Comparing \( \text{WO}(\omega^\omega) \) with \( \Sigma_2^0 \) induction

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
University Park, PA 16802, USA
http://www.math.psu.edu/simpson
simpson@math.psu.edu

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Abstract

Let \( \text{WO}(\omega^\omega) \) be the statement that the ordinal number \( \omega^\omega \) is well ordered. \( \text{WO}(\omega^\omega) \) has occurred several times in the reverse-mathematical literature. The purpose of this expository note is to discuss the place of \( \text{WO}(\omega^\omega) \) within the standard hierarchy of subsystems of second-order arithmetic. We prove that \( \text{WO}(\omega^\omega) \) is implied by \( I\Sigma_2^0 \) and independent of \( B\Sigma_2^0 \). We also prove that \( \text{WO}(\omega^\omega) \) and \( B\Sigma_2^0 \) together do not imply \( I\Sigma_2^0 \).

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1 Introduction

In the language of second-order arithmetic, let \( \text{WO}(\omega^\omega) \) be the statement that \( \omega^\omega \) is well ordered.\(^1\) In \([3, 4, 6]\) it was shown that several theorems of abstract algebra, including the Hilbert Basis Theorem, are reverse-mathematically equivalent to \( \text{WO}(\omega^\omega) \). It is therefore of interest to understand the place of \( \text{WO}(\omega^\omega) \) within the usual hierarchy of subsystems of second-order arithmetic \([7, 8]\).

In this expository note we prove the following results.

- \( \text{WO}(\omega^\omega) \) is provable from \( \text{RCA}_0 + \Sigma^0_2 \) induction.
- \( \text{WO}(\omega^\omega) \) and \( \Sigma^0_2 \) bounding are independent of each other over \( \text{RCA}_0 \).
- \( \Sigma^0_2 \) induction is not provable from \( \text{RCA}_0 + \text{WO}(\omega^\omega) + \Sigma^0_2 \) bounding.

These results are perhaps well known and implicit in the literature on fragments of arithmetic \([2, 5]\). Our reason for writing them up here is that, because of \([3, 4, 6]\), they deserve attention in the reverse-mathematical context \([7]\). I thank Keita Yokoyama for explaining these results to me during a visit to Penn State, July 11–16, 2015.

2 \( \Sigma^0_2 \) implies \( \text{WO}(\omega^\omega) \)

In this section we show that \( \text{WO}(\omega^\omega) \) is provable in \( \text{RCA}_0 + \Sigma^0_2 \) but not in \( \text{RCA}_0 + \text{BSigma}^0_2 \). Our arguments in this section have a proof-theoretical flavor.

Definition 2.1. Let \( \Phi \) range over \( \Sigma^0_k \) formulas in the language of second-order arithmetic. Note that \( \Phi \) may contain free number variables and free set variables. We consider the following schemes.

1. \( \Sigma^0_k \) is the \( \Sigma^0_k \) induction principle, i.e., the universal closure of
   \[
   (\Phi(0) \land \forall i (\Phi(i) \Rightarrow \Phi(i + 1))) \Rightarrow \forall i \Phi(i).
   \]
2. \( \text{BSigma}^0_k \) is the \( \Sigma^0_k \) bounding principle, i.e., the universal closure of
   \[
   (\forall i \exists j \Phi(i, j)) \Rightarrow \forall m \exists n (\forall j < m) (\exists j < n) \Phi(i, j).
   \]

Note that \( \Sigma^0_2 \) was called \( \Sigma^0_2 \)-IND in \([7, \text{Remark I.7.9}]\). It is known that \( \Sigma^0_{k+1} \) implies \( \text{BSigma}^0_{k+1} \) and \( \Sigma^0_{k+1} \) implies \( \Sigma^0_k \).

Theorem 2.2. \( \text{WO}(\omega^\omega) \) is provable in \( \text{RCA}_0 + \Sigma^0_2 \).

