Kirchhoff Index As a Measure of Edge Centrality in Weighted Networks: Nearly Linear Time Algorithms

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Abstract

Estimating the relative importance of vertices and edges is a fundamental issue in the analysis of complex networks, and has found vast applications in various aspects, such as social networks, power grids, and biological networks. Most previous work focuses on metrics of vertex importance and methods for identifying powerful vertices, while related work for edges is much lesser, especially for weighted networks, due to the computational challenge. In this paper, we propose to use the well-known Kirchhoff index as the measure of edge centrality in weighted networks, called $\theta$-Kirchhoff edge centrality. The Kirchhoff index of a network is defined as the sum of effective resistances over all vertex pairs. The centrality of an edge $e$ is reflected in the increase of Kirchhoff index of the network when the edge $e$ is partially deactivated, characterized by a parameter $\theta$. We define two equivalent measures for $\theta$-Kirchhoff edge centrality. Both are global metrics and have a better discriminating power than commonly used measures, based on local or partial structural information of networks, e.g. edge betweenness and spanning edge centrality.

Despite the strong advantages of Kirchhoff index as a centrality measure and its wide applications, computing the exact value of Kirchhoff edge centrality for each edge in a graph is computationally demanding. To solve this problem, for each of the $\theta$-Kirchhoff edge centrality metrics, we present an efficient algorithm to compute its $\epsilon$-approximation for all the $m$ edges in nearly linear time in $m$. The proposed $\theta$-Kirchhoff edge centrality is the first global metric of edge importance that can be provably approximated in nearly-linear time. Moreover, according to the $\theta$-Kirchhoff edge centrality, we present a $\theta$-Kirchhoff vertex centrality measure, as well as a fast algorithm that can compute $\epsilon$-approximate Kirchhoff vertex centrality for all the $n$ vertices in nearly linear time in $m$. 

1
1 Introduction

Most real networks (e.g. social networks) are massive and inhomogeneous [New10], where the roles of vertices/edges are often largely different, with peripheral vertices/edges having a limited effect on their function, while central vertices/edges having a strong impact on dynamical processes. Thus, it is of paramount importance to design both desirable metrics measuring the centrality or importance of vertices/edges and fast algorithms identifying vital vertices/edges [LM12]. In past decades, a lot of centrality measures have been presented to capture diverse aspects of the informal concept of importance, and various algorithms for these different metrics have been developed by researchers from interdisciplinary areas, such as computer science [WS03, BV14, BDFMR16], control science [YTQ17], and physics [LCR+16]. At present, it is still an active research topic in the scientific community.

Most previous work about centrality measures and algorithms concentrated on the vertex level, in spite of the fact that edge centrality plays an equally important role as its vertex counterpart. For example, edge centrality has been applied to detect communities of a network [GN02], which are dense subgraphs corresponding to functional units within the network. In addition, edge centrality is helpful to describe the intensity of social ties among individuals in social networks [DSD+11], and is also instrumental in revealing new knowledge in semantic web [BLHL01]. Last but not the least, edge centrality plays an indispensable role in designing or protecting infrastructure networks, e.g. power grids [BCH14]. Therefore, it is interesting to propose an edge centrality measure and develop algorithms characterizing the importance of an edge in networks relative to other edges.

Several measures for edge centrality have been proposed, including edge betweenness [Bra01, BKMM07, BP07, GSS08], spanning edge centrality [TMC+13, MGLKT15, HAY16], and current-flow centrality [BF05]. The betweenness of an edge is the fraction of shortest paths between vertex pairs that pass through the edge. The spanning edge centrality of an edge is defined as the probability that it is present in a randomly chosen spanning tree. While the current-flow centrality of an edge describes the amount of current flowing through it. Although these edge centrality metrics have been extensively studied, they themselves are subject to weakness. For example, edge betweenness only considers shortest paths and ignores those longer paths; spanning edge centrality cannot separate an edge linked to a leaf vertex and another cut edge connecting two large subgraphs, while the importance of these two edges are obviously different. Moreover, these measures are proposed for unweighted networks, and are either unapplicable to weighted networks, or have high computational complexity when applied to weighted networks.

In fact, it is very difficult to rigorously compare different measures of edge centrality, since the criteria of edge importance depend on real applications and the problems we are concerned with [YTQ17]. Hence, it is neither practical nor feasible to propose a universal measure that best quantifies the importance of edges for all situations. One should define the metric of edge centrality according to particular problems. In many real scenarios, the Kirchhoff index [KR93] of a network, defined as the sum of effective resistances over all vertex pairs, can be used as a unifying indicator to measure the interesting quantities associated with different problems in networks. For example, Kirchhoff index can be used to measure the mean cost of search in a complex network [FQY14], robustness of first-order consensus algorithm in noisy networks [PB14], the global utility of social recommender systems [WLC16], among others. Notwithstanding the relevance of the Kirchhoff index in various applications, there is a disconnect between this notion and efficiency of algorithms for estimating it.

The main purpose of this paper is to develop an edge centrality notion that not only has good discriminating power, but also can be evaluated using algorithms with good provable performances. For a connected graph, the popular Kirchhoff index follows Rayleigh’s monotonicity law [ESVM+11].
That is, the Kirchhoff index of a graph strictly increases when the weight of any edge is decreased. Based on this property, in this paper, we adopt the Kirchhoff index as an importance measure of edge in undirected weighted connected networks with a positive weight for each edge, which we call Kirchhoff edge centrality. In order to explore the role of an edge \( e \), we partially deactivate the edge \( e \) by changing its weight \( w(e) \) to \( \theta w(e) \), where \( 0 < \theta \leq 1/2 \) is a small scalar, and compute the Kirchhoff index of the resulting graph. The centrality of edge \( e \) is reflected in the Kirchhoff index of the new graph: the larger the Kirchhoff index is, the more important the edge \( e \) is. We define two equivalent metrics for edge centrality. One is the Kirchhoff index of the new graph after edge deactivation, the other is the difference of the Kirchhoff indices between the new graph and the original graph. For either edge centrality measure, we give a fast algorithm to compute the \( \epsilon \)-approximation for all the \( m \) edges in nearly linear time. Furthermore, based on the Kirchhoff edge centrality index, we propose a vertex importance measure, with the centrality of a vertex being defined as the Kirchhoff index of a new graph, where all edges incident to it are deactivated, and provide an efficient algorithm for estimating this new vertex centrality.

1.1 Related Works

Some edge centrality measures and related algorithms have been proposed. Here we give a brief introduction to these metrics and their computational complexity. Moreover, we simply describe some work or techniques that partially motivate this paper or relate to our algorithm.

Edge betweenness is probably the most popular and most studied measure of edge importance. It measures the probability that a shortest path between two vertices passes through a given edge. A fast algorithm for exact computation of edge betweenness was developed by Brandes [Bra01]. For a graph with \( n \) vertices and \( m \) edge, the complexity for this efficient technique is \( O(nm) \) and \( O(nm + n^2 \log n) \) for unweighted graphs and weighted graphs, respectively. In order to speed up the computation, some approximate algorithms have been proposed [BKMM07, BP07, GSS08]. All these approximate approaches aim at reducing the computation of shortest paths in different ways, without providing approximation guarantees.

Another edge importance measure is spanning edge centrality first introduced in [TMC+13]. The spanning edge centrality of an edge is equal to probability that the edge is used in a randomly selected spanning tree. The best known exact algorithm has a running time \( O(mn^{3/2}) \). In order to compute spanning edge centrality for massive networks, two fast approximation algorithms [MGLKT15, HAY16] have been designed, both having theoretical guarantees on their accuracy.

A third measure for edge importance is current-flow centrality introduced by Brandes and Fleischer [BF05]. An edge has relatively significant importance, if it participates in many short paths connecting pairs of vertices. Brandes and Fleischer [BF05] provided an algorithm with time complexity \( O(mn^{3/2} \log n) \), which can actually be dropped to \( O(mn \log n) \) as shown in [MGLKT15].

Both spanning edge centrality and current-flow centrality are closely related to effective resistance [MGLKT15]. In fact, the Kirchhoff edge centrality we propose belongs to the same class of electrical centrality measures. Moreover, this is the first definition of a global notion of centrality that can be provably approximated in nearly-linear time, which means that resistance based edge centrality for graphs may actually be easier to compute than other centrality measures based on discrete structures, e.g. triangles or shortest paths.

As many previous theoretical studies [BHNT15, KP17], our work is also motivated by graph mining applications. In [BHNT15], a streaming algorithm was developed for analyzing large-scale rapidly-changing graphs, which maintains densest subgraphs in one pass and achieves time and space efficiency, whereas in our case, effective resistances are persevered under updates. All the notions (dense subgraph [BHNT15], triangle [KP17], and Kirchhoff centrality) studied before or
in the present paper have vast applications in network analysis, and their related computational challenges fall within the scope of computer theory.

Our algorithms, in particular the resistance maintenance routines, closely build upon the sketching based inverse maintenance routine from [LSW15] and the computation of multiple partial states of Gaussian eliminations from [DKP+17]. The former maintains the inverse of a matrix under updates, and is a critical routine for many graph algorithms [San04, LS15, HX16]. However, this often leads to dense matrices, and we combine it with techniques from graph sparsification [SS11, ADK+16, KPPS17, LS17] to obtain our nearly-linear running times. The additional need to maintain dot-products against arbitrary vectors also leads us to incorporate iterative methods in our routine for approximating the vertex Kirchhoff centrality. This demonstrates the robustness of our algorithm in combining two different ways (Johnson-Lindenstrauss lemma and Schur complements) of computing effective resistances. We believe our results can be extended to provide more access to even more graph quantities motivated by practical problems on graphs.

1.2 Our Results

For a graph $G$, we write $G \setminus e$ to denote the graph obtained from $G$ by deactivating edge $e$, i.e., decreasing the weight of $e$ from $w(e)$ to $\theta w(e)$ for some small $0 < \theta \leq 1/2$. Let $L$ be the Laplacian matrix of $G$, and let $L \setminus e$ denote the Laplacian matrix of $G \setminus e$. Then we can define two metrics for $\theta$-Kirchhoff centrality of an edge $e$, denoted by $C_\theta(e)$ and $C_\theta^\Delta(e)$, respectively. $C_\theta(e)$ is the Kirchhoff index of the graph $G \setminus e$, i.e., the sum of effective resistances over all vertex pairs in $G \setminus e$, while $C_\theta^\Delta(e)$ is the difference between the Kirchhoff indices of graph $G \setminus e$ and graph $G$. Let $K(G)$ denote the Kirchhoff index of graph $G$, then we have $C_\theta(e) = K(G \setminus e)$ and $C_\theta^\Delta(e) = K(G \setminus e) - K(G)$.

As has been shown in [ESVM+11], the Kirchhoff index of a graph equals $n$ times $\text{Tr}(L^\dagger)$, where $L^\dagger$ is the pseudoinverse of the graph’s Laplacian matrix. Thus, we have $C_\theta(e) = n \text{Tr}((L \setminus e)^\dagger)$ and $C_\theta^\Delta(e) = n \text{Tr}((L \setminus e)^\dagger) - n \text{Tr}(L^\dagger)$. To compute the exact value of $\theta$-Kirchhoff centrality for each edge, a naive algorithm would invert the matrix $L \setminus e$ for all $e \in E$. Since a single inversion takes $O(n^\omega)$ time, where $\omega \approx 2.373$ is the matrix multiplication constant [Wil12], the naive algorithm runs in $O(n^{\omega}m)$ time for all the $m$ edges, which makes it untractable for large networks.

In this paper, we consider the scenario in which only approximate values of $\theta$-Kirchhoff centrality are needed. Such approximations are acceptable in many cases because we only need to estimate relative importance of edges. We give a randomized algorithm EDGECentComp1 that computes $\epsilon$-approximate Kirchhoff edge centrality $C_\theta(e)$ for all the $m$ edges in $\tilde{O}(me^{-4})$ time, and a randomized algorithm EDGECentComp2 that computes $\epsilon$-approximate Kirchhoff edge centrality $C_\theta^\Delta(e)$ for all the $m$ edges in $\tilde{O}(m\theta^{-2}e^{-2})$ time. The key ingredients of algorithm EDGECentComp1 are Schur complements and Cholesky factorizations, which have been used in various applications, such as solving linear systems in Laplacians [KLP+16, KS16] and counting and sampling spanning trees [DKP+17, DPPR17]. And the key technique for algorithm EDGECentComp2 is the combination of sketching with the Sherman-Morrison formula [SM50] from efficient maintenances of matrix inverses for optimization [LSW15].

The performance of the algorithm EDGECentComp1 is characterized in the following theorem.

**Theorem 1.1.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, positive edge weights $w : E \to \mathbb{R}_+$, and scalars $0 < \theta \leq 1/2$, $0 < \epsilon \leq 1/2$, the algorithm EDGECentComp1$(G = (V, E), w, \theta, \epsilon)$ returns a set of pairs $\hat{C} = \{(e, \hat{c}_e) \mid e \in E\}$. With high probability, the following
The total running time of this algorithm is bounded by \( O(m\epsilon^{-4}\log^2 m \log^7 n \text{polylog}(n)) \).

The proof of this theorem appears in Section 4.

In Theorem 1.1, \( \theta \) is arbitrary and can even depend on \( n \), e.g. \( 1/n \). When \( \theta \) is constant, we can give a simpler algorithm \textit{EdgeCentComp2} that approximates \( \hat{C}_\theta^\Delta \)-Kirchhoff edge centrality for all \( m \) edges in \( \tilde{O}(m\theta^{-2}\epsilon^{-2}) \) time. The idea is to use the Sherman-Morrison formula, which gives a fractional expression of the difference between \( L^\dagger \) and \( (L\setminus\theta e)^\dagger \), where we can approximate the numerator by the Johnson-Lindenstrauss lemma, and the denominator by estimating effective resistances. The technique is similar to the approach in [LSW15].

**Theorem 1.2.** Given a connected undirected graph \( G = (V, E) \) with \( n \) vertices, \( m \) edges, positive edge weights \( w : E \to \mathbb{R}_+ \), and scalars \( 0 < \theta \leq 1/2, 0 < \epsilon \leq 1/2 \), the algorithm \textit{EdgeCentComp2}(\( G = (V, E), w, \theta, \epsilon \)) returns a set of pairs \( \hat{C} = \{(e, \hat{c}_e^\Delta) \mid e \in E\} \). With high probability, the following statement holds: For \( \forall e \in E \),

\[
C_\theta^\Delta(e) \approx \hat{c}_e^\Delta,
\]

where

\[
C_\theta^\Delta(e) = \sum_{u, v \in V} R_{\text{eff}}^{G \setminus \theta e}(u, v)
\]

is the sum of effective resistances \( R_{\text{eff}}^{G \setminus \theta e}(u, v) \) over all vertex pairs \( u \) and \( v \) in graph \( G \setminus \theta e \). The total running time of this algorithm is bounded by \( O(m\theta^{-2}\epsilon^{-2}) \).

The proof of this theorem appears in Section 5. Its advantage is that for moderate values of \( \theta \), it can obtain a more accurate estimate of \( C_\theta^\Delta(e) \) even if \( K(G) \) is large. However, when \( \theta \) is small, the removal of high effective resistance edges can cause a large error in this routine, and we are not guaranteed to even get a good estimate of \( C_\theta(e) \) by adding this result to an estimate of \( K(G) \). As a result we believe both of our algorithms for estimating Kirchhoff edge centrality are of interest, and complement each other.

Based on the same idea of the definition for \( C_\theta^\Delta(e) \), we can define a centrality measure for any vertex \( v \), which is the difference of Kirchhoff indices between the new graph \( G \setminus \theta v \) and original graph \( G \), where \( G \setminus \theta v \) is obtained from \( G \) by multiplying the weights of all edges incident with \( v \) by \( \theta \). We write \( C_\theta^\Delta(v) \) to denote the \( \theta \)-Kirchhoff vertex centrality of \( v \). In this situation, the matrix perturbation caused by removing the neighborhood of \( v \) is no longer rank 1, and we need to leverage the approximate Schur complement routines from Section 4.2.2 to compute these intermediate matrices. Specifically, for constant \( \theta \), we give an algorithm \textit{VertexCentComp} that approximates \( \theta \)-Kirchhoff vertex centrality for all \( n \) vertices in \( \tilde{O}(m\theta^{-2}\epsilon^{-2}) \) time.

**Theorem 1.3.** Given a connected undirected graph \( G = (V, E) \) with \( n \) vertices, \( m \) edges, positive edge weights \( w : E \to \mathbb{R}_+ \), and scalars \( 0 < \theta \leq 1/2, 0 < \epsilon \leq 1/2 \), the algorithm \textit{VertexCentComp}(\( G = (V, E), w, \theta, \epsilon \)) returns a set of pairs \( \hat{C} = \{(v, \hat{c}_v^\Delta) \mid v \in V\} \). With high probability, the following statement holds: For \( \forall v \in V \),

\[
C_\theta^\Delta(v) \approx \hat{c}_v^\Delta,
\]
where
\[ C^\Delta_\theta (v) = \mathcal{K}(G\setminus_{\theta} E_v) - \mathcal{K}(G) \]
and \( E_v = \{(u, v) \mid u \sim v\} \) is the set of edges incident with \( v \). The total running time of this algorithm is bounded by \( O(m(\theta^{-2} \epsilon^{-4} \log^9 n + \theta^{-2.5} \epsilon^{-4} \log^6 n \log(1/\epsilon)) \text{polyloglog}(n)) \).

The proof of this theorem appears in Section 6.

1.3 Comparison With Other Measures

In addition to the low computational complexity, the \( \theta \)-Kirchhoff centrality is more discriminating than other edge centrality measures, such as edge betweenness centrality and spanning edge centrality. For example, in the graph in Figure 1, the importance of edge \( e_1 \) and edge \( e_2 \) are different, which can be seen by intuition. In fact, we can also understand this difference from the influences when the two edges are deleted. If \( e_1 \) is removed, the length of shortest path between vertices \( u \) and \( v \) increases by 6, while the removal of \( e_2 \) will increase the length of shortest path between any pair of vertices by at most 1. However, the betweenness centrality for \( e_1 \) and \( e_2 \) are the same, being equal to 18, implying that betweenness centrality cannot differentiate \( e_1 \) between \( e_2 \). However, these two edges can be discriminated by the \( \theta \)-Kirchhoff edge centrality. Exact computation shows that the 0.1-Kirchhoff edge centrality for \( e_1 \) and \( e_2 \) is \( C_{0.1}(e_1) = 132.65 \) and \( C_{0.1}(e_2) = 112.34 \), respectively. Thus, \( e_1 \) is relatively more important than \( e_2 \), which agrees with our human intuition.

