Limiting Behavior of Resistances in Triangular Graphs

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Abstract
Barrett et al. (2019) studied resistance labels of electrical circuits whose underlying graphs when embedded in the Cartesian plane has the form of an \(n\)-grid, \(n\) rows of upright triangles. Proofs in Barrett et al. (2019) introduced a row-reduction algorithm which uses series, \(\Delta\)-Y, and Y-\(\Delta\) electric transformations to transform an \(n\)-grid into an \(n-1\) grid with equivalent resistances between specified nodes. This paper explores this row-reduction algorithm computationally. The introductory part of the paper presents several conjectures supported by numerical evidence, showing that repeated application of the row-reduction algorithm to an initial \(n\)-grid uniformly labeled 1 asymptotically produces triangular grids whose sides are labeled with rational multiples of \(\frac{1}{e}\); moreover, the ratio of specified consecutive edges in the row-reduced grids are asymptotically described by four rational functions. The main part of this paper studies a family of graphs whose edge labels are determined using these limiting edge-ratios functions arising in the conjectures. The main result proven is that these \(n\)-grids and their repeated reductions under the row-reduction algorithm possess vertical and rotational symmetries and satisfy the relationships captured by the four edge-ratio functions. Thus, the limiting edge-ratio relationships are local algebraic relationships mirroring the global vertical and rotational symmetries possessed by the underlying graph. Additionally, because row-reduction is local (in contrast to the combinatoric Laplacian which is global) the paper is able to introduce a mechanical verification method of proof for assertions about effective resistance identities.

Keywords: electric resistance, effective resistance, circuit transformations, triangular \(n\)-grid, automated proof, symmetry, local-global

MSC Classification: 05C85, 68W05, 94C15, 05E99
1 Background and Goals

Several papers study resistance properties of graphical configurations where edges are labeled with one-ohm resistances Barrett et al. (2019, 2020a,b). These configurations are of interest in circuit theory and chemistry Bapat (1999); Berberler (2023); Bollobás (1998); Chu et al. (2020); Klein and Randić (1993); Jia-Bao Liu (2020); Yue Ma (2019); Stevenson (1975), combinatorial matrix theory Bapat (2014); Yang and Klein (2013), and spectral graph theory Bapat (1999); Bollobás (1998); Chen and Zhang (2007); Spielman and Srivastava (2011).

A recent paper Barrett et al. (2019) devoted to 2-linear trees, mentions, in passing, the triangular grid and offers a conjecture on the total resistance between any two corner vertices (vertices of degree 2). The proofs in Barrett et al. (2019) utilize a row-reduction algorithm which uses transformations that preserve electric resistance to reduce an \( n \)-row triangular grid to an \( n - 1 \) triangular grid.

The current paper explores computational uses of this row reduction algorithm. In the introductory part of the paper several conjectures are presented, supported by numerical evidence, showing that repeated application of the row-reduction algorithm to an initial \( n \)-grid uniformly labeled 1 asymptotically produces triangular grids whose sides are labeled with rational multiples of \( \frac{1}{e} \); moreover, the ratio of specified consecutive edges in these grids are asymptotically equal to the values of four rational functions.

Motivated by this conjecture, the main part of this paper studies a family of graphs whose edge labels are determined using the limiting edge-ratio rational functions arising in the conjectures. These graphs have interest independent of the conjectures which motivated them since the graphs of circuits in the literature typically are uniformly labeled one.

The main result of this paper is that the resulting \( n \)-grids including their repeated reductions under the row-reduction algorithm possess vertical and rotational symmetries and satisfy the relationships implied by the four limiting edge-ratio rational functions; in other words, these four limiting edge-ratio rational functions are local algebraic relationships that mirror the global vertical and rotational symmetries possessed by the underlying graph.

In the next section (Section 2) we introduce notations and conventions used throughout the paper as well as present the row-reduction algorithm. This is followed by the precise formulation of, and the numerical evidence for, the seven conjectures (Section 3). The rest of the paper motivates (Section 4) and states the three main theorems of the paper (Section 6) which are then proven in the concluding sections (Sections 7-10).

2 Notations, Conventions, and Algorithms

This section presents definitions, notations, and conventions for five items: the \( n \)-grid, a review of resistance-preserving electric-circuit transformations, the row-reduction algorithm, the four local edge functions, and proof methods. The definitions and notations are found in Evans and Hendel (2022) and therefore attribution will not be repeated on each item.
Definition 1 and Figure 1 present the definition of the $n$-grid and illustrate the conventions and notations associated with it.

**Definition 1.** A $n$-triangular-grid (usually abbreviated as an $n$-grid) is any graph that is (graph-) isomorphic to the graph, whose vertices are all integer pairs $(x,y) = (2r+s,s)$ in the Cartesian plane, with $r$ and $s$ integer parameters satisfying $0 \leq r \leq n, 0 \leq s \leq n - r$; and whose edges consist of any two vertices $(x,y)$ and $(x',y')$ with (i) $x' - x = 1, y' - y = 1$, (ii) $x' - x = 2, y' - y = 0$, or (iii) $x' - x = 1, y' - y = -1$.

**Fig. 1:** A 3-grid embedded in the Cartesian Plane constructed using Definition 1 (Left Panel) with the paper’s conventions on row and diagonal coordinates (Right Panel).

Recall that electric circuit theory introduces four transformations - series, parallel, $\Delta$-$Y$, and $Y$-$\Delta$ that preserve (electric) resistance. The series transformation, for example, takes a labeled 3-vertex subgraph with two edges and transforms (or replaces) it with a 2-vertex subgraph whose single edge is labeled with the sum of the labels (resistances) of the two edges of the 3-vertex subgraph. Thus, this transformation preserves resistance. Equation 1 and Figure 2 review the $\Delta$-$Y$ and $Y$-$\Delta$ circuit transformations which transform initial circuits into simpler circuits with equivalent electric resistances. The figure establishes several conventions about functional notation and labels used throughout the paper.

\[
\Delta(x,y,z) = \frac{xy}{x+y+z}, \quad Y(a,b,c) = \frac{ab+bc+ca}{a}.
\]

Throughout the paper, we refer to the edges of the triangles using the notation $T_{X,r,d}$ for some $X \in \{L,R,B\}$ which respectively stand for the left, right, and base side of the triangle. Note that we abuse notation; for example, $L$ refers both to the left edge and to the label of the left edge; however, meaning will always be clear from context. The notation for the $Y$ legs is similarly constructed: e.g. $Y_{12,1,2}$ is the 12 o’clock leg arising from applying a $Y$-$\Delta$ transformation to $T_{1,2}$.

Equation (1) implies the following scaling properties of these functions.

\[
\Delta(jx,jy,jz) = j\Delta(x,y,z), \quad Y(jx,jy,jz) = jY(x,y,z), \text{ for any real } j.
\]
Fig. 2: Definition and orientation conventions for the $\Delta$-$Y$ and $Y$-$\Delta$ circuit transformations. $S, T, U$ are vertex labels; $L, R, B$ are edge labels standing for left, right, and base edge. The $Y$ legs are labeled with an intuitive clock notation.

$$
R = Y(Y_8, Y_{12}, Y_4) \\
B = Y(Y_{12}, Y_5, Y_8) \\
L = Y(Y_4, Y_8, Y_{12}) \\
Y_{12} = \Delta(L, R, B) \\
Y_4 = \Delta(R, B, L) \\
Y_8 = \Delta(B, L, R)
$$

The reduction algorithm presented next is based on proofs in (Barrett et al., 2019, pg. 18) connected with linear 2-trees and accompanied by figures which are adapted and expanded here for application to the $n$-grid. The algorithm presented in Definition 2 has five steps which, in the sequel, will be referred to as Steps A-E. These steps are illustrated in the three rows of Figure 3.

Definition 2. Given a (parent) $n$-grid with labeled edges, the row-reduction of this labeled $n$-grid (to a labeled (child) $n-1$-grid) refers to the sequential performance of the following steps, the steps being illustrated by Figure 3.

- **Step A**: Start with an $n$-grid (Figure 3 illustrates with $n = 3$ and 2.).
- **Step B**: Apply a $\Delta-Y$ transformation to each upright triangle (a 3-loop) resulting in a grid of $n$ rows of out-stars.
- **Step C**: Ignore the corner tails, edges dotted in Step B with a vertex of degree one, since they do not affect the resistance labels of edges in the reduced two grid shown in Step 5. (However, these corner tails are useful for computing certain effective resistances as shown in Barrett et al. (2020a); Evans and Francis (2022)).
- **Step D**: Perform series transformations on all consecutive pairs of boundary edges (i.e., the dashed edges in Step C).
- **Step E**: Apply $Y-\Delta$ transformations to any remaining out-stars, transforming them into loops.

