UNSTABLE KODAIRA FIBRATIONS

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Abstract. Being inspired by Ross’ construction of unstable products of cer-
tain smooth curves, we show that the product $C \times C$ of every smooth curve $C$
of genus at least 2 is not slope semistable with respect to certain polarisations.
Besides, we produce examples of Kodaira-fibred surfaces of nonzero signature,
which are not slope semistable with respect to some polarisations, and so they
admit Kähler classes that do not contain any constant scalar curvature Kähler
metrics.

1. Introduction

The famous Calabi-Yau [2][18] theorem has told us that every compact complex
manifold $X$ with negative first Chern class admits a Kähler-Einstein metric in the
class $-c_1(X)$, which has a negative constant scalar curvature. By a deformation
argument due to LeBrun and Simanca [10], there exists a constant scalar curvature
Kähler(cscK) metric in every class near $-c_1(X)$. For a Kodaira-fibred surface
$\pi : X \to B$, Fine [7] found the existence of cscK metrics in the classes of $-c_1(X) - rc_1(B)$, which is far from $-c_1(X)$, for large $r$ via an adiabatic limit and the inverse
theorem of Banach space. It had been an open problem that whether every Kähler
class of a compact complex manifold $X$ with negative first Chern class contains
a cscK metric until Ross [16] constructed the first example by products of curves
which fails to have any cscK metric in some Kähler classes.

In Ross’ example [16], he shows that the product $C \times C$ is not semistable with
respect to certain polarisations if $C$ is a simple branched cover of $\mathbb{P}_1$ of degree $k$,
where $2 \leq k - 1 < \sqrt{\text{genus}(C)}$. In order to generalize this to the product of every
curve of genus at least 2, we consider different polarisations from the ones used in
[16]. But in both articles, these polarisations lie near the boundary of the ample
cone.

Later, by computing slopes, we also produce some Kodaira-fibred surfaces with
nonzero signature which is slope unstable with respect to either the polarisations
in [16] or in this paper. The main results are the following.

Theorem A. Let $C$ be a smooth curve of genus $q$ at least 2. Then $X = C \times C$ is
not semistable with respect to certain polarisations. Thus $X$ admits some Kähler
classes that do not contain cscK metrics.

Theorem B. (Cor. 5.11) There exist Kodaira-fibred surfaces $X$ with nonzero
signature, which are not semistable with respect to certain polarisations. Thus
these Kodaira-fibred surfaces $X$ admit some Kähler classes that do not contain
cscK metrics.

As consequences of Theorem A and B, we also have
Corollary C. Let $C$ be a smooth curve of genus $q$ of genus at least $2$. Then $X = C \times C$ is not asymptotically Hilbert semistable (resp. not asymptotically Chow semistable) with respect to certain polarisations.

Corollary D. There exist Kodaira-fibred surfaces $X$ with nonzero signature, which are not asymptotically Hilbert semistable (resp. not asymptotically Chow semistable) with respect to certain polarisations.

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2. SLOPE STABILITY

The concept of slope stability for a polarised manifold was introduced by Thomas and Ross in [14], and we will put it here for the sake of completeness of the paper. Let $(X, L)$ be a polarised manifold, and $Z$ be a subscheme of $(X, L)$. If we blow up $X$ along $Z$, we would get the exceptional divisor $E$ and the projection map $\pi : \tilde{X} \to X$, which is an isomorphism outside $E$. Now we can define the Seshadri constant of $Z$ as

$$\epsilon(Z, L) = \sup\{c \in \mathbb{Q} : \pi^*L - cE \text{ is ample}\}.$$ 

Let $n$ be the complex dimension of the manifold $X$, $x \in \mathbb{Q}$ with $kx \in \mathbb{N}$, we have the Hilbert polynomials of $L$ and $\pi^*L -xE$ as $\chi(kL) = a_0k^n + a_1k^{n-1} + O(k^{n-2})$ and $\chi(k(\pi^*L-xE)) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2})$. Note that $a_i(x)$ is a polynomial of degree at most $i$, so it can be extended to all real $x$. Let $\tilde{a_i(x)} = a_i - a_i(x).$ For $0 < c \leq \epsilon(Z, L)$, the slope of $X$ and the quotient slope of $Z$ are defined to be

$$\mu(X, L) = \frac{a_1}{a_0},$$

$$\mu_c(O_Z, L) = \frac{\int_0^c \tilde{a_1}(x) + \frac{\tilde{a_0'}(x)}{2} dx}{\int_0^c \tilde{a_0}(x)dx}.$$ 

These are well defined since by Riemann-Roch theorem, we have

$$a_0 = \frac{1}{n!} \int_X c_1(L)^n > 0, \quad \text{and}$$

$$a_0'(x) = -\frac{1}{(n-1)!} \int_X c_1(L-xE)^{n-1} \cdot E < 0.$$ 

The second equality implies $\int_0^c \tilde{a_0}(x)dx = ca_0 - \int_0^c a_0(x)dx > 0.$

Definition 2.1. A polarised manifold $(X, L)$ is called slope semistable with respect to a subscheme $Z$ if

$$\mu(X, L) \leq \mu_c(O_Z, L) \quad \text{for all } 0 < c \leq \epsilon(Z, L).$$

Furthermore, if it is slope semistable with respect to all subschemes, then it is called slope semistable.

