Spatial Concentration of Caching
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Abstract

We propose a decentralized caching policy for wireless networks that makes content placement decisions based on pairwise interactions between cache nodes. We call our proposed scheme γ-exclusion cache placement (GEC), where a parameter γ controls an exclusion radius that discourages nearby caches from storing redundant content. GEC takes into account item popularity and the nodes’ caching priorities and leverages negative dependence to relax the classic 0-1 knapsack problem to yield spatially balanced sampling across caches. We show that GEC guarantees a better concentration (reduced variance) of the required cache storage size than the state of the art, and that the cache size constraints can be satisfied with high probability. Given a cache hit probability target, we compare the 95% confidence intervals of the required cache sizes for three caching schemes: (i) independent placement, (ii) hard exclusion caching (HEC), and (iii) the proposed GEC approach. For uniform spatial traffic, we demonstrate that GEC provides approximately a 3x and 2x reduction in required cache size over (i) and (ii), respectively. For non-uniform spatial traffic based on realistic peak-hour variations in urban scenarios, the gains are even greater.

I. INTRODUCTION

Distributed caching is a powerful technique to minimize network latency [2], and to enable spectral reuse and throughput gain in networks [3]. The primary goal of cache placement in a wireless network is to maximize the cache hit probability, meaning the probability that a node in the network can obtain a desired item from a nearby cache within radio range. This then eliminates the need for the network, for example a base station or other form of infrastructure, to fetch and broadcast the content. The cache hit probability is affected by quantities such as the demand distribution, network topology, communication range, the cache storage size, and the topic of this paper, which is the policy for populating these small caches with content.

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A. Motivation and Objectives

Our focus is to devise a decentralized caching policy by exploiting the spatial distribution of wireless caches. We can attain a target cache hit rate performance by trading off the local cache space with the spatial diversity of the caching. The aggregate cache size that a user has access to grows with the diversity across the network. This can ensure that different demands can be satisfied with multiple caches, yielding an improved caching performance with the diversity.

We assume that the nodes are equipped with fixed size caches and are randomly and independently located. A baseline approach to placement is to independently populate the caches by capturing the popularity of the files. However, this approach may not perform well for the tail of the demand distribution when the number of files is large or when the spatial traffic is non-uniform. One way to improve on such a policy is to assume that a receiver can obtain a desired item as long as it is cached within its communication range, and devise a joint cache placement strategy across nodes so as to minimize the chance of cache misses. This will clearly outperform the independent placement policy, but it may be very complex to devise a spatial cache placement model that captures the joint interactions of all nodes.

In our prior work [4], which provides a baseline for the current paper, we considered a pairwise interaction model between nodes to make placement decisions, referred to as hard exclusion caching (HEC). HEC is based on Matérn hard-core process of type II (MatII) which is less regular than a lattice but more regular than a Poisson Point Process (PPP). More specifically in MatII, a node from the baseline process is retained – that is, selected to cache an item – only if there is no other node within a deterministic exclusion radius. The MatII makes binary placement decisions: it caches an item with probability 1 if the node separation is larger than the exclusion radius, else 0. Since each item has a different popularity, the exclusion radius is chosen to be a decreasing function of an item’s popularity, which introduces pairwise interactions between caches. While HEC captures node interactions, the pairwise placement decisions can eliminate caching a desired item within the communication range, rendering HEC sub-optimal. In addition, in heterogeneous networks where nodes have variable transmit powers and link qualities, intermittent node failures may occur, which further renders a fixed exclusion radius suboptimal. In addition, since the exclusion range is optimized for binary as opposed to probabilistic placement, HEC is not robust to changing or uncertain network conditions.

The above challenges motivate us to seek answers to the following questions: (Q1) How can
we devise a new decentralized caching policy such that the caching penalty is not as severe as
for the HEC policy with a fixed exclusion region? (Q2) Is there a class of probabilistic functions
that enables a desired caching configuration in light of the demand distribution and network
model? (Q3) Can we provide a continuous relaxation of the above 0-1 “knapsack problem”\(^1\)
as compared to HEC policy, while still spatially balancing content placement? (Q4) How much
gain can such a policy attain in terms of the cache hit versus cache size (provisioning) tradeoff?

B. Related Work

The modeling and analysis of caching gain have attracted significant interest in recent years,
and has been studied from many different perspectives. For example, some of the limits of
caching via exploiting the tradeoff between the network bandwidth usage and local caches have
been studied in [3], in which the content placement phase is carefully designed so that a single
coded multicast transmission can simultaneously satisfy multiple content requests. From the
viewpoint of capacity scaling laws, more specifically, the per-node capacity scaling law as the
number of nodes \( n \) grows to infinity, for the protocol model for wireless ad hoc networks in
[6], have been explored in the context of caching in [7], the [8], [9]. In [7], throughput of a
D2D one-hop caching network was shown to grow linearly with cache size, and be inversely
proportional to the number of files, but independent of \( n \), unlike the scaling behavior without
caching. Rate-memory and storage-latency tradeoffs for caching have been studied in [10], [11].
Caching has also been studied in the context of device-to-device (D2D) communications in [12],
[13], [14], and interference management in [15], [16], and in optimization of cloud and edge
processing for radio access networks in [17], [18].

Temporal caching models have been analyzed in [19] for popular cache replacement algo-
rithms, e.g. least recently used (LRU), least-frequently used (LFU), and most recently used
(MRU) cache update. Decentralized spatial LRU caching strategies, referred to as spatial multi-
LRU, have been developed in [20]. These combine the temporal and spatial aspects of caching,
and approach the performance of centralized policies as the coverage increases. However, they are
restricted to the LRU principle. A time-to-live (TTL) policy with a stochastic capacity constraint
and low variance has been proposed in [21]. The BitTorrent protocol employs the rarest first and

\(^1\)The 0-1 knapsack problem is a combinatorial optimization problem which is NP-hard [5]. In the context of caching, this is
equivalent to restricting the number of copies of each kind of item to zero or one.
choke algorithms to promote diversity among peers, and foster reciprocation, respectively. These have been demonstrated in the context of peer-to-peer (P2P) file replication in the Internet [22].

There also exist studies focusing on decentralized and geographic content placement policies such as [2], [23], [24], [25], [4]. The main focus of these works is to maximize the average cache hit probability subject to an average cache size constraint, relaxing the integrality constraints of cache capacities. This optimization problem can be solved as a convex program. However, to the best of our knowledge, the related literature does not provide performance guarantees, leading to these three performance questions. (P1) How far off is the optimized average cache size from the unique per-cache size requirement? (P2) How far off is the average cache hit rate from the attainable performance? (P3) How consistent and predictable is the cache hit probability across the caches? Cache size over-provisioning plays a critical role in addressing these issues, including for performance optimization in hybrid storage systems in computing. Hence, it is essential to devise content placement techniques that ensure the concentration of the cache size as well as provide a spatially balanced allocation and a lower cache hit variance across the nodes.

C. Key Aspects and Contributions

In this paper, we develop a decentralized spatial exclusion-based cache placement policy that exploits pairwise interactions. The intuition behind exclusion-based or negatively dependent caching models is to promote content diversity and reciprocation by ensuring caches storing the same item are never closer than some given distance. While any exclusion-based approach can yield a negatively dependent thinning of the baseline PPP, the nature of the thinning depends on whether the exclusion is determined in a strict sense, or in a probabilistic manner. For example, the HEC policy in [4] is strict: a subset of nodes is retained to cache an item based on binary decisions imposed by the Matérn’s deterministic exclusion range model.

Our proposed policy generalizes the HEC policy in two ways. First, the exclusion range is not fixed for a given item type. Second, the placement decision is probabilistic rather than binary. This is achieved by making the exclusion range at each node variable and determined by a probability measure γ for each item, which we refer to as γ-exclusion cache placement (GEC) policy. GEC retains a subset of the nodes of the baseline PPP to cache an item via this

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2Hybrid storage techniques combine NAND flash memory or solid-state drives (SSDs), with the hard disk drive (HDD) technology [26].

3The exclusion range for GEC will be gamma-distributed which is explained later in Sect. III.
probabilistic rule, as well as a combination of factors involving the nodes’ caching priorities, and a continuous function \( f(r, m, n) \) that makes the pairwise placement decisions corresponding to an item. This function is characterized by the distance \( r \) between the node pairs, which is symmetric with respect to the node exclusion radii \( m, n \) that are distributed according to \( \gamma \) that captures the item popularity. In that sense, \( \gamma \) is not a sufficient statistic for GEC as pairwise distances matter for making placement decisions.

Revising the HEC policy that induces a binary retaining of cache pairs for a content item, GEC enables the probability of placement decision to transition smoothly from 1 to 0 as a function of the pairwise distances and the exclusion radii of the nodes captured via \( f(r, m, n) \).

GEC roots in spatially balanced sampling, which is motivated by the request arrivals. For example, in P2P networking, the actual demand distribution is not known by nodes, and the cache updates in each peer are triggered by the requests. Furthermore, the traffic density in cellular networks is in general not uniform across the network, and the peak hour (PH) density can be approximated by a lognormal distribution [27]. Hence, instead of having a fixed exclusion range, it is desirable to have a variable exclusion range, depending on an item’s popularity. The GEC policy comes to the fore by putting a mark distribution on the exclusion range of an item based on its popularity. The marks may correspond to the detection ranges or the transmit powers of the nodes in heterogeneous networks. Motivated by the performance guarantee issues P1-P3 and answering design questions Q1-Q4, via GEC we aim to provide a better tradeoff between the cache hit rate and the cache size violation. Our main contributions and GEC’s use cases are:

i. GEC has desirable spatial and local properties. Via design of the pairwise placement function, GEC enables a spatially balanced sampling, yields an improved concentration of the cache size eliminating the need of over-provisioning, and enhances multi-hop connectivity.

ii. GEC yields a better cache hit rate versus provisioning tradeoff than the state of the art. GEC can provide about a 3x and 2x reduction in required cache storage size over independent placement [23] and HEC placement [4], respectively. The achievable gains are demonstrated for both uniform and non-uniform spatial traffic types.

iii. GEC has connections with rarest first caching as it promotes the item diversity and reciprocation among the nodes. Hence, it can be well-suited for P2P applications.

iv. GEC is suited for enabling proximity-based applications (such as D2D and P2P), and offloading mobile users in heterogeneous networks.
D. Organization and Notation

In Sect. II we formally define the cache hit probability as function of the placement configuration, and discuss the uniform and non-uniform spatial demand models we use. Sect. III contains our main technical contributions where we detail the construction of $\gamma$-exclusion caching model, and provide its characterization to answer Q1-Q4 and P1-P3. In Sect. IV we numerically evaluate these models and contrast their cache overprovisioning performance, for spatially uniform and non-uniform settings and demonstrate the performance gains of GEC over the baseline models.

