Non-Fermi liquid behavior in Kondo Models

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Despite the fact that the low energy behavior of the basic Kondo model cannot be studied perturbatively it was eventually shown by Wilson, Anderson, Nozières and others to have a simple “local Fermi liquid theory” description. That is electronic degrees of freedom become effectively non-interacting in the zero energy limit. However, generalized versions of the Kondo model involving more than one channel or impurity may exhibit low energy behavior of a less trivial sort which can, nonetheless, be solved exactly using either Bethe ansatz or conformal field theory and bosonization techniques. Now the low energy limit exhibits interacting many body behavior. For example, processes in which a single electron scatters off the impurity into a multi electron-hole state have a non-vanishing (and sometimes large) amplitude at zero energy. This corresponds to a rare solvable example of non-Fermi liquid behavior. Essential features of these phenomena are reviewed here.

I. INTRODUCTION

Kondo’s seminal paper of forty years ago showed that the low energy behavior of the Kondo model is fundamentally non-perturbative. I write the Hamiltonian as:

\[ H = \sum_{k \alpha} \psi_{k \alpha}^\dagger \psi_{k \alpha} \epsilon(k) + J \vec{S} \cdot \sum_{k \vec{k}} \psi_{k \vec{k}}^\dagger \vec{\sigma} \psi_{k \vec{k}} \]  

(1.1)

where \( \psi_{k \alpha} \)'s are conduction electron annihilation operators, (of momentum \( \vec{k} \), spin \( \alpha \)) and \( \vec{S} \) represents the spin of the magnetic impurity with

\[ [S^a, S^b] = i\epsilon^{abc} S^c. \]  

(1.2)

There is an implicit sum over electron spin indices, \( \alpha \) in the interaction term in Eq. (1.1). The dimensionless coupling constant is

\[ \lambda \equiv J \nu, \]  

(1.3)

where \( \nu \) is the density of states at the Fermi surface. As Kondo showed, perturbation theory in \( \lambda \) is infrared divergent at low \( T \). For instance, the temperature-dependent resistivity for a dilute array of impurities is given by a formula of the form:

\[ \rho(T) \sim [\lambda + \lambda^2 \ln \frac{D}{T} + ...]^2 \]  

(1.4)

Here \( D \) is the band-width. No matter how small the coupling constant, \( \lambda \), the higher order terms eventually overwhelm the lower order ones at low enough temperature. This result stimulated an enormous amount of theoretical work. As Nozières put it, “Theorists ‘diverged’ on their own, leaving the experiment realities way behind”.

As was realized later, the divergence of the resistivity formula has an elegant interpretation in terms of renormalization group (RG) concepts. The scale dependent effective coupling constant, \( \lambda(T) \) diverges as \( T \to 0 \):

\[ \lambda(T) \approx \lambda + \lambda^2 \ln \frac{D}{T} + ... \]  

(1.5)

The temperature at which the higher order terms overwhelm the lower order ones,

\[ T_K \approx D \exp[-1/\lambda], \]  

(1.6)

defines a fundamental energy scale. Perturbation theory can be applied for \( T >> T_K \) but not for \( T \leq T_K \). The low \( T \) behavior is fundamentally non-perturbative.

Nonetheless, the physics was eventually shown by Wilson, Anderson, Nozières and others to be simple at very low energies, \( E << T_K \). Only the intermediate energy range where \( E \) is \( O(T_K) \) defies a simple description. This simplicity at very low energies arises from the fact, that in a certain sense to be made precise below, \( \lambda(T) \to \infty \) at \( T \to 0 \). This infinite \( \lambda \) behavior is actually quite simple. (I begin with the case of an \( S = 1/2 \) impurity.) One electron forms a singlet with the impurity. The remaining low energy electronic degrees of freedom feel an infinite repulsion from the screened impurity which corresponds to a \( \pi/2 \) phase shift in the s-wave channel. The induced electron-electron interactions among these low energy degrees of freedom become increasingly unimportant as the energy scale decreases, corresponding to irrelevant interactions in the renormalization group sense. They lead to a simple dependence of physical quantities on \( T \) (or other energy scales) which can be Taylor expanded in powers of \( T/T_K \). From a renormalization group viewpoint, the low energy fixed point (\( \lambda \to \infty \)) is simply non-interacting electrons, like the high-energy fixed point (\( \lambda = 0 \)) except for the removal of the impurity spin and the presence of a modified boundary condition at the impurity location corresponding to the phase shift. The Kondo model thus provides a rare example of a renormalization group flow between two different fixed points, both of which are trivial.

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It is perhaps surprising that simple modifications of the basic Kondo model can completely change this trivial low energy behavior. The simplest such modification is to include several channels of electrons, changing the Hamiltonian of Eq. (1.1) to:

\[ H = \sum_{\vec{k},\alpha,i} \psi_{\vec{k}\alpha,i}^\dagger \psi_{\vec{k}\alpha,i} \epsilon(k) + J \vec{S} \cdot \vec{\sigma} \sum_{\vec{k}\vec{k}\prime} \psi_{\vec{k}\alpha,i}^\dagger \frac{1}{2} \psi_{\vec{k}\alpha,i} \]  

