Provably Convergent Off-Policy Actor-Critic with Function Approximation

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Abstract

We present the first provably convergent off-policy actor-critic algorithm (COF-PAC) with function approximation in a two-timescale form. Key to COF-PAC is the introduction of a new critic, the emphasis critic, which is trained via Gradient Emphasis Learning (GEM), a novel combination of the key ideas of Gradient Temporal Difference Learning and Emphatic Temporal Difference Learning. With the help of the emphasis critic and the canonical value function critic, we show convergence for COF-PAC, where the critics are linear and the actor can be nonlinear.

1. Introduction

The policy gradient theorem and the corresponding actor-critic algorithm (Sutton et al., 2000; Konda, 2002) have recently enjoyed great success in various domains, e.g., defeating the top human player in the game Go (Silver et al., 2016), achieving human-level control in Atari games (Mnih et al., 2016). The canonical actor-critic algorithm is provably convergent with function approximation in on-policy setting (Konda, 2002). However, it is unstable when used off-policy and suffers from significant data inefficiency (Mnih et al., 2016). While there have been efforts to combine actor-critic algorithms with off-policy learning (Degris et al., 2012; Imani et al., 2018; Maei, 2018; Zhang et al., 2019; Liu et al., 2019), none of the resulting off-policy actor-critic algorithms is provably convergent under function approximation.

In this paper, we present COF-PAC, the first provably convergent off-policy actor-critic algorithm with function approximation. COF-PAC builds on Actor-Critic with Emphatic weightings (ACE, Imani et al. 2018), which reweights policy updates with emphasis through the followon trace (Sutton et al., 2016). The emphasis adjusts the state distribution and the followon trace approximates the emphasis (see Sutton et al. 2016).† However, the followon trace can have unbounded variance (Sutton et al., 2016). Hence its approximation to the emphasis can have an arbitrarily large error, complicating the convergence analysis of ACE. Instead of using the trajectory-based followon trace, we present a novel transition-based learning method with function approximation, Gradient Emphasis Learning (GEM), to approximate the emphasis, inspired by Gradient TD methods (GTD, Sutton et al. 2009b;a; Maei 2011), Emphatic TD methods (ETD, Sutton et al. 2016), and reversed TD methods (Hallak & Mannor, 2017; Gelada & Bellemare, 2019). We prove the almost sure convergence of GEM with linear function approximation under a slowly changing target policy. Different from GEM, it is the conditional expectation of the followon trace, not the followon trace itself, that converges, and this convergence requires a fixed target policy, which is impractical in control. In previous actor-critic algorithms, we have only a value function critic. In COF-PAC, we introduce a new kind of critic, the emphasis critic, which is trained via GEM.

Our contributions are threefold: (1) We provide a novel transition-based stochastic approximation algorithm to approximate emphasis, which enjoys better theoretical properties than the previous trajectory-based followon trace approach. (2) We prove that GTD-style algorithms are able to track a slowly changing target policy, provided necessary regularization is enforced. (3) We prove the convergence of COF-PAC, where the policy parameterization can be nonlinear and the convergence level is the same as the canonical on-policy actor-critic (Konda, 2002).

2. Background

We use $||x||_{\Xi} = \sqrt{x^\top \Xi x}$ to denote the norm induced by a positive definite matrix $\Xi$, which induces the matrix norm $||A||_{\Xi} = \sup_{||x||_{\Xi} = 1} ||Ax||_{\Xi}$. To simplify notation, we write $|| \cdot ||$ for $|| \cdot ||_{I}$, where $I$ is the identity matrix. All vectors are column vectors. We use “0” to denote an all-zero vector and an all-zero matrix when the dimension can be easily deduced from the context, and similarly for “1”. When it does not cause confusion, we use vectors and functions interchangeably. Proofs of all lemmas, propositions, and theorems are in the appendix.

†We use emphasis to denote the limit of the expectation of the followon trace, which is slightly different from Sutton et al. (2016) and is clearly defined in the next section.
We consider a finite Markov Decision Process (MDP) with a finite state space $S$ with $|S|$ states, a finite action space $A$ with $|A|$ actions, a transition kernel $\rho : S \times A \times S \to \mathbb{R}$, and a discount factor $\gamma \in (0, 1)$. At time step $t$, an agent at state $S_t$ takes an action $A_t$ according to $\mu(\cdot | S_t)$, where $\mu : A \times S \to [0, 1]$ is a fixed behavior policy. The agent then proceeds to a new state $S_{t+1}$ according to $\rho(\cdot | S_t, A_t)$ and gets a reward $R_{t+1} \equiv r(S_t, A_t, S_{t+1})$. In the off-policy setting, the agent is interested in a target policy $\pi$. We use $G_t = \sum_{k=t}^{\infty} \gamma^{k-1} R_{t+k}$ to denote the return at time step $t$ when following $\pi$ instead of $\mu$. Consequently, we define the state value function $v_\pi$ and the state action value function $q_\pi$ as $v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$ and $q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$. We use $\rho(s, a) \equiv \frac{\pi(a|s)}{\mu(a|s)}$ to denote the importance sampling ratio and define $\rho_t \equiv \rho(S_t, A_t)$ (Assumption 1 below ensures $\rho$ is well-defined). We sometimes write $\rho$ as $\rho_\pi$ to emphasize its dependence on $\pi$.

**Policy Evaluation:** We consider linear function approximation for policy evaluation. Let $x : S \to \mathbb{R}^{|S|}$ be the state feature function, and $\bar{x} : S \times A \to \mathbb{R}^{|S| \times |A|}$ be the state-action feature function. We use $X \in \mathbb{R}^{N_{sa} \times |S|}$ and $\bar{X} \in \mathbb{R}^{N_{sa} \times |S| \times |A|}$ to denote feature matrices, where each row of $X$ is $x(s)$ and each row of $\bar{X}$ is $\bar{x}(s, a)$. Let $d_\mu \in \mathbb{R}^{|S|}$ be the stationary distribution of $\mu$; we define $d_\mu \in \mathbb{R}^{N_{sa}}$ where $d_\mu(s, a) \equiv d_\mu(s)\mu(a|s)$. We define $D \equiv \text{diag}(d_\mu) \in \mathbb{R}^{|S| \times |S|}$ and $\bar{D} \equiv \text{diag}(d_\mu) \in \mathbb{R}^{N_{sa} \times N_{sa}}$. Assumption 1 below ensures $d_\mu$ exists and $D$ is invertible, as well as $\bar{D}$. Let $P_\pi \in \mathbb{R}^{|S| \times |S|}$ be the state transition matrix and $P_{\pi*} \in \mathbb{R}^{N_{sa} \times N_{sa}}$ be the state-action transition matrix, i.e., $P_{\pi*}(s, s') \equiv \sum_{a} \pi(a|s) P_{\pi}((s, a), (s', a')) \equiv P(s'|s, a)\pi(a|s')$. From this we shall use on shorthand that $x_t \equiv x(S_t)$, $\bar{x}_t \equiv \bar{x}(S_t, A_t)$. We use $v \equiv X \nu, q \equiv \bar{X} \nu$ to denote estimates for $v_\pi, q_\pi$ respectively, where $\nu, u$ are learnable parameters.

We first consider GDT methods. For a vector $v \in \mathbb{R}^{|S|}$, we define a projection $\Pi v \equiv X y^*, y^* \equiv \arg \min_y ||X y - v||_F^2$. We have $X = X(X^T DX)^{-1}X^T$ and $\Pi v \equiv X \nu$. Similarly, for a vector $q \in \mathbb{R}^{N_{sa}}$, we define a projection $\bar{\Pi} q \equiv \bar{X} \nu + \gamma \bar{P}_\pi \bar{q}$ where $\bar{\Pi} q \equiv \sum_{s', a} \pi(a|s) P_{\pi*}(s', a|s) r(s, a, s')$. Similarly, $q_\pi$ is the unique fixed point of the Bellman operator $\bar{T} : T \nu \equiv \bar{r} + \gamma \bar{P}_\pi \bar{q}$, where $\bar{r} \in \mathbb{R}^{N_{sa}}$ and $\bar{r}(s, a) \equiv \sum_{s'} P_{\pi*}(s', a|s) r(s, a, s')$. GDT2 (Sutton et al., 2009a) learns the estimate $\nu$ for $v_\pi$ by minimizing $||\bar{T} \nu - v||_F^2$, GQ(0) (Maei, 2011) learns the estimate $q$ for $q_\pi$ by minimizing $||\bar{T} q - q||_F^2$.

Besides GDT methods, ETD methods are also used for off-policy policy evaluation. ETD(0) updates $\nu$ as

$$M_t \equiv i(S_t) + \gamma \rho_{t-1} M_{t-1},$$

$$\nu_{t+1} \equiv \nu_t + \alpha M_t \rho_t (R_{t+1} + \gamma x_{t+1}^\top \nu_t - x_t^\top \nu_t) x_t^T,$$

where $\alpha$ is a step size, $M_t$ is the followon trace, and $i : S \to [0, \infty)$ is the interest function reflecting the user’s preference for different states (Sutton et al., 2016). Similarly, we define $i_t \equiv i(S_t)$.

**Control:** Off-policy actor-critic methods (Degris et al., 2012; Imani et al., 2018) aim to maximize the excursion objective

$$J(\pi) \equiv \sum_{s} d_\mu(s) i(s) v_\pi(s)$$

by adapting the target policy $\pi$. We assume $\pi$ is parameterized by $\theta \in \mathbb{R}^k$, and use $\theta, \pi, \pi_\theta$ interchangeably in the rest of this paper when it does not cause confusion. All gradients are taken w.r.t. $\theta$ unless otherwise specified.

According to the off-policy policy gradient theorem (Imani et al., 2018), the policy gradient is $\nabla J(\theta) = \sum S \pi(s) \nabla \log \pi(a|s) v_\pi(s)$, where $\bar{m} \equiv (I - \gamma P^\top_\pi)^{-1} D_i \in \mathbb{R}^{|S|}$. We rewrite $\bar{m}$ as $D D^{-1} (I - \gamma P^\top_\pi)^{-1} D_i$ and define

$$m_\pi \equiv D^{-1} (I - \gamma P^\top_\pi)^{-1} D_i.$$ 

We therefore have $\bar{m} = D m_\pi$, i.e., $\bar{m} = d_\mu(s) m_\pi(s)$. Alternatively, we can rewrite $\nabla J(\theta)$ as

$$\nabla J(\theta) = \sum_{s} d_\mu(s) m_\pi(s) \sum_{a} \mu(a|s) \psi_\theta(s, a) q_\pi(s, a),$$

where $\psi_\theta(s, a) \equiv \rho_\pi(s, a) \nabla \log \pi(a|s) \in \mathbb{R}^{|S|}$. We refer to $m_\pi$ as the emphasis in the rest of this paper. To compute $\nabla J(\theta)$, we need $m_\pi$ and $q_\pi$, to which we typically do not have access. Degris et al. (2012) ignore the emphasis $m_\pi$ and update $\theta$ as $\theta_{t+1} \leftarrow \theta_{t} + \alpha \rho_\pi q_\pi(S_t, A_t) \nabla \log \pi(A_t|S_t)$ in the Off-Policy Actor-Critic (Off-PAC) algorithm, which is theoretically justified only in the tabular setting.* Imani et al. (2018) approximate $m_\pi(S_t)$ with the followon trace $M_t$, yielding the ACE update $\theta_{t+1} \leftarrow \theta_{t} + \alpha \rho_\pi q_\pi(S_t, A_t) \nabla \log \pi(A_t|S_t)$. Assuming $\lim_{t \to \infty} \mathbb{E}_\pi[M_t|S_t = s]$ exists and $\pi$ is fixed, Sutton et al. (2016) show that $\lim_{t \to \infty} \mathbb{E}_\pi[M_t|S_t = s] = m_\pi(s)$. The existence of this limit is later established in Lemma 1 in Zhang et al. (2019).

