The Rational Cherednik Algebra of Type $A_1$ with Divided Powers

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Abstract
Motivated by the recent developments of the theory of Cherednik algebras in positive characteristic, we study rational Cherednik algebras with divided powers. In our research we have started with the simplest case, the rational Cherednik algebra of type $A_1$. We investigate its maximal divided power extensions over $R[c]$ and $R$ for arbitrary principal ideal domains $R$ of characteristic zero. In these cases, we prove that the maximal divided power extensions are free modules over the base rings, and construct an explicit basis in the case of $R[c]$. In addition, we provide an abstract construction of the rational Cherednik algebra of type $A_1$ over an arbitrary ring, and prove that this generalization expands the rational Cherednik algebra to include all of the divided powers.

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1 Introduction
In this paper we study the rational Cherednik algebra of type $A_{n-1}$, which we denote by $\mathcal{H}_{t,c}(\mathfrak{S}_n, h)$. Cherednik algebras, also known as double affine Hecke algebras (DAHA), are a large family of algebras introduced by Cherednik in [Che93] to prove Macdonald’s conjectures concerning orthogonal polynomials for root systems. Since then Cherednik algebras have been discovered to be useful in many different contexts, most notably in the study of quantum Calogero-Moser systems (see [Eti07]). Cherednik algebras have also been applied to topology, harmonic analysis, Verlinde algebras, Kac-Moody algebras and more. For a thorough exposition of theory of DAHA in general, see [Che05]. Another good overview of the theory of rational Cherednik algebras is given in [EM10].

The representation theory of Cherednik algebras over fields of characteristic zero has been well studied (see [Gor03], [EM10]), but more recently a theory of Cherednik algebras in positive characteristic started to develop. Cherednik algebras in positive characteristic were investigated in [BC13] and [BF86]. In [La05], the case of rank one algebras was discussed. Later in [DS+14], [DS16], and [CK18] the Hilbert polynomials of some irreducible finite dimensional representations were calculated.

The current paper is a continuation of this research. Our main goal was to develop a theory of Cherednik algebras with divided powers in positive characteristic, so we have started with the simplest example, the rational Cherednik algebra of type $A_1$. To define the maximal divided power extension even in this case turned out to be an interesting problem. For more information on algebras with divided powers see [Jan07] and [Lon16]. The main reason for the study of this construction is the fact that naive reduction of the Cherednik algebra to positive characteristic makes the algebra “too small”, because a lot of operators become central and act by zero on important representations. To make representation theory richer one can work with the algebra extended by divided powers.
1.1 Main Results

In Section 1.1 we define the rational Cherednik algebra of type $A$, introduce our notion of divided power extensions, and show an example of this notion applied to an algebra of differential operators. In Section 1.2, we prove Theorem 1.2 and Theorem 1.3, which show the freeness of the maximal divided power extension of the rational Cherednik algebra of type $A_1$ over $R$ and $R[x]$, constructing a basis in the latter case. In Section 1.3 we construct the maximal divided power extension in an abstract way over an arbitrary ring, and prove equivalence in most cases.

1.2 The Rational Cherednik Algebra of Type $A$

In this section we will define the rational Cherednik algebra of type $A_{n-1}$, which we denote $\mathcal{H}_{t,c}(\mathfrak{S}_n, \mathfrak{h})$. In general we will work with the rational Cherednik algebra over an arbitrary ring, but here we introduce the standard notion over the field of complex numbers. Let $\mathfrak{S}_n$ be the symmetric group on $n$ elements and consider its permutation representation on $\mathfrak{h} = \mathbb{C}^n$ and its dual $\mathfrak{h}^*$. For any $1 \leq i \neq j \leq n$, let $s_{ij} \in \mathfrak{S}_n$ denote the reflection switching $i$ and $j$. For each reflection $s_{ij}$, let $P_{ij} \subset \mathfrak{h}$ be the hyperplane of fixed points of $s_{ij}$, i.e., $P_{ij} = \{ (\alpha_1, \ldots, \alpha_n) : \alpha_i = \alpha_j \}$. Let $\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{i<j} P_{ij}$ be the set of regular points of $\mathfrak{h}$, i.e., the set of points which are not fixed by any reflection.

Let $D(\mathfrak{h}_{\text{reg}})$ be the algebra of differential operators on the set $\mathfrak{h}_{\text{reg}}$. We have a natural action of $\mathfrak{S}_n$ on $\mathfrak{h}_{\text{reg}}$ and hence on $D(\mathfrak{h}_{\text{reg}})$. Note that $D(\mathfrak{h}_{\text{reg}})$ is isomorphic to the localization $\left\{ x_1 - x_j \right\}^{-1} \text{Diff}(\mathbb{C}[\mathfrak{h}])$ where $x_1, \ldots, x_n$ are the standard generators of $\mathbb{C}[\mathfrak{h}]$. The following results and definitions are taken from [EM10].

**Definition 1.1.** For any $1 \leq i \leq n$ and $t, c \in \mathbb{C}$, the Dunkl operator is defined as

$$D_i = t \frac{\partial}{\partial x_i} - c \sum_{j \neq i} \frac{1}{x_i - x_j} (1 - s_{ij}) \in D(\mathfrak{h}_{\text{reg}}) \rtimes \mathbb{C}[\mathfrak{S}_n].$$

**Proposition 1.2.** We have the following properties for Dunkl operators:

- For $\sigma \in \mathfrak{S}_n$, we have $\sigma D_i \sigma^{-1} = D_{\sigma(i)}$
- $[D_i, D_j] = 0$
- $[D_i, x_j] = cs_{ij}$
- $[D_i, x_i] = t - c \sum_{j \neq i} s_{ij}$

We can now define the rational Cherednik algebra of type $A$.

**Definition 1.3.** For any $t, c \in \mathbb{C}$ with $t \neq 0$, let $\mathcal{H}_{t,c}(\mathfrak{S}_n, \mathfrak{h})$ be the $\mathbb{C}$-subalgebra of $D(\mathfrak{h}_{\text{reg}}) \rtimes \mathbb{C}[\mathfrak{S}_n]$ generated by $\mathfrak{h}^*, \mathfrak{S}_n$, and $D_i$ for $i = 1, \ldots, n$. This is the rational Cherednik algebra of type $A_{n-1}$ associated to $t, c$.

**Proposition 1.4.** For any $t, c \in \mathbb{C}$ with $t \neq 0$, the algebra $\mathcal{H}_{t,c}(\mathfrak{S}_n, \mathfrak{h})$ is isomorphic to the quotient of the algebra $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes \mathbb{C}[\mathfrak{S}_n]$ by the relations

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_j] = cs_{ij}, \quad [y_i, x_i] = t - \sum_{j \neq i} cs_{ij}.$$

**Theorem 1.5** (PBW Theorem). Let $\text{Sym}(V)$ be the symmetric algebra of $V$. Let $x_1, \ldots, x_n$ be the standard basis for $\mathfrak{h}^*$ and let $y_1, \ldots, y_n$ be the corresponding basis of $\mathfrak{h}$. Then the map

$$\text{Sym}(\mathfrak{h}) \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}} \text{Sym}(\mathfrak{h}^*) \to \mathcal{H}_{t,c}(\mathfrak{S}_n, \mathfrak{h}),$$

which sends $y_i \otimes g \otimes x_i \mapsto D_i g x_i$, is an isomorphism of $\mathbb{C}$-vector spaces.

There is another useful algebra to consider when studying divided power extensions of $\mathcal{H}_{t,c}(\mathfrak{S}_n, \mathfrak{h})$. Consider the permutation representation of $\mathfrak{S}_n$ on $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, with bases $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ respectively. Consider the subrepresentation $1 = \text{Span}_C \{ y_i = y_i - y_1 : 1 < i \leq n \}$ and its dual $\Gamma' = \mathfrak{h}^*/\langle x_1 + x_2 + \cdots + x_n \rangle$. Let $T(1 \oplus \Gamma')$ be the tensor algebra of $1 \oplus \Gamma'$.

**Definition 1.6.** $\mathcal{H}_{t,c}(\mathfrak{S}_n, 1)$ is the $\mathbb{C}$-subalgebra of $\text{End}(\text{Sym}(\Gamma'))$ generated by $\Gamma', \mathfrak{S}_n$, and $D_i - D_1$.

