Multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields

JIXIA YUAN\textsuperscript{1,*} LIping Sun\textsuperscript{2,†} AND WENDE LIU\textsuperscript{3,‡}

\textsuperscript{1}School of Mathematical Sciences, Heilongjiang University
Harbin 150080, China

\textsuperscript{2}School of Applied Sciences, Harbin University of Science and Technology
Harbin 150080, China

\textsuperscript{3}School of Mathematical Sciences, Harbin Normal University
Harbin 150025, China

Abstract This paper considers the multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields with characteristic zero. The main result is that there is only the multiplicative Hom-Lie superalgebra structure on these Lie superalgebras.

Keywords: Lie superalgebra, Hom-Lie superalgebra structure, automorphism

MSC 2000: 17B40, 17B66

0. Introduction

Hom–Lie algebra structures were introduced and studied in \cite{1,4}. In 2008, Q. Jin and X. Li gave a description of Hom–Lie algebra structures of Lie algebras and determined the isomorphic classes of nontrivial Hom–Lie algebra structures of finite dimensional semisimple Lie algebras \cite{5}. The Hom–Lie algebras have been sufficiently studied in \cite{6,7}.

The theory of Lie superalgebras has seen a significant development. For example, V. G. Kac classified the finite dimensional simple Lie superalgebras and the infinite dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero \cite{8,9}. In 2010, F. Ammar and A. Makhlof generalized Hom–Lie algebras to Hom–Lie superalgebras \cite{10}. In 2012, B. T. Cao and L. Luo proved that there is only the trivial Hom–Lie superalgebra structure on a finite dimensional simple Lie superalgebra of characteristic zero \cite{11}.

\textsuperscript{*}Supported by the NSF of HLJ Provincial Education Department, China (12521158)
\textsuperscript{†}Supported by the NSF of HLJ Provincial Education Department (12511349), China
\textsuperscript{‡}Corresponding author. Email: wendeliu@ustc.edu.cn. Supported by the NSF of China (11171055)
This paper is motivated by the results and methods relative to finite dimensional simple Lie superalgebras with characteristic zero (cf. [11]). In Section 1 the notations of infinite dimensional simple Lie superalgebras of vector fields were introduced. In Section 2 the multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras of vector fields were studied. We proved that there is only the trivial multiplicative Hom-Lie superalgebra structure on infinite dimensional simple Lie superalgebras with characteristic zero (cf. [11]). In Section 1 the notations of infinite dimensional simple Lie superalgebras were studied. We proved that there is only the trivial multiplicative Hom-Lie superalgebra structures on infinite dimensional simple Lie superalgebras with characteristic zero (cf. [11]).

1. Preliminaries

Throughout \( \mathbb{F} \) is a field of characteristic zero. \( \mathbb{Z}_2 := \{0,1\} \) is the additive group of two elements. \( \mathbb{N} \) and \( \mathbb{N}_0 \) are the sets of positive integers and nonnegative integers, respectively. \( \mathbb{F}[x_1, \ldots, x_m] \) denotes the polynomial algebra over \( \mathbb{F} \) in even indeterminates \( x_1, x_2, \ldots, x_m \), where \( m \geq 3 \). For positive integers \( n \geq 3 \), let \( \Lambda(n) \) be the Grassmann superalgebra over \( \mathbb{F} \) in the \( n \) odd indeterminates \( x_{m+1}, x_{m+2}, \ldots, x_{m+n} \). Clearly, 
\[
\Lambda(m, n) := \mathbb{F}[x_1, \ldots, x_m] \otimes \Lambda(n).
\]
is an associative commutative superalgebra.

Let \( \partial_r \) be the superderivation of \( \Lambda(m, n) \) defined by \( \partial_r(x_s) = \delta_x \delta_r s, s \in \mathbb{N}_0, m + n. \) The generalized Witt superalgebra \( W(m, n) \) is \( \mathbb{F} \)-spanned by \( \{ f_r \partial_r \mid f_r \in \Lambda(m, n), r \in \mathbb{N}_0, m + n \} \). Note that \( W(m, n) \) is a free \( \Lambda(m, n) \)-module with basis \( \{ \partial_r \mid r \in \mathbb{N}, m + n \} \).