### 2.1. Assumptions

**Assumption 1.** The Markov chain induced by the behavior policy $\mu$ is ergodic, and $\psi(s, a), \mu(s, a) > 0$.

**Assumption 2.** The matrices $C \equiv X^T DX, \bar{C} \equiv \bar{X}^T \bar{D} \bar{X}$ are nonsingular.

*See Errata in Degris et al. (2012)
Assumption 3. There exists a constant $C_0 < \infty$ such that
\[
|\psi(s, a, \theta, \theta)|, \\
|\nabla \psi(s, a)| \leq C_0, \\
|\pi_\theta(a|s) - \pi_\theta(a|s)| \leq C_0 |\theta - \theta|, \\
|\nabla \psi(s, a) - \nabla \tilde{\psi}(s, a)| \leq C_0 |\theta - \tilde{\theta}|.
\]

Remark 1. The nonsingularity in Assumption 2 is commonly assumed in GTD methods (Sutton et al., 2009b; Maei, 2011) and can be satisfied by using linearly independent features. Assumption 3 contains common assumptions for policy parameterization akin to those of Sutton et al. (2000); Konda (2002).

Lemma 1. Under Assumptions (1, 3), there exists a constant $C_1 < \infty$ such that $\nabla J(\theta)$
\[
|\nabla J(\theta)| \leq C_1, \\
|\nabla J(\theta) - \nabla J(\theta)| \leq C_1 |\theta - \theta|, \\
|\partial^2 J(\theta)| \leq C_1.
\]

Lemma 2. Under Assumption 1, $P_\pi \leq |D^{-1} P_\pi^T D|_D$

3. Gradient Emphasis Learning

Motivation: To motivate, we first discuss the disadvantages of the followon trace $M_t$. The first problem is that it can have unbounded variance (Sutton et al., 2016). Empirically, it leads to problems when ETD is used for policy evaluation. For example, as pointed out in Sutton & Barto (2018), “it is nigh impossible to get consistent results in computational experiments” (for ETD) in Baird’s counterexample (Baird, 1995), a benchmark domain in measuring RL algorithms’ off-policy performance. Theoretically, this unbounded variance may preclude a convergent analysis for ACE, as unbounded variance of $M_t$ implies unbounded approximation error for $m_\pi(S_t)$.

The second problem is that $M_t$ is almost memoryless. $M_t$ is only a scalar random variable but we expect it to track $m_\pi$, a vector in $\mathbb{R}^{|S|}$. It is the expectation of $M_t$, not $M_t$ itself, that converges. However, in Eq (1), $M_{t+1}$ is bootstrapped by $M_t$, not its expectation, indicating this bootstrap for $M_{t+1}$ can be poor. By contrast, in canonical learning-based methods, e.g., the ETD value update in Eq (2), the approximation itself, e.g., $\nu_t$, converges (Yu, 2015) and we bootstrap via this approximation. The quality of this bootstrap is therefore likely to be high, which is particularly important when $\pi$ is changing, so that the approximator can adapt to the new policy quickly. The followon trace, however, does not provide a good bootstrap due to its lack of memory. It is questionable whether $M_t$ can track $m_\pi(S_t)$ under a changing $\pi$, which yields an obstacle in the convergence analysis for ACE. In this paper, we propose a novel stochastic algorithm to approximate $m_\pi$, and prove its ability to track a changing $\pi$.

Algorithm Design: We now present the Gradient Emphasis Learning (GEM) algorithm. We consider linear function approximation, and our estimate for $m_\pi$ is $\bar{m} = Xw$, where $w \in \mathbb{R}^{|S|}$ is the learnable parameters. For a vector $y \in \mathbb{R}^{|S|}$, we define an operator $\tilde{T}$ as $\tilde{T}y = i + \gamma D^{-1} P_\pi^T Dy$.

Proposition 1. $\tilde{T}$ is a contraction mapping w.r.t. some weighted maximum norm and $m_\pi$ is its fixed point.

Our operator $\tilde{T}$ has the same structure as the discounted COP-TD operator $Y_\gamma$ (Gelada & Bellemare, 2019), where $Y_\gamma y = (1 - \gamma)I + \gamma D^{-1} P_\pi^T Dy$ and $\gamma$ is a scalar similar to $\gamma$. They show that $Y_\gamma$ is contractive only when $\gamma$ is small enough. Here our Proposition 1 proves contraction for any $\gamma < 1$. Although $\tilde{T}$ and $Y_\gamma$ are similar, they are designed for different purposes. Namely, $Y_\gamma$ is designed to learn a density ratio while $\tilde{T}$ is designed to learn the emphasis. $\tilde{T}$ is a generalization of $Y_\gamma$ as users are free to choose $i$ in $\tilde{T}$.

Given Proposition 1, it is tempting to compose a semi-gradient update rule for updating $w$:
\[
w_{t+1} \leftarrow w_t + \alpha(t_{t+1} + \gamma\rho_t x_{t+1}^T w_t - x_{t+1} P_\pi w_t) x_{t+1},
\]

analogously to discounted COP-TD. This update rule, however, can diverge for the same reason as the divergence of off-policy linear TD: the key matrix $D(I - \gamma P_\pi)$ is not guaranteed to be negative semi-definite (see Sutton et al. (2016)). Motivated by GTD methods, we seek an approximate solution $m$ that satisfies $m = \Pi \tilde{T} m$ via minimizing a projected objective $||\Pi \delta w||_2^2$, where $\delta w = \tilde{T}(Xw) - Xw$. For reasons that will soon be clear, we also include ridge regularization, yielding the objective
\[
J^{\text{max}}(w) = \frac{1}{2}||\Pi \delta w||^2 + \frac{1}{2}\eta ||w||^2,
\]
where $\eta > 0$ is the weight of the ridge term. We can now compute $\nabla w J^{\text{max}}(w)$ following a similar routine as Sutton et al. (2009a). When sampling $\nabla w J^{\text{max}}(w)$, we use another set of parameters $\kappa \in \mathbb{R}^{|S|}$ to address the double sampling issue as proposed by Sutton et al. (2009a). See Sutton et al. (2009a) for details of the derivation. This derivation, however, provides only an intuition behind GEM and has little to do with the actual convergence proof for two reasons. First, in an actor-critic setting, $\pi$ keeps changing, as does $J^{\text{max}}$. It is usually hard to employ a Lyapunov approach to establish the boundedness of such time-varying linear systems (e.g., see DaCunha (2005)). Second, we consider sequential Markovian data $\{S_0, A_0, S_1, \ldots\}$. The proof in Sutton et al. (2009a) assumes i.i.d. data, i.e., each state $S_t$ is sampled from $d_{\mu}$ independently. Compared with the i.i.d. assumption, the Markovian assumption is more practical in RL problems. We now present the GEM algorithm, which
updates $\kappa$ and $w$ recursively as

$$
\delta_i \leftarrow i_{t+1} + \gamma \rho x_i^T w_i - x_i^T w_i,
$$

$$
\kappa_{t+1} \leftarrow \kappa_t + \alpha_t (\delta_t - x_i^T \kappa_t) x_{t+1},
$$

$$
w_{t+1} \leftarrow w_t + \alpha_t ((x_{t+1} - \gamma \rho x_i x_i^T \kappa_t - \eta w_t),
$$

where $\eta > 0$ is a constant, $\alpha_t$ is a deterministic sequence satisfying the Robbins-Monro condition (Robbins & Monro, 1951), i.e., $\alpha_t$ is non-increasing positive and $\sum \alpha_t = \infty, \sum \alpha_t^2 < \infty$. Similar to Sutton et al. (2009a), we define $d_t = [\kappa_t, w_t]$ and rewrite the GEM update as

$$
d_{t+1} = d_t + \alpha_t (h(Y_t) - G_{\theta_t}(Y_t) d_t),
$$

where $Y_t = (S_t, A_t, S_{t+1})$. With $y = (s, a, s')$, we define

$$
A_0(y) = x(s') (x(s') - \gamma \rho (s, a) x(s))^T,
$$

$$
C(y) = x(s') x(s')^T,
$$

$$
G_{\theta}(y) = C(y) - A_0(y)^T \eta I, h(y) = \begin{bmatrix} i(s') x(s') \\ 0 \end{bmatrix}.
$$

Let $d_y(y) = d_y(s) \mu(a|s) \pi(s'|s, a)$, the limiting behavior of GEM is then governed by

$$
A(\theta) = \mathbb{E}_{d_y} [A_0(y)] = X^T (I - \gamma P_0^\theta) DX,
$$

$$
G(\theta) = \mathbb{E}_{d_y} [G_{\theta}(y)] = \begin{bmatrix} C \\ -A(\theta)^T \eta I \end{bmatrix},
$$

$$
\bar{h} = \mathbb{E}_{d_y} [h(y)] = \begin{bmatrix} X^T D_i \\ 0 \end{bmatrix}.
$$

Readers familiar with GTD2 (Sutton et al., 2009a) may find that the $G(\theta)$ in Eq (7) is different from its counterpart in GTD2 in that the bottom right block of $G(\theta)$ is $\eta I$ while that block in GTD2 is $0$. This $\eta I$ results from the ridge regularization in the objective $J^{\tau\nu}$ in (6), and this block has to be strictly positive definite in order to ensure the positive definiteness of $G(\theta)$. In general, any regularization in the form of $\|w_i\|^2_2$ is sufficient. We assume ridge to simplify notation.

As we consider an actor-critic setting where the policy $\theta$ is changing every step, we pose the following condition on the changing rate of $\theta$:

**Condition 1.** (Assumption 3.1(3) in Konda (2002)) The random sequence $\{\theta_t\}$ satisfies $|\theta_{t+1} - \theta_t| \leq \beta_t H_t$, where $\{H_t\}$ is some nonnegative process with bounded moments and $\{\beta_t\}$ is a nonincreasing deterministic sequence satisfying the Robbins-Monro condition such that $\sum \beta_t \leq \infty$ for some $d > 0$.

When we consider a policy evaluation setting where $\theta$ is fixed, this condition is satisfied automatically. We show later that this condition is also satisfied in COF-PAC. We now characterize the asymptotic behavior of GEM.

**Theorem 1.** (Convergence of GEM) Under Assumptions (1, 2) and Condition 1, the iterate $\{d_t\}$ generated by (5) satisfies $\sup_t \|d_t\| < \infty$ and $\lim_{t \to \infty} \|G(\theta_t) d_t - \bar{h}\| = 0$ almost surely.

**Lemma 3.** Under Assumptions (1, 2), when $\eta > 0$, $G(\theta)$ is nonsingular and $\sup_d \|G(\theta)^{-1}\| < \infty$.

