Adiabatic quantization of Andreev levels

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We identify the time $T$ between Andreev reflections as a classical adiabatic invariant in a ballistic chaotic cavity (Lyapunov exponent $\lambda$), coupled to a superconductor by an $N$-mode constriction. Quantization of the adiabatically invariant torus in phase space gives a discrete set of periods $T_n$, which in turn generate a ladder of excited states $\epsilon_{nm} = (m+1/2)\pi\hbar/T_n$. The largest quantized period is the Ehrenfest time $T_0 = \lambda^{-1}\ln N$. Projection of the invariant torus onto the coordinate plane shows that the wave functions inside the cavity are squeezed to a transverse dimension $W/\sqrt{N}$, much below the width $W$ of the constriction.

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The notion that quantized energy levels may be associated with classical adiabatic invariants goes back to Ehrenfest and the birth of quantum mechanics. It was successful in providing a semiclassical quantization scheme for special integrable dynamical systems, but failed to describe the generic nonintegrable case. Adiabatic invariants play an interesting but minor role in the quantization of chaotic systems.

Since the existence of an adiabatic invariant is the exception rather than the rule, the emergence of a new one quite often teaches us something useful about the system. An example from condensed matter physics is the sorption of a Cooper pair by the superconductor. At the Fermi energy $E_F$ the dynamics of the point contact is increased. An electron trajectory is retraced by the hole that is produced upon absorption of a Cooper pair by the superconductor. At the Fermi energy $E_F$ the dynamics of the hole is precisely the time reverse of the electron dynamics, so that the motion is strictly periodic. The period from electron to hole and back to electron is twice the time $T$ between Andreev reflections. For finite excitation energy $\epsilon$ the electron (at energy $E_F + \epsilon$) and the hole (at energy $E_F - \epsilon$) follow slightly different trajectories, so the orbit does not quite close and drifts around in phase space. This drift has been studied in a variety of contexts, but not in connection with adiabatic invariants and the associated quantization conditions. It is the purpose of this paper to make that connection and point out a striking physical consequence: The wave functions of Andreev levels fill the cavity in a highly nonuniform “squeezed” way, which has no counterpart in normal state chaotic or regular billiards. In particular the squeezing is distinct from periodic orbit scarring and entirely different from the random superposition of plane waves expected for a fully chaotic billiard.

Adiabatic quantization breaks down near the excitation gap, and we will argue that random-matrix theory can be used to quantize the lowest-lying excitations above the gap. This will lead us to a formula for the gap that crosses over from the Thouless energy to the inverse Ehrenfest time as the number of modes in the point contact is increased.

To illustrate the problem we represent in Figs. 1 and 2 the quasiperiodic motion in a particular Andreev billiard. (It is similar to a Sinai billiard, but has a smooth potential $V$ in the interior to favor adiabaticity.) Fig. 1 shows a trajectory in real space while Fig. 2 is a section of phase space at the interface with the superconductor ($y = 0$). The tangential component $p_x$ of the electron momentum is plotted as a function of the coordinate $x$ along the interface. Each point in this Poincaré map corresponds to one collision of an electron with the interface. (The collisions of holes are not plotted.) The electron is retroreflected as a hole with the same $p_x$. At $\epsilon = 0$ the component $p_y$ is also the same, and so the hole retraces the path of the electron (the hole velocity being opposite to its momentum). At non-zero $\epsilon$ the retroreflection occurs with a slight change in $p_y$, because of the difference $2\epsilon$ in the kinetic energy of electrons and holes. The resulting slow drift of the periodic trajectory traces out a contour in the surface of section. The adiabatic invariant is the function of $x, p_x$ that is constant on the contour. We have found numerically that the drift follows isochronous contours $C_T$ of constant time $T(x, p_x)$ between Andreev reflections. Let us now demonstrate analytically that $T$ is an adiabatic invariant.