For a vector superspace \( V = V_0 \oplus V_1 \), we write \( |x| := \theta \) for the parity of a homogeneous element \( x \in V_0, \theta \in \mathbb{Z}_2 \). Once the symbol \( |x| \) appears, it will imply that \( x \) is a \( \mathbb{Z}_2 \)-homogeneous element.

The following symbols will be frequently used in this paper:

- \( i'_{H} = i'_{K} := \begin{cases} i + r, & \text{if } 1 \leq i \leq r \\ i - r, & \text{if } r < i \leq 2r \\ i, & \text{if } i \in m + 1, m + n, \end{cases} \) for \( m = 2r \) or \( m = 2r + 1 \);
- \( i'_{X} := \begin{cases} i + m, & \text{if } i \in \mathbb{N}_m \\ i - m, & \text{if } i \in m + 1, 2m \end{cases} \), where \( X = HO, KO, SHO \) and \( SKO \);
- \( \text{div}(f_r \partial_r) = (-1)^{|f|} f_r \partial_r(f_r) \), where \( \text{div} \) is a linear mapping from \( W(m, n) \) to \( \Lambda(m, n) \);
- \( \text{div}_{\lambda}(f) := (-1)^{|f|} \sum_{i=1}^{m+n} \partial_{i|_{SKO}}(f_\lambda) + (\mathcal{D} - m\lambda id_{\Lambda(m,n+1)}) \partial_{2m+1}(f) \), where \( f \in \Lambda(m, n) \) and \( \lambda \in \mathbb{F} \);
- \( D_{ij}(f) := (-1)^{|\partial_i|+|\partial_j|} \partial_i(f) \partial_j - (-1)^{|\partial_i|+|\partial_j|+|f|} \partial_j(f) \partial_i \), where \( f \in \Lambda(m, m) \);
- \( D_n(f) := \sum_{i=1}^{m+n} \tau(i)(-1)^{|f|} \partial_i(f) \partial_{i'_{H}}, \) where \( m = 2r \) and \( f \in \Lambda(m, m) \);
- \( D_K(f) := \sum_{i=1}^{m+n} (-1)^{|\partial_i|+|f|} (x_i \partial_m(f) + \tau(i'_{K}) \partial_{i'_{K}}(f)) \partial_i + \left(2f - \sum_{i=1}^{m+n} x_i \partial_i(f)\right) \partial_m, \) where \( m = 2r + 1 \) and \( f \in \Lambda(m, m) \);
2. Multiplicative Hom-Lie superalgebra

Definition 2.1. A multiplicative Hom-Lie superalgebra is a triple \((\mathfrak{g}, [\cdot, \cdot], \sigma)\) consisting of a \(\mathbb{Z}_2\)-graded vector space \(\mathfrak{g}\), a bilinear map \([\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}\) and an even linear map \(\sigma: \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying

\[
\sigma[x, y] = [\sigma(x), \sigma(y)],
\]
\[
[x, y] = -(-1)^{|x||y|} \sigma[y, x],
\]
\[
(-1)^{|x||z|}[\sigma(x), [y, z]] + (-1)^{|y||z|}[\sigma(y), [x, z]] + (-1)^{|z||y|}[\sigma(z), [x, y]] = 0,
\]

where \(x, y\) and \(z\) are homogeneous elements in \(\mathfrak{g}\).

For any simple Lie superalgebra \(\mathfrak{g}\), denote its Lie bracket by \([\cdot, \cdot]\) and take an even linear map \(\sigma: \mathfrak{g} \rightarrow \mathfrak{g}\). We say \((\mathfrak{g}, \sigma)\) is a multiplicative Hom-Lie superalgebra structure over the Lie superalgebra \(\mathfrak{g}\) if \((\mathfrak{g}, [\cdot, \cdot], \sigma)\) is a multiplicative Hom-Lie superalgebra. Suppose \(\sigma \neq 0\). Eq. (2.1) and the simplicity of \(\mathfrak{g}\) show that \(\sigma\) is a monomorphism of \(\mathfrak{g}\). In particularly, if \(\sigma = \text{id}\) or \(\sigma = 0\), the multiplicative Hom-Lie superalgebra structure is called trivial. Before consider the multiplicative Hom-Lie superalgebra structures on \(X(m, n)\), we introduce the gradations on them as in [2]. For any \((m + n)\)-tuple \(\underline{\alpha} := (\alpha_1, \ldots, \alpha_m | \alpha_{m+1}, \ldots, \alpha_{m+n}) \in \mathbb{N}^{m+n}\), we may define a gradation on \(W(m, n)\) by letting \(\deg x_{i} := \alpha_{i} = -\deg \partial_{i}\), where \(i \in 1, m + n\). Thus \(W(m, n)\) becomes a graded Lie superalgebra of finite depth, i.e., we have

