PARTIAL HYPERBOLICITY AND ERGODICITY IN
DIMENSION THREE.

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ABSTRACT. In [15] the authors proved the Pugh-Shub conjecture for partially hyperbolic diffeomorphisms with 1-dimensional center, i.e. stable ergodic diffeomorphism are dense among the partially hyperbolic ones. In this work we address the issue of giving a more accurate description of this abundance of ergodicity. In particular, we give the first examples of manifolds in which all conservative partially hyperbolic diffeomorphisms are ergodic.

1. INTRODUCTION

A diffeomorphism \( f : M \to M \) of a closed smooth manifold \( M \) is partially hyperbolic if \( TM \) splits into three invariant bundles such that two of them, the strong bundles, are hyperbolic (one is contracting and the other expanding) and the third, the center bundle, has an intermediate behavior (see the next section for a precise definition). An important tool in proving ergodicity of these systems is the accessibility property i.e. \( f \) is accessible if any two points of \( M \) can be joined by a curve that is a finite union of arcs tangent to the strong bundles.

In the last years, since the pioneer work of Grayson, Pugh and Shub [10], many advances have been made in the ergodic theory of partially hyperbolic diffeomorphisms. In particular, we want to mention the very recent works: of Burns and Wilkinson [4] proving that (essential) accessibility plus a bunching condition (trivially satisfied if center bundle is one dimensional) implies ergodicity and of the authors [15] obtaining the Pugh-Shub conjecture about density of stable ergodicity for conservative partially hyperbolic diffeomorphisms with one dimensional center. That is, ergodic diffeomorphisms contain an open and dense subset of the conservative partially hyperbolic ones. See [17] for a recent survey on the subject.

After these result a question naturally arises: Can we describe this abundance of ergodicity more accurately?

More precisely:
Question 1.1. Which manifolds support a non-ergodic partially hyperbolic diffeomorphism?

Somewhat surprisingly, studying this question we realize that, in dimension 3, there are strong obstructions for a partially hyperbolic diffeomorphism to fail to be ergodic. We conjecture that the answer to this question, in dimension 3, is that the only (orientable) such manifolds are the mapping tori of diffeomorphisms commuting with an Anosov one, namely, mapping tori of Anosov diffeomorphisms, $\mathbb{T}^3$, and the mapping torus of $-id$ where $id$ is the identity map of $\mathbb{T}^2$.

If this conjecture is true, then the fact of being partially hyperbolic will automatically imply ergodicity in many manifolds. We prove this for a family of manifolds:

Theorem 1.2. Let $f: N \to N$ be a conservative partially hyperbolic $C^2$ diffeomorphism where $N \neq \mathbb{T}^3$ is a compact 3-dimensional nilmanifold. Then, $f$ is ergodic.

Recall that a 3-dimensional nilmanifold is a quotient of the Heisenberg group (upper triangular $3 \times 3$ matrices with ones in the diagonal) by a discrete subgroup. Sacksteder [20] proved that certain affine diffeomorphisms of nilmanifolds are ergodic. These examples are partially hyperbolic.

Some of our results apply to other manifolds and we obtain, for instance, that every conservative partially hyperbolic diffeomorphism of $\mathbb{S}^3$ (or $\mathbb{S}^1 \times \mathbb{S}^2$) is ergodic but is this probably a theorem about the empty set.

The structure of the proof goes as follows: first of all the results in [4, 15] imply that it is enough to prove, in order to obtain ergodicity, that $f$ satisfies the accessibility property. Again in [15] is proved that, in our setting, accessibility classes are either open or part of a laminating class. In Section 3 we show that either there are periodic tori with Anosov dynamics or this laminating extends to a true foliation without compact leaves and we show that this last case is impossible for nilmanifolds. In Section 4 we prove that such a torus is incompressible (in fact this result is proved in a more general setting). Since 3-dimensional nilmanifolds do not support invariant tori with Anosov dynamics we arrive to a dichotomy: either $f$ is accessible or the accessibility classes are leaves of a codimension one foliation of $M$ without compact leaves. On one hand, results of Plante and Roussarie about codimension one foliations of three manifolds plus its $f$-invariance imply that the foliation is given by “parallel” cylinders (Section 7). On the other hand, we exploit the exponential growth of (Section 5) unstable curves to obtain that $f$ should be semiconjugated to a two dimensional Anosov diffeomorphism (Section 6). These two last facts lead us to a contradiction (Section 7).

All the results of the paper, but the ones that are specific for nilmanifolds, can be summarized in the following theorem. Recall that accessibility implies ergodicity (see Section 2).
Theorem 1.3. Let $f : M \rightarrow M$ a partially hyperbolic diffeomorphisms of an orientable 3-manifold $M$. Suppose that $E^e$ are also orientable, $\sigma = s, c, u$, and that $f$ is not accessible. Then one of the following possibilities holds:

(1) $f$ has a periodic incompressible torus tangent to $E^s \oplus E^u$.

