The Moduli of Singular Curves on K3 Surfaces

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Abstract

In this article we consider moduli properties of singular curves on K3 surfaces. Let $B_g$ denote the stack of primitively polarized K3 surfaces $(X, L)$ of genus $g$ and let $T^n_{g,k} \to B_g$ be the stack parametrizing tuples $[(f : C \to X, L)]$ with $f$ an unramified morphism which is birational onto its image, $C$ a smooth curve of genus $p(g, k) - n$ and $f_* C \subseteq |kL|$. We show that the forgetful morphism

$$\eta : T^n_{g,k} \to \mathcal{M}_{p(g,k) - n}$$

is generically finite on one component, for all but finitely many values of $p(g, k) - n$. We further study the Brill–Noether theory of those curves parametrized by the image of $\eta$, and find a Wahl-type obstruction for a smooth curve with an unordered marking to have a nodal model on a K3 surface in such a way that the marking is the divisor over the nodes.

1 Introduction

The aim of this article is to study the moduli of those singular curves which may be embedded into a K3 surface. Let $[C] \in \mathcal{M}_p$ be a point of the moduli space of smooth curves of genus $p$. We say $C$ admits a singular model lying on a K3 surface of genus $g$ if there exists a polarized K3 surface $(X, L)$ of genus $g$ and an integral curve $D \subseteq |kL|$ for some $k$ such that $C$ is isomorphic to the normalization of $D$. Broadly speaking, we wish to consider the following question:

Questions. What is the dimension of the locus of curves $[C] \in \mathcal{M}_p$ admitting a singular model lying on a K3 surface of genus $g$? Furthermore, what conditions must a curve $[C] \in \mathcal{M}_p$ satisfy in order to admit a singular model $D$ lying on a K3 surface?

In practice, one tends to put a condition on the singularities of the integral curve $D$ in order to approach the above question, as otherwise the deformation theory of the pair $(D, X)$ is hard to control. In [20], the above questions are considered under the hypothesis that $D$ is nodal. We will instead work with the much weaker hypothesis that the normalization morphism $f : C \to D$ is unramified. Note that if $D$ has ordinary singularities, then $f$ is unramified, whereas $f$ has ramification if $D$ has a cusp.
If $D$ is a singular, integral curve on a K3 surface $X$, let $\mu : C := \tilde{D} \to X$ denote the composition of the normalization $\tilde{D} \to D$ with the inclusion $D \hookrightarrow X$, and let $p$ be the arithmetic genus of $C := \tilde{D}$. This gives a one-to-one correspondence between pairs $(D, X)$, where $D$ is integral of geometric genus $p$, and morphisms $f : C \to X$ where $C$ is a smooth curve of genus $p$ and $f$ is birational onto its image. As is by now well-known, the deformation theory of the morphism $f$ is in many ways considerably easier to work with than the deformation theory of the pair $(D, X)$. We will take this viewpoint throughout this paper and formulate the above questions in terms of stable maps, see [21] or [1] for an excellent introduction to this topic.

1.1 The number of moduli of singular curves on K3 surfaces

Let $B_g$ denote the stack of pairs $((X, L)) \in B_g$ parametrizing all stable maps $f : C \to X$ with $f^*L \equiv kL$, where $C$ is a connected, nodal curve of arithmetic genus $p(g, k) − n$, with $p(g, k) := k^2(g − 1) + 1$. Denote by $\mathcal{T}_{g,k} \subseteq \mathcal{W}_{g,k}$ the open subset consisting of unramified stable maps $f : C \to X$ with $C$ integral and smooth such that $f$ is birational onto its image.

By the deformation theory of stable maps, $\mathcal{T}_{g,k}$ is a smooth stack of dimension $p(g, k) − n + 19$, and every component of $\mathcal{T}_{g,k}$ dominates $B_g$, cf. [6], [31]. Furthermore, $\mathcal{T}_{g,k}$ is nonempty, by [3].

For $p(g, k) − n \geq 2$, there is a forgetful morphism

$$\eta : \mathcal{T}_{g,k} \to \mathcal{M}_{p(g, k)−n}$$

defined by taking $[(f : C \to X, L)] \in \mathcal{T}_{g,k}$ to $[C]$, where $\mathcal{M}_{p(g, k)−n}$ denotes the stack of smooth curves of genus $p(g, k) − n$. A dimension count suggests that this might be dominant for $2 \leq p(g, k) − n \leq 11$ and generically finite for $p(g, k) − n \geq 11$. To ease the notation in the primitive case $k = 1$ write $\mathcal{T}_{g} := \mathcal{T}_{g,1}$.

The case $n = 0$ has been studied in depth. It was shown in [39] that if $n = 0$, $k = 1$ then the morphism

$$\eta : \mathcal{T}_{g} \to \mathcal{M}_{g}$$

is generically finite for $g \geq 13$ or $g = 11$. In the non-primitive case $k \geq 2$, a very different approach using the deformation theory of cones shows that $\eta$ is generically finite for $g \geq 7$ and $n = 0$, [12]. Our first result is an extension of the results on generic finiteness to the singular case $n > 0$. In the case $k = 1$, we show:

**Theorem 1.1.** Assume $g \geq 11$, $n \geq 0$, and let $0 \leq r(g) \leq 5$ be the unique integer such that

$$g − 11 = \left\lfloor \frac{g − 11}{6} \right\rfloor 6 + r(g).$$
Then there is a component $I \subseteq T^n$ such that
\[
\eta_I : I \to M_{g-n}
\]
is generically finite for $g - n \geq 15$. Furthermore, if $r(g) \neq 5$, the lower bound can be improved to $g - n \geq 13$, and if $r(g) = 0$ it can be improved to $g - n \geq 12$.

In the case $k \geq 2$ we show:

**Theorem 1.2.** Assume $k \geq 2$, $g \geq 8$. Then there is a component $I \subseteq T^n_{g,k}$ such that
\[
\eta_I : I \to M_{p(g,k)-n}
\]
is generically finite for $p(g,k) - n \geq 18$. Furthermore, in most cases the lower bound can be improved, see Section 3.

Setting $n = 0$, we recover the (optimal) statement in the smooth, primitive, case and all cases other than $g = 7$ in the nonprimitive case. In particular, this gives a new proof of the generic finiteness theorem for $n = 0$, $k \geq 2$, $g \geq 8$ which resembles the original approach of [39].

Denote by
\[
V^n_{g,k} \subseteq T^n_{g,k}
\]
the open substack parametrizing morphisms \([(f : C \to X,L)]\) with $f(C)$ nodal and set $V^n_g = V^n_{g,1}$. The following conjecture is found in [15]:

**Conjecture 1.3.** The moduli space $V^n_{g,k}$ is irreducible.

Let $I \subseteq T^n_{g,k}$ be an irreducible component and denote the fibre of $I \to B_g$ over $[(X,L)]$ by $I(X,L)$. If $(X,L)$ is general, then each component of $I(X,L)$ has dimension $p(g,k) - n$ and we have an injective morphism $I(X,L) \to [kL]$ sending $[f:C \to X]$ to the integral curve $[f(C)]$. Applying a theorem of Zariski as in [8, Lemma 3.1], it follows that if $p(g,k) - n > 0$, then $I(X,L) \cap V^n_{g,k}$ is nonempty. Thus $V^n_{g,k} \subseteq T^n_{g,k}$ is dense. In particular, we have:

**Corollary 1.4.** The restriction
\[
\eta_{|V^n_{g,k}} : V^n_{g,k} \to M_{p(g,k)-n}
\]
is generically finite on one component, for the same bounds on $p(g,k) - n$ as in Theorem 1.1 and 1.2.

We should also mention here that C. Ciliberto, F. Flamini, C. Galati and A. Knutsen have a very different approach using degenerations of sheaves to unions of rational scrolls, which when completed is likely to give another proof of Corollary 1.4 (for certain bounds on $p(g,k) - n$). Moreover, their approach may potentially prove the (local) surjectivity of $\eta$ on one component of $V^n_{g,k}$ for some cases within the range $p(g,k) - n \leq 11$, which is beyond the reach of our method.
1.2 An obstruction for a marked curve to admit a nodal model on a K3 surface

It is a natural question to study the image of $\eta$. In the case of smooth curves $n = 0$, there is a well-known conjectural characterization of the image $\eta$, due to Wahl [52]. He makes the following remarkable conjecture, which would give a complete characterization of those smooth curves that lie on a K3 surface:

**Conjecture 1.5 (Wahl).** Assume $C$ is a smooth curve of genus $g \geq 8$ which is Brill–Noether general. Then there exists a K3 surface $X \subseteq \mathbb{P}^g$ such that $C$ is a hyperplane section of $X$ if and only if the Wahl map $W_C$ is nonsurjective.

Here the Wahl map refers to the map $\Lambda^2 H^0(C, K_C) \to H^0(C, K_C^3)$ given by $s \wedge t \mapsto tds - sdt$. One side of this conjecture is well-known; indeed if $C \subseteq X$ is a smooth curve in a K3 surface then $W_C$ is nonsurjective, [50]. Furthermore, if $C$ is general and $Pic(X) \cong \mathbb{Z}C$, then $C$ is Brill–Noether general, [37]. In [20, Question 5.5], it was asked if there exists such a Wahl-type obstruction for a smooth curve to have a nodal model lying on a K3 surface. Let $\bar{M}_{p(g,k)-n,2n} := \bar{M}_{p(g,k)-n,2n}/S_{2n}$ denote the stack of curves with an unordered marking (or divisor). One may slightly alter the above question and ask if there exists an obstruction for a marked curve to have a nodal model lying on a K3 surface in such a way that the marking is the divisor over the nodes (when we forget about the ordering). For any positive integers $h, l$ and $[(C, T)] \in \bar{M}_{h,2l}$, one may consider the Gaussian

$$W_{C,T} : \Lambda^2 H^0(C, K_C(-T)) \to H^0(C, K_C^3(-2T))$$

which we will call the **marked Wahl map**, since it depends on both the curve and the marking. In Section 4 we show:

**Theorem 1.6.** Fix any integer $l \in \mathbb{Z}$. Then there exist infinitely many integers $h(l)$, such that the general marked curve $[(C, T)] \in \bar{M}_{h(l),2l}$ has surjective marked Wahl map.

On the other hand we show:

**Theorem 1.7.** Assume $g - n \geq 13$ for $k = 1$ or $g \geq 8$ for $k > 1$, and let $n \leq \frac{p(g,k) - 1}{5}$. Then there is an irreducible component $I^0 \subseteq \mathcal{V}^n_{g,k}$ such that for any $[(f : C \to X, L)] \in I^0$ the marked Wahl map $W_{C,T}$ is nonsurjective, where $T \subseteq C$ is the divisor over the nodes of $f(C)$.

1.3 Brill–Noether theory for nodal curves on K3 surfaces

In the last section we study the Brill–Noether theory of nodal curves on K3 surfaces. There are two related questions: for $[(f : C \to X, L)] \in \mathcal{V}^n_g$ general, one may firstly ask if the smooth curve $C$ is Brill–Noether general and secondly if the nodal curve $f(C)$ is Brill–Noether general. For the first question we show in Section 5.1:

**Proposition 1.8.** Assume $g - n \geq 8$. Then there exists a component $\mathcal{J} \subseteq \mathcal{V}^n_g$ such that for $[(f : C \to X, L)] \in \mathcal{J}$ general, $C$ is Brill–Noether–Petri general.

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1 An obstruction was also proposed in [20], the proof however seems flawed, see Remark 4.12.
The above result should not be expected to hold for all \([(f : C \to X, L)] \in \mathcal{J}\) (or even for all \([(f : C \to X, L)] \in \mathcal{J}\) with the general polarized K3 surface \((X, L)\) kept fixed), see [9, Thm. 0.1].

For the second question we again have a positive answer. For an integral nodal curve \(D\), we denote by \(\bar{J}_d(D)\) the compactified Jacobian of degree \(d\), rank one, torsion-free sheaves on \(D\). In Section 5.2 we show:

**Theorem 1.9.** Let \(X\) be a K3 surface with \(\text{Pic}(X) = \mathbb{Z} L\) and \((L \cdot L) = 2g - 2\). Suppose \(D \in [L]\) is a rational, nodal curve. Then

\[
\overline{W}_d(D) := \{ A \in \bar{J}_d(D) \text{ with } h^0(A) \geq r + 1 \}
\]

is either empty or is equidimensional of the expected dimension \(\rho(g, r, d)\).

As one may smoothen the nodes of a rational nodal curve \(D\) on a K3 surface to produce a curve with an arbitrary number of nodes, the above result immediately gives the following corollary:

**Corollary 1.10.** For any \(n \geq 0\), there is a component \(\mathcal{J} \subseteq \mathcal{V}_g^n\) such that if \([(f : C \to X, L)] \in \mathcal{J}\) is general and \(D = f(C)\) then

\[
\overline{W}_d(D) := \{ A \in \bar{J}_d(D) \text{ with } h^0(A) \geq r + 1 \}
\]

is either empty or is equidimensional of the expected dimension \(\rho(g, r, d)\).

In particular, if \(\rho(g, r, d) < 0\), \(\overline{W}_d(D) = \emptyset\) for \(D\) as in the above corollary; indeed this is well-known and follows from the arguments of [23, §3.2], [37, Cor. 1.4]. On the other hand, if \(\rho(g, r, d) \geq 0\), then \(\overline{W}_d(D) \neq \emptyset\) by deforming \(D\) to a smooth curve on \(X\) and semicontinuity.

We may summarize the above results as stating that there are no Brill–Noether obstructions for a curve to have a nodal model lying on a K3 surface. It would be interesting to find non-abelian, rank two, Brill–Noether obstructions for a curve to have a nodal model lying on a K3 surface, in analogy with the smooth case, [43], [2].

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## 2 Mukai’s theory for curves on K3 surfaces

In this section we will recall a construction of Mukai to construct loci \(Z \subseteq \mathcal{V}_g^0\) such that for \(x \in Z\) the fibre of \(\eta : \mathcal{V}_g^0 \to \mathcal{M}_g\) over \(\eta(x)\) is zero-dimensional at \(x\). This will be our basic tool for studying the generic finiteness of the morphism \(\eta : \mathcal{T}_{g,k} \to \mathcal{M}_{p(g,k)-n}\).
Let \( g \geq 5 \) be an integer, let \( 1 \leq d_1, d_2, \ldots, d_8 < \left\lfloor \frac{2g+1}{2} \right\rfloor \) be integers, and consider first the rank ten lattice \( \Omega_g \) with ordered basis \( \{ L, E, \Gamma_1, \ldots, \Gamma_8 \} \) and with intersection form given by:

- \( (L \cdot L) = 2g - 2 \)
- \( (L \cdot E) = \left\lfloor \frac{2g+1}{2} \right\rfloor \)
- \( (E \cdot E) = 0 \)
- \( (\Gamma_i)^2 = -2 \) for \( 1 \leq i \leq 8 \)
- \( (E \cdot \Gamma_i) = 0 \) for \( 1 \leq i \leq 8 \)
- \( (L \cdot \Gamma_i) = d_i \) for \( 1 \leq i \leq 8 \)
- \( (\Gamma_i \cdot \Gamma_j) = 0 \) for \( i \neq j, 1 \leq i, j \leq 8 \)

It is easily seen that the above lattice has signature \((1,9)\) and is even.

**Lemma 2.1.** Let \( g \geq 6 \) be an integer and choose \( 1 \leq d_1, \ldots, d_8 < \left\lfloor \frac{2g+1}{2} \right\rfloor \). There exists a K3 surface \( Y_{\Omega_g} \) with \( \text{Pic}(Y_{\Omega_g}) \simeq \Omega_g \). Furthermore, for any such K3 we may choose the ordered basis \( \{ L, E, \Gamma_1, \ldots, \Gamma_8 \} \) of \( \Omega_g \) in such a way that \( L - E \) is big and nef and with \( \Gamma_1 \) and \( E \) representable by smooth, integral curves for \( 1 \leq i \leq 8 \). Further there is a smooth rational curve \( \bar{\Gamma}_1 \in [E - \Gamma_1] \).

**Proof.** By the global Torelli theorem and from a result of Nikulin, the fact that this lattice has signature \((1,9)\) and is even implies that there exists a K3 surface \( Y_{\Omega_g} \) with \( \text{Pic}(Y_{\Omega_g}) \simeq \Omega_g \), [30, Cor. 14.3.1]. By performing Picard–Lefschetz reflections and a sign change, we may assume that \( L - E \) is big and nef, since \( (L - E \cdot L - E) > 0 \), [30, Cor. 8.2.10]. Next, \( (E \cdot E)^2 = 0 \) and \( (L - E \cdot E) = \left\lfloor \frac{2g+1}{2} \right\rfloor > 0 \) which implies that \( E \) is effective. We firstly claim that \( |E| \) is base-point free. It is enough to show that \( E \) is nef, [30, Prop. 2.3.10].

So it suffices to show there is no effective divisor \( R \) with \( (R)^2 = -2 \) and \( (E \cdot R) < 0 \). Suppose for a contradiction that such an \( R \) exists. Write \( R = xL + yE + \sum_{i=1}^{8} z_i \Gamma_i \) for integers \( x, y, z_i \). As \( (E \cdot R) = x \left\lfloor \frac{2g+1}{2} \right\rfloor < 0 \) we have \( x < 0 \). Then \( (R - xL)^2 = -2 \sum_{i=1}^{8} z_i^2 \leq 0 \). However,

\[
(R - xL)^2 = -2 + x^2(2g - 2) - 2x(L \cdot R)
= -2 + x^2(2g - 2 - 2 \left\lfloor \frac{g+1}{2} \right\rfloor) - 2x(L - E \cdot R)
> 0
\]

for \( x < 0 \) and \( g \geq 6 \). Thus \( |E| \) is an elliptic pencil.

