CLASSIFICATION OF TEN-DIMENSIONAL HETEROTIC STRINGS

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ABSTRACT

Progress towards the classification of the meromorphic $c = 24$ conformal field theories is reported. It is shown if such a theory has any spin-1 currents, it is either the Leech lattice CFT, or it can be written as a tensor product of Kac-Moody algebras with total central charge 24. The total number of combinations of Kac-Moody algebras for which meromorphic $c = 24$ theories may exist is 221. The next step towards classification is to obtain all modular invariant combinations of Kac-Moody characters. The presently available results are sufficient to obtain a complete list of all ten-dimensional heterotic strings. Furthermore there are strong indications for the existence of several (probably at least 20) new meromorphic $c = 24$ theories.
The purpose of this paper is to answer a question that arose five years ago. At that time nine heterotic string theories were known in ten dimensions: the supersymmetric $E_8 \times E_8$ and $O(32)$ theories [1], the tachyon-free non-supersymmetric $O(16) \times O(16)$ theory, and six other non-supersymmetric theories with tachyons [2-4]. Of these nine theories, eight have a rank 16 gauge group, and one has a gauge group of rank eight. The gauge groups of the former theories are embedded in a level 1, simply laced Kac-Moody algebra, whereas the latter theory has a Kac-Moody algebra $E_8,2$ (here and in the following we denote the level by a second subscript). All of these theories can be constructed using free fermions, and furthermore they are the only ten-dimensional heterotic strings that can be obtained this way. However, this does not prove that no other ten-dimensional heterotic string theories can exist. In less than 10 dimensions one can certainly not write all heterotic string theories in terms of free fermions. The purpose of this paper is to prove in a construction-independent way that the list of ten-dimensional heterotic string theories that we know since 1986 is indeed complete.

The proof is valid for the class of ten-dimensional critical heterotic strings satisfying the usual consistency conditions. Such a theory is defined by a conformal field theory consisting of a right-moving NSR sector and a left-moving bosonic sector. The right-moving super-conformal field theory is completely fixed by the requirement that the target-space has ten flat space-time dimensions. In the left-moving sector an “internal” $c = 16$ conformal field theory $C_{16}$ remains undetermined. Further restrictions arise from one-loop modular invariance. This imposes the condition the four spin-structures of the right-moving world-sheet fermions must be combined in a modular invariant way with representations of the left-moving internal $c = 16$ theory. This must be done in such a way that all space-time states have the correct spin-statistics relation, and that the ground state has multiplicity 1.

An extremely useful tool for addressing superstring classification problems is the bosonic string map [5]. This map replaces the NSR sector (including superghosts) of a heterotic (or type-II) string by a bosonic sector, in such a way that a modular invariant heterotic string satisfying the spin statistics condition is mapped to a modular invariant* bosonic string with positive multiplicities for all states. In $d$ dimensions, the light-cone

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* The fact that one-loop modular invariance is preserved was shown in the last paper of [5]. A generalization of this map to higher genus surfaces can be found in [6].
SO(d−2) algebra generated by the NSR fermions is replaced by an SO(d + 6) × E8 Kac-Moody algebra in the bosonic theory. Then a classification of all ten-dimensional heterotic strings is equivalent to a classification of all ten-dimensional bosonic strings with a right-moving Kac-Moody algebra \(D_{8,1} \times E_{8,1}\). The advantage of this map is that spin-statistic signs, as well as signs in the modular transformation of the world-sheet spin-\(\frac{3}{2}\) determinant are automatically taken care of.

It has been shown in [7] that the latter classification problem can be mapped to yet another classification problem, by dropping the \(E_8\) factor and replacing the right-moving \(D_8\) factor by a left-moving one, whose conjugacy classes are paired with the representations of \(C_{16}\) in exactly the same way. It is easy to see that this map preserves modular invariance. The result is a purely left-moving, unitary CFT which is modular invariant by itself, built out of representations of the tensor product \(D_{8,1} \times C_{16}\). This theory has central charge \(c = 24\), and, since it is modular invariant by itself, only a single primary field with respect to the appropriate chiral algebra (which is some extension of the chiral algebras of the factors \(D_{8,1}\) and \(C_{16}\)). Conformal field theories with only one primary field must have a central charge that is a multiple of 8, and are sometimes called meromorphic CFT’s [8].