(2) $f$ has an invariant lamination $\Gamma(f)$, tangent to $E^s \oplus E^u$, that trivially extends to a foliation without compact leaves of $M$. Moreover, the leaves of the accessible boundary are periodic, have Anosov dynamics and periodic points are dense.

(3) there is a Reebless foliation tangent to $E^s \oplus E^u$.

The assumption on the orientability of the bundles and $M$ is not essential, in fact, it can be achieved by a finite covering. We do not know any examples satisfying (2) in the theorem above (see Question 3.1).

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2. Preliminaries

Let $M$ be a compact Riemannian manifold. In what follows we shall consider a partially hyperbolic diffeomorphism $f$, that is, a diffeomorphism admitting a non trivial $Tf$-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) with $x \in M$ verify:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. It is also required that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$.

It is a known fact that there are foliations $W^\sigma$ tangent to the distributions $E^\sigma$ for $\sigma = s, u$ (see for instance [3]). A set $X$ will be called $\sigma$-saturated if it is a union of leaves of $W^\sigma$, $\sigma = s, u$. The accessibility class $AC(x)$ of the point $x \in M$ is the minimal $s$- and $u$-saturated set containing $x$. The diffeomorphism $f$ has the accessibility property if the accessibility class of some $x$ is $M$. We shall see that there are manifolds whose topology implies the accessibility property for all partially hyperbolic diffeomorphisms, then ergodicity will follow from the result below:

Theorem 2.1 ([4],[15]). If $f$ is a volume preserving partially hyperbolic diffeomorphism with the accessibility property and $\dim E^c = 1$, then $f$ is ergodic.

The leaf of $W^\sigma$ containing $x$ will be called $W^\sigma(x)$, for $\sigma = s, u$. The connected component containing $x$ of the intersection of $W^s(x)$ with a small $\varepsilon$-ball centered at $x$ is the $\varepsilon$-local stable manifold of $x$, and is denoted by $W^s_\varepsilon(x)$.

Suppose that $V$ is an open invariant set saturated by $s$- and $u$-leaves ($su$-saturated) and call $\partial^e V$ the accessible (by center curves) boundary of $V$. Let us
observe that ∂V is also an su-saturated set, which means that it is a union of accessibility classes. None of these classes can be open, so, due to Proposition A.3. of [15], it is laminated by codimension-one manifolds tangent to Eσ ⊕ Eu. Let us mention that the leaves of this lamination are C1 (see [6]).

Observe also that the proof of Proposition A.5 of [15] shows in fact that periodic points are dense in the accessibility classes of ∂cV endowed with its intrinsic topology. In other words, periodic points are dense in each plaque of the lamina of ∂V.

We will call \( U(f) = \{ x \in M; AC(x) \text{ is open} \} \) and \( \Gamma(f) = M \setminus U(f) \). Then the accessibility property of f is equivalent to \( \Gamma(f) = \emptyset \).

3. The su-lamination \( \Gamma(f) \)

In this section we will study the case when \( \emptyset \subsetneq \Gamma(f) \subsetneq M \).

By taking a finite covering we can suppose that the bundles \( E^\sigma (\sigma = s, c, u) \) and \( M \) are orientable.

Let \( \Lambda \subset \Gamma(f) \) be an invariant sublamination. Then its accessible boundary from the complement consists of periodic lamina with Anosov dynamics and density of periodic points (see Section 2). This implies, thanks to the local product structure, that stable and unstable leaves of periodic points are dense in each laminae (see [7]) and, as a consequence, that the restriction of an iterated of f that fixes such a laminae is transitive (always with the intrinsic metric).

Suppose that \( \Gamma(f) \) has a compact leaf \( F \). If \( \dim(M) = 3 \), it is clear that \( F \) must be a torus. Since all the leaves of the invariant sublamination \( \Lambda = \bigcup_{n \in \mathbb{Z}} f^n(F) \) are compact we obtain by the comments of the previous section that there exists a periodic torus leaf with an Anosov dynamics. We deal with this case in section 4.

In [17] (Theorem 4.11, Problem 22 and commentary below) we have announced that \( \Gamma(f) \) has always a torus leaf. Unfortunately our proof has a gap and the following question, up to our knowledge, remains open even in the codimension one case.

**Question 3.1.** Let \( f : M \to M \) be an Anosov diffeomorphism on a complete riemannian manifold \( M \). Is it true that if \( \Omega(f) = M \) then \( M \) is compact?

Then in this section we will assume that \( \Gamma(f) \) has no compact leaves (in fact, for our purposes it would be sufficient to assume that there exists an f-invariant sublamination of \( \Gamma(f) \) without compact leaves).

Let \( \Lambda \) be a lamination and let \( V \) be a component of \( M \setminus \Lambda \) and call \( V_c \) to the completion of \( V \). \( V_c \) is a manifold with boundary (we can identify \( \partial V_c \) with the leaves of \( \Lambda \) accessible from \( V \) and, sometimes, we will identify \( V_c \) with its image under the inclusion \( i_V : V_c \to M \)).

