The Lie algebraic structure of extended Sutherland models

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Abstract

We disclose the Lie algebraic structure of two extended Sutherland models. Their Hamiltonians are $BC_N$ and $A_N$ Sutherland Hamiltonians with some additional terms. We show that both Hamiltonians can be written in the quadratic forms of generators of the Lie algebra $gl(N+1)$.

Keywords: Quasi-exactly-solvability; Sutherland models; Lie algebra.

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1 Introduction

The quasi-exactly-solvable models have been developed considerabley over last decade. Ten years ago the crucial definition of quasi-exact-solvability was made\cite{1}. The author defined the quasi-exact-solvability as follows. Let $P_n$ be the space of polynomials of degree $\leq n$.

Definition 1 Let us name a linear differential operator of the $k$-th order, $T_k$ quasi-exactly-solvable, if it preserves the space $P_n$. Correspondingly, the operator $E_k$, which preserves the infinite flag $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots$ of spaces of all polynomials, is named exactly-solvable.

In particular, if operators can be represented by the elements of the enveloping algebra of a certain Lie algebras realized in terms of differential operators preserving the space $P_n$, they are quasi-exactly-solvable. Following this concept, various quasi-exactly-solvable models have been proposed. In reference\cite{2} the authors obtain the $sl_2$-based deformation of the Calogero models. $A_N$, $BC_N$, $B_N$, $C_N$, and $D_N$ Calogero and Sutherland Hamiltonian, as well as thier supersymmetric generalizations are shown to have expression in the quadratic polynomials in the generators of the Lie algebra or the Lie superalgebra for the supersymmetric case\cite{3}. Moreover the $sl(N+1)$ deformation of the Calogero models, whose Hamiltonians have the quadratic and sextic self-interaction terms, are shown to be quasi-exactly-solvable\cite{4}. Recently the quasi-exact-solvable models are classified
by using the new family of $A_N$-type Dunkl operators which includes the usual Dunkl operator. The purpose of this paper is to obtain the explicit Lie algebraic form of the extended Sutherland Hamiltonians, following the method explained in the reference. We consider here two kinds of Hamiltonians. One of them is an extended Hamiltonian of the $BC_N$-Sutherland model, which has an additional term:

$$
\sum_{i=1}^{N} \left\{ g_1 \cos^2 2x_i + 2g_2 \cos 2x_i \right\} .
$$

(1)

The other is an extended Hamiltonian of the $A_N$-Sutherland model, which has an additional term:

$$
\sum_{i=1}^{N} \left\{ g_1 \cos 4x_i + g_2 \cos 2x_i + g_3 \sin 4x_i + g_4 \sin 2x_i \right\} .
$$

(2)

These Hamiltonians are indicated in the reference. This paper is arranged as follows. In section 2, we consider an extended $BC_N$-Sutherland Hamiltonian. First we rotate it with the ground state eigenfunction as a gauge factor. Then we rewrite the gauge rotated Hamiltonian in terms of the elementary symmetric polynomials. Finally we arrive at the explicit form of the Hamiltonian in quadratic form of generators which generate the Lie algebra $gl(N+1)$. In section 3, we show an extended $A_N$-Sutherland Hamiltonian has Lie algebraic form as the same technique as in section 2.

2 Extended $BC_N$-Sutherland model

In this section we begin with the case of the extended $BC_N$-Sutherland Hamiltonian.

2.1 Hamiltonian

The Hamiltonian for the Sutherland model of type $BC_N$ is

$$
H = - \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \left\{ \frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right\} + 4g_3 \sum_{i=1}^{N} \frac{1}{\sin^2 2x_i} + g_4 \sum_{i=1}^{N} \frac{1}{\sin^2 x_i}
$$

(3)

where $g$, $g_3$ and $g_4$ are coupling constants. From the Hamiltonian we can deduce the Hamiltonian of the Sutherland models of type $B_N$, $C_N$, $D_N$ as follows:

- $g_3 = 0$ : $B_N$,
- $g_4 = 0$ : $C_N$,
- $g_3 = g_4 = 0$ : $D_N$.

