P-ADIC ELLIPTIC QUADRATIC FORMS, PARABOLIC-TYPE PSEUDODIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS, AND MARKOV PROCESSES.

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Abstract. In this article we study the Cauchy problem for a new class of parabolic-type pseudodifferential equations with variable coefficients for which the fundamental solutions are transition density functions of Markov processes in the four dimensional vector space over the field of p-adic numbers.

1. Introduction

The stochastic processes over the p-adics, or more generally over ultrametric spaces, have attracted a lot of attention during the last thirty years due to their connections with models of complex systems, such as glasses and proteins, see e.g. [2], [3], [4], [6], [10], [13], [14], [15], [23], [22], and the references therein. In particular, the Markov stochastic process over the p-adics whose transition density functions are fundamental solutions of pseudodifferential equations of parabolic-type have appeared in several new models of complex systems. In [12] Chapters 4, 5] A. N. Kochubei presented a general theory for one-dimensional parabolic-type pseudodifferential equations with variable coefficients, whose fundamental solutions are transition density functions for Markov processes in the p-adic line. Taking into account the physical motivations before mentioned, it is natural to try to develop a general n-dimensional theory for pseudodifferential equations of parabolic-type over the p-adics. In this article using the results of [5]-[12, Chapter 4] we present four-dimensional analogs of the pseudodifferential equations of parabolic-type with variable coefficients studied by A. N. Kochubei in [12], see also [11]. We should mention that other types of n-dimensional pseudodifferential equations have been studied in [23], [24], [18], [9], [21], [6], among others. To explain the novelty of our contribution, we begin by saying that the theory developed in [12] depends heavily on an explicit formula for the Fourier transform of the Riesz kernel associated to the symbol of the Vladimirov pseudodifferential operator. It turns out, that this formula is a particular case of the functional equation satisfied for certain local zeta functions (i.e. distributions in an arithmetical framework) attached to quadratic forms. This functional equation was established by S. Rallis and G. Schiffmann in [17], see also the references cited in [5]. In [5] the authors studied the Riesz kernels and pseudodifferential operators associated to elliptic quadratic forms of dimensions two and four. Among several results, we determined explicit formulas for the Fourier transform of the Riesz kernels. In this article we use these

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results jointly with classical techniques for parabolic equations to study the Cauchy problem for a new class of parabolic-type pseudodifferential equations for which the fundamental solutions are transition density functions of Markov processes in the four dimensional vector space over the field of \( p \)-adic numbers.

The article is organized as follows. In Section 2, we fix the notation and collect some results about Riesz kernels associated to elliptic quadratic forms of dimension four. In Section 3 we introduce the heat kernels and certain pseudodifferential operators attached to elliptic quadratic forms and show some basic results needed for the other sections. In Section 4 we study the Cauchy problem for pseudodifferential equations with constant coefficients, see Theorem 5.1. In Section 5 we study the existence and uniqueness of the Cauchy problem for pseudodifferential equations with variable coefficients, see Theorems 6.3, 6.4. We also discuss the probabilistic meaning of the fundamental solutions, see Theorem 6.5.

2. Preliminaries

In this section we fix the notation and collect some basic results on \( p \)-adic analysis that we will use through the article. For a detailed exposition on \( p \)-adic analysis the reader may consult [1], [20], [22].

2.1. The field of \( p \)-adic numbers. Along this article \( p \) will denote a prime number different from 2. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as

\[
|x|_p = \begin{cases} 
0 & \text{if } x = 0 \\
p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b},
\end{cases}
\]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma := \text{ord}(x) \), with \( \text{ord}(0) := +\infty \), is called the \( p \)-adic order of \( x \). We extend the \( p \)-adic norm to \( \mathbb{Q}_p^n \) by taking

\[
||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.
\]

We define \( \text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\} \), then \( ||x||_p = p^{-\text{ord}(x)} \). Any \( p \)-adic number \( x \neq 0 \) has a unique expansion \( x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j \), where \( x_j \in \{0, 1, 2, \ldots, p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the fractional part of \( x \in \mathbb{Q}_p \), denoted \( \{x\}_p \), as the rational number

\[
\{x\}_p = \begin{cases} 
0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\
p^{\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0.
\end{cases}
\]

For \( \gamma \in \mathbb{Z} \), denote by \( B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^\gamma\} \) the ball of radius \( p^\gamma \) with center at \( a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n \), and take \( B_\gamma^n(0) := B^n_\gamma \). Note that \( B_0^n(a) = B_1(a_1) \times \cdots \times B_1(a_n) \), where \( B_1(a_i) := \{x \in \mathbb{Q}_p : |x - a_i|_p \leq p^1\} \) is the one-dimensional ball of radius \( p^1 \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_0^n(0) \) is equals the product of \( n \) copies of \( B_0(0) := \mathbb{Z}_p \), the ring of \( p \)-adic integers. We denote by \( \Omega(||x||_p) \) the characteristic function of \( B^n_0(0) \). For more general sets, say Borel sets, we use \( 1_A(x) \) to denote the characteristic function of \( A \).
2.2. The Bruhat-Schwartz space and Fourier transform. A complex-valued function \( \varphi \) defined on \( \mathbb{Q}_p^n \) is called locally constant if for any \( x \in \mathbb{Q}_p^n \) there exist an integer \( l(x) \) such that
\[
\varphi(x + x') = \varphi(x) \quad \text{for} \quad x' \in B_l(x),
\]
(2.1)

A function \( \varphi : \mathbb{Q}_p^n \to \mathbb{C} \) is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The \( \mathbb{C} \)-vector space of Bruhat-Schwartz functions is denoted by \( S(\mathbb{Q}_p^n) := S \). For \( \varphi \in S(\mathbb{Q}_p^n) \), the largest of such number \( l = l(\varphi) \) satisfying (2.1) is called the exponent of local constancy of \( \varphi \).

Let \( S'(\mathbb{Q}_p^n) := S' \) denote the set of all functionals (distributions) on \( S(\mathbb{Q}_p^n) \). All functionals on \( S(\mathbb{Q}_p^n) \) are continuous.

Set \( \chi(y) = \exp(2\pi i (y)_p) \) for \( y \in \mathbb{Q}_p \). The map \( \chi(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuos map from \( \mathbb{Q}_p \) into the unit circle satisfying \( \chi(y_0 + y_1) = \chi(y_0) \chi(y_1), \ y_0, y_1 \in \mathbb{Q}_p \).

Given \( \xi = (\xi_1, \ldots, \xi_n) \) and \( x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n \), we set \( \xi \cdot x := \sum_{j=1}^n \xi_j x_j \). The Fourier transform of \( \varphi \in \phi(\mathbb{Q}_p^n) \) is defined as
\[
(F\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi(-\xi \cdot x)\varphi(x)d^n x \quad \text{for} \quad \xi \in \mathbb{Q}_p^n,
\]
where \( d^n x \) is the Haar measure on \( \mathbb{Q}_p^n \) normalized by the condition \( vol(B_1^n) = 1 \). The Fourier transform is a linear isomorphism from \( S(\mathbb{Q}_p^n) \) unto itself satisfying \( (F(F\varphi))(\xi) = \varphi(-\xi) \). We will also use the notation \( F_{x \to \xi} \varphi \) and \( \hat{\varphi} \) for the Fourier transform of \( \varphi \).

The Fourier transform \( F \) of a distribution \( f \in S'(\mathbb{Q}_p^n) \) is defined by
\[
(F[f], \varphi) = (f, F[\varphi]) \quad \text{for all} \quad \varphi \in S(\mathbb{Q}_p^n).
\]
The Fourier transform \( f \to F[f] \) is a linear isomorphism from \( S'(\mathbb{Q}_p^n) \) unto \( S'(\mathbb{Q}_p^n) \). Furthermore, \( f = F[F[f](\cdot)] \).

