BINOMIAL FIBERS AND INDISPENSABLE BINOMIALS

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Abstract. Let $I$ be an arbitrary ideal generated by binomials. We show that certain equivalence classes of fibers are associated to any minimal binomial generating set of $I$. We provide simple and efficient algorithms that compute the indispensable binomials of a general binomial ideal and detect binomial ideals generated by indispensable binomials.

Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. A pure difference binomial is a polynomial of the form $x^u - x^v$ where $u, v \in \mathbb{N}^n$. We say that the ideal $I$ of $R$ is a pure binomial ideal or just binomial ideal for short if $I$ is generated by pure difference binomials. Binomial ideals were first studied in [10] while the class of binomial ideals includes lattice ideals. Recall that if $L \subseteq \mathbb{Z}^n$ is a lattice, then the corresponding lattice ideal is defined as $I_L = \langle x^u - x^v : u - v \in L \rangle$ while the lattice is saturated exactly when the lattice ideal is toric, i.e. prime. The study of binomial ideals is a rich subject: the classical reference is [21] and we also refer to [15] for recent developments. It has applications in diverse areas in mathematics, such as algebraic statistics, integer programming, graph theory, computational biology, code theory, see [7, 9, 13, 17, 18, 22], etc.

A particular problem that arises is the efficient generation of binomial ideals by a set of binomials. Up to now, it has mainly been addressed for toric and lattice ideals, see [3, 4, 5, 11, 12, 21] among others. In Section 1 of this paper, we consider this problem in the case of general binomial ideals. For this we study the fibers of the binomial ideals: in [8, Proposition 2.4] an equivalence relation on $\mathbb{N}^n$ was introduced for any binomial ideal $I$ of $R$. Namely $u \sim_I v$ if $x^u - x^v \in I$. For each such equivalence class, we get a fiber on the set of monomials: the $I$-fiber of $x^u$ is the set $\{x^v : u \sim_I v\}$. When $I := I_L$ is the lattice ideal of $L$, the equivalence class of $u$ consists precisely of all $v$ such that $u - v \in L$ and the $I$-fibers are finite exactly when $L \cap \mathbb{N}^n = \{0\}$. In this case, for each $I$-fiber one can use a graph construction, [7, 4] that determines the $I$-fibers that appear as invariants associated to any minimal generating set of $I$. We also note that in [5] the fibers of $I_L$ were studied even when $L \cap \mathbb{N}^n \neq \{0\}$. In all cases finite or not, it is clear that divisibility of monomials does not induce necessarily a meaningful partial order on the set of $I$-fibers. In this paper for any binomial ideal $I$ we define an equivalence relation on the set of $I$-fibers and then order the equivalence classes of $I$-fibers, see Definition 1.6.

We note that this was first done in [5] for the case of $I = I_L$. This partial order permits us to prove that certain equivalence classes of $I$-fibers are an invariant associated to any generating set of $I$, Theorem 1.9. We note an added degree of complication in the general
case: the equivalence classes for lattice ideals have the same cardinality (finite or infinite), 
\cite[Propositions 2.3 and 3.5]{5} while this might not be the case for a general binomial ideal.

A related question that attracted a lot of interest in the recent years is whether there is a unique minimal binomial generating set for a binomial ideal. One of the first papers to deal with this question for lattice ideals from a purely theoretical point of view, was \cite{20}. As it turns out, the positive answer has applications to Algebraic Statistics: \cite{1, 2, 17}.

Thus in \cite{18} and \cite{2} the notions of indispensable monomials and binomials were defined. Let $I$ be a binomial ideal. A binomial is called indispensable if (up to a sign) it belongs to every minimal generating set of $I$ consisting of binomials. This implies of course that (up to a sign) it belongs to every binomial generating set of $I$. A monomial is called indispensable if it is a monomial term of at least one binomial of every system of binomial generators of $I$. How does one compute these elements?

When $I := I_L$ is a lattice ideal and $L \cap \mathbb{N}^n = \{0\}$ there are several works in the literature that deal with this problem. In particular, in \cite{18} it was shown that to compute the indispensable binomials of $I_L$, one computes all lexicographic reduced Gröbner bases and then their intersection: there are $n!$ such bases; a corresponding result for indispensable monomials was shown in \cite{2}. In \cite{19}, it was shown that to compute the indispensable binomials of $I_L$, it is enough to compute certain degree-reverse lexicographic reduced Gröbner bases of $I_L$ ($n$ of them), and then compute their intersection. In \cite[Proposition 3.1]{4}, it was shown that to find the indispensable monomials of $I_L$, it is enough to consider any one of the binomial generating sets of $I_L$. Moreover in \cite[Theorem 2.12]{4} it was shown that in order to find the indispensable binomials of $I_L$, it is enough to consider any minimal binomial generating set of $I_L$, assign $\mathbb{Z}^n/L$-degrees to the binomials of this set and to compute their minimal $\mathbb{Z}^n/L$-degrees. More recently in \cite[Theorem 1.1, Corollary 1.3]{14}, it was shown that if $I$ is a binomial ideal then there is a $d \in \mathbb{N}$ such that any $I$ is $A$-graded for some $A \subset \mathbb{Z}^d$: when $NA \cap (-NA) = \{0\}$ and all fibers are finite a sufficient condition was given in \cite{14} for the indispensable monomials and a characterization for the indispensable monomials, involving the $I$-fibers of a minimal generating set of $I$.

