Three-body dispersion-relation N/D equations for the coupled decay channels
\( \bar{p}p \ (J^{PC} = 0^{-+}) \rightarrow \pi^0 \pi^0 \pi^0, \ \eta \pi^0 \pi^0, \ \eta \eta \pi^0, \ \bar{K} K \pi^0 \)

A. V. Anisovich
Petersburg Nuclear Physics Institute
Gatchina, St. Petersburg, 188350 Russia
e-mail: aanisovi@thd.pnpi.spb.ru

Abstract

During several years the data on different channels \( \bar{p}p \ (J^{PC} = 0^{-+}) \)
\( \rightarrow \ 3 \) mesons presented by Crystal Barrel Collaboration were suc-
cessfully analyzed by extracting the leading amplitude singularities — pole singularities — with the aim to obtain information about two-meson resonances. But these analyses do not take into account three-body final state interactions (FSI) in an explicitly correct way. This paper is devoted to the consideration of this problem. Here I demonstrate how the coupled three-body equations may be written for the \( \pi^0 \pi^0 \pi^0, \ \eta \pi^0 \pi^0, \ \eta \eta \pi^0, \ \bar{K} K \pi^0 \) channels in the \( \bar{p}p \) annihilation at rest using three-body dispersion relation N/D-method.

1 Introduction

During several years Crystal Barrel Collaboration presents the high statistics data on three meson production from the \( \bar{p}p \) annihilation at rest. These data were successfully analyzed (see, for example, [1, 2, 3, 4]) with the aim to search for new meson resonances in the region 1000-1600 MeV using the K-matrix formalism or a simplified dispersion N/D-method. There is a strong expectation based on QCD [5] as well as on lattice calculations [6] that the lowest scalar glueball is located in this region. Thus, the identification of scalar resonances in the mass region 800-2000 MeV and their classification in \( \bar{q}q \)-nonets should be done to trap the lightest scalar glueball. The quark/qluon structure of these resonances may be determined from the analysis of coupling constants of these states to pseudoscalar mesons [7]. The question is: is the K-matrix approximation – or simplified N/D-method – sufficient for this purpose?

In this paper the three-body dispersion N/D-method is presented, which is based on the two-body unitarity condition and allows one to take into account FSI of three mesons. The basic principles of this technique were developed in 1960s and they were applied to the calculation of FSI in the K-meson decay into three pions (see [8, 9] and references
therein). Recently this technique was used for the calculation of the \( \eta \to 3\pi \) decay \[11\] and \( \phi \)-meson production in the \( pp \) annihilation at rest \[12\].

This paper is organized as follows. In Section 2 the two-particle discontinuity of the decay amplitude is written out, and the integral equation is derived. It is shown how to take into account not only the S-wave binary interactions but also interactions with higher angular momentum (P- and D-wave). In Section 3 this technique is generalized for the resonance and non-resonance two-particle interactions. The complete three-body dispersion equation is written, that may be used for the analysis of various three-particle reactions. In Section 4 it is shown how the coupled three-body equations could be written for \( \pi^0\pi^0\pi^0 \), \( \eta\pi^0\pi^0 \), \( \eta\eta\pi^0 \), \( \bar{K}K\pi^0 \) channels in the \( pp \) annihilation from the \( 0^- \) state.

2 Two-particle discontinuity of the decay amplitude

Let us start with the consideration of the decay of a scalar particle with the mass \( M \) and momentum \( P \) into three scalar particles with masses \( m_1, m_2 \) and \( m_3 \) and momenta \( k_1, k_2 \) and \( k_3 \). I would like to show that the energy dependence of this transition amplitude may be derived from the unitarity and analyticity. The imaginary part of the amplitude \( A_n \) for the transition \( 1 \to n \) is given by

\[
Im M_n = \frac{1}{2} \sum_{n'} (2\pi)^4 \delta(P - \sum_{i=1}^{n'} k_i) T_{n'n}^* M_{n'n},
\]

where \( T_{n'n} \) is the scattering matrix. Equation (1) may be simplified if the approximation is used where \( i \) binary interactions of particles are taken into account, three-particle forces being neglected; \( ii \) there are no transitions through intermediate states with more than three particles. It is also supposed that there are no transitions through intermediate states involving other particles; still, this is not a crucial restriction. Below it is shown how the production of new particles in the intermediate state may be taken into account. In this approximation one may write eq.(1) as follows:

\[
Im M_3 = \frac{1}{2} (2\pi)^4 \delta(P - k_1 - k_2 - k_3) T_{33}^* M_3.
\]

Three-particle forces being neglected, the amplitude \( M_3 \) for the three-particle production is a sum of four terms: \( i \) the direct production amplitude \( \lambda(s_{12}, s_{13}, s_{23}) \), which is assumed, for the sake of simplicity, to be free of singularities; \( ii \) the amplitude \( a_{12} \), where the last interaction is that of particles 1 and 2; likewise, there are similar terms \( iii \) \( a_{13} \) and \( iiii \) \( a_{23} \). I take into account FSI with different orbital momenta \( L \), hence the amplitude \( a_{ij} \) is written as

\[
a_{ij} = \sum_{L} F_{L} A_{ij}^{L}(s_{ij}).
\]

Here \( s_{ij} \) is the two-particle invariant mass squared, \( s_{ij} = (k_i + k_j)^2 \). The function \( F_{L} \) defines the angular distribution of decaying particles. In the case of the S-wave binary interaction \( F_{0} = 1 \), but for the higher-wave interaction the amplitude depends on the angle between interacting and spectator particles as well.
2.1 S-wave interaction

Let us begin with the simplest case of the S-wave pair interactions. The decaying amplitude is given by

\[ M(s_{12}, s_{13}, s_{23}) = \lambda(s_{12}, s_{13}, s_{23}) + A^0_{12}(s_{12}) + A^0_{13}(s_{13}) + A^0_{23}(s_{23}). \] (4)

The connected part of the matrix \( T_{33} \) is not known, so I cannot directly calculate the contribution from three-particle intermediate state into the imaginary part of \( M \) and write down corresponding dispersion integral. So, I will explore the two-particle unitarity condition to derive the integral equation for the amplitude \( A^0_{ij} \). This method was described in details in ref. [13]. The idea of such an approach is that one should consider the case of small external mass \( M < m_1 + m_2 + m_3 \), when only the scattering reactions are physically possible. A simple relation can be written in this case, and then I perform analytical continuation of the final equation over the mass \( M \) back to the decay region. Let us stress that the obtained expression differs from the contribution from the disconnected part of \( T \) into imaginary part \( M \): this result satisfies the three-body unitarity condition, in which three particles interact by pairs only [13].

