The Sturm-Liouville problem of two-body system

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Abstract
The two-body system is of great interest and complexity. To study the evolution of the two-body system under the influence of torque, a new Sturm-Liouville equation $\frac{d^2 u}{dx^2} + Lu = \frac{1}{L}$ is derived from the Newtonian mechanism. Its solutions and the eigenvalue problem are investigated.

1. Introduction
The two-body system has been studied extensively in physics, such as Kepler problem [1–3], Rutherford experiment [4, 5], Bohr model [6–8] and so forth. Based on the Newtonian mechanism, the trajectory of a particle in a two-body system is a conic section when angular momentum is a constant [9]. However, the evolution of the two-body system under the influence of torque is still improperly understood yet, which is of great interest and complexity. We introduce a new equation $\frac{d^2 u}{dx^2} + Lu = \frac{1}{L}$ derived from the Newtonian equation to extend the study of a two-body system under general circumstances. This equation turns out to be a Sturm-Liouville equation.

Sturm-Liouville equations are ubiquitous and continuously growing in mathematical physics [10]. Many second-order differential equations arise from a variety of phenomena can ultimately be written as Sturm-Liouville type equations. Based on the Sturm-Liouville theory, their eigenvalues can be ordered and eigenvectors span an orthogonal basis [11, 12]. It has become the primary technique for us to gain insight into the spectra of a dynamical system [13, 14]. This paper will investigate the properties of the Sturm-Liouville equation on a two-body system via eigenvalue problem and its solutions.

2. Derivation of Sturm-Liouville equation of Two-body system
It begins with two fundamental equations on the two-body system equations (1) and (2). Equation (1) is Newton’s law that the force between two bodies is proportional to the inverse square of the distance [7]. And equation (2) states the varying angular momentum caused by torque and mobility of angular momentum [9]. It is worthwhile noting that in this study, the angular momentum of the two-body system is not assumed to be a constant due to the influence of torque.

$$m\ddot{r} - m\dot{\theta}^2 r = -\frac{c}{r^2}, \quad \frac{\partial L}{\partial t} = \Gamma \quad (1)$$

where $c$ is a constant; $m$ is the particle’s mass; $r$ is the radius; $\theta$ is angular displacement; $t$ is time; $L$ is the orbital angular momentum; $\Gamma$ is the torque.

Before the detailed analysis, let’s simplify equation (1) into an ordinary differential equation. It is required to substitute $\dot{r}$ and $\dot{\theta}$ in equation (1) with $r = 1/u$ and $\dot{\theta} = Lu^2/m$. That is [9]

$$\dot{u} = -\frac{\dot{\theta}}{u^2} = \frac{L}{m} \frac{\partial u}{\partial \theta}$$

Thus, $\frac{d^2 u}{dx^2} + Lu = \frac{1}{L}$ is the final equation of the two-body system under the influence of torque.
With the above calculation, equation (1) reads as the following second order differential equation with
time-dependent coefficients:

\[
\dddot{r} = -\frac{L}{m} \dddot{\theta} - \frac{1}{m} \frac{\partial L}{\partial \theta} \ddot{\theta} = -\frac{L^2 u^2}{m^2} \dddot{u} + \frac{L^2}{m^2} \frac{\partial L}{\partial \theta} \dot{u}
\]

\[
\dddot{\theta} = \frac{L^2 u^3}{m^2}
\]

With the above calculation, equation (1) reads as the following second order differential equation with variable coefficients:

\[
\frac{L^2}{L^2} \ddot{u} + \frac{L}{L^2} \frac{\partial L}{\partial \theta} \ddot{u} + \frac{L}{m^2} \frac{\partial L}{\partial \theta} \dot{u} = 0
\]  

(3)

In this study, it will be assumed that there is a domain \( \Theta \) in which \( L \) is analytic. By multiplying the factor \( \frac{1}{L} \) to each term in equation (3), it can be written as a Sturm-Liouville equation.

\[
\frac{d}{d\theta} \left( L \frac{du}{d\theta} \right) + \frac{Lu}{L} \frac{mc}{L} = 0
\]  

(4)

Without loss of generality, let all constants (\( m \) and \( c \)) in the equation to be 1. Then we have

\[
(Lu')' + Lu = \frac{d}{d\theta}
\]  

(5)

3. Solutions of equation (5) \((Lu')' + Lu = \frac{1}{L}\)

As a deterministic problem, equation (5) can be solved with given initial value conditions or boundary value conditions in principle. When angular momentum is a constant, the equation turns to be a Kepler problem. For an attentive reader, it is straightforward to check that conic sections and Bohr orbits are solutions. In this paper, however, we pay particular attention to solutions with regard to varying angular momentum.