Since $\epsilon(Z, L^{\otimes m}) = m \cdot \epsilon(Z, L), \mu(X, L^{\otimes m}) = \frac{1}{m} \mu(X, L),$ and $\mu_c(O_Z, L^{\otimes m}) = \frac{1}{m} \mu_c(O_Z, L)$, slope semistability is preserved if $L$ is replacing by its power, and the notion of slope semistability can be extended to ample $\mathbb{Q}$-divisors.

Slope stability in fact gives an obstruction to the existence of cscK metrics. Donaldson [6], Chen-Tian [4] and Mabuchi [12] [13] have made substantial progress
on relating the existence and uniqueness of extremal Kähler metrics in Hodge Kähler classes to the K-stability of polarized projective varieties. In particular, it has been shown that K-stability is a necessary condition for the existence of cscK metrics for a polarized projective variety. Ross and Thomas [14] show that K-semistability implies slope semistability.

**Theorem 2.2.** ([15]) If $(X, L)$ is K-semistable, then it is slope semistable with respect to every subscheme $Z$.

*Proof.* See [15] Thm.4.18. When $X$ and $Z$ are smooth, this is also proved in [14] Thm. 4.2. □

Therefore, slope semistability provides an obstruction to the existence of cscK metrics. Furthermore, since asymptotic Hilbert (resp. Chow) semistability implies K-semistability, slope semistability also provides an obstruction to the notions of Hilbert (resp. Chow) semistability of a projective variety.

**Slope semistability for smooth complex surfaces.** In this article, we only consider the case when $X$ is a smooth compact complex surface and $Z$ is a smooth complex curve in $X$. Since the complex dimensions of $X$ and $Z$ are 2 and 1, the blow-up of $X$ along $Z$ would just be $X$ itself, and the Seshadri constant is

$$
\epsilon(Z, L) = \sup \{ c \in \mathbb{Q} : L - cZ \text{ is ample} \}.
$$

Let $K$ be the canonical divisor of $X$. We can express both the slope of $(X, L)$ and the quotient slope of $Z$ in terms of the intersection numbers ([14] Cor. 5.3):

$$
\mu(X, L) = \frac{K \cdot L}{L^2},
$$

$$
\mu_c(O_Z, L) = \frac{3(2L \cdot Z - c(K \cdot Z + Z^2))}{2c(3L \cdot Z - cZ^2)}.
$$

Therefore the Seshadri constant $\epsilon(Z, L)$, the slope $\mu(X, L)$, and the quotient slope $\mu_c(O_Z, L)$ depend only on the classes of $L$ and $Z$ modulo numerical equivalence. And we could extend the equations (2.3) to any $\mathbb{Q}$-divisor $L$ even if it is not ample. Nevertheless, this might cause a zero denominator in the computation of slopes.

3. **Bounds of the ample cone**

Let $C$ be a smooth curve of genus $g$, which is greater than 1, and $X = C \times C$. Let $p_i$ be the projection onto the $i$-th factor, $c$ be a fixed point in $C$, and $\gamma_i$ be the class of the fibre $p_i^{-1}(c)$ in the Néron-Severi group $N^1(X)_\mathbb{Q}$. The class of the canonical divisor of $X$ is $K_X = (2q - 2)(\gamma_1 + \gamma_2)$ which is ample. Let $f = \gamma_1 + \gamma_2$, $\delta$ be the class of the diagonal. For convenience, we make the change of variables $\delta' = \delta - f$. Then we have the following intersection numbers on $X$:

$$
f^2 = 2, \quad \delta' \cdot f = 0, \quad \text{and} \quad \delta'^2 = -2q.
$$

In this section, we consider the intersection of the ample cone and the $f, \delta'$ plane in the Néron-Severi group $N^1(X)_\mathbb{Q}$.

First of all, consider the $\mathbb{Q}$-divisor

$$
l_s = sf + \delta'
$$

which is ample for $s \gg 0$. To find the infimum of $s$ to make $l_s$ ample, we need the following tool:
Nakai’s criterion: Let $X$ be an algebraic compact complex surface, and $D$ be a divisor on $X$. Then $D$ is ample if and only if $D \cdot D > 0$ and $D \cdot C' > 0$ for each irreducible curve $C'$.

**Theorem 3.1.** $l_s$ is ample if and only if $s > q$.

*Proof.* The essential observation in the proof is the existence of an irreducible curve in the class $\delta = \delta' + f$. That is the diagonal curve $D = \{(x, x) : x \in C\} \subseteq C \times C$. If $l_s$ is ample, we have $l_s \cdot D = 2s - 2q$ is positive. Therefore $s > q$, as required. Now, suppose that $s > q$. One has $l_s \cdot l_s = 2(s^2 - q) > 0$, and $l_s \cdot D = 2s - 2q > 0$. Let $C'$ be any irreducible curve distinct from $D$ in $X$. Since the intersection pairing is nondegenerate, we could write

$$[C'] = x_1 \gamma_1 + x_2 \gamma_2 + y\delta' + \alpha,$$

where $\alpha \in N^1(X) \cap q$ is orthogonal to $\gamma_1, \gamma_2$, and $\delta'$. By intersecting with $\gamma_1, \gamma_2$, we find that $x_1, x_2 > 0$. Moreover, since $D$ and $C'$ are two distinct irreducible curves, $[D] \cdot [C'] \geq 0$, which yields

$$x_1 + x_2 - 2qy \geq 0.$$  

The direct computation shows that $l_s \cdot [C'] = s(x_1 + x_2) - 2gy > x_1 + x_2 - 2gy \geq 0$. Therefore a direct application of Nakai’s criterion implies that $l_s$ is ample. $\square$

