Existence results of the $m$–polyharmonic Kirchhoff problems

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Abstract
We extend and complement previous existence results in the literature to the following $m$–polyharmonic Kirchhoff problem:

$$
\begin{align*}
M(\|u\|_{rm}^m)\Delta_r^m u &= f(x,u) \quad \text{in } \Omega, \\
u &= \left(\frac{\partial}{\partial \nu}\right)^{k} u = 0, \quad \text{on } \partial \Omega, \quad k = 1, 2, \ldots, r - 1,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $r \in \mathbb{N}^*$, $m > 1$, $N \geq rm + 1$, $M$ is a Kirchhoff function and $\| \cdot \|_{r,m}$ is the norm of $W_0^{r,m}(\Omega)$. Our aim is to prove the existence of infinitely many solutions of (1.1) for some odd functions $f$ in $u$, without requiring any control on $f$ near 0. The new aspect here consists in employing the Schauder basis of $W_0^{r,m}(\Omega)$. We will also weaken the analogue of Ambrosetti-Rabinowitz condition, the standard subcritical polynomial growth and the strong $m\gamma$-superlinear conditions required in [6]. Similarly, we establish the existence of infinitely many solutions for the problem

$$
M(\|u\|_{rm}^m)\Delta_r^m u + a|u|^{m-2} u = K(x)f(u) \quad \text{in } \mathbb{R}^N,
$$

where $a$ is a nonnegative real number (which covers the $m\gamma$-zero mass case if $a = 0$), $K$ is a continuous positive weight function such that $K \in L^p(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p \geq 1$ or $K \to 0$ as $|x| \to \infty$.

In analogy with the first eigenvalue of the $m$-polyharmonic operator, we introduce a positive quantity $\lambda_M$ to find a mountain pass solution, we discuss also the $m\gamma$-sublinear-polyharmonic problem under large growth conditions at infinity and at zero in a bounded domain.

Keywords: Palais-Smale condition, Symmetric mountain pass theorem, Schauder basis, Krasnoselskii genus theory, $m$-polyharmonic operator, Kirchhoff equations, Zero mass case.
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1. Introduction
In this paper, we consider the following nonlocal Kirchhoff-type problem:

$$
\begin{align*}
M(\|u\|_{rm}^m)\Delta_r^m u &= f(x,u) \quad \text{in } \Omega, \\
u &= \left(\frac{\partial}{\partial \nu}\right)^{k} u = 0, \quad \text{on } \partial \Omega, \quad k = 1, 2, \ldots, r - 1,
\end{align*}
$$

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where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $r > N/m$, $N \geq rm + 1$, $M \in C([0, +\infty))$ is a non-negative function and $f \in C(\overline{\Omega} \times \mathbb{R})$. To describe our results more accurately, let us first introduce some notations and definitions. The $m$-polyharmonic operator $\Delta_m^r$ is defined by

$$
\Delta_m^r u = \begin{cases} 
-\text{div} \left( \Delta^{j-1} (\nabla \Delta^{j-1} u) \right), & \text{if } r = 2j - 1, \\
\Delta^{j} (|\Delta^j u|^{m-2} \Delta^j u), & \text{if } r = 2j, 
\end{cases} \quad j \in \mathbb{N}^*,
$$

which becomes the usual polyharmonic operator for $m = 2$, namely $(-\Delta)^r$. Define the main $r$-order differential operator by

$$
D_r u = \begin{cases} 
\nabla \Delta^{j-1} u, & \text{if } r = 2j - 1, \\
\Delta^j u, & \text{if } r = 2j, 
\end{cases} \quad j \in \mathbb{N}^*.
$$

Note that $D_r u$ is an $N$-vectorial operator when $r$ is odd and $N > 1$, while it is a scalar operator when $r$ is even.

In the sequel, if $\Omega$ is a bounded domain then $E_{r,m} = W_0^{r,m}(\Omega)$ and for $\Omega = \mathbb{R}^N$, $E_{r,m} = W_0^{r,m}(\mathbb{R}^N)$ if $a > 0$ (respectively $E_{r,m} = D^m(\mathbb{R}^N)$ if $a = 0$). In order to simplify the presentation we will denote the norm of $W_0^{r,m}(\Omega)$ (respectively $W_0^{r,m}(\mathbb{R}^N)$) by $\| \cdot \|$ instead of $\| \cdot \|_{W_0^{r,m}(\Omega)}$ (respectively $\| \cdot \|_{W_0^{r,m}(\mathbb{R}^N)}$). Then $E_{r,m}$ endowed with the norm which is equivalent to the standard one:

$$
\| u \|_E^{r,m} = \int_{\Omega} |D_r u|^m, \text{ (respectively } \| u \|_E^{r,m} = \int_{\mathbb{R}^N} (|D_r u|^m + a |u|^m)),
$$

so it is well known that $(E_{r,m}, \| \cdot \|)$ is a separable, uniformly convex, reflexive, real Banach space (see [3]). We will also use the Gagliardo-Nirenberg-Sobolev inequality

$$
\| u \|_{L^p(\Omega)} \leq C \| u \|_{E_{r,m}}, \quad \forall u \in E_{r,m},
$$

(1.2)

where $C$ is a positive constant and $p^* = \frac{mN}{N - rm}$ is the Sobolev critical exponent. Denote $E_{r,m}^*$ the dual space of $E_{r,m}$ and $q^* = \frac{p^*}{p^* - 1}$ the conjugate exponent of $p^*$.

If $\Omega$ is a bounded domain, $M \in C([0, +\infty))$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies the following large growth condition

$$
|f(x, s)| \leq C(1 + |s|^{p-1}) \text{ for all } (x, s) \in \overline{\Omega} \times \mathbb{R},
$$

then the Euler-Lagrange functional associated to problem (1.1) given by

$$
I(u) = \frac{1}{m} \overline{M}(\| u \|^m) - \int_{\Omega} F(x, u), \quad \forall u \in E_{r,m}
$$

(1.3)

is well defined and $I \in C^1(E_{r,m})$ with

$$
(I'(u), v) = M(\| u \|^m) \int_{\Omega} |D_r u|^{m-2} D_r u D_r v - \int_{\Omega} f(x, u)v, \quad \forall u, v \in E_{r,m}.
$$

(1.4)

So, $u \in E_{r,m}$ is a weak solution of (1.1) if and only if $u$ is a critical point of $I$. Problem (1.1) is called nonlocal due to the presence of the term

$$
M \left( \int_{\Omega} |D_r u(x)|^m \right) \text{ or } M \left( \int_{\mathbb{R}^N} |D_r u|^m + a |u|^m \right),
$$

which implies that the equation in (1.1) is no longer a pointwise identity. Recall now the Palais-Smale and the Cerami compactness conditions which are essential to apply the minimax methods.

\[\text{[1]}\text{Recall that } D^m(\mathbb{R}^N) \text{ is defined as the completion of } C^\infty(\mathbb{R}^N) \text{ with respect the norm } \| u \|^m = \int_{\mathbb{R}^N} |D_r u|^m.\]
Definition 1.1. 1. We say that \( \{u_n\} \) is a Palais-Smale sequence ("(PS) sequence" for short) of \( I \) if \( I(u_n) \) is bounded and \( I'(u_n) \to 0 \) as \( n \to \infty \) in \( E_{r,m} \);

2. we say that \( \{u_n\} \) is a Cerami sequence ("(C) sequence" for short) of \( I \) if \( I(u_n) \) is bounded and \( (1+\|u_n\||I'(u_n)||E_{r,m} \to 0 \) as \( n \to \infty \);

3. we say that \( I \) satisfies the (PS) condition (respectively the (C) condition) if any (PS) sequence (respectively (C) sequence) has a (strongly) convergent subsequence in \( E_{r,m} \).

Note that the (C) condition is weaker than the (PS) condition, and which allows rather general minimax results (see [3]).

In recent years, there has been an increasing interest in studying problem (1.1), which has a broad background in many different applications, such as game theory, mathematical finance, continuum mechanics, phase transition phenomena, population dynamics and minimal surface. The reader may consult [1, 6, 8, 12, 14, 15, 17, 24] and the references therein. The most recent papers for Kirchhoff problems deal only with the non-degenerate case, that is when \( M(\tau) \geq s > 0 \) for all \( \tau \in [0, +\infty) \) (see [8, 24]). Especially, by covering the degenerate case, Colasuonno-Pucci established in an elegant paper the existence of infinitely many solutions by using minimax approach [6].

