ASYMPTOTIC STABILITY OF TRAVELING FRONTS TO A CHEMOTAXIS MODEL WITH NONLINEAR DIFFUSION

MOHAMMAD GHANI, JINGYU LI AND KAIJUN ZHANG∗

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China

(Communicated by Zhian Wang)

Abstract. We are interested in the existence and stability of traveling waves of arbitrary amplitudes to a chemotaxis model with porous medium diffusion. We first make a complete classification of traveling waves under specific relations among the biological parameters. Then we show all these traveling waves are asymptotically stable under appropriate perturbations. The proof is based on a Cole-Hopf transformation and the energy method.

1. Introduction. In this paper, we consider the following PDE-ODE hybrid chemotaxis model

\[ \begin{cases} 
    u_t = D(u^m)_{xx} - \chi (u \ln c)_x, \\
    c_t = -uc + \beta c 
\end{cases} \tag{1} \]

with \( m > 0 \) and initial data

\[ (u, c)(x, 0) = (u_0, c_0)(x) \rightarrow (u_{\pm}, c_{\pm}) \text{ as } x \rightarrow \pm \infty. \tag{2} \]

System (1) could model the reinforced movement of cells (or bacterial) in porous media, where \( u \) is the population density of cells, and \( c \) is the concentration of chemical signals (e.g. nutrient) with growth rate \( \beta > 0 \). \( D > 0 \) is the diffusion rate of cells and \( \chi \) is the chemotactic coefficient. The chemotaxis is said to be attractive if \( \chi > 0 \) and repulsive if \( \chi < 0 \). The logarithmic sensitivity \( \ln c \) comes from the pervasiveness of Weber-Fechner law [9], and has been verified by experimental data [7].

When \( m = 1 \), system (1) is exactly the chemotaxis model proposed in [19] to describe the reinforced random walks. There are lots of interesting analytical works in this case. Othmer and Stevens [19] derived the model from random walk, and carried out the numerical simulations of the formation of spikes and blowup. Subsequently, Levine and Sleeman [10] presented analytical results supporting some numerical results in [19]. Yang etc. [25, 26] investigated the global existence and blowup of classical solutions on a bounded domain with no-flux boundary conditions. [13] and [27] further studied the global existence of smooth solutions and weak solutions to system (1) with Robin boundary condition, respectively. Global dynamics including well-posedness and large time behaviors of solutions in the whole

2020 Mathematics Subject Classification. Primary: 35A01, 35B40, 35Q92; Secondary: 92C17.

Key words and phrases. Chemotaxis, nonlinear diffusion, asymptotic stability, traveling fronts.

The third author is supported by National Science Foundation of China (No. 11771071).

∗ Corresponding author: zhangkj201@nenu.edu.cn.
In case (iii). Moreover, the wave speed $s$ in (i); $C_{\text{st}} > 0$. Hence, the wave patterns of $U$, $u$, and $c$ are different: $C$ is a front in case (i), while it is a pulse in case (ii). The traveling wave in case (iii) is a coexistence pattern, since at any finite spatial position, $U$ goes to $u_- > 0$, while $C$ shrinks to 0 as $t \to \infty$. However, the wave patterns of $C$ are different: $C$ is a front in case (i), while it is a pulse in case (ii). The traveling wave in case (iii) is a coexistence pattern, since at any finite spatial position, $U$ goes to $u_- > 0$ and $C$ goes to $c_- > 0$ as $t \to \infty$. This result indicates that the growth rate of signals has significant impact on the pattern formations of the chemotaxis model.

**Remark 2.** The chemotaxis model (1) does not have a traveling wave solution if $\chi < 0$. Otherwise, if $(U, C)$ is a traveling wave of (1) with $\chi < 0$, then equalities (13) and (54) still hold. If the wave speed $s > 0$, it then follows from (13) that $U' > 0$ and hence $u_- < u_+$. But owing to (54), to ensure $C$ is bounded, we need $u_+ \leq \beta$ and $u_- \geq \beta$, which leads to a contradiction that $\beta \leq u_- < u_+ \leq \beta$. Similarly, if $s < 0$, (13) implies $U' < 0$ and $u_- > u_+$. But (54) implies $u_- \leq \beta$ and $u_+ \geq \beta$, which also leads to a contradiction $\beta \geq u_- > u_+ \geq \beta$. **Theorem 1.1 (Existence).** Let $D > 0$, $m > 0$, $\chi > 0$, $\beta > 0$ and $u_+ > 0$. Assume that $(u_\pm, c_\pm)$ satisfies one of the following conditions:

(i) $u_- > u_+ = \beta$ and $0 = c_- < c_+$;
(ii) $u_- > \beta > u_+$ and $c_+ = 0$;
(iii) $\beta = u_- > u_+$ and $c_- > c_+ = 0$.

Then system (1)-(2) has a unique (up to a translation) traveling wave solution $(U, C)(x - st)$ with $U' < 0$. The chemical concentration $C$ satisfies: $C' > 0$ in case (i); $C' > 0$ on $(-\infty, z_0)$, $C' < 0$ on $(z_0, +\infty)$ for some number $z_0$ in case (ii); $C' < 0$ in case (iii). Moreover, the wave speed $s$ is given by $s = \sqrt{\chi(u_- + u_+ - \beta)}$. **Remark 1.** The traveling waves in cases (i) and (ii) are both invasion patterns, since at any finite spatial position, $U$ goes to $u_- > 0$, while $C$ shrinks to 0 as $t \to \infty$. However, the wave patterns of $C$ are different: $C$ is a front in case (i), while it is a pulse in case (ii). The traveling wave in case (iii) is a coexistence pattern, since at any finite spatial position, $U$ goes to $u_- > 0$ and $C$ goes to $c_- > 0$ as $t \to \infty$. This result indicates that the growth rate of signals has significant impact on the pattern formations of the chemotaxis model.
Denote by $H^m(\mathbb{R})$ the usual Sobolev space with norm $\|f\|_m := \sum_{k=0}^m \|\partial_x^k f\|$ and $\|f\| := \|f\|_{L^2(\mathbb{R})}$. Our main result on the stability of traveling wave is stated as follows.

