Concentration phenomena and pointwise description for a non-local sinh–Gordon model: a Dirichlet-to-Neumann approach

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Abstract

This work investigates a non-local elliptic sinh–Gordon equation with a singularly perturbed parameter in a ball. Under the Robin boundary condition, the solution asymptotically forms a quite steep boundary layer in a thin region (will be specifically described), and rapidly becomes a flat curve outside this region. Focusing more particularly on the refined structure of the thin layer in this region, the pointwise estimate with the precise boundary curvature effect is established. It should be stressed that, for this model, the standard argument of matching asymptotic expansions is limited because the model has a non-local coefficient depending on the unknown solution. A new approach relies on integrating ideas based on a Dirichlet-to-Neumann map in an asymptotic framework. The rigorous asymptotic expansions for the boundary layer structure also matches well with the numerical results. Furthermore, various boundary concentration phenomena of the thin layer are precisely demonstrated.

Keywords. Non-local sinh–Gordon, Pointwise structure, Dirichlet-to-Neumann approach, Curvature effects, Boundary concentration phenomenon.

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1 The model and an overview

Several important issues arising in plasma physics, electrochemistry and other topics lead to consider non-local models with singularly perturbed parameters; see, e.g., [1, 12, 14, 15, 16, 18, 30, 33, 34, 35] and references therein. Focusing particularly on the electrochemical phenomena near the charged particle immersed in symmetrical electrolytes [33, 35] as well as on related applications in colloidal systems [3, 21, 22], we are interested in a non-local sinh–Gordon equation

\[ \epsilon^2 \Delta U = \left( \int_{\Omega} \cosh U \, dx \right)^{-1} \sinh U \quad \text{in} \ \Omega \]

with the Robin boundary condition

\[ U + \gamma \epsilon \partial_{\bar{n}} U = a \quad \text{on} \ \partial\Omega. \]

Here \( 0 < \epsilon \ll 1 \) is a singular perturbation parameter scaled by length (see the related physical background below), \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) \((N > 1)\) with \( |\Omega| \) the standard Lebesgue measure, \( \Delta \) stands for the Laplace operator in \( \mathbb{R}^N \), \( \partial_{\bar{n}} := \bar{n} \cdot \nabla \) and \( \bar{n} := \bar{n}(x) \) is the outward unit normal vector at \( x \in \partial\Omega \) and

\[ \int_{\Omega} := |\Omega|^{-1} \int_{\Omega}. \]

Besides, \( \gamma > 0 \) is a constant independent of \( \epsilon \), and \( a := a(x) \neq 0 \) defined on \( \partial\Omega \) is a smooth function independent of \( \epsilon \). It should be stressed that the non-local coefficient \( \left( \int_{\Omega} \cosh U \, dx \right)^{-1} \) is a dimensionless variable because \( \int_{\Omega} \cosh U \, dx \) has the same physical dimension as the volume. Such a concept of dimensionless formulation plays a crucial role in connecting between the dimensionless model and the realistic physical phenomena; see, e.g., [35].

Equation (1.1) has various applications in the field of physics. When the non-local coefficient \( \left( \int_{\Omega} \cosh U \, dx \right)^{-1} \) is withdrawn, (1.1) becomes the standard elliptic sinh–Gordon equation describing a system of interacting charged particles for the thermal equilibrium of plasma at very high temperature (corresponding to the parameter \( \epsilon^{-2} \)); see, e.g., [11] and references therein. In such a situation, the physical background is usually set up in two dimensional domain \( \Omega \subset \mathbb{R}^2 \). Alternatively, (1.1) can be viewed as a sinh–Poisson equation [27] endowed with a “minus sign” on its Laplace operator. To distinguish between these models, in this case we shall call (1.1) a non-local sinh–Poisson type equation having a “positive sign”. On the other hand, on a formal level of a “stochastic” concept proposed in [24], (1.1) can be rewritten as independent identically distributed random variables with a Borel probability measure \( \mathcal{P} = \frac{1}{2} (\delta_{-1} + \delta_{+1}) \) defined on \([-1, 1]\); that is,

\[ \epsilon^2 \Delta U = 2 \left( \int_{\Omega} \int_{[-1,1]} e^{\mu U} \mathcal{P}(d\mu) \, dx \right)^{-1} \int_{[-1,1]} \mu e^{\mu U} \mathcal{P}(d\mu) \quad \text{in} \ \Omega, \]

where \( \mu \) is a positive parameter.
As $0 < \epsilon \ll 1$, $U$ develops a thin and quite steep layer near the boundary $\partial \Omega$; see Theorems 2.2 and 2.3 in Section 2.2 for the details of the layer structure.

where $\delta_{-1}$ and $\delta_{+1}$ are Dirac delta functions concentrated at $-1$ and $+1$, respectively. We further refer the reader to [4, 7, 23] and Section 3 of [6] for related theories and applications of this model.

Besides its traditional applications, recently this model has been used to simulate the ion transport and describe the structure and behavior of the thin electrical double layer (EDL) near the charged surface, particularly for that of spherical colloidal particle in a symmetrical electrolyte solution. We refer the reader to [20, 33, 35] and Section 2 for the specific detail. Hence, based on the related investigations in [33, 35], one has a strong motivation to study $(1.1) - (1.2)$ with small $\epsilon$ (corresponding to a small scaled Debye length), where $\Omega$ is set as a ball with the simplest geometry; see Section 2.1 for the setup. In particular, as $\epsilon$ approaches zero, the solution $U$ (corresponding to the electrostatic potential) is uniformly bounded to $\epsilon$ and exhibits a layer (corresponding to the EDL) with thickness of the order $\epsilon$ near the boundary. Recently, there is a vast literature concerning standard elliptic sinh–Gordon type equations and sinh–Poisson type equations (cf. [8, 27]). However, for non-local model $(1.1) - (1.2)$ with $0 < \epsilon \ll 1$, to the best of our knowledge the related concentration phenomena and the curvature effect on the asymptotics of solutions remain unclear. Our main interest will rely on its thin layer structure with the boundary curvature effect and establish various boundary concentration phenomena. The main results are stated in Section 2.2 and their proofs are put in Sections 4 and 5.

Before discussing the details of specific studies, let us sketch the basic property of such thin layers and point out the importance of analyzing its pointwise asymptotics (cf. Figure 1). Let $x_{1,\epsilon}$ and $x_{2,\epsilon}$ be two points located in this thin layer region and lying on the same direction of the outward normal to the boundary. So we have $\limsup_{\epsilon \downarrow 0} \epsilon^{-1} d(x_{1,\epsilon}) < \infty$, $i = 1, 2$, where $d(\cdot) := \text{dist}(\cdot, \partial \Omega)$ is the distance function to the boundary $\partial \Omega$. Note also that each $x_{i,\epsilon}$ approaches the boundary points as $\epsilon$ goes to zero. However, when $\lim_{\epsilon \downarrow 0} \epsilon^{-1} d(x_{1,\epsilon}) \neq \lim_{\epsilon \downarrow 0} \epsilon^{-1} d(x_{2,\epsilon})$, the height difference $|U(x_{1,\epsilon}) - U(x_{2,\epsilon})|$ of the thin layer profile at two points $x_{1,\epsilon}$ and $x_{2,\epsilon}$ does not tend to zero, and the difference between the slopes of the thin layer profile at these two points (in the direction of the outward normal to the same boundary point) will tend to infinity. Such a structure occurs in this quite thin region and is usually called the boundary layer. Outside this thin region, the whole profile exponentially decay to zero as $\epsilon$ approaches zero. Namely, the solution changes dramatically in this thin region, but merely makes a slight change outside this region. Without the pointwise asymptotic analysis at $x_\epsilon$ where $\limsup_{\epsilon \downarrow 0} \epsilon^{-1} d(x_\epsilon) < \infty$, we merely obtain a “one-point-jumping behavior” for the limiting profile of solutions at boundary points, and any information of the thin layer is hidden in the description.

Accordingly, we are devoted to pointwise asymptotics of solutions in order to better understand the structure of the whole thin layer. We develop a singular perturbation analysis for radially symmetric solutions (in the case that the domain $\Omega$ is a ball and $a(x)$ is a nonzero constant-valued function) and, more importantly, describe the effect of the boundary curvature on the thin layers precisely. A series of basic estimates will be introduced in Sections 3 and 4. The main concept is to establish a Dirichlet-to-Neumann type map in an asymptotic framework (cf. Theorem 3.1). This rigorously derives the expansion formulas with accurate first-two-term expansions (with respect to $\epsilon$) for the layered solution at each point which is sufficiently close to the boundary (in the sense that the distance between the point and the boundary has at most the order $\epsilon$). Furthermore, we show in Proposition 2.1 (see also, Lemma 3.4) that the second order term (the small perturbation term) of the asymptotic expansions of the non-local coefficient plays a key role in the structure of the thin layer because it involves the boundary curvature. As will be clarified in Lemma 3.2 and Theorem 2.3, the effect of the boundary curvature is significant in a thin region attaching to the boundary, but is quite slight outside this thin region.

It should be stressed that the application of the Dirichlet-to-Neumann map to singularly perturbed non-local elliptic model is novel and different from the method of matching asymptotic expansions (see, e.g., [2, 5, 10, 26]). To the best of our knowledge, the traditional approach of matching asymptotic expansions is actually not easy to deal with such a non-local model before we obtain the accurate asymptotics of its non-local coefficients. Accordingly, this new approach has some advantages in dealing with such a
We highlight them in turn here primarily for the reader to get a clear picture on this work (see (P1)–(P3) in Section 2.1 for the preliminary analysis).

(1) We first refer the reader to [25, 31, 32], showing that for some semilinear elliptic equations in a bounded smooth domain \( \Omega \), the mean curvature of \( \partial \Omega \) appears in the second term of asymptotic expansions of their layers. As a motivation, we consider the following non-local models which are generalized from (1.1) and study the structure of thin layers:

\[
e^2 \Delta u_i = C_{u_i} f(u_i) \quad \text{in } \Omega
\]

with the same boundary condition as (1.2), \( i = 1, 2 \), where \( C_{u_i} \) is a constant depending on unknown solution \( u_i \). By [25, 31], we assert that even if \( C_{u_1} - C_{u_2} \to 0 \), the different second order terms (tending to zero as \( \epsilon \downarrow 0 \)) of \( C_{u_1} \) and \( C_{u_2} \) results in different structures of their layers near the boundary. However, as \( \epsilon^2 \) is sufficiently small, the numerical solutions are not easy to show the difference. Hence, for (1.1)–(1.2) with small \( \epsilon^2 \), investigating the precise first two term of the non-local coefficient with respect to \( \epsilon \) and establishing the pointwise asymptotics and the boundary curvature effects is usually of a challenge and particularly interesting.

(2) In this work, we focus on the case of \( \Omega = B_R := \{ x \in \mathbb{R}^N : |x| < R \} \) a ball with the simplest geometry and \( a(x) \equiv a_0 \neq 0 \) a constant-valued function (cf. Section 2.1). Then the uniqueness of (1.1)–(1.2) (see Proposition 6.2 in the Appendix) implies that \( U \) is radially symmetric in \( B_R \). We develop a rigorous asymptotic analysis based on a Dirichlet-to-Neumann map in the asymptotic framework with \( 0 < \epsilon \ll 1 \) (cf. (2.8) and Theorem 3.1). Using such an approach, we establish precise first two terms of the non-local coefficient of (1.1). In particular, the second order term exactly involves the boundary curvature \( R^{-1} \) (cf. Proposition 2.1). Furthermore, we derive an ODE of \( U \) in an asymptotic framework involving the curvature effect (cf. (2.17) and Lemma 4.1). We show that as \( \epsilon \downarrow 0 \), \( U \) develops quite steep boundary layers in a thin region with thickness of the order \( \epsilon \) attaching to the boundary \( \partial \Omega \) (cf. Figure 1 and Theorem 2.2). We completely study the structure of the thin layer through establishing refined pointwise asymptotics of \( U \) in this thin region (cf. Theorem 2.3 and the proof in Section 4.2).

(3) An interesting outcome shows that the second order term of the asymptotics of \( U(x_\epsilon) \) is algebraically dependent on the first two order terms (with respect to \( \epsilon \)) of \( d(x_\epsilon) \), which are presented in (2.22) and Theorem 2.3. One may also find from Figure 2 and Table 1 that the rigorous asymptotics almost seems to match the numerical simulations of \( U \) corresponding to \( \epsilon = 10^{-3} \).

(4) Under the same boundary condition, a comparison between asymptotic solutions of the non-local sinh-Gordon equation and the standard sinh-Gordon equation is completely studied. Although these two solutions have the same leading order terms, their second order terms are totally different. The main difference comes from the second order term of the asymptotic expansion of the non-local coefficient of (1.1) (see Section 4.3). The conclusion supports the assertion in (1).

We also want to point out that the numerical solutions of these two models with \( \epsilon = 10^{-3} \) seem almost overlapping near the boundary (see Figure 3 in Section 4.3). However, a closer look at pointwise asymptotics of solutions reveals that the slopes of their solution curves near the boundary always have \( O(1) \) difference which does not tend to zero as \( \epsilon \) goes to zero (see Remark 4).

(5) Various boundary concentration phenomena for the thin boundary layer are established (cf. Theorem 2.4 and the proof in Section 5).

Finally, we shall emphasize that although this work focuses mainly on non-local sinh-Gordon equations of radial cases, the analysis technique can be generalized to a class of non-local elliptic equations (1.3) with \( C_{u_i} = \left( \int_{\Omega} F(u_i) \, dx \right)^l \) for positive function \( F \) and \( l \neq 0 \), which is one of our ongoing projects.

2 Problem formulation and the results

Let us start with an energy functional

\[
E_\epsilon[U] = \frac{\epsilon^2}{2} \int_{\Omega} |\nabla U|^2 \, dx + |\Omega| \log \int_{\Omega} \cosh U \, dx + \frac{\epsilon}{2} \int_{\partial\Omega} (U - a)^2 \, d\sigma_x, \quad U \in H^1(\Omega).
\]  

The singular perturbation parameter \( \epsilon \) can be regarded as a length-scale parameter. Thus, the standard dimension analysis immediately implies that the boundary term \( \frac{\epsilon}{2} \int_{\partial\Omega} (U - a)^2 \, d\sigma_x \) scales in the same way as the gradient term \( \frac{\epsilon^2}{2} \int_{\Omega} |\nabla U|^2 \, dx \) and the logarithm term \( |\Omega| \log \int_{\Omega} \cosh U \, dx \).

(1.1)–(1.2) results from applying variational calculus to functional \( E_\epsilon \) over \( H^1(\Omega) \), where the non-local form is obtained from the variation of the logarithm term of (2.1). Indeed, functional \( E_\epsilon \) is strictly convex and admits a unique minimizer in \( H^1(\Omega) \).
Estimates are more readily available.

Electrolyte involving, for example, electrostatic interactions in spherical colloidal systems; see the physical background in, e.g., \( \epsilon \) asymptotic expansions with the boundary curvature effect for the thin layer as \( \Omega \).

The uniqueness for solutions of (1.1)–(1.2) (cf. Proposition 6.2) asserts that \( U \) is a weak solution of (1.1) with the Robin boundary condition (1.2). In the present work, new analysis technique is developed to deal with the asymptotic behavior (cf. \([28, 35]\)). Accordingly, the physical setting of model (1.1) is different from that of (2.2). On the other hand, we also prove the uniqueness of the classical solution to (1.1) with the boundary condition (1.2) in Proposition 6.2(i). On the other hand, we also prove the uniqueness for the classical solutions of the equation (1.1) with the Dirichlet and the Neumann boundary conditions which are stated in Proposition 6.2(ii) and (iii).

Equation (1.1)–(1.2) has important applications in electrochemistry, biology and physiology. In the ion-conserving Poisson–Boltzmann theory for symmetrical electrolytes \([33]\), equation (1.1) has been derived under the assumption that \emph{the total density of all ion species are conserved.} Here \( U \) corresponds to the electrostatic potential, and the parameter \( \epsilon \) is a scaled Debye screening length \([17, 18]\). Physically, \( \Omega \) usually represents the bulk in which all ion species occupy, where \( \frac{1}{2} \left( \int_{\Omega} \cosh U \, dx \right)^{-1} e^U \) corresponds to the Boltzmann distribution of anion species with charge valence \( -e_0 \) (\( e_0 \) is the elementary charge), and \( \frac{1}{2} \left( \int_{\Omega} \cosh U \, dx \right)^{-1} e^{-U} \) corresponds to the Boltzmann distribution of cation species with charge valence \( +e_0 \). The boundary \( \partial \Omega \) is regarded as a charged surface. Moreover, the electric field driving the ions toward the charged surface creates the EDL. The Robin boundary condition (1.2) is derived from the capacitance effect of the EDL \([13]\), where \( \gamma \epsilon \) is a scaled length with respect to the Stern layer, and \( a := a(x) \) is an extra potential applied on the charged surface \( \partial \Omega \). In recent years, this model is used to simulate the behavior of the electrostatic potential in the EDL, and has many applications in colloidal systems. Hence, a boundary layer problem for the model (1.1)–(1.2) naturally arises in mathematics, and the rigorous analysis seems a challenge. According to this motivation, we are interested in the boundary layer problem for model (1.1)–(1.2), especially in the boundary concentration phenomena and the pointwise description of the thin layer structure.

