LIMIT OPERATOR THEORY FOR GROUPOIDS

KYLE AUSTIN AND JIAWEN ZHANG

Abstract. We extend the symbol calculus and study the limit operator theory for \(\sigma\)-compact, étale and amenable groupoids, in the Hilbert space case. This approach not only unifies various existing results which include the cases of exact groups and discrete metric spaces with Property A, but also establish new limit operator theories for group/groupoid actions and uniform Roe algebras of groupoids. In the process, we extend a monumental result by Exel, Nistor and Prudhon, showing that the invertibility of an element in the groupoid \(C^*\)-algebra of a \(\sigma\)-compact amenable groupoid with a Haar system is equivalent to the invertibility of its images under regular representations.

1. Introduction

Recall that Atkinson’s Theorem says that a bounded operator \(T\) on \(\ell^p(\mathbb{Z}^n)\) is Fredholm if and only if it is invertible modulo the compact operators. Thus, no finite portion of its matrix coefficients, thinking of \(T\) as a \(\mathbb{Z}^n\)-by-\(\mathbb{Z}^n\) matrix \((T_{x,y})_{x,y\in\mathbb{Z}^n}\), has any impact on the Fredholmness of \(T\). This suggests that the Fredholmness of an operator only depends on the asymptotic behaviour of its matrix coefficients. To carry out this idea, we mainly focus on band-dominated operators. Recall that an operator \(T\) on \(\ell^p(\mathbb{Z}^n)\) is a band operator if all non-zero entries in its matrix sit within a fixed distance from the diagonal, and that \(T\) is a band-dominated operator if it is a norm-limit of band operators. To study the asymptotic behaviour of a band-dominated operator \(T\), we associate a bounded operator \(T_\omega\) on \(\ell^p(\mathbb{Z}^n)\) to each limit point \(\omega\) in the Stone-Čech boundary \(\beta\mathbb{Z}^n \setminus \mathbb{Z}^n\), called the limit operator of \(T\) at \(\omega\). More precisely, given a net \(g_\alpha\) in \(\mathbb{Z}^n\) converging to \(\omega\), the operator \(T_\omega\) has \((x,y)\)-coefficient equal to \(\lim_\alpha -\omega T(g_\alpha + x, g_\alpha + y)\), i.e., \(T_\omega\) is the WOT-limit of the operators \(U_{g_\alpha}TU_{g_\alpha^{-1}}\) “at the \(\omega\)-direction”, where \(U_{g_\alpha}\) denotes the canonical shift operator on \(\ell^p(\mathbb{Z}^n)\) by \(g_\alpha\).

The above discussion leads to the central problem in limit operator theory: to study the Fredholmness of a band-dominated operator on the discrete domain \(\mathbb{Z}^n\) in terms of its limit operators. Major work has been done around it by Lange, Rabinovich, Roch, Roe and Silbermann [LR85, RRS98, RRS04, RRR04] with several recent contributions by Chandler-Wilde, Lindner, Seidel and others [Lin06, CWL11, LS14, Sei14]. The following fundamental theorem was proved by Rabinovich, Roch, Silbermann in the case of \(p \in (1, \infty)\) and later by Lindner filling the gaps for \(p \in \{0, 1, \infty\}\).

**Theorem 1.1** ([RRS04, Lin06]). Let \(T\) be a band-dominated operator on \(\ell^p(\mathbb{Z}^n)\), where \(p \in \{0\} \cup [1, \infty]\). Then \(T\) is Fredholm if and only if all the associated limit operators \(T_\omega\) for \(\omega \in \beta\mathbb{Z}^n \setminus \mathbb{Z}^n\) are invertible and their inverses are uniformly bounded in norm.

Concerning the uniform boundedness condition in the above theorem, it had been a longstanding question whether this can be removed, and recently Lindner and Seidel [LS14] provided an affirmative answer. Also notice that in some literature (for example [RRS04, CWL11]), Theorem 1.1 is stated in a Banach space valued version, making it possible to handle continuous underlying spaces as well.

Roe realised that it is essentially the coarse geometry of \(\mathbb{Z}^n\) that plays a key role in the limit operator theory. Note that for coarse geometers, the algebra of band-dominated operators is called the uniform Roe algebra. In [Roe05], he extended the symbol calculus and generalised limit operators in the Hilbert space case to all discrete groups, and proved Theorem 1.1 for exact discrete groups (also see [Wll09]). Following Roe’s philosophy, Georgescu [Geo11] generalised limit operator theory to metric spaces with...
Property A in the Hilbert case (also see [GI02, GI06]). Recall that the notion of Property A was introduced by Yu [Yu00], and is equivalent to exactness in the case of groups. Later on, Špakula and Willett [SW17] studied the general $\ell^p$-case for discrete metric spaces with Property A, and established the limit operator theory for $p \in (1, \infty)$. The remaining extreme cases of $p \in \{0, 1, \infty\}$ in the discrete metric setting was recently filled up by the second author [Zha18]. Notice that in Špakula, Willett and Zhang’s work [SW17, Zha18], they also showed that the condition of uniform boundedness can be removed, extending Lindler and Seidel’s result [LS14] mentioned above.

In this paper, we generalise the above idea and establish the limit operator theory in the Hilbert space case for a locally compact, $\sigma$-compact, étale and amenable groupoid with compact unit space. In the process, we extend the symbol calculus, using the unit space of the underlying groupoid as the “spectrum”. More precisely, we prove the following (details will be provided later):

**Theorem 1.2** (Theorem 4.4). Let $\mathcal{G}$ be a locally compact, $\sigma$-compact and étale groupoid with compact unit space $\mathcal{G}^{(0)}$, $X$ be an invariant open dense subset in $\mathcal{G}^{(0)}$ and $\partial X = \mathcal{G}^{(0)} \setminus X$. Suppose the reduction groupoid $\mathcal{G}(\partial X)$ is topologically amenable. Then for any element $T$ in the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G})$, the following conditions are equivalent:

1. $T$ is invertible modulo $C^*_r(\mathcal{G}(X))$.
2. The image of $T$ under the symbol morphism is invertible.
3. All limit operators $\lambda_\omega(T)$ for $\omega \in \partial X$ are invertible.

We use the formalism of groupoids to unify several existing limit operator theories in the Hilbert space case including those of exact groups and discrete metric spaces with Property A (see Section 6.1-6.3). Meanwhile, new limit operator theories are derived from the above theorem, including the case of group actions (Corollary 6.24) and uniform Roe algebras of groupoids (Corollary 6.30 and 6.31).

At this point, some readers might wonder why we regard the above as the limit operator theory for groupoids. Let us explain here briefly in the group case, and more details will be provided in Section 6. Let $G$ be a discrete group. It is well known that the uniform Roe algebra is isomorphic to the reduced groupoid $C^*$-algebra of the transformation groupoid $\beta G \rtimes G$, whose unit space has a natural decomposition: $\beta G = G \cup \partial G$. And it is a classic result that $G$ is exact if and only if $\beta G \rtimes G$ is amenable. Furthermore from definition, the reduced groupoid $C^*$-algebra of $G \rtimes G$ is exactly the algebra of compact operators on $\ell^2(G)$. Hence condition (1) in Theorem 1.2 is nothing but $T$ being Fredholm, and it is not hard to show that limit operators coincide with the ones defined by Roe [Roe05]. Therefore, Theorem 1.2 recovers the classic limit operator theory for exact groups. Similar explanation for the case of discrete metric spaces with Property A was also provided in [SW17, Appendix C].

We would also like to mention several related recent works on Fredholmness and groupoids, but with different focus from ours. In [CNQ17, CNQ18], Carvalho, Nistor and Qiao introduced Fredholm Lie groupoids to study the Fredholmness of pseudo-differential operators on open manifolds. And in [MN18a, MN18b], Măntoiu and Nistor studied the (essential) spectral theory by general twisted groupoid methods, which is closely related to [BLLS17] and [BBDN18] in the case of tail equivalence groupoids of dynamical systems over finitely generated groups. We apologise for the many missing references, but guide the interested readers to the listed literature and the references therein.

There are several novelties about our approach that do not exist in previous literatures. For one, we construct the receptacle for the generalised symbol map using the language of operator fiber spaces established in Section 3. Recall that in the case of groups, all limit operators live in the same space (i.e., the uniform Roe algebra of the group), and can be combined into the symbol morphism. Unfortunately, in the groupoid setting of Theorem 1.2, limit operators may not live in the same space any more, which already occurred in the case of discrete metric spaces [SW17]. To overcome, we introduce the notion of operator fibre spaces and construct the symbol of an operator as an element in the $C^*$-algebra of sections of a bundle, whose fibres are the uniform Roe algebras of the source fibers with respect to uniformly chosen coarse structures (Definition 4.2). In a future work, we aim to look at the index formulas for K-theory and KK-theory as investigated in [Wil09], especially in the case of a groupoid acting on its Higson fiberwise compactification (see Example 2.15).

Another novelty is the omission of the uniform bounded condition in Theorem 1.2, generalising Lindler, Seidel’s result for $\mathbb{Z}^n$, and Špakula, Willett and the second author’s result for discrete metric
spaces in the Hilbert space case. Their approach is more coarse geometric, depending heavily on Property A of the underlying spaces, while ours is quite different and more analytic, depending on an improvement of Exel’s result. Briefly speaking, Exel [Exe14] proved that for a second countable, étale and amenable groupoid with compact unit space, an element in the groupoid $C^*$-algebra is invertible if and only if its images under regular representations are invertible (later generalised by Nistor and Prudhon [NP17]). Applying his result directly, we may omit the uniform bounded condition in the case of second countability. Unfortunately, as we already pointed out, even the groupoid $\beta\mathbb{Z}^n \times \mathbb{Z}^n$ associated to $\mathbb{Z}^n$, is not second countable. Notice, however, that these groupoids are usually $\sigma$-compact (they always are when the group or the metric space is countable), so we strengthen the approximation results (see Theorem 5.25) by the first author and Georgescu [AG19] and prove the following:

**Theorem 1.3 (Theorem 5.3).** Let $\mathcal{G}$ be a locally compact, $\sigma$-compact and amenable groupoid with a Haar system, then the element $1+a \in C^*_r(\mathcal{G})^+$ is invertible if and only if $1+\lambda_x(a)$ is invertible for every $x \in \mathcal{G}^{(0)}$, where $\lambda_x$ is the regular representation at $x$.

The above theorem is more than enough for Theorem 1.2, and we choose to prove in its full generality since this result might be of independent interest. In fact, the property stated in Theorem 1.3 was first studied by Roch [Roc03] where he introduced the notion of invertibility sufficient family and strictly norming family. These are also closely related to the notion of exhausting family introduced in [NP17] and the generalized Effros-Hahn property [IW09]. Interested readers can refer to their literatures and the references therein.

We would also like to highlight the new limit operator theory for uniform Roe algebras of groupoids which we obtain as a consequence of Theorem 1.2 (see Section 6.5). Recall that the notion of uniform Roe algebras of groupoids was introduced by Anantharaman-Delaroche [AD16] in her study of the exactness for groupoids. Also notice that there is another closed related but quite different notion called the Roe $C^*$-algebras of groupoids recently introduced by Tang, Willett and Yao [TWY18] in their study of analytic indices of elliptic differential operators on Lie groupoids. Here we only focus on the former notion, and analogous to the classic case, we show in Corollary 6.30 that the Fredholmness of an operator in the sense of Hilbert module is equivalent to the invertibilities of all the associated limit operators. The technical part here is to provide a detailed and practical description for the limit operators (Lemma 6.29), rather than just regarding them as the images under regular representations. Although unintuitive phenomena might occur (see [AD14, Example 2.3]), we bypass these technical difficulties via the universal property of the Stone-Čech fibrewise compactification together with the technical Lemma 3.12.

Finally, let us mention some other potential generalisations which readers might wish to continue. Firstly, in this paper, we only consider the case of $p = 2$ while limit operator theories are usually establish for general $p \in \{0\} \cup [1, \infty]$, as well as for other classes of Banach spaces coefficients. The reason we discard the general case is that groupoid $C^*$-algebras have been developed mostly for the Hilbert space case. However, there should be alternative ways available in the general $p$-case, especially for the uniform Roe algebras of groupoids (our work already suggests what the limit operator theory might be). Another facet of generalisation is that one could look at other operator algebras which are naturally filtered by groupoids, rather than just the groupoid $C^*$-algebras. These algebras might include $C^\ast$-dynamics, twisted groupoid $C^\ast$-algebras, or even algebras of sections of a Fell bundle over a groupoid or a semigroupoid. The motivation comes from the fact that not every $C^\ast$-algebra is a groupoid $C^\ast$-algebra [BS17], however, every Elliott classifiable $C^\ast$-algebra is the twisted $C^\ast$-algebra of an amenable groupoid [Li17]. In other words, groupoid $C^\ast$-algebras do encompass a nice class of $C^\ast$-algebras, but there is still much room for generalisation.

**Outline.** The paper is organised as follows: In Section 2, we recall basic notions related to groupoids that we will use throughout the paper. In Section 3, we introduce the notion of operator fibre spaces which will be used as an engine for the symbol map, and study their topologies and other related properties. Then in Section 4, we construct the symbol morphism (Definition 4.2), state our limit operator theory for groupoids (Theorem 1.2), and prove the relatively easier part. Section 5 is mainly devoted to the rest of the proof that uniform bounded condition is unnecessary. We do so by generalising Exel’s result into a full generality (Theorem 1.3). Finally in Section 6, we apply Theorem 1.2 to recover


the limit operator theories in the Hilbert space case for exact groups and discrete metric spaces with Property A (Section 6.1-6.3), and also establish new theories for group actions (Section 6.4) and uniform Roe algebras of groupoids (Section 6.5).

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2. Preliminaries

2.1. Basic notions. Let us start by recalling some basic notions and terminology on groupoids. The readers are supposed to be familiar with basic definitions about groupoids, and details can be found in [Ren80], or [Sim17] in the étale case.

Definition 2.1. A groupoid is a small category, in which every morphism is invertible. More specifically, a groupoid consists of a set \( G \), a subset \( G(0) \) called the unit space, two maps \( s, r : G \to G(0) \) called the source and range maps respectively, a composition law

\[
G(2) := \{(\gamma_1, \gamma_2) \in G \times G : s(\gamma_1) = r(\gamma_2)\} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in G,
\]

and an inverse map \( \gamma \mapsto \gamma^{-1} \). These operations satisfy a couple of axioms, including associativity law and the fact that elements in \( G(0) \) act as units. For \( x \in G(0) \), denote \( G^x := r^{-1}(x) \) and \( G_x := s^{-1}(x) \). A groupoid morphism is a functor.

A locally compact groupoid is a groupoid equipped with a locally compact topology such that the structure maps (composition and inverse) are continuous with respect to the induced topologies, and the range map is open (which is equivalent to the source map being open). Note that the latter is a necessary condition for the existence of a Haar system of measures (see Definition 2.2 below).

Let \( Y \) be a subset in the unit space \( G(0) \) of a groupoid \( G \), and we set \( G(Y) := r^{-1}(Y) \cap s^{-1}(Y) \). Note that \( G(Y) \) is a subgroupoid of \( G \) (in the sense that it is stable under product and inverse), called the reduction of \( G \) by \( Y \). \( Y \) is said to be invariant if \( r^{-1}(Y) = s^{-1}(Y) \). When \( Y \) is an invariant locally compact subset of \( G(0) \), it is obvious that the reduction \( G(Y) \) is itself a locally compact groupoid.

From now on, we always assume locally compact spaces to be Hausdorff. Given a locally compact space \( Z \), denote \( C_0(Z) \) the \( C^* \)-algebra of bounded continuous complex-valued functions on \( Z \), \( C_r(Z) \) the \( C^* \)-subalgebra of functions vanishing at infinity, and \( C_c(Z) \) those with compact support. The support of a function \( f \) is denoted by \( \text{supp}(f) \).

Definition 2.2. Let \( G \) be a locally compact groupoid. A (right) Haar system of measures on \( G \) is a collection \( \{\mu_x : x \in G(0)\} \) of positive regular Radon\(^1\) measures on \( G \) such that:

1. The support of \( \mu_x \) is exactly \( G_x \), for all \( x \in G(0) \).
2. For all \( f \in C_c(G) \), the function \( x \to \int f(\gamma) d\mu_x(\gamma) \) is continuous on \( G(0) \).
3. For all \( \gamma \in G \) and \( f \in C_c(G) \), the following equality holds:

\[
\int_{G_{r(\gamma)}} f(\gamma') d\mu_{r(\gamma)}(\gamma') = \int_{G_{s(\gamma)}} f(\gamma') d\mu_{s(\gamma)}(\gamma').
\]

Definition 2.3. Let \( G, H \) be locally compact groupoids with Haar systems of measures \( \{\mu_x : x \in G(0)\} \) and \( \{\nu_y : y \in H(0)\} \). A groupoid morphism \( q : G \to H \) is said to be Haar system preserving if \( q \) is proper, continuous, and satisfying \( q_* \mu_x = \nu_y \) for any \( y \in H(0) \) and \( x \in g^{-1}(y) \). Recall that \( q_* \mu_x \) is the pushforward of \( \mu_x \), i.e., \( (q_* \mu_x) (A) := \mu_x (q^{-1}(A)) \) for any measurable set \( A \).

A locally compact groupoid is called étale (also called \( r \)-discrete) if the range (hence the source) map is a local homeomorphism. Clearly in this case, each fibre \( G^x \) (and \( G_x \) is discrete with the induced topology, and \( G(0) \) is clopen in \( G \). The notion of étaleness for a groupoid can be regarded as

\(^1\)By this, we mean that the measure is locally finite and is inner and outer regular with respect to subsets of finite measure.
an analogue of discreteness in the group case. Note that for a locally compact étale groupoid, we may always choose the counting measure on each fibre to form a canonical Haar system of measures.

2.2. Fibre spaces and fibrewise compactifications. Later on, we also need to consider more general notions than groupoids, called fibre spaces. Here we provide a brief introduction.

Definition 2.4. Let $X$ be a locally compact space. A fibre space over $X$ is a pair $(Y, p)$, where $Y$ is a locally compact space and $p$ is a continuous surjective map from $Y$ onto $X$. For each point $x \in X$, denote $Y^x$ the fibre $p^{-1}(x)$. We say $(Y, p)$ is fibrewise compact if $p$ is proper, i.e., $p^{-1}(K)$ is compact for any compact $K \subseteq X$.

Definition 2.5. A morphism between two fibre space $(Y_1, p_1)$, $(Y_2, p_2)$ over $X$ is a continuous map $\varphi : Y_1 \to Y_2$ such that $p_1 = p_2 \circ \varphi$.

Definition 2.6. A fibrewise compactification of a fibre space $(Y, p)$ over $X$ is a fibrewise compact fibre space $(Z, q)$ over $X$ together with a morphism $\iota : Y \to Z$ such that the image of $Y$ is an open dense subset in $Z$ and $\iota$ is a homeomorphism onto its image. Usually we regard $Y$ as a subset of $Z$ in this case, and the morphism $\iota$ is just the inclusion.

To understand fibrewise compactifications in a more precise way, we need to refer to Gelfand spectra. Let $(Y, p)$ be a fibre space over $X$. Denote

\[ p^* C_0(X) := \{ f \circ p : f \in C_0(X) \}, \]

and $C_0(Y, p)$ to the closure of

\[ C_{\iota}(Y, p) := \{ g \in C_b(Y) : \exists \text{ compact } K \subseteq X \text{ such that } g(x) = 0 \text{ for } x \notin p^{-1}(K) \}. \]

Equivalently, a function $g \in C_0(Y)$ belongs to $C_0(Y, p)$ if and only if for any $\varepsilon > 0$, there exists a compact $K \subseteq X$ such that $|g(x)| < \varepsilon$ for any $x \notin p^{-1}(K)$. And we have the following characterisation:

Proposition 2.7 (Proposition 1.2, [AD14]). For a fibre space $(Y, p)$ over $X$, its fibrewise compactifications are one-to-one corresponding to Gelfand spectra of $C^*$-subalgebras in $C_0(Y, p)$ containing $p^* C_0(X) + C_0(Y)$.

From the above proposition, there are two extreme cases $p^* C_0(X) + C_0(Y)$ and $C_0(Y, p)$. They correspond to the fibrewise Alexandroff compactification, denoted by $(Y^+, p^*)$, and the fibrewise Stone-Čech compactification, denoted by $(\beta_p Y, p_\beta)$. As the classic Stone-Čech compactifications, we have the following universal property:

Proposition 2.8 (Proposition 1.4, [AD14]). Let $(Y, p)$ and $(Y_1, p_1)$ be two fibre spaces over the same space $X$, and $(Y_1, p_1)$ is fibrewise compact. Then for any morphism $\varphi : Y \to Y_1$, there exists a unique morphism $\Phi : (\beta_p Y, p_\beta) \to (Y_1, p_1)$ extending $\varphi$.

Please be aware of the unintuitive phenomenon noticed by Anantharaman-Delaroche [AD14, Example 2.3]. Let $(Y, p)$ be a fibre space over $X$, and $(\beta_p Y, p_\beta)$ be its fibrewise Stone-Čech compactification. Given an $x \in X$, it might occur that the fibre $Y^x$ is not dense in the fibre $p^{-1}_\beta(x)$ of $\beta_p Y$.

2.3. Groupoid actions. Now we discuss groupoid actions, and several related notions. We start with the case of groups. Let $G$ be a discrete group acting on some locally compact space $X$ by homeomorphisms. The transformation groupoid $X \rtimes G$ is set-theoretically $X \times G$. The groupoid structure is given by $s(x, \gamma) = (\gamma^{-1} x, e)$, $r(x, \gamma) = (x, e)$ and $(x, \gamma) (\gamma^{-1} x, \gamma') = (x, \gamma \gamma')$. The topology on $X \rtimes G$ is nothing but the product topology, and it is clear that the groupoid $X \rtimes G$ is étale since the group $G$ is discrete.

Now we introduce groupoid actions and discuss the associated semi-direct products.

Definition 2.9. For two fibre spaces $(Y_i, p_i)_{i=1,2}$ over $X$, their fibred product $Y_1 \times_{p_1} p_2 Y_2$ is defined to be

\[ \{(y_1, y_2) \in Y_1 \times Y_2 : p_1(y_1) = p_2(y_2)\} \]

with the induce topology by the product topology.
Definition 2.10. Let $\mathcal{G}$ be a locally compact groupoid. A (left) $\mathcal{G}$-space is a fibre space $(Y, p)$ over $X = \mathcal{G}^{(0)}$, together with a continuous map $(\gamma, y) \mapsto \gamma y$ from $\mathcal{G}_{s*p} Y$ to $Y$, satisfying the following conditions:

1. $p(\gamma y) = r(\gamma)$ for $(\gamma, y) \in \mathcal{G}_{s*p} Y$, and $p(y)y = y$ for $y \in Y$;
2. $(\gamma_2\gamma_1)y = \gamma_2(\gamma_1 y)$ for $(\gamma_1, y) \in \mathcal{G}_{s*p} Y$ and $s(\gamma_2) = r(\gamma_1)$.

For a $\mathcal{G}$-space $(Y, p)$, now we define the semi-direct product groupoid $Y \rtimes \mathcal{G}$. As a topological space, it is $Y_{p*} \mathcal{G}$. The range of $(y, \gamma)$ is $r(y, \gamma) = (y, r(\gamma)) = (y, p(y))$, and its source is $s(y, \gamma) = (\gamma^{-1}y, s(\gamma))$. The product is given by

$$(y, \gamma)(\gamma^{-1}y, \gamma_1) = (y, \gamma\gamma_1)$$

and the inverse is given by

$$(y, \gamma)^{-1} = (\gamma^{-1}y, \gamma^{-1}).$$

Clearly, the unit space $(Y \rtimes \mathcal{G})^{(0)}$ can be identified with $Y$ via the homeomorphism sending $(y, p(y))$ to $y$. Hence from now on, we may regard $Y$ as the unit space of the groupoid $Y \rtimes \mathcal{G}$.

Note that when $\mathcal{G}$ is a group, the semi-direct product groupoid $Y \rtimes \mathcal{G}$ is nothing but the transformation groupoid mentioned above. As shown in [AD16, Proposition 1.4, 1.5], the range map $r : Y \rtimes \mathcal{G} \to Y$ is always open, and $Y \rtimes \mathcal{G}$ is étale when the groupoid $\mathcal{G}$ itself is.