1.1. The \( m \)-Superlinear case on a bounded domain.

Assume that \( M : [0, +\infty) \to [0, +\infty) \) a continuous function. We first relax the global structural assumption imposed on \( M \) in [6] into the following:

\( \left( M_1 \right) : \) there exist \( \tau_0 \geq 0 \) and \( \gamma \in (1, \frac{p^*}{m}) \) such that \( \tau M(\tau) \leq \gamma \widehat{M}(\tau), \forall \tau \geq \tau_0, \) where \( \widehat{M}(\tau) = \int_0^\tau M(z)dz. \)

As in [6] we assume:

\( \left( M_2 \right) : \) for each \( \eta > 0 \) there is \( m_\eta > 0 \) such that \( M(\tau) \geq m_\eta, \forall \tau \geq \eta. \)

For the nonlinearity \( f \) we need the following growth conditions at infinity which are somehow related to [5,11]:

\( \left( H_1 \right) : \) there exist \( s_0 > 0 \) and \( C > 0 \) such that \( C|f(x, s)|^p \leq sf(x, s) - myF(x, s), \forall s > s_0 \) and \( x \in \Omega, \)

where \( F(x, s) = \int_0^s f(x, t)dt; \)

\( \left( H_2 \right) : \) \( \lim_{s \to \infty} \frac{f(x, s)}{|s|^{m+1}} = 0, \) uniformly with respect to \( x \in \Omega; \)

\( \left( H_3 \right) : \) \( \lim_{s \to \infty} \frac{F(x, s)}{|s|^{m+1}} = \infty, \) uniformly in \( \Omega. \)

Note that for \( m \neq 2, \) the variational setting of [1,11] lacks an ordered Hilbert space structure which provokes some mathematical difficulties to obtain infinitely many solutions when \( f(x, \cdot) \) is an odd function. So, an adequate growth condition at zero was required to exhibit the mountain pass geometry related to the abstract Theorem 2.2 stated in [6] (see also [3]). In a very interesting paper [4], the authors developed a new variational method to obtain infinitely many solutions for the \( m \)-Laplacian equation (i.e., \( r = 1 \) and \( M = 1 \)) without any control on \( f \) at zero . Differently to [6,4,9], we will apply the more general version of the symmetric mountain pass theorem, that is:

Theorem A. ([25]). Let \( E \) be a real infinite dimensional Banach space and \( I \in C^1(E) \) satisfying the (PS) condition with \( I(0) = 0. \) Suppose \( E = E^- \oplus E^+, \) where \( E^- \) is finite dimensional, and assume the following conditions:

1. \( I \) is even;

2. for any finite dimensional subspace \( W \subset E \) there is \( R = R(W) \) such that \( I(u) \leq 0 \) for \( u \in W, \|u\| \geq R; \)

3. there exist \( \alpha > 0 \) and \( \rho > 0 \) such that \( I(u) \geq \alpha \) for any \( u \in E^+ \) with \( \|u\| = \rho; \)

then, \( I \) possesses an unbounded sequence of critical values.
To do so, we shall employ a Schauder basis of $E_{r,m}$ (see Corollary 3 in [10]) to show that only the quasicritical condition $(H_2)$ allows removal of any growth condition on $f$ near zero. More precisely, let $(e_j)_{j \in \mathbb{N}^+}$ be a Schauder basis of $E_{r,m}$, which means that each $x \in E_{r,m}$ has a unique representation $x = \sum_{i=1}^j a_ie_i$, where $a_i$ are real numbers. Set $E_j = \text{span}(e_1, e_2, \ldots, e_j)$. As a consequence, the linear projection onto $E_j$ i.e., $P_j : E_{r,m} \to E_j$, $P_j(x) = \sum_{i=1}^j a_ie_i$ is continuous for all $j \in \mathbb{N}^+$ (see [26]). Therefore, $F_j = N(P_j)$ (the kernel of $P_j$) is a topological complement of $E_j$, that is $E_j \oplus F_j = E_{r,m}$.

Fix $\rho \geq 0$ and set $S^+_j(\rho) = \{u \in F_j \text{ such that } \|u\| = \rho\}$ and $\beta_j := \sup_{s \in S^+_j(\rho)} \int_{\Omega} |F(x,u)|$. The following lemma is crucial to prove condition 3 of Theorem A.

**Lemma 1.1.** Assume that $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies $(H_2)$, then  

1. $\beta_j \to 0$, as $j \to +\infty$.

2. Consequently, for all $\rho \geq 0$, there exist $j_0$ and $\alpha > 0$ such that $I(u) \geq \alpha$, $\forall u \in S^+_j(\rho)$.

The two main assumptions that appeared in a rich literature ensuring the $(PS)$ condition are the following (see also [8, 18, 21, 25]):

$$(AR)_y$$ there are constants $\theta > my$ and $s_0 > 0$ such that $sf(x,s) \geq \theta F(x,s) > 0$, $\forall |s| > s_0$ and $\forall x \in \Omega$;

$$(SCP)$$ there exist $C > 0$ and $p$ satisfying $\theta \leq p < p^* - 1$ such that $|f(x,s)| \leq C|s|^p + 1$, $\forall (x,s) \in \Omega \times \mathbb{R}$.

In fact, $(AR)_y$ is the analogue of the Ambrosetti-Rabinowitz condition related to the Kirchhoff function $M$ and requires the following severe restriction called the strong $my$-superlinear condition, which is also useful to provide the mountain pass structure:

$$(SSL)$$ there exists $C > 0$ such that $F(x,s) \geq C|s|^p$, $\forall x \in \Omega$ and $\forall |s| \geq s_0$.

When $M = 1$ (and so $\gamma = 1$), some attempts were made to relax conditions $(AR)_1$ and $(SCP)$ (see [4, 5, 7, 11, 16, 18, 20, 22, 27] and the references therein). Obviously, $(SCP)$ implies $(H_2)$, also $(H_2)-(AR)_2$ imply $(H_1)$ (see Appendix (B)). Therefore $(H_1)-(H_2)$ are weaker than $(AR)_1-(SCP)$. Moreover $(H_1)$ will no longer require $(SSL)$ and so covers a large class of nonlinearities having an asymptotical behaviour at infinity such as $a|s|^q - s(\ln(|s|))$ or even $a|s|^q - 2s$ with $a > 0$ (see Appendix (B) for further comments).

**Proposition 1.1.** Assume that $f$ and $M$ verify $(H_1) - (H_2)$ and $(M_1) - (M_2)$ respectively, then

1) $I$ satisfies the Palais-Smale condition if $m \geq 2$.

2) $I$ satisfies the Cerami condition if $1 < m < 2$.

In addition we relax $(SSL)$ into $(H_2)$ to improve some multiplicities results in [6, 9, 11] and also in [4] where $(AR)_1$ was assumed and our proof is more easier since we will not here use any cut-off argument.

**Theorem 1.1.** Let $f \in C(\overline{\Omega} \times \mathbb{R})$ be such that $f(x,.)$ is an odd function for all $x \in \Omega$. Assume that $f$ satisfies $(H_1)-(H_2)$ and $M$ verifies $(M_1) - (M_2)$, then $I$ admits infinitely many distinct pairs $(u_j, -u_j)$, $j \in \mathbb{N}^+$, of critical points. Moreover, $I(u_j)$ is unbounded.

---

2 More precisely, $P_j$ are uniformly bounded, that is there exists $C > 0$ such that $\|P_j(x)\| \leq C\|x\|$ for each $j \in \mathbb{N}^+$ and all $x \in E_{r,m}$.

3 A simple examination of the proof of Proposition 1.1 shows that $I$ satisfies the Palais-Smale condition if $m > \frac{1}{q^*} + 1$ which is verified for $m \geq 2$ and $N > rm$. 
Mountain pass solution: To provide the mountain pass structure in the more familiar setting in the literature in which $M = 1$, we require that $F(x, s)$ grows less rapidly than $\frac{\lambda_1}{m}|s|^m$ near 0 and more rapidly than $\frac{\lambda_3}{m}|s|^m$ at infinity, where

$$\lambda_1 := \inf_{u \neq 0} \frac{\int_{\Omega} |Du|^m \, dx}{\int_{\Omega} |u|^m \, dx} > 0,$$

is the first “eigenvalue” of $\Lambda'_m$. By analogy with $\lambda_1$, we set

$$\lambda_M := \inf_{u \neq 0} \frac{\tilde{M}(|u|^m)}{\int_{\Omega} |u|^m \, dx},$$

with $1 < m\gamma < p^*$. Introduce now the following coercive condition on $M$

$$(M_3) : \text{there is a positive constant } C \text{ such that } C\tau^\gamma \leq \tilde{M}(\tau), \forall \tau \geq 0.$$

Then, we have (see the proof in Appendix (A)):

Lemma 1.2. (i) $\lambda_M$ is positive if and only if $M$ satisfies $(M_3)$;

(ii) if $M = C\tau^\gamma$, then $\lambda_M$ is attained. However, under $(M_3)$ in general, $\lambda_M$ is not attained.

In addition to $(H_1)$–$(H_2)$ we only need the following large growth conditions at infinity and at zero:

$$(H'_3) : \limsup_{s \to 0} \frac{F(x, s)}{|s|^m} < \frac{\lambda_M}{m} < \liminf_{s \to \infty} \frac{F(x, s)}{|s|^m}, \text{ uniformly in } \overline{\Omega}.$$

However, we have to assume that $(M_1)$ is global, i.e. $\tau_0 = 0.$ So, we have

Theorem 1.2. Under $(H_1)$–$(H_3)$–$(H'_3)$, $(M_1)$ (with $\tau_0 = 0$) and $(M_3)$, the problem (1.1) has a nontrivial mountain pass solution.

