Chiral Disorder in QCD

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Using the Gell-Mann-Oakes-Renner (GOR) relation and semi-classical arguments, we show that the bulk quark spectrum in QCD exhibits a variety of regimes including the ergodic one described by random matrix theory. We analyze the quark spectral form-factor in the diffusive and ballistic regime. We suggest that a class of chiral transitions in QCD is possibly of the metal-insulator type, with a universal spectral statistics at the mobility edge.

PACS numbers : 64.60.Cn, 11.30Rd, 12.38.Aw

1. The chiral properties of QCD in a small Euclidean volume are constrained by the way chiral symmetry is spontaneously broken in the infinite volume limit \cite{1}. This observation carries to the quark spectrum averaged over gauge configurations of fixed winding number, in the form of microscopic sum rules \cite{2}. The sum rules can be rederived using chiral random matrix models \cite{3}. The microscopic sum rules reflect on the universal properties of the eigenvalue distribution and correlations a quantum leap apart \cite{4}.

The interesting results achieved by random matrix models in describing the microscopic regime of chiral QCD, prompt us to ask whether we could think about other generic or universal aspects of the quark spectrum, aside from that of random matrix theory. In this letter, we would like to provide some new insights to the microscopic regime using semi-classical arguments. We will use these new insights to analyze the quark spectral form-factor in the diffusive and ballistic regimes (non-universal). We then argue that a number of chiral transitions in QCD are possibly of the metal-insulator type, those in the regime \( \tau_d < t < \tau_{\text{erg}} \) as ergodic, those in the regime \( \tau_d < t < \tau_{\text{erg}} \) as diffusive, and finally those in the regime \( t < \tau_d \) as ballistic. We note that the time scales are ordered as \( t_H \gg \tau_{\text{erg}} \gg \tau_d \) or \( V \gg \sqrt{V} \approx 1 \) in units where \( F = |\Sigma| = m_Q = 1 \). The virtual quark spectrum in QCD bears much resemblance to the real electronic spectrum in disordered metallic grains \cite{5}.

2. To be able to organize the various energy scales of interacting quarks in a finite Euclidean volume \( V \), we call upon the two-flavor GOR relation \cite{6}:

\[
F^2 = -\frac{m}{m^2} \left( \langle \bar{q}q \rangle \right) = -\frac{m}{m^2} \pi \rho(0),
\]

where \( F \) is the pion decay constant, \( m \) the pion mass and \( \langle \bar{q}q \rangle = -\Sigma \) is the quark condensate. The last identity in \cite{6} follows from the Banks-Casher relation \cite{7}, with \( \rho(0) \) the quark density of eigenvalues per unit four volume \( V = L^4 \) at zero virtuality (\( \lambda \approx 0 \)). Recall that in a metal, the Kubo formula for the dc-conductivity \( \sigma \) in terms of the diffusion constant \( D \) and the density of states \( \rho_F \) at the Fermi level is (in units of \( e^2/h ) : \sigma = D \rho_F \). By analogy, we see that in QCD, \( D \approx F^2/\Sigma \) plays the role of the diffusion constant while the Fermi level corresponds to the zero virtuality point. This physical analogy will be confirmed below by a direct calculation.

The appearance of a diffusion constant \( D \) allows us to organize the various stages of the disordered phase. For a quark with proper time \( t \sim 1/\lambda \), where \( \lambda \) is its virtuality, the relevant time scales are: the Heisenberg time, \( t_H = 1/\Delta \) with \( \Delta = 1/\rho V \) the typical quantum spacing; the ergodic time, \( \tau_{\text{erg}} = 1/E_c = L^2/D \), with \( E_c \) the Thouless energy \cite{7}; and the diffusion time \( \tau_d = 1/2m_Q \) with \( m_Q \) the constituent quark mass. For fixed \( V \), the Ohmic conductance is \( \sigma_I = E_c/\Delta \approx F^2L^2 \). The relation \( D = v_F^2 \tau_d/4 \) suggests that the virtual velocity of diffusive quarks at \( \lambda \approx 0 \) is about \( v_F \approx F(m_Q/\Sigma)^{1/2} \approx 1 \). Diffusive quarks are the analogue of constituent quarks.

