A generalization of the Ekeland variational principle

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Abstract

In this short communication, we present a generalization of the Ekeland variational principle. The main result is established through standard tools of functional analysis and calculus of variations. The novelty here is a result involving the second Gâteaux variation of the functional in question.

1 Introduction

In this article we present and prove a generalization of the Ekeland variational principle. A proof of the so far known principle may be found in Giusti, [3], pages 160-161. With slight improvements a similar result is presented in [2]. We also highlight details on the function spaces addressed may be found in [1]. At this point we state such a result.

Theorem 1.1 (Ekeland variational principle). Let $(U, d)$ be a complete metric space and let $F : U \rightarrow \mathbb{R} \equiv \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous bounded below functional taking a finite value at some point.

Let $\varepsilon > 0$. Assume for some $u \in U$ we have

$$F(u) \leq \inf_{u \in U} \{F(u)\} + \varepsilon.$$

Under such hypotheses, there exists $v \in U$ such that

1. $d(u, v) \leq 1$,
2. $F(v) \leq F(u)$,
3. $F(v) \leq F(w) + \varepsilon d(v, w)$, $\forall w \in U$.
The generalized Ekeland variational principle

In this section we state and prove the following new result, which the proof is based on the one presented in [3].

**Theorem 2.1** (Generalized Ekeland variational principle). Let \((U, d)\) be a complete metric space and let \(F : U \rightarrow \mathbb{R}\) be a lower semi continuous bounded below functional taking a finite value at some point.

Let \(\varepsilon > 0\). Assume for some \(u \in U\) we have

\[ F(u) \leq \inf_{w \in U} \{ F(w) \} + \varepsilon. \]

Under such hypotheses, there exists \(v \in U\) such that

1. \(d(u, v) \leq 1\),
2. \(F(v) \leq F(u)\),
3. \(F(v) \leq F(w) + \varepsilon d(v, w), \ \forall w \in U\).
4. Assuming \(U\) is a Banach space and \(F\) is Gâteaux differentiable, we have

\[ \| \delta F(v) \|_{U^*} \leq \varepsilon. \]

5. Finally, assuming also \(F\) is twice Fréchet differentiable, we have

\[ \delta^2 F(v, \varphi, \varphi) \geq -4\varepsilon\|\varphi\|_U - 2\frac{a(\varepsilon^2)}{\varepsilon^2}, \ \forall \varphi \in U, \]

where

\[ \frac{a(\varepsilon^2)}{\varepsilon^2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0^+. \]

**Proof.** Define the sequence \(\{u_n\} \subset U\) by:

\[ u_1 = u, \]

and having \(u_1, \ldots, u_n\), select \(u_{n+1}\) as specified in the next lines. First, define

\[ S_n = \{ w \in U \mid F(w) \leq F(u_n) - \varepsilon d(u_n, w) \}. \]

Observe that \(u_n \in S_n\) so that \(S_n\) in non-empty.

On the other hand, from the definition of infimum, we may select \(u_{n+1} \in S_n\) such that

\[ F(u_{n+1}) \leq \frac{1}{2} \left( F(u_n) + \inf_{w \in S_n} \{ F(w) \} \right). \tag{1} \]

Since \(u_{n+1} \in S_n\) we have

\[ \varepsilon d(u_{n+1}, u_n) \leq F(u_n) - F(u_{n+1}). \tag{2} \]

and hence

\[ \varepsilon d(u_{n+m}, u_n) \leq \sum_{i=1}^{m} \varepsilon d(u_{n+i}, u_{n+i-1}) \leq F(u_n) - F(u_{n+m}). \tag{3} \]
From (2), \( \{F(u_n)\} \) is a decreasing sequence bounded below by \( \inf_{u \in U} F(u) \) so that there exists \( \alpha \in \mathbb{R} \) such that
\[
F(u_n) \to \alpha \text{ as } n \to \infty.
\]
From this and (3), \( \{u_n\} \) is a Cauchy sequence, converging to some \( v \in U \).
Since \( F \) is lower semi-continuous we get,
\[
\alpha = \liminf_{m \to \infty} F(u_{n+m}) \geq F(v),
\]
so that letting \( m \to \infty \) in (3) we obtain
\[
\varepsilon d(u_n, v) \leq F(u_n) - F(v), \tag{4}
\]
and, in particular for \( n = 1 \) we get
\[
0 \leq \varepsilon d(u, v) \leq F(u) - F(v) - \inf_{u \in U} F(u) \leq \varepsilon.
\]
Thus, we have proven 1 and 2.
Suppose, to obtain contradiction, that 3 does not hold.
Hence, there exists \( w \in U \) such that
\[
F(w) < F(v) - \varepsilon d(w, v).
\]
In particular we have
\[
w \neq v. \tag{5}
\]
Thus, from this and (4) we have
\[
F(w) < F(u_n) - \varepsilon(u_n, v) - \varepsilon d(w, v) \leq F(u_n) - \varepsilon d(u_n, w), \forall n \in \mathbb{N}.
\]
Now observe that \( w \in S_n, \forall n \in \mathbb{N} \) so that
\[
\inf_{w \in S_n} \{F(w)\} \leq F(w), \forall n \in \mathbb{N}.
\]
From this and (1) we obtain,
\[
2F(u_{n+1}) - F(u_n) \leq F(w) < F(v) - \varepsilon d(v, w),
\]
so that
\[
2 \liminf_{n \to \infty} \{F(u_{n+1})\} \leq F(v) - \varepsilon d(v, w) + \liminf_{n \to \infty} \{F(u_n)\}.
\]
Hence,
\[
F(v) \leq \liminf_{n \to \infty} \{F(u_{n+1})\} \leq F(v) - \varepsilon d(v, w),
\]
so that
\[
0 \leq -\varepsilon d(v, w),
\]
which contradicts (5).
Thus 3 holds.
Assume now \( U \) is a Banach space, \( F \) is Gâteaux differentiable and \( \varphi \in U \). Fix \( t \in (0, 1) \).
Thus, from \(3\)
\[
F(v) - F(v + t\varphi) \leq \varepsilon \|t\varphi\|_U,
\]
so that
\[
\frac{F(v) - F(v + t\varphi)}{t} \leq \varepsilon \|\varphi\|_U,
\]
(6)