In Ross’ paper [16], he considers the case that $C$ is a simple branched cover of $\mathbb{P}_1$ of degree $k$, where $2 \leq k - 1 < \sqrt{q}$, and the $\mathbb{Q}$-divisor $L_i = tf - \delta'$. Let $s_C = \inf \{t : L_i \text{ is ample} \}$. Kouvidakis [8] shows that $s_C = \sqrt{q}$. Furthermore, Kouvidakis [8] also shows that when $C$ is a curve of general moduli, we have $\sqrt{q} \leq s_C \leq \sqrt{q}$. In particular, if $q$ is a perfect square, $s_C = \sqrt{q}$. By the previous discussion, we know that every class between the thick lines in figure 1 is ample.

![Diagram](image)

*Figure 1.* Ample cone in $f, \delta'$ plane

4. **UNSTABLE PRODUCTS OF CURVES**

In this section, we will show that for every smooth curve of genus $q$ greater than 1, the product $X = C \times C$ is not slope semistable with respect to certain polarisations. Ross [16] has shown that when $C$ admits a simple branched cover of $\mathbb{P}_1$ of degree $2 \leq k - 1 < \sqrt{q}$, $X = C \times C$ is not slope semistable with respect to the polarisations $L_t = tf - \delta'$ for $t$ sufficiently close to $s_C$. But here we consider general curves, and the polarisations $l_s = sf + \delta'$.

**Theorem 4.1.** Let $C$ be a smooth curve of genus $q$ at least 2. Then $X = C \times C$ is not semistable with respect to the polarisations $l_s$ for $s$ sufficiently close to $q$. 

Proof. By Thm. 3.1 let $s > q$ so that $l_s$ is ample. The canonical divisor of $X$ is $K_X = (2q - 2)f$, and

$$\mu(X, l_s) = \frac{K \cdot l_s}{l_s^2} = \frac{s(2q - 2)}{s^2 - q} \tag{4.2}$$

Let $D$ be the diagonal curve, whose class is $\delta = f + \delta'$. We now consider the Seshadri constant of $D$. We have that $l_s - cD = sf + \delta' - c(f + \delta') = (s - c)f + (1 - c)\delta'$. And by previous discussion, it is ample if and only if $c < \frac{s + \delta'}{s + 1} > 1$. (It is obvious to see $\epsilon(D, l_s) \geq 1$ since $l_s - D = (s - 1)f > 0$.)

To calculate the slope of $Z$ we need the quantities:

$$l_s \cdot D = (sf + \delta')(f + \delta') = 2s - 2q, \tag{4.3}$$

$$K \cdot D = (2q - 2)f \cdot (f + \delta') = 2(2q - 2),$$

$$D^2 = (f + \delta')^2 = 2 - 2q.$$  

Thus from (4.3),

$$\mu_c(O_D, l_s) = \frac{3(2s \cdot D - c(K \cdot D + D^2))}{2c(3l_s \cdot D - cD^2)} = \frac{3(4s - 4q - c(2q - 2))}{2c(6s - 6q - 2c + 2cq)}. \tag{4.4}$$

We claim that if $0 < c < \frac{3}{4}$, then $\mu_c(O_Z, l_s) < \mu(X, l_s)$ as $s$ tends to $q$ from above. Since this is an open condition it is sufficient to show that it holds when $s = q$. By (4.2),

$$\mu(X, l_q) = \frac{-q(2q - 2)}{q^2 - q} = -2,$$

$$\mu_c(O_Z, l_q) = \frac{-3c(2q - 2)}{2c(-2c + 2cq)} = \frac{-3}{2c} \tag{4.5}$$

Hence as $0 < c < \frac{3}{4}$, $\mu_c(O_Z, l_s) < \mu(X, l_s)$ as $s$ tends to $q$ from above, which proves that $(X, l_s)$ is not slope semistable. $\square$

**Corollary 4.6.** Let $C$ be a smooth curve of genus $q$ of genus at least 2. Then $X = C \times C$ admit some Kähler classes that do not contain cscK metrics.

**Proof.** It follows from (4.1) and (2.2). $\square$

This completes the proof of Theorem A in the introduction.

**Corollary 4.7.** Let $C$ be a smooth curve of genus $q$ of genus at least 2. Then $X = C \times C$ is not asymptotically Hilbert semistable (resp. not asymptotically Chow semistable) with respect to certain polarisations.

**Proof.** It follows from (4.1) and (2.2). $\square$

Related to the existence of cscK metric, Mabuchi introduces Mabuchi functional for a given Kähler class $\Omega$ on a complex manifold $X$. Chen-Tian [4] and Donaldson [6] show that the existence of cscK metrics in $\Omega$ implies that the Mabuchi functional is bounded from below. It is conjectured that the existence of the cscK metrics is equivalent to the properness of the Mabuchi functional. In the case that $X$ has negative first Chern Class, Chen [3] introduces the notion of $J$-flow, and show that
the convergence of $J$-flow implies lower boundedness of the Mabuchi functional. For a polarised surface $(X, L)$ with negative first Chern class, Weinkove [17] has the following theorem about sufficient condition of the convergence of the $J$-flow.