Here \( i \) labels the “channels” and runs from 1 to \( \vec{k} \) (not to be confused with the momentum label, \( \vec{k} \)). This fundamental change in low energy behavior was first demonstrated by Nozières and Blandin by simple, intuitive RG arguments. Some aspects of these models were later solved for exactly using the Bethe ansatz. This is a powerful method which allows calculations of thermodynamic quantities at all temperatures, enabling a study of the cross over between high energy and low energy fixed points. Later, Ludwig and I developed techniques to also calculate dynamical correlation functions, the resistivity and, in fact, any low energy property of these models. This amounted to an exact solution for the low energy fixed point which no longer has a simple Fermi liquid form. Since the interactions in these models only occur near the origin the low energy \textit{bulk} behavior is always that of free electrons. However, there is no guarantee that the behavior near the origin is also free electron like. This happens to be the case for the simplest Kondo model but is not so in general. What does turn out to be true in general is that the low energy behavior is given by a free electron model with some sort of conformally invariant boundary condition (BC) at the origin (that is, at the impurity location). In the simplest case this BC is of a trivial type which just corresponds to the phase shift mentioned above. In other cases, this BC encodes certain non-trivial electron-electron interactions which occur only near the origin. Indeed the problem of finding and studying such BC’s turns out to be surprisingly rich but fortunately was solved in an elegant and general way by Cardy. Ludwig and I were able to adapt Cardy’s methods to generalized Kondo models. [For a much more extensive review of this work than is given here, see Ref. (17).] Among low energy properties which can be solved for by these techniques are the S-matrix for low energy scattering off the impurity. In the single-channel model, an incoming electron at energy near \( E_F \) (the Fermi energy) only scatters into a single outgoing electron at low \( T \). All multi-electron-hole scattering processes have amplitudes which vanish as the energy and \( T \rightarrow 0 \). This is just like what happens in standard Landau Fermi liquid theory for electron-electron scattering. In some of the multi-channel models the amplitude for these multi-electron-hole processes is non-zero (and is sometimes large) in the zero energy limit. Thus, we speak of non-Fermi liquid fixed points.

Shortly after the general solution of the non-Fermi liquid Kondo fixed points using boundary condition techniques, was obtained, an alternative, simpler approach was developed by Emery and Kivelson for the particular, and most experimentally relevant case, of \( k = 2 \), \( S = 1/2 \). This is based on standard bosonization methods, and the study of a particular anisotropic model which can be mapped onto non-interacting fermions.

In this brief review, I will focus exclusively on the multi-channel Kondo model as a route to non-Fermi liquid behavior. However, another rather well-studied generalization of the basic Kondo model which exhibits non-Fermi liquid behavior is the 2 impurity Kondo model. Other generalizations with non-Fermi liquid behavior are also known. At this point in time, the multi-channel (in particular, 2 channel) Kondo model appears to offer the best hope for experimental realization.

In the next section I review the perturbative results on the multi-channel Kondo model and the Nozières-Blandin analysis of the strong coupling limit which first showed that the low energy fixed point was not of Fermi liquid type when the number of channels, \( k > 2S \), where \( S \) is the size of the impurity spin. In Sec. III I discuss the mapping onto a one-dimensional model which can be approximated as a Dirac fermion on the half-line with the impurity at the origin. I then discuss the \( \pi/2 \) phase shift as a simple boundary condition. In Sec. IV I sketch some aspects of Cardy’s general theory of conformally invariant BC’s and how it was applied to the multi-channel Kondo model. In Sec. V I discuss the stability of the non-Fermi liquid fixed point against various types of symmetry breaking interactions which may be present in the Hamiltonian. I also briefly review various proposed physical realizations.

II. PERTURBATIVE RESULTS AND STRONG COUPLING LIMIT

It is relatively straightforward to calculate the \( \beta \)-function, which gives the change in effective coupling constant, \( \lambda \), as the bandwidth of the fermions is lowered, integrating out modes away from the Fermi surface. Although a pseudo-fermion representation is sometimes introduced for the impurity spin, it is simpler to calculate directly expectation values of time-ordered products of spin operators in the non-interacting groundstate, using the spin commutation relations and \( S \cdot S = S(S+1)I \), where \( I \) is the identity matrix. The spin operators have no time-dependence in this non-interacting groundstate but the time-ordering introduces some minus signs due to the non-commutation of the spin operators. That is the time-ordered product:

\[ T[S^a(t_1)S^b(t_2)] = \theta(t_1-t_2)S^aS^b + \theta(t_2-t_1)S^bS^a. \]

Note that the operators have no time-dependence on the right hand side of this equation. In the \( S=1/2 \) case we may use:

\[ S^aS^b = (1/4)\delta^{ab}I + (i/2)\epsilon^{abc}S^c. \]
Combining these minus signs with the free fermion propagators which can be conveniently written in position space at \(x = 0\) it is straightforward to calculate the \(\beta\)-function. The diagrams, to fourth order, are shown in Fig. 1, and the result is \(O(\lambda^4)\).

\[
\frac{d\lambda}{d(\ln D')} = -[\lambda^2 - (k/2)\lambda^3 + O(\lambda^4)].
\] (2.3)

[See App. B of Ref. 12 for a quick derivation using the operator product expansion.] The lowest order term, essentially found by Kondo, tells us that as we lower the cut-off, \(D\), a small antiferromagnetic \((\lambda > 0)\) effective coupling begins to increase. Keeping only this leading term, we obtain the effective coupling:

\[
\lambda(D') \approx \frac{\lambda_0}{1 - \lambda_0 \ln(D/D')},
\] (2.4)

where \(\lambda_0\) is the bare coupling constant and \(D'\) is the reduced bandwidth. This simply tells us that the effective coupling becomes large at \(D' \approx T_K\), defined in Eq. 14, in the antiferromagnetic case, \(\lambda > 0\). Conversely, in the ferromagnetic case, \(\lambda < 0\), the effective coupling goes to zero at low energy scales. In the ferromagnetic case, a small bare coupling just keeps on getting smaller so that we never need to consider any terms beyond the first one in the \(\beta\)-function. In this ferromagnetic case, the low energy fixed point corresponds to an impurity spin which decouples from the non-interacting conduction electrons. Conversely, in the antiferromagnetic case, at energy scales of \(O(T_K)\) where \(\lambda\) becomes \(O(1)\) it is not sufficient to keep only the leading term in the \(\beta\)-function. In fact, normally all high order terms become important at this stage and we lose control over the calculation. This is what makes the simple assertion that, in some sense, \(\lambda \to \infty\) a non-obvious assumption in the simple, \(k = 1\) case.

