Quench dynamics of entanglement spectrum and topological superconducting phases in a long-range Hamiltonian

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We study the quench dynamics of entanglement spectrum (ES) in the Kitaev chain with variable-range pairing amplitudes quantified by power-law decay rate $\alpha$. Considering the post-quench Hamiltonians with flat bands, the degenerate behaviors of ES can be understood by the half-way winding number. We demonstrate that the ES crossings during its dynamics is able to characterize the topological phase transitions (TPTs) in both short-range ($\alpha > 1$) or long-range ($\alpha < 1$) sector. Novel properties of ES dynamics are revealed for the quench protocols in the long-range sector or with $\alpha$ as the quench parameter. Moreover, the characterization of TPTs via ES crossings is stable against energy dispersion in the long-range model.

I. INTRODUCTION

Topological superconductors have attracted considerable interests in recent years. The Majorana zero modes in those systems, being robust against disorder [1–4], play a key role in the realization of topological quantum computation [5–9]. One of the most intriguing topological superconductor is the Kitaev chain with long-range $p$-wave pairing terms, where novel topological phases with fractional winding numbers and massive Dirac edge states are found [10]. More importantly, the Hamiltonian can be realized in magnetic atomic chains [3, 11–13] with the long-range pairing induced by magnetic impurities [14–20].

The characterization of topological phase transitions (TPTs), beyond the Landau symmetry-breaking paradigm, is also of great significance. From the perspective of quantum information, it has been shown that the quantum coherence [21, 22], multipartite entanglement [23–26], bipartite entanglement entropy [27–29] and entanglement spectrum (ES) [30–32] of the ground state can detect the TPTs. Recently, the rapid developments of quantum simulation based on ultracold atoms [33–35], trapped ions [36] and superconducting qubits [37, 38] have stimulated the study of quench dynamics. It is therefore natural to extend the characterization of TPTs to an out-of-equilibrium regime [39–43]. For instance, the quench dynamics of ES in the topological insulators and superconductors with nearest-neighbor terms, involving the standard Su-Schrieffer-Heeger model [44] and Kitaev chain [45] are studied, suggesting its close relationship between TPTs [46–48]. Nevertheless, the investigation of the nonequilibrium behaviors of ES in long-range models remains limited, and the methods useful in short-range systems can be further generalized.

In this work, we explore the quench dynamics of ES in the Kitaev chain with long-range pairing, whose topological phase diagram is more complex than the previously studied models [10, 29]. The ES can be measured via the quantum state tomography efficiently implemented in various artificially-engineered platforms [49–51]. Therefore, our results can be tested by state-of-art quantum simulation experiments. The remainder is organized in what follows. In Sec. II, we briefly review the Hamiltonian and topological phases in this model, and the definition of ES. We also calculate the ES of ground state in this model and present a physical picture of our work. In Sec. III, we study the quench dynamics of ES in the Kitaev chain with variable range pairing, revealing several novel nonequilibrium properties of ES and demonstrating that the TPTs in the long-range Hamiltonian can be characterized by the quench dynamics of ES. In Sec. IV, we conclude and provide some outlooks.

II. PRELIMINARY

A. The model

We focus on the long-range Kitaev chain with power-law decay pairing terms [10, 29], as a generalization of the standard Kitaev chain with only nearest-neighbor terms [45]. The Hamiltonian reads

$$H = -\frac{t}{2} \sum_{i=1}^{N} (c_{i}^{\dagger} c_{i+1} + H.c.) - \mu \sum_{i=1}^{N} (c_{i}^{\dagger} c_{i} - \frac{1}{2})$$

$$+ \frac{\Delta}{2} \sum_{i=1}^{N} \sum_{l=1}^{N-l} \frac{1}{d_{l}^\alpha} (c_{i}^{\dagger} c_{i+l} + H.c.),$$

where $c_{i}$ is the fermion operator at site $i$, $t$ is the hopping parameter, $\mu$ is the chemical potential, $\Delta$ is the pairing amplitude and $\alpha$ is the power-law decay rate.

