathbb{P}^1 & & \\
\end{array}
$$

where $\lambda$ is a composition of a blow-up and an étale Galois map induced by $\hat{A}_d$ for certain $d$, verifying the following conditions:

- For $i = 1, \ldots, m$ and for any $\alpha \in \hat{A}$ we have $\lambda^*(L(-a_iF)) \otimes \alpha = W_i + N_{\alpha,i}$.
- $W_i$ does not depend on $\alpha$ and is the fixed component of the linear system.
- $N_{\alpha,i}$ induce a base point free linear system and for all $\alpha$ we have $N_{\alpha,i} = N_{0,i} \otimes \alpha$.

Furthermore, since the kernel of the map $\text{Pic}^0(X) \rightarrow \text{Pic}^0(F)$ is finite, we can choose $d$ in such a way that $\tilde{F} = \lambda^*F$ is a connected étale cover of $F$ and hence $\tilde{f}$ is a fibration.

Let $L = \lambda^*L$, and $r = \deg \mu$. By the election of $L$ and projection formula we have

$$
\tilde{E} = \tilde{f}_*\tilde{L} = E \oplus r = O_{p^1}(a_1)^{\oplus r} \oplus \ldots \oplus O_{p^1}(a_d)^{\oplus r}
$$

Applying Xiao’s method to $\tilde{f}$ we have that the vector bundles $\tilde{E}_j$ induce base point free linear systems $\tilde{N}_j$ on $\tilde{X}$ and $\tilde{P}_j$ on $\tilde{F}$. Among these we have for $i = 1, \ldots, m$

$$
\tilde{E}_{ri} = O_{p^1}(a_1)^{\oplus r} \oplus \ldots \oplus O_{p^1}(a_d)^{\oplus r}
$$

which induce precisely the linear systems $\tilde{N}_{ri} = N_{0,i}$ defined above. Then $\tilde{P}_{ri} = N_{0,i} \tilde{F}$ which by construction is of dimension at least $\text{rank} \tilde{E}_{ri} = ri$. Following the previous conventions recall that if $r(i-1) < j \leq ri$ then $\tilde{P}_j = \tilde{P}_{ri}$.

Summing up we obtain

**Proposition 5.6.**

(i) $h^0_a(\tilde{P}_{ri}) \geq ri$.

(ii) If $X$ is of maximal $a$-dimension, we can choose $\lambda$ in such a way that for all $j = 1, \ldots, rm$ the linear systems $[\tilde{P}_j]$ are generically finite.

**Proof.** (i) We can do the same construction for $\tilde{E}_\alpha = \tilde{f}_*(\tilde{L} \otimes \alpha)$. Similarly we obtain linear systems $\tilde{N}_{\alpha,ri} = N_{\alpha,i}$ such that when restricted to $\tilde{F}$ they induce $\tilde{P}_{ri}^\alpha$ of dimension at least $ri$. By construction for any $\alpha$ we have that $N_{\alpha,i} = N_{0,i} \otimes \alpha$ and hence the same happens when restricting to $\tilde{F}$. Hence for all $\alpha$ we have that $h^0(\tilde{F}, \tilde{P}_{ri} \otimes \alpha) = h^0(\tilde{F}, \tilde{P}_{ri}^\alpha) \geq ri$ and so $h^0_a(\tilde{P}_{ri}) \geq ri$.

(ii) We have that $\tilde{P}_j = \tilde{P}_{ri}$ for some $i$. By Theorem 3.5 we can modify $\lambda$ with a multiplication map with $d >> 0$ such that the maps induced by $N_{\alpha,i}$ are all generically finite. 

$\square$
5.2 The proof of Theorem 4.1

We will use freely the notations of subsection 5.1. Observe that if \( h^0_a(L) = 0 \) then the result is trivially true. From now on we will consider that \( h^0_a(L) \neq 0 \).

(i) Assume that \( X \) is of maximal \( a \)-dimension. We proceed by induction on \( n = \dim X \).

Case \( n = 1 \). Let \( X \) be a smooth curve of genus \( g \geq 1 \) and \( L \) a nef line bundle. If \( \deg L \leq \deg K_X = 2g - 2 \) and it is non-special, then \( \deg L \geq 2h^0(X, L) \) by Riemann-Roch theorem. If \( \deg L = r(2g - 2) \) with \( r \in \mathbb{Q}, r > 1 \) then again by Riemann-Roch theorem we obtain \( \deg L = 2d^2 h^0(X, L) \). It remains the case of a special divisor. Take the étale cover of \( C \) induced by \( \hat{A}_d \), say \( \mu : \hat{C} \to C \). Consider \( \tilde{L} = \mu^* L \) which is a special line bundle on \( \hat{C} \). Hence, we can apply Clifford’s theorem and obtain

\[
d^{2q} \deg L = \deg \tilde{L} \geq 2h^0(\hat{C}, \tilde{L}) - 2 \geq 2d^{2q} h^0_a(L) - 2
\]

where \( q = \dim A \). Since this holds for all \( d \) we obtain \( \deg L \geq 2h^0_a(L) \).

