Drift-diffusion current in organic diodes

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Because the conductivity of organic semiconductors is very low, a useful model for the organic diode consists of treating the organic layer as an insulator, an approximation often referred to as the metal-insulator-metal (MIM) model. Moreover, the dominant charge carrier injection process is diffusion, so that a modified Schottky’s theory can be used to derive a simple analytical equation for the current voltage curve of the diode. Here, we carried out a full analysis of the MIM model for the organic diode. We show that Schottky’s theory is only valid when charge injection is poor, that is, for high injection barriers. When the injection barrier is lowered, the current given by Schottky’s theory is still valid in the weak injection regime, when the applied potential is lower than the diffusion potential. However, it becomes largely overestimated in the strong injection regime. We also show that in the strong injection regime, the current given by the MIM model merges with Mott-Gurnery’s space-charge-limited regime.
I. INTRODUCTION

Three mechanisms are usually invoked to rationalize charge carrier injection in semiconductor diodes: Thermionic emission (TE), the drift-diffusion (DD) model and tunneling [1]. TE involves ballistic charge carrier transport through a depleted (aka space-charge) layer, and is the model of choice for silicon diodes. Because the mobility in organic semiconductors is currently several orders of magnitude lower than that in single crystal silicon, the DD model is generally recognized as more appropriate to describe the electrical current in organic diodes.

The DD model rests on two basic equations: Poisson’s equation (1) and the drift-diffusion equation (2).

\[
\frac{d^2 V}{dx^2} = -\frac{dF}{dx} = -\frac{qp(x)}{\varepsilon},
\]

\[
j = q\mu F - qD \frac{dp}{dx}.
\]

Here, the equations are written for positive charge carriers (holes). \( V \) is the electrical potential, \( F \) the electrical field, and \( p \) the hole density. \( \varepsilon \) is the permittivity of the semiconductor, \( j \) the electrical current density, \( q \) the elemental charge, \( \mu \) the hole mobility and \( D \) the hole diffusion coefficient. We will assume the validity of Einstein’s relation, which relates \( D \) and \( \mu \) through \( D = \mu kT/q \), where \( k \) is Boltzmann’s constant and \( T \) the absolute temperature.

In spite of their apparent simplicity, the exact resolution of these equations cannot be fully conducted by analysis; numerical calculations become a necessity at some stage, which tends to hinder the physical meaning of the results. Full calculations can be found in papers that date back to the early days of microelectronics [2–6], and in a more recent work by K. Seki [7]. At variance with this analytical approach, the current trend is to perform numerical resolutions through the finite element method (FEM) [8]. Various commercial packages are available for that purpose. One prominent advantage of the FEM is that it allows for various refinement in the calculation, e.g., including unconventional density of states (DOS) and non constant mobility. However, in spite of their usefulness for the physical understanding of the process, these simulations are less appropriate in terms of compact modeling, which requires the development of simple analytical equations.

The purpose of this paper is to delineate the various options to analytically resolve the drift-diffusion equation in organic semiconductors, which are characterized by an extremely low density of thermal charge carriers.

II. THEORETICAL BACKGROUND

All the equations in this section are written for hole only devices. The extension to electrons would be straightforward.

A. Schottky’s diffusion theory

The development of this theory can be found in textbooks [1]. The principle is to resolve Poisson’s and DD equations in sequence. In the first step, (1) is used to determine the shape of the potential in the diode. The result is expressed through the variation of the valence band edge \( E_v \) as a function of the distance \( x \) from the metal-semiconductor junction:

\[
E_v(x) = E_v(0) + \frac{q^2 N_A}{\varepsilon} \left( W_{sc} x - \frac{x^2}{2} \right),
\]

where :

\[
W_{sc} = \sqrt{\frac{2\varepsilon}{qN_A} \left( V_d - V_a - \frac{kT}{q} \right)},
\]

is the space charge layer width. \( N_A \) is the density of dopants (acceptors for a p-type semiconductor), \( V_d \) the diffusion (aka built-in) potential, defined as the difference between the work function of the metal and that of the semiconductor, and \( V_a \) the applied potential.