References:

[1]...
where \( c_i (c_i^\dagger) \) denotes the annihilation (creation) fermion operator at each site \( i \), \( N \) is the length of Kitaev chain, and \( t \) and \( \mu \) represent the hopping amplitude and the chemical potential, respectively. The amplitude of pairing \( \Delta \) decay with the parameter \( \alpha \) (the decay rate) of the distance \( d_i \). Here, the antiperiodic boundary conditions \( c_{i+N} = -c_i \), and the condition of closed chain, i.e., \( d_i = l \) for \( l \in [1, N/2] \), while \( d_i = N - l \) for \( l \in [N/2, N] \), are adopted.

By switching to the momentum space via the Fourier transformation, the Hamiltonian (1) can be written as

\[
H = \sum_k \epsilon_k \Psi_k^\dagger (d_k \cdot \sigma) \Psi_k
\]

with \( \Psi_k \) as the Nambu spinor, \( \sigma \) as the Pauli vector. The winding vector is

\[
d_k = (d_k^x, d_k^y, d_k^z) = (0, -0.5\Delta f_\alpha(k), - (\mu + t \cos k))
\]

where \( k = (2\pi/N)(n + 1/2) (n = 0, 1, \ldots, N - 1) \) and \( f_\alpha(k) = \sum_{\mu=1}^{N-1} \sin k/d_k^\mu \). The energy spectra is then \( \epsilon_k = |d_k| \). The topological phases in this model can be characterized by the \( Z \) topological invariant winding number \([52]\) defined as

\[
w = \frac{1}{2\pi} \oint dk \left( \frac{\partial_k d_k^x}{d_k^y} \right),
\]

which can be rewritten as \( w = (1/2\pi) \oint (ydz - zdy)/|d_k|^2 \) with \( y(z) \) as the \( y(z) \)-component of Eq. (2). Intuitively, it counts how many times \( d_k \) loops around the origin in the \( y-z \) plane. Thus, the winding number can also be obtained by simply plotting the trajectory of the winding vector Eq. (2). The phase diagram of the Hamiltonian (1) with \( \Delta = t = 1 \) (which is fixed in the rest of this work) is shown in Fig. 1(a), which differs from that of a conventional Kitaev chain in the short-range sector with \( \alpha < 1 \). In particular, the topological phase with a massive Dirac edge mode has winding number 1/2, while the winding number of the trivial phase is \(-1/2\).

Our quench protocol is as follows: The initial state is prepared as the ground state of a Hamiltonian \( H^f \) (the initial Hamiltonian). We then evolve it with the final Hamiltonian \( H^f \). Notice that the evolved state can be viewed as the ground state of the following Hamiltonian

\[
H(t) = e^{-iH^f t} H^i e^{iH^f t}
\]

In the momentum basis, this Hamiltonian can be similarly represented by its winding vector, whose dynamics reads \([46]\)

\[
\partial_t d_k(t) = 2d_k^\dagger \times d_k(t),
\]

which can be further solved as

\[
d_k(t) = \left[ 1 - \cos(2|d_k^\dagger(t)|) \right] [d_k^\dagger \cdot \hat{n}_k] n_k^I + \cos(2|d_k^\dagger(t)|) d_k^\dagger + \sin(2|d_k^\dagger(t)|) [d_k^\dagger \times \hat{n}_k] (\hat{n}_k^\dagger - d_k^\dagger/|d_k^\dagger|)
\]

with \( d_k^\dagger (d_k^\dagger) \) referring to the winding vector of \( H^f \) (\( H^i \)), and \( \hat{n}_k^I = -d_k^\dagger/|d_k^\dagger| \).

**FIG. 1.** (a) Phase diagram of the Hamiltonian (1) with \( \Delta = t = 1 \), characterized by the winding number and the trajectory of \( d_k \). (b) The first lowest ES \( \xi^{(1)} \) for the ground state of Hamiltonian (1) with \( \Delta = t = 1 \) as a function of \( \alpha \) and \( \mu \). (c) The energy spectrum for the Hamiltonian (1) with \( \Delta = t = 1 \) and \( \mu = 0 \). (d) is similar to (c) but with \( \mu = 0 \). Here, the system size is \( N = 200 \).

