OPTIMAL REPLENISHMENT, PRICING AND PRESERVATION TECHNOLOGY INVESTMENT POLICIES FOR NON-INSTANTANEOUS DETERIORATING ITEMS UNDER TWO-LEVEL TRADE CREDIT POLICY

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(Communicated by Aviv Gibali)

Abstract. In the business world, both the supplier and the retailer accept the credit to make their business position strong, because the credit not only strengthens their business relationships but also increases the scale of their profits. In this paper, we consider an inventory model for non-instantaneous deteriorating items with price sensitive demand, time varying deterioration rate under two-level trade credit policy. Besides, to reduce deterioration rate, retailers invest some cost to prevent product degradation/decay, known as preservation technology, is also inserted. Consumption of such items within shelf life prevents to deterioration, which can be achieved by bulk sale. In order to stimulate the selling, trade-credit policy is also considered here. In the sequel, not only the supplier would offer fixed credit period to the retailer, but retailer also adopt the trade credit policy to the customers in order to promote the market competition. The retailer can accumulate revenue and interest after the customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. The main objective is to determine the optimal replenishment, pricing and preservation technology investment strategies including whether or not invest in preservation technology and how much to invest in order to maximize the average profit of the system. It is proved that the optimal replenishment policy not only exists but is unique for any given selling price and preservation technology cost. An algorithm is presented to derive the optimal solutions of the model. Numerous theorems and lemmas have been inserted to obtain the optimal solution. Finally, numerical examples and managerial implications are incorporated to validate the proposed model.

1. Introduction. A basic assumption of classical EOQ model is that products are non-perishable and have an indefinite useful life. But in the real world, many products in inventory are subject to some risks such as decay, dryness, damage, evaporation, pilferage, and obsolescence, etc. Products like blood banks, chemicals, pharmaceuticals, food products, and volatile liquids deteriorate quickly over time. So, these products cannot serve its original purpose after a period of time and disposed of. Therefore, the impact of loss from deterioration cannot be ignored.

2020 Mathematics Subject Classification. Primary: 90B05; Secondary: 90C26.

Key words and phrases. Inventory, non-instantaneously deteriorating item, trade credit, pricing, preservation technology.

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The management of deteriorating inventory items has been a subject of interest since the publication of the seminal model by Ghare and Schrader [9]. Thereafter, a great deal of research efforts have been devoted to inventory models of deteriorating items, the details can be found in the review articles by Raafat [35] and Goyal and Giri [10]. They believed that deteriorations start instantaneously as soon as retailer receives items from supplier. But there are many products items which maintain their freshness for a short period. Wu et al. [48] defined this phenomenon as “non-instantaneous deterioration (NID)”. In real world, this type of phenomenon exists commonly such as firsthand vegetables, and fruits remain fresh for a short span of time, in which there is almost no spoilage. Afterwards, some of the items will start to decay. Researchers like Chang et al. [3], Maihami and Kamalabadi [29], Shah et al. [37], Maihami and Karimi [30], Jaggi et al. [15, 16], Shaikh et al. [39], Bhunia et al. [1] and Mishra et al. [31] have worked along these directions.

Raised demands of customers for top quality food items, fruits, vegetable, etc. merchandiser has magnified the interest in the development of latest food processing technologies over the past decade. In the food products process merchandiser has to face loss due to deterioration or spoilage of food items. To prevent this, preservation technology is the process designed to safeguard products from spoilage caused by microbes, enzymes, and autoxidation. Loss due to deterioration cannot be ignored by the companies and firms. They can invest in money in the storage equipment say refrigerators, freezing, cooling, salting, pickling, dehydrating, etc. to maintain the freshness of the products. However, this investment will increase the supplier’s total cost but simultaneously increased the life period of products, which raise the profit of the suppliers. The first inventory model with preservation technology introduced by Hsu et al. [12] with constant demand rate and exponential decay where the retailer is allowed to invest the preservation technology to reduce the deterioration rate. Later, some researcher Dye and Hsieh [7], Dye et al. [8], Kamma et al. [18], Priyamvada et al. [34], and Shastri et al. [40] has worked on the inventory model with preservation technology investment in deteriorating items.

Moreover, trade credit is a widespread common practice in today’s competitive business, confined to the main source of short-term financing. Recent studies indicate that 90% of buyers from the United States prefer a short-term, interest-free credit payment, that is, buy now and pay later. Normally, there are no interest charges if a buyer pays in full within the credit period. However, if the buyer fails to do so, the seller can charge interest on the outstanding balance. There is a vast amount of literature for inventory models with credit payments. The first review of inventory models under trade credits was given by Chang et al. [4]. A second, more recent review was provided by Seifert et al. [36]. To examine effects of trade credit toward the retailer’s optimal order policy, upstream or downstream credit period has been incorporated under various settings, such as price-dependent iso-elastic demand (Mahata et al. [28]), deteriorating imperfect quality items with selling price dependent demand and shortage backordering (Khanna et al. [20]), partial backordering (Jaggi et al. [14], Lashgari et al. [21]), in-transit deterioration (Feng et al. [23]), trade credit linked to order quantity (Tiwari et al. [46]), partial trade credit (Mahata [24]). Further, it should be noted that plenty of researches have been conducted ordering policy under two-level trade credit (Huang [13], Teng and Goyal [45], Mahata and Goswami [25], Liao [22], Teng [43], Mukherjee and Mahata [32]). Adopting discounted cash-flow analysis, Mahata and Mahata [26] built a production and payment policies for an imperfect manufacturing system with
two-level trade policy analysis in fuzzy random environments. Under trapezoidal-type demand, Jiang et al. [47] addressed inventory policies for deteriorating items with maximum lifetime and two-level trade credit. Under a two-level partial trade credit, Mahata et al. [27] focused on a supplier-retailer-customer supply chain for deteriorating items with expiration dates.

Most researchers studied inventory models for non-instantaneous deteriorating items (NIDIs) under one level trade credit policy (Ouyang et al. [33], Soni and Patel [42] and Soni [41], Shaikh et al. [38]) but in global practice it is unrealistic. Nevertheless, most of them failing to identify the effects of two-level trade credit on the retailer’s response when involving both quantity losses and preservation technology.

In the retail business, it is observed that the demand of an item is dependent on different factors. One such factor is the selling price of an item. Selling price of an item plays an important role in the demand of the product. In this context, Datta and Paul [5] proposed a finite time horizon inventory model with the variable demand dependent on price and stock of the product. Teng and Chang [44] solved a production inventory model under the price and stock dependent demand. Hou and Lin [11] proposed an inventory model with the demand dependent on selling price and stock of the product. Dye and Hsieh [6] solved an inventory model with price dependent demand under a shortage. Jaggi et al. [17] proposed a non-instantaneous inventory model with trade credit. Khan et al. [19] proposed deteriorating item with expiry date in an inventory model with pricing decision. Finally, we summarize the major assumptions and objectives used in the above research articles in Table 1 to make it easier for the readers to understand the contributions of our model.

Nevertheless, the use of above-mentioned models are very limited in virtuality, while some products such as food items, electronic components, and fashionable goods do not deteriorate instantaneously, though after a time period; those items have non-instantaneous deterioration. Pricing is always an important strategic instrument used to adjust consumer demand. We consider a price-dependent demand in the model. In this paper, we emphasize the importance of paying attention to trade credit and preservation technology investment when making lot sizing decisions. We integrate the effects of price dependent demand, NIDIs, preservation technology investment and two-level trade credit policy into the traditional deteriorating inventory model. The goal of this paper is to investigate a retailer’s optimal pricing, replenishment and preservation investment decisions, when he operates a NID inventory system under two-level trade credit policy. We assume a time-varying deterioration rate. The retailer invests in preservation technology to slower down the deterioration and prolongs the length of non-deterioration period. In addition, we discuss the combined problem where the replenishment policy, retail price and preservation technology cost are decision variables. Moreover, some useful theorems, and lemmas are developed for finding the optimal replenishment policy, retail price, and preservation technology strategies. The sensitivity analysis to understand how they depend on cost parameters and other parameters are discussed.

2. Assumptions and notations. We study an inventory system with a single deteriorating product and an infinite planning time horizon.

Notation.

$t_0$ Original non-deterioration period without preservation technology investment
| Authors                  | Demand factors | Demand patterns | Deterioration Pattern | Non-instantaneous | Trade Credit | Level of trade credit | Preservation technology |
|-------------------------|----------------|----------------|-----------------------|-------------------|--------------|------------------------|------------------------|
| Wu et al. [48]          | Inventory level | Linear         | Yes                   | Constant          | Yes          | No                     | No                     |
| Chang et al. [3]        | Constant       | -              | Yes                   | Constant          | Yes          | Yes                    | One                    | No                     |
| Hsu et al. [12]         | Constant       | Power form     | Yes                   | Constant          | No           | No                     | Yes                    | No                     |
| Shastri et al. [40]     | Selling price  | Power form     | Yes                   | Constant          | No           | Yes                    | Two                    | Yes                    |
| Dye & Hsieh [7]         | Constant       | -              | Yes                   | Constant          | No           | No                     | Yes                    | No                     |
| Mahata et al. [28]      | Selling price  | Iso-elastic     | Yes                   | Constant          | No           | Yes                    | One                    | No                     |
| Molkherjee & Mahata et al. [29] | Selling price | General        | Yes                   | Constant          | Yes          | No                     | -                      | No                     |
| Mukherjee & Mahata et al. [32] | Time & Credit period | General        | Yes                   | General type      | No           | Yes                    | Two                    | No                     |
| Jaggi et al. [17]       | Selling price  | Power form     | Yes                   | Constant          | Yes          | Yes                    | One                    | No                     |
| Soni [41]               | Selling price  | General        | Yes                   | Constant          | No           | Yes                    | One                    | No                     |
| Shah et al. [37]        | Advertisement of an item & selling price | power form | Yes                   | General type      | Yes          | No                     | -                      | Yes                    |
| Mishra et al. [31]      | Selling price  | Linear         | Yes                   | Constant          | No           | Yes                    | One                    | Yes                    |
| Yang et al. [49]        | Time & Credit period | General        | Yes                   | General type      | No           | Yes                    | One                    | Yes                    |
| Present paper           | Selling price  | General        | Yes                   | General type      | Yes          | Yes                    | Two                    | Yes                    |

$\xi$ Preservation technology investment per unit time (a decision variable).

$t_d(\xi)$ Non-deterioration period with preservation technology investment ($t_d(\xi) = t_0$)

$m(\xi)$ Proportion of reduced deterioration rate with preservation technology investment

$\theta(t)$ Deterioration rate of the on-hand inventory over $[t_d(\xi), T]$

$A$ Ordering cost per order.

$c$ Purchasing cost per unit

$p$ Retail price per unit (a decision variable)

$h(t)$ Unit holding cost per unit time at time $t$

$Q$ Ordering quantity per replenishment cycle

$T$ Length of replenishment cycle

$I(t)$ Inventory level at time $t$
\( Z(T, p, \xi) \) Average total profit per unit time of inventory system

**Assumptions.**

1. Based on traditional marketing and economic theory, the higher the price, the lower the demand. Hence, the selling price is a critical factor affecting demand. Generally, there are three simple ways to quantify the demand function of the retail price \( p \) (a decision variable): (a) a linear form such as \( D(p) = a - bp \), (b) a constant price elasticity such as \( D(p) = ap^{-b} \), and (c) an exponential pattern such as \( D(p) = ae^{-bp} \). For generality, we assume that the demand function is continuous, \( D(p) > 0 \) and the gross revenue, \( pD(p) \), is a strictly concave function of \( p \) (i.e., If \( 2D'(p) + pD''(p) < 0 \) or diminishing marginal revenue). This condition is common to many price dependent demand functions and similar to the condition of profit maximization with respect to the price \( p \) found in Dye et al. [16]. Also, note that if the gross revenue is increasing function of price \( p \), then price and gross revenue will always move in the same direction, hence retailer can realize infinite gross revenue by setting an infinite price. It is impossible. We also assume that \( c < p \leq \overline{p} \) and \( \lim_{p \to \overline{p}} D(p) = 0 \).

2. Preservation technology can lengthen the original non-deterioration period to \( t_d(\xi) \) (\( t_d(\xi) \geq t_0 \)). \( t_d(\xi) \) is continuous, \( t'_d(\xi) > 0 \), \( t''_d(\xi) \leq 0 \) with \( t_d(0) = t_0 \). For convenience, \( t_d(\xi) \) and \( t_d \) will be used interchangeably.

3. During the time interval \([0, t_d(\xi)]\) the product has no deterioration. After that the on-hand inventory items deteriorate at a time-varying of deterioration \( \theta(t) \), where \( 0 < \theta(t) < 1 \). \( \theta(t) \) is continuous, \( \theta'(. \geq 0 \) and \( \theta''(.) \geq 0 \) with \( \theta(0) = 0 \) and \( \theta'(0) = 0 \). Besides, there is no repair or replacement of deteriorated units during the inventory cycle.

4. The proportion of reduced deterioration rate by \( m(\xi) \) percent and lengthen the original non-deteriorating period to \( t_d \) (\( t_d \geq t_0 \)). \( m(\xi) \) is a continuous, concave, increasing function of retailer’s capital investment \( \xi \), where \( m(0) = 0 \) and \( \lim_{x \to \infty} m(\xi) = 1 \). Note that \( m'(\xi) > 0 \) so as to make the retailer lean to invest in it, and \( m''(\xi) < 0 \) to ensure diminishing return from capital investment in preservation.

5. \( M \) is the permissible delay period in payment for the retailer offered by the supplier (upstream credit). During the period, the retailer can use sales revenue to earn the interest with an annual rate \( I_p \) up to the end of \( M \). At time \( t = M \), the credit is settled and the retailer has to pay the interest at the rate of \( I_c \) for the items in stock. \( N \) is the permissible delay period in payment for the customer offered by the retailer (downstream credit). During the period, the retailer has to bear the opportunity cost of the revenue which is not settled in time \( N \) at the rate of \( I_p \).