So, I write down an ordinary unitarity condition for the scattering in the channel of particles 1 and 2. This means that \( (M + m_3) < s_{12} \) is assumed. The discontinuity of the amplitude is equal to

\[ disc_{12}M(s_{12}, s_{13}, s_{23}) = \]

\[ = \frac{1}{2} \int d\Phi_{12}(k_1, k_2) \left( \lambda + A^0_{12}(s_{12}) + A^0_{13}(s_{13}) + A^0_{23}(s_{23}) \right) A^0_{2 \to 2}(s_{12}), \] (5)

where \( d\Phi_{12}(k_1, k_2) \) is the phase volume of particles 1 and 2

\[ d\Phi_{12}(k_1, k_2) = (2\pi)^4 \delta^4(P - k_1 - k_2) \frac{d^4k_1 d^4k_2}{(2\pi)^6} \delta(m_1^2 - k_1^2) \delta(m_2^2 - k_2^2). \] (6)

\( A^0_{2 \to 2} \) is the S-wave two-particle scattering amplitude, which can be written in the dispersion N/D method as a series

\[ A^0_{2 \to 2}(s) = G^L_0(s)G^R_0(s) + G^L_0(s)B_0(s)G^R_0(s) + G^L_0(s)B_2^2(s)G^R_0(s) + \ldots = \]

\[ = \frac{G^L_0(s)G^R_0(s)}{1 - B_0(s)}, \] (7)

where \( G^L_0 \) and \( G^R_0 \) are left and right vertex functions and \( B(s) \) is the dispersion representation of the loop diagram:

\[ B_0(s) = \int_{(m_1 + m_2)^2}^{\infty} \frac{ds'}{\pi} \frac{G^L_0(s')\rho_{ij}(s')G^R_0(s')}{s' - s}, \] (8)

where \( \rho_{ij} \) is the two-particle phase space

\[ \rho_{ij}(s) = \frac{1}{16\pi s} \sqrt{[s - (m_1 + m_2)][s - (m_1 - m_2)]}. \] (9)
Vertex functions contain left-hand singularities related to the t-channel exchange diagrams, while B-function has singularities due to the elastic scattering. It is not specified from the consideration of the scattering amplitude $A^0_{2\rightarrow 2}$ of eq. (3) whether both vertices $G^L_0$ and $G^R_0$ have these singularities or only one of them. In the case of three-body decay the situation is quite opposite. On the first sheet the decay amplitude has only singularities at $s_{ij} = (m_i + m_j)^2$, which are associated with the elastic scattering in the subchannel of particles $i$ and $j$. This means that the vertex $G^R_0$ is analytical function. The simplest choice is

$$G^R_0 = 1$$

(10)

Now return to eq. (3). Since the functions $\lambda$ do not have the two-particle threshold singularity and thus do not have a discontinuity, the lhs of eq. (3) is

$$disc_{12} A^0_{12}(s_{12}) = disc_{12} A^0_{12}(s_{12}).$$

(11)

As was stressed in ref. [10], only one rescattering of particles 1 and 2 can be considered in the final state and a full set of binary rescatterings can be taken into account multiplying by $(1 - B_0(s_{12}))^{-1}$. Thus, the two-particle discontinuity in this special case of one rescattering is defined as

$$disc_{12} A^0_{12}(s_{12}) = \frac{1}{2} \int d\Phi_{12}(k_1, k_2) \left( \lambda(s_{12}, s_{13}, s_{23}) + A^0_{13}(s_{13}) + A^0_{23}(s_{23}) \right) G^L_0(s_{12}).$$

(12)

It is convenient to perform the phase space integration in eq. (12) in the center-of-mass system of particles 1 and 2. In this frame

$$s_{13} = m_1^2 + m_3^2 - 2k_{10}k_{30} + 2z_{13} \left| \vec{k}_1 \right| \left| \vec{k}_3 \right|,$$

$$k_{10} = \frac{s_{12} + m_1^2 - m_2^2}{2\sqrt{s_{12}}}, \quad \left| \vec{k}_1 \right| = \sqrt{k_{10}^2 - m_1^2},$$

$$k_{30} = \frac{s_{12} + m_3^2 - s}{2\sqrt{s_{12}}}, \quad \left| \vec{k}_3 \right| = \sqrt{k_{30}^2 - m_3^2},$$

(13)

where $z_{13} = \cos \theta_{13}$, and $\theta_{13}$ is the angle between particles 1 and 3 in the cms of particles 1 and 2, and $s = M^2$. The expression for $s_{23}$ is obtained from eq. (13) by the replacement $1 \leftrightarrow 2$. From eq. (12) one has:

$$disc_{12} A^0_{12}(s_{12}) = \left( \lambda_S(s_{12}) + \langle A^0_{13}(s_{13}) \rangle_0 + \langle A^0_{23}(s_{23}) \rangle_0 \right) G^L_0(s_{12}) \rho_{12}(s_{12}),$$

(14)

where the following notation is used:

$$\langle A^0_{i3}(s_{i3}) \rangle_0 = \int_{C_i(s_{12})} \frac{dz_{i3}}{2} A^0_{i3}(s_{i3}).$$

(15)

$\lambda_S$ is the S-wave projection of $\lambda$:

$$\lambda_S(s_{12}) := \int_{-1}^{1} \frac{dz_{13}}{2} \lambda(s_{12}, s_{13}, s_{23}).$$

(16)

Analytical continuation over external mass $M$ from the scattering to the decay region allows one to define correctly the rules of integration over $z$. This integration should be
carried out along the contour \( C_1(s_{12}) \), whose position at different \( s_{12} \) is described in detail in ref. [8]. Here I would like to note that only at small \( s_{12} \),

\[
(m_1 + m_2)^2 \leq s_{12} \leq \frac{m_1 s}{m_i + m_3} + \frac{m_3}{m_i + m_3} (m_1 + m_2 - m_i)^2 - m_i m_3,
\]

(17)
it coincides with the phase space integration contour

\[-1 \leq z_{i3} \leq 1,
\]

(18)
and it contains an additional piece at larger \( s_{12} \).

