Division Algebra Representations of SO(4, 2)

Joshua Kincaid and Tevian Dray
Department of Mathematics, Oregon State University, Corvallis, OR 97331
kincajos@math.oregonstate.edu, tevian@math.oregonstate.edu

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Abstract
Representations of SO(4, 2) are constructed using \(4 \times 4\) and \(2 \times 2\) matrices with elements in \(\mathbb{H}' \otimes \mathbb{C}\), and the known isomorphism between the conformal group and \(\text{SO}(4, 2)\) is written explicitly in terms of the \(4 \times 4\) representation.

1 Introduction
The (local) correspondence between the Lorentz group \(\text{SO}(3, 1)\) and \(\text{SL}(2, \mathbb{C})\) is well-known, and is a natural generalization of the correspondence between the rotation group \(\text{SO}(3)\) and the unitary group \(\text{SU}(2)\). Manogue and Schray \[1\] generalized this correspondence to the other division algebras, in particular providing an explicit construction of \(\text{SO}(9, 1)\) in terms of \(\text{SL}(2, \mathbb{O})\). In subsequent work \[2, 3\], Manogue and Dray outlined the implications of this mathematical description for the description of fundamental particles.

Here, we generalize this construction in a different direction, showing how to describe the conformal group \(\text{SO}(4, 2)\) as a matrix group over \(\mathbb{H}' \otimes \mathbb{C}\), along the way reinterpreting \(\text{SO}(3, 1)\) as a matrix group over \(\mathbb{C}' \otimes \mathbb{C}\). The resulting parameterization of certain orthogonal groups in terms of two division algebras is reminiscent of the Freudenthal-Tits magic square of Lie groups \[4, 5\], and our results should generalize to the corresponding “2 × 2” version of the magic square \[6\], shown in Figure H.

| \(\mathbb{R}'\) | \(\mathbb{C}'\) | \(\mathbb{H}'\) | \(\mathbb{O}'\) |
|---|---|---|---|
| \(\text{SO}(2) \equiv \text{SU}(2, \mathbb{R})\) | \(\text{SO}(3) \equiv \text{SU}(2, \mathbb{C})\) | \(\text{SO}(5) \equiv \text{SU}(2, \mathbb{H})\) | \(\text{SO}(9) \equiv \text{SU}(2, \mathbb{O})\) |
| \(\text{SO}(2, 1) \equiv \text{SL}(2, \mathbb{R})\) | \(\text{SO}(3, 1) \equiv \text{SL}(2, \mathbb{C})\) | \(\text{SO}(5, 1) \equiv \text{SL}(2, \mathbb{H})\) | \(\text{SO}(9, 1) \equiv \text{SL}(2, \mathbb{O})\) |
| \(\text{SO}(3, 2) \equiv \text{SP}(3, \mathbb{R})\) | \(\text{SO}(4, 2) \equiv \text{SU}(2, 2, \mathbb{C})\) | \(\text{SO}(6, 2)\) | \(\text{SO}(10, 2)\) |
| \(\text{SO}(5, 4)\) | \(\text{SO}(6, 4)\) | \(\text{SO}(8, 4)\) | \(\text{SO}(12, 4)\) |

Table 1: The “half-split” \(2 \times 2\) Lie group magic square. (The given equivalences are local, that is, up to double-cover.)
2 Split Quaternions

As can be seen from Table 1, the group SO(4, 2) is labeled by the complex numbers \( \mathbb{C} \) and the split quaternions \( \mathbb{H}' \). For compatibility with the notation in [2, 3], we write complex numbers \( a \in \mathbb{C} \) in the form

\[
a = x + y\ell
\]

with \( x, y \in \mathbb{R} \), and we introduce the notation

\[
A = z + qK + pKL + tL
\]

with \( p, q, t, z \in \mathbb{R} \) for split quaternions \( A \in \mathbb{H}' \). We have of course

\[
\ell^2 = -1
\]

and the split quaternionic multiplication table

\[
K^2 = -1; \quad L^2 = 1 = KL^2; \\
(K)(L) = KL; \quad L(KL) = -K; \quad (KL)K = L
\]

together with the usual anticommutativity of these unit elements. The split quaternions are associative, but not a division algebra, since for instance

\[
(1 + L)(1 - L) = 0
\]

so that there are zero divisors. We introduce separate conjugation operations

\[
\overline{a} = x - y\ell \\
A^* = z - qK - pKL - tL
\]

