THE SKELETON OF THE JACOBIAN, THE JACOBIAN OF THE SKELETON, AND LIFTING MEROMORPHIC FUNCTIONS FROM TROPICAL TO ALGEBRAIC CURVES

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ABSTRACT. Let $K$ be an algebraically closed field which is complete with respect to a nontrivial, non-Archimedean valuation and let $Λ$ be its value group. Given a smooth, proper, connected $K$-curve $X$ and a skeleton $Γ$ of the Berkovich analytification $X^{an}$, there are two natural real tori which one can consider: the tropical Jacobian $Jac(Γ)$ and the skeleton of the Berkovich analytification $Jac(X)^{an}$. We show that the skeleton of the Jacobian is canonically isomorphic to the Jacobian of the skeleton as principally polarized tropical abelian varieties. In addition, we show that the tropicalization of a classical Abel-Jacobi map is a tropical Abel-Jacobi map. As a consequence of these results, we deduce that $Λ$-rational principal divisors on $Γ$, in the sense of tropical geometry, are exactly the retractions of principal divisors on $X$. We actually prove a more precise result which says that, although zeros and poles of divisors can cancel under the retraction map, in order to lift a $Λ$-rational principal divisor on $Γ$ to a principal divisor on $X$ it is never necessary to add more than $g$ extra zeros and $g$ extra poles. Our results imply that a continuous function $F : Γ → \mathbb{R}$ is the restriction to $Γ$ of $− \log |f|$ for some nonzero meromorphic function $f$ on $X$ if and only if $F$ is a $Λ$-rational tropical meromorphic function, and we use this fact to prove that there is a rational map $f : X → \mathbb{P}^3$ whose tropicalization, when restricted to $Γ$, is an isometry onto its image.

Throughout this paper, $K$ denotes a field which is complete with respect to a nontrivial, non-Archimedean valuation $\text{val} : K → \mathbb{R} \cup \{∞\}$. Let $R$ be its valuation ring, let $k$ be its residue field, and let $Λ = \text{val}(K^{×})$ be its value group. Note that $Λ$ is either discrete or dense in $\mathbb{R}$. Let $|·| = \exp(−\text{val}(·))$ be an associated absolute value.

1. Introduction

In this section, we make the additional assumption that $K$ is algebraically closed. Let $X$ be a smooth, projective, connected $K$-curve and let $\breve{X}$ be a semistable $R$-model of $X$. Let $X^{an}$ be the analytification of $X$ in the sense of Berkovich $[\text{Ber90}]$. The incidence graph $Γ$ of the special fiber of $\breve{X}$ naturally has the structure of a finite metric graph with edge lengths in $Λ$; we let $Γ(Λ)$ denote the set of all points of $Γ$ whose distance from any vertex belongs to $Λ$. There is a natural inclusion $Γ(Λ) ↪ X^{an}$ and a deformation retraction $τ : X^{an} → Γ$. The metric graph $Γ$ is called the skeleton of $X$ (or $X^{an}$) associated to the model $X$. The retraction $τ$ takes $X(K)$ surjectively onto $Γ(Λ)$, hence defines (by linearity) a surjective homomorphism

$$τ_* : \text{Div}(X) → \text{Div}_Λ(Γ)$$

of divisor groups, where $\text{Div}_Λ(Γ)$ is the group of all $\mathbb{Z}$-linear combinations of points of $Γ(Λ)$. Since $τ_*$ preserves the degree of a divisor, it also induces a map on degree-zero divisors $τ_* : \text{Div}^0(X) → \text{Div}^0_Λ(Γ)$.

There is a well-developed theory of divisors and linear equivalence on graphs and on metric graphs recalled briefly here and in $[\text{Bak08}]$ see $[\text{Bak08}]$ and the references therein for details. A tropical meromorphic function on $Γ$ is a continuous function $F : Γ → \mathbb{R}$ which is piecewise affine with integer slopes. We say that $F$ is $Λ$-rational provided that $F(Γ(Λ)) ⊂ Λ$ and all points at which $F$ is not differentiable are contained in $Γ(Λ)$. For such an $F$ we define the divisor of $F$ to be $\text{div}(F) = \sum_{x ∈ Γ(Λ)} n_x(x)$, where $n_x$ is the sum of the outgoing slopes of $F$ at $x$. The group of ($Λ$-rational) principal divisors on $Γ$ is the subgroup $\text{Prin}_Λ(Γ) ⊂ \text{Div}^0_Λ(Γ)$ of divisors of $Λ$-rational tropical meromorphic functions.

The authors thank Grisha Mikhalkin for suggesting the “at most $g$ extra zeros and poles” assertion in Theorem 1.1. M.B. was partially supported by NSF grant DMS-1201473.
Let \( f \in K(X) \) be a nonzero rational function and let \( F \) be the restriction of \(- \log |f|\) to \( \Gamma \subset X^\text{an} \). It is a consequence of non-trivial results in non-Archimedean analysis (see Remark 5.4) that \( F \) is a \( \Lambda \)-rational tropical meromorphic function on \( \Gamma \) and that \( \tau_\ast(\text{div}(f)) = \text{div}(F) \). In particular, \( \tau_\ast \) takes principal divisors on \( X \) to \( \Lambda \)-rational principal divisors on \( \Gamma \). One of the main goals of this paper is to prove the following theorem:

**Theorem 1.1.** Let \( K \) be an algebraically closed complete non-Archimedean field, let \( X \) be a \( K \)-curve of genus \( g \), and let \( \Gamma \) be a skeleton of \( X \), as above. Then the map on principal divisors

\[
\tau_\ast : \text{Prin}(X) \longrightarrow \text{Prin}_\Lambda(\Gamma)
\]

is surjective. More precisely, for \( \overline{D} \in \text{Prin}_\Lambda(\Gamma) \), writing \( \overline{D} = \overline{D}_1 - \overline{D}_2 \) for effective divisors \( \overline{D}_1, \overline{D}_2 \in \text{Div}_\Lambda^\ast(\Gamma) \) of degree \( n \), there exist effective divisors \( D_1, D_2 \in \text{Div}^{\ast \cdot g}(X) \) of degree \( n + g \) such that \( D = D_1 - D_2 \in \text{Prin}(X) \) and \( \tau_\ast(D) = \overline{D} \).

The second assertion states that, although zeros and poles of divisors can “cancel out” under the retraction map \( \tau_\ast \), in order to lift a \( \Lambda \)-rational principal divisor on \( \Gamma \) to a principal divisor on \( X \) it is never necessary to add more than \( g \) zeros and poles. Theorem 1.1 is interesting for several reasons. From the viewpoint of tropical geometry, it says that the \( \Lambda \)-rational principal divisors on \( \Gamma \) (which one can think of as an “abstract complete tropical curve”) are exactly the retractions of the principal divisors on \( X \) for any curve \( X \) containing \( \Gamma \) as a skeleton. (Note that if we replace any with some then this weaker statement is a consequence of [ABBR13, Theorem 9.9].) This gives a precise and very close connection between the notions of principal divisors in tropical and algebraic geometry.

Theorem 1.1 is also interesting from the point of view of non-Archimedean analysis. The following corollary is immediate:

**Corollary 1.2.** In the situation of Theorem 1.1 let \( f : \Gamma \to \mathbb{R} \) be a continuous function. Then there exists a nonzero rational function \( f \in K(X) \) such that \( F = -\log |f| |_{\Gamma} \) if and only if \( F \) is a \( \Lambda \)-rational tropical meromorphic function.

Among other things, Corollary 1.2 can be used to simplify and sharpen the proofs of some of the “faithful tropicalization” results in [BPR11]. For example, we show in 5.8 that given a \( K \)-curve \( X \) together with a skeleton \( \Gamma \), there is a rational map \( f : X \dashrightarrow \mathbb{P}^1 \) whose tropicalization, when restricted to \( \Gamma \), is an isometry onto its image (with respect to lattice lengths in \( \mathbb{R}^2 \)). The main point is that Corollary 1.2 allows us to reduce such questions to purely combinatorial problems about metric graphs.

The other main goal of this paper is to prove a natural compatibility between classical and tropical Abel-Jacobi maps. This compatibility forms the backbone of our proof of Theorem 1.1 so our two main goals are closely related. To be more precise, let \( X \) be a \( K \)-curve as above and let \( \Gamma \) be a skeleton for \( X \). Choose a \( K \)-point \( P \in X(K) \) and let \( p = \tau(P) \in \Gamma(\Lambda) \). Let \( \alpha_P : X \to J \) and \( \alpha_P : \Gamma \to \text{Jac}(\Gamma) \) be the classical and tropical Abel-Jacobi maps based at \( P \) and \( p \), respectively. (The tropical Abel-Jacobi map is defined in 4.3 below.) We prove:

**Theorem 1.3.** There is a canonical isomorphism \( \text{Jac}(\Gamma) \cong \Sigma(J^\text{an}) \), where \( \Sigma(J^\text{an}) \) is the skeleton of \( J^\text{an} \) in the sense of Berkovich, such that (identifying \( \alpha_P \) with the corresponding map from \( \Gamma \) to \( \Sigma(J^\text{an}) \)) the following square commutes:

\[
\begin{array}{ccc}
X^\text{an} & \xrightarrow{\alpha_P} & J^\text{an} \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\alpha_P} & \Sigma(J^\text{an})
\end{array}
\]

where the vertical arrows are the natural retraction maps. Moreover, the isomorphism \( \text{Jac}(\Gamma) \cong \Sigma(J^\text{an}) \) is compatible with the natural principal polarizations on both sides (so defines an isomorphism of principally polarized tropical abelian varieties).

See Corollary 6.6 and and Proposition 6.1. The fact that \( \text{Jac}(\Gamma) \cong \Sigma(J^\text{an}) \) can be summarized as:
The skeleton of the Jacobian is the Jacobian of the skeleton.

A closely related result has recently been established in [Viv13, Theorem A].

2. The analogy with arithmetic geometry; strategy of proof

In order to explain how our two main goals are related, and to outline the proofs of Theorems 1.1 and 1.3 it is useful to make a digression into what was previously known in the analogous situation where \( K \) is discretely valued. This analogy with arithmetic geometry was one of the original motivations for this study.

2.1. The discretely valued situation. Suppose that the valuation on \( K \) is discrete with value group \( \Lambda = \mathbb{Z} \) and that the residue field \( k \) is algebraically closed. As in §1 we let \( X \) be a smooth, proper, geometrically connected \( K \)-curve with semistable \( R \)-model \( \bar{X} \). For simplicity we also assume that \( X \) is regular and that \( \bar{X}_k \) is (split) semistable with smooth irreducible components. In this case \( \Gamma(\Lambda) = \Gamma(\mathbb{Z}) \) is the set of vertices of \( \Gamma \), i.e. all edges have length one, so we may regard \( \Gamma \) as an ordinary (non-metric) graph \( G \); then the theory of \( \Lambda \)-rational divisors and linear equivalence on \( \Gamma \) reduces to Baker and Norine’s notions of divisors and linear equivalence on \( G \) from [BN07].

The abelian group

\[
\text{Jac}(G) = \text{Jac}_\mathbb{Z}(\Gamma) := \text{Div}_0^0(\Gamma)/\text{Prin}_\mathbb{Z}(\Gamma)
\]

is called the Jacobian of \( G \) (or the regularized Jacobian of \( \Gamma \)). It has the following alternative description. Let \( M = H_1(\Gamma, \mathbb{Z}) \) be the first homology group of (the topological realization of) \( \Gamma \) and let \( N = \text{Hom}(M, \mathbb{Z}) \) be its dual. We regard \( M \) as the group of cycles in the free abelian group \( C_1(G, \mathbb{Z}) \) generated by the edges of \( G \). Define a pairing \( \langle \cdot, \cdot \rangle \) on \( C_1(G, \mathbb{Z}) \) by

\[
\langle e, e' \rangle = \begin{cases} 
1 & \text{if } e = e' \\
0 & \text{otherwise.}
\end{cases}
\]

This restricts to a symmetric, positive-definite pairing \( \langle \cdot, \cdot \rangle : M \times M \to \mathbb{Z} \), hence defines an injective homomorphism \( \mu : M \to N \).

**Proposition 2.2.** There is a canonical short exact sequence

\[
0 \to M \xrightarrow{\mu} N \to \text{Jac}(G) \to 0.
\]

Proposition 2.2 should be viewed as a compatibility between the divisor-theoretic description of \( \text{Jac}_\mathbb{Z}(\Gamma) \), and a description in terms of “the dual of a space of one-forms modulo a period lattice.” See §3 for details.

Let \( J \) be the Jacobian of \( X \), let \( \bar{\mathfrak{g}} \) be its Néron model over \( R \), let \( \mathfrak{g}^0 \) be the (fiberwise) connected component of the identity in \( \bar{\mathfrak{g}} \), and let \( \Phi = \mathfrak{g}^0/\mathfrak{g}_k^0 = J(K)/\mathfrak{g}^0(R) \) be the component group (the second equality holds using [Liu02, Proposition 10.1.40(b)]). Since \( X \) has split semistable reduction, it is known that \( \mathfrak{g}_k^0 \) is a split semi-abelian variety.

