On the form factors of local operators in the Bazhanov-Stroganov and chiral Potts models

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Abstract

We consider general cyclic representations of the 6-vertex Yang-Baxter algebra and analyze the associated quantum integrable systems, the Bazhanov-Stroganov model and the corresponding chiral Potts model on finite size lattices. We first determine the propagator operator in terms of the chiral Potts transfer matrices and we compute the scalar product of separate states (including the transfer matrix eigenstates) as a single determinant formulae in the framework of Sklyanin’s quantum separation of variables. Then, we solve the quantum inverse problem and reconstruct the local operators in terms of the separate variables. We also determine a basis of operators whose form factors are characterized by a single determinant formulae. This implies that the form factors of any local operator are expressed as finite sums of determinants. Among these form factors written in determinant form are in particular those which will reproduce the chiral Potts order parameters in the thermodynamic limit. The results presented here are the generalization to the present models associated to the most general cyclic representations of the 6-vertex Yang-Baxter algebra of those we derived for the lattice sine-Gordon model.

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1 Introduction

In the article [1] we developed an approach in the framework of the quantum inverse scattering method (QISM) [2–14] to achieve the complete solution of lattice integrable quantum models by the exact characterization of their spectrum and the computation of the matrix elements of local operators in the eigenstates basis. This approach is addressed to the large class of integrable quantum models whose spectrum (eigenvalues and eigenstates) can be determined by implementing Sklyanin’s quantum separation of variables (SOV) method [15–17]. It can be considered as the generalization to this SOV framework of the Lyon group methods [166] for the computation of matrix elements of local operators in the algebraic Bethe ansatz settings. In [1] the approach has been developed for the lattice quantum sine-Gordon model [5,14] associated by QISM to particular cyclic representations [53] of the 6-vertex Yang-Baxter algebra. More in detail, in [54–56] the complete SOV spectrum characterization has been constructed for the lattice quantum sine-Gordon model while in [1] the scalar product of separate states and the matrix elements of local operators have been computed. In the present article we implement this approach for the quantum models associated by QISM to the most general cyclic representations of the 6-vertex Yang-Baxter algebra, i.e. the inhomogeneous Bazhanov-Stroganov model and subsequently the chiral Potts (chP) model [57–76], by exploiting the well known links between these two models [57]. We first build our two central tools for computing matrix elements of local operators, i.e. the expression of the scalar products of separate states in terms of a determinant formula and the local fields reconstruction in terms of quantum separate variables (by solving the so called quantum inverse scattering problem). Then, we use these results to compute the form factors of local operators on the transfer matrix eigenstates and to express them as sums of determinants given by simple deformations of the ones giving the scalar product of separate states.

1.1 Literature summary

Let us first summarize some known results concerning these quantum integrable models and that are relevant for our present work. In [57] the Bazhanov-Stroganov model was introduced from its Lax operator built as a general solution to the Yang-Baxter equation associated to the 6-vertex R-matrix. For a specific subset of cyclic representations, in which the parameters lie on the algebraic curves associated to the chP-model, the construction of the Baxter Q-operator allowed for the analysis of the spectrum (eigenvalues). This Q-operator was shown to coincide with the transfer matrix of the integrable $Z_p$ chP-model [60–69]; in this way a first remarkable connection between these two apparently very different models was established. Additional functional equations of fusion hierarchy type for commuting transfer ma-

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1This method has been introduced in [18] for the spin-1/2 XXZ quantum chain [19–27] with periodic boundaries and further developed in [28–40]. Its generalization to the higher spin XXX quantum chains and to the open spin-1/2 XXZ quantum chains [46–52] with diagonal boundary conditions has been respectively implemented in [41,42] and [43–45].

2Note that in a 2-dimensional statistical mechanics formulation both models have Boltzmann weights which satisfy the star-triangle equations. However, while the weights of the Bazhanov-Stroganov model satisfy the difference property in the rapidities those of the chP-model do not. In this respect, the link to classical integrable discrete models is quite illuminating [70–72]. It is worth recalling that the first solutions of the star-triangle equations with this non-difference property were obtained in [73–75] while in [76] the general solutions for the chP-model were derived.

3The approach of fusion hierarchy of commuting transfer matrices was first introduced in [77,78].
trices were then exhibited in [58]. Bethe ansatz type equations play an important role in the special sub-variety of the super-integrable chP-model as it was first shown in [60–62]. The connection between the Bazhanov-Stroganov model and the chP-model allowed to introduce rigorously [76] the description of the super-integrable chP spectrum using algebraic Bethe ansatz. The Bethe ansatz construction was applied to the transfer matrix $\tau_2$ of the Bazhanov-Stroganov model, thus obtaining in a different way the Baxter results [67] on the subset of the translation-invariant eigenvectors of the super-integrable chP-model. More recently, the extension of the eigenvalue analysis of the Bazhanov-Stroganov model to completely general cyclic representations was done by Baxter [59]. The main tool used there was the construction of a generalized Q-operator which satisfies the Baxter equation with the transfer matrix $\tau_2$ and the extension to these representations of the functional relations of the fused transfer matrices.

Another important feature of the chP-model which has been the subject of recent attention is the spontaneous magnetization. This order parameter was first described in [90] on the basis of perturbative calculations developed for the special class of super-integrable representations. The first non-perturbative derivation of this order parameter was achieved only recently by Baxter under some natural analyticity assumptions and the use of a technique introduced by Jimbo et al. [93]. More classical techniques, like the corner transfer matrix [94], could not be used, mainly because of the very nature of the chP-model [95]. The proof of the spontaneous magnetization formula [90] starting from direct computations on the finite lattice of matrix elements of the spin operators could only be achieved after the recent introduction by Baxter [96, 97] of a generalized version of the Onsager algebra for the special class of super-integrable representations of chP-model. The matrix elements used for this proof have been first analyzed by Au-Yang and Perk in a series of papers [79, 80, 98–100] for the case of the super-integrable chP-model. Their factorized form, first conjectured by Baxter [101], has been proven by Iorgov et al. [102] and used to derive the spontaneous magnetization formula conjectured in [90]. Finally, it is worth recalling that, in the algebraic framework of generalized Onsager algebra, Baxter has also first conjectured [105] and successively proven in [106] a determinant formula for the spontaneous magnetization of the super-integrable chP-model; this result is also used for a further derivation of the known formula of the order parameter in the thermodynamical limit.

1.2 Motivations for the use of SOV

Let us comment that in the literature we just recalled, the spectral analysis has usually one or more of the following problems: there is no eigenstates construction for the functional methods based only on the Baxter Q-operator and the fusion of transfer matrices. The ABA applies only to very special representations of the Bazhanov-Stroganov model as well as the algebraic framework of the generalized

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1. The transfer matrix of the Bazhanov-Stroganov model is the second element in this hierarchy, this explains the name $\tau_2$ given some times to this model.
2. For further analysis of the eigenstates of super-integrable chP-model see also [79, 80]. It is interesting to mention here also that in all these analysis the underlying Onsager algebra and realizations of the sl2 loop algebra, which are symmetries for these super-integrable representations [64, 65], have played fundamental roles.
3. This case both obeys Yang-Baxter integrability and has an underlying Onsager algebra [65].
4. Note that factorized formulas for the spin matrix elements exist also for the 2D Ising model [103] and for the quantum XY-chain [104].
Onsager algebra is proven to exist only in the class of super-integrable representations of chiral Potts model. The proof of the completeness of eigenstates is not ensured by these methods and it was so far missing in the general p-state chP-model and Bazhanov-Stroganov model. Existing results about this issue are mainly restricted to the case of the 3-state super-integrable chP-model \cite{107} and to the reduction of the 3-state Potts model to the trivial algebraic curve case \cite{108}, i.e. the Fateev-Zamolodchikov model \cite{109}, see also \cite{110} and \cite{111} for further applications of this method.

The circumstance interesting for us is that, in the case of the cyclic representations of the Bazhanov-Stroganov model for which the algebraic Bethe ansatz does not apply, Sklyanin’s quantum SOV can be developed to analyze the system. This means that, for most of the representations of this model, we have the opportunity to use the SOV method, which appears quite promising as it leads to both the eigenvalues and the eigenstates of the transfer matrix of the Bazhanov-Stroganov model with a complete spectrum construction if some simple conditions are satisfied. The SOV analysis of these representations was first introduced in \cite{112} and further developed in \cite{117}. Here we will use these SOV results as setup for the computation of the form factors of local operators. Let us recall that in \cite{117}, the functional equation characterization of the transfer matrix spectrum has been derived purely on the basis of the SOV spectrum characterization together with a first proof of the completeness of the system of equations of Bethe ansatz type for some classes of representations of Bazhanov-Stroganov model and chP-model and the simplicity of these transfer matrix spectra in the inhomogeneous models.

Beyond these motivations on the spectrum analysis, the summary presented in the previous subsection makes clear that the computations of matrix elements of local operators are so far mainly confined to the special class of super-integrable representations of chP-model as they were derived in the algebraic framework of the generalized Onsager algebra. This stresses the relevance of our approach using quantum separation of variables which leads to form factors of local operators and applies to generic representations of Bazhanov-Stroganov model and chiral Potts model to which the methods based on generalized Onsager algebra do not apply up to now.

1.3 Paper organization

In order to make the paper self-contained we dedicate Sections 2 and 3 to review the material presented in \cite{117} simultaneously integrating it with the presentation of new results needed for our purposes. In particular, Section 2 provides the definition of the Bazhanov-Stroganov model and the main results of \cite{117} on SOV while Subsection 2.3.1 and 2.4.2 contain new results on the SOV decomposition of the

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identity and the characterization of the transfer matrix eigenstates. Section 3 provides the definition of the chiral Potts model and the main results obtained by SOV method in [117]. The scalar products of separate states and the decomposition of the identity w.r.t. the transfer matrix eigenbasis are derived in Section 4. Section 5 contains the characterization of the propagator operator of the Bazhanov-Stroganov model in terms of the chiral Potts transfer matrices. The reconstruction of local operators in terms of separate variables is given in Section 6 while their form factors are expressed in terms of finite size determinants in Section 7. The last section addresses some comments on these results and a comparison with the existing literature.

2 The Bazhanov-Stroganov model

We use this section to give our notations and to briefly recall the main results derived in [117] on the spectrum description by SOV of the Bazhanov-Stroganov model and chiral Potts model that are useful for our purposes.

2.1 The Bazhanov-Stroganov model: definitions and first properties

We define in the N sites of the chain N local Weyl algebras \( \mathcal{W}_n \) and denote by \( u_n \) and \( v_n \) their generators:

\[
u_n v_m = q^{\delta_{n,m}} u_n \quad \forall n, m \in \{1, \ldots, N\}.
\]

(2.1)

The Lax operator of the Bazhanov-Stroganov model reads:

\[
L_n(\lambda) \equiv \begin{pmatrix}
\lambda \alpha_n v_n - \beta_n \lambda^{-1} v_{n-1} & u_n \left( q^{-1/2} a_n v_n + q^{1/2} d_n v_n^{-1} \right) \\
q^{1/2} e_n v_n + q^{-1/2} c_n v_n^{-1} & \gamma_n v_n / \lambda - \delta_n / v_n
\end{pmatrix},
\]

(2.2)

where \( \alpha_n, \beta_n, \gamma_n, \delta_n, a_n, b_n, c_n \) and \( d_n \) are constants associated to the site \( n \) of the chain subject to the relations:

\[
\alpha_n \gamma_n = a_n c_n, \quad \beta_n \delta_n = b_n d_n.
\]

(2.3)

The monodromy matrix of the model is defined in terms of the Lax operators by:

\[
M(\lambda) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix} \equiv L_N(\lambda) \cdots L_1(\lambda).
\]

(2.4)

It satisfies the quadratic Yang-Baxter relation:

\[
R(\lambda/\mu) \left( M(\lambda) \otimes 1 \right) \left( 1 \otimes M(\mu) \right) = \left( 1 \otimes M(\mu) \right) \left( M(\lambda) \otimes 1 \right) R(\lambda/\mu),
\]

(2.5)

driven by the six-vertex (standard) \( R \)-matrix:

\[
R(\lambda) = \begin{pmatrix}
q \lambda - q^{-1} \lambda^{-1} & \lambda - \lambda^{-1} & q - q^{-1} \\
q^{-1} - \lambda^{-1} & q - q^{-1} & \lambda - \lambda^{-1} \\
q \lambda - q^{-1} \lambda^{-1} & \lambda - \lambda^{-1} & q - q^{-1}
\end{pmatrix}.
\]

(2.6)

\[\text{Up to different notations, this Lax operator coincides with the one introduced in [57].}\]
Then the elements of $M(\lambda)$ generate a representation $\mathcal{R}_N$ of the so-called Yang-Baxter algebra. In particular, (2.5) yields the relation $[B(\lambda), B(\mu)] = 0$, for all $\lambda$ and $\mu$, and the mutual commutativity of the elements of the one parameter family of transfer matrix operators:

$$\tau_2(\lambda) \equiv \text{tr}_{C^2} M(\lambda) = A(\lambda) + D(\lambda).$$

(2.7)

Let us introduce the operator:

$$\Theta = \prod_{n=1}^{N} \nu_n,$$

(2.8)

which plays the role of a grading operator in the Yang-Baxter algebra.\(^\dagger\)

**Lemma 2.1.** (Lemma 1 of [117]) $\Theta$ commutes with the transfer matrix $T(\lambda)$. More precisely, its commutation relations with the elements of the monodromy matrix are:

$$\Theta C(\lambda) = q C(\lambda) \Theta, \quad [A(\lambda), \Theta] = 0,$$

(2.9)

$$B(\lambda) \Theta = q \Theta B(\lambda), \quad [D(\lambda), \Theta] = 0.$$

(2.10)

Besides, the $\Theta$-charge allows to express the following asymptotics in both $\lambda \to 0$ and $\lambda \to \infty$ of the leading operators of the Yang-Baxter algebras:

$$A(\lambda) = \left( \lambda^N \Theta \prod_{a=1}^{N} \alpha_a + (-1)^N \lambda^{-N} \Theta^{-1} \prod_{a=1}^{N} \beta_a \right) + \sum_{i=1}^{N-1} A_i \lambda^{-2i},$$

(2.11)

$$D(\lambda) = \left( \lambda^{-N} \Theta \prod_{a=1}^{N} \gamma_a + (-1)^N \lambda^N \Theta^{-1} \prod_{a=1}^{N} \delta_a \right) + \sum_{i=1}^{N-1} D_i \lambda^{-2i},$$

(2.12)

with $A_i$ and $D_i$ being operators, and so

$$\lim_{\log \lambda \to \mp \infty} \lambda^{ \mp N} \tau_2(\lambda) = \left( \Theta^{\mp 1} a_+ + \Theta^{\mp 1} d_+ \right),$$

(2.13)

where $\lim_{\log \lambda \to -\infty}$ means $\lim_{\lambda \to 0}$, $\lim_{\log \lambda \to +\infty}$ means $\lim_{\lambda \to \infty}$ and:

$$a_+ \equiv \prod_{a=1}^{N} \alpha_a, \quad a_- \equiv (-1)^N \prod_{a=1}^{N} \beta_a, \quad d_+ \equiv (-1)^N \prod_{a=1}^{N} \delta_a, \quad d_- \equiv \prod_{a=1}^{N} \gamma_a.$$

(2.14)

We only consider here representations for which the Weyl algebra generators $u_n$ and $v_n$ are unitary operators; then the following Hermitian conjugation properties of the generators of Yang-Baxter algebra hold:

**Lemma 2.2.** (Lemma 2 of [117]) Let $\epsilon \in \{+1, -1\}$, then under the following constrains on the parameters:

$$c_n = -\epsilon b_n^*, \quad d_n = -\epsilon a_n^*, \quad \beta_n = \epsilon (a_n^* b_n) / \alpha_n^*, \quad \gamma_n = \epsilon (b_n^* c_n) / \beta_n^*,$$

(2.15)

the generators of the Yang-Baxter algebra satisfy the following transformations under Hermitian conjugation:

$$M(\lambda)^\dagger = \begin{pmatrix} A^\dagger(\lambda) & B^\dagger(\lambda) \\ C^\dagger(\lambda) & D^\dagger(\lambda) \end{pmatrix} = \begin{pmatrix} D(\lambda^*) & -\epsilon C(\lambda^*) \\ -\epsilon B(\lambda^*) & A(\lambda^*) \end{pmatrix},$$

(2.16)

which, in particular, imply the self-adjointness of the transfer matrix $\tau_2(\lambda)$ for real $\lambda$.

\(^\dagger\)The proof of the lemma is given following the same steps of that of Proposition 6 of [54].
2.2 General cyclic representations

Here, we will consider general cyclic representations for which \( v_n \) and \( u_n \) have discrete spectra, and we will restrict our study to the case where \( q \) is a root of unity:

\[
q = e^{-i\pi\beta^2}, \quad \beta^2 = \frac{p'}{p}, \quad p, p' \in \mathbb{Z}^{\geq 0},
\]

with \( p \) odd and \( p' \) even being two co-prime numbers so that \( q^p = 1 \). The condition (2.17) implies that the powers \( p \) of the generators \( u_n \) and \( v_n \) are central elements of each Weyl algebra \( \mathcal{W}_n \). In this case, we fix them to the identity:

\[
v_n^p = 1, \quad u_n^p = 1.
\]

We associate to any site \( n \) of the chain a \( p \)-dimensional linear space \( \mathbb{R}_n \); we can define on it the following cyclic representation of \( \mathcal{W}_n \):

\[
v_n | k_n \rangle \equiv q^{k_n} | k_n \rangle, \quad u_n | k_n \rangle \equiv | k_n - 1 \rangle, \quad \forall k_n \in \{0, ..., p - 1\},
\]

with the following cyclic condition:

\[
| k_n + p \rangle \equiv | k_n \rangle.
\]

The vectors \( | k_n \rangle \) give a \( v_n \)-eigenbasis of the local space \( \mathbb{R}_n \). Let \( L_n \) be the linear space dual of \( \mathbb{R}_n \) and let \( \langle k_n | \) be the vectors of the dual basis defined by:

\[
\langle k_n | k'_n \rangle = (| k_n \rangle, | k'_n \rangle) \equiv \delta_{k_n, k'_n} \quad \forall k_n, k'_n \in \{0, ..., p - 1\}.
\]

The generators \( u_n \) and \( v_n \) being unitary, the covectors \( \langle k_n | \) define a \( v_n \)-eigenbasis in the dual space \( L_n \). This induces the following left representation of Weyl algebra \( \mathcal{W}_n \):

\[
\langle k_n | v_n = q^{k_n} \langle k_n |, \quad \langle k_n | u_n = \langle k_n + 1 |, \quad \forall k_n \in \{0, ..., p - 1\},
\]

with the cyclic condition:

\[
\langle k_n | = \langle k_n + p |.
\]

In the left and right linear spaces:

\[
\mathcal{L}_N \equiv \otimes_{n=1}^N L_n, \quad \mathcal{R}_N \equiv \otimes_{n=1}^N R_n,
\]

these representations of the Weyl algebras \( \mathcal{W}_n \) determine left and right cyclic representations of dimension \( p^N \) of the monodromy matrix elements, and therefore of the Yang-Baxter algebra. In the following, we will denote with \( \mathcal{R}_N^{S-\text{adj}} \) the sub-variety of the space of representations \( \mathcal{R}_N \) defined by the condition (2.15).

