Multiple Solutions for a Quasilinear Schrödinger Equation on $\mathbb{R}^N$

Claudianor O. Alves · Giovany M. Figueiredo

Received: 11 June 2013 / Accepted: 13 May 2014 / Published online: 27 May 2014 © Springer Science+Business Media Dordrecht 2014

Abstract The multiplicity of positive weak solutions is established for quasilinear Schrödinger equations

$$-L_p u + (\lambda A(x) + 1)|u|^{p-2} u = h(u) \text{ in } \mathbb{R}^N,$$

where $L_p u = \epsilon p \Delta_p u + \epsilon p \Delta_p (u^2) u$, $A$ is a nonnegative continuous function and nonlinear term $h$ has a subcritical growth. We achieved our results by using minimax methods and Lusternik-Schnirelman theory of critical points.

Keywords Quasilinear Schrödinger equation · Solitary waves, $p$-Laplacian · Variational method · Lusternik-Schnirelman theory

Mathematics Subject Classification (2000) 35J20 · 35J60 · 35Q55

1 Introduction

In this paper we establish existence and multiplicity of positive weak solutions for the following class of quasilinear Schrödinger equations:

$$-L_p u + (\lambda A(x) + 1)|u|^{p-2} u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \quad (P_{\epsilon,\lambda})$$

where

$$L_p u = \epsilon p \Delta_p u + \epsilon p \Delta_p (u^2) u,$$

$\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian, $\epsilon, \lambda$ are positive parameters, $1 < p < N$ and function $A: \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:
(A1) $A \in C^1(\mathbb{R}^N, \mathbb{R})$, $A(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\Omega = \text{int} A^{-1}(0)$ is a nonempty bounded open set with smooth boundary $\partial \Omega$ and $0 \in \Omega$. Moreover, $A^{-1}(0) = \overline{\Omega} \cup D$ where $D$ is a set of measure zero.

(A2) There exists $K_0 > 0$ such that

\[
\mu(\{x \in \mathbb{R}^N : A(x) \leq K_0\}) < \infty,
\]

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^N$.

On the nonlinearity $h$, we assume that it is of class $C^1$ and satisfies the following conditions:

(H1) $h'(s) = o(|s|^{p-2})$ at the origin;

(H2) $\lim_{|s| \to \infty} h'(s)|s|^{-q+2} = 0$ for some $q \in (2p, 2p^*)$ where $p^* = Np/(N - p)$;

(H3) There exists $\theta > 2p$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$.

(H4) The function $s \to h(s)/s^{2p-1}$ is increasing for $s > 0$.

A typical example of a function satisfying the conditions (H1)–(H4) is given by $h(s) = s^\mu$ for $s \geq 0$, with $2p - 1 < \mu < q - 1$, and $h(s) = 0$ for $s < 0$.

For $p = 2$, the solutions of $(P_{\epsilon, \lambda})$ are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

\[
i \partial_t \psi = -\Delta \psi + V(x) \psi - \tilde{h}(|\psi|^2) \psi - \kappa \Delta \left( \rho(|\psi|^2) \right) \rho'(\rho(|\psi|^2)) \psi,
\]

(1.1)

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V$ is a given potential, $\kappa$ is a real constant and $\rho, \tilde{h}$ are real functions. Quasilinear equations of the form (1.1) have been studied in relation with some mathematical models in physics. For example, when $\rho(s) = s$, the above equation is

\[
i \partial_t \psi = -\Delta \psi + V(x) \psi - \kappa \Delta \left( |\psi|^2 \right) \psi - \tilde{h}(|\psi|^2) \psi.
\]

(1.2)

It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.2) and numerics results on this equation are obtained in [9]. In [18], motivated by the nanotubes and fullerene related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.2) and numerics results have been done on domains of disc type, cylinder type and sphere type. The superfluid film equation in plasma physics has also the structure (1.1) for $\rho(s) = s$, see [21].

The general equation (1.1) with various form of quasilinear terms $\rho(s)$ has been derived as models of several other physical phenomena corresponding to various types of $\rho(s)$. For example, in the case $\rho(s) = (1 + s)^{1/2}$, (1.1) models the self-channeling of a high-power ultra short laser in matter, see [10] and [27]. Equation (1.1) also appears in fluid mechanics [20], in the theory of Heisenberg ferromagnets and magnons [35], in dissipative quantum mechanics and in condensed matter theory [25]. The Semilinear case corresponding to $\kappa = 0$ in whole $\mathbb{R}^N$ has been studied extensively in recent years, see for example [16, 19] and references therein.

Putting $\psi(t, x) = \exp(-iFt)u(x)$, $F \in \mathbb{R}$, into (1.2), we obtain a corresponding equation

\[
-\Delta u - \Delta (u^2) u + V(x) u = h(u)
\]

(1.3)

where we have renamed $V(x) - F$ to be $V(x)$, $h(u) = \tilde{h}(u^2)u$ and we assume, without loss of generality, that $\kappa = 1$. 

\[\text{Springer}\]
The quasilinear equation (1.3) in whole $\mathbb{R}^N$ has received special attention in the past several years, see for example the works [1, 4–6, 11–14, 22–24, 26, 33, 34] and references therein. In these papers, we find important results on the existence of nontrivial solutions of (1.3) and a good insight into this quasilinear Schrödinger equation. The main strategies used are the following: the first of them consists in by using a constrained minimization argument, which gives a solution of (1.3) with an unknown Lagrange multiplier $\lambda$ in front of the nonlinear term, see for example [26]. The other one consists in by using a special change of variables to get a new semilinear equation and an appropriate Orlicz space framework, for more details see [11, 13] and [23]. Variational methods combined with Lusternik-Schnirelman category has been considered in [1] and [2] to show existence, multiplicity and concentration of solutions.

In addition to these references, we would like to cite some recent papers that are related with problem $(P_{\epsilon,\lambda})$. For example, in [15] the authors develop a variational approach in $H^1(\mathbb{R}^N)$ for proving the existence of infinitely many solutions. In [17], using variational methods, the authors study results like multibump solutions for a system that is a version of problem $(P_{\epsilon,\lambda})$. In [30], the authors establish the existence of ground state solutions by a minimization argument. In [32] the authors study the asymptotic behavior of the ground state with general nonlinearities. In [36] the results is concerned with constructing nodal radial solutions in $\mathbb{R}^N$ with critical growth. Besides, using the variational method and the concentration compactness method in an Orlicz space, in [37] the authors obtain the existence of a least energy solution which localizes near a potential well.

The present paper was motivated by works [3, 7] and [8], where existence, multiple and concentration of solutions, by using the Lusternik-Schnirelman category, have been established for the following class of $p$-Laplacian equations

$$-\epsilon^p \Delta_p u + (\lambda A(x) + 1)|u|^{p-2} u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \quad (P)$$

by assuming that $A$ verifies conditions $(A_1)$–$(A_2)$, $h$ is a continuous functions with subcritical growth, $\epsilon$ is sufficiently small and $\lambda$ is large enough. In those papers, it is proved that there exists $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$ there exists $\lambda^*(\epsilon) > 0$ such that $(P)$ has at least $cat(\Omega)$ solutions for any $\lambda \geq \lambda^*(\epsilon)$. In [7, 8], the authors considered the case $p = 2$, while that [3] studied the case $p \geq 2$.

In this work, we show that the same type of results found in [3] for $p$-Laplacian also hold for operator $L_p$. However, due to the presence of the term $\Delta_p(u^2)u$ in $L_p u$, several estimates used [3] can not be repeated for the functional energy associated to $(P_{\epsilon,\lambda})$, given by

$$J_{\epsilon,\lambda}(u) = \frac{1}{p} \int_{\mathbb{R}^N} \epsilon^p (1 + 2^{p-1}|u|^p)|\nabla u|^p + \frac{1}{p} \int_{\mathbb{R}^N} (\lambda A(x) + 1)|u|^p - \int_{\mathbb{R}^N} H(u),$$

where $H(s) = \int_0^s h(t)dt$. As observed in [28] and [29], there are some technical difficulties to apply directly variational methods to $J_{\epsilon,\lambda}$. The main difficult is related to the fact that $J_{\epsilon,\lambda}$ is not well defined in $W^{1,p}(\mathbb{R}^N)$. By a direct computation, if $u \in C^1_0(\mathbb{R}^N \setminus \{0\})$ is defined by

$$u(x) = |x|^{(p-N)/2p} \quad \text{for } x \in B_1 \setminus \{0\},$$

then $u \in W^{1,p}(\mathbb{R}^N)$ while the function $|u|^p|\nabla u|^p$ does not belong to $L^1(\mathbb{R}^N)$. To overcome this difficulty, we use a change variable developed in [28] and [29], which generalizes one found in [23] and in [11] for the case $p = 2$. 

 Springer
Before to state our main result, we recall that if $Y$ is a closed set of a topological space $X$, we denote the Lusternik-Schnirelman category of $Y$ in $X$ by $\text{cat}_X(Y)$, which is the least number of closed and contractible sets in $X$ that cover $Y$. Hereafter, $\text{cat} X$ denotes $\text{cat}_X(X)$.

The main result that we prove is the following:

**Theorem 1.1** Suppose that $(A_1)$–$(A_2)$ and $(H_1)$–$(H_4)$ hold. Then there exists $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$ there exists $\lambda^*(\epsilon) > 0$ such that $(P_{\epsilon, \lambda})$ has at least $\text{cat}(\Omega)$ solutions for any $\lambda \geq \lambda^*(\epsilon)$.

The plan of this paper is as follows. In Sect. 2, we review some proprieties of the change variable that we will apply. Section 3 establishes a compactness result for the energy functional for all sufficiently large $\lambda$ and arbitrary $\epsilon$. In Sect. 4 we study of the behavior of the minimax levels with respect to parameter $\lambda$ and $\epsilon$. Section 5 offers the proof of our main result.

### 2 Variational Framework and Preliminary Results

Since we intend to find positive solutions, let us assume that

$$h(s) = 0 \quad \text{for all } s < 0.$$  

Moreover, hereafter, we will work with the following problem equivalent to $(P_{\epsilon, \lambda})$, which is obtained under change of variable $\epsilon z = x$

$$\begin{cases}
-\Delta_p u - \Delta_p (u^2)u + (\lambda A(\epsilon x) + 1)|u|^{p-2}u = h(u) & \text{in } \mathbb{R}^N \\
u \in W^{1,p}(\mathbb{R}^N) & \text{with } 1 < p < N, \\
u(x) > 0, & \forall x \in \mathbb{R}^N.
\end{cases} 

(P_{\epsilon, \lambda}^*)$$

In what follows, we use the change variable developed in [28] and [29] which generalizes one found in [23] and in [11] for the case $p = 2$. More precisely, let us introduce the change of variables $v = f^{-1}(u)$, where $f$ is defined by

$$f'(t) = \frac{1}{(1 + 2^{p-1}|f(t)|^p)^{1/p}} \quad \text{on } [0, +\infty),$$

$$f(t) = -f(-t) \quad \text{on } (-\infty, 0]. 

