Stochastic degenerate fractional conservation laws

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Abstract. We consider the Cauchy problem for a degenerate fractional conservation laws driven by a noise. In particular, making use of an adapted kinetic formulation, a result of the existence and uniqueness of the solution is established. Moreover, a unified framework is also established to develop the continuous dependence theory. More precisely, we demonstrate $L^1$-continuous dependence estimates on the initial data, the order of fractional Laplacian, the diffusion matrix, the flux function, and the multiplicative noise function present in the equation.

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1. Introduction

Stochastic degenerate parabolic-hyperbolic equations are one of the important branches of nonlinear stochastic PDEs. Equations of this type model the phenomenon of convection-diffusion of an ideal fluid in porous media. Therefore, these are in great demand in fluid mechanics. The study of this type of model equations with the nonlocal operator, considered in [8], is motivated by the anomalous diffusion encountered in many physical models. The nonlocal operator appears in mathematical phenomena for fluid flows and acoustic propagation in porous media, viscoelastic materials, and pricing derivative securities in financial markets [10]. The addition of a stochastic noise to this physical model is completely natural as it represents external perturbations or a lack of knowledge of certain physical parameters. In this paper, we are interested in the well-posedness theory for the degenerate fractional conservation laws driven by a Brownian noise in any space dimension. A formal description of our problem requires a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$. We
consider the following stochastic degenerate fractional conservation laws
\begin{align}
\begin{cases}
    du(x,t) + \text{div}(F(u(x,t)))dt + g^\lambda_x[u(x,t)]dt \\
    = \text{div}(A(u)\nabla u) dt + \Phi(x,u(x,t)) dB(t) & \text{in } \Pi_T, \\
    u(0,x) = u_0(x), & \text{in } \mathbb{T}^N,
\end{cases}
\end{align}
(1.1)

where $\Pi_T = \mathbb{T}^N \times (0,T)$ with $\mathbb{T}^N$ is the $N$-dimensional torus and $T > 0$ fixed, $u_0 : \Omega \times \mathbb{T}^N \to \mathbb{R}$ is the given initial random variable, $F: \mathbb{R} \to \mathbb{R}^N$ is a given (sufficiently smooth) vector valued flux function, $A$ is a diffusion matrix (possibly degenerate) and $g^\lambda_x$ denotes the fractional Laplace operator $(-\Delta)^\lambda$ of order $\lambda \in (0,1)$. Note that $B$ is a cylindrical Wiener process, $B = \sum_{k \geq 1} w_k \gamma_k$, where the coefficients $w_k$ are independent Brownian processes and $(\gamma_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space $\mathcal{X}$ and $\Phi: L^2(\mathbb{T}^N) \to L^2(\mathcal{X}; L^2(\mathbb{T}^N))$ is $L^2(\mathcal{X}; L^2(\mathbb{T}^N))$-valued function, where $L^2(\mathcal{X}; L^2(\mathbb{T}^N))$ denotes the collection of Hilbert–Schmidt operators from $\mathcal{X}$ to $L^2(\mathbb{T}^N)$ (see Sect. 2 for the complete list of assumptions).

### 1.1. Earlier works

In the absence of non-local term along with the case $\Phi = 0$ and $A = 0$, the Eq. (1.1) becomes a standard conservation laws. In the deterministic set-up, entropy solution for conservation laws was studied by Kruzhkov [30] and established well-posedness in $L^\infty$-framework. The entropy formulation of the parabolic-hyperbolic problem involving the Laray–Lions type operator has been treated by Carrillo [11]. We refer to Bendahmane and Karlsen [5] for the more delicate anisotropic diffusion case. We also mention works of Alibaud [1], Cifani et.al. [3,17] for deterministic fractal conservation laws and fractional degenerate convection-diffusion equations, respectively. The concept of the kinetic solution was first introduced by Lions, Perthame, and Tadmor in [32] for the scalar conservation laws. Chen and Perthame [12] developed a well-posedness theory for general degenerate parabolic-hyperbolic equations with non-isotropic non-linearity. Moreover, the well-posedness theory for the fractional conservation laws was studied by Alibaud et al. [2] and Wei et al. [39].

On the other hand, in stochastic set-up, the fundamental work of Kim [25] who defined entropy solutions for the stochastic conservation-laws and established well-posedness theory to one-dimensional conservation laws that are driven by additive Brownian noise and Vallet and Wittbold [38] to multidimensional Dirichlet problem. However, when the noise is of the multiplicative type we refer to Feng and Nualart [21] for one-dimensional balance laws. Debussche and Vovelle [18] introduced kinetic formulation of stochastic conservation laws and as result, they were able to establish the well-posedness of multidimensional stochastic balance laws via a kinetic approach. The more delicate anisotropic diffusion case has been treated by Debussche et.al. [19] using, in particular, the insight from the work [23] of Hofmanova. Chen and Pang recently studied the continuous dependence on parameters for nonlinear anisotropic degenerate parabolic-hyperbolic equations with stochastic forcing in [14]. We also refer to the work of Lv et al. [33] for stochastic nonlocal conservation laws in whole...
A number of authors have contributed in the area of stochastic conservation laws and we mention the works of Biswas et al. [9], Koley et al. [26,27], Bhauryal et al. [6,7]. In the stochastic setup, the main idea is to successfully capture the noise-noise interaction of the underlying problem since this plays an important role in the well-posedness theory for stochastic PDEs, for details see [6,7,15,16,24,26,28,29].

1.2. Scope and outline of this paper

Due to the presence of a nonlinear flux term, degenerate diffusion term, and a nonlocal term in Eq. (1.1), solutions to (1.1) are not necessarily smooth and weak solutions must be sought. In comparison to the notion of entropy solution introduced by Kruzhkov [30], the adapted notion of kinetic solutions seems to be better suited particularly for the degenerate fractional conservation laws, since this allows us to keep the precise structure of the parabolic dissipative measure, whereas in the case of entropy solution part of this information is lost and has to be recovered at some stage. To sum up, we aim at developing the following results related to (1.1):

1. Drawing primary motivation from [19,20,23], we establish the well-posedness theory of the kinetic solution to the Cauchy problem (1.1) by using vanishing viscosity method along with few a priori bounds. We also derive the contraction principle, in which we employ a Kruzhkov doubling of variable technique and attempt to bound the difference of their kinetic solutions (see Theorem 3.1).

2. For $\Phi = \Phi(u)$, making use of BV estimate, we also develop a unified framework to derive the continuous dependence estimates for the Cauchy problem (1.1) on initial data, order of fractional Laplacian, the flux function, and the multiplicative noise function (see Theorem 3.2). Whenever $\Phi = \Phi(u,x)$ has dependency on the spatial variable $x$, BV estimates are no longer valid even for the stochastic conservation laws (see [13]).

As an important part in this present study, formulation of the kinetic solution is weak in the space variable but strong in the time variable, as proposed in [20]. This allows us to obtain convergence of approximations for each time $t$ (see Remark 5.2). This formulation enjoys a càdlàg condition which ensures that the trajectories of solution to (1.1) are continuous in $L^p(T^N)$ (see Corollaries 4.2, 4.5). In the case of hyperbolic scalar conservation laws, Dotti and Vovelle [20] defined a notion of a generalized kinetic solution and obtained a comparison result showing that any generalized kinetic solution is actually a kinetic solution. Difficulties are quite different in our case. Indeed, due to the structure of kinetic measure corresponding to non-local and second-order terms, the comparison principle can be proved only for kinetic solutions, and therefore, strong convergence of approximate solutions is needed in order to prove existence. The main novelty of this work lies in successfully handling the nonlocal term (fractional Laplacian) in the proof of both the comparison principle and continuous dependence estimate (see step 2 of both proofs). We emphasize that the analysis presented in this manuscript differs significantly from the work in the stochastic nonlocal conservation laws on whole space.
[33], mainly because our formulation is different. Our method is inspired by the work [20], where the authors used the same notion of kinetic formulation for the stochastic conservation laws.

The paper is organized as follows. In Sect. 2 and Sect. 3, we give details of basic setting, define the notion of kinetic solution, and state our main results, Theorems 3.1 and 3.2. Section 4 is devoted to the proof of uniqueness part of Theorem 3.1 with the $L^1$-comparison principle, Theorem 4.3. The existence part of Theorem 3.1 is done in Sect. 5. The remainder of the paper deals with the BV estimate and continuous dependence estimate (proof of Theorem 3.2), given in Sect. 6. In the end, in “Appendix A and B”, the derivation of the kinetic formulation and the existence of viscous solution are given.

2. Assumptions and preliminaries

2.1. Hypotheses

We now give the precise assumptions on each of the term appearing in the equation (1.1).

**Flux function** Let $F = (F_1, F_2, \ldots, F_N): \mathbb{R} \to \mathbb{R}^N$ be $C^2(\mathbb{R}; \mathbb{R}^N)$-smooth flux function with a polynomial growth of its derivative, in the following sense: there exists $q^* \geq 1$, $C \geq 0$, such that

$$|F'(\xi)| \leq C(1 + |\xi|^{q^* - 1}),$$

$$\sup_{|\zeta| \leq \delta} |F'(\xi) - F'(\xi + \zeta)| \leq C(1 + |\xi|^{q^* - 1}) \delta, \quad (2.1)$$

**Diffusion matrix** Let the diffusion matrix $A = (A_{kl})_{k,l=1}^{N}: \mathbb{R} \to \mathbb{R}^{N \times N}$ be positive semidefinite and symmetric, then its square-root matrix, which is positive semidefinite and symmetric, is denoted by $\sigma$. Suppose that $\sigma$ is bounded and locally $\gamma$-Hölder continuous for some $\gamma > \frac{1}{2}$, i.e.

$$|\sigma(\zeta) - \sigma(\xi)| \leq C|\xi - \zeta|^{\gamma}, \quad \forall \xi, \zeta \in \mathbb{R}, \quad |\xi - \zeta| < 1. \quad (2.3)$$

**Stochastic term** Suppose $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (w_k(t))_{k \geq 1})$ is a stochastic basis with a complete, right continuous filtration. Suppose $\mathcal{P}_T$ indicates the predictable $\sigma$-algebra on $\Omega \times [0, T]$ associated to $(\mathcal{F}_t)_{t \geq 0}$. Assume the initial data is also random variable i.e. $\mathcal{F}_0$-measurable and we also assume $u_0 \in L^p(\Omega; L^p(\mathbb{T}^N))$ for all $p \in [1, +\infty)$. Without loss of generality we can assume that the $\sigma$-algebra $\mathcal{F}$ is countably generated and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $u_0$ and the Wiener process $B$. Here we define the canonical space $\mathcal{X} \subset \mathcal{X}_0$ via

$$\mathcal{X}_0 = \left\{ v = \sum_{k \geq 1} \lambda_k \gamma_k: \sum_{k \geq 1} \frac{\lambda_k^2}{k^2} < \infty \right\}$$

with the norm

$$\|v\|_{\mathcal{X}_0}^2 = \sum_{k \geq 1} \frac{\lambda_k^2}{k^2}, \quad v = \sum_{k \geq 1} \lambda_k \gamma_k.$$
Remark that the embedding $\mathfrak{X} \hookrightarrow \mathfrak{X}_0$ is Hilbert–Schmidt. $\mathbb{P}$-almost surely, trajectories of $B$ are in $C([0,T]; \mathfrak{X}_0)$. For all $u \in L^2(T^N)$ we consider a mapping $\Phi : \mathfrak{X} \rightarrow L^2(T^N)$ defined by $\Phi(u)\gamma_k = \beta_k(\cdot, u(\cdot))$. Thus we define

$$\Phi(x, u) = \sum_{k \geq 1} \beta_k(x, u)\gamma_k,$$

the action of $\phi(x, u)$ on $\gamma \in \mathfrak{X}$ is defined by $\langle \phi(x, u), \gamma \rangle_{\mathfrak{X}}$. We assume $\beta_k \in C(T^N \times \mathbb{R})$, with the following bounds

$$\beta^2(x, u) = \sum_{k \geq 1} |\beta_k(x, u)|^2 \leq D_0(1 + |u|^2), \quad (2.4)$$

$$\sum_{k \geq 1} |\beta_k(x, u) - \beta_k(y, v)|^2 = D_1(|x - y|^2 + |u - v|h(|u - v|)), \quad (2.5)$$

where $x, y \in T^N$, $u, v \in \mathbb{R}$ and $h$ is a non-decreasing continuous function on $\mathbb{R}_+$ satisfying, $h(0)=0$ and $0 \leq h(z) \leq 1$ for all $z \in \mathbb{R}_+$. Hypothesis (2.4) imply that $\Phi : L^2(T^N) \rightarrow L_2(\mathfrak{X}; L^2(T^N))$, where $L_2(\mathfrak{X}; L^2(T^N))$ refer to the space of Hilbert–Schmidt operators from $\mathfrak{X}$ to $L^2(T^N)$. Thus, for any $u \in L^2_{\mathcal{P}}(\Omega; L^2(0, T; L^2(T^N)))$, the Itô integral $t \rightarrow \int_0^t \Phi(u)dB$ is a process taking values in $L^2(T^N)$ (see [34] for detailed construction).

**Fractional term** $g^2$ is a fractional Laplace operator $(-\triangle)^{\lambda}$ for $\lambda \in (0, 1)$ properly defined at least on $C^\infty(T^N)$ by

$$g^2_x[\phi](x) := -P.V. \int_{\mathbb{R}^N} (\phi(x + z) - \phi(x))\mu(z) \, dz \quad \forall \, x \in T^N \quad (2.6)$$

where $\mu(z) = \frac{1}{|z|^{N+2\lambda}} \neq 0$ and $\mu(0) = 0$ (see [4,37] for details).

### 2.2. Assumptions for continuous dependence estimate

We are also interested to develop a general framework for the continuous dependence estimate. Our aim is to establish continuous dependence on the fractional exponent $\lambda$ and on the non-linearities, that is, on the flux function and noise coefficients. To achieve that, we proceed as follows. Consider the pair of the nonlinear equations:

$$\begin{cases}
    du(x, t) + \text{div}(F(u(x, t)))dt + g^2_x[u(x, t)]dt \\
    = \text{div}(A(u)\nabla u)dt + \Phi(u(x, t)) dB(t) \\
    u(0, x) = u_0(x), \quad x \in T^N, \, t \in (0,T) \quad (2.7)
\end{cases}$$

$$\begin{cases}
    dv(x, t) + \text{div}(G(v(x, t)))dt + g^2_x[v(x, t)]dt \\
    = \text{div}(B(v)\nabla v)dt + \Psi(v(x, t)) dB(t) \\
    v(0, x) = v_0(x), \quad x \in T^N, \, t \in (0,T) \quad (2.8)
\end{cases}$$

For continuous dependence estimate, in addition, we are assuming following assumptions on terms in above equations. For all $\xi, \zeta \in \mathbb{R}$,
\[
\sup_{|\zeta| \leq \delta} |F'(\zeta) - F'(\zeta + \xi)| \leq C(1 + |\xi|^{p_*-1})\delta^{\lambda_{F_1}}, \tag{2.9}
\]
\[
\sup_{|\zeta| \leq \delta} |G'(\zeta) - G'(\zeta + \xi)| \leq C(1 + |\xi|^{p_*-1})\delta^{\lambda_{G_1}}, \tag{2.10}
\]
\[
|\Phi(\xi) - \Phi(\zeta)|^2 \leq |\xi - \zeta|^{\lambda_{F_2}+1}, \tag{2.11}
\]
\[
|\Psi(\xi) - \Psi(\zeta)|^2 \leq |\xi - \zeta|^{\lambda_{G_2}+1}, \tag{2.12}
\]
\[
\|F' - G'\|_{L^\infty} \|\Phi - \Psi\|_{L^\infty} < \infty, \tag{2.13}
\]
\[
u_0, \nu_0 \in L^p(\Omega \times \mathbb{T}^N) \cap L^1(\Omega; BV(\mathbb{T}^N)) \tag{2.14}
\]
where \(\lambda_{F_1}, \lambda_{F_2}, \lambda_{G_1}\) and \(\lambda_{G_2}\) are positive constants. We still assume that \(\Phi, \Psi\) have at most linear growth (2.4). The diffusion matrices
\[
A = (A_{ij})_{i,j=1}^N, \quad B = (b_{ij})_{i,j=1}^N : \mathbb{R} \to \mathbb{R}^{N \times N}
\]
are symmetric and positive semidefinite. Its square-root matrices, which are also symmetric and positive semidefinite, is denoted by \(\sigma\).

2.3. Preliminary results on Young measure

Here we state some results regarding Young measure theory. We refer to [20, Section 2] for proof of results.

**Definition 2.1.** (Young measure) Let \((\mathcal{O}, \mathcal{F}, \lambda_1)\) be a finite measure space. A mapping \(\nu\) from \(\mathcal{O}\) to \(\mathcal{P}(\mathbb{R})\), the set of probability measures on \(\mathbb{R}\), is said to be a Young measure if, for all \(h \in C_b(\mathbb{R})\), the map \(y \to \nu_y(h)\) from \(\mathcal{O}\) into \(\mathbb{R}\) is \(\mathcal{F}\)-measurable. We say that a Young measure \(\nu\) vanishes at infinity if for all \(q \geq 1\),
\[
\int_\mathcal{O} \int_{\mathbb{R}} |\zeta|^q d\nu_y(\zeta) d\lambda_1(y) < \infty.
\]

**Definition 2.2.** (Kinetic function) Let \((\mathcal{O}, \mathcal{F}, \lambda_1)\) be finite measure space. A measurable function \(f : \mathcal{O} \times \mathbb{R} \to [0, 1]\) is said to be a kinetic function, if there exists a Young measure \(\nu\) on \(\mathcal{O}\) vanishing at infinity such that, for \(\lambda_1\)-a.e. \(y \in \mathcal{O}\), for all \(\xi \in \mathbb{R}\),
\[
f(y, \zeta) = \nu_y(\zeta, \infty).
\]

**Definition 2.3.** (Equilibrium) We say that \(f\) is an equilibrium, if there exists a measurable function \(u : \mathcal{O} \to \mathbb{R}\) such that \(f(y, \zeta) = \mathbb{1}_{u(y) > \zeta}\) almost everywhere, or, equivalently, \(\nu_y = \delta_{\zeta=u(y)}\) for almost every \(y \in \mathcal{O}\).

**Theorem 2.1.** (Compactness of Young measures) Let \((\mathcal{O}, \mathcal{F}, \lambda_1)\) be a finite measure space such that sigma algebra \(\mathcal{F}\) is countably generated. Let \((\nu^n)\) be a sequence of Young measures on \(\mathcal{O}\) satisfying uniformly for some \(q \geq 1\),
\[ \sup_{n} \int_{\mathcal{O}} \int_{\mathbb{R}} |\zeta|^q d\nu^n_y(\zeta) d\lambda_1(z) < +\infty. \]  

(2.17)

Then there exists a Young measure \( \mathcal{V} \) on \( \mathcal{O} \) and a subsequence still denoted \( (\mathcal{V}^n) \) such that, for all \( h \in L^1(\mathcal{O}) \), for all \( g \in C_b(\mathbb{R}) \),

\[ \lim_{n \to +\infty} \int_{\mathcal{O}} h(y) \int_{\mathbb{R}} g(\zeta) d\nu^n_y(\zeta) d\lambda_1(y) = \int_{\mathcal{O}} h(y) \int_{\mathbb{R}} g(\zeta) d\nu_y(\zeta) d\lambda_1(y). \]  

(2.18)

**Corollary 2.2.** (Compactness of kinetic functions) Let \( (\mathcal{O}, \mathbb{F}, \lambda_1) \) be a finite measure space such that \( \mathbb{F} \) is countably generated. Let \( (f^n) \) be a sequence of kinetic functions on \( \mathcal{O} \times \mathbb{R} \) :

\[ f^n(y, \zeta) = \nu^n_y(\zeta, +\infty), \]  

where \( \nu^n \) are Young measures on \( \mathcal{O} \) satisfying for some \( q \geq 1 \),

\[ \sup_{n} \int_{\mathcal{O}} \int_{\mathbb{R}} |\zeta|^q d\nu^n_y(\zeta) d\lambda_1(y) < +\infty. \]  

(2.19)

Then there exists a kinetic function \( f \) on \( \mathcal{O} \times \mathbb{R} \) (related to the Young measure \( \mathcal{V} \) by formula \( f(y, \zeta) = \nu_y(\zeta, +\infty) \)) such that, up to a subsequence, \( f^n \to f \) in \( L^\infty(\mathcal{O} \times \mathbb{R}) \) weak-*.

