BOGOMOLOV-GIESEKER TYPE INEQUALITIES ON RULED THREEFOLDS

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ABSTRACT. We strengthen a conjecture by the author. This conjecture is a Bogomolov-Gieseker type inequality involving the third Chern character of mixed tilt-stable complexes on fibred threefolds. We extend it from complexes of mixed tilt-slope zero to arbitrary relative tilt-slope. We show that this stronger conjecture implies the support property of Bridgeland stability conditions, and the existence of explicit stability conditions. We prove our conjecture for ruled threefolds, hence improving a previous result by the author.

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1. INTRODUCTION

Throughout this paper, we let \( f : \mathcal{X} \to C \) be a projective morphism from a complex smooth projective variety of dimension 3 to a complex smooth projective curve such that all scheme-theoretic fibers of \( f \) are integral and normal. We denote by \( F \) the general fiber of \( f \), and fix a nef and relative ample \( \mathbb{Q} \)-divisor \( H \) on \( \mathcal{X} \). When \( \mathcal{X} \) is ruled, i.e., \( \mathcal{X} = \mathbb{P}(E) \) for some rank three vector bundle \( E \) on \( C \), we prove a Bogomolov-Gieseker type inequality for the third Chern character of relative tilt-semistable objects on \( \mathcal{X} \). As a corollary, we construct a family of Bridgeland stability conditions on \( \mathcal{X} \) for which the central charge only depends on the degrees of the Chern character, and that satisfy the support property.

Stability conditions for triangulated categories were introduced by Bridgeland in [5]. The existence of stability conditions on three-dimensional varieties is often considered the biggest open problem in the theory of Bridgeland stability conditions. In [3, 2, 4], the authors introduced a conjectural construction of Bridgeland stability conditions for any projective threefold. Here the problem was reduced to proving...
Corollary 4.4. It seems stronger than the classical Bogomolov inequality. Theorem 1.1. The inequality holds on all fibers of \( \mathcal{X} \) in \( \mathbb{R}^3 \), hence improving a previous result by the author: X subscheme. We denote by Xb scheme satisfying the support property on product type varieties by a different method.

In [10], we give a relative version of the construction of Bayer, Macrì and Toda [3] on fibred threefolds. We prove Theorem 1.1 and conditions, propose Conjecture 3.6 and 3.7 and give a conjectural construction of fibred variety in [1] and [16]. Then in Section 3, we recall the definition of stability conditions on fibred threefolds. We prove Theorem 1.1 and review some basic results of relative slope-stability and relative tilt-stability on a

\[ \Delta_{H,F}(\xi) \geq \frac{H^3}{6H^2F} \sum_{H,F}(HF ch_1(\xi))^2. \]

Organization of the paper. Our paper is organized as follows. In Section 2 we review some basic results of relative slope-stability and relative tilt-stability on a fibred variety in [1] and [16]. Then in Section 3 we recall the definition of stability conditions, propose Conjecture 3.6 and 3.7 and give a conjectural construction of Bridgeland stability conditions on fibred threefolds. We prove Theorem 1.1 and Corollary 4.4 in Section 4

Notation. Let \( X \) be a smooth projective variety. We denote by \( T_X \) and \( \Omega_X^1 \) the tangent bundle and cotangent bundle of \( X \), respectively. \( K_X \) and \( \omega_X \) denote the canonical divisor and canonical sheaf of \( X \), respectively. We write \( c_i(X) := c_i(T_X) \) for the \( i \)-th Chern class of \( X \). We write \( NS(X) \) for the Néron-Severi group of divisors up to numerical equivalence. We also write \( NS(X)_\mathbb{Q}, NS(X)_\mathbb{R} \), etc. for \( NS(X) \otimes \mathbb{Q} \), etc. For a triangulated category \( \mathcal{D} \), we write \( K(\mathcal{D}) \) for its Grothendieck group. For a variety \( Y \), we denote by \( \text{Sing} Y \) the singular locus of \( Y \).

Let \( \pi : \mathcal{X} \to S \) be a flat morphism of Noetherian schemes and \( W \subset S \) be a subscheme. We denote by \( \mathcal{X}_W = \mathcal{X} \times S W \) the fiber of \( \pi \) over \( W \), and by \( i_W : \mathcal{X}_W \to \mathcal{X} \) the embedding of the fiber. In the case that \( S \) is integral, we write \( K(S) \)
for its fraction field, and $X_{K(S)}$ for the generic fiber of $\pi$. We denote by $D^b(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$. Given $E \in D^b(\mathcal{X})$, we write $E_W$ (resp., $E_{K(S)}$) for the pullback to $\mathcal{X}_W$ (resp., $\mathcal{X}_{K(S)}$).

Let $F$ be a coherent sheaf on $X$. We write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the cohomology groups of $F$ and write $D^j$ for the dimension of its support. We write $\text{Coh}(A) \subset \text{Coh}(X)$ for the subcategory of sheaves supported in dimension $\leq d$. Given a bounded t-structure on $D^b(X)$ with heart $A$ and an object $E \in D^b(X)$, we write $\mathcal{H}_A^j(E)$ ($j \in \mathbb{Z}$) for the cohomology objects with respect to $A$. When $\mathcal{A} = \text{Coh}(X)$, we simply write $\mathcal{H}^j(E)$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively. We write $\sqrt{\mathbb{Q}_{>0}}$ for the set $\{\sqrt{x} : x \in \mathbb{Q}_{>0}\}$.

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2. **Relative tilt-stability**

We will review some basic results on the relative slope-stability and relative tilt-stability in [1] and [10].

2.1. **Stability for sheaves.** For any $\mathbb{R}$-divisor $D$ on $\mathcal{X}$, we define the twisted Chern character $\text{ch}^D = e^{-D} \text{ch}$. More explicitly, we have

\[
\begin{align*}
\text{ch}_0^D &= \text{ch}_0 = \text{rk} \\
\text{ch}_1^D &= \text{ch}_1 - D \text{ch}_0 \\
\text{ch}_2^D &= \text{ch}_2 - D \text{ch}_1 + \frac{D^2}{2} \text{ch}_0 \\
\text{ch}_3^D &= \text{ch}_3 - D \text{ch}_2 + \frac{D^2}{2} \text{ch}_1 - \frac{D^3}{6} \text{ch}_0.
\end{align*}
\]

We define the relative slope $\mu_{H,F}$ of a coherent sheaf $E \in \text{Coh}(\mathcal{X})$ by

\[
\mu_{H,F}(E) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0(E) = 0, \\
\frac{F \text{ch}_1(E)}{F \text{ch}_2(E)}_\text{H^2\text{ch}_0(E)}, & \text{otherwise}.
\end{cases}
\]

**Definition 2.1.** A coherent sheaf $E$ on $X$ is $\mu_{H,F}$-(semi)stable (or relative slope-(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have

\[
\mu_{H,F}(F) < (\leq) \mu_{H,F}(E/F).
\]

Similarly, for any point $s \in C$, we can define $\mu_{H,s}$-stability (or slope-stability) of a coherent sheaf $G$ on the fiber $X_s$ over $s$ for the slope $\mu_{H,s}$:

\[
\mu_{H,s}(G) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0(G) = 0, \\
\frac{H_s \text{ch}_1(G)}{H_s \text{ch}_2(G)}_\text{H^2\text{ch}_0(G)}, & \text{otherwise}.
\end{cases}
\]

Here $\text{ch}_1(G)$ is defined as a Weil divisor up to linear equivalence such that

\[
\text{ch}_1(G)|_{X_s - \text{Sing} X_s} = \text{ch}_1(G)|_{X_s - \text{Sing} X_s}.
\]

Since, by our assumption, $\text{codim}(\text{Sing} X_s) \geq 2$, $\text{ch}_1(G)$ is well-defined. One sees that $\mu_{H,F}(E) = \mu_{H,s}(E_s)$.

The below lemma gives the relation between the Chern characters of objects on fibers and their pushforwards.

**Lemma 2.2.** Let $W$ be a closed subscheme of $C$, and $j$ be a positive integer.
(1) For any $E \in D^b(X)$ and $\mathbb{Q}$-divisor $D$ on $X$, we have
$$ch_j^D(i_{W*}E) = \chi_W^j ch^D_{j-1}(E).$$

(2) Assume that $W$ is a closed point of $C$. Then for any $Q \in D^b(X_W)$ we have
$$ch_j(i_{W*}Q) = i_{W*}ch_j(Q)$$
if $0 \leq j \leq 2$. Moreover, we have $ch_3(i_{W*}Q) = i_{W*}ch_2(Q)$ if $X_W$ is smooth.

Proof. See [16, Lemma 2.7] for part (1) and Part (2) in the case that $X_W$ is smooth. Part (2) for the case that $X_W$ is singular follows from applying the Grothendieck-Riemann-Roch theorem for the embedding $i_W|_{X_W-Sing X_W} : X_W-Sing X_W \hookrightarrow X$.

Definition 2.3. Let $A_C$ be the heart of a $C$-local $t$-structure on $D^b(X)$ (see [1, Definition 4.10]), and let $E \in A_C$.

(1) We say $E$ is $C$-flat if $E_c \in A_c$ for every point $c \in C$, where $A_c$ is the heart of the $t$-structure given by [1, Theorem 5.3] applied to the embedding $c \hookrightarrow C$.