Next \( \Gamma_1 \) is effective, since \( (\Gamma_1 \cdot L - E) > 0 \). We claim \( \Gamma_1 \) is integral. Otherwise, there would be an integral component \( R \) of \( \Gamma_1 \) with \( (R \cdot \Gamma_1) < 0 \), since \( (\Gamma_1)^2 = -2 \). Further, \( (R)^2 = -2 \), since \( R \) is not nef. Write \( R = xL + yE + \sum_{i=1}^{8} z_i \Gamma_i \). We have \( (R \cdot E) = x \left\lfloor \frac{2g+1}{2} \right\rfloor \geq 0 \) so \( x \geq 0 \). Assume \( x \neq 0 \). Then we have \( (R \cdot R + E) > 0 \) and \( (R + E)^2 > 0 \) so \( R + E \) is big and nef, which contradicts that \( (R + E \cdot \Gamma_1) = (R \cdot \Gamma_1) < 0 \). So \( x = 0 \). But then \( (R)^2 = -2 \) gives \( \sum_{i=1}^{8} z_i^2 = 1 \), and \( (R \cdot \Gamma_1) = -2z_1 < 0 \) so \( z_1 = 1 \) and \( z_i = 0, i > 1 \). Lastly, \( (R \cdot L) = d_1 + y \left\lfloor \frac{2g+1}{2} \right\rfloor \geq 0 \) so we must have \( y \geq 0 \) (as \( d_1 < \left\lfloor \frac{2g+1}{2} \right\rfloor \)). Since \( R \) is integral, we must then have \( y = 0 \) and \( R = \Gamma_1 \) (as
every smooth rational curve is extremal, \[30\] Rem. 8.3.4(iii)). Thus \(\Gamma_1\) is integral. Likewise, \(\Gamma_2, \ldots, \Gamma_8\) are integral.

Next, \(\Gamma_1\) is effective, since \((\tilde{\Gamma}_1)^2 = -2\) and \((\tilde{\Gamma}_1 \cdot L - E) > 0\). Let \(R\) be an integral component of \(\Gamma_1\) such that \((R \cdot \tilde{\Gamma}_1) < 0, (R)^2 = -2\). Writing \(R = xL + yE + \sum_{i=1}^{8} z_i \Gamma_i\), we see as above \(x = 0\) and we must have \(z_1 = -1\) and \(z_i = 0, i > 1\). Since \((R \cdot L) = -d_1 + y \left\lfloor \frac{g+1}{2} \right\rfloor \geq 0\) we must have \(y \geq 1\) and then \(R = \tilde{\Gamma}_1 + (y-1)E\). Since \(R\) is integral, this forces \(R = \tilde{\Gamma}_1\).

\[\Box\]

**Lemma 2.2.** Let \(Y_{\Omega_g}\) and \(\{L, E, \Gamma_1, \ldots, \Gamma_8\}\) be as in the previous lemma. If we assume \(g > 7\), then \(L - E\) is very ample (and hence \(L\) is also very ample).

**Proof.** Suppose the big and nef line bundle \(L - E\) is not very ample. Then there exists either a smooth rational curve \(R \subseteq Y_{\Omega_g}\) with \((L - E \cdot R) = 0\) or a smooth elliptic curve \(F \subseteq Y_{\Omega_g}\) with \(0 < (L - E \cdot F) \leq 2\) (or both exist), by \[33\] Thm. 1.1] (set \(k = 1\) in Knutsen’s theorem and note that \(L - E\) is primitive).

Assume firstly that \(R\) as above exists; we may write \(R = x_1L + y_1E + \sum_{i=1}^{8} z_{1,i} \Gamma_i\) for integers \(x_1, y_1, z_{1,i}\). We have

\[-2 \sum_{i=1}^{8} z_{1,i}^2 = (R - x_1L)^2\]

\[= -2 + x_1^2(2g - 2) - 2x_1(L \cdot R)\]

\[= -2 + x_1^2(2g - 2 - 2 \left\lfloor \frac{g+1}{2} \right\rfloor) - 2x_1(L - E \cdot R)\]

and so \(x_1 = 0\) and there exists some \(j\) such that \(z_{1,j} = \pm 1, z_{1,i} = 0\) for \(i \neq j\). Then \(0 = (L - E \cdot R) = \pm d_j + y_1 \left\lceil \frac{g+1}{2} \right\rceil\) which is impossible for \(1 \leq d_j < \left\lceil \frac{g+1}{2} \right\rceil\).

So now suppose there is some smooth elliptic curve \(F \subseteq Y_{\Omega_g}\) with \(0 < (L - E \cdot F) \leq 2\). We may write \(F = x_2L + y_2E + \sum_{i=1}^{8} z_{2,i} \Gamma_i\) for integers \(x_2, y_2, z_{2,i}\). We have \((F \cdot E) = x_2(L \cdot E)\) and hence \(x_2 > 0\) (as \(F \notin \{E\}\)). We calculate

\[-2 \sum_{i=1}^{8} z_{2,i}^2 = (F - x_2L)^2\]

\[= x_2^2(2g - 2) - 2x_2(L \cdot F)\]

\[= x_2(x_2((2g - 2 - 2 \left\lfloor \frac{g+1}{2} \right\rfloor) - 2(L - E \cdot F))\]

which is impossible for \(g > 7\), \(0 \leq (L - E \cdot F) \leq 2\). Thus \(L - E\) is very ample. Using Knutsen’s criterion again, and the fact that \(E\) is nef, we see that \(L\) is likewise very ample. \[\Box\]

For the rest of the section we will assume \(g\) is odd. The following technical lemma will be needed later in this section.

**Lemma 2.3.** Assume \(g \geq 11\) is odd and let \(Y_{\Omega_g}\) and \(\{L, E, \Gamma_1, \ldots, \Gamma_8\}\) be as in Lemma 2.2. Then \(L - 2E\) is not effective. Further \((L - E)^2 \geq 8\) and there exists no effective divisor \(F\) with \((F)^2 = 0\) and \((F \cdot L - E) = 3\).
Proof. Suppose \( L - 2E \) is an effective divisor and let \( D_1, \ldots D_k \) be its irreducible components. Write \( D_i = x_i L + y_i E + \sum_{j=1}^{8} z_{ij} \Gamma_j \) for integers \( x_i, y_i, z_{ij} \). Then \( 0 \leq (D_i \cdot E) = x_i(L \cdot E) \) so that \( x_i \geq 0 \) and \( \sum_i x_i = 1 \). Thus we may assume \( x_1 = 1 \) and \( x_i = 0 \) for all \( i \geq 2 \). Now let \( \tilde{D} \) be any irreducible curve of the form \( aE + \sum_{j=1}^{8} b_j \Gamma_j \) for integers \( a, b_j \) and suppose \( \tilde{D} \neq \Gamma_j, \forall 1 \leq j \leq 8 \). Then \( 0 \leq (\tilde{D} \cdot \Gamma_j) = -2b_j \) so \( b_j \leq 0 \) for all \( j \). Since \( (\tilde{D})^2 = -2 \sum_{j=1}^{8} b_j^2 \geq -2 \), there is at most one \( b_j \) such that \( b_j \neq 0 \). Suppose firstly that all \( b_j = 0 \). Then \( \tilde{D} \sim E \), since \( \tilde{D} \) is integral, and all effective divisors in \( |aE| \) are a sum of \( a \) divisors in \( |E| \). [46] Prop. 2.6(ii)]. Next suppose \( b_j = -1 \). Then \( \tilde{D} = aE - \Gamma_j \) and \( a \geq 1 \) since \( a \) is effective. Thus \( \tilde{D} = \Gamma_j + (a - 1)E \), and is smooth and rational, which implies \( a = 1 \) (as smooth rational curves are extremal).

Thus if \( i \geq 2 \), \( D_i \) is either \( E, \Gamma_j \) or \( \tilde{\Gamma}_j \), for some \( j \). Since \( \sum_i D_i = L - 2E \), we see \( D_1 = L - (2 + m')E - \sum_{j=1}^{8} n_{1,j} \Gamma_j - \sum_{j=1}^{8} n_{2,j} \tilde{\Gamma}_j \) for nonnegative integers \( m' \) and \( n_{1,j}, n_{2,j}, 1 \leq j \leq 8 \).

Since \( E = \Gamma_j + \tilde{\Gamma}_j \), we may rewrite \( D_1 \) in the form \( D_1 = L - (2 + m)E - \sum_{j=1}^{8} (n_{1,j} \Gamma_j + n_{2,j} \tilde{\Gamma}_j) \), where \( m, n_{1,j}, n_{2,j} \) are nonnegative integers and if \( n_{1,j} = 0 \) for some \( j \) then \( n_{2,j} = 0 \), and likewise if \( n_{2,j} = 0 \) then \( n_{1,j} = 0 \). But then one computes

\[
(D_1)^2 = (L - 2E - (mE + \sum_{j=1}^{8} n_{1,j} \Gamma_j + n_{2,j} \tilde{\Gamma}_j))^2
\]

\[
= -4 - 2 \sum_{j=1}^{8} (n_{1,j}^2 + n_{2,j}^2) - 2(L \cdot mE + \sum_{j=1}^{8} n_{1,j} \Gamma_j + n_{2,j} \tilde{\Gamma}_j)
\]

\[
\leq -4
\]

which is a contradiction (since \((D)^2 \geq -2\) for any integral curve \(D\)).

We have \((L - E)^2 = g - 3 \geq 8 \) for \( g \geq 11 \). For \( g \geq 11 \) and \( a' > 0 \) one has \((g - 3)a' - 6 > 0 \). From the proof of Lemma 2.2 this implies there is no effective \( F \) with \((F)^2 = 0 \) and \((F \cdot L - E) = 3 \).\[\Box\]

For any smooth curve \( C \) and \( M \in \text{Pic}(C) \) with \( \deg(M) = d \) and \( h^0(M) = r + 1 \), let \( \nu(M) := d - 2r \). The Clifford index of \( C \) is defined by

\[
\nu(C) := \min\{\nu(M) \mid M \in \text{Pic}(C) \text{ with } \deg(M) \leq g - 1, h^0(M) \geq 2\}.
\]

Clifford’s Theorem states that \( \nu(C) \geq 0 \) and \( \nu(C) = 0 \) if and only if \( C \) is hyperelliptic.

**Lemma 2.4.** Let \( D \in \Omega_y \) be an effective divisor with \((D)^2 \geq 0 \), and assume \( L - D \) is effective and \((L - D)^2 > 0 \). Then \( D = cE \) for some integer \( c \geq 0 \).

**Proof.** Write \( D = xL + yE + \sum_{i=1}^{8} z_i \Gamma_i \) for integers \( x, y, z_i \). One has \( 0 \geq (D-L \cdot E) = (x-1)(L \cdot E) \) so that \( x \leq 1 \). On the other hand \( 0 \leq (D \cdot E) = x(L \cdot E) \) so that \( x \geq 0 \). Thus \( x = 0 \) or \( x = 1 \).

Suppose firstly that \( x = 1 \). Then \( 0 < (L - D)^2 = -2 \sum_{i=1}^{8} z_i^2 \) which is a contradiction. Hence \( x = 0 \). Then \( 0 \leq (D)^2 = -2(\sum_{i=1}^{8} z_i^2) \) and so \( z_i = 0 \) for all \( i \), as required.\[\Box\]

**Lemma 2.5.** Let \( g \geq 11 \) be odd, let \( \{Y_{\Omega_y}, \ldots, \Gamma_8\} \) as in Lemma 2.3 and let \( C \in |L| \) be a smooth curve. Then \( \nu(C) = \frac{g+1}{2} - 2 \).
Lemma 2.3. Let $M$. Proof. Let $C \subseteq A$ represented by a smooth curve, we must have $D = Y_{\Omega_g}$ with $0 \leq (D)^2 < \nu(C) + 2$, $2(D)^2 < (E \cdot L)$ and $\nu(C) = (D \cdot L) - (D)^2 - 2$ (the sharp inequalities here are due to the fact that $L$ is primitive). We have

$$(D - L \cdot D) = (D)^2 - (D \cdot L) = -2 - \nu(C) \leq -2$$

and

$$(D - L)^2 = -6 - 2\nu(C) - (D)^2 + 2g > -8 - 3\nu(C) + 2g \geq 0,$$

as $g \geq 7$. Hence $L - D$ is effective and $D = cE$ for some $c \geq 0$ by Lemma [2.4]. As $D$ is represented by a smooth curve, we must have $D = E$. But then

$$\nu(C) = (E \cdot L) - (E)^2 - 2 = \frac{g + 1}{2} - 2,$$

giving a contradiction. \hfill \Box

Lemma 2.6. Assume $g \geq 11$ is odd and let $Y_{\Omega_g}$ be a K3 surface with $Pic(Y_{\Omega_g}) \simeq \Omega_g$ as in Lemma [2.3]. Let $M \in \Omega_g$ be an effective line bundle on $Y_{\Omega_g}$ satisfying

- $0 \leq (M)^2 < \frac{g + 1}{2}$
- $2(M)^2 < (M \cdot L)$
- $\frac{g + 1}{2} = (M \cdot L) - (M)^2$.

Then $M = E$.

Proof. The above inequalities give $(M - L \cdot M) < 0$ and

$$(M - L)^2 = (M)^2 + 2g - 2((M)^2 + \frac{g + 1}{2}) > 0$$

as $(M)^2 < \frac{g + 1}{2} \leq g - 3$ for $g \geq 7$. Thus $M = cE$ for $c > 0$ by Lemma [2.4]. The equation $\frac{g + 1}{2} = (M \cdot L) - (M)^2$ gives $c = 1$. \hfill \Box

Lemma 2.7. Let $g \geq 11$ be odd and let $Y_{\Omega_g}$ and $\{L, E, \Gamma_1, \ldots, \Gamma_8\}$ be as in Lemma [2.3] and let $C \in |L|$ be a smooth curve. Suppose $A \in Pic(C)$ has $h^0(A) = 2$ and $deg(A) = \frac{g + 1}{2}$. Then $A \simeq E_{|C|}$.

Proof. Let $A \in Pic(C)$ with $h^0(A) = 2$ and $deg(A) = \frac{g + 1}{2}$. Then by [17] Thm. 4.2], there is some effective divisor $D \subseteq Y_{\Omega_g}$ such that that $(D \cdot L) \leq g - 1$, $L - D$ is effective, dim $|D| \geq 1$, $D|_C$ achieves the Clifford index on $C$ and such that there exists a reduced divisor $Z_0 \in |A|$ of length $\frac{g + 1}{2}$ with $Z_0 \subseteq D \cap C$. Further from [17] Lem. 4.6] and [17] (1),(2) p. 60], we see $\frac{g + 1}{2} = (D \cdot L) - (D)^2$. It then follows that all the conditions of Lemma [2.6] are satisfied, so that $D \in |E|$. As $(E \cdot C) = \frac{g + 1}{2}$ and $Z_0$ is reduced of length $\frac{g + 1}{2}$, we have $Z_0 = D \cap C$. Thus $Z_0 \in |E_{|C|}|$ which forces $A \simeq E_{|C|}$. \hfill \Box
The following lemma may be extracted from work of Mukai, cf. [42, §3], [41, Lem. 2] (although it never appears in this precise form). Despite the simple proof, this lemma is actually rather fundamental, since it gives an example of a K3 surface $S$ and a divisor $D \subseteq S$ such that the K3 surface $S$ can be reconstructed merely from the curve $D$ together with a special divisor $A \in \text{Pic}(D)$.

Lemma 2.8 (Mukai). Let $g \geq 11$ be odd and let $S$ be a K3 surface with $L, M \in \text{Pic}(S)$ such that $L^2 = 2g - 2$, $M^2 = 0$ and $(L \cdot M) = \frac{g+1}{2}$. Let $D \in [L]$ be smooth and set $A := M_D$. Further, let $A^\dagger = (L - M)_{\vert_D}$ be the adjoint of $A$ and set $k := h^0(A^\dagger) - 1$. Assume $L - M$ is very ample with $(L - M)^2 \geq 8$ and that there is no integral curve $F \subseteq S$ with $(F)^2 = 0$ and $(F \cdot L - M) = 3$. Assume further that $M$ is represented by an integral curve and that $L - 2M$ is not effective. Then $A^\dagger$ is very ample, and $S$ is the quadric hull of the embedding $\phi_{A^\dagger} : D \to \mathbb{P}^k$ induced by $A^\dagger$.

Proof. As there exists an integral curve in $[M]$, we know $h^1(M) = 0$. From the exact sequence

$$0 \to M^* \to L \otimes M^* \to A^\dagger \to 0$$

we see that restriction induces an isomorphism $H^0(S, L \otimes M^*) \cong H^0(D, A^\dagger)$, which implies that $A^\dagger$ is very ample (as $L \otimes M^*$ is) and that $\phi_{A^\dagger} : D \to \mathbb{P}^k$ is the composition of $D \to S$ and $\phi_L - M : S \to \mathbb{P}^k$. Now let $Q \subseteq \mathbb{P}^k$ be a quadric containing $D$. Then we claim that $Q$ contains $S$. Indeed, otherwise there is some effective divisor $T \subseteq S$ such that $D + T = Q \cap S$ so that $D + T \in [2L - 2M]$. But then $T \in [L - 2M]$, contradicting that $L - 2M$ is not effective by assumption. Thus the quadric hull of $D$ coincides with the quadric hull of $S \to \mathbb{P}^k$. But the quadric hull of $S$ is simply $S$ from [40, Thm. 7.2]. The assumption $g \geq 11$ is needed to ensure $(L - M)^2 \geq 8$ which is required in Saint-Donat’s theorem.