2.3. Elliptic Quadratic Forms and Riesz Kernels. We set
\[
f(x) = x_1^2 - ax_2^2 - px_3^2 + apx_4^2, \quad f^o(\xi) = ap\xi_1^2 + p\xi_2^2 - a\xi_3^2 + \xi_4^2
\]
with \( a \in \mathbb{Z} \) a quadratic non-residue module \( p \). The form \( f(x) \) is elliptic, i.e. \( f(x) = 0 \iff x = 0 \), and \( f^o(\xi) = apf \left( \frac{\xi_1}{1}, \frac{\xi_2}{-a}, \frac{\xi_3}{-p}, \frac{\xi_4}{ap} \right) \), hence \( f^o(\xi) \) is also elliptic. The following estimate will be used frequently along the article: there exist positive constants \( A, B \) such that
\[
B \| x \|_p^2 \leq |f(x)|_p \leq A \| x \|_p^2 \quad \text{for any} \quad x \in \mathbb{Q}_p^n,
\]
c.f. [24] Lemma 1.

The Riesz kernels attached to \( f \) and \( f^o \) satisfy the following functional equation:
\[
\int_{\mathbb{Q}_p^n \setminus \{0\}} |f(z)|^{s-2}\tilde{\varphi}(z)d^n z = \frac{1 - p^{s-2}}{1 - p^{-s}} \int_{\mathbb{Q}_p^n \setminus \{0\}} |f^o(\xi)|_{p^{-s}}\varphi(\xi)d^4 \xi,
\]
(2.3)

for \( \varphi \in S(\mathbb{Q}_p^4) \) and \( s \in \mathbb{C} \), c.f. [5] Proposition 2.8]. For further details about Riesz kernels attached to quadratic forms we refer the reader to [5].
Lemma 2.1. If $\alpha > 0$, then
\[ |f^\circ(x)|^\alpha_p = \frac{1 - p^\alpha}{1 - p^{-\alpha/2}} \int_{\mathbb{Q}_p^d} |f(\xi)|^{-\alpha/2} [\chi(\xi \cdot x) - 1] d^d \xi. \]

Proof. The formula follows from [23] by using the argument given in [12] for Proposition 2.3.

\[ \square \]

3. Heat Kernels

We define the heat kernel attached to $f^\circ$ as
\[ Z(x, t) := Z(x, t; f^\circ, \alpha, \kappa) = \int \mathbb{Q}_p^d \chi(\xi \cdot x) e^{-\kappa t |f^\circ(\xi)|^\alpha_p} d^d \xi \]
for $x \in \mathbb{Q}_p^d$, $t > 0$, $\alpha > 0$, and $\kappa > 0$. When considering $Z(x, t)$ as a function of $x$ for $t$ fixed we will write $Z_t(x)$. The heat kernels considered in this article are a particular case of those studied in [24].

Given $M \in \mathbb{Z}$, we set
\[ Z_t^{(M)}(x) := \int \mathbb{Q}_p^d \Omega(p^{-M}||\xi||_p) \chi(\xi \cdot x) e^{-\kappa t |f^\circ(\xi)|^\alpha_p} d^d \xi \]
for $x \in \mathbb{Q}_p^d$, $t > 0$, $\alpha > 0$, and $\kappa > 0$. Taking into account that
\[ \left| \Omega(p^{-M}||\xi||_p) \chi(\xi \cdot x) e^{-\kappa t |f^\circ(\xi)|^\alpha_p} \right| \leq e^{-\kappa t |f^\circ(\xi)|^\alpha_p} \leq e^{-\kappa B^\alpha ||\xi||_p^2} \in L^1(\mathbb{Q}_p^d), \]
c.f. [22], the Dominated Convergence Theorem implies that
\[ \lim_{M \to \infty} Z_t^{(M)}(x) = Z_t(x) \text{ for } x \in \mathbb{Q}_p^d \text{ and for } t > 0. \]

Proposition 3.1. For $x, \xi \in \mathbb{Q}_p^d \setminus \{0\}$ the following assertions hold:
(i) $|f(x + \xi)|_p = |f(x)|_p$, for $||x||_p < p^{-1}||x||_p$;
(ii) $Z(x, t) = \sum_{m=1}^\infty \frac{(-1)^m}{m!} \left( \frac{1}{1 - p^{-\alpha} - 2} \right) \kappa^m \alpha^m |f(x)|_p^{-\alpha - 2}$;
(iii) $Z(x + \xi, t) = Z(x, t)$, for $||x||_p < p^{-1}||x||_p$;
(iv) $Z(x, t) \geq 0$, for $x \in \mathbb{Q}_p^d$ and $t > 0$.

Proof. (i) Set $x = p^{\text{ord}(x)} u$, $\xi = p^{\text{ord}(\xi)} v$ with $||u||_p = ||v||_p = 1$. Then
\[ |f(x + \xi)|_p = |f(p^{\text{ord}(x)} u + p^{\text{ord}(\xi)} v)|_p = |p^{2\text{ord}(x)} f(u + p^{\text{ord}(\xi) - \text{ord}(x)} v)|_p \]
\[ = p^{-2\text{ord}(x)} |f(u + p^{\text{ord}(\xi) - \text{ord}(x)} A)|_p \]
for some $A \in \mathbb{Z}_p$. Note that $|f(u)|_p \in \{1, p^{-1}\}$, then for $\text{ord}(\xi) - \text{ord}(x) > 1$, $|f(u + p^{\text{ord}(\xi) - \text{ord}(x)} A)|_p = |f(u)|_p$ and
\[ |f(x + \xi)|_p = p^{-2\text{ord}(x)} |f(u)|_p = |p^{2\text{ord}(x)} f(u)|_p = |f(x)|_p. \]

(ii) By using the Taylor expansion of $e^x$ and Fubini’s Theorem, $Z_t^{(M)}(x)$ can be rewritten as
\[ Z_t^{(M)}(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!} \kappa^m |f(x)|_p^m d^d \xi. \]
By using (2.3) with $m \neq 0$, we have

$$
I_M := \int_{Q_p^4} \Omega(p^{-M} ||\xi||_p) \chi(\xi \cdot x) |f^p(\xi)|_p^{m-2} d^4 \xi
$$

$$
= \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} \int_{Q_p^4} F_{\xi \cdot x} \left[ \Omega(p^{-M} ||\xi||_p) \chi(\xi \cdot x) \right] |f(z)|_p^{-m-2} d^4 z
$$

$$
= \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} p^{4M} \int_{Q_p^4} \Omega(p^M ||x - z||_p) |f(z)|_p^{-m-2} d^4 z.
$$

We now use the fact that $x \neq 0$ is fixed, and take $M > 1 + \text{ord}(x)$, changing variables as $z = x - p^M y$ in $I_M$, and using (i), we have

$$
I_M = \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} \int_{Q_p^4} \Omega(||y||_p) |f(x - p^M y)|_p^{-m-2} d^4 y
$$

$$
= \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} |f(x)|_p^{-m-2} \int_{Q_p^4} \Omega(||y||_p) d^4 y = \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} |f(x)|_p^{-m-2}.
$$

On the other hand, for $m = 0$,

$$
\lim_{M \to \infty} \int_{Q_p^4} \Omega(p^{-M} ||\xi||_p) \chi(\xi \cdot x) d^4 \xi = 0.
$$

Therefore

$$
Z(x, t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \kappa^m \left( \frac{1 - p^{om}}{1 - p^{-\alpha m - 2}} \right) |f(x)|_p^{-m-2} \text{ for } x \in Q_p^4 \setminus \{0\}.
$$

(iii) It is consequence of (ii) and (i).

(iv) See [24, Theorem 2].