**Theorem 1.2** (Stability). Assume that $D > 0$, $m > 0$, $\chi > 0$, $\beta > 0$ and $u_+ > 0$. Let $(U, C)(x - st)$ be a traveling wave obtained in Theorem 1.1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_2 + \|\ln c_0\|_2 + \|(\phi_0, \psi_0)\|_3 \leq \varepsilon_0$, then the Cauchy problem (1)-(2) has a unique global solution $(u, c)(x, t)$ satisfying

$$(u - U, (\ln c)_x - (\ln C)_x) \in C([0, \infty); H^2) \cap L^2([0, \infty); H^2),$$

and

$$\sup_{x \in \mathbb{R}} |(u, c)(x, t) - (U, C)(x - st)| \to 0 \text{ as } t \to +\infty.$$  

**Remark 3.** We need $u_+ > 0$ essentially to derive the existence and stability of traveling waves to system (1). It is interesting to investigate the dynamics of system (1) when $\beta = 0$ (and hence $u_+ = 0$). However, this problem is quite challenging since it has singularities if $m < 1$, while it has degeneracies if $m > 1$. We leave this problem for the future work.

To show Theorems 1.1 and 1.2, we take the Cole-Hopf transformation as in [6, 14],

$$v = -(\ln c)_x,$$  

(3)

to get the parabolic-hyperbolic system

$$\begin{cases}
u_t - \chi(vu)_x = D(u^m)_{xx}, \\
v_t - u_x = 0,
\end{cases}$$  

(4)

with initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u_+, v_+) \text{ as } x \to \pm\infty.$$  

(5)

We first prove the existence of traveling fronts to system (4) by the phase-plane analysis; and then we derive the stability of such traveling fronts via energy estimates; finally we transfer the results of system (4) back to the original chemotaxis model (1). During this process, one can see that under the influence of porous medium diffusion, we have to establish the estimate for the third order derivative to close the a priori estimates. Moreover, notice the transformed system (4) does not involve $\beta$, when we transfer the results from system (4) to system (1), certain conditions on $u_\pm, c_\pm$ and $\beta$ are needed to obtain biological meaningful results.

The rest of paper is organized as follows. In Section 2, we present the existence and some basic properties of traveling waves to the transformed parabolic-hyperbolic system (4), and derive the perturbation equations. Section 3 is devoted to establishing the a priori estimate. In Section 4, we prove the stability of traveling waves to the parabolic-hyperbolic system (4), and transfer the results to the original chemotaxis model (1).
2. Transformation of the problem. We first seek the traveling wave \((U, V)(x-st)\) of the parabolic-hyperbolic system (4). Substituting the following traveling wave ansatz 

\[(u,v)(x,t) = (U,V)(z), \quad z = x-st\]  

(6) into (4), where \(s\) denotes the traveling wave speed and \(z\) is the moving coordinate, we have

\[
\begin{cases}
-sU' - \chi(UV)' = D (U^m)', \\
-sV' = U', 
\end{cases}
\]

(7) where \(\cdot := \frac{d}{dz}\), and the boundary conditions are imposed as

\[(U,V)(z) \to (u_{\pm}, v_{\pm}) \text{ as } z \to \pm \infty.\]

(8)

Now, integrating (7) with respect to \(z\) and using the fact that \(U'(z) \to 0\) as \(z \to \pm \infty\), we get

\[
\begin{cases}
D (U^m)' = -sU - \chi UV + su_{\pm} + \chi u_{\pm}v_{\pm}, \\
-sV = U - sv_{\pm} - u_{\pm},
\end{cases}
\]

(9) and the Rankine-Hugoniot condition

\[s(u_+ - u_-) = \chi(u_-v_+ - u_+v_-),\]

\[s(v_+ - v_-) = u_- - u_+,\]

(10) which gives

\[s^2 + s\chi v_+ - \chi u_- = 0.\]

(11) Without loss of generality, we only consider the case \(s > 0\) and

\[s = -\frac{\chi v_+}{2} + \sqrt{\frac{\chi^2 v_+^2}{4} + 4\chi u_-}.\]

(12)

From (9) and (11), we get

\[U' = \frac{\chi U^{1-m}}{Dms} \cdot (U - u_-)(U - u_+).\]

(13)

Then employing the classical phase-plane analysis, one can easily prove the existence and uniqueness of traveling wave solutions to system (4).

**Lemma 2.1.** Assume that \(D > 0\), \(m > 0\), \(\chi > 0\), \(\beta > 0\) and \(u_+ > 0\). Suppose that \(u_\pm\) and \(v_\pm\) satisfy (10). Then there exists a monotone traveling wave solution \((U, V)(x-st)\) to system (7), which is unique up to a translation and satisfies \(U' < 0\), \(V' > 0\), where the wave speed \(s\) is given by (12). Moreover, \((U, V)\) decays exponentially fast with rates

\[U - u_\pm \sim e^{\lambda_\pm z}, V - v_\pm \sim e^{\lambda_\pm z} \text{ as } z \to \pm \infty,\]

where \(\lambda_\pm = \frac{\chi v_\pm}{Dms} \cdot (u_\pm - u_\pm).\)

