ENERGY CONCENTRATION OF THE FOCUSING ENERGY-CRITICAL FNLS

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Abstract. We consider the fractional nonlinear Schrödinger equation (FNLS) with general dispersion $|\nabla|^{\alpha}$ and focusing energy-critical nonlinearities $-|u|^{2\alpha-2}u$ and $-|x|^{-2\alpha}u^2$. By adopting Kenig-Tsutsumi [24], Kenig-Merle [19] and Killip-Visan [22] arguments, we show the energy concentration of radial solutions near the maximal existence time. For this purpose we use Sobolev inequalities for radial functions and establish strong energy decoupling of profiles. And we also show that when the kinetic energy is confined the maximal existence time is finite for some large class of initial data satisfying the initial energy $E(\phi)$ is less than energy of ground state $E(W_\alpha)$ but $\|\nabla|^{\frac{2}{d}-\frac{\alpha}{2}}\phi\|_{L^2} \geq \|\nabla|^{\frac{2}{d}}W_\alpha\|_{L^2}$.

1. Introduction

In this paper we consider the Cauchy problem of the focusing fractional nonlinear Schrödinger equations:

\begin{equation}
\begin{cases}
  i\partial_t u = |\nabla|^{\alpha}u - V(u)u, & \text{in } \mathbb{R}^{1+d}, d \geq 2, \\
  u(x,0) = \phi(x), & \text{in } \mathbb{R}^d,
\end{cases}
\end{equation}

where

$$V(u) = \begin{cases}
  |u|^{\frac{2\alpha}{d-\alpha}} (1 < \alpha < 2), \\
  (|x|^{-2\alpha} + |u|^2) (1 < \alpha < \min(2, \frac{d}{2})).
\end{cases}$$

The equation (1.1) is of $\dot{H}^{\frac{2}{d}}$-scaling invariance (so-called energy-critical). That is, if $u$ is a solution of (1.1), then for any $\lambda > 0$ the scaled function $u_\lambda$, given by

$$u_\lambda(t,x) = \lambda^{\frac{2}{d} - \frac{\alpha}{2}} u(\lambda^\alpha t, \lambda x),$$

is also a solution to (1.1).

The problem (1.1) can be easily shown to be well-posedness in $C((-T_*, T^*); \dot{H}^{\frac{2}{d}}_{\text{rad}})$ for $\alpha \in [\frac{2d}{d+1}, 2)$ and $d > \alpha$ in the case of power type ($d > 2\alpha$ in the case of Hartree type) through the radial Strichartz estimate. See Lemma 2.1 below for Strichartz estimate, and also see Theorem 4.10 of [17] and Theorem 5.2 of [5] for LWP and small data GWP. Here $-T_*, T^* \in [-\infty, +\infty]$ are the

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maximal existence times and $X_{\text{rad}}$ denotes the Banach space $X$ of radially symmetric functions. The solution $u$ satisfies the mass and energy conservation laws: for $t \in (-T^*, T^*)$

$$m(u(t)) = \|u(t)\|_{L^2}^2 = m(\varphi),$$

$$E(u(t)) = K(u(t)) + V(u(t)) = E(\varphi),$$

where

$$K(u) = \frac{1}{2} \int |\nabla \varphi^2 u(x)|^2 \, dx, \quad V(u) = -\frac{1}{\mu} \int V(u) |u|^2 \, dx,$$

$\mu = \frac{2d}{d-\alpha}$ for power type and $\mu = 4$ for Hartree type.

At this point due to the dependency on the profile which is the critical nature of (1.1) we do not know that $\limsup_{t \to T^*} \|\nabla \varphi^2 u(t)\|_{L^2}$ is infinity or not. In this paper we want to address some energy concentration phenomena for both cases. We first consider the concentration in the case of unconfined kinetic energy.

**Theorem 1.1 (Unconfined case).** Let $d > \alpha + 1$ in the case of power type ($d > 2\alpha$ in the case of Hartree type). Assume that $\varphi \in H_{\text{rad}}^{\frac{\alpha}{2}}$ and $u$ is the unique solution to (1.1) in $C([0, T^*); H_{\text{rad}}^{\frac{\alpha}{2}})$ with

$$\limsup_{t \to T^*} \|\nabla \varphi^2 u(t)\|_{L^2} = +\infty.$$

Then for any $R > 0$ we have

$$\limsup_{t \to T^*} \|\nabla \varphi^2 u(t)\|_{L^2(|x| \leq R)} = +\infty.$$

Moreover, if for $t < T^* \ u(t) \in L^\infty$, then

$$\limsup_{t \to T^*} \|u(t)\|_{L^\infty(|x| \leq R)} = +\infty.$$

The same result holds near $-T^*$.}

Next we deal with the confined case for which it is necessary to implement subtle estimate associated with ground state. The ground state of (1.1) plays an important role. It is a unique positive radial solution of

$$|\nabla|^\alpha W - V(W)W.$$  

In [4, 23, 14] the authors showed that any solution of the elliptic equation (1.3) is a constant multiple, dilation and translation of the function $W_\alpha(x) = C_1(1 + C_2 |x|^2)^{-\frac{d-\alpha}{2}}$ which is in $H_{\text{rad}}^{\frac{\alpha}{2}}$ for $0 < \alpha < \frac{d}{2}$, where $C_1, C_2$ depend on $d, \alpha$. The solution $W_\alpha$ is closely related to the best constant problem of the inequality

$$\int V(u) |u|^2 \, dx \leq C_{d, \alpha} \|\nabla \varphi^2 u\|_{L^2}^\mu.$$ 

Indeed, the maximizer $u \neq 0$ of the above inequality, that is,

$$\int V(u) |u|^2 \, dx = C_{d, \alpha} \|\nabla \varphi^2 u\|_{L^2}^\mu,$$ 

(1.4)
is characterized as \( u = e^{i\theta} \lambda^{\frac{d}{d-1}} W_\alpha(\lambda(x - x_0)) \) for some \( \theta \in [-\pi, \pi] \), \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \). See \([1, 27, 13]\) for power type. We will treat this problem for Hartree type in the appendix. Since \( W_\alpha \) is a solution of \((1.3)\), \( \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2}^2 = \int V(W_\alpha)|W_\alpha|^2 \, dx \). Thus \( \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2}^2 = C_{d,\alpha}^{\frac{d}{2}} \).

Let us denote by \( S_\alpha(I) \) for an interval \( I \) the spaces \( L_{\alpha,\alpha}^{2(d+\alpha)} \) for power type and \( L_{\alpha}^6 L_x^{\frac{2d}{d-\alpha}} \) for Hartree type. Then Lemma \((2.4)\) below states that LWP of \((1.1)\) in \( \dot{H}^{\frac{d}{2}} \) holds for some \( \alpha \) and \( d \) and also shows the blowup criterion that \( \|u\|_{S_\alpha((-T, T))} = +\infty \) and \( \|u\|_{S_\alpha([0, T^*))} = +\infty \) when \( T, T^* < +\infty \). Furthermore by following the arguments in \([19, 22, 26, 16]\) with profile decomposition developed in Section 3 below one can readily get the following theorem.

**Theorem 1.2.** Let \( d \geq 2 \), \( \frac{2d}{d-1} < \alpha < 2 \), \( \alpha < d \leq 2\alpha \) for power type \((d > 2\alpha \) for Hartree type) and let \( \varphi \in \dot{H}^{\frac{d}{2}} \). Assume that

\[
\sup_{t \in (-T, T^*)} \|\nabla^{\frac{d}{2}} u(t)\|_{L^2} < \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2}.
\]

Then \( T, T^* = +\infty \) and \( \|u\|_{S_\alpha(\mathbb{R})} < +\infty \).

As a corollary one can show that \( T = T^* = +\infty \) and \( \|u\|_{S_\alpha(\mathbb{R})} < +\infty \) if \( E(\varphi) < E(W_\alpha) \) and \( \|\nabla^{\frac{d}{2}} \varphi\|_{L^2} < \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2} \). The same result also holds for the defocusing case. The restriction \( \alpha \in (\frac{2d}{d-1}; 2) \) comes from the optimal range of Strichartz estimates (see Lemma \((2.1)\)). The condition \( \alpha \leq 2\alpha \) for power type is necessary to estimate perturbation like \( \|\sum_j f_j \|_{\dot{H}^{\frac{d}{2}}} \leq \sum_j |f_j|^{\frac{2\alpha}{d-\alpha}} \). For this see the arguments below \((3.6)\).

At this point one may expect the sharpness of Theorem \((1.2)\) and the blowup \((\|u\|_{S_\alpha((-T, T^*))} = +\infty)\) when \( E(\varphi) < E(W_\alpha) \) and \( \|\nabla^{\frac{d}{2}} \varphi\|_{L^2} \geq \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2} \). Unfortunately we do not know the complete answers. We think this is just a technical problem due to non-locality arising when treating \( |\nabla|^{\alpha} \). However, in case when kinetic energy is confined we can show the energy concentration near the maximal existence time and also find some class of initial data guaranteeing the finite time blowup. We first introduce the energy concentration.

**Theorem 1.3** (Confined case). Let \( d \geq 2 \), \( \frac{2d}{d-1} < \alpha < 2 \), \( \alpha < d \leq 2\alpha \) for power type \((d > 2\alpha \) for Hartree type) and let \( \varphi \in \dot{H}^{\frac{d}{2}} \). Assume that

\[
\|u\|_{S_\alpha([0, T^*))} = +\infty, \quad \sup_{t \in [0, T^*)} \|\nabla^{\frac{d}{2}} u(t)\|_{L^2} < +\infty.
\]

If \( T^* \) is finite, then there exists a sequence \( t_n \to T^* \) such that for any sequence \( R_n \in (0, \infty) \) obeying \((T^* - t_n)^{-\frac{1}{\alpha}} R_n \to \infty\),

\[
\limsup_{n \to \infty} \int_{|x| \leq R_n} \|\nabla^{\frac{d}{2}} u(t_n, x)\|^2 \, dx \geq \|\nabla^{\frac{d}{2}} W_\alpha\|_{L^2}^2.
\]

The same result also holds near \(-T^* \) if \( T^* < +\infty \).
The Schrödinger case was treated by Killip and Visan in [22]. Here we adapt their arguments to fractional case with nonlinear profile approximation. We want to mention that due to the lack of pseudo-conformal symmetry of the equation (1.1) we could not get the similar result when the solution blowup at time infinity.

From now on we try to demonstrate some evidence of the finite time blowup. Based on the virial argument the finite time blowup was shown for mass-critical Hartree type fractional Schrödinger equations in [6] and for fourth order power type NLS [11], where the mass-critical nature and radial symmetry are playing a crucial role in the proof of blowup. Those arguments cannot be applied to the power type mass-critical fractional NLS because of the lack of enough cancelation property of nonlinearity for virial argument to proceed. Since we do not know whether the kinetic energy is confined, it is hard to apply them to energy subcritical and mass supercritical or energy critical problem. However, if we are involved in energy critical problem and the energy is confined, then by using Sobolev inequality for radial functions [10] it is plausible to establish the virial argument to get finite time blowup for both power type and Hartree type. The following theorem leads us off the finite time blowup.

**Theorem 1.4.** Let \( \varphi \in H^{\frac{d}{2}}_{\text{rad}} \) and \( u \) be the unique solution of (1.1) in \( C([0,T^*]; H^{\frac{d}{2}}_{\text{rad}}) \) for the maximal existence time \( T^* \in (0, +\infty) \). Suppose that \( d \geq 2, \frac{4}{3} \leq \alpha < 2, \alpha < d < 3\alpha \) for power type and \( d > 2\alpha + 2, \frac{2d}{d-1} < \alpha < 2 \) for Hartree type. Then for any \( \varphi \) satisfying that \( (1.5) \) \( \| |x| \sqrt{1-\Delta} \varphi \|_{L^2} + \| |x|^2 \varphi \|_{L^2} < +\infty \), \( (1.6) \) \( E(\varphi) < E(W_{\alpha}), \quad \| |\nabla|^{\frac{\alpha}{2}} \varphi \|_{L^2} \geq \| |\nabla|^{\frac{\alpha}{2}} W_{\alpha} \|_{L^2}, \) if \( \sup_{0 \leq t < T^*} \| |\nabla|^{\frac{\alpha}{2}} u(t) \|_{L^2} < +\infty \), then \( T^* < \infty \).

The rest of paper is organized as follows: In Section 2 we gather some preliminary lemmas necessary for the proof of confined energy concentration including the profile decomposition in energy space. In Section 3 we show the energy concentration, Theorems 1.1 and 1.3. Section 4 is devoted to proving finite time blowup under energy confinement. In the last section we consider the best constant problem (1.4) for Hartree equation.