Proposition 3.2. The following assertions hold for any $x \in Q_p^4$, $t > 0$:

(i) there exists a positive constant $C_1$ such that $Z(x, t) \leq C_1 t^{1/2\alpha} + ||x||_p^{-2\alpha - 4}$;

(ii) $Z(\cdot, t) \in C^1 ((0, \infty), \mathbb{R})$ and $\frac{\partial Z(x, t)}{\partial t} = -\kappa \int_{Q_p^4} |f^p(\xi)|_p^2 e^{-\kappa t f^p(\xi)} \chi(x \cdot \xi) d^4 \xi$;

(iii) there exists a positive constant $C_2$ such that

$$
\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_2 \left( t^{1/2\alpha} + ||x||_p^{-2\alpha - 4} \right).
$$

Proof. (i) We first consider the case in which $t ||x||_p^{-2\alpha} \leq 1$, then by Proposition 3.1 (ii)-(iv) and by (2.2),

$$
Z(x, t) \leq |f(x)|_p^{-2} \sum_{m=1}^{\infty} \frac{C_0^m}{m!} (t ||x||_p^{-\alpha})^m \leq B^{-2} ||x||_p^{-2} \sum_{m=1}^{\infty} \frac{(C_0 B^{-\alpha})^m}{m!} (t ||x||_p^{-2\alpha})^m
$$

$$
= B^{-2} ||x||_p^{-4} \left( e^{C_0 B^{-\alpha} t ||x||_p^{-2\alpha}} - 1 \right) \leq C t ||x||_p^{-2\alpha - 4}.
$$
We now consider the case in which \( t > 0 \). Take \( k \) to be an integer satisfying \( p^{k-1} \leq t^{\frac{1}{2}} \leq p^k \). Then by Proposition 3.1 (iv), and by (2.2),

\[
|Z(x,t)| = Z(x,t) \leq \int_{Q_p^4} e^{-C_0 t \|\xi\|^{2\alpha} p} d^4 \xi \leq \int_{Q_p^4} e^{-C_0 t \|\xi\|^{2\alpha} p} d^4 \xi \leq \int_{Q_p^4} e^{-C_0 t \|\eta\|^{2\alpha} p} d^4 \eta \leq C t^{-2/\alpha}.
\]

By combining the above inequalities, see for instance the end of the proof of Proposition 4.5, we get the announced result.

(ii) The formula for \( \frac{\partial Z(x,t)}{\partial t} \) is obtained by a straightforward calculation. The continuity of \( Z(\cdot, t) \) is obtained from the formula for \( \frac{\partial Z(x,t)}{\partial t} \) by using the Dominated Convergence Theorem. (iii) This part is proved in the same way as (i). \( \square \)

The first part of Proposition 3.2 is a particular case of Theorem 1 in [24]. We include this proof here due to two reasons: first, it shows a very deep connection between the functional equation (2.3) and the heat kernels; second, we use this technique for bounding several types of oscillatory integrals in this article.

**Corollary 3.3.** (i) \( \int_{Q_p^4} Z_t(x) d^4 x = 1 \) for \( t > 0 \); (ii) \( Z_t(x) \in L^p \) for \( t > 0 \) and for \( 1 \leq p \leq \infty \).

**4. Some Results on Operators of Type \( f(\partial, \alpha) \)**

**4.1. The space \( M_\lambda \).** We denote by \( M_\lambda, \lambda \geq 0 \), the \( \mathbb{C} \)-vector space of locally constant functions \( \varphi(x) \) on \( \mathbb{R}^4_p \) such that \( |\varphi(x)| \leq C(1 + ||x||^\lambda) \), where \( C \) is a positive constant. If the function \( \varphi \) depends also on a parameter \( t \), we shall say that \( \varphi \in M_\lambda \) uniformly with respect to \( t \), if its constant \( C \) and its exponent of local constancy do not depend on \( t \).

**Lemma 4.1.** If \( \varphi \in M_{2\lambda}, \) with \( 0 \leq \lambda < \alpha \) and \( \alpha > 0 \), then

\[
\lim_{t \to 0^+} \int_{Q_p^4} Z(x - \xi, t) \varphi(\xi) d^4 \xi = \varphi(x).
\]
Proof. By Corollary 3.3 (i) and Proposition 3.2 (i), and the fact that \( \varphi \) is locally constant,

\[
I := \left| \int_{\mathbb{R}^4} Z(x - \xi, t) \varphi(\xi) d^4 \xi - \varphi(x) \right| = \left| \int_{\mathbb{R}^4} Z(x - \xi, t) [\varphi(\xi) - \varphi(x)] d^4 \xi \right|
\]

\[
= \left| \int_{\|x-\xi\|_p \geq p^L} Z(x - \xi, t) [\varphi(\xi) - \varphi(x)] d^4 \xi \right|
\]

\[
\leq C_1 t \int_{\|z\|_p \geq p^L} (t^{1/2\alpha} + \|z\|_p)^{-2\alpha - 4} |\varphi(z) - \varphi(x)| d^4 z.
\]

By applying the triangle inequality in the last integral and noticing that

\[
|\varphi(x)| \int_{\|z\|_p \geq p^L} (t^{1/2\alpha} + \|z\|_p)^{-2\alpha - 4} d^4 z \leq |\varphi(x)| \int_{\|z\|_p \geq p^L} \|z\|_p^{-2\alpha - 4} d^4 z \leq C_0 |\varphi(x)|,
\]

and

\[
\int_{\|z\|_p \geq p^L} (t^{1/2\alpha} + \|z\|_p)^{-2\alpha - 4} \|z\|_p^{2\lambda} d^4 z \leq \int_{\|z\|_p \geq p^L} \|z\|_p^{-2\alpha + 2\lambda - 4} d^4 z < \infty,
\]

we have

\[
\lim_{t \to 0^+} I \leq (C_1 + C_2 |\varphi(x)|) \lim_{t \to 0^+} t = 0.
\]

\[ \square \]

4.2. The Operator \( f(\partial, \alpha) \). Given \( \alpha > 0 \), we define the pseudodifferential operator with symbol \( |f^\alpha (\xi)|_p^\alpha \) by

\[
\mathbf{S} (Q^4_p) \to C (Q^4_p) \cap L^2 (Q^4_p)
\]

\[
\varphi \to (f(\partial, \alpha) \varphi)(x) = F_{\xi \to x}^{-1} \left( |f^\alpha (\xi)|_p^\alpha F_{x \to \xi} \varphi \right).
\]

This operator is well-defined since \( |f^\alpha (\xi)|_p^\alpha F_{x \to \xi} \varphi \in L^1 (Q^4_p) \cap L^2 (Q^4_p) \). By [5], Proposition 3.4 (iv),

\[
(f(\partial, \alpha) \varphi)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha - 2}} \int_{Q^4_p} \frac{\varphi(x - y) - \varphi(x)}{|f(y)|_{p^{\alpha + 2}}^{\alpha + 2}} d^4 y,
\]

for \( \varphi \in \mathbf{S} (Q^4_p) \). The operator \( f(\partial, \alpha) \) can be extended to any locally constant functions \( \varphi (x) \) satisfying

\[
\int_{\|x\|_p \geq p^m} \frac{|\varphi(x)|}{|f(x)|_{p^{\alpha + 2}}^{\alpha + 2}} d^4 x < \infty \text{ for some } m \in \mathbb{Z},
\]

c.f. [5] Lemma 4.1.
Note that
\[ Z_t^{(M)}(x) = \int_{||\eta||_p \leq p^M} \chi(x \cdot \eta)e^{-\kappa t|f^\alpha(\eta)|_p^\alpha} d^4 \eta, \text{ with } M \in \mathbb{N} \]
is a locally constant and bounded function, c.f. Proposition 3.2. Furthermore, by Proposition 3.2 and (2.2), \( Z_t^{(M)}(x) \) satisfies condition (4.2), for \( t > 0 \).

**Lemma 4.2.**

\[ (f(\partial, \gamma)Z_t^{(M)})(x) = \int_{||\eta||_p \leq p^M} \chi(x \cdot \eta)|f^\alpha(\eta)|_p^\alpha e^{-at|f^\alpha(\eta)|_p^\alpha} d^4 \eta, \]
for \( M \in \mathbb{N} \) and for \( t > 0 \).