It is worth stressing a similar model proposed in \([18, 19, 29, 35]\), e.g.,

\[
\epsilon^2 \Delta U = \left( \int_{\Omega} e^U \, dx \right)^{-1} e^U - \left( \int_{\Omega} e^{-U} \, dx \right)^{-1} e^{-U} \quad \text{in} \; \Omega. \tag{2.2}
\]

This model is a steady-state Poisson–Nernst–Planck equation for symmetric 1 : 1 electrolytes, assuming that \emph{the density of each ion species is conserved} (cf. \([28, 35]\)). Accordingly, the physical setting of model (1.1) is different from that of (2.2). On the other hand, from a mathematical perspective one finds that (1.1) does not satisfy the shift invariance and the integral of its right-hand side \( \left( \int_{\Omega} \cosh U \, dx \right)^{-1} \sinh U \) over \( \Omega \) is not a constant value. Such a property is totally different from that of (2.2), and may increase the difficulty on the analysis of solutions. In the present work, new analysis technique is developed to deal with the asymptotic behavior of solutions of (1.1)–(1.2) with small \( \epsilon > 0 \).

### 2.1 The radial configuration and preliminary techniques

For equation (1.1)–(1.2), the asymptotics of the non-local coefficient \( \left( \int_{\Omega} \cosh U \, dx \right)^{-1} \) may depend on the domain geometry. To see such effects in a simple way, we focus mainly on the case that \( \Omega \) is a ball with the simplest geometry, and establish fine asymptotic expansions with the boundary curvature effect for the thin layer as \( \epsilon \) approaches zero. This setup describes a realistic electrolyte involving, for example, electrostatic interactions in spherical colloidal systems; see the physical background in, e.g., \([21, 22, 35]\) and references therein. Mathematically, such a setup allows us to study radially symmetric solutions where precise estimates are more readily available.

Hence, we may set \( \Omega = B_R := \{ x \in \mathbb{R}^N : |x| < R \} \) for \( R > 0 \), and \( a(x) \equiv a_0 \) on \( \partial B_R \), where \( a_0 \in \mathbb{R} \) is a constant. Then the uniqueness for solutions of (1.1)–(1.2) (cf. Proposition 6.2) asserts that \( U(x) = u(r) \) with \( r = |x| \) is radially symmetric and (1.1)–(1.2) is equivalent to

\[
\epsilon^2 \left( u''(r) + \frac{N-1}{r} u'(r) \right) = \mathcal{C}_\epsilon(u) \sinh u, \quad r \in (0, R),
\]

\[
\mathcal{C}_\epsilon(u) = \left( \frac{N}{R^N} \int_0^R s^{N-1} \cosh u(s) \, ds \right)^{-1},
\]

\[
u'(0) = 0, \quad u(R) + \gamma u'(R) = a_0.
\]

Here we let the surface area of the unit sphere \( |\partial B_1| = 1 \) for the convenience. The solution \( u \) may depend on the parameter \( \epsilon \) and should be denoted as \( u_\epsilon \) but we denote it as \( u \) for a sake of simplicity.

When \( a_0 = 0 \), (2.3)–(2.5) merely has a trivial solution due to the uniqueness. To avoid the trivial case, without loss of generality we may assume \( a_0 > 0 \). We are devoted to the pointwise asymptotics and various boundary concentration phenomena of the solution \( u \) as \( 0 < \epsilon \ll 1 \).

In order to properly state the main results, we now introduce some notational conventions and definitions that will be used throughout the whole paper.
**Notations.** We abbreviate \( \leq C \) to \( \lesssim \), where \( C > 0 \) is a generic constant independent of parameter \( \epsilon \). \( O(1) \) is denoted by a bounded quantity independent of \( \epsilon \). \( o(1) \) is denoted by a small quantity tending towards zero as \( \epsilon \) approaches zero.

We can now make the following definitions.

**Definition 1.** Assume that \( f_\epsilon \) has an expansion \( f_\epsilon = \sum_{i \in \mathbb{N}} f_{i(\epsilon)} \epsilon^{\sigma_i} \), where \( f_{i(\epsilon)} \) and \( \sigma_i \) are real numbers independent of \( \epsilon \) and \( \sigma_i < \sigma_i+1 \). We define

\[
\left( f_\epsilon \right)_1 := f_{1(\epsilon)} \epsilon^{\sigma_1}, \quad \text{and} \quad \left( f_\epsilon \right)_2 := f_{1(\epsilon)} \epsilon^{\sigma_1} + f_{2(\epsilon)} \epsilon^{\sigma_2}
\]

which map \( f_\epsilon \) to its leading term and first two terms, respectively.

Next, to demonstrate the boundary concentration phenomena, we introduce a Dirac delta function \( \delta_R \) concentrated at the boundary point \( r = R \) as follows.

**Definition 2.** It is said that

\( f_\epsilon \rightharpoonup C \delta_R \) weakly in \( C([0, R]; \mathbb{R}) \)

with a weight \( C \neq 0 \) as \( \epsilon \downarrow 0 \) if there holds

\[
\lim_{\epsilon \downarrow 0} \int_0^R h(r) f_\epsilon(r) \, dr = C h(R)
\]

for any continuous function \( h : [0, R] \to \mathbb{R} \) independent of \( \epsilon \).

Since \( C_\epsilon(u) \) is positive and \( \sinh u \) is strictly increasing to \( u \), applying the standard elliptic PDE comparison to (2.3)–(2.5), we obtain that \( u \) and \( u' \) exponentially decay to zero in the interior domain \( (0, R) \) as \( \epsilon \downarrow 0 \) (see (P1) below for the interior estimate). One key point for studying boundary asymptotics of \( u \) is to transform (2.3) into an integro-ODE

\[
\frac{\epsilon^2}{2} u^2(t) + (N-1) \epsilon^2 \int_{R/2}^t \frac{1}{r} u^2(r) \, dr = C_\epsilon(u) \cosh u(t) + K_\epsilon, \quad t \in [0, R),
\]

where \( K_\epsilon \) is a constant depending on \( \epsilon \). Obviously, using the boundary condition (2.5) and (2.7), we can make appropriate manipulations to obtain \( C_\epsilon(u) \to 1 \) and \( K_\epsilon \to -1 \) as \( \epsilon \downarrow 0 \) and the exact leading–order terms of boundary asymptotic expansions of \( u(R) \) and \( u'(R) \) (see, e.g., the argument in [17, 18]). However, the leading order terms cannot show the effect of the domain geometry (e.g., boundary curvature \( R^{-1} \)) on the solution structure. To basically understand such an issue, investigating their first two term asymptotic expansions with respect to \( \epsilon \) is necessary.

There are two main difficulties requiring discussion. The first difficulty comes from the fact that \( C_\epsilon(u) \) depends on the unknown solution \( u \). Hence, as \( \epsilon \) approaches zero, the asymptotics of \( u \) and \( C_\epsilon(u) \) are influenced by each other. Such rigorous analysis will be clarified in Section 3. Particularly, for (2.7), we show in Lemma 3.3 that the leading order term of \( \frac{\epsilon^2}{2} u^2(t) + (N-1) \epsilon^2 \int_{R/2}^t \frac{1}{r} u^2(r) \, dr \)

exactly determines the second order term (with respect to \( \epsilon \)) of \( C_\epsilon(u) \), \( u(R) \) and \( u'(R) \) as \( 0 < \epsilon \ll 1 \). Based on such an observation, it suffices to establish the exact leading order term of \( \int_0^R g(r) \cdot c u^2(r) \, dr \) for any continuous function \( g \in C([0, R]) \). An interesting outcome shows that \( c u^2 \) behaves exactly as a Dirac delta function concentrated at boundary point \( r = R \) (cf. Lemma 3.4).

The other difficulty comes from the Robin boundary condition (2.5) at \( r = R \). As a technical idea for dealing with the asymptotics of the thin layer near the boundary \( r = R \), we establish a Dirichlet-to-Neumann type map (cf. Theorem 3.1),

\[
\Lambda_\epsilon : u(R) \mapsto u'(R)
\]

which maps \( u(R) \) to \( u'(R) \) in an asymptotic framework.

\[
\left( \Lambda_\epsilon(u(R)) \right)_2 = \frac{2}{\epsilon} \left( \sinh \frac{u(R)}{2} \right)_2 - \frac{2}{\epsilon} \left( \frac{N \cosh^2 \frac{u(R)}{2}}{2} - 1 \right) \tanh \left( \frac{u(R)}{4} \right)_2,
\]

\( O(1) \) term involving the curvature effect

as \( 0 < \epsilon \ll 1 \). Moreover,

\[
\left| u'(R) - \left( \Lambda_\epsilon(u(R)) \right)_2 \right| \lesssim \sqrt{\epsilon}.
\]

We stress that \( \left( \sinh \frac{u(R)}{2} \right)_2 \) involves the second order term of \( u(R) \). Combining (2.8) with the Robin boundary condition (2.5), we can determine the exact first two order expansions of \( C_\epsilon(u) \), \( u(R) \) and \( u'(R) \) with respect to \( \epsilon \), which are described as follows.
**Proposition 2.1.** For $\epsilon > 0$, let $u$ be the unique classical solution of (2.3)–(2.5), where $a_0$ and $\gamma$ are positive constants independent of $\epsilon$. Then as $0 < \epsilon \ll 1$, we have

$$\left( C_\epsilon(u) \right)_2 = 1 - \frac{2N}{R} \left( \cosh \frac{b}{2} - 1 \right) \epsilon, \quad (2.9)$$

$$\left( u(R) \right)_2 = b + \frac{2}{\epsilon^2} \cdot \frac{\gamma}{\cosh \frac{b}{2} + 1} \left( N \cosh^2 \frac{b}{2} - 1 \right) \tanh \frac{b}{2}, \quad (2.10)$$

$$\left( u'(R) \right)_2 = \frac{2}{\epsilon} \sinh \frac{b}{2} - \frac{2}{R} \cdot \frac{N \cosh^2 \frac{b}{2} - 1}{\cosh \frac{b}{2} + 1} \tanh \frac{b}{2}, \quad (2.11)$$

with an optimal error estimate

$$\left| C_\epsilon(u) - \left( C_\epsilon(u) \right)_2 \right| + \left| u(R) - \left( u(R) \right)_2 \right| + \epsilon \left| u'(R) - \left( u'(R) \right)_2 \right| \lesssim \epsilon^{3/2}, \quad (2.12)$$

where $b \in (0, a_0)$ uniquely solves

$$b + 2\gamma \sinh \frac{b}{2} = a_0. \quad (2.13)$$

Note that Proposition 2.1 precisely illustrates the effects of the coefficient $\gamma$ and the boundary curvature $R^{-1}$ on the boundary asymptotics of $u$. Particularly, the boundary curvature exactly appears in their second order terms, and the third order terms of $C_\epsilon(u)$, $e^{-1}u(R)$ and $u'(R)$ tend to zero as $\epsilon \downarrow 0$.

Throughout this work, we need some important estimates in investigating the asymptotic structure of solutions in the whole domain. We summarize some crucial estimates and properties of $u$ as follows.

(P1). (cf. Lemma 3.2) $u \in [0, a_0)$ and $u$ is convex and strictly increasing in $(0, R)$. Moreover, for $r_\epsilon \in (0, R]$ we have an interior estimate

$$|u(r_\epsilon)| + \epsilon |u'(r_\epsilon)| \lesssim e^{-M_1(R-r_\epsilon)}, \quad 0 < \epsilon \ll 1, \quad (2.14)$$

where $M_1$ is a positive constant independent of $\epsilon$.

By (P1), $\frac{R-r_\epsilon}{\epsilon} \to \infty$ implies $|u(r_\epsilon)| + \epsilon |u'(r_\epsilon)| \to 0$. However, when $r_\epsilon$ is sufficiently close to the boundary in the sense of $\limsup_{\epsilon \downarrow 0} \frac{R-r_\epsilon}{\epsilon} < \infty$, asymptotics of $u(r_\epsilon)$ and $u'(r_\epsilon)$ still remain unclear. Since Proposition 2.1 and (P1) indicate the existence of boundary layer, to see the refined structure of the boundary layer, we shall further consider a quite thin region attaching to the boundary:

$$\mathcal{B}_\theta^\epsilon := \left\{ r_\epsilon \in [0, R] : \frac{R-r_\epsilon}{\epsilon} = p + o_\epsilon(1) \quad \text{for some } p \geq 0 \text{ independent of } \epsilon \right\}. \quad (2.15)$$

It is apparent that as $0 < \epsilon \ll 1$, $\mathcal{B}_\theta^\epsilon$ is an interval contained in $(R - \epsilon^\kappa, R]$ for $\kappa \in (0, 1)$. Furthermore, one will see in Theorem 2.2 that for each $p \geq 0$ independent of $\epsilon$, there hold

$$\liminf_{\epsilon \downarrow 0} u(R - (p + o_\epsilon(1))\epsilon) > 0 \quad \text{and} \quad \liminf_{\epsilon \downarrow 0} \epsilon u'(R - (p + o_\epsilon(1))\epsilon) > 0. \quad (2.16)$$

As a consequence, this exactly shows that $u$ exhibits a quite steep boundary layer in the whole region of $\mathcal{B}_\theta^\epsilon$.

Note also that (2.16) cannot be obtained from the interior estimate (2.14). To get (2.16), we need more refined estimates which will be established in the proof of Theorem 2.2.

As a consequence, solution $u$ changes dramatically and develops quite thin and steep layers in $\mathcal{B}_\theta^\epsilon$ as $0 < \epsilon \ll 1$, and only makes a slight change outside $\mathcal{B}_\theta^\epsilon$. To better understand the structure of the thin layer in $\mathcal{B}_\theta^\epsilon$ and its dependence on the boundary curvature $R^{-1}$, establishing the following pointwise asymptotics is particularly important.

(P2). (cf. Lemmas 4.1 and 4.2) There holds

$$\limsup_{\epsilon \downarrow 0} \left| u'(r_\epsilon) - 2 \sinh \frac{u(r_\epsilon)}{2} \left[ \frac{1}{\epsilon} - \frac{1}{R} \left( 2N \sinh^2 \frac{b}{4} + \frac{N-1}{2} \frac{\text{sech}^2 \frac{u(r_\epsilon)}{4}}{\epsilon} \right) \right] \right| = 0. \quad (2.17)$$

Moreover, since $u$ is strictly increasing in $\mathcal{B}_\theta^\epsilon$, by (2.16) and (2.17) we have that, for any $r_\epsilon^* \in \mathcal{B}_\theta^\epsilon$,

$$\frac{u'(r_\epsilon^*)}{2 \sinh \frac{u(r_\epsilon^*)}{2}} = \frac{1}{\epsilon} - \frac{1}{R} \left( 2N \sinh^2 \frac{b}{4} + \frac{N-1}{2} \frac{\text{sech}^2 \frac{u(r_\epsilon^*)}{4}}{\epsilon} \right) + o_\epsilon(1). \quad (2.18)$$
For novel, and plays a crucial role in boundary concentration phenomena of $u$-expansions seems difficult to give refined asymptotics such as (2.9)–(2.11), (2.17) and (2.19). Moreover, the following formula is uniformly in $[r^*_c, R]$.

Integrating (2.18) over $[r, R]$ and rearranging terms yields a nonlinear algebraic equation of $u(r)$ with a small term $\epsilon \cdot \alpha_0(1)$ uniformly in $[r^*_c, R]$,

$$\frac{R-r}{\epsilon} = \left(1 + \frac{2N}{R} \epsilon \sinh^2 \frac{b}{4}\right) \cdot \left(1 + \frac{N-1}{2R} \left\{ \frac{\tanh \frac{u(R)}{4}}{\left(\tanh \frac{u(r)}{4}\right)^2} \right\} \right) + \frac{N-1}{4R} \left(2 \tanh^2 \frac{u(r)}{4} - \tanh^2 \frac{u(R)}{4} \right) + \epsilon \cdot \alpha_0(1).$$

(2.19)

Hence, for each $r_c \in \mathcal{B}_g$, by Proposition 2.1, (2.17) and (2.19), we can obtain precise formulas of $\left(\frac{R-r_c}{\epsilon}\right)_2$ and $\left(u'(r_c)\right)_2$ which are explicitly expressed by $\left(u(r_c)\right)_2$. The corresponding results are stated in Theorem 2.3.

We want to point out again that without the Dirichlet-to-Neumann type map, using the argument of formal matching asymptotic expansions seems difficult to give refined asymptotics such as (2.9)–(2.11), (2.17) and (2.19). Moreover, the following formula is novel, and plays a crucial role in boundary concentration phenomena of $u$.