**Notation.** Let $\Phi$ denote the mother point process (p.p.), and $\Phi_{th}$ be the child p.p. obtained via the thinning of $\Phi$. Let $\pi$ be a spatial caching policy that yields a set of child p.p.’s $\{\Phi_{th,i}\}$, where $\Phi_{th,i}$ is the set of retained points that cache item $i$. Let $A$ be a given bounded convex set in $\mathbb{R}^2$ containing the origin, and $rA$ be its dilation by the factor $r$. $\mathbb{1}\{A\}$ is the indicator of event $A$. Let $B$ be a bounded Borel set. Let $\Phi(B)$ be the random number of points of the spatial p.p. $\Phi$ which lie in $B$. Any receiver can obtain the desired item $i$ if it is within a critical communication range $R_c$. Assume that $B = B_0(R_c)$, where $B_0(r)$ is a ball in $\mathbb{R}^2$ with radius $r$, centered at origin. The notation for the caching network is detailed in Table I.

II. HOW TO OPTIMIZE THE CACHING GAIN

The locations of the nodes (caches) in the network are modeled by a homogeneous PPP $\Phi = \{x_k\}$ in $\mathbb{R}^2$ with intensity $\lambda$. There are $M$ items in the network, each having the same size, and each node has the same cache storage size $N < M$. Each user makes requests based on a Zipf popularity distribution over the set of the items. The probability mass function (pmf) of such requests (demand) is given by $p_r(i) = i^{-\gamma_r}/\sum_{j=1}^{M} j^{-\gamma_r}$, where $\gamma_r$ determines the tilt of the Zipf distribution. The demand profile is the Independent Reference Model (IRM), i.e., the standard synthetic traffic model in which the request distribution does not change over time [28]. We consider an isotropic demand model and a non-isotropic demand model. However, the request distribution does not change over time.

A. Uniform Demand

The request distribution is uniform across the network, i.e., isotropic. Hence, the intensity of the requests for item $i$, i.e. $\lambda_i$, is proportional to its demand probability $p_r(i)$. We assume $\lambda_i = p_r(i)$, and $I \sim p_r$ be the random variable that models the demand. Each node is associated with the variables $z_{xi} = \mathbb{1}\{i \in \text{Cache}(x)\}$ that denote whether item $i$ is available in its cache or
not. There is also a cost \( w_k \) associated with obtaining an item within the presence of \( k \) nodes in the range. Given these parameters, consider the caching gain function of the following form:

\[
F(Z) = \mathbb{E}_Z \left[ \sum_{k=1}^{\infty} w_k \left( 1 - \prod_{k'=1}^{k} \left( 1 - z_{p_{k'},i} \right) \right) \right] = \sum_{i \in \mathcal{R}} \lambda_i \sum_{k=1}^{\infty} w_k \left( 1 - \prod_{k'=1}^{k} \left( 1 - z_{p_{k'},i} \right) \right),
\]

(1)

where \( Z = [z_{x}]_{x \in \Phi, i=1,\ldots,M} \). Note that (1) can be used to model multi-hop coverage scenarios as in [24], [25], and Boolean coverage scenarios as in [23], [4], with the convention \( \prod_{k=1}^{0} a_k = 1 \).

We next justify these two different scenarios.

\( a) \) **Multi-hop scenario:** The network serves content requests routed over \( \Phi \). A request is determined by the item requested, and the path that the request follows. Given a path \( p \) that is a sequence \( \{p_1, p_2, \ldots, p_K\} \) of nodes with length \(|p| = K\) and a \( x \in p \), denote by \( k_p(x) \) the position of \( x \) in \( p \), i.e., \( k_p(x) \) equals \( k \in \{1, \ldots, |p|\} \) such that \( p_k = x \).

\[
F(Z) = \sum_{(i,p) \in \mathcal{R}} \lambda_{(i,p)} \sum_{k=1}^{|p|-1} w_{p_k} \left( 1 - \prod_{k'=1}^{k} \left( 1 - z_{p_{k'},i} \right) \right),
\]

(2)

where weight \( w_{p_k} \geq 0 \) associated with link \( (p_k, p_{k+1}) \) represents the cost of transferring an item across the link. If \( k^* \) is the first index where item \( i \) is stored, the caching gain for \( i \) is \( \sum_{k=k^*}^{|p|-1} w_{p_k} \).

| Definition | Function |
|------------|----------|
| Mother point process | \( \Phi = \{x_k\} \) |
| Intensity of \( \Phi \) | \( \lambda \) |
| Child point process | \( \Phi_{th} \) |
| Spatial caching policy that yields a set of child p.p.'s \( \{\Phi_{th,i}\}_i \) | \( \pi \) |
| Ball centered at origin with radius \( r \) | \( B_0(r) \) |
| Communication range | \( R_c \) |
| Catalog size | \( M \) |
| Set of items | \( \{i\} \) |
| Cache storage size | \( N \) |
| Request pmf in the uniform spatial demand setting | \( p_r \sim \text{Zipf}(\gamma_r) \) |
| Request traffic intensity pmf in the spatially heterogeneous (non-uniform demand) setting | \( \sim \text{lognormal}(\mu^*, \sigma) \) |
| Intensity of MatII as function of the exclusion radius \( r_i \) | \( \lambda_{\text{hcp}} \) |
| Bivariate marks of GEC | \( \{(m_k^{(i)}, v_k^{(i)})\} \) |
| Distribution of the mark exclusion radius \( m^{(i)} \) | \( \mu^{(i)} = \Gamma(\alpha, \beta) \) |
| Distribution of the weight of mark \( v^{(i)} \) | \( v_k^{(i)} \sim U[0,1] \) |
| Distance dependent pairwise deletion probability function for two points with marks \( m \) and \( v \) | \( f(., m, n) \) |

**TABLE I.**

**NOTATION.**
We assume that the path originates from the request centered at origin, and \( p_1 \) is the nearest node, \( p_2 \) is the second nearest node, and so on. To determine the weight \( w_{p_k} \) we consider a path loss-based model and let \( w_{p_k} = \mathbb{E}[[|p_k - p_{k+1}|^\alpha]] \), where \( \alpha \) is the path loss exponent. For \( \Phi \), let \( D_n \) be the distance of the \( n \)-th closest node. Then the joint distance distribution becomes

\[
f_{\vec{D}_n}(\vec{d}_n) = f_{D_1, \ldots, D_n}(d_1, \ldots, d_n) = (2\pi\lambda)^n \left( \prod_{i=1}^{n} d_i \right) \exp(-\lambda \pi d_n^2), \quad 0 \leq d_1 \leq d_2 \leq \ldots \tag{3}\]

From the path loss model and (3), and exploiting the fact that \( \Phi \) is isotropic, the weight satisfies

\[
w_{p_k} = \frac{1}{2\pi} \int_0^\infty \int_0^{d_{k-1}} \int_{d_k}^{\infty} (d_k^2 + d_{k+1}^2 - 2d_k d_{k+1} \cos(\theta))^{\alpha/2} f_{\vec{D}_{k+1}}(\vec{d}_{k+1})d\vec{d}_{k+1}d\theta,
\]

which can be plugged into (1) to determine the caching gain.

**b) Boolean coverage scenario:** The following function computes the cache hit probability:

\[
F(Z) = \sum_{i \in \mathcal{R}} \lambda_i \sum_{k=0}^{\infty} w_k \left( 1 - \prod_{k'=1}^{k} (1 - z_{p_{k'}}i) \right), \tag{4}
\]

where \( w_k = \mathbb{P}(\Phi(B) = k) \) is the probability that \( k \) caches of \( \Phi \) cover the typical receiver. We assume that \( \lambda_i = \lambda_{(i,p)} \) is the same across all paths. If \( k^* \) is the first index such that a cache has the desired item \( i \), then from (1), the caching gain for item \( i \) is given by the probability of having at least \( k^* \) caches within the communication range, i.e., \( \sum_{k=k^*}^{\infty} w_k = \mathbb{P}(\Phi(B) \geq k^*) \).

Since both the multi-hop and Boolean coverage scenarios are equivalent up to scaling, we focus on the second scenario in the rest of the paper. We have the following observations:

i. \( F(Z) \) is an increasing function of \( Z \).

ii. The product term in (4) satisfies the following relation:

\[
f_{z_i}(1, \ldots, k) = \prod_{k'=1}^{k} (1 - z_{p_{k'}}i) = \begin{cases} 
1, & z_{p_{k'}}i = 0, \forall k' \in \{1, \ldots, k\}, \\
0, & \text{otherwise}.
\end{cases} \tag{5}
\]
iii. From (5), the first-order and second-order properties of \( f_z(1, \ldots, k) \) are derived as

\[
\mathbb{E}\left[ f_z(1, \ldots, k) \right] = \mathbb{P}(\Phi_{th,i}(B) = 0).
\]

\[
\mathbb{E}\left[ f_z(1, \ldots, k) f_z(1, \ldots, k) \right] = \mathbb{P}\left( \sum_{k'=1}^{k'} \Phi_{th,k}(B) = 0 \right) - \mathbb{P}\left( \sum_{k'=1}^{k} z_{p_{kr}} = 0, \sum_{l'=1}^{l} z_{p_{lr}} = 0 \right).
\]

\[
\text{Cov}\left[ f_z(1, \ldots, k), f_z(1, \ldots, k) \right] = \mathbb{P}\left( \sum_{k'=1}^{k'} \Phi_{th,k}(B) = 0 \right) - \mathbb{P}\left( \sum_{k'=1}^{k} z_{p_{kr}} = 0 \right) \mathbb{P}\left( \sum_{l'=1}^{l} z_{p_{lr}} = 0 \right).
\]

B. Non-Uniform Demand

We next consider a more realistic demand model that varies geographically, inspired from the existing empirical models that demonstrate this [29], [30], [31], [27]. In [27], the geographical variation of traffic measurements were collected from commercial cellular networks, based on a large-scale measurement data set from commercial GPRS/EDGE cellular networks deployed in a major east province of China. They have demonstrated that the PH traffic density, i.e. the highest volume of the cell traffic load per unit area in kilobytes per square kilometer, both in a dense urban area and a rural scenario can be modeled by a lognormal distribution. In addition, authors in [30] have optimized the BS deployments for both urban and rural scenarios, where the deployments should be Matérn cluster and Strauss hard-core processes, respectively.