**Proposition 1.7.** The algebra $\mathcal{H}_{t,c}(\mathfrak{S}_n, 1)$ is the quotient of $T(1 \oplus \Gamma') \rtimes \mathbb{C}[\mathfrak{S}_n]$ by the relations:

- $[x_i, x_j] = 0$
- $[\hat{y}_i, \hat{y}_j] = 0$
- $[\hat{y}_i, x_i] = t - cs_{1i} - c \sum_{k \neq i} s_{ik}$
- $[\hat{y}_i, x_k] = cs_{ik} - cs_{ik}$ for $k \neq i, 1$
The algebras $\mathcal{H}_{t,c}(\mathbb{S}_n, h)$ and $\mathcal{H}_{t,c}(\mathbb{S}_n, t)$ are related in the following way. Let $z_1 = y_1 - y_2$, $z_2 = y_2 - y_3, \ldots, z_{n-1} = y_{n-1} - y_n$ and $Z = y_1 + \cdots + y_n$. Let $w_1 = x_1 - x_2, w_2 = x_2 - x_3, \ldots, w_{n-1} = x_{n-1} - x_n$ and $W = x_1 + \cdots + x_n$. Note that $[Z, x_i] = t$ and $[W, y_i] = -t$, it follows that $[Z, w_i] = [W, z_i] = 0$. Also $[Z, W] = n$. Furthermore, $[\sigma, Z] = [\sigma, W] = 0$ for all $\sigma \in \mathbb{S}_n$. So we have two subalgebras, one generated by $z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1}$ and $\mathbb{S}_n$ and the other generated by $Z$ and $W$. The first algebra is isomorphic to $\mathcal{H}_{t,c}(\mathbb{S}_n, 1)$, and the second algebra is isomorphic to $C[\sigma, \partial_\sigma]$, the subalgebra of $\text{End}(C[q])$ generated by $q$ and $\frac{\partial}{\partial q}$ for some formal variable $q$. By the PBW theorem, it follows that

$$\mathcal{H}_{t,c}(\mathbb{S}_n, h) \cong \mathcal{H}_{t,c}(\mathbb{S}_n, 1) \otimes C[q, \partial_q].$$

Another useful algebra to consider is the spherical subalgebra of $\mathcal{H}_{t,c}(\mathbb{S}_n, h)$, denoted by $B_{t,c}(\mathbb{S}_n, h)$.

**Definition 1.8.** Let $e_+ \in C[\mathbb{S}_n]$ be the symmetrizer, $e_+ = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \text{sgn}(\sigma) \sigma$. Let $e_-$ be the antisymmetrizer, $e_- = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \text{sgn}(\sigma)$ is the sign of a permutation.

**Note:** $e_+^2 = e_+$ and $e_-^2 = e_-$.

**Definition 1.9.** The spherical subalgebra of $\mathcal{H}_{t,c}(\mathbb{S}_n, h)$ is $B_{t,c}(\mathbb{S}_n, h) = e_+ \mathcal{H}_{t,c}(\mathbb{S}_n, h) e_+$. Let $B_{t,c}(\mathbb{S}_n, 1) = e_+ \mathcal{H}_{t,c}(\mathbb{S}_n, 1) e_+$.

Note that $e_+ (D(h_{reg}) \times C[\mathbb{S}_n]) e_+ = D(h_{reg})^{\mathbb{S}_n}$, i.e. the $\mathbb{S}_n$-invariant subspace of $D(h_{reg})$. This means that $B_{t,c}(\mathbb{S}_n, h) \subset D(h_{reg})^{\mathbb{S}_n}$. Since $\mathbb{S}_n$ acts trivially on $C[q, \partial_q]$, we have the decomposition

$$B_{t,c}(\mathbb{S}_n, h) \cong B_{t,c}(\mathbb{S}_n, 1) \otimes C[q, \partial_q].$$

### 1.3 Divided Power Extensions

We could not find a definition of divided powers in the existing literature which worked for our purposes, so we have developed our own framework.

Let $R$ be an integral domain of characteristic zero and let $V$ be a free $R$-module. Note that we have a canonical embedding $\text{End}_R(V) \hookrightarrow \text{End}_{R \otimes Q}(V \otimes Q)$.

**Definition 1.10.** For any submodule $A \subset \text{End}_R(V)$, the maximal divided power extension of $A$, denoted $A^{DP}$, is the submodule of $\text{End}_{R \otimes Q}(V \otimes Q)$ given by:

$$A^{DP} = (A \otimes Q) \cap \text{End}_R(V) \subset \text{End}_{R \otimes Q}(V \otimes Q).$$

Note that $A^{DP}$ is an $R$-module, and if $A$ is an $R$-algebra, then $A^{DP}$ is an $R$-algebra as well. Another insightful definition of $A^{DP}$ arises through the notion of divisibility of an operator.

**Definition 1.11.** For some operator $f \in \text{End}_R(V)$, and integer $n \in \mathbb{Z}_{\geq 1}$, we say that $n$ divides $f$ if $f \otimes (1/n) \in \text{End}_R(V)$. We write $n | f$.

The following definition is often easier to use in practice than Definition 1.10.

**Proposition 1.12.** $A^{DP} = \{ f \otimes (1/n) : f \in A, n \in \mathbb{Z}_{\geq 1}, n | f \}$.

**Proof.** This follows from the fact that every $a \in A \otimes Q$ can be uniquely expressed as $f \otimes (1/n)$, for some $f \in A$ and $n \in \mathbb{Z}$. \hfill \square

To show how this notion of divided power extensions applies to representation theory in characteristic $p$, suppose we had some faithful representation $\psi : A \rightarrow \text{End}_R(V)$. The naive reduction modulo $p$ gives a representation $A \otimes F_p \rightarrow \text{End}_{R \otimes F_p}(V \otimes F_p)$. The center of $A \otimes F_p$ can become large in characteristic $p$. Since central operators may act trivially on $V \otimes F_p$, this can become problematic. If we instead take the divided power extension, we have a representation $A^{DP} \otimes F_p \rightarrow \text{End}_{R \otimes F_p}(V \otimes F_p)$. This representation is faithful, since if the image of $Q \otimes 1$ was zero, then $Q = p^n L$ for some $n \geq 1$ and $L \in A^{DP}$ such that $L \otimes 1 \neq 0$ in $A^{DP} \otimes F_p$. This means that $Q \otimes 1 = 0$ in $A^{DP} \otimes F_p$, so the map is injective. In the cases when $R \otimes \mathbb{Z} = \{ \pm 1 \}$, $A^{DP} \otimes F_p$ contains a nonzero scaled copy of each nonzero operator in $A$. This can make the representation theory of $A^{DP}$ richer than that of $A$ in characteristic $p$.

When computing maximal divided power extensions of a ring, it often helps to decompose the ring into smaller pieces for which the maximal divided power extensions are already known.

**Proposition 1.13.** Suppose $\{ A_i \}_{i \in I}$ is a family of $R$-submodules of $\text{End}_R(V)$ and suppose that for any $a_i \in A_i$, $d \sum_{i \in I} a_i \text{ in } \text{End}_R(V)$ implies that $d | a_i$ for all $i \in I$. Then, in $\text{End}_{R \otimes Q}(V \otimes Q)$, we have $(\bigoplus_{i \in I} A_i)^{DP} = \bigoplus_{i \in I} A_i^{DP}$.

\footnote{Unless stated otherwise, all tensor are assumed to be taken over $\mathbb{Z}$.}
Note: The above divisibility condition implies that the sum $\bigoplus_{i \in I} A_i$ is direct.

Proof. For any $Q = \sum_{i \in I} a_i$, if $d(Q)$ then by assumption, $d|a_i$ for all $i \in I$ so $\frac{Q}{d} = \sum_{i \in I} \frac{a_i}{d} \in \bigoplus_{i \in I} A_i^{dp}$. So $(\bigoplus_{i \in I} A_i)^{dp} \subset \bigoplus_{i \in I} A_i^{dp}$. Conversely, if $Q = \sum_{i \in I} \frac{a_i}{d} \in \bigoplus_{i \in I} A_i^{dp}$ we have

$$Q = \sum_{i \in I} a_i \prod_{i \in I, j \neq i} d_i.$$ 

So $Q \in (\bigoplus_{i \in I} A_i)^{dp}$ and $(\bigoplus_{i \in I} A_i)^{dp} \supset \bigoplus_{i \in I} A_i^{dp}$, so the result follows. \hfill $\Box$

**Proposition 1.14.** Let $V$ and $W$ be free $R$ modules. Suppose that $A = \bigoplus_{i \in I} A_i \subset \text{End}_R(V)$ and $B = \bigoplus_{i \in I} B_i \subset \text{End}_R(W)$ satisfy the divisibility condition of Proposition 1.13. Furthermore, suppose that $A_i \cong B_i \cong A_i^{dp} \cong B_i^{dp} \cong R$. Finally, we make the additional requirement that $R^2 \cap \mathbb{Z} = \{ \pm 1 \}$. Then in $\text{End}_{R\otimes R}(V \otimes W \otimes Q)$,

$$(A \otimes_R B)^{dp} = A^{dp} \otimes_R B^{dp}.$$ 

Proof. First we claim that if $d$ divides $\sum_{i,j} a_{ij} \otimes b_{ij}$ then $d|a_{ij} \otimes b_{ij}$. Let $x_i$ be the basis element for $A_i^{dp}$ and let $y_j$ be the basis element for $B_j^{dp}$. Write $a_{ij} \otimes b_{ij} = k_{ij} x_i \otimes y_j$. If $d|\sum_{i,j} a_{ij} \otimes b_{ij}$, by definition there exists $q_{i,j}$ such that

$$\sum_{i,j} k_{ij} x_i \otimes y_j = d \sum_{i,j} q_{i,j} x_i \otimes y_j.$$ 

This implies $\sum_{i,j} (k_{ij} - dq_{i,j}) (x_i \otimes y_j) = 0$. By linear independence, $k_{ij} = dq_{i,j}$ and so $d|a_{ij} \otimes b_{ij}$.

