On finite simple and nonsolvable groups acting on homology 4-spheres

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Abstract. The only finite nonabelian simple group acting on a homology 3-sphere - necessarily non-freely - is the dodecahedral group $A_5 \cong \text{PSL}(2, 5)$ (in analogy, the only finite perfect group acting freely on a homology 3-sphere is the binary dodecahedral group $A_5^* \cong \text{SL}(2, 5)$). In the present paper we show that the only finite simple groups acting on a homology 4-sphere, and in particular on the 4-sphere, are the alternating or linear fractional groups groups $A_5 \cong \text{PSL}(2, 5)$ and $A_6 \cong \text{PSL}(2, 9)$. From this we deduce a short list of groups which contains all finite nonsolvable groups admitting an action on a homology 4-spheres.

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1. Introduction

We are interested in finite groups, and in particular in finite simple and nonsolvable groups admitting smooth orientation-preserving actions on integer homology spheres (in the present paper, simple group will always mean nonabelian simple group; also, all actions will be smooth and orientation-preserving). Let $G$ be a finite group acting on a homology $n$-sphere. If the action is free than $G$ has periodic cohomology, of period dividing $n+1$, and the groups of periodic cohomology have been classified by Zassenhaus and Suzuki; the only perfect groups among them are the linear groups $\text{SL}(2, p)$, for prime numbers $p \geq 5$. The cohomological period of $\text{SL}(2, p)$ is the least common multiple of 4 and $p-1$ (see [Sj],[Sw]), so the only finite perfect group acting freely on a homology 3-sphere is the binary dodecahedral group $A_5^* \cong \text{SL}(2, 5)$ (see also [Mn]).

In the present paper we are interested in arbitray, i.e. not necessarily free actions of finite nonsolvable groups on homology spheres. We note that a finite nonabelian simple group does not admit free actions on a homology sphere (any simple group has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ which does not act freely). It is shown in [Z1] that the only finite simple group acting on a homology 3-sphere is the dodecahedral group $A_5 \cong \text{PSL}(2, 5)$, and the finite nonsolvable groups acting on a homology 3-sphere are determined in [MeZ] and [Z2]
and are closely related to either $A_5$ or $A_5^*$. In the present paper, we determine the finite simple and nonsolvable groups which admit an orientation-preserving action on a homology 4-sphere (necessarily non-free by the Lefschetz fixed point theorem). Our first main result is the following.

1.1 Theorem The only finite nonabelian simple groups admitting an action on a homology 4-sphere, and in particular on the 4-sphere, are the alternating or linear fractional groups $A_5 \cong PSL(2,5)$ and $A_6 \cong PSL(2,9)$.

We note that the alternating group $A_6$ acts on the 5-simplex (permuting its vertices), and hence on its boundary homeomorphic to the 4-sphere.

Using Theorem 1.1 we will prove then the following.

1.2 Theorem Let $G$ be a finite non-solvable group acting orientation-preservingly on a homology 4-sphere. Then one of the following cases occurs:

a) $G$ contains a normal subgroup isomorphic to $(\mathbb{Z}_2)^4$, with factor group isomorphic to $A_5$ or the symmetric group $S_5$;

b) $G$ is isomorphic to $A_6$ or $S_6$;

c) $G$ contains, of index at most two, a subgroup isomorphic to one of the following groups:

- $A_5 \times C$ where $C$ is dihedral or cyclic;
- the central product $A_5^* \times_{\mathbb{Z}_2} A_5^*$;
- a central product $A_5^* \times_{\mathbb{Z}_2} C$ where $C$ is a solvable group that admits a free and orientation-preserving action on the 3-sphere.

We remark that the groups listed in part c) of the Theorem are close to the class of nonsolvable groups admitting an action already in dimension three, that is on a homology 3-sphere, see [MeZ] and [Z2] for lists of such groups. In contrast, the groups in a) and b) do not admit an action on a homology 3-sphere. The natural candidate for a group of type a) is obtained from the semidirect product $(\mathbb{Z}_2)^5 \rtimes S_5$ where $S_5$ acts on the normal subgroup $(\mathbb{Z}_2)^5$ by permuting the components (Weyl group or wreath product $\mathbb{Z}_2 \wr S_5$). The group $(\mathbb{Z}_2)^5 \rtimes S_5$ acts orthogonally on euclidean 5-space by inversion and permutation of coordinates, and the subgroup of index two of orientation-preserving elements is a semidirect product $(\mathbb{Z}_2)^4 \rtimes S_5$ which acts orthogonally on the 4-sphere. Concerning case b), the symmetric group $S_6$ acts on the 5-simplex by permuting its vertices, and on its boundary which is the 4-sphere; by composing the orientation-reversing elements with $-id_{S^4}$ one obtains an orientation-preserving action of $S_6$ on $S^4$. 

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We close the introduction with some remarks on the situation in higher dimensions. There is no finite perfect group acting freely on a homology 5-sphere (because none of the groups $\text{SL}(2,p)$ has period six: after $\text{SL}(2,5)$, of period four, the next cases are $\text{SL}(2,7)$ and $\text{SL}(2,13)$, of period 12). Considering the case of simple groups, besides $A_5$ and $A_6$ there are at least three other finite simple groups acting on a homology 5-sphere which are $A_7$, its subgroup $\text{PSL}(2,7)$, and the unitary group $U_4(2)$ (in the notation of [A]; this is a subgroup of index two in the Weyl or Coxeter group of type $E_6$ which has an 6-dimensional integral linear representation); all these groups act orthogonally on the 5-sphere. In dimension six, there are orthogonal actions on the 6-sphere of the simple groups $A_8$, PSL(2,8), PSL(2,13), $U_3(3)$ and of the symplectic group $S_6(2)$ (a subgroup of index two in the Weyl group of type $E_7$). We do not know if there are still other simple groups acting in these dimensions (but we suppose that the answer is no). This naturally leads to the following

