A NEW HOMOLOGICAL INVARIANT FOR MODULES

MOHAMMADALI IZADI

ABSTRACT. Let $R$ be a commutative Noetherian local ring with residue field $k$. Using the structure of Vogel cohomology, for any finitely generated module $M$, we introduce a new dimension, called $\zeta$-dimension, denoted by $\zeta\text{-dim}_RM$. This dimension is finer than Gorenstein dimension and has nice properties enjoyed by homological dimensions. In particular, it characterizes Gorenstein rings in the sense that: a ring $R$ is Gorenstein if and only if every finitely generated $R$-module has finite $\zeta$-dimension. Our definition of $\zeta$-dimension offers a new homological perspective on the projective dimension, complete intersection dimension of Avramov et al. and G-dimension of Auslander and Bridger.

1. INTRODUCTION

Let $M$ be a finitely generated module over a commutative Noetherian ring $R$. There are several homological invariants assigned to $M$. The most important one is projective dimension $\text{pd}_RM$. Auslander and Bridger [AB] singled out the class of $R$-modules of finite Gorenstein dimension ($G$-dimension) as a generalization of modules of finite projective dimension. Avramov et al. [AGP] introduced the concept of complete intersection (CI)-dimension. The purpose of this paper is to offer a new dimension that we believe gives a new homological perspective on the aforementioned concepts. We show that the above dimensions can be considered as special cases of a much more general dimension that we shall call it $\zeta$-dimension. To introduce this dimension, we use the structure of Vogel cohomology developed by Pierre Vogel in 1980. The cohomology theory that he developed, associates to each pair $(M,N)$ of modules, a sequence of $R$-modules $\tilde{\text{Ext}}^n_R(M,N)$ for $n \in \mathbb{Z}$, and comes equipped with a natural transformation $\zeta^*(M,N) : \text{Ext}^n_R(M,N) \to \tilde{\text{Ext}}^n_R(M,N)$ of cohomology functors. For any $R$-module $M$ and any integer $i$, we let $\zeta^i(M)$ denote the natural map $\text{Ext}^i_R(M,k) \to \tilde{\text{Ext}}^i_R(M,k)$, where $k$ is the residue field of $R$, and define $\zeta\text{-dim}_RM$ to be the infimum $n \in \mathbb{N}$, such that $\zeta^n(M)$ is epimorphism and $\zeta^n(M)$ is isomorphism for all $i > n$. Note that $\text{pd}_RM$, when it is finite, is equal to the supremum of $i$'s such that $\text{Ext}^i_R(M,k) \neq 0$. On the other hand, $\tilde{\text{Ext}}^i_R(M,k)$ will vanish, for all $i \in \mathbb{Z}$. So in fact $\text{pd}_RM$ is equal to the infimum of $i$'s such that $\text{Ext}^i_R(M,k) \cong \tilde{\text{Ext}}^i_R(M,K)$, that is $\text{pd}_RM = \zeta\text{-dim}_RM$. Moreover, if $G\text{-dim}_RM$ is finite, we shall show that it can be interpreted as the vanishing of certain $\zeta^i$'s (see Theorem 1.4).

Moreover we examine the ability of $\zeta$-dimension to detect Gorensteinness of the underlying ring: it is finite for all modules over a Gorenstein ring, conversely if $\zeta\text{-dim}_RM < \infty$ the ring is Gorenstein. $\zeta$-dimension shares many basic properties with other homological dimensions. In particular, it localizes. In an attempt to find a lower bound for $\zeta$-dimension, in Theorem 1.8, for any finitely generated $R$-module $M$.

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we obtain the following inequality \( \text{apd}_R M \leq \zeta \dim R M \), with equality if \( \zeta - \dim R M \) is finite. We recall that \( \text{apd}_R M \) is defined by the formula

\[
\text{apd}_R M = \sup \left\{ i \in \mathbb{N}_0 \mid \text{Ext}^i_R(M, T) \neq 0 \text{ for some finitely generated } R\text{-module } T \text{ with } \text{pd}_R T < \infty \right\}.
\]

So the place of \( \zeta - \dim R (M) \) in the hierarchy of homological dimensions is determined as

\[
\text{apd}_R M \leq \zeta - \dim R M \leq \text{G-dim}_R (M) \leq \text{CI-dim}_R (M) \leq \text{pd}_R (M),
\]

with equality to the left of any finite ones. No example of a module \( M \) with \( \zeta - \dim R M < \text{G-dim}_R (M) \) is known at present to the authors.