We can solve equation (3) subjected to some torque numerically. For instance, for a periodic torque \( \Gamma = \cos(kt) \), the ode4 solver implemented by Runge-Kutta method in MATLAB [15] can sketch the trajectory of the particle in the two-body system given the initial value conditions. As shown in figure 1, under some circumstances, solutions \( r(\theta) = \frac{1}{n} \) may look like a rotating elliptical orbit (leftmost and mid plots in figure 1) or a vibrating string (rightmost plot in figure 1).

For nonhomogeneous equation (5), the analytical solution constitutes two components that particular solution and homogeneous solution.

For particular solution, suppose \( L \) has the form \( L = c_1 e^{\alpha_1 \theta} \) and particular solution of \( u \) has the form \( u = c_1 e^{\alpha_1 \theta} \). After plugging in \( L \) and \( u \) back to equation (5), we get

\[
(Lu')' + Lu = (\alpha_1 + \alpha_2) \alpha_1 \alpha_2 e^{(\alpha_1 + \alpha_2) \theta} + \alpha_1 \alpha_2 e^{(\alpha_1 + \alpha_2) \theta} = \frac{1}{c_2} e^{-\alpha_2 \theta}
\]

By matching coefficients and exponential power of both sides, then there is

\[
\begin{align*}
((\alpha_1 + \alpha_2) \alpha_1 + 1) \alpha_1 \alpha_2 & = \frac{L}{c_2} \\
\alpha_1 + \alpha_2 & = -\alpha_2
\end{align*}
\]

Thus, as long as parameters \( c_1, \alpha_1, c_2 \) and \( \alpha_2 \) satisfy the following requirements, then \( u = c_1 e^{\alpha_1 \theta} \) is a particular solution of equation (5) with regard to \( L = c_2 e^{\alpha_2 \theta} \).
For the homogeneous solution which satisfies the equation \((Lu')' + Lu = 0\), we have

\[(Lu')' + Lu = Lu'' + L'u' + Lu = c_1 e^{\omega_1 \theta}(u'' + \alpha_2 u' + u) = 0\]

Hence,

\[u'' + \alpha_2 u' + u = 0\]  

(7)

The homogeneous solution of \(u\) with regard to \(L = c_1 e^{\omega_1 \theta}\) is

\[u = \begin{cases} 
    a_1 e^{-\alpha_2 e^{-\frac{\sqrt{\alpha_1^2 - 4}}{2} \theta}} + a_2 e^{-\alpha_2 e^{-\frac{\sqrt{\alpha_1^2 - 4}}{2} \theta}}, & \text{if } \alpha_2 = \pm 2 \\
    a_1 e^{\frac{\alpha_2}{2} \theta} + a_2 \theta e^{-\frac{\alpha_2}{2} \theta} + c_1 e^{\omega_1 \theta}, & \text{if } \alpha_2 = \pm 2
\end{cases}\]

In combination with the particular solution of \(u\), solution \(u\) with regard to \(L = c_1 e^{\omega_1 \theta}\) reads

\[u = \begin{cases} 
    a_1 e^{-\alpha_2 e^{-\frac{\sqrt{\alpha_1^2 - 4}}{2} \theta}} + a_2 e^{-\alpha_2 e^{-\frac{\sqrt{\alpha_1^2 - 4}}{2} \theta}} + c_1 e^{\omega_1 \theta}, & \text{if } \alpha_2 = \pm 2 \\
    a_1 e^{\frac{\alpha_2}{2} \theta} + a_2 \theta e^{-\frac{\alpha_2}{2} \theta} + c_1 e^{\omega_1 \theta}, & \text{if } \alpha_2 = \pm 2
\end{cases}\]

where parameters \(c_1, \alpha_1, c_2\), and \(\alpha_2\) satisfy the system of equation (6).

When \(\alpha_1\) is a real number, logarithm spiral could be solutions of equation (5). By changing the sign of \(\alpha_1\), it switch an inward logarithm spiral with an outward logarithm spiral. For instance \(\alpha_1 = -1\), \(u = \frac{1}{2} e^{-\frac{\alpha_1}{2} \theta}\) is logarithm spiral solution with increasing angular momentum \(L = e^{\frac{\alpha_1}{2} \theta}\). And if \(\alpha_1 = 1\), \(u = \frac{1}{2} e^{\frac{\alpha_1}{2} \theta}\) is the logarithm spiral with decreasing angular momentum \(L = e^{-\frac{\alpha_1}{2} \theta}\).

