Doppler shift generated by diffraction gratings under time-dependent incidence angle near a Wood anomaly

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Abstract

Diffraction gratings are famous for their ability to exhibit, near a Wood anomaly, an arbitrarily large angular dispersion, e.g., with respect to the incidence angle or wavelength. For a diffraction grating under incidence by a plane wave at a fixed frequency, by rotating the incidence angle at a given angular velocity, the field propagated by a nonzero diffraction order will rotate at increasingly fast angular velocity as the incidence angle approaches the angle where Wood anomaly occurs. Such a fast rotating diffracted field has the potential to generate a substantial Doppler shift. Indeed, under the assumption of a grating with infinite extent, the expression for the instantaneous frequency shift perceived by an observer, who is looking into the light radiated by the diffraction order, is derived and it is in full agreement with the prediction from an interpretation based on the Doppler shift generated by a rotation of light sources. In particular the classical (non-relativistic) Doppler shift can take arbitrarily high values as the incidence angle approaches a Wood anomaly. It is also found that gratings of a finite size can have a similar property. In order to have a physically detectable frequency shift, it is important to use a grating which can maintain a significant reflectance into higher diffraction orders near their Wood anomaly cut-off. Interestingly, we have found that the geometry of the nanostructures of a Morpho butterfly wing scale is aptly suited for such a function because it can strongly reflect into higher diffraction orders while minimising the reflection into the specular order.

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I. INTRODUCTION

Doppler shift has been successfully used to determine the frequency shift generated by moving diffraction gratings such as gratings in acousto-optic modulators [1, 2], or rotating gratings [3]. For instance the Doppler shift induced by an acousto-optic grating moving at the sound velocity $V$ is given by the expression [1, Eq. (3.29)] $\Delta \omega = k_0 V \sin \phi_p$, where $k_0$ is the wavenumber of the incident light and $\phi_p$ is the angle of propagation of the diffraction order $p$.

Grating-based modulators are generally operated at fixed incidence angle, so they do not really exploit the inherent property of diffraction gratings to exhibit arbitrarily high angular dispersion with respect to the incidence angle, near a Wood anomaly. Accordingly it may be worth investigating whether this property can be used to induce fast-rotating electromagnetic radiation which can generate a frequency shift. A practical implementation of such technique can be realised by varying the angle of incidence $\phi_0$ (e.g., by rotating the incident light or the grating) in a time-dependent manner near an angle where Wood anomaly occurs. In this work, it is assumed that the incidence angle is varied by rotating the incident beam while the grating remains fixed; the case of a rotating grating under incidence by a fixed light beam can be treated in a similar way although it will involve the use of an additional coordinate system which is attached to the moving grating. As the incidence angle $\phi_0$ rotates at a given angular velocity $\Omega_0$, the field propagated by a diffraction order $p \neq 0$ will rotate at increasingly fast angular velocity $\Omega_p$ as $\phi_0$ approaches the angle $\phi_p^W$ where Wood anomaly occurs for the order $p$. An observer, who is looking into the light radiated by the diffraction order $p$, can perceive a chirp signal as $\phi_0$ varies with time.

In addition to frequency modulation, another motivation for studying gratings under time-dependant incidence angle comes from the fact that natural diffraction gratings seem to be placed on surfaces designed to undergo oscillatory motion, for instance, wings of birds and butterflies, leaves and flowers of plants (wind-induced oscillations). Many species also display rapid cycles of movement of iridescent colours in ritualized dances during courtship, e.g., bird-of-paradise [4, 5], peacock spider [6]. Since the work of Robert Hooke [7] and Isaac Newton [8], it is known that the iridescent colours from some animals and plant species are generated by gratings structures, but the functional purpose of the oscillations of natural gratings is still not well understood [4, 9–11].
In the next section, a mathematical model for gratings under time-dependent incidence angle is presented and the instantaneous frequency will be determined. In the following section, light diffraction by gratings of finite size will be analysed by using a treatment based on a Doppler shift formula. In Section IV we will describe an example of grating which reflects light mostly into higher diffraction order and minimise coupling into the specular order (except for large angle of incidence).

II. A MATHEMATICAL MODEL FOR GRATINGS OF INFINITE SIZE UNDER A TIME-DEPENDENT INCIDENCE

A. The grating equation

As a model problem, we consider a one-dimensional diffraction grating as illustrated in Fig. 1. The grating is periodic in the $x$-direction, with a period $\Lambda$, invariant with respect to $y$ and has a finite thickness in the $z$-direction. The incident wave is a downward propagating monochromatic plane wave with an incidence angle $\phi_0$ and a wave vector $(k_x, k_y, k_z) = (k_0 \sin \phi_0, 0, -k_0 \cos \phi_0)$, i.e., propagation in the $(x, z)$-plane. The analysis can be carried out by splitting the incident fields into two polarisations [12]:

- $E_y$-polarisation: the incident electric field is parallel to the $y$-axis, i.e., $E(x, z, t) = (0, E_x(x, z, t), 0)$,

- $H_y$-polarisation: the incident magnetic field is parallel to the $y$-axis, i.e., $H(x, z, t) = (0, H_x(x, z, t), 0)$.

In both cases the corresponding scalar field is denoted $v_{\text{inc}}$:

$$v_{\text{inc}}(x, z, t) = \exp \left( i \left( k_0 \sin(\phi_0) x - k_0 \cos(\phi_0) z - \omega_0 t \right) \right) ,$$  \hspace{1cm} (1)

where $\omega_0$ and $k_0$ are respectively the angular frequency and the free-space wavenumber.