(2.1)$$

Therefore, using the above change of variables, we consider a new functional $I_{\epsilon, \lambda}$, given by

$$I_{\epsilon, \lambda}(v) = J_{\epsilon, \lambda}(f(v)) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p + \frac{1}{p} \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v)|^p - \int_{\mathbb{R}^N} H(f(v)). 

(2.2)$$

which is well defined on the Banach space $X_{\epsilon, \lambda}$ defined by

$$X_{\epsilon, \lambda} = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} A(\epsilon x)|f(u)|^p < \infty \right\}$$

endowed with the norm

$$\|u\|_{\epsilon, \lambda} = |\nabla u|_p + \inf_{\xi > 0} \frac{1}{\xi} \left[ 1 + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(\xi u)|^p \right].$$
Here, we would like point out that if $A = 0$, $X_{e,\lambda}$ becomes $W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$
\|u\| = |\nabla u|_p + \inf_{\xi > 0} \frac{1}{\xi} \left[ 1 + \int_{\mathbb{R}^N} |f(\xi u)|^p \right].
$$

Next, we will show that $\|\|$ is equivalent to usual norm $\|\|_{1,p}$ in $W^{1,p}(\mathbb{R}^N)$, given by

$$
\|v\|_{1,p} = \left[ \int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} |v|^p \right]^{\frac{1}{p}}.
$$

To see why, we need to recall some properties of function $f$, whose the proof can be found in [29].

**Lemma 2.1** The function $f$ and its derivative enjoy the following properties:

1. $f$ is uniquely defined, $C^2$ and invertible;
2. $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
3. $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
4. $f(t)/t \to 1$ as $t \to 0$;
5. $|f(t)| \leq 21/2^p |t|^{1/2}$ for all $t \in \mathbb{R}$;
6. $f(t)/2 \leq tf^{1/2}(t) \leq f(t)$ for all $t \geq 0$;
7. $f(t)/\sqrt{t} \to a > 0$ as $t \to +\infty$.
8. there exists a positive constant $C$ such that

$$
|f(t)| \geq \begin{cases} 
C|t|, & |t| \leq 1 \\
C|t|^{1/2}, & |t| \geq 1.
\end{cases}
$$

9. $|f(t)f'(t)| \leq 1/2^{(p-1)/p}$ for all $t \in \mathbb{R}$.

As a by product of the above lemma, we have the ensuing result

**Lemma 2.2** The norms $\|\|$ and $\|\|_{1,p}$ are equivalents in $W^{1,p}(\mathbb{R}^N)$.

**Proof** We will omit the proof that $\|\|$ is a norm, because we can repeat with few modifications, the same arguments used in [28]. From the hypotheses on $f$,

$$
0 \leq |f(t)| \leq |t| \quad \forall t \in \mathbb{R},
$$

this way

$$
\int_{\mathbb{R}^N} |f(\xi v)|^p \leq \xi^p \int_{\mathbb{R}^N} |v|^p \quad \forall \xi \geq 0,
$$

from where it follows that

$$
\inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} |f(\xi v)|^p \right\} \leq \inf_{\xi > 0} \left\{ \frac{1}{\xi} + L\xi^{p-1} \right\}
$$

where

$$
L = \int_{\mathbb{R}^N} |v|^p.
$$
Now, let us consider the function
\[ g(\xi) = \frac{1}{\xi} + L\xi^{p-1} \quad \text{for } \xi > 0. \]

A direct computation implies that \( g \) has a global minimum at some \( \xi_0 > 0 \), which satisfies
\[ g'(\xi_0) = 0 \iff -\xi_0^{-2} + (p-1)L\xi_0^{p-2} = 0. \]

Then,
\[ \xi_0 = \left( \frac{1}{(p-1)L} \right)^{\frac{1}{p}} \]
and so,
\[ g(\xi_0) = \left( L(p-1) \right)^{\frac{1}{p}} + L \left( \frac{1}{(p-1)L} \right)^{\frac{p-1}{p}} = CL^{\frac{1}{p}} \]
for some \( C > 0 \). Using these informations, it follows that
\[ \|v\| \leq |\nabla v|_p + C|v|_p \quad \forall v \in W^{1,p}(\mathbb{R}^N). \]

Hence, there is \( c_1 > 0 \) such that
\[ \|v\| \leq c_1 \|v\|_{1,p} \quad \forall v \in W^{1,p}(\mathbb{R}^N). \]

Since \((W^{1,p}(\mathbb{R}^N), \|\|)\) and \((W^{1,p}(\mathbb{R}^N), \|\|_{1,p})\) are Banach spaces, the last inequality together with Closed Graphic Theorem yields \( \|\| \) and \( \|\|_{1,p} \) are equivalent norms in \( W^{1,p}(\mathbb{R}^N) \). □

Considering the change of variables given in (2.1), and the functional \( I_{\epsilon,\lambda} \), we have the problem
\[
\begin{aligned}
-\Delta_p v + (\lambda A(\epsilon x) + 1) |f(v)|^{p-2} f(v) f'(v) &= h(f(v)) f'(v) \quad \text{in } \mathbb{R}^N \\
v(x) > 0, &\quad \forall x \in \mathbb{R}^N.
\end{aligned}
\]

\((S_{\epsilon,\lambda})\)

In whole this paper, a function \( v : \mathbb{R}^N \to \mathbb{R} \) is called a weak solution of \((S_{\epsilon,\lambda})\) if \( v \in X_{\epsilon,\lambda} \) and for all \( w \in W^{1,p}(\mathbb{R}^N) \),
\[ \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla w + (\lambda A(\epsilon x) + 1) |f(v)|^{p-2} f(v) f'(v) w = \int_{\mathbb{R}^N} h(f(v)) f'(v) w. \]

A direct computation shows that \( I_{\epsilon,\lambda} : X_{\epsilon,\lambda} \to \mathbb{R} \) is of class \( C^1 \) under the conditions \((A_1)-(A_2)\) and \((H_1)-(H_2)\) with
\[
I'_{\epsilon,\lambda}(v) w = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla w + (\lambda A(\epsilon x) + 1) |f(v)|^{p-2} f(v) f'(v) w - \int_{\mathbb{R}^N} h(f(v)) f'(v) w
\]
for \( v, w \in W^{1,p} (\mathbb{R}^N) \). Thus, the critical points of \( I_{\epsilon, \lambda} \) correspond exactly to weak solutions of the problem \((S_{\epsilon, \lambda})\).

The below proposition establishes a relation between the solutions of \((S_{\epsilon, \lambda})\) with one of \((P^*_\epsilon, \lambda)\):

**Proposition 2.3** If \( v \in X_{\epsilon, \lambda} \) is a critical point of \( I_{\epsilon, \lambda} \), then \( v \in W^{1,p} (\mathbb{R}^N) \cap L^\infty_{\text{loc}} (\mathbb{R}^N) \) and \( u = f(v) \) is a weak solution of \((P^*_\epsilon, \lambda)\).

**Proof** See [29]. □

From the above proposition, it is clear that to obtain a weak solution of \((P_{\epsilon, \lambda})\), it is sufficient to obtain a critical point of the functional \( I_{\epsilon, \lambda} \) in \( X_{\epsilon, \lambda} \).

In what follows, let us collect some properties of the change of variables \( f \) defined in (2.1), which will be usual in the sequel of the paper.

The next lemma can be found in [1], however for convenience of the reader we will write its proof.

**Lemma 2.4** Let \((v_n)\) be a sequence in \( W^{1,p} (\mathbb{R}^N) \) verifying
\[ \int_{\mathbb{R}^N} |f(\xi v_n)|^p \to 0 \quad \text{as} \quad n \to \infty. \]
Then,
\[ \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p \right\} \to 0 \quad \text{as} \quad n \to \infty. \]

**Proof** Hereafter, once that \( f \) is odd, we can assume without loss of generality that \( v_n \geq 0 \) for all \( n \in \mathbb{N} \). Since \( f(t)/t \) is nonincreasing for \( t > 0 \), for each \( \xi > 1 \),
\[ \frac{1}{\xi} + \frac{1}{\xi} \int_{\mathbb{R}^N} |f(\xi v_n)|^p \leq \frac{1}{\xi} + \xi^{p-1} \int_{\mathbb{R}^N} |f(v_n)|^p. \]
Hence, for each \( \delta > 0 \), fixing \( \xi_* \) sufficiently large such that \( \frac{1}{\xi_*} < \frac{\delta}{2} \), we get
\[ \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p \right\} \leq \frac{\delta}{2} + \xi_*^{p-1} \int_{\mathbb{R}^N} |f(v_n)|^p. \]
Thus,
\[ \limsup_{n \to \infty} \left( \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} |f(\xi v_n)|^p \right\} \right) \leq \frac{\delta}{2} \quad \text{for all} \quad \delta > 0, \]
which proves the lemma. □

Repeating the same type of arguments explored in the proof of the last lemma, we have the following result which will be used in the proof of Proposition 3.7, see Sect. 3.

**Corollary 2.5** Let \((\lambda_n)\) be a sequence with \( \lambda_n \to +\infty \) and \( v_n \in X_{\epsilon, \lambda_n} \) with
\[ \int_{\mathbb{R}^N} (\lambda_n A(\epsilon x) + 1) |f(v_n)|^p \to 0 \quad \text{as} \quad n \to +\infty. \]
Then
\[ \inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} (\lambda_n A(\epsilon x) + 1) |f(\xi v_n)|^p \right\} \to 0 \quad n \to +\infty. \]
The next lemma is related to a claim made in [1], which wasn’t proved in that paper. Here, we decide to show its proof.

**Lemma 2.6** Let \((v_n)\) be a sequence in \(W^{1,p}(\mathbb{R}^N)\) and set

\[
Q(v) := \int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} |f(v)|^p.
\]

Then, \(Q(v_n) \to 0\) if, and only if, \(\|v_n\| \to 0\). Moreover, \((v_n)\) is bounded in \((W^{1,p}(\mathbb{R}^N), \|\|\)) if, and only if, \((Q(v_n))\) is bounded in \(\mathbb{R}\).

**Proof** The first part of the lemma is an immediate consequence of Lemma 2.4, this way, we will prove only the second part of the lemma.