Since Steps B-E involve functions that produce equivalent electric resistances, the resistance between two corner vertices in the resulting $n-1$ grid is equal to the resistance between two corner vertices in the original $n$ grid.

Figure 3 also illustrates repeated row-reduction. The initial 3 grid uniformly labeled 1, transforms first to a 2-grid and then a 1-grid. The edge-labels throughout Figure 3 are justified by (1), for example, $\frac{1}{4} = \Delta(\frac{1}{2}, \frac{1}{2}, 1)$, and similarly, $\frac{1}{2} = \Delta(1, \frac{1}{2}, \frac{1}{2})$.

The sequence of steps in the algorithm naturally generate several single functions whose arguments are edge-label values in the parent grid and whose return-value is an edge-label in the child grid. To illustrate, for $1 \leq r \leq n-1$, define a function $P$ which returns the value of $T_{L,r,1}$ in a (child) reduced grid as a function of six edge values
Fig. 3: Illustration of two row reductions, Definition 2, on a 3-grid whose edges are uniformly labeled with 1.

\[
T_{L,r,1} = \mathcal{P}(T_{r,1}, T_{r+1,1}) = Y_{s,r,1} + Y_{12,r+1,1} = \Delta(T_{B,r,1}, T_{L,r,1}, T_{R,r,1}) + \Delta(T_{L,r+1,1}, T_{R,r+1,1}, T_{B,r+1,1}).
\]

While the notation can be made more precise using superscripts to indicate the parent and once-reduced grid, we simplify notation by using the convention that equations similar to (3) are interpreted so that triangle-edge values on the left-hand side refer to labels in the child (that is, the once-reduced \( n \)-) grid while those on the right-hand side refer to labels in the parent grid. Moreover, for readability, we have also indicated the arguments of \( \mathcal{R} \) using triangles. Note that, for example, if we let \( r = 1 \) in (3) we obtain an equation justifying the \( \frac{4}{7} \) label in Step D3.

Only four local-edge functions are needed for all computations. The other three local edge functions, \( \mathcal{L}, \mathcal{R}, \) and \( \mathcal{B} \) for the labels of a non-boundary left edge, right-edge, and base edge, respectively, with the convention just indicated that the left-hand and right-hand side refer to items in the child (reduced) and parent grids, are as follows.
For a given \( n \)-grid for \( 1 \leq r \leq n-2, 1 \leq d \leq r-1 \), the function \( R \) gives the label of a base-edge in a once-reduced \( n \)-grid.

\[
B(T_{r+2,d+1}, T_{r+1,d+1}) = T_{r,d,B} = Y(\Delta(T_{L,r+2,d+1}, T_{R,r+2,d+1}, T_{B,r+2,d+1}), \Delta(T_{R,r+1,d}, T_{B,r+1,d}, T_{L,r+1,d+1}, T_{L,r+1,d+1})). \tag{4}
\]

For an \( n \)-grid for \( 1 \leq r \leq n-2, 1 \leq d \leq r-1 \) the function \( R \) gives the label of a right-edge in a once-reduced \( n \)-grid.

\[
R(T_{r,d+1}, T_{r+1,d}) = T_{r,d,R} = Y(\Delta(T_{r,d+1}, T_{r,d+1}, T_{r,d+1}, T_{r,d+1}), \Delta(T_{r+1,d+1}, T_{r+1,d+1}, T_{r+1,d+1})). \tag{5}
\]

For a given \( n \)-grid with \( 2 \leq r \leq n-1, 2 \leq d \leq r \) the function \( L \) gives the label of a non-boundary left edge in a one reduced \( n \)-grid.

\[
L(T_{r,d-1}, T_{r+1,d}) = Y(\Delta(T_{r,d-1}, T_{r,d-1}, T_{r,d-1}, T_{r,d-1}), \Delta(T_{L,r,d}, T_{L,r,d}, T_{L,r,d}, T_{L,r,d})). \tag{6}
\]

In the sequel we will refer to (3)-(6) as the \textit{the four local functions}. As illustrated with \( P \), the proof that these functions accomplish their stated goals is provided by Definition 2.

The four local functions justify the following approach to proofs used throughout the paper. Traditionally, the combinatoric Laplacian is used to justify computational identities. But the combinatoric Laplacian is a global method involving the entire graph while the four local functions are computed locally (with at most 9 edge-labels as function arguments). Consequently, if a proof reduces to verification that two (rational) functions are equal, we will suffice with simply stating \textit{we may verify}, the verification being accomplishable either manually or by algebraic software. Although some proofs can easily be done without software, the narrative flows more naturally if the computational details of the proofs are always left to verification.

3 The Conjectures

Figure 3 can be used to motivate the seven conjectures presented in this paper.

\textbf{Example 1.} Figure 3 starts in Steps A1 and A2 with a 3-grid uniformly labeled 1, and then applies two repeated row reductions resulting in the 1-grid of Step D3 with edge label \( \frac{4}{7} \). But,

\[
\frac{4}{7} = 0.5714 \approx 0.5518 = \frac{3}{2e},
\]

with an error of 3.55%.

Continuing this example, the ignored tail in Step B3 has label \( \frac{4}{21} \). We have

\[
\frac{4}{21} = 0.1905 \approx 0.1839 = \frac{11}{2e}.
\]
Conjecture 1 will be formulated in terms of tails rather than edges. The reason for this is that in a 1-grid arising from repeated row-reduction, there is a nice relationship between the resistance distance of the edges of the 1-grid and the resistance distance of the ignored tail arising in the process of row reducing a 2-grid to a 1-grid. But in general, the edge labels of $c$-grids, $c > 1$, (arising from applying $n - c$ repeated row reductions to an initial $n$-grid uniformly labeled 1) do not have a nice relationship with tails arising in the row reduction process. To state the conjectures, we therefore need to create a notation for the resistance distance of tails.

**Definition 3.** $t(n, i)$ indicates the label of any tail in Step 2 of the $n - i$-th row reduction of an original $n$-grid.

**Example 2.** In Figure 3, $t(3, 2) = \frac{1}{2}$. Notice that $t(3, 2)$ depends on the labels in the initial $n$-grid.

Conjecture 1 generalizes and formalizes Example 1.

**Conjecture 1.** Consider a family of initial $n$-grids whose edges are uniformly labeled one. As $n$ goes to infinity,

$$t(n, 1) \approx \frac{1}{2e}.$$ 

**Conjecture 2.** Fix $i \geq 1$. With the conventions of Conjecture 1, as $n$ goes to infinity,

$$t(n, i) \approx \frac{1}{\frac{i}{2e}}.$$ 

Table 1 provides supportive numerical evidence for both conjectures.

| $i$   | Actual value of $t(150, i)$ | Conjectured value | Numerical Error (Ratio - 1) |
|-------|---------------------------|------------------|---------------------------|
| 1     | 0.183776286               | $\frac{1}{2e}$ = 0.183939721 | 0.0009                    |
| 2     | 0.091888653               | $\frac{1}{2e}$ = 0.09196986  | 0.0009                    |
| 3     | 0.061258443               | $\frac{1}{2e}$ = 0.06131324  | 0.0009                    |
| 4     | 0.045943311               | $\frac{1}{2e}$ = 0.04598493  | 0.0009                    |
| 5     | 0.036753773               | $\frac{1}{2e}$ = 0.036787944 | 0.0009                    |
| 6     | 0.030626823               | $\frac{1}{2e}$ = 0.03065662  | 0.001                     |
| 7     | 0.026249708               | $\frac{1}{2e}$ = 0.026277103 | 0.001                     |
| 8     | 0.02296602                | $\frac{1}{2e}$ = 0.022992465 | 0.0012                    |
| 9     | 0.020411064               | $\frac{1}{2e}$ = 0.020437747 | 0.0013                    |
| 10    | 0.018366                  | $\frac{1}{2e}$ = 0.018393972 | 0.0015                    |

The next four conjectures study patterns in the ratios of certain edge-labels in the $c$ grid resulting from applying $n - c$ row reductions to an $n$ grid whose edges are uniformly labeled 1. The conjectures should be read as defining four functions, $r_{R,L}, r_{B,L}, x, y$ and then conjecturing that the ratio of the labels specified in each conjecture have the indicated value of these functions. In the sequel, we will refer to these four functions as the (four) edge-ratio functions.
Conjecture 3. Fix $c \geq 1$. With the conventions of Conjecture 1, as $n$ goes to infinity,

$$\frac{T_{R,r,d}}{T_{L,r,d}} \approx \frac{2(r - d) + 1}{2d - 1} \approx r_{R,L}(r, d), \quad 1 \leq d \leq r \leq c.$$  \hspace{1cm} (7)

Conjecture 4. Fix $c \geq 1$. With the conventions of Conjecture 1, as $n$ goes to infinity,

$$\frac{T_{B,r,d}}{T_{L,r,d}} \approx \frac{2(c - r) + 1}{2d - 1} \approx r_{B,L}(c, r, d), \quad 1 \leq d \leq r \leq c.$$  \hspace{1cm} (8)

Conjecture 5. Fix $c \geq 1$. With the conventions of Conjecture 1, as $n$ goes to infinity,

$$\frac{T_{L,r,1}}{T_{L,r-1,1}} \approx \frac{r}{2(r - 1)} \frac{2(c - r) + 3}{(c - r) + 1} \approx x(c, r), \quad 2 \leq r \leq c.$$  \hspace{1cm} (9)

Conjecture 6. Fix $c \geq 1$. With the conventions of Conjecture 1, as $n$ goes to infinity,

$$\frac{T_{L,r,d}}{T_{L,r,d-1}} \approx \frac{d - 1}{2d - 3} \frac{2(r - d) + 3}{(r - d) + 1} \approx y(r, d), \quad 2 \leq d \leq r \leq c.$$  \hspace{1cm} (10)

Table 2 presents numerical evidence for Conjectures 3-6.

