Gauge-invariant perturbation theory for trans-Planckian inflation

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The possibility that the scale-invariant inflationary spectrum may be modified due to the hidden assumptions about the Planck scale physics — dubbed as trans-Planckian inflation — has received considerable attention. To mimic the possible trans-Planckian effects, among various models, modified dispersion relations have been popular in the literature. In almost all the earlier analyzes, unlike the canonical scalar field driven inflation, the trans-Planckian effects are introduced to the scalar/tensor perturbation equations in an ad hoc manner — without calculating the stress-tensor of the cosmological perturbations from the covariant Lagrangian. In this work, we perform the gauge-invariant cosmological perturbations for the single-scalar field inflation with the Jacobson-Corley dispersion relation by computing the fluctuations of all the fields including the unit time-like vector field which defines a preferred rest frame. We show that: (i) The non-linear effects introduce corrections only to the perturbed energy density. The corrections to the energy density vanish in the super-Hubble scales. (ii) The scalar perturbations, in general, are not purely adiabatic. (iii) The equation of motion of the Mukhanov-Sasaki variable corresponding to the inflaton field is different than those presumed in the earlier analyzes. (iv) The tensor perturbation equation remains unchanged. We perform the classical analysis for the resultant system of equations and also compute the power-spectrum of the scalar perturbations in a particular limit. We discuss the implications of our results and compare with the earlier results.

I. INTRODUCTION

Cosmological inflation is currently considered to be the best paradigm for describing the early stages of the universe [1, 2]. Inflation leads to the existence of a causal mechanism for producing fluctuations on cosmological scales (when measured today), which at the time of matter-radiation equality had a physical wavelength larger than the Hubble radius. Thus, it solves several conceptual problems of standard cosmology and leads to a predictive theory of the origin of cosmological fluctuations.

During inflation, the physical wavelength corresponding to a fixed comoving scale decreases exponentially as time decreases whereas the Hubble radius is constant. Thus, as long as the period of inflation is sufficiently long, all scales of cosmological interest today originate inside the Hubble radius during inflation. Recently, it has been realized that if inflation lasts slightly longer than the minimal time (i.e. the time it needs to last in order to solve the horizon problem and to provide a causal generation mechanism for CMB fluctuations), then the corresponding physical wavelength of these fluctuations at the beginning of inflation will be smaller than the Planck length. This is commonly referred to as the trans-Planckian problem of inflation [3–5].

Naturally, considerable amount of attention has been devoted to examine the possibility of detecting trans-Planckian imprints on the CMB [6–28]. Broadly, there have been two approaches in the literature in order to study these effects. In the first approach, the specific nature of trans-Planckian physics is not presumed, but is rather described by the boundary conditions imposed on the mode at the cut-off scale. In the second approach, which is of interest in this work, one incorporates quantum gravitational effects by introducing the fundamental length scale into the standard field theory in a particular fashion.

In almost all the analyses performed so-far in the literature, the trans-Planckian effects are introduced into the scalar and tensor perturbation equations in an ad hoc manner. In the standard inflation, the scalar and tensor perturbation equations are derived, from the first principles, in a gauge-invariant manner [29, 30]. However in the trans-Planckian inflationary scenario, to our knowledge, such a calculation has never been performed and the scalar/tensor power spectrum, with the trans-Planckian corrections, were obtained from the presumed perturbation equations.

With the possibility of detecting the trans-Planckian signatures in the current and future CMB experiments, it becomes imperative to obtain the scalar and tensor perturbation equation from the first principles. (For recent work on the trans-Planckian constraints from the CMB, see Refs. [31–35].) In this work, we perform the

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1 For the current status and developments of WMAP and PLANCK, see the following URLs: http://map.gsfc.nasa.gov/, http://astro.estec.esa.nl/SA-general/Projects/Planck
gauge-invariant cosmological perturbation theory for the single-scalar field inflation with the trans-Planckian corrections. The model we shall consider in this work is a self-interacting scalar field in (3 + 1)-dimensional space-time satisfying a linear wave equation with higher spatial derivative terms. The dispersion relation $[\omega = \omega(k)]$ thus differs at high wave-vectors from that of the ordinary wave equation. Such a model breaks the local Lorentz invariance explicitly while preserving the rotational and translational invariance. The particular dispersion relation we shall study in detail is

$$\omega^2(k) = |\vec{k}|^2 + b_1|\vec{k}|^4,$$

where $b_1$ is a dimensionful parameter. $b_1 < 0$ implies subluminal group velocity, while $b_1 > 0$ implies superluminal group velocity. The above dispersion relation is a subset of a general class of the form $\omega^2 = |\vec{k}|^2[1 + g(|\vec{k}|/k_0)]$, where $g$ is a function which vanishes as $k_0 \to \infty$, and $k_0$ is a constant which sets the scale for the deviation from Lorentz invariance. It has been suggested that these general dispersion relation might arise in loop quantum gravity [36, 37], or more generally from an unspecified modification of the short distance structure of space-time (see for example Refs. [38, 39]). Possible observational consequences have also been studied. For an up-to-date review, see Refs. [40, 41].

Even though the above dispersion relation breaks the local Lorentz invariance explicitly it has been shown that it is possible to write a Lagrangian for the above field in a generally covariant manner, consistent with spatial translation and rotation invariance, by introducing a unit time-like Killing vector field which defines a particular direction [11, 42, 43] (see Sec. (III) for more details). In this work, we use such a framework in-order to obtain the scalar/tensor perturbation equations for such a model during inflation.

Using the covariant Lagrangian used in Ref. [42], we obtain the perturbed stress-tensor for the scalar and tensor perturbations about the FRW background. We show that: (i) The non-linear effects introduce corrections to the perturbed energy density while the other components of the stress-tensor remains unchanged. (ii) The non-linear terms contributing to the stress-tensor are proportional to $k^2$. Hence in the super-Hubble scales the trans-Planckian contributions to the perturbed energy density, as expected, can be ignored. (iii) The spatial higher derivative terms appear only in the equation of the motion of the perturbed inflation field ($\delta\varphi$) and not in the equation of motion of the scalar perturbations ($\Phi$). (iv) Unlike the canonical scalar field inflation, the perturbations, in general, are not purely adiabatic. The entropic perturbations generated during the inflation, however, vanish at the super-Hubble scales. The speed of propagation of the perturbations is a constant and is less than the speed of light. (v) The tensor perturbation equation remain unchanged indicating that the well-know consistency relation between the scalar and tensor ratio will also be broken in this model.

We obtain the equation of motion of the Mukhanov-Sasaki variable corresponding to the inflaton field. We show that the equation of motion derived from the gauge-invariant perturbation theory is not same as those assumed in the earlier analyzes [3–5]. Later, we combine the system of differential equations into a single differential equation in $\Phi$ and obtain the solutions for the power-law inflation in different regimes. We also obtain the spectrum of scalar perturbations in a particular limit and compare with the earlier results.

This paper is organized as follows: In the following section, the theory of cosmological perturbations for the canonical single scalar field inflation is discussed and essential steps leading to the perturbation equation are reviewed. In Sec. (III), we discuss the general covariant formulation of the Lagrangian describing the scalar field with modified dispersion relation and derive the corresponding stress-tensor. In Sec. (IV), we obtain the perturbed stress-tensor for the scalar and tensor perturbations. In Sec. (V), we obtain the scalar perturbation equation and the equation of motion of the Mukhanov-Sasaki variable. In Sec. (VI), we perform the classical analysis and obtain the form of the scalar perturbations ($\Phi$) in the various regimes. In Sec. (VII), we solve the perturbation equations in a particular limit and obtain the power-spectrum of the perturbations. Our results are summarized and discussed in the last section. In Appendices (A, B) we derive the equations of motion of the fields in the FRW and perturbed FRW backgrounds. In Appendix (C), we obtain the equation of motion of the Bardeen potential in our model.

Through out this paper, the metric signature we adopt is $(+,-,-,-)$ [44], we set $\hbar = c = 1$ and $1/(8\pi G) = M_P^2$. The various physical quantities with the over-line refers to the values evaluated for the homogeneous and isotropic FRW background. A dot denotes derivative with respect to the cosmic time ($t$), a prime stands for a derivative with respect to conformal time ($\eta$) and a subscript $i$ denotes a derivative w.r.t space components. We follow the notation of Ref. [30] to provide easy comparison.

II. GAUGE-ININVARIANT PERTURBATION: CANONICAL SINGLE FIELD INFLATION

In this section, we obtain the scalar and tensor perturbation equations for the canonical single scalar field inflation. In the following subsection, we discuss key properties of the perturbed FRW metric and the “gauge problem” of the scalar perturbations. In the subsequent subsections, we discuss the matter Lagrangian and provide key steps in obtaining the scalar/tensor perturbation equations.
A. Perturbed FRW metric

We consider perturbations about a spatially flat (3+1)-dimensional FRW line element
\[
ds^2 = \mathbf{g}_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left( dx^2 = a^2(\eta) \left( d\eta^2 - dx^2 \right) \right),
\]
where \( t \) is the cosmic time, \( a(t) \) is the scale factor and \( \eta = \int \left[ dt/a(t) \right] \) denotes the conformal time. \( H \) is the Hubble parameter given by \( H \equiv \dot{a}/a \) while \( \mathcal{H} \equiv a'/a \) is related to the Hubble parameter by the relation \( \mathcal{H} = H a \). At the linear level, for the canonical single scalar field inflation, the metric perturbations \( (\delta g_{\mu\nu}) \) can be categorized into two distinct types — scalar and tensor perturbations. Thus, the perturbed FRW line-element can be written as
\[
ds^2 = a^2(\eta) \left[ (1 + 2\phi) d\eta^2 - 2\delta B dx^i dx^i - \left( (1 - 2\psi) \delta_{ij} + 2\theta \delta_{ij} \mathcal{E} + h_{ij} \right) dx^i dx^j \right],
\]
where the functions \( \phi, B, \psi \) and \( \mathcal{E} \) represent the scalar sector whereas the tensor \( h_{ij} \), satisfying \( \dot{h}_{ij} = \partial_i h_{ij} = 0 \), represent gravitational waves. Note that all these first-order perturbations are functions of \( (\eta, x) \). For convenience, we do not write the dependence explicitly.

The tensor perturbations do not couple to the energy density \( (\delta \rho) \) and pressure \( (\delta p) \) inhomogeneities. However, the scalar perturbations couple to the energy and pressure which lead to the growing inhomogeneities. At the linear level, the two types of perturbations decouple and can be treated separately.

The scalar and tensor perturbations have four and two degrees of freedom respectively. In the case of tensor perturbations, the two degrees of freedom correspond to the two polarizations of the gravitational waves and hence are physical. The scalar perturbations suffer from the gauge problem. (For a detailed discussion, see Refs. [29, 30].) However, it is possible to construct two gauge invariant variables, which characterize the perturbations completely, from the metric variables alone i. e.,
\[
\Phi \equiv \phi + \frac{1}{a} \left( (B - E') a' \right), \quad \Psi \equiv \psi - \mathcal{H} (B - E').
\]
Physically, \( \Phi \) corresponds to the Newtonian gravitational potential and is commonly referred to as the Bardeen potential while \( \Psi \) is related to the perturbations of the 3-space. For the single canonical scalar field scenario, we have \( \Phi = \Psi \).

\( \Phi \) and \( \Psi \) are related to the pressure and density perturbations of a generic perfect fluid via the perturbed Einstein’s equations. The pressure perturbations, in general, can be split into adiabatic and entropic (non-adiabatic) parts, by writing
\[
\delta p = c_s^2 \delta \rho + \overline{\rho} \Gamma,
\]
where \( c_s^2 \equiv \overline{p}/\overline{\rho} \) is the adiabatic sound speed [45–47]. The non-adiabatic part is \( \delta_{\text{had}} \equiv \overline{\rho} \Gamma \), and
\[
\Gamma \equiv \frac{\delta p}{\overline{\rho}} = \frac{\delta \rho}{\overline{\rho}}.
\]
The entropic perturbation \( \Gamma \), defined in this way, is gauge-invariant, and represents the displacement between hyper-surfaces of uniform pressure and uniform density. In the context of canonical single scalar field inflation, only adiabatic perturbations are present, i. e., \( \delta p = c_s^2 \delta \rho \) (where \( c_s^2 = 1 \)) or \( \Gamma = 0 \). However, as we shall see in the later sections, the trans-Planckian inflationary scenario introduces entropic perturbations as well.