We will consider a Hamiltonian with the external potential

$$
\sum_{i=1}^{N} \left\{ g_1 \cos^2 2x_i + 2g_2 \cos 2x_i \right\} ,
$$

(4)
that is,

\[
H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \left\{ \frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right\}
\]

\[
+ \sum_{i=1}^{N} \left( g_1 \cos^2 2x_i + 2g_2 \cos 2x_i + 4g_3 \frac{1}{\sin^2 2x_i} + g_4 \frac{1}{\sin^2 x_i} \right)
\]

where

\[
g = \nu(\nu - 1),
\]

\[
g_1 = -\nu_1^2,
\]

\[
g_2 = \nu_1 \{ 1 + 2n + 2(N-1)\nu + \nu_3 + \nu_4 \} \quad (n \in \mathbb{Z}),
\]

\[
g_3 = -\nu_3(\nu_3 - 1),
\]

\[
g_4 = \nu_3(\nu_3 - 1) - \nu_4(\nu_4 - 1).
\]

Let us call the Hamiltonian (5) the extended $BC_N$-Sutherland Hamiltonian.

### 2.2 Gauge rotation

First we make a gauge rotation of the Hamiltonian (5) with the ground state eigenfunction

\[
\mu = \left\{ \prod_{i<j} \sin(x_i - x_j) \sin(x_i + x_j) \right\}^\nu
\]

\[
\times \left\{ \prod_{i=1}^{N} \cos x_i \right\}^{\nu_3} \left\{ \prod_{i=1}^{N} \sin x_i \right\}^{\nu_4} \prod_{i=1}^{N} \exp \left( -\frac{\nu_1}{2} \cos 2x_i \right)
\]

as gauge factor, $\tilde{H} = \mu^{-1} H \mu$. Using the relation

\[
\sum_{i=1}^{N} \left[ \sum_{j(\neq i)} \left( \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} + \frac{\cos(x_i + x_j)}{\sin(x_i + x_j)} \right) \right]^2
\]

\[
= 2 \sum_{i<j} \left( \frac{\cos^2(x_i - x_j)}{\sin^2(x_i - x_j)} + \frac{\cos^2(x_i + x_j)}{\sin^2(x_i + x_j)} \right) + \text{constant},
\]

it is not difficult to obtain that

\[
\tilde{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N} \left\{ \nu \sum_{j(\neq i)} \left( \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} + \frac{\cos(x_i + x_j)}{\sin(x_i + x_j)} \right) \right\}
\]

\[
+ (\nu_3 + \nu_4) \frac{\cos 2x_i}{\sin 2x_i} - (\nu_3 - \nu_4) \frac{1}{\sin 2x_i} + \nu_1 \sin 2x_i \left\{ \frac{\partial}{\partial x_i} \right\}
\]

\[
+ 4\nu_1 \sum_{i=1}^{N} \cos 2x_i,
\]

hereafter we will omit the constant terms.
2.3 Lie algebraic form

In order to disclose the Lie algebraic structure of the gauge rotated Hamiltonian \( \tilde{H} \), it is expedient to rewrite the Hamiltonian \( \tilde{H} \) in terms of the elementary symmetric polynomials. Let \( e_i \) be the \( i \)-th elementary symmetric polynomials of \( \cos 2x \),

\[
e_1 = \sum_{i=1}^{N} \cos 2x_i,
\]

\[
e_2 = \sum_{i<j} \cos 2x_i \cos 2x_j,
\]

\[
e_3 = \sum_{i<j<k} \cos 2x_i \cos 2x_j \cos 2x_k,
\]

\[\vdots\]

\[
e_N = \cos 2x_1 \cos 2x_2 \cdots \cos 2x_N.
\]