However, something very nice and very special happens in the antiferromagnetic case, when \(k \gg 1\), as observed by Nozières and Blandin.\(^5\) If we keep only the first two terms then the \(\beta\) function vanishes at a fixed point coupling:

\[
\lambda_c \approx 2/k.
\] (2.5)

The third term in the \(\beta\)-function has no powers of \(k\). This can be seen from Fig. 1, observing that the powers of \(k\) come from fermion loops. Therefore, while the first two terms in the \(\beta\)-function are \(O(1/k^2)\) at the fixed point, the third term is \(O(1/k^4)\). In fact, all higher order terms are negligible compared to the first two at this fixed point at large \(k\). Thus, as we reduce the bandwidth, the effective coupling flows to an asymptotic small, non-zero value at low energies. This corresponds to a non-trivial critical behavior, different than the trivial strong coupling behavior occurring for \(k = 1\). One way of seeing this is to observe that the slope of the \(\beta\)-function at the critical point is:

\[
(d\beta/d\lambda)_{\lambda_c} \approx -2/k.
\] (2.6)

At this intermediate coupling fixed point, we expect to be able to introduce a new basis of local operators and corresponding coupling constants. i.e.

\[
H_{eff} = H_e + (\lambda - \lambda_c)O + \ldots
\] (2.7)

Here \(H_e\) is the fixed point Hamiltonian (i.e. free fermions together with a conformally invariant BC) and \(O\) is the leading irrelevant operator at the fixed point. The fact that the corresponding coupling constant, \(\lambda - \lambda_c\) has the \(\beta\)-function:

\[
d(\lambda - \lambda_c)/d\ln D = ((2/k)(\lambda - \lambda_c) + O[(\lambda - \lambda_c)^2]), \] (2.8)

tells us that the corresponding operator, \(O\) has a non-trivial scaling dimension of \(1 + 2/k\) and therefore this is not a Fermi liquid fixed point. (At Fermi liquid fixed points all operators can be written in terms of free fermion fields and therefore have scaling dimensions which are integers or \(1/2\)-integers. Furthermore \(H_{eff}\) only contains operators with even numbers of fermions which therefore have integer dimensions.)

Further insight into the situation can be obtained by considering the strong coupling limit. Let us first consider the case \(S = 1/2, k = 1\). It is convenient to consider a tight-binding model with a hopping term \(t\) in the limit \(t \ll J\) with the Kondo interaction on the site 0 only. Thus we consider a Hamiltonian:

\[
H = -t \sum_{<i,j>} \psi_i^\dagger \psi_j + J \psi_0^\dagger \frac{\sigma}{2} \psi_0 \cdot \vec{S}.
\] (2.9)

The first sum is over nearest neighbors on some lattice. (The details of the lattice are unimportant.) We consider the case of large antiferromagnetic Kondo coupling, \(J \gg t > 0\). The groundstate must have exactly one electron at the origin which forms a singlet with the impurity spin. Adding or removing one electron from the origin costs a large energy of \(O(J)\). All the other electrons can do anything they like as long as they stay away
from the origin. Thus, for a small non-zero \( t \), they will form a free electron groundstate on all sites but the origin. Low energy excitations simply correspond to the usual excitations of such a free fermion system with a vanishing boundary condition at the origin. In a spherically symmetric version of the problem, this BC corresponds to a \( \pi/2 \) phase shift in the s-wave channel.

It is important to consider the self-consistency of this strong coupling fixed point. Is it really plausible that \( \lambda \rightarrow \infty \) under renormalization? i.e. is the fixed point stable? We can answer this last question by considering all possible operators which could appear in the effective Hamiltonian at the fixed point which might potentially destabilize it. Importantly, these operators do not involve the impurity spin since it is screened and does not appear in the low energy \( H_{\text{eff}} \). We must form operators purely from the electron operators and these can be shown to be irrelevant at this fixed point. [Actually, they are only strictly irrelevant when exact particle-hole symmetry is present. Otherwise there is one exactly marginal operator. However, it is not important for our purposes.]