B. Canonical scalar field

The dominant matter component during inflation is a spatially homogeneous canonical scalar field \( \overline{\varphi}(\eta) \) (inflation). The Lagrangian density for the canonical scalar field \( \varphi(\eta, \mathbf{x}) \) propagating in a general curved background is given by
\[
\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - V(\varphi),
\]
where \( V(\varphi) \) is the self-interacting scalar field potential. The equation of motion and the stress tensor of the scalar field \( \varphi(\eta, \mathbf{x}) \) in the conformally flat FRW background (2) are
\[
\varphi'' + 2 \mathcal{H} \varphi' - \nabla^2 \varphi + a^2 V_{,\varphi}(\varphi) = 0,
\]
where \( V_{,\varphi}(\varphi) = (dV(\varphi)/d\varphi) \) and \( \nabla^2 \) refers to the Laplacian in the flat space. Let us consider a small inhomogeneous quantum fluctuations on top of a homogeneous and isotropic classical background. For the scalar field, we have
\[
\varphi(\eta, \mathbf{x}) = \overline{\varphi}(\eta) + \delta \varphi(\eta, \mathbf{x}),
\]
where one assumes that the perturbation \( \delta \varphi \) is small. The perturbed scalar field \( \delta \varphi(\varphi) \) and the perturbed stress-tensor of the scalar field \( \delta T_{\mu\nu}(\varphi) \), like the other scalar-type perturbation functions \( \delta \Phi, \delta \mathcal{H}, \delta \mathcal{E} \), suffer from the gauge problem. (For a detailed discussion, see Refs. [29, 30].) Similar to Eq. (4), it is possible to define a gauge-invariant quantity for the perturbed scalar field and the perturbed stress-tensor, i. e.,
\[
\delta \varphi(g_{\mu\nu}) \equiv \delta \varphi + \overline{\varphi} (B - E'),
\]
\[
\delta T_{0}^{\mu}(g_{\nu}) \equiv \delta T_{0}^{\mu} + \overline{\varphi} (B - E'),
\]
\[
\delta T_{i}^{\mu}(g_{\nu}) \equiv \delta T_{i}^{\mu} + \overline{\varphi} (B - E'),
\]
\[
\delta T_{0}^{\mu}(g_{\nu}) \equiv \delta T_{0}^{\mu} + \frac{1}{3} \overline{\varphi} (B - E'),.
\]
Separating the homogeneous and perturbed part from Eq. (8), we have
\[
\overline{\varphi}'' + 2 \mathcal{H} \overline{\varphi}' + a^2 V_{,\varphi} = 0,
\]
\[
\delta \varphi(g_{\mu\nu})'' + 2 \mathcal{H} \delta \varphi(g_{\nu})' - \nabla^2 \left( \delta \varphi(g_{\nu}) \right)
\]
\[
+ V_{,\varphi} a^2 \delta \varphi(g_{\nu}) - 4 \overline{\varphi} \Phi' + 2 V_{,\varphi} a^2 \Phi = 0.
\]
Similarly, separating the homogeneous and perturbed parts in the stress-tensor (9), we get
\[
\dot{T}^{(g)}_{ij} = \frac{1}{2} \left( \frac{\ddot{\varphi}^2}{a^2} + V(\varphi) \right) \delta_{ij} \quad \ddot{T}^{(\gamma)}_{ij} = \frac{1}{2} \left( \frac{\ddot{\varphi}^2}{a^2} - V(\varphi) \right) \delta_{ij} \quad \delta_{ij}
\]

\[
\dot{\delta\varphi}^{(g)}_{ij} = a^{-2} \left[ -\ddot{\varphi}^2 \Phi + \ddot{\Phi} \delta\varphi^{(g)} + V,_{\nu} a^3 \delta\varphi^{(g)} \right],
\]

\[
\dot{\delta\varphi}^{(\gamma)}_{ij} = a^{-2} \left[ -\ddot{\varphi}^2 \Phi - \ddot{\Phi} \delta\varphi^{(g)} + V,_{\nu} a^3 \delta\varphi^{(g)} \right] \delta_{ij}.
\]

C. Scalar and Tensor Perturbation equations

In the earlier subsections, we obtained the gauge invariant variables related to the scalar field \((\delta\varphi^{(g)}), (\delta\varphi^{(\gamma)})\) and metric perturbations \((\Phi, \Psi)\). In this subsection, we outline the essential steps leading to the scalar and tensor equations of motion. Even though, this is a standard result, and can be found in numerous review articles (see, for example, Refs. [29, 30]), we have given here the key steps for future reference.

From Eq. (16), it is easy to see that the non-diagonal space-space components of the stress-tensor are absent. This leads to the condition that \(\Phi = \Psi\). Thus, the Einstein’s equations for the perturbed FRW metric (3) in terms of the gauge invariant quantities are:

\[
\nabla^2 \Phi - 3H \Phi' - 3\dot{H} \Phi = \frac{1}{2M^2_{Pl}} a^2 \delta T_0^{(g)} \quad (17a)
\]

\[
\frac{(a\Phi)'_i}{a} = \frac{1}{2M^2_{Pl}} a^2 \delta T_i^{(g)} \quad (17b)
\]

\[
\Phi'' + 3H\Phi' + (2H' + \dot{H})\Phi = \frac{1}{2M^2_{Pl}} a^2 \delta T_i^{(g)} \quad (17c)
\]

where \(\delta T_i^{(g)}\) is given by Eq. (16). The three perturbed Einstein’s equations can be combined to form a single differential equation in \(\Phi\):

\[
\Phi'' - \nabla^2 \Phi + 2 \left( \frac{H - \frac{\ddot{\varphi}}{\varphi}}{\varphi} \right) \Phi' + 2 \left( \frac{\dot{H}}{H} - \frac{\ddot{\varphi}}{\varphi} \right) \Phi = 0. \quad (18)
\]

The system of perturbation equations (18, 17b, 14) is quite complex. In order to extract the physical content more transparent, these equations are expressed in terms of two new variables – \(Q\) and \(u\) – which are linearly related to \(\Phi\) and \(\delta\varphi^{(g)}\), i.e.,

\[
Q = a \left\{ \delta\varphi + \frac{\psi}{H} \right\} \quad u = a \frac{\psi}{\varphi} \Phi. \quad (19)
\]

\(Q\) is a gauge-invariant (Mukhanov-Sasaki) [29, 30] variable whose equation of motion is homogeneous. This ensures that one can quantify \(Q\) in the standard way using the Lagrangian associated with its equation of motion. At the early stages of inflation where the quantum effects are important, the equation of motion of \(Q\) helps in quantizing the fields and fixing the initial conditions. However, at the end stages of inflation where the relevant modes have crossed the Hubble radius and behave classically, it is easier to analyze the equation of motion of \(u\).

The equation of motion of \(Q\) is derived as follows: Substituting \(\delta \varphi\) in terms of \(Q\) in Eq. (14), and using the relations (18, 13, 17b), we get:

\[
Q'' - \nabla^2 Q - \frac{z''}{z} Q = 0, \quad (20)
\]

where

\[
z = a (\frac{H^2 - \dot{H}^2}{H c_s})^{1/2} = a \frac{\varphi'}{H}. \quad (21)
\]

The equation of motion of \(u\) is obtained by substituting the transformation (19) in Eq. (18):

\[
\ddot{u} - \nabla^2 u - \frac{\theta'}{\theta} u = 0, \quad (22)
\]

where

\[
\theta = \frac{H}{a} \left[ \frac{2}{3} (H^2 - \dot{H}^2) \right]^{-1/2} = \frac{H}{a \varphi}. \quad (23)
\]

\(Q\) is related to another gauge-invariant quantity \(\zeta(\equiv - (H/\dot{\varphi})\delta \varphi + \psi)\) by the relation \(Q = 2c_s \zeta\). The quantity \(\zeta\) is time-independent on scales larger than the Hubble radius and, more importantly, related to the large scale CMB anisotropies (via the Sachs-Wolfe effect) [2].

Decomposing \(Q\) into Fourier space, we have

\[
\mu_s'' + (k^2 - \frac{z''}{z}) \mu_s = 0, \quad (24)
\]

where \(k = |k|\) and \(\mu_s = -Q_k = -2c_s z \zeta\). The above equation is similar to a time-independent Schrödinger equation where the usual role of the radial coordinate is now played by the conformal time whose effective potential is \(U_s \equiv z''/z\) [48, 49]. The scalar perturbation spectrum per logarithmic interval can then be written in terms of the modes \(\mu_s\) as

\[
[k^3 \mathcal{P}_S(k)] = \left( \frac{k^3}{2\pi^2} \right) \left( \frac{|\mu_s|}{z} \right)^2, \quad (25)
\]

and the expression on the right hand side is to be evaluated when the physical wavelength \((k/a)^{-1}\) of the mode corresponding to the wavenumber \(k\) equals the Hubble radius \(H^{-1}\).

Before proceeding to the next section, we obtain the tensor perturbation equation in the FRW background. As we mentioned earlier, the tensor perturbations \(h_{ij}(\eta, x)\) do not couple to the energy density. These represent free gravitational waves and satisfy the equation:

\[
\mu_T'' + (k^2 - \frac{a''}{a}) \mu_T = 0. \quad (26)
\]
where $\mu_T \equiv a h_k$. This equation is very similar to the corresponding equation (24) for scalar gravitational inhomogeneities, except that in the effective potential $(U_\gamma \equiv a''/a)$ the scale factor $a(\eta)$ is replaced by $z(\eta)$.

III. MODIFIED DISPERSION RELATION LAGRANGIAN

In this section, we briefly discuss the general covariant formulation describing a scalar field with modified dispersion relation and derive the corresponding stress-tensor.

As discussed in the introduction, to keep the calculations tractable, we will assume that the scalar field with the high frequency dispersion relation is of the form

$$\omega^2 = |\vec{k}|^2 + b_{11} |\vec{k}|^4,$$

where $b_{11} > 0$. The above dispersion relation breaks the local Lorentz invariance explicitly while it preserves rotational and translational invariance. Even though, the modified dispersion relation breaks the local Lorentz invariance explicitly, it was shown in Ref. [50] that a covariant formulation of the corresponding theory can be carried out by introducing a unit time-like vector field $u^\mu$ which defines a preferred rest frame.

A. Covariant Lagrangian

The action for a scalar field with the modified dispersion relation takes the form [11, 50]

$$S = \int d^4x \sqrt{-g} \left( L_\phi + L_{\text{cor}} + L_u \right),$$

where $L_\phi$ is the standard Lagrangian of a minimally coupled scalar field given by Eq. (7). The last two terms $- L_{\text{cor}}$ and $L_u$ contribute to the modified dispersion relation of the scalar field. $L_{\text{cor}}$ corresponds to the nonlinear part of the dispersion relation while $L_u$ describes the dynamics of the vector field $u^\mu$. The two corrective Lagrangians have the form

$$L_{\text{cor}} = -b_{11} (D^2 \phi)^2,$$

$$L_u = -\lambda (g^{\mu\nu} u_\mu u_\nu - 1) - d_1 F^{\mu\nu} F_{\mu\nu},$$

where

$$F_{\mu\nu} \equiv \nabla_\mu u_\nu - \nabla_\nu u_\mu,$$

$$D^2 \phi = \perp_{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + u^\alpha \nabla_\alpha \phi \nabla_\beta u^\beta,$$

$$\perp_{\mu\nu} \equiv -g_{\mu\nu} + u_\mu u_\nu.$$

The covariant derivative associated with the metric $g_{\mu\nu}$ is $\nabla_\mu$ while $b_{11}$ and $d_1$ are arbitrary (dimensional) constants. The tensor $\perp_{\mu\nu}$ gives the metric on a slice of fixed time while $D^2$ is proportional to the Laplacian operator on the same surface. The fact that $u^\mu$ is a unit time-like vector $(u^\mu u_\mu = 1)$ is enforced by the Lagrange multiplier $\lambda$. In the above, $b_{11}, d_1$ have the dimensions of inverse mass square and mass square respectively while $u_\mu$ is dimensionless.

The equation of motion for $\phi$ and $u_\mu$ obtained by varying the action (28), respectively, are

$$\nabla^\mu \nabla_\mu \phi + V_\phi = 2b_{11} \left( \nabla_\mu \left( D^2 \phi \ n^\mu u^\nu \right) - \nabla_\mu \nabla_\nu \left( D^2 \phi \ n_\nu \right) \right),$$

$$2d_1 \nabla_\mu F^{\mu\nu} - \lambda u^\mu = -b_{11} \left[ \nabla_\mu \left( D^2 \phi \ n^\mu u_\nu \right) \right] - \nabla^\mu \nabla_\nu \phi \ n^\nu \ n^\mu u^2 \phi - \nabla_\nu \left( \nabla^\mu \phi n^\nu \right) D^2 \phi.$$

From the above equation, we get

$$\lambda = b_{11} u_\mu \left[ \nabla^\mu \left( D^2 \phi \ n^\mu u_\nu \partial_\nu \phi \right) - \nabla^\nu \nabla_\nu \phi \ n^\mu u_\mu D^2 \phi \right] - \nabla_\nu \left( \nabla^{\mu} \phi \ n^{\nu} \right) D^2 \phi + 2d_1 u_\mu \nabla_\nu F^{\nu\mu}.$$ 

Using the results of Appendix (A), it is easy to show that for the FRW background $\varphi$ satisfies Eq. (13) which is same as that of the canonical scalar field. The field equation for $u_\mu$ gives $\lambda = 0$.