Also Theorem 1.2 holds if we substitute assumptions $(M_1)$ and $(H'_3)$ respectively by $(M_2)$ and the following strong condition:

$$\limsup_{s \to 0} \frac{F(x, s)}{|s|^m} = 0 \text{ and } \liminf_{s \to \infty} \frac{F(x, s)}{|s|^m} = \infty, \text{ uniformly in } \overline{\Omega}.$$

1.2. The $m\gamma$–Superlinear case on $\Omega = \mathbb{R}^N$.

There are few papers considering Kirchhoff type problems on $\mathbb{R}^N$ see [12, 13, 17] and the references therein. In particular [2] a ground state positive solution was obtained for the following problem:

$$-\Delta u = K(x)f(u), \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$ and $K \in L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for some $p \geq 1$. Recently, Li-Li-Shi [17] studied the existence of positive solutions for the following nonlinear Kirchhoff type problem:

$$\begin{cases}
- \left( b + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u = K(x)f(u), & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}$$

where $N \geq 3$, $b > 0$, $\lambda \geq 0$ and $K$ is a weight function satisfies:

$K : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function and $\overline{K} \in [L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)] \setminus \{0\}$ for some $p \geq \frac{2N}{(N + 2)}.$
For \( r \in \mathbb{N}^* \), \( m > 1 \), \( N > rm \), we discuss the existence of infinity many solutions of the following nonlocal Kirchhoff \( m \)-polyharmonic equation:

\[
M(||u||^m)(\Delta_{m} u + a|\Delta u|^{m-2} u) = K(x)f(u) \quad \text{in } \mathbb{R}^N,
\]

where \( a \) is a nonnegative real number, \( M : [0, +\infty) \to [0, +\infty) \) is a continuous Kirchhoff function, \( f \in C(\mathbb{R}) \) and \( K \) is a continuous positive weight function satisfying

\((K_p)\) : there exists \( p \geq 1 \) such that \( K \in L^{\infty}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \) or \( (K_0) : K(x) \to 0 \) as \( |x| \to \infty \).

We shall discuss two cases, if \( a > 0 \) we assume:

\((h_0)\) : there exists \( C > 0 \) such that \( |f(s)| \leq C|s|^{m-1}, \forall s \in [-1, 1] \).

If \( a = 0 \) which covers the \( m'y \)-zero mass case, we need more restriction growth condition at zero:

\((h'_0)\) : there exists \( C > 0 \) such that \( |f(s)| \leq C|s|^{p-1} \).

Similarly to \((H_1)-(H_3)\) and in both cases we assume

\((h_1)\) : there exists \( C > 0 \) such that \( C|f(s)|^p \leq f(s)s - mf(s), \forall s \in \mathbb{R}; \)

\((h_2)\) : \( f \) is quasicritical at infinity, i.e. \( \lim_{|s| \to +\infty} \frac{f(s)}{|s|^p} = 0; \)

\((h_3)\) : \( f \) is \( m'y \)-superlinear at infinity in the sense that: \( \lim_{|s| \to +\infty} \frac{F(s)}{|s|^{m'y}} = +\infty. \)

Our multiplicity existence result reads as follows

**Theorem 1.3.** Suppose that:

- \( M \) verifies \((M_1)\) and \((M_2)\);

- \( f \) is an odd function satisfying \((h_1)-(h_2)\) and \((h_0)\) if \( a > 0 \) or \((h'_0)\) if \( a = 0 \);

- \( K \) is a positive weight function satisfying \((K_p)\) or \((K_0)\);

then, \( I_k \) admits infinitely many distinct pairs \((u_j, -u_j)\), \( j \in \mathbb{N}^* \), of critical points. Moreover, \( I_k(u_j) \) is unbounded.

Note that if \( f(s) > 0 \) for all \( s > 0 \), then \((h_3)\) implies \((h'_0)\); however from \((K_3)\) we have only \( f(s) > 0 \) for \( s > 0 \) large enough. Observe that \( K(x) = \frac{1}{\ln(|x|^2 + 2)} \) satisfies \((K_0)\) but not \((K_p)\).

### 1.3. The \( m'y \)-sublinear case in a bounded domain.

For \( 1 < m'y < p^* \), assume that \( M \) and \( f \) satisfy respectively \((M_3)\) and the following \( m'y \)-sublinear growth condition at infinity:

\( (H'_1) : \limsup_{s \to \infty} \frac{f(x, s)}{|s|^{m'y}} < \lambda_M. \)

Then, we have:

**Proposition 1.2.** Assume that \( f \) and \( M \) verify respectively \((H'_1)\) and \((M_3)\), then

1) \( I(u) \to \infty \) as \( ||u|| \to \infty. \)

2) \( I \) satisfies the Palais-Smale condition.

If in addition we assume the following growth condition at zero:

\( (H'_2) : \liminf_{s \to 0} \frac{F(x, s)}{|s|^{m'y}} = \infty, \) uniformly in \( \overline{\Omega}. \)

and the following assumption on \( M \):
\((M'_{2})\) : \(\tilde{M}(\tau) \leq \beta \tau^j\), for all \(\tau \geq 0\), where \(\beta\) is a positive constant.

We shall invoke the Krasnoselskii genus theory to prove the following multiplicity result:

**Theorem 1.4.** Assume that \(f(x, \cdot)\) is an odd function for all \(x \in \Omega\) satisfying \((H'_1)\) and \((H'_2)\) and \(M\) verifies \((M'_{2})\) and \((M'_{3})\), then \(I\) admits infinitely many distinct pairs \((u_j, -u_j)\), \(j \in \mathbb{N}^*\), of critical points.

Note that in Proposition 1.2 and Theorem 1.4 we may substitute \((H'_1)\) and \((M'_{3})\) only by the following strong assumption

\[
\lim_{s \to \infty} \frac{f(x, s)}{|s|^{p'}} = 0, \text{ uniformly in } \Omega.
\]

Next, if we relax \((H'_2)\) into the following condition

\[
(H''_2) : \frac{\lambda M}{m} < \liminf_{s \to 0} \frac{F(x, s)}{|s|^{p'}}, \text{ uniformly in } \Omega.
\]

Therefore, we shall apply the Ekeland variational principle to establish

**Theorem 1.5.** Assume that \(f\) satisfies \((H'_1)\) and \((H''_2)\) and \(M\) verifies \((M'_{3})\), then \(I\) is bounded from below and \(c = \inf\{h(u), u \in E_{r,m}\} < 0\). Consequently, \(I\) has a nontrivial critical point.

The organization of the rest of this paper is as follows. In Section 2, we give proofs of Lemma 1.1, Proposition 1.1, and Theorems 1.1-1.2. In Section 3, we prove Theorem 1.3. Section 4 is devoted to the proofs of Proposition 1.2, Theorems 1.4 and 1.5. Finally in the Appendix we develop the proof of Lemma 1.2 and we give some constructive examples of nonlinearities \(f\) and Kirchhoff functions \(M\) satisfying our assumptions.

In the following, \(|\cdot|\) denotes the Lebesgue measure in \(\mathbb{R}^N\) and \(C\) (respectively \(C_\epsilon\)) denotes always a generic positive constant independent of \(n\) and \(\epsilon\) (respectively independent of \(n\), even their value could be changed from one line to another one.

## 2. Proofs of Lemma 1.1, Proposition 1.1 and Theorems 1.1-1.2

### 2.1. Proof Lemma 1.1

**Proof of 1.** We argue by contradiction. Suppose that there exist \(m_0 > 0\) and a subsequence (denoted by \(\beta_j\)) such that \(m_0 < \beta_j, \forall j \in \mathbb{N}^*\). From the definition of \(\beta_j\), there exists \(u_j \in S_j^+ (\rho)\) such that

\[
m_0 < \int_{\Omega} |F(x, u_j)| \leq \beta_j. \tag{2.1}
\]

As \(||u_j|| = \rho\), \(u_j\) is bounded in \(L^p(\Omega)\), consequently there exist a subsequence (denoted by \(u_j\)) and \(u \in E_{r,m}\) such that \(u_j\) converges weakly to \(u\) and a.e \(\Omega\). Fix \(k \in \mathbb{N}^*\).