Towards the end of the paper, we deal with the resolving property of the category of modules of finite \( \zeta \)-dimension. Throughout the paper \((R, m, k)\) is a commutative Noetherian local ring with residue field \( k \).

2. \( \zeta \)-DIMENSION

We begin by recalling the construction of Vogel cohomology. First let us mention that, abusing notation we shall use the symbol \((A, B)\) for the graded Hom-functor applied to graded \( R \)-modules \( A \) and \( B \). Thus \((A, B)_n = \Pi_{i \in \mathbb{Z}} \text{Hom}_R(A_i, B_{i+n})\). The differential is defined on \((A, B)\) by the formula \( \partial(f)(x) = f(\partial(x)) - (-1)^{\deg f} f(\partial(x)) \), where \( x \in A \), thus making \((A, B)\) into a complex.

Let \( M \) and \( N \) be finitely generated \( R \)-modules and \( P_M \) and \( P_N \) denote their projective resolutions, respectively. We shall use \( P_M \) (resp. \( P_N \)) to denote the corresponding underlying graded module. The subset \((P_M, P_N)_b\) of bounded homogeneous maps (a homogeneous map is called bounded if only finitely many components of that map are non-zero) is a graded submodule of \((P_M, P_N)\). The restriction of \( \partial \) to \((P_M, P_N)_b\) make it into a subcomplex of \((P_M, P_N)\). The corresponding quotient complex will be of fundamental important to us. We denote by \((\tilde{P}_M, \tilde{P}_N)\) the quotient complex

\[
(\tilde{P}_M, \tilde{P}_N) = (P_M, P_N)/(P_M, P_N)_b.
\]

Passing on to cohomology we obtain Vogel cohomology. It will be denoted by \( \tilde{\text{Ext}}^*_R (M, N) \). Moreover the short exact sequence

\[
0 \to (P_M, P_N)_b \to (\tilde{P}_M, \tilde{P}_N) \to 0,
\]

where the cohomology of the middle term is just \( \tilde{\text{Ext}}^*_R (M, N) \), yields upon passing to corresponding long cohomology exact sequence, a natural transformation \( \zeta^*(M, N) : \text{Ext}^*_R (M, N) \to \tilde{\text{Ext}}^*_R (M, N) \).

By essentially following the same argument analogous to ordinary cohomology, one can see that \( \tilde{\text{Ext}}^*_R \) is a cohomological functor, independent of the choice of projective resolutions of \( M \) and \( N[G, I] \). The following result that will be used latter, is easy to see.

**Proposition 2.1.** Let \((R, m)\) be a commutative Noetherian local ring. Then for any \( R \)-module \( M \) the following are equivalent:

i) \( \text{pd}_R M \) is finite.

ii) \( \tilde{\text{Ext}}^i_R (M, ) = 0 \) for all integer \( i \).

iii) \( \tilde{\text{Ext}}^i_R (, M) = 0 \) for all integer \( i \).
For simplicity, for any \( R \)-module \( M \) and any integer \( i \) we let \( \zeta^i(M) \) denote the natural transformation \( \zeta^i(M, k) : \text{Ext}^i_R(M, k) \to \text{Ext}^i_R(M, k) \).

**Definition 2.2.** Let \( M \neq 0 \) be a finitely generated \( R \)-module. We assign an invariant to \( M \), called \( \zeta \)-dimension of \( M \), denoted \( \zeta \)-dim\(_R M \), by the formula

\[
\zeta \text{-dim}_R M = \inf \left\{ n \left| \begin{array}{c}
\zeta^n(M) \text{ is epimorphism and } \zeta^i(M) \text{ is isomorphism for all } i > n \\
\end{array} \right. \right\}.
\]

We complement this by setting \( \zeta \)-dim\(_R 0 = -\infty \).