In this work we will focus on the properties of the reflected fields although most of the results will also hold for the transmitted fields. The field reflected by the gratings can be written as a superposition of plane waves $v_p(x, z)$:

$$v_{\text{refl}}(x, z, t) = \exp(-i \omega_0 t) \sum_p \rho_p v_p(x, z)$$  \hspace{1cm} (2)
where, for all diffraction order $p \in \mathbb{Z}$, $\rho_p$ is the reflection coefficient into the order $p$ and $v_p(x, z)$ is the plane wave

$$v_p(x, z) = \exp(i k_0 (x \sin \phi_p + z \cos \phi_p)).$$

(3)

For periodic gratings, the conservation of tangential momentum states that the angle of propagation $\phi_p$ of the plane waves $v_p(x, z)$ must satisfy the grating equation [13]:

$$k_0 \sin \phi_p = k_0 \sin \phi_0 + \frac{2 \pi p}{\Lambda}.$$

(4)

B. Instantaneous frequency for gratings under time-dependent incidence angle

We now consider that the grating is operated by varying the angle of incidence $\phi_0$ in a time-dependent manner while the wavelength remains fixed, i.e., the wave vector of the incident plane wave is rotated around the $y$-axis. The angular velocity of $\phi_0(t)$ is denoted $\Omega_0(t)$. As a consequence of the change of the angle of incidence $\phi_0(t)$, the angle of diffraction $\phi_p$ is also time-dependent and we can obtain the associated angular velocity $\Omega_p(t)$ by differentiating Eq. (4):

$$\Omega_p(t) = \frac{d\phi_p(t)}{dt} = \frac{\cos \phi_0}{\cos \phi_p} \frac{d\phi_0(t)}{dt} = \frac{\Omega_0(t) \cos (\phi_0(t))}{\cos (\phi_p(t))}.$$

(5)

We note in passing that the time-derivative of Eq. (4), at fixed frequency, leads to the following angular momentum conservation relation:

$$\Omega_p(t) \cos (\phi_p(t)) = \Omega_0(t) \cos (\phi_0(t)).$$

(6)
We assume that for the operating wavelength and a given diffraction order $p \neq 0$, there exists an incidence angle $\phi_0 = \phi_p^W \in \mathbb{R}$ for which Wood anomaly occurs, i.e., $\phi_p = \pm \pi/2$ and if the diffracted order is propagating for $\phi_0 > \phi_p^W$ then it is evanescent for $\phi_0 < \phi_p^W$, and vice-versa. Since the denominator $\cos \phi_p$ in Eq. (5) takes the value zero at a Wood anomaly, the angular velocity $\Omega_p(t)$ tends to infinity when the angle of incidence $\phi_0(t)$ tends to $\phi_p^W$ (if $\Omega_0(t) \cos(\phi_0(t)) \neq 0$ at the Wood anomaly). This will result in a fast rotating plane wave field which can be expected to create a strong Doppler shift, assuming that the diffracted field carries enough energy (i.e., the reflection coefficient $\rho_p$ is large enough) to be detected by sensor. In this section we will analyse the frequency shift.

We assume that an observer at the position $\mathbf{r} = (x, z)$ is looking into the light radiated by a non-evanescent plane wave $\rho_p \exp(-i \omega_0 t) v_p(x, z)$. The instantaneous phase $\theta_p(t)$ perceived by the observer is:

$$\theta_p(t) = - \left( k_0 \begin{bmatrix} \sin (\phi_p(t)) \\ \cos (\phi_p(t)) \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ z \end{bmatrix} - \arg(\rho_p(t)) + \omega_0 t$$

and the corresponding instantaneous angular frequency $\omega(t)$ is

$$\omega(t) = \frac{d\theta_p(t)}{dt},$$

$$= -k_0 \frac{d\phi_p(t)}{dt} \begin{bmatrix} \cos (\phi_p(t)) \\ - \sin (\phi_p(t)) \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix} - \frac{d (\arg(\rho_p(t)))}{dt} + \omega_0.$$  

(9)

While the reflection coefficient $\rho_p(t)$ determines the magnitude of the field carried by the plane wave $v_p$, its contribution to the instantaneous angular frequency Eq. (9) is in general negligible in comparison to the other terms and will be removed from the expression of the frequency shift:

$$\omega_p(t) - \omega_0 = -k_0 \Omega_p(t) (x \cos (\phi_p(t)) - z \sin (\phi_p(t))).$$  

(10)

By using the angular velocity Eq. (5) and the relation $\omega_0 = k_0 c$, we get the relative angular frequency shift

$$\frac{\delta \omega_p}{\omega} = \frac{\omega_p(t) - \omega_0}{\omega_0} = - \frac{\Omega_0(t)}{c} \cos (\phi_0(t)) \left( x - z \tan (\phi_p(t)) \right),$$

(11)

where $c$ is the speed of light in vacuum (we have used $c = 3 \times 10^8 \text{m/sec.}$). The actual wave frequency is $\nu_0 = \omega_0/(2 \pi)$, so that we also have

$$\frac{\delta \nu_p}{\nu} = \frac{\nu_p(t) - \nu_0}{\nu_0} = - \frac{\Omega_0(t) \cos (\phi_0(t))}{c} \left( x - z \tan (\phi_p(t)) \right).$$

(12)
FIG. 2. The relative frequency shift Eq. (12): (a) for the specular order (b) for the first order. The angle of incidence varies as $\phi_0(t) = t$.