Since \(F_k = N(P_k)\) is the kernel of \(P_k\) and \(P_k \circ P_{k+1} = P_k\), then \(F_j \subset F_k\) for all \(j \geq k\). So as \(u_j \in F_j\), then \(P_k(u_j) = 0\), for all \(j \geq k\). Recall that \(P_k\) is a linear continuous operator, then \(P_k(u_j)\) converges weakly to \(P_k(u)\) which implies that \(P_k(u) = 0\) and also \(u = 0\) as \(u = \lim_{k \to \infty} \sum_{i=1}^k u_i e_i = \lim_{k \to \infty} P_k(u)\). Consequently, \(u_j\) converges to 0 a.e in \(\Omega\), and \(F(x, u_j) \to 0\) a.e in \(\Omega\) as \(F(x, 0) = 0\). By the virtue of \((H_2)\), we have for every \(\epsilon > 0\) there is \(C_\epsilon > 0\) such that

\[
|F(x, s)| \leq \epsilon |s|^{p'} + C_\epsilon, \quad \forall (x, s) \in \Omega \times \mathbb{R}. \tag{2.2}
\]

Hence, for each measurable set \(A \subset \Omega\) such that \(|A| < \frac{\epsilon}{C_\epsilon}\) (where \(|A|\) denotes the Lebesgue measure of \(A\)), we derive from (2.2) the following

\[
\int_A |F(x, u_j)| \leq \epsilon \int_A |u_j|^{p'} + C_\epsilon |A| \leq C\epsilon.
\]
Taking into account that $\Omega$ is a bounded domain, then Vitali’s theorem implies that $F(x, u_j) \to 0$ in $L^1(\Omega)$ and in view of (2.1), we obtain

$$0 < m_0 \leq 0.$$  

Thus, we reach a contradiction and Lemma 1.1 follows. \hfill \Box

**Proof of 2.** Let $\rho \geq 0$, then for all $u \in S^J(\rho)$, we have

$$I(u) \geq \frac{1}{m} \tilde{M}(||u||^m) - \beta_j.$$  

As $\beta_j$ converges to 0, we can choose $j = j_0$ large enough such that $\beta_{j_0} \leq \frac{1}{2m} \tilde{M}(||u||^m)$, then $I(u) \geq \frac{1}{2m} \tilde{M}(||u||^m)$. Therefore, condition 3 of Theorem A holds with $E^- = E_{j_0}$, $E^+ = F_{j_0}$, $||u|| = \rho$, and $\alpha = \frac{m\rho^m}{2m}$. \hfill \Box

Let us first recall some known results which will be useful to prove Proposition 1.1. Consider the functional $\psi(u) = \frac{1}{m} ||u||^m$, $u \in E_{rm}$, we have $\psi \in C^1(E_{rm})$ with Fréchet’s derivative $\langle \psi'(u), v \rangle = \int_{\Omega} |D_{t}u|^{m-2}D_{t}u \cdot D_{t}v$, $\forall u, v \in E_{rm}$. (respectively $\langle \psi'(u), v \rangle = \int_{\Omega} |D_{t}u|^{m-2}D_{t}uD_{t}v + a|u|^{m-2}uv$, $\forall u, v \in E_{rm}$ if $\Omega = \mathbb{R}^N$). Set $\varphi(t) = t^{\rho + 1}$, $t \geq 0$, clearly we have $\langle \psi'(u), u \rangle = \varphi(||u||)||u||$ and it follows from Hölder’s inequality that $||\psi'(u)||_{E_{rm}} = \varphi(||u||)$. Obviously, $\varphi$ is a normalization function and since $E_{rm}$ is locally uniformly convex and so uniformly convex and reflexive Banach space, then the corresponding duality mapping $J_\varphi$ is single valued (i.e., $J_\varphi = \psi'$) and satisfies the $S_\varphi$ condition (see Proposition 2 in [9]), respectively Lemma 3.2 in [19]:

$$\text{if } u_n \rightharpoonup u \text{ and } \limsup_{n \to \infty} \psi'(u)(u_n - u) \leq 0 \text{, then } u_n \to u. \quad (2.3)$$

2.2. **Proof of Proposition 1.1**

According to (M1) and (H1), there exists $C_0 > 0$ such that

$$C||f(x, s)||^p - C_0 \leq sf(x, s) - mfF(x, s), \forall (x, s) \in \Omega \times \mathbb{R}, \text{ and } -C_0 \leq \gamma \tilde{M}(\tau) - \tau M(\tau), \forall \tau \geq 0. \quad (2.4)$$

Since we assume that $N > rm$, we may easily see that

$$m > \frac{1}{q} + 1 \text{ if } m \geq 2. \quad (2.5)$$

Let $[u_n]$ be a (PS) sequence of $I$ if $m \geq 2$ (respectively (C) sequence if $1 < m < 2$). Two possible cases arise: either $u_n$ admits a subsequence which converges strongly to 0 in $E_{rm}$ and so we have done, or there exist $\eta_0 > 0$ and $n_0 \in \mathbb{N}$ such that $||u_n|| \geq \eta_0$ for all $n \geq n_0$. So according to (M2), there is $m_{\eta_0} > 0$ such that

$$M(||u_n||^m) \geq m_{\eta_0} > 0, \forall n \geq n_0. \quad (2.6)$$

**Step 1.** $[u_n]$ is bounded in $E_{rm}$. In fact, from (1.4) and (2.6), we have

$$m_{\eta_0} ||u_n||^m \leq M(||u_n||^m)||u_n||^m = \langle I'(u_n), u_n \rangle + \int_{\Omega} f(x, u_n)u_n.$$  

Apply Hölder’s inequality to the second term in the right-hand side and using (1.2), we obtain

$$m_{\eta_0} ||u_n||^m \leq \langle I'(u_n), u_n \rangle + C \left( \int_{\Omega} |f(x, u_n)|^q \right)^{\frac{1}{q}} ||u_n||. \quad (2.7)$$
Proof of Theorem 1.1.

Again using (2.6), yields

\[ \int_{\Omega} |f(x, u_n) - F(x, u_n)|^p \leq C(1 + \|u_n\|), \]  

(2.9)

Combining now (2.7) with (2.9), it follows

\[ \|u_n\|^m \leq C(1 + \|u_n\|^{1+\gamma}) \]  

if \( m \geq 2 \), (respectively \( \|u_n\|^m \leq C(1 + \|u_n\|) \) if \( 1 < m < 2 \)).

Clearly, the (C) sequence \( u_n \) is bounded in \( E_r,u \) if \( 1 < m < 2 \) and thanks to (2.5) the (PS) sequence is also bounded if \( m \geq 2 \).

Step 2. We shall prove that the bounded sequence \( \{u_n\} \) has a strong convergent subsequence in \( E_r,u \). Indeed, there exist a subsequence (denoted by \( u_n \)) and \( u \in E_r,u \) such that \( u_n \) converges to \( u \) weakly in \( E_r,u \) and strongly in \( L^1(\Omega) \). Also \( u_n \) and \( u_n - u \) are bounded in \( L^p(\Omega) \) and from (1.4), we get

\[ M(||u_n||^m) \int_{\Omega} |D_1u_n|^{m-2}D_1u_nD_1(u_n - u) = (I'(u_n), u_n - u) + \int_{\Omega} f(x, u_n)(u_n - u). \]  

(2.10)

By the virtue of the condition \((H_2)\), one has for every \( \epsilon \in (0, 1) \), there exists \( C_\epsilon > 0 \) such that

\[ |f(x, s)| \leq \epsilon |s|^{\nu-1} + C_\epsilon, \ \forall (x, s) \in \Omega \times \mathbb{R}. \]  

(2.11)

As \( u_n \) converges strongly in \( L^1(\Omega) \), there exists \( N_\epsilon \) such that \( \int_{\Omega} |u_n - u| \leq \frac{\epsilon}{C_\epsilon}, \ \forall n \geq N_\epsilon \). So, in view of (2.11) and Hölder’s inequality, we obtain

\[ \left| \int_{\Omega} f(x, u_n)(u_n - u) \right| \leq \epsilon \int_{\Omega} |u_n|^\nu |u_n - u| + C_\epsilon \int_{\Omega} |u_n - u| \]

\[ \leq \epsilon \left( \int_{\Omega} |u_n|^\nu \right)^{\frac{\nu-1}{\nu}} \left[ \left( \int_{\Omega} |u_n - u|^p \right)^{\frac{1}{p}} + C_\epsilon \int_{\Omega} |u_n - u| \right] \]

\[ \leq C_\epsilon, \ \forall n \geq N_\epsilon. \]

Consequently, \( \int_{\Omega} f(x, u_n)(u_n - u) \) converges to 0, and since \( I'(u_n) \to 0 \) in \( E_r,u \), and \( (u_n - u) \) is bounded in \( E_r,u \), we deduce from (2.11) that

\[ M(||u_n||^m) \int_{\Omega} |D_1u_n|^{m-2}D_1u_nD_1(u_n - u) \to 0. \]

Again using (2.6), yields

\[ \int_{\Omega} |D_1u_n|^{m-2}D_1u_nD_1(u_n - u) \]

converges to 0. Invoking now the \( S_+ \) property (see (2.3)), we conclude that \( \{u_n\} \) converges strongly to \( u \) in \( E_r,u \). \( \square \)

2.3. Proof of Theorem 1.1

We will show that the functional \( I \) satisfies all conditions of Theorem A. In fact, since \( f(x, \cdot) \) is odd and \( F(x, 0) = 0 \), \( I \) is even and \( I(0) = 0 \). According to Proposition 1.1, \( I \) satisfies the (PS) condition if \( m \geq 2 \) (respectively the (C) condition if \( 1 < m < 2 \)) and thanks to Lemma 1.1, condition 3 of Theorem A is well verified.

Lastly, it remains to show that condition 2 of Theorem A holds. In view of \( (H_3) \) we have for all \( A > 0 \), there is \( C_A > 0 \) such that

\[ F(x, s) \geq \langle A |s|^\nu - C_A, \forall (x, s) \in \Omega \times \mathbb{R}. \]
Using again \((M_1)\) we derive that
\[
\tilde{M}(\tau) \leq C_1 \tau^2 - C_2, \quad \forall \tau \geq 0,
\]
where \(C_1 = \tilde{M}(\tau_0)\) and \(C_2 > 0\).

