Dark energy and viscous cosmology

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Abstract

Singularities in the dark energy universe are discussed, assuming that there is a bulk viscosity in the cosmic fluid. In particular, it is shown how the physically natural assumption of letting the bulk viscosity be proportional to the scalar expansion in a spatially flat FRW universe can drive the fluid into the phantom region ($w < -1$), even if lies in the quintessence region ($w > -1$) in the non-viscous case.

KEY WORDS: dark energy, viscous cosmology, Big Rip

1 Introduction

Recent astrophysical data indicate the presence of a mysterious kind of energy (ideal fluid with negative pressure), contributing about 70\% of the total energy of the universe. It is quite possible that the equation of state parameter $w$ for dark energy is less than -1. If this is so, the universe shows some very strange properties such as the future finite singularity called Big Rip \textsuperscript{1}. In turn, this can lead to the occurrence of negative entropy \textsuperscript{2}. One may expect that natural effects, most likely of a quantum mechanical origin, may prevent the Big Rip, as was shown in

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Ref. [3]. Dependent on which sorts of dark energy are present, various types of singularities appear; for a classification, see Ref. [4]. In the present letter, we will consider the role of a bulk viscosity $\zeta$ in a universe having a Big Rip singularity. We take the universe to be spatially flat, with cosmological constant equal to zero. The shear viscosity $\eta$ will be put equal to zero, in conformity with usual practice. The latter assumption is however a non-trivial point, since the shear viscosity is usually so much greater than the bulk viscosity. At least, this is so in the early universe. For instance, in the plasma era after the time of recombination ($T \approx 4000$ K), the ratio $\eta/\zeta$ as calculated from kinetic theory is as large as about $10^{12}$. Therefore, even a slight anisotropy in the cosmic fluid would easily outweigh the influence from the bulk viscosity. We will below be concerned with the late universe and not the early one. However, we ought to bear in mind that the neglect of $\eta$ most likely rests on the tacit assumption about an extreme fine tuning of the spatial isotropy of the fluid.

In the next section we present the formalism for the FRW universe when the equation of state is of a general form, $w = w(\rho)$. Similarly, we admit a general form for the bulk viscosity, $\zeta = \zeta(\rho)$. Thereafter we discuss some special cases. Of main interest appears to be the case where $\zeta$ is proportional to the scalar expansion $\theta$, $\zeta = \tau \theta$, $\tau$ being a constant. Then, it turns out that the barrier $w = -1$ between the quintessence region ($w > -1$) and the phantom region ($w < -1$) can be crossed, as a consequence of the bulk viscosity. To our knowledge, this has not been pointed out before. The case $\zeta = \tau \theta$, implying proportionality of the bulk viscosity to the divergence of the fluid’s velocity vector, is physical natural, and has been considered earlier in an astrophysical context. Cf., for instance, the review article of Grøn [6].

2 General formalism

We assume the standard FRW metric,

$$ds^2 = -dt^2 + a^2(t) \, dx^2,$$  \hspace{1cm} (1)
and put the spatial curvature $k$ as well as the cosmological constant $\Lambda$ equal to zero. The Hubble parameter is $H = \dot{a}/a$, and the scalar expansion is $\theta \equiv U^\mu{}_{,\mu} = 3H$, $U^\mu$ being the four-velocity of the fluid. We assume the equation of state in the form

$$p = w(\rho)\rho,$$

where the thermodynamical variable $w(\rho)$ is an arbitrary function of the density. Similarly, we assume that the bulk viscosity is arbitrary, $\zeta = \zeta(\rho)$. The Friedmann equations are (cf., for instance, Ref. [5])

$$\theta^2 = 24\pi G\rho,$$

$$\dot{\rho} + (\rho + p)\theta = \zeta \theta^2,$$

$$\frac{\ddot{a}}{a} + \frac{\theta^2}{18} = -4\pi G(p - \zeta \theta).$$

The effective pressure is $\tilde{p} = p - \zeta \theta$. On thermodynamical grounds, in conventional physics $\zeta$ has to be a positive quantity. This is a consequence of the positive sign of the entropy change in an irreversible process (cf., for instance, Ref. [7]). We shall assume that $\zeta > 0$ also here, although although strictly speaking the usual thermodynamical relationships ought to be taken with some care when dealing with bizarre systems like a phantom fluid. The effect of the bulk viscosity is accordingly to reduce the thermodynamic pressure in the fluid. (The viscous Friedmann equations above were considered in a Cardy-Verlinde entropy context in Ref. [8].) Note that due to the time dependence of the viscosity the Friedmann equations above may also be interpreted as coming from a modification of gravity (for a recent discussion and a list of references, see Refs. [9] and [10]).

