Isochronous $n$-dimensional nonlinear PDM-oscillators: linearizability, exact solvability and $\dot{H}$-invariance

Omar Mustafa

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10 - Turkey, Tel.: +90 392 6301378; fax: +90 3692 365 1604.

Abstract: Within the standard Lagrangian settings (i.e., the difference between kinetic and potential energies), we discuss and report isochronicity, linearizability and exact solvability of some $n$-dimensional nonlinear position-dependent mass (PDM) oscillators. In the process, negative the gradient of the PDM-potential force field is shown to be no longer related to the time derivative of the canonical momentum, $p = m(\dot{x}) \dot{x}$, but it is rather related to the time derivative of the pseudo-momentum, $\pi(r) = \sqrt{m(r)} \dot{r}$ (i.e., Noether momentum). Moreover, using some point transformation recipe, we show that the linearizability of the $n$-dimensional nonlinear PDM-oscillators is only possible for $n = 1$ but not for $n \geq 2$. The Euler-Lagrange invariance falls short/incomplete for $n \geq 2$ under PDM settings. An alternative invariance is therefore sought in the so called, hereinafter, $n$-dimensional PDM $\dot{H}$-invariance (i.e., time derivative of the Hamiltonian). Such invariance, in addition to Newtonian invariance of Mustafa \cite{42} authorizes, in effect, the use of the exact solutions of one system to find the solutions of the other. A sample of isochronous $n$-dimensional nonlinear PDM-oscillators examples are reported.

Keywords: PDM-Lagrangians, PDM nonlinear oscillators, linearizability, isochronicity, $\dot{H}$-invariance, exact solvability.

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I. INTRODUCTION

A standard Lagrangian is the difference between kinetic and potential energies (otherwise the Lagrangian is a non-standard one). Likewise, the sum of the kinetic and potential energies represents a standard Hamiltonian, otherwise non-standard. In fact, such standard presentation for the Lagrangian/Hamiltonian renders the total energy to be an integral of motion (i.e., a constant of motion). However, it should be noted that the most prominent Mathews-Lakshmanan oscillators Lagrangians \cite{1}.

$$L(x, \dot{x}; t) = \frac{1}{2} \left( \dot{x}^2 - \omega^2 x^2 \right) ; \quad \dot{x} = \frac{dx}{dt}, \quad (1)$$

belongs, obviously, to the set of non-standard Lagrangians. That is, if in the standard textbook harmonic oscillator Lagrangian $L(x, \dot{x}; t) = m_0 \left( \dot{x}^2 - \omega^2 x^2 \right)/2$, the coordinate $x$ is transformed/deformed so that $x \rightarrow \sqrt{Q(u)} \dot{u}$, then the velocity $\dot{x}$ would be transformed/deformed in a completely different manner so that $\dot{x} \rightarrow \sqrt{m(u)} \ddot{u}$. Where, the relation between the dimensionless scalar functions $Q(u)$ and $m(u)$ would be determined in the process of enforcing the Euler-Lagrange invariance, or any physically feasible alternative invariance. Such non-standard Mathews-Lakshmanan Lagrangians (1) yield (using Euler-Lagrange equation of motion) the nonlinear dynamical equations

$$\ddot{x} = \frac{\lambda x}{1 + \lambda x^2} \dot{x}^2 + \frac{\omega^2}{1 + \lambda x^2} x = 0 , \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad (2)$$

that admit simple harmonic oscillators’ solutions

$$x = A \cos(\Omega t + \varphi) ; \quad \Omega^2 = \frac{\omega^2}{1 + \lambda A^2}.$$

Obviously, the conditions $\Omega^2 = \omega^2 / (1 \pm \lambda A^2)$ suggest that the dynamical equations (2) are conditionally-exactly solvable. Yet, such conditionally-exact solvability renders the oscillators’ frequencies $\Omega$ to be an amplitude-dependent frequency. Consequently, the nonlinear oscillators lose their isochronicity and are non-isochronous, therefore. Basically, if one defines $m(x) = m_0 / (1 \pm \lambda x^2)$, then Lagrangians (1) would (with $m_0 = 1$) read $L(x, \dot{x}; t) = m(x) \left( \dot{x}^2 - \omega^2 x^2 \right)/2$.

*Electronic address: omar.mustafa@emu.edu.tr
and may very well be, effectively and metaphorically speaking, classified as position-dependent mass (PDM) Lagrangians (but not within the standard Lagrangian settings).

Such non-standard Lagrangians’ structure have inspired a great research interest in PDM settings, both in classical and quantum mechanics (c.f., e.g., the sample of references [1][45]). Moreover, the nonlinear differential form of the PDM Euler-Lagrange equations of (2) represents some peculiar cases of the quadratic (i.e., with an \( \dot{x}^2 \) term) Liénard-type nonlinear differential equation

\[
\dddot{x} + f(x) \ddot{x}^2 + g(x) = 0.
\]

Which is, in fact, a very interesting equation for both physics and mathematics [1][11]. The linearizability and isochronicity of which have invited a vast number of interesting research studies in many fields (c.f., e.g., [35–45]). Tiwari et al. [2] and Lakshmanan and Chandrasekar [3], for example, have used Lie point symmetries and asserted that in the case of eight parameter symmetry group, the one-dimensional quadratic Liénard type equation (3) is linearizable and isochronic. It should be mentioned, nevertheless, that the Mathews-Lakshmanan oscillators (2) are linearizable via some nonlocal point transformations [3–5] but not isochronous.

In this work, however, we shall be interested in the generalization of such nonlinear PDM-oscillators for any physically viable PDM-settings. Therefore, we focus our attention on the class of standard PDM Lagrangians/Lagrangians and their linearizability that preserves isochronicity (i.e., with amplitude-independent frequencies) in the process (c.f., e.g., [35–45]). Consequently, we organize our paper in such a way that the current methodical proposal is made clear and comprehensive to serve for viable/feasible pedagogical implementations of isochronous nonlinear PDM-oscillators.