6. Replenishment rate is infinite, shortages are not allowed and the time horizon of the inventory system is infinite.

7. \( TP_1(p, T, \xi) \) is the total average profit which consists of (a) sales profit, (b) the cost of ordering (OC), (c) cost of holding inventory (HC) (excluding interest charges), (d) cost of deterioration (DC), (e) capital opportunity cost (IC), (f) interest earned from sales revenue (IE), \( i = 1, 2, 3 \), where \( i = 1 \) indicates \( M \leq N \), \( i = 2 \) indicates \( N \leq M \leq N + t_d \), and \( i = 3 \) indicates \( M \geq N + t_d \).
3. **Mathematical model.** The inventory system evolves as follows: \( Q \) units of items arrive at the inventory system at the beginning of each cycle. Based on values of \( T \) and \( t_d(\xi) \), two cases viz. \( T \leq t_d \) and \( T \geq t_d \) arise which are shown in Fig. 1. As shown in Fig. 1, each replenishment cycle starts by arriving items at the inventory system. Then the inventory level declines owing to demand and deterioration. Finally the inventory level reaches zero and the replenishment cycle terminates. Then the items are reordered for the next cycle and the next cycle commences. Each case is discussed in details as follows:

\[
\begin{align*}
\text{Case 1: } T & \leq t_d \\
\text{Case 2: } T & \geq t_d
\end{align*}
\]

**Figure 1. Inventory Level**

When \( T \leq t_d \), there is no deterioration in a single cycle. The inventory system is depicted in Fig. 1. The inventory level decreases owing to the constant demand rate during the whole cycle. It is given that

\[
I(t) = Q - D(p)t, \quad T \leq t_d
\]  

(1)

When \( T \geq t_d \), the inventory-holding period is greater than or equal to the non-deterioration period. As shown in Fig. 1(a), during time interval \([0, t_d]\), the inventory system exhibits no deterioration and the inventory level decreases owing to demand only. After time \( t = t_d \), the inventory starts to deteriorate and be disposed at a rate of \([1 - m(\xi)] \theta(t - t_d)\) for a given preservation investment \( \xi \). Subsequently the inventory level declines due to demand and deterioration during time interval \([t_d, T]\). Finally the replenishment cycle terminates as the inventory level reaches zero, and a new replenishment cycle starts. Let \( I_1(t) \) and \( I_2(t) \) denote the inventory levels at time \( t \in [0, t_d] \) and \( t \in [t_d, T] \), respectively. Based on this description, during time interval \([0, T]\), the changes of the inventory level per time unit is represented by the following differential equations:

\[
\begin{align*}
\frac{dI_1(t)}{dt} &= -D(p), \quad 0 \leq t \leq t_d(\xi) \\
\frac{dI_2(t)}{dt} &= -(1 - m(\xi)) \theta(t - t_d) I_2(t) - D(p), \quad t_d \leq t \leq T
\end{align*}
\]  

(2)

With boundary condition \( I_1(0) = Q \) and \( I_2(T) = 0 \). The solutions of Eqs. (2) are, respectively

\[
I_1(t) = Q - D(p)t, \quad 0 \leq t \leq t_d,
\]  

(3)
where, \( g(t) = \int_t^{T} \theta(u - t_d)du \).

The inventory level during the entire cycle time \([0, T]\) is given by,

\[
I(t) = \begin{cases} 
I_1(t) &= Q - D(p) t, \quad t \in [0, t_d] \\
I_2(t) &= D(p) e^{-(1-m(\xi))g(t)} \int_{t}^{T} e^{(1-m(\xi))g(u)}du, \quad t \in [t_d, T].
\end{cases}
\]

From the continuity of \(I_1(t)\) and \(I_2(t)\) at time \(t = t_d\), it follows from (3) and (4) that

\[
Q = D(p) \left( t_d + \int_{t_d}^{T} e^{(1-m(\xi))g(u)}du \right).
\]

Therefore equation (5) becomes,

\[
I(t) = \begin{cases} 
I_1(t) &= D(p) \left( t_d - t + \int_{t_d}^{T} e^{(1-m(\xi))g(u)}du \right), \quad t \in [0, t_d] \\
I_2(t) &= D(p) e^{-(1-m(\xi))g(t)} \int_{t}^{T} e^{(1-m(\xi))g(u)}du, \quad t \in [t_d, T].
\end{cases}
\]

The total annual relevant cost consists of the following five parts:

(a) Sales profit (SP):

\[
SP = (p - c) D(p)
\]

(b) Ordering cost per year (OC):

\[
OC = \frac{A}{T}
\]

(c) Cost of holding inventory (HC): There are two possible situations based on the value of \(T\) and \(t_d\). When \(T \leq t_d\), the inventory system is the first type shown in Fig. 1(a). When \(T \geq t_d\), the inventory system is the second type depicted in Fig. 1(b). Consequently, the inventory holding cost is given by

\[
HC = \begin{cases} 
\frac{h}{T} D(p) \left( \int_{0}^{T} I_1(t) dt \right), \quad T \leq t_d \\
\frac{h}{T} \left( \int_{0}^{t_d(\xi)} I_1(t) dt + \int_{t_d(\xi)}^{T} I_2(t) dt \right), \quad T \geq t_d
\end{cases}
\]

\[
= \begin{cases} 
\frac{hD(p)T}{2}, \quad 0 \leq t \leq t_d \\
\frac{hD(p)}{T} \left( \frac{t_d^2}{2} + t_d \int_{t_d(\xi)}^{T} e^{(1-m(\xi))g(u)}du + \int_{t_d(\xi)}^{T} \int_{t_d}^{T} e^{(1-m(\xi))g(u)}dudt \right), \quad t_d \leq t \leq T
\end{cases}
\]

(d) Cost of deterioration items (DC): For \(T \leq t_d\), there is no deterioration. For \(T \geq t_d(\xi)\), the cost of deteriorated items is \(c(Q - DT)/T\). So, the deterioration cost is given by

\[
DC = \begin{cases} 
0, \quad T \leq t_d \\
\frac{c(Q - DT)}{T}, \quad T \geq t_d
\end{cases}
\]

(e) When \(T \leq t_d(\xi)\), there is no deterioration in in-stock period and so \(\xi = 0\). When \(T \geq t_d(\xi)\), the preservation technology cost is \(\xi T\).

(f) Opportunity cost (IC) and interest earned from sales revenue (IE): In order to establish the total relevant inventory cost function, we consider three cases: Case 1. \(M \leq N\); Case 2. \(N \leq M \leq N + t_d\); and Case 3. \(M \geq N + t_d\).
Figure 2. Inventory system for Case 1 when $T \leq t_d$

Case 1 ($M \leq N$). In this case, there are two circumstances: $T \leq t_d$ and $T \geq t_d$. And when $M \leq N$, there is no interest earned by the retailer.

(1) $T \leq t_d$. The inventory system is depicted in Fig. 2. The retailer has the opportunity cost and has no interest earned. The opportunity cost is calculated as

$$IC_{11} = \frac{cI_c}{T} \left( (N - M)Q + \frac{D(p)T^2}{2} \right).$$

(12)

The total average profit function is

$$TP_{11}(p,T,\xi) = (p - c)D(p) - \frac{A}{T} - \xi - \frac{1}{2}(h + cI_c)D(p)T - cI_c(N - M)D(p).$$

(13)

(2) $T \geq t_d$. The inventory system is depicted in Fig. 3.

Figure 3. Inventory system for Case 1 when $T \leq t_d$

The retailer has the opportunity cost and has no interest earned. The opportunity cost is calculated as

$$IC_{12} = \frac{cI_c}{T} \left( (N - M)Q + \int_0^{t_d} I_1(t)dt + \int_{t_d}^{T} I_2(t)dt \right)$$
The total average profit function is

$$TP_{12}(p, T, \xi) = pD(p) - \frac{cD(p) t_d}{T} - \frac{1}{2} (h + cI_c) D(p) T^2 - cI_c (N - M) D(p) t_d - \xi$$

$$- \frac{A}{T} - \frac{(c + (h + cI_c) t_d + cI_c (N - M)) D(p)}{T} \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt$$

$$- \frac{(h + cI_c) D(p)}{T} \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u)-g(t))} dudt$$

(15)

The problem of Case 1 is to maximize the function

$$TP_1(p, T, \xi) = \begin{cases} T P_{11}(p, T, \xi), & T \leq t_d \\ T P_{12}(p, T, \xi), & T \geq t_d \end{cases}$$

(16)

**Case 2.** $N \leq M \leq N + t_d$ In this case, there are three circumstances: $T \leq M - N$, $M - N \leq T \leq t_d$, and $T \geq t_d$.

1. $T \leq M - N$. The inventory system is depicted in Fig. 4.

The retailer has no opportunity cost and only has the interest earned. The interest earned per cycle is calculated as

$$IE_{21} = \frac{pI_p}{T} \left( \frac{D(p)T^2}{2} + D(p)T(M - T - N) \right)$$

(17)
\[ TP_{21}(p, T, \xi) = (p - c)D(p) - \frac{A}{T} - \xi - \frac{hD(p)T}{2} + \frac{pI_p}{T} \left( \frac{D(p)T^2}{2} + D(p)T(M - T - N) \right) \]  

(18)

\[ (3) \ T \geq t_d. \]  

The opportunity cost per cycle is calculated as

\[ IC_{23} = \frac{cI_c}{T} \left( \int_{t_d(\xi)}^{t_d(N)} I_1(t)dt + \int_{t_d(\xi)}^{T} I_2(t)dt \right) \]

\[ = \frac{cI_cD(p)}{T} \left[ \frac{p^2}{2} - (M - N)t_d + \frac{(M - N)^2}{2} + (t_d - M + N) \int_{t_d}^{T} e^{(1-m(\xi))g(u)}du \right. \]

\[ + \left. \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u)-g(t))}dudt \right] . \]  

(22)

The interest earned per cycle is calculated as,

\[ IE_{23} = \frac{pI_p}{T} \frac{D(p)(M - N)^2}{2} . \]  

(23)
The total average profit function is

\[
TP_{23}(p, T, \xi) = pD(p) - \frac{cD(p)T}{2} - \frac{1}{2} \frac{(h + cI_c)D(p)T^2}{T} - \frac{(c + (h + cI_c)T + cI_c(N - M))D(p)}{2T}
\]

\[
+ cI_cD(p)(M - N)(2td + N - M)
\]

\[
- \frac{(h + cI_c)D(p)}{T}
\]

\[
\times \int_{td}^{T} \int_{td}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt - \xi - \frac{A}{T} + \frac{pI_p}{T} \frac{D(p)(M - N)^2}{2} \quad (24)
\]

The problem of Case 2 is to maximize the function

\[
TP_2(p, T, \xi) = \begin{cases} 
TP_{21}(p, T, \xi), & T \leq M - N \\
TP_{22}(p, T, \xi), & M - N \leq T \leq td \\
TP_{23}(p, T, \xi), & T \geq td(\xi)
\end{cases} \quad (25)
\]
Case 3. $M \geq N + t_d$. In this case, there are three circumstances: $T \leq t_d$, $t_d \leq T \leq M - N$, and $T \geq M - N$.

1. $T \leq t_d$. The inventory system is depicted in Fig. 7.
   There is no opportunity cost under this circumstance. The interest earned per cycle is calculated as,
   \[
   IE_{31} = \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + T(M - T - N) \right). \tag{26}
   \]
   The total average profit function is
   \[
   TP_{31}(p, T, \xi) = (p - c) D(p) - \frac{A}{T} - \xi - \frac{hD(p)T}{2} + \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + T(M - T - N) \right). \tag{27}
   \]

2. $t_d \leq T \leq M - N$. The inventory system is depicted in Fig. 8.
   \[
   IE_{32} = \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + T(M - T - N) \right). \tag{28}
   \]
   The total average profit function is
   \[
   TP_{32}(p, T, \xi) = SP - (OC + HC + DC - IE_{32})
   = pD(p) - \frac{cD(p) t_d}{T} - \frac{hD(p) t_d^2}{2T} - \frac{1}{T} (c + h t_d) D(p) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt - \xi - \frac{A}{T}
   + \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + (M - T - N) \right) - \frac{h}{T} D(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} dudt. \tag{29}
   \]

3. $T \geq M - N$. The inventory system is depicted in Fig. 9.
   The opportunity cost per cycle is calculated as,
   \[
   IC_{33} = \frac{cI_e}{T} \int_{M - N}^{T} I_2(t) dt = \frac{cI_eD(p)}{T} \int_{M - N}^{T} \int_{t}^{T} e^{(1-m)(g(u) - g(t))} dudt. \tag{30}
   \]

**Figure 8. Inventory system for Case 3 when $t_d \leq T \leq M - N$**

There is no opportunity cost under this circumstance. The interest earned per cycle is calculated as,
\[
IE_{32} = \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + T(M - T - N) \right). \tag{28}
\]

The total average profit function is
\[
TP_{32}(p, T, \xi) = SP - (OC + HC + DC - IE_{32})
= pD(p) - \frac{cD(p) t_d}{T} - \frac{hD(p) t_d^2}{2T} - \frac{1}{T} (c + h t_d) D(p) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt - \xi - \frac{A}{T}
+ \frac{pD(p)I_p}{T} \left( \frac{T^2}{2} + (M - T - N) \right) - \frac{h}{T} D(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} dudt. \tag{29}
\]
The interest earned per cycle is calculated as,

\[ IE_{33} = \frac{pL_p D(p)(M - N)^2}{T}. \] (31)

The total average profit function is

\[ TP_{33} (p, T, \xi) = SP - (OC + HC + DC + IC_{33} - IE_{33}) \]

\[ = pD(p) - \frac{cD(p)t_d}{T} - \frac{1}{2} \frac{hD(p)t_d^2}{T} - \frac{(c + h\ell_d) D(p)}{T} \int_{t_d}^{T} e^{(1-m(\xi))g(u)} du - \xi - \frac{A}{T} \]

\[- \frac{hD(p)}{T} \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt - \frac{cI_p D(p)}{T} \int_{M-N}^{T} \int_{T}^{T} e^{(1-m)(g(u) - g(t))} du dt \]

\[ + \frac{pL_p D(p)(M - N)^2}{2T}. \] (32)

**Figure 9.** Inventory system for Case 3 when \( T \leq M - N \)

The problem of Case 3 is to maximize the following function:

\[ TP_3 (p, T, \xi) = \begin{cases} 
TP_{31} (p, T, \xi), & T \leq t_d (\xi) \\
TP_{32} (p, T, \xi), & t_d (\xi) \leq T \leq M - N \\
TP_{33} (p, T, \xi), & T \geq M - N 
\end{cases} \] (33)

In next section, we have developed solution methodology along with theoretical results to identify global optimal solution for \((p, T, \xi)\) to maximize \(TP (p, T, \xi)\).