By simple block matrix inversion, Theorem 1 implies

$$
\lim_{t \to \infty} \|w_{\theta}^* (\eta) - w_i\| = 0,
$$

where

$$
\bar{w}_{\theta}^* (\eta) = (A(\theta)^T C^{-1} A(\theta) + \eta I)^{-1} A(\theta)^T C^{-1} X^T D_i.
$$

Konda (2002) provides a general theorem for stochastic approximation algorithms to track a slowly changing linear system. To prove Theorem 1, we verify that GEM indeed satisfies all the assumptions (listed in the appendix) in Konda’s theorem. Particularly, that theorem requires $G(\theta)$ to be strictly positive definite, which is impossible if $\eta = 0$. This motivates the introduction of the ridge regularization in $J^{\tau\nu}$ defined in Eq. (4). Namely, the ridge regularization is essential in the convergence of GEM under a slowly changing target policy. Introducing regularization in the GTD objective is not new. Mahadevan et al. (2014) introduce the proximal GTD learning framework to integrate GTD algorithms with first-order optimization-based regularization via saddle-point formulations and proximal operators. Yu (2017) introduces a general regularization term for improving robustness. Du et al. (2017) introduce ridge regularization to improve the convexity of the objective. However, their analysis is conducted with the saddle-point formulation of the GTD objective (Liu et al., 2015; Macua et al., 2014) and requires a fixed target policy, which is impractical in our control setting. We are the first to establish the tracking ability of GTD-style algorithms under a slowly changing target policy by introducing ridge regularization, which ensures the driving term $G(\theta)$ is strictly positive definite. Without this ridge regularization, we are not aware of any existing work establishing this tracking ability. Note our arguments do not apply when $\eta = 0$ and $\pi$ is changing, which is an open problem. However, if $\eta = 0$ and $\pi$ is fixed, we can use arguments from Yu (2017) to prove convergence. In this scenario, assuming $A(\theta)$ is nonsingular, $\{w_i\}$ converges to $w_{\theta}^*(0) = A(\theta)^{-1} X^T D_i$ and we have

**Proposition 2.** $\Pi^X (X w_{\theta}^*(0)) = X w_{\theta}^*(0)$.

Similarly, we introduce ridge regularization in the $q$-value analogue of GTD2, which we call GQ2. GQ2 updates $u$...
GQ2:
\[ \delta_t \leftarrow R_{t+1} + \gamma \rho_{t+1} x_{t+1}^\top u_t - x_t^\top u_t, \]
\[ \kappa_t + \kappa_t - \alpha_t (\delta_t - \tilde{x}_t^\top \kappa_t), \]
\[ u_{t+1} \leftarrow u_t + \alpha_t ((\tilde{x}_t - \rho_{t+1} x_{t+1}) x_t^\top \kappa_t - \eta u_t). \]  

Similarly, we define \( \bar{d}_t^\top \equiv [\kappa_t, u_t^\top], \)
\[ \bar{A}(\theta) = \bar{X}^\top \bar{D}(I - \gamma \bar{P}_b) \bar{X}, \]
\[ \bar{C}(\theta) \equiv \left[ \begin{array}{c} \bar{C}^\top - \bar{A}(\theta)^\top \eta I \\ -\bar{A}(\theta)^\top \end{array} \right], \bar{h} \equiv \left[ \begin{array}{c} \bar{X}^\top \bar{D} \bar{r} \\ 0 \end{array} \right]. \]

**Theorem 2.** (Convergence of GQ2) Under Assumptions (1, 2) and Condition 1, the iterate \( \{ \bar{d}_t \} \) generated by (9) satisfies \( \sup_t ||\bar{d}_t|| < \infty \) and \( \lim_{t \to \infty} ||\bar{C}(\bar{\theta}_t)\bar{d}_t - \bar{h}|| = 0 \) almost surely.

Similarly, we have
\[ \lim_{t \to \infty} ||u_{\bar{\theta}}^*(\eta) - u_t|| = 0, \text{ where} \]
\[ u_{\bar{\theta}}^*(\eta) \equiv (\bar{A}(\theta)^\top \bar{C}^{-1} \bar{A}(\theta) + \eta I)^{-1} \bar{A}(\theta)^\top \bar{C}^{-1} \bar{X}^\top \bar{D} \bar{r}. \]

Comparing the update rules of GEM and GQ2, it now becomes clear that GEM is “reversed” GQ2. Particularly, the \( \bar{A}(\theta) \) in GEM is the “transpose” of the \( \bar{A}(\theta) \) in GQ2. Such reversed TD methods have been explored by Hallak & Mannor (2017); Gelada & Bellemare (2019). All those reversed TD methods rely on the operator \( D^{-1} P_s^\top D \) introduced by Hallak & Mannor (2017). Previous methods implement this operator under the semi-gradient paradigm (Sutton, 1988). By contrast, GEM is full gradient. The techniques in GEM can be applied immediately to the discounted COP-TD (Gelada & Bellemare, 2019) to improve its convergence from small enough \( \gamma \) to any \( \gamma < 1 \). Applying GEM-style update to COP-TD (Hallak & Mannor, 2017) is still an open problem as COP-TD involves a nonlinear projection, whose gradient is hard to compute.

**4. Convergent Off-Policy Actor-Critic**

To estimate \( \nabla J(\theta) \), we use GEM and GQ2 to estimate \( m_\pi \) and \( q_\pi \) respectively, yielding Convergent Off-Policy Actor-Critic (COF-PAC, Eq (11)). In COF-PAC, we require both \( \{ \alpha_t \} \) and \( \{ \beta_t \} \) to be deterministic and nonincreasing and satisfy the Robbins-Monro condition. Furthermore, there exists some \( d, d \) such that \( \sum_t (\delta_t)^d < \infty \). These are common stepsize conditions in two-timescale algorithms (see Borkar (2009)). Like Konda (2002), we also use adaptive stepsizes \( \Gamma_i : \mathbb{R}^{K_i} \to \mathbb{R} \) and \( \Gamma_2 : \mathbb{R}^{K_2} \to \mathbb{R} \) to ensure \( \theta \) changes slowly enough. We now pose the same condition on \( \Gamma_i (t = 1, 2) \) as Konda (2002). There exist constants \( C_1, C_2 < \infty \) such that for any vector \( \tilde{d}, \tilde{d} \), the following properties hold: \( ||d|| \Gamma_i (d) \in [C_1, C_2] \), \( ||\Gamma_i (d) - \Gamma_i (\tilde{d})|| \leq \frac{C_2 ||d - \tilde{d}||}{1 + ||d + \tilde{d}||} \). Konda (2002) provides an example for \( \Gamma_i \). Let \( C_0 > 0 \) be some constant, then we define \( \Gamma_i \) as
\[ \Gamma_i (d) = \begin{cases} 1 & \text{if } ||d|| < C_0 \\ \frac{1 + C_0}{1 + ||d||} & \text{otherwise} \end{cases}. \]

It is easy to verify that the above conditions on stepsizes \( (\alpha_t, \beta_t, \Gamma_1, \Gamma_2) \), together with Assumptions (1, 3), ensure that \( \Gamma_1 (w_t) \Gamma_2 (u_t) \Delta_t \) is bounded. Condition 1 on the policy changing rate, therefore, indeed holds. Consequently, Theorems 1 and 2 hold when the target policy \( \pi \) is updated according to COF-PAC.

**COF-PAC:**
\[ \delta_t \leftarrow i_{t+1} + \gamma \rho_{t+1} x_{t+1}^\top w_t - x_{t+1}^\top w_t \]
\[ \kappa_{t+1} \leftarrow \kappa_t + \alpha_t (\delta_t - x_{t+1}^\top \kappa_t) x_{t+1} \]
\[ w_{t+1} \leftarrow w_t + \alpha_t ((x_{t+1} - \gamma \rho_{t+1} x_{t+1} \kappa_t - \eta u_t) \]
\[ \delta_t \leftarrow R_{t+1} + \gamma \rho_{t+1} \tilde{x}_{t+1}^\top u_t - \tilde{x}_{t+1}^\top u_t \]
\[ \kappa_{t+1} \leftarrow \kappa_t + \alpha_t (\delta_t - \tilde{x}_{t+1}^\top \kappa_t) \tilde{x}_{t+1} \]
\[ u_{t+1} \leftarrow u_t + \alpha_t ((\tilde{x}_t - \gamma \rho_{t+1} \tilde{x}_{t+1}) \tilde{x}_{t+1}^\top \kappa_t - \eta u_t) \]
\[ \Delta_t \leftarrow \rho_t (w_t^\top x_t)(u_t^\top \tilde{x}_t) \nabla \log \pi_\theta (A_t | S_t) \]
\[ \theta_{t+1} \leftarrow \theta_t + \beta_t \Gamma_1 (w_t) \Gamma_2 (u_t) \Delta_t \]  

We now characterize the asymptotic behavior of COF-PAC. The limiting policy update in COF-PAC is
\[ \hat{\theta} (\theta) \equiv \sum_s d_\mu (s) (x(s)^\top w_\theta^\top (\eta)) \sum_a \mu (a | s) \psi_\theta (s, a) (\tilde{x}(s, a)^\top u_\theta^\top (\eta)). \]

The bias introduced by the estimates \( m \) and \( q \) is
\[ b (\theta) \equiv \nabla J (\theta) - \hat{\theta} (\theta), \]
which determines the asymptotic behavior of COF-PAC.

**Theorem 3.** (Convergence of COF-PAC) Under Assumptions (1-3), the iterate \( \{ \theta_t \} \) generated by COF-PAC (Eq (11)) satisfies
\[ \liminf_t \left( ||\nabla J (\theta_t)|| - ||b (\theta_t)|| \right) \leq 0, \]
almost surely, i.e., \( \{ \theta_t \} \) visits any neighborhood of the set \( \{ \theta : ||\nabla J (\theta)|| \leq ||b (\theta)|| \} \) infinitely many times almost surely.

The proof is inspired by Konda (2002). According to Theorem 3, COF-PAC reaches the same convergence level as the canonical on-policy actor-critic (Konda, 2002). Together with the fact that \( \nabla J (\theta) \) is Lipschitz continuous and \( \beta_t \) is diminishing, it is easy to see \( \theta_t \) will eventually remain in
the neighborhood in Theorem 3 for arbitrarily long time. When $\pi_\theta$ is close to $\mu$ in the sense of the following Assumption 4(a), we can provide an explicit bound for the bias $b(\theta)$. However, failing to satisfy Assumption 4 does not necessarily imply the bias is large. The bound here is indeed loose and is mainly to provide an intuition for the source of the bias.

**Assumption 4.** (a) The following two matrices are positive semidefinite:

\[
\begin{align*}
F_\theta &= \begin{bmatrix} C & X^T P_\theta X \\ X^T D P_\theta X & C \end{bmatrix}, \\
F_\bar{\theta} &= \begin{bmatrix} \bar{C} & X^T \bar{D} P_\theta X \\ X^T \bar{D} X & \bar{C} \end{bmatrix}.
\end{align*}
\]

(b) $\inf_\theta |\det(A(\theta))| > 0$, $\inf_\theta |\det(A(\theta))| > 0$.

(c) The Markov chain induced by $\pi_\theta$ is ergodic.

**Remark 2.** Part (a) is from Kolter (2011), which ensures $\pi_\theta$ is not too far away from $\mu$. The non-singularity of $A(\theta)$ and $\bar{A}(\theta)$ for each fixed $\theta$ is commonly assumed (Sutton et al., 2009a; b; Maei, 2011). In part (b), we make a slightly stronger assumption that their determinants do not approach 0 during the optimization of $\theta$.

**Proposition 3.** Under Assumptions (1-4), let $d_\theta$ be the stationary distribution under $\pi_\theta$ and define $\bar{d}_\theta(a|s) = \bar{d}_\theta(s)\pi_\theta(a|s)$, $D_\theta = diag(d_\theta)$, $\bar{D}_\theta = diag(\bar{d}_\theta)$, we have

\[
||b(\theta)||_D \leq C_0 \eta + C_1 \frac{1 + \gamma \kappa(D - \frac{2}{1-\gamma} D^2)}{1-\gamma} ||\mu - \bar{\mu}||_D
\]

\[
+ C_2 \frac{1 + \gamma \kappa(D - \frac{2}{1-\gamma} D^2)}{1-\gamma} ||\bar{\mu} - \bar{\pi}||_D,
\]

where $\kappa(\cdot)$ is the condition number of a matrix w.r.t. $\ell_2$ norm and $C_0$, $C_1$, $C_2$ are some positive constants.