We will refer to the quarks in the regime \( t > t_H \) as quantum, those in the regime \( \tau_d < t < \tau_{\text{erg}} \) as ergodic, those in the regime \( \tau_d < t < \tau_{\text{erg}} \) as diffusive, and finally those in the regime \( t < \tau_d \) as ballistic.

3. An important tool for studying the quark level correlations in the QCD vacuum is the averaged two-level correlation function \cite{8}:

\[
R_A,E(s) \equiv R(s) = \frac{1}{\nu^2} \langle \nu(\lambda_+)\nu(\lambda_-) \rangle - 1
\]

measuring correlation between two eigenvalues \( \lambda_\pm \) in the virtuality window around \( \Delta \) with width \( E \), being apart from each other by \( \Delta s. \nu = \sqrt{V} \) is the mean density in the band \( E \). The averaging in \cite{9} is over the gauge configurations \( A \) with the QCD measure. The quark eigenvalues in a fixed gauge configuration \( A \) are denoted by \( \nu(A)q_n = \lambda_n(A)q_n \), and the unaveraged density of states is

\[
\nu(\lambda) = \frac{1}{V} \sum_k \delta(\lambda - \lambda_k(A)).
\]

In particular \( \rho(0) = \langle \nu(0) \rangle \), Eq. \cite{9} is related to the two-point connected correlation function discussed in \cite{10} for constant \( \nu_A \). It satisfies the sum rule \( \int ds R(s) ds = 0 \) and its Fourier transform is the spectral form-factor.
\[ K(t) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} R(\lambda/\Delta) e^{-i\lambda t}. \] (4)

Following the general semi-classical arguments by Gutzwiller [1], Berry [12] and others [13], we suggest that for QCD
\[ K(t) \approx \frac{2|\Delta|^2}{(2\pi)^2} p(t). \] (5)

where \( p(t) \) is the quark return probability for fixed proper time \( t \). In QCD with \( N_c \geq 3 \) (number of colors) the quarks are in the (complex) fundamental representation. Their evolution operator along the proper time does not enjoy time-reversal invariance in the presence of chromomagnetic interactions. In the semi-classical approximation leading to (5) this corresponds to \( \beta = 2 \) (only the diagonal terms in the closed orbit indices are retained). In QCD with \( N_c = 2 \) the quark orbits are paired by the conjugation operator \( C \), where \( C \) is charge conjugation and \( K \) complex conjugation, hence \( \beta = 1 \) (diagonal and off-diagonal terms within the pair retained). In QCD with \( N_f \) adjoint and real Majorana quarks, the orbits are paired by \( CK \). If only one pair of orbits is retained (square root of the theory) then \( \Sigma = \pi \rho(0)/2 \) and \( \beta = 4 \). A similar observation applies to Kogut-Suskind quarks on the lattice.

The normalized return probability \( p(t) \) of a quark from \( x(0) \) back to \( x(t) \) for fixed proper time \( t \) in a four volume \( V \), is
\[ p(t) = \frac{V^2}{N} \lim_{y \to x} \int \frac{d\lambda_1 d\lambda_2}{(2\pi)^2} e^{-i(\lambda_1 - \lambda_2)|t|} \langle \text{Tr} (\Sigma(x, y; z_1) \Sigma^\dagger(x, y; z_2)) \rangle \] (6)

with \( z_{1,2} = m - i\lambda_{1,2} \), and
\[ S(x, y; z) = \langle x | \frac{1}{i\nabla + iz} | y \rangle \] (7)

where \( N \) is the number of quark states in \( V \). A proper regularization of the short paths is assumed. Setting \( \lambda_{1,2} = \frac{\Lambda \pm \alpha}{2} \) with \( \Lambda \approx 0 \) (near the zero virtuality point) [4], then \( z_1 = z = m - i\lambda/2 \) and \( z_2 = z^* \). In this case, the quark propagator satisfies
\[ S^\dagger(x, y; z^*) = -\gamma_5 S(y, x; z) \gamma_5 \] (8)

and (3) is seen to relate to the pion correlation function
\[ \langle x^0 y^4 z \rangle = \langle \text{Tr} (\Sigma(x, y; z) i\gamma_5 \tau^a S(y, x; z) i\gamma_5 \tau^a) \rangle \] (9)