If we take a nice enough covering \( \mathcal{U} = \{ U_1 \ldots U_n \} \) by charts of the lamination \( \Lambda \) we have three possible types of components of \( i_V^{-1}(U_j) \):
• $i_{-1}^{-1}(U_j)$ is diffeomorphic to the open three ball, in other words $U_j \subset V$.
• $i_{-1}^{-1}(U_j)$ is diffeomorphic to $[0, 1) \times D^2$, where $D^2$ is the open 2-dimensional disk.
• $i_{-1}^{-1}(U_j)$ is diffeomorphic to $[0, 1] \times D^2$.

Since $\mathcal{U}$ is finite there are only a finite number of components of the first two types. The union of all components of the third type forms a finite number of connected $I$-bundles ($I = [0, 1]$). The union of this $I$-bundles that are not relatively compact in $V_c$ will be called the interstitial region of $V$. The union of the other components (a connected and relatively compact region) will be called the gut of $V$. Of course this definitions depends on $\mathcal{U}$. For the basic concepts about laminations see [9].

Observe that, when $\Lambda$ is a sublamination of $\Gamma(f)$, $E^c$ is naturally defined in $V_c$.

We will prove that the complement of $\Lambda$ consists of $I$-bundles. For this end we need the following proposition.

**Proposition 3.2.** Let $f$ be a partially hyperbolic diffeomorphism of a 3-manifold $M$. Let $\Lambda \subset \Gamma(f)$ be an nonempty $f$-invariant sublamination without compact leaves and $V$ a component of $M \setminus \Lambda$. Then, $E^c$ is uniquely integrable in $V_c$.

**Proof.** Call $\mathcal{I}$ the interstitial region of $V_c$. By Remark 3.7 of [10] there exists $\delta > 0$ satisfying that if the central bundle is not uniquely integrable at $x$ and $\gamma$ is a center curve through it, there exists $n_\gamma > 0$ such that the length of $f^n(\gamma)$ or $f^{-n}(\gamma)$ is greater than $\delta$ for all $n > n_\gamma$. Take the charts of the lamination in such a way that the interstitial thickness is small enough (at least less than $\delta$). Then, it is not difficult to see that if $p$ is a periodic point in $\mathcal{I}$ then $E^c$ is uniquely integrable at $p$ and, furthermore, at every point of the connected component of $W^c(p) \cap V_c$ that contains $p$ (observe that the lamina in $\partial V_c$ are periodic). This is easily obtained because a maximal center interval inside $V_c$ containing the periodic point cannot growth greater than the interstitial thickness. In the non-periodic case the proof go through similar lines. Suppose that at $x \in \mathcal{I}$ there are two center curves. By projecting along the strong stable or strong unstable foliations we can suppose that these curves are contained in a local center stable or local center unstable manifold. We assume without lose of generality that both curves are in the local center unstable manifold of $x$. Then, we have a “triangle” formed by the two center curves and a strong unstable curve. Now take a small neighborhood $U$ in the interior of the triangle and very close to one of the vertices different of $x$. Since the “normal” direction to $U$ is stable and $\Omega(f) = M$ there exists a point $y \in U$ that returns very close to $U$ for an arbitrarily large $n > 0$. By observing that $y$ is the vertex of a triangle that is very near the previous one (and with $x$ as a vertex) we obtain that if $\gamma$ is the center curve joining $x$ and $y$, $f^n(\gamma)$, for $n$ large enough, have length greater than $\delta$ which is impossible because $f^n(y) \in \mathcal{I}$.

Now take $z \in \mathcal{I}$. Then the connected component of $W^c(z) \cap V_c$ containing $z$ (it is unique) has two extremes $z_i \in F_i$, with $F_i$ leaves of $\Lambda$, $i = 1, 2$. By taking an
iterate suppose that $F_1$ (and then $F_2$) is invariant by $f$. Since $f|_{F_1}$ is transitive we have that for an open and dense set of points of $F_1$ there is a unique center curve joining this point and a point in $F_2$ (observe that the same is true for $F_2$).

We will prove that $\mathcal{V} = \cup_{n \in \mathbb{Z}} f^n(\mathcal{I})$ is dense in $V_c$. If this is true the argument of Proposition 1.6 of [4] can be straightforward adapted to this case implying unique integrability of the center unstable and center stable bundles and the proposition.

Observe that $\mathcal{V}$ is su-saturated. $W^s(x), W^u(x) \subset \mathcal{V}$ for all $x \in W^c(y)$ with $y \in F_1$ with dense backward and forward orbit. The density of these kind of points implies the su-saturation of $\mathcal{V}$.

Then, if $\mathcal{V} \neq V_c$, $\mathcal{V}$ has nonempty boundary. This boundary consists of (complete) su-leaves. Close to it there are points of $\mathcal{V}$ that have center curves that go from $F_1$ to $F_2$. Since these center curves are contained in $\mathcal{V}$ we arrive to a contradiction.