Next we claim that $(A \otimes_R B)_i^{dp} = A_i^{dp} \otimes_R B_i^{dp}$. To show $A_i^{dp} \otimes_R B_i^{dp} \subset (A \otimes_R B)_i^{dp}$, let $\sum_{i,j} a_{ij} \otimes b_{ij} \in (A \otimes_R B)_i^{dp}$ for some $a_{ij} \in A_i, b_{ij} \in B_j$, and $d_i, d_j \in \mathbb{Z}_{>0}$. Then

$$d_i \otimes d_j \in (A_i \otimes_R B_j)^{dp} \subset (A \otimes_R B)_i^{dp}.$$ 

Now to show that $(A \otimes_R B)_i^{dp} \subset A_i^{dp} \otimes_R B_i^{dp}$, suppose $d|a_{ij} \otimes b_{ij}$ for some $j \in \mathbb{Z}_{>0}$. For an operator $f$ on some space $Z$, let $N_f = \{ n : n|f(z)$ for some $z \in Z \}$. Note that $N_x \cdot N_y \subset N_{x \otimes y}$. We claim that $\gcd(N_{x \otimes y}) = \gcd(N_{xy}) = 1$. Indeed, if $d|N_{xy}$, then $\frac{1}{d}x \in A_i^{dp}$, and so $\frac{1}{d} \in R$, a contradiction unless $d = 1$. The same argument shows that $\gcd(N_{x \otimes y}) = 1$. We claim that $\gcd(N_{x \otimes y}) = 1$. Indeed, if $d|N_{x \otimes y}$, then $d|N_{x \otimes y}$. Pick some $\ell \in N_{xy}$. Then $d|\ell N_{xy}$, but since $\gcd(\ell N_{xy}) = \ell$ it follows that $d|\ell$. Since $\ell$ was arbitrary, $d|N_{xy}$, which implies that $d = 1$. Now since $d|k_{ij} x_i \otimes y_j$, we have $d|k_{ij} N_{x_i \otimes y_j}$, so by the previous argument, $d|k_{ij}$. So

$$\frac{a_{ij} \otimes b_{ij}}{d} = \frac{k_{ij} x_i \otimes b_{ij}}{d} \in A_i^{dp} \otimes_R B_j^{dp}.$$ 

Now to combine the above claims, we have

$$(A \otimes_R B)^{dp} = \left( \bigoplus_{(i,j) \in I \times J} A_i \otimes_R B_j \right)^{dp} = \bigoplus_{(i,j) \in I \times J} (A_i \otimes_R B_j)^{dp} = \bigoplus_{(i,j) \in I \times J} A_i^{dp} \otimes_R B_j^{dp} = A^{dp} \otimes_R B^{dp},$$

where the second equality follows by the first claim and Proposition 1.13. This completes the proof. \hfill $\Box$

### 1.4 Polynomial Differential Operators

To show a known example of divided power extensions, we consider the integral Weyl algebra $W(\mathbb{Z}) = \mathbb{Z}(x,y)/(yx - xy - 1)$ and its faithful polynomial representation in $\text{End}(\mathbb{Z}[x])$ given by $x \mapsto x \times$ (i.e. multiplication by $x$) and $y \mapsto \partial_x$, where $\partial_x = \frac{d}{dx}$. Let $\mathbb{Z}[x, \partial_x] \subset \text{End}(\mathbb{Z}[x])$ be the image of this representation.

We call this the ring of integral polynomial differential operators. Similarily define $\mathbb{Q}[x, \partial_x]$. The results of this section aren’t original. Nonetheless we decided to include their proofs, adapted to fit within our framework of divided power extensions, because they illustrate a simple example of the methods we use in the case of Cherednik algebra in Section 2.1.

**Definition 1.15.** Let $\binom{i}{j} \in \mathbb{Q}[t]$ be the polynomial $\binom{i}{j} = \frac{(t+1)\cdots(t+k+1)}{k!} \in \mathbb{Q}[t]$, and $P_k(t) = k! \binom{i}{j} \in \mathbb{Z}[t]$. Let $\mathcal{D}_x$ be the Hasse derivative, whose action is given on the basis by $\mathcal{D}_x^k x^n = \binom{n}{k} x^{n-k}$. Let $\mathcal{D}_x^n$ extend linearly.

**Proposition 1.16** (Newton’s Interpolation Formula). Define the zeroth order forward difference operator as $\Delta^0 f(n) = f(n)$, and define the higher order operators as $\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n)$. Let $f(t)$ be a polynomial. Then $f(t) = \sum_{k \geq 0} \binom{t}{k} \Delta^k f(0)$. 


Lemma 1.17. Let \( f \) be some integer-valued polynomial, and write \( f(n) = \sum_{k \geq 0} \alpha_k \binom{n}{k} \) for some integer coefficients \( \alpha_k \). If \( d|f(n) \) for all \( n \in \mathbb{Z}_{\geq 0} \), then \( d|\alpha_k \) for all \( k \geq 0 \).

Proof. Suppose \( f(n) \equiv 0 \mod d \) for all \( n \). Let \( N = \deg f \), so \( \alpha_n = 0 \) whenever \( n > N \). By Newton’s Interpolation formula, \( LA = F \equiv 0 \mod d \) where \( (L)_i = \binom{i}{j} \) is the \((N + 1) \times (N + 1)\) lower triangular Pascal matrix, \( A = (\alpha_0, \ldots, \alpha_N) \), and \( F = (f(0), \ldots, f(N)) \). Note that \( \det L = 1 \). Multiplying both sides by \( L^{-1} \), we get that \( \alpha_k \equiv 0 \mod d \) for all \( 0 \leq k \leq N \). It follows that \( \alpha_k \equiv 0 \mod d \) for all \( k \geq 0 \).

The above lemma implies the following classical result.

Proposition 1.18 (Newton). Let \( \text{Int}(\mathbb{Z}[x]) = \{ f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subset \mathbb{Z} \} \). Then \( \text{Int}(\mathbb{Z}[x]) \) is a free \( \mathbb{Z} \)-module generated by the polynomials \( \binom{x}{k} \).

Proposition 1.19. For any \( n \geq 0 \), let \( D[n] = \mathbb{Z}[x] \) and for \( n < 0 \), let \( D[n] = P_{-n}(x)\mathbb{Z}[x] \). Consider the map \( \psi_n : D[n] \to \operatorname{End}(\mathbb{Z}[x]) \) where \( f(t) \in D[n] \) is sent to the operator which acts on \( x^t \) by sending it to \( f(t)x^{t+n} \). There is an isomorphism of \( \mathbb{Z} \)-modules, \( \psi : \bigoplus_{n \geq 0} D[n] \to \mathbb{Z}[x, \partial_x] \), where \( \psi(D[n]) = \psi_n \) for all \( n \in \mathbb{Z} \).

Proof. Consider the \( \mathbb{Z} \)-grading on \( \mathbb{Z}[x, \partial_x] \) given by \( \partial_x \mapsto -1 \) and \( x \mapsto 1 \). Let \( P[n] \) be the set of homogeneous elements of degree \( n \). Since \( \{ x^d \partial_x^k \}_{k \geq 0} \) is a basis for \( \mathbb{Z}[x, \partial_x] \) as a \( \mathbb{Z} \)-module, we have an isomorphism \( \mathbb{Z}[x, \partial_x] \cong \bigoplus_{n \geq 0} P[n] \). We claim that \( \psi_n : D[n] \to P[n] \) is an isomorphism. First, note that \( \text{Im } (\psi_n) \subset P[n] \). This is clear if \( n \geq 0 \). Indeed, let \( f(x) \in D[n] = \mathbb{Z}[x] \) be some polynomial, say \( f(x) = \sum_{i=0}^d \alpha_i x^i \). Then

\[
\psi_n(f(x)) = x^n \sum_{i=0}^d \alpha_i (x \partial_x)^i \in P[n].
\]

Similarly, if \( n < 0 \), let \( P_{-n}(x)f(x) \in D[n] = P_{-n}(x)\mathbb{Z}[x] \) be arbitrary, with \( f(x) = \sum_{i=0}^d \alpha_i x^i \). Then

\[
\psi_n(P_{-n}(x)f(x)) = \partial_x^n \psi_n(f(x)) \in P[n].
\]

To show surjectivity, we consider the cases \( n \geq 0 \) and \( n < 0 \) separately. If \( n \geq 0 \), this map is surjective, since \( \psi_n(P(t)) = x^{t+n} \partial_x^n \) and \( x^{t+n} \partial_x^n \) generate \( P[n] \). If \( n < 0 \), by the grading, every \( Q \in P[n] \) can be expressed as \( Ld_x^{-n} \) for some \( L \in P[0] \). So \( \psi_n((t+x+n)P_{-n}(x)) = Q \) where \( t(x) \) is the polynomial representing the action of \( L \). Since \( L \in P[0] \) is arbitrary, it follows that \( \text{Im } (\psi_n) = \text{Im } (\psi_n(P_{-n}(x)\mathbb{Z}[x]) = P[n] \). So for any \( n \in \mathbb{Z} \), the map \( \psi_n : D[n] \to P[n] \) is a surjection, hence an isomorphism. We have the desired isomorphism \( \psi \) by the definition of direct sum.