Proof. We reason in \( \text{RCA}_0 + \Sigma^0_2 \). Assume that \( f \) is a descending sequence through \( \omega^\omega \). Consider the \( \Pi^0_2 \) formula \( \Phi(n, f) \equiv \forall \alpha (\text{if } \exists i (f(i) < \alpha + \omega^n) \text{ then } \exists i (f(i) < \alpha)) \). By \( \Pi^0_2 \) induction on \( n \) we prove \( \forall n \Phi(n, f) \). Trivially \( \Phi(0, f) \) holds. Assume inductively that \( \Phi(n, f) \) holds, and let \( \alpha \) be such that \( \exists i (f(i) < \alpha + \omega^{n+1}) \). Then we have \( \exists m \exists i (f(i) < \alpha + \omega^{n+1} \cdot m) \), so by \( \Pi^0_1 \) induction there is a least such \( m \). If \( m = 0 \) then \( \exists i (f(i) < \alpha) \) and we are done. If \( m = l + 1 \) then \( \exists i (f(i) < \alpha + \omega^n \cdot l + \omega^n) \), so by \( \Phi(n, f) \) we have \( \exists i (f(i) < \omega^\omega \cdot l) \) contradicting our choice of \( m \). We now see that \( \forall n \Phi(n, f) \) holds. For \( \alpha = 0 \) this says that \( \forall n (\text{if } \exists i (f(i) < \omega^n) \text{ then } \exists i (f(i) < 0)) \), or in other words \( \forall n \forall i (f(i) \geq \omega^n) \), contradicting the fact that \( \omega^\omega = \sup_n \omega^n \).

Theorem 2.3. \( \text{WO}(\omega^\omega) \) is not provable in \( \text{RCA}_0 + \text{BSigma}^0_2 \).

\(^1\)More precisely, \( \text{WO}(\omega^\omega) \) is the statement that the standard set of Cantor normal form notations for the ordinal numbers less than \( \omega^\omega \) is well ordered.
Proof. It is known [7, §IX.3] that the provably total recursive functions of RCA$_0$ are just the primitive recursive functions. In particular, totality of the Ackermann function is not provable in RCA$_0$. It is also known [2, Theorem IV.1.59] that RCA$_0$ + B$\Sigma^0_2$ is conservative over RCA$_0$ for $\Pi^0_2$ sentences. Therefore, totality of the Ackermann function is not provable in RCA$_0$ + B$\Sigma^0_2$. On the other hand, totality of the Ackermann function is straightforwardly provable in RCA$_0$ + WO($\omega^\omega$).

Remark 2.4. More generally, for each $k \geq 2$, letting $\omega_k$ be a stack of $\omega$’s of height $k$, it is known that WO($\omega_k$) is provable in RCA$_0$ + I$\Sigma^0_k$ and not provable in RCA$_0$ + B$\Sigma^0_k$. These results belong to Gentzen-style proof theory.

3 WO($\omega^\omega$) does not imply B$\Sigma^0_2$

In this section we show that B$\Sigma^0_2$ is not provable in RCA$_0$ + WO($\omega^\omega$). Our arguments in this section and the next have a model-theoretical flavor.

Definition 3.1. I$\Sigma_k$ and B$\Sigma_k$ consist of basic arithmetic plus the respective restrictions of I$\Sigma^0_k$ and B$\Sigma^0_k$ to the language of first-order arithmetic [2]. It is known that I$\Sigma_{k+1}$ implies B$\Sigma_{k+1}$ and B$\Sigma_{k+1}$ implies I$\Sigma_k$.

Remark 3.2. In the language of first-order arithmetic, let $\Phi(x)$ be a $\Sigma_{k+1}$ formula with a distinguished free variable $x$. Write $\Phi(x)$ as $\exists y \Theta(x,y)$ where $\Theta(x,y)$ is a $\Pi_k$ formula. Let $\overline{\Phi}(x)$ be the $\Sigma_{k+1}$ formula

$$\exists z ((z)_1 = x \land \Theta((z)_1, (z)_2) \land \neg(\exists w < z) \Theta(((w)_1, (w)_2)))$$

The universal closures of the following are provable in I$\Sigma_k$.

1. $\forall x (\overline{\Phi}(x) \Rightarrow \Phi(x))$.
2. $\forall x \forall x' ((\overline{\Phi}(x) \land \overline{\Phi}(x')) \Rightarrow x = x')$.
3. $(\exists x \Phi(x)) \Rightarrow (\exists x \overline{\Phi}(x))$.

