An energy and dynamical systems perspective to the
damping-induced self recovery phenomenon

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1 Introduction

The damping-induced self-recovery phenomenon, partially inspired by Andy Ruina [1], was proved by
Chang, et al. [2, 3], by introducing a new conserved quantity - the damping-induced momentum. If an
unactuated variable is under the influence of linear viscous damping, and the system begins from rest,
then the cyclic variable always goes back to its initial state asymptotically, as the actuated variables
are brought to a halt. This phenomenon is a fascinating one, and defies any kind of intuition, from
pre-existing well-known momenta conservation. A popular example exhibiting this phenomenon is that
of a person sitting on a rotating stool with linear damping, holding a wheel horizontally, which can be
spun and stopped at will. Initially, the system begins from rest, but when the person starts spinning
the wheel (say anticlockwise), the stool begins moving in the expected direction (i.e., clockwise). But
as soon as the wheel is brought to a halt, the stool begins moving backward in the opposite direction
(i.e., anticlockwise), and reaches its initial state asymptotically! Although this kind of behaviour has
been proved using momentum conservation principles, an understanding of the phenomenon using energy
dynamics and a dynamical systems perspective has not been investigated. This is the objective of the
current article.

In the video demonstrating the experiment of the stool-wheel system, by Chang, et al. [2, 3], when
the wheel is stopped, the stool begins moving in the opposite direction and oscillates about its initial
position, reaching its initial state asymptotically. During this motion, the entire system comes to rest
momentarily, at the extreme positions of these oscillations. In order for the system to move back in
the opposite direction from a momentary rest state, energy has to be stored. An initial guess is that,
this “stored energy” is due to damping in the stool, but in this document, we show that no such stored
energy exists. We conclude that this model cannot be used to explain the momentary rest state in the
experiment, and that further investigation is required.

However, the damping-induced self-recovery can be explained via the first order dynamics of the damping-
induced momentum equation. We show that the recovery phenomenon arises from a bifurcation of this
first order system associated with the damping factor $k$. Our motivation for this claim comes from the fact
that the final equilibrium state for the no-damping case and damping case are drastically different. In the
no-damping case, the stool is neutrally stable, whereas in the damping case, the stool is asymptotically
stable.

2 Model and Energy Description

We analyze a simplified, idealized version of the person with a wheel in hand, sitting on a rotatable stool
whose motion is opposed by linear viscous damping. We assume two flat disks, one for the wheel and
one for the stool-person mass as shown in Fig. 1. The stool consists of an internal motor, that actuates
the wheel, while the motor-rod-stool setup rotates as one piece (henceforth just called the stool). There

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is linear viscous damping present in the rotational motion of the stool, with damping coefficient $k$. The force imparted on the wheel by the motor is denoted by $u(t)$.

The inertia matrix of the system is given by

$$ (m_{ij}) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} I_w + I_s & I_w \\ I_w & I_w \end{pmatrix}, \quad (1) $$

where $I_w$ and $I_s$ are the moments of inertia of the wheel and stool respectively. The kinetic energy of the described system is

$$ K.E.(t) = \frac{1}{2} I_w (\dot{\theta}_w + \dot{\phi}_s)^2 + \frac{1}{2} I_s \dot{\phi}_s^2, \quad (2) $$

where $\theta_w$ denotes the angle rotated by the wheel relative to the stool, and $\phi_s$ denotes the angle rotated by the stool relative to the ground frame. Since there is no external potential in our system, the Lagrangian only comprises of the total kinetic energy. The Euler-Lagrange equations for the system are given by

$$ \begin{align*}
(I_w + I_s) \ddot{\phi}_s + I_w \ddot{\theta}_w &= -k \dot{\phi}_s \\
I_w \ddot{\phi}_s + I_w \ddot{\theta}_w &= u 
\end{align*} \quad (3a) $$

$$ \begin{align*}
(I_w + I_s) \ddot{\phi}_s + I_w \ddot{\theta}_w &= -k \dot{\phi}_s \\
I_w \ddot{\phi}_s \dot{\phi}_s + I_w \ddot{\theta}_w \dot{\theta}_w &= u \dot{\theta}_w 
\end{align*} \quad (3b) $$

In order to derive the energy equations of the system, $(3a)$ is multiplied by $\dot{\phi}_s$, and $(3b)$ with $\dot{\theta}_w$, to yield

$$ \begin{align*}
(I_w + I_s) \dot{\phi}_s \dot{\phi}_s + I_w \dot{\theta}_w \dot{\phi}_s &= -k \dot{\phi}_s \dot{\phi}_s \\
I_w \dot{\phi}_s \dot{\phi}_s + I_w \dot{\theta}_w \dot{\theta}_w &= u \dot{\theta}_w 
\end{align*} \quad (4a) $$

$$ \begin{align*}
(I_w + I_s) \dot{\theta}_w \dot{\phi}_s + I_w \dot{\theta}_w \dot{\theta}_w &= -k \dot{\phi}_s \dot{\phi}_s \\
I_w \dot{\phi}_s \dot{\phi}_s + I_w \dot{\theta}_w \dot{\theta}_w &= u \dot{\theta}_w 
\end{align*} \quad (4b) $$