**Notations.** We will use the notations:

- \( |\nabla| = \sqrt{-\Delta}, \Delta = |\nabla|^{-s} L^r, \dot{H}^s = \dot{H}^s_{2}, H^s = (1-\Delta)^{-s/2} L^r, H^s = H^s_{2}, L^r = L^r_x(\mathbb{R}^d) \) for some \( s \in \mathbb{R} \) and \( 1 \leq r \leq \infty \).
- We use the following mixed-norm notations \( L^q_1 L^r = L^q_t(I; L^r_x(\mathbb{R}^d)), L^q_{1,x} = L^q_t L^q \) and \( L^q_1 L^r = L^q_{1,x} \).
- \( \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx \).
- For any dyadic number \( N \) we denote frequency localization of function \( f \) by \( f_N \), which is defined by \( \hat{f}_N(\xi) = \mathcal{F}^{-1}_N \hat{f}(\xi) = \beta(\xi/N) \hat{f} \) for a fixed Littlewood-Paley function \( \beta \in C^0_{0,\text{rad}} \) with \( \beta \beta = \beta \) and...
$P_N \tilde{P}_N = P_N$, where $\tilde{\beta}(\xi) = \beta(\xi/2) + \beta(\xi) + \beta(2\xi)$ and $\tilde{P}_N = P_{N/2} + P_N + P_{2N}$.

- As usual different positive constants are denoted by the same letter $C$, if not specified.
- $[A, B]$ denotes the commutator $AB - BA$ for any operators $A$ and $B$ defined on suitable Banach spaces.
- $\langle u, v \rangle = \int_{\mathbb{R}^d} u \overline{v} \, dx$ and $\langle f ; g \rangle = \sum_{1 \leq j \leq d} \langle f_j, g_j \rangle$ for $f = (f_1, \ldots, f_d), g = (g_1, \ldots, g_d)$.

2. Preliminary lemmas

We define the linear propagator $U(t)$ of the linear equation $iu_t = |\nabla|^{\alpha} u$ with initial datum $f$. Then it is formally given by

$$U(t)f = e^{-it|\nabla|^{\alpha} f} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-\xi-t|\xi|^{\alpha})} \hat{f}(\xi) \, d\xi. \tag{2.1}$$

We have Strichartz estimates for radial functions (see [9] and [17, 18]) as follows.

**Lemma 2.1.** Suppose that $d \geq 2$, $\frac{2d}{2d-1} \leq \alpha < 2$ and $f, F$ are radial. Then there hold

$$\|U(t)f\|_{L_t^q L_x^r} \leq C\|f\|_{L^2}, \quad \| \int_0^t U(t-t')F(t') \, dt' \|_{L_t^q L_x^r} \leq C\|F\|_{L_t^q L_x^r}$$

for the pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ such that

$$\frac{\alpha + d}{q} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad (q, r) \neq (2, \frac{4d-2}{2d-3}).$$

Such pairs are said to be $\alpha$-admissible.

Then we have the following inverse Strichartz estimate.

**Lemma 2.2.** Fix $d \geq 2$ for power type and $d > 2\alpha$ for Hartree type. Let $f \in \dot{H}^{\frac{\alpha}{2}}$ rad and $\eta > 0$ such that

$$\|U(t)f\|_{S_{\alpha}(t)} \geq \eta$$

for some interval $I \subset \mathbb{R}$. Then there exists $\tilde{C} = \tilde{C}(\|\nabla|^{\frac{\alpha}{2}} f\|_{L^2}, \eta)$, and $J \subset I$ so that

$$\int_{|x| \leq \tilde{C}|J|^{\frac{2}{\alpha}}} \left| U(t)|\nabla|^{\frac{\alpha}{2}} f \right|^2 \, dx \geq \tilde{C}^{-1} \text{ for all } t \in J.$$

Here $\tilde{C}$ does not depend on $I$ or $J$.

**Proof of Lemma 2.2.** For simplicity we only consider the Hartree type, the power type can be treated similarly to [22]. We will show that

$$\|U(t)f_M\|_{S_{\alpha}(t)} \geq C^{-1} \eta^{\frac{2d}{2d-1}} \|\nabla|^{\alpha} f\|_{L^2}^{1-\frac{2d}{2d-1}}$$

for some dyadic $M \geq A|I|^{-\frac{1}{d}}$ and some (depending on $d$ and $\alpha$). Here $A = C\|\nabla|^{\alpha} f\|_{L^2}^{-D_1}\eta^{D_2}$ and the constants $D_1, D_2$ will be specified later.
We assume that (2.2) is true. By Strichartz estimate we have
\[
\|U(t)f_M\|_{L_t^q \cap L_x^r} \leq CM^{-\frac{d}{2}}\|\nabla_{\frac{s}{2}} f\|_{L^2}.
\]
Combining this with (2.2), we get by Hölder’s inequality that
\[
\|U(t)f_M\|_{L_{tx}^r} \geq C^{-1}\|\nabla_{\frac{s}{2}} f\|_{L^2}^{1-\frac{3d}{2r\eta}}M^\frac{d}{2r\eta}.
\]
From this with the fact that the kernel of $M^\frac{s}{2}\nabla^{-\frac{s}{2}}P_M$ is integrable and its value is independent of $M$ we deduce that
\[
C^{-1}\|\nabla_{\frac{s}{2}} f\|_{L^2}^{1-\frac{3d}{2r\eta}}M^\frac{d}{2r\eta} \leq \|(M^\frac{s}{2}\nabla^{-\frac{s}{2}}P_M)(U(t)\nabla_{\frac{s}{2}} f_M)\|_{L_{tx}^r} \leq C\|U(t)\nabla_{\frac{s}{2}} f_M\|_{L_{tx}^r}.
\]
Thus there exist $(t_0, x_0) \in I \times \mathbb{R}^d$ so that
\[
(2.3) \quad |(U(t_0)\nabla_{\frac{s}{2}} f_M(x_0))| \geq A_0 M^\frac{d}{2},
\]
where $A_0 = C^{-1}\|\nabla^{\frac{s}{2}} f\|_{L^2}^{1-\frac{3d}{2r\eta}}M^\frac{d}{2r\eta}$. Let $A_1 = \frac{A_0}{2C\|\nabla^{\frac{s}{2}} f\|_{L^2}}$. Then for $|x - x_0| \leq A_1M^{-1}$ and $|t - t_0| \leq A_1M^{-\alpha}$ we have
\[
|U(t_0)\nabla_{\frac{s}{2}} f_M(x_0) - U(t)\nabla_{\frac{s}{2}} f_M(x)| \leq \frac{1}{2}A_0 M^\frac{d}{2}
\]
and thus
\[
|U(t)\nabla_{\frac{s}{2}} f_M(x)| \geq \frac{1}{2}A_0 M^\frac{d}{2}.
\]
This yields for all $t \in J = \{t \in I : |t - t_0| \leq A_1M^{-1}\}$
\[
\int_{|x - x_0| \leq A_1M^{-1}} |U(t)\nabla_{\frac{s}{2}} f_M(x)|^2 \, dx \geq \frac{s_d^d}{4}A_0^2 A_1^d,
\]
where $s_d$ is the measure of the unit ball. By convexity we have
\[
|U(t)\nabla_{\frac{s}{2}} f_M|^2 \leq C_0|U(t)\nabla_{\frac{s}{2}} f|^2 \beta_M^*,
\]
where $\beta_M^*(x) = M^d|\hat{\beta}(Mx)|$ and $C_0 = \int \beta^* \, dx$. And also
\[
\int_{|x - x_0| \leq A_1M^{-1}} |U(t)\nabla_{\frac{s}{2}} f_M(x)|^2 \, dx \leq C_0\langle|U(t)\nabla_{\frac{s}{2}} f|^2, \beta_M^* \chi_{|x-x_0| \leq A_1M^{-1}}\rangle.
\]
We divide inner product into two parts as follows:
\[
\langle|U(t)\nabla_{\frac{s}{2}} f|^2, \beta_M^* \chi_{|x-x_0| \leq A_1M^{-1}}\rangle \leq I + II,
\]
where
\[
I = \langle|U(t)\nabla_{\frac{s}{2}} f|^2, \chi_{|x-x_0| \leq A_2A_1M^{-1}}\beta_M^* \chi_{|x-x_0| \leq A_1M^{-1}}\rangle,
\]
\[
II = \langle|U(t)\nabla_{\frac{s}{2}} f|^2, \chi_{|x-x_0| > A_2A_1M^{-1}}\beta_M^* \chi_{|x-x_0| \leq A_1M^{-1}}\rangle.
\]
Now we can find a constant $A_2 = A_2(||\nabla|^n f||_{L^2},\eta) > 1$ such that $C_0 I \leq \frac{s_d}{8} A_0^2 A_1^d$. Then $I \leq C_0 \int_{|x-x_0| \leq A_2 A_1 M^{-1}} |U(t)| \nabla^\frac{n}{2} f(x)^2 \, dx$ and thus we obtain

\begin{equation}
\int_{|x-x_0| \leq A_2 A_1 M^{-1}} |U(t)| \nabla^\frac{n}{2} f(x)^2 \, dx \geq \frac{s_d}{8 C_0^2} A_0^2 A_1^d.
\end{equation}

On the other hand, since $f$ is radial, we use the Sobolev inequality \[10\] that

\begin{equation}
\sup_{x \in \mathbb{R}^d} |x| \frac{d-\alpha}{2} f(x) | \leq C ||\nabla|^\frac{n}{2} f||_{L^2} \text{ a.e.}
\end{equation}

together with (2.3) to get

\[ A_0 (M|x_0|)^{\frac{d-\alpha}{2}} \leq |x_0|^{\frac{d-\alpha}{2}} M^{-\frac{\alpha}{2}} |U(t_0)| \nabla^\frac{n}{2} f_M(x_0) | \leq C ||\nabla|^\frac{n}{2} f||_{L^2}, \]

which means that

\begin{equation}
|x_0| \leq A_0^{-\frac{2}{d-\alpha}} M^{-1}.
\end{equation}

Combining (2.6) with (2.4), we have

\[ \int_{|x| \leq (A_0^{-\frac{2}{d-\alpha}} + A_2 A_1) M^{-1}} |U(t)| \nabla^\frac{n}{2} f(x)^2 \, dx \geq \frac{s_d}{8 C_0^2} A_0^2 A_1^d. \]

Since $|I| \geq A^\alpha M^{-\alpha}$, we deduce that $|J| \geq \frac{1}{2} \min(A^\alpha, A_1) M^{-\alpha}$. By letting $\bar{C} = \max(2(A_0^{-\frac{2}{d-\alpha}} + A_1 A_2) \max(A^{-1}, A_1^{-\frac{1}{2}}, \frac{8 C_0^2}{s_d A_0^2 A_1^d})$ we get the desired result.

Now we show (2.2). By Littlewood-Paley theory and Hölder’s inequality we have

\[ \eta^6 \leq ||U(t)f||_{S_{\eta}(I)}^6 \]

\[ \leq C \int_I \left( \int \sum_{N} |U(t)f_N|^2 \right)^{d-4\alpha/3} \, dt \]

\[ \leq C \sum_{M \leq N} \int_I \left( \int |U(t)f_M|^{\frac{d-4\alpha}{2\alpha}} |U(t)f_N|^{\frac{d-4\alpha}{2\alpha}} \right)^{\frac{3d-4\alpha}{d}} \, dt \]

\[ \leq C \sum_{M \leq N} \int_I \|U(t)f_M\|^{\frac{3d-4\alpha}{2\alpha}} L^{2d-2\alpha} \|U(t)f_M\|^{\frac{4\alpha}{2\alpha}} L^{2d-4\alpha} \|U(t)f_N\|^{\frac{4\alpha}{4\alpha}} L^{2d-4\alpha} \|U(t)f_N\|^{\frac{3d-4\alpha}{2\alpha}} \, dt \]

\[ \leq C \sum_{M \leq N} \left( \frac{M}{N} \right)^{\frac{\alpha(3d-4\alpha)}{2\alpha}} (||\nabla|^\frac{n}{2} f_M||_{L^2} ||\nabla|^\frac{n}{2} f_N||_{L^2})^{\frac{3d-4\alpha}{d}} (||U(t)f_M||_{S_{\eta}(I)} ||U(t)f_N||_{S_{\eta}(I)}) \]

\[ \leq C \sup_M \|U(t)f_M\|_{S_{\eta}(I)}^{\frac{8\alpha}{d}} \left( \sum_{M \leq N} \left( \frac{M}{N} \right)^{\frac{\alpha}{2\alpha}} ||\nabla|^\frac{n}{2} f_M||_{L^2} ||\nabla|^\frac{n}{2} f_N||_{L^2} \right)^{\frac{3d-4\alpha}{d}} \]

\[ \leq C \sup_M \|U(t)f_M\|_{S_{\eta}(I)}^{\frac{8\alpha}{d}} ||\nabla|^\frac{n}{2} f||_{L^2}^{\frac{6d-8\alpha}{d}} . \]

\[ ^1 \text{We may choose } A_2 = A_2 > 1 + \max(A_1^{-1}, \frac{C_0 C_2 A_1^{-2}}{8 A_0^2}), \text{ where } C_0 \text{ is the constant satisfying } \beta^*(x) \leq C_0 (1 + |x|)^{-1}. \]
From the Sobolev inequality it follows that
\[ \|U(t)f_M\|_{S_\alpha(I)} \leq C(\|M\|^\frac{1}{2}) \|\nabla \hat{\phi} f\|_{L^2}. \]
Thus we conclude that there exists \( M \geq A\|M\|^{-\frac{1}{2}} \) so that (2.2) holds with \( A = C\|\nabla \hat{\phi} f\|_{L^2}^{-\frac{2\alpha}{2\alpha^2 - 9\alpha^2}} \).