3 The Clifford Algebra \( \mathcal{C}\ell(4, 2) \)

We consider matrices of the form

\[
X = \begin{pmatrix} A & \overline{a} \\ a & -A^* \end{pmatrix} = \begin{pmatrix} z + qK + pKL + tL & x - y\ell \\ x + y\ell & -z + qK + pKL + tL \end{pmatrix}
\]

with \( A \in \mathbb{H}' \) and \( a \in \mathbb{C} \). Then \( X \) can be written as

\[
X = x^a SS_a
\]

where

\[
\{x^a\} = \{x, y, z, t, p, q\}
\]
and there is an implicit sum over the repeated index \( a \). Equation (9) defines the *generalized Pauli matrices*

\[
SS_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad SS_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
SS_y = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}, \quad SS_t = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \\
SS_q = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad SS_p = \begin{pmatrix} KL & 0 \\ 0 & KL \end{pmatrix}
\]

(11)

which are given this name because \( SS_x, SS_y, \) and \( SS_z \) are just the usual Pauli spin matrices.

We now consider the matrix

\[
P = \begin{pmatrix} 0 & X & \tilde{X} \end{pmatrix} = x^a \Gamma_a,
\]

(12)

where tilde represents trace reversal,

\[
\tilde{X} = X - \text{tr}(X) I,
\]

(13)

and where the gamma matrices \( \Gamma_a \) are implicitly defined by (12). Explicitly,

\[
\Gamma_a = \begin{cases} 
SS_x \otimes SS_a & m \in \{x, y, z\} \\
\ell SS_y \otimes SS_a & m \in \{p, q, t\}
\end{cases}
\]

(14)

or, in more traditional notation,

\[
\Gamma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \Gamma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\Gamma_y = \begin{pmatrix} 0 & 0 & 0 & -\ell \\ 0 & 0 & \ell & 0 \\ 0 & -\ell & 0 & 0 \\ \ell & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_t = \begin{pmatrix} 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \\ -L & 0 & 0 & 0 \\ 0 & -L & 0 & 0 \end{pmatrix},
\]

\[
\Gamma_q = \begin{pmatrix} 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \\ -K & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \end{pmatrix}, \quad \Gamma_p = \begin{pmatrix} 0 & 0 & KL & 0 \\ 0 & 0 & 0 & KL \\ -KL & 0 & 0 & 0 \\ 0 & -KL & 0 & 0 \end{pmatrix}
\]

(15)

A straightforward computation using the commutativity of \( \mathbb{C} \) with \( \mathbb{H}' \) now shows that

\[
\{\Gamma_a, \Gamma_b\} = 2g_{ab} I
\]

(16)
where \( I \) is the identity matrix and
\[
g_{ab} = \begin{cases} 
0 & a \neq b \\
1 & a = b \in \{x, y, z, q\} \\
-1 & a = b \in \{p, t\}
\end{cases}
\] (17)

These are precisely the anticommutation relations necessary to generate a representation of the (real) Clifford algebra \( \mathbb{C}l(4, 2) \), so \( P \) represents an arbitrary element of the vector space underlying \( \mathbb{C}l(4, 2) \).

### 4 Division Algebra Representations of SO(4, 2)

It is now straightforward to use our representation of the Clifford algebra \( \mathbb{C}l(4, 2) \) to construct a representation of \( \text{SO}(4, 2) \). The homogenous quadratic elements of \( \mathbb{C}l(4, 2) \) act as generators of \( \text{SO}(4, 2) \) via the map

\[
P \mapsto M_{ab}P M_{ab}^{-1}
\] (18)

where

\[
M_{ab} = \exp \left( -\Gamma_a \Gamma_b \frac{\theta}{2} \right)
\] (19)

and \( P = x^a \Gamma_a \) as above.

In the following, we assume \( a, b, c \) are distinct but otherwise arbitrary indices. The properties below follow from the Clifford algebra anticommutation relation (16):

\[
\Gamma_a \Gamma_a = \pm I,
\]
\[
(\Gamma_a \Gamma_b) \Gamma_c = \Gamma_c (\Gamma_a \Gamma_b),
\]
\[
(\Gamma_a \Gamma_b) \Gamma_b = (\Gamma_b)^2 \Gamma_a = g_{ab} \Gamma_a,
\]
\[
(\Gamma_a \Gamma_b) \Gamma_a = -(\Gamma_a)^2 \Gamma_b = -g_{aa} \Gamma_b,
\]
\[
(\Gamma_a \Gamma_b)^2 = -\Gamma_a^2 \Gamma_b^2 = \pm I.
\] (24)