For the parabolic-hyperbolic system (4), we define

\[(\phi_0, \psi_0)(z) := \int_{-\infty}^{z} (u_0 - U, v_0 - V)(y)dy,\]

which is the so-called zero mass perturbation (see [8, 16]). Then we have the following stability result.
Theorem 2.2. Assume that $D > 0$, $m > 0$, $\chi > 0$, $\beta > 0$ and $u_0 > 0$. Let $(U,V)(x-st)$ be the traveling wave obtained in Lemma 2.1. Then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_2 + \|v_0 - V\|_2 + \|\phi_0, \psi_0\|_3 \leq \varepsilon_0$, then the Cauchy problem (4)-(5) has a unique global solution $(u,v)(x,t)$ satisfying

$$(u - u, v - V) \in C([0,\infty); H^2) \cap L^2([0,\infty); H^2),$$

and

$$\sup_{x \in \mathbb{R}} |(u,v)(x,t) - (U,V)(x-st)| \to 0 \quad \text{as} \quad t \to +\infty.$$ 

By changing the variables $(x,t) \rightarrow (z = x-st,t)$, system (4) becomes

$$\begin{aligned}
\begin{cases}
  u_t - su_z - \chi(\phi u)_z = D(u^m)_z, \\
  v_t - sv_z - u_z = 0.
\end{cases}
\end{aligned}$$

We decompose the solution $(u,v)$ of (4) as

$$(u,v)(z,t) = (U,V)(z) + (\phi_z, \psi_z)(z,t).$$

Then

$$\phi(z,t) = \int_{-\infty}^{z} (u(y,t) - U(y))dy, \quad \psi(z,t) = \int_{-\infty}^{z} (v(y,t) - V(y))dy.$$ 

Substituting (15) into (14) and integrating the resultant equation with respect to $z$, one has

$$\begin{aligned}
\begin{cases}
  \phi_t - (s + \chi V)\phi_z - \chi U\psi_z = Dm(U^{m-1}\phi_z)_z + G + \chi\phi_z\psi_z, \\
  \psi_t - sv_z - \phi_z = 0,
\end{cases}
\end{aligned}$$

where $G = D((U + \phi_z)^m - U^m - mU^{m-1}\phi_z)_z$. The initial datum of $(\phi, \psi)$ is given by

$$(\phi, \psi)(z,0) = (\phi_0, \psi_0)(z) = \int_{-\infty}^{z} (u_0 - U, v_0 - V)dy$$

with $(\phi_0, \psi_0)(\pm \infty) = 0$. We seek the solution of reformulated problem (17)-(18) in the space

$$X(0, T) := \{(\phi, \psi) \in C([0,T), H^3) : \phi_z \in L^2((0,T); H^3), \psi_z \in L^2((0,T); H^2) \}$$

with $0 < T \leq +\infty$. Let

$$N(t) := \sup_{0 \leq \tau \leq t} \{\|\phi(\cdot,\tau)\|_3 + \|\psi(\cdot,\tau)\|_3\}.$$ 

From the Sobolev inequality $\|f\|_{L^\infty} \leq \sqrt{2}\|f\|_{L^2}^2\|f_x\|_{L^2}^2$, it follows that

$$\sup_{\tau \in [0,t]} \{\|\phi(\cdot,\tau)\|_{W^{2,\infty}}, \|\psi(\cdot,\tau)\|_{W^{2,\infty}} \} \leq N(t).$$

For system (17)-(18), we have the following global well-posedness.

Theorem 2.3. There exists a constant $\delta_1 > 0$ such that if $N(0) \leq \delta_1$, then the Cauchy problem (17)-(18) has a unique global solution $(\phi, \psi) \in X(0, +\infty)$ such that

$$\|\phi(\cdot, \tau)\|_3 + \int_0^t (\|\phi_z(\cdot, \tau)\|_3^2 + \|\psi_z(\cdot, \tau)\|_3^2) \, d\tau \leq C(\|\phi_0\|_3^2 + \|\psi_0\|_3^2)$$

for any $t > 0$. Moreover, it holds that

$$\sup_{z \in \mathbb{R}} |(\phi_z, \psi_z)(z,t)| \to 0 \quad \text{as} \quad t \to +\infty.$$
According to the classical works (see [14]), the global smooth solution can be constructed by the local wellposedness, the a priori estimate and an extension procedure. Since the local wellposedness can be proved in a standard way (e.g. see [17]), we need to establish the following a priori estimate.

**Proposition 1.** Assume that \( (\phi, \psi) \in X(0,T) \) is a solution of (17)-(18) for some time \( T > 0 \). Then there is a constant \( \varepsilon_1 > 0 \), independent of \( T \), such that if \( N(T) < \varepsilon_1 \), then \( (\phi, \psi) \) satisfies (19) for any \( 0 \leq t \leq T \).

### 3. Energy estimates.

In this section, we establish the a priori estimates for solution \( (\phi, \psi) \) of (17)-(18), and hence prove Proposition 1. We first derive the basic \( L^2 \) estimate.