By a general theorem of Grothendieck [GRR72, Exposé IX, 11.5], the component group \( \Phi \) fits into an exact sequence of the form

\[
0 \to M' \to N' \to \Phi \to 0,
\]

where \( M' \) is the character lattice of the toric part of \( \mathfrak{g}_k^0 \), \( N' \) is its dual, and \( \mu_{\text{mon}} \) is the natural map derived from the monodromy pairing \( \langle \cdot, \cdot \rangle_{\text{mon}} : M' \times M' \to \mathbb{Z} \).

A proof of the following result can be found in [BL02]; see also [GRR72, III.91].

**Theorem 2.3.** The character lattice \( M' \) of the toric part of \( \mathfrak{g}_k^0 \) is canonically isomorphic to \( M \), and the pairing \( \langle \cdot, \cdot \rangle \) defined above coincides with Grothendieck’s monodromy pairing under this isomorphism.
The previously published proofs of this result use arithmetic intersection theory and rest on a Picard-Lefschetz argument. We are able to derive (the second part of) Theorem 2.3 from our results, thus giving a very different proof; see Remark 6.4.

As an immediate consequence of Proposition 2.2 and Theorem 2.3 one has the following discrete analogue of the second part of Theorem 1.3.

**Corollary 2.4.** There is a canonical isomorphism $\text{Jac}(G) \cong \Phi$.

Corollary 2.4 is a reformulation of Raynaud’s famous description of the component group $\Phi$ in terms of the intersection matrix of the special fiber of $X$ [Ray70, Proposition 8.1.2], since the latter coincides with the Laplacian matrix of $G$. Raynaud’s result plays an important technical role in many key papers on the arithmetic of modular curves, including for example [Maz77, Rib90].

We next turn to a discrete analogue of Theorem 1.1. For this, we define the specialization map $\rho : \text{Div}(X) \to \text{Div}_Z(\Gamma)$ from divisors on $X$ to divisors on $G$ as follows. For $x \in \Gamma(\mathbb{Z})$ let $C_x$ be the associated irreducible component of $X_k$, considered as a divisor on $X$. If $D \in \text{Div}(X)$ is a divisor then its Zariski closure $\mathcal{O}$ in $X$ is again a Weil divisor; we set

$$\rho(D) = \sum_{x \in \Gamma(\mathbb{Z})} (C_x \cdot \mathcal{O})(x),$$

where $C_x \cdot \mathcal{O}$ is the intersection number $\deg((C_x \cdot \mathcal{O})|_{C_x})$. (Note that this map $\rho$ does not coincide with the extension by linearity of the retraction map $\tau : X(\mathbb{K}) \to \Gamma$, as the image of the latter is not contained in $\Gamma(\mathbb{Z})$. However, $\tau_*(D)$ and $\rho(D)$ are always linearly equivalent divisors: see [Bak08, Remark 2.10].) By [Bak08, Lemma 2.1], $\rho$ takes principal divisors to principal divisors, i.e., restricts to a map $\rho_0 : \text{Prin}(X) \to \text{Prin}_Z(\Gamma)$.

**Theorem 2.5.** The following diagram

\[
\begin{array}{ccc}
0 \rightarrow & \text{Prin}(X) \rightarrow & \text{Div}^0(X) \rightarrow & J(K) \rightarrow & 0 \\
& \downarrow \rho_0 \downarrow \rho & & \downarrow & \\
0 \rightarrow & \text{Prin}_Z(\Gamma) \rightarrow & \text{Div}^0_Z(\Gamma) \rightarrow & \text{Jac}_Z(\Gamma) = \Phi \rightarrow & 0
\end{array}
\]

is commutative, where the right vertical arrow is the canonical quotient map $J(K) \to J(K)/\mathfrak{J}^0(R) = \Phi$.

**Theorem 2.5** is a consequence of a theorem of Raynaud [BLR90, Theorem 9.6.1], as was observed by the first author [Bak08, Appendix A].

The specialization map from $X(K)$ to the smooth locus of $X_k(k)$ is surjective (see for instance [Liu02, Proposition 10.1.40(b)]); hence $\rho : \text{Div}^0(X) \to \text{Div}^0_Z(\Gamma)$ is surjective. Applying the snake lemma to the morphism of short exact sequences (2.5.1), we obtain an exact sequence

$$0 \rightarrow \text{Prin}^0(X) \rightarrow \text{Div}^0(X) \rightarrow \mathfrak{J}^0(R) \rightarrow \text{coker}(\rho_0) \rightarrow 0,$$

where $\text{Div}^0(X) = \ker(\rho)$ and $\text{Prin}^0(X) = \ker(\rho_0) = \text{Div}^0(X) \cap \text{Prin}(X)$. A celebrated theorem of Raynaud [Ray70, BLR90, Theorem 9.5.4] asserts that $\mathfrak{J}^0$ is the relative Jacobian of $X$. That is to say, $\mathfrak{J}^0$ represents the functor of line bundles of total degree zero, which implies that $\mathfrak{J}^0(R)$ is the group of those line bundles on $J$ represented by divisors $D$ such that $\rho(D) = 0$. Therefore we have $\mathfrak{J}^0(R) = \text{Div}^0(X)/\text{Prin}^0(X)$, so we obtain the discretely-valued version of Theorem 1.1.

**Theorem 2.6.** ([Bak08, Appendix A]) With the above hypotheses, the specialization map

$$\rho_0 : \text{Prin}(X) \rightarrow \text{Prin}_Z(\Gamma)$$

is surjective.

**2.7. Strategy of the proofs.** Much of this paper is devoted to developing analogues of the tools and language used above for more general (not necessarily discretely valued) non-Archimedean fields $K$. Our “tropical” point of view on these results forms an attractive framework which should be useful in its own right. The primary difficulty is that Néron models are not available in the non-Noetherian setting, so we are forced to work with the special fiber of the reduction of $X$ at primes of $K$.
situation. Instead we use the non-Archimedean uniformization theory of Jacobians, which was worked out by Bosch, Lütkebohmert, and Raynaud in this generality.

Let $K$ be any complete non-Archimedean field and let $X$ be a smooth, projective, geometrically connected $K$-curve endowed with a split semistable $R$-model $\mathfrak{X}$. As above we let $\Gamma \subset X^\an$ be the associated skeleton, with retraction map $\tau : X^\an \to \Gamma$. The $R$-model $\mathfrak{X}$ defines a weighted graph model $G$ for $\Gamma$ in the sense of (3.1) below; for ease of terminology we will refer to vertices and edges of $\Gamma$ instead of $G$.

Recall that $\operatorname{Div}^0(\Gamma)$ is the group of degree-zero divisors on $\Gamma$ supported on $\Gamma(\Lambda)$ and that $\operatorname{Prin}_\Lambda(\Gamma)$ is the group of divisors of $\Lambda$-rational tropical meromorphic functions on $\Gamma$. The set of $\Lambda$-points of the Jacobian of $\Gamma$ is defined to be the abelian group

$$\operatorname{Jac}_\Lambda(\Gamma) = \operatorname{Div}^0(\Gamma) / \operatorname{Prin}_\Lambda(\Gamma).$$

This group has the following alternative description. Let $M = H_1(\Gamma, \mathbb{Z})$ and let $N = \operatorname{Hom}(M, \mathbb{Z})$. The edge length pairing $\langle \cdot, \cdot \rangle$ on the free abelian group $C_1(\Gamma, \mathbb{Z})$ generated by the edges of $\Gamma$ is defined on edges by

$$(2.7.1) \quad \langle e, e' \rangle = \begin{cases} \ell(e) & \text{if } e = e' \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell(e)$ is the length of $e$. Regarding $M = H_1(\Gamma, \mathbb{Z})$ as a subgroup of $C_1(\Gamma, \mathbb{Z})$, we restrict $\langle \cdot, \cdot \rangle$ to a symmetric, positive-definite pairing on $M$. This is exactly analogous to (2.1.1). Since $\langle \cdot, \cdot \rangle$ takes values in $\Lambda$, we obtain a map $\mu : M \to N^\ast_\Lambda = \operatorname{Hom}(M, \Lambda) \subset N^\ast_R$. We have the following metric graph generalization of Proposition 2.2.

**Proposition 2.8.** There is a canonical short exact sequence

$$0 \to M \xrightarrow{\ell} N^\ast_\Lambda \to \operatorname{Jac}_\Lambda(\Gamma) \to 0.$$ 

See Proposition 3.2. We define the Jacobian of $\Gamma$ to be the real torus $\operatorname{Jac}(\Gamma) := \operatorname{Jac}_R(\Gamma) = N^\ast_R / \mu(M)$. Note that $\operatorname{Jac}_\Lambda(\Gamma) \subset \operatorname{Jac}(\Gamma)$.

Let $J$ be the Jacobian of $X$ and let $E$ be the universal cover of $J^\an$. This topological space has the structure of a $K$-analytic group making the structure map $E \to J^\an$ into a $K$-analytic group homomorphism. Its kernel $H_1(J^\an, \mathbb{Z}) \subset E(K)$ is canonically isomorphic to $M = H_1(\Gamma, \mathbb{Z}) = H_1(X^\an, \mathbb{Z})$. There exists a continuous map $\operatorname{trop} : E \to N^\ast_R = \operatorname{Hom}(M, R)$, which restricts to an isomorphism $\operatorname{trop} : M \xrightarrow{\sim} \operatorname{trop}(M)$ onto a full-rank lattice $\operatorname{trop}(M) \subset N^\ast_R$. Letting $\Sigma$ be the real torus $N^\ast_R / \operatorname{trop}(M)$, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & M & \xrightarrow{\ell} & E & \to & J^\an & \to & 0 \\
& \downarrow{\text{trop}} & & \downarrow{\text{trop}} & & \downarrow{\tau} & & \\
0 & \to & \operatorname{trop}(M) & \to & N^\ast_R & \to & \Sigma & \to & 0
\end{array}$$

The real torus $\Sigma$ is called the skeleton of the abelian variety $J$. Berkovich has shown [Ber90, §6.5] that there is an inclusion $\Sigma \hookrightarrow J^\an$, and that $\tau : J^\an \to \Sigma$ is a deformation retraction.

The map $\operatorname{trop} : M \hookrightarrow E^\an \to N^\ast_R$ defines a bilinear pairing $\langle \cdot, \cdot \rangle^\an : M \times M \to R$ which makes the triple $(\Sigma, M, \langle \cdot, \cdot \rangle^\an)$ into a principally polarized tropical abelian variety in the sense of (3.7). One of the main results of this paper is the following analogue of Theorem 2.3.

**Theorem 2.9.** The pairings $\langle \cdot, \cdot \rangle^\an$ and $\langle \cdot, \cdot \rangle$ coincide. As a consequence, there is a canonical isomorphism of principally polarized tropical abelian varieties $\Sigma \cong \operatorname{Jac}(\Gamma)$.

In the discretely-valued case, Theorem 2.9 is exactly Theorem 2.3. In fact, it is easy to derive Theorem 2.3 from Theorem 2.9 thus giving a very different proof; see Remark 6.4. In the special case where $X$ is a Mumford curve, Theorem 2.9 has also been proved by van der Put [vdP92, Theorem 6.4], using the theory of $p$-adic theta functions. Our proof in the general case is based on Theorem 1.3 (the compatibility between the algebraic Abel-Jacobi map and its tropical analogue).
Remark 2.10. Theorem 2.9 has the following interpretation. Let \( g \geq 2 \). There is a map \( \text{trop} : M_g^\text{an} \to \coprod_{g' \leq g} M_{g'}^\text{trop} \) from the analytification of the moduli space of genus \( g \) curves to the moduli space of metric graphs of genus \( g' \leq g \), which takes a curve \( X \) to its minimal skeleton. There is also a map \( \text{trop} : A_g^\text{an} \to \coprod_{g' \leq g} A_{g'}^\text{trop} \) from the moduli space of principally polarized abelian varieties of genus \( g \) to the moduli space of principally polarized tropical abelian varieties of dimension \( g' \leq g \), taking an abelian variety to its skeleton in the sense of Berkovich. There are Torelli maps \( M_g \to A_g \) and \( M_{g'}^\text{trop} \to A_{g'}^\text{trop} \) which take a curve (resp. graph) to its Jacobian. Theorem 2.9 is equivalent to the statement that the following square commutes:

\[
\begin{array}{ccc}
M_g^\text{an} & \xrightarrow{\text{trop}} & M_g^\text{trop} \\
\downarrow & & \downarrow \\
A_g^\text{an} & \xrightarrow{\text{trop}} & A_g^\text{trop}
\end{array}
\]

See also [Viv13, Theorem A], and compare with Remark 6.7.

