A new analytical approximation for a light sterile neutrino oscillation in matter

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ABSTRACT: The existence of light sterile neutrinos is a long standing question which has haunted particle physics. Several experimental “anomalies” could be explained by introducing eV mass scaled sterile neutrinos which mix with the active ones. Many short and long-baseline neutrino oscillation experiments are actively searching for light sterile neutrinos at various mass ranges. Matter effects need to be treated carefully for precise neutrino oscillation probability calculation, especially in long baseline neutrino oscillation experiments. Both charged current and neutral current coherent forward scattering could change sterile neutrino oscillation behaviour when they propagate through terrestrial matter. In this manuscript we introduce a Jacobi-like method to diagonalize the Hermitian Hamiltonian matrix and derive the analytically simplified neutrino oscillation probabilities for 3 (active) + 1 (sterile)-neutrino mixing in a constant matter density. When active and sterile neutrino mixing is small, the approximation formula can reach high numerical accuracy. Given its analytical simplicity and fast computing speed, this method could be very useful for current and future medium and long baseline neutrino oscillation experiments.

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1 Introduction

Neutrino oscillation has been indisputably established by atmospheric, solar, reactor and accelerator experimental results [1]. After recent reactor experiments [2–4] discovered the last unknown mixing angle $\theta_{13}$ in a 3-neutrino mixing framework, neutrino oscillation experiments entered the precision era. Massive neutrinos provide convincing evidence of new physics beyond the standard model. Introducing right-handed neutrinos is a natural way to explain non-zero neutrino mass. In the standard electro-weak V-A theory, right-handed neutrinos cannot couple with W and Z bosons. Electron collider experimental data [5] constrain the number of active light neutrino flavors to three, other new types of light neutrinos must be sterile. Currently there is no constraint on sterile neutrino mass. They could be
very massive $(10^{15}$ GeV) as suggested by the see-saw mechanism; They also could be dark matter in the keV mass range; And they also might be as light as sub-eV which would match the CMB measurement.

Searching for sterile neutrinos is a very active field in neutrino physics. If light sterile neutrinos mix with the active neutrinos, their signature could be observed by neutrino oscillation experiments. LSND observed $87.9 \pm 22.4 \pm 6.0 \ {
u_e}$ signal events from $\nu_\mu$ source from $\mu^+$ decay at rest, which suggests a sterile neutrino with mass greater than 0.4 eV [6, 7]. Recently MiniBooNE experiments reported a $4.7\sigma$ excess of electron-like events when combining both the $\nu_\mu$ and $\bar{\nu}_e$ beam configurations. The significance of the combined LSND and MiniBooNE excesses can even reach $6\sigma$ [8], although the source of the low energy excess from MiniBooNE is still unclear. Experimental hints of the existence of eV mass scaled sterile neutrinos also come from short baseline reactor neutrino experiments [9–11]. However, the uncertainty of theoretical reactor neutrino flux calculation might be underestimated, giving an observed excess of anti-neutrino events at 4-6 MeV relative to predictions [12–16]. It is worth mentioning, although eV-scale sterile neutrinos could help to explain several experimental anomalies, they are in tension with the muon neutrino disappearance results, especially for recent results from IceCube [17] and MINOS/MINOS+ [18]. The existence of eV mass scale sterile neutrinos therefore needs further evidence. Many reactor and accelerator neutrino experiments are actively searching for sterile neutrinos at various mass scales [19–23].

For long baseline accelerator neutrino experiments [24, 25], neutrino matter effects play an important role in neutrino mass hierarchy [26, 27] and CP violation [28] measurements. As first pointed out by Wolfenstein, neutrinos propagating in matter will oscillate differently from those in a vacuum [29]. The presence of electrons in matter changes the energy levels of propagation eigenstates of neutrinos due to charged current coherent forward scattering of the electron neutrinos. Later on, Mikheyev and Smirnov [30] further noticed the matter effect can produce resonant maximal flavor transition when neutrinos propagate through matter at certain electron densities. Super-Kamiokande observes an indication of different solar neutrino flux during the night and day for solar neutrinos passing through additional terrestrial matter in the earth at different periods [31]. For sterile neutrino and other new physics searches, matter effects have to be calculated carefully and precisely, especially for long baseline neutrino oscillation experiments.

Neutrino oscillation in matter can be solved accurately using numerical calculation with a complex matrix diagonalization algorithm. However, it is usually quite computationally time consuming. In practice, analytic approximations are more commonly used in neutrino experiments and make it easier for people to understand the oscillation features. For simplicity, these analytic expressions usually keep the same oscillation formula for neutrinos propagating in matter as they have in a vacumm with the effective neutrino mixing angles and mass-squared differences in matter. High precision analytical expressions for 3-neutrino oscillation in matter has been throughly studied [32–45] and most of them utilize perturbation theory to solve the problem. However, these become much more complicated if additional light sterile neutrinos exist [46]. Compared with standard 3-neutrino mixing, the simplest 3 (active) + 1 (sterile)-neutrino mixing needs 3 additional mixing angles (i.e.
θ_{14}, \theta_{24} \text{ and } \theta_{34}) \text{ and 2 additional CP phases (i.e. } \delta_{24} \text{ and } \delta_{34}). \text{ In addition, matter effects are also different for sterile neutrinos. Since sterile neutrinos do not interact with matter, the neutral current potential for active neutrinos needs be taken into account. Ref. } [47] \text{ suggests a method of transforming 3+1-neutrino mixing with matter effects into a Non-Standard Interaction (NSI) problem in the 3-neutrino mixing case. Previously we derived the exact analytical expressions } [50] \text{ for 3+1-neutrino oscillation in matter. Here we further derived analytical approximations using the extended Jacobi-like method } [51–53], \text{ which is able to diagonalize the hermitian complex matrix. With this approach, the analytical expressions of neutrino oscillation probabilities can achieve very good numerical accuracy and fast calculation speed. This could be very useful for current and near future neutrino oscillation experiments.}

This paper starts with the section 2 and introduces neutrino mixing and oscillation, including sterile neutrinos and matter effects. The basic idea of the Jacobi-like method and the derivation of analytical approximations for sterile neutrino oscillation probabilities are presented in section 3. In the end, the accuracy of this work is shown in section 4 with two near future long baseline accelerator neutrino experiments as demonstrated. More details about the Jacobi-like method and formula derivation are listed in the appendix.

2 Theoretical framework

2.1 Neutrino oscillation

In the standard neutrino mixing paradigm, three neutrino flavor eigenstates \( (\nu_e, \nu_\mu, \nu_\tau) \) are superpositions of three neutrino mass eigenstates \( (\nu_1, \nu_2, \nu_3) \).

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = U 
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]  

(2.1)

Here \( U \) is the so-called PMNS (Pontecorvo-Maki-Nakawaga-Sakata) mixing matrix [54–56], which can be parametrized as

\[
U = R_{23}(\theta_{23}, 0)R_{13}(\theta_{13}, \delta_{13})R_{12}(\theta_{12}, 0)
= R_{ij}(\theta_{ij}, \delta_{ij})
\]

(2.2)

where \( R_{ij}(\theta_{ij}, \delta_{ij}) \) denotes a counterclockwise rotation in the complex \( ij \)-plane through a mixing angle \( \theta_{ij} \) and a CP phase \( \delta_{ij} \) with \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \). This work adopts the conventions \( 0 \leq \theta_{ij} \leq \pi/2 \) and \( 0 \leq \delta_{ij} \leq 2\pi \).

Under the plane wave assumption, the general oscillation probability from \( \alpha \)-flavor type neutrinos to \( \beta \)-flavor type neutrinos can be expressed as

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{i>j} \Re (U_{i\beta} U_{\alpha i}^* U_{j\beta}^* U_{\beta j}) \sin^2 \Delta_{ij} \pm 2 \sum_{i>j} \Im (U_{i\beta} U_{\alpha i}^* U_{j\beta}^* U_{\beta j}) \sin 2\Delta_{ij}, \quad (i, j = 1, 2, 3)
\]  

(2.3)
where the upper and lower sign is for the neutrino and antineutrino cases respectively. $\Delta_{ij}$ stands for

$$
\Delta_{ij} \equiv \frac{\Delta m_{ij}^2 L}{4E} = 1.267 \left( \frac{\Delta m_{ij}^2}{\text{eV}^2} \right) \left( \frac{\text{GeV}}{E} \right) \left( \frac{L}{\text{km}} \right),
$$

(2.4)

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ is the mass-squared difference between neutrino mass eigenstates $\nu_i$ and $\nu_j$.

According to eq. (2.2) and eq. (2.3), 3-flavor neutrino oscillation is described with six parameters, including two independent neutrino mass squared differences ($\Delta m_{21}^2$ and $\Delta m_{32}^2$), three mixing angles ($\theta_{12}$, $\theta_{13}$ and $\theta_{23}$) and one leptonic CP phase ($\delta_{13}$). Following the same convention, the 4-flavor neutrino mixing matrix can be parametrized as

$$
U = R_{34}(\theta_{34}, \delta_{34})R_{24}(\theta_{24}, \delta_{24})R_{14}(\theta_{14}, 0)R_{13}(\theta_{13}, \delta_{13})R_{12}(\theta_{12}, 0),
$$

(2.5)

with six additional neutrino oscillation parameters: $\theta_{14}$, $\theta_{24}$, $\theta_{34}$, $\delta_{24}$, $\delta_{34}$ and $\Delta m_{41}^2$. The exact parameterization expression for each mixing element is listed in the appendix A. The general expression for the neutrino oscillation probabilities still follow eq. (2.3) by simply increasing the total number of neutrino flavors and mass eigenstates to 4.