(P3). For $r_c \xrightarrow{\epsilon \downarrow 0} R$, there holds

$$\frac{1}{\epsilon} \int_{r_c}^{R} h(r)f(u(r))\, dr = h(R) \int_{u(r_c)}^{b} \frac{f(t)}{2\sinh \frac{t}{2}} \, dt + \mathcal{O}(1) \cdot (R-r_c),$$

(2.20)

as $0 < \epsilon \ll 1$, where $h : [0, R] \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a bounded function, and both functions are independent of $\epsilon$. (2.20) can be directly obtained from (2.10), (2.17) and $u(r_c) \leq u(r) < a_0$ for $r \in [r_c, R]$ (see (P1)). In particular, when $r^*_c \in \mathcal{B}_g$ satisfies $u(r^*_c) \xrightarrow{\epsilon \downarrow 0} k \in (0, b)$, by (2.15)–(2.16) and (2.20) we arrive at

$$\frac{1}{\epsilon} \int_{r^*_c}^{R} h(r)f(u(r))\, dr = h(R) \int_{k}^{b} \frac{f(t)}{2\sinh \frac{t}{2}} \, dt + \alpha_0(1).$$

This shows that the concentration phenomenon of $\frac{L(u(r))}{\epsilon}$ occurs in the quite thin region $[r^*_c, R] \subset \mathcal{B}_g$ provided $\int_{k}^{b} \frac{f(t)}{2\sinh \frac{t}{2}} \, dt \neq 0$.
Furthermore, (P3) gives an intuition to study the boundary concentration phenomena of solutions of \((2.3)-(2.5)\) as \(\epsilon \downarrow 0\). As an example, one may use Proposition 2.1 and (P1) to check that as \(\epsilon \downarrow 0\), both \(\epsilon^{-1} u\) and \(\epsilon u'\) exponentially decay to zero at any interior point independent of \(\epsilon\), and \(||\epsilon^{-1} u||_{L^1([0,R])}\) and \(||\epsilon u'||_{L^1([0,R])}\) stay bounded as \(\epsilon \downarrow 0\). But, they have the order \(\epsilon^{-1}\) at the boundary point. In Example 2 (see Section 2.2), we describe precisely the boundary concentration phenomena of \(\epsilon^{-1} u\) and \(\epsilon u'\) in the sense of Definition 2. Various boundary concentration phenomena are established in Theorem 2.4.

### 2.2 Statement of the main theorems

As \(\epsilon \downarrow 0\), the limiting profile of \(u\) becomes flat in \([0, R] = B_0^\epsilon\) (due to \(\lim_{\epsilon \downarrow 0} (u(r) + \epsilon u'(r)) = 0\) for \(r \not\in B_0^\epsilon\); see (P1)), and develops boundary layer near the boundary point (cf. Proposition 2.1). Accordingly, the thin region that may have two possibilities:

"either in the whole region of \(B_0^\epsilon\) or only in a partial region of \(B_0^\epsilon\)."

Notice that in the second situation, there exists \(r_0 \in B_0^\epsilon\) such that both \(u(r_0)\) and \(\epsilon u'(r_0)\) approach zero as \(\epsilon\) goes to zero. In such an issue, it is difficult to judge from the numerical solution of \((2.3)-(2.5)\) so a rigorous mathematical assertion is necessary.

The following theorem makes a specific presentation to assert that \(u\) indeed exhibits a quite steep boundary layer in the whole region of \(B_0^\epsilon\) as \(\epsilon \downarrow 0\), which is in extreme contrast with the behavior of \(u\) in the region \([0, R] = B_0^\epsilon\).

**THEOREM 2.2.** For \(\epsilon > 0\), let \(u\) be the unique classical solution of \((2.3)-(2.5)\), where \(a_0\) and \(\gamma\) are positive constants independent of \(\epsilon\). Then for \(r_\epsilon \in [0, R]\),

\[
\limsup_{\epsilon \downarrow 0} \frac{R - r_\epsilon}{\epsilon} < \infty \text{ if and only if } \begin{cases}
\liminf_{\epsilon \downarrow 0} u(r_\epsilon) > 0, \\
\liminf_{\epsilon \downarrow 0} \epsilon u'(r_\epsilon) > 0.
\end{cases}
\]  

(2.21)

The proof of Theorem 2.2 is stated in Section 4.1.

Moreover, to get the refined structure of the thin layer in \(B_0^\epsilon\), we focus on those points \(r_\epsilon^{p,q} \in B_0^\epsilon\) satisfying

\[
\left( \frac{R - r_\epsilon^{p,q}}{\epsilon} \right)_2 = p + \frac{q}{R}\epsilon, \quad \text{where } p \geq 0 \text{ and } q \in \mathbb{R} \text{ are independent of } \epsilon.
\]  

(2.22)

The setting of (2.22) with specific orders of \(\epsilon\) is mainly due to the boundary asymptotic expansions of \(u(R)\) and \(u'(R)\) in Proposition 2.1 so that we can compare them with \(u(r_\epsilon^{p,q})\) and \(u'(r_\epsilon^{p,q})\) in a direct way. The following theorem reveals that the leading order terms of \(u(r_\epsilon^{p,q})\) and \(u'(r_\epsilon^{p,q})\) are uniquely determined by \(p\) and the second order terms of that depend on both \(p\) and \(q\). Moreover, the effect of boundary curvature appearing in their second order terms are precisely described.

**THEOREM 2.3** (Pointwise descriptions with curvature effects in \(B_0^\epsilon\)). Under the same hypotheses as in Theorem 2.2, as \(0 < \epsilon \ll 1\), for \(r_\epsilon^{p,q} \in B_0^\epsilon\) obeying (2.22), the precise first two terms of \(u(r_\epsilon^{p,q})\) and \(u'(r_\epsilon^{p,q})\) are depicted as follows:

\[
\left( u(r_\epsilon^{p,q}) \right)_2 = k(p) + \frac{\epsilon}{R} \mathcal{H}_{p,q}^b \sinh \frac{k(p)}{2},
\]

(2.23)

\[
\left( u'(r_\epsilon^{p,q}) \right)_2 = 2 \sinh \frac{k(p)}{2} \left[ \frac{1}{\epsilon} - \frac{1}{R} \left( 2N \sinh^2 \frac{b}{4} + \frac{N - 1}{2} \text{sech}^2 \frac{k(p)}{4} - \frac{\mathcal{H}_{p,q}^b}{2} \cosh \frac{k(p)}{2} \right) \right],
\]

(2.24)

where \(k(p) \in (0, b]\) is uniquely determined by

\[
\left( 1 + \frac{N - 1}{2R} \right) \log \frac{\tanh \frac{b}{4}}{\tanh \frac{k(p)}{4}} + \frac{N - 1}{4R} \left( \tanh^2 \frac{k(p)}{4} - \tanh^2 \frac{b}{4} \right) = p,
\]

and

\[
\mathcal{H}_{p,q}^b := \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \text{sech}^2 \frac{b}{4}}{\gamma \cosh \frac{b}{2} + 1} \frac{1 + \frac{N - 1}{2RT} \text{sech}^2 \frac{b}{4}}{1 + \frac{N - 1}{2RT} \text{sech}^2 \frac{k(p)}{4}} - \frac{2q - 4Np \sinh^2 \frac{b}{4}}{1 + \frac{N - 1}{2RT} \text{sech}^2 \frac{k(p)}{4}}.
\]

Moreover, the convergence

\[
\frac{1}{\epsilon} \left| u(r_\epsilon^{p,q}) - \left( u(r_\epsilon^{p,q}) \right)_2 \right|_2 \rightarrow 0, \quad \left| u'(r_\epsilon^{p,q}) - \left( u'(r_\epsilon^{p,q}) \right)_2 \right|_2 \rightarrow 0
\]

(2.27)

is uniformly as \(p\) is located in a bounded subinterval of \([0, \infty)\).
The uniqueness of (2.25) is trivially due to the fact that \( (1 + N \frac{1}{2R}) \log \tanh \frac{b}{4} + N \frac{1}{4R} (\tanh^2 \frac{b}{4} - \tanh^2 \frac{b}{4}) \) is strictly decreasing to \( k \) in \( (0, b] \). The proof of Theorem 2.3 is stated in Section 4.2.

Theorem 2.3 establishes a rigorous analysis technique for rendering the curvature effect on the thin boundary layer of \( u \). Moreover, we can calculate the precise first two terms of \( \frac{d^2u}{dr^2}(r_{\gamma,q}^\epsilon) \) and \( u(R) \cdot (r_{\gamma,q}^\epsilon - R) \). One may obtain the combination of Proposition 2.1 and the formal Taylor approximation since \( r_{\gamma,q}^\epsilon \) approaches to the boundary point as \( \epsilon \downarrow 0 \). We shall stress that such a method does not work in general situations.

**Remark 1.** Assume \( p > 0 \) in (2.22). Using the formal Taylor approximation, we have

\[
 u(r_{\gamma,q}^\epsilon) = u(R) + \sum_{i=1}^{\infty} \frac{d^i u}{dr^i}(R) \cdot (r_{\gamma,q}^\epsilon - R)^i.
\]  

(2.28)

On the other hand, by (2.3) and (2.11), one may use mathematical induction to prove that the leading term of \( \frac{d^i u}{dr^i}(R) \) is exactly the order of \( \epsilon^{-4} \), \( \forall i \in \mathbb{N} \). Along with (2.22), we have \( \lim_{\epsilon \downarrow 0} \frac{d^i u}{dr^i}(R) \cdot (r_{\gamma,q}^\epsilon - R)^i \neq 0 \) for \( i \in \mathbb{N} \) since \( p > 0 \). (We want to stress that although the case \( p = 0 \) implies \( \lim_{\epsilon \downarrow 0} \frac{d^i u}{dr^i}(R) \cdot (r_{\gamma,q}^\epsilon - R)^i = 0 \), \( \forall i \in \mathbb{N} \), by (2.28) we merely make sure that \( u(r_{\gamma,q}^\epsilon) \) and \( u(R) \) have the same leading order term.) Accordingly, it seems that such an idea is not easy to get the exact first two order terms of \( u(r_{\gamma,q}^\epsilon) \) and \( u'(r_{\gamma,q}^\epsilon) \).

Here we give an application as follows.

**Example 1.** We establish asymptotics of \( u(r_{\gamma}) \), \( u'(r_{\gamma}) \) and \( r_{\gamma} \) as \( \epsilon \) tends to zero, where

\[
 |u(R) - u(r_{\gamma})| = \frac{1}{2} |u(R) - b|
\]  

(2.29)

and \( b = \lim_{\epsilon \downarrow 0} u(R) \) (cf. Proposition 2.1). Firstly, by (2.10) and (2.29), it yields that

\[
 \left( u(r_{\gamma}) \right)^2 = b + \frac{\epsilon}{R} \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{\gamma \cosh \frac{b}{2} + 1},
\]  

(2.30)

which shares the same leading order term with \( u(R) \), and \( |u(r_{\gamma}) - u(R)| \) is merely of the order \( \epsilon \). Hence, by the comparison of (2.23) and (2.30), one may obtain \( k(p) = b \) and \( \mathcal{H}^{\gamma,b} = \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \sech^2 \frac{b}{4}}{2(\gamma \cosh \frac{b}{2} + 1)}. \) Along with (2.25)–(2.26), it turns out that

\[
 p = 0 \quad \text{and} \quad q = \frac{\gamma}{4} \left( 1 + \frac{N - 1}{2R} \sech \frac{b}{4} \right) \frac{(N \cosh^2 \frac{b}{2} - 1) \sech^2 \frac{b}{2}}{\gamma \cosh \frac{b}{2} + 1}.
\]

The conclusion is \( R - r_{\gamma} \sim \epsilon^2 \) with asymptotics

\[
 \frac{R - r_{\gamma}}{\epsilon^2} = \frac{\gamma}{4R} \left( 1 + \frac{N - 1}{2R} \sech \frac{b}{4} \right) \frac{(N \cosh^2 \frac{b}{2} - 1) \sech^2 \frac{b}{2}}{\gamma \cosh \frac{b}{2} + 1} + o(1)
\]  

(2.31)

and \( u'(r_{\gamma}) \sim \epsilon^{-1} \) with asymptotics

\[
 \left( u'(r_{\gamma}) \right)^2 = \frac{2}{\epsilon} \sinh \frac{b}{4} \frac{1}{1 - \frac{2N}{R} \left( \cosh \frac{b}{2} - 1 + o(1) \right) \epsilon} \sinh u(r_{\gamma})
\]  

(2.32)

Finally, by (2.3) (2.9), (2.30) and (2.32), one immediately gets

\[
 u''(r_{\gamma}) = - (N - 1) \left( \frac{1}{R} + O(1) \epsilon^2 \right) u'(r_{\gamma}) + \frac{1}{\epsilon^2} \left[ 1 - \frac{2N}{R} \left( \cosh \frac{b}{2} - 1 + o(1) \right) \epsilon \right] \sinh u(r_{\gamma})
\]  

\[
 = \sinh \frac{b}{2} + N \left( \cosh \frac{b}{2} - 1 \right) \sinh \frac{b}{2} - \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{\gamma \cosh \frac{b}{2} + 1} + o(1)\epsilon.
\]

Here we have calculated

\[
 \sinh u(r_{\gamma}) = \sinh \left( b + \frac{\epsilon}{R} \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{\gamma \cosh \frac{b}{2} + 1} + o(1)\epsilon \right)
\]  

\[
 = \sinh b + \frac{\epsilon}{R} \frac{\gamma (N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4} \cosh b}{\gamma \cosh \frac{b}{2} + 1} + o(1)\epsilon.
\]
It seems that (2.31) and (2.32) are not easy to be obtained via the method of matching asymptotic expansions because it involves the exact second order terms of the asymptotics of the non-local coefficient $C_{\epsilon}(u)$.

For thin layered solutions of (2.3)–(2.5), we are also interested in its boundary concentration phenomenon. To see such phenomena, let $F \in C^0_{\alpha}([0, R]; R)$, $\tau \in (0, 1]$, be a locally Hölder (or Lipschitz) continuous function with exponent $\tau$ that is independent of $\epsilon$. Then both $\epsilon^{-1}|F(\epsilon u'(-r)) - F(0)|$ and $\epsilon^{-1}|F(\epsilon u(r)) - F(0)|$ blows up asymptotically near the boundary point. Indeed, by Lemma 3.2(ii), we obtain that for $r \in [0, R)$ and $0 < \epsilon \ll 1$,

$$|F(\epsilon u'(-r)) - F(0)| + |F(\epsilon u(r)) - F(0)| \lesssim \epsilon^\tau |u'(r)|^\tau + |u(r)|^\tau \lesssim \epsilon^{-\frac{\alpha}{\alpha+1}}r(R-r).$$

(2.33)

In particular, as $\epsilon \downarrow 0$, both $\epsilon^{-1}|F(\epsilon u'(-r)) - F(0)|$ and $\epsilon^{-1}|F(\epsilon u(r)) - F(0)|$ are uniformly bounded to $\epsilon$ in $L^1([0, R])$, and converge to zero uniformly in any compact subset of $[0, R]$. However, by Proposition 2.1, $\epsilon^{-1}|F(\epsilon u'(-r)) - F(0)|$ and $\epsilon^{-1}|F(\epsilon u(r)) - F(0)|$ diverge to infinity (note that $b > 0$). This also asserts the boundary concentration phenomenon of $\epsilon^{-1}|F(\epsilon u'(-r)) - F(0)|$ and $\epsilon^{-1}|F(\epsilon u(r)) - F(0)|$.

The following theorem precisely describes their boundary concentration phenomena via Dirac delta functions concentrated at boundary points (see Definition 2).

**Theorem 2.4 (Boundary concentration phenomenon).** Assume again that the same hypotheses as in Theorem 2.2 hold. Then for $F \in C^0_{\alpha}(\mathbb{R}; \mathbb{R})$ independent of $\epsilon$, as $\epsilon \downarrow 0$, $\frac{F(\epsilon u') - F(0)}{\epsilon}$ and $\frac{F(\epsilon u) - F(0)}{\epsilon}$ have boundary concentration phenomena described as follows:

**(I-i)** If $\int_{0+}^{b} \frac{F(2 \sinh \frac{1}{2} - F(0))}{2 \sinh \frac{1}{2}} \, dt \neq 0$, there holds

$$\frac{F(\epsilon u') - F(0)}{\epsilon} \rightharpoonup \left(\int_{0+}^{b} \frac{F(2 \sinh \frac{1}{2} - F(0))}{2 \sinh \frac{1}{2}} \, dt\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}).$$

(2.34)

**(I-ii)** If $\int_{0+}^{b} \frac{F(\tau - F(0))}{2 \sinh \frac{1}{2}} \, dt \neq 0$, there holds

$$\frac{F(\epsilon u) - F(0)}{\epsilon} \rightharpoonup \left(\int_{0+}^{b} \frac{F(\tau - F(0))}{2 \sinh \frac{1}{2}} \, dt\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}).$$

(2.35)

Moreover, for $r_p \in \mathbb{B}_\delta$ with $\lim_{\epsilon \downarrow 0} \frac{R - r_p}{\epsilon} = p$, as $\epsilon \downarrow 0$ we have

**(II-i)** If $\int_{k(p)}^{b} \frac{F(2 \sinh \frac{1}{2} - F(0))}{2 \sinh \frac{1}{2}} \, dt \neq 0$, there holds

$$\frac{F(\epsilon u')}{\epsilon} \chi_{[r_p, R]} \rightharpoonup \left(\int_{k(p)}^{b} \frac{F(2 \sinh \frac{1}{2})}{2 \sinh \frac{1}{2}} \, dt\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}),$$

(2.36)

where characteristic function $\chi_{[r, R]}$ is defined by $\chi_{[r, R]}(r) = 1$ for $r \in [r_p, R]$, and $\chi_{[r_p, R]}(r) = 0$ for $r \notin [r_p, R]$, and $k(p) \in (0, b)$ is uniquely determined by (2.25).