4. **Solution procedure.** In order to determine the optimal selling price \( p \), preservation technology investment \( \xi \), and cycle time \( T \) which maximize the total profit per unit time \( TP (p, T, \xi) \), we first find the optimal solutions for every case, respectively. Due to the complexity of the problem, it is very difficult to prove analytically that \( TP (p, T, \xi) \) is jointly concave in \((p, T, \xi)\). In this section, we will explore the mathematical properties of the model to assist in proposing an algorithm to search for the optimal solution. Proofs of the results in the paper are all relegated to the Appendix.

**Case 1.** \((M \leq N)\) The problem is to maximize function (16). It can be calculated that \( TP_{11} (p, T, \xi) = TP_{12} (p, T, \xi) \) at \( T = t_d \). So function (16) is continuous at point \( T = t_d \).
\(1.1\) \(T \leq t_d\)

Taking the first- and second-order partial derivatives of \(TP_{11}(p, T)\) with respect to \(p\) and \(T\), and simplifying terms, we get:

\[
\frac{\partial TP_{11}(p, T, \xi)}{\partial p} = D(p) + (p - c) D'(p) - \frac{(h + cI_e) D'(p) T}{2}
\]

\[
-cI_e (N - M) D'(p),
\]

\[
\frac{\partial^2 TP_{11}(p, T, \xi)}{\partial p^2} = \left(2D'(p) + pD''(p)\right)
\]

\[
-D''(p) \left\{c + \frac{(h + cI_e) T}{2} + cI_e(N - M)\right\},
\]

\[
\frac{\partial TP_{11}(p, T, \xi)}{\partial T} = \frac{A}{T^2} - \frac{(h + cI_e) D'(p)}{2},
\]

\[
\frac{\partial^2 TP_{11}(p, T, \xi)}{\partial T^2} = \frac{-2A}{T^3} < 0,
\]

Consequently, \(TP_{11}(T, \xi|p)\) is a concave function of \(T\). Thus, there exists a unique value of \(T\) (say \(T_{11}\)) which maximize \(TP_{11}(T|p)\) as

\[
T_{11} = \sqrt{\frac{2A}{(h + cI_e) D(p)}}.
\]

To ensure \(T \leq t_d\), we substitute (38) into inequality \(T \leq t_d\) and we obtain

\[
0 < 2A \leq (h + cI_e) D(p)t_d^2.
\]

Also,

\[
\frac{\partial^2 TP_{11}(p, T, \xi)}{\partial p\partial T} = -\frac{(h + cI_e) D'(p)}{2}
\]

We may assume without loss of generality that \(\frac{\partial^2 TP_{11}(p, T, \xi)}{\partial p^2} < 0\), if \(2D'(p) + pD''(p) < 0\). As a result, we have following theoretical results.

**Theorem 1.** If the gross revenue, \(pD(p)\), is a strictly concave function of \(p\) (i.e., If \(2D'(p) + pD''(p) < 0\) or diminishing marginal revenue), the determinant of the Hessian Matrix is positive, then the annual total profit function, \(TP_{11}(p, T, \xi)\), is joint concave function in \(p\) and \(T\).

**Proof.** See Appendix A.

\(1.2\) \(T \geq t_d\)

For any given \(T\), the necessary conditions for the optimality of the annual total profit function \(TP_{12}(p, T, \xi)\) are \(\frac{\partial TP_{12}(p, T, \xi)}{\partial p} = 0\) and \(\frac{\partial TP_{12}(p, T, \xi)}{\partial \xi} = 0\). That is,

\[
pD'(p) + D(p) = \frac{cD'(p)t_d}{T} - \frac{(cI_e + h) D'(p) t_d^2}{2T} - \frac{cI_e (N - M) D'(p) t_d}{T}
\]

\[
-\frac{(c + (cI_e + h) t_d + cI_e (N - M)) D'(p)}{T} \int_{t_d}^T e^{(1 - m(\xi)g(t))} dt
\]

\[
-\frac{(cI_e + h) D'(p)}{T} \int_{t_d}^T \int_t^T e^{(1 - m(\xi))g(u) - g(t))} dudt = 0,
\]

(41)
Theorem 2

We have following theoretical results.

Proof. See Appendix B.

Lemma 1. For given \( p \) and \( \xi \), when \( M \leq N \),

1. If \( 2A \geq (h + cL_c) D(p) t_d^2 \), then the solution of \( T \in [t_d, \infty) \) (say \( T_{12} \)) in (43) not only exists but also is unique.
2. If \( 0 < 2A < (h + cL_c) D(p) t_d^2 \), then the solution of \( T \in [t_d, \infty) \) in (43) does not exist.
Lemma 2. For given $p$ and $\xi$, when $M \leq N$,
1. If $2A \geq (h + cI_c)D(p)t_3^2$, then the annual total average profit $TP_{12}(T|p,\xi)$ has the global maximum value at $T = T_{12}$, where $T_{12} \in [t_4, \infty)$ and satisfies (43).
2. If $0 < 2A < (h + cI_c)D(p)t_3^2$, then the annual total relevant profit $TP_{12}(T|p,\xi)$ has maximum value at the boundary point $T = t_4$.

Proof. See Appendix D.

For notational convenience, we denote $\Delta_1 = (h + cI_c)D(p)t_3^2$. Combining the above mentioned inequality (39), Lemmas 1 and 2 and the assumption $M \leq N$, we can obtain the following theorem

Theorem 3. For given $p$ and $\xi$, when $M \leq N$,
1. If $2A < \Delta_1$, then $TP_1(T^*|p) = TP_{11}(T_{11}|p)$ and $T^* = T_{11}$.
2. If $2A \geq \Delta_1$, then $TP_1(T^*|p) = \max (TP_{12}(T_{12}|p), TP_{12}(t_4|p))$. Hence $T^*$ is $T_{12}$ or $t_4$ associated with lower total average profit.

Case 2. $N \leq M \leq N + t_4$

(2.1) $T \leq M - N$.

The problem is to maximize function (25). For $T \leq M - N$, taking the first-and second-order partial derivatives of $TP_{21}(p,T,\xi)$ with respect to $p$ and $T$, and simplifying terms, we get:

$$
\frac{\partial TP_{21}(p,T,\xi)}{\partial p} = (p - c)D'(p) + D(p) - \frac{hD'(p)T}{2}
+ I_p \left(D(p) + pD'(p)\right) \left(M - \frac{T}{2} - N\right),
$$

(44)

$$
\frac{\partial^2 TP_{21}(p,T,\xi)}{\partial p^2} = \left(2D'(p) + pD''(p)\right) - D''(p) \left(c + \frac{hT}{2}\right)
+ I_p \left(2D'(p) + pD''(p)\right) \left(M - \frac{T}{2} - N\right),
$$

(45)

$$
\frac{\partial TP_{21}(p,T,\xi)}{\partial T} = \frac{A}{T^2} - \frac{hD}{2} - \frac{pI_pD}{2},
$$

(46)

$$
\frac{\partial^2 TP_{21}(p,T,\xi)}{\partial T^2} = \frac{-2A}{T^3} < 0.
$$

(47)

Consequently, $TP_{21}(T,\xi|p)$ is a concave function of $T$. Thus, there exists a unique value of $T$ (say $T_{21}$) which maximize $TP_{21}(T,\xi|p)$ as

$$
T_{21} = \frac{2A}{(h + pI_p)D(p)}.
$$

(48)

To ensure $T \leq M - N$, we substitute (58) into inequality $T \leq M - N$ and we obtain

$$
0 < 2A \leq (h + pI_p)D(p)(M - N)^2.
$$

(49)

Also,

$$
\frac{\partial^2 TP_{21}(p,T,\xi)}{\partial p \partial T} = -\frac{(h + pI_p)D'(p)}{2} - \frac{I_pD(p)}{2}.
$$

(50)
We may assume without loss of generality that \( \frac{\partial^2 TP_21(p,T,\xi)}{\partial p^2} < 0 \), if \( 2D'(p) + pD''(p) < 0 \). As a result, we have following theoretical results.

**Theorem 4.** If the gross revenue, \( pD(p) \), is a strictly concave function of \( p \) (i.e., If \( 2D'(p) + pD''(p) < 0 \) or diminishing marginal revenue), the determinant of the Hessian Matrix is positive, then the annual total profit function, \( TP_21(p,T,\xi) \), is joint concave function in \( p \) and \( T \).

**Proof.** Same as Theorem 1.

(2.2) \( M - N \leq T \leq t_d \).

Taking the first and second order partial derivatives of \( TP_{22}(p,T,\xi) \) with respect to \( p \), and \( T \) and simplifying terms, we get:

\[
\frac{\partial TP_{22}(p,T,\xi)}{\partial p} = (p - c) D'(p) + D(p) - \frac{hD'(p)}{2} - \frac{21}{2T} + \left( \frac{D(p) + pD'(p)}{2T} \right) I_p(M - N)^2,
\]

\[
\frac{\partial^2 TP_{22}(p,T,\xi)}{\partial p^2} = \frac{2D'(p) + pD''(p)}{2T} - \frac{cI_c D'(p)}{2T} \left( T + N - M \right)^2 + \frac{2D'(p) + pD''(p)}{2T} I_p(M - N)^2,
\]

\[
\frac{\partial TP_{22}(p,T,\xi)}{\partial T} = \frac{A}{2T^2} - \frac{hD(p)}{2} - \frac{cI_c D(p)}{2T^2} \left( T^2 - (N - M)^2 \right) - \frac{pI_p D(p)(M - N)^2}{2T^2},
\]

and

\[
\frac{\partial^2 TP_{22}(p,T,\xi)}{\partial T^2} = \frac{2A}{T^3} - \frac{(cI_c - pI_p) D(p)(M - N)^2}{T^3} < 0.
\]

Consequently, \( TP_{22}(T|p,\xi) \) is a concave function of \( T \). Thus, there exists a unique value of \( T \) (say \( T_{22} \)) which maximize \( TP_{22}(T|p,\xi) \) as

\[
T_{22} = \sqrt{\frac{2A + (cI_c - pI_p)(M - N)^2 D(p)}{(h + cI_c) D(p)}}.
\]

To ensure \( M - N < T \leq t_d(\xi) \), we substitute (55) into inequality \( M - N < T \leq t_d(\xi) \) and we obtain

\[
(h + cI_c) D(p) (M - N)^2 < 2A \leq (h + cI_c) D(p) t_d(\xi)^2 - (cI_c - pI_p)(M - N)^2 D(p).
\]

We may assume without loss of generality that \( \frac{\partial^2 TP_{22}(p,T,\xi)}{\partial p^2} < 0 \), if \( 2D'(p) + pD''(p) < 0 \). As a result, we have following theoretical results.

**Theorem 5.** If the gross revenue, \( pD(p) \), is a strictly concave function of \( p \) (i.e., If \( 2D'(p) + pD''(p) < 0 \) or diminishing marginal revenue), the determinant of the Hessian Matrix is positive, then the annual total profit function, \( TP_{22}(p,T,\xi) \), is joint concave function in \( p \) and \( T \).

**Proof.** Same as Theorem 1.
Following the same treatment as in $\text{TP}_1$, the necessary conditions for the optimality of the annual total profit function $\text{TP}_{23}(p, T, \xi)$ are $\frac{\partial \text{TP}_{23}(p, T, \xi)}{\partial p} = 0$ and $\frac{\partial \text{TP}_{23}(p, T, \xi)}{\partial \xi} = 0$. That is,

$$D(p) + pD'(p) - \frac{cD'(p)t_d}{T} - \frac{1}{2} \left( h + cI_c \right) \frac{D'(p) t_d^2}{T}$$

$$- \frac{c + (h + cI_c) t_d + cI_c(N - M)}{T} \int_t^T e^{(1-m(\xi))g(t)} dt$$

$$+ \frac{cI_cD'(p)(M - N)(2t_d + N - M)}{2T}$$

$$- \frac{(h + cI_c) D'(p)}{T} \int_{t_d}^T \int_t^T e^{(1-m(\xi))(g(u) - g(t))} dudt = 0,$$

$$\frac{-D(p)}{T} \left[ c t_d' + (h + cI_c) t_d t_d' + cI_c(N - M) t_d' + (h + cI_c) t_d' \int_{t_d}^T e^{(1-m(\xi))g(t)} dt \right]$$

$$+ (c + (h + cI_c) t_d + cI_c(N - M)) \left\{ \int_{t_d}^T e^{(1-m(\xi))g(t)} \left( -m' g(t) + (1 - m) \right) \right\}$$

$$\times \frac{dg(t)}{d\xi} dt - t_d' \right\} + (h + cI_c) \int_{t_d}^T \int_t^T \left\{ -m' (g(u) - g(t)) \right\} e^{(1-m(\xi))(g(u) - g(t))} dudt = 0.$$ (58)

Following the same treatment as in $\text{TP}_{12}(p, T, \xi)$, we have following theoretical results.