We continue to show that the \( \theta \)-Kirchhoff centrality is also more discriminating than the spanning edge centrality. By intuition, the importance for the two edges \( e_3 \) and \( e_4 \) of the graph illustrated in Figure 2 are distinct. Unfortunately, spanning edge centrality cannot distinguish these two edges, since their spanning edge centrality is identical, both equalling 1. In contrast, the 0.1-Kirchhoff edge centrality of these two edges \( e_3 \) and \( e_4 \) is \( C_{0.1}(e_3) = 467.33 \) and \( C_{0.1}(e_4) = 197.33 \), respectively. This implies that \( e_3 \) plays a relatively more significant role than \( e_4 \), which is consistent with our intuition.

To further show the capability of our \( \theta \)-Kirchhoff edge centrality to discriminate between different edges, we experimentally compare our measure \( C^\Delta_\theta \) with other metrics, including edge centrality, spanning edge centrality, and current-flow edge centrality. For each measure, we numerically evaluate the importance of each edge for some classic real-world networks\(^1\) in Table 1. The data sets are from

\[^1\] All data can be found at http://www-personal.umich.edu/~mejn/netdata/
Table 1: Some classic real networks.

| Network name | Number of vertices | Number of edges |
|--------------|--------------------|-----------------|
| Karate [Zac77] | 34                | 78              |
| Lesmis [Knu93] | 77                | 254             |
| Adjnoun [New06] | 112              | 425             |
| Dolphins [LSB’03] | 62               | 159             |
| Celegansneural [WS98] | 297       | 2148            |

Figure 3: Relative standard deviation for different edge centrality measures.

published data mining related papers [Zac77, Knu93, New06, LSB’03, WS98]. Based on which we then compute the relative standard deviation for each centrality measure (as the authors did in [BWLM16]), where the relative standard deviation is defined as the standard deviation divided by the average. Figure 3 shows the relative standard deviation for all the centrality measures. It is always significantly higher for $\theta$-Kirchhoff edge centrality than it is for other measures, meaning that our measure has a better capability to distinguish between different edges.

1.4 Organization

The remaining part of the paper is organized as follows. In Section 2, we present the background and formulate the problem of computing $\theta$-Kirchhoff centrality. In Section 3, we introduce Schur complements and partial Cholesky factorizations, and a lemma with regard to the performance of the approximate partial Cholesky factorization algorithm in [DKP’17]. In Section 4, we introduce our algorithm EDGECENTCOMP1 that approximates $C_{\theta}(e)$. In Section 5, we introduce our algorithm EDGECENTCOMP2 that approximates $C_{\theta}^\Delta(e)$. In Section 6, we introduce our algorithm...
VertexCentComp that approximates $C^\Delta_\theta(v)$. In Section 7, we give our conclusion and discuss some directions for future works.

2 Background and the Problem

2.1 Multiplicative Approximation of Scalars and Matrices

We use the notion of $\epsilon$-approximation in [PS14].

Let $a, b \geq 0$ be two nonnegative scalars. We say $a$ is an $\epsilon$-approximation of $b$ if

$$\exp(-\epsilon) a \leq b \leq \exp(\epsilon) a.$$ (4)

We write $a \approx_\epsilon b$ to denote Eq. (4).

For two matrices $A$ and $B$, we write $A \preceq B$ to indicate that $B - A$ is positive semidefinite. We say $A$ is an $\epsilon$-spectral approximation of $B$ if

$$\exp(-\epsilon) A \preceq B \preceq \exp(\epsilon) A.$$ (5)

We write $A \approx_\epsilon B$ to denote Eq. (5).

Note that these two relations are symmetric. Namely, $a \approx_\epsilon b$ implies $b \approx_\epsilon a$ and $A \approx_\epsilon B$ implies $B \approx_\epsilon A$.

The following facts are basic properties of $\epsilon$-approximation:

**Fact 2.1.** For nonnegative scalars $a, b, c, d \geq 0$, positive semidefinite matrices $A, B, C, D$,

1. if $a \approx_\epsilon b$, then $a + c \approx_\epsilon b + c$;
2. if $a \approx_\epsilon b$ and $c \approx_\epsilon d$, then $a + c \approx_\epsilon b + d$;
3. if $a \approx_{\epsilon_1} b$ and $b \approx_{\epsilon_2} c$, then $a \approx_{\epsilon_1 + \epsilon_2} c$;
4. if $a$ and $b$ are positive such that $a \approx_\epsilon b$, then $1/a \approx_\epsilon 1/b$;
5. if $a \approx_\epsilon b$, then $ac \approx_\epsilon bc$;
6. if $A \approx_\epsilon B$, then $A + C \approx_\epsilon B + C$;
7. if $A \approx_\epsilon B$ and $C \approx_\epsilon D$, then $A + C \approx_\epsilon B + D$;
8. if $A \approx_{\epsilon_1} B$ and $B \approx_{\epsilon_2} C$, then $A \approx_{\epsilon_1 + \epsilon_2} C$;
9. if $A$ and $B$ are positive definite matrices such that $A \approx_\epsilon B$, then $A^{-1} \approx_\epsilon B^{-1}$;
10. if $A \approx_\epsilon B$ and $V$ is a matrix, then $V^\top AV \approx_\epsilon V^\top BV$.

2.2 Graphs and Laplacians

We consider a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, and positive edge weights $w : E \to \mathbb{R}_+$. For a pair of vertices $u, v \in E$, we write $u \sim v$ to denote $(u, v) \in E$. The Laplacian matrix of $G$ is an $n \times n$ matrix $L$ with the entry on its $u^{th}$ row and $v^{th}$ column being

$$L(u, v) = \begin{cases} -w(u, v) & \text{if } u \sim v, \\ \deg(u) & \text{if } u = v, \\ 0 & \text{otherwise}, \end{cases}$$
where \( \deg(u) = \sum_{u \sim v} w(u, v) \). If \( A \) and \( B \) are two sets of vertices in \( G \), we write \( L_{AB} \) to denote the submatrix of \( L \) with rows corresponding to \( A \) and columns corresponding to \( B \).

Let \( e_i \) denote the \( i \)th standard basis vector, and \( b_{u,v} = e_u - e_v \). We fix an arbitrary orientation of the edges in \( G \). For each edge \( e \in E \), we define \( b_e = b_{u,v} \), where \( u \) and \( v \) are head and tail of \( e \), respectively. It is easy to show that \( L = \sum_{e \in E} w(e) b_e b_e^\top \). We refer to \( w(e) b_e b_e^\top \) as the Laplacian of \( e \).

It is immediate that \( L \) is positive semidefinite since
\[
\sum_{e \in E} w(e) b_e b_e^\top \sum_{e \in E} w(e) b_e b_e^\top = \sum_{e \in E} w(e) \left( x^\top b_e \right)^2 \geq 0
\]
holds for any \( x \in \mathbb{R}^n \).

### 2.3 The Pseudoinverse, Effective Resistances, and Kirchhoff Index

Since \( L \) is positive semidefinite, we can diagonalize it and write
\[
L = \sum_{i=1}^{n-1} \lambda_i v_i v_i^\top,
\]
where \( \lambda_1, \ldots, \lambda_{n-1} \) are the nonzero eigenvalues of \( L \) and \( v_1, \ldots, v_{n-1} \) are the corresponding orthonormal eigenvectors. The pseudoinverse of \( L \) is defined as
\[
L^\dagger = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i v_i^\top.
\]

It is not hard to show that if \( L \) and \( H \) are Laplacian matrices of connected graphs and \( L \approx \epsilon H \), then \( L^\dagger \approx \epsilon H^\dagger \).

We then give the definitions of effective resistance and Kirchhoff index:

**Definition 2.2** (Effective Resistance). For a connected undirected graph \( G = (V, E) \), the effective resistance between \( u \) and \( v \) is defined as
\[
R_{\text{eff}}^G(u, v) = b_{u,v}^\top L^\dagger b_{u,v}.
\]

**Definition 2.3** (Kirchhoff Index). The Kirchhoff index \( K(G) \) of a graph \( G = (V, E) \) is defined as the sum of effective resistances over all vertex pairs. Namely,
\[
K(G) = \sum_{u,v \in V} R_{\text{eff}}^G(u, v).
\]

For a graph, Kirchhoff index is a measure of its overall connectedness. A graph with smaller Kirchhoff index is better connected on an average. It is known that the Kirchhoff index of a graph equals \( n \) times the sum of reciprocals of nonzero eigenvalues of \( L \) [ESVM11], and hence also equals \( n \) times the trace of \( L^\dagger \). We give this relation in the following Fact:

**Fact 2.4.** Let \( \lambda_1, \ldots, \lambda_{n-1} \) be the nonzero eigenvalues of \( L \). The Kirchhoff index of graph \( G \) satisfies
\[
K(G) = n \sum_{i=1}^{n-1} \frac{1}{\lambda_i} = n \text{Tr} \left( L^\dagger \right).
\]

By Rayleigh’s Monotonicity Law [ESVM11], the effective resistance between any pair of vertices can only increase when edges are deleted or edge weights are decreased. Since the Kirchhoff index is the sum of effective resistances over all vertex pairs, we have the following Fact:

**Fact 2.5.** The Kirchhoff index of a graph does not decrease when edges are deleted or edge weights are decreased.
2.4 Kirchhoff Edge Centrality

With Fact 2.5, it is reasonable to measure the importance of an edge $e$ in graph $G$ by the Kirchhoff index of the new graph in which $e$ is deactivated. We formalize the notion of edge deactivation by defining $\theta$-deletion of an edge.

**Definition 2.6 ($\theta$-Deletion).** Let $0 < \theta \leq 1/2$ be a scalar. For an edge $e \in E$, the $\theta$-deletion of $e$ is to decrease its weight from $w(e)$ to $\theta w(e)$. If we use $G_{\theta} e$ to denote the graph obtained from $G$ by $\theta$-deleting edge $e$, and $L_{\theta} e$ to denote the Laplacian matrix corresponding to $G_{\theta} e$, we have

$$L_{\theta} e = L - (1 - \theta) w(e) b_e b_e^T.$$

We can also define $\theta$-Deletion of an edge set $B \subset E$, which is to decrease the weight of each edge $e \in B$ from $w(e)$ to $\theta w(e)$. Similar notations are $G_{\theta} B$ and

$$L_{\theta} B = L - (1 - \theta) \sum_{e \in B} w(e) b_e b_e^T.$$

We then give the definitions of $\theta$-Kirchhoff edge centrality $C_{\theta}(e)$ and $C_{\theta}^\Delta(e)$.

**Definition 2.7 ($\theta$-Kirchhoff Edge Centrality $C_{\theta}$).** Let $0 < \theta \leq 1/2$ be a scalar. For an edge $e \in E$, its $\theta$-Kirchhoff edge centrality $C_{\theta}(e)$ is defined as the Kirchhoff index of the graph obtained from $G$ by $\theta$-deleting $e$. Namely,

$$C_{\theta}(e) = \mathcal{K}(G_{\theta} e).$$

**Definition 2.8 ($\theta$-Kirchhoff Edge Centrality $C_{\theta}^\Delta$).** Let $0 < \theta \leq 1/2$ be a scalar. For an edge $e \in E$, its $\theta$-Kirchhoff edge centrality $C_{\theta}^\Delta(e)$ is defined as the increase of the Kirchhoff index of the graph upon edge $e$’s $\theta$-deletion. Namely,

$$C_{\theta}^\Delta(e) = \mathcal{K}(G_{\theta} e) - \mathcal{K}(G).$$

Clearly, these two definitions of Kirchhoff edge centrality lead to the same ranking of edges.

Following the above two definitions of Kirchhoff edge centrality, we can also define $\theta$-centrality of a vertex.

**Definition 2.9 ($\theta$-Kirchhoff Vertex Centrality).** Let $0 < \theta \leq 1/2$ be a scalar. For a vertex $v \in V$, its $\theta$-Kirchhoff vertex centrality $C_{\theta}^\Delta(v)$ is defined as the increase of the Kirchhoff index of the graph upon $\theta$-deletion of its incident edges. Namely,

$$C_{\theta}^\Delta(v) = \mathcal{K}(G_{\theta} E_v) - \mathcal{K}(G),$$

where $E_v = \{(u, v) \mid u \sim v\}$ is the set of edges incident with $v$.

We now formulate the core problems of approximating $\theta$-Kirchhoff centrality:

**Problem 1.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, and positive edge weights $w : E \to \mathbb{R}_+$, and scalars $0 < \theta \leq 1/2$, $0 < \epsilon \leq 1/2$, for each $e \in E$, find an $\epsilon$-approximation of its $\theta$-Kirchhoff edge centrality $C_{\theta}(e)$.

**Problem 2.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, and positive edge weights $w : E \to \mathbb{R}_+$, and scalars $0 < \theta \leq 1/2$, $0 < \epsilon \leq 1/2$, for each $e \in E$, find an $\epsilon$-approximation of its $\theta$-Kirchhoff edge centrality $C_{\theta}^\Delta(e)$.

**Problem 3.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, and positive edge weights $w : E \to \mathbb{R}_+$, and scalars $0 < \theta \leq 1/2$, $0 < \epsilon \leq 1/2$, for each $v \in V$, find an $\epsilon$-approximation of its $\theta$-Kirchhoff vertex centrality $C_{\theta}^\Delta(v)$. 
3 Schur Complements and Partial Cholesky Factorizations

In this section, we introduce Schur complements and partial Cholesky factorizations, which are key techniques in our algorithm.

3.1 Preliminaries

We first give the definitions of Schur complements and partial Cholesky factorizations according to [KS16, DKP+17].

Definition 3.1 (Schur Complement). Suppose $L$ is the Laplacian of an undirected positive-weighted connected graph, and $L(:,i)$ is the $i^{th}$ column of $L$. For a vertex $v_1$,

$$S^{(1)}(v_1) \equiv L - \frac{1}{L(v_1,v_1)} L(:,v_1) L(:,v_1)^\top$$

is called the Schur complement of $L$ with respect to vertex $v_1$. The operation of subtracting $\frac{1}{L(v_1,v_1)} L(:,v_1) L(:,v_1)^\top$ from $L$ is called the elimination of vertex $v_1$. Suppose we perform a sequence of eliminations, where in the $i^{th}$ step, we select a vertex $v_i \in V \setminus \{v_1,\ldots,v_{i-1}\}$ and eliminate the vertex $v_i$. We define

$$\alpha_i = S^{(i-1)}(v_i,v_i),$$

$$c_i = \frac{1}{\alpha_i} S^{(i-1)}(:,v_i),$$

$$S^{(i)} = S^{(i-1)} - \alpha_i c_i c_i^\top.$$

Then $S^{(i)}$ is called the Schur complement with respect to vertices $\{v_1,\ldots,v_i\}$. Let $C = \{v_1,\ldots,v_i\}$, then we also write $\text{Sc}(L,C) = S^{(i)}_{CC}$ to denote the Schur complement of $L$ onto $C$.

Definition 3.2 (Partial Cholesky factorization). Suppose we eliminate a sequence of vertices $v_1,\ldots,v_i$. Let $L$ be the $n \times i$ matrix with $c_j$ as its $j^{th}$ column, and $D$ be the $i \times i$ diagonal matrix $D(j,j) = \alpha_j$, then

$$L = S^{(i)} + \sum_{j=1}^i \alpha_j c_j c_j^\top = S^{(i)} + L D L^\top.$$

Let us write $F = \{v_1,\ldots,v_i\}$, and $C = V \setminus F$. Let $S$ be the submatrix of $S^{(i)}$ with rows and columns corresponding to vertices in $C$, i.e., $S = S^{(i)}_{CC}$. Since $S^{(i)}_{CC}$ contains all nonzero entries of $S^{(i)}$, we can write $L = \begin{pmatrix} L_{FF} & L_{CF} \\ L_{CF} & L_{CC} \end{pmatrix}$ and

$$L = \begin{pmatrix} L_{FF} \\ L_{CF} \end{pmatrix} D \begin{pmatrix} L_{FF} & L_{CF} \end{pmatrix}^\top + \begin{pmatrix} 0_{FF} & 0_{FC} \\ 0_{CF} & S \end{pmatrix} = \begin{pmatrix} L_{FF} & 0 \\ L_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} L_{FF} & 0 \\ L_{CF} & I_{CC} \end{pmatrix}^\top. \quad (6)$$

Here $\begin{pmatrix} L_{FF} & 0 \\ L_{CF} & I_{CC} \end{pmatrix}$ is a lower triangular matrix up to row exchanges, and $D$ is diagonal. Eq. (6) is known as partial Cholesky factorization.