Equation (14) allows us to write down the dispersion integral for the amplitude with one pair rescattering in the final state:

\[
A_{12}^0(s_{12}) = \lambda_S(s_{12}) B(s_{12}) + \int_0^\infty \frac{d s'_{12}}{\pi} \frac{\rho_{12}(s'_{12}) G_{1}^0(s'_{12})}{s'_{12} - s_{12}} (\langle A_{13}^0(s'_{13}) \rangle_{0} + \langle A_{23}^0(s'_{23}) \rangle_{0}).
\]

(19)
Here I exclude \( \lambda_S \) from the dispersion integral, but it is also possible to include it: in both cases the unitarity is satisfied but with different behaviour of the amplitude at the infinity, which cannot be defined by the unitarity and analyticity only. After the transition from one rescattering to the full set of binary interactions in the final state one has

\[
A_{12}^0(s_{12}) = \frac{\lambda_S(s_{12})}{1 - B_0(s_{12})} + \frac{1}{1 - B_0(s_{12})} \int_0^\infty \frac{d s'_{12}}{\pi} \frac{\rho_{12}(s'_{12}) G_{1}^0(s'_{12})}{s'_{12} - s_{12}} (\langle A_{13}^0(s'_{13}) \rangle_{0} + \langle A_{23}^0(s'_{23}) \rangle_{0}).
\]

(20)
Let us now check that the extraction of the final state interaction does not violate the unitarity condition (4). To calculate the lhs of eq. (4), the equation (20) is rewritten as follows:

\[
A_{12}^0(s_{12}) = \frac{1 - B_0(s_{12})}{1 - B_0(s_{12})} \left( b^\lambda(s_{12}) + J(s_{12}) \right),
\]

(21)
where

\[
b^\lambda(s_{12}) = \lambda_S(s_{12}) B_0(s_{12}),
\]

(22)
\[
J(s_{12}) = \int_0^\infty \frac{d s'_{12}}{\pi} \frac{\rho_{12}(s'_{12}) G_{1}^0(s'_{12})}{s'_{12} - s_{12}} (\langle A_{13}^0(s'_{13}) \rangle_{0} + \langle A_{23}^0(s'_{23}) \rangle_{0}).
\]

(23)
Thus,

\[
\text{disc}_{12} A_{12}^0(s_{12}) = \frac{1}{1 - B_0(s_{12})} \left( \text{Im} B_0(s_{12}) b^\lambda(s_{12}) + \langle 1 - B_0(s_{12}) \rangle \text{Im} b^\lambda(s_{12}) + \text{Im} B_0(s_{12}) J(s_{12}) + (1 - B_0(s_{12})) \text{disc} J(s_{12}) \right).
\]

(24)
Taking into account that

\[
\text{disc} B_0(s_{12}) = \text{Im} B_0(s_{12}) = \rho_{12}(s_{12}) G_{1}^0(s_{12}),
\]

\[
\text{disc} b^\lambda(s_{12}) = \rho_{12}(s_{12}) G_{1}^0(s_{12}) \lambda_S,
\]

(25)
\[
\text{disc} J(s_{12}) = \rho_{12}(s_{12}) G_{1}^0(s_{12}) (A_{13}^0 + A_{23}^0),
\]

I get the rhs of eq. (4), hence, the unitarity condition is fulfilled.


2.2 P-wave interaction in the final state

First, determine the structure of the amplitude, where particles 1 and 2 interact in the P-wave and the particle 3 is spectator. The easiest way to do this is to transform the decay amplitude into scattering amplitude. To perform this transformation the antiparticle 3 with the momentum \((-k_3)\) is considered instead of particle 3 itself. The assumed form of the amplitude is \(O_{1\mu}Q_{1\mu}A_{12}(s_{12})\). Operator \(Q_1\) describes the P-wave angular distribution of particles 1 and 2 in the final state, it is defined as relative momentum of these particles

\[
Q_{1\mu} = k_{1\mu} - k_{2\mu} - \frac{m_1^2 - m_2^2}{s_{12}}(k_1 + k_2)_\mu.
\]  

(26)

Operator \(O_1\) should be constructed as relative momentum of initial state and antiparticle 3. Taking into account that \(Q_1\) turns into \(\vec{k}_{12}\), \(O_1\) is defined as

\[
O_{1\mu} = k_{3\mu}.
\]  

(27)

As is seen from eq. (2),

\[
F_1 = O_{1\mu}Q_{1\mu},
\]  

(28)

and it is easy to find out that \(F_1\) is proportional to \(z_{13}\) in the cms of particles 1 and 2.

Hereafter the procedure of the above section is used to write down the two-particle unitarity condition and the integral equation for \(A_1\). The P-wave two-particle scattering amplitude of particles 1 and 2 in the N/D method can be written as

\[
A_{12}(s_{12}) = Q_{1\mu} \frac{G_L^T(s_{12})}{1 - B_1(s_{12})} Q_{1\mu},
\]  

(29)

where the B-function is equal to

\[
B_1(s_{ij}) = \int \frac{ds'}{(m_i + m_j)^2} \frac{G_L^T(s')\rho_{ij}(s')\langle Q_{1\mu}Q_{1\mu}\rangle}{\pi s' - s_{ij}},
\]  

(30)

and \(\langle . . . \rangle\) means the averaging over space angle.

As the first step, one should consider triangle diagram with one P-wave rescattering of particles 1 and 2. The discontinuity of this diagram is equal to:

\[
disc_{12} A_{12}(s_{12}) = \frac{1}{2} \int d\Phi_{12}(k_1, k_2)a_{13}(s_{13}, z_{13})Q_{1\mu}G_L^T(s_{12}) =
\]

\[
= G_L^T(s_{12})\rho_{12}(s_{12}) \int \frac{d\Omega}{4\pi}Q_{1\mu}a_{13}(s_{13}, z_{13}).
\]  

(31)

The integration over space angle is performed in the c.m. frame of particles 1 and 2. In this frame \(Q_1\) turns into \(\vec{k}_{12}\). The z-axis being directed along \(\vec{k}_3\), the components of \(\vec{k}_{12}\) are equal to

\[
k_{12x} = k_{12} \sin \theta_{13} \cos \phi, \quad k_{12y} = k_{12} \sin \theta_{13} \sin \phi, \quad k_{12z} = k_{12} \cos \theta_{13}
\]

\[
k_{12} = \sqrt{\frac{1}{s_{12}} [s_{12} - (m_1 + m_2)^2][s_{12} - (m_1 - m_2)^2]}.
\]  

(32)
where $\phi$ is azimuthal angle. The integration over $\phi$ in eq. (31) keeps the components of $Q_{1\mu}$ with $\mu = z$ only:

$$
\int \frac{d\Omega}{4\pi} Q_{1x} \ldots = \int \frac{d\Omega}{4\pi} Q_{1y} \ldots = 0,
$$
$$
\int \frac{d\Omega}{4\pi} Q_{1z} \ldots = k_{12} \int \frac{dz_{13}}{2} z_{13} \ldots
$$

Let us introduce the vector $k_{3\mu}^{\perp}$ with only $z$ component unequal to zero in the cms:

$$
k_{3\mu}^{\perp} = k_{3\mu} - p_{\mu} \frac{(p \cdot k_3)}{p^2},
$$

where $p = k_1 + k_2$. Then it follows from eq. (31)

$$
disc = k_{3\mu}^{\perp} \rho_{12}(s_{12}) G_{1}^{L}(s_{12}) \frac{k_{12}}{\sqrt{-(k_{3}^{\perp})^2}} \int_{C_{3}(s_{12})} \frac{dz_{13}}{2} z_{13} a_{13}(s_{13}, z_{13}),
$$

where

$$
-(k_{3}^{\perp})^2 = \frac{[(M - m_3)^2 - s_{12}][(M + m_3)^2 - s_{12}]}{4s_{12}}.
$$