2.2.1 Centrality of operator averages

We define the average value \( \mathcal{O} \) of any operator matrix element \( \mathcal{O} \) of the monodromy matrix \( M(\Lambda) \) by

\[
\mathcal{O}(\Lambda) = \prod_{k=1}^p \mathcal{O}(q^k \Lambda), \quad \Lambda = \lambda^p,
\]

then the commutativity of each family of operators \( A(\lambda), B(\lambda), C(\lambda) \) and \( D(\lambda) \) implies that the corresponding average values are functions of \( \Lambda \).
Proposition 2.1. (Proposition 1 of [117])

a) The average values of the monodromy matrix entries, $A(\Lambda)$, $B(\Lambda)$, $C(\Lambda)$, $D(\Lambda)$, are central elements. They also satisfy, in the case of self-adjoint representations $R_N^{S-adj}$, the following relations under complex conjugation:

$$ (A(\Lambda))^* \equiv D(\Lambda^*), \quad (B(\Lambda))^* \equiv -\epsilon C(\Lambda^*), \quad (2.26) $$

b) Let

$$ M(\Lambda) \equiv \begin{pmatrix} A(\Lambda) & B(\Lambda) \\ C(\Lambda) & D(\Lambda) \end{pmatrix} \quad (2.27) $$

be the $2 \times 2$ matrix made of the average values of the elements of the monodromy matrix $M(\lambda)$, then it holds:

$$ M(\Lambda) = L_N(\Lambda) L_{N-1}(\Lambda) \ldots L_1(\Lambda), \quad (2.28) $$

where:

$$ L_n(\Lambda) \equiv \begin{pmatrix} \Lambda \alpha_n - \beta_n / \Lambda & q^{p/2}(\alpha_n + B_n) \\ q^{p/2}(\alpha_n + B_n) & \gamma_n / \Lambda - \Lambda \delta_n^p \end{pmatrix}, \quad (2.29) $$

is the $2 \times 2$ matrix made of the average values of the elements of the Lax matrix $L_n(\lambda)$.

2.2.2 Quantum determinant

The following linear combination of products of the Yang-Baxter generators:

$$ \det_q M(\lambda) \equiv A(\lambda) D(\lambda/q) - B(\lambda) C(\lambda/q), \quad (2.30) $$

is called quantum determinant and it is central in this algebra. It admits the following factorized form:

$$ \det_q M(\lambda) = \prod_{n=1}^{N} \det_q L_n(\lambda), \quad (2.31) $$

in terms of the local quantum determinants:

$$ \det_q L_n(\lambda) \equiv (L_n(\lambda))_{11} (L_n(\lambda/q))_{22} - (L_n(\lambda/q))_{12} (L_n)_{21}. \quad (2.32) $$

In the Bazhanov-Stroganov model it reads:

$$ \det_q M(\lambda) = \prod_{n=1}^{N} k_n(\lambda, \mu_n^+, \mu_n^-) \frac{\lambda}{\mu_n^+} - \frac{\mu_n^-}{\lambda} \frac{\lambda}{\mu_n^-} \frac{\lambda}{\mu_n^+} - \frac{\mu_n^-}{\lambda} \frac{\lambda}{\mu_n^-} \frac{\lambda}{\mu_n^+} \frac{\lambda}{\mu_n^-} = (-q)^N \prod_{n=1}^{N} \frac{\beta_n \alpha_n \epsilon_n}{\alpha_n} (1 + q^{-1} \frac{\beta_n \alpha_n}{\alpha_n} \Lambda)(1 + q^{-1} \frac{\alpha_n \beta_n}{\epsilon_n} \Lambda), \quad (2.33) $$

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14The centrality of the quantum determinant in the Yang-Baxter algebra was first discovered in [124], see also [125].
where:

\[ k_n \equiv (a_n b_n c_n d_n)^{1/2}, \quad \mu_{n,h} \equiv \begin{cases} i q^{1/2} \left( \frac{a_n b_n}{c_n d_n} \right)^{1/2} & h = +, \\ i q^{1/2} \left( \frac{c_n d_n}{a_n b_n} \right)^{1/2} & h = -. \end{cases} \]  

Moreover, for the representations that satisfy (2.15) the quantum determinant reads\(^\text{15}\):

\[
\det_q M(\lambda) = q^N \prod_{n=1}^{N} \left| \frac{\alpha_n^2}{|\alpha_n|^2} \right|^2 \left( \frac{1}{\lambda} + \epsilon q^{-1} \frac{|\alpha_n|^2}{|\alpha_n|^2} \lambda \right)^{1/2} \left( \frac{1}{\lambda} + \epsilon q^{-1} \frac{|\alpha_n|^2}{|\alpha_n|^2} \lambda \right).
\]  

Let us define the following functions that will be crucial in the rest of the paper:

\[
\bar{A}(\lambda) \equiv A(\lambda), \quad \bar{D}(\lambda) \equiv \alpha^{-1}(q\lambda)D(\lambda)
\]  

where:

\[
A(\lambda) \equiv \prod_{n=1}^{N} \left( \frac{\beta_n \alpha_n}{\beta_n \alpha_n} \right)^{1/2} \left( \frac{\lambda}{\mu_{n,+}} - \frac{\mu_{n,+}}{\lambda} \right), \quad D(\lambda) \equiv \prod_{n=1}^{N} \left( \frac{\alpha_n b_n c_n d_n}{\alpha_n b_n c_n d_n} \right)^{1/2} \left( \frac{q\lambda}{\mu_{n,-}} - \frac{\mu_{n,-}}{q\lambda} \right).
\]  

They always satisfy the condition:

\[
\det_q M(\lambda) = \bar{A}(\lambda) \bar{D}(\lambda/q),
\]  

while the function \( \alpha(\lambda) \) is defined by the requirement:

\[
\prod_{n=1}^{p} \bar{A}(\lambda q^n) + \prod_{n=1}^{p} \bar{D}(\lambda q^n) = A(\Lambda) + D(\Lambda).
\]  

Note that this last condition is a second order equation in the average \( \prod_{n=1}^{p} \alpha(q^n \lambda) \) and then we have only two possible choices for the averages of the functions \( \bar{A}(\lambda) \) and \( \bar{D}(\lambda) \):

\[
\prod_{n=1}^{p} \bar{A}(\lambda q^n) = \Omega_+ (\Lambda), \quad \prod_{n=1}^{p} \bar{D}(\lambda q^n) = \Omega_- (\Lambda),
\]  

where \( \epsilon = \mp \) and \( \Omega_\pm \) are the two eigenvalues of the \( 2 \times 2 \) matrix \( M(\Lambda) \) composed by the averages of the Yang-Baxter generators.

### 2.3 SOV-representations and the Yang-Baxter algebra

The spectral problem of the transfer matrix \( \tau_2(\lambda) \) admits a separate variables representation in the basis which diagonalize the commutative family of operators \( B(\lambda) \) as generally argued by Sklyanin in \([15–17]\). In \([117]\) it has been proven:

**Theorem 2.1. (Theorem 1 of \([117]\))** For almost all the values of the parameters of the representation, there exists a SOV representation for the Bazhanov-Stroganov model; in this case \( B(\lambda) \) is diagonalizable and has simple spectrum.

\(^{15}\)Remark that it depends on the parameters in Lax operators only through their modules.
Let us recall here the left SOV-representations of the generators of the Yang-Baxter algebra for the Bazhanov-Stroganov model. Let \( \langle \eta_k \rangle \) be the generic element of a basis of eigenvectors of \( B(\lambda) \):

\[
\langle \eta_k \rangle \mid B(\lambda) = \eta_N b_{\eta_k}(\lambda) \langle \eta_k \rangle, \quad b_{\eta_k}(\lambda) = \prod_{a=1}^{N-1} \left( \lambda/\eta_a^{(k_a)} - \eta_a^{(k_a)}/\lambda \right),
\]

and

\[
\eta_k \in \mathbb{Z}_B \equiv \left\{ (\eta_1^{(k_1)} \equiv q^{k_1} \eta_1^{(0)}, \ldots, \eta_N^{(k_N)} \equiv q^{k_N} \eta_N^{(0)}); \ k \equiv (k_1, \ldots, k_N) \in \mathbb{Z}_p^N \right\},
\]

where \( \eta_a^{(0)} \) are fixed constants of the representations. For simplicity, when possible we will omit the subscript \( k \) in \( \langle \eta_k \rangle \). The action of the remaining generators of the Yang-Baxter algebra on arbitrary states \( \langle \eta \rangle \) reads:

\[
\langle \eta \rangle \mid A(\lambda) = b_\eta(\lambda) \left[ \lambda \eta_A^{(+)} \langle q^{-\delta a} \eta \rangle + \lambda^{-1} \eta_A^{(-)} \langle q^{\delta a} \eta \rangle \right] + \sum_{a=1}^{N-1} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_a/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} a^{(SOV)}(\eta_a) \langle q^{-\delta a} \eta \rangle,
\]

\[
\langle \eta \rangle \mid D(\lambda) = b_\eta(\lambda) \left[ \lambda \eta_D^{(+)} \langle q^{\delta a} \eta \rangle + \lambda^{-1} \eta_D^{(-)} \langle q^{-\delta a} \eta \rangle \right] + \sum_{a=1}^{N-1} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_a/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} d^{(SOV)}(\eta_a) \langle q^{\delta a} \eta \rangle,
\]

where:

\[
\eta_A^{(+)} = (\pm 1)^{N-1} a_1^{+1} \prod_{n=1}^{N-1} \eta_n^{+1}, \quad \eta_A^{(-)} = (\pm 1)^{N-1} a_1^{-1} \prod_{n=1}^{N-1} \eta_n^{-1}, \quad \eta_D^{(+)} = (\pm 1)^{N-1} d_1^{+1} \prod_{n=1}^{N-1} \eta_n^{+1}, \quad \eta_D^{(-)} = (\pm 1)^{N-1} d_1^{-1} \prod_{n=1}^{N-1} \eta_n^{-1},
\]

and the states \( \langle q^{\pm \delta a} \eta \rangle \) are defined by:

\[
\langle q^{\pm \delta a} \eta \rangle \equiv \langle \eta_1, \ldots, q^{\pm 1} \eta_a, \ldots, \eta_N \rangle.
\]

Finally, the quantum determinant relation defines uniquely \( C(\lambda) \). The expressions (2.43) and (2.44) contain complex-valued coefficients \( a^{(SOV)}(\eta_a) \) and \( d^{(SOV)}(\eta_a) \) which completely characterize the SOV representation. These coefficients have to be solution of the quantum determinant conditions:

\[
\det_q M(\eta_r) = a^{(SOV)}(\eta_r) d^{(SOV)}(q^{-1} \eta_r), \quad \forall r = 1, \ldots, N - 1,
\]

and of the average conditions:

\[
A(Z_r \equiv \eta^p_r) \equiv \prod_{k=1}^{p} a^{(SOV)}(q^k \eta_r), \quad D(Z_r) \equiv \prod_{k=1}^{p} d^{(SOV)}(q^k \eta_r), \quad \forall r \in \{1, \ldots, N - 1\}.
\]

\(^{16}\)Here, the simplicity of the spectrum of \( B(\lambda) \) is equivalent to the requirement \( \langle \eta_a^{(0)} \rangle^p \neq \langle \eta_b^{(0)} \rangle^p \) for any \( a \neq b \in \{1, \ldots, N - 1\} \).
In a SOV representation, some freedom is left in the choice of \( a^{(SOV)}(\eta_r) \) and \( d^{(SOV)}(\eta_r) \). It can be parametrized by the gauge transformation written in terms of an arbitrary function \( f \):
\[
\bar{a}^{(SOV)}(\eta_r) = a^{(SOV)}(\eta_r) \frac{f(\eta_r q^{-1})}{f(\eta_r)}, \quad \bar{d}^{(SOV)}(\eta_r) = d^{(SOV)}(\eta_r) \frac{f(\eta_r q)}{f(\eta_r)}; \tag{2.49}
\]
which just amounts to the following change of normalization for the states of the B-eigenbasis:
\[
\langle \eta \rangle \rightarrow \prod_{r=1}^{N-1} f^{-1}(\eta_r) \langle \eta \rangle. \tag{2.50}
\]
Similarly, we can construct a right SOV-representation of the Yang-Baxter generators by the following actions:
\[
B(\lambda)|\eta\rangle = |\eta\rangle \eta_N b_\eta(\lambda), \tag{2.51}
\]
\[
A(\lambda)|\eta\rangle = \left[ |q^{b_\eta(\lambda)}\eta_A^{(+)}\rangle + |q^{-b_\eta(\lambda)}\eta_A^{(-)}\rangle \right] b_\eta(\lambda) + \sum_{a=1}^{N-1} |q^{b_\eta(\lambda)}\rangle \prod_{b \neq a} \frac{\langle \lambda/\eta_b - \eta_b/\lambda \rangle}{\eta_a/\eta_b - \eta_b/\eta_a} \bar{a}^{(SOV)}(\eta_a), \tag{2.52}
\]
\[
D(\lambda)|\eta\rangle = \left[ |q^{-b_\eta(\lambda)}\eta_D^{(+)}\rangle + |q^{b_\eta(\lambda)}\eta_D^{(-)}\rangle \right] b_\eta(\lambda) + \sum_{a=1}^{N-1} |q^{-b_\eta(\lambda)}\rangle \prod_{b \neq a} \frac{\langle \lambda/\eta_b - \eta_b/\lambda \rangle}{\eta_a/\eta_b - \eta_b/\eta_a} \bar{d}^{(SOV)}(\eta_a), \tag{2.53}
\]
where \(|\eta\rangle \in \mathcal{R}_N\) is the right B-eigenstate corresponding to the generic \( \eta \in \mathbb{Z}_B \). The coefficients \( \bar{a}^{(SOV)}(\eta_a) \) and \( \bar{d}^{(SOV)}(\eta_a) \) are solutions of the same average \( \langle 2.48 \rangle \) and quantum determinant:
\[
\text{det}_q \mathcal{M}(\eta_r) = \bar{d}^{(SOV)}(\eta_r) \bar{a}^{(SOV)}(q^{-1}\eta_r), \quad \forall r = 1, \ldots, N - 1 \tag{2.54}
\]
conditions while \( C(\lambda) \) is uniquely defined by the quantum determinant relation \( \langle 2.49 \rangle \).

### 2.3.1 SOV-decomposition of the identity

The diagonalizability of the Yang-Baxter generator \( B(\lambda) \) and the simplicity of its spectrum imply the following spectral decomposition of the identity \( \mathbb{1} \) in terms of the B-eigenbasis:
\[
\mathbb{1} \equiv \sum_{\mathbb{k} \in \mathbb{Z}_p} \mu_\mathbb{k} \langle \eta_\mathbb{k} | \eta_\mathbb{k} \rangle, \tag{2.55}
\]
where:
\[
\mu_\mathbb{k} \equiv \langle \eta_\mathbb{k} | \eta_\mathbb{k} \rangle^{-1} \quad \forall \mathbb{k} \in \mathbb{Z}_p^N, \tag{2.56}
\]
is the equivalent of the so-called Sklyanin’s measure\(^{17}\). The non-Hermitian character of the operator family \( B(\lambda) \) clearly implies that, for generic \( \mathbb{k} \in \mathbb{Z}_p^N, \langle | \eta_\mathbb{k} \rangle \rangle \) and \( \langle | \eta_\mathbb{k} \rangle \rangle \) are not proportional covectors in

---

\(^{17}\)Sklyanin’s measure has been first introduced by Sklyanin in his article \( [15] \) on quantum Toda chain, \( [127, 128] \); see also \( [130] \) and \( [131] \) for further discussions on the measure in the quantum Toda chain and in the sinh-Gordon model, respectively.

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Proposition 2.2. The following identities hold:

\[
\langle \eta_k | \eta_h \rangle = \langle \eta_h | \eta_h \rangle \prod_{j=1}^N \delta_{k_j, h_j}, \quad \forall \ k, h \in \mathbb{Z}_p^N, \quad (2.57)
\]

\[
\mu_h = \frac{\prod_{1 \leq a < b \leq N-1} ((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2)}{C_N \prod_{a=1}^{N-1} \omega_a(\eta_a^{(h_a)})}, \quad \forall h \in \mathbb{Z}_p^N, \quad (2.58)
\]

where:

\[
\omega_a(\eta_a^{(h_a)}) = \left( \eta_a^{(h_a)} \right)^{N-1} \prod_{l_a=1}^{h_a} \overline{a}^{\text{SOV}}(\eta_a^{(l_a)}) / \overline{a}^{\text{SOV}}(\eta_a^{(l_a)-1}) \quad (2.59)
\]

are gauge dependent parameters and \(C_N\) in the formula for \(\mu_h\) is a constant w.r.t. \(h \in \mathbb{Z}_p^N\). Then, the SOV-decomposition of the identity explicitly reads:

\[
\mathbb{I} = \sum_{h_1, \ldots, h_N = 1}^{p} \prod_{1 \leq a < b \leq N-1} \frac{((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2)}{C_N \prod_{b=1}^{N-1} \omega_b(\eta_b^{(h_b)})}. \quad (2.60)
\]

Note that the constant \(C_N\) can be put equal to one by a trivial (constant) gauge transformation that does not affect the functions \(\overline{a}^{\text{SOV}}\) and \(\overline{\overline{a}}^{\text{SOV}}\).

Proof. Computing \(\langle \eta_k | B(\lambda) | \eta_h \rangle\), we get:

\[
(b_{\eta_k}(\lambda) - b_{\eta_h}(\lambda)) \langle \eta_k | \eta_h \rangle = 0 \quad \forall \lambda \in \mathbb{C}, \quad \forall \ k, h \in \mathbb{Z}_p^N \quad (2.61)
\]

and then the simplicity of the spectrum of \(B(\lambda)\) implies (2.61). To compute \(\mu_h\), we compute the following matrix elements \(\theta_a = \langle \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)}, \ldots, \eta_N^{(h_N)} | A(\eta_a^{(h_a)-1}) | \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)}, \ldots, \eta_N^{(h_N)} \rangle\), by using first the left action of \(A(\eta_a^{(h_a)-1})\), then the right action of \(A(\eta_a^{(h_a)})\) together with (2.57) and finally equating the two results we get:

\[
\frac{\langle \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)}, \ldots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)}, \ldots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)-1}, \ldots, \eta_N^{(h_N)}, \eta_1^{(h_1)}, \ldots, \eta_a^{(h_a)-1}, \ldots, \eta_N^{(h_N)} \rangle} = \delta_{a,N} + (1 - \delta_{a,N}) \frac{\overline{a}^{\text{SOV}}(\eta_a^{(h_a)})}{\overline{\overline{a}}^{\text{SOV}}(\eta_a^{(h_a)-1})} \prod_{b \neq a, b=1}^{N-1} \frac{(\eta_a^{(h_a)}/\eta_b^{(h_b)} - \eta_b^{(h_b)}/\eta_a^{(h_a)})}{(\eta_a^{(h_a)}/\eta_b^{(h_b)} - \eta_b^{(h_b)}/\eta_a^{(h_a)})},
\]

from which (2.58) simply follows.