Putting all the pieces together, we have the following proposition:

Proposition 2.9. Let $g \geq 11$ be odd. Let $T$ be a smooth and irreducible scheme with base point $0 \in T$. Let $Z \to T$ be a flat family of K3 surfaces together with an embedding $\phi : Z \to T \times \mathbb{P}^g$. Assume $\phi_0 : Z_0 \to \mathbb{P}^g$ is the embedding $Y_{\Omega_0} \to \mathbb{P}^g$ induced by $[L]$, where $Y_{\Omega_0}$ is as in Lemma 2.3. Let $H \subseteq \mathbb{P}^g$ be a fixed hyperplane and $C := H \cap Y_{\Omega_0}$ a smooth curve. Assume that for all $t \in T$, we have $Z_t \cap H \simeq C$. Then for $t \in T$ general there is an isomorphism $\psi_t : Z_t \simeq Y_{\Omega_t}$ and an automorphism $f_t \in \text{Aut}(\mathbb{P}^g)$ such that $f_t \circ \psi_t = \phi_0 \circ \psi_t$. Furthermore, $f_t$ preserves $H$.

Proof. For $t \in T$ general we have a primitive embedding $j : \text{Pic}(Z_t) \to \text{Pic}(Y_{\Omega_t})$ with $j(\mathcal{O}_{Z_t}(1)) = L$. Each fibre $Z_t$ contains the curve $C$ which has Clifford index $\frac{2g+1}{2} - 2$ by Lemma 2.3. By [35, Lem. 8.3] there is a smooth and irreducible divisor $M' \in \text{Pic}(Z_t)$ satisfying $0 \leq (M')^2 < \frac{2g+1}{2}$, $2(M')^2 < (M' \cdot \mathcal{O}_{Z_t}(1))$ and $\frac{2g+1}{2} = (M' \cdot \mathcal{O}_{Z_t}(1)) - (M')^2$. As in the proof of Lemma 2.3 these conditions ensure that $\mathcal{O}_{Z_t}(1) - M'$ is effective. Then $M = j(M')$ satisfies the conditions of Lemma 2.6 so that $M = E$. Thus $(M')^2 = 0$ and hence $M'$ is a smooth elliptic curve. By lemmas 2.2 and 2.3 $L - E$ is very ample, $(L - E)^2 \geq 8$, $L - 2E$ is not effective and there exists no effective $F$ with $(F)^2 = 0$ and $(F \cdot L - E) = 3$; thus the same holds for $\mathcal{O}_{Z_t}(1) - M' \in \text{Pic}(Z_t)$ for $t$ close to $0$. Thus by Lemma 2.8 and since $(\mathcal{O}_{Z_t}(1) - M')_{\vert_C} \simeq K_C(E_{\vert_C}^*)$ by Lemma 2.7 we have $Z_t \simeq Y_{\Omega_t}$. Furthermore, the embedding $C \to Z_t$ is induced by the natural embedding of $C$ into the quadric hull of the
embedding $\phi_{(L-E)|_C} : C \to \mathbb{P}^k$. Thus for the general $t$ there is some $f_t \in \text{Aut}(\mathbb{P}^g) = \text{Aut}([L])$ with $f_t(Z_t) = Y_{\Omega_g} \subseteq \mathbb{P}^g$ such that $f_t$ preserves $H$.

As a direct consequence we have the following corollary.

**Corollary 2.10.** Let $g \geq 11$ be odd and let $Y_{\Omega_g}$ and $\{L,E,\Gamma_1,\ldots,\Gamma_k\}$ be as in Lemma 2.3 and let $C \in [L]$ be a smooth curve. Then the fibre of the morphism $\eta : T^0_g \to \mathcal{M}_g$ over $[C]$ is zero-dimensional at $[(i : C \hookrightarrow Y_{\Omega_g},L)]$.

**Remark 2.11.** In fact, $\eta : T^0_g \to \mathcal{M}_g$ is birational onto its image for $g \geq 11, g \neq 12$ [10]. Also see [43, §10], [2] for an approach in the odd genus case which more closely resembles the above.

## 3 Generic finiteness of the morphism $\eta$

In this section we will investigate the generic finiteness of the morphism of stacks

$$\eta : T^m_{g,k} \to \mathcal{M}_{p(g,k)-n}$$

defined by sending $[(f : B \to X,L)]$ to $[B]$. Then $\eta$ can be extended to a morphism of stacks

$$\mathcal{W}^n_{g,k} \to \overline{\mathcal{M}}_{p(g,k)-n},$$

by sending a pair $[(f : B \to X,L)]$ to $[\hat{B}]$, where $\hat{B}$ denotes the stabilization of the nodal curve $B$; this works in families from the proof of [34, Prop. 2.1](see also [21, §1.3]). By abuse of notation we will continue to denote this extension by $\eta$.

We start by recalling the basic deformation theory of stable maps. For the construction of $\mathcal{W}^m_{g,k}$ as a Deligne–Mumford stack and its elementary deformation theory, we recommend [1, §10]. The following is [32, Prop. 4.1]:

**Proposition 3.1.** Each component of $\mathcal{W}^m_{g,k}$ has dimension at least $p(g,k) - n + 19$.

The criterion below has been used several times in the literature, see [6], [31].

**Proposition 3.2.** Let $[(f : B \to X,L)] \in \mathcal{W}_{g,k}$ represent an unramified stable map such that $h^0(N_f) \leq p(B)$, where $p(B) = p(g,k) - n$ denotes the arithmetic genus of $B$. Then the projection $\pi : \mathcal{W}^n_{g,k} \to \mathcal{B}_g$ is dominant near $[(f : B \to X,L)]$.

**Proof.** The fibre $\pi^{-1}([(X,L)])$ may be identified with the space of stable maps into the fixed surface $X$, and thus each component $J$ near $[(f : B \to X,L)]$ has dimension at most $h^0(N_f) \leq p(B)$, [47, §3.4.2]. Thus $\dim \pi(J) = \dim \mathcal{B}_g$, so $\pi : \mathcal{W}^m_{g,k} \to \mathcal{B}_g$ is dominant near $[(f : B \to X,L)]$ (note that this also forces the equality $h^0(N_f) = p(B)$).

The following is a generalization of [31, Prop. 2.3].

---

There is a minor mistake in the proof of [31, Prop. 2.3]: the claimed isomorphism $\Omega_{\mathcal{D}_{\mathbb{P}^n}} \cong \mathcal{O}(-1)$ should be replaced with $\text{det}(\Omega_{\mathcal{D}_{\mathbb{P}^n}}) \cong \mathcal{O}(-1)$, as the sheaf $\Omega_{\mathcal{D}_{\mathbb{P}^n}}$ is not torsion free.

---
Lemma 3.3. Let $f : B \to X$ be an unramified morphism from a connected nodal curve to a K3 surface, and let $N_f$ denote the normal bundle of $f$. Assume that the irreducible components $Z_1, \ldots, Z_s$ of $B$ are smooth. Assume further that we may label the components such that $\cup_{i=1}^s Z_i$ is connected for all $j \leq s$. Then $h^0(N_f) \leq p(B)$, where $p(B)$ denotes the arithmetic genus of $B$.

Proof. We will prove this by induction on $s$. If $B$ is irreducible, then by assumption $B$ is smooth, so we have a short exact sequence

$$0 \to T_B \to f^*T_X \to N_f \to 0$$

and taking determinants gives $N_f \simeq \omega_B$. Thus $h^0(N_f) = h^0(\omega_B) = p(B)$. Now let $T := B \setminus Z_s = \cup_{i=1}^{s-1} Z_i$; this is connected by assumption. Let $\{p_1, \ldots, p_r\} = Z_s \cap T$. We have a short exact sequence

$$0 \to N_{f|T}(-p_1 - \ldots - p_r) \to N_f \to N_{f|Z_s} \to 0.$$

If $A \subseteq B$ is a connected union of components, and $f_A := f_{|A}, Y := A \cap (B \setminus A)$, then $N_{f_A}(Y) = N_{f|A}$, from [24, §2]. Thus

$$h^0(N_f) \leq h^0(N_{f_T}) + h^0(\omega_{Z_s}(p_1 + \ldots + p_r)).$$

By induction, $h^0(N_{f_T}) \leq p(T)$, and further $h^1(\omega_{Z_s}(p_1 + \ldots + p_r)) = h^0(\mathcal{O}_{Z_s}(-p_1 - \ldots - p_r)) = 0$, so Riemann–Roch gives $h^0(\omega_{Z_s}(p_1 + \ldots + p_r)) = p(Z_s) + r - 1$. Thus the claim follows from $p(B) = p(T) + p(Z_s) + r - 1$. $\Box$

Remark 3.4. It follows from the proof that the above result may be generalized as follows. Suppose $f : B \to X$ be an unramified morphism from a connected nodal curve to a K3 surface, and $B = \cup_{i=1}^r Z_i$ where $Z_1$ is connected, but not necessarily irreducible or smooth, and with $Z_2, \ldots, Z_s$ smooth (and with $s > 1$). Assume $\cup_{i=1}^s Z_i$ is connected for all $j \leq s$, and $h^0(N_{f|j}) \leq p(Z_1)$, where $f_j := f_{|Z_i}$. Then $h^0(N_f) \leq p(B)$.

Lemma 3.5. Let $g : B \to X$ be an unramified morphism from an integral, nodal curve $B$ to a K3 surface, with $[(g : B \to X, L)] \in \mathcal{W}_{g,k}^n$ and assume that $g$ is birational onto its image. Then $[(g : B \to X, L)]$ lies in the closure of $\mathcal{T}_{g,k}^n$.

Proof. Suppose there was a component of $J$ of $\mathcal{W}_{g,k}^n$ containing $[(g : B \to X, L)]$ such that if $[(g' : B' \to X', L')]$ is general, then $B'$ is nodal with at least $m > 0$ nodes. Replace $J$ with the dense open subset parametrizing unramified maps, which are birational onto the image and such that the base $B'$ is integral with exactly $m$ nodes. Composing $g'$ with the normalization $\tilde{B} \to B'$ gives an unramified stable map $\tilde{h} : \tilde{B} \to X'$; thus we have a map of sets $G : J(Spec(\mathbb{C})) \to \mathcal{T}_{g,k}^n(Spec(\mathbb{C}))$. The fact that $g$ is unramified implies that for a general $y \in \text{Im}(G)$, $G^{-1}(y)$ is a finite set.

In order to make a dimension estimate, we need to argue that $G$ is induced by a morphism of stacks. After replacing $J$ with a small analytic open set, we may assume it comes with a universal family. Let $J' \to J$ be the normalization of $J$. Pulling back the universal family on $J$ gives a family of stable maps over $J'$. In particular, passing to an open set of $J'$ if necessary, we have a flat family $B \to J'$ of nodal curves with exactly $m$ nodes specializing to
Let $G' : J'(\text{Spec}(\mathbb{C})) \to \mathcal{T}^{n+m}_{g,k}(\text{Spec}(\mathbb{C}))$ be the composition of $J' \to J$ with $G$. Replacing $J'$ with another small analytic open set if necessary, we may simultaneously resolve the $m$ nodes of the fibres of $B$ to produce a family $\widetilde{B} \to J'$ of smooth curves, together with a morphism $\widetilde{B} \to B$ restricting to the normalization over each point in $J'$, [49, p. 80]. By composing with the universal family of maps $B \to \mathcal{X}$, where $\mathcal{X}$ is a family of K3 surfaces, we produce the family of maps $\widetilde{B} \to \mathcal{X}$ over $J'$. By the universal property of $\mathcal{T}^{n+m}_{g,k}$, this produces a morphism of analytic stacks $J' \to \mathcal{T}^{n+m}_{g,k}$ which coincides with $G'$ on the level of points. As this is generically finite and meets the open subset $\mathcal{T}^{n+m}_{g,k}$, the dimension of $\mathcal{T}^{n+m}_{g,k}$ is at least $\dim J \geq p(B) + 19$. But $\mathcal{T}^{n+m}_{g,k}$ is smooth of dimension $p(B) - m + 19$, so this is a contradiction. □

**Proposition 3.6.** Let $[(f : B \to X, L)] \in \mathcal{W}^n_{g,k}$ represent an unramified stable map such that $h^0(N_f) \leq p(B)$. Assume furthermore that there is no decomposition

$$B = \bigcup_{i=1}^{t} B_i$$

for $t > 1$ with each $B_i$ a connected union of irreducible components of $B$ such that $B_i$ and $B_j$ meet transversally for all $i \neq j$ and such that for all $1 \leq i \leq t$, $f_i(B_i) \in |m_i L|$ for a positive integer $m_i > 0$ (this is automatic if $k = 1$). Lastly, assume that there is some component $B_j$ such that $f_j|B_j$ is birational onto its image, and if $B_i \neq B_j$ is any component, $f(B_i)$ and $f(B_j)$ intersect properly. Then $[(f : B \to X, L)]$ lies in the closure of $\mathcal{T}^n_{g,k}$.

**Proof.** We need to show that we may deform $[(f : B \to X, L)]$ to an unramified stable map $[(f' : B' \to X', L')]$ with $B'$ irreducible and smooth. We will firstly show that $[(f : B \to X, L)]$ deforms to a stable map with irreducible base.

From Proposition 3.2, $\pi : \mathcal{W}^n_{g,k} \to B$ is dominant near $[(f : B \to X, L)]$. Thus we may deform $[(f : B \to X, L)]$ to an unramified stable map $[(f' : B' \to X', L')]$, where $\text{Pic}(X') \simeq \mathbb{Z}L'$. For any irreducible component $Z \subseteq B'$, $f'_*(Z) \in |a_i L'|$ for some integer $a_i > 0$. Given any one-parameter family $\gamma(t) = [(f_t : B_t \to X_t, L_t)]$ of stable maps with $\gamma(0) = [(f : B \to X, L)]$, then after performing a finite base change about $0$ if necessary, the irreducible components of $B_t$ for generic $t$ deform to a connected union of irreducible components of $B$ as $t \to 0$ [48, Tag 0551] (note that a projective, one parameter family of curves with general fibre integral specializes to a connected curve [38, Cor. 8.3.6]). Thus the condition on $B$ ensures that $[(f : B \to X, L)]$ deforms to an unramified stable map of the form $[(f' : B' \to X', L')]$ with $B'$ integral and nodal.

We next show that $B'$ can be assumed smooth after deforming. If $\mu : \tilde{B} \to B'$ is the normalization of $B'$, then $[(f \circ \mu : \tilde{B} \to X', L')]$ is an unramified morphism representing a point in $\mathcal{W}^m_{g,k}$, where $m$ is the number of nodes of $B'$. Since $\dim_{f_0} \mathcal{W}^m_{g,k} = p(g, k) - n - m + 19 < \dim \mathcal{W}^n_{g,k} = p(g, k) - n + 19$, a dimension count as in Lemma 3.3 shows that $[(f : B \to X, L)]$ deforms to an unramified stable map of the form $[(f' : B' \to X', L')]$ with $B'$ integral and smooth.

We will lastly show that $[(f : B \to X, L)]$ deforms to an unramified stable map of the form $[(f' : B' \to X', L')]$ with $B'$ integral and smooth and such that $f'$ is birational onto its image. Let $S$ be a smooth, irreducible, one-dimensional scheme with base point 0, and
suppose we have a diagram

\[
\begin{array}{ccl}
B & \xrightarrow{g} & X \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
S & & 
\end{array}
\]

with \( g \) proper, \( \pi_1, \pi_2 \) flat and with \( g_s : B_s \to X_s \) an unramified stable map to a K3 surface for all \( s \), with \( g_0 = f \) and such that \( B_s \) is smooth for \( s \neq 0 \). Assume further that \( \mathcal{L} \) is an \( S \)-flat line bundle on \( X \), with \( \mathcal{L}_0 = L \) and that the cycle \( g_* (B) \sim \mathcal{L} \) is a relatively effective (Cartier) divisor. So the cycle \( g_* (B) \) may be considered as an \( S \)-relatively effective divisor \( \overline{B} \subseteq X \).

Then \( B \) is an irreducible surface and so \( \overline{B} \) is irreducible. For \( x \in B_j \subseteq B_0 \) general, \( \overline{B} \to S \) is smooth near \( g(x) \), by the assumptions on \( f = g_0 \). Thus \( \overline{B} \to S \) is smooth in an open subset about \( g(x) \), and in particular is reduced. Thus the generic fibre of \( \overline{B} \to S \) is reduced and in particular \( g_s \) is birational for \( s \) near 0 (as if \( \deg(g_s) = d \), \( \overline{B}_s = dg_s(B_s) \)).

We next aim to reduce the study of generic finiteness of \( \eta \) to that of \( \eta \) for \( m >> n \).

**Lemma 3.7.** Assume that \( n \leq m \) and that there is a component \( I_m \subseteq T_{g,k}^m \) such that 

\[ \eta_{|I_m} : I_m \to \mathcal{M}_{p(g,k)-m} \]

is generically finite. Then there exists a component \( I_n \subseteq T_{g,k}^n \) such that 

\[ \eta_{|I_n} : I_n \to \mathcal{M}_{p(g,k)-n} \]

is generically finite.