From the above equations we derive the following differential equation for the scalar expansion:

$$\dot{\theta} + \frac{1}{2}(1 + w)\theta^2 - 12\pi G\zeta \theta = 0$$

which, in view of the relationship $\dot{\theta} = \sqrt{6\pi G}\dot{\rho}/\sqrt{\rho}$, can be written as a differential equation for $\rho$. Since we will be interested
in the region around $w = -1$, it is convenient, following Nojiri et al. [4], to introduce the function $f(\rho)$ defined by

$$1 + w(\rho) = -f(\rho)/\rho,$$  

(7)

implying that $f(\rho) = 0$ for a ”vacuum” fluid. Then, the equation takes the form

$$\dot{\rho} - \sqrt{24\pi G\rho f(\rho) - 24\pi G\zeta(\rho)\rho} = 0.$$  

(8)

We shall be interested in the development of the late universe, from $t = t_0$ onwards. For simplicity, we put $t_0 = 0$. The corresponding starting value of $\rho$ will be denoted by $\rho_0$. From Eq. (8) we obtain

$$t = \frac{1}{\sqrt{24\pi G}} \rho \int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\rho f(\rho)} \left[ 1 + \sqrt{24\pi G\zeta(\rho)\rho} f(\rho) \right]}.$$  

(9)

This is the general relation between the cosmological time $t (> 0)$ and the density $\rho$.

3 The case when $w$ is a constant

This is thermodynamically the simplest case. Let us put

$$f(\rho) = \alpha \rho,$$  

(10)

where $\alpha$ is a constant. This means that $p \equiv w \rho = -(1 + \alpha)\rho$.

We investigate in the following some different choices for $\zeta$.

(i) $\zeta = 0$. In this non-viscous case Eq. (9) yields

$$t = \frac{1}{\sqrt{24\pi G\alpha}} \frac{2}{\sqrt{\rho_0}} \left( \frac{1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho}} \right).$$  

(11)

Thus, if $\alpha > 0$ a finite value of the time $t$ is compatible with a Big Rip singularity ($\rho = \infty$). This is the conventional phantom case, corresponding to $w < -1$. The variation of the density with time is

$$\rho(t) = \rho_0 \left( 1 - \frac{1}{2\alpha \theta_0 t} \right)^{-2},$$  

(12)
where the initial scalar expansion is
\[ \theta_0 = \sqrt{24\pi G \rho_0}. \] (13)

Similarly, the scalar expansion is
\[ \theta(t) = \theta_0 \left(1 - \frac{1}{2} \alpha \theta_0 t\right)^{-1}, \] (14)

and the scale factor is
\[ a(t) = a_0 \left(1 - \frac{1}{2} \alpha \theta_0 t\right)^{-2/3}. \] (15)

One may note here that \( \rho/\rho_0 = (a/a_0)^{3\alpha} \). Thus all the quantities \( \rho(t), \theta(t), a(t) \) diverge at the Big Rip. The above expressions are in agreement with earlier results [1, 5]. It ought to be mentioned that for \( w < -1 \) the dark energy behaves in close analogy with quantum fields [11].

(ii) \( \zeta \) equal to a constant. Equation (9) yields in this case
\[ t = \frac{1}{\sqrt{24\pi G}} \frac{1}{\alpha} \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^{3/2} \left[1 + \sqrt{24\pi G \zeta/(\alpha \sqrt{\rho})}\right]}. \] (16)

It is however in this case most convenient to go back to Eq. (6) and solve it with respect to \( \theta \) (cf. Ref. [5]):
\[ \theta(t) = \frac{\theta_0 e^{t/t_c}}{1 - \frac{1}{2} \alpha \theta_0 t_c (e^{t/t_c} - 1)}, \] (17)

where
\[ t_c = (12\pi G \zeta)^{-1}. \] (18)

Correspondingly,
\[ a(t) = a_0 \left[1 - \frac{1}{2} \alpha \theta_0 t_c (e^{t/t_c} - 1)\right]^{-2/3}. \] (19)

From Eq. (8) then
\[ \rho(t) = \rho_0 \frac{e^{2t/t_c}}{\left[1 - \frac{1}{2} \alpha \theta_0 t_c (e^{t/t_c} - 1)\right]^2}. \] (20)
Thus, at the time $t = t_s$, where

$$t_s = t_c \ln \left[ 1 + \frac{2}{\alpha \theta_0 t_c} \right], \quad (21)$$

there occurs a Big Rip; both $\theta(t), a(t)$ and $\rho(t)$ diverge. As before, $\alpha$ has to be a positive quantity, corresponding to $w = -1 - \alpha < -1$. When $\zeta \to 0$, the results of the previous case (i) are recovered.