The current is now established by rewriting (2) as :

\[
j = \mu kT \left( \frac{p}{kT} \frac{dE_v}{dx} - \frac{dp}{dx} \right),
\]
which is next integrated using \( \exp(E_v/kT) \) as an integrating factor:

\[
j \int_0^{W_{sc}} \exp \left( -\frac{E_v}{kT} \right) \, dx = -\mu kT \left[ p \exp \left( -\frac{E_v}{kT} \right) \right]_0^{W_{sc}}.
\]

Using the Fermi level of the metal as the reference energy, the boundary conditions are given by:

\[
E_v(0) = -E_{bp}, \quad (7)
\]
\[
E_v(W_{sc}) = -E_p + q(V_d - V_a), \quad (8)
\]
\[
p(0) = N_v \exp \left( -\frac{E_{bp}}{kT} \right), \quad (9)
\]
\[
p(W_{sc}) = N_v \exp \left( -\frac{E_p}{kT} \right). \quad (10)
\]

\( E_{bp} \) is the hole barrier height at the metal-semiconductor interface, and \( E_p \) the energy difference between the valence band edge and the Fermi level in the bulk of the semiconductor. \( N_v \) is the effective density of state at valence band edge. An energy diagram of the junction and the relevant parameters are shown in Fig. 1.

\[
\text{Figure 1. Energy diagram of a Schottky contact with a p-type semiconductor at equilibrium. Relevant parameters are: } E_{bp}: \text{ hole energy barrier; } E_p: \text{ difference between the Fermi level and the valence band edge in the bulk of the semiconductor; } V_d: \text{ diffusion (or built-in) potential; } W_{sc}: \text{ space-charge layer width.}
\]

Combining all the above leads to:

\[
j = j_d \left( \exp \left( \frac{qV_a}{kT} \right) - 1 \right), \quad (11)
\]
\[
j_d = q\mu N_v F(0) \exp \left( -\frac{E_{bp}}{kT} \right), \quad (12)
\]

where \( F(0) \) is the electric field at the metal-semiconductor interface \( (x = 0) \).

In an organic diode, the dopant density and semiconductor thickness are so small that it is generally accepted that the space charge layer extends over the whole semiconductor layer, which is referred to as the full-depletion or metal-insulator-metal (MIM) model. Under such circumstances, the potential at equilibrium varies linearly with distance, and the electric field is constant. Attempts to adapt the Schottky model to such a geometry have been recently made [9, 10]. The new energy diagram is shown in Fig. 2.

Now we have to consider both sides of the device; the hole injecting electrode is called anode, and the cathode is hole blocking. Here, we restrict to a hole only diode, when the electron injection barriers are so high that only holes can be injected at both electrodes.

The variation of the valence band edge is now given by:

\[
E_v(x) = E_v(0) - q(V_d - V_a) \frac{x}{d}, \quad (13)
\]

where \( d \) is the thickness of the semiconductor. \( V_d \) is the diffusion potential, that is, the energy difference between the work function of both electrodes, and \( V_a \) is the voltage difference applied between the anode and the cathode.
Equation (6) must now be integrated over the whole thickness of the semiconductor, thus leading to:

\[ j = q\mu p_0 \frac{V_d - V_a}{d} \frac{\exp(qV_a/kT) - 1}{\exp(qV_d/kT) - \exp(qV_a/kT)}, \]

where \( p_0 = N_v \exp(-E_{\text{an}}^{\text{nn}}/kT) \) is the hole density at the anode \((x = 0)\), \( E_{\text{an}}^{\text{nn}} \) being the hole injection barrier at the anode.

The main assumption that leads to (14) is that the shape of the potential profile (quadratic for a Schottky diode, linear for a MIM diode) remains unchanged when a voltage is applied. This is basically true in a Schottky diode, as shown in Equation (3). However, as will be shown in the following, the assumption only verifies in a MIM when the injection barrier at the anode is high.