**Theorem 4.8.** (17) Let $(X, L)$ be a polarised surface with negative first Chern Class. Let the divisor $\alpha$ be defined by

$$\alpha = 2(K.L)L - (L^2)K.$$  

If $\alpha$ is ample, then the $J$-flow converges and the Mabuchi functional is proper on the class $c_1(L)$.

When $C$ is a curve of genus $q$ at least 2, $X = C \times C$ and $L = l_s = sf + \delta'$ with $s > q$ it is easy to determine when $\alpha$ is ample.

**Lemma 4.9.** Let $L = l_s = sf + \delta'$ with $s > q$. Then $\alpha = 2(K.L)L - (L^2)K$ is ample if and only if $s > q + \sqrt{q^2 - q}$.

**Proof.** Since $l_s^2 = 2s^2 - 2q$ and $K \cdot l_s = 2s(2q - 2)$, we have

$$\alpha = 4s(2q - 2)(sf + \delta') - (2s^2 - 2q)(2q - 2)f$$

$$= 2(2q - 2)((s^2 + q)f + 2s\delta'),$$

which is ample if and only if $s^2 + q > 2qs$. The conclusion follow. $\square$

**Corollary 4.10.** Let $C$ be a smooth curve of genus at least 2, and $X$ be the product $X = C \times C$. Then the Mabuchi functional is proper on the polarised surface $(X, l_s)$ if $s > q + \sqrt{q^2 - q}$.

**Proof.** It follows from Theorem 4.8 and lemma 4.9 $\square$

5. **Unstable Kodaira Fibrations of Nonzero Signature**

In this section, we give a short sketch of an explicit construction of Kodaira fibrations with nonzero signature. Then we will show Kodaira fibrations constructed in this way are not slope semistable with respect to certain polarisations.

**Definition 5.1.** A Kodaira fibration is a holomorphic submersion $\pi : X \to B$ from a compact complex surface $X$ to a compact complex curve $B$, with base $B$ and fibre $F = f^{-1}(z)$ both have genus $\geq 2$.

Clearly, $\pi$ is locally a trivial fibre bundle in the smooth sense. Nevertheless, the complex structures of all fibres may vary. A surface admitting a Kodaira fibration is called a Kodaira-fibred surface. Every Kodaira-fibred surface $X$ is algebraic since by the adjunction formula, $K_X \cdot F = -\chi(F) \geq 2$, and $c_1^2(K_X + kF) > 0$ for $k$ large enough, where $F$ is the class of the fibre. On the other hand, $X$ could not contain any rational or elliptic curves because if $C$ is a curve in $X$ with genus less than two, then by the fact that a holomorphic map from a curve of lower genus to a curve of higher genus must be constant, we have $\pi(C)$ is a point. Therefore $C$ lies in a fibre, which is absurd since the fibre has the genus greater than or equal to 2. The Kodaira-Enrique classification theorem henceforth tells us that $X$ is a minimal surface of general type. Furthermore, since $X$ contains no $(-2)$ rational curves, we have the canonical divisor $K_X > 0$. 

A product $B \times F$ of two complex curves of genus $\geq 2$ is certainly Kodaira fibred, but such a product would have signature $\tau = 0$ since it admits an orientation-reversing diffeomorphism. We can construct Kodaira fibrations of nonzero signature in the following way:

Let $C$ be a compact complex curve of genus $\geq 2$, $G$ be a finite group of order divisible by $r$, which acts effectively on $C$. We can define a homomorphism by the composition $\pi_1(C) \to H_1(C, \mathbb{Z}) \to H_1(C, \mathbb{Z}_r)$. Since $H_1(C, \mathbb{Z}_r)$ has finite order, there exists an unbranched finite cover $h : B \to C$ such that $h_*(\pi_1(B)) = \ker[\pi_1(C) \to H_1(C, \mathbb{Z}_r)]$. Clearly, the genus of $B$ is greater than or equal to the genus of $C$. Let $\Sigma \subset B \times C$ be the union of the graphs of $g \circ h : B \to C$, where $g$ runs over $G$.

**Lemma 5.2.** The homology class of $\Sigma$ in $H_2(B \times C, \mathbb{Z})$ is divisible by $r$. Hence there exists a cyclic $r$-cover $X$ of $B \times C$ branched over $\Sigma$.

**Proof.** (see [1]). Since $c_1(\mathcal{O}(\Sigma))$ is the Poincaré duality of the fundamental class of $\Sigma$, it suffices to show that $c_1(\mathcal{O}(\Sigma))$ is divisible by $r$. That is, we have to show the intersection pairing $(c_1(\mathcal{O}(\Sigma)), \alpha) \equiv 0 \pmod{r}$ for all $\alpha \in H^2(B \times C, \mathbb{Z})$. Let $p_1 : B \times C \to B$ and $p_2 : B \times C \to C$ be the projections. Using Künneth's formula, we deal it with three cases: $\alpha \in p_1^*(H^2(B, \mathbb{Z}))$, $\alpha \in p_2^*(H^2(C, \mathbb{Z}))$, and $\alpha = p_1^*(H^1(B, \mathbb{Z})) \cup p_2^*(H^1(C, \mathbb{Z}))$. The first two cases follow from the fact that $r$ divides the order of $G$. The last case can be done by applying the projection formula

$$(c_1(\mathcal{O}(\Sigma)), p_1^*\beta \cup p_2^*\gamma) = (p_1^*\beta, c_1(\mathcal{O}(\Sigma)) \cup p_2^*\gamma) = \sum_{g \in G} (\beta, (g \circ h)^*\gamma) \equiv 0 \pmod{r}.$$ 