Items 1 and 2 are trivial, and for item 3 we use I$\Sigma_k$ to prove the existence of $z$. See also the discussion of “special” $\Sigma_{k+1}$ formulas in [2, §IV.1(d)]. The passage from $\Phi(x)$ to $\overline{\Phi}(x)$ will be referred to as uniformization with respect to the variable $x$.

Lemma 3.3. In the language of first-order arithmetic, let $\Psi$ be a $\Pi_3$ sentence. If I$\Sigma_1$ + $\Psi$ is consistent, then I$\Sigma_1$ + $\Psi$ does not prove B$\Sigma_2$.

Proof. Let $M$ be a nonstandard model of I$\Sigma_1$ + $\Psi$. Fix a nonstandard element $c \in M$. By Remark 3.2 we know that every nonempty subset of $M$ which is $\Sigma_2(M)$-definable from $c$ contains an element which is $\Sigma_2(M)$-definable from $c$. Hence

$$M_2 = \{ x \in M \mid x \text{ is } \Sigma_2(M)\text{-definable from } c \}$$

is a $\Sigma_2$-elementary submodel of $M$. Therefore, since $\Psi$ is a $\Pi_3$ sentence, $M_2$ satisfies $\Psi$. And likewise, since I$\Sigma_1$ is axiomatized by $\Pi_3$ sentences, $M_2$ satisfies I$\Sigma_1$. We shall finish the proof by showing that $M_2$ does not satisfy B$\Sigma_2$.

Let $\Phi(e,x,c)$ be a $\Sigma_2$ formula which is universal in sense that, as $e$ ranges over the natural numbers, $\Phi(e,x,c)$ ranges over all $\Sigma_2$ formulas with one free variable $x$ and one parameter $c$. For each $x \in M_2$ we know that $x$ is $\Sigma_2(M_2)$-definable from $c$,

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2See also the proofs of Lemma 4.1 and Theorem 4.2 below.
\[ \Phi(x, y, z) \]

i.e., there exists a natural number \( e \) such that \( x \) is the unique element of \( M_2 \) such that \( M_2 \) satisfies \( \Phi(e, x, c) \). Moreover, since \( e \) is a natural number and \( c \) is nonstandard, we have \( e < c \). Uniformizing with respect to \( x \), we see that \( M_2 \) satisfies \( \overline{\Phi}(e, x, c) \) for all such pairs \( e, x \). Uniformizing again with respect to \( e \), we see that for each \( x \in M_2 \) there is exactly one \( e = e_x \in M_2 \) such that \( e < c \) and \( M_2 \) satisfies \( \Phi(e, x, c) \). We now have a mapping \( x \mapsto e_x \) which is \( \Delta_2(M_2) \)-definable from \( c \) and maps \( M_2 \) one-to-one into \( \{ e \in M_2 \mid e < c \} \). If \( M_2 \) were a model of \( \mathcal{B}_2 \), then the restriction of \( x \mapsto e_x \) to \( \{ x \in M_2 \mid x \leq c \} \) would be \( M_2 \)-finite, so we would have an \( M_2 \)-finite mapping of the \( M_2 \)-finite set \( \{ x \in M_2 \mid x \leq c \} \) into its \( M_2 \)-finite proper subset \( \{ e \in M_2 \mid e < c \} \). This contradiction shows that \( M_2 \) cannot satisfy \( \mathcal{B}_2 \). \( \Box \)

**Theorem 3.4.** In the language of second-order arithmetic, let \( \exists X \forall Y \Psi(X, Y) \) be a \( \Sigma^2_2 \) sentence such that \( \Psi(X, Y) \) is \( \Pi^1_3 \). If \( \text{RCA}_0 + \exists X \forall Y \Psi(X, Y) \) is consistent, then \( \text{RCA}_0 + \exists X \forall Y \Psi(X, Y) \) does not prove \( \mathcal{B} \Sigma^0_2 \).