As a trivial example, for a groupoid $\mathcal{G}$, $(\mathcal{G}^{(0)}, \text{id})$ is a fibre space over $\mathcal{G}^{(0)}$ itself. And it is obvious that the associated semi-direct product $\mathcal{G}^{(0)} \rtimes \mathcal{G}$ is isomorphic to the original groupoid $\mathcal{G}$. Hence there is no difference to study semi-direct product groupoids and general groupoids.

Now we discuss the notion of equivariant fibrewise compactifications associated to a groupoid action. Briefly speaking, it is a special class of fibrewise compactifications which are compatible with the groupoid action. To be more precise, we start with the following notion.

Definition 2.11. A morphism $\varphi : (Y, p) \to (Z, q)$ between two $\mathcal{G}$-spaces is said to be $\mathcal{G}$-equivariant if $\varphi(\gamma y) = \gamma \varphi(y)$ for any $(\gamma, y) \in \mathcal{G}_{s*p} Y$.

Definition 2.12. A $\mathcal{G}$-equivariant fibrewise compactification of a $\mathcal{G}$-space $(Y, p)$ is a fibrewise compactification $(Z, q)$ of $(Y, p)$ such that the associated morphism $\iota : Y \to Z$ is $\mathcal{G}$-equivariant.

As before, we would also like to provide a characterisation in terms of the Gelfand spectra. Recall from Proposition 2.7 that fibrewise compactifications of a fibre space $(Y, p)$ can be characterised in terms of certain $C^*$-subalgebras in $C_b(Y)$. In the equivariant case, they can be characterised by certain invariant subalgebras in the following sense.

To simplify the discussion, we only focus on the étale case. Let $(Y, p)$ be a $\mathcal{G}$-space where $\mathcal{G}$ is a locally compact and étale groupoid. Given any $g \in C_c(\mathcal{G})$ and $f \in C_b(Y)$, define the convolution product $g * f$ by:

$$g * f(x) = \sum_{\gamma \in r^{-1}(p(x))} g(\gamma)f(\gamma^{-1}x).$$

As shown in [AD14], $g * f \in C_c(Y, p)$ defined in (2.1). A $C^*$-subalgebra $\mathcal{A}$ in $C_b(Y)$ is said to be stable under convolution of $C_c(\mathcal{G})$, if $g * f \in \mathcal{A}$ for any $g \in C_c(\mathcal{G})$ and $f \in \mathcal{A}$. Now we have the following:

Proposition 2.13 (Corollary 2.6, [AD14]). Let $(Y, p)$ be a $\mathcal{G}$-space, where $\mathcal{G}$ is a locally compact and étale groupoid. Then there is a one-to-one correspondence between $\mathcal{G}$-equivariant fibrewise compactifications of $Y$ and those subalgebras of $C_0(Y, p)$ that contain $p^*C_0(X) + C_0(Y)$ and are stable under convolution of $C_c(\mathcal{G})$.

By the above proposition, it is easy to check that for a $\mathcal{G}$-space $(Y, p)$ where $\mathcal{G}$ is a locally compact and étale groupoid, the fibrewise Alexandroff compactification $(Y^+, p^*)$ and the fibrewise Stone-Cech compactification $(\beta_p Y, p_\beta)$ are $\mathcal{G}$-equivariant. As before we still have the following universal property:

Proposition 2.14 (Proposition 2.8, [AD14]). Let $\mathcal{G}$ be a locally compact and étale groupoid, $(Y, p)$ and $(Y_1, p_1)$ be two $\mathcal{G}$-spaces. Suppose $(Y_1, p_1)$ is fibrewise compact, and $\varphi : Y \to Y_1$ is a $\mathcal{G}$-equivariant morphism. Then the unique morphism $\Phi : (\beta_p Y, p_\beta) \to (Y_1, p_1)$ extending $\varphi$ is also $\mathcal{G}$-equivariant.
As in Proposition 2.13, the Stone-Čech and the Alexandroff fiberwise compactifications are the “largest” and the “smallest” equivariant fiberwise compactification of a fiber space being acted upon by an étale groupoid. Here is an example of yet another natural equivariant fiberwise compactification.

Example 2.15. Let $\mathcal{G}$ be a locally compact and étale groupoid. Let $C_h(\mathcal{G})$ denote the set of bounded continuous functions on $\mathcal{G}$ such that for every $\epsilon > 0$ and every compact subset $K \subset \mathcal{G}$, there exists a compact subset $\hat{K} \subset \mathcal{G}$ such that if $g, h \in \mathcal{G} \setminus \hat{K}$ with $t(g) = t(h)$ and $gh^{-1} \in K$, then we have $|f(g) - f(h)| \leq \epsilon$. Elements in $C_h(\mathcal{G})$ are called the Higson functions on $\mathcal{G}$. The readers can verify that $C_h(\mathcal{G})$ is a closed subalgebra of $C_0(\mathcal{G}, s)$ that contains $s^*C_0(G^{(0)}) + C_0(\mathcal{G})$, and is stable under convolution. Hence it determines a $\mathbb{G}$-equivariant fiberwise compactification of $\mathcal{G}$, denote by $h\mathcal{G}$ and called the Higson fiberwise compactification of $\mathcal{G}$. It is our intention of further investigating the index theory of a groupoid acting on its Higson fiberwise compactifications.

2.4. Groupoid $C^*$-algebras. Here we recall some $C^*$-algebras associated to a groupoid $\mathcal{G}$ with a (right) Haar system $\{\mu_x : x \in \mathcal{G}^{(0)}\}$. First note that the space $C_c(\mathcal{G})$ is a $\ast$-involutive algebra with respect to the following operations: for $f, g \in C_c(\mathcal{G})$,

$$(f \ast g)(\gamma) = \int_{\alpha \in \mathcal{G}_{x(\gamma)}} f(\gamma\alpha^{-1})g(\alpha)d\mu_{x(\gamma)}(\alpha),$$

$$f^*(\gamma) = \overline{f(\gamma)}.$$

In the special case that $\mathcal{G}$ is étale with the Haar system consisting of counting measures on each fibre, the above convolution can be simplified as follows:

$$(f \ast g)(\gamma) = \sum_{\alpha \in \mathcal{G}_{x(\gamma)}} f(\gamma\alpha^{-1})g(\alpha).$$

Consider the following algebraic norm on $C_c(\mathcal{G})$ defined by:

$$\|f\|_I := \max \left\{ \sup_{x \in \mathcal{G}^{(0)}} \int |f|d\mu_x, \sup_{x \in \mathcal{G}^{(0)}} \int |f^*|d\mu_x \right\}.$$  

The completion of $C_c(\mathcal{G})$ with respect to the norm $\| \cdot \|_I$ is denoted by $L^1(\mathcal{G})$.

The maximal (full) groupoid $C^*$-algebra $C^*_\text{max}(\mathcal{G})$ is defined to be the completion of $C_c(\mathcal{G})$ with respect to the norm

$$\|f\|_{\text{max}} := \sup \|\pi(f)\|,$$

where the supremum is taken over all contravariant $\ast$-representations $\pi$ of $L^1(\mathcal{G})$.

In order to define the reduced counterpart, we recall that for each $x \in \mathcal{G}^{(0)}$ the regular representation at $x$, denoted by $C_{\ast} : C_c(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x; \mu_x))$, is defined as follows:

$$(\lambda_x f)(\xi)(\gamma) := \int_{\alpha \in \mathcal{G}_x} f(\gamma\alpha^{-1})\xi(\alpha)d\mu_x(\alpha), \quad \text{where } f \in C_c(\mathcal{G}) \text{ and } \xi \in L^2(\mathcal{G}_x; \mu_x).$$

Again in the special case that $\mathcal{G}$ is étale with the Haar system consisting of counting measures on each fibre, the regular representation $\lambda_x : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$ can be simplified as follows:

$$(\lambda_x f)(\xi)(\gamma) := \sum_{\alpha \in \mathcal{G}_x} f(\gamma\alpha^{-1})\xi(\alpha), \quad \text{where } f \in C_c(\mathcal{G}) \text{ and } \xi \in \ell^2(\mathcal{G}_x).$$

It is routine work to check that $\lambda_x$ is a well-defined $\ast$-homomorphism. The reduced norm on $C_c(\mathcal{G})$ is

$$\|f\|_\ast := \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\|,$$

and the reduced groupoid $C^*$-algebra $C^*_\ast(\mathcal{G})$ is defined to be the completion of the $\ast$-algebra $C_c(\mathcal{G})$ with respect to this norm. Clearly, each regular representation $\lambda_x$ can be extended to a homomorphism $\lambda_x : C^*_\ast(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x; \mu_x))$ automatically. It is also a routine work to check that there is a canonical surjective homomorphism from $C^*_\text{max}(\mathcal{G})$ to $C^*_\ast(\mathcal{G})$.

Now we discuss an alternative way to define the reduced groupoid $C^*$-algebra $C^*_\ast(\mathcal{G})$ in terms of the Hilbert module language as follows. (For those who are not familiar with Hilbert modules, we refer
to [Lan95].) Let \( L^2(\mathcal{G}) \) be the Hilbert module over \( C_0(\mathcal{G}(0)) \) obtained by taking completion of \( C_c(\mathcal{G}) \) with respect to the \( C_0(X) \)-valued inner product
\[
\langle \xi, \eta \rangle(x) := \int_{\gamma \in G_x} \overline{\xi(\gamma)} \eta(\gamma) d\mu_x(\gamma),
\]
and the right \( C_0(X) \)-module structure is given by
\[
(\xi f)(\gamma) := \xi(\gamma) f(s(\gamma)).
\]
Denote \( \mathfrak{B}(L^2(\mathcal{G})) \) the \( C^* \)-algebra of all adjointable operators on the Hilbert module \( L^2(\mathcal{G}) \).

Note that all the regular representations \( \lambda_x \) defined in (2.2) can be put together to a single representation \( \Lambda : C_c(\mathcal{G}) \rightarrow \mathfrak{B}(L^2(\mathcal{G})) \) by the formula:
\[
(\Lambda f)(\gamma) := \int_{\alpha \in G_{s(\gamma)}} f(\gamma \alpha^{-1}) \xi(\alpha) d\mu_{s(\gamma)}(\alpha) = ((\lambda_{s(\gamma)} f)\xi|_{G_{s(\gamma)}})(\gamma).
\]
And it is easy to check that for \( f \in C_c(\mathcal{G}) \), we have \( \|f\|_r = \|\Lambda(f)\|_{\mathfrak{B}(L^2(\mathcal{G}))} \). Therefore, \( \Lambda \) can be extended to a faithful representation \( \Lambda : C^*_c(\mathcal{G}) \rightarrow \mathfrak{B}(L^2(\mathcal{G})) \), which is called the regular representation.

2.5. Amenability. Amenable groupoids compose a large class of groupoids with relatively nice properties, and they are the central objects of our paper. Literally, they are the analogue of amenable groups in the world of groupoids. However unlike the case of groups, there are different versions of amenability (for example measurewise, topological and Borel amenabilities) which might not be equivalent for general groupoids. Fortunately, they behave quite well under the restriction of the étaleness. Here we mainly focus on topological amenability. A standard reference is [ADR00] and another reference for just étale groupoids is [BO08, Chapter 5.6].

**Definition 2.16.** Recall that a locally compact groupoid \( \mathcal{G} \) is *topologically amenable* if there exists a *topological approximate invariant mean*, i.e. a sequence \( \{m^{(n)}\}_{n \in \mathbb{N}} \) of families of positive and finite Radon measures \( m^{(n)} = \{m^{(n)}_x : x \in \mathcal{G}(0)\} \) satisfying

1. \( \|m^{(n)}_x\|_1 \leq 1 \) and \( m^{(n)}_x(\mathcal{G} \setminus s^{-1}(x)) = 0 \), for all \( x \in \mathcal{G}(0) \) and \( n \in \mathbb{N} \);
2. for all \( n \in \mathbb{N} \) and \( f \in C_c(\mathcal{G}) \), the function \( x \mapsto \int f dm^{(n)}_x \) is continuous on \( \mathcal{G}(0) \);
3. \( \|m^{(n)}_x\|_1 \rightarrow 1 \) as \( n \rightarrow \infty \), uniformly on any compact subset of \( \mathcal{G}(0) \);
4. \( \|m^{(n)}_{s(\gamma)} - \gamma m^{(n)}_{r(\gamma)}\|_1 \rightarrow 0 \) as \( n \rightarrow \infty \), uniformly on any compact subset of \( \mathcal{G} \).

Note that for a locally compact groupoid equipped with a Haar system of measures, topological amenability can also be characterised in terms of topological approximate invariant densities. We omit the details, and guide the readers to [ADR00, Proposition 2.2.13] for details.

In the case of étaleness, amenabilities behave quite well as in the case of groups. We recall the following result which is crucial in the proof our main result.

**Proposition 2.17** (Corollary 5.6.17, Theorem 5.6.18, [BO08]). Let \( \mathcal{G} \) be a locally compact and étale groupoid. Then \( \mathcal{G} \) is topologically amenable if and only if the reduced groupoid \( C^*_r(\mathcal{G}) \) is nuclear. In this case, the natural quotient \( C^*_r(\mathcal{G}) \rightarrow C^*_r(\mathcal{G}) \) is an isomorphism.

Now we provide two examples which will be further discussed in Section 6.

**Example 2.18.** A discrete group \( G \) can be regarded as a groupoid with the unit space consisting of a single point. In this case, topological amenability is nothing but the classic notion of amenability for a group.

Now suppose a discrete group \( G \) acts on a compact Hausdorff topological space \( X \) by homeomorphisms. As discussed at the beginning of Section 2.3, we consider the transformation groupoid \( X \times G \). It is not hard to check directly by definition that \( X \times G \) is topologically amenable *if and only if* the action is amenable (see for example [BO08]). And it also follows directly by definition that a group \( G \) is amenable *if and only if* the action induced by the left multiplication on its Alexandroff one point compactification is amenable. Finally recall from [BO08, Theorem 5.1.7] that for a discrete group \( G \), the following are equivalent:

- \( G \) is exact;
The action induced by the left multiplication of $G$ on its Stone-Čech compactification $\beta G$ is amenable;

- $G$ acts amenably on some compact Hausdorff topological space.

Example 2.19. Given a discrete metric space $X$ with bounded geometry, as introduced in [STY02] we may associate the coarse groupoid $G(X)$ (see Section 6.3 for more details). It is shown in [Roe03, Theorem 10.29] that the reduced groupoid $C^*$-algebra $C^*(G(X))$ is isomorphic to the uniform Roe algebra $C^*_u(X)$. Therefore from Proposition 2.17 and [BO08, Theorem 5.5.7], amenability of the coarse groupoid $G(X)$ is equivalent to the space $X$ having Property A.

Finally, we discuss briefly on the amenability of groupoid actions and the notion of exactness for groupoids. A full detailed discussion is provided in [AD16].

Given a locally compact groupoid $\mathcal{G}$ and a $\mathcal{G}$-space $(Y, p)$. Following [AD16, Definition 2.5], we say that the action is amenable, or $Y$ is an amenable $\mathcal{G}$-space, if the associated semi-direct product $Y \rtimes \mathcal{G}$ is topologically amenable. By [ADR00, Corollary 2.2.10], if $\mathcal{G}$ is an amenable locally compact groupoid, then for every $\mathcal{G}$-space the action is amenable. As in the group case, an étale groupoid $\mathcal{G}$ is amenable if and only if the fibrewise Alexandroff compactification $\mathcal{G}^\beta$ is amenable (see [AD16, Proposition 3.3]).

In the group case as we see in Example 2.18, amenable actions of groups on compact spaces have close relation with the exactness of the given group. Unfortunately, things become complicated for general groupoids. We start with the following definition.

Definition 2.20 ([AD16]). We say that a locally compact groupoid $\mathcal{G}$ is amenable at infinity if there is an amenable $\mathcal{G}$-space $(Y, p)$ such that $Y$ is fibrewise compact. If the space $(Y, p)$ can be chosen to be the fibrewise Stone-Čech compactification $(\beta \mathcal{G}, r_{\beta})$ and the action is induced by the left multiplication, then we say the groupoid $\mathcal{G}$ is strongly amenable at infinity.

Now we introduce the notion of exactness for groupoids, analogous to the case of groups.

Definition 2.21 ([AD16]). A locally compact groupoid $\mathcal{G}$ with a Haar system of measures is called $C^*$-exact if the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G})$ is exact.

Proposition 2.22 ([AD16], Corollary 6.4). Let $\mathcal{G}$ be a locally compact, second countable and étale groupoid. We consider the following:

1. $\mathcal{G}$ is strongly amenable at infinity;
2. $\mathcal{G}$ is amenable at infinity;
3. $\beta_\ast \mathcal{G} \rtimes \mathcal{G}$ is topologically amenable;
4. $\mathcal{G}$ is $C^*$-exact.

Then we have $1 \Rightarrow 2 \Rightarrow 4$, and $1 \Rightarrow 3 \Rightarrow 4$.

The converse direction of the above proposition is also discussed in [AD16]. In order to state their result, we need an extra notion as follows:

Definition 2.23 ([AD16]). We say that a locally compact groupoid $\mathcal{G}$ is weakly inner amenable if for any compact subset $K \subseteq \mathcal{G}$ and any $\varepsilon > 0$, there exists a continuous bounded positive definite function $f$ on the product groupoid $\mathcal{G} \times \mathcal{G}$, properly supported, such that $|f(\gamma, \gamma) - 1| < \varepsilon$ for all $\gamma \in K$.

Proposition 2.24 ([AD16], Theorem 7.6). Let $\mathcal{G}$ be a second countable weakly inner amenable étale groupoid. Then conditions (1) to (4) in Proposition 2.22 are all equivalent.

3. Operator Fibre Spaces

Recall that in Section 2.4, for an étale groupoid $\mathcal{G}$ and any point $x \in \mathcal{G}^{(0)}$, we define the regular representation $\lambda_x : C^*_r(\mathcal{G}) \to \mathcal{B}(l^2(\mathcal{G}_x))$. Fix an element $T \in C^*_r(\mathcal{G})$ we obtain a map $x \mapsto \lambda_x(T)$, which we will show can be regarded as a section of a bundle of operators on the unit space $\mathcal{G}^{(0)}$. In this section, we will generalise this observation and introduce the notion of operator fibre spaces, which serves as the receptacle for the symbol map we will establish later. Again in this section we only consider the étale case.
Fix a locally compact and étale groupoid $\mathcal{G}$ with unit space $\mathcal{G}^{(0)}$. Consider the space

$$E := \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x)).$$

For each element $T$ in $\mathcal{B}(\ell^2(\mathcal{G}_x)) \subseteq E$, we write $T_x$ to indicate the fibre it lives in. We define the projection map

$$p : E \to \mathcal{G}^{(0)} \text{ by } T_x \mapsto x.$$

Now we endow a topology on $E$ as follows: a net $\{T_{x_i}\}_{i \in I}$ converges to $T_x$ if and only if $x_i \to x$, and for any $\gamma'_i \to \gamma'$, $\gamma''_i \to \gamma''$ with $s(\gamma'_i) = x_i = s(\gamma''_i)$ (which implies that $s(\gamma') = x = s(\gamma'')$), we have

$$\langle T_{x_i}, \delta_{\gamma'_i}, \delta_{\gamma''_i} \rangle = \langle T_x, \delta_{\gamma'}, \delta_{\gamma''} \rangle.$$

**Definition 3.1.** Given a locally compact and étale groupoid $\mathcal{G}$, the space $E$ defined in (3.1) equipped with the above topology is called the operator fibre space associated to $\mathcal{G}$.

Now we would like to provide a description for a local basis of a given point $T_x \in E$. First let us fix some notations. For a topological space $X$ and a point $x \in X$, denote $\mathcal{N}_x$ the set of all the neighbourhoods of $x$. Since $\mathcal{G}$ is étale, for any $\gamma', \gamma'' \in \mathcal{G}_x$, there exist neighbourhoods $V_{\gamma'} \in \mathcal{N}_{\gamma'}$ and $V_{\gamma''} \in \mathcal{N}_{\gamma''}$ such that the restriction of $s$ on $V_{\gamma'}$, $V_{\gamma''}$ are homeomorphisms with open images $s(V_{\gamma'})$, $s(V_{\gamma''})$. Take $U_{\gamma', \gamma''} = s(V_{\gamma'}) \cap s(V_{\gamma''})$ and

$$\zeta'_{\gamma'} := (s|_{U_{\gamma', \gamma''}})^{-1} : U_{\gamma', \gamma''} \to V_{\gamma'} \cap s^{-1}(U_{\gamma', \gamma''}),$$

$$\zeta''_{\gamma''} := (s|_{U_{\gamma', \gamma''}})^{-1} : U_{\gamma', \gamma''} \to V_{\gamma''} \cap s^{-1}(U_{\gamma', \gamma''}).$$

For any $\varepsilon > 0$ and $U \in \mathcal{N}_x$ with $U \subseteq U_{\gamma', \gamma''}$, we define

$$W_{T_x}(\varepsilon; \gamma', \gamma'') := \{T_y \in E : y \in U \text{ and } |\{T_y \delta_{\zeta'_{\gamma'}(y)}, \delta_{\zeta''_{\gamma''}(y)}\} - \{T_x \delta_{\gamma'}, \delta_{\gamma''}\}| < \varepsilon\}.$$

**Lemma 3.2.** Let $\mathcal{G}$ be a locally compact and étale groupoid and $E$ be the associated operator fibre space. Then

$$\{W_{T_x}(\varepsilon; \gamma', \gamma'') : \varepsilon > 0, \gamma', \gamma'' \in \mathcal{G}_x, \text{ and open } U \subseteq U_{\gamma', \gamma''}\}$$

is a local basis of $T_x$.

**Proof.** By definition, a net $\{T_{x_i}\}_{i \in I}$ converges to $T_x$ in $E$ if and only if for any $\varepsilon > 0$, $\gamma', \gamma'' \in \mathcal{G}_x$ and $U \in \mathcal{N}_x$, there exists $i_0 \in I$, $V' \in \mathcal{N}_{\gamma'}$ and $V'' \in \mathcal{N}_{\gamma''}$ such that for any $i > i_0$, we have $x_i \in U$ and for any $V' \cap \mathcal{G}_x$, $V'' \cap \mathcal{G}_x$, we have

$$|\{T_{x_i} \delta_{\zeta'_{\gamma'}(x_i)}, \delta_{\zeta''_{\gamma''}(x_i)}\} - \{T_x \delta_{\gamma'}, \delta_{\gamma''}\}| < \varepsilon.$$

Since $\mathcal{G}$ is étale, we can shrink $V'$, $V''$ if necessary to ensure that $V' \subseteq V_{\gamma'}$ and $V'' \subseteq V_{\gamma''}$, which implies that $V' \cap \mathcal{G}_x = \{\zeta'_{\gamma'}(x_i)\}$ and $V'' \cap \mathcal{G}_x = \{\zeta''_{\gamma''}(x_i)\}$. Hence, $T_{x_i} \to T_x$ if and only if for any $\varepsilon > 0$, $\gamma', \gamma'' \in \mathcal{G}_x$ and $U \in \mathcal{N}_x$ with $U \subseteq U_{\gamma', \gamma''}$, there exists $i_0 \in I$ such that for any $i > i_0$, we have $x_i \in U$ and

$$|\{T_{x_i} \delta_{\zeta'_{\gamma'}(x_i)}, \delta_{\zeta''_{\gamma''}(x_i)}\} - \{T_x \delta_{\gamma'}, \delta_{\gamma''}\}| < \varepsilon.$$

So we finish the proof. \(\square\)

**Lemma 3.3.** The projection map $p : E \to \mathcal{G}^{(0)}$ is an open and continuous surjection.

**Proof.** Clearly, $p$ is continuous and surjective. To see $p$ is open, note that $p(W_{T_x}(\varepsilon; \gamma', \gamma'')) = U$ for any $\varepsilon > 0$, $\gamma', \gamma'' \in \mathcal{G}_x$ and $U \subseteq \mathcal{N}_{\gamma', \gamma''}$. \(\square\)

**Remark 3.4.** Although $E$ is called an operator fibre space, it might not be a genuine fibre space in the sense of Definition 2.4. From the above, the only obstruction is that we don’t know whether the topology is locally compact in general. However in some special case, we do know certain subspace of $E$ is locally compact or even fibrewise compact (see Lemma 3.11).