The bias $b(\theta)$ comes from the bias of both the $q_\pi$ estimate and the $m_\pi$ estimate. The bound of the $q_\pi$ estimate follows directly from Kolter (2011). The proof from Kolter (2011), however, can not be applied to analyze the $m_\pi$ estimate until Lemma 2 is established.

**Compatible Features:** One possible approach to eliminate the bias $b(\theta)$ is to consider compatible features as in the canonical on-policy actor-critic (Sutton et al., 2000; Konda, 2002). Let $\Psi$ be a subspace and $\prec \gamma, \cdot, \cdot \succ_\Psi$ be an inner product, which induces a norm $||\cdot||_\Psi$. We define a projection $\Pi_\Psi$ as $\Pi_\Psi y = \arg \min_{y \in \Psi} ||y - \bar{y}||_\Psi^2$. For any vector $y$ and a vector $\bar{y} \in \Psi$, we have $y - \Pi_\Psi y, \bar{y} >_{\Psi} \bar{y} = 0$ by the Pythagorean. Based on this equality, Konda (2002) designs compatible features for an on-policy actor-critic. Inspired by Konda (2002), we now design compatible features for COF-PAC.

Let $\hat{m}_\theta, \hat{q}_\theta$ be estimates for $m_\pi, q_\pi$. With slight abuse of notations, we define

\[
\hat{g}(\theta) = \sum \mu(s) \hat{m}(s) \sum a \mu(a|s) \psi(a, s, a) \hat{q}_\theta(s, a),
\]

which is the limiting policy update. The bias $\nabla J(\theta) - \hat{g}(\theta)$ can then be decomposed as $b_1(\theta) + b_2(\theta)$, where

\[
b_1(\theta) = \sum d_\mu(s)(m_\pi(s) - \hat{m}_\theta(s))\phi_0^2(s),
\]

\[
b_2(\theta) = \sum d_\mu(s)(m_\pi(s)\phi_0(s, a) \hat{q}_\theta(s, a),
\]

\[
d_\mu(s)\phi_0(s, a) \hat{q}_\theta(s, a).
\]

We consider $\phi_{1, i}^\theta \in \mathbb{R}^{|S|}$, where $\phi_{1, i}^\theta(s)$ is the $i$-th element of $\phi_{1}^\theta(s) \in \mathbb{R}^K$. Let $\Psi_1$ denote the subspace in $\mathbb{R}^{|S|}$ spanned by $\{\phi_{1, i}^\theta\}_{i=1,...,K}$. We define an inner product $<, , >_{\Psi_1} = \sum_{s,a} d_\mu(s)\pi(s, a)\phi_0(s, a) \phi_{1, i}^\theta(s, a)$. Then we can write $b_{1, i}(\theta)$, the $i$-the element of $b_1(\theta)$, as

\[
b_{1, i}(\theta) = ||m_\pi - \hat{m}_\theta, \phi_{1, i}^\theta >_{\Psi_1}.
\]

If our estimate $\hat{m}_\theta$ satisfies $\hat{m}_\theta = \Pi_{\Psi_1} m_\pi$, we have $b_1(\theta) = 0$. This motivates learning the estimate $\hat{m}_\theta$ by minimizing $J_{\Psi_1} = ||\Pi_{\Psi_1} m_\pi - \hat{m}_\theta||_{\Psi_1}$. One possibility is to consider linear function approximation for $\hat{m}_\theta$ and use $\{\phi_{1, i}^\theta\}$ as features. Similarly, we consider the subspace $\Psi_2$ in $\mathbb{R}^{N_{\pi}}$ spanned by $\{\phi_{2, j}^\theta\}$ and define the inner product according to $d_{\mu,m}$. Then we aim to learn $\hat{q}_\theta$ via minimizing $J_{\Psi_2} = ||\Pi_{\Psi_2} q_\pi - \hat{q}_\theta||_{\Psi_2}$. Again, we can consider linear function approximation with features $\{\phi_{2, j}^\theta\}$. In general, any feature, whose feature space contains $\Psi_1$ or $\Psi_2$, are compatible features. Due to the change of $\theta$, compatible features usually change every time step (Konda, 2002). Note if we consider a state value critic instead of a state-action value critic, the computation of compatible features will involve the transition kernel $p$, to which we do not have access.

In the on-policy setting, Monte Carlo or TD(1) can be used to train a critic with compatible features (Sutton et al., 2000; Konda, 2002). In the off-policy setting, one could consider a GEM analogue of GTD($\lambda$) (Yu, 2017) with $\lambda = 1$ to minimize $J_{\Psi_1}$. To minimize $J_{\Psi_2}$, one could consider a $q$-value analogue of ETD($\lambda$) (Yu, 2015) with $\lambda = 1$. We leave the convergent analysis for those analogues under a changing target policy for future work.

### 5. Experiments

We design experiments to answer the following questions: (a) Can GEM approximate the emphasis as promised? (b) Can the GEM-learned emphasis boost performance compared with the followon trace?

**Approximating Emphasis:** We consider variants of Baird’s counterexample (Baird, 1995; Sutton & Barto, 2018) as shown in Figure 1. There are two actions and the behavior policy $\mu$ always chooses the dashed action with probability $\frac{1}{2}$. The initial state is chosen from all the states with equal probability, and the interest $i$ is 1 for all states. We consider
A variant of Baird’s counterexample. This figure is adapted from Sutton & Barto (2018). The solid action always leads to the state 7 and a reward 0, and the dashed action leads to states 1 - 6 with equal probability and a reward +1.

We consider two target policies: \( \pi \). As shown in Figure 2, the GEM approximation enjoys lower variance than the followon trace approximation and has emphasis approximation error in (Eq (1)). We report the emphasis approximation error in variance than the followon trace approximation and has lower approximation error under all four sets of features. The quantities of interest, e.g., \( m_\pi \) and \( v_\pi \), can be expressed accurately under all the three sets of features. In the fourth set of features, we consider state aliasing. Namely, we still consider the original features but now the feature of the state 7 is modified to be identical as the feature of the state 6. The last two dimensions of features then become identical for all states and we therefore removed them, resulting in features lying in \( \mathbb{R}^6 \). Now the quantities of interest may not lie in the feature space.

In this section, we compare the accuracy of approximating the emphasis \( m_\pi \) with GEM (Eq (5)) and the followon trace (Eq (1)). We report the emphasis approximation error in Figure 2. At time step \( t \), the emphasis approximation error is computed as \( |M_t - m_\pi(S_t)| \) and \( |w_t^\top x(S_t) - m_\pi(S_t)| \) for the followon trace and GEM respectively, where \( m_\pi \) is computed analytically, \( M_{-1} = 0 \), and \( w_0 \) is drawn from a unit normal distribution. For GEM, we consider a fixed learning rate \( \alpha \) and tune it from \( \{0.1 \times 2^0, \ldots, 0.1 \times 2^{-6}\} \). We consider two target policies: \( \pi(\text{solid} | \cdot) = 0.1 \) and \( \pi(\text{solid} | \cdot) = 0.3 \).

As shown in Figure 2, the GEM approximation enjoys lower variance than the followon trace approximation and has lower approximation error under all four sets of features. Interestingly, when the original features are used, the C matrix is indeed singular, which violates Assumption 2. However, the algorithm does not diverge. This may suggest that the Assumption 2 can be relaxed in practice.

**Policy Evaluation**: The followon trace \( M_t \) is originally used in ETD to reweight updates (Eq (1) and Eq (2)). Here we compare ETD(0) with GEM-ETD(0), where the latter updates \( \nu \) as

\[
\nu_{t+1} = \nu_t + \alpha_2 \hat{M}_t \rho_t(R_{t+1} + \gamma x_{t+1}^\top \nu_t - x_t^\top \nu_t)x_t^\top,
\]

where \( \hat{M}_t = w_t^\top x_t \) and \( w_t \) is updated according to GEM (Eq (5)) with a fixed learning rate \( \alpha_1 \). If we assume \( m_\pi \) lies in the column space of \( X \), a convergent analysis of GEM-ETD(0) is straightforward.

We consider a target policy \( \pi(\text{solid} | \cdot) = 0.05 \). We report the root mean squared value error (RMSVE) at each time step during training in Figure 3(a). RMSVE is computed as \( \|v - v_\pi\|_D \), where \( v_\pi \) is computed analytically. For ETD(0), we tune the learning rate \( \alpha \) from \( \{0.1 \times 2^0, \ldots, 0.1 \times 2^{-19}\} \). For GEM-ETD(0), we set \( \alpha_1 = 0.025 \) and tune \( \alpha_2 \) in the same range as \( \alpha \). For both algorithms, we report the results with learning rates that minimized the area under curve (AUC) in the solid lines in Figure 3. In our policy evaluation experiments, GEM-ETD(0) has a clear win over ETD(0) under all four sets of features. Note the AUC-minimizing learning rate for ETD(0) is usually several orders smaller than that of GEM-ETD(0), which explains why ETD(0) curves tend to have smaller variance than GEM-ETD(0) curves. When we decrease the learning rate of GEM-ETD(0) (as indicated by the red dashed lines in Figure 3), the variance of GEM-ETD(0) can be reduced and the AUC is still smaller than that of ETD(0).

ETD(0) is a special case of ETD(\( \lambda, \beta \)) (Hallak et al., 2016), where \( \lambda \) and \( \beta \) are used for bias-variance trade-off. Similarly, we can have GEM-ETD(\( \lambda, \beta \)) by introducing \( \lambda \) and \( \beta \) to our GEM operator \( T \) analogously to ETD(\( \lambda, \beta \)). A comparison between ETD(\( \lambda, \beta \)) and GEM-ETD(\( \lambda, \beta \)) is a possibility for future work.

GEM-ETD is indeed a way to trade off bias and variance. If the states are heavily aliased, the GEM emphasis estimation may be heavily biased, as will GEM-ETD. We do not claim that GEM-ETD is always better than ETD. For example,
when we set the target policy to $\pi(\text{solid}) = 1$, there was no observable progress for both GEM-ETD(0) and ETD(0) with reasonable computation resources.\footnote{This target policy is problematic for GEM-ETD(0) mainly because the magnitude of $d_t$ in Eq (5) varies dramatically at different states, which makes the supervised learning of $\kappa$ hard.} When it comes to the bias-variance trade-off, the optimal choice is usually task dependent. Our empirical results suggest GEM-ETD is a promising approach for this trade-off.

Control: We benchmarked COF-PAC and ACE in Reacher-v2 from OpenAI Gym (Brockman et al., 2016). Our implementation is based on Zhang et al. (2019) and we inherited their hyperparameters. Like Gelada & Bellemare (2019); Zhang et al. (2019), we consider uniformly random behavior policy. Neural networks are used to parameterize $\pi, v, m$. A semi-gradient version of GEM is used to train $m$ inspired by the success of semi-gradient methods in large scale RL (Mnih et al., 2015). Details are provided in the appendix. We trained both algorithms for $5 \times 10^4$ steps and evaluate $J(\pi)$ every $10^4$ steps. According to Figure 3(b), COF-PAC solves the task faster than ACE.

6. Related Work

Off-PAC has inspired the invention of many other off-policy actor-critic algorithms (Silver et al., 2014; Lillicrap et al., 2015; Wang et al., 2016; Gu et al., 2017; Cloeck & Whiteson, 2017; Espholt et al., 2018), all of which, like Off-PAC, ignore emphasis. Another line of policy-based off-policy algorithms involves reward shaping via policy entropy (Nachum et al., 2017a; O’Donoghue et al., 2016; Schulman et al., 2017; Nachum et al., 2017b; Haarnoja et al., 2017; 2018; Dai et al., 2017) and is orthogonal to our work.