**Problems**

i) What is the minimal dimension of a homology sphere on which PSL(2, $q$) acts? (or a metacyclic group $H(p : q)$, see Lemma 2.2; there should be some analogy here with the case of free actions of the groups SL(2, $p$) where a lower bound comes from the cohomological dimension)

ii) Show that in every dimension there are only finitely many groups PSL(2, $q$), and more generally only finitely many finite simple groups acting on an integer homology sphere.

iii) Show that every group PSL(2, $q$), with $q$ an odd prime power, acts on a $\mathbb{Z}_2$-homology 3-sphere (i.e., with coefficients in the integers mod two, see [MeZ],[Z3])

**2. Preliminary results**

The first step of the proof of Theorem 1.1 is the following

**2.1 Proposition** Let $G$ be a finite group acting orientation-preservingly on a homology 4-sphere.

a) If $G$ is a linear fractional group PSL(2, $q$) then $q \leq 5$ or $q = 9$.

b) If $G$ is a linear group SL(2, $q$) then $q \leq 5$.

We start with the following two Lemmas.

**2.2 Lemma** For a prime $p$ and an integer $q \geq 2$, let $H = H(p : q)$ be a metacyclic group, with normal subgroup $\mathbb{Z}_p$ and factor group $\mathbb{Z}_q$, acting orientation-preservingly on a homology $m$-sphere.

a) If $m = 3$ then any element of $\mathbb{Z}_q$ acts by ±identity on $\mathbb{Z}_p$.

b) If $m = 4$ then the square of any element of $\mathbb{Z}_q$ acts by ±identity on $\mathbb{Z}_p$.

**Proof.** a) Follows from [Z1,proof of Proposition 1].
b) By [Bo, chapter IV.4], the fixed point set of the normal subgroup \( Z_p \) of \( H \) is a homology sphere of even codimension, so the fixed point set is a 0-sphere or a 2-sphere.

Suppose first that the fixed point set of \( Z_p \) is a 2-sphere \( S^2 \). Then, locally in 4-dimensional space, \( Z_p \) acts as rotations around \( S^2 \); also, \( S^2 \) is invariant under the action of \( H \). Any element of \( H \) conjugates a rotation in \( Z_p \) of minimal angle around \( S^2 \) to a rotation of minimal angle, and hence induces \( \pm \) identity on \( Z_p \) (see [Br, chapter VI.2] for the existence of equivariant tubular neighbourhoods).

Suppose then that the fixed point set of \( Z_p \) is a 0-sphere \( S^0 \). Again \( S^0 \) is invariant under the action of \( H \), and a subgroup \( H_0 \) of index one or two in \( H \) fixes both points of \( S^0 \). The boundary of an \( H_0 \)-invariant neighbourhood of one of these two fixed points is homeomorphic to a 3-sphere, so \( H_0 \) acts on a homology 3-sphere. Now b) follows from a).

**2.3 Lemma** For a prime \( p \), let \( A \) be an elementary abelian \( p \)-group acting orientation-preservingly on a homology \( m \)-sphere.

a) If \( m = 3 \) then \( A \) has rank at most two if \( p > 2 \), and rank at most three if \( p = 2 \).

b) If \( m = 4 \) then \( A \) has rank at most two if \( p > 2 \), and rank at most four if \( p = 2 \).

**Proof.** a) See [MeZ, Proposition 4].

b) Consider a subgroup \( Z_p \) of \( A \). The fixed points set of \( Z_p \) is a 0-sphere \( S^0 \) or a 2-sphere \( S^2 \), invariant under the action of \( A \).

Suppose first that the fixed point set of \( Z_p \) is \( S^2 \). Any finite orientation-preserving group fixing \( S^2 \) pointwise acts as rotations around \( S^2 \) and hence is cyclic; this implies that the factor group \( A/Z_p \) acts faithfully on \( S^2 \). The elementary abelian \( p \)-groups acting on a 2-sphere are cyclic if \( p > 2 \), and of rank at most three if \( p = 2 \). Thus \( A \) is as stated in the Lemma.

Now suppose that the fixed point set of \( Z_p \) is a 0-sphere \( S^0 \). Then a subgroup \( A_0 \) of index one or two fixes both points of \( S^0 \), and \( A_0 \) maps a 3-sphere to itself which is the boundary of a regular invariant neighbourhood of one of the two fixed points. By a), the only elementary abelian group of rank \( \geq 3 \) acting orientation-preservingly on a homology 3-sphere is the group \((\mathbb{Z}_2)^3\) which implies the statement of the Lemma.

**Proof of Proposition 2.1**

a) Suppose that \( G = \text{PSL}(2, q) \) acts on a homology 3-sphere \( M \), for a prime power \( q = p^n \). We will show that \( q \leq 5 \) or \( q = 9 \).

Let \( A \cong (\mathbb{Z}_p)^n \) be the subgroup of all elements of \( G \) represented by upper triangular matrices of the form

\[
\begin{pmatrix}
1 & y \\
0 & 1
\end{pmatrix},
\]

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where \( y \in GF(q) \) (so \( A \) is isomorphic to the additive group of the finite Galois field \( GF(q) \)). For a generator \( x \) of the multiplicative group of the field \( GF(q) \), let \( B \) be the cyclic subgroup of \( G \) generated by the matrix

\[
\begin{pmatrix}
x & 0 \\
0 & x^{-1}
\end{pmatrix},
\]

of order \((q - 1)/2\) if \( p > 2 \), and of order \( q - 1 \) if \( p = 2 \). The cyclic group \( B \) normalizes \( A \), so the subgroup \( H \) generated by \( A \) and \( B \) is a semidirect product of these two groups (metacyclic if \( A \) is cyclic). It is easy to see that no nontrivial element of \( B \) operates trivially on \( A \) by conjugation, or in other words that the action of \( B \) on \( A \) is faithful.