**Proof.** Consider the \( \Pi^1_3 \) formula \( \overline{\Psi}(X) \equiv \forall Y \left( Y \leq_T X \Rightarrow \Psi(X, Y) \right) \). We may view \( \overline{\Psi}(X) \) as a \( \Pi^1_3 \) sentence in the language of first-order arithmetic with an extra unary predicate \( X \). Let \( (M, X_M) \) be a nonstandard model of \( \Sigma_1(X) + \overline{\Psi}(X) \). As in the proof of Lemma 3.3, fix a nonstandard \( c \in M \) and let \( M_2 = \{ x \in M \mid x \text{ is } \Sigma_2(M, X_M) \text{-definable from } c \} \). Also as in the proof of Lemma 3.3, we have that \( (M_2, X_M \cap M_2) \) satisfies \( \Sigma_1(X) + \overline{\Psi}(X) \) and does not satisfy \( \mathcal{B}_2(X) \). Passing to the language of second-order arithmetic, it follows by [7, §IX.1] that \( (M_2, \Delta_1(M_2, X_M \cap M_2)) \) satisfies \( \text{RCA}_0 + \exists X \forall Y \Psi(X, Y) \) and does not satisfy \( \mathcal{B} \Sigma^0_2 \). \( \Box \)

**Corollary 3.5.** \( \text{RCA}_0 + \text{WO}(\omega^\omega) \) does not prove \( \mathcal{B} \Sigma^0_2 \). More generally, for any primitive recursive linear ordering \( \alpha \) of the natural numbers, if \( \text{RCA}_0 + \text{WO}(\alpha) \) is consistent then \( \text{RCA}_0 + \text{WO}(\alpha) \) does not prove \( \mathcal{B} \Sigma^0_2 \).

**Proof.** \( \text{WO}(\alpha) \) can be written in the form \( \forall Y \Psi(X, Y) \) where \( \Psi(X, Y) \) is as in the hypothesis of Theorem 3.4. Our corollary is then a special case of Theorem 3.4. \( \Box \)

### 4 \( \text{WO}(\omega^\omega) + \mathcal{B} \Sigma^0_2 \) does not imply \( \Sigma^0_2 \)

In this section we show that \( \Sigma^0_2 \) is not provable in \( \text{RCA}_0 + \text{WO}(\omega^\omega) + \mathcal{B} \Sigma^0_2 \).

**Lemma 4.1.** In the language of first-order arithmetic, let \( \Psi \) be a \( \Pi^1_3 \) sentence. If \( \mathcal{B}_2 + \Psi \) is consistent, then \( \mathcal{B}_2 + \Psi \) does not prove \( \Sigma_2 \).

**Proof.** Let \( M \) be a nonstandard model of \( \mathcal{B}_2 + \Psi \). As in the proof of Lemma 3.3, fix a nonstandard element \( e \in M \) and consider the \( \Sigma_2 \)-elementary submodel \( M_2 = \{ a \in M \mid a \text{ is } \Sigma_2(M) \text{-definable from } c \} \). We may safely assume\(^3\) that \( M_2 \) is not cofinal in \( M \). We shall show that the submodel

\[
\bar{M}_2 = \{ x \in M \mid (\exists a \in M_2) (x < a) \}
\]

satisfies \( \mathcal{B}_2 + \Psi + \neg \Sigma_2 \).

Claim 1: \( \bar{M}_2 \) is a \( \Sigma_1 \)-elementary submodel of \( M \). To see this, let \( \Phi(x) \) be a \( \Sigma_1 \) formula with no free variables other than \( x \). Given \( u \in \bar{M}_2 \) such that \( M_2 \) satisfies \( \Phi(u) \), we need to show that \( \bar{M}_2 \) satisfies \( \Phi(u) \). Write \( \Phi(x) \) as \( \exists y \Theta(x, y) \) where \( \Theta(x, y) \) is

\(^3\)For instance, this would be the case if \( M \) is countably saturated, or if \( M \) satisfies \( \Sigma_2 \).
3.3

that with no free variables other than \( x \). The assumption that used in those proofs, but it will be used in the proof of Claim 4.

suffice to show that \((\exists y < b) \Phi(x,y)\) (where \(\Phi(x,y)\) is \(\Pi_0\)) gives a \(\Sigma_2\)-definable mapping from a bounded subset of \( M \) onto \( M \). Since \( M \) is a cofinal \(\Sigma_2\)-elementary submodel of \( \hat{M} \), this same formula gives a \( \Sigma_2(\hat{M}_2)\)-definable mapping from a bounded subset of \( \hat{M}_2 \) onto an unbounded subset of \( \hat{M}_2 \). This implies that \( \hat{M}_2 \) does not satisfy \( \Sigma_2 \).