\[
W(m, n) = \bigoplus_{j=-h}^{\infty} W(m, n)_{\underline{\alpha}[j]},
\]
Multiplicative Hom-Lie superalgebra structures

where \( h \) is a positive integer. Put

\[
\gamma := 1 + \delta_{X,K} e_m + \delta_{X,KO} e_{2m+1} + \delta_{X,SKO} e_{2m+1} \in \mathbb{N}^{m+n}
\]

and sometimes omit the subscript \( \gamma \). Putting

\[
X(m, n)_{\gamma, [i]} := X(m, n) \cap W(m, n)_{\gamma, [i]},
\]

one sees that \( X(m, n) \) is graded by \( (X(m, n)_{\gamma, [i]} )_{i \in \mathbb{Z}} \). In particular,

- \( X(m, n)_{[-2]} = F \cdot D_X(1) \), where \( X := K, KO \) or \( SKO \);
- \( X(m, n)_{[-1]} = \text{span}_F \{ \partial_i \mid i \in \mathbb{I}, m+n \} \), where \( X := W \) or \( S \);
- \( X(m, n)_{[0]} = \text{span}_F \{ D_X(x_i) \mid \nu \neq i \in \mathbb{I} \}, \) where \( X := H, K, HO, KO, SHO \) or \( SKO \);
- \( W(m, n)_{[0]} = \text{span}_F \{ x_i \partial_j \mid i, j \in \mathbb{I}, m + n \} \);
- \( S(m, n)_{[0]} = \text{span}_F \{ x_i \partial_j + x_j \partial_i \mid \nu \neq j \in \mathbb{I}, m+n \} \);
- \( X(m, n)_{[0]} = \text{span}_F \{ D_X(x_i x_j), \delta_{X,K} D_H(x_m) + \delta_{X,KO} D_H(x_{2m+1}) \mid i, j \in \mathbb{I}, m+n \}, \) where \( X := H, K, HO \) or \( KO \);
- \( X(m, n)_{[0]} = \text{span}_F \{ D_X(x_i x_j), D_X(x_i x_j - x_j x_i), D_X(x_{2m+1} + \delta_{X,SKO} n x_i x_j) \mid \nu \neq j \in \mathbb{I}, m+n \}, \) where \( X := SHO \) or \( SKO \).

Next we give an equation and several lemmas needed in the sequel. The verifications are straightforward. The equation will be used without notice: for \( f, g \in \Lambda(m, n) \),

\[
[D_X(f), D_X(g)] = D_X \left( D_X(f) (g) - 2 \left( \delta_{X,K} - (-1)^{|f|} \delta_{X,KO} \right) \partial_\nu (f) g \right).
\]

**Lemma 2.2.** (cf. [26]) The \( \mathbb{Z} \)-graded Lie superalgebra \( X(m, n) \) is transitive, that is, if \( x \in \mathfrak{g} \) with \( i \geq 0 \) and \( [x, \mathfrak{g}_{[-1]}] = 0 \), then \( x = 0 \).

**Lemma 2.3.** For \( \mathbb{Z} \)-graded Lie superalgebra \( X(m, n) \), we have

\[
\ker(\text{ad} \partial_i) \cap X(m, n)_{[0]} = \text{span}_F \{ x_j \partial_k \mid j, k \in \mathbb{I}, m + n, i \neq j \} \cap X(m, n)_{[0]}
\]

and

\[
[\ker(\text{ad} \partial_i) \cap X(m, n)_{[0]}, \ker(\text{ad} \partial_j) \cap X(m, n)_{[0]}] = \ker(\text{ad} \partial_i) \cap X(m, n)_{[0]},
\]

where \( i \in \mathbb{I}, m + n \setminus \nu \).