A straightforward computation gives

\[
\|v\| \leq \left(\frac{1}{p} Q(v) + 1\right)^{\frac{1}{p}} \forall v \in W^{1,p}(\mathbb{R}^N),
\]

from where it follows that if \((Q(v_n))\) is bounded, then \((v_n)\) is also bounded. On the other hand, by Lemma 2.2, \((v_n)\) is a bounded sequence in \((W^{1,p}(\mathbb{R}^N), \|\|_1, p)\), Hence, there is \(M > 0\) such that

\[
\int_{\mathbb{R}^N} |\nabla v_n|^p \leq M \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^p \leq M \quad \forall n \in \mathbb{N}.
\]

Recalling that

\[
|f(t)| \leq |t| \quad \forall t \geq 0,
\]

we have the estimate

\[
\int_{\mathbb{R}^N} |f(v_n)|^p \leq \int_{\mathbb{R}^N} |v_n|^p \leq M \quad \forall n \in \mathbb{N},
\]

which guarantees the boundedness of \((Q(v_n))\). \(\square\)

The next three results will be used to study the behavior of Palais-Smale sequences associated with \(I_{e,\lambda}\).

**Lemma 2.7** The function \(|f|^p\) is a convex function, and so,

\[
(\frac{1}{p} - 2)f(t)f'(t) - |f(t)|^{p-2} f(s) f'(s) (t - s) \geq 0 \quad \forall t, s \in \mathbb{R}.
\]

**Proof** A direct computation shows that second derivative of the function

\[
Q(t) = |f(t)|^p \quad \text{for} \ t \in \mathbb{R}
\]

satisfies the equality

\[
Q''(t) = \frac{p|f(t)|^{p-2} f'(t)^2 ((p - 1) + (p - 2)2^{p-1}|f(t)|^p)}{1 + 2^{p-1}|f(t)|^p} > 0 \quad \forall t \in \mathbb{R} \setminus \{0\},
\]

\(\square\)
implying that $Q$ is a convex function. From this,

\[(Q'(t) - Q'(s))(t - s) \geq 0 \quad \forall t, s \in \mathbb{R}\]

that is,

\[\left(\left|f(t)\right|^{p-2}f(t)f'(t) - \left|f(s)\right|^{p-2}f(s)f'(s)\right)(t - s) \geq 0 \quad \forall t, s \in \mathbb{R},\]

finishing the proof. \[\square\]

**Lemma 2.8** Let $(v_n) \subset W^{1,p}(\mathbb{R}^N)$ be a sequence of nonnegative functions such that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$, $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ and

\[
\int_{\mathbb{R}^N} \left(\left|f(v_n)\right|^{p-2}f(v_n)f'(v_n) - \left|f(v)\right|^{p-2}f(v)f'(v)\right)(v_n - v) \to 0 \quad \text{as } n \to +\infty.
\]

Then,

\[
\int_{\mathbb{R}^N} \left|f(v_n - v)\right|^p \to 0 \quad \text{as } n \to +\infty.
\]

**Proof** By hypothesis,

\[
\int_{\mathbb{R}^N} \left(\left|f(v_n)\right|^{p-2}f(v_n)f'(v_n) - \left|f(v)\right|^{p-2}f(v)f'(v)\right)(v_n - v) = o_n(1)
\]

or equivalently,

\[
\int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v_n = \int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v
\]

\[+ \int_{\mathbb{R}^N} \left|f(v)\right|^{p-2}f(v)f'(v)(v_n - v) + o_n(1).
\]

Once that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$,

\[
\int_{\mathbb{R}^N} \left|f(v)\right|^{p-2}f(v)f'(v)(v_n - v) = o_n(1)
\]

and so,

\[
\int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v_n = \int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v + o_n(1).
\]

Recalling that

\[
\left|f(t)\right| \leq |t| \quad \text{and} \quad \left|f'(t)\right| \leq 1 \quad \forall t \in \mathbb{R},
\]

it follows that $(\left|f(v_n)\right|^{p-2}f(v_n)f'(v_n))$ is bounded sequence in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$. Hence,

\[
\int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v \to \int_{\mathbb{R}^N} \left|f(v)\right|^{p-2}f(v)f'(v)v
\]

which gives

\[
\int_{\mathbb{R}^N} \left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v_n \to \int_{\mathbb{R}^N} \left|f(v)\right|^{p-2}f(v)f'(v)v.
\]
From Lemma 2.1,
\[ |f(t)|^p \leq 2|f(t)|^{p-2}f(t)f'(t)t \quad \forall t \geq 0 \]
then,
\[ |f(v_n)|^p \leq |f(v_n)|^{p-2}f(v_n)f'(v_n)v_n \quad \forall n \in \mathbb{N}. \]
Using the above informations together with Lebesgue’s Theorem, we deduce
\[ \int_{\mathbb{R}^N} |f(v_n)|^p \to \int_{\mathbb{R}^N} |f(v)|^p. \]
On the other hand, since \(|f'(t)| \leq 1\) for all \(t \in \mathbb{R}\), we have the inequality
\[ |f(v_n - v)| = f(|v_n - v|) \leq f(|v_n| + |v|) \leq f(|v_n|) + |v| \quad \forall n \in \mathbb{N} \]
which gives
\[ |f(v_n - v)|^p \leq 2^p(|f(v_n)|^p + |v|^p) \quad \forall n \in \mathbb{N}. \]
Combining the last inequality with Lebesgue’s Theorem, we get
\[ \int_{\mathbb{R}^N} |f(v_n - v)|^p \to 0, \]
concluding the proof of the lemma. \(\square\)

**Corollary 2.9** Let \((v_n) \subset W^{1,p}(\mathbb{R}^N)\) be a sequence of nonnegative functions such that \(v_n \to v\) in \(W^{1,p}(\mathbb{R}^N)\), \(v_n(x) \to v(x)\) a.e in \(\mathbb{R}^N\) and the below limits hold
\[ \int_{\mathbb{R}^N} \left( |f(v_n)|^{p-2}f(v_n)f'(v_n) - |f(v)|^{p-2}f(v)f'(v) \right)(v_n - v) \to 0 \quad \text{as } n \to +\infty \quad (2.3) \]
and
\[ \int_{\mathbb{R}^N} \left| \nabla v_n \right|^{p-2}\nabla v_n - \left| \nabla v \right|^{p-2}\nabla v, \nabla v_n - \nabla v \right| \to 0 \quad \text{as } n \to +\infty. \quad (2.4) \]
Then, \(v_n \to v\) in \(W^{1,p}(\mathbb{R}^N)\).

**Proof** By Lemma 2.8, the limit (2.3) leads to
\[ \int_{\mathbb{R}^N} |f(v_n - v)|^p \to 0 \quad \text{as } n \to +\infty. \]
On the other hand, the limit (2.4) implies that
\[ \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \to 0 \quad \text{as } n \to +\infty. \]
The above limits give
\[ Q(v_n - v) \to 0 \quad \text{as } n \to +\infty, \]
and so, by Lemma 2.6
\[ \|v_n - v\| \to 0 \quad \text{as } n \to +\infty \]
or equivalently,

\[ v_n \to v \quad \text{in} \quad W^{1,p}(\mathbb{R}^N), \]

proving the lemma.

\[ \square \]

3 The Palais-Smale Condition

In this section, the main goal is to show that \( I_{\epsilon,\lambda} \) satisfies the Palais-Smale condition. To this end, we have to prove some technical lemmas.

**Lemma 3.1** Suppose that \( h \) satisfies \((H_1)-(H_3)\). Let \( (v_n) \subset X_{\epsilon,\lambda} \) be a \((PS)_c\) sequence for \( I_{\epsilon,\lambda} \). Then there exists a constant \( K > 0 \), independent of \( \epsilon \) and \( \lambda \), such that

\[
\limsup_{n \to \infty} \|v_n\|_{\epsilon,\lambda} \leq K
\]

for all \( \epsilon, \lambda > 0 \). Moreover, \( c \geq 0 \), and if \( c = 0 \), we have that \( v_n \to 0 \) in \( X_{\epsilon,\lambda} \).

**Proof** Using \((H_3)\) and Lemma 2.1(6),

\[
c + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq \left( \frac{1}{p} - \frac{2}{\theta} \right) \left[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(x) + 1) |f(v_n)|^p \right]. \tag{3.1}
\]

where \( o_n(1) \to 0 \) as \( n \to \infty \). Recalling that \( |\nabla v_n|_p \leq 1 + |\nabla v_n|^p, \)

\[
c + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq \frac{\theta - 2p}{p\theta} (|\nabla v_n|^p - 1 + \int_{\mathbb{R}^N} (\lambda A(x) + 1) |f(v_n)|^p) \tag{3.2}
\]

from where it follows the inequality

\[
c_1 + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq \frac{\theta - 2p}{p\theta} (|\nabla v_n|^p + 1 + \int_{\mathbb{R}^N} (\lambda A(x) + 1) |f(v_n)|^p)
\]

\[
\geq \frac{\theta - 2p}{p\theta} \|v_n\|_{\epsilon,\lambda}.
\]

Thus,

\[
\limsup_{n \to \infty} \|v_n\|_{\epsilon,\lambda} \leq c_1 \frac{p\theta}{\theta - 2p} := K.
\]

Moreover, using (3.1) we get

\[
c + o_n(1)\|v_n\|_{\epsilon,\lambda} = I_{\epsilon,\lambda}(v_n) - \frac{2}{\theta} I'_{\epsilon,\lambda}(v_n)v_n \geq \left( \frac{1}{p} - \frac{2}{\theta} \right) \|v_n\|_{\epsilon,\lambda} \geq 0 \tag{3.3}
\]

that is

\[
c + o_n(1)\|v_n\|_{\epsilon,\lambda} \geq 0.
\]

The boundedness of \( (v_n) \) in \( X_{\epsilon,\lambda} \) gives \( c \geq 0 \) after passage to the limit as \( n \to \infty \). If \( c = 0 \), the inequality (3.3) gives \( v_n \to 0 \) in \( X_{\epsilon,\lambda} \) as \( n \to \infty \), finishing the proof of Lemma 3.1. \( \square \)
Lemma 3.2 Suppose that \( h \) satisfies (H1)–(H3). Let \( c > 0 \) and \( (v_n) \) be a \((PS)_c\) sequence for \( I_{\epsilon, \lambda} \). Then, there exists \( \delta > 0 \) such that
\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} |f(v_n)|^q \geq \delta,
\]
with \( \delta \) being independent of \( \lambda \) and \( \epsilon \).