**B. Entanglement spectrum**

Next, we present the definition of ES. In a fermionic model, the reduced density matrix of a subsystem \( A \) has the form \([53]\)

\[
\rho_A \propto \exp(-\sum_q \Omega_q \gamma^\dagger_q \gamma_q),
\]

which is closely related to the single-particle ES defined as \([54]\)

\[
\xi_q = 1/\left[ 1 + \exp(-\Omega_q) \right]
\]

with \( \Omega_q > 0 \). The method of calculating the \( \Omega_q \) for the system Eq. (1) and its dynamics Eq. (4) is presented in Appendix A. In this work, we focus on the first and second lowest ES denoted as \( \xi^{(1)} \) and \( \xi^{(2)} \).
An exact correspondence between the ES and the spectrum of physical edge modes is given in Ref. [32]. Specifically, ES can be casted into the form

$$\xi_{\mu} = \frac{1}{2} + \frac{\lambda_{q}}{2},$$  \hspace{1cm} (9)

assuming that all ES are larger than 1/2. The $\lambda_{q}$ equals the energy spectrum of the corresponding spectrally flattened Hamiltonian. Since band-flattening in general does not change the topology of the system, topological edge modes can be directly read off by looking at the low-lying ES. As an example, topological superconductors in the BDI class is characterized by a Z topological index [55]. This topological invariant is directly related to the winding number $w$, and gives the number of massless edge modes on one edge. As a consequence, the lowest $w$ ES will be 1/2, and we refer to this phenomena as the ES crossing(s).

Before we study the quench dynamics of ES, we first illustrate the properties of ES for the ground states as a benchmark. As shown in Fig. 1(b), in the short-range sector $(\alpha > 1)$, the topological phase can be characterized by the ES for the ground states. The ES $\xi^{(1)} \approx 0.5$ for the topological phase while $\xi^{(1)} \approx 1$ in the trivial phase, corresponding to the presence or absence of massless edge mode. This difference is however less prominent in the long-range sector $(\alpha < 1)$. Since the long range topological phase features a massive edge mode, the lowest ES in general is not close to 0.5. Nevertheless, we could still pinpoint the phase boundary $\mu_c = 1$ by the sharp change of ES.

We also plot the energy spectrum as a function of $\alpha$ for $\mu = 0$ and $\mu = -3$ in Fig. 1(c) and (d) respectively. With $\mu = 0$, massive Dirac fermions are observed when $\alpha < 1$ and one massless edge state when $\alpha > 1$. However, with $\mu = -3$, there is no massless edge state when $\alpha > 1$. The results of energy spectrum reveal the mechanism of the TPTs driven by $\alpha$. It can be recognized that at the critical point $\alpha_c = 1$, there is no degeneracy of energy spectrum, which can explain that the phase boundary $\alpha_c = 1$ is less distinguishable, and the change of ES is continuous when crossing the phase boundary (shown in Fig. 1(b)). In addition, the more obvious boundary $\mu_c = 1$ in the long-range sector also corresponds to the behaviors of energy spectrum. In Ref. [10], it is seen that there is a degeneracy of energy spectrum at the critical point $\mu_c = 1$ for both short and long-range sector.

C. Physical picture

To focus on the topological properties of the quench dynamics, we can restrict ourselves to the case where $H^f$ is a band-flattened Hamiltonian. For concreteness, we can take the length of winding vector $\epsilon_k = |d_k|$ to be 1. With this condition the winding vectors $d_k(t)$ will process at the same velocity. The Hamiltonian $H(t)$ is thus time-periodic. It has been shown that for short-range systems, ES crossings will appear half-way through the time evolution, and the number of crossings are related to the dynamical topological indices. Here, we present a physical picture to relate the number of crossings to the topological indices of $H^t$ and $H^f$. Utilizing this picture, we will then present and analyze our result on the long range Kitaev chains and explain how the behaviors of the ES differ from the short range case.

We pay attention to the topological superconductors in BDI class. The winding vectors of systems belonging to BDI class will lie on the $y-z$ plane due to the symmetry constraints. In our quench protocol, $d'$ and $d''$ will satisfy this condition, while $d(t)$ will not lie in one plane for an arbitrary time instant. The only exception will be $t = \pi/2$ and $t = 0(\pi)$, while the latter case is simply $d''$ itself. The half-way ES crossings is then associated with the half-way winding vector, i.e., $d(\pi/2)$. Since the $d(\pi/2)$ still possesses the symmetry constraints, the number of its edge modes can be directly counted by the winding number. We can then view the half-way ES as characterizing the topology of this half-way Hamiltonian.