**Lemma 2.4.** For \( i, j, k, l \in \mathbb{I}, m + n \), we have that

\[
x_k \partial_l \in [\ker(\text{ad} \partial_i) \cap X(m, n)_{[0]}, \ker(\text{ad} \partial_j) \cap X(m, n)_{[0]}],
\]

where \( k \neq j, l \);

\[
D_X(x_i x_j) \in [\ker(\text{ad} D_X(x_i x_j)) \cap X(m, n)_{[0]}, \ker(\text{ad} D_X(x_i x_j)) \cap X(m, n)_{[0]}],
\]

where \( k \neq l \in \mathbb{I}, m + n \setminus \nu \).

The next proposition is essential for the main result in this paper.
Proposition 2.5. If \((X(m,n), \sigma)\) is a multiplicative Hom-Lie superalgebra structure and \(\sigma \neq 0\), then

\[\sigma |_{X(m,n)_{[-1]}} = \text{id} |_{X(m,n)_{[-1]}}.\]

Proof. Case 1: \(X := W\) or \(S\). By Eq. (2.2), we have

\[
0 = (-1)^{|\partial_i||\partial_j|}\sigma(\partial_i), [\partial_j, x_j\partial_k]] + (-1)^{|\partial_i||\partial_j|}\sigma(\partial_j), [x_j\partial_k, \partial_i] \\
+ (-1)^{|\partial_i||\partial_j|}\sigma(\partial_j), [\partial_i, \partial_j] \\
= (-1)^{|\partial_i||\partial_j|}\sigma(\partial_i), \partial_k],
\]

where \(j \neq k, k \in m + n\). By Lemma 2.2, we have

\[
\sigma(X(m,n)_{[-1]}) = X(m,n)_{[-1]}.
\]

Then for any \(i \in m + n\), one may suppose \(\sigma(\partial_i) = \sum_{l=1}^{m+n} a_l \partial_l\), where \(a_l \in F\). Lemma 2.3 and Eq. (2.2) imply that \(\sigma(\partial_i) = a_i \partial_i\). For distinct \(i, j, k \in m + n\), put \(x = x_j \partial_j - x_k \partial_k\), \(y = \partial_i\) and \(z = x_i \partial_j\). Then Eq. (2.2) implies that

\[
[\sigma(x_j \partial_j - x_k \partial_k), \partial_j] + [\sigma(\partial_i), x_i \partial_j] = 0. \tag{2.3}
\]

Suppose \(\sigma^{-1}\) is an left linear inverse of \(\sigma\) (vector space). Then

\[
\sigma^{-1}([\sigma(x_j \partial_j - x_k \partial_k), \partial_j]) = [x_j \partial_j - x_k \partial_k, \sigma^{-1}(\partial_j)] = [x_j \partial_j - x_k \partial_k, a_j^{-1} \partial_j] = -a_j^{-1} \partial_j.
\]

Hence

\[
[\sigma(x_j \partial_j - x_k \partial_k), \partial_j] = -\partial_j.
\]

By Eq. (2.3), we have \(a_i = 1\), where \(i \in m + n\). That is

\[
(\sigma - \text{id}) |_{X(m,n)_{[-1]}} = 0.
\]

Case 2: \(X := H, K, HO, KO, SHO\) or \(SKO\). For \(i, j, k \in m + n\), Eq. (2.2) implies that

\[
0 = (-1)^{|D_X(x_i)||D_X(x_j' x_k')}[\sigma(D_X(x_i)), [D_X(x_j), D_X(x_j' x_k')]] \\
+ (-1)^{|D_X(x_i)||D_X(x_j')}[\sigma(D_X(x_j)), [D_X(x_j' x_k'), D_X(x_i)]] \\
+ (-1)^{|D_X(x_j') x_k'|}[\sigma(D_X(x_j' x_k'), [D_X(x_i), D_X(x_j)])].
\]

It follows that

\[
[\sigma(D_X(x_i)), D_X(x_{j' k'})] = 0, \quad i \neq j, j', k \tag{2.4}
\]

and

\[
[\sigma(D_X(x_i)), D_X(x_{j' k'})] = [\sigma(D_X(x_j')), D_X(x_{j' k'})], \quad i \neq j, j'. \tag{2.5}
\]

By Eq. (2.4), (2.5) and Lemma 2.2, it is easy to obtain that

\[
\sigma(D_X(x_i)) = a_i D_X(1) + \sum_{\nu \neq i} a_{i\nu} D_X(x_i)
\]
for some $a_i, a_{il} \in \mathbb{F}$.