(2) An object $F \in D^b(X)$ is called $C$-torsion if it is the pushforward of an object in $D^b(X_W)$ for some proper closed subscheme $W \subset C$.

(3) $E$ is called $C$-torsion free if it contains no nonzero $C$-torsion subobject.

We denote by $A_{C-tor}$ the subcategory of $C$-torsion objects in $A_C$, and by $A_{C-tf}$ the subcategory of $C$-torsion free objects. We say $A_C$ has a $C$-torsion theory if the pair of subcategories $(A_{C-tor}, A_{C-tf})$ forms a torsion pair in the sense of [1, Definition 4.6].

Lemma 2.4. Let $E \in A_C$ be as in Definition 2.3. Then

(1) $E$ is $C$-flat if and only if $E$ is $C$-torsion free;

(2) $E$ is $C$-torsion if and only if $E_{|_{K(C)}} = 0$.

Proof. See [1, Lemma 6.12 and Lemma 6.4].

Since $\text{Coh}(X)$ is the heart of the natural $C$-local $t$-structure on $D^b(X)$, one can applies the above definition and lemma to coherent sheaves. The following lemma shows the relation of the relative slope-stability to the slop-stability.

Lemma 2.5. Let $E$ be a $C$-torsion free sheaf on $X$. Then $E$ is $\mu_{H,F}$-(semi)stable if and only if there exists an open subset $U \subset C$ such that $E_s$ is $\mu_{H,-}$-(semi)stable for any point $s \in U$.

Proof. See [16, Lemma 2.4].

We recall the classical Bogomolov inequality:

Theorem 2.6. Assume that $E$ is a $\mu_{H,F}$-semistable torsion free sheaf on $X$. Then we have

$$F \Delta(E) := \mu(\chi^2(E) - 2 \chi_0(E) \chi_2(E)) \geq 0;$$
$$H \Delta(E) := \mu(\chi^2(E) - 2 \chi_0(E) \chi_2(E)) \geq 0.$$ 

Proof. See [16, Theorem 3.2].
Let $\beta$ be a real number. For brevity, we write $\text{ch}^\beta$ for the twisted Chern character $\text{ch}^{\beta H}$. A short calculation shows

$$\Delta(\mathcal{E}) := (\text{ch}_1(\mathcal{E}))^2 - 2\text{ch}_0(\mathcal{E})\text{ch}_2(\mathcal{E})$$

$$= (\text{ch}^\beta_1(\mathcal{E}))^2 - 2\text{ch}^\beta_0(\mathcal{E})\text{ch}^\beta_2(\mathcal{E}).$$

**Definition 2.7.** We define the generalized relative discriminants

$$\Sigma^H_{H,F} := (HF \text{ch}_1^\beta(\mathcal{E}))^2 - 2H^2F \text{ch}_0^\beta(F \text{ch}_2^\beta)$$

and

$$\tilde{\Sigma}^H_{H,F} := (HF \text{ch}_1^\beta)(H^2 \text{ch}_1^\beta) - H^2F \text{ch}_0^\beta(H \text{ch}_2^\beta).$$

A short calculation shows

$$\Sigma^H_{H,F} = (HF \text{ch}_1)^2 - 2H^2F \text{ch}_0(F \text{ch}_2) = \Sigma_{H,F}.$$  

Hence the first generalized relative discriminant $\Sigma^H_{H,F}$ is independent of $\beta$. In general $\tilde{\Sigma}^H_{H,F}$ is not independent of $\beta$, but we have

$$\tilde{\Sigma}^H_{H,F}(\mathcal{E} \otimes \mathcal{O}(mF)) = \tilde{\Sigma}^H_{H,F}(\mathcal{E}),$$

for any $\mathcal{E} \in \mathbf{D}^b(\mathcal{X})$ and $m \in \mathbb{Z}$.

**Theorem 2.8.** Assume that $\mathcal{E}$ is a $\mu_{H,F}$-semistable torsion free sheaf on $\mathcal{X}$. Then we have $\Sigma^H_{H,F}(\mathcal{E}) \geq 0$ and $\tilde{\Sigma}^H_{H,F}(\mathcal{E}) \geq 0$.

**Proof.** See [16, Theorem 3.10].

**Definition 2.9.** Let $p$ be a point of $C$. We say the Bogomolov inequality holds on the fiber $\mathcal{X}_p$, if

$$(H \text{ch}_2(i_p(E))^2 \geq 2H^2 \text{ch}_1(i_p(E))\text{ch}_3(i_p(E)))$$

for any $\mu_{H_p}$-semistable sheaf $E \in \text{Coh}(\mathcal{X}_p)$.

A short computation shows

$$(H \text{ch}_2(i_p(E))^2 - 2H^2 \text{ch}_1(i_p(E))\text{ch}_3(i_p(E)))$$

$$= (H \text{ch}_2^\beta(i_p(E))^2 - 2H^2 \text{ch}_1^\beta(i_p(E))\text{ch}_3^\beta(i_p(E))).$$

By Lemma 2.2 one sees

$$(H \text{ch}_2(i_p(E))^2 - 2H^2 \text{ch}_1(i_p(E))\text{ch}_3(i_p(E)))$$

$$= (H_p \text{ch}_{\mathcal{X}_p,1}(E))^2 - 2H^2_p \text{ch}_{\mathcal{X}_p,0}(E)\text{ch}_{\mathcal{X}_p,2}(E)$$

if $\mathcal{X}_p$ is smooth. Hence the Bogomolov inequality holds on every smooth fiber of $f$.

Another important notion of stability for a sheaf on a fibration is the stability introduced in [11, Example 15.3]. We define the slope $\mu_C$ of a coherent sheaf $\mathcal{E} \in \text{Coh}(\mathcal{X})$ by

$$\mu_C(\mathcal{E}) = \begin{cases} 
\frac{HF \text{ch}_1(\mathcal{E})}{HF \text{ch}_2(\mathcal{E})}, & \text{if } \text{ch}_0(\mathcal{E}) \neq 0, \\
\frac{HF \text{ch}_1(\mathcal{E})}{H^2 \text{ch}_1(\mathcal{E})}, & \text{if } \mathcal{E}_{K(C)} = 0 \text{ and } H^2 \text{ch}_1(\mathcal{E}) \neq 0, \\
+\infty, & \text{otherwise.}
\end{cases}$$

We can define $\mu_C$-stability as in Definition 2.1.
Definition 2.10. A coherent sheaf $\mathcal{E}$ on $\mathcal{X}$ is $\mu_C$-(semi)stable if, for all non-zero subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$, we have

$$\mu_C(\mathcal{F}) < (\leq) \mu_C(\mathcal{E}/\mathcal{F}).$$

Like the slope-stability and the relative slope-stability, the $\mu_C$-stability also satisfies the following weak see-saw property and the Harder-Narasimhan property (see [11 Proposition 16.6] and [16 Proposition 2.6]).

Proposition 2.11. Let $\mathcal{E} \in \text{Coh}(\mathcal{X})$ be a non-zero sheaf.

1. For any short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

in $\text{Coh}(\mathcal{X})$, we have

$$\mu_C(\mathcal{F}) \leq \mu_C(\mathcal{E}) \leq \mu_C(\mathcal{G}) \text{ or } \mu_C(\mathcal{F}) \geq \mu_C(\mathcal{E}) \geq \mu_C(\mathcal{G}).$$

2. There is a filtration (called Harder-Narasimhan filtration)

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}$$

such that: $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\mu_C$-semistable, and $\mu_C(\mathcal{G}_1) > \cdots > \mu_C(\mathcal{G}_m)$.

We write $\mu_C^+(\mathcal{E}) := \mu_C(\mathcal{G}_1)$ and $\mu_C^-(\mathcal{E}) := \mu_C(\mathcal{G}_m)$.

Lemma 2.5 says that the usual notion of relative slope-stability for a torsion free sheaf is equivalent to the slope-stability of the general fiber of the sheaf. In contrast, $\mu_C$-stability requires stability for all fibers:

Proposition 2.12. Let $\mathcal{E}$ be a $C$-torsion free sheaf on $\mathcal{X}$. Then $\mathcal{E}$ is $\mu_C$-semistable if and only if $\mathcal{E}$ is $\mu_{H,F}$-semistable and for any closed point $p \in C$ and any quotient $\mathcal{E}_p \to \mathcal{Q}$ in $\text{Coh}(\mathcal{X}_p)$ we have $\mu_{H_p}(\mathcal{E}_p) \leq \mu_{H_p}(\mathcal{Q})$.

Proof. See [11 Lemma 15.7].

2.2. Relative tilt-stability. Let $\beta$ be a real number and $\alpha$ be a positive real number. There exists a torsion pair $(\mathcal{T}_{\beta_H}, \mathcal{F}_{\beta_H})$ in $\text{Coh}(\mathcal{X})$ defined as follows:

$$\mathcal{T}_{\beta_H} = \{ \mathcal{E} \in \text{Coh}(\mathcal{X}) : \mu_C^-(\mathcal{E}) > \beta \}$$

$$\mathcal{F}_{\beta_H} = \{ \mathcal{E} \in \text{Coh}(\mathcal{X}) : \mu_C^+(\mathcal{E}) \leq \beta \}.$$

Equivalently, $\mathcal{T}_{\beta_H}$ and $\mathcal{F}_{\beta_H}$ are the extension-closed subcategories of $\text{Coh}(\mathcal{X})$ generated by $\mu_C$-stable sheaves with $\mu_C$-slope $> \beta$ and $\leq \beta$, respectively.