Next theorem is our first main result.

**Theorem 2.3.** Let \( (R, m, k) \) be a commutative Noetherian local ring. Let \( n \in \mathbb{N} \). Then the following conditions are equivalent:

i) \( \zeta \)-dim\(_R M \leq n \), for all finitely generated \( R \)-module \( M \).

ii) \( \zeta \)-dim\(_R k \leq n \).

iii) \( \zeta^i(k) \) is epimorphism, for all \( i \geq n \).

**Proof.** The implications (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are trivially hold.

(ii) \( \Rightarrow \) (i). Let \( M \) be a finitely generated \( R \)-module. We induce on \( \dim M \). Suppose first \( \dim M = 0 \). So \( l(M) \), the lengths of \( M \) is finite, say \( s \). We use induction on \( s \) to prove the result in this case. If \( s = 1 \), there is nothing to prove. So let \( s > 1 \), and consider the short exact sequence

\[
0 \rightarrow k \rightarrow M \rightarrow N \rightarrow 0,
\]

where \( l(N) = s - 1 \). For any integer \( i \), there exists a commutative diagram of \( R \)-modules and \( R \)-homomorphisms

\[
\begin{array}{ccccccccc}
\cdots & \rightarrow & \text{Ext}^{i-1}_R(k, k) & \rightarrow & \text{Ext}^i_R(N, k) & \rightarrow & \text{Ext}^i_R(M, k) & \rightarrow & \text{Ext}^i_R(k, k) & \rightarrow & \cdots \\
\downarrow \zeta^{i-1}(k) & & \downarrow \zeta^i(N) & & \downarrow \zeta^i(M) & & \downarrow \zeta^i(k) \\
\cdots & \rightarrow & \tilde{\text{Ext}}^{i-1}_R(k, k) & \rightarrow & \tilde{\text{Ext}}^i_R(N, k) & \rightarrow & \tilde{\text{Ext}}^i_R(M, k) & \rightarrow & \tilde{\text{Ext}}^i_R(k, k) & \rightarrow & \cdots \\
\end{array}
\]

By induction assumption, \( \zeta^i(k) \) and \( \zeta^i(N) \) are both epimorphism for \( i = n \) and isomorphism for \( i > n \). This by a simple diagram chasing, in view of the Five Lemma, will implies that \( \zeta^i(M) \) is epimorphism for \( i = n \) and isomorphism for \( i > n \). This completes the proof in this case. Now suppose, inductively, that \( \dim M = n > 0 \) and the result has been proved for all \( R \)-modules of dimension less than \( n \). Consider the short exact sequence

\[
0 \rightarrow \Gamma_m(M) \rightarrow M \rightarrow M/\Gamma_m(M) \rightarrow 0
\]

of \( R \)-modules, where \( \Gamma_m(M) \) denotes the \( m \)-torsion functor \( \cup_{n \in \mathbb{N}} \text{Tor}_n(M, m^n) \). This in turn induces, for any integer \( i \) a commutative diagram of \( R \)-modules and \( R \)-homomorphisms

\[
\begin{array}{ccccccccc}
\text{Ext}^{i-1}_R(\Gamma_m(M), k) & \rightarrow & \text{Ext}^i_R(M/\Gamma_m(M), k) & \rightarrow & \text{Ext}^i_R(M, k) & \rightarrow & \text{Ext}^i_R(\Gamma_m(M), k) \\
\downarrow \zeta^{i-1}(\Gamma_m(M)) & & \downarrow \zeta^i(M/\Gamma_m(M)) & & \downarrow \zeta^i(M) & & \downarrow \zeta^i(\Gamma_m(M)) \\
\tilde{\text{Ext}}^{i-1}_R(\Gamma_m(M), k) & \rightarrow & \tilde{\text{Ext}}^i_R(M/\Gamma_m(M), k) & \rightarrow & \tilde{\text{Ext}}^i_R(M, k) & \rightarrow & \tilde{\text{Ext}}^i_R(\Gamma_m(M), k) \\
\end{array}
\]
By inductive assumption, $\zeta^n(\Gamma_m(M))$ is epimorphism and $\zeta^i(\Gamma_m(M))$ is isomorphism for all $i > n$. So $\zeta\text{-dim}_RM \leq n$ if and only if $\zeta\text{-dim}_R(M/\Gamma_m(M)) \leq n$. We can therefore assume, in the inductive step that there exists an element $r \in R$ which is a non-zero divisor on $M$. The exact sequence