Motivated by [27], we characterize the PH traffic density by \( D = \exp(S) \) where \( S \sim \mathcal{N}(\mu^*, \Sigma) \), where the entries of the covariance matrix satisfy the relation \( \Sigma_{ij} = \sigma^2 \exp(-d_{ij}/r) \) that is obtained exploiting the exponential variogram model where the correlation is a function of the pairwise distances between the receivers [27]. The variogram has an appropriate scaling \( r \) that is determined by range\(^4\). Numerical values of the distribution parameters are given in Sect. IV.

For the non-uniform traffic model, the effective intensity of the requests for item \( i \) at \( x \in \Phi \) is

\[
\lambda_i^x = \frac{D_x}{\mathbb{E}[D]} \lambda_i = \frac{D_x}{\mathbb{E}[D]} p_r(i),
\]

where \( \lambda_i \) is the intensity of the requests for item \( i \) in the uniform spatial traffic model, and \( D_x \) is the traffic density at \( x \in \Phi(B) \) and \( \mathbb{E}[D] = \frac{1}{|\Phi(B)|} \sum_{y \in \Phi(B)} D_y \). Note that the isotropic demand result \( p_r(i) \) in Sect. II-A is can be obtained when \( D_x = \mathbb{E}[D] \).

\(^4\) The finite lag distance at which a variogram reaches its sill is called range. The PH traffic of the two positions beyond this range is almost uncorrelated.
We model the non-uniform spatial request distribution at \( x \in \Phi(B) \) by exponential tilting of the isotropic demand random variable \( I \), which is parameterized by \( \theta_x \):

\[
p_x^r(i) = \frac{e^{i\theta_x} p_r(i)}{\mathbb{E}_x[e^{i\theta_x}]} = \frac{e^{i\theta_x} p_r(i)}{\sum_{m=1}^{M} e^{m\theta_x} p_r(m)}, \quad i = 1, \ldots, M. \tag{7}
\]

From (6) and (7) we choose the tilting parameter \( \theta_x \) such that

\[
e^{i\theta_x} = e^{i \log \left( \frac{D_x}{\mathbb{E}[D]} \right)}, \quad i = 1, \ldots, M. \tag{8}
\]

Note that if the original distribution for the intensity of the requests for item \( i \) is Poisson \( (\lambda_i) \), its tilted distribution parameterized by \( \theta_x \) will also be a Poisson distribution with parameter \( \lambda_i^x = e^{\theta_x} \lambda_i \). Hence, exponential tilting ensures the validity of (6). When the original distribution is different from Poisson, \( \theta_x \) has to be set to ensure that the local request distribution at \( x \in \Phi(B) \) is valid. For example, if \( p_r \sim \text{Zipf}(\gamma_r) \), then \( e^{i\theta_x} \) equals \( (1/i)^{\gamma_{\theta_x}} \) for some \( \gamma_{\theta_x} \) so that the tilted distribution is also a Zipf distribution with parameter \( \gamma_r + \gamma_{\theta_x} \).

Another choice of the tilting parameter is based on a weighted version of exponential tilting such that the PH traffic density is higher for only a subset of items:

\[
e^{i\theta_x} = \begin{cases} 
\sum_{x \in \Phi(B), D_x > \mathbb{E}[D]} \frac{D_x}{\left| \Phi(B) \right| \mathbb{E}[D]} &, i = 1, \ldots, M/2 \\
1, &, i = M/2 + 1, \ldots, M. 
\end{cases} \tag{9}
\]

Provided that non-uniform spatial traffic satisfies \( \mathbb{E}_x[\lambda_i^x] = \lambda_i \), the average caching gain over all the users remains unchanged from the uniform model. However, non-uniform demand also affects the second order properties of caching, such as its spatial distribution and variance across nodes, which play a key role in the performance.

III. THE \( \gamma \)-EXCLUSION CACHING MODEL

The \( \gamma \)-exclusion caching policy is constructed from the underlying PPP \( \Phi \) with intensity \( \lambda \), by removing certain nodes depending on the positions of the neighboring nodes, and on the marks and weights attached to them. It is based on a generalization of the Matérn II hard-core p.p. (MatII). The MatII process is obtained via the thinning of \( \Phi \), where each node is equipped with different exclusion radii (deterministic marks) for distinct items according to their popularity [4]. For this process, as the exclusion radius increases, the intensity \( \lambda_{hcp}(i) < \lambda \) corresponding to
the set of nodes caching item $i$ decreases. The GEC policy is more general than MatII such that there is a distinct distribution that models the exclusion radius of each item. This can enable a softer version of thinning to ensure a more effective content placement.

For each item $i$, let $\tilde{\Phi}_i = \{(x_k, m^{(i)}_k, v^{(i)}_k)\}_k$ be a homogeneous independently marked PPP with intensity $\lambda$, and i.i.d. $\mathbb{R}^2$-valued marks, where $\Phi = \{x_k\}$, and $\{(m^{(i)}_k, v^{(i)}_k)\}$ is the random bivariate mark. The first component $m^{(i)}_k$ of the bivariate mark is referred to as mark, and has distribution $\mu^{(i)}$. The mark of item $i$, i.e., $m^{(i)}_k$, denotes its exclusion radius, and depends on its popularity in the network. If item $i$ is more popular than item $j$, then $m^{(i)}_k$ is stochastically dominated by $m^{(j)}_k$. The second component $v^{(i)}_k$ of the bivariate mark is weight, where $v^{(i)}_k$ serves as a weight of the thinning procedure, and has distribution $\nu^{(i)}_{m^{(i)}}$ which might depend on $m^{(i)}$.

Let $\Phi_{th,i}$ be a child p.p. that denotes the set of points that cache item $i$. The cache placement model is such that item $i$ is stored in cache $x_k \in \Phi$ if and only if cache $x_k$ is kept as a point of $\Phi_{th,i}$. Equivalently, we have

$$z_{xki} = 1\{i \in \text{Cache}(x_k)\} = 1\{x_k \in \Phi_{th,i}\}. \quad (10)$$

Node $x_k$ is retained as a point of $\Phi_{th,i}$ with probability $\mathbb{E}[z_{xki}] = p(x_k, m^{(i)}_k, v^{(i)}_k, \Phi)$. The weights are i.i.d. and uniformly distributed, i.e. $v^{(i)}_k \sim U[0, 1]$, for each node $x_k$ and item $i$. The marks $m^{(i)}_k$ are distributed according to $\mu^{(i)}$ for each $x_k$, and $i$. For the special case of MatII, i.e., when the marks are fixed, we optimized the exclusion radii in [4].

The number of items in cache $x_k$ is the sum of the individual items’ indicator functions $C(x_k) = \sum_i 1\{i \in \text{Cache}(x_k)\}$. The cache size constraint has to be satisfied on average, i.e.

$$N = \mathbb{E}[C(x_k)] = \sum_i p(x_k, m^{(i)}_k, v^{(i)}_k, \Phi), \quad x_k \in \Phi. \quad (11)$$

We next detail the dependent thinning procedure, and investigate the relationship between the mother p.p. $\Phi$ and the child p.p.’s $\Phi_{th,i}$, $i = \{1, \ldots, M\}$.

### A. Dependent Sampling of Nodes for Placement

In this section and onwards, for brevity of notation, we omit the index $i$, and consider the generic thinned process $\Phi_{th}$, derived from $\Phi$ by applying the following probabilistic dependent thinning rule. Assume that mark $m$ has a distribution $\mu$, and $\vec{m} = \{m\}$ is the set of marks for

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5 $X$ is stochastically dominated by $Y$, which is denoted by $X \leq_{st} Y$, if for all increasing functions $g$, $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$. 

---
all points in $\tilde{\Phi}$, where $m \sim \mu$ and $\bar{m} = \mathbb{E}_m [m]$. Assume that weight $\nu_m$ does not depend on the mark $m$. The marked point $(x, m, v) \in \tilde{\Phi}$ is retained as a point of $\Phi_{th}$ with probability

$$p(x, m, v, \Phi) = p_0 \prod_{(y, n, w) \in \Phi, y \neq x} \left[ 1 - \mathbb{1}\{v \geq w\}f(||x - y||, m, n) \right]$$

(12)

independently from deleting or retaining other points of $\Phi$. In other words, a node $x \in \Phi$ is retained to cache item $i$ with probability $p_0$, if it has the lowest weight among all the points within its exclusion range. In (12), $p_0 \in (0, 1]$, $f : [0, \infty] \times \mathbb{R}^2 \rightarrow [0, 1]$ is a deterministic function satisfying $f(\cdot, m, n) = f(\cdot, n, m)$ for all $m, n \in \mathbb{R}$. This means that if two points with marks $m$ and $n$, and weights $v \geq w$ are a distance $r > 0$ apart, then the point with weight $v$ is deleted by the other point with probability $f(r, m, n)$. Additionally, each surviving point is then again independently $p_0$-thinned. Conditional on retaining $(y, n, w) \in \tilde{\Phi}$ such that $y \neq x$, $1 - f(||x - y||, m, n)$ represents the pairwise retaining probability of $x \in \Phi$ given that $v \geq w$.

The function $f(||x - y||, m, n)$ for GEC in (12) should be determined to ensure that (11) holds. Inspired from [32], we assume that $f$ is continuous and symmetric, and satisfies

$$f(r, m, n) = \exp (-c |r - m - n|_+), \quad r \geq 0,$$

(13)

where $[x]_+ = \max\{0, x\}$. Fig. 1 clarifies the distinction of GEC and the independent caching and HEC policies, via indicating their pairwise retaining probabilities for $1 - f(r, 2, 2)$. 