**Lemma 2.3.** (Convergence to an equilibrium) Let \( (\mathcal{O}, \mathbb{F}, \lambda_1) \) be a finite measure space, Let \( q > 1 \). Let \( (f^n) \) be a sequence of kinetic functions on \( \mathcal{O} \times \mathbb{R} \) : \( f^n(y, \zeta) = \nu^n_y(\zeta, +\infty) \) where \( \nu^n \) are Young measures on \( \mathcal{O} \) satisfying for some \( q > 1 \),

\[ \sup_{n} \int_{\mathcal{O}} \int_{\mathbb{R}} |\zeta|^q d\nu^n_y(\zeta) d\lambda_1(y) < +\infty. \]  

(2.20)

Let \( f \) be a kinetic function on \( \mathcal{O} \times \mathbb{R} \) such that \( f^n \to f \) in \( L^\infty(\mathcal{O} \times \mathbb{R}) \) weak-*.

Assuming that \( f \) is an equilibrium, \( f(y, \zeta) = 1_{u(y) > \zeta} \), and letting

\[ u_n(y) = \int_{\mathbb{R}} \zeta d\nu^n_y(\zeta) \]

then, for all \( 1 \leq p < q \), \( u_n \to u \) in \( L^p(\mathcal{O}) \).

### 2.4. Sobolev space

Let us denote Sobolev space \( H^\lambda(\mathbb{T}^N) \), for \( \lambda \in \mathbb{R}_+ \), as the subspace of \( L^2(\mathbb{T}^N) \) for which the norm

\[ \|u\|^2_{H^\lambda(\mathbb{T}^N)} = \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^{\lambda} |\hat{u}(n)|^2 \]

is finite. Here \( \hat{u}(n) \) denotes the Fourier coefficient of \( u \).

### 3. Definitions and main results

Here, we introduce the kinetic formulation to (1.1) as well as the basic definitions concerning the notion of kinetic solution.
3.1. Random measure, kinetic solution

**Definition 3.1.** *(Random measure with finite first-order moment)* A mapping $m$ from $\Omega$ to $\mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, the set of bounded Borel signed measures over $\mathbb{T}^N \times [0, T] \times \mathbb{R}$, is said to be a random measure with finite first-order moment provided:

(i) $m$ is measurable in the following sense: for each $h \in C_0(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, the mapping $m(h) : \Omega \to \mathbb{R}$ is measurable,
(ii) $\mathbb{E}|m(\mathbb{T}^N \times [0, T] \times \mathbb{R})| < \infty$.

**Definition 3.2.** *(Kinetic solution)* A $L^1(\mathbb{T}^N)$-valued stochastic process $(u(t))_{t \in [0, T]}$ is said to be a kinetic solution to (1.1) with initial datum $u_0$, if $(u(t))_{t \in [0, T]}$ and $f(t) := 1_{u(t) > \zeta}$ have the following properties:

1. $u \in L^p_{\text{loc}}(\mathbb{T}^N \times [0, T] \times \Omega) \cap L^2(\Omega; L^2([0, T]; H^\lambda(\mathbb{T}^N)))$, $\forall p \in [1, +\infty)$,
2. for all $\varphi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$, $\mathbb{P}$-almost surely, $t \mapsto \langle f(t), \varphi \rangle$ is càdlàg,
3. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that
\[
\mathbb{E}(\sup_{0 \leq t \leq T} \|u(t)\|_{L^p(\mathbb{T}^N)}) \leq C_p, \tag{3.1}
\]
4. $\text{div} \int_0^u \sigma(\zeta) d\zeta \in L^2(\Omega \times [0, T] \times \mathbb{T}^N)$,
5. for any $\varphi \in C_b(\mathbb{R})$ the following chain rule formula holds true: $\mathbb{P}$-almost surely, almost every $t \in [0, T]$
\[
\text{div} \int_0^u \varphi(\zeta)\sigma(\zeta) d\zeta = \varphi(u)\text{div} \int_0^u \sigma(\zeta) d\zeta \quad \text{in } D'(\mathbb{T}^N) \text{ a.e.} \tag{3.2}
\]
6. Let $\eta_1, \eta_2 : \Omega \to \mathcal{M}^+(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ be defined as follows:
\[
\eta_1(x, t, \xi) = \int_{\mathbb{R}^N} |u(x + z, t) - \zeta| \mathbb{1}_{\text{Conv}\{u(x,t),u(x+z,t)\}}(\zeta) \mu(z) dz,
\]
and
\[
\eta_2(x, t, \zeta) = |\text{div} \int_0^u \sigma(\zeta) d\zeta|^2 \delta_{u(x,t)}(\zeta).
\]

There exists a non-negative random measure $m$ with finite first-order moment in the sense of Definition 3.1 such that $\mathbb{P}$-a.s., $m \geq \eta_1 + \eta_2$, and for all $\varphi \in C^2_c(\mathbb{T}^N \times \mathbb{R})$, $t \in [0, T]$,
\[
\langle f(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), F'(\zeta) \cdot \nabla \varphi \rangle ds + \int_0^t \langle f(s), A(\zeta) : D^2 \varphi \rangle ds - \int_0^t \langle f(s), g^\lambda_x(\varphi) \rangle ds
\]
\[
+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^N} \beta_k(x, u(x, s)) \varphi(x, u(x, s)) dx dw_k(s)
\]
\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_x \varphi(x, u(x, s)) \beta^2(x, u(x, s)) dx ds - m(\partial_x \varphi)([0, t])
\]
\[
\tag{3.3}
\]
$\mathbb{P}$-almost surely, where $f_0(x, \zeta) = 1_{u_0 > \zeta}$, $\beta^2 := \sum_{k \geq 1} |\beta_k|^2$. 
Here we have used the brackets $\langle \cdot, \cdot \rangle$, to indicate the duality between $C^\infty_c(T^N \times \mathbb{R})$ and the space of distributions over $T^N \times \mathbb{R}$. We have used the shorthand $m(\varphi)$ to indicate the Borel measure on $[0, T]$ defined by

$$m(\varphi): A \mapsto \int_{T^N \times \mathbb{R}} \varphi(x, \xi)dm(x, t, \zeta), \varphi \in C_b(T^N \times \mathbb{R})$$

for all $A$ Borel subset of $[0, T]$, and

$$\text{Conv}\{a, b\} := (\min\{a, b\}, \max\{a, b\}).$$

We have used the notation $A : B = \sum_{i,j} a_{ij} b_{ij}$ for two matrices $A = (a_{ij})$, $B = (b_{ij})$ of the same size.

### 3.2. The main results

In this subsection, we record the statements of main results. We have càdlàg condition in formulation of kinetic solutions which ensure that almost surely, trajectories of solution $u$ are continuous in $L^p(T^N)$. We have two results as follows.

**Theorem 3.1.** (Existence and uniqueness) Under the assumptions (2.1)–(2.5), there exists a unique kinetic solution $(u(t))_{t \in [0, T]}$ to (1.1) which has $\mathbb{P}$-almost surely continuous trajectories in $L^p(T^N)$, for all $p \in [1, +\infty)$. Moreover, if $(u_1(t))_{t \in [0, T]}, (u_2(t))_{t \in [0, T]}$ are kinetic solutions to (1.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t \in [0, T]$,

$$E \|u_1(t) - u_2(t)\|_{L^1(T^N)} \leq E \|u_{1,0} - u_{2,0}\|_{L^1(T^N)}.$$  \hfill (3.4)

We also develop a general framework for the continuous dependence estimate of kinetic solutions. Note that the $L^1$- contraction (3.4) gives the continuous dependence on the initial data. However, we intend to establish continuous dependence on the order of fractional Laplacian, the flux function, the diffusion matrix and the multiplicative noise present in equation (1.1).

**Theorem 3.2.** (Continuous dependence estimate) Let assumptions (2.9)–(2.16) holds. Let $(u(t))_{t \in [0, T]}$ be a kinetic solution to (2.7) with initial data $u_0$, and let $(v(t))_{t \in [0, T]}$ be a kinetic solution to (2.8) with initial data $v_0$. Then, the following continuous dependence estimate holds: for all $t \in [0, T]$,

$$E \int_{T^N} |u(x, t) - v(x, t)|dx \leq C_T \left( E \int_{T^N} |v_0(x) - u_0(x)|dx + \|F' - G'\|_{L^\infty(\mathbb{R})} \right)$$

$$+ \left( \|\Phi - \Psi\|_{L^\infty(\mathbb{R})} + \sqrt{\int_{|z| \leq r_1} |z|^2 d|\mu_\lambda - \mu_\beta|(z)} \right)$$

$$+ \|\sigma - \tau\|_{L^\infty(\mathbb{R})} \min \left\{ \frac{1}{2}, \frac{\lambda G_1}{2}, \frac{\lambda G_2}{2} \right\}$$

$$+ \int_{|z| > r_1} E(\|u_0(\cdot + z) - u_0\|_{L^1(T^N)})$$

$$+ \|v_0(\cdot + z) - v_0\|_{L^1(T^N)}) d|\mu_\lambda - \mu_\beta|(z).$$
where constants $C > 0$ (depending on $T, u_0, v_0, G$) and $r_1 > 0$. Here $d\mu_\lambda(z) := \frac{dz}{|z|^{d+r}}$.

**Remark 3.1.** Throughout this paper, the letter $C$ to denote various generic constant. There are situations where constant may change from line to line, but the notation is kept unchanged, so long as it does not impact central idea.

4. Proof of uniqueness and continuity part of Theorem 3.1

4.1. Left limit representation

Since we need some technical results to prove the contraction principle, we first state those technical results and then move on to the proof of uniqueness. Here we closely follow the approach of [20] and obtain a canonical property of kinetic solution, which is useful to show that the kinetic solution, $u$ has almost surely continuous trajectories. In the following proposition, we show that the almost surely property to be càdlàg is independent from test function, and limit from the left at any point $t_\ast \in (0, T]$ is also represented by a kinetic function.

**Proposition 4.1.** Let $u_0$ be a initial data. Let $(u(t))_{t \in [0,T]}$ be a solution to (1.1) with initial data $u_0$, then the following conditions hold,

1. there exists a measurable subset $\Omega_1 \subset \Omega$ of full probability such that, for all $\omega \in \Omega_1$, for all $\varphi \in C_c(\mathbb{T}^N \times \mathbb{R}), t \rightarrow \langle f(\omega, t), \varphi \rangle$ is càdlàg.

2. there exists an $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0, 1])$-valued process $(f^- (t))_{t \in [0,T]}$ such that:

   for all $t \in (0, T]$, for all $\omega \in \Omega_1$ for all $\varphi \in C^2_c(\mathbb{T}^N \times \mathbb{R})$, $f^-(t)$ is a kinetic function on $\mathbb{T}^N$ which represents the left limit of $s \rightarrow \langle f(s), \varphi \rangle$ at $t$:

   $$\langle f^-(t), \varphi \rangle = \lim_{s \rightarrow t^-} \langle f(s), \varphi \rangle. \tag{4.1}$$

**Proof.** For a proof, one can follow similar lines as proposed in [20, Proposition 2.10]. □

**Bounds** By construction, we note that $\mathcal{V}^- = -\partial_\zeta f^-$ satisfies the following bounds: for all $\omega \in \Omega_1$,

$$\sup_{t \in [0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\zeta|^p d\mathcal{V}^-_{x,t}(\zeta) dx \leq C_p(\omega), \quad \mathbb{E} \left( \sup_{t \in [0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\zeta|^p d\mathcal{V}^-_{x,t}(\zeta) dx \right) \leq C_p. \tag{4.2}$$

We obtain (4.2) using Fatou’s lemma and (3.1).

**Equation for $f^-$:** For all $\varphi \in C^2_c(\mathbb{T}^N \times \mathbb{R})$, $\mathbb{P}$-a.s., for all $t \in [0, T]$,

$$\langle f^-(t), \varphi \rangle = \langle f(0), \varphi \rangle + \int_0^t \langle f(s), F'(\zeta) \cdot \nabla_x \varphi \rangle ds + \int_0^t \langle f(s), A : D^2 \varphi \rangle ds$$

$$- \int_0^t \langle f(s), g^x_\lambda[\varphi] \rangle ds$$
\[ + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_\mathbb{R} \beta_k(x, \xi) \varphi(x, \zeta) d\mathcal{V}_{x,s}(\zeta) dx dw_k(s) \]
\[ + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_\mathbb{R} \beta^2(x, \xi) \partial_\zeta \varphi(x, \zeta) d\mathcal{V}_{x,s}(\zeta) dx ds - m(\partial_\zeta \varphi)([0, t])). \tag{4.3} \]

In particular, we have, \(\mathbb{P}\)-almost surely, for all \(t \in [0, T]\),
\[ \langle f(t) - f^-(t), \varphi \rangle = -m(\partial_\zeta \varphi)(\{t\}). \tag{4.4} \]

It implies that outside the set of atomic points (at most countable points) of \(A \mapsto m(\partial_\zeta \varphi)(A)\), we get \(\langle f(t), \varphi \rangle = \langle f^-(t), \varphi \rangle\). It shows that \(\mathbb{P}\)-almost surely, \(f = f^\pm\) a.e. \(t \in [0, T]\).

In particular, equation (4.3) gives us the following equation on \(f^-\); for all \(\varphi \in C^2_c(\mathbb{T}^N \times \mathbb{R})\), \(\mathbb{P}\)-a.s., for all \(t \in [0, T]\),
\[ \langle f^-(t), \varphi \rangle = \langle f(0), \varphi \rangle + \int_0^t \langle f^-(s), F'(\zeta) \cdot \nabla_x \varphi \rangle ds + \int_0^t \langle f^-(s), A : D^2 \varphi \rangle ds \]
\[ - \int_0^t \langle f^-(s), g^\beta_x[\varphi] \rangle ds \]
\[ + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_\mathbb{R} \beta_k(x, \xi) \varphi(x, \zeta) d\mathcal{V}_{x,s}^-x(s) dx d\zeta \]
\[ + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_\mathbb{R} \beta^2(x, \xi) \partial_\zeta \varphi(x, \zeta) d\mathcal{V}_{x,s}^-x(s) dx ds - m(\partial_\zeta \varphi)([0, t])). \tag{4.5} \]

We have càdlàg condition in formulation of kinetic solutions which ensure that almost surely, trajectories of solution \(u\) are right continuous in \(L^p(\mathbb{T}^N)\). For that purpose, here we state the following result.

**Corollary 4.2.** Let \((u(t))_{t \in [0,T]}\) be a solution to (1.1) with initial datum \(u_0\). Then, for all \(p \in [1, +\infty)\), for all \(\omega \in \Omega_1\) (given in Proposition 4.1), the map \(t \mapsto u(t)\) from \([0, T]\) to \(L^p(\mathbb{T}^N)\) is continuous from the right.

**Proof.** We refer to [20, Corollary 2.13] for a proof. \(\square\)

### 4.2. Doubling of variables

As a next step towards the proof of uniqueness, we need a technical proposition relating two kinetic solutions to (1.1). We will also use the following notation:
If \(f : X \times \mathbb{R} \to [0, 1]\) is kinetic function, we indicate by \(\bar{f}\) the conjugate function \(\bar{f} = 1 - f\). We indicate by \(f^+\) the right limit, which is simply \(f\), that is \(f^+(t) := f(t)\). From now on, we will work with two fixed representatives of \(f (f^+ \text{ and } f^-)\) and we can take any of them in integral with respect to time or in a stochastic integral. We need the following technical Proposition to prove uniqueness of kinetic solution. We follow similar lines as proposed in [20] for the proof of following proposition. Here, we give the details for the sake of completeness.
Proposition 4.3. Let \((u_1(t))_{t \in [0,T]}\) and \((u_2(t))_{t \in [0,T]}\) be kinetic solutions to (1.1) with initial data \(u_{1,0}\) and \(u_{2,0}\), respectively and denote \(f_1(t) = \mathbb{1}_{u_1(t) > \xi} \& f_2(t) = \mathbb{1}_{u_2(t) > \xi}\). Then, for all \(t \in [0,T]\) and non-negative test functions \(\theta \in C^\infty(T^N), \kappa \in C^\infty_c(\mathbb{R})\), we have

\[
\begin{align*}
\mathbb{E}\left[ \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_1^{\pm}(x,t,\xi) f_2^{\pm}(y,t,\zeta) d\xi d\zeta dx dy \right] \\
\leq \mathbb{E}\left[ \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_{1,0}(x,\xi) f_{2,0}(y,\zeta) d\xi d\zeta dx dy \\
+ \mathcal{R}_\theta + \mathcal{R}_\kappa + J + K \right],
\end{align*}
\]

(4.6)

where

\[
\begin{align*}
\mathcal{R}_\theta &= \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) f_2(y, s, \zeta)(F'(\xi) - F'(\zeta))\kappa(\xi - \zeta) d\xi d\zeta \cdot \nabla \theta(x-y) dx dy ds, \\
\mathcal{R}_\kappa &= \frac{1}{2} \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y) \int_0^t \int_{\mathbb{R}^2} \kappa(\xi - \zeta) \sum_{k \geq 1} |\beta_k(x, \xi)|^2 d\nu_{x,s}^1(\xi, \zeta) dx dy ds, \\
J &= -2 \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) \kappa(\xi - \zeta) g_{x,s}^1(\theta(x-y)) d\xi d\zeta dx dy ds \\
&\quad - \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \partial_\xi \kappa(\xi - \zeta) \theta(x-y) dy_{2,2}(y, s, \zeta) dx d\xi \\
&\quad + \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} \bar{f}_2(y, s, \zeta) \partial_\xi \kappa(\xi - \zeta) \theta(x-y) dy_{1,2}(x, s, \xi) dy d\zeta, \\
K &= \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2(A(\xi) + A(\zeta)) : D_x^2 \theta(x-y)\kappa(\xi - \zeta) d\xi d\zeta dx dy ds \\
&\quad - \int_0^t \int_{(T^N)^2} \theta(x-y)\kappa(\xi - \zeta) d\nu_{x,s}^1(\xi) dx dy_{2,3}(y, s, \zeta) \\
&\quad - \int_0^t \int_{(T^N)^2} \theta(x-y)\kappa(\xi - \zeta) d\nu_{y,s}^2(\zeta) dy dy_{1,3}(x, s, \xi).
\end{align*}
\]

Remark 4.1. Let us fix some notation corresponding kinetic solutions \(u_1\) and \(u_2\). Let \(m_1\) and \(m_2\) be are kinetic measures corresponding \(u_1\) and \(u_2\) respectively, satisfying \(m_1 \geq \eta_{1,2} + \eta_{1,3}\) and \(m_2 \geq \eta_{2,2} + \eta_{2,3}\), \(\mathbb{P}\)-almost surely, where

\[
\begin{align*}
\eta_{1,2}(x, t, \xi) &= \int_{\mathbb{R}^N} |u_1(x + z, t) - \xi| \mathbb{1}_{\text{Conv}(u_1(x,t), u_1(x+z,t))}(\xi) \mu(z) dz, \\
\eta_{1,3}(x, t, \xi) &= |\text{div} \int_0^{u_1(x, t)} \sigma(s) ds|^2 \delta_{u_1(x, t)}(\xi),
\end{align*}
\]
\[ \eta_{2,2}(y, t, \zeta) = \int_{\mathbb{R}^N} |u_2(y + z, t) - \zeta| \mathbb{I}_{\text{Conv}\{u_2(y, t), u_2(y+z, t)\}}(\zeta) \mu(z) dz, \]

\[ \eta_{2,3}(y, t, \zeta) = |\text{div} \int_{0}^{u_2(y, t)} \sigma(s) ds|^{2} \delta_{u_2(y, t)}(\zeta). \]

We can write \( m_1 = m_{1,1} + \eta_{1,2} + \eta_{1,3} \) and \( m_2 = m_{2,1} + \eta_{2,2} + \eta_{2,3} \) for some non-negative measures \( m_{1,1} \) and \( m_{2,1} \) respectively.

**Proof.** We define \( \bar{\beta}_1^2(x, \xi) = \sum_{k \geq 1} |\beta_k(x, \xi)|^2 \), and \( \bar{\beta}_2^2(y, \zeta) = \sum_{k \geq 1} |\beta_k(y, \zeta)|^2 \).