The retraction map \( \tau : X(K) \to \Gamma(\Lambda) \) extends by linearity to give a map \( \tau_* : \text{Div}_K^0(X) \to \text{Div}_\Lambda^0(\Gamma) \), where \( \text{Div}_K^0(X) \) is the group of degree-zero divisors on \( X \) supported on \( X(K) \). Let \( \text{Prin}_K(X) \) be the kernel of the natural homomorphism \( \text{Div}_K^0(X) \to J(K) \). One checks after passing to the completion of the algebraic closure of \( K \) that \( \tau_*(\text{Prin}_K(X)) \subset \text{Prin}_\Lambda(\Gamma) \). We will prove the following analogue of Theorem 2.5, which says that the retraction map \( \tau : J^\text{an} \to \Sigma = \text{Jac}(\Gamma) \) is compatible with the descriptions of \( J(K) \) and \( \text{Jac}_\Lambda(\Gamma) \) in terms of divisors modulo principal divisors:

**Theorem 2.11.** We have \( \tau(J(K)) \subset \text{Jac}_\Lambda(\Gamma) \), and the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \xrightarrow{\text{Prin}_K(X)} & \text{Div}_K^0(X) & \xrightarrow{\tau_*} & J(K) \\
\downarrow & & \downarrow & \downarrow \tau_* & \\
0 & \xrightarrow{\text{Prin}_\Lambda(\Gamma)} & \text{Div}_\Lambda^0(\Gamma) & \xrightarrow{\tau} & \text{Jac}_\Lambda(\Gamma) & \xrightarrow{\tau} & 0
\end{array}
\]

Note that the arrow \( \text{Div}_K^0(X) \to J(K) \) is surjective when \( K \) is algebraically closed. The first part of Theorem 1.1 follows from Theorem 2.11 in the same way that Theorem 2.6 follows from Theorem 2.5 (applying the snake lemma to the above diagram), with a result of Bosch and Lütkebohmert playing the role of Raynaud’s theorem that the connected component of the Néron model of \( J \) represents the relative Jacobian. The second assertion in Theorem 1.1 requires a separate argument.

**Remark 2.12.** In fact we will prove a preliminary version of Theorem 2.11 before proving Theorem 2.9, see Proposition 5.3. This is necessary because Theorem 2.9 is proven using a compatibility of Abel-Jacobi maps, which already depends on the divisor-theoretic description of the retraction map \( \tau \).

**3. The Jacobian of a metric \( \Lambda \)-graph**

We briefly review the theory of metric graphs and their Jacobians, following [MZ08] (see also [BF11]).

3.1. Let \( \Lambda \) be a nonzero additive subgroup of \( \mathbf{R} \) and let \( \Gamma \) be a \( \Lambda \)-metric graph in the sense of [ABBR13, §1.2]. A \( \Lambda \)-rational model \( \Gamma \) is a weighted graph \( G \) with edge lengths in \( \Lambda \) whose geometric realization is \( \Gamma \); see [ABBR13, §1.2] or [BF11, §3]. Recall that \( \Gamma(\Lambda) \) denotes the subset of \( \Lambda \)-rational points of \( \Gamma \), which formally means the set of points of \( \Gamma \) whose distance to any vertex of \( G \) belongs to \( \Lambda \), for any \( \Lambda \)-rational model \( G \) for \( \Gamma \). We define \( \text{Div}_\Lambda(\Gamma) \) to be the free abelian group on \( \Gamma(\Lambda) \) and \( \text{Div}_\Lambda^0(\Gamma) \) to be its degree-zero subgroup. The group \( \text{Prin}_\Lambda(\Gamma) \) consisting of divisors of \( \Lambda \)-rational tropical meromorphic functions on \( \Gamma \) is a subgroup of \( \text{Div}_\Lambda^0(\Gamma) \), and we define \( \text{Pic}_\Lambda^0(\Gamma) \) to be the quotient group:

\[ \text{Pic}_\Lambda^0(\Gamma) = \text{Div}_\Lambda^0(\Gamma)/\text{Prin}_\Lambda(\Gamma). \]
We also set $\text{Pic}^0(\Gamma) = \text{Pic}^0_\mathbb{R}(\Gamma)$.

Following [MZ08] Definition 4.1 and [BF11] §2.1, §3, we define the group $\Omega_{\Gamma(\Lambda)}(\Gamma)$ of $\Lambda$-harmonic 1-forms on $\Gamma$ to be the direct limit of $\Omega(\Gamma, G)$ over all $\Lambda$-rational weighted graph models $G$ for $\Gamma$. (This coincides with the subgroup of $\Lambda$-rational 1-forms in the sense of Mikhalkin-Zharkov by [BF11] Footnote 5.) We can view an element $\gamma \in H_1(\Gamma, \mathbb{Z})$ as an element of

$$\Omega_{\Gamma(\Lambda)}(\Gamma)^* = \text{Hom}(\Omega_{\Gamma(\Lambda)}(\Gamma), \Lambda)$$

via the integration pairing $\mu : \gamma \mapsto \int_\gamma$. We define

$$\text{Jac}_\Lambda(\Gamma) = \Omega_{\Gamma(\Lambda)}(\Gamma)^*/\mu(H_1(\Gamma, \mathbb{Z})) \quad \text{and} \quad \text{Jac}(\Gamma) = \text{Jac}_\mathbb{R}(\Gamma).$$

If we set $M = H_1(G, \mathbb{Z})$ then it is formal to check that if we identify $\Omega_{\Gamma(\Lambda)}(\Gamma)$ with $H_1(\Gamma, \Lambda)$ in the obvious way and set $N_\Lambda = \text{Hom}(M, \Lambda)$ then the above definition becomes

$$\text{Jac}_\Lambda(\Gamma) = N_\Lambda/\mu(H_1(\Gamma, \mathbb{Z})),$$

where $\mu$ now denotes the map from $H_1(\Gamma, \mathbb{Z})$ to $N_\Lambda$ given by the edge length pairing. The following result, a reformulation of Proposition 2.8 was first proved for $\Lambda = \mathbb{R}$ in [MZ08] Theorem 6.2. An alternate proof, which easily generalizes to the present situation in which $\Lambda$ is an arbitrary subgroup of $\mathbb{R}$, is given in [BF11] Theorem 3.4.

**Proposition 3.2.** There is a canonical isomorphism $\text{Pic}^0_\Lambda(\Gamma) \cong \text{Jac}_\Lambda(\Gamma)$.

In particular, when $\Lambda = \mathbb{R}$ there is a canonical isomorphism between the degree-zero Picard group $\text{Pic}^0(\Gamma)$ and the $g$-dimensional real torus $\text{Jac}(\Gamma)$, where $g = \dim_{\mathbb{R}} H^1(\Gamma, \mathbb{R})$.

**3.3.** If we fix a base point $q \in \Gamma$, we obtain a corresponding (tropical) *Abel-Jacobi map*

$$\alpha = \alpha_q : \Gamma \to \text{Jac}(\Gamma)$$

taking $\Gamma(\Lambda)$ to $\text{Jac}_\Lambda(\Gamma)$ if $q \in \Gamma(\Lambda)$. The map $\alpha$ is defined by

$$\alpha(p) = \int_q^p \in \Omega_{\Gamma(\mathbb{R})}(\Gamma)^*,$$

where the integral is computed along any path in $\Gamma$ from $q$ to $p$. Different choices of paths give rise to elements of $\Omega_{\Gamma(\mathbb{R})}(\Gamma)^*$ which differ by an element of $\mu(H_1(\Gamma, \mathbb{Z}))$. From this description, it is straightforward to check (just as in the classical case of Riemann surfaces) that we have:

**Lemma 3.4.** The Abel-Jacobi map $\alpha : \Gamma \to \text{Jac}(\Gamma)$ induces an isomorphism

$$\alpha_* : H_1(\Gamma, \mathbb{Z}) \iso H_1(\text{Jac}(\Gamma), \mathbb{Z})$$

on singular homology groups.

We remark that under the canonical isomorphism afforded by Proposition 3.2, the map $\alpha$ corresponds to the map $\Gamma \to \text{Pic}^0(\Gamma)$ given by $p \mapsto [(p) - (q)]$.

**3.5.** As in classical algebraic geometry, there is a canonical bijection between isomorphism classes of $\Lambda$-line bundles on $\Gamma$ and elements of $\text{Pic}_\Lambda(\Gamma)$. Following [MZ08] Definition 4.4, a *line bundle* on $\Gamma$ is by definition a fiber bundle $\pi : L \to \Gamma$ whose fibers are isomorphic to the tropical affine line $T$ and whose transition maps are harmonic $\Lambda$-rational tropical meromorphic functions on $\Gamma$. Isomorphism classes of $\Lambda$-line bundles on $\Gamma$ are classified by the Čech cohomology group $H^1(\Gamma, \mathcal{O}_\Lambda^*)$, where $\mathcal{O}_\Lambda^*$ is the sheaf of harmonic $\Lambda$-rational tropical meromorphic functions on $\Gamma$.

Given an open subset $U \subset \Gamma$ which locally trivializes $L$, one says that a section $s : U \to \pi^{-1}(U)$ is regular (resp. rational) if for any open subset $V \subset U$ and any trivialization $\Phi : \pi^{-1}(V) \cong V \times T$, $s$ becomes a regular (resp. rational) function on $V$. A global rational section of $L$ gives rise to a well-defined divisor on $\Gamma$, independent of the choice of trivialization. By [MZ08] Proposition 4.6, every divisor defines a line bundle together with a rational section which is well-defined up to an additive (i.e., tropically multiplicative) constant, every line bundle has a rational section, and the divisors of any two rational sections are linearly equivalent.
If \( \{ U_i \} \) is an open cover of \( \Gamma \) trivializing \( L \), then the map sending \( [D] \) to the 1-cocycle whose value on \( U_i \) is (some choice of) a local section \( s_i \) associated to \( D \) induces a natural map from \( \text{Pic}_A(\Gamma) \) to \( H^1(\Gamma, \mathcal{O}_\Lambda^*) \). By the proof of [MZ08, Proposition 4.6], we have:

**Proposition 3.6.** The natural map

\[
\text{Pic}_A(\Gamma) \to H^1(\Gamma, \mathcal{O}_\Lambda^*)
\]

is an isomorphism.

3.7. A **principally polarized tropical abelian variety** \( A \) of dimension \( g \) is a triple \( (A, L, \langle \cdot, \cdot \rangle) \), where \( A \) is a \( g \)-dimensional real torus, \( L \) is a rank-\( g \) lattice, and \( \langle \cdot, \cdot \rangle : L \times L \to \mathbb{R} \) is a bilinear form whose associated quadratic form is positive definite, together with an isomorphism \( A \cong \text{Hom}(L, \mathbb{R})/\mu(L) \) where \( \mu : L \to \text{Hom}(L, \mathbb{R}) \) is the natural map derived from the bilinear form. (This is more or less equivalent to the definition given in [MZ08, §5], and slightly less general than the one used in [Viv13, Definition 3.1.1] and the references therein.) An isomorphism \( (A, L, \langle \cdot, \cdot \rangle) \to (A', L', \langle \cdot, \cdot \rangle') \) of principally polarized tropical abelian varieties is an isomorphism \( L \to L' \) which transforms the bilinear form on \( L \) to the one on \( L' \), together with the isomorphism \( A \to A' \) induced by the resulting identification of \( \text{Hom}(L, \mathbb{R})/\mu(L) \) with \( \text{Hom}(L', \mathbb{R})/\mu'(L') \).

There is a canonical principal polarization on the Jacobian \( \text{Jac}(\Gamma) \) of a metric graph \( \Gamma \) defined by taking \( L = H_1(\Gamma, \mathbb{Z}) \) and \( \langle \cdot, \cdot \rangle \) to be the edge length pairing \((2.7.1)\).

## 4. Non-Archimedean uniformization of abelian varieties

Let \( K \) be any complete non-Archimedean field. In this section we recall some results of Bosch, Bosch-Lütkebohmert, and Raynaud, in Berkovich’s language; see [Bos76, BL84, BL91] and [Ber90, §6.5]. We also prove a compatibility result involving the Abel-Jacobi map.