This observation was used in [7] to show that the list of ten-dimensional heterotic strings with a rank-16 gauge group is complete. If \(C_{16}\) contains a Kac-Moody algebra of rank 16, it can be written entirely in terms of free bosons, and the same is then true for the tensor product \(C_{24} = D_{8,1} \times C_{16}\). Hence \(C_{24}\) is given by a 24-dimensional even self-dual lattice, and since all such lattices have been classified, the problem is solved. All solutions can be obtained by considering all possible embeddings of \(D_{8,1}\) in the Kac-Moody algebras that appear in the list of Niemeier lattices [9]. The \(E_{8,2}\) theory cannot be obtained in this way, but one can invert the argument to conclude that there must exist a meromorphic \(c = 24\) theory built out of the tensor product \(B_{8,1} \times E_{8,2}\) [10]. Conversely, if a list of all meromorphic \(c = 24\) theories were available, one could read off from it all ten-dimensional heterotic strings by considering all embeddings of \(D_{8,1}\) in the Kac-Moody algebras of those theories.

Unfortunately, unlike the Niemeier lattices, a classification of the more general meromorphic \(c = 24\) CFT’s was not available so far. As a classification problem this is of considerable interest in its own right. Of course one should keep in mind that the \(c = 24\)
Theories are only the third member of an infinite series $c = 8k$, and that the number of free bosonic solutions (i.e. even self-dual lattices) increases extremely rapidly with $k$. Nevertheless, the value $c = 24$ is special, being the smallest value for which the list of solutions is interesting, and the largest for which a listing is practically possible. Furthermore, the $c = 24$ theories have intriguing connections with other mathematical concepts, such as the monster group. Finally, as the second example in [10] shows, from a list of meromorphic $c = 24$ theories we can get some information about solutions to another interesting and unsolved problem, the classification of modular invariants of Kac-Moody algebras.

The real purpose of this paper is to report progress towards the classification of the $c = 24$ meromorphic theories. Although this classification is not yet finished, enough of it is now known to carry out the enumeration of the ten-dimensional heterotic strings, along the lines explained above.

The basic idea is to use the results of [11] on the relation between modular invariance and chiral anomaly cancellation in effective field theories of string theories. Note that this argument can be used to show that the list of supersymmetric heterotic string theories is complete. This sort of argument is far less powerful for the non-supersymmetric string theories for two reasons: first of all there is no constraint on the size of the gauge group because there are no gaugino contributions to the gravitational anomaly, and secondly one can have cancelling contributions to the chiral anomaly due to Weyl fermions with opposite chirality.

However, the cancellation of chiral anomalies is only a small part of the information contained in the “character valued partition function” constructed in [11]. Rather than trying to get constraints on the $d = 10$ heterotic strings or $c = 24$ CFT’s by applying the condition of anomaly cancellation in some string theory, it turns out to be much more effective to study directly the character valued partition function of the $c = 24$ CFT’s.

A character-valued partition function can be written down if there is at least one spin-1 operator $J$ in the theory. It can be shown [15], (and is anyhow not surprising)

\* *In [11] the character valued partition functions used to prove Green-Schwarz factorization [12,13] of anomalies in string theory were explicitly constructed for theories constructed out of free bosons or complex fermions. The generalization to higher level Kac-Moody algebras or non-simply laced ones, using the Weyl-Kac character formula, was found later by Ginsparg, Moore and Vafa, and is described in [14].
that such operators either generate Kac-Moody algebras (by definition non-abelian) or $U(1)$’s, i.e. their operator products are restricted to the following form

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{1}{z-w}if^{abc}J^c(w) + \text{finite terms} \quad .$$

(1)

Then one can organize the states at any level according to representations of the Lie-algebras generated by the zero-modes of the currents $J^a$. The character-valued partition function has the form

$$P(q, F) = \text{Tr} e^{F \cdot J_0} q^{L_0 - c/24} = \sum_{n=-\infty}^{\infty} q^n \text{Tr} e^{F_n} .$$

where $F \cdot J = \sum_a F^a J^a$ and $F^a$ is an arbitrary set of real coefficients. In the second line $F_n$ denotes $F \cdot J$ evaluated in the matrix representation of the $n^{\text{th}}$ level.

