HEAPS AND TWO EXPONENTIAL STRUCTURES

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ABSTRACT. Let $Q = (Q_1, Q_2, \ldots)$ be an exponential structure, let $M(n)$ be the number of minimal elements of $Q_n$ and set $M(0) = 1$. Then a sequence of numbers \( r_n(Q_n) \) defined by the equation

$$
\sum_{n \geq 1} r_n(Q_n) \frac{z^n}{n! M(n)} = -\log \left( \sum_{n \geq 0} \frac{(-1)^n z^n}{n! M(n)} \right).
$$

Let $\widetilde{Q}_n$ denote the poset $Q_n$ with a 0 adjoined, $\mu_{Q_n}$ be the Möbius function on the poset $Q_n$. Stanley proved $r_n(Q_n) = (-1)^n \mu_{Q_n}(0, 1)$. This implies the numbers $r_n(Q_n)$ are integers. In this paper we study the case $Q_n = \Pi_n^{(r)}$ where $\Pi_n^{(r)}$ is the poset of set partitions of $[rn]$ whose block sizes are divisible by $r$ and the case $Q_n = \mathcal{Q}_n$ where $\mathcal{Q}_n$ is the poset of $r$-partitions of $[n]$. In both cases we give combinatorial interpretations of $r_n(Q_n)$ in terms of heaps by applying Cartier-Foata monoid identity, and further prove $r_n(\Pi_n^{(r)})$ are the generalized Euler numbers $E_{rn-1}$, $r_n(Q_n^{(r)})$ are the numbers of complete non-ambiguous trees by using bijections. This gives a new proof of Welker’s theorem that $r_n(\Pi_n^{(r)}) = E_{rn-1}$ and implies the construction of $r$-dimensional complete non-ambiguous trees. As a bonus of applying the theory of heaps, we give a bijection between the set of complete non-ambiguous forests and the set of pairs of permutations with no common rise. This answers an open question raised by Aval et al.

1. INTRODUCTION

We denote by $\Pi_n$ the poset of all the set partitions of $[n]$ ordered by refinement, that is, define $\sigma \leq \pi$ if every block of $\sigma$ is contained in a block of $\pi$. Let $\rho \in \Pi_n$ be the minimal element of $\Pi_n$, i.e., $\rho = \{\{1\}, \{2\}, \ldots, \{n\}\}$. Consider an interval $[\sigma, \pi]$ in the poset $\Pi_n$, suppose $\pi = \{B_1, B_2, \ldots, B_k\}$ and $B_i$ is partitioned into $\lambda_i$ blocks in $\sigma$, we have $[\sigma, \pi] \cong \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}$. For the particular case $\sigma = \rho$, we have $[\rho, \pi] \cong \Pi_{|B_1|} \times \Pi_{|B_2|} \times \cdots \times \Pi_{|B_k|}$. If we set $a_j = |\{i : \lambda_i = j\}|$ for every $j$, then we can rewrite

$$
[\sigma, \pi] \cong \Pi_{a_1}^{a_1} \times \Pi_{a_2}^{a_2} \times \cdots \times \Pi_{a_n}^{a_n}.
$$

The poset $\Pi_n$ of set partitions is the archetype of exponential structures. The concept of exponential structure was introduced by Stanley as a generalization of compositional and exponential formulas [6, 7, 3]. An exponential structure is a sequence $Q = (Q_1, Q_2, \ldots)$ of posets such that

1. for each $n \in \mathbb{N}^+$, $Q_n$ is finite and has a unique maximal element $\hat{1}$, and every maximal chain of $Q_n$ has $n$ elements,
2. for $\pi \in Q_n$, the interval $[\pi, \hat{1}]$ is isomorphic to the poset $\Pi_k$ of set partitions for some $k$.
3. the subposet $\Lambda_\pi = \{\sigma \in Q_n : \sigma \leq \pi\}$ of $Q_n$ is isomorphic to $Q_1^{a_1} \times Q_2^{a_2} \times \cdots \times Q_n^{a_n}$ for unique $a_1, a_2, \ldots, a_n \in \mathbb{N}$.

Suppose $\pi \in Q_n$ and $\rho$ is a minimal element of $Q_n$ satisfying $\rho \leq \pi$. By (1) and (2), $[\rho, \hat{1}] \cong \Pi_n$. It follows from eq. (1.1) that $[\rho, \pi] \cong \Pi_{a_1}^{a_1} \times \Pi_{a_2}^{a_2} \times \cdots \times \Pi_{a_n}^{a_n}$ for

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unique \(a_1, a_2, \ldots, a_n \in \mathbb{N}\) satisfying \(\sum_i i a_i = n\) and \(\sum_i a_i = |\pi|\). In particular, if \(\rho_1\) is another minimal element of \(Q_n\) satisfying \(\rho_1 \leq \pi\), then we have \([\rho_1, \pi] \cong [\rho, \pi]\).