**Subcase 2.1:** $X := H, HO,$ or SHO. Lemma 2.3 and Eq. (2.2) imply that $a_{ik} = 0$ for all $i \neq k \in 1, m + n \setminus \nu$. Thus,

$$(\sigma - \text{id})|_{X(m,n)_{[-1][\text{-}1]}} = 0.$$ 

**Subcase 2.2:** $X := K, KO$ or SKO. Put $x = D_X(x_i), y = D_X(x_j)$ and

$$z = D_X(x_{j_X} x_{x} + \delta_{X, SKO}(-1)|^{x_{j_X} x_{1}}(m\lambda - 1)x_kx_{k_X}x_{j_X}.$$ 

Eq. (2.2) implies that

$$0 = (-1)^{|D_X(x_i)| ||z|}[\sigma(D_X(x_i)), D_X(x_{x} + \delta_{X, SKO}(m\lambda - 1)x_kx_{k_X} + x_jx_{j_X})]$$

$$+ (-1)^{|D_X(x_i)| |D_X(x_j)|}[\sigma(D_X(x_j)), D_X(x_{j_X} x_{j_X})].$$

Hence $a_i = 0$. Take $i, j, k \in 1, m + n \setminus \nu$ and $i \neq j, j_X', k, k_X$. Put $x = D_X(x_j x_k) \in X(m, n)_0$, $y = D_X(x_i)$ and $z = D_X(x_{i_X} x_{j_X})$. By Eq. (2.2) again, we have

$$0 = (-1)^{|\partial_x| ||z_x|}[\sigma(D_X(x_j x_k)), D_X(x_{j_X})]$$

$$+ (-1)^{|\partial_x| ||z_x| ||z_{x_{j_X}}| + |\partial_x| ||z_{x_{j_X}}|}[\sigma(D_X(x_i)), D_X(x_{i_X} x_k)].$$

Suppose $\sigma^{-1}$ is a left inverse of $\sigma$. Then

$$\sigma^{-1}([\sigma(D_X(x_j x_k)), D_X(x_{j_X})]) = [D_X(x_j x_k), \sigma^{-1}(D_X(x_{j_X})]$$

$$= [D_X(x_j x_k), a_{j_X j_X}^{-1} D_X(x_{j_X})]$$

$$= (-1)^{|\partial_x| ||z_x|} a_{j_X j_X}^{-1} D_X(x_k).$$

Hence

$$- (-1)^{|\partial_x| ||z_x|} a_{j_X j_X}^{-1} D_X(x_k) = [\sigma(D_X(x_j x_k)), D_X(x_{j_X})]$$

$$= (-1)^{|\partial_x| ||z_x| + |\partial_x| ||z_x|}[\sigma(D_X(x_i)), D_X(x_{i_X} x_k)]$$

