Complementarity and the pathological statistics of the quantum impossible

Irene Bartolomé and Alfredo Luis
Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
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I. INTRODUCTION

Complementarity is a distinguished quantum feature that precludes from the start the simultaneous exact observation of conjugate observables. This makes impossible even the mere conception of a joint probability distribution. But nothing prevents their less than perfect observation, providing us with an operational joint distribution for complementary variables [1]. Then, as far as we know all the details of our measuring scheme, we know the way this instrumental uncertainty has been added to each variable. Thus we can remove the effect of the measurement from both observables and get their exact distributions. This inversion can be then applied to the operational joint distribution. In classical physics this program works giving their bona fide exact joint distribution [2]. But in the quantum domain this is an attempt to obtain the quantum impossible, so quantum mechanics manifests in the form of pathological distributions, that nonetheless have perfect marginals for both observables. We may say that this is actually the hallmark of quantumness and the way nonclassical states are defined in quantum optics [3].

We apply the above program to a seminal example of complementarity: the single-particle Young interferometer. The two conjugate observables are the slit crossed and the interference, or phase difference. Their joint observation will be allowed by marking the slit crossed in the polarization or spin state of the interfering particle. Then the interference is observed keeping track of the polarization/spin state when recording the interference. As mentioned above, this program requires that the path observation must be less than perfect so that the interference is not fully destroyed [4].

II. EXACT, UNOBSERVED STATISTICS

We address here the basic definitions of states and observables and their exact statistics before observation.

Let us express the state at the plane of the apertures of a Young interferometer by a two-dimensional complex vector \(|\psi\rangle = (\alpha, \beta)|\rangle\), where \((1, 0)\) means particle in the upper aperture and \((0, 1)\) particle at the lower aperture with \(|\alpha|^2 + |\beta|^2 = 1\). This is to say that we can represent the path observable by the third Pauli matrix \(Z\), and denote as \(z = 1\) the particle in the upper aperture and \(z = -1\) as the particle in the lower aperture. The corresponding statistics are
\[
P_Z(z = 1) = |\alpha|^2, \quad P_Z(z = -1) = |\beta|^2, \quad (2.1)
\]
or equivalently
\[
P_Z(z = \pm 1) = \frac{1}{2}(1 + z\langle Z\rangle), \quad \langle Z\rangle = |\alpha|^2 - |\beta|^2. \quad (2.2)
\]
The conjugate observable is slightly more difficult to be described, as far as one expects interferograms with a continuous distribution in the form of fringes. More rigorously the phase/interference observable should involve both the other two Pauli matrices \(X\) and \(Y\). But for the sake of simplicity let us first properly represent phase/interference by just one of them, say \(X\). Deep down, this corresponds to say that \(\langle Y\rangle = 0\), or in any case, to represent phase difference by the unitary operator exponential of phase introduced in Ref. [5].

The statistics of \(X\) is given by projection of the following vectors, expressed in the same basis used up to now:
\[
| x = \pm 1 \rangle = \frac{1}{\sqrt{2}} (1, \pm 1), \quad (2.3)
\]
and then
\[
P_X(x) = \frac{1}{2}(1 + x \langle X\rangle), \quad \langle X\rangle = \alpha \beta^* + \alpha^* \beta. \quad (2.4)
\]

In a later section we will consider the alternative approach where phase is represented by the positive operator valued measure given by projection on the nonorthogonal phase states [6]
\[
| \phi \rangle = \frac{1}{\sqrt{2\pi}} (1, e^{i\phi}), \quad (2.5)
\]
so that the exact phase distribution is
\[
P_{\phi}(\phi) = \frac{1}{2\pi} (1 + \cos \phi \langle X\rangle + \sin \phi \langle Y\rangle), \quad (2.6)
\]
where \(\langle Y\rangle = i(\alpha \beta^* - \alpha^* \beta)\).
III. JOINT OBSERVATION: DISCRETE PHASE

To perform a simultaneous observation of $X$ and $Z$ we must involve additional degrees of freedom. For example let us transfer path information to the polarization or spin state of the interfering particle. This can be easily achieved in practice with a half-wave plate in the case of photons, and a suitable arranged magnetic field in the case of massive particles. Let us describe the spin state by two orthogonal base vectors $| \rightarrow \rangle$ and $| \uparrow \rangle$. The particle is initially prepared in the state $| \rightarrow \rangle$. The corresponding marginals are:

\[
P_\psi(x, z') = \sum_{x', z} \mu(x, x') \mu(z, z') \tilde{P}(x', z'),
\]

leading to

\[
P(x, z) = \frac{1}{4} \left[ 1 + x \delta(z) \langle X \rangle + z \langle Z \rangle \right],
\]

where

\[
\delta(1) = \frac{\sin(2\theta)}{\cos(\theta) \sin(2\theta)}, \quad \delta(-1) = \frac{\sin(2\theta - 2\phi)}{\cos(\theta) \sin(2\theta - 2\phi)}.
\]