Theories that do not have any spin-1 operators fall outside the scope of this paper. So far only one such theory is known, the so-called “monster module”, which can be obtained as a modular invariant of the $\mathbb{Z}_2$-twisted Leech lattice CFT, and which is conjectured to be the only $c = 24$ meromorphic theory without spin-1 operators. In all other cases one may group all spin-1 operators into the adjoint representations of some semi-simple Lie-algebra, plus possibly some additional $U(1)$ factors.

As in the case of anomaly cancellation, we are interested in the modular transformation properties of the character valued partition function. They follow directly from the fact that, for $F = 0$, the partition function is modular invariant, i.e.

$$P \left( \frac{a\tau + b}{c\tau + d}, 0 \right) = P(\tau, 0), \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad ,$$

where, for convenience we have traded the variable $q$ for $\tau$, with $q = e^{2\pi i \tau}$. If we focus on one Kac-Moody algebra or $U(1)$ factor, we can express $P$ in terms of the characters $X_i(\tau)$ of the representations of that algebra

$$P(\tau, 0) = \sum_i X_i(\tau) P_i^f(\tau) ,$$

(2)

where $P_i^f$ is a character of the “unknown” part of the theory.
If we choose $F$ entirely within the Kac-Moody or $U(1)$ factor under consideration, then the dependence on $F$ enters only through the characters $X_i$. The modular transformation properties of “character valued” Kac-Moody characters is known. Under the two generating transformations of the modular group they transform as follows \[ \tau \rightarrow \tau + 1 : \quad X_i(\tau + 1, F) = e^{2\pi i(h_i - c/24)} X_i(\tau, F) \] \[ \tau \rightarrow -1/\tau : \quad X_i(-1/\tau, F) = e^{-i8\pi \gamma k/g} \text{Tr} F^2 S_{ij} X_j(\tau, F) . \] (3)

Here $g$ is the dual Coxeter number of the Kac-Moody algebra. The normalization of the Kac-Moody generators is defined by (1) plus the requirement that the smallest allowed value for $k$ must be 1. The Lie-algebra generators $J_{0a}$ are taken to be hermitean, and the real, anti-symmetric structure constants $f_{abc}$ satisfy $f_{abc} f_{abe} = 2g_\delta_{ce}$. In (3) the representation in which the trace is evaluated is the adjoint representation. Any other non-trivial representation may be used, provided one changes the factor multiplying the trace (for example, to compare with [11] one should use the vector representation of $SO(2N)$, and compensate the change in normalization by omitting the factor $1/g$; furthermore one should replace $F$ by $iF/2\pi$). For $U(1)$ factors the adjoint representation cannot be used, since it is trivial. Instead, any representation with a non-vanishing quadratic trace can be used, in which case one must replace $k/g \text{Tr} F^2$ in (3) by $N \text{Tr} F^2$.

Although it is not difficult to obtain the correct normalization factor $N$, for our purposes it turns out to be irrelevant. In the following all results will be presented explicitly for non-abelian factors only, but it is straightforward to make the appropriate changes for abelian ones. the appropriate changes for abelian ones.