Consequently, we obtain
\[
I(u) \leq \frac{C_1}{m} \|u\|_m^{my} - A \|u\|_m^{my} + C_A - C_2.
\]

Let \(W\) be a fixed finite dimensional subspace of \(E_{r,m}\), as \(||.||\) and \(||u||_{L^m(\Omega)}\) are equivalent norms on \(W\), there is \(C_W > 0\) such that \(I(u) \leq \left(\frac{C_1}{m} - AC_W\right) \|u\|_m^{my} + C_A - C_2\). Choosing \(A = \frac{2C_1}{mc_W}\), so we may find \(R = R(W) > 0\) such that \(I(u) < 0\) for all \(u \in W\) and \(||u|| \geq R\). This completes the proof of Theorem 1.1.

\(\square\)

2.4. \textbf{Proof of Theorem 1.2}

First of all observe that \((M_1)\) (with \(\tau_0 = 0\)) implies that for each \(\tau_1 > 0\), we have
\[
\frac{\tilde{M}(\tau)}{\tau^2} \leq \frac{\tilde{M}(\tau_1)}{\tau_1^2}, \quad \forall \tau \geq \tau_1.
\]

To prove Theorem 1.2 we shall verify the validity of the conditions of the standard mountain pass theorem \([25]\). Since \((M_3)\) implies \((M_2)\), Proposition 1.1 holds. Consequently, \(I\) satisfies the \((PS)\) condition if \(m > 2\) (respectively the \((C)\) condition if \(1 < m < 2\)). By combining \((H_2)\) and \((H_3)\) (at 0), we can find \(\epsilon_0 > 0\) small enough and \(C_0 > 0\) such that \(F(x, s) \leq \left(\frac{\tilde{A}}{m} - \epsilon_0\right) s|s|^{p_y} + C_0|s|^{p^*}\) for all \((x, s) \in \Omega \times \mathbb{R}\). Also recall that \((M_3)\) implies \((i)\) of Lemma 1.2 which with (1.2) implies
\[
I(u) \geq \frac{1}{m} \tilde{M}(||u||^m) - \left(\frac{\tilde{A}}{m} - \epsilon_0\right) \int_{\Omega} |u|^{py} - C_0 \int_{\Omega} |u|^{p^*}
\geq \frac{\epsilon_0}{\tilde{A}} \tilde{M}(||u||^m) - C_0 ||u||_{m, \Omega}^{p^*}, \quad C_0 > 0.
\]

Set \(||u||_{m, \Omega} = \rho\) with \(0 < \rho \leq 1\), thus using \((M_3)\), we deduce
\[
I(u) \geq \frac{C \epsilon_0}{\tilde{A}} \rho^{p^*} - C_0 \rho^{p^*} \geq \rho^{p^*} (\frac{C \epsilon_0}{\tilde{A}} - C_0 \rho^{p^* - p^*}).
\]

Choose \(\rho = \inf(1, \left(\frac{C \epsilon_0}{2C_0 \tilde{A}}\right)^{p^*})\) and \(C = \frac{C \epsilon_0}{2C_0 \tilde{A}}\rho^{p^*} > 0\), then we have \(I(u) \geq \alpha\) for all \(||u|| = \rho\).

On the other hand, using \((H_3')\) (at infinity) and (1.2) then for \(\epsilon_0 > 0\) small enough, we can find a positive constant \(\alpha_0\) and \(B \in E_{r,m} \setminus \{0\}\) such that
\[
|F(x, s)| \geq \left(\frac{\tilde{A}}{m} + 2\epsilon_0\right) s|s|^{p^*} - C_0, \quad \forall (x, s) \in \Omega \times \mathbb{R},
\]
and
\[
\lambda \int_{\Omega} |\varphi|^{p^*} \leq \tilde{M}(||\varphi||_m^m) \leq (\lambda \epsilon + m \epsilon_0) \int_{\Omega} |\varphi|^{p^*}.
\]

Set \(\varphi = \varphi, \ t \geq 1\) and using (2.13), we obtain
\[
I(v) \leq \frac{1}{m} \tilde{M}(t^{p^*} ||\varphi||^m) - \left(\frac{\tilde{A}}{m} + 2\epsilon_0\right) t^{p^*} \int_{\Omega} |\varphi|^{p^*} + C_0 |\Omega|
\leq \frac{1}{m} \left(\lambda \epsilon + m \epsilon_0\right) t^{p^*} - \lambda \epsilon_0 t^{p^*} \int_{\Omega} |\varphi|^{p^*} - C_0 |\Omega|.
\]
Inequalities (3.5) and (3.6) will be useful to prove Theorem 1.3 under the assumption (1.2).

Combining the above inequalities, we derive (see the introduction for the definition of $\beta$)

$$\frac{\tilde{M}(\|\varphi\|^m)}{p^m} \leq \tilde{M}(\|\varphi\|^m), \ \forall t \geq 1.$$  \hspace{1cm} (2.16)

So, from (2.14) and (2.15), we derive

$$I(t\varphi) \leq \frac{1}{m} \tilde{M}(\|\varphi\|^m) - \left(\lambda_m + m\epsilon_0\right) \int_{\Omega} |\varphi|^m - \epsilon_0 m \int_{\Omega} |\varphi|^m + C_0|\Omega| \leq -\epsilon_0 m \int_{\Omega} |\varphi|^m + C_0|\Omega|.$$  

Choose $t$ large enough, we deduce that $I(t\varphi) < 0$. In conclusion, $I$ satisfies the mountain pass geometry which ends the proof of Theorem 1.3.

\[ \square \]

3. Proof of Theorem 1.3.

We assume here that $K$ satisfies $(K_p)$. The case of $K$ satisfying $(K_0)$ is similar and more easier; hence we omit the proof here. First of all since we assume that $K \in L^\infty(\mathbb{R}^N)$ we may assume that $K$ satisfies $(K_p)$ with $p > 1$.

The case of $a > 0$. From $(h_0)$ and $(h_2)$ we can find a positive constant $C_0$ such that

$$|f(s)| \leq C_0(|s|^{p-1} + |s|^m), \ |F(s)| \leq C_0(|s|^{p'} + |s|^m), \ \forall s \in \mathbb{R}.$$  \hspace{1cm} (3.1)

Consequently, the associate energy functional to problem (1.8)

$$I_K(u) = \frac{1}{m} \tilde{M}(\|u\|^m) - \int_{\mathbb{R}^N} K(x)F(u), \ \forall u \in E_{r,m},$$  \hspace{1cm} (3.2)

is well defined and $I_K \in C^1(E_{r,m})$ with

$$\langle I_K'(u), v \rangle = M(\|u\|^m) \left( \int_{\mathbb{R}^N} |D_x u|^{m-2} D_x u D_x v + a \int_{\mathbb{R}^N} |u|^{m-2} uv \right) - \int_{\mathbb{R}^N} K(x)F(u)v, \ \forall u, v \in E_{r,m}. \hspace{1cm} (3.3)$$

Clearly, $(h_2)$ implies that $\lim_{s \to +\infty} \frac{F(s)}{|s|^{p'}} = 0$. Thus, for every $\epsilon > 0$, there exists $s_\epsilon > 0$ such that

$$|f(s)| \leq \epsilon |s|^{p-1} \quad \text{and} \quad |F(s)| \leq \epsilon |s|^{p'} \quad \forall |s| > s_\epsilon.$$  

Set $p' = \frac{p}{p-1}$ the conjugate exponent of $p$ (which appeared in assumption $(K_p)$). As $p' > 1$, $(h_0)$ implies that there is $C_\epsilon > 0$ such that

$$|f(s)| \leq C_\epsilon |s|^{\frac{p'}{2}}, \quad \text{and} \quad |F(s)| \leq C_\epsilon |s|^{\frac{p'}{2}}, \quad \forall |s| \leq s_\epsilon.$$  

Combining the above inequalities, we derive

$$|f(s)| \leq \epsilon |s|^{p-1} + C_\epsilon |s|^{\frac{p'}{2}}, \quad \forall s \in \mathbb{R}, \hspace{1cm} (3.4)$$

$$|F(s)| \leq \epsilon |s|^{p'} + C_\epsilon |s|^{\frac{p'}{2}}, \quad \forall s \in \mathbb{R}. \hspace{1cm} (3.5)$$

Multiplying (3.4) by $|s - s_0|$ and applying Young’s inequality, then for $s_0 \in \mathbb{R}$ that

$$|f(s)(s - s_0)| \leq \epsilon(|s|^{p'} + |s - s_0|^{p'} + |s - s_0|^m) + C_\epsilon |s|^\frac{p'}{2}, \ \forall s \in \mathbb{R}. \hspace{1cm} (3.6)$$

Inequalities (3.5) and (3.6) will be useful to prove Theorem 1.3 under the assumption $(K_p)$ and $a > 0$.

Let $(e_j)_{j \in \mathbb{N}}$ be a Schauder basis of $E_{r,m}$ (see the introduction of [10]). As in Lemma 1.1 we define

$$S_j^+(\rho) = \{u \in F_j \text{ such that } \|u\| = \rho\} \quad \text{and} \quad \beta_j := \sup_{S_j^+(\rho)} \int_{\mathbb{R}^N} K(x)F(u),$$

(see the introduction for the definition of $F_j$). We have

**Lemma 3.1.** Assume that $f \in C(\mathbb{R})$ satisfies $(h_0)$ and $(h_2)$, then:

1. $\beta_j \to 0$, as $j \to +\infty$.