Definition 1.20. Let \( R \) be an integral domain of characteristic zero, and suppose \( A \) is a submodule of \( \text{Fun}(\mathbb{Z}, R) \), the \( \mathbb{Z} \)-module of set-theoretic functions from \( \mathbb{Z} \) to \( R \). The ring of \( R \)-valued elements of \( A \) is

\[
\text{Int}(R) = \{ f/d : f \in A, d|f, d \in \mathbb{Z}_{\geq 1} \},
\]

where \( d|f \) if \( f/d \in \text{Fun}(\mathbb{Z}, R) \subset \text{Fun}(\mathbb{Z}, R \otimes \mathbb{Q}) \). Note that this agrees with our earlier definition of \( \text{Int}(\mathbb{Z}[x]) \). We write \( \text{Int}(A) \) if \( R = \mathbb{Z} \).

Proposition 1.21. We have an isomorphism of \( \mathbb{Z} \)-modules, \( \mathbb{Z}[x, \partial_x]^{DP} \cong \bigoplus_{n \in \mathbb{Z}} \text{Int}(D[n]) \). In particular, this implies that as a \( \mathbb{Z} \)-module, \( \mathbb{Z}[x, \partial_x]^{DP} \) is spanned by \( x^k \partial_x^l \) for all \( k, l \geq 0 \). Furthermore these are \( \mathbb{Z} \)-linearly independent.

Proof. To apply Proposition 1.18, we must prove the divisibility condition. So suppose \( d|\sum_{n \in \mathbb{Z}} Q_n \) where \( \deg Q_n = n \). Then for all \( t \geq 0 \), \( d\left( \sum_{n \in \mathbb{Z}} Q_n \right) x^t = \sum_{n \in \mathbb{Z}} f_n(t)x^{t+n} \). Therefore \( d|f_n(t) \) for all \( t \geq 0 \), and so \( d|Q_n \).

So by Proposition 1.18, we have the equality \( \bigoplus_{n \in \mathbb{Z}} P[n]^{DP} = \mathbb{Z}[x, \partial_x]^{DP} \). Note however that \( d|Q \in P[n] \), if and only if \( d|q(t) \in D[n] \) for all \( t \geq 0 \), where \( q(t) \) is the polynomial representing the action of \( Q \) on \( x^t \). So we have an isomorphism \( \psi_n^{DP} : \text{Int}(D[n]) \to P[n]^{DP} \) for each \( n \), defined similarly to \( \psi_n \). Combining these, we get an isomorphism \( \psi^{DP} : \mathbb{Z}[x, \partial_x]^{DP} \cong \bigoplus_{n \in \mathbb{Z}} \text{Int}(D[n]) \).

Next we claim that \( \mathbb{Z}[x, \partial_x]^{DP} \) is generated by \( x^k \partial_x^l \) for all \( k, l \geq 0 \) as a \( \mathbb{Z} \)-module. It suffices to consider \( \text{Int}(D[n]) \), so first assume that \( n \geq 0 \). By Corollary 1.16, \( \text{Int}(\mathbb{Z}[t]) \) is generated by \( \binom{t}{k} \). So the image of \( \text{Int}(D[n]) \) in \( \mathbb{Z}[x, \partial_x]^{DP} \) is generated by \( x^{n+k} \partial_x^{n+k} \), since \( x^{n+k} \partial_x^{n+k} x^t = \binom{t}{k} x^{t+n+k} \). If \( n < 0 \), note that by Lemma 1.17, \( \text{Int}(D[n]) = \text{Int}(P_{-n}(t)\mathbb{Z}[t]) \) is generated by \( \binom{t}{k} \) for \( k \geq 0 \). This is because if \( d|P_{-n}(t) \sum_{j \geq 0} \alpha_j P_j(t-n) = \sum_{j \geq 0} \alpha_j (j-n+1) \binom{t}{k} \), then \( d|\alpha_j (j-n+1) \) for all \( j \). This basis for \( \text{Int}(P_{-n}(t)\mathbb{Z}[t]) \) corresponds to \( x^k \partial_x^{k-n} \), where \( k \geq 0 \). The \( \mathbb{Z} \)-linear independence follows from linear independence of \( \binom{t}{k} \) in \( \mathbb{Z}[t] \).

Corollary 1.22. \( \mathbb{Z}[x, \partial_x]^{DP} \) is a free \( \mathbb{Z}[x] \)-module, freely spanned by \( D_x^k \) for \( k \geq 0 \).
2  Maximal Divided Power Extensions of $\mathcal{H}_{t,c}(\mathfrak{S}_2, h)$

To apply the notion of divided powers to $\mathcal{H}_{t,c}(\mathfrak{S}_2, h)$, we must introduce an integral version of this algebra. Before we do this, we use the tensor decomposition given in Section 1.2 to reduce the size of the algebra. Let $n = 2$, and consider $\mathcal{H}_{t,c}(\mathfrak{S}_2, h)$. This is a subalgebra of $(x_1 - x_2)^{-1}\text{Diff}(\mathbb{C}[x_1, x_2]) \otimes \mathbb{C}[\mathfrak{S}_2]$ generated by $x_1, x_2, s_{12}$.

$$D_1 = \frac{\partial}{\partial x_1} - \frac{1}{x_1 - x_2}(1 - s_{12}) \quad \text{and} \quad D_2 = \frac{\partial}{\partial x_2} + \frac{1}{x_1 - x_2}(1 - s_{12}).$$

$\mathcal{H}_{t,c}(\mathfrak{S}_2, l)$ is the subalgebra of $\text{End}(\mathbb{C}[x])$ generated by $s$ and $t$, where $\frac{\partial}{\partial x} = \frac{1}{x}$ and $\frac{\partial}{\partial s} = \frac{1}{s}$. Here $x = x_1 - x_2$ and $s = s_{12}$. Note that $\mathcal{H}_{t,c}(\mathfrak{S}_2, h) \cong \mathcal{H}_{t,c}(\mathfrak{S}_2, l) \otimes \mathbb{C}[q, \partial_q]$. By definition, $\mathcal{H}_{t,c}(\mathfrak{S}_2, h) \subset \text{End}(\mathbb{C}[h]) = \text{End}(\mathbb{C}[l]) \otimes \text{End}(\mathbb{C}[q])$, where $q$ is some formal variable. Since $\mathcal{H}_{t,c}(\mathfrak{S}_2, l) \subset \text{End}(\mathbb{C}[l])$ and $\mathbb{C}[q, \partial_q] \subset \text{End}(\mathbb{C}[q])$, Proposition 1.13 implies that to study divided power extensions of $\mathcal{H}_{t,c}(\mathfrak{S}_2, h)$, it suffices to study divided power extensions of $\mathcal{H}_{t,c}(\mathfrak{S}_2, l)$ and $\mathbb{C}[q, \partial_q]$ separately. The conditions of Proposition 1.14 are shown to be satisfied by the results of Section 1.4 and Section 2. Since divided power extensions of $\mathbb{C}[q, \partial_q]$ are known (see Section 1.4), we only need to consider $\mathcal{H}_{t,c}(\mathfrak{S}_2, l)$.

Using the canonical isomorphism $\mathcal{H}_{t,c}(\mathfrak{S}_2, l) \rightarrow \mathcal{H}_{t,h}(\mathfrak{S}_2, l)$ for any $\lambda \in \mathbb{C}^\times$, we can normalize $t = 0$ or $t = 1$. In this paper, we only consider the case when $t = 1$.

**Definition 2.1.** For any domain of characteristic zero $R$ and $c \in R$, let $H_{t,c}(R)$ be the subalgebra of $\text{End}_R(R[x])$ generated by $e_\pm, x$ and $D = \frac{\partial}{\partial x} - 2x e_-$. Note that $e_+ = 1 - e_-$. In particular note that $H_{t,c}(\mathbb{C}) = H_{t,c}(\mathfrak{S}_2, l)$.