Now denote $\Gamma(E)$ the set of all continuous sections of $E$. Note that for a general element $\varphi$ in $\Gamma(E)$, the norms $\{\|\varphi(x)\| : x \in \mathcal{G}^{(0)}\}$ may not have an upper bound. To overcome, we introduce the following:
**Definition 3.5.** Let $\mathcal{G}$ be a locally compact and étale groupoid. For each $k \in \mathbb{N}$, we define the $k$-bounded operator fibre space to be

$$E_k := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x))_k \ (\subseteq E),$$

where $\mathcal{B}(\ell^2(\mathcal{G}_x))_k$ is the $k$-ball at the origin. Denote $\Gamma(E_k)$ the set of all continuous sections of $E_k$, and

$$\Gamma_k(E) := \bigcup_{k \in \mathbb{N}} \Gamma(E_k).$$

Endow a norm on $\Gamma_k(E)$ by $\|\varphi\| := \sup_{x \in \mathcal{G}^{(0)}} \|\varphi(x)\|$ for $\varphi \in \Gamma_k(E)$. It is routine to check that equipped with the norm and the point-wise addition, multiplication and adjoint, $\Gamma_k(E)$ is a $C^*$-algebra, which is called the $C^*$-algebra of bounded sections of $E$.

Now we go back to the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G})$. Given an element $T \in C^*_r(\mathcal{G})$ with norm $\|T\| = k$ and any $x \in \mathcal{G}^{(0)}$, we have an operator $\lambda_x(T) \in \mathcal{B}(\ell^2(\mathcal{G}_x))_k$ defined in (2.2). In other words, we obtain a section of $E_k$ by $x \mapsto \lambda_x(T)$.

**Lemma 3.6.** The section $x \mapsto \lambda_x(T)$ defined above is continuous.

**Proof.** It suffices to prove the lemma for any element $f \in C_c(\mathcal{G})$. Suppose the net $\{x_i\}_{i \in I}$ converges to $x$ in $\mathcal{G}^{(0)}$, and we need to show that $\lambda_{x_i}(f) \to \lambda_x(f)$ in $E_k$. For any $\gamma' \to \gamma$, $\gamma'' \to \gamma''$ with $s(\gamma') = x_i = s(\gamma'')$, we have

$$(\lambda_{x_i}(f) \delta_{\gamma', \gamma''}) = f(\gamma'' \cdot \gamma'^{-1}) \to f(\gamma'' \cdot \gamma'^{-1}) = (\lambda_x(f) \delta_{\gamma', \gamma''}).$$

So the lemma holds.

Therefore, we obtain a $C^*$-homomorphism $\lambda : C^*_r(\mathcal{G}) \to \Gamma_k(E)$ defined by $\lambda(T)(x) := \lambda_x(T)$. Now we would like to provide a more detailed picture for the image $\mathrm{Im}\lambda$.

**Definition 3.7.** A section $\varphi$ in $\Gamma(E)$ is called $\mathcal{G}$-equivariant, if for any $\gamma \in \mathcal{G}$ we have

$$\varphi(r(\gamma)) = R_\gamma^* \varphi(s(\gamma)) R_\gamma,$$

where the operator $R_\gamma : \ell^2(\mathcal{G}_{r(\gamma)}) \to \ell^2(\mathcal{G}_{s(\gamma)})$ is defined by $\delta_\alpha \mapsto \delta_{\alpha \gamma}$ for any $\alpha \in \mathcal{G}_{r(\gamma)}$. Denote $\Gamma(E)^G$, $\Gamma(E_k)^G$ and $\Gamma_k(E)^G$ the subset of $\Gamma(E)$, $\Gamma_k(E)$ and $\Gamma_k(E)$ consisting of $\mathcal{G}$-equivariant sections, respectively. Clearly, $\Gamma_k(E)^G$ is a $C^*$-subalgebra of $\Gamma_k(E)$, called the $C^*$-algebra of $\mathcal{G}$-equivariant bounded sections of $E$.

**Lemma 3.8.** Notations as above. We have $\lambda(C^*_r(\mathcal{G})) \subseteq \Gamma_k(E)^G$.

**Proof.** It suffices to show that for any $f \in C_c(\mathcal{G})$ and $\gamma$, we have $\lambda_{r(\gamma)}(f) = R_\gamma^* \lambda_s(\gamma)(f) R_\gamma$ in $\mathcal{B}(\ell^2(\mathcal{G}_{r(\gamma)}))$. For any $\alpha', \alpha'' \in \mathcal{G}_{r(\gamma)}$, we have

$$\langle R_\gamma^* \lambda_s(\gamma)(f) R_\gamma, \delta_{\alpha', \alpha''} \rangle = \langle \lambda_s(\gamma)(f) \delta_{\alpha' \gamma, \alpha'' \gamma}, \delta_{\alpha', \alpha''} \rangle = \langle f(\alpha'' \gamma (\alpha' \gamma)^{-1}), f(\alpha'' \alpha'^{-1}) \rangle = \langle \lambda_{r(\gamma)}(f) \delta_{\alpha', \alpha''} \rangle.$$

So the lemma holds.

**Definition 3.9.** For $x \in \mathcal{G}^{(0)}$, we define the following $*$-algebra

$$\mathbb{C}[\mathcal{G}_x] := \{T = (T_{\gamma', \gamma''})_{\gamma', \gamma'' \in \mathcal{G}_x} \in \mathcal{B}(\ell^2(\mathcal{G}_x)) : \exists \text{ compact } K \subseteq \mathcal{G}, \text{ such that } T_{\gamma', \gamma''} \neq 0 \text{ implies } \gamma'' \gamma'^{-1} \in K\}.$$

The uniform Roe algebra at $x$ is defined to be $C^*_u(\mathcal{G}_x) := \overline{\mathbb{C}[\mathcal{G}_x]}$.

It is obvious that for each $T \in C^*_r(\mathcal{G})$ and $x \in \mathcal{G}^{(0)}$, the operator $\lambda_x(T)$ belongs to the uniform Roe algebra $C^*_u(\mathcal{G}_x)$. This suggests us to consider the following:
Definition 3.10. Let $\mathcal{G}$ be an étale groupoid. The uniform Roe fibre space is defined to be

$$E_u := \bigcup_{x \in \mathcal{G}^{(0)}} C_u^*(\mathcal{G}_x) \quad (\in E).$$

Similarly, for each $k \in \mathbb{N}$, we define the $k$-bounded uniform Roe fibre space $E_{u,k} \subseteq E_k$ to be

$$E_{u,k} := \bigcup_{x \in \mathcal{G}^{(0)}} C_u^*(\mathcal{G}_x)_k \quad (\in E).$$

Note that $E_u$ may not be close in $E$, neither does $E_{u,k}$. Denote all continuous sections of $E_u, E_{u,k}$ by $\Gamma(E_u), \Gamma(E_{u,k})$, and set

$$\Gamma_b(E_u) := \bigcup_{k \in \mathbb{N}} \Gamma(E_{u,k}).$$

Note that $\Gamma_b(E_u)$ is a $C^*$-subalgebra in $\Gamma_b(E)$, called the $C^*$-algebra of bounded uniform Roe sections of $E$. And the intersection

$$(3.2) \quad \Gamma_b(E_u)^G := \Gamma_b(E_u) \cap \Gamma_b(E)^G$$

is also a $C^*$-subalgebra in $\Gamma_b(E)$, called the $C^*$-algebra of $\mathcal{G}$-equivariant bounded uniform Roe sections of $E$.

Proposition 3.11. Let $\mathcal{G}$ be an étale groupoid. Then the regular representations $x \mapsto \lambda_x$ induce a $C^*$-monomorphism $\lambda : C^*_u(\mathcal{G}) \rightarrow \Gamma_b(E_u)^G$.

Finally, we study the following nice property for bounded operator fibre spaces when the underlying unit space is first countable. This result plays an important role when we study the uniform Roe algebra of a groupoid later in Section 6.5. Note that restricted on each fibre, the topology is nothing but the weak operator topology, hence compact by Banach-Alaoglu Theorem.

Lemma 3.12. Let $\mathcal{G}$ be a locally compact, $\sigma$-countable and étale groupoid with first countable unit space $\mathcal{G}^{(0)}$. Then for each $k \in \mathbb{N}$, the $k$-bounded operator fibre space $E_k$ is fibrewise compact.

Proof. By definition, we need show that for any compact $K \subseteq \mathcal{G}^{(0)}$, $p^{-1}(K)$ is compact in $E_k$. Since $\mathcal{G}^{(0)}$ is first countable and $\mathcal{G}$ is $\sigma$-compact, we know that $E_k$ is also first countable by Lemma 3.2. So it suffices to prove that for any sequence $\{T_n\}_{n \in \mathbb{N}}$ in $p^{-1}(K)$, we can find a subsequence $\{T_{n_k}\}_{k \in \mathbb{N}}$ such that it converges in $p^{-1}(K)$.

Consider the sequence $\{x_n := p(T_n)\}_{n \in \mathbb{N}} \subseteq K$. Since $K$ is compact and first countable, there exists a subsequence which converges to some point $x \in K$. After taking the subsequence, we may assume that $x_n \rightarrow x$. For any $\gamma', \gamma'' \in \mathcal{G}_x$, take $U_{\gamma', \gamma''}, \zeta_{\gamma', \gamma''}$ and $\zeta_{\gamma''}$ as in the paragraph before Lemma 3.2. Then there exists $n_{\gamma', \gamma''} \in \mathbb{N}$ such that

$$\{x_n : n \geq n_{\gamma', \gamma''}\} \subseteq U_{\gamma', \gamma''}.$$

Now consider the sequence:

$$\{(T_{x_n} \delta_{\zeta_{\gamma', \gamma''}}(x_n), \delta_{\zeta_{\gamma''}}(x_n))\}_{n \geq n_{\gamma', \gamma''}},$$

which is contained in the compact set $\{z \in \mathbb{C} : |z| \leq k\}$. Hence there exists a subsequence $I_{\gamma', \gamma''} \subseteq \mathbb{N}$ such that $\{(T_{x_n} \delta_{\zeta_{\gamma', \gamma''}}(x_n), \delta_{\zeta_{\gamma''}}(x_n)) : n \in I_{\gamma', \gamma''}\}$ converges to a complex number $\kappa_{\gamma', \gamma''}$ with $|\kappa_{\gamma', \gamma''}| \leq k$. Since the set $\mathcal{G}_x \times \mathcal{G}_x$ is countable, using a Cantor diagonal argument, we may find a subsequence $\{n_k\}_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that for any $(\gamma', \gamma'') \in \mathcal{G}_x \times \mathcal{G}_x$, we have

$$\lim_{k \rightarrow \infty} (T_{x_{n_k}} \delta_{\zeta_{\gamma', \gamma''}}(x_{n_k}), \delta_{\zeta_{\gamma''}}(x_{n_k})) = \kappa_{\gamma', \gamma''}.$$

We claim: there exists $T_x \in \mathfrak{B}(\ell^2(\mathcal{G}_x))$ such that $\langle T_x \delta_{\gamma'}, \delta_{\gamma''} \rangle = \kappa_{\gamma', \gamma''}$. In fact, consider the following sesquilinear map

$$F : C_c(\mathcal{G}_x) \times C_c(\mathcal{G}_x) \rightarrow \mathbb{C}$$

defined by

$$F\left(\sum_{i=1}^l c_i^j \delta_{\gamma_i^j}, \sum_{j=1}^l c_j^j \delta_{\gamma_j^j}\right) := \sum_{i,j=1}^l c_i^j c_j^j \kappa_{\gamma_i^j, \gamma_j^j} = \sum_{i,j=1}^l c_i^j \lim_{k \rightarrow \infty} \langle T_{x_{n_k}} \delta_{\zeta_{\gamma_i^j}}(x_{n_k}), \delta_{\zeta_{\gamma_j^j}}(x_{n_k}) \rangle$$

$$= \lim_{k \rightarrow \infty} \langle T_{x_{n_k}} \left(\sum_{i=1}^l c_i^j \delta_{\zeta_{\gamma_i^j}}(x_{n_k})\right), \sum_{j=1}^l c_j^j \delta_{\zeta_{\gamma_j^j}}(x_{n_k}) \rangle.$$
Since all $T_{x_\lambda}$’s have norm at most $k$, we obtain that
\[
\sup \{ \| F(\xi', \xi'') \| : \xi', \xi'' \in C_c(G_\omega) \text{ with } \| \xi' \|_2 \leq 1, \| \xi'' \|_2 \leq 1 \}
\]
does not exceed $k$. Hence $F$ can be extended to a bounded sesquilinear map on $\ell^2(G_\omega) \times \ell^2(G_\omega)$ with norm at most $k$, which implies there exists $T_x \in \mathfrak{B}(\ell(G_\omega))_k$ such that
\[
(T_x \xi', \xi'') = F(\xi', \xi'')
\]
for any $\xi', \xi'' \in \ell(G_\omega)$. Hence the claim holds, and we finish the proof. \qed

4. **Main Theorem**

Having introduced sufficient background tools in previous sections, now we are in the position to establish the limit operator theory for groupoids. This follows the same philosophy as the existing theories in the Hilbert space case of groups and metric spaces. We start with the following setting.

Let $\mathcal{G}$ be a locally compact and étale groupoid with compact unit space $\mathcal{G}^{(0)}$. Suppose that there is a dense invariant open subset $X$ in $\mathcal{G}^{(0)}$, then its complement $\partial X := \mathcal{G}^{(0)} \setminus X$ is an invariant closed subset in $\mathcal{G}^{(0)}$. Concerning the associated groupoid reductions, we have the following decomposition:

\[
\mathcal{G} = \mathcal{G}(X) \sqcup \mathcal{G}(\partial X).
\]

Note that $\mathcal{G}(X)$ is open in $\mathcal{G}$, while $\mathcal{G}(\partial X)$ is closed. Clearly, we have the following short exact sequence induced by (4.1):

\[
0 \to C_c(\mathcal{G}(X)) \to C_c(\mathcal{G}) \to C_c(\mathcal{G}(\partial X)) \to 0,
\]

where the map $C_c(\mathcal{G}(X)) \to C_c(\mathcal{G})$ is the inclusion, and $C_c(\mathcal{G}) \to C_c(\mathcal{G}(\partial X))$ is the restriction.

We may complete this sequence with respect to the reduced norms and obtain the following sequence:

\[
0 \to C^*_r(\mathcal{G}(X)) \to C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}(\partial X)) \to 0.
\]

By constructions, $i$ is injective and $q$ is surjective, so we may regard $C^*_r(\mathcal{G}(X))$ as an ideal of $C^*_r(\mathcal{G})$. However, in general (4.3) fails to be exact at the middle item, which is crucial in HLS02 [MRW96, Lemma 2.10] to study the counterexample to the Baum-Connes conjecture. We may also complete the sequence (4.2) with respect to the maximal norms and obtain the following sequence:

\[
0 \to C^*_{\max}(\mathcal{G}(X)) \to C^*_{\max}(\mathcal{G}) \to C^*_{\max}(\mathcal{G}(\partial X)) \to 0,
\]

which is easy to check by definition to be exact automatically (see for example [MRW96, Lemma 2.10]).

**Definition 4.1.** Let $\mathcal{G}, X, \partial X$ be as above and $T \in C^*_r(\mathcal{G})$. For each $\omega \in \partial X$, we define the limit operator of $T$ at $\omega$ to be $\lambda_\omega(T) \in \mathfrak{B}(\ell^2(G_\omega))$, where $\lambda_\omega : C^*_r(\mathcal{G}) \to \mathfrak{B}(\ell^2(G_\omega))$ is the regular representation at $\omega$.

Note that for $\omega \in \partial X$, we have $\mathcal{G}_\omega = \mathcal{G}(\partial X)_\omega$. Hence from the definition we have

\[
\| \lambda_\omega(T) \| = \sup_{\omega \in \partial X} \| \lambda_\omega(T) \|
\]

for any $T \in C^*_r(\mathcal{G})$, where $q : C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}(\partial X))$ is the quotient homomorphism from (4.3).

As Roe did in [Roe05], we would like to compose all limit operators into a single homomorphism. Unfortunately limit operators $\lambda_\omega$’s do not live in the same space generally, hence we have to appeal to the language of operator fibre spaces established in Section 3.

For the given locally compact étale groupoid $\mathcal{G}$, let $\lambda : C^*_r(\mathcal{G}) \to \Gamma_b(E_u)\mathcal{G}$ be the $C^*$-monomorphism established in Proposition 3.11. Denote the operator fibre space associated to the reduction $\mathcal{G}(\partial X)$ by

\[
E^\partial := \bigcup_{x \in \partial X} \mathfrak{B}(\ell^2(G_x)).
\]

Similarly, denote $E^\partial_k$ and $E^\partial_u$ the $k$-bounded version and the uniform Roe version respectively, and their intersection by $E^\partial_{u,k}$, as we did in Section 3. Recall that the $C^*$-algebra of $\mathcal{G}(\partial X)$-equivariant bounded uniform Roe sections of $E^\partial$ is defined in (3.2):

\[
\Gamma_b(E^\partial_{u,k})\mathcal{G}(\partial X) = \bigcup_{k \in \mathbb{N}} \Gamma(E^\partial_{u,k})\mathcal{G}(\partial X).
\]
3.11. For the reduction groupoid 

\( G(\partial X) \), we obtain the following lemma.

\textbf{Definition 4.2.} Let \( G, X, \partial X \) be as above and \( \lambda, \text{Res} \) defined in Proposition 3.11 and (4.5). We define the symbol morphism to be the composition

\[ \varsigma = \text{Res} \circ \lambda: C^*_r(G) \rightarrow \Gamma_b(E^0_{\partial})G(\partial X). \]

It is easy to check that for each \( \omega \in \partial X \), the regular representation \( \lambda_\omega: C^*_r(G) \rightarrow \mathfrak{B}(\ell^2(G_\omega)) \) factors through \( C^*_r(G(\partial X)) \). More precisely, we have the following commutative diagram:

\[
\begin{array}{ccc}
C^*_r(G) & \overset{\lambda_\omega}{\longrightarrow} & \mathfrak{B}(\ell^2(G_\omega)) \\
\downarrow q & & \downarrow \left\| \right. \\
C^*_r(G(\partial X)) & \overset{\lambda_\omega^\partial}{\longrightarrow} & \mathfrak{B}(\ell^2(G(\partial X)_\omega))
\end{array}
\]

where the bottom map \( \lambda_\omega^\partial \) is the regular representation for the reduction \( G(\partial X) \) at \( \omega \). Hence we have

\[ \lambda_\omega(T) = \lambda_\omega^\partial(q(T)). \]

Consequently, we obtain the following lemma.

\textbf{Lemma 4.3.} Notations as above. The following diagram commutes:

\[
\begin{array}{ccc}
C^*_r(G) & \overset{\varsigma}{\longrightarrow} & \Gamma_b(E^0_{\partial})G(\partial X) \\
\downarrow q & & \downarrow \left\| \right. \\
C^*_r(G(\partial X)) & \overset{\lambda^\partial}{\longrightarrow} & \Gamma_b(E^0_{\partial})G(\partial X)
\end{array}
\]

where \( \lambda^\partial \) is the monomorphism defined as in Proposition 3.11 for the reduction groupoid \( G(\partial X) \).

Now we are in the position to state our main result:

\textbf{Theorem 4.4.} Let \( G \) be a locally compact, \( \sigma \)-compact and étale groupoid with compact unit space \( G^{(0)}, X \) be an invariant open dense subset in \( G^{(0)} \) and \( \partial X = G^{(0)} \setminus X \). Suppose the reduction groupoid \( G(\partial X) \) is topologically amenable. Then for any element \( T \) in the reduced groupoid \( C^* \)-algebra \( C^*_r(G) \), the following conditions are equivalent:

1. \( T \) is invertible modulo \( C^*_r(G(X)) \).
2. The image of \( T \) under the symbol morphism, \( \varsigma(T) \), is invertible in \( \Gamma_b(E^0_{\partial})G(\partial X) \).
3. For each \( \omega \in \partial X \), the limit operator \( \lambda_\omega(T) \) is invertible, and

\[ \sup_{\omega \in \partial X} \| \lambda_\omega(T)^{-1} \| < \infty. \]

4. For each \( \omega \in \partial X \), the limit operator \( \lambda_\omega(T) \) is invertible.

Here we only prove the equivalence among conditions (1), (2) and (3). The equivalence between (3) and (4) is left to the next section after some other technical tools are developed.

\textit{Proof of Theorem 4.4.} "(1) \( \iff \) (2) \( \iff \) (3)". First we show that the sequence (4.3) is exact (this is well known, for example see [Ren91, Remark 4.10]). Combining (4.3) and (4.4), we have the following commutative diagram induced by the natural homomorphism from maximal groupoid \( C^* \)-algebras to their reduced counterparts:

\[
\begin{array}{cccc}
0 & \longrightarrow & C^*_\text{max}(G(X)) & \longrightarrow & C^*_\text{max}(G) & \longrightarrow & C^*_\text{max}(G(\partial X)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^*_r(G(X)) & \longrightarrow & C^*_r(G) & \longrightarrow & C^*_r(G(\partial X)) & \longrightarrow & 0
\end{array}
\]

As explained before, the top row is exact and the bottom row is exact at the second and the fourth items. Since \( G(\partial X) \) is amenable, the third vertical map is an isomorphism by Proposition 2.17. Via
an elementary diagram chasing argument, it is easy to see that the bottom row is exact at the third item as well.

Now consider the following commutative diagram coming from Lemma 4.3:

\[
\begin{array}{c}
0 \rightarrow C^*_r(G(X)) \rightarrow C^*_r(G) \xrightarrow{\xi} \Gamma_b(E^0_\partial(G(\partial X)) \xrightarrow{\lambda^0} 0.
\end{array}
\]

Note that the lower horizontal sequence is exact and \( \lambda^0 \) is injective, hence the upper horizontal sequence is also exact. Therefore for \( T \in C^*_r(G) \), we have that \( T \) is invertible modulo \( C^*_r(G(X)) \) if and only if \( \varphi(T) \) is invertible, which proves \( "(1) \Leftrightarrow (2)" \).

Now we move on to \( "(2) \Leftrightarrow (3)" \). Consider the following \( C^* \)-homomorphism

\[ \iota: \Gamma_b(E^0_\partial(G(\partial X)) \rightarrow \prod_{\omega \in \partial X} \mathfrak{G}(\ell^2(G_\omega)) \]

defined by \( \iota(\xi) = (\xi(x))_{x \in \partial X} \), which is clearly injective. Hence for any \( T \in C^*_r(G) \), \( \varphi(T) \) is invertible if and only if \( \iota \circ \varphi(T) \) is invertible in \( \prod_{\omega \in \partial X} \mathfrak{G}(\ell^2(G_\omega)) \), hence the above is also equivalent to the condition that each \( \lambda_\omega(T) \) is invertible and their inverses have uniform bounded norms.

\[ \Box \]

Remark 4.5. Note that the hypothesis of the unit space \( G^{(0)} \) being compact ensures that the reduced groupoid \( C^* \)-algebra \( C^*_r(G) \) is unital. Although the main theorem can be modified to hold in the general case as well, here we only focus on the compact case to simplify the arguments. This also due to the fact that all the examples we study in Section 6 have compact unit spaces.

Remark 4.6. It is clear from Definition 2.16 that if \( G \) is amenable, then both of the reductions \( G(X) \) and \( G(\partial X) \) are amenable. Conversely when \( G(X) \) is amenable, \( G \) is amenable if and only if \( G(\partial X) \) is amenable from the Five Lemma. In Section 6, most of the examples satisfy the condition that \( G(X) \) is amenable, so there is no difference between the hypothesis that \( G \) is amenable and \( G(\partial X) \) is amenable.

Remark 4.7. From the above proof of \( "(1) \Leftrightarrow (2) \Leftrightarrow (3)" \), we know that the assumption of topological amenability is only used to show that the short sequence \( (4.3) \) is exact. Hence we may weaken the amenability hypothesis in Theorem 4.4 to the exactness of \( (4.3) \), and the result that \( "(1) \Leftrightarrow (2) \Leftrightarrow (3)" \) still holds. This observation will be used later in Section 6.2.