---

**Fig. 1.** An example contrasting the probability of retaining a node of the baseline PPP for the independent caching, HEC and GEC policies as function of the pairwise distance.
Denote by \( \text{GEC}[\lambda, \mu, (\nu_m)_{m \in \mathbb{R}}, p_0, f] \) the distribution of \( \Phi_{th} \). We next give its intensity, i.e.,

\[
\lambda_{th} = \lambda \mathbb{E}[p(x, m, v, \Phi)].
\]

**Theorem 1.** [32, Theorem 12] The intensity of p.p. \( \Phi_{th} \sim \text{GEC}[\lambda, \mu, (\nu_m)_{m \in \mathbb{R}}, p_0, f] \) is given by

\[
\lambda_{th} = \lambda p_0 \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \left( -\lambda \int_{\mathbb{R}} F_{\nu_n}(w) \int_{\mathbb{R}^2} f(||x||, m, n) dx \mu(\text{d}n) \right) \nu_m(\text{d}w) \mu(\text{d}m),
\]

where \( F_{\nu_n}(w) = \nu_m((-\infty, w]), w \in \mathbb{R} \) is the cumulative distribution function (CDF) of \( \nu_m \).

**Proof.** The probability generating functional (PGFL) [33] of the PPP states for function \( f(x) \) that \( \mathbb{E}\left[ \prod_{x \in \Phi} f(x) \right] = \exp(-\lambda \int_{\mathbb{R}^2} (1 - f(x)) dx) \). We can compute the intensity of \( \Phi_{th} \) using

\[
\lambda_{th} = \lambda p_0 \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \left( -\lambda \mathbb{E}[\mathbb{1} \{ \nu_n \leq w \} \int_{\mathbb{R}^2} f(||x||, m, n) dx \right) \nu_m(\text{d}w) \mu(\text{d}m).
\]

We obtain \( \lambda_{th} \) using the PGFL and computing

\[
\mathbb{E}[\mathbb{1} \{ \nu_n \leq w \}] = \int_{\mathbb{R}} \mathbb{P}(\nu_n \leq w) \mu(\text{d}n) = \int_{\mathbb{R}} F_{\nu_n}(w) \mu(\text{d}n),
\]

along with incorporating the retaining probability given in (12). \( \square \)

**Corollary 1.** Letting \( f(\cdot, m, n) = 1_{[0,m+n]}(\cdot) \), and \( m = n = R/2 \), the intensity of MatII equals

\[
\lambda_{th} = \frac{1 - \exp(-\lambda \pi R^2)}{\pi R^2}.
\]
In Fig. 2, we illustrate different realizations of GEC $\Phi_{th}$ formed by thinning $\Phi$ for different mark radii where the mark distributions are gamma-distributed. A gamma-distributed random variable $X$ is characterized by a shape parameter $\alpha$ and a rate parameter $\beta$, and is denoted as $X \sim \Gamma(\alpha, \beta) \equiv \text{Gamma}(\alpha, \beta)$. The corresponding probability density function is
\[
f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)} \quad \text{for } x > 0, \quad \alpha, \beta > 0,
\]
where its mean is $\alpha/\beta$, variance is $\alpha/\beta^2$, and $\Gamma(\alpha)$ is the gamma function. As the mark variance increases, the packing becomes denser, which is desired for spatially balanced caching.

B. Spherical Contact Distribution Function

Our goal in this section is to relate the cache hit probability distribution to the contact distribution function. We next formally define the spherical contact distribution function.

**Definition 1.** Spherical contact distribution function [33]. The spherical contact distribution function (SCDF) of the p.p. $\Xi$ is the conditional distribution function of the distance from a point chosen randomly outside $\Xi$ (i.e. 0), to the nearest point of $\Xi$ given $0 \notin \Xi$. It is given by
\[
H(r) = \mathbb{P}(R_{\text{Sph}} \leq r | R_{\text{Sph}} > 0), \quad r \geq 0,
\]
where $R_{\text{Sph}} = \inf\{s : \Xi \cap sA \neq \emptyset\}$, $A = B_0(1)$ is the unit ball in $\mathbb{R}^2$ containing the origin, and $rA$ is the dilation of the set $A$ by the factor $r$, $\Xi_0$ is the typical grain of $\Xi$, $\mathbb{E}(\nu_2(\Xi_0 \oplus rA))$ is the mean volume of the typical grain, $\Xi_0 \oplus rA$ denotes the Minkowski addition, and $\nu_2$ is the Lebesgue measure on $[\mathbb{R}^2, \mathcal{B}^2]$ where the $\sigma$-algebra $\mathcal{B}^2$ contains all the subsets of $\mathbb{R}^2$ that can be constructed from the open subsets by the basic set operations and by limits [33].

In Fig. 3 we illustrate the SCDF for the Boolean model with random spherical grains [34, Ch. 3.1]. For the special case of PPP-MatII, the spherical grains have the same radius.

**Theorem 2.** The average cache hit probability of policy $\pi$ is
\[
\mathbb{E}_\pi[F(Z)] = \mathbb{E}_X[H_{\pi, X}(R_c)], \quad (17)
\]
\[\text{If the caching gain function satisfies (1), then this result holds.}\]
where $H_{\pi,\mathcal{I}}(R_c)$ is the SCDF of the thinned p.p. $\Phi_{th,\mathcal{I}}$ for $\mathcal{I}$.

Proof. Let $B = B_0(R_c)$ and $\Phi_{th,i}(B) = \sum_{x \in \Phi_{th,i}} 1(x \in B)$ be the number of transmitters containing item $i$ within a circular region of radius $R_c$ around the origin. Then we have

$$F(Z) = \sum_i p_r(i) \mathbb{1}(\Phi_{th,i}(B) > 0).$$

The average cache hit probability is given by $\mathbb{E}[F(Z)] = \sum_i p_r(i) \mathbb{P}(\Phi_{th,i}(B) > 0)$, where defining $R_{\text{Sph}} = \inf \{s : \Phi_{th,i}(B_0(s)) \neq 0\}$, given $0 \notin \Phi_{th,i}$ we have that

$$\mathbb{P}(\Phi_{th,i}(B) > 0) = \mathbb{P}(R_{\text{Sph}} \leq R_c | R_{\text{Sph}} > 0),$$

which is the SCDF of $\Phi_{th,i}$ evaluated at $R_c$.

The variance of $F(Z)$ across the nodes satisfies the following additive relation

$$\text{Var}_\pi[F(Z)] = \sum_i p_r^2(i) H_{\pi,i}(R_c)(1 - H_{\pi,i}(R_c))$$

because $\Phi_{th,i}$ across $i = \{1, \ldots, M\}$ are independent of each other. Under the IRM and a Zipf popularity model, $\text{Var}_\pi[F(Z)]$ decreases with increasing variance of marks when $\mathbb{E}[C(x)]$ is held constant [21] Prop. 1. A spatially balanced sampling yields a low $\text{Var}_\pi[F(Z)]$ as expected.

C. Migration to the Child Process: Effective Thinning

Consider the pair $\Phi - \Phi_{th}$ of mother and child p.p.’s. The spherical contact distance denotes the distance between a typical point in $\Phi$ and its nearest neighbor from the thinned process $\Phi_{th}$.
Using (16), the SCDF for the p.p. \( \Phi \) can be written as:

\[
H_\pi(R) = 1 - \exp \left( - \int_0^R 2\pi r \lambda \eta_\pi(r, \delta) dr \right),
\]

(20)

where \( \eta_\pi(r, \delta) \) is the conditional thinning Palm-probability (CTPP), i.e. the probability of the point \( x \in \Phi \) migrating to \( \Phi_{th} \) under policy \( \pi \), with a fixed (exclusion) radius \( \delta \). It equals

\[
\eta_\pi(r, \delta) = \mathbb{P}(x \in \Phi_{th} | \Phi_{th} \cap B_{x_0}(r) = \emptyset, x_0 \in \Phi).
\]

(21)

**Remark 1.** The more effective the thinning (caching) policy \( \pi \) is, the larger its CTPP \( \eta_\pi(r, \delta) \) and the SCDF (20) is. From Theorem 2, \( \mathbb{E}_\pi[F(Z)] \) is improved if \( \pi \) is more effective.

We next compute the CTPP for the GEC policy.

**Proposition 1.** The CTPP for PPP-GEC is given as

\[
\eta_{SSCC}(r, \bar{\delta}) = \int_\mathbb{R} \int_0^1 \exp \left( -u \lambda \int_\mathbb{R} \int_\mathbb{R}^2 h(||x||, m, n) dx \mu(dm) \right) du \mu(dm),
\]

where given radius marks \( m, n \), \( h(||x||, m, n) \) satisfies the relation

\[
\int_\mathbb{R}^2 h(||x||, m, n) dx = \pi (m + n)^2 - l_2(r, n),
\]

(22)

where \( l_2(r, \delta) \) is the area of the intersection of \( B_{x_0}(r) \) and \( B_x(\delta) \). It is given by

\[
l_2(r, \delta) = \begin{cases} 
\pi r^2, & 0 < r < \frac{\delta}{2} \\
\frac{\delta^2}{2} \left( 1 - \frac{\delta^2}{2r^2} \right) + \delta^2 \cos^{-1} \left( \frac{\delta}{2r} \right) - \frac{\delta}{2} \sqrt{4r^2 - \delta^2}, & r \geq \delta/2.
\end{cases}
\]

(23)

**Proof.** The proof follows from generalizing [35, Eq. (15)]. \qed

Special cases of Prop. 1 corresponding to PPP-PPP and PPP-MatII thinnings that give significant simplications are given next.

**Corollary 2.** The homogeneous PPP \( \Phi_{th} \) with intensity \( \lambda_{th} \) is obtained via independent thinning of \( \Phi \), exploiting that \( \delta = 0 \), where

\[
\eta_{\text{Indep}}(r, \delta) = \int_0^1 \frac{\lambda_{th}}{\lambda} \exp (-t \lambda_\pi 0^2) dt = \frac{\lambda_{th}}{\lambda}.
\]

(24)

Using (20) and (24), \( H_{\text{Indep}}(R) = 1 - \exp (-\pi \lambda_{th} R^2) \).
Corollary 3. The contact distribution function for MatII, i.e., the CDF of the contact distance from a typical point in MatII to its nearest point in the same process, is derived in \cite[Eq. (10)]{35}. The CDF of the PPP-MatII contact distance is larger than the MatII-MatII contact distance.

The CTPP for PPP-MatII is given as
\[
\eta_{\text{MatII}}(r, \delta) = \int_0^1 \exp \left( -u \lambda(\pi \delta^2 - l_2(r, \delta)) \right) du = \frac{1 - \exp \left( -\lambda(\pi \delta^2 - l_2(r, \delta)) \right)}{\lambda(\pi \delta^2 - l_2(r, \delta))}. \quad (25)
\]

Using (20) and (25), $H_{\text{MatII}}(R)$ can be computed.