Because $P(\tau, 0)$ is modular invariant, the phase $e^{2\pi i(h_i - c/24)}$ and the matrix $S_{ij}$ are compensated by the transformation of $P'$. From this we can deduce how $P(\tau, F)$ transforms

\[
P \left( \frac{a\tau + b}{c\tau + d}, \frac{F}{c\tau + d} \right) = \exp \left[ -i c \frac{k_{\ell}}{8\pi (c\tau + d)} \sum_{\ell} \frac{k_{\ell}}{g_{\ell}} \text{Tr} F_{\ell}^2 \right] P(\tau, F) . \] (4)

Here we have allowed for the possibility that $F$ has components $F_{\ell}$ in several simple factors of the Kac-Moody algebra. To simplify the notation we define

\[
F^2 \equiv \sum_{\ell} \frac{k_{\ell}}{g_{\ell}} \text{Tr} F_{\ell}^2 .
\]

Now we expand $P(q, F)$ in traces of $F$, and make use of the theory of modular
functions to constrain the coefficients, which are functions of $q$. A few facts about modular functions will be mentioned below, but for more details see [11], or references therein. In the absence of the exponential factor in (4), the coefficient function of a trace of order $n$ is a modular function of weight $n$. Such functions can all be expressed in terms of the Eisenstein functions $E_4$ and $E_6$,

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n},$$
$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n},$$

which have weight 4 and 6 respectively. Any entire modular function (which, as a function of $\tau$, has no poles in the complex upper half-plane including the point $\tau = i\infty$) has non-negative and even modular weight, and can be written as a polynomial in $E_4$ and $E_6$. The number of parameters in such a polynomial is $k$ for weight $12k$, as well as for weights $12k + 4$, $12k + 6$, $12k + 8$ and $12k + 10$, and $k - 1$ for weights $12k + 2$.

The coefficient functions appearing in the expansion of $c = 24$ partition function are in general not entire functions, because of the “tachyon” pole at $q = 0$ ($\tau = i\infty$). To obtain modular functions with poles at $q = 0$ one defines first

$$\Delta(q) = \frac{1}{1728} \left( (E_4)^3 - (E_6)^2 \right) = \eta(q)^{-24}.$$

The most general modular function with a single pole at $q = 0$ and none elsewhere in the $\tau$ upper half-plane is a combination of the form $(G_4)^m(G_6)^n/\Delta$, which has weight $4m + 6n - 12$. The number of different functions is for weights $12k + l$ is now $k + 1$ if $l = 0, 4, 6, 8, 10$, and $k$ if $l = 2$.

To use these results we need to know that the coefficient functions do not have spurious poles elsewhere in the positive upper half-plane. In general, characters $X_i$ of unitary conformal field theories do not have such poles if their chiral algebra is generated by a finite number of currents $N_J$ (each of which may have an infinite number of modes) because they are bounded by $X_i \leq q^{h_i - c/24 + N_J/24} N_0 \eta^{-N_J}$, where $N_0$ is the ground state multiplicity. The inequality is saturated if all $N_J$ currents act independently without null-vectors (i.e. as $N_J$ free bosons). Null-vectors reduce the number of states at each
level and hence reduce the value of the left-hand side with respect to the right-hand side. Since \( \eta^{-1} \) has no poles in the upper half plane outside \( \tau = i\infty \), \( X_i \) is well-behaved as well. Thus we will assume that the chiral algebra is finitely generated, or in other words that beyond a certain spin all currents can be expressed in terms of lower spin currents. Then not only the partition function \( P(\tau, 0) \) is free of spurious poles, but also the coefficient functions of higher traces, since in (2) the Kac-Moody characters \( X_i(F, \tau) \) are explicitly known and well-behaved, and \( P'_i(\tau) \) is free of spurious poles by the same assumption.

The term of order zero in the expansion in \( F \) is the partition function, \( P(q, 0) \). This is a modular function of weight zero with a single pole at \( \tau = i\infty \). We conclude that it must be equal to

\[
P(q, 0) = J(q) + N,
\]

where \( J(q) \) is the absolute modular invariant

\[
J(q) = \frac{1}{q} + 196884q + \ldots,
\]

and \( N \) is the number of spin-1 operators in the theory. This is not determined by the requirement of modular invariance of the partition function.