For \( z = m \), pion-pole dominance (long paths) yields
\[ C_{\pi}(x, y; m) \approx \frac{1}{V} \sum_Q e^{iQ(x-y)} |\langle y|q \rangle|^2 \frac{1}{F^2 Q^2 + m^2} \] (10)

where the sum is over the pion momenta \( Q \), and the analytical continuation \( m \rightarrow m - i\lambda/2 \), we find
\[ C_{\pi}(x, y; z) \approx \frac{1}{V} \sum_Q e^{iQ(x-y)} |\langle y|q \rangle|^2 \frac{2|\langle y|q \rangle|^2}{-i\lambda + 2m + DQ^2} \] (11)

with the diffusion constant \( D = 2F^2/\Sigma \). Inserting (11) into (13), and noting that \( \int d\lambda \sim E \) with \( E/\Delta = N \) and \( \rho = 1/\Delta V \), with \( \Sigma = |\langle y|q \rangle|^2 = \pi \rho \), we conclude after a contour integration that
\[ p(t) = e^{-2m|t|} \sum_Q e^{-DQ^2|t|} \] (12)

The prefactor \( e^{-2m|t|} \) is the expected damping for zero mode quarks. Heavy quarks with bare masses \( 2m > D(2\pi/L)^2 \) (symmetric box) do not diffuse.

The present arguments will allow us to analyze the spectral rigidity (12)
\[ \Sigma_2(N) = \int_{-N}^{N} ds \langle N - |s| \rangle R(s) . \] (13)

This is the variance in the number of virtual quark levels in an energy band \( E = N\Delta \), averaged over the QCD gauge configurations. Its dependence on \( N \) or \( E \) changes qualitatively with the virtuality band considered as we now discuss.

4. In the ergodic regime \( \tau_{\text{erg}} < t < t_H \) the result (12) is dominated by the constant mode \( n = 0 \). Hence the return probability is \( p(t) = e^{-2m|t|} \) and
\[ R(s) = \frac{1}{\beta \pi^2} \frac{\alpha^2 - s^2}{(\alpha^2 + s^2)^2} \] (14)

with \( \alpha = 2m/\Delta \). The corresponding spectral rigidity is
\[ \Sigma_2(N) = \frac{1}{\beta \pi^2} \ln \left( \frac{N^2 + \alpha^2}{\alpha^2} \right) \] (15)

for \( N \gg 1 \). For quark masses in the range \( \alpha \sim 1 \) or \( 0 < m/EV \leq 1 \) (V \sim N) [13], the spectral rigidity is universal, \( \Sigma_2(N) = 2\ln N/((\beta \pi^2)^2) \). This is in agreement with the result of standard [13] and chiral [10] random matrix theory. For quark masses \( m \sim 1 \) in units where \( F = \Sigma = 1 \), the spectral rigidity [14] deviates from random matrix theory. In QCD this mass range is already sensitive to strangeness.

The result is also valid for quenched QCD as [11] and the GOR relation hold in this case as well. For \( \alpha \sim 1 \), the asymptotic of (14) is enough [18] to show that the tail of the bulk level spacing distribution follows that of a Wigner surmise [15]. The arguments in [17] can be easily extended to general \( \alpha \).

The present arguments only yield the perturbative part of the two-level correlation function (2). The oscillatory part well-known from random matrix theory [15], requires a reassessment of (3) including the short paths which we have not done in this work. In this context an interesting observation was made recently by Agam,
Altshuler and Andreev \cite{18}, relating not only the perturbative part of \cite{13} but also the oscillatory part to the spectrum of the diffusion operator (in our case \cite{21}), and more generally to the Perron-Frobenius operator. A similar observation should hold in the context of QCD, where the short paths are just controlled by QCD perturbation theory, thereby establishing the chaotic character of the quark spectrum in QCD from first principles.