Suppose first that \( p \) is an odd prime. For the subgroup \( PSL(2, p) \) of \( PSL(2, p^n) \), the subgroup \( A \) is cyclic of order \( p \) and \( B \) cyclic of order \((p - 1)/2\), so Lemma 2.2 implies \( p = 3 \) or \( p = 5 \). For \( G = PSL(2, p^n) \), we have \( A \cong (Z_p)^n \), and Lemma 2.3 implies \( n \leq 2 \). This leaves us with the two groups \( PSL(2, 9) \) (which admits an action on the 4-sphere) and \( PSL(2, 25) \).

The group \( PSL(2, 25) \) can be excluded by applying the Borel formula [Bo, Theorem XIII.2.3] to the subgroup \( A \cong (Z_5)^2 \). The Borel formula states that

\[
4 - r = \Sigma_H (n(H) - r)
\]

where the sum is taken over the six different subgroups \( H \cong Z_5 \) of \( A \), \( n(H) \) denotes the dimension of the fixed point set of a subgroup \( H \) and \( r \) the dimension of the fixed point set of \( A \cong (Z_5)^2 \) (equal to -1 if the fixed point set is empty). We note that all elements of order five in \( PSL(2, 25) \) are conjugate, so the formula becomes \( 4 + 5r = 6n(H) \) which admits no integer solution \( n(H) \) for \(-1 \leq r \leq 3 \) and hence excludes \( PSL(2, 25) \).

Now suppose that \( q = 2^n \) is a power of two. Then \( PSL(2, q) \) has a subgroup \( A \cong (Z_2)^n \) and Lemma 2.3 implies \( n \leq 4 \). The groups \( PSL(2, 2) \cong S_3 \) and \( PSL(2, 4) \cong A_5 \) act on the 4-sphere which leaves us with the groups \( PSL(2, 8) \) and \( PSL(2, 16) \).

We apply again the Borel formula to the subgroup \( A \cong (Z_2)^3 \) of \( PSL(2, 8) \) where the sum is now taken over the seven subgroups \( H \cong (Z_2)^2 \) of index two in \( A \). The subgroup \( A \cong (Z_2)^3 \) of \( PSL(2, 8) \) is normalized by a cyclic subgroup \( B \) of order seven which acts transitively on the seven involutions of \( A \); then \( B \) acts transitively also on the seven subgroups \( H \cong (Z_2)^2 \) of index two in \( A \), and the Borel formula becomes \( 4 + 6r = 7n(H) \); again this has no solution for \(-1 \leq r \leq 3 \), so \( PSL(2, 8) \) does not act.

Finally, we consider \( PSL(2, 16) \) and its subgroup \( A \cong (Z_2)^4 \). The group \( B \) is cyclic of order 15 now and acts transitively on the 15 involutions in \( A \) which hence are all conjugate. Then also all 15 subgroups \( H \cong (Z_2)^3 \) of index two in \( A \) are conjugate, and the Borel formula reads \( 4 + 14r = 15n(H) \) which again has no solution. This excludes also \( PSL(2, 16) \) and finishes the proof of part a) of the Proposition.
b) Suppose that $G = \text{SL}(2, q)$ acts on a homology 4-sphere $M$, for a prime power $q = p^n$. By [Bo, chapter IV.4], the central involution $z$ of $\text{SL}(2, q)$ has fixed point set $S^0$ or $S^2$. Suppose first that it has fixed point set $S^2$. Then $S^2$ is invariant under the action of $G$, and $G$ induces an action of $\text{SL}(2, q)/z \cong \text{PSL}(2, q)$ on $S^2$ which implies $q \leq 5$. Now suppose that $z$ has fixed point set $S^0$. Then $G$ fixes each of the two points in $S^0$ and acts on a regular neighbourhood of each of them which is a 3-sphere. By [Z2] or [MeZ], the only perfect group of type $\text{SL}(2, q)$ acting on a homology 3-sphere is the binary dodecahedral group $A_5^* \cong \text{SL}(2, 5)$, so $q \leq 5$.

This finishes the proof of Proposition 2.1. The proof of part b) generalizes to give the following Lemma which will be used in the proof of Theorem 1.2 (the 3-dimensional case follows from [Z2] or [MeZ]).

2.4 Lemma Let $G$ be a central extension, with nontrivial center, of a nonabelian simple group $\bar{G}$. If $G$ acts on a homology 3- or 4-sphere then $\bar{G}$ is isomorphic to the dodecahedral group $A_5 \cong \text{PSL}(2, 5)$.

3. Proof of Theorem 1.1

We will show that a finite simple group acting on a homology 4-sphere has sectional 2-rank at most four (i.e., every 2-subgroup is generated by at most four elements), and then apply the Gorenstein-Harada classification of such groups. Then, by suitable subgroup considerations using Proposition 2.1 and Lemmas 2.2 - 2.4, all groups of the Gorenstein-Harada list can be excluded except $A_5$ and $A_6$ (we note that this works also for the full list of the finite simple groups).

3.1 Proposition

Let $S$ be a finite 2-group acting orientation-preservingly on a $\mathbb{Z}_2$-homology 4-sphere $M$. Then $S$ is generated by at most four elements (i.e., has rank at most four).

Proof. Let $h$ be a central involution in $S$. By Smith fixed point theory ([Br]), the fixed point set of $\text{Fix}(h)$ of $h$ is a 2-sphere $S^2$ or a 0-sphere $S^0$ (since $h$ is orientation preserving, the codimension of the fixed point set is even); since $h$ is central, $\text{Fix}(h)$ is invariant under the action of $S$.