**Theorem 3.3.** Let $f$ be a partially hyperbolic diffeomorphism of an orientable 3-manifold $M$. Suppose that $\emptyset \subseteq \Lambda \subset \Gamma(f)$ is an orientable and transversely orientable $f$-invariant sublamination without compact leaves such that $\Lambda \neq M$. Then, for all component $V$ of $M \setminus \Lambda$, $V_c$ is an I-bundle.

The proof of Theorem 3.3 involves well known techniques of codimension one foliations of three manifolds whose presentation exceeds the purpose of this paper. The reader may found them, for instance, in [5] and [12].

**Proof.** Proposition 3.2 implies the existence of center stable and center unstable foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ in $V_c$. These foliations cannot have Reeb components (see, for instance, [2]). Observe that, despite the non-compactness of $V_c$, the existence of a vanishing cycle implies the existence of a Reeb component. This is a consequence of the fact that center stable or center unstable disks (whose boundary does not intersect $\partial V_c$) cannot intersect $\partial V_c$ because there are no compact stable manifolds. Then the exploding disks of the vanishing cycle cannot go inside $\mathcal{I}$, which implies that are all contained in the gut. This gives us a Reeb component.

Now use the notation of Proposition 3.2. Since $F_i \setminus \mathcal{I}$ is compact and stable and unstable manifolds of periodic points of $f$ in $F_i$ are dense, $K_i = F_i \setminus \cup_{n \in \mathbb{Z}} f^n(\mathcal{I})$ is a compact totally disconnected set, $i = 1, 2$. Take a sequence of closed curves in $F_1, C_n \subset \mathcal{V}$, converging to a point $p \in K_1$. By taking the other extremes of the center curves we obtain a sequence of curves in $F_2$ that are also in the gut (it is clear that they cannot be in $\mathcal{I}$). Taking a convergent subsequence we have that the limit must be a point of $K_2$ otherwise $K_2$ would contain a nontrivial connected set. Then, for $n$ large enough, its image in $F_2, D_n$, bounds a disk. Consider the sphere formed by the center curves beginning in $C_n$ and ending in $D_n$ and the two disks of $F_1$ and $F_2$ bounded by these two curves. This sphere is
homotopically trivial because if this not were the case we would obtain a vanishing cycle. Then it bounds a region with trivial fundamental group (in fact, it bounds a ball). This implies that a center curve starting in $K_1$ must cut $K_2$ (if not we obtain again a vanishing cycle). Then $V_c$ is an $I$-bundle.

**Remark 3.4.** Observe that Theorem 3.3 implies that we can complete $\Lambda$ to a foliation of $M$ without compact leaves (in particular, Reebless) This implies that the inclusion injects the fundamental group of any leaf in the fundamental group of $M$. If $M$ is a nilmanifold then the fundamental group (of any and then, in particular) of a boundary leaf (periodic with Anosov dynamics and with density of periodic points) is isomorphic to a subgroup of a nilpotent group; then itself is nilpotent. Since the leaf is an orientable surface this implies that it is either a sphere, or a cylinder, or a torus. It is not difficult to prove that neither a sphere nor a cylinder support such a dynamics. Then, if $M$ is a 3-dimensional nilmanifold and $\emptyset \subset \Gamma(f) \subset M$ then $\Gamma(f)$ has a periodic torus leaf with Anosov dynamics. In next sections we will see that this situation is also impossible unless the nilmanifold $M$ is $T^3$.

4. Invariant tori

In this section we show that invariant tori for a diffeomorphism $f$ with restricted dynamics homotopic to Anosov are incompressible. Recall that a two-sided embedded closed surface $S \subset M^3$ other than the sphere is *incompressible* if and only if the homomorphism induced by the inclusion $i_\# : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, by the Loop Theorem, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \sim 0$ in $S$ (see, for instance, [11]).

**Theorem 4.1.** Let $f$ be a diffeomorphism of a three-dimensional orientable closed manifold $M$ and $T \subset M$ an $f$-invariant embedded torus. If $f|_T$ is isotopic to Anosov, then $T$ is incompressible.

**Proof.** Let $D^2 \subset M$ be an embedded disc such that $D \cap T = \partial D$ and $\partial D \sim 0$ in $T$. Then, by splitting $M$ along $T$ we obtain a manifold with boundary $\overline{M}$ such that $\partial \overline{M} = T_1 \cup T_2$ where $T_i, i = 1, 2$, are two tori and at least one of them, say $T_1$, verifies that the homomorphism induced by the inclusion $i_\# : \pi_1(T_1) \hookrightarrow \pi_1(M)$ is not injective.

The diffeomorphism $f$ naturally induces a diffeomorphism $\overline{f} : \overline{M} \to \overline{M}$ fixing $T_1$ (take $f^2$ if necessary) and such that $(\overline{f}|_{T_1})_\# : \pi_1(T_1) \to \pi_1(T_1)$ is a hyperbolic linear automorphism. Moreover, $(\overline{f}|_{T_1})_\#$ leaves $\ker(i_\#)$ invariant.