Before we proceed to the computation of the stress-tensor, it is important to know the current astrophysical constraints on the parameters of the model [40]. The constraints of the parameter $d_1$ comes from the big-bang nucleosynthesis [51] and the solar system tests of general relativity [52]. These give:

$$0 < \frac{d_1}{M_{Pl}^2} < \frac{1}{7}. \quad (36)$$

The constraints on the parameter $b_{11}$ comes from the observations of highest energy cosmic rays [53]. Using effective field theory with higher-dimensional operators, resulting in the modified field theory with a dispersion relation, it was shown that for various standard model particles

$$b_{11} M_{Pl}^2 < 5 \times 10^{-5}. \quad (37)$$

B. Stress tensor

In this subsection, we obtain the stress-tensor for the scalar field defined in Eq. (28). Formally, the stress-tensor for a general Lagrangian containing up to first order derivative in the metric is given by (cf. Ref. [44], p. 272)

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{2}{\sqrt{-g}} \partial_\rho \left( \sqrt{-g} \partial^{\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\rho g_{\mu\nu})} \right). \quad (38)$$

For simplicity, we will separate the contributions from the three different Lagrangians defined in Eq. (28), i. e.,

$$T_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(u)} + T_{\mu\nu}^{(cor)}.$$

$$T_{\mu\nu}^{(\phi)} = \nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \nabla_\alpha \phi,$$

$$T_{\mu\nu}^{(u)} = g_{\mu\nu} \nabla^2 \phi + \nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\beta \nabla_\beta \phi.$$
The stress-tensor $T_{\mu\nu}^{(u)}$ corresponding to the canonical scalar field Lagrangian is given in Eq. (9). The stress-tensor corresponding to the Lagrangian $\mathcal{L}_u$ can easily be obtained and is given by

$$T_{\mu\nu}^{(u)} = d_1 g_{\mu\nu} F_{\alpha\beta}^\alpha F_{\epsilon\kappa}^\epsilon - 4 d_4 F_{\mu\nu} F_{\rho\kappa} g_{\rho\kappa} + \lambda [g_{\mu\nu}(g_{\alpha\beta} u^\alpha u^\beta - 1) - 2 u_{\mu} u_{\nu}].$$ (40)

However, the stress-tensor corresponding to $\mathcal{L}_{cor}$ is much more involved. For the sake of continuity we give below only the final result while the steps leading to the result is given in Appendix (A). We get,

$$T_{\mu\nu}^{(cor)} = b_{41} g_{\mu\nu} (D^2 \phi)^2 + b_{41} E_{\mu\nu} D^2 \phi + 4 b_1 C_{\mu\nu}^\alpha \partial_\alpha \phi \partial_\beta [D^2 \phi],$$ (41)

where,

$$E_{\mu\nu} = 4 \left[ \partial_\rho \ln (\sqrt{-g}) \right] C_{\mu\nu}^\rho \partial_\rho \phi + \partial_\rho \left( C_{\mu\nu}^\rho \partial_\rho \phi \right) - A_{\rho\mu\nu} \partial_\rho \partial_\alpha \phi - B_{\rho\mu\nu} \partial_\alpha \phi,$$ (42)

$$C_{\mu\nu}^\alpha = \frac{1}{2} \left[ g_{\mu\nu} g^{\alpha\rho} - \delta^\alpha_\mu \delta_\mu^\nu - \delta^\alpha_\nu \delta_\nu^\mu + u_\mu \delta^\alpha_\nu u^\mu - u_\nu \delta^\alpha_\mu u^\nu + \delta^\alpha_\mu \delta_\mu^\nu \right].$$ (43)

Before proceeding with the evaluation of the perturbed stress-tensor we would like to mention the following point: Using the results of Appendix (A), it is clear that $T_{\mu\nu}^{(u)}$ and $T_{\mu\nu}^{(cor)}$ vanishes in the unperturbed FRW background. Hence, the equations determining the evolution of the scale factor, i.e.,

$$3 \mathcal{H}^2 = \frac{a^2}{M_{Pl}^2} T_0^{(\phi)}; \quad 2 \mathcal{H}' + \mathcal{H}^2 = \frac{a^2}{3 M_{Pl}^2} T_i^{(\phi)},$$ (44)

remain the same as in the canonical scalar field inflation. It is also worth mentioning that the trans-Planckian corrections do not play any role on the expansion of the FRW background while, as we will see in the next section, the trans-Planckian corrections affect the metric and inflaton perturbations.

In this work, we will focus on the power-law inflation, for which, the scale factor is given by

$$a(t) = (a_0 t^p)^{(\beta+1)} \quad \text{or} \quad a(\eta) = \left( \frac{\eta}{-\eta_0} \right)^{(\beta+1)},$$ (45)

where $p > 1$, $\beta \leq -2$ ($\beta = -2$ corresponds to de Sitter), $a_0$ is a constant,

$$\beta = -\left( \frac{2p - 1}{p - 1} \right) \quad \text{and} \quad (-\eta_0) = \frac{a_0}{(p - 1)}. \quad (46)$$

The scalar field potential and other background field parameters are given by ($q = \sqrt{2/p}$)

$$V = v_\nu M_{Pl}^4 \exp \left( -q \frac{\phi}{M_{Pl}} \right); \quad \mathcal{H} = \left( -\eta \right)^(-1 + \beta),$$ (47)

$$\mathcal{V} = \sigma_\phi M_{Pl} \ln \left( -\eta \right); \quad \sigma_\phi = \sqrt{2(\beta + 1)}; \quad a_0 = -\frac{\sqrt{(\beta + 1)(1 + 2\beta)}}{\sqrt{v_\nu \eta_0 M_{Pl}}}. \quad (48)$$

IV. PERTURBED STRESS TENSOR

In this section, we will obtain the perturbed stress-tensor for the scalar field with modified dispersion relation (28) in the perturbed FRW background (3). As in the previous section, we will separate the contributions to the perturbed stress-tensor from the three different Lagrangians defined in Eq. (28), i.e.,

$$\delta T_\mu^\nu = \delta T_\mu^\nu (\phi) + \delta T_\mu^\nu (u) + \delta T_\mu^\nu (cor).$$ (49)

The first term in the RHS of the above expression corresponds to the perturbed stress-tensor of the canonical scalar field Lagrangian and is given by Eqs. (16). In the rest of the section, we will obtain contributions from the other two terms.

Perturbing Eq. (40) and using the fact that $F_{\mu\nu}$ vanishes for the FRW background [See Appendix (A)], we get

$$\delta T_\mu^\nu (u) = -2 \delta_\mu^\rho \delta_\nu^\rho (\lambda),$$ (50)

where $\lambda$ is given by Eq. (B14). Using the fact that $D^2 (\phi)$ and $\partial_\mu D^2 (\phi)$ vanish for the FRW background [See Appendix (A)], the perturbation of Eq. (41) takes the following simple form:

$$\delta T_\mu^\nu (cor) = 4 b_{11} \left[ F_{\mu\nu} \delta (D^2 \phi) + C_{\mu\nu}^\rho \partial_\rho (\partial D^2 \phi) \right].$$ (51)

Substituting the relation (B11) for $\delta (D^2 \phi)$ and Eqs. (B7) in the above expression, we get 

$$\delta T_\mu^\nu (cor) = \frac{2 b_{11}}{a^4} \left[ \frac{5}{a^2} \mathcal{V}^2 \partial^2 \xi (g_{ij}) - \frac{1}{a^2} \mathcal{V}^2 \partial^2 \xi (g_{ij}) \right] \delta_\mu^\rho \delta_\nu^\rho,$$ (52)

where $\xi$ is defined in Eq. (B22). Substituting Eqs. (16, 50, 52) in Eq. (49), we obtain the perturbed gauge-invariant stress-tensor to be:

$$\delta T_{\mu}^{(gi)} = \frac{1}{a^2} \left[ -\mathcal{V}^2 \Phi + \mathcal{V}^2 \partial^2 \phi (g_{ij}) + V_{\phi} a^2 \delta \phi (g_{ij}) \right]$$

$$+ \frac{4 d_4}{a^2} \left[ \mathcal{V}^2 \Phi - \frac{1}{a^2} \mathcal{V}^2 \partial^2 \xi (g_{ij}) \right],$$ (53a)

$$\delta T_{\mu}^{(gi)} = \left[ -\mathcal{V}^2 \Phi + \mathcal{V}^2 \partial^2 \phi (g_{ij}) + V_{\phi} a^2 \delta \phi (g_{ij}) \right] \delta_\mu^\rho \delta_\nu^\rho, \quad (53b)$$

$$\delta T_{\mu}^{(gi)} = a^{-\mathcal{V}^2} \mathcal{V}^2 \partial^2 \phi (g_{ij}),$$ (53c)

Following points are worth-mentioning regarding the above result: Firstly, the two corrective Lagrangians $-\mathcal{L}_u$ and

$^2$ The mixed stress-tensor $\delta T_\mu^\nu$ is given by

$$\delta T_\mu^\nu \equiv \delta (g_{\mu\nu} T_\mu) = \mathcal{V}^2 \left[ (\delta T_\mu^\rho + \partial_\rho \delta T_\mu) \right].$$ (48)
perturbations. (See, for example, Ref. [54, 55]) We will discuss more on these in the following sections.

Thirdly, it is interesting to note that the trans-Planckian contributions to the energy density go as $k^2$. Hence as one would expect, in the super-Hubble scales, only the canonical scalar field contributes significantly in these scales. Lastly, these can have two significant implications on the perturbation spectrum: (a) The speed of propagation of the perturbations ($c_s^2$) can be different from that of the standard single-scalar field inflation. In the case of single scalar field inflation, we know that $c_s^2 = 1$. However, due to the extra contributions to the energy density, this can no longer be true. (b) The perturbations need not be purely adiabatic. ξ can act as an extra scalar field during the inflation and hence can act as a source. This can introduce non-adiabatic (entropic) perturbations. (See, for example, Ref. [54, 55]) We will discuss more on these in the following sections.

V. SCALAR PERTURBATION EQUATION

Substituting Eqs. (53) in (17), the first-order perturbed Einstein’s equations take the following form,

$$\nabla^2 \Phi - 3H \dot{\Phi}' - 3H^2 \Phi = \frac{2d_i}{M_{p1}^2} \left[ \nabla^2 \Phi - \frac{1}{a} \nabla^2 \xi^{(gi)} \right]$$

$$+ \frac{1}{2M_{p1}^2} \left[ -\nabla^2 \Phi + \nabla^2 \xi^{(gi)} + V_{g} \dot{a}^2 \xi^{(gi)} \right]$$

(54a)

$$\mathcal{H} \Phi + \Phi' = \frac{1}{2M_{p1}^2} \nabla^2 \Phi^{(gi)}$$

(54b)

$$\Phi'' + 3H \dot{\Phi}' + (2H' + H^2) \Phi = \frac{1}{2M_{p1}^2} \left[ \nabla^2 \Phi - \nabla^2 \xi^{(gi)} + V_{g} \dot{a}^2 \xi^{(gi)} \right]$$

(54c)

As in the canonical scalar field inflation, the three perturbed Einstein’s (54) can be combined to give

$$\Phi'' - \left( 1 - \frac{M_{p1}^2}{2d_i} \right) \nabla^2 \Phi + 2 \left( H - \frac{\dot{\Phi}'}{\Phi} \right) \Phi'$$

$$+ 2 \left( H' - \mathcal{H} \frac{\Phi''}{\Phi} \right) \Phi = \frac{2d_i}{M_{p1}^2} \frac{1}{a} \nabla^2 \xi^{(gi)}.$$  

(55)

Perturbing the field equations (33, 34), we get

$$\delta \varphi^{(gi)}'' + 2H \delta \varphi^{(gi)}' - \nabla^2 \left( \delta \varphi^{(gi)} \right) + V_{,\varphi} \dot{a}^2 \delta \varphi^{(gi)} = 0$$

$$-4\mathcal{H} \Phi' + 2V_{,\varphi} \dot{a}^2 + \frac{2b_{11}}{a^2} \left[ \nabla^2 \delta \varphi^{(gi)} - \frac{\nabla^2}{\mathcal{H}} \nabla^2 \xi^{(gi)} \right] = 0,$$

$$\partial_m \left[ \left( 1 - \frac{c_1}{M_{p1}^2} \nabla^2 \right) \Phi' - 2H \left( 1 + \frac{c_1}{2M_{p1}^2} \nabla^2 \right) \Phi' \right]$$

$$- \frac{\partial_m}{a} \left[ \nabla^2 \xi^{(gi)} - 3H \eta^{(gi)} - \frac{b_{11}}{2d_i} \nabla^2 \nabla^2 \xi^{(gi)} \right] = 0,$$

(57)

where $c_1 = (M_{p1}^2 b_{11})/d_1$ is a dimensionless constant.