With these observations, we are prepared to see how \( \text{SO}(4, 2) \) is generated by the matrices \( \{\Gamma_a\} \). We compute

\[
M_{ab}P M_{ab}^{-1} = \exp \left( -\Gamma_a \Gamma_b \frac{\theta}{2} \right) (x^c \Gamma_c) \exp \left( \Gamma_a \Gamma_b \frac{\theta}{2} \right).
\] (25)

From (20), if \( a = b \), then \( M_{ab} \) is a real multiple of the identity matrix, which therefore leaves \( P \) unchanged under the action (13). On the other hand, if \( a \neq b \), properties (21)–(23) imply that \( M_{ab} \) commutes with all but two of the matrices \( \Gamma_c \). We therefore have

\[
\Gamma_c M_{ab}^{-1} = \begin{cases} 
M_{ab} \Gamma_c, & c = a \text{ or } c = b \\
M_{ab}^{-1} \Gamma_c, & a \neq c \neq b
\end{cases}
\] (26)
so that the action of $M_{ab}$ on $P$ affects only the $ab$ plane. To see what that action is, we first note that if $A^2 = \pm I$ then

$$\exp (A\alpha) = Ic(\alpha) + A\, s(\alpha) = \begin{cases} \exp (\cosh(\alpha)) + A\, \sinh(\alpha), & A^2 = I \\ \exp (\cos(\alpha)) + A\, \sin(\alpha), & A^2 = -I \end{cases}$$

(27)

where the second equality serves to define the functions $c$ and $s$. Inserting (26) and (27) into (25), we obtain

$$M_{ab} \left( x^a \Gamma_a + x^b \Gamma_b \right) M_{ab}^{-1} = \left( M_{ab} \right)^2 \left( x^a \Gamma_a + x^b \Gamma_b \right)$$

$$= \exp (-\Gamma_a \Gamma_b \theta) \left( x^a \Gamma_a + x^b \Gamma_b \right)$$

$$= (Ic(\theta) - \Gamma_a \Gamma_b \, s(\theta)) \left( x^a \Gamma_a + x^b \Gamma_b \right)$$

$$= (x^a c(\theta) - x^b s(\theta) g_{ab}) \Gamma_a + (x^b c(\theta) + x^a s(\theta) g_{aa}) \Gamma_b.$$  

(28)

Thus, the action (18) is either a rotation from $a$ to $b$ or a boost in the $ab$-plane, depending on whether

$$(\Gamma_a \Gamma_b)^2 = \pm I$$

(29)

that is, on which version of (27) is required. It follows from (12) that

$$\text{tr}(P) = 0$$

(30)

$$P^2 = -\left( \det X \right) I$$

(31)

so that the characteristic equation for $P$ implies that

$$\det P = \left( \det X \right)^2 = (x^2 + y^2 + z^2 + p^2 - q^2 - t^2)^2$$

(32)

which can also be verified by direct computation. Since transformations of the form (18) preserve the determinant of $P$, it is clear that we have constructed $SO(4, 2)$. The fifteen independent group generators $M_{ab}$ are given explicitly in the appendix.

So far we have considered transformations of the form (18) acting on $P$, but we can also consider the effect (18) has on $X$. First, we observe that trace-reversal of $X$ corresponds to conjugation in $\mathbb{H}^\prime$, that is,

$$\overline{SS_a} = SS_a^*.$$  

(33)

In light of the off-diagonal structure of the matrices $\Gamma_a$, the matrices $\Gamma_a \Gamma_b$ then take the block diagonal form

$$\Gamma_a \Gamma_b = \begin{pmatrix} SS_a SS_b^* & 0 \\ 0 & SS_a^* SS_b \end{pmatrix}$$

(34)

and, in particular,

$$\exp \left( \frac{\Gamma_a \Gamma_b \theta}{2} \right) = \begin{pmatrix} \exp \left( SS_a SS_b^* \frac{\theta}{2} \right) & 0 \\ 0 & \exp \left( SS_a^* SS_b \frac{\theta}{2} \right) \end{pmatrix}. $$