**Proof.** Note that if \( ||\xi||_p \leq p^{-M} \), then \( Z_t^{(M)}(x - \xi) = Z_t^{(M)}(x) \). In addition, since \( Z_t^{(M)}(x) \) satisfies condition (4.2), we can use formula (4.1) to compute \( (f(\partial, \gamma)Z_t^{(M)})(x) \) as follows:

\[
(f(\partial, \gamma)Z_t^{(M)})(x) = \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{Q^4_k} |f(\xi)|_p^{-\gamma - 2} \left[ Z_t^{(M)}(x - \xi) - Z_t^{(M)}(x) \right] d^4 \xi
\]

\[
= \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{||\xi||_p > p^{-M}} |f(\xi)|_p^{-\gamma - 2} \left[ Z_t^{(M)}(x - \xi) - Z_t^{(M)}(x) \right] d^4 \xi
\]

\[
\quad + \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{||\xi||_p \leq p^{-M}} |f(\xi)|_p^{-\gamma - 2} \left[ Z_t^{(M)}(x - \xi) - Z_t^{(M)}(x) \right] d^4 \xi
\]

\[
= \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{||\xi||_p > p^{-M}} |f(\xi)|_p^{-\gamma - 2} \int_{||\eta||_p \leq p^M} e^{-at|f^\alpha(\eta)|_p^\alpha} \chi(x \cdot \eta)(\chi(\xi \cdot \eta) - 1)d^4 \eta d^4 \xi
\]

\[
= \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{||\eta||_p \leq p^M} e^{-at|f^\alpha(\eta)|_p^\alpha} \chi(x \cdot \eta) \left( \int_{Q^4_k} |f(\xi)|_p^{-\gamma - 2} \chi(\xi \cdot \eta) - 1 \right) d^4 \xi d^4 \eta
\]

\[
= \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{||\eta||_p \leq p^M} e^{-at|f^\alpha(\eta)|_p^\alpha} \chi(x \cdot \eta) \left( \left( \int_{Q^4_k} |f(\xi)|_p^{-\gamma - 2} \chi(\xi \cdot \eta) - 1 \right) \right) d^4 \eta
\]

\[
= \int_{||\eta||_p \leq p^M} |f^\alpha(\eta)|_p^\alpha e^{-at|f^\alpha(\eta)|_p^\alpha} \chi(x \cdot \eta) d^4 \eta, \text{ c.f. Lemma 2.1. }
\]

By Proposition 3.1 (iii) and Proposition 3.2 (i), \( (f(\partial, \gamma)Z_t)(x) \) is well-defined for \( x \neq 0 \) and for \( 0 < \gamma \leq \alpha \).

**Proposition 4.3.**

\[ (f(\partial, \gamma)Z_t)(x) = \int_{Q^4_k} |f^\alpha(\eta)|_p^\alpha e^{-\kappa t|f^\alpha(\eta)|_p^\alpha} \chi(x \cdot \eta) d^4 \eta, \text{ for } 0 < \gamma \leq \alpha, t > 0. \]
Proof. By \( \text{(2.2)} \), \(|f^p(x)|_p^\gamma e^{-\kappa t|f^p(x)|_p^\gamma} \in L^1(\mathbb{Q}^4_p)\) for \( t > 0 \), then from \( \text{(4.3)} \), by the Dominated Convergence Theorem, we obtain
\[
\lim_{M \to \infty} (f(\partial, \gamma)Z_t(M))(x, t) = \int_{\mathbb{Q}^4_p} \chi(x \cdot \eta)|f^p(\eta)|_p^\gamma e^{-\kappa t|f^p(\eta)|_p^\gamma} d^4\eta, \text{ for } t > 0.
\]

On the other hand, fixing an \( x \neq 0 \), by Proposition \( \text{(3.1)} \) (i),
\[
(f(\partial, \gamma)Z_t(M))(x) = \frac{1-p^\gamma}{1-p^\gamma} - \int_{||\xi||_p > p^{-1}||x||_p} |f(\xi)|_p^{\gamma-2} \left[ Z_t^p(M)(x - \xi) - Z_t^p(M)(x) \right] d^4\xi.
\]

Finally, by the Dominated Convergence Theorem and \( \text{(4.5)} \), we have
\[
\lim_{M \to \infty} (f(\partial, \gamma)Z_t(M))(x, t) = \int_{\mathbb{Q}^4_p} \chi(x \cdot \eta)|f^p(\eta)|_p^\gamma e^{-\kappa t|f^p(\eta)|_p^\gamma} d^4\eta.
\]

Finally, we note that the right-hand side of \( \text{(4.4)} \) is continuous at \( x = 0 \). \( \square \)

Corollary 4.4. \( \frac{\partial Z_t(x, t)}{\partial t} = -\kappa(f(\partial, \alpha)Z_t(x, t)) \) for \( t > 0 \).

Proof. The formula follows from Propositions \( \text{(4.3)} \) and \( \text{(3.2)} \) (ii). \( \square \)

Proposition 4.5. If \( 0 < \gamma \leq \alpha \), then
\[
|f(\partial, \gamma)Z_t(x)| \leq C(t^{1/2\alpha} + ||x||_p)^{-2\gamma-4}, \text{ for } x \in \mathbb{Q}^4_p \text{ and for } t > 0.
\]

Proof. By reasoning as in the proof of Proposition \( \text{(3.1)} \) (ii), we have
\[
f(\partial, \gamma)Z_t(x) = \sum_{m=1}^\infty \frac{(-1)^m}{m!} \kappa^m t^m \left( \frac{1-p^\gamma}{1-p^\gamma} \right) |f(x)|_p^{-m\alpha-\gamma-2}.
\]

If \( t||x||^{-2\alpha} \leq 1 \), from \( \text{(4.6)} \) and \( \text{(2.2)} \), we obtain
\[
|f(\partial, \gamma)Z_t(x)| \leq |f(x)|_p^{\gamma-2} \sum_{m=1}^\infty \frac{C_m}{m!} (t||f(x)||^{-\alpha})^m \leq C_1||x||^{-2\gamma-4}.
\]

On other hand, take \( k \) such that \( p^{k-1} \leq t^{1/2\alpha} \leq p^k \). From \( \text{(4.4)} \) by using \( \text{(2.2)} \), we get
\[
|f(\partial, \gamma)Z_t(x)| \leq A_\gamma \int_{\mathbb{Q}^4_p} ||\eta||_p^{2\gamma} |f^p(\eta)|_p^{\gamma} d\eta \leq A_\gamma \int_{\mathbb{Q}^4_p} ||\eta||_p^{2\gamma} e^{-aB^\alpha ||\eta||_p^{2\alpha}} d\eta
\]
\[
= A_\gamma p^{-4(k-1)-2\gamma} \int_{\mathbb{Q}^4_p} ||\xi||_p^{2\gamma} e^{-aB^\alpha} d^4\xi \leq C\gamma^{4-2\gamma/2\alpha}.
\]

The announced results follows from inequalities \( \text{(4.7)} \)-\( \text{(4.8)} \). Indeed, \( t||x||^{-2\alpha} \leq 1 \) implies that \( ||x||_p \geq \frac{||x||_p}{2} + \frac{t^{1/2\alpha}}{2} \), and hence
\[
||x||_p^{2\gamma-4} \leq 2^{2\gamma+4} \left( ||x||_p + t^{1/2\alpha} \right)^{-2\gamma-4}.
\]

Now, if \( t||x||^{-2\alpha} > 1 \), then \( t^{1/2\alpha} > \frac{1}{2} + \frac{||x||_p}{2} \) and
\[
t^{-4-2\gamma/2\alpha} < 2^{2\gamma+4} \left( \frac{1/2 + ||x||_p/2}{2} \right)^{-2\gamma-4}.
\]

\( \square \)
Corollary 4.6. \[
\int_{\mathbb{Q}^4_p} (f(\partial, \gamma)Z_t)(x)d^4x = 0 \text{ for } t > 0.
\]

5. The Cauchy problem

Along this section, we fix the domain \((\text{Dom } \mathbb{f})\) of the operator \(\mathbb{f}(\partial, \alpha)\) to be the \(\mathbb{C}\)-vector space of locally constant functions satisfying \((1.2)\), and \(\mathbb{f}(\partial, \alpha)\varphi\) is given by \((1.1)\) for \(\varphi \in \text{Dom } \mathbb{f}\). Note that \(\mathbb{M}_{2\lambda} \subset \text{Dom } \mathbb{f}\) for \(\lambda < \alpha\).

In this section we study the following Cauchy problem:

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} + \kappa \mathbb{f}(\partial, \gamma)u(x,t) = g(x,t), \quad x \in \mathbb{Q}^4_p, \quad 0 < t \leq T, \\
u(x,0) = \varphi(x)
\end{cases}
\]  
(5.1)

where \(\kappa > 0, \alpha > 0, T > 0, \varphi \in \mathbb{M}_{2\lambda}, g(x,t) \in \mathbb{M}_{2\lambda}\) uniformly in \(t, 0 \leq \lambda < \alpha, g(x,t)\) is continuous in \((x,t), u : \mathbb{Q}^4_p \times [0,T] \rightarrow \mathbb{C}\) is an unknown function. We say that \(u(x,t)\) is a solution of Cauchy problem \((5.1)\) if \(u(x,t)\) is continuous in \((x,t)\), \(u(\cdot, t) \in \text{Dom } \mathbb{f}\) for \(t \in [0,T]\), \(u(x, \cdot)\) is continuously differentiable for \(t \in (0,T]\), \(u(x, t) \in \mathbb{M}_{2\lambda}\) uniformly in \(t\), and \(u\) satisfies \((5.1)\) for all \(t > 0\).