**(II-ii)** If $\int_{k(p)}^{b} \frac{F(\tau - F(0))}{2 \sinh \frac{1}{2}} \, dt \neq 0$, there holds

$$\frac{F(u)}{\epsilon} \chi_{[r_p, R]} \rightharpoonup \left(\int_{k(p)}^{b} \frac{F(\tau - F(0))}{2 \sinh \frac{1}{2}} \, dt\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}).$$

(2.37)

**Remark 2.** Integrals $\int_{0+}^{b} \frac{F(2 \sinh \frac{1}{2} - F(0))}{2 \sinh \frac{1}{2}} \, dt$ and $\int_{0+}^{b} \frac{F(\tau - F(0))}{2 \sinh \frac{1}{2}} \, dt$ are finite due to the estimate

$$\frac{|F(2 \sinh \frac{1}{2} - F(0))|}{2 \sinh \frac{1}{2}} + \frac{|F(\tau - F(0))|}{2 \sinh \frac{1}{2}} \lesssim t^{\tau-1}$$

for $t > 0$ and $0 < \tau \leq 1$, which can be checked via the elementary inequality $\sinh \frac{1}{2} \geq \frac{1}{2}$ for $t \geq 0$.

We will give the proof of Theorem 2.4 in Section 5. The following example describing the boundary concentration phenomena of $\epsilon(u')^2$ and $\epsilon^{-1}u$ is a direct result of Theorem 2.4.

**Example 2.** Both $\epsilon(u')^2$ and $\epsilon^{-1}u$ have boundary concentration phenomena in the following senses:

$$\epsilon(u')^2 \rightharpoonup 4 \left(\cosh \frac{b}{2} - 1\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}),$$

$$\epsilon^{-1}u \rightharpoonup \left(\int_{0}^{b} \frac{t}{2 \sinh \frac{1}{2}} \, dt\right) \delta_R \quad \text{weakly in } C([0, R]; \mathbb{R}), \quad \text{as } \epsilon \downarrow 0.$$
3 The Dirichlet-to-Neumann approach

Let \( u \in C^\infty((0, R)) \cap C^1([0, R]) \) be the unique classical solution of (2.3)–(2.5) (cf. Proposition 6.2). Since \( a_0, \gamma \) and \( \mathcal{C}_e(u) \) are positive, and \( \sinh u \) is increasing to \( u \), the standard maximum principle immediately implies

\[
0 \leq u(r) \leq a_0, \quad \forall r \in (0, R].
\]

In this section, we shall establish a Dirichlet-to-Neumann map at the boundary point \( r = R \),

\[
\Lambda_e(u(R)) = u'(R),
\]

in an asymptotic framework involving the curvature \( R^{-1} \) as \( 0 < \epsilon \ll 1 \), which plays a crucial role in the asymptotics of the non-local coefficient \( \mathcal{C}_e(u) \) and the proof of Theorem 2.4. The asymptotics of \( \Lambda_e(u(R)) \) is depicted as follows.

**Theorem 3.1.** Under the same hypotheses as in Proposition 2.1, we assume

\[
\liminf_{\epsilon \downarrow 0} u(R) > 0.
\]

Then, as \( 0 < \epsilon \ll 1 \) we have

\[
\left( \Lambda_e(u(R)) \right)_2 = \frac{2}{\epsilon} \left( \sinh \frac{u(R)}{2} \right)^2 - \frac{2}{R} \tanh \left( \frac{u(R)}{4} \right) \left( N \cosh^2 \frac{u(R)}{2} \right)^\frac{1}{4} - 1
\]

and

\[
\left| u'(R) - \left( \Lambda_e(u(R)) \right) \right|_2 \leq \epsilon^{1/2}.
\]

3.1 Proof of Theorem 3.1

To prove Theorem 3.1, we need some lemmas. Firstly, we establish crucial interior estimates as follows.

**Lemma 3.2** (Interior estimates). Assume \( a_0 > 0 \). For \( \epsilon > 0 \) and \( \gamma > 0 \), let \( u \) be the unique classical solution of (2.3)–(2.5). Then

(i) For \( \epsilon > 0 \) fixed, \( u(r) \) and \( u'(r) \) are strictly positive and \( u''(r) \geq 0 \) in \( (0, R] \).

(ii) As \( 0 < \epsilon \ll 1 \), there hold

\[
\max \{|u(r)|, \gamma \epsilon|u'(r)|\} \leq 2a_0 e^{-\frac{1}{2} (\cosh a_0)^{-1/2} (R - r)}, \quad r \in [0, R].
\]

**Proof.** Note that (3.1) implies

\[
\mathcal{C}_e(u) \geq (\cosh a_0)^{-1}.
\]

Thus by (2.3) and (3.7), we have

\[
\epsilon^2 \left( u'' + \frac{N - 1}{r} u' \right) \geq (\cosh a_0)^{-1} u.
\]

Hence, by Proposition 2.1. of [18], (2.3) is a second order elliptic equation and the solution \( u \) satisfies the unique continuation property. Now we give the proof of (i). Suppose by contradiction that there exists \( r_0 \in (0, R) \) such that \( u'(r_0) = 0 \). Then, multiplying (3.8) by \( r^{N-1} \), integrating the expression over \((0, r_0)\) and using \( u''(0) = u''(r_0) = 0 \) immediately give

\[
\int_0^{r_0} u(r) \, dr = 0.
\]

Along with (3.1) implies \( u \equiv 0 \) in \([0, r_0]\), and then the unique continuation property shows that \( u \) is trivial in \([0, R]\), a contradiction. Consequently, \( u' > 0 \) in \((0, R]\). Similarly, by (3.1) and unique continuation property, we obtain \( u > 0 \) in \((0, R]\).

Differentiating (2.3) to \( r \) and using (3.7) and \( u, u' > 0 \) in \((0, R]\), we have

\[
\epsilon^2 \left( u'' + \frac{N - 1}{r} u' \right) = \left( \frac{(N - 1) \epsilon^2}{r^2} + \mathcal{C}_e(u) \cosh u \right) u' \geq (\cosh a_0)^{-1} u' > 0 \text{ in } (0, R).
\]

For \( \epsilon > 0 \) fixed, multiplying (3.9) by \( r^{N-1} \), one arrives at \( \epsilon^2 (r^{N-1} u''(r))' > 0 \). Hence, for \( r \in (0, R) \) we have

\[
r^{N-1} u''(r) \geq \liminf_{s \downarrow 0+} s^{N-1} u''(s) = \liminf_{s \downarrow 0+} \left( \frac{\mathcal{C}_e(u(s))}{\epsilon^2} s^{N-1} \sinh u(s) - (N - 1) s^{N-2} u'(s) \right) > 0.
\]
Here we have used the facts \( \sinh u(s) \geq 0, \ u'(0) = 0 \) and \( N > 1 \) to verify (3.10). This implies \( u''(r) \geq 0 \) for \( r \in (0, R) \), and completes the proof of (i).

Now we want to prove (ii). Multiplying (3.8) by \( u \), one may check that
\[
e^2(u^2)'' \geq 2 \left( e^2u'^2 - \frac{(N-1)e^2}{r} u u' + (\cosh a_0)^{-1} u^2 \right).
\]
In particular, for \( r \in [\frac{R}{2}, R] \), by (i) one has \( e^2u'^2 - \frac{(N-1)e^2}{r} u u' \geq e^2u'^2 - \frac{2(N-1)e^2}{R} u u' \geq \frac{(N-1)^2e^2}{R^2} u^2 \). This concludes
\[
e^2(u^2)'' \geq 2 \left[ \frac{(N-1)^2e^2}{R^2} + (\cosh a_0)^{-1} \right] u^2 \geq (\cosh a_0)^{-1} u^2, \quad \text{as } 0 < \epsilon < \epsilon^*(R),
\]
where
\[
e^*(R) = R(N-1)^{-1}(2 \cosh a_0)^{-1/2}.
\]
Applying elliptic comparison arguments to (3.11) and using (3.1), we obtain
\[
0 \leq u(r) \leq \tilde{a}_0 \left( e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(r-\frac{R}{2}) + e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r) \right), \quad \forall r \in [\frac{R}{2}, R],
\]
as \( 0 < \epsilon < \epsilon^*(R) \). As a consequence,
(a1). When \( r \geq \frac{3}{4} R \), i.e., \( R - r \leq r - \frac{R}{2} \), we have
\[
0 \leq u(r) \leq 2a_0e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r).
\]
(a2). When \( r \in [0, \frac{3}{4} R] \), by \( u' \geq 0 \) we have
\[
0 \leq u(r) \leq \frac{3R}{4} \leq 2a_0e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}} \leq 2a_0e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r).
\]
One can conclude from (a1) and (a2) that
\[
0 \leq u(r) \leq 2a_0e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r), \quad \forall r \in [0, R],
\]
as \( 0 < \epsilon < \epsilon^*(R) \).

Now we deal with the estimate of \( u' \). Multiplying (3.9) by \( u' \) and using \( u' \geq 0 \), one may check that, for \( r \in [\frac{R}{2}, R] \),
\[
e^2(u'^2)'' \geq (\cosh a_0)^{-1} u'^2, \quad \text{as } 0 < \epsilon < \epsilon^*(R).
\]
Hence, following similar argument of (3.11)–(3.13), we have
\[
0 \leq u'(r) \leq 2u'(R)e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r), \quad \forall r \in [0, R],
\]
as \( 0 < \epsilon < \epsilon^*(R) \). Moreover, by the boundary condition (2.5) and (3.1) we have \( u'(R) \leq \frac{a_0}{\gamma} \). Along with (3.15) yields
\[
0 \leq u'(r) \leq \frac{2a_0}{\gamma} e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}(R-r), \quad \forall r \in [0, R],
\]
as \( 0 < \epsilon < \epsilon^*(R) \). Therefore, by (3.13) and (3.16) we get (3.6).

Therefore, we complete the proof of Lemma 3.2.

\[\Box\]

**Lemma 3.3.** Under the same hypotheses as in Proposition 2.1, as \( 0 < \epsilon \ll 1 \) we have
\[
\left| \frac{e^2}{2}u'^2(t) + (N-1)e^2 \int_0^t \frac{1}{r} u'^2(r) \, dr - C_\epsilon(u)(\cosh u(t) - 1) \right| \lesssim e^{-\frac{2a_0}{\gamma} e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}},
\]
\[
\left| \epsilon u'(t) - 2\sqrt{C_\epsilon(u)} \sinh \frac{u(t)}{2} \right| \lesssim \sqrt{\gamma}, \quad t \in (0, R],
\]
and
\[
\left| C_\epsilon(u) - 1 + 2 \int_{\frac{R}{2}}^R \left( \frac{N-1}{r} - \frac{N-2}{2RN} u'^2(r) \right) \, dr \right| \lesssim e^{-\frac{a_0}{\gamma} e^{-\frac{(\cosh a_0)^{-1/2}}{\epsilon^*(R)}}}.
\]
Proof. Putting $t = \frac{R}{2}$ into (2.7) and using (3.6) and (3.7), one may check that

$$|\mathcal{C}_e(u) + K_e| \leq \mathcal{C}_e(u) \left| \cosh u\left(\frac{R}{2}\right) - 1 \right| + e^2 u^2\left(\frac{R}{2}\right) \lesssim \frac{u(R) + e^2 u^2(R)}{2} \lesssim e^{-\frac{u}{4}(\cosh a_0)^{-1/2}}, \text{ as } 0 < \epsilon \ll 1. \quad (3.20)$$

Along with (2.7) immediately yields (3.17).

Moreover, by (3.6) and (3.17), one may check that

$$|e^2 u^2(t) - 2C_e(u) (\cosh u(t) - 1)| \lesssim e^2 \int_0^t \frac{1}{r^2} u^2(r) \, dr + e^{-\frac{u}{4}(\cosh a_0)^{-1/2}} \lesssim \epsilon. \quad (3.21)$$

On the other hand, since $u \geq 0$ and $u' \geq 0$, one finds

$$|e^2 u^2(t) - 2C_e(u) (\cosh u(t) - 1)| = |e^2 u^2(t) - 2C_e(u) \sinh^2 \frac{u(t)}{2}| \geq \left( u' \left( t - \sqrt{2C_e(u) \sinh \frac{u(t)}{2}} \right) \right)^2. \quad (3.22)$$

Along with (3.21), we get (3.18).

It remains to prove (3.19). Multiplying (2.7) by $t^{N-1}$ and integrating the result over $(0, R)$, we have

$$\frac{\epsilon^2}{2} \int_0^R u^2(t) t^{N-1} \, dt + (N-1) e^2 \int_0^R t^{N-1} \int_0^t \frac{1}{r^2} u^2(r) \, dr \, dt = \frac{R^N}{N} (1 + K_e). \quad (3.23)$$

By a simple calculation, we obtain

$$\frac{\epsilon^2}{2} \int_0^R u^2(t) t^{N-1} \, dt + (N-1) e^2 \int_0^R t^{N-1} \int_0^t \frac{1}{r^2} u^2(r) \, dr \, dt = \frac{\epsilon^2}{N} \int_0^R \left( (N-1) \frac{R^N}{t} - \frac{N-2}{2} t^{N-1} \right) u^2(t) \, dt \quad (3.24)$$

Combining (3.20)–(3.24) and using the gradient estimate in (3.6), it follows

$$\left| \mathcal{C}_e(u) - 1 + e^2 \int_0^R \left( \frac{N-1}{t} - \frac{N-2}{2R^N} t^{N-1} \right) u^2(t) \, dt \right| \leq |\mathcal{C}_e(u) + K_e| + \frac{N-2}{2R^N} e^2 \int_0^R t^{N-1} u^2(t) \, dt \lesssim e^{-\frac{u}{4}(\cosh a_0)^{-1/2}}, \text{ as } 0 < \epsilon \ll 1. \quad (3.25)$$

This proves (3.19) and completes the proof of Lemma 3.3.

\[ \square \]

**Lemma 3.4.** Under the same hypotheses as in Proposition 2.1, as $0 < \epsilon \ll 1$ we have

$$\left| \epsilon \int_0^R g(r) u^2(r) \, dr - 4g(R) \left( \frac{\cosh u(R)}{2} - 1 \right) \right| \leq \max_{[R-\sqrt{\tau}, R]} |g(r) - g(R)| + \epsilon \to 0, \quad (3.26)$$

where $g \in C([0, R])$ is a continuous function independent of $\epsilon$. Moreover, there holds

$$\left| \mathcal{C}_e(u) - 1 + 2 \frac{N}{R} \left( \frac{\cosh u(R)}{2} - 1 \right) \epsilon \right| \lesssim \epsilon^{3/2}. \quad (3.27)$$

**Proof.** We write the integral in (3.26) as

$$\epsilon \int_0^R g(r) u^2(r) \, dr = \epsilon \left( \int_0^{R-\sqrt{\tau}} + \int_{R-\sqrt{\tau}}^R \right) g(r) u^2(r) \, dr, \quad 0 < \epsilon \ll 1. \quad (3.28)$$

Note that $\sup_{[0, R]} |g|$ is finite and independent of $\epsilon$. Thus, using (3.6) and passing through simple calculations, one finds

$$\epsilon \left| \int_0^{R-\sqrt{\tau}} g(r) u^2(r) \, dr \right| \lesssim e^{-\frac{1}{4}(\cosh a_0)^{-1/2}}. \quad (3.29)$$

On the other hand, by (3.1), Lemma 3.2(i) and (3.18), we have

$$\left| \epsilon \int_{R-\sqrt{\tau}}^R g(r) u^2(r) \, dr - 2 \sqrt{\mathcal{C}_e(u)} \int_{R-\sqrt{\tau}}^R g(r) \sinh \frac{u(r)}{2} u'(r) \, dr \right|$$
\[
\sqrt{C_\epsilon(u)} \int_{R-\sqrt{\epsilon}}^{R} g(r) \sinh \frac{u'(r)}{2} \, dr = \int_{R-\sqrt{\epsilon}}^{R} \left( \sqrt{C_\epsilon(u)} (g(r) - g(R)) + \left( \sqrt{C_\epsilon(u)} - 1 \right) g(R) \right) \sinh \frac{u'(r)}{2} \, dr + 2g(R) \left( \cosh \frac{u(R)}{2} - \cosh \frac{u(R-\sqrt{\epsilon})}{2} \right).
\]