**Proof.** Let \( [(f : B \to X, L)] \in I_m \) be a general point. There is an integral, nodal curve \( B' \) with \( m - n \) nodes with normalization \( \mu : B \to B' \) such that \( f \) factors through a morphism \( g : B' \to X \). The fibre of \( \eta \) over \( [B'] \in \mathcal{M}_{p(g,k)-n} \) is zero-dimensional near \( [(g, L)] \) as otherwise we could compose with \( \mu \) to produce a one dimension family near \( [(f : B \to X, L)] \) in the fibre of \( \eta \) over \( [B] \). Since \( [(g, L)] \) lies in the closure of \( T_{g,k}^n \) by Lemma 3.5, we see that there exists a component \( I_n \subseteq T_{g,k}^n \) such that 

\[ \eta_{|I_n} : I_n \to \mathcal{M}_{p(g,k)-n} \]

is generically finite. □

The following proposition gives a criterion for generic finiteness of the morphism 

\[ \eta : T_{g,k}^n \to \mathcal{M}_{p(g,k)-n} \]

on one component \( I \). The idea is to assume we have an unramified map \( f_0 : C_0 \to X \) representing a point in \( W_{g,n}^{m'} \) such that finiteness of \( \eta \) holds near the point representing \( f_0 \). If we then build a new morphism \( f : C_0 \cup \mathbb{P}^1 \to X \) by finding a rational curve \( f(\mathbb{P}^1) \) in \( X \), and if we further assume \( C_0 \cup \mathbb{P}^1 \) is a stable curve (i.e. \( \mathbb{P}^1 \) intersects \( C_0 \) in at least three points), then by rigidity of rational curves in \( X \), one sees easily that finiteness of \( \eta \) holds near the point representing \( f \), where \( n, k \) are such that \( f \) represents a point in \( W_{g,k}^n \).
Proposition 3.8. Assume there exists a polarized K3 surface $(X, L)$ and an unramified stable map $f : B \to X$ with $f_*(B) \in |kL|$. Assume:

1. $[(f : B \to X, L)]$ lies in the closure of $\mathcal{T}_{g,k}$.

2. There exists an integral, nodal component $C \subseteq B$ of arithmetic genus $p' \geq 2$ such that $f_{|C}$ is an unramified morphism $j : C \to X$, birational onto its image. Let $k'$ be an integer such that there is a big and nef line bundle $L'$ on $X$ with $j_*(C) \in |k'L'|$, and let $g' = \frac{1}{2}(L')^2 + 1$.

3. The fibre of the morphism $\eta : \mathcal{W}_{g',k'} \to \overline{\mathcal{M}}_{p'}$ over $[C]$ is zero-dimensional near $[(j : C \to X, L')]$, where $n' = p(j(C)) - p'$.

4. If $D \subseteq B$ is a component, $D \neq C$, then $D$ has geometric genus zero.

5. The stabilization morphism $B \to \hat{B}$ is an isomorphism in an open subset $U \subseteq B$ such that $C \subseteq U$.

Then there exists a component $I \subseteq \mathcal{T}_{g,k}$ such that $\eta_!$ is generically finite and $[(f : B \to X, L)]$ lies in the closure of $I \subseteq \mathcal{W}_{g,k}$.

Proof. We have a morphism $\eta : \mathcal{W}_{g,k} \to \overline{\mathcal{M}}_{p(k)}$. By assumption 1, it suffices to show $\eta^{-1}([\hat{B}])$ is zero-dimensional near $[(f : B \to X, L)]$. Let $S$ be a smooth, irreducible, one-dimensional scheme with base point 0, and suppose we have a diagram

$$
\begin{array}{ccc}
B & \xrightarrow{g} & X \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
S & & 
\end{array}
$$

with $g$ proper, $\pi_1, \pi_2$ flat and with $g_s : B_s \to X_s$ an unramified stable map to a K3 surface for all $s$, with $g_0 = f$. Further assume $\hat{B}_s \simeq \hat{B}$. For any $s \in S$, we have open subsets $U_s \subseteq B_s$, $V_s \subseteq \hat{B}$ with the stabilization map inducing isomorphisms $U_s \simeq V_s$ and such that $\hat{B} \setminus V_s$ has zero-dimensional support. By assumption 5, $C \subseteq V_0$, and thus for $s$ near 0, $C \subseteq V_s \simeq U_s$. Thus, after performing a finite base change about 0 $\in S$, there exists an irreducible component $C \subseteq B$, such that $C_s \simeq C$, for all $s$ near 0. We have a diagram

$$
\begin{array}{ccc}
C & \xrightarrow{h} & X \\
\downarrow{\pi_{1|C}} & & \downarrow{\pi_2} \\
S & & 
\end{array}
$$

where $\pi_{1|C}$ is flat and $h := g_{|C}$. Since $h_0 = j$, assumption 3 gives $X_s \simeq X$ and $h_s = j$ for all $s$. In particular, $g_s$ is a one-dimensional family of unramified morphisms into a fixed K3 surface. From assumption 4, the fact that rational curves on a K3 surface are rigid and since $g_s$ is unramified, we conclude that $B_s \simeq B$ and $g_s : B \to X$ is independent of $s$. Thus $\eta^{-1}(\hat{B})$ is zero-dimensional near $[(f : B \to X, L)]$. \qed
We will apply the above criterion to prove generic finiteness of \( \eta \) on one component, for various bounds on \( p(g,k) - n \). We first consider the case \( k = 1 \). To begin, we will need an easy lemma. Let \( p > h \geq 8 \) be integers, and let \( l, m \) be nonnegative integers with

\[
p - h = \left\lceil \frac{h + 1}{2} \right\rceil l + m
\]

and \( 0 \leq m < \left\lfloor \frac{h+1}{2} \right\rfloor \). Define:

\[
s_1 := \begin{cases} p - h - 1, & \text{if } m = 0 \text{ or } m = \left\lfloor \frac{h+1}{2} \right\rfloor - 1 \\ p - h + 1, & \text{otherwise.} \end{cases}
\]

Let \( P_{p,h} \) be the rank three lattice generated by elements \( \{M, R_1, R_2\} \) and with intersection form given with respect to this ordered basis by:

\[
\begin{pmatrix}
2h - 2 & s_1 & 3 \\
-2 & 0 & 0 \\
3 & 0 & -2
\end{pmatrix}
\]

**Lemma 3.9.** Let \( p > h \geq 8 \). There exists a K3 surface \( S_{p,h} \) with \( \text{Pic}(S_{p,h}) \simeq P_{p,h} \) as above such that the classes \( M, R_1, R_2 \) are each represented by integral curves and with \( M \) very ample. If \( h \) is odd and at least 11, then for \( D \in |M| \) general the fibre of \( \eta : T^0_{h} \to \mathcal{M}_h \) is zero-dimensional at \( [(i : D \cong S_{p,h}, M)] \). Furthermore, any divisor of the form \(-xM + yR_1 + zR_2\) for integers \( x, y, z \) with \( x > 0 \) is not effective.

**Proof.** Consider the K3 surface \( Y_{1h} \) from Lemma 2.1. We choose \( d_1 := (L \cdot \Gamma_1) \) to be \( m + 1 \) if \( 0 < m < \left\lfloor \frac{h+1}{2} \right\rfloor - 1 \) and \( d_1 = \left\lceil \frac{h+1}{2} \right\rceil - 1 \) if \( m = 0 \) and \( d_1 = \left\lfloor \frac{h+1}{2} \right\rfloor - 2 \) if \( m = \left\lfloor \frac{h+1}{2} \right\rfloor - 1 \). Further set \( d_2 = 3 \) and let all other \( d_i \) be arbitrary integers in the range \( 1 \leq d_i < \left\lceil \frac{h+1}{2} \right\rceil \). We define a primitive embedding \( j : S_{p,h} \hookrightarrow Y_{1h} \) as follows. If \( m = 0 \) (so that \( l \geq 1 \) as \( p > h \)), we define the embedding via \( M \mapsto L, R_1 \mapsto (l - 1)E + \Gamma_1, R_2 \mapsto \Gamma_2 \). If \( m \neq 0 \) we define the embedding via \( M \mapsto L, R_1 \mapsto IE + \Gamma_1, R_2 \mapsto \Gamma_2 \). Let \( M_{S_{p,h}} \) be the moduli space of ample, \( S_{p,h} \)-polarized K3 surfaces as constructed in [16], and let \( M_1 \) be a component containing \( Y_{1h} \). Then the general point of \( M_1 \) represents a K3 surface with \( \text{Pic}(S_{p,h}) \simeq P_{p,h} \). Further, \( M, R_2 \) are each represented by integral curves and \( M \) is very ample by Lemmas 2.1 and 2.2. If \( h \) is odd and at least 11, the statement about the fibres of \( \eta \) follows from Corollary 2.10.

We now claim that any divisor of the form \(-xM + yR_1 + zR_2\) for integers \( x, y, z \) with \( x > 0 \) is not effective. By degenerating \( S_{p,h} \) to \( Y_{1h} \) as above, it suffices to show that \(-xL + yj(R_1) + z\Gamma_2 \in \text{Pic}(Y_{1h})\) is not effective. But this is clear, since the rank ten lattice \( \text{Pic}(Y_{1h}) \) contains the class of a smooth, integral, elliptic curve \( E \) with \((E \cdot -xL + yj(R_1) + z\Gamma_2) = -x(E \cdot L) < 0 \).

The \(-2\) class \( R_1 \) is effective since \((R_1)^2 = -2, (R_1 \cdot M) > 0 \). It remains to show that \( R_1 \) is integral. Let \( D \) be any integral component of \( R_1 \) with \((D)^2 = -2, (D \cdot R_1) < 0 \). Writing \( D = xM + yR_1 + zR_2 \) we see that \( x = 0 \) (as \( R_1 - D \) is effective). Thus \((D \cdot R_1) < 0 \) implies \( y > 0 \) and \(-2 = (D)^2 = -2(y^2 + z^2) \) forces \( z = 0 \); thus \( D = R_1 \) is integral. \( \square \)

**Theorem 3.10.** Assume \( g \geq 11, n \geq 0 \), and let \( 0 \leq r(g) \leq 5 \) be the unique integer such that

\[
g - 11 = \left\lfloor \frac{g - 11}{6} \right\rfloor 6 + r(g).
\]

Define
• \(l_g = 12\), if \(r(g) = 0\).
• \(l_g = 13\), if \(1 \leq r(g) < 5\).
• \(l_g = 15\) if \(r(g) = 5\).

Then there is a component \(I \in T_g^n\) such that

\[
\eta_I : I \to M_{g-n}
\]

is generically finite for \(g - n \geq l_g\).

**Proof.** From Lemma 3.7 it suffices to prove the result for the maximal value of \(n\). Assume \(g - n = l_g\) if \(r(g) \neq 0\) and for \(g - n = 15\) if \(r(g) = 0\). Set \(p = g, \ h = 11\) and consider the lattice \(P_{g,11}\) and K3 surface \(S_{g,11}\) from Lemma 3.9. Set \(\epsilon = 1\) if \(r(g) = 0\) or \(r(g) = 5\) and \(\epsilon = 0\) otherwise. Then \((M + R_1 + \epsilon R_2)^2 = 2g - 2\) and \(M + R_1 + \epsilon R_2\) is ample. Let \(D \in |M|\) be general and consider the curve \(D_1 = D \cup R_1 \cup \epsilon R_2\) where all intersections are transversal. Choose any subset of \(s_1 - 3\) distinct points of \(D \cap \Gamma_1\) and let \(f : B \to D_1\) be the partial normalization at the chosen points. Then \(B\) has arithmetic genus \(l_g\) if \(r(g) \neq 0\) and genus 15 if \(r(g) = 0\). Further, \(f\) satisfies the conditions of Proposition 5.8. Then there is a component \(I \in T_g^n\) such that

\[
\eta_I : I \to M_{g-n}
\]

is generically finite for \(g - n \geq l_g\) if \(r(g) \neq 0\) and for \(g - n \geq 15\) if \(r(g) = 0\).

We now wish to improve the bound in the case \(r(g) = 0\). Recall from Section 2 the lattice \(\Omega_{11}\) with ordered basis \(\{L, E, \Gamma_1, \ldots, \Gamma_8\}\). Thus the general \(C \in |L|\) is a smooth, genus 11 curve and \((L \cdot E) = 6\). Note that \(\Gamma_1 \sim L - E\) is a class satisfying \((\Gamma_1)^2 = 2, (\Gamma_1 \cdot E) = 0, (\Gamma_1 \cdot \Gamma_1) = 2\). From Lemma 2.1 \(\Gamma_1\) is represented by an integral class. Further \(\Gamma_1 + \Gamma_i\) is an \(I_2\) singular fibre of \(|E|\) (the union of two smooth rational curves intersecting transversally in two points). We will denote by \(x_i\) and \(y_i\) the two nodes of \(\Gamma_1 + \Gamma_i\).

Set \(m := \left\lfloor \frac{2g - 11}{6} \right\rfloor\) and assume \(r(g) = 0\). Consider the primitive, ample line bundle \(H := L + \left\lfloor \frac{g-11}{6} \right\rfloor E\), which satisfies \((H)^2 = 2g - 2\). Let \(C \in |L|\) be a general smooth curve which meets \(\Gamma_1\) transversally. Let \(B\) be the union of \(C\) with \(2m\) copies of \(\mathbb{P}^1\) as in the diagram below.

Thus \(B = C \cup R_{1,1} \cup R_{1,2} \ldots \cup R_{m,1} \cup R_{m,2}\), where each \(R_{i,j}\) is smooth and rational, all intersections are transversal and described as follows for \(m \geq 2\): \(R_{i,j} \cap C = \emptyset\) unless \((i, j) = (1, 1), (1, 2), \ldots, (m, 1), (m, 2)\).
Lemma 3.11. Let $R_{i,j}$ intersects $R_{k,l}$ in at most one point, with intersections occurring if and only if, after swapping $R_{i,j}$ and $R_{k,l}$ if necessary, we have $(i = k$ and $l = j + 1)$, $(k = i + 1$ and $l = j)$ or $((i,j) = (1,1)$ and $(j,k) = (m,2))$. The arithmetic genus of $B$ is 12. For $m = 1$, $B = C \cup R_{1,1} \cup R_{1,2}$ where $C$ intersects $R_{1,1}$ transversely in one point and $R_{1,1} \cap C = \emptyset$ and $R_{1,1}$ intersects $R_{1,2}$ transversally in two points.

There is then a unique unramified morphism $f : B \to Y_{\Omega_{11}}$ such that $f_{|C}$ is a closed immersion, $f_{|R_{1,1}}$ is a closed immersion with image $\Gamma_1$ and $f_{|R_{1,2}}$ is a closed immersion with image $\Gamma_1$, for $1 \leq i \leq m$ and where $(f(p), f(q)) = (x_1, y_1)$. Thus $f_* B \in |H|$. Note that the stabilization $\bar{B}$ has two components, namely it is the union of $C$ with a rational curve with one node. Thus the claim holds in the $r = 0$ case by Proposition 3.3 and Corollary 2.10.

We now turn to the nonprimitive case.

**Lemma 3.11.** Let $Z_a$ be a general K3 surface with Picard lattice $\Lambda_a$ generated by elements $D, F, \Gamma$ giving the intersection matrix

$$
\begin{pmatrix}
2a - 2 & 6 & 1 \\
6 & 0 & 0 \\
1 & 0 & -2
\end{pmatrix}
$$

Assume that $14 \leq a \leq 19$. Then we may pick the basis such that $D, F, \Gamma$ are all effective and represented by integral, smooth curves with $D$ ample. Further, there is an unramified stable map $f_a : B_a \to Z_a$, birational onto its image, with $B_a$ an integral, nodal curve of arithmetic genus 13 for $14 \leq a \leq 15$ and 15 for $16 \leq a \leq 19$, such that $f_a(B_a) \in |D|$ and $f_a$ satisfies the conditions of Proposition 3.8. Further, there is an integral, rational nodal curve $F_0 \in |F|$ which meets $f_a(B_a)$ transversally, and $\Gamma$ meets $f_a(B_a)$ transversally in one point.

**Proof.** Let $M_{\Lambda_a}$ be the moduli space of pseudo-ample, $\Lambda_a$-polarized K3 surfaces, [16]. This has at most two components, which locally on the period domain are interchanged via complex conjugation. Consider the lattice $\Omega_{11}$ with ordered basis $\{L, E, \Gamma_1, \ldots, \Gamma_8\}$ and set $d_8 = 1$. For $d_1, d_2 \geq 3$, let $H$ be the primitive, ample line bundle $H = L + \Gamma_1 + \epsilon \Gamma_2$, where $\epsilon = 0$ for $14 \leq a \leq 15$ and $\epsilon = 1$ for $16 \leq a \leq 19$. Choose $3 \leq d_1, d_2 \leq 5$ such that $H^2 = 2a - 2$; it is easily checked that all six possibilities can be achieved. There is a primitive lattice embedding

$$
\Lambda_a \hookrightarrow \Omega_{11}
$$

defined by $D \mapsto H$, $\Gamma \mapsto \Gamma_8$, $F \mapsto E$.

Let $Y_{\Omega_{11}}$ be any K3 surface with $Pic(Y_{\Omega_{11}}) \simeq \Omega_{11}$, and choose the basis $\{L, E, \Gamma_1, \ldots, \Gamma_8\}$ as in Lemma 2.11. Consider the curve $C \cup \Gamma_1 \cup \epsilon \Gamma_2 \in |H|$, where $C \in |L|$ is a general smooth curve. By partially normalizing at all nodes other than three on $C \cup \Gamma_1$ and three on $C \cup \Gamma_2$, we construct an unramified stable map $\bar{f}_a : \bar{B}_a \to Y_{\Omega_{11}}$, birational onto its image and satisfying the conditions of Proposition 3.8. Note that $B_a$ has arithmetic genus 13 for $14 \leq a \leq 15$ and 15 for $16 \leq a \leq 19$. After deforming $\bar{f}_a$ to an unramified stable map $f_a : B_a \to Z_a$, we find $B_a$ must become integral, since it is easily checked that $Z_a$ contains no smooth rational curves $R$ with $(R \cdot F) = (R \cdot \Gamma) = 0$. 