Here, we emphasize that although the following results are based on the band-flattened Hamiltonian, as shown in Appendix B, the TPTs can still be characterized by the ES crossings for the post-quench Hamiltonian without flat bands.

III. RESULTS

A. Chemical potential $\mu$ as the quench parameter

We first focus on the dynamical properties of ES with the quench protocols where the chemical potential $\mu$ is chosen as the quench parameter, i.e., $\mu_i = \mu_f$, and other parameters are fixed. The $\mu_i$ and $\mu_f$ refer to the chemical potential of the initial and final Hamiltonian respectively.

As a warm up, we review the results in the systems with nearest-neighbor interactions. Taking $H^t$ to be in the trivial phase, there will always be two degenerate ES crossings as long as $H^f$ belongs to the topological phase. If $H^f$ is also in the trivial phase, even the lowest ES is far away from 1/2. Thus the ES crossing provides a distinctive signature for diagnosing topological phases [46–48]. We now show that similar behaviors of ES can also be observed in our Hamiltonian (1) in the short-range sector $(\alpha > 1)$. We study the quench dynamics with the parameters in (1) as $\Delta = t = 1$, $\alpha = 2$ and $\mu_t = 3$. The results are depicted in Fig. 2(a). It is seen that the lowest ES $\xi^{(1)}$ ap-
Different from the previous protocols where the ground state is obtained for both $\mu_f = 1.5$ and $\mu_i = 3$, and novel behaviors of ES are observed. As shown in Fig. 2(c), the ES crossing is observed when quenching across the critical line $\mu_c = 1$ ($\mu_f = 0.5$), and is absent when staying in the same phase ($\mu_f = 1.5$). However, different from the results in Fig. 2(a), the degeneracy of $\xi^{(1)}$ and $\xi^{(2)}$ is destroyed. This can be traced to the fact that the half-way winding vector is different from that of the short-range case. In the long-range sector, the half-way winding number is $3/2$ (see Fig. 2(d)), and there is only one edge mode, corresponding to the non-degenerate ES.

Conventionally, the initial state is chosen as a topologically trivial ground state $[46–48]$. It has also been demonstrated that the quench dynamics of ES with a non-trivial $H^\sigma$, such as one with $w = 1$, can not reveal the signatures of TPTs $[46]$, which is still hold for the short-range sector of the Hamiltonian Eq. (1) (see Appendix B). To investigate if the above statements still hold in the long-range sector, we choose our $H^\sigma$ to be in the topologically nontrivial phase with massive Dirac edge states and winding number $w = 1/2$, while our $H^f$ is a trivial one with $w = -1/2$. Specifically, we quench the chemical potential $\mu = -2 \rightarrow -1$ and $2$ separately with $\alpha = 0.5$. It is quite remarkable that as shown in Fig. 2(e), the TPT can still be characterized by the dynamics of ES in this case. Actually, the half-way winding number of the quench in Fig. 2(e) is $-3/2$ (see Fig. 2(f)), different from that of the initial Hamiltonian ($w = 1/2$). On the contrary, in the short-range sector, the half-way winding number would still be $1$, same as that of the original Hamiltonian. Consequently, one can see that the characterization of TPTs via ES dynamics tightly depends on the difference between the winding number of initial Hamiltonian and the half-way winding number. Moreover, similar to Fig. 2(c) and (d), the non-degenerate property of ES can also be explained by the winding number $w = -3/2$.

In addition, we study the above quench protocols in a more detailed method. We fix the initial state as a ground state in the phases with $w = 1$, $w = 1/2$ and $w = -1/2$, i.e., $(\alpha_i, \mu_i) = (2, 3)$, $(0.5, 3)$ and $(0.5, -3)$ respectively, and explore the dependence of the ES $\xi^{(1)}(t)$ at $t = \pi/2$ (denoted as $\xi^{(1)}(t = \pi/2)$) and $\mu_f$. In Fig. 3(a) and (b), with the initial Hamiltonian in the trivial topological phase, $\xi^{(1)}(t = \pi/2)$ shows non-analytical behaviors at the critical points. In Fig. 3(c), one can see that $\xi^{(1)}(t = \pi/2)$ seems to gradually approach $0.5$ as $\mu_f$ gets closer to the critical point $\mu_c = 1$. However, at the critical point $\mu_c = 1$, there is a distinctive behavior of $\xi^{(1)}$.