In so doing, we recollect, in section II, some preliminaries on the Mathews-Lakshmanan nonlinear oscillators (1) (within their non-standard Lagrangians/Hamiltonians presentations) so that their generalization to any PDM \( m(x) \) settings is made feasible and safe. We also summarize their exact and conditionally-exact, non-oscillatory and oscillatory, feasible solutions. This would allow the reader to clearly figure out the difference between our current standard Lagrangians/Hamiltonians proposal and the non-standard Mathews-Lakshmanan nonlinear oscillators (1), along with their generalized PDM settings. Within the standard Lagrangian settings, we discuss and report, in section III, the correlation between negative the \( n \)-dimensional gradient of the PDM potential force field (i.e., the \( n \)-dimensional PDM force) and the pseudo-momentum \( \pi(r) = \sqrt{m(r)} \dot{r} \) (i.e., Noether momentum [6]). We show that negative the gradient of the PDM potential force field is no longer the time derivative of the canonical momentum, \( p = m(r) \dot{r} \), but it is rather related with the time derivative of the pseudo-momentum, \( \pi(r) = \sqrt{m(r)} \dot{r} \) (as in (33) below). In the same section, we introduce a new concept to be called, hereinafter, the \( n \)-dimensional PDM \( \dot{H} \)-invariance. Where, we show that the connection between constant mass settings and PDM settings may very well be established through some point transformation (c.f., e.g., [3][15][16][42][43]). In this case, the Euler-Lagrange invariance is shown to be satisfied in \( n = 1 \) dimension but falls short and incomplete for \( n \geq 2 \) dimensions. Alternatively, one may appeal to the so called Newtonian invariance [42] or the PDM \( \dot{H} \)-invariance of the current methodical proposal. Consequently, we discuss the linearizability and \( \dot{H} \)-invariance of some isochronous \( n \)-dimensional PDM oscillators in section IV. Such invariance allows us to use the well known exact solutions of constant mass oscillators and reflect/connect these solutions with isochronous nonlinear PDM oscillators. This is illustrated through the two sets of examples in section V, where we consider a set of one-dimensional PDM Euler-Lagrange equations of motion and an \( n \)-dimensional one. Therein, isochronicity, linearizability and exact solvability of the samples of nonlinear PDM-oscillators are made clear in step-by-step procedures. Our concluding remarks are given in section VI.

II. PRELIMINARIES ON MATHEWS-LAKSHMANAN NONLINEAR PDM-OSCILLATORS: RECOLLECTED AND PDM GENERALIZED

In the generalization of the non-standard Mathews-Lakshmanan oscillators Lagrangian (1) to cover PDM settings, one should keep in mind that Lagrangian (1) is rewritten as

\[
L = \frac{1}{2} m(x) \dot{x}^2 - \frac{1}{2} m(x) \omega^2 x^2; \quad m(x) = \frac{1}{1 \pm \lambda x^2}, \quad \dot{x} = \frac{dx}{dt}. \tag{4}
\]

This would imply the Euler-Lagrange dynamical system

\[
\ddot{x} + \frac{m'(x)}{2m(x)} \dot{x}^2 + \left(1 + \frac{m'(x)}{2m(x)} \right) \omega^2 x = 0; \quad m'(x) = \frac{dm(x)}{dx}. \tag{5}
\]

Obviously, only under the assumption that

\[
\left(1 + \frac{m'(x)}{2m(x)} \right) = m(x), \tag{6}
\]

...
would the PDM function read
\[
m(x) = \frac{1}{1 \pm \beta^2 x^2}.
\] (7)

Which is indeed the PDM used in the Mathews-Lakshmanan oscillator (1), with \( \lambda = \beta^2 \) for the convenience of the current methodical proposal.

However, the linearization of such dynamical system, (5) along with (6), into simple harmonic oscillator
\[
d^2U \over d\tau^2 + \omega^2 U = 0 ; \ U = A \cos (\omega \tau + \varphi)
\] (8)

may be achieved through two nonlocal transformations (to the best of our knowledge). The first of which (c.f., e.g., [3, 4]) suggests that
\[
U = \sqrt{m(x)x} ; \ d\tau = m(x) dt \Leftrightarrow dU \over d\tau = \frac{1}{m(x)} \left(1 + \frac{m'(x)}{2m(x)} x\right) \sqrt{m(x)x} \dot{x} = \sqrt{m(x)x} \dot{x},
\] (9)

and consequently
\[
d^2U \over d\tau^2 = \frac{1}{\sqrt{m(x)}} \left(\ddot{x} + \frac{m'(x)}{2m(x)} \dot{x}^2\right),
\] (10)

to imply that (8) be rewritten as
\[
\ddot{x} + \frac{m'(x)}{2m(x)} \dot{x}^2 + m(x) \omega^2 x = 0.
\] (11)

This is a valid result if and only if condition (6) is satisfied to yield the PDM in (7). The second nonlocal transformation (e.g., [5]) suggests that
\[
dU = \sqrt{g(x)} dx , \ d\tau = f(x) dt ; \ U = \sqrt{m(x)x}.
\] (12)

Under such nonlocal transformation setting, we get
\[
dU \over d\tau = \sqrt{g(x)} \left(1 + \frac{m'(x)}{2m(x)} x\right) \sqrt{m(x)x} \dot{x} \Leftrightarrow \sqrt{g(x)} = \left(1 + \frac{m'(x)}{2m(x)} x\right) \sqrt{m(x)}.
\] (13)

and
\[
d^2U \over d\tau^2 = \frac{g(x)}{f(x)^2} \left[\ddot{x} + \left(\frac{g'(x)}{2g(x)} - \frac{f'(x)}{f(x)}\right) \dot{x}^2\right].
\] (14)