Let \( \phi_1 \in C_c^{\infty}(\mathbb{T}_N \times \mathbb{R}_\xi) \) and \( \phi_2 \in C_c^{\infty}(\mathbb{T}_N \times \mathbb{R}_\zeta) \). For \( f_1 = f_1^+ \) we have

\[ \langle f_1^+(t), \phi_1 \rangle = \langle \mu_1^*, \phi_1 \rangle + M_1(t) \]

with

\[ M_1(t) = \sum_{k \geq 1} \int_{0}^{t} \int_{\mathbb{T}_N} \int_{\mathbb{R}} \beta_k(x, \xi) \phi_1(x, \xi) d\mathcal{V}_{x,s}^{1}(\xi) dx dw_k(s) \]

and

\[ \langle \mu_1^*, \phi_1 \rangle([0, t]) = \langle f_{1,0}, \phi_1 \rangle \delta_0([0, t]) + \int_{0}^{t} \langle f_1, F' \cdot \nabla \phi_1 \rangle ds \]

\[ + \int_{0}^{t} \langle f_1(s), A(\xi) : D_y^2 \phi_1 \rangle ds \]

\[ - \int_{0}^{t} \langle f_1, g_{\lambda}^y[\phi_1] \rangle ds \]

\[ + \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}_N} \int_{\mathbb{R}} \partial_\xi \phi_1 \bar{\beta}_1^2(x, \xi) d\mathcal{V}_{x,s}^{1}(\xi) ds - m_1(\partial_\xi \phi_1)([0, t]). \]

We have \( m_1(\partial_\xi \phi_1)(\{0\}) = 0 \), and value of \( \langle \mu_1^*, \phi_1 \rangle(\{0\}) \) is \( \langle f_{1,0}, \phi_1 \rangle \). Similarly,

\[ \langle \bar{f}_2^+(t), \phi_2 \rangle = \langle \bar{\mu}_2^*, \phi_2 \rangle([0, t]) + \bar{M}_2(t) \]

with

\[ \bar{M}_2(t) = \sum_{k \geq 1} \int_{0}^{t} \int_{\mathbb{T}_N} \int_{\mathbb{R}} \beta_k(y, \zeta) \phi_2(y, \zeta) d\mathcal{V}_{y,s}^{1}(\zeta) ds dw_k(s) \]

and

\[ \langle \bar{\mu}_2^*, \phi_2 \rangle([0, t]) = \langle \bar{f}_{2,0}, \phi_1 \rangle \delta_0([0, t]) + \int_{0}^{t} \langle \bar{f}_2, F' \cdot \nabla \phi_2 \rangle ds \]

\[ + \int_{0}^{t} \langle \bar{f}_2(s), A(\zeta) : D_y^2 \phi_2 \rangle ds - \int_{0}^{t} \langle \bar{f}_2, g_{\lambda}^y[\phi_2] \rangle ds \]

\[ - \frac{1}{2} \int_{0}^{t} \int_{\mathbb{T}_N} \int_{\mathbb{R}} \partial_\zeta \phi_2 \bar{\beta}_2^2(y, \zeta) d\mathcal{V}_{y,s}^{2}(\zeta) ds + m_2(\partial_\zeta \phi_2)([0, t]), \]

where \( \langle \bar{\mu}_2^*, \phi_2 \rangle(\{0\}) = \langle \bar{f}_{2,0}, \phi_2 \rangle \). Let \( \varphi(x, \xi, y, \zeta) = \phi_1(x, \xi) \phi_2(y, \zeta) \). Using Itô formula for \( M_1(t) \bar{M}_2(t) \), and integration by parts for functions of finite variation for \( \langle \mu_1^*, \partial_\xi \phi_1 \rangle([0, t]) \langle \bar{\mu}_2^*, \partial_\zeta \phi_2 \rangle([0, t]) \), (see [36, Chapter 0]) which gives
\[
\langle \mu_1^*, \phi_1([0,t]) \rangle \langle \mu_2^*, \phi_2([0,t]) \rangle \\
= \langle \mu_1^*, \phi_1(\{0\}) \rangle \langle \mu_2^*, \phi_2(\{0\}) \rangle + \int_{(0,t]} \langle \mu_1^*, \phi_1([0,s]) \rangle d\langle \mu_2^*, \phi_2 \rangle(s) \\
+ \int_{(0,t]} \langle \mu_2^*, \phi_2([0,s]) \rangle d\langle \mu_1^*, \phi_1 \rangle(s)
\]
and the following formula
\[
\langle \mu_1^*, \phi_1([0,t]) \rangle \bar{M}_2(t) = \int_0^t \langle \mu_1^*, \phi_1([0,s]) \rangle d\bar{M}_2(s) + \int_0^t \bar{M}_2(s) \langle \mu_1^*, \phi_1 \rangle(ds),
\]
which is easy to obtain since \( \bar{M}_2 \) is continuous and a similar formula for \( \langle \mu_2^*, \partial_\zeta \phi_2 \rangle \bar{M}_1(t) \), we get that
\[
\langle f_1^+(t), \phi_1 \rangle \langle \bar{f}_2^+(t), \phi_2 \rangle = \langle \langle f_1^+(t), \bar{f}_2^+(t), \varphi \rangle \rangle.
\]
It implies that
\[
\mathbb{E}(\langle f_1^+(t), \bar{f}_2^+(t), \varphi \rangle) = \mathbb{E}(\langle f_{1,0}, \bar{f}_{2,0}, \varphi \rangle)
\]
\[
+ \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 f_2 F'(\xi) \cdot \nabla_x + F'(\zeta) \cdot \nabla_y \varphi d\xi d\zeta dx dy ds \\
+ \mathbb{E} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} f_1 f_2 (A(\xi) + A(\zeta)) : D^2[\varphi] d\xi d\zeta dx dy ds \\
- \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 f_2 (g_x^2 + g_y^2)[\varphi] d\xi d\zeta dx dy ds \\
+ \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\xi \varphi f_2(s) \beta_1^2(x, \xi) d\mathcal{V}^1_{x,s}(\xi) d\xi d\zeta dx dy ds \\
- \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\zeta \varphi f_1(s) \beta_2^2(y, \zeta) d\mathcal{V}^2_{y,s}(\zeta) d\xi d\zeta dx ds \\
- \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \beta_{1,2} \varphi d\mathcal{V}_{x,s}(\xi) d\mathcal{V}^2_{y,s}(\zeta) dx dy ds \\
- \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_1^-(s) \partial_\xi \varphi dm_1(x, s, \xi) d\zeta dy \\
+ \mathbb{E} \int_{(0,t]} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\zeta \varphi dm_2(y, s, \zeta) d\xi dx
\]
(4.7)
where \( \beta_{1,2}(x, y; \xi, \zeta) := \sum_{k \geq 1} \beta_k(x, \xi) \beta_k(y, \zeta) \) and \( \langle \cdot, \cdot \rangle \) indicates the duality distribution over \( \mathbb{T}^N_x \times \mathbb{R}_\xi \times \mathbb{T}^N_y \times \mathbb{R}_\zeta \). Equation (4.7) also hold for any test function \( \varphi \in C_c^\infty(\mathbb{T}^N_x \times \mathbb{R}_\xi \times \mathbb{T}^N_y \times \mathbb{R}_\zeta) \) by a density argument. The assumption that \( \varphi \) is compactly supported can be relaxed thanks to the condition at infinity on \( m_i \) and \( \mathcal{V}_i \), \( i = 1, 2 \). Using truncation argument for \( \varphi \), we obtain that equation (4.7) is also true if \( \varphi \in C_c^\infty(\mathbb{T}^N_x \times \mathbb{R}_\xi \times \mathbb{T}^N_y \times \mathbb{R}_\zeta) \) is compactly supported in a neighborhood of the diagonal \( \{ (x, \xi, x, \xi); x \in \mathbb{T}^N, \xi \in \mathbb{R} \} \). We then take \( \varphi = \theta \kappa \) where \( \theta = \theta(x - y), \kappa = \kappa(\xi - \zeta) \). We use the following identities
\[
(\nabla_x + \nabla_y) \varphi = 0, \quad (\partial_\xi + \partial_\zeta) \varphi = 0,
\]
to obtain
\[
\mathbb{E}\left[ \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_1^\pm(x, t, \xi) f_2^\pm(y, t, \zeta) \, d\xi \, d\zeta \, dx \, dy \right] =
\mathbb{E}\left[ \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_{1,0}(x, \xi) f_{2,0}(y, \zeta) \, d\xi \, d\zeta \, dx \, dy + J + K' + \mathcal{R}_\theta + \mathcal{R}_\kappa + K \right],
\]

where
\[
K' = \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_1^-(x, s, \xi) \partial_\zeta \varphi \, dm_{2,1}(y, s, \zeta) - \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_2^+(y, s, \zeta) \partial_\xi \varphi \, dm_{1,1}(x, s, \xi)
\]
\[
= - \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_1^-(x, s, \xi) \partial_\zeta \varphi \, dm_{2,1}(y, s, \zeta) + \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_2(y, s, \zeta) \partial_\xi \varphi \, dm_{1,1}(x, s, \xi)
\]
\[
= - \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_1^-(x, s, \xi) \partial_\zeta \varphi \, dm_{2,1}(y, s, \zeta) - \int_{[0, t]} \int_{T^N} \int_{\mathbb{R}} f_2(y, s, \zeta) \partial_\xi \varphi \, dm_{1,1}(x, s, \xi)
\]
\[
\leq 0.
\]

Consequently we have the required estimate: for all \( t \in [0, T] \)
\[
\mathbb{E}\left[ \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_1^\pm(x, t, \xi) f_2^\pm(y, t, \zeta) \, d\xi \, d\zeta \, dx \, dy \right]
\leq \mathbb{E}\left[ \int_{(T^N)} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_{1,0}(x, \xi) f_{2,0}(y, \zeta) \, d\xi \, d\zeta \, dx \, dy + \mathcal{R}_\theta + \mathcal{R}_\kappa + J + K \right].
\]

\[\square\]

**Remark 4.2.** One can easily notice that, if \( f_1^\pm = f_2^\pm \), then inequality (4.6) holds pathwise, that is, \( \mathbb{P} \)-almost surely, for all \( t \in [0, T] \),
\[
\int_{(T^N)^2} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_1^\pm(x, t, \xi) f_2^\pm(y, t, \zeta) \, d\xi \, d\zeta \, dx \, dy
\leq \int_{(T^N)} \int_{\mathbb{R}^2} \theta(x-y)\kappa(\xi - \zeta) f_{1,0}(x, \xi) f_{2,0}(y, \zeta) \, d\xi \, d\zeta \, dx \, dy + \mathcal{R}_\theta + \mathcal{R}_\kappa + J + K,
\]

where \( \mathcal{R}_\theta, \mathcal{R}_\kappa, J, \) and \( K \) are defined as above in Proposition 4.3. For proof of this, there is no need to take expectation after use of Itô formula and integration by part formula in the proof of Proposition 4.3, since after the
steps of approximation of test functions $\theta(x-y)$, and $\kappa(\xi-\zeta)$, the contributed martingale terms will cancel.

**Theorem 4.4. (Contraction principle)** If $(u_1(t))_{t \in [0,T]}, (u_2(t))_{t \in [0,T]}$ are kinetic solutions to (1.1) with initial data $u_{1,0}$ and $u_{2,0}$ respectively, then for all $t \in [0,T]$,

$$E\|u_1(t) - u_2(t)\|_{L^1(T^N)} \leq E\|u_{1,0} - u_{2,0}\|_{L^1(T^N)}.$$  

Moreover, let $(u(t))_{t \in [0,T]}$ be a solution to (1.1), then there exists a $L^1(T^N)$-valued process $(u^{-}(t))_{t \in [0,T]}$ such that $\mathbb{P}$-almost surely, for all $t \in [0,T]$, $f^{-}(t) = \mathbb{1}_{u^{-}(t) > \xi}$.

**Proof.** Let $(\theta_\epsilon), (\kappa_\delta)$ be approximations to the identity on $T^N$ and $\mathbb{R}$, respectively, that is, $\theta \in C^\infty(T^N)$ and $\kappa \in C^\infty_c(\mathbb{R})$ be symmetric non-negative functions such that $\int_{T^N} \theta(x)dx = 1$, $\int_{\mathbb{R}} \kappa(\xi)d\xi = 1$ and $\text{supp} \kappa \subset (-1,1)$. We define $\theta_\epsilon = \frac{1}{\epsilon} \theta(\frac{x}{\epsilon})$, and $\kappa_\delta(\xi) = \frac{1}{\delta} \kappa(\frac{\xi}{\delta})$. Then we follow proof of [20, Theorem 3.2] to conclude

$$E\int_{T^N} \int_{\mathbb{R}} f^\pm_1(x,t,\xi) \bar{f}^\pm_2(x,t,\xi) d\xi dx = E\int_{T^N} ^2 \theta_\epsilon(x-y)\kappa_\delta(\xi-\zeta) f_1(x,t,\xi) \bar{f}_2(y,t,\zeta) d\xi d\zeta dy + \eta_\epsilon(\epsilon,\delta)$$

where $\lim_{\epsilon, \delta \to 0} \eta_\epsilon(\epsilon,\delta) = 0$ (see [20, Estimates (3.14)–(3.16)]). With regard to Proposition 4.3, we need to find suitable bounds for terms I, J, K. We define $\Phi_\delta(w) = \int_{-\infty}^{w} \kappa_\delta(\xi)d\xi$ and note that $\Phi_\delta$ is non decreasing function. We shall estimate each terms in the following several steps.

**Step 1** It follows from proof of [18, Theorem 15] that $\mathbb{P}$-almost surely,

$$|\mathcal{R}_\theta| \leq C'(\omega)\epsilon^{-1} \delta.$$  \n
where $C'(\omega) = \sup_{t \in [0,T]} \|u_1(w,t)\|_{L^{q_\ast}(T^N)}^q + \|u_2(w,t)\|_{L^{q_\ast}(T^N)}^q$.

**Step 2** In order to estimate the term $J$, we observe that

$$J = -2 \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x,s,\xi) \bar{f}_2(y,s,\zeta) \kappa_\delta(\xi-\zeta) g^x_\epsilon(\theta_\epsilon(x-y)) d\xi d\zeta dx dy ds$$

$$- \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} \partial_\xi \kappa_\delta(\xi-\zeta) \theta_\epsilon(x-y) \bar{f}_2(y,s,\zeta) d\eta_{1,1}(x,s,\xi) dy d\zeta$$

$$+ \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} \partial_\zeta \kappa_\delta(\xi-\zeta) \theta_\epsilon(x-y) f_1(x,s,\xi) d\eta_{2,1}(y,s,\zeta) dx d\xi$$

$$:= J_1 + J_2 + J_3,$$

where

$$J_1 = -2 \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x,s,\xi) \bar{f}_2(y,s,\zeta) \kappa(\xi-\zeta) g^x_\epsilon(\theta(x-y)) d\xi d\zeta dx dy ds,$$

$$J_2 = - \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} \partial_\xi \kappa_\delta(\xi-\zeta) \theta_\epsilon(x-y) \bar{f}_2(y,s,\zeta) d\eta_{1,1}(x,s,\xi) dy d\zeta,$$

$$J_3 = \int_{0}^{t} \int_{(T^N)^2} \int_{\mathbb{R}^2} \partial_\zeta \kappa_\delta(\xi-\zeta) \theta_\epsilon(x-y) f_1(x,s,\xi) d\eta_{2,1}(y,s,\zeta) dx d\xi.$$
We have the following identity in the sense of weak derivative
\[
\partial_\xi \left[ (\tau_z u_1(x, s) - \xi)(s) \right]_{\text{Conv} \{ u_1(x, s), \tau_z u_1(x, s) \}} = \left( \tau_z u_1(x, s) - u_1(x, s) \right) \delta_{\xi = u_1(x, s)} + \text{sign} \left( u_1(x, s) - \tau_z u_1(x, s) \right) \mathbb{1}_{\text{Conv} \{ u_1(x, s), \tau_z u_1(x, s) \}}(\xi).
\]
(4.11)

Thus, by using the above identity we have \( \mathbb{P} \)-a.s., for all \( t \in [0, T] \),
\[
J_2 = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^{N+1}} \left\{ \left( \tau_z u_1(x, s) - u_1(x, s) \right) \kappa_\delta(u_1(x, s) - \xi) \right. \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1} U \int_{\mathbb{R}^N}} \left\{ \int_{\mathbb{R}^N} \kappa_\delta(\xi - \zeta) d\xi \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) \mu(z) dz d\zeta dx dy ds \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{-\infty}^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) d\zeta dx dy ds \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{-\infty}^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{-\infty}^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{-\infty}^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds \\
+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{-\infty}^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds,
\]
where interchange of integration is justified by the fact that, \( \mathbb{P} \)-almost surely,
\[
\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^{N+2}} |\partial_\xi \kappa_\delta(\xi - \zeta) \theta_\varepsilon(x - y) f_2(y, s, \xi) \eta_{1,1}| \mu(z) dz d\zeta dx dy ds \\
\leq C \|\eta_{1,1}\|_{L^1(\mathbb{T}^N \times \mathbb{R} \times [0, T])}.
\]

Similarly, for other term \( J_3 \), we can conclude that
\[
J_3 = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^{N+2}} \left\{ \left( \tau_z u_1(x, s) - \xi)(s) \right) \right. \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_{\mathbb{R}^N} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) d\zeta dx dy ds \\
- \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_0^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds \\
+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^{N+1}} \left\{ \int_0^{u_1(x, s)} \kappa_\delta(\xi - \zeta) d\zeta \right\} \theta_\varepsilon(x - y) f_2(y, s, \xi) dz d\zeta dx dy ds.
\]
We conclude that
\[ J_2 + J_3 = - \int_0^t \int_{(\mathbb{T}^N)^2} \left[ g_x^\lambda(u_1(x, s)) - g_y(u_2(x, y)) \right] \Phi_\delta(u_1(x, s) - u_2(y, s)) dxdyds \]
\[ - u_2(y, s)) \theta_\epsilon(x - y) dxdyds \]
\[ + 2 \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \kappa_\delta(\xi - \zeta) g_x^\lambda(\theta_\epsilon)(x - y) \bar{f}_2(y, s, \zeta) dxdyd\zeta d\xi ds \]
\[ := I_1 - J_1, \]
where
\[ I_1 = - \int_0^t \int_{(\mathbb{T}^N)^2} \left[ g_x^\lambda(u_1(x, s)) - g_y(u_2(y, s)) \right] \Phi_\delta(u_1(x, s) - u_2(y, s)) dxdyds \]
\[ - u_2(y, s)) \theta_\epsilon(x - y) dxdyds \]
\[ = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^N} \Phi_\delta(u_1(x, s) - u_2(y, s)) \left[ (\tau_2(u_1(x, s) - u_2(y, s)) \theta_\epsilon(x - y) \mu(z) dxdy dxdyds \right. \]
\[ - u_2(y, s)) - (u_1(x, s) - u_2(y, s)) \theta_\epsilon(x - y) \mu(z) dxdy dxdyds \]
\[ = - \frac{1}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^N} \left[ \tau_2(\Phi_\delta(u_1(x, s) - u_2(y, s)) - \Phi_\delta(u_1(x, s) - u_2(y, s))) \right] dxdy dxdy dxdy \]
\[ \leq 0, \]
where in the second line from the last we used the change of variable \( z \to -z \) and in the last line, we used the fact that \( \Phi_\delta \) is a non-decreasing function which satisfies \( (\Phi_\delta(\xi) - \Phi_\delta(\zeta))(\xi - \zeta) \geq 0 \) for all \( \xi, \zeta \in \mathbb{R} \). Finally it shows clearly, \( \mathbb{P} \)-almost surely,
\[ J \leq 0. \tag{4.12} \]

**Step 3** In order to estimate the term \( K \), we closely follow proof of [19, Theorem 3.3]. We observe that
\[ K = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2(\sigma(\xi) - \sigma(\zeta))^2 \cdot D_2^2 \theta_\epsilon(x - y) \phi_\delta(\xi - \zeta) dxdy dxdy dxdy ds \]
\[ + 2\mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2(\xi)(\xi) \cdot D_2^2 \theta_\epsilon(x - y) \phi_\delta(\xi - \zeta) dxdy dxdy dxdy ds \]
\[ - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) \phi_\delta(\xi - \zeta) d\mathcal{N}_{\bar{x}, s}^1 dxdy d\eta_{2,3}(y, x, \zeta) \]
\[ - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) \phi(\xi - \zeta) d\mathcal{N}_{\bar{y}, s}(\xi) dxdy d\eta_{1,3}(x, s, \xi) \]
\[ = K_1 + K_2 + K_3 + K_4 \]

Since \( \sigma \) is locally \( \gamma \)-Hölder continuous due to (2.3), it holds
\[ |K_1| \leq C t \delta^{2\gamma} \epsilon^{-2} \]
From the definition of the parabolic dissipative measure in Definition 3.2, we have $K_2 + K_3 + K_4 \leq 0$ (for details see [19, Theorem 3.3]). It concludes that $\mathbb{P}$-almost surely,

$$|K| \leq C(\omega)t\delta^{2\gamma}e^{-2} \quad (4.13)$$

**Step 4** In this step we estimate Itô correction terms as follows:

$$\mathcal{R}_\kappa \leq D_1 \int_0^t \int_{(T^N)^2} \theta_\epsilon(x-y)|x-y|^2 \int_{\mathbb{R}^2} \kappa_\delta(\xi - \zeta)d\mathcal{V}^1_{x,s}d\mathcal{V}^2_{y,s}(\zeta)dxdyds$$

$$+ D_1 \int_0^t \int_{(T^N)^2} \theta_\epsilon(x-y)$$

$$\times \int_{\mathbb{R}^2} \kappa_\delta(\xi - \zeta)|\xi - \zeta|h(|\xi - \zeta|)d\mathcal{V}^1_{x,s}(\xi)d\mathcal{V}^2_{y,s}(\zeta)dxdyds$$

$$\leq D_1 t\delta^{-1}e^2 + D_1 th(\delta). \quad (4.14)$$

**Step 5** As a consequence of previous steps, we deduce for all $t \in [0, T]$,

$$\mathbb{E} \int_{T^N} \int_{\mathbb{R}} f^\pm_1(x,t,\xi)f^\pm_2(x,t,\xi)d\xi dx$$

$$\leq \mathbb{E} \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta_\epsilon(x-y)\phi_\delta(\xi - \zeta)f_{1,0}\bar{f}_{2,0}(y,\zeta)d\xi d\zeta dxdy$$

$$+ C_T(\delta\epsilon^{-1} + \delta^{-1}e^2 + h(\delta) + \delta^{2\gamma}e^{-2}) + \eta_t(\epsilon, \delta).$$

Taking $\delta = \epsilon^a$ with $a \in (\frac{1}{\gamma}, 2)$ and letting $\epsilon \to 0$ yields, for all $t \in [0, T]$

$$\mathbb{E} \int_{T^N} \int_{\mathbb{R}} f_1(x,t,\xi)\bar{f}_2(x,t,\xi)d\xi dx \leq \mathbb{E} \int_{T^N} \int_{\mathbb{R}} f_{1,0}(x,\xi)\bar{f}_{2,0}(x,\xi)d\xi dx. \quad (4.15)$$

Since we have

$$\int_{\mathbb{R}} \mathbb{I}_{u_1(t) > \xi} \mathbb{I}_{u_2(t) > \xi} d\xi = (u_1(t) - u_2(t))^+. \quad (4.16)$$

It implies that contraction principle (4.9) holds.