Proof From (H1)–(H2), there exists a constant \( C > 0 \) such that
\[
|h(t)t| \leq \frac{1}{2}|t|^p + C|t|^q \quad \forall t \in \mathbb{R}. \tag{3.4}
\]
Now, \( I'_{\epsilon, \lambda}(v_n)v_n = o_n(1) \) and Lemma 2.1(6) give
\[
\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \leq \int_{\mathbb{R}^N} h(f(v_n))f(v_n). \tag{3.5}
\]
Combining (3.4) with (3.5), we get
\[
\frac{1}{2} \left[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \right] \leq C \int_{\mathbb{R}^N} |f(v_n)|^q,
\]
that is,
\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \leq C \int_{\mathbb{R}^N} |f(v_n)|^q. \tag{3.6}
\]
On the other hand, we have the equality
\[
\frac{1}{p} \left[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \right] = I_{\epsilon, \lambda}(v_n) + \int_{\mathbb{R}^N} H(f(v_n))
\]
which combined with (H3) and \( I_{\epsilon, \lambda}(v_n) = c + o_n(1) \) leads to
\[
\liminf_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1)|f(v_n)|^p \right] \geq pc > 0. \tag{3.7}
\]
Now, the lemma follows from (3.6) and (3.7). \( \Box \)

Lemma 3.3 Suppose that \( h \) satisfies (H1)–(H3) and \( A \) satisfies (A1)–(A2). Let \( d > 0 \) be an arbitrary number. Given any \( \epsilon > 0 \) and \( \eta > 0 \), there exist \( \Lambda_\eta > 0 \) and \( R_\eta > 0 \), which are independent of \( \epsilon \), such that if \( (v_n) \) is a \((PS)_c\) sequence for \( I_{\epsilon, \lambda} \) with \( c \leq d \) and \( \lambda \geq \Lambda_\eta \), then
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_{R_\eta}(0)} |f(v_n)|^q < \eta.
\]

Proof Given any \( R > 0 \), define
\[
X(R) = \{ x \in \mathbb{R}^N : |x| > R; A(\epsilon x) \geq K_0 \}
\]
and
\[
Y(R) = \{ x \in \mathbb{R}^N : |x| > R; A(\epsilon x) < K_0 \},
\]
where \( K_0 \) is the constant that appears in the hypothesis (A2). Observe that

\[
\int_{\mathbb{R}^N} |f(v_n)|^p \leq \frac{1}{\lambda K_0 + 1} \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1) |f(v_n)|^p.
\]

Once that \( (\|v_n\|_{\epsilon, \lambda}) \) is bounded in \( \mathbb{R} \), that is, there is \( K > 0 \) such that

\[
\|v_n\|_{\epsilon, \lambda} \leq K \quad \forall n \in \mathbb{N}.
\] (3.8)

As a consequence of the above boundedness, we have the following claim.

**Claim 3.4** There exists a constant \( K_1 > 0 \), independent of \( \epsilon \), such that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1) |f(v_n)|^p \leq K_1.
\]

Indeed, by definition of \( I_{\epsilon, \lambda} \),

\[
\int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1) |f(v_n)|^p \leq p \left( I_{\epsilon, \lambda}(v_n) + \int_{\mathbb{R}^N} H(f(v_n)) \right) \quad (3.9)
\]

From (H1)–(H3), there are \( c_1, c_2 > 0 \) such that

\[
\int_{\mathbb{R}^N} H(f(v_n)) \leq c_1 \int_{\mathbb{R}^N} |v_n|^p + c_2 \int_{\mathbb{R}^N} |v_n|^{\frac{q}{2}} \quad \forall n \in \mathbb{N}.
\]

Using the continuous Sobolev embedding, there are \( c_3, c_4 > 0 \) verifying

\[
\int_{\mathbb{R}^N} H(f(v_n)) \leq c_3 \|v_n\| + c_4 \|v_n\|^{\frac{q}{2}} \leq c_3 \|v_n\|_{\epsilon, \lambda}^p + c_4 \|v_n\|_{\epsilon, \lambda}^{\frac{q}{2}}.
\]

Thus, by (3.8), there is \( \tilde{K} > 0 \) independent of \( \epsilon \), such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} H(f(v_n)) \leq \tilde{K}. \quad (3.10)
\]

Now, since \( (v_n) \) is a \((PS)_c\) sequence, (3.9) and (3.10) load to

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} (\lambda A(\epsilon x) + 1) |f(v_n)|^p \leq p(c + \tilde{K}),
\]

finishing the proof of the claim.

From Claim 3.4, there exists \( K_1 > 0 \) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |f(v_n)|^p \leq \frac{K_1}{\lambda K_0 + 1}. \quad (3.11)
\]

On the other hand, by Hölder inequality

\[
\int_{\mathbb{R}^N} |f(v_n)|^p \leq \left( \int_{\mathbb{R}^N} |f(v_n)|^{p^*} \right)^{\frac{p}{p^*}} \left( \mu(\mathbb{R}) \right)^{\frac{1}{p^*}}.
\]
Arguing as in (3.11), there exists a constant \( \hat{K} > 0 \) such that
\[
\limsup_{n \to \infty} \int_{Y(R)} |f(v_n)|^p \leq \hat{K} \left( \mu(Y(R)) \right)^\frac{p}{N},
\] (3.12)
where the constant \( \hat{K} \) is uniform on \( c \in [0, d] \). Since
\[ Y(R) \subset \{ x \in \mathbb{R}^N : A(\epsilon x) \leq K_0 \}, \]
it follows from (A2)
\[
\lim_{R \to \infty} \mu(Y(R)) = 0.
\] (3.13)
Now, since \( p < q < 2p^* \) and \( (f(v_n)) \subset L^p(\mathbb{R}^N) \cap L^{2p^*}(\mathbb{R}^N) \), by interpolation,
\[
|f(v_n)|_{L^q(\mathbb{R}^N \setminus B_R(0))} \leq |f(v_n)|_{L^p(\mathbb{R}^N \setminus B_R(0))}^{\alpha} |f(v_n)|_{L^{2p^*}(\mathbb{R}^N \setminus B_R(0))}^{1-\alpha},
\]
for some \( \alpha \in (0, 1) \). Hence, by Lemma 2.1(3),
\[
|f(v_n)|_{L^q(\mathbb{R}^N \setminus B_R(0))} \leq |f(v_n)|_{L^p(\mathbb{R}^N \setminus B_R(0))}^{\alpha} |v_n|_{L^{2p^*}(\mathbb{R}^N \setminus B_R(0))}^{1-\alpha}.
\]
Now, using Lemma 3.1, there exists a constant \( \hat{K} > 0 \) such that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^q \leq \hat{K} \left( \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^p \right)^\frac{q}{p}. \] (3.14)
Combining (3.11) with (3.12) and (3.13), given \( \eta > 0 \), we can fix \( R = R_\eta \) and \( \Lambda_\eta > 0 \) such that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^p \leq \left( \frac{\eta}{2 \hat{K}} \right)^\frac{p}{q}, \] (3.15)
for all \( \lambda \geq \Lambda_\eta \). Consequently, from (3.14) and (3.15),
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^q \leq \eta
\]
concluding the proof of the lemma.

As a first consequence of the last lemma, we have the following result

**Corollary 3.5** If \( (v_n) \) is a \((PS)_c\) sequence for \( I_{\epsilon, \lambda} \) and \( \lambda \) is large enough, then its weak limit is nontrivial provided that \( c > 0 \).

The next result shows that \( I_{\epsilon, \lambda} \) satisfies the Palais-Smale condition for \( \lambda \) sufficiently large and \( \epsilon \) arbitrary.

**Proposition 3.6** Suppose that (H1)–(H3) and (A1)–(A2) hold. Then for any \( d > 0 \) and \( \epsilon > 0 \) there exists \( \Lambda > 0 \), independent of \( \epsilon \), such that \( I_{\epsilon, \lambda} \) satisfies the \((PS)_c\) condition for all \( c \leq d, \lambda \geq \Lambda \) and \( \epsilon > 0 \). That is, any sequence \( (v_n) \subset X_{\epsilon, \lambda} \) satisfying
\[
I_{\epsilon, \lambda}(v_n) \to c \quad \text{and} \quad I'_{\epsilon, \lambda}(v_n) \to 0,
\] (3.16)
for \( c \leq d \), has a strongly convergent subsequence in \( X_{\epsilon, \lambda} \).
Proof Given any \( d > 0 \) and \( \epsilon > 0 \), take \( c \leq d \) and let \((v_n)\) be a \((PS)_\epsilon\) sequence for \( I_{\epsilon,\lambda} \). From Lemma 3.1, there are a subsequence still denoted by \((v_n)\) and \( v \in X_{\epsilon,\lambda} \) such that \((v_n)\) is weakly convergent to \( v \) in \( X_{\epsilon,\lambda} \). If \( \tilde{v}_n = v_n - v \), arguing as in [1] and [31], it follows that

\[
I_{\epsilon,\lambda}(\tilde{v}_n) = I_{\epsilon,\lambda}(v_n) - I_{\epsilon,\lambda}(v) + o_n(1)
\]

and

\[
I'_{\epsilon,\lambda}(\tilde{v}_n) \to 0.
\]

Once that \( I'_{\epsilon,\lambda}(v) = 0 \), \((H_3)\) gives

\[
I_{\epsilon,\lambda}(v) = I_{\epsilon,\lambda}(v) - \frac{2}{\theta} I'_{\epsilon,\lambda}(v)v \geq \left( \frac{1}{p} - \frac{2}{\theta} \right) \|v\|_{\epsilon,\lambda}^p \geq 0.
\]

Setting \( c' = c - I_{\epsilon,\lambda}(v) \), by (3.17)–(3.19), we deduce that \( c' \leq d \) and \((\tilde{v}_n)\) is a \((PS)_{c'}\) sequence for \( I_{\epsilon,\lambda} \), thus by Lemma 3.1, we have \( c' \geq 0 \). We claim that \( c' = 0 \). On the contrary, suppose that \( c' > 0 \). From Lemma 3.2, there is \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |f(\tilde{v}_n)|^q > \delta.
\]

Letting \( \eta = \frac{\delta}{2} \) and applying Lemma 3.3, we get \( \Lambda > 0 \) and \( R > 0 \) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(\tilde{v}_n)|^q < \frac{\delta}{2}
\]

for the corresponding \((PS)_{c'}\) sequence for \( I_{\epsilon,\lambda} \) for all \( \lambda \geq \Lambda \). Combining (3.20) with (3.21) and using the fact that \( \tilde{v}_n \rightharpoonup 0 \) in \( X_{\epsilon,\lambda} \), we derive

\[
\delta \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |f(\tilde{v}_n)|^q \leq \limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(\tilde{v}_n)|^q \leq \frac{\delta}{2}
\]

which is impossible, then \( c' = 0 \). Thereby, by Lemma 3.1, \( \tilde{v}_n \to 0 \) in \( X_{\epsilon,\lambda} \), that is, \( v_n \to v \) in \( X_{\epsilon,\lambda} \) and the proof of Proposition 3.6 is complete. \( \square \)