Finally, we note that the relation \cite{4} in combination with \cite{12} allows us to write

\[ R(s) = -\frac{\Delta^2}{\beta\pi^2} \text{Re} \sum_q \frac{1}{(s\Delta + iDQ^2 + i2m)^2} \]  

in agreement with the perturbative result derived by Altshuler and Shklovskii \cite{14} for disordered electrons in metallic grains. In our case, the squared denominator in \cite{14} follows from the exchange of 2-pions in the double ring diagram corresponding to the density-density correlation function \cite{4}. This diagram is forced by G-parity and is dominant at large \( t \). So QCD with \( N_c \geq 3 \) corresponds to the exchange of diffusions \cite{13}, hence \( \beta = 2 \). QCD with \( N_c = 2 \) and \( N_f \geq 2 \) allows for massless pions and baryons. In this case, the baryons are the analogue of the cooperons \cite{19}, hence \( \beta = 1 \).

6. In the diffusive regime \( \tau_d < t < \tau_{\text{erg}} \) the result \cite{12} receives contribution from all modes in the box \( V \). The outcome is still a classically diffusive motion of the quark in \( d=4 \), with a return probability

\[ p(t) = e^{-2m|t|} \frac{V}{(4\pi Dt)^2} \]  

Setting \( 1/m > \tau_{\text{erg}} \) that is \( \Sigma/V < m < D/\sqrt{V} \) and dialing the virtuality \( \lambda \) such that \( 1/\tau_{\text{erg}} < \lambda < 1/\tau_d \) will allow us to probe the diffusive part of the quark spectrum in QCD. In particular, the spectral form-factor \cite{4} can be readily found. Its Fourier transform is seen to diverge for short times. Setting a short time cutoff at \( \tau_d \) (the diffusion time), we obtain

\[ R(s) = -\frac{V\Delta^2}{(2\pi)^4D^2\beta} \left( C + \ln(\Delta\tau_d |s|) \right) \]  

where \( C = 0.577 \) is Euler’s constant. We note that \( \Delta\tau_d = \tau_d/\tau_H \gg 1 \). In the diffusive regime the spectral rigidity is

\[ \Sigma_2(N) = \frac{V\Sigma^2}{4\beta(2\pi)^4D^4} \left( 3 - C - \ln(N\Delta\tau_d) \right)(N\Delta)^2 \]  

where \( N = E/\Delta \gg 1 \) is the mean number of states in the energy band \( E \). In the diffusive regime \( E/\tau_d < 1 \) so the spectral rigidity is always positive. However, it decreases with increasing \( N \) or \( E = N\Delta \). Although not universal, this result should be specific to the QCD quark spectrum in the quoted regime, and through the \( 1/F^4 \) behavior amenable to chiral power counting \cite{24}.

7. In the ballistic regime \( t < \tau_d \) the classical trajectories of the quarks are shorter than the typical mean free path \( l = v_f\tau_d = v_F/2m_Q \), which is about the constituent quark Compton wavelength. Although in this regime the density of states vary with the virtuality \( \lambda \), we will for simplicity ignore the variations in our case \cite{21}. For short times, the return probability is about constant but small, say \( p(t) \sim p + O(t^2/\tau_d^2) \). Hence,

\[ R(s) = -\frac{p}{\beta(\pi s)^2} \left( 1 - O \left( \frac{1}{(s\Delta\tau_d)^2} \right) \right) \]  

A direct calculation of \( p \) may be averted by noticing that \cite{13} implies

\[ \frac{d\Sigma_2(N)}{dN} = \int_{-N}^{N} ds R(s) \]  

for a fixed \( N = E/\Delta \), in the ballistic regime. Since \cite{2} obeys the null sum rule on the entire support of eigenvalues as required by the conservation of the number of energy levels, it follows that the contribution from the ballistic regime balances the contribution from the diffusive regime for \( L \gg l \), thereby fixing \( p \). Setting the cutoff between the ballistic and diffusive regimes at \( \sigma_1 = D/(l^2\Delta) \), we obtain \cite{21}

\[ R(s) = -\frac{V}{(2\pi l)^4\beta} \left( 1 - C + \ln \left( \frac{l^2}{D\tau_d} \right) \right) \frac{1}{8\pi^2} \]  