(iii) The case $\zeta = \tau \theta$. This is the most interesting case. As $\zeta$ is assumed to be positive, the constant $\tau$ has to be positive. In view of Eq. (3) we can alternatively write the bulk viscosity as a function of the density:

$$\zeta(\rho) = \tau \sqrt{24\pi G \rho}. \quad (22)$$

Equation (9) yields in this case

$$t = \frac{1}{\sqrt{24\pi G}} \frac{2}{\alpha + 24\pi G \tau} \left( \frac{1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho}} \right). \quad (23)$$

From this condition it follows that the condition for a Big Rip ($\rho = \infty$) to occur in a finite time $t$ is that the prefactor is positive,

$$\alpha + 24\pi G \tau > 0. \quad (24)$$

This is the most important result of the present paper. Even if we start from a situation where $w > -1$ (i.e., $\alpha < 0$), corresponding to the quintessence region for an ideal fluid, the presence of a sufficiently large bulk viscosity will make the condition (24) satisfied and thus drive the fluid into the Big Rip singularity. Somewhat surprisingly, it is physically the reduction of the thermodynamical pressure generated by the viscosity which in turn causes the barrier $w = -1$ to be crossed.

The system is actually most easily analyzed by going back to Eq. (6) for the scalar expansion. Introducing the effective new parameter

$$\tilde{\alpha} = \alpha + 24\pi G \tau, \quad (25)$$

we can write the equation as

$$\dot{\theta} - \frac{1}{2} \tilde{\alpha} \theta^2 = 0. \quad (26)$$
From this it follows that \( \rho(t), \theta(t) \) and \( a(t) \) can actually be found from the non-viscous expressions \((12)-(15)\), only with the replacement \( \alpha \to \tilde{\alpha} \).

In general, the existence of viscosity coefficients in a fluid is due to the thermodynamic irreversibility of the motion. If the deviation from reversibility is small, the momentum transfer between various parts of the fluid (i.e., the stress tensor) can be taken to be linearly dependent on the velocity derivatives \( \partial_k u_i \). This is the conventional case, corresponding to constant viscosity coefficients. The present case \( \zeta = \tau \theta \) means that we go step further; it means that the momentum transfer involves second order quantities in the deviation from reversibility, still maintaining the scalar property of \( \zeta \). As mentioned above, the ansatz \( \zeta = \tau \theta \) has been considered earlier in a cosmological context \([6]\).

### 4 Remarks on the case \( w = w(\rho) \)

In this general case we have to go back to the expression \((9)\). As the ansatz \( \zeta = \tau \theta \) appears to be that of main interest, we consider this case first. From Eq. \((9)\) we get

\[
t = \frac{1}{\sqrt{24\pi G}} \int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\rho f(\rho)[1 + 24\pi G \tau \rho/f(\rho)]}}. \tag{27}
\]

Thus, if \( f(\rho) \to \alpha \rho \) in the limit of large \( \rho \), the Big Rip singularity is allowed. The necessary condition on the values of \( \alpha \) and \( \tau \) are again given by Eq. \((24)\).

Another choice for \( f(\rho) \) that may seem natural, is to take

\[
f(\rho) = A \rho^\beta \tag{28}
\]

for all \( \rho \), where \( A \) and \( \beta \) are constants (cf. Refs. \([4, 12, 13]\)). Thus \( p = -\rho - A \rho^\beta \). We shall here take \( A \) and \( \beta \) to be positive. From Eq. \((27)\) we get

\[
t = \frac{1}{\sqrt{24\pi G}} \frac{1}{A} \int_{\rho_0}^{\rho} \frac{\rho^{-\beta - 1/2}}{1 + (24\pi G \tau / A) \rho^{1-\beta}}. \tag{29}
\]
If $\beta > 1$ the influence from viscosity is seen to fade away for large $\rho$, and the Big Rip is allowed. If $\beta < 1$ the viscosity term becomes dominant for large $\rho$, and if we perform the integration over a region $\rho \in [\rho_0, \rho]$ where $\rho$ is large, we get

$$t \approx \frac{2}{(24\pi G)^{3/2}} \frac{1}{\tau} \left( \frac{1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho}} \right). \quad (30)$$

We can thus have Big Rip also when $\beta < 1$. Close to the singularity, the time $t$ depends on viscosity through the factor $\tau$ but is independent of $A$ and $\beta$. Recall that the formalism so far in this section assumes that $\zeta = \tau \theta$.