Since $\tan(\phi_p(t))$ tends to infinity as $\phi_0(t)$ approaches a Wood anomaly angle $\phi_p W$ (i.e., $\phi_p(t)$ tends to $\pm \pi/2$), the frequency shift created by a propagating order $p \neq 0$ will formally tend to infinity (the sign of which depends on the sign of $\Omega_0(t)$).

Figure 2 illustrates the relative frequency shift detected by an observer looking into the reflected specular order $p = 0$ (see Fig. 2(a)) and the reflected first order $p = 1$ (see Fig. 2(b)). The specular order $p = 0$ is representative of the response of a perfectly flat reflecting mirror under oscillating incidence angle while the first order $p = 1$ illustrates the additional properties of a diffraction grating reflector. For the numerical simulation results in Fig. 2, the time dependent incident angle is set to $\phi_0(t) = t$ so that its angular velocity is $\Omega_0(t) = 1$ rad/sec. The grating period is $\Lambda = 0.5 \mu m$. The wavelength is set to $\lambda = \Lambda$ (which corresponds to the frequency $\nu_0 = 6 \times 10^{14}$ Hz), in particular, the Wood anomaly for the diffraction order $p = 1$ occurs at $\phi_0(0) = \phi_1 W = 0$. The coordinates of the observer position is $(x, z) = (0, 2 \times 10^6 \Lambda) = (0, 1 \text{ m})$. Note that the first order is evanescent for $t \in ]0, \pi/2]$ and this is why the curve in 2(b) is plotted only for $t \in [-\pi/2, 0]$. We can see that $\delta \nu_1/\nu$ converges to infinity as $t$ approaches $t = 0$.

With the specular order, the maximum value of the relative frequency shift is $\delta \nu_0/\nu = \ldots$
\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
0 & $\pi/2 - \phi_1(t)$ & $\delta\nu_1/\nu$ & $\delta\nu_1$ \\
\hline
$-0.282486$ sec & $0.765183$ rad & $3.333 \times 10^{-9}$ & $2 \times 10^6$ Hz \\
$-5.55551 \times 10^{-6}$ sec & $3.33332 \times 10^{-3}$ rad & $10^{-6}$ & $6 \times 10^8$ Hz \\
$-5.55556 \times 10^{-12}$ sec & $3.33333 \times 10^{-6}$ rad & $10^{-3}$ & $6 \times 10^{11}$ Hz \\
$-5.55556 \times 10^{-14}$ sec & $3.33 \times 10^{-7}$ rad & $10^{-2}$ & $6 \times 10^{12}$ Hz \\
$-5.55556 \times 10^{-16}$ sec & $3.332 \times 10^{-8}$ rad & $10^{-1}$ & $6 \times 10^{13}$ Hz \\
\hline
\end{tabular}
\caption{Relative frequency shift $\delta\nu_1/\nu$ (see Eq. (12) and Fig. 2) and absolute frequency shift $\delta\nu_1$ (see Eq. (12)) created by the first order as the time-dependent angle of incidence is given by $\phi_0(t) = t$. The frequency of the incident light is $\nu_0 = 6 \times 10^{14}$ Hz.}
\end{table}

As shown in Table I, the first order can give much higher values. It appears that the duration of high frequency shift regime is very short; for instance the relative frequency shift is higher than 0.1% only for $t \in [-5.556 \times 10^{-12}, 0]$. This is partly due to the low value of the incident angular velocity $\Omega_0(t) = 1$ rad/sec. For instance, with the angular velocity $\Omega_0(t) = 10$ rad/sec and $\Omega_0(t) = 100$ rad/sec., a frequency shift higher than 0.1% is reached respectively for $t \in [-5.556 \times 10^{-10}, 0]$ and $t \in [-5.556 \times 10^{-8}, 0]$. Indeed, as we now show, the length of the interval where $\delta\nu_1/\nu$ is higher than 0.1% is approximately proportional to the square of $\Omega_0(t)$. For an angle of incidence $\phi_0$ near $\phi_1^W$, we have the Taylor approximation

$$\sin \phi_0 = \sin ((\phi_0 - \phi_1^W) + \phi_1^W)$$
$$\approx \sin \phi_1^W + (\phi_0 - \phi_1^W) \cos \phi_1^W$$

and the substitution into the expression of the angle of diffraction of the first order

$$\phi_1 = \arcsin \left( \sin \phi_0 + \frac{2\pi}{k_0 \Lambda} \right)$$
leads to
\[ \phi_1 \approx \arcsin \left( \sin \phi_1^W + \frac{2 \pi}{k_0 \Lambda} + (\phi_0 - \phi_1^W) \cos \phi_1^W \right). \] (20)

Since the first order Wood anomaly occurs at the angle of incidence \( \phi_0 = \phi_1^W \), we have
\[ \sin \phi_1^W + \frac{2 \pi}{k_0 \Lambda} = 1, \] (21)
and by taking into account the following asymptotic for small values of \( x \):
\[ \arcsin (1 - x) \approx \frac{\pi}{2} - \sqrt{2} x, \] (22)

Eq. (20) becomes
\[ \phi_1 \approx \frac{\pi}{2} - \sqrt{-2 (\phi_0 - \phi_1^W) \cos \phi_1^W}, \] (23)
and, by applying the relation \( \cos(\pi/2 - x) = \sin x \approx x \), we obtain
\[ \cos \phi_1 \approx \sqrt{-2 (\phi_0 - \phi_1^W) \cos \phi_1^W}. \] (24)