Following points are interesting to note regarding the above results: (i) The spatial higher derivatives appear only in the equation of motion of $\delta \varphi$ and not in metric perturbation equation $\Phi$. Unlike the standard inflation, the perturbations are not purely adiabatic and the speed of propagation of the perturbations is less than unity i. e. $c_s^2 = 1 - 2d_i/M_{p1}^2$. (ii) In the case of canonical scalar-field inflation, the two dynamical variables $\Phi$ and $\xi^{(gi)}$ are related by the constraint equation (17b). In our model, $\Phi$ and $\delta \varphi^{(gi)}$ are again related by the same constraint equation, however $\xi$ is related to the other fields via the equations of motion. Hence, unlike the standard inflation, we have two sets of independent variables. (iii) In the case of canonical scalar-field inflation, $\Phi$ and $\xi^{(gi)}$ can be combined into a single variable — Mukhanov-Sasaki ($Q$) variable — in terms of which we can obtain the perturbation spectrum. However, in this model, as in the case of multi-field inflation models [54], the equations of motion in terms of the Mukhanov-Sasaki variables are coupled. In the rest of the section, we derive the equation of motion of the Mukhanov-Sasaki variables corresponding to $\delta \varphi^{(gi)}$ and $\xi$.

Substituting $\delta \varphi$ in-terms of $Q$ in Eq. (56), and using the relations (55, 13, 17b), we get

$$Q'' - \left( 1 - \frac{2b_{11}}{a^2} \nabla^2 \right) \nabla^2 Q - \frac{z''}{z} Q = \frac{2d_i}{M_{p1}^2} \nabla^2 S(\eta),$$

(58)

where

$$S = \mathcal{H} \left[ \frac{Q^{(2)}_\xi}{\mathcal{H}} + \frac{c_1}{a^{1/2}} \nabla^2 Q^{(1)}_\xi \right],$$

(59)

and $Q^{(1)}_\xi, Q^{(2)}_\xi$ are the gauge-invariant variables associated with $\xi$ and are given by:

$$Q^{(1)}_\xi = a^{-3/2} \left[ \xi + \frac{a}{\mathcal{H}} \dot{\varphi} \right]; Q^{(2)}_\xi = \xi' - a \dot{\varphi}.$$  

(60)

Substituting for $\xi$ in-terms of $Q^{(1)}_\xi$ in (57), we get,

$$Q^{(1)}_\xi'' + \left[ \frac{3}{2} \mathcal{H}' - \frac{9}{4} \mathcal{H}^2 - \frac{b_{11}}{2d_i} a^{-2} \nabla^2 \right] Q^{(1)}_\xi$$

$$- \frac{\partial_m}{a} \left[ \frac{\nabla^2}{\mathcal{H}} - \mathcal{H}' \dot{\mathcal{H}} - \frac{c_1 \mathcal{H}}{2M_{p1}^2 a^2} \nabla^2 \right] Q - \frac{a}{\mathcal{H}} \nabla^2 \Phi,$$

(61)

$m = 0$. 

$$m = 0$. 

Decomposing $Q, Q^{(1)}_\xi, Q^{(2)}_\xi$ into Fourier space, we have
\[ \mu_\xi'' + \left[ k^2 + \frac{2b_1}{a^3} + \frac{a}{z} \right] \mu_\xi = -\frac{2d_1}{M^2} k^2 S_k(\eta) \tag{62} \]
\[ \mu_\xi' + \frac{3}{2} H' - \frac{9}{4} H^2 + \frac{b_{11} \Phi^2}{2d_1 a^2} k^2 \] \[ \mu_\xi = a^{-3/2} \left[ \frac{a k^2}{H} \Phi_k \right] \tag{63} \]
where the Fourier transform of $Q, Q^{(1)}_\xi, Q^{(2)}_\xi$, respectively, are $\mu_\xi, \mu_\xi', Q^{(2)}_k$ and
\[ S_k(\eta) = \frac{\varphi}{H} \left[ \frac{Q^{(2)}_k}{\varphi} - \frac{c_1 H}{2M^2 a^2 k^2} \right] . \tag{64} \]

Eqs. (62, 63) are the main results of our paper, regarding which we would like to stress the following points: Firstly, in the earlier analyses, the equation of motion of the Mukhanov-Sasaki variable ($Q$) was assumed to be satisfy the differential equation Eq. (62) in which the source term was assumed to be zero. We have shown explicitly from the gauge invariant perturbation theory that, in general, this is not true. The RHS of (62) vanishes in the super-Hubble scales (i.e. $k \to 0$) where the perturbations can be treated classical. Hence, as expected, the trans-Planckian effects are negligible. Secondly, it is clear from Eq. (62) that the terms in the RHS will dominate during the trans-Planckian regime and can have interesting consequences on the primordial spectrum. Lastly, the perturbations (in general) are not purely adiabatic, i.e., it contains isocurvature perturbations. However, these perturbations do not contribute significantly in the super-Hubble scales. Taking the Fourier transformation of the non-adiabatic part of the pressure perturbation ($\delta p_{\text{nad}}$), we have
\[ F(\delta p_{\text{nad}}) = -4d_1 \frac{k^2}{a^3} Q^{(2)}_k . \tag{65} \]

From the above expression, it is straightforward to see that, in the super-horizon scales, the entropic perturbations vanish. Following Refs. [45, 46], we can assume that, on large scales, the total curvature perturbation $\zeta$ is conserved. As mentioned earlier, in the FRW background only the canonical scalar field contributes to the stress-tensor. Following Ref.[47], it is possible to show that, on large scales, only the curvature perturbation associated to $\delta \phi$ contributes to the total curvature perturbation. Hence, it is sufficient to calculate the power-spectrum associated to the scalar-field perturbation ($\delta \phi$). This will be discussed in Sec. (VII)

VI. CLASSICAL ANALYSIS

In this section, we combine Eqs. (55, 56, 57) to obtain a single differential equation of $\Phi$. We show that the resultant differential equation of $\Phi$ is different from that of the standard canonical scalar field driven inflation. More importantly, the differential equation of $\Phi$ in our model is fourth order while in the standard canonical scalar field it is second order. We obtain the solutions of $\Phi$ in three regimes — trans-Planckian (I), linear (II) and super-Hubble (III) — for the power-law inflation. In the following section, we obtain the power-spectrum of the scalar perturbations, in a particular limit, for the power-law inflation.

A. The Power law inflation.

Let us decompose the fields in their Fourier modes:
\[ \Phi(\eta, \vec{x}) = \Phi_k(\eta) e^{i \vec{k} \cdot \vec{x}}, \delta \phi(\eta, \vec{x}) = \delta \phi_k(\eta) e^{i \vec{k} \cdot \vec{x}}. \tag{66} \]

We have dropped the superscript indicating that the quantities are gauge invariant. Combining the equations, we end up with a fourth order differential equation in the Bardeen potential. To keep the presentation light, the derivation of the equation is given in Appendix C. This equation, for the power-law inflation (45, 48), reads
\[ \Phi_k^{(4)} + \frac{\Gamma_1}{\eta} \Phi_k^{(3)} + \frac{\Gamma_2}{\eta^2} \left[ 1 + \frac{\Gamma_3}{(-\eta)^{2+3\beta}} \right] k^2 + \frac{\Gamma_4}{(-\eta)^{2(1+2\beta)}} \Phi_k^{(2)} \]
\[ + \frac{\Gamma_5}{\eta^2} + \frac{\Gamma_6}{(-\eta)^{3+2\beta}} + \frac{\Gamma_7}{\eta} k^2 + \frac{\Gamma_8}{(-\eta)^{4+2\beta}} k^4 \right] \Phi_k' \]
\[ + \frac{\Gamma_9}{\eta^3} + \frac{\Gamma_{10}}{\eta^2} k^2 + \left[ \frac{\Gamma_{11}}{(-\eta)^{5+2\beta}} + \frac{\Gamma_{12}}{(-\eta)^{5+3\beta}} \right] k^3 \Phi_k = 0 . \tag{67} \]

The constants $\Gamma_i$ depend on the background and the fundamental constants in the following way
\[ \Gamma_1 = 4(2 + \beta), \quad \Gamma_2 = 12 + 13\beta + 3\beta^2 - 6\sigma^2, \]
\[ \Gamma_3 = -3 \left( \frac{1}{2} - \frac{3\beta}{a^2} \right) \frac{b_{11}}{d_1} M^2, \]
\[ \Gamma_4 = 2 \left( -\frac{1}{2} + \frac{2\beta}{a^2} \right) b_{11}, \quad \Gamma_5 = -18(1 + 3\beta) \sigma^2, \]
\[ \Gamma_6 = 3 \left( \frac{1}{2} - \frac{3\beta}{a^2} \right) \frac{b_{11}}{d_1} M^2, \quad \Gamma_7 = 6 + 4\beta, \]
\[ \Gamma_8 = -4(2 + \beta) \left( \frac{1}{2} - \frac{3\beta}{a^2} \right) b_{11}, \]
\[ \Gamma_9 = 6 \sigma^2 \left( \frac{2q^2}{24 + q^2} \right) \frac{M^2}{d_1}, \quad \Gamma_{10} = 4 + 7\beta + 3\beta^2, \]
\[ \Gamma_{11} = 2(1 + 3\beta) \left( \frac{1}{2} - \frac{3\beta}{a^2} \right) \frac{b_{11}}{d_1} (2d_1 - M^2) \]. \tag{68}
B. The zeroth order approximation

We can find approximate solutions for the power law inflation in the following way. Introducing the quantity $\epsilon$ by

$$\beta = -2 - \epsilon,$$  \hfill (69)

($\epsilon$ vanishes on the de Sitter space) we can make Taylor expansions of the coefficients $\Gamma_i$ and postulate the same for the Bardeen potential

$$\Phi_k(\eta) = \sum_{m=0} \epsilon^m \Phi_{k,m}(\eta) \ .$$  \hfill (70)

The outcome is the following. Each component $\Phi_{k,m}(\eta)$ obeys a differential equation which is inhomogeneous, the source term depending on the preceding components $\Phi_{k,0}(\eta), \ldots, \Phi_{k,m-1}(\eta)$.

As discussed in Ref. [20], we have to be careful in the limit of $\epsilon \rightarrow 0$ in the sense that it does not give the perturbation corresponding to the de Sitter space. This is related to the fact that on this background the inflaton field is constant so that the quantities $X_i, Y_i, Z_i$ are undetermined. To work out what happens for the de Sitter space, one would have to consider the earlier equations where divisions by the derivatives of the inflaton was not performed yet.

The zeroth order contribution is a solution of the equation

$$\Phi_{k,0}^{(4)}(\eta) + \left( k^2 - \frac{2}{\eta^2} + \gamma^2 k^4 \eta^2 \right) \Phi_{k,0}(\eta) - 2 \frac{k^2}{\eta^2} \Phi_{k,0}'(\eta) + 2 \frac{k^2}{\eta^2} \Phi_{k,0}'(\eta) = 0 \ .$$  \hfill (71)

We now introduce the dimensionless variable $x$ defined by $x = k\eta$ and the function $f(x)$ given by

$$\Phi_{k,0}(\eta) = f(x) \ .$$  \hfill (72)

The fourth order equation takes the form

$$f^{(4)}(x) + \left( 1 - \frac{2}{x^2} + \gamma^2 x^2 \right) f''(x) - \frac{2}{x} f'(x) + \frac{2}{x^2} f(x) = 0,$$  \hfill (73)

where,

$$\gamma = \sqrt{2\hbar v_0 M_{\text{pl}}} ,$$

As mentioned earlier, we obtain approximate solutions to the above differential equation in three different regions.