In practice, when sterile neutrinos are much heavier than active neutrinos ($|\Delta m_{41}^2| \gg |\Delta m_{31}^2|$), due to finite detector space and energy resolution, the rapid oscillation frequency associated with large mass-squared differences between the 4th and the other mass eigenstates $\Delta m_{4k}^2(k = 1, 2, 3)$ will be averaged out, leading to $\langle \sin^2 \Delta_{4k} \rangle \approx \frac{1}{2}$. The neutrino oscillation equation can then be simplified to

$$
P_{\nu_{\alpha} \to \nu_{\beta}} = \delta_{\alpha\beta} - 4 \sum_{i>j} R \left( U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \right) \sin^2 \Delta_{ij}
$$

$$
\pm 2 \sum_{i>j} \Im \left( U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \right) \sin 2 \Delta_{ij} (i, j = 1, 2, 3)
$$

(2.6)

with $\sin^2 2\theta_{\alpha\beta} = 4|U_{\alpha 4}|^2 (\delta_{\alpha 4} - |U_{\beta 4}|^2)$. In this paper we prefer to use the full oscillation formula to preserve the rapid oscillation induced by sterile neutrinos.

### 2.2 Matter effects

When active neutrinos propagate through matter, the evolution equation is modified by coherent interaction potentials, which are generated through coherent forward elastic weak charged-current (CC) and the neutral-current (NC) scattering in a medium. All active neutrinos can interact with electrons, neutrons and protons in matter through the exchange of a $Z$ boson in the NC process. However, only electron neutrinos participate in the CC process with electrons through the exchange of $W^\pm$.

For electron neutrinos, CC potential is proportional to electron number density. $V_{CC} = \sqrt{2} G_F N_e$, where $G_F$ is the Fermi coupling constant, $N_e$ is the electron number density. The NC potentials caused by electrons and protons will cancel each other because they have

*This is equivalent to using $\delta_{14}$ and $\delta_{24}$, or $\delta_{14}$ and $\delta_{34}$ for the additional CP phases.
opposite signs and the number densities of electrons and protons are basically the same in the earth. The net NC potential, \( V_{NC} = -\frac{\sqrt{2}}{2} G_F N_n \), is only sensitive to the neutron number density, \( N_n \). Both \( V_{CC} \) and \( V_{NC} \) need to swap signs for antineutrinos.

For 3-flavor neutrino oscillation, only CC potential needs to be considered for the electron neutrino eigenstate, while the NC potential is a common term for all neutrino flavors and has no net effect on neutrino oscillation. However, the NC potential cannot be neglected in 3+1-flavor neutrino case, since sterile neutrinos do not interact with matter.

The effective Hamiltonian in the mass eigenstate representation for 3+1-flavor neutrino mixing is

\[
H = H_v + U^\dagger V U = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & \Delta m_{21}^2 & 0 & 0 \\
  0 & 0 & \Delta m_{31}^2 & 0 \\
  0 & 0 & 0 & \Delta m_{41}^2
\end{bmatrix} + U^\dagger \begin{bmatrix}
  A_{CC} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & A_{NC}
\end{bmatrix} U, \tag{2.7}
\]

where \( H_v \) is the neutrino Hamiltonian in vacuum and \( V \) is the matter effect potential. \( A_{CC} \) and \( A_{NC} \) for neutrinos are given by

\[
A_{CC} = 2E V_{CC} = 7.63 \times 10^{-5}(eV^2)(\frac{\rho}{\text{g/cm}^3})(\frac{E}{\text{GeV}}), \tag{2.8a}
\]

\[
A_{NC} = -2E V_{NC} = 3.815 \times 10^{-5}(eV^2)(\frac{\rho}{\text{g/cm}^3})(\frac{E}{\text{GeV}}), \tag{2.8b}
\]

respectively, where \( \rho \) is the mass density. Similarly to \( V_{CC} \) and \( V_{NC} \), both \( A_{CC} \) and \( A_{NC} \) have to swap signs for antineutrinos. In this work, we assume a constant \( \rho \). If there is no special declaration, \( \rho \) will be set to 2.6 g/cm\(^3\) as default. For simplicity, write the effective Hamiltonian as

\[
H = \begin{bmatrix}
  H_{11} & H_{12} & H_{13} & H_{14} \\
  H_{21} & H_{22} & H_{23} & H_{24} \\
  H_{31} & H_{32} & H_{33} & H_{34} \\
  H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}, \tag{2.9}
\]

where the Hermitian matrix element \( H_{ij} \) yields

\[
H_{ij} = \begin{cases}
  A_{CC} U_{ei}^* U_{sj} + A_{NC} U_{si}^* U_{sj} & (i \neq j) \\
  \Delta m_{21}^2 + A_{CC} |U_{ei}|^2 + A_{NC} |U_{si}|^2 & (i = j)
\end{cases}. \tag{2.10}
\]

The evolution of neutrino flavor state \( \Psi_\alpha \) can be calculated using Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \Psi_\alpha = \frac{H}{\hbar} \Psi_\alpha. \tag{2.11}
\]

After diagonalizing the effective Hamiltonian matrix \( H \), we can calculate neutrino oscillation probability in matter through the equation

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{i>j} \Re \left( \tilde{U}_{\beta i} \tilde{U}_{\alpha i}^* \tilde{U}_{\beta j}^* \tilde{U}_{\alpha j} \right) \sin^2 \tilde{\Delta}_{ij} + 2 \sum_{i>j} \Im \left( \tilde{U}_{\beta i} \tilde{U}_{\alpha i}^* \tilde{U}_{\beta j}^* \tilde{U}_{\alpha j} \right) \sin 2\tilde{\Delta}_{ij}, \tag{2.11}
\]

with the effective mixing matrix \( \tilde{U} \) and effective mass-squared differences \( \tilde{\Delta} m_{ij}^2 (i, j = 1, 2, 3, 4) \).
3 The analytical approximation

As shown in ref. [50], the exact solution for the effective mixing matrix $\tilde{U}$ and effective mass-squared differences $\Delta \tilde{m}_{ij}^2 (i,j = 1,2,3,4)$ can be obtained analytically. However, to obtain higher precision analytical approximations for neutrino oscillation in matter would be more convenient and time-saving. Here we would like to introduce a Jacobi-like method, which is a unitary transformation operation method to diagonalize the complex Hermitian matrix. Then we present the effective mixing matrix and effective mass-squared differences of the 3+1-flavor neutrino mixing framework for both neutrinos and antineutrinos. As a result, high accuracy can be obtained for the calculation of neutrino oscillation probabilities in matter.

3.1 Jacobi-like method: Diagonalization of a $2 \times 2$ hermitian matrix

The Jacobi-like method, which originates from the Jacobi eigenvalue algorithm, is an effective matrix rotation approach to a diagonalize complex hermitian matrix. Here we start with an example of solving a $2 \times 2$ Hermitian matrix. A hermitian matrix

$$M = \begin{bmatrix} \alpha & \beta \\ \beta^* & \gamma \end{bmatrix} \quad (\alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}) \quad (3.1)$$

can be diagonalized as

$$M' = R^\dagger(\omega, \phi) M R(\omega, \phi) = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix} \quad (3.2)$$

with a rotation matrix

$$R(\omega, \phi) = \begin{bmatrix} \cos \omega & \sin \omega e^{-i\phi} \\ -\sin \omega e^{i\phi} & \cos \omega \end{bmatrix}, \quad (\omega, \phi \in \mathbb{R}) \quad (3.3)$$

where $e^{i\phi} = \frac{A}{\beta}$, $A = \pm |\beta|$ and $\tan \omega = \frac{2A}{\gamma - \alpha \pm \sqrt{(\gamma - \alpha)^2 + 4A^2}}$. The choice of $\pm$ sign for $A$ is optional. For simplicity, we choose it to be the same sign as $A_{CC}$ and $A_{NC}$ in eq. (2.8) for matter effects in the 3+1 framework. The $\pm$ sign in the denominator of $\tan \omega$ is correlated with the exchange of the values of $\lambda_-$ and $\lambda_+$ in eq. (3.2). In this work we adopt $+$ for the i-j submatrix diagonalization if $\Delta m_{ij}^2 > 0$ ($\Delta m_{ij}^2 < 0$). After rotation, the eigenvalues of $M$ can be obtained as

$$\lambda_- = \frac{\alpha + \gamma \tan^2 \omega - 2A \tan \omega}{1 + \tan^2 \omega}, \quad \lambda_+ = \frac{\alpha \tan^2 \omega + \gamma + 2A \tan \omega}{1 + \tan^2 \omega}. \quad (3.4)$$

In a summary, this method is easily used to diagonalize a complex hermitian matrix through rotation, in which the complex factor $\phi$ is used to deal with the complex diagonalization.

3.2 The application of Jacobi-like method on 3+1-flavor neutrino mixing

To accurately diagonalize the $4 \times 4$ neutrino Hamiltonian Hermitian matrix using the Jacobi-like method, in principle, we need to perform infinite iterations of $2 \times 2$ submatrix rotation.
However, in practice, with only two continuous rotations on the effective Hamiltonian, we already can get analytical approximations for neutrino oscillation in matter with very high accuracy. The diagonalized Hamiltonian yields

\[
\tilde{H} \approx R^{2 \dagger} R^{1 \dagger} H R^1 R^2 = \tilde{U}^\dagger (U \mathcal{H} U^\dagger + V) \tilde{U},
\]  

(3.5)

where \( \tilde{U} = U R^1 R^2 \), and \( R^1 \) and \( R^2 \) are the rotation matrices. After some mathematical simplifications, \( \tilde{U} \) can be expressed as \( R_{34} R_{24} R_{14} R_{23} R_{13} R_{12} \), which has the same form as standard neutrino mixing \( U \). For simplicity, we just show the major results of \( \tilde{U} \) and \( \Delta \tilde{m}_{ij}^2 \) \((i, j = 1, 2, 3, 4) \) in this section. The complete derivations are shown in appendix B.1 and B.2.

With two continuous rotations on the effective Hamiltonian \( H \), we can obtain the effective neutrino mixing matrix \( \tilde{U} \)

\[
\tilde{U} \approx R_{34}(\theta_{34}, \delta_{34}) R_{24}(\theta_{24}, \delta_{24}) R_{14}(\theta_{14}, 0) R_{23}(\theta_{23}, 0) R_{13}(\tilde{\theta}_{13}, \tilde{\delta}_{13}) R_{12}(\tilde{\theta}_{12}, \tilde{\delta}_{12}).
\]  

(3.6)

It is very similar to the one in vacuum (i.e. eq. (2.5)), except that there is one additional effective phase \( \tilde{\delta}_{12} \) in the submatrix \( R_{12} \). \( \tilde{\theta}_{12} \), \( \tilde{\theta}_{13} \), \( \tilde{\delta}_{13} \) and \( \tilde{\delta}_{12} \) are the effective angles and phases as functions of \( E \) in \( R_{13} \) and \( R_{12} \). And \( R_{34} \), \( R_{24} \), \( R_{14} \) and \( R_{23} \) are the same as in vacuum. In the diagonalization process, it is always better to first apply a rotation to the submatrix which has the largest absolute ratio of the off-diagonal element to the diagonal one. Since \( \Delta m_{31}^2 \) is the smallest mass-squared difference compared with others, we can start with \( R_{12} \) submatrix rotation first.