2. For all $\rho \geq 0$, there exist $j_0$ and $\alpha > 0$ such that $I(u) \geq \alpha$, $\forall u \in S_j^+(\rho)$. 

3.1. Proof of Lemma 3.1

We will give a brief proof of point 1 (please see the proof of Lemma 3.1 for more details). Suppose not, then there exist \( m_0 > 0 \), \( u_j \in S_j^+ (\rho) \) such that modulo a subsequence we have

\[
m_0 < \int_{\mathbb{R}^N} K(x) |F(u_j)| \leq \beta_j.
\]

Also \( u_j \) is bounded in \( L^{\rho'} (\mathbb{R}^N) \) and up to subsequence \( u_j \) converges to 0 and so

\[
K(x) F(u_j) \to 0 \text{ a.e. in } \mathbb{R}^N.
\]

For any measurable set \( A \subset \mathbb{R}^n \) such that \( |A|^\frac{1}{\rho'} < \frac{\epsilon}{K_0 C_\rho} \), using (3.5) and Hölder’s inequality we derive

\[
\int_A |K(x) F(u_j)| \leq \epsilon K_0 \int_A |u_j|^{\rho'} + \epsilon K_0 C_\rho \int_A |u_j|^\frac{\rho}{\rho'}
\]

\[
\leq C \epsilon + K_0 C_\rho |A| \left( \int_A |u_j|^\rho \right)^{\frac{1}{\rho'}} \leq C \epsilon.
\]

By the virtue of \((K_{\rho})\) we have \( K(x)|u_j|^{\frac{\rho}{\rho'}} \in L^1 (\mathbb{R}^N) \), and there is \( R_\epsilon > 0 \) such that \( \left( \int_{|x| > R_\epsilon} K^\rho (x) \right)^{\frac{1}{\rho'}} \leq \frac{\epsilon}{C_\rho} \). As above, we have

\[
\int_{|x| > R_\epsilon} |K(x) F(u_j)| \leq \epsilon K_0 \int_{|x| > R_\epsilon} |u_j|^{\rho'} + C_\rho \int_{|x| > R_\epsilon} |K(x)| |u_j|^{\frac{\rho}{\rho'}}
\]

\[
\leq C \epsilon + C_\rho \left( \int_{|x| > R_\epsilon} K^\rho (x) \right)^{\frac{1}{\rho'}} \left( \int_{|x| > R_\epsilon} |u_j|^\rho \right)^{\frac{1}{\rho'}} \leq C \epsilon.
\]

Apply now Vitali’s theorem to deduce that \( K(x) F(u_j) \to 0 \) in \( L^1 (\mathbb{R}^N) \). Hence, we reach a contradiction from (3.7) since 0 < \( m_0 \leq 0 \).

The proof of point 2 of Lemma 3.1 is similar to the same one of Lemma 1.1. \( \square \)

We divide the rest of the proof into two steps:

Step 1. \( I_k \) satisfies the Palais-Smale condition if \( m \geq 2 \) (respectively \( C \) condition if \( 1 < m < 2 \)). Let \( u_n \) be a \((PS) \) sequence if \( m \geq 2 \) (respectively \( C \) sequence if \( 1 < m < 2 \)). We shall first verify that \( u_n \) is bounded. Similarly to the proof of Proposition 1.1 we may assume there exist \( \eta_0 > 0 \) and \( n_0 \in \mathbb{N} \) such that \( |u_n|^m \geq \eta_0 \) for all \( n \geq n_0 \).

Combining (3.2) with (3.3), so in view of \((P_{11})\) and \((M_{1})\), there exists \( C_0 > 0 \) such that

\[
C \int_{\mathbb{R}^N} K(x) |f(u_n)|^q \leq \int_{\mathbb{R}^N} K(x) (f(u_n) u_n - m y F(u_n)) \leq m y I_k (u_n) - \langle I_k' (u_n), u_n \rangle - C_0.
\]

Using again (3.3), then Hölder’s inequality and (1.2) yield

\[
M(||u_n||^m) ||u_n||^m \leq \langle I_k' (u_n), u_n \rangle + C \left( \int_{\mathbb{R}^N} K(x) |f(u_n)|^q \right)^{\frac{1}{q'}} ||u_n||^m.
\]

where \( C = K_0^\frac{1}{\rho'} \). The last inequality together with (3.3) and (M2) imply

\[
||u_n||^m \leq C (1 + ||u_n||^{\frac{1}{\rho'} + 1}) \text{ if } m > 2, \text{ (respectively } ||u_n||^m \leq C (1 + ||u_n||) \text{ if } 1 < m < 2).\]
As in the proof of step 1 of Proposition 1.1, we deduce that \( u_n \) is bounded in \( E_{r,m} \). Consequently, there exist a subsequence (noted again \( u_n \)) and \( u \in E_{r,m} \) such that \( u_n \rightharpoonup u \) in \( E_{r,m}, u_n \to u \) a.e. in \( \mathbb{R}^N \), also \( u_n - u \) are bounded in \( L^p(\mathbb{R}^N) \) and \( L^m(\mathbb{R}^N) \).

Apply now (3.6) with \( s = u_n, \ s = u \), we obtain
\[
|f(u_n)(u_n - u)| \leq \epsilon(|u_n|^{p^*} + |u_n - u|^{p^*} + |u_n - u|^m) + C_p|u_n|^\frac{m}{p^*}.
\]

Let \( A \subset \mathbb{R}^n \) be measurable set such that \( |A|^\frac{1}{p} < \frac{\epsilon}{K_\alpha C_\epsilon} \), integrate the above inequality over \( A \), then from Hölder’s inequality, we have
\[
\int_A K(x)|f(u_n)(u_n - u)| \leq C_\epsilon + K_\alpha C_p|A|^\frac{1}{p} \left( \int_A |u_n|^{m} \right)^\frac{1}{m} \leq C_\epsilon.
\]

Invoking now the assumption \((K_p)\), we may choose \( R_\epsilon > 0 \) large enough such that \( \left( \int_{|x| > R_\epsilon} K^p(x) \right)^\frac{1}{p} \leq \frac{\epsilon}{C_\epsilon} \), thus as above we derive
\[
\int_{|x| > R_\epsilon} K(x)|f(u_n)(u_n - u)| \leq C_\epsilon + C_\epsilon \int_{|x| > R_\epsilon} K(x)|u_n|^{\frac{m}{p}} \leq C_\epsilon + C_\epsilon \left( \int_{|x| > R_\epsilon} K^{p'}(x) \right)^{\frac{1}{p'}} \left( \int_{|x| > R_\epsilon} |u_n|^{m} \right)^{\frac{1}{m}} \leq C_\epsilon.
\]

Since \( K(x)f(u_n)(u_n - u) \to 0 \) a.e. in \( \mathbb{R}^N \), applying Vitali’s theorem it follows from (3.10) and (3.11) that
\[
\int_{\mathbb{R}^N} K(x)f(u_n)(u_n - u) \to 0.
\]

As \( I'_k(u_n) \) converges to 0 in \( E_{r,m}^* \) and \( (u_n - u) \) is bounded in \( E_{r,m} \), from (3.3) and (3.12), we have
\[
M(||u_n||^{m}) \int_{\mathbb{R}^N} \left( |D_1 u_n|^{m-2} D_1 u_n D_1 (u_n - u) + a|u_n|^{m-2} u(u_n - u) \right) \to 0.
\]

Using again \((M_2)\), we derive \( \int_{|x| > R_\epsilon} \left( |D_1 u_n|^{m-2} D_1 u_n D_1 (u_n - u) + a|u_n|^{m-2} u(u_n - u) \right) \to 0 \) and by the virtue of the \( S_+ \) condition (see (2.3)), we conclude that \( u_n \) converges strongly to \( u \) in \( E_{r,m} \). This completes the proof of step 1. \( \square \)

**Step 2.** In view of the assumptions \((h_1)\) and \((h_0)\), we have for any \( A > 0 \) there are \( s_A > 0 \) and \( C_A > 0 \) such that
\[
F(s) \geq A|s|^{m^*}, \ \forall |s| > s_A \text{ and } |F(s)| \leq C_A|s|^m, \ \forall |s| \leq s_A.
\]

Thus, we have
\[
F(s) \geq A|s|^{m^*} - C_A|s|^m, \ \forall s \in \mathbb{R}.
\]

Using now \((M_1)\) (with \( \tau_0 = 0 \)) then we can find \( C_1 > 0 \) such that \( \tilde{M}(\tau) \leq C_1 \tau^\gamma \) for all \( \tau \geq 1 \). Consequently, from the above inequality and for \( ||u|| \geq 1 \) we derive
\[
I_k(u) \leq \frac{C_1}{m}||u||^{m^*} - A \int_{\mathbb{R}^N} K(x)|u|^{m^*} + C_A K_\infty \int_{\mathbb{R}^N} |u|^m.
\]

Let \( W \) be a finite dimensional subspace of \( E_{r,m} \). Taking into account that \( \left( \int_{\mathbb{R}^N} K(x)|u|^{m^*} \right)^{\frac{1}{m^*}} \) is a norm and since all norms on \( W \) are equivalent, there is \( C_W > 0 \) such that
\[
\int_{\mathbb{R}^N} K(x)|u|^{m^*} \geq C_W||u||^{m^*} \quad \text{and} \quad \frac{1}{C_W}||u||^m \geq \int_{\mathbb{R}^N} |u|^m.
\]
Hence, for any $u \in E_{r,m}$, we get
\[ I_K(u) \leq \left( \frac{C_1}{m} - AC_W \right) ||u||^m + \frac{K_\infty C_2}{C_W} ||u||^m. \]

Choosing $A = \frac{2C_1}{mC_W}$, as $my > m$ then we may find $R = R(W) > 1$ large enough such that $I(u) < 0$ for all $||u|| \geq R$, $u \in W$.

