Nonstandard $GL_h(n)$ quantum groups and contraction of covariant $q$-bosonic algebras

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Abstract

$GL_h(n) \times GL_h(m)$-covariant $h$-bosonic algebras are built by contracting the $GL_q(n) \times GL_q(m)$-covariant $q$-bosonic algebras considered by the present author some years ago. Their defining relations are written in terms of the corresponding $R_h$-matrices. Whenever $n = 2$, and $m = 1$ or 2, it is proved by using $U_h(sl(2))$ Clebsch-Gordan coefficients that they can also be expressed in terms of coupled commutators in a way entirely similar to the classical case. Some $U_h(sl(2))$ rank-1/2 irreducible tensor operators, recently constructed by Aizawa in terms of standard bosonic operators, are shown to provide a realization of the $h$-bosonic algebra corresponding to $n = 2$ and $m = 1$.

1 Introduction

It is well known that the Lie group $GL(2)$ admits, up to isomorphism, only two quantum group deformations with central determinant: the standard deformation $GL_q(2)$, and the Jordanian deformation $GL_h(2)$ \footnote{Presented at the 7th Colloquium “Quantum Groups and Integrable Systems”, Prague, 18–20 June 1998}. The quantum group $GL_h(2)$, or $SL_h(2)$, and the dual quantum algebra of the latter, $U_h(sl(2))$ \footnote{Directeur de recherches FNRS; E-mail: cquesne@ulb.ac.be}, have been the subject of many recent investigations, among which one may quote the determination of the $U_h(sl(2))$ universal $R$-matrix \footnote{1}. Two useful tools have been devised for the Jordanian deformation study. One of them is a contraction procedure that allows one to construct the latter from the standard deformation \footnote{2}. In other words, $GL_h(2)$ can be obtained from $GL_q(2)$ by a singular limit
of a similarity transformation. Such a technique has been generalized by Alishahiha to higher-dimensional quantum groups \[3\].

The other tool is a nonlinear invertible map between the generators of \( U_h(\mathfrak{sl}(2)) \) and \( \mathfrak{sl}(2) \) \[4\], yielding an explicit and simple method for constructing the finite-dimensional irreducible representations (irreps) of \( U_h(\mathfrak{sl}(2)) \). In addition, it has provided an explicit formula for \( U_h(\mathfrak{sl}(2)) \) Clebsch-Gordan coefficients (CGC) \[7\], as well as bosonic or fermionic realizations of irreducible tensor operators (ITO) for \( U_q(\mathfrak{sl}(2)) \) \[8\].

As a result, we will obtain \( \mathrm{GL}_h(\mathfrak{sl}(2)) \) algebras constructed by the present author some years ago \[9\], and recently rederived by Van der Jeugt on CGC for \( U_h(\mathfrak{sl}(2)) \) \[8\].

The purpose of the present communication is to apply the contraction procedure of Ref. \[4\], as generalized by Alishahiha \[2\], to the \( \mathrm{GL}_q(n) \times \mathrm{GL}_q(m) \)-covariant \( q \)-bosonic algebras constructed by the present author some years ago \[9\], and recently rederived by Fiore by another procedure \[10\]. As a result, we will obtain \( \mathrm{GL}_h(n) \times \mathrm{GL}_h(m) \)-covariant \( h \)-bosonic algebras. We will then consider the cases where \( n = 2 \), \( m = 1 \), and \( n = m = 2 \) in more detail, and establish some relations with the works of Aizawa on ITO \[8\], and of Van der Jeugt on CGC for \( U_h(\mathfrak{sl}(2)) \) \[10\].

## 2 Contraction of \( \mathrm{GL}_q(N) \)

The quantum group \( \mathrm{GL}_q(N) \) is defined by the \( RTT \)-relations, \( RTT' = T_1 T_2 = T_2' T_1' R' \), where \( T' = (T'_{ij}) \in \mathrm{GL}_q(N) \), \( T_1 = T' \otimes I \), \( T_2 = I \otimes T' \), and

\[
R' = R'_q = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i<j} e_{ii} \otimes e_{jj} + \left( q - q^{-1} \right) \sum_{i<j} e_{ij} \otimes e_{ji},
\]

with \( i, j \) running over 1, 2, \ldots, \( N \), and \( e_{ij} \) denoting the \( N \times N \) matrix with entry 1 in row \( i \) and column \( j \), and zeros everywhere else. An equivalent form of the \( RTT \)-relations is obtained by replacing \( R' = R'_{12} \) by \( R'_{21} \). Throughout this communication, \( q \)-deformed objects will be denoted by primed quantities, whereas unprimed ones will represent \( h \)-deformed objects.