Theorem 5.1. The function

\[
u(x,t) = \int_{\mathbb{Q}^4_p} Z(x-y, t)\varphi(y)d^4y + \int_0^t \left( \int_{\mathbb{Q}^4_p} Z(x-y, t-\theta)g(y, \theta)d^4y \right) d\theta
\]

is a solution of Cauchy problem \((5.1)\).

The proof of the theorem will be accomplished through the following lemmas.

Lemma 5.2. Assume that \(g \in \mathbb{M}_{2\lambda}, 0 \leq \lambda < \alpha,\) uniformly with respect to \(\theta\). Then the function

\[
u_2(x, t, \tau) := \int_\tau^t \left( \int_{\mathbb{Q}^4_p} Z(x-y, t-\theta)g(y, \theta)d^4y \right) d\theta
\]

belongs to \(\mathbb{M}_{2\lambda}\) uniformly with respect to \(t\) and \(\tau\).

Proof. We first note that \(u_2(x, t, \tau)\) has the same exponent of local constancy as \(g\), and thus it does not depend on \(t\) and \(\tau\). We now show that \(|u_2(x, t, \tau)| \leq C_0(1 + ||x||_{2\lambda}^2)\). By Proposition 5.2 (i),

\[
|u_2(x, t, \tau)| \leq \int_\tau^t \left( \int_{\mathbb{Q}^4_p} Z(x-y, t-\theta)||g(y, \theta)||d^4y \right) d\theta
\]

\[
\leq C_1 \int_\tau^t (t-\theta)^{1/2\alpha} \left( \int_{\mathbb{Q}^4_p} ((t-\theta)^{1/2\alpha} + ||x-y||_p)^{2\alpha-4} (1 + ||y||_{2\lambda}^2)dy \right) d\theta.
\]

Now the result follows from the following estimation.
**Proof.** Set $h \in (0, 1)$ where $t > 0$. The first integral contains no singularity at $t = \theta$ due to Proposition 3.2 (iii) and the local constancy of $g$. By Proposition 5.3 (i) and Corollary 5.3 (i), the second integral is equal to zero. The third integral can be written as the sum of the integrals over $\{\xi \in \mathbb{R}^n \mid \|x - \xi\|_p \leq \rho^M\}$, where $M$ is the exponent of local constancy of $g$, and the complement of this set. The first integral tends to zero when $h \to 0^+$ due to the uniform local constancy of $g$, while the other tends to zero when $h \to 0^+$ due to Proposition 5.3 (i) and condition $\lambda < \alpha$. Finally, the fourth integral tends to $g(x, t)$ as $h \to 0^+$, c.f. Lemma 4.1. 

**Lemma 5.3.** Assume that $g \in \mathcal{M}_{2\lambda}$, $0 \leq \lambda < \alpha$, uniformly with respect to $\theta$. Then

$$
\frac{\partial u_2(x, t, \tau)}{\partial t} = g(x, t) + \int t \left( \int \frac{\partial Z(x - \xi, t - \theta)}{\partial t} [g(\xi, \theta) - g(x, \theta)]d^4\xi \right) d\theta.
$$

**Proof.** Set

$$
u_{2,h}(x, t, \tau) := \int t \int \int \int Z(x - \xi, t - \theta) g(\xi, \theta) d^4\xi,$$

where $h$ is a small positive number. By differentiating $u_h$ under the sign of integral

$$
\frac{\partial u_{2,h}}{\partial t} = \int t \int \int \frac{\partial Z(x - \xi, t - \theta)}{\partial t} [g(\xi, \theta) - g(x, \theta)]d^4\xi + \int Z(x - \xi, t) g(\xi, \theta) d^4\xi
$$

$$
+ \int t \int g(x, \theta) d\theta \int \frac{\partial Z(x - \xi, t - \theta)}{\partial t} d^4\xi + \int Z(x - \xi, h) [g(\xi, t - h) - g(\xi, t)]d^4\xi
$$

$$
+ \int Z(x - \xi, h) g(\xi, t) d^4\xi.
$$

The first integral contains no singularity at $t = \theta$ due to Proposition 5.3.2 (iii) and the local constancy of $g$. By Proposition 5.3.2 (i) and Corollary 5.3 (i), the second integral can be written as the sum of the integrals over $\{\xi \in \mathbb{R}^n \mid \|x - \xi\|_p \leq \rho^M\}$, where $M$ is the exponent of local constancy of $g$, and the complement of this set. The first integral tends to zero when $h \to 0^+$ due to the uniform local constancy of $g$, while the other tends to zero when $h \to 0^+$ due to Proposition 5.3.2 (i) and condition $\lambda < \alpha$. Finally, the fourth integral tends to $g(x, t)$ as $h \to 0^+$, c.f. Lemma 4.1.

For $\varphi \in \mathcal{M}_{2\lambda}$, $0 \leq \lambda < \alpha$, we set

$$
u_1(x, t) := \int Z(x, y, t) \varphi(y) dy$$

for $t > 0$.

**Lemma 5.4.** Assume that $\varphi \in \mathcal{M}_{2\lambda}$, $0 \leq \lambda < \alpha$, then the following assertions hold:

(i) $\nu_1(x, t)$ belongs to $\mathcal{M}_{2\lambda}$ uniformly with respect to $t$;

(ii) $\frac{\partial \nu_1}{\partial t}(x, t) = \int \int Z(x - y, t) \varphi(y) dy$ for $t > 0$.
Proof. (i) The proof is similar to that of Lemma 5.2
(ii) By Proposition 3.2 (ii),
\[
\lim_{h \to 0} \frac{u_1(x, t + h) - u_1(x, t)}{h} = \lim_{h \to 0} \int_{Q^4_p} \left[ \frac{Z(x - y, t + h) - Z(x - y, t)}{h} \right] \phi(y) d^4 y
\]
\[
= \lim_{h \to 0} \int_{Q^4_p} \frac{\partial Z}{\partial t} (x - y, \tau) \phi(y) d^4 y
\]
where \( \tau \) is between \( t \) and \( t + h \). Now the result follows from Proposition 3.2 (iii) by applying the Dominated Convergence Theorem.

Lemma 5.5. Assume that \( \lambda < \gamma \leq \alpha \). Then
\[
(f(\partial, \gamma) u_1)(x, t) = \int (f(\partial, \gamma) Z_t)(x - y) \phi(y) d^4 y \text{ for } t > 0.
\]

Proof. By Lemma 5.4 (i), \( u_1(x, t) \) belongs to the domain of \( f(\partial, \gamma) \) for \( t > 0 \) and for \( \lambda < \gamma \leq \alpha \), then for any \( L \in \mathbb{N} \), the following integral exists:
\[
\frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_{p^{-\gamma - 2}} [u_1(x - y, t) - u_1(x, t)] d^4 y
\]
\[
= \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_{p^{-\gamma - 2}} \left[ \int_{Q^4_p} [Z_t(x - y - \xi) - Z_t(x - \xi)] \phi(\xi) d^4 \xi \right] d^4 y.
\]
By using Fubini’s Theorem, see 222, Proposition 3.2 (i),
\[
\int_{Q^4_p} \left[ \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_{p^{-\gamma - 2}} [Z_t(x - y - \xi) - Z_t(x - \xi)] d^4 y \right] \phi(\xi) d^4 \xi
\]
\[
= \int_{Q^4_p} \phi(\xi) Z_t^{(\gamma, L)}(x - \xi) d^4 \xi.
\]
(5.3)
By fixing a positive integer \( M \), the last integral in (5.3) can be expressed as
\[
\int_{|x - \xi|_p \geq p^{-M}} \phi(\xi) Z_t^{(\gamma, L)}(x - \xi) d^4 \xi + \int_{|x - \xi|_p < p^{-M}} \phi(\xi) Z_t^{(\gamma, L)}(x - \xi) d^4 \xi.
\]
Note that if \( |x|_p \geq p^{-M} \) and \( M < L - 1 \), then, by Proposition 3.1 (iii), \( Z_t^{(\gamma, L)} = (f(\partial, \gamma) Z_t)(x) \), and
\[
\lim_{L \to \infty} \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_{p^{-\gamma - 2}} [u_1(x - y, t) - u_1(x, t)] d^4 y
\]
\[
= \int_{|x - \xi|_p \geq p^{-M}} \phi(\xi) (f(\partial, \gamma) Z_t)(x - \xi) d^4 \xi + \lim_{L \to \infty} \int_{|x - \xi|_p < p^{-M}} \phi(\xi) Z_t^{(\gamma, L)}(x - \xi) d^4 \xi,
\]
for $M < L - 1$. Now by using twice Fubini's theorem, see \cite[Proposition 3.1 (i), and Proposition 3.1 (iii)], we have