□

Next we introduce the tightness of trajectories of solution. The proof is almost same as the one of Proposition 2.13 in [22] and so we omit it.

**Lemma 2.3.** Let \( v : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) be a radial solution to (1.1) with \( \|v\|_{S_\alpha(I)} < \infty \). Suppose that
\[ \int_{|x| \leq r_k} |U(t_k)|^2 |\nabla \hat{\phi} v(\tau_k)|^2 \, dx \geq \varepsilon \]
for some \( \varepsilon > 0 \), \( r_k > 0 \), and bounded sequences \( t_k \in \mathbb{R} \) and \( \tau_k \in I \). Then
\[ \left| \|\nabla \hat{\phi} v(\tau_k)\|^2 - \int_{|x| \leq a_k r_k} |U(t_k)|^2 |\nabla \hat{\phi} v(\tau_k)|^2 \, dx \right| \rightarrow 0 \]
for any sequence \( a_k \rightarrow +\infty \).

We close this section by introducing local well-posedness and stability. Since the proof is quite standard, we omit the details (for instance see [19 5]).

**Lemma 2.4.** Let \( \alpha \in (\frac{2d}{2d-1}, 2) \) and \( \alpha < d < 3\alpha \) for power type \( (d > 2\alpha \) for Hartree type), and let \( \varphi \in \dot{H}^\frac{\alpha}{2d} \cap \dot{H}^\frac{\alpha}{2d} \leq A \). Then there exists \( \delta = \delta(A) \) such that if \( \|U(t-t_0)\varphi\|_{S_\alpha(I)} \leq \delta \), \( t_0 \in I \), there exists a unique solution \( u \in C(I; \dot{H}_\alpha^\frac{\alpha}{2d}) \) to (1.1) with
\[ \sup_I \|u(t)\|_{\dot{H}_\alpha^\frac{\alpha}{2d}} + \|\nabla \hat{\phi} u\|_{X_\alpha(I)} \leq C(A), \quad \|u\|_{S_\alpha(I)} \leq 2\varepsilon. \]
Here \( X_\alpha(I) = L^\frac{2(d+\alpha)}{2d} \cap L^\frac{2(d+\alpha)}{d+\alpha} \) for power type and \( L^\frac{2(d+\alpha)}{d+\alpha} \) for Hartree type. Moreover, \( \varphi \mapsto u \in C(I; \dot{H}_\alpha^\frac{\alpha}{2d}) \) is Lipschitz. If \( A \) is sufficiently small, then \( I = \mathbb{R} \).

**Lemma 2.5.** Assume that \( \alpha \in (\frac{2d}{2d+1}, 2) \) and \( \alpha < d < 3\alpha \) for power type \( (d > 2\alpha \) for Hartree type). Let \( I = [0, L] \), \( L \leq +\infty \), and let \( \tilde{u} \) be radial and defined on \( \mathbb{R}^d \times I \) be such that
\[ \sup_{t \in I} \|\tilde{u}(t)\|_{\dot{H}_\alpha^\frac{\alpha}{2d}} \leq A, \quad \|\tilde{u}\|_{S_\alpha(I)} \leq M, \quad \|\nabla \hat{\phi} \tilde{u}\|_{X_\alpha(I)} < \infty \]
for some constants \( A \) and \( M \), and \( \tilde{u} \) verifies in the sense of integral equation
\[ i\tilde{u}_t = |\nabla|^\alpha \tilde{u} - V(\tilde{u})\tilde{u} + \varepsilon \]
for some function \( \varepsilon \). Let \( \varphi \in \dot{H}_\alpha^\frac{\alpha}{2d} \) be such that \( \|\varphi - \tilde{u}(0)\|_{\dot{H}_\alpha^\frac{\alpha}{2d}} \leq A' \). Suppose there exists \( \varepsilon_0 = \varepsilon_0(M, A, A') \) such that if \( 0 < \varepsilon \leq \varepsilon_0 \) and
\[ \|\nabla \hat{\phi} \varepsilon\|_{X_\alpha(I)} \leq \varepsilon, \quad \|U(t)(\varphi - \tilde{u}(0))\|_{S_\alpha(I)} \leq \varepsilon, \]
then there exists a unique radial solution \( u \) on \( \mathbb{R}^d \times I \) to \( (1.1) \) such that

\[
\| u \|_{S_{\alpha}(I)} + \sup_I \| u(t) - \bar{u}(t) \|_{\dot{H}^\alpha_x} \leq C(M, A, A').
\]

Here \( Y_{\alpha}(I) = L^2_I L_x^{\frac{2d}{d-\alpha}} \) for both power type and for Hartree type.

Now we consider the profile decomposition in energy space. Most of them are standard and thus we only show the energy decoupling of Hartree case.

**Lemma 2.6** (see Theorem 1 of [12]). Let \( \{t_n\} \) be sequence in \( \mathbb{R} \). Suppose \( \lim_{n \to \infty} |t_n| = \infty \), then for any \( f \in C_0^\infty \)

\[
\| U(t_n)f \|_{L^p} \to 0 \quad \text{as} \quad n \to \infty,
\]

when \( p > 2 \).

The profile decomposition of \( U(t) \) for mass critical case was already verified for radial data in [7] (see also [8]). From that decomposition, one can easily prove the following profile decomposition for the energy critical case:

**Lemma 2.7.** Let \( d \geq 2 \), \( \frac{2d}{2d-1} < \alpha < 2 \), and \( (q, r) \) be \( \alpha \)-admissible pairs with \( 2 < q, r < \infty \). Suppose that \( \{u_n\}_{n \geq 1} \) is a sequence of complex-valued radial functions bounded in \( \dot{H}^\frac{\alpha}{2} \). Then up to a subsequence, for any \( J \geq 1 \), there exist a sequence of radial functions \( \{\phi^j\}_{1 \leq j \leq J} \subset \dot{H}^\frac{\alpha}{2} \), \( \omega_n^j \in \dot{H}^\frac{\alpha}{2} \) and a family of parameters \((h_n^j, t_n^j)_{1 \leq j \leq J, n \geq 1}\) such that

\[
(2.7) \quad u_n(x) = \sum_{1 \leq j \leq J} U(t_n^j)[(h_n^j)^{-d/2+\alpha/2}\phi^j(-/h_n^j)](x) + \omega_n^j(x)
\]

and the following properties are satisfied:

\[
(2.8) \quad \lim_{J \to \infty} \limsup_{n \to \infty} \| \nabla \hat{U}(\cdot) \omega_n^j \|_{L^q_x L^r_x} = 0,
\]

and for \( j \neq k \), \((h_n^j, t_n^j)_{n \geq 1}\) and \((h_n^k, t_n^k)_{n \geq 1}\) are asymptotically orthogonal in the sense that

\[
(2.9) \quad \text{either} \quad \limsup_{n \to \infty} \left( \frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} \right) = \infty,
\]

\[
\text{or} \quad (h_n^j) = (h_n^k) \quad \text{and} \quad \limsup_{n \to \infty} \frac{|t_n^j - t_n^k|}{(h_n^j)^{\alpha}} = \infty,
\]

and for each \( J \)

\[
(2.10) \quad \lim_{n \to \infty} \left[ \| u_n \|_{\dot{H}^\frac{\alpha}{2}}^2 - \left( \sum_{1 \leq j \leq J} \| \phi^j \|_{\dot{H}^\frac{\alpha}{2}}^2 + \| \omega_n^j \|_{\dot{H}^\frac{\alpha}{2}}^2 \right) \right] = 0.
\]

**Remark 1.** Since the space and frequency translations do not appear in the profile decomposition due to the radial symmetry, it is possible to get the strong convergence of remainder term in \( L^q_x \hat{H}^\frac{\alpha}{2} \) as in \( (2.8) \) not in \( L^q_x L_x^{\frac{2d}{d-\alpha}} \) norm. It plays a crucial role in the proof of Theorem 1.3.
From energy critical profile decomposition, we prove some useful corollaries.

**Corollary 2.8.** Suppose that \( \{u_n\}_{n \geq 1} \) is a sequence of complex-valued radial functions bounded in \( \dot{H}^{\frac{d}{2}} \). Let \( \{\phi^j\}_{1 \leq j \leq J} \subset \dot{H}^{\frac{d}{2}} \), \( \omega_n^j \in \dot{H}^{\frac{d}{2}} \) and a family of parameters \((h_n^j, t_n^j)_{1 \leq j \leq J, n \geq 1}\) from Lemma 2.7. Define group operator \( G_n^j \) as \( G_n^j(f) = U(t_n^j)((h_n^j)^{-d/2+\alpha/2} f(\cdot/h_n^j))(x) \). Then we have

\[
(G_n^j)^{-1}(\omega_n^j) \rightharpoonup 0 \quad \text{weakly in } \dot{H}^{\frac{d}{2}} \quad \text{as } n \to \infty,
\]

\[
(G_n^j)^{-1}(u_n) \rightharpoonup \phi^j \quad \text{weakly in } \dot{H}^{\frac{d}{2}} \quad \text{as } n \to \infty.
\]

**Proof.** We first prove

\[
(G_n^j)^{-1}U(t)(|\nabla|^{\frac{d}{2}} u_n) \rightharpoonup U(t)|\nabla|^{\frac{d}{2}} \phi^j \quad \text{weakly in } L^{\frac{2(2+\alpha)}{d}}_{t,x} \quad \text{as } n \to \infty.
\]

Applying \((G_n^j)^{-1}U(t)\) to (2.7), we obtain

\[
(G_n^j)^{-1}(U(t)|\nabla|^{\frac{d}{2}} u_n) = U(t)|\nabla|^{\frac{d}{2}} \phi^j + \sum_{j' \neq j} (G_n^{j'})^{-1}(G_n^{j'})^{-1}(U(t)|\nabla|^{\frac{d}{2}} \phi^j) + (G_n^j)^{-1}U(t)|\nabla|^{\frac{d}{2}} \omega_n^j.
\]

From the pairwise orthogonality of the family of parameters, we have

\[
(G_n^j)^{-1}(G_n^{j'})^{-1}(U(t)|\nabla|^{\frac{d}{2}} \phi^j) \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2(2+\alpha)}{d}}_{t,x} \quad \text{as } n \to \infty
\]

for every \( j' \neq j \). Let \( \omega^J \) be the weak limit of \( \{(G_n^j)^{-1}U(t)|\nabla|^{\frac{d}{2}} \omega_n^j\} \). Then

\[
(G_n^j)^{-1}(U(t)|\nabla|^{\frac{d}{2}} u_n) \rightharpoonup U(t)|\nabla|^{\frac{d}{2}} \phi^j + \omega^J.
\]

Since the weak limit is unique, \( \omega^J \) does not depend on \( J \). And from

\[
\| (G_n^j)^{-1}U(t)|\nabla|^{\frac{d}{2}} \omega_n^j \|_{L^{\frac{2(2+\alpha)}{d}}_{t,x}} \leq \limsup_{n \to \infty} \| U(t)|\nabla|^{\frac{d}{2}} \omega_n^j \|_{L^{\frac{2(2+\alpha)}{d}}_{t,x}} \to 0 \quad \text{as } J \to \infty,
\]

we have \( \omega^J = 0 \) for every \( J \geq 1 \). So we have

\[
(G_n^j)^{-1}U(t)(|\nabla|^{\frac{d}{2}} u_n) \rightharpoonup U(t)|\nabla|^{\frac{d}{2}} \phi^j \quad \text{weakly in } L^{\frac{2(2+\alpha)}{d}}_{t,x} \quad \text{as } n \to \infty.
\]

Then following lemma gives the conclusion.