Now consider the case of general \( S \) and \( k \) and again assume \( J >> t \). Now the groundstate will have \( k \) electrons at the origin, one from each channel. They will lock into a spin \( k/2 \) configuration. This spin \( k/2 \) couples antiferromagnetically to the impurity spin \( S \). The resulting groundstate will have a residual impurity spin of magnitude \( |S - k/2| \). This is non-zero in general, except in the case \( S = k/2 \). This is referred to as underscreening when \( k/2 < S \) and overscreening when \( k/2 > S \). A crucial question is whether this effective spin couples ferromagnetically or anti-ferromagnetically to the conduction electrons. This is a straightforward calculation and the sign can actually be deduced from the well-known result that the induced Heisenberg exchange from the Hubbard model is antiferromagnetic. For similar reasons, the induced exchange interaction between the spin of the electrons trapped at 0 and the electron spins on the surrounding sites is antiferromagnetic. However, this is not the end of the story. We must consider the sign of the projection of the electron spin at the origin onto the effective spin (of size \( |S - k/2| \)). This is positive in the overscreened case, \( k/2 > S \) and negative in the underscreened case, \( S > k/2 \). Thus the effective Kondo interaction between the over (or under) screened spin at the origin and the conduction electron spins on the surrounding sites is antiferromagnetic in the overscreened case (but ferromagnetic in the underscreened case). This effective Kondo coupling is of order:

\[
J_{\text{eff}} \propto t^2/J, \quad (2.10)
\]

which is \( \ll 1 \) in magnitude, for strong coupling. Thus we may analyse its RG flow using weak coupling perturbation theory, Eq. (2.2). The conclusion is that the strong coupling fixed point is stable in the underscreened case, where the partially screened spin of size \( S - k/2 \) asymptotically decouples from the free conduction electrons. On the other hand, the strong coupling fixed point is unstable in the overscreened case where the antiferromagnetic Kondo coupling of the overscreened effective spin to the surrounding conduction electrons is relevant. We can summarize this situation by the RG flow diagram in Fig. 2. Both zero coupling and strong coupling fixed points are unstable and the RG flows from either weak or strong coupling go to the intermediate coupling fixed point at low energies. These arguments don’t depend on \( k \) being large and we expect them to be qualitatively correct in general. Thus we conclude that the multi-channel Kondo model has a non-Fermi liquid stable low energy fixed point whenever \( k > 2S \).

Unfortunately, these arguments don’t tell us much about the low energy physics, except in the case \( k >> 1 \) where some information can be extracted from weak coupling perturbation theory since \( \lambda_c << 1 \) in that case. Therefore, we turn to more powerful methods.

III. 1 DIMENSIONAL PHYSICS

The Kondo model is fundamentally 1-dimensional. This is a fortunate observation because there are special techniques, applicable only in 1 dimension (1D), which can be brought to bear on the problem. To derive the 1-dimensionality in a simple way, we begin with a 3 dimensional model with a spherically symmetric dispersion relation and assume that the Kondo interaction is a spatial \( \delta \)-function. If we then expand the electron operators in spherical harmonics, only the s-wave harmonic couples to the impurity. The s-wave theory is equivalent to a 1D model defined on the 1/2-line \( x > 0 \). Upon linearizing the dispersion relation, valid for weak Kondo coupling where we are only concerned with low energy states, the Hamiltonian becomes:

\[
H = \frac{iv_F}{2\pi} \int_0^\infty dx \left[ \psi_L^d d\psi_L - \psi_R^d d\psi_R \right] + v_F \lambda \psi_L^d(0) \frac{\sigma^z}{2} \psi_L(0) \cdot \vec{S}, \quad (3.1)
\]

with the BC:

\[
\psi_R(0) = \psi_L(0). \quad (3.2)
\]

[See, for example, App. A of Rev. 12 for a detailed derivation.] This corresponds to a relativistic Dirac fermion, on the 1/2-line, interacting with the impurity spin at the origin. (We henceforth generally set the Fermi velocity, \( v_F \), which plays the role of velocity of light, to 1.)
A phase shift by $\delta$ (at all wave-vectors) corresponds to the BC:

$$\psi_{R}(0) = e^{2i\delta}\psi_{L}(0).$$  \hspace{1cm} (3.3)

In particular, for $\delta = \pi/2$ we get a new effective BC at the strong coupling fixed point:

$$\psi_{R}(0) = -\psi_{L}(0).$$  \hspace{1cm} (3.4)

For any finite coupling, the phase shift would vary with $k$ but in the strong coupling limit, in a theory with a reduced bandwidth from RG transformations, we may consider the phase shift to be $\pi/2$ at all $k$. We then obtain this simple change of BC’s. So, the low energy effective Hamiltonian in the Fermi liquid case is simply:

$$H = \frac{i}{2\pi} \int_{0}^{\infty} dx \left[ \psi_{L}^{\dagger} \frac{d}{dx} \psi_{L} - \psi_{R}^{\dagger} \frac{d}{dx} \psi_{R} \right],$$  \hspace{1cm} (3.5)

with the new BC of Eq. (3.4). The impurity spin has disappeared from the low energy Hamiltonian and we are left with free fermions with a modified BC as the fixed point Hamiltonian.

Starting with a weak bare coupling, the only leading irrelevant operator can be shown to be:

$$H_{int} = \frac{1}{T_{K}} \bar{J}(0)^{2},$$  \hspace{1cm} (3.6)

where,

$$\bar{J}(0) \equiv \psi_{L}^{\dagger}(0) \frac{\sigma}{2} \psi(0),$$  \hspace{1cm} (3.7)

and $T_{K}$ appears here simply as a coupling constant. The free fermion propagator is:

$$\langle \psi(\tau, 0) \psi^{\dagger}(0, 0) \rangle \propto \frac{1}{\tau}.$$  \hspace{1cm} (3.8)

Consequently the fermion operator has an RG scaling dimension of $1/2$ and the $\bar{J}(0)$ interaction has dimension 2. The corresponding coupling constant thus has dimensions of inverse energy which justifies our calling it $1/T_{K}$ in Eq. (3.6). [It follows from standard scaling arguments that this coupling constant, with dimensions of inverse energy should be of $O(T_{K})$. That it should be precisely $1/T_{K}$ is just a matter of definition of $T_{K}$.] It is straightforward to do perturbation theory in $1/T_{K}$ when calculating low energy (eg. low temperature) quantities. This generates a series in $T/T_{K}$ (or more generally $E/T_{K}$ where $E$ is the energy at which a physical quantity is being calculated).