$$0 \to M \xrightarrow{\iota} M \to M/rM \to 0$$

induces for any integer $i$, a commutative diagram of $R$-modules and $R$-homomorphisms

$$\begin{array}{cccccc}
\text{Ext}^i_R(M/rM, k) & \to & \text{Ext}^i_R(M, k) & \to & \text{Ext}^i_R(M, k) & \to & \text{Ext}^{i+1}_R(M/rM, k) \\
\downarrow \zeta^i(M/rM) & & \downarrow \zeta^i(M) & & \downarrow \zeta^i(M) & & \downarrow \zeta^{i+1}(M/rM)
\end{array}$$

Since $\dim M/rM < \dim M$, by inductive assumption, $\zeta\text{-dim}_R(M/rM) \leq n$. So $\zeta^n(M/rM)$ is epimorphism and $\zeta^i(M/rM)$ is isomorphism for all $i > n$. But $r \in m$ and so each element of $\text{Ext}^i_R(M, k)$ is annihilated by multiplication by $r$. This fact in conjunction with the latter diagram, will implies that $\zeta^n(M)$ is epimorphism and $\zeta^i(M)$ is isomorphism for all $i > n$.

(iii) $\Rightarrow$ (ii). If $R$ is regular, the result is clear, because $\text{pd}_R k$ is finite and so $\text{Ext}^i_R(k, k) = 0$ for all integer $i$. So let $R$ is non-regular. Then by [M2, Theorem 6], $\zeta^i(k)$ is monomorphism for all integer $i$. This follows the result.\[\square\]

Following result shows that $\zeta\text{-dim}_R M$ is a refinement of $G$-dimension. We prepare it by recalling the structure of Tate cohomology, introduced through complete resolutions, that has been the subject of several recent expositions, in particular by Buchweitz [B] and Cornick and Kropholler [CK]. Let $M$ be a finite $R$-module of finite $G$-dimension. Choose a complete resolution $T \xrightarrow{\nu} P \xrightarrow{\pi} M$ of $M$ (see for instance [AM, Sec.5]). Then for each $R$-module $N$ and for each $n \in \mathbb{Z}$, Tate cohomology group is defined by the equality

$$\text{Ext}^{-n}_R(M, N) = H^n\text{Hom}_R(T, N).$$

**Theorem 2.4.** For any finitely generated $R$-module $M$, there is an inequality

$$\zeta\text{-dim}_R M \leq G\text{-dim}_R M$$

with equality, when $G\text{-dim}_R M$ is finite.

**Proof.** Without loss of generality, we may assume that $G\text{-dim}_R M = g$ is finite. With this assumption, by [M1, 2], for any integer $i$, there is a natural isomorphism of cohomology functors $\text{Ext}^i_R(M, k) \cong \text{Hom}_R(M, k)$, compatible with the maps coming from $\text{Ext}^i_R(M, k)$. If $g = 0$, using the definition of Tate cohomology it is easily seen that $\text{Ext}^0_R(M, k) \cong \text{Hom}_R(M, k)/N$, for suitable $R$-module $N$. So $\zeta^0(M)$ is always epimorphism. Moreover it follows from [AM, 5.2(2)] that $\zeta^i(M)$ is isomorphism for all $i > 0$. Hence $\zeta\text{-dim}_R M = 0$. Now let $g > 0$. It follows from [AM, 5.2(2)] that $\zeta^i(M)$ is isomorphism for all $i > g$ and follows from [AM, 7.1] that $\zeta^0(M)$ is epimorphism. So $\zeta\text{-dim}_R M \leq G\text{-dim}_R M$. For equality, consider a Gorenstein resolution of $M$, say