In this paper, we significantly improve all previously known results regarding indispensable binomials. Moreover our results apply to the general case of all binomial ideals. In Section 2 we show that as in \cite{14}, the indispensable monomials are the elements of the minimal generating set of the monomial ideal of $I$, Remark 2.3. Then we go on and are able to express this condition into three necessary and equivalent conditions involving a graph with vertices the (possibly infinitely many) elements of the fiber, Theorem 2.5. This result is then applied to provide sufficient and necessary conditions for a binomial in $I$ to be indispensable, Theorem 2.6.

In Section 3, we prove that any system of binomial generators of $I$ is actually enough and gives all the necessary information needed to decide whether a given binomial is indispensable, see Theorem 3.3. We apply this to give Algorithm 1 an algorithm that given any system of binomial generators of $I$ computes the indispensable binomials and monomials of $I$ without having to compute any reduced Gröbner basis of $I$, nor the fibers of $I$. We also give Algorithm 2 which performs a check whether the binomial ideal $I$ is generated by indispensables binomials, simply knowing any minimal system of binomial generators of $I$. Both algorithms are extremely fast and efficient in their simplicity.

In Section 4 we generalize the notion of primitive elements to general binomial ideals. The set of all primitive elements is the Graver basis of $I$. This set is extremely important in the theory and all computations involving lattice ideals, see \cite{21}. We prove that the Graver basis of any binomial ideal is finite, Proposition 4.4 and show that it includes as
a subset the universal Gröbner basis of $I$, Theorem 1.3. We show that a Lawrence lifting construction gives a binomial ideal generated by indispensable binomials, Theorem 1.5 strongly connected to the Graver basis of the original ideal. When $I$ is a lattice ideal, the Lawrence lifting construction gives an algorithm for computing the Graver basis of $I$, this was shown in [21]. We comment on the difficulties encountered in the general case.

1. FIBERS OF A BINOMIAL IDEAL

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. We denote by $\mathbb{T}^n$ the set of monomials of $R$ including $1 = x^0$, where $x^u = x_1^{u_1} \cdots x_n^{u_n}$. If $J$ is a monomial ideal of $R$ we denote as usual by $G(J)$ the unique set of minimal monomial generators of $J$. For $B = x^u - x^v$, we let $\text{supp}(B) := \{x^u, x^v\}$.

**Definition 1.1.** Let $I$ be a binomial ideal of $R$. We say that $F \subset \mathbb{T}^n$ is an $I$-fiber if there exists a $x^u \in \mathbb{T}^n$ such that $F = \{x^v : v \sim_I u\}$. If $x^u \in \mathbb{T}^n$, and $F$ is an $I$-fiber containing $x^u$ we write $F_u$ or $F_{x^u}$ for $F$. If $B \in I$ and $B = x^u - x^v$ we write $F_B$ for $F_u$.

It is trivial that $|F_u| = 1$, that is $F_u$ is a singleton, if and only if there is no binomial $0 \neq B \in I$ such that $x^u \in \text{supp}(B)$. If $J \subset I$ gives the containment between two binomial ideals and $F$ is a $J$-fiber, then clearly $F$ is contained in an $I$-fiber.

**Example 1.2.**

a) Let $a \in \mathbb{N}$, $r \in \mathbb{Z}_{\geq 1}$, $L = r\mathbb{Z}$ and $I_1 = (x^a - x^{a+r}) = x^aI_L$. It is immediate that $x^{a+j} - x^{a+nr+j} \in I_1$ for all $j, n \in \mathbb{N}$. The $I_1$-fibers are either singletons or infinite. There are exactly $a$ singletons and $r$ distinct infinite $I_1$-fibers of the form $F_{a+j} = \{x^{a+j+nr} : n \in \mathbb{N}\}$, for $0 \leq j \leq r - 1$. Moreover since

$$x^a - x^{a+r} = (1 - x^r)(x^a - x^{a+3r}) + x^{3r-1}(x^{a+1} - x^{a+r+1}),$$

$I_1 = (x^a - x^{a+3r}, x^a - x^{a+r+1})$. Thus $I_1$ has no indispensable binomials. It is clear that the only indispensable monomial of $I_1$ is $x^a$, see also [5, Theorem 4.17].

b) Let $I_2 = (y - x^6y, y^3 - xy^5, y^4 - y^6, y^7 - y^8)$. The $I_2$-fibers are as follows:

- $F_{x^i} = \{x^i\}$ for all $i \in \mathbb{N}$
- $F_{y^i} = \{yx^{2n} : n \in \mathbb{N}\}$
- $F_{yx} = \{yx^{2n+1} : n \in \mathbb{N}\}$
- $F_{y^2x} = \{y^2x^{2n+1} : n \in \mathbb{N}\}$
- $F_{y^2y} = \{y^2x^{2n} : n \in \mathbb{N}\}$
- $F_{y^3x} = \{y^3x^n : n \in \mathbb{N}\}$
- $F_{y^4} = \{y^{4+m}x^n : m, n \in \mathbb{N}\}$