It is known that Schur complements of a Laplacian are also Laplacians:

Fact 3.3 (Fact 5.1 of [DKP+17]). The Schur complement of a Laplacian w.r.t. vertices $v_1,\ldots,v_i$ is a Laplacian.
3.2 Commutativity With Edge Deletions

According to [KS16], we can write the Schur complement w.r.t. a vertex \( v_1 \) as

\[
S^{(1)} = \sum_{e \in E : e \not\ni v_1} w(e) b_e b_e^\top + \sum_{u \sim v_1} \sum_{v \sim v_1} \frac{w(u, v_1)w(v, v_1)}{\deg(v_1)} b_{u,v} b_{u,v}^\top,
\]

(7)

where the first term on rhs is the Laplacian corresponding to the edges not incident with \( v_1 \), and the second term on rhs is a Laplacian whose edges are supported on \( V \setminus \{v_1\} \). Thus, \( S^{(1)} \) can be seen as a multigraph obtained by adding edges to \( G[V \setminus \{v_1\}] \), the induced graph of \( G \) on \( V \setminus \{v_1\} \). By induction, for all \( i \), \( S^{(i)} \) can be seen as a multigraph obtained by adding edges to \( G[V \setminus \{v_1, \ldots, v_i\}] \), the induced graph of \( G \) on \( V \setminus \{v_1, \ldots, v_i\} \). Also, by Eq. (7), edges added to \( G[V \setminus \{v_1\}] \) to obtain \( S^{(1)} \) are fully determined by edges incident with \( v_1 \) in the original graph \( G \). By induction, for all \( i \), edges added to \( G[V \setminus \{v_1, \ldots, v_i\}] \) to obtain \( S^{(i)} \) are fully determined by edges incident with \( \{v_1, \ldots, v_i\} \) in the original graph \( G \). Thus, deletions (or \( \theta \)-deletions) performed to edges with both endpoints in \( V \setminus \{v_1, \ldots, v_i\} \) commute with taking partial Cholesky factorization. Therefore, we have the following lemma:

**Lemma 3.4.** Given a connected undirected graph \( G = (V, E) \), with positive edge weights \( w : E \to \mathbb{R}_+ \), and associated Laplacian \( L \), a set of vertices \( F = \{v_1, \ldots, v_i\} \subset V \). Let \( C = V \setminus F \), and the partial Cholesky factorization of \( L \) be

\[
L = \left( \begin{array}{c|c} \mathcal{L}_{FF} & \mathcal{L}_{FC} \\ \hline \mathcal{L}_{CF} & \mathcal{L}_{CC} \end{array} \right) \mathcal{D} \left( \begin{array}{c|c} \mathcal{L}_{FF} & \mathcal{L}_{FC} \\ \hline \mathcal{L}_{CF} & \mathcal{L}_{CC} \end{array} \right)^\top + \left( \begin{array}{c|c} 0_{FF} & 0_{FC} \\ \hline 0_{CF} & S \end{array} \right) \left( \begin{array}{c|c} \mathcal{D} & 0 \\ \hline 0 & \mathcal{S} \end{array} \right) \left( \begin{array}{c|c} \mathcal{L}_{FF} & 0 \\ \hline \mathcal{L}_{CF} & \mathcal{I}_{CC} \end{array} \right)^\top.
\]

For any edge \( e \) whose endpoints are both in \( C \), any \( 0 \leq \theta < 1 \),

\[
L_{\theta e} = \left( \begin{array}{c|c} \mathcal{L}_{FF} & \mathcal{L}_{FC} \\ \hline \mathcal{L}_{CF} & \mathcal{L}_{CC} \end{array} \right) \mathcal{D} \left( \begin{array}{c|c} \mathcal{L}_{FF} & \mathcal{L}_{FC} \\ \hline \mathcal{L}_{CF} & \mathcal{L}_{CC} \end{array} \right)^\top + \left( \begin{array}{c|c} 0_{FF} & 0_{FC} \\ \hline 0_{CF} & S_{\theta e} \end{array} \right) \left( \begin{array}{c|c} \mathcal{D} & 0 \\ \hline 0 & \mathcal{S}_{\theta e} \end{array} \right) \left( \begin{array}{c|c} \mathcal{L}_{FF} & 0 \\ \hline \mathcal{L}_{CF} & \mathcal{I}_{CC} \end{array} \right)^\top.
\]

3.3 Approximate Partial Cholesky Factorization Algorithm

In [DKP+17], the authors give an approximate partial Cholesky factorization algorithm, whose performance can be characterized in the following lemma:

**Lemma 3.5** (Lemma 5.7 of [DKP+17], paraphrased). There is an algorithm \( \text{APXPartialCholesky}(L, C, \epsilon) \) that when given a connected undirected graph \( G = (V, E) \), with positive edge weights \( w : E \to \mathbb{R}_+ \), and associated Laplacian \( L \), a set of vertices \( C \subset V \), and a scalar \( 0 < \epsilon \leq 1/2 \), returns a decomposition \( (\tilde{L}, \tilde{\mathcal{D}}, \tilde{S}) \). With high probability, the following statement hold:

\[
L \approx_\epsilon \tilde{L},
\]

(8)

where \( F = V \setminus C \) and

\[
\tilde{L} = \left( \begin{array}{c|c} \tilde{\mathcal{L}}_{FF} & \tilde{\mathcal{L}}_{FC} \\ \hline \tilde{\mathcal{L}}_{CF} & \tilde{\mathcal{L}}_{CC} \end{array} \right) \tilde{\mathcal{D}} \left( \begin{array}{c|c} \tilde{\mathcal{L}}_{FF} & \tilde{\mathcal{L}}_{FC} \\ \hline \tilde{\mathcal{L}}_{CF} & \tilde{\mathcal{L}}_{CC} \end{array} \right)^\top + \left( \begin{array}{c|c} 0_{FF} & 0_{FC} \\ \hline 0_{CF} & \tilde{S} \end{array} \right) \left( \begin{array}{c|c} \tilde{\mathcal{D}} & 0 \\ \hline 0 & \tilde{\mathcal{S}} \end{array} \right) \left( \begin{array}{c|c} \tilde{\mathcal{L}}_{FF} & 0 \\ \hline \tilde{\mathcal{L}}_{CF} & \mathcal{I}_{CC} \end{array} \right)^\top.
\]

(9)

Here \( \tilde{S} \) is a Laplacian matrix whose edges are supported on \( C \) such that \( \tilde{S} \approx_\epsilon \text{Sc}(L, C) \). Let \( k = |C| = n - |F| \). The total number of non-zero entries in \( \tilde{S} \) is \( O(k\epsilon^{-2}\log n) \). \( \left( \begin{array}{c|c} \tilde{\mathcal{L}}_{FF} & 0 \\ \hline \tilde{\mathcal{L}}_{CF} & \mathcal{I}_{CC} \end{array} \right) \) is a
lower triangular matrix up to row exchanges. The total number of non-zero entries in \( \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \) is \( O(m + n\epsilon^{-2}\log^3 n) \). \( \tilde{\mathcal{D}} \) is a diagonal matrix.

For any vector \( b \in \mathbb{R}^n \), one can evaluate \( \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix}^{-1} b \) in \( O(m + n\epsilon^{-2}\log^3 n) \) time. For any vector \( c \in \mathbb{R}^{|F|} \), one can evaluate \( (\tilde{\mathcal{D}})^{-1} c \) in \( O(|F|) \) time.

The total running time is bounded by \( O((m\log^3 n + n\epsilon^{-2}\log^5 n)\text{polylog}(n)) \).

Comparing to [DKP+17], Lemma 3.5

1. removes the failure probability factor \( \delta \) and just claims high probability. The running time of the algorithm \textsc{ApxPartialCholesky}(\( L, C, \epsilon \)) in [DKP+17] is \( O((m\log n\log^2(n/\delta) + n\epsilon^{-2}\log n\log^4(n/\delta))\text{polylog}(n)) \). To gain high probability, we just set the failure probability \( \delta \) to \( 1/n^c \) for an arbitrary constant \( c > 0 \). Then we have the running time bounded by \( O((m\log^3 n + n\epsilon^{-2}\log^5 n)\text{polylog}(n)) \).

2. emphasizes that inverses of matrices \( \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \) and \( \tilde{\mathcal{D}} \) can both be applied quickly, as they can be treated as lower triangular matrix and diagonal matrix, respectively.

The following lemma shows that edge additions performed within \( C \) commute with taking approximate partial Cholesky factorization:

**Lemma 3.6.** Given a connected undirected multi-graph \( G = (V, E) \), with positive edge weights \( w : E \to \mathbb{R}_+ \), and associated Laplacian \( L \), a set of vertices \( C \subset V \), and an approximate partial factorization of \( L \):

\[
L \approx_{\epsilon} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{D}} & 0 \\ 0 & \tilde{S} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix}^\top,
\]

where \( F = V \setminus C \). For any edge \( e \) (not necessarily in \( E \)) with both endpoints in \( C \) and a positive scalar \( w_e > 0 \),

\[
L + w_e b_e b_e^\top \approx_{\epsilon} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{D}} & 0 \\ 0 & \tilde{S} \end{pmatrix} \begin{pmatrix} 0 \\ (w_e b_e b_e^\top)_{CC} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix}^\top.
\]

**Proof.** As multiplicative approximations are preserved under additions, by adding \( w_e b_e b_e^\top \) to both sides of Eq. (10) we have

\[
L + w_e b_e b_e^\top \approx_{\epsilon} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{D}} & 0 \\ 0 & \tilde{S} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} & 0 \\ \tilde{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix}^\top + w_e b_e b_e^\top
\]

\[
= \begin{pmatrix} \tilde{\mathcal{L}}_{FF} \\ \tilde{\mathcal{L}}_{CF} \end{pmatrix} \tilde{\mathcal{D}} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} \\ \tilde{\mathcal{L}}_{CF} \end{pmatrix}^\top + \begin{pmatrix} 0_{FF} \\ 0_{CF} \end{pmatrix} \tilde{S} + \begin{pmatrix} 0_{FC} \\ (w_e b_e b_e^\top)_{CC} \end{pmatrix}
\]

\[
= \begin{pmatrix} \tilde{\mathcal{L}}_{FF} \\ \tilde{\mathcal{L}}_{CF} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{D}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (w_e b_e b_e^\top)_{CC} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}_{FF} \\ \tilde{\mathcal{L}}_{CF} \end{pmatrix}^\top.
\]

\( \square \)
4 Algorithm for Approximating $\theta$-Kirchhoff Edge Centrality $C_{\theta}(e)$

4.1 Turning the Kirchhoff Index Into Quadratic Forms of $L^{\dagger}$

By Fact 2.4, the Kirchhoff Index of a graph equals $n$ times the trace the Laplacian’s pseudoinverse. Although the explicit pseudoinverse of $L$ is hard to compute, by taking approximate Cholesky factorizations [KS16, DKP+17], one can approximate $z^{\top}L^{\dagger}z$ for a $z \in \mathbb{R}^n$ quickly. Thus, we can use Monte-Carlo methods to estimate trace of $L^{\dagger}$.

The standard Monte-Carlo method for estimating the trace of an implicit matrix $A$ is due to Hutchinson [Hut89]. The idea is to estimate the trace of $A$ by $\frac{1}{M} \sum_{i=1}^{M} z_i^{\top} Az_i$, where the $z_i$’s are random $\pm 1$ vectors (i.e., independent Bernoulli entries). Since there is $E \left[ z_i^{\top} Az_i \right] = \text{Tr} \left( A \right)$, by the law of large numbers, $\frac{1}{M} \sum_{i=1}^{M} z_i^{\top} Az_i$ should be close to $\text{Tr} \left( A \right)$ when $M$ is large. [AT11] gives a rigorous bound on the number of Monte-Carlo samples required to achieve a maximum error $\epsilon$ with probability at least $1 - \delta$.

**Lemma 4.1** (Theorem 7.1 of [AT11], paraphrased). Let $A$ be a positive semidefinite matrix with rank $\text{rank}(A)$. Let $z_1, \ldots, z_M$ be independent random $\pm 1$ vectors. Let $\epsilon, \delta$ be scalars such that $0 < \epsilon \leq 1/2$ and $0 < \delta < 1$. For any $M \geq 24\epsilon^{-2} \ln(2\text{rank}(A)/\delta)$, the following statement holds with probability at least $1 - \delta$:

$$\frac{1}{M} \sum_{i=1}^{M} z_i^{\top} Az_i \approx \epsilon \text{Tr} \left( A \right).$$

**Remark 4.2.** We remark that the Hutchinson’s method can be seen as Johnson-Lindenstrauss Lemma [JL84] in some sense. The reason is that since $A$ is positive semidefinite, one can write its trace as

$$\text{Tr} \left( A \right) = \text{Tr} \left( A^{1/2} A^{1/2} \right) = \left\| A^{1/2} \right\|_F^2,$$

where $\left\| A^{1/2} \right\|_F^2$ can be seen as a sum of the squared lengths of the rows of $A^{1/2}$. By the discrete version of Johnson-Lindenstrauss Lemma from [Ach01], we can use a $n \times k$ random $\pm 1$ matrix $Q$, where $k = O(\epsilon^{-2} \log n)$, to reduce the dimensions:

$$\left\| A^{1/2} \right\|_F^2 \approx \epsilon \frac{1}{k} \left\| A^{1/2} Q \right\|_F^2.$$

This in turn implies

$$\text{Tr} \left( A \right) \approx \epsilon \frac{1}{k} \sum_{j=1}^{k} q_j^{\top} A q_j,$$

(12)

where $q_j$ is the $j^{th}$ column of $Q$. The rhs of (12) can be seen as Hutchinson’s method. Indeed, [AT11] used the discrete Johnson-Lindenstrauss Lemma from [Ach01] to prove their bound.

Since $L^{\dagger}$ is positive semidefinite and $\text{rank}(L^{\dagger}) = n - 1$, by letting $\delta = 1/n$, we have the following bound on the number of Monte-Carlo samples required to achieve an $\epsilon$-approximation of $\text{Tr} \left( L^{\dagger} \right)$ with high probability:
Lemma 4.3. Let $L$ be a Laplacian matrix. Let $z_1, \ldots, z_M$ be independent random $\pm 1$ vectors. Let $\epsilon$ be a scalar such that $0 < \epsilon \leq 1/2$. For any $M \geq 48\epsilon^{-2}\ln(2n)$, the following statement holds with probability at least $1 - 1/n$:

$$
\frac{1}{M} \sum_{i=1}^{M} z_i^\top L z_i \approx_\epsilon \text{Tr} \left( L^\dagger \right).
$$

A direct conclusion of Lemma 4.3 is that for an edge $e \in E$, its $\theta$-Kirchhoff edge centrality satisfies

$$
C_\theta(e) = \mathcal{K}(G\setminus\theta e) = n\text{Tr}(L/\theta e) \approx_\epsilon \frac{1}{M} \sum_{i=1}^{M} z_i^\top (L/\theta e)^\dagger z_i.
$$

Thereby, the task of approximating the $\theta$-Kirchhoff edge centrality for all $e \in E$ can be divided into $O(\epsilon^{-2}\log n)$ independent tasks, each of which is to compute quadratic forms $z^\top (L/\theta e)^\dagger z$ for a fixed $z \in \mathbb{R}^n$ for all $e \in E$. We formulate these tasks in the following problem:

**Problem 4.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, positive edge weights $w : E \to \mathbb{R}_+$, and associated Laplacian $L$, a set of edges $E^Q \subset E$ such that every vertex in $V$ is incident to some edge $e \in E^Q$, a scalar $0 < \theta \leq 1/2$, and a vector $z \in \mathbb{R}^n$, find (approximately) $z^\top L^\dagger z$ for all $e \in E^Q$.

### 4.2 Computing Quadratic Forms of $L^\dagger$ Upon Edge Deactivation

The idea of solving Problem 4 is to use recursions based on partial Cholesky factorizations. We summarize the key steps in the following enumeration:

1. If $L$ only have $O(1)$ vertices, invert $L/\theta e$ to compute $z^\top (L/\theta e)^\dagger z$ for all $e \in E^Q$ and return.
2. Divide edges in $E^Q$ into $E^{(1)}, E^{(2)}$ with equal sizes.
3. Let $C$ denote endpoints of edges in $E^{(1)}$ and $F = V \setminus C$.
4. By taking (approximate) partial Cholesky factorization of $L$, find a vector $y = \begin{pmatrix} y_F \\ y_C \end{pmatrix}$, a diagonal matrix $D_{|F| \times |F|}$, and a Laplacian matrix $S$ whose edges are supported on $C$, such that for each edge $e \in E^{(1)}$, $z^\top (L/\theta e)^\dagger z$ can be evaluated by computing $y_F^\top D^{-1} y_F + y_C^\top (S/\theta e)^\dagger y_C$.
5. Compute $y_F^\top D^{-1} y_F$ by inverting $D$ and $y_C^\top (S/\theta e)^\dagger y_C$ for all $e \in E^{(1)}$ by recursion, then use $y_F^\top D^{-1} y_F + y_C^\top (S/\theta e)^\dagger y_C$ to evaluate $z^\top (L/\theta e)^\dagger z$ for all $e \in E^{(1)}$.
6. Repeat steps 3-5 to $E^{(2)}$ to compute $z^\top (L/\theta e)^\dagger z$ for all $e \in E^{(2)}$.

The reason that in Step 5 we can compute $y_C^\top (S/\theta e)^\dagger y_C$ for each $e \in E^{(1)}$ by recursion is that $S$ is a Laplacian matrix whose edges are supported on $C$, and hence to compute $y_C^\top (S/\theta e)^\dagger y_C$ for all $e \in E^{(1)}$ is just a smaller-sized version of Problem 4 in which $L = S$ and $E^Q = E^{(1)}$.

In the rest of this subsection we give first an algorithm that solves Problem 4 exactly and then an algorithm that solves Problem 4 approximately.
4.2.1 Computing Exact Quadratic Forms of $L^\dagger$ Upon Edge Deactivation

We first give an algorithm $\textsc{ExactQuad}(L, E^Q, w, z, \theta)$ that computes the exact value of $z^\top (L\setminus e)^\dagger z$ for a fixed $z \in \mathbb{R}^n$ for all $e \in E^Q$ (Here $w$ is the edge weight function).

In this algorithm, we find $y$, $D$, and $S$ in step 4 by eliminating vertices in $F$ and obtain an exact partial Cholesky factorization of $L$. The following Lemma shows how to find them when an exact partial Cholesky factorization of $L$ is given:

Lemma 4.4. For a graph $G = (V, E)$ with associate Laplacian $L$ and a set of vertices $C \subset V$. Let $F = V \setminus C$, and the partial Cholesky factorization of $L$ be

$$L = \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^\top.$$

Let $y = \begin{pmatrix} y_F \\ y_C \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^{-1} z$, then for each edge $e \in E$ with both endpoints in $C$ the following statement holds:

$$z^\top (L\setminus e)^\dagger z = y_F^\top (D^{-1}) y_F + y_C^\top (S\setminus e)^\dagger y_C.$$

Proof. By Lemma 3.4, $\theta$-deletions performed to edges with both endpoints in $C$ commute with taking partial Cholesky factorization. Thus, for each $e \in E$ with both endpoints in $C$, we have

$$L\setminus e = \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & S\setminus e \end{pmatrix} \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^\top.$$

Inverting both sides of Eq. (13) leads to

$$(L\setminus e)^\dagger = \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^{-\top} \begin{pmatrix} D^{-1} & 0 \\ 0 & (S\setminus e)^\dagger \end{pmatrix} \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^{-1}.$$

Substituting $y = \begin{pmatrix} y_F \\ y_C \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix}^{-1} z$, we obtain

$$z^\top (L\setminus e)^\dagger z = y_F^\top (D^{-1}) y_F + y_C^\top (S\setminus e)^\dagger y_C.$$

This completes the proof. \hfill \Box

We give the pseudocode for $\textsc{ExactQuad}$ in Algorithm 1. Its performance is characterized in Lemma 4.5.

Lemma 4.5. Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, positive edge weights $w : E \to \mathbb{R}_+$, and associated Laplacian $L$, a set of edges $E^Q \subset E$ such that every vertex in $V$ is incident with some edge $e \in E^Q$, a vector $z \in \mathbb{R}^n$, and a scalar $0 < \theta \leq 1/2$, the algorithm $\textsc{ExactQuad}(L, E^Q, w, z, \theta)$ returns a set of pairs $N = \{(e, n_e) \mid e \in E^Q\}$, where

$$n_e = z^\top (L\setminus e)^\dagger z.$$

The total running time of this algorithm is bounded by $O(n^{\omega-1}m)$. 