18
Thus the claim on $f_a$ follows from the proof of Proposition 3.8. Note that the $I_2$ fibre $\Gamma_2 + \tilde{\Gamma}_2$ must deform to an integral, nodal, rational curve on $Z_a$, since $Z_a$ contains no smooth rational curves which avoid $F$ and $\Gamma$. If $\Omega_{11} \to Pic(Y_{\Omega_{11}})$ is the embedding as in Lemma 2.1 and if $Y_{\Omega_{11}}^c$ is the conjugate K3 surface, then we obviously have an embedding $\Omega_{11} \to Pic(Y_{\Omega_{11}}^c)$ satisfying the conclusions of Lemma 2.1. Thus the claim holds for the general pseudo-ample, $\Lambda_a$-polarized K3 surface.

**Remark 3.12.** In the notation of the above proof, we have $h^0(N_{f_a}) \leq p(B_a)$ from Lemma 3.8. It thus follows that $h^0(N_{f_a}) \leq p(B_a)$ by semicontinuity.

**Lemma 3.13.** Let $1 \leq d \leq 5$ be an integer and consider the rank five lattice $K_d$ with basis \{A, B, \Gamma_1, \Gamma_2, \Gamma_3\} giving the intersection matrix

\[
\begin{pmatrix}
-2 & 6 & 3 & 2 & d \\
6 & 0 & 0 & 0 & 0 \\
3 & 0 & -2 & 0 & 0 \\
2 & 0 & 0 & -2 & 0 \\
d & 0 & 0 & 0 & -2
\end{pmatrix}.
\]

Then $K_d$ is an even lattice of signature $(1, 4)$. There exists a K3 surface $Y_{K_d}$ with $Pic(Y_{K_d}) \approx K_d$, and such that the classes \{A, B, \Gamma_1, \Gamma_2, \Gamma_3\} are all represented by nodal, reduced curves such that the nodal curve $A$ meets $\Gamma_1, \Gamma_2$ and $\Gamma_3$ transversally. Further, $A+B$ is big and nef.

**Proof.** Let $Y_1, Y_2$ be smooth elliptic curves and consider the Kummer surface $\tilde{Z}$ associated to $Y_1 \times Y_2$. Let $P_1, P_2, P_3, P_4$ be the four 2-torsion points of $Y_1$ and let $Q_1, Q_2, Q_3, Q_4$ be the 2-torsion points of $Y_2$. Let $E_{i,j} \subseteq \tilde{Z}$ denote the exceptional divisor over $P_i \times Q_j$, let $T_i \subseteq \tilde{Z}$ denote the strict transform of $(P_i \times Y_2)/\pm$ and let $S_j$ denote the strict transform of $(Y_1 \times Q_j)/\pm$. We also denote by $F$ a smooth elliptic curve of the form $x \times Y_2$, where $x \in Y_1$ is a non-torsion point. It may help the reader to consult the diagram on [39, p. 344], to see the configuration of these curves. We set

\[
\begin{align*}
\tilde{A} &:= S_1 + E_{1,1} + T_1 + E_{1,2} + S_2 + E_{1,3} + S_3 \\
\tilde{B} &:= F + S_4 + E_{2,4} + T_2 + E_{2,1} + E_{2,2} + E_{2,3} \\
\tilde{\Gamma}_1 &:= T_3 + E_{3,1} + E_{3,2} + E_{3,3} \\
\tilde{\Gamma}_2 &:= T_4 + E_{2,1} + E_{2,2} \\
\tilde{\Gamma}_3 &:= \begin{cases} 
T_4 + \sum_{i=1}^{d} E_{4,i} & \text{if } 1 \leq d \leq 3 \\
T_4 + \sum_{i=1}^{d-3} E_{4,i} + E_{4,4} + S_4 + E_{2,4} + T_2 + E_{2,1} + E_{2,2} + E_{2,3} & \text{if } 4 \leq d \leq 5.
\end{cases}
\end{align*}
\]

Then $\tilde{A}, \tilde{B}, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$ generate $K_d$ (to simplify the computations, use that a tree of $-2$ curves has self-intersection $-2$). To see that this gives a primitive embedding of $K_d$ in $Pic(\tilde{Z})$ we compute the intersections with elements of $Pic(\tilde{Z})$; for $J \in Pic(\tilde{Z})$, define $(J \cdot K_d)$ to be the quintuple $((J \cdot \tilde{A}), (J \cdot \tilde{B}), (J \cdot \tilde{\Gamma}_1), (J \cdot \tilde{\Gamma}_2), (J \cdot \tilde{\Gamma}_3))$. Then one computes $(T_1 \cdot K_d) = (1, 0, 0, 0, 0), (T_3 \cdot K_d) = (0, 1, 0, 0, 0), (E_{4,1} \cdot K_d) = (1, 0, 0, 0, -1), (E_{2,4} \cdot K_d) = (0, 0, 0, 1, 0), (E_{2,3} \cdot K_d) = (1, -1, 0, 1, c)$, where $c$ is either 0 or $-1$, depending on $d$. Thus $K_d$ is primitively embedded in $Pic(\tilde{Z})$. Further, all intersections of $\tilde{B}, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$ with $\tilde{A}$ are transversal. Note that for any (rational) component $R \subseteq \tilde{A} + \tilde{B}$, $(R \cdot \tilde{A} + \tilde{B}) \geq 0$. Thus $\tilde{A} + \tilde{B}$ is big and nef. Hence the claim holds by degenerating to $\tilde{Z}$. \hfill \qed
Lemma 3.14. Let $1 \leq d \leq 5$ be an integer and consider the K3 surface $Y_{K_d}$ from Lemma 3.13. Then the classes $\{A, B, \Gamma_1, \Gamma_2, \Gamma_3\}$ are all represented by integral curves.

Proof. We will firstly show that $B$ is nef, and hence base point free. Indeed, there would otherwise exist an effective divisor $R = xA + yB + z\Gamma_1 + w\Gamma_2 + u\Gamma_3$ for integers $x, y, z, w, u,$ with $(R)^2 = -2$ and $(R \cdot B) < 0$, i.e. $x < 0$. Thus $R - xA$ is effective and $(R - xA \cdot A) = (R - xA \cdot A + B) \geq 0$, since $A + B$ is nef. But then

$$-2 = (R)^2 = ((R - xA) + xA)^2 = -2(z^2 + w^2 + u^2 + x^2) + 2x(R - xA \cdot A)$$

So we must have $z = w = u = 0$, $x = -1$ and $(R + A \cdot A) = 0$. But then $R = -A + yB$ and $(R + A \cdot A) = 0$ gives $y = 0$. But this contradicts that $A$ is effective.

We next show that each $\Gamma_i$ is integral. Let $R$ be any irreducible component of $\Gamma_i$ with $(R \cdot \Gamma_i) < 0$, $(R)^2 = -2$ (such a component exists). Write $R = xA + yB + z\Gamma_1 + w\Gamma_2 + u\Gamma_3$. We have $x \geq 0$ as $(R \cdot B) \geq 0$. Assume $x \neq 0$. Then $(R \cdot R + B) > 0$, $(R + B)^2 > 0$, so that $R + B$ is big and nef, contradicting that $(R + B \cdot \Gamma_i) = (R \cdot \Gamma_i) < 0$. Thus $x = 0$ and $R = yB + z\Gamma_1 + w\Gamma_2 + u\Gamma_3$. Since $(R)^2 = -2$, $(R \cdot \Gamma_i) < 0$, we have $R = yB + \Gamma_i$. Since $(R \cdot A + B) = (R \cdot A) \geq 0$, we must have $y \geq 0$ (for $i = 3$, we need here that $d < 6$). Since smooth rational curves are extremal, we must have $y = 0$. Thus $\Gamma_i = R$ is integral.

To show that $A$ is integral, let $R_1, \ldots, R_s$ be the components of the effective $-2$ curve $A$, and write $R_i = x_iA + y_iB + z_i\Gamma_1 + w_i\Gamma_2 + u_i\Gamma_3$ for integers $x_i, y_i, z_i, w_i, u_i$. Intersecting with $B$ shows we have $x_i \geq 0$ for all $i$. There is precisely one component, say $R_1$ with $x_1 \neq 0$ and further $x_1 = 1$. Now, choose a component $R_i$ with $(R_i)^2 = -2$, $(R_i \cdot A) < 0$. Firstly assume $i \neq 1$, so that $x_i = 0$. Intersecting with the integral curves $\Gamma_j$, $1 \leq j \leq 3$ (and noting $R_i \neq \Gamma_j$ as $(R_i \cdot A) < 0$), we have $z_i, w_i, u_i \leq 0$. From $(R_i)^2 = -2$, we see $R_i = y_iB - \Gamma_j$ for some $1 \leq j \leq 3$. Intersecting with $A$ gives $6y_i - k < 0$ for some $1 \leq k \leq 5$, and thus $y_i \leq 0$ which is a contradiction to the effectivity of $R_i$.

In the second case, assume $(R_1)^2 = -2$, $(R_1 \cdot A) < 0$, with $R_1 = A + y_1B + z_1\Gamma_1 + w_1\Gamma_2 + u_1\Gamma_3$. We compute

$$-2 = (R_1)^2 = ((R_1 - A) + A)^2 = -2(y_1^2 + z_1^2 + w_1^2 + 1) + 2((R_1 \cdot A) + 2) = -2(y_1^2 + z_1^2 + w_1^2) + 2((R_1 \cdot A) + 1).$$

Thus we have either $(R_1 \cdot A) = -2$ and $R_1 = A + y_1B$ or $(R \cdot A) = -1, R_1 = A + y_1B \pm \Gamma_j$ for some $1 \leq j \leq 3$. In the first case, $(A + y_1B \cdot A) = -2$ implies $y_1 = 0$ so $A = R_1$ is integral. In the second case, $(A + y_1B \pm \Gamma_j \cdot A) = -1$ implies $-1 = -2 + 6y_1 \pm k$, for $1 \leq k = (A \cdot \Gamma_j) \leq 5$. The only possibilities are $y_1 = 0$, $k = 1, R_1 = A + \Gamma_j$, contradicting extremality of $R_1$, or $y_1 = 1$, $k = 5, R_1 = A + B - \Gamma_j$. Since $(B - \Gamma_j)^2 = -2, (B - \Gamma_j \cdot A + B) > 0$, we have that $B - \Gamma_j$ is effective, so once again this contradicts extremality of $R_1$. \qed

Lemma 3.15. Let $M_A$ denote the moduli space of pseudo-ample $\Lambda_a$-polarized K3 surfaces, with the lattice $\Lambda_a$ as in Lemma [3.17] for $14 \leq a \leq 19$. Then there is a nonempty open subset $U \subseteq M_A$ such that for $[Y_a] \in U$ we may pick the basis $\{D, F, \Gamma\}$ such that there is an integral, nodal, rational curve $R_a \in \{D - 2F - \Gamma\}$ such that $R_a$ meets $\Gamma$ transversally in three points.
Proof. Set \( y := a - 14 \), so that \( 0 \leq y \leq 5 \) by assumption. If we change the basis of \( \Lambda_a \) to \( \{ D, 2F, \Gamma, F, \Gamma \} = \{ X, Y, Z \} \), we see \( \Lambda_a \) is isometric to the lattice \( \Lambda_a \) with intersection matrix

\[
\begin{pmatrix}
2y-2 & 6 & 3 \\
6 & 0 & 0 \\
3 & 0 & -2
\end{pmatrix}.
\]

Let \( K_d \) be the lattice from Lemma 3.13. For appropriate choices of \( d \) there is a primitive lattice embedding \( \tilde{\Lambda}_a \to K_d \), given by

\[
\begin{align*}
X & \mapsto A + \epsilon_1 \Gamma_3 + \epsilon_2 \Gamma_2 \\
Y & \mapsto B \\
Z & \mapsto \Gamma_1
\end{align*}
\]

where if \( a = 14 \) we set \( \epsilon_1 = \epsilon_2 = 0 \) and \( d \) arbitrary, if \( 15 \leq a < 19 \) we set \( d = y + 1, \epsilon_1 = 1, \epsilon_2 = 0 \) and if \( a = 19 \) we set \( d = 5 \) and \( \epsilon_1 = \epsilon_2 = 1 \). By Lemmas 3.13, 3.14, we have that \( A, \Gamma_2, \Gamma_3 \) are represented by smooth rational curves intersecting transversally on \( Y_{K_d} \). In all cases, the divisor \( D = X + 2Y + Z \) is big and nef, since if \( R \in \{ \Gamma_1, \epsilon_1 \Gamma_3, \epsilon_2 \Gamma_2, A \} \), \( (R \cdot D) \geq 0 \). After deforming \( Y_{K_d} \) to a K3 surface \( Y_a \) with \( Pic(Y_a) \simeq \Lambda_a \), we can deform \( A + \epsilon_1 \Gamma_3 + \epsilon_2 \Gamma_2 \) to a nodal rational curve \( \tilde{R}_a \) which meets \( \Gamma \) transversally in three points (the arguments of 3.1, \S 2] can be adapted to this case, even though the class \( A + \epsilon_1 \Gamma_3 + \epsilon_2 \Gamma_2 \) is not necessarily even nef thanks to 3.1, \S 2.1, Remark 3.1], and using that \( (A + \epsilon_1 \Gamma_3 + \epsilon_2 \Gamma_2 \cdot B) > 0 \). Furthermore, \( \tilde{R}_a \) is integral, since \( \Lambda_a \) contains no \(-2\) curves which have zero intersection with \( F, \Gamma \).

\[\Box\]

Theorem 3.16. Assume \( k \geq 2 \), \( g \geq 8 \). Set \( m := \left\lfloor \frac{g - 5}{6} \right\rfloor \) and let \( 0 \leq r(g) \leq 5 \) be the unique integer such that

\[g - 5 = 6m + r(g)\]

Define:

- \( l_g := 15 \), if \( r(g) = 3, 4, m \) even.
- \( l_g := 16 \), if \( r(g) = 3, 4, m \) odd.
- \( l_g := 17 \), if \( r(g) = 3, 4, m \) even.
- \( l_g := 18 \), if \( r(g) = 3, 4, m \) odd.

Then there is a component \( I \subseteq \mathcal{I}^{n}_{g,k} \) such that

\[\eta_I : I \to \mathcal{M}_{p(g,k) - n}\]

is generically finite for \( p(g, k) - n \geq l_g \).

Proof. Consider the \( \Lambda_a \)-polarized K3 surface \( Y_a \) from Lemma 3.15 and let \( \{ D, F, \Gamma \} \) be as in the lemma. Set \( m := \left\lfloor \frac{g - 5}{6} \right\rfloor \geq 0 \) and consider the primitive, ample line bundle \( H = D + (m - 1)F \). By varying \( a \) we can achieve all values of \( (H)^2 = 2g - 2 \) for \( g \geq 8 \). Let \( f_a : Y_a \to Y_a \) resp. \( \tilde{R}_a \) be the unramified stable map, resp. rational curve from lemmas 3.11 resp. 3.15. Set \( l = k(m - 1) + 2(k - 1) \) which is strictly positive for \( k \geq 2 \). We have an effective decomposition

\[kH \sim f_a(B_a) + (k - 1)R_a + (k - 1)\Gamma + lF_0,\]

21
where $F_0 \in |F|$ is an integral, nodal rational curve as in lemma 3.11. We will prove the result by constructing an unramified stable map $f : B \to Y_a$ with $f_*(B) = f_*(B_a) + (k - 1)R_a + (k - 1)\Gamma + lF_0$ satisfying the conditions of Proposition 3.8.

Assume firstly $m$ is odd, so $l$ is even, and set $s = l/2$. Let $\mathbb{P}^1 \to F_0$ be the normalization morphism and let $p, q$ be the points over the node. Let $x$ be the point of intersection of $f_*(B_a)$ and $\Gamma$ and let $y, z$ be distinct points in $\Gamma \cap R_a$. We may pick the points to ensure $y \neq x, z \neq x$. Define $B$ as the union of $B_a$ with $l + 2(k - 1)$ copies of $\mathbb{P}^1$ and with transversal intersections as in the following diagram:

![Diagram](image)

Then there is a unique unramified morphism $f : B \to Y_a$ with $f_*(B) = f_*(B_a) + (k - 1)R_a + (k - 1)\Gamma + lF_0$ which restricts to the normalization $\mathbb{P}^1 \to R_a$ on all components marked $R_{a,i}$, restricts to $f_a$ on $B_a$, sends all components marked $R_{a,j}$ to $F_0$: all components marked $\Gamma_i$ to $\Gamma$ and which takes points marked $x$ (resp. $y, z, p, q$) to $x$ (resp. $y, z, p, q$).

We now claim that if $B_0 \subseteq B$ is a connected union of components containing $R_{a,k-1}$ with $f_*(B_0) \in |nH|$ then $n = k$ and $B_0 = B$. If $c_1D + c_2F + c_3\Gamma$ is a divisor linearly equivalent to $nH$, then intersecting with $F$ shows $c_1 = n$. Now $R_a \in |D - 2F - \Gamma|$, whereas $H = D + (m - 1)F$ for $m \geq 0$. Thus one sees readily that $B_0$ must contain $\sum_{i=1}^{k-1}(\Gamma_i + R_{a,i}) + B_a$. We then get $n = k$ as required, which forces $B_0 = B$.

Using Remarks 3.4 and 3.12 one sees $h^0(N_f) \leq p(B)$. For any component $C \neq B_a \subseteq B$, $f(C)$ meets $f(B_a)$ properly. Thus it follows from Proposition 3.16 that the conditions of Proposition 3.8 are met. Note that the arithmetic genus of $B$ is $l_g$.

Now assume $m$ is even. Let $a, b, c \in f_a(B_a) \cap F_0$ be distinct points. Let $B$ be as in the diagram below.

![Diagram](image)
Then as before there is an unramified morphism \( f : B \to \mathcal{Y}_a \) with \( f_*(B) = f_a(B_a) + (k - 1)R_a + (k - 1)\Gamma + lF_0 \) satisfying the conditions of Proposition 3.8 and \( B \) has arithmetic genus \( l_g \). □

The following lemma will be needed for Theorem 4.11.