To characterize the spatial performance and correlation in different caching models, we provide the functions describing the second-order behavior of the point processes in Appendix A.

The next propositions provide bounds on average cache hit rate and its variance for MatII.

Proposition 2. A lower bound on average cache hit rate for MatII is given as follows:
\[
F_{\text{LB}}(Z) = \sum_{i=1}^{m_c} p_r(i) \left[ 1 - \exp(-\lambda\text{hcp}(i)\pi R_c^2) \right] + \sum_{i=m_c+1}^M p_r(i) \lambda\text{hcp}(i)\pi R_c^2,
\]
where $i \leq m_c : r_i < R_c$, i.e., the set $i = \{1, \ldots, m_c\}$ corresponds to the set of files within the communications range. An upper bound on average cache hit rate for MatII is given as follows:
\[
F_{\text{UB}}(Z) = \sum_{i=1}^{m_c} p_r(i) \left[ 1 - \exp(-\lambda\text{hcp}(i)\pi R_c^2) \right] + \lambda^{-1} \int_{B_0(R_c)} \rho_i^{(2)}(x) dx + \sum_{i=m_c+1}^M p_r(i) \lambda\text{hcp}(i)\pi R_c^2.
\]

Proof. See Appendix C.

Proposition 3. The variance of the cache hit probability of MatII is upper bounded as
\[
\text{Var}[F(Z)] \leq \sum_{i=1}^M p_r^2(i) \left( \frac{1}{\epsilon} + \pi \lambda p_r^2(i)(R_c^2 - r_i^2) \right). \quad (26)
\]

Proof. See Appendix D.

We next show that having a CDF on marks yields a more effective thinning than MatII does.

Theorem 3. The CTPPs of GEC and MatII satisfy the relation
\[
\eta_{\text{SSCC}}(r, \delta) \geq \eta_{\text{MatII}}(r, \bar{m}),
\]
where $\delta = \{m\}$ is the set of marks in $\tilde{\Phi}$, with $\bar{m} = \mathbb{E}_m[m]$.

Proof. See Appendix E.
Exploiting Theorem 3, $\eta_{SSCC}(r, \delta)$ can be improved using a mixture of marks. The variable exclusion range model can suit to the case of cellular networks where demand is not uniform across the network [27]. In [31], the authors use the Ginibre p.p. to model random phenomena of BS positions with repulsion, where repulsion means that the node locations of a real deployment usually appear to form a more regular (or more clustered) point pattern than the homogeneous PPP. We numerically investigate the performance of the non-uniform spatial demand in Sect. IV.

D. Spatial Characterization: Variances of Point Counts

As a baseline model, we consider independent thinning. Then, the resulting point process for item $i$ is PPP with intensity $\lambda p_c(i)$, and from the Slivnyak-Mecke theorem [33, Ch. 4.5], if $B = B_0(R_c)$, it is satisfied that $\lambda K(R_c) = \mathbb{E}[\Phi_{\text{ppp},i}(B)] = \text{Var}[\Phi_{\text{ppp},i}(B)] = \lambda p_c(i) \pi R_c^2$. Optimal caching distribution for PPP that maximizes the cache hit probability is computed in [23]. In this setup, authors have proposed a probabilistic allocation model that places exactly $N$ items at each node $x \in \Phi_{\text{ppp},i}$. Hence, it is guaranteed that the cache constraint is satisfied with equality, i.e. $\mathbb{E}[C(x)] = \sum_i p_c(i) = N$ and $\text{Var}[C(x)] = 0$.

As the second baseline model, we consider the MatII model with intensity $\lambda_{hcp}(i) = \frac{1 - \exp(-\lambda \pi r_i^2)}{\pi r_i^2}$, i.e. the caching probability for item $i$ is $p_c(i) = \frac{\lambda_{hcp}(i)}{\lambda}$. In this model, since the placement across the nodes is independent across the items, at each node $x \in \Phi_{\text{th},i}$, we have that $\mathbb{E}[C(x)] = \sum_i p_c(i) = N$, and $\text{Var}[C(x)] = \sum_i p_c(i)(1 - p_c(i))$.

Next result puts an upper bound on the variances of point-counts of $\Phi_{\text{th},i}$ when $R_c$ is large.

**Proposition 4.** [33, Ch. 4] The variances of point-counts of $\Phi_{\text{th},i}$ for large $B$ is bounded as

$$\frac{\text{Var}[\Phi_{\text{th},i}(B)]}{\pi R_c^2} \leq \lambda_{hcp}(i) \exp(-\lambda \pi r_i^2). \quad (27)$$

**Proof.** See Appendix F

From (27) spatial variance decays exponentially fast in $r_i^2$. This yields a more deterministic placement of rare items. Hence, MatII is spatially balanced, unlike independent placement.

E. Negative Association and Further Implications in Caching Optimization

Let $\{Z_i, 1 \leq i \leq n\}$ be a feasible negatively associated (NA) sequence, and let $\{Z^*_i, 1 \leq i \leq n\}$ be a sequence of independent random variables such that $Z^*_i$ and $Z_i$ have the same distribution for each $i = 1, \ldots, n$. 
Definition 2. Negative association [36]. A set of random variables \( \{ Z_i, 1 \leq i \leq n \} \) is said to be NA if for any two disjoint index sets \( I, J \subseteq [n] \) and two functions \( f, g \) both monotone increasing or both monotone decreasing, it holds

\[
\mathbb{E}[f(Z_i : i \in I) \cdot g(Z_j : j \in J)] \leq \mathbb{E}[f(Z_i : i \in I)] \cdot \mathbb{E}[g(Z_j : j \in J)].
\] (28)

The next result bounds the cost of NA caching that is a result of a coupling.

Theorem 4. [37, Theorem 1] For any convex function \( f \) on \( \mathbb{R}^1 \)
\[
\mathbb{E} \left[ f \left( \sum_{i=1}^{n} Z_i \right) \right] \leq \mathbb{E} \left[ f \left( \sum_{i=1}^{n} Z_i^* \right) \right]
\] (29)
if the expectation on the right hand side of (29) exists. If \( f \) is also non-decreasing, then
\[
\mathbb{E} \left[ f \left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} Z_i \right) \right] \leq \mathbb{E} \left[ f \left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} Z_i^* \right) \right]
\] (30)
provided that the expectation on the right hand side of (30) exists.

The following observation is immediate.

Proposition 5. \( F(Z) \) is convex if \( z_{p_{k,i}} \)'s are NA for \( k' \in \{1, \ldots, k\} \), for all \( i \in \{1, \ldots, M\} \) [38].

Proof. Exploiting the caching gain function in (1), observe that the following relation holds:
\[
\mathbb{E}[F(Z)] = \mathbb{E}_Z \left[ \sum_{k=0}^{\infty} w_k (1 - \mathbb{E}[f_{z_{k}}(1, \ldots, k)]) \right] \\
\geq \mathbb{E}_Z \left[ \sum_{k=0}^{\infty} w_k \left( 1 - \prod_{k'=1}^{k} (1 - \mathbb{E}[z_{p_{k,i}}]) \right) \right] = F(\mathbb{E}[Z]), \quad (31)
\]
where (a) is due to that \( \mathbb{E}[f_{z_{k}}(1, \ldots, k)] \leq \prod_{k'=1}^{k} (1 - \mathbb{E}[z_{p_{k,i}}]) \) as \( z_{p_{k,i}} \)'s are NA.

From Prop. 5, \( \mathbb{E}[F(Z)] \geq F(\mathbb{E}[Z]) \). The expected cache hit probability obtained via NA placement upper bounds the independent placement solution with probabilities \( \mathbb{E}[z_{p_{k,i}}] \). NA has desirable properties in terms of sampling and concentration. Some important results that hold for independent variables, e.g., the Chernoff-Hoeffding bounds, and the Kolmogorov’s inequality [38], [39], also hold for NA variables. Hence, Prop. 5 justifies that NA across caches is desirable.
Proposition 6. The average cache hit probabilities of NA and independent caching policies satisfy the relation \( \mathbb{E}[F(Z)] \geq \mathbb{E}[F(Z^*)] \).

Proof. If \( \{Z_i, 1 \leq i \leq n\} \) satisfy the NA condition, then for any non-decreasing (or non-increasing) functions \( g_j, j \in [n] \) \[38, Lemma 2\], \( \mathbb{E} \left[ \prod_{j \in [n]} g_j(Z_j) \right] \leq \prod_{j \in [n]} \mathbb{E}[g_j(Z_j)] \). The proof follows from contrasting the caching gain function in (1) of \( \{Z_i\} \) versus \( \{Z^*_i\} \). \( \square \)

The following corollary follows from \[39\], and demonstrates that NA placement policies have lower variance across the nodes, hence are more stable than independent placement policies.

Corollary 4. \[39\]. Let \( \{Z_i, 1 \leq i \leq n\} \) be a NA sequence. Then, \( \text{Var} \left[ \sum_{i=1}^{n} Z_i \right] \leq \sum_{i=1}^{n} \text{Var}[Z_i] \).

From Cor. 4 we have that \( \text{Var} \left[ \sum_{k'=1}^{k} z_{p_{k'i}} \right] \leq \sum_{k'=1}^{k} \text{Var}[z_{p_{k'i}}] \). Since variance is sub-additive for NA variables, upper bounding the number of times a network caches the same item in a Boolean coverage setting or in a given path in the multi-hop scenario, implies less redundancy. For example, a more balanced allocation is given in \[40\].

Remark 2. Use for optimization \[39\]. If \( \{Z_i, 1 \leq i \leq n\} \) is a feasible NA solution that minimizes the convex function \( f \left( \sum_{i=1}^{n} Z_i \right) \), then the expected value of this solution is upper bounded by the expected value of \( f \left( \sum_{i=1}^{n} Z^*_i \right) \), which is easier to compute since \( Z^*_i \) are independent.

Although there might be multiple policies that maximize \( \mathbb{E}[F(X)] \), an allocation that is NA can have a smaller variance. Variance of a caching policy characterizes the loss of the algorithm in terms of cache over-provisioning (stability). We investigate over-provisioning in Sect. III-F.

Proposition 7. The variances of the cache hit probabilities of NA and independent caching policies satisfy the relation \( \text{Var}[F(Z)] \leq \text{Var}[F(Z^*)] \).