---

Let us recall that this measure has been first derived in [114] for cyclic representations of Bazhanov-Stroganov model [57, 59] through the recursion in the construction of left and right SOV-basis.
2.4 SOV-characterization of the spectrum

Let us denote with $\Sigma_{r_2}$ the set of eigenvalue functions $t(\lambda)$ of the transfer matrix $\tau_2(\lambda)$. We have then:

$$\Sigma_{r_2} \subset \mathbb{C}_{\text{even}}[\lambda, \lambda^{-1}]_N \text{ for } N \text{ even, } \quad \Sigma_{r_2} \subset \mathbb{C}_{\text{odd}}[\lambda, \lambda^{-1}]_N \text{ for } N \text{ odd},$$

(2.63)

where $\mathbb{C}_\epsilon[x, x^{-1}]_M$ denotes the linear space in the field $\mathbb{C}$ of the Laurent polynomials of degree $M$ in the variable $x$ which are even or odd as stated in the index $\epsilon$. The $\Theta$-charge naturally induces the grading $\Sigma_{r_2} = \bigcup_{k=0}^{2l} \Sigma^k_{r_2}$, where:

$$\Sigma^k_{r_2} \equiv \left\{ t(\lambda) \in \Sigma_{r_2} : \lim_{\log \lambda \to \mp \infty} \lambda^{\mp k} t(\lambda) = \left( q^{\mp k} a_{\mp} + q^{\pm k} d_{\pm} \right) \right\}.$$  

(2.64)

This simply follows from the commutativity of $\tau_2(\lambda)$ with $\Theta$ and from its asymptotics. In particular, any $t_k(\lambda) \in \Sigma^k_{r_2}$ is a $\tau_2$-eigenvalue corresponding to simultaneous eigenstates of $\tau_2(\lambda)$ and $\Theta$ with $\Theta$-eigenvalue $q^k$.

2.4.1 Eigenvalues and wave-funtions

In the SOV representations, the spectral problem for $\tau_2(\lambda)$ is reduced to the following discrete system of Baxter-like equations in the wave-function $\Psi_t(\eta) \equiv \langle \eta | t \rangle$ of a $\tau_2$-eigenstate $\langle t \rangle$:

$$t(\eta_r) \Psi_t(\eta) = a^{(SOV)}(\eta_r) \Psi_t(q^{-\delta_r} \eta) + d^{(SOV)}(\eta_r) \Psi_t(q^{\delta_r} \eta) \quad \forall r \in \{1, \ldots, N - 1\},$$

(2.65)

plus the following equation in the variable $\eta_N$:

$$\Psi_t(q^{\delta_N} \eta) = q^{-k} \Psi_t(\eta), \quad \text{where } \quad q^{\pm \delta_r} \eta \equiv (\eta_1, \ldots, q^{\pm 1} \eta_r, \ldots, \eta_N),$$

(2.66)

for $t(\lambda) \in \Sigma^k_{r_2}$ with $k \in \{0, \ldots, 2l\}$. Let us introduce the one parameter family $D(\lambda)$ of $p \times p$ matrix:

$$D(\lambda) \equiv \begin{pmatrix}
  t(\lambda) & -\bar{\lambda}(\lambda) & 0 & \cdots & 0 & -\bar{\lambda}(\lambda) \\
  -\bar{\lambda}(q\lambda) & t(q\lambda) & -\bar{\lambda}(q\lambda) & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  t^k: & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & -\bar{\lambda}(q^{2l-1}\lambda) & t(q^{2l-1}\lambda) & -\bar{\lambda}(q^{2l-1}\lambda) \\
  -\bar{\lambda}(q^{2l}\lambda) & 0 & \cdots & 0 & -\bar{\lambda}(q^{2l}\lambda) & t(q^{2l}\lambda)
\end{pmatrix},$$

(2.67)

then when we make the following choice of gauge for the left SOV-representation:

$$a^{(SOV)}(\lambda) \equiv \bar{\lambda}(\lambda), \quad \quad d^{(SOV)}(\lambda) \equiv \bar{\lambda}(\lambda),$$

(2.68)

it holds:
The SOV-decomposition of the identity

2.4.2 Eigenvectors and eigencovectors

Stroganov representation, the spectrum of \( \text{Theorem 2.2.} \) For almost all the values of the parameters of a Bazhanov-Stroganov model under the further constrains:

\[ \text{Theorem 2, 3 and 4 of [117]} \]

I) \( \Sigma_{\tau_2} \) coincides with the set of functions in (2.63) which are solutions of the functional equation:

\[ \det_{p} D(\Lambda) = 0, \quad \forall \Lambda \in \mathbb{C}. \]  

(2.69)

Then, up to an overall normalization, we can fix the \( \tau_2 \)-eigenstate corresponding to \( t_k(\lambda) \in \Sigma_{\tau_2} \) by:

\[ \Psi_k(\eta) \equiv \langle \eta_1, ..., \eta_N | t_k \rangle = \eta_N^{-k} \prod_{r=1}^{N-1} Q_{t_k}(\eta_r), \]  

(2.70)

where \( Q_{t_k}(\lambda) \) is the only solution (up to quasi-constants) corresponding to \( t_k(\lambda) \) of the Baxter equation:

\[ t_k(\lambda)Q_{t_k}(\lambda) = \bar{\lambda}(\lambda)Q_{t_k}(\lambda/q) + \bar{D}(\lambda)Q_{t_k}(q\lambda). \]  

(2.71)

II) In the self-adjoint representations of the Bazhanov-Stroganov model under the further constrains:

\[ \prod_{h=1}^{N} \frac{\alpha_h^*}{\alpha_h} = 1, \quad \frac{b_n}{a_n^*} = \frac{a_n}{a_n^*}, \quad \frac{\alpha_n^* \alpha_{n+1}^*}{\alpha_{n+1} \alpha_n} = \frac{b_{n+1} b_n}{b_{n+1}^* b_n^*}, \quad \forall n \in \{1, ..., N\}, \]  

(2.72)

the functions \( \bar{\lambda}(\lambda) \) and \( \bar{D}(\lambda) \) are gauge equivalent to the Laurent polynomials:

\[ a(\lambda) \equiv i^{N} \prod_{n=1}^{N} \frac{\beta_n}{\lambda} (1 - i^{(1+\epsilon)/2} q^{-1/2} |a_n|^{-1}/\lambda) (1 - i^{(1-\epsilon)/2} q^{-1/2} |a_n|^{-1}/\lambda), \quad d(\lambda) \equiv q^N a(-\lambda q), \]  

(2.73)

respectively, and for any \( t_k(\lambda) \in \Sigma_{\tau_2} \), we can construct uniquely up to quasi-constants a \( \epsilon \)-real polynomial\footnote{i.e. it satisfies the following complex-conjugation conditions: \((Q_{t}(\lambda))^* \equiv Q_{t}(\epsilon \lambda^*) \quad \forall \lambda \in \mathbb{C}.\)}

\[ Q_{t_k}(\lambda) = \lambda^{a_{t_k}} \prod_{h=1}^{2N-(b_{t_k} + a_{t_k})} (\lambda h - \lambda), \quad 0 \leq a_{t_k} \leq 2l, \quad 0 \leq b_{t_k} + a_{t_k} \leq 2lN, \]  

(2.74)

which is a solution of the Baxter functional equation (2.71) in the gauge (2.73) and:

\[ a_{t_k} = \pm k \mod p, \quad b_{t_k} = \pm k \mod p. \]  

(2.75)

2.4.2 Eigenvectors and eigencovectors

The SOV-decomposition of the identity (2.60) and the results of the previous subsections imply that the state:

\[ |t_k \rangle = \sum_{h_1, ..., h_N=1}^{p} q^{b_{h_N}} p^{l/2} \prod_{a=1}^{N-1} Q_{t_k}(\eta_a^{(h_a)}) \prod_{1 \leq a < b \leq N-1} \left((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2\right) \prod_{b=1}^{N-1} \omega_b(\eta_b^{(h_b)}), \]  

(2.76)

\footnote{Note that \( Q_{t}(\lambda) \) has been constructed in terms of the cofactors of the matrix \( D(\Lambda) \) in Theorem 3 of [117].}
is, up to an overall normalization, the only right \(\tau_2\)-eigenstate associated to \(t_k(\lambda) \in \Sigma_T^k\). Here, \(Q_{tk}(\lambda)\) is the only solution (up to quasi-constants) of the Baxter equation:

\[
t_k(\lambda)Q_{tk}(\lambda) = \bar{\lambda}(\lambda)Q_{tk}(\lambda q^{-1}) + \bar{D}(\lambda)Q_{tk}(\lambda q),
\]
as defined in Theorem \[2.2\]. Similarly, we can prove that the state:

\[
\left\langle t_k \right| = \sum_{h_1, \ldots, h_N=1}^P \frac{q^{kh_N}}{p^{1/2}} \prod_{a=1}^{N-1} \bar{Q}_{tk}(\eta_a^{(h_a)}) \prod_{1 \leq a < b \leq N-1} \left( (\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2 \right) \prod_{b=1}^{N-1} \omega_b(\eta_b^{(h_b)}),
\]
is, up to an overall normalization, the only left \(\tau_2\)-eigenstate associated to \(t_k(\lambda) \in \Sigma_T^k\). Here, \(\bar{Q}_{tk}(\lambda)\) is the only solution (up to quasi-constants) of the Baxter equation:

\[
t_k(\lambda)\bar{Q}_{tk}(\lambda) = \bar{D}(\lambda/q)\bar{Q}_{tk}(\lambda/q) + \bar{\lambda}(\lambda q)\bar{Q}_{tk}(\lambda q),
\]
when we make the following choice of gauge for the right SOV-representation:

\[
\bar{\lambda}^{(SOV)}(\lambda) \equiv \bar{\lambda}(\lambda q), \quad \bar{\lambda}^{(SOV)}(\lambda) \equiv \bar{D}(\lambda/q).
\]

## 3 The inhomogeneous chiral Potts model

### 3.1 Definitions and first properties

The connections between the integrable chiral Potts model and the Bazhanov-Stroganov model restricted to parametrization by points on the algebraic curves \(C_k\) were first remarked in \[57\]. We can summarize them as follows:

I) the fundamental R-matrix intertwining the Bazhanov-Stroganov Lax operator in the quantum space is given by the product of four chiral Potts Boltzmann weights;

II) the transfer matrix of the chiral Potts model is a Baxter Q-operator for the Bazhanov-Stroganov model.

Let us recall here how the spectrum of the inhomogeneous chiral Potts transfer matrix is characterized by SOV construction thanks to the property (II). The algebraic curve \(C_k\) of modulus \(k\) is by definition the locus of the points \(p = (a_p, b_p, c_p, d_p) \in \mathbb{C}^4\) which satisfy the equations:

\[
x_p^p + y_p^p = k(1 + x_p^p y_p^p), \quad k x_p^p = 1 - k' s_p^{-p}, \quad k y_p^p = 1 - k' s_p^p,
\]

where:

\[
x_p \equiv a_p/d_p, \quad y_p \equiv b_p/c_p, \quad s_p \equiv d_p/c_p, \quad t_p \equiv x_p y_p, \quad k^2 + (k')^2 = 1.
\]

Let us introduce the following cyclic dilogarithm functions\[21\]; here we use the notation:

\[
\frac{W_{qp}(z(n))}{W_{qp}(z(0))} = (s_p q^n) \prod_{k=1}^n \frac{y_p - q^{-2k}x_q}{y_q - q^{-2k}x_p}, \quad \frac{W_{qp}(z(n))}{W_{qp}(z(0))} = (s_p s_q^n) \prod_{k=1}^n \frac{q^{-2k}x_q - q^{-2k}x_p}{y_p - q^{-2k}y_q},
\]

\[21\]They are the Boltzmann weights of the chiral Potts model \[64\], see also \[132, 133, 141\] for the study of the properties of dilogarithm functions.
where \( z(n) = q^{-2n}, n \in \{0, ..., 2l\} \). They are solutions of the following recursion relations:

\[
\frac{W_{qp}(zq)}{W_{qp}(zq^{-1})} = -z \frac{s_p x_p}{s_q y_p} q^{-1} \frac{1 - \frac{y_q}{y_p} q^{-1} z}{1 - \frac{x_p}{x_q} y_p^{-1} z}, \quad \frac{W_{qp}(z)}{W_{qp}(zq^{-1})} = -q^{-1} \frac{y_p}{s_q y_p} x_p \frac{1 - \frac{x_p}{x_q} y_p^{-1} z}{1 - \frac{x_q}{x_p} y_p^{-1} z^{-1}}.
\]

(3.4)

If the points \( p \) and \( q \) belong to the curves \( C_k \), they are well defined functions of \( z \in S_p \equiv \{q^{2n}; n = 0, ..., 2l\} \) which satisfy the cyclicity condition:

\[
\frac{W_{qp}(z(p))}{W_{qp}(z(0))} = 1, \quad \frac{W_{qp}(z(p))}{W_{qp}(z(0))} = 1. \quad (3.5)
\]

Then, in the left and right \( u_n \)-eigenbasis, the transfer matrix \( T_{\lambda}^{chp} \) of the inhomogeneous chiral Potts model is characterized by the following kernel:

\[
T_{\lambda}^{chp}(z, z') \equiv \langle z | T_{\lambda}^{chp} | z' \rangle = \prod_{n=1}^{N} W_{q_n, p}(z_n / z_n') W_{r_n, p}(z_n / z_n' + 1), \quad (3.6)
\]

where:

\[
\lambda = t_p^{-1/2} c_0, \quad p, r_n, q_n \in C_k, c_0 \in \mathbb{C}. \quad (3.7)
\]

Let us denote with \( R_{chp}^N \) the sub-variety of the representations defined by the following parametrization of the Bazhanov-Stroganov Lax operator in terms of points of the curve:

\[
\alpha_n = -b_{q_n}^2 / c_0, \quad \beta_n = -q_{l_n} / q = -a_n d_{q_n} / q^{3/2}, \quad (3.8)
\]

\[
\beta_n = -c_0 d_{q_n}^2, \quad \nu_n = -a_n q = b_{q_n} c_{q_n} q^{1/2}, \quad (3.9)
\]

and \( q_n \in C_k, k \in \mathbb{C}. \) \( T_{\lambda}^{chp} \) is then a Baxter Q-operator w.r.t. the transfer matrix of the Bazhanov-Stroganov model in \( R_{chp}^N \):

\[
\tau_2(\lambda) T_{\lambda}^{chp} = \Theta_{BS}(\lambda) T_{\lambda}^{chp} + d_{BS}(\lambda) T_{\lambda}^{chp}, \quad (3.10)
\]

\[
[\tau_2(\lambda), T_{\lambda}^{chp}] = 0, \quad [\Theta, T_{\lambda}^{chp}] = 0, \quad [T_{\lambda}^{chp}, T_{\mu}^{chp}] = 0 \quad \forall \lambda, \mu \in \mathbb{C}, \quad (3.11)
\]

with \( \Theta_{BS} \) and \( d_{BS} \) defined in (5.8) and (5.9) of [117].

### 3.2 SOV-spectrum characterization

**Theorem 3.1. (Proposition 3, Theorem 5 and Lemma 13 of [117])** For almost all the representations in \( R_{chp}^N \) the spectrum of the chiral Potts transfer matrix \( T_{\lambda}^{chp} \) is simple. Moreover:

---

\footnote{For a direct comparison see formula (4.12) of [96] with the following identifications:

\[ z_j \equiv q^{2r_j}, \quad z'_{j} \equiv q^{2s_j} \quad \forall j \in \{1, ..., N\}. \]

Note that \( T_{\lambda}^{chp} \) is well defined since the \( W \)-functions are cyclic functions of their arguments.

\footnote{It is worth pointing out that while the Baxter equation holds in the general inhomogeneous representations the commutativity properties are proven only under the further restrictions \( q_n \equiv r_n \quad \forall n \in \{1, ..., N\} \) under which is characterized \( R_{chp}^N \).}

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I) All right and left eigenstates of the chiral Potts transfer matrix $T^{hp}_\lambda$ are eigenstates of $\tau_2(\lambda)$ and they admit the SOV construction presented in point I) of Theorem 2.2. The solution $Q_1(\lambda)$ of the functional Baxter equation (4.7.11) is gauge equivalent to the corresponding $T^{hp}_\lambda$-eigenvalue $Q^{hp}_\lambda$ being the coefficients $a_{BS}(\lambda)$ and $d_{BS}(\lambda)$ of (3.10) gauge equivalent to the SOV-ones:

$$a_{BS}(\lambda) = h_{BS}(\lambda)\lambda(\lambda) \quad d_{BS}(\lambda) = h_{BS}^{-1}(\lambda q)\lambda(\lambda).$$ (3.12)

Here $h_{BS}(\lambda)$ is a function whose average value is 1 for any $\lambda \in \mathbb{C}$.

II) In the sub-variety $\mathcal{R}_{N}^{hp,adj} = \mathcal{R}_{N}^{hp} \cap \mathcal{R}_{N}^{S,adj}$, characterized by (3.8)-(3.9) under the following constraints:

$$q_n = (\alpha_n, \epsilon_n, \alpha^{*}_n) \in \mathcal{C}_k, \quad \epsilon_0, n = \pm 1, \quad k^* = e^k, \quad (3.13)$$

the operator $T^{hp}_\lambda$ is normal and $\tau_2(\lambda)$ is self-adjoint. Then, point I) of Theorem 2.2 allows to construct the full simultaneous $(T^{hp}_\lambda, \tau_2(\lambda), \Theta)$-eigenbasis associating to any $t(\lambda) \in \Sigma_{T_2}$ the corresponding eigenstate.

