A Tighter Relation Between Hereditary Discrepancy & Determinant Lower Bound

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Joint work with Haotian Jiang

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Outline of the talk

- Introduction to discrepancy and the determinant lower bound
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- Summary of previous results
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▶ Introduction to discrepancy and the determinant lower bound
▶ Summary of previous results
▶ Our contribution
▶ Open problems
Discrepancy as rounding

- Given $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^n$, how can we round $y$ to $x \in \mathbb{Z}^n$ so that $Ax \approx Ay$?
Discrepancy as rounding

- Given $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^n$, how can we round $y$ to $x \in \mathbb{Z}^n$ so that $Ax \approx Ay$?

- How much error do we necessarily incur in this rounding?
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- Denote $\|z\|_\infty := \max_{i \in [n]} |z_i|$ and define the linear discrepancy

\[
\text{lindisc}(A) := \max_{y \in \mathbb{R}^n} \text{lindisc}(A, y) := \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{Z}^n} \|Ax - Ay\|_\infty
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(Li-Nikolov, 2020) NP-hard to compute. Can we approximate it?
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Determinant lower bound

\[ \text{lindisc}(A) = \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{Z}^n} \|Ax - Ay\|_\infty \]

**Theorem (Lovász-Spencer-Vesztergombi, 1986)**

For any square \( A \in \mathbb{R}^{n \times n} \) we have \( \text{lindisc}(A) \geq \frac{1}{2} |\det(A)|^{1/n} \).
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- By definition, \( \text{lindisc}(A) \cdot K + \mathbb{Z}^n = \mathbb{R}^n \implies \text{vol}_n(\text{lindisc}(A) \cdot K) \geq 1 \)
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- It remains to note \( \text{vol}_n(K) = |\det(A)|^{-1} \cdot \text{vol}([-1, 1]^n) = 2^n / |\det(A)| \).
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What about an upper bound?
Hereditary discrepancy

Define the *hereditary discrepancy* 

$$\text{herdisc}(A) := \max_{S \subseteq [n]} \min_{x \in \{-1, 1\}^S} \| A_S x \|_\infty,$$

where $A_S$ is the submatrix of $A$ with columns from $S$. 
Hereditary discrepancy

Define the hereditary discrepancy

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For any \(A \in \mathbb{R}^{m \times n}\) we have \(\text{lindisc}(A) \leq \text{herdisc}(A)\).
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- First step: \( \text{lindisc}(A, y) \leq \frac{1}{2} \text{herdisc}(A) \) for \( y \in \frac{1}{2} \mathbb{Z}^n \)
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- Round non-integer coordinates \(S := \{i : y_i \notin \mathbb{Z}\}\) based on \(x \in \{-1,1\}^S\).
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Alternatively: \(\frac{1}{2} \mathbb{Z}^n \subseteq \frac{1}{2} \text{herdisc}(A) \cdot K + \mathbb{Z}^n\)
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Alternatively: \( \frac{1}{2} \mathbb{Z}^n \subseteq \frac{1}{2} \text{herdisc}(A) \cdot K + \mathbb{Z}^n \)

- Second step: Show this implies \( \mathbb{R}^n \subseteq \text{herdisc}(A) \cdot K + \mathbb{Z}^n \).
\( l \text{indisc} \leq \text{herdisc} \)

Second step: for any closed convex \( K \subset \mathbb{R}^n \),

\[
\frac{1}{2} \mathbb{Z}^n \subseteq K + \mathbb{Z}^n \implies \mathbb{R}^n \subseteq 2K + \mathbb{Z}^n.
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lindisc \leq herdisc

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$$\frac{1}{2} \mathbb{Z}^n \subseteq K + \mathbb{Z}^n \implies \mathbb{R}^n \subseteq 2K + \mathbb{Z}^n.$$
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Corollary

- We showed for any matrix $A$ we have $\text{herdisc}(A) \geq \text{lindisc}(A)$
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For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \geq \frac{1}{2} \cdot \max_{(S, T) \subseteq [m] \times [n] \atop |S|=|T|=k} |\det(A_{S,T})|^{1/k}$. 
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- Denote $\text{detLB}(A) := \max_{k \in \mathbb{N}} \max_{(S,T) \subseteq [m] \times [n]} |\det(A_{S,T})|^{1/k}$
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- How tight is the bound $\text{herdisc} \gtrsim \text{detLB}$?
Corollary

- We showed for any matrix $A$ we have $\text{herdisc}(A) \geq lindisc(A)$
- In particular, $\text{herdisc}(A) \geq lindisc(A_{S,T})$ for every $(S, T) \subseteq [m] \times [n]$

**Corollary**

For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \geq \frac{1}{2} \cdot \max_{k \in \mathbb{N}} \max_{(S, T) \subseteq [m] \times [n]} |\det(A_{S,T})|^{1/k}$.

- Denote $\detLB(A) := \max_{k \in \mathbb{N}} \max_{(S, T) \subseteq [m] \times [n]} |\det(A_{S,T})|^{1/k}$
- How tight is the bound $\text{herdisc} \succeq \detLB$?

**Theorem (Matoušek, 2011)**

For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \lesssim \sqrt{\log m \cdot \log n \cdot \detLB(A)}$. 
Matoušek’s Bound

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Combination of two results involving the *hereditary vector discrepancy*:
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**Theorem (Bansal, 2010)**

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**Matoušek’s lemma (2011)**

For any $A \in \mathbb{R}^{m \times n}$ we have $\detLB(A) \gtrsim \text{hervecdisc}(A)/\sqrt{\log n}$. 
Our contribution

**Theorem (Jiang-R., 2021)**
For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \lesssim \sqrt{\log m \cdot \log n} \cdot \text{detLB}(A)$.