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Let \( \rho \) denote the number of vertices in the original graph, i.e., the graph corresponding to \( L \). Let \( \omega \) denote endpoints of edges in \( L \). Thus, we have \( \rho = 2 \). Otherwise when \( \rho > 2 \), the algorithm goes to Line 3, and hence we have \( T(m) = O(1) \). Otherwise, the algorithm goes to Lines 4–11, among which the most time-consuming work is eliminating \( F \), inverting \( \begin{pmatrix} \mathcal{L}_{FF} & 0 \\ \mathcal{L}_{CF} & I_{CC} \end{pmatrix} \) and \( \mathcal{D} \), and recursively calling EXACTQUAD. The first two both run in \( O(\rho^\omega) \) time, and the third runs in \( 2T(m/2) \) time. When \( m > \rho \), we can bound the number of vertices in the current call by \( n_{\text{cur}} = O(\rho) \); otherwise when \( m \leq \rho \), we can bound the number of vertices in the current call by \( n_{\text{cur}} = O(m) \). Thus, We have

\[
T(m) = \begin{cases} 
2T(m/2) + O(n^\omega), & m > \rho, \\
2T(m/2) + O(m^\omega), & m \leq \rho.
\end{cases}
\]  

Eq. (14) leads to \( T(m) = O(n^\omega m) \).

\[\]
4.2.2 Approximating Quadratic Forms of $L^\dagger$ Upon Edge Deactivation

Clearly, if we only want to approximately compute the quadratic forms, we can use the approximate partial Cholesky algorithm in Lemma 3.5 to speed up. Thereby, we give an approximation algorithm QUADEST($L, E, w, z, \theta, \epsilon$), that computes an $\epsilon$-approximation of $z^\top (L\setminus e)^\dagger z$ for a fixed $z \in \mathbb{R}^n$ for all $e \in E^Q$. We also make a few modifications to maintain the error and further speed up. We list the modifications in QUADEST below:

1. In step 4, instead of computing the exact partial Cholesky factorization of $L$, we use the algorithm APXPARTIALCHOLESKY in Lemma 3.5 to obtain an approximate partial Cholesky factorization of $L$. However, if we pass the whole $L$ to APXPARTIALCHOLESKY, it may change the edges in $E^{(1)}$, to which we need to perform $\theta$-deletions when deactivating them. Thus, instead, we first delete all edges in $E^{(1)}$ and pass the resulting $L$ to APXPARTIALCHOLESKY, and then add those edges back to the approximate Schur complement $\tilde{S}$ returned by it. This modification is feasible since adding edges with both endpoints in $C$ commutes with taking approximate partial Cholesky factorization (Lemma 3.6). This modification is addressed on Lines 7-9 of Algorithm 2.

2. By Lemma 3.5, matrices $\tilde{D}$ and $\tilde{L}$ returned by APXPARTIALCHOLESKY satisfy that $\tilde{D}$ is diagonal, $\tilde{L}$ is sparse, and $(\tilde{L}_{FF} 0 0 \tilde{L}_{CF} I_{CC})$ is a lower triangular matrix up to row exchanges.

Therefore, by applying inverses of diagonal matrix and lower triangular matrix quickly, we can compute $y = \begin{pmatrix} y_F \\ y_C \end{pmatrix} = (\tilde{L}_{FF} 0 0 \tilde{L}_{CF} I_{CC})^{-1} z$ and $y_F^\top (\tilde{D})^{-1} y_F$ in linear time of the number of nonzero entries. This is addressed on Lines 10-11 of Algorithm 2.

3. Since errors may accumulate among different levels of the recursion, we bound the error by $\epsilon / \log |E^Q|$ when taking approximate partial Cholesky factorization (Line 8 of Algorithm 2), and bound the error by $\epsilon - \epsilon / \log |E^Q|$ when recursively calling QUADEST (Line 12 of Algorithm 2). Thereby, the errors add up to $\epsilon$ as required, and only an extra $\log^2 m$ factor is added to the running time (see Lemma 4.7 and its proof for details).

According to the first modification, in this algorithm, we find $y$, $D$, and $S$ in step 4 by taking approximate partial Cholesky factorization. The following Lemma shows how to find them when an approximate partial Cholesky factorization is given.

Lemma 4.6. For a graph $G = (V, E)$ with associate Laplacian $L$ and a set of vertices $C \subset V$. Let $F = V \setminus C$. Let $E^{(1)} \subset E$ be a set of edges with both endpoints in $C$, and $H$ be the Laplacian matrix corresponding to edges in $E^{(1)}$, i.e., $H = \sum_{e \in E^{(1)}} w(e) b_e b_e^\top$. Clearly $L - H$ is also a Laplacian. Let an approximate partial factorization of $L - H$ be

$$L - H \approx \epsilon \begin{pmatrix} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} \tilde{D} & 0 \\ 0 & \tilde{S} \end{pmatrix} \begin{pmatrix} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{pmatrix}^\top. \quad (15)$$

Let $y = \begin{pmatrix} y_F \\ y_C \end{pmatrix} = (\tilde{L}_{FF} 0 0 \tilde{L}_{CF} I_{CC})^{-1} z$ and $\tilde{S} = \tilde{S} + H_{CC}$, then for each edge $e \in E^{(1)}$ the following statement holds:

$$z^\top (L\setminus e)^\dagger z \approx \epsilon y_F^\top (\tilde{D})^{-1} y_F + y_C^\top (\tilde{S} - e)^\dagger y_C.$$
Proof. By Lemma 3.6, adding edges with both endpoints in \( C \) commutes with taking approximate partial Cholesky factorization. Thus, for each edge \( e \in E^{(1)} \), by adding first edges in \( E^{(1)} \) \{ \( e \) \} and then the deactivated edge \( e \) (i.e., edge \( e \) with weight \( \theta w(e) \)), we have

\[
L - H + (H \setminus \theta e) \approx \epsilon \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right) \left( \begin{array}{cc} \tilde{D} & 0 \\ 0 & \tilde{S} + (H_{CC} \setminus \theta e) \end{array} \right) \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right)^\top.
\]

Substituting \( (\tilde{S} \setminus \theta e) = \tilde{S} + (H_{CC} \setminus \theta e) \) and \( L \setminus \theta e = L - H + (H \setminus \theta e) \) leads to

\[
L \setminus \theta e \approx \epsilon \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right) \left( \begin{array}{cc} \tilde{D} & 0 \\ 0 & (\tilde{S} \setminus \theta e) \end{array} \right) \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right)^\top.
\] (16)

Note that \( \tilde{S} \) is a Laplacian since it is a sum of two Laplacians. Inverting both sides of Eq. (16) leads to

\[
(L \setminus \theta e)^\dagger \approx \epsilon \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right)^{-\top} \left( \begin{array}{cc} (\tilde{D})^{-1} & 0 \\ 0 & (\tilde{S} \setminus \theta e)^\dagger \end{array} \right) \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right)^{-1}.
\] (17)

Multiplying both sides of Eq. (17) by \( z^\top \) on the left and \( z \) on the right and substituting

\[
y = \begin{pmatrix} y_F \\ y_C \end{pmatrix} = \left( \begin{array}{cc} \tilde{L}_{FF} & 0 \\ \tilde{L}_{CF} & I_{CC} \end{array} \right)^{-1}\, z,
\]
gives

\[
z^\top (L \setminus \theta e)^\dagger z \approx \epsilon \begin{pmatrix} y_F & y_C \end{pmatrix} \left( \begin{array}{cc} (\tilde{D})^{-1} & 0 \\ 0 & (\tilde{S} \setminus \theta e)^\dagger \end{array} \right) \begin{pmatrix} y_F \\ y_C \end{pmatrix} = y_F^\top (\tilde{D})^{-1} y_F + y_C^\top (\tilde{S} \setminus \theta e)^\dagger y_C.
\] (18)

This completes the proof. \( \square \)

The pseudocode for QUADEST is given in Algorithm 2. Its performance is characterized in Lemma 4.7.

**Lemma 4.7.** Given a connected undirected graph \( G = (V, E) \) with \( n \) vertices, \( m \) edges, positive edge weights \( w : E \to \mathbb{R}_+ \), and associated Laplacian \( L \), a set of edges \( E^Q \subseteq E \) such that every vertex in \( V \) is incident with some edge \( e \in E^Q \), a vector \( z \in \mathbb{R}^n \), and scalars \( 0 < \theta \leq 1/2, 0 < \epsilon \leq 1/2 \), the algorithm QUADEST\((L, E^Q, w, z, \theta, \epsilon)\) returns a set of pairs \( \hat{N} = \{(e, \hat{n}_e) | e \in E^Q\} \). With high probability, the following statement holds: For \( \forall e \in E^Q \),

\[
n_e \approx \epsilon \hat{n}_e,
\] (19)

where

\[
n_e = z^\top (L \setminus \theta e)^\dagger z.
\]

The total running time of this algorithm is bounded by \( O(m\epsilon^{-2} \log^2 m \log^6 n \text{polyloglog}(n)) \).
Algorithm 2: QUADEst(\(L, E^Q, w, z, \theta, \epsilon\))

**Input**
- \(L\): A graph Laplacian.
- \(E^Q\): A set of edges supported on vertices in \(L\).
- \(w\): An edge weight function.
- \(z\): A vector whose dimension matches the number of vertices in \(L\).
- \(\theta\): An edge \(e\)'s weight should be temporarily reduced to \(\theta w(e)\) when deactivating it.
- \(\epsilon\): Error of the estimates.

**Output**
\(\tilde{N} = \{(e, \hat{n}_e) \mid e \in E^Q\}\): \(\hat{n}_e\) is an estimate of \(n_e = z^\top (L \setminus \theta e)^\dagger z\).

1. Let \(V\) denote the vertex set of \(L\).
2. **if** \(|V| = 2** then
   3. For every edge \(e \in E^Q\), compute exact \(\hat{n}_e = n_e = z^\top (L - (1 - \theta)w(e)b_eb_e^\top)^\dagger z\), then combine the results and return \(\tilde{N} = \{(e, \hat{n}_e) \mid e \in E^Q\}\).
4. Partition \(E^Q\) into \(E^{(1)}, E^{(2)}\) with \(|E^{(1)}| = \left\lfloor \frac{|E^Q|}{2} \right\rfloor\) and \(|E^{(2)}| = |E^Q| - \left\lfloor \frac{|E^Q|}{2} \right\rfloor\).
5. For \(i = 1\) to 2 do
   6. Let \(C\) denote endpoints of edges in \(E^{(i)}\) and \(F = V \setminus C\).
   7. Let \(H\) denote the Laplacian matrix corresponding to edges in \(E^{(i)}\), i.e., \(H \leftarrow \sum_{e \in E^{(i)}} w(e)b_e b_e^\top\).
   8. \((\~{\mathcal{L}}, \~{\mathcal{D}}, \~{\mathcal{S}}) \leftarrow \text{APXPARTIALCHOLESKY}(L - H, C, \epsilon/\log |E^Q|)\)
   9. Add edges in \(E^{(i)}\) back to \(\~{\mathcal{S}}\) and store the resulting Laplacian in \(\~{\mathcal{S}}'\), i.e., \(\~{\mathcal{S}}' \leftarrow \~{\mathcal{S}} + H_{CC}\).
   10. Compute \(y = y_F \begin{pmatrix} y_F \\ y_C \end{pmatrix} = \begin{pmatrix} \~{\mathcal{L}}_{FF} \ 0 \\ \~{\mathcal{L}}_{CF} \ 0 \end{pmatrix}^{-1} z\) in linear time.
   11. Compute \(f = y_F^\top (\~{\mathcal{D}})^{-1} y_F\) in linear time.
   12. Call QUADEst\((\~{\mathcal{S}}', E^{(i)}, y_C, \theta, \epsilon/\log |E^Q|)\) to get an estimate \(\hat{n}_e^{(i)}\) of \(n_e = y_C^\top (\~{\mathcal{S}}' \setminus \theta e)^\dagger y_C\) for all \(e \in E^{(i)}\) and store \((e, f + \hat{n}_e^{(i)})\) in \(\tilde{N}^{(i)}\).
13. **return** \(\tilde{N} = \tilde{N}^{(1)} \cup \tilde{N}^{(2)}\).

**Proof of Lemma 4.7.**
We first prove the error bound (i.e., Eq. (19)) by induction on the size of \(E^Q\).

For \(|E^Q| = 1\), we have \(|V| = 2\). Hence, the algorithm QUADEst will go into Line 3 and returns an \(\hat{n}_e = n_e\) holds for any \(\epsilon > 0\).

Suppose Eq. (19) holds for all \(1 \leq |E^Q| \leq k, k \geq 1\). We now prove that it holds for \(|E^Q| = k + 1\), too. Clearly, by symmetry, it suffices to show that \(n_e \approx \epsilon \hat{n}_e\) holds for each \(e \in E^{(1)}\). By Lemma 3.5, matrices \(\~{\mathcal{L}}, \~{\mathcal{D}}, \~{\mathcal{S}}\) on Line 8 satisfy

\[
L - H \approx_{\epsilon/\log |E^Q|} \begin{pmatrix} \~{\mathcal{L}}_{FF} & 0 \\ \~{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix} \begin{pmatrix} \~{\mathcal{D}} & 0 \\ 0 & \~{\mathcal{S}} \end{pmatrix} \begin{pmatrix} \~{\mathcal{L}}_{FF} & 0 \\ \~{\mathcal{L}}_{CF} & I_{CC} \end{pmatrix}^\top.
\]

By Lemma 4.6, we have

\[
z^\top (L \setminus \theta e)^\dagger z \approx_{\epsilon/\log |E^Q|} y_F^\top (\~{\mathcal{D}})^{-1} y_F + y_C^\top (\~{\mathcal{S}}' \setminus \theta e)^\dagger y_C,
\]

(21)
where $\tilde{S}' = \tilde{S} + H_{CC}$ (Line 9). Since $|E^{(1)}| = \left\lceil \frac{|E^Q|}{2} \right\rceil \leq k$, by inductive assumption, each $\tilde{n}_e^{(1)}$ returned by the recursive call QUAD EST on Line 12 satisfies $y_C^\top (\tilde{S}'_{\setminus \theta} e) y_C \approx \varepsilon - \varepsilon / \log |E^Q| \tilde{n}_e^{(1)}$, which when adding $f = y_F^\top (D^{-1}) y_F$ (Line 11) to its both sides turns into

$$y_F^\top (D^{-1}) y_F + y_C^\top (\tilde{S}'_{\setminus \theta} e) y_C \approx \varepsilon - \varepsilon / \log |E^Q| f + \tilde{n}_e^{(1)}.$$  \hfill (22)

Combining Eq. (21) and Eq. (22) and substituting $n_e = z^\top (L_{\setminus \theta} e)^\top z$, we have $n_e \approx f + \tilde{n}_e^{(1)}$. Thus, Eq. (19) holds for $|E^Q| = k + 1$, too. By induction, it holds for all $|E^Q|$.

We then prove the running time of the algorithm.

Let $T(m, \epsilon)$ denote the running time of QUAD EST($L, E^Q, w, z, \theta, \epsilon$), where $m = |E^Q|$ and $\epsilon$ is the error of estimates. Let $n$ denote the number of vertices in the original graph, i.e., the graph corresponding to $L$ in the earliest call to QUAD EST. Let $n_{\text{cur}}$ and $m_{\text{cur}}$ denote the number of vertices and the number of edges in $L$ in the current call, respectively. In each call other than the earliest call, the Laplacian $L$ equals $\tilde{S}'$ on Line 9 of the parent call (i.e., the call that invoked current call), where we have the total number of edges in $\tilde{S}'$ being the number of edges in $\tilde{S}$ plus the number of edges in $H$. Since $H$ is the Laplacian corresponding to edges in $E^{(i)}$, which is precisely $E^Q$ in the current call, we have the number of edges in $H$ equaling $m = |E^Q|$. By Lemma 3.5, the number of edges in $\tilde{S}$ is $O(n_{\text{cur}} \epsilon^{-2} \log^2 m \log n)$, where there is an extra $\log^2 m$ factor because the error is set to $\epsilon / \log |E^Q|$ when calling AXP PARTIAL CHOLESKY on Line 8. Hence, the number of edges in $L$ in the current call is bounded by $m_{\text{cur}} = O(m + n_{\text{cur}} \epsilon^{-2} \log^2 m \log n)$.

If $n_{\text{cur}} = 2$, the algorithm goes to Line 3, and hence we have $T(m, \epsilon) = O(1)$. Otherwise, the algorithm goes to Lines 4 - 13, among which the most time-consuming work can be divided into three parts:

1. The first part is computing $\left( \tilde{L}_{FF} 0 \begin{array}{c} \tilde{L}_{CF} \end{array} I_{CC} \right)^{-1} z$ and $y_F^\top (D)^{-1} y_F$. Since $\left( \tilde{L}_{FF} 0 \begin{array}{c} \tilde{L}_{CF} \end{array} I_{CC} \right)$ is a lower triangular matrix up to row exchanges and $D$ is diagonal, their inverse can be applied in linear time of the number of nonzero entries. By Lemma 3.5, this part runs in $O(m_{\text{cur}} + n_{\text{cur}} (\epsilon / \log m)^{-2} \log^3 n) = O(m + n_{\text{cur}} \epsilon^{-2} \log^2 m \log^3 n)$ time.

2. The second part is taking approximate partial Cholesky factorization, which by Lemma 3.5 runs in $O((m_{\text{cur}} \log^3 n + n_{\text{cur}} (\epsilon / \log m)^{-2} \log^5 n) \text{polylog}(n)) = O(m \log^3 n + n_{\text{cur}} \epsilon^{-2} \log^2 m \log^5 n \text{polylog}(n))$ time.

3. The third part is recursively calling QUAD EST($\tilde{S}', E^{(i)}, y_C, \theta, \epsilon - \epsilon / \log |E^Q|$), which runs in $2T(m/2, \epsilon - \epsilon / \log m)$ time.

The first two parts add up to a running time of $O(m \log^3 n + n_{\text{cur}} \epsilon^{-2} \log^2 m \log^5 n \text{polylog}(n))$. When $m > n$, we can bound the number of vertices in the current call by $n_{\text{cur}} = O(n)$; otherwise when $m \leq n$, we can bound the number of vertices in the current call by $n_{\text{cur}} = O(m)$. Thus, We have

$$T(m, \epsilon) = \begin{cases} 2T(m/2, \epsilon - \epsilon / \log m) + O(m \log^3 n + n \epsilon^{-2} \log^2 m \log^5 n \text{polylog}(n)), & m > n \\ 2T(m/2, \epsilon - \epsilon / \log m) + O(m \log^3 n + m \epsilon^{-2} \log^2 m \log^5 n \text{polylog}(n)), & m \leq n \end{cases}.$$  \hfill (23)

Eq. (23) leads to $T(m, \epsilon) = O(m \epsilon^{-2} \log^2 m \log^6 n \text{polylog}(n))$. 