2.1  Freeness of $H_{t,c}^{DP}(R)$

In this section, we prove the following theorem:

**Theorem 2.2.** Let $R$ be a PID of characteristic zero. Then for any $c \in R$, $H_{t,c}^{DP}(R)$ is a free $R$-module.

**Proof.** For now, let $R$ be an arbitrary domain of characteristic zero. For any $k \geq 0$, consider the polynomials $D_k^+(t) = \prod_{i=0}^{k-1}(2t - i - 2cp_i)$ and $D_k^-(t) = \prod_{i=0}^{k-1}(2t + 1 - i - 2cp_{i+1})$ where $p_i = 0$ if $i$ is even and 1 otherwise. Note that $D_k^+e_+x^{2n} = D_k^+(n)x^{2n-k}$ and $D_k^+e_-x^{2n+1} = D_k^-(n)x^{2n+1-k}$. Now consider the $R$-modules

$$H^+[n] = \begin{cases} R[2t] & n \geq 0 \\ D_n^-(t)R[2t] & n < 0 \end{cases} \quad \text{and} \quad H^-[n] = \begin{cases} R[2t + 1] & n \geq 0 \\ D_n^-(t)\mathbb{Z}[2t + 1, \ell] & n < 0 \end{cases}.$$

**Note:** We are aware that $R[2t + 1] = R[2t]$; this distinction is purely to motivate the connection between these sets and $H_{t,c}(R)$.

We have a $\mathbb{Z}$-grading on $H_{t,c}(R)$, given by $D \mapsto -1$, $x \mapsto 1$ and $s \mapsto 0$. By the PBW theorem, there is an isomorphism $H_{t,c}(R) \rightarrow \bigoplus_{n \in \mathbb{Z}} P[n]$ where $P[n]$ is the module of homogeneous elements of $H_{t,c}(R)$ of degree $n$. For all $Q \in H_{t,c}(R)$, we have $Q = Qe_+ + Qe_- + H_{t,c}(R)e_+ \cap H_{t,c}(R)e_- = \{0\}$, so it follows that $P[n] = P[n]e_+ \oplus P[n]e_-$. We claim that $\psi^+_n : H^+[n] \rightarrow P[n]e_+$ and $\psi^-_n : H^-[n] \rightarrow P[n]e_-$ are isomorphisms, where $\psi$ sends $f(t)$ to the operator which maps $x^t$ to $e_\pm f(t)x^{t+1}$. Note that this operator acts by zero on odd powers of $x$ in the $e_-$ case, and by zero on even powers of $x$ in the $e_+$ case. Im $(\psi^+_n) \subset P[n]e_\pm$ and the surjectivity of these maps follows by a similar argument to the proof of Proposition 1.19 and from the fact that $Q \in P[n]e_\pm$ for $n < 0$ implies that $Q = LD^{-n}e_\pm$ for some $L \in P[0]$. Combining these maps gives an isomorphism of $R$-modules:

$$\psi : \bigoplus_{n \in \mathbb{Z}} (H^+[n] \oplus H^-[n]) \xrightarrow{\sim} H_{t,c}(R).$$

We can consider this direct sum as a subring of $\text{End}_R(R[x])$ given by the action of $H^+[n] \oplus H^-[n]$ on $x^n$, defined by $(f^+, f^-)x^n = f^+(t)x^{2n} + n$ and $(f^+, f^-)x^{2n+1} = f^-(t)x^{2n+1} + n$. Note that by Proposition 1.13 and the argument used in Proposition 1.21 there is an induced isomorphism:

$$\psi^{DP} : \bigoplus_{n \in \mathbb{Z}} (\text{Int}_R(H^+[n]) \oplus \text{Int}_R(H^-[n])) \xrightarrow{\sim} H_{t,c}^{DP}(R).$$

So to understand $H_{t,c}^{DP}(R)$, it suffices to understand $\text{Int}_R(H^+[n])$. By assumption, $R$ is a PID. Since $\text{Int}_R(H^+[n])$ is a submodule of a free module, $R[t]$, it is free. By the isomorphism $\psi^{DP}$, it follows that $H_{t,c}^{DP}(R)$ is free as well. \qed
Proposition 2.3. Let $R$ be a PID, and fix some $c \in R$. Then there exist coefficients $\alpha_{i,j,k}^{\pm} \in R$ and integers $d_{k,j}^{\pm} \in \mathbb{Z}_{\geq 1}$ yielding operators

$$
\Delta_{k_1,k_2}^{\pm} = \begin{cases} 
\frac{D^{k_1} \sum_{i=0}^{k_2-1} \alpha_{i,k_1,k_2}^{\pm} (L^\pm)^i}{2^{\varepsilon_{k_1,k_2}} e_\pm} & \text{if } k_1 > 0, \\
\frac{d_{k_1,k_2}^{\pm} (L^\pm - 2t)}{2^{\varepsilon_{k_1,k_2}} e_\pm} & \text{if } k_1 = 0,
\end{cases}
$$

where $L^+ = xD$ and $L^- = xD + 2c - 1$. Then, the set $\{\Delta_{n,k}^{\pm}, x^{n+1} \Delta_{0,k}^{\pm}\}_{n,k \geq 0}$ is a basis for $H_{1,c}^{DP}(R)$.

Proof. To obtain this basis for $H_{1,c}^{DP}(R)$, we first construct a basis for $\bigoplus_{n \in \mathbb{Z}} (\text{Int}_R(H^+[n]) \oplus \text{Int}_R(H^-[n]))$. First suppose $n \geq 0$. In this case, $H^+[n] = R[2t]$, and so the binomial coefficients $\binom{n}{k}$ form a basis for $\text{Int}_R(H^+[n])$. Since for every $k \geq 0$, $\psi^{DP}_{\text{Int}(H^+[n])} \left( \binom{n}{k} \right) = x^n \Delta_{0,k}^{\pm}$, it follows that $\{x^n \Delta_{0,k}^{\pm}\}_{n,k \geq 0}$ spans the set of non-negatively graded operators in $H_{1,c}^{DP}(R)$.

Now suppose $n < 0$. It follows that $\text{Int}_R(H^+[n])$ has a basis of the form $\left\{ \frac{1}{d_{-n,k}} D_{-n}^{\pm} (t) \sum_{i=0}^{k-1} \alpha_{i,-n,k}^{\pm} (2t)^i \right\}_{k \geq 0}$. Since

$$
\psi^{DP}_{\text{Int}(H^+[n])} \left( \frac{1}{d_{-n,k}} D_{-n}^{\pm} (t) \sum_{i=0}^{k-1} \alpha_{i,-n,k}^{\pm} (2t)^i \right) = \Delta_{-n,k}^{\pm},
$$

It follows that $\{\Delta_{n,k}^{\pm}\}_{n,k \geq 0}$ spans the set of negatively graded operators in $H_{1,c}^{DP}(R)$, completing the proof. \hfill \Box

2.2 Basis for $H_{1,c}^{DP}(R[c])$

In this section, we prove a similar result for $H_{1,c}^{DP}(R[c])$. In this case, we can even construct an explicit basis for $H_{1,c}^{DP}(R[c])$ as an $R[c]$-module.

Theorem 2.4. For any integers $k_1, k_2 \geq 0$, consider the operators

$$
\Delta_{k_1,k_2}^{\pm} = \begin{cases} 
\frac{D^{k_1} \prod_{i=0}^{k_2-1} (xD - 2(1 + m_1(k_1)))}{2^{m_1(k_1)+k_2} (m_1(k_1) + k_2)!} e_+, \\
\frac{D^{k_1} \prod_{i=0}^{k_2-1} (xD - 2c - 1 - 2(i + m_0(k_1)))}{2^{m_0(k_1)+k_2} (m_0(k_1) + k_2)!} e_-, \n\end{cases}
$$

where $m_0(k_1) = \left\lceil \frac{k_1}{2} \right\rceil$ for $\delta = 0, 1$. Then the set $\{\Delta_{n,k}^{\pm}, x^{n+1} \Delta_{0,k}^{\pm}\}_{n,k \geq 0}$ is an $R[c]$-basis for $H_{1,c}^{DP}(R[c])$.

Recall in the proof of Theorem 2.2 we proved that $H_{1,c}^{DP}(R)$ is isomorphic to the direct sum $\bigoplus_{n \in \mathbb{Z}} (\text{Int}_R(H^+[n]) \oplus \text{Int}_R(H^-[n]))$ for any domain $R$. To prove Theorem 2.4 we make use of this fact by constructing a basis for $\text{Int}_R(H^+[n])$.

Proposition 2.5. $(\psi^{DP})^{-1} \left( \{\Delta_{n,k}^{\pm}, x^{n+1} \Delta_{0,k}^{\pm}\}_{n,k \geq 0} \right)$ is a basis for $\bigoplus_{n \in \mathbb{Z}} (\text{Int}_{R[c]}(H^+[n]) \oplus \text{Int}_{R[c]}(H^-[n]))$ as an $R[c]$-module.