**Lemma 2.9** (Lemma 3.63 in [25]). Let \( \{v_n\} \) and \( v \) be in \( L^2 \). The following statements are equivalent.

1. \( v_n \rightharpoonup v \) weakly in \( L^2 \).
2. \( U(t)v_n \rightharpoonup U(t)v \) weakly in \( L^{\frac{2(2+\alpha)}{d}}_{t,x} \).

\[\square\]

**Proposition 2.10.** Let \( \{u_n\}_{n \geq 1} \) be a sequence of complex-valued radial functions satisfying

\[
\| u_n \|_{\dot{H}^{\frac{d}{2}}} \leq A \quad \text{and} \quad \| U(t)u_n \|_{S_n(\mathbb{R})} \geq \delta.
\]
Suppose \( \{\phi^j\}_{1 \leq j \leq J} \subset \dot{H}^\frac{d}{2} \) be linear profiles obtained in Lemma 2.7. Then there exist at least one linear profile \( \phi^{j_0} \) such that
\[
\|U(t)\phi^{j_0}\|_{\mathcal{S}_\alpha(\mathbb{R})} \geq C(A, \delta).
\]

**Proof.** By Lemma 2.7, we have
\[
U(t)(u_n)(x) = \sum_{1 \leq j \leq J} U(t - t^j_n)[(h^j_n)^{-d/2 + \alpha/2}\phi^j(\cdot/h^j_n)](x) + U(t)\omega^J_n(x)
\]
with
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|\nabla |\hat{\cdot}|^2 \dot{U}(t)\omega^J_n\|_{X_\alpha(\mathbb{R})} = 0,
\]
and for each \( J \)
\[
\lim_{n \to \infty} \left[ \|u_n\|^2_{L^2_{t,x}} - \left( \sum_{1 \leq j \leq J} \|\phi^j\|^2_{H^\frac{d}{2}} + \|\omega^J_n\|^2_{H^\frac{d}{2}} \right) \right] = 0.
\]
From the orthogonality (see Lemma 3.3 in [7]), we get
\[
\lim_{n \to \infty} \left\| \sum_{j=1}^J U(t - t^j_n)[(h^j_n)^{-d/2 + \alpha/2}\phi^j(\cdot/h^j_n)](x) \right\|^4_{\mathcal{S}_\alpha(\mathbb{R})} = \sum_{j=1}^J \|U(t)(\phi^j)(x)\|^4_{\mathcal{S}_\alpha(\mathbb{R})}
\]
for every \( J \geq 1 \). However,
\[
\limsup_{n \to \infty} \left\| U(t)(u_n)(x) - \sum_{j=1}^J U(t - t^j_n)[(h^j_n)^{-d/2 + \alpha/2}\phi^j(\cdot/h^j_n)](x) \right\|_{\mathcal{S}_\alpha(\mathbb{R})} \\
\leq \limsup_{n \to \infty} \left\| U(t)\omega^J_n \right\|_{\mathcal{S}_\alpha(\mathbb{R})} \leq \limsup_{n \to \infty} \|\nabla |\hat{\cdot}|^2 \dot{U}(t)\omega^J_n\|_{X_\alpha(\mathbb{R})} \to 0 \text{ as } J \to \infty.
\]
So we obtain
\[
\limsup_{n \to \infty} \|U(t)(u_n)(x)\|^4_{\mathcal{S}_\alpha(\mathbb{R})} = \sum_{j=1}^J \|U(t)(\phi^j)(x)\|^4_{\mathcal{S}_\alpha(\mathbb{R})}.
\]
And Strichartz estimate gives
\[
\sum_{j=1}^J \|U(t)(\phi^j)(x)\|^4_{\mathcal{S}_\alpha(\mathbb{R})} \leq C \left( \sup_{j \geq J} \|U(t)(\phi^j)(x)\|^2_{\mathcal{S}_\alpha(\mathbb{R})} \right) \sum_{j \geq 1} \|\phi^j\|^2_{H^\frac{d}{2}}.
\]
Since \( \sum_{j \geq 1} \|\phi^j\|^2_{H^\frac{d}{2}} \leq \limsup_{n \to \infty} \|u_n\|^2_{H^\frac{d}{2}} \leq A^2 \), we have
\[
\sup_{j \geq 1} \|U(t)(\phi^j)(x)\|^2_{\mathcal{S}_\alpha(\mathbb{R})} \geq \frac{\delta^4}{A^2}.
\]
In particular, we can find \( j_0 \) such that
\[
\|U(t)(\phi^{j_0})(x)\|^2_{\mathcal{S}_\alpha(\mathbb{R})} \geq \frac{\delta^4}{A^2}.
\]
\[\square\]
Proposition 2.11. Let \( \{u_n\}_{n \geq 1} \) be a sequence of complex-valued radial functions bounded in \( \dot{H}^{\frac{d}{2}} \). Suppose \( \{\phi^j\}_{1 \leq j \leq J} \subset \dot{H}^{\frac{d}{2}} \) be linear profiles obtained in Lemma 2.7. Then for each \( J \),

\[
\lim_{n \to \infty} \left( E(u_n) - \sum_{1 \leq j \leq J} E(U(t_n^{j})(h_n^{j} - \frac{d}{2} + \frac{1}{2})\phi^j(c/h_n^{j}))(x) - E(\phi^j) \right) = 0.
\]

Proof. For the power type we refer the readers to the proof in [21]. We only consider the Hartree case. Also see [26] for NLS with Hartree nonlinearity.

Thanks to the kinetic energy decoupling (2.10), it suffices to show

\[
\lim_{n \to \infty} \left( \int |u_n|^2 |x|^{-2\alpha} |u_n|^2 dx - \sum_{1 \leq j \leq J} \int |G_n^j(\phi^j)|^2 |x|^{-2\alpha} |G_n^j(\phi^j)|^2 dx \right.
\]

\[
- \int |\omega_n^j|^2 |x|^{-2\alpha} |\omega_n^j|^2 dx \bigg) = 0.
\]

We first prove

\[
\lim_{n \to \infty} \left( \int |u_n|^2 |x|^{-2\alpha} |u_n|^2 dx - \int |u_n - G_n(\phi^1)|^2 |x|^{-2\alpha} |u_n - G_n(\phi^1)|^2 dx \right.
\]

\[
- \int |G_n(\phi^1)|^2 |x|^{-2\alpha} |G_n(\phi^1)|^2 dx \bigg) = 0.
\]

Then repeated arguments give the conclusion.

When \( \lim_{n \to \infty} \frac{t_n^1}{(h_n^1)^{\alpha}} \) = \( \infty \), we obtain

\[
\lim_{n \to \infty} \int |G_n(\phi^1)|^2 |x|^{-2\alpha} |G_n(\phi^1)|^2 dx = \lim_{n \to \infty} \|G_n(\phi^1)\|_{L^{2d/\alpha}}^4 = \lim_{n \to \infty} \|U(t_n^1(\alpha))\|_{L^{2d/\alpha}}^4 = 0
\]

by using Hölder inequality, fractional integration, scaling and Lemma 2.6. Similarly, one can prove

\[
\lim_{n \to \infty} \left( \int |u_n|^2 |x|^{-2\alpha} |u_n|^2 dx - \int |u_n - G_n(\phi^1)|^2 |x|^{-2\alpha} |u_n - G_n(\phi^1)|^2 dx \right) = 0.
\]

Now we handle the case \( \lim_{n \to \infty} \frac{t_n^1}{(h_n^1)^{\alpha}} \) < \( \infty \). By taking subsequence we may assume that \( \lim_{n \to \infty} \frac{t_n^1}{(h_n^1)^{\alpha}} = t_{\infty} \). Let \( S_n^1(u_n) := (h_n^1)^{\frac{d}{2} - \frac{\alpha}{2}} u_n(h_n^1 \cdot) \). Then we have

\[
S_n^1(u_n) \to U(t_{\infty})\phi^1 \text{ weakly in } \dot{H}^{\frac{d}{2}} \text{ as } n \to \infty,
\]

and \( G_n^1(S_n^1(\phi^1)) \to U(t_{\infty})\phi^1 \text{ strongly in } L^{\frac{2d}{\alpha}} \text{ as } n \to \infty. \)
The scaling symmetry yields
\[
\int |u_n|^2(|x|^{-2\alpha} + |u_n|^2)dx - \int |u_n - G_n^1(\phi^1)|^2(|x|^{-2\alpha} + |G_n^1(\phi^1)|^2)dx
- \int |G_n^1(\phi^1)|^2(|x|^{-2\alpha} + |G_n^1(\phi^1)|^2)dx
= \int \left| S_n^1(u_n) - G_n^1(\phi^1) \right|^2(|x|^{-2\alpha} + |S_n^1(u_n) - G_n^1(\phi^1)|^2)dx
- \int \left| G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |G_n^1(S_n^1(\phi^1))|^2)dx
\]
=: I_n + II_n + III_n,
where
\[
I_n = \int \left| S_n^1(u_n) - U(t_\infty)\phi^1 \right|^2(|x|^{-2\alpha} + |S_n^1(u_n) - U(t_\infty)\phi^1|^2)dx
- \int \left| S_n^1(u_n) - U(t_\infty)\phi^1 \right|^2(|x|^{-2\alpha} + |S_n^1(u_n) - U(t_\infty)\phi^1|^2)dx
- \int \left| U(t_\infty)\phi^1 \right|^2(|x|^{-2\alpha} + |U(t_\infty)\phi^1|^2)dx,
\]
\[
II_n = \int \left| S_n^1(u_n) - G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |S_n^1(u_n) - G_n^1(S_n^1(\phi^1))|^2)dx
- \int \left| S_n^1(u_n) - G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |S_n^1(u_n) - G_n^1(S_n^1(\phi^1))|^2)dx
- \int \left| G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |G_n^1(S_n^1(\phi^1))|^2)dx
\]
\[
III_n = \int \left| G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |G_n^1(S_n^1(\phi^1))|^2)dx
- \int \left| G_n^1(S_n^1(\phi^1)) \right|^2(|x|^{-2\alpha} + |G_n^1(S_n^1(\phi^1))|^2)dx
\]
\[
I_n \to 0 \text{ by Lemma 2.12 below. And by using Hölder inequality, fractional integration and Lemma 2.6 again, we also obtain}
\[
\lim_{n \to \infty} (II_n + III_n) = 0.
\]

\[\square\]

**Lemma 2.12.** Let \(\{f_n\}\) be bounded sequence in \(\dot{H}^{\frac{\alpha}{2}}\). If \(f_n\) weakly converges to \(f\), then for some subsequence \(\{f_n\}\),
\[
\int |f_n|^2(|x|^{-2\alpha} + |f_n|^2) - |f_n - f|^2(|x|^{-2\alpha} + |f_n - F|^2) - |f|^2(|x|^{-2\alpha} + |f|^2)dx
\]
\[
\to 0 \text{ as } n \to \infty.
\]

**Proof.** Assume that \(\|\nabla\frac{\alpha}{2} f_n\|_{L^2} \leq M\) for all \(n \geq 1\). Since \(C_0^\infty\) is dense in \(\dot{H}^{\frac{\alpha}{2}}\), one can find \(\beta \in C_0^\infty\) such that \(\|\beta - f\|_{\dot{H}^{\frac{\alpha}{2}}} < \frac{\varepsilon}{12C(1+M)^2}\) for some constant \(C\). And since the multiplication operator
Then by using Hölder’s inequality and Sobolev embedding, we have a linear profile \( \phi \) and its associated data \( \{t_n, \phi_n\} \) as obtained in Lemma 2.7. Then by refining the sequence \( \{t_n\} \) to \( \{t^{\epsilon}_n\} \) and using the standard time-translation and absorbing error we may assume that for each \( j \) the sequence \( \{t^{\epsilon}_n\} \) converges to \( t^j \in [-\infty, +\infty) \). By using the standard time-translation and absorbing error we may assume that \( t^j := 0 \) and either \( t^{\epsilon}_n := 0 \) or \( t^{\epsilon}_n \to \pm\infty \).

As stated in [19] the nonlinear profiles \( v^j : I^j \times \mathbb{R}^d \to \mathbb{C} \) associated with \( \phi^j \) and \( t^j \) always exist and they can be summarized as follows: If \( t^{\epsilon}_n = 0 \), then \( v^j \) is the maximal solution to (1.1) with initial data \( v^j(0) = \phi^j \). If \( t^{\epsilon}_n \to \pm\infty \), then \( v^j \) is the maximal solution to (1.1) that scatters forward/backward in time to \( U(t)\phi^j \).