\[
\lim_{L \to \infty} \int_{\|x - \xi\| < p^{-M}} \varphi(\xi) Z^{(\gamma, L)}(x - \xi) \, d^4 \xi = \lim_{L \to \infty} \int_{\|x - \xi\| < p^{-M}} \varphi(\xi) \times \\
\left[ \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{\|y\| > p^{-L}} |f(y)|_{p}^{-\gamma - 2} [Z_t(x - y) - Z_t(x - \xi)] \, d^4 y \right] \, d^4 \xi
\]

\[
= \lim_{L \to \infty} \int_{\|y\| > p^{-L}} |f(y)|_{p}^{-\gamma - 2} \times \\
\left[ \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{\|z - \xi\| < p^{-M}} \varphi(z) [Z_t(x - y) - Z_t(x - \xi)] \, d^4 z \right] \, d^4 y
\]

\[
= \int_{\|x - \xi\| < p^{-M}} \varphi(\xi) \left[ \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{\|y\| > p^{-L}} |f(y)|_{p}^{-\gamma - 2} [Z_t(x - y) - Z_t(x - \xi)] \, d^4 y \right] \, d^4 \xi
\]

\[
= \int_{\|x - \xi\| < p^{-M}} (f(\vartheta, \gamma) Z_t)(x - \xi) \varphi(\xi) \, d^4 \xi.
\]

\[\square\]

**Lemma 5.6.** If $\lambda < \gamma \leq \alpha$, then

\[
(f(\vartheta, \gamma) u_2)(x, t, \tau) = \int_{\tau}^{t} \left( \int_{\mathbb{R}^4_p} (f(\vartheta, \gamma) Z)(x - y, t - \vartheta) g(y, \vartheta) \, d^4 y \right) \, d\vartheta \quad \text{for } t > 0.
\]

**Proof.** Let

\[
u_{2, \lambda}(x, t, \tau) = \int_{\tau}^{t-h} \left( \int_{\mathbb{R}^4} Z(x - y, t - \theta) g(y, \vartheta) \, d^4 y \right) \, d\theta
\]

where $h$ is a small positive number such that $0 < h < t - \tau$. Set

\[
Z^{(\gamma, L)}(x, t) = \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{\|y\| > p^{-L}} |f(y)|_{p}^{-\gamma - 2} [Z(x - y, t) - Z(x, t)] \, d^4 y.
\]
By the Fubini Theorem
\[
\frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_p^{-\gamma - 2} \left[ u_{2,h}(x - y, t, \tau) - u_{2,h}(x, t, \tau) \right] dy
\]
(5.5) \[= \int_t^{t-h} \int_{Q^L_p} Z^{(\gamma,L)}(x - \xi, t - \theta) g(\xi, \theta) d\xi d\theta. \]

Note that
\[
Z^{(\gamma,L)}(x, t) = \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} \times \int_{Q^L_p} |f(y)|_p^{-\gamma - 2} \left[ \chi(\xi \cdot x) \left[ \chi(-\xi \cdot y) - 1 \right] e^{-\kappa t|\tau(\xi)|_p} d\xi \right] dy
\]
\[= \int_{Q^L_p} \chi(\xi \cdot x) e^{-\kappa t|\tau(\xi)|_p} \left[ \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_p^{-\gamma - 2} \left[ \chi(-\xi \cdot y) - 1 \right] dy \right] d\xi
\[= \int_{Q^L_p} \chi(\xi \cdot x) e^{-\kappa t|\tau(\xi)|_p} P_L(\xi) d\xi,
\]
where
\[P_L(\xi) = \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \int_{|y|_p > p^{-L}} |f(y)|_p^{-\gamma - 2} \left[ \chi(-\xi \cdot y) - 1 \right] dy.
\]
On the other hand, by (2.2),
\[|P_L(\xi)| \leq B^{-\gamma - 2} \left| \frac{1 - p^\gamma}{1 - p^{-\gamma - 2}} \right| \int_{|y|_p > p^{-L}} ||y||_p^{-\gamma - 4} |\chi(-\xi \cdot y) - 1| dy,
\]
and by using a similar reasoning to the one used in [12, p. 142], we have
\[|P_L(\xi)| \leq C||\xi||_p^{2\gamma}
\]
whence
(6.6) \[\left| Z^{(\gamma,L)}(x, t) \right| \leq \int_{Q^L_p} e^{-\kappa t|\tau(\xi)|_p} |P_L(\xi)| d\xi \leq C \int_{Q^L_p} e^{-\kappa t|\tau(\xi)|_p} ||\xi||_p^{2\gamma} d\xi \leq C',
\]
where \(C'\) is a positive constant, which not depend on \(x, t \geq h + \tau, L\).

By writing the right-hand side of (5.5) as
(5.7) \[\int_t^{t-h} \int_{|\xi - \xi|_p \geq p^{-K}} Z^{(\gamma,L)}(x - \xi, t - \theta) g(\xi, \theta) d\xi d\theta
\[+ \int_t^{t-h} \int_{|\xi - \xi|_p < p^{-K}} Z^{(\gamma,L)}(x - \xi, t - \theta) g(\xi, \theta) d\xi d\theta,
\]
where $K$ is a fixed natural number. Now the result follows by taking limit $L \to \infty$ in (5.5). Indeed, for the first integral in (5.7), if $|x-\xi|_p \geq p^{-K}$ and $L > K + 1$, then $Z(\gamma, L)(x-\xi, t-\theta) = (a(\partial, \gamma)Z)(x-\xi, t-\theta)$. For the second integral in (5.7), we use (5.6) and the Dominated Convergence Theorem.

\begin{proof}[Proof of Theorem 5.1] By Lemmas 5.2 and 5.4 (i), $u(x,t) \in \mathcal{M}_{2\lambda}$ uniformly with respect to $t$, and by Lemma 4.1 $u(x,t)$ satisfies the initial condition. By Lemmas 5.3, 5.4 and Corollary 4.4, $u(x,t)$ is a solution of the Cauchy problem (5.1).

5.1. **Proof of Theorem 5.1**

By Lemmas 5.2 and 5.4 (i), $u(x,t) \in \mathcal{M}_{2\lambda}$ uniformly with respect to $t$, and by Lemma 4.1 $u(x,t)$ satisfies the initial condition. By Lemmas 5.3, 5.4 and Corollary 4.4, $u(x,t)$ is a solution of the Cauchy problem (5.1).

6. PARABOLIC-TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

Fix $n + 1$ positive real numbers satisfying

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha,$$

with $\alpha > 1$. We also fix $n + 2$ functions $a_k(x,t), \ k = 0, \ldots, n$ and $b(x,t)$ from $\mathbb{Q}_p^4 \times [0,T]$ to $\mathbb{R}$, here $T$ is a fixed positive constant. We assume that $a_k(x,t), \ k = 0, \ldots, n$ and $b(x,t)$ satisfy the following conditions:

(i) they belong to $\mathcal{M}_0$, with respect to $x$, uniformly in $t \in [0,T]$;

(ii) they satisfy the Hölder condition in $t$ with exponent $\nu \in (0,1]$ uniformly in $x$.

In addition we assume the uniform parabolicity condition $a_0(x,t) \geq \mu > 0$.