After the first rotation with \( R^1 = R_{12}(\omega_1, \phi_1) \) submatrix, we can obtain the effective angle \( \tilde{\theta}_{12} \) and effective phase \( \tilde{\delta}_{12} \) represented as functions of \( \omega_1 \) and \( \phi_1 \) through the combination of \( R_{12}(\theta_{12}, 0) R_{12}(\omega_1, \phi_1) \):

\[
\sin \tilde{\theta}_{12} \approx \frac{|c_{12} \tan \omega_1 e^{i\phi_1} + s_{12}|}{\sqrt{1 + \tan^2 \omega_1}}, \quad \cos \tilde{\theta}_{12} \approx \frac{|c_{12} - s_{12} \tan \omega_1 e^{i\phi_1}|}{\sqrt{1 + \tan^2 \omega_1}},
\]

(3.7a)

\[
e^{i\tilde{\delta}_{12}} \approx \frac{(c_{12} \tan \omega_1 e^{i\phi_1} + s_{12})(c_{12} - s_{12} \tan \omega_1 e^{-i\phi_1})}{\cos \tilde{\theta}_{12} \sin \tilde{\theta}_{12}(1 + \tan^2 \omega_1)},
\]

(3.7b)

in which \( \tan \omega_1 = \frac{2A_{\omega_1}}{(H_{12} - H_{41}) + \sqrt{(H_{22} - H_{11})^2 + 4A_{\omega_1}^2}} \), \( A_{\omega_1} = \pm |H_{12}| \) and \( e^{i\phi_1} = \frac{A_{\omega_1}}{m_{12}} \). The + and - signs in \( A_{\omega_1} \) are for the neutrino and antineutrino cases respectively. After the first rotation ((B.3) and (B.26)), we can obtain the eigenvalues of the effective Hamiltonian submatrix

\[
\lambda_- = \frac{H_{11} + H_{22} \tan^2 \omega_1 - 2A_{\omega_1} \tan \omega_1}{1 + \tan^2 \omega_1}, \quad \lambda_+ = \frac{H_{11} \tan^2 \omega_1 + H_{33} + 2A_{\omega_1} \tan \omega_1}{1 + \tan^2 \omega_1}.
\]

(3.8)

After partial diagonalization on the 1-2 submatrix, the off-diagonal elements of 2-3 (1-3) submatrix become the largest of the rest of the submatrices for neutrinos (antineutrinos) due to the smallness of the sterile neutrinos mixing angles (i.e. \( \theta_{14}, \theta_{24}, \theta_{34} \)). In the second rotation, we adopt \( R^2 = R_{23}(\omega_2, \phi_2) \left( R^2 = R_{13}(\omega_2, \phi_2) \right) \) rotation matrix for the neutrino

\footnote{Rotation is chosen by considering convenience of calculations shown in B.1.2 and B.2.2.}
(antineutrino) case. After the second rotation, we obtain the $\tilde{\theta}_{13}$ and $\tilde{\delta}_{13}$ as the functions of $\omega_2$ and $\phi_2$:

$$\sin \tilde{\theta}_{13} \approx \frac{|c_{13} \tan \omega_2 e^{i\phi_2} + s_{13} e^{i\delta_{13}}|}{\sqrt{1 + \tan^2 \omega_2}}, \quad \cos \tilde{\theta}_{13} \approx \frac{|c_{13} - s_{13} \tan \omega_2 e^{i(\delta_{13} - \phi_2)}|}{\sqrt{1 + \tan^2 \omega_2}}, \quad (3.9a)$$

$$e^{i\tilde{\delta}_{13}} \approx \frac{(c_{13} \tan \omega_2 e^{i\phi_2} + s_{13} e^{i\delta_{13}}) (c_{13} - s_{13} \tan \omega_2 e^{i(\delta_{13} - \phi_2)})}{\cos \tilde{\theta}_{13} \sin \tilde{\theta}_{13} (1 + \tan^2 \omega_2)}, \quad (3.9b)$$

in which $\tan \omega = \frac{2 A_{\omega_2}}{(|H_{232} - \lambda_\pm| + \sqrt{(H_{332} - \lambda_\pm)^2 + 4 A_{\omega_2}^2})}$. In the equation for $\tan \omega$, the upper (lower) sign in front of $\sqrt{(H_{332} - \lambda_\pm)^2 + 4 A_{\omega_2}^2}$ is for NH (IH) case, and $\lambda_+ (\lambda_-)$ is for neutrino (antineutrino) case. In the above equations, $A_{\omega_2}$ and $e^{i\phi_2}$ have different expressions for neutrinos and antineutrinos. For the neutrino case,

$$A_{\omega_2} = |H'_{23}|, \quad e^{i\phi_2} = \frac{A_{\omega_2}}{H'_{23}}, \quad H'_{23} = \frac{H_{13} \tan \omega_1 e^{i\phi_1} + H_{23}}{\sqrt{1 + \tan^2 \omega_1}}. \quad (3.10)$$

While for the antineutrino case,

$$A_{\omega_2} = -|H'_{13}|, \quad e^{i\phi_2} = \frac{A_{\omega_2}}{H'_{13}}, \quad H'_{13} = \frac{H_{13} - H_{23} \tan \omega_1 e^{-i\phi_1}}{\sqrt{1 + \tan^2 \omega_1}}. \quad (3.11)$$

In this rotation, we can diagonalize 2-3 (1-3) submatrix for neutrinos (antineutrinos) in eq. (B.7) (eq. (B.30)), resulting in two eigenvalues $\lambda'_{\pm}$. The formula for $\lambda'_{\pm}$ are

$$\lambda'_{\pm} = \frac{\lambda_+ + H_{33} \tan^2 \omega_2 - 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}, \quad \lambda'_+ = \frac{\lambda_+ \tan^2 \omega_2 + H_{33} + 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}.$$

(3.12)

for the neutrino case, and

$$\lambda'_{\pm} = \frac{\lambda_- + H_{33} \tan^2 \omega_2 - 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}, \quad \lambda'_- = \frac{\lambda_- \tan^2 \omega_2 + H_{33} + 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}.$$

(3.13)

for the antineutrino case.

When the mixing between sterile neutrinos and active neutrinos is relatively small and neutrino beam energy is $E < 100$ GeV, the off-diagonal elements in the effective Hamiltonian will be very small compared with the diagonal ones after two of the above rotations are performed. Namely the effective Hamiltonian is approximately diagonalized. So far, all of the effective parameters (i.e. $\tilde{\theta}_{12}$, $\tilde{\delta}_{12}$, $\tilde{\theta}_{13}$ and $\tilde{\delta}_{13}$) in $\tilde{U}$ are presented. The diagonal terms in the effective Hamiltonian in the new representation can be treated as $\tilde{m}_i^2 (i = 1, 2, 3, 4)$. After subtracting the smallest neutrino (antineutrino) mass $\lambda_- (\lambda'_-)$, we can get the effective neutrino (antineutrino) mass-squared difference $\Delta \tilde{m}_{ij}^2$ as

$$\Delta \tilde{m}_{21}^2 \approx \lambda'_- - \lambda_- \quad \Delta \tilde{m}_{31}^2 \approx \lambda'_+ - \lambda_- \quad \Delta \tilde{m}_{41}^2 \approx H_{44} - \lambda_- \quad (3.14)$$

for the neutrino case, and

$$\Delta \tilde{m}_{21}^2 \approx \lambda_+ - \lambda'_- \quad \Delta \tilde{m}_{31}^2 \approx \lambda'_+ - \lambda'_- \quad \Delta \tilde{m}_{41}^2 \approx H_{44} - \lambda'_- \quad (3.15)$$

for the antineutrino case.
for the antineutrino case.

Up to now all the effective parameters in 4-flavor neutrino oscillation have been provided, and hence the neutrino oscillation probabilities can be easily calculated using eq. 2.11. Since both CC and NC potentials in matter are proportional to neutrino energy, the values of those effective parameters in $\tilde{U}$ and $\tilde{m}_i^2$ ($i = 1, 2, 3, 4$) are also energy dependent, as shown in figure 1, 2.

**Figure 1**: The values of $\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\delta}_{12}$ and $\tilde{\delta}_{13}$ with respect to neutrino energy. The solid and dashed lines are the effective angles and phases respectively. In this figure, we assume $\Delta m_{41}^2 = 0.1 \text{ eV}^2$, $\sin^2 \theta_{14} = 0.019$, $\sin^2 \theta_{24} = 0.015$, $\sin^2 \theta_{34} = 0$ [57], $\delta_{13} = 218^\circ$ [58] and $\delta_{24} = \delta_{34} = 0^\circ$.

**Figure 2**: The values of $\Delta \tilde{m}_i^2$ ($i = 2, 3, 4$) with respect to neutrino energy. The solid and dashed lines are for NH and IH. In this figure, we assume $\Delta m_{41}^2 = 0.1 \text{ eV}^2$, $\sin^2 \theta_{14} = 0.019$, $\sin^2 \theta_{24} = 0.015$, $\sin^2 \theta_{34} = 0$ [57], $\delta_{13} = 218^\circ$ [58] and $\delta_{24} = \delta_{34} = 0^\circ$. 
3.3 Discussion

The effective matrix $\tilde{U}$ in matter has introduced two effective mixing angles $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$, two effective CP phases $\tilde{\delta}_{12}$ and $\tilde{\delta}_{13}$, and effective mass-squared differences $\Delta \tilde{m}_{ij}^2$, in which $\tilde{\delta}_{12}$ is an additional parameter introduced from the Jacobi-like method. These effective parameters are clearly energy dependent, as shown in figure 1, 2.