Suppose first that $\text{Fix}(h)$ is a 2-sphere $S^2$. Taking an equivariant tubular neighbourhood, the subgroup $F$ of $S$ which fixes $S^2$ pointwise acts orientation-preservingly on an orthogonal 1-sphere $S^1$ and hence is cyclic. The factor group $S/F$ acts effectively on $S^2$; such an action is conjugate to an orthogonal one, in particular $S/F$ is isomorphic to a finite subgroup of the orthogonal group $O(3)$. The finite subgroups of $O(3)$ are well-known (cyclic, dihedral, tetrahedral, octahedral, dodecahedral groups and 2-fold extensions of such groups); these groups have rank at most three, and hence $S$ has rank at most four.
Now suppose that Fix(h) is a 0-sphere $S^0$ (i.e., two points). By [DH] the finite 2-group $S$ admits also an orthogonal action on the 4-sphere $S^4$ such that the dimension of the fixed point set of any subgroup of $S$ coincides for the actions of $S$ on $M$ and $S^4$ (the two actions have the same dimension function for the fixed point set of subgroups of $S$).

We consider such an orthogonal action of $S$ on $S^4$. The involution $h$ fixes a 0-sphere $S^0$ which consists of two antipodal points of $S^4$, and we denote $S^3$ the corresponding equatorial 3-sphere between these two antipodal points. Note that with $S^0$ also $S^3$ is invariant under the action of $S$, and hence $S$ is isomorphic to a subgroup of the orthogonal group O(4) of $S^3$. A list of the finite subgroups of O(4) can be found in [DV]. One way to proceed now is to identify the finite 2-groups among these groups and prove one by one that they are generated by at most four elements. Since the lists in [DV] are rather technical and a proof of completeness is basically not given, we prefer to present a direct argument avoiding such a list.

We can assume that $S$ is a finite subgroup of the orthogonal group O(4); we will prove that $S$ has rank at most four.

If every abelian normal subgroup of $S$ is cyclic then, by [Su2,4.4.3], $S$ is a cyclic, dihedral, quaternion or quasi-dihedral 2-group; since each of these groups has rank at most two, this finishes the proof in this case.

We can assume then that $S$ has a non-cyclic abelian normal subgroup. By [Su2,4.4.5], $S$ has also a normal subgroup $U = \mathbb{Z}_2 \times \mathbb{Z}_2$. The group $S$ acts by conjugation on the three involutions of $U$; since $S$ is a finite 2-group, each involution in $U$ is fixed by a subgroup of index one or two in $S$. If some involution in $U$ has fixed point set $S^2$ then $S^2$ is invariant under $S$ or a subgroup of index two; since every finite 2-subgroup of O(3) has rank at most three, $S$ has rank at most four. Similarly, if some involution in $U$ has fixed point set $S^0$ (two antipodal points of $S^3$) then again $S$ or a subgroup of index two leaves invariant $S^0$ and the corresponding equatorial 2-sphere $S^2$, so again $S$ has rank at most four.

Since $U = \mathbb{Z}_2 \times \mathbb{Z}_2$ does not act freely on $S^3$, we can assume then that some involution $u$ in $U$ has fixed point set $S^1$. If $u$ is central in $S$ then $F$ is invariant under $S$ and it is again easy to see that $S$ has rank at most four. Now the remaining case is the following: two involutions $u_1$ and $u_2$ in $U$ have fixed point set $F_1 \cong S^1$ and $F_2 \cong S^1$ and are exchanged by some element of $S$, their product is central in $S$ and acts freely. Note that $F_1$ and $F_2$ are great circles in $S^3$; decomposing $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$, these are obtained by intersecting $S^3$ with the two orthogonal planes $\mathbb{R}^2$ of such a decomposition. Denoting by $D_n$ a rotation of $\mathbb{R}^2$ of order $n$, one has $u_1 = (D_1, D_2)$ and $u_2 = (D_2, D_1)$.

Let $S_0 = S \cap \text{SO}(4)$ be the orientation-preserving subgroup, of index one or two in $S$, with $U \subset S_0$. An element of $S$ either exchanges $F_1$ and $F_2$, or leaves invariant both $F_1$ and $F_2$; if an element of $S_0$ maps $F_1$ to itself then it acts as a rotation around $F_1$ (fixing $F_1$ pointwise), or as a rotation along $F_1$, or as strong inversion of $F_1$. Let $A$ be
the subgroup of elements of $S_0$ which are rotations of $F_1$. Then $A$ contains exactly the three involutions $u_1, u_2$ and $u_1u_2$. Every element of $A$ acts as a rotation also on $F_2$, and a strong inversion of $F_1$ is also a strong inversion of $F_2$. Note that $A$ is normal in $S$ and a direct product of two nontrivial cyclic groups. If there are no strong inversions or no elements in $S_0$ which exchange $F_1$ and $F_2$ then $S_0$ has rank at most three and we are done. So we can assume that there is a strong inversion $r$ of $F_1$, and also an element $s$ in $S_0$ exchanging $F_1$ and $F_2$. Note that $r$ and $s$ together with $A$ generate $S_0$. If $S = S_0$ then $S$ has rank at most four and we are done, so we can assume that there is also an orientation-reversing element $t$ in $S$. By eventually composing $t$ with $s$, we can assume that $t$ leaves invariant both $F_1$ and $F_2$. By eventually composing $t$ with $r$, we can assume that $t = (D, R)$ where $D$ denotes a rotation and $S$ a reflection of $\mathbb{R}^2$.