4.1. Let \( A \) be an abelian variety over \( K \) with split semi-abelian reduction. This means that there is a unique compact analytic domain \( A_0 \subset A^\text{an} \) which is a formal \( K \)-analytic subgroup in the sense of [Bos76], whose special fiber \( \mathcal{A}_0 \) is an extension of an abelian variety \( \mathcal{B} \) by a split torus \( \mathcal{T} \). Such an abelian variety admits a non-Archimedean uniformization, in the following sense. Let \( E^\text{an} \) be the topological universal cover of \( A^\text{an} \). Then \( E^\text{an} \) has a unique structure of \( K \)-analytic group (a group object in the category of \( K \)-analytic spaces) such that the structure map \( \pi : E^\text{an} \to A^\text{an} \) is a homomorphism of \( K \)-analytic groups. In fact \( E^\text{an} \) is the analytification of an algebraic group \( E \), although \( \pi \) is not algebraic. As \( \pi \) is a local isomorphism, its kernel \( M' \coloneqq \ker(\pi) \cong H_1(A^\text{an}, \mathbb{Z}) \cong \pi_1(A^\text{an}) \) is a discrete subgroup of \( E^\text{an}(K) \), so we have an exact sequence of \( K \)-analytic groups

\[
0 \longrightarrow M' \longrightarrow E^\text{an} \longrightarrow \pi \longrightarrow A^\text{an} \longrightarrow 0. 
\]

The algebraic group \( E \) is an extension of an abelian variety \( B \) with good reduction by a split torus \( T \):

\[
0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0.
\]

More precisely, \( B \) has an abelian scheme \( R \)-model \( \mathcal{B} \) with special fiber \( \mathcal{B} \), and \( T \) is the generic fiber of the unique \( R \)-torus \( \mathcal{T} \) with special fiber \( \mathcal{T} \) (so \( T \) has the same character lattice as \( \mathcal{T} \)). Let \( T_0 \) denote the affinoid torus inside of \( T^\text{an} \), i.e. the locus of all points in \( T^\text{an} \) with a well-defined specialization in \( \mathcal{T} \). The short exact sequence \( 0 \to \mathcal{T} \to \mathcal{A}_0 \to \mathcal{B} \to 0 \) lifts to a short exact sequence

\[
0 \longrightarrow T_0 \longrightarrow A_0 \longrightarrow B^\text{an} \longrightarrow 0,
\]

and (4.1.2) is the (algebraization of the) push-out of (4.1.3) with respect to the inclusion \( T_0 \to T^\text{an} \). In particular, \( A_0 \) is an analytic domain in \( E^\text{an} \). The sequence (4.1.2) is unique since \( T \) is the maximal torus in \( E \) (the image of any morphism \( \mathbb{G}_m \to E \) is contained in \( T \)). The above uniformization
data \([4.1.1], [4.1.2]\) are often combined into the following diagram, known as the Raynaud uniformization cross:

\[(4.1.4)\]

\[
\begin{array}{ccc}
T^{an} & \longrightarrow & E^{an} \\
\downarrow & & \downarrow \\
M' & \longrightarrow & A^{an}
\end{array}
\]

The formation of \((4.1.4)\) is compatible with extension of the ground field in the obvious way.

4.2. Let \(M\) be the character lattice of \(T\) and let \(N = \text{Hom}(M, \mathbb{Z})\), with \((\cdot, \cdot) : N \times M \to \mathbb{Z}\) the evaluation pairing. For \(u \in M\) we let \(\chi_u \in K[M]\) denote the corresponding character of \(T\). We have a tropicalization map \(\text{trop} : T^{an} \to N_{\mathbb{R}} = \text{Hom}(M, \mathbb{R})\) defined by \((\text{trop}(\parallel \cdot \parallel), u) = -\log \parallel \chi_u \parallel\), where we regard \(T^{an}\) as a space of semi-norms on \(K[M]\). The map \(\text{trop}\) is surjective, continuous, and proper; in fact the affinoid torus \(T_0\) is equal to \(\text{trop}^{-1}(0)\). We extend \(\text{trop}\) to all of \(E^{an}\) by declaring that \(\text{trop}^{-1}(0) = A_0\), which makes sense because

\[
\begin{array}{ccc}
T_0 & \longrightarrow & A_0 \\
\downarrow & & \downarrow \\
T^{an} & \longrightarrow & E^{an}
\end{array}
\]

is a push-out square. If \(K'\) is any valued field extension of \(K\) with value group \(\Lambda'\) then \(\text{trop}(E(K')) \subset N_{\Lambda'} = \text{Hom}(M, \Lambda')\), and the map \(\text{trop} : E(K') \to N_{\Lambda'}\) is a surjective group homomorphism. The restriction of \(\text{trop}\) to \(M' \subset E(K)\) is injective, and its image \(\text{trop}(M') \subset N_{\Lambda}\) is a full-rank lattice in the real vector space \(N_{\mathbb{R}}\). Let \(\Sigma = \Sigma(A)\) be the real torus \(N_{\mathbb{R}}/\text{trop}(M)\). Since \(A^{an}\) is the quotient of \(E^{an}\) by the action of \(M'\), there exists a unique map \(\tau : A^{an} \to \Sigma\) making the following diagram commute:

\[(4.2.1)\]

\[
\begin{array}{ccc}
0 & \longrightarrow & M' \\
\downarrow \cong & & \downarrow \text{trop} \\
0 & \longrightarrow & \text{trop}(M') \\
\downarrow \cong & & \downarrow \text{trop} \\
0 & \longrightarrow & N_{\Lambda} & \longrightarrow & A^{an} & \longrightarrow & 0
\end{array}
\]

Berkovich [Ber90, Theorem 6.5.1] has shown that there exists a section \(\iota : \Sigma \hookrightarrow A^{an}\) of \(\tau\), and that \(\iota \circ \tau\) is a deformation retraction. For this reason \(\Sigma\) is called the skeleton of \(A\). Letting \(\Sigma_{\Lambda} = N_{\Lambda}/\text{trop}(M)\) and taking \(K\)-points, we have a surjective homomorphism of short exact sequences

\[(4.2.2)\]

\[
\begin{array}{ccc}
0 & \longrightarrow & M' \\
\downarrow \cong & & \downarrow \text{trop} \\
0 & \longrightarrow & \text{trop}(M') \\
\downarrow \cong & & \downarrow \text{trop} \\
0 & \longrightarrow & N_{\Lambda} & \longrightarrow & \Sigma_{\Lambda} & \longrightarrow & 0
\end{array}
\]

Both sequences \((4.2.1)\) and \((4.2.2)\) are compatible with extension of the ground field.

4.3. Uniformization and duality. Let \(T'\) be the split \(K\)-torus with character group \(M'\). As explained in [BL91, §6], the Raynaud cross \((4.1.4)\) gives rise to a dual cross

\[(4.3.1)\]

\[
\begin{array}{ccc}
T'^{an} & \longrightarrow & \bar{A}^{an} \\
\downarrow & & \downarrow \\
M & \longrightarrow & E'^{an}
\end{array}
\]

where \(\bar{A}\) (resp. \(\bar{B}\)) is the dual abelian variety of \(A\) (resp. \(B\)), and \(E'^{an}\) is the universal cover of \(\bar{A}^{an}\). Note that we have identifications \(M = H_1(\bar{A}^{an}, \mathbb{Z})\) and \(M' = H_1(A^{an}, \mathbb{Z})\), so duality interchanges character groups and uniformizing lattices. In the sequel we will need the following explicit description of the composite homomorphism \(T(K) \to E(K) \to A(K)\) in terms of the data \((4.3.1)\).
We have $\text{Pic}(\bar{A}^\text{an}) = \text{Pic}(\bar{A})$ by analytic GAGA theorems: see [BL84 §1]. Hence we have a map on sheaf cohomology groups

$$H^1(\bar{A}^\text{an}, \mathcal{K}^\times) \to H^1(\bar{A}^\text{an}, \mathcal{O}_{\bar{A}^\text{an}}^\times) = \text{Pic}(\bar{A}^\text{an}) = \text{Pic}(\bar{A})$$

induced by the inclusion of the constant sheaf $\mathcal{K}^\times$ into $\mathcal{O}_{\bar{A}^\text{an}}^\times$. By [BL91] p.658, the line bundle on $\bar{A}$ corresponding to the image of a class in $H^1(\bar{A}^\text{an}, \mathcal{K}^\times)$ is translation-invariant, so we get a homomorphism

$$H^1(\bar{A}^\text{an}, \mathcal{K}^\times) \to A(K).$$

(4.3.2)

Identifying the sheaf cohomology group $H^1(\bar{A}^\text{an}, \mathcal{K}^\times)$ with the singular cohomology group

$$H^1(\bar{A}^\text{an}, \mathcal{K}^\times) = \text{Hom}(H_1(\bar{A}^\text{an}, \mathcal{Z}), \mathcal{K}^\times) = \text{Hom}(M, \mathcal{K}^\times) = T(K)$$

and composing with (4.3.2) gives a homomorphism

$$\theta : T(K) \to A(K).$$

(4.3.3)

**Lemma 4.4.** The composition $T(K) \to E(K) \to A(K)$ coincides with the homomorphism $\theta$ of (4.3.3).

**Proof.** Unwrapping the construction of [BL91] §4, one finds that the line bundle on $\bar{A}^\text{an}$ corresponding to the image of a point $x \in T(K) = \text{Hom}(M, \mathcal{K}^\times)$ under the composition $T(K) \to E(K) \to A(K)$ is obtained by descending the trivial bundle on $E^\text{an}$ through the exact sequence $0 \to M \to E^\text{an} \to \bar{A}^\text{an} \to 0$ using the homomorphism $x : M \to K^\times$. Regarding $L = \theta(x)$ as a line bundle on $\bar{A}^\text{an}$, it is elementary to check (as for topological line bundles) that the canonical action of $M'$ on the pullback of $L$ to $E'$ is again given by the homomorphism $\theta$. ■

### 4.5. Uniformization of Jacobians.

Let $X$ be a smooth, proper, geometrically connected $K$-curve endowed with a split semistable $R$-model $\mathcal{X}$. Let $J$ be the Jacobian of $X$. Since $X$ has split semistable reduction, $J$ has split semi-abelian reduction, so we have dual non-Archimedean uniformizations

$$T^\text{an} \quad \begin{array}{c} T'\text{an} \\ \downarrow \quad \downarrow \end{array} \quad \begin{array}{c} M' \to E^\text{an} \to J^\text{an} \\ \downarrow \quad \downarrow \end{array} \quad \begin{array}{c} M \to E^\text{an} \to J^\text{an} \\ \downarrow \quad \downarrow \end{array} \quad \begin{array}{c} B^\text{an} \\ \end{array}$$

(4.5.1)

Recall that $M = H_1(\bar{J}^\text{an}, \mathcal{Z})$ is the character lattice of $T$ and $M' = H_1(\bar{J}^\text{an}, \mathcal{Z})$ is the character lattice of $T'$.

#### 4.5.2.
Bosch and L"utebohmert [BL84 §§6–7] construct the left square of (4.5.1) directly in terms of the curve $X$. In their construction, the torus $T$ has character lattice $H_1(\mathcal{X}^\text{an}, \mathcal{Z})$, i.e. $M = H_1(\mathcal{X}^\text{an}, \mathcal{Z})$. The composition $\theta : T(K) \to E(K) \to J(K)$ can be described in a manner analogous to Lemma 4.4 as the composition

$$T(K) = \text{Hom}(H_1(\mathcal{X}^\text{an}, \mathcal{Z}), \mathcal{K}^\times) = H^1(\mathcal{X}^\text{an}, K^\times) \to H^1(\mathcal{X}^\text{an}, \mathcal{O}_{\mathcal{X}^\text{an}}^\times) = \text{Pic}(\mathcal{X}^\text{an}),$$

(4.5.3)

which factors through $J(K) \subset \text{Pic}(X) = \text{Pic}(\mathcal{X}^\text{an})$. See Proposition 1.5 and Theorems 6.2 and 7.5 in loc. cit. for details.

#### 4.6. The Abel-Jacobi map.
Suppose that $X(K) \neq \emptyset$, and let $\alpha : X \to J$ be an Abel-Jacobi map. The morphism $\alpha^* : J \to J$ induced by Picard functoriality is the (inverse of the) canonical principal polarization of $J$. If $X$ has no $K$-rational points then $\alpha^*$ is defined by descent theory after passing to a finite extension of $K$. In either case, $\alpha^*$ lifts in a unique way to an isomorphism $\tilde{\alpha}^* : E^\text{an} \to E^\text{an}$, which then restricts to an isomorphism $\tilde{\alpha}^* : M \simto M'$. In other words, we have an isomorphism of
short exact sequences

\[(4.6.1)\]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & E'_{an} & \longrightarrow & \tilde{J}_{an} & \longrightarrow & 0 \\
\quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong \\
0 & \longrightarrow & M' & \longrightarrow & E_{an} & \longrightarrow & J_{an} & \longrightarrow & 0
\end{array}
\]

We have the following Berkovich-analytic version of the classical consequence of Abel's theorem which says that the Abel-Jacobi map realizes the isomorphism \( H_1(X(C), \mathbb{Z}) \sim H_1(J(C), \mathbb{Z}) \) in the complex setting.

**Proposition 4.7.** Suppose that \( X(K) \neq \emptyset \), and let \( \alpha : X \to J \) be an Abel-Jacobi map. The homomorphism on singular homology groups

\[\alpha_* : H_1(X_{an}, \mathbb{Z}) \longrightarrow H_1(J_{an}, \mathbb{Z}) = M'\]

coincides with the isomorphism \( \tilde{\alpha}^* : M \sim M' \) of \((4.6)\) under the identification \( M = H_1(X_{an}, \mathbb{Z}) \) of \((4.5.2)\). In particular, \( \alpha_* \) is an isomorphism.

**Proof.** The polarization \( \tilde{\alpha} : \tilde{J} \sim J \) also induces an isomorphism of Raynaud extensions

\[(4.7.1)\]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & (T')_{an} & \longrightarrow & E' & \longrightarrow & \tilde{B}_{an} & \longrightarrow & 0 \\
\quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong & \quad \cong \\
0 & \longrightarrow & T_{an} & \longrightarrow & E & \longrightarrow & B_{an} & \longrightarrow & 0
\end{array}
\]

Since polarizations are symmetric, by [BL91, Proposition 6.10(c)] the map on character groups \( \beta : M \sim M' \) induced by the map \( \beta^* : T' \sim T \) coincides with \( \tilde{\alpha}^* \) (note \( \beta^* \) is algebraic because analytic characters are algebraic). Therefore we may equivalently prove that this homomorphism \( \beta \) on character groups coincides with \( \alpha_* \).