**Definition 2.13.** Let \( \{(h_n, t_n)\} \) be a family of parameters and \( \{t_n\} \) have a limit in \([0, \infty) \). Given a linear profile \( \phi \in \dot{H}^{\frac{d}{2}} \) with \( \{(h_n, t_n)\} \), we define the nonlinear profile associated with it to be the maximal solution \( v \) to (1.1) which is in \( C([-T_{\min}, T_{\max}); \dot{H}^{\frac{d}{2}}] \) satisfying an asymptotic condition: For the sequence \( \{t_n\} \),

\[
\lim_{n \to \infty} \|U(t_n)\phi - v(t_n)\|_{H^{\frac{d}{2}}} = 0.
\]

**Remark 2.** Let \( \{u_n\}_{n \geq 1} \) be a sequence of complex-valued radial functions bounded in \( \dot{H}^{\frac{d}{2}} \) and \( \{\phi^j\}_{1 \leq j \leq J} \subset \dot{H}^{\frac{d}{2}} \) be the corresponding linear profiles obtained in Lemma 2.7. Then by refining subsequence and using diagonal argument we may assume that for each \( j \) the sequence \( \{t^{\epsilon}_n\} \) converges to \( t^j \in [-\infty, +\infty) \). By using the standard time-translation and absorbing error we may assume that \( t^j := 0 \) and either \( t^{\epsilon}_n := 0 \) or \( t^{\epsilon}_n \to \pm\infty \).

As stated in [19] the nonlinear profiles \( v^j : I^j \times \mathbb{R}^d \to \mathbb{C} \) associated with \( \phi^j \) and \( t^j \) always exist and they can be summarized as follows: If \( t^{\epsilon}_n = 0 \), then \( v^j \) is the maximal solution to (1.1) with initial data \( v^j(0) = \phi^j \). If \( t^{\epsilon}_n \to \pm\infty \), then \( v^j \) is the maximal solution to (1.1) that scatters forward/backward in time to \( U(t)\phi^j \).
3. Energy concentration

In this section we show Theorems 1.1 and 1.3 by following the arguments as in [24] and [22], respectively.

3.1. Unconfined kinetic energy: Proof of Theorem 1.1 Let $\beta$ be a $C^\infty_0$-bump function which is 1 for $|x| \leq 1$ and 0 for $|x| > 1$. Then we have from Lemma 4.2 and mass conservation that

$$
\|\beta(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}}|_{L^{\frac{2d}{d-\alpha}}} \leq C\|\nabla|\frac{\alpha}{d}|f(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}} + C\|\nabla|\frac{\alpha}{d}|u\|_{L^{\frac{2d}{d-\alpha}}}
\leq C\|\nabla|\frac{\alpha}{d}|(\beta(\cdot/R)u)\|_{L^{\frac{2d}{d-\alpha}}} + C\|\nabla|\frac{\alpha}{d}|u\|_{L^{\frac{2d}{d-\alpha}}}
\leq CR^{-\frac{\alpha}{d}}\|u\|_{L^{\frac{2d}{d-\alpha}}} + C\|\nabla|\frac{\alpha}{d}|u\|_{L^{\frac{2d}{d-\alpha}}}
\leq C(R^{-\frac{\alpha}{d}}\|\varphi\|_{L^2}^2) + C\|\nabla|\frac{\alpha}{d}|u\|_{L^{\frac{2d}{d-\alpha}}}
\leq A + \||\nabla|\frac{\alpha}{d}|u\|_{L^{\frac{2d}{d-\alpha}}}
$$

for some $A = A(R, \|\varphi\|_{L^2})$. Using the endpoint Sobolev inequality (Proposition 2 of [10]) and real interpolation [2] that

$$
|x|^{\frac{d-1}{2}}|f(x)| \leq C\|f\|_{B^{\frac{\alpha}{d}}_{2,1}} \leq C\|f\|_{B^{\frac{\alpha}{d}}_{2,1}} \||\nabla|\frac{\alpha}{d}|f\|_{L^2}^\frac{1}{2},
$$

we have

$$
\| (1 - \beta(\cdot/R))u \|_{L^{\frac{2d}{d-\alpha}}}^2 = \int (1 - \beta(\cdot/R))|u|^{\frac{2d}{d-\alpha}}|u|^2 dx \leq CR^{-\frac{\alpha}{d}}\|u\|_{L^2}^{2(d-1)}\||\nabla|\frac{\alpha}{d}|u\|_{L^2}^\frac{2}{d-\alpha} \leq A\||\nabla|\frac{\alpha}{d}|u\|_{L^2}^{\frac{2}{d-\alpha}},
$$

where $\|f\|_{B^{\frac{\alpha}{d}}_{2,1}} := \sum N^{\frac{d}{2}}\|f_N\|_{L^2}$ is the homogeneous Besov norm.

On the other hand, for Hartree type we have that

$$
\int V(u)|\beta(x/R)u|^{2} dx
\leq C\|u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|\beta(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}}^2
\leq C\|\beta(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}}^2 + C\|(1 - \beta(\cdot/R))u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|\beta(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}}^2
\leq A + \||\nabla|\frac{\alpha}{d}|u\|_{L^2(|x| \leq 2R)}^2 + A\||\nabla|\frac{\alpha}{d}|u\|_{L^2(|x| \leq 2R)}^2
$$

and

$$
\int V(u)|(1 - \beta(\cdot/R))u|^{2} dx
\leq C\|u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|(1 - \beta(\cdot/R))u\|_{L^{\frac{2d}{d-\alpha}}}^2
\leq C\|(1 - \beta(\cdot/R))u\|_{L^{\frac{2d}{d-\alpha}}}^4 + C\|(1 - \beta(\cdot/R))u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|\beta(\cdot/R)u\|_{L^{\frac{2d}{d-\alpha}}}^2
\leq A\||\nabla|\frac{\alpha}{d}|u\|_{L^2}^2 + A\||\nabla|\frac{\alpha}{d}|u\|_{L^2}^2 + A\||\nabla|\frac{\alpha}{d}|u\|_{L^2(|x| \leq 2R)}^2
$$

and
Let \( y(t) = \| \nabla \tilde{u}(t) \|_{L^2}^2 \) and \( z(t) = \| \nabla \tilde{u}(t) \|_{L^2(\|x\| \leq 2R)}^2 \). Then from the above estimates we have
\[
y(t) \leq C \left\{ A + \varepsilon^{\alpha} + \frac{y^{d-1}}{\alpha} \right\} + E(\phi).
\]

Since \( \limsup_{t \to T^*} y(t) = +\infty \) and \( d > \alpha + 1 \) for power type \((d > 2\alpha \text{ for Hartree type})\), we conclude that \( \limsup_{t \to T^*} z(t) = +\infty \).

If \( u(t) \in L^\infty \) for all \( t < T^* \), then since \( \| \beta(\cdot / R)u(t) \|_{L^{\frac{2d}{d-\alpha}}} \leq CR^{\frac{d-\alpha}{\alpha}} \| u \|_{L^\infty(\|x\| \leq 2R)}, \) by replacing \( \| \nabla \tilde{u}(t) \|_{L^2(\|x\| \leq 2R)}^2 \) with \( \| u \|_{L^\infty(\|x\| \leq 2R)} \) in the above estimates we get the desired.

### 3.2. Confined kinetic energy: Proof of Theorem 1.3

Choose a sequence \( t_n \to T^* \) and let \( u_n \) be the solution on \([0, T^* - t_n)\) to (1.1) with initial data \( u(t_n) \). Then since \( \sup_{0 < t < T^*} \| \nabla \tilde{u}(t) \|_{L^2} =: M < +\infty \), by Lemma 2.7 we can decompose each \( u_n(0) \) by
\[
u_n(0) = \sum_{j=1}^{J} G_n^j \phi_j^j + \omega_n^j.
\]

We denote the symmetry operator \( g_n^j \) by \( g_n^j f(t, x) = (\tilde{h}_n^j)^{-\frac{d-\alpha}{\alpha}} f(t / (\tilde{h}_n^j)^{\alpha}, x / \tilde{h}_n^j) \). Then \( G_n^j \phi_j^j = g_n^j U(t_n^j) \phi_j^j \). Let \( \phi_j : I^j \times \mathbb{R}^d \to \mathbb{C} \) be nonlinear profile associated with \( \phi_j \) and \( (\tilde{h}_n^j, t_n^j) \) as stated in Remark 2. For each \( j, n \geq 1 \), we define \( \nu_n^j : I^j_n \times \mathbb{R}^d \to \mathbb{C} \) by
\[
\nu_n^j(t) := g_n^j \nu_j^j (\cdot + t_n^j)(t),
\]
where \( I^j_n = \{ t \in \mathbb{R} : (\tilde{h}_n^j)^{-\alpha} t + t_n^j \in I^j \} \). Then \( \nu_n^j \) is also a solution to (1.1) with initial data \( \nu_n^j(0) = g_n^j \nu_j^j (h_n^j) \) and maximal time interval \( I^j_n = (- T_{n,j}^-, T_{n,j}^+) \) for \( 0 < T_{n,j}^-, T_{n,j}^+ < +\infty \). By the kinetic energy decoupling (2.10), there exists \( J_0 = J_0(\delta_0) \geq 1 \) such that
\[
\| \nabla \tilde{u}^j \|_{L^2} \leq \delta_0 \text{ for all } j \geq J_0.
\]

For sufficiently small \( \delta_0 \), Lemma 2.4 yields that \( \nu_n^j \) are global and satisfy that
\[
\| \nabla \tilde{u}^j \|_{L^2} + \| \nu_n^j \|_{S_n(\mathbb{R})} \leq C \| \nabla \tilde{u}^j \|_{L^2}.
\]

Now we can find a so-called bad profile \( \phi_{j_0} \), \( 1 \leq j_0 < J_0 \) such that
\[
\limsup_{n \to \infty} \| u_{j_0} \|_{S_n([0, T^* - t_n])} = +\infty,
\]
Proof of (3.2). We will actually show that

\[ \limsup_{n \to \infty} \| v_n^j \|_{S_u([0,T_n^+]))} = +\infty, \]

where \( T_n^+ = \min_{1 \leq j < J_0} (T^* - t_n, T_{n,j}^+) \). Suppose that \( \limsup_{n \to \infty} \| v_n^j \|_{S_u([0,T_n^+]))} < +\infty \) for all \( 1 \leq j < J_0 \). Then this implies that \( T^* - t_n \leq T_{n,j}^+ \) for all \( 1 \leq j < J_0 \) if \( n \) is large. If \( T_{n,j}^+ \leq T^* - t_n \) for some \( j \), then since \( \limsup_{n \to \infty} \| v_n^j \|_{S_u([0,T_{n,j}^+]))} < +\infty \), the maximality means that \( T_{n,j}^+ = +\infty \) for sufficiently large \( n \). This contradicts the fact \( T^* < +\infty \). Then from this together with (3.1) and (2.10) it follows that

\[ \sum_{j \geq 1} \| v_n^j \|_{S_u([0,T_n^+]))}^2 \leq C(1 + \sum_{j \geq J_0} \| \nabla |\hat{\phi}| \theta_j \|_{L^2}^2) \leq C(1 + M^2) \]

for any \( J \) and for sufficiently large \( n \). We now define functions \( u_n^j \) on \([0,T_{n,0}^+]\) approximating \( u_n \) by

\[ u_n^j = \sum_{j=1}^J v_n^j + U(t) \omega_n^j. \]

Since \( v^j \) are nonlinear profile associated with \((\phi^j, t_n^j)\), we have

\[ \| u_n^j(0) - u_n(0) \|_{\dot{H}^{\frac{1}{2}}} = \| \sum_{j=1}^J (v_n^j(t_n^j) - g_n^j U(t_n^j) \phi^j) \|_{\dot{H}^{\frac{1}{2}}} \leq \sum_{j=1}^J \| v_n^j(t_n^j) - U(t_n^j) \phi^j \|_{\dot{H}^{\frac{1}{2}}} \to 0 \]

as \( n \to \infty \). By (2.8) and (3.4) we also have

\[ \limsup_{n \to \infty} \| u_n^j \|_{S_u([0,T_n^+]))} \leq \limsup_{n \to \infty} (\| \sum_{j} v_n^j \|_{S_u([0,T_n^+]))} + \| U(t) \omega_n^j \|_{S_u([0,T_n^+]))}) \leq C(1 + M^2). \]

By the local well-posedness we deduce that

\[ \limsup_{n \to \infty} \| u_n^j \|_{L^\infty_{[0,T_n^+])} \dot{H}^{\frac{1}{2}} + \| \nabla |\hat{\phi}| u_n^j \|_{X_u([0,T_n^+]))} \leq C(M). \]