We set

$$F(\partial, \alpha_1, \alpha_2, \cdots, \alpha_n) := \sum_{k=1}^n a_k(x,t)f(\partial, \alpha_k) + b(x,t)I.$$ 

Note that $F(\partial, \alpha_1, \alpha_2, \cdots, \alpha_n) \mathcal{M}_{2\lambda} \subset \mathcal{M}_{2\lambda}$ for $\lambda < \alpha_1$, c.f. (4.1).

In this section we study the following initial value problem:

$$
\begin{align*}
\frac{\partial u(x,t)}{\partial t} + a_0(x,t)(f(\partial, \alpha)u)(x,t) + (F(\partial, \alpha_1, \alpha_2, \cdots, \alpha_n)u)(x,t) &= g(x,t) \\
\quad u(x,0) &= \varphi(x), \ x \in \mathbb{Q}_p^4, \ t \in (0,T],
\end{align*}
$$

where $x \in \mathbb{Q}_p^4$, $t \in (0,T], \varphi \in \mathcal{M}_{2\lambda}$ with $0 \leq \lambda < \alpha_1$ (if $a_1(x,t) = \cdots = a_n(x,t) \equiv 0$, then we shall assume that $0 \leq \lambda < \alpha_1$), and $g(x,t)$ is continuous in $(x,t)$, and $g(x,t) \in \mathcal{M}_{2\lambda}$ uniformly in $t \in [0,T], 0 \leq \lambda < \alpha_1$.

In this section we find the solution of the general problem (6.1). The technique used here is an adaptation of the classical Archimedean techniques, see e.g. [8], [16]. In the $p$-adic setting the technique was introduced by A. N. Kochubei in [11].

Our presentation is highly influenced by Kochubei’s book [12]. The proofs of some theorems are very similar to the corresponding in [12] for this reason we will omit them.

The first step of the construction of a fundamental solution is to study the parametrized fundamental solution $Z(x,t,y,\theta)$ for the Cauchy problem:

$$
\begin{align*}
\frac{\partial u(x,t)}{\partial t} + a_0(y, \theta)(f(\partial, \alpha)u)(x,t) &= 0 \\
\quad u(x,0) &= \varphi(x),
\end{align*}
$$

where $y \in \mathbb{Q}_p^4$ and $\theta > 0$ are parameters. By the results of Section 3 we have

$$Z(x,t,y,\theta) = \int_{\mathbb{Q}_p^4} \chi(\xi \cdot x)e^{-a_0(y,\theta)|f(\xi)(\xi)|_p^\alpha} d^4\xi,$$
and if $x \neq 0$, then

$$Z(x, t, y, \theta) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left( \frac{1 - p^m}{1 - p^{-\alpha m - 2}} \right) (a_0(y, \theta)t)^m |f(x)|_p^{-\alpha m - 2},$$

c.f. Proposition 3.1 (ii). By the Propositions 3.2, 4.3, 4.5, and Corollaries 3.3 (i) and 4.6, we have

$$Z(x, t, y, \theta) \leq Ct(t^{1/2\alpha} + ||x||_p)^{-2\alpha - 4}, \quad x \in \mathbb{Q}_p^4, \ t > 0,$$

$$|\langle f(\partial, \gamma)Z \rangle(x, t, y, \theta)| \leq C(t^{1/2\alpha} + ||x||_p)^{-2\gamma - 4}, \quad x \in \mathbb{Q}_p^4, \ t > 0,$$

$$\frac{\partial Z(x, t, y, \theta)}{\partial t} = -a_0(y, \theta) \int_{\mathbb{Q}_p^4} |f^z(\eta)|_p^\alpha e^{-a_0(y, \theta)|f^z(\eta)|^\alpha} \chi(x \cdot \eta) d^4 \eta,$$

$$\left| \frac{\partial Z(x, t, y, \theta)}{\partial t} \right| \leq C \left( t^{1/2\alpha} + ||x||_p \right)^{-2\alpha - 4},$$

$$\langle f(\partial, \gamma)Z \rangle(x, t, y, \theta) = \int_{\mathbb{Q}_p^4} |f^z(\eta)|_p^\alpha e^{-a_0(y, \theta)|f^z(\eta)|^\alpha} \chi(x \cdot \eta) d^4 \eta, \quad x \neq 0, \ 0 < \gamma \leq \alpha,$$

$$\int_{\mathbb{Q}_p^4} Z(x, t, y, \theta) d^4 x = 1,$$

$$\int_{\mathbb{Q}_p^4} \langle f(\partial, \gamma)Z \rangle(x, t, y, \theta) d^4 x = 0,$$

where the constants do not depend on $y, \theta$.

**Lemma 6.1.** There exists a positive constant $C$, such that

$$\int_{\mathbb{Q}_p^4} \left| \frac{\partial Z(x - y, t, y, \theta)}{\partial t} \right| d^4 y \leq C.$$

**Proof.** The proof follows from (6.7), (6.4), (6.5) by using the reasoning given in [12] for Lemma 4.5. □

Consider the parametrized heat potential

$$u(x, t, \tau) := \int_{\theta}^{t} \int_{\mathbb{Q}_p^4} Z(x - y, t - \theta, y, \theta) g(y, \theta) d^4 y d\theta,$$

with $g \in \mathcal{M}_{2\lambda}$, $0 \leq \lambda < \alpha_1$, uniformly with respect to $\theta$, and continuous in $(y, \theta)$. By using (6.2) we obtain as Lemma 5.2 that $u(x, t, \tau)$ is locally constant and belongs
to $M_2$, uniformly with respect to $t$, $\tau$.

Also we have

$$\frac{\partial u}{\partial t}(x, t, \tau) = g(x, t) + \int_{\tau}^{t} \int_{Q_p^4} \frac{\partial Z(x - y, t - \theta, y, \theta)}{\partial t} [g(y, \theta) - g(x, \theta)] d^4 y d\theta$$

(6.10)

$$+ \int_{\tau}^{t} \int_{Q_p^4} \frac{\partial Z(x - y, t - \theta, y, \theta)}{\partial t} g(x, \theta) d^4 y d\theta,$$

see Lemma 6.1, and

(6.11) \((f(\partial, \gamma)u)(x, t, \tau) = \int_{\tau}^{t} \int_{Q_p^4} Z^{(\gamma)}(x - y, t - \theta, y, \theta) g(y, \theta) d^4 y d\theta,\)

with $Z^{(\gamma)}(x, t, y, \theta) := f(\partial, \gamma)Z(x, t, y, \theta)$, for $\lambda < \gamma < \alpha$, and

(6.12) \((f(\partial, \alpha)u)(x, t, \tau) = \int_{\tau}^{t} \int_{Q_p^4} Z^{(\alpha)}(x - y, t - \theta, y, \theta) [g(y, \theta) - g(x, \theta)] d^4 y d\theta$$

As in [12] we look for a fundamental solution of (6.1) of the form

(6.13) \(\Gamma(x, t, \xi, \tau) = Z(x - \xi, t - \tau, \xi, \tau) + \int_{\tau}^{t} \int_{0}^{Q_p^4} Z(x - \eta, t - \theta, \eta, \theta) \phi(\eta, \theta, \xi, \tau) d^4 \eta d\theta.\)

By using formally the formulas given above, we can see that $\phi(x, t, \xi, \tau)$ is a solution of the integral equation

(6.14) \(\phi(x, t, \xi, \tau) = R(x, t, \xi, \tau) + \int_{0}^{t} \int_{Q_p^4} R(x, t, \eta, \theta) \phi(\eta, \theta, \xi, \tau) d^4 \eta d\theta\)

where

$$R(x, t, \xi, \tau) = [a_0(\xi, \tau) - a_0(x, t)] Z^{(\alpha)}(x - \xi, t - \tau, \xi, \tau)$$

$$- \sum_{k=1}^{n} a_k(x, t) Z^{(\alpha_k)}(x - \xi, t - \tau, \xi, \tau) - b(x, t) Z(x - \xi, t - \tau, \xi, \tau).$$

Integral equation (6.14) can be solved by the methods of successive approximations:

(6.15) \(\phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} R_m(x, t, \xi, \tau)\)

where

$$R_1(x, t, \xi, \tau) := R(x, t, \xi, \tau)$$
and
\[ R_{m+1}(x, t, \xi, \tau) := \int_0^t \int_{Q_4^4} R(x, t, \eta, \theta) R_m(\eta, \theta, \xi, \tau) d^4 \eta d\theta. \]

In order to prove the convergence of series (6.15), we need the following lemma which is an easy variation of the Lemmas 6 and 7 given in [19].