We obtain
\[
\left| \sqrt{C_\epsilon(u)} \int_{R-\sqrt{\epsilon}}^{R} g(r) \sinh \frac{u'(r)}{2} \, dr - 2g(R) \left( \cosh \frac{u(R)}{2} - 1 \right) \right| \leq 4 \left( \sqrt{C_\epsilon(u)} \max_{[R-\sqrt{\epsilon}, R]} |g(r) - g(R)| + \left( \sqrt{C_\epsilon(u)} - 1 \right) g(R) \right) \sinh \frac{u(R)}{2} + 2|g(R)| \left( \cosh \frac{u(R-\sqrt{\epsilon})}{2} - 1 \right) + \epsilon \frac{(\cosh \frac{u(R)}{2})^{1/2}}{\sqrt{\epsilon}}.
\]

Here we have used (3.1) and (3.6) to get \( \cosh \frac{u(R-\sqrt{\epsilon})}{2} - 1 = 2 \sinh^2 \frac{u(R-\sqrt{\epsilon})}{4} \lesssim u^2(R-\sqrt{\epsilon}) \lesssim \epsilon \frac{(\cosh \frac{u(R)}{2})^{1/2}}{\sqrt{\epsilon}} \) which asserts the last inequality of (3.32). Since \( g \) is continuous and independent of \( \epsilon \) and \( R - \sqrt{\epsilon} \to R \) as \( \epsilon \downarrow 0 \), by (3.6), (3.19), (3.28)–(3.30) and (3.32), we get
\[
\epsilon \int_{0}^{R} g(r) u''(r) \, dr = 4g(R) \left( \cosh \frac{u(R)}{2} - 1 \right) + o_\epsilon(1).
\]

In particular, when we set a function \( g \in C([0, R]) \) satisfying
\[
g(r) = 0 \text{ for } r \in \left[ 0, \frac{R}{4} \right]; \quad g(r) = \frac{N-1}{r} - \frac{N-2}{2RN} r^{N-1} \text{ for } r \in \left[ \frac{R}{2}, R \right],
\]
by (3.6), (3.19) and (3.33), we have
\[
C_\epsilon(u) = 1 - \epsilon^2 \int_{\frac{R}{2}}^{R} \left( \frac{N-1}{r} - \frac{N-2}{2RN} r^{N-1} \right) u^2(r) \, dr + o_\epsilon(1)
= 1 - 4g(R) \left( \cosh \frac{u(R)}{2} - 1 + o_\epsilon(1) \right) \epsilon
= 1 - \frac{2N}{R} \left( \cosh \frac{u(R)}{2} - 1 + o_\epsilon(1) \right) \epsilon.
\]

Hence, we have \( \left| \sqrt{C_\epsilon(u)} - 1 \right| \lesssim \epsilon \). Along with (3.32), we arrive at (3.26). Moreover, by (3.19) and (3.26), (3.34) can be improved by
\[
\left| C_\epsilon(u) - 1 + \frac{2N}{R} \left( \cosh \frac{u(R)}{2} - 1 \right) \epsilon \right|
\leq \left| C_\epsilon(u) - 1 + \epsilon^2 \int_{\frac{R}{2}}^{R} \left( \frac{N-1}{r} - \frac{N-2}{2RN} r^{N-1} \right) u^2(r) \, dr \right|
+ \epsilon \left| \frac{N}{r} \left( \frac{N-1}{r} - \frac{N-2}{2RN} r^{N-1} \right) u^2(r) \, dr - \frac{2N}{R} \left( \cosh \frac{u(R)}{2} - 1 \right) \epsilon \right|
\lesssim \epsilon \frac{\#_{\epsilon} (\cosh a_\epsilon)^{-1/2}}{\sqrt{\epsilon}} + \epsilon^{3/2} \lesssim \epsilon^{3/2}.
\]

Therefore, we prove (3.27) and completes the proof of Lemma 3.4. \( \square \)
Remark 3. Assume $g \in C^\alpha([0, R])$ is Hölder continuous with exponent $\alpha \in (0, 1)$. Owing to (3.27) and (3.32), (3.26) can be improved by

$$
eq 0 \cdot \epsilon^{1/2}.$$

Using Lemmas 3.3 and 3.4, we now give the proof of Theorem 3.1 as follows.

**Proof of Theorem 3.1.** Setting $t = R$ in (3.17) gives

$$
\left| \frac{\epsilon^2}{2} u''(R) + (N - 1) \epsilon^2 \int_{R}^{1} \frac{1}{r} u''(r) \ dr - C_e(u)(\cosh u(R) - 1) \right| \leq \epsilon. 
$$

Next, consider a continuous function $g$ with $g(r) = \frac{1}{r}$ for $r \in [R, \infty]$ and $g(r) = 0$ near $r = 0$ in (3.26). Using (3.6), one immediately finds $\left| \frac{\epsilon^2}{2} \int_{R}^{1} \frac{1}{r} u''(r) \ dr - \cosh u(R) - 1 \right| \leq \epsilon^{1/2}$. Note also the estimate of $C_e(u)$ in (3.27). As a consequence, after making appropriate manipulations we obtain

$$
\left| \frac{\epsilon^2}{2} u''(R) - 2 \sinh \frac{u(R)}{2} \left[ 1 - \frac{4}{R} \frac{\sinh^2 \frac{u(R)}{4}}{\sinh^2 \frac{u(R)}{4}} \left( N \cosh^2 \frac{u(R)}{2} - 1 \right) \right] \right| 
\leq \epsilon^2 u''(R) + (N - 1) \epsilon^2 \int_{R}^{1} \frac{1}{r} u''(r) \ dr - C_e(u)(\cosh u(R) - 1) 

+ (N - 1) \epsilon^2 \int_{R}^{1} \frac{1}{r} u''(r) \ dr - \frac{8}{R} \epsilon^2 \sinh^2 \frac{u(R)}{4} 

+ C_e(u)(\cosh u(R) - 1) - 2 \sinh \frac{u(R)}{2} \left( \frac{1}{R} - 4N \cosh \frac{u(R)}{4} \right) 
\lesssim \epsilon^{1/2}. 
$$

Since $u(R)$ is uniformly bounded to $\epsilon$ (cf. Lemma 3.2(i)), together with assumption (3.3) we conclude

$$
\left| u'(R) - 2 \sinh \frac{u(R)}{2} \left[ \frac{1}{\epsilon} - \frac{2}{R} \frac{\sinh^2 \frac{u(R)}{4}}{\sinh^2 \frac{u(R)}{4}} \left( N \cosh^2 \frac{u(R)}{2} - 1 \right) \right] \right| \lesssim \epsilon^{1/2}, 
$$

together with (3.2), we get (3.4) and (3.5) and complete the proof of Theorem 3.1.

3.2 Proof of Proposition 2.1

The proof of Proposition 2.1 is stated as follows.

By the Robin boundary condition (2.5) and Theorem 3.1, we have

$$
\left| u(R) + 2\gamma \sinh \frac{u(R)}{2} - \epsilon \frac{2\gamma}{R} \tanh \frac{u(R)}{4} \left( N \cosh^2 \frac{u(R)}{2} - 1 \right) - a_0 \right| \leq \epsilon^{3/2}, 
$$

as $0 < \epsilon \ll 1$. It is apparent that $\left| (u(R) - b) + 2\gamma \left( \sinh \frac{u(R)}{2} - \sinh \frac{b}{2} \right) \right| \lesssim \epsilon$, where $b$ is uniquely determined by (2.13). Since $s + 2\gamma \sinh \frac{s}{2}$ is increasing to $s$, by the mean value theorem and (3.1) it immediately follows

$$
|u(R) - b| \lesssim \epsilon. 
$$

Consequently, by (3.1), (3.27) and (3.36), we obtain

$$
\left| C_e(u) - 1 + \frac{2N}{R} \left( \cosh \frac{b}{2} - 1 \right) \right| \lesssim \epsilon^{3/2} + \left| \cosh \frac{b}{2} - \cosh \frac{u(R)}{2} \right| \epsilon 
\lesssim \epsilon^{3/2} + \left| b - u(R) \right| \epsilon \lesssim \epsilon^{3/2}. 
$$

This gives (2.9) and

$$
\left| C_e(u) - \left( C_e(u) \right)_2 \right| \lesssim \epsilon^{3/2}. 
$$

Now we shall prove (2.10) and (2.11). By (3.36), we set

$$
\begin{align*}
\{ u(R) = b + \tilde{b}, 
\tilde{b} \text{ is uniquely determined by (2.13), and } |\tilde{b}| \lesssim \epsilon \text{ as } \epsilon \downarrow 0. 
\end{align*}
$$
It suffices to calculate the leading order term and an optimal second order error of $\tilde{b}_e$ with respect to $\epsilon$ as $0 < \epsilon \ll 1$. Notice the relation $b + 2\gamma \sinh \frac{b}{2} = a_0$. Putting (3.38) into (3.35) and passing through simple calculations, one arrives at

$$\tilde{b}_e = \epsilon^2 \left( \frac{2\gamma}{R} \left( N \cosh^2 \frac{b}{2} - 1 \right) \tanh \frac{b}{2} \right) \lesssim \epsilon^{3/2}. \quad (3.39)$$

Here we have used (2.13) to get approximations

$$\left| \sinh \frac{b + \tilde{b}_e}{2} - \sinh \frac{b}{2} \right| \lesssim \epsilon^2, \quad (3.40)$$

$$\left| \tanh \frac{b + \tilde{b}_e}{4} - \tanh \frac{b}{4} \right| \lesssim \epsilon^2, \quad (3.41)$$

$$\left| \cosh^2 \frac{b + \tilde{b}_e}{2} - \cosh^2 \frac{b}{2} \right| \lesssim \epsilon^2, \quad (3.42)$$

and

$$\left| \tanh \frac{b + \tilde{b}_e}{4} \left( N \cosh^2 \frac{b}{2} - 1 \right) - \tanh \frac{b}{4} \left( N \cosh^2 \frac{b}{2} - 1 \right) \right| \lesssim \epsilon. \quad (4.3)$$

As a consequence,

$$\left| u(R) - b - \epsilon^2 \left( \frac{2\gamma}{R} \left( N \cosh^2 \frac{b}{2} - 1 \right) \tanh \frac{b}{2} \right) \right| \lesssim \epsilon^{3/2}. \quad (4.4)$$

Along with the Robin boundary condition (2.5) yields

$$\left| u'(R) - \frac{2}{\epsilon} \sinh \frac{b}{2} + \frac{2}{R} \cdot \left( N \cosh^2 \frac{b}{2} - 1 \right) \tanh \frac{b}{2} \right| \lesssim \epsilon^{1/2}. \quad (4.5)$$

Therefore, we get (2.10) and (2.11) and

$$\left| u(R) - u(R) \right| \lesssim \epsilon^{3/2}, \quad \left| u'(R) - u'(R) \right| \lesssim \epsilon^{1/2}. \quad (4.6)$$

(2.12) immediately follows from (3.37) and (3.44), and, therefore, the proof of Proposition 2.1 is completed.

## 4 Curvature effects on the thin layer: Pointwise asymptotics

### 4.1 Proof of Theorem 2.2

Recall $0 < u(r) \leq b$ and $u'(r) > 0$ in $(0, R]$ (cf. Lemma 3.2(i)). To prove Theorem 2.2, we need the following estimate.

**Claim 1.** For $r_e \in B_0^B$, as $0 < \epsilon \ll 1$ there holds

$$\left| \log \left( \frac{\tanh \frac{u(R)}{4}}{\tanh \frac{u(r_e)}{4}} \right) - \sqrt{C_\epsilon} \frac{R - r_e}{\epsilon} \right| \leq \frac{\sqrt{\epsilon}}{\sinh \frac{u(r_e)}{2}} \cdot \sup_{0 < \epsilon \ll 1} \frac{R - r_e}{\epsilon}. \quad (4.1)$$

Moreover,

$$\frac{R - r_e}{\epsilon} < \infty \quad \text{and} \quad \lim_{\epsilon \downarrow 0} u(r_e) = 0, \quad \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{\epsilon}} \sinh \frac{u(r_e)}{2} = 0. \quad (4.2)$$

**Proof of Claim 1.** By (3.1) and (3.18) we have

$$\frac{\epsilon u(t)}{\sinh \frac{u(t)}{2}} - 2 \sqrt{C_\epsilon} \frac{u(t)}{2} \lesssim \frac{\sqrt{\epsilon}}{\sinh \frac{u(t)}{2}}. \quad (4.3)$$

for $t \in [r_e, R]$. Integrating (4.3) over $[r_e, R]$ and using

$$\int_{r_e}^R \frac{u'(r)}{\sinh \frac{u(t)}{2}} \, dt = 2 \log \left| \frac{\tanh \frac{u(R)}{4}}{\tanh \frac{u(r_e)}{4}} \right|, \quad (4.4)$$

one immediately obtains (4.1).
Next, we assume that \( r_\epsilon \in \mathcal{B}_0^\epsilon \) satisfies \( \lim_{\epsilon \downarrow 0} u(r_\epsilon) = 0 \). Then by (2.10) and \( b > 0 \), we have \( \lim_{\epsilon \downarrow 0} \log \frac{\tanh u(r_\epsilon)}{\tanh \frac{u(R)}{4}} = \infty \), together with (2.9) and (4.1), we conclude

\[
0 \leq \limsup_{\epsilon \downarrow 0} \frac{1}{\sqrt{\epsilon}} \sinh \frac{u(r_\epsilon)}{2} \leq \limsup_{\epsilon \downarrow 0} \left( \log \left| \frac{\tanh u(R)}{\tanh \frac{u(r_\epsilon)}{4}} \right| - \sqrt{c_\epsilon(u)} \frac{R - r_\epsilon}{\epsilon} \right)^{-1} = 0.
\]

This proves (4.2) and completes the proof of Claim 1.

Before proving Theorem 2.2, we notice that by (2.9) and (3.18), there must hold

\[
\liminf_{\epsilon \downarrow 0} u(r_\epsilon) > 0 \iff \liminf_{\epsilon \downarrow 0} \epsilon u'(r_\epsilon) > 0. \tag{4.5}
\]

We are now turning to the proof of Theorem 2.2.

**Proof of Theorem 2.2.** By Lemma 3.2(ii) and (4.5), it suffices to prove \( \liminf_{\epsilon \downarrow 0} u(r_\epsilon) > 0 \) for \( r_\epsilon \in \mathcal{B}_0^\epsilon \) satisfying \( \limsup_{\epsilon \downarrow 0} \frac{R - r_\epsilon}{\epsilon} < \infty \). Suppose by contradiction that there exists \( \tilde{r}_\epsilon \in \mathcal{B}_0^\epsilon \) with \( \frac{R - \tilde{r}_\epsilon}{\epsilon} = \tilde{p} + o_\epsilon(1) \) such that \( \lim_{\epsilon \downarrow 0} u(\tilde{r}_\epsilon) = 0 \), where \( \tilde{p} \geq 0 \) is independent of \( \epsilon \). Then by (4.2) we have

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{\epsilon}} \sinh \frac{\tilde{u}(\tilde{r}_\epsilon)}{2} = 0. \tag{4.6}
\]

On the other hand, we set

\[
\tilde{c}_\epsilon = 2 \log(\sqrt{\epsilon} + \sqrt{\epsilon + 1}). \tag{4.7}
\]

By (2.10), (4.6) and (4.7) we have \( u(\tilde{r}_\epsilon) < \tilde{c}_\epsilon < u(R) \) as \( 0 < \epsilon \ll 1 \) due to

\[
0 < \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{\epsilon}} \sinh \frac{u(\tilde{r}_\epsilon)}{2} < \frac{1}{\sqrt{\epsilon}} \sinh \frac{\tilde{c}_\epsilon}{2} = 1 < \frac{1}{\sqrt{\epsilon}} \sinh \frac{u(R)}{2} \text{ as } \epsilon \downarrow 0 \to \infty,
\]

and the fact that \( \sinh s \) is a strictly increasing function. By the intermediate value theorem there exists \( \tilde{r}_\epsilon \in [\tilde{r}_\epsilon, R] \subset \mathcal{B}_0^\epsilon \) such that \( u(\tilde{r}_\epsilon) = \tilde{c}_\epsilon \). In particular, \( \tilde{r}_\epsilon \in \mathcal{B}_0^\epsilon \) satisfies

\[
\limsup_{\epsilon \downarrow 0} \frac{R - \tilde{r}_\epsilon}{\epsilon} \leq \tilde{p}, \text{ lim } u(\tilde{r}_\epsilon) = 0, \text{ and } \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{\epsilon}} \sinh \frac{u(\tilde{r}_\epsilon)}{2} \neq 0,
\]

contradicting to (4.2). As a consequence,

\[
\liminf_{\epsilon \downarrow 0} u(r_\epsilon) > 0 \text{ for each } r_\epsilon \in \mathcal{B}_0^\epsilon. \tag{4.8}
\]

Along with (4.5), therefore, we get \( \liminf_{\epsilon \downarrow 0} \epsilon u'(r_\epsilon) > 0 \) and complete the proof of Theorem 2.2.

\[\square\]

### 4.2 Proof of Theorem 2.3

To prove Theorem 2.3, we need to collect some preliminary estimates. Firstly, based on Theorem 3.1, we shall generalize the concept of the Dirichlet-to-Neumann map at \( r_\epsilon \in \mathcal{B}_0^\epsilon \), and establish more refined asymptotic approximations of \( u'(r_\epsilon) \) and \( (R - r_\epsilon)/\epsilon \) as \( 0 < \epsilon \ll 1 \).