In figure 1, when $E < 100$ MeV, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ are very close to $\theta_{12}$ and $\theta_{13}$ values in vacuum. The value of $\tilde{\theta}_{12}$ increases (decreases) rapidly up to the maximum $\frac{\pi}{2}$ (the minimum 0) in the neutrino (antineutrino) energy range from 100 MeV to 10 GeV, leading to $\sin \tilde{\theta}_{12} \rightarrow 1$ ($\sin \tilde{\theta}_{12} \rightarrow 0$). While $\tilde{\theta}_{13}$ starts to change after $E > 1$ GeV. It can go up to $\frac{\pi}{2}$ assuming NH for neutrinos and IH for antineutrinos when $E > 100$ GeV; While it will go down to 0 for the other two combinations. When $E < 1$ GeV, both effective CP phases are close to their corresponding vacuum oscillation values ($\tilde{\delta}_{12} \rightarrow 0$ and $\tilde{\delta}_{13} \rightarrow \delta_{13}$). When energy increases above 1 GeV, the influence of matter effects on $\tilde{\delta}_{12}$ and $\tilde{\delta}_{13}$ is not negligible.

In figure 2, the effect of matter also changes the values of effective neutrino mass-squared differences $\Delta \tilde{m}_{ij}^2$. When $E < 100$ MeV, $\Delta \tilde{m}_{21}^2$, $\Delta \tilde{m}_{31}^2$ and $\Delta \tilde{m}_{41}^2$ are close to their vacuum values. $\Delta \tilde{m}_{21}^2$ begins to vary when $E > 100$ MeV, while $\Delta \tilde{m}_{31}^2$ starts to change values after $E > 1$ GeV. In the case of $\Delta m_{41}^2 = 0.1$ eV$^2$ with current sterile neutrino limits, $\Delta \tilde{m}_{41}^2$ is insensitive to matter effects when $E < 100$ GeV. As neutrino energy increases, matter effects shift the values of effective $\Delta \tilde{m}_{21}^2$ more than $\Delta \tilde{m}_{31}^2$ and $\Delta \tilde{m}_{41}^2$ when $E < 100$ GeV. It should be noticed that $|\Delta \tilde{m}_{31}^2|$ has a dip structure around 10 GeV for the antineutrino IH case. This feature also shows up in the 3-flavor neutrino case.

In general, as shown in figure 1 and 2, matter effects are negligible on both $\Delta \tilde{m}_{31}^2$ and 1-2 neutrino mixing when $A_{CC}$ ($A_{NC}$) $\ll \Delta m_{21}^2 \ll \Delta m_{31}^2$ (or equivalently $E \ll 100$ MeV). When energy increases, it is clear that the 1-2 neutrino mixing submatrix is affected from matter much more than other submatrices, as well as $\Delta \tilde{m}_{21}^2$. However, $\Delta \tilde{m}_{31}^2$ and 1-3 mixing neutrino mixing still hold stable when $A_{CC}$ ($A_{NC}$) $\ll \Delta m_{31}^2$ (or equivalently $E \ll 1$ GeV). Furthermore, mixing between active and sterile neutrinos has little impact.

From a mathematical point of view, in the function of rotation angles yielding $\tan \theta = \frac{2A}{\gamma - \alpha + \sqrt{(\gamma - \alpha)^2 + 4A^2}}$, $A$ is proportional to the values of $\theta_{14}$, $\theta_{24}$ and $\theta_{34}$. Hence, the smallness of those mixing angles can effectively suppress the values of the corresponding rotation angles to a negligible level in the submatrices. Therefore, after the rotations on 1-2 and 2-3 (1-3) submatrices of the neutrino Hamiltonian, the effective Hamiltonian matrix is approximately diagonal.

What we have discussed is for the general feature of our derived oscillation formula. In some particular cases, the oscillation formula can be simplified:

- No CP violations ($\delta_{13} = \delta_{24} = \delta_{34} = 0/\pi$)

In this case, $\tilde{\theta}_{12} = \theta_{12} + \omega_1$, $\tilde{\theta}_{13} = \theta_{13} + \omega_2$, $\tilde{\delta}_{12} = 0$ and $\tilde{\delta}_{13} = \delta_{13}$. The neutrino oscillation form is the same as the vacuum case, which allows it to be greatly simplified due to $\tilde{\delta}_{12} = 0$ in eq. (3.6).

- No active-sterile neutrino mixing ($\theta_{14} = \theta_{24} = \theta_{34} = \delta_{24} = \delta_{34} = 0$)

The analytical approximations will reduce to 3-flavor neutrino oscillations.
4 The accuracy of the approximations

Section 3.2 details effective $\tilde{U}$ and $\Delta m^2_{ij}$. We can utilize eq. (2.11) to calculate all neutrino oscillation probabilities. In this section, we want to highlight the loss of accuracy due to these approximations. Firstly, the accuracy will be calculated across a range of energies and baselines. Subsequently, the accuracy of this work applied to the specific cases of T2HK and DUNE will be presented as two examples of medium and long baseline accelerator experiments respectively.

4.1 The general accuracy

![Figure 3](image_url)

Figure 3: The accuracies in different channels with $\Delta P_{\nu_\alpha \rightarrow \nu_\beta}$. All parameters except $\Delta m^2_{41}$ assume $\theta_{14} = \theta_{24} = \theta_{34} = 10^\circ$ and $\delta_{13} = \delta_{24} = \delta_{34} = 90^\circ$. $\Delta m^2_{41} = 0.1$ eV$^2$ is for the upper row and $\Delta m^2_{41} = 0.0001$ eV$^2$ is for the lower one.

Figure 3 concerns the case of neutrino oscillation with normal hierarchy and follows the results from section 3.2. In figure 3 we set all unknown angles to $10^\circ$ and unknown phases to $90^\circ$ for different oscillation probability channels (distinguished by neutrino energy and baseline). $10^\circ$ is large enough for sterile neutrino angles according to current sterile neutrino limits. Precision is quantified by defining $\Delta P_{\nu_\alpha \rightarrow \nu_\beta}$ as the numerical difference between the approximate and exact solutions

$$\Delta P_{\nu_\alpha \rightarrow \nu_\beta} = \left| P_{\nu_\alpha \rightarrow \nu_\beta}^{\text{Exact}} - P_{\nu_\alpha \rightarrow \nu_\beta}^{\text{Approximate}} \right|. \tag{4.1}$$

We find the accuracies of all $\Delta P_{\nu_\alpha \rightarrow \nu_\beta}$ are generally very good in current and near future experiments for the purpose of searching for sterile neutrinos. In the first column of figure 3, the accuracy of $\Delta P_{\nu_\alpha \rightarrow \nu_e}$ can reach $10^{-4}$ in the majority of energy and baseline ranges.
Accelerator experiments are constructed to detect $P_{\nu_{\mu} \rightarrow \nu_{e}}$ and $P_{\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}}$ with possible $P_{\nu_{\mu} \rightarrow \nu_{\tau}}$ in neutrino mode. For $P_{\nu_{\mu} \rightarrow \nu_{e}}$ in the second column, the accuracy can be at least $10^{-3}$ for long baseline accelerator experiments. For $P_{\nu_{\mu} \rightarrow \nu_{e}}$ as well as $P_{\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}}$, which can contribute to $\nu_{\mu}$ disappearance, the error is larger than $P_{\nu_{\mu} \rightarrow \nu_{e}}$ due to the large scale oscillations from 0 to 1. However the accuracy can also reach $10^{-3}$ in long baseline accelerator experiments. Obviously when $\Delta m_{41}^{2} = 0.1 \text{eV}^2$ the accuracies improve. Although, we only provide the accuracies within limited energy and baseline ranges, there are likely ranges of energy and baseline outside these limits where this approach can be applied.

4.2 T2HK and DUNE

As shown in eq. (2.7) and (2.8), matter effects are proportional to neutrino beam energy and its propagation length. Hence matter effects can significantly modify neutrino oscillation features for long-baseline neutrino oscillation experiments, such as T2HK and DUNE. Those experiments have relative high energy beams at $\sim \text{GeV}$ and baselines of hundreds and thousands of kilometers respectively. Here we would like to use them as examples to demonstrate 3+1-neutrino oscillation and check the accuracy of our approximations.

Deep Underground Neutrino Experiment (DUNE) is the next generation on-axis long-baseline accelerator neutrino experiment. It is proposed to use Liquid Argon (LAr) detectors located deep underground 1300 km away from the beam source. Its main physics goals are to solve three challenging issues in the neutrino sector, neutrino mass hierarchy, CP asymmetry and the octant of $\theta_{23}$. It can look for electron and tau neutrino (anti-neutrino) appearance and muon neutrino (antineutrino) disappearance channels from both neutrino and antineutrino beam modes.

Tokai-to-Hyper-Kamiokande (T2HK) is a proposed long-baseline experiment, which has a primary objective of measuring CP asymmetry. The far detector is 295 km away and $2.5^\circ$ off-axis from the J-PARC beam in Japan using water Cherenkov detector. The accuracies of T2HK are similar to DUNE.

Suppose the existence of the sterile neutrinos with relatively large mass-squared difference $\Delta m_{41}^{2} = 0.1 \text{eV}^2$, the high frequency oscillation feature is clearly shown in the muon neutrino disappearance and electron neutrino appearance modes in figure 4. Given matter effects and CP-violation phases, the electron neutrino and antineutrino appearance probabilities $P_{\nu_{\mu} \rightarrow \nu_{e}}$ and $P_{\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}}$ are very different. Compared with numerical calculation, the accuracy of the analytical approximations can reach $10^{-4}$ and $10^{-5}$ for neutrinos and antineutrinos respectively for appearance mode. For disappearance mode, the accuracies of $P_{\nu_{\mu} \rightarrow \nu_{\mu}}$ and $P_{\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{\mu}}$ are less than $10^{-3}$.