Combination of two results involving partial *hereditary vector discrepancy*:

**Theorem (Bansal, 2010), slight adaptation**
For any $A \in \mathbb{R}^{m \times n}$ we have $\text{herdisc}(A) \lesssim \sqrt{\log m \cdot \log n} \cdot \text{herpvdisc}(A)$.

**Key lemma**
For any $A \in \mathbb{R}^{m \times n}$ we have $\text{detLB}(A) \gtrsim \text{herpvdisc}(A)$. 
Partial vector discrepancy

Given $A \in \mathbb{R}^{m \times n}$, the partial vector discrepancy is given by the SDP

$$\begin{align*}
\min \lambda \\
\left\| \sum_{j=1}^{n} a_{ij} v_j \right\|_2 \leq \lambda \quad \forall i \in [m] \\
\sum_{j=1}^{n} \|v_j\|_2^2 \geq n/2 \\
\|v_j\|_2^2 \leq 1 \quad \forall j \in [n].
\end{align*}$$
Partial vector discrepancy

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\| \sum_{j=1}^{n} a_{ij} v_j \|_2 & \leq \lambda \quad \forall \ i \in [m] \\
\sum_{j=1}^{n} \| v_j \|_2^2 & \geq n/2 \\
\| v_j \|_2^2 & \leq 1 \quad \forall \ j \in [n].
\end{align*}
\]

In order to show \( \det LB \gtrsim \lambda \), suffices to beat any dual feasible solution
Dual partial vector discrepancy SDP

The dual SDP is given by

\[
\begin{align*}
\max & \quad n\gamma - \sum_{j=1}^{n} z_j \\
\sum_{i=1}^{m} w_i a_i a_i^\top + \sum_{j=1}^{n} z_j e_j e_j^\top & \succeq 2\gamma \cdot I_n \\
\sum_{i=1}^{m} w_i & = 1 \\
w, z & \geq 0.
\end{align*}
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Here \( \lambda^2 = n\gamma - \sum_{j=1}^{n} z_j \) for some feasible \((w, z, \gamma)\).
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Idea: find a submatrix with large singular values, therefore large det
Proof sketch

The dual SDP is given by

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\sum_{i=1}^{m} w_i = 1 \\
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Here \( \lambda^2 = n \gamma - \sum_{j=1}^{n} z_j \) for some feasible \((w, z, \gamma)\).

\( J := \{j \in [n] : z_j < 1.5 \gamma\} \) so that \(|J| \geq n/3\) and \(2 \gamma - z_j > 0.5 \gamma\) for \(j \in J\).
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It follows all eigenvalues of $\sum_{i=1}^{m} w_i a_{i,J} a_{i,J}^\top$ are $> 0.5 \gamma \geq 0.5 \cdot \lambda^2 / n$. 
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- Therefore $\det(\sum_{i=1}^{m} w_i a_{i,J} a_{i,J}^T) \geq (0.5\lambda^2 / n)^{|J|}$
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- Cauchy-Binet also gives

$$\det(\sum_{i=1}^{m} w_i a_{i,J} a_{i,J}^T) = \sum_{I \subseteq \{m\}, |I| = |J|} \det(A_{I,J})^2 \prod_{i \in I} w_i$$
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$$\leq \detLB(A)^{2|J|} \cdot \sum_{I \subseteq \{m\}, |I|=|J|} \prod_{i \in I} w_i$$
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\leq \detLB(A)^{2|J|} \cdot \frac{1}{|J|!} \cdot \left( \sum_{i=1}^{m} w_i \right)^{|J|}
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\[=1\]
Proof sketch

- It follows all eigenvalues of $\sum_{i=1}^{m} \omega_i a_{i,J} a_{i,J}^\top$ are $> 0.5\gamma \geq 0.5 \cdot \lambda^2 / n$.

- Therefore $\det(\sum_{i=1}^{m} \omega_i a_{i,J} a_{i,J}^\top) \geq (0.5\lambda^2 / n)^{|J|}$.

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$$\leq \detLB(A)^{2|J|} \cdot \sum_{I \subseteq \{m\}, |I| = |J|} \prod_{i \in I} \omega_i$$

$$\leq \detLB(A)^{2|J|} \cdot \frac{1}{|J|!} \cdot \left(\sum_{i=1}^{m} \omega_i\right)^{|J|}$$

- Combining the two inequalities, $\detLB(A) \gtrsim \lambda \cdot \sqrt{|J|/n} \gtrsim \lambda$. 
Open problems

▶ Is it possible to approximate $\text{detLB}$ up to $\Theta(1)$ in poly time?

▶ We showed $\text{detLB} \gtrapprox \text{herpvdisc}$. Is it true $\text{detLB} \lesssim \text{herpvdisc}$?

▶ Is it true that $\text{herdisc}(A) \lesssim (p \log m + \log n) \cdot \text{detLB}(A)$?
Open problems

- Is it possible to approximate $\text{detLB}$ up to $\Theta(1)$ in poly time?
- We showed $\text{detLB} \succeq \text{herpvdisc}$. Is it true $\text{detLB} \preceq \text{herpvdisc}$?
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- We showed $\text{detLB} \gtrapprox \text{herpvdisc}$. Is it true $\text{detLB} \lesssim \text{herpvdisc}$?
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Thanks for your attention!