\hfill □
4.3 Approximating $C_{\theta}(e)$

We are now ready to give the algorithm $\text{EdgeCentComp1}(G = (V, E), w, \theta, \epsilon)$, which computes an $\epsilon$-approximation of the $\theta$-Kirchhoff edge centrality $C_{\theta}(e)$ for all $e \in E$. The pseudocode for $\text{EdgeCentComp1}$ is given in Algorithm 3. Its performance is characterized in Theorem 1.1.

**Algorithm 3: EdgeCentComp1** $(G = (V, E), w, \theta, \epsilon)$

- **Input**: $G = (V, E)$, $w$: A connected undirected graph with positive edges weights $w : E \to \mathbb{R}_+$.
  - $\theta$: An edge $e$’s weight should be temporarily reduced to $\theta w_e$ when deactivating it.
  - $\epsilon$: Error of the centrality estimate per edge.
- **Output**: $\hat{C} = \{(e, \hat{c}_e) \mid e \in E\}$: $\hat{c}_e$ is an estimate of $C_{\theta}(e)$, the $\theta$-Kirchhoff edge centrality of $e$.

1. Let $z_1, \ldots, z_M$ be independent random $\pm 1$ vectors, where $M = \lceil 192\epsilon^{-2}\ln(2n) \rceil$.
2. for $i = 1$ to $M$ do
   3. Call $\text{QuadEst}(L^G, E, w, z_i, \epsilon, \frac{\theta}{\epsilon})$ to get an estimate $\hat{n}_e$ of $n_e = z_i^\top (L \setminus \theta e)^\dagger z_i$ for each $e \in E$, and store each $\hat{n}_e$ in $\hat{n}_e^{(i)}$.
4. For each $e \in E$ compute $\hat{c}_e = \frac{n}{M} \sum_{i=1}^M \hat{n}_e^{(i)}$ and return $\hat{C} = \{(e, \hat{c}_e) \mid e \in E\}$.

**Proof of Theorem 1.1.**

The running time is the total cost of $O(\epsilon^{-2}\log n)$ calls to $\text{QuadEst}$, each of which runs in $O(m\epsilon^{-2}\log^2 m \log^6 n \polylog(n))$ time according to Lemma 4.7.

Since $M = \lceil 192\epsilon^{-2}\ln(2n) \rceil \geq 48 \left(\frac{\epsilon}{2}\right)^{-2}\ln(2n)$, by Lemma 4.3, for each $e \in E$, there is

$$\frac{1}{M} \sum_{i=1}^M z_i^\top (L \setminus \theta e)^\dagger z_i \approx \frac{\epsilon}{2} \text{Tr} (L \setminus \theta e).$$

Multiplying both sides by $n$ and substituting $C_{\theta}(e) = n\text{Tr} (L \setminus \theta e)$, we have

$$\frac{n}{M} \sum_{i=1}^k z_i^\top (L \setminus \theta e)^\dagger z_i \approx \frac{n}{2} C_{\theta}(e). \tag{24}$$

By Lemma 4.7, each $\hat{n}_e^{(i)}$ on Line 3 satisfies

$$n_e^{(i)} \approx \frac{\epsilon}{2} \hat{n}_e^{(i)}, \tag{25}$$

where $n_e^{(i)} = z_i^\top (L \setminus \theta e)^\dagger z_i$. Summing Eq. (25) over $i = 1, \ldots, M$ and multiplying both sides by $\frac{n}{M}$ lead to

$$\frac{n}{M} \sum_{i=1}^M z_i^\top (L \setminus \theta e)^\dagger z_i \approx \frac{n}{2} \sum_{i=1}^M \hat{n}_e^{(i)}.$$

Combining with Eq. (24) and substituting $\hat{c}_e = \frac{n}{M} \sum_{i=1}^M \hat{n}_e^{(i)}$ (Line 4) lead to $C_{\theta}(e) \approx \epsilon \hat{c}_e$. 

\[\square\]
5 Algorithm for Approximating $\theta$-Kirchhoff Edge Centrality $C^\Delta_\theta(e)$

By Sherman-Morrison formula, for an edge $e \in E$ and a scalar $0 < \theta < 1$, we have

$$
(L \setminus \theta e)^\dagger = (L - (1 - \theta)w(e)b_e b_e^\top)^\dagger = \hat{L}^\dagger + (1 - \theta)\frac{w(e)\hat{b}_e b_e^\top \hat{L}^\dagger}{1 - (1 - \theta)w(e)\hat{b}_e^\top \hat{L}^\dagger \hat{b}_e}. 
$$

Since the off-the-shelf Sherman-Morrison formula is for full rank matrices, we give the detailed proof of Equation (26) in Appendix A.

Since by our definition $C^\Delta_\theta(e) = \mathcal{K}(G \setminus \theta e) - \mathcal{K}(G) = n\left(\operatorname{Tr}\left(\left((L \setminus \theta e)^\dagger\right) - \operatorname{Tr}(\hat{L}^\dagger)\right)\right)$, it follows that

$$
C^\Delta_\theta(e) = n(1 - \theta)\frac{w(e)\operatorname{Tr}(\hat{L}^\dagger \hat{b}_e b_e^\top \hat{L}^\dagger)}{1 - (1 - \theta)w(e)\hat{b}_e^\top \hat{L}^\dagger \hat{b}_e}. \tag{27}
$$

The numerator of (27) is the trace of an implicit matrix, and hence can be approximated by Hutchinson’s [AT11, Hut89] Monte-Carlo method. To apply $L^\dagger$, we can utilize nearly-linear time solvers for Laplacian systems [ST14, CKM+14]. We will use the solver from [CKM+14], whose performance can be characterized in the following lemma.

**Lemma 5.1** (Theorem 1.1 of [CKM+14], paraphrased). There is an algorithm $y = \text{LAPLSOLVE}(L^G, z, \delta)$ which takes a Laplacian matrix $L^G$ of a graph $G$ with $n$ vertices and $m$ edges, a vector $z \in \mathbb{R}^n$, and a scalar $\delta > 0$, and returns a vector $y \in \mathbb{R}^n$ such that with high probability the following statement holds:

$$
\left\| y - L^\dagger z \right\|_L \leq \delta \left\| L^\dagger z \right\|_L,
$$

where $\left\| x \right\|_L = \sqrt{x^\top Lx}$. The algorithm runs in expected time $O(m \log^{0.5} n \log(1/\delta) \text{polylog}(n))$.

To track the error for the solver, we will need the following two lemmas, whose proofs are deferred to Appendix C.1.

**Lemma 5.2.** Let $L$ be the Laplacian of a graph with all weights in the range $[1, U]$, and $z$ be any vector such that $\left\| z \right\|^2 \leq n$. Suppose $y$ is a vector such that $\left\| y - L^\dagger z \right\|_L \leq \delta \left\| L^\dagger z \right\|_L$ for some $0 < \delta < 1$. For any edge $e$ of the graph, we have

$$
\left\| y^\top b_e b_e^\top y - z^\top L^\dagger b_e b_e^\top L^\dagger z \right\| \leq 6\delta n^5 U^2. \tag{28}
$$

**Lemma 5.3.** Let $L$ be the Laplacian of a graph with all weights in the range $[1, U]$. For any edge $e$ of the graph, we have

$$
\operatorname{Tr}\left( L^\dagger b_e b_e^\top L^\dagger \right) \geq \frac{2}{n^2 U^2}.
$$

The denominator of (27) is just $1 - (1 - \theta)w(e)\mathcal{R}_{\text{eff}}(e)$. Since $w(e)\mathcal{R}_{\text{eff}}(e)$ is between 0 and 1 and $\theta$ is positive, $(1 - \theta)w(e)\mathcal{R}_{\text{eff}}(e)$ is strictly bounded away from 1. Thus, we can multiplicatively approximate the denominator by approximating $\mathcal{R}_{\text{eff}}(e)$, for which we can use the random projection in [SS11]. By Using the solvers from [CKM+14] in the effective resistance estimation procedure of [SS11], we immediately have the following lemma:

**Lemma 5.4.** There is an algorithm $\text{EREST}(G, e)$ that when given a graph $G = (V, E)$, returns an estimate $\hat{r}_e$ of $\mathcal{R}_{\text{eff}}(e)$ for all $e \in E$ in $O(m \epsilon^{-2} \log^{2.5} n \text{polylog}(n))$ time. With high probability, $\hat{r}_e \approx \epsilon \mathcal{R}_{\text{eff}}(e)$ holds for all $e \in E$. 

23
As these estimates are approximate, we will need to bound their approximations when subtracted from 1. Here we use the fact that \( 0 < \theta < 1 \), and that the weight times effective resistance of an edge, \( w(e) b_e^T L^b_e \) is between 0 and 1. Since we will also need the matrix version of this type of approximation propagation when computing Kirchhoff vertex centralities in Section 6, we will state the more general version here.

**Lemma 5.5.** If \( A \) and \( B \) are matrices such that \( 0 \preceq A \preceq I \), and \( A \approx_{\epsilon} B \) for some \( 0 < \epsilon \leq 1/2 \), then for any \( 0 < \theta \leq 1/2 \) such that \( \epsilon/\theta \leq 1/10 \), we have

\[
I - (1 - \theta) A \approx_{3\epsilon/\theta} I - (1 - \theta) B.
\]

The proof is deferred to Appendix B.

We then give an algorithm `EdgeCentComp2` to approximate the \( \theta \)-edge Kirchhoff centrality \( C_\Delta^\theta \) for all \( m \in E \). The pseudocode for `EdgeCentComp2` is given in Algorithm 4. The performance of `EdgeCentComp2` is characterized in Theorem 1.2.

**Algorithm 4: `EdgeCentComp2(G, \theta, \epsilon)`**

- **Input**:
  - \( G \): A graph.
  - \( \theta \): a scalar between 0 and \( 1/2 \).
  - \( \epsilon \): the error parameter.

- **Output**:
  - \( \hat{C}_\Delta^\theta \) = \[ (e, \hat{c}_\Delta^e) \mid e \in E \].

1. Let \( z_1, \ldots, z_M \) be independent random \( \pm 1 \) vectors, where \( M = \lceil 432\epsilon^{-2}\ln(2n) \rceil \).
2. for \( i = 1 \) to \( M \) do
3.   \( y_i \leftarrow \text{LaplSolve}(L^G, z_i, \frac{1}{36}\epsilon n^{-7}U^{-4}) \)
4.   for each \( e \in E \) do
5.     Compute \( \hat{c}_\Delta^{\theta}(i) = y_i^T b_e b_e^T y_i \).
6. \( \hat{r}_e \leftarrow \text{EREst}(G, \theta \epsilon/9) \)
7. Compute \( \hat{c}_\Delta^\theta = (1 - \theta) \frac{n}{M} w(e) \sum_{i=1}^{M} \hat{c}_\Delta^{\theta}(i) / (1 - (1 - \theta)w(e)\hat{r}_e) \) for each \( e \).

**Proof of Theorem 1.2.** The running time is the total cost of \( O(\epsilon^{-2}\log n) \) calls to LAPLsolve each of which runs in \( O(m \log 1.5 n \log(1/\epsilon) \text{polyloglog}(n)) \) time, and a call to ERest which runs in \( O(m\theta^{-2}\epsilon^{-2} \log^{2.5} n \text{ polyloglog}(n)) \) time.

Since \( M = \lceil 432\epsilon^{-2}\ln(2n) \rceil \geq 48 (\epsilon/3)^{-2}\ln(2n) \), by Lemma 4.3, we have

\[
\frac{1}{M} \sum_{i=1}^{M} z_i^T L^b_e b_e^T L^b_e z_i \approx_{\epsilon/3} \text{Tr} \left( L^b_e b_e^T L^b_e \right). \tag{29}
\]

By Lemma 5.3, we have

\[
\text{Tr} \left( L^b_e b_e^T L^b_e \right) \geq \frac{2}{n^2 U^2},
\]

and hence

\[
\frac{1}{M} \sum_{i=1}^{M} z_i^T L^b_e b_e^T L^b_e z_i \geq \exp(-\epsilon/3) \frac{2}{n^2 U^2} \geq \frac{1}{n^2 U^2}, \tag{30}
\]

where the second inequality follows by \( 0 < \epsilon \leq 1/2 \).
Since we set $\delta = \frac{1}{36} \epsilon n^{-7} U^{-4}$ when invoking LAPL.SOLVE, by Lemma 5.1,
\[
\|y_i - L^\dagger z_i\|_L \leq \frac{1}{36} \epsilon n^{-7} U^{-4} \|L^\dagger z_i\|_L
\]
holds for each $i$. Then, by Lemma 5.2, we have that
\[
\left|y_i^\top b_e b_e^\top y_i - z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i\right| \leq \frac{1}{6} \epsilon n^{-2} U^{-2}
\]
holds for each $i$. We then have
\[
\left|\frac{1}{M} \sum_{i=1}^M y_i^\top b_e b_e^\top y_i - \frac{1}{M} \sum_{i=1}^M z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i\right|
\leq \frac{1}{6} \epsilon n^{-2} U^{-2}
\leq \frac{1}{6} \epsilon \left(\frac{1}{M} \sum_{i=1}^M z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i\right),
\]
where the last inequality follows by (30). Thus,
\[
(1 - \epsilon/6) \frac{1}{M} \sum_{i=1}^M z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i \leq \frac{1}{M} \sum_{i=1}^M y_i^\top b_e b_e^\top y_i \leq (1 + \epsilon/6) \frac{1}{M} \sum_{i=1}^M z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i,
\]
which implies
\[
\frac{1}{M} \sum_{i=1}^M z_i^\top L^\dagger b_e b_e^\top L_i^\dagger z_i \approx \epsilon/3 \frac{1}{M} \sum_{i=1}^M y_i^\top b_e b_e^\top y_i.
\]
By Lemma 5.4, we have $\hat{r}_e \approx \theta_{c/e} R_{\text{eff}}(e)$, which by Lemma 5.5 implies
\[
1 - (1 - \theta)w(e)\hat{r}_e \approx \epsilon/3 1 - (1 - \theta)w(e)R_{\text{eff}}(e).
\]
Combining Equations (29), (31) and (32), we have
\[
\frac{1}{M} \sum_{i=1}^M y_i^\top b_e b_e^\top y_i \approx \epsilon \frac{\text{Tr} \left( L^\dagger b_e b_e^\top L_i^\dagger \right)}{1 - (1 - \theta)w(e)\hat{r}_e} \approx \epsilon \frac{\text{Tr} \left( L^\dagger b_e b_e^\top L_i^\dagger \right)}{1 - (1 - \theta)w(e)R_{\text{eff}}(e)}.
\]
which together with Equation (27) proves this theorem.

### 6 Algorithm for Approximating $\theta$-Kirchhoff Vertex Centrality

We now combine the projection based approximation algorithm from Section 5 with the recursive Schur complement approximation algorithm to produce a routine for estimating Kirchhoff vertex centrality as defined in Definition 2.9 in nearly-linear time.
6.1 Turning to Low-rank Updates

We will treat the $\theta$-deletion of a vertex as $\theta$-deleting a batch of edges from the graph, which in turn corresponds to a high rank update to the graph Laplacian. Specifically, we can define the matrix $B_{E_v}$ as the $\deg^u(v) \times n$ edge-vertex incidence matrix containing the edges incident to $v$, and $W_{E_v}$ as the corresponding $\deg^a(v) \times \deg^a(v)$ diagonal edge weight matrix. Here we use $E_v \defeq \{(u, v) \mid u \sim v\}$ to denote the set of edges incident with $v$, and $\deg^u(v) \defeq |E_v|$ to denote the number of edges incident with $v$. Then the graph Laplacian with edges in $E_v$ $\theta$-deleted is

$$L \setminus_\theta E_v = L - (1 - \theta) B_{E_v}^\top W_{E_v} B_{E_v}.$$

By Lemma 4.3, our goal becomes solving Problem 4 on the difference between the pseudoinverses of these matrices. Specifically, computing the value of

$$z^\top \left( (L - (1 - \theta) B_{E_v}^\top W_{E_v} B_{E_v})^\dagger - L^\dagger \right) z$$

for a vector $z$. For this we once again turn to low-rank updates, specifically the Woodbury formula.

**Lemma 6.1** (Derived from Woodbury formula). Given an edge set $T \subset E$ supported on vertex set $C$, and a scalar $0 < \theta < 1$. Let $B_T$ be the $|T| \times n$ edge-vertex incidence matrix corresponding to edges in $T$, and $W_T$ be the $|T| \times |T|$ diagonal edge weight matrix corresponding to edges in $T$. The following statement holds:

$$(L \setminus_\theta T)^\dagger = L^\dagger + (1 - \theta) L^\dagger B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1} W_T^{1/2} B_T^\dagger L^\dagger. \quad (33)$$

Since the off-the-shelf Woodbury formula is for full rank matrices, we give the detailed proof of Equation (33) in Appendix A.

Note that this formula applies to any subset of $T$ edges. The only property of evaluating Kirchhoff vertex centrality we need is that the total size of such $T$s over all vertices is $2m$.

This means just as in Section 5, the problem reduces to estimating

$$z^\top L^\dagger B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1} W_T^{1/2} B_T^\dagger z.$$

Furthermore, since we can compute $L^\dagger z$ to high accuracy via a single solver to a linear system in a graph Laplacian, and $W_T^{1/2} B_T$ is $|T| \times n$ matrix with $2|T|$ nonzero entries, we can compute for each set $T$ the vector $W_T^{1/2} B_T^\dagger z$ in $O(|T|)$ time (after $O(m)$ preprocessing time to compute an approximation to $L^\dagger z$). To track the error for the solver, we will need to following two lemmas, which we prove in Appendix C.2.

**Lemma 6.2.** Let $L$ be the Laplacian of a graph with all weights in the range $[1, U]$, and $z$ be any vector such that $\|z\|^2 \leq n$. Suppose $y$ is a vector such that $\|y - L^\dagger z\|_L \leq \delta \|L^\dagger z\|_L$ for some $0 < \delta < 1$. For any edge set $T \subset E$, we have:

$$\left| z^\top L^\dagger B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1} W_T^{1/2} B_T^\dagger z \right| - y^\top B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1} W_T^{1/2} B_T y \leq 6\theta^{-1} \delta n^5 U^2. \quad (34)$$

**Lemma 6.3.** Let $L$ be the Laplacian of a graph with all weights in the range $[1, U]$. For any edge set $T \subset E$ of the graph, we have

$$\text{Tr} \left( L^\dagger B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1} W_T^{1/2} B_T L^\dagger \right) \geq \frac{2|T|}{n^2 U^2}.$$
6.2 Approximating Quadratic Forms

Since we can utilize nearly-linear time solvers for Laplacian linear systems to compute high accuracy approximations to the vector $L^\dagger z$, the problem is further reduced to estimating quadratic forms of

$$\left(I - (1 - \theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}\right)^{-1}.$$

Since the edges in $T$ form a subgraph of $L$, we have $B_T^\top W_T B \preceq L$, and in turn

$$0 \preceq W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2} \preceq I$$

This coupled with the assumption that $0 < \theta$ means that the eigenvalues of matrix $(1 - \theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}$ are bounded away from 1. Therefore, we can use iterative methods to solve the resulting system. As we work entirely with matrix approximations, we will use the following matrix-based version of Chebyshev iteration. More details on these iterative methods can be found in Section 11.2 of [GVL12].