Maei (2018) proposes the Gradient Actor-Critic algorithm under a different objective, $\sum_s d_\mu(s)v(s)$, for off-policy learning with function approximation. This objective is different from the excursion objective in that it replaces the true value function $v_\pi$ with an estimate $v$. Furthermore, the policy gradient estimator Maei (2018) proposed is also based on followon trace. That estimator tracks the true gradient only in a limiting sense under a fixed $\pi$ (see Theorem 2 in Maei (2018)) and has potentially unbounded variance, similar to how $M_t$ tracks $m_\pi(S_t)$. It is unclear whether that policy gradient estimator can track the true policy gradient under a changing $\pi$.

Liu et al. (2019) propose to reweight the Off-PAC update via the density ratio between $\pi$ and $\mu$. This density ratio can be learned by either Liu et al. (2018) as Liu et al. (2019) did or Hallak & Mannor (2017); Gelada & Bellemare (2019); Nachum et al. (2019); Uehara & Jiang (2019). The convergence of those density ratio learning algorithms under a slowly changing target policy is, however, unclear, and this reweighted Off-PAC optimizes a different objective than the excursion objective. Zhang et al. (2019) propose a new objective based on the density ratio from Gelada & Bellemare (2019), yielding Generalized Off-Policy Actor-Critic (Geoff-PAC), whose convergence is also unclear. Composing a convergent Geoff-PAC with GEM-style updates is a possibility for future work.

Our work relies on results from Konda (2002) with fundamental differences: (1) The original method by Konda (2002) focuses on only the on-policy setting and cannot be naturally extended to the off-policy counterpart, while we work on the off-policy setting by incorporating state-of-the-art techniques such as GTD, emphatic learning, and reversed TD. (2) The learning architecture has substantial differences in that Konda (2002) considers one TD critic while we consider two GTD-style critics, and the structures of TD algorithms and GTD-style algorithms are dramatically different.

7. Conclusion

We have presented the first provably convergent off-policy actor-critic with function approximation via introducing the emphasis critic and establishing the tracking ability of GTD-style algorithms under a slowly changing target policy. A possibility for future work is to extend COF-PAC with nonlinear critics via considering projection onto tangent plane as Maei (2011).
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A. Proofs

A.1. Proof of Lemma 1

**Lemma 1.** Under Assumptions (1, 3), there exists a constant $C_1 < \infty$ such that $\forall (\theta, \tilde{\theta})$

\[
||\nabla J(\theta)|| \leq C_1, \\
||\nabla J(\theta) - \nabla J(\tilde{\theta})|| \leq C_1||\theta - \tilde{\theta}||, \\
||\frac{\partial^2 J(\theta)}{\partial \theta \partial \theta^T}|| \leq C_1.
\]

**Proof.** Using $||\cdot||\infty$ to denote infinity norm. It is a well-known fact that $||P_\theta||\infty = 1$. For any $y$, either a vector or a square matrix, we have

\[
||(I - \gamma P_\theta)^{-1}y||_\infty = ||\sum_{i=0}^{\infty} \gamma^i P_\theta^i y||_\infty \leq \sum_{i=0}^{\infty} \gamma^i ||P_\theta||_\infty ||y||_\infty = \frac{||y||_\infty}{1 - \gamma},
\]

implying

\[
||(I - \gamma P_\theta)^{-1}y|| \leq \frac{\sqrt{|S||y||_\infty}}{1 - \gamma}. \tag{12}
\]

(i) Applying Eq (12) to $m_\pi$ and $v_\pi$ yields

\[
||m_\pi|| = ||D^{-1}(I - \gamma P_\theta^T )^{-1}Di|| \leq ||D^{-1}(I - \gamma P_\theta^T )^{-1}|| ||Di||
= ||(I - \gamma P_\theta)^{-1}D^{-1}|| ||Di|| \leq ||Di|| \frac{\sqrt{|S|}||D^{-1}||_\infty}{1 - \gamma},
\]

\[
||v_\pi|| = ||(I - \gamma P_\theta)^{-1}r_\pi|| \leq \frac{\sqrt{|S|}\max_{s,a,a'} r(s,a,a')}{1 - \gamma}.
\]

According to the analytical expression of $\nabla_\theta J(\theta)$ in Eq (3), it follows easily that $\sup_\theta ||\nabla_\theta J(\theta)|| < \infty$.

(ii) For the sake of clarity, in this part use $\nabla_\theta$ to denote the gradient w.r.t. one dimension of $\theta$. We first show $\nabla_\theta v_\pi(s)$ is bounded. As $v_\pi = r_\pi + \gamma P_\pi v_\pi$, we have

\[
\nabla_\theta v_\pi = \nabla_\theta r_\pi + \gamma P_\pi \nabla_\theta v_\pi + \gamma \nabla_\theta P_\pi v_\pi,
\]

\[
\nabla_\theta v_\pi = (I - \gamma P_\pi)^{-1}(\nabla_\theta r_\pi + \gamma \nabla_\theta P_\pi v_\pi).
\]

According to Assumptions (1, 3), $\sup_\theta ||\nabla_\theta r_\pi + \gamma \nabla_\theta P_\pi v_\pi|| < \infty$, Eq (12) then implies $\sup_\theta ||\nabla_\theta v_\pi|| < \infty$.

We then show $\nabla_\theta m_\pi(s)$ is bounded. We have

\[
i + \gamma D^{-1}P_\pi^T Dm_\pi = i + \gamma D^{-1}P_\pi^T (I - \gamma P_\pi^T)^{-1}Di
= \left(D^{-1}(I - \gamma P_\pi^T)+ \gamma D^{-1}P_\pi^T\right)(I - \gamma P_\pi^T)^{-1}Di
= D^{-1}(I - \gamma P_\pi^T)^{-1}Di = m_\pi. \tag{13}
\]

Taking gradients in both sides,

\[
\nabla_\theta m_\pi = \gamma D^{-1}\nabla_\theta P_\pi^T Dm_\pi + \gamma D^{-1}P_\pi^T D\nabla_\theta m_\pi,
\]

\[
\nabla_\theta m_\pi = (I - \gamma D^{-1}P_\pi^T D)^{-1}\gamma D^{-1}\nabla_\theta P_\pi^T Dm_\pi
= \left(D^{-1}(I - \gamma P_\pi^T)D\right)^{-1}\gamma D^{-1}\nabla_\theta P_\pi^T Dm_\pi
= D^{-1}(I - \gamma P_\pi^T)^{-1}\gamma D^{-1}\nabla_\theta P_\pi^T Dm_\pi
= \gamma D^{-1}(I - \gamma P_\pi^T)^{-1}\nabla_\theta P_\pi^T Dm_\pi.
\]

\[
||\nabla_\theta m_\pi|| \leq \gamma||\nabla_\theta P_\pi^T Dm_\pi|| ||(I - \gamma P_\pi^T)^{-1}D^{-1}||
\]
Eq (12) then implies $\sup_{\theta} \nabla_{\theta} m_{\pi} < \infty$. We now take gradients w.r.t. $\theta$ in both sides of Eq (3) and use the product rule of calculus, it follows easily that $\sup_{\theta} \| \frac{\partial^2 J(\theta)}{\partial \theta^2} \| < \infty$.

(iii) The bounded Hessian of $J(\theta)$ in (ii) implies $\nabla_{\theta} J(\theta)$ is Lipschitz continuous.

\[\Box\]

A.2. Proof of Lemma 2

**Lemma 2.** Under Assumption 1, $\|P_{\pi}\|_D = \|D^{-1}P_{\pi}^T D\|_D$

**Proof.** This proof is inspired by Kolter (2011).

\[
\|P_{\pi}\|_D = \sup_{\|x\|_D = 1} \|P_{\pi}x\|_D = \sup_{\|x\|_D = 1} \sqrt{x^T P_{\pi}^T D P_{\pi} x}
\]

\[
= \sup_{\|y\|_1 = 1} \sqrt{y^T D^{-\frac{1}{2}} P_{\pi}^T D P_{\pi} D^{-\frac{1}{2}} y} = \|D^{-\frac{1}{2}} P_{\pi} D^{-\frac{1}{2}}\|_\infty
\]

\[
\|D^{-1}P_{\pi}^T D\|_D = \sup_{\|y\|_1 = 1} \sqrt{y^T D^{-\frac{3}{2}} D P_{\pi}^T D^{-1} D D^{-1} P_{\pi} D D^{-\frac{3}{2}} y} = \|D^{-\frac{3}{2}} P_{\pi} D^{-\frac{3}{2}}\|
\]

The rest follows from the well-known fact that $\ell_2$ matrix norm is invariant under matrix transpose. \[\Box\]

A.3. Proof of Lemma 3

**Lemma 3.** Under Assumptions (1,2), when $\eta > 0$, $\bar{G}(\theta)$ is nonsingular and $\sup_{\theta} \| \bar{G}(\theta)^{-1} \| < \infty$.

**Proof.** By rule of block matrix determinant, we have

\[
\det(\bar{G}(\theta)) = \det(\eta I) \det(C + \frac{1}{\eta} A(\theta) A(\theta)^T)
\]

\[
\geq \det(\eta I) \left[ \det(C) + \det\left(\frac{1}{\eta} A(\theta) A(\theta)^T\right) \right]
\]

\[
\geq \det(\eta I) \det(C),
\]

where both inequalities result from the positive semi-definiteness of $A(\theta) A(\theta)^T$ and Assumption 2. $\bar{G}(\theta)$ is therefore nonsingular. Recall

\[
\bar{G}(\theta)^{-1} = \frac{\text{adj}(\bar{G}(\theta))}{\det(\bar{G}(\theta))},
\]

where $\text{adj}(\bar{G}(\theta))$ is the adjoint matrix of $\bar{G}(\theta)$, whose elements are polynomials of elements of $\bar{G}(\theta)$. It is trivial to see $\sup_{\theta} \| \bar{G}(\theta) \| < \infty$. So $\sup_{\theta} \| \text{adj}(\bar{G}(\theta)) \| < \infty$. It follows immediately that $\sup_{\theta} \| \bar{G}(\theta)^{-1} \| < \infty$ as $\inf_{\theta} \det(\bar{G}(\theta)) > 0$. \[\Box\]

A.4. Proof of Proposition 1

**Proposition 1.** $\hat{T}$ is a contraction mapping w.r.t. some weighted maximum norm and $m_{\pi}$ is its fixed point.

**Proof.** $\hat{T} m_{\pi} = m_{\pi}$ follows directly from Eq (13). Given any two square matrices $A$ and $B$, the products $AB$ and $BA$ have the same eigenvalues (see Theorem 1.3.22 in Horn & Johnson (2012)). Therefore $\gamma D^{-1}(P_{\pi}^T D)$ has the same eigenvalues as $\gamma (P_{\pi}^T D) D^{-1}$. Consequently, they have the same spectral radius, i.e., $\rho(\gamma D^{-1} P_{\pi}^T D) = \rho(\gamma P_{\pi}) < 1$. Here we use $\rho(\cdot)$ to denote the spectral radius of a matrix and $\rho(\gamma P_{\pi}) < 1$ is a well-known fact. Obviously, $\gamma D^{-1} P_{\pi}^T D$ is a non-negative matrix. Then Corollary 6.1 in Bertsekas & Tsitsiklis (1989) implies that $\hat{T}$ is a contraction mapping w.r.t. some weighted maximum norm. \[\Box\]
A.5. Proof of Theorem 1

Theorem 1. (Convergence of GEM) Under Assumptions (1, 2) and Condition 1, the iterate \( \{d_t\} \) generated by (5) satisfies

\[
\sup_t ||d_t|| < \infty \quad \text{and} \quad \lim_{t \to \infty} ||\hat{G}(\theta_t)d_t - \bar{h}|| = 0 \quad \text{almost surely.}
\]

Proof. We first rephrase Theorem 3.1 in Konda (2002), which is a general theorem about the tracking ability of stochastic approximation algorithms under a slowly changing linear system. The original theorem adopts a stochastic process taking value in a Polish space. Here we rephrase it for our finite MDP setting.

