Correlation analysis among vorticity, \( Q \) method and Liutex *

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Abstract: Influenced by the fact that vorticity represents rotation for rigid body, people believe this idea also works for fluid flow. However, the vortex predictions by vorticity do not match experimental results, which drove scientists to look for more appropriate methods to identify vortex. All vortex identification methods can be categorized into three generations. The vorticity-based method is classified as the first generation. Methods relying on eigenvalues of velocity gradient tensor are considered as the second generation. People still believe vorticity is vortex since vorticity theory looks correct in mathematics, but all other methods are only scalars and unable to indicate the swirl direction. Recently, a new vortex identification method called Liutex is innovated. It is regarded as the third-generation method, not only overcoming all previous methods’ drawbacks but also having a clear physical meaning. The direction of Liutex represents the swirl axis of rotation, and its strength is equal to twice the angular speed. In this paper, we did a correlation analysis between vorticity, \( Q \), \( \lambda_\alpha \), \( \lambda_2 \) methods and Liutex based on a direct numerical simulation (DNS) case of boundary layer transition. The results show that the correlation between vorticity and Liutex is very small or even negative in strong shear regions, which demonstrates that using vorticity to detect vortex lacks scientific foundation and vorticity is not appropriate to represent vortex. The correlation analysis also shows that the second generation is contaminated too by shear and thus is not accurate to identify the vortex structure.

Key words: Correlation, Liutex, vorticity, \( Q \) method, \( \lambda_\alpha \) method, \( \lambda_2 \) method

Vorticity is an indicator of angular speed in solid mechanics. This fact has resulted in the assumption that vorticity could be used in the same way for fluid mechanics. In 1858, Helmholtz\(^1\) suggested use a vorticity tube to display vortex structure. However, many experimental results show that vorticity does not match the actual results. The famous 2-D Couette flow\(^2\) is a type of laminar flow where no vortex exists. However, the vorticity is non-zero or even very large near the wall surface, conflicting with the non-rotational fact. direct numerical simulation (DNS) research\(^3\) has found some places, where the rotation is strong but the vorticity magnitude is small. All these counterexamples drove researchers to find more appropriate vortex identification methods.

During the past four decades, several vortex identification methods have been proposed, including \( Q \) criterion\(^4\), \( \Delta \) criterion\(^5\), \( \lambda_\alpha \) method\(^6\) and \( \lambda_2 \) method\(^7\). All of them are scalars, making iso-surface the only way to display the vortex structure. This causes the problem by selection of a proper threshold value, which is in general empirical and some kind arbitrary. In addition, because these methods are scalar, it is unable to locate the swirling axis. Liu et al.\(^8\) classified these eigenvalue-based methods as the second-generation methods and further founded a new physical quantity-Liutex\(^9-10\), whose direction is the swirling axis and strength is the twice angular speed of the rigid rotation. Liutex is proved to be unique and Galilean invariant\(^11\). Many Liutex-related theories have been developed and gradually form a theoretical system. Right now, this system includes Liutex similarity\(^12\), Liutex core line\(^13-14\), Liutex-Omega method\(^15-16\), Objective Liutex\(^17\) as well as the Principal Coordinate and Principal Decomposition\(^18\). The Principal Coordinate is a unique coordinate under which rotation, shear, and stretching can be easily and correctly decomposed. Principal decomposition is the decomposition under the Principal Coordinate.

Many text-books\(^19\) still consider vorticity as the fundamental vortex identification method. So, in this letter, more evidences will be provided from correlation analysis perspective. Correlation analysis,
a data-based method, can be used to test the extent that two groups of data are related. If two variables are 100% relevant, then their correlation coefficient should be 1. On the other hand, if they are 100% irrelevant, their correlation coefficient should be near zero.

Suppose the velocity gradient tensor under Principal Coordinate\(^{(20)}\) of a point is

\[
\nabla V = \begin{bmatrix}
\lambda_{cr} & -\frac{1}{2}R_{r} & 0 \\
\frac{1}{2}R + \epsilon & \lambda_{cr} & 0 \\
\frac{1}{2}\xi & \eta & \lambda_{cr}
\end{bmatrix}
\]

The magnitude of vorticity can be written as

\[
|\nabla \times V| = \sqrt{\eta^2 + \xi^2 + (R + \epsilon)^2}
\] (2)

Apply Cauchy-Stokes decomposition on Eq. (1)

\[
\nabla V = \begin{bmatrix}
\lambda_{cr} & \frac{1}{2}\epsilon & \frac{1}{2}\xi \\
\frac{1}{2}\epsilon & \lambda_{cr} & \frac{1}{2}\eta \\
\frac{1}{2}\xi & \frac{1}{2}\eta & \lambda_{cr}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -\frac{1}{2}R - \frac{1}{2}\epsilon & -\frac{1}{2}\xi \\
\frac{1}{2}R + \frac{1}{2}\epsilon & 0 & -\frac{1}{2}\eta \\
\frac{1}{2}\xi & \frac{1}{2}\eta & 0
\end{bmatrix} = A + B
\] (3)