Hence, equation (8) reads
\[
\ddot{x} + \left(\frac{g'(x)}{2g(x)} - \frac{f'(x)}{f(x)}\right) \dot{x}^2 + \left(\frac{f(x)^2}{\sqrt{g(x)}}\right) \sqrt{m(x)x} \omega^2 x = 0.
\] (15)

Which when compared with (5) implies that
\[
\frac{g'(x)}{2g(x)} - \frac{f'(x)}{f(x)} = \frac{m'(x)}{2m(x)} \Leftrightarrow \sqrt{g(x)} = f(x) \sqrt{m(x)} \Leftrightarrow f(x) = \left(1 + \frac{m'(x)}{2m(x)} x\right).
\] (16)

As a result, equation (15) collapses into (5). Moreover, if one chooses to work with \( f(x) = m(x) \) then necessarily \( m(x) = 1/(1 \pm \beta^2 x^2) \) as in (6) and (7) above. One may conclude that the current PDM generalization of the Mathews-Lakshmanan oscillator (1) is now safe and clear. The above PDM settings are, therefore, applicable to some general PDM \( m(x) \) in principle (without condition (6) of course), but not within the standard Lagrangian presentation.

Nevertheless, one should be aware that for the PDM in (7) our dynamical equation of (5) (or equivalently, equation (2) with \( \lambda = \beta^2 \)) admits non-oscillatory exact solutions in the forms of
\[
x_1(t) = \frac{1}{2\alpha \beta^2} \left[-(\alpha^2 \beta^2 + \omega^2) e^{-\alpha \beta (\alpha_2 + t)} \pm e^{\alpha \beta (\alpha_2 + t)}\right],
\] (17)
and/or

\[ x_2(t) = \pm \frac{1}{2\alpha_1 \beta^2} \left[ -\left( \alpha_2^2 \beta^2 + \omega^2 \right) e^{\alpha_1 \beta (\alpha_2 + t)} \pm e^{-\alpha_1 \beta (\alpha_2 + t)} \right]. \]  

(18)

Moreover, a set of conditionally non-oscillatory exact solutions are feasible too. Amongst are

\[ x_3(t) = A \cosh (\Omega_3 t + \varphi) ; \quad \Omega_3^2 = \pm \frac{\omega^2}{1 + \beta^2 A^2}, \]

(19)

and/or

\[ x_4(t) = A \sinh (\Omega_4 t + \varphi) ; \quad \Omega_4^2 = \pm \frac{\omega^2}{1 + \beta^2 A^2}. \]

(20)

Yet, the prominent conditionally exact Mathews-Lakshmanan oscillators solutions are given by

\[ x_5(t) = A \cos (\Omega_5 t + \varphi) ; \quad \Omega_5^2 = \pm \frac{\omega^2}{1 + \beta^2 A^2}, \]

(21)

and/or

\[ x_6(t) = A \sin (\Omega_6 t + \varphi) ; \quad \Omega_6^2 = \pm \frac{\omega^2}{1 + \beta^2 A^2}. \]

(22)

It is obvious that the Mathews-Lakshmanan oscillators’ frequencies, in (21) and (22), are amplitude-dependent frequencies and are non-isochronous oscillators, therefore. Hereby, we argue that, although the nonlocal point transformation recipes in (9) and (12) serve to secure invariance between the dynamical systems of (5) and (8), they render the oscillators non-isochronous. This is, in fact, a natural consequence of the position-dependent deformation of the time elements of (9) and (12) (i.e., \( d\tau = m(x) \, dt \) in (9) and \( d\tau = f(x) \, dt \) in (12), where \( f(x) \) is given in (16)). This would necessarily mean that, if the oscillators isochronicity is the sought-after objective then the time element should not be a position-dependent deformed one.

In what follows, we abort the non-standard Lagrangian presentations and discuss some standard classical mechanical textbook Lagrangians under PDM settings. This would be very interesting for any viable/feasible pedagogical implementations of PDM Lagrangians/Hamiltonians.

### III. n-DIMENSIONAL GRADIENT OF PDM-POTENTIAL ENERGY AND PDM \( \dot{\mu} \)-INVARINCE

Apriori, it is known that under constant mass setting, the force is the time derivative of the canonical momentum and is given by negative the gradient of the potential force field, i.e.,

\[ F = \frac{d\mathbf{p}}{dt} = -\nabla V(\mathbf{r}) ; \quad \nabla = \sum_{j=1}^{3} \partial_{x_j} \hat{x}_j, \quad \mathbf{r} = \sum_{j=1}^{3} x_j \hat{x}_j, \quad F = \sum_{j=1}^{3} F_j \hat{x}_j, \quad r = \sqrt{\sum_{j=1}^{3} x_j^2}. \]

(23)

Under PDM settings, however, negative the gradient of the potential force field is no longer given by the time derivative of the canonical momentum. In the one-dimensional case, for example, Mustafa [3] has asserted that the relation between the force and the potential force field is rather given by

\[ F = \sqrt{m(x)} \frac{d}{dt} \left( m(x) \dot{x} \right) = -V'(q(x)), \]

(24)

where \( V(q(x)) \) is the PDM-deformed potential force field and

\[ V'(q(x)) = \frac{dV(q(x))}{dx} ; \quad q(x) = \int \sqrt{m(x)} dx. \]

Equation (24) is a documentation that, in the one-dimensional case, negative the gradient of the potential force field is not equal to the time derivative of the canonical momentum \((i.e., \frac{dp}{dt} \neq -V'(x))\), where \( p = m(x) \dot{x} \). So is the \( n \)-dimensional case. Consequently, the underlying \( n \)-dimensional dynamics of the PDM systems have to be clarified in advance. Namely, one has to answer the question as to "what would negative the gradient of the \( n \)-dimensional PDM-deformed potential force field yield to?" That would be the sought-after net PDM force vector.
A. Negative the gradient of the PDM potential force field