Let us consider the similarity transformation \( R'' = (g^{-1} \otimes g^{-1}) R'(g \otimes g) \), \( T'' = g^{-1} T' g \), where \( g \) is the \( N \times N \) matrix defined by \( g = \sum_i e_{ii} + \eta e_{1N} \), in terms of some parameter \( \eta = h/(q - 1) \) \[4\] \[5\]. The \( RTT \)-relations simply become \( RTT'_{12}' = T_2' T_1'' R'' \).

Whenever \( q \) goes to 1, although \( \eta \) becomes singular, the latter have a definite limit \( RT_1 T_2 = T_2' T_1 R \), where \( T = \lim_{q \to 1} T'' \), and

\[
R = R_h = \lim_{q \to 1} R'' = \sum_{ij} e_{ii} \otimes e_{jj} + h \left[ e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11} + e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N} + 2 \sum_{i=2}^{N-1} (e_{ii} \otimes e_{iN} - e_{iN} \otimes e_{ii}) \right] + h^2 e_{1N} \otimes e_{1N}. \tag{2}
\]

The resulting \( R \)-matrix is triangular, i.e., it is quasi-triangular and \( R_{12}^1 = R_{21} \), showing that the two equivalent forms of \( RTT \)-relations for \( \mathrm{GL}_q(N) \) have actually the same contraction limit. The matrix elements \( T_{ij} \) generate \( \mathrm{GL}_h(N) \).
3 \( \text{GL}_q(n) \times \text{GL}_q(m) \)-covariant \( q \)-bosonic algebras

Let us consider two different copies of \( \text{GL}_q(N) \), corresponding to possibly different dimensions \( n, m \), and let us denote quantities referring to \( \text{GL}_q(n) \) by ordinary letters (\( R', T', \ldots \)), and quantities referring to \( \text{GL}_q(m) \) by script ones (\( \mathcal{R}', \mathcal{T}', \ldots \)). The elements \( T'_{ij} \), \( i, j = 1, 2, \ldots, n \), of \( \text{GL}_q(n) \), and \( T'_{st} \), \( s, t = 1, 2, \ldots, m \), of \( \text{GL}_q(m) \) are assumed to commute with one another.

In Ref. [3], \( q \)-bosonic creation and annihilation operators \( A'_{is}^+, \tilde{A}'_{is}, i = 1, 2, \ldots, n \), \( s = 1, 2, \ldots, m \), that are double ITO of rank \( (1|0)_n, (0|1)_m \), with respect to \( U_q(gl(n)) \times U_q(gl(m)) \), respectively, were constructed in terms of standard \( q \)-bosonic operators \( \prod a_{is}^+, a_{is}' \), \( i = 1, 2, \ldots, n \), \( s = 1, 2, \ldots, m \), acting in a tensor product Fock space \( F = \Pi_{i=1}^n \Pi_{s=1}^m F_{is} \). The annihilation operators \( A'_{is} \) contragredient to \( A_{is}^+ \) were also considered. Both sets of annihilation operators \( A'_{is} \) and \( A_{is} \), \( i = 1, 2, \ldots, n \), \( s = 1, 2, \ldots, m \), are related through the equation \( A'_{is} = A'_{is} \tilde{C}'_{is} \), where \( C' = C'C', \tilde{C}' = \sum_{i} (-1)^{n-i} q^{-(n-2i+1)/2} \epsilon_{ii}' \), and \( C' = \sum_{s} (-1)^{m-s} q^{-(m-2s+1)/2} \epsilon_{ss}' \), with \( i' = n - i + 1 \), \( s' = m - s + 1 \).

The operators \( A'_{is}, A_{is} \) or \( A'_{is}^+, \tilde{A}'_{is} \) generate with \( I = I I \) a \( U_q(gl(n)) \times U_q(gl(m)) \)-module algebra or \( \text{GL}_q(n) \times \text{GL}_q(m) \)-comodule algebra, whose \( q \)-commutation relations can be compactly written in coupled form by using \( U_q(gl(n)) \times U_q(gl(m)) \) CGC. When rewritten in componentwise form, such relations can be expressed in terms of the \( \text{GL}_q(n) \) and \( \text{GL}_q(m) \) \( R \)-matrices as [3]

\[
R'A_1^+A_2^+ = A_2^+A_1^+R', \quad R'A_2A_1 = A_1A_2R',
\]