We note that \( l^2/(D\tau_d) \sim 4 \), and the prefactor yields \( p \approx V/l^4 > 0 \), which is expected from free phase space in \( d = 4 \). In the virtuality range \( \lambda \gg 1/\tau_d \sim m_Q \), the spectral rigidity for fixed \( E/\Delta \gg 1 \), is

\[ \Sigma_2(N) = \frac{2V}{(2\pi l)^4} \left( 1 - C + \ln \left( \frac{l^2}{D\tau_d} \right) \right) \ln(N\Delta\tau_d) \]  

in overall agreement with the result of Gefen and Altland \cite{21} for two-dimensional non-diffusive systems.

8. In models of the QCD vacuum such as the instanton liquid model, a chiral transition may be envisioned by varying the density of instantons in fixed \( V \). The low density phase is characterized by instanton clusters where the quarks and antiquarks are localized over the cluster sizes. At high instanton density, the quarks are delocalized and chiral symmetry is spontaneously broken. These transitions are actually quite generic and may be triggered by changing the external parameters in QCD, including the shape and size of the Euclidean box \( V \). For instance, the formation of quark clusters is strongly flavor dependent, so it is natural to assume that such a transition may take place by just varying the number of flavors \cite{22}. In the following, we will assume that these transitions are of the metal-insulator type and proceed to analyze their signature on the quark spectrum.

Let \( \xi \) be the correlation or localization length in the cross-over region. For \( \xi \leq L \), the diffusion becomes anomalous with a scale dependent diffusion constant \cite{26,27}. The scale dependence may be estimated
using renormalization group arguments \[ \frac{1}{\sigma} \approx \frac{1}{\sigma_0} \]. At the critical point with \( \xi \sim L \) and \( s \gg 1 \), the level correlation asymptotes \( R(s) \sim s^{-1-1/(\nu d)} \). The critical exponent \( \nu \) sets the rate of divergence of \( \xi \) in terms of the critical conductance \( \sigma_\nu \), \( \xi/L \approx 1 - \sigma_\nu / \sigma_\nu \). In \[ \mathbf{23} \] this result was reached by resumming the multi-2-diffuson diagrams and using renormalization group analysis. Their arguments carry to QCD by trading 2-diffusons with 2-grams and using renormalization group arguments \[ \mathbf{8,28} \]. In particular arguments extend to matter since (1) is protected by symmetry, or the Nambu-Jona-Lasinio model. They can be readily extended to continuum models such as the instanton liquid model \[ \mathbf{22} \] or the Wigner-Dyson and Poisson \[ \mathbf{26,27,29} \]. This result is a modified Wigner surmise that is intermediate between Wigner-Dyson and Poisson \[ \mathbf{8} \]. The spectral rigidity in the regime \( \Delta \approx \frac{1}{s} \) with \( \Delta \) follows from the null spectral sum rule through the ballistic contribution \[ \mathbf{23} \].

Finally, the asymptotics of \( R(s) \) for \( s \gg 1 \) in the critical regime, implies that the level spacing distribution \( P(s) \) \[ \mathbf{8} \] in QCD changes at the mobility edge. The result is a modified Wigner surmise that is intermediate between Wigner-Dyson and Poisson \[ \mathbf{22,24,27,29} \]. This transition between chaos and integrability suggests that in the same regime the quark wavefunctions in fixed \( \nu \) are multi-fractal \[ \mathbf{31} \] with a multi-fractal exponent \( \eta \approx 8 \chi \).

The present results may be checked numerically by studying quark spectra using lattice QCD simulations, or continuum models such as the instanton liquid model or the Nambu-Jona-Lasinio model. They can be readily extended to matter since \[ \mathbf{1} \] is protected by symmetry, and supersymmetric gauge theories with flavor \[ \mathbf{32} \].

IZ would like to thank Igor Aleiner for discussions. This work was supported in part by the US DOE grant DE-FG-88ER40388, by the Polish Government Project (KBN) grants 2P03B04412 and 2P03B00814 and by the Hungarian grants FKFP-0126/1997 and OTKA-F026622.