4 Decomposition of the identity in the transfer matrix eigenbasis

4.1 Action of left separate states on right separate states

Here we compute the action of covectors on vectors which in the left and right SOV-basis have a separate form similar to that of the transfer matrix eigenstates. To be more precise, let us give the following definition of a left $\langle \alpha_k \rangle$ and a right $|\beta_k \rangle$ separate states characterized by the given arbitrary set of functions $\alpha_a$ and $\beta_a$:

$$\langle \alpha_k \rangle = \sum_{h_1, \ldots, h_N=1}^p \frac{q^{kh_N}}{p^{1/2}} \prod_{a=1}^{N-1} \alpha_a(\eta^{(h_a)}) \prod_{1 \leq a < b \leq N-1} \left( \frac{(\eta^{(h_a)})^2 - (\eta^{(h_b)})^2}{\prod_{b=1}^{N-1} \omega_a(\eta^{(h_b)})} \right).$$ (4.1)

$$|\beta_k \rangle = \sum_{h_1, \ldots, h_N=1}^p \frac{q^{-kh_N}}{p^{1/2}} \prod_{a=1}^{N-1} \beta_a(\eta^{(h_a)}) \prod_{1 \leq a < b \leq N-1} \left( \frac{(\eta^{(h_a)})^2 - (\eta^{(h_b)})^2}{\prod_{b=1}^{N-1} \omega_a(\eta^{(h_b)})} \right).$$ (4.2)

Proposition 4.1. The action of the left separate state $\langle \alpha_k \rangle$ of form (4.1) on the right separate state $|\beta_k \rangle$ of form (4.2) reads:

$$\langle \alpha_k | \beta_k \rangle = \delta_{k,h} \det_{N-1} |M^{(\alpha,\beta)}_{a,b}| \quad \text{with} \quad M^{(\alpha,\beta)}_{a,b} \equiv \left( \eta^{(0)}_{\alpha} \right)^{2(b-1)} \sum_{h=1}^p \frac{\alpha_a(\eta^{(h)}_{\alpha}) \beta_a(\eta^{(h)}_{\beta}) \omega_a(\eta^{(h)}_{\alpha})}{\omega_a(\eta^{(h)}_{\beta})} q^{2(b-1)h}. (4.3)$$

Proof. The SOV-decomposition of these states implies:

$$\langle \alpha_k | \beta_k \rangle = \sum_{h_N=1}^p \frac{q^{(k-h)h_N}}{p} \sum_{h_1, \ldots, h_{N-1}=1}^p \frac{V((\eta^{(h_1)}_{\alpha})^2, \ldots, (\eta^{(h_{N-1})}_{\alpha})^2)}{\omega_a(\eta^{(h_N)}_{\alpha})} \prod_{a=1}^{N-1} \frac{\alpha_a(\eta^{(h_a)}_{\alpha}) \beta_a(\eta^{(h_a)}_{\beta}) \omega_a(\eta^{(h_a)}_{\alpha})}{\omega_a(\eta^{(h_a)}_{\beta})},$$ (4.4)

where $V(x_1, \ldots, x_N) \equiv \prod_{1 \leq a < b \leq N-1} (x_a - x_b)$ is the Vandermonde determinant. Then from the identity:

$$\delta_{k,h} = \sum_{h_N=1}^p \frac{q^{(k-h)h_N}}{p} \quad \text{when} \quad q \text{ is a } p\text{-root of unit and } h, k \in \mathbb{Z}_p$$ (4.5)
and by using the multilinearity of the determinant w.r.t. the rows we prove the proposition.

It is worth remarking that the previous determinant formulae define also scalar products for vectors in $\mathcal{R}_N$ which have a separate form in the right $B$-eigenbasis and in the dual of the left $B$-eigenbasis. Indeed, $(\langle \alpha_k \rangle)^\dagger \in \mathcal{R}_N$ is a separate vector in the basis of $\mathcal{R}_N$ formed out of the $(\langle \eta_k \rangle)^\dagger$ dual states of the left $B$-eigenbasis. Then these results represent the SOV analogue of the scalar product formulæ \[18, 126\] computed for Bethe states in the framework of the algebraic Bethe ansatz. Note that this formula is not restricted to the case in which one of the two states is an eigenstate of the transfer matrix. It is also interesting to remark that the previous scalar product formulæ allow to prove directly, as in the case of the sine-Gordon model, that the action of a transfer matrix eigencovector on an eigenvector corresponding to different eigenvalue is zero.

**Corollary 4.1.** Let $t_h(\lambda)$ and $t'_h(\lambda) \in \Sigma^h_2$ and $\langle t_h \rangle$ and $|t'_h\rangle$ the $\tau_2$-eigenstates defined in Section 2.4.2 then for $t_h(\lambda) \neq t'_h(\lambda)$ the $N \times N$ matrix $M_{a,b}^{(t_h,t'_h)}$ has rank equal or smaller than $N - 1$. Indeed, the non-zero $N \times 1$ vector $V^{(t_h,t'_h)}$ defined by:

$$V^{(t_h,t'_h)}_b = c_b - c_b \quad \forall b \in \{1, \ldots, N\},$$

where:

$$t_h(\lambda) = \sum_{c=\pm 1} \left( q^{c\cdot \alpha_c} + q^{-\cdot \alpha_c} \right) \lambda^{b} + \sum_{b=1}^{N-1} c_b \lambda^{-N-2+2b},$$

$$t'_h(\lambda) = \sum_{c=\pm 1} \left( q^{c\cdot \alpha_c} + q^{-\cdot \alpha_c} \right) \lambda^{b} + \sum_{b=1}^{N-1} c_b \lambda^{-N-2+2b},$$

is an eigenvector of $\|M_{a,b}^{(t_h,t'_h)}\|$ corresponding to the eigenvalue zero.

**Proof:** Note that under the choice (2.68) for the left gauge and (2.80) for the right gauge, it holds:

$$\omega_a(\eta^{(h)}_a) = (\eta^{(h)}_a)^N,$$

and then by the definitions (4.6), (4.7) and (4.8) it holds:

$$\sum_{b=1}^{N} M_{a,b}^{(t_h,t'_h)} V^{(t_h,t'_h)}_b = \sum_{h=0}^{2N-1} Q_{t_h(\eta^{(h)}_a)}(t'_h(\eta^{(h)}_a))(t'_h(\eta^{(h)}_a)) - t_h(\eta^{(h)}_a)).$$

The desired result:

$$\sum_{b=1}^{N} \Phi_{a,b}^{(t_h,t'_h)} V^{(t_h,t'_h)}_b = 0 \quad \forall a \in \{1, \ldots, N\},$$

then follows as the Baxter equations (2.77) and (2.79) allow to write:

$$Q_{t'_h(\eta^{(k)}_a)}(t'_h(\eta^{(k)}_a))(t'_h(\eta^{(k)}_a)) = (\bar{D}(\eta^{(k+1)}_a)Q_{t'_h(\eta^{(k+1)}_a)}(t'_h(\eta^{(k+1)}_a)) + \bar{A}(\eta^{(k+1)}_a)Q_{t'_h(\eta^{(k+1)}_a)}(t'_h(\eta^{(k+1)}_a)))\bar{Q}_{t_h(\eta^{(k)}_a)},$$

which substituted in (4.10) implies (4.11).
4.2 Decomposition of the identity in transfer matrix eigenbasis

In the representations for which $\tau_2(\lambda)$ is diagonalizable then the simplicity of its spectrum plus the explicit characterizations of its left and right eigenstates allows to write the following decomposition of the identity:

$$\mathbb{I} = \sum_{k=0}^{p-1} \sum_{\tau(\lambda) \in \Sigma_{\tau_2}} \frac{|t_k\rangle\langle t_k|}{\langle t_k|t_k\rangle},$$

(4.13)

where

$$\langle t_k|t_k\rangle = \det_N \left| \mathcal{M}_{a,b}^{(t_k,t_k)} \right|$$

with

$$\mathcal{M}_{a,b}^{(t_k,t_k)} = (\eta_a^{(c)})^2 (b-1) \sum_{c=1}^p \frac{Q_{t_k}(\eta_a^{(c)})}{\omega_a(\eta_a^{(c)})} q^{2(b-1)c},$$

(4.14)

is the action of the covector $\langle t_k|$ on the vector $|t_k\rangle$, both defined in Section 5.2.2. Note that in the representations which define a normal $\tau_2(\lambda)$ the simplicity of the spectrum implies the following identity:

$$\langle |t_k\rangle \rangle^\dagger \equiv \alpha_{t_k} \langle t_k|$$

where $\alpha_{t_k} = \frac{||t_k||^2}{\langle t_k|t_k\rangle} \in \mathbb{C}$

(4.15)

for any eigenvector $|t_k\rangle$ of $\tau_2(\lambda)$. For these special representations, this stresses the interest in computing the norm $||t_k||$ as it allows to write left and right $\tau_2$-eigenstates as one the exact dual of the other.

5 Propagator for the Bazhanov-Stroganov model

In this section we construct the propagator operator along the chain of the Bazhanov-Stroganov model for the representations parametrized by points on the chP curves.

5.1 Fundamental R-matrix of the Bazhanov-Stroganov model

In the next proposition we report adapting to our notations a fundamental result of the paper [57].

Proposition 5.1 ([57]). Let $S_{(q_1,r_1|q_2,r_2)}$ be the operator defined on the tensor product of two p-dimensional spaces by:

$$\langle z_1, z_2 | S_{(q_1,r_1|q_2,r_2)} | z'_1, z'_2 \rangle \equiv \bar{W}_{q_2 q_1}(z_1/z'_1)W_{r_2 r_1}(z_2/z'_2)W_{q_2 r_1}(z_2/z_1),$$

(5.1)

Then, $S_{(q_1,r_1|q_2,r_2)}$ is the fundamental R-matrix intertwining the Bazhanov-Stroganov Lax operator in the quantum space, i.e. it holds:

$$L_{02}(\lambda|q_2, r_2)L_{01}(\lambda|q_1, r_1)S_{(q_1,r_1|q_2,r_2)} = S_{(q_1,r_1|q_2,r_2)}L_{01}(\lambda|q_1, r_1)L_{02}(\lambda|q_2, r_2).$$

(5.2)

Proof. Let us just point out that the proof can be obtained by proving it for any matrix element $(i_1, i_2) \in \{1, 2\} \times \{1, 2\}$. Indeed, taking the matrix elements on the quantum states $\langle z_1, z_2 |$ and $| z''_1, z''_2 \rangle$, the
proposition simply follows from the identities:

\[
\sum_{z_1', z_2' \in S_{p,j}} (L_0)^{j_1 j_2} (z|q_2, r_2) (L_0)^{j_1 j_2} (z'|q_1, r_1) \langle z_1', z_2'| S_{q_1, r_1 | q_2, r_2} | z_1'', z_2'' \rangle = \\
\sum_{z_1', z_2' \in S_{p,j}} \langle z_1, z_2| S_{q_1, r_1 | q_2, r_2} | z_1', z_2' \rangle (L_0)^{j_1 j_2} (z'|q_1, r_1) (L_0)^{j_1 j_2} (z|q_2, r_2),
\]

(5.3)

once the elements of \( L_{0i} \) are rewritten in terms of the points of \( C_k \) and we use the definition of the functions \( W \) and \( W' \).

\[ \square \]

5.2 Propagator for the Bazhanov-Stroganov model

The first transfer matrix of the chP-model has been defined in (5.6) while the second chP-transfer matrix reads:

\[
\hat{T}^{chP}_{\lambda_q, (p|\{q_n, r_n\})} (z, z') \equiv \langle z| \hat{T}^{chP}_{\lambda_q, (p|\{q_n, r_n\})} | z' \rangle = \prod_{n=1}^{N} W_{n p} (z_{n+1}/z_n) W_{q_n, p} (z_n/z'_{n}).
\]

(5.4)

Let us recall that the propagator operator \( U_n \) along the Bazhanov-Stroganov chain is defined by:

\[
U_n M_{1, ..., N} (\lambda) U_n^{-1} = M_{n, ..., N, 1, ..., n-1} (\lambda) \equiv L_{n-1} (\lambda) L_1 (\lambda) L_N (\lambda) \cdots L_n (\lambda),
\]

(5.5)

then we can prove:

**Proposition 5.2.** The propagator operator \( U_n \) has the following representation in terms of the chP-transfer matrices:

\[
U_n^{-1} \equiv T^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} \hat{T}^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} \cdots T^{chP}_{\lambda_{q_{m-1}, l_{m-1}}(\{q_{n+m-1}, r_{n+m-1}\})} \hat{T}^{chP}_{\lambda_{q_{m-1}, l_{m-1}}(\{q_{n+m-1}, r_{n+m-1}\})}.
\]

(5.6)

**Proof.** The previous proposition implies that the operator \( S_{q_1, r_1 | q_2, r_2} \) satisfies the following equation

\[
(S_{q_1, r_1 | q_2, r_2})^{-1} L_0 (\lambda|q_2, r_2) L_0 (\lambda|q_1, r_1) S_{q_1, r_1 | q_2, r_2} = L_0 (\lambda|q_1, r_1) L_0 (\lambda|q_2, r_2),
\]

(5.7)

then it is simple to verify that:

\[
(S_{q_1, r_1 | q_2, r_2} S_{q_1, r_1 | q_3, r_3} \cdots S_{q_1, r_1 | q_{n}, r_{n}})^{-1} L_0 (\lambda|q_n, r_n) \cdots L_0 (\lambda|q_2, r_2) L_0 (\lambda|q_1, r_1) \times (S_{q_1, r_1 | q_2, r_2} S_{q_1, r_1 | q_3, r_3} \cdots S_{q_1, r_1 | q_{n}, r_{n}}) = L_0 (\lambda|q_1, r_1) L_0 (\lambda|q_n, r_n) \cdots L_0 (\lambda|q_2, r_2).
\]

(5.8)

Let us compute the matrix elements:

\[
\langle z| T^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} \hat{T}^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} | z'' \rangle = \sum_{z'} \langle z| T^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} | z' \rangle \langle z'| \hat{T}^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} | z'' \rangle.
\]

(5.9)

Using the relations \( W_{pp}(z/z') = \delta_{z,z'} \) and \( W_{p}(z) W_{pq}(z) = 1 \), we get:

\[
\langle z| T^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} \hat{T}^{chP}_{\lambda_{q_1, l_1}(\{q_n, r_n\})} | z'' \rangle = \sum_{z_1, z_2} \delta_{z_1, z_2} \delta_{z_1, z''} \prod_{n \geq 2} \langle z'_1, z_2 | S_{q_1, r_1 | q_2, r_2} | z'_1, z_2 \rangle \langle z_1', z_2'| S_{q_1, r_1 | q_2, r_2} | z_1', z_2' \rangle.
\]

(5.10)

\[
= \langle z_1, \ldots, z_{N} | S_{q_1, r_1 | q_2, r_2} \cdots S_{q_1, r_1 | q_{n}, r_{n}} | z_1'', \ldots, z_{N} \rangle.
\]

(5.11)
Let us use the notation $\bar{S}_i = T^{\text{chP}}_{\lambda_{n_i},\{r_i,\{q_{i,n_i}\}})} \bar{T}^{\text{chP}}_{\lambda_{n_i},\{q_{i,n_i}\}}$, then (5.8) can be rewritten as it follows:

$$S^{-1}_i L_{0N}(\lambda|q_{N},r_N) \cdots L_{02}(\lambda|q_2, r_2) L_{01}(\lambda|q_1, r_1) S_1 = L_{01}(\lambda|q_1, r_1) L_{0N}(\lambda|q_N, r_N) \cdots L_{02}(\lambda|q_2, r_2)$$

and acting similarly with the others $\bar{S}_n$ with $n > 1$ it holds:

$$\bar{S}^{-1}_n L_{0n-1}(\lambda|q_{n-1}, r_{n-1}) \cdots L_{01}(\lambda|q_1, r_1) L_{0N}(\lambda|q_N, r_N) \cdots L_{0n+1}(\lambda|q_{n+1}, r_{n+1}) L_{0n}(\lambda|q_n, r_n) \bar{S}_n$$

$$= L_{0n}(\lambda|q_n, r_n) \cdots L_{01}(\lambda|q_1, r_1) L_{0N}(\lambda|q_N, r_N) \cdots L_{0n+2}(\lambda|q_{n+2}, r_{n+2}) L_{0n+1}(\lambda|q_{n+1}, r_{n+1})$$

from which defining:

$$U^{-1}_n = \bar{S}_1 \bar{S}_2 \cdots \bar{S}_{n-1}$$

$$= T^{\text{chP}}_{\lambda_{n_1},\{r_1,\{q_{1,n_1}\}})} \bar{T}^{\text{chP}}_{\lambda_{n_1},\{q_{1,n_1}\}} \cdots T^{\text{chP}}_{\lambda_{n_{n-1}},\{r_{n-1},\{q_{n-1,n_{n-1}}\}})} \bar{T}^{\text{chP}}_{\lambda_{n_{n-1}},\{q_{n-1,n_{n-1}}\}}$$

$U_n$ surely satisfies the equation (5.5) which defines the propagator. \[\]

It is worth noticing that the eigenvalues of the two chP-transfer matrices on the eigenstates of the $\tau_2$ transfer matrix are characterized according to the discussion made in Section 5.2, then the eigenvalues of $U_n$ are also known. Moreover, let us point out that:

$$\lambda_{q_n} = t \left( q \frac{\alpha_n \beta_n}{\alpha_n \beta_n} \right)^{1/2}$$

$$\lambda_{r_n} = t \left( q \frac{\alpha_n \beta_n}{\alpha_n \beta_n} \right)^{1/2}$$

i.e. we are computing the Q-operators, $T^{\text{chP}}_{\lambda_{q_n}} T^{\text{chP}}_{\lambda_{r_n}}$, in the zeros of the quantum determinant of the $\tau_2$ model. In the case of self-adjoint representations on trivial curves (like for sine-Gordon model) we have up to an overall constant:

$$U^{-1}_n = Q_{\lambda_{q_n}} Q_{\lambda_{q_1}} \cdots Q_{\lambda_{q_{m-1}}} Q_{\lambda_{q_{m-1}}}$$

(5.17)

The case of Bethe anzatz representations correspond to the case $q_n = r_n$, i.e. the two zeros of the quantum determinant coincide up to $p$-roots of units. In this case and in the homogeneous case we reproduce the known result of [142] for the propagator.

### 6 Representation of local operators by separate variables

The results on the scalar product formulae define one of the main steps to compute matrix elements of local operators. The other one is to reconstruct local operators by using the generators of the Yang-Baxter algebra, namely to invert the map from the local operators in the Lax matrices to the monodromy matrix elements. This inverse problem solution makes possible to compute the action of local operators on transfer matrix eigenstates in this way leading to the determination of form factors of local operators once the scalar product formulae are used.

In [18] the first solution of this inverse problem has been obtained for the XXZ spin 1/2 chain and then in [28] it has been generalized to all fundamental lattice models having isomorphic auxiliary and local quantum spaces characterized by a Lax operator matrix coinciding with the permutation operator for a
special value of the spectral parameter. This reconstruction can be also used for non-fundamental lattice models, as derived in [28] for the higher spin XXX chains by using the fusion procedure [77]. For the Bazhanov-Stroganov model we still don’t know how to achieve this type of reconstruction and the known results reduce to those given by T. Oota [143]. However, Oota’s results lead only to reconstruct some local operators of the Bazhanov-Stroganov model. We will explain in this section how to complete the Oota’s reconstruction for all the local operators of the Bazhanov-Stroganov model associated to the most general cyclic representations of the 6-vertex Yang-Baxter algebra. The procedure developed here is the natural generalization to these representations of the one for the special subclass presented in our previous paper [11]. The new technical tools required to handle these general representations will be also introduced in the next subsections.