The exponential factor in (4) is cancelled if instead of \( P(q, F) \) we consider

\[
\tilde{P}(q, F) = e^{-\frac{1}{48}E_2(q)F^2} P(q, F),
\]

(5)

where

\[
E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.
\]

This function transforms as a modular function of weight 2, plus an “anomalous” extra term,

\[
E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d).
\]

Now we can apply the foregoing arguments to \( \tilde{P}(q, F) \), and from the expansion of \( \tilde{P} \) derive that of \( P \) by multiplying with the inverse of the correction factor in (5). An example of such a computation (for the \( O(32) \) heterotic superstring) can be found in appendix C of [17].
The resulting expression has many undetermined parameters, some of which can be determined from our knowledge of the zeroth and first level of the theory. Since the ground state is a singlet representation, all coefficient functions of non-trivial traces must be free of poles. This condition leads to the following parametrization of $P(q, F)$

$$P(q, F) = \exp\left(\frac{1}{48}E_2(q)F^2\right)$$

$$\times \left\{J(q) + N + \left(\frac{E_4}{\Delta}\right)^3[\cosh(\frac{1}{48}\sqrt{E_4(q)}F^2) - 1]ight.$$  

$$- \sqrt{E_4(q)}E_4(q)E_6(q)[\sinh(\frac{1}{48}\sqrt{E_4(q)}F^2)]$$

$$+ \sum_{n=4}^{\infty} E_n a_{n,i} F^n_i \right\}. \tag{6}$$

Here $F^n_i$ denotes traces of $n^{th}$ order in $F$. In general there are many possible combinations of traces of a given order (e.g. at fourth order one can have $\text{Tr} F^4, (\text{Tr} F^2)^2$ or $\text{Tr}(F_1)^2 \text{Tr}(F_2)^2$), and the different types are labelled by $i$. Each different trace appears with a parameter $a_{n,i}$. The functions $E_n$ are just the Eisenstein functions or polynomials of such functions, normalized to have a first coefficient equal to 1: $E_8 = (E_4)^2$, $E_{10} = E_4E_6$, $E_{12} = \alpha E_4^3 + (1 - \alpha)E_6^2$, etc. Note that extra parameters start appearing in the definition of $E_{12}$. The purpose of the “cosh” and “sinh” terms is simply to cancel the $\frac{1}{q}$ poles in coefficient functions of combinations of quadratic traces.

Now compare this expression with the exactly known expansion of $P(q, F)$ to the first level:

$$P(q, F) = \frac{1}{q} + \sum \text{Tr} e^{F\ell} + \text{higher order terms in } q.$$

Here all traces must be evaluated in the representations of the first level, i.e. the adjoint representations of the Kac-Moody and $U(1)$ factors. Comparing the quadratic traces in both expressions we find immediately

$$\frac{k_\ell}{g_\ell} \left(\frac{N}{48} - \frac{1}{2}\right) = \frac{1}{2}, \tag{7}$$

for non-abelian factors, and

$$N_\ell \left(\frac{N}{48} - \frac{1}{2}\right) = 0 \tag{8}$$

for $U(1)$ factors, because then $F_\ell$ is trivial in the adjoint representation.
This leads first of all to the conclusion that the value of $\frac{g}{k}$ must be the same for every Kac-Moody algebra appearing in a given $c = 24$ theory, and furthermore that abelian and non-abelian factors cannot be mixed (this is already known to be true for the Niemeier lattices, see e.g. [18]). Secondly, we learn that the value of $\frac{g}{k}$ is determined by the total number of spin-1 operators. If $N = 24$ only $U(1)$ factors are allowed, and then we have no choice but to saturate $N$ entirely with generators of $U(1)$'s i.e. free bosons. The resulting theory falls thus within the classification of Niemeier, and is in fact the conformal field theory of the Leech lattice. But also for larger values of $N$ we arrive at an interesting conclusion. The total central charge of the Kac-Moody system is

$$c_{KM} = \sum_\ell \frac{k_\ell \dim_\ell}{k_\ell + g_\ell},$$

where $\dim_\ell$ is the dimension of the adjoint representation of the $\ell^{th}$ factor. Substituting (7) we find

$$c_{KM} = \frac{24}{N} \sum_\ell \dim_\ell = 24,$$

so that the entire central charge is saturated by the Kac-Moody part of the theory. Hence we conclude that if a $c = 24$ theory has any spin-1 operators, it is either the Leech lattice CFT or can be entirely written as a tensor product of Kac-Moody algebras.