Hereby, we consider the $n$-dimensional PDM Lagrangian

$$L (\mathbf{r}, \dot{\mathbf{r}}; t) = \frac{1}{2} m_\circ m (\mathbf{r}) \dot{\mathbf{r}}^2 - V (\mathbf{q} (\mathbf{r})) = \frac{1}{2} m_\circ m (\mathbf{r}) \sum_{j=1}^{n} \dot{x}_j^2 - V (\mathbf{q} (\mathbf{r})),$$  \hspace{1cm} (25)

where

$$\mathbf{q} (\mathbf{r}) = \sqrt{Q (\mathbf{r})} \mathbf{r} = \sum_{j=1}^{n} q_j (\mathbf{r}) \hat{x}_j,$$  \hspace{1cm} (26)

and $Q (\mathbf{r})$ is some PDM-deformation function manifested by the PDM-deformation $m (\mathbf{r})$ of the kinetic energy term. Obviously, the velocity vector $\dot{\mathbf{r}}$ in the kinetic energy term of $L (\mathbf{r}, \dot{\mathbf{r}}; t)$ in (25), practically and intuitively, is assumed to be transformed as $\dot{\mathbf{r}} \rightarrow \sqrt{m (\mathbf{r})} \dot{\mathbf{r}}$ under PDM settings. It is, therefore, convenient and sufficient to assume that the coordinates would transform in a different way as $\mathbf{r} \rightarrow \sqrt{Q (\mathbf{r})} \mathbf{r}$, as in (26), under the same settings. As long as the relation between $m (\mathbf{r})$ and $Q (\mathbf{r})$ is to be determined in the process, this assumption remains valid and sufficient.

We now use the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0; \ i = 1, 2, \cdots, n \in \mathbb{N},$$  \hspace{1cm} (27)

to obtain (with $m_\circ = 1$ throughout) $n$ PDM Euler-Lagrange equations

$$m (\mathbf{r}) \ddot{x}_i + \dot{m} (\mathbf{r}) \dot{x}_i - \frac{1}{2} \sum_{i=1}^{n} \partial_x m (\mathbf{r}) \ddot{x}_j = - \partial_x V (\mathbf{q} (\mathbf{r})).$$  \hspace{1cm} (28)

Next, we multiply each term by $\dot{x}_i$ and sum over $i = 1, 2, \cdots, n$ to get the corresponding Newtonian dynamical equation

$$m (\mathbf{r}) \sum_{i=1}^{n} \ddot{x}_i \dot{x}_i + \dot{m} (\mathbf{r}) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i - \frac{1}{2} \sum_{i=1}^{n} \partial_x m (\mathbf{r}) \sum_{j=1}^{n} \ddot{x}_j \dot{x}_j = - \sum_{i=1}^{n} \dot{x}_i \partial_x V (\mathbf{q} (\mathbf{r})) = - \nabla V (\mathbf{q} (\mathbf{r})).$$  \hspace{1cm} (29)

To avoid mathematical complexities, we may assume that $m (\mathbf{r}) = m (\mathbf{r})$ and $Q (\mathbf{r}) = Q (\mathbf{r})$ where $\mathbf{r}$ is readily defined in (23). This would allow us to represent (29) as

$$m (\mathbf{r}) \ddot{\mathbf{r}} + \dot{m} (\mathbf{r}) \dot{\mathbf{r}} - \frac{1}{2} \sum_{i=1}^{n} \partial_x m (\mathbf{r}) \ddot{x}_i \dot{x}_i = - \nabla V (\mathbf{q} (\mathbf{r})).$$  \hspace{1cm} (30)

However, one may express $\dot{m} (\mathbf{r})$, with $\partial_r m (\ddot{r}) = \partial m (\ddot{r}) / \partial r$, as

$$\dot{m} (\mathbf{r}) = \sum_{k=1}^{n} \partial_x m (\mathbf{r}) \dot{x}_k = \frac{\partial_x m (\mathbf{r})}{\mathbf{r}} \sum_{k=1}^{n} \dot{x}_k \dot{x}_k = \frac{\partial_x m (\mathbf{r})}{\mathbf{r}} \left( \mathbf{r} \cdot \dot{\mathbf{r}} \right).$$  \hspace{1cm} (31)

and

$$\sum_{i=1}^{n} \partial_x m (\mathbf{r}) \dot{x}_i = \frac{\partial_x m (\mathbf{r})}{\mathbf{r}} \sum_{i=1}^{n} \dot{x}_i \dot{x}_k = \frac{\partial_x m (\mathbf{r})}{\mathbf{r}} \mathbf{r} = \frac{m' (\mathbf{r})}{\mathbf{r}},$$  \hspace{1cm} (32)

so that equation (30) reads, with $\mathbf{r} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = (\mathbf{r} \cdot \dot{\mathbf{r}}) \dot{\mathbf{r}}$ (i.e., no rotational effects under consideration and $\mathbf{r} || \dot{\mathbf{r}}$, therefore),

$$m (\mathbf{r}) \ddot{\mathbf{r}} + \frac{\dot{m} (\mathbf{r})}{2} \dot{\mathbf{r}} = - \nabla V (\mathbf{q} (\mathbf{r})) \iff \mathbf{F} = \sqrt{m (\mathbf{r})} \frac{d}{dt} \left( \sqrt{m (\mathbf{r})} \dot{\mathbf{r}} \right) = - \nabla V (\mathbf{q} (\mathbf{r})).$$  \hspace{1cm} (33)

This result would, in fact, represent the $n$-dimensional PDM Newtonian dynamics. It suggests that in a free force field (i.e., $V (\mathbf{q} (\mathbf{r})) = 0$), the canonical momentum $\mathbf{p} = m (\mathbf{r}) \dot{\mathbf{r}}$ is no longer a conserved quantity but rather the PDM pseudo-momentum $\mathbf{p} (\mathbf{r}) = \sqrt{m (\mathbf{r})} \dot{\mathbf{r}}$ (or in the Cariñena et al’s [3] language, the "Noether momentum") is the conserved quantity. Moreover, it is now obvious that, under PDM setting, negative the gradient of the potential force field is no longer the same as the time derivative of the canonical momentum $\mathbf{p}$. Yet it recovers the constant mass settings for $m (\mathbf{r}) = 1$ to yield the usual textbook relation $m_\circ \ddot{\mathbf{r}} = - \nabla V (\mathbf{r})$. 