For powers $x, y$ and $z$ of two, we denote by $h = (D_1, D_x)$ a rotation of maximal order (or minimal angle) around $F_1$ (i.e., with fixed point set $F_1$), and by $k = (D_y, D_z)$ a rotation of minimal translation length along $F_1$ (so $k^y$ is the minimal power of $k$ which fixes $F_1$ pointwise). Note that $h$ and $k$ generate $A$. If $x \geq z$ then we can choose $k = (D_y, D_1)$; since $s$ exchanges $F_1$ and $F_2$ this implies $x = y$. In this case $s$ and $h$ generate $k$, and $h, r, s$ and $t$ generate $S$.

So we can assume that $x < z$. Since $A$ is normal in $S$, we have $tkt^{-1} = (D_y, D_z^{-1}) = k(D_1, D_z^{-2}) = kh^q$, for a primitive power $h^q$ of $h$. Then $t$ and $k$ generate $h$, and $k, r, s$ and $t$ generate $S$.

This finishes the proof of Proposition 3.1.

**Proof of Theorem 1.1.** We apply the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four (see [G,p.6] or [Su2, chapter 6, Theorem 8.12]). By Proposition 3.1, $G$ has sectional 2-rank at most four and hence is one of the groups in the Gorenstein-Harada list; the groups are the following:

- $PSL(n, q), PSU(n, q)$ ($n \leq 5, q$ odd),
- $G_2(q), 3D_4(q), PSp(4, q)$ ($q$ odd), $2G_2(3^{2m+1})$,
- $PSL(2, 8), PSL(2, 16), PSL(3, 4), PSU(3, 4), Sz(8)$,
- $A_l$ ($7 \leq l \leq 11$), $M_i$ ($i \leq 23$), $J_i$ ($i \leq 3$), $Mc, Ly$

The alternating group $A_7$ has the linear fractional group $PSL(2, 7)$ (which has a subgroup $S_4$ of index seven) as a subgroup, so by Proposition 2.1 it does not act on a homology 4-sphere. This excludes $A_n$ for $n \geq 7$. Also, the sporadic groups $M_i, J_i, Mc$ and $Ly$ in the Gorenstein-Harada list have metacyclic subgroups $H$ excluded by Lemma 2.2 or linear fractional subgroups $PSL(2, q)$ excluded by Proposition 2.1, so they do not
act on a homology 4-sphere (see [A] for information about the maximal subgroups of the sporadic groups).

All other groups in the list except PSL(2, 5) and PSL(2, 9) can be excluded along similar lines. For example, concerning the linear groups $L_m(q) = PSL(m, q)$, $q = p^n$, we note that $L_2(p)$ is a subgroup of $L_2(q)$; also, for $m > r$, the linear group $SL(r, q)$ is a subgroup of the linear fractional group $L_m(q) = PSL(m, q)$ (see also [Su2, chapter 6.5] where the centralizers of involutions in the classical groups are determined). Applying Proposition 2.1 and Lemma 2.4, it suffices then to exclude the groups $L_3(2)$, $L_3(3)$ and $L_3(5)$. But $L_3(2)$ is isomorphic to $L_2(7)$, the group $L_3(3)$ has a metacyclic subgroup $H(13:3)$ excluded by Lemma 2.2 ([A] or [Su2, p.530]), and $L_3(5)$ a metacyclic subgroup $H(31:3)$. Thus among the linear fractional groups there remain only $L_2(5) \cong A_5$ and $L_2(9) \cong A_6$.

We will not repeat the arguments for the other groups: the most interesting remaining case is the Suzuki group $Sz(8)$ which has one conjugacy class of involutions and a subgroup $(Z_2)^3$; applying the Borel formula to this subgroup (similar as for $PSL(2, 8)$ in the proof of Proposition 2.1) shows that $Sz(8)$ does not act on a homology 4-sphere.

This finishes the proof of Theorem 1.1 (we remark that in fact all finite simple groups except $A_5$ and $A_6$ can be excluded in the same way).

4. Proof of Theorem 1.2

In the proof of Theorem 1.2 we need some extra informations about the elementary abelian groups acting on homology 4-spheres; we summarize them in the following lemma. We recall that the fixed point set of a group of prime order $Z_p$, that acts orientation-preservingly on a homology 4-sphere, is a homology sphere of dimension zero or two ([Bo, chapter IV.4]). To obtain Lemma 4.1, we use again the Borel Formula but in a more technical way.

4.1 Lemma For a prime $p$, let $A$ be an elementary abelian $p$-group acting orientation-preservingly on a homology 4-sphere.

a) If $A$ has rank two and $p$ is odd, then $A$ contains exactly two cyclic subgroups with 0-dimensional fixed point set.

b) If $A$ has rank two and $p = 2$, then $A$ contains at least one involution with 2-dimensional fixed point set; in the case of three involutions with 2-dimensional fixed point set the group $A$ has a fixed-point set of dimension one.

c) If $A$ has rank three and $p = 2$, then $A$ can contain either one or three involutions with 0-dimensional fixed point set.

d) If $A$ has rank four and $p = 2$, then $A$ contains exactly five involutions with 0-dimensional fixed-point set.
Proof. For an abelian $p$-group $A$ acting on a homology 4-sphere the Borel Formula appears as follow:

$$4 - r = \Sigma_{H}(n(H) - r).$$

The sum is taken over the subgroups $H$ of index $p$ in $A$, $n(H)$ denotes the dimension of the fixed point set of $H$ and $r$ the dimension of the fixed point set of $A$ (equal to -1 if the fixed point set is empty).

If $A$ has rank two we have $p + 1$ cyclic subgroups of index $p$; if $H$ is cyclic then either $n(H) = 0$ or $n(H) = 2$. Considering all the possibilities we obtain the situations described in points a) and b). In particular for the 2-groups we have: if $r = 1$ then $A$ contains three involutions with 2-dimensional fixed point set, if $r = 0$ then $A$ contains two involutions with 2-dimensional fixed point set, if $r = -1$ then $A$ contains one involution with 2-dimensional fixed point set (the case $r = 2$ can not occur). We remark that the dimension of the global fixed point set of the group determines the fixed point sets of the involutions contained in the group.