It is straightforward to show that the square

\[
\begin{array}{ccc}
T'(K) & \xrightarrow{\beta^*} & \tilde{J}(K) \\
\gamma \downarrow & & \alpha^* \downarrow \\
T(K) & \xrightarrow{\theta} & J(K)
\end{array}
\]

is commutative, where \( \gamma : T' \to T \) is the homomorphism induced by the map \( \alpha_* : H_1(X_{an}, \mathbb{Z}) \to H_1(J_{an}, \mathbb{Z}) \) on characters, and the maps \( \theta \) are defined in \((4.3.3)\) and \((4.5.3)\). On the other hand, the diagram

\[
\begin{array}{ccc}
T'(K) & \xrightarrow{\beta} & E'(K) \\
\beta^* \downarrow & & \tilde{\alpha}^* \downarrow \\
T(K) & \xrightarrow{\theta} & E(K)
\end{array}
\]

made from \((4.6.1)\) and \((4.7.1)\) is also commutative, and the top (resp. bottom) row is again \( \theta \) by Lemma\[4.4\] (resp. \((4.5.2)\)). It follows that \( \varphi := \gamma - \beta^* : T' \to T \) takes \( T'(K) \) into the discrete subgroup \( \ker(\theta) = T(K) \cap M' \). Since \( M' \) is a lattice in \( E \), we have \( \varphi(T(K)) \cap T_0(K) = \{0\} \), where \( T_0 \subset T_{an} \) is the affinoid torus \( \text{trop}^{-1}(0) \). It is an elementary exercise to show that any algebraic homomorphism of tori \( \varphi : T' \to T \) such that \( \varphi(T'(K)) \cap T_0(K) = \{0\} \) is trivial, so \( \beta^* = \gamma \) and hence \( \beta = \alpha_* \). \[ \square \]

### 5. Retraction of divisors and divisor classes

Recall that \( K \) denotes a complete non-Archimedean field.

#### 5.1. Curves and their skeleta

At this point it is necessary to recall some of the analytic structure of \( K \)-curves. The original reference is Berkovich’s book [Berk90 Chapter 4]. Our main reference is [BPR13]; see also [Tem], Thuillier’s thesis [Thu05], and Ducros’ forthcoming book [Duc]. Let \( X \) be a smooth, proper, geometrically connected \( K \)-curve endowed with a split semistable \( R \)-model \( \mathcal{X} \). This means that the special fiber \( \mathcal{X}_k \) of our model \( \mathcal{X} \) is geometrically reduced, geometrically connected, and...
generically smooth, and for any singular point \( x \in X \), the completed local ring \( \hat{O}_{X,x} \) is isomorphic to \( R[s,t]/(st - x) \) for some nonzero element \( s \) of \( R \) with \( \text{val}(x) > 0 \). The incidence graph \( \Gamma = \Gamma_\chi \) associated to \( X \) is the metric \( \Lambda \)-graph (together with an associated \( \Lambda \)-rational model \( G \)) constructed as follows. The vertices of \( \Gamma \) (or more properly, of \( G \)) are the irreducible components of \( X \), and for each node \( x \in X \), there is an edge \( e_x \) of length \( \ell(e_x) := \text{val}(x) \) in \( \Gamma \) connecting the irreducible components containing \( x \) (this is a loop edge if \( x \) is contained in a single irreducible component). There is a canonical inclusion \( \Gamma \hookrightarrow X^{\text{an}} \) and a deformation retraction \( \tau : X^{\text{an}} \to \Gamma \). We identify \( \Gamma \) with its image in \( X^{\text{an}} \), called the skeleton of \( X \) (or of \( X^{\text{an}} \)) associated to \( X \). We have \( \tau(X(K)) \subset \Gamma(\Lambda) \), and \( \tau : X(K) \to \Gamma(\Lambda) \) is surjective when \( k \) is algebraically closed (or more generally when every irreducible component of \( X \) contains a smooth \( k \)-rational point; see [Liu02, Proposition 10.1.40(b)]).

Let \( \text{Div}_K(X) \) be the group of divisors of \( X \) supported on \( X(K) \). Extending \( \tau \) by linearity gives a homomorphism

\[
\tau_* : \text{Div}_K(X) \longrightarrow \text{Div}_\Lambda(\Gamma).
\]

The above constructions are compatible with extension of the ground field, in the following sense. Let \( K' \) be a complete valued field extension of \( K \) and let \( \hat{K}' \) be its valuation ring. Then \( X_{\hat{K}'} \) a split semistable \( \hat{K}' \)-model of \( X' \). One checks that the associated skeleton \( \Gamma' \subset X'_{\hat{K}'} \) maps isomorphically onto \( \Gamma \) under the structure morphism \( X'_{\hat{K}'} \to X_{\hat{K}'}^{\text{an}} \), and that the following squares commute:

\[
\begin{array}{ccc}
X'_{\hat{K}'} & \xrightarrow{\cong} & X_{\hat{K}'}^{\text{an}} \\
\downarrow \cong & & \downarrow \cong \\
\Gamma' & \xrightarrow{\cong} & X_{\hat{K}'}^{\text{an}}
\end{array}
\]

5.2. Let \( J \) be the Jacobian of \( X \). It has a uniformization

\[
\begin{array}{ccc}
T^{\text{an}} & \xrightarrow{\tau} & J^{\text{an}} \\
\downarrow & & \downarrow \\
M' & \xrightarrow{p} & J^{\text{an}}
\end{array}
\]

\[
\begin{array}{ccc}
\text{as in} \ (4.5), \text{ where} \ T \text{ has character lattice} \ M = H_1(X^{\text{an}}, \mathbb{Z}) = H_1(\Gamma, \mathbb{Z}). \text{ Likewise the Jacobian of} \ \Gamma \text{ has a uniformization of the form} \ 0 \to M \to N_R \xrightarrow{\partial} \text{Jac}(\Gamma) \to 0, \text{ where} \ N = \text{Hom}(M, \mathbb{Z}) = H^1(\Gamma, \mathbb{Z}) \text{ as usual; see Proposition 3.2.} \end{array}
\]

The goal of this section is to define a specialization map \( \tau' : J^{\text{an}} \to \text{Jac}(\Gamma) \) which is compatible with the complementary descriptions of \( J \) and of \( \text{Jac}(\Gamma) \) in terms of divisor classes and uniformizations. This will be a key ingredient in the proof of Theorem 2.9, which says that \( \text{Jac}(\Gamma) \cong \Sigma(J) \) canonically; under this identification, we will have \( \tau' = \tau \) (see Corollary 6.6).

**Proposition 5.3.** There exists a unique surjective homomorphism \( \tau' : J(K) \to \text{Jac}_\Lambda(\Gamma) \) making the following two squares commute:

\[
\begin{array}{ccc}
\text{Div}_K^0(X) & \xrightarrow{\tau} & J(K) \\
\downarrow & & \downarrow \\
\text{Div}_\Lambda^0(\Gamma) & \xrightarrow{\tau} & \text{Jac}_\Lambda(\Gamma)
\end{array}
\]

\[
\begin{array}{ccc}
E(K) & \xrightarrow{p} & J(K) \\
\downarrow \tau & & \downarrow \tau \\
N_\Lambda & \xrightarrow{q} & \text{Jac}_\Lambda(\Gamma)
\end{array}
\]

**Proof.** Suppose first that \( K \) is algebraically closed, so that \( \text{Div}_K(X) = \text{Div}(X) \). Let \( \text{Prin}(X) = \ker(\text{Div}^0(X) \to J(K)) \) denote the group of principal divisors. We have \( \tau_* (\text{Prin}(X)) \subset \text{Prin}_\Lambda(\Gamma) \) by the Poincaré-Lelong formula (see Remark 5.4 below), so there exists a unique map \( \tau' : J(K) \to \text{Jac}_\Lambda(\Gamma) \) making the left square in (5.3.1) commute. We claim that \( \tau' \circ p = q \circ \text{trop} \), so that the right square commutes as well. We have \( E(K) = T(K) + J_0(K) \) (where \( J_0 = \tau^{-1}(0) \) as in (4.2)), so it suffices to show \( \tau'(p(x)) = q(\text{trop}(x)) \) for \( x \in T(K) \) and \( x \in J_0(K) \).
5.3.2. Let $x \in J_0(K) \subset J(K)$ and let $\mathcal{L}$ be a line bundle representing $x$. Then $\mathcal{L}$ is represented by a divisor $D$ such that $\tau_*(D) = 0$ by [BL84] Theorem 5.1(c)], so $\mathcal{T}(x) = 0$. By definition we have $\tau(x) = 0$ as well.

5.3.3. Choose a $\Lambda$-rational model $G$ of $\Gamma$ with no loop edges (5.1), and let $\mathcal{V}(G)$ (resp. $\mathcal{E}(G)$) denote the set of vertices (resp. edges) of $G$. For $v \in \mathcal{V}(G)$ let $U_v$ be the open star around $v$ in $\Gamma$, i.e. the union of $\{v\}$ and all open edges adjacent to $v$. Let $U$ denote the open cover $\{U_v : v \in \mathcal{V}(G)\}$ of $\Gamma$. Let $U'_v = \tau^{-1}(U_v)$; this is an open analytic domain in $X^{an}$, and $U'_e = \{U'_v : v \in \mathcal{V}(G)\}$ is an open cover. Choose orientations of the edges of $G$, so that $G$ becomes a simplicial complex. Since each $U_v$ and $U'_v$ is contractible, we have canonical isomorphisms

$$\tilde{H}^1(U', K^\times) = H^1(X^{an}, K^\times) \quad \text{and} \quad \tilde{H}^1(U, \Lambda) = H^1(\Gamma, \Lambda).$$

Let $H \in \check{C}^1(U', K^\times)$ be a Čech 1-cocycle, so $H$ amounts to a function $H : \mathcal{E}(G) \to K^\times$ and defines a class $[H] \in H^1(X^{an}, K^\times) = \text{Hom}(\mathcal{M}, K^\times) = T(K)$. Extending $H$ linearly defines a homomorphism $H : C_1(\mathcal{V}(G), \mathbb{Z}) \to K^\times$ on simplicial 1-chains; the restriction of this homomorphism to $H_1(\Gamma, \mathbb{Z})$ is the element of $\text{Hom}(H_1(\Gamma, \mathbb{Z}), K^\times) = H^1(\Gamma, K^\times) = H^1(X^{an}, K^\times)$ corresponding to the class of $H$. Let $h = \text{val} : H : \mathcal{E}(G) \to \Lambda$. This is a cocycle in $\check{C}^1(U, \Lambda)$, and we have an equality of cohomology classes $[h] = \text{trop}(H) \in H^1(\Gamma, \Lambda) = N_\Lambda$.

Let $\mathcal{L}$ be the algebraization of the analytic line bundle on $X$ given by the cocycle $H \in \check{C}^1(U', K^\times) \subset \check{C}^1(U', \mathcal{O}^{\times}_{X^{an}})$. In other words, $\mathcal{L}$ is the image of $[H]$ under the composition

$$H^1(X^{an}, K^\times) \to H^1(X^{an}, \mathcal{O}^{\times}_{X^{an}}) = \text{Pic}(X^{an}) = \text{Pic}(X).$$

Regarding $[H]$ as an element of $T(K)$, by (4.5.2) we have that $p([H])$ is the isomorphism class of $\mathcal{L}$. Since $\mathcal{L}^{an}$ was constructed using the Čech cocycle $H$, we have distinguished trivializations $\mathcal{O}_{\mathcal{L}^{an}}|_{U'_v} = \mathcal{O}_{U_v}$ for each $v \in \mathcal{V}(G)$, where $\mathcal{O}_{\mathcal{L}^{an}}$ is the sheaf of sections of $\mathcal{L}^{an}$; the cocycle $H$ determines the transition functions on $U_v \cap U'_v$. Similarly, let $L$ be the line bundle on $\Gamma$ given by the cocycle $h \in \check{C}^1(U, \Lambda) \subset \check{C}^1(U, \mathcal{O}_{\Lambda})$, so $q([h])$ is the isomorphism class of $L$; see Proposition 5.3.6. We have distinguished trivializations $\mathcal{O}_L|_{U_v} = \mathcal{O}_L|_{U'_v}$ for each $v \in \mathcal{V}(G)$, and $h$ determines the transition functions.

Let $f$ be a nonzero meromorphic section of $\mathcal{L}$ and let $D = \text{div}(f)$, so $\mathcal{L} = p([H]) = [D]$. We may identify $f$ with a collection $\{f_v : v \in \mathcal{V}(G)\}$, where $f_v$ is a meromorphic function on $U_v$ and for each edge $e = \overrightarrow{vw}$ we have $H(e)(f_v) = f_w$ on $\tau^{-1}(e^\circ)$. Let $F_{w\to v}$ be the restriction of $-\log |f_w|$ to $\Gamma$. By the Poincaré-Lelong formula (see Remark 5.4 below), $F_w$ is a tropical meromorphic function on $U_v$, and $\text{div}(F_w) = \tau_e(\text{div}(f_v))$. For $e = \overrightarrow{vw}$ we have $h(e) + F_w = F_w$ on $e^\circ$, so the collection $\{F_w : v \in \mathcal{V}(G)\}$ glues to a meromorphic section $F$ of $L$. By construction we have $\tau_*(D) = \text{div}(F)$. By definition $[L] = [\text{div}(F)]$, so

$$q(\text{trop}([H])) = q([h]) = [L] = [\tau_*(D)] = \mathcal{T}([D]) = \mathcal{T}(p([H])).$$

Therefore the right square in (5.3.1) commutes.