1. J. Gasser and H. Leutwyler, Phys. Lett. \textbf{B184} (1987) 83; Phys. Lett. \textbf{B188} (1987) 477; Nucl. Phys. \textbf{B307} (1988) 763.

2. H. Leutwyler and A. Smilga, Phys. Rev. \textbf{D46} (1992) 5607.

3. E. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. \textbf{A560} (1993) 306.

4. J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. \textbf{70} (1993) 3852.

5. M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. \textbf{175} (1968) 2195.

6. T. Banks and A. Casher, Nucl. Phys. \textbf{B169} (1980) 103.

7. D.J. Thouless, Phys. Rep. \textbf{13} (1979) 93.

8. P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. \textbf{57} (1985) 287.

9. K.B. Efetov, Adv. Phys. \textbf{32} (1983) 53.

10. M.A. Nowak, J.J.M. Verbaarschot and I. Zahed, Phys. Lett. \textbf{B217} (1989) 157; Nucl. Phys. \textbf{B325} (1989) 581.

11. M.C. Gutzwiller, J. Math. Phys. \textbf{8} (1967) 1979; \textbf{10} (1969) 1004; \textbf{12} (1971) 343.

12. M.V. Berry, Proc. Roy. Soc. \textbf{A400} (1985) 229.

13. J.H. Hannay and A.M. Ozorio de Almeida, J. Phys. \textbf{A17} (1984) 3429; N. Argaman, Y. Imry and U. Smilansky, Phys. Rev. \textbf{B47} (1993) 4440.

14. The general case with \( \Lambda \) finite will be discussed elsewhere.

15. M.L. Mehta, “Random Matrices”, Academic Press, New York (1991).

16. J.J.M. Verbaarschot, Phys. Rev. Lett. \textbf{72} (1994) 2531.

17. R.A. Jalabert, J.L. Pichard and C.W.J. Beenakker, Euro. Lett. \textbf{24} (1993) 1.

18. O. Agam, B.L. Altshuler and V. Andreev, cond-mat/9509102.

19. B.L. Altshuler and B.I. Shklovskii, Sov. Phys. JETP \textbf{64} (1986) 127.

20. J. Gasser and H. Leutwyler, Ann. Phys. \textbf{158} (1984) 142.

21. The variations in the density of states can be taken into account by using unfolding procedures \[ \mathbf{33} \].

22. The sensitivity of the result to the cutoff \( \sigma_\nu \) is logarithmically weak.

23. A. Altland and Y. Gefen, Phys. Rev. Lett. \textbf{71} (1993) 3339.

24. T. Banks and A. Zaks, Nucl. Phys. \textbf{B196} (1982) 189.

25. E. Shuryak, Phys. Rev. \textbf{D52} (1995) 5370.

26. V. E. Kravtsov, I. V. Lerner, B. L. Altshuler, and A. G. Aronov, Phys. Rev. Lett. \textbf{72} (1994) 888.

27. A.G. Aronov, V.E. Kravtsov and I.V. Lerner, JETP Lett. \textbf{59} (1994) 40.

28. A.J. McKane and M. Stone, Ann. Phys. \textbf{131} (1981) 36.

29. B.L. Altshuler, I.K. Zharekheshev, S.A. Kotochigova and B.I. Shklovskii, Sov. Phys. JETP \textbf{67} (1988) 625; B.I. Shklovskii, B. Shapiro, B.R. Sears, P. Lambrianides and H.B. Shore, Phys. Rev. \textbf{B47} (1993) 11487.

30. A.G. Aronov and A.D. Mirlin, Phys. Rev. \textbf{B51} (1995) 6131.

31. J.T. Chalker, I.V. Lerner and R.A. Smith, Phys. Rev. Lett. \textbf{77} (1996) 554; and JETP Lett. \textbf{64} (1996) 386; V.E. Kravtsov and K.A. Muttalib, Phys.Rev.Lett. \textbf{79} (1997) 1913.

32. N. Seiberg, Nucl. Phys. \textbf{B435} (1995) 129.