So we apply the formula to $A = (\mathbb{Z}_2)^3$; in this case we have seven subgroups of index two that are elementary abelian 2-groups of rank two. A priori $r$ can be equal to 1, 0 or $-1$ and the possibilities for $n(H)$ are given by the previous case. We explain the situation for $r = 0$; in this case we obtain from the formula that we can have either four subgroups with 1-dimensional fixed point set and three subgroups with 0-dimensional fixed point set or five subgroups with 1-dimensional fixed point set, one subgroup with 0-dimensional fixed point set and one subgroup with empty fixed point set (it is possible to see also that the second case can not occur but it is not necessary for the proof).

The fixed point set of a subgroup of index two determines the fixed point sets of the involutions contained in the subgroup; since any involution is contained in three different subgroups of index two, we can compute that in both cases we have six involutions with 2-dimensional fixed point set and one involution with 0-dimensional fixed point set. Analogously we can analyze the case $r = -1$; the case $r = 1$ can not occur.

For the case of elementary abelian group of rank four we can repeat a similar computation referring to the results for elementary abelian group of rank three; this concludes the proof.

Recall that a finite $Q$ group is quasisimple if it is perfect (the abelianized group is trivial) and the factor group of $Q$ by its center is a non-abelian simple group. A finite group $E$ is semisimple if it is perfect and the factor group of $E$ by its center is a direct product of simple non-abelian groups (see [Su2, chapter 6.6]). A semisimple group is a central product of quasisimple groups that are uniquely determined. Any finite group $G$ contains a unique maximal semisimple normal group $E(G)$ (the subgroup $E(G)$ may be trivial); the subgroup $E(G)$ is characteristic in $G$ and the quasisimple factors of $E(G)$ are called the components of $G$. To prove the Theorem 1.2 we consider first the case of groups with trivial maximal normal semisimple subgroup.

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4.2 Lemma Let $G$ be a group acting orientation-preservingly on a homology $m$-sphere such that $G$ has trivial maximal normal semisimple subgroup $E(G)$.

a) If $m = 3$ then $G$ is solvable.

b) If $m = 4$ either $G$ is solvable or $G$ contains a normal subgroup isomorphic to $(\mathbb{Z}_2)^4$ with factor group isomorphic to $A_5$ or $S_5$.

Proof. a) This is [MeZ, Proposition 8].

b) We consider first the case of $G$ containing a normal non-trivial cyclic subgroup $H$ and we prove that in this case if $E(G)$ is trivial then $G$ is solvable. We can suppose that $H$ has prime order $p$ so each element of $H$ has the same fixed point-set; since $G$ normalizes $H$ then $G$ fixes setwise the fixed point set of $H$.

If the fixed point set of $H$ is $S^0$ there exists a subgroup $G_0$ of index at most two in $G$ such that $G_0$ fixes both points of $S^0$; the subgroup $G_0$ acts faithfully on a 3-sphere, that is the boundary of a regular invariant neighbourhood of one of the two fixed points. Since $E(G_0)$ is trivial we obtain by a) that $G_0$, and consequently $G$, are solvable.

If the fixed point set is $S^2$ we consider in $G$ the normal subgroup $K$ of elements fixing pointwise $S^2$; the subgroup $K$ contains $H$ and since $K$ acts locally as a rotation around $S^2$ then $K$ is cyclic. The factor group $G/K$ acts faithfully on $S^2$. If $G/K$ is solvable, we get the thesis; otherwise we can suppose that $G/K$ is isomorphic either to $A_5$ or to $\mathbb{Z}_2 \times A_5$ because these are the only non-solvable finite groups acting on $S^2$ (the action on $S^2$ is not necessarily orientation-preserving). In both cases the action of $A_5$ by conjugation on $K$ is trivial because $K$ is cyclic and its automorphism group is abelian; then $G$ contains with index at most two $G_0$ a subgroup that is a central extension of $A_5$. The derived group $G'_0$ is a quasisimple normal subgroup of $G_0$ (see [Su1, Theorem 9.18, pag.257]) and this fact implies that $E(G)$ is not trivial in contradiction with our hypothesis.

The proof of this particular case is now complete and in the following we can use this fact.

Fact: if a subgroup $N$ of $G$ contains a non-trivial cyclic normal subgroup then either $N$ is solvable or $E(N)$ is not trivial.

We consider now the general case. We denote by $F$ the Fitting subgroup of $G$ (the maximal nilpotent normal subgroup of $G$). Since $E(G)$ is trivial, the Fitting subgroup $F$ coincides with the generalized Fitting subgroup that is the product of the Fitting subgroup with the maximal semisimple normal subgroup. The generalized Fitting subgroup $F$ contains its centralizer in $G$ and $F$ is not trivial ([Su2, Theorem 6.11, pag.452]). Since $F$ is nilpotent it is the direct product of its Sylow $p$-subgroups. In particular any Sylow subgroup of $F$ is normal in $G$; since $F$ is not trivial we have $P$ a non-trivial $p$-subgroup normal in $G$. We consider $Z$ the maximal elementary abelian $p$-subgroup contained in the center of $P$; this subgroup is not trivial and it is normal in $G$.

Suppose first that we can chose $p$ odd (the order of $F$ is not a power of two); then, by Lemma 2.3, $Z$ has rank one or two. If $Z$ is cyclic, by the first part of the proof, $G$ is
solvable. If $Z$ has rank two by Lemma 4.1 it contains exactly two cyclic subgroups $H$ and $H'$ with 0-dimensional fixed point set; $G$ acts by conjugation on the set $\{H, H'\}$ and $N_G H$ has index at most two in $G$. Since $E(G)$ is trivial then the maximal normal semisimple subgroup of $N_G(H)$ is trivial; we obtain that $N_G(H)$, and consequently $G$, are solvable. This concludes this case.