5.3.4. Now suppose that $K$ is not necessarily algebraically closed. Let $K'$ denote the completion of an algebraic closure of $K$ and let $\Lambda' = \text{val}(K'^\times)$. Define $\mathcal{T}' : J(K) \to \text{Jac}_{\Lambda'}(\Gamma)$ to be the composition

$$J(K) \hookrightarrow J(K') \xrightarrow{\mathcal{T}} \text{Pic}^0_{\Lambda'}(\Gamma).$$

Then the following two diagrams commute:

\[
\begin{align*}
\text{Div}^0_K(X) & \quad \xrightarrow{\tau} \quad J(K) \\
\downarrow & \quad \downarrow \\
\text{Div}_{\Lambda'}(X) & \quad \xrightarrow{\mathcal{T}} \quad \text{Jac}_{\Lambda'}(\Gamma)
\end{align*}
\begin{align*}
\text{Div}^0_K(X) & \quad \xrightarrow{\tau} \quad J(K) \\
\downarrow & \quad \downarrow \\
\text{Div}_{\Lambda'}(X) & \quad \xrightarrow{\mathcal{T}} \quad \text{Jac}_{\Lambda'}(\Gamma)
\end{align*}
\begin{align*}
\text{Div}^0_K(X) & \quad \xrightarrow{\tau} \quad J(K) \\
\downarrow & \quad \downarrow \\
\text{Div}_{\Lambda'}(X) & \quad \xrightarrow{\mathcal{T}} \quad \text{Jac}_{\Lambda'}(\Gamma)
\end{align*}
\begin{align*}
\text{Div}^0_K(X) & \quad \xrightarrow{\tau} \quad J(K) \\
\downarrow & \quad \downarrow \\
\text{Div}_{\Lambda'}(X) & \quad \xrightarrow{\mathcal{T}} \quad \text{Jac}_{\Lambda'}(\Gamma)
\end{align*}
Since \( \text{trop}(E(K)) = N_{\Lambda} \) and \( N_{\Lambda} \) surjects onto \( \text{Jac}_\Lambda(\Gamma) \), the image of \( \overline{\tau} : J(K) \to \text{Jac}_{\Lambda'}(\Gamma) \) is equal to the subgroup \( \text{Jac}_\Lambda(\Gamma) \). Therefore \( \overline{\tau} : J(K) \to \text{Jac}_\Lambda(\Gamma) \) makes the squares (5.3.1) commute. The uniqueness of \( \overline{\tau} \) follows from the surjectivity of \( E(K) \to J(K) \).

**Remark 5.4.** In the above proof we used the non-Archimedean Poincaré-Lelong formula, which can be formulated as follows. Let \( X \) be a smooth curve and \( \Gamma \) a skeleton of \( X^{\text{an}} \), as above. Let \( f \) be a nonzero meromorphic function on \( X \) and let \( F = -\log |f| \) restricted to \( \Gamma \). Then \( F \) is a \( \Lambda \)-rational meromorphic function on \( \Gamma \) and \( \text{div}(F) \) is the retraction of \( \text{div}(f) \). A statement and proof in our language can be found in [BPR13, §5], where it is called the slope formula. The result is originally due to Thuillier [Thu05, Proposition 3.3.15], where it is stated in the language of non-Archimedean potential theory. Thuillier’s theorem is also proved in greater generality; in particular it applies to all smooth analytic curves \( X \), not just algebraic ones.

5.5. The second square in (5.3.1) extends to a homomorphism of short exact sequences

\[
\begin{array}{ccccccc}
0 & \to & M' & \to & E(K) & \overset{p}{\to} & J(K) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M & \to & N_{\Lambda} & \overset{q}{\to} & \text{Jac}_{\Lambda}(\Gamma) & \to & 0 \\
\end{array}
\]

In particular \( \text{trop}(M') \subset M \), so there is a natural homomorphism

\[
\pi : \Sigma(J) = N_{\mathbb{R}}/\text{trop}(M') \to N_{\mathbb{R}}/M = \text{Jac}(\Gamma)
\]

such that \( \pi \circ \overline{\tau} = \overline{\tau} \). Composing with \( N_{\mathbb{R}} \to \Sigma(J) \), we have proved:

**Corollary 5.6.** There is a unique surjective map \( \overline{\tau} : J^{\text{an}} \to \text{Jac}(\Gamma) \) making the square

\[
\begin{array}{ccccccc}
E^{\text{an}} & \overset{p}{\to} & J^{\text{an}} \\
\downarrow & & \downarrow \\
N_{\mathbb{R}} & \overset{q}{\to} & \text{Jac}(\Gamma)
\end{array}
\]

commute. Restricting to \( K \)-points yields the second square in (5.3.1).

6. The skeleton of the Jacobian is the Jacobian of the skeleton

We continue to use the notations in §5. In particular, \( X \) is a curve with split semistable reduction, \( J \) is its Jacobian, and \( M = H_1(X^{\text{an}}, \mathbb{Z}) = H_1(J^{\text{an}}, \mathbb{Z}) \) is the character lattice of the torus in the uniformization of \( J \). Let \( \Gamma \subset X^{\text{an}} \) be the skeleton of \( X \) associated to a split semistable model \( X \), and let \( \tau : X^{\text{an}} \to \Gamma \) be the retraction map.

The following compatibility of Abel-Jacobi maps is the key ingredient in the proof of Theorem 2.9. Its proof uses Proposition 5.3 in an essential way.

**Proposition 6.1.** Suppose that \( X(K) \neq \emptyset \). Let \( P \in X(K) \) and let \( p = \tau(P) \in \Gamma(\Lambda) \). Let \( \alpha_P : X \to J \) and \( \alpha_p : \Gamma \to \text{Jac}(\Gamma) \) be the Abel-Jacobi maps based at \( P \) and \( p \), respectively. Then the following square commutes:

\[
\begin{array}{ccccccc}
X^{\text{an}} & \overset{\alpha_P}{\to} & J^{\text{an}} \\
\downarrow & & \downarrow \overline{\tau} \\
\Gamma & \overset{\alpha_p}{\to} & \text{Jac}(\Gamma)
\end{array}
\]

where \( \overline{\tau} \) is the map of Corollary 5.6.

**Proof.** Let \( K' \) be a complete valued field extension of \( K \). Define \( \overline{\alpha}_P : X(K') \to \text{Div}^0_{K'}(X_{K'}) \) by \( Q \mapsto (Q) - (P) \) and \( \overline{\alpha}_p : \Gamma \to \text{Div}^0(\Gamma) \) by \( q \mapsto (q) - (p) \). The commutativity of the left square in the
is obvious, and the commutativity of the right square follows from the commutativity of the left square in (5.3.1) after extending the ground field. The compositions of the horizontal arrows in the above diagram are the maps $\alpha_p$ and $\alpha_p^*$, so the outer square and the top square of the diagram

$$
\begin{array}{ccc}
X(K') & \xrightarrow{\alpha_p} & J(K') \\
\downarrow \tau & & \downarrow \tau \\
\Gamma & \xrightarrow{\alpha_p} & \text{Div}^0(\Gamma) \rightarrow \text{Jac}(\Gamma)
\end{array}
$$

are commutative. It follows that the bottom square commutes as well, since any point $x \in X^\text{an}$ is the image of a morphism $\text{Spec}(K') \rightarrow X^\text{an}$ for some $K'$.

6.2. If $X(K) \neq \emptyset$ we let $\alpha : X \rightarrow J$ be an Abel-Jacobi map, and in general we let $\alpha^* : \tilde{J} \xrightarrow{\sim} J$ be the canonical principal polarization, as in (4.6). Let $\tilde{\alpha}^* : E^\text{an} \xrightarrow{\sim} E^\text{an}$ be the lift of $\alpha^*$ to universal covers; this restricts to an isomorphism $\tilde{\alpha}^* : M \xrightarrow{\sim} M' = H_1(J^\text{an}, Z)$ on homology groups. Letting $N = \text{Hom}(M, Z)$, the composition $\text{trop} \circ \tilde{\alpha}^* : M \rightarrow N_\Lambda$ defines a pairing

$$
(\cdot, \cdot)_{\text{an}} : M \times M \rightarrow \Lambda,
$$

which we call the analytic monodromy pairing.

Since $M = H_1(X^\text{an}, Z) = H_1(\Gamma, Z)$, the Jacobian of $\Gamma$ has a uniformization of the form

$$
0 \rightarrow M \rightarrow N_{\mathbb{R}} \rightarrow \text{Jac}(\Gamma) \rightarrow 0,
$$

where the homomorphism $M \rightarrow N_{\mathbb{R}}$ comes from the edge length pairing $(\cdot, \cdot) : M \times M \rightarrow \Lambda$ of (27.1).

Remark 6.3. The analytic monodromy pairing $(\cdot, \cdot)_{\text{an}}$ coincides with Grothendieck’s monodromy pairing in the case $\Lambda = Z$. This is proved in [Col00]. Note that this fact is unrelated to the more difficult Theorem 2.3 which says that the edge length pairing $(\cdot, \cdot)$ coincides with the monodromy pairing. Indeed, the former fact holds for arbitrary abelian varieties, and the standard proofs of the latter make no reference to the non-Archimedean uniformization theory of $J$.

We are now able to prove Theorem 2.9.

Theorem. (Theorem 2.9) The analytic monodromy pairing $(\cdot, \cdot)_{\text{an}}$ defined in (4.6) coincides with the edge length pairing $(\cdot, \cdot)$ under the identification $M = H_1(X^\text{an}, Z) = H_1(\Gamma, Z)$ of (4.5.2).

Proof. We observed in (5.5) that $\text{trop} : E(K) \rightarrow N_{\Lambda}$ takes $M'$ into $M = H_1(\Gamma, Z)$. The Theorem is equivalent to the assertion that the composition $\text{trop} \circ \tilde{\alpha}^* : M \rightarrow M$ is the identity, which can be checked after extending the ground field. Hence we may assume that $X$ has a $K$-rational point $P$. Let $p = \tau(P) \in \Gamma(\Lambda)$ and let $\alpha_p : X \rightarrow J$ (resp. $\alpha_p : \Gamma \rightarrow \text{Jac}(\Gamma)$) be the Abel-Jacobi map based at $P$ (resp. at $p$). Applying $H_1(\cdot, Z)$ to (6.1.1), we get the commutativity of the left square in

$$
\begin{array}{ccc}
H_1(X^\text{an}, Z) & \xrightarrow{(\alpha_p)_*} & H_1(J^\text{an}, Z) \\
\downarrow \tau_* & & \downarrow \tau_* \\
H_1(\Gamma, Z) & \xrightarrow{(\alpha_p)_*} & H_1(\text{Jac}(\Gamma), Z) \rightarrow N_{\mathbb{R}}
\end{array}
$$

The right square commutes by basic point-set topology: note that the map $\text{trop} : E^\text{an} \rightarrow N_{\mathbb{R}}$ is the (unique) lift of $\tau' : J^\text{an} \rightarrow \text{Jac}(\Gamma)$ to universal covers.
The map $(\alpha_P)_* : M \to M' = H_1(J^{an}, \mathbb{Z})$ by Proposition 4.7. One checks that the composition of the bottom horizontal arrows is the homomorphism $M \to N_R$ defined by the edge length pairing: see (3.3) and Lemma 3.4. Hence $\text{trop} \circ \alpha^*$ takes $H_1(X^{an}, \mathbb{Z})$ into $H_1(\Gamma, \mathbb{Z})$ via the isomorphism $\tau$, functorially induced by the retraction $\tau$. We are using $\tau$ to identify $H_1(\Gamma, \mathbb{Z})$ with $M = H_1(X^{an}, \mathbb{Z})$, so $\text{trop} \circ \alpha^*$ is the identity map under our identifications.

Remark 6.4. Suppose that $\Lambda = \mathbb{Z}$. Theorem 2.9 says that the edge length pairing $(\cdot, \cdot)$ coincides with the analytic monodromy pairing $(\cdot, \cdot)_{an}$, which is the same as Grothendieck’s monodromy pairing by Remark 6.3. We thus have a new proof of Theorem 2.3, which is very different from the standard ones as it does not use arithmetic intersection theory or the Picard-Lefschetz formula.