The $I_2$-fibers are depicted in the left of Figure 1.

c) Consider the ideal $I_3 = (y^5 - xy^6, x^2y^5 - x^3y^3, x^3y^3 - x^5y^2, x^6y - x^8)$. Its fibers are depicted in the right part of Figure 1. There are 29 singleton $I$-fibers, depicted by dots. The other fibers are:

- $F_{xy^6} = \{xy^6, y^8\}$
- $F_{x^3y^3} = \{x^3y^3, x^2y^5, x^5y^2\}$
- $F_{x^6y} = \{x^6y, x^5\}$
- $F_{xy^7} = \{xy^7, y^9\}$
- $F_{x^7y} = \{x^7y, x^9\}$
- $F_{x^4y^3} = \{x^4y^3 : (a, b) \in \mathbb{N}^2 \text{ and } (a, b) \in \{(4, 3) + \mathbb{N}(-1, 2) + \mathbb{N}(2, -1)\}\}$
- $F_{x^3y^4} = \{x^3y^4 : (a, b) \in \mathbb{N}^2 \text{ and } (a, b) \in \{(3, 4) + \mathbb{N}(-1, 2) + \mathbb{N}(2, -1)\}\}$
one can extend Proposition 2.6 of [5], by essentially the same proof.

If \( G \subset \mathbb{T}^n \) and \( t \in \mathbb{N}^n \) we let \( x^t G := \{ x^t u : u \in G \} \).

**Theorem 1.3.** Let \( F \) be a partition of \( \mathbb{T}^n \). There exists a binomial ideal \( I \) such that \( F \) is the set of \( I \)-fibers if and only if for any \( u \in \mathbb{N}^n \) and any \( F \in F \) there exists a \( G \in F \) such that \( x^u F \subset G \).

*Proof.* Let \( I \) be a binomial ideal. Let \( F := F_t \) be an \( I \)-fiber and let \( u \in \mathbb{N}^n \). It is clear \( x^u F \subset F_{t+u} \). For the converse we let \( F = (F_i : i \in \Lambda) \) and \( I \) be the ideal generated by the set \( \{ x^u - x^v : u, v \in F_i, i \in \Lambda \} \). Consider the set \( G \) of \( I \)-fibers. We will show that \( G = F \). Indeed let \( G \) be an \( I \)-fiber and \( x^u \in G \). Since \( F \) is a partition of \( \mathbb{T}^n \), there is an \( F \in F \) such that \( x^u \in F \). From the definition of \( I \), it is clear that \( F \subset G \).

For the converse inclusion suppose that \( x^v \in G \) and thus \( x^y - x^u \in I \). It follows that \( x^y - x^u = \sum_{k=1}^{l} c_k x^{w_k} (x^{v_k} - x^{u_k}) \) where \( x^{u_k}, x^{v_k} \in F_{i_k}, i_k \in \Lambda, c_k \in K \), and \( l \geq 1 \). Therefore for some \( k \), \( x^{w_k} x^{u_k} = x^u \in F \). By the hypothesis on the elements of \( F \) it follows that \( x^{w_k} x^{v_k} \in F \). An easy induction on \( l \) finishes the proof. \( \square \)

A vector \( u \in \mathbb{Z}^n \) is called pure if \( u \in \mathbb{N}^n \) or \(-u \in \mathbb{N}^n \). If \( B = x^u - x^v \in I \), we let \( \mathbf{v}(B) = u - v \). If \( F \) is a fiber of a binomial ideal \( I \), we let \( L_F := \langle \mathbf{v}(B) : F_B = F \rangle \subset \mathbb{Z}^n \).

We also consider \( L_{\text{pure},F} \), the sublattice of \( L_F \) generated by the set
\[
\{ w \in \mathbb{N}^n : \exists x^u, x^v \in F \text{ such that } w = u - v \},
\]
and denote by \( L_{\text{pure},F}^+ \) the semigroup generated by the same set. Finally we let \( M_F \) be the monomial ideal of \( R \) generated by the elements of \( F \). In some cases it might be that the monomials in \( M_F \) are precisely the elements of \( F \), for example if \( I = (1-x) \) in \( K[x] \), but usually this is far from being the case. If \( I \) is a lattice ideal and \( I = I_L \) then \( L_F \subset L \) and by [5] Proposition 2.3 or [10] Theorem 8.6 it follows that \( F \) is an infinite fiber if and only if \( L_F \) contains a nonzero pure element. In the case of an arbitrary binomial ideal \( I \), one can extend Proposition 2.6 of [5], by essentially the same proof.
Proposition 1.4. Let I be a binomial ideal and F be an I-fiber. If \( \{x^{a_1}, \ldots, x^{a_s}\} \) is the minimal monomial generating set of \( M_F \) then

\[
F = \bigcup_{i=1}^{s} \{x^{a_i}x^w : w \in L_{\text{pure},F}^+\}.
\]

In particular, F is infinite if and only if \( L_{\text{pure},F} \neq 0 \).