In closing this section, we proceed with the study of \((PS)_{c,\infty}\) sequences, that is, sequences \((v_n)\) in \( W^{1,p}(\mathbb{R}^N) \) verifying:

(i) \( v_n \in X_{\epsilon,\lambda_n} \)
(ii) \( \lambda_n \to \infty \)
(iii) \((I_{\epsilon,\lambda_n}(v_n))\) is bounded
(iv) \( \|I'_{\epsilon,\lambda_n}(v_n)\|_{\epsilon,\lambda_n}^* \to 0 \)

where \( \|\|_{\epsilon,\lambda_n}^* \) is defined by

\[
\|\phi\|_{\epsilon,\lambda_n}^* = \sup\{ |\phi(u)| ; u \in X_{\epsilon,\lambda_n} , \|u\|_{\epsilon,\lambda_n} \leq 1 \} \quad \text{for} \ \phi \in X_{\epsilon,\lambda_n}^*.
\]

**Proposition 3.7** Suppose that \((H_1)–(H_3)\) and \((A_1)–(A_2)\) hold. Assume that \((v_n) \subset W^{1,p}(\mathbb{R}^N)\) is a \((PS)_{c,\infty}\) sequence. Then for each \( \epsilon > 0 \) fixed, there exists a subsequence still denoted by \((v_n)\) and \( v_{\epsilon} \in W^{1,p}(\mathbb{R}^N) \) such that
(i) \( v_n \to v_\epsilon \) in \( W^{1,p}(\mathbb{R}^N) \). Moreover, \( v_\epsilon = 0 \) on \( \Omega_\epsilon^c \) and \( v_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N) \) is a solution of

\[
\begin{aligned}
-\Delta_p v + |f(v)|^{p-2}f(v)f'(v) &= h(f(v))f'(v), & \text{in } \Omega_\epsilon \\
v > 0 & \text{ in } \Omega_\epsilon \text{ and } v = 0 \text{ on } \partial \Omega_\epsilon
\end{aligned}
\]

where \( \Omega_\epsilon = \Omega^c \).

(ii) \( \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x)|f(v_n)|^p \to 0 \).

(iii) \( \|v_n - v\|_{\epsilon, \lambda_n} \to 0 \).

**Proof** As in the proof of Lemma 3.1, the sequence \((\|v_n\|_{\epsilon, \lambda_n})\) is bounded in \( \mathbb{R} \), that is, there is \( K > 0 \) such that

\[
\|v_n\|_{\epsilon, \lambda_n} \leq K \forall n \in \mathbb{N}.
\]

This way, following the same steps of the Claim 3.4, it is easy to prove the ensuing claim

**Claim 3.8** There exists a constant \( K_1 > 0 \), independent of \( \epsilon \), such that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} \left( \lambda_n A(\epsilon x) + 1 \right) |f(v_n)|^p \leq K_1.
\]

Once that \( \|u\| \leq \|u\|_{\epsilon, \lambda_n} \) for all \( u \in X_{\epsilon, \lambda_n} \), \((\|v_n\|)\) is bounded in \( \mathbb{R} \). Thus, we can extract a subsequence of \((v_n)\), still denote by itself, such that \( v_n \rightharpoonup v_\epsilon \) weakly in \( W^{1,p}(\mathbb{R}^N) \). For each \( m \in \mathbb{N} \), we define the set

\[
C_m = \left\{ x \in \mathbb{R}^N : A_\epsilon(x) \geq \frac{1}{m} \right\}, \text{ where } A_\epsilon(x) = A(\epsilon x).
\]

Then,

\[
\int_{C_m} |f(v_n)|^p \leq m \int_{C_m} A_\epsilon(x) |f(v_n)|^p \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (1 + \lambda_n A_\epsilon(x)) |f(v_n)|^p,
\]

and by Claim 3.8,

\[
\limsup_{n \to +\infty} \int_{C_m} |f(v_n)|^p \leq \frac{mK_1}{\lambda_n} \text{ for } n \in \mathbb{N},
\]

for some constant \( K_1 > 0 \). Hence by Fatou’s Lemma,

\[
\int_{C_m} |f(v_\epsilon)|^p = 0
\]

after to passage to the limit as \( n \to +\infty \). Thus \( f(v_\epsilon) = 0 \) almost everywhere in \( C_m \). Once that \( f(t) = 0 \) if, and only if \( t = 0 \), it follows that \( v_\epsilon = 0 \) almost everywhere in \( C_m \). Observing that

\[
\mathbb{R}^N \setminus A_\epsilon^{-1}(0) = \bigcup_{m=1}^{\infty} C_m,
\]

we deduce that \( v_\epsilon = 0 \) almost everywhere in \( \mathbb{R}^N \setminus A_\epsilon^{-1}(0) \). Now, recalling that \( A_\epsilon^{-1}(0) = \Omega_\epsilon \cup D_\epsilon \) and \( \mu(D_\epsilon) = \mu(D) = 0 \), it follows that \( v_\epsilon = 0 \) almost everywhere in \( \mathbb{R}^N \setminus \overline{\Omega_\epsilon} \).

As \( \partial \Omega_\epsilon \) is a smooth set, let us conclude that \( v_\epsilon \in W^{1,p}_0(\Omega_\epsilon) \).
Arguing as in Lemma 3.3, we can assert that given any \( \eta > 0 \) there exists \( R > 0 \) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^p < \eta \tag{3.23}
\]

and

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^q < \eta. \tag{3.24}
\]

From \((H1)-(H2)\), for each \( \tau > 0 \) there exists \( C_\tau > 0 \) such that

\[
|h(s)| \leq \tau |s|^{p-1} + C_\tau |s|^{q-1} \quad \text{for all } s \in \mathbb{R}.
\]

This inequality combined with Sobolev’s embeddings and limits \((3.23)\) and \((3.24)\) yields there is a subsequence, still denoted by \((v_n)\), such that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n = \int_{\mathbb{R}^N} h(f(v_\epsilon)) f'(v_\epsilon) v_\epsilon, \tag{3.25}
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} h(f(v_n)) f'(v_\epsilon) v_\epsilon = \int_{\mathbb{R}^N} h(f(v_\epsilon)) f'(v_\epsilon) v_\epsilon \tag{3.26}
\]

In the sequel, define

\[
P_n = \int_{\mathbb{R}^N} \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla v_n - \nabla v_\epsilon \rangle
\]

\[
+ \int_{\mathbb{R}^N} \left( |f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) \right) (v_n - v_\epsilon)
\]

and observe that

\[
P_n \leq \int_{\mathbb{R}^N} \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla v_n - \nabla v_\epsilon \rangle
\]

\[
+ \int_{\mathbb{R}^N} \left( \lambda_\epsilon A_\epsilon(x) + 1 \right) \left( |f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) \right) (v_n - v_\epsilon)
\]

\[
= I_{\epsilon,\lambda_\epsilon}(v_n) v_n - I_{\epsilon,\lambda_\epsilon}(v_\epsilon) v_\epsilon + \int_{\mathbb{R}^N} h(f(v_n)) |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n
\]

\[
- \int_{\mathbb{R}^N} h(f(v_\epsilon)) |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) v_\epsilon + o_n(1).
\]

Thus, by \((3.25)\) and \((3.26)\), it follows that \( P_n = o_n(1) \), that is,

\[
\int_{\mathbb{R}^N} \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla v_n - \nabla v_\epsilon \rangle = o_n(1) \tag{3.27}
\]

and

\[
\int_{\mathbb{R}^N} \left( |f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) \right) (v_n - v_\epsilon) = o_n(1). \tag{3.28}
\]
The limits (3.28) and (3.27) combined with Corollary 2.9 give

\[ v_n \to v_\epsilon \quad \text{strongly in } W^{1,p}(\mathbb{R}^N). \]  

(3.29)

Now, using the fact that \( (\lambda_n A_\epsilon(x) + 1) v_\epsilon(x) = v_\epsilon(x) \) a.e in \( \mathbb{R}^N \) and that for each \( \phi \in C_0^\infty(\Omega_\epsilon), I'_{\epsilon, \lambda_n}(v_\epsilon) \phi = o_n(1) \), we have that

\[
\int_{\mathbb{R}^N} \left( |\nabla v_n|^{p-2} \nabla v_n \nabla \phi + |f(v_n)|^{p-2} f(v_n) f'(v_n) \phi \right) = \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) \phi + o_n(1).
\]

This together with (3.29) yields

\[
\int_{\Omega_\epsilon} \left( |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \nabla \phi + |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) \phi \right) = \int_{\Omega_\epsilon} h(f(v_\epsilon)) f'(v_\epsilon) \phi
\]

and hence

\[
\int_{\Omega_\epsilon} \left( |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon \nabla w + |f(v_\epsilon)|^{p-2} f(v_\epsilon) f'(v_\epsilon) w \right) = \int_{\Omega_\epsilon} h(f(v_\epsilon)) f'(v_\epsilon) w,
\]

(3.30)

for all \( w \in W^{1,p}_0(\Omega_\epsilon) \). Arguing as [1, Proposition 3.6], we can prove that \( v_\epsilon \in L^\infty(\mathbb{R}^N) \). Thereby, \( v_\epsilon \) is a solution of

\[
\begin{cases}
-\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), & \text{in } \Omega_\epsilon \\
v > 0 & \text{in } \Omega_\epsilon \text{ and } u = 0 & \text{on } \partial \Omega_\epsilon,
\end{cases}
\]

and the proof of (i) is complete.

To deduce (ii), we start observing that

\[
\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n + \lambda_n \int_{\mathbb{R}^N} A_\epsilon |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \\
= \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n + o_n(1).
\]

The last equality, combined with (3.29) and (3.30) leads to

\[
\lim_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x) |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n = 0.
\]

(3.31)

This limit together with Lemma 2.1(6) implies that

\[
\lim_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x) |f(v_n)|^p = 0,
\]

proving (ii).