B. Full MIM model

We now focus on the following equation, which results of a combination of (1), (2) and Einstein’s relation:

\[ j = \varepsilon \mu \left( F \frac{dF}{dx} - \frac{kT}{q} \frac{d^2F}{dx^2} \right). \]

Following earlier works \[5, 6\], we will use the dimensionless reduced variables defined as follows:

\[ x_r = \frac{x}{d}, \quad V_r = \frac{V}{V_T}, \quad F_r = \frac{d}{V_T} F, \quad p_r = \frac{qF^2}{\varepsilon V_T} p, \quad j_r = \frac{d^3}{\varepsilon \mu V_T^2} j, \]

where \( V_T = kT/q \). (15) now writes:

\[ \frac{d^2F_r}{dx_r^2} - F_r \frac{dF_r}{dx_r} + j_r = 0. \]

1. The diode without current

Integrating (17) with \( j_r = 0 \) leads to:

\[ \frac{dF_r}{dx_r} - \frac{F_r^2}{2} + 2g^2 = 0, \]

where \( g \) is an integration constant. The solution of this equation is given by:

\[ F_r = -2g \coth(gx_r + \text{arg sinh} gx_{r0}). \]

The electrical potential at a point \( x_r \) between the anode and the cathode is obtained by integrating (19) between the anode \((x_r = 0)\) and \( x_r \), thus leading to:

\[ V_r = 2 \ln \left( \cosh gx_r + \sqrt{1 + \frac{p_{r0}}{2g^2} \text{sinh} gx_r} \right). \]

Here, \( p_{r0} = p_r(0) \) is the reduced density of holes at the anode. The integration constant \( g \) can be calculated by writing that the reduced potential at the cathode \((x = d, x_r = 1)\) is equal to to the reduced diffusion potential \( V_{r,d} \).
2. Solution of the equation with current

The full DD Equation (17) has the following analytical solution:

\[
F_r(x_r) = -2\alpha \frac{c_2A'_i(c_1 + \alpha x_r) + B'_i(c_1 + \alpha x_r)}{c_2A_i(c_1 + \alpha x_r) + B_i(c_1 + \alpha x_r)},
\]

where \(\alpha = \sqrt{\frac{j_r}{2}}\). \(A_i\) and \(B_i\) are Airy’s functions, and \(A'_i\) and \(B'_i\) their first derivative; \(c_1\) and \(c_2\) are integration constants. Although (21) looks analytical, it does not allow for a direct computation of the current-voltage curve of the diode, because the integration constants \(c_1\) and \(c_2\) must be estimated for each value of the reduced current \(j_r\).

From the following relationship between Airy’s functions

\[
A''_i(z) = zA_i(z), \quad B''_i(z) = zB_i(z),
\]

the reduced hole density \(p_r = dF_r/dx_r\) can be written as:

\[
p_r = \frac{F_r^2}{2} - j_r x_r - 2\alpha^2 c_1.
\]

Equation (23) can now be used to estimate the integration constants. In a first step, we write the values of the reduced hole density at the anode and cathode. Assuming a quasi-equilibrium, we postulate that these values are those at thermodynamic equilibrium (no overall current). This leads at the anode \((x_r = 0)\):

\[
p_{r0} = 2\alpha^2 \left\{ \left[ \frac{c_2A'_i(c_1) + B'_i(c_1)}{c_2A_i(c_1) + B_i(c_1)} \right]^2 - c_1 \right\},
\]

or:

\[
c_2 = -\frac{B'_i(c_1) - P_0 B_i(c_1)}{A'_i(c_1) - P_0 A_i(c_1)},
\]

\[
P_0 = \pm \sqrt{c_1 + \frac{p_{r0}}{2\alpha^2}}.
\]