5. An Extension of a Result of Exel

This section is mainly devoted to the proof of \( "(3) \Leftrightarrow (4)" \) in Theorem 4.4, based on our generalisation of Exel’s result: Theorem 5.3. Let us explain the idea first. Recall Exel originally proved the following:

Proposition 5.1 ( [Exe14]). Let \( G \) be a locally compact, second countable, étale and amenable groupoid with compact object space, then an element \( a \in C^*_r(G) \) is invertible if and only if \( \lambda_x(a) \) is invertible for every \( x \in G^{(0)} \), where \( \lambda_x \) is the regular representation at \( x \).

Consequently when the groupoid \( G \) in Theorem 4.4 is additionally assumed to be second countable, we may apply Exel’s result to obtain the equivalence between \( (3) \) and \( (4) \) directly. Unfortunately as we will see in Section 6, many groupoids coming from interesting examples, even including the one from the classic limit operator theory for groups (Section 6.1), are not second countable. However all of them are \( \sigma \)-compact, so we would like to study Exel’s type result for \( \sigma \)-compactness.

Our generalisation is more than enough to prove Theorem 4.4 and, as far as we know, it is the most general among these types of results. We choose to prove in its full generality since it might have its own interest. Let us first recall that Nistor and Prudhon extended Exel’s result to the following:

Proposition 5.2 ( [NP17]). Let \( G \) be a locally compact, second countable and amenable groupoid with a Haar system, then an element \( 1 + a \in C^*_r(G)^+ := C^*_r(G) \oplus \mathbb{C} \) is invertible if and only if \( 1 + \lambda_x(a) \) is invertible for every \( x \in G^{(0)} \), where \( \lambda_x \) is the regular representation at \( x \).
And here what we would like to prove is the following:

**Theorem 5.3.** Let $\mathcal{G}$ be a locally compact, $\sigma$-compact and amenable groupoid with a Haar system, then an element $1 + a \in C^*_r(\mathcal{G})^+$ is invertible if and only if $1 + \lambda_x(a)$ is invertible for every $x \in \mathcal{G}^{(0)}$.

Note that the only difference among the above three results is marked by Italic font. Our approach relies on the result from [AG19] that the topology of a $\sigma$-compact groupoid may be approximated in a very controlled way by pseudo-metric topologies. The key idea is that a locally compact pseudo-metric space is $\sigma$-compact if and only if it is second countable. The reader can compare our techniques here with [AD16, Proposition 3.8]. Before we get into the proof of Theorem 5.3, let us show how to use it to finish the proof of Theorem 4.4.

**Proof of Theorem 4.4, “(3) $\iff$ (4)”:** It suffices to show that “(4) $\Rightarrow$ (1)”. Since $\mathcal{G}$ is a locally compact, $\sigma$-compact and étale groupoid with compact unit space $\mathcal{G}^{(0)}$ and $\partial X$ is a closed subset in $\mathcal{G}^{(0)}$, the reduction $\mathcal{G}(\partial X)$ is also locally compact, $\sigma$-compact and étale. Notice that the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G}(\partial X))$ is unital. Moreover by assumption, $\mathcal{G}(\partial X)$ is topologically amenable. Given $T \in C^*_r(\mathcal{G})$ and applying Theorem 5.3 to the groupoid $\mathcal{G}(\partial X)$, condition (4) implies that $q(T)$ is invertible in $C^*_r(\mathcal{G}(\partial X))$. As shown in the proof of “(1) $\iff$ (2)” in Theorem 4.4, we know that the short sequence (4.3) is exact. So condition (1) holds, and we finish the proof.

The rest of this section is devoted to the proof of Theorem 5.3, and divided into three parts. As we need to strengthen the approximations given in [AG19] for our purpose, we go over the notions and basic techniques in the first two subsections, where the theory of uniform spaces is recalled first as the major tool to construct the approximations; then we state and prove our more controlled approximation result and finish the proof of Theorem 5.3.

### 5.1. Review of uniform spaces

Here we recall briefly the theory of uniform spaces, mainly focusing on the fact that they can be presented as inverse limits of metrisable spaces. This is the fundamental building block for the approximations results in [AG19]. We suggest [lsb68] as a standard reference on uniform spaces.

Let $X$ be a set and $U, V$ be covers of $X$. $U$ is said to refine $V$ (equivalently, $V$ coarsens $U$), written $U \triangleleft V$, if each element in $U$ is contained in some element of $V$. For a subset $A \subset X$, we define the star of $A$ against $U$, denoted by $st(A, U)$, to be the set $\bigcup \{U \in U : U \cap A \neq \emptyset\}$. We say that $U$ star refines $V$, written $U \leq V$, if $\{st(U, U) : U \in U\}$ refines $V$. Now we recall the notion of uniform spaces, and a prototypical example is a metric space with covers of positive Lebesgue number.

**Definition 5.4.** A uniform space is a set $X$ with a collection of covers $\mathcal{C}$ of $X$, called the uniform structure on $X$, that is closed under coarsening and such that if $U, V \in \mathcal{C}$ then there exists $W \in \mathcal{C}$ such that $W$ star refines both $U$ and $V$. Elements in $\mathcal{C}$ are called uniform covers. A function $f : X \to Y$ between uniform spaces is uniformly continuous if the pre-image of uniform covers are uniform covers.

If $X$ is a set, we call a sequence of uniform covers $U_0 \geq U_1 \geq U_2 \geq \ldots$ a normal sequence of covers. Note that every normal sequence of covers defines a uniform structure on $X$ by taking all coarsenings of covers from the sequence and, furthermore, every “minimal” uniform structure is obtained in this fashion from a normal sequence of covers. One nice feature of normal sequences is that they correspond exactly to the pseudo-metrizable uniform structures on $X$. We give an outline for reader’s convenience.

For elements $x, y \in X$, let $n(x, y)$ denote the maximum integer $k$ such that $x$ and $y$ are both contained in an element of $U_k$, and $\infty$ if no such maximum exists. Let $\rho : X \times X \to [0, 1)$ be defined by $\rho(x, y) = 2^{-n(x, y)}$, with the convention that $2^{-\infty} = 0$. We observe that $\rho$ itself is not necessarily a pseudo-metric (because it may not satisfy the triangle inequality), but can be mollified to a pseudo-metric $d$ via $d(x, y) = \inf \sum_{i=1}^{n} \rho(x_i, x_{i+1})$, where the infimum is taken over all chains $x = x_1, x_2, \ldots, x_n = y$ in $X$. Write $(X, \{\{U_n\}\})$ to denote the resulting uniform/pseudo-metric structure on $X$, and $X_{(U_n)}$ the resulting metric quotient.

**Definition 5.5.** Let $\{U_n\}_n$ and $\{V_n\}_n$ be two normal sequences of covers. We say that $\{V_n\}_n$ cofinally refines $\{U_n\}_n$, if for every $m \geq 0$ there exists $k(m)$ such that $V_{k(m)} \leq U_m$. 


Notice that \( \{U_n\} \) cofinally refines \( \{V_n\} \) if and only if the identity map \( \text{Id} : (X, \{\{U_n\}\}) \to (X, \{\{V_n\}\}) \) is uniformly continuous; moreover this implies that the canonical map \( X_{\{U_n\}} \to X_{\{V_n\}} \) is uniformly continuous. It turns out that the collection of normal sequences of uniform covers form a directed set under cofinal refinement, and we have the following:

**Proposition 5.6** ([Isb64]). Any uniform space is the inverse limit of metrisable uniform spaces; the inverse system being indexed by the collection of normal sequences with cofinal refinement being the partial ordering.

**Remark 5.7** (Remark on associated topologies). Every uniform structure on a set \( X \) induces a topology by saying that a set \( A \subseteq X \) is a neighborhood of a point \( x \in X \) if there exists a uniform cover \( U \) such that \( st(x, U) \subseteq A \). The uniform spaces we are interested in come from topology, and it is well known that the topologies which are induced by uniform structures are exactly the completely regular topologies. What is less well known is that there is a more canonical class of topological spaces which are intimately linked with uniform structures, namely the paracompact spaces. Indeed, one can define a space to be paracompact if the collection of open covers forms a base for a uniform structure (one must restrict to only finite open covers for completely regular spaces). In fact, by identifying paracompact spaces with this particular uniform structure, one can easily see that continuous maps of paracompact spaces correspond exactly to uniformly continuous maps. The paracompact spaces we are interested in here are locally compact and \( \sigma \)-compact spaces. The strategy in [AG19] and this paper is actually to approximate the underlying uniform structure for a locally compact and \( \sigma \)-compact groupoid, which consequently approximates its induced topological structure.

The following lemma will be used later:

**Lemma 5.8** ([AG19]). Let \( X \) be a locally compact, \( \sigma \)-compact topological space. Then \( X \) is paracompact, hence every open cover admits an open start refinement.

### 5.2. Review of Austin-Georgescu’s approximation result

Now we focus on groupoids, and recall the explicit approximations constructed in [AG19]. We start with the following notion:

**Definition 5.9** ([AG19]). Let \( G \) be a locally compact topological groupoid. An inverse approximation of \( G \) is an inverse system \( \{G_\alpha, q_\alpha^\beta : G_\alpha \to G_\beta\}_{\alpha \in A} \) where each \( G_\alpha \) is a locally compact groupoid and the index set \( A \) is directed, satisfying:

1. for each \( \alpha \geq \beta \in A \), the map \( q_\alpha^\beta : G_\alpha \to G_\beta \) is a proper continuous and surjective groupoid morphism, and moreover, \( q_\alpha^\beta \circ q_\beta^\gamma = q_\alpha^\gamma \) whenever \( \alpha \geq \beta \geq \gamma \);
2. \( q_\alpha^\alpha = id_{G_\alpha} \) for all \( \alpha \in A \); and
3. \( \lim_{\alpha} G_\alpha = G \) in the category of topological groupoids with proper continuous morphisms.

**Remark 5.10.** We denote the canonical projections from \( G \) to the inverse system by \( q_\alpha : G \to G_\alpha \).

As indicated in Theorem 5.3, we are mostly interested in \( \sigma \)-compact spaces. In order to approximate them, the following concept is required and useful:

**Definition 5.11** ([AG19]). Let \( X \) be a locally compact and \( \sigma \)-compact space. We say that a collection \( \{K_n : n \in \mathbb{N}\} \) is an exhaustion of \( X \) by compact subsets, or simply an exhaustion, if each \( K_n \) is a compact neighbourhood in \( X \), \( K_n \subseteq \text{int}(K_{n+1}) \) and such that \( \bigcup_n \text{int}(K_n) = X \).

In the case that \( G \) is a locally compact and \( \sigma \)-compact groupoid, for any exhaustion \( \{K_n\} \) of \( G \), the sequence \( \{K_n^* := K_n \cup r(K_n) \cup s(K_n)\} \) is also an exhaustion of \( G \), and \( \{K_n^* |_{0^{(0)}}\} \) is an exhaustion of the unit space \( G^{(0)} \). We call an exhaustion obtained in this manner a groupoid exhaustion.

Also recall from [AG19] that an open cover of a groupoid \( G \) is a pair \( (\mathcal{W}^1, \mathcal{W}^0) \), where \( \mathcal{W}^1 \) is an open cover of \( G \) and \( \mathcal{W}^0 \) is an open cover of \( G^{(0)} \). The reason for having to take covers of both is that a very important part of a groupoid is its structure as a fibration over its unit space.

Now assume \( G \) is a locally compact and \( \sigma \)-compact groupoid with a fixed groupoid exhaustion \( \{K_n\} \). A crucial technical part in [AG19] is to construct normal sequences of open covers of \( G \) satisfying certain natural but delicately-designed conditions, such that each of the resulting metrisable quotients (see Section 5.1) possesses a compatible second countable and locally compact groupoid structure, and the
quotient maps are proper, continuous and surjective groupoid morphisms ([AG19, Proposition 6.5, Theorem 6.8]). Furthermore, these kind of normal sequences are cofinal in the collection of all normal sequences, thus Proposition 5.6 shows that \( G \) admits an inverse approximation by second countable and locally compact groupoids ([AG19, Theorem 6.10]). Additionally, if the given groupoid \( G \) possesses a Haar system, then the normal sequences above to be modified to ensure that each quotient also possess a Haar system, and the quotients are Haar system preserving ([AG19, Theorem 6.9]).

We would like to study this inverse approximations further, so we have to refer to those specific normal sequences of covers mentioned above. However, we decide not to present all the precise details since they are very technical and require more notions which are only used to ensure that the resulting metrisable quotient possess a groupoid structure. Therefore, we choose the following neutral way to put them into a “black box”, and guide the interested readers to the original paper [AG19] for details.

**Definition 5.12** ([AG19, Definition 6.6]. A normal sequence of open covers which satisfies Properties (1), (3)-(7) in [AG19, Proposition 6.5] is called a groupoid normal sequence for \( G \).

Now the above analysis can be converted into the following result:

**Proposition 5.13** ([AG19, Theorem 6.8]. Let \( G \) be a locally compact and \( \sigma \)-compact groupoid. For any normal sequence \( \{W_n\} \) of open covers of \( G \), there exists a groupoid normal sequence \( \alpha = \{U_n\} \) of open covers of \( G \) cofinally refining \( \{W_n\} \), and the induced metrisable quotient \( G_\alpha \) is a locally compact and second countable groupoid. Moreover, the quotient map \( q_\alpha : G \to G_\alpha \) is proper and continuous, and the pre-image of the cover of \( G_\alpha \) by balls of radius \( \frac{1}{2^n} \) refines \( W_n \).

Via a classic diagonal argument, we obtain the following version dealing with a countable family of normal sequences simultaneously. This result will be used several times later.

**Proposition 5.14.** Let \( G \) be a locally compact and \( \sigma \)-compact groupoid. For each \( l \in \mathbb{N} \), suppose \( \{W_n^l\}_n \) is a normal sequence of open covers of \( G \). Then there exists a groupoid normal sequence \( \alpha = \{U_n\} \) of open covers of \( G \) cofinally refining \( \{W_n^l\}_n \) for every \( l \), and the induced metrisable quotient \( G_\alpha \) is a locally compact and second countable groupoid. Moreover, the quotient map \( q_\alpha : G \to G_\alpha \) is proper and continuous, and the sequence of open covers consisting of pre-image of the cover of \( G_\alpha \) by balls of radius \( \frac{1}{2^n} \) cofinally refines \( \{W_n^l\}_n \) for each \( l \).

### 5.3. Strengthened Approximation Theorem

Now we would like to study the amenabilities of the inverse approximations given in [AG19], and use them to prove Theorem 5.3. Unfortunately, the groupoids \( G_\alpha \)'s in the inverse approximation being amenable does not automatically follow from the fact that they are quotients (i.e. topological quotients such that the quotient map is a groupoid morphism) of the amenable groupoid \( G \). The following example provides a counterexample in the general case.

**Example 5.15.** Let \( \Gamma \) be a countable discrete and exact group that is not amenable (a free group on two generators will do). Notice that the Stone-Čech compactification \( \beta \Gamma \) equivariantly quotients to the Alexandroff one-point compactification \( \Gamma^+ \) and hence the transformation groupoid \( \beta \Gamma \times \Gamma \) quotients to \( \Gamma^+ \times \Gamma \). Note that \( \Gamma^+ \times \Gamma \) is not amenable while \( \beta \Gamma \times \Gamma \) is.

However in the setting of Proposition 5.14, we may modify the groupoid normal sequences of covers to ensure that each quotient is still amenable. The idea is inspired by [AG19, Theorem 6.9] where it is proved that a Haar system of measures on \( G \) can be pushed forward to a Haar system on the metric quotient induced by the groupoid normal sequences. To do so, we generalise the result to a sequence of systems of measures which were introduced by Renault in [Ren87].

**Definition 5.16.** Let \( X, Y \) be locally compact spaces, and \( \pi : X \to Y \) be a continuous open surjective map. A \( \pi \)-system of measures is a collection of positive Radon measures \( \{m_y : y \in Y\} \) on \( X \) such that \( m_y \) is supported on \( \pi^{-1}(y) \) which is continuous in the sense that for every \( f \in C_0(X) \), the function \( y \mapsto f (y) \) is continuous on \( Y \).

**Definition 5.17.** Let \( X_i, Y_i \) be locally compact spaces, \( \pi_i : X_i \to Y_i \) be continuous open surjective maps for \( i = 1, 2 \), and \( \{m_y : y \in Y_1\} \), \( \{m_y : y \in Y_2\} \) be \( \pi_i \)-system of measures, respectively. Suppose \( f : X_1 \to X_2 \), \( g : Y_1 \to Y_2 \) are proper and continuous functions satisfying \( \pi_2 \circ f = g \circ \pi_1 \). Then \( f \) is said to be measure-preserving if for any \( y \in Y_1 \), we have \( \int_X f \, dm_y = \int_Y f \circ \pi \, dm_y \).
The $\pi$-system we are interested in comes from the source map $s$ of groupoids. Notice that a Haar system is an example of an $s$-system of measures. When the groupoid has a topological approximate invariant mean $\{m^{(k)}\}_{k \in \mathbb{N}}$ (Definition 2.16), then each $m^{(k)}$ is also an $s$-system of measures. Now we state the following technical lemma, dealing with a sequence of $\pi$-systems simultaneously. The proof follows almost the same as that of [AG19, Theorem 6.5] together with a diagonal argument, so we only provide the sketch here.

**Lemma 5.18.** Let $\mathcal{G}$ be a locally compact and $\sigma$-compact groupoid with a fixed groupoid exhaustion $\{K_n\}$. For each $l \in \mathbb{N}$, let $\{W^l_n\}_n$ be a normal sequence of open covers of $\mathcal{G}$. For each $k \in \mathbb{N} \cup \{0\}$, suppose $m^{(k)} = \{m^{(k)}_x : x \in \mathcal{G}^{(0)}\}$ is a $\pi$-system of measures for some open surjective continuous map $\pi : \mathcal{G} \to \mathcal{G}^{(0)}$. Then the groupoid normal sequence $\{\mathcal{U}_n = (\mathcal{U}^0_n, \mathcal{U}^1_n)\}$ in Proposition 5.14 can be modified to satisfy the following additional condition:

- Fix $\{f_j^n : j \in J_n\}$ a finite partition of unity of $K_n$ whose supports refine $\mathcal{U}^1_n$. Let $(\lambda_j)_j \in \mathbb{C}$ be any sequence with $|\lambda_j| < n$. Then for each open set $U \in \mathcal{U}^1_{n+1}$ and any $x, y \in s(U)$ and for all $k \leq n$, we have

$$\left| \int_{\mathcal{G}} \left( \sum_j \lambda_j f_j^n \right) dm^{(k)}_x - \int_{\mathcal{G}} \left( \sum_j \lambda_j f_j^n \right) dm^{(k)}_y \right| < \frac{1}{n}.$$ 

**Sketch of the proof:** Given $n \in \mathbb{N}$, suppose $\mathcal{U}_1, \ldots, \mathcal{U}_n$ have been chosen to satisfy the requirements for a groupoid normal cover (Definition 5.12) and the above condition such that for $k = 1, \ldots, n$, each $\mathcal{U}_k$ refines $\{W^l_n\}$ for $l \leq k$. Note that the maps $x \mapsto \int_{\mathcal{G}} f_j^n dm^{(k)}_x$ is continuous for each $j$ and $k = 1, \ldots, n$. Hence we can find a cover $V_0^n$ of $\mathcal{G}^{(0)}$ such that for each $x, y \in V_0^n$ and $|\lambda_j| < n$, we have

$$\left| \int_{\mathcal{G}} \left( \sum_j \lambda_j f_j^n \right) dm^{(k)}_x - \int_{\mathcal{G}} \left( \sum_j \lambda_j f_j^n \right) dm^{(k)}_y \right| < \frac{1}{n}$$

for all $k \leq n$. Define $V^1_n := s^{-1}(V_0^n)$, and we obtain an open cover $V_n = (V_0^n, V^1_n)$ of $\mathcal{G}$. Furthermore, after shrinking $V_n$ if necessary, we may assume that $V_n$ refines $\{W^l_n\}$ for $l \leq n+1$. Now we can construct an open cover $\mathcal{U}_{n+1}$ of $\mathcal{G}$ such that it refines $V_n$ and satisfies the condition for groupoid normal sequences, as done in [AG19, Theorem 6.5].

Now we are ready to prove the following approximation result for systems of measures, which generalises [AG19, Theorem 6.9] where they studied the case of Haar system.

**Proposition 5.19.** Let $\mathcal{G}$ be a locally compact and $\sigma$-compact groupoid, and $\{W^l_n\}_{n \in \mathbb{N}}$ be a normal sequence of open covers on $\mathcal{G}$ for each $l \in \mathbb{N}$. Suppose for each $k \in \mathbb{N}$, $m^{(k)} = \{m^{(k)}_x : x \in \mathcal{G}^{(0)}\}$ is an $\bar{s}$-system of measures on $\mathcal{G}$. Then the metrisable quotient $\mathcal{G}_a$ in Proposition 5.14 can be arranged so that for each $k \in \mathbb{N}$, there is an $\bar{s}$-system of measures $\overline{m}^{(k)} = \{\overline{m}^{(k)}_{\bar{x}} : \bar{x} \in \mathcal{G}^{(0)}\}$ (here $\bar{s}$ is the source map of $\mathcal{G}_a$) and the quotient map $q_a$ is measure preserving.

**Proof.** Fix a groupoid exhaustion $\{K_n\}$ of $\mathcal{G}$, and assume $\{\mathcal{U}_n\}$ is the groupoid normal sequences obtained in Lemma 5.18. Given $f \in C_c(\mathcal{G}_a)$ and following the argument in [AG19, Theorem 6.9], we know that for any $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists an $n$ such that if $x, y \in U \in \mathcal{U}^0_n$, then

$$\left| \int (q_a)^*(f) dm^{(k)}_x - \int (q_a)^*(f) dm^{(k)}_y \right| < \varepsilon.$$ 

Hence $q_a(x) = q_a(y)$ implies $\int (q_a)^*(f) dm^{(k)}_x = \int (q_a)^*(f) dm^{(k)}_y$. For each $\bar{x} \in \mathcal{G}_a$, take a pre-image $x \in \mathcal{G}$, then the positive linear functional on $C_c(\mathcal{G}_a)$ by $f \mapsto \int (q_a)^*(f) dm^{(k)}_{\bar{x}}$ provides a unique positive Radon measure $\overline{m}^{(k)}_{\bar{x}}$ on $\mathcal{G}_a$ supported on $(\mathcal{G}_a)_{\bar{x}}$ for each $k \in \mathbb{N}$, by Riesz-Markov-Kakutani Theorem. From the same argument as in [AG19], it is routine to check that $\overline{m}^{(k)}_{\bar{x}}$ satisfies the requirements.

As a direct corollary, we obtain the following:

**Corollary 5.20.** Let $\mathcal{G}$ and $\{W^l_n\}_{n \in \mathbb{N}}$ be as above. Suppose $\{m^{(k)}\}$ is a topological approximation invariant mean on $\mathcal{G}$, then $\{\overline{m}^{(k)}\}$ in the above proposition forms a topological approximation invariant mean on $\mathcal{G}_a$. Consequently, when $\mathcal{G}$ is amenable, the approximation $\mathcal{G}_a$ can be made amenable as well.
Also note that we recover [AG19, Theorem 6.9] as a special case when taking the Haar system. Furthermore in the ´etale case, we may arrange the quotients to satisfy the following stronger condition. Note that it was already proved in [AG19] that the approximation \( \mathcal{G}_\alpha \) can be made ´etale in this case.

**Proposition 5.21.** Let \( \mathcal{G} \) be a locally compact, \( \sigma \)-compact and ´etale groupoid, and \( \{ \mathcal{W}_n \}_{n \in \mathbb{N}} \) be a normal sequence of open covers on \( \mathcal{G} \) for each \( l \in \mathbb{N} \). Then the metrisable quotient \( \mathcal{G}_\alpha \) in Proposition 5.14 can be arranged so that \( \mathcal{G}_\alpha \) is ´etale, and the quotient map \( q_\alpha \) induces a bijection between fibres \( \mathcal{G}_\alpha \) and \( (\mathcal{G}_\alpha)_{q_\alpha(x)} \) for any \( x \in \mathcal{G}^{(0)} \).