The next example comments on certain subtleties of the above proposition.

Example 1.5. Let \( I_3 \) be the ideal of Example 1.2 c). The fiber \( F_1 = F_{x^3y^3} \) is finite even though \( L_{F_1} = \langle (2,-1),(-1,2) \rangle \) contains \( (1,1) \). Note that \( (1,1) \notin L_{\text{pure},F_1} \) and \( L_{\text{pure},F_1} = (0) \). For the fiber \( F_2 = F_{x^4y^3} \), one can see that

\[
M_{F_2} = (x^{10}, x^6y, x^6y^2, x^4y^3, x^3y^5, x^2y^7, xy^9, y^{11}),
\]

thus \( L_{\text{pure},F_2} = \langle (1,1), (0,3) \rangle \) while \( L_{\text{pure},F_2}^+ = (0,3)\mathbb{N} + (1,1)\mathbb{N} + (3,0)\mathbb{N} \). It follows that \( \{x^{10}, x^{6+3c}y^{3a+b} : a, b, c \in \mathbb{N}\} \subset F_2 \).

Let I be a binomial ideal. We define an equivalence relation \( \equiv \) on the set of I-fibers and a partial order \( < \) of the equivalence classes which generalize those from [5]:

Definition 1.6. If \( F, G \) are I-fibers, we let \( F \equiv G \) if there exist \( u, v \in \mathbb{N}^n \) such that \( x^uF \subset G \) and \( x^vG \subset F \) and denote by \( \overline{F} \), the equivalence class of \( F \). We set \( \overline{F} \leq \overline{G} \) if there exists \( u \in \mathbb{N}^n \) such that \( x^uF \subset G \). We write \( \overline{F} < \overline{G} \) if \( \overline{F} \leq \overline{G} \) and \( \overline{F} \neq \overline{G} \).

In [5] Proposition 3.5 it was shown for lattice ideals that any two equivalence classes of fibers have the same cardinality. This is no longer necessarily true for an arbitrary binomial ideal I as you can see in the Example 1.7 b), but the cardinality of \( \overline{F} \) for an I-fiber \( F \) can be computed similarly by replacing \( L_F \) with \( L_{\text{pure},F} \).

Example 1.7.

a) Let \( I_1 \) be the ideal of Example 1.2 a). The infinite \( I_1 \)-fibers are equivalent.

b) Let \( I_2 \) be the ideal of Example 1.2 b). We note that

- \( F_y = \{F_y,F_{xy}\} \),
- \( F_{y^2} = \{F_{y^2},F_{xy^2}\} \),
- \( F_{y^3} = \{F_{y^3}\} \),
- \( F_{y^4} = \{F_{y^4}\} \).

The set of equivalence classes of \( I_2 \)-fibers is totally ordered, the maximal element being \( \overline{F}_{y^4} \) and the minimal one \( \overline{F}_1 \), where \( F_1 = \{1\} \).

c) Let \( I_3 \) be the ideal of Example 1.2 c). The three infinite fibers \( F_{x^4y^3}, F_{x^5y^3}, F_{x^4y^4} \) are equivalent. The equivalence classes \( \overline{F}_{x^4y^3}, \overline{F}_{xy^3} \) and \( \overline{F}_{x^3y^3} \) are incomparable and minimal when restricting to equivalence classes of fibers of cardinality greater than one.

The proof of [5] Theorem 3.8] applies for an arbitrary binomial ideal I and the following holds:

Theorem 1.8. Let I be a binomial ideal. Any descending chain of equivalence classes of fibers

\[
\overline{F} > \cdots > \overline{F}_k > \overline{F}_{k+1} > \cdots
\]

terminates.
It is easy to see that if $F$ is finite then $\overline{F} = \{F\}$. In the next example we compute the equivalence classes of the infinite fibers for the binomial ideals of Example 1.2.

Let $F$ be a fiber of a binomial ideal $I$. We define two binomial ideals contained in $I$

$$I_{\leq F} = (B \in I : B \text{ binomial}, \overline{F_B} < \overline{F}), \quad I_{\leq F} = (B \in I : \overline{F_B} \leq \overline{F}).$$

We say that $F$ is a Markov fiber if there exists a minimal binomial generating set $S$ for $I$ and $B \in S$ such that $F_B \equiv F$. The definition of the equivalence relation among the fibers together with the induced partial order allows us to identify the equivalent classes of the Markov fibers and to prove that the set of equivalence classes of Markov fibers is an invariant of the ideal. This is the content of the next theorem, whose proof is omitted since it is a direct generalization of the proof of [5, Proposition 3.12].

Theorem 1.9. Let $I$ be a binomial ideal, $S$ a binomial generating set of $I$ and $F$ an $I$-fiber. Then

$$I_{\leq F} = (B : B \in S, \overline{F_B} < \overline{F}) \text{ and } I_{\leq F} = (B : B \in S, \overline{F_B} \leq \overline{F}).$$

Thus $F$ is a Markov fiber of $I$ if and only if $I_{\leq F} \neq I_{\leq F}$. Moreover the set $\{\overline{F_B} : B \in S\}$ is an invariant of $I$.