For second remaining part, making use of inequality (4.8) and pathwise estimates (4.10)–(4.14), we can similarly conclude that $\mathbb{P}$-almost surely, for all $t \in [0, T]$,

$$\int_{T^N} \int_{\mathbb{R}} f^-(x,t,\xi)\bar{f}^-(x,t,\xi)d\xi dx \leq \int_{T^N} \int_{\mathbb{R}} f_{0}(x,\xi)\bar{f}_0(\xi,x)d\xi dx. \quad (4.16)$$

We have the identity $f_0(x,\xi)\bar{f}_0(\xi,x) = 0$ and therefore, $\mathbb{P}$-almost surely, for all $t \in [0, T], f^-(x,t,\xi)(1 - f^-(x,t,\xi)) = 0$ a.e. $(x, \xi)$. The fact that $f^-$ is a kinetic function and then Fubini’s theorem imply that, $\mathbb{P}$-almost surely, for all $t \in [0, T], there exists a set $S \subset T^N$ of full measure such that, for $x \in S, f^-(x,t,\xi) \in \{0, 1\}$ for a.e. $\xi \in \mathbb{R}$. Therefore, for all $t \in [0, T]$, there exists $u^-(t) : \Omega \to L^1(T^N)$ such that $\mathbb{P}$-almost surely, for all $t \in [0, T] f^-(t) = \mathbb{I}_{u^-(t) > \xi}$. \hfill \Box

**Continuity in time** As a consequence, we get the continuity of trajectories in $L^p(T^N)$ whose proof is similar to the proof of [20, Corollary 3.3].
Corollary 4.5. Let \( u_0 \in L^p(\Omega; L^p(\mathbb{T}^N)) \) for all \( p \in [1, +\infty) \). Then, the solution \( (u(t))_{t \in [0,T]} \) to (1.1) with initial data \( u_0 \) has \( \mathbb{P} \)-almost surely continuous trajectories in \( L^p(\mathbb{T}^N) \).

Remark 4.3. (No atomic point) Since for all \( t \in [0,T] \), \( \mathbb{P} \)-almost surely, \( f(t) = f^-(t) \) (cf. [20, Proposition 2.11]), then identity (4.4) implies that set of atomic point, \( B_{\text{at}} = \{ t \in [0,T]; \mathbb{P}(\pi_m(t) > 0 > 0) \} \), is empty.

5. Proof of existence part of Theorem 3.1

In this section, first we prove the existence part of Theorem 3.1 for the initial condition \( u_0 \in L^p(\Omega; C^5(\mathbb{T}^N)) \). Here we apply the vanishing viscosity method, while using also some appropriately chosen approximations \( F^\tau \) of \( F \). These equations have weak solutions and consequently passage to the limit gives the existence of a kinetic solution (1.1). Here, the limit argument is quite technical and has to be done in many steps. We consider a truncation \( (S^\tau_{\eta}) \) on \( \mathbb{R} \) and approximations \( (\kappa^\tau) \) to the identity on \( \mathbb{R} \). The regularization of \( F \) is defined in the following way

\[
F_j^\tau(\zeta) = ((F_j * \kappa^\tau)S^\tau)(\zeta) \quad j = 1, \ldots, N,
\]

Consequently, we set \( F^\tau = (F_1^\tau, \ldots, F_N^\tau) \). It is clear that approximations \( F^\tau \) is of class \( C^\infty \) with the compact support therefore Lipschitz continuous. Also the polynomial growth of \( F \) remains valid for \( F^\tau \) and holds uniformly in \( \tau \).

\[
du^\tau(x, t) + \text{div}(F^\tau(u^\tau(x, t)))dt + g^\tau_x[u^\tau(x, t)]dt = \text{div}(A(u^\tau) \nabla u^\tau) + \tau \Delta u^\tau + \Phi(x, u^\tau(x, t))dB(t) \quad x \in \mathbb{T}^N, \quad t \in (0, T).
\]  

(5.1) 

\( u^\tau(0) = u_0 \).

There exists a unique weak solution \( u^\tau \) to (5.1) (see “Appendix B”) such that

\( u^\tau \in L^2(\Omega; C([0, T]; L^2(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0, T; H^1(\mathbb{T}^N))) \).

Convergence of viscous solutions: The convergence of viscous solutions is based on the computations of Proposition 4.3 and Theorem 4.4.

Step 1 We can easily conclude that for all \( t \in [0, T] \),

\[
\mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\varepsilon(x - y)f^\tau(x, t, \zeta)\tilde{f}^\tau(y, t, \zeta)d\zeta dxdy
\]

\[
\leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R}^2)^2} \theta_\varepsilon(x - y)\kappa_\delta(\zeta - \zeta)f^\tau(x, t, \zeta)\tilde{f}^\tau(y, t, \zeta)d\zeta d\zeta dxdy + \delta
\]

\[
\leq \mathbb{E} \left[ \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R}^2)^2} \theta_\varepsilon(x - y)\kappa_\delta(\zeta - \zeta)f_0(x, \zeta)\tilde{f}_0(y, \zeta)d\zeta d\zeta dxdy + \delta
\]

\[
+ \mathcal{R}_\delta + \mathcal{R}_\kappa + J^\tau + K^\tau + J^\tau \right]
\]

\[
\leq \mathbb{E} \left[ \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R}^2)^2} \theta_\varepsilon(x - y)f_0(x, \zeta)\tilde{f}_0(y, \zeta)d\zeta dxdy + 2\delta + \mathcal{R}_\delta + \mathcal{R}_\kappa + J^\tau + K^\tau + J^\tau \right],
\]
where \( R_N^{\tau}, \mathcal{R}_N^{\tau}, J^{\tau}, K^{\tau} \) are introduced similarly to Proposition 4.3, \( J^{\tau} \) is regarding to the second order term \( \tau \Delta u^{\tau} \):

\[
\mathbb{E}[J^{\tau}] = 2\tau \mathbb{E} \int_0^T \int_{(T_N)^2} \int_{\mathbb{R}^2} f^{\tau} \tilde{f}^{\tau} \Delta_t \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) d\zeta d\zeta dxdyds \\
- \mathbb{E} \int_0^T \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) dV_{x,s}^\tau(\zeta) dx dy ds \\
- \mathbb{E} \int_0^T \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) dV_{y,s}^\tau(\zeta) dy ds \\
= -\tau \mathbb{E} \int_0^T \int_{(T_N)^2} \theta^\varepsilon(x-y) \kappa_\delta(u^{\tau}(x)-u^{\tau}(y)) |\nabla_x u^{\tau} - \nabla_y u^{\tau}|^2 dxdy \leq 0,
\]

and for the error term \( \delta \), we refer to \([19, \text{Estimate} \ 6.4]\). By the proof of Theorem 4.4, we can conclude that for all \( t \in [0,T] \)

\[
\mathbb{E} \int_{(T_N)^2} \theta^\varepsilon(x-y)|u^{\tau}(x,t) - u^{\tau}(y,t)|dxdy \\
\leq \mathbb{E} \int_{(T_N)^2} \theta^\varepsilon(x-y)|u_0(x) - u_0(y)|dxdy \\
+ C_T(\delta + \delta \varepsilon^{-1} + \delta^2 \varepsilon^{-2} + \varepsilon^2 \delta^{-1} + h(\delta)). \tag{5.2}
\]

**Step 2** By similar techniques as in the proof of Proposition 4.3, we obtain for any two viscous solution \( u^{\tau}, u^\sigma \), for all \( t \in [0,T] \),

\[
\mathbb{E} \int_T (u^{\tau}(t) - u^{\sigma}(t))^+ dx = \mathbb{E} \int_{T_N} \int_{\mathbb{R}^2} f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(x,t,\zeta) d\zeta dx \\
= \mathbb{E} \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(y,t,\zeta) d\zeta d\zeta dxdy + \eta_1(\tau, \sigma, \varepsilon, \delta). \tag{5.3}
\]

We want to show that the error term \( \eta_t(\tau, \sigma, \varepsilon, \delta) \) is independent of \( \tau, \sigma \) as follows:

\[
\eta_t(\tau, \sigma, \varepsilon, \delta) = \mathbb{E} \int_{T_N} \int_{\mathbb{R}^2} f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(x,t,\zeta) d\zeta dx \\
- \mathbb{E} \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(y,t,\zeta) d\zeta d\zeta dxdy \\
= \left( \mathbb{E} \int_{T_N} \int_{\mathbb{R}^2} f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(x,t,\zeta) d\zeta dx \\
- \mathbb{E} \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(y,t,\zeta) d\zeta dxdy \right) \\
+ \left( \mathbb{E} \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(y,t,\zeta) d\zeta dxdy \\
- \mathbb{E} \int_{(T_N)^2} \int_{\mathbb{R}^2} \theta^\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) f^{\tau}(x,t,\zeta) \tilde{f}^{\sigma}(y,t,\zeta) d\zeta d\zeta dxdy \right) \\
= H_1 + H_2,
\]
where

\[ |H_1| = \left| \mathbb{E} \int_{(\mathbb{T}^N)^2} \theta_\varepsilon(x-y) \int_{\mathbb{R}} \mathbb{1}_u^\ast(x) > \zeta \left[ \mathbb{1}_u^\ast(x) \leq \zeta - \mathbb{1}_w^\ast(y) \leq \zeta \right] \, d\zeta \, dx \, dy \right| \]

\[ = \left| \mathbb{E} \int_{(\mathbb{T}^N)^2} \theta_\varepsilon(x-y) (u^\ast(y) - u^\ast(x)) \, dx \, dy \right| \]

\[ \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \theta_\varepsilon(x-y) |u_0(x) - u_0(y)| \, dx \, dy + C_T (\delta + \delta \varepsilon^{-1} + \delta^2 \gamma \varepsilon^{-2} + \varepsilon^2 \delta^{-1} + h(\delta)) \]

due to (5.2) and |H_2| \leq \delta, see [19, Estimate 6.4]. Come back to (5.3) and using the same computations as in the proof of Theorem 4.4, we obtain for all \( t \in [0, T] \),

\[ \mathbb{E} \int_{\mathbb{T}^N} \left( u^\tau(t) - u^\sigma(t) \right)^+ \, dx \leq \left[ \mathbb{E} \int_{(\mathbb{T}^N)^2} \theta_\varepsilon(x-y) |u_0(x) - u_0(y)| \, dx \, dy \right. \]

\[ + C_T (\delta + \delta \varepsilon^{-1} + \delta^2 \gamma \varepsilon^{-2} + \varepsilon^2 \delta^{-1} + h(\delta)) + 2\delta \mathcal{R}_\theta + \mathcal{R}_\kappa + J + K + J^\# \bigg] . \]

The terms \( \mathcal{R}_\theta, \mathcal{R}_\kappa, J \) and \( K \) are defined similarly to Proposition 4.3. The term \( J^\# \) is defined as

\[ \mathbb{E}[J^\#] = (\tau + \sigma) \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f^\tau f^\sigma \Delta_x \theta_\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) d\zeta d\zeta dxdyds - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) dV^\tau_{x,s} (\zeta) dxdy \eta^2(y,s,\zeta) - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\varepsilon(x-y) \kappa_\delta(\zeta - \zeta) dV^\sigma_{y,s} (\zeta) dyd\eta^2(s,x,\zeta) . \]

We follow [19, Section 6] to estimate \( J^\# \) as follows:

\[ \|\mathbb{E}[J^\#]\| \leq C \left( \sqrt{\tau} - \sqrt{\sigma} \right)^2 \varepsilon^{-2} , \]

Consequently, we see that for all \( t \in [0, T] \),

\[ \mathbb{E} \int_{\mathbb{T}^N} \left( u^\tau(t) - u^\sigma(t) \right)^+ \, dx \, dt \leq \left( \mathbb{E} \int_{(\mathbb{T}^N)^2} \theta_\varepsilon(x-y) |u_0(x) - u_0(y)| \, dx \, dy \right. \]

\[ + C_T (\delta + \delta \varepsilon^{-1} + \delta^2 \gamma \varepsilon^{-2} + \varepsilon^2 \delta^{-1} + h(\delta)) + \left( \sqrt{\tau_n} - \sqrt{\tau_k} \right)^2 \varepsilon^{-2} \bigg) . \] (5.4)

Choose \( \delta = \varepsilon^\beta \) with \( \beta \in \left( \frac{1}{\gamma}, 2 \right) \). then we conclude, for all \( t \in [0, T] \),

\[ \mathbb{E}\|u^{\tau_n}(t) - u^{\tau_k}(t)\|_{L^1(\mathbb{T}^N)} \to 0 \quad \text{as} \quad \tau_n, \tau_k \to 0. \]

It shows that sequence of viscous solutions \( u^{\tau_n} \) is Cauchy sequence in \( L^p_{\mathcal{P}_T}(\Omega \times [0, T] \times \mathbb{T}^N) \). Therefore \( u^{\tau_n} \) converges to \( u \in L^p_{\mathcal{P}_T}(\Omega \times [0, T] \times \mathbb{T}^N) \) as \( \tau_n \to 0. \)
5.1. Approximate kinetic solutions

Here we apply the technique of “Appendix A” to derive the kinetic formulation to (5.1) that satisfied by $f^{τ_n}(t) = 1_{u^{τ_n}(t)} > \zeta$ in the sense of $D'(\mathbb{T}^N \times \mathbb{R})$. It reads as follows:

$$
df^{τ_n}(t) + F^{τ_n} \cdot \nabla f^{τ_n}(t) dt - A : D^2 f^{τ_n}(t) dt - \tau_n \Delta f^{τ_n}(t) dt + g_x^{λ}(f^{τ_n}(t)) dt = \delta_{u^{τ_n}(t)=\zeta} \Phi dW + \partial_\zeta (m^{τ_n} - \frac{1}{2} \beta^2 \delta_{u^{τ_n}(t)=\zeta}) dt,
$$

where

$$
m^{τ_n}(x, t, \zeta) \geq m_1^{τ_n}(x, t, \zeta) + \eta_1^{τ_n}(x, t, \zeta) + \eta_2^{τ_n}(x, t, \zeta),
$$

$$
dm_1^{τ_n}(x, t, \zeta) = \tau_n |\nabla u^{τ_n}|^2 \delta_{u^{τ_n} = \zeta}, dxdtdt
$$

$$
d\eta_1^{τ_n}(x, t, \zeta) = \int_{\mathbb{R}^N} |u^{τ_n}(x + z) - \zeta| \mathbb{1}_{\text{Conv}\{u^{τ_n}(x), u^{τ_n}(x+z)\}}(\zeta) \mu(z) dz d\zeta dx dt
$$

$$
d\eta_2^{τ_n}(x, t, \zeta) = |\text{div} \int_0^{τ_n} \sigma(\zeta)d\zeta|^2 \delta_{u^{τ_n}(x, t)}(\zeta) dx dt.
$$

It implies that for all $φ \in C^2_c(\mathbb{T}^N \times \mathbb{R})$, $\mathbb{P}$-almost surely, for all $t \in [0, T]$, 

$$
\langle f^{τ_n}(t), φ \rangle = \langle f_0^φ, φ \rangle + \int_0^t \langle f^{τ_n}(s), F^{τ_n}(\zeta) \cdot \nabla_x ϕ \rangle ds + \int_0^t \langle f(s), A : D^2 ϕ \rangle ds
$$

$$
- \int_0^t \langle f^{τ_n}(s), g_x^{λ}(ϕ) \rangle ds
$$

$$
+ \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \beta_k(x, ζ) ϕ(x, ζ) dV_x s(ζ) dx dw_k(s) + \tau_n \int_0^t \langle f^{τ_n}(s), \Delta ϕ \rangle ds.
$$

$$
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \beta^2(x, ζ) \partial_ζ ϕ(x, ζ) dV_x s(ζ) dx ds - m^{τ_n}(\partial_ζ ϕ)([0, t]).
$$

(5.5)

**Proposition 5.1.** The viscous solution $u^{τ_n}$ to (5.1) satisfies the following estimates: for all $p \in [2, \infty)$,

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u^{τ_n}(t)\|_{L^p(\mathbb{T}^N)}^p \leq C(1 + \mathbb{E}\|u_0\|_{L^p(\mathbb{T}^N)}^p),
$$

(5.6)

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u^{τ_n}(t)\|_{L^p(\mathbb{T}^N)}^p + \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |ζ|^{p-2} dm^{τ_n}(x, s, ζ) \leq C(1 + \mathbb{E}\|u_0\|_{L^p(\mathbb{T}^N)}^p),
$$

(5.7)

where the constant $C$ does not depend on $τ_n$.