**Lemma 3.1.** Under the same assumptions of Proposition 1, if \( N(t) \ll 1 \), then
\[
\| (\phi, \psi)(\cdot, t) \|^2 + \int_0^t \| \phi_z (\cdot, \tau) \|^2 d\tau \leq C(\| \phi_0 \|^2 + \| \psi_0 \|^2) + CN(t) \int_0^t \int (\phi_z^2 + \psi_z^2).
\] (21)

**Proof.** Multiplying (17) by \( \phi_z \) and (17) by \( \chi \psi \), adding them, and integrating the resulting equations, we have
\[
\frac{1}{2} \frac{d}{dt} \int \left( \frac{\phi_z^2}{U} + \chi \psi^2 \right) + Dm \int U^{m-2} \phi_z^2 = - \int \phi_z^2 \left( \frac{s + \chi V}{U} \right)_z - Dm \left( U^{m-1} \frac{1}{U} \right)_z \frac{\phi_z^2}{U} + \int \left( \frac{G\phi}{U} + \chi \psi_z \phi \right).
\] (22)

By (12), \( s + \chi v_+ > 0 \). It then follows from the first equation of (9) and \( U_z < 0 \) that
\[
\left( \frac{s + \chi V}{U} \right)_z - Dm \left( U^{m-1} \frac{1}{U} \right)_z = \left( \frac{s + \chi V}{U} - DmU^{m-1} \frac{1}{U} \right)_z = \left( \frac{su_+ + \chi u_+ v_+}{U^2} \right)_z = - \frac{2u_+(s + \chi v_+)U_z}{U^3} > 0.
\] (23)

Noting that \( U \geq u_+ > 0 \) and \( \| \phi_z (\cdot, t) \|_{L^\infty} \leq N(t) \ll 1 \), it holds
\[
|G| \leq C(|\phi_{zz}||\phi_z| + |\phi_z|^2).
\] (24)

And then by Young’s inequality, we have
\[
\left| \int \frac{G\phi}{U} \right| \leq CN(t) \int (\| \phi_z^2 \| + \| \phi_z \|^2),
\] (25)

where we have used \( \| \phi(\cdot, t) \|_{L^\infty} \leq N(t) \). Similarly,
\[
\left| \int \frac{\phi_z \psi_z \phi}{U} \right| \leq CN(t) \int (\phi_z^2 + \psi_z^2).
\] (26)

Substituting (23), (25)-(26) into (22) and using \( U \geq u_+ > 0 \), we get (21). \( \square \)

The next lemma gives the estimate of the first order derivative of \( (\phi, \psi) \).

**Lemma 3.2.** Under the same assumptions of Proposition 1, if \( N(t) \ll 1 \), it holds that
\[
\| (\phi, \psi)(\cdot, t) \|^2 + \int_0^t (\| \phi_z (\cdot, \tau) \|^2 + \| \psi_z (\cdot, \tau) \|^2) d\tau \leq C(\| \phi_0 \|^2 + \| \psi_0 \|^2).
\] (27)
Proof. Differentiating (17) in $z$ gives
\[
\begin{align*}
\frac{\partial}{\partial t} \chi U \psi_z - Dm(U^{m-1}\phi_z)z &= Dm((U^{m-1})_z \psi_z) + \chi U \psi_z + ((s + \chi V)\phi_z)_z + (G + \chi \phi_z \psi_z)_z, \quad (28) \\
\psi_z t - s \psi_{zz} - \phi_{zz} &= 0.
\end{align*}
\]
Multiplying $(28)_1$ by $\frac{\partial}{\partial t}$ and $(28)_2$ by $\chi \psi_z$, we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \int \left( \frac{\partial^2}{\partial t} + \chi \psi_z^2 \right) + Dm \int U^{m-2} \phi_{zz}^2 \\
&= \int \frac{\partial^2}{\partial t} \left( Dm \left((U^{m-1})_z \psi_z \right) + U^{m-1} \left( \frac{1}{U} \phi_z \right)_z \right) - \left( s + \chi V \right)_z \phi_z + \chi \int \frac{U \psi_z \phi_z}{U} - \int (G + \chi \phi_z \psi_z) \frac{\phi_z}{U}. \\
&= \int \left( \frac{\partial^2}{\partial t} + \chi \psi_z^2 \right) + 2Dm \int_0^t \int U^{m-2} \phi_{zz}^2 \\
&\leq C \|(\phi_0, \psi_0)_z\|^2 + C \left( 1 + \frac{1}{\delta} \right) \int_0^t \int \phi_z^2 + \frac{\delta}{\chi} \int_0^t \int U \psi_z^2 \\
&+ C \int_0^t \int \left| (G + \chi \phi_z \psi_z) \right| \left( |\phi_z| + |\phi_{zz}| \right).
\end{align*}
\]
By Young’s inequality,
\[
\chi \int \frac{U \psi_z \phi_z}{U} \leq \frac{\delta}{2} \int U \psi_z^2 + \frac{\chi}{2\delta} \int U^2 \phi_z^2,
\]
where $\delta$ is a small constant to be determined later. Substituting this inequality into (29) leads to
\[
\begin{align*}
\int \left( \frac{\partial^2}{\partial t} + \chi \psi_z^2 \right) + 2Dm \int_0^t \int U^{m-2} \phi_{zz}^2 \\
&\leq C \|(\phi_0, \psi_0)_z\|^2 + C \left( 1 + \frac{1}{\delta} \right) \int_0^t \int \phi_z^2 + \frac{\delta}{\chi} \int_0^t \int U \psi_z^2 \\
&+ C \int_0^t \int \left| (G + \chi \phi_z \psi_z) \right| \left( |\phi_z| + |\phi_{zz}| \right) + \frac{\delta}{\chi} \int \phi_z^2.
\end{align*}
\]
It remains to estimate the term $\int_0^t \int U \psi_z^2$. Multiplying the first equation of (17) by $\psi_z$ yields
\[
\chi U \psi_z^2 = \phi_t \psi_z - (s + \chi V)\phi_z \psi_z - Dm \left((U^{m-1})_z \psi_z \right) - (G + \chi \phi_z \psi_z). \quad (31)
\]
By the second equation of (28), we have
\[
\phi_t \psi_z = (\phi \psi_z)_t - \phi \psi_{zt} = (\phi \psi_z)_t - \phi(s \psi_{zz} + \phi_{zz}) = (\phi \psi_z)_t - s(\phi \psi_z)_z + s \phi \psi_z - (\phi \phi_z) + \phi_z^2. \quad (32)
\]
Combining (31) with (32) and integrating the results, we get
\[
\begin{align*}
\chi \int_0^t \int U \psi_z^2 &= \int \phi \psi_z - \int \phi_0 \psi_0 + \int \phi_z^2 - Dm \int \left((U^{m-1})_z \psi_z \right) \\
&= \chi \int_0^t \int V \phi \psi_z - \int \int (G + \chi \phi \psi_z) \psi_z.
\end{align*}
\]
By Young’s inequality, noting $u_- \geq U \geq u_+ > 0$, we have
\[
-Dm \int \left((U^{m-1})_z \psi_z \right) \psi_z = -Dm \int U^{m-1} \phi_z \psi_z - Dm \int \left((U^{m-1})_z \phi_z \psi_z \right) \\
\leq \frac{\chi}{4} \int U \psi_z^2 + \frac{Dm^2 A_m}{\chi} \int U^{m-2} \phi_z^2 + C \int \phi_z^2,
\]
In view of (24), by Young’s inequality, the fact that
\[
\|V \phi_z \psi_z \| \leq \frac{\lambda}{4} \int U \psi_z^2 + \lambda \int V^2 \phi_z^2.
\]
Thus,
\[
\chi \int_0^t \int U \psi_z^2 \leq \int \phi_z^2 + \int \psi_z^2 + 2 \int |\phi_0 \psi_0| + \frac{2D^2 m^2 A_m}{\chi} \int_0^t \int U^{m-2} \phi_{zzz}^2
\]
\[
+ C \int_0^t \int \phi_z^2 + C \int_0^t \int |G + \chi \phi_z \psi_z| \psi_z.
\]
Substituting (33) into (30), and choosing \(\delta = \min \{\frac{\lambda}{2Dm A_m}, \frac{\lambda}{2}\}\), by Lemma 3.1, when \(N(t) \ll 1\), we have
\[
\int (\phi_z^2 + \chi \psi_z^2) \leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C t \int |G + \chi \phi_z \psi_z| \left( \|\phi_z\|_U + \frac{|\phi_{zzz}|}{U} + |\psi_z| \right).
\]
Substituting (34) into (33) gives
\[
\int_0^t \int \psi_z^2 \leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C t \int |G + \chi \phi_z \psi_z| \left( \|\phi_z\|_U + \frac{|\phi_{zzz}|}{U} + |\psi_z| \right),
\]
which in combination with (34) further leads to
\[
\int (\phi_z^2 + \psi_z^2) \leq C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C t \int |G + \chi \phi_z \psi_z| \left( \|\phi_z\|_U + \frac{|\phi_{zzz}|}{U} + |\psi_z| \right).
\]
In view of (24), by Young’s inequality, the fact that \(\|\phi(\cdot, t)\|_{L^\infty} \leq N(t)\) and Lemma 3.1, we get
\[
\int |G + \chi \phi_z \psi_z| \left( \|\phi_z\|_U + \frac{|\phi_{zzz}|}{U} + |\psi_z| \right) \leq C N(t) \int_0^t (\phi_{zzz}^2 + \psi_z^2).
\]
Finally, substituting this inequality into (36), when \(N(t) \ll 1\), we obtain the desired (27).