**Theorem 6.**

1. For any given $T$ and $\xi$, if the gross revenue, $pD(p)$, is a strictly concave function of $p$ (i.e., $2D'(p) + pD''(p) < 0$ or diminishing marginal revenue), then the annual total profit function, $\text{TP}_{23}(p, T, \xi)$, is a strictly concave function with respect to $p$.
2. For any given $p$ and $T$, then the annual total profit function, $\text{TP}_{23}(p, T, \xi)$, is a strictly concave function with respect to $\xi$.
3. For any given $p$ and $\xi$, then the annual total profit function, $\text{TP}_{23}(p, T, \xi)$, is a strictly concave function in $T$, and hence has a unique global optimal solution $T^*_{23}$.

**Proof.** Proof is same as Theorem 2.

For any given $p$ and $\xi$, taking the first-order partial derivative of $\text{TP}_{23}(p, T, \xi)$ with respect to $T$, and setting the result to zero, we obtain:

$$A + D(p) \left[ ct_d + \frac{(h + cI_c) t_d^2}{2} + cI_c(N - M)(2t_d + N - M) + (c + (h + cI_c) t_d \right.$$}

$$+ cI_c(N - M)) \times \int_{t_d}^T e^{(1-m(\xi))g(t)} dt + (h + cI_c) \int_{t_d}^T \int_t^T e^{(1-m(\xi))(g(u) - g(t))} dudt$$

$$- \frac{pI_p(M - N)^2}{2} \right] - D(p) \left[ (c + (h + cI_c) t_d + cI_c(N - M)) e^{(1-m(\xi))g(T)} \right.$$}

$$\left. - \frac{(h + cI_c) D'(p)(M - N)(2t_d + N - M)}{2T} \int_{t_d}^T \int_t^T e^{(1-m(\xi))(g(u) - g(t))} dudt \right] = 0.$$ (57)
It is not easy to find a closed form solution of \( T \) from (59). But we can show that the value of \( T \) satisfy (59) not only exists but also is unique. So we have the following lemma.

**Lemma 3.** For given \( p \) and \( \xi \), when \( N \leq M \leq N + t_d \),

1. If \( 2A \geq (h + cI_c)D(p)T_d^2 - (cI_c - pI_c)D(p)(M - N)^2 \), then the solution of \( T \in [t_d, \infty) \) (say \( T_{23} \)) in (59) not only exists but is also unique.
2. If \( 2A < (h + cI_c)D(p)T_d^2 - (cI_c - pI_c)D(p)(M - N)^2 \), then the solution of \( T \in [t_d, \infty) \) in (59) does not exist.

**Proof.** See Appendix E.

According to Lemma 3, we have the following results.

**Lemma 4.** For given \( p \) and \( \xi \), when \( N \leq M \leq N + t_d \)

1. If \( 2A \geq (h + cI_c)D(p)T_d^2 - (cI_c - pI_c)D(p)(M - N)^2 \), then the annual total average profit \( TP_{23}(T|p,\xi) \) has the global maximum value at \( T = T_{23} \), where \( T_{23} \in [t_d, \infty) \) and satisfies (59).
2. If \( 2A < (h + cI_c)D(p)T_d^2 - (cI_c - pI_c)D(p)(M - N)^2 \), then the annual total relevant profit \( TP_{23}(T|p,\xi) \) has maximum value at the boundary point \( T = t_d \).

**Proof.** See Appendix F.

For notational convenience, we denote

\[
\Delta_2 = (h + pI_p)D(p)(M - N)^2, \\
\Delta_3 = (h + cI_c)D(p)T_d^2 - (cI_c - pI_c)D(p)(M - N)^2.
\]

Combining the aforementioned equations (56) and (60), Lemmas 4 and 5, and the assumption \( N \leq M \leq N + t_d \), we can obtain the following theorem.

**Theorem 7.** For given \( p \) and \( \xi \), when \( N \leq M \leq N + t_d \)

1. If \( 0 < 2A < \Delta_2 \), then \( TP_2(T^*|p,\xi) = \max (TP_{21}(T_{21}|p,\xi), TP_{21}(M - N|p,\xi)) \). Hence \( T^* \) is \( T_{21} \) or \( M - N \) associated with higher total average profit.
2. If \( \Delta_2 < 2A < \Delta_3 \), then \( TP_2(T^*|p,\xi) = \max (TP_{22}(T_{22}|p,\xi), TP_{22}(t_d|p,\xi)) \). Hence \( T^* \) is \( T_{22} \) or \( t_d \) associated with higher total average profit.
3. If \( 2A \geq \Delta_3 \), then \( TP_2(T^*|p,\xi) = TP_{23}(T_{23}|p,\xi) \). Hence \( T^* = T_{23} \).

**Case 3** \( M \geq N + t_d \)

(3.1) \( T \leq t_d \)

The problem is to maximize function (33). For \( T \leq t_d \), taking the first- and second-order partial derivatives of \( TP_{31}(p,T,\xi) \) with respect to \( p \) and \( T \), we get

\[
\frac{\partial TP_{31}(p,T,\xi)}{\partial p} = D(p) + (p - c)D'(p) - \frac{hD'(p)T}{2} + \left( D(p) + pD'(p) \right) I_p \left( M - \frac{T}{2} - N \right),
\]

\[
+ (h + cI_c) \int_{t_d}^{T} e^{(1-m(\xi))(g(T) - g(t))} dt = 0. \tag{59}
\]
\[
\frac{\partial^2 TP_{31}(p,T,\xi)}{\partial p^2} = \left(2D'(p) + pD''(p)\right) - cD''(p) - \frac{hD''(p)T}{2} + \left(2D'(p) + pD''(p)\right)I_p \left(M - \frac{T}{2} - N\right),
\]

(62)

\[
\frac{\partial TP_{31}(p,T,\xi)}{\partial p} = \frac{A}{T^2} - \frac{hD(p)}{2} - \frac{pI_pD(p)}{2},
\]

(63)

and

\[
\frac{\partial^2 TP_{31}(p,T,\xi)}{\partial T^2} = -\frac{2A}{T^3} < 0.
\]

(64)

Consequently, \(TP_{31}(T|p,\xi)\) is a concave function of \(T\). Thus, there exists a unique value of \(T\) (say \(T_{31}\)) which maximize \(TP_{31}(T|p,\xi)\) as

\[
T_{31} = \sqrt{\frac{2A}{(h + pI)p} D(p)}.
\]

(65)

To ensure \(T \leq t_d\), we substitute (65) into inequality \(T \leq t_d\) and we obtain

\[
0 < 2A \leq (h + pI)p|T_d|^2.
\]

(66)

We may assume without loss of generality that \(\frac{\partial^2 TP_{31}(p,T,\xi)}{\partial p^2} < 0\), if \(2D'(p) + pD''(p) < 0\). As a result, we have following theoretical results.

**Theorem 8.** If the gross revenue, \(pD(p)\), is a strictly concave function of \(p\) (i.e., \(2D'(p) + pD''(p) < 0\) or diminishing marginal revenue), the determinant of the Hessian Matrix is positive, then the annual total profit function, \(TP_{31}(p,T,\xi)\), is a joint concave function in \(p\) and \(T\).

**Proof.** Same as Theorem 1.

\[(3.2)\] \(t_d \leq T \leq M - N\)

Taking the first- and second-order partial derivatives of \(TP_{32}(p,T,\xi)\) with respect to \(p\) and \(\xi\) for any given \(T\), and simplifying terms, we get:

\[
\frac{\partial TP_{32}(p,T,\xi)}{\partial p} = D(p) + pD'(p) - cD'(p) t_d - \frac{hD'(p)t_d^2}{T}
\]

\[
-\frac{(c + h t_d)D'(p)}{T} \int_{t_d}^T e^{(1-m(\xi))g(t)} dt + \frac{\left(D(p) + pD'(p)\right)}{T} I_p \left(\frac{T^2}{2} + T(M - T - N)\right)
\]

\[
- \frac{hD'(p)}{T} \int_{t_d}^T \int_t^T e^{(1-m(\xi))(g(u) - g(t))} dudt,
\]

(67)

\[
\frac{\partial TP_{32}(p,T,\xi)}{\partial \xi} = -\frac{D(p)}{T}[c t_d't_d + h t_d t_d' + \int_{t_d}^T e^{(1-m(\xi))g(t)} dt]
\]

\[
+ (c + h t_d) \left\{ \int_{t_d}^T e^{(1-m)(g(t))} \left[-m'(g(t)) + (1 - m)\frac{dg(t)}{d\xi}\right] dt - t_d' \right\}
\]

\[
+ h \int_{t_d}^T \int_t^T \left[-m'(g(u) - g(t)) + (1 - m)\left(\frac{dg(u)}{d\xi} - \frac{dg(t)}{d\xi}\right)\right]
\]

\[
\times e^{(1-m)(g(u) - g(t))} dudt\right] - 1.
\]

(68)
Following the same treatment as in $TP_{12}(p, T, \xi)$, we have following theoretical results.

**Theorem 9.**

1. For any given $T$ and $\xi$, if the gross revenue, $pD(p)$, is a strictly concave function of $p$ (i.e., $2D'(p) + pD''(p) < 0$ or diminishing marginal revenue), then the annual total profit function, $TP_{32}(p, T, \xi)$, is a strictly concave function with respect to $p$.

2. For any given $p$ and $T$, then the annual total profit function, $TP_{32}(p, T, \xi)$, is a strictly concave function with respect to $\xi$.

3. For any given $p$ and $\xi$, then the annual total profit function, $TP_{32}(p, T, \xi)$, is a strictly concave function in $T$, and hence has a unique global optimal solution $T^*_p$.

**Proof.** Proof is same as Theorem 2.

For any given $p$ and $\xi$, taking the first-order partial derivative of $TP_{32}(p, T, \xi)$ with respect to $T$, and setting the result to zero, we obtain:

$$A + D(p) \left[ ct_d + \frac{ht_d^2}{2} + (c + ht_d) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt + h \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u)-g(t))} du dt \right]$$

$$-TD(p) \left[ (c + ht_d) e^{(1-m(\xi))g(T)} + h \int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))} dt \right]$$

$$-\frac{pI_pD(p) T^2}{2} = 0$$

(69)

Let

$$\Delta_4 = (h + pI_p) D(p)t_d^2,$$

$$\Delta_5 = A + D(p) \left[ ct_d + \frac{ht_d^2}{2} + (c + ht_d) \int_{t_d}^{M-N} e^{(1-m(\xi))g(t)} dt + h \int_{t_d}^{M-N} \int_{t}^{M-N} e^{(1-m(\xi))(g(u)-g(t))} du dt \right]$$

$$-(M-N) D(p) \left[ (c + ht_d) e^{(1-m(\xi))g(M-N)} + h \int_{t_d}^{M-N} e^{(1-m(\xi))(g(M-N)-g(t))} dt \right]$$

$$-\frac{pI_pD(p) (M-N)^2}{2}.$$

(70)

Then we have the following Lemma.

**Lemma 5.** For given $p$ and $\xi$, when $M \geq N + t_d$,

1. If $\Delta_4 \leq 2A \leq \Delta_5$, then the solution of $T \in [t_d, M-N]$ (say $T_{32}$) in (69) not only exists but also is unique,

2. If $2A < \Delta_4$ or $2A > \Delta_5$, then the solution $T \in [t_d, M-N]$ in (69) does not exist.

**Proof.** See Appendix G.

According to Lemma 5, we have the following result.

**Lemma 6.** For given $p$ and $\xi$, when $M \geq N + t_d$,
1. If $\Delta_d \leq 2A \leq \Delta_5$, then the annual total average profit $TP_{32}(T|p,\xi)$ has the global maximum value at $T = T_{32}$, where $T_{32} \in [t_d, M - N]$ and satisfies (69).
2. If $2A < \Delta_4$, then the annual total relevant profit $TP_{32}(T|p,\xi)$ has the maximum value at the boundary point $T = t_d$.
3. If $2A > \Delta_5$, then the total average profit $TP_{32}(T|p,\xi)$ has the maximum value at the boundary point $T = M - N$.

Proof. See Appendix H.

(3.3) $T \geq M - N$

For any given $T$, the necessary conditions for the optimality of the annual total profit function $TP_{33}(p, T, \xi)$ are $\frac{\partial TP_{33}(p, T, \xi)}{\partial p} = 0$ and $\frac{\partial TP_{33}(p, T, \xi)}{\partial \xi} = 0$. That is,

$$D(p) + pD'(p) - \frac{cD'(p)t_d}{T} - \frac{1}{2} hD'(p) \frac{t_d^2}{T} - \left(\frac{c + ht_d}{T} \right) \frac{D'(p)}{T} \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt$$

$$- \frac{hD'(p)}{T} \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt$$

$$- \frac{cI_cD'(p)}{T}$$

$$\times \int_{M-N}^{T} \int_{t}^{T} e^{(1-m)(g(u) - g(t))} du dt + \frac{\left(D(p) + pD'(p)\right) I_p (M - N)^2}{2T} = 0, \quad (72)$$

$$\frac{-D(p)}{T} \left[ c t_d' + ht_d t_d' + ht_d \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt \right]$$

$$+(c + ht_d) \left\{ \int_{t_d}^{T} e^{(1-m)(\xi)g(t)} \left( -m' g(t) + (1 - m) \frac{dg(t)}{d\xi} \right) dt - t_d' \right\}$$

$$+ cI_c \int_{M-N}^{T} \int_{t}^{T} \left\{ -m' (g(u) - g(t)) + (1 - m) \left( \frac{dg(u)}{d\xi} - \frac{dg(t)}{d\xi} \right) \right\}$$

$$\times e^{(1-m)(g(u) - g(t))} du dt + h \int_{t_d}^{T} \int_{t}^{T} \left\{ -m' (g(u) - g(t)) + (1 - m) \left( \frac{dg(u)}{d\xi} - \frac{dg(t)}{d\xi} \right) \right\}$$

$$\times e^{(1-m)(g(u) - g(t))} du dt$$

$$- \frac{dg(t)}{d\xi} \right\} e^{(1-m)(g(u) - g(t))} du dt \right] - 1 = 0. \quad (73)$$

Following the same treatment as in $TP_{33}(p, T, \xi)$, we have following theoretical results.