Thanks to Lemma 3.1 we may apply Theorem $A$ to conclude our Theorem 1.3 when $a > 0$. \hfill \Box

The zero mass case $a = 0$. As above in view of $(h'_0)$ and $(f_0)$ we can find a positive constant $C_0$ such that
\[ |f(s)| \leq C_0 |s|^{p^*-1} \quad \text{and} \quad |F(s)| \leq C_0 |s|^{p^*}, \quad \forall s \in \mathbb{R}. \tag{3.14} \]

Consequently, the associate energy functional to problem (1.8)
\[ I_K(u) = \frac{1}{m} \bar{M}(||u||^m) - \int_{\mathbb{R}^N} K(x)f(u), \tag{3.15} \]

is well defined in $E_{r,m} = D^{m,\prime}(\mathbb{R}^N)$ and $I_K \in C^1(E_{r,m})$ with
\[ \langle I'_K(u), v \rangle = M(||u||^m) \int_{\mathbb{R}^N} |D_s u|^{m-2} D_s u D_s v - \int_{\mathbb{R}^N} K(x)f(u)v, \quad \forall u, v \in E_{r,m}. \tag{3.16} \]

Using again $(h'_0)$ and $(f_0)$, then we can establish the analogue of inequalities (3.5) and (3.6) used above
\[ |F(s)| \leq \epsilon |s|^{p^*-1} + C_\epsilon |s|^{p^*}, \quad \forall s \in \mathbb{R}, \]

and
\[ |f(s)(s - s_0)| \leq \epsilon |s|^{p^*-1} + |s - s_0|^{p^*} + C_\epsilon |s|^{p^*}. \]

Also note that inequality (3.13) (see the end of the proof of the case $a > 0$) becomes
\[ F(s) \geq A |s|^{m^*} - C_A |s|^{p^*}, \quad \forall s \in \mathbb{R}. \]

Substituting (3.5), (3.6) and (3.13) by the above inequalities, therefore the proof of the case $a = 0$ is exactly the same one of the case that $a > 0$. \hfill \Box

4. Proofs of Proposition 1.2, Theorems 1.4 and 1.5

4.1. Proof of Proposition 1.2

In view of $(H'_1)$ and for $\epsilon_0 > 0$ small enough, there exists $C_0 > 0$ such that $F(x, s) \leq \left( \frac{\lambda M}{m} - \epsilon_0 \right) |s|^{m^*} + C_0$ for all $(x, s) \in \Omega \times \mathbb{R}$. According to $(M_3)$ we derive
\[ I(u) \geq \frac{1}{m} \bar{M}(||u||^m) - \left( \frac{\lambda M}{m} - \epsilon_0 \right) \int_{\Omega} |u|^{m^*} - C_0|\Omega| \geq \frac{\epsilon_0}{\lambda M} \bar{M}(||u||^m) - C_0|\Omega| \geq C||u||^{m^*} - C_0|\Omega|. \]

Consequently, $I(u) \to \infty$ as $||u|| \to \infty$. Then any $(PS)$ sequence is bounded. Since $(H'_1)$ implies $(H_2)$ as $1 < my < p^*$, then from the proof of step 2 of Proposition 1.1 we deduce that any bounded $(PS)$ sequence has a strong convergent subsequence. So $I$ verifies the $(PS)$ condition. \hfill \Box
4.2. Proof of Theorem 1.4

Recall first some basic notations of Krasnoselskii’s genus, which can be found in [25]. Let $E$ be a Banach space and

$$
\Sigma = \{ A \subset X - \{ 0 \} : A \text{ is closed in } X \text{ and symmetric with respect to } 0 \}.
$$

**Definition 4.1.** (See [25]) For $A \in \Sigma$, we say genus of $A$ is $n$ denoted by $\gamma(A) = n$ if there is an odd map $\phi \in C(\Sigma, R^n \setminus \{0\})$ and $n$ is the smallest integer with this property.

We invoke the following abstract theorem based on the Krasnoselskii genus theory to prove Theorem 1.4.

**Theorem B.** (See [25]) Let $I \in C^1(E)$ be an even functional satisfying the $(PS)$ condition. For $c \in \mathbb{R}$ and each $n \in \mathbb{N}$, set $K_c = \{ u \in E : I(u) = 0 \text{ and } I(u) = c \}$ and $\Sigma_n = \{ A \in \Sigma : \gamma(A) \geq n \}$, $c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u)$. 

(i) If $\Sigma_n \neq \emptyset$ and $-\infty < c_n < 0$, then $c_n$ is a critical value of $I$.

(ii) If there exists $\varrho \in \mathbb{N}$ such that $c_n = c_{n+1} = \ldots = c_{n+\varrho} = c \in \mathbb{R}$, and $c \neq I(0)$, then $\gamma(K_c) \geq \varrho + 1$. 

We will verify the conditions of Theorem B.

Clearly, $I$ is even with $I(0) = 0$ and from Proposition 1.2 $I$ satisfies the $(PS)$ condition. We shall now prove that for any $n \geq 2$, $n \in \mathbb{N}$, one has $\Sigma_n \neq \emptyset$. In fact, we can find $\phi_1, \phi_2, \ldots, \phi_n \in C^2_c(\Omega)$ satisfying $\| \phi_i \|_{L^2(\Omega)} = 1$ and $\text{supp}(\phi_i) \cap \text{supp}(\phi_j) = \emptyset$ if $i \neq j$, $1 \leq i, j \leq n$. Set

$$
E_n = \text{span}(\phi_1, \phi_2, \ldots, \phi_n) = \{ u = \sum_{i=1}^n \lambda_i \phi_i, \lambda_i \in R \} \subset E_{2n} \cap L^2(\Omega),
$$

and for $0 < \sigma < 1$

$$
S_{\sigma}^n = \{ u \in E_n : \| u \|_{L^2(\Omega)} = \sigma \} = \{ u \in E_n, \sum_{i=1}^n \lambda_i^2 = \sigma^2 \}.
$$

Consider the map $h : S_{\sigma}^n \to S^{n-1}$ defined by:

$$
h(u) = \left( \frac{\lambda_1}{\sigma}, \frac{\lambda_2}{\sigma}, \ldots, \frac{\lambda_n}{\sigma} \right), \ u \in S_{\sigma}^n,
$$

where $S^{n-1} = \{ (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n : \sum_{i=1}^n \beta_i^2 = 1 \}$ is the sphere of dimension $n - 1$. Clearly $h$ is an homeomorphic odd map, which means that $\gamma(S_{\sigma}^n) = n$ (see [25]), then $S_{\sigma}^n \in \Sigma_n$ and the claim is well proved.

Therefore, $c_n$ is well defined and since point 1 of Proposition 1.2 implies that $I$ is bounded from below, so $-\infty < c_n$.

In order to apply Theorem B we have to prove that $c_n < 0$. Fix $n \in \mathbb{N}$. Indeed, since all norms in $E_n$ are equivalent, there exists $C_n > 0$ such that

$$
\frac{1}{C_n} \| u \| \leq \| u \|_{L^2(\Omega)} \leq C_n \| u \|_{L^{2n}(\Omega)}. \tag{4.1}
$$

According to $(H^*_2)$ one has for every $A > 0$ there is $s_A > 0$ such that

$$
F(x, s) \geq A|s|^m, \forall x, \ |s| \leq s_A. \tag{4.2}
$$

Set $M_n = \max(\| \phi_i \|_{L^{2n}(\Omega)}, 1 \leq i \leq n)$, then for $\sigma = \inf\left( \frac{1}{2}, \frac{s_A}{2nM_n} \right)$ we have $\| u \|_{L^{2n}(\Omega)} \leq \frac{s_A}{2}, \forall u \in S_{\sigma}^n$. Choose now $A = \beta C_n^{2m}$, so by combining $(M^*_2)$, (4.1) and (4.2) we deduce that

$$
I(u) = \frac{1}{m} M(\| u \|^m) - \int_{\Omega} F(x, u) \leq \frac{\beta}{mC_n} \left( C_n^2 \| u \|_{L^2(\Omega)}^m - \frac{mA}{\beta} \| u \|_{L^{2n}(\Omega)}^m \right) \leq \frac{\beta(1-m)C_n^{2m}}{m} \sigma^m < 0, \forall u \in S_{\sigma}^n.
$$

As $m > 1$, then $\frac{\beta(1-m)C_n^{2m}}{m} \sigma^m < 0$, thus $\sup I(u) < 0$ and $c_n = \inf_{u \in S_{\sigma}^n} \sup_{u \in S_{\sigma}^n} I(u) < 0$. In conclusion, from point 1 of Theorem B we derive that $c_n$ is a critical value of $I$ and since $n$ is arbitrary, then point 2 of Proposition 1.2 implies that $I$ admits infinitely many nontrivial critical points. The proof is completed.
4.3. Proof of Theorem.\[1,5