Definition 2.13. We let $\text{Coh}^\beta_{\mu_C}(\mathcal{X}) \subset \text{D}^b(\mathcal{X})$ be the extension-closure

$$\text{Coh}^\beta_{\mu_C}(\mathcal{X}) = \langle \mathcal{T}_{\beta_H}, \mathcal{F}_{\beta_H}[1] \rangle.$$

By the general theory of torsion pairs and tilting [7], $\text{Coh}^\beta_{\mu_C}(\mathcal{X})$ is the heart of a bounded t-structure on $\text{D}^b(\mathcal{X})$; in particular, it is an abelian category. For any point $s \in C$ such that $\mathcal{X}_s$ is smooth, similar as Definition 2.13 one can define the subcategory

$$\text{Coh}^\beta_{\mu_{H_s}}(\mathcal{X}_s) = \langle \mathcal{T}_{\beta_{H_s}}, \mathcal{F}_{\beta_{H_s}}[1] \rangle \subset \text{D}^b(\mathcal{X}_s)$$

via the $\mu_{H_s}$-stability (see [11 Section 14.2]). For any $\mathcal{E} \in \text{Coh}^\beta_{\mu_C}(\mathcal{X})$, its relative tilt-slope $\nu^\alpha_{\mu_{H,F}}$ is defined by

$$\nu^\alpha_{\mu_{H,F}}(\mathcal{E}) = \begin{cases} +\infty, & \text{if } FH \text{ ch}_1^\beta(\mathcal{E}) = 0, \\ \frac{F \text{ ch}_1^\beta(\mathcal{E}) - \frac{1}{2}FH^2 \text{ ch}_1^\alpha(\mathcal{E})}{HF \text{ ch}_1^\alpha(\mathcal{E})}, & \text{otherwise.} \end{cases}$$
Definition 2.14. An object $\mathcal{E} \in \text{Coh}^{\beta H}_C(\mathcal{X})$ is $\nu^{\alpha,\beta}_{H,F}$-(semi)stable (or relative tilt-(semi)stable) if, for all non-zero subobjects $\mathcal{F} \hookrightarrow \mathcal{E}$, we have

$$\nu^{\alpha,\beta}_{H,F}(\mathcal{F}) < (\leq) \nu^{\alpha,\beta}_{H,F}(\mathcal{E}/\mathcal{F}).$$

We can also consider the tilt-stability on the smooth fibers of $f$. For any point $s \in C$ such that $\mathcal{X}_s$ is smooth, the tilt-slope $\nu^{\alpha,\beta}_s$ of an object $\mathcal{G} \in \text{Coh}^{\beta H_s}(\mathcal{X}_s)$ is defined by

$$\nu^{\alpha,\beta}_s(\mathcal{G}) = \begin{cases} +\infty, & \text{if } H_s \text{ ch}^2_{\mathcal{X}_s,1}(\mathcal{G}) = 0, \\ \frac{\text{ch}^2_{\mathcal{X}_s,2}(\mathcal{G}) - \frac{1}{4} s^2 \text{ ch}^4_{\mathcal{X}_s,0}(\mathcal{G})}{H_s \text{ ch}^2_{\mathcal{X}_s,1}(\mathcal{G})}, & \text{otherwise}. \end{cases}$$

This gives the tilt-stability condition on $\mathcal{X}_s$ defined in [32]. One sees

$$\nu^{\alpha,\beta}_s(\mathcal{E}_s) = \nu^{\alpha,\beta}_{H,F}(\mathcal{E}).$$

Relative tilt-stability gives a notion of stability, in the sense that Harder-Narasimhan filtrations exist:

Lemma 2.15. Let $\mathcal{E}$ be an object in $\text{Coh}^{\beta H}_C(\mathcal{X})$. Then there is a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}$$

such that: $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\nu^{\alpha,\beta}_{H,F}$-semistable, and $\nu^{\alpha,\beta}_{H,F}(\mathcal{G}_1) > \cdots > \nu^{\alpha,\beta}_{H,F}(\mathcal{G}_m)$. We write $\nu^{\alpha,\beta,+}_{H,F}(\mathcal{E}) := \nu^{\alpha,\beta}_{H,F}(\mathcal{G}_1)$ and $\nu^{\alpha,\beta,-}_{H,F}(\mathcal{E}) := \nu^{\alpha,\beta}_{H,F}(\mathcal{G}_m)$.

Proof. Since one has the standard exact sequence $\mathcal{H}^{-1}(\mathcal{E})[1] \hookrightarrow \mathcal{E} \rightarrow \mathcal{H}^0(\mathcal{E})$ in $\text{Coh}^{\beta H}_C(\mathcal{X})$, by [H] Lemma 6.17, one sees that $\mathcal{E}$ admit a unique maximal $C$-torsion subobject, namely $\text{Coh}^{\beta H}_C(\mathcal{X})$ has a $C$-torsion theory. We denote by $\mathcal{E}_{C,\text{tor}}$ the maximal $C$-torsion subobject of $\mathcal{E}$ and $\mathcal{E}_{C,tf} = \mathcal{E}/\mathcal{E}_{C,\text{tor}}$ the $C$-torsion free quotient of $\mathcal{E}$. It will be enough to show that the Harder-Narasimhan filtration exists for $\mathcal{E}_{C,tf}$.

By [2] Appendix 2, one sees that the Harder-Narasimhan filtration exists for $(\mathcal{E}_{C,tf})_{K(C)}$ with respect to $\nu^{\alpha,\beta}_{K(C)}$-stability. We denote it by

$$0 = \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}_1 \subset \cdots \subset \widetilde{\mathcal{E}}_m = (\mathcal{E}_{C,tf})_{K(C)}.$$

One can lift it to a filtration in $\text{Coh}^{\beta H}_C(\mathcal{X})$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}_{C,tf}$$

from [1] Lemma 4.16.(3). Replacing $\mathcal{E}_j$ by the kernel of $\mathcal{E}_{j+1} \rightarrow (\mathcal{E}_{j+1}/\mathcal{E}_j)_{C,tf}$, one can assume that $\mathcal{E}_{j+1}/\mathcal{E}_j$ is $C$-torsion free. Hence $\mathcal{E}_{j+1}/\mathcal{E}_j$ is $\nu^{\alpha,\beta}_{H,F}$-semistable by the following lemma and

$$\nu^{\alpha,\beta}_{H,F}(\mathcal{E}_j/\mathcal{E}_{j-1}) = \nu^{\alpha,\beta}_{K(C)}(\widetilde{\mathcal{E}}_j/\mathcal{E}_{j-1}) > \nu^{\alpha,\beta}_{K(C)}(\widetilde{\mathcal{E}}_{j+1}/\mathcal{E}_j) = \nu^{\alpha,\beta}_{H,F}(\mathcal{E}_{j+1}/\mathcal{E}_j).$$

This completes the proof. □

Lemma 2.16. Let $\mathcal{E} \in \text{Coh}^{\beta H}_C(\mathcal{X})$ be a $C$-torsion free object. Then the following conditions are equivalent:

1. $\mathcal{E}$ is $\nu^{\alpha,\beta}_{H,F}$-(semi)stable;
2. $\mathcal{E}_{K(C)}$ is $\nu^{\alpha,\beta}_{K(C)}$-(semi)stable.

Proof. The proof is the same as that of [16] Lemma 3.15. □
Proposition 2.17. Assume that $\beta \in \mathbb{Q}$. Then the category $\text{Coh}_C^B(X)$ is noetherian.

Proof. The conclusion was showed in the proof of [16, Theorem 3.11] in the case that $f$ is smooth.

For our case, let $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \cdots$ be an infinite sequence of surjections in $\text{Coh}_C^B(X)$. By the proof of Lemma 2.15, one sees that $\text{Coh}_C^B(X)$ has a $C$-torsion theory. Hence the induced sequence of surjections $(\mathcal{E}_1)_{\mathcal{K}(C)} \rightarrow (\mathcal{E}_2)_{\mathcal{K}(C)} \rightarrow \cdots$ stabilizes by the proof of [3, Lemma 3.2.4], in other words we may assume that the kernel $\mathcal{K}_i$ of every surjection $\mathcal{E}_i \rightarrow \mathcal{E}_i$ is $C$-torsion. Then we obtain a chain of injections

$$0 = \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{E}_{1,C}-\text{tor},$$

where $\mathcal{E}_{1,C}-\text{tor}$ is the maximal $C$-torsion subobject of $\mathcal{E}_1$. It induces the chain of injections

$$0 \subset \mathcal{H}^{-1}(\mathcal{K}_2) \subset \mathcal{H}^{-1}(\mathcal{K}_3) \subset \cdots \subset \mathcal{H}^{-1}(\mathcal{E}_{1,C}-\text{tor}).$$

Thus we can assume that $\mathcal{H}^{-1}(\mathcal{K}_2) = \mathcal{H}^{-1}(\mathcal{K}_i)$ for $i \geq 2$ as $\text{Coh}(X)$ is noetherian.