1. The first region

In this region, the term $\gamma^2 x^2$ dominates. In other words, the trans-Planckian effects are dominant and we are dealing with large values of $x$. Using the fact that $f(x) = x$ is a solution of the full equation, one introduces the function $h(x)$ by the relation

$$f(x) = x \int h(\zeta) d\zeta$$  \hfill (74)

and obtains the third order equation

$$2(-1 + \gamma^2 x^2) h(x) + \gamma^2 x^3 h'(x) + 4h''(x) + xh'''(x) = 0$$  \hfill (75)

We can get rid of the second derivative by the change of function

$$h(x) = \frac{1}{x^{4/3}} R(x) \ ;$$  \hfill (76)

using the fact that we are in the region given by large values of $x$, one ends up with the differential equation

$$R'''(x) + \gamma^2 x^2 R'(x) + \frac{2}{3} \gamma^2 x R(x) = 0$$  \hfill (77)

whose solution is a combination of generalized hypergeometric functions multiplied by polynomials:

$$R(x) = C_1 F_{pq} \left[ \begin{array}{c} \{1, 3 \} \ \{1/2, 3/4 \} \ \{1/16, \gamma^2 x^2 \} \\ \{5/12, 3/4, 1/16 \} \\ \{1/2, 1/4 \} \end{array} \right]$$

$$+ C_2 \sqrt{x} F_{pq} \left[ \begin{array}{c} \{5/12, 3/4, 1/16 \} \\ \{1/2, 1/4 \} \end{array} \right]$$

$$+ C_3 \gamma^2 x^2 F_{pq} \left[ \begin{array}{c} \{2, 3/4 \} \ \{1/2, 1/4 \} \ \{1/16, \gamma^2 x^4 \} \\ \{3, 3, 1 \} \ \{1/2, 1/4 \} \end{array} \right]$$  \hfill (78)

where $C_i$'s are constants to be determined and $F_{pq}$ are the generalized Hypergeometric functions. Thus, the solution to the differential equation (73) is

$$f(x) = C_1(k) x$$

$$+ C_2(k) x^{2/3} F_{pq} \left[ \begin{array}{c} \{-1/12, 1/6 \} \ \{1/2, 3/4, 1/16 \} \ \{1/16, \gamma^2 x^4 \} \\ \{5/12, 3/4, 1/16 \} \ \{1/2, 1/4 \} \end{array} \right]$$

$$+ C_3(k) x^{5/3} F_{pq} \left[ \begin{array}{c} \{5/12, 3/4, 1/16 \} \\ \{1/2, 1/4 \} \end{array} \right]$$

$$+ C_4(k) x^{8/3} F_{pq} \left[ \begin{array}{c} \{5/12, 3/4, 1/16 \} \\ \{1/2, 1/4 \} \end{array} \right]$$  \hfill (79)

where $C_i$'s are related to $C_i$'s. These generalized hypergeometric functions have a few properties which are worth mentioning. First, they are highly oscillating. For example, the function

$$F_{pq} \left[ \begin{array}{c} \{-1/12, 1/6 \} \ \{1/2, 3/4, 1/16 \} \ \{1/16, \gamma^2 x^4 \} \\ \{5/12, 3/4, 1/16 \} \ \{1/2, 1/4 \} \end{array} \right]$$  \hfill (80)

goes from $7.7 \times 10^{32}$ to $1.4 \times 10^{194}$ when $x$ goes from $x = 50$ to $x = 100$, fixing $\gamma = 1/10$ for illustrative purposes.

Let us now say a few words about these generalized hypergeometric functions; this will help us to quantify
their oscillatory behavior. They are special cases of Meijer functions which can be defined by integrals on the complex plane [56]:

\[ G_{p,q}^{m,n}(z|a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q) = \frac{1}{2\pi i} \int_C \chi(s) z^{-s} ds \quad (81) \]

where

\[ \chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{j=m+1}^{n} \Gamma(1 - b_j - s) \prod_{j=n+1}^{\infty} \Gamma(a_j + s)} \quad (82) \]

and three possibilities are allowed for the contour \( C \), according to some conditions on the parameters \( a_i, b_j, m, n, p, q \) [56]. Our solutions correspond to \( m = n = 0 \).

The asymptotic behavior which is relevant here is the following. For large values with \( -\nu^* + 1)\pi < \arg z < 0 \), the dominant part is roughly given by

\[ G_{p,q}^{m,n}(z|a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q) \sim \rho(z) \exp(i\pi \mu^*) \quad (83) \]

where

\[ \mu^* = q - m - n, \quad \nu^* = -p + m + n, \]

\[ H_{pq}(z) = \exp((p - q)z \frac{1}{\pi} \gamma), \quad \text{and} \]

\[ \rho^* = \frac{1}{q - p} \left( \sum_{j=1}^{p} b_j - \sum_{j=1}^{p} a_j + p - q + \frac{1}{2} \right) \quad (84) \]

In our case, one has to make the replacements

\[ z = -\frac{1}{16} \gamma^2 k^4 \eta^4, \quad q = 3, \quad p = 2 \quad (85) \]

so that

\[ \Phi_{k,0}(\eta) = C_0(k) k \eta + \sum_{i=1}^{3} C_i(k) (k \eta)^{\sigma_i} \]

\[ \times \left( -\frac{1}{16} \gamma^2 k^4 \eta^4 \right)^{\mu^*_i} \exp \left( \frac{1}{16} \gamma^2 k^4 \eta^4 \exp(i\pi \mu^*_i) \right), \quad (86) \]

where \( \sigma_1 = 2/3, \sigma_2 = 5/3, \sigma_3 = 8/3 \). From the above expression, it is easy to see that the solution in the Bardeen potential \( \Phi \) is oscillating in this region. The choice of the constants \( C_i(k) \) correspond to different choices of initial conditions and thus, in principle, to different choices of vacua. We will come back to this later.

2. The second region

In the intermediary region, 1 dominates over \( \gamma^2 x^2 \). The solution in this region is

\[ f(x) = D_1(k) x + D_2(k) x^2 \]

\[ + D_3(k) \left[ e^{-ix} (-1 + ix) - x^2 Ei(-ix) \right] \]

\[ + D_4(k) \left[ e^{ix} (i - x) + ix^2 Ei(ix) \right] \quad (87) \]

where \( Ei(x) \) refers to the exponential integral. Using the asymptotic behavior of the exponential integral (cf. Ref. [57], p. 231), we get

\[ f(x) = \frac{1}{2} \left[ H_1(k) x + \frac{1}{2} H_2(k) (2 - 2x + x^2) \right] \]

\[ + \frac{1}{2} H_3(k) \left( e^{-ix} x - iEi(-ix) \right) \]

\[ + \frac{1}{2} H_4(k) \left( e^{ix} x - iEi(ix) \right) \quad (88) \]

As we can see, the Bardeen potential is a sum of plane-waves.

3. The third region

When the term \( -2/x^2 \) dominates in the coefficient of the second derivative, the solution can be found and is given by

\[ f(x) = G_1(k) + G_2(k) x + G_3(k) x^2 + G_4(k) x \ln x \quad (89) \]

From the above expression, we see that in the super-Hubble scales the scalar perturbations has a constant term which is identical to the canonical scalar field inflation.

To finish this section, let us remark that in the non-Planckian region, i.e when

\[ 1 - \frac{2}{x^2} >> \gamma^2 x^2, \]

the solution to the differential equation (73) can be obtained and is given by

\[ f(x) = H_1(k) x + \frac{1}{2} H_2(k) (2 - 2x + x^2) \]

\[ + \frac{1}{2} H_3(k) \left( e^{-ix} x - iEi(-ix) \right) \]

\[ + \frac{1}{2} H_4(k) \left( e^{ix} x - iEi(ix) \right) \quad (90) \]

this approximation covers the II and III region simultaneously.

C. The first order approximation

The first order contribution obeys the equation

\[ \Phi_{k,1}^{(4)}(\eta) + \left( k^2 - \frac{2}{\eta^2} + \gamma^2 k^4 \eta^2 \right) \Phi_{k,1}^{(2)}(\eta) - \frac{2k^2}{\eta} \Phi_{k,1}'(\eta) \]

\[ + 2\frac{k^2}{\eta^2} \Phi_{k,1}(\eta) = S_k(\eta), \quad (91) \]
Let us concentrate on the second and third region for

tions that this can be achieved by the following system of equa-

\[ \Phi = \sum_{a=1}^{4} L_a Y_a + \frac{b_{11} M_p^2 \nu_0^4 \eta^2}{d_1} k^2 \eta \]

This equation is exactly the one obeyed by the zeroth order contribution, except for the source term which is known since we obtained the approximations of the zeroth order in the three regions. Let us specialize to one of the regions and call \( Y_1(\eta) \), \( Y_2(\eta) \), \( Y_3(\eta) \), \( Y_4(\eta) \) the four different solutions of the homogeneous equation given in Eq (71):

The equation being linear and knowing the complete solution of the homogeneous equation \( \Phi = \sum_{a=1}^{4} L_a Y_a \) (with \( L_a \) constants) solution, we can solve it using the method of the variation of the constants. One can show that this can be achieved by the following system of equations

\[ \sum_{a=1}^{4} L'_a Y_a = 0 \quad \sum_{a=1}^{4} L'_a Y_1 = 0 \]

\[ \sum_{a=1}^{4} L'_a Y_3 = 0 \quad \sum_{a=1}^{4} L'_a Y_2 = S(\eta). \quad (92) \]

Let us concentrate on the second and third region for example (the non trans-Planckian zone). One has

\[ \Phi_{k,1}(\zeta) = k \zeta \int_0^\zeta d\eta \left[ \frac{2i + 2k \eta - ik^2 \eta^2}{2k^2 \eta} Ei(ik \eta) \right. \]

\[ \frac{-2i + 2k \eta + ik^2 \eta^2}{2k^2 \eta} Ei(-ik \eta) \left. S(\eta) \right] \]

\[ + \left[ 1 - k \zeta + \frac{1}{2} k^2 \zeta^2 \right] \int_0^\zeta d\eta \frac{1}{k^4 \eta} S_k(\eta) \]

\[ \frac{-1}{2} i e^{-ik \zeta} + \frac{1}{2} x Ei(-ik \zeta) \]

\[ \times \int_0^\zeta d\eta j \frac{-(2 + 2ik \eta + k^2 \eta^2)}{k^4 \eta} e^{ik \eta} S(\eta) \]

\[ + \frac{1}{4} i e^{ik \zeta} - \frac{1}{4} i k \zeta Ei(ik \zeta) \]

\[ \times \int_0^\zeta d\eta j \frac{2(2 - 2ik \eta + k^2 \eta^2)}{k^4 \eta} e^{-ik \eta} S(\eta). \quad (93) \]

A similar treatment can be applied to the trans-

Planckian region but the formulas are too lengthy and

will not be recorded here.

Using the analysis discussed in this section, the power spectrum of the perturbations can be obtained upto a \( k \) dependent constant factor. In order to obtain the exact power spectrum, we need to quantize the theory and fix the initial state of the field [30]. In the following section, we obtain exact power spectrum of the perturbations in a particular limit.

**VII. POWER-SPECTRUM OF THE PERTURBATIONS – QUANTUM ANALYSIS**

In this section, we calculate the power-spectrum corresponding to \( \mu_s \) during the power-law inflation using the following approach: (i) We assume that the quantum field \( \mu_s \) is coupled to an external, classical source field \( S_k(\eta) \) which is determined by solving the coupled differential equations (55, 57). (ii) We solve the equation of motion of \( \mu_s \) in three regions – Trans-Planckian (I), linear (II) and super-Hubble (III) – separately [4]. We further assume that \( S_k(\eta) \) will contribute significantly in the trans-Planckian region while it can be neglected in the linear and super-Hubble region. (iii) The power-spectrum at the super-Hubble scales is determined by performing the matching of the modes and its derivatives at the times of transition between regions I and II \( - \eta \eta_0^{1+\beta} \equiv (\omega k^2)^{-1/2} \) and regions II and III \( \eta_0 \equiv (1 + \beta)/k \). We assume that the quantum field \( \mu_s \) is in a minimum energy state at \( \eta = \eta_0 \) [58].

Region (I) corresponds to the limit where the non-linearities of the dispersion relation play a dominant role, i. e. \( 2 b_{11}/k/a(\eta)^2 \gg 1 \) and \( k \eta \gg 1 \). Region (II) corresponds to the limit where the non-linearities of the modes are negligible i. e. \( \omega \simeq k \) and \( k \eta \gg 1 \). Region (III) corresponds to the limit where \( k \eta \ll 1 \). In three regions, the equation of motion of \( \mu_s \) (62) reduces to:

\[ \mu_s^{(I)} + \omega^2(\eta) \mu_s^{(I)} \simeq \frac{-2d_1}{M_{\text{pl}}} k^2 S_k(\eta), \quad (94a) \]

\[ \mu_s^{(II)} + k^2 \mu_s^{(II)} \simeq 0, \quad (94b) \]

\[ \mu_s^{(III)} - \frac{\beta(1+\beta)}{\eta^2} \mu_s^{(III)} \simeq 0, \quad (94c) \]

where

\[ \omega(\eta) = \omega_0 \frac{k^2}{(-\eta)^{1+\beta}}; \quad \omega_0 = (2b_{11})^{1/2}(-\eta_0)^{(1+\beta)}, \quad (95) \]

and \( S_k \) is given by Eq. (64). The general solution to the differential equation (94a) is given by

\[ \mu_s^{(I)}(\eta) = A_1(k) (-\eta)^{1/2} H_v^{(1)}(\alpha(\eta)) \]

\[ + A_2(k) (-\eta)^{1/2} H_v^{(2)}(\alpha(\eta)) + \mu_p(\eta), \quad (96) \]

where \( \mu_p(\eta) \) is the particular solution to the inhomogeneous part of the differential equation and is given by (cf. Ref. [59], p. 529)
\[ \mu_\nu (\eta) = \frac{i\pi}{2} \frac{d_1 k^2}{\beta M_{Pl}^2} (-\eta)^{1/2} \left[ H^{(1)}_\nu (\alpha(\eta)) \int_{\eta_1}^{\eta} (-s)^{1/2} H^{(2)}_\nu (\alpha(s)) S_\alpha(s) ds + H^{(2)}_\nu (\alpha(\eta)) \int_{\eta_1}^{\eta} (-s)^{1/2} H^{(1)}_\nu (\alpha(s)) S_\alpha(s) ds \right] \] (97)

and \( \nu = \frac{1}{2\beta} \); \( \alpha(\eta) = \alpha_0 (-\eta)^{-\beta} \); \( \alpha_0 = \omega_0 k^2 / -\beta \), (98)