To prove (iii), we observe that

\[
\lambda_n \int_{\mathbb{R}^N} A_\epsilon(x) |f(v_n - v_\epsilon)|^p = \lambda_n \int_{\Omega_\epsilon} A_\epsilon(x) |f(v_n)|^p + \lambda_n \int_{\Omega_\epsilon} A_\epsilon(x) |f(v_n - v_\epsilon)|^p \\
\leq \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x) |f(v_n)|^p
\]
because \( v_\varepsilon = 0 \) in \( \Omega_\varepsilon^c \) and \( A_\varepsilon = 0 \) in \( \Omega_\varepsilon \). Hence, using (ii)

\[
\lambda_n \int_{\mathbb{R}^N} A_\varepsilon(x) \left| f(v_n - v_\varepsilon) \right|^p \to 0 \quad \text{as } n \to +\infty
\]

from where it follows that

\[
\int_{\mathbb{R}^N} (1 + \lambda_n A_\varepsilon(x)) \left| f(v_n - v_\varepsilon) \right|^p \to 0 \quad \text{as } n \to +\infty. \tag{3.32}
\]

The last limit combined with the fact that \( v_n \to v \) in \( W^{1,p}(\mathbb{R}^N) \) and Corollary 2.5 yields

\[
\lim_{n \to \infty} \| v_n - v_\varepsilon \|_{\varepsilon, \lambda_n} = 0,
\]

which proves (iii), and the proof of Proposition 3.7 is complete. \( \square \)

**Corollary 3.9** Suppose that (A1)–(A2) and (H1)–(H4) hold. Then for each \( \epsilon > 0 \) and a sequence \( (v_n) \) of solutions of \( (P_{\epsilon, \lambda_n}) \) with \( \lambda_n \to \infty \) and \( \limsup_{n \to \infty} I_{\epsilon, \lambda_n}(v_n) < \infty \), there exists a subsequence that converges strongly in \( W^{1,p}(\mathbb{R}^N) \) to a solution of the problem

\[
\begin{cases}
-\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), & \text{in } \Omega_\epsilon \\
v > 0 & \text{in } \Omega_\epsilon \text{ and } v = 0 & \partial \Omega_\epsilon.
\end{cases}
\]

**Proof** By assumptions, there exist \( c \in \mathbb{R} \) and a subsequence of \( (v_n) \), still denoted by \( (v_n) \), such that \( (v_n) \) is a \((PS)_{c, \infty} \) sequence. The rest of the proof follows from Proposition 3.7. \( \square \)

### 4 Behavior of Minimax Levels

This section is devoted to the study of the behavior of the minimax levels with respect to parameter \( \lambda \) and \( \epsilon \). For this purpose, we introduce some notations. In the next, \( \mathcal{M}_{\epsilon, \lambda} \) denotes the Nehari manifold associated to \( I_{\epsilon, \lambda} \), that is,

\[
\mathcal{M}_{\epsilon, \lambda} = \{ v \in X_{\epsilon, \lambda} : v \neq 0 \text{ and } I'_{\epsilon, \lambda}(v)v = 0 \}
\]

and

\[
c_{\epsilon, \lambda} = \inf_{v \in \mathcal{M}_{\epsilon, \lambda}} I_{\epsilon, \lambda}(v).
\]

From (H1)–(H4), as proved in [1, Lemma 3.3], the number \( c_{\epsilon, \lambda} \) is the mountain pass minimax level associated with \( I_{\epsilon, \lambda} \). The below proposition is a version of Proposition 3.6 and its proof follows the same steps found in [1], then we will omit its proof.

**Proposition 4.1** Suppose that (H1)–(H3) and (A1)–(A2) hold. Then for any \( d > 0 \) and \( \epsilon > 0 \) there exists \( \Lambda > 0 \), independent of \( \epsilon \), such that \( I_{\epsilon, \lambda} \) satisfies the \((PS)_c \) condition for all \( c \leq d, \lambda \geq \Lambda \) and \( \epsilon > 0 \) on \( \mathcal{M}_{\epsilon, \lambda} \). That is, any sequence \( (v_n) \subset \mathcal{M}_{\epsilon, \lambda} \) satisfying

\[
I_{\epsilon, \lambda}(v_n) \to c \quad \text{and} \quad I'_{\epsilon, \lambda}|_{\mathcal{M}_{\epsilon, \lambda}}(v_n) \to 0, \tag{4.1}
\]

for \( c \leq d \), has a strongly convergent subsequence in \( X_{\epsilon, \lambda} \).
Corollary 4.2 If \( v \) is a critical point of \( I_{\epsilon,\lambda} \) on \( \mathcal{M}_{\epsilon,\lambda} \), then \( v \) is a critical point of \( I_{\epsilon,\lambda} \) on \( X_{\epsilon,\lambda} \).

Proof See [1]. □

On account of the proof of Proposition 3.7, when \( \lambda \) is large, the following problem can be seen as a limit problem of \((S_{\epsilon,\lambda})\) for each \( \epsilon > 0 \):

\[
\begin{cases}
-\Delta_p v + \left| f(v) \right|^{p-2} f(v) f'(v) = h(f(v)) f'(v), & \text{in } \Omega_{\epsilon} \\
v > 0 & \text{in } \Omega_{\epsilon} \\
v = 0 & \text{on } \partial \Omega_{\epsilon},
\end{cases}
\]

whose corresponding functional is given by

\[
E_{\epsilon}(v) = \frac{1}{p} \int_{\Omega_{\epsilon}} (|\nabla v|^p + |f(v)|^p) - \int_{\Omega_{\epsilon}} H(f(v))
\]

for every \( v \in W^{1,p}_0(\Omega_{\epsilon}) \). Here and subsequently, \( \mathcal{M}_{\epsilon} \) denotes the Nehari manifold associated to \( E_{\epsilon} \) and

\[
c(\epsilon, \Omega) = \inf_{v \in \mathcal{M}_{\epsilon}} E_{\epsilon}(v)
\]

stands for the mountain pass minimax associated with \( E_{\epsilon} \). Since \( 0 \in \Omega \), there is \( r > 0 \) such that \( B_r = B_r(0) \subset \Omega \) and \( B_{r/\epsilon} = B_{r/\epsilon}(0) \subset \Omega_{\epsilon} \). We will denote by \( E_{\epsilon,B_r} : W^{1,p}_0(B_{r/\epsilon}(0)) \to \mathbb{R} \) the functional

\[
E_{\epsilon,B_r}(v) = \frac{1}{p} \int_{B_{r/\epsilon}} (|\nabla v|^p + |f(v)|^p) - \int_{B_{r/\epsilon}} H(f(v)).
\]

Furthermore, we write \( \mathcal{M}_{\epsilon,B_r} \) the Nehari manifold associated to \( E_{\epsilon,B_r} \) and

\[
c(\epsilon, B_r) = \inf_{v \in \mathcal{M}_{\epsilon,B_r}} E_{\epsilon,B_r}(v).
\]

Once that \( B_{r/\epsilon} \subset \Omega_{\epsilon} \), we have \( c(\epsilon, \Omega) \leq c(\epsilon, B_r) \) for every \( \epsilon > 0 \).

Here it is important the number \( c_{\infty} \), which denotes the mountain minimax value associated to

\[
I_{\infty}(v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + |f(v)|^p) - \int_{\mathbb{R}^N} H(f(v)) \quad \text{for all } v \in W^{1,p}(\mathbb{R}^N),
\]

whose existence is guaranteed by [2, Lemma 3.1]. Since \( I_{\epsilon,\lambda}(tv) \geq I_{\infty}(tv) \) for all \( t > 0 \) and \( v \in W^{1,p}(\mathbb{R}^N) \),

\[
c_{\epsilon,\lambda} \geq c_{\infty}.
\]

Proposition 4.3 Suppose \((H_1)-(H_4)\) and \((A_1)-(A_2)\) hold. Let \( \epsilon > 0 \) be an arbitrary number. Then,

\[
\lim_{\lambda \to \infty} c_{\epsilon,\lambda} = c(\epsilon, \Omega).
\]

Proof By Proposition 3.6 and Mountain Pass Theorem, we can assume that there are two sequences, \( \lambda_n \to \infty \) and \( v_n \in X_{\epsilon,\lambda_n} \), such that

\[
I_{\epsilon,\lambda_n}(v_n) = c_{\epsilon,\lambda_n} > 0 \quad \text{and} \quad I'_{\epsilon,\lambda_n}(v_n) = 0.
\]
From definitions of $c_{\epsilon,\lambda_n}$ and $c(\epsilon, \Omega)$,
\[ c_{\epsilon,\lambda_n} \leq c(\epsilon, \Omega) \quad \text{for all } n \in \mathbb{N}, \]
which implies
\[ 0 \leq I_{\epsilon,\lambda_n}(v_n) \leq c(\epsilon, \Omega) \quad \text{and} \quad I_{\epsilon,\lambda_n}'(v_n) = 0. \]
Thus, for some subsequence $(v_{n_j})$, there exists $c \in [0, c(\epsilon, \Omega)]$ such that
\[ I_{\epsilon,\lambda_{n_j}}(v_{n_j}) = c_{\epsilon,\lambda_{n_j}} \to c \quad \text{and} \quad I_{\epsilon,\lambda_{n_j}}'(v_{n_j}) \to 0, \]
showing that $(v_{n_j})$ is a $(PS)_{c,\infty}$, and so,
\[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} (\lambda_n A_\epsilon(x) + 1)|f(v_n)|^p \geq pc_{\epsilon,\lambda_n} \geq pc_{\infty} > 0 \quad \forall n \in \mathbb{N}. \]
By Proposition 3.7,
\[ \lambda_n \int_{\mathbb{R}^N} A_\epsilon(x) |f(v_n)|^p \to 0 \quad \text{as } n \to +\infty \]
then,
\[ \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} |f(v_n)|^p \geq pc_{\infty} > 0 + o_n(1) \quad \forall n \in \mathbb{N}, \quad (4.2) \]
implying that any subsequence of $(v_n)$ does not converge to zero in $W^{1,p}(\mathbb{R}^N)$. From Proposition 3.7, there exist a subsequence $(v_{n_{jk}})$ and $v \in W^{1,p}(\mathbb{R}^N)$ such that
\[ v_{n_{jk}} \to v \quad \text{strongly in } W^{1,p}(\mathbb{R}^N) \quad \text{and} \quad v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_\epsilon. \quad (4.3) \]
From (4.2) and (4.3), $v \neq 0$ in $W^{1,p}_0(\Omega_\epsilon)$ and $v$ satisfies
\[ \begin{cases} -\Delta_p u + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v), & \text{in } \Omega_\epsilon, \\ v > 0 & \text{in } \Omega_\epsilon \quad \text{and} \quad v = 0 \quad \text{on } \partial \Omega_\epsilon, \end{cases} \]
from where it follows that
\[ E_\epsilon(v) \geq c(\epsilon, \Omega). \quad (4.4) \]
On the other hand,
\[ E_\epsilon(v) = \lim_{k \to \infty} I_{\epsilon,\lambda_{n_{jk}}}(v_{n_{jk}}) = \lim_{k \to \infty} c_{\epsilon,\lambda_{n_{jk}}} = c \leq c(\epsilon, \Omega). \quad (4.5) \]
Therefore, (4.4) and (4.5) give
\[ \lim_{k \to \infty} c_{\epsilon,\lambda_{n_{jk}}} = c(\epsilon, \Omega). \]
As a result, $c_{\epsilon,\lambda} \to c(\epsilon, \Omega)$ as $\lambda \to \infty$, and the lemma follows. \hfill \square