From the substitution of Eq. (24) into the relative frequency shift Eq. (12) and the application of the relations \( \sin x \approx x \) and \( \cos x \approx 1 \), for small values of \( x \), we can derive the following asymptotic relation:
\[ \frac{\delta \nu_1}{\nu} \approx - \frac{\Omega_0(t) \cos (\phi_0(t))}{c} \left( x - \frac{z}{\sqrt{-2 (\phi_0 - \phi_1^W) \cos \phi_1^W}} \right). \] (25)

If \( \phi_0 \) is close enough to \( \phi_1^W \), then we can write
\[ \frac{\delta \nu_1}{\nu} \approx \frac{\Omega_0(t) \cos \phi_1^W}{c} \frac{z}{\sqrt{-2 (\phi_0 - \phi_1^W) \cos \phi_1^W}}. \] (26)

In particular, for a real number \( f > 0 \), we have
\[ \left| \frac{\delta \nu_1}{\nu} \right| \geq f \implies (\phi_0 - \phi_1^W) \in \left[ -\frac{\cos \phi_1^W}{2} \left( \frac{z \Omega_0(t)}{c f} \right)^2, 0 \right], \] (27)
i.e., the length of the interval where \( \delta \nu_1/\nu \) is higher than \( f \) is approximately proportional to the square of \( \Omega_0(t) \) and \( z \).
FIG. 3. We have $\tan \phi_p = (x - x_S)/z$.

C. Physical interpretation of the frequency shift

The frequency shift Eq. (12) can be stated in a way which is similar to that of gratings in acousto-optic modulators [1, see Eq. (3.29)]. For an acousto-optic grating moving horizontally at the sound velocity $V$, an observer looking at the light diffracted into an order $p$, sees a light source moving with the velocity $V$ and the velocity component in the observer’s direction is given by $V \sin \theta_p$ so that the Doppler shift is

$$\Delta \nu_p = \nu \frac{V \sin \theta_p}{c}. \quad (28)$$

For the case where the plane wave order $p$ is rotating around the $y$-axis with an angular velocity $\Omega_p$, the velocity of a source located on the grating interface $z = 0$ and at the position $x = x_S$ is $V = -x_S \Omega_p$ and its direction is parallel to the $z$-axis ($V$ is perpendicular to the radial direction). Note that the negative sign in the expression $V = -x_S \Omega_p$ comes from the fact that when $\Omega_p > 0$, the points $x_S$ on the left of the rotation axis move upward while they move downward on the right side. Again, an observer looking into the light radiated from the position $(x, z) = (x_S, 0)$, will sees a light source moving with a velocity $V$ and, here, the velocity component in the observer’s direction is given by $V \cos \phi_p$ so that the Doppler shift is

$$\Delta \nu_p = \nu \frac{V \cos \phi_p}{c} = -\nu \frac{x_S \Omega_p \cos \phi_p}{c} = -\nu \frac{x_S \Omega_0 \cos \phi_0}{c}. \quad (29)$$

The radiations from the other positions $(x, 0)$ such that $x \neq x_S$ do not contribute to the frequency shift because they cancel each other by destructive interference at the observation point $P$. This fact can also be explained by applying the principle of stationary phase to an integral representation (e.g., Rayleigh-Sommerfeld integral) of the field at the point $P$.

Interestingly, the frequency shift given by Eq. (29) is equal to the one in Eq. (12). Indeed
the substitution of the following relation (see Fig. 3)
\[ \tan \phi_p = \frac{x - x_S}{z} \]  
into Eq. (12) leads to Eq. (29), since \((x - z \tan \phi_p) = x_S\).

III. DIFFRACTION BY GRATINGS OF FINITE SIZE

For a grating of finite size, the formulas Eq. (29) for Doppler shift can still be accurate if the observer is close enough to the grating, although in far field the diffraction from the boundary of the grating must also be taken into account [2, Section 3.9]. The linear speed of the left and right boundaries is proportional to the angular velocity \(\Omega_p\), which can take large values as the incidence angle \(\phi_0\) moves toward a Wood anomaly. Thus the spreading wavefront from the edges of the grating can be expected to generate a significant Doppler shift which can be perceived over a vast viewing angle. The finite grating is supposed to be a rectangle aperture \(A = [x_{\text{rect}}, x_{\text{rect}} + D_x] \times [-D_y/2, D_y/2]\) which is sitting on the horizontal plane \(z = 0\), and the wavevector of the incident plane wave can rotate around the \(y\)-axis.

The linear speed at diffracted field at \((x, y) \in A\)
\[ V_p = -x \Omega_p = -x \frac{\Omega_0 \cos \phi_0}{\cos \phi_p}. \]  
and the frequency shift perceived by an observer looking at the light radiated from the source point \((x, y) \in A\), at an azimuthal angle \(\psi\), is
\[ \frac{\Delta \nu_p}{\nu} = -\frac{x \Omega_0 \cos \psi}{c} \frac{\cos \phi_0}{\cos \phi_p}. \]  
If the observation angle \(\psi\) happens to be equal to the diffraction angle \(\phi_p\), then the shift is same as the one given by the expression Eq. (29) for infinite gratings.