Actually, there is another operator allowed in $H_{eff}$ when particle-hole symmetry is broken, $\psi^{\dagger} \sigma_{a} \psi(0)$. This has dimension 1 and is marginal. However it can be shown to be strictly marginal; i.e. it does not renormalize at all. It generally has a small value and can be ignored.

In this simple case the new BC at the low energy fixed point is the trivial one of Eq. (3.4). This turns out to be a fortuitously simple example of a much more general phenomenon to which we now turn.

IV. CONFORMALLY INVARIANT BOUNDARY CONDITIONS

In quantum field theory in 1 space and 1 time dimensions, it is sometimes convenient to combine space and imaginary time ($\tau$) variables into a complex co-ordinate:

$$z \equiv \tau + i x.$$  \hspace{1cm} (4.1)

The conformal group is the set of analytic tranformations:

$$z \rightarrow w(z).$$  \hspace{1cm} (4.2)

The fact that this group is infinite dimensional in 2 space-time dimensions leads to many remarkable results. A quantum field theory defined on the half line, $x > 0$, i.e. the upper half $z$-plane in the space-time picture, cannot be invariant under the full conformal group. However, it can be invariant under the infinite sub-group which leaves the real axis (i.e. the boundary) invariant:

$$w(\tau)^{*} = w(\tau).$$  \hspace{1cm} (4.3)

These transformations include time-translations, $z \rightarrow z + \tau_{0}$ scale transformations, $z \rightarrow az$ ($a \in R^{+}$) and an infinite set of others. Cardy developed powerful methods for classifying, discovering and studying all conformally invariant BC’s.

An essential point to grasp about Cardy’s general theory of conformally invariant BC’s is that generally it is not possible to state explicitly what the BC is. Fortunately this is not necessary and all physical implications of a given BC can be extracted by ingenious indirect methods. It is useful to consider a complete set of conformally invariant BC’s for a specified bulk conformal field theory (CFT). For the Kondo model the bulk CFT is simply the free fermion Hamiltonian (with 2 spin components and $k$ channels.)

The study of the finite size spectrum (FSS) plays a central role in conformal field theory. In the case of boundary CFT (BCFT) it is very useful to study the FSS on a strip of length $l$ with conformally invariant BC’s $A$ and $B$ at $x = 0$ and $x = l$. The complete set of finite size spectra (for a given bulk CFT) with an arbitrary pair of conformally invariant BC’s actually encodes all information about all the boundary conditions. For instance, all Green’s functions on the semi-infinite line with an arbitrary BC at $x = 0$ can then be calculated both close to and far from the boundary for an arbitrary BC. (Here we assume that the bulk Green’s functions, in the absence of a boundary, are already known.)

The method that Ludwig and I used to find the correct BC’s corresponding to the overscreened Kondo models was actually based on first finding the FSS by a plausible conjecture. Apart from its fundamental role in BCFT, this FSS is also important because it is commonly calculated by Wilson’s numerical renormalization group (NRG) method for Kondo models. Comparisons
of the numerical FSS with the BCFT prediction provides a powerful check on a conjectured BC.\[^{22}\]

Let us first consider the FSS with the BC’s:

\[
\psi_L(0) = \psi_R(0) \\
\psi_L(l) = -\psi_R(l).
\]

(4.4)

Since these BC’s are true at all times, and since \(\psi_L\) is a function only of \(t+x\) while \(\psi_R\) is a function only of \(t-x\), it is possible to regard the right movers as the reflection of the left movers about either boundary, i.e.

\[
\psi_R(x) = \psi_L(-x) = -\psi_L(2l-x) \quad (0 < x < l).
\]

(4.5)

Thus we may equivalently work with left-moving fermions only on a circle of radius \(2l\) with anti-periodic BC’s. The corresponding free fermion FSS can be decomposed into multiplets of spin, \(SU(2)\) and channel symmetry, \(SU(k)\). Furthermore all states have a definite charge, as measured from that of the ground state. In fact, using conformal field theory techniques it is possible to decompose the spectrum into charge, spin and channel factors in a much more powerful way. Any state can be regarded as a produce of spin, channel and charge states with the excitation energy a sum of spin, channel and charge energies. Furthermore, all spin states can be grouped into a finite number of “conformal towers”. These have the properties that the energies of all states in a conformal tower have the form:

\[
E - E_0 = \frac{\pi}{\pi} [\Delta + n],
\]

(4.6)

where \(n\) is a non-negative integer which generally takes all values from 0 to \(\infty\) in each conformal tower and \(\Delta\) is a non-negative real number which is fixed for a given conformal tower. The spin quantum numbers of the infinite set of states in a given conformal tower are not fixed but range from a lowest value, \(j\) to infinity. The spin conformal towers are conveniently labelled by the spin of the lowest energy state in the tower. [A similar structure also exists for channel and charge conformal towers but is not relevant to the present discussion.] It turns out that, for \(k\) channels of fermions, there is a total of \(k+1\) spin conformal towers labelled by spins \(j = 0, 1/2, \ldots k/2\).