**Corollary 4.4** Suppose that $(A_1)$–$(A_2)$ and $(H_1)$–$(H_4)$ hold. Then for each $\epsilon > 0$ and a sequence $(v_n)$ of least energy solutions of $(S_{\epsilon,\lambda_n})$ with $\lambda_n \to \infty$ and
\[ \limsup_{n \to \infty} I_{\epsilon, \lambda_n}(v_n) < \infty, \] there exists a subsequence that converges strongly in \( W^{1,p}(\mathbb{R}^N) \) to a least energy solution of the problem
\[
\begin{aligned}
-\Delta_p u + |f(v)|^{p-2} f(v) f'(v) &= h(f(v)) f'(v), & \text{in } \Omega_{\epsilon} \\
u > 0 & \text{ in } \Omega_{\epsilon} \quad \text{and} \quad u = 0 & \text{ on } \partial \Omega_{\epsilon}.
\end{aligned}
\]

**Proof** The proof is a consequence of Propositions 3.7 and 4.3. \(\square\)

Hereafter, \( r > 0 \) denotes a number such that \( B_r(0) \subset \Omega \) and the sets
\[
\Omega_+ = \{ x \in \mathbb{R}^N : d(x, \overline{\Omega}) < r \}
\]
and
\[
\Omega_- = \{ x \in \Omega : d(x, \partial \Omega) > r \}
\]
are homotopically equivalent to \( \Omega \). The existence of this \( r \) is given by condition \( (A_1) \). For each \( v \in W^{1,p}(\mathbb{R}^N) \) whose positive part \( v_+ = \max\{v, 0\} \) is different from zero and has a compact support, we consider the center mass of \( v \)
\[
\beta(v) = \frac{\int_{\mathbb{R}^N} x v_+^p}{\int_{\mathbb{R}^N} v_+^p}.
\]

Consider \( R > 0 \) such that \( \Omega \subset B_R(0) \), thus \( \Omega_{\epsilon} \subset B_{\frac{R}{\epsilon}}(0) \), and define the auxiliary function
\[
\xi_{\epsilon}(t) = \begin{cases} 1, & 0 \leq t \leq \frac{R}{\epsilon} \\
\frac{R}{\epsilon t}, & \frac{R}{\epsilon} \leq t \leq 1.
\end{cases}
\]

For \( v \in W^{1,p}(\mathbb{R}^N), v_+ \neq 0 \), define
\[
\beta_{\epsilon}(v) = \frac{\int_{\mathbb{R}^N} x \xi_{\epsilon}(|x|) v_+^p}{\int_{\mathbb{R}^N} v_+^p}.
\]

Now for each \( y \in \mathbb{R}^N \) and \( R > 2\text{diam}(\Omega) \), we fix
\[
A_{\frac{R}{\epsilon}, \frac{R}{\epsilon}, y} = \left\{ x \in \mathbb{R}^N : \frac{R}{\epsilon} \leq |x - y| \leq \frac{R}{\epsilon} \right\}.
\]

We observe that if \( y \notin \frac{1}{\epsilon} \Omega_+ \) then \( \overline{\Omega}_{\epsilon} \cap B_{\frac{R}{\epsilon}}(y) = \emptyset \). As a consequence
\[
\overline{\Omega}_{\epsilon} \subset A_{\frac{R}{\epsilon}, \frac{R}{\epsilon}, y} \tag{4.6}
\]
for every \( y \notin \frac{1}{\epsilon} \Omega_+ \). Moreover, for \( y \in \mathbb{R}^N \), \( \alpha(R, r, \epsilon, y) \) denotes the number
\[
\alpha(R, r, \epsilon, y) = \inf \left\{ \tilde{J}_{\epsilon, y}(v) : \beta(v) = y \quad \text{and} \quad v \in \tilde{M}_{\epsilon, y} \right\}
\]
where
\[
\tilde{J}_{\epsilon, y}(v) = \frac{1}{p} \int_{A_{\frac{R}{\epsilon}, \frac{R}{\epsilon}, y}} (|\nabla v|^p + |f(v)|^p) - \int_{A_{\frac{R}{\epsilon}, \frac{R}{\epsilon}, y}} H(f(v))
\]
\(\square\) Springer
and
\[ \widehat{\mathcal{M}}_{\epsilon, y} = \{ v \in W^{1,p}_0(\mathbb{R}^N_+, A_{\epsilon, x, y}) : v \neq 0 \text{ and } \widehat{J}_{\epsilon, y}(v) v = 0 \} . \]

From now on, we will write \( \alpha(R, r, \epsilon), \widehat{J}_{\epsilon} \) and \( \widehat{\mathcal{M}}_{\epsilon} \) to denote \( \alpha(R, r, \epsilon, 0), \widehat{J}_{\epsilon, 0} \) and \( \widehat{\mathcal{M}}_{\epsilon, 0} \) respectively.

**Lemma 4.5** Assume that \((H_1)-(H_4)\) hold. Then, there exists \( \epsilon^* > 0 \) such that
\[ c(\epsilon, \Omega) < \alpha(R, r, \epsilon) \quad \forall \epsilon \in (0, \epsilon^*) . \]

**Proof** Invoking \([2, \text{Proposition 4.1}]\), we assert that
\[ \lim_{\epsilon \to 0} \alpha(R, r, \epsilon) > c_\infty . \]
Thus, there exists \( \epsilon_1 > 0 \) such that
\[ \alpha(R, r, \epsilon) > c_\infty + \delta \]  \( (4.7) \)
for all \( 0 < \epsilon < \epsilon_1 \), for some \( \delta > 0 \). On the other hand, arguing as in \([2, \text{Proposition 4.2}]\),
\[ \lim_{\epsilon \to 0} c(\epsilon, B_r) = c_\infty . \]
Therefore, there exists \( \epsilon_2 > 0 \) such that
\[ c(\epsilon, B_r) < c_\infty + \frac{\delta}{2} \quad \text{for all } 0 < \epsilon < \epsilon_2 . \]  \( (4.8) \)
For \( \epsilon^* = \min\{\epsilon_1, \epsilon_2\} \), \((4.7)\) and \((4.8)\) lead to
\[ c(\epsilon, B_r) < \alpha(R, r, \epsilon) \]
for every \( \epsilon \in (0, \epsilon^*) \). Now, the lemma follows of the inequality \( c(\epsilon, \Omega) \leq c(\epsilon, B_r) \). \( \square \)

To conclude this section, we establish a result about the center of mass of some functions that are in the Nehari manifold \( \mathcal{M}_{\epsilon, \lambda, \cdot} \).

**Lemma 4.6** Suppose \((H_1)-(H_4)\) and \((A_1)-(A_2)\) hold. Let \( \epsilon^* > 0 \) given by Lemma 4.5. Then for any \( \epsilon \in (0, \epsilon^*) \), there exists \( \lambda^* > 0 \) which depends on \( \epsilon \) such that
\[ \beta_{\epsilon}(v) \in \frac{1}{\epsilon} \Omega_+ \]
for all \( \lambda > \lambda^* \), \( 0 < \epsilon < \epsilon^* \) and \( v \in \mathcal{M}_{\epsilon, \lambda, \cdot} \) with \( I_{\epsilon, \lambda}(v) \leq c(\epsilon, B_r) \).

**Proof** Suppose by contradiction that there exists a sequence \((\lambda_n)\) with \( \lambda_n \to \infty \) such that
\( v_n \in \mathcal{M}_{\epsilon, \lambda_n, \cdot} \), \( I_{\epsilon, \lambda_n}(v_n) \leq c(\epsilon, B_r) \)
and
\[ \beta_{\epsilon}(v_n) \notin \frac{1}{\epsilon} \Omega_+ . \]  \( (4.9) \)
Repeating the same arguments used in the proofs of Lemma 3.3 and Proposition 3.7, \((\|v_n\|_{\epsilon,\lambda_n})\) is a bounded sequence in \(\mathbb{R}\) and there exists \(v \in W^{1,p}(\mathbb{R}^N)\) such that \(v_n \rightharpoonup v\) weakly in \(W^{1,p}(\mathbb{R}^N)\), \(v = 0\) in \(\mathbb{R}^N \setminus \Omega_{\epsilon}\) and for each \(\eta > 0\) there exists \(R > 0\) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |f(v_n)|^p < \eta.
\]

This fact implies that

\[
f(v_n) \to f(v) \quad \text{strongly in } L^p(\mathbb{R}^N).
\]

Hence by interpolation,

\[
f(v_n) \to f(v) \quad \text{strongly in } L^t(\mathbb{R}^N) \quad \text{for all } t \in [p, p^*).
\]

On the other hand, since \(v_n \in \mathcal{M}_{\epsilon,\lambda_n}\), from (4.2),

\[
0 < pc_{\infty} \leq \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n, \quad \text{for all } n \in \mathbb{N},
\]

from where it follows that

\[
0 < pc_{\infty} \leq \int_{\mathbb{R}^N} h(f(v)) f'(v) v,
\]

which yields

\[
v \neq 0, \quad E'_{\epsilon}(v) v \leq 0 \quad \text{and} \quad \lim_{n \to \infty} \beta_{\epsilon}(v_n) = \beta(v).
\] (4.10)

From (4.9) and (4.10), \(y = \beta(v) \notin \frac{1}{\epsilon} \Omega_+\), \(\Omega_\epsilon \subset \mathcal{A}_{r, \epsilon, y}\) and there exists \(\tau \in (0, 1]\) such that \(\tau v \in \mathcal{M}_{\epsilon, y}\). Thereby,

\[
\hat{J}_{\epsilon, y}(\tau v) = E_{\epsilon}(\tau v) \leq \liminf_{n \to \infty} I_{\epsilon, \lambda_n}(\tau v_n) \leq \liminf_{n \to \infty} I_{\epsilon, \lambda_n}(v_n) \leq c(\epsilon, B_r)
\]

which implies

\[
\alpha(R, r, \epsilon, y) \leq c(\epsilon, B_r).
\]

On the other hand, since

\[
\alpha(R, r, \epsilon, y) = \alpha(R, r, \epsilon)
\]

we have

\[
\alpha(R, r, \epsilon) \leq c(\epsilon, B_r),
\]

contrary to Lemma 4.5, and the proof is complete. \(\square\)