The energy carried by a diffracted wave is also an important quantity. The flux of the real-valued Poynting vector \(S_p = \text{Re} \left( E_p \times H_p^* \right) / 2 \) across the aperture \(A\) characterizes the amount of energy radiated by a diffraction order \(p\):
\[ \mathcal{E}_p = \int_A S_p \cdot z \, dx \, dy. \]  
For the \(E_y\)-polarised plane wave \(E_p(x, y, z) = (0, \rho_p v_p(x, z), 0)\) and \(H_y\)-polarised plane wave \(H_p(x, y, z) = (0, \rho_p v_p(x, z), 0)\) we have respectively
\[ \mathcal{E}_p = \frac{|\rho_p|^2 D_x D_y}{2 \mu} \cos \phi_p \quad \text{and} \quad \mathcal{E}_p = \frac{|\rho_p|^2 D_x D_y}{2 \varepsilon} \cos \phi_p. \]  

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The derivation of Eq. (34) can be established by using the relations \( \mathbf{H} = -\nabla \times \mathbf{E} / (i k_0 \mu) \) and \( \mathbf{E} = \nabla \times \mathbf{H} / (i k_0 \varepsilon) \) from Maxwell’s equations, where the magnetic field \( \mathbf{H} \) has been rescaled as \( Z_0 \mathbf{H} \rightarrow \mathbf{H} \) with \( Z_0 = \sqrt{\mu_0 / \varepsilon_0} \), the impedance of free space.

For scattering problem the diffraction efficiency is the energetic quantity of interest. For a propagating plane wave order \( p \), the diffraction efficiency \( \chi_p \) (also referred to as the reflectance into the order \( p \)) is the ratio between ratio energy flux carry by the plane wave order \( p \) and the incident plane wave Eq. (1):

\[
\chi_p = \frac{E_p}{E_{inc}} = \frac{|\rho_p|^2 \cos \phi_p}{\cos \phi_0}. \tag{35}
\]

When the incident angle \( \phi_0 \) is near the Wood anomaly angle \( \phi_p^W \), we have \( \cos \phi_0 \approx \cos \phi_p^W \), and the following approximation (which can be derived in a similar way as Eq. (24)) is also valid:

\[
\cos \phi_p \approx \sqrt{-2 (\phi_0 - \phi_p^W) \cos \phi_p^W}, \quad \text{for } p > 0, \tag{36}
\]

so that Eq. (31), Eq. (32) and Eq. (35) have the following asymptotic expressions:

\[
V_p \approx -x \Omega_0 \sqrt{-2 (\phi_0 - \phi_p^W) / \cos \phi_p^W}, \tag{37}
\]

\[
\frac{\Delta \nu_p}{\nu} \approx \frac{x \Omega_0 \cos \psi}{c} \sqrt{\frac{\cos \phi_p^W}{-2 (\phi_0 - \phi_p^W)}}, \tag{38}
\]

\[
\chi_p \approx |\rho_p|^2 \sqrt{-2 (\phi_0 - \phi_p^W) / \cos \phi_p^W}. \tag{39}
\]

In particular as \( \phi_0 \) approaches \( \phi_p^W \), the frequency shift tends to infinity while the energy carried by the diffracted order converges to zero. Figure 4 illustrates the dynamic of frequency shift as \( \phi_0 \) converges to the Wood anomaly angle \( \phi_1^W = \pi / 4 \). The following parameter values are used to generate the curves of Fig. 4: the diffraction order is \( p = 1 \), the light source position is \((x, y, z) = (2 \text{ cm}, 0, 0)\), the observation angle \( \psi = \pi / 4 \) and the intensity of the reflected field is set arbitrarily to \( |\rho_1|^2 = 1 \), in Eq. (39). The intensity varies \( |\rho_1|^2 \) slowly near a Wood anomaly (see for example Fig. 14) so that its variation, as \( \phi_0 \) converges to \( \phi_1^W \), is neglected in Eq. (39).

Note that for \( p < 0 \), the angle of diffraction \( \phi_p \) tends to \(-\pi / 2\) at a Wood anomaly, i.e., \( \phi_p \approx -\pi / 2 + (2 (\phi_0 - \phi_p^W) \cos \phi_p^W)^{1 / 2} \), so that Eq. (36) becomes \( \cos \phi_p \approx (2 (\phi_0 - \phi_p^W) \cos \phi_p^W)^{1 / 2} \) and Eqs. (37)- (39) must be changed accordingly.
FIG. 4. The sub-figures (a), (b) and (c) show the linear speed Eq. (37), the relative frequency shift Eq. (38) and the diffraction efficiency Eq. (39) for the first order $p = 1$.

IV. NUMERICAL SIMULATIONS OF SOME EXAMPLES OF GRATINGS

In order to have a physically detectable frequency shift, it is important that the diffracted field carries a sufficiently high electromagnetic energy near the Wood anomaly cut-off. Thus it is essential to use gratings which can provide a strong reflectance into the higher diffraction orders, especially near their Wood anomaly cut-off. In this section we will plot the reflectance contour plots in order to analyse the distribution of the reflected energy among the diffraction orders. The diffraction problem is solved numerically using a finite element method [12] and the results for $E_y$-polarised incident fields will be presented in this section. Note that we did not observe any significant qualitative difference between the reflectance of the $E_y$ and $H_y$ polarisations.
A. An example of a square lattice of circular cylinders