In view of Proposition\[1,2\] \(I\) satisfies the (PS) condition and \(I\) is bounded from below. Also, we claim that \(c = \inf\{I(u), u \in E_{r,m}\} < 0\). Indeed, as \(f\) verifies \((H'_{2})\) at 0, so for \(\epsilon_0 > 0\) small enough there exists \(s_0 > 0\) such that
\[
F(x,s) \geq \frac{\lambda_M}{m} + 2\epsilon_0 |s|^m \quad \text{for all } (x,s) \in \Omega \times [-s_0, s_0].
\]
(4.3)

Taking into account that \(C^r(\Omega) = E_{r,m}\), and according to\[1,2\] there is \(\phi \in C^r(\Omega) \setminus [0]\) such that
\[
\widetilde{M}(||\phi||^m) \leq (\lambda_M + m\epsilon_0) \int_\Omega |\phi|^m.
\]
(4.4)

Using now (2.12) (which is a consequence of (M3)), we derive
\[
\frac{\tilde{M}(t^m||\phi||^m)}{t^m} \leq \tilde{M}(||\phi||^m), \quad \forall t \leq 1.
\]
(4.5)

Set \(t = \inf\left(\frac{s_0}{||\phi||_{L^\infty(\Omega)}}, \frac{1}{2}\right)\). So \(t\phi \in [-s_0, s_0]\) and by combining (4.3)-(4.5), we get
\[
I(t\phi) \leq \frac{t^m}{m} \left(\tilde{M}(||\phi||^m) - (\lambda_M + m\epsilon_0) \int_\Omega |\phi|^m\right) - \epsilon_0 t^m \int_\Omega |\phi|^m < 0.
\]
(4.6)

Therefore, \(c = \inf\{I(u), u \in E_{r,m}\} < 0\), invoking Ekeland’s variational principle, we deduce that \(c\) is a nontrivial critical value which achieves the proof of Theorem\[1,5\].

Appendix.

Appendix (A)

We prove here Lemma\[1,2\] stated in Section 2.

Proof of (i): Assume that \(M\) satisfies (M3). As \(1 < m\gamma < p^*\), Sobolev’s inequality implies \(\int_{\Omega} |u|^{\gamma m} \leq C||u||_{L^{\gamma m}(\Omega)}^{\gamma m}\), which combined with (M3) yields \(\lambda_M > 0\).

Conversely, if \(M\) does not satisfies (M3), then there is a sequence \(\tau_i \to 0\) such that \(\tau_i^{-\gamma} \tilde{M}(\tau_i) \to 0\). Consider \(\varphi \in E_{r,m}\) such that \(||\varphi|| = 1\), set \(v_i = \tau_i^{\gamma/m} \varphi\), then \(||v_i||^m = \tau_i\). So
\[
\frac{\tilde{M}(||u||^m)}{\int_{\Omega} |u||^m} = \frac{\tilde{M}(\tau_i)}{\tau_i^{-\gamma}} \times \frac{1}{\int_{\Omega} |\varphi_i|^m} \to 0.
\]

Therefore, \(\lambda_M = 0\).

\(\Box\)

Proof of (ii): (a) We will first prove that if \(M = C \tau^{\gamma-1}\), then \(\lambda_M\) is attained. Without losing any generality, we may assume that \(C = \gamma\). Let
\[
\lambda_M := \inf_{u \in E_{r,m}} \frac{\tilde{M}(||u||^m)}{\int_{\Omega} |u||^m} = \inf_{u \in E_{r,m}} \frac{C}{\gamma} ||u||^{\gamma m} = \inf \{||u||^{\gamma m}, \text{such that } ||u||_{L^{\gamma m}(\Omega)} = 1\}.
\]

Let \(u_n\) be a minimizing sequence, i.e., \(||u_n||_{L^{\gamma m}(\Omega)} = 1\) and \(||u_n||^{\gamma m} \to \lambda_M\), so \(u_n\) is bounded in the \(E_{r,m}\) norm. Therefore, as \(1 < m\gamma < p^*\), there is \(u \in E_{r,m}\) and a subsequence (still denoted by \(u_n\)) such that \(u_n\) converges weakly to \(u\) in \(E_{r,m}\), \(||u||_{L^{\gamma m}(\Omega)} \to ||u||_{L^{\gamma m}(\Omega)}\), and
\[
\lambda_M = \lim \inf_{n \to +\infty} ||u||^{\gamma m} \geq ||u||^{\gamma m}.\]

Consequently, \(||u||_{L^{\gamma m}(\Omega)} = 1\) and \(||u||^{\gamma m} \geq \lambda_M\) which implies that \(||u||^{\gamma m} = \lambda_M\).

Moreover, there exists a Lagrange multiplier \(\mu\) such that
\[ m M(||u||^m) \int_\Omega |D_i u|^{m-2} D_i u D_j \nabla v = \mu m y \int_\Omega |u|^{m-1} v, \ \forall v \in E_{m}. \]

If \( v = u \) we have \( m \lambda_M = \mu m y \), that is \( \lambda_M = \mu m y \) and so

\[
\begin{cases}
M(||u||^m) \Delta u = \lambda_M ||u||^{m-1} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \quad k = 1, 2, \ldots, r - 1,
\end{cases}
\]

(b) In general \( \lambda_M \) is not attained if \( M \) satisfies (M3). The typical example is \( \bar{m} (t) = t^\gamma H(t) \), where \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) is strictly decreasing with \( \lim_{t \to 0^+} H(t) = t > 0 \) and \( t^\gamma H(t) \) is strictly increasing on \( \mathbb{R}_+ \). So, \( M \) is positive and satisfies (M3). Set

\[ E(u) = \frac{\bar{M}(||u||^m)}{\int_\Omega |u|^m dx}. \]

The monotonicity of \( H \) involving \( E(\alpha u) < E(u) \) for all \( \alpha > 1 \), and \( u \neq 0 \). It means clearly that \( \lambda_M \) is attained. More exactly, one can find easily examples of \( H \) such as \( H(t) = 1 + \beta (t+1)^{-1} \) (with \( \beta > 0 \)) and which also satisfy (M1).

**Appendix (B)** Let us first show that (H2)-(AR), imply (H1). In fact, as \( p^* - 1 = \frac{1}{q' - 1} \), (H2) implies that there is \( s_0' > s_0 \) such that \( |f(x, s)|^q \leq sf(x, s) \), for all \( |s| \geq s_0' \) and \( x \in \Omega \). From (AR), we have

\[ 0 < (1 - \frac{m y}{\rho}) sf(x, s) < sf(x, s) - my F(x, s), \ \text{for all } |s| > s_0 \text{ and } x \in \Omega. \]

Hence, (H1) follows from the above inequalities.

Now, we collect some remarks and constructive examples of nonlinearities \( f \) and Kirchhoff function to understand the improvement brought by our assumptions.

1. Let \( \gamma_1 \in (1, \frac{p^*}{m}), \ \gamma_2 \geq \frac{p^*}{m} \). Consider the degenerate Kirchhoff function \( M(\tau) = \tau^{\gamma_1 - 1} \) if \( \tau \geq 1 \) and \( M(\tau) = \tau^{\gamma_2 - 1} \) if \( \tau \leq 1 \). We can see that \( M \) satisfies (M1)-(M2) but not the global assumption (M) required in [6].

2. Consider the following Kirchhoff function introduced in [6]:

\[ M(\tau) = a \tau^{\gamma_1 - 1} + b \tau^{\gamma_2 - 1} \]

with \( a \geq 0, b > 0 \) and \( 1 \leq \gamma_1 \leq \gamma_2 < \frac{p^*}{m} \).

Then \( M \) satisfies (M1)-(M2) and also assumption (M3). Moreover \( M \) is degenerate if \( a = 0 \) or \( a \neq 0 \) and \( \gamma_1 \neq 1 \).

3. Since we assume \( \gamma < \frac{p^*}{m} \), then \( (my - 1)q' \leq my \), so for \( (my - 1)q' - 1 \leq \alpha < 1 \) and \( a > \gamma \lambda_M \), then a simple computation shows that \( f_1(s) = a|s|^{my-2} - |s|^{q'-1} \) satisfies (H1) (and (H2)-(H3)) but never (AR), and nor (SSL).

4. In [27], (AR)_1 was relaxed into one of the following conditions (for \( m = 2 \) and \( M = 1 \)): there are constants \( \theta > 2 \) and \( C > 0 \) such that

\[ |\theta F(x, s) - sf(x, s)| \leq C(1 + s^2), \ \forall (x, s) \in \Omega \times \mathbb{R}, \]

or the *global* convexity condition

\[ H(x, s) := sf(x, s) - 2F(x, s) \]

is convex in \( s \), \( \forall x \in \Omega \).

However, the following nonlinearity \( f_2(s) = |s|^{my-2} \ln^q(|s|) \) with \( q \geq 1 \) verifies (H1)-(H3) and (H3) but does not satisfies assumptions (4.1) and (4.2) if \( q > 1 \).
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