Theorem 4. (Konda, 2002) Let \( \{Y_t\} \) be a Markov chain with a finite state space \( \mathcal{Y} \) and a transition kernel \( P_{Y} \in \mathbb{R}^{|\mathcal{Y}| \times |\mathcal{Y}|} \), consider an iterate \( \{d_t\} \) in \( \mathbb{R}^n \) evolving according to

\[
d_{t+1} = d_t + \alpha_t (h_{\theta_t}(Y_t) - G_{\theta_t}(Y_t)d_t),
\]

where \( \{\theta_t\} \) is another iterate in \( \mathbb{R}^m \), \( h_{\theta} : \mathcal{Y} \to \mathbb{R}^n \) and \( G_{\theta} : \mathcal{Y} \to \mathbb{R}^{n \times n} \) are vector-valued and matrix-valued functions parameterized by \( \theta \), \( \alpha_t \) is a positive step size, assume \(^5\)

Assumption 5.

1. (Stepsize) The sequence \( \{\alpha_t\} \) is deterministic, non-increasing and satisfies the Robbins-Monro condition

\[
\sum_t \alpha_t = \infty, \quad \sum_t \alpha_t^2 < \infty
\]

2. (Parameter Changing Rate) The random sequence \( \{\theta_t\} \) satisfies

\[
||\theta_{t+1} - \theta_t|| \leq \beta_t H_t,
\]

where \( \{H_t\} \) is some nonnegative process with bounded moments and \( \{\beta_t\} \) is a deterministic sequence such that

\[
\sum_t (\frac{\beta_t}{\alpha_t})^d < \infty \quad \text{for some} \quad d > 0.
\]

3. (Poisson Equation) There exists \( \hat{h}_\theta : \mathcal{Y} \to \mathbb{R}^n, \hat{G}_\theta : \mathcal{Y} \to \mathbb{R}^{n \times n}, \bar{G}(\theta) \in \mathbb{R}^{n \times n} \) such that

\[
\hat{h}_\theta(y) = h_\theta(y) - \bar{h}(\theta) + \sum_{y'} P_{\mathcal{Y}}(y, y') \hat{h}_\theta(y'),
\]

\[
\hat{G}_\theta(y) = G_\theta(y) - \bar{G}(\theta) + \sum_{y'} P_{\mathcal{Y}}(y, y') \hat{G}_\theta(y')
\]

4. (Boundedness) There is a constant \( C_0 < \infty \) such that \( \forall \theta, y, \)

\[
\max(||\hat{h}(\theta)||, ||\hat{G}(\theta)||, ||\hat{h}_\theta(y)||, ||h_\theta(y)||, ||G_\theta(y)||) \leq C_0
\]

5. (Lipschitz Continuity) There is a constant \( C_0 < \infty \) such that \( \forall \theta, \bar{\theta}, y \)

\[
\max(||\hat{h}(\theta) - \hat{h}(\bar{\theta})||, ||\hat{G}(\theta) - \hat{G}(\bar{\theta})||) \leq C_0||\theta - \bar{\theta}||, \quad ||f_\theta(y) - f_\bar{\theta}(y)|| \leq C_0||\theta - \bar{\theta}||
\]

where \( f_\theta \) represents any of \( \hat{h}_\theta, h_\theta, \hat{G}_\theta, G_\theta \).

6. (Uniformly Positive Definite) There exists a \( \eta_0 > 0 \) such that \( \forall \theta, d, \)

\[
d^T \bar{G}(\theta)d \geq \eta_0||d||^2.
\]

then \(^6\)

\[
\sup_t ||d_t|| < \infty, \quad \lim_t ||G(\theta_t)d_t - \bar{h}(\theta_t)|| = 0 \quad a.s.
\]

\(^5\)The Assumptions 5(4,5) imply the Assumptions 3.1(7, 9) in Konda (2002)

\(^6\)The boundedness is from Lemma 3.9 in Konda (2002)
We now prove Theorem 1 by verifying that GEM indeed satisfies Assumptions 5(1-6). Assumption 5(1) is satisfied by the requirement of \( \{ \alpha_t \} \) in GEM. Assumption 5(2) is satisfied by Condition 1.

We now verify Assumption 5(3). Let \( Y \) be \( S \times A \times S \), \( Y \) be the transition kernel of \( \{ Y_t \} \), it is easy to verify that \( P_{Y_t}(s, a, s') = I_{s'=s_2} \mu(a|s_2)p(s'|s_2, a_2) \). According to Assumption 1, the chain \( \{ Y_t \} \) is ergodic. Let \( d_Y \in \mathbb{R}^{|Y|} \) be its stationary distribution, we have \( d_Y(y) = d_\mu(s|a)\mu(a|s)p(s'|s, a) \). For two fixed integers \( i \) and \( j \) in \([1, 2K_1]\), we consider an MRP with the state space \( Y \), the transition kernel \( P_Y \) and the reward function \( G_{\theta, ij} : Y \to \mathbb{R} \), where \( G_{\theta, ij}(y) \) is the \((i, j)\)-th element of the matrix \( G_\theta(y) \) (defined in Eq (6)). Alternatively, we can view \( G_{\theta, ij} \) as a vector in \( \mathbb{R}^{|Y|} \). The average reward of this MRP is then \( \bar{G}_{ij}(\theta) \), the \((i, j)\)-indexed element in \( G(\theta) \) (defined in Eq (7)). We consider the differential value function (see Sutton & Barto (2018)) \( \dot{G}_{\theta, ij} \in \mathbb{R}^{|Y|} \) of this MRP, where

\[
\dot{G}_{\theta, ij}(y) = \mathbb{E}_Y \sum_{t=0}^\infty (G_{\theta, ij}(Y_t) - \bar{G}_{ij}(\theta)) |Y_0 = y|.
\]

These differential value functions define a matrix-valued function \( \dot{G}_\theta : Y \to \mathbb{R}^{2K_1 \times 2K_1} \). Assumption 5(3) is then satisfied according to the Bellman equation of differential value function (see Sutton & Barto (2018)). Moreover, according to the standard Markov chain theory (e.g., Section 8.2.1 in Puterman 2014), we have

\[
\dot{G}_{\theta, ij} = H_Y \dot{G}_{\theta, ij},
\]

where \( H_Y = (I - P_Y + P_\mu)^{-1} = (I - P_Y) \in \mathbb{R}^{|Y| \times |Y|} \) is the fundamental matrix which depends only on \( \mu \) and \( p \). Here each row of \( P_Y \) is \( d_Y \). Similarly, we define \( \dot{h} : Y \to \mathbb{R}^{2K_1} \) by defining its \( i \)-th component as

\[
\dot{h}_i = H_Y \dot{h}_i.
\]

Now \( \dot{h} \) defined in (6), and \( \dot{h} \) defined in (7) satisfy Assumption 5(3). Note they are independent of \( \theta \).

It is trivial to see that \( \sup_\theta \| \dot{G}(\theta) \| < \infty \) and \( \| \dot{G}(\theta) \| \) is Lipschitz continuous in \( \theta \). Assuming Assumption 1, \( \rho_\theta \) is bounded. It follows easily that \( \| \sup_\theta \dot{G}(y) \| < \infty \). As \( Y \) is finite, \( \sup_{\theta, y} \| \dot{G}(y) \| < \infty \). Similarly, \( \| \dot{G}(y) \| \) is Lipschitz in \( \theta \) and the Lipschitz constant is independent on \( y \). Assumptions 5(4, 5) are now satisfied.

We now verify Assumption 5(6). For any \( d^T = [\kappa^T, w^T] \), it is easy to verify that

\[
d^T \dot{G}(\theta)d = \kappa^T C\kappa + \eta w^Tw.
\]

According to Assumption 2, for any \( \kappa \neq 0 \), there exists a \( \bar{\eta} > 0 \) such that \( \kappa^T C\kappa \geq \bar{\eta} \| \kappa \|^2 \), we therefore have

\[
d^T \dot{G}(\theta)d \geq \min(\bar{\eta}, \eta) \| d \|^2, \quad \forall \theta, d.
\]

Assumption 5(6) is then satisfied. Note here it is important to have \( \eta > 0 \). If \( \eta = 0 \) as in the original GTD2, \( d^T \dot{G}(\theta)d \) will not depend on \( w \). Consequently, we can increase \( \| d \| \) arbitrarily while keeping \( d^T \dot{G}(\theta)d \) unchanged by increasing \( \| w \| \). Assumption 5(6) then cannot be satisfied.

**A.6. Proof of Proposition 2**

**Proposition 2.** \( \Pi \tilde{T}(X w^*_\theta(0)) = X w^*_\theta(0) \).

**Proof.** Let \( b \doteq X^TDi \) and use similar techniques as Hallak & Mannor (2017), we have

\[
\Pi \tilde{T}(X w^*_\theta(0)) = X(X^TDX)^{-1}X^TD \left( i + \gamma (DX)^{-1}P_x^DAX w^*_\theta(0) \right)
\]

\[
= X(X^TDX)^{-1}b + \gamma X(X^TDX)^{-1}X^TP_x^DAX A(\theta)^{-1}b
\]

\[
= X(X^TDX)^{-1} \left( A(\theta) + \gamma X^TP_x^DAX \right) A(\theta)^{-1}b
\]

\[
= X(X^TDX)^{-1}X^TDXA(\theta)^{-1}b \quad \text{(Definition of } A(\theta))
\]

\[
= X w^*_\theta(0).
\]

\( \square \)
A.7. Proof of Theorem 2

**Theorem 2.** (Convergence of GQ2) Under Assumptions (1, 2) and Condition 1, the iterate \( \{ \hat{d}_t \} \) generated by (9) satisfies \( \sup_t \| \hat{d}_t \| < \infty \) and \( \lim_{t \to \infty} \| \hat{G}(\theta_t)\hat{d}_t - \hat{h} \| = 0 \) almost surely.

**Proof.** The proof is the same as the proof of Theorem 1 up to a change of notations. \( \square \)

A.8. Proof of Theorem 3

**Theorem 3.** (Convergence of COF-PAC) Under Assumptions (1-3), the iterate \( \{ \theta_t \} \) generated by COF-PAC (Eq (11)) satisfies

\[
\lim \inf_t \left( \| \nabla J(\theta_t) \| - \| b(\theta_t) \| \right) \leq 0,
\]

almost surely, i.e., \( \{ \theta_t \} \) visits any neighborhood of the set \( \{ \theta : \| \nabla J(\theta) \| \leq \| b(\theta) \| \} \) infinitely many times almost surely.