6.1 Reconstruction of a class of local operators

The results of Oota’s paper [143] are here reproduced for the more general cyclic representations associated to the the Bazhanov-Stroganov model; this leads to the reconstruction of a subclass of local operators. In terms of quantum projectors, when computed in the zeros $\mu_{n,\pm}$ of the quantum determinant, the Lax operator $L_n(\lambda)$ has the following factorization:

$$L_n(\mu_{n,+}) = \left( \begin{array}{c} (L_n)_{12}^{-1/2} f_n^{-1} \\ (L_n)_{21}^{-1/2} f_n^{-1} \end{array} \right) \left( \begin{array}{cc} u_n^{-1/2} & u_n^{1/2} f_n^{-1} \\ u_n^{1/2} f_n & u_n^{-1/2} f_n^{-1} \end{array} \right),$$

(6.1)

$$L_n(\mu_{n,-}) = \left( \begin{array}{c} g_n u_n^{1/2} \\ g_n^{-1} u_n^{-1/2} \end{array} \right) \left( \begin{array}{cc} (L_n)_{21}^{-1/2} & g_n^{-1} u_n^{1/2} (L_n)_{12}^{-1/2} \\ g_n u_n^{1/2} (L_n)_{21} & g_n^{-1} u_n^{-1/2} (L_n)_{12} \end{array} \right),$$

(6.2)

where $\langle L_n \rangle_{ij}$ stays for the matrix element $i, j$ of the Lax operator and:

$$f_n \equiv \left( -\frac{\alpha_n \beta_n}{\alpha_n \beta_n} \right)^{1/4}, \quad g_n \equiv \left( -\frac{\alpha_n \beta_n}{\alpha_n \beta_n} \right)^{1/4}.$$  

(6.3)

These factorizations properties were used by Oota’s to reconstruct local operators as it follows:

**Proposition 6.1.** The following reconstructions of local operators hold:

$$u_n^{-1} = \left( -\frac{a_n b_n}{\alpha_n \beta_n} \right)^{1/2} U_n B^{-1}(\mu_{n,+}) A(\mu_{n,+}) U_n^{-1} = \left( -\frac{a_n b_n}{\alpha_n \beta_n} \right)^{1/2} U_n D^{-1}(\mu_{n,+}) C(\mu_{n,+}) U_n^{-1},$$

(6.4)

$$\alpha_{0,n} = U_n A^{-1}(\mu_{n,-}) B(\mu_{n,-}) U_n^{-1} = U_n C^{-1}(\mu_{n,-}) D(\mu_{n,-}) U_n^{-1},$$

(6.5)

where we have defined:

$$\alpha_{0,n} \equiv \left( \frac{1 + q^{-1}(\alpha_n b_n) v_n^2}{1 + q^{-1}(\alpha_n b_n) v_n^2} \right)^{1/2} u_n.$$  

(6.6)

Oota’s formulae (6.4)-(6.5) clearly allow to reconstruct all the powers $u_n^{-k} = U_n (B^{-1}(\mu_{n,+}) A(\mu_{n,+}))^k U_n^{-1}$; however the local operators $v_n^k$ do not admit direct reconstructions as only rational functions like $(1 + q^{-1}(\alpha_n b_n) v_n^2) / (1 + q^{-1}(\alpha_n b_n) v_n^2)$ are reconstructed.
6.2 Reconstruction of all local operators

Here, we solve the inverse problem for the local operators $v_{k,n}^k$ in this way completing the reconstruction of local operators. The cyclicity of the representations of the Bazhanov-Stroganov model will be the main property here used. Let us define the following local operators:

$$\beta_{k,n} \equiv (U_nA^{-1}(\mu_{n,+})B(\mu_{n,+})U_n^{-1})^{-k-1} \alpha_{0,n} (U_nA^{-1}(\mu_{n,+})B(\mu_{n,+})U_n^{-1})^k$$  \hspace{1cm} (6.7)

then it holds:

**Proposition 6.2.** For the cyclic representations of the Bazhanov-Stroganov model we consider, the local operators $v_{2k,n}$ have the following reconstructions:

$$v_{2k,n} = \frac{1}{p} \left( -\frac{\text{d}n_0}{\text{d}n} \right)^k \frac{1 + (\varepsilon_n/\text{d}n)^p}{(\text{d}n \varepsilon_n / \text{a}_n \text{d}n)^{1/2} - (\text{a}_n \text{d}n / \text{b}_n \varepsilon_n)^{1/2}} \sum_{a=0}^{p-1} q^{k(2a+1)} \beta_{a,n}.$$ \hspace{1cm} (6.8)

**Proof.** By definition in our cyclic representations the powers $u_n^p$ and $v_n^p$ are central elements of the algebra coinciding with 1. Then it holds:

$$\frac{1 + (\varepsilon_n/\text{d}n)^p}{1 + q^{-2k-1}(\varepsilon_n/\text{d}n)v_n^2} = \sum_{i=0}^{p-1} (-q^{-2k-1}(\varepsilon_n/\text{d}n)v_n^2)^i.$$ \hspace{1cm} (6.9)

The previous formula and the reconstruction (6.4)-(6.5) allow to rewrite $\beta_{k,n}$ as the following finite sum in powers of $v_n^2$:

$$\beta_{k,n} = \frac{(\text{d}n_0 \varepsilon_n / \text{a}_n \text{d}n)^{1/2} + (\text{a}_n \text{d}n / \text{d}n_0 \varepsilon_n)^{1/2} (\varepsilon_n/\text{d}n)^p}{1 + (\varepsilon_n/\text{d}n)^p} + \frac{(\text{d}n_0 \varepsilon_n / \text{a}_n \text{d}n)^{1/2} - (\text{a}_n \text{d}n / \text{d}n_0 \varepsilon_n)^{1/2}}{1 + (\varepsilon_n/\text{d}n)^p} \sum_{a=1}^{p-1} (-1)^a q^{-a(2k+1)} \left( \frac{\varepsilon_n}{\text{d}n} \right)^a v_n^{2a},$$ \hspace{1cm} (6.10)

then, taking a discrete Fourier transformation, the reconstructions (6.8) is obtained together with the following sum rules

$$\sum_{a=0}^{p-1} \beta_{a,n} = p \left( (\text{d}n_0 \varepsilon_n / \text{a}_n \text{d}n)^{1/2} + (\text{a}_n \text{d}n / \text{d}n_0 \varepsilon_n)^{1/2} (\varepsilon_n/\text{d}n)^p \right) / (1 + (\varepsilon_n/\text{d}n)^p).$$ \hspace{1cm} (6.11)

The formulae in (6.8) lead to the reconstruction of all the powers $v_n^k$ for $k \in \{1, ..., p - 1\}$ as it follows from the identities $v_n^k = v_n^{2h}$, for $k = 2h - p$ odd integer smaller than $p$. Hence, as desired, all the local operators of the cyclic representations of the Bazhanov-Stroganov model are reconstructed by using the above proposition and the Oota’s reconstructions.
6.3 Separate variables representations of all local operators

To compute the action of the local operators \( v^k_n \) and \( u^k_n \) on eigenstates of the transfer matrix and then their form factors we need before to determine their SOV-representations. These SOV-representations are obtained from the above solution of the inverse problem. To this aim we first prove two lemmas that are important to overcome the combinatorial problem associated to the computation of the SOV-representations of the local operators \((6.4)-(6.5)\).

Let us introduce the coordinate operators \( \hat{\eta}_i \) for \( i \in \{1, \ldots, N\} \), \( \hat{\eta}_A^\pm \) and \( \hat{\eta}_D^\pm \) such that:

\[
\langle \eta | \hat{\eta}_i \equiv \eta_i \langle \eta |, \quad \langle \eta | \hat{\eta}_A^\pm \equiv \eta_A^{\pm} \langle \eta |, \quad \langle \eta | \hat{\eta}_D^\pm \equiv \eta_D^{\pm} \langle \eta |.
\]

and the operator \( T_i^\pm \) are defined on the left and right SOV-representations by \[24\]:

\[
\langle \eta | T_i^\pm \equiv \langle q^{\pm \delta_i} \eta |, \quad T_i^\pm | \eta \rangle \equiv | q^{\mp \delta_i} \eta \rangle
\]

and clearly the commutation relations hold:

\[
T_i^\pm \hat{\eta}_j = q^{\pm \delta_i \delta_j} \hat{\eta}_j T_i^\pm.
\]

Lemma 6.1. We have the expansion

\[
(\hat{\Omega}(f))^k = \sum_{\alpha = \{\alpha_1, \ldots, \alpha_{N-1}\}} \sum_{\sum \alpha_i = k} \left[ \frac{k}{\alpha} \right] \prod_{i=1}^{N-1} \left( \frac{\alpha_i}{h} \right) \prod_{i=1}^{N-1} f(\hat{\eta}_i) \prod_{j \neq i} \frac{1}{q^{\alpha_j - h} \hat{\eta}_i / \hat{\eta}_j - q^{-\alpha_j + h} \hat{\eta}_j / \hat{\eta}_i} \prod_{i=1}^{N-1} (T_i^-)^{\alpha_i}
\]

for the operator

\[
\hat{\Omega}(f) = \sum_{a=1}^{N-1} \prod_{b \neq a} \frac{1}{\eta_a / \eta_b - \eta_b / \eta_a} f(\hat{\eta}_a) T_a^-,
\]

with

\[
\left[ \frac{k}{\alpha} \right] = \frac{[k]!}{\prod_{j=1}^{N-1} \alpha_j !}, \quad [k]! = [k][k-1] \cdots [1], \quad [\alpha] = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.
\]

Proof. The lemma holds for \( k = 1 \) and we prove it by induction for \( k > 1 \). Let us take \( N-1 \) integers \( \alpha_i \):

\[
\sum_{i=1}^{N-1} \alpha_i = k,
\]

from which we define the set of integers \( I = \{i \in \{1, \ldots, N-1\} : \alpha_i \neq 0\} \) and \( \hat{C}_{\alpha}^{(k)} \) as the operator coefficient of \( \prod T_i^{-\alpha_i} \) (put to the left) in the expansion of the \( k \)-th power of \( \hat{\Omega}(f) \). By writing \((\hat{\Omega}(f))^k = \cdots \) 

\[24\] It is worth remarking that from the definition of the SOV-representations of the generators of the Yang-Baxter algebra, given in Section 2.3 and the definitions in \((6.13)\), it follows that the SOV-representation of the charge \( \Theta \) coincides with the operator \( T_N \).
$(\hat{\Omega}(f))^{k-1}\hat{\Omega}(f)$ and by using the induction hypothesis for the power $k-1$ of $\hat{\Omega}(f)$, we have:

\[
\hat{C}^{(k)}_{\alpha} = \sum_{a \in I} \left[ \frac{\alpha - \delta_a}{\alpha} \right] \prod_{j=1}^{k-1} \prod_{h=0}^{N-1} \left( f(q^{-h} \hat{\eta}_j) \times \prod_{i \neq j, i=1}^{N-1} \frac{1}{q_{\alpha_j-\delta_a,h} \hat{\eta}_j / \hat{\eta}_i - q_{\alpha_j-\delta_a,h} \hat{\eta}_j / \hat{\eta}_i} \right) \times f(\hat{\eta}_a q^{-\alpha_a+1}) \prod_{i \in I(a)} \frac{q^{\alpha_a} \hat{\eta}_i / \hat{\eta}_a - \hat{\eta}_i / q^{\alpha_a} \hat{\eta}_i}{q^{\alpha_a-\alpha_i} \hat{\eta}_a - \hat{\eta}_a / q^{\alpha_a-\alpha_i} \hat{\eta}_i},
\]

with $\delta_a \equiv (\delta_{1,a}, \ldots, \delta_{N,a})$. The first term in r.h.s. is the coefficient of $\prod_i T_i^{-\alpha_i+\delta_{i,a}}$ in $(\hat{\Omega}(f))^{k-1}$ and the second is the coefficient of $T_a^{-1}$ in $\hat{\Omega}(f)$ once the commutations between $\prod_i T_i^{-\alpha_i+\delta_{i,a}}$ and the $\hat{\eta}_i$ have been performed. Hence we get:

\[
\hat{C}^{(k)}_{\alpha} = \frac{\left[ \alpha - \delta_a \right]}{\left[ \alpha \right]} \prod_{j=1}^{k-1} \prod_{h=0}^{N-1} \left( f(q^{-h} \hat{\eta}_j) \times \prod_{i \neq j, i=1}^{N-1} \frac{1}{q_{\alpha_j-\delta_a,h} \hat{\eta}_j / \hat{\eta}_i - q_{\alpha_j-\delta_a,h} \hat{\eta}_j / \hat{\eta}_i} \right) f(\hat{\eta}_a q^{-\alpha_a+1}) \prod_{i \in I(a)} \frac{q^{\alpha_a} \hat{\eta}_i / \hat{\eta}_a - \hat{\eta}_i / q^{\alpha_a} \hat{\eta}_i}{q^{\alpha_a-\alpha_i} \hat{\eta}_a - \hat{\eta}_a / q^{\alpha_a-\alpha_i} \hat{\eta}_i},
\]

which leads to our result by using the relation:

\[
\sum_{a=1}^{n} \left[ \alpha_a \right] \prod_{i \neq a} \frac{q^{\alpha_a} \hat{\eta}_i / \hat{\eta}_a - \hat{\eta}_i / q^{\alpha_a} \hat{\eta}_i}{q^{\alpha_a-\alpha_i} \hat{\eta}_a - \hat{\eta}_a / q^{\alpha_a-\alpha_i} \hat{\eta}_i} = \left[ \sum_{a=1}^{n} \alpha_a \right].
\]

Note that the above formula holds for any $n$, for any set of numbers $\eta_i$ and for any non-negative integers $\alpha_i$. This is proven by studying the analytical properties of the function

\[
g(z) = \frac{1}{z} \prod_{i=1}^{\infty} \frac{z - \eta_i^2}{z - q^{-\alpha_i} \eta_i^2}.
\]

\[
\text{Lemma 6.2.} \ \text{The SOV-representation of the powers of } B^{-1}(\lambda)A(\lambda) \text{ are given by}
\]

\[
(B^{-1}(\lambda)A(\lambda))^m = \sum_{i+j+k=m} (-1)^j \left( \frac{\lambda}{\eta} \right)^{i-j} a_i^j q^{i(i-1)-j(j-1)} \left[ \begin{array}{c} m \end{array} \right] \hat{\sigma}(\lambda)^{k-j} T_{N}^{-i-1}
\]

\[
\hat{\sigma}(\lambda) = \sum_{a=1}^{N-1} \frac{1}{\eta_a / \eta_b - \eta_b / \eta_a \lambda / \eta_a / \eta_a / \lambda} \hat{a}^{(SOV)}(\eta_a) T_a^{-1},
\]

where the powers of $\hat{\sigma}(\lambda)$ are given by the previous lemma.

\[
\text{Proof.} \ \text{Let } \hat{a}, \hat{b} \text{ and } \hat{c} \text{ be three operators satisfying the relations}
\]

\[
\hat{b}\hat{a} = q^{-2}\hat{a}\hat{b}, \hat{c}\hat{b} = q^{2}\hat{b}\hat{c}, \hat{c}\hat{a} = q^{-2}\hat{a}\hat{c}
\]
It is easy to prove by induction that
\[
\left( \hat{a} + \hat{b} + \hat{c} \right)^m = \sum_{i+j+k=m} q^{k(i-j)-ij} \left[ \begin{array}{c} m \\ i, j, k \end{array} \right] \hat{a}^i \hat{b}^j \hat{c}^k
\]  
(6.26)

The SOV-representation of $B^{-1}(\lambda)A(\lambda)$ is the sum of three main terms,
\[
\hat{a} = \prod_{i=1}^{N-1} \eta_i \lambda a_+ T_N
\]  
(6.27)
\[
\hat{b} = -\prod_{i=1}^{N-1} \eta_i^{-1} \lambda^{-1} a_- T_N^+
\]  
(6.28)
\[
\hat{c} = \frac{1}{\eta_N} \sum_{a=1}^{N-1} \prod_{b \neq a} 1 \frac{1}{\eta_a/\eta_b - \eta_b/\eta_a} \frac{\alpha^{(SOV)}(\eta_a)}{\lambda - \eta_a/\lambda} T_a
\]  
(6.29)

Since they satisfy the commutation relations (6.25), the power of $B^{-1}(\lambda)A(\lambda)$ can be computed using the formula (6.26), which ends the proof.

\[\square\]

**Remark 1.** The quantum multinomials have the property
\[
\begin{pmatrix} p \\ \alpha \end{pmatrix} = \begin{cases} 1 & \text{if } \exists i \in \{1, \ldots, N-1\} : \alpha_i = p\delta_{a,i} \forall a \in \{1, \ldots, N-1\}, \\ 0 & \text{otherwise}, \end{cases}
\]  
(6.30)

This property yields that the power $p$ of $B^{-1}(\lambda)A(\lambda)$ is a central element of the Yang-Baxter algebra and it reads:
\[
(B^{-1}(\lambda)A(\lambda))^p = B(\lambda)^{-1}A(\lambda),
\]  
(6.31)
result which is consistent with the commutations relations:
\[
B^{-1}(q\lambda)A(q\lambda) = A(\lambda)B^{-1}(\lambda).
\]  
(6.32)

The two previous lemmas allow to expand the SOV-representation of the operators $u_k^n$. However, they do not apply directly to the expansion of $v_n$. The aim of the following lemma is to transform the operators $\beta_k,n$, whose linear combination gives the powers of $v_n$.

**Lemma 6.3.** The operator $\beta_k,n$ has the following expansion:
\[
\beta_{k,n} = \frac{B(\mu_{n,-}^p)}{A(\mu_{n,+})B(\mu_{n,+}^p) q^{k \mu_{n,+}/\mu_{n,-} - \mu_{n,-}/\mu_{n,+}}} \times B^{-1}(\mu_{n,+})A(\mu_{n,+}) \prod_{i=1}^{p-k} B(q^{-i} \mu_{n,+}) (B^{-1}(\mu_{n,-})A(\mu_{n,-}))^{p-1} \prod_{i=p-k+1}^p B(q^{-i} \mu_{n,+}) + \frac{q^k - q^{-k}}{q^k \mu_{n,+}/\mu_{n,-} - q^{-k} \mu_{n,-}/\mu_{n,+}}
\]  
(6.33)
The form factors of some local operators written as single determinants are here provided.

7.1 Form factors of our paper [1] defines one peculiar and evident instance of this universality.

In this section we present the main results of our paper on the form factors of the local operators. One of the main peculiarities emerging in quantum separate variables is a feature of universality in the representation of these dynamical observables. In fact, the comparison between the results presented here for the most general cyclic representations of the 6-vertex Yang-Baxter algebra and those previously derived in our paper [1] defines one peculiar and evident instance of this universality.

7 Form factors of local operators

Proof: A simple induction on the Yang-Baxter relation \(B(\lambda)A(q^{-1}\lambda) = A(\lambda)B(q^{-1}\lambda)\) shows that

\[
(A^{-1}(\lambda)B(\lambda))^k = \prod_{i=1}^{k} B(q^{-i}\lambda) \prod_{i=1}^{k} A^{-1}(q^{i-1}\lambda) = \prod_{i=0}^{k-1} A^{-1}(q^{i}\lambda) \prod_{i=0}^{k-1} B(q^{i}\lambda) . \tag{6.34}
\]

From the definition of the average values of operators, we get

\[
(A^{-1}(\lambda)B(\lambda))^p = A^{-1}(\lambda)^p B(\lambda), \tag{6.35}
\]

\[
(A^{-1}(\lambda)B(\lambda))^{-k} = B(\lambda)^{-1} \prod_{i=1}^{p-k} B(q^{-i}\lambda) \prod_{i=p-k+1}^{p} A(q^{-i}\lambda) . \tag{6.36}
\]

It also yields

\[
(A^{-1}(\lambda)B(\lambda))^p = A^{-1}(\lambda)B(\lambda) \left( B^{-1}(\lambda)A(\lambda) \right)^{p-1} . \tag{6.37}
\]

Standard arguments give the relation

\[
B(\mu_{n,-}) \prod_{i=1}^{p-k} A(q^{-i}\mu_{n,+}) = \frac{q^{k} - q^{-k}}{q^{k}\mu_{n,+} - q^{-k}\mu_{n,-}} A(\mu_{n,-}) \prod_{i=1}^{p-k} A(q^{-i}\mu_{n,+}) B(q^{k}\mu_{n,+})
+ \frac{\mu_{n,+} - \mu_{n,-}}{q^{k}\mu_{n,+} - q^{-k}\mu_{n,-}} \prod_{i=1}^{p-k} A(q^{-i}\mu_{n,+}) B(\mu_{n,-}) . \tag{6.39}
\]

Eventually, the use of these relations proves the lemma. □

7.1 Form factors of \(u_{n}^{-1}\) and \(\alpha_{n}^{-1}\)

The form factors of some local operators written as single determinants are here provided.