Next, we estimate the second order derivative of \((\phi, \psi)\).

**Lemma 3.3.** If \(N(t) \ll 1\), then it follows that
\[
\|\phi_{zzz}(\cdot, t)\|_2^2 + \int_0^t (\|\phi_z(\cdot, \tau)\|_2^2 + \|\psi_z(\cdot, \tau)\|_2^2) \leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2).
\]

**Proof.** Differentiating (28) with respect to \(z\) gives
\[
\begin{align*}
\phi_{zzt} - \chi U \psi_{zzz} & = Dm \left( U^{m-1} \phi_{zzz} \right) \\
& = Dm(2(U^{m-1})_z \phi_{zzz} + (U^{m-1})_{zzz} \phi_z) + \chi(2U_z \psi_{zzz} + U_{zz} \psi_z) \\
& + ((s + \chi V) \phi_z)_{zzz} + (G + \chi \phi_z \psi_z)_{zzz}, \\
\psi_{zzt} & = s \psi_{zzz} - \phi_{zzz} = 0.
\end{align*}
\]
Similarly, \(\delta\)

By Young’s inequality,

\[
\chi \left| \left( 2U_z \psi_{zz} + U_{zz} \psi_z \right) \frac{\phi_{zz}}{U} \right| \leq \frac{\delta}{2} U_{zz}^2 + \chi \left( \frac{2}{\delta} \frac{U_z^2}{U^2} + \frac{U_{zz}^2}{U^2} \right) \phi_{zz}^2 + \chi \psi_{zz}^2, \tag{40}
\]

where \(\delta\) is a small constant. Noting

\[
G_z = Dm((U + \phi_z)m-1 - U^{m-1})\phi_{zz} + Dm(m - 1)(U + \phi_z)^{m-2}\phi_{zz}^2
+ Dm(m - 1)U_z^2((U + \phi_z)^{m-2} - U^{m-2} - (m - 2)U^{m-3}\phi_z)
+ DmU_{zz}((U + \phi_z)^{m-1} - U^{m-1} - (m - 1)U^{m-2}\phi_z)
+ 2Dm(m - 1)U_z((U + \phi_z)^{m-2} - U^{m-2})\phi_{zz},
\]

we have

\[
\int (G + \chi\phi_z \psi_z) z \left( \frac{\phi_{zz}}{U} \right) z \leq CN(t) \int (\phi_z^2 + \phi_{zz}^2 + \phi_{zz}^2 + \psi_{zz}^2), \tag{42}
\]

where we have used \(\|\phi_z(\cdot, t)\|_{L^\infty}, \|\psi_z(\cdot, t)\|_{L^\infty}, \|\phi_{zz}(\cdot, t)\|_{L^\infty} \leq N(t)\). Substituting (40) and (42) into (39), by (27), we get

\[
\int \left( \frac{\phi_{zz}^2}{U} + \psi_{zz}^2 \right) + 2Dm \int_0^t \int U^{m-2}\phi_{zz}^2
\leq C(\|\phi_0\|_2^2 + \|\psi_0\|_2^2) + \delta \chi \int_0^t \int U\psi_{zz}^2 + CN(t) \int_0^t \int (\psi_{zz}^2 + \phi_{zz}^2).
\]