5 Proof of Theorem 1.1

For \(r > 0\) and \(\epsilon > 0\), let \(v_{r\epsilon} \in W^{1,p}_0(B_r(0))\) be a nonnegative radially symmetric function such that

\[
E_{\epsilon, B_r}(v_{r\epsilon}) = c(\epsilon, B_r) \quad \text{and} \quad E'_{\epsilon, B_r}(v_{r\epsilon}) = 0,
\]
whose existence is proved in [2, Proposition 4.4]. For $r > 0$ and $\epsilon > 0$, define $\Psi_r : \frac{1}{\epsilon} \Omega_- \rightarrow W^{1, p}_0(\Omega_\epsilon)$ by

$$
\Psi_r(y)(x) = \begin{cases} 
vr_{\epsilon}(|x - y|), & x \in B_{\frac{\epsilon}{r}}(y) \\
0, & x \notin B_{\frac{\epsilon}{r}}(y).
\end{cases}
$$

It is immediate that $\beta_{\epsilon}(\Psi_r(y)) = y$ for all $y \in \frac{1}{\epsilon} \Omega_-$. In the sequel, we denote by $I_{\epsilon, \lambda}^{c(e, B_r)}$ the set

$$
I_{\epsilon, \lambda}^{c(e, B_r)} = \{ v \in M_{\epsilon, \lambda} : I_{\epsilon, \lambda}(v) \leq c(\epsilon, B_r) \}.
$$

We claim that

$$
cat I_{\epsilon, \lambda}^{c(e, B_r)} \geq cat(\Omega) \tag{5.1}
$$

for all $\epsilon \in (0, \epsilon^*)$ and $\lambda \geq \lambda^*$. In fact, suppose that

$$
I_{\epsilon, \lambda}^{c(e, B_r)} = \bigcup_{i=1}^{n} O_i
$$

where $O_i, i = 1, \ldots, n$, is closed and contractible in $I_{\epsilon, \lambda}^{c(e, B_r)}$, that is, there exists $h_i \in C([0, 1] \times O_i, I_{\epsilon, \lambda}^{c(e, B_r)})$ such that, for every, $v \in O_i$,

$$
h_i(0, v) = v \quad \text{and} \quad h_i(1, u) = w_i
$$

for some $w_i \in I_{\epsilon, \lambda}^{c(e, B_r)}$. Consider

$$
B_i = \Psi_r^{-1}(O_i), \quad i = 1, \ldots, n.
$$

The sets $B_i$ are closed and

$$
\frac{1}{\epsilon} \Omega_- = B_1 \cup \cdots \cup B_n.
$$

Consider the deformation $g_i : [0, 1] \times B_i \rightarrow \frac{1}{\epsilon} \Omega_+$ given

$$
g_i(t, y) = \beta_{\epsilon}(h_i(t, \Psi_r(y))).
$$

From Lemma 4.6, the function $g_i$ is well defined. Thus, $B_i$ is contractile in $\frac{1}{\epsilon} \Omega_+$. Hence,

$$
cat(\Omega) = cat(\Omega_\epsilon) = cat \left( \frac{1}{\epsilon} \Omega_+ \right) \leq cat I_{\epsilon, \lambda}^{c(e, B_r)}
$$

which verifies (5.1).

Now, we are ready to conclude the proof of Theorem 1.1. From Proposition 4.1, there is $\lambda^* > 0$, such that $I_{\epsilon, \lambda}$ satisfies the $(PS)_d$ condition on $M_{\epsilon, \lambda}$ for all $d \in (-\infty, c(\epsilon, B_r)]$ provided that $\lambda \geq \lambda^*$. Thus, by Lusternik-Schirelman theory, the functional $I_{\epsilon, \lambda}$ has at least $cat(\Omega)$ critical points for all $\epsilon \in (0, \epsilon^*)$ where $\epsilon^* > 0$ is given by Lemma 4.5. The proof is complete.

\[\square\]

Acknowledgements The authors thank the referee for his/her useful suggestions and comments. Research of C.O. Alves partially supported by INCT-MAT, PROCAD and CNPq/Brazil 303080/2009-4. Research of G.M. Figueiredo partially supported by supported by CNPq/Brazil 300705/2008-5.
References

1. Alves, C.O., Figueiredo, G.M., Severo, U.B.: Multiplicity of positive solutions for a class of quasilinear problems. Adv. Differ. Equ. 14, 911–942 (2009)
2. Alves, C.O., Figueiredo, G.M., Severo, U.B.: A result of multiplicity of solutions for a class of quasilinear equations. In: Proceedings of the Edinburgh Mathematical Society, vol. 55, pp. 291–309 (2012)
3. Alves, C.O., Soares, S.H.M.: Multiplicity of positive solutions for a class of nonlinear Schrödinger equations. Adv. Differ. Equ. 11, 1083–1102 (2010)
4. Alves, C.O., Miyagaki, O.H., Soares, S.H.M.: Multi-bump solutions for a class of quasilinear equations in \( \mathbb{R} \). Commun. Pure Appl. Anal. 11, 829–844 (2012)
5. Alves, C.O., Miyagaki, O.H., Soares, S.H.M.: On the existence and concentration of positive solutions to a class of quasilinear elliptic problems on \( \mathbb{R} \). Math. Nachr. 1, 1–12 (2011)
6. Alves, M.J., Carrião, P.C., Miyagaki, O.H.: Soliton solutions to a class of quasilinear elliptic equations on \( \mathbb{R} \). Adv. Nonlinear Stud. 7, 579–597 (2007)
7. Barstch, T., Wang, Z.Q.: Existence and multiplicity results for some superlinear elliptic problem on \( \mathbb{R}^N \). Commun. Partial Differ. Equ. 20(9–10), 1725–1741 (1995)
8. Barstch, T., Wang, Z.Q.: Multiple positive solutions for a nonlinear Schrödinger equation. Z. Angew. Math. Phys. 51(3), 366–384 (2000)
9. Brizhik, L., Eremko, A., Piette, B., Zakrzewski, W.J.: Static solutions of a \( D \)-dimensional modified nonlinear Schrödinger equation. Nonlinearity 16, 1481–1497 (2003)
10. Borovskii, A., Galkin, A.: Dynamical modulation of an ultrashort high-intensity laser pulse in matter. JETP Lett. 77, 562–573 (1983)
11. Colin, M., Jeanjean, L.: Solutions for a quasilinear Schrödinger equation: a dual approach. Nonlinear Anal. 56, 213–226 (2004)
12. Colin, M., Jeanjean, L., Squassina, M.: Stability and instability results for standing waves of quasi-linear Schrödinger equations. Nonlinearity 23, 1353–1385 (2010)
13. do Ó, J.M.B., Severo, U.B.: Quasilinear Schrödinger equations involving concave and convex nonlinearities. Commun. Pure Appl. Anal. 8, 621–644 (2009)
14. do Ó, J.M.B., Miyagaki, O.H., Soares, S.M.H.: Soliton solutions for quasilinear Schrödinger equations: the critical exponential case. Nonlinear Anal. 67, 3357–3372 (2007)
15. Fang, X., Szulkin, A.: Multiple solutions for a quasilinear Schrödinger equation. J. Differ. Equ. 254(4), 2015–2032 (2013)
16. Floer, A., Weinstein, A.: Nonsqueezing wave packets for the packets for the cubic Schrödinger with a bounded potential. J. Func. Anal. 69, 397–408 (1986)
17. Guo, Y., Tang, Z.: Multibump bound states for quasilinear Schrödinger systems with critical frequency. J. Fixed Point Theory Appl. 12(1–2), 135–174 (2012)
18. Hartmann, B., Zakrzewski, W.J.: Electrons on hexagonal lattices and applications to nanotubes. Phys. Rev. B 68, 184302 (2003)
19. Jeanjean, L., Tanaka, K.: A positive solution for a nonlinear Schrödinger equation on \( \mathbb{R}^N \). Indiana Univ. Math. J. 54, 443–464 (2005)
20. Kosevich, A.M., Ivanov, B.A., Kovalev, A.S.: Magnetic solitons in superfluid films. J. Phys. Soc. Jpn. 50, 3262–3267 (1981)
21. Kurihara, S.: Large-amplitude quasi-solitons in superfluids films. J. Phys. Soc. Jpn. 50, 3262–3267 (1981)
22. Liu, J., Wang, Z.Q.: Soliton solutions for quasilinear Schrödinger equations I. Proc. Am. Math. Soc. 131(2), 441–448 (2002)
23. Liu, J., Wang, Y., Wang, Z.: Soliton solutions for quasilinear Schrödinger equations II. J. Differ. Equ. 187, 473–493 (2003)
24. Liu, J., Wang, Y., Wang, Z.Q.: Solutions for quasilinear Schrödinger equations via the Nehari method. Commun. Partial Differ. Equ. 29, 879–901 (2004)
25. Makhanov, V.G., Fedyanin, V.K.: Non-linear effects in quasi-one-dimensional models of condensed matter theory. Phys. Rep. 104, 1–86 (1984)
26. Poppenberg, M., Schmitt, K., Wang, Z.Q.: On the existence of soliton solutions to quasilinear Schrödinger equations. Calc. Var. Partial Differ. Equ. 14, 329–344 (2002)
27. Ritchie, B.: Relativistic self-focusing and channel formation in laser-plasma interactions. Phys. Rev. E 50, 687–689 (1994)
28. Severo, U.B.: Estudo de uma classe de equações de Schrödinger quase-lineares. Doct. dissertation, Unicamp (2007)
29. Severo, U.B.: Existence of weak solutions for quasilinear elliptic equations involving the p-Laplacian. Electron. J. Differ. Equ. 56, 1–16 (2008)
30. Shaoxiong, C.: Existence of positive solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}^N$. J. Math. Anal. Appl. 405(2), 595–607 (2013)
31. Mercuri, C., Willem, M.: A global compactness result for the $p$-Laplacian involving critical nonlinearities. Discrete Contin. Dyn. Syst. 28, 469–493 (2010)
32. Shinji, A., Masataka, S., Tatsuya, W.: Asymptotic behavior of positive solutions for a class of quasilinear elliptic equations with general nonlinearities. Commun. Pure Appl. Anal. 13(1), 97–118 (2014)
33. Silva, E.A.B., Vieira, G.F.: Quasilinear asymptotically periodic Schrödinger equations with critical growth. Calc. Var. 39, 1–33 (2010)
34. Silva, E.A.B., Vieira, G.F.: Quasilinear asymptotically periodic Schrödinger equations with subcritical growth. Nonlinear Anal. 72, 2935–2945 (2010)
35. Takeno, S., Homma, S.: Classical planar Heisenberg ferromagnet, complex scalar fields and nonlinear excitations. Prog. Theor. Phys. 65, 172–189 (1981)
36. Yinbin, D., Shuangjie, P., Jixiu, W.: Nodal soliton solutions for quasilinear Schrödinger equations with critical exponent. J. Math. Phys. 54(1), 011504 (2013)
37. Yujuan, J.: Least energy solutions for a quasilinear Schrödinger equation with potential well. Bound. Value Probl. 9, 17 (2013)