**Lemma 3.17.** Assume \( p(g, k) \) and \( n \) are such that there is a component \( I \subseteq T^a_{g,k} \) such that the morphism \( \eta_{l_i} : I \to \mathcal{M}_{p(g,k) - n} \) is generically finite. Then for the general \( [(f : C \to X, L)] \in I \), we have

\[ H^0(C, f^*(T_X)) = 0. \]

**Proof.** Let \( T^a_{g,k}(X, L) \) denote the fibre of \( T^a_{g,k} \to \mathcal{B}_g \) over \( [(X, L)] \). The finiteness of \( \eta_{l_i} \) at \( [(f : C \to X, L)] \) obviously implies that the morphism

\[ r_{n,k} : T^a_{g,k}(X, L) \to \mathcal{M}_{p(g,k) - n} \]

\[ [f : C \to X] \mapsto [D] \]

is finite near \([f]\). The claim \( H^0(C, f^*(T_X)) = 0 \) then follows from the exact sequence of sheaves on \( C' \)

\[ 0 \to T_C \to f^*(T_X) \to N_f \to 0 \]

and the fact that the coboundary morphism \( H^0(C, N_f) \to H^1(C, T_C) \) corresponds to the differential of \( r_{n,k} \). □

**Remark.** For \( g \geq 51, n \leq \frac{g^4 - 250}{4} \) then one has \( H^0(C, f^*(T_X)) = 0 \) for the general \( [(f : C \to X, L)] \in I \) in every component of \( \mathcal{V}^n_g \subseteq T^a_{g,n} \), by [20]. Similar bounds for the case \( k \geq 2 \) are also stated. In this paper, generic finiteness results for \( \mathcal{V}^n_{g,k} \to \mathcal{M}_{p(g,k) - n} \) were also claimed, the proof however seems flawed, see Remark 4.12.

### 4 The marked Wahl map

Recall the following definition from [51]: let \( V \) be any smooth projective variety, and let \( R \) be a line bundle on \( V \). Then there is a linear map, called the Gaussian:

\[ \Phi_R : \bigwedge^2 H^0(V, R) \to H^0(V, \Omega_V(R^2)) \]

\[ s \wedge t \mapsto sdt - tds. \]

In the case \( R = \omega_V \), this map is called the Wahl map. For \( V = C \) a smooth curve, and \( T \subseteq C \) a marking, we call \( \Phi_{\omega_C(-T)} \) the marked Wahl map. In this section we will use an approach inspired by [13] to study Gaussians in the case where \( V \) is a general curve and \( R \) is a twist of the canonical bundle.

We begin with the following lemma, which is a special case of [25 Lem. 3.3.1]:

**Lemma 4.1.** Let \( x_1, \ldots, x_n, y_1, \ldots, y_m \) be distinct, generic points of \( \mathbb{P}^2 \) and let \( d \) be a positive integer satisfying

\[ 3n + 6m < \frac{d^2 + 6d - 1}{4} - \left\lfloor \frac{d}{2} \right\rfloor. \]

Then there exists an integral curve \( C \subseteq \mathbb{P}^2 \) of degree \( d \) with nodes at \( x_i \), ordinary singular points of multiplicity 3 at \( y_j \) for \( 1 \leq i \leq n, 1 \leq j \leq m \) and no other singularities.
Let $C \subset \mathbb{P}^2$ be an integral curve of degree $d$ with nodes at $x_i$, ordinary singular points of multiplicity three at $y_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and no other singularities, as in the lemma above. Let $\pi : S \to \mathbb{P}^2$ be the blow-up at $x_1, \ldots, x_n, y_1, \ldots, y_m$, let $E_X$ be the sum of the exceptional divisors over $x_i$ for $1 \leq i \leq n$ and let $E_Y$ be the sum of the exceptional divisors over $y_j$ for $1 \leq j \leq m$. Denote by $D$ the strict transform of $C$, and let $T \subset D$ be the marking $E_X \cap D$. Note that $D$ is smooth, since all singularities are ordinary. Set $M = \mathcal{O}_S((d - 3)H - 2E_X - 2E_Y)$, where $H$ denotes the pull-back of the hyperplane of $\mathbb{P}^2$. Note that

$$K_D \sim (D + K_S)_D \sim (dH - 2E_X - 3E_Y) + (E_X + E_Y - 3H).$$

We therefore have the following commutative diagram

$$\begin{array}{ccc}
\wedge^2 H^0(S, M) & \xrightarrow{\Phi_M} & H^0(S, \Omega_S(M^2)) \\
\downarrow & & \downarrow \phi \\
\wedge^2 H^0(D, K_D(-T)) & \xrightarrow{W_{D,T}} & H^0(D, K_D^3(-2T)).
\end{array}$$

where $\Phi_M$ is the Gaussian, [51] §1 and $W_{D,T}$ is the marked Wahl map of $(D, T)$. Here $g$ denotes the composition of the natural maps $H^0(S, \Omega_S(M^2)) \to H^0(C, \Omega_S(M^2)_C)$ and $H^0(C, \Omega_S(M^2)_C) \to H^0(C, \Omega_C(M^2))$. We aim to show that $W_{D,T}$ is surjective. We will firstly show that $g$ is surjective. The main tool we will need is the Hirschowitz criterion, [28]:

**Theorem 4.2** (Hirschowitz). Let $p_1, \ldots, p_t$ be generic points in the plane, and assume $m_1, \ldots, m_t, d$ are nonnegative integers satisfying

$$\sum_{i=1}^t m_i(m_i + 1) < \left[ \frac{(d+3)^2}{4} \right].$$

Then $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes I_{p_1}^{m_1} \otimes \ldots \otimes I_{p_t}^{m_t}) = 0$.

**Lemma 4.3.** Let $M$ be as above and assume in addition $d \geq 8$, $3n + m < \left[ \frac{(d-5)^2}{4} \right]$. Then $H^1(S, \Omega_S((d-6)H - 2E_X - E_Y)) = 0$.

**Proof.** The relative cotangent sequence twisted by $(d - 6)H - 2E_X - E_Y$ gives

$$0 \to \pi^*\Omega_{p_2}((d-6)H - 2E_X - E_Y) \to \Omega_S((d-6)H - 2E_X - E_Y) \to \omega_{E_X}(2) \oplus \omega_{E_Y}(1) \to 0.$$ 

Thus it suffices to show

$$H^1(\mathbb{P}^2, \Omega_{p_2}(d-6) \otimes I_X^2 \otimes I_Y) = H^1(S, \pi^*\Omega_{p_2}((d-6)H - 2E_X - E_Y)) = 0$$

where $I_X = I_{x_1} \otimes \ldots \otimes I_{x_n}$ and $I_Y = I_{y_1} \otimes \ldots \otimes I_{y_m}$. Twisting the Euler sequence by $\omega_{p_2}$ gives a short exact sequence

$$0 \to \omega_{p_2} \to \mathcal{O}_{p_2}(-2)^{\oplus 3} \to \Omega_{p_2} \to 0.$$ 

As $H^2(\mathbb{P}^2, \omega_{p_2}(d-6) \otimes I_X^3 \otimes I_Y) = 0$ for $d \geq 7$, it suffices to show $H^1(\mathbb{P}^2, \mathcal{O}_{p_2}(d-8) \otimes I_X^3 \otimes I_Y) = 0$. This follows from Theorem 4.2 and the assumption $d \geq 8$, $3n + m < \left[ \frac{(d-5)^2}{4} \right]$. 

\[\square\]
Lemma 4.4. Let $M, D$ be as above and assume $d \geq 6$, $3n + m < \left\lfloor \frac{(d-5)^2}{4} \right\rfloor$. Assume further that $m \geq 10$. Then

$$H^1(D, \mathcal{O}_D((d-6)H - 2E_X - E_Y)) = 0.$$  

Proof. We have an exact sequence

$$0 \to \mathcal{O}_S(-6H + 2E_Y) \to \mathcal{O}_S((d-6)H - 2E_X - E_Y) \to \mathcal{O}_D((d-6)H - 2E_X - E_Y) \to 0.$$  

By the Hirschowitz criterion, $H^1(S, \mathcal{O}_S((d-6)H - 2E_X - E_Y)) = 0$, as we are assuming $d \geq 6$, $3n + m < \left\lfloor \frac{(d-3)^2}{4} \right\rfloor$. Thus it suffices to show $H^2(S, \mathcal{O}_S(-6H + 2E_Y)) = 0$. By Serre duality, $h^2(S, \mathcal{O}_S(-6H + 2E_Y)) = h^0(S, \mathcal{O}_S(3H + E_X - E_Y))$. We have

$$0 \to \mathcal{O}_S(3H - E_Y) \to \mathcal{O}_S(3H + E_X - E_Y) \to \mathcal{O}_{E_X}(-1) \to 0$$  

and so it suffices to show $H^0(S, \mathcal{O}_S(3H - E_Y)) = 0$. But $H^0(S, \mathcal{O}_S(3H - E_Y)) = H^0(\mathbb{P}^2, \mathcal{O}(3) \otimes I_Y) = 0$, since $m \geq 10$, and any ten general points do not lie on any plane cubic (as the space of plane cubics has dimension nine).

Corollary 4.5. Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be distinct, general points of $\mathbb{P}^2$ with $m \geq 10$ and let $d \geq 8$ be a positive integer satisfying

$$3n + 6m < \left\lfloor \frac{(d-5)^2}{4} \right\rfloor.$$  

Let $C \subseteq \mathbb{P}^2$ be an integral curve as in Lemma 4.4. Let $S \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at $x_1, \ldots, x_n, y_1, \ldots, y_m$, and let $D \subseteq S$ denote the strict transform of $C$. Then the map $g$ from Diagram (7) is surjective.

Proof. Note that $\left\lfloor \frac{(d-5)^2}{4} \right\rfloor < \frac{d^2 + 6d - 1}{4} - \left\lfloor \frac{d}{2} \right\rfloor$ for $d \geq 5$ so that such a curve $C$ exists. We have short exact sequences

$$0 \to \Omega_S((d-6)H - 2E_X - E_Y) \to \Omega_S(M^2) \to \Omega_S(M^2)|_D \to 0$$  

and

$$0 \to \mathcal{O}_D((d-6)H - 2E_X - E_Y) \to \Omega_S(M^2)|_D \to \mathcal{O}_D(M^2) \to 0.$$  

The map $f$ is the composition of the natural maps $H^0(S, \Omega_S(M^2)) \to H^0(S, \Omega_S(M^2)|_D)$ and $H^0(D, \Omega_S(M^2)|_D) \to H^0(D, \Omega_D(M^2))$, so the claim follows from lemmas [4,3] and [4,4].

We now wish to show that the Gaussian $\Phi_M$ from Diagram (11) is surjective. We start by recalling one construction of Gaussian maps from [51, §1]. Let $X$ be a smooth, projective variety, and $L \in \text{Pic}(X)$ a line bundle. Let $Y \to X \times X$ be the blow-up of the diagonal $\Delta$, and let $F$ denote the exceptional divisor. There is a short exact sequence of sheaves on $X \times X$

$$0 \to I^4_\Delta \to I_\Delta \to \Delta_* \mathcal{O}_X \to 0.$$  

Twisting the above sequence by $L \boxtimes L$ produces a short exact sequence

$$0 \to I^4_\Delta(L \boxtimes L) \to I_\Delta(L \boxtimes L) \to \Delta_* \mathcal{O}_X(L^2) \to 0$$  

25
and upon taking cohomology we get a map

$$\tilde{\Phi}_L : H^0(X \times X, I_{\Delta}(L \otimes L)) \to H^0(X, \Omega_X(L^2)).$$

Now $H^0(X \times X, I_{\Delta}(L \otimes L))$ may be identified with the kernel $\mathcal{R}(L, L)$ of the multiplication map $H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^2)$, and we have $\wedge^2 H^0(X, L) \subseteq \mathcal{R}(L, L)$ by sending $s \wedge t$ to $s \otimes t - t \otimes s$. Further, $\Phi_L$ is the restriction of $\tilde{\Phi}_L$ to $\wedge^2 H^0(X, L)$, and it is easily verified that both $\Phi_L$ and $\tilde{\Phi}_L$ have the same image in $H^0(X, \Omega_X(L^2))$. Thus, to verify the surjectivity of $\Phi_L$, it suffices to show

$$H^1(X \times X, I^3_\Delta(L \otimes L)) = H^1(Y, L_1 + L_2 - 2F) = 0,$$

where $L_1$ and $L_2$ denote the pull-backs of $L$ via the projections $pr_i : Y \to X \times X \to X$, for $i = 1, 2$.

Following [11, 13], we now wish to use the Kawamata–Viehweg vanishing theorem to show $H^1(Y, L_1 + L_2 - 2F) = 0$.

**Proposition 4.6 ([11]).** Let $X$ be a smooth projective surface, which is not isomorphic to $\mathbb{P}^2$. Assume $L \in \text{Pic}(X)$ is a line bundle such that there exist three very ample line bundles $M_1, M_2, M_3$ with $L - K_X \sim M_1 + M_2 + M_3$. Then the Gaussian $\Phi_L$ is surjective.

**Proof.** For a line bundle $A$ on $X$, we denote by $A_i \in \text{Pic}(X)$ the pullback via the projection $pr_i : Y \to X \times X \to X$, for $i = 1, 2$. By the above discussion, it suffices to show $H^1(Y, L_1 + L_2 - 2F) = 0$. As the diagonal $\Delta \subseteq X \times X$ has codimension two, we have $K_Y \simeq g^*K_X + F$. [27, Exercise II.8.5] Thus we see $H^1(Y, L_1 + L_2 - 2F) = H^1(Y, (L-K_X)_1 + (L-K_X)_2 - 3F + K_Y)$, and so by the Kawamata–Viehweg vanishing theorem it suffices to show $(L-K_X)_1 + (L-K_X)_2 - 3F$ is big and nef. Since we have

$$(L-K_X)_1 + (L-K_X)_2 - 3F = (M_{i_1} + M_{i_2} - F) + (M_{i_2} + M_{i_3} - F),$$

it suffices to show that $M_{i_1} + M_{i_2} - F$ is big and nef for $1 \leq i \leq 3$. Now $H^0(Y, M_{i_1} + M_{i_2} - F)$ is the kernel $\mathcal{R}(M_{i_1}, M_{i_2})$ of the multiplication map $H^0(X, M_{i_1}) \otimes H^0(X, M_{i_2}) \to H^0(X, M_{i_1}M_{i_2})$, and we have an injective map $\wedge^2 H^0(X, M_{i_1}) \hookrightarrow \mathcal{R}(M_{i_1}, M_{i_2})$ sending $s \wedge t$ to $s \otimes t - t \otimes s$. Thus $\wedge^2 H^0(X, M_{i_1})$ induces a sublinear system of $|M_{i_1} + M_{i_2} - F|$ which induces a rational map

$$\psi_1 : Y \to \text{Gr}(1, \mathbb{P}(H^0(M_{i_1})))$$

$$(x, y) \mapsto \phi_i(x) \phi_i(y)$$

where $\phi_i : X \to \mathbb{P}(H^0(M_{i_1}))$ is the embedding induced by $M_{i_1}$, and where $\phi_i(x) \phi_i(y)$ denotes the line through $\phi_i(x)$ and $\phi_i(y)$. By viewing $(x, y) \in F$ as a pair $x \in X$, $y \in T_{X,x}$, one sees that the map $\psi_1$ is in fact globally defined, and hence $M_{i_1} + M_{i_2} - F$ is nef. To see that it is big, it suffices to show that $\psi_1$ is generically finite, i.e. we need to show that there exist points $x, y \in X$ such that $\phi_i(X)$ does not contain the line $\phi_i(x) \phi_i(y)$. But if this were not the case $\phi_i(X)$ would be a linear space, contrary to the hypotheses. 

We now return to the situation of the blown-up plane. We start with the following:
Theorem 4.7 ([14]). Let \( p_1, \ldots, p_k \) be generic distinct points in the plane, and let \( \pi : S \to \mathbb{P}^2 \) be the blow-up. Let \( E \subseteq S \) be the exceptional divisor, and let \( H \) be the pull-back of the hyperplane class on \( \mathbb{P}^2 \). If we assume \( d \geq 5 \) and \( k + 6 \leq \frac{(d+1)(d-2)}{2} \), then \( dH - E \) is very ample on \( S \).

Putting everything together, we have the main result of this section:

**Proposition 4.8.** Let \( x_1, \ldots, x_n, y_1, \ldots, y_m \) be distinct, generic points of \( \mathbb{P}^2 \) with \( m \geq 10 \) and let \( d \geq 10 \) be a positive integer satisfying

\[
3n + 6m < \left\lfloor \frac{(d-5)^2}{4} \right\rfloor.
\]

Let \( C \subseteq \mathbb{P}^2 \) be an integral curve of degree \( d \) with nodes at \( x_i \), ordinary singular points of multiplicity 3 at \( y_j \) for \( 1 \leq i \leq n, 1 \leq j \leq m \) and no other singularities. Let \( S \to \mathbb{P}^2 \) denote the blow-up of \( \mathbb{P}^2 \), and let \( D \subseteq S \) denote the strict transform of \( C \). Then the marked Wahl map \( W_{D,T} \) is surjective, where \( T \) is the divisor over the nodes of \( C \). Furthermore, \( h^0(D, \mathcal{O}_D(T)) = 1 \).