**Lemma 1** In these variables \( \{e_1, e_2, \ldots, e_N\} \), the gauge rotated Hamiltonian \( \tilde{H} \) becomes

\[
\tilde{H} = -4 \sum_{k,l=1}^{N} \left\{ Ne_{k-1}e_{l-1} - \sum_{i \geq 0} (k-i)e_{k-i}e_{l+i} + (l-i)e_{k-1-i}e_{l+1+i} \right\} \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_l} \\
+ 4 \sum_{k=1}^{N} \left\{ \nu(N-k+2)(N-k+1)e_{k-2} \\
+ \left[ \nu k(2N-k-1) + 2\nu_3k + k \right] e_k \\
+ \nu_1 \left[ -e_1e_k + (N-k+1)e_{k-1} + (k+1)e_{k+1} \right] \right\} \frac{\partial}{\partial e_k} \\
+ 4n\nu_1 e_1.
\]

**Proof.** After the change of variables \( x_i \to e_i \), the Hamiltonian \( \tilde{H} \) is

\[
\tilde{H} = \sum_{k,l=1}^{N} \sum_{i=1}^{N} \frac{\partial e_k}{\partial x_i} \frac{\partial e_l}{\partial x_i} \frac{\partial}{\partial e_k} - \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 e_k}{\partial x_i^2} \frac{\partial}{\partial e_k} \\
- \nu \sum_{k=1}^{N} \sum_{i \neq j}^{N} \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} \left( \frac{\partial e_k}{\partial x_i} - \frac{\partial e_k}{\partial x_j} \right) \\
+ \frac{\cos(x_i + x_j)}{\sin(x_i + x_j)} \left( \frac{\partial e_k}{\partial x_i} + \frac{\partial e_k}{\partial x_j} \right) \frac{\partial}{\partial e_k} \\
- 2(\nu_3 + \nu_4) \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{\cos 2x_i}{\sin 2x_i} \frac{\partial e_k}{\partial x_i} \frac{\partial}{\partial e_k}
\]
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\[+2(\nu_3 - \nu_4) \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{1}{\sin 2x_i} \frac{\partial e_k}{\partial x_i} \frac{\partial}{\partial e_k} - 2\nu_1 \sum_{k=1}^{N} \sum_{i=1}^{N} \sin 2x_i \frac{\partial e_k}{\partial x_i} \frac{\partial}{\partial e_k} + 4n\nu_1 \sum_{i=1}^{N} \cos 2x_i.\]

As an example, let us demonstrate how to express

\[X_k = \sum_{i=1}^{N} \sin 2x_i \frac{\partial e_k}{\partial x_i}\]  \hspace{1cm} (11)

in terms of the elementary symmetric polynomials. Let us consider the generating function of \(X_k\):

\[X(t) = \sum_{k=0}^{N} X_k t^k.\]

Note that the generating function of elementary symmetric polynomials \(e_k\) is

\[E(t) = \sum_{k=0}^{N} e_k t^k = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} p_n(\cos 2x) t^n \right], \hspace{1cm} (12)\]

where \(p_n\) are the power sums:

\[p_n(\cos 2x) = \sum_{i=1}^{N} \cos^n 2x_i.\]

Calculating the generating function \(X(t)\), we have

\[X(t) = \sum_{k=0}^{N} \left[ \sum_{i=1}^{N} \sin 2x_i \frac{\partial e_k}{\partial x_i} \right] t^k\]  \hspace{1cm} (13)

\[= \sum_{i=1}^{N} \sin 2x_i \sum_{k=0}^{N} \frac{\partial e_k}{\partial x_i} t^k\]

\[= \sum_{i=1}^{N} \sin 2x_i \frac{\partial E(t)}{\partial x_i}\]

\[= 2 \left\{ \sum_{n=1}^{\infty} (-1)^n p_{n-1}(\cos 2x) t^n - \sum_{n=1}^{\infty} (-1)^n p_{n+1}(\cos 2x) t^n \right\} E(t).\]