Case \( n \geq 2 \). The key argument here is to mix the idea of Pardini’s proof in [20] with the application of Xiao’s method on a suitable étale cover as in Subsection 5.1. The basic construction needs two steps.

**Step 1. Claim.** Given a base point free linear system \( |M| \) on \( X \) such that \( a(M) \) generates \( A \), we have

\[
L^n + n\delta(L) L^{n-1} M \geq \delta(L) n h^0_a(L).
\]

**Proof of Claim.** Consider the base point free linear system given by \( |M| \) on \( X \). The general member of this linear system is smooth and irreducible since we are assuming \( n \geq 2 \). Take two general smooth members \( F, F' \in |M| \). Consider a blow up \( \epsilon : Y \to X \) in order to get a fibration \( f : Y \to \mathbb{P}^1 \) induced by \( F \) and \( F' \). Since the formula we want to prove is invariant in the algebraic class of \( L \), we change \( L \) by \( L \otimes \beta \) in such a way that Proposition 5.5 applies for \( \epsilon^* L \). Moreover we can get that \( h^0(X, L) = h^0_a(L) \) and that \( h^0(M, L|_M) = h^0_a(L|_M) \).

Then we can apply the construction of Subsection 5.1 to \( f : Y \to \mathbb{P}^1 \) and get

\[
\begin{array}{c}
A \\
\downarrow a \\
\tilde{X} \\
\downarrow \tilde{\epsilon} \\
\tilde{Y} \\
\downarrow f \\
\mathbb{P}^1
\end{array}
\]

where \( a \) is of degree \( r = d^{2q} \) (\( q = \dim A \)). Let \( \tilde{L} = \lambda^* (\epsilon^* L) \) and \( \tilde{F} = \lambda^* F \), which is an irreducible étale cover of \( F \) of degree \( r \). We can apply Proposition 5.6 (ii) and so all the induced maps are generically finite and hence by (5.3) we obtain

\[
\tilde{L}^n \geq n[\tilde{\beta}_r^{n-1}(a_1 - a_2) + \ldots + \tilde{\beta}_r^{n-1} a_m]
\]

Observe that \( \tilde{F} \) is \((n - 1)\)-dimensional, of maximal \( a_\tilde{F} \)-dimension and the map \( \tilde{a}_{\tilde{F}} \) is generating. So we apply induction hypotheses for the nef line bundles \( \tilde{\beta}_i \) on \( \tilde{F} \), and for all \( i = 1, \ldots, m \) we get
The last inequality holding by Remark 2.10. By Proposition 5.6 (i) we have that
\[ h_0^0(\tilde{P}_{ri}) \geq ri \]
Thus
\[ rL^n = \tilde{L}^n \geq r\delta(L) n!(a_1 + ... + a_m) \geq r\delta(L) n!(h_0^0(L) - h_0^0(\epsilon^*L_{|F})) \]
Using again induction for \( \epsilon^*L_{|F} \) we have
\[ (\epsilon^*L_{|F})^{n-1} \geq \delta(\epsilon^*L_{|F})(n-1)!h_0^0(\epsilon^*L_{|F}) \geq \delta(L) (n-1)!h_0^0(\epsilon^*L_{|F}) \]
and summing up, since \( \epsilon_*F = M \) we finally obtain
\[ L^n + n\delta(L) L^{n-1}M \geq \delta(L) n!h_0^0(L). \]

**Step 2.** Let us apply now Pardini’s covering trick to prove the statement. Consider again \( d \in \mathbb{N} \) and the étale Galois map induced by multiplication

\[
\begin{array}{ccc}
A & \xrightarrow{\mu} & A \\
\bar{a} & \downarrow & \alpha \\
\bar{X} & \xrightarrow{\mu} & X
\end{array}
\]

Let \( H \) be a fixed very ample line bundle on \( A \) and let \( M = \alpha^*H, \tilde{M} = \bar{a}^*(H) \). By [11] Ch.2. Prop. 3.5 we have
\[ \mu^*H \equiv d^2H \]
and so
\[ \tilde{M} \equiv \frac{1}{d^2}\mu^*M. \quad (5.7) \]
Define again
\[ \tilde{L} := \mu^*L \]
We have
- \( h_0^0(\tilde{L}) = d^2h_0^0(L) \) by Proposition 2.7, and \( \delta(\tilde{L}) = \delta(L) \) by Remark 2.10.
- For all \( i = 0, ..., n \) \( \tilde{L}^{n-i}\tilde{M}^i = d^{2n-2i}L^{n-i}M^i \) by (5.7).