Let us consider the sequence of surjections

$$\mathcal{E}_{1,C}-\text{tor} \rightarrow \mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_2 \rightarrow \cdots \rightarrow \mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i \rightarrow \cdots.
$$

Then we have the chain of surjections

$$\mathcal{H}^{0}(\mathcal{E}_{1,C}-\text{tor}) \rightarrow \mathcal{H}^{0}(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_2) \rightarrow \cdots \rightarrow \mathcal{H}^{0}(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i) \rightarrow \cdots.
$$

So we may assume that $\mathcal{H}^{0}(\mathcal{E}_{1,C}-\text{tor}) = \mathcal{H}^{0}(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i)$ for $i \geq 1$. By the construction of $\text{Coh}_C^B(X)$, one sees $H\text{ch}_2^B(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i) \geq 0$. Thus we may also assume that $H\text{ch}_2^B(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i)$ is independent of $i$ from the discreteness of $H\text{ch}_2^B$. This implies that $H\text{ch}_2^B(\mathcal{K}_i)$ is 0 for any $i \geq 1$. By the following lemma, one sees $\mathcal{H}^{0}(\mathcal{K}_i)$ is a torsion sheaf supported in dimension zero. By setting $V = \mathcal{H}^{-1}(\mathcal{E}_{1,C}-\text{tor})/\mathcal{H}^{-1}(\mathcal{K}_i)$, we have the exact sequence

$$0 \rightarrow V \rightarrow \mathcal{H}^{-1}(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i) \rightarrow \mathcal{H}^{0}(\mathcal{K}_i) \rightarrow 0.
$$

Let us assume without loss of generality that $V = i_p^*(V')$, $\mathcal{H}^{-1}(\mathcal{E}_{1,C}-\text{tor}/\mathcal{K}_i) = i_p^*(E'_i)$ and $\mathcal{H}^{0}(\mathcal{K}_i) = i_p^*(Q'_i)$ for $V', E'_i, Q'_i \in \text{Coh}(X_p)$. By the construction of $\text{Coh}_C^B(X)$, one sees that $E'_i$ is torsion free. So is $V'$. The integrality of $X_p$ implies that $V'$ is a subsheaf of

$$(V')^\vee := \text{Hom}(\text{Hom}(V', \mathcal{O}_{X_p}), \mathcal{O}_{X_p}).$$

By the normality of $X_p$, one sees that $(V')^\vee$ is reflexive and the quotient $(V')^\vee/V'$ is supported on $X_p$ in dimension zero. It follows that $Q'_i$ is a subshae of $(V')^\vee/V'$. In particular the length of $\mathcal{H}^{0}(\mathcal{K}_i)$ is bounded. Therefore $\mathcal{H}^{0}(\mathcal{K}_i) = \mathcal{H}^{0}(\mathcal{K}_{i+1})$ for large $i$, and the sequence terminates. This completes the proof. \hfill $\square$

Lemma 2.18. Let $\mathcal{E}$ be an object in $\text{Coh}_C^B(X)$.

(1) We have $HF\text{ch}_1^B(\mathcal{E}) \geq 0$.

(2) If $HF\text{ch}_1^B(\mathcal{E}) = 0$, then one has $HF\text{ch}_2^B(\mathcal{E}) \geq 0$, $F\text{ch}_2(\mathcal{E}) \geq 0$ and $\text{ch}_0(\mathcal{E}) \leq 0$.

(3) If $HF\text{ch}_1^B(\mathcal{E}) = \text{ch}_0(\mathcal{E}) = H\text{ch}_2^B(\mathcal{E}) = 0$, then $\mathcal{H}^{0}(\mathcal{E}) \in \text{Coh}_{C^0}(X)$, $\mathcal{H}^{-1}(\mathcal{E})$ is a $C$-torsion $\mu_C$-semistable sheaf with $\mu_C(\mathcal{H}^{-1}(\mathcal{E})) = \beta$. Moreover, if the Bogomolov inequality holds on every fiber of $f$, then $\text{ch}_2^B(\mathcal{E}) \geq 0$.

Proof. The proof is the same as that of [16, Lemma 3.14]. \hfill $\square$
2.3. Properties of relative tilt-stability. The relative tilt-stability is unchanged under tensoring with line bundles \(O_X(aF)\) for any \(a \in \mathbb{Z}\):

**Lemma 2.19.** If \(E \in \text{Coh}_C^{\beta H}(\mathcal{X})\) is \(v_{H,F}^{\alpha,\beta}\)-semistable, then \(E(aF) \in \text{Coh}_C^{\beta H}(\mathcal{X})\) and \(E(aF)\) is also \(v_{H,F}^{\alpha,\beta}\)-semistable for any \(a \in \mathbb{Z}\).

**Proof.** The conclusion follows directly from the definition of \(\text{Coh}_C^{\beta H}(\mathcal{X})\) and \(v_{H,F}^{\alpha,\beta}\).

**Proposition 2.21.** Let \(E\) be an object in \(D^b(\mathcal{X})\) and \(u \in \Lambda\) a fixed class.

1. Numerical walls with respect to \(u\) are either semicircles with centers on the \(\beta\)-axis or rays parallel to the \(\alpha\)-axis.
2. If two numerical walls intersect, then the two walls are completely identical.
3. If \(E\) is \(v_{H,F}^{\alpha,\beta}\)-semistable with \(v_{H,F}^{\alpha,\beta}(E) \neq +\infty\), then the object \(E\) is \(v_{H,F}^{\alpha,\beta}\)-semistable along the semicircle \(C_{\alpha,\beta}(E)\) in the \((\alpha, \beta)\)-plane with center \((0, \beta + v_{H,F}^{\alpha,\beta}(E))\) and radius \(\sqrt{\alpha^2 + (v_{H,F}^{\alpha,\beta}(E))^2}\).
4. If there is an exact sequence of \(v_{H,F}^{\alpha,\beta}\)-semistable objects \(F \hookrightarrow E \twoheadrightarrow G\) such that \(v_{H,F}^{\alpha,\beta}(F) = v_{H,F}^{\alpha,\beta}(G)\), then \(\Delta_{H,F}(F) + \Delta_{H,F}(G) \leq \Delta_{H,F}(E)\). Moreover, equality holds if and only if either \(v(F) = 0\), \(v(G) = 0\), or both \(\Delta_{H,F}(E) = 0\) and \(\Delta_{H,F}(F)\) and \(\Delta_{H,F}(G)\) are all proportional.
5. If \(\Delta_{H,F}(E) = 0\) for a tilt semistable object \(E\), then \(E\) can only be destabilized at the unique numerical vertical wall.
6. Let \(F\) be a \(\mu_C\)-stable locally free sheaf on \(X\) with \(\Delta_{H,F}(F) = 0\). Then \(F\) or \(F[1]\) is a \(v_{H,F}^{\alpha,\beta}\)-stable object in \(\text{Coh}_C^{\beta H}(\mathcal{X})\).
7. Let \(F\) be a \(\mu_{H,F}\)-stable torsion free sheaf on \(\mathcal{X}\). If \(\mu_{C}(F) > \beta\), then \(F \in \text{Coh}_C^{\beta H}(\mathcal{X})\) and it is \(v_{H,F}^{\alpha,\beta}\)-stable for \(\alpha \gg 0\).

In order to reduce the relative tilt-stability to small \(\alpha\), we need the relative \(\beta\)-stability which is a relative version of \(\beta\)-stability in [2 Section 5].
Definition 2.22. For any \( E \in \text{Coh}_{\beta H}^\beta (\mathcal{X}) \), we define
\[
\beta(H) = \begin{cases} 
HF \text{ch}_1(E) - \sqrt{\Delta_{H,F}(E)}, & \text{if } \text{cho}(E) \neq 0, \\
F \text{ch}_0(E), & \text{otherwise}.
\end{cases}
\]
Moreover, we say that \( E \) is relative \( \beta \)-(semi)stable, if it is (semi)stable in a neighborhood of \((0, \beta(E))\).

By this definition we have \( F \text{ch}_2(\beta(E)) = 0 \).

3. Conjectures and constructions

In this section, we recall the definition of stability conditions on triangulated category introduced by Bridgeland in [5] and generalize [16, Conjecture 5.2] to arbitrary relative tilt-semistable objects with more precise form. Using this conjecture, we then give a construction of stability conditions on \( \mathcal{X} \).

3.1. Stability conditions. Let \( D \) be a triangulated category, for which we fix a finitely generated free abelian group \( \Lambda \) and a group homomorphism \( v : K(D) \to \Lambda \).