As mentioned earlier, Barrett et al. (2019), in introducing the triangular grid also presented a conjecture (Barrett et al., 2019, Conjecture 7.8) that if $r_n$ denotes the resistance distance between any two corners (degree two vertices) of the $n$-grid, then as $n$ approaches infinity $e^{r_{n+1}} - e^{r_n} = C$, for some constant $C$.

It is straightforward to show that the resistance between any two degree-2 nodes in an $n$-grid is exactly equal to $2 \times \sum_{i=1}^{n} t(n, i)$. But by Conjecture 1, $t(n, i) \approx \frac{1}{i} t(n, 1)$ provided $n$ is large enough and $i$ is independent of $n$. Motivated by these considerations, Evans and I were able to refine the conjectured asymptotic limit stated in Barrett et al. (2019).

Conjecture 7.

The resistance distance between two corner nodes of the $n$-grid $\approx \sum_{i=1}^{n} \frac{1}{i}$.

None of these conjectures will be proven in this paper. Although they are interesting, they are presented because they naturally suggest other avenues of exploration. In the remainder of this paper we study defining the labels of an initial $n$-grid using the four edge-ratio functions. This contrasts with most examples in the literature which almost always present circuits uniformly labeled with ones.

This study will result in three main theorems which collectively state that both the initial $n$-grid defined by the four edge-ratio functions as well as all $n-i$ grids arising from $i, 1 \leq i \leq n-1$, repeated row reductions of the initial $n$ grid possess vertical and rotational symmetry and satisfy the relationships implied by the four edge-ratio functions. The tails arising in the row reductions manifest behavior similar to that stated in Conjecture 2.
Table 2: Numerical evidence for Conjectures 3-6 based on 140 consecutive reductions of an initial 150-grid uniformly labeled 1.

| Edge ratio | Actual value of edge ratios | Equation reference | Predicted edge ratio Value | Error |
|------------|-----------------------------|--------------------|---------------------------|-------|
| $T_{L,1,1}$ | 7.00101503391               | (7)                | $r_{R,L}(4,1) = \frac{7}{1}$ | -0.00014 |
| $T_{L,4,1}$ | 1.6667476811                | (7)                | $r_{R,L}(4,2) = \frac{5}{3}$ | -0.00005 |
| $T_{R,6,1}$ | 0.7142421068                | (7)                | $r_{R,L}(6,4) = \frac{2}{3}$ | -0.00006 |
| $T_{L,1,1}$ | 19.0125047453               | (8)                | $r_{R,L}(10,1,1) = 19$    | -0.00066 |
| $T_{L,5,3}$ | 2.20046025                  | (8)                | $r_{R,L}(10,5,3) = \frac{11}{7}$ | -0.00021 |
| $T_{R,6,4}$ | 1.2858046383                | (8)                | $r_{R,L}(10,6,4) = \frac{3}{7}$ | -0.00007 |
| $T_{L,1,1}$ | 0.7036757582                | (9)                | $x(10,2) = \frac{12}{7}$   | 0.00004 |
| $T_{L,3,1}$ | 0.8499690996                | (9)                | $x(10,3) = \frac{17}{12}$  | 0.00004 |
| $T_{L,1,1}$ | 0.9183432041                | (9)                | $x(10,4) = \frac{15}{7}$   | 0.00003 |
| $T_{L,1,1}$ | 2.3334191779                | (10)               | $y(4,2) = \frac{5}{3}$    | -0.00004 |
| $T_{L,4,1}$ | 1.6667476811                | (10)               | $y(4,3) = \frac{5}{3}$    | -0.00005 |
| $T_{L,4,2}$ | 1.500093074                 | (10)               | $y(5,4) = \frac{1}{3}$    | -0.00006 |

1The interpretation of the first table row states that the quotient of the labels of the right and left edge of triangle $T_{L,1,1}$ in the 10 grid arising from applying 140 row reductions to an initial 150-grid uniformly labeled 1 is $7.001 \ldots$. However, (7) predicts a ratio of $r_{R,L}(10,4,1) = 7$, implying a very small error.

These theorems shed light on the four edge-ratio functions. They are local algebraic relationships which correspond to the global symmetries present in the initial and reduced grids.

4 A Motivating Example

This section motivates the idea of an initial c-grid whose edges are not uniformly labeled 1. The section provides definitions, a worked out example, and notes various patterns which will be formalized in the three main theorems of the paper.

The $c$ grid considered in this section uses the four edge ratio functions to label edges. More specifically, given a $c$-grid we use the following equations to define labels.

$$T_{L,1,1} = 1. \quad (11)$$

$$T_{L,r,1} = x(c,r)T_{L,r-1,1}, \quad 2 \leq r \leq c;$$

$$T_{L,r,d} = y(r,d)T_{L,r,d-1}, \quad 2 \leq d \leq r;$$

$$T_{R,r,d} = r_{R,L}(r,d)T_{L,r,d}, \quad 1 \leq r \leq c, 1 \leq d \leq r \quad (12)$$
\[ T_{B,r,d} = r_{B,L}(c, r, d) T_{L,r,d}, \quad 1 \leq r \leq c, 1 \leq d \leq r. \]

In the sequel, we will refer to (12) as the (four) edge-ratio relationships. Figure 4 illustrates the application of (11)-(12) to completely label a 3-grid.

**Fig. 4**: A 3 grid labeled using (11)-(12)

An important point, is that the four edge-ratio functions can be used to describe the ratio of any two edges in the 3-grid. To illustrate this, we introduce the heuristic of travel along the edges of the \( c \)-grid. Heuristically, the edge ratio functions \( x, y, r_{R,L}, r_{B,L} \) are used to describe vertical, horizontal, and rotational travel, respectively. The next two examples clarify how this concept of travel will be used.

**Example 3.** Referring to Figure 4

\[ T_{R,3,3} = r_{R,L}(3, 3) \cdot \left( y(3,2)y(3,3) \right) \cdot \left( x(3,2)x(3,3) \right) \cdot T_{L,1,1}. \]

In this equation we start with \( T_{L,1,1} \) then travel down the left side of the grid to \( T_{L,3,1} \) by multiplying \( T_{L,1,1} \) by \( x(3,2)x(3,3) \), then travel horizontally across the bottom row from \( T_{L,3,1} \) to \( T_{L,3,2} \) by multiplying by \( y(3,2)y(3,3) \) and finally travel from \( T_{L,3,3} \) to \( T_{R,3,3} \) by multiplying by \( r_{R,L}(3,3) \).

**Example 4.** The next example will be used to motivate the definition of the function \( z \) in Section 5.

\[ T_{L,3,2} = y(3,2) \cdot x(3,3) \cdot \frac{1}{y(2,2)} \cdot T_{L,2,2}. \]  

(13)

In this example, we start at \( T_{L,2,2} \), travel horizontally backward to \( T_{L,2,1} \) by multiplying by \( \frac{1}{y(2,2)} \), then travel vertically downward one row to \( T_{L,3,1} \) using \( x(3,3) \), and then travel horizontally on row three from \( T_{L,3,1} \) to \( T_{L,3,2} \) using \( y(3,2) \).

In the sequel, when giving such equations, we may simply say they are justified by general travel considerations.

Figure 4 motivates the following formal definition of the initial non-one \( c \) grid and related concepts.

**Definition 4.** Given \( c \geq 1 \), an initial non-one \( c \) grid is a triangular \( c \)-grid, satisfying Definition 1, with the edge labels satisfying (11)-(12).
A non-one \( c \)-grid (without the adjective initial) is a triangular \( c \)-grid, satisfying Definition 1, with the edge labels satisfying the four edge-ratio relationships.