B. $n$-dimensional PDM $\dot{H}$-invariance

Let us consider a standard $n$-dimensional constant mass Lagrangian

$$L(q, \dot{q}; t) = \frac{1}{2} m \sum_{j=1}^{n} \dot{q}_j^2 - V(q); \quad \dot{q}_j = \frac{dq_j}{dt}; \quad j = 1, 2, \ldots, n \in \mathbb{N},$$

(34)

Then the corresponding $n$ Euler-Lagrange equations (with $m_o = 1$) are given by

$$\ddot{q}_i + \partial_{q_i} V(q) = 0; \quad i = 1, 2, \ldots, n \in \mathbb{N}.$$

(35)

Under a point transformation in the form of

$$dq_i = \sqrt{m(r)} dx_i \iff \partial_{x_i} q_i = \frac{\partial q_i}{\partial x_i} = \sqrt{m(r)} \iff \ddot{q}_i = \sqrt{m(r)} \ddot{x}_i \iff \dot{q} = \sqrt{m(r)} \dot{r},$$

(36)

and the assumption that

$$q = \sqrt{Q(r)} r \iff \dot{q} = \sqrt{Q(r)} \left(1 + \frac{Q'(r)}{2Q(r)} r^2\right) \dot{r},$$

(37)

the comparison between (36) and (37) would imply that

$$\sqrt{m(r)} = \sqrt{Q(r)} \left(1 + \frac{Q'(r)}{2Q(r)} r^2\right).$$

(38)

The connection between $m(r)$ and $Q(r)$ is clear, therefore. We may now proceed with (35) and use

$$\ddot{q}_i = \sqrt{m(r)} \ddot{x}_i \iff \ddot{q}_i = \sqrt{m(r)} \left[\ddot{x}_i + \frac{\dot{m}(r)}{2m(r)} \dot{x}_i \right]$$

(39)

in (35), along with $\partial_{q_i} = (\partial x_i / \partial q_i) \partial_{x_i} = m(r)^{-1/2} \partial_{x_i}$, to obtain

$$m(r) \ddot{x}_i + \frac{1}{2} \dot{m}(r) \dot{x}_i + \partial_{x_i} V(q) = 0$$

(40)

Which, when compared with (28), suggests that the invariance between (28) and (35) is still far beyond reach at this stage. However, if we multiply (28) by $\dot{x}_i$ and sum over $i = 1, 2, \ldots, n$ we get

$$m(r) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \dot{m}(r) \sum_{i=1}^{n} \dot{x}_i^2 - \frac{1}{2} \left( \sum_{i=1}^{n} \dot{x}_i \partial_{x_i} m(r) \right) \sum_{j=1}^{n} \dot{x}_j^2 + \sum_{i=1}^{n} \dot{x}_i \partial_{x_i} V(q(r)) = 0,$$

(41)

and consequently with $\dot{m}(r)$ in (31) it reads

$$m(r) \sum_{i=1}^{n} \ddot{x}_i \dot{x}_i + \frac{1}{2} \dot{m}(r) \sum_{i=1}^{n} \dot{x}_i^2 + \ddot{V}(q(r)) = 0 \quad ; \quad \ddot{V}(q(r)) = \sum_{i=1}^{n} \ddot{x}_i \partial_{x_i} V(q(r)).$$

(42)

Similarly, equation (40) would yield

$$m(r) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \frac{1}{2} \dot{m}(r) \sum_{i=1}^{n} \dot{x}_i^2 = \frac{d}{dt} \left( \frac{1}{2} m(r) \sum_{i=1}^{n} \dot{x}_i^2 \right).$$

(43)

Now we got a clear invariance between (42) and (43), through the point transformation of (36). However, a question of delicate nature arises in the process as to "what kind of invariance we got here?"

Let us rearrange the first two term of either (42) or (43) so that

$$m(r) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \frac{1}{2} \dot{m}(r) \sum_{i=1}^{n} \dot{x}_i^2 = \frac{d}{dt} \left( \frac{1}{2} m(r) \sum_{i=1}^{n} \dot{x}_i^2 \right).$$

(44)
Then equation (42) and (43) are nothing but the time derivative of the total energy of the PDM system. That is, 
they can both be expressed as
\[
\frac{d}{dt} \left( \frac{1}{2} \sum_{j=1}^{n} \dot{q}_j^2 - V(q) \right) = 0 = \frac{d}{dt} \left( \frac{\mathbf{p}^2}{2m(r)} + V(q(r)) \right),
\]
(45)
where \( \mathbf{p} = m(r) \mathbf{\dot{r}} \) is the canonical momentum. Consequently, one may safely write
\[
\dot{H}(q, \dot{q}; t) = 0 = \dot{H}(r, \dot{r}; t).
\]
Therefore, we may now conclude that \( \dot{H}(q, \dot{q}; t) \) is invariant with \( \dot{H}(r, \dot{r}; t) \) and hence the notion \( \dot{H} \)-invariance is a proper terminology to be used hereinafter. Such invariance, however, gives us the authority to use the exact solutions of one system and map it into the other. This would consequently enrich the class of exactly solvable dynamical systems within the standard Lagrangian/Hamiltonian settings. Moreover, one should be aware that when equation (35) is multiplied by \( \dot{q}_i \) and summed over \( i = 1, 2, \cdots, n \), it would yield (43) or equivalently (45).