On the other hand, \( u_n^j \) satisfy that

\[ i \partial_t u_n^j = |\nabla|^{\alpha} u_n^j - V(u_n^j) u_n^j + e, \]

where \( e = e_1 + e_2 \),

\[ e_1 = V(u_n^j) - V(\sum_{j=1}^J v_n^j)(\sum_{j=1}^J v_n^j) \]

and

\[ e_2 = V(\sum_{j=1}^J v_n^j)(\sum_{j=1}^J v_n^j) - \sum_{j=1}^J V(v_n^j) v_n^j. \]
We first show that \( \limsup_{n \to \infty} \| | \nabla |^{\frac{d}{d-\alpha}} e_2 \|_{Y_\alpha([0,T^n_\ast])} = 0 \). In fact, from direct calculation we get that for power type

\[
e_2 = \frac{2\alpha}{d-\alpha} \text{Re} \sum_{j \neq j'} v_n^j v_n^j \int_0^1 |s| \sum_{j' \neq j} v_n^{j'} + v_n^j |^{\frac{4\alpha-2d}{d-\alpha}} (s \sum_{j' \neq j} v_n^{j'} + v_n^j) \, ds
\]

and for Hartree type

\[
e_2 = \sum_{j' \neq j} (|x|^{-2\alpha} |v_n^{j'}|^2) v_n^j + \sum_j \sum_{j' \neq j_2} (|x|^{-2\alpha} (v_n^{j_1} v_n^{j_2}^*) v_n^j.
\]

Since \( \alpha < d \leq 2\alpha \) for power type, we have

\[
\| | \nabla |^{\frac{d}{d-\alpha}} e_2 \|_{Y_\alpha([0,T^n_\ast])} \leq C \sum_{j \neq j'} \left( \| | \nabla |^{\frac{d}{d-\alpha}} (v_n^{j'} v_n^j) \|_{L_{\alpha}^{\frac{d}{d-\alpha}}([0,T^n_\ast])} \| \sum_{j=1}^J \| v_n^{j'} \|_{S_{\alpha}([0,T^n_\ast])}^{\frac{3\alpha-d}{3\alpha}} \right)
\]

\[
+ \left( \sum_{j=1}^J \| | \nabla |^{\frac{d}{d-\alpha}} (v_n^{j'} v_n^j) \|_{X_\alpha([0,T^n_\ast])} \right) \left( \sum_{j=1}^J \| v_n^{j'} \|_{S_{\alpha}([0,T^n_\ast])}^{\frac{4\alpha-2d}{d-\alpha}} \| v_n^j \|_{L_{\alpha}^{\frac{3\alpha}{d-\alpha}}([0,T^n_\ast])} \right)
\]

Thus the orthogonality \([2.9]\) gives

\[
\limsup_{n \to \infty} \| | \nabla |^{\frac{d}{d-\alpha}} e_2 \|_{L_{\alpha}^{\frac{d}{d-\alpha}}([0,T^n_\ast])} = 0.
\]

For Hartree type by the orthogonality \([2.9]\) and the argument used for the proof of Lemma 3.3 in \([7]\) one can easily get

\[
\limsup_{n \to \infty} \| | \nabla |^{\frac{d}{d-\alpha}} e_2 \|_{L_{\alpha}^{\frac{d}{d-\alpha}}([0,T^n_\ast])} = 0.
\]

Now let us consider \( e_1 \). Let \( V_n^J = \sum_{j=1}^J v_n^j \) and let us invoke that \( \mu = \frac{2d}{d-\alpha} \) for power type and \( \mu = 4 \) for Hartree type. Then we have

\[
\| | \nabla |^{\frac{d}{d-\alpha}} e_1 \|_{Y_\alpha([0,T^n_\ast])}
\]

\[
\leq C \left( \| | \nabla |^{\frac{d}{d-\alpha}} u_n^J \|_{X_\alpha([0,T^n_\ast])} + \| | \nabla |^{\frac{d}{d-\alpha}} V_n^J \|_{X_\alpha([0,T^n_\ast])} \right) \left( \| u_n^J \|^\mu \|_{S_{\alpha}([0,T^n_\ast])} \right) \left( \| V_n^J \|^\mu \|_{S_{\alpha}([0,T^n_\ast])} \right)
\]

\[
+ C \left( \| u_n^J \|^\mu \|_{S_{\alpha}([0,T^n_\ast])} \right) \left( \| V_n^J \|^\mu \|_{S_{\alpha}([0,T^n_\ast])} \right) \| | \nabla |^{\frac{d}{d-\alpha}} U(t) \omega_n^J \|_{X_\alpha([0,T^n_\ast])}.
\]

By \([2.8]\) we get

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| | \nabla |^{\frac{d}{d-\alpha}} e_1 \|_{Y_\alpha([0,T^n_\ast])} = 0.
\]

We apply Lemma \([2.5]\) with \( \tilde{u} = u_n^J \) and \( u = u_n \) to conclude that

\[
\| u_n \|_{S_{\alpha}([0,T^n_\ast-t_n])} < +\infty \text{ for sufficiently large } n.
\]

This contradicts that \( u \) blows up within finite time \( T^\ast \). \(\square\)
By reordering we may assume that \( \limsup_{n \to \infty} \| v^1_n \|_{S^m_{\alpha, (0, T^* - t_n)}} = +\infty \) and that there exists \( 1 \leq J_1 < J_0 \) such that
\[
\limsup_{n \to \infty} \| v^2_n \|_{S^m_{\alpha, (0, T^* - t_n)}} = \infty \quad (j \leq J_1) \quad \text{and} \quad \limsup_{n \to \infty} \| v^2_n \|_{S^m_{\alpha, (0, T^* - t_n)}} < \infty \quad (j > J_1).
\]
Then for each \( m, n \geq 1 \), there exist \( 1 \leq j(m, n) \leq J_1 \) and \( 0 < T^m_n < T^* - t_n \) such that
\[
(3.7) \quad \sup_{1 \leq j \leq J_1} \| v^j_n \|_{S^m_{\alpha, (0, T^m_n)}} = \| v_n^{j(m, n)} \|_{S^m_{\alpha, (0, T^m_n)}} = m.
\]
By using the pigeonhole principle and then reordering, we may assume that \( j(m, n) = 1 \) for infinitely many \( m, n \). Then by Theorem \( \ref{thm:local_wellposedness} \) there exists \( 0 \leq \tau^m_n \leq T^m_n \) such that
\[
(3.8) \quad \limsup_{m \to \infty} \limsup_{n \to \infty} \| \nabla |\tilde{\alpha} v^1_n(\tau^m_n)\|_{L^2} \geq \| \nabla |\tilde{\alpha} W_\alpha\|_{L^2} - \varepsilon \quad \text{for infinitely many } n.
\]
For any \( \varepsilon > 0 \) we can find \( m_0 = m_0(\varepsilon) \) such that
\[
\| \nabla |\tilde{\alpha} v^1_n(\tau^m_n)\|_{L^2} \geq \| \nabla |\tilde{\alpha} W_\alpha\|_{L^2} - \varepsilon \quad \text{for infinitely many } n.
\]
Passing to a subsequence we may have that
\[
(3.9) \quad \| v^1_n \|_{S^m_{\alpha, ([\tau^-_n, \tau^+_n])}} = \eta.
\]
Using local well-posedness (Lemma \( \ref{lem:local_wellposedness} \)) we get
\[
\| U(t) v^1_n(\tau^m_n) \|_{S^m_{\alpha, ([\tau^-_n - \tau^-m_n, \tau^+_n - \tau^-m_n])}} \geq C \eta \tilde{D}
\]
for some dimension-dependent constant \( \tilde{D} \). By Lemma \( \ref{lem:energy_concentration} \) there exists \( \tau^-_n - \tau^-m_n \leq s_n \leq \tau^+_n - \tau^-m_n \) such that
\[
(3.10) \quad \int_{|x| \leq \tilde{C} |T^* - t'_n|^{\frac{1}{\alpha}}} |U(s_n)| \nabla |\tilde{\alpha} v^1_n(\tau^m_n)|^2 dx \geq \tilde{C}^{-1},
\]
where \( \tilde{C} = \tilde{C}(d, M, \eta) \) and \( t'_n = t_n + s_n + \tau^m_n \).

From the definition of \( v^1_n \) and \( (3.10) \) we deduce that
\[
\int_{|y| \leq \tilde{C}(h'_n)^{-\frac{1}{\alpha}} |T^* - t'_n|^{\frac{1}{\alpha}}} |U(s_n(h^1_n))^{-\alpha} (|\nabla |\tilde{\alpha} v^1_n((h^1_n)^{-\alpha} \tau^m_n + t'_n, y)|^2) dy \geq \tilde{C}^{-1}.
\]
By applying Lemma \( \ref{lem:local_wellposedness} \) and rescaling we have
\[
(3.11) \quad \| |\tilde{\alpha} v^1_n(\tau^m_n)|^2 \|_{L^2} \to \int_{|x| \leq R_n} |U(s_n) v^1_n(\tau^m_n)|^2 dx \to 0
\]
for any sequence \( R_n \in (0, \infty) \) such that \( (T^* - t'_n)^{-\frac{1}{d}} R_n \to \infty \) as \( n \to \infty \). Let \( u_n^J \) be the approximate functions defined on \([0, T^*_{n_{\text{mo}}}]\) as above. Then in view of the proof of (3.2) and (3.7) we can deduce that

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|© u_n^J(s_n + \tau_{n_{\text{mo}}}) - u(t'_n)\|_{L^2} = 0.
\]

Using (2.9) and Corollary 2.8 we have

\[
\limsup_{n \to \infty} \langle \nabla| \hat{u}_n^J(s_n + \tau_{n_{\text{mo}}}), \nabla| \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}}) \rangle = \limsup_{n \to \infty} \|© \hat{u}_n^J(s_n + \tau_{n_{\text{mo}}})\|^2_{L^2}
\]

for all \( J \geq 1 \). Thus we obtain

\[
\limsup_{n \to \infty} \langle \nabla| \hat{u}_n^J(t'_n), \nabla| \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}}) \rangle = \limsup_{n \to \infty} \|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}})\|^2_{L^2}.
\]

From (3.9) and Strichartz estimate it follows that

\[
\|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}}) - U(s_n)\|_{L^2} \leq C \eta^{\mu - 2}.
\]

So, if \( \eta \) is sufficiently small, then we get

\[
\limsup_{n \to \infty} \langle \nabla| \hat{u}_n^J(t'_n), U(s_n)\|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}}) \rangle \geq \limsup_{n \to \infty} \|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}})\|^2_{L^2} - \eta^{D'},
\]

for some \( D' < \mu - 2 \). Therefore by Cauchy-Schwarz inequality and (3.8) we obtain that

\[
\limsup_{n \to \infty} \int_{|x| \leq R_n} \|© u(t'_n)\|^2 dx \geq \frac{(\limsup_{n \to \infty} \|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}})\|^2_{L^2} - \eta^{D'})^2}{\limsup_{n \to \infty} \|© \hat{v}_n^J(s_n + \tau_{n_{\text{mo}}})\|^2_{L^2}} \geq \|© W_\alpha\|^2_{L^2} - \varepsilon - 2\eta^{D'} + \eta^{2D'}/M^2.
\]

Since \( \varepsilon \) and \( \eta \) can be taken arbitrarily small, we get the desired result.

## 4. Proof of finite time Blowup

Let us denote \( \sup_{0 \leq t < T^*} \|© u(t)\|_{L^2} \) by \( M \) and \( \|© u\|_{L^2} \) by \( m \). We will show that \( T^* = T^*(\varphi, M) < +\infty \). From the regularity persistence it follow that if \( \varphi \in H^2 \), then \( u \in C([0, T^*); H^2) \) (this is the case for the power type since \( \alpha < d < 3\alpha \) and thus \( \frac{2\alpha}{d - \alpha} > 1 \)). Since the maximal existence time \( T^* = T^*(\varphi) \) is lower semi-continuous, that is, if \( \varphi_k \to \varphi \) in \( H^2 \), then \( T^*(\varphi) \leq \liminf_{k \to \infty} T^*(\varphi_k) \), we may assume that \( u \in C([0, T^*); H^2) \) and \( \varphi \) satisfies the condition (1.6).