### B. Decay rate $\alpha$ as the quench parameter

To further explore the dynamics of ES in the long-range system, we now turn to consider the quench protocols with the decay rate $\alpha$ as the quench parameter, i.e., $\alpha = \alpha_i \rightarrow \alpha_f$, and other parameters are fixed. The $\alpha_i$ and $\alpha_f$ refer to the decay rate of the initial and final Hamiltonian respectively.

Fig. 4(a)-(e) show the quench dynamics of ES with the protocol $\alpha = 6 \rightarrow 0.1$ and $2$. For $\mu = 3$ ($-3$), the half-way winding number is $+1$ ($-1$) and the ES is non-degenerate. Thus, we only focus on the lowest ES $\xi^{(1)}$, whose dynamical behaviors are shown in Fig. 4(a) and (b). The half-way cross of ES, i.e., $\xi^{(1)}(t = \pi/2) = 0.5$ can still characterize the TPTs with the critical line $\alpha_c = 1$. Next, we study the quench protocol $\alpha = 6 \rightarrow 0.1$ with $\mu = 0$. Different from the previous protocols where the ground state
FIG. 3. (a) The value of ES for the quench state at \( t = \pi/2 \), i.e., \( \xi^{(1)}(t = \pi/2) \), with the quench protocol \( \mu = 3 \rightarrow \mu_f \) as a function of \( \mu_f \) in the Hamiltonian (1) with \( \alpha = 2 \). Here, the critical points are \( \mu_c = \pm 1 \). (b) is similar to (a) but in the Hamiltonian (1) with \( \alpha = 0.5 \), and the critical point is \( \mu_c = 1 \). (c) is similar to (a) but with the quench protocol \( \mu = -3 \rightarrow \mu_f \) in the Hamiltonian (1) with \( \alpha = 0.5 \), and the critical point is \( \mu_c = 1 \).

FIG. 4. (a) Time evolution of the lowest ES \( \xi^{(1)} \) with the quench protocol \( \alpha = 6 \rightarrow 2 \) and \( 0.1 \) in the Hamiltonian (1) with \( \mu = 3 \) and \( \Delta = t = 1 \). (b) is similar to (a) but in the Hamiltonian (1) with \( \mu = -3 \). (c) Time evolution of the first and second lowest ES \( \xi^{(1)} \) and \( \xi^{(2)} \) with the quench protocol \( \alpha = 6 \rightarrow 0.1 \) in the Hamiltonian (1) with \( \mu = 0 \) and \( \Delta = t = 1 \). (d) and (e) are similar to (c) but with the quench protocol \( \alpha = 6 \rightarrow 2 \). (f) Time evolution of the lowest ES \( \xi^{(1)} \) with the quench protocol \( \alpha = 0.1 \rightarrow 6 \) and \( 0.4 \) in the Hamiltonian (1) with \( \mu = 0 \) and \( \Delta = t = 1 \). (g) and (h) are similar to (f) but in the Hamiltonian (1) with \( \mu = \pm 3 \).

state ES \( \xi^{(1)} \) of \( H^i \) is larger than that of \( H^f \), such as the quench protocols in Fig. 4(a) and (b), the ground state ES \( \xi^{(1)} \) of \( H^i \) in the current protocol is smaller than that of \( H^f \). In fact, the \( \xi^{(1)} \) of \( H^i \) is equal to 0.5. Remarkably, as shown in Fig. 4(c), (d) and (e), instead of the ED crossing \( \xi = 1/2 \), the TPTs can also be characterized by a novel behavior of the ES, i.e., \( \xi^{(1)}(t = \pi/2) = \xi^{(2)}(t = \pi/2) \) when quenching across the critical line \( \alpha_c = 1 \) \((\alpha_f = 0.1)\), while there is a large discrepancy between \( \xi^{(1)}(t = \pi/2) \) and \( \xi^{(2)}(t = \pi/2) \) when staying the same phase \( \alpha_f = 2 \).