**Proof.** Following [AG19, Section 6.1.3], recall that a groupoid \( \mathcal{G} \) is ´etale if and only if \( \mathcal{G}^{(0)} \) is open in \( \mathcal{G} \) and \( \mathcal{G} \) admits a Haar system. Since \( \mathcal{G} \) is ´etale, there exists an open cover \( \mathcal{V} \) of \( \mathcal{G} \) consisting of bisections (recall that a bisection is a subset \( V \) that the source and range maps are homeomorphisms restricted on \( V \)). Since \( \mathcal{G}^{(0)} \) is clopen, we may also assume that any element \( V \) in \( \mathcal{V} \) has the property that either \( V \subseteq \mathcal{G}^{(0)} \) or \( V \subseteq (\mathcal{G} \setminus \mathcal{G}^{(0)}) \). Hence by Lemma 5.8, we may shrink each \( \mathcal{W}_n \) to ensure that \( \mathcal{W}_n \) refines \( \mathcal{V} \).

Now applying Proposition 5.19 to a Haar system, we obtain a groupoid normal sequence \( \mathcal{U}_n \) of open covers of \( \mathcal{G} \) satisfying the conditions therein. Consequently, \( \mathcal{G}_\alpha \) has an induced Haar system. Note that each \( \mathcal{U}_n \) refines \( \mathcal{V} \), so \( \mathcal{G}^{(0)} \) is also open in \( \mathcal{G}_\alpha \). Hence we obtain that \( \mathcal{G}_\alpha \) is ´etale as well. Moreover, since different points on each fibre cannot be identified under the quotient map (this is due to the fact that \( \mathcal{U}_n \) consists of bisections for all \( n \)), so the restriction \( q_\alpha|_\mathcal{G}_\alpha : \mathcal{G}_\alpha \rightarrow (\mathcal{G}_\alpha)_{q_\alpha(x)} \) is injective for any \( x \in \mathcal{G}^{(0)} \).

As for surjection, note that \( q_\alpha \) is measure preserving, hence \( q_\alpha(x) = q_\alpha(y) \) implies \( \sum_{\gamma \in \mathcal{G}_\alpha} ((q_\alpha)_* f)(\gamma) = \sum_{\gamma \in \mathcal{G}_\alpha} ((q_\alpha)_* f)(\gamma) \) for any \( f \in C_c(\mathcal{G}_\alpha) \). Consequently, we obtain that \( q_\alpha(\mathcal{G}_\alpha) = q_\alpha(\mathcal{G}_\alpha) \) provided \( q_\alpha(x) = q_\alpha(y) \). Note that

\[
(\mathcal{G}_\alpha)_{q_\alpha(x)} = \bigcup_{y \in (\mathcal{G}_\alpha)_{q_\alpha(x)}} q_\alpha(\mathcal{G}_y),
\]

so the result holds. \( \square \)

It is not hard to see from the above proofs that Corollary 5.20, Proposition 5.21 and [AG19, Theorem 6.9] can be dealt with simultaneously. In fact, we may apply Proposition 5.19 for the sequence \( \{m^{(k)}\} \) consisting of the Haar system and the topological approximation invariant mean at the same time. This is the reason how we design Proposition 5.19. Consequently, we obtain the following:

**Proposition 5.22.** Let \( \mathcal{G} \) be a locally compact, \( \sigma \)-compact and amenable groupoid with a Haar system. Let \( \{ \mathcal{W}_n \}_{n \in \mathbb{N}} \) be a normal sequence of open covers of \( \mathcal{G} \) for each \( l \). Then the metrisable quotient \( \mathcal{G}_\alpha \) obtained in Proposition 5.14 can be arranged so that \( \mathcal{G}_\alpha \) is amenable, and the Haar system on \( \mathcal{G} \) can be pushforwarded to a Haar system on \( \mathcal{G}_\alpha \), such that the quotient map \( q_\alpha \) is measure preserving. Additionally if \( \mathcal{G} \) is ´etale, then \( \mathcal{G}_\alpha \) can be arranged to be ´etale as well and \( q_\alpha \) induces bijections between the corresponding fibres.

Finally we focus on the associated groupoid C*-algebras. Roughly speaking, as the given groupoid \( \mathcal{G} \) is the inverse limit of the metrisable quotients \( \mathcal{G}_\alpha \), the groupoid C*-algebra \( \mathcal{C}^*(\mathcal{G}) \) is the inductive limit of \( \mathcal{C}^*(\mathcal{G}_\alpha) \). However, in order to prove Theorem 5.3, we need to show that \( \mathcal{C}^*(\mathcal{G}) \) is exactly the union of the images of \( \mathcal{C}^*(\mathcal{G}_\alpha) \), which might be anti-intuitive at first glance. The following proposition (which is a minor adaptation to [AG19, Proposition 5.11]) will give us the necessary equipment for this to happen.

**Lemma 5.23.** Let \( X \) be a locally compact and \( \sigma \)-compact space, and \( \{X_\alpha, q_\alpha^\beta, A\} \) be an inverse approximation of \( X \) by metric spaces. For every \( \alpha \), denote \( q_\alpha \) the canonical projection map \( X \rightarrow X_\alpha \), and for each \( n \), let \( \mathcal{V}_n \) be the cover of \( X \) given by the pre-images of the cover of \( X_\alpha \) by \( \frac{1}{2^n} \)-balls. Suppose for every countable family of normal sequences \( \{\{\mathcal{V}_n^l\}_{n \in \mathbb{N}} : l \in \mathbb{N}\} \) in \( X \), there exists an \( \alpha \) such that the normal sequence \( \{\mathcal{V}_n^l\}_{n \in \mathbb{N}} \) cofinally refines \( \{\mathcal{V}_n^l\}_{n \in \mathbb{N}} \) for every \( l \in \mathbb{N} \). Then for any countable family of continuous functions \( f_l : X \rightarrow Y_l \) where \( Y_l \) is a second countable metric space, there exists an \( \alpha \) such that each \( f_l \) is the pullback of a uniformly continuous function \( f_{l,\alpha} : X_\alpha \rightarrow Y_l \), i.e. \( f_l = f_{l,\alpha} \circ q_\alpha \).

**Proof.** Notice that each \( f_{l,\alpha} \) induces a normal sequence \( \{\mathcal{W}_n^l\}_{n \in \mathbb{N}} \) of open covers on \( X \) by taking the pre-image of the normal sequence of open covers of \( Y_l \) given by \( \frac{1}{2^n} \)-balls. Let \( \alpha \in A \) be provided from the hypothesis with respect to the family \( \{\mathcal{W}_n^l\}_{n \in \mathbb{N}} \), such that the normal sequence \( \{\mathcal{V}_n^\alpha\}_{n \in \mathbb{N}} \) cofinally
refines all \(\{W_n^l\}_{n\in\mathbb{N}}\). It follows that if \(q_n(x) = q_n(y)\) for \(x, y \in X\), then \(f_l(x) = f_l(y)\) for all \(l \in \mathbb{N}\). Thus we may define \(f_{l, \alpha} : X_{\alpha} \to Y_l\) by \(f_{l, \alpha}(x_{\alpha}) = f_l(x)\). It is routine to check that \(f_{l, \alpha}\) is continuous for each \(l \in \mathbb{N}\) by the choice of \(\alpha\).

The above lemma suggests the following notion:

**Definition 5.24.** An inverse approximation \(\{X_{\alpha}, q_{\alpha}^0, A\}\) of a uniform space \(X\) by metric spaces is called a **strong inverse approximation**, if for every countable family of normal sequences \(\{\{W_n^l\}_{n \in \mathbb{N}} : l \in \mathbb{N}\}\) of uniform covers, there exists an \(\alpha\) such that the normal sequence \(\{V_n^l\}_{n \in \mathbb{N}}\) obtained as in Lemma 5.23 cofinally refines \(\{W_n^l\}_{n \in \mathbb{N}}\) for every \(l \in \mathbb{N}\).

Consequently, we are ready to prove the following strengthened version of the approximation result:

**Theorem 5.25.** Let \(G\) be a locally compact, \(\sigma\)-compact and amenable groupoid with a Haar system. Then there exists a strong inverse approximation \(\{G_{\alpha}, q_{\alpha}^0, A\}\) of \(G\) by locally compact, second countable, amenable and metrisable groupoids with Haar systems such that the quotient \(q_{\alpha} : G \to G_{\alpha}\) is Haar measure preserving. Moreover, for the induced direct system of topological star algebras \(\{C_{c}(G_{\alpha}), (q_{\alpha}^0)^*, A\}\), we have that \(C_{c}(G) = \bigcup_\alpha (q_{\alpha})^* C_{c}(G_{\alpha})\) and \(C_{c}(G)^* = \bigcup_\alpha (q_{\alpha})^* C_{c}(G_{\alpha})^*\).

**Proof.** We follow the proof of [AG19, Theorem 6.10], and let \(A\) denote all groupoid normal sequences that satisfy the condition in Lemma 5.18. By Proposition 5.22, we only need to prove the last statement.

Let \(a \in C_{c}(G)^*\) and \(a_{\alpha} \in C_{c}(G_{\alpha})\) such that \(a\) is the max-norm limit of the \(a_{\alpha}\)'s. Since the approximation is a strong groupoid approximation, the conclusion of Lemma 5.23 holds. Thus there exists \(\alpha \in A\) such that \(a_{\alpha} \in (q_{\alpha})^* C_{c}(G_{\alpha})\). As the map \((q_{\alpha})^* : C_{c}(G_{\alpha}) \to C_{c}(G)^*\) is a norm-preserving isometric embedding, we obtain that \(1 + a_{\alpha}\) is invertible in \(\mathcal{B}(L^2(G_{\alpha}; \mu_{\alpha}))\), and we would like to show that \(1 + a_{\alpha}\) converges to some element \(\tilde{a}\) in \(C_{c}(G)^*\). Then we have

\[
a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} (q_{\alpha})^*(\tilde{a}_n) = (q_{\alpha})^*(\tilde{a}).\]

It follows that \(C_{c}(G)^* = \bigcup_\alpha (q_{\alpha})^* C_{c}(G_{\alpha})^*\), so we finish the proof.

**Proof of Theorem 5.3.** Suppose \(\{\mu_x : x \in G^{(0)}\}\) is the given Haar system on \(G\). By Theorem 5.25, we may take a strong inverse approximation \(\{G_{\alpha}, q_{\alpha}^0, A\}\) of \(G\) by locally compact, second countable and amenable groupoids with Haar systems \(\{\mu_x : x \in G_{\alpha}^{(0)}\}\) such that the quotient map \(q_{\alpha} : G \to G_{\alpha}\) is measure preserving and induces an isometric embedding

\[
U_x : L^2((G_{\alpha}); \mu_{q_{\alpha}(x)}) \longrightarrow L^2(G_{\alpha}; \mu_{\alpha}).
\]

Furthermore for the given \(a \in C_{c}(G)^*\), there exists \(\alpha \in A\) such that \(a = (q_{\alpha})^*(\tilde{a})\) for some \(\tilde{a} \in C_{c}(G_{\alpha})\).

Now fix \(x \in G^{(0)}\) and denote \(\tilde{x} := q_{\alpha}(x) \in G_{\alpha}^{(0)}\). By assumption, we know that \(1 + \lambda_{\alpha}(a)\) is invertible in \(\mathcal{B}(L^2(G_{\alpha}; \mu_{\alpha}))\). In order to do so, we make the following claim:

**Claim:** The subspace \(\text{Im} U_x\) is reduced with respect to \(\lambda_x(a)\), i.e., both \(\text{Im} U_x\) and its orthogonal complement are invariant under \(\lambda_x(a)\). Furthermore, \(\lambda_{\alpha}(a)\) coincides with \(U_{\tilde{x}}(\tilde{a})U_{\tilde{x}}^*\) on \(\text{Im} U_x\).

Before we prove the claim, first let us use it to finish the proof of the theorem. It is implied by the claim that \(1 + \lambda_x(a)\) has a block diagonal form with respect to the following decomposition:

\[
L^2(G_x; \mu_x) = \text{Im} U_x \oplus (\text{Im} U_x)^\perp.
\]

And the block corresponding to \(\text{Im} U_x\) is unitary equivalent to \(1 + \lambda_{\tilde{x}}(\tilde{a})\) via the unitary \(U_{\tilde{x}}\) onto its image. Hence \(1 + \lambda_{\tilde{x}}(\tilde{a})\) is invertible in \(\mathcal{B}(L^2((G_{\alpha}); \mu_{\alpha}))\), and we may apply Proposition 5.2 to \(G_{\alpha}\) directly since it is a locally compact, second countable and amenable groupoid with a Haar system of measures. Therefore, we reobtain that \(1 + \tilde{a}\) is invertible in \(C_{c}(G_{\alpha})^*\). Since \((q_{\alpha})^* : C_{c}(G_{\alpha}) \to C_{c}(G)^*\) is an isometric embedding, we obtain that \(1 + a = (q_{\alpha})^*(1 + \tilde{a})\) is invertible in \(C_{c}(G)^* = C_{c}(G)^*\) as well. This finishes the proof, so long as the claim is correct.
To prove the claim: first consider \( a = (q_\alpha)^*(\tilde{a}) \in C_c(\mathcal{G}) \) where \( \tilde{a} \in C_c(\mathcal{G}_\alpha) \). For any \( \xi = U_x(\tilde{\xi}) \) with \( \tilde{\xi} \in L^2((\mathcal{G}_\alpha)\xi; \mu_{\xi}) \) and any \( \gamma \in \mathcal{G}_x \), we calculate:

\[
(\lambda_x(a))(\xi)(\gamma) = \int_{\mathcal{G}_x} a(\gamma^{1}\xi(\delta))d\mu_x(\delta)
= \int_{\mathcal{G}_x} \tilde{a}(q_\alpha(\gamma)q_\alpha(\delta)^{-1})\xi(q_\alpha(\delta))d\mu_x(\delta)
= \int_{(\mathcal{G}_\alpha)_x} \tilde{a}(q_\alpha(\gamma)\delta^{-1})\tilde{\xi}(\tilde{\delta})d\mu_{\tilde{\xi}}(\tilde{\delta})
= U_x(\lambda_x(\tilde{a})(\tilde{\xi}))(\gamma),
\]

where we use the fact the \( q_\alpha \) is measure preserving in the third inequality. This implies that

\[
\lambda_x(a) \circ U_x = U_x \circ \lambda_x(\tilde{a}).
\]

By extension, the above holds for \( a = (q_\alpha)^*(\tilde{a}) \in q_\alpha^*(C^*_\text{max}(\mathcal{G}_\alpha)) \) where \( \tilde{a} \in C^*_\text{max}(\mathcal{G}) \). Note that the algebra \( (q_\alpha)^*(C^*_\text{max}(\mathcal{G}_\alpha)) \) is a \( C^* \)-algebra, hence the claim holds and we finish the proof. \( \square \)

Remark 5.26. Notice that by Proposition 5.21, the above proof simplifies considerably when the groupoid \( \mathcal{G} \) is étale since \( U_x : L^2((\mathcal{G}_\alpha)_{\mu_{\alpha}(x)}/(\mathcal{G}_\alpha)_{\mu_{\alpha}(x)}) = L^2(\mathcal{G}_x; \mu_x) \) is an isometric isomorphism and, furthermore, \( U_x \) establishes a unitary equivalence between the representations \( \lambda_x \) and \( \lambda_{\mu_{\alpha}(x)} \) on \( C^*_r(\mathcal{G}_\alpha) \).

6. Applications

In Section 4, we established the limit operator theory for groupoids (Theorem 4.4). Now we provide various applications to different groupoids, recovering the classic limit operator theory in the Hilbert space case for exact groups [Roe05, RRS04], for spaces with Property A [SW17], and also establishing new limit operator theories for amenable group actions, uniform Roe algebras for certain groupoids, and amenable groupoid actions.

6.1. Discrete group case. Our first application is to recover the limit operator theory for exact groups studied by Roe [Roe05], which includes the classic limit operator theory of Rabinovich, Roch and Silbermann [RRS04] in the Hilbert space case.

Let \( G \) be a finitely generated discrete exact group, and \( \beta G \) be its Stone-Čech compactification. Clearly, the left multiplication of \( G \) on itself extends naturally to an action of \( G \) on \( \beta G \) by homeomorphisms. We consider the associated transformation groupoid \( \mathcal{G} := \beta G \rtimes G \). Its unit space \( \mathcal{G}^{(0)} = \beta G \) can be decomposed into

\[
\beta G = G \sqcup \partial G,
\]

where \( \partial G \) is the Stone-Čech boundary. Clearly, \( G \) is an open dense invariant subset in \( \beta G \) and we have

\[
\mathcal{G}(G) = G \times G, \quad \text{and} \quad \mathcal{G}(\partial G) = \partial G \times G.
\]

Since \( G \) is exact, the groupoid \( \beta G \rtimes G \) is topologically amenable (see Example 2.18). Therefore we may apply Theorem 4.4 to the groupoid \( \mathcal{G} = \beta G \rtimes G \) together with the decomposition (6.1), and obtain the associated limit operator theory for \( \mathcal{G} \). Now we would like to provide a more detailed picture and compare it with the classic limit operator theory for exact groups studied by Roe [Roe05].

First we focus on the groupoid \( C^* \)-algebra \( C^*_r(\mathcal{G}) = C^*_r(\beta G \rtimes G) \). Equipping \( G \) with a proper right-invariant metric, we write \( [G] \) to denote the associated metric space. For any \( \gamma \in G \), let \( L_{\gamma}, R_{\gamma} \) be the unitary operators on \( \ell^2(G) \) induced by left and right multiplication by \( \gamma \). Denote \( \mathbb{C}[\mathcal{G}] \) the \( * \)-subalgebra in \( \mathcal{B}(\ell^2(G)) \) generated by the unitaries \( L_{\gamma} \) and the diagonal matrices \( \ell^\infty(G) \), and \( C^*_r([G]) \) its norm closure in \( \mathcal{B}(\ell^2(G)) \). For coarse geometers, elements in \( \mathbb{C}[\mathcal{G}] \) are said to have finite propagation, and \( C^*_r([G]) \) is called the uniform Roe algebra of \( G \).

It is not hard to see that the groupoid \( C^* \)-algebra \( C^*_r(\mathcal{G}) = C^*_r(\beta G \rtimes G) \) is isomorphic to the uniform Roe algebra \( C^*_u([G]) \). More precisely, we have the following algebraic \( * \)-isomorphism \( \theta : C^*_r(\beta G \rtimes G) \to \mathbb{C}[\mathcal{G}] \) defined by

\[
\theta(f)(\gamma, \gamma_1) := f(\gamma, \gamma_1^{-1}).
\]
One can verify that $\theta$ is an isometry with respect to the reduced norm on $C_{c}^{\ast}(\beta G \times G)$ and the operator norm on $\mathbb{C}[[G]]$ (see for example [Roe03]). Hence $\theta$ can be extended to an isomorphism

$$\Theta : C^{\ast}_{u}(\beta G \times G) \rightarrow C^{\ast}_{u}(G).$$

Furthermore, it is not hard to see that $\Theta(C^{\ast}_{u}(G \times G)) = \mathfrak{R}(\ell^{2}(G))$. Consequently for $T \in C^{\ast}_{u}(\beta G \times G)$, $T$ is invertible modulo $C^{\ast}_{u}(G \times G)$ if and only if $\Theta(T)$ is Fredholm.

Now we move on to the discussion of limit operators. Let us first recall the classic definition of limit operators introduced by Roe, which is also equivalent to the one defined by Rabinovich, Roch and Silbermann in the case of $\mathbb{Z}^{n}$. Following [Roe05], given $T \in C^{\ast}_{u}([G])$ the map

$$G \rightarrow C^{\ast}_{0}([G]), \quad \gamma \mapsto R_{\gamma}^{\ast}TR_{\gamma}$$

has range in a WOT-compact set and hence extends uniquely to a WOT-continuous map

$$\sigma(T) : \partial G \rightarrow C^{\ast}_{u}([G]).$$

So we obtain a $C^{\ast}$-homomorphism $\sigma : C^{\ast}_{u}(G) \rightarrow C^{\ast}_{WOT}(\partial G, C^{\ast}_{u}([G]))$. Furthermore, as Willett pointed out in his thesis [Wil09], the image of $\sigma$ sits in the set of $G$-equivariant functions $C^{\ast}_{WOT}(\partial G, C^{\ast}_{u}([G]))^{G}$ in the following sense: a function $F \in C^{\ast}_{WOT}(\partial G, C^{\ast}_{u}([G]))^{G}$ is $G$-equivariant if $F(\gamma, \omega) = R_{\gamma}^{\ast}F(\omega)R_{\gamma}$ for any $\gamma \in G$ and $\omega \in \partial G$. Therefore, we obtain a $C^{\ast}$-homomorphism, also called the symbol morphism $^{2}$,

$$\sigma : C^{\ast}_{u}(G) \rightarrow C^{\ast}_{WOT}(\partial G, C^{\ast}_{u}([G]))^{G}.$$

For any $\omega \in \partial G$, the limit operator of $T$ at $\omega$ is defined to be $\sigma_{\omega}(T) := \sigma(T)(\omega)^{3}$. It is obvious that the map $T \mapsto \sigma_{\omega}(T)$ is an endomorphism of $C^{\ast}_{u}(G)$.

On the other hand, we have the symbol morphism associated to $\mathcal{G} = \partial G \times G$ from Definition 4.2:

$$\zeta : C^{\ast}_{u}(\mathcal{G}) \rightarrow \Gamma_{b}(E_{\omega}^{\partial G \times G}).$$

Now we would like to compare the above two morphisms $\sigma$ and $\zeta$, and show that they coincide under the isomorphism $\Theta$. Fix $\omega \in \partial G \subset \mathcal{G}^{(0)}$, and note that

$$\mathcal{G}_{\omega} = \{(g\omega, g) \in \mathcal{G} : g \in G\}$$

is bijective to $G$ via the map $(g\omega, g) \mapsto g$. This induces a unitary $U_{\omega} : \ell^{2}(\mathcal{G}_{\omega}) \cong \ell^{2}(G)$, and hence an isomorphism $\text{Ad}_{U_{\omega}} : \mathfrak{B}(\ell^{2}(\mathcal{G}_{\omega})) \cong \mathfrak{B}(\ell^{2}(G))$. These isomorphisms can be put together and provide a map between the operator fibre spaces as follows:

$$\text{Ad}_{U} = \bigcup_{\omega \in \partial G} \text{Ad}_{U_{\omega}} : E^{\partial} = \bigcup_{\omega \in \partial G} \mathfrak{B}(\ell^{2}(G)) \rightarrow \partial G \times \mathfrak{B}(\ell^{2}(G)),$$

where the final item is regarded as a trivial fibre space over $\partial G$, and the topology coincides with the product topology. It is not hard to check that $\text{Ad}_{U}$ is a homeomorphism, hence an isomorphism between operator fibre spaces over $\partial G$. Note that continuous sections of $\partial G \times \mathfrak{B}(\ell^{2}(G))$ coincides with $C^{\ast}_{WOT}(\partial G, \mathfrak{B}(\ell^{2}(G)))$, hence $\text{Ad}_{U}$ induces the following $C^{\ast}$-isomorphism:

$$(\text{Ad}_{U})_{\ast} : \Gamma(E^{\partial}) \rightarrow \Gamma(\partial G \times \mathfrak{B}(\ell^{2}(G))) = C^{\ast}_{WOT}(\partial G, \mathfrak{B}(\ell^{2}(G))).$$

Note that elements in $C^{\ast}_{WOT}(\partial G, \mathfrak{B}(\ell^{2}(G)))$ have uniformly bounded norms, hence $\Gamma(E^{\partial}) = \Gamma_{b}(E^{\partial})$.

**Lemma 6.1.** $(\text{Ad}_{U})_{\ast} \Gamma(E^{\partial})^{G \times G} = C^{\ast}_{WOT}(\partial G, \mathfrak{B}(\ell^{2}(G)))^{G}$.