Definition 3.1. A stability condition on \( D \) is a pair \( \sigma = (Z, A) \), where \( A \) is the heart of a bounded t-structure on \( D \), and \( Z : \Lambda \to \mathbb{C} \) is a group homomorphism (called central charge) such that
\[
Z(v(E)) \in \{ re^{i\pi\phi} : r > 0, 0 < \phi \leq 1 \}.
\]
\( (Z, A) \) satisfies the Harder-Narasimhan property: every object of \( A \) has a Harder-Narasimhan filtration in \( A \) with respect to \( \nu_\sigma \)-stability, here the slope \( \nu_\sigma \) of an object \( E \in A \) is defined by
\[
\nu_\sigma(E) = \begin{cases} 
+\infty, & \text{if } \Im Z(v(E)) = 0, \\
\frac{\Re Z(v(E))}{\Im Z(v(E))}, & \text{otherwise}.
\end{cases}
\]
We say \( E \in A \) is \( \nu_\sigma \)-(semi)stable if for any non-zero subobject \( F \subset E \) in \( A \), we have
\[
\nu_\sigma(F) < (\leq) \nu_\sigma(E/F).
\]
The Harder-Narasimhan filtration of an object \( E \in A \) is a chain of subobjects
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_m = E
\]
in \( A \) such that \( G_i := E_i/E_{i-1} \) is \( \nu_\sigma \)-semistable and \( \nu_\sigma(G_1) > \cdots > \nu_\sigma(G_m) \). We set \( \nu_\sigma^+(E) := \nu_\sigma(G_1) \) and \( \nu_\sigma^-(E) := \nu_\sigma(G_m) \).

Definition 3.2. For a stability condition \((Z, A)\) on \( D \) and for \( 0 < \phi \leq 1 \), we define the subcategory \( P(\phi) \subset D \) to be the category of \( \nu_\sigma \)-semistable objects \( E \in A \) satisfying \( \tan(\pi\phi) = -1/\nu_\sigma(E) \). For other \( \phi \in \mathbb{R} \) the subcategory \( P(\phi) \) is defined by the rule:
\[
P(\phi + 1) = P(\phi)[1].
\]
The objects in \( P(\phi) \) is still called \( \nu_\sigma \)-semistable objects.
For an interval $I = (a, b) \subset \mathbb{R}$, we denote by $\mathcal{P}(I)$ the extension-closure of

$$
\bigcup_{\phi \in I} \mathcal{P}(\phi) \subset \mathcal{D}.
$$

$\mathcal{P}(I)$ is a quasi-abelian category when $b - a < 1$ (cf. [5, Definition 4.1]). If we have a distinguished triangle

$$A_1 \xrightarrow{h} A_2 \xrightarrow{g} A_3 \rightarrow A_1[1]$$

with $A_1, A_2, A_3 \in \mathcal{P}(I)$, we say $h$ is a strict monomorphism and $g$ is a strict epimorphism. Then we say that $\mathcal{P}(I)$ is of finite length if $\mathcal{P}(I)$ is Noetherian and Artinian with respect to strict epimorphisms and strict monomorphisms, respectively.

**Definition 3.3.** A stability condition $\sigma = (Z, \mathcal{A})$ is called locally finite if there exists $\varepsilon > 0$ such that for any $\phi \in \mathbb{R}$, the quasi-abelian category $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length.

**Definition 3.4.** We say a stability condition $\sigma = (Z, \mathcal{A})$ satisfies the support property if there is a quadratic form $Q$ on $\Lambda$ satisfying $Q(v(\mathcal{E})) \geq 0$ for any $\nu_\sigma$-semistable object $\mathcal{E} \in \mathcal{A}$, and $Q|_{\ker Z}$ is negative definite.

**Remark 3.5.** The local finiteness condition automatically follows if the support property is satisfied (cf. [3, Section 1.2] and [6, Lemma 4.5]).

3.2. Conjectures. We now generalize [16, Conjecture 5.2] to arbitrary relative tilt-semistable objects:

**Conjecture 3.6.** Assume that $\mathcal{E} \in \text{Coh}^H_{b, \mathcal{C}}(\mathcal{X})$ is $\nu_{H,F}^{\alpha,\beta}$-semistable for some $\alpha, \beta$ in $\mathbb{R}_{>0} \times \mathbb{R}$ with $\nu_{H,F}^{\alpha,\beta}(\mathcal{E}) \neq +\infty$. Then

$$
\begin{align*}
(1) & \quad \left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}^\alpha_0(\mathcal{E}) \right) \left( H \text{ch}^\beta_2(\mathcal{E}) - \frac{H^3}{3 H^2 F} H F \text{ch}^\beta_2(\mathcal{E}) \right) \\
& \quad \geq \left( \text{ch}^\beta_3(\mathcal{E}) - \frac{\alpha^2}{2} H^2 \text{ch}^\beta_1(\mathcal{E}) + \frac{\alpha^2 H^3}{3 H^2 F} H F \text{ch}^\beta_1(\mathcal{E}) \right) H F \text{ch}^\beta_1(\mathcal{E}).
\end{align*}
$$

We will show that Conjecture 3.6 follows from a more natural and seemingly weaker statement:

**Conjecture 3.7.** Assume that $\mathcal{E} \in \text{Coh}^H_{b, \mathcal{C}}(\mathcal{X})$ is $\nu_{H,F}^{\alpha,\beta}$-semistable for some $\alpha, \beta$ in $\mathbb{R}_{>0} \times \mathbb{R}$ with $\nu_{H,F}^{\alpha,\beta}(\mathcal{E}) = 0$. Then

$$
(2) \quad \text{ch}^\beta_3(\mathcal{E}) \leq \alpha^2 \frac{H^2}{2} \text{ch}^\beta_1(\mathcal{E}) - \frac{\alpha^2 H^3}{3 H^2 F} H F \text{ch}^\beta_1(\mathcal{E}).
$$

**Theorem 3.8.** Conjecture 3.6 holds if and only if Conjecture 3.7 holds for all $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$.

**Proof.** The proof is similar to that of [2, Theorem 4.2]. Consider the below statement:

(*) Assume that $\mathcal{E}$ is $\nu_{H,F}^{\alpha,\beta}$-semistable $\nu_{H,F}^{\alpha,\beta}(\mathcal{E}) \neq +\infty$. Let $\beta' := \beta + \nu_{H,F}^{\alpha,\beta}(\mathcal{E})$. Then

$$
(3) \quad \text{ch}^\beta_{3'}(\mathcal{E}) \leq (\alpha^2 + \nu_{H,F}^{\alpha,\beta}(\mathcal{E}))^2 \left( \frac{1}{2} H^2 \text{ch}^{\beta'}_1(\mathcal{E}) - \frac{H^3}{3 H^2 F} H F \text{ch}^{\beta'}_1(\mathcal{E}) \right).
$$
Obviously, Conjecture 3.7 is a special case of (*). Conversely, consider the assumptions of (*). By Proposition 2.21 (3), $\mathcal{E}$ is $\nu^\alpha,\beta H,F$-semistable, where $\alpha^2 = \nu^\alpha,\beta H,F \mathcal{E}^2$. A simple computation shows $\nu^\alpha,\beta H,F \mathcal{E} = 0$. Thus Conjecture 3.7 implies the statement (*).

Finally, a straightforward computation shows that the inequalities (3.3) and (3.1) are equivalent. For this purpose, let us use the abbreviations $\nu = \nu_H,F \mathcal{E}$ and $e_i = \text{ch}_i \mathcal{E}$ for $1 \leq i \leq 3$. Expanding inequality (3.3), one sees that

$$e_3 - \nu H e_2 + \frac{\nu^2 H^2}{2} e_1 - \frac{\nu^3 H^3}{6} e_0 \leq \frac{1}{2} (\alpha^2 + \nu^2)(H^2 e_1 - \nu H^3 e_0)$$

$$- \frac{\alpha^2}{3H^2 F}(\alpha^2 + \nu^2)(HF e_1 - \nu H^2 F e_0).$$

Collecting related terms, one obtains

$$e_3 - \nu H e_2 \leq \frac{\alpha^2}{2} H^2 e_1 - \frac{\alpha^2}{6} \nu H^3 e_0.$$ Substituting $\nu = (Fe_2 - \frac{1}{2} \alpha^2 H^2 F e_0)/HF e_1$ and multiplying with $HF e_1$ yields:

$$(HF e_1)e_3 - (Fe_2 - \frac{1}{2} \alpha^2 H^2 F e_0)HF e_2 \leq \frac{\alpha^2}{2} (H^2 e_1)(HF e_1) - \frac{\alpha^2 H^3}{3H^2 F}(HF e_1)^2$$

$$- \frac{H^3}{3H^2 F}(Fe_2 - \frac{1}{2} \alpha^2 H^2 F e_0)^2$$

$$- \frac{\alpha^2}{6} (Fe_2 - \frac{1}{2} \alpha^2 H^2 F e_0)H^3 e_0.$$ This simplifies to (3.1). □

Considering the limit as $\alpha \to +\infty$, Conjecture 3.6 gives a Bogomolov type inequality:

**Proposition 3.9.** Let $\mathcal{E}$ be a $\mu_{H,F}$-semistable torsion free sheaf on $X$. Assume that Conjecture 3.6 holds for all $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. Then

$$\nabla_{H,F}(\mathcal{E}) := \frac{1}{3} H^3 \text{ch}_0(\mathcal{E}) F \text{ch}_2(\mathcal{E}) - \frac{2H^3}{3H^2 F}(HF \text{ch}_1(\mathcal{E}))^2$$

$$+ H^2 \text{ch}_1(\mathcal{E}) HF \text{ch}_1(\mathcal{E}) - H^2 F \text{ch}_0(\mathcal{E}) H \text{ch}_2(\mathcal{E})$$

$$= \Delta_{H,F}(\mathcal{E}) - \frac{H^3}{6H^2 F} \nabla_{H,F}(\mathcal{E}) - \frac{H^3}{2H^2 F}(HF \text{ch}_1(\mathcal{E}))^2$$

$$\geq 0.$$ 

**Proof.** If $\mathcal{E}$ is not $\mu_{H,F}$-stable, we let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be the stable factors of $\mathcal{E}$. Then one has

$$\mu_{H,F}(\mathcal{F}_1) = \cdots = \mu_{H,F}(\mathcal{F}_m) = \mu_{H,F}(\mathcal{E}) = \mu,$$

and thus

$$\nabla_{H,F}(\mathcal{E}) = \frac{H^3}{3H^2 F} \text{ch}_0(\mathcal{E}) F \text{ch}_2(\mathcal{E}) - \frac{2\mu H^3}{3H^2 F} HF \text{ch}_1(\mathcal{E}) + \mu H^2 \text{ch}_1(\mathcal{E}) - H \text{ch}_2(\mathcal{E})$$

$$= \nabla_{H,F}(\mathcal{F}_1) + \cdots + \nabla_{H,F}(\mathcal{F}_m).$$

It follows that $\nabla_{H,F}(\mathcal{E}) \geq 0$, if $\nabla_{H,F}(\mathcal{F}_1), \ldots, \nabla_{H,F}(\mathcal{F}_m) \geq 0$. Therefore we can reduce to the case that $\mathcal{E}$ is $\mu_{H,F}$-stable.
By Proposition 2.21(7), one sees that $\mathcal{E}$ is $\nu_{H,F}^{\alpha,\beta}$-stable for $\alpha \gg 0$ and for some $\beta \in \mathbb{R}$, and hence our assumption implies

$$\left( F \, \text{ch}^2_{\beta}(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \, \text{ch}_0(\mathcal{E}) \right) \left( H \, \text{ch}^2_{\beta}(\mathcal{E}) - \frac{H^3}{3H^2 F} F \, \text{ch}^3_{\beta}(\mathcal{E}) \right) \geq \left( \text{ch}^3_{\beta}(\mathcal{E}) - \frac{\alpha^2}{2} H^2 \, \text{ch}^1_{\beta}(\mathcal{E}) + \frac{\alpha^2 H^3}{3H^2 F} H F \, \text{ch}^3_{\beta}(\mathcal{E}) \right) H F \, \text{ch}^2_{\beta}(\mathcal{E}).$$

Taking $\alpha \to +\infty$, one deduces

$$-\frac{1}{2} H^2 F \, \text{ch}_0(\mathcal{E}) \left( H \, \text{ch}^2_{\beta}(\mathcal{E}) - \frac{H^3}{3H^2 F} F \, \text{ch}^3_{\beta}(\mathcal{E}) \right) \geq -\frac{1}{2} H^2 \, \text{ch}^1_{\beta}(\mathcal{E}) + \frac{H^3}{3H^2 F} H F \, \text{ch}^3_{\beta}(\mathcal{E}) \right) H F \, \text{ch}^2_{\beta}(\mathcal{E}),$$

i.e.,

$$\nabla_{H,F}^{\beta}(\mathcal{E}) \ := \ \frac{1}{3} H^3 \, \text{ch}_0(\mathcal{E}) F \, \text{ch}^2_{\beta}(\mathcal{E}) - \frac{2H^3}{3H^2 F} (H F \, \text{ch}^3_{\beta}(\mathcal{E}))^2 + H^2 \, \text{ch}^1_{\beta}(\mathcal{E}) H F \, \text{ch}^3_{\beta}(\mathcal{E}) - H^2 F \, \text{ch}_0(\mathcal{E}) H \, \text{ch}^2_{\beta}(\mathcal{E})$$

$$= \ \Delta_{H,F}^{\beta}(\mathcal{E}) - \frac{H^3}{2H^2 F} (H F \, \text{ch}^3_{\beta}(\mathcal{E}))^2 \geq 0.$$

A straightforward calculation shows that $\nabla_{H,F}^{\beta}(\mathcal{E}) = \nabla_{H,F}(\mathcal{E})$ for any $\beta \in \mathbb{R}$. This finishes the proof. \hfill \Box

**Remark 3.10.** It seems that the inequality (3.4) is highly non-trivial even for line bundles. In fact, when $\mathcal{E} = \mathcal{O}(L)$ for some divisor $L$, the inequality (3.4) becomes

$$(H^2 L)(H F L) - \frac{1}{2}(H^2 F)(H L^2) + \frac{1}{6} H^3 (F L)^2 - \frac{2H^3}{3H^2 F} (H F L)^2 \geq 0.$$

It seems not easy to prove.

The following weaker inequality is more convenient to use to construct stability conditions.

**Proposition 3.11.** Assume that Conjecture 3.6 holds for some $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. For any $\nu_{H,F}^{\alpha,\beta}$-semistable object $\mathcal{E}$, we have

$$\left( F \, \text{ch}^2_{\beta}(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \, \text{ch}_0(\mathcal{E}) \right) \left( H \, \text{ch}^2_{\beta}(\mathcal{E}) \right) \geq \left( \text{ch}^3_{\beta}(\mathcal{E}) - \frac{\alpha^2}{2} H^2 \, \text{ch}^1_{\beta}(\mathcal{E}) + \frac{\alpha^2 H^3}{4H^2 F} H F \, \text{ch}^3_{\beta}(\mathcal{E}) \right) H F \, \text{ch}^2_{\beta}(\mathcal{E}).$$
Proof. When $HF \text{ch}^\beta_1(\mathcal{E}) = 0$, the inequality (3.5) immediately follows from Lemma 2.18. By Theorem 2.20, one sees that

\[
\begin{align*}
\left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}_0(\mathcal{E}) \right) \left( H \text{ch}^\beta_2(\mathcal{E}) - \frac{H^3}{3 H^2 F} F \text{ch}^\beta_2(\mathcal{E}) \right) \\
= \left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}_0(\mathcal{E}) \right) H \text{ch}^\beta_2(\mathcal{E}) - \frac{H^3}{3 H^2 F} (F \text{ch}^\beta_2(\mathcal{E}))^2 \\
+ \frac{\alpha^2 H^3}{6 H^2 F} H^2 F \text{ch}_0(\mathcal{E}) F \text{ch}^\beta_2(\mathcal{E}) \\
\leq \left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}_0(\mathcal{E}) \right) H \text{ch}^\beta_2(\mathcal{E}) + \frac{\alpha^2 H^3}{12 H^2 F} (HF \text{ch}^\beta_1(\mathcal{E}))^2.
\end{align*}
\]

Therefore, if $HF \text{ch}^\beta_1(\mathcal{E}) \neq 0$, our assumption implies

\[
\begin{align*}
\left( \text{ch}^\beta_3(\mathcal{E}) - \frac{\alpha^2}{2} H^2 \text{ch}^\beta_1(\mathcal{E}) + \frac{\alpha^2 H^3}{3 H^2 F} HF \text{ch}^\beta_1(\mathcal{E}) \right) HF \text{ch}^\beta_1(\mathcal{E}) \\
\leq \left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}_0(\mathcal{E}) \right) H \text{ch}^\beta_2(\mathcal{E}) + \frac{\alpha^2 H^3}{12 H^2 F} (HF \text{ch}^\beta_1(\mathcal{E}))^2.
\end{align*}
\]

This simplifies to (3.5). \qed

In order to apply Conjecture 3.6 to construct stability conditions on $\mathcal{X}$, we now introduce the mixed tilt-stability with a slight modification of the notion in [16, Section 4]. Let $t$ be a positive rational number. For any $\mathcal{E} \in \text{Coh}^{\beta H}(\mathcal{X})$, its mixed tilt-slope $\nu_{\alpha, \beta, t}$ is defined by

\[
\nu_{\alpha, \beta, t}(\mathcal{E}) = \begin{cases} 
+ \infty, & \text{if } HF \text{ch}^\beta_1(\mathcal{E}) = 0, \\
\left( \frac{(H+tF) \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} F H^2 \text{ch}^\beta_1(\mathcal{E})}{HF \text{ch}^\beta_1(\mathcal{E})} \right), & \text{otherwise}.
\end{cases}
\]

The $\nu_{\alpha, \beta, t}$-stability of $\mathcal{E}$ is defined as before in Definition 2.11. By [16, Theorem 4.1], $\nu_{\alpha, \beta, t}$-stability gives a notion of stability when $(\alpha, \beta, t) \in \sqrt{\mathbb{Q}_{>0}} \times \mathbb{Q} \times \mathbb{Q}_{>0}$, in the sense that it satisfies the weak see-saw property and the Harder-Narasimhan property. We also call it mixed tilt-stability.

**Theorem 3.12.** Assume that Conjecture 3.6 holds for some $(\alpha, \beta) \in \sqrt{\mathbb{Q}_{>0}} \times \mathbb{Q}$ and $\mathcal{E}$ is $\nu_{\alpha, \beta, t}$-semistable. Then

\[
(3.6) \quad \left( F \text{ch}^\beta_2(\mathcal{E}) - \frac{\alpha^2}{2} H^2 F \text{ch}_0(\mathcal{E}) \right) H \text{ch}^\beta_2(\mathcal{E}) \\
\geq \left( \text{ch}^\beta_3(\mathcal{E}) - \frac{\alpha^2}{2} H^2 \text{ch}^\beta_1(\mathcal{E}) + \frac{\alpha^2 H^3}{4 H^2 F} HF \text{ch}^\beta_1(\mathcal{E}) \right) HF \text{ch}^\beta_1(\mathcal{E}).
\]

Moreover, we have $\text{ch}^\beta_3(\mathcal{E}) \leq \frac{\alpha^2}{2} H^2 \text{ch}^\beta_1(\mathcal{E}) - \frac{\alpha^2 H^3}{4 H^2 F} HF \text{ch}^\beta_1(\mathcal{E})$ if $\nu_{\alpha, \beta, t}(\mathcal{E}) = 0$. 