Next we estimate \(\int_0^t \int U\psi_{zz}^2\). Multiplying (28)1 by \(\psi_{zz}\), we get

\[
\chi U\psi_{zz}^2 = \phi_{zz} \psi_{zz} - Dm(U^{m-1}\phi_z) z \psi_{zz} = Dm((U^{m-1})_z \phi_z)_z \psi_{zz} - (\chi U\psi_z + ((s + \chi V)\phi_z)_z + (G + \chi\phi_z \psi_z)_z) \psi_{zz}. \tag{44}
\]

By the second equation of (38),

\[
\phi_{zz} \psi_{zz} = (\phi_z \psi_{zz})_t - \phi_z \psi_{zz}
= (\phi_z \psi_{zz})_t - s\phi_z \psi_{zz} - \phi_z \phi_{zz}
= (\phi_z \psi_{zz})_t - s(\phi_z \psi_{zz})_z + s\phi_{zz} \psi_{zz} - (\phi_z \phi_{zz})_z + \phi_{zz}^2.
\]

By Young’s inequality,

\[
Dm((U^{m-1})_z \phi_z)_z \psi_{zz} \leq \chi U\psi_{zz}^2 + \frac{2D^2m^2U^{2m-3}\phi_{zz}^2}{\chi} + \frac{2D^2m^2U^{m-1})^2\phi_{zz}^2}{\chi}.
\]

Similarly,

\[
|Dm((U^{m-1})_z \phi_z)_z \psi_{zz} + (\chi U\psi_z + ((s + \chi V)\phi_z)_z| \leq \frac{\chi U\psi_{zz}^2}{4} + CN(\psi_{zz}^2 + \phi_{zz}^2 + \phi_{zz}^2).
\]
In view of (41), since \( \|\phi_z(\cdot,t)\|_{L^\infty}, \|\psi_z(\cdot,t)\|_{L^\infty}, \|\phi_{zzz}(\cdot,t)\|_{L^\infty} \leq N(t) \), we get
\[
|(G + \chi\phi_z\psi_z)\phi_{zzz}| \leq CN(t)(\psi_{zzz}^2 + \phi_z^2 + \phi_{zzz}^2 + \phi_z^2).
\]
Thus, integrating (44) gives
\[
\chi \int_0^t \int U\psi_{zzz}^2 \leq \int \left( \frac{1}{\chi} \phi_z^2 + \phi_{zzz}^2 + \phi_z^2 + \phi_{zzz}^2 \right) + \frac{2D^2m^2}{\chi} \int_0^t \int U^{2m-3} \phi_{zzzz}^2 + C \int_0^t \int (\phi_z^2 + \phi_z^2 + \phi_{zzz}^2) + CN(t) \int_0^t \int (\psi_{zzz}^2 + \phi_{zzzz}^2). \tag{45}
\]
Substituting (45) into (43), choosing \( \delta \ll 1 \) and \( N(t) \ll 1 \), since \( U \geq u_+ > 0 \), by Lemmas 3.1 and 3.2, we have
\[
\int (\phi_{zzz}^2 + \chi\psi_{zzz}^2) + \int_0^t \int \phi_{zzzz}^2 \leq C\|\phi_0, \psi_0\|_{\dot{H}^2}, \tag{46}
\]
which in combination with (45) further gives
\[
\int_0^t \int \phi_{zzz}^2 \leq C\|\phi_0, \psi_0\|_{\dot{H}^2}. \tag{47}
\]
The desired estimate (37) finally follows from (46) and (47).

Under the influence of nonlinear diffusion, we need to estimate the third order derivative of \((\phi, \psi)\) so as to close the energy estimates.

**Lemma 3.4.** If \( N(t) \ll 1 \), we have
\[
\|\phi(\cdot,t)\|_{H^3}^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{H^3}^2 + \|\psi_z(\cdot,\tau)\|_{H^3}^2 \leq C(\|\phi_0\|_{H^3}^2 + \|\psi_0\|_{H^3}^2). \tag{48}
\]

**Proof.** Differentiating (38) with respect to \( z \) gives
\[
\begin{aligned}
&\phi_{zzzt} - \chi U\psi_{zzzz} - Dm\ (U^{m-1}\phi_{zzzzz})_z \\
&= Dm(3(U^{m-1})_{zzz}\phi_z + 3(U^{m-1})_z\phi_{zzz} + (U^{m-1})_{zzz}\phi_z)_z \\
&\quad + \chi(3U_z\psi_{zzz} + 3Uzz\psi_{zzz} + U_{zzzz}\psi_z) + ((s + \chi V)\phi_z)_{zzz} + (G + \chi\phi_z\psi_z)_{zzz}, \\
&\psi_{zzzt} - s\psi_{zzzz} - \phi_{zzzz} = 0.
\end{aligned} \tag{49}
\]
Multiplying (49)_1 by \( \phi_{zzz} \) and (49)_2 by \( \chi\psi_{zzz} \), applying the same argument as that of Lemma 3.3, one can get the third order estimate (48). We omit the details here.