Now, since the eigenspaces of $(\overline{f}|_{T_1})_\#$ have irrational slope, it is not difficult to find $(j, 0), (0, k) \in \ker(i_\#)$ such that $j, k \in \mathbb{N} \setminus \{0\}$. Let $\alpha, \beta$ be two simple closed curves in $T_1$ such that $\alpha$ and $\beta$ meet only at a single point and $\alpha^2$ is in the class $(j, 0)$ and $\beta^k$ in the class $(0, k)$. If we delete all $\partial \overline{M}$ but a small tubular
neighborhood of $\alpha$, the Loop Theorem gives us a disc $D_\alpha$ embedded in $\overline{M}$ with boundary $\alpha$ and in the same way we obtain $D_\beta$ with boundary $\beta$ (this implies $(1,0),(0,1) \in \ker(i_\#)$ and, obviously, $\ker(i_\#) = \pi_1(T_1)$) Since we can assume that $D_\alpha$ and $D_\beta$ are transversal, this leads to a contradiction with the fact that the intersection between $\alpha$ and $\beta$ consist of one point.

\section{Growth of curves}

Let us recall the following consequence of Novikov’s and Reeb stability theorems (see [15]).

\begin{theorem}
Let $\mathcal{F}$ be a codimension one foliation of a compact 3-manifold $M$, and let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal covering of $M$, $\tilde{M}$. If $\mathcal{F}$ is Reebless, then either $M$ is $S^2 \times S^1$ and $\mathcal{F}$ is the product foliation or $\tilde{M} = \mathbb{R}^3$ and $\tilde{\mathcal{F}}$ is a foliation by planes. Moreover, in this last case, there is $\epsilon_0 > 0$ such that $L \cap B_{\epsilon_0}(x)$ is connected for every $x \in \tilde{M}$, for every leaf $L$ of $\tilde{\mathcal{F}}$.
\end{theorem}

We shall use Theorem 5.1 following the idea in [2], but in our setting. Given a compact manifold $M$ and $x \in \tilde{M}$, let us define $v_x(r) = \text{vol}(B(x,r))$. Notice that there is $C > 0$ such that $v_x(r) \leq C v_y(r)$ for any two point $x$ and $y$. So let us fix $x_0 \in \tilde{M}$ and call $v(r) = v_{x_0}(r)$.

\begin{proposition}
Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a three dimensional manifold. Assume that either $E^s \oplus E^u$ or $E^c \oplus E^u$ is tangent to an invariant foliation $\mathcal{F}$. Then there is a constant $C > 0$ such that if $I \subset \tilde{M}$ is an unstable arc then $\text{length}(I) \leq C v(\text{diam}(I))$.
\end{proposition}

\begin{proof}
First of all, observe that $\mathcal{F}$ is a Reebless foliation. This is a consequence of Proposition 2.1 of [2] (see also Theorem 4.1) because the boundary of a Reeb component should be periodic.

Given an $\epsilon > 0$ small enough ($\epsilon < \epsilon_0$ given by Theorem 5.1 and such that the disk of radius $\epsilon$ in a leaf is contained in a chart of the unstable foliation) there exists $\delta > 0$ verifying that the distance between the extremes of an unstable arc of length greater than $\delta$ is greater than $\epsilon$. If this were not the case we have two possibilities:

1. The two extremes are in different plaques of $\tilde{\mathcal{F}}$: this would imply the existence of a closed transversal in $\tilde{M}$ contradicting that $\mathcal{F}$ is Reebless.
2. The two extremes are in the same plaque of $\tilde{\mathcal{F}}$: this would imply that the leaf containing $I$ has nontrivial homotopy contradicting again that $\mathcal{F}$ is Reebless (by closing $I$ with a stable or a center arc we obtain an essential closed curve in the leaf).
\end{proof}
Given an unstable arc $I$ there exist at least $\text{length}(I)/2\delta$ disjoint 3-balls of radius $\varepsilon/2$ with center in a point of $I$. Clearly, the union of these balls is contained in $B(x, \text{diam}(I))$ for some $x$ which implies:

$$\text{length}(I) \leq \frac{2\delta}{\min\{v_y(\varepsilon/2); y \in M\}} v_x(\text{diam}(I)) \leq C v(\text{diam}(I)).$$

As nilmanifolds have polynomial growth of volume (of degree at most 4) we get:

**Corollary 5.3.** *In the setting of the proposition above, if $M$ is a nilmanifold then there is $C > 0$ such that $\text{length}(I) \leq C (\text{diam}(I))^4$.***

**6. Nilmanifolds in dimension 3**

Let $\mathcal{H}$ be the group of upper triangular matrices with ones in the diagonal. This is the non-abelian nilpotent simply connected three dimensional Lie group. We may identify $\mathcal{H}$ with the pairs $(x, t)$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$, $(x, t) \cdot (y, s) = (x + y, t + s + x_1y_2)$ and $(x, t)^{-1} = (-x, x_1x_2 - t)$. We have the projection $p : \mathcal{H} \to \mathbb{R}^2$, $p(x, t) = x$ which is also an homomorphism.