For a lattice ideal $I$ we have even more: if $S_1, S_2$ are two minimal binomial generating sets of $I$ of minimal cardinality, then $\{\overline{F_B} : B \in S_1\} = \{\overline{F_B} : B \in S_2\}$ where the equality holds for the multisets (i.e. sets together with the multiplicities of their elements), see [5, Corollary 4.14].

Example 1.10. Let $I_2$ be the ideal of Example 1.2 b), which is minimally generated by the four binomials. The set of equivalence classes of Markov fibers of $I_2$ is the set $\{\overline{F_y}, \overline{F_y}, \overline{F_y}, \overline{F_y}\}$. It is not hard to see that $I_2$ has also a minimal generating set of cardinality three: $I_2 = (y - x^2y, y^3 - xy^3, y^4 - y^5)$.

2. Indispensable Monomials and Binomials

Let $I$ be a binomial ideal and $F$ an $I$-fiber. We consider the monomial ideals $M_{\overline{F}}$ generated by the monomial in all fibers $G$ where $G \equiv F$, and $M_I$ generated by all monomials $x^u \in \text{supp}(B)$, where $B \in I$ is a nonzero binomial. It is clear that $M_F \subseteq M_{\overline{F}}$. We note the following:

Lemma 2.1. If $I = (x^{u_1} - x^{v_1}, \ldots, x^{u_s} - x^{v_s})$ then $M_I = (x^{u_1}, x^{v_1}, \ldots, x^{u_s}, x^{v_s})$.

Proof. One inclusion is immediate. For \(\subseteq\) suppose that $x^u - x^v \in I$. It follows that there are $f_j \in R$ such that

$$x^u - x^v = \sum_{j=1}^s f_j(x^{u_j} - x^{v_j}).$$

Thus for some $i, k \in [s]$, $x^{u_i}$ or $x^{v_i}$ divides $x^u$, and $x^{u_k}$ or $x^{v_k}$ divides $x^v$. Therefore $x^u, x^v$ belong to $(x^{u_1}, x^{v_1}, \ldots, x^{u_s}, x^{v_s})$. $\square$

We also note that if $x^u$ is a minimal monomial generator of $M_I$ then $x^u$ is also a minimal monomial generator of $M_{\overline{F_u}}$ and of $M_{F_u}$.

Example 2.2. Let $I_2$ be the ideal of Example 1.7 b). Then $M_{I_2} = (y)$. Also, if $F = F_{xy^2}$ then $M_F = (xy^2)$ and $M_{\overline{F}} = (y^2)$. If $G = F_y$ then $M_G = M_{\overline{F}} = M_{I_2}$. 

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Theorem 2.6. Let $I$ be a binomial ideal and $S$ a system of binomial generators of $I$. The indispensable monomials of $I$ are precisely the elements of $G(M_I)$. Moreover $G(M_I)$ comes from $\bigcup_{B \in S} \text{supp}(B)$ by keeping the minimal elements according to divisibility.

Next, for $F$ an $I$-fiber, we define the graph $\Gamma_F$ (on possibly infinitely many vertices):

Definition 2.4. $\Gamma_F$ is the graph with vertices the elements of $F$ and edges $\{x^u, x^v\}$ whenever $x^u - x^v \in \mathbb{I}_{<F}$.

The connected components of $\Gamma_F$ determine the indispensable monomials and binomials as the next two theorems show.

Theorem 2.5. Let $I$ be a binomial ideal. The monomial $x^u$ is a minimal monomial generator of $M_I$ (and thus an indispensable monomial) if and only if the following conditions hold simultaneously

(a) $|F_u| \geq 2$,
(b) $x^u$ is an isolated vertex of $\Gamma_{F_u}$,
(c) $x^u$ is a minimal generator of $M_{F_u}$.

Proof. Assume first that $x^u$ is a minimal monomial generator of $M_I$. Conditions (a) and (c) follow immediately. For condition (b) we suppose that $x^u$ is not an isolated vertex of $\Gamma_{F_u}$: there exists $x^v \in F_u$ so that $\{x^u, x^v\}$ is an edge of $\Gamma_{F_u}$ and thus $x^u - x^v \in \mathbb{I}_{<F_u}$. By Theorem 1.9, $x^u - x^v$ is not part of any minimal binomial generating set of $I$, and using Lemma 2.1 we obtain that $x^u$ is not a minimal generator of $M_I$, a contradiction.