We can apply now the Claim of Step 1 to \((\tilde{M}, \bar{X}, \tilde{L})\):
\[ (\tilde{L})^n + n\delta(\tilde{L})(\tilde{L})^{n-1}\tilde{M} \geq \delta(\tilde{L}) n! \]
And hence (we use that \( \delta \) is bounded by 2):
\[ d^{2n}L^n + nd^{2n-2}L^{n-1}M \geq d^{2n}\delta(L) n!h_0^0(L) \]
which holds for all \( d \). Hence
$$L^n \geq \delta(L) n! h^0_a(L).$$

(ii) Assume now that $1 \leq k = \dim a(X) < n$ and that $M|G$ is big, being $G$ a general fibre of the algebraic fibre space induced by $a$ and $M$ the continuous moving part of the linear system $|L|$.

Following Theorem 3.5 (ii), up to a composition of a blow-up and an étale cover, we have $\lambda^*L = W + N$ with $|N|$ base point free and generically finite, $h^0_a(N) = (\deg \mu) h^0_a(L)$ and $\delta(N) \geq \delta(L)$. Since $L$ and $N$ are nef we have $L^n = (\lambda^*L)^n \geq N^n$. Hence its enough to prove the statement for $N$.

Take general elements $N_1,...,N_{n-k} \in |N|$ and let $T = N_1 \cap \ldots \cap N_{n-k}$. We have that $T$ is smooth, $k$-dimensional and dominates $a(X)$ since $N$ is transversal to a general $G$ (it induces a generically finite map).

Hence $T$ is of maximal $a$-dimension. Let $N_T = N|_T$. We have $h^0_a(N_T) \geq h^0_a(N)$ (by Proposition 2.7 (iii)) and $\delta(N_T) \geq \delta(N)$ (by Remark 2.9). Then we apply (i) to the pair $(T, N_T)$ with respect to $a$ and obtain

$$N^n = (N_T)^k \geq \delta(N_T) k! h^0_a(N_T) \geq \delta(N) k! h^0_a(N).$$

(iii) As in (ii) up to an étale cover and blow-up we obtain $|\lambda^*L| = W + |N|$. But in this case $N = \phi^* R$ where $\phi : \tilde{X} \longrightarrow Z$ is the algebraic fibre space induced by $|N|$, dim$Z \geq k$ and the linear system $|R|$ on $Z$ is generically finite. Up to blow-ups on $\tilde{X}$ and $Z$ we can assume that $Z$ is smooth. $L$ is big so also $\lambda^*L$ is. Contrary to (ii), in this case there is no clear relation between $\delta(R)$ and $\delta(N) \geq \delta(\lambda^*L) \geq \delta(L)$, so we only use $\delta(R) \geq 1$.

Let $r = \dim Z$ and let $\overline{G}$ be the fibre of $\phi$. By nefness of $\lambda^*L$ and $N$ and bigness of $(\lambda^*L)|_G$ we have

$$L^n = (\lambda^*L)^n \geq (\lambda^*L)^{n-r} \phi^*(R)^r = ((\lambda^*L)_{\overline{G}})^{n-r} R^r \geq R^r.$$ 

If $\dim Z = k$ we just apply (i) to the pair $(Z, R)$. If $\dim Z > k$, since $|R|$ is base point free on $Z$ and induces a generically finite map, we can apply to it the same argument as in (ii). In any case we obtain

$$R^r \geq \delta(R) k! h^0_{\overline{G}}(R) \geq k! h^0_{\overline{G}}(R) = k! h^0_a(L).$$

**Remark 5.8.** As a corollary of proof, observe that in the proof of (iii) we can obtain the same inequality than in (ii) provided that $((\lambda^*L)_{\overline{G}})^{n-r} \geq 2$. In other words, if (ii) does not hold, then $\tilde{X}$ is fibred by a family of varieties $\overline{G}$ with $((\lambda^*L)_{\overline{G}})^{n-r} = 1$.

When $X$ is smooth and minimal, $L = K_X$ and $a = alb_X$, observe that this implies that the $\tilde{a}$-fibres of $\tilde{X}$ are fibred by $\overline{G}$ such that $K^{\omega_{\overline{G}}}_{\overline{G}} = 1$. Since the general fibre of the algebraic fibre space induced by $\tilde{a}$ is isomorphic to the general fibre of the algebraic fibre space induced by alb$_X$ on $X$, then we can conclude that if $K^2_X < 2k! \chi(\omega_{\overline{G}})$, then alb$_X$ factorizes through a fibration with general fibre $G$ such that $K^{\omega_{\overline{G}}}_{\overline{G}} = 1$. In particular, either $G$ is regular or $\chi(\omega_{\overline{G}}) \leq 1$. Observe also this can not happen when dim$alb(X) = n - 1$.

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