We now consider the FSS that occurs when we maintain the trivial BC of Eq. (4.3) at \(x = l\) but introduce a non-trivial BC at \(x = 0\) corresponding to the Kondo model fixed point for spin \(S\) and \(k\) channels. We found that the FSS in this case can be obtained from the spectrum with the free BC’s of Eq. (4.3) by application of a general method introduced by Cardy (in this boundary context) and known as “fusion”. The fusion rules give an algorithm for generating a new spectrum. In this spin context the fusion rules are a generalization of the ordinary angular momentum addition rules. The algorithm states that for each spin conformal tower occurring in the spectrum with the BC’s of Eq. (4.3) with spin \(j\), a set of conformal towers occur in the Kondo spectrum with spin

\[
j' = |j - S|, |j - S| + 1, \ldots, \min(j + S, k - j - S).
\]

(4.7)

Here the last quantity represents the minimum of \(j + S\) and \(k - j - S\). To take a simple example, for \(S = 1/2\), \(k = 2\), the 3 conformal towers have \(j = 0, 1/2, \text{and } 1\). The fusion rules replace these conformal towers by:

\[
0 \rightarrow 1/2 \\
1/2 \rightarrow 0, 1 \\
1 \rightarrow 1/2.
\]

(4.8)

In general, in the spectrum for the trivial BC’s of Eq. (4.3) each time the spin \(j\) conformal tower from the spin sector occurs together with some product of channel and charge conformal towers we must replace it by a sum of new products of conformal towers with \(j\) replaced by the set of values of \(j'\) given in Eq. (4.7). This finite size spectrum was compared to NRG results, in the case \(k = 2\), \(S = 1/2\) with excellent results.\[^{22}\]

Generalizing this approach, we can generate spectra corresponding to any pair from a complete set of BC’s corresponding to a spin \(S\) impurity at \(x = 0\) and a spin \(S'\) impurity at \(x = l\) with \(S, S' \leq k/2\). Furthermore, this represents a complete set of spectra which contains enough information to determine the boundary Green’s functions on the semi-infinite line with a Kondo BC at \(x = 0\).

Note that we have turned the intuitive physical idea that the spin-\(S\) impurity is somehow screened by the conduction electrons into a precise algorithm which gives the FSS and all other critical properties.

Also note that since the spin conformal towers are labelled by spins from \(j = 0\) to \(j = k/2\) only, in the underscreened case, \(S < k/2\), it is not possible to construct the new spectrum as outlined above. In the underscreened case \(S\) is replaced by \(k/2\) in Eq. (4.3) and a decoupled impurity of size \(S_{\text{imp}} = S - k/2\) remains in the low energy spectrum. It can be shown that fusion with the maximal spin conformal tower, \(j = k/2\), always gives a trivial spectrum corresponding to the new BC:

\[
\psi_R(0) = -\psi_L(0).
\]

(4.9)

Thus we get a Fermi liquid groundstate with a decoupled impurity spin, consistent with the naive strong coupling picture discussed in the previous section.

Without attempting to give more details on the method, we now summarize some of the results which emerge from this BCFT approach.

One of the most intriguing results is an exact formula for the equal-time correlation function of the left and right moving fermions:\[^{14}\]

\[
\langle \psi_L^{\text{iso}}(x) \psi_R j \beta(x) \rangle = \frac{\cos[\pi(2S + 1)/(2 + k)]}{\cos[\pi/(2 + k)]} \cdot \frac{\delta_j \delta_\beta}{2x}.
\]

(4.10)

(Here, and in the formulas below it is assumed that \(S \leq k/2\). Otherwise \(S\) must be replaced by \(k/2\).) The free bulk Green’s function is:

\[
\langle \psi_L^{\text{iso}}(x) \psi_L j \beta(x) \rangle = \frac{\delta_j \delta_\beta}{2x}.
\]

(4.11)
In the exactly screened, or overscreened case, $S = k/2$, simple algebra shows that Eq. (4.11) reduces to:

$$\langle \psi_{L}^\dagger(x) \psi_{Rj\beta}(x) \rangle = -\frac{\delta_{j,\beta}^i}{2x}. \quad (4.12)$$

Comparing to Eq. (4.11), we see that the change of sign just reflects the Fermi liquid BC, $\psi_{R}(0) = -\psi_{L}(0)$, i.e. the $\pi/2$ phase shift. In general, we may write:

$$\langle \psi_{L}^\dagger(x) \psi_{Rj\beta}(x) \rangle = S^{(1)} \frac{\delta_{j,\beta}^i}{2x}, \quad (4.13)$$

where $S^{(1)}$ is the $S$-matrix amplitude for an incoming electron to scatter into 1 electron (and no holes) at zero temperature and right at the Fermi surface. In the Fermi liquid case $|S^{(1)}| = 1$. In the non-Fermi liquid case, $|S^{(1)}| < 1$. Since the $S$-matrix must be unitary when all processes are included, this implies that the probability of producing multi-particle final states is finite, even at $T = 0$ and right at the Fermi surface in the overscreened case. This is perhaps the clearest demonstration that the overscreened fixed points correspond to non-Fermi liquid groundstates. From this single fermion Green’s function, in the presence of a single impurity at the origin, we can find the self-energy in the case of a dilute random array of impurities, and hence the resistivity. This gives the $T = 0$ dc resistivity:

$$\rho(0) = \frac{3n_{i}}{k\pi(e\nu v_{F})^{2}} \left[ 1 - \frac{S^{(1)}}{2} \right]. \quad (4.14)$$

Here $n_{i}$ is the density of impurities. In the case $S^{(1)} = -1$, ($\pi/2$ phase shift) we obtain the “unitary limit” resistivity. In non-Fermi liquid cases the resistivity is reduced. A useful check is to consider the large $k$ limit. In this case, $S^{(1)} \to 1$ and we recover the Green’s function with the original BC $\psi_{R}(0) = \psi_{L}(0)$. In this limit, the resistivity goes to zero. This corresponds to a weak critical Kondo coupling, $\lambda_{c} = 2/k \to 0$, from Eq. (2.5). The case $k = 2$, $S = 1/2$ (or, in general $k = 4S$) is especially interesting because now $S^{(1)} = 0$. In this case all the scattering is multi-particle with zero amplitude for a single electron to scatter into a single electron!