**Proof.** In this proof, we use the generalized Itô formula [19, Appendix A]. Since we can not apply the generalized Itô formula directly to $ϕ(ζ) = |ζ|^p$, $p \in [2, \infty)$, and $κ(x) = 1$, we introduce functions $φ_l \in C^2(\mathbb{R})$ that approximate $ϕ$ and have quadratic growth at infinity as required by [19, Appendix A]. We define $φ_l$ as follows:

$$
φ_l(ζ) = \begin{cases} 
|ζ|^p, & |ζ| \leq l, \\
p^{(p-1)}(p-2)|ζ|^2 - p(p-2)|ζ| + \frac{(p-1)^2(p-2)}{2} l^2, & |ζ| > l.
\end{cases}
$$
We can easily see that
\[
\begin{align*}
|\zeta \phi'_l(\zeta)| & \leq \phi_l(\zeta), \\
|\phi'_l(\zeta)| & \leq p(1 + \phi_l(\zeta)), \\
|\phi''_l(\zeta)| & \leq |\zeta| \phi''(\zeta), \\
\zeta^2 \phi''_l(\zeta) & \leq p(p - 1) \phi_l(\zeta), \\
\phi''_l(\zeta) & \leq p(p - 1)(1 + \phi_l(\zeta))
\end{align*}
\]
holds true for all $\zeta \in \mathbb{R}$, $l \in \mathbb{N}$, $p \in [2, \infty)$, then by generalized Itô formula [19, Appendix A] we have $\mathbb{P}$-almost surely, for all $t \in [0, T]$

\[
\int_{T N} \phi_l(u^{T_n}(t)) dx = \int_{T N} \phi_l(u_0) dx - \int_0^t \left\langle \phi'_l(u^{T_n}), \text{div}(F^{T_n}(u^{T_n})) \right\rangle ds \\
- \int_0^t \left\langle \phi'_l(u^{T_n}), g^l_x(u^{T_n}) \right\rangle ds \\
+ \int_0^t \left\langle \phi'_l(u^{T_n}), \text{div}(A(u^{T_n}) \nabla u^{T_n}) \right\rangle ds + \int_0^t \left\langle \phi'_l(u^{T_n}), \tau_n \Delta u^{T_n} \right\rangle ds \\
+ \sum_{k \geq 1} \int_0^t \left\langle \phi'_l(u^{T_n}) \beta_k(u^{T_n}, x) \right\rangle dw_k(s) + \frac{1}{2} \int_0^t \left\langle \phi''_l(u^{T_n}), \beta^2(u^{T_n}) \right\rangle ds.
\]

Let us define as $H^n(\zeta) = \int_0^\zeta \phi'_l(\zeta) F^{T_n}(\zeta) d\zeta$, then it is easy to conclude that the second term on the right-hand side vanishes due to the boundary conditions. The following terms are non positive:

\[
\int_0^t \left\langle \phi'_l(u^{T_n}), \text{div}(A(u^{T_n}) \nabla u^{T_n}) \right\rangle ds = - \int_0^t \int_{T N} \phi''_l(u^{T_n}) |\sigma(u^{T_n}) \nabla u^{T_n}|^2 dx ds, \\
- \int_0^t \left\langle \phi'_l(u^{T_n}), g^l_x(u^{T_n}) \right\rangle ds \\
= - \lim_{\delta \rightarrow 0} \int_0^t \left\langle \phi'_l(u^{T_n, \delta}), g^l_x(u^{T_n, \delta}) \right\rangle ds \\
= - \lim_{\delta \rightarrow 0} \int_0^t \int_{T N} \int_{\mathbb{R}} \phi''_l(\zeta) \eta_{1, \delta}(x, \zeta, s), \\
\int_0^t \left\langle \phi'_l(u^{T_n}), \tau_n \Delta u^{T_n} \right\rangle ds = - \int_0^t \int_{T N} \phi''_l(u^{T_n}) \tau_n |\nabla u^{T_n}|^2 dx ds.
\]

For above second equality we refer to discussion in “Appendix A”. Here $u^{T_n, \delta}$ is pathwise mollification of $u^{T_n}$ in space variable such that $u^{T_n, \delta} \rightarrow u^{T_n}$ in $L^2(\Omega; L^2([0, T]; H^1(\mathbb{T}^N)))$ with $u^{T_n, \delta} \in L^2(\Omega; L^2([0, T]; C^\infty(\mathbb{T}^N)))$ and $\eta_{1, \delta}$ is measure associated to $u^{T_n, \delta}$. After this, we can follow proof of [19, Proposition 4.3] to conclude our result. □

**Remark 5.1.** (Uniform bound in $H^\lambda$) We have uniform $H^\lambda$-bound of viscous solutions as consequence of proof of above proposition. In particular for $p=2$, we get $\mathbb{P}$-almost surely, for all $t \in [0, T]$
\[
\frac{1}{2} \int_{T^N} |u^{\tau_n}(t)|^2 \, dx = \frac{1}{2} \int_{T^N} |u_0|^2 \, dx - \int_0^t \langle u^{\tau_n}, \text{div}(F^{\tau_n}(u^{\tau_n})) \rangle \, ds \\
+ \int_0^t \langle u^{\tau_n}, \text{div}(A(u^{\tau_n})\nabla u^{\tau_n}) \rangle \, ds \\
- \int_0^t \langle u^{\tau_n}, g^\lambda_x(u^{\tau_n}) \rangle \, ds + \int_0^t \langle u^{\tau_n}, \tau \Delta u^{\tau_n} \rangle \, ds \\
+ \sum_{k \geq 1} \int_0^t \langle u^{\tau_n}, \beta_k(u^{\tau_n}, x) \rangle \, dw_k(s) + \int_0^t \langle \beta^2(x, u^{\tau_n}), 1 \rangle \, ds.
\]

A simple consequence of BDG inequality and Gronwall’s inequality implies that, for all \( t \in [0, T] \)
\[
\mathbb{E} \int_{T^N} |u^{\tau_n}(t)|^2 \, dx + \mathbb{E} \int_0^T \int_{T^N} |g^\lambda_x(u(x))|^2 \, dx \leq C \mathbb{E} \int_{T^N} |u_0(x)|^2 \, dx.
\]
It shows that \( u^{\tau_n} \) uniformly bounded in \( L^2(\Omega; L^2([0, T]; H^\lambda(\mathbb{T}^N))) \).

5.2. Convergence of approximate kinetic functions

In this subsection, we shall use the following notions: we say that a sequence \((\mathcal{V}^{\tau_n})\) of Young measures converges to \( \mathcal{V} \) in \( \mathcal{Y}^1 \) if (2.18) is satisfied. A random Young measure is a \( \mathcal{Y}^1 \)-valued random variable by definition. We define Young measures on \( \mathbb{T}^N \times [0, T] \) as following \( \mathcal{V}^{\tau_n} = \delta_{u^{\tau_n}=\zeta}, \mathcal{V} = \delta_{u=\zeta} \) and sequence \((\mathcal{V}^{\tau_n})\) of Young measures satisfies
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\zeta|^p \, d\mathcal{V}^{\tau_n}_{x,t}(\zeta) \, dx \right] \leq C_p. \tag{5.8}
\]

**Proposition 5.2.** It holds true (up to subsequences)

1. sequence of Young measures \( \mathcal{V}^n \) converge to \( \mathcal{V} \) in \( \mathcal{Y}^1 \), \( \mathbb{P} \)-almost surely.
2. \( \mathcal{V} \) satisfies
\[
\mathbb{E} \left( \sup_{K \subset [0, T]} \frac{1}{|K|} \int_K \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\zeta|^p \, d\mathcal{V}^{\tau_n}_{x,t}(\zeta) \, dx \, dt \right) \leq C_p, \tag{5.9}
\]
where the supremum in (5.9) is a countable supremum over all open intervals \( K \subset [0, T] \) with rational end points. Furthermore, if \( f^{\tau_n}, f : \mathbb{T}^N \times [0, T] \times \mathbb{R} \times \Omega \to [0, 1] \) are defined by
\[
f^{\tau_n}(x, t, \zeta) = \mathcal{V}^{\tau_n}_{x,t}(\zeta, +\infty), \quad f(x, t, \zeta) = \mathcal{V}_{x,t}(\zeta, +\infty),
\]
then \( f^{\tau_n} \to f \) in \( L^\infty(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \)-weak*- \( \mathbb{P} \)-almost surely.
3. There exists a full measure subset \( \mathcal{B} \) of \( [0, T] \) containing 0 such that for all \( t \in \mathcal{B} \)
\[
f^{\tau_n} \to f \text{ in } L^\infty(\Omega \times \mathbb{T}^N \times \mathbb{R}).
\]

**Proof.** Markov inequality and strong convergence of \( u^{\tau_n} \) in \( L^1(\Omega \times [0, T] \times \mathbb{T}^N) \), implies that \( \mathbb{P} \)-a.s. \( u^{\tau_n} \) (up to subsequence) converges \( u \) in \( L^1([0, T] \times \mathbb{T}^N) \). It implies that, \( \mathbb{P} \)-almost surely, sequence of Young measures \( \mathcal{V}^{\tau_n} \) converges to
\( \mathcal{V} \) in \( \mathcal{Y} \). For second point, we follow proof of [20, Proposition 4.3]. Since the map
\[
\kappa_p : \mathcal{Y} \to [0, +\infty], \quad \mathcal{V} \mapsto \sup_{J \subset [0, T]} \frac{1}{|J|} \int_J \int_{\mathbb{T}^N} \int_{\mathbb{R}} \left| \zeta \right|^p d\mathcal{V}_{x,t}(\zeta) \, dx \, dt
\]
is lower semi-continuous, we have
\[
E \kappa_p(\mathcal{V}) \leq \liminf_{\tau_n \to 0} E \kappa_p(\mathcal{V}^{\tau_n}) \leq C_p.
\]
Consequently, the \( \mathcal{V} \) satisfies the condition
\[
E \left( \sup_{J \subset [0, T]} \frac{1}{|J|} \int_J \int_{\mathbb{T}^N} \int_{\mathbb{R}} \left| \zeta \right|^p d\mathcal{V}_{x,t}(\zeta) \, dx \, dt \right) \leq C_p.
\]
If we apply Corollary 2.2, we obtain that \( f^{\tau_n} \to f \) in \( L^\infty(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \)-weak-* \( \mathbb{P} \)-almost surely. By using similar argument, there exists full measure subset \( B \) of \( [0, T] \) such that for all \( t \in B \)
\[
f^{\tau_n} \to f \text{ in } L^\infty(\Omega \times \mathbb{T}^N \times \mathbb{R}).
\]

5.3. Limit of the random measures
Suppose \( \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \) denotes the space of bounded Borel measures on \( \mathbb{T}^N \times [0, T] \times \mathbb{R} \) equipped norm is given by the total variation of measures. Let \( C_0(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \) be the space of all continuous functions vanishing at infinity equipped with the supremum norm. \( \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \) is the dual space to \( C_0(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \). \( C_0(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \) is separable, so the following duality holds for \( r, r^* \in (1, \infty) \) being conjugate exponents
\[
( L^{r^*}(\Omega; C_0(\mathbb{T}^N \times [0, T] \times \mathbb{R}))^* \simeq L^r_w(\Omega; \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})),
\]
where the space \( L^r_w(\Omega; \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})) \) is collection of all weak*-measurable functions \( \eta : \Omega \to \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \) such that
\[
E \| \eta \|_{L^r_w} < \infty.
\]

Lemma 5.3. There exists a kinetic measure \( m \) such that
\[
m^{\tau_n} \overset{w^*}{\to} m \quad \text{in} \quad L^2_w(\Omega; \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})) - \text{weak}^*.
\]
Moreover, \( m \) can be rewritten as \( m_1 + m_2 + m_1 \), where
\[
d\eta_1(x, t, \zeta) = \int_{\mathbb{R}^N} |u(x + z, t) - \zeta| 1_{\text{Conv}(u(x+z,t),u(x,t))}(\zeta) \mu(z) \, dz \, dx \, dt,
\]
\[
d\eta_2(x, t, \zeta) = |\text{div} \int_0^{\tau_n} \sigma(\zeta) d\zeta|^2 \delta_{u(x,t)}(\zeta) \, dx \, dt,
\]
and \( m_1 \) is almost surely a non-negative measure over \( \mathbb{T}^N \times [0, T] \times \mathbb{R} \).

Proof. From the computations used in the proof of 5.1 we deduce that \( \mathbb{P} \)-almost surely,
\[
\int_{0}^{T} \int_{T}^{T+N} \int_{\mathbb{R}} m_{\tau_{n}}(x, t, \zeta) \, d\zeta \, dx \, dt \\
\leq C \|u_{0}\|_{L^{2}(T)}^{2} + C \sum_{k \geq 1} \int_{0}^{T} \int_{T}^{T+N} u_{\tau_{n}} \beta_{k}(x, u_{\tau_{n}}) \, dx \, dw_{k}(t) \\
+ C \int_{0}^{T} \int_{T}^{T+N} \beta^{2}(x, u_{\tau_{n}}) \, dx \, ds.
\]

Taking square and expectation and finally by using of the Ito isometry, we deduce that

\[
\mathbb{E}|m_{\tau_{n}}([0, T] \times T^{N} \times \mathbb{R})|^{2} \leq C. \tag{5.10}
\]

Therefore, sequence of approximate kinetic measures, \(\{m_{\tau_{n}}; n \in \mathbb{N}\}\), is bounded in \(L^{2}_{w}(\Omega; \mathcal{M}_{b}(T^{N} \times [0, T] \times \mathbb{R}))\) and, making use of the Banach-Alaoglu theorem, it give a weak* convergent subsequence, still denoted by \(\{m_{\tau_{n}}, n \in \mathbb{N}\}\). There exists a kinetic measure \(m\) such that

\[
m_{\tau_{n}} \rightharpoonup m \quad \text{in} \quad L^{2}_{w}(\Omega; \mathcal{M}_{b}(T^{N} \times [0, T] \times \mathbb{R})).
\]

Finally, we conclude that there exist measures \(\mu_{1}, \mu_{2}\) and \(\mu_{3}\) such that

\[
m_{\tau_{n}^{1}} \rightharpoonup \mu_{1}, \quad \eta_{\tau_{n}^{1}} \rightharpoonup \mu_{2}, \quad \eta_{\tau_{n}^{2}} \rightharpoonup \mu_{3} \quad \text{in} \quad L^{2}_{w}(\Omega; \mathcal{M}_{b}(T^{N} \times [0, T] \times \mathbb{R})).
\]

The first point about measurability of \(m\) in Definition 3.1 is direct. The second point of Definition 3.1 is valid from uniform bound (5.10). Then weak* limit \(m\) is a random measure with finite first-order moment. On the other hand, from (5.7) we obtain

\[
\mathbb{E} \int_{0}^{T} \int_{T}^{T+N} \left| \text{div} \int_{0}^{u_{\tau_{n}}} \sigma(\xi) \, d\xi \right|^{2} \, dx \, dt \leq C.
\]

Therefore as consequence of the Banach-Alaoglu theorem we get that \(\text{div} \int_{0}^{u_{\tau_{n}}} \sigma(\xi) \, d\xi\) converges weakly in \(L^{2}(\Omega \times [0, T] \times T^{N})\) (up to subsequence). From the fact that \(\sigma \in C_{b}(\mathbb{R})\), using integration by parts formula, and the strong convergence of \(u_{\tau_{n}}\), we can conclude that for all \(\kappa \in C^{1}(T^{N} \times [0, T])\), \(\mathbb{P}\)-almost surely,

\[
\int_{0}^{T} \int_{T}^{T+N} \left( \text{div} \int_{0}^{u_{\tau_{n}}} \sigma(\xi) \, d\xi \right) \kappa(x, t) \, dx \, dt \rightarrow \int_{0}^{T} \int_{T}^{T+N} \left( \text{div} \int_{0}^{u} \sigma(\xi) \, d\xi \right) \kappa(x, t) \, dx \, dt,
\]

and therefore \(\mathbb{P}\)-a.s.

\[
\text{div} \int_{0}^{u_{\tau_{n}}} \sigma(\xi) \, d\xi \rightharpoonup \text{div} \int_{0}^{u} \sigma(\xi) \, d\xi \quad \text{in} \quad L^{2}([0, T] \times T^{N}). \tag{5.11}
\]

Making use the fact that norm is weakly lower semi continuous, we conclude that for all \(\kappa \in C_{0}(T^{N} \times [0, T] \times \mathbb{R})\) and fixed \(\zeta \in \mathbb{R}\), \(\mathbb{P}\)-almost surely,
\[
\int_0^T \int_{\mathbb{T}^N} \left| \operatorname{div} \int_0^u \sigma(\xi) d\xi \right|^2 \kappa^2(x,t,\zeta) \, dx \, dt \\
\leq \liminf_{\tau_n \to 0} \int_0^T \int_{\mathbb{T}^N} \left| \operatorname{div} \int_0^{u^{\tau_n}} \sigma(\xi) d\xi \right|^2 \kappa^2(x,t,\zeta) \, dx \, dt.
\]

As an application of the Fatou’s lemma, we have \(\mathbb{P}\)-a.s.

\[
\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \left| \operatorname{div} \int_0^u \sigma(\xi) d\xi \right|^2 \kappa^2(x,t,\zeta) \, d\delta_{u=\zeta} \, dx \, dt \\
\leq \liminf_{\tau_n \to 0} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \left| \operatorname{div} \int_0^{u^{\tau_n}} \sigma(\xi) d\xi \right|^2 \kappa^2(x,t,\zeta) \, d\delta_{u^{\tau_n}=\zeta} \, dx \, dt
\]

It shows that \(\mathbb{P}\)-a.s., \(\eta_2 \leq \mu_3\). Recall that, up to subsequence, \(\mathbb{P}\)-almost surely, \(u^{\tau_n}(x,t) \to u(x,t)\) for \((x,t) \notin N\) with \(N \subset \mathbb{T}^N \times [0,T]\) negligible subset. \(\mathbb{P}\)-almost surely, fixing \(z \in \mathbb{R}^N\), we thus have also \(u^{\tau_n}(x+z,t) \to u(x+z,t)\) for any \((x,t)\) not in some negligible subset \(N_z\) of \(\mathbb{T}^N \times [0,T]\). Hence we have

\[
|u^{\tau_n}(x+z,t)) - \zeta|_{\operatorname{Conv}\{u^{\tau_n}(x,t),u^{\tau_n}(x+z,t)\}}(\zeta) \to |u(x+z,t)) - \zeta|_{\operatorname{Conv}\{u(x,t),u(x+z,t)\}}(\zeta)
\]

as \(\tau_n \to 0\). For any \((x,t,\zeta) \in \mathbb{T}^N \times [0,T] \times \mathbb{R}\) such that \((x,t) \notin N \cup N_z\) and \(\zeta \neq u(t,x)\). Note that latter subset of \(\mathbb{T}^N \times [0,T] \times \mathbb{R}\), on which previous convergence does not hold, has Lebesgue measure zero. We can then use Fatou’s lemma to conclude \(\mathbb{P}\)-a.s., \(\eta_1 \leq \mu_2\).

Hence \(\mathbb{P}\)-almost surely,

\[
m \geq \mu_1 + \mu_2 + \mu_3 \geq \eta_1 + \eta_2.
\]

Regarding the chain rule formula (3.2), making use of the regularity of \(u^{\tau_n}\), we can conclude that, for all \(u^{\tau_n}\) and for any \(\kappa \in C_b(\mathbb{R})\),

\[
\operatorname{div} \int_0^{u^{\tau_n}} \kappa(\zeta) \sigma(\zeta) d\zeta = \kappa(u^{\tau_n}) \operatorname{div} \int_0^{u^{\tau_n}} \sigma(\zeta) d\zeta \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^N), \text{ a.e. } (\omega,t).
\]

Furthermore, it is easy to obtain (5.11) with the integrant \(\sigma\) replaced by \(\kappa \sigma\), we can pass to the limit on the left hand side and making use of the strong-weak convergence, we can also pass to the limit on the right hand side of (5.12). The proof is complete. \(\square\)

### 5.4. Kinetic solution

As a consequence of convergence results, stated in previous subsections and as proposed in [20, Lemma 2.1, Proposition 4.9], we can pass to the limit in all the terms of (5.5). Convergence of the stochastic integral can be verified easily using strong convergence and uniform \(L^p\)-bound of \(u^{\tau_n}\).

For all \(\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})\), for \(\mathbb{P}\)-almost surely, there exists a negligible set \(N_0 \subset [0,T]\) such that for all \(t \in [0,T] \setminus N_0\),
\[
\langle f(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), F'(\zeta) \cdot \nabla \varphi \rangle \, ds \\
+ \int_0^t \langle f(s), A : D^2 \varphi \rangle \, ds - \int_0^t \langle f(s), g_x^k[\varphi] \rangle \, ds \\
+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \beta_k(x, \zeta) \varphi(x, \zeta) d\mathcal{N}_{x,t}(\zeta) dx dw_k(s) \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_{\zeta} \varphi(x, \zeta) \beta^2(x, \zeta) d\mathcal{N}_{x,t}(\zeta) \, ds - m(\partial_{\zeta} \varphi)([0, t])
\] 

(5.13)

We now want to show that the above formulation holds for all time \( t \in [0, T] \). For that, we use the following Proposition.