A similar equation is obtained at the cathode \((x_r = 1)\):

\[
c_2 = -\frac{B'_i(c_1 + \alpha) - P_1 B_i(c_1 + \alpha)}{A'_i(c_1 + \alpha) - P_1 A_i(c_1 + \alpha)},
\]

\[
P_1 = \pm \sqrt{c_1 + \frac{j_r + p_{r1}}{2\alpha^2}} = \pm \sqrt{c_1 + \alpha + \frac{p_{r1}}{2\alpha^2}}.
\]

The sign in front of the square root in (26) and (28) depends on the orientation of the electric field at the anode \((x_r = 0)\) and cathode \((x_r = 1)\). The constant \(c_1\) is now obtained by eliminating \(c_2\) between (25) and (27):

\[
[B'_i(c_1) - P_0 B_i(c_1)] [A'_i(c_1 + \alpha) - P_1 A_i(c_1 + \alpha)] - [A'_i(c_1) - P_0 A_i(c_1)] [B'_i(c_1 + \alpha) - P_1 B_i(c_1 + \alpha)] = 0.
\]

The electrical potential is calculated by integrating (21) between 0 and \(x_r\):

\[
V_r = 2\ln \frac{c_2A_i(c_1 + \alpha x_r) + B_i(c_1 + \alpha x_r)}{c_2A_i(c_1) + B_i(c_1)}.
\]

Replacing \(c_2\) by its value in (25) leads to:

\[
V_r = 2\ln \pi \left\{ [B'_i(c_1) - P_0 B_i(c_1)] A_i(c_1 + \alpha x_r) - [A'_i(c_1) - P_0 A_i(c_1)] B_i(c_1 + \alpha x_r) \right\},
\]

where we used the identity \(A_i(z)B'_i(z) - A'_i(z)B_i(z) = 1/\pi\).

The applied voltage \(V_a\) is connected to the reduced potential at \(x_r = 1\) through \(V(d) = V \frac{V_r(1)}{V} = V_a - V_d\).
C. Space-charge limited current

A useful approximation of the DD model was first introduced by Mott [11], which consists of neglecting the diffusion component of the current. The SCLC regime becomes valid at high applied voltage, and also requires strong charge carrier injection at the anode.

Neglecting the diffusion term leads to the following equation:

\[ j = \varepsilon \mu F \frac{dF}{dx}, \]  

which can be integrated to:

\[ F^2 = \frac{2j}{\varepsilon \mu} x + C. \]  

The integration constant \( C \) can be estimated by establishing the hole density at the anode to \( p(0) = p_0 = N_v \exp(-E_b^0/kT) \):

\[
\begin{align*}
    p(x) &= \frac{\varepsilon}{q} \frac{dF}{dx} = \frac{j}{q \mu} \left( \frac{2j}{\varepsilon \mu} x + C \right)^{-1/2}, \\
    p(0) &= \frac{j}{q \mu \sqrt{C}}, \\
    C &= \left( \frac{j}{q \mu p_0} \right)^2. 
\end{align*}
\]  

The potential a point at a distance \( x \) of the anode is obtained by integrating the electric field from the anode to this point:

\[
V(x) = \int_0^x F(t) dt = \frac{\varepsilon \mu}{3j} \left[ \left( \frac{2j}{\varepsilon \mu} x + C \right)^{3/2} - C^{3/2} \right].
\]  

Mott-Gurney’s model requires no limitation to charge carrier injection, so \( C \to 0 \) \( (p_0 \to \infty) \) and the potential profile becomes:

\[ V(x) = \sqrt{\frac{8j}{9\varepsilon \mu}} x^{3/2}. \]  

Writing (36) at \( x = d \) leads to the well-known equation:

\[ j = \frac{9}{8} \varepsilon \mu \frac{(V_a - V_d)^2}{d^3}, \]  

where the applied voltage is defined as \( V_a = V(d) + V_d \).