Lemma 6.4 (Chebyshev iteration). There is an algorithm $\text{ChebSolve}(P, \kappa, \epsilon, b)$ such that for any positive definite matrix $P$ along with $\kappa$ such that $\frac{1}{\kappa} I \preceq P \preceq I$, $\text{ChebSolve}(P, \kappa, \epsilon, b)$ corresponds to a linear operator on $b$ such that the matrix $Z_{\text{ChebSolve}}$ realizing this operator satisfies

$$Z_{\text{ChebSolve}} \approx \epsilon P^{-1},$$

and the cost of the algorithm is $O(\sqrt{\kappa} \log(1/\epsilon))$ matrix-vector multiplications involving $P$.

Therefore the main difficulty becomes finding the matrix

$$W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}.$$

Note that while $B_T$ has up to $n$ columns, most of these columns are 0s. So it means that we can only consider the entries corresponding to $V(T)$, the set of vertices incident to at least one edge in $T$, using the following fact about Schur complements.

Fact 6.5. Let $L$ be a Laplacian matrix, and $C$ be a subset of vertices. Then, we have

$$\left(L^\dagger\right)_{CC} = \text{Sc}(L, C)^\dagger.$$

However, we only have approximate Schur complements. To bound this also, we once again invoke the bound about preservations of approximations when subtracting matrices from $I$ from Lemma 5.5.

Lemma 6.6. There is an algorithm $\bar{x} = \text{QuadSolve}(B_T, W_T, b, \theta, \epsilon, \tilde{S})$ which takes an edge-vertex incidence matrix $B_T$ corresponding to edges in $T \subset E$ with edge weight matrix $W_T$ supported on vertex set $V(T)$, a vector $b \in \mathbb{R}^n$, scalars $0 < \theta \leq 1/2$ and $0 < \epsilon < 1/2$, and a Laplacian matrix $\tilde{S}$ whose edges are supported on $V(T)$ such that $\tilde{S} \approx_{\epsilon \theta/9} \text{Sc}(L, V(T))$, and returns a value $\bar{x}$ satisfying

$$\bar{x} \approx \epsilon b^\top \left(I - (1 - \theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}\right)^{-1} b.$$ 

The algorithm runs in time $O(\text{nnz}(\tilde{S})\theta^{-0.5} \log^3 n \log(1/\epsilon) + |T| \theta^{-2.5} \epsilon^{-2} \log^5 n \log(1/\epsilon) \text{polyloglog}(n))$, where $\text{nnz}(\tilde{S})$ is the number of nonzero entries in $\tilde{S}$. 

Theorem 6.7. For any $\theta < 1/2$, we can compute a matrix $(1 - \theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}$ from $B_T$, $W_T$, and $b$ in $\text{nnz}(\tilde{S}) + O(\text{nnz}(\tilde{S}) \theta^{-0.5} \log^3 n \log(1/\epsilon) + |T| \theta^{-2.5} \epsilon^{-2} \log^5 n \log(1/\epsilon) \text{polyloglog}(n))$ time using $\text{QuadSolve}(B_T, W_T, b, \theta, \epsilon, \tilde{S})$. 

Proof. The bounds follow directly from Lemma 6.6.
Also, since \(B_T\) is only non-zero on the entries corresponding to \(V(T)\), Fact 6.5 gives
\[
W_{T}^{1/2} B_T \tilde{L} B_T^\top W_{T}^{1/2} = W_{T}^{1/2} B_{T,V(T)} \mathcal{S} c (L, V(T))^\top B_{T,V(T)}^\top W_{T}^{1/2},
\]
and hence Fact 2.1 Part 10 gives
\[
W_{T}^{1/2} B_T \tilde{L} B_T^\top W_{T}^{1/2} \approx_{\epsilon/9} W_{T}^{1/2} B_{T,V(T)} \tilde{S} B_{T,V(T)}^\top W_{T}^{1/2}. \tag{35}
\]
Also, since \(T\) is a subset of edges,
\[
\left(W_{T}^{1/2} B_T\right)^\top \left(W_{T}^{1/2} B_T\right) \preceq L,
\]
which in turn implies
\[
W_{T}^{1/2} B_T \tilde{L} B_T^\top W_{T}^{1/2} \preceq I,
\]
and
\[
\theta I \preceq I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \mathcal{S} c (L, V(T))^\top B_{T,V(T)}^\top W_{T}^{1/2} \preceq I.
\]
Combining this with the approximation factor above from Equation (35) and Lemma 5.5 then gives
\[
I - (1 - \theta) W_{T}^{1/2} B_T \tilde{L} B_T^\top W_{T}^{1/2} \approx_{\epsilon/3} I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \tilde{S} B_{T,V(T)}^\top W_{T}^{1/2}. \tag{36}
\]
To apply \(\tilde{S}^\top\), we can invoke the algorithm \texttt{ApxPartialCholesky}(\tilde{S}, \{v\}, \epsilon \theta/9) in Lemma 3.5 for an arbitrary vertex \(v\) to get an \(\epsilon \theta/9\)-approximate sparse \textit{complete} Cholesky factorization of \(\tilde{S}\) and then apply its inverse quickly. Suppose the Cholesky factorization returned is \(L \tilde{D} \tilde{C}^\top \approx_{\epsilon \theta/9} \tilde{S}\), then again by Lemma 5.5 we have
\[
I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \tilde{S} B_{T,V(T)}^\top W_{T}^{1/2} \approx_{\epsilon/3} I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \left(\tilde{L} \tilde{D} \tilde{C}^\top\right) B_{T,V(T)}^\top W_{T}^{1/2}. \tag{37}
\]
Combining Equation (36) and (37) leads to
\[
I - (1 - \theta) W_{T}^{1/2} B_T \tilde{L} B_T^\top W_{T}^{1/2} \approx_{2\epsilon/3} I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \left(\tilde{L} \tilde{D} \tilde{C}^\top\right) B_{T,V(T)}^\top W_{T}^{1/2}, \tag{38}
\]
which also means that all the eigenvalues of \(I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \left(\tilde{L} \tilde{D} \tilde{C}^\top\right) B_{T,V(T)}^\top W_{T}^{1/2}\) are between \(\exp(-2\epsilon/3)\theta\) and 1. Therefore by Lemma 6.4, we can access a linear operator \(Z_{\text{Solve}}\) such that
\[
Z_{\text{ChebSolve}} \approx \epsilon/3 \left(I - (1 - \theta) W_{T}^{1/2} B_{T,V(T)} \left(\tilde{L} \tilde{D} \tilde{C}^\top\right) B_{T,V(T)}^\top W_{T}^{1/2}\right)^{-1}, \tag{39}
\]
whose cost is $O(\theta^{-0.5} \log(1/\epsilon))$ matrix-vector multiplications involving
\[ I - (1 - \theta) W^{1/2}_T B_{T,V(T)} \left( \tilde{L}D\tilde{L} \right)^\dagger B_{T,V(T)}^\dagger W^{1/2}_T. \] Here $I$, $W^{1/2}$ and $B_{T,V(T)}$ can all be applied in $O(|T|)$ time. By Lemma 3.5, $\left( \tilde{L}D\tilde{L} \right)^\dagger$ can be applied in $O(mnz(\tilde{S}) + |T|\theta^{-2}\epsilon^{-2} \log^3 n)$ time, and $\text{APXPartialCholesky}(\tilde{S}, \{v\}, \epsilon\theta/9)$ runs in $O(mnz(\tilde{S}) \log^3 n + |T|\theta^{-2}\epsilon^{-2} \log^5 n \text{polyloglog}(n))$ time.

Inverting both sides of Equation (38) and then combining it with Equation (39) gives
\[ Z_{\text{Cholesky}} \approx \epsilon \left( I - (1 - \theta) W^{1/2}_T B_{T}^\dagger B_{T} B^{1/2}_T \right)^{-1}, \] so we can set $\tilde{x} = b^\dagger Z_{\text{Cholesky}} b$ and return it as our overall estimate.

Thus, the problem becomes efficiently approximating Schur complements onto subsets of edges. We give an algorithm $\text{QuadApprox}$ that first computes approximate Schur complements onto neighbors of each vertex and then uses the algorithm $\text{QuadSolve}$ in Lemma 6.6 to compute
\[ z^\top B^\dagger_{E_v} W^{1/2}_v (I - (1 - \theta) W^{1/2}_v B_{E_v} L^\dagger B_{E_v}^\dagger W^{1/2}_v) \left( I - (1 - \theta) W^{1/2}_v B_{E_v} L^\dagger B_{E_v}^\dagger W^{1/2}_v \right)^{-1} W^{1/2}_v B_{E_v} z \]
for some vector $z$. The pseudocode for $\text{QuadApprox}$ is given in Algorithm 5. Note that in the pseudocode we use $G[C]$ to denote $G$’s induced graph on a subset of vertices $C$, $z_C$ to denote a $|C|$-dimensional vector obtained from $z$ by taking entries corresponding to vertices in $C$, and $\deg^n(v)$ to denote the number of edges incident with $v$. The performance of $\text{QuadApprox}$ is characterized in Lemma 6.7.

**Lemma 6.7.** Given a connected undirected graph $G = (V, E)$ with $n$ vertices, $m$ edges, positive edge weights $w : E \to \mathbb{R}_+$, and associated Laplacian $L$, a set of vertices $V^Q \subset V$ such that $V = \{N(v) \mid v \in V^Q\} \cup V^Q$, a vector $z \in \mathbb{R}^n$, and scalars $0 < \theta \leq 1/2$, $0 < \epsilon \leq 1/2$, the algorithm $\text{QuadApprox}(G, L, V^Q, z, \theta, \epsilon\theta/9, \epsilon)$ returns a set of pairs $\hat{N}^\Delta = \{(v, \hat{n}_v^\Delta) \mid v \in V^Q\}$. With high probability, the following statement holds: For $\forall v \in V^Q$,
\[ \hat{n}_v^\Delta \approx \epsilon \hat{n}_v^\Delta, \tag{40} \]
where
\[ \hat{n}_v^\Delta = z^\top B^\dagger_{E_v} W^{1/2}_v (I - (1 - \theta) W^{1/2}_v B_{E_v} L^\dagger B_{E_v}^\dagger W^{1/2}_v) \left( I - (1 - \theta) W^{1/2}_v B_{E_v} L^\dagger B_{E_v}^\dagger W^{1/2}_v \right)^{-1} W^{1/2}_v B_{E_v} z, \]
and $E_v = \{(u, v) \mid u \sim v\}$ is the set of edges incident with $v$. The total running time of this algorithm is bounded by $O(m(\theta^{-2} \epsilon^{-2} \log^8 n + \theta^{-2.5} \epsilon^{-2} \log^5 n \log(1/\epsilon)) \text{polyloglog}(n)).$

**Proof of Lemma 6.7.** Let $\text{volume}(V^Q)$ denote the quantity $\text{vol}$ on Line 5. We first observe that every time we recursively call $\text{QuadApprox}$, one of the following two events occurs:

1. $\text{volume}(V^Q)$ becomes no more than its $3/4$ (Lines 14 and 19), or
2. $|V^Q|$ becomes 1 (Lines 11).

When $|V^Q| = 1$, the algorithm will go to Lines 2-4, and hence the recursion depth is only 1. Then, as we set $V^Q = V$ in the earliest call to $\text{QuadApprox}$, we have that the total recursion depth is no more than $\log_4 \text{volume}(V) = \log_4 2m$. 

29
Algorithm 5: QUADAPPROX(G, S, V^Q, z, θ, ε_1, ε_2)

Input: G = (V, E): A graph.
S: A graph Laplacian whose edges are supported on V.
V^Q ⊂ V: a set of vertices
z ∈ R^|V|: a vector.
θ: a scalar between 0 and 1/2.
ε_1: the error parameter for Schur complement.
ε_2: the error parameter for QUAD SOLVE.

Output: \( \hat{N}^\Delta = \{(v, \hat{n}_v^\Delta) \mid v \in V^Q\} \).

1 if |V^Q| = 1 then
2 Let n and m be the number of vertices and edges in G, respectively.
3 Let B be the m × n edge-vertex incidence matrix of G, and W be the m × m diagonal edge weight matrix of G.
4 Let \( \hat{n}_v^\Delta = \text{QUAD SOLVE}(B, W, W^{1/2}Bz, \theta, \epsilon_2, S) \) and return \{ (v, \hat{n}_v^\Delta) \} for the only vertex v ∈ V^Q.
5 Let vol = \( \sum_{v \in V^Q} \deg u(v) \), and set \( \epsilon_{\text{schur}} = \epsilon_1 / \log \frac{4}{3} \text{vol} \).
6 Let V_4 be vertices in V^Q with \( \deg u(v) \geq \frac{\text{vol}}{4} \).
7 if V_4 ≠ ∅ then
8 for each v ∈ V_4 do
9 Let C denote v and its neighbors.
10 \( (\tilde{L}, \tilde{D}, \tilde{S}) \leftarrow \text{APX PARTIAL CHOLESKY}(S, C, \epsilon_{\text{schur}}) \)
11 \( \hat{N}^\Delta(v) \leftarrow \text{QUADAPPROX}(G[C], \tilde{S}, \{v\}, z_C, \theta, \epsilon_1 - \epsilon_{\text{schur}}, \epsilon_2) \)
12 Let C denote vertices in V^Q \ V_4 and their neighbors.
13 \( (\tilde{L}, \tilde{D}, \tilde{S}) \leftarrow \text{APX PARTIAL CHOLESKY}(S, C, \epsilon_{\text{schur}}) \)
14 return \text{QUADAPPROX}(G[C], \tilde{S}', V^Q \ V_4, z_{C'}, \theta, \epsilon_1 - \epsilon_{\text{schur}}, \epsilon_2) \cup \left( \bigcup_{v \in V_4} \hat{N}^\Delta(v) \right) \)
15 Divide V^Q into two parts V(1) and V(2) such that both \( \sum_{v \in V(1)} \deg u(v) \) and \( \sum_{v \in V(2)} \deg u(v) \) are in the range \( \left[ \frac{1}{4} \text{vol}, \frac{3}{4} \text{vol} \right] \).
16 for i = 1 to 2 do
17 Let C denote vertices in V(i) and their neighbors.
18 \( (\tilde{L}, \tilde{D}, \tilde{S}) \leftarrow \text{APX PARTIAL CHOLESKY}(S, C, \epsilon_{\text{schur}}) \)
19 \( \hat{N}^\Delta(i) \leftarrow \text{QUADAPPROX}(G[C], \tilde{S}, V(i), z_C, \theta, \epsilon_1 - \epsilon_{\text{schur}}, \epsilon_2) \)
20 return \( \hat{N}^\Delta(1) \cup \hat{N}^\Delta(2) \).
We then give guarantees for our approximations. Note that we set \( \epsilon_{\text{schur}} = \epsilon_1 / \log \frac{3}{2} \text{vol} \) (Line 5), and when recursively calling QUADAPPROX we set the \( \epsilon_1 \) of the recursive call to \( \epsilon_1 - \epsilon_{\text{schur}} \) (Line 11, 14 and 19). Then, since we set \( \epsilon_1 = \epsilon \theta / 9 \) in the earliest call to QUADAPPROX, we have that \( \epsilon_{\text{schur}} \leq \frac{\theta^2}{\pi^2} \log \frac{3}{2} 2m \) always holds. Coupled with the fact that the total recursion depth is no more than \( \log \frac{3}{2} 2m \), on Line 4 we have that

\[
S \approx_{\epsilon \theta / 9} \text{Sc}(L^G, V(E_v))
\]

holds for the only vertex \( v \in V^Q \), where \( L^G \) is the Laplacian matrix of the graph in the earliest call to QUADAPPROX (i.e., the original graph). Then by Lemma 6.6, \( \hat{n}_v^L \) on Line 4 satisfies

\[
\hat{n}_v^L \approx_{\epsilon} z^\top B_{E_v} W_{E_v}^{1/2} \left( I - (1 - \theta) W_{E_v}^{1/2} B_{E_v} L^1 B_{E_v} W_{E_v}^{1/2} \right)^{-1} W_{E_v}^{1/2} B_{E_v} z.
\]

We now analyze the running time.

Let \( T(m) \) denote the running time of QUADAPPROX \((G, S, V^Q, z, \theta, \epsilon_1, \epsilon_2)\), where \( m = \text{volume}(V^Q) \). Let \( n_{\text{cur}} \) and \( m_{\text{cur}} \) denote the number of vertices and the number of edges in \( L \) in the current call, respectively. We first assume QUADSOLVE to be an \( O(1) \) operation, and hence \( T(m) = O(1) \) for \( |V^Q| = 1 \). For \( |V^Q| > 1 \), we consider the set \( V_4 \) on Line 6:

1. If \( V_4 \) is not empty, the algorithm goes to Lines 7-14. Since there are at most 4 vertices in \( V_4 \), and by our assumption the recursive calls to QUADAPPROX on Line 11 all run in \( O(1) \) time, we have by Lemma 3.5 Lines 7-13 runs in total \( O((m_{\text{cur}} \log^3 n + n_{\text{cur}} \epsilon_{\text{schur}}^2 \log^5 n) \text{polyloglog}(n)) \) time. Since \( V_4 \) is not empty, we have volume\((V \setminus V_4) \leq \frac{3}{4} \text{volume}(V) \). Hence, the running time of the recursive call to QUADAPPROX on Line 14 is at most \( T(3m/4) \).

2. If \( V_4 \) is empty, the algorithm goes to Lines 15-20. By Lemma 3.5, the calls to APXPARTIALCHOLESKY on Line 18 run in total \( O((m_{\text{cur}} \log^3 n + n_{\text{cur}} \epsilon_{\text{schur}}^2 \log^5 n) \text{polyloglog}(n)) \) time. The running time of the recursive calls to QUADAPPROX on Line 19 is \( T(m_1) + T(m - m_1) \), where \( m_1 = \text{volume}(V(1)) \) is in the range \([m/4, 3m/4]\).