Since $\delta(1) + \delta(-1) = 2$ we can appreciate that $P(x, z)$ provides the correct exact marginals for both observables. In particular extremely simple expressions are obtained in the case $\theta \rightarrow 0$ leading to

\[
P(x, z) = \frac{1}{4} \left[ 1 + z \langle Z \rangle + x \langle X \rangle \right].
\]

The main conclusion is that for every state $| \psi \rangle$ we can suitable chose angles $\theta$ and $\phi$ so that $P(x, z)$ takes negative values. Focusing in the simplest case the minimum value is

\[
P_{\text{min}} = \frac{1}{4} (1 - |\langle Z \rangle| - |\langle X \rangle|).
\]

We have that

\[
|\langle Z \rangle| + |\langle X \rangle| \geq \langle Z \rangle^2 + \langle X \rangle^2 = 1,
\]

where we have taken into account that we are working with pure states with $\langle Y \rangle = 0$. Therefore $P_{\text{min}} < 0$ and $P(x, z)$ can no longer be probabilities in the standard sense. This does not mean that they are meaningless. For example they reveal that every state $| \psi \rangle$ is nonclassical.

IV. DATA INVERSION AND IMPOSSIBLE STATISTICS: DISCRETE PHASE

It is possible to obtain the exact $P_A$ statistics from the operational ones $\tilde{P}A$ in Eqs. (3.7) and (3.8), $A = X, Z$ in an extremely simple linear way as

\[
P_A(a) = \sum_{a'} \mu_A(a, a') \tilde{P}_A(a'),
\]

with

\[
\mu_X(x, x') = \frac{1}{2} \left[ 1 + xx' \frac{1}{\cos \theta} \right],
\]

and

\[
\mu_Z(z, z') = \frac{\sin \left( \frac{\theta}{2} \right)}{\sin \left( \frac{\theta - \phi}{2} \right)}.
\]
V. JOINT OBSERVATION AND DATA INVERSION: CONTINUOUS PHASE

An alternative and maybe more intuitive approach to interference or phase difference is provided by the phase states (2.3). With the help of them the joint distribution for the simultaneous observation of path and phase/interference is

\[ \tilde{P}(\phi, z) = \left| \langle \tilde{z} | \langle \phi | \tilde{\psi} \rangle \right|^2, \tag{5.1} \]

\[ \tilde{P}(\phi, z) = \frac{1}{2\pi} \left[ \gamma_0(z) + \gamma_X(z) \cos \phi \langle X \rangle + \gamma_Y(z) \sin \phi \langle Y \rangle + z \gamma_Z(z) \langle Z \rangle \right], \tag{5.2} \]

with the same parameters \( \gamma_A \) in Eq. (3.6). In comparison with the discrete case the only essential difference is that \( \phi \) assumes a continuous and 2\( \pi \)-periodic range of variation.

The marginal for \( Z \) is the same in Eq. (3.8), so the inversion is performed by the same matrix \( \mu_Z \) in Eq. (4.3). On the other hand, the marginal for the phase is

\[ \tilde{P}_\Phi(\phi) = \frac{1}{2\pi} \left[ 1 + \cos \theta (\cos \phi \langle X \rangle + \sin \phi \langle Y \rangle) \right], \tag{5.3} \]

that can be inverted as

\[ P_\Phi(\phi) = \int \, d\phi' \mu_\Phi(\phi, \phi') \tilde{P}_\Phi(\phi'), \tag{5.4} \]

with

\[ \mu_\Phi(\phi, \phi') = \frac{1}{2\pi} \left[ 1 + \frac{2}{\cos \theta} \cos(\phi - \phi') \right]. \tag{5.5} \]

When applying the \( Z \) and \( \Phi \) inversions simultaneously to the joint distribution we get

\[ P(\phi, z) = \frac{1}{4\pi} \left[ 1 + \delta(z) (\cos \phi \langle X \rangle + \sin \phi \langle Y \rangle) + z \langle Z \rangle \right], \tag{5.6} \]

for the same \( \delta(z) \) in Eq. (4.6). In the limit \( \theta \to 0 \) it becomes

\[ P(x, z) = \frac{1}{4\pi} \left[ 1 + \cos \phi \langle X \rangle + \sin \phi \langle Y \rangle + z \langle Z \rangle \right]. \tag{5.7} \]

The conclusions about the lack of positivity are the same as obtained above. The minimum value in Eq. (5.7) is

\[ P_{\min} = \frac{1}{4\pi} \left( 1 - |\langle Z \rangle| - \sqrt{\langle X \rangle^2 + \langle Y \rangle^2} \right). \tag{5.8} \]

Since

\[ |\langle Z \rangle| + \sqrt{\langle X \rangle^2 + \langle Y \rangle^2} \geq \langle Z \rangle^2 + \langle X \rangle^2 + \langle Y \rangle^2 = 1, \tag{5.9} \]

where the last equality holds for pure states, we get again \( P_{\min} < 0 \) and the same conclusion as above.