An arbitrary \( c \)-grid is a triangular \( c \)-grid, satisfying Definition 1.

In the rest of the paper, an initial non-one \( c \)-grid is notationally indicated by \( T_c \); \( T_{c,i} \), \( 1 \leq i \leq c \) indicates the non-one \( i \) grid obtained by applying \( c - i \) consecutive row-reductions to \( T_c \) (note, \( T_c = T_{c,c} \)).

We can further use this example to illustrate all three of the Main Theorems. Figure 5 shows the results of applying the row-reduction algorithm, Definition 2, or the four local functions, to the initial non-one 3-grid of Figure 4. The computations are similar to those presented in Figure 3.

Fig. 5: Two applications of row reduction to an initial non-one 3-grid (Figure 4) resulting in a non-one 2-grid and 1 grid.

![Diagram](image)

We observe the following, each of which will be generalized and proven later in the paper. The first main result generalizes the following example. For any \( X \in \{L, R, B\} \), we have

\[
T^{3,2}_{X,r,d} = \frac{15}{14} T^{2}_{X,r,d}, \quad 1 \leq r \leq 2; 1 \leq d \leq r \\
T^{3,1}_{X,r,d} = \frac{9}{7} T^{1}_{X,r,d}, \quad 1 \leq r \leq 1; 1 \leq d \leq r.
\]

Equation (14) immediately implies the following result which the second main result generalizes: The reduced grids \( T^{3,i}, i = 1, 2 \) satisfy the four edge-ratio relationships.

By Definition 3 as illustrated by Example 2 we have

\[
t(3, 3) = \Delta(1, 1, 5) = \frac{1}{7}, \quad t(3, 2) = \Delta(\frac{15}{14}, \frac{15}{14}, \frac{45}{14}) = \frac{3}{14}, \quad t(3, 1) = \Delta(\frac{9}{7}, \frac{9}{7}, \frac{9}{7}) = \frac{3}{7}.
\]

Equation (15) implies

\[
t(3, 2) = \frac{1}{2} \cdot t(3, 1) \text{ and } t(3, 3) = \frac{1}{3} \cdot t(3, 1).
\]
relationships which mirror those found in Conjecture 2. The third main result makes this resemblance precise and generalizes it.

5 Needed Functions

Besides the four local functions, to formulate and prove the three main theorems we need three additional functions, \( f, g, z \), presented in this section.

For positive integer \( c \), define

\[
f(c, i) = 1 + \frac{c - i}{i} \cdot \frac{1}{2c + 1}, \quad 1 \leq i \leq c - 1,
\]

\[
g(c) = \frac{c}{2c - 1} \cdot \frac{2c - 1}{2c + 1}.
\]

Example 5. \( f(3, 2) = \frac{15}{14}, f(3, 1) = \frac{9}{7} \) corresponding to the ratios found in (14). We will formally prove in one of the main theorems that \( f(c, i) = \frac{T_{c, i}}{T_{i, 1}} \).

Proposition 1. For integer \( c \geq 1 \) and \( 1 \leq j \leq c - 1 \),

\[
f(c, j) = \prod_{i=j+1}^{c} g(i).
\]

Proof. The proof is by induction on \( j \) with arbitrary \( c \) fixed. For the base case, \( j = c - 1 \), we verify that \( f(c, c - 1) = g(c) \). For an induction assumption we assume that for some \( s \), \( 1 \leq s \leq c - 2 \), that

\[
f(c, s) = \prod_{i=s+1}^{c} g(i).
\]

For an induction step, to complete the proof, we need to prove

\[
f(c, s + 1) = \prod_{i=s+2}^{c} g(i).
\]

Dividing both sides of (18) and (19) we see that it suffices to verify

\[
\frac{f(c, s)}{f(c, s + 1)} = g(s + 1).
\]

This verification is an assertion that two rational functions are equal which can be verified manually or by algebraic software. (Recall (Section 2) that to facilitate the flow of the narrative, proofs in this paper are accomplished by reducing assertions to the verification that two rational functions are equal, or equivalently that their difference is 0 or their ratio is 1.) Since \( c \) was arbitrary, this completes the proof.
Define a function \( z \), by

\[
z(r,d) = \prod_{i=2}^{d} y(r+1,i) / \prod_{i=2}^{r} y(r,i), \quad \text{for } 1 \leq r \leq c-1, 2 \leq d \leq r.
\] (21)

As illustrated in (13) and the surrounding narrative, we have

\[
T_{L,r+1,d} = x(c,r+1)z(r,d)T_{L,r,d}.
\] (22)

However \( z \) as defined in (21) is not closed which would not allow using algebraic verification as a proof method since this method requires closed forms. Fortunately, there is an alternate closed form for \( z(r,d) \).

**Proposition 2.** With \( z \) defined by (21), for \( 1 \leq d \leq r \),

\[
z(r,d) = 1 - \frac{d-1}{r} \times \frac{1}{2(r-d)+3} \pm z_1(r,d),
\] (23)

where we have defined \( z_1(r,d) \) as indicated.

**Proof.** The proof is by induction on \( d \) with an arbitrary \( r \) fixed. For a base case, \( d = 2 \), by (23), (21), and (10), we may verify \( z(r,2) = y(r+1,2) y(r,2) = z_1(r,2) \).

For an induction assumption, we assume the following holds for some \( s, 2 \leq s \leq r-1 \).

\[
\prod_{i=2}^{r} y(r+1,i) / \prod_{i=2}^{r} y(r,i) = z_1(r,s)
\] (24)

For the induction step, which will complete the proof, we must, given (24), prove

\[
\prod_{i=2}^{s+1} y(r+1,i) / \prod_{i=2}^{s+1} y(r,i) = z_1(r,s+1).
\] (25)

Dividing the right and left-hand sides of (24) and (25) (and performing routine cancellations) we see that it suffices to verify \( y(r+1,s+1) y(r,s+1) = z(r,s+1)/z(r,s) \). This completes the proof.

In the sequel we will use \( z \) to refer either to \( z \) proper or to \( z_1 \).

6 The Three Main Theorems

Section 4 motivated with illustrative examples the main theorems of this paper. In stating the main theorems we use the language presented in Definition 4 and the notation indicated immediately after it.

To state the theorem about the symmetry of reduced initial \( c \) grids we have to formally define what we mean by vertical, rotational, and slide symmetry. We provide a graph theoretic definition, independent of an embedding of the graph in the Cartesian plane.
Definition 5. An arbitrary \( c \)-grid is said to possess vertical symmetry if
\[
T_{L,r,d} = T_{R,r+1-d,2}, \quad T_{B,r,d} = T_{B,r+1-d,2}, \quad 1 \leq d \leq r \leq c.
\]

An arbitrary \( c \)-grid is said to possess rotational symmetry (by \( \frac{\pi}{3} \)) if for \( 1 \leq d \leq r \leq c \),
\[
T_{L,r,d} = T_{R,n+1-r,n+1-d}, \quad T_{R,r,d} = T_{B,n+1-r,n+1-d}, \quad T_{B,r,d} = T_{L,n+1-r,n+1-d}.
\]

An arbitrary \( c \)-grid is said to possess slide symmetry if for \( 1 \leq d \leq r \leq c \),
\[
T_{L,r,d} = T_{L,n+1-r,n+1-d}, \quad T_{R,r,d} = T_{R,n+1-r,n+1-d}, \quad T_{B,r,d} = T_{B,n+1-r,n+1-d}.
\]

These symmetries are amply illustrated in Figures 4 and 5.

Remark 1. It is straightforward to verify that an assumption of vertical and slide symmetry is equivalent to an assumption of vertical and rotational symmetry. Therefore, in the sequel the statement that a \( c \)-grid possesses symmetry will mean that it possesses vertical, rotational, and slide symmetry.

Theorem 3 (Symmetry). For integer \( c > 0 \), \( 1 \leq i \leq c \) \( T^c_i \) possesses vertical and slide symmetry.

Theorem 4 (Edge ratios). For integer \( c > 0 \), and an initial non-one \( c \)-grid \( T^c_1 \).

i) For \( 1 \leq i \leq c \),
\[
t(c, i) = \frac{1}{i} \cdot t(c, 1)
\]

with
\[
t(c, 1) = \frac{c}{2c + 1}.
\]

ii) For \( 1 \leq i \leq c \) the four edge-factor relationships hold in \( T^c_i \).

iii) For \( 1 \leq i \leq c - 1 \),
\[
T^c_{i,r,d} = f(c, i) \cdot T^c_{i,r,d}, \quad \text{for} \quad 1 \leq r \leq i, 1 \leq d \leq r, X \in \{L, R, B\}.
\]

To formulate the third main theorem we need to introduce the concept of the upper-left half of a \( c \)-grid.