The invariant part of the P-wave amplitude can be written as a dispersion integral over the energy squared of particles 1 and 2 in the intermediate state, while $k_{3}^{\perp}$ in eq. (37) defines the operator structure of the P-wave amplitude. Thus, the P-wave triangle diagram is equal to

$$
k_{3\mu}^{\perp} \int_{(m_1 + m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{\rho_{12}(s'_{12}) G_{1}^{L}(s'_{12})}{s'_{12} - s_{12}} \langle a_{13} \rangle_1,
$$

where

$$
\langle a_{13} \rangle_1 = \frac{k_{12}}{\sqrt{-(k_{3}^{\perp})^2}} \int_{C_{3}(s_{12})} \frac{dz}{2} z a_{13}(s_{13}, z).
$$

Equation (37) should be multiplied by the operator $Q_1$ which describes angular distribution of particles 1 and 2 in the final state. Using $k_{3\mu}^{\perp} Q_{1\mu} = O_{1\mu} Q_{1\mu}$, one may conclude that the operator part of $a_{12}$ is correctly reconstructed. To take into account binary rescattering in the final state it is necessary to multiply eq. (37) by the factor $(1 - B_1(s_{12}))^{-1}$.

The same steps should be done, if $a_{13}$ is replaced by the direct production term $\lambda(s_{12}, s_{13}, s_{23})$. Still, the energy dependence of $\lambda$ is not usually known, so use a simpler assumption is used. Let us introduce the constants $\lambda_{ij}^P$, which define the direct production amplitude of particles $i$ and $j$ in the P-wave. Then the following integral equation for the amplitude $A_{12}$ can be written:

$$
A_{12}(s_{12}) = \frac{\lambda_{12}^P}{1 - B_1(s_{12})} + \frac{1}{1 - B_1(s_{12})} \int_{(m_1 + m_2)^2}^{\infty} \frac{ds'_{12}}{\pi} \frac{\rho_{12}(s'_{12}) G_{1}^{L}(s'_{12})}{s'_{12} - s_{12}} \left(\langle a_{13} \rangle_1 + \langle a_{23} \rangle_1\right).
$$
2.3 D-wave interaction in the final state

Likewise the case of D-wave interaction in the final state may be investigated. The amplitude, where particles 1 and 2 have the last interaction in the D-wave, has a form $O_2 Q_2 A_{12}^2(s_{12})$. The operator $Q_2$ is a traceless tensor of the range 2 constructed of relative momenta of particles 1 and 2:

$$Q_{2\mu\nu} = k_{12\mu} k_{12\nu} - \frac{1}{3} k_{12}^2 g_{\mu\nu}^{\perp},$$  \hspace{1cm} (40)

where

$$k_{12\mu} = k_{1\mu} - k_{2\mu} - \frac{m_1^2 - m_2^2}{s_{12}} (k_1 + k_2)_\mu,$$  \hspace{1cm} (41)

and

$$g_{\mu\nu}^{\perp} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \hspace{1cm} (42)$$

Another operator, $O_2$, can be defined as

$$O_{2\mu\nu} = k_{3\mu} k_{3\nu}. \hspace{1cm} (43)$$

Below it will be proved that $O_2$ can be defined in this way. Let us note here that the operator part of the D-wave decaying amplitude is proportional to $3 z_2^2 - 1$.

Let us define the D-wave scattering amplitude of particles 1 and 2 in N/D representation as

$$A_{2 \to 2}^2(s_{12}) = Q_{2\mu\nu} \frac{G_L^T(s_{12})}{1 - B_2(s_{12})} Q_{2\mu\nu}. \hspace{1cm} (44)$$

The discontinuity of the triangle diagram with the D-wave rescattering of particles 1 and 2 is equal to

$$\text{disc}_{12} A_{12}^2(s_{12}) = \frac{1}{2} \int d\Phi_{12}(k_1, k_2) a_{13}(s_{13}, z_{13}) Q_{2\mu\nu} G_L^T(s_{12}) = \frac{d\Omega}{4\pi} Q_{2\mu\nu} a_{13}(s_{13}, z_{13}). \hspace{1cm} (45)$$

The integration over space angle is performed in the c.m. frame of particles 1 and 2. Equation (32) is used, and only the following components of tensor $k_{12\mu} k_{12\nu}$ are not equal to zero:

$$\int \frac{d\Omega}{4\pi} k_{12x} k_{12x} \ldots = \int \frac{d\Omega}{4\pi} k_{12y} k_{12y} \ldots = k_{12}^2 \int \frac{dz_{13}}{4} (1 - z_{13}^2) \ldots$$

$$\int \frac{d\Omega}{4\pi} k_{12z} k_{12z} \ldots = k_{12}^2 \int \frac{dz_{13}}{2} z_{13}^2 \ldots \hspace{1cm} (46)$$

To perform the space integration in the invariant form, let us define the tensor $g_{\mu\beta}^{\perp\perp}$ as

$$g_{\mu\nu}^{\perp\perp} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{k_{13\mu} k_{13\nu}}{(k_3^2)^2}. \hspace{1cm} (47)$$
In the c.m. frame of particles 1 and 2 the tensor $g^\perp_\mu\nu$ has only two non-zero diagonal elements:

$$g^\perp_\mu\beta \rightarrow \text{diag}(0, -1, -1, 0),$$ \hspace{1cm} (48)

Thus, eq. (46) can be rewritten as

$$\int \frac{d\Omega}{4\pi} k_{12\mu} k_{12\nu} \ldots = -g^\perp_\mu_\nu k_{12}^2 \int \frac{dz_{13}}{4} (1 - z_{13}^2) \ldots - \frac{k_{13}^\perp k_{3\nu}}{(k_3^\perp)^2} k_{12}^2 \int \frac{dz_{13}}{2} z_{13}^2 \ldots$$ \hspace{1cm} (49)

Equation (49) should be placed into eq. (45) for the discontinuity of triangle diagrams. Taking into account that this expression must be multiplied by the external operator $Q_2$, one can see that only the third term in eq. (49) contributes into eq. (45). Moreover, one could replace $k_3^\perp$ by $k_3$. Finally, as follows from eq. (45),

$$\text{disc} = k_{3\mu} k_{3\nu} \rho_{12}(s_{12}) G^L_{2}(s_{12}) \langle a_{13} \rangle_2,$$ \hspace{1cm} (50)

where

$$\langle a_{13} \rangle_2 = \frac{k_{12}^2}{(k_{3}^\perp)^2} \int_{C_{3}(s_{12})} \frac{dz}{4} (1 - 3z^2) a_{13}(s_{13}, z).$$ \hspace{1cm} (51)