**Proof.** We will firstly show that \( W_{D,T} \) is surjective. We have already seen in Corollary 4.3 that the map \( f \) from Diagram 1 is surjective. Thus it suffices to show that \( \Phi_M \) is surjective, where \( M = \mathcal{O}_S((d-3)H - 2E_X - 2E_Y) \). Since \( M - K_S \sim (d-6)H - 3E_X - 3E_Y \), and \( 3n + 6m < \left\lfloor \frac{(d-5)^2}{4} \right\rfloor \) implies \( n + m + 6 \leq \left\lfloor \frac{(d-5)(d-4)}{6} \right\rfloor \), then Theorem 4.7 shows that \( M - K_S \) may be written as a sum of three very ample line bundles. Thus Proposition 4.6 implies that the Gaussian \( \Phi_M \) is surjective.

For the second statement, note that the short exact sequence

\[
0 \to \mathcal{O}_S \to \mathcal{O}_S(E_X) \to \mathcal{O}_{E_X}(-1) \to 0
\]

gives that \( h^0(S, \mathcal{O}_S(E_X)) = 1 \). From the sequence

\[
0 \to \mathcal{O}_S(3E_X + 3E_Y - dH) \to \mathcal{O}_S(E_X) \to \mathcal{O}_D(T) \to 0
\]

it suffices to show \( H^1(S, \mathcal{O}_S(3E_X + 3E_Y - dH)) = 0 \). But \( dH - 3E_X - 3E_Y \) is a sum of three very ample line bundles, so it is big and nef, so that this follows from the Kawamata–Viehweg vanishing theorem.

As an immediate consequence we have:

**Theorem 4.9.** Fix any integer \( l \in \mathbb{Z} \). Then there exist infinitely many integers \( h(l) \), such that the general marked curve \( [(C, T)] \in \bar{M}_{h(l), 2l} \) has surjective marked Wahl map.

**Proof.** Consider the curve \( D \subseteq S \) from Proposition 4.8 applied to \( n = l \) and \( m = 10 \) (choose any \( d \) satisfying the hypotheses of the proposition). Let \( h(l) \) denote the genus of \( D \). In an open subset about \( [(D, T)] \in \bar{M}_{h(l), 2l} \), we have \( h^0(D, \mathcal{O}_D(T)) = 1 \) and thus \( h^0(D, K_D(T)) = \chi(K_D(T)) + 1 \) is locally constant. Further, the equality \( h^0(D, K_D^3(-2T)) = \chi(K_D^3(-2T)) \) holds, since \( \deg(K_D^3(-2T)) > 2h(l)-2 \). Thus the claim follows immediately from Proposition 4.8 and semicontinuity.

\( \square \)
Remark 4.10. In our example $(D,T)$, we have that $l$ is of the order $\frac{h}{5}$, where $h$ is the genus of $D$. Thus one would expect that the marked Wahl map of a general marked curve in $\bar{M}_{h,2l}$ is surjective, so long as $l$ is at most of the order $\frac{h}{5}$.

We will now study the marked Wahl map for curves arising via the normalization of nodal curves on K3 surfaces.

Theorem 4.11. Assume $g - n \geq 13$ for $k = 1$ or $g \geq 8$ for $k > 1$, and let $n \leq \frac{p(a,k) - 2}{8}$. Then there is an irreducible component $I^0 \subseteq \mathcal{V}_{g,k}^n$ such that for any $[(f : C \rightarrow X, L)] \in I^0$ the marked Wahl map $W_{C,T}$ is nonsurjective, where $T \subseteq C$ is the divisor over the nodes of $f(C)$.

Proof. Let $I \subseteq \mathcal{I}_{g,k}^n$ be the irreducible component from Theorem 3.10 (in the case $k = 1$) or Theorem 3.16 (in the case $k > 1$) and set $I^0 = I \cap \mathcal{V}_{g,k}^n$, which is nonempty from [8, Lemma 3.1]. Let $\pi : Y \rightarrow X$ be the blow up of the K3 surface $X$ at the nodes of $f(C)$. There is a natural closed immersion $C \subseteq Y$. Let $E \subseteq Y$ denote the sum of the exceptional divisors, and let $M = \mathcal{O}_Y(C)$. We have $K_C = (M + E)|_C$ by the adjunction formula. Consider the following commutative diagram:

$$
\begin{array}{cccc}
\wedge^2 H^0(Y, M) & \xrightarrow{\Phi_M} & H^0(Y, \Omega_Y(M^2)) & \\
\downarrow h & & \downarrow g & \\
\wedge^2 H^0(C, K_C(-T)) & \xrightarrow{W_{C,T}} & H^0(C, K_C^3(-2T)). & \\
\end{array}
$$

where the top row is a Gaussian on $Y$. The map $h$ is surjective, as $H^1(Y, \mathcal{O}_Y) = 0$. Suppose for a contradiction that the marked Wahl map $W_{C,T}$ were surjective. Then $g$ would be surjective, and hence the natural map

$$
H^0(C, \Omega_{\mathcal{V}|_C}(M^2)) \rightarrow H^0(C, K_C^3(-2T))
$$

would also be surjective. Now consider the short exact sequence

$$
0 \rightarrow M|_C \xrightarrow{} \Omega_{\mathcal{V}|_C}(M^2) \rightarrow K_C^3(-2T) \rightarrow 0.
$$

Since

$$
H^1(C, M) = H^1(C, K_C(-E)) = H^0(C, E) \neq 0,
$$

the surjectivity of $g$ would imply that $h^1(C, \Omega_{\mathcal{V}|_C}(M^2)) = h^0(C, \mathcal{T}_{\mathcal{V}|_C}(2E - K_C)) \neq 0$. However

$$
H^0(C, \mathcal{T}_{\mathcal{V}|_C}(2E - K_C)) \subseteq H^0(C, f^*(T_X)(2E - K_C)) \subseteq H^0(C, f^*(T_X)) = 0
$$

from Lemma 3.17 and since $K_C - 2E$ is effective for $n \leq \frac{p(a,k) - 2}{8}$. So this is a contradiction and hence $W_{C,T}$ is nonsurjective. \qed

Remark 4.12. In the paper [26], claims are made about the generic finiteness of $\eta : \mathcal{V}_{g,k}^n \rightarrow \mathcal{M}_{p(g,k) - n}$ and the nonsurjectivity of the Wahl map for curves parametrized by the image of $\eta$. The proof of the first statement, [26 Theorem 3.1], seems flawed to us. Indeed, the statement in Step 1 that $s = (s_0, 0)$ is trivial, as $s$ defines the splitting. The conclusion in Step 2 that family (3.1) is trivial in a small analytic open subset seems likewise rather obvious, but in any case does not have the consequence claimed. The proof of Theorem 4.1 also seems incorrect. Namely, the last row in diagram in (A.1) should be twisted by $-2E$, but then Lemma A.2 fails.

\footnote{This was pointed out to us by Stefan Schreieder.}
5 Brill–Noether theory for nodal curves on K3 surfaces

In this section we consider two related questions on the Brill–Noether theory of nodal curves on a K3 surface. Let $D \subseteq X$ be a nodal curve on a K3 surface, and let $C := \tilde{D}$ be the normalization of $D$. In the first part, we consider the Brill–Noether theory of the smooth curve $C$, whereas in the second part we consider the Brill–Noether theory for the nodal curve $D$.

5.1 Brill–Noether theory for smooth curves with a nodal model on a K3 surface

In this section we will apply an argument from [37] to the K3 surface $S_{p,h}$ as in Lemma 3.9 in order to study the Brill–Noether theory for smooth curves with a primitive nodal model on a K3 surface.

**Lemma 5.1.** Consider the K3 surface $S_{p,h}$ as in Lemma 3.9. There is no expression $M = A_1 + A_2$, where $A_1$ and $A_2$ are effective divisors with $h^0(Y, \mathcal{O}(A_1)) \geq 2$ and $h^0(Y, \mathcal{O}(A_2)) \geq 2$.

**Proof.** We first claim that any effective divisor of the form $D = aR_1 + bR_2$, for integers $a, b$, must have $a, b \geq 0$. Suppose for a contradiction that $a < 0$. Clearly we must have $b > 0$. Thus there is some integral component $D_1$ of $D$ with $(D_1 \cdot R_2) < 0$, as $(D \cdot R_2) = -2b < 0$. Thus $D_1 \sim R_2$. Repeating this argument on $D - R_2$, we see that $bR_2$ is a summand of $D$. But then $D - bR_2 = aR_1$ is effective, which is a contradiction as $a < 0$. Thus $a \geq 0$. Likewise $b \geq 0$.

Furthermore, this argument also shows that all integral components of any effective divisor of the form $D = aR_1 + bR_2$ are linearly equivalent to either $R_1$ or $R_2$. In particular, $D$ is rigid.

Suppose $M = A_1 + A_2$ is an expression as above. Write $A_1 = x_1M + \sum_{i=1}^{2} y_{1,i}R_i$ and $A_2 = x_2M + \sum_{i=1}^{2} y_{2,i}R_i$ for integers $x_i, y_{i,j}$ for $i = 1, 2$, $1 \leq j \leq 2$. We have $x_1, x_2 \geq 0$ by Lemma 3.9 and $x_1 + x_2 = 1$, and assume $x_1 \geq x_2$. Thus we must have $x_2 = 0$, which gives $h^0(Y, \mathcal{O}(A_2)) \leq 1$ (as the divisor $\sum_{i=1}^{2} y_{2,i} \Gamma_i$ is rigid if $y_{2,i} \geq 0$ for all $i$ and not effective if there is some $j$ with $y_{2,j} < 0$).

Let $C \subseteq X$ be a smooth curve on a K3 surface $X$. Let $M \in \text{Pic}(C)$ be a globally generated line bundle such that $\omega_C \otimes M^*$ is also globally generated. We denote by $F_{C,M}$ the vector bundle on $X$ defined as the kernel of the evaluation map $H^0(C, M) \otimes_{\mathcal{O}} \mathcal{O}_X \rightarrow M$. Let $G_{C,M}$ be the dual bundle of $F_{C,M}$, this is globally generated from the exact sequence

$$0 \rightarrow H^0(M) \otimes \mathcal{O}_X \rightarrow G_{C,M} \rightarrow \omega_C \otimes M^* \rightarrow 0$$

(using $H^1(\mathcal{O}_X) = 0$). The following generalization of [37] Lemma 1.3 is well-known, see [19] Remark 3.1.

**Lemma 5.2.** In the above situation, assume further that there is no expression $\mathcal{O}(C) \cong L_1 \otimes L_2$, where $L_1$ and $L_2$ are effective line bundles on $X$ with $h^0(L_i) \geq 1 + s_i$ for $i = 1, 2$, where $s_i \geq 1$ are integers satisfying $s_1 + s_2 = h^0(C, A)$. Then $F_{C,M}$ is a simple vector bundle.
Proof. We follow the proof of [30, Ch.7, Prop.2.2]. The bundle $F_{C,M}$ is simple if and only if its dual $G_{C,M}$ is simple. Suppose $G_{C,M}$ were not simple. Then there would exist a nontrivial endomorphism $\psi : G \to G$ with nontrivial kernel. Set $K := \ker(\psi)$, $L_1 := \det(K)$ and $L_2 := \det((G/K)/T)$, where $T$ is the maximal torsion subsheaf of $G/K$. Clearly $s_1 \geq 1$ for $i = 1, 2$ and $s_1 + s_2 = \text{rank } (G) = h^0(C, M)$. So it suffices to prove $h^0(L_i) \geq 1 + s_i$ for $i = 1, 2$ (as $c_1(T)$ is effective). As is explained in [30, Sec.7, Prop.2.2], if we pick a sufficiently positive divisor $D$ on $X$ we have $h^0(D, L_i|_D) \geq s_i + 1$. On the other hand, if $D$ is sufficiently positive then $D - L_i$ is big and nef, so that $H^0(X, L_i) \to H^0(D, L_i|_D)$ and thus $h^0(L_i) \geq 1 + s_i$ for $i = 1, 2$.

**Corollary 5.3.** Consider a K3 surface $S_{p,h}$ as in Lemma 5.2. Let $C \in |M|$ be a smooth curve. Then $C$ is Brill–Noether–Petri general.

**Proof.** This follows immediately from the proof of the main theorem in [37] and the above lemma.

Putting all the pieces together, we get the following result.

**Proposition 5.4.** Assume $g - n \geq 8$. Then there exists an irreducible component $J \subseteq \mathcal{V}_g$ such that for $[(f : D \to X, M)] \in J$ general, $D$ is Brill–Noether–Petri general.

**Proof.** Set $h = g - n$, $p = g$. The case $n = 0$ is [37], so we may assume $p > h$. Let $l, m$ be the unique nonnegative integers such that

$$p - h = \left\lfloor \frac{h + 1}{2} \right\rfloor l + m$$

and $0 \leq m < \left\lfloor \frac{h + 1}{2} \right\rfloor$. Set $\epsilon = 1$ if $m = 0$ or $m = \left\lfloor \frac{h + 1}{2} \right\rfloor - 1$ and $\epsilon = 0$ otherwise. Then $(M + R_1 + \epsilon R_2)^2 = 2g - 2$, where $M, R_1, R_2$ are a basis of $P_{p,h}$ as in Lemma 5.3. The claim then follows from the proof of Theorem 5.10 by deforming to the curve

$$R := D \cup R_1 \cup \epsilon R_2$$

on $S_{p,h}$, where $D \in |M|$ is general, marked at all nodes other than one point from $D \cap R_i$ for $i = 1, 2$. Note that the partial normalization of $R$ at the marked nodes is an unstable curve, and the stabilization is isomorphic to $D$, which is Brill–Noether–Petri general by Corollary 5.3.

### 5.2 Brill–Noether theory for nodal rational curves on K3 surfaces

In this section we will denote by $X$ a K3 surface with $\text{Pic}(X) \cong \mathbb{Z}L$, $(L)^2 = 2g - 2$ with $g \geq 2$, and $C \in |L|$ will denote a fixed rational curve (not necessarily nodal). Let $\bar{J}^d(C)$ denote the compactified Jacobian of degree $d$, rank one torsion free sheaves and consider the **generalized Brill–Noether loci**

$$\overline{W}_d(C) := \{ A \in \bar{J}^d(C) \text{ with } h^0(A) \geq r + 1 \}$$

which can be given a determinantal scheme structure, see [5]. There is an open subset $W_d(C) \subseteq \overline{W}_d(C)$ parametrizing line bundles. We will denote by $\rho(g, r, d)$ the Brill–Noether number $g - (r + 1)(g - d + r)$.

The following comes from the proof of [5, Remark 2.3(i)] (although it may have been known to experts earlier). The proof is essentially the same as in the smooth case.
**Theorem 5.5.** Each irreducible component of $\overline{W}_d(C)$ has dimension at least $\rho(g, r, d)$.

In the case $\rho(g, r, d) > 0$, $\overline{W}_d(C)$ is nonempty, [5] Thm. 3.1. If $\rho(g, r, d) > 0$, then under our hypotheses $\overline{W}_d(C)$ is connected, [23] Thm. 1.

Let $V_{d,r} \subseteq \overline{W}_d(C)$ be the open locus parametrizing sheaves $A$ which are globally generated and with $h^0(A) = r + 1$. Assume $V_{d,r} \neq \emptyset$. We will begin by proving that $\dim V_{d,r} \leq \rho(g, r, d)$ (in particular $V_{d,r} = \emptyset$ if $\rho(g, r, d) < 0$).

Fix a vector space $\mathbb{H}$ of dimension $r + 1$ and let $P^r_d \to V_{d,r}$ parametrize pairs $(A, \lambda)$ where $A \in V_{d,r}$ and $\lambda$ is a surjection of $\mathcal{O}_X$ modules
\[ \lambda : \mathbb{H} \otimes \mathcal{O}_X \to A \]
inducing an isomorphism $\mathbb{H} \cong H^0(A)$. Two such surjections are identified if they differ by multiplication by a nonzero scalar. Thus $P^r_d$ is a PGL$(r+1)$ bundle over $V_{d,r}$.

Let $(A, \lambda) \in P^r_d$. Then $\text{Ker} \lambda$ is a vector bundle $F$ of rank $r + 1$, [23] §3.2. We have $\det(F) \cong L^*$, $\deg(c_2(F)) = d$, $h^0(F) = h^1(F) = 0$ and $h^2(F) = r + 1 + (g - d + r)$, cf. [37] §1. Note that $g - d + r = h^1(A) \geq 0$. Further, for any rank one, torsion-free sheaf $A$ on $C$ we may define an ‘adjoint’ $A^1$, which is a rank one torsion-free sheaf with $(A^1)^* = A$. From the short exact sequence
\[ 0 \to F \to \mathbb{H} \otimes \mathcal{O}_X \to A \to 0 \]
we may form the dual sequence
\[ 0 \to \mathbb{H}^* \otimes \mathcal{O}_X \to F^* \to A^1 \to 0. \]

The following lemma is a slight generalization of [30] Cor. 9.3.2:

**Lemma 5.6.** Assume $\text{Pic}(X) \cong \mathbb{Z}L$ as above and let $(A, \lambda) \in P^r_d$. Then the vector bundle $F = \text{Ker} \lambda$ is stable.

**Proof.** For any vector bundle $H \subseteq \mathcal{O}_X^{\oplus b}$ and any integer $s \geq 1$, we have $h^0(\wedge^s H^*) \geq 1$. Indeed, we have $\wedge^s H \subseteq \wedge^s \mathcal{O}_X^{\oplus b} \cong \mathcal{O}_X^{\oplus \tilde{b}}$ for some integer $\tilde{b}$ and then $\text{End}\mathcal{O}_X(\wedge^s H) \cong \wedge^s H \otimes \wedge^s H^* \subseteq (\wedge^s H^*)^{\oplus \tilde{b}}$. Taking global sections gives $h^0(\wedge^s H^*) \geq 1$ (as $id \in H^0(\text{End}\mathcal{O}_X(\wedge^s H))$).