**Proof.** This proof is inspired by Konda (2002). With \( \psi_t \equiv \rho_t \nabla \log \pi(A_t|S_t), w_t^* \equiv w_\eta^* (\eta), u_t^* \equiv u_\eta^* (\eta) \), we rewrite the update \( \Gamma_1(w_t)\Gamma_2(u_t)\Delta_t \) as

\[
\Gamma_1(w_t)\Gamma_2(u_t)\Delta_t = \Gamma_1(w_t)(w_t^T x_t)\Gamma_2(u_t)(u_t^T \hat{x}_t)\psi_t = e_t^{(1)} + e_t^{(2)} + g_t^*,
\]

where

\[
e^{(1)}_t = (\Gamma_1(w_t)w_t - \Gamma_1(w_t^*)w_t^*)^T x_t \Gamma_2(u_t)u_t^T \hat{x}_t \psi_t,
\]

\[
e^{(2)}_t = \Gamma_1(w_t^*)w_t^T x_t(\Gamma_2(u_t)u_t^T - \Gamma_2(u_t^*)u_t^*)^T \hat{x}_t \psi_t,
\]

\[
g_t^* = \Gamma_1(w_t^*)(x_t^T w_t^*)\Gamma_2(u_t^*)(\hat{x}_t^T u_t^*)\psi_t.
\]

Then

\[
\theta_{t+1} = \theta_t + \beta_t \Gamma_1(w_t)\Gamma_2(u_t)\Delta_t
\]

\[
= \theta_t + \beta_t e^{(1)}_t + \beta_t e^{(2)}_t + \beta_t g_t^* + \beta_t \Gamma_1(w_t^*)\Gamma_2(u_t^*) \left( \nabla J(\theta_t) - \hat{g}(\theta_t) - b(\theta_t) \right)
\]

Using the second order Taylor expansion and Cauchy-Schwarz inequality, we have

\[
J(\theta_{t+1}) \geq J(\theta_t) + \beta_t \Gamma_1(w_t^*)\Gamma_2(u_t^*) \left( \| \nabla J(\theta_t) \| - \| b(\theta_t) \| \right)
\]

\[
+ \beta_t \nabla J(\theta_t)^T (g_t^* - \Gamma_1(w_t^*)\Gamma_2(u_t^*)\hat{g}(\theta_t))
\]

\[
+ \beta_t \nabla J(\theta_t)^T e_t^{(1)}
\]

\[
+ \beta_t \nabla J(\theta_t)^T e_t^{(2)}
\]

\[
- \frac{1}{2} C_0 \| \beta_t \Gamma_1(w_t)\Gamma_2(u_t)\Delta_t \|^2,
\]

where \( C_0 \) reflects the bound of the Hessian. We will prove in following subsections that all noise terms in Eq (14) are negligible. Namely,

**Lemma 4.** \( \lim_{t \to \infty} e^{(i)}_t = 0 \) a.s. \( (i = 1, 2) \)

**Lemma 5.** \( \sum_t \| \beta_t \Gamma_1(w_t)\Gamma_2(u_t)\Delta_t \|^2 \) converges a.s.

**Lemma 6.** \( \sum_t \beta_t \nabla J(\theta_t)^T (g_t^* - \Gamma_1(w_t^*)\Gamma_2(u_t^*)\hat{g}(\theta_t)) \) converges a.s.

Same as the section “Proof of Theorem 5.5” in Konda (2002), we now consider a sequence \( \{ k_i \} \) such that

\[
k_0 = 0, \quad k_{i+1} = \min \{ k \geq k_i \mid \sum_{l=k_i}^{k} \beta_l \geq T \} \quad (i > 0)
\]
for some constant $T > 0$. Iterating Eq (14) yielding

$$J(\theta_{k_{i+1}}) \geq J(\theta_{k_i}) + \delta_i + \sum_{t=k_i}^{k_{i+1}-1} \beta_i \Gamma_1(w_t^*)\Gamma_2(u_t^*) ||\nabla J(\theta_t)||\left(||\nabla J(\theta_t)|| - ||b(\theta_t)||\right),$$

(15)

where

$$\delta_i \doteq \sum_{t=k_i}^{k_{i+1}-1} \left[ \beta_i \nabla J(\theta_t)^T (g_t^* - \Gamma_1(w_t^*)\Gamma_2(u_t^*)\hat{g}(\theta_t)) + \beta_i \nabla J(\theta_t)^T (e_t^{(1)} + e_t^{(2)}) - \frac{1}{2} C_0 ||\beta_t \Gamma_1(w_t)\Gamma_2(u_t)\Delta_t||^2 \right].$$

Lemmas (1 - 6) and the selection of $\{k_i\}$ imply $\lim_{k \to \infty} \delta_i = 0$. Theorems 1 and 2 imply that $w_t^*$ and $u_t^*$ are bounded. Consequently, $\Gamma_1(w_t^*)$ and $\Gamma_2(u_t^*)$ are bounded below by some positive constant. If $\lim \inf \left[ ||\nabla J(\theta_t)|| - ||b(\theta_t)|| \right] \leq 0$ does not hold, there must exist $t_0$ and $\epsilon > 0$ such that $\forall t > t_0$, $||\nabla J(\theta_t)|| - ||b(\theta_t)|| > \epsilon$. So the summation in Eq (15) is bounded below. Iterating Eq (15) then implies $J(\theta_t)$ is unbounded, which is impossible as $\gamma < 1$ and $r$ is bounded.

A.8.1. PROOF OF LEMMA 4

Proof. It is easy to verify that $\Gamma_1$ is continuous. According to Assumptions (2, 3), $x_t, \bar{x}_t, \psi_t$ are bounded. The reset follows immediately from Eq (8) and Eq (10).

A.8.2. PROOF OF LEMMA 5

Proof. It follows from the proof of Lemma 4 that $||\Gamma_1(w_t)\Gamma_2(u_t)\Delta_t||$ is bounded by some constant $C_0$. Then $\sum_t ||\beta_t \Gamma_1 \Gamma_2 || \leq C_0 \sum_t \beta_t^2 < \infty$.

A.8.3. PROOF OF LEMMA 6

Proof. In this subsection we write $w_0^* (\eta)$ as $w_0^*$ for simplifying notations and define

$$g_t^* = \Gamma_1(w_t^*)\Gamma_2(u_t^*) (x^T w_t^* \hat{x}) v_t(s, a).$$

(16)

So $g_t^* = g_t^*(S_t, A_t)$. Assumptions (2,3) imply that there is a constant $C_0 < \infty$ such that

$$\forall (t, s, a), ||g_t^*(s, a)|| < C_0.$$

We first make a transformation of the original noise using Poisson equation as in Section A.5. Similarly, let $Y = S \times A, Y_t \doteq (S_t, A_t), y \doteq (s, a)$ and $P_\delta$ be the transition kernel of $\{Y_t\}$, i.e., $P_\delta (\{(s', a') \doteq p(s'|s, a)\mu(a'|s')$. For every integer $i$ in $[1, K]$, we consider the MRP with the reward function $g_{t, i}^* : Y \to \mathbb{R}$, where $g_{t, i}^*(y)$ is the $i$-th element of $g_t^*(y)$. Again, we view $g_{t, i}$ as a vector in $\mathbb{R}^{|Y|}$. The average reward is therefore $g_{t, i}(\theta)$, which is the $i$-th element of $\bar{g}(\theta) \doteq \Gamma_1(w_t^*)\Gamma_2(u_t^*)\bar{g}(\theta) \in \mathbb{R}^K$. With $H_Y$ denoting the fundamental matrix of this MRP, the differential value function of this MRP is then

$$\hat{v}_{t, i} = H_Y g_{t, i}^* \in \mathbb{R}^{|Y|}.$$ (17)

These differential value functions define a vector-valued function $\hat{v}(\theta) : Y \to \mathbb{R}^K$, which satisfies $\sup_{\theta, s, a} \hat{v}(\theta, s, a) < \infty$ as it is just linear transformation of $g_0^*$. According to the Bellman equation of differential value function, we have

$$\hat{v}(y) = g^*_0(y) - \bar{g}(\theta) + \sum_{y'} P_\delta (y, y') \hat{v}(y').$$ (18)

Now we are ready to decompose the noise $\nabla J(\theta_t)^T (g_t^* - 1 \Gamma_1(w_t^*)\Gamma_2(u_t^*)\hat{g}(\theta_t)) as$

$$\nabla J(\theta_t)^T (g_t^* - \Gamma_1(w_t^*)\Gamma_2(u_t^*)\hat{g}(\theta_t))$

$$= \nabla J(\theta_t)^T (\hat{g}_t^*(S_t, A_t) - \bar{g}(\theta_t)) \quad \text{(Definition of } g_{t, i}^* \text{ and } \bar{g}(\theta))$$

$$= \nabla J(\theta_t)^T \left( \hat{v}_{t, i}(S_t, A_t) - \sum_{s', a'} p(s'|S_t, A_t)\mu(a'|s')\hat{v}_{t, i}(s', a') \right) \quad \text{(Eq (18))}$$

$$= \sum_{i=1}^4 e_i^{(i)},$$
where
\[ \epsilon_t^{(1)} = \nabla J(\theta_t)^\top \left( \hat{\nu}_{\theta_t}(S_{t+1}, A_{t+1}) - \sum_{s',a'} p(s'|S_t, A_t) \mu(a'|s') \hat{\nu}_{\theta_t}(s', a') \right), \]
\[ \epsilon_t^{(2)} = \frac{\beta_{t-1} \nabla J(\theta_{t-1})^\top \hat{\nu}_{\theta_{t-1}}(S_t, A_t) - \beta_t \nabla J(\theta_t)^\top \hat{\nu}_{\theta_t}(S_{t+1}, A_{t+1})}{\beta_t}, \]
\[ \epsilon_t^{(3)} = \frac{\beta_t - \beta_{t-1}}{\beta_t} \nabla J(\theta_{t-1})^\top \hat{\nu}_{\theta_{t-1}}(S_t, A_t), \]
\[ \epsilon_t^{(4)} = \nabla J(\theta_t)^\top \hat{\nu}_{\theta_t}(S_t, A_t) - \nabla J(\theta_{t-1})^\top \hat{\nu}_{\theta_{t-1}}(S_t, A_t). \]

We now show that \( \sum_t \beta_t \epsilon_t^{(i)} \) converges almost surely for \( i = 1, 2, 3, 4 \).

(1) We first state a Martingale Convergence Theorem (see Proposition 4.3 in Bertsekas & Tsitsiklis 1996).

**Lemma 7.** Assuming \( \{M_t\}_{t=1}^\infty \) is a Martingale sequence and there exists a constant \( C_0 < \infty \) such that \( \forall l, \mathbb{E}[|M_l|^2] < C_0 \), then \( \{M_t\} \) converges almost surely.

Let \( F \) be a \( \sigma \)-algebra and \( M_t = \sum_{\ell=0}^t \beta_\ell \epsilon_{\ell+1} \). It is easy to see that \( M_t \) is adapted to \( F \). Due to Lemma 1 and boundedness of \( \hat{\nu} \), \( |\epsilon_{\ell}^{(i)}| \leq C_1 \) for some constant \( C_1 \), implying \( \mathbb{E}[|M_l|] < \infty \) holds for any fixed \( l \). Moreover,
\[ \mathbb{E}[M_{t+1} | F_t] = M_t + \mathbb{E}_{\theta_{t+1}, S_{t+2}, A_{t+2}} [\beta_{t+1} \epsilon_{t+1}^{(1)} | F_t], \]
\[ = M_t + \beta_{t+1} \mathbb{E}_{\theta_{t+1}} \left[ \mathbb{E}_{S_{t+2}, A_{t+2}} [\epsilon_{t+1}^{(1)} | \theta_{t+1}, F_t] \right], \]
\[ = M_t + \beta_{t+1} \mathbb{E}_{\theta_{t+1}} [0] = M_t. \]

\( M_t \) is therefore a Martingale. We now verify that \( M_t \) has bounded second moments, then \( \{M_t\} \) converges according to Lemma 7. For any \( t_1 < t_2 \), we have
\[ \mathbb{E}[\epsilon_{t_1}^{(1)} \epsilon_{t_2}^{(1)}] = \mathbb{E} \left[ \mathbb{E}[\epsilon_{t_1}^{(1)} \epsilon_{t_2}^{(1)} | F_{t_1-1}] \right] = \mathbb{E} \left[ \epsilon_{t_1}^{(1)} \mathbb{E}[\epsilon_{t_2}^{(1)} | F_{t_2-1}] \right] = \mathbb{E} \left[ \epsilon_{t_1}^{(1)} 0 \right] = 0. \]

Consequently,
\[ \forall l, \quad \mathbb{E}[|M_l|^2] = \mathbb{E} \left[ \sum_{t=0}^l \beta_t^2 (\epsilon_{t+1}^{(1)})^2 \right] \leq C_1^2 \sum_{t=0}^\infty \beta_t^2. \]

(2) \( \sum_{t=1}^l \beta_t \epsilon_t^{(2)} = \beta_0 \nabla J(\theta_0)^\top \hat{\nu}_{\theta_0}(S_1, A_1) - \beta_t \nabla J(\theta_t)^\top \hat{\nu}_{\theta_t}(S_{t+1}, A_{t+1}) \). The rest follows from the boundedness of \( \nabla J(\theta) \) and \( \hat{\nu}(s,a) \) and \( \lim_{t \to \infty} \beta_t = 0 \).