We first consider the case of a square lattice of circular cylinders, a geometry which has been widely studied. Figure 5(a) illustrates a grating consisting of 8 layers of a square lattice of circular cylinders and Fig. 5(b) shows the corresponding contour plot of the total reflectance, as a function of the wavelength and incidence angle; the incident plane wave is $E_y$-polarised. Figure 6 shows the contribution of the diffraction orders $p = 0$, $p = 1$, $p = 2$ and $p = 3$ to the total reflectance. The dashed black lines in Fig. 6 indicate the points $(\lambda, \phi^W_p)$ where Wood anomaly appears. The reflectance into the first order can be as high as $\chi_1 = 0.26$ for wavelengths around $\lambda = 800\text{ nm}$, and $\lambda = 1200\text{ nm}$; it can remain high for angles of incidence close to Wood anomaly. The maximum reflectance is smaller for the second and third diffraction orders. However, for most of the plotting domain, the reflectance into the specular order is larger than the reflectance into the higher orders; this is not favourable for frequency shifting because the specular order does not amplify the angular speed of the incident light. This issue may be addressed by searching for a grating geometry which increases the coupling into higher diffraction orders and reduces the coupling into the specular order. It is not easy for us to do such optimisation for a square lattice of circular cylinders because we do not have a clear understanding of the optical mechanisms which control the occurrence of its peak reflectance.

B. Examples of multilayer gratings

In order to develop an efficient design principle which leads to significant reflectance into higher diffraction orders while minimising coupling into the specular, it is important to have a good understanding of the mechanisms which control the reflection. For layered structures, we will describe a design strategy in which the reflectance into the specular order is minimised and this turns out to be beneficial to the reflectance into higher diffraction orders. The design is inspired by the geometry of the nanostructures of *Morpho* butterfly wing scales [14–17].

We first start with a classical multilayer film, as illustrated in Fig. 7(a), and by applying some basic rules of optics, the multilayer film will be transformed into the tree-like structure depicted in Fig. 7(d), a geometry which models the nanostructures found in *Morpho rhetenor*.
FIG. 5. (a) Diagram of a unit cell of a grating consisting of a square lattice of circular cylinders. The lattice constant of the square array is $d = 746$ nm and the cylinder radius is $R = 0.3d$. The refractive index of the cylinders is $n = 1.56 + 0.06i$ (chitin) and the background material is air.

(b) Contour plot of the total reflectance as a function of the wavelength $\lambda \in [100\text{ nm}, 1500\text{ nm}]$ and the incidence angle $\phi_0 \in [-89\text{ degrees}, 89\text{ degrees}]$. Wing scales (see Fig. 9 in [14]). The contour plots of the total reflectance of the 4 gratings in Fig. 7 are shown in Fig 8; the contour plots are not exactly similar, but, as we will see, the most dramatic difference between them is rather the way the reflected energy is distributed among the diffracted orders. The reflectance corresponds to an incidence by a plane wave with $E_y$ polarisation.

Most of the material and geometric parameters used for the calculations are taken from [14, 16], where the *Morpho rhetenor* scales have been modelled: the structures consist of 8 layers which are made of a material with refractive index $n = 1.56 + 0.06i$ (chitin) and are surrounded by air; the layer thickness is $h = 62$ nm and the distance between the centres of two consecutive layers is $d_z = 207$ nm (periodicity in the longitudinal direction). For an incidence in the normal direction, a peak reflectance wavelength occurs when there is a constructive interference in reflection:

$$2(\text{Re}(n)h + (d_z - h)) = m\lambda,$$

where $m$ is integer. For instance, with $m = 1$, the peak reflectance wavelength is $\lambda = 483.44$ nm. Figure 9(a) show the reflectance curve of the multilayer structure in Fig. 7(a);
FIG. 6. Square lattice of circular cylinders: the sub-figures (a), (b), (c) and (d) show respectively the contour plots of the reflectance into the orders $p \in \{0, 1, 2, 3\}$. The dashed black lines indicate the Wood anomaly curves, i.e., curves consisting of points $(\lambda, \phi_p^W)$ where Wood anomaly occurs. The two dashed black lines in panel (a) indicate the line of first Wood anomaly for $p \in \{1, -1\}$.

For the calculated results, the peak reflectance wavelength is $\lambda = 446.65$ nm and the difference between the two results is mainly due to the fact that the imaginary part of refractive index $n$ is neglected in the asymptotic formula Eq. (40).

Since the layers in Fig. 7(a) are invariant in the $x$-direction, the specular order $p = 0$ carries 100% of the reflected energy. The reflectance into the specular order can be substantially reduced by the grating with alternating lamellae depicted Fig. 7(b): the period in the $x$-direction is $d_x = 746$ nm and, inside each unit cell, the left half of the initial layers of Fig. 7(a) are shifted downward by a distance $d_z/2 = 103.5$ nm. Figure 9(b) shows the reflectance curves of the grating with alternating lamellae, we can observe that the reflectance into the specular order (thin red curve) is significantly lower than the reflectance into the
FIG. 7. Transformation of the simple multilayer film in sub-figure (a) into gratings with alternating lamellae which reflect strongly into higher diffraction orders. The periods in the $x$ and $z$ directions are $d_x = 746$ nm and $d_z = 207$ nm. The length and thickness of the lamellae are respectively $L_0 = 308$ nm and $h = 62$ nm; the thickness of the ridge in (c) and (d) is $h_R = 60$ nm. For the grating in panel (d), a linear apodisation is applied to the length of the top 7 lamellae: $L_m = L_0 - 40m$, for $m \in \{1, 2, \ldots, 7\}$. The grating geometry in panel (d) models the nanostructures found in *Morpho rhetenor* wing scales [14, see Fig. 9].

higher diffraction orders, especially in the high reflectance band $\lambda \in [400 \text{ nm}, 510 \text{ nm}]$.