Definition 6. The upper left half of an arbitrary \( c \)-grid, consists of the triangles
\[
T_{r,d}, \quad d = 1, \ldots, \left\lfloor \frac{c + 2}{3} \right\rfloor, \quad r = 2d - 1, \ldots, \left\lfloor \frac{c + d}{2} \right\rfloor.
\]

Figure 6 illustrates the definition.

Remark 2. In the sequel it will be convenient to be able to refer to the i) upper right half, ii) lower left half and iii) lower right half of an arbitrary \( c \)-grid. These respectively correspond to the triangles \( T_{r,d} \) with \( (r, d) \) in i) \( d = \left\lfloor \frac{c + 2}{3} \right\rfloor, \ldots, r = 2d - 1, \ldots, \left\lfloor \frac{c + d}{2} \right\rfloor \), ii) \( d = 1, \ldots, \left\lfloor \frac{c + 2}{3} \right\rfloor, \quad r = \left\lfloor \frac{c + d}{2} \right\rfloor, \ldots, c \), iii) \( d = \left\lfloor \frac{c + 2}{3} \right\rfloor, \ldots, r = \left\lfloor \frac{c + d}{2} \right\rfloor, \ldots, c \). Notice that these regions are not mutually disjoint. We may further use the terms i) top corner, ii) 4 O’Clock corner, and iii) 8 O’Clock corner of the \( c \)-grid to respectively
Fig. 6: The upper left half of this 4-grid consists of the triangles $T_{1,1}, T_{2,1}, T_{3,2}$. 

Proposition 5. The edge labels in the upper left half of an arbitrary $c$-grid possessing symmetry determine all remaining edge-labels in the $c$-grid.

Proof. Vertical symmetry maps the upper left half to the upper right half showing that the edges in the top corner of the $c$-grid are determined. Two applications of rotational symmetry then map this top corner to the 4 O’Clock and 8 O’Clock corners showing that the edge labels of the entire triangular grid are determined from the edge labels in the upper left half.

We provide one more definition.

Definition 7. Let $T$ be a $c$-grid. For $1 \leq t \leq \lfloor \frac{c+2}{3} \rfloor$, the $t$-rim of a $c$-grid is a union of the following 3 sets of triangles: 

$T_{r,1}, 2t-1 \leq r \leq c+1-t, T_{c+1-t,d}, t \leq d \leq c-2(t-1)$

$T_{r,r-(t-1)}, 2t-1 \leq r \leq c+1-2t$.

Example 6. If $c=4$, as in Figure 6, then the 2-rim consists of the singleton triangle $T_{3,2}$ while the 1-rim consists of the 9 triangles $\{T_{r,1}\}_{1 \leq r \leq 4} \cup \{T_{4,d}\}_{1 \leq d \leq 4} \cup \{T_{r,r}\}_{1 \leq r \leq 4}$. Heuristically, the $c$ grid may be perceived as a collection of concentric rims.

We may now state the third main theorem.

Theorem 6 (Edge-Ratio Symmetry duality). For an arbitrary $c$ grid possessing symmetry, if the four edge-ratio relationships are satisfied for edges of triangles in the upper left half, then the four edge-factor relationships are satisfied in the entire $c$-grid.

The name given to the theorem reflects the heuristic of its insights. The theorem coupled with Proposition 5 intuitively says that the local edge-ratio relationships mirror global symmetry in the sense that (with the conditions as indicated in the theorem) the edge ratio relationships imply symmetry and symmetry in turn implies the edge-factor relationships.

Section 4 presented the non-one initial 3-grid. Similar methods can be used to calculate the initial non-one $c$-grids, $c = 1, 2, 4$. The theorems of this paper can be manually checked for these small values.

Proposition 7. To prove Theorem 4(ii) it suffices to prove the required equations for any $c$ grid with $c \geq 5$.

An advantage of restricting ourselves to $c \geq 5$ is the following proposition.
Proposition 8. For \( c \geq 5 \) the restrictions on using the four local functions are satisfied in the upper left half of the initial \( c \)-grid.

Proof. By (3)-(6), to use the four local functions, requires,

\[
\begin{align*}
  r &\leq c - 2, \quad d \leq r - 1.
\end{align*}
\]

By Definition 6, \((r, d)\) lies in the upper left half if

\[
T_{r,d}, \quad d = 1, \ldots, \left\lceil \frac{c + 2}{3} \right\rceil, r = 2d - 1, \ldots, \left\lfloor \frac{c + d}{2} \right\rfloor.
\]

It is straightforward to check that the solution to these simultaneous inequalities is \( c \geq 5 \).

7 Proof of the Symmetry Theorem

Theorem 3 states that both \( T^c \) and its reductions \( T^{c,i}, 1 \leq i \leq c \) possess symmetry. However, in terms of proofs, we first prove the theorem for the case \( i = c \). We then, (Proposition 9), prove that the reduced grids inherit, one reduction at a time, the symmetries of the parent grid, completing the proof of the theorem.

By Definition 5, to prove Theorem 3 for the case \( i = c \) we must prove vertical and slide symmetry for \( T^c \).

Proof of vertical symmetry. We prove

\[
T_{L,r,\frac{r}{2} - s} = T_{R,r,\frac{r}{2} + s + 1}, \quad \text{for } 0 \leq s \leq \frac{r}{2} - 1,
\]

the corresponding proof for the base edges being similar and hence omitted. The proof, depends on the parity of \( r \). We assume \( r \) even, the proof for the \( r \) odd case being similar and hence omitted.

For the base case we prove (30) for the case \( s = 0 \). Travel from \( T_{L,r,\frac{r}{2} - s} \) to \( T_{L,r,\frac{r}{2} + s + 1} \) is accomplished by \( y \) while travel from \( T_{L,r,\frac{r}{2} + s + 1} \), to \( T_{R,r,\frac{r}{2} + s + 1} \), is accomplished by \( r_{R,L} \). Hence the proof for the case \( s = 0 \) is completed by verifying

\[
y(r, \frac{r}{2} + 1) \cdot r_{R,L}(r, \frac{r}{2} + 1) = 1.
\]

For an induction assumption we assume (30) true for some \( s \geq 0 \). Then, to complete the proof, we must prove an induction step, that given (30) we have

\[
T_{L,r,\frac{r}{2} - s - 1} = T_{R,r,\frac{r}{2} + s + 2}.
\]

The travel from i) \( T_{L,r,\frac{r}{2} - s - 1} \) to \( T_{L,r,\frac{r}{2} - s} \), then from ii) \( T_{L,r,\frac{r}{2} - s} \), to \( T_{R,r,\frac{r}{2} + s + 1} \), then iii) from \( T_{R,r,\frac{r}{2} + s + 1} \) to \( T_{L,r,\frac{r}{2} + s + 1} \), then iv) from \( T_{L,r,\frac{r}{2} + s + 1} \), to \( T_{L,r,\frac{r}{2} + s + 2} \), and finally (v) from \( T_{L,r,\frac{r}{2} + s + 2} \), to \( T_{R,r,\frac{r}{2} + s + 2} \), is respectively accomplished by i) \( \frac{1}{y(r,\frac{r}{2} - s)} \), ii) \( y(r, \frac{r}{2} + 1) \cdot r_{R,L}(r, \frac{r}{2} + 1) \) (which by (31) equals 1), iii) \( \frac{1}{r_{R,L}(r,\frac{r}{2} + s + 1)} \), iv) \( y(r, \frac{r}{2} + s + 2) \),
and v) $r_{R,L}(r, \frac{r}{2} + s + 2)$. Consequently the proof of (32) given (31) is completed by verification that

$$\frac{1}{y(r, r/2 - s)} = \frac{1}{r_{R,L}(r, r/2 + s + 1)} y(r, r/2 + s + 2) r_{R,L}(r, r/2 + s + 2).$$

**Proof of Slide Symmetry.** By Definition 5 we must prove slide symmetry for all three sides. We suffice with dealing just with the left-hand side, that is, with proving,

$$T_{L, c + d - \frac{1}{2} - i - s, d} = T_{L, c + d - \frac{1}{2} + 1 + i, d},$$

for $1 \leq d \leq c, 0 \leq i \leq \frac{c - d - 1}{2}$, (33)

the proof for the other sides being similar and hence omitted. In the proof, we assume $c + d$ odd, the proof for the even case being similar and hence omitted.