As shown using the above experiments, our approximations can extend to the other long baseline accelerator experiments with accuracy under current sterile neutrino limits. To be noted, our approximations are more general. It can not only be applied to long baseline experiments, but also short and medium baseline neutrino experiments, such as JUNO [62], although their matter effects are usually quite small and can be neglected.
Figure 4: The left plots are $P_{\nu_\mu \rightarrow \nu_\mu}$ and the right plots are $P_{\bar{\nu}_\mu \rightarrow \bar{\nu}_e}$, assuming $\Delta m^2_{41} = 0.1$ eV$^2$, $\sin^2 \theta_{14} = 0.019$, $\sin^2 \theta_{24} = 0.015$, $\sin^2 \theta_{34} = 0$, $\delta_{13} = 218^\circ$ and $\delta_{24} = \delta_{34} = 0^\circ$ in the case of NH. The appendant plots are accuracies of their corresponding channels.

5 Summary

The search for light sterile neutrinos is an area of interest in the neutrino field. Many long baseline neutrino experiments are also very actively searching for light sterile neutrinos in various mass regions. Both charged-current and neutral-current induced matter effects are quite important for those experiments. Although the numerical method can accurately calculate neutrino oscillation probability in matter, it is usually very computationally time consuming. Analytical approximations of neutrino oscillation are more commonly used in experimental neutrino research because it saves considerable time and is helpful for understanding neutrino oscillation features.
In this manuscript we introduced a Jacobi-like method to derive simplified analytical expressions with excellent accuracy for neutrino oscillation in matter. The compact expressions of the effective mixing matrix $\tilde{U}$ and the effective mass-squared differences $\Delta \tilde{m}^2_{ij}(i,j = 1,2,3,4)$ are presented. The accuracy of this work is sufficient for various long baseline neutrino experiments.

In addition, the Jacobi-like method is a general method to diagonalize complex hermitian matrices. It also can be extended into other physics topics, such as 3 (active) + N (sterile)-neutrino mixing, Neutrino Non-Standard Interactions, etc.

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A The parameterization of mixing matrix in vacuum

In the $3 + 1$ framework, neutrino mixing can be written as a $4 \times 4$ matrix \((2.5)\). 6 rotation angles with 3 Dirac phases\(^\dagger\) are found in this matrix. All the elements of the mixing matrix are listed in table 1. Indeed, if we set the angles and Dirac phases introduced by sterile neutrinos to 0, the $4 \times 4$ matrix will reduce to 3-flavor neutrino mixing.

B Jacobi-like method

Matter effects for the $3+1$ framework are more difficult to calculate than for the 3 neutrino framework because additional parameters are involved in neutrino Hamiltonian, which is

\(^\dagger\)In this table, we adopt $\delta_{24}$ and $\delta_{34}$ as the sterile phases. It is equivalent to using $\delta_{14}$ and $\delta_{24}$, or $\delta_{14}$ and $\delta_{34}$ for additional CP phases.
should diagonalize the 1-2 submatrix first. In the case of neutrinos, we find the absolute values of elements are the relatively largest off-diagonal ones because of the smallness of \( \Delta m^{2}_{21} \). Hence, we should diagonalize the 1-2 submatrix first. In this subsection, we show how to diagonalize the effective Hamiltonian with matter effects and simplify the expressions of the effective mixing and mass-squared differences for the neutrino case.

### B.1 Neutrino case

In this subsection, we show how to diagonalize the effective Hamiltonian with matter effects and simplify the expressions of the effective mixing and mass-squared differences for the neutrino case.

#### B.1.1 Diagonalization process

In the case of neutrinos, we find the absolute values of elements \( H_{12} \) and \( H_{21} \) in eq. (2.9) are the relatively largest off-diagonal ones because of the smallness of \( \Delta m^{2}_{21} \). Hence, we should diagonalize the 1-2 submatrix first.

| \( \alpha \) | \( U_{\alpha 1} \) |
|--------|----------------|
| \( e \) | \( e \) |
| \( U_{e 1} \) | \( C_{12}C_{13}C_{14} \) |
| \( U_{e 2} \) | \( C_{13}C_{14}s_{12} \) |
| \( U_{e 3} \) | \( C_{14}s_{13}e^{-i\theta_{13}} \) |
| \( U_{e 4} \) | \( s_{14} \) |
| \( \mu \) | \( \mu \) |
| \( U_{\mu 1} \) | \( -s_{12}c_{23}c_{24} - c_{12}(s_{13}c_{24}s_{23}e^{i\theta_{13}} + c_{13}s_{14}s_{24}e^{-i\theta_{24}}) \) |
| \( U_{\mu 2} \) | \( c_{12}c_{23}c_{24} - s_{12}(s_{13}c_{24}s_{23}e^{i\theta_{13}} + c_{13}s_{14}s_{24}e^{-i\theta_{24}}) \) |
| \( U_{\mu 3} \) | \( c_{13}c_{24}s_{23} - s_{13}s_{14}s_{24}e^{-i\theta_{13}}e^{-i\theta_{24}} \) |
| \( U_{\mu 4} \) | \( c_{14}s_{24}e^{-i\theta_{24}} \) |
| \( \tau \) | \( \tau \) |
| \( U_{\tau 1} \) | \( C_{12}[s_{13}(s_{23}s_{24}s_{34}e^{i\theta_{24}} - i\delta_{34} - c_{23}c_{34})e^{i\theta_{13}} - c_{13}c_{24}s_{14}s_{34}e^{-i\theta_{34}}] + s_{12}(c_{34}s_{23} + c_{23}s_{24}s_{34}e^{i\theta_{24}}e^{-i\delta_{34}}) \) |
| \( U_{\tau 2} \) | \( s_{12}[s_{13}(s_{23}s_{24}s_{34}e^{i\theta_{24}} - i\delta_{34} - c_{23}c_{34})e^{i\theta_{13}} - c_{13}c_{24}s_{14}s_{34}e^{-i\theta_{34}}] - c_{12}(c_{34}s_{23} + c_{23}s_{24}s_{34}e^{i\theta_{24}}e^{-i\delta_{34}}) \) |
| \( U_{\tau 3} \) | \( c_{13}(c_{23}s_{34} - s_{23}s_{24}s_{34}e^{i\theta_{24}}e^{-i\delta_{34}}) - s_{13}s_{24}s_{14}s_{34}e^{-i\theta_{13}}e^{-i\delta_{34}} \) |
| \( U_{\tau 4} \) | \( c_{14}c_{24}s_{34}e^{-i\delta_{34}} \) |
| \( s \) | \( s \) |
| \( U_{s 1} \) | \( C_{12}[s_{13}(c_{34}s_{23}s_{24}e^{i\theta_{24}} + c_{23}s_{34}e^{i\theta_{34}})e^{i\theta_{13}} - c_{13}c_{24}s_{14}s_{34}] + s_{12}(c_{23}c_{34}s_{24}e^{i\theta_{24}} - s_{23}s_{34}e^{i\delta_{34}}) \) |
| \( U_{s 2} \) | \( s_{12}[s_{13}(c_{34}s_{23}s_{24}e^{i\theta_{24}} + c_{23}s_{34}e^{i\theta_{34}})e^{i\theta_{13}} - c_{13}c_{24}s_{14}s_{34}] + c_{12}(s_{23}s_{34}e^{i\theta_{34}} - c_{23}c_{34}s_{24}e^{i\delta_{24}}) \) |
| \( U_{s 3} \) | \( -c_{13}(c_{34}s_{23}s_{24}e^{i\theta_{24}} + c_{23}s_{34}e^{i\theta_{34}}) - c_{24}s_{14}s_{14}e^{-i\theta_{13}} \) |
| \( U_{s 4} \) | \( c_{14}c_{24}s_{34} \) |

**Table 1:** The elements of the 4-flavor mixing matrix.
**First rotation:** The rotation matrix can be written as

$$ R^1 = R_{12}(\omega_1, \phi_1) \equiv \begin{bmatrix} c_{\omega_1} & s_{\omega_1} e^{-i\phi_1} & 0 & 0 \\ -s_{\omega_1} e^{i\phi_1} & c_{\omega_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (c_{\omega_1} = \cos \omega_1, s_{\omega_1} = \sin \omega_1) \quad (B.1) $$

We utilize the Jacobi-like method to derive $\omega_1$ yielding

$$ \tan \omega_1 = \frac{2A_{\omega_1}}{(H_{22} - H_{11}) + \sqrt{(H_{22} - H_{11})^2 + 4A_{\omega_1}^2}}, \quad (0 < \omega_1 < \frac{\pi}{2} - \theta_{12}) \quad (B.2) $$

with $A_{\omega_1} = |H_{12}|$ and $e^{i\phi_1} = \frac{A_{\omega_1}}{H_{12}}$. Here $A_{\omega_1}$ is the amplitude of $H_{12}$. After the rotation by $R_{12}(\omega_1, \phi_1)$, we rewrite the Hamiltonian in the new representation as

$$ H' = R_{12}^T(\omega_1, \phi_1) H R_{12}(\omega_1, \phi_1) = \begin{bmatrix} \lambda_- & H_{13}' & H_{14}' \\ 0 & \lambda_+ & H_{23}' \\ H_{31}' & H_{32}' & H_{33}' \end{bmatrix}, \quad (B.3) $$

with the eigenvalues of the 1-2 submatrix

$$ \lambda_- = \frac{H_{11} + H_{22} \tan^2 \omega_1 - 2A_{\omega_1} \tan \omega_1}{1 + \tan^2 \omega_1}, \lambda_+ = \frac{H_{11} \tan^2 \omega_1 + H_{22} + 2A_{\omega_1} \tan \omega_1}{1 + \tan^2 \omega_1}. \quad (B.4) $$

$H_{23}' = \frac{H_{13} \tan \omega_1 e^{i\phi_1} + H_{32}}{\sqrt{1 + \tan^2 \omega_1}}$ is useful for the further calculations. We diagonalize the 2-3 submatrix second because it has a relatively big off-diagonal element, and also it is useful to simplify the effective matrix $U$ through eq. (B.15).