As a point of interest, note that our proofs of Claims 1 through 3 used only the assumption that \( M \) satisfies \( \Sigma_1 \). The assumption that \( M \) satisfies \( \Sigma_2 \) was not used in those proofs, but it will be used in the proof of Claim 4.

Claim 4: \( \hat{M}_2 \) satisfies \( \Theta \). To see this, write \( \Psi \) as \( \forall x \exists y \Phi(x,y) \) (where \( \Phi(x,y) \) is \(\Pi_1\)) with no free variables other than \( x \) and \( y \). We need to show that \( \forall x (\exists y \in \hat{M}_2) \Phi(x,y) \) (where \( \Phi(x,y) \) is \(\Pi_1\)). By Claim 1 plus the fact that \( M \) is cofinal in \( \hat{M}_2 \), it will suffice to show that \( \forall x \in \hat{M}_2 \exists y \in \hat{M}_2 \exists a \in M \exists \Phi(x,y) \). Fix \( a \in M \). Since \( M \) satisfies \( \Sigma_2 + \forall x \exists y \Phi(x,y) \), there exists \( b \in M \) such that \( M \) satisfies \( \forall x < a \exists y < b \Phi(x,y) \). But then, because \( M \) is a \(\Sigma_2\)-elementary submodel of \( M \) and \( b \) belongs to \( M \), there exists such a \( b \) which also belongs to \( M \).

**Theorem 4.2.** In the language of second-order arithmetic, let \( \exists X \forall Y \Psi(X,Y) \) be a \( \Sigma_2 \) sentence such that \( \Psi(X,Y) \) is a \( \Pi_0^0 \) formula. If \( \text{RCA}_0 + \text{BS}_2^0 + \exists X \forall Y \Psi(X,Y) \) is consistent, then \( \text{RCA}_0 + \text{BS}_2^0 + \exists X \forall Y \Psi(X,Y) \) does not prove \( \Sigma_2^0 \).

**Proof.** As in the proof of Theorem 3.4, we may view the \( \Pi_0^0 \) formula \( \bar{\Psi}(X) \equiv \forall Y (Y \leq_T X \Rightarrow \Psi(X,Y)) \) as a \( \Pi_1 \) sentence in the language of first-order arithmetic with an extra unary predicate \( X \). As in the proof of Lemma 4.1, let \( (M, X_M) \) be a nonstandard model of \( \text{BS}_2(X) + \bar{\Psi}(X) \), fix a nonstandard \( c \in M \), let \( M_2 = \{ a \in M \mid a \in \Sigma_3(\hat{M}_2, X_M) \text{definable from } c \} \), and let \( \hat{M}_2 = \{ x \in M \mid (\exists a \in M_2) (x < a) \} \). Also as in the proof of Lemma 4.1, we have that \((\hat{M}_2, X_M \cap \hat{M}_2) \) satisfies \( \text{BS}_2(X) + \bar{\Psi}(X) + \neg \Sigma_2(X) \). Passing to the language of second-order arithmetic, it follows by [7, §IX.1] that \((\hat{M}_2, X_M \cap \hat{M}_2) \) satisfies \( \text{RCA}_0 + \text{BS}_2^0 + \exists X \forall Y \Psi(X,Y) \).}

**Corollary 4.3.** \( \text{RCA}_0 + \text{BS}_2^0 + \text{WO}(\omega) \) does not prove \( \Sigma_2^0 \). More generally, for any primitive recursive linear ordering \( \alpha \) of the natural numbers, if \( \text{RCA}_0 + \text{BS}_2^0 + \text{WO}(\alpha) \) is consistent then \( \text{RCA}_0 + \text{BS}_2^0 + \text{WO}(\alpha) \) does not prove \( \Sigma_2^0 \).
**Proof.** \( \text{WO}(\alpha) \) can be written as \( \forall Y \Psi(Y,Y) \) where \( \Psi(X,Y) \) is as in the hypothesis of Theorem 4.2. Our corollary is then a special case of Theorem 4.2.

**Remark 4.4.** Theorems 3.4 and 4.2 and Corollaries 3.5 and 4.3 hold more generally, for all \( k \geq 2 \), replacing \( \Sigma^0_2 \) by \( \Sigma^0_k \) and \( \Pi^0_3 \) by \( \Pi^0_{k+1} \), with essentially the same proofs.

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