\( \eta_i < \eta_i \) is the epoch in which the integrals in (97) vanish. The quantities \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) in the above solution are the Hankel functions of the first and the second kind (of order \( \nu \)), respectively, and the \( k \)-dependent constants \( A_1(k) \) and \( A_2(k) \) are to be fixed by the initial conditions for the modes at \( \eta_i \). Unlike the canonical scalar field inflation, where one assumes that the field is in a Bunch-Davies vacuum at \( \eta_i \), it is not possible to assume such an initial condition due to the non-linearities of the modes. As mentioned earlier, we assume that the field is in the minimum energy vacuum state at \( \eta_i \), i.e.,

\[ \mu_S(\eta_i) = \frac{1}{\sqrt{2\omega(\eta_i)}} \; \mu_S' (\eta_i) = \pm i \sqrt{\frac{\omega(\eta_i)}{2}}. \] (99)

We thus get

\[ A_1(k) = \frac{i\pi\alpha(\eta_i)}{4} (-\eta_i)^{-1/2} \mu_S(\eta_i) H^{(2)}_{\nu - 1}[\alpha(\eta_i)] \times \left[ 1 + (\eta_i)^{\beta+1} \mu_S(\eta_i) H^{(2)}_{\nu - 1}[\alpha(\eta_i)] \right] \] (100a)

\[ A_2(k) = -\frac{i\pi\alpha(\eta_i)}{4} (-\eta_i)^{-1/2} \mu_S(\eta_i) H^{(1)}_{\nu - 1}[\alpha(\eta_i)] \times \left[ 1 + (\eta_i)^{\beta+1} \mu_S(\eta_i) H^{(1)}_{\nu - 1}[\alpha(\eta_i)] \right] \] (100b)

\[ \frac{\exp(-ik\eta_i)}{(-\eta_i)^{1/2}} B_1 = \frac{A_1}{2} H^{(1)}_\nu [\alpha(\eta_i)] \left[ 1 + \frac{i\beta \alpha_0}{k(-\eta_i)^{\beta+1}} H^{(1)}_{\nu - 1}[\alpha(\eta_i)] \right] + \frac{A_2}{2} H^{(2)}_\nu [\alpha(\eta_i)] \left[ 1 + \frac{i\beta \alpha_0}{k(-\eta_i)^{\beta+1}} H^{(2)}_{\nu - 1}[\alpha(\eta_i)] \right], \] (102a)

\[ \frac{\exp(ik\eta_i)}{(-\eta_i)^{1/2}} B_2 = \frac{A_1}{2} H^{(1)}_\nu [\alpha(\eta_i)] \left[ 1 - \frac{i\beta \alpha_0}{k(-\eta_i)^{\beta+1}} H^{(1)}_{\nu - 1}[\alpha(\eta_i)] \right] + \frac{A_2}{2} H^{(2)}_\nu [\alpha(\eta_i)] \left[ 1 - \frac{i\beta \alpha_0}{k(-\eta_i)^{\beta+1}} H^{(2)}_{\nu - 1}[\alpha(\eta_i)] \right]. \] (102b)

In region III, the solution is

\[ \mu^{(III)} = C(k) \alpha(\eta), \] (103)

where \( C(k) \) is constant (not to be confused with the constants used in the previous section) whose modulus square gives the power spectrum of the density perturbations and is determined by performing the matching of the modes \( \mu_S^{(I)} \) and \( \mu_S^{(II)} \) at \( \eta_i \equiv (1 + \beta)/k \). We thus get,

\[ C(k) = \frac{\eta_i k}{1 + \beta} \left[ B_1(k) \exp(-ik\eta_i) + B_2(k) \exp(ik\eta_i) \right]. \] (104)

The spectrum of the perturbations (25) reduce to

\[ [k^3 P_S(k)] = \left( \frac{1}{4\pi^2 M_{Pl}^2} \frac{(\beta + 1)}{\beta} \right) k^3 |C(k)|^2. \] (105)

We are interested in the leading order behavior of the primordial power-spectrum and the possible modifications to the primordial spectrum due to the trans-Planckian effects. In order to do that, we need to obtain the leading order behavior of the constants \( A_1, A_2, B_1 \) and \( B_2 \). Using the fact that \( k\eta \gg 1 \) and the asymptotic behavior
of the Hankel functions, viz. (cf. Ref. [57], p. 364)

$$\lim_{z \to \infty} H_{\nu}^{(1/2)}(z) \to \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z-(\pi\nu/2)-(\pi/4))},$$

we get

$$A_1(k) \approx A_0 \mu_s(\eta) \exp(-ix_1) (1 + \mathcal{I}) ,$$

$$A_2(k) \approx A_0 \mu_s(\eta) \exp(ix_1) (1 + \mathcal{I}),$$

where

$$A_0 = \left(\frac{\pi a_0}{\sqrt{8}}\right)^{1/2} (\eta)^{-(\beta+1)/2}; \quad x_1 = \alpha(\eta) - \frac{\pi \nu}{2} - \frac{\pi}{4},$$

$$\mathcal{I} = \frac{1 - \mu'_s(\eta)/\mu_s(\eta)}{1 - \mu'_s(\eta)/\mu_s(\eta)}. \quad (108)$$

Having obtained $A_1, A_2$ in the limit of $k\eta \gg 1$. Our next step is to evaluate $B_1(k), B_2(k)$ in the same limit. In order to do that, we need to know the correct matching time $\eta_m$. Demanding $\omega^2(\eta_m) = k^2$ gives $(-\eta_m)^{1+\beta} = \omega^{-1/2}$ and $k$. We thus get,

$$B_1 \approx A_1 \left(\frac{-2\beta}{\pi k}\right)^{1/2} \exp(ix_{P}) \quad (109)$$

$$B_2 \approx A_2 \left(\frac{-2\beta}{\pi k}\right)^{1/2} \exp(-ix_{P}) \quad (110)$$

where $x_{P} = k\eta_P(\beta+1)/\beta - \pi \nu/2 - \pi/4$. Thus, we get,

$$[\xi^{3}{P}_S(k)] \approx C_0 k^{2(\beta+2)} \left[1 - \frac{\mu'_s(\eta)}{\mu_s(\eta)}\right] \left[1 + 2 \cos(x_{\eta}) - 2 Im[\mathcal{I}] \sin(x_{\eta})\right].$$

where

$$C_0 = \left(\frac{1}{16\pi^2 M_p^2}\right)^{\beta+1} \left(\frac{\eta_0 \eta}{1 + \beta}\right)^{2(1+\beta)};$$

$$x_{\eta} = (1 + \beta - x_{P} + x_{1}). \quad (112)$$

and we have neglected higher order terms like $|\mathcal{I}|^2$. It is interesting to note that in the limit of $S_k(\eta) \to 0$, the power-spectrum is same as that of the standard power-law inflation spectrum with small oscillations. In this limit, we recover the result of Refs. [4, 7]. In order to obtain the exact form of the power-spectrum, we need to evaluate $\mu_{\varphi}$ which requires the knowledge of $S_k(\eta)$.

In the rest of this section, we evaluate the power-spectrum in a particular limit $(1/c_1 \to 0)$. We, first, obtain the form of $S_k(\eta)$ by solving the system of coupled differential equations (55, 57) in two - sub-Hubble and super-Hubble - regimes. As mentioned earlier, the two differential equations (55, 57) do not contain higher order spatial derivatives. Hence, it is sufficient to obtain solutions in these two regimes. Performing the following transformations

$$u = \frac{a(\eta)}{\varphi(\eta)} \Phi; \quad \xi^{(gi)} = a^{3/2}(\eta) \tilde{\xi}, \quad (113)$$

and taking the Fourier transform, Eqs. (55, 57), reduce to

$$u'' + \left(c_2^2 k^2 - \frac{\theta''}{\theta}\right) u_k = -\frac{2 d_1 k^2}{M_p^{2}\eta^2} \left(a^{3/2} \tilde{\xi}_k\right)' \quad (114)$$

$$\tilde{\xi}'' + \left[\frac{-9}{4} \xi' + \frac{3}{2} \xi' + c_1 k^2 \frac{\varphi'^2}{a^2 M_p^2 \eta^2}\right] \xi \quad (115)$$

$$= a^{1/2} \left[1 + \frac{c_1 k^2}{a^2 M_p^2 \eta^2}\right] \Phi_k - 2 \eta \left[1 - \frac{c_1 k^2}{2a^2 M_p^2 \eta^2}\right] \Phi_k. \quad (116)$$

In the limit of $1/c_1 \to 0$ (i.e., $d_1/M_p^2 \ll b_1 M_p^2$), the above differential equations can be solved exactly. In this limit, the above differential equations become:

$$u_k'' + \left(k^2 - \frac{\theta''}{\theta}\right) u_k = 0 \quad \frac{\varphi'^2}{2M_p^2} \xi_k^{(gi)} = a(\varphi' u_k)'. \quad (117)$$

For the sub-Hubble scales, during the power-law inflation, we get

$$u_k = D_1(\eta) \exp(-i\eta) + D_2(\eta) \exp(i\eta)$$

$$\xi_k^{(gi)} = i k \left((\eta) (\beta + 2) + \left(\frac{2 M_p^2}{\beta + 1}\right) \right) \left[D_1(\eta) \exp(i\eta) - D_2(\eta) \exp(-i\eta)\right], \quad (117)$$

where we have neglected the terms of the order $1/(\eta)$ and $D_1(\eta), D_2(\eta)$ are $k$-dependent constants with the dimensions of length squared ($k^{-2}$). Using the condition that the modes are outgoing, we set $D_2(\eta) = 0$. In the super-Hubble scales, we have

$$u_k \approx D_3(\eta) a(\eta); \quad \xi_k^{(gi)} = D_3(\eta) \sqrt{\frac{2}{\beta + 1}} a^2(\eta), \quad (118)$$

where $D_3(\eta)$ is a constant. In the sub-Hubble scales, we have

$$S_k(\eta) = 4 i M_p^2 b_1 d_1(\eta) k^5 \left(\frac{-\eta}{\eta - \eta}\right)^{\beta+1} \exp(-i\eta). \quad (119)$$

Our next task is to obtain $\mu_{\varphi}$ and the power-spectrum of the scalar perturbations. From Eq. (97) using the asymptotic limit of Hankel functions, we get

$$\mu_s(\eta) = \left(\frac{b_1}{\beta^2}\right)^{\beta/2} \left(\frac{-\eta}{\eta - \eta}\right)^{\beta+1} M_p^2 k^{1/\beta}$$

$$\times \cos\left[\alpha(\eta) + \ln\left(\left(\frac{\beta - 1}{2\beta}\right) a(\eta)\right)\right], \quad (120)$$

where we have set $D_1 \propto 1/k^2$. Substituting the above expression in Eq. (111), we get

$$[k^3 P_S(k)] = C_0 k^{2(\beta+2)} \left[1 - C_1 k^{(1+\beta)}\right]^2 \times [1 + 2 \cos(x_{\eta}) - 2 Im[\mathcal{I}] \sin(x_{\eta})], \quad (121)$$
In this work, we have computed the gauge-invariant cosmological perturbation for the single scalar field inflation with the trans-Planckian effects introduced via the Jacobson-Corley dispersion relation. Even though the dispersion relation breaks the local Lorentz invariance, a covariant formulation of the corresponding theory can be carried out by introducing a unit time-like vector field.

Using the covariant Lagrangian, we have obtained the perturbed stress-tensor for the scalar and tensor perturbations around the FRW background. We have shown the following: (i) The non-linear effects introduce corrections to the perturbed energy density while the other components of the perturbed stress-tensor remains unchanged. Thus, for the trans-Planckian scenario, we have shown that $\Phi = \Psi$ and the constraint equation (17b) remains unchanged. (ii) The non-linear terms contributing to the stress-tensor are proportional to $k^2$ and hence in the super-Hubble scales, as expected, the contribution to the perturbed energy density can be ignored. (iii) The spatial higher derivative terms appear only in the equation of the motion of the perturbed inflation field ($\delta \varphi$) while the speed of propagation of the perturbations [in the equation of motion of the scalar perturbations ($\Phi$)] is different from that of the standard inflation. (iv) The speed of propagation of the perturbations ($c_s^2$) is different from that of the canonical single scalar field inflation. (v) The perturbations are not purely adiabatic. $\xi$ act as an extra scalar field during inflation and hence can act as a source. This introduces non-adiabatic (entropic) perturbations. (vi) Since, the trans-Planckian corrections do not change the pressure perturbations, the perturbation equations for the tensor modes do not change. Hence, the tensor perturbation equation remain unchanged.

Following points are to be noted regarding the above results: (i) We have obtained the general power-spectrum (111) of the scalar perturbations assuming that the scalar field is in the minimum energy state and that the contribution of the unit vector field to the energy density can be neglected. We have shown that the power spectrum depends on the form of the source term $S_k$ which can be solved analytically in some particular limits. (ii) We have computed the power-spectrum of perturbations in a particular limit i.e. $(1/c_1 \rightarrow 0)$. In this limit, we recover the result of Refs. [4, 7].