\[
A_2A_1 = I_{21} + R'^{t_1}R'^{t_1} A_2^+A_2,
\]

or

\[
R'A_1^+A_2^+ = A_2^+A_1^+R', \quad R'A_1 = A_1^+R',
\]

\[
\tilde{A}_2^+\tilde{A}_1 = C_{12} + q^2 A_1^+\tilde{A}_2 R^{-1}\tilde{R}',
\]

where \( t_1 \) (resp. \( t_2 \)) denotes transposition in the first (resp. second) space of the tensor product, \( \tilde{R}' \) is defined by \( \tilde{R}' = qC_1'(R'^{-1})^t C'^{-1}_1 = qC_2'(R'^{-1})^{-1} C'^{-1}_2 \), and similar relations hold for \( \tilde{R}' \). The transformations leaving Eqs. (3) and (4) invariant are \( \varphi' \left( A^+ \right) = A^+T'^{t}T', \)

\[
\varphi' \left( \tilde{A}' \right) = T'^{-1}T'^{t} \tilde{A}', \quad \text{and } \varphi' \left( A^+ \right) = A^+T'^{t}T', \quad \varphi' \left( \tilde{A}' \right) = \tilde{A}' T'^{t} \tilde{A}',
\]

respectively. Here \( T' \) and \( T' \) are defined by \( T = C'^{-1}(T'^{-1})^t C' \), and \( \tilde{T}' = C'^{-1}(T'^{-1})^t C' \).

There exists another independent set of \( \text{GL}_q(n) \times \text{GL}_q(m) \)-covariant \( q \)-bosonic operators, which satisfy equations similar to Eq. (3) or (4), but with \( R'_{12} \rightarrow R'^{-1}_{21}, R'_{12} \rightarrow R'^{-1}_{21} \), implying \( q^{-1}R'^{-1}_{12} \rightarrow qR'_{21}, q^{-1}R'_{12} \rightarrow qR'_{21} \).

4 \( \text{GL}_h(n) \times \text{GL}_h(m) \)-covariant \( h \)-bosonic algebras

Let us apply the contraction procedure of Sec. 2 to the \( \text{GL}_q(n) \times \text{GL}_q(m) \)-covariant \( q \)-bosonic algebras, given in two equivalent forms in Eqs. (3) and (4), respectively. Since we now have
two copies of \( \text{GL}_q(N) \), we have to consider two transformation matrices 
\[ g = \sum e_{ii} + \eta e_{1n}, \]
and \( \bar{g} = \sum e_{is} + \eta e_{1m}, \) acting on \( \text{GL}_q(n) \) and \( \text{GL}_q(m) \), respectively.

Let us first consider Eq. (3), and introduce transformed \( q \)-bosonic operators defined by 
\[ A''^+ = A'^+ g, \ A'' = g^{-1} A', \]
where \( g = g g \). By using the property \( R'_{12} = R'^{12} \), and a similar one for \( R' \), it is straightforward to show that Eq. (3) becomes

\[
\begin{align*}
A''^+ A''^+ &= A''^+ A''^+ R'^{-1} R'^{12}, & A''^+ A''^+ &= R'^{-1} R'^{12} A''^+ A''^+,
A'' A''^+ &= I_{21} + R'^{m} R'^{m} A'' A''^+.
\end{align*}
\]

(5)

Since \( R \) and \( R' \) are triangular, in the \( q \to 1 \) limit the \( h \)-bosonic operators 
\[ A_i^+ = \lim_{q \to 1} A_i^+ A_i^+, \]
\[ A_i = \lim_{q \to 1} A_i^+ A_i^+ A_i^+ A_i^+ \]
satisfy the relations

\[
\begin{align*}
A_1^+ A_2^+ &= A_1^+ A_2^+ R R, & A_1 A_2 &= R R A_2 A_1, \\
A_2 A_1^+ &= I_{21} + R^T R^T A_1^+, A_2 A_1^+.
\end{align*}
\]

(6)

defining a \( \text{GL}_h(n) \times \text{GL}_h(m) \)-comodule algebra. The transformation \( \varphi(A^+) = A^+ T^T \),
\( \varphi(A) = T^{-1} A^{-1} T \), where \( T_{ij} \in \text{GL}_h(n) \), \( T_{st} \in \text{GL}_h(m) \), leaves Eq. (3) invariant.