For the converse, assume that $x^u$ satisfies conditions (a)-(c). Suppose that $x^u$ is not a minimal monomial generator of $M_I$. Since $|F_u| \geq 2$, we conclude that there exists $x^\nu \in \mathbb{N}^n$ such that $0 \neq x^u - x^\nu \in I$. Since $x^u \notin G(M_I)$ we conclude that there exists a minimal binomial system of generators $S$ of $I$ and $B \in S$ so that $x^u = x^w x^u'$ with $x^u' \in \text{supp}(B)$ and $x^w \neq 1$. In particular we get that $\text{F}_{w'} \leq \text{F}_u$. We first examine the case where $B \in \mathbb{I}_{<F_u}$. If $\text{supp}(B) = \{x^u', x^v\}$ we let $x^w = x^v \cdot x^w \in F_u$. It follows that $\{x^u, x^w\}$ is an edge of $\Gamma_{F_u}$, contradicting condition (b). If $B \notin \mathbb{I}_{<F_u}$ then $F_u \equiv F_{u'}$ and since $x^u'$ properly divides $x^u$ it follows that $x^u$ is not a minimal generator of $M_{F_u}$, a contradiction to (c).

The description of the indispensable monomials in terms of the graph of their fibers will be useful in the next theorem.

Theorem 2.6. Let $I$ be a binomial ideal. The binomial $B \in I$ is indispensable if and only if the graph $\Gamma_{F_B}$ consists of two isolated vertices.

Proof. Let $F = F_B$, $\text{supp}(B) = \{x^u, x^v\}$. Assume first that the graph $\Gamma_F$ consists of two isolated vertices, that is $F = \text{supp}(B)$ and $B \notin \mathbb{I}_{<F}$. By Theorem 1.9 any generating set of $I$ contains a binomial $B'$ such that $F_{B'} \equiv F$. However since $F$ is finite it follows that $F = \{F\}$ and since $F = \text{supp}(B)$ it must be that $B' = \pm B$. Hence $B$ is indispensable.

For the other direction, assume that $B$ is indispensable. Thus the elements of $\text{supp}(B)$ are indispensable monomials. By Theorem 2.5 the elements of $\text{supp}(B)$ are isolated vertices of $\Gamma_F$ and $B \notin \mathbb{I}_{<F}$. It remains to show that $F = \text{supp}(B)$. Suppose that $x^w \notin \text{supp}(B)$
is also in $F$. Let $S$ be any minimal generating set of $I$. It necessarily contains $B$. The set $S \setminus \{B\} \cup \{x^u - x^w, x^v - x^w\}$ is a system of generators of $I$ and when we minimize it we obtain a minimal system of generators of $I$ not containing $B$, a contradiction since $B$ is indispensable.

Example 2.7. Let $I_3$ be the ideal of Example 1.2 c). Since

$G(M_{I_3}) = \{x^8, x^6 y, x^5 y^2, x^3 y^3, x^2 y^5, x y^6, y^8\}$

it follows that $I_3$ has seven indispensable monomials, by Remark 2.3. This in terms implies that a minimal generating set of $I_3$ can have cardinality no less than four. The indispensable binomials of $I_3$ are $x^6 y - x^8$ and $x y^6 - y^8$ as follows from Theorem 2.6 and the study of the fibers of the indispensable monomials, see Examples 1.2 c), 1.7 c).

The following result generalizes [14, Corollary 1.11].

Corollary 2.8. Let $J \subset I$ be two binomial ideals. If $B$ is an indispensable binomial of $I$ and $B \in J$ then $B$ is indispensable in $J$.

Proof. By Theorem 2.6 we have that the $I$-fiber $F_B$ is equal to $\text{supp}(B)$. Since the fiber of $B$ in $J$ is a subset of $F_B$ and contains $\text{supp}(B)$, it follows that it equals $F_B$. Moreover since $J_{\leq F_B} \subset I_{\leq F_B}$ it follows that $B \notin J_{\leq F_B}$. The conclusion now is obtained by applying Theorem 2.6 one more time. \qed

3. Computing indispensable binomials of $I$

Let $I$ be a binomial ideal and $S$ a system of binomial generators of $I$. In this section we provide an algorithm for finding the indispensable binomials of $I$. This algorithm generalizes by far the three algorithms known to us which are given in the restrictive case of positively $A$-graded toric ideals, (i.e. $NA \cap (-NA) = \{0\}$). We recall that:

- the algorithm in [13, Theorem 2.4] implies computation of $n!$ reduced Gröbner bases with respect to the lexicographic orders,
- the algorithm in [4, Theorem 3.4] implies the computation of one Gröbner basis and the knowledge of the minimal elements in the set of $I$-fibers,
- the algorithm in [19, Theorem 13] implies computation of $n$ reduced Gröbner basis with respect to $n$ degree reverse lexicographic orders.

The simplicity and the swiftness of the algorithm we propose in this section depends only on the information given by $S$. First we define a graph with vertices the (at most $2|S|$) elements in the union of the supports of the binomials of $S$.

Definition 3.1. Let $\mathcal{F}(S)$ be the graph whose vertices are the monomials in $\bigcup_{B \in S} \text{supp}(B)$ and edges $\{x^u, x^v\}$ whenever $x^u - x^v \in S$.

Note that if $S$ is a minimal binomial generating set then $\mathcal{F}(S)$ is a forest.