**Proposition 5.4.** There exists a measurable subset \( \tilde{\Omega} \) of \( \Omega \) of probability one, a random Young measure \( \tilde{\mathcal{V}} \) on \( \mathbb{T}^N \times (0, T) \) such that

1. for all \( \omega \in \tilde{\Omega} \), for almost every \( (x, t) \in \mathbb{T}^N \times (0, T) \), the probability measures \( \tilde{\mathcal{V}}_{x,t} \) and \( \mathcal{V}_{x,t} \) coincide.
2. the kinetic function \( \tilde{f}(x, t, \zeta) := \tilde{\mathcal{V}}_{x,t}(\zeta, +\infty) \) satisfies: for all \( \omega \in \tilde{\Omega} \), for all \( \varphi \in C_c(\mathbb{T}^N \times (0, T)) \), \( t \mapsto \langle \tilde{f}(t), \varphi \rangle \) is càdlàg.
3. The random Young measure \( \tilde{\mathcal{V}} \) satisfies

\[
\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\zeta|^p d\tilde{\mathcal{V}}_{x,t}(\zeta) dx \leq C_p.
\]

**Proof.** For proof, we refer to [20, Proposition 4.8]. \( \square \)

We will now consider only the càdlàg version. By making use of [20, Lemma 4.14] for measurability issue and Itô isometry for Itô integration term, we replace \( \mathcal{V} \) by \( \tilde{\mathcal{V}} \), \( f \) by \( \tilde{f} \). Let us recall the fact that (5.13) is now true for all \( t \). For all \( t \in [0, T] \), for all \( \varphi \in C^2_c(\mathbb{T}^N \times \mathbb{R}) \)

\[
\langle \tilde{f}(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle \tilde{f}(s), F'(\zeta) \cdot \nabla \varphi \rangle \, ds \\
+ \int_0^t \langle \tilde{f}(s), A : D^2 \varphi \rangle \, ds - \int_0^t \langle \tilde{f}(s), g_x^k[\varphi] \rangle \, ds \\
+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \beta_k(x, \zeta) \varphi(x, \zeta) d\tilde{\mathcal{V}}_{x,t}(\zeta) dx dw_k(s) \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_{\zeta} \varphi(x, \zeta) \beta^2(x, \zeta) d\tilde{\mathcal{V}}_{x,t}(\zeta) \, ds - m(\partial_{\zeta} \varphi)([0, t])
\] 

(5.14)

\( \mathbb{P} \)-a.s.. Proposition 5.4 implies that \( \mathbb{P} \)-almost surely, almost every \( t \in [0, T] \), \( \tilde{f}(x, t, \xi) = \mathbb{1}_{u(x,t)}(\xi) \), almost every \( (x, \xi) \in \mathbb{T}^N \times \mathbb{R} \). It shows that \( \mathbb{P} \)-almost surely, almost every \( t \in [0, T] \), \( \tilde{f}(t) \) is an equilibrium. Now, our aim is to
prove that $\mathbb{P}$-almost surely, $\tilde{f}$ is an equilibrium for all $t \in [0,T]^1$. To obtain an equilibrium form of $\tilde{f}$ for all $t \in [0,T]$, we can follow similar argument as in the proof of Theorem 4.4 (as used to prove $f^{-}(t)$ is equilibrium) for $\tilde{f}$ with $m \geq \eta_1 + \eta_2$ and equation (5.14). Thus we obtain $\mathbb{P}$-almost surely, for all $t \in [0,T]$ 

$$
\tilde{f}(t) = \mathbb{I}_{\tilde{u}(t)>\zeta}
$$

where

$$
\tilde{u}(t) = \int_{\mathbb{R}} (\tilde{f}(t) - \mathbb{I}_{0>\zeta}) \, d\zeta.
$$

Since $\mathbb{P}$-almost surely, $u(x,t) = \tilde{u}(x,t)$ almost $(x,t) \in \mathbb{T}^N \times [0,T]$, therefore $(\tilde{u}(t))_{t \in [0,T]}$ and $\tilde{f}(t) = \mathbb{I}_{\tilde{u}(t)>\zeta}$, satisfy the all points of Definition 3.2 of kinetic solution. It shows that $(\tilde{u}(t))_{t \in [0,T]}$ is kinetic solution.

5.5. Existence-general initial data

In this final subsection, we prove the existence for the general case of $u_0 \in L^p(\Omega; L^p(\mathbb{T}^N))$, for all $p \in [1, +\infty)$. It is direct consequence of the previous subsections. We approximate the initial data by a sequence $\{u_0^\delta\} \subset L^p(\Omega; C^\infty(\mathbb{T}^N))$, $p \in [1, +\infty)$ such that $u_0^\delta \to u_0$ in $L^1(\Omega; L^1(\mathbb{T}^N))$. That is, the initial data $u_0^\delta$ can be defined as a pathwise mollification of $u_0$ which satisfies the following estimate

$$
\|u_0^\delta\|_{L^p(\Omega; L^p(\mathbb{T}^N))} \leq \|u_0\|_{L^p(\Omega; L^p(\mathbb{T}^N))}.
$$

(5.15)

According to the previous subsections, for each $\delta \in (0,1)$, there exists a kinetic solution $u^\delta$ to (1.1) with initial data $u_0^\delta$. As application of the contraction principle, for all $t \in [0,T]

$$
\mathbb{E}\|u^\delta_1(t) - u^\delta_2(t)\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|u_0^\delta_1 - u_0^\delta_2\|_{L^1(\mathbb{T}^N)} \quad \delta_1, \delta_2 \in (0,1).
$$

It shows that there exists $u \in L^p_{P^T}(\Omega \times [0,T] \times \mathbb{T}^3)$ such that $u^\delta$ converges to $u$ in $L^1_{P^T}(\Omega \times [0,T] \times \mathbb{T}^3)$ as $\delta \to 0$. By (5.15) and (5.6), we still have uniform spatial regularity bound and the uniform energy estimate, that is for $p \in [1, +\infty),

$$
\mathbb{E}\sup_{0 \leq t \leq T}\|u^\delta(t)\|_{L^p(\mathbb{T}^N)} \leq C(T, u_0),
$$

$$
\mathbb{E}\left[\|u^\delta\|^2_{L^2(0,T; H^\lambda(\mathbb{T}^N))}\right] \leq C(T, u_0),
$$

as well as (using the usual notation)

$$
\mathbb{E}|m^\delta(\mathbb{T}^N \times [0,T] \times \mathbb{R})^2 \leq C_{T,u_0}.
$$

We can also conclude that Lemma 5.3 also true for sequence of kinetic measure $m^\delta$. With these information in hand, we can pass the limit in (3.3) and conclude that there exists a kinetic solution $(u(t))_{t \in [0,T]}$ to (1.1). The proof of Theorem 3.1 is complete.

---

1In the case of stochastic conservation laws, Dotti and Vovelle prove the existence of solution from reduction of generalized solution (cf. [20, Proposition 2.8 & Theorem 3.2]).
Remark 5.2. (Convergence of approximations at fixed time $t$) We define $\mathbb{P}$-almost surely.

We define $\mathbb{P}$-almost surely

$$B^\omega_{at} = \{ t \in [0,T] ; m(T^N \times \{ t \} \times \mathbb{R}) > 0 \}.$$ 

Thus we have $\mathbb{P}$-almost surely, for all $t \in [0,T] \setminus B^\omega_{at}$, for all $\varphi \in C_c(T^N \times \mathbb{R})$, $\langle f^\varepsilon(t), \varphi \rangle \to \langle f(t), \varphi \rangle$ (cf. [20, Proposition 4.9]). By making use of [20, Lemma 2.6] we conclude that $\mathbb{P}$-almost surely, for all $t \in [0,T] \setminus B^\omega_{at}$, $u^\varepsilon(t) \to u(t)$ in $L^p(T^N)$. We have $\mathbb{P}$-almost surely, for all $t \in [0,T]$, $f(t) = f^-(t)$ (cf. [20, Proposition 2.11]). Equation (4.4) implies that $B^\omega_{at}$ is empty. This gives that $\mathbb{P}$-almost surely, for all $t \in [0,T]$, $u^\varepsilon(t) \to u(t)$ in $L^p(T^N)$.

6. Continuous dependence estimate: proof of Theorem 3.2

In this section, we develop a general framework for the continuous dependence estimate by using $BV$ estimate of kinetic solutions.

6.1. BV estimate

**Theorem 6.1.** Let $(u(t))_{t \in [0,T]}$ be a kinetic solution to (2.7) with initial data $u_0 \in BV(T^N)$. Then the following $BV$-estimate holds: for all $t \in [0,T]$

$$ \mathbb{E}[\|u(t)\|_{BV}] \leq \mathbb{E}[\|u_0\|_{BV}]. \quad (6.1) $$

**Proof.** By the substitution $z = x + h$ in (2.7), it implies that, if $u(x,t)$ solves (2.7) with initial data $u_0(x)$, then $u(z,t) = u(x + h,t)$ solves (1.1) with initial data $u_0(x + h)$. From estimates given in the proof of Theorem 4.4, we have for all $t \in [0,T]$,

$$ \mathbb{E} \int_{(T^N)^2} \int_{\mathbb{R}^2} f(x,t,\xi)\tilde{f}(y+h,t,\zeta)\kappa_\delta(\xi - \zeta)\theta_\epsilon(x-y)d\xi d\zeta dx dy $$

$$ \leq \mathbb{E} \int_{(T^N)^2} \int_{\mathbb{R}^2} \theta_\epsilon(x-y)\kappa_\delta(\xi - \zeta)f_0(x,\xi)\tilde{f}_0(y+h,\zeta)d\xi d\zeta dx dy $$

$$ + C t \delta^{\lambda_F_1} \epsilon^{-1} + C t \delta^{\lambda_F_2} + C t \epsilon^{-2} \delta^{2\gamma_a}. $$

Taking $\delta \to 0$, then for all $t \in [0,T]$,

$$ \mathbb{E} \int_{(T^N)^2} (u(y+h,t) - u(x,t))_+ \theta_\epsilon(x-y)dx dy \leq \mathbb{E} \int_{(T^N)^2} (u_0(y+h) - u_0(x))_+ \theta_\epsilon(x-y)dx dy. $$

Letting $\epsilon \to 0$, then it implies that for all $t \in [0,T]$,

$$ \mathbb{E} \int_{T^N} \frac{|u(x+h,t) - u(x,t)|}{|h|} dx \leq \mathbb{E} \int_{T^N} \frac{|u_0(x+h) - u_0(x)|}{|h|} dx. \quad (6.2) $$

By using Fatou’s lemma and (6.2), we have for all $t \in [0,T]$,

$$ \mathbb{E} \liminf_{h \to 0^+} \int_{T^N} \frac{|u(x+h,t) - u(x,t)|}{|h|} dx \leq \liminf_{h \to 0^+} \mathbb{E} \int_{T^N} \frac{|u_0(x+h) - u_0(x)|}{|h|} dx, \quad (6.3) $$

where $\epsilon_n(t) = \epsilon_n(x)$.
\begin{align}
\limsup_{h \to 0^+} \mathbb{E} \int_{\mathbb{T}^N} \frac{|u_0(x + h, t) - u_0(x, t)|}{|h|} dx & \leq \mathbb{E} \limsup_{h \to 0^+} \int_{\mathbb{T}^N} \frac{|u_0(x + h) - u_0(x)|}{|h|} dx.
\end{align}

(6.4)

By making use of [31, Theorem 13.48], we have \(\mathbb{P}\)-almost surely,

\begin{align}
\limsup_{h \to 0^+} \int_{\mathbb{T}^N} \frac{|u_0(x + h) - u_0(x)|}{|h|} dx & \leq TV_x(u_0).
\end{align}

(6.5)

From (B.2)–(6.5), we conclude that for all \(t \in [0, T]\),

\begin{align}
\mathbb{E} \liminf_{h \to 0^+} \int_{\mathbb{T}^N} \frac{|u(x + h, t) - u(x, t)|}{|h|} dx & \leq \mathbb{E}[TV_x(u_0)].
\end{align}

(6.6)

Again by making use of [31, Theorem 13.48], we conclude that for all \(t \in [0, T]\),

\(\mathbb{E}[TV_x(u(\cdot, t))] \leq \mathbb{E}[TV_x(u_0)].\)

This finishes the proof. \(\square\)

Now we apply Kruzhkov’s doubling of variable technique and try to bound the difference of kinetic solutions. Proof of following proposition is similar as Proposition 4.3, so we left it to reader.

**Proposition 6.2.** Let \((u(t))_{t \in [0,T]}\) be a kinetic solution to (2.7) with initial data \(u_0\), and let \((v(t))_{t \in [0,T]}\) be a kinetic solution to (2.8) with initial data \(v_0\). Then, for all \(t \in [0, T]\) and non-negative test functions \(\theta \in C^\infty(\mathbb{T}^N), \, \kappa \in C^\infty_c(\mathbb{R})\), we have

\begin{align}
\mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R}^2)^2} \theta(x - y) \kappa(\xi - \zeta) f_1(x, t, \xi) \bar{f}_2(y, t, \zeta) d\xi d\zeta dx dy & \
\leq \mathbb{E} \left[ \int_{(\mathbb{T}^N)^2} \int_{(\mathbb{R}^2)^2} \theta(x - y) \kappa(\xi - \zeta) f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) d\xi d\zeta dx dy + \mathcal{R}_\theta \right] \
& \quad + \mathcal{R}_\kappa + J + K,
\end{align}

(6.7)

where

\begin{align*}
\mathcal{R}_\theta & = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (F'(\xi) - G'(\zeta)) \kappa(\xi - \zeta) dx dy ds,
\end{align*}

\begin{align*}
\mathcal{R}_\kappa & = \frac{1}{2} \int_{(\mathbb{T}^N)^2} \theta(x - y) \int_0^t \int_{\mathbb{R}^2} \kappa(\xi - \zeta) \Phi(\xi) - \Psi(\zeta)^2 G_{x}^\lambda(\theta(x - y))) d\xi d\zeta dx dy ds,
\end{align*}

\begin{align*}
J & = - \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) \kappa(\xi - \zeta) g^\lambda_x(\theta(x - y))) d\xi d\zeta dx dy ds \
& \quad - \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) \kappa(\xi - \zeta) g^\lambda_y(\theta(x - y))) d\xi d\zeta dx dy ds \
& \quad - \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \partial_x \kappa(\xi - \zeta) \theta(x - y) d\eta_{x,2}(y, s, \zeta) d\xi \
& \quad + \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2(y, s, \zeta) \partial_x \kappa(\xi - \zeta) \theta(x - y) d\eta_{y,2}(x, s, \xi) dy d\zeta,
\end{align*}

\begin{align*}
K & = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2(\Lambda(\xi) + B(\zeta)) : D^2 \theta(x - y) \kappa(\xi - \zeta) d\xi d\zeta dx dy ds.
\end{align*}
\[ - \int_0^t \int_{\mathbb{T}^2} \theta(x - y) \kappa(\zeta) d\nu_{x,s}(\zeta) \, dy \, d\eta_{x,s}(y, s, \zeta) \]

\[ - \int_0^t \int_{\mathbb{T}^2} \theta(x - y) \kappa(\zeta) d\nu_{y,s}(\zeta) \, dy \, d\eta_{y,s}(x, s, \xi). \]

### 6.2. Proof of Theorem 3.2

**Proof.** Idea of this proof is similar to the proof of Theorem 4.4. Here we define \( \kappa_\delta, \theta_\epsilon \) and \( \Phi_\delta \) similarly as in the proof of Theorem 4.4. We will estimate each term of inequality (6.7) separately in the following steps.

**Step 1** We estimate \( \mathcal{R}_\theta \) as follows:

\[
|\mathbb{E}[\mathcal{R}_\theta]| = |\mathbb{E}\left[ \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (F'(\xi) - G'(\xi)) \kappa_\delta(\xi - \zeta) d\xi d\zeta \right. \\
- \nabla \theta_\epsilon (x - y) dxdyds | \\
\leq |\mathbb{E}\left[ \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (F'(\xi) - G'(\xi)) \kappa_\delta(\xi - \zeta) \right. \\
\cdot \nabla_x \theta_\epsilon (x - y) dxdyds | + C t \epsilon^{-1} \delta^{\lambda \gamma_1} \\
\leq |\mathbb{E}\left[ \int_0^t \int_{(T^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \nabla v \\
\cdot (F(\xi) - G(\xi)) \theta_\epsilon (x - y) \kappa_\delta(\xi - \zeta) \delta_{\zeta=v} d\xi d\zeta dxdyds | \\
+ C t \epsilon^{-1} \delta^{\lambda \gamma_1} \\
\leq ||F' - G'||_{L^\infty(\mathbb{R})} |\mathbb{E}\left[ \int_0^t \int_{(T^N)^2} |\nabla v| \theta_\epsilon (x - y) \Phi_\delta(u - v) dxdyds | \\
+ C t \epsilon^{-1} \delta^{\lambda \gamma_1} \\
\leq C t \mathbb{E}[||v_0||_{BV}] ||F' - G'||_{L^\infty(\mathbb{R})} + C t \epsilon^{-1} \delta^{\lambda \gamma_1}. \]

**Step 2** To improve the readability of the presentation in remaining paper, we make use of the following notation:

\[
g^\lambda_x [\varphi](x) = -\text{P.V.} \int_{\mathbb{R}^N} (\varphi(x + z) - \varphi(x)) \, d\mu_\lambda(z), \\
g^\lambda_{x,r} [\varphi](x) = - \int_{|z| < r} (\varphi(x + z) - \varphi(x)) \, d\mu_\lambda(z), \\
\text{and} \\
g^\lambda_{x,r} [\varphi](x) = - \int_{|z| > r} (\varphi(x + z) - \varphi(x)) \, d\mu_\lambda(z), \]
where $d\mu_\lambda(z) := \frac{dz}{|z|^{N+2}}$. Note that $\mu_\lambda$ is a non-negative Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying
\[
\int_{\mathbb{R}^N \setminus \{0\}} (|z|^2 \wedge 1) \, d\mu_\lambda(z) < +\infty. \tag{6.8}
\]
For technical purposes (see [3,6]), it is necessary to split Radon measures $\mu_\lambda, \mu_\beta$ as follows: Let $S^{\pm}$ be the sets such that
\[
\begin{align*}
S^{\pm} &\subseteq \mathbb{R}^N \setminus \{0\} \text{ are Borel sets}, \\
\bigcup_{\pm} S^{\pm} &\subseteq \mathbb{R}^N \setminus \{0\}, \quad \cap_{\pm} S^{\pm} = \emptyset, \tag{6.9}
\end{align*}
\]
and we denote $\mu_{\beta^{\pm}}$ and $\mu_{\lambda^{\pm}}$ as the restrictions of $\mu_{\beta}$ and $\mu_{\lambda}$ to $K^{\pm}$, respectively. Then it is clear to see that
\[
\begin{cases}
\mu_\lambda = \sum_{\pm} \mu_{\lambda^{\pm}}, \quad \text{and} \quad \mu_\beta = \sum_{\pm} \mu_{\beta^{\pm}} \\
\pm (\mu_{\lambda^{\pm}} - \mu_{\beta^{\pm}}) = (\mu_\lambda - \mu_\beta)^{\pm}. \\
\mu_{\lambda^{\pm}}, \mu_{\beta^{\pm}}, \text{ and } \pm (\mu_{\lambda^{\pm}} - \mu_{\beta^{\pm}}) \\
\text{all are non-negative Radon measures satisfying (6.8)}. \tag{6.10}
\end{cases}
\]
We will now estimate $J$. Here we follow similar idea as used in estimate of term $J$ in proof of Theorem 4.3, we have
\[
\mathbb{E}[J] = \mathbb{E} \int_0^t \int_{(\mathbb{T}^2)^2} \int_{\mathbb{R}^N} \Phi_\delta(u(x, s) - v(y, s)) \left[ (\tau_z u(x, s) - u(x, s)) \mu_\lambda(z) - (\tau_z v(y, s) - v(y, s)) \mu_\beta(z) \right] \theta_\epsilon(x - y) \, dz \, dx \, dy \, ds \\
= J_r + J^r
\]
where
\[
J_r = -\mathbb{E} \int_0^t \int_{(\mathbb{T}^2)^2} \left[ g_{x,r}^\lambda(u) - g_{y,r}^\beta(v) \right] \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds,
\]
and
\[
J^r = -\mathbb{E} \int_0^t \int_{(\mathbb{T}^2)^2} \left[ g_{x}^{\lambda,r}(u) - g_{y}^{\beta,r}(v) \right] \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds.
\]
First we try to get bound on term $J^r$ as follows:
\[
J^r = \sum_{\pm} \mathbb{E} \left[ \int_0^t \int_{(\mathbb{T}^2)^2} \left( g_{y}^{\pm,r}(v) - g_{x}^{\pm,r}(u) \right) \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds \right]
\]
\[
= \mathbb{E} \left[ \int_0^t \int_{(\mathbb{T}^2)^2} \left( g_{y}^{\pm,r}(v) - g_{x}^{\pm,r}(u) \right) \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds \right]
\]
\[
+ \mathbb{E} \left[ \int_0^t \int_{(\mathbb{T}^2)^2} \left( g_{y}^{\beta,r}(v) - g_{x}^{\lambda,r}(u) \right) \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds \right]
\]
\[
+ \mathbb{E} \left[ \int_0^t \int_{(\mathbb{T}^2)^2} \left( g_{y}^{\lambda,r}(v) - g_{x}^{\beta,r}(u) \right) \Phi_\delta(u(x, s) - v(y, s)) \theta_\epsilon(x - y) \, dx \, dy \, ds \right].
\]
\[
\begin{align*}
\gamma_
 & \quad \frac{-\gamma^2}{2} + \mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} (g^\lambda_{\gamma,r}(v) - g^\lambda_{\gamma,r}(u)) \Phi_\delta(u(x,s) - v(y,s)) \theta(x-y) dxdyds \right] \\
& := J_+ + J_2 + J_3 + J_4.
\end{align*}
\]