Then, \(Q\) can be expressed as

\[
Q = \frac{1}{2}\|\nabla \|_{r}^{2} - \|\nabla^{\prime}\|_{r}^{2} = \frac{1}{2}\left[2\left(\frac{R}{2} + \epsilon\right)^{2} + 2\left(\frac{\xi}{2}\right)^{2} + 2\left(\frac{\eta}{2}\right)^{2}\right] - \frac{1}{2}\left[2\lambda_{cr}^{2} + \lambda_{cr}^{2} + 2\left(\frac{\xi}{2}\right)^{2} + 2\left(\frac{\eta}{2}\right)^{2}\right]
\]

\[
= \frac{(R\epsilon)^{2}}{2} + \frac{\epsilon R}{2} - \lambda_{cr}^{2} = \frac{\lambda_{cr}^{2}}{2}
\] (4)

\(\lambda_{ci}\) is another vortex identification method, using the imaginary part of the conjugate complex eigenvalues of the velocity gradient tensor to detect vortex. The characteristic equation of (1) is

\[
(\lambda - \lambda_{cr})\left[(\lambda - \lambda_{cr})^2 + \frac{R}{R + 2\epsilon}\right] = 0
\] (5)

So, its three eigenvalues are

\[
\lambda_{1} = \lambda_{cr}
\]

\[
\lambda_{2} = \lambda_{cr} + i\sqrt{\frac{R}{2}(\frac{R}{2} + \epsilon)}
\]

\[
\lambda_{3} = \lambda_{cr} - i\sqrt{\frac{R}{2}(\frac{R}{2} + \epsilon)}
\] (6)

Since the rotation matrix is orthogonal, the eigenvalues of velocity gradient tensor do not change, i.e.

\[
\lambda_{3} = \lambda_{cr} + i\sqrt{\frac{R}{2}(\frac{R}{2} + \epsilon)} = \lambda_{cr} + i\lambda_{ci}
\] (7)

Here, \(\lambda_{2}\) is the second eigenvalue only, which is not related to the \(\lambda_{2}\) - method. Thus

\[
\lambda_{ci} = \sqrt{\frac{R}{2}(\frac{R}{2} + \epsilon)}
\] (8)

From the above analysis, it is easy to find from Eqs. (2), (4) and (10) that vorticity, \(Q\) and \(\lambda_{ci}\) cannot be used to represent the rotation or vortex strength as Liutex has been proved as the exact strength of fluid rotation but vorticity, \(Q\) and \(\lambda_{ci}\) all have shear terms.

In a DNS research of boundary layer transition\(^{(21)}\), three different \(x\) positions, 402.8 (Fig. 1), 500.7 (Fig. 3) and 815.5 (Fig. 5), which are in the area of laminar, transitional, and turbulent flow respectively, are selected. For each \(x\) position, ten different \(z\) positions are chosen, which can be found in Table 1. 400 snapshots in time are used to calculate the correlation coefficient. The selection of sample points can be found inside of the red circle of Figs. 2, 4 and 6.

![Figure 1](Color online) Points selected at \(x = 402.8\) (global)
Fig. 2 (Color online) Points (black in a red circle) selected at $x = 402.8$

Fig. 3 (Color online) Points selected at $x = 500.7$ (global)

Fig. 4 (Color online) Points (black in a red circle) selected at $x = 500.7$

Fig. 5 (Color online) Points selected at $x = 815.5$ (global)

Fig. 6 (Color online) Points (black in a red circle) selected at $x = 815.5$

The positions of $z$ are shown in Table 1.
The correlation coefficients for $x = 402.8$ and $y = 4.98$ are shown in Table 2 below.
The correlation coefficients for $x = 500.7$ and $y = 4.98$ are shown in Table 3 below.
The correlation coefficients for $x = 815.5$ and $y = 4.98$ are shown in Table 4 below.

Correlations between Liutex and vorticity, $Q$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ are depicted in Figs. 7, 8 and 9 at $x = 402.8$, 500.7 and 815.5 respectively.