Now let $F' \subseteq F \subseteq \mathbb{H} \otimes \mathcal{O}_X$ be a locally free subsheaf of $F$ with $rk(F') = r' < r + 1$. From the above, $h^0(det(F'^*)) \geq 1$ and $h^0(\wedge^{r'-1} F'^*) \geq 1$ (if $r' > 1$). As $\text{Pic}(X) \cong \mathbb{Z}L$, we have $det(F') = kL^*$ for some $k \geq 0$. We claim $k = 0$. If $k = 0$ and $r' = 1$, then $F' \cong \mathcal{O}_X$ which contradicts that $h^0(F) = 0$. If $k = 0, r' > 0$, then $F' \cong \wedge^{r'-1} F'^* \otimes det(F')$ gives $h^0(F') \geq 1$ which is again a contradiction. So we have $det(F') = kL^*$ for $k > 0$ and $det(F) = L^*$, which implies $\deg(F')/rk(F') < \deg(F)/rk(F)$ as required.

Let $M_v$ be the moduli space of stable sheaves on $X$ with Mukai vector $v = (r+1, L, g-d+r)$. We have a morphism
\[ \psi_C : P^r_d \to M_v \]
\[ (A, \lambda) \mapsto (\text{Ker} \lambda)^* \]
where $(\text{Ker} \lambda)^*$ denotes the dual bundle to $\text{Ker} \lambda$. Let $M_C$ be the closure of the image of $\psi_C$, with the induced reduced scheme structure. By the description of $F$, if $[F^*] \in \text{Im}(\psi_C)$,
\[ c_2(F^*) \sim d c_X \]

31
where \( c_X \) is the rational equivalence class of a point lying on a rational curve as defined in [44].

There is a natural symplectic form \( \alpha \) on \( M_v \) defined in [40].

**Proposition 5.7.** Let \( \alpha \) be the natural symplectic form on \( M_v \) and \( i : M_C^0 \to M_v \) the inclusion, where \( M_C^0 \) is the smooth locus of \( M_C \). Then \( i^*\alpha = 0 \).

**Proof.** Since \( g - d + r \geq 0 \), [45] Thm. 0.6(1) applies and for any \([G] \in M_v\), there is an effective, degree \( \rho(g,r,d) \) zero-cycle \( Z \) with \( c_2(G) \sim [Z] + a_cX \) for some \( a \in \mathbb{Z} \). Following [45] Prop. 1.3 there is then a smooth quasi-projective variety \( \tilde{M}_v \) with morphisms \( q : \tilde{M}_v \to M_v \), \( p : \tilde{M}_v \to X^{[\rho(g,r,d)]} \) such that \( q \) is surjective and generically finite, and with the property that if \( x = [F^*] \in \text{Im}(\psi_C) \) and \( y \in q^{-1}(x) \), then we have the rational equivalence

\[
p(y) + (d - \rho(g,r,d))c_X \sim c_2(F^*) \sim dc_X.
\]

Further if \( \beta \) is the symplectic form on the Hilbert scheme of points \( X^{[\rho(g,r,d)]} \), we have \( q^*\alpha = kp^*\beta \) for some nonzero constant \( k \in \mathbb{C} \). Let \( \tilde{M}_C \subseteq \tilde{M}_v \) denote the preimage of \( M_C \) under \( q \) with the induced reduced scheme structure, and let \( p_C \) respectively \( q_C \) be the restriction of \( p \) respectively \( q \) to the smooth locus of \( \tilde{M}_C \). Then \( p_C(x) \) is rationally equivalent to \( p_C(y) \) for all \( x \) and \( y \), from (2). Thus \( p_C(\beta) = 0 \) by [44]. Hence \( q_C(\alpha) = 0 \) and since \( q \) is surjective, \( i^*\alpha = 0 \).

**Corollary 5.8.** We have \( \dim M_C \leq \rho(g,r,d) \).

**Proof.** Indeed \( \dim M_v = 2\rho(g,r,d) \) from [40] Thm. 0.1 so this follows from the proposition above.

**Corollary 5.9.** If \( V_{d,r} \) is nonempty then \( \dim V_{d,r} = \rho(g,r,d) \).

**Proof.** If \( V_{d,r} \) is nonempty then \( \dim V_{d,r} \geq \rho(g,r,d) \) by Theorem 5.5 so it suffices to show \( \dim V_{d,r} \leq \rho(g,r,d) \). It then suffices to show that \( \psi_C : P_d^r \to M_v \) has fibres of dimension \( \dim PGL(r+1) \). In other words, we need to show that for each fixed \( F \in M_v \) there are only finitely many \( A \in V_{d,r} \) fitting into an exact sequence \( 0 \to F \to H \otimes O_X \to A \to 0 \). But this follows immediately from the fact that in our circumstances the degeneracy locus map \( Gr(r+1,H^0(F^*)) \to |L| \) is globally defined and finite, see [45] §2] (recall that all such \( A \) are supported on a fixed \( C \) by definition).

**Remark.** Assume \( V_{d,r} \) is nonempty. We have \( \dim M_C = \dim V_{d,r} = \rho(g,r,d) \) from the above Corollary. Thus \( M_C \) is a (possibly singular) Lagrangian subvariety of \( M_v \).

**Corollary 5.10.** Let \( X \) be a K3 surface with \( \text{Pic}(X) \cong \mathbb{Z}L \) and \( (L \cdot L) = 2g - 2 \). Let \( C \in |L| \) be rational and assume \( \rho(g,r,d) < 0 \). Then

\[
\overline{W}_d(C) := \{ A \in \bar{J}_d(C) \text{ with } h^0(A) \geq r + 1 \}
\]

is empty.

**Proof.** Assume for a contradiction that \( A \in \overline{W}_d(C) \). Let \( A' \) be the image of the evaluation morphism \( H^0(A) \otimes O_C \to A \). Then \( A' \) is a globally generated, torsion free, rank one sheaf of degree \( d' \leq d \) with \( r' + 1 \geq r + 1 \) sections, and thus \( A' \in V_{d',r'} \). But \( \rho(g,r',d') \leq \rho(g,r,d) < 0 \) for \( d' \leq d, \ r' \geq r \) and thus \( V_{d',r'} \) is empty by Corollary 5.9. This is a contradiction.
To proceed we need two technical lemmas.

**Lemma 5.11.** Let \( C \) be an arbitrary integral nodal curve. Suppose \( A' \) is a rank one torsion free sheaf on \( C \) and let \( k(p) \) be the length one skyscraper sheaf on \( C \) supported at a node \( p \in C \). Then if \( Z \subseteq \overline{W}_d(C) \) is an irreducible family of rank one torsion free sheaves such that we have an exact sequence

\[
0 \to A' \to A \to k(p) \to 0
\]

for all \( A \in Z \), then \( \dim Z \leq 1 \).

**Proof.** It suffices to show \( \dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), A') \leq 2 \). We have

\[
\text{Ext}^1_{\mathcal{O}_C}(k(p), A') \cong \text{Ext}^1_{\mathcal{O}_C}(k(p), A'(n)) \quad \text{for any } n \in \mathbb{Z}
\]

\[
\cong H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(k(p), A'(n))) \quad \text{for } n \gg 0
\]

\[
\cong H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(k(p), A'))
\]

where the second line follows from [27, Prop III.6.9]. The sheaf \( \mathcal{E}xt^1_{\mathcal{O}_C}(k(p), A') \) is a skyscraper sheaf supported at \( p \). If \( A' \) is nonsingular at \( p \) then

\[
\dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), A') = \dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), \omega_C) = 1,
\]

by Serre duality.

Suppose now \( A' \) is singular at \( p \) and let \( \pi : C' \to C \) be the normalization of \( C \). Then \( \dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), A') = \dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), \pi_*(\mathcal{O}_{C'})) \). Since \( A'_p \cong m_p \cong \pi_*(\mathcal{O}_{C'})_p \) where \( m_p \) is the maximal ideal of \( p \), by [18, III.1][4]. But \( \dim_C \text{Ext}^1_{\mathcal{O}_C}(k(p), \pi_*(\mathcal{O}_{C'})) = 2 \) as required, by [3, Prop. 2.3]. \( \square \)

**Lemma 5.12.** Let \( C \) be an arbitrary integral nodal curve. Suppose \( A' \) is a rank one torsion free sheaf and let \( Q \) be a sheaf with zero-dimensional support such that \( \text{supp}(Q) \subseteq C_{\text{sing}} \), where \( C_{\text{sing}} \) is the singular locus of \( C \). Then if \( Z \subseteq \overline{W}_d(C) \) is an irreducible family of rank one torsion free sheaves such that we have an exact sequence

\[
0 \to A' \to A \to Q \to 0
\]

for all \( A \in Z \), then \( \dim Z \leq l(Q) \), where \( l(Q) \) denotes the length of \( Q \).

**Proof.** We will prove the result by induction on \( l(Q) \). When \( l(Q) = 1 \) the result holds from Lemma [5.11][4]. Suppose \( Q \) has length \( r \) and choose a sheaf \( Q' \) with zero-dimensional support and length \( r - 1 \) such that we have a surjection \( \phi : Q \to Q' \). For any

\[
0 \to A' \to A \to Q \to 0,
\]

\( \phi \) then induces a short exact sequence

\[
0 \to A'' \to A \to Q' \to 0
\]

\[\text{Note that } m_p \text{ is a degree } -1, \text{ rank one t.f. sheaf, and } \pi_*(\mathcal{O}_{C'}) \text{ is a t.f. sheaf of strictly positive degree, so although these sheaves are locally isomorphic, they are not globally isomorphic.} \]
where $A''$ fits into the exact sequence

$$0 \to A' \to A'' \to \text{Ker}(\phi) \to 0.$$ 

Now let $\pi: T \to Z$ be the moduli space with fibre over $A \in Z$ parametrising all extensions $0 \to A' \to A \to Q \to 0$; this can be constructed from [33, §4.1]. After replacing $T$ with an open set we have a morphism $\psi: T \to \overline{W}_d^0(C)$ for some $d'$, $r'$, which sends a point representing the exact sequence $0 \to A' \to A \to Q \to 0$ to $A'' := \text{Ker}(A \to Q')$. By Lemma 5.11 the image of $\psi$ is at most one dimensional as $l(\text{Ker}(\phi)) = 1$. Further, $\pi(\psi^{-1}(A''))$ is at most $l(Q') = l(Q) - 1$ dimensional for any $A'' \in \text{Im}(\psi)$ by the induction hypothesis. It then follows that $\dim Z \leq l(Q)$ as required. \hfill \square

**Lemma 5.13.** Let $C$ be an integral, nodal curve. Then

$$\overline{W}_d^0(C) := \{ A \in \tilde{J}^d(C) \text{ with } h^0(A) \geq 1 \}$$

is irreducible of dimension $\rho(g, 0, d) = d$.

**Proof.** Let $U_d \subseteq \overline{W}_d^0(C)$ be the open subset consisting of line bundles. Let $V_d := \text{Div}_d(C)$ denote the scheme parametrizing zero-dimensional schemes $Z \subseteq C$ such that the ideal sheaf $I_Z$ is invertible of degree $d$; i.e $V_d$ is the scheme of effective Cartier divisors. Let $\tilde{C} \to C$ denote the normalization. From [33, Thm. 2.4], pullback induces a birational morphism $V_d \to \text{Div}_d(\tilde{C})$, and thus $\dim V_d = d$. We have a morphism $V_d \to \tilde{J}^d(C)$ with image $U_d$, which sends a scheme $Z$ to the effective line bundle $I_Z^\perp$. Thus $U_d$ of dimension at most $d$. Since each component of $\overline{W}_d^0(C)$ has dimension at least $d$ by Theorem 5.5, we see $\dim(U_d) = d$.

Let $I$ be an irreducible component of $\overline{W}_d^0(C) \setminus U_d$; we need to prove $\dim(I) < d$. There is a nonempty open set $I^0$ of $I$, an integer $d' < d$ and a partial normalization $\mu: C' \to C$ such that for each $A \in I^0$ there exists a unique effective line bundle $B \in \text{Pic}^{d'}(C')$ with $\mu_*(B) \simeq A$ by $[22]$ Prop. 3.4. Since the dimension of the moduli space of effective line bundles of degree $d'$ on $C'$ has dimension $d'$ by the above, we see that $\dim(I) \leq d' < d$. \hfill \square

We now prove the main result of this section.

**Theorem 5.14.** Let $X$ be a K3 surface with $\text{Pic}(X) \simeq \mathbb{Z}L$ and $(L \cdot L) = 2g - 2$. Suppose $C \in |L|$ is a rational, nodal curve. Then

$$\overline{W}_d^0(C) := \{ A \in \tilde{J}^d(C) \text{ with } h^0(A) \geq r + 1 \}$$

is either empty or is equidimensional of the expected dimension $\rho(g, r, d)$.

**Proof.** By Corollary 5.10 the theorem holds whenever $\rho(g, r, d) < 0$. Thus it suffices to prove the theorem for $\rho(g, r, d) \geq -1$. We will proceed by induction on $\rho(g, r, d)$ starting from the case $\rho(g, r, d) = -1$.

Choose nonnegative integers $r, d$ and suppose the claim holds for all $r', d'$ such that $\rho(g, r', d') < \rho(g, r, d)$. We know the claim holds for $r = 0$ by Lemma 5.13, so we may suppose $r > 0$. Let $I$ be an irreducible component of $\overline{W}_d^0(C) \setminus V_{d,r}$; from Theorem 5.5 it suffices to show $\dim(I) < \rho(g, r, d)$. For all $A \in I$, we denote by $A'$ the globally generated part of $A$, and so on.
i.e. the image of the evaluation morphism $H^0(A) \otimes \mathcal{O}_C \to A$. There is an open dense subset $I^0 \subseteq I$ such that $\deg(A') = d'$, $h^0(A') = r'$ is constant for all $A \in I^0$. Replacing $I^0$ by a smaller open set if necessary, we have a morphism

$$f : I^0_{red} \to \overline{W^r_{d'}(C)}$$  
$$A \mapsto A'.$$

Indeed, let $S$ be an integral, locally Noetherian scheme over $\mathbb{C}$, let $\pi : C \times S \to S$ be the projection, and let $A$ be an $S$ flat family of rank one torsion free sheaves $A_s$ on $C$, with $\deg(A_s) = d$, $h^0(A_s) = r' + 1$ constant. Replacing $S$ with an open subset, we may assume that $\pi_*A$ is a trivial vector bundle of rank $r' + 1$, and that the image $A'$ of the evaluation morphism

$$H^0(A) \otimes \mathcal{O}_{C \times S} \to A$$

is flat over $S$. Replacing $S$ with another open subset, we may further assume $\pi_*A'$ is a trivial vector bundle of rank $r' + 1$. We claim that $A'_s$ is the base-point free part of $A_s$. Let $B_s$ denote the base point free part of $A_s$. The surjection $H^0(A_s) \otimes \mathcal{O}_C \to A'_s$ shows $A'_s \subseteq B_s \subseteq A$. Then the exact sequence

$$0 \to A'_s \to B_s \to F \to 0,$$

where $F$ has zero-dimensional support, and the equality $h^0(A'_s) = h^0(B_s) = r + 1$ implies $F$ is the zero sheaf (since $B_s$ is base point free). Thus if $d' := \deg(A'_s)$, $A'$ is a flat family of rank one, torsion free sheaves on $C$ of degree $d'$ with $r' + 1$ sections, so the universal property of $\overline{W^r_{d'}(C)}$ induces a morphism $S \to \overline{W^r_{d'}(C)}$.

We next claim that $f$ has fibres of dimension at most $d - d'$. This will then imply the result as $\rho(g, r', d') \leq \rho(g, r, d') = \rho(g, r, d) - (r + 1)(d - d')$ so that $\dim(I^0) < \rho(g, r, d)$ for $r \neq 0$. For any $A \in I^0$ we have a sequence

$$0 \to f(A) \to A \to Q_A \to 0$$

where $Q_A$ has zero-dimensional support. We have a canonical decomposition $Q_A = Q_{A_{\text{sm}}} \oplus Q_{A_{\text{sing}}}$ with $\text{Supp}(Q_{A_{\text{sm}}}) \subseteq C_{\text{sm}}$ and $\text{Supp}(Q_{A_{\text{sing}}}) \subseteq C_{\text{sing}}$, where $C_{\text{sm}}$ is the smooth locus of $C$ and $C_{\text{sing}} = C - C_{\text{sm}}$. Replacing $I^0$ with a dense open set we may assume $Q' := Q_{A_{\text{sing}}}$ is independent of $A \in I^0$. Let $e := l(Q')$. For any $A \in I^0$, there is a unique effective line bundle $M$ of degree $d - d' - e$ such that we have a short exact sequence

$$0 \to f(A)(M) \to A \to Q' \to 0.$$  

We have a morphism

$$g : I^0 \to \overline{W^r_{d'-e}(C)}$$  
$$A \mapsto f(A)(M).$$

By Lemma 5.12, $g$ has fibres of dimension at most $e$. For any $A'$ in the image of $f$ consider

$$g|_{f^{-1}(A')} : f^{-1}(A') \to \overline{W^r_{d'-e}(C)}.$$  

35
The image of $g|_{f^{-1}(A')}$ is a subset of the space of tuples $A' \otimes M$ for $M \in \text{Pic}^{d-d'-(d-e)}(C)$ effective. The moduli space of effective line bundles in $\text{Pic}^{d-d'-(d-e)}(C)$ may be identified with the image of the natural map $C_{\text{sm}}^{(d-d'-(d-e))} \to \text{Pic}^{d-d'-(d-e)}(C)$, where $C_{\text{sm}}$ is the smooth locus of $C$, and thus has dimension at most $d - d' - e$. Thus $\dim f^{-1}(A') \leq d - d'$ as required.

Remark. It is clear from the proof that the theorem would hold for any constant cycle curve $C \in |L|$ such that $C$ is integral and nodal, see [29].

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37
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