### 4.1. Moment estimates.

**Proposition 4.1.** If \( \varphi \) satisfies the condition (1.5), then the solution \( u \in C([0, T^*); H^2) \) satisfies that for each \( t \in (0, T^*) \)

\[
\|© x u(t)\|_{L^2} \leq CMt + \|© x \varphi\|_{L^2}, \quad \|© x|\nabla|^{\alpha - 1} u(t)\|_{L^2} + \|© |x|^2 u(t)\|_{L^2} < +\infty.
\]
Proof. For a fixed radial bump function $\psi \in C_0^\infty$ with $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when $|x| \geq 2$ we denote $\psi(\frac{x}{\lambda})$ by $\psi_\lambda$ for $\lambda \geq 1$. Then we can define moments $m_{1,\lambda}, \tilde{m}_{1,\lambda}, m_{2,\lambda}$ by

$$m_{1,\lambda} := \langle x\psi_\lambda u; x\psi_\lambda u \rangle,$$

$$\tilde{m}_{1,\lambda} := \langle x\psi_\lambda \nabla u; x\psi_\lambda |\nabla u| \rangle,$$

$$m_{2,\lambda} := \langle |x|^2 \psi_\lambda u; |x|^2 \psi_\lambda u \rangle.$$

Differentiating $m_{1,\lambda}$ w.r.t $t$, we have

$$\frac{d}{dt} m_{1,\lambda}^2 = 2\text{Im} \langle |\nabla|^{-\frac{d}{2}} x\psi_\lambda(|\nabla|^\alpha u - V(u)u); |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla|^\alpha u \rangle = 2\text{Im} \langle |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla|^\alpha u; |\nabla|^{-\frac{d}{2}} x\psi_\lambda u \rangle$$

$$= 2 \sum_{j=1}^d \text{Im} \langle x_j \psi_\lambda, |\nabla|^\alpha u, x_j \psi_\lambda u \rangle \leq 2 \left( \sum_j \| x_j \psi_\lambda, |\nabla|^\alpha u \|_{L^2}^2 \right)^{\frac{1}{2}} m_{1,\lambda}.$$

In order to estimate the last term we use the following lemma.

**Lemma 4.2.** Let $\beta_\lambda(x) = \beta(\frac{x}{\lambda})$ for $\beta \in C_0^\infty$. If $s \geq 1$ for any $f \in H^{s-1}$ we have

$$\| [\beta_\lambda, |\nabla|^s] f \|_{L^2} \leq C_\beta \lambda^{-1} \| f \|_{H^{s-1}}.$$  

If $0 < s < 1$, then for any $f \in L^2$ we have

$$\| [\beta_\lambda, |\nabla|^s] f \|_{L^2} \leq C_\beta \lambda^{-s} \| f \|_{L^2}.$$

From the above lemma it follows that

$$\frac{d}{dt} m_{1,\lambda}^2 \leq 2\lambda \left( \sum_j \| x_j \psi_\lambda, |\nabla|^\alpha u \|_{L^2}^2 \right)^{\frac{1}{2}} m_{1,\lambda} \leq C \| u \|_{H^{s-1}} m_{1,\lambda} \leq C M m_{1,\lambda}$$

and thus $\frac{d}{dt} m_{1,\lambda} \leq C M$. Integrating over $[0, t]$, we have

$$m_{1,\lambda}(t) \leq C M t + \| x|\psi_\lambda\|_{L^2}.$$

Letting $\lambda \to +\infty$, by Fatou’s lemma we get the desired result.

Next we estimate $\tilde{m}_{1,\lambda}$ as follows.

$$\frac{d}{dt} \tilde{m}_{1,\lambda}^2 = 2\text{Im} \langle |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla|(|\nabla|^\alpha - V(u)u); |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla| u \rangle$$

$$= 2\text{Im} \langle |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla|^\alpha u; |\nabla|^{-\frac{d}{2}} x\psi_\lambda |\nabla| u \rangle - 2\text{Im} \langle x\psi_\lambda |\nabla| (V(u)u); x\psi_\lambda |\nabla| u \rangle$$

$$= 2 \sum_j \text{Im} \langle x_j \psi_\lambda, |\nabla|^\alpha |\nabla| u, x_j \psi_\lambda |\nabla| u \rangle$$

$$+ 2 \sum_j \text{Im} \langle (x_j \psi_\lambda |\nabla| |\nabla|^{-1}((\nabla V(u))u + V(u)\nabla u), x_j \psi_\lambda |\nabla| u \rangle$$

$$\leq C \| u \|_{H^{s-1}} \tilde{m}_{1,\lambda} + C \| x((\nabla V(u))u + V(u)\nabla u) \|_{L^2} \tilde{m}_{1,\lambda}.$$
Lemma 4.2 shows that $\|x\|((\nabla V(u))u + V(u)\nabla u)\|_{L^2} \leq C\|\|x\|^{2\alpha} - 2|d|2^{\alpha}|\nabla V(u)|u\|_{L^2} + \|\|x\|^{\alpha} - 2|d|2^{\alpha}\|\nabla u\|_{L^2}
\leq C\|\|x\|^{2\alpha} - 2|d|2^{\alpha}\|u\|_{H^{1+\frac{1}{2}}} + \|\|x\|^{2\alpha}|\nabla |u\|_{L^2}
\] and thus by integrating over $[0, t]$

$$\tilde{m}_{1, \lambda}(t) \leq \|x\|\nabla \varphi\|_{L^2} + C(1 + M)^{2\alpha} \int_0^t \|u(t')\|_{H^{1+\frac{1}{2}}} dt'.$$

If $V(u) = |x|^{-2\alpha} + |u|^2$, then from the fractional integration for radial function that

$$|x|^\delta(|x|^{-\gamma} |f|) \leq C\|\|x\|^\delta |f|\|_{L^1} \ (0 < \delta \leq \gamma < d - 1)$$

we get

$$\|x\|\nabla V(u)|u\|_{L^2} \leq \|\nabla V(u)\|_{L^2} + \|x|^{2\alpha} \|\nabla V(u)\|_{L^2}
\leq \|\nabla V(u)\|_{L^2} + \|x|^{2\alpha} \|\nabla V(u)\|_{L^2} + C\|\|x\|^2 |\nabla u\|_{L^2}
\leq C\|\|x\|^2 \|_{H^{1+\frac{1}{2}}}
$$

and

$$\|x\|\nabla u\|_{L^2} \leq \|V(u)\|_{L^\infty} \|\nabla u\|_{L^2} + \|x|^{2\alpha} \|\nabla u\|_{L^2} \leq C\|u\|^3_{H^{2\gamma}}.
$$

Thus

$$\tilde{m}_{1, \lambda}(t) \leq \|x\|\nabla \varphi\|_{L^2} + C\int_0^t \|u(t')\|^3_{H^{1+\frac{1}{2}}} dt'.$$

Fatou’s lemma yields the desired results.

Similarly to the estimate of $m_{1, \lambda}$ we have for $m_{2, \lambda}$ that

$$\frac{d}{dt} m_{2, \lambda} = 2 \text{Im}\langle |x|^2 \varphi |(\nabla |^\alpha u - (V(u))u)| |x|^2 \psi u \rangle
\leq 2 \text{Im}\langle |x|^2 \varphi |\nabla |^\alpha u, |x|^2 \psi u \rangle = 2 \text{Im}\langle \varphi x \cdot (|\nabla |^\alpha x + \alpha |\nabla |^\alpha - 2 |\nabla |u) , |x|^2 \psi u \rangle$$

$$= 2 \text{Im}\langle \varphi \psi x \cdot |\nabla |^\alpha x u, |x|^2 \psi u \rangle + 2 \alpha \text{Im}\langle \varphi x \cdot \nabla / |\nabla | |\nabla |^{\alpha - 1} u, |x|^2 \psi u \rangle
= 2 \sum_j \text{Im}\langle [x_j \varphi x, |\nabla |^\alpha x] j u, |x|^2 \psi u \rangle + 2 \alpha \text{Im}\langle \varphi x \cdot \nabla / |\nabla | |\nabla |^{\alpha - 1} u, |x|^2 \psi u \rangle.
$$

Lemma 4.2 shows that

$$\frac{d}{dt} m_{2, \lambda} \leq C\|\|x\|_{H^{\alpha - 1}} + \|\|x\| \nabla |^{\alpha - 1} u\|_{L^2} m_{2, \lambda} \leq C\|u\|_{H^{\alpha - 1}} + \|\|x\| \nabla |^{\alpha - 1} u\|_{L^2} m_{2, \lambda},$$
which implies that
\[ \| |x|^2 u(t)\|_{L^2} \leq \| |x|^2 \varphi\|_{L^2} + C \int_0^t (\|u\|_{H^{s-1}} + \| |\nabla|^{\alpha-1} u\|_{L^2}) \, dt'. \]
This completes the proof of Proposition \ref{prop}.

\[ \square \]

**Proof of Lemma \ref{lem}**. We show the first inequality. By Plancherel’s theorem it suffices to show that \( \|Tg\|_{L^2} \leq C \| (1 + |\xi|^{s-1})g \|_{L^2} \), where
\[ Tg(\xi) = \lambda^d \int \hat{\beta}(\lambda(\xi - \zeta))(|\zeta|^s - |\xi|^s)g(\zeta) \, d\zeta. \]

In fact,
\[
|Tg(\xi)| \leq s\lambda^d \int |\hat{\beta}(\lambda(\xi - \zeta))|(|\zeta|^{s-1} + |\xi|^{s-1})|\xi - \zeta||g(\zeta)| \, d\zeta \\
\leq \frac{s \lambda^d}{2} \int |\hat{\beta}(\lambda(\xi - \zeta))||\xi - \zeta||g(\zeta)| \, d\zeta + 2s \lambda^d \int |\hat{\beta}(\lambda(\xi - \zeta))||\zeta|^{s-1}|\xi - \zeta||g(\zeta)| \, d\zeta \\
= \frac{\lambda^{-s} s \lambda^d}{2} \int |\hat{\beta}(\lambda(\xi - \zeta))||\lambda(\xi - \zeta)||\xi - \zeta||g(\zeta)| \, d\zeta \\
+ 2s \lambda^d \int |\hat{\beta}(\lambda(\xi - \zeta))||\lambda(\xi - \zeta)||\zeta|^{s-1}g(\zeta)| \, d\zeta.
\]

Since \( \hat{\beta}(\xi)(|\xi|^\alpha + |\xi|) \) is integrable and \( s > 1 \), we get
\[
\|Tg\|_{L^2} \leq C\beta \lambda^{-1} \| (1 + |\xi|^{s-1})g \|_{L^2}.
\]

Similarly for \( 0 < s < 1 \) we have
\[
|Tg(\xi)| \leq \lambda^d \lambda^{-s} \int |\hat{\beta}(\lambda(\xi - \zeta))|\lambda(\xi - \zeta)|^{s-1}|g(\zeta)| \, d\zeta
\]

and thus
\[
\|Tg\|_{L^2} \leq C\beta \lambda^{-s} \|g\|_{L^2}.
\]

This completes the proof of lemma. \[ \square \]

**4.2. Virial argument.** Here we consider the virial inequality through the moment estimates above. Let us define two quantities associated with dilation and virial operators respectively by
\[ \mathcal{A}(u) := -\text{Im} \langle u, x \cdot \nabla u \rangle, \quad \mathcal{M}(v) := \langle |\nabla|^{1-\frac{\alpha}{2}}(xu); |\nabla|^{1-\frac{\alpha}{2}}(xu) \rangle. \]

From the regularity and moment estimates we can differentiate them w.r.t time.
\[
\frac{d}{dt} \mathcal{A}(u(t)) = \text{Re} \langle (|\nabla|^\alpha u - V(u)u), x \cdot \nabla u \rangle - \text{Re} \langle u, x \cdot \nabla (|\nabla|^\alpha u - V(u)u) \rangle.
\]

By integration by parts, we have
\[
\frac{d}{dt} \mathcal{A}(u(t)) = \text{Re} \langle x|\nabla|^\alpha u; \nabla u \rangle + d\langle u, V(u)u \rangle - \text{Re} \langle u, x \cdot \nabla (|\nabla|^\alpha u) \rangle \\
+ 2\text{Re} \langle u, x \cdot \nabla (V(u)u) \rangle.
\]
Using the identity \(x|\nabla|^{\beta} = |\nabla|^{\beta} x + \beta |\nabla|^{\beta - 2}\nabla\) for \(0 < \beta < 2\), we have

(4.3) \[ \frac{d}{dt} A(u(t)) = \alpha \langle u, |\nabla|^{\alpha} u \rangle + 2\text{Re} \langle u, x \cdot \nabla (V(u)) \rangle + d \langle u, V(u)u \rangle. \]

We first consider the power type. If \(V(u) = |u|^{2\alpha - d}\), then by direct calculation we get that

\[ 2\text{Re} \langle u, x \cdot \nabla (V(u)) \rangle = -(d + \alpha) \text{Re} \langle u, V(u)u \rangle. \]

Plugging this into (4.3), we have

(4.4) \[ \frac{d}{dt} A(u(t)) = \alpha \left( \langle u, |\nabla|^{\alpha} u \rangle - \langle u, V(u)u \rangle \right). \]

Now we consider the Hartree case \(V(u) = |x|^{-2\alpha} |u|^2\). Using integration by parts, we also get

\[ 2\text{Re} \langle u, x \cdot \nabla (V(u)) \rangle = \text{Re} \langle u, (x \cdot \nabla V(u))u \rangle - d \langle u, V(u)u \rangle. \]

Since \(d > 2\alpha + 1\), by direct differentiation we have

\[ \text{Re} \langle u, (x \cdot \nabla V(u))u \rangle = -2\alpha \langle u, V(u)u \rangle - 2\alpha \int |u(x)|^2 |x - y|^{-2\alpha - 1} |x - y| |u(y)|^2 \, dx \, dy. \]

In fact, from change of variables we deduce that

\[ 2\alpha \int |u(x)|^2 |x - y|^{-2\alpha - 1} y \cdot \frac{x - y}{|x - y|} |u(y)|^2 \, dx \, dy = \text{Re} \langle u, (x \cdot \nabla V(u))u \rangle. \]

So, we have

\[ \text{Re} \langle u, (x \cdot \nabla V(u))u \rangle = -\alpha \langle u, V(u)u \rangle. \]

Putting all together, we finally have

(4.5) \[ \frac{d}{dt} A(u(t)) = \alpha \left( \langle u, |\nabla|^{\alpha} u \rangle - \langle u, V(u)u \rangle \right). \]

To deal with the RHS of (4.4) and (4.5) we introduce the following lemma to be shown in appendix.