6.5. We can rephrase Theorem 2.9 by saying that the uniformizations of $J$ and of $\text{Jac}(\Gamma)$ are “compatible under tropicalization” in the sense that the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \to & M & \to & E^{an} & \to & J^{an} & \to & 0 \\
& & \downarrow \text{trop} & & \tau & & \\
0 & \to & M & \to & N_R & \to & \text{Jac}(\Gamma) & \to & 0
\end{array}
\]

Here we have used $\alpha^*$ to identify $M' = H_1(J^{an}, \mathbb{Z})$ with $M = H_1(X^{an}, \mathbb{Z}) = H_1(\Gamma, \mathbb{Z})$. Since $\Sigma(J) = N_R/\text{trop}(M)$, the map $\pi : \Sigma(J) \to \text{Jac}(\Gamma)$ of (5.5.2) is an isomorphism, so we have proved:

Corollary 6.6. There is a canonical isomorphism of the skeleton $\Sigma(J)$ of the Jacobian of $X$ with the Jacobian $\text{Jac}(\Gamma)$ of the skeleton of $X$ as principally polarized real tori, with the polarizations defined by the analytic monodromy pairing and the edge length pairing, respectively. The map $\varphi' : J^{an} \to \text{Jac}(\Gamma)$ of Corollary 6.6 coincides with the retraction to the skeleton $\varphi : J^{an} \to \Sigma(J)$ under this identification.

Remark 6.7. Suppose that $X$ is a Mumford curve, so that its Jacobian $J$ has totally degenerate reduction. In this case $E = T$ and the map $M \to T(K) = \text{Hom}(M, K^\times)$ is given by a pairing $[\cdot, \cdot] : M \times M \to K^\times$. Choosing a basis for $M$, one can represent this pairing by a matrix $m$ with entries in $K^\times$. We call $m$ the period matrix of $J$. Similarly, we call the matrix for the pairing $(\cdot, \cdot)$ the period matrix of the principally polarized tropical abelian variety $\text{Jac}(\Gamma)$. We have $(\cdot, \cdot)_{an} = \text{val} \circ [\cdot, \cdot]$ by definition, so Theorem 2.9 says that the period matrix of $\text{Jac}(\Gamma)$ is the valuation of the period matrix of $J$. Cf. Remark 2.10.

Corollary 6.8. The inverse image of zero under the specialization map $\varphi' : J^{an} \to \text{Jac}(\Gamma)$ is equal to the analytic domain $J_0 \subset J^{an}$ of (4.2).

Proof. By definition $J_0 = \varphi'^{-1}(0)$, so this follows from Corollary 6.6.

7. Surjectivity of principal divisors

In this section we assume that $K$ is an algebraically closed complete non-Archimedean field. Otherwise we keep the notations of §5.5.

7.1. Since $K$ is algebraically closed we have $\text{Div}_K^0(X) = \text{Div}^0(X)$, which surjects onto $J(K)$ and onto $\text{Div}_\Lambda^0(X)$; taking kernels of the left square in (5.3.1), we get a homomorphism of short exact sequences

\[
\begin{array}{cccccc}
0 & \to & \text{Prin}(X) & \to & \text{Div}^0(X) & \to & J(K) & \to & 0 \\
& & \downarrow \tau_s & & \tau_s & & \varphi & \\
0 & \to & \text{Prin}_\Lambda(\Gamma) & \to & \text{Div}_\Lambda^0(\Gamma) & \to & \text{Jac}_\Lambda(\Gamma) & \to & 0
\end{array}
\]

with surjective middle and right vertical arrows. By the Poincaré-Lelong formula (Remark 5.4), if $f$ is a nonzero meromorphic function on $X$ and $F = -\log |f|_T$, then $F$ is a $\Lambda$-rational meromorphic function on $\Gamma$ and $\tau_s(\text{div}(f)) = \text{div}(F)$; this explains the left vertical arrow. We can now prove the first part of Theorem 1.1.
Theorem. (Theorem 1.1) The map on principal divisors $\tau_* : \text{Prin}(X) \to \text{Prin}_\Lambda(\Gamma)$ is surjective.

Proof. Let $\text{Div}^0(X) = \ker(\tau_* : \text{Div}^0(X) \to \text{Div}^0(\Gamma))$. By Corollary 6.8 we have $J_0(K) = \ker(\tau')$. Applying the snake lemma to the sequence (7.1.1), it is enough to prove that the map $\text{Div}^0(X) \to J_0(K)$ is surjective. But this is exactly [BL84, Theorem 5.1(c)]. ■

Corollary. (Corollary 1.2) Let $F : \Gamma \to \mathbb{R}$ be a continuous function. Then there exists a nonzero rational function $f \in K(X)$ such that $F = -\log|f|_{\Gamma}$ if and only if $F$ is a $\Lambda$-rational tropical meromorphic function.

Proof. We have already remarked that the restriction of $-\log|f|$ to $\Gamma$ is meromorphic and $\Lambda$-rational. Conversely, let $F$ be a $\Lambda$-rational tropical meromorphic function on $\Gamma$ and let $D = \text{div}(F)$. By Theorem 1.1 there exists a nonzero meromorphic function $f$ on $X$ such that $\tau_*(\text{div}(f)) = D$. Letting $F' = -\log|f|_{\Gamma}$, we have $\text{div}(F') = D = \text{div}(F)$, so $F - F' = \lambda \in \Lambda$ is a constant function. Choosing $\alpha \in K^\times$ with valuation $\lambda$, we have $F = -\log|\alpha f|_{\Gamma}$. ■

7.2. Let $\overline{D} = \overline{D}_1 - \overline{D}_2 \in \text{Prin}_\Lambda(\Gamma)$, where $\overline{D}_1$ and $\overline{D}_2$ are effective divisors of degree $n \geq 0$. The surjectivity of $\tau_* : \text{Prin}(X) \to \text{Prin}_\Lambda(\Gamma)$ does not imply that there exist effective divisors $D_1, D_2$ on $X$ of degree $n$ such that $D = D_1 - D_2$ is principal and $\tau_*(D_i) = \overline{D}_i$ for $i = 1, 2$ (see Remark 7.4 for a counterexample). This is because the map $\tau_* : \text{Div}^0(X) \to \text{Div}^0(\Gamma)$ allows for cancellation: for example, if $x, y \in X(K)$ retract to the same point of $\Gamma$ then $\tau_*(x - y) = 0$. The second part of Theorem 1.1 states that we can limit the amount of cancellation that happens:

Theorem. (Theorem 1.1 continued) Suppose that $X$ has genus $g$. Let $\overline{D} = \overline{D}_1 - \overline{D}_2 \in \text{Prin}_\Lambda(\Gamma)$, where $\overline{D}_1, \overline{D}_2 \in \text{Div}_\Lambda^0(\Gamma)$ are effective divisors of degree $n$. There exist effective divisors $D_1, D_2 \in \text{Div}^{n+g}(X)$ of degree $n + g$ such that $D = D_1 - D_2 \in \text{Prin}(X)$ and $\tau_*(D) = \overline{D}$.

Proof. We will prove the following more precise statement. Let $D' \in \text{Div}^0(X)$ be any divisor such that $\tau_*(D') = \overline{D}$. We claim that there exist effective divisors $E_1, E_2 \in \text{Div}(X)$ such that:

1. $\deg(E_1) = \deg(E_2) \leq g$,
2. $\tau_*(E_1 - E_2) = 0$, and
3. the divisor $D := D' + E_1 - E_2 \in \text{Div}^0(X)$ is principal.

Chasing $D'$ around the diagram (7.1.1) and using Corollary 6.8 one sees that the claim amounts to proving that any element of $J_0(K)$ has the form $[E_1 - E_2]$ for $E_1, E_2 \in \text{Div}(X)$ satisfying (1) and (2).

Let $X$ be a semistable model of $X$ whose special fiber has smooth irreducible components. As explained in [4.1] and [BL84 §5], the $K$-analytic group $J_0$ has semi-abelian reduction; in fact its special fiber identified with the generalized Jacobian $\text{Jac}(X_k)$ of the special fiber $X_k$. Hence there is a reduction map $\text{red} : J_0(K) \to \text{Jac}(X_k)(k)$. Let $\overline{C}_1, \ldots, \overline{C}_r$ be the irreducible components of $X_k$. There is a canonical short exact sequence

$$0 \to \overline{T} \to \text{Jac}(X_k) \xrightarrow{\rho} \prod_{i=1}^r \text{Jac}(\overline{C}_i) \to 0,$$

where $\overline{T}$ is the split $k$-torus with character lattice $M$; its dimension is equal to the cyclomatic number $t := \dim_{\mathbb{Q}} H_1(\Gamma, \mathbb{Q})$.

Let $e_1, \ldots, e_t$ be a set of edges for some $\Lambda$-rational model of $\Gamma$ whose complement is a maximal spanning tree. For $i = 1, \ldots, t$ choose a $\Lambda$-point $\zeta_i$ in the interior of $e_i$, and set $R = \prod_{i=1}^t r^{-1}(\zeta_i)$. Then $R$ is a product of $t$ affinoid annuli of modulus zero, so its canonical reduction $\overline{R}$ is isomorphic to $\mathbb{G}_{m,k}^t$. For $i = 1, \ldots, r$ let $\zeta_i \in X^\text{an}$ be the unique point reducing to the generic point of $\overline{C}_i$, and let $C_i = \tau^{-1}(\zeta_i)$, so $C_i$ is the set of all points of $X^\text{an}$ reducing to smooth points of $\overline{C}_i$. As in [BL84 Situation 3.4′] we define

$$R \times C^{(\epsilon)} := R \times \prod_{i=1}^r C_i^{(\epsilon_i)} \subset (X^\text{an})^{(\epsilon)}$$
where \( g_i \) is the genus of \( C_i \) and the superscripts denote symmetric powers. This is an affinoid space with canonical reduction \( (R \times C^{(s)})^- = G_{m,k}^t \times \prod_{i=1}^s \overline{C}_i^{(s)}(g_i) \). Points of \( R \times C^{(s)}(K) \) can be interpreted as degree-\( g \) effective divisors on \( X \), all of which have the same retraction \( \sum_{i=1}^s (e_i) + \sum_{j=1}^n g_i (\xi_j) \) to \( \Gamma \). For a divisor \( E_2 \in R \times C^{(s)}(K) \) let \( \varphi_{E_2} : R \times C^{(s)} \to \mathcal{A}_n \) be the morphism defined on \( K \)-points by \( E_1 \mapsto [E_1 - E_2] \). The image of \( \varphi_{E_2} \) is contained in \( J_0 = \ker(\tau) \) by (7.13) and Corollary 6.8 since \( E_1 - E_2 \in \ker(\tau) \) for all \( E_1 \in R \times C^{(s)}(K) \). We will prove that for a suitable divisor \( E_2 \in R \times C^{(s)}(K) \), the map \( \varphi_{E_2} : R \times C^{(s)}(K) \to J_0(K) \) is surjective.

Let \( E_2 \in R \times C^{(s)}(K) \). Define a map \( \overline{\varphi}_{E_2} : (R \times C^{(s)})^- (k) \to \text{Jac}(X_k)(k) \) as follows. Let \( X'_k \) be the semistable model of \( X \) obtained by blowing up the nodes of \( X_k \) in such a way that each \( e_i \) reduces to a generic point of \( X_k \). Then the points of \( (R \times C^{(s)})^- (k) \) can be interpreted as degree-\( g \) divisors on \( X'_k \) with support on the smooth locus. Letting \( E_2 \in \text{Div}^g(X'_k) \) be the reduction of \( E_2 \), we set \( \overline{\varphi}_{E_2}(E_1) = [E_2 - E_1] \in \text{Jac}(X'_k) = \text{Jac}(X_k) \). By [BL84] Theorem 5.1(c) we have \( \overline{\varphi}_{E_2} \circ \text{red} = \text{red} \circ \varphi_{E_2} \).

Write \( E_2 = (E_2, E_2') \in (R \times C^{(s)})^- (k) \), where \( E_2 \in \mathcal{R}(k) = G_{m,k}^t(k) \) and \( E_2' \in \prod_{i=1}^s \overline{C}_i^{(s)}(g_i)(k) \). The map \( \overline{\varphi}_{E_2} \) identifies \( G_{m,k}^t(k) = \mathcal{R}(k) \times \prod_{i=1}^s \overline{C}_i^{(s)}(g_i)(k) \) with the toric part \( \overline{T}(k) \) of \( \text{Jac}(X_k) \); see the end of §4 in [BL84]. Therefore \( \overline{\varphi}_{E_2} \) is surjective for generic \( E_2 \). We fix a divisor \( E_2 \) whose reduction \( \overline{E}_2 \) is generic in this sense.