We consider the Poincaré map $C_T \to C(\epsilon, T)$ at energy $\epsilon$. If $\epsilon = 0$ the Poincaré map is the identity, so $C(0, T) = C_T$. For adiabatic invariance we need to prove that $\lim_{\epsilon \to 0} dC/\epsilon = 0$, so that the difference between $C(\epsilon, T)$ and $C_T$ is of higher order than $\epsilon$. Since the contour $C(\epsilon, T)$ can be locally represented by a function $p_x(x, \epsilon)$, we need to prove that $\lim_{\epsilon \to 0} \partial p_x(x, \epsilon)/\epsilon = 0$.

In order to prove this, it is convenient to decompose the map $C_T \to C(\epsilon, T)$ into three separate stages, starting out as an electron (from $C_T$ to $C_+$), followed by
Andreev reflection ($C_+ \rightarrow C_-$), and then concluded as a hole [from $C_- \rightarrow C(\varepsilon, T)$]. Andreev reflection introduces a discontinuity in $p_y$ but leaves $p_x$ unchanged, so $C_+ = C_-$. The flow in phase space as electron (+) or hole (−) at energy $\varepsilon$ is described by the action $S_\pm(q, \varepsilon)$, such that $p^\pm = (p_x, p_y)$ on position $\mathbf{q} = (x, y)$. The derivative $\partial S_\pm / \partial \varepsilon = t_\pm(q, \varepsilon)$ is the time elapsed since the previous Andreev reflection. Since by construction $t_\pm(x, y = 0, \varepsilon = 0) = T$ is independent of the position $x$ of the end of the trajectory, we find that $\lim_{\varepsilon \rightarrow 0} \partial p_x^\pm(x, y = 0, \varepsilon) / \partial \varepsilon = 0$, completing the proof.

The drift $(\delta x, \delta p_x)$ of a point in the Poincaré map is perpendicular to the vector $(\partial T / \partial x, \partial T / \partial p_x)$. Using also that the map is area preserving, it follows that

$$ (\delta x, \delta p_x) = \varepsilon f(T)(\partial T / \partial p_x, -\partial T / \partial x) + O(\varepsilon^2), \quad (1) $$

with a prefactor $f(T)$ that is the same along the entire contour.

The adiabatic invariance of isochronous contours may alternatively be obtained from the adiabatic invariance of the action integral $I$ over the quasiperiodic motion from electron to hole and back to electron:

$$ I = \int p\,dq = \varepsilon \int \frac{dq}{q} = 2\varepsilon T. \quad (2) $$

Since $\varepsilon$ is a constant of the motion, adiabatic invariance of $I$ implies adiabatic invariance of the time $T$ between Andreev reflections. This is the way in which adiabatic invariance is usually proven in text books. Our proof explicitly takes into account the fact that phase space in the Andreev billiard consists of two sheets, joined in the constriction at the interface with the superconductor, with a discontinuity in the action on going from one sheet to the other.

The contours of large $T$ enclose a very small area. This will play a crucial role when we quantize the billiard, so let us estimate the area. It is convenient for this estimate to measure $p_x$ and $x$ in units of the Fermi momentum $p_F$ and width $W$ of the constriction to the superconductor. The highly elongated shape evident in Fig. 2 is a consequence of the exponential divergence in time of nearby trajectories, characteristic of chaotic dynamics. The rate of divergence is the Lyapunov exponent $\lambda$. (We consider a fully chaotic phase space.) Since the Hamiltonian flow is area preserving, a stretching $\ell_+(t) = \ell_+(0)e^\lambda t$ of the dimension in one direction needs to be compensated by a squeezing $\ell_-(t) = \ell_-(0)e^{-\lambda t}$ of the dimension in the other direction. The area $A \approx \ell_+ \ell_-$ is then time independent. Initially, $\ell_+(0) < 1$. The constriction at the superconductor acts as a bottleneck, enforcing $\ell_+(T) < 1$. These two inequalities imply $\ell_+(t) < e^{\lambda(t-T)}$, $\ell_-(t) < e^{-\lambda t}$. 

FIG. 1: Classical trajectory in an Andreev billiard. Particles in a two-dimensional electron gas are deflected by the potential $V = [1 - (r/L)^2]V_0$ for $r < L$, $V = 0$ for $r > L$. (The dotted circles are equipotentials.) There is specular reflection at the boundaries with an insulator (thick solid lines) and Andreev reflection at the boundary with a superconductor (dashed line). The trajectory follows the motion between two Andreev reflections of an electron near the Fermi energy $E_F = 0.84 V_0$. The Andreev reflected hole retraces this trajectory in opposite direction.