**Lemma 4.1.** Let \( r_\epsilon \in \mathcal{B}_0^\epsilon \). Under the same hypotheses as in Theorem 2.2, as \( 0 < \epsilon \ll 1 \) we have

\[
\left| u'(r_\epsilon) - 2 \sinh \frac{u(r_\epsilon)}{2} \left[ \frac{1}{\epsilon} \frac{1}{R} \left( 2N \sinh^2 \frac{b}{4} + \frac{N - 1}{2} \text{sech}^2 \frac{u(r_\epsilon)}{4} \right) \right] \right| \lesssim \sqrt{\epsilon} \tag{4.9}
\]

and

\[
\left| \frac{R - r_\epsilon}{\epsilon} - \left( 1 + \epsilon \cdot \frac{2N}{R} \sinh^2 \frac{b}{4} \right) \left( 1 + \frac{N - 1}{2R} \log \left| \frac{\tanh(u(R)/4)}{\tanh(u(r_\epsilon)/4)} + \frac{N - 1}{4R} \left( \frac{\tanh^2 u(r_\epsilon)/4}{4} \right) - \frac{\tanh^2 u(R)/4}{4} \right) \right| \lesssim \epsilon^{3/2}. \tag{4.10}
\]

**Proof.** We first deal with (4.9) via the concept of Proposition 2.1 and Lemma 3.4. Note that \( r_\epsilon \in \mathcal{B}_0^\epsilon \) implies \( \frac{1}{r_\epsilon} = \frac{1}{R} + O(1) \cdot \epsilon \) and \( R - \sqrt{\epsilon} < r_\epsilon \leq R \) as \( 0 < \epsilon \ll 1 \). Hence, for function \( g \in C([\frac{R}{\sqrt{\epsilon}}, R]) \) we may follow the same argument of (3.28)--(3.32) to get

\[
\epsilon \int_{\frac{R}{\sqrt{\epsilon}}}^{r_\epsilon} g(r) u^2(r) \, dr = \epsilon \left( \int_{\frac{R}{\sqrt{\epsilon}}}^{R - \sqrt{\epsilon}} + \int_{R - \sqrt{\epsilon}}^{r_\epsilon} \right) g(r) u^2(r) \, dr = 8g(R) \sinh^2 \frac{u(r_\epsilon)}{4} + \pi_{c_\epsilon g}(r_\epsilon) \tag{4.11}
\]
with
\[ |\pi_{c,g}(r_\epsilon)| \lesssim \max_{r \in [R - \sqrt{\epsilon}r_\epsilon]} |g(r) - g(R)| + \epsilon \xrightarrow{\epsilon \downarrow 0} 0. \]

(4.12)

Since \( \liminf u(r_\epsilon) > 0 \) (by Theorem 2.2), using (3.17), (4.11), Proposition 2.1 and Lemma 3.2(i) we can derive a relationship between \( u(r_\epsilon) \) and \( u'(r_\epsilon) \) as follows:

\[
eu'(r_\epsilon) = \left[ 2\mathcal{C}_\epsilon(u) (\cosh u(r_\epsilon) - 1) - 2(N - 1)\epsilon^2 \int_0^{r_\epsilon} \frac{1}{r} u'^2(r) \, dr + \mathcal{O}(1) \cdot e^{-\frac{1}{4} (\cosh a_0)^{-1/2}} \right]^{1/2}
\]

\[
= 4 \left( 1 - \frac{4N}{R} \sinh^2 \frac{b}{4} \right) \sinh^2 \frac{u(r_\epsilon)}{2} - \epsilon \frac{16(N - 1)}{R} \sinh^2 \frac{u(r_\epsilon)}{4} + \mathcal{O}(1) \cdot \epsilon^{3/2}
\]

\[= 2 \sinh \frac{u(r_\epsilon)}{2} \left[ 1 - \epsilon \left( \frac{2N}{R} \sinh^2 \frac{b}{4} + \frac{N - 1}{2R} \frac{\text{sech}^2 u(r_\epsilon)}{4} + \mathcal{O}(1) \cdot \sqrt{\epsilon} \right) \right]. \quad (4.13)
\]

We stress that \( \mathcal{O}(1) \) is uniformly bounded to \( r_\epsilon \) because \( |\pi_{c,g}(r_\epsilon)| \lesssim \sqrt{\epsilon} \) for \( g(r) = \frac{1}{r} \). Therefore, (4.9) follows from (3.1) and (4.13).

For the convenience, in (4.13) we replace \( r_\epsilon \) with \( r \) and obtain

\[
\frac{u'(r)}{2 \sinh \frac{u(r)}{2}} = \frac{1}{\epsilon} - \frac{2N}{R} \sinh^2 \frac{b}{4} - \frac{N - 1}{2R} \frac{\text{sech}^2 u(r)}{4} + \mathcal{O}(1) \cdot \sqrt{\epsilon}. \quad (4.14)
\]

Integrating (4.14) over \([r_\epsilon, R]\) immediately gives

\[
\frac{1}{2} \int_{r_\epsilon}^{R} \frac{u'(r)}{\sinh \frac{u(r)}{2}} \, dr = \left( \frac{1}{\epsilon} - \frac{2N}{R} \sinh^2 \frac{b}{4} + \mathcal{O}(1) \cdot \sqrt{\epsilon} \right) (R - r_\epsilon) - \frac{N - 1}{2R} \int_{r_\epsilon}^{R} \frac{\text{sech}^2 u(r)}{4} \, dr. \quad (4.15)
\]

Note also that \( u \) is uniformly bounded to \( \epsilon \) and
\[
\frac{u'(r)}{2 \sinh \frac{u(r)}{2}} = 1 + \mathcal{O}(1) \cdot \epsilon \quad \text{(by (4.14))}. \]

Hence, following the argument of (2.20), one may check that

\[
\int_{r_\epsilon}^{R} \frac{\text{sech}^2 u(r)}{4} \, dr = \int_{r_\epsilon}^{R} \frac{\text{sech}^2 u(r)}{4} \left( \frac{\nu'(r)}{2 \sinh \frac{\nu(r)}{2}} + \mathcal{O}(1) \cdot \epsilon \right) \, dr
\]

\[
= \log \left[ \frac{\tanh \frac{u(R)}{4}}{\tanh \frac{u(r_\epsilon)}{4}} \right] + \frac{1}{2} \left( \tanh^2 \frac{u(r_\epsilon)}{4} - \tanh^2 \frac{u(R)}{4} + \mathcal{O}(1) \cdot (R - r_\epsilon) \epsilon. \quad (4.16)
\]

Combining (4.4) with (4.15)–(4.16) and passing a calculation directly, one finds

\[
\left( 1 - \epsilon \left( \frac{2N}{R} \sinh^2 \frac{b}{4} + \mathcal{O}(1) \cdot \sqrt{\epsilon} \right) \right) \frac{R - r_\epsilon}{\epsilon} = \left( 1 + \frac{N - 1}{2R} \right) \log \left[ \frac{\tanh \frac{u(R)}{4}}{\tanh \frac{u(r_\epsilon)}{4}} \right] + \frac{N - 1}{4R} \left( \tanh^2 \frac{u(r_\epsilon)}{4} - \tanh^2 \frac{u(R)}{4} \right) + \mathcal{O}(1) \cdot \epsilon^{3/2}. \]

After making appropriate manipulations, we obtain

\[
\frac{R - r_\epsilon}{\epsilon} = \left( 1 + \frac{N - 1}{2R} \right) \log \left[ \frac{\tanh \frac{u(R)}{4}}{\tanh \frac{u(r_\epsilon)}{4}} \right] + \frac{N - 1}{4R} \left( \tanh^2 \frac{u(r_\epsilon)}{4} - \tanh^2 \frac{u(R)}{4} \right) + \mathcal{O}(1) \cdot \epsilon^{3/2}. \quad (4.17)
\]

Here we have used \( \liminf u(r_\epsilon) > 0 \) again to get the refined asymptotics of \( (R - r_\epsilon)/\epsilon \). Therefore, we prove (4.10) and completes the proof of Lemma 4.1. \( \square \)

Note that by Proposition 2.1 and Lemma 3.2, there exists \( r_{\epsilon}^{k, \tilde{k}} \xrightarrow{\epsilon \downarrow 0} R \) with \( \limsup \frac{R - r_{\epsilon}^{k, \tilde{k}}}{\epsilon} < \infty \) such that

\[
u(r_{\epsilon}^{k, \tilde{k}}) = k + \left( \frac{\tilde{k}}{R} + o_\epsilon(1) \right) \epsilon \leq u(R) \quad \text{as} \quad 0 < \epsilon \ll 1,
\]

(4.18)

where \( k \) and \( \tilde{k} \) are real numbers independent of \( \epsilon \) and satisfy one of the following conditions:

\[
\begin{cases}
(a) \ 0 < k < b \quad \text{and} \quad \tilde{k} \in \mathbb{R};
(b) k = b \quad \text{and} \quad \tilde{k} \leq \frac{2N(N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{2}}{\gamma \cosh^2 \frac{b}{2} + 1}.
\end{cases}
\]

(4.19)
Here the second order term having the order $\epsilon$ is a natural consideration due to the rigorous derivation of $u(R)$ and $u'(R)$ in Proposition 2.1. Since $r^k_{e,\tilde{k}}$ may depend on $k$ and $\tilde{k}$, we shall establish asymptotics of $r^k_{e,\tilde{k}}$ such that
\[
\left( u(r^k_{e,\tilde{k}}) \right)_2 = k + \frac{\tilde{k}}{R} \epsilon, \quad \text{as } 0 < \epsilon \ll 1.
\] (4.20)

More precisely, for $r^k_{e,\tilde{k}} \in B_{\phi}^k$ admitting (4.20), asymptotics of $\left( (R - r^k_{e,\tilde{k}})/\epsilon \right)_2$ and $\left( u'(r^k_{e,\tilde{k}}) \right)_2$ are uniquely determined by $k$ and $\tilde{k}$, which can be precisely depicted as follows.

**Lemma 4.2.** Under the same hypotheses as in Theorem 2.3, if $r^k_{e,\tilde{k}} \in B_{\phi}^k$ satisfies (4.20) as $0 < \epsilon \ll 1$, then
\[
\left( u'(r^k_{e,\tilde{k}}) \right)_2 = \frac{2}{\epsilon} \sinh k \left( 1 - \frac{1}{R} \right) \left( 4N \sinh k \left( \frac{b}{2} \right) - 2(1 - 1/\gamma) \tanh \left( \frac{k}{4} \right) \left( \frac{e}{4} - \tilde{k} \cosh \frac{k}{2} \right) \right).
\] (4.21)

Moreover, the exact leading order term $O(1)$ and the second order term $O(1) \cdot \epsilon$ of $\left( (R - r^k_{e,\tilde{k}})/\epsilon \right)_2$ is described by
\[
\left( \frac{R - r^k_{e,\tilde{k}}}{\epsilon} \right)_2 = A^k + \frac{\epsilon}{2R} \left( 4N A^k \sinh \frac{b}{4} + B^k_{\tilde{k}} \right),
\] (4.22)

where
\[
A^k = \left( 1 + \frac{N - 1}{2R} \right) \log \frac{\tanh \frac{b}{4}}{\cosh \frac{k}{2}} + \frac{N - 1}{4R} \left( \tanh^2 \frac{k}{4} - \tanh^2 \frac{b}{4} \right),
\] (4.23)

and
\[
B^k_{\tilde{k}} = \frac{\gamma(N \cosh^2 \frac{b}{2} - 1) \coth^2 \frac{b}{4}}{\gamma \cosh \frac{k}{2} + 1} \left( 1 + \frac{N - 1}{2R} \coth^2 \frac{b}{4} - \frac{\tilde{k} \cosh \frac{k}{2}}{4} \right) \left( 1 + \frac{N - 1}{2R} \coth^2 \frac{k}{4} \right).\] (4.24)

**Proof.** We calculate asymptotics of $u'(r^k_{e,\tilde{k}})$ and $(R - r^k_{e,\tilde{k}})/\epsilon$ as follows, respectively.

**(b1).** Plug (4.18) into (4.9).
\[
u'(r^k_{e,\tilde{k}}) = \frac{2}{\epsilon} \left( \sinh k + \left( \frac{k}{R} + o(1) \right) \right) \left( 1 - 2 \frac{N}{R} \sinh^2 \frac{b}{4} - \frac{N - 1}{2R} \coth^2 \frac{k}{4} + \left( \frac{k}{R} + o(1) \right) \right)
\]
\[
= \left( 2 \sinh k \left( 1 + \epsilon \left( \cos k + o(1) \right) \right) \right) \times \left[ \left( 1 - 2 \frac{N}{R} \sinh^2 \frac{b}{4} \right) - \frac{N - 1}{2R} \left( 1 - \tilde{k} \epsilon \tanh \frac{k}{4} \right) \coth^2 \frac{k}{4} + o(1) \right]
\]
\[
= \frac{2}{\epsilon} \left( 2 \sinh k \left( 1 - \frac{1}{R} \right) \left( 4N \sinh \frac{k}{2} \sinh \frac{b}{4} + 2(1 - 1/\gamma) \tanh \left( \frac{k}{4} \right) \left( \frac{e}{4} - \tilde{k} \cosh \frac{k}{2} \right) + o(1) \right)
\]

Here we have used similar approximations as (3.40) and (3.42) to deal with the asymptotics of $u'(r^k_{e,\tilde{k}})$.

**(b2).** Putting (4.18) into (4.10) and using asymptotics of $u(R)$ (see (2.10)), one can check that
\[
\frac{R - r^k_{e}}{\epsilon}
\]
\[
= \left( 1 + \epsilon \cdot \frac{2N}{R} \sinh^2 \frac{b}{4} \right) \left( 1 + \frac{N - 1}{2R} \right) \log \frac{\tanh \left( \frac{b}{4} \right) + \left( \frac{e}{2} + \frac{\gamma(N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{2R(\gamma \cosh \frac{b}{2} + 1)} \right) + o(1) \cdot \epsilon}{\tanh \left( \frac{b}{4} \right) + \frac{\gamma(N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{2R(\gamma \cosh \frac{b}{2} + 1)} + o(1) \cdot \epsilon}
\]
\[
+ \frac{N - 1}{4R} \left( \tanh^2 \left( \frac{k}{4} \right) + o(1) \cdot \epsilon \right) - \tanh^2 \left( \frac{b}{4} \right) - \frac{2\gamma(N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{2R(\gamma \cosh \frac{b}{2} + 1)} + o(1) \cdot \epsilon \right) \right] + O(1) \cdot \epsilon^{3/2}
\]
\[
= \left( 1 + \epsilon \cdot \frac{2N}{R} \sinh^2 \frac{b}{4} \right) \left( 1 + \frac{N - 1}{2R} \right) \log \frac{\tanh \frac{b}{4}}{\tanh \frac{k}{4} + \frac{N - 1}{4R} \left( \tanh^2 \frac{k}{4} - \tanh^2 \frac{b}{4} \right)}
\]
\[
= A^k \quad \text{(see (4.23))}
\]
\[
+ \frac{\epsilon}{2R} \left( 1 + \frac{N - 1}{2R} \right) \left( \frac{2\gamma(N \cosh^2 \frac{b}{2} - 1) \tanh \frac{b}{4}}{2R(\gamma \cosh \frac{b}{2} + 1) \sinh \frac{b}{2}} \right) \frac{\tilde{k}}{\sinh \frac{k}{2}}
\]
\[
= I_1 + I_2
\] (4.25)
Since (cf. (4.24))
\[ \gamma(N \cosh^2 \frac{b}{2} - 1) \tanh^2 \frac{b}{4} \left( 1 + \frac{N - 1}{2R} \right) \]

\[ - \frac{k}{\sinh \frac{k}{4} - 1 \cosh \frac{k}{2} + 1} \]

we have applied some elementary approximations
\[ \lim_{k \to 0} \frac{1}{\sinh \frac{k}{2}} \left( 1 + \frac{N - 1}{2R} \tanh^2 \frac{k}{4} \right) = 1. \]

Therefore, we prove (4.21) and (4.22) and complete the proof of Lemma 4.2.