For the base case we prove (33) for $i = 0$. By (22) it suffices to verify

$$x(c, \frac{c + d - 1}{2} + 1) \cdot z(c + d - 1, d) = 1.$$

**Induction Assumption.** For an induction assumption, we assume that for some $s, 0 \leq s \leq \frac{c - d - 1}{2} - 1$ that

$$T_{L, c + d - \frac{1}{2} - s, d} = T_{L, c + d - \frac{1}{2} + 1 + s, d}.$$  (34)

To complete the proof, we must, using an induction step, prove

$$T_{L, c + d - \frac{1}{2} - s - 1, d} = T_{L, c + d - \frac{1}{2} + 2 + s, d}.$$  (35)

Travel i) from $T_{L, c + d - \frac{1}{2} - s - 1, d}$ to $T_{L, c + d - \frac{1}{2} - s, d}$, then ii) to $T_{L, c + d - \frac{1}{2} + 1 + s, d}$, then iii) to $T_{L, c + d - \frac{1}{2} + 2 + s, d}$, is, by (22), respectively accomplished by i) $\left(x(c, \frac{c + d - 1}{2} - s) \cdot z(\frac{c + d - 1}{2} - s - 1, d)\right)$, ii) 1 (by the induction assumption, (34)), and iii) $\left(x(c, \frac{c + d - 1}{2} + s + 2) \cdot z(\frac{c + d - 1}{2} + (s + 1), d)\right)$. Hence, the proof is completed by verifying

$$\left(x(c, \frac{c + d - 1}{2} - s) \cdot z(\frac{c + d - 1}{2} - s - 1, d)\right) \cdot \left(x(c, \frac{c + d - 1}{2} + s + 2) \cdot z(\frac{c + d - 1}{2} + (s + 1), d)\right) = 1.$$

This completes the proof of slide symmetry and the symmetry theorem for the case $i = c$. To prove the symmetry theorem for all $i$ we need the following Proposition.

**Proposition 9.** If an arbitrary $c$ grid possesses symmetry then the $c - 1$ grid arising from one reduction of it also possesses symmetry.

**Proof.** In the obvious manner we may speak about a triangular grid of $Y$s possessing vertical, rotational, and slide symmetry. Since the $Y$s arising from a $\Delta$ - $Y$ transformation are determined by the sides of the underlying triangle, the $Y$s will have the same
symmetries as the original configuration of triangles. Walking through the definition of reduction, Definition 2, we see that discarding tails, performing series transformation on adjacent edges, and performing Y-∆ transformations preserve vertical and slide symmetry. Hence the once-reduced grid will possess the symmetries of the parent grid.

**Corollary 1.** The symmetry theorem is true for all $i, 1 \leq i \leq c$.

### 8 Proof of the Edge-Factor Symmetry Duality Theorem

The proof of Theorem 6 consists of a series of propositions and corollaries, each proposition-corollary pair focusing on one edge-factor relationship. Throughout this section statements of satisfaction of the edge-ratio relationships or of edges belonging to the upper left half have the formal meaning given in (12) and Definition 6 respectively.

**Proposition 10.** For an arbitrary $c$ grid possessing symmetry and for an arbitrary fixed row, $r, 1 \leq r \leq c$:

a) If the **right-left** edge ratio relationship $\frac{T_{r,d,R}}{T_{r,d,L}} = r_{R,L}(r, d)$ holds for $1 \leq d \leq \lfloor \frac{r+1}{2} \rfloor$, then it holds for $1 \leq d \leq r$.

b) If the **base-left** edge ratio relationship $\frac{T_{r,d,B}}{T_{r,d,L}} = r_{B,L}(r, d)$ holds for $1 \leq d \leq \lfloor \frac{r+1}{2} \rfloor$, then it holds for $1 \leq d \leq r$.

c) If the **horizontal** edge ratio relationship $\frac{T_{r,d,L}}{T_{r,d,L}} = y(r, d)$ holds for $2 \leq d \leq \lfloor \frac{r+1}{2} \rfloor$, then it holds for $1 \leq d \leq r$.

**Proof.** a) Vertical symmetry implies that the numerators and denominators of the following two fractions are equal (and hence their ratios are equal).

$$\frac{T_{r,d,R}}{T_{r,d,L}} = \frac{T_{r,r+1-d,R}}{T_{r,r+1-d,L}}$$

By the hypothesis of the Proposition that the edge-ratio relationships hold in the upper left half, for $1 \leq d \leq \lfloor \frac{r+1}{2} \rfloor$,

$$r_{R,L}(r, d) = \frac{T_{r,d,R}}{T_{r,d,L}}$$

We can algebraically verify that for all $d, 1 \leq d \leq r$,

$$r_{R,L}(r, d) = \frac{1}{r_{R,L}(r, r+1-d)}$$

Combining these equations we infer

$$\frac{T_{r,r+1-d,R}}{T_{r,r+1-d,L}} = r_{R,L}(r, r+1-d)$$
as required.

b) Vertical symmetry implies

$$\frac{T_{r,d,B}}{T_{r,d,L}} = \frac{T_{r,r+1-d,B}}{T_{r,r+1-d,R}} = \frac{T_{r,r+1-d,L}}{T_{r,r+1-d,R}}.$$

By the hypothesis of the Proposition, for $1 \leq d \leq \lfloor \frac{r+1}{2} \rfloor$,

$$\frac{T_{r,d,B}}{T_{r,d,L}} = r_{B,L}(c,r,d).$$

By part (a)

$$\frac{T_{r,r+1-d,L}}{T_{r,r+1-d,R}} = \frac{1}{r_{R,L}(r+1-d)}.$$

We can algebraically verify

$$r_{B,L}(c,r,d) = r_{B,L}(c,r+1-d) \frac{1}{r_{R,L}(r+1-d)}.$$

Combining these equations we infer, as required, that

$$\frac{T_{r,r+1-d,B}}{T_{r,r+1-d,L}} = r_{B,L}(c,r+1-d).$$

(c) Vertical symmetry implies

$$\frac{T_{r,d,L}}{T_{r,d-1,L}} = \frac{T_{r,r+1-d,R}}{T_{r,r+2-d,R}} = \frac{T_{r,r+1-d,L}}{T_{r,r+2-d,R}} \frac{T_{r,r+1-d,R}}{T_{r,r+2-d,R}}.$$

By the hypothesis of the Proposition and for $d$ in the range indicated

$$y(r,d) = \frac{T_{r,d,L}}{T_{r,d-1,L}}.$$

By Part (a)

$$\frac{T_{r,r+1-d,R}}{T_{r,r+1-d,L}} \frac{T_{r,r+2-d,L}}{T_{r,r+2-d,R}} = \frac{r_{R,L}(r+1-d)}{r_{R,L}(r+2-d)}.$$

We can algebraically verify

$$y(r,d) = \frac{1}{y(r+2-d)} \frac{r_{R,L}(r+1-d)}{r_{R,L}(r+2-d)}.$$
Combining these last 3 equations we infer, as required, that
\[
\frac{T_{r,r+2-d,L}}{T_{r,r+1-d,L}} = y(r, r + 2 - d).
\]

\[\square\]

**Corollary 2.** For an arbitrary \(c\)-grid possessing symmetry, if the right-left, base-left, and horizontal edge ratios hold in the upper left half then they hold in the entire upper half.

**Proposition 11.** For an arbitrary \(c\)-grid possessing symmetry:

a) If the vertical edge ratio relationship holds in the upper half, then it holds in the entire \(c\)-grid.

b) If the right-left and base-left edge ratio relationships hold in the upper half, then they hold in the entire \(c\)-grid.

**Proof.**

a) Slide symmetry applied to diagonal \(d = 1\), implies \(\frac{T_{r+1-L}}{T_{r-L}} = \frac{T_{r+1-d,L}}{T_{r+1-d,L}}\).

But \(x(c, r) = \frac{T_{r+1-L}}{T_{r-L}}\). We may algebraically verify \(x(c, r) = \frac{1}{x(c, c + 2 - r)}\), \(2 \leq r \leq c\).

Combining these equations we infer \(T_{c+1-r,d,R} = x(c, c + 2 - r)\) as required.

b) For any diagonal \(d\), slide symmetry requires that for \(d \leq r \leq c\),

\[
\frac{T_{r,d,R}}{T_{r,d,L}} = \frac{T_{c+d-r,d,B}}{T_{c+d-r,d,L}} \quad (36)
\]

We may algebraically verify

\[
r_{R,L}(r, d) = r_{B,L}(c, c + d - r, d). \quad (37)
\]

We can now combine these two equations in two ways.

(i) If \(r\) is in the upper half, then by hypothesis, \(r_{R,L}(r, d) = \frac{T_{r,d,R}}{T_{r,d,L}}\). Combining this equation with (36)-(37) we infer \(r_{B,L}(c, c + d - r, d) = \frac{T_{c+d-r,d,B}}{T_{c+d-r,d,L}}\), with \(c + d - r\) in the lower half as required.