Therefore, letting \(t \to 0^+\), we get
\[
-\langle \delta F(v), \varphi \rangle_U \leq \varepsilon \|\varphi\|_U.
\]
(7)

Similarly, for \(t \in (0, 1)\),
\[
F(v) - F(v + t(-\varphi)) \leq \varepsilon \|t\varphi\|_U,
\]
so that
\[
\frac{F(v) - F(v + t(-\varphi))}{t} \leq \varepsilon \|\varphi\|_U.
\]
(9)

Letting \(t \to 0^+\), we obtain
\[
\langle \delta F(v), \varphi \rangle_U \leq \varepsilon \|\varphi\|_U,
\]
(10)

so that
\[
|\langle \delta F(v), \varphi \rangle_U| = \varepsilon \|\varphi\|_U, \ \forall \varphi \in U.
\]
(11)

Thus,
\[
\|\delta F(v)\|_{U^*} \leq \varepsilon.
\]
(12)

Assume here, in addition, \(F\) is twice Fréchet differentiable in \(U\). From \(3\) with \(\varepsilon^2\) replacing \(\varepsilon\) in the previous items, we have
\[
F(v + \varepsilon\varphi) - F(v) \geq -\varepsilon^2\|\varepsilon\varphi\|_U,
\]
so that from this and the twice Fréchet differentiability hypothesis, we get
\[
\varepsilon\langle \delta F(v), \varphi \rangle_U + \frac{1}{2}\varepsilon^2\delta^2 F(v, \varphi, \varphi) + o(\varepsilon^2) \geq -\varepsilon^3\|\varphi\|_U,
\]
so that, from this and
\[
|\langle \delta F(v), \varphi \rangle_U| \leq \varepsilon^2\|\varphi\|_U,
\]
we obtain
\[
\frac{1}{2}\delta^2 F(v, \varphi, \varphi) \geq -\varepsilon\|\varphi\|_U - \varepsilon \frac{|\langle \delta F(v), \varphi \rangle_U|}{\varepsilon^2} - \frac{o(\varepsilon^2)}{\varepsilon^2}
\]
\[
\geq -2\varepsilon\|\varphi\|_U - \frac{o(\varepsilon^2)}{\varepsilon^2}.
\]
(13)
Hence,
\[ \delta^2 F(v, \varphi, \varphi) \geq -4 \varepsilon \| \varphi \|_U - 2 \frac{o(\varepsilon^2)}{\varepsilon^2}, \forall \varphi \in U, \]

where
\[ \frac{o(\varepsilon^2)}{\varepsilon^2} \to 0, \text{ as } \varepsilon \to 0^+. \]

The proof is complete.

**Remark 2.2.** We may introduce in $U$ a new metric given by $d_1 = \varepsilon^{1/2}$. We highlight that the topology remains the same and also $F$ remains lower semi-continuous. Under the hypotheses of the last theorem, for a not relabeled metric $d$, if $u \in U$ is such that $F(u) < \inf_{u \in U} F(u) + \varepsilon^2$, then there exists $v \in U$ such that

1. $d(u, v) \leq \varepsilon^{1/2}$,
2. $F(v) \leq F(u),$
3. $F(v) \leq F(w) + \varepsilon^{3/2}d(v, w), \forall w \in U.$
4. Assuming $U$ is a Banach space and $F$ is Gâteaux differentiable, we have
\[ \|\delta F(v)\|_{U^*} \leq \varepsilon^{3/2}. \]
5. Finally, assuming also $F$ is twice Fréchet differentiable, we have
\[ \delta^2 F(v, \varphi, \varphi) \geq -4 \varepsilon^{1/2} \| \varphi \|_U - 2 \frac{o(\varepsilon^2)}{\varepsilon^2}, \forall \varphi \in U, \]

where
\[ \frac{o(\varepsilon^2)}{\varepsilon^2} \to 0, \text{ as } \varepsilon \to 0^+. \]

**References**

[1] R.A. Adams and J.F. Fournier, *Sobolev Spaces*, second edition, Elsevier (2003).

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