Proof of Theorem 2.3. Let
\[ \overline{r} := R \left( 1 - \frac{p}{R} - \frac{q}{R^2} \right) \in B^\prime. \]

Then for any \( r_{p,q} \in B^\prime \), we have
\[ \lim_{\epsilon \to 0} \epsilon^{-2} (r_{p,q} - \overline{r}) = 0. \]

Since \( u \) and \( \epsilon u' \) are uniformly bounded to \( \epsilon \) (by (3.6)), together with (4.27) one immediately finds
\[ \left( u(r_{p,q}) \right)_2 = \left( u(\overline{r}) \right)_2 \]

and
\[ \left( u'(r_{p,q}) \right)_2 = \left( u'((\overline{r}) \right)_2 \]

as \( 0 < \epsilon \ll 1 \). Hence, to prove Theorem 2.3, it suffices to establish asymptotics of
\[ \left( u(r_{p,q}) \right)_2 = \left( u(\overline{r}) \right)_2 \]

and
\[ \left( u'(r_{p,q}) \right)_2 = \left( u'((\overline{r}) \right)_2 \]

Regarding \( A^k \) (defined in (4.23)) as a function of \( k \) in \( (0, b) \), one may check that \( \lim_{k \to 0} A^k = \infty \), \( A^b = 0 \) and \( A^k \) is strictly decreasing to \( k \) in \( (0, b) \). Note also that \( B^{k,\overline{k}} \) (defined in (4.24)) is a linear function of \( \overline{k} \). Hence by (4.22)–(4.24), we obtain that for any \( p \geq 0 \) and \( q \in \mathbb{R} \) there uniquely exist \( k = k(p) \) and \( \overline{k} = \overline{k}(p, q) \) satisfying (4.19) such that (2.22) and (4.20) hold, i.e.,
\[ A^k(p) = p \quad \text{and} \quad \frac{1}{2} \left( 4Np \sinh^2 \frac{b}{4} + k^2(p)(\overline{k}(p,q)) \right) = q \]

and
\[ \left( u(r_{p,q}) \right)_2 = k(p) + \frac{\epsilon}{R} \overline{k}(p, q) \quad \text{as} \quad 0 < \epsilon \ll 1. \]

From the second equation of (4.28), one gets
\[ \overline{k}(p, q) = \mathcal{H}_{p,q} \sinh \frac{k(p)}{2}, \]

where (4.24)
\[ B^{k,\overline{k}} := I_3 + J_3 = \frac{\gamma(N \cosh^2 \frac{b}{2} - 1) \tanh^2 \frac{b}{4}}{\gamma \cosh \frac{b}{2} + 1} \left( 1 + \frac{N - 1}{2R} \tanh^2 \frac{k}{4} \right) - \frac{k}{\sinh \frac{k}{2}} \left( 1 + \frac{N - 1}{2R} \tanh^2 \frac{k}{4} \right). \]
where $H_{p,q}^{\gamma,b}$ is defined in (2.26). Moreover, by (4.26)–(4.28) and (4.29)–(4.30) we have

$$
\left( R - r_e^{(p):k(p,q)} \right) = \epsilon \left( R - r^{(p):k(p,q)} \right)
$$

and

$$
\left( u(r_e^{(p):k(p,q)}) \right) = k(p) + \frac{\epsilon}{R} H_{p,q}^{\gamma,b} \sinh \frac{k(p)}{2}.
$$

As a consequence, $r^{(p):k(p,q)} - r_e^{(p):k(p,q)} = \epsilon^2 \cdot o_e(1)$ and

$$
u(R^{(p)} - r_e^{(p):k(p,q)}) = u(R^{(p)} - r_e^{(p):k(p,q)}) = k(p) + \frac{\epsilon}{R} H_{p,q}^{\gamma,b} \sinh \frac{k(p)}{2} + o_e(1),
$$

(4.31)

where $\theta^e$ lies between $r^{(p):k(p,q)}$ and $r_e^{(p):k(p,q)}$. Here we have used (3.6) to assert

$$
u'(\theta^e) \left( R^{(p)} - r_e^{(p):k(p,q)} \right) = \epsilon \cdot o_e(1).
$$

Hence, (2.23) follows from (4.31) and

$$
\left( u(R^{(p)} - r_e^{(p):k(p,q)}) \right) = \left( u(r_e^{(p):k(p,q)}) \right).
$$

Comparing (4.20) to the first two order terms of (2.23), we shall put $k = k(p)$ and $\tilde{k} = \frac{H_{p,q}^{\gamma,b}}{R} \sinh \frac{k(p)}{2}$ into (4.21), and therefore obtain

$$
u'(r^{(p)} - r_e^{(p):k(p,q)}) = \frac{2}{\epsilon} \sinh \frac{k(p)}{2} - \frac{1}{R} \left( 4N \sinh^2 \frac{b}{4} \sinh \frac{k(p)}{2} \right.

+ \left. 2(N-1) \tan \frac{k(p)}{4} - \frac{1}{2} \frac{H_{p,q}^{\gamma,b}}{R} \sinh k(p) \right) + o_e(1).
$$

Therefore, we get (2.24).

Finally, we need to check their uniform convergence when $p$ is located in a bounded interval $[0, p^*]$. By (2.22), for any $q \in \mathbb{R}$ one finds $\lim_{\epsilon \downarrow 0} \sup_{p \in [0, p^*]} R - r_e^{(p):k(p,q)} \leq p^*$. Along with Theorem 2.2, we have

$$
\lim_{\epsilon \downarrow 0} \inf_{p \in [0, p^*]} \sinh u(r^{(p):k(p,q)}) \geq \lim_{\epsilon \downarrow 0} \inf_{p \in [0, p^*]} \sinh u(r^{(p):k(p,q)}) > 0.
$$

As a consequence, $\sup_{p \in [0, p^*]} \frac{1}{\epsilon} \sinh u(r^{(p):k(p,q)})$ is uniformly bounded to $\epsilon$ as $0 < \epsilon \ll 1$, and all arguments involving the pointwise estimates of $u(r^{(p):k(p,q)})$ and $u'(r^{(p):k(p,q)})$ can be improved so that the convergence (2.27) is uniformly in $[0, p^*] \times \mathbb{R}$. This completes the proof of Theorem 2.3.

4.3 Comparison of non-local and standard elliptic sinh–Gordon equations

For (2.3)–(2.5), recall the non-local coefficient $C_e(u) \sim 1$ as $0 < \epsilon \ll 1$. Hence, as $\epsilon \downarrow 0$, $u$ formally approaches the solution $v$ of the standard elliptic sinh–Gordon equation

$$
\epsilon^2 \left( v''(r) + \frac{N-1}{r} v'(r) \right) = \sinh v, \quad r \in (0, R),
$$

(4.32)

$$
v'(0) = 0, \quad v(R) + \gamma \epsilon v'(R) = a_0,
$$

(4.33)

where the condition of $\gamma$ and $a_0$ are same as that in (2.5). For the sake of completeness, we shall compare the pointwise asymptotics of $u$ and $v$ in the whole domain $[0, R]$.

Following the same argument of Lemma 3.2, it is easy to obtain

$$
|u - v|(r) + \epsilon |(u - v)'(r)| \lesssim e^{-\frac{1}{\epsilon} \left( \frac{\cosh a_0}{2} - 1 \right)}(R - r), \quad r \in [0, R].
$$

However, as $0 < \epsilon \ll 1$, $u$ and $v$ have different asymptotic behavior near the boundary since $C_e(u)$ is not identically equal to 1. Alternatively, note that making the following replacements in (2.3)–(2.5):

$$
\epsilon \mapsto \epsilon \sqrt{C_e(u)} \quad \uparrow \quad (2.9) \quad \mapsto \epsilon \left( 1 - \frac{N}{R} \left( \cosh \frac{b}{2} - 1 \right) \epsilon + o_e(1) \epsilon \right),
$$

$$
\gamma \mapsto \gamma \sqrt{C_e(u)} \quad \downarrow \quad \gamma \left( 1 + \frac{N}{R} \left( \cosh \frac{b}{2} - 1 \right) \epsilon + o_e(1) \epsilon \right),
$$

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one can transform (2.3) and (2.5) into (4.32) and (4.33), respectively. Accordingly, it is expected that asymptotic expansions of $u(R)$ and $v(R)$ with respect to $\epsilon$ have different second order terms. To see the difference, we can use the same arguments in Sections 3 and 4.1–4.2 to get the asymptotics of $v$ and $v'$ in $[0, R] - \mathcal{B}_0^+$ and $\mathcal{B}_0^-$, respectively. The following lemmas can be proved following the same arguments in Lemmas 4.1 and 4.2 so we omit the proof. The reader can compare these results (of $v$) with Lemmas 4.1 and 4.2 (of $u$) directly.

**Lemma 4.3.** For $\epsilon > 0$, let $v$ be the unique classical solution of (4.32)–(4.33), where $a_0$ and $\gamma$ are positive constants independent of $\epsilon$. Then $v$ and $v'$ are strictly positive in $(0, R]$, and $v''(r) \geq 0$ for $r \in (0, R]$. In addition, for $r_\epsilon \in \mathcal{B}_0^+$, as $0 < \epsilon \ll 1$ there hold

$$
\left| v'(r_\epsilon) - 2 \sinh \left( \frac{v(r_\epsilon)}{2} \right) \right| \lesssim \sqrt{\epsilon}
$$

and

$$
\left| \frac{R - r_\epsilon}{\epsilon} \right| \left[ \left( 1 + \frac{N - 1}{2R} \right) \log \left( \frac{\tanh \frac{v(r_\epsilon)}{2}}{\tanh \frac{v(R)}{2}} \right) + \frac{N - 1}{4R} \left( \frac{\tanh^2 \frac{v(r_\epsilon)}{4} - \tanh^2 \frac{v(R)}{4}}{4} \right) \right] \lesssim \epsilon^{3/2}.
$$

**Lemma 4.4.** Under the same hypotheses as in Lemma 4.3, if $r_\epsilon^{k;\bar{k}} \in \mathcal{B}_0^+$ satisfies

$$
\left( v(r_\epsilon^{k;\bar{k}}) \right)_2 = \left( k + \frac{k}{R} \right) \epsilon \text{ as } 0 < \epsilon \ll 1,
$$

then

$$
\left( v'(r_\epsilon^{k;\bar{k}}) \right)_2 = \frac{2}{\epsilon} \sinh \left( \frac{k}{2} \right) - \frac{1}{R} \left( 2(N - 1) \tanh \frac{k}{4} - \bar{k} \cosh \frac{k}{2} \right).
$$

Moreover, the exact leading order term $\mathcal{O}(1)$ and the second order term $\mathcal{O}(1) \cdot \epsilon$ of $(R - r_\epsilon^{k;\bar{k}})/\epsilon$ is described by

$$
\left( \frac{R - r_\epsilon^{k;\bar{k}}}{\epsilon} \right)_2 = A^k + \frac{B_{k;\bar{k}}}{2R} \epsilon,
$$

where $A^k$ is defined in (4.23) and

$$
B_{k;\bar{k}} = \frac{\gamma(N - 1) \sech^2 \frac{b}{2} - \gamma \cosh \frac{b}{2} + 1}{\gamma \cosh \frac{b}{2} + 1} \left( 1 + \frac{N - 1}{2R} \sech^2 \frac{b}{4} \right) = \frac{\bar{k}}{2} \sinh \left( \frac{k}{2} \right) \left( 1 + \frac{N - 1}{2R} \sech^2 \frac{k}{4} \right).
$$

Using Lemmas 4.3–4.4 and following the similar arguments in the proof of Theorem 2.3, we establish refined structure of the boundary layer of $v$ in $[0, R]$ as follows.

**Theorem 4.5.** Under the same hypotheses as in Lemma 4.3, as $0 < \epsilon \ll 1$, we have

$$
\max \{ |v(r)|, \gamma |v'(r)| \} \leq 2a_0 e^{-\frac{b}{2R} \sech^2 \frac{b}{4}} (\cosh \gamma \cosh \frac{b}{2})^{-1/2} (R - r), \quad \forall r \in [0, R],
$$

and

$$
\left( v(R) \right)_2 = b + \frac{N - 1}{R} \epsilon - \frac{2\gamma}{\gamma \cosh \frac{b}{2} + 1} \tanh \frac{b}{2},
$$

$$
\left( v'(R) \right)_2 = \frac{2}{\epsilon} \sinh \left( \frac{b}{2} \right) - \frac{N - 1}{R} \epsilon - \frac{2\gamma}{\gamma \cosh \frac{b}{2} + 1} \tanh \frac{b}{2} + 1.
$$

Moreover, for $r_{p,q}^\epsilon \in \mathcal{B}_0^+$ obeying (2.22), we have

$$
\left( v(r_{p,q}^\epsilon) \right)_2 = k(p) + \frac{\gamma}{R} \gamma_{p,q;\#} \sinh \left( \frac{k(p)}{2} \right)
$$

and

$$
\left( v'(r_{p,q}^\epsilon) \right)_2 = 2 \sinh \left( \frac{k(p)}{2} \right) \left[ \frac{1}{\epsilon} - \frac{1}{R} \left( \frac{N - 1}{2R} \sech^2 \frac{k(p)}{4} - \gamma_{p,q;\#} \cosh \frac{k(p)}{2} \right) \right],
$$

where $k(p)$ is uniquely determined by (2.25) and

$$
\gamma_{p,q;\#} = \frac{\gamma(N - 1) \sech^2 \frac{b}{2} + 1}{\gamma \cosh \frac{b}{2} + 1} - \frac{N - 1}{2R \sech^2 \frac{k(p)}{4}}.
$$
Due to (5.2), we shall consider a decomposition of (5.1) as follows:

Thus, the main difficulty is to deal with the estimate $F_{\epsilon u_0 - F(0)} h(r) dr$, as $0 < \epsilon \ll 1$.

Note that (2.33) implies

$$|F(\epsilon u'(r)) - F(0)| \lesssim \epsilon^{-1} e^{-\frac{4\alpha}{\epsilon} \tau(R-r)}, \quad r \in [0, R].$$

Thus, the main difficulty is to deal with the estimate $F(\epsilon u'(r)) - F(0)$ as $r$ is quite close to the boundary. Note also that $0 < \tau \leq 1$. Due to (5.2), we shall consider a decomposition of (5.1) as follows:

$$\int_{0}^{R} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} h(r) dr \in \int_{0}^{R_{-\epsilon-\tau/2}} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} h(r) dr + \int_{R_{-\epsilon-\tau/2}}^{R} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} (h(r) - h(R)) dr + h(R) \int_{R_{-\epsilon-\tau/2}}^{R} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} dr.$$
Then by (5.2) one may check that
\[
\left| \int_0^{R-\epsilon^{1-\tau/2}} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} h(r) \, dr \right| \lesssim \epsilon^{-1} \left( \max_{[0, R]} |h| \right) \int_0^{R-\epsilon^{1-\tau/2}} e^{-\frac{M_1 \tau}{\epsilon} (R-r)} \, dr \lesssim e^{-\frac{M_1 \tau}{\epsilon}},
\]
and
\[
\left| \int_{R-\epsilon^{1-\tau/2}}^R \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} (h(r) - h(R)) \, dr \right| \lesssim \max_{r \in [R-\epsilon^{1-\tau/2}, R]} |h(r) - h(R)|.
\]
As a consequence,
\[
\lim_{\epsilon \downarrow 0} \left( \int_0^{R-\epsilon^{1-\tau/2}} \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} h(r) \, dr + \int_{R-\epsilon^{1-\tau/2}}^R \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} (h(r) - h(R)) \, dr \right) = 0. \tag{5.4}
\]
To deal with the rightmost-hand side of (5.3), we first notice that by (2.9), (3.6) and (3.18), there holds \( |\epsilon u'(r) - 2 \sinh \frac{u(r)}{2} | \leq C \sqrt{\epsilon} \) uniformly in \([0, R]\) as \( 0 < \epsilon \ll 1 \), where \( C > 0 \) is independent of \( \epsilon \). For the sake of convenience, we express it by
\[
\epsilon u'(r) = 2 \sinh \frac{u(r)}{2} + o_\epsilon(1), \quad \text{as } 0 < \epsilon \ll 1. \tag{5.5}
\]
Note also that \( u(R) \to b \) as \( \epsilon \downarrow 0 \), \( u(r) > 0 \) and \( u'(r) > 0 \) for \( r \in (0, R) \) (cf. Lemma 3.2(i)). Hence, by (5.5),
\[
\int_{R-\epsilon^{1-\tau/2}}^R \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} \, dr = \int_{u(R)-\epsilon^{1-\tau/2}}^{u(R)} \frac{F(2 \sinh \frac{u(r)}{2} + o_\epsilon(1)) - F(0)}{2 \sinh \frac{u(r)}{2} + o_\epsilon(1)} \, dr \drule{u(R)}{\epsilon}
\]
\[
= \left\{ \int_{u(R)-\epsilon^{1-\tau/2}}^{0+} + \int_{b}^{u(R)} \right\} \frac{F(2 \sinh \frac{t}{2} + o_\epsilon(1)) - F(0)}{2 \sinh \frac{t}{2} + o_\epsilon(1)} \, dt \drule{b}{u(R)}{\epsilon}
\]
\[
= \left\{ \int_{0+}^{0+} + \int_{0+}^{b} \right\} \frac{F(2 \sinh \frac{t}{2} + o_\epsilon(1)) - F(0)}{2 \sinh \frac{t}{2} + o_\epsilon(1)} \, dt \quad \text{(which is finite)}. \tag{5.6}
\]
Obviously, as \( \epsilon \downarrow 0 \),
\[
\int_{0+}^{b} \frac{F(2 \sinh \frac{t}{2} + o_\epsilon(1)) - F(0)}{2 \sinh \frac{t}{2} + o_\epsilon(1)} \, dt \to \int_{0+}^{b} \frac{F(2 \sinh \frac{t}{2}) - F(0)}{2 \sinh \frac{t}{2}} \, dt \quad \text{(which is finite)}. \tag{5.7}
\]
We further check
\[
\left| \left\{ \int_{u(R)-\epsilon^{1-\tau/2}}^{0+} + \int_{b}^{u(R)} \right\} \frac{F(2 \sinh \frac{t}{2} + o_\epsilon(1)) - F(0)}{2 \sinh \frac{t}{2} + o_\epsilon(1)} \, dt \right| \lesssim \left\{ \int_{0+}^{u(R)-\epsilon^{1-\tau/2}} + \int_{b}^{u(R)} \right\} \left( 2 \sinh \frac{t}{2} \right)^{-1} \, dt + o_\epsilon(1) \drule{u(R)}{\epsilon}
\]
\[
\lesssim \left\{ \int_{0+}^{u(R)-\epsilon^{1-\tau/2}} + \int_{b}^{u(R)} \right\} t^{\tau-1} \, dt + o_\epsilon(1) \drule{0+}{\epsilon}
\]
\[
\lesssim \left( u^\tau (R - \epsilon^{1-\tau/2}) + u^\tau (R) - b^\tau + o_\epsilon(1) \right) \to 0 \quad \text{as } \epsilon \downarrow 0.
\]
Here we have used \( 0 < \tau \leq 1, \sinh \frac{t}{2} \geq \frac{t}{2} \) for \( t \geq 0 \), \( u(R - \epsilon^{1-\tau/2}) \to 0 \) (by (3.6)) and \( u(R) \to b \) as \( \epsilon \downarrow 0 \). So
\[
\lim_{\epsilon \downarrow 0} \int_{R-\epsilon^{1-\tau/2}}^R \frac{F(\epsilon u'(r)) - F(0)}{\epsilon} \, dr = \int_{0+}^{b} \frac{F(2 \sinh \frac{t}{2}) - F(0)}{2 \sinh \frac{t}{2}} \, dt \tag{5.9}
\]
immediately follows from (5.6)–(5.8). Combining (5.3)–(5.4) and (5.9), we get (2.34) and complete the proof of Theorem 2.4(I-i).