It is not difficult to show that

\[X(t) = 2 \left( p_1(\cos 2x) - Nt + t^2 \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \right) E(t). \hspace{1cm} (14)\]

Substituting \(E(t) = \sum_{k=0}^{N} e_k t^k\) in (14), we find that

\[X_k = 2 \left\{ e_1 e_k - (N - k + 1) e_{k-1} - (k + 1) e_{k+1} \right\}. \hspace{1cm} (15)\]

One can show the other parts similarly. \(\square\)
Next we will realize the Lie algebra $gl(N+1)$ in terms of elementary symmetric polynomials. One of the simplest realization of $gl(N + 1)$ is the differential realization. The generators can be represented in the following form:

$$J_i^- = \frac{\partial}{\partial e_i} \quad (i = 1, 2, \ldots, N), \quad (16)$$

$$J_{ij}^0 = e_i \frac{\partial}{\partial e_j} \quad (i, j = 1, 2, \ldots, N), \quad (17)$$

$$J^0 = n - \sum_{i=1}^{N} e_i \frac{\partial}{\partial e_i} \quad (18)$$

$$J_i^+ = e_i J^0 \quad (i = 1, 2, \ldots, N), \quad (19)$$

where the parameter $n$ is an integer.

If $n$ is a non-negative integer, the generators act on the representation space of polynomials in $N$ variables of the following type

$$\mathcal{P}_n = \text{span} \left\{ e_1^{n_1} e_2^{n_2} \cdots e_N^{n_N} \middle| 0 \leq \sum_{i=1}^{N} n_i \leq n \right\}. \quad (20)$$

With simple algebraic transformations from (9), we obtain the following result.

**Proposition 1** We obtain the Lie algebraic form of the gauge rotated extended $BC_N$-Sutherland Hamiltonians (4) as follows:

$$\tilde{H} = -4 \sum_{k,l=1}^{N} \left\{ NJ^0_{k-1,l} J^0_{l-1,k} - \sum_{i \geq 0} \left[ \left( k - i \right) \left( J^0_{k-l+1,i} J^0_{l,i,k} - \delta_{l0} J^0_{k,k} \right) \right. \right. \right.$$

$$\left. \left. + \left( l - 1 + i \right) \left( J^0_{k-1-i,l} J^0_{l+1+i,k} - \delta_{i1} J^0_{k-2,k} \right) \right] \right. \right.$$

$$\left. - \left( k - 2 - i \right) \left( J^0_{k-2-i,l} J^0_{l+i+1,k} - \delta_{i0} J^0_{k-2,k} \right) - (l + 1 + i) J^0_{k-1-l,i} J^0_{l+1+i,k} \right) \bigg\}$$

$$+ 4 \sum_{k=1}^{N} \left\{ \nu(N - k + 2)(N - k + 1) J^0_{k-2,k} + \nu(N - k + 1) J^0_{k-1,k} \right. \right.$$

$$\left. + \left[ \nu k (2N - k - 1) + 2 \nu k + k \right] J^0_{k,k} + \nu (k + 1) J^0_{k+1,k} \right\}$$

$$+ 4 \nu \ J^+_1. \quad (21)$$

Here we define

$$J^0_{0i} = J_i^- \quad (i = 1, 2, \ldots, N).$$

3 **Extended $A_N$-Sutherland Hamiltonian**

In this section we devote to a consideration of the extended $A_N$-Sutherland Hamiltonian with using the same approach as in the previous section.
3.1 Hamiltonian

A second Hamiltonian we deal with is an extension of $A_N$-Sutherland model:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \frac{1}{\sin^2(x_i - x_j)}. \quad (22)$$