We set \( \alpha_{n+1} := \alpha (1 - \nu) > \alpha_n. \)

**Lemma 6.2.** The following estimations hold:
\[ |R(x, t, \xi, \tau)| \leq C \sum_{k=1}^{n+1} \left( (t - \tau)^{1/2\alpha} + \|x - \xi\|_p \right)^{-2\alpha_k - 4} \]
and
\[ |R_m(x, t, \xi, \tau)| \leq D M^m \Gamma \left( \frac{\nu}{2\alpha} \right) \frac{m}{\Gamma \left( \frac{m
u}{2\alpha} \right)} \sum_{k=1}^{n+1} \left( (t - \tau)^{1/2\alpha} + \|x - \xi\|_p \right)^{-2\alpha_k - 4} \]
where \( C, M \) and \( D \) are positive constants and \( \Gamma (\cdot) \) is the Archimedean Gamma function.

Lemma 6.2 also implies
\[ |\phi(x, t, \xi, \tau)| \leq C \sum_{k=1}^{n+1} \left( (t - \tau)^{1/2\alpha} + \|x - \xi\|_p \right)^{-2\alpha_k - 4}. \]

**Theorem 6.3.** The function
\[ u(x, t) = \int_{Q_4^4} \Gamma(\xi, \eta, 0) \varphi(\xi) d^4\xi + \int_0^t \int_{Q_4^4} \Gamma(x, t, \xi, \tau) g(\xi, \tau) d^4\xi d\tau \]
which is continuous on \( Q_n^4 \times [0; T]\), continuously differentiable in \( t \in (0, T]\), and belonging to \( M_{2\lambda} \) uniformly with respect to \( t \) is a solution of Cauchy problem (6.7).

The fundamental solution \( \Gamma(x, t, \xi, \tau), \xi, \tau \in Q_n^4, 0 \leq \tau < t \leq T, \) is of the form
\[ \Gamma(x, t, \xi, \tau) = Z(x - \xi, t, \xi, \tau) + W(x, t, \xi, \tau) \]
with
\[ |W(x, t, \xi, \tau)| \leq C (t - \tau)^{1+\nu} \left[ (t - \tau)^{1/2\alpha} + \|x - \xi\|_p \right]^{-2\alpha - 4} \]
\[ + C (t - \tau) \sum_{k=1}^{n+1} \left[ (t - \tau)^{1/2\alpha} + \|x - \xi\|_p \right]^{-2\alpha_k - 4}. \]

Furthermore \( Z(x, t, y, \theta) \) satisfies the estimates (6.2), (6.3), (6.5), (6.9).

**Proof.** Denote for \( u_1(x, t) \) and \( u_2(x, t) \) the first and second summands in the right-hand side of (6.17). Substituting (6.13) into (6.17) we get and
\[ u_1(x, t) = \int_{Q_4^4} Z(x - \xi, t, \xi, 0) \varphi(\xi) d^4\xi \]
\[ + \int_0^t \int_{Q_4^4} Z(x - \eta, t - \theta, \eta, \theta) G(\eta, \theta) d^4\eta d\theta, \]
where
\[
G(\eta, \theta) = \int_{Q^4_p} \phi(\eta, \theta, \xi, 0) \varphi(\xi) d^4 \xi d\tau.
\]

and
\[
(6.21) \quad u_2(x, t) = \int_0^t \int_{Q^4_p} Z(x - \xi, t - \tau, \xi, \tau) g(\xi, \tau) d^4 \xi d\tau + \int_0^t \int_{Q^4_p} Z(x - \eta, t - \theta, \eta, \theta) F(\eta, \theta) d^4 \eta d\theta,
\]

where
\[
F(\eta, \theta) = \int_0^d \int_{Q^4_p} \phi(\eta, \theta, \xi, \tau) g(\xi, \tau) d^4 \xi d\tau.
\]

Now by (6.16) and (5.2),
\[
|F(\eta, \theta)| \leq C, \quad \text{and} \quad |G(\eta, \theta)| \leq C \theta^{-\alpha_n + 1/\alpha}
\]
for all \(\eta \in Q^4_p\) and \(\theta \in (0, T]\). In addition the functions \(F\) and \(G\) are uniformly locally constant. Indeed, by the recursive definition of the function \(\phi\), we see that if \(N\) is a local constancy exponent for all the functions \(g, \varphi, a_k, b, Z^{(\alpha_k)}\) and \(Z\), and if \(|\delta| \leq p^{-N}\), then
\[
\phi(\eta + \delta, \theta, \xi + \delta, \tau) = \phi(\eta, \theta, \xi, \tau),
\]
therefore
\[
F(\eta + \delta, \theta) = F(\eta, \theta), \quad \text{and} \quad G(\eta + \delta, \theta) = G(\eta, \theta).
\]

Thus, the potentials in the expressions for \(u_1(x, t)\) and \(u_2(x, t)\) satisfy the conditions under which the differentiation formulas (6.10), (6.11), (6.12) were obtained. By using these formulas one verifies after some simple transformations that \(u(x, t)\) is a solution of the equation in (6.11).

We now show that \(u(x, t) \to \varphi(x)\) as \(t \to 0^+\). Due to (6.20) and (6.21), it is sufficient to verify that
\[
v(x, t) = \int_{Q^4_p} Z(x - \xi, t, \xi, 0) \varphi(\xi) d^4 \xi \to \varphi(x) \text{ as } t \to 0^+.
\]

By virtue of equation (6.7),
\[
v(x, t) = \int_{Q^4_p} [Z(x - \xi, t, \xi, 0) - Z(x - \xi, t, x, 0)] \varphi(\xi) d^4 \xi
\]
\[
+ \int_{Q^4_p} Z(x - \xi, t, x, 0) [\varphi(\xi) - \varphi(x)] d^4 \xi + \varphi(x).
\]

We now use that \(Z\) (as a function of the its third argument) and \(\varphi\) are locally constant, then the integrals in the previous formula are performed over the set
\[
\{ \xi \in Q^4_p; ||x - \xi||_p \geq p^{-N} \} \quad \text{for some } N \in \mathbb{N}.
\]

By applying (6.2) we see that both integrals tend to zero as \(t \to 0^+\). \(\square\)
6.1. The uniqueness of the solution of the Cauchy problem. The technique used in [12, Theorem 4.5] for establishing the uniqueness of the solution of the Cauchy problem associated with perturbations of the Vladimirov operator can be used to show the uniqueness of the Cauchy problem (6.1). Then we state here the corresponding result without proof.

**Theorem 6.4.** Assume that the coefficients $a_k(x, t)$, $k = 0, \ldots, n$ are non-negative, bounded, continuous functions and that $b(x, t)$ is bounded, continuous function. Take $0 \leq \lambda < \alpha_1$ (if $a_1(x, t) = \cdots = a_n(x, t) \equiv 0$, then we shall assume that $0 \leq \lambda < \alpha$). If $u(x, t)$ is a solution of (6.1) with $g(x, t) \equiv 0$, such that $u \in \mathcal{M}_{2\lambda}$ uniformly with respect to $t$, and $\varphi (x) \equiv 0$, then $u(x, t) \equiv 0$.

6.2. Probabilistic interpretation. The fundamental solution of (6.1) $\Gamma(x, t, \xi, \tau)$ is the transition function for a random walk on $\mathbb{Q}_p^4$. This result can be established by using classical results on stochastic processes see e.g. [7] and the techniques given in [12, pp. 161-162]. Formally we have

**Theorem 6.5.** Assume that the coefficients $a_k(x, t)$, $k = 0, \ldots, n$ and $b(x, t)$ are non-negative, bounded, continuous functions. The fundamental solution $\Gamma(x, t, \xi, \tau)$ is the transition density of a bounded right-continuous Markov process without second kind discontinuities.

6.3. Quadratic forms of dimension two. All the results presented in this article are valid if for the quadratic forms of type $\xi_1^2 - \tau \xi_2^2$ where $\tau \in \mathbb{Q}_p \setminus \{0\}$ is a not square of an element of $\mathbb{Q}_p$, see [5].

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