From proof of Theorem 4.4, it is clear that \( J_2 \) and \( J_4 \) is non-positive. Finally, we are left with two terms \( J_1 \) and \( J_3 \). Consider in the sequel \( r_1 > r \). We have
\[
\begin{align*}
J_1 &= \mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} (g^\beta_{\gamma,r,r_1}(u) - g^\lambda_{\gamma,r,r_1}(u)) \Phi_\delta(u(x,s) - v(y,s)) \theta(x-y) dxdyds \right] \\
& \quad + \mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} (g^\beta_{\gamma,r,r_1}(u) - g^\lambda_{\gamma,r,r_1}(u)) \Phi_\delta(u(x,s) - v(y,s)) \theta(x-y) dxdyds \right] \\
& := I_1 + I_2.
\end{align*}
\]

where the notation \( g^\lambda_{\gamma,r,r_1} \) means that the nonlocal integration is understood in the set \( \{ r < |z| \leq r_1 \} \) (resp. with \( \mu_{\lambda_+} \)). Let \( \gamma' = \Phi_\delta \), then
\[
\begin{align*}
\int_{(\mathbb{T}^N)^2} (g^\beta_{\gamma,r,r_1}(u(x,s)) - g^\lambda_{\gamma,r,r_1}(u(x,s))) \Phi_\delta(u(x,s) - v(y,s)) \theta(x-y) dxdyds \\
& = \int_{(\mathbb{T}^N)^2} \int_{r < |z| \leq r_1} (u(x + z, s) - u(x, s)) \Phi_\delta(u(x,s)) \\
& \quad - v(y,s)) \theta(x-y)d(\mu_{\lambda_+} - \mu_{\beta_+})(z)dxdyds \\
& = \int_{(\mathbb{T}^N)^2} \int_{r < |z| \leq r_1} [(u(x + z, s) - v(y,s)) - (u(x, s) - v(y,s))] \Phi_\delta(u(x,s)) \\
& \quad - v(y,s)) \theta(x-y)d(\mu_{\lambda_+} - \mu_{\beta_+})(z)dxdyds \\
& \leq \int_0^t \int_{(\mathbb{T}^N)^2} \int_{r < |z| \leq r_1} \left( \frac{\gamma(u(x+z,s) - v(y,s))}{\gamma(u(x,s) - v(y,s))} \right) \theta(x-y)d(\mu_{\lambda_+} - \mu_{\beta_+})(z)dxdyds \\
& \quad - \frac{\gamma(u(x,s) - v(y,s))}{\gamma(u(x,s) - v(y,s))} \left( \left( g^\beta_{\gamma,r,r_1}(u) - g^\lambda_{\gamma,r,r_1}(u) \right) \theta(x-y) dxdyds \right)
\end{align*}
\]

where to derive the penultimate inequality, we have used the fact that \( \gamma(b) - \gamma(a) \geq \gamma'(a)(b-a) \) with \( a = (u(x,s) - v(y,s)) \) and \( b = u(x+z) - v(y,s) \). For the last equality, we have performed a change of coordinates for first integral \( x \to x + z, z \to -z \). To estimate \( I_1 \), we observe that, by construction of the measures, the nonlocal domain of integration is always radial symmetric. It implies that
\[
\int_{r < |z| \leq r_1} z \cdot \nabla_x \theta(x-y)d(\mu_{\lambda_+} - \mu_{\beta_+})(z) = 0.
\]

With the help of above identity, we estimate term \( I_1 \) as follows:
\[
\begin{align*}
I_1 \leq -\mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} \gamma(u(x,s) - v(y,s)) \left( \frac{g^\beta_{\gamma,r,r_1}(u) - g^\lambda_{\gamma,r,r_1}(u)}{(\theta)(x-y)} \right) dxdyds \right] \\
& = -\mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} \gamma(u(x,s) - v(y,s)) \right] \times \left\{ \int_{r < |z| \leq r_1} \left( \theta(x-y+z) - \theta(x-y) \right) d(\mu_{\lambda_+} - \mu_{\beta_+})(z) \right\} dxdyds
\end{align*}
\]

\( \text{NoDEA Stochastic degenerate fractional conservation laws} \)
\[
\begin{align*}
&= -\mathbb{E} \int_0^t \int_{|x| \leq r_1} \gamma(u(x, s) - v(y, s)) \\
&\quad \times \left\{ \int_{r_1 < |x| \leq r_1} (\theta_\varepsilon(x-y+z) - \theta_\varepsilon(x-y) - z \cdot \nabla \theta_\varepsilon(x-y)) \ d(\mu_{\lambda_+} - \mu_{\beta_+})(z) \right\} \ dx dy ds \\
&= -\mathbb{E} \int_0^t \int_{|x| \leq r_1} \gamma(u(x, s) - v(y, s)) \\
&\quad \times \left\{ \int_0^1 (1 - \tau) z^T \cdot \text{Hess}_x \theta_\varepsilon(x + \tau z - y) \cdot z \ d\tau \right\} \ d(\mu_{\lambda_+} - \mu_{\beta_+})(z) \ dx dy ds \\
&\leq -\mathbb{E} \left[ \frac{1}{\epsilon} \int_0^t \int_{|x| \leq r_1} \int_{|x| \leq r} |\nabla \theta_\varepsilon(x - \tau z - y) \cdot z| \ |\nabla \theta_\varepsilon(x - \tau z - y) \cdot z| \left| \int_0^1 (1 - \tau) \gamma(u(x, s) - v(y, s)) \right| \ d\tau \right] \\
&\quad \times \left\{ \int_0^1 (1 - \tau) z^T \cdot \text{Hess}_x \theta_\varepsilon(x + \tau z - y) \cdot z \ d\tau \right\} \ d(\mu_{\lambda_+} - \mu_{\beta_+})(z) \ dx dy ds \\
&\leq \frac{C t}{\epsilon} \mathbb{E}[\|u_0\|_{BV}] \int_{r_1 < |x| \leq r_1} |z|^2 d(\mu_{\lambda_+} - \mu_{\beta_+})(z) \\
&\leq \frac{C t}{\epsilon} \mathbb{E}[\|u_0\|_{BV}] \int_{|x| \leq r_1} |z|^2 d(\mu_{\lambda_+} - \mu_{\beta_+})(z),
\end{align*}
\]

where we have also used the fact that for any Lipschitz continuous function \( \gamma \) (with Lipschitz constant 1), \(|D\gamma(u)| \leq |Du|\). On the other hand, to handle the other term we proceed as follows:

\[
I_2 \leq C t \int_{|z| > r_1} \mathbb{E}[\|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{T}^N)} \ d(\mu_{\lambda_+} - \mu_{\beta_+})],
\]

thanks to the contraction principle of Theorem 4.4 and the fact that \( u(\cdot, \cdot + z) \) is the solution corresponding to the initial condition \( u_0(\cdot + z) \). Note that exact same calculations will help us to estimate \( J_3 \). Indeed, we have

\[
J_3 \leq \frac{C t}{\epsilon} \mathbb{E}[\|v_0\|_{BV}] \int_{|z| \leq r_1} |z|^2 d(\mu_{\mu_+} - \mu_{\beta_+})(z)
\]

\[
+ C t \int_{|z| > r_1} \mathbb{E}[\|v_0(\cdot + z) - v_0\|_{L^1(\mathbb{T}^N)} \ d(\mu_{\mu_+} - \mu_{\beta_+})(z),
\]

Now, we will try to estimate \( J_r \) when \( r \to 0 \). First note that,

\[
|g_{x,r}^\lambda(\theta_\varepsilon)(x-y)| \leq \begin{cases} \\
\|D\theta_\varepsilon\|_{L^\infty} \int_{|z| \leq r} \frac{|z|^2}{|z| + 2x} \ dz, & \lambda \in (0, 1/2) \\
\|D^2\theta_\varepsilon\|_{L^\infty} \int_{|z| \leq r} \frac{|z|^2}{|z| + x} \ dz, & \lambda \in [1/2, 1)
\end{cases}
\]
Thus we see that in both cases, \(|g_{x,r}^\lambda(\theta)(x - y)| \leq cr^s\) for some \(s > 0\) and 
\[\lim_{r \to 0} |g_{x,r}^\lambda(\theta)(x - y)| = 0.\]

By Making use of computations as we used in estimating term \(I_1\), we obtain
\[
|J_r| \leq \mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^N)^2} |\gamma(u(x,s) - v(y,s))||g_{x,r}^\lambda(\theta)(x - y)|
+ |g_{y,r}^\beta(\theta)(x - y)|dxdyds \right]
\leq \mathbb{E}\left[ \int_0^t \int_{(\mathbb{T}^2)^2} |u(x,s) - v(y,s)|( |g_{x,r}^\lambda(\theta)(x - y)|
+ |g_{y,r}^\beta(\theta)(x - y)|)dxdyds \right] \longrightarrow 0.
\]

**Step 3** We estimate \(R_\kappa\) as follows: \(\mathbb{P}\)-almost surely, for all \(t \in [0, T]\),
\[
R_\kappa = \frac{1}{2} \int_{(\mathbb{T}^N)^2} \theta_\epsilon(x - y) \int_0^t \int_{\mathbb{R}^2} \kappa_\delta(\xi - \zeta)|\Phi(\xi)
- \Psi(\xi) + \Psi(\zeta)|^2d\mathcal{V}_{x,s}^1 \oplus \mathcal{V}_{y,s}^2(\xi, \zeta)dxdyds
\leq C t (\delta^{-1}||\Phi - \Psi||^2_{L^\infty} + \delta^{\lambda c_2}).
\]

**Step 4** In this step we estimate \(K\) term by using similar argument as estimate \(K\) term in proof of Theorem 4.4. We have
\[
A(\xi) + B(\zeta) = (\sigma(\xi) - \tau(\xi)) (\sigma(\xi) - \tau(\zeta)) + (\sigma(\xi)\tau(\zeta) + \tau(\zeta)\sigma(\xi)).
\]

Then we conclude that \(\mathbb{P}\)-almost surely, for all \(t \in [0, T]\)
\[
\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi)f_2(y, s, \zeta)(\sigma(\xi)\tau(\zeta)
+ \tau(\zeta)\sigma(\xi)): D^2_x \theta_\epsilon(x - y)\kappa_\delta(\xi - \zeta)d\xi d\zeta dxdyds
= 2 \int_0^t \int_{(\mathbb{T}^N)^2} \theta_\epsilon(x - y)\phi_\delta(u - v) \times \text{div}_x \int_0^u \sigma(\xi)d\xi \cdot \text{div}_y \int_0^v \tau(\zeta)d\zeta dxdyds,
\]
and by using chain rule 3.2 we get
\[
- \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta_\epsilon(x - y)\kappa_\delta(\xi - \zeta)d\mathcal{V}_{x,s}^1(\xi)dxd\eta_{u,3}(y, s, \xi)
- \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \theta(x - y)\kappa(\xi - \zeta)d\mathcal{V}_{y,s}^2(\xi)dy d\eta_{u,3}(x, s, \xi)
= - \int_0^t \int_{(\mathbb{T}^N)^2} \theta_\epsilon(x - y)\phi_\delta(u_1 - u_2)|\text{div}_y \int_0^{u_2} \sigma(\zeta)d\zeta|^2dxdyds
- \int_0^t \int_{(\mathbb{T}^N)^2} \theta_\epsilon(x - y)\phi_\delta(u_1 - u_2)|\text{div}_x \int_0^{u_1} \sigma(\xi)d\xi|^2dxdyds.
\]
So we can conclude that
\[
K \leq \int_0^t \int_{(TN)^2} \int_{R^2} f_1(x, s, \xi) f_2(y, s, \zeta)(\sigma(\xi) - \tau(\zeta)) d\xi d\zeta dy ds
- \tau(\zeta)(\sigma(\xi) - \tau(\zeta)) : D^2_x \theta(x - y) \kappa_\delta(\xi - \zeta) d\xi d\zeta dxdy ds
= \int_0^t \int_{(TN)^2} \int_{R^2} f_1(x, s, \xi) f_2(y, s, \zeta)(\sigma(\xi) - \tau(\zeta)) d\xi d\zeta dxdy ds
- \tau(\zeta)(\sigma(\xi) - \tau(\zeta)) : D^2_x \theta(x - y) \kappa_\delta(\xi - \zeta) d\xi d\zeta dxdy ds
+ \int_0^t \int_{(TN)^2} \int_{R^2} f_1(x, s, \xi) f_2(y, s, \zeta)(\sigma(\xi) - \tau(\zeta)) d\xi d\zeta dxdy ds
- \tau(\zeta)(\sigma(\xi) - \tau(\zeta)) : D^2_x \theta(x - y) \kappa_\delta(\xi - \zeta) d\xi d\zeta dxdy ds
+ \int_0^t \int_{(TN)^2} \int_{R^2} f_1(x, s, \xi) f_2(y, s, \zeta)(\tau(\xi) - \tau(\zeta)) (\tau(\xi) - \tau(\zeta)) : D^2_x \theta(x - y) \kappa_\delta(\xi - \zeta) d\xi d\zeta dxdy ds
- \tau(\zeta)(\sigma(\xi) - \tau(\zeta)) : D^2_x \theta(x - y) \kappa_\delta(\xi - \zeta) d\xi d\zeta dxdy ds
=: K_1 + K_2 + K_3 + K_4
\]

By similar calculation as proof in Theorem 4.4, we can get the following estimate

\[
K_1 \leq C t \epsilon^{-2} \|\sigma - \tau\|_{L^\infty_1},
K_2 \leq C t \|\sigma - \tau\|_{L^\infty_1} \delta^{7\epsilon} \epsilon^{-2},
K_3 \leq C t \|\sigma - \tau\|_{L^\infty_1} \delta^{7\epsilon} \epsilon^{-2},
K_4 \leq C t \delta^{2\gamma} \epsilon^{-2}.
\]

**Step 5** From previous steps we have for all \( t \in [0, T] \)

\[
\mathbb{E} \int_{(TN)^2} \int_{(R^2)^2} \theta(x - y) \kappa_\delta(\xi - \zeta) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\zeta dxdy
\leq \mathbb{E} \int_{(TN)^2} \int_{(R^2)^2} \theta(x - y) \kappa_\delta(\xi - \zeta) f_{1,0}(x, \xi) f_{2,0}(y, \zeta) d\xi d\zeta dxdy
+ C t \|F' - G\|_{L^\infty_1} \mathbb{E}[\|v_0\|_{BV}]
+ C t \epsilon^{-1} \delta^{\lambda_c_1}
+ C t (\delta^{-1} \|\Phi - \Psi\|_{L^\infty} + \delta^{\lambda_c_2}) + \frac{C t \epsilon}{\mu} \mathbb{E}[\|u_0\|_{BV}]
+ \|v_0\|_{BV} \int_{|z| \leq r_1} |z|^2 d|\mu\lambda - \mu_\beta| (z)
\]
\[ + C t \int_{|z| > r_1} \mathbb{E}(\|u_0(\cdot + z) - u_0\|_{L^1(T^N)} + \|v_0(\cdot + z) - v_0\|_{L^1(T^N)} + \mu_\lambda - \mu_\beta(z) \\
- v_0\|_{L^1(T^N)} \|d\mu_\lambda - \mu_\beta(z) \\
+ C t \varepsilon^{-2}\|\sigma - \tau\|_{L^\infty(R)}^2 + C t \varepsilon^{-2}f + C t \frac{\varepsilon^{2\gamma_n}}{\varepsilon^{-2}}. \quad (6.11) \]

Finally, we will now estimate \( \mathbb{E} \int_{T^N} (u(x) - v(x))^+ dx \) in terms of given data as follows:

\[ \mathbb{E} \int_{T^N} (u(x, t) - v(x, t))^+ dx = \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} f_1(x, t, \xi) f_2(x, t, \zeta) d\xi dx \\
= \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} \theta_\varepsilon(x - y) \kappa_\delta(\xi - \zeta) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\zeta d\chi dy \]

where

\[ \eta_\varepsilon(u, v, \epsilon, \delta) = \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} f_1(x, t, \xi) f_2(x, t, \zeta) d\xi dx \\
- \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) \kappa_\delta(\xi - \zeta) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\zeta d\chi dy \\
= \left( \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} f_1(x, t, \xi) f_2(x, t, \zeta) d\chi dx \\
- \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) f_1(x, t, \xi) f_2(y, s, \xi) d\xi d\chi d\chi dy \right) \\
+ \left( \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) f_1(x, t, \xi) f_2(y, s, \xi) d\xi d\chi dy \right) \\
- \mathbb{E} \int_{T^N} \int_{\mathbb{R}^2} \theta_\epsilon(x - y) \kappa_\delta(\xi - \zeta) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\zeta d\chi dy \]

\[ =: H_1 + H_2 \]

where

\[ |H_1| = \left| \mathbb{E} \int_{T^N} \theta_\epsilon(x - y) \int_{\mathbb{R}} 1_{u(x) > \chi} \left[ 1_{u(x) \leq \chi} - 1_{v(y) \leq \chi} \right] d\xi d\chi dy \right| \\
= \left| \mathbb{E} \int_{T^N} \theta_\epsilon(x - y) (v(x) - v(y)) d\xi d\chi dy \right| \\
\leq \varepsilon \mathbb{E} \|v_0\|_{BV}, \quad (6.12) \]

The error term will estimate as follows:

\[ \left| \int_{T^N} \theta_\epsilon(x - y) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\chi dy \right| \\
- \left| \int_{T^N} \theta_\epsilon(x - y) \kappa_\delta(\xi - \zeta) f_1(x, t, \xi) f_2(y, t, \zeta) d\xi d\zeta d\chi dy \right| \\
\]

\[ = \left| \int_{T^N} \theta_\epsilon(x - y) 1_{u(x) > \chi} \int_{\mathbb{R}} \kappa_\delta(\xi - \zeta) \left[ 1_{v(y) \leq \chi} - 1_{v(y) \leq \chi} \right] d\zeta d\chi dy \right|. \]
\[
\leq \int_{(\mathbb{T}^{N})^2} \int_{\mathbb{R}} \theta_{\epsilon} (x-y) \mathbb{1}_{u(y) > \xi} \int_{\xi - \delta}^{\xi} \kappa_{\delta} (\xi - \zeta) \mathbb{1}_{\zeta < v(y) \leq \xi} \zeta d\zeta dx dy \\
+ \int_{(\mathbb{T}^{N})^2} \int_{\mathbb{R}} \theta_{\epsilon} (x-y) \mathbb{1}_{u(y) > \xi} \int_{\xi}^{\xi + \delta} \kappa_{\delta} (\xi - \zeta) \mathbb{1}_{\zeta < v(y) \leq \xi} \zeta d\zeta dx dy \\
\leq \frac{1}{2} \int_{(\mathbb{T}^{N})^2} \theta_{\epsilon} (x-y) \int_{v(y)}^{\min \{ u(x), v(y) + \delta \}} d\zeta dx dy \\
+ \frac{1}{2} \int_{(\mathbb{T}^{N})^2} \theta_{\epsilon} (x-y) \int_{v(y) - \delta}^{\min \{ u(x), v(y) \}} d\zeta dx dy \\
\leq \delta.
\]

It gives that

\[|H_2| \leq \delta, \quad (6.13)\]

and

\[\eta_t (u, v, \epsilon, \delta) \leq \epsilon \mathbb{E} \| v_0 \|_{BV} + \delta \quad (6.14)\]

Finally, we conclude that for all \( t \in [0, T] \)

\[
\mathbb{E} \int_{\mathbb{T}^N} (u(x, t) - v(x, t))_+ dx \\
\leq \mathbb{E} \left[ \int_{(\mathbb{T}^{N})^2} \int_{\mathbb{R}^2} \theta_{\epsilon} (x-y) \kappa_{\delta} (\xi - \zeta) f_{1,0} (x, \xi) \tilde{f}_{2,0} (y, \zeta) d\zeta d\xi dx dy \right] \\
+ \epsilon \mathbb{E} \| v_0 \|_{BV} + \delta + C t \| F' - G' \|_{L_{\infty} (\mathbb{R})} \mathbb{E} \| v_0 \|_{BV} + C t \epsilon^{-1} \delta^{-\lambda G_1} \\
+ C t \left( \delta^{-1} \left\| \Phi - \Psi \right\|_{L_{\infty}} + \delta^{-\lambda G_2} \right) \\
+ \frac{C t}{\epsilon} (\mathbb{E} \| u_0 \|_{BV} + \| v_0 \|_{BV}) \int_{|z| \leq r_1} |z|^2 d\mu_\lambda - \mu_\beta (z) \\
+ C t \int_{|z| > r_1} \mathbb{E} \left( \| u_0 (\cdot + z) - u_0 \|_{L^1 (\mathbb{T}^N)} \right) \\
+ \| v_0 (\cdot + z) - v_0 \|_{L^1 (\mathbb{T}^N)} d\mu_\lambda - \mu_\beta (z), \\
+ C t \epsilon^{-2} \| \sigma \|_{L_{\infty} (\mathbb{R})} + C t \| \sigma \|_{L_{\infty} (\mathbb{R})} \\
- \tau \| v_0 \|_{L_{\infty} (\mathbb{R})}^{\delta^{\gamma b}} \epsilon^{-2} + C t \delta^{2 \gamma b} \epsilon^{-2}.
\]