Proof. See Appendix B. \( \square \)

Note that the caching gain function \( F(Z) \) given in (1) is not concave. Following the approximation technique in \[41\], let \( L(Z) \) be the concave approximation of \( F(Z) \) given by

\[
L(Z) = \sum_{(i,p) \in \mathcal{R}} \lambda_{(i,p)} \sum_{k=1}^{w_{pk+1p}^{-1}} \min \left\{ 1, \sum_{l=1}^{k} z_{p_{l1}} \right\}.
\]

We then have the following result.
Proposition 8. The approximate caching gain function satisfies \( L(Z^*) \leq L(Z) \).

Proof. This result follows from Theorem 4.

F. Cache Over-Utilization

The cache placement requires \( C(x) = \sum_i \mathbb{1}_{x \in \Phi_{th,i}} \leq N \), for all \( x \in \Phi \), where \( N \) is finite. This constraint is satisfied on average, i.e. \( N = \mathbb{E}[C(x)] = \sum_i p(x, m_i, v, \Phi), x \in \Phi \). However, the p.p.’s \( \{\Phi_{th,i}\}_i, i = 1, \ldots, M \) might overlap. We need to make sure that the cache capacities are not over-utilized. Hence, the intersection of the sampled processes, i.e. \( \cap_i \Phi_{th,i} \), should not include any \( x \in \Phi \) more than \( N \) times with high probability. We next provide an upper bound for the violation probability of the cache size for GEC, exploiting the Chernoff bound.

Proposition 9. The cache violation probability of GEC is upper bounded as

\[
\mathbb{P}(C(x) > C) \leq \exp \left( \epsilon - (N + \epsilon) \log \left( 1 + \frac{\epsilon}{N} \right) \right),
\]

where \( C = N + \epsilon \), for \( \epsilon \) arbitrarily small.

Proof. We can rewrite the cache violation probability as

\[
\mathbb{P}(C(x) > C) = \mathbb{P} \left( \sum_i \mathbb{1}_{x \in \Phi_{th,i}} > C \right) = \sum_{A \subseteq S, |A| > C} \mathbb{P}(x \in \Phi_{th,i}, i \in A, x \notin \Phi_{th,j}, j \in A^c)
\]

\[\overset{(a)}{=} \sum_{A \subseteq S, |A| > C} \prod_{i \in A} \mathbb{P}(x \in \Phi_{th,i}) \prod_{j \in A^c} (1 - \mathbb{P}(x \in \Phi_{th,j})) \]

\[\overset{(b)}{\leq} \exp \left( C - \sum_i p(x, m_i, v, \Phi) - C \log \left( \frac{C}{\sum_i p(x, m_i, v, \Phi)} \right) \right),
\]

where \( C = N + \epsilon \) for \( \epsilon \) arbitrarily small, \( S \) is the set of all subsets of \( \{1, \ldots, M\} \), and \( A \) is a subset of \( S \) and \( A^c \) is its complement, and (a) is due to the independent sampling of points for each \( i \), i.e., \( \mathbb{P}(x \in \Phi_{th,i}, x \in \Phi_{th,j}) = \mathbb{P}(x \in \Phi_{th,i}) \mathbb{P}(x \in \Phi_{th,j}) \) for all \( i \neq j \), and (b) follows from employing the Chernoff bound.

As the Chernoff bound does not capture the second-order characteristics of the sampling, we next provide another bound based on the Bernstein inequalities.
Fig. 4. The normalized cache size $N/M$ versus average cache hit rate for different placement policies. (Left) $R_c = 3$, (Right) $R_c = 10$. The intensity for the baseline process is $\lambda = 0.1$. The request process is isotropic. The parameter $c$ for function $f(r, m, n)$ in (13) of GEC is selected as indicated. The gamma-distributed mark distribution satisfies $\Gamma(\alpha = \mathbb{E}[RI]/\beta, \beta)$ with $\beta = 1$, where $RI = \{R_i\}_i$ are the exclusion radii optimized for MatII. Note that the dotted curves correspond to the 95% confidence intervals. The remaining curves correspond to the average of the normalized cache size $N/M$ for different content placement techniques. On the figures, we have marked the 95th percentile of the cache sizes $N$ required to achieve an average cache hit probability of 0.7.

**Proposition 10.** The cache violation probability of GEC is upper bounded as

$$
\mathbb{P}(C(x) > C) \leq \exp \left( -\frac{(C - N)^2}{\text{Var}[C(x)] + \frac{1}{3}(C - N)} \right),
$$

where $\text{Var}[C(x)] = \sum_{i=1}^{M} \text{Var}[z_{x_i}]$ since the placement is independent across items, where

$$
\text{Var}[z_{x_i}] = \mathbb{E}[z_{x_i}^2] - \mathbb{E}[z_{x_i}]^2 = p(x, m_i, v, \Phi)(1 - p(x, m_i, v, \Phi)), \quad i \in \{1, \ldots, M\}, \quad x \in \Phi.
$$

**Proof.** It follows from employing Bernstein inequality as $\mathbb{1}_{x \in \Phi_{th,i}}$ are independent across $i$. □

As $\text{Var}[C(x)]$ drops, the upper bound in (34) drops. Hence, the cache violation probability of GEC is negligible if the cache placement strategy has a very low-variance. While it is nontrivial to design probabilistic and NA placement techniques to satisfy the cache size constraint with probability 1, in Sect. IV we demonstrate that for GEC the cache violation probability can be made negligibly small, and contrast its performance with independent caching and HEC.

**IV. Numerical Simulations**

The nodes live in a square region of the Euclidean plane with area $L^2$ where $L = 100$. To avoid edge effects, we evaluate the performance only for the middle square region with area...
The network parameters are $\lambda = 0.1$ and $R_c \in \{3, 10\}$. We initially assume that the request process is isotropic and Zipf distributed with parameter $\gamma_r = 0.1$ over $M = 100$ items.

For MatII, there is a fixed exclusion range for a given item, and we have derived the optimal exclusion radii in [4]. Let $r_i$ be the optimized exclusion range for item $i$ for MatII. For GEC, the weights $v_i \sim U[0, 1]$ are i.i.d. for each node $x$ and each item $i$. We assume that the marks $m^{(i)}$ for item $i$ (exclusion radii) are distributed according to a gamma distribution $\mu^{(i)} = \Gamma(0.7r_i, 1)$ for each $x \in \Phi$, and all items $i$, where we choose its parameters such that the average value of the radius mark for item $i$ equals $\bar{m}(i) = 0.7r_i$. Hence, $\Phi_{\text{th}} \sim \text{GEC}[0.1, \Gamma(0.7r_i, 1), U[0, 1], 1, f]$ with $c = 10$, where $c$ is a parameter of $f(r, m, n)$ in (13) that determines its decay rate. We can observe that the GEC model with gamma-distributed exclusion radius can be used to optimize the cache hit probability-cache violation probability tradeoff. However, as variance of exclusion range increases, the violation probability might also increase for a cache hit probability target. We leave the optimization of the distributions of the marks $\mu^{(i)}$ (across all $i$) over a class of distributions and the study of the fundamental performance limits of GEC as future work.

We numerically investigate how much cache over-provisioning is required for different spatial cache placement policies: spatially independent [23], MatII [4], and GEC cache placement. In Fig. 4, we investigate the required cache size $N$ (normalized) of each policy given that the probability of cache violation is small such that $\mathbb{P}(|C(x) - N| \leq \epsilon) > 0.95$ to characterize the required cache size for a given average cache hit probability. We also illustrate the 95% confidence intervals represented by the shaded regions, and mark the cache sizes for different policies when the average cache hit probability is $\mathbb{E}_x[F(Z)] = 0.7$. For example, when $R_c = 3$, for the 95% confidence interval, the excess cache ratio for independent placement in [23], and MatII placement in [4] with respect to the GEC policy is 142%, and 93%, respectively. When $R_c = 10$, the respective excess ratios for the independent and MatII placement policies are 188% and 109%, which are illustrated on the plots. GEC yields a better concentration of the required cache size, which is desired. Hence, it can be exploited to ensure that the cache does not overrun or underrun its capacity constraint. The performance of caching can be improved if the variance of the marks are higher to better exploit the geographic variation of the demand. In this case, the packing will be denser (see Fig. 2), which is desired for a spatially balanced caching design.

We next generate spatially varying request distributions. We use two sets of model parameters for an urban area and a rural area. The parameters are chosen in accordance with the downlink parameters in [27] such that the logarithm of the PH traffic density is characterized by a normal
distribution. The non-isotropic request (demand) distribution model for urban and rural areas, with the normal distribution parameters \((\mu^* = 15.999, \sigma = 1.4116)\), and \((\mu^* = 10.2496, \sigma = 1.3034)\), respectively, are shown in Fig. 5. These maps show the PH traffic densities in \((\log(Z))(\text{Kbytes/km}^2)\), i.e. the colorbar values are in \(\log(\text{Kbytes/km}^2)\), where the averages are \(\sim 10^7\) (Kbytes/km\(^2\)) for urban area and \([0.8 \times 10^4, 2 \times 10^4]\) (Kbytes/km\(^2\)) for rural area \([27]\). The parameter \(r\) for the exponential variogram model is chosen to be one third of the range, where ranges for urban and rural areas are 0.0154(km) and 1.7139(km), respectively.

The performance of the non-isotropic request model for (9) and (8) are shown in Fig. 6. For the non-uniform spatial traffic, we used Fig. 5 urban area model which contains 120 \(\times\) 120 pixels. We randomly picked a 10 \(\times\) 10 pixel region and averaged the performance of the spatial request distribution models in (9) and (8), where the request distribution is tilted as function of the PH demand intensity. From Fig. 6 (left), we observe that the tilted request model in (9) provides a reduction in the required cache size \(N\) for any average cache hit probability desired \(E_{\pi}[F(Z)]\), compared to the uniform request model in Fig. 4 (left). Different from the uniform demand model, we see that the scaling for the three different models is similar for \(E_{\pi}[F(Z)] > 0.9\). This is because under this spatially correlated demand model, the tail of the demand is negligible compared to the uniform request model such that less cache over-provisioning is required to capture requests that have lower popularities. While the concentration results are improved because the demand is more correlated rather than being uniform at random, over-provisioning is still required to improve the cache hit performance beyond \(E_{\pi}[F(Z)] > 0.8\).
Note that for the same amount of provisioning the threshold required for the uniform demand model is approximately $\mathbb{E}_x[F(Z)] > 0.6$, as shown in Fig. 4 (left). From Fig. 6 (right), we observe that for the tilted request model in (8), the concentration results are not improved. The performances of the independent and MatII models match the results obtained for the uniform spatial traffic model because the tilting of the demand within a $10 \times 10$ pixel region does not modify the demand as the intensity does not fluctuate much within an urban region (see Fig. 5 (Left)). However, the GEC scheme requires more over-provisioning to achieve a similar average performance. While it is not clear whether among all classes of mark distributions that yield exclusion-based placement configurations, the gamma-distributed marks can achieve the best provisioning performance, the mark distribution parameters could be fine tuned to improve the tail performance. Hence, more analytical investigation is needed to realize the attainable performance of the GEC model under different mark distribution models.