This phenomenon can be explained as follows: the vertical shift is equal to one-quarter of the wavelength $\lambda = 2d_z = 414$ nm, and for a normal incidence, this introduces a $\pi$ phase shift between the light reflected into the specular order by the left side and right side lamellae. So the grating with alternating lamellae can create a destructive interference between the wave reflected by the alternating column of lamellae, as a consequence the reflected light couples poorly into the specular order at $\lambda = 414$ nm; actually, as shown in the contour plot Fig. 10(b), this poor coupling persists over a wide range of wavelengths and incidence angles, forcing the reflected light into higher diffraction orders (see Figs. 11(a), 12(a) and 13(a)). This shows that the alternation of lamellae can be an efficient strategy to design gratings which strongly reflect light into higher diffraction orders. For practical implementation, the
FIG. 8. The sub-figures (a), (b), (c) and (d) represent the contour plots of the total reflectance, as a function of $\lambda \in [200\text{ nm}, 600\text{ nm}]$ and $\phi_0 \in [-89\text{ degrees}, 89\text{ degrees}]$, for the gratings shown respectively in the sub-figures (a), (b), (c) and (d) of Fig. 7. The incident plane wave is $E_y$-polarised.

lamellae can be attached to a ridge structure as shown in Fig. 7(c). The reflection curve of the ridge structure is plotted in Figs. 9(c), and its reflection contour plots are shown in Figs. 8(c), 11(b), 12(b) and 13(b). It appears that the presence of the ridges perturbs only slightly the reflectance of the idealised grating with alternating lamellae of Fig. 7(b).

Note that the arguments used to explain the poor reflection into the specular order does not apply to the transmission into the order $p = 0$ because at the lower interface the wave transmitted by the right side lamellae will also acquire a $\pi$ phase shift which will cancel the initial phase shift of the left side lamellae. Indeed we have observed that the shapes of the transmittance contour plots of the two alternating lamellae structures in Fig. 7(b) and Fig. 7(c) look like a moderate perturbation of the contour plot of the transmittance of the original multilayer in Fig. 7(a). For wavelengths where there is a
FIG. 9. The sub-figures (a), (b), (c) and (d) represent the reflectance curves of the gratings shown respectively in the sub-figures (a), (b), (c) and (d) of Fig. 7. The angle of incidence is normal. The dotted black curve indicates the total reflectance while the thin red curves, blue solid curves and magenta dashed curves correspond respectively to the reflectance into the orders \( p = 0 \), \( p = 1 \) and \( p = -1 \). Note that the reflectance into the orders \( p = \pm 2, \pm 3 \) are not shown.

Resonance in reflection, most of the incident wave is reflected before it reaches the lower interface so that the relative position, at the lower interface, of the left and right columns of lamellae is irrelevant. However, the phase shift between the left and right columns of lamellae will cancel again for a reflected field which reaches the lower interface before it bounces back and as a consequence such a reflected wave can still couple efficiently into the specular order. This kind of Fabry-Perot cavity effect is well known in fibre Bragg grating applications where it leads to the appearance of secondary maxima (sidelobes) on both sides of a resonance peak; the classical method to suppress these sidelobes consists in using apodised gratings. Apodisation [18, 19] is a technique where the strength of a grating is slowly reduced over a transition region, near an output or input end.

Indeed, the grating structure in Fig. 7(d) can be seen as an apodised version of the one in Fig. 7(c). The length of the unperturbed lamellae is \( L_0 = 308 \text{ nm} \). A linear apodisation is applied to the length of the top 7 lamellae: \( L_m = L_0 - 40 \, m \): \( L_1 = 268 \text{ nm} \), \( L_2 = 228 \text{ nm} \),

\[ \]
FIG. 10. The sub-figures (a), (b), (c) and (d) represent the contour plots of the reflectance into the specular order $p = 0$ from the gratings shown respectively in the sub-figures (a), (b), (c) and (d) of Fig. 7. The dashed black curves indicate the points $(\lambda, \phi^W_p)$ where first order Wood anomaly occurs, i.e., for $p \in \{1, -1\}$.

$L_3 = 188$ nm, $L_4 = 148$ nm, $L_5 = 108$ nm, $L_6 = 68$ nm and $L_7 = 28$ nm. The rate of decrease of the apodised lamella length is 40 nm as in [14, see Fig. 9]. As shown in Figs 9(d) and 10(d), the apodisation has further reduced the reflectance into the specular order and increased the reflection into the second and third diffraction orders (see Figs. 12(c) and 13(c)). This result suggests that, in 10(b) and 10(c), most of the energy reflected into the specular order is caused by a resonant cavity effect between the upper and lower interfaces, when angle of incidence $\phi_0$ is between -70 degrees and 70 degrees.

Figure 14 shows the contour plot of $|\rho_p|^2$, where $\rho_p$ is the reflection coefficient into the order $p$, for the apodised gratings. We can notice that the largest values of $|\rho_p|^2$ appear near a Wood anomaly. This can be explained by the fact that the principle of energy
FIG. 11. The sub-figures (a), (b) and (c) represent the contour plots of the reflectance into the first order $p = 1$ from the gratings shown respectively in the sub-figures (b), (c) and (d) of Fig. 7. In all cases the reflectance can still be significant near the Wood anomaly lines (dashed lines).

conservation \[20\] requires the maximum value of the diffraction efficiency $|\rho_p|^2 \cos \phi_p / \cos \phi_0$ to be 1, i.e., $|\rho_p|^2 \leq \cos \phi_0 / \cos \phi_p$; but this maximum bound is not restrictive enough near a Wood anomaly because the denominator $\cos \phi_p$ tends to zero and, as a consequence, $|\rho_p|^2$ can take large values in this case.