If we denote with $\mathfrak{h}$ the Lie algebra of $\mathcal{H}$, then we may also identify $\mathfrak{h}$ with the pairs $(x, t)$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$. We have the exponential map $\exp : \mathfrak{h} \to \mathcal{H}$ given by $\exp(x, t) = (x, t + \frac{1}{2}x_1x_2)$, $\exp$ is one to one and onto; and its inverse, the logarithm, $\log : \mathcal{H} \to \mathfrak{h}$ is given by $\log(x, t) = (x, t - \frac{1}{2}x_1x_2)$.

The homomorphisms from $\mathcal{H}$ to $\mathcal{H}$ are of the form $L(x, t) = (Ax, l(x, t))$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad l(x, y) = \alpha x_1 + \beta x_2 + \det(A) t + \frac{ac}{2} x_1^2 + \frac{bd}{2} x_2^2 + bcx_1x_2.$$

If we denote with $\hat{L} : \mathfrak{h} \to \mathfrak{h}$, $\hat{L} = D_0 L$, $\hat{L}$ is induced by the matrix

$$\hat{L} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \alpha & \beta & \det(A) \end{pmatrix}$$

and $\exp(\hat{L}(x, t)) = L(\exp(x, t))$.

The centralizer of $\mathcal{H}$ is exactly $\mathcal{H}_1 = [\mathcal{H}, \mathcal{H}]$ which consist the elements of the form $(0, t)$. Any homomorphism from $\mathcal{H}$ to $\mathcal{H}$ must leave $\mathcal{H}_1$ invariant.

Any lattice in $\mathcal{H}$ is isomorphic to $\Gamma_k = \{(x, y) : x \in \mathbb{Z}^2, y \in \mathbb{Z}/k\}$, for $k$ a positive integer.

The automorphisms of $\mathcal{H}$ are exactly the ones with $\det(A) \neq 0$ and the automorphisms leaving $\Gamma_k$ invariant are the ones with $A \in GL(2, \mathbb{Z})$ (the matrices with integral entries and determinant $\pm 1$) and $\alpha, \beta \in \mathbb{Z}/k$. On the other hand, every automorphisms of $\Gamma_k$ extends to an automorphisms of $\mathcal{H}$. 
Lemma 6.1. If $S$ is a subgroup of $\Gamma_k$ isomorphic to $\mathbb{Z}^2$, then $S \cap \mathcal{H}_1 \neq \{(0,0)\}$.

Proof. Let $(x, t)$ and $(y, s)$ generate $S$. Then, $(x, t) \cdot (y, s) = (y, s) \cdot (x, t)$ implies $x_1y_2 = x_2y_1$. So, $y = \frac{q}{q}x$. Now, $(x, t)^p \cdot (y, s)^{-q} = (0, u)$ for some $u \in \mathbb{R}$. The fact that $(x, t)$ and $(y, s)$ generate $S$ implies $u \neq 0$, so $(0, u) \in \mathcal{H}_1 \cap S$ \hfill $\square$

We define the quotient compact nilmanifold $N_k = \mathcal{H}/\Gamma_k$ by the relation $(x, t) \sim (y, s)$ iff $(x, t)^{-1} \cdot (y, s) \in \Gamma_k$. The first homotopy group is $\pi_1(N_k) = \Gamma_k$ and two maps of $N_k$ to itself are homotopic if and only if their action on $\Gamma_k$ coincide. Moreover, any map from $N_k$ to itself is homotopic to an automorphism as the described above leaving $\Gamma_k$ invariant. Given $f : N_k \to N_k$, with induced automorphism on $\Gamma_k$, $f^\# = L$, if $F : \mathcal{H} \to \mathcal{H}$ is a lift of $f$ to $\mathcal{H}$, then $F(zn) = F(z)L(n)$ for every $n \in \Gamma_k$, $z \in \mathcal{H}$. Moreover, $F(z) = u(z)L(z)$, where $u : \mathcal{H} \to \mathcal{H}$ is such that $u(zn) = u(z)$ for every $n \in \Gamma_k$, $z \in \mathcal{H}$ and for $k > 0$, $F^k(z) = u_k(z)L^k(z)$ where

$$u_{k+1}(z) = u\left(F^k(z)\right)L\left(u\left(F^{k-1}(z)\right)\right)\ldots L^k\left(u(z)\right)L^{k+1}(z)$$

The projection $p : N_k \to \mathbb{T}^2$, $p((x, t) \cdot \Gamma_k) = x + \mathbb{Z}^2$ induces an isomorphism $p_* : H_1(N_k) \to H_1(\mathbb{T}^2)$. Let $d$ be a right invariant metric on $\mathcal{H}$, for instance $d((x, t), 0) = |x| + |t - \frac{1}{2}x_1x_2|$ where $x = (x_1, x_2)$. The exponential $\exp : \mathfrak{h} \to \mathcal{H}$ become an isometry if the metric in $\mathfrak{h}$ is $|(x, t)| = |x| + |t|$.