FIG. 2: Poincaré map for the Andreev billiard of Fig. 1. Each dot represents a starting point of an electron trajectory, at position $x$ (in units of $L$) along the interface $y = 0$ and with tangential momentum $p_x$ (in units of $\sqrt{mV_0}$). The inset shows the full surface of section, while the main plot is an enlargement of the central region. The drifting quasiperiodic motion follows contours of constant time $T$ between Andreev reflections. The cross marks the starting point of the trajectory shown in the previous figure, having $T = 18$ (in units of $\sqrt{mL^2/V_0}$).
The enclosed area, therefore, has upper bound
\[ A_{\text{max}} \simeq \rho_T W e^{-\lambda T} \simeq \hbar N e^{-\lambda T}, \]  
where \( N \simeq p_T W / \hbar \gg 1 \) is the number of channels in the point contact.

We now continue with the quantization. The two invariants \( \varepsilon \) and \( T \) define a two-dimensional torus in the four-dimensional phase space. Quantization of this adiabatically invariant torus proceeds following Einstein-Brillouin-Keller \[ 3 \], by quantizing the area
\[ \oint p\,dq = 2\pi \hbar (m + \nu/4), \quad m = 0, 1, 2, \ldots \]  
enclosed by each of the two topologically independent contours on the torus. Eq. \[ 4 \] ensures that the wavefunctions are single valued. (See Ref. \[ 16 \] for a derivation in a two-sheeted phase space.) The integer \( \nu \) counts the number of caustics (Maslov index) and in our case should also include the number of Andreev reflections.

The first contour follows the quasiperiodic orbit of Eq. \[ 2 \], leading to
\[ \varepsilon T = (m + \frac{1}{2})\pi \hbar, \quad m = 0, 1, 2, \ldots \]  
The quantization condition \[ 5 \] is sufficient to determine the smoothed probability distribution \( \rho(\varepsilon) \), using the classical probability distribution \( P(T) \propto e^{(T_N \delta / \hbar)} \) \[ 16 \] for the time between Andreev reflections. (We denote by \( \delta \) the level spacing in the isolated billiard.) The density of states
\[ \rho(\varepsilon) = N \int_0^\infty dT \, P(T) \sum_{m=0}^\infty \delta(\varepsilon - (m + \frac{1}{2})\pi \hbar / T) \]  
has no gap, but vanishes smoothly \( \propto e^{N \delta / 4\varepsilon} \) at energies below the Thouless energy \( N \delta \). This “Bohr-Sommerfeld approximation” \[ 12 \] has been quite successful \[ 14, 15, 19 \], but it gives no information on the location of individual energy levels — nor can it be used to determine the wave functions.

To find these we need a second quantization condition, which is provided by the area \( \oint_T p_x dx \) enclosed by the contours of constant \( T(x, p_x) \),
\[ \oint_T p_x dx = 2\pi \hbar (n + \nu/4), \quad n = 0, 1, 2, \ldots \]  
Eq. \[ 7 \] amounts to a quantization of the period \( T \), which together with Eq. \[ 5 \] leads to a quantization of \( \varepsilon \).

For each \( T_n \) there is a ladder of Andreev levels \( \varepsilon_{nm} = (m + \frac{1}{2})\pi \hbar / T_n \).

While the classical \( T \) can become arbitrarily large, the quantized \( T_n \) has a cutoff. The cutoff follows from the maximal area \[ 3 \] enclosed by an isochronous contour. Since Eq. \[ 7 \] requires \( A_{\text{max}} > 2\pi \hbar \), we find that the longest quantized period is \( T_0 = \lambda^{-1} \ln N + O(1) \).

The lowest Andreev level associated with an adiabatically invariant torus is therefore
\[ \varepsilon_{00} = \frac{\pi \hbar}{2T_0} = \frac{\pi \hbar \lambda}{2 \ln N}. \]  
The time scale \( T_0 \propto | \ln \hbar | \) represents the Ehrenfest time of the Andreev billiard, which sets the scale for the excitation gap in the semiclassical limit \[ 21, 22 \].