IV. ISOCHRONOUS \( n \)-DIMENSIONAL PDM HARMONIC OSCILLATORS: LINEARIZABILITY AND \( \dot{H} \)-INVARIANCE

Having had settled down the technical mathematical issues in the preceding section, we may now proceed to discuss the \( n \)-dimensional PDM harmonic oscillators linearizability, \( \dot{H} \)-invariance and isochronicity.

We begin with the \( n \)-dimensional PDM oscillator Lagrangian
\[
L(r, \dot{r}; t) = \frac{1}{2} m(r) \dot{r}^2 - V(r) = \frac{1}{2} m(r) \sum_{j=1}^{n} \dot{x}_j^2 - \frac{1}{2} \omega^2 Q(r) \sum_{j=1}^{n} x_j^2,
\]
(47)
where the oscillator potential is now assumed to be PDM-deformed in such a way that \( r \rightarrow \sqrt{Q(r)} \mathbf{r} \) as a consequence of the PDM-deformation of the velocity vector \( \dot{\mathbf{r}} \rightarrow \sqrt{\frac{m(r)}{r}} \mathbf{\dot{r}} \). The substitution of the PDM oscillator Lagrangian (47) in the \( n \) Euler-Lagrange equations of motion (27) would result
\[
\ddot{x}_i + \frac{\dot{m}(r)}{m(r)} \dot{x}_i - \frac{m'(r)}{2m(r)} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + \sqrt{\frac{Q(r)}{m(r)}} \omega^2 x_i = 0,
\]
(48)
where we have used the relation (38) in the process. On the other hand, the \( n \)-dimensional constant mass oscillator Lagrangian
\[
L(q, \dot{q}; t) = \frac{1}{2} m_0 \dot{q}^2 - \frac{1}{2} m_0 \omega^2 \dot{q}^2 = \frac{1}{2} m_0 \sum_{j=1}^{n} \dot{q}_j^2 - \frac{1}{2} m_0 \omega^2 \sum_{j=1}^{n} q_j^2,
\]
(49)
yields the \( n \) Euler-Lagrange linear differential equations (with \( m_0 = 1 \))
\[
\ddot{q}_i + \omega^2 q_i = 0,
\]
(50)
that admit exact sinusoidal oscillatory solutions
\[
q_i = A_i \cos (\omega t + \varphi).
\]
(51)
Using our point transformation of (36)-(39) in (50) one obtains
\[
\sqrt{m(r)} \left[ \ddot{x}_i + \frac{\dot{m}(r)}{2m(r)} \dot{x}_i \right] + \sqrt{Q(r)} \omega^2 x_i = 0 \iff \ddot{x}_i + \frac{\dot{m}(r)}{2m(r)} \dot{x}_i + \sqrt{\frac{Q(r)}{m(r)}} \omega^2 x_i = 0.
\]
(52)
This result clearly suggests that, under the current point transformation, the linearizability of (48) into (50) is only possible for the one-dimensional case. Whereas, for the \( n \)-dimensional case we observe that the invariance could not be established and the linearization is not feasible. Nevertheless, the two systems are readily \( \dot{H} \) invariant. That is, if the exact solutions of one of the systems is known, then we may reflect/map it (through the current point transformation) into the exact solutions of the other system. This would, in effect, authorize the use of the exact solutions (51) of (50) to find the solutions of (48). This is illustrated in the sample of examples below.
V. ISOCHRONOUS n-DIMENSIONAL NONLINEAR PDM OSCILLATORS: ILLUSTRATIVE EXAMPLES

A. One-dimensional isochronous nonlinear PDM oscillators

For the one-dimensional case one should be aware that the dynamical equations in (48) and (52), associated with the one-dimensional PDM-oscillators Lagrangians (47)

\[ L = \frac{1}{2} m(x) \dot{x}^2 - \frac{1}{2} Q(x) \omega^2 x^2, \]

are identical and the Euler-Lagrange invariance is very well established. Moreover, the linearizability of (48) into (50) is possible and straightforward.

1. A PDM without singularity: \( m(x) = \frac{1}{1 + \lambda^2 x^2} \)

A PDM in the form of

\[ m(x) = \frac{1}{1 + \lambda^2 x^2}, \]  

would result, by (38), in the coordinate deformation

\[ \sqrt{Q(x)} = \frac{1}{\lambda x} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right). \]

Consequently, the dynamical equation (48), or (52), for the one-dimensional PDM-oscillator Lagrangian

\[ L = \frac{\dot{x}^2}{2 (1 + \lambda^2 x^2)} - \frac{\omega^2}{2 \lambda} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right)^2 \omega^2, \]

in (47) reads

\[ \ddot{x} - \frac{\lambda^2 x}{1 + \lambda^2 x^2} \dot{x}^2 + \frac{1 + \lambda^2 x^2}{\lambda} \ln \left( \lambda x + \sqrt{1 + \lambda^2 x^2} \right) \omega^2 = 0, \]

Where its exact solution is inherited from (51) along with (37) so that

\[ q = A \cos (\omega t + \varphi) = \sqrt{Q(x)x} \iff x = \frac{1}{2A} \left( e^{\lambda A \cos (\omega t + \varphi)} - e^{-\lambda A \cos (\omega t + \varphi)} \right), \]

which exactly satisfies (56) and forms its exact isochronous (i.e., \( \omega \) is amplitude-independent and no constraints are imposed upon it) nonlinear PDM-oscillators solutions, therefore.