Example 3.2. Let $I_3$ be as in Example 1.2 c), generated by $S = \{y^8 - xy^6, x^2 y^5 - x^3 y^3, 3x^3 y^3 - x^5 y^2, x^6 y - x^8\}$. The graph $\mathcal{F}(S)$ is depicted below:

$$
\begin{array}{c}
y^8 & xy^6 & x^3 y^3 & x^2 y^5 & x^8 & x^6 y \\
x^5 y^2 & & & & & \\
\end{array}
$$

$\mathcal{F}(S)$
Theorem 3.3. Let $S$ be a binomial generating set of $I$. The binomial $B \in S$ is 
dispensable if and only if $\text{supp}(B) \subset G(M_I)$ and $\text{supp}(B)$ is a connected 
component of $\mathcal{F}(S)$.

Proof. If $B \in S$ is indispensable then the first assertion follows from Remark 2.3. Moreover 
by Theorem 2.6 $|F_B| = 2$. Thus $\text{supp}(B)$ is necessarily a connected component of $\mathcal{F}(S)$.

For the converse, assume that $\text{supp}(B) = \{x^u, x^v\} \subset G(M_I)$ and $\text{supp}(B)$ is a connected 
component of $\mathcal{F}(S)$. The last condition implies that $\text{supp}(B) \cap \text{supp}(B') = \emptyset$ for all 
$B' \in S$, $B' \neq B$. We can assume that $S = \{B_1, B_2, \ldots, B_s\}$ (where $B = B_1$) and that 
$B_j = x^{u_j} - x^{v_j}$. Suppose now that $x^w \in F_B \setminus \text{supp}(B)$. Thus 
\[(1) \quad x^u - x^w = \sum_{j,t} c_{a_j,t} x^{a_j,t} B_j,\]

where the monomials $x^{a_j,t}$ are pairwise different and $c_{a_j,t} \in \mathbb{K}$. Since $x^u \in G(M_I)$ and 
$x^u \notin \text{supp}(B_j)$ for $j \neq 1$ we can assume that $c_{a_{1,1}} = 1$ and $x^{a_{1,1}} = 1$. By our assumption 
$x^w \neq x^v = x^v$. Thus $x^v$ must appear at least twice in the RHS of Equation 1. It follows 
that $x^v$ is divisible by $x^{u_j}$ or $x^{v_j}$ for some $j \neq 1$. Since $x^v \in G(M_I)$ this would imply 
that $x^v = x^{u_j}$ or $x^v = x^{v_j}$. However this is impossible since $\text{supp}(B)$ is a connected 
component of $\mathcal{F}(S)$. Therefore our assumption is wrong and $F_B = \text{supp}(B)$. Assume now 
that $B \in I_{<F_B}$. By Theorem 2.6 
\[(2) \quad x^u - x^v = \sum_{j,t} c_{a_j,t} x^{a_j,t} B_j,\]

where the binomials $B_j$ that appear in the RHS have the property that $\overline{F_{B_j}} \subset F_B$. Thus 
x^u is properly divisible by some monomial in $\text{supp}(B_j)$ for $j \neq 1$. Since $x^u \in G(M_I)$ this 
is a contradiction. Thus $B \notin I_{<F_B}$ and $\Gamma_{F_B}$ consists of two isolated vertices. The desired 
conclusion follows from Theorem 2.6. \hfill \Box

The following algorithm is an immediate application of Theorem 3.3.

Algorithm 1 Computing the indispensable binomials of a binomial ideal $I$

\[\text{Input: } F = \{B_1, \ldots, B_s\} \subseteq \mathbb{K}[X], \text{ with } B_i = x^{u_i} - x^{v_i} \text{ for } i \in [s].\]

\[\text{Output: } F' \subset F, \text{ the set of indispensable binomials of } I.\]

1: Compute $G(M_I)$, a subset of $\{x^{u_1}, x^{v_1}, \ldots, x^{u_s}, x^{v_s}\}$, and set $T = \{i : \{x^{u_i}, x^{v_i}\} \subset G(M_I)\}$.

2: If $T = \emptyset$ then $F' = \emptyset$.

3: Otherwise, for every $i \in T$ check whether $x^{u_i} \in \text{Supp}(B_j)$ or $x^{v_i} \in \text{Supp}(B_j)$ for some 
   $j \neq i$.

4: $F' = \{x^{u_i} - x^{v_i} : i \in T, x^{u_i} \notin \text{Supp}(B_j), x^{v_i} \notin \text{Supp}(B_j), \text{ for all } j \neq i\}$.

We note that step 1 of Algorithm involves checking the divisibility relations of the 
elements of $\{x^{u_1}, x^{v_1}, \ldots, x^{u_s}, x^{v_s}\}$.

Corollary 3.4. Let $I$ be a binomial ideal minimally generated by $s$ binomials. $I$ is generated 
by indispensable binomials if and only if $|G(M_I)| = 2s$.