Cache size over-provisioning is required for performance optimization in hybrid storage techniques. Our results indicate that the over-provisioning requirement can be made much smaller than the existing spatial caching models, for both the spatially uniform demand and the non-uniform demand. While having no over-provisioned space decreases the performance due to the update cost, having too much over-provisioning can cause performance decrease due to less data caching [26]. Our GEC approach shows that it is possible to decrease both the average cache space needed and the over-provisioning required. Via a more effective use of cache capacity, the maintenance costs can be significantly reduced and the lifetime of caches can be prolonged.
V. CONCLUSIONS

We devised a spatial $\gamma$-exclusion caching model (GEC) in which nodes can cooperate via proximity-based techniques to decide how to populate their caches. Our approach exploits the second order properties of the placement configuration and the variability of the demand in a geographic setting. We demonstrate that our approach provides significantly better concentration of the cache storage size compared to non-cooperative or simple cooperative techniques that do not make a full use of the correlations in content placement. Our results suggest guarantees on the over-provisioned cache space, which can help improve the performance of caching networks. We believe that GEC gives insights into not only how to cache the content, but also how to effectively sample in spatial settings. GEC is suited for enabling proximity-based applications such as D2D and P2P as it promotes the item diversity and reciprocation.

Extensions of this work include devising more balanced algorithms exploiting strong Rayleigh measures that imply negative association and have the stochastic covering property [42, Sect. 4.2]. They also include the joint optimization of the spatio-temporal demand dynamics, and employment of exclusion-based models to optimize the performance of time-to-live caches.

APPENDIX

A. Functions Describing the Second-Order Behavior of the Point Processes

The spatial regularity and second-order properties of $\Phi$ can be characterized by the reduced second moment function which is known as Ripley’s $K$-function $K(r), r \geq 0$. The mean number of points of $\Phi$ within a ball of radius $r$ and centred at the typical point, which is not itself counted in the mean is given by $\lambda K(r) = \mathbb{E}^{i_0} [\Phi(B_0(r))]$ [33, Ch. 2.3]. While the $K$ function for GEC is not easy to characterize, some special cases are easy to analyze.

Using the Campbell’s theorem [34, Ch. 1.4], we deduce that the average number of transmitters of the stationary point process $\Phi_{th,i}$ – conditioned on there being a point at the origin but not counting it – contained in the ball $B = B_0(R_c)$ is given by

$$\lambda K(B_0(R_c)) = \mathbb{E}^{i_0}[\Phi_{th,i}(B)] = \lambda^{-1} \int_{B_0(R_c)} \rho_i^{(2)}(x) dx = \lambda \int_{B_0(R_c)} k_i(x) dx,$$

(35)

where $\rho_i^{(2)}(r)$ is the second-order product density (SOPD) corresponding to $\Phi_{th,i}$, and $k_i(r) = \lambda^{-2} \rho_i^{(2)}(r)$ is the two-point Palm probability that two points of $\Phi$ separated by distance $r$ are
both retained to store item \( i \). For a stationary p.p. \( \Phi_{th} \), the SOPD is the joint probability that there are two points of \( \Phi_{th} \) at locations \( x, y \) in the infinitesimal volumes \( dx, dy \) \cite[Ch. 5.4]{33}.

We next give the SOPD for MatII.

**Proposition 11.** For a stationary p.p. \( \Phi_{th} \) that is MatII, the SOPD is given by \cite[Ch. 5.4]{33}

\[
\rho_{i}^{(2)}(r) = \begin{cases} 
\lambda_{hcp}^2, & r \geq 2r_i \\
\frac{2V_{r_i}(r)[1-\exp(-\lambda\pi r_i^2)]-2\pi r_i^2[1-\exp(-\lambda V_{r_i}(r))]}{\pi r_i^2 V_{r_i}(r)[V_{r_i}(r)-\pi r_i^2]}, & r_i < r < 2r_i, \\
0, & r \leq r_i, 
\end{cases}
\]

where \( V_{r_i}(r) = 2\pi r_i^2 - l_2(r_i, r) = 2\pi r_i^2 - 2r_i^2 \cos^{-1}\left(\frac{r}{2r_i}\right) + r\sqrt{r_i^2 - \frac{r^2}{4}} \) \cite[Eqn. (5.60)]{33} is the area of the union of two circles with radii \( r_i \) that are separated by \( r \). Pairwise correlations between the points separated by \( r > r_i \) are modeled using the SOPD \(-\rho_{i}^{(2)}(r)\) for item \( i \) of the MatII hard-core point process.

A special case of MatII is MatI in which all nodes have the same mark values. Hence, each node is retained as long as there is no other node within the exclusion radius, and therefore we have \( \lambda_{hcp-I} = \lambda \exp(-\lambda\pi r_i^2) \). The \( K \) function for MatI can be given using \cite[35]{35}, where any pair of points of \( \Phi \) separated by less than a critical distance \( r_i \) are deleted. Hence, the second-order product density for MatI is given as

\[
\rho_{i}^{(2)}(r) = \begin{cases} 
\lambda^2 \exp(-\lambda(2\pi r_i^2 - V_{r_i}(r))), & r \geq r_i, \\
0, & r < r_i. 
\end{cases}
\]

The covariance or two-point probability function of a random set with volume fraction \( p \) is

\[
C(r) = \mathbb{P}(0 \in B, r \in B) = \mathbb{P}(0 \notin B, r \notin B) + 2p - 1, \quad r \in \mathbb{R}^2. 
\]

For example, if \( \Xi \) is a Boolean model with typical grain \( \Xi_0 \), then its covariance is

\[
C(r) = 2p - 1 + (1 - p)^2 \exp(\lambda\mathbb{E}[\gamma_{\Xi_0}(r)]), 
\]

where \( \gamma_B(r) = \nu_2(B \cap (B - r)) \) is the set covariance of the convex set \( B \), where \( r \in \mathbb{R}^2 \) with \( \|r\| = r \), and \( \nu_2 \) is the 2-dimensional Lebesgue measure, i.e. the area measure \cite[1.7.2]{33}.

The definitions of the functions discussed in the section for the baseline PPP \( \Phi \) and MatII \( \Phi_{th, i} \) models are summarized in Table II.
| Definition               | Function                          | Poisson p.p. $\Phi$ | Matérn II hard-core p.p. $\Phi_{th,i}$ |
|-------------------------|-----------------------------------|---------------------|----------------------------------------|
| Volume fraction         | $p$                               | $p = 1$             | $p = \lambda_{hcp}(i)/\lambda$        |
| Covariance              | $C(r) = C_B(r)$                   | $p^2$               | $\approx p(1 - p) \exp(-\alpha r) + p^2$ [43], [44] |
| Set covariance (Variogram) | $\gamma_B(r) = C(0) - C(r)$      | 0                   | $p - C(r)$                             |
| Covariance function     | $k(r) = C(r) - p^2$               | 1                   | $\lambda^{-2} p^{(2)}(r)$             |
| Ripley’s $K$ function   | $K(r)$                            | $\pi r^2$           | $\lambda^{-1} E_i[\Phi_{th,i}(B)]$    |
| Second-order product density | $\rho^{(2)}(r) = \lambda^2 \frac{dK(r)}{dr} / 2\pi r$ | $\lambda^2$         | Eq. (36)                               |
| Pair correlation function | $g(r) = \frac{E[r_C(r)]}{\lambda^2} \approx C(r) / \rho^2$ | 1                   | Eq. (36) / $\lambda^2$                |

**TABLE II**

Functions describing the second-order behavior of the point processes in $\mathbb{R}^2$ [33, Ch. 4].

---

**B. Proof of Proposition 7**

The variance of $F(Z)$ is given as

$$
\text{Var}[F(Z)] = \mathbb{E}\left[ \left( \sum_{(i,p) \in R} \lambda_{(i,p)} \sum_{k=1}^{[p]-1} w_{p_{k+1}p_k} (1 - f_{z_i}(1, \ldots, k)) \right)^2 \right] - \mathbb{E}[F(Z)]^2
$$

$$
= \sum_{(i,p) \in R} \sum_{(i',p') \in R} \sum_{k=1}^{[p]-1} \sum_{k'=1}^{[p']-1} \lambda_{(i,p)} \lambda_{(i',p')} w_{p_{k+1}p_k} w_{p'_{k'+1}p'_k} \times

\left\{ \mathbb{E}[f_{z_i}(1, \ldots, k)] f_{z_{i'}}(1, \ldots, k') - \mathbb{E}[f_{z_i}(1, \ldots, k)] \mathbb{E}[f_{z_{i'}}(1, \ldots, k')] \right\}.
$$

Note that the above expression can be rewritten as

$$
\text{Var}[F(Z)] = \sum_{(i,p) \in R} \sum_{k=1}^{[p]-1} \lambda_{(i,p)}^2 w_{p_{k+1}p_k}^2 \left\{ \mathbb{E}[f_{z_i}(1, \ldots, k)]^2 - \mathbb{E}[f_{z_i}(1, \ldots, k)]^2 \right\} +

\sum_{(i,p) \in R} \sum_{(i',p') \in R} \sum_{(i',p') \neq (i,p)} \sum_{k=1}^{[p]-1} \sum_{k'=1}^{[p']-1} \lambda_{(i,p)} \lambda_{(i',p')} w_{p_{k+1}p_k} w_{p'_{k'+1}p'_k} \times

\left\{ \mathbb{E}[f_{z_i}(1, \ldots, k)] f_{z_{i'}}(1, \ldots, k') - \mathbb{E}[f_{z_i}(1, \ldots, k)] \mathbb{E}[f_{z_{i'}}(1, \ldots, k')] \right\}.
$$

Note also that the variance of $F(Z^*)$ is given as

$$
\text{Var}[F(Z^*)] = \sum_{(i,p) \in R} \sum_{k=1}^{[p]-1} \lambda_{(i,p)}^2 w_{p_{k+1}p_k}^2 \left\{ \prod_{l=1}^{k} \mathbb{E}\left[(1 - z_{p_{l(i)}}^*)^2\right] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z_{p_{l(i)}}^*)^2\right] \right\}.
$$

For the exponential covariance approximation for MatII given above, $\alpha \in \mathbb{R}^+$ describes the degree of variability in $B$ [43].
Given feasible NA sequences $Z$, we have that
\[
\left\{ \mathbb{E}[f_{\tilde{z}}(1, \ldots, k) f_{\tilde{z}'}(1, \ldots, k')] - \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] \mathbb{E}[f_{\tilde{z}'}(1, \ldots, k')] \right\} \leq 0,
\]
and equality is satisfied for $Z = Z^*$.