(35)
so we can write
\[
\exp \left(-\Gamma_a \Gamma_b \frac{\theta}{2}\right) P \exp \left(\Gamma_a \Gamma_b \frac{\theta}{2}\right) = \begin{pmatrix}
0 & \exp \left(-SS_a SS_b^* \frac{\theta}{2}\right) X \exp \left(SS_a^* SS_b \frac{\theta}{2}\right) \\
\exp \left(-SS_a^* SS_b \frac{\theta}{2}\right) \tilde{X} \exp \left(SS_a SS_b^* \frac{\theta}{2}\right) & 0
\end{pmatrix}.
\]

The $4 \times 4$ action (18) acting on $P$ is thus equivalent to the $2 \times 2$ action
\[
X \mapsto \exp \left(-SS_a SS_b^* \frac{\theta}{2}\right) X \exp \left(SS_a^* SS_b \frac{\theta}{2}\right).
\]
on $X$. However, these transformations do not appear to have the general form
\[
X \mapsto MXM^\dagger,
\]
even if we restrict the dagger operation to include conjugation in just one of $\mathbb{H}'$ or $\mathbb{C}$. Nonetheless, since $X$ is Hermitian with respect to $\mathbb{C}$, and since that condition is preserved by (37), we will refer to our $2 \times 2$ representation of $\text{SO}(4, 2)$ as $\text{SU}(2, \mathbb{H}' \otimes \mathbb{C})$.

## 5 A Real Representation of $\text{SO}(4, 2)$

We seek now to identify a real representation of $\text{SO}(4, 2)$ that satisfies certain “nice” conditions. We would like our representation to contain a representation of $\text{SO}(3, 1)$ in an obvious way and be linked explicitly to $\text{SU}(2, \mathbb{H}' \otimes \mathbb{C})$. Ideally, the construction developed here will extend naturally to the other groups in Table 1 and admit an analog that can be applied to the Freudenthal-Tits magic square.

We are seeking a real representation, so we require a way to express these as real matrices while retaining the essential anticommutation relations. The solution is provided by finding suitable representations for $\mathbb{C}$ and $\mathbb{H}'$ in the form of $2 \times 2$ real matrices. We can do this by making use of the Pauli spin matrices, using the facts that
\[
(\ell SS_y)^2 = -I, \tag{39}
SS_z^2 = SS_x^2 = I, \tag{40}
\]
and that all three of these matrices are real. If we map
\[
1 \to I \tag{41}
\]
\[
\ell \to \ell SS_y
\]
for $\{1, \ell\} \subset \mathbb{C}$ and
\[
1 \to I \tag{42}
\]
\[
L \to SS_z
\]
\[
K \to -\ell SS_y
\]
\[
KL \to SS_x,
\]
for \{1, L, K, KL\} \subset \mathbb{H}', then the appropriate multiplication tables are preserved. Thus, we can write elements of \(\mathbb{H}' \otimes \mathbb{C}\) by taking tensor products of these representations. We can now write the matrices \(\{\Gamma_a\}\) as

\[
\Gamma_x = SS_x \otimes SS_x \otimes I \otimes I, \quad \Gamma_y = SS_x \otimes -\ell SS_y \otimes I \otimes \ell SS_y, \quad \Gamma_z = SS_x \otimes SS_x \otimes I \otimes I, \\
\Gamma_t = \ell SS_y \otimes I \otimes SS_y \otimes I, \quad \Gamma_q = \ell SS_y \otimes I \otimes -\ell SS_y \otimes I, \quad \Gamma_p = \ell SS_y \otimes I \otimes SS_x \otimes I.
\]

In this form, the fifteen generators of \(SO(4, 2)\) are

\[
\Gamma_t \Gamma_x = SS_z \otimes SS_x \otimes SS_z \otimes I, \quad \Gamma_t \Gamma_y = -SS_z \otimes \ell SS_y \otimes SS_z \otimes \ell SS_y, \quad \Gamma_t \Gamma_z = SS_z \otimes SS_z \otimes SS_z \otimes I, \\
\Gamma_x \Gamma_y = I \otimes SS_z \otimes I \otimes \ell SS_y, \quad \Gamma_y \Gamma_z = I \otimes SS_z \otimes I \otimes \ell SS_y, \quad \Gamma_z \Gamma_x = I \otimes \ell SS_y \otimes I \otimes I, \\
\Gamma_q \Gamma_x = -SS_z \otimes SS_z \otimes \ell SS_y \otimes I, \quad \Gamma_q \Gamma_y = SS_z \otimes \ell SS_y \otimes SS_z \otimes \ell SS_y, \quad \Gamma_q \Gamma_z = -SS_z \otimes SS_z \otimes \ell SS_y \otimes I, \\
\Gamma_p \Gamma_x = SS_z \otimes SS_x \otimes SS_z \otimes I, \quad \Gamma_p \Gamma_y = -SS_z \otimes \ell SS_y \otimes SS_z \otimes \ell SS_y, \quad \Gamma_p \Gamma_z = SS_z \otimes SS_z \otimes SS_x \otimes I, \\
\Gamma_t \Gamma_p = -I \otimes I \otimes \ell SS_y \otimes I, \quad \Gamma_t \Gamma_q = I \otimes I \otimes SS_x \otimes I, \quad \Gamma_p \Gamma_q = -I \otimes I \otimes SS_z \otimes I. \quad (43)
\]