**Theorem 10.**

1. For any given $T$ and $\xi$, if the gross revenue, $pD(p)$, is a strictly concave function of $p$ (i.e., $2D'(p) + pD''(p) < 0$ or diminishing marginal revenue), then the annual total profit function, $TP_{33}(p, T, \xi)$, is a strictly concave function with respect to $p$.
2. For any given $p$ and $T$, then the annual total profit function, $TP_{33}(p, T, \xi)$, is a strictly concave function with respect to $\xi$.
3. For any given $p$ and $\xi$, then the annual total profit function, $TP_{33}(p, T, \xi)$, is a strictly concave function in $T$, and hence has a unique global optimal solution $T_{33}$. 
Lemma 7. For given $p$ and $\xi$, when $M \geq N + t_d$, 

1. If $2A \geq \Delta_5$, then the solution of $T \in [M - N, +\infty]$ (say $T_{33}$) in (74) not only exists but also is unique,
2. If $0 < 2A < \Delta_5$, then the solution of $T \in [M - N, +\infty]$ in (74) does not exists.

Proof. See Appendix I.

According to Lemma 7, we have the following result.

Lemma 8. For given $p$ and $\xi$, when $M \geq N + t_d$, 

1. If $2A \geq \Delta_5$, then the annual total average profit $TP_{33}(T|p, \xi)$ has the global maximum value at $T = T_{33}$, where $T_{33} \in [M - N, +\infty]$ and satisfies (74),
2. If $0 < 2A < \Delta_5$, then the annual total average profit $TP_{33}(T|p, \xi)$ has the global maximum value at point $T = M - N$, where $T_{33} \in [M - N, +\infty)$.

Proof. See Appendix J.

Combining the aforementioned equation (66), Lemmas 5 - 8, and the assumption $M \geq N + t_d$, we can obtain the following theorem.

Theorem 11. For given $p$ and $\xi$, when $M \geq N + t_d$, 

1. If $0 < 2A < \Delta_4$, then $TP_3(T^* | p, \xi) = \max(TP_{31}(T_{31} | p, \xi), TP_{31}(t_d | p, \xi))$. Hence $T^*$ is $T_{31}$ or $t_d$ associated with higher total average profit.
2. If $2A \leq \Delta_4$, then $TP_3(T^* | p, \xi) = \max(TP_{33}(T_{33} | p, \xi), TP_{32}(M - N | p, \xi))$.
3. If $2A \geq \Delta_5$, then $TP_3(T^* | p, \xi) = TP_{33}(T_{33} | p, \xi)$, and $T^* = T_{33}$.

Algorithm

Based on the propositions above, we propose the following algorithm to obtain the optimal solution.

Step 1. Set $i = 1$ and initialize the value of $\xi = \xi^{(i)} \geq 0$.

Step 2. Set $j = 1$ and initialize the value of $p = p^{(j)} \in (c, \bar{p})$.

Step 3. Compare the values of $M$, $N$, $t_d$.

1. If $M \leq N$, then go to step 4;
2. If $N \leq M \leq N + t_d$, then go to step 8,
3. If $M \geq N + t_d$, then go to step 12.
Step 4. Calculate \( \Delta_1 \),
1. If \( 2A < \Delta_1 \), then \( T^* = T_{11} \) and \( TP_1 (T^* \mid p, \xi) = TP_{11}(T_{11} \mid p, \xi). \)
2. If \( 2A \geq \Delta_1 \), and
3. \( TP_2 (T_2 \mid p, \xi) > TP_2 (T_d \mid p, \xi) \), then \( T^* = T_{12} \),
   \[ TP_1 (T^* \mid p) = TP_1 (T_{12} \mid p, \xi), \]
4. \( TP_2 (T_2 \mid p, \xi) < TP_2 (T_d \mid p, \xi) \), then \( T^* = T_d, \) \( TP_1 (T^* \mid p) = TP_1 (T_d \mid p, \xi). \)

Step 5.
1. If \( 2A < \Delta_1 \), using the result \( T^* \) gained from Step 4, and then determine the optimal \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_{21}(p, T, \xi)}{\partial p} = 0. \)
2. If \( 2A \geq \Delta_1 \), using the result \( T^* \) gained from Step 4 and \( \xi = \xi^{(i)} \), we get \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_{22}(p, T, \xi)}{\partial p} = 0. \) Furthermore, substituting \( T = T^* \) and \( p = p^{(j+1)} \), we get \( \xi = \xi^{(i+1)} \) by solving \( \frac{\partial TP_{22}(p, T, \xi)}{\partial \xi} = 0. \)

Step 6. If \( |p^{(j+1)} - p^{(j)}| \leq 10^{-4} \) (small enough), then go to Step 7. Otherwise, set \( j = j + 1 \) and go back to Step 4.

Step 7. If \( |\xi^{(i+1)} - \xi^{(i)}| \leq 10^{-4} \) (small enough), then \( (T^*, p^*, \xi^*) = (T^*, p^{(j+1)}, \xi^{(i+1)}) \). Otherwise, set \( i = i + 1 \) and go back to Step 4.

Step 8. Calculate \( \Delta_2 \) and \( \Delta_3 \)
1. If \( 2A < \Delta_2 \), then \( T^* = T_{21} \) and \( TP_2 (T^* \mid p, \xi) = TP_{21}(T_{21} \mid p, \xi). \)
2. If \( \Delta_2 \leq 2A < \Delta_3 \), and
   i.) \( TP_2 (T_{22} \mid p, \xi) > TP_2 (M - N \mid p, \xi) \), then \( T^* = T_{22}, \) \( TP_2 (T^* \mid p, \xi) = TP_{22}(T_{22} \mid p, \xi), \)
   ii.) \( TP_2 (T_{22} \mid p, \xi) < TP_2 (M - N \mid p, \xi) \), then \( T^* = M - N, \) \( TP_2 (T^* \mid p, \xi) = TP_{22}(M - N \mid p, \xi). \)
3. If \( 2A \geq \Delta_3 \), and
   i.) \( TP_2 (T_{23} \mid p, \xi) > TP_2 (t_d \mid p, \xi) \), then \( T^* = T_{23}, \) \( TP_2 (T^* \mid p, \xi) = TP_{23}(T_{23} \mid p, \xi), \)
   ii.) \( TP_2 (T_{22} \mid p) < TP_2 (M - N \mid p) \), then \( T^* = M - N, \) \( TP_2 (T^* \mid p) = TP_{23}(t_d \mid p, \xi). \)

Step 9.
1. If \( 2A < \Delta_2 \), using the result \( T^* \) gained from Step 8, and then determine the optimal \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_{21}(p, T, \xi)}{\partial p} = 0. \)
2. If \( \Delta_2 \leq 2A < \Delta_3 \), using the result \( T^* \) gained from Step 8 and \( \xi = \xi^{(i)} \), we get \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_{22}(p, T, \xi)}{\partial p} = 0. \)
3. If \( 2A \geq \Delta_3 \), using the result \( T^* \) gained from Step 8 and \( \xi = \xi^{(i)} \), we get \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_{22}(p, T, \xi)}{\partial p} = 0. \) Furthermore, substituting \( T = T^* \) and \( p = p^{(j+1)} \), we get \( \xi = \xi^{(i+1)} \) by solving \( \frac{\partial TP_{22}(p, T, \xi)}{\partial \xi} = 0. \)

Step 10. If \( |p^{(j+1)} - p^{(j)}| \leq 10^{-4} \) (small enough), then go to Step 9. Otherwise, set \( j = j + 1 \) and go back to Step 8.

Step 11. If \( |\xi^{(i+1)} - \xi^{(i)}| \leq 10^{-4} \) (small enough), then \( (T^*, p^*, \xi^*) = (T^*, p^{(j+1)}, \xi^{(i+1)}) \). Otherwise, set \( i = i + 1 \) and go back to Step 8.

Step 12. Calculate \( \Delta_4 \) and \( \Delta_5 \)
1. \( 2A < \Delta_4 \), then \( T^* = T_{31} \) and \( TP_3 (T^* \mid p) = TP_{31}(T_{31} \mid p). \)
2. If \( \Delta_1 \leq 2A < \Delta_5 \), and
   i.) \( TP_{32} (T_{32} \mid p, \xi) > TP_{32} (t_d \mid p, \xi) \), then \( T^* = T_{32}, \text{ } TP_3 (T^* \mid p, \xi) = TP_{32} (T_{32} \mid p, \xi) \).
   ii.) \( TP_{32} (T_{32} \mid p, \xi) < TP_{22} (t_d \mid p, \xi) \), then \( T^* = t_d, \text{ } TP_3 (T^* \mid p, \xi) = TP_{32} (t_d \mid p, \xi) \).
3. If \( 2A \geq \Delta_5 \), and
   i.) \( TP_{33} (T_{33} \mid p, \xi) > TP_{33} (M - N \mid p, \xi) \), then \( T^* = T_{33}, \text{ } TP_3 (T^* \mid p, \xi) = TP_{33} (T_{33} \mid p, \xi) \).
   ii.) \( TP_{33} (T_{33} \mid p, \xi) < TP_{33} (M - N \mid p, \xi) \), then \( T^* = M - N, \text{ } TP_3 (T^* \mid p, \xi) = TP_{33} (M - N \mid p, \xi) \).

**Step 13.**
1. If \( 2A < \Delta_4 \), using the result \( T^* \) gained from Step 12, and then determine the optimal \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_3 (p, T, \xi)}{\partial p} = 0 \).
2. If \( \Delta_4 \leq 2A < \Delta_5 \), using the result \( T^* \) gained from Step 12 and \( \xi = \xi^{(i)} \), we get \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_3 (p, T, \xi)}{\partial p} = 0 \).
3. If \( 2A \geq \Delta_5 \), using the result \( T^* \) gained from Step 12 and \( \xi = \xi^{(i)} \), we get \( p = p^{(j+1)} \) by solving \( \frac{\partial TP_3 (p, T, \xi)}{\partial \xi} = 0 \). Furthermore, substituting \( T = T^* \) and \( p = p^{(j+1)} \), we get \( \xi = \xi^{(i+1)} \) by solving \( \frac{\partial TP_3 (p, T, \xi)}{\partial \xi} = 0 \).

**Step 14.** If \( |p^{(j+1)} - p^{(j)}| \leq 10^{-4} \) (small enough), then go to Step 15. Otherwise, set \( j = j + 1 \) and go back to Step 12.

**Step 15.** If \( |\xi^{(i+1)} - \xi^{(i)}| \leq 10^{-4} \) (small enough), then \( (T^*, p^*, \xi^*) = (T^*, p^{(j+1)}, \xi^{(i+1)}) \). Otherwise, set \( i = i + 1 \) and go back to Step 12.

5. **Numerical examples.** To validate the proposed model, numerical examples are considered and solved by the proposed algorithm.

**Example 1.** In order to illustrate the model, we consider an inventory system with the following data: \( A = $120/\text{order}, \text{ } c = $20/\text{per unit}, \text{ } h = $3/\text{per unit/\text{per year}}, \text{ } M = 0.3 \text{ year}, \text{ } N = 0.2 \text{ year}, \text{ } D(p) = 1000 - 10p, \text{ } \theta (t) = 0.2 + 0.3 (t - t_d (\xi)), \text{ } t_d (\xi) = t_0 + \gamma (1 - e^{-\alpha \xi}) \) with \( t_0 = 0.1, \text{ } \gamma = 1.5 \text{ and } \alpha = 0.006, \text{ } m (\xi) = 1 - e^{-\beta \xi} \) with \( \beta = 0.5 \).

Based on the algorithm, the optimal length of replenishment cycle \( T^* = 0.2930 \) year, Optimal selling price \( p^* = 60.2669 \), the optimal preservation technology cost \( \xi^* = 5.7565 \), optimal order quantity per replenishment \( Q^* = 116.1991 \) units and the maximum total average profit is \( TP^* = 153777.67 \).