The invariant part of the D-wave amplitude can be written as a dispersion integral and a full set of binary rescatterings in the final state are defined by the factor $(1 - B_2(s_{12}))^{-1}$. As a result, I come to the following integral equation for the amplitude $A_{12}^2$:

$$A_{12}^2(s_{12}) = \lambda_{ij}^D \frac{B_2(s_{12})}{1 - B_2(s_{12})} +$$

$$\frac{1}{1 - B_2(s_{12})} \int_{(m_1 + m_2)^2}^{\infty} \frac{ds_{12}}{\pi} \frac{\rho_{12}(s_{12}) G^L_{2}(s_{12})}{s_{12} - s_{12}} (\langle a_{13} \rangle_2 + \langle a_{23} \rangle_2),$$ \hspace{1cm} (52)

where the constant $\lambda_{ij}^D$ stands for the direct production amplitude of particles $i$ and $j$ in the D-wave.

3 Dispersion equations with resonance and non-resonance production of particles

In this Section more realistic case is considered, when there are resonance and non-resonance interactions between two particles. This situation happens, for example, in the $0^{++}$ wave of the pion-pion amplitude, where there is a non-resonance background and a set of resonances at the energies above 1 GeV.

I start with the dispersion representation of the two-particle amplitude for this particular case. As the simplest case, consider the S-wave one-channel amplitude. The first resonance term of the amplitude can be written as

$$\sum_a \frac{g_a(s)}{M_a^2 - s}.$$ \hspace{1cm} (53)

where $M_\alpha$ is a pure (non-physical) mass of resonance $\alpha$, and the function $g^\alpha(s)$ describes its decay into two particles. Generally, these functions can depend on $s$, so the following form is suggested:

$$g^\alpha(s) = g^\alpha \phi(s).$$

(54)

The second term of the amplitude with one virtual loop is equal to

$$\sum_{\alpha \alpha'} \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s} g^{(\alpha')} b(s) \frac{g^{(\alpha')2}(s)}{M_{\alpha'}^2 - s},$$

(55)

where $b(s)$ is loop diagram with the cutoff function. The behaviour of $\phi(s)$ at large $s$ (or at small distances $r$) is not known, and the simplest way to avoid these uncertainties is to assume that the contribution from $r < r_0$ is equal to zero. Then the cutoff function $\Lambda(s)$ is defined as follows:

$$\Lambda(s) = \int d^3r \ e^{ik\cdot\vec{r}} \Theta(r - r_0) \int \frac{d^3k'}{(2\pi)^3} \ e^{-ik'\cdot\vec{r}} \phi^2(s'),$$

(56)

and $b(s)$ is equal to

$$b(s) = i\rho(s) + P \int \frac{ds'}{\pi} \ \rho(s') \ \Lambda(s').$$

(57)

Summing up the terms with different number of loops, one obtains the following expression for the amplitude

$$A = \frac{\sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s}}{1 - b(s) \sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s}}.$$

(58)

For non-resonance interaction of particles defined by vertex function $f(s)$, eq. (58) should be rewritten as follows:

$$A = \frac{\sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s} + f(s)}{1 - \left\{ b(s) \sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s} + b_f(s) \right\}},$$

(59)

where

$$b_f(s) = i\rho(s) + P \int \frac{ds'}{\pi} \ \rho(s') \ \Lambda_f(s'),$$

(60)

and $\Lambda_f(s)$ is given by eq. (56) with the replacement of $\phi^2(s)$ by $f(s)$. It should be noted here that if $\phi$ and $f$ are constant, then cutoff functions are equal to zero, and dispersion representation coincides with the K-matrix approach.

Equation (59) can be easily generalized for the case of the two-particle interaction with orbital momentum $L$:

$$A_L = Q_{L,\mu} \frac{\sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s} + f(s)}{1 - \left\{ b(s) \sum_\alpha \frac{g^{(\alpha)2}(s)}{M_\alpha^2 - s} + b_f(s) \right\}},$$

(61)

where $b(s)$ and $b_f(s)$ are given by eqs. (57) and (60), with the cutoff functions $\Lambda(s)$ and $\Lambda_f(s)$, correspondingly, which are equal to

$$\Lambda(s) = \int d^3r \ e^{ik\cdot\vec{r}} \Theta(r - r_0) \int \frac{d^3k'}{(2\pi)^3} \ e^{-ik'\cdot\vec{r}} \phi^2(s') \langle Q_{L,\mu} Q_{L,\mu} \rangle,$$

(62)
\[ \Lambda_f (s) = \int d^3 r \, e^{i\vec{k}' \vec{r}} \Theta(r - r_0) \int \frac{d^3 k'}{(2\pi)^3} \, e^{-i\vec{k}' \vec{r}} f(s') \langle Q_{L,\mu} Q_{L,\nu} \rangle \]  

(63)

The same formalism can be applied to the case of multichannel amplitude. The decay of resonance \( \alpha \) into particles \( m \) and \( n \) is given by the function \( g_{mn}^{(\alpha)}(s) = g_{mn}^{(\alpha)}(s) \) and the non-resonance transition from the channel with particles \( m \) and \( n \) to the channel with particles \( m' \) and \( n' \) is given by \( f_{mn;m'n'}(s) \). Here the expression for multichannel amplitude is not given in an explicit form, the recipe of its construction can be found, for example, in [14]. I would like only to note that the denominator in eq. (63), which describes the rescattering of particles in multichannel case, has the matrix form \((\hat{I} - \hat{B})^{-1}\), where \( \hat{I} \) is a unit matrix and the B-matrix element is equal to

\[ B_{mn;m'n'} = b_{mn;m'n'}(s) \sum_\alpha \frac{g_{mn}^{(\alpha)} g_{m'n'}^{(\alpha)}}{M_\alpha^2 - s} + \beta_{mn;m'n'}(s), \]  

(64)

\[ b_{mn;m'n'}(s) = i \rho_{mn}(s) + P \int \frac{ds'}{\pi} \frac{\rho_{mn}(s')}{s' - s} \Lambda_{mn;m'n'}(s'), \]  

(65)

\[ \beta_{mn;m'n'}(s) = i \rho_{mn}(s) + P \int \frac{ds'}{\pi} \frac{\rho_{mn}(s')}{s' - s} \Lambda_{f,mn;m'n'}(s'). \]  

(66)