Proposition 7.1. Let us denote with \(\varphi_{n}^{(t_{k})}\) and \(\varphi_{n}^{(t'_{k'})}\) the eigenvalues of the shift operator \(U_{n}\) respectively on the left \(\langle t_{k}\rangle\) and right \(\langle t'_{k'}\rangle\) eigenstates of the transfer matrix \(\tau_{2}(\lambda)\), then the following determinant formula is verified:

\[
\langle t_{k}|u_{n}^{-1}|t'_{k'}\rangle = \left( -\frac{a_{n}b_{n}}{\alpha_{n}\beta_{n}} \right)^{1/2} \frac{\varphi_{n}^{(t_{k})}}{\varphi_{n}^{(t'_{k'})}} \delta_{k,k'-1} \det (|\varphi_{a,b}^{(t_{k},t'_{k'})}(\mu_{n,+})|). \tag{7.1}
\]
Here, $|U^{(t_k,t_k')}_{a,b} (\lambda)|$ is the $(N-1) \times (N-1)$ matrix defined by:

$$U^{(t_k,t_k')}_{a,b} (\lambda) \equiv M^{(t_k,t_k')}_{a,b+1/2} \text{ for } b \in \{1,\ldots,N-2\},$$

$$U^{(t_k,t_k')}_{a,N-1} (\lambda) = \frac{1}{\eta_{N}^{(0)}} \sum_{h=1}^{p} \frac{\eta_{h}^{(h)}}{\omega_{a}(\eta_{h}^{(h)})} Q_{t_k}(\eta_{h}^{(h)}) \left[ \frac{\tilde{Q}_{t_k}(\eta_{a}^{(h+1)})}{(\lambda/\eta_{a}^{(h+1)} - \eta_{a}^{(h+1)}/\lambda)} \hat{a}(SOV)(\eta_{a}^{(h)}) + \tilde{Q}_{t_k}(\eta_{a}^{(h)}) \right].$$

(7.2)  (7.3)

Proof. The operator $B^{-1}(\lambda)A(\lambda)$ admits the following SOV-representation:

$$B^{-1}(\lambda)A(\lambda) = \frac{1}{\eta_{N}} \left( \lambda \eta_{N}^{+} T_{N} + \eta_{N}^{-} \eta_{N}^{+} T_{N} \right) + \sum_{a=1}^{N-1} T_{a}^{-} \eta_{N}(\lambda/\eta_{a} q - \eta_{a} q/\lambda) \prod_{a \neq b} \frac{1}{(\eta_{a}/\eta_{b} - \eta_{b}/\eta_{a})}.$$  (7.4)

For brevity we denote with $|B^{-1}(\lambda)A(\lambda)|$ the sum on the r.h.s. of (7.4). Then, from the SOV-decomposition of the $\tau_{2}$-eigenstates, it holds:

$$\langle t_k||B^{-1}(\lambda)A(\lambda)||t_k' \rangle = \sum_{h=1}^{p} \frac{q^{(k+1-k)h_{N}}}{\eta_{N}^{(0)}} \sum_{a=1}^{N-1} \sum_{h_{1},\ldots,h_{N-1}=1}^{p} V\left( \left( \eta_{1}^{(h_{1})} \right)^{2}, \ldots, \left( \eta_{N-1}^{(h_{N-1})} \right)^{2} \right)$$

$$\times \prod_{a \neq b=1}^{N-1} \frac{\eta_{b}^{(h_{a})} Q_{t_k}(\eta_{h_{a}}) \tilde{Q}_{t_k}(\eta_{h_{a}})}{\omega_{a}(\eta_{h_{a}})} \left( \frac{\tilde{Q}_{t_k}(\eta_{a}^{(h_{a})}) Q_{t_k}(\eta_{a}^{(h_{a})})}{\omega_{a}(\eta_{a}^{(h_{a})})} \right) \left( \frac{(\lambda/\eta_{b} q^{h_{a}+1} - \eta_{b} q^{h_{a}+1}/\lambda)}{\hat{a}(SOV)(\eta_{a}^{(h_{a})})} \right).$$

(7.5)

and so:

$$\langle t_k||B^{-1}(\lambda)A(\lambda)||t_k' \rangle = \frac{\delta_{k,k'-1}}{\eta_{N}^{(0)}} \sum_{a=1}^{N-1} \sum_{h_{1},\ldots,h_{N-1}=1}^{p} \hat{V}_{a} \left( \left( \eta_{1}^{(h_{1})} \right)^{2}, \ldots, \left( \eta_{N-1}^{(h_{N-1})} \right)^{2} \right)$$

$\text{ (The row } a \text{ is removed.})$

$$\times \prod_{b \neq a=1}^{N-1} \frac{\eta_{b}^{(h_{a})} Q_{t_k}(\eta_{h_{a})}) \tilde{Q}_{t_k}(\eta_{h_{a})})}{\omega_{b}(\eta_{h_{a})})} \times (-1)^{(N-1+a)} \sum_{h_{a}=1}^{p} \frac{\tilde{Q}_{t_k}(\eta_{a}^{(h_{a})}) q^{h_{a}+1}) Q_{t_k}(\eta_{a}^{(h_{a})}) \left( \eta_{a}^{(h_{a})} \right)^{(N-2)} \hat{a}(SOV)(\eta_{a}^{(h_{a})})}{\omega_{a}(\eta_{a}^{(h_{a})})(\lambda/\eta_{a}^{(h_{a})} q^{h_{a}+1}) \eta_{a}^{(h_{a})}/\lambda).$$

(7.6)

inserting the sum over $(h_{1},\ldots,h_{a},\ldots,h_{N-1})$ in the Vandermonde determinant $\hat{V}_{a}$, the above expression reduces to the expansion of the following determinant:

$$\langle t_k||B^{-1}(\lambda)A(\lambda)||t_k' \rangle = \delta_{k,k'-1} \frac{\det}{N-1} \left( |U^{(t_k,t_k')}_{a,b} (\lambda)| \right).$$

(7.7)
where \( \mathcal{M}_{\alpha_{b+1/2}}^{(t_k,t_{k'})} \) is just \( \mathcal{M}_{\alpha_{b+1/2}}^{(t_k,t_{k'})} \) for \( b \in \{1, ..., N-2\} \), while:

\[
\mathcal{M}_{\alpha_{b+1/2}}^{(t_k,t_{k'})}(\lambda) = \frac{1}{\eta_0^{(0)}} \sum_{h=1}^{p} q^{(N-2)h} Q_{t_{k'}}(\eta_{h}^{(h)}) Q_{t_k}(\eta_{a}^{(h+1)}) \mathcal{A}^{(SOV)}(\eta_{a}^{(h)}). \tag{7.8}
\]

We compute now the matrix elements:

\[
\langle t_k | \hat{n}_N^{-1} \hat{\eta}_{A}^{(\pm)} T_{N}^{*} | t_{k'} \rangle = \frac{\pm a \pm q^{\pm k'}}{p \eta_{0}^{(0)}} \sum_{h=1}^{p} q^{(k+1-k')h} \sum_{h_1,...,h_{N-1}=1}^{p} V\left(\left(\eta_{1}^{(h_{1})}\right)^{2}, ..., \left(\eta_{N-1}^{(h_{N-1})}\right)^{2}\right) \times \prod_{b=1}^{N-1} \left(\eta_{b}^{(h_{b})}\right)^{\pm 1} Q_{t_{k'}}(\eta_{b}^{(h_{b})}) Q_{t_k}(\eta_{b}^{(h_{b})}) \times \frac{\omega_{b}^{(h_{b})}}{\eta_{b}^{(h_{b})}}, \tag{7.9}
\]

hence leading to:

\[
\langle t_k | \hat{n}_N^{-1} \hat{\eta}_{A}^{(\pm)} T_{N}^{*} | t_{k'} \rangle = \frac{\pm a \pm q^{\pm k'}}{p \eta_{0}^{(0)}} \sum_{h=1}^{p} q^{(k+1-k')h} \delta_{k,k'}-1 \det_{N-1}^{(\mathcal{M}_{\alpha_{b+1/2}}^{(t_k,t_{k'})})} \tag{7.10}
\]

Then our result follows as the matrices of formula (7.7) and (7.10) have \( N-2 \) common columns. Let us note that the above formula holds for any value of \( \lambda \).

\[ \square \]

**Remark 2.**

1) The matrix elements \( \langle t_k | \alpha_{\alpha_{n}}^{-1} | t_{k'} \rangle \) of the local operators \( \alpha_{\alpha_{n}}^{-1} \) are given by:

\[
\langle t_k | \alpha_{\alpha_{n}}^{-1} | t_{k'} \rangle = \frac{\hat{\phi}_{t_k}^{(t_k)}}{\hat{\phi}_{t_{k'}}^{(t_{k'})}} \delta_{k,k'}-1 \det_{N-1}^{(\mathcal{M}_{\alpha_{b+1/2}}^{(t_k,t_{k'})})} \tag{7.11}
\]

2) In the case of general representations \( R_{N} \) the matrix elements \( \langle t_k | u_{k}^{-1} | t_{k'} \rangle \) can be computed by using the reconstruction:

\[
u_{n} = \left(-\frac{\alpha_{n}\beta_{n}}{\alpha_{n}\beta_{n}}\right)^{1/2} U_{n}^{-1}C^{-1}(\mu_{n,+})D(\mu_{n,+})U_{n}^{-1}, \tag{7.12}
\]

in the SOV C-representation. Here, we do not make this explicitly as the result will have the same type of form presented for \( \langle t_k | u_{k}^{-1} | t_{k'} \rangle \); the difference will be that all the quantities will be written in the SOV C-representation.

### 7.2 Determinant representations of form factors for a suitable basis of operators

In this section we construct an operator basis for which the form factors of any operator in this basis is written by a one determinant formula. For this reason we will refer to it as the basis of elementary operators. The idea of the construction goes back to the sine-Gordon case \[1\].

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7.2.1 Introduction of the basis of elementary operators

**Lemma 7.1.** Let us define the operators:

\[ \mathcal{O}_{a,k} = \frac{\mathcal{B}(\eta_a^{(p+k-1)})\mathcal{B}(\eta_a^{(p+k-2)}) \cdots \mathcal{B}(\eta_a^{(k)})\mathcal{A}^{(k)}}{\prod_{b \neq a, b=1}^{N-1}(Z_a/Z_b - Z_b/Z_a)} \quad \text{with } k \in \{0, \ldots, p-1\}, \]  

(7.13)

then they satisfy the following properties:

\[ \mathcal{O}_{a,k}\mathcal{O}_{a,h} \text{ is non-zero if and only if } h = k - 1, \]  

(7.14)

and

\[ \mathcal{O}_{a,k}\mathcal{O}_{a,k-1} \cdots \mathcal{O}_{a,k+1-p}\mathcal{O}_{a,k-p} = \frac{\mathcal{A}(Z_a)}{\prod_{b \neq a, b=1}^{N-1}(Z_a/Z_b - Z_b/Z_a)}\mathcal{O}_{a,k}. \]  

(7.15)

The following commutation relations are furthermore satisfied:

\[ \hat{\eta}_A^{(\pm)}\mathcal{O}_{a,k} = q^{\mp 1}\mathcal{O}_{a,k}\hat{\eta}_A^{(\pm)}, \quad [\hat{\eta}_N, \mathcal{O}_{a,k}] = [T_N, \mathcal{O}_{a,k}] = 0, \]  

(7.16)

and

\[ \mathcal{O}_{a,k}\mathcal{O}_{b,h} = \frac{(\eta_a^{(k-h+1)}/\eta_b^{(0)}) - (\eta_a^{(0)})/\eta_b^{(k-h+1)}}{(\eta_a^{(k-h-1)}/\eta_b^{(0)}) - (\eta_a^{(0)})/\eta_b^{(k-h-1)}}\mathcal{O}_{b,h}\mathcal{O}_{a,k}. \]  

(7.17)

for \( a \neq b \in \{1, \ldots, N-1\} \).

**Proof.** Since \( \mathcal{B}(Z_a) = 0 \) with \( \mathcal{B}(\lambda) \) the average value of the operator \( \mathcal{B}(\lambda) \), the first is quite immediate. Moreover, the following identity:

\[ \langle \eta_1, \ldots, \eta_a^{(h)}, \ldots, \eta_N | \mathcal{O}_{a,k} = \frac{a(\eta_a^{(k)})/\delta_{h,k}}{\prod_{b \neq a, b=1}^{N-1}(\eta_a^{(k)}/\eta_b - \eta_b/\eta_a^{(k)})} \langle \eta_1, \ldots, \eta_a^{(k-1)}, \ldots, \eta_N |, \]  

(7.18)

is a direct consequence of the definition of the operators \( \mathcal{O}_{a,k} \) so that the second identity of the lemma follows. Now by using the following Yang-Baxter commutation relation:

\[ (\lambda/\mu - \mu/\lambda)\mathcal{A}(\lambda)\mathcal{B}(\mu) = (\lambda/q\mu - \mu/q\lambda)\mathcal{B}(\mu)\mathcal{A}(\lambda) + (q - q^{-1})\mathcal{B}(\lambda)\mathcal{A}(\mu) \]  

(7.19)

and moving the \( \mathcal{A}(\eta_a^{(k)}) \) to the right through all the \( \mathcal{B}(\eta_b^{(j)}) \), for \( j \neq h \), remarking that only the first term of the r.h.s of (7.19) survives, and after moving the \( \mathcal{A}(\eta_a^{(h)}) \) to the left, we get the last identity of the lemma.

Now we define **elementary operators** by the following monomials:

\[ \mathcal{E}_k^{(\alpha_1, \ldots, \alpha_r)}_{\eta_1, \ldots, \eta_N} = \hat{\eta}_N^{(-)\eta_A^{(+)T_N^{(+)}}} \eta_1^{(\alpha_1)0} \cdots \eta_N^{(\alpha_r)0} \mathcal{O}_{a_1,k_1} \cdots \mathcal{O}_{a_r,k_r}, \]  

(7.20)

where \( \sum_{h=1}^r \alpha_h \leq p, k; k_i \in \{0, \ldots, p-1\} \), \( a_i < a_j \in \{1, \ldots, N-1\} \) for \( i < j \in \{1, \ldots, N-1\} \) and:

\[ \mathcal{O}_{a,k}^{(\alpha)} = \mathcal{O}_{a,k}\mathcal{O}_{a,k-1} \cdots \mathcal{O}_{a,k+1-\alpha}, \text{ with } \alpha \in \{1, \ldots, p\}. \]  

(7.21)
Lemma 7.2. Once the set of the elementary operators is dressed by the shift operator $U_n$ as it follows:
\[ U_n e^{\alpha_1,\ldots,\alpha_r}_{k,k_0,(a_1,k_1),\ldots,(a_r,k_r)} U_n^{-1}, \]  
(7.22)
a basis is defined in the space of the local operators at the quantum site $n$, $\forall n \in \{1,\ldots,N\}$.

Proof. In order to prove the lemma the local operators in site $n$ generated by $u_n^k$ and $v_n^k$ for $k \in \{1,\ldots,p-1\}$ have to be written as linear combinations of the dressed elementary operators (7.20) and thanks to Proposition 6.2 this is equivalent to prove the same statement for the following basis of local operators:
\[ u_n^{-k} = U_n \left( B^{-1}(\mu_{n,+})A(\mu_{n,+}) \right)^k U_n^{-1}, \]  
(7.23)
\[ \tilde{\beta}_{k,n} = U_n \left( B^{-1}(\mu_{n,+})A(\mu_{n,+}) \right)^k B^{-1}(\mu_{n,-})A(\mu_{n,-}) \left( B^{-1}(\mu_{n,+})A(\mu_{n,+}) \right)^{p-1-k} U_n^{-1}. \]  
(7.24)
The operator $B^{-1}(\lambda)$ is invertible for $\lambda^p \neq Z_a$ with $a \in \{1,\ldots,N-1\}$ so that the centrality of the average values implies:
\[ B^{-1}(\lambda)A(\lambda) = \frac{B(\lambda q^{-1})B(\lambda q^{-2})\cdots B(\lambda q)A(\lambda)}{B(\lambda)}. \]  
(7.25)
The monomial $B(\lambda q^{-1})B(\lambda q^{-2})\cdots B(\lambda q)A(\lambda)$ is an even Laurent polynomial of degree $p(N-1)+1$ in $\lambda$ and so we can write:
\[ B^{-1}(\lambda)A(\lambda) = \frac{1}{\eta_N} \left( \frac{\lambda \eta_A^{(+)}}{T_N^+} - \frac{\lambda \eta_A^{(-)}}{T_N^-} \right) + \frac{1}{\eta_N} \sum_{a=1}^{N-1} \sum_{k=0}^{p-1} \frac{O_{a,k} \eta_a^{(k)}}{(\lambda - \eta_a^{(k)})}. \]  
(7.26)
It is then clear that the local operators $u_n^{-k}$ and $\tilde{\beta}_{k,n}$ are linear combinations of the monomials:
\[ U_n \eta_n^{-h} \left( \eta_A^{(+)}/T_N^+ \right)^{h_0} O_{a_1,h_1} \cdots O_{a_s,h_s} U_n^{-1} \]  
(7.27)
for $s \leq p$, $a_i \in \{1,\ldots,N-1\}$ and $h, h_i \in \{0,\ldots,p-1\}$. The commutation rules (7.17) allow to rewrite any monomial $O_{a_1,h_1} \cdots O_{a_s,h_s}$ in a way that operators with the same index $a$ are adjacent and those with different $a$ are ordered in a way $a_i < a_j$ for $i < j \in \{1,\ldots,N-1\}$. Then the rule (7.14) tell us if the monomial is zero or non-zero. The property (7.15) finally implies:
\[ O_{a,k}^{(p+\alpha)} = \frac{A(Z_a)}{\prod_{b \neq a,b=1}^N (Z_a/Z_b - Z_b/Z_a)} O_{a,k}^{(\alpha)}, \]  
(7.28)
and so that all the non-zero monomials $O_{a_1,h_1} \cdots O_{a_s,h_s}$ are rewritable in the form (7.20). $\square$