Suppose now that the order of $F$ is a power of two; in this case $F = P$ is a 2-group and $Z$ is an elementary abelian 2-group of rank at most four (by Lemma 2.3).

If $Z$ has rank one by the first part of the proof $G$ is solvable.

If $Z$ has rank two we consider $C_G(Z)$ the centralizer of $Z$ in $G$ that is normal because $Z$ is normal; $C_G(Z)$ contains a non-trivial normal cyclic subgroup and it is solvable. The factor $G/C_G(Z)$ is isomorphic to a subgroup of $GL(2,2)$, the automorphism group of an elementary abelian 2-group of rank two; since $GL(2,2)$ is a solvable group we can conclude that $G$ is solvable.

Suppose that $Z$ has rank three. In this case the factor group $G/C_G(Z)$ is isomorphic to a subgroup of $GL(2,3)$, the automorphism group of an elementary abelian 2-group of rank three; $GL(2,3)$ has order $2^3 \cdot 3 \cdot 7$ and any element of order seven permutes cyclically all the involutions of $(\mathbb{Z}_2)^3$. The group $G/C_G(Z)$ can not contain element of order 7 otherwise all involutions in $Z$ are conjugated and this is impossible by Lemma 4.1; so the group $G/C_G(Z)$ has order $2^a3^b$ and it is solvable. This fact implies that $G$ is solvable.

It remains the case $Z$ of rank four; by Lemma 4.1 the group $Z$ contains at least one involution with fixed point set $S^0$. The group $P$ fixes setwise $S^0$ and we have a subgroup $P_0$ of index at most two that fixes both the points in $S^0$. The center of $P_0$ contains an elementary abelian group of rank at least three (the group $Z \cap P_0$) and $P_0$ acts faithfully on a 3-sphere that is the boundary of a regular invariant neighbourhood of one of the two fixed points; the only 2-group acting on the 3-sphere with this property is $(\mathbb{Z}_2)^3$ (see [MeZ, Proposition 2 and Proposition 3]). In this case the generalized Fitting subgroup $F = P$ is an elementary abelian group of rank four, we recall that $F$ contains its centralizer in $G$. By Lemma 4.1 in $F$ are contained exactly five involutions with 0-dimensional fixed point set; they generate the group $F$ because by Lemma 4.1 the subgroups of index two contain at most three of such involutions. We can conclude that $G/F$ acts faithfully on the set of the five involutions of $F$ with 0-dimensional fixed point set and $G/F$ is isomorphic to a subgroup of $S_5$. In particular if $G$ is not solvable we obtain that $G/F$ is isomorphic either to $A_5$ or to $S_5$ (the only non-solvable subgroups of $S_5$). This fact concludes the proof.

Now we consider the case of semisimple group.

4.3 Lemma Let $G$ be a finite semisimple group acting orientation-preservingly on a homology 4-sphere, then $G$ is isomorphic to one of the following group:

$$A_5, A_6, A_5^*, A_5^* \times \mathbb{Z}_2, A_5^*.$$

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Proof. By Theorem 1.1 and Lemma 2.4, if $G$ is quasisimple, then $G$ is isomorphic to $A_5$, $A_6$ or to $A_5^* \cong \text{SL}(2,5)$ that is the unique perfect central extension of $A_5$.

We consider now the case of $G$ with two quasisimple components; since in our list of quasisimple groups $A_5^*$ is the unique group with non-trivial center, then either $G \cong A_5^* \times \mathbb{Z}_2 A_5^*$ or $G$ is the direct product of two quasisimple subgroups. We have to exclude this second possibility thus we consider $G \cong Q \times Q'$ with $Q$ and $Q'$ isomorphic to $A_5$, $A_6$ or $A_6^*$.

Suppose first that one of the two components, say $Q$, is isomorphic to $A_6$. Let $f$ be a non-trivial element of $Q'$, since $Q$ commutes elementwise with $f$, the subgroup $Q$ fixes setwise the fixed point set of $f$.

If the fixed point set of $f$ is a 0-sphere there exists a subgroup of $Q$ of index at most two that acts faithfully on a 3-sphere that is the boundary of a regular invariant neighbourhood of one of the two fixed points. The group $A_6$ has no subgroup of index two and $A_6$ can not act faithfully on a homology 3-sphere (see [MeZ, Theorem 2]).

If the fixed point set of $f$ is a 2-sphere there exists a factor group of $Q$ by a cyclic subgroup that acts faithfully on a 2-sphere. Since $A_6$ is simple and $A_6$ can not act on $S^2$ also this case can not occur.

Suppose now that one component is isomorphic to $A_5^*$, in this case the center of $G$ is not trivial because it contains the involution that is in the center of $A_5^*$. The same argument used for $A_6$ applies to $G$ and we can exclude $A_5^*$ as component.

The only case that remains is $G \cong A_5 \times A_5$. We denote by $A$ the Sylow 2-subgroup of the first component. The subgroup $A$ is elementary abelian of rank two and the three involutions $\{t_1, t_2, t_3\}$ in $A$ are all conjugated; by Lemma 4.1 the only possibilities is that the fixed point set of each involution in $A$ is a 2-sphere. We consider the group generated by $t_1$ and by the second component; this group is isomorphic to $\mathbb{Z}_2 \times A_5$ and acts faithfully on the 2-sphere that is the fixed point set of $t_2$. Since $t_1$ commutes with $A_5$, the action of $t_1$ on the fixed point set of $t_2$ is free in contradiction with Lemma 4.1.