In the reverse case, when the hole density at the anode becomes low, we can develop the first term in the bracket in the right side of Equation (35) to the first power of \( x \):

\[ V(x) = \frac{\varepsilon \mu}{3j} C^{3/2} \left[ \frac{3}{2} \frac{2jx}{\varepsilon \mu C} \right] = \sqrt{C} x, \]  

so the current now writes:

\[ j = q p_0 \mu \frac{V_a - V_d}{d}. \]  

A similar result is reported in Ref. [12].

Note that both (37) and (39) are only valid in the strong injection regime, when \( V_a > V_d \).
III. RESULTS

A. Potential profile

We conducted a numerical resolution of the full MIM model with the commercial package Mathcad. The first step consisted of calculating the electric field profile \( F(x) \) for given values of the current, from which the potential and charge carrier density profiles were obtained through numerical integration and derivation, respectively. The parameters used for the calculations are gathered in Table I.

Table I. Parameters used for the calculation of the voltage profile of a MIM diode. \( \varepsilon_0 \) is the permittivity of free space.

| Parameter                                      | Value                                      |
|------------------------------------------------|--------------------------------------------|
| Temperature                                    | \( T = 300 \text{ K} \)                   |
| Permittivity                                   | \( \varepsilon = 4 \times \varepsilon_0 \) |
| Mobility                                       | \( \mu = 1 \text{ cm}^2/\text{Vs} \)      |
| Density of states at valence band edge         | \( N_v = 10^{20} \text{ cm}^{-3} \)      |
| Semiconductor thickness                        | \( d = 100 \text{ nm} \)                 |
| Hole injection barrier at the anode            | \( E_{an}^{bb} = 0 \) or 0.3 eV          |
| Diffusion potential at the anode               | \( V_d = 0.6 \text{ V} \)                |

The calculated potential profiles of the MIM diode for various values of the applied voltage are shown in Fig. 3. It clearly appears that the MIM approximation, in which the potential linearly varies with distance, is only valid in the case of a high injection barrier (\( E_{an}^{bb} = 0.3 \text{ eV} \)). When the injection barrier is lower (0.1 eV), a slight curvature appears at the anode (\( x = 0 \)), which is usually interpreted in terms of accumulation of holes at this electrode. Moreover, the potential in the direct current regime (\( V_{\text{applied}} > V_d \)) is no longer a straight line; instead, it presents an upward curvature. Comparing with the voltage profile described by Equation (36), this can be interpreted in terms of space charge limited regime, as will be confirmed in the following.

Figure 3. Calculated potential profile of a MIM diode with the parameters listed in Table I at various applied voltages and for an injection barrier at the anode of 0.3 (a) and 0.1 eV (b).

B. Current-voltage curves

Calculated current-voltage curves of MIM diodes with an injection barrier at the anode of 0.3 and 0.1 eV are drawn in semi-log plot in Figures 4 and 5, respectively.

The exact MIM model is in good agreement with Schottky’s theory for a hole injection barrier of 0.3 eV, that is, when injection efficiency is poor. When improving charge carrier injection by lowering the injection barrier down to
Figure 4. IV curve for a MIM diodes with the parameters in Table I and a hole injection barrier of 0.3 eV. Filled circles correspond to data numerically calculated from the full MIM model, and the dashed line to the Schottky theory.

0.1 eV, the agreement remains good in the weak injection regime (applied voltage lower than the diffusion potential \(V_d\)). However, a discrepancy of nearly two order of magnitude is observed under strong injection, when the applied voltage is in excess of the diffusion potential.

A log-log plot of the current-voltage curves is shown in Figure 5. Here, we have also calculated the space charge limited current through Equations (34) and (37). Interestingly, the exact MIM curve now merges with the SCLC at high voltages (strong injection regime). We also note that Schottky’s current is linear with the applied voltage in the strong injection regime.
IV. DISCUSSION AND CONCLUSION

Two distinct cases can be separated, depending on the injection barrier height: poor injection (high barrier) and good injection (low barrier). In the former case, the organic semiconductor behaves as a perfect insulator; the voltage profile remains perfectly linear, including in the direct bias regime, when current is flowing through the diode. In this particular case, Schottky’s theory leads to a current that is in perfect agreement with the full MIM calculation in all regimes. It is worth pointing out that, in agreement with our calculation, poor injection also prevents SCLC to occur at high applied voltages. Accordingly, the current at high voltage is proportional to the voltage rather than the voltage to the square.