VIII. DISCUSSION AND CONCLUSION

In this work, we have computed the gauge-invariant cosmological perturbation for the single scalar field inflation with the trans-Planckian effects introduced via the Jacobson-Corley dispersion relation. Even though the dispersion relation breaks the local Lorentz invariance, a
to an external, classical source field $S_b(\eta)$ which is determined by solving the coupled differential equations (55, 57) (ii) the quantum field is initially in a minimum energy state and (iii) $d_1/M_{Pl}^2 \ll b_{i_1}M_{Pl}^2$. We have shown that in this particular limit, the power-spectrum is same as that obtained in Refs. [4, 7].

The work suggests various possible directions for further study:

- We have obtained the power-spectrum analytically in the limit of $d_1/M_{Pl}^2 \ll b_{i_1}M_{Pl}^2$. The trans-Planckian corrections in this limit are small to be observed in the present or the future CMB experiments. It would be interesting to obtain the power-spectrum by solving the system of differential equations numerically and obtain the leading order trans-Planckian corrections in these models.

- As we have mentioned earlier, this model introduces non-adiabatic perturbations which can lead to the non-Gaussianity in the CMB. Recently in Refs. [31–35], trans-Planckian constraints from the CMB was studied in detail. It would be interesting to do a similar analysis for this scenario. The non-Gaussian signatures may place stringent and independent constraints on the parameters $b_{i_1}, d_1$.

- In this work, we have ignored the solenoidal part of the perturbed $u$ field. The solenoidal part contributes to the vector perturbations. It would be interesting to see whether the solenoidal part of the perturbed unit-time like vector field can lead to the growing large-scale vorticity and hence the production of large-scale primordial magnetic field.

- In this work, we have ignored the back-reaction of the field excitations on the perturbed FRW background. There have been claims in the literature [6, 9] that trans-Planckian modes may effect the evolution of cosmological fluctuations in the early stages of cosmological inflation in a non-trivial way. In Ref. [24], the authors have discussed in detail the backreaction problem of the trans-Planckian inflation in a toy model and have shown that the backreaction of the trans-Planckian modes may lead to a renormalization of the cosmological constant driving inflation. It would be interesting to perform a similar analysis for this model.

We hope to return to study some of these issues in the near future.

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APPENDIX A: THE BACKGROUND.

In this appendix, we give key steps in obtaining the stress-tensor corresponding to the two corrective Lagrangians (29a, 29b), and the equations of motion of the scalar field and the unit vector field. Having obtained these, we discuss their properties in the FRW background.

This question has been addressed in Ref. [11]. Our treatment differs with the one followed in that paper by the fact that we do not make the decomposition in time and space-like components. Our condensed formulas will prove very useful when computing the perturbations.

In order to do that, it proves easier to go back to the action. Let us first specialize to the contribution of the non-linear part of the Lagrangian (29a):

$$S_{\text{nc}} = -b_1 \int d^4x \sqrt{-g} (D^2 \varphi)^2 \ . \ (A1)$$

Using the definition given in Eq.(31), the variation of $D^2 \varphi$ can be written as

$$\delta (D^2 \varphi) = \tilde{A}_{\mu \nu} \delta g^{\mu \nu} + \tilde{B}_{\mu \nu} \partial_{\sigma} \delta g^{\mu \nu} + \tilde{C}_{\mu} \delta u_{\mu} + \tilde{D}_{\mu \nu} \partial_{\sigma} \delta u_{\mu} \ . \ + \tilde{E}_{\mu} \partial_{\sigma} \varphi \ + \tilde{F}_{\mu \nu} \partial_{\nu} \partial_{\sigma} \varphi \ . \ \ (A2)$$

The quantities $\tilde{A} \cdots \tilde{H}$ can be written explicitly. For example,

$$\tilde{A}_{\mu \nu} = \frac{\partial}{\partial g_{\mu \nu}} D^2 \varphi \ , \ \tilde{H}_{\mu \nu} = \frac{\partial}{\partial (\partial_{\mu} \partial_{\nu} \varphi)} D^2 \varphi \ . \ \ (A3)$$

These partial derivatives have to be kept consistently in mind the choice of variables made in this work. In order to be consistent, we choose the following set of independent variables

$$\varphi, \partial_{\beta} \varphi, \partial_{\alpha} \partial_{\beta} \varphi, \partial_{\alpha} \partial_{\sigma} \partial_{\beta} \varphi, g^{\mu \nu}, \partial_{\lambda} g^{\mu \nu}, \partial_{\beta} g^{\mu \nu}, \partial_{\sigma} g^{\mu \nu} \ , \ u_{\sigma}, \partial_{\beta} u_{\sigma}, \partial_{\alpha} \partial_{\beta} u_{\lambda} \ . \ \ (A4)$$

Using the relations

$$\partial_{\alpha} (g_{\kappa \sigma} g^{\kappa \sigma}) = 0 \ ; \ \frac{\partial}{\partial g_{\alpha \beta}} (g_{\kappa \sigma} g^{\kappa \sigma}) = 0 \ , \ \ (A5)$$

we get,

$$\frac{\partial g^{\alpha \beta}}{\partial g^{\sigma \tau}} = \delta^{\alpha}_\sigma \delta^{\beta}_\tau \ , \ \frac{\partial g_{\alpha \beta}}{\partial g^{\sigma \tau}} = -g_{\rho \sigma} g_{\beta \tau} \ , \ \partial_{\kappa} g_{\alpha \beta} = -g_{\alpha \beta} \partial_{\kappa} g^{\rho \sigma} \ . \ \ (A6)$$

The expressions giving the quantities $\tilde{A} \cdots \tilde{H}$ are very long and time consuming to obtain and will not be displayed here. One of the most important aspects of this way we will present our results is that for our purposes we only need to know their values on the FRW background. This is the subject of the next appendix.
Substituting Eq. (A2) in (A1) and integrating by parts the resultant expression, we obtain
\[ \delta S_{cor} = -b_{11} \int d^4x \left[ \sqrt{-g}\left( \frac{1}{2} g_{\mu\nu}(D^2 \varphi)^2 - 2 \tilde{\Lambda}_{\mu\nu} D^2 \varphi \right) + \partial_\nu \left( \sqrt{-g} D^2 \varphi \tilde{B}_{\mu\nu} \right) \right] \delta g^{\mu\nu} 
+ 2b_{11} \int d^4x \left[ \sqrt{-g} D^2 \varphi \tilde{C}^{\mu} - \partial_\nu \left( \sqrt{-g} D^2 \varphi \tilde{D}^\nu \right) \right] \delta u_\mu 
- 2b_{11} \int d^4x \left[ -\partial_\nu \left( \sqrt{-g} D^2 \varphi \tilde{E}^\nu \right) \right] \delta \varphi . \]  

(A7)

From the above expression, it is easy to infer the contribution of the corrective Lagrangian to the stress-energy tensor as well as to the field equations of the inflaton and the vector field \( u_\mu \), i.e.,
\[ T^{(cor)}_{\mu\nu} = -b_{11} g_{\mu\nu}(D^2 \varphi)^2 - 4b_{11} \tilde{B}_{\mu\nu} \partial_\nu(D^2 \varphi) \]
\[ + b_{11} \left( 4\tilde{\Lambda}_{\mu\nu} - 2\frac{\partial_\eta \tilde{B}_{\mu\rho} - 4\partial_\mu \tilde{B}_{\rho\nu}}{g} \right) D^2 \varphi \]
\[ \equiv b_{11} g_{\mu\nu}(D^2 \varphi)^2 + b_{11} E_{\mu\nu} D^2 \varphi + 4b_{11} C_{\mu\nu} \partial_\nu \varphi \partial_\nu [D^2 \varphi], \]

(eq. \( \varphi \))
\[ + \frac{2b_{11}}{\sqrt{-g}} \left[ \partial_\nu \left( \sqrt{-g} E^\nu \right) - 2\tilde{\Lambda}_0 \partial_\nu \left( \sqrt{-g} H^{\mu\nu} \right) \right] D^2 \varphi \]
\[ + \frac{1}{\sqrt{-g}} \left[ \sqrt{-g} E^\nu - \partial_\nu \left( \sqrt{-g} H^{\mu\nu} \right) - \partial_\nu \left( \sqrt{-g} H^{\mu\nu} \right) \right] \partial_\nu D^2 \varphi \]
\[ - \sqrt{-g} \tilde{H}_{\mu\nu} \partial_\nu \partial_\lambda D^2 \varphi , \]

(A9)

\[ \delta T^{(cor)}_{\mu\nu} = \left[ 4\tilde{\Lambda}_{\mu\nu} - 2\frac{\partial_\eta \tilde{B}_{\mu\rho} - 4\partial_\mu \tilde{B}_{\rho\nu}}{g} \right] \delta D^2 \varphi \]
\[ - 4 \left( \frac{\tilde{B}_{\mu\nu}^{\alpha}}{\alpha} \right) \partial_\rho \delta D^2 \varphi . \]

(B2)

From the above expression, we see that it is enough to know the quantities \( A, \cdots, \tilde{H} \) on the background. Going back to Eq. (A2), we have to compute the variation of the vector field, the Christoffel symbol and other quantities. Let us begin by the connection coefficient:
\[ \Gamma^{\alpha}_{\rho\sigma} = \frac{1}{2} g^{\alpha \tau} (-\partial_\tau g_{\rho\sigma} + \partial_\rho g_{\sigma\tau} + \partial_\sigma g_{\tau\rho}) . \]

(B3)

Due to our choice of independent variables, we need to express the variations of the covariant components of the metric in terms of the contravariant ones. This is achieved simply:
\[ \delta (g_{\alpha \beta} g^{\beta \gamma}) = 0 \implies \delta g_{\alpha \beta} = -g_{\alpha \gamma} g^{\beta \sigma} \delta g_{\gamma \sigma} . \]

(B4)

One then obtains
\[ \delta T^{\alpha}_{\rho\sigma} = a \left( -\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\nu^\rho \right) \delta_{\mu\sigma} + 2\delta_\mu^\rho \delta_\nu^\sigma \delta_{\mu\sigma} + \frac{a^2}{2} \left( \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} \right) \delta_{\mu\rho} \delta_{\nu\sigma} . \]

(B5)
Similarly, we get
\[ \delta \rightarrow^{\mu \nu} = \left( -\delta^{\mu}_{\nu} \delta_{\gamma}^{\lambda} + \delta^{\mu}_{\gamma} \delta_{\nu}^{\lambda} + \delta_{\mu}^{\lambda} \delta_{\nu}^{\gamma} + \delta_{\nu}^{\lambda} \delta_{\mu}^{\gamma} \right) \delta g^{\mu \nu} + \left( \delta^{\mu}_{\nu} \eta^{\rho \sigma} + \delta_{\mu}^{\rho} \eta_{\nu}^{\sigma} \right) \delta u_{\mu} \].  
(B6)

After a long but straightforward calculation, one obtains the result given in Eqs. (42, 43) with the following expressions for the background:
\[ (\tilde{A}_{\mu \nu})_{\mu} = \left( \tilde{\varphi} \rightarrow^{\mu \nu} + 3 \frac{a}{a} \tilde{\varphi} \right) \delta^{\mu \nu}_{\sigma}, \quad (\tilde{B}_{\mu \nu})_{\mu} = \frac{1}{2} \tilde{\varphi} \delta^{\mu \nu}_{\sigma}, \delta_{\nu}^{\sigma} \]
\[ (\tilde{C}^{\mu})_{\mu} = \frac{1}{a^{3}} \left( 2 \tilde{\varphi} \rightarrow^{\mu \nu} + \frac{a}{a} \tilde{\varphi} \right) \delta^{\mu \nu}_{\sigma}, \quad (\tilde{D}^{\mu \nu})_{\mu} = \frac{1}{a^{3}} \tilde{\varphi} \eta^{\mu \nu}, \]
\[ (\tilde{H})_{\mu} = \frac{1}{a^{3}} \left( -\eta^{\mu \nu} + \delta^{\mu \nu}_{\sigma} \delta^{\alpha \beta}_{\sigma} \right), \quad (\tilde{E})_{\mu} = 0. \]  
(B7)

At this point it is worth noticing that \( 0 - 0 \) is the only non-zero component of the stress-tensor. Since the trans-Planckian corrections do not change the pressure perturbations, the perturbation equation for the tensor modes do not change. Hence, the tensor perturbation equations is given by Eq. (26). Recently, Lim [43] had show that general Lorentz violating models (with out taking into account higher derivatives of the scalar field) can modify the pressure perturbations and hence the tensor perturbation equations. However, in our specific Lorentz violating model, this is not the case.