Three properties of Eq. (4) are worth noting: (1) Had we started instead from the second form of Eq. (3) corresponding to the substitutions \( R'_{12} \to R'^{-1} \), \( R'^{12} \to R^{12} \), we would have obtained the same contraction limit (3), owing to the triangularity of \( R \) and \( R' \). (2) Contrary to what happens in the \( q \)-bosonic case, \( A_i^+ \) can never be considered as the adjoint of \( A_i^+ \), since no \( * \)-structure is known on \( \text{GL}_h(N) \). (3) For \( m = 1 \), Eq. (3) is consistent with the general form of \( \mathcal{H} \)-covariant deformed bosonic algebras for triangular \( \mathcal{H} \), obtained by Fiore [12].

Let us next consider Eq. (4), and define \( A''^+ = A'^+ g, \ A'' = A' g \), where \( g \) is the same as before. Compatibility of the \( A'' \) and \( A'' \) definitions with \( A'' = A'' C'' \), where \( C'' = C'' C'' \), leads to \( C'' = g C' g, \ C'' = g C' g \). A simple calculation shows that for \( n > 1 \), a contraction limit of \( C'' \) only exists for even \( n \) values, and is given by 
\[ C = \lim_{q \to 1} C'' = \sum_i (-1)^i e_{ii} + (n - 1) h e_{nn}, \]
Similar results hold for \( C = \lim_{q \to 1} C'' \).

Restricting the range of \( n, m \) values to \( \{1, 2, 4, 6, \ldots \} \), we obtain that after transformation, Eq. (4) contracts into

\[
\begin{align*}
A_1^+ A_2^+ &= A_1^+ A_2^+ R R, & \tilde{A}_1^+ \tilde{A}_2^+ &= \tilde{A}_1^+ \tilde{A}_1^+ R R, \\
\tilde{A}_2 A_1^+ &= C_{12} + A_1^+ \tilde{A}_2 \tilde{A}_1^+ \tilde{A}_1^+.
\end{align*}
\]

(7)

where \( C = C C \), \( \tilde{R} = \lim_{q \to 1} (g^{-1} \otimes g^{-1}) \tilde{R} (g \otimes g) = C_{12}^{-1} (R^{-1} T^1) C_1 = C_2^{-1} (R^2 T^1) C_2, \) and similarly for \( \tilde{R} \). For such restricted \( n, m \) values, Eq. (7) yields another form of the \( \text{GL}_h(n) \times \text{GL}_h(m) \)-covariant \( h \)-bosonic algebra defined in Eq. (3) for arbitrary \( n, m \) values. The transformation leaving Eq. (7) invariant is \( \varphi(A^+) = A^+ T T, \) \( \varphi(A) = \tilde{A} \tilde{T} \tilde{T}, \) where \( \tilde{T} = C^{-1} (T^{-1}) \). However, for \( n \) and/or \( m \in \{3, 5, 7, \ldots \} \), the contraction procedure does not preserve the equivalence between Eqs. (3) and (4), since only the former has a limit.
5 \( GL_h(2) \) and \( GL_h(2) \times GL_h(2) \)-covariant \( h \)-bosonic algebras

For \( n = 2, m = 1 \), by making the substitutions

\[
R = \begin{pmatrix}
1 & h & -h & h^2 \\
0 & 1 & 0 & h \\
0 & 0 & 1 & -h \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & -1 \\
1 & h
\end{pmatrix}, \quad R = C = 1, 
\]

(8)

into Eqs. (3) and (4), we obtain that \( A^+_1, A^+_2, A_1, A_2 \) satisfy the commutation relations

\[
\begin{align*}
[A^+_1, A^+_2] &= h \left( A^+_1 \right)^2, \quad [A_1, A_2] = hA^+_2, \\
[A_2, A^+_1] &= 0, \quad [A_1, A^+_2] = h \left( -A^+_1 A_1 - A^+_2 A_2 + hA^+_1 A^+_2 \right), \\
[A_1, A^+_1] &= [A^+_2, A^+_2] = I + hA^+_1 A_2,
\end{align*}
\]

(9)

while \( A^+_1, A^+_2, \tilde{A}_1, \tilde{A}_2 \) fulfil

\[
\begin{align*}
[A^+_1, A^+_2] &= h \left( A^+_1 \right)^2, \quad [\tilde{A}_1, \tilde{A}_2] = h\tilde{A}^+_1, \\
[\tilde{A}_1, A^+_1] &= 0, \quad [\tilde{A}_2, A^+_2] = h(I - A^+_1 \tilde{A}_2 + A^+_2 \tilde{A}_1 + hA^+_1 \tilde{A}_1), \\
[\tilde{A}_1, A^+_2] &= -[\tilde{A}_2, A^+_1] = I + hA^+_1 \tilde{A}_1.
\end{align*}
\]