For a diode with good hole injection, the agreement of Schottky’s theory with the full MIM model restricts to weak injection, when the applied voltage is lower than the diffusion potential. At higher voltages, the current predicted by Schottky’s theory is overestimated by a factor of nearly 100. This discrepancy is accompanied by two important points. First, apart from a slight curvature near the anode due to hole accumulation, the voltage profile only remains linear in the weak injection regime ($V_{\text{applied}} < V_d$). In the strong injection regime, the profile presents an upward curvature. This observation can be associated with the fact that under strong injection, the MIM current merges with the SCLC regime, with a current that is now proportional to the voltage to the square.

As a final remark, we note that the model developed here assumes that the organic semiconductor follows a non-degenerate statistics. This assumption was used when estimating the density of holes at the electrodes. We have recently shown that this assumption is not fulfilled in the case of a Gaussian density of states[13], which best describes the vast majority of disordered organic solids. Further work will therefore be necessary to extend the model to a degenerate statistics.

ACKNOWLEDGMENTS

I am profoundly grateful to Prof. Yvan Bonnassieux, Dr. Chang Hyun Kim and Sungyeop Jung for their constant support during this work.

Appendix: Approximation at low current

The use of the exact equations [29] and [31] for the electrical potential and integration constant, respectively, becomes problematic at low current because the value of the constant $c_1$ becomes positive and large, so Airy’s
function $B_i$ and its first derivative diverge. To work around this issue, we develop in this appendix an analytical approximation of the equations at low current.

First, let us recall the asymptotic form of Airy’s functions:

\[ A_i(z) \sim \frac{\exp \left( -\frac{2}{3}z^{3/2} \right)}{2\sqrt{\pi} \sqrt{z}}, \quad (A.1) \]

\[ A'_i(z) \sim -\frac{\sqrt{z}}{2\sqrt{\pi}} \exp \left( -\frac{2}{3}z^{3/2} \right), \quad (A.2) \]

\[ B_i(z) \sim \frac{\exp \left( \frac{4}{3}z^{3/2} \right)}{\sqrt{\pi} \sqrt{z}}, \quad (A.3) \]

\[ B'_i(z) \sim \frac{\sqrt{z}}{\sqrt{\pi}} \exp \left( \frac{2}{3}z^{3/2} \right). \quad (A.4) \]

The asymptotic form of the ratios $A'_i(z)/A_i(z)$ et $B'_i(z)/B_i(z)$ write:

\[ \frac{A'_i(z)}{A_i(z)} \sim -\sqrt{z}, \quad (A.5) \]

\[ \frac{B'_i(z)}{B_i(z)} \sim \sqrt{z}. \quad (A.6) \]

We can also write the asymptotic forms of the products of Airy functions as:

\[ A_i(z)B_i(z) \sim \frac{1}{2\pi \sqrt{z}}, \quad (A.7) \]

\[ A'_i(z)B_i(z) \sim -\frac{1}{2\pi}, \quad (A.8) \]

\[ A_i(z)B'_i(z) \sim \frac{1}{2\pi}, \quad (A.9) \]

\[ A'_i(z)B'_i(z) \sim -\frac{\sqrt{z}}{2\pi}. \quad (A.10) \]