There are two steps in this construction at which we can project onto a representation of \(SO(3, 1)\) by projecting from \(\mathbb{H}'\) to \(\mathbb{C}'\). First, in our definition of \(X\) we can set \(p = q = 0\), which is equivalent to restricting \(A\) to be in \(\mathbb{C}' \subset \mathbb{H}'\). Calling this projection \(\pi\), we then have

\[
\pi(\Gamma_p) = \pi(\Gamma_q) = 0
\]

and we are left with only the \(t, x, y,\) and \(z\) elements. The anticommutation relations are obviously still satisfied, and the remaining matrices \(\{\Gamma_a\}\) still generate \(SO(3, 1)\) in the obvious way.

On the other hand, we can make the projection from \(\mathbb{H}'\) to \(\mathbb{C}'\) in the final step by restricting to elements where the third factor is \(1 = I\) or \(L = SS_z\). However, in this case we get an extra generator, namely \(\Gamma_p \Gamma_q\). Exponentiating \(\Gamma_p \Gamma_q\) to find the corresponding group element \(M_{pq}\), we get

\[
M_{pq} = e^{-L\theta/2}I, \quad (44)
\]

which clearly commutes with \(M_{ab} \in SO(3, 1)\), so in fact this is a projection onto

\[
SO(3, 1) \times \mathbb{R} \subset SO(4, 2) \quad (45)
\]

### 6 The Conformal Group

We now show explicitly how to interpret \(SO(4, 2)\) as the conformal group by transforming the representation constructed in Section 4 into one in which the conformal operations are explicit. The conformal group expands the Lorentz group, which consists of rotations and boosts, by adding translations, conformal translations (translations after inverting through the unit sphere; see (70) below), and a dilation (rescaling). We address each of these types of transformations in turn.
Let $V = \text{span}(\{\Gamma_a\})$ and consider $P \in V$ as defined in (12). We also impose the constraints $p + q \neq 0$ and
\[
|P|^2 = \langle P, P \rangle = -t^2 + x^2 + y^2 + z^2 - p^2 + q^2 = 0.
\]
where we have introduced the inner product
\[
\langle A, B \rangle = \frac{1}{8} \text{tr}(AB + BA)
\]
We then define
\[
Q = T \Gamma_t + X \Gamma_x + Y \Gamma_y + Z \Gamma_z,
\]
with
\[
T = \frac{t}{p + q} = \frac{\langle \Gamma_t, P \rangle}{\langle \Gamma_p + \Gamma_q, P \rangle},
\]
\[
X = \frac{x}{p + q} = \frac{\langle \Gamma_x, P \rangle}{\langle \Gamma_p + \Gamma_q, P \rangle},
\]
\[
Y = \frac{y}{p + q} = \frac{\langle \Gamma_y, P \rangle}{\langle \Gamma_p + \Gamma_q, P \rangle},
\]
\[
Z = \frac{z}{p + q} = \frac{\langle \Gamma_z, P \rangle}{\langle \Gamma_p + \Gamma_q, P \rangle},
\]
so that
\[
P = Q(p + q) + p \Gamma_p + q \Gamma_q.
\]
We now consider how $Q$ changes when elements of $\text{SO}(4,2)$ act on $P$. As a first observation, when the rotations ($M_{xy}$, $M_{yz}$, and $M_{zx}$) and boosts ($M_{tx}$, $M_{ty}$, and $M_{tz}$) act on $P$, the effect on $Q$ is the same, since $p + q$ is unaffected. The effect of $M_{pq}$ on $p + q$ is given by
\[
p + q \mapsto p \cosh \theta + q \sinh \theta + q \cosh \theta + p \sinh \theta = (p + q)(\cosh \theta + \sinh \theta),
\]
so that
\[
Q \mapsto Q/(\cosh \theta + \sinh \theta) = Q e^{-\theta}.
\]
since $x$, $y$, $z$, and $t$ are unaffected. This rescaling of $Q$ by $M_{pq}$ represents the dilation.