Since \( \epsilon_{\text{schur}} = O(\theta \epsilon / \log n) \), \( n_{\text{cur}} = O(m) \), and by Lemma 3.5 \( m_{\text{cur}} = O(n_{\text{cur}} \epsilon_{\text{schur}}^2 \log n) = O(m \theta^{-2} \epsilon^{-2} \log^3 n) \), we have in the worst case

\[
T(m) = T(3m/4) + T(m/4) + O(m \theta^{-2} \epsilon^{-2} \log^7 n \text{polyloglog}(n)),
\]

which gives \( T(m) = O(m \theta^{-2} \epsilon^{-2} \log^8 n \text{polyloglog}(n)) \).

Note that we get this running time under the assumption that QUADSOLVE is an \( O(1) \) operation. Thus, we also need to analyze the total running time of the calls to QUADSOLVE on Line 4.

By Lemma 6.6, the QUADSOLVE \((B, W, W^{1/2} B z, \theta, \epsilon_2, S)\) on Line 4 runs in \( O(\text{nnz}(S) \theta^{-0.5} \log^3 n \log(1/\epsilon) + \text{deg}^u(v) \theta^{-2.5} \epsilon^{-2} \log^5 n \log(1/\epsilon) \text{polyloglog}(n)) \) time, where \( v \) indicates the only vertex in \( V^Q \) and \( \text{deg}^u(v) \) is the number of edges incident to \( v \). By Lemma 3.5, we have \( \text{nnz}(S) = O(\text{deg}^u(v) \epsilon_{\text{schur}}^2 \log n) = O(\text{deg}^u(v) \theta^{-2} \epsilon^{-2} \log^3 n) \). Then, summing this running time over all vertices gives \( O(m \theta^{-2} \epsilon^{-2} \log^5 n \log(1/\epsilon) \text{polyloglog}(n)) \), which plus \( T(m) \) gives the overall running time of this algorithm.

6.3 Approximating \( C^\Delta_\theta(v) \)

We give the pseudocode of the algorithm VERTEXCENTCOMP which approximates \( \theta \)-Kirchhoff vertex centrality \( C^\Delta_\theta(v) \) for all \( v \in V \) in Algorithm 6. Note that in this algorithm we once again invoke the Laplacian solver of [CKM+14]. The performance of VERTEXCENTCOMP is characterized in Theorem 1.3. Analyzing this algorithm gives the main result for estimating vertex centralities.
where the second inequality follows by Lemma 6.3, we have

\[ \text{Algorithm 6: VertexCentComp}(G = (V, E), w, \theta, \epsilon) \]

**Input**: \( G = (V, E), w, \): A connected undirected graph with positive edges weights \( w : E \to \mathbb{R}_+ \),
\( \theta \): A scalar between 0 and 1/2.
\( \epsilon \): Error of the centrality estimate per vertex.

**Output**: \( \hat{C}_\Delta = \{(v, \hat{c}_{\Delta v}^z) \mid v \in V \} \).

1. Let \( z_1, \ldots, z_M \) be independent random \( \pm 1 \) vectors, where \( M = \lceil 432 \epsilon^{-2} \ln(2n) \rceil \).
2. for \( i = 1 \) to \( M \) do
3. \( y_i \leftarrow \text{LaplSolve}(L^G, z_i, \frac{1}{36} \theta \epsilon n^{-7} U^{-4}) \)
4. \( \left( \hat{N}(\Delta) = \{(v, \hat{n}_{\Delta v}(i)) \mid v \in V \} \right) \leftarrow \text{QuadApprox}(L^G, V, y_i, \theta / 27, \epsilon / 3) \)
5. For each \( v \in V \) compute \( \hat{c}_{\Delta v} = (1 - \theta) \frac{1}{M} \sum_{i=1}^{M} \hat{n}_{\Delta v}(i) \) and return \( \hat{C}_\Delta = \{(v, \hat{c}_{\Delta v}) \mid v \in V \} \).

**Proof of Theorem 1.3.** The running time is the total cost of \( O(\epsilon^{-2} \log n) \) calls to LapseSolve each of which runs in \( O(m \log^{1.5} n \log \frac{1}{\theta} \epsilon n^{-7} U^{-4}) \) time, and \( O(\epsilon^{-2} \log n) \) calls to QuadApprox each of which runs in \( O(m(\theta^{-2} \epsilon^{-2} \log^8 n + \theta^{-2.5} \epsilon^{-2} \log^5 n \log(1/\epsilon)) \text{polyloglog}(n)) \) time.

In the rest of this proof, we will use the matrix \( C_v \), defined as

\[
C_v \overset{\text{def}}{=} B_{E_v}^T W_{E_v}^{1/2} \left( I - (1 - \theta) W_{E_v}^{1/2} B_{E_v} L^v B_{E_v}^T W_{E_v}^{1/2} \right)^{-1} W_{E_v}^{1/2} B_{E_v},
\]

to simplify notation.

Since \( M = \lceil 432 \epsilon^{-2} \ln(2n) \rceil \geq 48 (\epsilon / 3)^{-2} \ln(2n) \), by Lemma 4.3, we have

\[
\frac{1}{M} \sum_{i=1}^{M} z_i^T L^v C_v L^v z_i \approx \epsilon / 3 \text{ Tr} \left( L^v C_v L^v \right). \tag{41}
\]

By Lemma 6.3, we have

\[
\text{Tr} \left( L^v C_v L^v \right) \geq \frac{2}{n^2 U^2},
\]

and hence

\[
\frac{1}{M} \sum_{i=1}^{M} z_i^T L^v C_v L^v z_i \geq \exp(-\epsilon / 3) \frac{2}{n^2 U^2} \geq \frac{1}{n^2 U^2}, \tag{42}
\]

where the second inequality follows by \( 0 < \epsilon \leq 1/2 \).

Since we set \( \delta = \frac{1}{36} \theta en^{-7} U^{-4} \) when invoking LapseSolve, by Lemma 5.1,

\[
\| y_i - L^v z_i \|_L \leq \frac{1}{36} \theta en^{-7} U^{-4} \| L^v z_i \|_L,
\]

holds for each \( i \). Then, by Lemma 6.2, we have that

\[
\left| y_i^T C_v y_i - z_i^T L^v C_v L^v z_i \right| \leq \frac{1}{6} \epsilon n^{-2} U^{-2}
\]

32
holds for each $i$. We then have

$$
\left| \frac{1}{M} \sum_{i=1}^{M} y_i^\top C_v y_i - \frac{1}{M} \sum_{i=1}^{M} z_i^\top L_i^\top C_v L_i z_i \right|
$$

$$
\leq \frac{1}{M} \sum_{i=1}^{M} \left| y_i^\top C_v y_i - z_i^\top L_i^\top C_v L_i^\top z_i \right|
$$

$$
\leq \frac{1}{M} \sum_{i=1}^{M} \left| z_i^\top L_i^\top C_v L_i^\top z_i \right|
$$

$$
\leq \frac{1}{M} \sum_{i=1}^{M} \left| y_i^\top C_v y_i - z_i^\top L_i^\top C_v L_i^\top z_i \right|
$$

where the last inequality follows by (42). Thus,

$$
(1 - \epsilon/6) \frac{1}{M} \sum_{i=1}^{M} z_i^\top L_i^\top C_v L_i^\top z_i \leq \frac{1}{M} \sum_{i=1}^{M} y_i^\top C_v y_i \leq (1 + \epsilon/6) \frac{1}{M} \sum_{i=1}^{M} z_i^\top L_i^\top C_v L_i^\top z_i,
$$

which implies

$$
\frac{1}{M} \sum_{i=1}^{M} z_i^\top L_i^\top C_v L_i^\top z_i \approx \frac{1}{M} \sum_{i=1}^{M} y_i^\top C_v y_i. \quad (43)
$$

By Lemma 6.7, we have

$$
\hat{n}_v^{\Delta(i)} \approx \epsilon/3 \ y_i^\top C_v y_i. \quad (44)
$$

Combining Equation (41), (43) and (44), we have

$$
\frac{1}{M} \sum_{i=1}^{M} \hat{n}_v^{\Delta(i)} \approx \epsilon \ Tr \left( L_i^\top C_v L_i^\top \right),
$$

which coupled with the fact that

$$
C^\Delta_\theta(v) = \mathcal{K} (G \setminus \theta E_v) - \mathcal{K} (G) \quad \text{by definition}
$$

$$
= n \left( Tr \left( (L \setminus \theta E_v)^\top \right) - Tr \left( L_i^\top \right) \right) \quad \text{by Fact 2.4}
$$

$$
= n(1 - \theta) \left( Tr \left( L_i^\top C_v L_i^\top \right) \right) \quad \text{by Equation (33)}
$$

gives the guarantee of our approximation.

## 7 Conclusion and Future Work

The Kirchhoff index arises in many applications such as noisy consensus problems [PB14] and social recommender systems [WLC16]. It is a global index, and any changes of network structure, e.g. weight of edges, can be reflected in this popular index. In this paper, we proposed to use Kirchhoff index as a global metric of the importance of edges in weighted undirected networks. For any network, when the weight of any edge $e$ is changed from $w(e)$ to $\theta w(e)$, the Kirchhoff index of the resulting graph will strictly increase, with the increase deciphering the importance of edge $e$. We used the Kirchhoff index of the new graph, or its increment with respect to the original graph,
as the $\theta$-Kirchhoff edge centrality. We demonstrated experimentally that this new global measure of edge centrality has a more discriminating power than edge betweenness, spanning edge centrality, and current-flow centrality.

However, the time cost of exactly computing the $\theta$-Kirchhoff edge centrality is prohibitive. To overcome this weakness, we introduced two approaches that estimate the $\theta$-Kirchhoff edge centrality for all edges in nearly linear time. Our proposed centrality metrics are the first global measure of centrality that can be estimated in nearly linear time. Our algorithms combine techniques from several recent works on graph algorithms [LSW15, DKP+17]. We also extend these ideas to develop efficient algorithms for estimating $\theta$-Kirchhoff vertex centrality, as well as estimating the Kirchhoff edge centrality to a set of edges. This raises the possibility of designing highly efficient algorithms that can detect the set of $k$ most influential edges, that is, the $k$ edges whose $\theta$-deletion leads to the largest increase of the Kirchhoff index.

Despite the advantages of our algorithms, their theoretical performance still has much room for improvement, both in the overhead of logarithmic factors and the dependency on $\theta$. The latter is particularly interesting because our two algorithms for estimating edge centrality can perform better under different regimes of $\theta$. On the other hand, the importance of centrality measures in graph mining means it is just as, if not more, interesting to study the practical behaviors of our algorithms. Specifically, to see if they are reasonably fast and accurate on massive networks with millions of vertices and edges. Recent packages for solving large scale linear systems and related tasks [LB12, KMT11, SSM14, Spi17] should greatly facilitate such a study. Moreover, the significantly higher deviations from our experiments suggest the question of whether it is possible to theoretically model the advantages/disadvantages of the many centrality measures.

Finally, it should be mentioned that as an application of the introduced edge centrality, we studied the vertex centrality based on the idea of the definition for $C^2_{\theta}(e)$. Actually, we can also define the centrality of a vertex $v$ as the Kirchhoff index of the graph $G\setminus \theta E_v$, the algorithm for the $\epsilon$-approximation of which is similar to EdgeCentComp1. We thus omit the algorithmic details of this version of vertex centrality for the lack of space. Another reason for ignoring this algorithm is that our main focus is the edge centrality.
References

[Ach01] Dimitris Achlioptas. Database-friendly random projections. In Proceedings of the 20th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS), 2001.

[ADK+16] Ittai Abraham, David Durfee, Ioannis Koutis, Sebastian Krinninger, and Richard Peng. On fully dynamic graph sparsifiers. In Proceedings of IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 335–344, 2016.

[AT11] Haim Avron and Sivan Toledo. Randomized algorithms for estimating the trace of an implicit symmetric positive semi-definite matrix. Journal of ACM, 58(2):8:1–8:34, 2011.

[BCH14] Daniel Bienstock, Michael Chertkov, and Sean Harnett. Chance-constrained optimal power flow: Risk-aware network control under uncertainty. SIAM Review, 56(3):461–495, 2014.

[BDFMR16] Francesco Bonchi, Gianmarco De Francisci Morales, and Matteo Riondato. Centrality measures on big graphs: Exact, approximated, and distributed algorithms. In Proceedings of the 25th International Conference Companion on World Wide Web (WWW), pages 1017–1020, 2016.

[BF05] Ulrik Brandes and Daniel Fleischer. Centrality measures based on current flow. In Proceedings of the 22nd Annual Symposium on Theoretical Aspects of Computer Science (STACS), volume 3404, pages 533–544, 2005.

[BHNT15] Sayan Bhattacharya, Monika Henzinger, Danupon Nanongkai, and Charalampos Tsourakakis. Space-and time-efficient algorithm for maintaining dense subgraphs on one-pass dynamic streams. In Proceedings of the 47th annual ACM Symposium on Theory of Computing (STOC), pages 173–182, 2015.

[BKMM07] David A Bader, Shiva Kintali, Kamesh Madduri, and Milena Mihail. Approximating betweenness centrality. In Proceedings of the 5th International Conference on Algorithms and Models for the Web-Graph (WAW), volume 4863, pages 124–137, 2007.

[BLHL01] Tim Berners-Lee, James Hendler, and Ora Lassila. The semantic web. Scientific American, 284(5):28–37, 2001.

[BP07] Ulrik Brandes and Christian Pich. Centrality estimation in large networks. International Journal of Bifurcation and Chaos, 17(07):2303–2318, 2007.

[Bra01] Ulrik Brandes. A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, 25(2):163–177, 2001.

[BV14] Paolo Boldi and Sebastiano Vigna. Axioms for centrality. Internet Mathematics, 10(3-4):222–262, 2014.

[BWLM16] Elisabetta Bergamini, Michael Wegner, Dimitar Lukarski, and Henning Meyerhenke. Estimating current-flow closeness centrality with a multigrid laplacian solver. In Proceedings of the 7th SIAM Workshop on Combinatorial Scientific Computing (CSC), pages 1–12, 2016.
[CKM+14] Michael B. Cohen, Rasmus Kyng, Gary L. Miller, Jakub W. Pachocki, Richard Peng, Anup Rao, and Shen Chen Xu. Solving SDD linear systems in nearly $m \log^{1/2} n$ time. In Proceedings of the 46th annual ACM Symposium on Theory of Computing (STOC), pages 343–352, 2014.

[DKP+17] David Durfee, Rasmus Kyng, John Peebles, Anup B. Rao, and Sushant Sachdeva. Sampling random spanning trees faster than matrix multiplication. In Proceedings of the 49th annual ACM Symposium on Theory of Computing (STOC), pages 730–742, 2017.

[DPPR17] David Durfee, John Peebles, Richard Peng, and Anup B. Rao. Determinant-preserving sparsification of SDDM matrices with applications to counting and sampling spanning trees. CoRR, abs/1705.00985, 2017.

[DSD+11] Li Ding, Dana Steil, Brandon Dixon, Allen Parrish, and David Brown. A relation context oriented approach to identify strong ties in social networks. Knowledge-Based Systems, 24(8):1187–1195, 2011.

[ESVM+11] W Ellens, FM Speiksma, P Van Mieghem, A Jamakovic, and RE Kooij. Effective graph resistance. Linear Algebra and its Applications, 435(10):2491–2506, 2011.

[FQY14] Minyu Feng, Hong Qu, and Zhang Yi. Highest degree likelihood search algorithm using a state transition matrix for complex networks. IEEE Transactions on Circuits and Systems I: Regular Papers, 61(10):2941–2950, 2014.

[GN02] Michelle Girvan and Mark EJ Newman. Community structure in social and biological networks. Proceedings of the National Academy of Sciences, 99(12):7821–7826, 2002.

[GSS08] Robert Geisberger, Peter Sanders, and Dominik Schultes. Better approximation of betweenness centrality. In Proceedings of the Meeting on Algorithm Engineering & Experiments, pages 90–100, 2008.

[GVL12] Gene H Golub and Charles F Van Loan. Matrix computations, volume 3. JHU Press, 2012.

[HAY16] Takanori Hayashi, Takuya Akiba, and Yuichi Yoshida. Efficient algorithms for spanning tree centrality. In Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI), pages 3733–3739, 2016.

[Hut89] MF Hutchinson. A stochastic estimator of the trace of the influence matrix for Laplacian smoothing splines. Communications in Statistics-Simulation and Computation, 18(3):1059–1076, 1989.

[HX16] Nicholas JA Harvey and Keyulu Xu. Generating random spanning trees via fast matrix multiplication. In Proceedings of Latin American Symposium on Theoretical Informatics, pages 522–535, 2016.

[JL84] William B Johnson and Joram Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. Contemporary Mathematics, 26(189-206):1, 1984.

[KLP+16] Rasmus Kyng, Yin Tat Lee, Richard Peng, Sushant Sachdeva, and Daniel A. Spielman. Sparsified Cholesky and multigrid solvers for connection Laplacians. In Proceedings of
the 48th annual ACM Symposium on Theory of Computing (STOC), pages 842–850, 2016.

[KMT11] Ioannis Koutis, Gary L. Miller, and David Tolliver. Combinatorial preconditioners and multilevel solvers for problems in computer vision and image processing. Computer Vision and Image Understanding, 115(12):1638–1646, 2011.

[Knu93] Donald Ervin Knuth. The Stanford GraphBase: a Platform for Combinatorial Computing, volume 37. Addison-Wesley Reading, 1993.

[KP17] John Kallaugher and Erie Price. A hybrid sampling scheme for triangle counting. In Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1778–1797, 2017.

[KPPS17] Rasmus Kyng, Jakub Pachocki, Richard Peng, and Sushant Sachdeva. A framework for analyzing resparsification algorithms. In Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2032–2043, 2017.

[KR93] Douglas J Klein and Milan Randić. Resistance distance. Journal of Mathematical Chemistry, 12(1):81–95, 1993.

[KS16] Rasmus Kyng and Sushant Sachdeva. Approximate gaussian elimination for laplacians - fast, sparse, and simple. In Proceedings of IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 573–582, 2016.

[LB12] Oren E. Livne and Achi Brandt. Lean algebraic multigrid (LAMG): fast graph laplacian linear solver. SIAM Journal on Scientific Computing, 34(4), 2012.

[LCR+16] Linyuan Lü, Duanbing Chen, Xiao-Long Ren, Qian-Ming Zhang, Yi-Cheng Zhang, and Tao Zhou. Vital nodes identification in complex networks. Physics Reports, 650:1–63, 2016.

[LM12] Amy N Langville and Carl D Meyer. Who’s # 1?: the science of rating and ranking. Princeton University Press, 2012.

[LS15] Yin Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In Proceedings of IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS), pages 230–249, 2015.

[LS17] Yin Tat Lee and He Sun. An SDP-based algorithm for linear-sized spectral sparsification. In Proceedings of the 49th Annual ACM Symposium on Theory of Computing (STOC), pages 678–687, 2017.