5.1. **Sensitivity analysis.** In this subsection, we mainly study the impact of the parameters associated with the model on the optimal solutions. The values of other parameters keep the same as in Example 1 when each of parameters varies. Table 2 presents the observed results with various parameters. On the basis of the results of this table, the following managerial insights can be achieved:
1. \( p^*, \xi^*, T^*, \text{ and } Q^* \) are increasing in \( A \), while \( TP^* \) is decreasing in \( A \). With a higher ordering cost, the retailer tends to order more units per cycle and charge a higher price. The retail price and ordering quantity are not very sensitive to \( A \). Notice that when the ordering cost increases, the retailer will invest in preservation technology and the optimal investment is increasing in \( A \). When the ordering cost is low enough, the retailer could place orders more
Table 2. Sensitivity analysis with respect to different parameters

| Parameter | Values | $P^*$  | $\xi^*$ | $T^*$  | $Q^*$  | $TP^*$ |
|-----------|--------|--------|---------|--------|--------|--------|
| A         | 80     | 60.0939| 4.7922  | 0.2097 | 83.1619| $TP^*_{23} = 15543.44$ |
|           | 100    | 60.1823| 4.9543  | 0.2514 | 99.7122| $TP^*_{23} = 15456.70$ |
|           | 120    | 60.2498| 5.1039  | 0.2868 | 113.7114| $TP^*_{23} = 15382.39$ |
|           | 140    | 60.3056| 5.2469  | 0.3180 | 126.0302| $TP^*_{23} = 15316.27$ |
|           | 160    | 60.3540| 5.3864  | 0.3463 | 137.1341| $TP^*_{23} = 15256.08$ |
| c         | 10     | 55.2849| 5.3094  | 0.3289 | 146.8334| $TP^*_{23} = 19629.55$ |
|           | 15     | 57.7692| 5.1910  | 0.3053 | 128.6614| $TP^*_{23} = 17441.33$ |
|           | 20     | 60.2498| 5.1039  | 0.2868 | 113.7114| $TP^*_{23} = 15382.39$ |
|           | 25     | 62.7284| 5.0393  | 0.2725 | 101.2497| $TP^*_{23} = 13451.34$ |
|           | 30     | 65.2071| 4.9925  | 0.2618 | 90.7764 | $TP^*_{23} = 11647.08$ |
| h         | 1      | 60.2001| 5.4634  | 0.3623 | 144.0668| $TP^*_{23} = 15509.14$ |
|           | 2      | 60.2272| 5.2450  | 0.3185 | 126.4466| $TP^*_{23} = 15441.97$ |
|           | 3      | 60.2498| 5.1039  | 0.2868 | 113.7114| $TP^*_{23} = 15382.39$ |
|           | 4      | 60.2691| 5.0048  | 0.2725 | 101.2497| $TP^*_{23} = 13451.34$ |
|           | 5      | 60.2859| 4.9310  | 0.2618 | 90.7764 | $TP^*_{23} = 11647.08$ |
| $t_0$     | 0.02   | 60.2890| 5.5741  | 0.3011 | 119.4436| $TP^*_{23} = 15371.08$ |
|           | 0.05   | 60.2768| 5.3873  | 0.2964 | 117.5873| $TP^*_{23} = 15374.53$ |
|           | 0.10   | 60.2498| 5.1039  | 0.2868 | 113.7114| $TP^*_{23} = 15382.39$ |
|           | 0.15   | 60.2115| 4.8468  | 0.2742 | 108.6560| $TP^*_{23} = 15393.68$ |
|           | 0.20   | 60.1570| 4.6081  | 0.2580 | 102.1061| $TP^*_{23} = 15409.61$ |

frequently and order less units per cycle so that he can reduce the loss from deterioration with minimum investing in preservation technology. In practice, large orders increase the amount of inventory available, which is costly, but may provide financial benefit by reducing ordering cost and providing volume discounts.

2. $p^*$ and $\xi^*$ are increasing in $c$, while $T^*$, $Q^*$, and $TP^*$ are decreasing in $c$. The impact of $c$ on $p^*$, $\xi^*$, $T^*$, $Q^*$, and $TP^*$ is quite intuitive and actually quite significant. An increase in $c$ leads to a significant increase in $p^*$, and hence a significant decrease in demand. Thus $T^*$ and $Q^*$ are decreasing with increasing in $c$. When the purchasing cost per unit $c$ increases, the retailer’s deterioration cost increases. Hence, the retailer would decrease the length of the inventory period and increase the preservation technology cost to reduce deterioration loss. Therefore, the optimal service rate decreases and the optimal preservation technology cost increases, respectively.

3. $p^*$ is increasing in $h$, while $\xi^*$, $T^*$, $Q^*$ and $TP^*$ are decreasing in $h$. $\xi^*$ is non-increasing in $h$ and very sensitive to a change in $h$. We observe that it is optimal for the retailer not to invest in preservation technology when $h$ is sufficiently high. As the holding cost per unit per unit time increases, the retailer tends to order less per cycle, and invest less in preservation technology. When the holding cost per unit per unit time is very high, the benefits from the preservation technology investment are not sufficient to offset the increase of the holding cost, and thereby the retailer does not invest. In practice, smaller and more frequent order quantities translate into fewer inventories.
Frequent orders are costly to process, and the resulting low inventory levels may increase the probability of stock-outs, leading to loss of customers.

4. $p^*, \xi^*, T^*$, and $Q^*$ are decreasing with increasing in $t_0$, while $TP^*$ is increasing in $t_0$. It is intuitive that the deterioration of products is less with a long non-deterioration period than with a short one, so do the costs caused by deterioration. A longer non-deterioration period benefits the retailer by reducing investment in preservation technology and decrease the length of inventory period simultaneously. Therefore, the optimal preservation technology cost decreases.

We now perform sensitivity analysis for different values of trade credit periods. Table 3 shows the numerical results. If the value of supplier’s trade credit to the retailer $M$ increases, $p^*, \xi^*, T^*$, and $Q^*$ stay at the same threshold, but it is trivial, that the retailer’s optimal total profit increases; since the retailer has more time to accumulate the sales revenue and earns interest. When the length of credit period increases, it is reasonable to shorten the replenishment cycle time to get the benefit of trade credit more frequently. Consequently, the company does not need to make more preservation effort because the ordering quantity decreases. To acquire more profit, industrial managers have to concentrate on their credit period tactics and cycle length.

If $N = M = 0$, the model becomes no-trade credit policy system and the optimal total profit is $15263.37$. This means that the trade credit policy has a significant effect on the retailer’s total profit. Moreover, when $N = M = 0.6$, the retailer’s total profit is $15301.15$. This implies that even the retailer adopts the credit period to his/her customer equal to supplier’s credit period to the retailer, the retailer’s total profit is greater than the no-trade credit policy system.

If $N$ increases, $p^*, \xi^*, T^*$, and $Q^*$ increases slightly, whereas $TP^*$ decreases slightly. Due to the trade period for the customers, measure of total profit for the retailer is slightly smaller. However, the retailer’s optimal selling price $p^*$ is slightly larger. This indicates that the increase of the retailer’s selling price cannot offset the decrease of profit due to the trade credit period $N$ for the customers and the reduced demand. As the retailer’s order quantity $Q^*$ increases with $N$ increases, the retailer could increase the sales quantity by adopting policy of delay in payment for his/her customer. We explain this phenomenon that when $N$ increases, the retailer will order more items to accumulate more interest to make restitution the loss of interest earned. However, when the optimal order quantity increases the total profit decreases.

6. Conclusion. Financing tools play a more and more important role in business today, which provide us with a new method to study the inventory problems. In the inventory problems, credit can have significant influence on the inventory decisions, that is, ordering quantity and ordering cycle length. The retailer invests in preservation technology to reduce losses due to deterioration. We argue that preservation technology can both reduce the deterioration rate and lengthen the non-deterioration period. In the present article, we have developed an inventory model for non-instantaneous deteriorating items with price-dependent demand rate, time-varying deterioration rate and two-level trade credit. The proposed model is developed for retailers, where the supplier offers retailer trade-credit (upstream trade credit) for settlement of the account and the retailer in turn provides trade credit (downstream) to the customers. The analytical mathematical formulations
of the problem on the comprehensive structure have been given. In the proposed model we have discussed the combined problem where the replenishment policy, retail price and preservation technology cost are decision variables. In addition, some useful theorems, and lemmas for finding the optimal replenishment policy, price, and preservation technology strategies is given. Moreover, we have analyzed the effects of some major parameter as a form of sensitivity analysis.

This model has several future works for further researches. A potential extension is to consider the finite replenishment, the quantity discount, introduced inflation rate, multi-echelon, generalized for multi-items, and etc.

**Acknowledgment.** The authors are thankful to the honorable Editor-in-Chief and the anonymous reviewers for their valuable comments and suggestions to improve the quality of this paper.

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**Table 3. The numerical results for different values of M and N**

| M   | N   | $P^*$  | $\xi^*$ | $T^*$  | $Q^*$  | $TP^*$          |
|-----|-----|--------|---------|--------|--------|-----------------|
| 0.0 | 0.0 | 60.4012| 5.1072  | 0.2875 | 112.4756| $TP^*_{23} = 15263.37$ |
| 0.2 | 0.4 | 61.0067| 5.1207  | 0.2906 | 113.0335| $TP^*_{12} = 14791.82$ |
| 0.6 | 0.6 | 61.3095| 5.1276  | 0.2921 | 113.3073| $TP^*_{12} = 14558.77$ |
| 0.8 | 0.8 | 61.6124| 5.1345  | 0.2937 | 113.5776| $TP^*_{12} = 14327.53$ |
| 0.2 | 0.2 | 60.0984| 5.1006  | 0.2860 | 112.7563| $TP^*_{33} = 15501.87$ |
| 0.4 | 0.4 | 60.4212| 5.1085  | 0.2879 | 113.0335| $TP^*_{23} = 15287.52$ |
| 0.6 | 0.6 | 61.0067| 5.1139  | 0.2892 | 113.3073| $TP^*_{12} = 14791.82$ |
| 0.8 | 0.8 | 61.3095| 5.1276  | 0.2921 | 113.5776| $TP^*_{12} = 14558.77$ |
| 0.0 | 0.4 | 59.7957| 5.0941  | 0.2845 | 113.0335| $TP^*_{33} = 15742.19$ |
| 0.2 | 0.2 | 60.0984| 5.1006  | 0.2860 | 113.3073| $TP^*_{33} = 15501.87$ |
| 0.4 | 0.4 | 60.4302| 5.1092  | 0.2881 | 113.5776| $TP^*_{33} = 15293.23$ |
| 0.6 | 0.6 | 60.7039| 5.1139  | 0.2892 | 113.8444| $TP^*_{12} = 15026.68$ |
| 0.8 | 0.8 | 61.0067| 5.1207  | 0.2906 | 114.1078| $TP^*_{12} = 14791.82$ |
| 0.4 | 0.0 | 59.4930| 5.0877  | 0.2831 | 113.3073| $TP^*_{32} = 15984.32$ |
| 0.2 | 0.2 | 59.7957| 5.0941  | 0.2845 | 113.5776| $TP^*_{32} = 15742.19$ |
| 0.4 | 0.4 | 60.0984| 5.1006  | 0.2860 | 113.8444| $TP^*_{33} = 15501.87$ |
| 0.6 | 0.6 | 60.4612| 5.1102  | 0.2878 | 114.1078| $TP^*_{33} = 15301.15$ |
| 0.8 | 0.8 | 60.7039| 5.1139  | 0.2890 | 114.3679| $TP^*_{33} = 15026.68$ |
| 0.0 | 0.8 | 59.1903| 5.0814  | 0.2816 | 113.5776| $TP^*_{32} = 16228.27$ |
| 0.2 | 0.2 | 59.4930| 5.0877  | 0.2831 | 113.8444| $TP^*_{32} = 15984.32$ |
| 0.4 | 0.4 | 59.7957| 5.0941  | 0.2845 | 114.1078| $TP^*_{32} = 15742.19$ |
| 0.6 | 0.6 | 60.0984| 5.1006  | 0.2867 | 114.3679| $TP^*_{33} = 15501.87$ |
| 0.8 | 0.8 | 60.4825| 5.1172  | 0.2885 | 114.6246| $TP^*_{33} = 15374.12$ |
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\[ \text{Appendix A. Let Hessian Matrix of } TP_1(p, T) \text{ be} \]

\[ H_1 = \begin{bmatrix}
\frac{\partial^2 TP_1(p, T)}{\partial p^2} & \frac{\partial^2 TP_1(p, T, \xi)}{\partial p \partial T} \\
\frac{\partial^2 TP_1(p, T, \xi)}{\partial T \partial p} & \frac{\partial^2 TP_1(p, T, \xi)}{\partial T^2}
\end{bmatrix}
\]

From (35) and the assumption of \(2D' (p) + pD'' (p) < 0\) implies that \(\frac{\partial^2 TP_1(p, T)}{\partial p^2} < 0\).

Now, the determinant of \(H_1\) is

\[ |H_1| = \frac{\partial^2 TP_1(p, T)}{\partial p^2} \frac{\partial^2 TP_1(p, T, \xi)}{\partial T^2} - \left[ \frac{\partial^2 TP_1(p, T, \xi)}{\partial T \partial p} \right]^2 \]

\[ = -\frac{2A}{T^3} \left[ (2D' (p) + pD'' (p)) - D'' (p) \left\{ c + \frac{(h + cI_c)T}{2} + cI_c(N - M) \right\} \right] \]

\[ - \frac{(h + cI_c) D' (p)}{2} \times \frac{h + cI_c) D' (p)}{2} \]

\[ = -\frac{2A}{T^3} \left[ \left\{ 2D' (p) + pD'' (p) \right\} - pD'' (p) - D'' (p) \left\{ c + cI_c(N - M) \right\} \right] \]

\[ - \left( \frac{2A (h + cI_c)T}{2} \right) - \frac{2AD' (p)}{T^3 D(p)} \times \frac{(h + cI_c) D' (p) T}{4} \]

\[ = -\frac{2A}{T^3} \left[ 2D' (p) - D'' (p) \left\{ c + cI_c(N - M) \right\} \right] \]

\[ + (h + cI_c)TD' (p) \left\{ \frac{1}{D(p)} + \frac{D' (p)}{2D(p)} \right\} \]

\[ - \frac{1}{D(p)} + \frac{D' (p)}{2D(p)} \]

\[ = \left( D (p) + \left\{ p - c - cI_c(N - M) \right\} D'' (p) \right) \left\{ \frac{1}{D(p)} + \frac{D' (p)}{2D(p)} \right\} , \]
Note that if \( D' (p) < 0 \), therefore, the determinant of \( H_1 \) is positive then \( H_1 \) is negative definite. This completes the Proof.