**Proof.** Given $\varphi \in \Gamma(E^{\partial})$, by definition $\varphi$ is $\partial G \times G$-equivariant if and only if for any $(g\omega, g) \in \partial G \times G$, we have

$$\varphi(g\omega) = R_{(g\omega, g)}^{\ast} \varphi(\omega) R_{(g\omega, g)}.$$

---

$^{2}$As we will see in Proposition 6.4, this coincides with the symbol morphism we defined in Definition 4.2 up to isomorphisms. The same happens in the next four subsections and we will not explain anymore

$^{3}$As we will see in Lemma 6.3, this coincides with the limit operator we defined in Definition 4.1 up to isomorphisms. The same happens in the next four subsections and we will not explain anymore
Note that here $R_{(gω, g)} : ℓ^2(Γ_gω) → ℓ^2(Γ_ω)$ is defined by $δ_{(hω, h)} → δ_{(hω, hg)}$, which induces the following commutative diagram:

$$
\begin{array}{ccc}
ℓ^2(Γ_gω) & \xrightarrow{R_{(gω, g)}} & ℓ^2(Γ_ω) \\
U_{gω} \cong & \equiv & Uω \\
\downarrow & & \downarrow \\
ℓ^2(G) & \xrightarrow{R_g} & ℓ^2(G)
\end{array}
$$

Therefore, we have $((Ad_U)_*φ)(gω) = R^*_g((Ad_U)_*φ)(ω)R_g$. So the lemma holds. □

**Lemma 6.2.** For each $ω ∈ ∂G$ and $g ∈ G$, we have $Ad_U_*(C^*_u(Γ_gω)) = C^*_u(|G|)$. Hence we have $\langle (Ad_U)_*Γ(E_a^0)|G × G\rangle = C_{WOT}(∂G, C^*_u(|G|))G$.

**Proof.** Recall that by definition, $T = (T(g′ω, g′), (g^ω, g^′))g′, g^′ ∈ G ∈ B(ℓ^2(G_x))$ belongs to $C[Γ_ω]$ if and only if there exists a compact set $K ∈ βG × G$ such that $T(g′ω, g′)g^ω, g^′ = 0$ implies that $(g^ω, g^′)(g′ω, g′)^{-1} ∈ K$. Since $βG$ is compact and $(g^ω, g^′)(g′ω, g′)^{-1} = (g^ω, g^′)^{-1}$, the above is equivalent to the condition that there exists a finite set $F ⊆ G$ such that $Ad_U_*(T)g^ω, g^′ = 0$ implies that $g^ωg^′ ∈ F$. In other words, we have $Ad_U_*(C[Γ_ω]) = C[|G|]$. Taking completions on both sides, the lemma holds. □

**Lemma 6.3.** For $ω ∈ ∂G$, the following diagram commutes.

$$
\begin{array}{ccc}
C^*_r(βG × G) & \xrightarrow{λ_ω} & C^*_u(Γ_ω) \\
\Theta \equiv & \equiv & Ad_Uω \\
C^*_u(|G|) & \xrightarrow{σ_ω} & C^*_u(|G|)
\end{array}
$$

**Proof.** It suffices to show that $f ∈ C_c(βG × G)$, we have $Ad_Uω ◦ λ_ω(f) = σ_ω ◦ Θ(f)$. For any $g, h ∈ G$, we have

$$
(λ_ω(f)δ_{(hω, h)})(gω, g) = \sum_{γ ∈ G} f((gω, g)(γω, γ)^{-1})δ_{(hω, h)}(γω, γ) = f(gω, gh^{-1}).
$$

Hence,

$$
(Ad_Uω ◦ λ_ω(f)δ_h, δ_g) = (λ_ω(f)δ_{(hω, h)}, δ_{(gω, g)}) = f(gω, gh^{-1}) = \lim_{α→ω} f(gα, gh^{-1}),
$$

$$
(σ_ω ◦ Θ(f)δ_h, δ_g) = \lim_{α→ω} (R^*_αΘ(f)R_αδ_h, δ_g) = \lim_{α→ω} (Θ(f)δ_{ha}, δ_{ga}) = \lim_{α→ω} f(gα, gh^{-1}).
$$

So we finish the proof. □

Combining Lemma 6.1, 6.2 and 6.3 together, we obtain the following:

**Proposition 6.4.** Notations as above. The following diagram commutes:

$$
\begin{array}{ccc}
C^*_r(βG × G) & \xrightarrow{ζ} & Γ_k(E_a^0)|G × G \\
\Theta \equiv & \equiv & (Ad_U)_* \\
C^*_u(|G|) & \xrightarrow{σ} & C_{WOT}(∂G, C^*_u(|G|))G
\end{array}
$$

Consequently, we recover the classic limit operator theory for exact groups [Roe05] as a corollary of Theorem 4.4:

**Corollary 6.5.** Let $G$ be a finitely generated discrete exact group. For any operator $T ∈ C^*_u(|G|)$, the following are equivalent:

1. $T$ is Fredholm.
2. $σ(T)$ is invertible in $C_{WOT}(∂G, C^*_u(|G|))G$. 

$$
\frac{24}{24}
$$
(3) For each $\omega \in \partial G$, the limit operator $\sigma_\omega(T)$ is invertible, and
\[ \sup_{\omega \in \partial G} \| \sigma_\omega(T)^{-1} \| < \infty. \]

(4) For each $\omega \in \partial G$, the limit operator $\sigma_\omega(T)$ is invertible.

6.2. General compactifications of group case. In the previous subsection, we recover the classic limit operator theory for exact groups which characterises Fredholmness for operators in the uniform Roe algebra. And as we observed, the uniform Roe algebra of a group corresponds to the action on its Stone-Čech compactification. Now we study the limit operator theory associated to general compactifications recovering [Wil09, Chapter 3.2].

Let $G$ be a finitely generated discrete exact group, and $Y$ be an equivariant compactification of $G$ in the sense that $Y$ is a compact space containing $G$ as an open and dense subset, and the action of the left multiplication of $G$ on itself extends to an action on $Y$. As before, we consider the transformation groupoid $\mathcal{G} = Y \rtimes G$, whose unit space is $Y$ with a natural decomposition:

\[ Y = G \cup \partial Y G. \]

And we have
\[ \mathcal{G}(G) = G \rtimes G, \quad \text{and} \quad \mathcal{G}(\partial Y G) = \partial Y G \rtimes G. \]

Note that although $G$ is exact, the action on $Y$ is in general not amenable, i.e., the groupoid $\mathcal{G} = Y \rtimes G$ might not be amenable. In fact, when $Y$ is the Alexandroff one-point compactification, the amenability of the action is equivalent to the amenability of the group itself, and there exist non-amenable but exact groups (e.g. free groups). However we still have the following:

**Lemma 6.6.** Let $G$ be a finitely generated discrete exact group and $Y$ be an equivariant compactification of $G$. Then the following short sequence is exact:

\[ 0 \longrightarrow C^*_r(G \rtimes G) \longrightarrow C^*_r(Y \rtimes G) \longrightarrow C^*_r(\partial Y G \rtimes G) \longrightarrow 0. \]

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C^*_r(G \rtimes G) & \longrightarrow & C^*_r(Y \rtimes G) & \longrightarrow & C^*_r(\partial Y G \rtimes G) & \longrightarrow & 0 \\
& | & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(G) \rtimes_r G & \longrightarrow & C(Y) \rtimes_r G & \longrightarrow & C(\partial Y G) \rtimes_r G & \longrightarrow & 0,
\end{array}
\]

where all vertical maps are $C^*$-isomorphisms (this is well-known, and the readers can refer to Section 6.4 for details where we deal with general group actions). By [BO08, Theorem 5.1.10], we know the bottom line is exact, so is the top one. \qed

Therefore by Remark 4.7, we know that part of our main result: “(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)” in Theorem 4.4 holds directly for the transformation groupoid $\mathcal{G} = Y \rtimes G$, which establishes part of the limit operator theory in this case. Now we would like to provide a more detailed picture and compare it with the limit operator theory studied in Section 6.1.

First we focus on the groupoid $C^*$-algebra $C^*_r(\mathcal{G}) = C^*_r(Y \rtimes G)$. Equipping $G$ with a proper right-invariant metric, and denote the uniform Roe algebra by $C^*_u([G])$ as in Section 6.1. From the universal property of the Stone-Čech compactification, there is a $G$-equivariant surjection $\phi : \beta G \twoheadrightarrow Y$, which induces an embedding $C_c(Y \rtimes G) \hookrightarrow C_c(\beta G \rtimes G)$ and hence a $C^*$-monomorphism
\[ \phi^* : C^*_r(Y \rtimes G) \longrightarrow C^*_r(\beta G \rtimes G). \]

Recall that in (6.2) we provide an isomorphism $\Theta : C^*_r(\beta G \rtimes G) \to C^*_u([G])$. Combining them together, we have an embedding
\[ \Theta \circ \phi^* : C^*_r(Y \rtimes G) \longrightarrow C^*_u([G]), \]
whose image is denoted by $A_Y$. Consequently, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
C^*_r(G \times G) & \xrightarrow{\phi^*} & C^*_r(B \times G) \\
\cong & \cong & \cong \\
\otimes (\ell^2(G))' & \xrightarrow{A_Y} & C^*_u(|G|).
\end{array}
\]

Note that for $T \in A_Y \subseteq C^*_u(|G|)$, we already established the limit operator theory in Corollary 6.5. Here we provide another intrinsic viewpoint to gain a more precise description of the limit operator theory. Recall that we have the symbol morphism associated to $G = Y \times G$ from Definition 4.2:

$\gamma_Y : C^*_r(Y \times G) \rightarrow \Gamma(E^G_{Y,u})^{\partial_Y G \times G}$,

where $E^G_{Y,u}$ is the operator fibre space associated to the reduction groupoid $\partial_Y G \times G$. Now we would like to compare $\gamma_Y$ with $\zeta, \sigma$ in Section 6.1.

Fix $\omega \in \partial_Y G$ and note that $(Y \times G)_\omega = \{(g\omega, q) \in Y \times G : q \in G\}$ is bijective to $G$ via the map $(g\omega, q) \rightarrow q$. This induces a unitary $U_{Y,\omega} : \ell^2((Y \times G)_\omega) \cong \ell^2(G)$, and hence an isomorphism $\text{Ad}_{U_{Y,\omega}} : \mathcal{B}(\ell^2((Y \times G)_\omega)) \cong \mathcal{B}(\ell^2(G))$. These isomorphisms can be put together and provide a map between operator fibre spaces:

$\text{Ad}_{U_Y} = \bigcup_{\omega \in \partial_Y G} \text{Ad}_{U_{Y,\omega}} : E^G_Y = \bigcup_{\omega \in \partial_Y G} \mathcal{B}(\ell^2((Y \times G)_\omega)) \rightarrow \partial_Y G \times \mathcal{B}(\ell^2(G))$.

It is easy to check that $\text{Ad}_{U_Y}$ is a homeomorphism, hence an isomorphism between operator fibre spaces over $\partial_Y G$. So $\text{Ad}_{U_Y}$ induces the following $C^*$-isomorphism:

$\Gamma(E^G_Y) = \Gamma(E^G_{Y,u})^{\partial_Y G \times G}$.

On the other hand, from Lemma 6.2 we have the following isomorphism for $\beta G \times G$:

$\phi^*_{WOT} : C^*_{\text{WOT}}(\partial_Y G, C^*_u(|G|)) \rightarrow C^*_{\text{WOT}}(\partial G, C^*_u(|G|))$.

Furthermore, the map $\phi : \beta G \rightarrow Y$ induces an injective $C^*$-morphism:

$\phi^*_{WOT} : C^*_{\text{WOT}}(\partial_Y G, C^*_u(|G|)) \rightarrow C^*_{\text{WOT}}(\partial G, C^*_u(|G|))$.

**Lemma 6.7.** The following diagram commutes:

\[
\begin{array}{ccc}
C^*_r(\beta G \times G) & \xrightarrow{\zeta} & \Gamma(E^G_{Y,u})^{\partial_Y G \times G} \\
\phi^* & \cong & \phi^*_{WOT} \\
\otimes (\ell^2(G))' & \xrightarrow{A_Y} & C^*_u(|G|).
\end{array}
\]

**Proof.** Note that for $\omega \in \partial_Y G$ and any of its inverse $\overline{\omega}$ under $\phi$, the following diagram commutes:

\[
\begin{array}{ccc}
C^*_r(\beta G \times G) & \xrightarrow{\lambda_{\overline{\omega}}} & C^*_u((\beta G \times G)_{\overline{\omega}}) \\
\phi^* & \cong & \phi^*_{WOT} \\
\otimes (\ell^2(G))' & \xrightarrow{A_{Y,\overline{\omega}}} & C^*_u(|G|).
\end{array}
\]

So the lemma holds. \qed
Proposition 6.8. There exists a unique morphism \( \sigma_Y : \mathcal{A}_Y \rightarrow C_{WOT}(\partial_Y G, C_u^*(|G|))^G \) (also called the symbol morphism) such that the following diagram commutes:

\[
\begin{array}{ccc}
C^*_r(\beta G \rtimes G) & \overset{\zeta}{\longrightarrow} & \Gamma(E^\partial_G) \rtimes G \\
\downarrow \phi^* \hspace{1cm} & & \downarrow (Ad_U)_* \\
C^*_u(|G|) & \overset{\sigma}{\longrightarrow} & C_{WOT}(\partial G, C^*_u(|G|))^G \\
\downarrow \Theta \hspace{1cm} & & \downarrow (Ad_U)_* \\
C^*_r(Y \rtimes G) & \overset{\varphi^*_Y}{\longrightarrow} & \Gamma(E^\partial_{Y,G}) \rtimes G \\
\end{array}
\]

For \( T \in \mathcal{A}_Y \) and \( \omega \in \partial_Y G \), we define the limit operator of \( T \) at \( \omega \) to be \( \sigma_{Y,\omega}(T) := \sigma_Y(T)(\omega) \in C^*_u(|G|) \).

**Proof.** Define \( \sigma_Y = (Ad_{U,Y})_* \circ \gamma_Y \circ (\Theta \circ \phi^*)^{-1} : \mathcal{A}_Y \rightarrow C_{WOT}(\partial_Y G, C^*_u(|G|))^G \). For any \( \omega \in \partial_Y G \), take one of its inverses \( \bar{\omega} \) under \( \phi \). From Lemma 6.7, we have the following commutative diagram:

\[
\begin{array}{ccc}
C^*_r(\beta G \rtimes G) & \overset{\lambda_{\bar{\omega}}}{\longrightarrow} & C^*_u((\beta G \rtimes G)_{\bar{\omega}}) \\
\downarrow \phi^* \hspace{1cm} & & \downarrow (Ad_{U,Y})_* \\
C^*_u(|G|) & \overset{\sigma_{\bar{\omega}}}{\longrightarrow} & C^*_u(|G|) \\
\downarrow \Theta \hspace{1cm} & & \downarrow (Ad_{U,Y})_* \\
C^*_r(Y \rtimes G) & \overset{\lambda_{\omega}}{\longrightarrow} & C^*_u((Y \rtimes G)_{\omega}) \\
\end{array}
\]

So the result holds. \( \square \)

**Remark 6.9.** Note that the symbol morphism \( \sigma_Y \) defined above is the same as the morphism \( \sigma \) in [Wil09, Proposition 3.2.1].

Consequently, we obtain the limit operator theory for general equivariant compactifications:

**Corollary 6.10.** Let \( G \) be a finitely generated discrete exact group, and \( Y \) be a \( G \)-equivariant compactification. Let \( \phi : \beta G \rightarrow Y \) be the induced surjection, and \( Z \) be a subset in \( \partial G \) such that \( \phi(Z) = \partial_Y G \). Suppose \( \mathcal{A}_Y \) is the associated \( C^* \)-subalgebra of \( C^*_u(|G|) \) defined above. Then for any \( T \in \mathcal{A}_Y \), the following are equivalent:

1. \( T \) is Fredholm.
2. \( \sigma_Y(T) \) is invertible in \( C_{WOT}(\partial_Y G, C^*_u(|G|))^G \).
3. For each \( \bar{\omega} \in Z \), the limit operator \( \sigma_{\bar{\omega}}(T) \) is invertible, and
   \[
   \sup_{\bar{\omega} \in Z} \| \sigma_{\bar{\omega}}(T)^{-1} \| < \infty.
   \]
4. For each \( \bar{\omega} \in Z \), the limit operator \( \sigma_{\bar{\omega}}(T) \) is invertible.

**Proof.** From the above analysis, we know “(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3)” holds, so it suffices to prove “(4) \( \Rightarrow \) (3)”.

Unfortunately since the action of \( G \) on \( Y \) is not amenable in general, we cannot refer to Theorem 4.4 directly. However, from the proof of Proposition 6.8, we know that for any \( \bar{w}_1, \bar{w}_2 \in \beta G \) with \( \phi(\bar{w}_1) = \phi(\bar{w}_2) \), we have that \( \sigma_{\bar{w}_1}(T) = \sigma_{\bar{w}_2}(T) \). So condition (4) implies that for any \( \bar{\omega} \in \beta G \), the limit operator \( \sigma_{\bar{\omega}}(T) \) is invertible. Hence from “(4) \( \Rightarrow \) (3)” in Corollary 6.5, we know that \( \| \sigma_{\bar{\omega}}(T)^{-1} \| \) is uniformly bounded for all \( \bar{\omega} \in \beta G \). As a special case, condition (3) holds. \( \square \)
Remark 6.11. Note that the key difference between Corollary 6.5 and the above is that in order to obtain the Fredholmness for operators in the subalgebra $A_Y$, we do not need to check the invertibility of limit operators for all limit points in $\partial G$, but only those related to the given compactification ($\mathcal{Z}$, in the previous corollary).

Example 6.12. Suppose $G = \mathbb{Z}$, and $Y = \mathbb{Z} \cup \{\pm \infty\}$ be the two-point compactification of $\mathbb{Z}$ (more explicitly, the topology on $\mathbb{Z}$ is discrete, $\lim_{n \to +\infty} n = +\infty$ and $\lim_{n \to -\infty} n = -\infty$). In this case, the above algebra $A_Y$ can be determined as follows:
\[ A_Y = \{ T \in C^*_{\ell_1}(\mathbb{Z}) : \lim_{n \to +\infty} T_{n,m+n} \text{ and } \lim_{n \to -\infty} T_{n,m+n} \text{ exist, for all } m \in \mathbb{Z} \}. \]
And the limit operator of $T$ over $+\infty$ is the operator whose entries on the $m$-th diagonal
\[ \{(m+n,n) : n \in \mathbb{Z}\} \]
are all equal to $\lim_{n \to +\infty} T_{n,m+n}$. Similar formula also holds for the limit operator over $-\infty$. Under the Fourier transformation $\ell^2(\mathbb{Z}) \cong L^2(S^1)$, these two limit operators correspond to multiplication operators by two continuous functions, denoted by $f^{+\infty}$ and $f^{-\infty}$ respectively. Hence the limit operator theorem in the case says that $T$ is Fredholm if and only if $f^{+\infty}, f^{-\infty}$ are invertible (i.e., point-wise nonzero). We remark that the Gohberg-Krein index theorem says that in this case, when $T$ is Fredholm, then
\[ \text{Index}(T) = -(\text{winding number})(f^{-\infty}) - (\text{winding number})(f^{+\infty}). \]

6.3. Discrete metric space case. The next application recovers (in the Hilbert space case) the limit operator theory for metric spaces introduced by Špakula and Willett in [SW17]. In fact, part of the proof (excluding the omission of uniform bounded condition) using the groupoid language was already mentioned in [SW17, Appendix C]. For the convenience to the readers, we recall the details again.

Let $(X,d)$ be a uniformly discrete metric space with bounded geometry and propert $\mathcal{A}$. The coarse groupoid $G(X)$ on $X$ was introduced by Skandalis, Tu and Yu in [STY02] (also see [Ro03, Chapter 10]). As a set,
\[ G(X) := \bigcup_{r>0} \overline{E_r}^{\beta X \times \beta X} \]
where $E_r := \{(x,y) \in X \times X : d(x,y) \leq r\}$. The groupoid structure is just the restriction of the pair groupoid $\beta X \times \beta X$. On each $\overline{E_r}$, the topology of $G(X)$ agrees with the subspace topology of $\beta X \times \beta X$; and generally, a subset $U$ in $G(X)$ is open if and only if the intersection $U \cap \overline{E_r}$ is open in $\overline{E_r}$ for each $r > 0$. Equipped with this topology, $G(X)$ is a locally compact, $\sigma$-compact, principal and étale groupoid. Clearly, the unit space of $G(X)$ is $\beta X$, and we consider the following decomposition:
\[ \beta X = X \sqcup \partial X, \]
where $\partial X$ is the Stone-Čech boundary. Obviously $X$ is an open invariant dense subset in $\beta X$, hence the above decomposition induces the following:
\[ G(X) = (X \times X) \sqcup G_\infty(X), \]
where $G_\infty(X)$ is the reduction of $G(X)$ by $\partial X$.

Recall from [STY02] that the space $X$ has Property $\mathcal{A}$ if and only if the coarse groupoid $G(X)$ is topologically amenable. Hence we may apply Theorem 4.4 to $\mathcal{G} = G(X)$ together with the decomposition (6.3), and obtain the associated limit operator theory. Now we would like to follow [SW17, Appendix C] to provide a more detailed picture.

As before, we start with the reduced groupoid $C^*$-algebra $C^*_r(G(X))$. For any $f \in C_c(G(X))$, by definition $f$ is a continuous function supported in $\overline{E_r}$ for some $r > 0$; equivalently, we may interpret $f$ as a bounded continuous function on $E_r$. Hence we can define an operator $\theta(f)$ on $\ell^2(X)$ by setting its matrix coefficients to be $\theta(f)_{x,y} := f(x,y)$, where $x,y \in X$. And we have the following lemma:

Lemma 6.13 ([Ro03], Proposition 10.29). The map $\theta$ provides a $\ast$-isomorphism from $C_c(G(X))$ to $C^*[X]$, and extends to a $C^*$-isomorphism $\Theta$ from $C^*_r(G(X))$ to the uniform Roe algebra $C^*_u(G(X))$, which maps the $C^*$-subalgebra $C^*_r(X \times X)$ onto the compact operators $\mathfrak{K}(\ell^2(X))$. Consequently $T \in C^*_r(G(X))$ is invertible modulo $C^*_r(G \times G)$ if and only if $\Theta(T)$ is Fredholm.
Now we move on to the discussion of limit operators. Let us first recall the notion of limit operators for metric spaces introduced by Spakula and Willett [SW17]. A function \( t : D \to R \) with \( D, R \subseteq X \) is called a partial translation if \( t \) is a bijection from \( D \) to \( R \), and \( \sup_{x \in X} d(x, t(x)) \) is finite.

**Definition 6.14** ([SW17]). Fix an ultrafilter \( \omega \in \beta X \). A partial translation \( t : D \to R \) on \( X \) is compatible with \( \omega \) if \( \omega(D) = 1 \). In this case, regarding \( t \) as a function from \( D \) to \( \beta X \), we define

\[
 t(\omega) := \lim_{\omega} t \in \beta X.
\]

An ultrafilter \( \alpha \in \beta X \) is compatible with \( \omega \) if there exists a partial translation \( t \) compatible with \( \omega \) and \( t(\omega) = \alpha \). Denote \( X(\omega) \) the collection of all ultrafilters on \( X \) compatible with \( \omega \). A compatible family for \( \omega \) is a collection of partial translations \( \{t^\alpha\}_{\alpha \in X(\omega)} \) such that each \( t^\alpha \) is compatible with \( \omega \) and \( t^\alpha(\omega) = \alpha \).

Fix an ultrafilter \( \omega \) on \( X \), and a compatible family \( \{t^\alpha\}_{\alpha \in X(\omega)} \). Define a function \( d_\omega : X(\omega) \times X(\omega) \to [0, \infty) \) by

\[
 d_\omega(\alpha, \beta) := \lim_{x \to \omega} d(t^\alpha(x), t^\beta(x)).
\]

It is shown in [SW17, Proposition 3.7] that \( d_\omega \) is a uniformly discrete metric of bounded geometry on \( X(\omega) \) which does not depend on the choice of \( \{t^\alpha\} \).