It is also possible to calculate the leading $T$ dependence of the resistivity at low $T^{13}$. This requires doing perturbation theory in the leading irrelevant coupling constant at the low energy fixed point. Our solution for the non-Fermi liquid BC determines this coupling constant to have renormalization group eigenvalue:

$$y = 2/(2 + k). \quad (4.15)$$

It then follows by a standard scaling argument that this coupling constant is $1/T_{K}^{y}$ times a constant of order 1. Thus the resistivity, at low $T$ goes as:

$$\rho(0) \to \frac{3n_{i}}{k\pi(e\nu v_{F})^{2}} \left[ 1 - \frac{S^{(1)}}{2} \right] \left[ 1 + c(T/T_{K})^{y} \right]. \quad (4.16)$$

Here $c$ is a constant of $O(1)$. In the Fermi liquid case, the leading irrelevant operator is $J_{2}^{2}(0)$ as mentioned in the previous section and the corresponding coupling constant has dimension 1. However, in this case it contributes to the resistivity only beginning in second order or perturbation theory giving:

$$\rho(0) \to \frac{3n_{i}}{k\pi(e\nu v_{F})^{2}} \left[ 1 - c(T/T_{K})^{2} \right]. \quad (4.17)$$

In particular the $T$ dependence of $\rho$ is quadratic for $k = 1$ but square root for $k = 2$ and $S = 1/2$.

The impurity entropy also shows interesting behavior. In the Fermi liquid case, at $T = 0$, this is simply:

$$S_{\text{imp}}(0) = 2S_{\text{eff}} + 1, \quad (4.18)$$

where $S_{\text{eff}} = S - k/2$, is the size of the underscreened spin. (This becomes zero for exact screening.) This is simply the entropy of a decoupled spin of size $S_{\text{eff}}$ and reflects the asymptotic decoupling of the electronic degrees of freedom from the partially screened spin at $T \to 0$. On the other hand, from the non-Fermi liquid BC for overscreening, we find:

$$S_{\text{imp}}(0) = \frac{\sin(\pi/2 + 1)/(2 + k)}{\sin(\pi/(2 + k))}. \quad (4.19)$$

This is always between 1 and $2S + 1$, corresponding to non-trivial partial screening of the impurity spin. It is generally not an integer. This result was first obtained from the Bethe ansatz solution of the multi-channel Kondo problem$^{7,8}$ and thus provides a valuable check on the BCFT approach.

The $T$-dependence of the impurity entropy at low $T$, i.e. the specific heat, can also be calculated from lowest order perturbation theory in the irrelevant operator. It can be found at all $T$ from the Bethe ansatz, giving compatible results. The same is true of the impurity susceptibility. These quantities exhibit fractional power law behavior in the non-fermi liquid case$^{12}$:

$$C_{\text{imp}}(T) \to (T/T_{K})^{2y}$$

$$\chi_{\text{imp}}(T) \to (T/T_{K})^{2y-1}. \quad (4.20)$$

The behavior is different in the special case $k = 2$, $S = 1/2$ where $2y = 1$. We now obtain:

$$C_{\text{imp}}(T) \to (T/T_{K}) \ln(T/K)$$

$$\chi_{\text{imp}}(T) \to \ln(K/T). \quad (4.21)$$

V. SYMMETRY BREAKING PERTURBATIONS AND PHYSICAL REALIZATIONS

The multi-channel Kondo model has an exact $SU(k)$ channel symmetry in addition to exact $SU(2)$ spin symmetry. In possible physical realizations this symmetry is usually partly broken by extra terms in the Hamiltonian.
For instance, the channel symmetry might be broken with 1 channel coupling more strongly to the impurity that the rest. The interaction term of Eq. (1.7) is then modified to:

\[ H = \sum_{\vec{k}_\alpha,i} \psi_\alpha^\dagger \psi_{\alpha i} \epsilon(k) + \vec{S} \cdot \sum_{\vec{k}_\beta,i} J \psi_\beta^\dagger \frac{\sigma}{2} \psi_{\beta i} \]  

(5.1)

It is easy to see how this effects the \( \beta \) functions for the \( k \) different couplings, \( \lambda_i \). The quadratic term is self-interaction whereas the cubic term contains a Fermion loop which implies a sum over all channels:

\[ d\lambda_i / d(\ln D') = -\lambda_i^2 + (k/2)\lambda_i \sum_{j=1}^{k} \lambda_j^2 + O(\lambda^4) \]  

(5.2)

If one of the coupling constants, say \( \lambda_1 \), is initially larger than the rest, then Eq. (5.2) implies that it grows faster than the rest. Indeed, the other couplings eventually start to decrease under renormalization. This begins to happen when the \( j = 1 \) term in the cubic term in \( \beta \) function for \( \lambda_1 \) (with \( i > 1 \)) starts to dominate over the quadratic (and other) terms. The simple physical picture, in the \( S = 1/2 \) case, is that one channel screens the impurity and the other \( k - 1 \) channels asymptotically decouple. Thus we effectively get a single channel fixed point which is of Fermi liquid type.