Proof. For any $k \geq 0$, consider the polynomials

$$
L^+(t) = \prod_{i=0}^{m_1(k)-1} (2t - 2i - 1 - 2c), \quad \text{and} \quad L^-(t) = \prod_{i=0}^{m_0(k)-1} (2t - 2i + 1 - 2c).
$$

Borrowing notation from the proof of Theorem 2.2 note that $D^+_k(t) = 2^{m_1(k)} m_1(k)! L^+_k(t)$ and $D^-_k(t) = 2^{m_0(k)} m_0(k)! L^-_k(t)$. Note that the statement of the Proposition is equivalent to the following four statements:

1. The set $\{ \binom{t}{k} \}_{k \geq 0}$ is an $R[c]$-basis for $\text{Int}_{R[c]}(H^+[n])$ for $n \geq 0$.
2. The set $\left\{ L^+_n(k+m_1(-n)) \right\}_{k \geq 0}$ is an $R[c]$-basis for $\text{Int}_{R[c]}(H^+[n])$ for $n < 0$.
3. The set $\{ \binom{t}{k} \}_{k \geq 0}$ is an $R[c]$-basis for $\text{Int}_{R[c]}(H^-[n])$ for $n \geq 0$.
4. The set $\left\{ L^-_n(k+m_0(-n)) \right\}_{k \geq 0}$ is an $R[c]$-basis for $\text{Int}_{R[c]}(H^-[n])$ for $n < 0$. 

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We will only prove (1) and (2), since (3) and (4) are proved similarly. For (1), assume \( n \geq 0 \) and let \( f(2t) \in R[2t] \). By induction, we can find coefficients \( \alpha \in R[c] \) such that \( f(t) = \sum_{k \geq 0} \alpha_k \prod_{i=0}^{k-1} (t - 2i) \). Then \( f(2t) = \sum_{k \geq 0} \alpha_k 2^k k! \binom{t}{k} \). By Lemma 1.17 if \( d \mid f(2n) \) for all \( n \) then \( d \alpha_k 2^k k! \). This means that \( \frac{f(t)}{d} = \sum_{k \geq 0} \frac{\alpha_k 2^k k!}{d} \binom{t}{k} \). Since \( H^+[n] = R[2t] \) when \( n \geq 0 \), it follows that \( \{ \binom{t}{k} \}_{k \geq 0} \) is a basis for \( \text{Int}_{R[c]}(H^+[n]) \).

For (2), assume \( n < 0 \) and let \( m = m_1(-n) \). Note that \( D^+_{-n}(t) = L^+_{-n}(t) \prod_{i=0}^{m-1} (2t - 2i) \). Let \( f(2t) \in R[2t] \) be arbitrary and suppose \( d|L^+_{-n}(t) \prod_{i=0}^{m-1} (2t - 2i) f(t) \) since \( L^+_{-n}(t) \) is a primitive polynomial, it follows that \( d|\prod_{i=0}^{m-1} (2t - 2i) f(t) \). Writing \( f(t) = \sum_{j \geq 0} \alpha_j t^j \) we have \( d|\prod_{i=0}^{m-1} (2t - 2i) f(t) = \sum_{j \geq 0} 2^{m+j}(m+j)\alpha_j \binom{t}{m+j} \). By Lemma 1.17
\[
\frac{D^+_{-n}(t)f(t)}{d} = \frac{L^+_{-n}(t) \prod_{i=0}^{m-1} (2t - 2i) f(t)}{d} = \sum_{j \geq 0} 2^{m+j}(m+j)\alpha_j \binom{t}{m+j}.
\]
The claim follows, since \( H^+[n] = D^+_{-n}(t)R[2t] \) when \( n < 0 \).

The above proposition immediately implies Theorem 2.4.

### 2.3 Hilbert Series for \( H^{DP}_{1,c}(R) \)

**Definition 2.6.** Let \( M \) be a module over a domain \( R \) and suppose we have a filtration \( M = \bigcup_{i \geq 0} M_i \). Let \( \text{gr}(M) \) be the associated graded module of \( M \) with respect to the filtration, i.e. \( \text{gr}(M) = M_0 \oplus \bigoplus_{i \geq 1} (M_i/M_{i-1}) \). Let \( \text{gr}_n(M) \) be the \( n \)-th graded component of \( \text{gr}(M) \). The Hilbert series of \( M \) is defined as
\[
\text{HS}_M(z) = \sum_{n \geq 0} \dim_R(\text{gr}_n(M)) z^n.
\]

In the following proposition, we show that the Hilbert series of the rational Cherednik algebra of type \( A_1 \) remains unchanged after the divided power extension construction.

**Proposition 2.7.** Let \( R \) be a principal ideal domain. Then:
1. \( \text{HS}_{H_{1,c}(R)}(z) = \frac{2}{(1-z)^2} \).
2. \( \text{HS}_{H^{DP}_{1,c}(R[c])}(z) = \frac{2}{(1-z)^2} \).
3. For any \( c \in R \), \( \text{HS}_{H^{DP}_{1,c}(R)}(z) = \frac{2}{(1-z)^2} \).

**Proof.** (1) immediately follows from the PBW Theorem, since \( H_{1,c}(R) \) is generated by elements of the form \( x^iD^e \). This implies that \( \dim_R(\text{gr}_n(H_{1,c}(R))) = 2(n+1) \). (2) follows from a similar argument, since by Theorem 2.4 \( \dim_R(\text{gr}_n(H_{1,c}(R[c]))) = 2(n+1) \). (3) is the same as (2), since Proposition 2.3 shows that the basis for \( H^{DP}_{1,c}(R) \) has the same degree as the basis for \( H^{DP}_{1,c}(R[c]) \).

### 2.4 The Lie Algebra \( \mathfrak{sl}_2 \)

**Definition 2.8.** A triple of operators \( E, H, F \) is said to be an \( \mathfrak{sl}_2 \)-triple if:
- \([H, E] = 2E\]
- \([H, F] = -2F\]
- \([E, F] = H\]

**Proposition 2.9.** In \( H_{1,c}(R[c]) \) let \( H = (xD + \frac{1-2c}{c})e_+ \), \( E = -\frac{1}{2}x^2e_+ \), and \( F = \frac{1}{2}D^2e_+ \). Then \( E, H, F \) form an \( \mathfrak{sl}_2 \)-triple. It follows that \( e_+H_{1,c}(R[c])e_+ \) is isomorphic to a quotient of \( U(\mathfrak{sl}_2) \) by the central character \( (C + \frac{1-2c}{c})e_+ \), where \( C \) is the Casimir operator \( C = EF + FE + \frac{H^2}{2} \).

This map suggests a divided power structure on this quotient of \( U(\mathfrak{sl}_2) \). An immediate corollary to Theorem 2.3 states:

**Corollary 2.10.** The set \( \{ \Delta_{2n,k}, x^{2n+2} \Delta_{2n+2,k} \}_{n,k \geq 0} \) is an \( R[c] \)-basis for \( e_+H^{DP}_{1,c}(R[c])e_+ \).

Writing this basis in \( \mathfrak{sl}_2 \)-triple gives us a basis for a divided power structure on \( U(\mathfrak{sl}_2) \). Let
\[
\Sigma_{a,b,c} = \frac{(-2E)^2(2F)^2 \prod_{i=0}^{m-1} (H - \frac{1-2c}{c} - 2(i + m_1(2b))}{2m_1(2b)+(c!)} \in U(\mathfrak{sl}_2(\mathbb{Q})).
\]

Then the set \( \{ \Sigma_{a,0,k}, \Sigma_{a+1,0,k} \}_{a,k \geq 0} \) is a basis for a divided power structure on a quotient of \( U(\mathfrak{sl}_2) \).

**Note:** This basis of divided powers is different from the basis given in [Jan07]. Indeed the basis given there is symmetric, containing both divided powers of \( E \) and \( F \). Our divided power extension contains no divided powers of \( E \) (indeed the denominator above does not depend on \( a \) at all), but it has more divided powers of \( F \).
3 Abstract Construction of $H_{1,c}^{DP}(R)$

In this section, we prove Theorem \ref{thm:abstractconstruction}, which takes some setup to properly state.

3.1 Grothendieck Differential Operators

Before stating the main theorem, we recall a purely algebraic notion of differential operators due to Grothendieck. The results from this section can be found in \cite{jeffery99}.

**Definition 3.1** (Grothendieck Differential Operators). Let $R \subset A$ be a pair of commutative rings. For any $a \in A$, let $\mathfrak{p}$ be the multiplication by $a$ operator on $A$. We define the $R$-linear differential operators on $A$ of order at most $i$, denoted $\text{Diff}_R(A)$, inductively in $i$.