Proof. Assume that $I$ is minimally generated by the binomials $x^{u_1} - x^{v_1}, \ldots, x^{u_s} - x^{v_s}$. 
If the binomials are indispensable then $|G(M_I)| = 2s$ by Theorem 3.3. Conversely, if $|G(M_I)| = 2s$ we obtain the desired conclusion from Algorithm 1. \hfill \Box
This as before leads to a contradiction.

\[ S \text{ then an arbitrary element of } f \in G \]

We argue by contradiction: assume that
\[ \text{Example 4.2. Let } I \text{ be a binomial ideal. A binomial } 0 \neq x^u - x^v \in I \text{ is called a primitive binomial of } I \text{ if there exists no other binomial } 0 \neq x^{u'} - x^{v'} \in I \text{ such that } x^{u'} \text{ divides } x^u \text{ and } x^{v'} \text{ divides } x^v. \text{ The set of all primitive binomials of } I \text{ is called the Graver basis of } I, \text{ and denoted by } \text{Gr}(I). \]

\[ \text{Example 4.2. Let } I_2 \text{ be the ideal of Example 1.2. It can be easily seen that Gr}(I_2) = \{ y - x^2y, y^3 - xy^3, y^4 - y^5 \}. \]

As in the case of lattice ideals, one can generalize [21, Lemma 4.6] to show that all elements of the universal Gröbner basis of \( I \) are primitive. Below we include the proof for completeness.

**Proposition 4.3.** Let \( I \) be a binomial ideal. Every binomial in the universal Gröbner basis of \( I \) is contained in Gr\((I)\). In particular, Gr\((I)\) is a generating set for the ideal \( I \).

**Proof.** We argue by contradiction: assume that \( f = x^u - x^v \notin \text{Gr}(I) \) while \( f \in G_\prec \), the reduced Gröbner basis of \( I \) according to the monomial order "$<". Moreover suppose that \( \text{in}(f) = x^u > x^v \), i.e. \( x^u \in \text{Gr}(\text{in}(I)) \). Since \( f \) is not primitive, there exists \( g = x^{u'} - x^{v'} \in I \) such that \( f \neq g \) and \( x^{u'} | x^u, x^{v'} | x^v \). If \( \text{in}(g) = x^{v'} \) then \( x^{v'} \in \text{in}(I) \) and thus \( x^v \) is divisible by an element of \( \text{Gr}(\text{in}(I)) \). This leads to a contradiction since \( f \in G_\prec \). If \( \text{in}(g) = x^{u'} \) then \( x^{u'} = x^u \). Thus \( f - g = x^v - x^{v'} \in I \) and \( \text{in}(x^v - x^{v'}) = x^v \). This as before leads to a contradiction. \( \Box \)

To prove that \( \text{Gr}(I) \) is finite, first we remark that if \( S \) is an infinite set of monomials then an arbitrary element of \( S \) is divisible by some element of \( G(S) \). Thus if whenever \( m, m' \in S \) we have that \( m \) does not divide \( m' \) neither \( m' \) divides \( m \) then \( S \) is necessarily finite.

**Proposition 4.4.** Let \( I \) be a binomial ideal. The Graver basis of \( I \), Gr\((I)\), is a finite set.
Proof. We consider the set \( S = \{ x^u y^v, x^v y^u : \pm x^u - x^v \in \text{Gr}(I) \} \). It is immediate that there are no divisibility relations among distinct elements of \( S \). Thus \( S \) is finite. It follows that \( \text{Gr}(I) \) is finite. \( \Box \)

One can think of the pairs that form \( S \) in the above proof, as the support of binomials that generate an ideal closely resembling the binomial ideal of the Lawrence lifting of \( I \), see [21, Theorem 7.1]. In the next theorem we study this ideal.

**Theorem 4.5.** Let \( I \) be a nonzero binomial ideal and let \( \Lambda(I) := \{ x^u y^v - x^v y^u : \pm x^u - x^v \in \text{Gr}(I) \} \subset K[x, y] \).

Then \( \{ x^u y^v - x^v y^u | x^u - x^v \in \text{Gr}(I) \} \) is a minimal system of generators of \( \Lambda(I) \) consisting of indispensable binomials.

**Proof.** The conclusion follows immediately by Algorithm 2 and Proposition 4.4. \( \Box \)

**Remark 4.6.** Note that the conclusion of Theorem 4.5 holds for any binomial ideal \( J \), where \( J = \{ x^u y^v - x^v y^u : \pm x^u - x^v \in A, \emptyset \neq A \subset \text{Gr}(I) \} \).

We also remark that when \( I \) is a toric ideal, then the ideal \( \Lambda(I) \) is precisely equal to the ideal of [21, Theorem 7.1]: the generating set of \( \Lambda(I) \) is a Graver basis of \( \Lambda(I) \) and a universal Gröbner basis of \( \Lambda(I) \) at the same time. However in the general case we need to emphasize that the set of Theorem 4.5 is not a Graver basis of \( \Lambda(I) \), nor a reduced Gröbner basis or the universal Gröbner basis of \( \Lambda(I) \). Moreover there is no one to one correspondence between the elements of \( \text{Gr}(\Lambda(I)) \) and the elements of \( \text{Gr}(I) \).

**Example 4.7.** Let \( I_3 \) be the ideal of Example 1.2. Then \( \text{Gr}(I_3) \) has 35 elements, while the Graver basis of \( \Lambda(I_3) \) consists of at least 113 elements. Moreover the reduced Gröbner bases of \( \Lambda(I_3) \) with respect to lex and degrevlex order differ.

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