(ii) On the other hand, if \(r\) is in the lower half, then \(c + d - r\) is in the upper half and by hypothesis \(r_{B,L}(c, c + d - r, d) = \frac{T_{c+d-r,d,B}}{T_{c+d-r,d,L}}\). Combining the two equations we infer \(r_{R,L}(r, d) = \frac{T_{r,d,R}}{T_{r,d,L}}\) with \(r\) in the lower half as required. \[\square\]

**Corollary 3.** For an arbitrary \(c\)-grid possessing symmetry, if the four edge-ratio relationships hold in the upper left half then i) the vertical, right-left, and base-edge ratio relationships hold in the entire \(c\)-grid, and ii) the horizontal edge ratio relationships holds in the upper half.

Thus, to complete the proof of Theorem 6 we must show that the horizontal edge ratio relationship holds in the lower half. The next Proposition proves this inductively, the various Proposition components corresponding to the parts of the inductive proof and themselves requiring their own inductive proofs.
Proposition 12. For an arbitrary \( c \) grid possessing symmetry, if the edge ratio relationships hold in the upper left half, then:

a) The horizontal edge ratio relationship, \( y(c, r) \) holds for the bottom row, \( r = c \).

b) Induction Base Case: For \( d = 2 \), i) the horizontal edge ratio relationships hold (that is, \( y(r, 2) = \frac{T_{c,r}}{T_{c,r-1}} \), \( 2 \leq r \leq c \)), and ii) \( x(c, r) \cdot z(r-1, 2) = \frac{T_{c,r} \cdot y(r-1, 2)}{T_{c,r-1}} \) for \( 3 \leq r \leq c \).

c) The Induction Step. For each diagonal \( d > 2 \), i) the horizontal edge ratio relationship holds for diagonal \( d \) and ii) \( x(c, r) \cdot z(r-1, d) = \frac{T_{c,r} \cdot y(r-1, d)}{T_{c,r-1,d}} \) for \( d + 1 \leq r \leq c \).

d) The horizontal edge ratio holds for the entire \( c \)-grid.

Proof. Clearly, d) follows from a), b), and c). Thus we need only prove a)-c). Proposition 10(c) (which was stated for arbitrary \( r \)) asserts that to prove the horizontal edge ratio is satisfied in the entire \( c \)-grid it suffices to show it is satisfied in the left half. Accordingly throughout the proof we assume \( 1 \leq r \leq \lfloor \frac{c+2}{2} \rfloor \). Note that the set of \( r \) satisfying \( 1 \leq r \leq \lfloor \frac{c+2}{2} \rfloor \) is non-empty as, by Proposition 7, it contains 3.

Proof of Part (a). By rotational symmetry, for \( 2 \leq r \leq c \), \( \frac{T_{c,r,b}}{T_{c,r-1,b}} = \frac{T_{c,r,b}}{T_{c,r-1,b}} \frac{T_{c,r-1,L}}{T_{c,r-1,L}} = \frac{T_{c,r,b}}{T_{c,r-1,b}} \frac{T_{c,r-1,L}}{T_{c,r-1,L}} = \frac{T_{c,r,b}}{T_{c,r-1,b}} \). Previous results in this section have established that \( x(c, r) = \frac{T_{c,r}}{T_{c,r-1}} \) and \( \frac{T_{c,r,b}}{T_{c,r-1,b}} = \frac{T_{c,r,b}}{T_{c,r-1,b}} \). Moreover, we may verify \( x(c, r) = r_{b,L}(c, c, r) \frac{g(c, r)}{g(c, r)} \). Combining these equations we infer \( y(c, r) = \frac{T_{c,r,1-L}}{T_{c,r-1,1-L}} \) as was to be proved.

Proof of Part (b). For the proof we first need to verify the following.

\[
x(c, r) \cdot z(r-1, 2) = \frac{1}{x(c, c+3-r) z(c+2-r, 2)} = \frac{1}{x(c, c+3-r)} \frac{y(c+2-r, 2)}{y(c+3-r, 2)}.
\]

By slide symmetry
\[
\frac{T_{c,2,2,L}}{T_{c,1,2,L}} = \frac{T_{c,1,1,L}}{T_{c,1,1,L}} \frac{T_{c,2,2,L}}{T_{c,1,2,L}} = x(c, r) \frac{y(r, 2)}{y(r-1, 2)} = x(c, r) \cdot z(r-1, 2),
\]

the last equality following from (21)-(22).

By results previously established in this section, for the left-hand side of (39)

\[
\frac{T_{c,2,2,L}}{T_{c,1,2,L}} = \frac{T_{c,1,1,L}}{T_{c,1,1,L}} \frac{T_{c,2,2,L}}{T_{c,1,2,L}} = x(c, r) \frac{y(r, 2)}{y(r-1, 2)} = x(c, r) \cdot z(r-1, 2),
\]

the last equality following from (21)-(22).

By results previously established in this section, for the right-hand side of (39),

\[
\frac{T_{c+2-r,2,L}}{T_{c+3-r,2,L}} = \frac{T_{c+2-r,1,L}}{T_{c+3-r,1,L}} \frac{T_{c+2-r,2,L}}{T_{c+2-r,1,L}} = \frac{1}{x(c, c+3-r)} \frac{T_{c+3-r,1,L}}{T_{c+2-r,1,L}} \frac{T_{c+2-r,2,L}}{T_{c+3-r,2,L}}.
\]

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Combining (40)-(41) with (39) yields

\[ x(c, r) \cdot z(r - 1, 2) = \frac{1}{x(c, c + 3 - r) T_{c+3-r,1,L} T_{c+2-r,2,L}} \]  

(42)

We now proceed inductively. For \( r = 3 \), by Proposition 12(a), \( \frac{T_{c+3-r,1,L}}{T_{c+3-r,2,L}} = y(c, 2) \). Combining this equation for \( y(c, 2) \) with (42) and (38), we conclude that \( \frac{T_{c+3-r,2,L}}{T_{c+3-r,1,L}} = y(c - 1, 2) \). Letting \( r = 4 \), we combine this equation for \( y(c - 1, 2) \) with (42), and (38) and conclude \( \frac{T_{c+2-r,1,L}}{T_{c+2-r,2,L}} = y(c - 2, 2) \). Letting \( r = 5 \) we combine this equation for \( y(c - 2, 2) \) with (42), and (38) and conclude \( \frac{T_{c+1-r,1,L}}{T_{c+1-r,2,L}} = y(c - 3, 2) \). Thus, proceeding inductively, we prove Proposition 12(b). (Note: For every \( r \) in the upper half, \( c - r \) is in the lower half. So we only need to use this inductive argument while \( r \) is in the upper half.)

**Proof of Part (c).** Let \( d > 2 \). We may inductively assume that i) the horizontal edge ratio relationship holds for diagonal \( d = 1 \), and ii) \( x(c, r) \cdot z(r - 1, d - 1) = \frac{T_{c+d-r,1,L}}{T_{c+d-1,r,1,L}} \), the base case of this induction assumption proven in Part b) for \( d = 2 \).

For the proof we must first verify

\[ x(c, r)z(r - 1, d - 1) \cdot \frac{y(r, d)}{y(r - 1, d)} = 1/(x(c, c + d + 1 - r) z(c + d - r, d - 1)) \cdot \frac{y(c + d - r, d)}{y(c + d + 1 - r, d)} \]

By slide symmetry,

\[ \frac{T_{r,d,L}}{T_{r-1,d,L}} = \frac{T_{c+d-r,d,L}}{T_{c+d-1-r,d,L}} \]  

(43)

Using results previously proven in this section, the left-hand side of (44) satisfies

\[ \frac{T_{r,d,L}}{T_{r-1,d,L}} = \frac{T_{r,d-1,L}}{T_{r-1,d-1,L}} = \frac{T_{c+d-r,d,L}}{T_{c+d-1-r,d,L}} = x(c, r)z(r - 1, d - 1) \cdot \frac{y(r, d)}{y(r - 1, d)} \]

(45)

Using the induction assumption, the right-hand side of (44) satisfies

\[ \frac{T_{c+d-r,d,L}}{T_{c+d-1-r,d,L}} = \frac{T_{c+d-r,d-1,L}}{T_{c+d-1-r,d-1,L}} = \frac{T_{c+d+1-r,d,L}}{T_{c+d+1-r,d-1,L}} \]

(46)

Combining (45)–(46) with (44) we obtain,

\[ x(c, r)z(r - 1, d - 1) \cdot \frac{1}{y(r - 1, d)} = \frac{1}{x(c, c + d - r) z(c + d - 1 - r, d - 1))} \cdot \frac{T_{c+d-r,d,L}}{T_{c+d-1-r,d,L}} \]