(3) \[ \sum_{t=1}^l |\beta_t \epsilon_t^{(3)}| \leq \sum_{t=1}^l |\beta_t - \beta_{t-1}| \left| \nabla J(\theta_{t-1})^\top \hat{\nu}_{\theta_{t-1}}(S_t, A_t) \right| \]
\[ \leq C_1 \sum_{t=1}^l (\beta_{t-1} - \beta_t) \leq C_1 (\beta_0 - \beta_t) < C_1 \beta_0 \quad \text{a.s.} \]

It follows easily that \( \sum_t \beta_t \epsilon_t^{(3)} \) converges absolutely, thus converges.

(4) Lemma 1 implies \( \nabla J(\theta) \) is bounded and Lipschitz continuous in \( \theta \), if we are able to show \( \forall (s,a,t) \), there exists a constant \( C_0 \) such that
\[ \| \hat{\nu}_{\theta_t}(S_t, A_t) - \hat{\nu}_{\theta_{t-1}}(S_t, A_t) \| \leq C_0 |\theta_t - \theta_{t-1}| \],
According to Assumption 3, we will have
\[ |\epsilon_t^{(4)}| \leq C_1 |\theta_t - \theta_{t-1}| = C_1 |\beta_t \Gamma_1(w_t) \Gamma_2(u_t) \Delta_t| \leq \beta_t C_2. \]

Consequently,
\[ \sum_{t=1}^{l} |\beta_t \epsilon_t^{(4)}| \leq C_2 \sum_{t=1}^{l} \beta_t^2 < C_2 \sum_{t=1}^{\infty} \beta_t^2 \quad a.s. \]

Thus \( \sum_{t} \beta_t \epsilon_t^{(4)} \) converges. We now proceed to show Eq (19) does hold. According to Eq (17), it suffices to show \( \forall (s, a, \theta, \bar{\theta}) \), there exists a constant \( C_0 \) such that
\[ ||g_\theta^*(s, a) - g_\theta^*(s, a)|| \leq C_0 ||\theta - \bar{\theta}||. \]

According to Assumption 3, \( \psi_\theta(s, a) \) is bounded and Lipschitz continuous in \( \theta \). It is easy to verify the function \( ||w \Gamma_2(w)|| \) is bounded and Lipschitz continuous in \( w \). According to the definition of \( g_\theta^*(s, a) \) in Eq (16), it then suffices to show \( w_\theta^* \) and \( u_\theta^* \) is Lipschitz continuous in \( \theta \). Recall by definition \( w_\theta^* \) is the second half of \( \tilde{G}(\theta)^{-1} \), it then suffices to show \( \tilde{G}(\theta)^{-1} \) is Lipschitz continuous in \( \theta \). Using the fact
\[ ||B_1^{-1} - B_2^{-1}|| = ||B_1^{-1}(B_1 - B_2)B_2^{-1}|| \leq ||B_1^{-1}|| \ ||B_1 - B_2|| \ ||B_2^{-1}||, \]
we have
\[ ||\tilde{G}(\theta)^{-1} - \tilde{G}(\theta)^{-1}|| \leq ||\tilde{G}(\theta)^{-1}|| \ ||\tilde{G}(\theta) - \tilde{G}(\theta)|| \ ||\tilde{G}(\theta)^{-1}||. \]

The rest follows from the fact that \( \tilde{G}(\theta) \) is Lipschitz continuous in \( \theta \) and Lemma 3. Similarly, we can establish the Lipschitz continuity of \( u_\theta^* \), which completes the proof. \( \square \)

**A.9. Proof of Proposition 3**

**Proposition 3.** Under Assumptions (1-4), let \( \alpha_\theta \) be the stationary distribution under \( \pi_\theta \) and define \( \tilde{\alpha}_\theta(s, a) \doteq d_\theta(s) \pi_\theta(a|s), D_\theta \doteq \text{diag}(d_\theta), \tilde{D}_\theta \doteq \text{diag}(\tilde{d}_\theta) \), we have
\[ ||b(\theta)||_D \leq C_0 \eta + C_1 \frac{1 + \kappa(D - \tilde{D})^2 + \tilde{D}^2}{1 - \gamma} ||m_{\pi_\theta} - \tilde{\Pi}^m_{\pi_\theta}||_D \]
\[ + C_2 \frac{1 + \kappa(D - \tilde{D})^2 + \tilde{D}^2}{1 - \gamma} ||\tilde{\Pi} q_{\pi_\theta}||_D, \]

where \( \kappa(\cdot) \) is the condition number of a matrix w.r.t. \( \ell_2 \) norm and \( C_0, C_1, C_2 \) are some positive constants.

**Proof.** We first decompose \( b(\theta) \) as \( b(\theta) = b_1(\theta) + b_2(\theta) \), where
\[ b_1(\theta) \doteq \sum_s d_\mu(s) \left( m_{\pi_\theta}(s) - x(s)^T w_\theta^*(\eta) \right) \sum_a \mu(a|s) \psi_\theta(s, a) \left( \tilde{x}(s, a)^T u_\theta^*(\eta) \right) \]
\[ b_2(\theta) \doteq \sum_s d_\mu(s) m_{\pi_\theta}(s) \sum_a \mu(a|s) \psi_\theta(s, a) \left( q_{\pi_\theta}(s, a) - \tilde{x}(s, a)^T u_\theta^*(\eta) \right) \]

The boundedness of \( m_{\pi_\theta}(s) \) and \( \psi_\theta(s, a) \) implies
\[ ||b_2(\theta)||_D \leq C_2 ||q_{\pi_\theta} - \tilde{X} u_\theta^*(\eta)||_D \leq C_2 ||q_{\pi_\theta} - \tilde{X} u_\theta^*(0)||_D + C_2 ||\tilde{X} u_\theta^*(\eta) - \tilde{X} u_\theta^*(0)||_D \]
for some constant \( C_2 \). Theorem 2 in Kolter (2011) states
\[ ||q_{\pi_\theta} - \tilde{X} u_\theta^*(0)||_D \leq \frac{1 + \gamma \kappa(D - \tilde{D})^2 + \tilde{D}^2}{1 - \gamma} ||q_{\pi_\theta} - \tilde{\Pi} q_{\pi_\theta}||_D. \]
Moreover, we have
\[
\|\hat{X}u_\theta^*(\eta) - \hat{X}u_\theta^*(0)\|_D \leq \|\hat{X}\|_D \|u_\theta^*(\eta) - u_\theta^*(0)\|_D \\
\leq \|\hat{X}\|_D \|(\hat{A}(\theta)\hat{C}^{-1}\hat{A}(\theta) + \eta I)^{-1}\|_D \|I\|_D \|(\hat{A}(\theta)\hat{C}^{-1}\hat{A}(\theta))^{-1}\|_D \|\hat{A}(\theta)\hat{C}^{-1}\hat{X}\hat{D}\|_D,
\]
where the last inequality results from Eq (20). The boundedness of \(\|(\hat{A}(\theta)\hat{C}^{-1}\hat{A}(\theta) + \eta I)^{-1}\|_D\) can be established with the same routine as the proof of Lemma 3. The boundedness of \(\|(\hat{A}(\theta)\hat{C}^{-1}\hat{A}(\theta))^{-1}\|_D\) can be established with this routine and Assumption 4(b), which completes the boundedness of \(b_1(\theta)\).

The same routine can be used to bound \(b_2(\theta)\). Particularly, we need to show
\[
\|m_{\pi_o} - Xw_\theta^*(0)\|_D \leq \frac{1 + \gamma \kappa (D^{-\frac{1}{2}}D_\theta^2)}{1 - \gamma} \|m_{\pi_o} - \Pi m_{\pi_o}\|_D.
\]
We omit the proof for this inequality as it is mainly verbatim repetition of the proof of Theorem 2 in Kolter (2011) except two main differences: (1) The positive semidefiniteness of \(F_\theta\) in Assumption 4 ensures \(\|D^{-\frac{1}{2}}D_\theta^2 Xw\|_D \leq \|Xw\|_D \forall w\), analogously to Eq (10) in Kolter (2011). We will then replace the occurrence of \(P_\pi\) in Kolter (2011) with \(D^{-\frac{1}{2}}D_\theta^2 D\). (2) When we reach the place to compute \(\|D^{-\frac{1}{2}}D_\theta^2 D\|_D\), analogously to Eq (17) in Kolter (2011), Lemma 2 implies we need only \(\|P_\pi\|_D\), which has been computed by Kolter (2011).

**B. Experiment Details**

**B.1. Original Features of Baird’s Counterexample**

According to (Sutton & Barto, 2018), we have
\[
\begin{align*}
x(s_1) & \doteq [2, 0, 0, 0, 0, 0, 0, 1]^T \\
x(s_2) & \doteq [0, 2, 0, 0, 0, 0, 0, 1]^T \\
x(s_3) & \doteq [0, 0, 2, 0, 0, 0, 0, 1]^T \\
x(s_4) & \doteq [0, 0, 0, 2, 0, 0, 0, 1]^T \\
x(s_5) & \doteq [0, 0, 0, 0, 2, 0, 0, 1]^T \\
x(s_6) & \doteq [0, 0, 0, 0, 0, 2, 0, 1]^T \\
x(s_7) & \doteq [0, 0, 0, 0, 0, 0, 1, 2]^T
\end{align*}
\]

**B.2. Mujoco Experiments**

To evaluate \(J(\pi)\), we first sample a state from \(d_\mu\) and then follow \(\pi\) until episode termination, which we call an excursion. We use the averaged return of 10 excursions as an estimate of \(J(\pi)\).

Our implementation is based on the open-sourced implementation from Zhang et al. (2019). We, therefore, refer the reader to Zhang et al. (2019) for full details. In particular, two-hidden-layer networks are used to parameterize \(\pi, \nu, m_\pi, \) and the target policy \(\pi\) is a Gaussian policy. We use a semi-gradient rule to learn \(m_\pi\). Assuming \(m_\pi\) is parameterized by \(\theta_m\), we then update \(\theta_m\) to minimize
\[
\left(i(S_{t+1}) + \gamma \rho(S_t, A_t)m_\pi(S_t; \hat{\theta}_m) - m_\pi(S_{t+1}; \theta_m)\right)^2,
\]
where \(\hat{\theta}_m\) indicates the parameters of the target network. Our COF-PAC implementation inherits the hyperparameters from the ACE implementation of Zhang et al. (2019) without further tuning.