The translations and conformal translations are best understood by considering null rotations generated by
\[
a_a = \Gamma_p \Gamma_a - \Gamma_q \Gamma_a
\]
and
\[
b_a = \Gamma_p \Gamma_a + \Gamma_q \Gamma_a
\]
First, observe that
\[
(\Gamma_p \Gamma_a \pm \Gamma_q \Gamma_a)^2 = (\Gamma_p \Gamma_a)^2 + (\Gamma_q \Gamma_a)^2 \pm \Gamma_p \Gamma_a \Gamma_p \Gamma_a \Gamma_q \Gamma_a \Gamma_q \Gamma_a = 0,
\]
where in the last equality we have employed the anticommutation relations (16). As a result,

\[ \exp \left( \pm a_x \frac{\theta}{2} \right) = I \pm a_x \frac{\theta}{2}, \]  

(56)

and

\[ \exp \left( \pm b_x \frac{\theta}{2} \right) = I \pm b_x \frac{\theta}{2}. \]  

(57)

Next, we compute, as an example, the action of \( a_x \) on \( P \). To begin, observe that \( a_x \) involves only \( \Gamma_p, \Gamma_x, \) and \( \Gamma_q \), so that \( t, y, \) and \( z \) will be unaffected. Thus

\[
\exp \left( a_x \frac{\theta}{2} \right) \Gamma_x \exp \left( -a_x \frac{\theta}{2} \right) = \left( I + \frac{\theta}{2} \Gamma_p \Gamma_x - \frac{\theta}{2} \Gamma_q \Gamma_x \right) \Gamma_x \left( I - \frac{\theta}{2} \Gamma_p \Gamma_x + \frac{\theta}{2} \Gamma_q \Gamma_x \right) \\
= \left( \Gamma_x + \frac{\theta}{2} \Gamma_p \right) \left( I - \frac{\theta}{2} \Gamma_p \Gamma_x + \frac{\theta}{2} \Gamma_q \Gamma_x \right) \\
= \Gamma_x + \frac{\theta}{2} \Gamma_p - \frac{\theta}{2} \Gamma_q + \frac{\theta^2}{4} \Gamma_x \\
+ \frac{\theta^2}{2} \Gamma_p \Gamma_x - \frac{\theta^2}{2} \Gamma_q \Gamma_x - \frac{\theta^2}{4} \Gamma_x \\
= \Gamma_x + \theta \Gamma_p - \theta \Gamma_q.
\]

(58)

A similar calculation shows that

\[
\Gamma_p \mapsto \theta \Gamma_x - \frac{\theta^2}{2} \Gamma_q + \left( 1 + \frac{\theta^2}{2} \right) \Gamma_p,
\]

(59)

and

\[
\Gamma_q \mapsto \Gamma_q + \theta \Gamma_x + \frac{\theta^2}{2} \Gamma_p.
\]

(60)

Combining (58), (59), and (60), we find

\[
x \Gamma_x + p \Gamma_p + q \Gamma_q \mapsto (x + (p + q) \theta) \Gamma_x + \left( p + \frac{\theta^2}{2} + q + \frac{\theta^2}{2} + x \theta \right) \Gamma_p + \left( q - x \theta - \frac{\theta^2}{2} \right) \Gamma_q.
\]

(61)

Applying (59) to (61),

\[
X' = \frac{x}{p+q} + \theta = X + \theta.
\]

(62)

In other words, \( a_x \) acting on \( P \) has the effect of translating \( Q \) by \( \theta \) in the \( \Gamma_x \) direction. Similar calculations show that \( a_y, a_z \), and \( a_z \) yield corresponding translations.