Let us now specialize to the scalar perturbations. The covariant components read
\[ \delta g^{\mu \nu} = a^{2}(\eta) \left( \begin{array}{cc} 2\phi & -\partial_{\mu}B \\ -\partial_{\nu}B & 2\psi \delta^{ij} - 2\partial_{i}\partial_{j}E \end{array} \right) \].  
(B8)

Using the identities given in Eq. (A5), we get the contravariant components
\[ \delta g^{00} = -\frac{2\phi}{a^{2}(\eta)}, \quad \delta g^{0i} = \frac{\partial_{i}B}{a^{2}(\eta)}, \quad \delta g^{ij} = \frac{(2\partial_{i}\partial_{j}E - 2\psi \delta_{ij})}{a^{2}(\eta)}. \]  
(B9)

Using the fact that \( u_{\mu} \) is a unit vector, we have
\[ \delta(g^{0i}u_{a}u_{i}) = 0 \implies \delta u_{0} = a\phi . \]  
(B10)

Using the form of the scalar perturbations Eq. (B8) and the results obtained in Eq. (B7), we obtain the following simple formula
\[ \delta(D^{2}\varphi) = \frac{1}{a^{2}} \left( -\frac{1}{a} \tilde{\varphi} \delta^{ij} \partial_{i}\partial_{j}u_{j} + \nabla^{2}\delta \varphi \right) . \]  
(B11)

Let us now turn our attention to the equation of motion obeyed by the perturbations. Concerning the inflaton field, the first two contributions to the formula given in Eq. (A9) vanish and the result is
\[ \delta g_{\eta_{i},\varphi} = \frac{2b_{11}}{a^{4}} \left[ -\nabla^{4}(\delta \varphi) + \frac{1}{a} \tilde{\varphi} \nabla^{2}(\delta^{ij} \partial_{i}\partial_{j}u_{j}) \right] . \]  
(B12)

In the same way, we get
\[ b_{11} \frac{1}{a^{4}} \frac{\delta^{0}_{\mu}}{a} \left( 10 \frac{a}{a} \tilde{\varphi} \delta^{ij} \partial_{i}\partial_{j}u_{j} - (2\tilde{\varphi}'' + 8\tilde{\varphi}')a\nabla^{2}\varphi \right) + b_{11} \frac{1}{a^{6}} \eta^{\mu \nu} \tilde{\varphi} (2\tilde{\varphi} \delta_{\mu} \partial_{\nu} \varphi - 2\tilde{\varphi'} \delta_{\nu} \partial_{\mu} \delta u_{j}) - \frac{1}{a^{6}} \delta_{\mu} \delta_{\lambda} \]
\[ + 4d_{1} \frac{1}{a^{4}} \left( \eta^{\mu \nu} \eta^{\mu \nu} - \eta^{\mu \nu} \eta^{\nu \mu} \right) \partial_{\mu} \partial_{\nu} \delta u_{\sigma} = 0 . \]  
(B13)

From the temporal index \( (\mu = 0) \), we obtain the variation of the Lagrange multiplier, i.e.,
\[ \delta \lambda = 2d_{1} \frac{1}{a^{3}} \delta^{ij} \partial_{j}(u_{i}') + \frac{b_{11}}{a^{4}} \left[ a\tilde{\varphi} \nabla^{2} \delta \varphi - \tilde{\varphi}^{2} \delta^{ij} \partial_{i}u_{j}' + 5H(\tilde{\varphi}')^{2} \partial_{i}u_{i} - a \right] \tilde{\varphi}' + 4H\tilde{\varphi} \right) \nabla^{2} \delta \varphi . \]  
(B14)

From the spatial indices \( \mu = k \), we obtain
\[ b_{11} \left( -2a\tilde{\varphi} \nabla^{2} \delta \varphi + 2(\tilde{\varphi}')^{2} \delta^{ij} \partial_{i}\partial_{j}u_{j}' \right) + 4d_{1} a^{2} \left[ -\delta u_{k}' + \nabla^{2} \delta u_{k} + \partial_{k}(a\phi)' - \partial_{k}\partial_{l}u_{l} \right] = 0 . \]  
(B15)

Having these results, especially the variation of the Lagrange multiplier, we obtain the expression for the non-vanishing component of the stress-tensor:
\[ \delta T_{00}^{(cor)} = \frac{2b_{11}}{a^{2}} \left( \frac{H}{a} \tilde{\varphi}^{2} \partial_{i}u_{i} - \frac{1}{a} \tilde{\varphi}^{2} \partial_{i}u_{i}' \right) \]  
\[ - (\tilde{\varphi}' + 4H\tilde{\varphi}') \nabla^{2} \delta \varphi + \tilde{\varphi}' \nabla^{2} \delta \varphi' . \]  
(B16)

\[ \delta T_{ij}^{0} = -2a^{2} \delta \lambda . \]  
(B17)

We now wish to express all the quantities in terms of gauge invariant quantities. For the vector field, we have
\[ \delta u_{i} = \delta u_{i}^{(gr)} - u_{0} \partial_{i}(B - E) . \]  
(B18)

The equation of the vector field now becomes
\[ -b_{11} \left( a\tilde{\varphi} \nabla^{2} \partial_{k} \partial_{i}u_{j}^{(gr)} - \tilde{\varphi}^{2} \delta^{ij} \partial_{k} \partial_{i}u_{j}^{(gr)} \right) + 2d_{1} a^{2} \left[ -\delta u_{k}^{(gr)} + \nabla^{2} \delta u_{k}^{(gr)} + \partial_{k}(a\phi)' + \partial_{k}\partial_{l}u_{l}^{(gr)} \right] = 0 , \]  
(B19)

while for the inflaton one obtains
\[ 2V_{\varphi} a^{2} \Phi - 4\tilde{\varphi} \Phi' + a^{2} V_{\varphi} \partial_{k} \partial_{i}u_{j}^{(gr)} + 2H \delta \varphi^{(gr)} + \partial_{k}(a\phi)' \]  
\[ - \nabla^{2} \delta \varphi^{(gr)} + \frac{2b_{11}}{a^{2}} \left[ -\nabla^{2} \delta \varphi^{(gr)} + \frac{1}{a} \tilde{\varphi} \nabla^{2} \delta^{ij} \partial_{i}u_{j}^{(gr)} \right] = 0 . \]  
(B20)

Similarly, we can obtain expressions for the stress-energy tensor (53) in terms of the gauge-invariant variables.

Until this point we have not assumed any specific form of the spatial part of the perturbed unit-vector field. In general, the time-dependent spatial components of the perturbed \( u \) field can be expressed as a sum of irrotational and solenoidal parts, i.e.
\[ \delta u_{i} = (\nabla \xi)_{i} + (\nabla \times \omega)_{i} . \]  
(B21)
The irrotational part of the perturbed $u$ field will contribute to the scalar perturbations while the solenoidal part of the perturbed $u$ field contributes to the vector perturbations. Since we are interested in the scalar perturbations, for the rest of the calculations, we ignore the solenoidal part. Thus, we get
\[ \delta u_i^{(g1)} = \partial_i \xi^{(g1)}. \] (B22)

Substituting this in the Einstein and the field equations, we obtain Eqs. (53).

**APPENDIX C: THE FOURTH ORDER EQUATION**

In this appendix, we provide key steps in obtaining the equation of motion of Bardeen potential ($\Phi$), in a general FRW background, by combining Eqs. (55, 56, 57).

From the constraint equation we deduce that
\[ \delta \varphi_k = \frac{2}{3} M_p^2 \left( \frac{1}{\varphi} \varphi' k + \frac{1}{\varphi} \frac{\mathcal{H}}{k} \right). \] (C1)

Substituting this expression and its derivatives into the equation of motion of $\delta \varphi_k$, one obtains $\xi_k$, i.e.,
\[ \xi_k = \frac{M_p}{k^2} \left( Q_3 \Phi'''_k + Q_2 \Phi''_k + Q_1 \Phi'_k + Q_0 \Phi_k \right), \] (C2)

where
\begin{align*}
Q_3 &= \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} k^2, \\
Q_2 &= \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} \left[ \frac{2}{\varphi} \left( \frac{1}{\varphi} \right)' + \frac{1}{k} \right] k^2, \\
Q_1 &= \left[ \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} \right] \left( \frac{1}{\varphi''} + \frac{2}{b_{11} a^3} \right) \frac{1}{k^2}, \\
+ \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} \frac{2}{3} \frac{M_p^2}{a^2} \frac{a^3}{\varphi^2} k^2, \\
Q_0 &= \left[ \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} \left( \frac{1}{\varphi''} + \frac{1}{b_{11} a^3} \varphi \right) \frac{1}{k^2} + \frac{M_p^2}{3 b_{11}} \frac{a^3}{\varphi^2} \mathcal{H} \right] \frac{1}{k^2} + \frac{2}{3} \frac{M_p^2}{a^2} \frac{a^3}{\varphi^2} \mathcal{H} k^2. \] (C3)

Substituting the above expressions in the field equation of the Bardeen potential, we get,
\begin{align*}
\Phi'''_k + X_3 \Phi''_k + (X_2 + Y_2 k^2 + Z_2 k^4) \Phi'_k \\
+ (X_1 + Y_1 k^2 + Z_1 k^4) \Phi_k + (X_0 + Y_0 k^2 + Z_0 k^4) \Phi_k = 0, \quad \text{(C4)}
\end{align*}

where
\begin{align*}
X_3 &= \frac{2}{a^3} \left( \frac{a^3}{\varphi^2} \right)' + \frac{2}{a^3} \left( \frac{1}{\varphi} \right)' + \frac{\mathcal{H}}{\varphi}, \\
X_2 &= \frac{2}{a^3} \left[ \frac{a^3}{\varphi^2} \left( \frac{1}{\varphi} \right)'' + \frac{1}{b_{11} a^3} \right] k^2, \\
+ \frac{2}{a^3} \left( \frac{1}{\varphi} \right)' + \frac{\mathcal{H}}{\varphi}, \\
Y_2 &= 3 b_{11} \frac{a^2}{a^3} + 1, \quad Z_2 = 2 b_{11} \frac{a^3}{a^3}, \\
X_1 &= \frac{2}{a^3} \left[ \frac{a^3}{\varphi^2} \left( \frac{1}{\varphi} \right)'' + \frac{1}{b_{11} a^3} \right] k^2, \\
Y_1 &= \frac{2}{a^3} \left[ \frac{a^3}{\varphi^2} \left( \frac{1}{\varphi} \right)'' + \frac{1}{b_{11} a^3} \right] k^2, \\
Z_1 &= 2 b_{11} \frac{a^2}{a^3} \left[ \frac{a^3}{\varphi^2} + \frac{a^2}{a^3} \mathcal{H} \right] + \frac{3 b_{11} \varphi^2}{a^3} \left( \frac{a^3}{\varphi^2} \right)', \\
X_0 &= \frac{2}{a^3} \left[ \frac{a^3}{\varphi^2} \left( \frac{4}{\varphi} \right)' + \frac{3}{a^3} \frac{1}{M_p^2} \varphi \right] k^2, \\
Y_0 &= \frac{2}{a^3} \left[ \frac{a^3}{\varphi^2} \mathcal{H} \right] + \frac{3 b_{11} \varphi^2}{a^3} \left( \mathcal{H'} - \frac{\varphi}{\varphi''} \right), \\
Z_0 &= 2 b_{11} \frac{a^2}{a^3} \left[ \frac{a^2}{\varphi^2} \right] + \frac{3 b_{11} \varphi^2}{a^3} \left( \frac{a^3}{\varphi^2} \right)' + \frac{3}{2 b_{11} \left( \frac{1}{a^2} - 2 \frac{1}{M_p^2} \right) \varphi^2}{a^3}. \end{align*}
[15] U. H. Danielsson, Phys. Rev. D66, 023511 (2002), hep-th/0203198.
[16] R. Brandenberger and P.-M. Ho, Phys. Rev. D66, 023517 (2002), hep-th/0203119.
[17] S. F. Hassan and M. S. Sloth, Nucl. Phys. B674, 434 (2003), hep-th/0204110.
[18] R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D66, 023518 (2002), hep-th/0204129.
[19] N. Kaloper, M. Kleban, A. E. Lawrence, and S. Shenker, Phys. Rev. D66, 123510 (2002), hep-th/0305161.
[20] J. Martin and R. Brandenberger, Phys. Rev. D66, 023517 (2002), hep-th/0203198.
[21] R. Brandenberger and P.-M. Ho, Phys. Rev. D66, 023517 (2002), hep-th/0203119.
[22] S. F. Hassan and M. S. Sloth, Nucl. Phys. B674, 434 (2003), hep-th/0204110.
[23] R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D66, 023518 (2002), hep-th/0204129.
[24] N. Kaloper, M. Kleban, A. E. Lawrence, and S. Shenker, Phys. Rev. D66, 123510 (2002), hep-th/0305161.
[25] R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D66, 023517 (2002), hep-th/0203198.
[26] N. Kaloper, M. Kleban, A. E. Lawrence, and S. Shenker, Phys. Rev. D66, 123510 (2002), hep-th/0305161.
[27] R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D66, 023518 (2002), hep-th/0204129.