In addition, we also study the inverse of above protocols. Fig. 4 (f)-(h) show the quench dynamics of ES with the protocol \( \alpha = 0.1 \rightarrow 6 \) and \( 0.4 \). For \( \mu = 0 \), the ground state ES \( \xi^{(1)} \) of \( H^i \) is larger than that of \( H^f \), and the dynamical properties of ES are similar to Fig. 4(a) and (b). For \( \mu = \pm 3 \), the quench dynamics are predicted to be trivial since the winding number of initial Hamiltonian is equal to the half-way winding number. Indeed, even if the quenches cross the critical line \( \alpha_c = 1 \), \( \xi^{(1)}(t = \pi/2) = \xi^{(2)}(t = \pi/2) \) is not satisfied.

We then fix the initial Hamiltonian with the parameters \((\alpha_c, \mu_c) = (6, 3), (6, 0)\) and \((0.1, 0)\), and explore the ES at \( t = \pi/2 \) as a function of \( \alpha_f \). In Fig. 5(a), the dependence of the ES \( \xi^{(1)}(t = \pi/2) \) and \( \alpha_f \) with the quench protocols \( \alpha = 6 \rightarrow \alpha_f \) and \( \mu = 3 \) is presented. One can see that the results of system size \( N = 200 \) suffers from finite-size effect. Thus, we further calculate the results of larger system size \( N = 500 - 4000 \). When increasing \( N \), the change of \( \xi^{(1)}(t = \pi/2) \) becomes more dramatic at the critical point \( \alpha_c = 1 \). To study another quench protocols \( \alpha = 6 \rightarrow \alpha_f \) with \( \mu = 0 \), we focus on the difference between the first and second lowest ES at \( t = \pi/2 \), i.e., \( \Delta \xi = \xi^{(2)} - \xi^{(1)} \). The difference of ES \( \Delta \xi(t = \pi/2) \) as a function of \( \alpha_f \) is shown in Fig. 5(b). Similar to the results in Fig. 5(a), with the increase of \( N \), the critical behavior of the ES becomes more obvious. In Fig. 5(c), we present the results of the quench protocols \( \alpha = 0.1 \rightarrow \alpha_f \) with \( \mu = 0 \), as the quenches with opposite direction of these in Fig. 5(b). On this condition, the finite-size effect is smaller, and it can be directly inferred that the ES \( \xi^{(1)}(t = \pi/2) \) has finite value for \( \alpha_f < \alpha_c \) while vanishes for \( \alpha_f > \alpha_c \) \((\alpha_c = 1)\) when \( N \rightarrow \infty \).

IV. SUMMARY AND OUTLOOK

Recent works [46-48] show that the TPTs in the systems with nearest-neighbor interactions can be character-
ized by the quench dynamics of ES. By studying the out-of-equilibrium properties of ES in the long-range Kitaev chain with power-law decay pairing terms, we have found that: (i) in the short-range sector (decay rate $\alpha > 1$), the behaviors of ES are similar to those in the conventional Kitaev chain [48], i.e., when quenching across the critical line and the initial Hamiltonian is topologically trivial, the ES crossing $\xi = 1/2$ is observed. (ii) In the long-range sector (decay rate $\alpha < 1$), for both topologically trivial or non-trivial initial Hamiltonian, the ES crossings can still characterize the TPTs. However, the ES crossing becomes unstable for topologically non-trivial initial Hamiltonian in the short-range sector. (iii) For the quench protocols with decay rate $\alpha$ as the quench parameter, the ES crossings can diagnose TPTs when the ES of initial Hamiltonian is larger than that of final Hamiltonian. On the other hand, the TPTs can also be detected by studying the difference between the first and second lowest ES when the ES of initial Hamiltonian is smaller than that of final Hamiltonian. In a word, the characterization of TPTs via the quench dynamics of ES could be well generalized to long-range systems.