**Second rotation:** The second rotation matrix yields

$$ R^2 = R_{23}(\omega_2, \phi_2) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\omega_2} & s_{\omega_2} e^{-i\phi_2} & 0 \\ 0 & -s_{\omega_2} e^{i\phi_2} & c_{\omega_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (c_{\omega_2} = \cos \omega_2, s_{\omega_2} = \sin \omega_2) \quad (B.5) $$

The diagonal angle $\omega_2$ is compatible with

$$ \tan \omega_2 = \frac{2A_{\omega_2}}{(H_{33} - \lambda_+) \pm \sqrt{(H_{33} - \lambda_+)^2 + 4A_{\omega_2}^2}}, \quad (B.6) $$

where the upper sign is for NH ($0 < \omega_2 < \frac{\pi}{2} - \theta_{13}$), and the lower sign is for IH ($0 > \omega_2 > -\theta_{13}$). $A_{\omega_2}$ is the amplitude of $H_{23}$ in eq. (B.3) yielding $A_{\omega_2} = |H_{23}|$. The additional complex factor is given by $e^{i\phi_2} = \frac{A_{\omega_2}}{H_{23}}$. After two rotations, we obtain

$$ H'' = R_{23}^T(\omega_2, \phi_2) H' R_{23}(\omega_2, \phi_2) = \begin{bmatrix} \lambda_- & H_{12}' & H_{13}' & H_{14}' \\ H_{21}' & \lambda_- & 0 & H_{24}' \\ H_{31}' & 0 & \lambda_+ & H_{34}' \\ H_{41}' & H_{42}' & H_{43}' & H_{44}' \end{bmatrix}. \quad (B.7) $$
where the diagonal terms $\lambda_-$ and $\lambda_+$ obey

$$
\lambda_- = \frac{\lambda_+ + H_{33} \tan^2 \omega_2 - 2A_2 \tan \omega_2}{1 + \tan^2 \omega_2}, \quad \lambda_+ = \frac{\lambda_+ \tan^2 \omega_2 + H_{33} + 2A_2 \tan \omega_2}{1 + \tan^2 \omega_2}.
$$

(B.8)

For now, the off-diagonal terms of $H''$ are negligible as a good approximation in current sterile neutrino bounds. The effective neutrino mixing matrix and mass-squared differences are given by

$$
\tilde{U} \approx U R_{12}(\omega_1, \phi_1)R_{23}(\omega_2, \phi_2) = \underbrace{R_{34}R_{24}R_{14}R_{23}R_{13}}_{\tilde{U}} R_{12}(\omega_1, \phi_1)R_{23}(\omega_2, \phi_2),
$$

(B.9a)

$$
\Delta \tilde{m}_{21}^2 \approx \lambda_-' - \lambda_-, \quad \Delta \tilde{m}_{31}^2 \approx \lambda_+' - \lambda_-, \quad \Delta \tilde{m}_{41}^2 \approx H_{44} - \lambda_-. \quad \text{ (B.9b)}
$$

If $\delta_{13} = \delta_{24} = \delta_{34} = 0$, $\phi_1$ and $\phi_2$ will be 0. For the sake of beauty in mathematical form (It is convenient for $\tilde{U}$ to have the same form with $U$.) as well as a good understanding of matter effects on oscillation parameters, we continue to simplify the effective mixing matrix $\tilde{U}$ below.

**B.1.2 Simplification**

Firstly, we find there are two 1-2 submatrix rotations $R_{12}(\theta_{12}, 0)$ and $R_{12}(\omega_1, \phi_1)$ next to each other in eq. (B.9b). Therefore we combine them into one submatrix. The processes are given by

$$
R_{12}(\theta_{12}, 0)R_{12}(\omega_1, \phi_1) = \begin{bmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\omega_1} & s_{\omega_1}e^{-i\phi_1} & 0 & 0 \\ -s_{\omega_1}e^{i\phi_1} & c_{\omega_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R_{12}(\bar{\theta}_{12}, \bar{\delta}_{12})D_{12}(e^{i\Theta_{12}}, e^{-i\Theta_{12}}, 1, 1)
$$

(B.10)

Then we obtain

$$
\tilde{U} \approx R_{34}R_{24}R_{14}R_{23}R_{13} \tilde{R}_{12} R_{23}(\omega_2, \phi_2).
$$

(B.11)

We find there is a new $\tilde{R}_{12}$ consisting of a rotation matrix with $\bar{\theta}_{12}$ and $\bar{\delta}_{12}$ and one diagonal unitary matrix with a phase $\Theta_{12}$. Those new items yield

$$
\bar{s}_{12} = \sin \tilde{\theta}_{12} = \frac{|c_{12} \tan \omega_1 e^{i\phi_1} + s_{12}|}{\sqrt{1 + \tan^2 \omega_1}}, \quad \bar{c}_{12} = \cos \tilde{\theta}_{12} = \frac{|c_{12} - s_{12} \tan \omega_1 e^{i\phi_1}|}{\sqrt{1 + \tan^2 \omega_1}},
$$

(B.12a)

$$
e^{i\bar{\delta}_{12}} = \frac{(c_{12} \tan \omega_1 e^{i\phi_1} + s_{12})(c_{12} - s_{12} \tan \omega_1 e^{-i\phi_1})}{\cos \tilde{\theta}_{12} \sin \tilde{\theta}_{12}(1 + \tan^2 \omega_1)},
$$

(B.12b)

$$
e^{i\Theta_{12}} = \frac{c_{12} - s_{12} \tan \omega_1 e^{i\phi_1}}{\cos \tilde{\theta}_{12} \sqrt{1 + \tan^2 \omega_1}}.
$$

(B.12c)
Here, we set the same limit on $\tilde{\theta}_{12}$ in $[0, \frac{\pi}{2}]$ with $\theta_{12}$. The energy dependent $\tilde{\theta}_{12}$ is shown in figure 1. When neutrino energy is small, matter effects are slight, leading to $\omega_2 \to 0$. Hence, we get $\tilde{R}_{12} R_{23}(\omega_2, \phi_2) \approx R_{13}(\omega_2, \phi_2) \tilde{R}_{12}$. When $E$ goes up, we find $\sin \tilde{\theta}_{12} \to 1$, leading to

$$\tilde{R}_{12} \approx \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$  \hspace{1cm} (B.13)

which can be used in the following operation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ c_{\omega_2} & s_{\omega_2} e^{-i\phi_2} & 0 \\ 0 & -s_{\omega_2} e^{i\phi_2} & c_{\omega_2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} c_{\omega_2} & 0 & s_{\omega_2} e^{-i\phi_2} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{\omega_2} e^{i\phi_2} & c_{\omega_2} & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (B.14)

This is to say $\tilde{R}_{12} R_{23}(\omega_2, \phi_2) \approx R_{13}(\omega_2, \phi_2) \tilde{R}_{12}$. Therefore, we can use such an approximation within a different $E$ range:

$$\tilde{R}_{12} R_{23}(\omega_2, \phi_2) \approx R_{13}(\omega_2, \phi_2) \tilde{R}_{12}.$$  \hspace{1cm} (B.15)

Subsequently, we obtain

$$\tilde{U} \approx R_{34} R_{24} R_{14} R_{23} \tilde{R}_{12} R_{23}(\omega_2, \phi_2) = R_{34} R_{24} R_{14} R_{13} R_{13}(\omega_2, \phi_2) \tilde{R}_{12},$$  \hspace{1cm} (B.16)

where

$$R_{13}(\theta_{13}, 0) R_{13}(\omega_2, \phi_2)$$

$$= \begin{bmatrix} c_{\tilde{\theta}_{13}} & 0 & s_{\tilde{\theta}_{13}} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{\tilde{\theta}_{13}} e^{i\delta_{13}} & c_{\tilde{\theta}_{13}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\omega_2} & 0 & s_{\omega_2} e^{-i\phi_2} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{\omega_2} e^{i\phi_2} & c_{\omega_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \tilde{c}_{\tilde{\theta}_{13}} & 0 & s_{\tilde{\theta}_{13}} e^{-i\delta_{13}} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{\tilde{\theta}_{13}} e^{i\delta_{13}} & \tilde{c}_{\tilde{\theta}_{13}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\Theta_{13}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-i\Theta_{13}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$= R_{13}(\tilde{\theta}_{13}, \tilde{\delta}_{13}) D_{13}(e^{i\Theta_{13}}, 1, e^{-i\Theta_{13}}, 1)$$

The items above can be written as

$$\tilde{s}_{13} = \sin \tilde{\theta}_{13} = \frac{|c_{13} \tan \omega_2 e^{i\phi_2} + s_{13} e^{i\delta_{13}}|}{\sqrt{1 + \tan^2 \omega_2}}, \hspace{1cm} \tilde{c}_{13} = \cos \tilde{\theta}_{13} = \frac{|c_{13} - s_{13} \tan \omega_2 e^{i(\delta_{13} - \phi_2)}|}{\sqrt{1 + \tan^2 \omega_2}},$$  \hspace{1cm} (B.18a)

$$e^{i\tilde{\theta}_{13}} = \frac{(c_{13} \tan \omega_2 e^{i\phi_2} + s_{13} e^{i\delta_{13}})(c_{13} - s_{13} \tan \omega_2 e^{i(\delta_{13} - \phi_2)})}{\cos \tilde{\theta}_{13} \sin \tilde{\theta}_{13}(1 + \tan^2 \omega_2)},$$  \hspace{1cm} (B.18b)

$$e^{i\Theta_{13}} = \frac{c_{13} - s_{13} \tan \omega_2 e^{-i(\delta_{13} - \phi_2)}}{\cos \tilde{\theta}_{13} \sqrt{1 + \tan^2 \omega_2}}.$$  \hspace{1cm} (B.18c)
Equally, we set limit on $\tilde{\theta}_{13}$ with $[0, \frac{\pi}{2}]$. We find
\[
\tilde{U} \approx R_{34} R_{24} R_{14} R_{23} R_{13} (\tilde{\theta}_{13}, \tilde{\delta}_{13}) D_{13} \tilde{R}_{12}.
\] (B.19)

Using the inverse of the method shown in eq. (B.15), we obtain
\[
D_{13}(e^{i\Theta_{13}}, 1, e^{-i\Theta_{13}}, 1) \tilde{R}_{12} \approx \tilde{R}_{12} D_{23}(1, e^{i\Theta_{13}}, e^{-i\Theta_{13}}, 1),
\] (B.20)
which leads to
\[
\tilde{U} \approx R_{34} R_{24} R_{14} R_{23} R_{13} (\tilde{\theta}_{13}, \tilde{\delta}_{13}) R_{12}(\tilde{\theta}_{12}, \tilde{\delta}_{12}) D_{123},
\] (B.21)
with
\[
D_{123} = D_{123}(e^{i\Theta_{12}}, e^{i(\Theta_{13}-\Theta_{12})}, e^{-i\Theta_{13}}, 1).
\] (B.22)