- $\text{Diff}_R(A)^0 = \text{Hom}_A(A, A) = \{\tau : a \in A\}$
- $\text{Diff}_R(A)^i = \{f \in \text{Hom}_R(A, A) : [f, \tau] \in \text{Diff}_R(A)^{i-1}, \forall a \in A\}$

Let $\text{Diff}_R(A) = \bigcup_{i=0}^{\infty} \text{Diff}_R(A)^i \subset \text{Hom}_R(A, A)$ be the algebra of differential operators of $A$ over $R$. When $R$ is clear, we simply write $\text{Diff}(A)$ to denote $\text{Diff}_R(A)$.

For the results in Section \ref{3.2}, it suffices to consider differential operators of polynomial algebra. The following results describe the structure of the ring of differential operators completely.

**Definition 3.2.** For any $\lambda \in \mathbb{N}^n$ let $\partial^\lambda$ be the Hasse derivative, i.e. the $R$-linear operator on $R[x_1, \ldots, x_n]$, given on the basis by

$$
\partial^\lambda(x_1^{\beta_1} \cdots x_n^{\beta_n}) = \left(\frac{\beta_1}{\lambda_1}\right) \cdots \left(\frac{\beta_n}{\lambda_n}\right) x_1^{\beta_1-\lambda_1} \cdots x_n^{\beta_n-\lambda_n}.
$$

In rings where $\lambda_1! \cdots \lambda_n! \in R^\times$, the operator $\partial^\lambda$ is simply $\partial^\lambda = \frac{1}{\lambda_1! \cdots \lambda_n!} \partial^{\lambda_1} \cdots \partial^{\lambda_n}$.

**Proposition 3.3.** Let $A = R[x_1, \ldots, x_n]$. Then $\text{Diff}_R(A) = \bigoplus_{\lambda \in \mathbb{N}^n} A\partial^\lambda$, where multiplication is given by composition of operators.

Since we are dealing with differential operators defined on a punctured line, we need to consider rings of differential operators over localized polynomial rings as well.

**Proposition 3.4.** Let $R \subset A$ be rings where $A$ is of finitely generated over $R$. Let $W \subset A$ be a multiplicative subset. Then $W^{-1}\text{Diff}_R(A) \cong \text{Diff}_R(W^{-1}A)$.

**Corollary 3.5.** $\text{Diff}_R(R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) = \bigoplus_{\lambda \in \mathbb{N}^n} R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\partial^\lambda$, where multiplication is given by composition of operators.

3.2 Abstract Construction

In this section, we would like to naturally define the ring $H_{1,c}^{DP}(R[c])$ as a space of differential operators preserving some sets of the form $x^k|x^r R[x]$, for some $k \in \mathbb{Z}$ and $r \in R$. Here $|x^r| = |x|^r$ is fixed by the action of $\mathfrak{S}_2$, and $\frac{\partial}{\partial |x^r|} = r |x|^{-2} \partial |x|^{-2}$. We will denote this ring as $\mathcal{H}_c(R)$, and its definition should be purely algebraic, similar to the definition of $\text{Diff}_R(A)$. First, we need a nice space of differential operators to work in.

**Definition 3.6.** For any domain of characteristic zero $R$, let $\mathcal{D}(R)$ be the ring

$$
\mathcal{D}(R) = \text{Diff}_R(R[x^{\pm 1}] \otimes_R \mathfrak{S}(R))
$$

where $\mathfrak{S}(R) = \text{Re}_+ \oplus \text{Re}_-$ is the ring acting on $R[x^{\pm 1}]$ the canonical way. Note that $\mathcal{D}(R) = \left(\text{Diff}_R(R[x^{\pm 1}]) \times R[\mathfrak{S}_2]\right)^{DP}$.

Our main theorem of the section can then be stated:

**Theorem 3.7.** For a domain of characteristic zero $R$ and $c \in R$, consider

$$
\mathcal{H}_c(R) = \{Q \in \mathcal{D}(R) : Q \text{ fixes } R[x] \text{ and } x^{-1}|x|^{1+2c} R[x]\}
$$

Then, $\mathcal{H}_c(R) \cong H_{1,c}^{DP}(R)$ if $c \notin \frac{1}{2} + \mathbb{Z}$.

To prove this theorem, it is useful to decompose $H_{1,c}(R[c])$ in the following way:

$$
H_{1,c}(R[c]) = e_+ H_{1,c}(R[c]) e_+ \oplus e_+ H_{1,c}(R[c]) e_- \oplus e_- H_{1,c}(R[c]) e_+ \oplus e_- H_{1,c}(R[c]) e_-
$$

Expressing each of these summands in a similar way to $\mathcal{H}_c(R)$ helps with the proof. Note that $e_{\pm} H_{1,c}^{DP}(R)e_{\pm} = (e_{\pm} H_{1,c}(R)e_{\pm})^{DP}$, where $e_{\pm}$ can be either $e_+$ or $e_-$. 
Definition 3.8. For a domain of characteristic zero \( R \) and \( c \in R \), consider the following sets:

- \( B_1(R) = \{ Q \in e_+ \mathcal{D}(R)e_+ : Q \text{ fixes } R[x] \text{ and } Q \text{ fixes } |x|^{1+2c}R[x] \} \).
- \( \overline{B_1(R)} = \{ Q \in e_- \mathcal{D}(R)e_- : x^{-1}Qx \text{ fixes } R[x] \text{ and } xQx^{-1} \text{ fixes } |x|^{1+2c}R[x] \} \).
- \( A_\epsilon(R) = \{ Q \in e_+ \mathcal{D}(R)e_+ : Q \text{ fixes } R[x] \text{ and } xQx \text{ fixes } |x|^{1+2c}R[x] \} \).
- \( A_\overline{\epsilon}(R) = \{ Q \in e_+ \mathcal{D}(R)e_+ : Qx \text{ fixes } R[x] \text{ and } xQx^{-1} \text{ fixes } |x|^{1+2c}R[x] \} \).

Proposition 3.9. If \( r \notin \frac{1}{2} + \mathbb{Z} \) then \( B_1(R) = e_+H_{1,c}^{DP}(R)e_+ \), \( \overline{B_1(R)} = e_-H_{1,c}^{DP}(R)e_- \), \( A_\epsilon(R) = e_+H_{\epsilon,c}^{DP}(R)e_+ \), \( \overline{A_\epsilon(R)} = e_-H_{\epsilon,c}^{DP}(R)e_- \), and \( A_\overline{\epsilon}(R) = e_+H_{\overline{\epsilon},c}^{DP}(R)e_+ \).

Proof. We will only prove the first equality, \( B_1(R) = e_+H_{1,c}^{DP}(R)e_+ \), the rest follow similarly. First, we show that \( e_+H_{1,c}^{DP}(R)e_+ \subset B_1(R) \). Let \( Q \in e_+H_{1,c}^{DP}(R)e_+ \) be some operator. If we write \( Q = \sum_{n \in \mathbb{Q}} Q_n \), where \( \deg Q_n = n \), it suffices to check that \( Q_n \in B_1(R) \). So without loss of generality, assume \( Q \) is graded of degree \( n \). If \( n \geq 0 \), clearly \( Q \in B_1(R) \). If \( n < 0 \), then \( Q \) can be expressed as \( Q = e_+ED^{-n}e_+ / d \) for some \( L \) of degree 0 and \( d \in \mathbb{Z} \).

To check that \( Q \) fixes \( R[x] \) and \( |x|^{1+2c}R[x] \), it suffices to check the action of \( Q \) on monomials. To start, let’s consider the action of \( Q \) on \( x^k \) for some \( k \geq 0 \). If \( k \) is odd, \( Qx^k = 0 \in R[x] \). If \( k \) is even, there are two cases. If \( k \geq -n \), then \( Qx^k = \lambda D_\alpha x \) \( x^{k+n} / d \in R[x] \) since \( k + n \geq 0 \). Recall notation from the proof of Theorem 2.2. If \( k < -n \), note that \( D_\alpha x (k) = 0 \), so \( Qx^k = 0 \in R[x] \). A similar thing happens for \( |x|^{1+2c}R[x] \), since \( D_\alpha = (k + 1 + 2c) = 0 \) for even \( k < -n \). This shows that \( e_+H_{1,c}^{DP}(R)e_+ \subset B_1(R) \).