$$= (-1)^{|\partial_x| ||z_x|} a_{i_X} D_X(x_k).$$

The arbitrariness of $j$ and $k$ implies that $a_{ii} = 1$. Thus,

$$(\sigma - \text{id})|_{X(m,n)_{[-1][\text{-}1]}} = 0.$$ 

**Proposition 2.6.** If $(X(m, n), \sigma)$ is a multiplicative Hom-Lie superalgebra structure and $\sigma \neq 0$, then

$$\sigma|_{X(m,n)_{[0][0]}} = \text{id}|_{X(m,n)_{[0][0]}}.$$ 

**Proof.** Case 1: $X := W$ or S. Put $x \in X(m, n)_{[0][0]}$. Then by Proposition 2.5 we have

$$[\sigma(x), \partial_l] = [\sigma(x), \sigma(\partial_l)] = \sigma([x, \partial_l]) = [x, \partial_l]$$
for all $i \in [1, m + n]$. By Lemma 2.2 we may write
\[ \sigma(x_i \partial_j) = x_i \partial_j + \sum_{s=1}^{m+n} a_{ij}s \partial_s, \]
where $a_{ij}s \in \mathbb{F}$. By Lemma 2.4 and Eq. (2.2), we have
\[ a_{ijk} \partial_l = \sum_{s=1}^{m+n} a_{ij}s \partial_s, x_k \partial_l = 0 \]
for $k, l \in [1, m + n]$ and $k \neq l$. By the arbitrariness of $l$, we know $a_{ijk} = 0$ for all $j \neq k \in [1, m + n]$. Put $x = x_i \partial_j$, $y = x_j \partial_k$, $z = x_i \partial_l - (-1)^{(|x|+|x|)} x_s \partial_s$. By Eq. (2.2) we have that
\[ [\sigma(x_i \partial_j), x_j \partial_k] + [\sigma(x_i \partial_l - (-1)^{(|x|+|x|)} x_s \partial_s), [x_i \partial_j, x_j \partial_k]] = 0. \]
Furthermore,
\[ [a_{ijj} \partial_j, x_j \partial_l] + [a_{llj} \partial_l - a_{sj}s \partial_s, x_i \partial_l] = 0. \]
Then $a_{ijj} = 0$. Summarizing, we have $\sigma(x_i \partial_j) = x_i \partial_j$.

**Case 2**: $X := H, K, HO, KO, SHO$ or $SKO$. For $x \in X(m, n)\{0\}$, by Proposition 2.5 we have
\[ [\sigma(x), D_X(x_i)] = [\sigma(x), (D_X(x_i))] = [\sigma(x), D_X(x_i)] = [x, D_X(x_i)] \quad (2.6) \]
where $i \in [1, m + n] \setminus \nu$. By Lemma 2.2 we may write
\[ \sigma(D_X(x_i x_j)) = D_X(x_i x_j) + \sum_{s \neq 1}^{m+n} a_{ij}s D_X(x_s) + a_{ij} D_X(1), \]
where $D_X(x_i x_j) \in X(m, n)\{0\}$ and $a_{ij}, a_{ij}s \in \mathbb{F}$. By Lemma 2.4 and Eq. (2.2) we have that
\[ \pm a_{ijk} x_i X = \pm a_{ij} X x_i D_X(x_k) = \left[ \sum_{s=1}^{m+n} a_{ij}s D_X(x_s), D_X(x_k x_i) \right] = 0, \]
where $k \neq l \in [1, m + n] \setminus \nu$ and $k, l \neq i', j'$. That is $a_{ij} = 0$ for all $i, j \neq s \in [1, m + n] \setminus \nu$.

Then
\[ \sigma(D_X(x_i x_j)) = D_X(x_i x_j) + a_{ij} D_X(x_i) + a_{ijj} D_X(x_j) + a_{ij} D_X(1). \]
If $k = i'$ or $k = j'$, by Eq. (2.2) we have
\[ \pm a_{ijk} D_X(x_i) = [a_{ij} D_X(x_i) + a_{ijj} D_X(x_j), D_X(x_k x_i)] = 0 \]
Hence
\[ \sigma(D_X(x_i x_j)) = D_X(x_i x_j) + a_{ij} D_X(1). \quad (2.7) \]

**Subcase 2.1**: $X := H$ or $HO$. From Eq. (2.7), we have
\[ (\sigma - \text{id}) |_{X(m, n)\{0\}} = 0. \]
Subcase 2.2: $X := \text{SHO}$. By Eq. \eqref{eq:2.7} again, for $i \neq j \in 1, m \setminus \nu$ we can obtain
\[
\sigma(D_X(x_i x_{\nu}) - x_j x_{j_{k}^\nu})) = \sigma(D_X(x_i x_j), D_X(x_{i_{k}^\nu} x_{j_{k}^\nu})) = \sigma(D_X(x_i x_j)) \sigma(D_X(x_{i_{k}^\nu} x_{j_{k}^\nu})) = D_X(x_i x_{i_{k}^\nu} - x_j x_{j_{k}^\nu}).
\]

From Eq. \eqref{eq:2.7} and \eqref{eq:2.8}, we know
\[
(\sigma - \text{id}) \mid_{X(m,n)[0]} = 0.
\]