7.2.2 Determinant representation of elementary operator form factors

Lemma 7.3. The elementary operators admit the following simple characterizations for their form factors:
\[ \langle t_k | e^{\alpha_1,\ldots,\alpha_r}_{(h,h_0,(a_1,k_1),\ldots,(a_r,k_r)} | t_{k'} \rangle = \frac{\delta_{k,k' + h,a_0}^{h_0,k_0} \delta_{k_0}^{h_0,k'}}{(\eta_N^{(0)})^h} f_{h_0,(\alpha),(a)}^0 \det_{N-1+p} (||O_{a,b}^{(h_0,(\alpha),(a))}||). \]  
(7.29)
Here, \( \langle t_k \rangle \) and \( |t'_k\rangle \) are two eigenstates of the transfer matrix \( \tau_2(\lambda) \) and \( \|O^{(a)}_{a,b}(a,\{a\})\| \) is the \((N - 1 + rp - g) \times (N - 1 + rp - g)\) matrix of elements:

\[
O^{(a)}_{a,b}(a,\{a\}) = \left( \eta_{am}^2 q^{2j_m} \right)^{(a-1)} f_{m} \quad \text{for } j_m \in \{0, \ldots, p - \alpha_m\}, \ m \in \{1, \ldots, r\},
\]

\[
O^{(a,b)}_{a,b}(a,\{a\}) = \Phi_{b_i(a,b)+g/2} \quad \text{for } i \in \{1, \ldots, N - 1 - r\}, \ g \equiv \sum_{h=1}^{r} \alpha_h,
\]

for any \( a \in \{1, \ldots, N - 1 + rp - g\} \). Moreover, we have used the following notations \( \{b_1, \ldots, b_{N-1-r}\} = \{1, \ldots, N - 1\} \setminus \{a_1, \ldots, a_r\} \) where the elements are ordered by \( b_i < b_j \) for \( i < j \),

\[
f^{(a,b)}_{a,b}(a,\{a\}) = \frac{\prod_{i=1}^{r} Q_{i}(\eta_{ai} q^{-\alpha_i}) \tilde{Q}_{i}(\eta_{ai}) \left( \eta_{ai}^{b_0+\alpha_i(N-1-r)}/\omega_{ai}(\eta_{ai}) \right) \prod_{h=0}^{\alpha_i-1} a(\eta_{ah} q^{-h})}{\prod_{i=1}^{r} \prod_{h=0}^{\alpha_i-1} \prod_{j=1}^{1}(q_{i+j}^{\alpha_j} - \eta_{ai}/\omega_{ai}), \prod_{i=1}^{r} (q_{i+j}^{1} q_{i+j}^{\alpha_j} - \eta_{ai}/\omega_{ai})} \times \prod_{i=1}^{r} \prod_{j=1}^{N-1-r} (Z_{ai}^{2} - Z_{bj}^{2}) V(\eta_{ai}^{2}, \eta_{ai}^{q_{2}^{2}}, \ldots, \eta_{ai}^{q_{2}^{2}(p-\alpha_i)}, \ldots, \eta_{ai}^{2}, \eta_{ai}^{q_{2}^{2}}, \ldots, \eta_{ai}^{2} q_{2}^{2(p-\alpha_r)}),
\]

\[
V(x_1, \ldots, x_N) = \prod_{1 \leq a < b \leq N} (x_a - x_b) \text{ is the Vandermonde determinant and for brevity:}
\]

\[
\eta_{am} \equiv \eta_{am}^{(h_m)}.
\]

**Proof.** The following actions hold:

\[
\langle t_k \rangle \hat{\eta}_{N}^{-h} |\eta_{A}^{(+)} T_{N}\rangle^{(h_0)} = \frac{h_a q^{h_a(k-h)}}{q^{(h_0)}} \sum_{h_1, \ldots, h_N=1}^{p} \frac{q^{(k-h)h_N}}{p^{1/2}} \prod_{a=1}^{N-1} \left( \eta_{a}^{(h_a)} \right) \tilde{Q}_{i}(\eta_{ai}) \times \prod_{1 \leq a < b \leq N-1} ((\eta_{a}^{(h_a)})^2 - (\eta_{b}^{(h_b)})^2) \frac{\eta_{1}^{(h_1)} \ldots \eta_{N}^{(h_N)} \omega_{b}(\eta_{b}^{(h_b)})}{\prod_{b=1}^{N-1} \omega_{b}(\eta_{b}^{(h_b)})}.
\]

From the formula (7.18), it follows:

\[
\langle \eta_{1}, \ldots, \eta_{N}^{(f)} | O^{(a_1)}_{a_1,h_1}, \ldots, O^{(a_r)}_{a_r,h_r} \rangle_{a_1, h_1} = \frac{\prod_{h=0}^{N-1} \sum_{a=1}^{p} a(\eta_{a} q^{-h}) \delta_{f,h_i} \langle \eta_{1}, \ldots, \eta_{N} q^{-\alpha_i}, \ldots, \eta_{N} \rangle}{\prod_{1 \leq a_1 \neq a_2 \leq N} (\eta_{a_1} q^{-h} / \eta_{a_2} - \eta_{a_2} / \eta_{a_1} q^{-h})},
\]

where \( \eta_{a} \) is defined in (7.33). The action of \( O^{(a_1)}_{a_1,h_1} \ldots O^{(a_r)}_{a_r,h_r} \) can be computed now taking into account
the order of the operators which appear in the monomial then by using the scalar product formula we get:

\[
\langle t_k | \mathcal{E}^{(\alpha_1, \ldots, \alpha_r)}_{(h, h_0, (\alpha_1, h_1), \ldots, (\alpha_r, h_r))} | t'_{k'} \rangle = \frac{a^{h_0 q^{(k-h)}}}{(\eta_0(q))^{h_0}} \sum_{k_1, \ldots, k_N=1}^{p} \frac{q^{[(k-h)-k']h_0}}{p} \prod_{a=1}^{N-1} (\eta_a^{(h_a)})^{h_0} \\
\times \prod_{i=1}^{r} \prod_{j=1}^{N-1} \prod_{h=0}^{N-1} \frac{a(q^{h-\alpha_j}) \delta_{k_{a_i}, h_i}}{(\eta_a q^{-h}/\eta_{a_j} - \eta_{a_j}/\eta_a q^{-h})} \\
\times \prod_{i=1}^{r} \prod_{j=i+1}^{r} \prod_{h=0}^{N-1} \frac{Q_{t'}(\eta_{b_j}) Q_t(\eta_{b_j})}{\omega_{b_j}^{(k_{b_j})}} \prod_{i=1}^{r} \frac{Q_{t'}(\eta_{b_j} q^{-\alpha_i}) Q_t(\eta_{b_j})}{\omega_{a_i}^{(k_{a_i})}} V(\eta_1^2, \ldots, \eta_{N-1}^2). 
\]

The presence of the \( \prod_{i=1}^{r} \delta_{k_{a_i}, h_i} \) reduces the sum \( \sum_{k_1, \ldots, k_N=1}^{p} \) to \( \delta_{k, k'+h} \) times the sum \( \sum_{k_{b_1}, \ldots, k_{b_{N-(r+1)}}}^{p} \), where:

\[
\{\alpha_1, \ldots, \alpha_r\} \cup \{b_1, \ldots, b_{N-(r+1)}\} = \{1, \ldots, N-1\}. 
\]

We get our formula (7.29) multiplying each term of the sum by:

\[
1 = \prod_{\epsilon=\pm1}^{r} \prod_{i=1}^{N-1-r} \prod_{j=1}^{N-1} \prod_{h=-p+\alpha_i}^{h=-p+\alpha_i} \frac{\eta_0^2 q^{-2h} - (\eta_{b_j})^2}{\eta_{a_i}^2 q^{2(p-\alpha_i)} - \eta_{a_i}^2 q^{2(p-\alpha_i)}}. 
\]

Indeed, the power +1 leads to the construction of the Vandermonde determinant:

\[
V(\eta_1^2, \ldots, \eta_{N-1}^2, \eta_1^{2(p-\alpha_1)}, \ldots, \eta_1^{2(p-\alpha_r)}, \eta_2^{2(p-\alpha_1)}, \ldots, \eta_2^{2(p-\alpha_r)}, \ldots, \eta_{N-1}^{2(p-\alpha_1)}, \ldots, \eta_{N-1}^{2(p-\alpha_r)})^{(N-1-r) columns}
\]

and the sum \( \sum_{k_{b_1}, \ldots, k_{b_{N-(r+1)}}}^{p} \) becomes sum over columns which can be brought inside the determinant.

\[\square\]

### 7.3 The chiral Potts model order parameters

The results presented in the previous subsections are as well results for the matrix elements of local operators in the inhomogeneous chiral Potts model. In particular, let \( |t_k\rangle \) and \( |t'_{k'}\rangle \) be two eigenstates of the chiral Potts transfer matrix, then the matrix elements:

\[
\langle t_k | u_n^{-1} | t'_{k'} \rangle, \quad \langle t_k | \alpha_0^{-1} | t'_{k'} \rangle \quad \text{and} \quad \langle t_k | \mathcal{E}^{(\alpha_1, \ldots, \alpha_r)}_{(h, h_0, (\alpha_1, h_1), \ldots, (\alpha_r, h_r))} | t'_{k'} \rangle
\]
are given respectively by the formulae (7.1), (7.11) and (7.29). Furthermore, in the representations \( R_{N_{p,S-adj}} \) the formulae (7.1), (7.11) and (7.29) are always matrix elements of the corresponding local operators on chiral Potts eigenstates. As clarified below, some of these matrix elements generate the chiral Potts order parameters under the homogeneous and thermodynamic limits.

### 7.3.1 Local Hamiltonians and order parameters

It is worth recalling that the following local quantum Hamiltonians:

\[
H ≡ H_0 + k H_1, \quad H_0 ≡ \sum_{n=1}^{N} \left[ p - 1 \sum_{r=1}^{p} f_r(\theta) u_n^r u_{n+1}^r \right], \quad H_1 ≡ \sum_{n=1}^{N} \left[ p - 1 \sum_{r=1}^{p} f_r(\overline{\theta}) v_n^r \right],
\]

(first constructed by von Gehlen and Rittenberg \[65\]), commute with the homogeneous \( Z_p \) chP transfer matrices. Indeed, they are generated by derivative of these transfer matrices w.r.t. the spectral parameter, see for example \[60\] for a derivation. Then the order parameters associated to the homogeneous \( Z_p \) chP models:

\[
\mathcal{M}_r ≡ \frac{\langle g.s. | u_1^r | g.s. \rangle}{\langle g.s. | g.s. \rangle}, \quad \forall r ∈ \{1, \ldots, p - 1\}
\]

admit a natural interpretation as spontaneous magnetizations in terms of the spin chain formulation associated to these local Hamiltonians. They have been mainly analyzed in the special representations associated to the super-integrable \( Z_p \) chP model, characterized by the following constraints:

\[
x_{q_n}^p = y_{q_n}^p = x_{r_n}^p = y_{r_n}^p = 1 + \frac{k'}{k}, \quad \forall n ∈ \{1, \ldots, N\} \rightarrow \overline{\theta} = \theta = \pi/2.
\]

In these special representations the \( Z_p \) chP model also has an underlying Onsager algebra \[63\] generated by the two components \( H_0 \) and \( H_1 \) of the local quantum Hamiltonians. The following thermodynamic limits:

\[
\mathcal{M}_r = (1 - k^2)^{r(p-r)/2p^2}, \quad \forall r ∈ \{1, \ldots, p - 1\}
\]

have been first argued by perturbative computations in \[90\] and then proven with techniques \[25\] which apply only starting from finite lattice computations in the super-integrable case. Nevertheless, as argued in \[99\], the formulae (7.44) should hold true for the general homogeneous \( Z_p \) chP models. It is then relevant pointing out that our approach should give us the possibility to prove this statement for general representations without the need to be restricted to the super-integrable case and our SOV results already provide simple determinant formulae for the matrix elements associated to \( \mathcal{M}_{p-1} \) in the finite size and inhomogeneous regime.

\[25\]See Section 1.1 for an historical recall.
8 Conclusion and outlook

8.1 Conclusions

In this article we have considered general cyclic representations of the 6-vertex Yang-Baxter algebra on N-sites finite lattices and analyzed the associated Bazhanov-Stroganov model and consequently the chiral Potts model. We have derived a reconstruction for all local operators in terms of standard Sklyanin’s quantum separate variables and characterized by one determinant formulae of $N\times N$ matrices the scalar products of separate states. These findings imply that the action of any local operator on transfer matrix eigenstates reduces to a finite sum of separate states which allows to characterize matrix elements of any local operator as finite sum of determinants of the scalar product type. Moreover, we have obtained: form factors of the local operators $u_n^{-1}$ and $\alpha^{-1}_{0,n}$ expressed by one determinant formulae obtained by modifying a single row in the scalar product matrices; form factors of a basis of operators expressed by one determinant formulae obtained by modifying the scalar product matrices by introducing rows which coincide with those of Vandermonde’s matrix computed in the spectrum of the separate variables.

Let us comment that it would be desirable to get also for the generators $v_n$ of the local Weyl algebras simple one determinant formulae as for the generators $u_n$ (at this moment we have expressed its form factors as finite sums of determinants); this interesting issue is currently under investigation. One important motivation to derive form factors of local operators by simple determinant formulae is for their use as efficient tools for the computations of correlation functions. The decomposition of the identity (11.13) allows to write correlation functions in spectral series of form factors and so it allows to analyzed numerically them mainly by the same tools developed in [144] in the ABA framework and used in the series of works [144]-[150]. Indeed, in our SOV framework we have determinant representations of the form factors and eventually complete characterization of the transfer matrix spectrum in terms of the solutions of a system of Bethe equations type. Let us mention that in a recent series of papers [158]-[168] the problem to compute the asymptotic behavior of correlation functions has been successfully addressed with a method which is in principle susceptible to be extended to any (integrable) quantum model possessing determinant representations for the form factors of local operators [167] and so also to the models analyzed by our approach in the SOV framework.

Finally, let us remark that the originality and interest of our current results are also due to the fact that matrix elements of local operators were so far mainly confined to the special class of super-integrable representations of $Z_p$ chiral Potts model. As these representations can be obtained by taking well defined limits on the parameters of a generic (non-super-integrable) representation to which SOV applies, it is then an interesting issue to investigate how from our form factor results one can reproduce also those known in the super-integrable case. About this point it is worth mentioning that in the special case ($p=2$) of the generalized Ising model, it was already remarked in [102] that the matrix elements of the local spin operators obtained in the SOV framework in [115] admit factorized forms similar to those conjectured in the numerical approach. Relevant physical observables (like the so called dynamical structure factors) were evaluated and successfully compared with the measurements accessible by neutron scattering experiments [151]-[157].

These results have been also successfully compared with those obtained previously with a method relying mainly on the Riemann-Hilbert analysis of related Fredholm determinants [169]-[171].
and proven in [102] for the super-integrable \( Z_p \) cases for general \( p \geq 2 \).

In a future paper, we will analyze the homogeneous and thermodynamic limits focusing on the derivation of the order parameter formulae for the general homogeneous \( Z_p \) chiral Potts models. These formulae were proven with techniques working only in the super-integrable case but they are expected to be true [99] for the general homogeneous \( Z_p \) chiral Potts models. Our approach should give access to a proof of this statement from the finite lattice in general representations and we find encouraging the fact that the matrix element describing the order parameter:

\[
M_{p-1} \equiv \frac{\langle \text{g.s.}|u_{p-1}|\text{g.s.}\rangle}{\langle \text{g.s.|g.s.\rangle}}
\]  

admits simple determinant formula in our approach.

### 8.2 Outlook

It is worth recalling that in the literature of quantum integrable models there exist some results on form factors derived by different applications of separation of variable methods. For a more detailed analysis of the most relevant preexisting results and an explicit comparison with those obtained by our method in separation of variables we address the reader to [1]. Here, we want just recall the Smirnov’s results [130], in the case of the integrable quantum Toda chain [13], [127]-[129] and those of Babelon, Bernard and Smirnov [172, 173], in the case of the restricted sine-Gordon at the reflectionless points. In both these cases form factors of local operators were argued to have a determinant form. A strong similarity in the form of the results appears: the elements of the matrices whose determinants give the form factors are expressed as “convolutions”, over the spectrum of each separate variable, of the product of the corresponding separate components of the wave functions times contributions associated to the action of local operators. It is then remarkable that also our results fall in this general form. This observation and the potential generality of the SOV method leads to the expectation of an universality in the SOV characterization of form factors.

A natural project is then to develop explicitly our method for a set of fundamental integrable quantum models providing determinant representations for form factors. This SOV method is not restricted to the case of cyclic representation and applies to a large class of integrable quantum models which were not tractable with other methods and in particular by algebraic Bethe ansatz. There exist already several key integrable quantum models associated by QISM to highest weight representations of the Yang-Baxter algebras and generalization of it for which this program has been developed. In [178]-[182] and [183] our approach has been respectively implemented for the spin-1/2 XXZ and the spin-s XXX inhomogeneous quantum chains with antiperiodic boundary conditions, for the spin-1/2 XXZ and XYZ open quantum chains with general non-diagonal integrable boundary conditions [46]-[52] and finally for the spin-1/2

\[28\] The absence of a direct reconstruction of the local operators in terms of the Sklyanin’s quantum separate variables was the motivation in [130, 172] to use some well-educated guess relying on counting arguments for the characterization of local operators basis and to use semi-classical arguments relying on the classical SOV-reconstruction for the identification of primary fields [172, 173]. Note that a reconstructions of local operators in the lattice Toda model have been achieved in [174] in terms of a set of quantum separate variables defined by a change of variables in terms of the original Sklyanin’s quantum separate variables. Recent analysis of this reconstruction problem for the lattice Toda model appear also in [175] and [176].
representations of highest weight type of the dynamical 6-vertex Yang-Baxter algebra. In all these models the universality we just discussed in the structure of the matrix elements of local operator has been verified.

Acknowledgements

N. G. is supported by the University of Cergy-Pontoise. He acknowledges the support of ENS Lyon and ANR grant ANR-10-BLAN-0120-04-DIADEMS during his thesis when most of this work was done. N. G. would also like to thank the YITP Institute of Stony Brook for hospitality. J. M. M. is supported by CNRS, ENS Lyon and by the grant ANR-10-BLAN-0120-04-DIADEMS. G. N. gratefully thanks Barry McCoy for his teachings on pre-existing results on the Bazhanov-Stroganov model, in particular about the order parameters. G. N. is supported by National Science Foundation grants PHY-0969739. G. N. gratefully acknowledge the YITP Institute of Stony Brook for the opportunity to develop his research programs. G. N. would like to thank the Theoretical Physics Group of the Laboratory of Physics at ENS-Lyon for hospitality, under support of ANR-10-BLAN-0120-04-DIADEMS, which made possible this collaboration.