Proof. We use the notations by Liu in [12].

\[
\begin{align*}
\alpha(E) &= -F \cdot \chi^2_2(E) + \frac{\alpha^2}{2} H^2 F \cdot ch_0(E) \\
\beta(E) &= -\chi^3_3(E) + \frac{\alpha^2}{2} H^2 \cdot ch^3_1(E) - \frac{\alpha^2 H^3}{4H^2F} H F \cdot ch^3_1(E) \\
\gamma(E) &= HF \cdot ch^3_1(E) \\
\delta(E) &= H \cdot ch^3_1(E).
\end{align*}
\]

Then inequality (3.19) and (3.20) become \(b(E) c(E) \geq a(E) d(E)\) and \(\nu_{\alpha,\beta,t}(E) = \frac{d(E) - ta(E)}{c(E)}\).

If \(c(E) = 0\), by Lemma 2.18 one sees that \(a(E) \leq 0\) and \(d(E) \geq 0\). Hence \(b(E) c(E) \geq a(E) d(E)\) in this case. The proof of the case of \(c(E) > 0\) is the same as that of [12, Theorem 5.4].

If \(\nu_{\alpha,\beta,t}(E) = 0\), one sees that \(d(E) = ta(E)\) and \(c(E) > 0\). Hence the inequality \(b(E) c(E) \geq a(E) d(E)\) implies \(b(E) \geq 0\). This completes the proof. \(\square\)

3.3. Constructions of stability conditions. We give the construction of the heart \(A_{\alpha,\beta}^t(\mathcal{X})\) of a bounded \(t\)-structure on \(D^b(\mathcal{X})\) as a tilt starting from \(\text{Coh}^H_c(\mathcal{X})\).

We consider the torsion pair \((T'_t, F'_t)\) in \(\text{Coh}^H_c(\mathcal{X})\) as follows:

\[
\begin{align*}
T'_t &= \{E \in \text{Coh}^H_c(\mathcal{X}) : \text{any quotient } E \to \mathcal{G} \text{ satisfies } \nu_{\alpha,\beta,t}(\mathcal{G}) > 0\} \\
F'_t &= \{E \in \text{Coh}^H_c(\mathcal{X}) : \text{any subobject } K \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta,t}(K) \leq 0\}.
\end{align*}
\]

Definition 3.13. We define the abelian category \(A_{\alpha,\beta}^t(\mathcal{X}) \subset D^b(\mathcal{X})\) to be the extension-closure

\[
A_{\alpha,\beta}^t(\mathcal{X}) = \langle T'_t, F'_t[1] \rangle.
\]

For \(s, t \in \mathbb{Q}_{>0}\), consider the following central charge

\[
\begin{align*}
\nu_{s,t} &= (s - \frac{\alpha^2 H^3}{4H^2F}) H F \cdot \chi^3_1 - \chi^3_3 + \frac{\alpha^2}{2} H^2 \cdot \chi^3_1 + t F \cdot \chi^3_1 - \frac{\alpha^2}{2} H^2 F \cdot ch_0.
\end{align*}
\]

We think of it as the composition

\[
\begin{align*}
\nu_{s,t} : \text{K}(D^b(\mathcal{X})) &\xrightarrow{\nu} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \frac{1}{6} \mathbb{Z} \xrightarrow{Z_{s,t}} \mathbb{C},
\end{align*}
\]

where the first map is given by

\[
\nu(\mathcal{E}) = (\chi_0(\mathcal{E}), F H \chi_1(\mathcal{E}), H^2 \chi_1(\mathcal{E}), F \chi_2(\mathcal{E}), H \chi_2(\mathcal{E}), \chi_3(\mathcal{E})).
\]

Theorem 3.14. Assume that Conjecture 3.6 holds for some \((\alpha, \beta) \in \sqrt{\mathbb{Q}_{>0}} \times \mathbb{Q}\) and the Bogomolov inequality holds on every fiber of \(f\), then the pair \((Z_{s,t}, A^t_{\alpha,\beta}(\mathcal{X}))\) is a stability condition on \(\mathcal{X}\) satisfying the support property for \(s, t \in \mathbb{Q}_{>0}\).

Proof. The proof of the positivity property and the Harder-Narasimhan property is the same as that of [16, Theorem 5.3]. Indeed, the key ingredients in the proof of [16, Theorem 5.3] are Proposition 2.17, Lemma 2.18, [10, Theorem 4.3] and [16, Proposition 4.7] which still hold in our case. The support property follows from Theorem 3.12 and [12, Lemma 5.6]. \(\square\)
4. Stability conditions on ruled threefolds

Throughout this section we let \( E \) be a locally free sheaf on \( C \) with \( \text{rk} E = 3 \) and \( \mathcal{X} := \mathbb{P}(E) \) be the projective bundle associated to \( E \) with the projection \( f : \mathcal{X} \to C \) and the associated relative ample invertible sheaf \( \mathcal{O}_X(1) \). Since \( \mathbb{P}(E) \equiv \mathbb{P}(E \otimes L) \) for any line bundle \( L \) on \( C \), we can assume that \( H := c_1(\mathcal{O}_X(1)) \) is nef. One sees that \( H^3 = \deg E \) and \( H^2F = 1 \). We freely use the notations in previous sections.

Lemma 4.1. Denote by \( g \) the genus of \( C \). Then we have

\[
\begin{align*}
    c_1(T_X) &= -f^*K_C - f^*c_1(E) + 3H \\
    c_2(T_X) &= 3H^2 - (6g - 6 + 2\deg E)HF.
\end{align*}
\]

In particular, for any divisor \( D \) on \( \mathcal{X} \) we have

\[
\begin{align*}
    DFc_1(T_X) &= 3DHF \\
    DHC_1(T_X) &= 3DH^2 - (2g - 2 + H^3)DHF \\
    D\left(c_1(T_X) + c_2(T_X)\right) &= 12DH^2 - (18g - 18 + 8H^3)DHF \\
    \chi(O_X) &= \frac{1}{24}c_1(T_X)c_2(T_X) = 1 - g.
\end{align*}
\]

Proof. See [16, Lemma 6.1] \( \square \)

Proposition 4.2. Let \( \mathcal{E} \) be a relative \( \beta \)-semistable object on \( \mathcal{X} = \mathbb{P}(E) \) with \( HF \text{ch}^\alpha_1(\mathcal{E}) \neq 0 \) for some \( \beta_0 \in \mathbb{R} \). Then we have

\[
HF \text{ch}_1(\mathcal{E}) = F \text{ch}_2(\mathcal{E}) = H \text{ch}_2(\mathcal{E}) = 0 \quad \text{and} \quad \text{ch}_3(\mathcal{E}) \leq 0.
\]

Proof. Tensoring with lines bundles \( \mathcal{O}_X(aH) \) for \( a \in \mathbb{Z} \) make it possible to reduce to the case of \( 0 \leq \beta(\mathcal{E}) < 1 \). If \( \Sigma_{H,F}(\mathcal{E}) = 0 \), from \( HF \text{ch}^\alpha_1(\mathcal{E}) \neq 0 \), it follows that \( \text{ch}_3(\mathcal{E}) \neq 0 \). This implies that

\[
\nu_{H,F}^{\alpha,\beta}(\mathcal{E}) = \frac{F \text{ch}^\beta_2(\mathcal{E}) - \alpha^2}{HF \text{ch}^\alpha_1(\mathcal{E})} \geq \frac{(\beta(\mathcal{E}) - \beta)^2 - \alpha^2}{2(\beta(\mathcal{E}) - \beta)} \to 0
\]

for \( (\alpha, \beta) \to (0, \beta(\mathcal{E})) \). Since \( HF \text{ch}^\beta_1(\mathcal{E}) > 0 \) when \( \Sigma_{H,F}(\mathcal{E}) > 0 \), we have \( \nu_{H,F}^{\alpha,\beta}(\mathcal{E}) \to 0 \) for \( (\alpha, \beta) \to (0, \beta(\mathcal{E})) \) in any case.

