Influence of the Properties of the Plate Surface on the Natural Oscillations of the Clamped Drop

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Abstract. We consider free oscillations of a clamped liquid drop. An incompressible fluid of different density surrounds the drop. In equilibrium, the drop has the form of a circular cylinder bounded axially by the parallel solid planes, the contact angle is right. These plates have different surface (wetting etc.) properties. The solution is represented as a Fourier series in eigenfunctions of the Laplace operator. The resulting system of complex equations for unknown amplitudes was solved numerically. The fundamental frequency of free oscillations can vanish in a certain interval of values of the Hocking parameter. The length of this interval depends on the ratio of the drop dimensions. Frequencies of other drop eigenmodes decrease monotonically with increasing Hocking parameter.

1. Introduction

The properties of the substrate surface have a significant effect on the movement of the contact line of the three media. Attention to such phenomena is caused not only by the study of physical processes, but also by applied technological problems.

For a nonstationary contact angle, i.e. in the case of dynamic motion of the contact line, the effective boundary condition is applied most frequently, which was used in [1] to study waves on the surface of a liquid between two vertical walls. It assumes that the velocity of the contact-line motion is related linearly to the deviation of the contact angle from its equilibrium value (as a rule, the equilibrium contact angle is assumed to be right for the sake of simplicity):

\[
\frac{\partial \zeta^*}{\partial t} = \Lambda \mathbf{k} \cdot \nabla \zeta^*,
\]

where \(t^*\) is the time, \(\zeta^*\) is the deviation of the surface from the equilibrium state, \(\Lambda^*\) is the phenomenological (Hocking) constant, and \(\mathbf{k}\) is the normal vector to the solid surface. Note that the conditions of a fixed contact line and a constant contact angle are special cases of boundary condition (1), has been \(\zeta^* = 0\) and \(\mathbf{k} \cdot \nabla \zeta^* = 0\), respectively. Condition (1) was used in many works, e.g., in the studies of oscillations of a semispherical drop of incompressible liquid on a substrate [2], a semispherical gas bubble in a finite-depth liquid on a substrate [3], a liquid (capillary) bridge [4], a cylindrical drop [5], an oblate drop [6] and a cylindrical bubble [7]. The parameter \(\Lambda^*\) was a real and constant in all the articles listed above. In [8], it was assumed that a change in the contact line does not necessarily occur in one phase with a contact angle, i.e., the Hocking constant \(\Lambda^*\) is complex. Also, different parameters \(\Lambda^*\) for parallel plates are used in a number of articles for a cylindrical drop (bubble) [9-11].

Altering the surface wettability has received great attention over the past years. Drop dynamics has been intensively studied on different materials: homogeneous plates [12], heterogeneous/chemically patterned substrates [13] and on gradient and soft surfaces [14]. In [15], the role of the correlation length
in the behavior of a drop on noisy surfaces is investigated numerically. The static contact angle is related to the substrate density through model [16]. Also important topics are the interaction of a drop (bubble) with a solid wall [17] and the fall of a drop (rigid body) onto the surface [18]. The influence of identical heterogeneous plates was considered in [19, 20] in contrast to homogeneous ones in [2]. A new equation-based model (1) was proposed in these articles: parameter $\lambda'$ was depends on the polar angle, i.e. the coefficient of the interaction between the plate and the fluid (the contact line) is a function of the plane coordinates. In this paper, we study the effect of the properties of the substrate surfaces on the free oscillations of a liquid drop. This drop is sandwiched between these substrates and surrounded by another liquid of a different density. In contrast to the previous works, in this case the surfaces of the plates not only differ from each other, but also are heterogeneous.

2. Problem formulation
We consider the dynamic behavior of the incompressible liquid drop of density $\rho^*$ surrounded by another liquid of density $\rho^*_s$ (here and in the following, the quantities with subscript $i$ refer to the drop, and those with subscript $e$ – to the surrounding liquid). The system is bounded by two parallel solid surfaces (see figure 1) with the inter-plate spacing $h^*$. The equilibrium contact angle $\theta_i$ between the drop surface and the solid surface is equal to $0.5\pi$.