[LSB+03] David Lusseau, Karsten Schneider, Oliver J Boisseau, Patti Haase, Elisabeth Slooten, and Steve M Dawson. The bottlenose dolphin community of doubtful sound features a large proportion of long-lasting associations. Behavioral Ecology and Sociobiology, 54(4):396–405, 2003.

[LSW15] Yin Tat Lee, Aaron Sidford, and Sam Chiu-wai Wong. A faster cutting plane method and its implications for combinatorial and convex optimization. In Proceedings of IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS), pages 1049–1065, 2015.
Charalampos Mavroforakis, Richard García-Lebron, Ioannis Koutis, and Evimaria Terzi. Spanning edge centrality: Large-scale computation and applications. In *Proceedings of the 24th International Conference on World Wide Web (WWW)*, pages 732–742, 2015.

Mark EJ Newman. Finding community structure in networks using the eigenvectors of matrices. *Physical review E*, 74(3):036104, 2006.

Mark Newman. *Networks: An introduction*. Oxford university press, 2010.

Stacy Patterson and Bassam Bamieh. Consensus and coherence in fractal networks. *IEEE Transactions on Control of Network Systems*, 1(4):338–348, 2014.

Richard Peng and Daniel A Spielman. An efficient parallel solver for SDD linear systems. In *Proceedings of the 46th annual ACM Symposium on Theory of Computing (STOC)*, pages 333–342, 2014.

Piotr Sankowski. Dynamic transitive closure via dynamic matrix inverse. In *Proceedings of 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 509–517, 2004.

Jack Sherman and Winifred J Morrison. Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. *The Annals of Mathematical Statistics*, 21(1):124–127, 1950.

Dan Spielman. Laplacians.jl. [https://github.com/danspielman/Laplacians.jl](https://github.com/danspielman/Laplacians.jl), 2017.

Daniel A. Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. *SIAM Journal of Computing*, 40(6):1913–1926, 2011.

Christian Staudt, Aleksejs Sazonovs, and Henning Meyerhenke. Networkit: An interactive tool suite for high-performance network analysis. *CoRR*, abs/1403.3005, 2014.

D. Spielman and S. Teng. Nearly linear time algorithms for preconditioning and solving symmetric, diagonally dominant linear systems. *SIAM Journal on Matrix Analysis and Applications*, 35(3):835–885, 2014.

Andrea Sofia Teixeira, Pedro T Monteiro, João A Carriço, Mário Ramirez, and Alexandre P Francisco. Spanning edge betweenness. In *Proceedings of 11st Workshop on Mining and Learning with Graphs*, volume 24, pages 27–31, 2013.

Virginia Vassilevska Williams. Multiplying matrices faster than Coppersmith-Winograd. In *Proceedings of the 44th annual ACM Symposium on Theory of Computing (STOC)*, pages 887–898, 2012.

Felix Ming Fai Wong, Zhenming Liu, and Mung Chiang. On the efficiency of social recommender networks. *IEEE/ACM Transactions on Networking*, 24(4):2512–2524, 2016.

Duncan J Watts and Steven H Strogatz. Collective dynamics of ‘small-world’ networks. *Nature*, 393(6684):440–442, 1998.
Scott White and Padhraic Smyth. Algorithms for estimating relative importance in networks. In *Proceedings of the 9th ACM International Conference on Knowledge Discovery and Data Mining (KDD)*, pages 266–275, 2003.

Keyou You, Roberto Tempo, and Li Qiu. Distributed algorithms for computation of centrality measures in complex networks. *IEEE Transactions on Automatic Control*, 62(5):2080–2094, 2017.

Wayne W Zachary. An information flow model for conflict and fission in small groups. *Journal of Anthropological Research*, 33(4):452–473, 1977.
A Proofs of Our Version of Sherman-morrison and Woodbury Formulas

In this section, we give detailed proofs for the Sherman-Morrision and Woodbury formulas we used, i.e., Equations (26) and (33).

In the proofs, we will use the matrix $\Pi$ defined as

$$\Pi \overset{\text{def}}{=} LL^\dagger = I - \frac{1}{n} 11^\dagger,$$

where $1$ is the vector with all entries being 1.

**Proof of Equation (26).** First, we have

$$b_e \left( 1 - (1 - \theta)w(e)b_e^\dagger L^\dagger b_e \right) = b_e - (1 - \theta)w(e)b_e b_e^\dagger L^\dagger b_e = \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right) L^\dagger b_e,$$

where the second equality follows by $b_e = \Pi b_e = LL^\dagger b_e$.

Since $\theta < 1$, we have that $L - (1 - \theta)w(e)b_e b_e^\dagger$ is a Laplacian matrix and $1 - (1 - \theta)w(e)b_e^\dagger L^\dagger b_e$ is strictly positive. Thus, Equation (45) implies

$$\left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger b_e = \frac{L^\dagger b_e}{1 - (1 - \theta)w(e)b_e L^\dagger b_e}.$$

Then, we have

$$L^\dagger = \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right) L^\dagger$$

$$= \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger \left( \Pi - (1 - \theta)w(e)b_e b_e^\dagger L^\dagger \right)$$

$$= \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger - (1 - \theta)w(e) \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger b_e b_e^\dagger L^\dagger$$

$$= \left( L - (1 - \theta)w(e)b_e b_e^\dagger \right)^\dagger - (1 - \theta) \frac{w(e)L^\dagger b_e b_e^\dagger L^\dagger}{1 - (1 - \theta)w(e)b_e L^\dagger b_e},$$

which implies Equation (26). \hfill \square

**Proof of Equation (33).** First, we have

$$B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)$$

$$= B_T^\dagger W_T^{1/2} - (1 - \theta) B_T^\dagger W_T B_T L^\dagger B_T^\dagger W_T^{1/2}$$

$$= \left( L - (1 - \theta) B_T^\dagger W_T B_T \right) L^\dagger B_T^\dagger W_T^{1/2},$$

(46)

where the second equality follows by $B_T^\dagger = \Pi B_T^\dagger = LL^\dagger B_T^\dagger$.

Since $\theta < 1$, we have that $I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2}$ is positive definite and $L - (1 - \theta) B_T^\dagger W_T B_T$ is a Laplacian matrix. Thus, Equation (46) implies

$$\left( L - (1 - \theta) B_T^\dagger W_T B_T \right)^\dagger B_T^\dagger W_T^{1/2} = L^\dagger B_T^\dagger W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\dagger W_T^{1/2} \right)^{-1}.$$
Then, we can write $L^\dagger$ as
\[
\left( L - (1 - \theta)B_T^\top W_T B_T \right)^\dagger \left( L - (1 - \theta)B_T^\top W_T B_T \right) L^\dagger
\]
\[
= \left( L - (1 - \theta)B_T^\top W_T B_T \right)^\dagger \left( \Pi - (1 - \theta)B_T^\top W_T B_T \right) L^\dagger
\]
\[
= \left( L - (1 - \theta)B_T^\top W_T B_T \right)^\dagger - (1 - \theta) \left( L - (1 - \theta)B_T^\top W_T B_T \right)^\dagger B_T^\top W_T B_T L^\dagger
\]
\[
= \left( L - (1 - \theta)B_T^\top W_T B_T \right)^\dagger - (1 - \theta) L^\dagger B_T^\top W_T^{1/2} \left( I - (1 - \theta) W_T^{1/2} B_T L^\dagger B_T^\top W_T^{1/2} \right)^{-1} W_T^{1/2} B_T L^\dagger,
\]
which implies Equation (33).

\[ \square \]

B Approximations When Subtracted From Identity Matrix

In this section, we bound the transfer of approximations between $A$ and $B$ to approximations between $I - (1 - \theta)A$ and $I - (1 - \theta)B$.

\textit{Proof of Lemma 5.5.}\ The given condition with the approximation can be written as:
\[
(1 - 2\epsilon) A \preceq B \preceq (1 + 2\epsilon) A,
\]
which implies
\[
I - (1 - \theta) (1 + 2\epsilon) A \preceq I - (1 - \theta) B \preceq I - (1 - \theta) (1 - 2\epsilon) A.
\]
Since $0 \preceq A \preceq I$, we have the following lower bound:
\[
I - (1 - \theta) (1 + 2\epsilon) A = \left( 1 - \frac{2\epsilon}{\theta} \right) (I - (1 - \theta) A) + \frac{2\epsilon}{\theta} I - (1 - \theta) \left( 2\epsilon + \frac{2\epsilon}{\theta} \right) A
\]
\[
\succeq \left( 1 - \frac{2\epsilon}{\theta} \right) (I - (1 - \theta) A) + \left[ \frac{2\epsilon}{\theta} - (1 - \theta) \left( 2\epsilon + \frac{2\epsilon}{\theta} \right) \right] A.
\]
The coefficient on the trailing $A$ in turn simplifies to $2\epsilon - (1 - \theta)2\epsilon \geq 0$.

Similarly for the upper bound we get:
\[
I - (1 - \theta) (1 - 2\epsilon) A = \left( 1 + \frac{2\epsilon}{\theta} \right) (I - (1 - \theta) A) - \frac{2\epsilon}{\theta} I + (1 - \theta) \left( 2\epsilon + \frac{2\epsilon}{\theta} \right) A
\]
\[
\preceq \left( 1 + \frac{2\epsilon}{\theta} \right) (I - (1 - \theta) A).
\]
Then the final bound involving $\exp(3\epsilon/\theta)$ follows from the condition of $\epsilon/\theta$ being small.

\[ \square \]

C Error Tracking for Laplacian Solvers

In this section, we provide more details on error tracking for Laplacian solvers in Section 5 and 6 in a way similar to Section 4 of [SS11].

We first give bounds on eigenvalues of $L$. Let $L$ be the Laplacian matrix of a graph $G = (V,E)$ with $n$ vertices, $m$ edges and edge weights all in the range $[1, U]$. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $L$, and $0 = \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$ be the eigenvalues of the normalized Laplacian matrix, $N \overset{\text{def}}{=} D^{-1/2}LD^{-1/2}$, of $G$. It is easy to verify that $\nu_i \leq \lambda_i \leq nU \nu_i$ holds for all $i$. Let
\[
\phi_G = \min_{S \subset V} \frac{|\partial(S)|}{\min (d(S), d(V \setminus S))}
\]

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be the conductance of $G$, where $|\partial(S)|$ denotes the total weights of edges with one endpoint in $S$ and the other endpoint in $V \setminus S$, and $d(S)$ denotes the total degree of vertices in $S$. Then, we can bound $\lambda_2$ by

$$\lambda_2 \geq \nu_2 \geq \phi^2_G / 2 \quad \text{by Cheeger’s inequality}$$

$$\geq \left( \frac{1}{n^2 U} \right)^2 / 2 \quad \text{since all edge weights are in } [1, U]$$

$$= \frac{1}{2n^4 U}. \quad (47)$$

We then bound $\lambda_n$ using the fact that $L^G \preceq U L^{K_n}$, where $K_n$ is the complete graph of $n$ vertices. Thus,

$$\lambda^G_n \leq \lambda^{K_n}_n = nU. \quad (48)$$

From (47) and (48) it is immediate that

$$\frac{1}{2n^4 U^2} \Pi \preceq L \preceq nU I \quad \text{and} \quad \frac{1}{nU} \Pi \preceq L^\dagger \preceq 2n^4 U^2 I$$

hold, where $\Pi \overset{\text{def}}{=} LL^\dagger = I - \frac{1}{n} \mathbf{1} \mathbf{1}^\top$.

We will also need to use the inequality

$$\left| x^2 - y^2 \right| \leq (2|y| + |x - y|)|x - y|$$

for scalars $x, y$, which follows by

$$\left| x^2 - y^2 \right| \leq (|x| + |y|)|x - y| \leq (|y| + |y + (x - y)|)|x - y| \leq (2|y| + |x - y|)|x - y|.$$

C.1 Error Tracking for the Laplacian Solver in Section 5

Proof of Lemma 5.2. The lhs of inequality (28) can be written as

$$\left| \|y\|^2_{b_e b^\top_e} - \|L^\dagger z\|^2_{b_e b^\top_e} \right|.$$  

We first bound the value $\left| \|y\|^2_{b_e b^\top_e} - \|L^\dagger z\|^2_{b_e b^\top_e} \right|$ by

$$\left| \|y\|^2_{b_e b^\top_e} - \|L^\dagger z\|^2_{b_e b^\top_e} \right| \leq \left| y - L^\dagger z \right|_{b_e b^\top_e} \quad \text{by the triangle inequality of norms}$$

$$\leq \left\| y - L^\dagger z \right\|_L \quad \text{since } b_e b^\top_e \preceq L$$

$$\leq \delta \left\| L^\dagger z \right\|_L = \delta \sqrt{\mathbf{z}^\top L^\dagger L^\dagger \mathbf{z}}$$

$$\leq \delta n^{0.5} \sqrt{\frac{\mathbf{z}^\top L^\dagger L^\dagger \mathbf{z}}{\mathbf{z}^\top \mathbf{z}}} \quad \text{since } \|\mathbf{z}\|^2 \leq n$$

$$\leq \sqrt{2}\delta n^{2.5} U \quad \text{since } L^\dagger \preceq 2n^4 U^2 I.$$
We then use the inequality $|x^2 - y^2| \leq (2|y| + |x - y|)|x - y|$ to bound $\|y\|_{b_e b_e^\top}^2 - \|L^\dagger z\|_{b_e b_e^\top}^2$:

$$\left|\|y\|_{b_e b_e^\top}^2 - \|L^\dagger z\|_{b_e b_e^\top}^2\right| \leq \left(2\|L^\dagger z\|_{b_e b_e^\top} + \|y\|_{b_e b_e^\top} - \|L^\dagger z\|_{b_e b_e^\top}\right)\|y\|_{b_e b_e^\top} - \|L^\dagger z\|_{b_e b_e^\top} \leq \left(2\|L^\dagger z\|_L + \sqrt{2}\delta n^{-2.5}U\right)\sqrt{2}\delta n^{-2.5}U \quad \text{by } b_e b_e^\top \leq L \text{ and the above bound}$$

$$\leq \left(2\sqrt{2}\delta n^{-2.5}U + \sqrt{2}\delta n^{-2.5}U\right)\sqrt{2}\delta n^{-2.5}U \quad \text{since } \|z\|^2 \leq n \text{ and } L^\dagger \leq 2n^4U^2 I$$

$$\leq 6\delta n^5U^2 \quad \text{by } \delta < 1.$$

\[ \square \]

**Proof of Lemma 5.3.**

$$\text{Tr} \left( L^\dagger b_e b_e^\top L^\dagger \right) = b_e^\top L^\dagger b_e \quad \text{by cyclicity of trace}$$

$$= 2\frac{b_e^\top \left(L^\dagger\right)^2 b_e}{b_e^\top b_e} \quad \text{since } \|b_e\|^2 = 2$$

$$\geq \frac{2}{n^2U^2} \quad \text{since } \left(L^\dagger\right)^2 \geq \frac{1}{n^2U^2} \Pi.$$

\[ \square \]

### C.2 Error Tracking for the Laplacian Solver in Section 6

**Proof of Lemma 6.2.** The lhs of inequality (34) can be seen as the difference between the following two values:

$$\|y\|_{B_T W_T^{1/2}}^2 \left(I - (1-\theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}\right)^{-1} W_T^{1/2}B_T$$

$$\|L^\dagger z\|_{B_T W_T^{1/2}}^2 \left(I - (1-\theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}\right)^{-1} W_T^{1/2}B_T.$$

By the triangle inequality of norms, the difference between the square roots of these two values is at most

$$\left\|y - L^\dagger z\right\|_{B_T W_T^{1/2}}^2 \left(I - (1-\theta)W_T^{1/2}B_T L^\dagger B_T^\top W_T^{1/2}\right)^{-1} W_T^{1/2}B_T \leq \theta^{-0.5} \left\|y - L^\dagger z\right\|_{B_T W_T B_T} \quad \text{since } B_T^\top W_T B_T \leq L$$

$$\leq \theta^{-0.5} \left\|y - L^\dagger z\right\|_L \quad \text{since } B_T^\top W_T B_T \leq L$$

$$\leq \theta^{-0.5} \delta \left\|L^\dagger z\right\|_L = \theta^{-0.5} \delta \sqrt{z^\top L^\dagger z} \leq \theta^{-0.5} \delta n^{0.5} \sqrt{z^\top L^\dagger z} \quad \text{since } \|z\|^2 \leq n$$

$$\leq \sqrt{2} \theta^{-0.5} \delta n^{2.5} U \quad \text{since } L^\dagger \leq 2n^4U^2 I.$$
Then by the inequality $|x^2 - y^2| \leq (2|y| + |x - y|)|x - y|$, the lhs of (34) is at most

$$\left( 2 \left\| L^\dagger z \right\|_{B_T^\dagger W_1^{1/2}} (I - (1 - \theta) W_1^{1/2} B_T L^\dagger B_T^\top W_1^{1/2})^{-1} W_1^{1/2} B_T + \sqrt{2} \theta^{-0.5} \delta n^{2.5} U \right) \sqrt{2} \theta^{-0.5} \delta n^{2.5} U$$

$$\leq \left( 2 \theta^{-0.5} \left\| L^\dagger z \right\|_L + \sqrt{2} \theta^{-0.5} \delta n^{2.5} U \right) \sqrt{2} \theta^{-0.5} \delta n^{2.5} U$$

$$\leq \frac{1}{\theta} (2 \sqrt{2} n^{2.5} U + \sqrt{2} \delta n^{2.5} U) \sqrt{2} \delta n^{2.5} U \quad \text{since } \|z\|^2 \leq n \text{ and } L^\dagger \preceq 2 n^4 U^2 I$$

$$\leq 6 \theta^{-1} \delta n^5 U^2 \quad \text{by } \delta < 1.$$  

\[\square\]

**Proof of Lemma 6.3.**

$$\text{Tr} \left( L^\dagger B_T^\top W_1^{1/2} (I - (1 - \theta) W_1^{1/2} B_T L^\dagger B_T^\top W_1^{1/2})^{-1} W_1^{1/2} B_T L^\dagger \right)$$

$$\geq \text{Tr} \left( L^\dagger B_T^\top W_T B_T L^\dagger \right) \quad \text{since } (I - (1 - \theta) W_1^{1/2} B_T L^\dagger B_T^\top W_1^{1/2})^{-1} \preceq I$$

$$= \text{Tr} \left( L^\dagger \left( \sum_{e \in T} w(e) b_e b_e^\top \right) L^\dagger \right)$$

$$= \sum_{e \in T} w(e) \text{Tr} \left( L^\dagger b_e b_e^\top L^\dagger \right)$$

$$\geq \frac{2 |T|}{n^2 U^2} \quad \text{by Lemma 5.3 and } w(e) \geq 1.$$  

\[\square\]