Equations $(4a)$ and $(4b)$ are integrated, and then summed up, resulting in the following energy balance (in the ground frame)

$$ \int_0^t u \dot{\theta}_w \, dt = K.E.(t) + \int_0^t k \dot{\phi}_s^2 \, dt \quad (5) $$

The second term on the right hand side of Eq. $(5)$ represents the cumulative energy lost due to damping losses up to time $t$. The energy lost in damping is precisely equal to the work done by the damping force, that is

$$ L.E.(t) = \int_0^{\phi_s(t)} k \dot{\phi}_s \, d\phi_s = \int_0^t k \dot{\phi}_s^2 \, dt. \quad (6) $$
The term on the left hand side of Eq. (5) represents the total energy pumped into the system by the motor, up to time \( t \). The power input into the system by the motor is the sum of power imparted to the wheel by the actuation force \( u(t) \), and the power imparted to the stool by the reaction force. The cumulative input energy (denoted by I.E.(t)) is given by

\[
I.E.(t) = \int_0^t u(\dot{\theta}_w + \dot{\phi}_s) \, dt + \int_0^t (-u)\dot{\phi}_s \, dt = \int_0^t u\dot{\theta}_w \, dt
\]  

(7)

We can simplify the energy balance (5) as

\[
I.E.(t) = K.E.(t) + L.E.(t)
\]  

(8)

We see that there is no stored energy term in the energy balance, i.e. the input energy is either utilised to change K.E., or is lost due to damping losses.

3 Bifurcation Investigation

The damping-induced momentum equation, as derived by Chang, et al. \[2, 3\], is

\[
(I_w + I_s)\ddot{\phi}_s + I_w\ddot{\theta}_w(t) + k\dot{\phi}_s = 0
\]  

(9)

where \( \dot{\theta}_w(t) \) is written as a function of time, since it is an external driving force for the stool dynamics. (Using partial feedback linearization, the wheel can be controlled independently of the stool). Rewritting Eq. (9) as a first order system in \( \phi_s \)

\[
\dot{\phi}_s = \frac{-k}{I_w + I_s}\phi_s + \frac{-I_w}{I_w + I_s}\ddot{\theta}_w(t) = -\alpha\phi_s - \beta\dot{\theta}_w(t) = f(\phi_s, t)
\]  

(10)

where \( \alpha, \beta \) are constants. Since Eq. (10) is a nonautonomous system, we rewrite it in a higher dimensional autonomous form as

\[
\dot{\phi}_s = -\alpha\phi_s - \beta\dot{\theta}_w(t)  
\]

\[
i = 1
\]

(11a)

(11b)

In the differential equation for the stool, for a fixed speed of the wheel, say \( \dot{\theta}_w(t) = v_w = \) constant, the solution is

\[
\phi_s(t) = -\beta \int_0^t [e^{-\alpha(t-s)}]v_w \, ds = -\beta v_w \frac{1 - e^{-\alpha t}}{\alpha}
\]  

(12)

and as \( t \to \infty \), we have

\[
\phi_s^* = \lim_{t \to \infty} \phi_s(t) = -\frac{\beta v_w}{\alpha} = -\frac{I_w v_w}{k}
\]

This corresponds to the damping-induced boundedness phenomenon, and is especially relevant in the figures that we illustrate. Now we get to the dynamical systems aspect of our discussion.

A fixed point for system (11) does not exist since \( i \neq 0 \) for any state \( (\phi_s, t) \). We will consider \( (\phi_s^*, t^*) \) to be a fixed point if \( f(\phi_s^*, t^*) = 0 \).

When \( k = 0 \), we see that the stool dynamics has a fixed point, only when the wheel is brought to a rest. Since \( \phi_s \) does not appear in the fixed point equation \( f(\phi_s^*, t^*) = 0 \), the stool is neutrally stable.

However, when \( k > 0 \), the fixed point for the stool dynamics is

\[
\phi_s^* = -\frac{I_w}{k}\dot{\theta}_w(t^*)
\]  

(13)

We define \( \phi_s^* \) as a fixed point only if it is constant. This can happen when the wheel has attained a constant velocity, i.e. \( \phi_s^* = -\frac{I_w v_w}{k} \) where \( v_w \) is the constant speed of the wheel (this corresponds to the damping-induced boundedness phenomenon). The other case corresponds to when the wheel is brought to a halt. Clearly, if the wheel is braked, the stool must go back to its initial state, i.e. \( \phi_s^* = 0 \), since

\[
\dot{\phi}_s = -\alpha\phi_s
\]  

(14)
This is the damping-induced self recovery equilibrium state.