We now consider the effect of \( b_x \) acting on \( P \). Proceeding as for (58), one finds

\[
\exp \left( b_x \frac{\theta}{2} \right) \Gamma_x \exp \left( -b_x \frac{\theta}{2} \right) = \Gamma_x + \frac{\theta}{2} \Gamma_p + \frac{\theta}{2} \Gamma_q,
\]

(63)

\[
\exp \left( b_x \frac{\theta}{2} \right) \Gamma_q \exp \left( -b_x \frac{\theta}{2} \right) = -\theta \Gamma_x - \frac{\theta^2}{2} \Gamma_p + \left( 1 - \frac{\theta^2}{2} \right) \Gamma_q.
\]

(64)
and,
\[
\exp \left( \frac{b_x \theta}{2} \right) \Gamma_p \exp \left( -\frac{b_x \theta}{2} \right) = \theta \Gamma_x + \left( 1 + \frac{\theta}{2} \right) \Gamma_p + \frac{\theta^2}{2} \Gamma_q. \tag{65}
\]
Taken together, (63), (64), and (65) yield
\[
x \Gamma_x + p \Gamma_p + q \Gamma_q \mapsto (x + (p - q) \theta) \Gamma_x + \left( x \theta + p + p \frac{\theta^2}{2} - q \frac{\theta^2}{2} \right) \Gamma_p
\]
\[
+ \left( x \theta + q - q \frac{\theta^2}{2} + p \frac{\theta^2}{2} \right) \Gamma_q, \tag{66}
\]
from which it follows that
\[
X' = \frac{x + (p - q) \theta}{x + p + p \frac{\theta^2}{2} - q \frac{\theta^2}{2} + x \theta + q - q \frac{\theta^2}{2} + p \frac{\theta^2}{2}} \Gamma_x
\]
\[
= \frac{X + \frac{p - q}{p + q} \theta}{2X \theta + 1 + \frac{p - q}{p + q} \theta^2}
\]
\[
= \frac{X + |Q|^2 \theta}{2X \theta + 1 + |Q|^2 \theta^2}, \tag{67}
\]
where in the last line we have used (46) to write
\[
\frac{p - q}{p + q} \frac{p^2 - q^2}{(p + q)^2} = \frac{-t^2 + x^2 + y^2 + z^2}{(p + q)^2}
\]
\[
= -T^2 + X^2 + Y^2 + Z^2
\]
\[
= \langle Q, Q \rangle \equiv |Q|^2. \tag{68}
\]
To see why this is a conformal translation, we note that an element \( v \) in an inner product space \( V \) satisfies
\[
v^{-1} = \frac{v}{|v|^2}. \tag{69}
\]
Then a conformal translation of \( v \) in the direction of the vector \( \alpha \) is given by
\[
v \mapsto (v^{-1} + \alpha)^{-1} = \frac{v + \alpha |v|^2}{1 + 2 \langle v, \alpha \rangle + |\alpha|^2 |v|^2}. \tag{70}
\]
Taking \( v = Q \) and assuming \( \alpha = \theta \Gamma_x \), the \( \Gamma_x \) component of (70) becomes precisely (67). Again, a similar calculation shows that \( b_y, b_z, \) and \( b_t \) are the other conformal translations.

The fifteen elements of \( \text{SO}(4,2) \) therefore act on \( Q \) via (18) and (50) as the conformal group, with 3 rotations, 3 boosts, a dilation, 4 translations, and 4 conformal translations.
7 Conclusion

As shown in Section 4, we obtain a $2 \times 2$ representation of $SO(4, 2)$ over $\mathbb{H}' \otimes \mathbb{C}$ simply by restricting the $4 \times 4$ representation via (18) to one of the $2 \times 2$ blocks of $P$, say $X$. The resulting action via (37) reproduces that of Manogue and Schray [12] when both are restricted to $SO(3, 1)$, that is, when the matrix elements are complex. However, as already noted, it is not possible to express (37) in the simple form involving Hermitian conjugation that was used by Manogue and Schray in all cases. Nonetheless, we have shown that

$$SO(4, 2) \equiv SU(2, \mathbb{H}' \otimes \mathbb{C}) \quad (71)$$

(up to double-cover issues). This is our primary result.

We expect the construction given in Sections 3 and 4 to carry over with minimal modification to any combination of division algebras, yielding a representation of each group in Table 1 of the form $SU(2, \mathbb{K}' \otimes \mathbb{K})$. For further details, see [7]. It is hoped that this construction can be further generalized to the Freudenthal-Tits magic square itself, leading to an interpretation of each group in that magic square of the form $SU(3, \mathbb{K}' \otimes \mathbb{K})$, and providing new insight into the exceptional groups $E_6$, $E_7$, and $E_8$.