Considering the external potential

$$2 \sum_{i=1}^{N} \left\{ g_1 \cos 4x_i + 2g_2 \cos 2x_i + 2g_3 \sin 4x_i + 2g_4 \sin 2x_i \right\}, \quad (23)$$

we define a Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \frac{1}{\sin^2(x_i - x_j)} + 2 \sum_{i=1}^{N} \left\{ g_1 \cos 4x_i + 2g_2 \cos 2x_i + 2g_3 \sin 4x_i + 2g_4 \sin 2x_i \right\} \quad (24)$$

where coupling constants $g, g_1, g_2, g_3, g_4$ satisfy

$$g = \nu(\nu - 1),$$
$$g_1 = (\beta - \gamma)(\beta + \gamma),$$
$$g_2 = (\gamma + \alpha\beta + n\gamma + \nu\gamma(N - 1)), $$
$$g_3 = \beta\gamma,$$
$$g_4 = (\alpha\gamma - \beta - n\beta - \nu\beta(N - 1)).$$

We call the Hamiltonian (24) the extended $A_N$-Sutherland Hamiltonian.

3.2 Gauge rotation

Similarly to what was done in section 2 for the extended $BC_N$-Sutherland Hamiltonian, we first make a gauge rotation of (24) with the ground state eigenfunction

$$\mu(x) = \prod_{i<j} \sin^\nu(x_i - x_j) \prod_{i=1}^{N} \cos^n x_i \exp \left[ \alpha x_i + \beta \sin 2x_i - \gamma \cos 2x_i \right] \quad (25)$$

as a gauge factor $\bar{H} = \mu^{-1} H \mu$. It is not difficult to obtain

$$\bar{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N} \left\{ \nu \sum_{i \neq j} \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} - n \frac{\sin x_i}{\cos x_i} \right\} \frac{\partial}{\partial x_i} \quad (26)$$

$$-n(n - 1) \sum_{i=1}^{N} \tan^2 x_i + 2n \sum_{i<j} \tan x_i \tan x_j$$

$$+2n(\alpha - 2\beta) \sum_{i=1}^{N} \tan x_i.$$
3.3 Lie algebraic form

Next we rewrite the gauge rotated Hamiltonian \( \tilde{H} \) in terms of the elementary symmetric polynomials of \( \tan x \).

\[
e_1 = \sum_{i=1}^{N} \tan x_i, \\
e_2 = \sum_{i<j} \tan x_i \tan x_j, \\
e_3 = \sum_{i<j<k} \tan x_i \tan x_j \tan x_k, \\
\vdots \\
e_N = \tan x_1 \tan x_2 \cdots \tan x_N.
\]

Numerous but straightforward calculations similar to what we made in the proof of lemma 1 leads to the following representation.

\[
\tilde{H} = -N \sum_{k,l=1}^{N} \left\{ N e_{k-1} e_{l-1} - e_1 e_{k+1} e_l - e_1 e_k e_{l+1} + (e_1^2 - 2e_2)e_k e_l \\
+ \sum_{i \geq 0} \left[ (k-2-i)e_{k-2-i} e_{l+i} - (l-1+i)e_{k-1-i} e_{l+1+i} \\
+ 2(k-i)e_{k-i-1} e_{l+i} - 2(l+1+i)e_{k-1-i} e_{l+1+i} \\
+ (k+2-i)e_{k+2-i} e_{l+i} - (l+3+i)e_{k-1-i} e_{l+3+i} \right] \right\} \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_l} \\
+ \nu \sum_{k=1}^{N} \left\{ (N-k+2)(N-k+1) e_{k-2} \\
- 2 \left[ k(N-k) + e_2 \right] e_k + (k+1) k e_{k+2} \right\} \frac{\partial}{\partial e_k} \\
- \sum_{k=1}^{N} \left\{ 2 \left[ (k+2) e_{k+2} - e_1 e_{k+1} + (k+e_1^2-2e_2)e_k \right] \right\} \frac{\partial}{\partial e_k} \\
+ 2n \sum_{k=1}^{N} \left\{ (k+2) e_{k+2} - e_1 e_{k+1} + (k+e_1^2-2e_2)e_k \right\} \frac{\partial}{\partial e_k} \\
- 2n \sum_{k=1}^{N} \left\{ \left[ N-(k-1) \right] e_{k-1} + e_1 e_k - (k+1) e_{k+1} \right\} \frac{\partial}{\partial e_k} \\
- 4\beta \sum_{k=1}^{N} \left\{ \left[ N-(k-1) \right] e_{k-1} - e_1 e_k + (k+1) e_{k+1} \right\} \frac{\partial}{\partial e_k} \\
- 4\gamma \sum_{k=1}^{N} 2k e_k \frac{\partial}{\partial e_k} \\
n(n-1)(e_1^2 - 2e_2) + 2\nu e_2 + 2n(\alpha - 2\beta)e_1.
\]