The last equality is because $h_*(\pi_1(B)) = \ker[\pi_1(C) \to H_1(C, \mathbb{Z}_r)]$. □

The explicit construction of the surface $X$ in the lemma [5.2] is as follows. Since the homology class of $\Sigma$ in $H_2(B \times C, \mathbb{Z})$ is divisible by $r$, we can have a line bundle $\mathcal{L}$ on $B \times C$ such that $\mathcal{O}(\Sigma) = \mathcal{L}^\otimes r$ and a section $s \in \Gamma(B \times C, \mathcal{O}(\Sigma))$, which vanishes exactly on $\Sigma$. Now take $X$ to be the zero set of the section $s$ and $\Sigma$ is the disjoint union of $|G|$ smooth curves, $X$ is a smooth surface. Accordingly, $X$ inherits a natural projection $\pi : X \to B$. This projection $\pi$ is a submersion since it admits a local section everywhere. The signature of $X$ can be computed as follows: Let $p = \text{genus of } B$, $q = \text{genus of } C$, and $d = \text{degree of } h$. By the Riemann-Hurwitz formula, we have the Euler number

$$\chi(X) = \chi(B)(r\chi(C) - (r - 1)|G|) = 4r(p - 1)(q - 1) + 2(p - 1)(r - 1)|G| > 0.$$ 

On the other hand, we have the canonical divisor $K_X = \pi^*K_{B \times C} + R$, where $R$ is the ramification divisor. To compute the self-intersection number of the canonical line bundle $K_X$, we need the following lemma.
Lemma 5.3. Let $M, N$ be two compact complex manifolds of the same dimension $m$, and $f : M \to N$ a smooth map of degree $d$. Then the self intersection number of the graph $\Gamma$ of $f$ in $M \times N$ is given by $d \cdot \chi(N)$.

Proof. Let $p_1, p_2$ be the projection of $M \times N$ to $M$ and $N$, respectively. Then the normal bundle $N_{\Gamma}$ of $\Gamma$ is isomorphic to the pullback bundle $p_2^*T_N|_{\Gamma}$ of the tangent bundle of $N$ restricted to $\Gamma$ since we have the following commutative diagram between two exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_M|_{\Gamma} & \longrightarrow & (T_M \oplus T_N)|_{\Gamma} & \longrightarrow & T_N|_{\Gamma} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & & & \\
0 & \longrightarrow & T_{\Gamma} & \longrightarrow & (T_M \oplus T_N)|_{\Gamma} & \longrightarrow & N_{\Gamma} & \longrightarrow & 0
\end{array}
$$

Let $eu(V)$ denote the Euler class of the vector bundle $V$. We have

$$
\Gamma \cdot \Gamma = \int_{\Gamma} eu(N_{\Gamma}) \\
= \int_{\Gamma} p_2^*eu(T_N) \quad \text{by naturality of Euler class} \\
= \int_{M} f^*eu(T_N) \\
= \int_{M} d \cdot \frac{\chi(N)}{\chi(M)} eu(T_M) \\
= d \cdot \chi(N).
$$

\[ \square \]

Using $K_X = \pi^*K_{B \times C} + R$, where $R = \frac{\pi^*\Sigma}{r}$ is the ramification divisor, we have

$$
K_X^2 = rK_{B \times C}^2 + 2(r - 1)K_{B \times C} \cdot \Sigma + \frac{(r - 1)^2}{r} \Sigma \cdot \Sigma \\
= 2r(2q - 2)(2p - 2) + 2(r - 1)((2p - 2)|G| + (2q - 2)d|G|) + \frac{(r - 1)^2}{r} d(2 - 2q)|G| \\
= 8r(p - 1)(q - 1) + 4(r - 1)(p - 1)|G| + 2 \frac{r^2 - 1}{r} (q - 1)d|G| > 0.
$$

Using Signature formula, we have $\tau(X) = \frac{1}{3}(K_X^2 - 2\chi(X)) = \frac{2(r^2 - 1)}{3r}(q - 1)d|G| > 0$ as a result.