So we have that, if $G$ has two components, $G$ is isomorphic to $A_5^* \times \mathbb{Z}_2 A_5^*$.

Finally we exclude the possibility to have more then two components. By the previous cases we can argue that all the components are isomorphic to $A_5^*$, but in this case the center of $G$ is not trivial and referring again either to three dimensional case ([MeZ, Theorem 2]) or to two dimensional case we can exclude these groups.

Proof of Theorem 1.2 If the maximal semisimple normal subgroup $E$ of $G$ is trivial we apply Lemma 4.2 and we obtain that either $G$ is solvable or we are in case a).

Suppose then that $E$ is not trivial, then $E$ is one of the groups presented in Lemma 4.1. Since $E$ is normal, its centralizer $C = C_G(E)$ in $G$ is also normal. We denote by $\tilde{E}$ the subgroup generated by $C$ and $E$; by definition $\tilde{E}$ is a normal subgroup of $G$ that is a central product of $C$ and $E$. The factor group $G/\tilde{E}$ is a subgroup of the outer automorphism group $\text{Aut}(E)/\text{Inn}(E)$ of $E$. 

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The maximal semisimple normal subgroup of $C$ is trivial otherwise $E$ is not maximal in $G$. So by Lemma 4.2 either $C$ is solvable or $C$ contains an elementary normal 2-subgroup of rank four and $C$ has trivial center; the second case can not occur otherwise $\tilde{E}$ would contain an elementary abelian group of rank five in contradiction with Lemma 2.3.

We consider first the case $E \cong A_6$; the same arguments used in Lemma 4.3 to exclude $A_6 \times Q$ apply and we obtain that $C$ is trivial. The outer automorphism group of $A_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [Su1,p.300]), so there are three 2-fold extensions of $A_6 \cong \text{PSL}(2,9)$ corresponding to the three subgroups $\mathbb{Z}_2$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$. One of these extensions is $S_6$ which acts orientation-preservingly and orthogonally on the 4-sphere. Another 2-fold extension is $\text{PGL}(2,9)$. The Sylow 3-subgroup of $\text{PGL}(2,9)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. In $\text{PGL}(2,9)$ all elements of order three are conjugate so their fixed point sets have the same dimension but this is impossible by Lemma 4.1. Hence $\text{PGL}(2,9)$ does not act on a homology 4-sphere. A similar argument works for the third 2-fold extension of $\text{PSL}(2,9)$ (the Matthieu group $M_{10}$, see [A]) in which also all elements of order three are conjugate. Now also the $\mathbb{Z}_2 \times \mathbb{Z}_2$-extension of $\text{PSL}(2,9)$ does not act, and we are left with the two groups $A_6$ and $S_6$ in case b) of Theorem 2.1.

Suppose now that $E \cong A_5^* \times \mathbb{Z}_2 A_5^*$. We denote by $t$ the central involution in $E$ that is central in $G$ because $E$ is characteristic in $G$. The fixed point set of $t$ is not a 2-sphere otherwise a factor of $A_5^* \times \mathbb{Z}_2 A_5^*$ by a cyclic group would act faithfully on the 2-sphere. The fixed point set of $t$ is $S^0$ and we get $G_0$ a subgroup of $G$ of index at most two acting faithfully and orientation-preservingly on a 3-sphere, that is the boundary of a regular invariant neighbourhood of one of the two fixed points. Since $A_5^* \times \mathbb{Z}_2 A_5^*$ is maximal between the groups acting on the 3-sphere (see [MeZ, Theorem 2]) we obtain that $G_0 \cong A_5^* \times \mathbb{Z}_2 A_5^*$.

Next we consider the case of $E \cong A_5$. A Sylow 2-subgroup $S$ of $E$ is an elementary abelian group of rank two and the three involutions $\{t_1, t_2, t_3\}$ in $S$ are all conjugated; by Lemma 4.1 we have that all the three involutions have 2-dimensional fixed point set. We consider $S^2$ the fixed point set of $t_1$, the action of $t_2$ on $S^2$ is orientation reversing and then $C$ acts orientation-preservingly and faithfully on $S^2$. Suppose $C$ isomorphic to $A_5$, $S_4$ or $A_4$, since $t_2$ commutes with $C$, then $t_2$ has to act freely on $S^2$ in contradiction with Lemma 4.1. We can conclude that the group $C$ can be either dihedral or cyclic. The outer automorphism group of $A_5$ has order two and the factor group $G/\tilde{E}$ has order at most two.

Suppose finally that $E \cong A_5^*$; we denote by $t$ the central involution in $E$, $t$ is central also in $G$ since $E$ is characteristic. Suppose that the fixed point set of $t$ is $S^0$, a 0-sphere. In this case we get $G_0$ a subgroup of $G$ with index at most two, that acts faithfully and orientation-preservingly on a 3-sphere. By [MeZ, Theorem 2] we can conclude that $G_0$ is isomorphic to $G_0 \cong A_5^* \times \mathbb{Z}_2 C$ where $C$ is a solvable group acting freely on a 3-sphere (for further details see also [Z2]).
If the fixed point set of $t$ is $S^2$ we denote by $K$ the cyclic group of elements that fix pointwise $S^2$. The factor $G/K$ acts faithfully on $S^2$, then $G/K$ contains with index at most two $A_5$: since $K$ is cyclic the action by conjugation of $A_5$ is trivial on $K$. In this case $G$ contains a subgroup of index at most two isomorphic to $A_5 \times \mathbb{Z}_2 \mathbb{Z}_2$. Any cyclic group admits a free and orientation preserving action on the 3-sphere (but in this case the action is not naturally related to the action on the homology 4-sphere).

We have considered all the possible cases for $E$ and the proof is finished.

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