**Lemma 4.3.** If \(E(\varphi) \leq (1 - \delta_0)E(W_\alpha)\) and \(\|\nabla |\varphi|^2\|_{L^2} \geq \|\nabla |\varphi| W_\alpha\|_{L^2}\) for some \(0 < \delta_0 < 1\), then there exists a positive \(\overline{\delta}\) such that \(\|\nabla |\varphi|^2(u(t))\|_{L^2} \geq (1 + \overline{\delta})\|\nabla |\varphi| W_\alpha\|_{L^2}\) for all \(t \in (0, T^*)\).

From Proposition ?? it follows that

\[ \langle u, |\nabla|^{\alpha} u \rangle - \langle u, V(u)u \rangle = \mu \left( \frac{1}{2} \|\nabla |\varphi|^2 u\|_{L^2}^2 - \frac{1}{\mu} \int V(u)|u|^2 \, dx \right) - \frac{\mu - 2}{2} \|\nabla |\varphi|^2 u\|_{L^2}^2 \]

\[ = \mu E(\varphi) - \frac{\mu - 2}{2} \|\nabla |\varphi|^2 u\|_{L^2}^2 \leq \mu E(W_\alpha) - \frac{\mu - 2}{2} (1 + \overline{\delta}) \|\nabla |\varphi| W_\alpha\|_{L^2}^2 \]

\[ = -\frac{\mu - 2}{2} \overline{\delta} C_{d,\alpha} \mu \leq -\delta_0 < 0. \]

Thus integrating (4.4) and (4.5) over \([0, t]\) we get

(4.6) \[ A(u(t)) \leq A(\varphi) - \alpha \delta_0 t. \]
On the other hand by differentiating $\mathcal{M}$ and using the identity $x|\nabla|^\beta = |\nabla|^\beta x + \beta |\nabla|^{\beta - 2} \nabla$ for $\beta = \alpha$ and $2 - \alpha$, we get
\[
\frac{d}{dt} \mathcal{M}(u) = 2 \text{Im} \langle |\nabla|^{1-\alpha} x (|\nabla|^\alpha u - V(u) u); |\nabla| x u \rangle \\
= 2 \text{Im} \langle |\nabla|^{1-\alpha} x |\nabla|^\alpha u; |\nabla| x u \rangle - 2 \text{Im} \langle x V(u) u; |\nabla|^{2-\alpha} x u \rangle \\
= -2\alpha \text{Im} \langle x \cdot \nabla u \rangle - 2 \text{Im} \langle |x|^2 V(u) u; |\nabla|^{2-\alpha} u \rangle - 2(2 - \alpha) \text{Im} \langle x V(u) u; |\nabla|^{-\alpha} \nabla u \rangle \\
= 2 \alpha \mathcal{A}(u) - 2 \text{Im} \langle |x|^2 V(u) u; |\nabla|^{2-\alpha} u \rangle - 2(2 - \alpha) \text{Im} \langle x V(u) u; |\nabla|^{-\alpha} \nabla u \rangle.
\]

Since $|x|^2 u \in L^2$, $V(u) \in L^\infty$ and $u \in H^{1+\frac{\alpha}{2}}$, the second term of last line is at least well-defined. Actually, it is possible to get a better estimate as below.

If $V(u) = |u|^{\frac{2\alpha}{2-\alpha}}$, then since $\frac{4}{3} \leq \alpha < 2$ we have
\[
-2 \text{Im} \langle |x|^2 V(u) u; |\nabla|^{2-\alpha} u \rangle \leq 2 \||\nabla|^\alpha V(u)\||L^\infty| \||x|^2\|_{L^2} \||\nabla|^{2-\alpha} u\||L^2 \\
\leq CM \frac{2^{\frac{4-2\alpha}{\alpha}}}{\alpha} M^{\frac{2\alpha}{\alpha}} + \frac{4-2\alpha}{\alpha} \||x|^2 u\||L^2^{2-\alpha}.
\]

From Lemma 4.2 it follows that
\[
-2 \text{Im} \langle |x|^2 V(u) u; |\nabla|^{2-\alpha} u \rangle \leq CM \frac{2^{\frac{4-2\alpha}{\alpha}}}{\alpha} M^{\frac{2\alpha}{\alpha}} + \frac{4-2\alpha}{\alpha} (M t + m_1)^{2-\alpha},
\]
where $m_1 = \||x|\varphi\||L^2$.

On the other hand, the last term is bounded by
\[
(4.7) \quad C \int |\nabla|^{-(\alpha - 1)} (| \cdot |^{-(\alpha - 1)} g) (x) f(x) \, dx = C \int \int \frac{f(x) g(y)}{|x-y|^{-(d-(\alpha-1))}} |y|^{\alpha-1} \, dx dy,
\]
where $f = |\nabla|/|\nabla| u$ and $g = x |\nabla|^{-1} V(u) u$. For this we use Stein-Weiss inequality that
\[
(4.8) \quad \left| \int \int \frac{f(x) \overline{g(y)}}{|x|^{\theta_1} |x-y|^{\theta_2} |y|^{\theta_2}} \, dx dy \right| \leq C \||f||L^{p_1}||g||L^{p_2},
\]
provided that $1 < p_1, p_2 < \infty$, $\theta_1, \theta_2 \geq 0$, $0 < \theta < d$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{\theta_1 + \theta_2}{d} = 2$ and $\theta_1 < \frac{d}{p_1}$, $\theta_2 < \frac{d}{p_2}$. Let $p_1 = p_2 = 2$ and $\theta_1 = 0, \theta_2 = \alpha - 1, \theta = d - (\alpha - 1)$. Then (4.8) implies that
\[\text{(4.7)} \leq C m^2 M \frac{2\alpha}{d-\alpha}.
\]

These estimates lead us to
\[
(4.9) \quad \mathcal{M}(u(t)) \leq -\alpha^2 \epsilon_0 t^2 + \left( C(m, M) (Mt + m_1)^{3-\alpha} + (C m^2 M \frac{2\alpha}{d-\alpha} + \mathcal{A}(\varphi)) t \right) + \mathcal{M}(\varphi).
\]

We then consider the Hartree case. We follow the same strategy as in [6]. To begin with let us observe that
\[
2 \text{Im} \langle |x|^2 V(u) u; |\nabla|^{2-\alpha} u \rangle = \text{Im} \langle [|\nabla|^{2-\alpha}, g] u, u \rangle,
\]
where $g = |x|^2 V(u)$. Then by the commutator estimate of [6] one can get
\[
\||[|\nabla|^{2-\alpha}, g] u\||_{L^2} \leq C m \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^{2-\alpha}}.
\]
If $x \neq y$, then
$$|g(x) - g(y)| \leq |x - y| \int_0^1 |\nabla g(z_s)| ds, \ z_s = x + s(y - x).$$

Since $|\nabla g(z_s)| \leq 2|z_s|V(u) + 2\alpha |z_s|^2 \int |z_s - y|^{-(2\alpha + 1)} |u(y)|^2 dy$ and $d > 2\alpha + 2$, by (4.11) and Hardy-Sobolev inequality we have
$$|\nabla g(z_s)| \leq C|z_s|^{1-\alpha}(||x|^{-2\alpha + \alpha} |u|^2|_{L^1} + ||x|^{-2\alpha + 1 + \alpha} |u|^2||_{L^1}) \leq C|z_s|^{1-\alpha}\|\nabla \frac{\alpha}{2} u\|_{L^2}^2 \leq CM^2|z_s|^{1-\alpha}.$$ 

Thus $|g(x) - g(y)| \leq C|x - y|^{2-\alpha}M^2$, which implies that
$$\text{Im}(|x|^2V(u)|\nabla|^{2-\alpha}u|) \leq CM^2m^2.$$ 

Moreover, since by (4.11) $|x|^\alpha V(u) \leq C||x|^{-\alpha} |u|^2||_{L^2} \leq CM^2$, from (4.7) and (4.8) we have
$$\text{Im}\langle xV(u)|\nabla|^{2-\alpha}u\rangle \leq CM^2m^2.$$ 

Therefore we get
$$M(u(t)) \leq -\alpha^2\varepsilon_t t^2 + (Cm^2M^2 + A(\varphi))t + M(\varphi).$$

Since $M(u)$ is non-negative, by (4.9) and (4.10) we deduce that $T^* < +\infty$.

5. Appendix

We consider the characterization of maximizer of (1.1) only for Hartree equation. For this we study a minimization problem:

$$m = \inf_{u \in H^{1/2}_x} \int_{V(u)|u|^2dx=0} I(u), \quad I(u) := \frac{\|\nabla^{1/2} u\|^4_{L^2}}{\int V(u)|u|^2dx}.$$ 

This is equivalent to the constrained minimization problem:

$$m = \inf_{u \in H^{1/2}_x} \int_{V(u)|u|^2dx=1} J(u), \quad J(u) := \|\nabla^{1/2} u\|^4_{L^2}.$$ 

By Sobolev embedding one can observe that $m > 0$. Suppose that $u \in H^{1/2}_x$ is a minimizer of (5.2).

Then since $J$ is Fréchet differentiable on $H^{1/2}_x$, for any $\phi \in C_0^\infty$ $J$ should satisfy that
$$\frac{d}{d\varepsilon} J(v_\varepsilon) \bigg|_{\varepsilon=0} = 0, \text{ where } v_\varepsilon = \frac{u + \varepsilon \phi}{(\int V(u + \varepsilon \phi)|u + \varepsilon \phi|^2 dx)^{1/4}}.$$ 

By direct calculation we conclude that
$$\langle |\nabla|^\alpha u - m^{2/3}V(u)u, \phi \rangle = 0.$$ 

which means $u$ is a solution to $|\nabla|^\alpha w - m^{2/3}V(w)w$. By using a change of variables it is also a solution to (1.3). Thus the minimizer $u$ is $e^{i\theta} \lambda^{\frac{d-\alpha}{d}} W_\alpha(\lambda(x - x_0))$. Here we note that $W_\alpha \in H^{1/2}_x$ because $d > 2\alpha$. Now it remains to show that $J$ attains $m$ in $H^{1/2}_x$. In fact, the minimizer can be found in $H^{1/2}_rad$. Choose a minimizing sequence $u_j \in H^{1/2}_rad$ with $\int V(u_j)|u_j|^2 dx = 1$. Then it is bounded in $H^{1/2}_rad$ and thus we can take a subsequence converging weakly to $u$. According to Lemma 5.2 of [14]...
\[ \int V(u_j)|u_j|^2 \, dx \to \int V(u)|u|^2 \, dx \] due to the radial symmetry, which implies that \[ \int V(u)|u|^2 \, dx = 1. \] By the lower semi-continuity we deduce that \[ m \leq \left\| \nabla \frac{u}{2} \right\|_{L^2}^4 \leq \liminf_{j \to \infty} \left\| \nabla \frac{u_j}{2} \right\|_{L^2}^4 = m. \] Therefore \( u \) is a minimizer.

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