Our goal is to prove that the following relation holds:
\[
\mathbb{E}[f_{\tilde{z}}(1, \ldots, k)^2] - \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)]^2 \leq \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})^2\right] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right]^2.
\]
Since $(1 - z^*_{p_l i})^2 = (1 - z^*_{p_l i})$ for binary variables, we need to show that
\[
\mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \leq \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)]^2 - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right]^2
\]
\[
= \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right]
\leq \left( \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \right) \left( \mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] + \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \right).
\]
Since $\mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] - \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \leq 0$, the above condition is true when $\mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] + \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \in [0, 1]$. Since $\mathbb{E}[f_{\tilde{z}}(1, \ldots, k)] \leq \prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right]$, it is sufficient to have $\prod_{l=1}^{k} \mathbb{E}\left[(1 - z^*_{p_l i})\right] \in [0, 1/2]$. This is true whenever $\mathbb{E}[z^*_{p_l i}] \leq 1/2$ for all $p_l \in p$.

C. **Proof of Prop. 2**

For the special case of MatII (Corollary 1), we have that
\[
\mathbb{P}(\Phi_{th,i}(B) > 0) = \lambda_{hcp}(i) \pi R_c^2, \quad r_i \geq R_c.
\]
The cache hit probability for $r_i < R_c$ can be bounded below as
\[
\mathbb{P}(\Phi_{th,i}(B) > 0) \geq 1 - \exp(-\lambda_{hcp}(i) \pi R_c^2), \quad r_i < R_c,
\]
which is due to NA property of MatII. Hence the miss probability is lower than that of the independently thinning of the original PPP $\Phi$ with intensity $\lambda$ with probability $\lambda_{hcp}(i)/\lambda$. 
Similarly, the cache hit probability for $r_i < R_c$ can be bounded above as
\[
P(\Phi_{th,i}(B) > 0) \leq 1 - \exp(-\lambda_{hc}(i)\pi R_c^2) + \lambda^{-1} \int_{B_0(R_c)} \rho_i^{(2)}(x)dx, \quad r_i < R_c, \quad (42)
\]
where the upper bound follows from Markov’s inequality.

\section*{D. Proof of Prop. \[3\]}

Using the notation $A_i = \mathbbm{1}(\Phi_{th,i}(B) > 0)$ for $i = 1, \ldots, M$, the variance of $F(Z)$ satisfies
\[
\text{Var}[F(Z)] = \sum_i p_r^2(i) \text{Var}[A_i] + 2 \sum_{1 \leq i < j \leq M} p_r(i)p_r(j) \text{Cov}[A_i, A_j]
\]
\[
\overset{(a)}{=} \sum_i p_r^2(i) \text{Var}[A_i] = \sum_i p_r^2(i)P(\Phi_{th,i}(B) > 0)(1 - P(\Phi_{th,i}(B) > 0))
\]
\[
\overset{(b)}{\leq} \sum_{i=1}^{m_c} p_r^2(i) \left[ 1 - e^{-\lambda_{hc}(i)\pi R_c^2} + \lambda^{-1} \int_{B_0(R_c)} \rho_i^{(2)}(x)dx \right] e^{-\lambda_{hc}(i)\pi R_c^2}
\]
\[
+ \sum_{i=m_c+1}^{M} p_r^2(i)\lambda_{hc}(i)\pi R_c^2(1 - \lambda_{hc}(i)\pi R_c^2)
\]
\[
\overset{(a)}{=} \sum_{i=1}^{m_c} p_r^2(i) \lambda_{hc}(i)\pi R_c^2 e^{-\lambda_{hc}(i)\pi R_c^2} + \sum_{i=m_c+1}^{M} p_r^2(i)\lambda_{hc}(i)\pi R_c^2 e^{-\lambda_{hc}(i)\pi R_c^2}
\]
\[
= \sum_{i=1}^{M} p_r^2(i) \lambda_{hc}(i)\pi R_c^2 e^{-\lambda_{hc}(i)\pi R_c^2} + \sum_{i=1}^{m_c} p_r^2(i) \left[ \lambda^{-1} \int_{B_0(R_c)} \rho_i^{(2)}(x)dx \right] e^{-\lambda_{hc}(i)\pi R_c^2}
\]
\[
\overset{(b)}{\leq} \frac{1}{e} \sum_{i=1}^{M} p_r^2(i) + \frac{2\pi}{\lambda} \sum_{i=1}^{m_c} p_r^2(i) \int_{r_i}^{R_c} \rho_i^{(2)}(r)rdr \leq \frac{1}{e} \sum_{i=1}^{M} p_r^2(i) + \frac{\pi}{\lambda} \sum_{i=1}^{m_c} p_r^2(i) \left( \frac{1 - e^{-\lambda r_i^2}}{\pi^2 r_i^4} \right) (R_c^2 - r_i^2)
\]
\[
= \frac{1}{e} \sum_{i=1}^{M} p_r^2(i) + \pi \lambda \sum_{i=1}^{m_c} p_r^2(i)p_c^2(i)(R_c^2 - r_i^2) \leq \sum_{i=1}^{M} p_r^2(i) \left( \frac{1}{e} + \pi \lambda p_c^2(i)(R_c^2 - r_i^2) \right),
\]
where (a) is due to the independent placement of each item across caches. (b) is from (41)-(42).

\section*{E. Proof of Theorem \[3\]}

From Prop. \[1\] we have that
\[
\eta_{SSCC}(r, \delta) = \int_{R} \int_{0}^{1} \exp \left( -u\lambda \int_{R} (\pi(m+n)^2 - l_2(r,n)) \mu(dn) \right) du \mu(dm)
\]
\[
= \mathbb{E}_m \left[ \mathbb{E}_U \left[ \exp \left( -U q(\lambda, r, m) \right) \right] \right] = \mathbb{E}_m \left[ \frac{1 - \exp \left( -q(\lambda, r, m) \right)}{q(\lambda, r, m)} \right]. \quad (43)
\]
In (43), \( U \sim U[0, 1] \) is a uniformly distributed random variable, and
\[
q(\lambda, r, m) = \lambda \mathbb{E}_m \left[ \pi (m + m_2)^2 - l_2(r, m_2) \right] = \lambda \pi (m^2 + 2m\bar{m}_2) + \lambda \mathbb{E}_m [\pi m_2^2 - l_2(r, m_2)].
\]

In (43), let \( f(m) = \exp(-U \lambda \pi (m^2 + 2m\bar{m}_2)) \). Hence, \( f' = (-U \lambda \pi (2m + 2\bar{m}_2)) f < 0 \), and \( f'' = (-2U \lambda \pi) f + (U \lambda \pi (2m + 2\bar{m}_2))^2 f \geq 0 \) that is convex in \( m \) provided that \( m + \bar{m}_2 \geq \sqrt{\frac{1}{2U \lambda \pi}} \).

Since the baseline process \( \Phi \) is a homogeneous PPP with intensity \( \lambda \), we have \( \bar{m}_2 \geq \frac{1}{2\sqrt{\lambda}} \). If \( m + m_2 \geq \frac{1}{\sqrt{\lambda}} \) is satisfied, then the condition \( U > \frac{1}{2\pi} \) is sufficient for the convexity. In addition, we can show that \( \eta_{\text{SSCC}}(r, \vec{d}) = \mathbb{E}_m [g(m)] \), with \( g = \frac{f^{-1}(\text{exp}(x))}{x} \), where \( x = U \lambda \pi (m^2 + 2m\bar{m}_2) \). Then \( g' = \frac{\text{exp}(x)(x+1)-1}{x^2}, \) and \( g'' = \frac{2-\text{exp}(x)x^2+2x+2}{x^3} > 0 \). Hence,
\[
\eta_{\text{SSCC}}(r, \vec{d}) = \mathbb{E}_m [g(m)] \geq g(\bar{m}) = \eta_{\text{MatII}}(r, \bar{m}).
\]

F. Proof of Prop. 4

From [33, Ch. 4], the variances of point-counts of \( \Phi_{\text{th},i} \) can be calculated as
\[
\text{Var}[\Phi_{\text{th},i}(B)] = \lambda_{\text{hcp}}^2(i) \int_0^\infty \gamma_{B}(r) dK(r) + \lambda_{\text{hcp}}^2(i) \pi R_c^2 \left(-\lambda_{\text{hcp}}(i) \pi R_c^2 \right)^2,
\]
where \( \gamma_{B}(r) \) is the set covariance of the convex set \( B \) for MatII (see Appendix A Table II).

For large \( B = B_0(R_c) \), i.e., spherically infinite according to [45], we have that
\[
\frac{\text{Var}[\Phi_{\text{th},i}(B)]}{\pi R_c^2} \approx \lambda_{\text{hcp}}(i) + 2\pi \lambda_{\text{hcp}}^2(i) \int_0^\infty (g_i(r) - 1) r dr
\]
\[
= \lambda_{\text{hcp}}(i) - 4\pi \lambda_{\text{hcp}}^2(i) r_i^2 + 2\pi \int_{r_i}^{2r_i} \rho_i^{(2)}(r) r dr \overset{(a)}{\leq} \lambda_{\text{hcp}}(i) - \pi \lambda_{\text{hcp}}^2(i) r_i^2 \overset{(b)}{\leq} \lambda_{\text{hcp}}(i) e^{-\pi r_i^2},
\]
where \( g_i(r) \) is the pair correlation function of \( \Phi_{\text{th},i} \) (see Appendix A Table II). \( (a) \) follows from observing that \( \rho_i^{(2)}(r) \leq \lambda_{\text{hcp}}^2(i) \) from (36), and \( (b) \) follows from \( 1 - x - \exp(-x) \leq 0 \) for \( x \geq 0 \).

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