Now let \( z \in J_0(K) \) and let \( \overline{z} \) be its reduction in \( \text{Jac}(X_k) \). By the above we know that there exists \( E_1 \in R \times C^{(s)}(K) \) such that \( \overline{\varphi}_{E_2}(E_1) = z \). Hence \( \varphi_{E_2}(E_1) \) and \( z \) are both contained in \( \text{red}^{-1}(\overline{z}) \). This formal fiber is isomorphic to an open \( g \)-ball \( B(1,g^r) \); fix an isomorphism \( \text{red}^{-1}(\overline{z}) \cong B(1,g^r) \) such that \( \varphi_{E_2}(E_1) \) is identified with zero. Since \( z \) is contained in some smaller closed ball \( B(\rho)^g \subset B(1,g^r) \), there exists a morphism \( \psi : B(\rho)^g \to \text{red}^{-1}(\overline{z}) \) such that \( \psi(0) = \varphi_{E_2}(E_1) = [E_1 - E_2] \) and whose image contains \( z \). By Bosch and Lütkebohmert’s Homotopy Theorem [BL84] Theorem 3.5], for generic \( E_2 \) the map \( \psi \) lifts through \( R \times C^{(s)} \); that is, there exists a map \( \psi' : B(\rho)^g \to R \times C^{(s)} \) such that \( \varphi_{E_2} \circ \psi' = \psi \). Choosing \( E_2 \) so that \( \overline{E}_2 \) satisfies all necessary genericity conditions, we have that \( z \) is contained in the image of \( \varphi_{E_2} \), as desired.

The following geometric lemma was used in the above proof.

**Lemma 7.3.** Let \( k \) be an algebraically closed field and let \( C \) be a smooth, proper, connected \( k \)-curve of genus \( g \). Let \( D \subset C \) be a finite set of points and let \( \mathcal{L} \) be a degree-zero line bundle on \( C \). For a generic effective divisor \( E_2 \in \text{Div}^g(C \setminus D) \), there exists an effective divisor \( E_1 \in \text{Div}^g(C \setminus D) \) such that \( \mathcal{L} \cong \Theta(E_1 - E_2) \).

**Proof.** Write \( D = \{x_1, \ldots, x_n\} \). For any effective divisor \( E_2 \in \text{Div}^g(C) \) we have \( h^0(\mathcal{L}(E_2)) > 0 \) by the Riemann-Roch theorem, which means that \( \mathcal{L} \) admits a meromorphic section \( s \) with poles bounded by \( E_2 \); hence \( \text{div}(s) = E_1 - E_2 \) for some effective divisor \( E_1 \in \text{Div}^g(C) \). We need to show that for generic \( E_2 \), we can choose \( E_1 \) with support disjoint from \( D \). It is enough to prove that a generic effective divisor \( E_2 \in \text{Div}^g(C \setminus D) \) satisfies \( h^0(\mathcal{L}(E_2 - (x_i))) = 0 \) for all \( i = 1, \ldots, n \) since then no global section of \( \mathcal{L}(E_2) \) can have a zero at any \( x_i \).

A line bundle \( \mathcal{L}' \) on \( C \) of degree \( g - 1 \) is called special if \( h^0(\mathcal{L}') \) > 0. The special line bundles are exactly those contained in the theta divisor \( \Theta \subset \text{Pic}^{g-1}(C) \), the image of the natural map \( C^{(g-1)} \to \text{Pic}^{g-1}(C) \). Hence we want to prove that for generic \( E_2 \), the line bundle \( \mathcal{L}(E_2 - (x_i)) \) is not special for all \( i = 1, \ldots, n \). Let \( \mu_i : \text{Pic}^g(C) \to \text{Pic}^{g-1}(C) \) be the isomorphism defined on \( K \)-points by \( \mathcal{M} \mapsto \mathcal{M}(-x_i) \) and let \( U = \text{Pic}^g(C) \setminus \bigcup_{i=1}^n \mu_i^{-1}(\Theta) \). This is a dense open subset of \( \text{Pic}^g(C) \), and for a line bundle \( \mathcal{M} \in U(K) \), by construction \( \mathcal{M}(-x_i) \) is not special for all \( i \). Any divisor \( E_2 \) mapping to \( U \) under the proper birational morphism \( C^{(g)} \to \text{Pic}^g(C) \) given by \( E \mapsto \mathcal{L}(E) \) satisfies our requirements.

**Remark 7.4.** With the notation in the above theorem, it is not true in general that there exist \( D_1, D_2 \in \text{Div}^m(X) \) such that \( D := D_1 - D_2 \) is principal and \( \tau_*(D) = \overline{T} \). For example, let \( X \) be an elliptic curve with good reduction. Then \( X \) has a skeleton \( \Gamma \) which is a line segment, corresponding
to a blow-up of its smooth model along a closed point on the special fiber. Let $P_1, P_2$ be the endpoints of $\Gamma$. Then $(P_1) - (P_2)$ is a principal divisor on $\Gamma$, but a divisor of the form $(p_1) - (p_2) \in \text{Div}^0(X)$ is principal if and only if $p_1 = p_2$.

8. An application to tropical geometry

We conclude with an application of Theorem 1.1 (or more precisely of Corollary 1.2) to “faithful tropical representations” of skeleta in the sense of [BPR11].

8.1. In order to state the result, we briefly recall some terminology from tropical geometry. Let $K$ be a complete and algebraically closed non-trivial non-Archimedean field. Let $X$ be a variety over $K$ and let $\mathfrak{X}$ be a toric variety over $K$ with dense torus $T = \text{Spec} K[M]$, where $M \cong \mathbb{Z}^n$ is the character lattice of $T$. If $f : X \to Y$ is any rational map which restricts to a morphism $U \to T$ with $U$ a Zariski-dense open affine subset of $X$, we can define a tropicalization map $\text{trop}(f) : U^{an} \to N_\mathbb{R} := \text{Hom}(M, \mathbb{R}) \cong \mathbb{R}^n$ similarly to (4.2) by setting

$$(\text{trop}(f)(x))(u) := -\log |\chi^u \circ f|_x,$$

where $\chi^u$ is the character of $T$ corresponding to $u \in M$ and $| \cdot |_x$ is the multiplicative seminorm on $K[U]$ corresponding to $x \in U^{an}$. In particular, if $X$ is a smooth proper $K$-curve and $\Gamma$ is a skeleton of $X$, then $\Gamma$ is contained in $U^{an}$ for any Zariski-dense open subset of $X$ and thus there is a well-defined restriction map

$$\text{trop}(f)|_\Gamma : \Gamma \to N_\mathbb{R}.$$ 

Since $\text{trop}(f)(\Gamma)$ has rational slopes, there is a natural lattice length metric on $\text{trop}(f)(\Gamma)$ (see [BPR11] Paragraph 6.2)) and since $\Gamma$ itself is naturally a metric space, it makes sense to ask whether $\text{trop}(f)|_\Gamma$ is an isometry. By [BPR11] Theorem 6.20, there exists a closed immersion $f : X \to Y_\Delta$ into some toric variety $Y_\Delta$ such that $\text{trop}(f)|_\Gamma$ is an isometry. However, the toric variety $Y_\Delta$ constructed in loc. cit. has very large dimension in general. Using Corollary 1.2 we can cut down the dimension of $Y$ significantly, at the expense of replacing the closed immersion $f$ with a rational map.

**Theorem 8.2.** If $X$ is a smooth proper $K$-curve and $\Gamma$ is any skeleton of $X$, then there is a rational map $f : X \to \mathbb{P}^3$ such that the restriction of $\text{trop}(f)$ to $\Gamma$ is an isometry onto its image (with respect to the lattice length metric on $\text{trop}(f)(\Gamma) \subset \mathbb{R}^3$).

**Proof.** Let $G$ be a $\Lambda$-rational model for $\Gamma$ in the sense of (3.1). By Corollary 1.2, it suffices to prove that there are $\Lambda$-rational tropical meromorphic functions $F_1, F_2, F_3$ on $G$ which are linear on each edge of some $\Lambda$-rational subdivision $G''$ of $G$ such that (i) the $F_i$ separate points of $\Gamma$ and (ii) for each edge $e$ of $G''$, the greatest common divisor of the absolute values of the slopes of the $F_i$ along $e$ is equal to 1. (Condition (ii) guarantees that $\text{trop}(f) : \Gamma \to \mathbb{R}^3$ is a local isometry; c.f. [BPR11] Definition 6.5 and Remark 6.6), and condition (i) guarantees that it is injective.)

We define such functions $F_1, F_2, F_3$ as follows. Fix a loopless model $G$ for $\Gamma$ with no multiple edges and label the vertices of $G$ as $v_1, \ldots, v_n$ and the edges of $G$ as $e_1, \ldots, e_m$. Let $G'$ be the ‘barycentric subdivision’ of $G$ in which a new vertex is inserted in the middle of each edge $e$, thereby replacing $e$ with two new edges of half the original length. For each edge $e = vw$ of $G$ and each $x \in e$, let $\tilde{x}$ be the reflection of $x$ with respect to the midpoint $\tilde{e}$ of $e$.

1. Let $F_1$ be the function which, for each edge $e = vw$ of $G'$, increases linearly with slope 1 from $v$ to $\tilde{e}$ and then decreases with slope 1 from $\tilde{e}$ to $w$, with $F_1(v) = F_1(w) = 0$ (so that $F_1$ is linear on each edge of $G'$).

2. Choose distinct positive integers $\sigma_1, \ldots, \sigma_m$ and let $F_2$ be the function which, for each edge $e_i = vw$ of $G'$, increases linearly with slope $\sigma_i$ from $v$ to $\tilde{e}_i$ and then decreases with slope $\sigma_i$ from $\tilde{e}_i$ to $w$, with $F_1(v) = F_1(w) = 0$ (so that $F_2$ is linear on each edge of $G'$).

3. Choose distinct positive real numbers $\alpha_1, \ldots, \alpha_n \in \Lambda$ and let $F_3$ be any $\Lambda$-rational tropical meromorphic function on $\Gamma$ such that $F_3(v_i) = \alpha_i$ for all $i$, and which is asymmetric in the sense that $F_3(x) \neq F_3(\tilde{x})$ for all $x \in \Gamma \setminus V(G')$. (For example, one can choose $F_3$ so that along
In the latter case, since $F_i$ has slope zero in some neighborhoods of $v$ and $w$ and slope $\pm \mu$ in some neighborhood of $\hat{\epsilon}$ for an integer $\mu$ with $|\mu|$ sufficiently large.)

See Figure 1. Let $G''$ be a refinement of $G'$ such that $F_3$ is linear along each edge of $G''$. We claim that $F_1, F_2, F_3$ satisfy conditions (i) and (ii) above. Condition (ii) is clear, since $F_1$ has slope $\pm 1$ along each edge of $G''$. To verify condition (i), suppose $x, y \in \Gamma$ and that $F_i(x) = F_i(y)$ for $i = 1, 2, 3$. We want to show that $x = y$. Suppose $x \in e_i$ and $y \in e_j$, and identify $e_i$ (resp. $e_j$) with $[0, \ell(e_i)]$ (resp. $[0, \ell(e_j)]$) via an isometry $\iota_i$ (resp. $\iota_j$) in such a way that $\iota_i(x) \in [0, \ell(e_i)/2]$ (resp. $\iota_j(y) \in [0, \ell(e_j)/2]$). Since $F_1(x) = F_1(y)$, we must have $\iota_i(x) = \iota_j(y)$. Since $F_2(x) = F_2(y)$, we must have $\sigma_{\iota_i}(x) = \sigma_{\iota_j}(y)$, and therefore either $i = j$ or $\iota_i(x) = \iota_j(y) = 0$ (i.e., $x, y$ are vertices of $G$).

In the latter case, since $F_3$ takes distinct values on distinct vertices of $G$ we conclude that $x = y$ as desired. So suppose that $i = j$ and that $x, y$ are not vertices of $G$. From the relation $\iota_i(x) = \iota_j(y)$ we conclude that either $x = y$ or $x = \hat{y}$. Since $F_3$ is asymmetric, the relation $F_3(x) = F_3(y)$ implies that $x = y = \hat{\epsilon}_i$.

**Remark 8.3.** The proof of Theorem 8.2 implies the purely combinatorial fact that any metric graph $\Gamma$ admits a quasi-balanced isometric embedding in $\mathbb{R}^3$, where ‘quasi-balanced’ means that one can add infinite rays (with multiplicities) to $\Gamma$ in such a way that the resulting one-dimensional weighted polyhedral complex $\tilde{\Gamma}$ satisfies the balancing condition from tropical geometry. (See for example [MS09, §3.4] for a discussion of the balancing condition.)

**Remark 8.4.** If $X$ is an elliptic curve with bad reduction and $\Gamma$ is the minimal skeleton of $X$ (which is homeomorphic to a circle), then by [BPR11, Theorem 7.2] or [CS12] there is a closed immersion $f : X \to \mathbb{P}^2$ such that the restriction of $\text{trop}(f)$ to $\Gamma$ is an isometry. However, it is clear that such a result cannot hold for arbitrary curves $X$ since (a) not every algebraic curve can be embedded in $\mathbb{P}^2$ and (b) not every graph is planar.

**Remark 8.5.** We conjecture that Theorem 8.2 can be strengthened by requiring $f$ to be a closed immersion of $X$ in a 3-dimensional projective toric variety $Y_\Delta$ (in which case $\Gamma$ is “faithfully represented” in the sense of [BPR11, Paragraph 6.15.2]). One might even be able to always take $Y_\Delta = \mathbb{P}^3$.

**Remark 8.6.** Corollary 1.2 can also be used to streamline the proof of [BPR11, Theorem 6.20], reducing the number of rational functions required to get a faithful representation. In particular, Corollary 1.2 obviates the need for Lemmas 6.18 and 6.19 in [BPR11], and provides a powerful generalization of such results.

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