We now turn from the energy levels to the wave functions. The wave function has electron and hole components \( \psi_\pm(x, y) \), corresponding to the two sheets of phase space. By projecting the invariant torus in a single sheet onto the \( x-y \) plane we obtain the support of the electron or hole wave function. This is shown in Fig. \[ 4 \] for the same billiard presented in the previous figures. The curves are streamlines that follow the motion of individual electrons, all sharing the same time \( T \) between Andreev reflections. (A single one of these trajectories was shown in Fig. \[ 1 \].)

Together the streamlines form a flux tube that represents the support of \( \psi_+ \). The width \( \delta W \) of the flux tube is of order \( W \) at the constriction, but becomes much smaller in the interior of the billiard. Since \( \delta W / W < \ell_k + \ell_+ < e^{\lambda(t-T)} + e^{-\lambda t} \) (with \( 0 < t < T \)), we conclude that the flux tube is squeezed down to a width
\[ \delta W_{\text{min}} \simeq W e^{-\lambda T/2}. \]  
The flux tube for the level \( \varepsilon_{00} \) has a minimal width \( \delta W_{\text{min}} \simeq W / \sqrt{N} \). Particle conservation implies that \( |\psi_+|^2 \propto 1 / \delta W \), so that the squeezing of the flux tube
is associated with an increase of the electron density by a factor of $\sqrt{N}$ as one moves away from the constriction.

Let us examine the range of validity of adiabatic quantization. The drift $\delta x$, $\delta p_x$ upon one iteration of the Poincaré map should be small compared to $W, p_F$. We estimate

$$\frac{\delta x}{W} \approx \frac{\delta p_x}{p_F} \approx \frac{\varepsilon_n m}{\hbar N} e^{\lambda T_n} \approx (m + \frac{1}{2}) \frac{e^{-\lambda(T_0 - T_n)}}{\lambda T_n}. \quad (10)$$

For low-lying levels ($m \sim 1$) the dimensionless drift is $\ll 1$ for $T_n < T_0$. Even for $T_n = T_0$ one has $\delta x/W \approx 1/\ln N \ll 1$.

Semiclassical methods allow to quantize only the trajectories with periods $T \leq T_0$. The part of phase space with longer periods can be quantized by random-matrix theory (RMT), according to which the excitation gap $E_1$ is associated with an increase of the electron density by a factor of $\gamma$. For low-lying levels ($\gamma \ll 1$) the Lyapunov time). The semiclassical (large-$N$) limit of $E_1$, $\lim_{N \to \infty} E_1 = 0.30 \hbar/T_0$ is a factor of $5$ below the lowest adiabatic level, $\varepsilon_0 = 1.6 \hbar/T_0$, so that indeed the energy range near the gap is not accessible by adiabatic quantization.\footnote{1}

Up to now we considered 2-dimensional Andreev billiards. Adiabatic quantization may equally well be applied to 3-dimensional systems, with the area enclosed by an isochronous contour as the second adiabatic invariant. For a fully chaotic phase space with two Lyapunov exponents $\lambda_1, \lambda_2$, the longest quantized period is $T_0 = \frac{1}{2}(\lambda_1 + \lambda_2)^{-1} \ln N$. We expect interesting quantum size effects on the classical localization of Andreev levels discovered in Ref. \footnote{2}, which should be measurable in a thin metal film on a superconducting substrate.

One important challenge for future research is to test the adiabatic quantization of Andreev levels numerically, by solving the Bogoliubov-De Gennes equation on a computer. The characteristic signature of the adiabatic invariant that we have discovered, a narrow region of enhanced intensity in a chaotic region that is squeezed as one moves away from the superconductor, should be readily observable and distinguishable from other features that are unrelated to the presence of the superconductor, such as scars of unstable periodic orbits.\footnote{3}

Experimentally these regions might be observable using a scanning tunnelling probe, which provides an energy and spatially resolved measurement of the electron density.

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\end{thebibliography}