2. Two coordinate deformations without/with singularity: \( Q(x) = \frac{1}{1 \pm \lambda^2 x^2} \)

Position-dependent coordinate deformations in the form of

\[ \sqrt{Q(x)} = \sqrt{\frac{1}{1 \pm \lambda^2 x^2}}, \]

would imply, by (38), two PDM functions given by

\[ m(x) = \left( \frac{1}{1 \pm \lambda^2 x^2} \right)^{\frac{3}{2}}. \]

Then the dynamical equation (48), or (52), for the one-dimensional PDM-oscillator Lagrangian (47)

\[ L = \frac{1}{2} \left[ \frac{\dot{x}^2}{(1 \pm \lambda^2 x^2)^{\frac{3}{2}}} - \frac{\omega^2 x^2}{1 \pm \lambda^2 x^2} \right], \]
yields
\[ \ddot{x} \mp \frac{3\lambda^2 x}{1 \pm \lambda^2 x^2} \dot{x}^2 + (1 \pm \lambda^2 x^2) \omega^2 x = 0, \quad (61) \]
that admits, using (51) and (37), exact solution in the form of
\[ q = A \cos (\omega t + \varphi) = \sqrt{Q(x)x} \iff x = \frac{A \cos (\omega t + \varphi)}{\sqrt{1 \pm \lambda^2 A^2 \cos^2 (\omega t + \varphi)}}, \quad (62) \]
which exactly satisfy the dynamical systems in (61) and hence represent their exact isochronous nonlinear PDM-oscillators solutions.

3. A coordinate deformation with a singularity: \( Q(x) = 1/(1 - \lambda x) \)

A coordinate deformation in the form of
\[ \sqrt{Q(x)} = \sqrt{\frac{1}{1 - \lambda x}}, \quad (63) \]
would imply that the PDM function is
\[ m(x) = -\frac{1}{4} \frac{(\lambda x - 2)^2}{(\lambda x - 1)^3}. \quad (64) \]
Using (51) and (37) one obtains
\[ q = A \cos (\omega t + \varphi) = \sqrt{1 - \lambda x} x \iff x = \frac{A}{2} \cos (\omega t + \varphi) \left[ -\lambda A \cos (\omega t + \varphi) \pm \sqrt{\lambda^2 A^2 \cos^2 (\omega t + \varphi) + 4} \right], \quad (65) \]
which satisfies the corresponding dynamical equation, (52),
\[ \ddot{x} - \frac{\lambda (\lambda x - 4)}{2(\lambda x - 1)(\lambda x - 2)} \dot{x}^2 + \frac{2(\lambda x - 1)}{\lambda x - 2} \omega^2 x = 0, \quad (66) \]
and represents its exact isochronous nonlinear PDM-oscillator solution.

4. A power-law type PDM: \( m(x) \sim x^{2\nu} \)

A power-low type coordinate deformation
\[ \sqrt{Q(x)} = ax^\nu, \quad (67) \]
would result the power-law type PDM function
\[ m(x) = a^2 (\nu + 1)^2 x^{2\nu}. \quad (68) \]
Hence, using (51) and (37), the exact isochronous nonlinear PDM-oscillator solution would be
\[ q = A \cos (\omega t + \varphi) = a x^{\nu+1} \iff x = \left[ \frac{A}{a} \cos (\omega t + \varphi) \right]^{1/(\nu + 1)}, \quad (69) \]
that satisfies the dynamical equation, (52),
\[ \ddot{x} + \frac{\nu}{x} \dot{x}^2 + \frac{1}{\nu + 1} \omega^2 x = 0; \quad \nu \neq -1. \quad (70) \]
5. An exponential-type PDM: \( m(x) = e^{2\lambda x} \)

An exponential-type PDM

\[
m(x) = e^{2\lambda x}
\]  

would imply, by (38), that the coordinate deformation is

\[
\sqrt{Q(x)} = \frac{e^{\lambda x}}{\lambda x} (1 - e^{-\lambda x}).
\]  

Which when substituted in the dynamical equation (52) yields

\[
\ddot{x} + \lambda \dot{x}^2 + \frac{\omega^2}{\lambda} (1 - e^{-\lambda x}) = 0.
\]  

Using (51) and (37), one finds that it admits exact isochronous nonlinear PDM-oscillator solution

\[
q = A \cos (\omega t + \varphi) = \frac{1}{\lambda} (1 - e^{\lambda x}) \iff x = \frac{1}{\lambda} \ln (1 - \lambda A \cos (\omega t + \varphi)).
\]  

B. \( n \)-dimensional isochronous nonlinear PDM oscillators

For the \( n \)-dimensional PDM-oscillators Lagrangian (47) case, we shall recollect that the Euler-Lagrange invariance falls short and incomplete. One has therefore to appeal to \( \dot{H} \)-invariance and use the exact solution (51) of (50) to extract exact solutions for (48), where the linearizability of (48) into (50) turned out to be not feasible.

1. Two coordinate deformations without/with singularity: \( Q(r) = 1/ (1 \pm \lambda^2 r^2) \)

The coordinate deformations of the form

\[
\sqrt{Q(r)} = \sqrt{\frac{1}{1 \pm \lambda^2 r^2}} : r = \sqrt{\sum_{j=1}^{n} x_j^2},
\]  

would result, by (38), two PDM function

\[
m(r) = \frac{1}{(1 \pm \lambda^2 r^2)^{\lambda}}.
\]  