Let \( n \) be an integer satisfying \( 1 \leq n \leq 2 \). By Proposition [2.21(6)], one sees that \( \mathcal{O}_X(nH) \) and \( \mathcal{O}_X(K_X + nH)[1] \) are \( \nu_{H,F}^{\alpha,\beta} \)-stable for all \( \alpha > 0 \) and \( 0 \leq \beta < 1 \). For \( (\alpha, \beta) \to (0, \beta(\mathcal{E})) \), we have

\[
\nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(nH)) \to \frac{n - \beta(\mathcal{E})}{2} > 0
\]

and therefore

\[
\nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(nH)) = \nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(K_X + nH)[1]) \to \frac{n - 3 - \beta(\mathcal{E})}{2} < 0.
\]

and therefore

\[
\nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(nH)) > \nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(K_X + nH)).
\]
Applying the standard Hom-vanishing between stable objects and Serre duality, we conclude

\[
\text{Hom}(\mathcal{O}_X(nH), \mathcal{E}) = 0 \quad \text{and} \quad \text{Ext}^2(\mathcal{O}_X(nH), \mathcal{E}) = 0.
\]

The Hirzebruch-Riemann-Roch Theorem implies

\[
0 \geq \chi(\mathcal{O}_X(H), \mathcal{E})
\]

\[
= \frac{c_1(X)}{2} \text{ch}_2^H(\mathcal{E}) + \frac{c_2(X) + c_2^2}{12} \text{ch}_1^H(\mathcal{E}) + \chi(\mathcal{O}_X) \text{ch}_0(\mathcal{E})
\]

\[
= \frac{c_1(X)}{2} \text{ch}_2^H(\mathcal{E}) + \left(3H - (2g - 2 + H^3)F\right) \text{ch}_2^H(\mathcal{E}) + (1 - g) \text{ch}_0(\mathcal{E})
\]

\[
+ \left(H^2 - \frac{3}{2}g - \frac{3}{2} + \frac{2}{3}HF\right) \text{ch}_1^H(\mathcal{E})
\]

\[
= \text{ch}_3(\mathcal{E}) + \frac{1}{2}H \text{ch}_2(\mathcal{E}) - (g - 1 + \frac{H^3}{2})F \text{ch}_2(\mathcal{E}) - \left(\frac{1}{2}g - \frac{1}{2} + \frac{1}{6}H^3\right)HF \text{ch}_1(\mathcal{E})
\]

and

\[
0 \geq \chi(\mathcal{O}_X(2H), \mathcal{E})
\]

\[
= \frac{c_1^2(X)}{2} \text{ch}_2^{2H}(\mathcal{E}) + \left(3H - (2g - 2 + H^3)F\right) \text{ch}_2^{2H}(\mathcal{E}) + (1 - g) \text{ch}_0(\mathcal{E})
\]

\[
+ \left(H^2 - \frac{3}{2}g - \frac{3}{2} + \frac{2}{3}HF\right) \text{ch}_1^{2H}(\mathcal{E})
\]

\[
= \text{ch}_3(\mathcal{E}) - \frac{1}{2}H \text{ch}_2(\mathcal{E}) - (g - 1 + \frac{H^3}{2})F \text{ch}_2(\mathcal{E}) + \left(\frac{1}{2}g - \frac{1}{2} + \frac{1}{3}H^3\right)HF \text{ch}_1(\mathcal{E}).
\]

Thus one obtains

\[
\text{ch}_3(\mathcal{E}) \leq -\frac{1}{2}H \text{ch}_2(\mathcal{E}) + (g - 1 + \frac{H^3}{2})F \text{ch}_2(\mathcal{E}) + \left(\frac{1}{2}g - \frac{1}{2} + \frac{1}{6}H^3\right)HF \text{ch}_1(\mathcal{E})
\]

and

\[
\text{ch}_3(\mathcal{E}) \leq \frac{1}{2}H \text{ch}_2(\mathcal{E}) + (g - 1 + \frac{H^3}{2})F \text{ch}_2(\mathcal{E}) - \left(\frac{1}{2}g - \frac{1}{2} + \frac{1}{3}H^3\right)HF \text{ch}_1(\mathcal{E}).
\]

By Lemma 2.19, one sees that the above two inequalities also hold for \(\mathcal{E}(aF)\) for any \(a \in \mathbb{Z}\). This implies that

\[
F \text{ch}_2(\mathcal{E}) = -\frac{1}{2}HF \text{ch}_1(\mathcal{E}) = \frac{1}{2}HF \text{ch}_1(\mathcal{E}).
\]

Therefore one gets

\[
F \text{ch}_2(\mathcal{E}) = HF \text{ch}_1(\mathcal{E}) = 0.
\]

It follows that \(\text{ch}_3(\mathcal{E}) \leq -\frac{1}{6}HF \text{ch}_2(\mathcal{E})\) and \(\text{ch}_3(\mathcal{E}) \leq \frac{1}{3}HF \text{ch}_2(\mathcal{E})\). They imply

\[
\text{ch}_3(\mathcal{E}) \leq 0 \quad \text{and} \quad H \text{ch}_2(\mathcal{E}) = 0.
\]

This completes the proof. □

We show that Conjecture 3.7 holds for \(X = \mathbb{P}(\mathcal{E})\) and any \((\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}\).

**Theorem 4.3.** Let \(\mathcal{E}\) be a \(\nu_{H,F}^{\alpha,\beta}\)-semistable object on \(X = \mathbb{P}(\mathcal{E})\) with \(\nu_{H,F}^{\alpha,\beta}(\mathcal{E}) = 0\). Then we have

\[
\text{ch}_3^\beta(\mathcal{E}) \leq \frac{\alpha^2}{2}H^2 \text{ch}_1^\beta(\mathcal{E}) - \frac{\alpha^2 H^3}{3HF}HF \text{ch}_1^\beta(\mathcal{E}).
\]
Proof. We proceed by induction on \( \Delta_{H,F}(E) \), which by Theorem [2.20] is a non-negative integer-valued function on objects of \( \text{Coh}^H_X \).

If \( \Delta_{H,F}(E) = 0 \), then \( E \) is relative \( \beta \)-semistable by Proposition [2.21] (3). From Proposition [4.2] one sees that

\[
(4.1) \quad H F \, \text{ch}_1(E) = F \, \text{ch}_2(E) = H \, \text{ch}_2(E) = 0 \quad \text{and} \quad \text{ch}_3(E) \leq 0,
\]

and hence

\[
(4.2) \quad \text{ch}_3^\beta(E) = \text{ch}_3(E) - \beta H \, \text{ch}_2(E) + \frac{\beta^2}{2} H^2 \, \text{ch}_1(E) - \frac{\beta^3}{6} H^3 \, \text{ch}_0(E)
\]

\[
\leq \frac{\beta^2}{2} H^2 \, \text{ch}_1(E) - \frac{\beta^3}{6} H^3 \, \text{ch}_0(E)
\]

\[
= \frac{\beta^2}{2} H^2 \, \text{ch}_1^\beta(E) + \frac{\beta^3}{2} H^3 \, \text{ch}_0(E) - \frac{\beta^3}{6} H^3 \, \text{ch}_0(E)
\]

\[
= \frac{\beta^2}{2} H^2 \, \text{ch}_1^\beta(E) + \frac{\beta^3}{3} H^3 \, \text{ch}_0(E).
\]

On the other hand, since \( \nu^{\alpha,\beta}_{H,F}(E) = 0 \), one infers

\[
0 < HF \, \text{ch}_1^\beta(E) = HF \, \text{ch}_1(E) - \beta HF \, \text{ch}_0(E) = -\beta HF \, \text{ch}_0(E)
\]

and

\[
\frac{\alpha^2}{2} H^2 F \, \text{ch}_0(E) = F \, \text{ch}_2^\beta(E)
\]

\[
= F \, \text{ch}_2(E) - \beta HF \, \text{ch}_0(E) + \frac{\beta^2}{2} H^2 F \, \text{ch}_0(E)
\]

\[
= \frac{\beta^2}{2} H^2 F \, \text{ch}_0(E).
\]

Thus \( \text{ch}_0(E) \neq 0 \) and \( \beta^2 = \alpha^2 \). Equality (4.1) implies that \( \beta = -\frac{HF \, \text{ch}_1^\beta(E)}{HF \, \text{ch}_0(E)} \). Therefore one deduces

\[
\beta^3 = \alpha^2 \beta = -\frac{HF \, \text{ch}_1^\beta(E)}{HF \, \text{ch}_0(E)}.
\]

Substituting it into (4.2), one concludes that

\[
\text{ch}_3^\beta(E) \leq \frac{\alpha^2}{2} H^2 \, \text{ch}_1^\beta(E) - \frac{\alpha^2 H^3}{3HF} HF \, \text{ch}_1^\beta(E).
\]

Now we assume that \( \Delta_{H,F}(E) > 0 \). One sees that \( E \) is not relative \( \beta \)-semistable. Otherwise we obtain the conclusion of Proposition [4.2] which contradicts \( \Delta_{H,F}(E) \geq 0 \). Hence \( E \) is destabilized along a wall between \( (\alpha,\beta) \) and \( (0,\beta(E)) \). Let \( F_1, \ldots, F_m \) be the stable factors of \( E \) along this wall. By induction, the desired inequality holds for \( F_1, \ldots, F_m \) and so it does for \( E \) by the linearity of Chern character. \( \square \)

Corollary 4.4. Let \( E \) be a \( \mu_{H,F} \)-semistable torsion free sheaf on \( X = \mathbb{P}(E) \). Then

\[
\Delta_{H,F}(E) \geq \frac{H^3}{6HF} \Delta_{H,F}(E) + \frac{H^3}{2HF} (HF \, \text{ch}_1(E))^2.
\]

Proof. The conclusion follows from Proposition [3.9] and Theorem 4.3. \( \square \)

Corollary 4.5. There exist stability conditions satisfying the support property on \( X = \mathbb{P}(E) \).
Proof. The conclusion follows from Theorem 3.8, Theorem 1.13 and Theorem 3.14.

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