**Appendix B.** Taking the second-order partial derivatives of \( TP_{12}(p,T,\xi) \) with respect to \( p \) and \( \xi \) for any given \( T \), and simplifying terms, we get:

\[
\frac{\partial^2 TP_{12}(p,T,\xi)}{\partial p^2} = \left( 2D' (p) + pD'' (p) \right) - \frac{D'' (p)}{T} [(c + cI_e (N - M)) t_d +
\]

\[
(cI_e + h) \left\{ \frac{t_d^2}{2} \int_{t_d}^{T} \int_{t_d}^{T} e^{(1 - m(\xi))(g(\xi) - g(t))} du dt + (c + (cI_e + h) t_d + cI_e (N - M)) \right\} + (c + (cI_e + h) t_d + cI_e (N - M)) \int_{t_d}^{T} e^{(1 - m(\xi))g(t)} dt, \tag{B.1}\]

Note that if \( 2D'' (p) + pD'' (p) < 0 \), then \( \frac{\partial^2 TP_{12}(p,T,\xi)}{\partial p^2} < 0 \). Therefore, \( TP_{12}(p,T,\xi) \) is concave in \( p \). This completes the Proof of Part (1).

\[
\frac{\partial^2 TP_{12}(p,T,\xi)}{\partial \xi^2} = -\frac{D'(p)}{T} \left[ (c + (h + cI_e) t_d + cI_e (N - M)) \int_{t_d}^{T} e^{(1-m(\xi))g(u)} \right.
\]

\[
\times \left\{ \left( -m' g (u) + (1 - m) \frac{dg (u)}{d\xi} \right)^2 - m'' g (u) - 2m' \frac{dg (u)}{d\xi} \right.
\]

\[
+ (1 - m) \frac{d^2 g (u)}{d\xi^2} \right\} du + (h + cI_e) \int_{t_d}^{T} \int_{t_d}^{T} e^{(1-m)(g(\xi) - g(t))} \left\{ (-m' (g (u)
\]

Using \( \frac{\partial TP_{12}(p,T)}{\partial p} = 0 \) from Eq. (34)}
Lemma 1, Part (a): Motivated by (43), we define a new function
Appendix C. the Proof of Part (3).
Taking the first- and second-order derivatives of
From $T_P$ and the assumptions of $\theta(.) \geq 0$, $\theta'(.) \geq 0$, $\theta''(.) \geq 0$ and $t_d'(\xi) \leq 0$, it is easy to verify that Eqs. (B.3)-(B.7) all hold. In addition, $m(\xi) > 0$ and $m''(\xi) < 0$. Thus $\frac{d^2}{d\xi^2}TP_{12}(p,T,\xi) < 0$, which implies $TP_{12}(p,T,\xi)$ is concave in $\xi$. This completes the Proof of Part (2).
To prove $TP_{12}(p,T,\xi)$ has a unique optimal solution, by using Eq. (15), we define

\[
\begin{align*}
f_{12}(T) &= pTD(p) - cD(p)t_d - \frac{(h + cI_c)D(p)t_d^2}{2} - cI_c(N - M)D(p)t_d - \xi T - A - (c + (h + cI_c)t_d + cI_c(N - M))D(p)\int_{t_d}^{T} e^{(1-m(\xi))g(t)}dt \\
&= - (h + cI_c)D(p)\int_{t_d}^{T} e^{(1-m(\xi))(g(u)-g(t))}du dt \\
&= g_{12}(T) = T.
\end{align*}
\]

Taking the first- and second-order derivatives of $f_{12}(T)$ with respect to $T$, we get

\[
\begin{align*}
\frac{df_{12}(T)}{dT} &= -D(p) - (c + (h + cI_c)t_d + cI_c(N - M))D(p)e^{(1-m(\xi))g(T)} \\
&= - (h + cI_c)D(p)\int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))}dt - \xi \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{d^2f_{12}(T)}{dT^2} &= -D(p)\left[\left(c + (h + cI_c)t_d + cI_c(N - M)\right)\left(1 - m(\xi)\right)g'(T)e^{(1-m(\xi))g(T)} + \left(h + cI_c\right)\left(\int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))}(1 - m(\xi))g'(T)dt + 1\right)\right] < 0.
\end{align*}
\]

By applying the theoretical result in Cambini and Martein [49] (2009, p. 245), we know that $TP_{12}(p,T,\xi) = \frac{f_{12}(T)}{g_{12}(T)}$ is strictly pseudo-concave in $T$. This completes the Proof of Part (3).

Appendix C. Lemma 1, Part (a): Motivated by (43), we define a new function

\[
F_{12}(T) = A + cD(p)t_d + \frac{(h + cI_c)D(p)t_d^2}{2} + cI_c(N - M)D(p)t_d
\]
Moreover, we have

\[ F(T) = (c + (h + cI_c) t_d + cI_c (N - M)) D(p) \left\{ \int_{t_d}^{T} e^{(1-m)(g(t))} dt - T e^{(1-m)(g(T))} \right\} \]
\[ + (cI_c + h) D(p) \left\{ \int_{t_d}^{T} \int_{t_d}^{T} e^{(1-m)(g(u)-g(t))} du dt - T \int_{t_d}^{T} e^{(1-m)(g(T)-g(t))} dt \right\} \]

for \( T \in [t_d, \infty) \).

Since the first derivative of \( F(T) \) with respect to \( T \in [t_d, \infty) \) is

\[
\frac{dF_{12}(T)}{dT} = -D(p) T [(c + (h + cI_c) t_d + cI_c (N - M)) (1-m(\xi)) g'(T)] e^{(1-m(\xi))} g(T) + (cI_c + h) \left\{ \int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))} (1-m(\xi)) g'(T) dt + 1 \right\} < 0
\]

We obtain that \( F_{12}(T) \) is a strictly decreasing function of \( T \) in the interval \([t_d, \infty)\). Moreover, we have \( F_{12}(T)|_{T \to \infty} = -\infty \), and

\[
F_{12}(T)|_{T \to t_d} = A - \frac{(h+cI_c)D(p)t_d^2}{2}.
\]

Therefore, if \( 2A \geq (h+cI_c)D(p)t_d^2 \), then \( F_{12}(T)|_{T \to t_d} \geq 0 \). According to the intermediate value theorem, there exists a unique \( T_{12} \in [t_d, \infty) \) such that \( F_{12}(T_{12}) = 0 \).

Part (b) If \( 0 < 2A < (h+cI_c)D(p)t_d^2 \), then from (A.3), \( F_{12}(T)|_{T \to t_d} < 0 \). Since \( F_{12}(T) \) is a strictly decreasing function of \( T \) in the interval \([t_d, \infty)\), thus there is no value of \( T \in [t_d, \infty) \) such that \( F_{12}(T) = 0 \).

Appendix D. Lemma 2, Part (a): When \( 2A \geq (h+cI_c)D(p)t_d^2 \), \( T_{12} \) is the unique solution of (51) from Lemma 1. Taking the second derivative of \( TP_{12}(p, T, \xi) \) with respect to \( T \) and finding the value of the function at \( T_{12} \), we obtain

\[
TP_{12}(T|p, \xi) = \frac{1}{T} \left\{ p D(p) T - c D(p) t_d - \frac{(h+cI_c)D(p)t_d^2}{2} - cI_c (N-M) D(p)t_d \
- \xi T - A - (c+(h+cI_c) t_d + cI_c (N-M)) D(p) \int_{t_d}^{T} e^{(1-m)(g(t))} dt \right\}
\]

\[
\frac{dTP_{12}(T|p, \xi)}{dT} = \frac{1}{T} \left\{ p D(p) - \xi - (c+(h+cI_c) t_d + cI_c (N-M)) D(p) \right\}
\]

\[
- \left. e^{(1-m(\xi))} g(T) - (h+cI_c) D(p) \int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))} dt \right] 
\]

\[
- \frac{1}{T^2} [p D(p) T - c D(p) t_d - \frac{(h+cI_c)D(p)t_d^2}{2} - cI_c (N-M) D(p)t_d - \xi T \]

\[
- A - (c+(h+cI_c) t_d + cI_c (N-M)) D(p) \int_{t_d}^{T} e^{(1-m(\xi))} g(T) dt \\
- (h+cI_c) D(p) \int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))} dt \right] = 0
\]

\[
\frac{d^2TP_{12}(T|p, \xi)}{dT^2} \bigg|_{T=T_{12}} = -\frac{D(p)}{T_{12}} \left( (c+(h+cI_c) t_d + cI_c (N-M)) e^{(1-m(\xi))} g(T_{12}) \right)
\]
Moreover, we have

\[
\times (1 - m(\xi)) g'(T_{12}) + (h + cI_c) \left\{ \int_{t_d}^{T_{12}} e^{(1-m(\xi))(g(T_{12}) - g(t))} \right. \\
\times (1 - m(\xi)) g'(T_{12}) dt + 1 \left\} < 0
\]

Thus \( T_{12} \) is the global maximum point of \( TP_{12}(T|p, \xi) \).

**Lemma 2, Part (b)** From the proof of Lemma 1(b), we know that if \( 0 < 2A < (h + cI_c) D(p) t_d^2 \), then \( F_{12}(T) < 0 \), for all \([t_d, \infty)\). Thus we have

\[
\frac{d}{dT} TP_{12}(T|p, \xi) = \frac{F_{12}(T)}{T^2} < 0,
\]

for all \( T \in [t_d, \infty) \), which implies that \( TP_{12}(T|p, \xi) \) is a strictly decreasing function of \( T \in [t_d, \infty) \). So, \( TP_{12}(T|p, \xi) \) has a maximum value at the boundary point \( T = t_d \).

**Appendix E. Lemma 3, Part (a):** Motivated by (59), we define a new function

\[
F_{23}(T) = A + D(p) \left[ ct_d + \frac{(h + cI_c) t_d^2}{2} + \frac{cI_c}{2} (N - M) (2t_d + N - M) \right. \\
+ (c + (h + cI_c) t_d + cI_c (N - M)) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt \\
+ (h + cI_c) \int_{t_d}^{T} \int_{t_d}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt - \frac{pI_p(M - N)^2}{2} \left. \right]
\]

\[-D(p)T \left[ (c + (h + cI_c) t_d + cI_c (N - M)) e^{(1-m(\xi))g(T)} \right. \\
+ (h + cI_c) \int_{t_d}^{T} e^{(1-m(\xi))(g(T) - g(t))} dt \left. \right]
\]

for \( T \in [t_d, \infty) \).

Since the first derivative of \( F_{23}(T) \) with respect to \( T \in [t_d, \infty) \) is

\[
\frac{dF_{23}(T)}{dT} = -D(p)T [(c + (h + cI_c) t_d + cI_c (N - M)) (1 - m(\xi)) g'(T) \\
\times e^{(1-m(\xi))g(T)} + (cI_c + h) \left\{ \int_{t_d}^{T} e^{(1-m(\xi))(g(T) - g(t))} \right. \\
\times (1 - m(\xi)) g'(T) dt + 1 \left\} < 0 \right.
\]

We obtain that \( F_{23}(T) \) is a strictly decreasing function of \( T \) in the interval \([t_d, \infty)\). Moreover, we have \( F_{23}(T)|_{T \rightarrow \infty} = -\infty \), and

\[
F_{23}(T)|_{T \rightarrow t_d} = A - \frac{(h+cI_c)D(p)t_d^2 - (cI_c + pI_c)D(p)(M-N)^2}{2}.
\]

Therefore, if \( 2A \geq (h + cI_c) D(p) t_d^2 - (cI_c - pI_c) D(p)(M - N)^2 \), then \( F_{23}(T)|_{T \rightarrow t_d} \geq 0 \). According to the intermediate value theorem, there exists a unique \( T_{23} \in [t_d, \infty) \) such that \( F_{23}(T_{23}) = 0 \).

**Lemma 3, Part (b):** If \( 2A < (h + cI_c) D(p) t_d^2 - (cI_c - pI_c) D(p)(M - N)^2 \), then from (E.3), \( F_{23}(T)|_{T \rightarrow t_d} < 0 \). Since \( F_{23}(T) \) is a strictly decreasing function of \( T \) in the interval \([t_d, \infty)\), thus there is no value of \( T \in [t_d, \infty) \) such that \( F_{23}(T) = 0 \).
Appendix F. Lemma 4, Part (a) When $2A \geq (h + cI_c) D(p) t_d^2 - (cI_c - pI_c) D(p)(M - N)^2$, $T_{23}$ is the unique solution of (59) from Lemma 3(a). taking the second order derivative of $TP_{23}(T|p, \xi)$ with respect to $T$ and finding the value of the function at $T_{23}$, we obtain

$$TP_{23}(T|p, \xi) = \frac{1}{T} \left[ pD(p) T - cD(p) t_d - \frac{(h + cI_c) D(p) t_d^2}{2} - (c + (h + cI_c) t_d + cI_c (N - M)) D(p) \int_{t_d}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt - \xi T - A + \frac{pI_c D(p)(M-N)^2}{2} \right]. \tag{F.1}$$

$$\frac{dTP_{23}(T|p, \xi)}{dT} = \frac{1}{T} \left[ pD(p) - (c + (h + cI_c) t_d + cI_c (N - M)) D(p)e^{(1-m(\xi))g(T)} \right] = \frac{1}{T^2} \left[ pD(p) T - cD(p) t_d - \frac{(h + cI_c) D(p) t_d^2}{2} - (c + (h + cI_c) t_d + cI_c (N - M)) D(p) \right] \times \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt + \frac{cI_c D(p) (M - N) (2t_d + N - M)}{2} \times (h + cI_c) D(p) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt + (c + h + cI_c) (N - M) \int_{t_d}^{T} e^{(1-m(\xi))g(T)} dt - \xi T - A + \frac{pI_c D(p)(M-N)^2}{2} = 0, \tag{F.2}$$

$$\frac{d^2TP_{23}(T|p, \xi)}{dT^2} \bigg|_{T=T_{23}} = -\frac{D(p)}{T_{23}^2} \left[ (c + (h + cI_c) t_d + cI_c (N - M)) e^{(1-m(\xi))g(T_{23})} \right] \times (1 - m(\xi)) g'(T_{23}) + (h + cI_c) \left\{ \int_{t_d}^{T_{23}} e^{(1-m(\xi))g(T_{23})} dt \right\} < 0. \tag{F.3}$$

Thus $T_{23}$ is the global maximum point of $TP_{23}(T|p, \xi)$.