Now we proceed to another interpretation of the phenomenon. A dynamical system whose phase behaviour changes significantly beyond a particular value of a parameter of the system, is said to exhibit a bifurcation at this value. In our case, the system undergoes a bifurcation, when $k$ is varied from 0 to something positive. Eq. (13), rewritten as

$$\phi_s^*k = -I_w\dot{\theta}_w(t^*)$$

is the equation of a hyperbola (in $\phi_s^*$ and $k$), and as $k \to 0$, $\phi_s^* \to \infty$, i.e. it blows up. Since $k > 0$, we know that the recovery phenomenon occurs (implying that the stool must return to its initial position regardless how many rotations it has completed). But for $k = 0$, we know that the stool is neutrally stable, so there is a discontinuity in behaviour. (Note that as $\phi_s$ is $2\pi-$periodic, one should consider $\phi_s \text{ mod } 2\pi$ to obtain the actual position of the stool. However, not doing so, gives us the number of rotations the stool completes. For e.g., if $\phi_s^* = 8\pi$, then the damping-induced bound is attained after four rotations)

4 Qualitative interpretation of the damping-induced momentum equation to explain the change in direction of the stool

To get a qualitative picture of why the stool rotates backwards when the wheel is brought to a halt, we consider a case of impulsive acceleration to get the wheel spinning, and impulsive deceleration, to bring the wheel to a halt. Consider the damping-induced momentum equation (9),

$$I_w(\ddot{\theta}_w + \dot{\phi}_s) + I_s(\ddot{\phi}_s) + k\phi_s = 0$$

During the period of initial impulse (to get the wheel spinning), $\phi_s$ does not change much, as shown in Figs. 2, 3. In particular, if we assume the initial acceleration to be Dirac delta-like, we can ignore the $\phi_s$ term in the equation since it is zero. Thus, we get
Figure 3: Phase portrait of stool dynamics with varying values of steady state wheel velocity $v_w$. Innermost to outermost loop: $v_w = 1, 2, 3$ ($k = 1$ for all cases).

\[ I_w(\dot{\theta}_w + \dot{\phi}_s) + I_s(\dot{\phi}_s) = 0 \]  

This is just the usual momentum conservation. Hence, in the this case, during the initial acceleration, the velocity of the stool reaches a velocity as governed by the usual momentum conservation, and the steady state velocity of the wheel. Let this velocity be $\dot{\phi}_s^f$.

In the case of braking, once again, $\dot{\phi}_s$ is approximately constant during the impulsive deceleration. Again, if we assume the impulsive braking to be Dirac delta-like, the system follows

\[ I_w(\dot{\theta}_w + \dot{\phi}_s) + I_s(\dot{\phi}_s) = C \]  

where $C$ is a constant ($C = -k\phi_s$). This can again be interpreted as our usual momentum conservation.

Now we examine three distinct cases:

Case 1 (No damping): If the speed of the stool just before braking was the same as attained at the end of the initial impulse, i.e. $\dot{\phi}_s^f$, then the final velocity of the stool would be zero at the end of the braking. (Recall $k = 0$ in this case). Refer to Fig. 6.

Case 2 (Damping present but damping induced boundedness not yet reached): In the time between the two impulses, the damping force decreases the (magnitude of) velocity of the stool. However, the momentum equation at time of braking is still the same. Thus the change in the momentum of the stool would be the same, as in case 1. But as the magnitude of momentum just before braking has decreased, the final momentum of the stool ends up overshooting over zero, and the stool ends up changing direction. Refer to Fig. 7.

Case 3 (Damping present and damping-induced boundedness has been attained) This is just a special instance of case 2, where in the interval between the initial impulse and braking, the damping reduces the speed of the stool to zero. Everything else follows the same way, as in case 2. Refer to Fig. 8.

Thus, by looking at the damping-induced momentum equation, we can intuitively explain why the
direction of the stool changes upon braking.

References

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Figure 5: Comparison of the three cases before and after braking. Blue dots: velocity of stool just before braking; red dots: velocity of stool immediately after braking. For cases 2, 3, the (magnitude of) velocity of the stool before braking decreases due to damping; hence momentum transferred by the wheel overshoots the stool velocity to something positive. This explains why the wheel changes direction.

Figure 6: Case 1. $k = 0$, $t_{stop} = 5$, $v_w = 2$. This case corresponds to the usual momentum conservation.
Figure 7: Case 2. \( k = 1, t_{stop} = 1, v_w = 2 \). In this case, damping-induced boundedness is not yet reached, but the recovery phenomenon can be seen.

Figure 8: Case 3. \( k = 1, t_{stop} = 5, v_w = 2 \). In this case, both—damping-induced boundedness and recovery phenomenon can be seen.