Proposition 1 follows from Lemma 3.1 to Lemma 3.4.

4. **Proof of main results.** We now prove the main theorems in the current paper. Owing to the transformation (15), Theorem 2.2 is a consequence of Theorem 2.3.

**Proof of Theorem 2.3.** The a priori estimate (19) guarantees that \( N(t) \) is small if \( N(0) \) is small enough. Thus, applying the standard extension procedure, we get the global well-posedness of (17)-(18) in \( X(0, +\infty) \).

Next, we prove the convergence (20). Owing to the global estimate (19), we get
\[
\int_0^t \int_0^\infty \phi_z^2(z, \tau) dz \ d\tau \leq C(\|\phi_0\|_{H^3}^2 + \|\psi_0\|_{H^3}^2) < \infty, \ \forall \ t > 0. \tag{50}
\]
In view of the first equation of (17), by Young’s inequality,
\[
\frac{d}{dt} \int_{-\infty}^{\infty} \phi_z^2(z, t) dz = -2 \int_{-\infty}^{\infty} \phi_t \phi_{zz} dz = -2 \int_{-\infty}^{\infty} \phi_{zz} (D_m (U^{m-1})_z + (s + \chi V) \phi_z + \chi U \psi + G + \chi \phi_z \psi_z)
\leq C \int_{-\infty}^{\infty} (\phi_{zz}^2 + \phi_z^2 + \psi^2_z).
\]

It then follows from the global estimate (19) that
\[
\int_0^{\infty} \left| \frac{d}{dt} \int_{-\infty}^{\infty} \phi_z^2(z, t) dz \right| \leq C \int_0^{\infty} \left( \int_{-\infty}^{\infty} (\phi_{zz}^2 + \phi_z^2 + \psi^2_z) \right) \leq C (\|\phi_0\|_3^2 + \|\psi_0\|_3^2) < \infty.
\]
(51)

By (50) and (51), we get
\[
\int_{-\infty}^{\infty} \phi_z^2(z, t) dz \to 0 \text{ as } t \to +\infty.
\]

By Cauchy-Schwarz inequality, we further have
\[
\phi_z^2(z, t) = 2 \int_{-\infty}^{z} \phi_z \phi_{zz}(y, t) dy 
\leq 2 \left( \int_{-\infty}^{\infty} \phi_z^2(y, t) dy \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} \phi_{zz}^2(y, t) dy \right)^{\frac{1}{2}} 
\leq C \left( \int_{-\infty}^{\infty} \phi_z^2(y, t) dy \right)^{\frac{1}{2}} 
\to 0 \text{ as } t \to +\infty.
\]

Applying the same argument to \( \psi_z \) yields
\[
\sup_{z \in \mathbb{R}} |\psi_z(z, t)| \to 0 \text{ as } t \to +\infty.
\]
(52)

Hence (20) is proved.

Proof of Theorem 1.1. The existence of \( U \) follows from Lemma 2.1. We next show the existence of \( C \). The second equation of (1) implies
\[
sC'' = C(U - \beta),
\]
(53)
from which one can easily calculate that
\[
C(z) = C(0) e^{\frac{s}{2} \int_z^U (U(y) - \beta) dy}.
\]
(54)

Since \( U \) converges to \( u_\pm \) exponentially fast as \( z \to \pm \infty \), and \( s > 0 \), to ensure \( C \) is bounded, we need
\[
u_+ \leq \beta, \quad u_- \geq \beta.
\]

(i) If \( u_+ = \beta \), since \( U'' < 0 \), we have \( u_- > U > \beta \). It then follows from (53) that \( C' > 0 \). Noting that (53) also implies
\[
c_\pm (u_\pm - \beta) = 0,
\]
(55)
we further get \( c_- = 0 \). Hence \( u_- > u_+ = \beta \) and \( 0 = c_- < c_+ \).
(ii) If \( u_- > \beta > u_+ > 0 \), then we get from (55) that \( c_\pm = 0 \). Since \( U \) is monotone, there is only one point \( z_0 \) such that \( U(z_0) = \beta \). Hence by (53), \( C'(z) > 0 \) on \((-\infty, z_0)\) and \( C'(z) < 0 \) on \((z_0, +\infty)\).

(iii) If \( u_- = \beta \), then \( \beta = u_- > U > u_+ > 0 \). We get from (53) that \( C' < 0 \). (55) also implies \( c_+ = 0 \). Hence, \( \beta = u_- > u_+ > 0 \) and \( c_- > c_+ = 0 \).

We next compute the wave speed \( s \). Owing to (53) and the Cole-Hopf transformation (3), by the second equation of (9), we get \( \beta = sv_\pm + u_\pm \), which together with (11) gives
\[
s^2 + \chi (\beta - u_+ - u_-) = 0.
\]
Hence, \( s = \sqrt{\chi(u_+ + u_- - \beta)} \).