Lemma 4, Part (b) From the proof of Lemma 3(b), we know that if $0 < 2A < (h + cI_c) D(p) t_d^2 - (cI_c - pI_c) D(p)(M - N)^2$, then $F_{23}(T) < 0$, for all $T \in [t_d, \infty)$. Thus we have

$$\frac{dTP_{23}(T|p, \xi)}{dT} < 0, \text{ for all } T \in [t_d, \infty), \tag{F.4}$$

which implies that $TP_{23}(T|p, \xi)$ is a strictly decreasing function of $T \in [t_d, \infty)$. So, $TP_{23}(T|p, \xi)$ has a maximum value at the boundary point $T = t_d$.

Appendix G. Proof of Lemma 5, Part (a). Motivated by (69), we define a function $F_{32}(T)$ as follows:

$$F_{32}(T) = A + D(p) \left[ ct_d + \frac{ht_d^2}{2} + (c + ht_d) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt \right] + h \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))g(u) - g(t)} du dt \right] - TD(p) \left[ (c + ht_d) e^{(1-m(\xi))g(T)} \right] + h \int_{t_d}^{T} e^{(1-m(\xi))g(T) - g(t)} dt] - \frac{pI_c D(p)T^2}{2} \tag{G.1}$$

For $T \in [t_d, M - N]$.
Since the first derivative of $F_{32}(T)$ with respect to $T \in [t_d, M - N]$ is

$$\frac{dF_{32}(T)}{dT} = -D(p)T \left[ (c + ht_d) e^{(1-m(\xi))g(T)} (1 - m(\xi)) g'(T) \right]$$

$$+ h \left\{ \int_{t_d}^{T} e^{(1-m(\xi))(g'(T)-g(t))} (1 - m(\xi)) g'(T) dt + 1 \right\} + pI_p < 0 \quad \text{(G.2)}$$

We obtain that $F_{32}(T)$ is a strictly decreasing function of $T$ in the interval $[t_d, M - N]$. Moreover,

$$F_{32}(T)|_{T=t_d} = A + D(p) \left( c + \frac{ht_d^2}{2} \right) - (c + ht_d) t_d D(p) - \frac{pI_pD(p)t_d^2}{2}, \quad \text{(G.3)}$$

$$F_{32}(T)|_{T=M-N} = A + D(p) \left[ c + \frac{ht_d^2}{2} + (c + ht_d) \int_{t_d}^{M-N} e^{(1-m(\xi))g(t)} dt \right]$$

$$+ h \int_{t_d}^{M-N} \int_{t}^{M-N} e^{(1-m(\xi))(g(u)-g(t))} du dt \right] - (M-N) D(p) \left[ (c + ht_d)$$

$$\times e^{(1-m(\xi))(g(M-N) - g(t))} + h \int_{t_d}^{M-N} e^{(1-m(\xi))(g(M-N)-g(t))} dt \right] - \frac{pI_pD(p)(M-N)^2}{2} \quad \text{(G.4)}$$

According to the intermediate value theorem, $\Delta_4 \leq 2A \leq \Delta_5$, then $F_{32}(T)|_{T=M-N} \leq 0$ and $F_{32}(T)|_{T=t_d} \geq 0$, so there exists a unique $T_{32} \in [t_d, M - N]$ such that $F_{32}(T_{32}) = 0$.

**Proof of Lemma 5, Part (b).** If $2A < \Delta_4$ or $2A > \Delta_5$, then $F_{32}(T)|_{T=t_d} < 0$ or $F_{32}(T)|_{T=M-N} > 0$. Since $F_{32}(T)$ is a strictly decreasing function of $T$ in the interval $[t_d, M - N]$. Thus, there is no value of $T \in [t_d, M - N]$ such that $F_{32}(T) = 0$.

**Appendix H.** Proof of Lemma 6, Part (a). When $\Delta_4 \leq 2A \leq \Delta_5$, $T_{32}$ is the unique solution of (69) from Lemma 5(a). taking the second derivative of $TP_{32}(p, T, \xi)$ with respect to $T$ and finding the value of the function at the point $T_{32}$, we obtain

$$TP_{32}(T | p, \xi) = \frac{1}{T} \left[ pD(p) T - cD(p) t_d - \frac{hD(p)t_d^2}{2} - (c + ht_d) D(p) \right]$$

$$\times \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt - \xi T - A + pD(p) I_p T \left( \frac{T}{2} + (M - T - N) \right)$$

$$- hD(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u)-g(t))} du dt \right] \quad \text{(H.1)}$$

$$\frac{dTTP_{32}(T | p, \xi)}{dT} = \frac{1}{T} \left[ pD(p) - (c + ht_d) D(p) e^{(1-m(\xi))g(T)} - \xi + pD(p) \right]$$

$$\times I_p \left( M - T - N \right) - hD(p) \int_{t_d}^{T} e^{(1-m(\xi))(g(T)-g(t))} dt \right] - \frac{1}{T^2} \left[ pD(p) T$$

$$- cD(p) t_d - \frac{hD(p)t_d^2}{2} - (c + ht_d) D(p) \int_{t_d}^{T} e^{(1-m(\xi))g(t)} dt \right.$$\n
$$- \xi T - A + pD(p) I_p T \left( \frac{T}{2} + (M - T - N) \right)$$

$$- hD(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u)-g(t))} du dt \right] = 0 \quad \text{(H.2)}$$

$$\frac{d^2TP_{32}(T|p, \xi)}{dT^2} \bigg|_{T=T_{32}} = - \frac{D(p)}{T_{32}^2} \left[ (c + ht_d) e^{(1-m(\xi))g(T_{32})} (1 - m(\xi)) g'(T_{32}) \right)$$
+pI_p + h \left \{ \int_{t_d}^{T_{32}} e^{(1-m(t))(g(T_{32})-g(t))} \left ( 1-m(t) \right ) g'(T_{32}) dt + 1 \right \} < 0. \quad \text{(H.3)}

Thus, $T_{32}$ is the global maximum point of $TP_{32}(T \mid p, \xi)$.

**Proof of Lemma 6, Part (b).** From the proof of Lemma 5(b), we know that if $2A < \Delta_t$, then $F_{32}(T) < 0$ for all $T \in [t_d, M - N]$. Thus, we have

$$\frac{dTP_{32}(T \mid p, \xi)}{dT} = \frac{F_{32}(T)}{T} < 0, \quad \text{for all } T \in [t_d, M - N] \quad \text{(H.4)}$$

which implies that $TP_{32}(T \mid p, \xi)$ is a strictly decreasing function of $T \in [t_d, M - N]$.

So,

$$\max TP_{32}(T \mid p, \xi) = TP_{32}(t_d \mid p, \xi).$$

**Proof of Lemma 6, Part (c).** From the proof of Lemma 5(b), we know that if $2A > \Delta_t$, then $F_{32}(T) > 0$ for all $T \in [t_d, M - N]$. Thus, we have

$$\frac{dTP_{32}(T \mid p, \xi)}{dT} = \frac{F_{32}(T)}{T} > 0, \quad \text{for all } T \in [t_d, M - N] \quad \text{(H.5)}$$

which implies that $TP_{32}(T \mid p, \xi)$ is a strictly increasing function of $T \in [t_d, M - N]$.

So,

$$\max TP_{32}(T \mid p, \xi) = TP_{32}(M - N \mid p, \xi).$$

**Appendix I. Proof of Lemma 7, Part (a).** Motivated by (74), we define a new function $F_{33}(T)$ as follows:

$$F_{33}(T) = A - \left \{ \frac{pI_p D(p)(M - N)^2}{2} + D(p) \left [ ct_d + \frac{ht_d^2}{2} + (c + ht_d) \int_{t_d}^{T} e^{(1-m(t))g(t)} dt \right ] \right \}$$

$$+ h \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(t))(g(u)-g(t))} duds + cI_c \int_{M - N}^{T} \int_{t}^{T} e^{(1-m(t))(g(u)-g(t))} duds$$

$$- TD(p) \left [ (c + ht_d) e^{(1-m(t))g(T)} + h \int_{t_d}^{T} e^{(1-m(t))(g(T)-g(t))} dt \right ]$$

$$+ cI_c \int_{M - N}^{T} e^{(1-m(t))(g(T)-g(t))} dt \quad \text{(I.1)}$$

Since the first derivative of $F_{33}(T)$ with respect to $T \in [M - N, +\infty]$ is

$$\frac{dF_{33}(T)}{dT} = -TD(p) \left [ (c + ht_d) e^{(1-m(t))g(T)} (1 - m(t)) g'(T) \right ]$$

$$+ h \left \{ \int_{t_d}^{T} e^{(1-m(t))(g(T)-g(t))} (1 - m(t)) g'(T) dt + 1 \right \}$$

$$+ cI_c \left \{ \int_{M - N}^{T} e^{(1-m(t))(g(T)-g(t))} (1 - m(t)) g'(T) dt + 1 \right \} < 0. \quad \text{(I.2)}$$

We obtain that $F_{33}(T)$ is a strictly decreasing function of $T$ in the interval $[M - N, +\infty]$. Moreover, we have

$$F_{33}(T) \mid_{T = M - N} = -\infty$$

and

$$F_{33}(T) \mid_{T = M - N} = A - \left \{ \frac{pI_p D(p)(M - N)^2}{2} + D(p) \left [ ct_d + \frac{ht_d^2}{2} + (c + ht_d) \right ] \right \}$$

$$\times \int_{t_d}^{M - N} e^{(1-m(t))g(t)} dt + h \int_{t_d}^{M - N} \int_{t}^{M - N} e^{(1-m(t))(g(u)-g(t))} duds$$

$$- (M - N) D(p) \left [ (c + ht_d) e^{(1-m(t))g(M - N)} \right ]$$

$$+ h \int_{t_d}^{M - N} e^{(1-m(t))(g(M - N)-g(t))} dt \quad \text{(I.3)}$$
Thus, if $2A \geq \Delta_5$, then $F_{33}(T)|_{T=M-N} \geq 0$. According to the intermediate value theorem, there exists a unique $T_{33} \in [M-N, +\infty)$ such that $F_{33}(T_{33}) = 0$.

**Lemma 7. Part (b)** If $0 < 2A < \Delta_5$, then from (D.3) $F_{33}(M-N) < 0$. Since $F_{33}(T)$ is a strictly decreasing function of $T$ in the interval $[M-N, +\infty)$. Thus, there is no value of $T \in [M-N, +\infty)$ such that $F_{33}(T) = 0$.

**Appendix J. Proof of Lemma 8, Part (a).** When $2A \geq \Delta_5$, $T_{33}$ is the unique solution of (74) from Lemma 6(a). Taking the second derivative of $TP_{33}(p, T, \xi)$ with respect to $T$ and finding the value of the function at the point $T_{33}$, we obtain

\[
TP_{33}(T|p, \xi) = \frac{1}{T} [pD(p)T - cD(p)t_d - \frac{hD(p)t_d^2}{2} - (c + ht_d)D(p)\
\times \int_{t_d}^{T} e^{(1-m(\xi))g(t)}dt - \xi T - A - hD(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt\
- cI_cD(p) \int_{M-N}^{T} e^{(1-m(\xi))(g(T) - g(t))} dt + cI_cD(p) \int_{M-N}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt] (J.1)
\]

\[
\frac{dT P_{33}(T|p, \xi)}{dT} = \frac{1}{T} [pD(p) - (c + ht_d)D(p)e^{(1-m(\xi))g(T)} - \xi - hD(p)
\times \int_{t_d}^{T} e^{(1-m(\xi))(g(T) - g(t))} dt - cI_cD(p) \int_{M-N}^{T} e^{(1-m(\xi))(g(T) - g(t))} dt - \xi T - A - hD(p) \int_{t_d}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(u) - g(t))} du dt\
- cI_cD(p) \int_{M-N}^{T} \int_{t}^{T} e^{(1-m(\xi))(g(T) - g(t))} du dt + \frac{pI_cD(p)(M-N)^2}{2}] = 0 (J.2)
\]

\[
\frac{d^2TP_{33}(T|p, \xi)}{dT^2} \bigg|_{T=T_{33}} = -\frac{D(p)}{T_{33}} [c + ht_d] e^{(1-m(\xi))g(T_{33})} (1 - m(\xi)) g'(T_{33})\
+ cI_c \left\{ \int_{M-N}^{T_{33}} e^{(1-m(\xi))(g(T_{33}) - g(t))} (1 - m(\xi)) g'(T_{33}) dt + 1 \right\} (J.3)
\]

Thus $T_{33}$ is the global maximum point of $TP_{33}(T|p, \xi)$.

**Proof of Lemma 8, Part (b).** From the proof of Lemma 7(b), we know that if $0 < 2A < \Delta_5$, then $F_{33}(T) < 0$ for all $T \in [M-N, +\infty)$. Thus, we have

\[
\frac{dT P_{33}(T|p, \xi)}{dT} = \frac{F_{33}(T)}{T} < 0, \text{ for all } T \in [M-N, +\infty) (J.4)
\]

which implies that $TP_{33}(T|p, \xi)$ is a strictly decreasing function of $T \in [M-N, +\infty)$. So, $TP_{33}(T|p, \xi)$ has a maximum value at the boundary point $T = M-N$ for $T \in [M-N, +\infty)$.

Received July 2020; 1st revision October 2020; final revision March 2021; early access July 2021.

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