Subcase 2.3: $X := K, KO$ or $SKO$. Take $x = D_X(x_i x_j), y = D_X(x_k)$ and $z = D_X(x_{k_{l}^t} x_{v} + (-1)^{|x_{k_{l}^t}| |(n\lambda - 1) x_{j} x_{j_{k}^\nu}}$, where $i, j, k, k_{l}^t, l, l_{t}^l \in 1, m + n$ are distinct. By Eq. \eqref{eq:2.2}, we have
\[
-2a_{ij} = [\sigma(x), [y, z]] = (1 - 1)^{|x_{k_{l}^t}| |(n\lambda - 1) x_{j} x_{j_{k}^\nu} + (-1)^{|x_{k_{l}^t}| |(n\lambda - 1) x_{j} x_{j_{k}^\nu}}] = 0.
\]

Hence
\[
\sigma(D_X(x_i x_j)) = D_X(x_i x_j).
\]

By Eq. \eqref{eq:2.7}, we can write
\[
\sigma(D_X(x_v + \delta_{X,SKO} \lambda x_j x_{j_{k}^\nu})) = D_X(x_v + \delta_{X,SKO} \lambda x_j x_{j_{k}^\nu}) + \sum_{\nu \neq \nu, \nu, \nu = 1} a_{\nu j} D_X(x_s) + a_{\nu j} D_X(1).
\]

Using the method above, we can obtain easily
\[
\sigma(D_X(x_v + \delta_{X,SKO} \lambda x_j x_{j_{k}^\nu})) = D_X(x_v + \delta_{X,SKO} \lambda x_j x_{j_{k}^\nu}).
\]

By Eq. \eqref{eq:2.7}, \eqref{eq:2.8} and \eqref{eq:2.9}, we have
\[
(\sigma - \text{id}) \mid_{X(m,n)[0]} = 0.
\]

The proof is complete. \hfill $\Box$

Theorem 2.7. There is only the trivial multiplicative Hom-Lie superalgebra structure on the infinite dimensional simple Lie superalgebras of vector fields.

Proof. Let $(X(m, n), \sigma)$ be a multiplicative Hom-Lie superalgebra structure and $\sigma \neq 0$. By Propositions \ref{prop:2.6} and \ref{prop:2.7} we have
\[
\sigma \mid_{X(m,n)[-1] \oplus X(m,n)[0]} = \text{id} \mid_{X(m,n)[-1] \oplus X(m,n)[0]} \cdot \sigma(x - x, [y, z] = 0.
\]

Then $\sigma(x) - x = 0$. We get $\sigma = \text{id}$. The proof is complete. \hfill $\Box$
References

[1] J. T. Hartwig, D. Larsson, S. D. Silvesrov. Derformations of the Lie algebras using $\sigma$-derivations. *J. Algebra*, 295 (2007): 314–361.

[2] D. Larsson, S. D. Silvesrov. Quasi-hom-Lie algebras, Central extensions and 2-cocycle-like identities. *J. Algebra*, 288 (2005): 321–344.

[3] D. Larsson, S. D. Silvesrov. Quasi-derformations of $\text{sl}_2(\mathbb{F})$ using twisted derivations. *Comm. Algebra*, 35 (2007): 4303–4318.

[4] D. Larsson, S. D. Silvesrov. Graded quasi-Lie algebras. *Czechoslovak J. Phys.*, 55 (2005): 1473–1478.

[5] Q. Jin, X. Li. Hom-Lie algebra stuctures on semi-simple Lie algebras. *J. Algebra*, 319 (2008): 1398–1408.

[6] S. Benayadi, A. Makhlouf. Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. *arXiv: 1009.4226*, (2010).

[7] Y. H. Sheng. Representations of Hom-Lie algebras. *Algebr. Represent. Theory*, DOI: 10.1007/s10468-011-9280-8.

[8] V. G. Kac. Lie superalgebras. *Adv. Math.*, 26 (1977): 8–96.

[9] V. G. Kac. Classification of infinite dimensional simple linearly compact Lie superalgebras. *Adv. Math.*, 139 (1998): 1–55.

[10] F. Ammar, A. Makhlouf. Hom-Lie superalgebras and Hom-Lie admissible superalgebras. *J. Algebra*, 324 (2010): 1513–1528.

[11] B. T. Cao, L. Luo. Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras. *arXiv:1203.0136* (2012).