Here $D_{123}(e^{i\Theta_{12}}, e^{i(\Theta_{13}-\Theta_{12})}, e^{-i\Theta_{13}}, 1)$ can be cancelled in the neutrino oscillation paradigm like Majorana phases. Finally we obtain the last expression for the neutrino mixing matrix:
\[
\tilde{U} \approx R_{34} R_{24} R_{14} R_{23} R_{13} (\tilde{\theta}_{13}, \tilde{\delta}_{13}) R_{12}(\tilde{\theta}_{12}, \tilde{\delta}_{12}),
\] (B.23)
which has almost the same form with the standard mixing matrix $U$ except for an additional $\tilde{\delta}_{12}$. We conclude that if $\tilde{\delta}_{13} = \delta_{24} = \delta_{34} = 0$, we can get $\tilde{\theta}_{12} = \theta_{12} + \omega_1$, $\tilde{\theta}_{13} = \theta_{13} + \omega_2$, $\tilde{\delta}_{13} = \delta_{13}$ and $\tilde{\delta}_{12} = \Theta_{12} = \Theta_{13} = 0$. The effective Hamiltonian, meanwhile, becomes a real Hermitian matrix. In that case, our method reduces to the Jacobi method, which is a way to deal with the diagonalization of real Hermitian matrices.

### B.2 Antineutrino case

In this subsection, we diagonalize the effective Hamiltonian in matter and simplify the expressions of the effective mixing and mass-squared differences for the antineutrino case.

#### B.2.1 Diagonalization process

In the case of antineutrinos, we rotate the 1-2 submatrix first, similar to the neutrino case.

**First rotation:** The rotation is
\[
R^1 = R_{12}(\theta, \phi_1) \equiv \begin{bmatrix}
c_{\omega_1} & s_{\omega_1} e^{-i\phi_1} & 0 & 0 \\
-s_{\omega_1} e^{i\phi_1} & c_{\omega_1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (c_{\omega_1} = \cos \omega_1, s_{\omega_1} = \sin \omega_1)
\] (B.24)

where $\omega_1$ is compatible with
\[
\tan \omega_1 = \frac{2 A_{\omega_1}}{(H_{22} - H_{11}) + \sqrt{(H_{22} - H_{11})^2 + 4 A^2_{\omega_1}}}, \quad (0 > \omega_1 > -\theta_{12})
\] (B.25)

with $A_{\omega_1} = -|H_{12}|$ and the additional complex factor $e^{i\phi_1} = \frac{A_{\omega_1}}{H_{12}}$. After the first rotation we get the new effective Hamiltonian
\[
H' = R^1_{12}(\omega_1, \phi_1) H R_{12}(\omega_1, \phi_1) = \begin{bmatrix}
\lambda_- & 0 & H'_{13} & H'_{14} \\
0 & \lambda_+ & H'_{23} & H'_{24} \\
H'_{31} & H'_{32} & H_{33} & H_{34} \\
H'_{41} & H'_{42} & H_{43} & H_{44}
\end{bmatrix}.
\] (B.26)
The eigenvalues of $H'$ in 1-2 submatrix are

$$
\lambda_- = \frac{H_{11} + H_{22} \tan^2 \omega_1 - 2 A_\omega \tan \omega_1}{1 + \tan^2 \omega_1}, \quad \lambda_+ = \frac{H_{11} \tan^2 \omega_1 + H_{22} + 2 A_\omega \tan \omega_1}{1 + \tan^2 \omega_1}.
$$

$H'_{13}$ is useful for the further calculation, yielding

$$
H'_{13} = \frac{H_{13} - H_{33} \tan \omega_1 e^{-i \phi_1}}{\sqrt{1 + \tan^2 \omega_1}}.
$$

Then we diagonalize the 1-3 submatrix, which not only has relatively big off-diagonal terms, but is helpful for the further transformation in eq. (B.37) as well.

**Second rotation:** The rotation matrix is

$$
R^2 = R_{13}(\omega_2, \phi_2) \equiv \begin{bmatrix}
    c_{\omega_2} & 0 & s_{\omega_2} e^{-i \phi_2} & 0 \\
    0 & 1 & 0 & 0 \\
    -s_{\omega_2} e^{i \phi_2} & 0 & c_{\omega_2} & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}, \quad (c_{\omega_2} = \cos \omega_2, s_{\omega_2} = \sin \omega_2)
$$

(B.28)

The rotation angle $\omega_2$ yields

$$
\tan \omega_2 = \frac{2 A_{\omega_2}}{(H_{33} - \lambda_-) \pm \sqrt{(H_{33} - \lambda_-)^2 + 4 A_{\omega_2}^2}}.
$$

(B.29)

where the upper sign is for NH ($0 > \omega_2 > -\theta_{13}$), and the lower sign is for IH ($0 < \omega_2 < \frac{\pi}{2} - \theta_{13}$) with $A_{\omega_2} = -|H'_{13}|$ and a corresponding complex factor $e^{i \phi_2} = \frac{A_{\omega_2}}{H'_{13}}$. After two operations, we get the new effective Hamiltonian in the new representation:

$$
H'' = R^1_{13}(\omega_2, \phi_2) H' R_{13}(\omega_2, \phi_2) = \begin{bmatrix}
    \lambda_- & H''_{12} & 0 & H''_{14} \\
    H''_{12} & \lambda_+ & H''_{23} & H''_{24} \\
    0 & H''_{32} & \lambda_+ & H''_{34} \\
    H''_{41} & H''_{42} & H''_{43} & H_{44}
\end{bmatrix},
$$

(B.30)

with

$$
\lambda_- = \frac{\lambda_- + H_{33} \tan^2 \omega_2 - 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}, \quad \lambda_+ = \frac{\lambda_- \tan^2 \omega_2 + H_{33} + 2 A_{\omega_2} \tan \omega_2}{1 + \tan^2 \omega_2}.
$$

(B.31)

So far we get the effective mixing matrix $\tilde{U}$ and effective mass-squared $\Delta \tilde{m}^2_{ii}$ ($i = 2, 3, 4$) with some negligible off-diagonal terms in $H''$ (B.30). Using the above rotations, we obtain

$$
\tilde{U} \approx U R_{12}(\omega_1, \phi_1) R_{23}(\omega_2, \phi_2) = \frac{R_{34} R_{24} R_{14} R_{23} R_{13} R_{12} (\omega_1, \phi_1) R_{13}(\omega_2, \phi_2)}{\tilde{U}},
$$

(B.32a)

$$
\Delta \tilde{m}^2_{21} \approx \lambda_+ - \lambda_-', \quad \Delta \tilde{m}^2_{31} \approx \lambda_+ - \lambda_-, \quad \Delta \tilde{m}^2_{41} \approx H_4 - \lambda_{-}'.
$$

(B.32b)
B.2.2 Simplification

Similar to the neutrino case we combine $R_{12}(\theta_{12}, 0)$ and $R_{12}(\omega_1, \phi_1)$ into one submatrix below:

$$
R_{12}(\theta_{12}, 0)R_{12}(\omega_1, \phi_1)
= \begin{bmatrix}
c_{12} & s_{12} & 0 & 0 \\
-s_{12} & c_{12} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_{\omega_1} & s_{\omega_1} e^{-i\phi_1} & 0 & 0 \\
-s_{\omega_1} e^{i\phi_1} & c_{\omega_1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
c_1 & \tilde{s}_{12} e^{-i \tilde{\theta}_{12}} & 0 & 0 \\
- \tilde{s}_{12} e^{i \tilde{\theta}_{12}} & \tilde{c}_{12} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
e^{i\Theta_{12}} & 0 & 0 & 0 \\
0 & e^{-i\Theta_{12}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \hat{R}_{12}
$$

with

$$
\tilde{s}_{12} = \sin \tilde{\theta}_{12} = \frac{|c_{12} \tan \omega_1 e^{i\phi_1} + s_{12}|}{\sqrt{1 + \tan^2 \omega_1}},
\tilde{c}_{12} = \cos \tilde{\theta}_{12} = \frac{|c_{12} - s_{12} \tan \omega_1 e^{i\phi_1}|}{\sqrt{1 + \tan^2 \omega_1}},
$$

$$
e^{i \tilde{\theta}_{12}} = \frac{(c_{12} \tan \omega_1 e^{i\phi_1} + s_{12})(c_{12} - s_{12} \tan \omega_1 e^{-i\phi_1})}{\cos \tilde{\theta}_{12} \sin \tilde{\theta}_{12}(1 + \tan^2 \omega_1)},
$$

$$
e^{i \Theta_{12}} = \frac{c_{12} - s_{12} \tan \omega_1 e^{i\phi_1}}{\cos \Theta_{12} \sqrt{1 + \tan^2 \omega_1}}.
$$

Therefore, we get

$$
\tilde{U} \approx R_{34} R_{24} R_{14} R_{23} R_{13} \hat{R}_{12} R_{13}(\omega_2, \phi_2).
$$

Here, we set a constraint on $\tilde{\theta}_{12}$ within $[0, \pi]$.