VI. CONCLUSIONS

Most practical and meaningful observations in quantum and classical physics are indirect in the sense that the desired information is retrieved after a suitable data analysis. This idea allows us to approach the statistics of conjugate observables by removing the instrumental effects of their imperfect simultaneous measurement. In classical physics this protocol always works providing \emph{bona fide} joint probabilities.

Here is where relies the most significant difference between classical and quantum physics. This is that in quantum mechanics this protocol fails, say it becomes a kind of \emph{ghost protocol}, as a clear quantum signature.

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[1] W. M. Muynck, \emph{Foundations of Quantum Mechanics, an Empiricist Approach}, (Kluwer Academic Publishers, 2002); W. M. de Muynck and H. Martens, Neutron interferometry and the joint measurement of incompatible phase states [2-3].
observables, Phys. Rev. A 42, 5079–5085 (1990); W. M. de Muynck, Information in neutron interference experiments, Phys. Lett. A 182, 201–206 (1993); W. M. de Muynck, An alternative to the Lüders generalization of the von Neumann projection, and its interpretation, J. Phys. A: Math. Gen. 31, 431–444 (1998).

[2] A. Luis, Nonclassical states from the joint statistics of simultaneous measurements, [http://arxiv.org/abs/1506.07680](http://arxiv.org/abs/1506.07680); A. Luis, Nonclassical light revealed by the joint statistics of simultaneous measurements, Opt. Lett. 41, 1789 (2016); A. Luis, All states are nonclassical: entanglement of joint statistics, [arXiv:1606.01478](http://arxiv.org/abs/1606.01478) [quant-ph]; A. Luis and L. Monroy, Coherent and thermal states are nonclassical: entanglement of joint statistics, [arXiv:1707.09318](http://arxiv.org/abs/1707.09318) [quant-ph].

[3] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995); M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).

[4] W. K. Wootters and W. H. Zurek, Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of Bohr’s principle, Phys. Rev. D 19, 473 (1979); D. M. Greenberger and A. Yasin, Simultaneous wave and particle knowledge in a neutron interferometer, Phys. Lett. A 128, 391 (1988); K. Koide and Y. Toyozawa, The Effect of Path Observation on Quantum Interference Pattern, J. Phys. Soc. Japan 62, 3395 (1993); G. Jaeger, A. Shimony, and L. Vaidman, Two interferometric complementarities, Phys. Rev. A 51, 54 (1995); B.-G. Englert, Fringe Visibility and Which-Way Information: An Inequality, Phys. Rev. Lett. 77, 2154 (1996); N. Erez, D. Jacobs and G. Kurizki, Operational pathphase complementarity in single-photon interferometry, J. Phys. B: At. Mol. Opt. Phys. 42, 1 (2009); A. Luis and I. Gonzalo, Single-particle interference only when imperfect path detectors click simultaneously, Eur. J. Phys C 6, 025012 (2015).

[5] A. Luis and L. L. Sánchez-Soto, Phase-difference operator, Phys. Rev. A 48, 4702 (1993); Quantum phase difference, phase measurements and Stokes operators, *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 2000) 41, 421 (2000).

[6] J.M. Lévy-Leblond, Who is afraid of nonhermitian operators? A quantum description of angle and phase, Ann. Phys. (N.Y.) 101, 319 (1976); M. Grabowski, Spin phase, Int. J. Theor. Phys. 28, 1215 (1989); On the phase operator, Rep. Math. Phys. 29, 377 (1991); J. Bergou and B.-G. Englert, Operators of the phase–fundamentals, Ann. Phys. (N. Y.) 209, 479 (1991); R. Lynch, The quantum phase problem: a critical review, Phys. Rep. 256, 367 (1995); V. Perinová, A. Lukš and J. Perina, *Phase in Optics* (World Scientific, Singapore, 1998).

[7] R. P. Feynman, Negative Probability, en Quantum Implications: Essays in Honour of David Bohm. Routledge and Kegan Paul Ltd. (1987).