Following the same argument, we can prove Theorem 2.4(I-ii).

Now we want to prove (2.36). By (2.21), we have \( \lim_{\epsilon \downarrow 0} \inf_{[\epsilon R, R]} u > 0 \) and \( \lim_{\epsilon \downarrow 0} \inf_{[\epsilon R, R]} \epsilon u' > 0 \). Along with (4.14) yields
\[
\frac{u'(r)}{2 \sinh \frac{u(r)}{2}} - \frac{1}{\epsilon} = O(1) \quad \text{uniformly in } [\epsilon R, R]. \tag{5.10}
\]
Since \( |R - \epsilon R| \approx \epsilon R \), by putting (5.10) into (2.36) and using the same argument of (5.3)–(5.8), after making appropriate manipulations we obtain
\[
\int_0^R \frac{F(\epsilon u'(r))}{\epsilon} \chi_{[\epsilon R, R]}(r) h(r) \, dr = h(\epsilon R) \int_{\epsilon R}^{R} \frac{F(2 \sinh \frac{u(r)}{2})}{2 \sinh \frac{u(r)}{2}} u'(r) \, dr + o_\epsilon(1)
\]
= h(R) \int_{a(t^2)}^{b} \frac{F(2 \sinh \frac{t}{2})}{2 \sinh \frac{t}{2}} \, dt + o(1) \\
= h(R) \int_{c(t^2)}^{d} \frac{F(2 \sinh \frac{t}{2})}{2 \sinh \frac{t}{2}} \, dt + o(1) \quad \text{as } 0 < \epsilon \ll 1,

yielding (2.36). Similarly, we can prove (2.37). Therefore, the proof of Theorem 2.4 is completed.

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6 Appendix: Uniqueness result of (1.1) with three type boundary conditions

In this section, we show the strictly convexness of the functional (2.1).

**Proposition 6.1.** For any \( U_1, U_2 \in H^1(\Omega) \) with \( U_1 \neq U_2 \), we have

\[
\mathcal{E}_1[\epsilon U_1 + (1-t)U_2] < t \mathcal{E}_1[U_1] + (1-t) \mathcal{E}_1[U_2], \quad \forall t \in (0,1).
\]

**Proof.** Since \( \int_{\Omega} |\nabla U|^2 \, dx \) and \( \int_{\partial \Omega} (U-a)^2 \, d\sigma \) are convex functionals, it suffices to show that \( \mathcal{E}_1[U] := \log \int_{\Omega} \cosh U \, dx \) is strictly convex. We need the following elementary inequality:

\[
(A+1)^t (B+1)^{1-t} \leq A^t B^{1-t} + 1, \quad \text{for } A, B > 0 \text{ and } t \in (0,1),
\]

and the equality holds if and only if \( A = B \).

Note that \( \frac{1}{t}, \frac{1}{1-t} > 1 \). Applying (6.2) with \( A = e^{2U_1} \) and \( B = e^{2U_2} \) and the Hölder inequality to \( \mathcal{E}_1 \), one may check that

\[
\mathcal{E}_1[\epsilon U_1 + (1-t)U_2] = \log \int_{\Omega} \frac{1}{2} e^{-(\epsilon U_1 + (1-t)U_2)} \left( e^{\epsilon U_1 + (1-t)U_2} + 1 \right) \, dx \\
\leq \log \int_{\Omega} \frac{1}{2} e^{-(\epsilon U_1 + (1-t)U_2)} \left( e^{\epsilon U_1 + 1} \right)^t \left( e^{2U_2 + 1} \right)^{1-t} \, dx \\
= \log \left( \int_{\Omega} \frac{e^{U_1} + e^{-U_1}}{2} \, dx \right)^t \left( \int_{\Omega} \frac{e^{U_2} + e^{-U_2}}{2} \, dx \right)^{1-t} \\
= t \mathcal{E}_1[U_1] + (1-t) \mathcal{E}_1[U_2].
\]

Moreover, the equality of (6.3) holds if and only if \( e^{2U_1} = e^{2U_2} \) (by (6.2) and the second line of (6.3)) and \( \cosh \frac{U_1}{\cosh U_2} \) is constant almost everywhere in \( \Omega \) (from the fourth line of (6.3) and the condition for equality to hold), which implies \( U_1 = U_2 \). As a consequence, we have \( \mathcal{E}_1[\epsilon U_1 + (1-t)U_2] < t \mathcal{E}_1[U_1] + (1-t) \mathcal{E}_1[U_2] \) for \( U_1 \neq U_2 \). Therefore, we get (6.1) and complete the proof of Proposition 6.1.

On the other hand, one finds \( \inf_{H^1(\Omega)} \mathcal{E}_\epsilon \geq |\Omega| \log 1 = 0 \) since \( \cosh U \geq 1 \). Along with Proposition 6.1, we may use the Direct method to show that \( \mathcal{E}_\epsilon \) has a unique minimizer \( U^* \) in \( H^1(\Omega) \), and \( U^* \) is a weak solution of (1.1)–(1.2). Then the standard elliptic regularity theory immediately shows that the minimizer \( U^* \in C^2(\overline{\Omega}) \cap C^1(\Omega) \) is a class solution of (1.1)–(1.2) for bounded domain \( \Omega \) with smooth boundary.

We now shall prove the uniqueness of the model (1.1) with three type boundary conditions: the Robin boundary condition (1.2), the Dirichlet boundary condition

\[
U = \overline{a}(x) \quad \text{on } \partial \Omega,
\]

and the Neumann boundary condition

\[
\partial_\nu U = \overline{a}(x) \quad \text{on } \partial \Omega,
\]

where \( \overline{a} \) is a smooth function on \( \partial \Omega \).

**Proposition 6.2.** The model (1.1) with the following boundary conditions has a unique classical solution.

(i) The Robin boundary condition (1.2).

(ii) The Dirichlet boundary condition (6.4).

(iii) The Neumann boundary condition (6.5).

**Proof.** The main argument is based on the proof of Theorem 1.1 in [14]. For convenience, we let

\[
\mathcal{C}_i = \left( \int_{\Omega} \cosh U^i \, dx \right)^{-1}, \quad i = 1, 2.
\]
Therefore, we get $U^1$ and $U^2$ are two distinct classical solutions of (1.1) with the Dirichlet boundary condition (6.4). Following the same argument of (6.6), we arrive at

\[-e^2 \int_\Omega |\nabla(U^1 - U^2)|^2 \, dx - \frac{\epsilon}{\gamma} \int_{\partial \Omega} (U^1 - U^2)^2 \, d\sigma_x = 0.
\]

Here we have applied the integration by parts and the boundary constraint $(U^1 - U^2) + \gamma e \partial_n(U^1 - U^2) = 0$ to the left-hand side of (6.6), and the elementary inequality $(e^A - e^B)(A - B) \geq 0$ for $A, B \in \mathbb{R}$ to the fourth line of (6.6). Note that $\gamma > 0$. Thus, (6.6) gives $\int_\Omega |\nabla(U^1 - U^2)|^2 \, dx = \int_{\partial \Omega} (U^1 - U^2)^2 \, d\sigma_x = 0$, which immediately implies

\[\nabla(U^1 - U^2) = 0 \quad \text{in} \ \Omega, \quad \text{and} \quad U^1 - U^2 = 0 \quad \text{on} \ \partial \Omega.
\]

Therefore, we get $U^1 = U^2$ in $\overline{\Omega}$ (also leads to a contradiction) and complete the proof of (i).

**Proof of (ii).** Suppose that $U^1$ and $U^2$ are two distinct classical solutions of (1.1) with the Dirichlet boundary condition (6.4). Following the same argument of (6.6), we arrive at

\[-e^2 \int_\Omega |\nabla(U^1 - U^2)|^2 \, dx \geq \frac{1}{2} \int_\Omega \left( e^{U^1 + \log C^1} - e^{U^2 + \log C^2} \right) \left[ (U^1 + \log C^1) - (U^2 + \log C^2) \right] \, dx
\]

\[+ \frac{1}{2} \int_\Omega \left( e^{-U^1 + \log C^1} - e^{-U^2 + \log C^2} \right) \left[ (-U^1 + \log C^1) - (-U^2 + \log C^2) \right] \, dx \geq 0.
\]

Hence, in $\Omega$ we must have $\nabla U^1 = \nabla U^2$ and

\[e^{U^1 + \log C^1} - e^{U^2 + \log C^2} = 0,
\]

\[e^{-U^1 + \log C^1} - e^{-U^2 + \log C^2} = 0.
\]

This implies $U^1 + \log C^1 = U^2 + \log C^2$ and $-U^1 + \log C^1 = -U^2 + \log C^2$, i.e., $U^1 = U^2$ in $\Omega$. Along with $U^1 = U^2$ on $\partial \Omega$, we get $U^1 = U^2$ in $\overline{\Omega}$ and complete the proof of (ii).

It remains to prove (iii). When both $U^1$ and $U^2$ are solutions of (1.1) with the boundary condition (6.5), it is easy to check that (6.7) still holds. Hence, we immediately get $U^1 = U^2$ in $\overline{\Omega}$ and complete the proof of (iii). Therefore, the proof of Proposition 6.2 is done. □

7 References

[1] M. AL-REFAI, N.I. KAVALLARIS AND M. ALI HAJJI, Monotone iterative sequences for non-local elliptic problems, Euro. J. Appl. Math., 22 (2011), 533–552.

[2] U. ASCHER AND R.D. RUSSELL, Reformulation of boundary value problems in “standard” form, SIAM Rev., 23 (1981), 238–254.

[3] G.V. BOSSA, B.K. BERNTSON AND S. MAY, Curvature Elasticity of the Electric Double Layer, Phys. Rev. Lett. 120 (2018), 215502.

[4] D. CARPENTIER AND P. LE DOUSSAL, Glass transition of a particle in a random potential, front selection in nonlinear renormalization group and entropic phenomena in Liouville and sinh–Gordon models, Phys. Rev. E 63 (2001), 026110.

[5] P.C. FIFE, Semilinear elliptic boundary value problems with small parameters, Arch. Rational Mech. Anal. 52 (1973), 205–232.

[6] V. FATEEV, D. FRADKIN, S.L. LUKYANOV, A. ZAMOLODCHIKOV AND A. ZAMOLODCHIKOV, Expectation values of descendent fields in the sine–Gordon model, Nuclear Phys. B 540 (1999), 587–609.

[7] D. GAIOTTO , G.W. MOORE AND A. NEITZKE, Wall-crossing, Hitchin systems, and the WKB approximation, Adv. Math. 234 (2013), 239–403.

[8] M. GROSSI AND A. PISTOIA, Multiple blow-up phenomena for the sinh–Poisson equation, Arch. Ration. Mech. Anal. 209 (2013), 287–320.

[9] D. GILBARG AND N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer, Berlin, 2001.

[10] F.A. HOWES, Singularly perturbed semilinear elliptic boundary value problems, Commun. Partial Differ. Equ. 4 (1979), 1–39.

[11] M. JAWORSKI AND D. KAUP, Direct and inverse scattering problem associated with the elliptic sinh–Gordon equation, Inverse Problems 6 (1990) 543–556.
[12] A. Khare and A. Saxena, Periodic and hyperbolic soliton solutions of a number of nonlocal nonlinear equations, J. Math. Phys. 56 (2015), 032104.

[13] M.S. Kilic, M.Z. Bazant and A. Aidari, Steric effects in the dynamics of electrolytes at large applied voltages. I. Double-layer charging, Phys. Rev. E 75 (2007), 021502.

[14] C.-C. Lee, The charge conserving Poisson–Boltzmann equations: Existence, uniqueness and maximum principle, J. Math. Phys. 55 (2014), 051503.

[15] C.-C. Lee, Asymptotic analysis of charge conserving Poisson–Boltzmann equations with variable dielectric coefficients, Discrete Contin. Dyn. Syst. 36 (2016) 3251–3276.

[16] C.-C. Lee, Thin layer analysis of a non-local model for the double layer structure, J. Differential Equations 266 (2019) 742–802.

[17] C.-C. Lee, Statistical mechanics of the N-point vortex system with random intensities on a bounded domain, Ann Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 381–399.

[18] C.-C. Lee and R.J. Ryham, Boundary asymptotics for a non-neutral electrochemistry model with small Debye length, Z. Angew. Math. Phys. 69 (2018) 41.

[19] R. Messina, Image charges in spherical geometry: Application to colloidal systems, J. Chem. Phys. 117 (2002), 11062.

[20] R. Messina, C. Holm and K. Kremer, Strong electrostatic interactions in spherical colloidal systems, Phys. Rev. E 64 (2001), 021405.

[21] B.M. McCoy, C.A. Tracy and T.T. Wu, Painleve functions of the third kind, J. Math. Phys. 18 (1977), 1058.

[22] N. Neri, Statistical mechanics of a N-point vortex system on a closed domain, Comm. Math. Phys. 247 (2004), 381–399.

[23] W.-M. Ni and I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993), 247–281.

[24] R.E. O'Malley, Topics in singular perturbations, Adv. Math. 2 (1968), 365–470.

[25] A. Pistoia and T. Ricciardi, Sign-changing tower of bubbles for a sinh-Poisson equation with asymmetric exponents, Discrete Contin. Dyn. Syst. 37 (2017) 5651–5692.

[26] R. Ryham, C. Liu and L. Zikatanov, Mathematical Models for the Deformation of Electrolyte Droplets, Discrete Contin. Dyn. Syst. Ser. B 8 (2007), 649–661.

[27] R. Ryham, C. Liu and Z.Q. Wang, On electro-kinetic fluids: one dimensional configurations, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), 357–371.

[28] C.S.Z. Redwan, J.R. Santos Júnior and Antonio Suárez, Existence and uniqueness of solution for a nonhomogeneous non-local problem, Z. Angew. Math. Phys. 68 (2017), 144.

[29] T. Shibata, The steepest point of the boundary layers of singularly perturbed semilinear elliptic problems, Tran. Amer. Math. Soc. 356 (2004), 2123–2135.

[30] T. Shibata, Three-term asymptotics for the boundary layers of semilinear elliptic eigenvalue problems, Nonlinear differ. equ. appl. NoDEA 13 (2006) 23–41.

[31] H. Sugio, Ion-conserving Poisson–Boltzmann theory, Phys. Rev. E 86 (2012), 016318.

[32] L. Sun, J. Shi and Y. Wang, Existence and uniqueness of steady state solutions of a non-local diffusive logistic equation, Z. Angew. Math. Phys. 64 (2013), 1267–1278.

[33] L. Wang, S. Xu, M. Liao, C. Liu and P. Sheng, Self-consistent approach to global charge neutrality in electrokinetics: A surface potential trap model, Phys. Rev. X 4 (2014), 011042.