Table 1 Selected points’ $z$ positions

|   | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_8$ | $Z_9$ | $Z_{10}$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| $Z$ | 0.26   | 0.38   | 0.50   | 0.65   | 0.81   | 1.08   | 1.35   | 1.70   | 2.17   | 2.77     |

Table 2 Correlation between Liutex and vorticity, $Q$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ at $x = 402.8$

|   | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_8$ | $Z_9$ | $Z_{10}$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| $\rho(R,Vor)$ | -0.59 | -0.61 | -0.24 | 0.43  | -0.26 | -0.51 | -0.62 | -0.43 | 0.89  | 0.93     |
| $\rho(R,Q)$    | 0.71  | 0.63  | 0.54  | 0.70  | 0.56  | 0.92  | 0.84  | 0.78  | 0.80  | 0.94     |
| $\rho(R,\hat{\lambda}_1)$ | 0.95 | 0.94  | 0.92  | 0.88  | 0.81  | 0.94  | 0.89  | 0.84  | 0.81  | 0.97     |
| $\rho(R,\hat{\lambda}_2)$ | 0.80  | 0.73  | 0.47  | 0.60  | 0.66  | 0.93  | 0.90  | 0.90  | 0.85  | 0.95     |
Generally, vorticity is the worst way to detect vortex, as seen in each figure, and the line of correlation between Liutex and vorticity is always at the bottom. The reason for this phenomenon can be founded from Eq. (2),

$$2 = + + ( \cdot \cdot )$$

vorticity is contaminated by shear from three different directions, while the other methods are only contaminated by shear from one or two directions. An apparent tendency observed from the figures is that with the increase of distance from the boundary, vorticity’s correlation goes up, corresponding to the fact that shears are strong near boundary region and become weaker when moving away from the wall boundary. This also shows the correctness of Eq. (2).

Although \( Q \) and \( \lambda_2 \) methods perform better than vorticity, these two methods share one significant problem that is their dimensions are not correct. From Eq. (4)

$$2 = (R/2)^2 + (R\epsilon/2) - \lambda_{\nu}^2 - (\lambda_2^2/2),$$

it can be seen that \( Q \) is roughly related to \( R^2 \), which is the square of the angular speed, rather than exactly rotation speed. So the quantity given by these two methods lacks correct physical meaning. An interesting phenomenon is that the correlation between vorticity and Liutex could be negative in some places, which seems to conflict with \( R \cdot S \)
decomposition \( \omega = R + S \). However, indeed, it does

| Table 3 Correlation between Liutex and vorticity, \( Q \), \( \lambda_{ci} \) and \( \lambda_2 \) at \( x = 500.7 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \rho(R, V) \) | -0.05 | 0.007 | -0.06 | 0.26 | 0.42 | 0.58 | 0.56 | 0.57 | 0.83 | 0.79 |
| \( \rho(R, Q) \) | 0.77 | 0.730 | 0.72 | 0.83 | 0.32 | 0.86 | 0.85 | 0.81 | 0.92 | 0.88 |
| \( \rho(R, \lambda_{ci}) \) | 0.91 | 0.870 | 0.86 | 0.91 | 0.92 | 0.93 | 0.95 | 0.93 | 0.98 | 0.96 |
| \( \rho(R, -\lambda_2) \) | 0.81 | 0.800 | 0.82 | 0.86 | 0.63 | 0.91 | 0.89 | 0.85 | 0.92 | 0.90 |

| Table 4 Correlation between Liutex and vorticity, \( Q \), \( \lambda_{ci} \) and \( \lambda_2 \) at \( x = 815.5 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \rho(R, V) \) | 0.10 | 0.03 | 0.08 | 0.30 | 0.55 | 0.73 | 0.41 | 0.81 | 0.71 | 0.47 |
| \( \rho(R, Q) \) | 0.90 | 0.86 | 0.70 | 0.69 | 0.83 | 0.79 | 0.69 | 0.85 | 0.84 | 0.68 |
| \( \rho(R, \lambda_{ci}) \) | 0.91 | 0.90 | 0.92 | 0.88 | 0.90 | 0.94 | 0.90 | 0.96 | 0.95 | 0.93 |
| \( \rho(R, -\lambda_2) \) | 0.90 | 0.87 | 0.80 | 0.75 | 0.89 | 0.88 | 0.78 | 0.89 | 0.87 | 0.74 |
not. The key point is that \( \omega = R + S \) is a vector relation, which has the possibility that when \( R \) becomes larger \( \omega \) could be smaller as \( R \) and \( S \) are not orthogonal. This is another proof that vorticity is not vortex. Therefore, it is of considerable importance to correct the concept that vorticity is vortex, which has been accepted by many research papers and textbooks for centuries.

In conclusion, Liutex extracts the rigid rotation part from the fluid motion. It makes much more sense to use Liutex to represent fluid rotation or vortex than those previous methods that cannot measure the flow rotation speed or vortex strength\(^{22-23}\). This letter explains from a correlation analysis perspective, both theoretically and numerically, that vorticity and all the other previous vortex identification methods cannot represent vortex properly.

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