It shows that for all \( t \in [0, T] \)

\[
\mathbb{E} \int_{\mathbb{T}^N} |u(x, t) - v(x, t)| dx \leq C \left( \mathbb{E} \int_{\mathbb{T}^N} |v_0 (x) - u_0 (x)| dx \right) \\
+ 2 \epsilon \left( \mathbb{E} \| v_0 \|_{BV} + \| u_0 \|_{BV} \right) + 2 \delta \\
+ t \| F' - G' \|_{L_{\infty} (\mathbb{R})} \mathbb{E} \| v_0 \|_{BV} + t \epsilon^{-1} \delta^{-\lambda G_1} + t \left( \delta^{-1} \left\| \Phi - \Psi \right\|_{L_{\infty}} + \delta^{-\lambda G_2} \right) \\
+ \mathbb{E} \| u_0 \|_{BV} + \| v_0 \|_{BV} \int_{|z| \leq r_1} |z|^2 d\mu_\lambda - \mu_\beta (z) \\
+ t \int_{|z| > r_1} \mathbb{E} \left( \| u_0 (\cdot + z) - u_0 \|_{L^1 (\mathbb{T}^N)} + \| v_0 (\cdot + z) - v_0 \|_{L^1 (\mathbb{T}^N)} \right) d\mu_\lambda - \mu_\beta (z)
\]
We can choose \( \lambda_2 = \max\{ \frac{2}{\lambda G_1}, \frac{2}{\gamma b}, 2 \} \) and set
\[
\delta = e^{\lambda_2} = \| \Phi - \Psi \|_{L^\infty(\mathbb{R})} + \frac{1}{\varepsilon} \int_{[0,T]} (|z|^2 d|\mu\lambda - \mu\beta| + \| \sigma - \tau \|_{L^\infty(\mathbb{R})}) d\varepsilon.
\]

If we assume differences are small, then we can conclude that for all \( t \in [0,T] \)
\[
\mathbb{E} \left[ \left| u(x,t) - v(x,t) \right| dx \leq C_T \left( \mathbb{E} \left[ \int_{T^N} |v_0(x) - u_0(x)| dx \right] + \| F' - G' \|_{L^\infty(\mathbb{R})} \right.ight.
\]
\[
+ \left( \| \Phi - \Psi \|_{L^\infty(\mathbb{R})} + \frac{1}{\varepsilon} \int_{[0,T]} (|z|^2 d|\mu\lambda - \mu\beta| + \| \sigma - \tau \|_{L^\infty(\mathbb{R})}) \right)
\]
\[
+ \min \left\{ \frac{1}{2}, \frac{\lambda G_1}{2}, \frac{\lambda G_2}{2}, \frac{\gamma b}{2} \right\}
\]
\[
+ \int_{|z| > r_1} \mathbb{E} (\| u_0(\cdot + z) - u_0 \|_{L^1(T^N)} + \| v_0(\cdot + z) - v_0 \|_{L^1(T^N)} d|\mu\lambda - \mu\beta| (z).
\]
This finishes the proof of Theorem 3.2. \( \square \)

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A. Derivation of the kinetic formulation

In this “Appendix”, we briefly derive the kinetic formulation of equation \((1.1)\) with Lipschitz flux, because we work with the approximations \((5.1)\) of equation \((1.1)\). We show that if \( u \) is a weak solution to \((1.1)\) such that \( u \in L^2(\Omega; C([0,T]; L^2(T^N))) \cap L^2(\Omega; L^2(0,T; H^1(T^N))) \) then \( f(t) = 1_{u(t) > \xi} \) satisfies
\[
df(t) + F' \cdot \nabla f(t) dt - A : D^2 f(t) dt + g_x^\lambda(f(t)) dt = \delta_{\xi}(\lambda_2 = \max\{ \frac{2}{\lambda G_1}, \frac{2}{\gamma b}, 2 \}) \frac{1}{\varepsilon} \Phi dW(t)
\]
\[
+ \partial_\xi (\eta - \frac{1}{2} \beta^2 \delta_{\xi}(\lambda_2 = \max(\{ \frac{2}{\lambda G_1}, \frac{2}{\gamma b}, 2 \})) dt
\]
in the sense of \( \mathcal{D}'(\mathbb{T}^N \times \mathbb{R}) \) where \( \mathbb{P} \)-almost surely \( \eta \geq \eta_1 + \eta_2 \) with

\[
d\eta_1(x, t, \xi) = \int_{\mathbb{R}^N} |u(x + z) - \xi| \mathbb{1}_{\text{Conv} \{u(x,t),u(x+z,t)\}}(\xi) \mu(z) dz d\xi dxt,
\]

\[
d\eta_2(x, t, \xi) = |\text{div} \int_0^u \sigma(\zeta) d\zeta|^2 \delta_{u(x,t)}(\zeta) dxt.
\]

Indeed, it follows from generalized Itô formula [19, Appendix A], for \( \phi \in C^2_b(\mathbb{R}) \) such that \( \phi(-\infty) = 0 \) & \( \phi' \in C^\infty(\mathbb{R}) \), and \( \kappa \in C^2(\mathbb{T}^N) \), \( \mathbb{P} \)-almost surely,

\[
\langle \phi(u(t)), \kappa \rangle = \langle \phi(u_0), \kappa \rangle - \int_0^t \langle \phi'(u) \text{div}(F(u)), \kappa \rangle ds - \int_0^t \langle \phi'(u) g^\kappa_x(u), \kappa \rangle ds
\]

\[- \int_0^t \langle \phi''(u) \nabla u \cdot (A(u) \nabla u), \kappa \rangle ds + \int_0^t \langle \text{div}(\phi'(u) A(u) \nabla u), \kappa \rangle ds
\]

\[+ \sum_{k \geq 1} \int_0^t \langle \phi'(u) \beta_k(x, u, \kappa) \rangle dw_k(s) + \frac{1}{2} \int_0^t \langle \phi''(u) \beta^2(x, u, \kappa) \rangle ds.
\]

Since \( g^{\lambda / 2} = (-\Delta)^{\lambda / 2} \) is continuous linear operator from \( H^1(\mathbb{T}^N) \) to \( L^2(\mathbb{T}^N) \), it implies that

\[
\langle \phi'(u) \rangle g^{\lambda / 2}_x(u, \kappa) = \langle g^{\lambda / 2}_x(\phi'(u) \kappa), g^{\lambda / 2}_x(u) \rangle = \lim_{\delta \to 0} \langle g^{\lambda / 2}_x(\phi'(u) \kappa), g^{\lambda / 2}_x(u^\delta) \rangle
\]

\[= \lim_{\delta \to 0} \langle \phi'(u^\delta) \rangle g^{\lambda / 2}_x(u^\delta, \kappa),
\]

whenever \( u^\delta \to u \) in \( H^1(\mathbb{T}^N) \) with \( u^\delta \in C^\infty(\mathbb{T}^N) \). We obtain \( \mathbb{P} \)-almost surely, almost every \( t \in [0, T] \),

\[
\langle \phi'(u(\cdot, t)) \rangle g^{\lambda / 2}_x(u(\cdot, t), \kappa) = \lim_{\delta \to 0} \langle \phi'(u^\delta(\cdot, t)) \rangle g^{\lambda / 2}_x(u^\delta(\cdot, t), \kappa),
\]

Here \( u^\delta \) is pathwise mollification of \( u \) in space variable. By making use of Taylor’s identity, we have

\[
\langle \phi'(u^\delta(\cdot, t)) \rangle \int_{\mathbb{R}^N} (u^\delta(\cdot + z, t) - u^\delta(\cdot, t)) \mu(z) dz, \kappa
\]

\[= \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \phi'(u^\delta(x, t))(u^\delta(x + z, t) - u^\delta(x, t)) \kappa(x) \mu(z) dz dx
\]

\[= \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \kappa(x) \left( \int_{\mathbb{R}} (\phi'(\xi) \mathbb{1}_{u^\delta(x+z,t) > \xi} - \phi'(\xi) \mathbb{1}_{u^\delta(x,t) > \xi}) d\xi
\]

\[- \int_{\mathbb{R}} \phi''(\xi) u^\delta(x + z, t) - \xi \mathbb{1}_{\text{Conv} \{u^\delta(x+z,t),u^\delta(x,t)\}} \right) \mu(z) dz dx
\]

\[= \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \kappa(x) \phi''(\xi) \int_{\mathbb{R}} \mu(z) dz dx
\]

\[- \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} \kappa(x) \phi''(\xi) \int_{\mathbb{R}} \mu(z) dz dx,
\]

It implies that \( \mathbb{P} \)-almost surely, for all \( t \in [0, T] \)
\[
\int_0^t \langle \phi'(u(x,s))g^\lambda_x(u)(x,s), \kappa \rangle ds \\
= - \lim_{\delta \to 0} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \phi'(\xi) \mathbb{I}_{u^\delta(x,s) > \xi} \int_{\mathbb{R}^N} (\kappa(x + z) - \kappa(x)) \mu(z) d\xi dz dx ds \\
+ \lim_{\delta \to 0} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \kappa(x) \phi''(\xi) \int_{\mathbb{R}^N} \vert u^\delta(x + z, s) \\
- \xi \mathbb{I}_{\text{Conv}(u^\delta(x+z,s), u^\delta(x,s))} (\xi) \mu(z) d\xi dz dx ds \\
= - \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \phi'(\xi) \mathbb{I}_{u(x,t) > \xi} \int_{\mathbb{R}^N} (\kappa(x + z) - \kappa(x)) \mu(z) d\xi dz dx ds \\
+ \lim_{\delta \to 0} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \kappa(x) \phi''(\xi) d\eta^\delta(x, \xi, s) \\
= \int_0^t \langle \mathbb{I}_{u(x,s) > \xi}, \phi' \rangle ds - \langle \kappa(x) \phi'(\xi), \partial_\xi \eta' \rangle [0, t]
\]
where \( \eta' \) is weak-* limit of \( \eta^\delta = \int_{\mathbb{R}^N} \vert u^\delta(x + z, t) - \xi \mathbb{I}_{\text{Conv}(u^\delta(x+z,t), u^\delta(x,t))} (\xi) \mu(z) d\xi \) in \( \mathcal{M}^+(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \). Note that above convergence holds for all \( t \in [0, T] \) due to \( u \in L^2(\Omega; \mathcal{C}([0, T]; L^2(\mathbb{T}^N))) \). As consequence of Fatou’s lemma we have \( \mathbb{P} \)-almost surely, \( \eta' \geq \eta_1 \). Now we proceed remaining terms and apply the chain rule for functions from Sobolev spaces. We conclude that the following identity are true in \( \mathcal{D}'(\mathbb{T}^N) \),
\[
\langle \mathbb{I}_{u(x,t) > \xi}, \phi' \rangle = \int_{\mathbb{R}} \mathbb{I}_{u(x,t) > \xi} \phi'(\xi) d\xi = \phi(u(x,t)), \\
\phi'(u(x,t)) \text{div}(F(u(x,t))) = \phi'(u(x,t))F'(u(x,t)) \nabla u(x,t) \\
\quad = \text{div} \left( \int_{-\infty}^{u(x,t)} F'(\xi) \phi'(\xi) \right) = \text{div} \langle F' \mathbb{I}_{u(x,t) > \xi}, \phi' \rangle, \\
\phi''(u) \nabla u \cdot (A(u) \nabla u) = - \langle \partial_\xi \eta_2, \phi' \rangle, \\
\text{div}(\phi'(u)A(u) \nabla u) = D^2 : \left( \int_{-\infty}^{u} A(\xi) \phi'(\xi) d\xi \right) = D^2 : \langle A \mathbb{I}_{u(x,t) > \xi}, \phi' \rangle, \\
\phi'(u(x,t)) \beta_k(x, u(x,t)) = \langle \beta_k \delta_{u(x,t) = \xi}, \phi' \rangle, \\
\phi''(u(x,t)) \beta^2(x, u(x,t)) = \langle \beta^2 \delta_{u(x,t) = \xi}, \phi'' \rangle = - \langle \partial_\xi (\beta^2 \delta_{u(x,t) = \xi}), \phi' \rangle.
\]
Therefore, we define \( \phi(\xi) = \int_{-\infty}^{\xi} \theta(\xi) d\zeta \), for some \( \theta \in C_c^\infty(\mathbb{R}) \) and \( \eta = \eta' + \eta_2 \) to obtain the result.

**B. Existence of viscous solution**

In this section, we discuss the existence of the viscous solution. We approximate viscous equation (5.1) by the following regularized equation
\[
du^{\tau,\varepsilon}(x,t) + \text{div}(F^\tau(u^{\tau,\varepsilon}(x,t))) dt + g^\lambda_x[u^{\tau,\varepsilon}(x,t)] dt \\
= \text{div} \left( A^{\tau,\varepsilon}(u^{\tau,\varepsilon}) \nabla u^{\tau,\varepsilon} \right) - \varepsilon \Delta^2 u^{\tau,\varepsilon}
\]
\[ A^{\tau,\varepsilon}(u^{\tau,\varepsilon}) = A^{\varepsilon}(u^{\tau,\varepsilon}) + \tau \Delta u^{\tau,\varepsilon}, \quad A^{\varepsilon} \text{ and } \Phi^{\varepsilon} \text{ are smooth approximations of } A \text{ and } \Phi, \text{ respectively, with bounded derivatives.} \]

**Theorem B.1.** (Existence) Let \( u_0 \in L^p(\Omega; C^5(T^N)) \), for all \( p \geq 1 \). For any \( \tau, \varepsilon > 0 \), there exists a unique strong solution \( u^{\tau,\varepsilon} \) to (B.1) such that
\[
\begin{align*}
\|u^{\tau,\varepsilon}\|_{L^p(\Omega; C^{4,\alpha}(T^N))}, \quad \forall \alpha(0,1), \quad \forall p \in [1,\infty).
\end{align*}
\]

**Proof.** The second order term can be rewritten in the following manner
\[
\text{div}(A^{\tau,\varepsilon}(u^{\tau,\varepsilon}) \nabla u^{\tau,\varepsilon}) = \sum_{i,j=1}^N \partial^2_{x_i x_j} A^{\tau,\varepsilon,1}_{i,j}(u^{\tau,\varepsilon}), \quad A^{\tau,\varepsilon,1}_{i,j}(\xi) = \int_0^\xi A^{\tau,\varepsilon}_{i,j}(\zeta) d\zeta.
\]

To prove the existence and uniqueness of \( u^{\tau,\varepsilon} \), one can follow similar lines as in the proof of [22, Corollary 2.2] and [19, Theorem 4.1]. Here, we can notice that there is an additional fractional Laplacian term \( g^\lambda_x(u^{\tau,\varepsilon}) \) compared to [19, equation 4.1]. Therefore, we have to do a little more work to get the required estimates for fixed point argument (cf. [22, Proposition 4.1]) and regularity results (cf. [22, Propositions 4.2 & 4.3]). Here we are giving only the details of estimates which are corresponding to the fractional Laplacian term. The following additional estimates are enough to get desired results. For details of notations we also refer to [22]. In our case, \(-A = \Delta^2, l = 2, \delta = \frac{2l-1}{2l} = \frac{3}{4}\) and
\[
\mathcal{F}(u) = -\text{div}(A^{\tau,\varepsilon}(u) \nabla u) + g^\lambda_x(u) + \text{div}(F^\tau(u)) = -\sum_{i,j=1}^N \partial^2_{x_i x_j} A^{\tau,\varepsilon,1}_{i,j}(u) + \sum_{i=1}^N \partial_{x_i} F^\tau_i(u) + g^\lambda_x(u),
\]

Let \( S_p \) denotes the semigroup generated by \( A \) on \( L^p(T^N) \) for all \( p \geq 2 \).

**Estimates:** Operators
\[
(-A)^{-\delta} : L^p(T^N) \to W^{2l\delta,p}(T^N), \quad v \to (-A)^{-\delta}(v)
\]
and
\[
g^\lambda_x = (-\Delta)^\lambda : W^{2\lambda,p}(T^N) \to L^p(T^N), \quad v \to (-\Delta)^\lambda(v)
\]
are bounded (see, [35, Section 2.6]). Therefore these implies that there exists a positive constant \( C \) such that for all \( v \in L^p(T^N) \),
\[
\|(-A)^{\delta}(v)\|_{W^{2l\delta,p}(T^N)} \leq C\|v\|_{L^p(T^N)},
\]
and for all \( v \in W^{2\lambda,p}(T^N) \),
\[
\|(-\Delta)^\lambda(v)\|_{L^p(T^N)} \leq C\|v\|_{W^{2\lambda,p}(T^N)}.
\]
By making use of above bounds and [22, inequality (2.1)], we conclude that for all \( m \in \mathbb{N} \cup \{0\}, \) for all multi-indices \( \beta \) with \( |\beta| = m, \) for all \( p \geq 2, \) for all \( u \in W^{m,p}(\mathbb{T}^N), \) for all \( t \in [0,T], \)

\[
\| D^\beta S_p(t)(-\Delta)^\lambda u \|_{L^p(\mathbb{T}^N)} = \| D^\beta S_p(t)(-A)^\delta (-\Delta)^\lambda u \|_{L^p(\mathbb{T}^N)} \\
= \| (-A)^\delta S_p(t)(-\Delta)^\lambda (A)^{-\delta} D^\beta u \|_{L^p(\mathbb{T}^N)} \\
\leq C t^{-\delta} \| (-\Delta)^\lambda (A)^{-\delta} D^\beta u \|_{L^p(\mathbb{T}^N)} \\
\leq C t^{-\delta} \| (A)^{-\delta} D^\beta u \|_{W^{2\lambda,p}(\mathbb{T}^N)} \\
\leq C t^{-\delta} \| D^\beta u \|_{L^p(\mathbb{T}^N)}.
\]

It implies that for all \( m \in \mathbb{N} \cup \{0\}, \) for all \( p \geq 2, \) for all \( t \in [0,T], \)

\[
\| S_p(t)(-\Delta)^\lambda u \|_{W^{m,p}(\mathbb{T}^N)} \leq C t^{-\delta} \| u \|_{W^{m,p}(\mathbb{T}^N)}.
\]

Bound (B.2) is enough to get the required estimates for fixed point argument and regularity results. The other details are left to the interested reader. \( \square \)

**B.1. Passing to limit as \( \varepsilon \to 0 \)**

Passing to the limit as \( \varepsilon \to 0 \) in equation (B.1) gives the existence of unique weak-solution to (5.1). The proof of [19, Propositions 4.4] as well as all the proofs in [19, Subsections 4.1, 4.2, 4.3] can be repeated with only minor modifications and consequently the following result deduced.

**Theorem B.2.** For any \( \tau > 0, \) there exists a unique weak solution \( u^\tau \) to (5.1) such that

\[
\begin{align*}
  u^\tau &\in L^2(\Omega; C([0,T]; L^2(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; H^1(\mathbb{T}^N))) \\
  &\quad \cap L^p(\Omega; L^{\infty}_w(0,T; L^p(\mathbb{T}^N))).
\end{align*}
\]

**Proof.** Here we are giving an only sketch of proof for the sake of completeness. Following the approach of [19, Section 4], we obtain

(1) for all \( r \in (0,1/2), \) there exists constant \( C>0, \) independent of \( \varepsilon \) such that

\[
\mathbb{E} \| u^{\tau,\varepsilon} \|_{C^r([0,T]; H^{-3}(\mathbb{T}^N))} \leq C, \quad \mathbb{E} \| u^{\tau,\varepsilon} \|_{L^2(0,T; H^1(\mathbb{T}^N))} \leq C,
\]

(2) for any \( \tau > 0, \) the laws of \( \{u^{\tau,\varepsilon}; \varepsilon \in (0,1)\} \) form a tight sequence on

\[
L^2(0,T; L^2(\mathbb{T}^N)) \cap C([0,T]; H^{-4}(\mathbb{T}^N)),
\]

by using compact embeddings,

(3) there exists \( ((\tilde{\Omega}, \tilde{F}, (\tilde{F}_t), \tilde{P}), \tilde{W}, \tilde{u}^\tau) \) that is weak martingale solution to (5.1) by using stochastic compactness,

(4) Gyöngy-Krylov characterization implies that there exists

\[
  u^\tau \in L^2(\Omega; C([0,T]; L^2(\mathbb{T}^N))) \cap L^2(\Omega; L^2(0,T; H^1(\mathbb{T}^N)))
\]

that is weak solution to (5.1). \( \square \)
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