(10)

Both sets of operators \( (A^+_1, A^+_2) \) and \( (\tilde{A}_1, \tilde{A}_2) \) may be considered as the components \( m = 1/2 \) and \( m = -1/2 \) of ITO of rank \( 1/2 \), or spinors, with respect to the quantum algebra \( U_h(\mathfrak{sl}(2)) \). By considering the adjoint action of the \( U_h(\mathfrak{sl}(2)) \) generators on such spinors, Aizawa [8] recently realized them in terms of standard bosonic operators \( a^+_1, a^+_2, a_1, a_2, \)

\[
A^+_1 = \left( 1 - \frac{h}{2} J_+ \right)^{-1} a^+_1, \quad A^+_2 = \left( 1 - \frac{h}{2} J_+ \right) a^+_2 + \frac{h}{2} \left( A^+_1 - 2a^+_2 J_0 \right), \\
\tilde{A}_1 = \left( 1 - \frac{h}{2} J_+ \right)^{-1} a_2, \quad \tilde{A}_2 = - \left( 1 - \frac{h}{2} J_+ \right) a_1 + \frac{h}{2} \left( \tilde{A}_1 - 2a_2 J_0 \right),
\]

(11)

where \( J_+ = a^+_1 a_2 \), and \( J_0 = (a^+_1 a_1 - a_2^2 a_2) / 2 \) are \( \mathfrak{sl}(2) \) generators. As can be easily checked, the operators (11) satisfy Eq. (10), as it should be.

Equation (10) can be recast into an alternative form by using coupled commutators

\[
\left[ U^{j_1}, V^{j_2} \right]_m^j = \left[ U^{j_1} \times V^{j_2} \right]_m^j - (-1)^\epsilon \left[ V^{j_2} \times U^{j_1} \right]_m^j, \quad \epsilon = j_1 + j_2 - j, 
\]

(12)

where \( U^{j_1} \) and \( V^{j_2} \) denote two ITO of rank \( j_1 \) and \( j_2 \) with respect to \( U_h(\mathfrak{sl}(2)) \), respectively,

\[
\left[ U^{j_1} \times V^{j_2} \right]_m^j = \sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 | jm \rangle \left[ U^{j_1} \right]_{m_1} \left[ V^{j_2} \right]_{m_2}.
\]

(13)
and $\langle , | \rangle_h$ denotes a $U_h(\text{sl}(2))$ CGC, as determined in Ref. [7]. The results read

$$[A^+, A^+_0]_0 = [\tilde{A}, \tilde{A}]_0 = [\tilde{A}, A^+]_m = 0, \quad [\tilde{A}, A^+]_0 = \sqrt{2}I.$$  \hfill (14)

For $n = m = 2$, $R$ and $C$ take the same form as $R$ and $C$ in Eq. (8). Relations similar to those in Eqs. (9) and (10) can be easily written. The operators $A^i_{is}, \tilde{A}^i_{is}, i, s = 1, 2$, may now be considered as the components of double spinors with respect to $U_h(\text{sl}(2)) \times U_h(\text{sl}(2))$, and they satisfy the coupled commutation relations

$$[A^+, A^+]_{m,0}^1 = [A^+, A^+]_{0,m'}^1 = [\tilde{A}, \tilde{A}]_{m,0}^1 = [\tilde{A}, \tilde{A}]_{0,m'}^1 = 0,$$

$$[\tilde{A}, A^+]_{j,j'}_{m,m'} = 2\delta_{j,0}\delta_{j',0}\delta_{m,0}\delta_{m',0}I,$$ \hfill (15)

where in the definition of coupled commutators there now appear two $\epsilon$ phases, and two $U_h(\text{sl}(2))$ CGC.

It is remarkable that both Eqs. (14) and (15) are formally identical with those for $\text{sl}(2)$ and $\text{sl}(2) \times \text{sl}(2)$, respectively. Contrary to what happens in the $q$-bosonic case where the commutators are $q$-deformed, here all the dependence upon the deforming parameter $h$ is contained in the CGC.

6 Conclusion

In this communication, we showed that $\text{GL}_h(n) \times \text{GL}_h(m)$-covariant $h$-bosonic algebras can be obtained by contracting $\text{GL}_q(n) \times \text{GL}_q(m)$-covariant $q$-bosonic ones. Some extensions of the present work to $h$-fermionic and multiparametric algebras are under current investigation.

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