We note in passing that this observation may be important in explaining why single-channel Kondo behavior is apparently observed quite commonly in metals with dilute magnetic impurities. Our discussion so far has always assumed that the Kondo interaction was a spatial \( \delta \) function, so that only the s-wave degrees of freedom of the electrons couple to the impurity. Allowing for a finite range interaction, the other harmonics will couple as well. We can still map to a 1-dimensional model but with many channels, all with different Kondo couplings. Since the s-wave will generally couple most strongly the previous naive 1-channel analysis still applies.

To actually verify that channel symmetry breaking is a relevant perturbation at the non-Fermi liquid fixed point it is necessary to calculate the scaling dimension of the most relevant new operator which appears in the Hamiltonian when channel symmetry is broken. The conclusion is that this operator has dimension, \( \Delta = k/(2+k) \), and thus is relevant.

Another possible type of symmetry breaking interaction is exchange anisotropy which preserves channel symmetry but breaks ordinary spin-rotation symmetry. This is found to be irrelevant at both Fermi and non-Fermi liquid fixed points.

Finally, we can consider a local magnetic field, acting only on the impurity spin. This is found to be relevant at the non-Fermi liquid fixed points. In particular, the corresponding boundary operator has dimension \( 2/(2+k) \).

There have been several proposals for experimental realizations of non-Fermi liquid Kondo behavior, primarily in the simplest case \( k = 2, S = 1/2 \). An early proposal involves an atom in a double well potential coupled to conduction electrons. In this case, the two different locations of the atom correspond to \( S^z = \pm 1/2 \) for the impurity “spin”. A simplified treatment keeps only 2 angular momentum states of the conduction electrons, the s-wave and the p-wave (with aximuthal component, \( m = 0 \)). These two angular momentum components of the conduction electrons play the role of conduction electron spin. A very anisotropic Kondo interaction exists between the conduction electrons and the atom. The \( \sigma^z S^z \) term represents electrons in either channel scattering off the atom in either location. The \( \sigma^z S^z \) term represents “electron-assisted tunnelling” in which the atom tunnels from one location to the other while the scattering electron changes angular momentum states. It is crucial to note that the real electron spin is assumed to be a completely passive quantum number in the scattering/tunnelling process. Thus it plays the role of the “channel” in the 2-channel Kondo model. Since spin anisotropy is irrelevant, as mentioned above, this model would renormalize to the non-Fermi liquid fixed point. However, there are 2 other interactions which must be included. One of these is direct tunnelling of the atom between the 2 locations (without any electron involvement). This corresponds to a term \( \Delta \cdot S^z \) in the Hamiltonian, where \( \Delta \) is a parameter with dimensions of energy. Finally, the energies of the atom at the 2 locations are generally unequal, leading to a term \( \Delta_0 \cdot S^z \). Both of these terms correspond to a local magnetic field acting on the impurity only, which is relevant, as mentioned above. Only if

\[ \Delta, \Delta_0 \ll T_K \]  

(5.3)

does non-Fermi liquid behavior occur. In that case we expect we expect to see approximate non-Fermi liquid behavior over an intermediate energy range:

\[ \Delta, \Delta_0 \ll E \ll T_K \]  

(5.4)

It is not obvious that the condition of Eq. (5.3) ever occurs in any real system although various proposals have been made. A relatively recent proposal involved electrons tunnelling through a nano-constriction. The proposal is that, in some cases, the tunnelling goes through localized states in the nano-constriction which can exist in 2 different configurations, corresponding to “impurity spin”. A closely related proposal involves crystal field states of an atom in a crystal.

Another pair of proposals involves the Coulomb blockade in a quantum dot. In one scenario the voltage on the quantum dot is tuned to a critical point where \( 2 \) possible charges of the dot have the same energy. In this case the 2 charge states of the dot behave as the effective \( S^z \) eigenstates. Again the electron spin plays the role of the 2 “channels” in the effective 2-channel Kondo model. An alternative scenario involves a quantum dot where the voltage is such that there is an odd number of electrons on
the dot which is then assumed to have a spin-1/2 ground state. This is closer to the original formulation of the Kondo model in which it is real electron spin which couples to the impurity. In the usual transport experiments, where 2 normal leads are connected to the quantum dot, a single channel Kondo model is appropriate. This single channel corresponds to the symmetric linear combination formed from the 2 leads ("s-wave"). In fact, even if a larger number of normal leads is connected to the dot, the single channel Kondo model is still the appropriate description because only a single symmetrized channel, a symmetric linear combination formed from all the leads, couples to the spin of the quantum dot. A 2 channel model can be obtained if one of the leads is replaced by a mesoscopic sized island of electrons, a "large dot". In this case tunnelling of electrons onto or off of this island may be suppressed by its Coulomb blockade. If only Kondo coupling terms are kept in the Hamiltonian which preserve the number of electrons on this island, it can acts as a second channel leading to 2 channel behavior. However, this requires that the finite-size level spacing of spin excitations of the island be small compared to the Kondo temperature which itself must be small compared to the Coulomb blockade energy. Furthermore, a gate voltage must be fine-tuned in order to make the Kondo couplings to the two leads equal.

Thus, it remains an interesting open question which or not non-Fermi liquid behavior in a Kondo model has been or will soon be observed experimentally.

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