Let us generalize the results of the previous sections and write down the dispersion equations for the three particle decay amplitude. The amplitude for the production of three different particles \( k, m, n \), which is denoted as \( M_{kmn} \), is the following

\[ M_{kmn} = a_{km;n}(s_{12}, z) + a_{nm;k}(s_{13}, z) + a_{kn;m}(s_{23}, z), \]  

(67)

where the first term in the rhs gives the amplitude with the last interaction between particles \( k \) and \( m \), and so on. The interaction of two particles, for example, \( n \) and \( m \), can happen in different channels. Here I neglect the isospin (let \( I = 0 \)), so the two-particle momentum describes the type of interaction. Hence, the amplitude \( a_{nm;k} \) can be written:

\[ a_{nm;k} = \sum_J F_J A_{nm;k}^{0J}, \]  

(68)

where \( A_{nm;k}^{0J} \) is the amplitude with the last interaction of particles \( n \) and \( m \) with the momentum \( J \) and isospin \( I = 0 \). The integral equation for \( A_{nm;k}^{0J} \) can be written as a generalization of eqs. (20), (39) and (52). I start with the term, where only particles \( m \) and \( n \) interact with each other and the particle \( k \) is spectator. The resonance and non-resonance production of these particles from the initial state is

\[ \lambda_{mn;k}^{0J}(s) = \sum_\alpha \frac{C_{k}^{(\alpha)} g_{mn}^{(\alpha)}(s)}{M_\alpha^2 - s} + \Phi_{mn;k}^{0J}, \]  

(69)

where \( C_{k}^{(\alpha)} \) is the production constant of resonance \( \alpha \) and particle \( k \), \( \Phi_{mn;k}^{0J} \) corresponds to the direct production of particles \( m \) and \( n \) with relative momentum \( J \). The rescattering of particles \( m \) and \( n \) gives the following amplitude:

\[ \sum_{(n'm')} \lambda_{n'm';k}^{0J} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{n'm';nm}^{0J}, \]  

(70)
where the summation over intermediate states with the production of \( n' \) and \( n'' \) particles is performed. The expression for triangle diagram can be written in the same way. Finally, I get the following three-body dispersion relation:

\[
A_{nm,k}^{0J} = \sum_{(n'm')} \lambda_{n'm';k}^{0J} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{n'm'n}^{0J} + \sum_{(n'm')} \sum_{(n''m'')} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{n'm'n}^{0J},
\]

\[
\int \frac{ds_{12}}{\pi} \rho_{m'n'}(s_{12}'|s_{12})\frac{N_{m'n'm'n}^{0J}(s_{12}'|s_{12})}{s_{12}' - s_{12}} \langle (a_{n'k;n'}(s_{13}';z) \rangle + \langle a_{n'k;n'}(s_{23}';z) \rangle, J),
\]

where

\[
N_{m'n'm'n}(s_{12}') = \Lambda_{j,m'n'm'n} + \sum_{\alpha} \frac{g_{m'n}^{(\alpha)} g_{mn}^{(\alpha)}(s)}{M_{\alpha}^2 - s} \tilde{\Lambda}(s'),
\]

and \( \tilde{\Lambda}(s') \) is the cutoff function obtained in eq. (72) with the help of \( \phi_{m'n'}(s) \).

### 4 Dispersion equations for the coupled decay channels \( \bar{p}p \) \((0^{-+}) \rightarrow \pi^0\pi^0\pi^0, \eta\pi^0\pi^0, \eta\eta\pi^0, K\bar{K}\pi^0 \)

The proton-antiproton annihilation in the state \( J^{PC} = 0^{-+} \) can originate from the two isospin states \( I = 0 \) and \( I = 1 \). Due to isospin conservation in strong interactions there exist two integral equations for the following decay channels:

- \( \bar{p}p \) \((IJ^{PC} = 10^{-+}) \rightarrow \pi^0\pi^0\pi^0, \eta\pi^0\pi^0, K\bar{K}\pi^0 \)
- \( \bar{p}p \) \((IJ^{PC} = 00^{-+}) \rightarrow \eta\pi^0\pi^0, K\bar{K}\pi^0 \)

Let us write integral equations for the decay from \( I = 1 \) state.

#### a) Reaction \( \bar{p}p \) \((10^{-+}) \rightarrow \pi^0\pi^0\pi^0 \)

For this reaction the only difference from the analysis given above is that one should take into account the isospin structure of the amplitude. The same analysis has been done in refs. [11], [12], where the unitarity condition in the \( \pi\pi \) channel was used to derive the amplitude for the \( \eta \rightarrow 3\pi^0 \) decay. This decay goes with violation of isospin symmetry, so in both cases initial states have the same quantum numbers. Because of that, here the results of refs. [10] and [11] are reproduced, and the contribution from the \( \eta\eta \) and \( K\bar{K} \) states is also taken into account. The annihilation amplitude into \( 3\pi^0 \) is

\[
M_{\pi^0\pi^0\pi^0} = a_{\pi^0\pi^0;\pi^0}(s_{12}, z) + a_{\pi^0\pi^0;\pi^0}(s_{13}, z) + a_{\pi^0\pi^0;\pi^0}(s_{23}, z),
\]

where

\[
a_{\pi^0\pi^0;\pi^0} = a_{\pi^0\pi^0;\pi^0}^{00} + \frac{4}{3} a_{\pi^0\pi^0;\pi^0}^{2},
\]

and \( a_{\pi^0\pi^0;\pi^0}^{2} \) is the amplitude with the last pions interacting in the isospin state \( I \). It should be noted that, due to the C-invariance, there is no term in eq. (74) with pion interactions in the state \( I = 1 \). For the channel with isospin \( I = 0 \) S- and D-wave interactions should be taken into account; this allows one to calculate properly the production of \( f_0 \) and \( f_2 \) resonances, so

\[
a_{\pi^0\pi^0;\pi^0}^{00} = A_{\pi^0\pi^0;\pi^0}^{00} + F_a A_{\pi^0\pi^0;\pi^0}^{02},
\]
In the channel with isospin $I = 2$ only S-wave interaction should be accounted for, therefore

$$a_0^{2\pi;\pi^0} = A_0^{2\pi;\pi^0}. \quad (76)$$

Integral equations for $A_{\pi\pi;\pi^0}^{0J}$ have the following form:

$$A_{\pi\pi;\pi^0}^{0J} = \lambda_{\pi\pi;\pi^0}^{0J}\{(\hat{I} - \hat{B})^{-1}\}_{\pi\pi;\pi^0}^{0J} + \{(\hat{I} - \hat{B})^{-1}\}_{\pi\pi;\pi^0}^{0J} \int \frac{ds'_{12} \rho_{\pi\pi}(s'_{12}) N_{\pi\pi;\pi^0}^{0J}(s'_{12}, s_{12})}{s'_{12} - s_{12}} \times$$