(47)
We now proceed inductively. If \( r = d + 1 \), by Proposition 12(b), \( \frac{T_{c,d+1-r,d,l}}{T_{c,d+1-r,d,l+1}} = \frac{1}{y(c,d)} \). Combining this equation with (47) and (43) we infer \( \frac{T_{c,d+1-r,d,L}}{T_{c,d+1-r,d,L+1}} = y(c-1,d) \). If \( r = d + 2 \), then combining this equation for \( y(c-1,d) \) with (47) and (43) implies \( \frac{T_{c,d+1-r,d,L}}{T_{c,d+1-r,d,L+1}} = y(c-2,d) \). If \( r = d + 3 \), then combining this equation for \( y(c-2,d) \) with (47) and (43) implies \( \frac{T_{c,d+1-r,d,L}}{T_{c,d+1-r,d,L+1}} = y(c-3,d) \). Proceeding inductively, we prove that the horizontal edge ratio holds on diagonal \( d \), completing the inductive proof of Proposition 12(c), implying completion of the proof of Proposition 12, so that the proof of Theorem 6 is complete. \( \square \)

9 Proof of Theorem 4(ii), Case \( i = c - 1 \).

This section proves Theorem 4(ii), for the case \( i = c - 1 \), the proof for \( 1 \leq i \leq c - 2 \), as well as proof of the other assertions of Theorem 4 being completed in the next section. Throughout this section, we assume that \( c \geq 5 \), and that \( T^{c,-1} \) possesses symmetry, justified by Proposition 7, and Theorem 3 respectively.

Theorem 4(ii) requires proving that all four edge ratio relationships are satisfied in \( T^{c,-1} \). We however suffice with the proof for the base-edge ratio relationship, the proof of the others being similar and hence omitted. By Proposition 5, to prove that \( T^{c,-1} \) satisfies the base-edge ratio relationship, it suffices to prove that the base edge ratio holds in its upper left half. To prove satisfaction in the upper left half, we prove that the base edge ratio relationship holds in all rows \( r \), \( 1 \leq r \leq c - 1 \) but only in diagonals \( d, 2 \leq d \leq \lceil \frac{c-3}{2} \rceil \), the proof for the \( d = 1 \) case being highly similar and hence omitted. In summary, in this section, we prove

\[
\frac{T^{3(c,c-1)}_{B,r,d}}{T^{3(c,c-1)}_{L,r,d}} = r_{B,L}(c-1,r,d), \quad 3 \leq r \leq c-2, 2 \leq d \leq r.
\]  

(48)

Figure 7 based on (4) and the four edge ratio relationships, present the 9 edge-value arguments needed to compute \( T^{3(c,c-1)}_{B,r,d} \). Similarly, Figure 8 based on (6) and the four edge ratio relationships, present the 9 edge-value arguments needed to compute \( T^{3(c,c-1)}_{L,r,d} \).

The “work” is done in the Figures. The proof of (48) is then simply completed by plugging these arguments into the base and edge local functions and verifying (48) is true.

10 Completion of proof of Theorem 4

Section 9 proves Theorem 4(ii) for the special case, \( i = c - 1 \). This section provides proofs for the cases \( 1 \leq i \leq c - 2 \). This section also proves Theorem 4(i). The proofs will be accomplished by a sequence of Propositions. Proposition 13 was illustrated and motivated by (14).

**Proposition 13.**

\[
T^{c,-1}_{X,r,d} = g(c)T^{c,-1}_{L,r,d}, \quad X \in \{L,R,B\}, 1 \leq r \leq c-1, 1 \leq d \leq r.
\]  

(49)
Fig. 7: The left-hand panel presents the edges of $T^c$ that are arguments of (4) used to compute $T_{B,r,d}^{(c,c-1)}$. The right-hand panel contains the values of these arguments.

$$A = \prod_{i=2}^{r} x(c,i) \cdot \prod_{i=2}^{d} y(r,i)$$
$$B = A \cdot r_{R,L}(r,d)$$
$$C = A \cdot r_{B,L}(c,r,d)$$
$$D = A \cdot x_{c,r+1} \cdot z_{r,d}$$
$$E = D \cdot r_{R,L}(r+1,d)$$
$$F = D \cdot y_{r+1,d+1}$$
$$G = F \cdot r_{R,L}(r+1,d+1)$$
$$H = D \cdot r_{B,L}(c,r+1,d)$$
$$I = F \cdot r_{B,L}(c,r+1,d+1)$$
$$J = F \cdot x_{c,r+2} \cdot z_{r+1,d+1}$$
$$K = J \cdot r_{R,L}(r+2,d+1)$$
$$L = J \cdot r_{B,L}(c,r+2,d+1)$$

Proof. For given $c$ we can compute $T_{L,1,1}^{c,c-1}$ by (3) and then verify that $g(c) = T_{L,1,1}^{c,c-1}$. By Definition 4, $T_{L,1,1}^{c-1,c-1} = 1$, and therefore using Definition 2 or the four local functions

$$T_{L,1,1}^{c,c-1} = g(c)T_{L,1,1}^{c-1,c-1}.$$  \hspace{1cm} (50)
Both $T^{c,-1}$ and $T^{c,c,-1}$ satisfy the four edge-ratio relationships, $T^{c,-1}$ by definition, and $T^{c,c,-1}$ by the result of Section 9. Therefore, (49) follows from (50) because all edges are identical multiples (using the four edge-ratio relationships) of the left edge of the top corner triangle.

\[ \text{Proposition 14.} \]

\[ T^{c,i}_{X,r,d} = f(c,i) \times T^{i}_{X,r,d}, \quad 1 \leq r \leq c - i, 1 \leq d \leq r, X \in \{L, R, B\}. \quad (51) \]

\[ \text{Proof.} \] We iterate (49).

If we start with an initial $c - 1$ grid and row reduce it once, then by Proposition 13 we have $T^{c,-1}_{L,1,1} = g(c - 2)T^{c-2}_{L,1,1}$. But row reducing $T^{c}$ once and then row reducing $T^{c,1,-1}$ once is the same as row reducing $T^{c}$ twice. In other words, $T^{c,c,-2}_{L,1,1} = g(c) \times g(c - 1) \times T^{c-2}_{L,1,1}$ and hence, since $T^{c,c,-2}$ and $T^{c-2}$ both satisfy the edge-ratio relationships, $T^{c,c,-2}_{X,r,d} = g(c) \times g(c - 1) \times T^{c-2}_{X,r,d}$ for $1 \leq r \leq c - 2, 1 \leq d \leq r, X \in \{L, R, B\}$.

Inductively iterating the above argument we have, for $1 \leq i \leq c - 1$, that $T^{c,c,-i}_{L,1,1} = \prod_{j=0}^{i-1} g(c - j) \times T^{c-1}_{L,1,1} = f(c, -i) \times T^{c-1}_{L,1,1}$, the last equality following from Proposition 1. Since, $T^{c,c,-1}$ and $T^{c-1}$ both satisfy the edge-ratio relationships, we then also have $T^{c,c,-1}_{X,r,d} = f(c, i) \times T^{c-1}_{X,r,d}$ for $1 \leq r \leq c - i, 1 \leq d \leq r, X \in \{L, R, B\}$. \[ \square \]

Propositions 13 and 14, complete the proof of Theorem 4(ii). We proceed to prove Theorem 4(i).

\[ \text{Proposition 15.} \]

\[ t(c,i) = f(c,i)\Delta(1,1,r_{B,L}(i,1,1)) \]

\[ \text{Proof.} \] By Definition 3, the tail distance $t(c,i)$ is simply the left leg of the $Y$ arising from applying the $Y$ transformation to $T^{1}_1$. By Proposition 14, $T^{1}_{i,1,X} = f(c,i) \cdot T^{i}_{1,1,X}$, for $X \in \{L, R, B\}$. By Definition 4, $\{T^{i}_{1,1,L}, T^{i}_{1,1,R}, T^{i}_{1,1,B}\} = \{1,1,r_{B,L}(i,1,1)\}$. The proof now follows by substitution and verification. \[ \square \]

\[ \text{Proposition 16.} \]

\[ \text{i) } t(c,1) = \frac{c}{2c + 1} \]
\[ \text{ii) } t(c,i) = \frac{c}{2c + 1} \]
\[ \text{iii) } \frac{t(c,1)}{t(c,i)} = \frac{1}{i}. \]

\[ \text{Proof.} \] By Proposition 15 and a straightforward verification. \[ \square \]

But Proposition 16 is Theorem 4(i). Therefore, the proof of Theorem 4 is complete.

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