This work may enlighten further investigations on the quench dynamics of ES in several long-range systems, for instance, the characterization of the topological phases with higher winding number in the longer-range Kitaev chains [56, 57], the TPTs in the two-dimensional topological superconductors with long-range interactions [58], the conventional quantum phase transitions [59] or dynamical phase transitions [60] in the long-range Ising chains.

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**Appendix A: The details of calculating the quench dynamics of entanglement spectrum**

After obtaining the winding vector at arbitrary time $t$ during the quench dynamics according to Eq. (6), we can diagonalize the Hamiltonian at $t$, i.e., $H(t) = \sum_k \epsilon_k \Psi_k^\dagger(d_k(t) \cdot \sigma)\Psi_k$, via the Bogoliubov transformation

$$\Psi_k^\dagger(d_k(t) \cdot \sigma)\Psi_k = \Psi_k^\dagger V_k(t)\Lambda_k(t)V_k^\dagger(t)\Psi_k,$$  \hspace{2cm} (A1)

giving the Bogoliubov fermion operators at $t$, i.e., $(\alpha_{-k}(t), a_{+k}^\dagger(t)) = \Psi_k^\dagger V_k(t)\Lambda_k(t)^{-1}$, which satisfies $a_{\pm k}(t)|\text{Vac}\rangle = 0$. The two-dimension unitary matrix $V_k$, whose elements are denoted as $v_{ij}(k)$ ($i,j \in \{1,2\}$), is the key to calculate the correlation matrix

$$C_{mn} = \langle \text{Vac}|c_{m}^\dagger c_n|\text{Vac}\rangle$$

$$= \frac{1}{N} \sum_k e^{-i k (m-n)} v_{22}(k) v^*_{22}(k)$$

$$+ e^{i k (m-n)} v_{11}(k) v^*_{11}(k),$$  \hspace{2cm} (A2)

and anomalous correlation matrix,

$$F_{mn} = \langle \text{Vac}|c_{m}^\dagger c_n^\dagger|\text{Vac}\rangle$$

$$= \frac{1}{N} \sum_k e^{-i k (m-n)} v_{22}(k) v_{12}^*(k)$$

$$+ e^{i k (m-n)} v_{21}(k) v_{11}^*(k),$$  \hspace{2cm} (A3)

which are useful for obtaining the ES.

The value of $\Omega_\ell$ in Eq. (8) can be obtained by the diagonalization of a matrix composed of the correlation matrix and anomalous correlation matrix, i.e.,

$$\begin{pmatrix}
I - C_A^\dagger & -F_A^\dagger \\
F_A & C_A
\end{pmatrix} = P^\dagger \begin{pmatrix}
\Xi^- & 0 \\
0 & \Xi^+
\end{pmatrix} P,$$  \hspace{2cm} (A4)
with $\Xi^\pm$ as diagonal matrix whose elements are $1/(1 + e^{i\Omega x})$.

**Appendix B: Results of the post-quench Hamiltonian without flat bands**

In this Appendix, we present the quench dynamics of ES with the same quench protocols in Fig. 2(a), (c), (e) and Fig. 4. The results are shown in Fig 6, indicating that the conclusions made in the flat-band case are stable. Indeed, it has been shown that the ES crossing is stable for the conventional Kitaev chain with nearest interactions in class D [47]. Here, we demonstrate that the stability of ES can be generalized to the long-range Kitaev chain.

It is noted that a distinctive fast oscillating behavior of ES can be observed when the initial or final Hamiltonian is in the long-range sector. For instance, the oscillation of ES is more dramatic in Fig. 6(b) than that in Fig. 6(a). The oscillating behavior may be related to the divergence of quasiparticle energy since for the Hamiltonian (1), the divergence of $\epsilon_k$ can occur when $k = 0$ or $2\pi$ in the long-range sector $\alpha < 1$.
FIG. 6. The quench dynamics of ES for the post-quench Hamiltonian without flat bands. The quench protocol in (a)-(e) is the same as that in Fig. 2(a), Fig. 2(c), Fig. 2(e), Fig. 4(a) and Fig. 4(b), respectively. The quench protocol in (f) is the same as that in Fig. 4(c)-(e). The quench protocol in (g) is the same as that in Fig. 4(f). The quench protocol in (h) is the same as that in Fig. 4(g) and (h).

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