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Appendix

We list here the fifteen $4 \times 4$ matrices over $\mathbb{H}' \otimes \mathbb{C}$ that generate $SO(4, 2)$ under the action (18):

$$M_{xy} = \begin{pmatrix} e^{i\phi/2} & 0 & 0 & 0 \\ 0 & e^{-i\phi/2} & 0 & 0 \\ 0 & 0 & e^{i\phi/2} & 0 \\ 0 & 0 & 0 & e^{-i\phi/2} \end{pmatrix},$$

$$M_{yz} = \begin{pmatrix} \cos \frac{\phi}{2} & i \sin \frac{\phi}{2} & 0 & 0 \\ i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\phi}{2} & i \sin \frac{\phi}{2} \\ 0 & 0 & i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix},$$

$$M_{zx} = \begin{pmatrix} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} & 0 & 0 \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ 0 & 0 & -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix},$$

$$M_{zz} = \begin{pmatrix} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} & 0 & 0 \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ 0 & 0 & -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}.$$
\[
M_{qx} = \begin{pmatrix}
\cos \frac{\phi}{2} & K \sin \frac{\phi}{2} & 0 & 0 \\
K \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cos \frac{\phi}{2} & -K \sin \frac{\phi}{2} \\
0 & 0 & -K \sin \frac{\phi}{2} & \cos \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{qy} = \begin{pmatrix}
\cos \frac{\phi}{2} & -K \otimes i \sin \frac{\phi}{2} & 0 & 0 \\
K \otimes i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cos \frac{\phi}{2} & K \otimes i \sin \frac{\phi}{2} \\
0 & 0 & -K \otimes i \sin \frac{\phi}{2} & \cos \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{qz} = \begin{pmatrix}
e^{K \phi/2} & 0 & 0 & 0 \\
e^{-K \phi/2} & 0 & 0 & 0 \\
0 & 0 & e^{K \phi/2} & 0 \\
0 & 0 & 0 & e^{-K \phi/2}
\end{pmatrix},
\]
\[
M_{tx} = \begin{pmatrix}
\cosh \frac{\phi}{2} & L \sinh \frac{\phi}{2} & 0 & 0 \\
L \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cosh \frac{\phi}{2} & -L \sinh \frac{\phi}{2} \\
0 & 0 & -L \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{ty} = \begin{pmatrix}
\cosh \frac{\phi}{2} & L \sinh \frac{\phi}{2} & 0 & 0 \\
L \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cosh \frac{\phi}{2} & -L \otimes i \sinh \frac{\phi}{2} \\
0 & 0 & -L \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{tz} = \begin{pmatrix}
e^{KL \phi/2} & 0 & 0 & 0 \\
e^{-KL \phi/2} & 0 & 0 & 0 \\
0 & 0 & e^{KL \phi/2} & 0 \\
0 & 0 & 0 & e^{-KL \phi/2}
\end{pmatrix},
\]
\[
M_{p} = \begin{pmatrix}
\cosh \frac{\phi}{2} & K \otimes i \sinh \frac{\phi}{2} & 0 & 0 \\
K \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cosh \frac{\phi}{2} & -K \otimes i \sinh \frac{\phi}{2} \\
0 & 0 & -K \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{px} = \begin{pmatrix}
\cosh \frac{\phi}{2} & -KL \otimes i \sinh \frac{\phi}{2} & 0 & 0 \\
KL \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2} & 0 & 0 \\
0 & 0 & \cosh \frac{\phi}{2} & KL \otimes i \sinh \frac{\phi}{2} \\
0 & 0 & -KL \otimes i \sinh \frac{\phi}{2} & \cosh \frac{\phi}{2}
\end{pmatrix},
\]
\[
M_{pz} = \begin{pmatrix}
e^{KL \phi/2} & 0 & 0 & 0 \\
e^{-KL \phi/2} & 0 & 0 & 0 \\
0 & 0 & e^{KL \phi/2} & 0 \\
0 & 0 & 0 & e^{-KL \phi/2}
\end{pmatrix},
\]
\[
M_{py} = \begin{pmatrix}
e^{L \phi/2} & 0 & 0 & 0 \\
e^{-L \phi/2} & 0 & 0 & 0 \\
0 & 0 & e^{L \phi/2} & 0 \\
0 & 0 & 0 & e^{-L \phi/2}
\end{pmatrix},
\]
\[
M_{pq} = \begin{pmatrix}
e^{L \phi/2} & 0 & 0 & 0 \\
e^{-L \phi/2} & 0 & 0 & 0 \\
0 & 0 & e^{L \phi/2} & 0 \\
0 & 0 & 0 & e^{-L \phi/2}
\end{pmatrix},
\]
(72)
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