$$0 \to P_g \to P_{g-1} \to \cdots \to P_1 \to G_0 \to M \to 0$$

with all the $P_i$’s finitely generated and projective and with $G_0$ of Gorenstein dimension zero. But then we can also assume that each $P_i \to P_{i-1}$ gives a projective cover of
the image of $P_i$ in $P_{i-1}$. So in particular this means that $P_g \subseteq mP_{g-1}$. But this implies that $\text{Hom}_R(P_{g-1}, k) \to \text{Hom}_R(P_g, k)$ is the zero map. So if $P_g \neq 0$ we see that $\text{Ext}^g_R(M, k) \neq 0$. Here we are tacitly assuming $g \geq 2$. If $g = 1$, then the resolution of $M$ looks like $0 \to F \to G \to M \to 0$. Here $F$ is a finitely generated free $R$-module. We can assume that $F \subseteq mG$, for if not one can use Nakayama’s lemma to get a copy of $R$ in $F$ a direct summand of $G$. But then we can go modulo this copy of $R$. So repeating if necessary, we finally get that $F \subseteq mG$. But then if $F \neq 0$, by the same type argument as above we get that $\text{Ext}^1_R(M, k) \neq 0$. Hence by [AM, 7.1], either $\zeta^g(M)$ is not injective or $\zeta$-$\dim^{g-1}(M)$ is not epimorphism. So $\zeta$-$\dim_RM \geq g$. This completes the proof.

Now we are in position to put all our results together to present a characterization for Gorenstein rings in terms of $\zeta$-dimension. We need the following proposition.

**Proposition 2.5.** Let $M$ be a finitely generated $R$-module of finite $\zeta$-dimension, say $n$. Then for any finitely generated $R$-module $N$, $\zeta^n(M, N)$ is epimorphism and $\zeta^i(M, N)$ is isomorphism for all $i > n$.

**Proof.** Let $N$ be a finitely generated $R$-module. By following the same type argument as we have used for the proof of Theorem 1.3, we may assume inductively that $\dim N > 0$, the result holds for all finitely generated modules of dimension less than $\dim N$ and also there exists a non-zerodivisor $r$ on $N$. So we have a short exact sequence $0 \to N \xrightarrow{r} N \to N/rN \to 0$. This induces, for any integer $i$, a commutative diagram of $R$-modules and $R$-homomorphisms

\[
\begin{array}{cccccc}
\text{Ext}_R^i(M, N/rN) & \longrightarrow & \text{Ext}_R^{i+1}(M, N) & \longrightarrow & \text{Ext}_R^{i+1}(M, N/rN) & \longrightarrow \\
\gamma(M, N/rN) & \downarrow & \gamma^{i+1}(M, N) & \downarrow & \gamma^{i+1}(M, N/rN) & \\
\text{Ext}_R^i(M, N/rN) & \longrightarrow & \text{Ext}_R^{i+1}(M, N) & \longrightarrow & \text{Ext}_R^{i+1}(M, N/rN) & \longrightarrow \\
\end{array}
\]

Since for any $i \geq n$, $\zeta^i(M, N/rN)$ is epimorphism and $\zeta^{i+1}(M, N/rN)$ is isomorphism, by a diagram chasing one can see that the multiplication map by $r$ restricted to $\text{Ker}\zeta^{i+1}(M, N)$ is epimorphism. So using Nakayama’s Lemma, for any $i \geq n$, we get that the map $\zeta^{i+1}(M, N)$ is monomorphism. Now consider the commutative diagram

\[
\begin{array}{cccccc}
\text{Ext}_R^i(M, N) & \longrightarrow & \text{Ext}_R^i(M, N) & \longrightarrow & \text{Ext}_R^i(M, N/rN) & \longrightarrow \\
\gamma(M, N) & \downarrow & \gamma(M, N) & \downarrow & \gamma(M, N/rN) & \downarrow \\
\text{Ext}_R^i(M, N) & \longrightarrow & \text{Ext}_R^i(M, N) & \longrightarrow & \text{Ext}_R^i(M, N/rN) & \longrightarrow \\
\end{array}
\]