Example. Let $C$ be a curve of genus $3$ with a holomorphic involution $\iota : C \to C$ without fixed points; one may visualize such an involution as a $180^\circ$ rotation of a 5-holed doughnut about an axis which passes through the middle hole, without meeting the doughnut. Let $h : B \to C$ be the unique 64-fold unbranched cover with $f_4[\pi_1(B)] = \ker(\pi_1(C) \to H_1(C, \mathbb{Z}_2))$; thus $B$ is a complex curve of genus 129. Let $\Sigma \subset B \times C$ be the union of the graphs of $f$ and $\iota \circ h$. Then the homology class of $\Sigma$ is divisible by 2. We may therefore construct a ramified double cover $X \to B \times C$ branched over $\Sigma$. The projection $X \to B$ is then a Kodaira fibration, with fiber $F$ of genus 6. The projection $X \to C$ is also a Kodaira fibration, with fiber of genus 321. The signature of this doubly Kodaira-fibres complex surface is $\tau(M) = 256 > 0$. 
Recall that in Ross’ construction \cite{Ross} of unstable products of curves, \( C \) is a compact complex curve of genus \( q \geq 2 \), which admits a simple branched covering map to \( \mathbb{P}_1 \) of degree \( k \), for \( 2 \leq k-1 < \sqrt{q} \). Let \( X_0 = C \times C \). Consider the \( \mathbb{Q} \)-divisor \( L_t = tf - \delta' \), where \( f = \gamma_1 + \gamma_2 \) is the class of the sum of fibres in two directions, and \( \delta' = \delta - f \) with \( \delta \) the class of diagonal. It is shown in \cite{Miura} that \( L_t \) is ample if \( t > \frac{2q}{k-1} \). Let \( Z = C \times \mathbb{P}_1 \) \( C - \delta \) be the residual divisor of the diagonal in the fibre product. By computing the slope of \( X_0 \) and \( Z_0 \), Ross \cite{Ross} shows that \((X_0, L_t)\) is destabilized by \( Z \) for \( t \) sufficiently close to \( \frac{2q}{k-1} \) from above.

Now let \( X_1 \) be an unbranched \( d \)-covering of \( X_0 \) with the covering map \( \pi_1 : X_1 \to X_0 \). Let \( L_{1,t} = \pi_1^*L_t \), and \( Z_1 = \pi_1^*Z \), then the Seshadri constant \( \epsilon(Z_1, X_1, L_{1,t}) = \epsilon(Z, X_0, L_t) \). Since \( K_{X_1} = \pi_1^*K_{X_0} \), and the intersection pairings \( \pi_1^*D \cdot \pi_1^*D' = d(D \cdot D') \) for \( D, D' \) any divisors on \( X_0 \), we have the slopes \( \mu(X_1, L_{1,t}) = \mu(X_0, L_t) \), and the quotient slopes \( \mu_c(O_{Z_1}, L_{1,t}) = \mu_c(O_Z, L_t) \). It follows that \((X_1, L_{1,t})\) is destabilized by \( Z_1 \) for \( t \) sufficiently close to \( \frac{2q}{k-1} \).

Here we take \( X_1 = B \times C, \) a \( d \) to \( 1 \) unbranched cover of \( C \times C \), where \( B \) and \( C \) satisfy all hypotheses in the previous construction of Kodaira fibrations. Let \( B_0 = B \times c, C_0 = b \times C \) be the classes of the fibres of the projection onto \( C \)-factor and \( B \)-factor, respectively, which is independent of the choices of the points \( b \in B, c \in C \). Let \( X_2 \) be the constructed Kodaira fibration, which is a cyclic cover of \( X_1 \) with nonzero signature. Then we have \( \pi_2 : X_2 \to X_1 \) the covering map branched over \( \Sigma \) of degree \( r \), and \( \pi_1 : X_1 \to X_0 \) the unbranched covering map of degree \( d \). Let \( f_2 = (\pi_1 \circ \pi_2)^*f = \pi_1^*(B_0 + dC_0) \), \( \delta_2 = (\pi_1 \circ \pi_2)^*\delta = \pi_2^*(\text{graph of } h) \). We see that \( f_2^2 = 2rd, f_2 \cdot \delta_2 = 2rd, \delta_2^2 = rd(2 - 2q) \). For convenience make the change of variables \( \delta_2' = \delta_2 - f_2 \). Then we have the following intersection numbers on \( X_2 \):

\[
\delta_2'f_2 = 0, \quad \delta_2'^2 = -2rdq.
\]

Now consider the \( \mathbb{Q} \)-divisor
\[
L_{2,t,\varepsilon} = (\pi_1 \circ \pi_2)^*L_t + \varepsilon K_{X_2} = tf_2 - \delta_2' + \varepsilon K_{X_2}
\]
which is ample for \( \varepsilon > 0 \) and \( t \gg 0 \). Here the canonical divisor is \( K_{X_2} = \pi_2^*(K_{X_1}) + R \) with \( R \) the ramification divisor. We define
\[
s_\varepsilon = \inf\{t : L_{2,t,\varepsilon} \text{ is ample}\}.
\]
Clearly \( s_\varepsilon \leq \frac{2q}{k-1} \) for \( L_{1,t,\varepsilon} \) is numerically effective (see \cite{Asahi}) and \( K_{X_2} \) is ample.

**Theorem 5.4.** Let the divisor \( Z_2 \) be defined by \( Z_2 = \pi_2^*Z_1 = (k - 1)f_2 - \delta_2' \). Then \( X_2 \) is not slope semistable with respect to \( Z_2 \) for the polarisations \( L_{2,t,\varepsilon} \) if \( t \) is sufficiently close to \( \frac{2q}{k-1}, \) and \( \varepsilon \) is small enough.