It is now straightforward to obtain a complete list of all Kac-Moody algebras that can appear. In total there are 221 combinations (this does not include $N = 0$ (no spin-1 algebra) and $N = 24$ ($U(1)^{24}$)). This list contains of course all 39 presently known cases: the 23 algebras of the Niemeier lattices [9], the 14 additional ones obtained from the $\mathbb{Z}_2$-twisted Niemeier lattices [19,20],* and the 2 examples of [10].

The next task is to try and find modular invariant combinations of the characters of each combination of Kac-Moody algebras. Of the 221 combinations 24 require no further discussion because the only valid character combinations are those of the Niemeier

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* One combination listed in [20], namely $(A_3)^4(A_2)^2$, is incorrect, and should be $(A_3)^4(A_1)^4$. 
lattices.† For the other 17 Kac-Moody combinations for which modular invariants are known this is not quite true, because there might still exist other modular invariant combinations.

Modular invariant combinations of characters will certainly not exist for all 221 combinations. This can especially not expected to be true for small $N$, where often many algebras with small dimensions occur, and accidental solutions to the condition $\sum_\ell \dim_\ell = N$ are likely. One way to eliminate such accidental combinations is to use the other trace identities one can obtain from (6). If the functions $E_n$ have no free parameters themselves, then the coefficients $a_{n,i}$ can be determined at the first level. One then gets predictions for traces of representations at the second level. If there are no representations with conformal spin 2 that can satisfy these identities, then there cannot exist a meromorphic modular invariant. If there is a solution with positive integer coefficients one can either try to go on to the third level, or directly use the matrix $S$ to determine the remaining unknown coefficients, and prove that indeed one gets an invariant. This will in any case have to be done to be certain that one has indeed a solution.

In general, the total number of spin-2 representations is very large, but so is the number of trace identities. The traces $F_i^n$ of the second level representations are completely determined by modular invariance for $n = 0, 2, 4, 6, 8, 10$ and $n = 14$ (odd traces must vanish). If there are several Kac-Moody factors there are of course mixed traces with components in more than one factor. So far our analysis is based only upon traces that are of at most order 2 in each factor. This eliminates at least 90 of the initial 221 combinations. For another 24 combinations a solution to the trace identities exists (not including the 39 already known solutions). Although a priori the coefficients of each representation are allowed to be any positive integer, in these solutions they are (up to unresolved ambiguities) either 0 or 1, which is a strong indication that these solutions are not accidental. Of course, further checks are required. Finally, for the remaining combinations the existence of solutions is still unclear, because the number of fields is

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† In principle only even self-dual lattices have been classified in [9], and this is not necessarily the same as classifying all modular invariants. There might exist character combinations that do not have a lattice interpretation, for example because there are multiplicities higher than 1. However, all primary fields of level-1 simply-laced Kac-Moody algebras are simple currents. All closed chiral algebra that can be built out of simple currents and that can appear in modular invariant partitions have been classified in [21]. This rules out any “non-lattice” solutions.
too large in comparison with the number of trace identities. The use of higher order traces might improve this very significantly. In any case, the classification of $c = 24$ theories (or at least the corresponding modular invariant combinations of Kac-Moody algebras) with at least one spin-1 current is now reduced to a finite algorithm.

The values of $N$ for which $c = 24$ theories may exist are $0, 24, 28, 30, 32, 36, 40, 42, 48, 56, 72, 84, 96, 112, 120, 144, 168, 216, 224, 234, 240, 248, 252, 260, 276, 280, 288, 300, 308, 312, 320, 324, 336, 348, 360, 372, 384, 390, 396, 400, 408$, and $1128$. For all combinations with $N \geq 300$ solutions are already known. A new modular invariant appears for $N = 288$ with a Kac-Moody algebra $B_{6,1}C_{10,1}$. This invariant can easily be understood in the following way. Because of rank-level duality for the $C_{n,k}$ algebras, the well-known invariant of $SU(2)$ level 10 ($i.e. C_{1,10}$) implies the existence of a similar invariant for $C_{10,1}$ [22]. The latter has three primary fields which turn out to have the fusion rules of the Ising model, or of $B_{6,1}$. Glueing these theories together in the obvious way yields a meromorphic $c = 24$ invariant.