After the absorption of two rotations in the first step, we get an effective $\tilde{R}_{12}$. The energy dependent $\tilde{\theta}_{12}$ is shown in figure 1. When $E$ is small, matter effects are slight, leading to $\omega_2 \to 0$. Therefore, we get $\tilde{R}_{12} R_{13}(\omega_2, \phi_2) \approx R_{13}(\omega_2, \phi_2) \tilde{R}_{12}$. When $E$ goes up, $\cos \tilde{\theta}_{12} \to 1$, leading to a unitary matrix $\tilde{R}_{12}$

$$
\tilde{R}_{12} \approx \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

Then we get $\tilde{R}_{12} R_{13}(\omega_2, \phi_2) \approx R_{13}(\omega_2, \phi_2) \tilde{R}_{12}$ as well. Subsequently, we obtain the following approximate simplification over different $E$ ranges:

$$
\tilde{U} \approx R_{34} R_{24} R_{14} R_{13} \tilde{R}_{12} R_{13}(\omega_2, \phi_2) = R_{34} R_{24} R_{14} R_{13} R_{13}(\omega_2, \phi_2) \tilde{R}_{12}.
$$
We notice that $R_{13}(\theta_{13}, 0)R_{13}(\omega_2, \phi_2)$ can be simplified as

$$R_{13}(\theta_{13}, 0)R_{13}(\omega_2, \phi_2) = \begin{bmatrix} c_{\theta_{13}} & s_{\theta_{13}}e^{-i\delta_{13}} & 0 \\ 0 & 1 & 0 \\ -s_{\theta_{13}}e^{i\delta_{13}} & 0 & c_{\theta_{13}} \end{bmatrix} \begin{bmatrix} c_{\omega_2} & 0 & s_{\omega_2}e^{-i\phi_2} \\ 0 & 1 & 0 \\ -s_{\omega_2}e^{i\phi_2} & 0 & c_{\omega_2} \end{bmatrix},$$  \quad \text{(B.38)}

with

$$\tilde{s}_{13} = \sin \tilde{\theta}_{13} = \frac{c_{\theta_{13}} \tan \omega_2 e^{i\phi_2} + s_{\theta_{13}} e^{i\delta_{13}}}{\sqrt{1 + \tan^2 \omega_2}}, \quad \tilde{c}_{13} = \cos \tilde{\theta}_{13} = \frac{|c_{\theta_{13}} - s_{\theta_{13}} \tan \omega_2 e^{i(\delta_{13} - \phi_2)}|}{\sqrt{1 + \tan^2 \omega_2}},$$  \quad \text{(B.39a)}

$$e^{i\delta_{13}} = \frac{(c_{\theta_{13}} \tan \omega_2 e^{i\phi_2} + s_{\theta_{13}} e^{i\delta_{13}})(c_{\theta_{13}} - s_{\theta_{13}} \tan \omega_2 e^{i(\delta_{13} - \phi_2)})}{\cos \tilde{\theta}_{13} \sin \tilde{\theta}_{13}(1 + \tan^2 \omega_2)},$$  \quad \text{(B.39b)}

$$e^{i\Theta_{13}} = \frac{c_{\theta_{13}} - s_{\theta_{13}} \tan \omega_2 e^{-i(\delta_{13} - \phi_2)}}{\cos \tilde{\theta}_{13} \sqrt{1 + \tan^2 \omega_2}}.$$  \quad \text{(B.39c)}

We also set a bound on $\tilde{\theta}_{13}$ within $[0, \frac{\pi}{2}]$. Subsequently, we obtain

$$\tilde{U} \approx R_{34}R_{24}R_{14}R_{13}(\tilde{\theta}_{13}, \tilde{\delta}_{13})D_{13}(e^{i\Theta_{13}}, 1, e^{-i\Theta_{13}}, 1).$$  \quad \text{(B.40)}

Using the approach from eq. (B.37), we get

$$D_{13}(e^{i\Theta_{13}}, 1, e^{-i\Theta_{13}}, 1) \tilde{R}_{12} \approx \tilde{R}_{12}D_{13}(e^{i\Theta_{13}}, 1, e^{-i\Theta_{13}}, 1).$$  \quad \text{(B.41)}

Consequently, the effective mixing matrix can be written as

$$\tilde{U} \approx R_{34}R_{24}R_{14}R_{13}(\tilde{\theta}_{13}, \tilde{\delta}_{13})R_{12}(\tilde{\theta}_{12}, \tilde{\delta}_{12})D_{123}$$  \quad \text{(B.42)}

with

$$D_{123} = D_{123}(e^{i(\Theta_{12} + \Theta_{13})}, e^{-i\Theta_{12}}, e^{-i\Theta_{13}}, 1).$$  \quad \text{(B.43)}

Similarly, here $D_{123}(e^{i(\Theta_{12} + \Theta_{13})}, e^{-i\Theta_{12}}, e^{-i\Theta_{13}}, 1)$ can be cancelled like Majorana phases when neutrinos oscillate. Eventually, the effective mixing matrix $\tilde{U}$ is identical to the vacuum case $U$ except for an additional $\tilde{\delta}_{12}$. $\tilde{U}$ can be written as

$$\tilde{U} \approx R_{34}R_{24}R_{14}R_{13}(\tilde{\theta}_{13}, \tilde{\delta}_{13})R_{12}(\tilde{\theta}_{12}, \tilde{\delta}_{12}).$$  \quad \text{(B.44)}

Similarly, We conclude that $\tilde{\theta}_{12} = \theta_{12} + \omega_1$, $\tilde{\theta}_{13} = \theta_{13} + \omega_2$, $\tilde{\delta}_{13} = \delta_{13}$ and $\tilde{\delta}_{12} = \Theta_{12} = \Theta_{13} = 0$ when $\delta_{13} = \delta_{24} = \delta_{34} = 0$. It is much easier to diagonalize the effective Hamiltonian than before because it is a real Hermitian matrix. For real matrices the Jacobi-like method reduces to the Jacobi method.
B.3 Effective mixing matrix for 3+1 flavor neutrino in matter

In this subsection, we summarize the effective mixing matrix $\tilde{U}$ for both the neutrino and antineutrino cases from the results of B.1 and B.2 in table 2. Using the neutrino oscillation probability functions in eq. (2.11) and the elements in table 2, every neutrino oscillation probability is available.

| $\alpha$ | $U_{\alpha\beta}$ |
| --- | --- |
| $e$ | $\tilde{c}_{12}\tilde{c}_{13}\tilde{c}_{14}$ |
| $U_e$ | $\tilde{c}_{13}\tilde{c}_{14}s_{12}e^{-i\delta_{12}}$ |
| $U_\mu$ | $c_{14}s_{13}e^{-i\delta_{13}}$ |
| $U_\tau$ | $s_{14}$ |
| $\mu$ | $-\tilde{s}_{12}c_{23}s_{24}e^{i\delta_{12}} - \tilde{c}_{12}(\tilde{s}_{13}c_{24}s_{23}e^{i\delta_{13}} + \tilde{c}_{13}s_{14}s_{24}e^{-i\delta_{24}})$ |
| $U_{\mu 1}$ | $\tilde{c}_{12}c_{23}s_{24} - \tilde{s}_{12}(\tilde{s}_{13}c_{24}s_{23}e^{i\delta_{13}} + \tilde{c}_{13}s_{14}s_{24}e^{-i\delta_{24}})e^{-i\delta_{12}}$ |
| $U_{\mu 2}$ | $\tilde{c}_{13}c_{24}s_{23} - \tilde{s}_{13}s_{14}s_{24}e^{-i\delta_{13}}e^{-i\delta_{24}}$ |
| $U_{\mu 3}$ | $c_{14}s_{24}e^{-i\delta_{24}}$ |
| $\tau$ | $\tilde{c}_{12}\left[s_{13}(s_{23}s_{24}s_{34}e^{i\delta_{24}}e^{-i\delta_{34}} - c_{23}s_{34})e^{i\delta_{13}} - \tilde{c}_{13}c_{24}s_{14}s_{34}e^{-i\delta_{34}}\right] + \tilde{s}_{12}(c_{34}s_{23} + c_{23}s_{24}s_{34}e^{i\delta_{24}}e^{-i\delta_{34}})e^{i\delta_{12}}$ |
| $U_{\tau 1}$ | $\tilde{s}_{12}\left[s_{13}(s_{23}s_{24}s_{34}e^{i\delta_{24}}e^{-i\delta_{34}} - c_{23}s_{34})e^{i\delta_{13}} - \tilde{c}_{13}c_{24}s_{14}s_{34}e^{-i\delta_{34}}\right] - \tilde{c}_{12}(c_{34}s_{23} + c_{23}s_{24}s_{34}e^{i\delta_{24}}e^{-i\delta_{34}})$ |
| $U_{\tau 2}$ | $\tilde{c}_{13}(c_{23}s_{34} - s_{23}s_{24}s_{34}e^{i\delta_{24}}e^{-i\delta_{34}}) - \tilde{s}_{13}c_{24}s_{14}s_{34}e^{-i\delta_{13}}e^{-i\delta_{34}}$ |
| $U_{\tau 3}$ | $c_{14}c_{24}s_{34}e^{-i\delta_{34}}$ |
| $s$ | $\tilde{c}_{12}\left[s_{13}(c_{34}s_{23}s_{24}e^{i\delta_{24}} + c_{23}s_{34}e^{i\delta_{34}})e^{i\delta_{13}} - \tilde{c}_{13}c_{24}s_{14}s_{34}\right] + \tilde{s}_{12}(c_{34}s_{23}s_{24}e^{i\delta_{24}} - s_{23}s_{34}e^{i\delta_{34}})e^{i\delta_{12}}$ |
| $\tilde{U}_{s 1}$ | $\tilde{s}_{12}\left[s_{13}(c_{34}s_{23}s_{24}e^{i\delta_{24}} + c_{23}s_{34}e^{i\delta_{34}})e^{i\delta_{13}} - \tilde{c}_{13}c_{24}s_{14}s_{34}\right] - \tilde{c}_{12}(s_{23}s_{34}e^{i\delta_{34}} - c_{23}s_{34}s_{24}e^{i\delta_{24}})$ |
| $\tilde{U}_{s 2}$ | $\tilde{c}_{13}(c_{34}s_{23}s_{24}e^{i\delta_{24}} + c_{23}s_{34}e^{i\delta_{34}}) - c_{24}c_{14}s_{14}e^{-i\delta_{13}}$ |
| $U_{s 1}$ | $c_{14}c_{24}s_{34}$ |
| $U_{s 2}$ | $c_{14}c_{24}s_{34}$ |
| $U_{s 3}$ | $c_{14}c_{24}s_{34}$ |

Table 2: The elements of the effective mixing matrix for the 3+1-neutrino case in matter based on the results from section B.1 and B.2. If we set the sterile parameters to zero, this mixing will reduce to the effective mixing matrix for 3-flavor neutrinos with matter effects.