**Proof.** Let \( t > s_\varepsilon \) so \( L_{2,t,\varepsilon} \) is ample for any \( \varepsilon > 0 \). The canonical divisor of \( X_2 \) is \( K_{X_2} = \pi_2^*K_{X_1} + R \), where \( R \) is the ramification divisor, and \( \pi_2^*\Sigma = \frac{2q}{k-1}R \). From Lemma \ref{lem:intersection} we can compute the intersection numbers

\[
R \cdot \pi_2^*L_{1,t} = r\frac{r-1}{r}\Sigma \cdot L_{1,t}
= (r - 1)\Sigma \cdot \left( (t + 1)(dC_0 + B_0) - \text{graph of } h \right)
= 2(r - 1)d((t - 1)|G| + q - 1),
\]
\[
R \cdot Z_2 = (r - 1)\Sigma \cdot (k - 1)f_2 - \delta_2'^2
= (r - 1)(2kd|G| - d(2 - 2q))
= 2d(r - 1)(k|G| - 1 + q),
\]
and we have

$$\mu(X_2, L_{2,t,\varepsilon}) = \frac{-K_{X_2} \cdot L_{2,t,\varepsilon}}{L_{2,t,\varepsilon}^2} = - \frac{(\pi_2^* K_{X_1} + R) \cdot (\pi_2^* L_{1,t} + \varepsilon K_{X_2})}{(\pi_2^* L_{1,t} + \varepsilon K_{X_2})^2}$$

$$= - \frac{r K_{X_1} \cdot L_{1,t} + R \cdot \pi_2^* L_{1,t} + O(\varepsilon)}{r L_{1,t}^2 + O(\varepsilon)}$$

$$= - \frac{K_{X_1} \cdot L_{1,t}}{L_{1,t}^2} \frac{2(r-1)d((t + 1) |G| + q - 1)}{2rd(t^2 - q)} + O(\varepsilon)$$

$$= \mu(X_1, L_{1,t}) - \frac{(r - 1)((t + 1) |G| + q - 1)}{r(t^2 - q)} + O(\varepsilon). \quad (5.6)$$

Recall that $L_t - Z = (t-k+1)f$ is ample when $2 \leq k - 1 \leq \sqrt{q}$ (see [10]). Therefore $L_{2,t,\varepsilon} - Z_2 = (\pi_1 \circ \pi_2)^*(L_t - Z) + \varepsilon K_{X_2}$ is ample, and $\epsilon(2, Z_{2,2,t,\varepsilon}) \geq 1$ for any positive $\varepsilon$.

To calculate the quotient slope of $Z_2$, we have from (2.3) and (5.5),

$$\mu_1(O_{Z_2}, L_{2,t,\varepsilon}) = \frac{3(2L_{2,t,\varepsilon} \cdot Z_2 - (K_{X_2} \cdot Z_2 + Z_2^2))}{2(3L_{2,t,\varepsilon} \cdot Z_2 - Z_2^2)}$$

$$= \frac{3(2r L_{1,t} \cdot Z_1 - r K_{X_1} \cdot Z_1 - r Z_1^2) - 3R \cdot \pi_2^* Z_1 + O(\varepsilon)}{2(3r L_{1,t} \cdot Z_1 - r Z_1^2) + O(\varepsilon)}.$$}

$$= \mu_1(O_{Z_1}, L_{1,t}) - \frac{3R \cdot \pi_2^* Z_1}{2(3r L_{1,t} \cdot Z_1 - r Z_1^2)} + O(\varepsilon).$$

$$= \mu_1(O_{Z_1}, L_{1,t}) - \frac{6d(r - 1)(k|G| + q - 1)}{4rd(3t(k - 1) - (k - 1)^2 - 2q)} + O(\varepsilon).$$

$$= \mu_1(O_{Z_1}, L_{1,t}) - \frac{3(r - 1)(k|G| + q - 1)}{2r(3t(k - 1) - (k - 1)^2 - 2q)} + O(\varepsilon). \quad (5.7)$$

Since $\mu_1(O_{Z_1}, L_{1,t}) < \mu(X_1, L_{1,t})$ near $t = \frac{k-1}{k}$ from Ross' computation [10], by (2.10) and (5.7), it suffices to show

$$\frac{(r - 1)((t + 1) |G| + q - 1)}{r(t^2 - q)} < \frac{3(r - 1)(k|G| + q - 1)}{2r(3t(k - 1) - (k - 1)^2 - 2q)}.$$}

Because the slopes depend continuously on $t$, we just need to show the inequality holds at $t = \frac{k}{k-1}$. That is, $2 \frac{(k-1)^2}{4} (\frac{q}{k-1} + 1) |G| + q - 1 < 3(k|G| + q - 1)$. Using the assumption $2 \leq k - 1 < \sqrt{q}$, we have

$$3(k|G| + q - 1) - 2 \frac{(k-1)^2}{q} (\frac{q}{k-1} + 1) |G| + q - 1$$

$$= 3(k|G| + q - 1) - 2((k - 1) + \frac{(k-1)^2}{q}) |G| + \frac{(k-1)^2(q - 1)}{q}$$

$$> k|G| + q - 1 - 2((k - 1) + |G| + (q - 1))$$

$$= k|G| + q - 1 > 0.$$}

Thus $\epsilon(2, L_{2,t,\varepsilon}) \geq 1$ and $\mu_1(O_{Z_2}, L_{2,t,\varepsilon}) < \mu(X_2, L_{2,t,\varepsilon})$ as $t$ is sufficiently close to $\frac{k}{k-1}$, and $\varepsilon$ tends to 0 from above, which proves that $(X_2, L_{2,t,\varepsilon})$ is not slope semistable. \qed

Instead of the polarisations $L_{2,t,\varepsilon} = t f_2 - \delta_2 + \varepsilon K_{X_2}$, we could also consider the polarisations $l_2, s, \varepsilon = s f_2 + \delta_2^* + \varepsilon K_{X_2}$, where $\varepsilon > 0$. Let $D$ be the diagonal in $X = C \times C$, $D_1 = \pi_1^* D$, and $D_2 = \pi_2^* D_1$. 
Theorem 5.8. \( X_2 \) is not slope semistable with respect to the curve \( D_2 \) for the polarisations \( l_{2,s,\varepsilon} \) if \( s \) is sufficiently close to \( q \), and \( \varepsilon \) is small enough.