Since for any integer $i \geq n$, $\zeta^i(M, N/rN)$ is epimorphism and $\zeta^{i+1}(M, N)$ is monomorphism, the restriction of the multiplication map $r$ to $\text{Coker}\zeta^i(M, N)$ is surjective, and so by Nakayama’s Lemma, $\text{Coker}\zeta^i(M, N) = 0$. Therefore $\zeta^i(M, N)$ is epimorphism for all $i \geq n$. This completes the proof. \hfill \Box

**Theorem 2.6.** The following conditions are equivalent:

i) $R$ is Gorenstein.
Proof. (i) \( \Rightarrow \) (ii). Since \( R \) is Gorenstein, by [AB], G-dim\(_R\)M < \( \infty \), for all finitely generated \( R \)-module \( M \). So by Theorem 1.4, \( \zeta \)-dim\(_R\)M < \( \infty \).

(ii) \( \Rightarrow \) (iii). This trivially holds.

(iii) \( \Rightarrow \) (i). Let \( \zeta \)-dim\(_R\)k < \( \infty \). By Proposition 1.5, \( \zeta^i(k, R) \) is isomorphism for all integer \( i > \zeta \)-dim\(_R\)k. But \( \tilde{\text{Ext}}^i_R(k, R) = 0 \) for all \( i \). This implies that \( \text{Ext}^i_R(k, R) = 0 \) for all \( i \) large enough. So \( R \) is Gorenstein.

\( \square \)

**Proposition 2.7.** Let \( \mathfrak{p} \) be a prime ideal in Spec(\( R \)). Then for any finitely generated \( R \)-module \( M \),

\[ \zeta \text{-dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \zeta \text{-dim}_{R} M. \]

Proof. For proof just one should note that both functors \( \text{Ext}^*_R \) and \( \tilde{\text{Ext}}^*_R \) are well behaved under localization. \( \square \)

Now we aim to give a lower bound for \( \zeta \)-dim\(_R\)M. There is a refinement of projective dimension of \( M \) denoted by apd\(_R\)M, defined by the formula

\[ \text{apd}_R M = \sup \left\{ i \in \mathbb{N}_0 \mid \text{Ext}^i_R(M, T) \neq 0 \text{ for some finitely generated } R \text{-module } T \text{ with pd}_RT < \infty \right\}. \]

It is proved in [AB, Theorem 4.13] that apd\(_R\) is also a refinement of G-dimension G-dim\(_R\). We shall show that apd\(_R\)M \( \leq \) \( \zeta \)-dim\(_R\)M, with equality where \( \zeta \)-dim\(_R\)M is finite.

**Theorem 2.8.** Let \( M \) be a finitely generated \( R \)-module. Then

\[ \text{apd}_R M \leq \zeta \text{-dim}_{R} M \]

with equality if \( \zeta \)-dim\(_R\)M is finite.

Proof. We may (and do) assume that \( \zeta \)-dim\(_R\)M = \( n \) is finite. So by Proposition 1.5, for any finitely generated \( R \)-module \( N \), \( \zeta^n(M, N) \) is epimorphism and \( \zeta^i(M, N) \) is isomorphism for all \( i > n \). Now let \( N \) be an \( R \)-module of finite projective dimension.

By Proposition 1.1, \( \tilde{\text{Ext}}^i_R(M, N) = 0 \) for all integer \( i \). So we get \( \text{Ext}^i_R(M, N) = 0 \) for all \( i > n \). This implies that apd\(_R\)M \( \leq \) \( \zeta \)-dim\(_R\)M. Now let apd\(_R\)(\( M \)) = \( s \) and \( s < n \).