$$\times \left\{ \frac{2}{3} a_0^{0\pi;\pi^0}(s'_{13}, z) \right\}_J + \left\{ \frac{20}{9} a_0^{2\pi;\pi^0}(s'_{13}, z) \right\}_J + \left\{ \frac{4}{3} a_0^{1\pi;\pi^0}(s'_{23}, z) \right\}_J \right\} J + \Delta_{\eta\eta} + \Delta_{KK}, \quad (77)$$

where the contributions from the $\eta\eta$ and $KK$ intermediate states are:

$$\Delta_{\eta\eta} = \lambda_{\eta\eta;\pi^0}^{0J}\{(\hat{I} - \hat{B})^{-1}\}_{\eta\eta;\pi^0}^{0J} +$$

$$\left\{ \frac{\rho_{\eta\eta}}{(\hat{I} - \hat{B})^{-1}} \right\}_{\eta\eta;\pi^0}^{0J} \int \frac{ds'_{12} \rho_{\eta\eta}(s'_{12}) N_{\eta\eta;\pi^0}^{0J}(s'_{12}, s_{12})}{s'_{12} - s_{12}} \left\{ 2 a_{0\eta\eta}(s'_{13}, z) \right\}_J, \quad (78)$$

$$\Delta_{KK} = \lambda_{KK;\pi^0}^{0J}\{(\hat{I} - \hat{B})^{-1}\}_{KK;\pi^0}^{0J} +$$

$$\left\{ \frac{\rho_{KK}}{(\hat{I} - \hat{B})^{-1}} \right\}_{KK;\pi^0}^{0J} \int \frac{ds'_{12} \rho_{KK}(s'_{12}) N_{KK;\pi^0}^{0J}(s'_{12}, s_{12})}{s'_{12} - s_{12}} \left\{ 2 a_{0KK}(s'_{13}, z) \right\}_J. \quad (79)$$

The amplitude $a_0^{1\pi;\pi^0}$ in eq. (77) can be found only from the $\bar{p}p$ annihilation into charged pions, so in our approach it could be simply replaced by the direct production amplitude of $\rho^+\pi^-$. In the integral equation for $A_{\pi\pi;\pi^0}^{20}$ only $\pi\pi$ intermediate states are taken into account, so one has:

$$A_{\pi\pi;\pi^0}^{20} = \lambda_{\pi\pi;\pi^0}^{20}\{(\hat{I} - \hat{B})^{-1}\}_{\pi\pi;\pi^0}^{20} + \{(\hat{I} - \hat{B})^{-1}\}_{\pi\pi;\pi^0}^{20} \times$$

$$\times \int \frac{ds'_{12} \rho_{\pi\pi}(s'_{12}) N_{\pi\pi;\pi^0}^{20}(s'_{12}, s_{12})}{s'_{12} - s_{12}} \left\{ a_0^{0\pi;\pi^0}(s'_{13}, z) \right\}_0 + \left\{ \frac{1}{3} a_0^{2\pi;\pi^0}(s'_{23}, z) \right\}_0 + \left\{ a_0^{1\pi;\pi^0}(s'_{23}, z) \right\}_0 \right\} (80)$$

b) Reactions $\bar{p}p (0^{-+}) \rightarrow \eta\eta\pi^0$

In the $\eta\pi^0$ channel the S- and D-wave interaction with the production of $a_0$ and $a_2$ resonances are taken into consideration. The annihilation amplitude is the following:

$$M_{\eta\pi^0} = a_{\eta\pi;\pi^0}(s_{12}, z) + a_{\eta\pi;\eta}(s_{13}, z) + a_{\pi\eta;\eta}(s_{23}, z), \quad (81)$$

13
where
\[a_{\eta\pi;\pi^0} = A_{\eta\pi;\pi^0}^{00} + F_2 A_{\eta\pi;\pi^0}^{02},\]  
(82)
\[a_{\eta\pi^0;\eta} = A_{\eta\pi^0;\eta}^{00} + F_2 A_{\eta\pi^0;\eta}^{12}.\]  
(83)

The integral equation for the \(A^{0J}_{\eta\pi;\pi^0}\) amplitude has the following form:
\[A^{0J}_{\eta\pi;\pi^0} = \lambda^{0J}_{\eta\pi;\pi^0} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi;\pi^0}^{0J} + \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi;\eta}^{0J} \times \]
\[\times \int \frac{ds'}{\pi} \rho_{\pi\pi}(s') N^{0J}_{\pi\pi;\eta}(s,s') \left( \frac{2}{3} a^{0\eta}_{\pi\pi;\eta}(s', z) \right) J + \left( \frac{20}{9} a^{2\pi}_{\pi\pi;\eta}(s', z) \right) J + \]
\[\left( \frac{4}{3} a^{0\eta}_{\pi\pi;\eta}(s, z) \right) J + \Delta_{\eta\pi} + \Delta_{KK},\]  
(84)
where contributions from the \(\eta\pi\) and \(KK\) intermediate states are:
\[\Delta_{\eta\pi} = \lambda^{0J}_{\eta\pi;\pi^0} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi;\eta}^{0J} + \]
\[\left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi;\eta}^{0J} \int \frac{ds'}{\pi} \rho_{\eta\pi}(s') N^{0J}_{\eta\pi;\eta}(s,s') \left( 2 a^{0\eta}_{\eta\pi;\eta}(s', z) \right) J ,\]  
(85)
\[\Delta_{KK} = \lambda^{0J}_{KK;\pi^0} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{KK;\eta}^{0J} + \]
\[\left\{ (\hat{I} - \hat{B})^{-1} \right\}_{KK;\eta}^{0J} \int \frac{ds'}{\pi} \rho_{KK}(s') N^{0J}_{KK;\eta}(s,s') \left( 2 a^{0KK}_{KK;\pi^0}(s', z) \right) J .\]  
(86)

The integral equation for the \(A^{1J}_{\eta\pi^0;\eta}\) amplitude is:
\[A^{1J}_{\eta\pi^0;\eta} = \lambda^{1J}_{\eta\pi^0;\eta} \left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi^0;\eta}^{1J} + \]
\[\left\{ (\hat{I} - \hat{B})^{-1} \right\}_{\eta\pi^0;\eta}^{1J} \int \frac{ds'}{\pi} \rho_{\eta\pi^0}(s') N^{1J}_{\eta\pi^0;\eta^0}(s,s') \left( (a^{0\eta^0}_{\eta^0;\eta^0}(s', z) \right) J + (a^{0\eta^0}_{\eta^0;\eta^0}(s', z), J) .\]  
(87)

b) Reactions \(\bar{p}p (0^{-+}) \rightarrow KK\pi^0\)