Proof. Let \( s > q \) so that \( l_{2,s,\varepsilon} \) is ample for any \( \varepsilon > 0 \). The canonical divisor of \( X_2 \) is \( K_{X_2} = \pi_2^* K_{X_1} + R \), where \( R \) is the ramification divisor. From (5.8) and (4.5), we have

\[
R \cdot \pi_2^* l_{1,s} = r \left( \frac{r - 1}{r} \Sigma \right) \cdot l_{1,s} = (r - 1) \Sigma \cdot ((s - 1)(dC_0 + B_0) + \text{graph of } h) = 2(r - 1)d((s - 1)|G| - q + 1),
\]

and

\[
\mu(X_2, l_{2,s,\varepsilon}) = \frac{K_{X_2} \cdot l_{2,s,\varepsilon}}{l_{2,s,\varepsilon}^2} = \frac{(\pi_2^* K_{X_1} + R) \cdot (\pi_2^* l_{1,s} + \varepsilon K_{X_2})}{(\pi_2^* l_{1,s} + \varepsilon K_{X_2})^2} = \frac{rK_{X_1} \cdot l_{1,s} + R \cdot \pi_2^* l_{1,s} + O(\varepsilon)}{rl_{1,s}^2 + O(\varepsilon)} = \mu(X_1, l_{1,s}) - \frac{2(r - 1)d((s - 1)|G| - q + 1)}{2rd(s^2 - q)} + O(\varepsilon) = \frac{-s(2q - 2)}{s^2 - q} - \frac{(r - 1)((s - 1)|G| - q + 1)}{r(s^2 - q)} + O(\varepsilon).
\]

To bound the Seshadri constant of \( D_2 \), we have \( c(l_{2,s,\varepsilon}, D_2) \geq 1 \) since \( l_{2,s,\varepsilon} - D_2 = (\pi_1 \circ \pi_2) l_{s} + \varepsilon K_{X_2} \) is ample if \( s > q \). To calculate the quotient slope of \( D_2 \), we need the quantities:

\[
l_{2,s,\varepsilon} \cdot D_2 = r(s f_1 + \delta_1')(f_1 + \delta_1') + O(\varepsilon)
\]

\[
R \cdot D_2 = (r - 1)\Sigma(f_1 + \delta_1') = (r - 1)d(2 - 2q),
\]

\[
D_2^2 = rD_1^2 = rd(2 - 2q).
\]

Thus from (2.3) and (4.5),

\[
\mu_c(O_{D_2}, l_{2,s,\varepsilon}) = \frac{3(2l_{2,s,\varepsilon} \cdot D_2 - c(K_{X_2} \cdot D_2 + D_2^2))}{2c(3l_{2,s,\varepsilon} \cdot D_2 - cD_2^2)} = \frac{3cR \cdot D_2}{2c(3l_{2,s,\varepsilon} \cdot D_2 - cD_2^2)} + O(\varepsilon).
\]

\[
= \frac{3(4s - 4q - c(2q - 2))}{2c(6s - 6q - 2c + 2cq)} - \frac{3cd(r - 1)(2 - 2q)}{2rc(6ds - 6dq - cd(2 - 2q))} + O(\varepsilon)
\]

\[
= \frac{3(r - 1)(1 - q)}{2r(3s - 3q - c(1 - q))} + O(\varepsilon).
\]

We claim that \( \mu_c(O_{D_2}, l_{2,s,\varepsilon}) < \mu(X, l_{2,s,\varepsilon}) \) as \( s \) tends to \( q \) from above. Since this is an open condition, it suffices to show it holds when \( s = q \). By (5.9) and (5.10),

\[
\mu(X_2, l_{2,q,\varepsilon}) = -2 - \frac{(r - 1)(|G| - 1)}{rq} + O(\varepsilon),
\]

\[
\mu_c(O_{D_2}, l_{2,q,\varepsilon}) = -3 + \frac{3(r - 1)}{2rc} + O(\varepsilon).
\]
Hence as $c < \frac{3}{4} + 2(\varepsilon - 1)(|G| - 1) < \frac{3}{4}$, $s$ close to $q$, and $\varepsilon$ small enough, $\mu_c(\mathcal{O}_{D_2}, l_{2,s,\varepsilon}) < \mu(X_2, l_{2,s,\varepsilon})$, which proves that $(X_2, l_{2,s,\varepsilon})$ is not slope semistable.

**Corollary 5.11.** There exist Kodaira-fibred surfaces $X$ with nonzero signature, which admit some Kähler classes that do not contain cscK metrics.

**Proof.** It follows from (5.4), (5.8), and (2.2).

**Corollary 5.12.** There exist Kodaira-fibred surfaces $X$ with nonzero signature, which are not asymptotically Hilbert semistable (resp. not asymptotically Chow semistable) with respect to certain polarisations.

**Proof.** It follows from (5.4), (5.8), and (2.2).

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