This would allow us to write (37) as

\[
q = A \cos (\omega t + \varphi) = \sqrt{Q(r)} r \iff r = \frac{q}{\sqrt{1 \pm \lambda^2 q^2}} \iff x_i = \frac{A_i \cos (\omega t + \varphi)}{\sqrt{1 \pm \lambda^2 A_i^2 \cos^2 (\omega t + \varphi)}}; A = \sqrt{\sum_{j=1}^{n} A_j^2}.
\]  

which satisfy our dynamical equations of (48)

\[
\ddot{x}_i \pm \frac{6\lambda^2}{1 \pm \lambda^2 r^2} \left( \sum_{j=1}^{n} x_j \dot{x}_j \right) \dot{x}_i \pm \frac{3\lambda^2}{1 \pm \lambda^2 r^2} \left( \sum_{j=1}^{n} \dot{x}_j \right) x_i + (1 \pm \lambda^2 r^2) \omega^2 x_i = 0,
\]  

and forms their exact \( n \)-dimensional isochronous nonlinear PDM-oscillators solutions, therefore.
Consider a power-law type coordinate deformation
\[ \sqrt{Q(r)} = a r^v, \] (79)
which in turn implies a PDM function
\[ m(r) = a^2 (v + 1)^2 r^{2v}; \quad v \neq -1. \] (80)
Consequently, with \( q = A \cos(\omega t + \varphi), \) equation (37) yields
\[ q = a r^v r \iff r = \left( \frac{q-1}{a} \right)^{1/(v+1)} \quad q \iff x_i = A, \cos(\omega t + \varphi) \left( \frac{[A \cos(\omega t + \varphi)]-1}{a} \right)^{1/(v+1)}, \] (81)
as the exact \( n \)-dimensional isochronous nonlinear PDM-oscillators solutions for the dynamical equations (48)
\[ \ddot{x}_i + \frac{2v}{r^2} \left( \sum_{j=1}^{n} x_j \dot{x}_j \right) \dot{x}_i - \frac{v}{r^2} \left( \sum_{j=1}^{n} x_j^2 \right) x_i + \frac{\omega^2}{v+1} x_i = 0; \quad r^2 = \sum_{j=1}^{n} x_j^2. \] (82)

In the sample of illustrative example discussed above, we notice that there are no constraints on the frequencies of the nonlinear PDM oscillators considered. Such frequencies are clearly amplitude-independent and are isochronic. Therefore, all our examples are isochronous nonlinear PDM oscillators.

VI. CONCLUDING REMARKS

In this work, we have considered the \( n \)-dimensional PDM-Lagrangians in their standard form (i.e., the difference between kinetic and potential energies). However, in order to make our study comprehensive and self-contained, we have recollected and elaborated on the solvability (exact and conditionally exact) and linearizability of the non-standard Mathews-Lakshmanan nonlinear oscillators (2). The generalization of such nonlinear oscillators (2) to any PDM, \( m(r) \), settings is also discussed and reported in section II. Yet we have asserted that the position-dependent deformation of time (manifested by the nonlocal point transformations in (9) or (12)) renders such PDM nonlinear oscillators non-isochronous so that their frequencies become amplitude-dependent.

To preserve isochronicity of the PDM nonlinear oscillators, we had to return back to the standard Lagrangians form to obtain an interesting sets of isochronous PDM nonlinear oscillators. In so doing, we have shown/emphasized (in section III) that negative the gradient of the PDM potential force field (i.e., the force vector associated with PDM settings) is no longer given by the time derivative of the canonical momentum, \( p(r) = m(r) \dot{r} \), but it is rather given in terms of the pseudo-momentum, \( \pi(r) = \sqrt{m(r)} \dot{r} \) [3, 4] (or the Noether momentum as in [3]). That is,
\[ -\nabla V(q(r)) = F = \sqrt{m(r)} \frac{d}{dt} \left( \sqrt{m(r)} \dot{r} \right), \]
where \( q(r) = \sqrt{Q(r)} r \), with \( m(r) \) and \( Q(r) \) satisfy the correlation
\[ \sqrt{m(r)} = \sqrt{Q(r)} \left( 1 + \frac{Q'(r)}{2Q(r)} r \right). \]

In the same section, moreover, we have shown that the connection between constant mass settings and PDM settings is feasible through some point transformation, where the time is kept as is (i.e., no position-dependent deformation of time). Hereby, the Euler-Lagrange invariance is shown satisfactory for \( n = 1 \) but unsatisfactory/incomplete for \( n \geq 2 \). Hence, an alternative invariance is sought through the time derivative of the PDM-Hamiltonian. Consequently, in addition to Newtonian invariance of Mustafa [12], we have introduced yet another type of invariance to be called, hereinafter, \( H \)-invariance (where \( H = dH/dt \)). Moreover, such invariance goes alongside with the fact that the total energy is a conserved quantity (documented in (40)) and is a constant of motion (i.e., integral of motion), therefore. This result allowed us to use, in section IV and V, the well known exact solutions (51) of the linear oscillator (50) along with our point transformation (37) to obtain exact solutions for a set of \( n \)-dimensional isochronous nonlinear

2. A power-law type PDM: \( m(x) \sim r^{2v} \).
PDM oscillators. This is documented in the illustrative examples of section V, where a set of one-dimensional and a set of \( n \)-dimensional isochronous nonlinear PDM oscillators are reported.

In the light of our experience in the current methodical proposal, we argue that the linearizability of the equations of motion (48) of the standard PDM oscillators Lagrangians (47) is only possible for the one-dimensional systems. Whereas, for the \( n \)-dimensional case we observe that the invariance could not be established and the linearization is not feasible (documented in (48) to (52)). Nevertheless, the constant mass and the PDM systems (49) and (47), respectively) are readily \( \dot{H} \) invariant (reported in (34) to (46)). This would, in effect, authorize the use of the exact solutions (51) of (50) to find the solutions of (48). In a more general language, if the exact solution of the constant mass system in (34) is known, then we may reflect/map it (through the current point transformation) into the solution of the corresponding PDM system in (25) (the other way around is also true, of course). To the best of our knowledge, such results and/or methodical proposal have never been reported elsewhere in the literature.
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