We seek for a contradiction. The short exact sequence \( 0 \to m \to R \to k \to 0 \) induces the following commutative diagram

\[
\begin{array}{cccccc}
\text{Ext}^{n−1}_R(M, k) & \longrightarrow & \text{Ext}^n_R(M, m) & \longrightarrow & \text{Ext}^n_R(M, k) & \longrightarrow & \text{Ext}^{n+1}_R(M, m) \\
\downarrow \zeta^{n−1}(M, k) & & \downarrow \zeta^n(M, m) & & \downarrow \zeta^n(M, k) & & \downarrow \zeta^{n+1}(M, m) \\
\tilde{\text{Ext}}^{n−1}_R(M, k) & \longrightarrow & \tilde{\text{Ext}}^n_R(M, m) & \longrightarrow & \tilde{\text{Ext}}^n_R(M, k) & \longrightarrow & \tilde{\text{Ext}}^{n+1}_R(M, m)
\end{array}
\]

Since \( s < n \), \( \text{Ext}^i_R(M, R) = 0 \) for all \( i \geq n \). So since \( \zeta^{n+1}(M, m) \) is isomorphism, we get \( \zeta^n(M, k) \) is isomorphism and since \( \zeta^n(M, m) \) is epimorphism, we get \( \zeta^{n−1}(M, k) \) is epimorphism. Therefore \( \zeta \)-dim\(_R\)M \( \leq n − 1 \). This is the desired contradiction. So \( s = n \).

\( \square \)
Corollary 2.9. For any finitely generated $R$-module $M$,
\[ \text{apd}_R(M) \leq \zeta \text{-dim}_R M \leq G \text{-dim}_R M \leq \text{CI} \text{-dim}_R M \leq \text{pd}_R M, \]
with equality to the left of any finite ones.

Finally we show that the category of modules of finite $\zeta$-dimension has resolving property. Let $\mathcal{Z}$ (resp. $\tilde{\mathcal{Z}}$) denotes the full subcategory of $\mathcal{F}$, the category of finitely generated $R$-modules and $R$-homomorphisms, whose objects are modules of $\zeta$-dimension zero (resp. of finite $\zeta$-dimension).

Proposition 2.10. The category $\mathcal{Z}$ is closed under extension and kernels of epimorphisms. Moreover $\mathcal{Z}$ contains $\mathcal{P}$, the category of finite projective $R$-modules.

Proof. Let $0 \to E \to L \to M \to 0$ be an exact sequence of $R$-modules with $M \in \mathcal{Z}$. We shall show that $L \in \mathcal{Z}$ if and only if $E \in \mathcal{Z}$. The above short exact sequence induces for any integer $i$, a commutative diagram of $R$-modules and $R$-homomorphisms
\[ \begin{array}{cccccc}
\text{Ext}^i_R(M,k) & \longrightarrow & \text{Ext}^i_R(L,k) & \longrightarrow & \text{Ext}^i_R(E,k) & \longrightarrow & \text{Ext}^{i+1}_R(M,k) \\
\downarrow^{\zeta(M)} & & \downarrow^{\zeta(L)} & & \downarrow^{\zeta(E)} & & \downarrow^{\zeta^{i+1}(M)} \\
\tilde{\text{Ext}}^i_R(M,k) & \longrightarrow & \tilde{\text{Ext}}^i_R(L,k) & \longrightarrow & \tilde{\text{Ext}}^i_R(E,k) & \longrightarrow & \tilde{\text{Ext}}^{i+1}_R(M,k) 
\end{array} \]

It is now easy to deduce the first assertion, by a simple diagram chasing and also five lemma. The last assertion is elementary. □

Proposition 2.11. The category $\tilde{\mathcal{Z}}$ is closed under extension, kernel of epimorphisms and cokernel of monomorphisms. Moreover $\tilde{\mathcal{Z}} \supseteq \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}}$ denotes the subcategory of $\mathcal{F}$ whose objects are finitely generated $R$-modules of finite $G$-dimension.

Proof. Last assertion follows from Theorem 1.4. The other ones are easy to see. □
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School of Science and Environment (Mathematics), Grenfell Campus, Memorial
University of Newfoundland, Corner Brook, NL, A2H 6P9, Canada
E-mail address: mizadi@grenfell.mun.ca