The contour plot of $|\rho_p|^2$ has a smooth shape, in a contrast to the complicated shapes of the diffraction efficiency contour plots in Figs 11, 12 and 13. For a given frequency, as the incidence angle $\phi_0$ moves toward $\phi_p^W$, the energy reflected into the order $p$ is proportional to $|\rho_p|^2 |\phi_0 - \phi_p^W|^{1/2}$ (see Eq. 39). Thus incident wavelengths, for which $|\rho_p|^2$ has relatively small values near a Wood anomaly, will shift only a small amount of their energy into other wavelengths perceived by an observer. But incident wavelengths which have relatively large values of $|\rho_p|^2$, near a Wood anomaly, will shift much more of their energy into other
FIG. 12. The sub-figures (a), (b) and (c) represent the contour plots of the reflectance into the second order \( p = 2 \) from the gratings shown respectively in the sub-figures (b), (c) and (d) of Fig. 7.

perceived wavelengths. Thus, for a grating which is oscillating under the diffuse skylight (i.e., simultaneous incidence from multiple wavelengths and multiple angles), the transfer of reflected energy between wavelengths is non-uniform and it is possible that this transfer of energy can add up constructively for some wavelengths, and an observer who is watching the grating at such a wavelength may see a very bright grating colour.

From the results in this section, we can conclude that the nanostructures of *Morpho* butterfly wings are remarkably well-suited for the purpose of Wood anomaly-based frequency shifting. As a *Morpho* butterfly flaps its wings under the diffuse skylight, Wood anomaly will occur continuously for various wavelengths. Since the specular order cannot provide the high angular dispersion required for frequency shifting, it is undesirable that this order carries any significant amount of energy over the operating ranges of wavelength and incidence angle; by design, the array of apodised alternating lamellae conveniently reduces the coupling into
FIG. 13. The sub-figures (a), (b) and (c) represent the contour plots of the reflectance into the third order $p = 3$ from the gratings shown respectively in the sub-figures (b), (c) and (d) of Fig. 7.

the specular order. The apodised array can efficiently couple energy into first diffraction orders and even higher orders, which means that Wood anomaly-based frequency shifting can occur over a larger range of incidence angle; this is an advantage when the grating is operated under the diffuse skylight, where light is incident from a wide range of angles.

Naturally, all these facts raise the question about the biological function of the nanosstructures of Morpho butterflies. A highly reflecting surface can provide a better visibility. As shown in the reflectance contour plots of Fig. 8, each of the 4 grating geometries in Fig. 7 can deliver quite a high reflectivity over a vast range of frequencies and incidence angles. The apodised grating of alternating lamellae is apparently the most complicated to fabricate and its presence on Morpho wings suggests that this grating structure must provide some benefits which the other gratings cannot offer. The blue colour of Morpho butterflies is due to diffraction from an apodised grating and this colour is so bright that it can be seen from
FIG. 14. The sub-figures (a), (b) and (c) represent respectively the contour plots of the reflection intensity $|\rho_p|^2$ for the orders $p = 1$, $p = 2$ and $p = 3$. The apodised gratings shown in Fig. 7(d) is used for the calculation.

half a kilometre away or even from a low-flying aircraft \cite{21}. It would be interesting to investigate if the Wood anomaly-based frequency shifting has played any role in this remarkable performance.

V. CONCLUSION

Near a Wood anomaly, a diffracted field can amplify the angular speed of the incident field by an arbitrarily large factor. As a consequence, grating under oscillating incidence angle can act as a “super modulator” of frequency. For a grating with infinite extent, as the angle of incidence approaches a Wood anomaly with an angular velocity $\Omega_0 \neq 0$, the instantaneous frequency shift perceived by an observer tends to infinity; the frequency shift can be explained by the classical Doppler shift, in a very similar way to that of the traditional
explanation for acousto-optic modulators. For a grating of finite size, the analytical treatment based on the concept of instantaneous frequency becomes difficult to apply because of the appearance of spreading wavefront from the edges of the grating; a treatment based on the Doppler shift is more convenient.

For a grating under incidence by a single monochromatic plane wave (with time-dependent incidence angle), the duration of high frequency shift regime is very short because the large angular dispersion occurs very close to the Wood anomaly angle. For a grating under incidence from the diffuse skylight, Wood anomaly can take place continuously as the grating rotates because the light is incident from a wide range of angles and a wide frequency band.

This work has been made easier by the availability of a well-documented example of a grating which turns out to be well-suited for Wood anomaly-based frequency shifting. Indeed, the nanostructures in *Morpho rhetenor* wing scales are remarkable for their ability to efficiently reflect light into higher diffraction orders while reducing substantially the light coupled into the specular order. It also raises the question as to whether frequency shifting plays any role in the intense blue colour which can be observed as the *Morpho* butterfly is flying. Oscillating gratings are ubiquitous in nature and it would be interesting to further investigate whether the oscillatory motion of naturally occurring gratings can generate a frequency shift which can affect the behaviour of animals and plants.

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