**Proof of Theorem 1.2.** The stability of \( u \) has been proved in Theorem 2.2. It remains to pass the results from \( v \) to \( c \). In view of the transformations (3) and (15), we have
\[
c(x, t) = \frac{e^{\int_{x-st}^{x} (V(y-st) - v(y,t))dy}}{C(x-st)} = e^{\psi(x,t)}.
\]
By Cauchy-Schwarz inequality, the global estimate (19) and (52), we get
\[
\sup_{x \in \mathbb{R}} \psi^2(x, t) = 2 \sup_{x \in \mathbb{R}} \int_{-\infty}^{x} \psi \psi_y(y, t)dy
\leq 2 \left( \int_{\mathbb{R}} \psi^2(y, t)dy \right)^{1/2} \left( \int_{\mathbb{R}} \psi_y^2(y, t)dy \right)^{1/2}
\leq C \|\psi_x(t, \cdot)\|
\rightarrow 0 \text{ as } t \rightarrow \infty.
\]
Then for all \( x \in \mathbb{R} \),
\[
|c(x, t) - C(x - st)| = |C(x-st)e^{\psi(x,t)} - C(x - st)|
\leq C|1 - e^{\psi(x,t)}|
\rightarrow 0 \text{ as } t \rightarrow \infty.
\]
The proof is completed. \( \square \)

**Acknowledgements.** The authors are grateful to the referee’s insightful comments and suggestions, which lead to improvements of this manuscript.

**REFERENCES**

[1] M. Burger, M. Di Francesco and Y. Dolak-Strub, *The Keller-Segel model for chemotaxis with prevention of overcrowding: Linear vs. nonlinear diffusion*, *SIAM J. Math. Anal.*, 38 (2006), 1288–1315.

[2] S.-H. Choi and Y.-J. Kim, *Chemotactic traveling waves with compact support*, *J. Math. Anal. Appl.*, 488 (2020), 124090, 21 pp.

[3] C. Deng and T. Li, *Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the Sobolev space framework*, *J. Differential Equations*, 257 (2014), 1311–1332.

[4] T. Hillen and K. Painter, *Global existence for a parabolic chemotaxis model with prevention of overcrowding*, *Adv. Appl. Math.*, 26 (2001), 280–301.

[5] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. II, *Jahresber. Dtsch. Math.-Ver.*, 106 (2004), 51–69.

[6] H.-Y. Jin, J. Li and Z.-A. Wang, *Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity*, *J. Differential Equations*, 255 (2013), 193–219.
[7] Y. V. Kalinin, L. Jiang, Y. Tu and M. Wu, Logarithmic sensing in Escherichia coli bacterial chemotaxis, *Biophys. J.*, 96 (2009), 2439–2448.
[8] S. Kawashima and A. Matsumura, Stability of shock profiles in viscoelasticity with non-convex constitutive relations, *Comm. Pure Appl. Math.*, 47 (1994), 1547–1569.
[9] E. F. Keller and L. A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis, *J. Theor. Biol.*, 26 (1971), 235–248.
[10] H. A. Levine and B. D. Sleeman, A system of reaction diffusion equations arising in the theory of reinforced random walks, *SIAM J. Appl. Math.*, 57 (1997), 683–730.
[11] D. Li, R. Pan and K. Zhao, Quantitative decay of a hybrid type chemotaxis model with large data, *Nonlinearity*, 28 (2015), 2181–2210.
[12] J. Li and Z. Wang, Convergence to traveling waves of a singular PDE-ODE hybrid chemotaxis system in the half space, *J. Differential Equations*, 268 (2020), 6940–6970.
[13] T. Li, R. H. Pan and K. Zhao, Global dynamics of a chemotaxis model on bounded domains with large data, *SIAM J. Appl. Math.*, 72 (2012), 417–443.
[14] T. Li and Z.-A. Wang, Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis, *SIAM J. Appl. Math.*, 70 (2010), 1522–1541.
[15] V. R. Martinez, Z.-A. Wang and K. Zhao, Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology, *Indiana Univ. Math. J.*, 67 (2018), 1383–1424.
[16] A. Matsumura and K. Nishihara, On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, 2 (1985), 17–25.
[17] T. Nishida, *Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics*, Publications Mathématiques d’Orsay 78-02, Département de Mathématique, Université de Paris-Sud, Orsay, France, 1978.
[18] M. Olson, R. Ford, J. Smith and E. Fernandez, Quantification of bacterial chemotaxis in porous media using magnetic resonance imaging, *Environ. Sci. Technol.*, 38 (2004), 3864–3870.
[19] H. G. Othmer and A. Stevens, Aggregation, blowup, and collapse: The ABCs of taxis in reinforced random walks, *SIAM J. Appl. Math.*, 57 (1997), 1044–1081.
[20] Y. Tao and M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, *Discrete Contin. Dyn. Syst. Ser. A*, 32 (2012), 1901–1914.
[21] F. Valdés-Parada, M. Porter, K. Narayanaswamy, R. Ford and B. Wood, Upscaling microbial chemotaxis in porous media, *Adv. Water Resour.*, 32 (2009), 1413–1428.
[22] Z.-A. Wang, Mathematics of traveling waves in chemotaxis: A review paper, *Discrete Contin. Dyn. Syst. Ser. B*, 18 (2013), 601–641.
[23] Z.-A. Wang and T. Hillen, Shock formation in a chemotaxis model, *Math. Methods Appl. Sci.*, 31 (2008), 45–70.
[24] Z.-A. Wang, Z. Xiang and P. Yu, Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis, *J. Differential Equations*, 260 (2016), 2225–2258.
[25] Y. Yang, H. Chen and W. Liu, On existence of global solutions and blow-up to a system of the reaction-diffusion equations modelling chemotaxis, *SIAM J. Math. Anal.*, 33 (2001), 763–785.
[26] Y. Yang, H. Chen, W. Liu and B. D. Sleeman, The solvability of some chemotaxis systems, *J. Differential Equations*, 212 (2005), 432–451.
[27] M. Zhang and C. J. Zhu, Global existence of solutions to a hyperbolic-parabolic system, *Proc. Amer. Math. Soc.*, 135 (2007), 1017–1027.

Received August 2020; revised November 2020.

E-mail address: jian111@nenu.edu.cn
E-mail address: lijy645@nenu.edu.cn
E-mail address: zhangkj201@nenu.edu.cn