We have the following proposition:

Proposition 6.2. Let $f : N_k \to N_k$ be a partially hyperbolic diffeomorphism. If $f$ has not the accessibility property then $E^s \oplus E^u$ integrates to a foliation such that any closed nonempty $f$-invariant set saturated by leaves is the whole manifold $M$. Moreover, the action of $f_* = A$ on $H_1(N_k, \mathbb{Z})$ is hyperbolic and hence there is a semiconjugacy $h : N_k \to \mathbb{T}^2$ homotopic to $p$ such that $h \circ f = Ah$.

Proof. Using Remark 3.4 and Theorem 4.1 we have that either $E^s \oplus E^u$ integrates to a foliation with the minimality property above or there is a torus $T$ invariant by $f^k$ whose homotopy group injects on $\pi_1(N_k)$ and such that $f^k_{\#} \pi_1(T)$ is hyperbolic.

If this last were the case, as $\pi_1(T) \sim \mathbb{Z}^2$, using Lemma 6.1 we get that $\pi_1(T) \cap \mathcal{H}_1 \neq \{0\}$. On the other hand, for any automorphism $L$ of $\Gamma_k$, $L(x, t) = (x, t)^{\pm 1}$ for every $(x, t) \in \mathcal{H}_1$ which gives a contradiction.

So we have that $E^s \oplus E^u$ integrates to a foliation. Let us call $L = f^\#$. Notice that $L(x, t) = (Ax, l(x, t))$ where $A = f_*$ the action of $f$ on $H_1(N_k, \mathbb{Z})$. Thus, if $A$ is not hyperbolic, then it is not hard to see that for $k > 0$, $d(L^kz, 0) \leq Ck^2d(z, 0)$ for some constant $C > 0$. Take $F(z) = u(z)Lz$ a lift of $f$ to $\mathcal{H}$. Then, using
Formula (6.1) and that $d$ is a right invariant metric, we have that for every $z \in \mathcal{H}$,

$$d \left( F^k(z), 0 \right) \leq \sum_{i=0}^{k-1} d \left( L^i u \left( f^{k-1-i}(z) \right), 0 \right) + d \left( L^k(z), 0 \right)$$

$$\leq d \left( u \left( f^{k-1}(z) \right), 0 \right) + \sum_{i=1}^{k-1} Ci^2 d \left( u \left( f^{k-1-i}(z) \right), 0 \right) + Ck^2 d(z, 0)$$

$$\leq Ck^3 + Ck^2 d(z, 0)$$

since $d(u(z), 0) \leq C$ for every $z \in \mathcal{H}$.

Thus, if $I$ is an unstable arc in $\mathcal{H}$, then for $k > 0$, $\text{diam} \left( F^k(I) \right) \leq C(I)k^3$ for some constant $C(I)$ that does not depend on $k$. On the other hand, by Corollary 5.3 the growth of $F^k(I)$ should be exponential. Thus we get a contradiction and hence $A$ must be hyperbolic. The existence of $h$ follows from standard arguments (see [7]).

\[ \square \]

7. Proof of Theorem 1.2

As was shown in Proposition 6.2 if a conservative partially hyperbolic diffeomorphism $f : N_k \rightarrow N_k$ has not the accessibility property then $E^s \oplus E^u$ integrate to a foliation $\mathcal{F}^{su}$ with the following minimality property: any closed, nonempty, $f$-invariant and saturated by leaves set is the whole manifold $M$. In this last section we shall prove the existence of such a foliation leads us to a contradiction. Without lose of generality we may assume, by taking a double covering if necessary, that $\mathcal{F}^{su}$ is transversely orientable. Observe that the double covering of a nilmanifold is again a nilmanifold.

Observe that $\mathcal{F}^{su}$ has no compact leaves. On one hand there are not periodic compact leaves by the minimality property of the foliation. On the other hand, if there is a compact noninvariant leaf $T$ (it must be a torus) then $\{ f^n(T); n \in \mathbb{Z} \} = M$. This implies that all the leaves are tori and $M$ is the mapping torus of a linear automorphism of $\mathbb{T}^2$ that commutes with a hyperbolic one. The only three dimensional nilmanifold satisfying this is $T^3$.

We need the following theorem whose proof is essentially in [13] (see also [12]).

**Theorem 7.1.** Let $\mathcal{F}$ be a codimension one $C^0$-foliation without compact leaves of a three dimensional compact manifold $M$. If $\pi_1(M)$ has non-exponential growth the number $k$ of minimal sets that support an $\mathcal{F}$-invariant measure satisfies $0 < k < +\infty$.