References

[1] N. Grosjean, J. M. Maillet, G. Niccoli, J. Stat. Mech. P10006 (2012).
[2] E.K. Sklyanin and L.D. Faddeev, Sov. Phys. Dokl. 23 (1978) 902.
[3] L.D. Faddeev and L.A. Takhtajan, Russ. Math. Surveys, 34: 5 (1979) 11
[4] P.P. Kulish and E.K. Sklyanin, Phys. Lett. A 70 (1979) 461
[5] L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan, Theor. Math. Phys. 40 (1979) 688
[6] L.D. Faddeev, Sov. Sci. Rev. C: Math. Phys. 1 (1980) 107-155
[7] E.K. Sklyanin, J. Sov. Math. 19 (1982) 1546-1596
[8] P.P. Kulish and E.K. Sklyanin, Lect. Notes Phys. 151 (1982) 61
[9] L. D. Fadeev, Integrable models in 1 + 1 dimensional quantum field theory. In J.-B. Zuber and R. Stora, editors, Recent Advances in Field Theory and Statistical Mechanics, Les Houches, Session XXXIX, pages 561-608. Amsterdam: North Holland Publishing Company, June 1984. ISBN: 0444866752.
[10] L.D. Fadeev, How Algebraic Bethe Ansatz works for integrable model, [hep-th/9605187v1].
[11] M. Jimbo, Yang Baxter Equation in Integrable Systems, Adv. Ser. Math. Phys. 10, Singapore: World Scientific, 1990. ISBN: 978-981-02-0120-3.
[12] B. S. Shastry, S. S. Jha and V. Singh, Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory, Lect. Notes Phys. 242 (1985).
[13] H. B. Thacker, Rev. Mod. Phys. 53 (1981) 253.
[14] A. G. Izergin and V. E. Korepin, Nucl. Phys. B 205 (1982) 401-413.
[15] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196-233.
[16] E. K. Sklyanin. Quantum Inverse Scattering Method. Selected Topics. In M.-L. Ge, editor, Quantum Group and Quantum Integrable Systems: Nankai Lectures on Mathematical Physics. Singapore: World Academic, July 1992. ISBN: 978-9810207458. [hep-th/9211111]
[17] E. K. Sklyanin, Prog. Theor. Phys. Suppl. [118 (1995) 35-60].
[18] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B 554 (1999) 647.
[19] W. Heisenberg, Z. Phys. 49 (1928) 619.
[20] H. Bethe, Z. Phys. 71 (1931) 205.
[21] L. Hulten, Ark. Mat. Astron. Fys. 26 (1938) 1.
[22] R. Orbach, Phys. Rev. 112 (1958) 309.
[23] L. R. Walker, Phys. Rev. 116 (1959) 1089.
[24] C. N. Yang and C. P. Yang, Phys. Rev. 150 (1966) 321.
[25] C. N. Yang and C. P. Yang, Phys. Rev. 150 (1966) 327.
[26] M. Gaudin, La Fonction d’onde de Bethe, Paris, Masson, 1983. ISBN: 9782225796074.
[27] E. H. Lieb and D. C. Mattis, Mathematical Physics in One Dimension, New-York: Academic, 1966. ISBN: 978-0124487505.
[28] J. M. Maillet and V. Terras, Nucl. Phys. B 575 (2000) 627.
[29] A. G. Izergin, N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B 554 (1999) 679.
[30] N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B 567 (2000) 554.
[31] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B 641 (2002) 487.
[32] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B 642 (2002) 433.
[33] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A 35 (2002) L385.
[34] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A 35 (2002) L753.
[35] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B 712 (2005) 600.
[36] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B 729 (2005) 558.
[37] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A 38 (2005) 7441.
[38] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Stat. Mech. L09002 (2005).
[39] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, On the algebraic Bethe Ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain, in Recent Progress in Solvable lattice Models, RIMS Sciences Project Research 2004 on Method of Algebraic Analysis in Integrable Systems, RIMS, Kyoto, Kokyuroku, 1480, 14 (2006); hep-th/0505006.
[40] N. Kitanine, K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. P01022 (2007).
[41] N. Kitanine, J. Phys. A: Math. Gen. 34 (2001) 8151.
[42] O. A. Castro-Alvaredo, J. M. Maillet, J. Phys. A 40 (2007) 7451.
[43] N. Kitanine, K. K. Kozlowski, J.M. Maillet, G. Niccoli, N.A. Slavnov, V. Terras, J. Stat. Mech. P10009 (2007).
[44] K. K. Kozlowski, J. Stat. Mech. P02006 (2008).
[45] N. Kitanine, K. Kozlowski, J. M. Maillet, G. Niccoli, N. A. Slavnov, V. Terras, J. Stat. Mech. P07010 (2008).
[46] E. K. Sklyanin, J. Phys. A: Math. Gen. 21 (1988) 2375.
[47] I. V. Cherednik, Theor. Math. Phys. 61 (1984) 977.
[48] P. P. Kulish and E. K. Sklyanin, J. Phys. A: Math. Gen. 24 (1991) L435.
[49] L. Mezincescu and R. Nepomechie, Int. J. Mod. Phys. A 6 (1991) 5231.
[50] P. P. Kulish and E.K. Sklyanin, J. Phys. A: Math. Gen. 25 (1992) 5963.
[51] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A 9 (1994) 3841.
[52] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A 9 (1994) 4353
[53] V. Tarasov, Int. J. Mod. Phys. A 07 (1992) 963
[54] G. Niccoli and J. Teschner, J. Stat. Mech. P09014 (2010)
[55] G. Niccoli, Nucl. Phys. B 835 (2010) 263
[56] G. Niccoli, JHEP03(2011)123
[57] V.V. Bazhanov, Yu. G. Stroganov, J. Stat. Phys. 59 (1990) 799
[58] R.J. Baxter, V.V. Bazhanov and J.H.H. Perk, Int. J. Mod. Phys. B 4 (1990) 803
[59] R.J. Baxter, J. Stat. Phys. 117 (2004) 1
[60] G. Albertini, B.M. McCoy and J.H.H. Perk, Eigenvalue spectrum of the super-integrable chiral Potts model, 1989 Integrable Systems in Quantum Field Theory and Statistical Mechanics (Adv. Stud. Pure Math. vol 19) ed M Jimbo, T Miwa and A Tsuchiya (Tokyo: Kinokuniya) pp 1-55. ISBN: 9780123853424
[61] G. Albertini, B.M. McCoy and J.H.H. Perk, Phys. Lett. A 135 (1989) 159
[62] G. Albertini, B.M. McCoy and J.H.H. Perk, Phys. Lett. A 139 (1989) 204
[63] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M-L, Phys. Lett. A 123 (1987) 219
[64] R.J. Baxter, J.H.H. Perk and H. Au-Yang, Phys. Lett. A 128 (1988) 138
H. Au-Yang and J. H. H. Perk, Onsagers star triangle equation: master key to integrability, 1989 Integrable Systems in Quantum Field Theory and Statistical Mechanics (Adv. Stud. Pure Math. vol 19) ed M Jimbo, T Miwa and A Tsuchiya (Tokyo: Kinokuniya) pp 57-94. ISBN: 9780123853424
[65] G. von Gehlen and V. Rittenberg, Nucl. Phys. B 257 (1985) 351
[66] J. H. H. Perk, Star-triangle equations, quantum Lax pairs, and higher genus curves, 1989 Proc. 1987 Summer Research Institute on Theta Functions (Proc. Symp. Pure Math. vol 49) ed R C Gunning and L Ehrenpreis (Providence, RI: American Mathematical Society) pp 341-354. ISBN: 9780821814857
[67] R.J. Baxter, Phys. Lett. A 133 (1989) 185
[68] R.J. Baxter, J. Stat. Phys. 57 (1989) 1
[69] R. J. Baxter, V. V. Bazhanov and J. H. H. Perk, Functional relations for transfer matrices of the chiral Potts model, Int. J. Mod. Phys. B, 4 (1990), 803870.
[70] V. V. Bazhanov, A. Bobenko, N. Reshetikhin, Comm. Math. Phys. 175 (1996) 377.
[71] V. V. Bazhanov, Adv. in Pure Math. 61 (2011) 91-123.
[72] V. V. Bazhanov, S. Sergeev, Adv. in Theor. and Math. Phys. 16 (2012) 65-95.
[73] H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. Yan, Phys. Lett. A 123 (1987) 219
[74] B. M. McCoy, J. H. H. Perk, S. Tang, and C. H. Sah, Phys. Lett. A 125 (1987) 9
[75] H. Au-Yang, B. M. McCoy, J. H. H. Perk, and S. Tang, Solvable models in statistical mechanics and Riemann surfaces of genus greater than one, 1988 Papers Dedicated to Professor Mikio Sato on the Occasion of His Sixtieth Birthday vol I, ed M Kashiwara and T Kawai (San Diego: Academic) pp 29-40. ISBN: 9780124004658
[76] V. O. Tarasov Phys. Lett. A 147 (1990) 487
[77] P. P. Kulish, N. Y. Reshetikhin, and E. K. Sklyanin, Lett. Math. Phys. 5 (1981) 393
[78] A. N. Kirillov and N. Y. Reshetikhin, J. Phys. A: Math Gen. 20 (1987) 1565
[79] H. Au-Yang and J. H. H. Perk, J. Phys. A: Math. Theor. 41 (2008) 275201
[80] H. Au-Yang and J. H. H. Perk, J. Phys. A: Math. Theor. 42 (2009) 375208
[81] A. Nishino and T. Deguchi, J. Stat. Phys. (2008) 133 587.
[82] S. S. Roan, *Eigenvalues of an arbitrary Onsager sector in super-integrable $\tau_2$ model and chiral Potts model*, (2010) arXiv:1003.3621.

[83] L. Onsager, Phys. Rev. 65 (1944) 117.

[84] K. Fabricius and B. M. McCoy, *Evaluation parameters and Bethe roots for the six-vertex model at roots of unity*, 2001 MathPhys Odyssey (Progress in Math. Phys. vol 23) ed M Kashiwara and T Miwa (Basel: Birkhauser) pp 119.

[85] B. Davies, J. Phys. A: Math. Gen. 23 (1990) 2245.

[86] E. Date and S. S. Roan, Czech. J. Phys. 50 (2000) 37.

[87] S. S. Roan, J. Stat. Mech. (2005) P09007.

[88] A. Nishino and T. Deguchi, Phys. Lett. A 356 (2006) 366.

[89] S. S. Roan, J. Phys. A: Math. Theor. 40 (2007) 1481, J. Stat. Mech. (2009) P08012.

[90] G. Albertini, B. M. McCoy, J. H. H. Perk and S. Tang, Nucl. Phys. B 314 (1989) 741.

[91] R. J. Baxter, Phys. Rev. Lett. 94 (2005) 130602.

[92] R. J. Baxter, J. Stat. Phys. 120 (2005) 1.

[93] M. Jimbo, T. Miwa and A. Nakayashiki, J. Phys. A: Math. Gen. 26 (1993) 2199.

[94] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, London: Academic, 1982. ISBN: 0120831821. See also here.

[95] R. J. Baxter, J. Phys. A: Math. Theor. 40 (2007) 12577.

[96] R. J. Baxter, J. Stat. Phys. 132 (2008) 959.

[97] R. J. Baxter, J. Stat. Phys. 137 (2009) 798.

[98] H. Au-Yang and J. H. H. Perk, J. Phys. A: Math. Theor. 43 (2010) 025203.

[99] H. Au-Yang and J. H. H. Perk, J. Phys. A: Math. Theor. 44 (2011) 025205.

[100] H. Au-Yang and J. H. H. Perk, *super-integrable chiral Potts model: Proof of the conjecture for the coefficients of the generating function $G(t,u)$*, 1108.4713v1.

[101] R. J. Baxter, J. Stat. Phys. 132 (2008) 983.

[102] N. Iorgov, S. Pakuliak, V. Shadura, Yu Tykhyy and G. von Gehlen, J. Stat. Phys. 139 (2009) 743.

[103] A. Bugrij, O. Lisovyy, Theor. Math. Phys. 140 (2004) 987.

[104] N. Iorgov, J. Phys. A: Math. Theor. 44 (2011) 335005.

[105] R. J. Baxter, J. Phys. A: Math. Theor. 43 (2010) 145002.

[106] R. J. Baxter, ANZIAM J. 51 (2010) 309.

[107] S. Dasmahapatra, R. Kedem and B. McCoy, Nucl. Phys. B 396 (1993) 506.

[108] G. Albertini, S. Dasmahapatra and B. McCoy, Int. J. Mod. Phys. A 7:supp01a (1992) 1.

[109] V. A. Fateev and A. B. Zamolodchikov, Phys. Lett. A 92 (1982) 37.

[110] K. Fabricius and B. McCoy, J. Stat. Phys. 103 (2001) 647.

[111] R. I. Nepomechie and F. Ravanini, J. Phys. A 36 (2003) 11391.

[112] G. von Gehlen, N. Iorgov, S. Pakuliak and V. Shadura, J. Phys. A: Math. Gen. 39 (2006) 7257.

[113] N. Iorgov, SIGMA 2 (2006) 019.

[114] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura and Yu Tykhyy, J. Phys. A: Math. Theor. 40 (2007) 14117.

[115] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura and Yu Tykhyy, J. Phys. A: Math. Theor. 41 (2008) 095003.
[116] G von Gehlen, N Iorgov, S Pakuliak, V Shadura, J. Phys. A: Math. Theor. 42 (2009) 304026
[117] N. Grosjean and G. Niccoli, J. Stat. Mech. P11005 (2012).
[118] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, J. Phys. A 20 (1987) 6397
[119] N.Y. Reshetikhin, Lett. Math. Phys. 7 (1983) 205
[120] N.Y. Reshetikhin, JETP 57 (1983) 691
[121] E. Mukhin, V. Tarasov, A. Varchenko, Comm. Math. Phys. 288 (2009) 1
[122] D. Orlando, S. Reffert and N. Reshetikhin, *On domain wall boundary conditions for the XXZ spin Hamiltonian*, arXiv:0912.0348
[123] C. Korff, Comm. Math. Phys. 318 (2013) 173
[124] A. G. Izergin and V. E. Korepin, Doklady Akademii Nauk 259 (1981) 76, also available on arXiv 0910.0295
[125] A. G. Izergin and V. E. Korepin, *A lattice model related to the nonlinear Schroedinger equation*, arXiv: 0910.0295
[126] N. A. Slavnov, Theor. Math. Phys. 79 (1989) 502
[127] M. Gutzwiller, Ann. of Phys. 133 (1981) 304
[128] V. Pasquier and M. Gaudin, J. Phys. A 25 (1992) 5243
[129] S. Kharchev, D. Lebedev, Lett. Math. Phys. 50 (1999) 53
[130] F. Smirnov, J. Phys. A: Math. Gen. 31 (1998) 8953
[131] A. Bytsko, J. Teschner, J. Phys. A 39 (2006) 12927
[132] L.D. Faddeev and R.M. Kashaev, Mod. Phys. Lett. A 9 (1994) 427
[133] L.D. Faddeev, Lett. Math. Phys. 34 (1995) 249
[134] S.N.M. Ruijsenaars, J. Math. Phys. 38 (1997) 1069
[135] S.L. Woronowicz, Rev. Math. Phys. 12 (2000) 873
[136] B. Ponsot and J. Teschner, Commun. Math. Phys. 224 (2001) 613
[137] R.M. Kashaev, J. Stat. Phys. 102 (2001) 923
[138] R.M. Kashaev. *The quantum dilogarithm and Dehn twists in quantum Teichm"uller theory*, 2001 Integrable structures of exactly solvable two-dimensional models of quantum field theory (Nato Science Series II: (closed), vol. 35) ed Pakuliak S and von Gehlen G (Dordrecht : Kluwer) pp 211-221. ISBN: 978-0-7923-7183-0
[139] A. Bytsko and J. Teschner, Comm. Math. Phys. 240 (2003) 171
[140] J. Teschner, Class. Quant. Grav. 18 (2001) R153
[141] J. Teschner, Int. J. Mod. Phys. A 19:supp02 (2004) 436
[142] A. Yu. Volkov, Comm. Math. Phys. 258 (2005) 257
[143] V.O. Tarasov, I. A. Takhtadzhyan and L.D. Faddeev, Theo. Math. Phys. 57 2 (1983) 1059
[144] T. Oota, J. Phys. A: Math. Gen. 37 (2004) 441
[145] J.-S. Caux, J.-M. Maillet, Phys. Rev. Lett. 95 (2005) 077201.
[146] J.-S. Caux, R. Hagemans, J.-M. Maillet, J. Stat. Mech. P09003 (2005)
[147] R. Hagemans, J.-S. Caux and J. M. Maillet. *How to Calculate Correlation Functions of Heisenberg Chains* in Proceedings of the "Tenth Training Course in the Physics of Correlated Electron Systems and High-Tc Superconductors", Salerno, Oct 2005. AIP Conference Proceedings 846 (2006) 245

43
[148] R. G. Pereira, J. Sirker, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, J. Stat. Mech. P08022 (2007).

[149] J. Sirker, R. G. Pereira, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, Boson decay and the dynamical structure factor for the XXZ chain at finite magnetic field in Proceedings SCES ’07, Houston, Physica B 403 (2008) 1520.

[150] J. S. Caux, P. Calabrese and N. A. Slavnov, J. Stat. Mech. P01008 (2007).

[151] F. Bloch, Phys. Rev. 50 (1936) 259.

[152] J. S. Schwinger, Phys. Rev. 51 (1937) 544.

[153] O. Halpern and M. H. Johnson, Phys. Rev. 55 (1938) 898.

[154] L. Van Hove, Phys. Rev. 95 (1954) 249.

[155] L. Van Hove, Phys. Rev. 95 (1954) 1374.

[156] W. Marshall and S. W. Lovesey, Theory of Thermal Neutron Scattering, Oxford: Clarenton Press, 1971. ISBN: 9780198512547.

[157] R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, New York: Wiley, 1975. ISBN: 978-0471046004.

[158] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Math. Phys. 50 (2009) 095209.

[159] K. K. Kozlowski, J. Math. Phys. 50 (2009) 095205.

[160] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. P05028 (2011).

[161] K.K. Kozlowski, J.M. Maillet, N.A. Slavnov, J. Stat. Mech. P03018 (2011).

[162] K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, J. Stat. Mech. P03019 (2011).

[163] K. K. Kozlowski, Lett. Math. Phys. (2013) DOI: 10.1007/s11005-013-0654-1.

[164] K. K. Kozlowski, J. Math. Phys. 52 (2011) 083302.

[165] K. K. Kozlowski, Large-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model, arXiv: 1101.1626.

[166] K. K. Kozlowski, V. Terras, J. Stat. Mech. P09013 (2011).

[167] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. P12010 (2011).

[168] K. K. Kozlowski, B. Pozsgay, J. Stat. Mech. P05021 (2012).

[169] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, Comm. Math. Phys. 291 (2009) 691.

[170] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. P04003 (2009).

[171] K. K. Kozlowski, Adv. Theor. Math. Phys. 15 (2011) 1655.

[172] O. Babelon, D. Bernard, F. Smirnov, Comm. Math. Phys. 182 (1996) 319.

[173] O. Babelon, D. Bernard, F. Smirnov, Comm. Math. Phys. 186 (1997) 601.

[174] O. Babelon, J. Phys. A 37 (2004) 303.

[175] E. Sklyanin, J. Phys. A: Math. Theor. 46 382001.

[176] K. K. Kozlowski, Aspects of the inverse problem for the Toda chain, arXiv:1307.4052.

[177] F. Smirnov, Quasi-classical Study of Form Factors in Finite Volume, arXiv:hep-th/9802132.

[178] G. Niccoli, Nucl. Phys. B 870: (2013) 397.

[179] G. Niccoli, J. Math. Phys. 54 (2013) 053516.

[180] G. Niccoli, J. Stat. Mech. (2012) P10025.
[181] S. Faldella, N. Kitanine, G. Niccoli, *Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms*, arXiv:1307.3960.

[182] S. Faldella, G. Niccoli, *SOV approach for integrable quantum models associated to general representations on spin-1/2 chains of the 8-vertex reflection algebra*, arXiv:1307.5533.

[183] G. Niccoli, *J. Phys. A: Math. Theor.* **46** (2013) 075003.