(27)
Now we use the same realization\cite{16-19} of Lie algebra $gl(N + 1)$. With simple algebraic transformations from (27), we derive the following result.

**Proposition 2** We obtain the Lie algebraic form of the gauge rotated extended $A_N$-Sutherland Hamiltonian \cite{27} as follows:

\[
\hat{H} = \sum_{k,l=1}^{N} N J_{k-1,l}^{0} J_{l-1,k}^{0} + \sum_{i \geq 0}^{N} \left\{ (k - 2 - i)(J_{k-2-i,l}^{0} J_{l+i,k}^{0} - \delta_{i0} J_{k-2,k}^{0}) - (l - 1 + i)(J_{k-1-i,l}^{0} J_{l+i,k}^{0} - \delta_{i0} J_{k,k}^{0}) + 2(k - i)(J_{k-i,l}^{0} J_{l+i,k}^{0} - \delta_{i0} J_{k,k}^{0}) - 2(l + 1 + i)(J_{k-1-i,l}^{0} J_{l+i+1,k}^{0}) + (k + 2 - i)(J_{k+2-i,l}^{0} J_{l+i,k}^{0} - \delta_{i0} J_{k+2,k}^{0}) - (l + 3 + i)(J_{k-1-i,l}^{0} J_{l+i+3,k}^{0}) \right\}
\]

\[+ \sum_{k=1}^{N} \left\{ \nu(N - k + 1)(N - k + 2) J_{k-2,k}^{0} - 2 \nu k(N - k + 2) J_{k,k}^{0} \right\}
\]

\[-2(\alpha + 2\beta)(N - k + 1) J_{k-1,k}^{0} \]

\[+ \left\{ \nu k(k+1) + 2(n - 1)(k+2) \right\} J_{k+2,k}^{0} \]

\[-2 \sum_{k=1}^{N} J_{k+1,k}^{0} J_{1}^{0} + 2(n + \nu + 1) J_{1}^{0} + 2(\alpha - 2\beta) J_{1}^{0} - 2 \sum_{k=1}^{N} J_{k,k}^{0} J_{2}^{0} - J_{1}^{0} J_{1}^{0}. \]

Here we define $J_{0i}^{0} = J_{i}^{0}$ ($i=1,2,\ldots,N$).

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**References**

1. A. Turbiner, *Contemp.Math.* **Vol.160** 263-310 (1994).
2. A. Minzoni, M. Rosenbaum and A. Turbiner, *Mod.Phys.Lett.* **A11** 1977-1984 (1996).
3. L. Brink, A. Turbiner and N. Wyllard, *J.Math.Phys.* **39** 1285-1315 (1998).
4. Xinrui Hou and M. Shifman, *Int.J.Mod.Phys.* **A14** 2993-3004 (1999).
5. F. Finkel, D. Gómez-Ullate, A. González-Lopéz, M. A. Rodríguez and R. Zhdanov, *Comm.Math.Phys.* **221** 472-496 (2001).
6. Toshiaki Tanaka, *Annals Phys.* **309** 239-280 (2004).