Next, we show that \( B_1(R) \subset e_+H_{1,c}^{DP}(R)e_+ \). As before, we can assume that \( Q \) is graded of degree \( n \). Let \( f(t) \) be the polynomial representing the action of \( Q \), \( Qx^k = f(k)x^{k+n} / d \). If \( n \geq 0 \), write \( f(t) = \sum_{j \geq 0} \alpha_j t^j \) for some \( \alpha_j \in R \otimes Q \). This tensor product with \( Q \) arises from the fact that \( e_+\mathcal{D}(R)e_+ = \{ e_+\operatorname{Diff}_R(R[x]) \}^{DP} \), hence operators might have coefficients in \( R \otimes \mathbb{Q} \). Then

\[
Q = e_+x^n \sum_{j \geq 0} \alpha_j (xD)^j e_+ \in e_+H_{1,c}^{DP}(R)e_+.
\]

Now suppose \( n < 0 \). Notice that \( f(k) = f(k + 1 + 2c) = 0 \) for all even \( k \) satisfying \( 0 \leq k < -n \), so \( \prod_{j=0}^{n-2} (t-2j)(t-2j-1-2c) \) divides \( f(t) \). This is exactly the action of the Dunkl operator \( D_\alpha e_+ \). Also note that this depends on the fact that \( e \notin \frac{1}{2} + \mathbb{Z} \), otherwise, the linear factors could overlap. Let \( L(t) = \sum_{j \geq 0} \beta_j t^j \) be the quotient of this division for some \( \beta_j \in R \otimes \mathbb{Q} \). Then

\[
Q = e_+D^{-n} \sum_{j \geq 0} \beta_j (xD)^j e_+,
\]

completing the proof.

Proposition 3.10. \( \mathcal{H}_c(R) \cong B_1(R) \oplus \overline{B_1(R)} \oplus A_\epsilon(R) \oplus \overline{A_\epsilon(R)} \).

Proof. Let \( H = B_1(R) \oplus \overline{B_1(R)} \oplus A_\epsilon(R) \oplus \overline{A_\epsilon(R)} \). Consider both \( \mathcal{H}_c(R) \) and \( H \) as subrings of \( \operatorname{End}_R(R[x]) \). First we show that \( H \subset \mathcal{H}_c(R) \). Let \( Q \in H \) be a graded operator, say \( Q = e_+Qe_+ + e_-Qe_- + e_+Qe_- + e_-Qe_+ \). First we show that \( Q \) fixes \( R[x] \). By Proposition 3.3

\[
Q(R[x]) = e_+Qe_+(R[x]) + e_-Qe_-(R[x]) + e_+Qe_-(R[x]) + e_-Qe_+(R[x])
= e_+Qe_+(R[x]) + e_-Qe_-(-xR[x]) + e_+Qe_+(xR[x]) + e_-Qe_+(xR[x])
\subset R[x] + R[x] + R[x] + \subset R[x]
\]

because \( x^{-1}e_-Qe_-(xR[x]) \subset R[x] \) implies that \( e_-Qe_-(xR[x]) \subset R[x] \). Let \( y = x^{-1}|x|^{1+2c} \). By Proposition 3.3 we have

\[
Q(yR[x]) = e_+Qe_+(yR[x]) + e_-Qe_-(yR[x]) + e_+Qe_-(yR[x]) + e_-Qe_+(yR[x])
= e_+Qe_+(xyR[x]) + e_-Qe_-(-yR[x]) + e_+Qe_-(yR[x]) + e_-Qe_+(yR[x])
\subset yR[x] + R[x] + yR[x] + \subset yR[x].
\]

So \( H \subset \mathcal{H}_c(R) \). To show that \( \mathcal{H}_c(R) \subset H \), suppose \( Q \in \mathcal{H}_c(R) \) is some graded operator. If \( \deg Q \) is even, then \( Q = e_+Qe_+ + e_-Qe_- \). Since \( Q(R[x]) \subset R[x] \), \( Q(R[x]) = e_+Qe_+(R[x]) + e_-Qe_-(R[x]) \subset R[x] \), and \( e_+Qe_+ + e_-Qe_- \) act non-trivially on only even and odd degrees of \( x \) respectively, it follows that \( e_+Qe_+(R[x]) \subset R[x] \) and \( x^{-1}e_-Qe_-(xR[x]) \subset R[x] \). Similarly, we can deduce that \( e_+Qe_+(|x|^{1+2c}R[x]) \subset |x|^{1+2c}R[x] \) and \( e_-Qe_-(x^{-1}|x|^{1+2c}R[x]) \subset x^{-1}|x|^{1+2c}R[x] \). So \( e_+Qe_+ \subset B_1(R) \) and \( e_-Qe_- \subset B_1(R) \). Similarly, in the case when \( \deg Q \) is odd we can show that \( e_+Qe_+ \in A_\epsilon(R) \) and \( e_-Qe_- \in A_\epsilon(R) \). This shows that \( \mathcal{H}_c(R) \subset H \), completing the proof.
To prove Theorem \[\text{[3.11]}\] note that by Proposition \[\text{[1.13]}\] $H_{1,c}^{DP}(R) \cong e_+ H_{1,c}^{DP}(R) e_+ \oplus e_- H_{1,c}^{DP}(R) e_- \oplus e_0 H_{1,c}^{DP}(R) e_0$. If $c \notin \frac{1}{2} + \mathbb{Z}$, by Proposition \[\text{[3.9]}\] and Proposition \[\text{[1.10]}\] we have
\[
\mathcal{H}_c(R) \cong \mathcal{B}_c(R) \oplus \mathcal{B}_c(R) \oplus \mathcal{A}_c(R) \oplus \mathcal{A}_c(R)
\cong e_+ H_{1,c}^{DP}(R) e_+ \oplus e_- H_{1,c}^{DP}(R) e_- \oplus e_0 H_{1,c}^{DP}(R) e_0
\cong H_{1,c}^{DP}(R).
\]
This concludes the proof.

### 3.3 The case $c \in \frac{1}{2} + \mathbb{Z}$

Interestingly, the case $c \in \frac{1}{2} + \mathbb{Z}$ appears throughout the theory of Cherednik algebras. In the case of our construction, this exception appears because the polynomial representing the action of the Dunkl operator has multiplicity two zeroes, when our construction can only encode multiplicity one zeroes. A future direction would be to extend our construction so that it works even when $c \in \frac{1}{2} + \mathbb{Z}$. Pavel Etingof suggested that the construction should preserve an infinite family of subsets of functions in $x$ involving $|x|$ which converge to some set of functions involving $|x|$ and $\log(x)$ as $c$ approaches a half-integer. This is useful by the following proposition:

**Proposition 3.11.** For $f(t) \in \mathbb{Z}[t]$ and $F \in \mathbb{Z}[x, \partial_x]$ the operator mapping $x^n$ to $f(n)x^{n+\frac{d}{2}}$ for some $d \in \mathbb{Z}$,
\[
F(x^n \log(x)) = \frac{df}{dt}(n)x^{n+\frac{d}{2}} + f(n)x^{n+\frac{d}{2}} \log(x).
\]

Here we let $\partial_x(\log x) = \frac{1}{x}$.

So using $\log(x)$, we can encode information about the multiplicity-two roots about the polynomial which represents the action of the operator. Since the Dunkl operator has roots of at most multiplicity two, there is a construction which should work in all cases.

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### References

[BC13] Martina Balagović and Harrison Chen. Representations of rational Cherednik algebras in positive characteristic. *Journal of pure and applied algebra*, 217(4):716–740, 2013.

[BFG06] Roman Bezrukavnikov, Michael Finkelberg, and Victor Ginzburg. Cherednik algebras and hilbert schemes in characteristic $p$. *Representation Theory of the American Mathematical Society*, 10(11):254–298, 2006.

[Che93] Ivan Cherednik. The Macdonald constant-term conjecture. *International Mathematics Research Notices*, 1993(6):165–177, 1993.

[Che05] Ivan Cherednik. *Double affine Hecke algebras*, volume 319. Cambridge University Press, 2005.

[CK18] Merrick Cai and Daniil Kalinov. The Hilbert series of the irreducible quotient of the rational Cherednik algebra of type $A_{n-1}$ in characteristic $p$ for $p | n - 1$. *arXiv preprint arXiv:1811.04910*, 2018.

[DS+14] Sheela Devadas, Steven V Sam, et al. Representations of rational Cherednik algebras of $G(m, r, n)$ in positive characteristic. *Journal of Commutative Algebra*, 6(4):525–559, 2014.

[DS16] Sheela Devadas and Yi Sun. The polynomial representation of the type $A_n$ rational Cherednik algebra in characteristic $p/n$. 2016.

[EM10] Pavel Etingof and Xiaoguang Ma. Lecture notes on Cherednik algebras. *arXiv preprint arXiv:1001.0432*, 2010.

[Eti07] Pavel Etingof. *Calogero-Moser systems and representation theory*, volume 4. European Mathematical Society, 2007.

[Gor03] Iain Gordon. Baby Verma modules for rational Cherednik algebras. *Bulletin of the London Mathematical Society*, 35(3):321–336, 2003.
[Jan07] Jens Carsten Jantzen. *Representations of algebraic groups*. Number 107. American Mathematical Soc., 2007.

[Jef18] Jack Jeffries. Topics in commutative algebra: Differential operators. 2018.

[Lat05] Frédéric Latour. Representations of rational cherednik algebras of rank one in positive characteristic. *Journal of Pure and Applied Algebra*, 195(1):97–112, 2005.

[Lon16] Gus Lonergan. A strong splitting of the frobenius morphism on the algebra of distributions of $sl_2$. *arXiv preprint arXiv:1611.07512*, 2016.