When \(\alpha_1\) is a complex number, for instance \(\alpha_1 = -i\), \(u = 2 e^{-\frac{\alpha_1}{2} \theta}\) and \(L = e^{i \frac{\alpha_1}{2} \theta}\) satisfies equation (5). There is a great potential success for these solutions, however, also a challenge to interpret their physical meaning.

### 4. Eigenvalue equation of two-body system

If we write equation (5) as an eigenvalue equation of a Sturm-Liouville operator \(T\), such that

\[T(u) = (Lu')' + Lu = \lambda \omega(\theta) u\]

it requests to introduce an additional equation \(\frac{1}{\lambda} = \lambda \omega(\theta) u\) such that

\[\begin{cases} 
    T(u) = (Lu')' + Lu = \lambda \omega(\theta) u \\
    \frac{1}{\lambda} = \lambda \omega(\theta) u
\end{cases}\]

(8)

where the \(\lambda\) is spectral parameter, and the \(\omega(\theta)\) is weight function. The set of eigenvalue \(\lambda\) is the spectrum of the two-body system. Cautiously, as a side effect, the introduction of an additional equation may narrow down the solutions set.

Considering their physical meaning, \(u\) and \(L\) are not independent of each other. Different from most Sturm-Liouville equations in which eigenvectors share the same variable coefficients and weight function, solutions of the system of equation (8) have their own distinct variable coefficients and weight functions. If we plug in the solutions \(u\) and \(L\) back to the system of equation (8), this construction of solutions would help us investigate the eigenvalue and weight function of the equation.

\[(\alpha_1 + \alpha_2) c_1 c_2 e^{\omega_1 \theta + \omega_2 \theta} + c_1 c_2 e^{\omega_1 \theta + \omega_2 \theta} = \lambda \omega(\theta) c_1 e^{\omega_1 \theta}\]

(9)

Then, the most reasonable pair of eigenvalue and weight function in the system of equation (8) are

\[\begin{cases} 
    \lambda = 1 + \frac{\omega_1}{2} \\
    \omega(\theta) = c_2 e^{\omega_2 \theta} = L
\end{cases}\]

(10)

With the auxiliary equation \(\frac{1}{\lambda} = \lambda u\), we can rewrite the system of equation (8) with eigenvalue \(\lambda\) and weight function \(\omega(\theta) = L\),

\[\begin{cases} 
    (Lu')' + Lu = \lambda Lu \\
    \frac{1}{\lambda} = \lambda Lu
\end{cases}\]

(11)
Then $L = (\lambda u)^{-\frac{1}{2}}$ and substitute $L$ with $(\lambda u)^{-\frac{1}{2}},$

$((\lambda u)^{-\frac{1}{2}}u')' + (\lambda u)^{-\frac{1}{2}}u = \lambda(\lambda u)^{-\frac{1}{2}}u$

With the replacement $y = u^2,$ we obtain

$y'' = \frac{\lambda - 1}{2}y$  \hspace{1cm} (12)

Solving the equation we get

$y = \begin{cases} 
  c_1 e^{\beta y} + c_2 e^{-\beta y}, & \text{if } \lambda \neq 1 \\
  c_1 + c_2 \theta, & \text{if } \lambda = 1
\end{cases}$ \hspace{1cm} (13)

where $\beta = \left(\frac{\lambda - 1}{2}\right)^{\frac{1}{2}}.$ Thus,

$L = (\lambda u)^{-\frac{1}{2}} = \begin{cases} 
  \lambda^{\frac{1}{2}}(c_1 e^{\beta y} + c_2 e^{-\beta y})^{-1}, & \text{if } \lambda \neq 1 \\
  \lambda^{\frac{1}{2}}(c_1 + c_2 \theta)^{-1}, & \text{if } \lambda = 1
\end{cases}$ \hspace{1cm} (14)

And,

$u = y^2 = \begin{cases} 
  (c_1 e^{\beta y} + c_2 e^{-\beta y})^2, & \text{if } \lambda \neq 1 \\
  (c_1 + c_2 \theta)^2, & \text{if } \lambda = 1
\end{cases}$ \hspace{1cm} (15)

5. Conclusion

This paper presents a Sturm-Liouville equation $(Lu')' + Lu = \frac{1}{4}$ that calculates the two-body system under the influence of torque. The investigation of its solutions and eigenvalue problem shed light on the behavior and spectra of the two-body system.

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