These annihilation amplitudes are:
\[M_{\pi^0KK} = a^{0KK}_{KK}(s_{12}, z) + a^{0KK}_{KK}(s_{13}, z) + a^{0KK}_{KK}(s_{23}, z),\]  
(88)
As before, in the $K\bar{K}$ channel the S- and D-wave interactions and $K^*$ resonances in the $K\pi$ channel are accounted for. So, one has:

$$a_{KK;\pi^0} = A_{KK;\pi^0}^{00} + F_2 A_{KK;\eta^I}^{02} \tag{89}$$

$$a_{K\pi^0;\bar{K}} = A_{K\pi^0;\bar{K}}^{1/21} \tag{90}$$

For these amplitudes one can get the following integral equations:

$$A_{K\pi^0;\bar{K}}^{1/21} = \lambda_{K\pi^0;\bar{K}}^{1/21} \left( (\hat{I} - \hat{B})^{-1} \right)^{1/21} \pi^0_{\bar{K}K} \times$$

$$\times \int \frac{ds_{12}'}{\pi} \frac{\rho_{\pi^0 K}(s_{12}') N^{1/21}_{\pi^0 K;\pi^0 K}(s_{12}', s_{12})}{s_{12}' - s_{12}} (a_{KK;\pi^0}(s_{13}') J + a_{K\pi^0;\bar{K}}(s_{13}', z))^J \right) . \tag{91}$$

The integral equation for the $A_{KK;\pi^0}^{0J}$ amplitude has the following form:

$$A_{KK;\pi^0}^{0J} = \lambda_{\pi^0;\bar{K}K}^{0J} \left( (\hat{I} - \hat{B})^{-1} \right)^{0J} \pi^0_{\bar{K}K} \times$$

$$\times \int \frac{ds_{12}'}{\pi} \frac{\rho_{\pi^0 K}(s_{12}') N^{0J}_{\pi^0 K;\pi^0 K}(s_{12}', s_{12})}{s_{12}' - s_{12}} (\frac{2}{3} a_{\pi^0;\pi^0}(s_{13}', z))^J + \frac{20}{9} (\frac{a_{\pi^0;\pi^0}(s_{13}', z))^J +} \tag{92}$$

$$\langle \frac{4}{3} a_{\pi^0;\pi^0}(s_{13}', z))^J \rangle_J + \Delta_{\eta\pi} + \Delta_{KK},$$

where contributions from the $\eta\pi$ and $K\bar{K}$ intermediate states are as follows:

$$\Delta_{\eta\pi} = \lambda_{\eta\pi;\pi^0 K}^{0J} \left( (\hat{I} - \hat{B})^{-1} \right)^{0J} \eta_{\pi^0 K}$$

$$\left( (\hat{I} - \hat{B})^{-1} \right)^{0J} \eta_{\pi^0 K} \int \frac{ds_{12}'}{\pi} \frac{\rho_{\eta\pi}(s_{12}') N^{0J}_{\eta\pi;\pi^0 K}(s_{12}', s_{12})}{s_{12}' - s_{12}} \langle 2 a_{\pi^0;\eta\pi}(s_{13}', z))^J , \tag{93}$$

$$\Delta_{KK} = \lambda_{KK;\pi^0}^{0J} \left( (\hat{I} - \hat{B})^{-1} \right)^{0J} \pi^0_{\bar{K}K}$$

$$\left( (\hat{I} - \hat{B})^{-1} \right)^{0J} \pi^0_{\bar{K}K} \int \frac{ds_{12}'}{\pi} \frac{\rho_{KK}(s_{12}') N^{0J}_{KK;KK}(s_{12}', s_{12})}{s_{12}' - s_{12}} \langle 2 a_{\pi^0;KK}(s_{13}', z))^J , \tag{94}$$

A set of integral equations for the reactions $\bar{p}p (IJPC = 00^-) \rightarrow \eta\pi^0\pi^0, K\bar{K}\pi^0$ may be written in the same way.
5 Conclusion

To summarize, the two-particle discontinuity of the decay amplitude is written, and the integral equation is obtained which takes into account not only the S-wave binary interactions but also interactions with higher angular momenta (P- and D-wave). These equations are generalized for the resonance and non-resonance type of two-particle interactions. A complete three-body dispersion equation is derived, which may be used for the analysis of various three-particle reactions. I demonstrate how the coupled three-body equations can be written for the $\pi^0\pi^0\pi^0$, $\eta\pi^0\pi^0$, $\eta\eta\pi^0$, $KK\pi^0$ channels in the $\bar{p}p$ annihilation at rest.

This work is supported by the grants INTAS-93-0283 and RFFI 96-02-17934.

References

[1] C.Amsler et al., Phys. Lett B355 (1995) 425; B342 (1995) 433.
[2] C.Amsler et al., Phys. Lett B333 (1994) 277;
   V.V.Anisovich et al., Phys. Lett B323 (1994) 233.
[3] V.V.Anisovich, D.V.Bugg, A.V.Sarantsev and B.S.Zou, Phys. Rev. D50 (1994) 4412.
[4] V.V.Anisovich and A.V.Sarantsev, ”K-matrix analysis of 00++ amplitude in the mass region up to 1550 MeV”, preprint hep-ph/9603276.
[5] J.Paton and N.Isgur, Phys. Rev. D31 (1985) 2910;
   J.F.Donoghue, K.Johnson and B.A.Li, Phys. Lett. B99 (1981) 416;
   R.L.Jaffe and K.Johnson, Phys. Lett. B60 (1976) 201.
[6] G.S.Bali et al., Phys. Lett. B309 (1993) 378;
   II.Chen et al., Nucl. Phys. B34 (1994) 357.
[7] S.S.Gershtein, A.K.Likhoded and Yu.D. Prokoshkin, Z. Phys. C24 (1984) 305;
   C.Amsler and F.E.Close, Phys. Lett. B353 (1995) 385;
   V.V.Anisovich, Phys. Lett. B364 (1995) 195.
[8] V.V.Anisovich and A.A.Anselm, Sov. Phys. Usp. 88 (1966) 117.
[9] I.J.R.Aitchison, Phys. Rev. 137 (1965) 1070.
[10] A.V.Anisovich, Yad. Fiz. 58 (1995) 1467, [Phys. of Atomic Nuclei, 58 (1995) 1383].
[11] A.V.Anisovich and H.Leutwyler, Phys. Lett. B375 (1996) 335.
[12] A.V.Anisovich and E.Klempt, Z. Phys. A354 (1996) 197.
[13] I.J.R.Aitchison and R.Pasquier, Phys. Rev. 152 (1966) 1274.
[14] V.V.Anisovich, D.V.Bugg, A.V.Sarantsev, Nucl. Phys. A537 (1992) 501.