Finally, developing $(c_1 + \alpha x_r)^{3/2}$ in first of order of $\alpha/c_1$ leads to:

\[ (c_1 + \alpha x_r)^{3/2} \sim c_1^{3/2} + \frac{3}{2} \cosh c_1 \alpha x_r, \quad (A.11) \]

so we can write the asymptotic form of Airy function at $c_1 + \alpha$ as:

\[ A_i(c_1 + \alpha x_r) \sim A_i(c_1) \exp \left( -\sqrt{c_1} \alpha x_r \right), \quad (A.12) \]

\[ A'_i(c_1 + \alpha x_r) \sim A'_i(c_1) \exp \left( -\sqrt{c_1} \alpha x_r \right), \quad (A.13) \]

\[ B_i(c_1 + \alpha x_r) \sim B_i(c_1) \exp \left( \sqrt{c_1} \alpha x_r \right), \quad (A.14) \]

\[ B'_i(c_1 + \alpha x_r) \sim B'_i(c_1) \exp \left( \sqrt{c_1} \alpha x_r \right). \quad (A.15) \]

**Approximate electrical potential**

Using the asymptotic forms of Airy’s functions, Equation (31) writes:

\[ V_r(x_r) = 2 \ln \pi \left\{ \left[ B'_i(c_1) - P_0B_i(c_1) \right] A_i(c_1 + \alpha x_r) - \left[ A'_i(c_1) - P_0A_i(c_1) \right] B_i(c_1 + \alpha x_r) \right\}, \]

\[ = 2 \ln \pi \left[ (\sqrt{c_1} - P_0) B_i(c_1) A_i(c_1 + \alpha x_r) - (-\sqrt{c_1} - P_0) A_i(c_1) B_i(c_1 + \alpha x_r) \right], \]

\[ = 2 \ln \frac{1}{2\pi} \left[ \left( \frac{P_0}{\sqrt{c_1}} + 1 \right) e^{\sqrt{c_1} \alpha x_r} - \left( \frac{P_0}{\sqrt{c_1}} - 1 \right) e^{-\sqrt{c_1} \alpha x_r} \right], \]

\[ V_r(x_r) = 2 \ln \left[ \cosh \sqrt{c_1} \alpha x_r + \sqrt{1 + \frac{P_0}{2\sqrt{c_1} \alpha}} \sinh \sqrt{c_1} \alpha x_r \right]. \quad (A.16) \]

Equation (A.16) is similar to the potential at zero current (20) where the integration constant $g$ is replace by $\sqrt{c_1} \alpha$. Hence we deduce that as the current tends to zero, $c_1 \alpha^2 \to g^2$ and $c_1$ tends to infinity.
At low current, the value of $c_1$ becomes large and that of the current reduces; we can therefore neglect $\alpha$ and rewrite Equation (29) as follows:

$$\left[ \frac{B'_i(c_1)}{B_i(c_1)} - P_0 \right] \left[ \frac{A'_i(c_1)}{A_i(c_1)} - P_1 \right] B_i(c_1) A_i(c_1 + \alpha) = \left[ \frac{A'_i(c_1)}{A_i(c_1)} - P_0 \right] \left[ \frac{B'_i(c_1)}{B_i(c_1)} - P_1 \right] A_i(c_1) B_i(c_1 + \alpha). \quad (A.17)$$

Next, we approximate Equation (28) as:

$$P_1 \simeq \sqrt{c_1} \sqrt{\frac{p_e}{2\alpha^2 c_1}} + 1 \simeq \sqrt{c_1} \left( 1 + \frac{p_e}{4\alpha^2 c_1} \right). \quad (A.18)$$

At this stage, we also need a development at a higher order of the asymptotic form of $B'_i(z)/B_i(z)$. A useful form was recently derived Kearney and Martin [14]:

$$\frac{B'_i(z)}{B_i(z)} \sim \sqrt{z} - \frac{1}{4z}, \quad (A.19)$$

which leads to the final result:

$$(P_0 - \sqrt{c_1}) \frac{e^{-\sqrt{c_1}\alpha}}{\pi} = (P_0 + \sqrt{c_1}) \left( \frac{p_e}{\alpha^2} + \frac{1}{\sqrt{c_1}} \right) \frac{e^{\sqrt{c_1}\alpha}}{8\pi \sqrt{c_1}}. \quad (A.20)$$