A second new example is $A_{5,1} \times E_{7,3}$, obtained by glueing the characters of the first factor to those of a non-diagonal modular invariant of the second. This $E_{7,3}$ invariant is apparently not yet known, and has the form

$$|X_{0000000} + X_{0000001}|^2 + |X_{0000001} + X_{0000030}|^2 + 2|X_{00000100}|^2 + 2|X_{1000010}|^2.$$

After resolving the two fixed points one obtains a total of six fields, with the same fusion rules as $A_{5,1}$. The rest of the discussion is completely analogous as that of the $A_{2,2} \times F_{4,6}$ example of [10]. Note that in all these cases one discovers extensions of the chiral algebra (of $C_{10,1}, E_{7,3}$ and $F_{4,6}$ respectively) by operators with spin larger that one, since we have already accounted for all spin-1 operators by satisfying the sum rule $\sum_\ell \dim_\ell = N$ at the first level. Therefore one always finds extension that can not be obtained from conformal embeddings. (This should not be confused with the fact that Kac-Moody algebras of all examples of [20] can be conformally embedded in the Kac-Moody algebras of Niemeier theories. In those cases the extra spin-1 currents have been removed by the $\mathbb{Z}_2$ twist). These two examples are only new solutions of the 19 candidates mentioned above for which modular invariance has been checked so far, but there are several others that should not be too difficult to verify.

A modular invariant combination of Kac-Moody characters is not yet a conformal field theory, but it is an important step in that direction. The next step could be either
to find an explicit realization, or to try to compute correlation functions or higher genus partition functions. It should be emphasized that there is a crucial difference between knowing the field content of a $c = 24$ CFT in terms of representations of a $c = 24$ Kac-Moody sub-algebra of the full chiral algebra and just knowing the partition function $P(q, 0)$. This point was apparently not appreciated by the authors of [20], who remark that one could not distinguish the $E_8 \times E_8$ and $O(32)$ theories in this way. One certainly cannot distinguish those theories if one only knows $P(q, 0)$, but if one knows $P(q, F)$ and the two different ways of writing it in terms of characters of $E_8 \times E_8$ or $O(32)$ respectively, then one has certainly made the distinction. The point is that $P(q, 0)$ contains only information about Virasoro representations, and the Virasoro algebra is too small to determine correlation functions except when $c < 1$. If $P(q, 0)$ is all one knows about a $c = 24$ theory one cannot even begin to compute operator products or correlators. On the other hand, if the full $c = 24$ theory is covered by a combination of Kac-Moody algebras, one already knows a large enough sub-algebra of the chiral algebra to determine correlators and operator products, although the actual computations become extremely tedious due to the fact that one has an off-diagonal invariant with a (known!) extension of the chiral algebra. Inconsistencies and ambiguities in these computations are not a priori ruled out. In any case, since the number of allowed Kac-Moody combinations is only 221, a good strategy is to try first to find modular invariant character combinations, and worry about further consistency checks later. This is certainly sufficient for the classification of $d = 10$ heterotic strings, to which we return now.

All possible Kac-Moody algebras in which $D_{8,1}$ can be embedded occur for $N \geq 360$. Since no new theories appeared for $N \geq 300$ it follows that there are no more ten-dimensional heterotic strings than the already known ones (for the only non-lattice theory, $B_{8,1}E_{8,2}$, one has to verify that there is no other modular invariant, but this is trivial). Thus the main goal of this paper has been accomplished.

In a forthcoming paper the classification of the remaining $c = 24$ theories will be discussed, and, depending on the complexity of the calculations and the available computer time, a more or less complete classification of the modular invariants will be presented.

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