**Definition 6.15** ([SW17]). For each non-principal ultrafilter \( \omega \) on \( X \), the metric space \( (X(\omega), d_\omega) \) is called the limit space of \( X \) at \( \omega \). Fix a compatible family \( \{t^\alpha\}_{\alpha \in X(\omega)} \) for \( \omega \), and suppose \( A \in C^*_u(X) \). The limit operator of \( A \) at \( \omega \), denoted by \( \Phi^\omega(A) \), is an \( X(\omega) \)-by-\( X(\omega) \) indexed matrix defined by

\[
 \Phi^\omega(A)_{\alpha, \beta} := \lim_{x \to \omega} A(t^\alpha(x), t^\beta(x)).
\]

It was studied in [SW17, Chapter 4] that the above definition does not depend on the choice of compatible family \( \{t^\alpha\}_{\alpha \in X(\omega)} \) for \( \omega \). Furthermore, the limit operator \( \Phi^\omega(A) \) is indeed a bounded operator on \( \ell^2(X(\omega)) \), and belongs to the uniform Roe algebra \( C^*_u(X(\omega)) \).

On the other hand, we have the symbol morphism associated to \( G = G(X) \) from Definition 4.2:

\[
 \zeta : C^*_r(G(X)) \to \Gamma_0(E^u_\omega)^{G(\omega)(X)}.
\]

Now we would like to compare \( \zeta \) with limit operators \( \Phi^\omega(A) \) in Definition 6.15, and show that they coincide under the isomorphism \( \Theta \). In the process, we will compose all limit operators into a single homomorphism in terms of the language of operator fibre spaces established in Section 3.

**Lemma 6.16.** [SW17, Lemma C.3] Given \( \omega \in \partial X \), we define

\[
 F : X(\omega) \to G(X)_\omega, \alpha \mapsto (\alpha, \omega).
\]

Then \( F \) is a bijection. Let \( W^\omega : \ell^2(G(X)_\omega) \to \ell^2(X(\omega)) \) be the unitary operator induced by \( F \). Then we have the following commutative diagram:

\[
 \begin{array}{ccc}
 C^*_u(G(X)) & \xrightarrow{\lambda_\omega} & \mathfrak{B}(\ell^2(G(X)_\omega)) \\
 \Theta \downarrow & \approx & \approx \downarrow \text{Ad}_{W^\omega} \\
 C^*_u(X) & \xrightarrow{\Phi^\omega} & \mathfrak{B}(\ell^2(X(\omega))).
\end{array}
\]

The above lemma suggests us to consider the following operator fibre space over \( \partial X \):

\[
 \mathcal{E}^\omega := \bigsqcup_{\omega \in \partial X} \mathfrak{B}(\ell^2(X(\omega))),(\omega) ,
\]

with the topology defined as follows: a net \( \{T^\omega_i\}_{i \in I} \) converges to \( T^\omega \) if and only if \( \omega_i \to \omega \), and for any \( \alpha'_i \to \alpha', \alpha''_i \to \alpha'' \) with \( \alpha'_i, \alpha''_i \in X(\omega_i) \) and \( \alpha', \alpha'' \in X(\omega) \), we have

\[
 (T^\omega_i, \delta_{\alpha'_i}, \delta_{\alpha''_i}) \to (T^\omega, \delta_{\alpha'}, \delta_{\alpha''}).
\]
Applying the unitary operators $W_\omega$ in Lemma 6.16 together, we obtain an isomorphism between the operator fibre spaces as follows:

\[
\Ad W = \bigsqcup_{\omega \in \partial X} \Ad W_\omega : E^0 = \bigsqcup_{\omega \in \partial X} \mathfrak{B}(\ell^2(G(X)_\omega)) \longrightarrow \widetilde{E}^0 = \bigsqcup_{\omega \in \partial X} \mathfrak{B}(\ell^2(X(\omega))).
\]

Denote the subspace

\[
\widetilde{E}^0_u := \bigsqcup_{\omega \in \partial X} C_u^*(X(\omega)).
\]

A continuous section $\varphi$ of $\widetilde{E}^0$ is called \textit{coarsely constant} (c.c.) if for any $\omega \in \partial X$ and $\alpha \in X(\omega)$, we have $\varphi(\alpha) = \varphi(\omega)$. Denote the set of all the bounded coarsely constant sections of $\widetilde{E}^0_u$ by $\Gamma_b(\widetilde{E}^0_u)^{c.c.}$.

**Lemma 6.17.** The isomorphism $\Ad W$ induces a $C^*$-isomorphism:

\[
(\Ad W)_* : \Gamma_b(E^0_u)_{G_\infty(X)} \cong \Gamma_b(\widetilde{E}^0_u)^{c.c.}.
\]

**Proof.** For a bounded section $\varphi$ of $E^0$, it is $G_\infty(X)$-equivariant if and only if for any $(\alpha, \omega) \in G_\infty(X)$, we have

\[
\varphi(\alpha) = R^*_y(\omega)\varphi(\omega)R_y(\omega).
\]

This is equivalent to the condition that for any $\omega \in \partial X$ and $\alpha \in X(\omega)$, we have

\[
(6.4) \quad \Ad W_\omega(\varphi(\alpha)) = \Ad W_\omega(R^*_y(\omega)\varphi(\omega)R_y(\omega)).
\]

Note that $X(\alpha) = X(\omega)$ and the following diagram commutes:

\[
\begin{array}{ccl}
\ell^2(G(X)_\alpha) & \xrightarrow{R_y(\omega)} & \ell^2(G(X)_\omega) \\
W_\omega \cong & \equiv & W_\omega \\
\ell^2(X(\alpha)) & \xrightarrow{\text{Id}} & \ell^2(X(\omega)).
\end{array}
\]

Hence (6.4) is equivalent to $((\Ad W)_* \varphi)(\alpha) = ((\Ad W)_* \varphi)(\omega)$, i.e., $(\Ad W)_* \varphi$ is coarsely constant.

On the other hand, given a point $\omega \in \partial X$, we claim $\Ad W_\omega(C[G(X)_\omega]) = C[X(\omega)]$. By definition, an operator $T = (T(\alpha', \omega),(\alpha'', \omega)) \in \ell^2(G(X)_\omega)$ belongs to $C[G(X)_\omega]$ if and only if there exists a compact set $K \subseteq G(X)$ such that $T(\alpha', \omega),(\alpha'', \omega) \neq 0$ implies that $(\alpha', \alpha'') \in K$. By definition, there exists some $r > 0$ such that $K \subseteq E_r$. Therefore, the above is equivalent to the condition that there exists some $r > 0$ such that $(\alpha', \alpha'') \in E_r$, which is equivalent to $d_\omega(\alpha', \alpha'') \leq r$. In other words, this is equivalent to $\Ad W_\omega(T) \in C[X(\omega)]$. Taking completion, we have $\Ad W_\omega(C^*_u(G(X)_\omega)) = C^*_u(X(\omega))$. \hfill \Box

Now we define the following morphism (also called the \textit{symbol morphism})

\[
\Phi := (\Ad W)_* \circ \Theta^{-1} : C^*_u(X) \longrightarrow \Gamma_b(\widetilde{E}^0_u)^{c.c.}.
\]

In other words, the following diagram commutes:

\[
\begin{array}{ccl}
C^*_u(G(X)) & \xrightarrow{\Theta} & \Gamma_b(E^0_u)^{G(X)} \\
\cong \uparrow & \equiv & \cong \uparrow \\
C^*_u(X) & \xrightarrow{\Phi} & \Gamma_b(\widetilde{E}^0_u)^{c.c.}
\end{array}
\]

And Lemma 6.16 implies that $\Phi(T)(\omega) = \Phi_\omega(T)$ for any $T \in C^*_u(X)$ and $\omega \in \partial X$. Consequently, combining the above analysis we recover the limit operator theory for metric spaces in the Hilbert space case with complex number values as a corollary of Theorem 4.4:

**Corollary 6.18.** Let $X$ be a uniformly discrete metric space with bounded geometry and Property A. For any operator $T \in C^*_u(X)$, the following are equivalent:

1. $T$ is Fredholm.
2. $\Phi(T)$ is invertible in $\Gamma_b(\widetilde{E}^0_u)^{c.c.}$. 
(3) For each $\omega \in \partial X$, the limit operator $\Phi_\omega(T)$ is invertible, and
$$\sup_{\omega \in \partial X} \|\Phi_\omega(T)^{-1}\| < \infty.$$

(4) For each $\omega \in \partial X$, the limit operator $\Phi_\omega(T)$ is invertible.

Example 6.19. Let us recall an interesting example from [SW17, Example 4.13]. Take $X = \mathbb{N}$ with the usual metric, and identify the Hilbert space $\ell^2(\mathbb{N})$ with the Hardy space $H^2$ of the disc. All limit spaces are isometric to $\mathbb{Z}$ with the usual metric. Let $T_f$ be the Toeplitz operator on $H^2$ with continuous symbol $f : S^1 \to \mathbb{C}$. Then all limit operators of $T_f$ correspond to the symbol $f$, acting as the multiplication operator on $L^2(S^1) \cong \ell^2(\mathbb{Z})$ via Fourier transformation. Hence we recover the classical theory in Toeplitz operators that limit spaces are isometric to $\mathbb{Z}$.

6.4. Group action case. In Section 6.2, we studied the limit operator theory for groups acting on their equivariant compactifications. Now we move on to general group actions and establish the associated limit operator theory.

Let $G$ be a countable discrete exact group acting on a compact space $X$ by homeomorphisms. Let $Y$ be a $G$-invariant open dense subset in $X$, and denote $\partial Y := X \setminus Y$ its boundary. Clearly, both $Y$ and $\partial Y$ are locally compact and $G$ acts on them by homeomorphisms respectively. We consider the associated transformation groupoid $\mathcal{G} = X \rtimes G$. Clearly, both $Y$ and $\partial Y$ are invariant in $X$.

As we pointed out in Section 6.2, although $G$ is exact, the action on $X$ is in general not amenable. However we still have the following, and proof is the same as Lemma 6.6 hence omitted.

Lemma 6.20. Let $G$ be a countable discrete exact group acting on a compact space $X$, and $Y$ be an invariant open dense subset in $X$. Then the following short sequence is exact:
$$0 \to C_c^*(Y \rtimes G) \to C_c^*(X \rtimes G) \to C_c^*(\partial Y \times G) \to 0.$$

Therefore by Remark 4.7, part of our main result: “(1) $\iff$ (2) $\iff$ (3)” in Theorem 4.4 holds directly for the transformation groupoid $\mathcal{G} = X \rtimes G$, and they are also equivalent to condition (4) if the action is additionally assumed to be amenable. This establishes the limit operator theory for group actions. Now we would like to provide a more detailed and practical picture.

First we focus on the groupoid $C^*$-algebra $C_c^*(X \rtimes G)$, which is isomorphic to the reduced cross product $C(X) \rtimes_r G$. To make it more precise, let us recall the definition of reduced cross product (see [BO08, Chapter 4.1] for more details). Denote $C_c(G, C(X))$ the linear space of finitely supported functions on $G$ with values in $C(X)$, equipped with the twisted convolution product and the $*$-operation. The reduced cross product $C(X) \rtimes_r G$ is the norm closure with respect to a regular representation of $C_c(G, C(X))$ (see [BO08, Definition 4.1.4]). We have the following algebraic $*$-isomorphism:
$$\theta : C_c(X \rtimes G) \to C_c(G, C(X))$$
given by
$$F \mapsto \sum_{\gamma \in G} f_\gamma \gamma, \quad \text{where } f_\gamma \in C(X) \text{ is defined by } f_\gamma(x) = F(x, \gamma).$$

It is known that $\theta$ can be extended to be an $C^*$-isomorphism:
$$\Theta : C_c^*(X \rtimes G) \to C_c(G, C(X) \rtimes_r G),$$
which maps $C_c^*(Y \times G)$ to $C_c(Y \rtimes_r G)$.

Now we move on to the discussion of limit operators. Recall that we have the symbol morphism associated to $\mathcal{G} = X \rtimes G$ from Definition 4.2:
$$\varsigma : C_c^*(X \rtimes G) \to \Gamma_b(E^\partial \mathcal{G}^{\partial Y \times G}),$$
where $E^\partial$ is the operator fibre space associated to the reduction groupoid $\mathcal{G}(\partial Y) = \partial Y \times G$. Now we would like to offer a more detailed description for the symbol morphism. For each $\omega \in \partial Y$, the fibre
$$(X \times G)_\omega = \{ (\gamma \omega, \gamma) : \gamma \in G \}$$
is bijective to $G$ via the map $(\gamma \omega, \gamma) \mapsto \gamma$, and induces a unitary:
$$V_\omega : \ell^2((X \times G)_\omega) \cong \ell^2(G).$$
These isomorphisms can be put together to provide a map between operator fibre spaces as follows:

$$\text{Ad}_y = \bigsqcup_{\omega \in \partial Y} \text{Ad}_{y\omega} : E^y = \bigsqcup_{\omega \in \partial Y} \mathfrak{B}(\ell^2((X \times G)_\omega)) \rightarrow \partial Y \times \mathfrak{B}(\ell^2(G)),$$

where the final item can be regarded as a trivial fibre space over \(\partial Y\), and the topology coincides with the product topology. It is easy to check that \(\text{Ad}_y\) is a homeomorphism, hence an isomorphism between operator fibre spaces over \(\partial Y\). Note that continuous sections of \(\partial Y \times \mathfrak{B}(\ell^2(G))\) coincides with \(C_{\text{WOT}}(\partial Y, \mathfrak{B}(\ell^2(G)))\), hence \(\text{Ad}_y\) induces the following \(C^*\)-isomorphism:

$$(\text{Ad}_y)_* : \Gamma(E^y) \rightarrow C_{\text{WOT}}(\partial Y, \mathfrak{B}(\ell^2(G))).$$

Note that elements in \(C_{\text{WOT}}(\partial Y, \mathfrak{B}(\ell^2(G)))\) have uniform bounded norms, hence \(\Gamma(E^y) = \Gamma_b(E^y)\). As in Section 6.2, we obtain an isomorphism:

$$(\text{Ad}_y)_* : \Gamma_b(E^y \mid \partial Y) \rightarrow C_{\text{WOT}}(\partial Y, C^*_u(G))^G.$$

Therefore, we define the following morphism (also called the symbol morphism):

$$\sigma := (\text{Ad}_y)_* \circ \circ \sigma^{-1} : C(X) \times_r G \rightarrow C_{\text{WOT}}(\partial Y, C^*_u(G))^G.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc}
C(X) \times_r G & \xrightarrow{\sigma} & C_{\text{WOT}}(\partial Y, C^*_u(G))^G \\
\Theta \cong & & \cong (\text{Ad}_y)_* \\
\end{array}$$

Now we would like to provide a more concrete formula for the above symbol morphism \(\sigma\). Fix a boundary point \(\omega \in \partial Y\) and consider the following representations:

$$M_\omega : C(X) \rightarrow \mathfrak{B}(\ell^2(G)) \quad \text{by} \quad M_\omega(f)\delta_\gamma := f(\gamma \omega)\delta_\gamma,$$

and

$$\rho : G \rightarrow \mathfrak{B}(\ell^2(G)) \quad \text{by} \quad \rho(\gamma)\delta_\gamma := \delta_{\gamma \gamma}.$$ 

It is routine to check that \(\rho(\gamma)^* M_\omega(f) \rho(\gamma) = M_\omega(\gamma^{-1} f)\). Hence they induce a \(*\)-representation

$$M_\omega \times \rho : C_c(G, C(X)) \rightarrow \mathfrak{B}(\ell^2(G)) \quad \text{by} \quad \sum_{\gamma \in G} f_\gamma \mapsto \sum_{\gamma \in G} M_\omega(f_\gamma) \rho(\gamma).$$

**Lemma 6.21.** \(M_\omega \times \rho\) induces a \(C^*\)-representation: \(C(X) \times_r G \rightarrow \mathfrak{B}(\ell^2(G))\) whose image is contained in the uniform Roe algebra \(C^*_u(|G|)\), still denoted by \(M_\omega \times \rho\).

**Proof.** First note that the \(*\)-representation \(M_\omega \times \rho\) extends naturally to the maximal crossed product \(C(X) \times_{\text{max}} G\). So, when the action is amenable, we have \(C(X) \times_{\text{max}} G = C(X) \times_r G\) and the lemma holds.

To deal with the general case, note that the restriction map \(C(X) \rightarrow C(\overline{G \omega})\) is \(G\)-equivariant and contractively completely positive (c.c.p), hence induces a c.c.p map

$$\theta_\omega : C(X) \times_r G \rightarrow C(\overline{G \omega}) \times_r G.$$

Consider the faithful representation \(M'_\omega : C(\overline{G \omega}) \rightarrow \mathfrak{B}(\ell^2(G))\) defined by \(M'_\omega(f)\delta_\gamma := f(\gamma \omega)\delta_\gamma\), which induces another representation

$$\pi_\omega : C(\overline{G \omega}) \rightarrow \mathfrak{B}(\ell^2(G) \otimes \ell^2(G))$$

defined by

$$(\pi_\omega(f))(\delta_\alpha \otimes \delta_\gamma) := M'_\omega(\gamma^{-1} f)(\delta_\alpha) \otimes \delta_\gamma = f(\gamma \alpha \omega) \delta_\alpha \otimes \delta_\gamma.$$

Hence we obtain a faithful covariant representation

$$\pi_\omega \times (1 \otimes \rho) : C_c(G, C(\overline{G \omega})) \rightarrow \mathfrak{B}(\ell^2(G) \otimes \ell^2(G)),$$

and the reduced cross product \(C(\overline{G \omega}) \times_r G\) is the completion of \(C_c(G, C(\overline{G \omega}))\) with respect to the norm of its image.
Now consider the unitary $U : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$ defined by $U(\delta_\alpha \otimes \delta_\gamma) = \delta_\alpha \otimes \delta_{\gamma \alpha}$. It is routine to check that $\text{Ad} U \circ \pi_\omega(f) = \text{Id} \otimes M'_\omega(f)$ for any $f \in C(G \omega)$, and $\text{Ad} U \circ (\text{Id} \otimes \rho)(\gamma) = (\text{Id} \otimes \rho)(\gamma)$ for any $\gamma \in G$. Hence, we have
\[
\text{Ad} U \circ \left( \pi_\omega \times (1 \otimes \rho) \right) : C(G \omega) \rtimes r \to C(\text{Id} \otimes C^*(\{M'_\omega(f), \rho : f \in C(G \omega), \gamma \in G\}) \subseteq C(\text{Id} \otimes C^*_u(G)) \cong C^*_u(G).
\]
In conclusion, we obtain a $C^*$-representation:
\[
\text{Ad} U \circ \left( \pi_\omega \times (1 \otimes \rho) \right) \circ \varrho_\omega : C(X) \rtimes r \to C^*_u(G)
\]
satisfying the requirements. □

**Definition 6.22.** Notations as above. For each $\omega \in \partial Y$ and $T \in C(X) \rtimes r$, we define the **limit operator** of $T$ at $\omega$ to be $\sigma_\omega(T) := (M_\omega \times \rho)(T) \in C^*_u(G)$.

**Lemma 6.23.** For $\omega \in \partial Y$, we have $\sigma(T)(\omega) = \sigma_\omega(T)$, i.e., the following diagram commutes:
\[
\begin{array}{ccc}
C^*_u(X \rtimes G) & \xrightarrow{\lambda_\omega} & C^*_u((X \rtimes G)_\omega) \\
\Theta \downarrow & & \downarrow \text{Ad}_V \omega \\
C(X) \rtimes r & \xrightarrow{\sigma_\omega} & C^*_u([G]).
\end{array}
\]

**Proof.** It suffices to show that for $f \in C_c(X \rtimes G)$, we have $\text{Ad}_V \omega \circ \lambda_\omega(f) = \sigma_\omega \circ \Theta(f)$. By definition, we may assume that $\Theta(f) = \sum_{\gamma \in G} f_{\gamma} \gamma$. For any $g, h \in G$, we have
\[
(\lambda_\omega(f)\delta_{(h,\omega)})(g, \omega) = \sum_{\gamma \in G} f((g, \omega), (\gamma \omega, \gamma)^{-1})\delta_{(h,\omega)}(\gamma \omega, \gamma) = f(g, gh^{-1}) = f_{gh^{-1}}(g, \omega).
\]
Hence,
\[
\begin{align*}
\langle \text{Ad}_V \omega \circ \lambda_\omega(f) \delta_h, \delta_g \rangle &= \langle \lambda_\omega(f)\delta_{(h,\omega)}(\delta_{(g,\omega)}), \delta_{(gh^{-1})}\rangle = f_{gh^{-1}}(g, \omega), \\
\langle \sigma_\omega \circ \Theta(f) \delta_h, \delta_g \rangle &= \langle \sigma_\omega(\sum_{\gamma \in G} f_{\gamma} \gamma)\delta_h, \delta_g \rangle = \sum_{\gamma \in G} (M_\omega(f_{\gamma})\rho(\gamma)\delta_h, \delta_g) = \sum_{\gamma \in G} (f(\gamma h, \omega)\delta_{\gamma h}, \delta_g) \\
&= f_{gh^{-1}}(g, \omega).
\end{align*}
\]
So we finish the proof. □

Combining the above analysis, we establish the limit operator theory for group actions as a corollary of Theorem 4.4:

**Corollary 6.24.** Let $G$ be a countable discrete exact group acting on a compact space $X$, $Y$ be a $G$-invariant open dense subset in $X$ and $\partial Y = X \setminus Y$. For any $T \in C(X) \rtimes r$, the following are equivalent:

1. $T$ is invertible modulo $C_0(Y) \rtimes r$.
2. $\sigma(T)$ is invertible in $C_{\text{WOT}}(\partial Y, C^*_u(G))^G$.
3. For each $\omega \in \partial Y$, the limit operator $\sigma_\omega(T)$ is invertible, and
\[
\sup_{\omega \in \partial Y} \|\sigma_\omega(T)^{-1}\| < \infty.
\]

Furthermore if the action of $G$ on $X$ is amenable, then the above are also equivalent to the following:

4. For each $\omega \in \partial Y$, the limit operator $\sigma_\omega(T)$ is invertible.

6.5. **Groupoid case.** Recall that in Section 6.1, we studied the classic limit operator theory for discrete exact groups, characterising the Fredholmness of operators in the uniform Roe algebra of the group in terms of invertibilities of their limit operators. Now we would like to study the uniform Roe algebra of a general groupoid, and establish the associated limit operator theory.

Let $G$ be a locally compact, $\sigma$-compact, second countable and étale groupoid with unit space $G^{(0)}$. Suppose $G$ is strongly amenable at infinity (see Definition 2.20), or weakly inner amenable and $C^*$-exact (see Definition 2.21 and 2.23). Recall from Section 2.3 and 2.4, the left action of $G$ on itself
can be extended to its fibrewise Stone–Čech compactification \( \beta \mathcal{G} \). Hence we may form the semi-direct product groupoid \( \beta \mathcal{G} \rtimes \mathcal{G} \), whose unit space \( \beta \mathcal{G} \) has a natural decomposition:

\[
(6.6) \quad \beta \mathcal{G} = \mathcal{G} \sqcup \partial \mathcal{G},
\]

where \( \mathcal{G} \) is open and dense in \( \beta \mathcal{G} \), and \( \partial \mathcal{G} := \beta \mathcal{G} \setminus \mathcal{G} \) is the fibrewise Stone–Čech boundary of \( \mathcal{G} \). It is clear that both \( \mathcal{G} \) and \( \partial \mathcal{G} \) are invariant.

Since \( \mathcal{G} \) is strongly amenable at infinity, or weakly inner amenable and \( C^* \)-exact, the groupoid \( \beta \mathcal{G} \rtimes \mathcal{G} \) is topologically amenable by Proposition 2.22 or 2.24. Hence we may apply Theorem 4.4 to the groupoid \( \beta \mathcal{G} \rtimes \mathcal{G} \) together with the decomposition (6.6), and obtain the associated limit operator theory for \( \beta \mathcal{G} \rtimes \mathcal{G} \). Now we would like to provide a more detailed picture.