The final point that we shall deal with, is the following question: what form for the function $f(\rho)$ does the ansatz

$$a(t) = a_0 \left( \frac{t}{t_s - t} \right)^n \quad (31)$$

correspond to? Here $n$ is a positive constant and $t_s$ is the Big Rip time. The ansatz (31) is simple and suggestive, and has been dealt with before in the non-viscous case [4]. From Eqs. (31) and (3) we get

$$\theta(t) = 3n \left( \frac{1}{t} + \frac{1}{t_s - t} \right), \quad (32)$$

$$\rho(t) = \frac{3n^2}{8\pi G} \left( \frac{1}{t} + \frac{1}{t_s - t} \right)^2, \quad (33)$$

leading to

$$\dot{\rho}(t) = \pm 2\rho \left\{ \frac{8\pi G}{3} \frac{\rho}{n^2} - \frac{4}{nt_s} \sqrt{\frac{8\pi G}{3}} \sqrt{\rho} \right\}^{1/2}. \quad (34)$$

The doubly-valued equation of state is typical for a first-order transition (cf. a more extensive discussion on this point in Ref. [4]). These equations are the same as in the non-viscous case. But there is a difference arising from Eq. (4): we get

$$f(\rho) = \pm \frac{2\rho}{3n} \left\{ 1 - \frac{4n}{t_s} \sqrt{\frac{3}{8\pi G \rho}} \right\}^{1/2} - 3\zeta(\rho) \sqrt{\frac{8\pi G}{3}} \rho, \quad (35)$$
showing the influence from viscosity in the last term. Note that \( \zeta = \zeta(\rho) \) is here an arbitrary function. Thus, the initial simple ansatz (31) for \( a(t) \) leads to a rather complicated form for \( f(\rho) \).

It would be interesting to generalize the present work taking into account quantum effects together with viscosity. This may be done in close analogy with previous attempts in this direction, for an asymptotically de Sitter universe [14].

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References

[1] Caldwell, R. R., Kamionkowski, M. and Weinberg, N. N. (2003). Phys. Rev. Lett. 91, 071301; McInnes, B. (2002). JHEP 0208, 029. Barrow, J. D. (2004). Class. Quant. Grav. 21, L79.

[2] Brevik, I., Nojiri, S., Odintsov, S. D. and Vanzo, L. (2004). Phys. Rev. D 70, 043520.

[3] Elizalde, E., Nojiri, S. and Odintsov, S. D. (2004). Phys. Rev. D 70, 043539; Nojiri, S. and Odintsov, S. D. (2004) Phys. Lett. B595, 1.

[4] Nojiri, S., Odintsov, S. D. and Tsujikawa, S. (2005). Preprint hep-th/0501025, to appear in Phys. Rev. D.

[5] Brevik, I. and Heen, L. T. (1994). Astrophys. Space Sci. 219, 99; Brevik, I. (2005). Preprint gr-qc/0404095, to appear in Horizons in World Physics, Vol. 246, Ch. 06, Nova Sci. Publ.

[6] Grøn, Ø. (1990). Astrophys. Space Sci. 173, 191.

[7] Landau, L. D. and Lifshitz, E. M. (1987). Fluid Mechanics, 2nd ed., Pergamon Press, Oxford, sect. 49.

[8] Brevik, I. and Odintsov, S. D. (2002). Phys. Rev. D 65, 067302.

[9] Nojiri, S. and Odintsov, S. D. (2003). Phys. Rev. D 68, 123512.

[10] Allemandi, G., Borowiec, A. and Francaviglia, M. (2004). Phys. Rev. D 70, 103503.

[11] Nojiri, S. and Odintsov, S. D. (2003). Phys. Lett. B562, 147.

[12] Nojiri, S. and Odintsov, S. D. (2004). Phys. Rev. D 70, 103522.
[13] Stefancic, H. (2005). *Phys. Rev. D* **71**, 084024.

[14] Brevik, I. and Odintsov, S. D. (1999). *Phys. Lett.* **B455**, 104; Nojiri, S. and Odintsov, S. D. (2001). *Phys. Lett.* **B519**, 145.