Because of the problem symmetry it is convenient to introduce the cylindrical coordinates $r^*, \alpha^*, z^*$. The azimuthal angle $\alpha^*$ is reckoned from the $x$-axis. Let the surface of the droplet be described by the equation $r^* = R_0^* + \zeta^*(\alpha^*, z^*, t^*)$. Assuming a potential liquid motion, we introduce the velocity potential $\mathbf{v} = \nabla \varphi^*$. Taking length $R_0^*$, height $h^*$, density $\rho^*_s + \rho^*$, time $t^*$, velocity potential $A'\sqrt{\sigma} \left( (\rho^*_s + \rho^*) R_0^* \right)^{-1/2}$, pressure $A' \sigma \left( R_0^* \right)^{-2}$, and deviation of the surface $\Lambda'$ as characteristic quantities, we pass to dimensionless variables and obtain the following linear problem

$$p_j = -\rho_j (\varphi_j + \alpha^* \omega^* e^{\omega^* t^*}), \quad \Delta \varphi_j = 0, \quad j = i, e,$$

$$A = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \alpha^*}, \quad r = 1: \ [\varphi] = 0, \quad \zeta^* = \varphi^* + \zeta^*_0 + \int \zeta^*, \quad \Lambda = \int \Lambda_{\alpha} \left( \alpha \right) \zeta^*,$$

$$\zeta = \pm \frac{1}{2}, \quad \varphi = 0,$$

$$r = 1, \quad z = \pm \frac{1}{2} \quad \zeta = \pm \Lambda_{\alpha} \left( \alpha \right) \zeta^*,$$

where $\Lambda_{\alpha} (\alpha)$ and $\Lambda_{\psi} (\alpha)$ are the wetting parameters (the Hocking parameters), which describes the inhomogeneity of the upper $(z = 0.5)$ and bottom $(z = -0.5)$ plates, respectively, $\rho$ is the fluid pressure, the square brackets denote a jump in the quantity at the interface between the external liquid and the drop. The boundary condition (5) (see details [19, 20]) is a modified Hocking condition (1). The boundary-value problem (2) – (5) involves five parameters: the oscillation amplitude, the aspect ratio, the dimensionless density, the wetting parameter:

$$A = \frac{A'}{R_0^*}, \quad b = \frac{R_0^*}{h^*}, \quad \rho_i = \frac{\rho^*_s + \rho^*}{\rho^*_s + \rho^*}, \quad \rho_s = \frac{\rho^*_s + \rho^*}{\rho^*_s + \rho^*}, \quad \Lambda = b A' \left( (\rho^*_s + \rho^*) R_0^* \right)^{-1/2}.$$
3. Natural oscillations

Let us consider a particular case of heterogeneous plates: $\Lambda_{\alpha k} (\alpha) = \lambda_{\alpha k} \mid \cos (\alpha) \mid$. The function $\Lambda (\alpha)$ is represented as a Fourier series in eigenfunctions of the Laplace operator. In this case only even azimuthal modes will be more important in the spectrum. Therefore, further we will study only them. The natural oscillations of a cylindrical drop between homogeneous surfaces were studied in [21]. The fields of velocity potentials and surface deviation can be represented as

$$\phi_1 (r, \alpha, z, t) = \Re \left[ \Omega \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left( a_{nl} R_{nl}^* (r) \sin \left( (2k + 1) \pi z \right) + a_{2nl} R_{2nl}^* (r) \cos \left( 2k \pi z \right) \cos (2m\alpha) \right) e^{i\omega t} \right],$$

$$\phi_2 (r, \alpha, z, t) = \Re \left[ \Omega \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left( b_{nl} R_{nl}^* (r) \sin \left( (2k + 1) \pi z \right) + b_{2nl} R_{2nl}^* (r) \cos \left( 2k \pi z \right) \cos (2m\alpha) \right) e^{i\omega t} \right],$$

$$\zeta (\alpha, z, t) = \Re \left[ \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left( c_{nl} \sin \left( (2k + 1) \pi z \right) + c_{2nl} \cos \left( 2k \pi z \right) \right) \cos (2m\alpha) + d_{10} \sin \left( \frac{z}{b} \right) + d_{30} \cos \left( \frac{z}{b} \right) \right],$$

$$R_{nl}^* (r) = I_n \left( \sqrt{\frac{(2k + 1) \pi b r}} \right), \quad R_{2nl}^* (r) = r^n, \quad R_{2nl}^* (r) = I_n \left( 2k \pi b r \right), \quad R_{2nl}^* (r) = r^n,$$

where $I_n$ and $K_n$ are the modified Bessel functions of $m$-th order. Substituting solutions (6) – (8) into (2) – (5), we obtain a spectral-amplitude problem, whose eigenvalues are values of the natural oscillation frequency $\Omega$. These complex algebraic equations have complex solutions, which lead to damping of oscillations. This effect is caused only by the condition on the contact line, not by viscosity. The equations of our spectral-amplitude problem were solved numerically by the two-dimensional secant method. For convenience we will denote the frequencies of the even modes as $\Omega_{n, 2k}$ (where $k = 0, 1, 2, \ldots$), and the frequencies of the odd modes as $\Omega_{n, 2k + 1}$ (where $k = 0, 1, 2, \ldots$). Here, the first index $m$ is a azimuthal number and the second index $2k$ (or $2k + 1$) is wavenumber. Thus, the frequencies $\Omega_{m, n}$ of the natural oscillations with the odd index $n$ will correspond to the odd modes and an even index $n$ to the even mode.
Figure 2. Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter $\lambda_b$ for $\Omega_{01}$ ($b=1$, $\rho=0.7$). $\lambda_b = 0$ – dotted, $\lambda_b = 5$ – dashed, $\lambda_b = 10$ – dot-dashed, $\lambda_b = 100$ – solid line.