First we focus on the groupoid \( C^* \)-algebra \( C^*_r(\beta \mathcal{G} \rtimes \mathcal{G}) \), which is isomorphic to the uniform Roe algebra \( C^*_u(\mathcal{G}) \) of \( \mathcal{G} \) introduced in [AD16]. To recall the definition and the isomorphism, denote

\[
\mathcal{G} \ast_s \mathcal{G} := \{ (\gamma, \gamma_1) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = s(\gamma_1) \}.
\]

A tube is a subset of \( \mathcal{G} \ast_s \mathcal{G} \) whose image by the map \( (\gamma, \gamma_1) \mapsto \gamma \gamma_1^{-1} \) is relatively compact in \( \mathcal{G} \). We denote by \( C_t(\mathcal{G} \ast_s \mathcal{G}) \) the space of continuous bounded functions on \( \mathcal{G} \ast_s \mathcal{G} \) with support in a tube. Recall that \( L^2(\mathcal{G}) \) is the Hilbert \( C_0(\mathcal{G}^{(0)}) \)-module (see Section 2.4), and we consider the map \( \Phi : C_t(\mathcal{G} \ast_s \mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G})) \) defined by

\[
((\Phi f)\xi)(\gamma) := \sum_{\alpha \in \mathcal{G}(\gamma)} f(\gamma, \alpha)\xi(\alpha), \quad \text{where } \xi \in L^2(\mathcal{G}) \text{ and } \gamma \in \mathcal{G}.
\]

Clearly \( \Phi \) is injective so we may pull back the \(*\)-algebraic structure of \( \mathcal{B}(L^2(\mathcal{G})) \) to \( C_t(\mathcal{G} \ast_s \mathcal{G}) \). Intuitively, this is nothing but the \( \langle (\mathcal{G} \times \mathcal{G}) \rangle \)-matrix structure”. Restricted on each fibre: for any \( x \in \mathcal{G}^{(0)} \), we have the induced \(*\)-representation \( \Phi_x : C_t(\mathcal{G} \ast_s \mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x)) \) by

\[
((\Phi_x f)\eta)(\gamma) := \sum_{\alpha \in \mathcal{G}_x} f(\gamma, \alpha)\eta(\alpha), \quad \text{where } \eta \in L^2(\mathcal{G}_x) \text{ and } \gamma \in \mathcal{G}_x.
\]

**Definition 6.25.** [AD16, Definition 5.1] The uniform Roe algebra of a groupoid \( \mathcal{G} \) is the \( C^* \)-completion of \( C_t(\mathcal{G} \ast_s \mathcal{G}) \) with respect to the operator norm of \( \Phi(C_t(\mathcal{G} \ast_s \mathcal{G})) \) in \( \mathcal{B}(L^2(\mathcal{G})) \). Hence \( \Phi \) extends to a faithful representation from \( C^*_r(\mathcal{G}) \) to \( \mathcal{B}(L^2(\mathcal{G})) \), still denoted by \( \Phi \). For each \( x \in \mathcal{G}^{(0)} \), \( \Phi_x \) extends to a \(*\)-homomorphism from \( C^*_u(\mathcal{G}_x) \) to the classic uniform Roe algebra \( C^*_u(\mathcal{G}_x) \), still denoted by \( \Phi_x : C^*_u(\mathcal{G}_x) \to C^*_u(\mathcal{G}_x) \).

Now we consider the following algebraic \(*\)-isomorphism \( \theta : C_c(\beta \mathcal{G} \rtimes \mathcal{G}) \to C_t(\mathcal{G} \ast_s \mathcal{G}) \) defined by

\[
\theta(f)(\gamma, \gamma_1) = f(\gamma, \gamma_1^{-1})
\]

for \( f \in C_c(\beta \mathcal{G} \rtimes \mathcal{G}) \). It is proved in [AD16, Theorem 5.3] that \( \theta \) extends to a \( C^* \)-isomorphism

\[
\Theta : C^*_r(\beta \mathcal{G} \rtimes \mathcal{G}) \xrightarrow{\cong} C^*_u(\mathcal{G}).
\]

What does the image \( \Theta(C^*_r(\mathcal{G} \times \mathcal{G})) \) look like in the uniform Roe algebra \( C^*_u(\mathcal{G}) \)? Denote \( \mathcal{R}(L^2(\mathcal{G})) \) the algebra of compact operators on \( L^2(\mathcal{G}) \) (recall that an adjointable operator is said to be compact if it can approximated by finite-rank operators), and we have the following lemma:

**Lemma 6.26.** Notations as above. We have that \( \Phi \circ \Theta(C^*_r(\mathcal{G} \times \mathcal{G})) = \mathcal{R}(L^2(\mathcal{G})) \). Consequently, \( T \in C^*_r(\beta \mathcal{G} \times \mathcal{G}) \) is invertible modulo \( C^*_r(\mathcal{G} \times \mathcal{G}) \) if and only if \( \Phi \circ \Theta(T) \) is invertible modulo \( \mathcal{R}(L^2(\mathcal{G})) \).

**Proof.** The proof is almost the same as the group case, but we need to modify slightly since we are working in Hilbert modules. By definition, \( \Theta(C^*_r(\mathcal{G} \times \mathcal{G})) \) is the norm closure of \( \theta(C_c(\mathcal{G} \times \mathcal{G})) = C_c(\mathcal{G} \ast_s \mathcal{G}) \), where the norm of \( f \in C_c(\mathcal{G} \ast_s \mathcal{G}) \) is the operator norm of \( \Phi f \) acting on \( L^2(\mathcal{G}) \) defined by

\[
((\Phi f)\xi)(\gamma) := \sum_{\alpha \in \mathcal{G}(\gamma)} f(\gamma, \alpha)\xi(\alpha)
\]

On the other hand, for any \( \xi, \eta \in L^2(\mathcal{G}) \), denote by \( T_{\xi, \eta} \) the rank one operator \( \zeta \mapsto \xi(\eta, \zeta) \). Hence every finite rank operator has the following form:

\[
\left( \sum_{i=1}^N T_{\xi_i, \eta_i} \right)(\zeta)(\gamma) = \sum_{i=1}^N \xi_i(\gamma)\overline{\eta_i(\alpha)}\zeta(\alpha),
\]
for some \( \xi_i, \eta_i \in L^2(G) \); \( i = 1, \ldots, N \). Therefore, \( \sum_{i=1}^N T_{\xi_i, \eta_i} = \Phi(g) \) for \( g \in C_c(G \rtimes_s G) \) defined by \( g(\gamma, \alpha) = \sum_{i=1}^N \xi_i(\gamma) \eta_i(\alpha) \).

Now we only need to show that for a given \( f \in C_c(G \rtimes_s G) \), the operator \( \Phi(f) \) can be approximated by a sequence of finite rank operators. Since \( f \) has compact support, there exists \( M \in \mathbb{N} \) such that

\[
\sup_{x \in G^{(0)}} |\{ \gamma \in G_x : 3n \in G_x \text{ such that } f(\gamma, n) \neq 0 \}| \leq M,
\]

and

\[
\sup_{x \in G^{(0)}} |\{ \alpha \in G_x : 3n \in G_x \text{ such that } f(\gamma, n) \neq 0 \}| \leq M.
\]

Recall that for two locally compact Hausdorff topological spaces \( X \) and \( Y \), we have that \( C_0(X \times Y) \cong C_0(X) \otimes C_0(Y) \). The same argument can be used to show that for any \( \varepsilon > 0 \), there exist \( \xi_1, \ldots, \xi_N \) and \( \eta_1, \ldots, \eta_N \) in \( C_c(G) \) such that

\[
\sup_{(\gamma, \alpha) \in \mathcal{G} \rtimes_s \mathcal{G}} |f(\gamma, \alpha) - \sum_{i=1}^N \xi_i(\gamma) \eta_i(\alpha)| \leq \frac{\varepsilon}{M \|f\|_\infty}.
\]

Consequently, we have that

\[
\|\Phi f - \sum_{i=1}^N T_{\xi_i, \eta_i}\| \leq \varepsilon.
\]

Therefore, \( \Phi f \in \mathcal{A}(L^2(G)) \) and we finish the proof. \( \square \)

Now we move on to the discussion of limit operators. Recall that we have the symbol morphism associated to \( \beta, G \times G \) from Definition 4.2:

\[
\varsigma : C^*_r(\beta, G \rtimes G) \longrightarrow \Gamma_b(E^\beta_{\omega}(G \rtimes G)),
\]

where \( E^\beta \) is the operator fibre space associated to the reduction groupoid \( \beta, G \rtimes G \). We would like to offer a more detailed description for \( \varsigma \). Given a boundary point \( \omega \in \partial_r G = \beta, G \rtimes G \), note that

\[
(\beta, G \rtimes G)_\omega = \{(\gamma, \omega, \alpha) : s(\gamma) = \beta(\omega)\}
\]

is bijective to \( G_{\beta(\omega)} \) via the map \( \phi_\omega : (\gamma, \omega, \alpha) \mapsto \gamma \), and induces a unitary

\[
U_\omega : \ell^2((\beta, G \rtimes G)_\omega) \cong \ell^2(G_{\beta(\omega)}).
\]

Hence we obtain an isomorphism:

\[
(6.7) \quad \text{Ad}_{U_\omega} : \mathcal{B}(\ell^2((\beta, G \rtimes G)_\omega)) \cong \mathcal{B}(\ell^2(G_{\beta(\omega)})).
\]

This isomorphism suggests us to consider the following operator fibre space

\[
E^\beta_{\omega} := \bigcup_{\omega \in \partial_r G} \mathcal{B}(\ell^2(G_{\beta(\omega)})),
\]

with the topology defined as follows: a net \( \{T_\omega\}_{\iota \in I} \) converges to \( T_\omega \) if and only if \( \omega_\iota \to \omega \), and for any \( \gamma_1' \to \gamma' \), \( \gamma''_1 \to \gamma'' \) in \( G \) with \( s(\gamma_1') = r_\beta(\omega_1) = s(\gamma''_1) \) (which implies that \( s(\gamma') = r_\beta(\omega) = s(\gamma'') \)), we have

\[
\{T_\omega \delta_{\gamma_1'}, \delta_{\gamma_1''} \} \to \{T_\omega \delta_{\gamma'}, \delta_{\gamma''}\}.
\]

Combining the unitary operators \( \text{Ad}_{U_\omega} \) in (6.7) together, we obtain an isomorphism between the operator fibre spaces as follows:

\[
\text{Ad}_U = \bigcup_{\omega \in \partial_r G} \text{Ad}_{U_\omega} : E^\beta := \bigcup_{\omega \in \partial_r G} \mathcal{B}(\ell^2(G_{\beta(\omega)})) \to E^\beta_{\omega} := \bigcup_{\omega \in \partial_r G} \mathcal{B}(\ell^2(G_{\beta(\omega)})).
\]

Denote the sub-fibre space

\[
E^\beta_{\omega, u} := \bigcup_{\omega \in \partial_r G} C^*_u(G_{\beta(\omega)}),
\]

where \( C^*_u(G_{\beta(\omega)}) \) is the uniform Roe algebra defined in Definition 3.9. A section \( \mathcal{F} \) of \( E^\beta_{\omega} \) is called \( G \)-equivariant if for any \( \omega \in \partial_r G \) and \( \gamma \in G \) with \( s(\gamma) = r_\beta(\omega) \), we have \( \mathcal{F}(\gamma \omega) = R_\gamma \mathcal{F}(\omega) R_\gamma \), where \( R_\gamma : \ell^2(G_{r(\gamma)}) \to \ell^2(G_{s(\gamma)}) \) is the right multiplication as in Definition 3.7. Denote the \( C^* \)-algebra of bounded \( G \)-equivariant sections of \( E^\beta_{\omega, u} \) by \( \Gamma_b(E^\beta_{\omega, u}) \). Then we have the following result, whose proof is straightforward as we did in Lemma 6.2, hence omitted.
Lemma 6.27. The isomorphism $\text{Ad}_U$ induces a $C^*$-isomorphism:

$$(\text{Ad}_U)_* : \Gamma_b(E^\partial_\beta G \rtimes G) \cong \Gamma_b(E^\partial_\beta, (\text{Ad}_U)_*).$$

Now we define the following morphism (also called the symbol morphism)

$$\sigma := (\text{Ad}_U)_* \circ \Theta^{-1} : C^*_u(G) \to \Gamma_b(E^\partial_\beta, (\text{Ad}_U)_*).$$

In other words, the following diagram commutes:

$$
\begin{array}{ccc}
C^*_u(\beta_r G \times G) & \xrightarrow{\sigma} & \Gamma_b(E^\partial_\beta, \rho \rtimes G) \\
\Theta \cong & & \cong (\text{Ad}_U)_* \\
C^*_u(G) & \xrightarrow{\tau} & \Gamma_b(E^\partial_\beta, (\text{Ad}_U)_*). \\
\end{array}
$$

We are looking for a more practical formula of $\sigma$. Fix an element $T \in C^*_u(G)$ with $|T| \leq k$ for some $k \in \mathbb{N}$. Recall from Section 3, the $k$-bounded operator fibre space associated to $G$ is

$$E_k = \bigcup_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x))_k.$$

Consider the following map $\tau_T$ between the fibre space $(G, r)$ and $(E_k, p)$ over $G^{(0)}$:

$$\tau_T : G \to E_k, \quad \gamma \mapsto R^*_\gamma \Phi_{s(\gamma)}(T) \gamma,$$

where

$$R_\gamma : \ell^2(G_{r(\gamma)}) \to \ell^2(G_{s(\gamma)}), \quad \delta_\alpha \mapsto \delta_{\alpha\gamma}.$$ 

It is obvious that $r = p \circ \tau_T$ and $\tau_T$ is continuous, hence $\tau_T$ is a morphism of fibre spaces over $G^{(0)}$. Since $G$ is $\sigma$-compact and second countable, we know that $(E_k, p)$ is fibrewise compact by Lemma 3.12. Hence by the universal property of $\beta_r G$ (Proposition 2.8), $\tau_T$ can be extended to a morphism:

$$\tau_T : \beta_r G \to E_k.$$

Definition 6.28. Let $G$ be as above, and $\omega \in \partial\beta G$. We define the limit operator of $T$ at $\omega$ to be the operator $\tau_T(\omega)$ in $\mathcal{B}(\ell^2(G_{r(\omega)})$. Also define $\tau_\omega : C^*_u(G) \to \mathcal{B}(\ell^2(G_{r(\omega)})$ by $\tau_\omega(T) := \tau_T(\omega)$.

Lemma 6.29. For $T \in C^*_u(G)$ and $\omega \in \partial\beta G$, we have $\sigma(T)(\omega) = \tau_\omega(T)$.

Proof. It suffices to prove the following diagram commutes:

$$
\begin{array}{ccc}
C^*_u(\beta_r G \times G) & \xrightarrow{\lambda_\omega} & \mathcal{B}(\ell^2((\beta_r G \times G)_{\omega})) \\
\Theta \cong & & \cong \text{Ad}_U_{\omega} \\
C^*_u(G) & \xrightarrow{\tau_\omega} & \mathcal{B}(\ell^2(G_{r(\omega)})). \\
\end{array}
$$

By density, we only need to prove the lemma for $T \in C_c(G \ast_x G)$. To simplify notations, let $x = r_\beta(\omega)$. Recall that $\theta : C_c(\beta_r G \times G) \to C_c(G \ast_x G)$ is an isomorphism, so we may assume that $T = \theta(f)$ for some $f \in C_c(\beta_r G \times G)$. For any $\xi \in \ell^2(G_x)$ and $\gamma \in G_x$, we have:

$$
(\sigma(T)(\omega)\xi)(\gamma) = (\text{Ad}_U_{\omega} \circ \lambda_\omega \circ \Theta^{-1}(\theta(f))\xi)(\gamma) = (\text{Ad}_U_{\omega} \circ \lambda_\omega(f)\xi)(\gamma) = \sum_{\alpha \in \mathcal{G}_x} f(\gamma\omega, \gamma^{-1}\alpha^{-1})\xi(\alpha) = \lim_{\gamma_1 \to \omega} \sum_{\alpha_{\beta, \gamma_1} \in \mathcal{G}_x} f(\gamma_1, \alpha_{\beta, \gamma_1})\xi(\alpha) = \lim_{\gamma_1 \to \omega} (\theta(f))(\gamma_1, \alpha_{\beta, \gamma_1})\xi(\alpha).
$$

Hence for any $\gamma, \alpha \in G_x$, we have

$$
\langle \sigma(T)(\omega)\delta_{\alpha}, \delta_{\gamma} \rangle = \lim_{\gamma_1 \to \omega} (\theta(f))(\gamma_1, \alpha_{\beta, \gamma_1}).
$$
On the other hand, for any $\gamma, \alpha \in \mathcal{G}_x$, we have

$$\langle \tau_\omega(T)\delta_\alpha, \delta_\gamma \rangle = \lim_{\gamma_1 \to \omega} \langle R^\ast_{\gamma_1} \Phi_{s(\gamma_1)}(T)R_{\gamma_1}\delta_\alpha, \delta_\gamma \rangle = \lim_{\gamma_1 \to \omega} \langle \Phi_{s(\gamma_1)}(T)\delta_{\alpha\gamma_1}, \delta_{\gamma_1} \rangle$$

$$= \lim_{\gamma_1 \to \omega} \langle (\theta f)(\gamma_1, \alpha\gamma_1) \rangle.$$

Hence we finish the proof. \qed

Consequently, we obtain the associated limit operator theory as a corollary of Theorem 4.4:

**Corollary 6.30.** Let $\mathcal{G}$ be a locally compact, $\sigma$-compact, second countable and étale groupoid. Suppose $\mathcal{G}$ is either strongly amenable at infinity, or weakly inner amenable and $C^\ast$-exact. Then for any $T$ in the uniform Roe algebra $C^\ast_u(\mathcal{G})$, the following are equivalent:

1. $\Phi(T)$ is invertible modulo $\mathcal{R}(L^2(\mathcal{G}))$.
2. $\sigma(T)$ is invertible in $\Gamma_b(F^0_{\beta,u})^\mathcal{G}$.
3. For each $\omega \in \partial_x \mathcal{G}$, the limit operator $\tau_\omega(T)$ is invertible, and
   $$\sup_{\omega \in \partial_x \mathcal{G}} \|\tau_\omega(T)^{-1}\| < \infty.$$
4. For each $\omega \in \partial_x \mathcal{G}$, the limit operator $\tau_\omega(T)$ is invertible.

The equivalence between the first three conditions also holds for groupoids with a weaker property: $KW$-exactness (for abbreviation of Kirchberg-Wassermann exactness). This notion is introduced in [AD16], and roughly speaking it means that the cross product of the groupoid preserves short equivariant exact sequences (see [AD16, Definition 6.6]). It is also shown in [AD16] that strongly amenability at infinity implies $KW$-exactness, and $KW$-exactness implies $C^\ast$-exactness (Definition 2.21) for second countable locally compact groupoids with Haar system. Furthermore, $KW$-exactness and $C^\ast$-exactness coincide when the groupoid is further assumed to be weakly inner amenable and étale, hence are also equivalent to the strongly amenability at infinity by Proposition 2.24. The proof for the equivalence of the first three conditions in the case of $KW$-exactness follows similarly as we did in Section 6.2, based on Remark 4.7. Consequently, we have an alternative version of Corollary 6.30 as follows:

**Corollary 6.30'.** Let $\mathcal{G}$ be a locally compact, $\sigma$-compact, second countable, étale and $KW$-exact groupoid. Then for any $T$ in the uniform Roe algebra $C^\ast_u(\mathcal{G})$, the following are equivalent:

1. $\Phi(T)$ is invertible modulo $\mathcal{R}(L^2(\mathcal{G}))$.
2. $\sigma(T)$ is invertible in $\Gamma_b(F^0_{\beta,u})^\mathcal{G}$.
3. For each $\omega \in \partial_x \mathcal{G}$, the limit operator $\tau_\omega(T)$ is invertible, and
   $$\sup_{\omega \in \partial_x \mathcal{G}} \|\tau_\omega(T)^{-1}\| < \infty.$$
4. For each $\omega \in \partial_x \mathcal{G}$, the limit operator $\tau_\omega(T)$ is invertible.

Finally, we mention briefly some other equivariant fibrewise compactification of groupoids and study the associated limit operator theory, as we did in Section 6.2. We only provide the sketch and omit the details, since the proofs are similar.

Let $\mathcal{G}$ be a locally compact, $\sigma$-compact, second countable and étale groupoid. Let $(Y, p)$ be a $\mathcal{G}$-equivariant fibrewise compactification of the $\mathcal{G}$-space $(\mathcal{G}, r)$, and $\partial_y \mathcal{G} := Y \setminus \mathcal{G}$. By the universal property of $\beta_y \mathcal{G}$ (Proposition 2.8), there exists a $\mathcal{G}$-equivariant surjective morphism $\phi : \beta_y \mathcal{G} \to Y$. Hence, we obtain an embedding $C_c(Y \times \mathcal{G}) \to C_c(\beta_y \mathcal{G} \times \mathcal{G})$, which induces an injective $*$-homomorphism:

$$\phi^* : C^\ast_c(Y \times \mathcal{G}) \to C^\ast_c(\beta_y \mathcal{G} \times \mathcal{G}).$$

Denote $\mathcal{A}_Y$ the subalgebra $\Theta \circ \phi^* (C^\ast_c(Y \times \mathcal{G}))$ in $C^\ast_c(\mathcal{G})$. We may also define the associated fibre space $E^0_Y := \sqcup_{x \in \mathcal{G}} \mathcal{B}(\ell^2(\mathcal{G}_{p(x)}))$, and the symbol morphism $\sigma_Y : \mathcal{A}_Y \to \Gamma_b(E^0_{Y,u})^\mathcal{G}$. Then we have:
Corollary 6.31. Let $\mathcal{G}$ be a locally compact, $\sigma$-compact, second countable and étale groupoid which is either strongly amenable at infinity, or weakly inner amenable and $C^*$-exact. Let $(Y, p)$ be a $\mathcal{G}$-equivariant fibrewise compactification of $(\mathcal{G}, r)$, $\phi : \beta \mathcal{G} \to Y$ be the induced surjection, and $Z$ be a subset in $\partial \mathcal{G}$ such that $\phi(Z) = \partial_Y \mathcal{G} = Y \setminus \mathcal{G}$. Suppose $A_Y$ is the associated $C^*$-subalgebra of $C^*_u(\mathcal{G})$ as defined above. Then for any $T \in A_Y$, the following are equivalent:

1. $\Phi(T)$ is invertible modulo $\mathbb{R}(L^2(\mathcal{G}))$.
2. $\sigma_Y(T)$ is invertible in $\Gamma_b(E_{Y,u})^0$.
3. For each $\omega \in Z$, the limit operator $\tau_\omega(T)$ is invertible, and
\[
\sup_{\omega \in Z} \| \tau_\omega(T)^{-1} \| < \infty.
\]
4. For each $\omega \in Z$, the limit operator $\tau_\omega(T)$ is invertible.

As noted before, the first three conditions also hold when $\mathcal{G}$ is KW-exact. Consequently, we have the following alternative version:

Corollary 6.31'. Let $\mathcal{G}$ be a locally compact, $\sigma$-compact, second countable, étale and KW-exact groupoid. Let $(Y, p)$ be a $\mathcal{G}$-equivariant fibrewise compactification of $(\mathcal{G}, r)$, $\phi : \beta \mathcal{G} \to Y$ be the induced surjection, and $Z$ be a subset in $\partial \mathcal{G}$ such that $\phi(Z) = \partial_Y \mathcal{G} = Y \setminus \mathcal{G}$. Suppose $A_Y$ is the associated $C^*$-subalgebra of $C^*_u(\mathcal{G})$ as defined above. Then for any $T \in A_Y$, the following are equivalent:

1. $\Phi(T)$ is invertible modulo $\mathbb{R}(L^2(\mathcal{G}))$.
2. $\sigma_Y(T)$ is invertible in $\Gamma_b(E_{Y,u})^0$.
3. For each $\omega \in Z$, the limit operator $\tau_\omega(T)$ is invertible, and
\[
\sup_{\omega \in Z} \| \tau_\omega(T)^{-1} \| < \infty.
\]
4. For each $\omega \in Z$, the limit operator $\tau_\omega(T)$ is invertible.

Furthermore, if in addition $\mathcal{G}$ is weakly inner amenable, then the above are also equivalent to:

(4) For each $\omega \in Z$, the limit operator $\tau_\omega(T)$ is invertible.

Remark 6.32. Finally we remark that one may also study the limit operator theory for groupoid actions, which is analogous to the one studied in Section 6.4. However, here we choose not to mention it any further since more preliminary notions would be required, and this would definitely make the current paper more complicated.

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Weizmann Institute of Science and Technology

*E-mail address*: ksaustin88@gmail.com

University of Southampton

*E-mail address*: jiawen.zhang@soton.ac.uk