**Proof.** Theorems 7.3, 4.1 and 6.3 of [13] implies the existence of a minimal set supporting an $\mathcal{F}$-invariant measure and Proposition 8.5 of the same paper implies finiteness. Observe that the $C^1$ hypothesis of Theorem 7.3 comes from Novikov’s Theorem that is valid in the $C^0$ setting ([21]).

\[ \square \]
Lemma 7.2. $\mathcal{F}^{su}$ is minimal.

Proof. $\mathcal{F}^{su}$ has not compact leaves and the fundamental group of a nilmanifold has polynomial growth. Call $K_1, \ldots, K_k$ to the minimal sets given by Theorem 7.1. $K = K_1 \cup \cdots \cup K_k$ is an $\mathcal{F}^{su}$-saturated set and it is $f$-invariant. Thus $K = M$ proving the minimality of $\mathcal{F}^{su}$. \hfill \Box

As a consequence of the existence of an invariant transversal measure, we obtain that the holonomy of $\mathcal{F}^{su}$ is trivial (Theorem 2.3.1 of Chapter X of [12]).

Since $\mathcal{F}^{su}$ has no compact leaves, an incompressible torus can be isotoped in such a way it becomes everywhere transversal to it (see [19]). Roussarie result is stated with the assumption of some extra differentiability for the foliation but the general position arguments on which his result is based are valid for $C^0$ foliations (see [12] and, also, [8]). Now cutting along the transverse torus and arguing as in Theorem 4.1 of [14] we obtain that the foliation on $\mathbb{T}^2 \times [0, 1]$ may be isotoped to a foliation obtained from a foliation of the torus $\mathcal{P}$ covered by parallel lines of the plane taking as leaves the surfaces (leaf in $\mathbb{T}^2 \times [0, 1]$). In order to obtain the original nilmanifold we have to identify the two boundary tori by a homeomorphism isotopic to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. This implies that the lines of $\mathcal{P}$ are parallel to the vector $(1, 0)$, thus $\mathcal{P}$ is a foliation by circles. Moreover, this curves correspond to the homotopy class of the center $\mathcal{H}_1$.

Remark 7.3. Observe that the leaves of $\mathcal{F}^{su}$ are subfoliated by circles that are (uniformly) isotopic to the corresponding to $x = \text{constant}$ where $(x, y) \in \mathcal{H}$.

Let $h : N_k \to \mathbb{T}^2$ be the semiconjugacy given by Proposition 6.2.

Lemma 7.4. There is $w \in N_k$ such that $h(W^\sigma(w)) = W^\sigma(h(w))$ with $\sigma = u$ or $s$.

Proof. First of all we claim that there exists $x$ such that $h(W^u(x))$ or $h(W^s(x))$ contains more than one point. If for all $y \in N_k$ we have that $h(W^u(y)) = h(W^s(y)) = q_y \in \mathbb{T}^2$ then $h(AC(x)) = q_x$ and the minimality of $\mathcal{F}^{su}$ implies that $AC(x)$ is dense and $h(N_k) = q_x$ contradicting the sobjectivity of $h$.

Take $x \in N_k$ such that $h(W^u(x))$ is a nontrivial interval of $W^u(h(x))$ and $z \in W^u(x)$ such that $h(z)$ is an interior point of $h(W^u(x))$. A standard argument shows that any point $w \in \omega(z)$ satisfies that $h(W^u(w)) = W^u(h(w))$. \hfill \Box

Take a point $w$ satisfying the lemma (suppose that it works in the $u$ case) and call $F$ to the $\mathcal{F}^{su}$-leaf through $w$. There exists an injective immersion $i : \mathbb{R} \times S^1 \to N_k$ such that $i(\mathbb{R} \times S^1) = F$ and consider $j : h \circ i : \mathbb{R} \times S^1 \to \mathbb{T}^2$. The previous considerations about $\mathcal{F}^{su}$ implies that $j_\# = h_\# \circ i_\# = p_\# \circ i_\# : \pi_1(\mathbb{R} \times S^1) \to \pi_1(\mathbb{T}^2)$ is trivial. Then there exists $\tilde{j} : \mathbb{R} \times S^1 \to \mathbb{R}^2$ such that $j = \pi \circ \tilde{j}$ where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is a covering projection.
Remark 7.3 implies that $\tilde{j}(\mathbb{R} \times S^1)$ is contained in bounded neighborhood of $W^u(\tilde{j}(i^{-1}(w))) \subset \tilde{j}(\mathbb{R} \times S^1)$.

Consider now $i^{-1}(W^s(w))$. Since the foliation $\mathcal{F}^s$ has neither singularities nor compact leaves, $i^{-1}(W^s(w))$ is unbounded. It is not difficult to see that this implies that $\tilde{j}(i^{-1}(W^s(w)))$ is unbounded but it is at bounded distance of $W^u(\tilde{j}(i^{-1}(w)))$ which is a contradiction. This proves Theorem 1.2.

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