Figures 2–4 show the real part $\Omega = \text{Re}(\Omega)$ (oscillation frequency) and imaginary part $\Omega = \text{Im}(\Omega)$ (damping ratio) of the complex natural frequency. Typical dependencies are shown in the figure 2: the frequency decreases monotonically with increasing parameter $\lambda_b$, and the damping rate is maximum for a finite wetting parameter and decreases in the limiting cases of the free or fixed contact line. Note, that changes in the parameter $\lambda_u$ (or $\lambda_s$) are symmetric relative to each other, i.e. you can change one with a fixed other.

Figure 3. Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter $\lambda_u$ for $\Omega_{01}$ ($b=0.32$, $\rho=0.7$). $\lambda_u = 0$ – dotted, $\lambda_u = 5$ – dashed, $\lambda_u = 10$ – dot-dashed, $\lambda_u = 100$ – solid line.
The total damping rate is determined by the sum of the individual coefficients for each plate. This fact determines the finite value of the damping parameter at small \( \lambda \) (see figure 2b). Surface inhomogeneity changes the monotonicity of curves and leads to the appearance of local extrema. In a certain range of \( \lambda \), the real part of the frequency \( \text{Re}(\Omega_0) \) can vanish. It’s depending on the value of the ratio \( b \) and Hocking parameter \( \lambda \) (figure 3a). The vanishing of \( \text{Re}(\Omega_0) \) corresponds to the bifurcation of the branch of the increment \( \text{Im}(\Omega_0) \) (figure 3b). The reason for the frequency disappearance is the magnitude of dissipation (parameters \( \lambda \) and \( \lambda_i \) are finite) which is so large that the oscillations cannot arise. In this case the dissipation is proportional to the area of the lateral surface of the drop for the surface harmonics of natural oscillations (the larger surface is required the more energy for the excitation of the surface waves and the energy of drop oscillations is proportional to the volume). Consequently, increase of the aspect ratio \( b \) corresponds to a decreased side surface of the drop at its constant volume, i.e., dissipation reduces the surface oscillations (figure 1, \( b = 1 \)). Note, for \( b < \pi^{-1} \) the damping ratio of the first shape frequency becomes negative, which corresponds to the occurrence of the Rayleigh–Plateau instability.

Figure 4 shows the oscillation frequency and damping ratio of the complex natural frequency \( \Omega_{20} \) for the second azimuthal mode. The real part of the frequency \( \text{Re}(\Omega_{20}) \) can vanish in a certain range of \( \lambda \) (figure 4a) as well as \( \text{Re}(\Omega_{01}) \) (figure 3a). But, this range is absent for small values of geometry ratio \( b \). The dissipation is proportional to the length of the contact line in this case, because this is just the interaction between the contact line and the solid plate that causes the energy dissipation. Therefore, growing of parameter \( b \) increases the length of the contact line at constant drop volume, i.e., it increases the energy dissipation. The break of the damping decrement occurs at the point of equality of two different oscillation modes (figure 4b).

Figure 4. Frequency (a) and damping ratio (b) of natural oscillations vs wetting parameter \( \lambda \) for \( \Omega_{20} \) (\( b = 4 \), \( \rho_i = 0.7 \)). \( \lambda_i = 0 \) – dotted, \( \lambda_i = 5 \) – dashed, \( \lambda_i = 10 \) – dot-dashed, \( \lambda_i = 100 \) – solid line.

4. Conclusions
The natural oscillations of a cylindrical drop confined between solid plates has been considered taking into account the dynamics of the contact line. The heterogeneous solid plates have different Hocking parameters. The solid plates have non-uniform surfaces described by the function \( \Lambda_{u,b}(\alpha) = \Lambda_{u,b} \cos(\alpha) \). The boundary condition imposed on the contact line leads to the damping of
oscillations. In addition, there is a phase shift between the oscillations of different parts of the liquid, which leads to the appearance of traveling capillary surface waves. The energy dissipation is due to the Hocking condition, even despite the model of an inviscid fluid. This allows us to use the potential flow according to the Kelvin’s circulation theorem.

It is shown that the summary damping rate is determined by the sum of the individual coefficients for each plate. Fundamental frequencies can vanish depending on the parameters of the problem, which is associated with monotonic damping of free oscillations. It’s corresponds to the bifurcation of the branch of the damping rate. Surface heterogeneity changes the monotonicity of curves and leads to the appearance of local extrema.

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