We will define the numbers \(r_n(Q_n)\) associated with an exponential structure \(Q = (Q_1, Q_2, \ldots)\) in the following way. Let \(M(n)\) be the number of minimal elements of \(Q_n\) for \(n \geq 1\), and set \(M(0) = 1\). Then a sequence of numbers \(\{r_n(Q_n)\}_{n \geq 1}\) is defined by the equation

\[
(1.2) \quad \sum_{n \geq 1} r_n(Q_n) \frac{z^n}{n! M(n)} = -\log \left( \sum_{n \geq 0} (-1)^n \frac{z^n}{n! M(n)} \right).
\]

Let furthermore \(\tilde{Q}\) be the poset \(Q_n\) with a \(0\) adjoined, \(\mu \tilde{Q}\) be the Möbius function on the poset \(\tilde{Q}_n\), then from Chapter 5.5 of [7], we have

\[
(1.3) \quad \sum_{n \geq 1} \mu(Q_n)(0, \hat{1}) \frac{z^n}{n! M(n)} = -\log \left( \sum_{n \geq 0} \frac{z^n}{n! M(n)} \right),
\]

and therefore \(r_n(Q_n) = (-1)^n \mu \tilde{Q}_n(0, \hat{1})\). This implies the numbers \(r_n(Q_n)\) are integers for any exponential structure \(Q = (Q_1, Q_2, \ldots)\). In the case \(Q_n = \Pi_n\), the number \(M(n)\) of minimal elements in the poset \(\Pi_n\) is 1, then from eq. (1.3) and eq. (1.2) we get \(r_n(\Pi_n) = \mu \Pi_n(0, \hat{1}) = 0\) if \(n \geq 2\) and \(r_1(\Pi_1) = -\mu \Pi_1(0, \hat{1}) = 1\). There are three other classical examples of poset whose sequence forms an exponential structure. They are the poset \(Q_n(q)\) of direct sum decompositions of the \(n\)-dimensional vector space \(V_n\) over the finite field \(\mathbb{F}_q\), the poset \(\Pi_n^{(r)}\) of set partitions of \([rn]\) whose block sizes are divisible by \(r\) and the poset \(Q_n^{(r)}\) of \(r\)-partitions of \([n]\).

The poset \(Q_n(q)\) of direct sum decompositions of the \(n\)-dimensional vector space \(V_n\) over the finite field \(\mathbb{F}_q\) had been studied by Welker [10], who used the theory of free monoid to give an expression of the Möbius function \(\mu Q_n(q)(0, \hat{1})\), see Theorem 4.4 in [10], thereby an expression of \(r_n(Q_n(q))\). Here we aim to prove the numbers \(r_n(\Pi_n^{(r)})\) are the generalized Euler numbers \(E_{rn-1}\) and \(r_n(Q_n^{(2)})\) are the numbers of complete non-ambiguous trees. First we interpret the numbers \(r_n(\Pi_n^{(r)})\) and \(r_n(Q_n^{(2)})\) in terms of pyramids by applying Cartier-Foata monoid identity on the posets \(\Pi_n^{(r)}\) and \(Q_n^{(r)}\). Then we build a bijection between the set of the pyramids counted by \(r_n(\Pi_n^{(r)})\) and the set of permutations of \([rn-1]\) with descent set \([r, 2r, \ldots, rn-r]\), and a bijection between the set of the pyramids counted by \(r_n(Q_n^{(2)})\) and the set of complete non-ambiguous trees of size \(2n-1\). This gives a new proof of Welker’s theorem that \(r_n(\Pi_n^{(r)}) = E_{rn-1}\). Welker [10] proved \(r_n(\Pi_n^{(r)}) = |\mu \Pi_n^{(r)}(0, \hat{1})| = E_{rn-1}\) by showing \(\Pi_n^{(r)}\) is a CL-shellable (chain lexicographic shellable) poset and the absolute value of its Möbius number \(\mu \Pi_n^{(r)}(0, \hat{1})\) is equal to the number of descending chains in the poset \(\Pi_n^{(r)}\). Furthermore, our combinatorial interpretation of \(r_n(Q_n^{(r)})\) implies the construction of \(r\)-dimensional complete non-ambiguous trees. We also give a bijection between the set of complete non-ambiguous forests and the set of pairs of permutations with no common rise by using heaps as intermediate objects. This answers an open question raised by Aval et al.

This paper is organized as follows. In section 2 we introduce heap, pyramid and Cartier-Foata identity where a heap is connected with a unique acyclic orientation of a graph. In section 3 and 4 we prove our main results: The numbers \(r_n(\Pi_n^{(r)})\) are the generalized Euler numbers \(E_{rn-1}\) and \(r_n(Q_n^{(2)})\) are the numbers of complete non-ambiguous trees. In subsection 4.2 we show the bijection between the set
of complete non-ambiguous forests and the set of pairs of permutations with no common rise.

2. Heap, Monoid and Cartier-Foata identity

The theory of heaps was introduced by Viennot to interpret the elements of the Cartier-Foata monoid in a geometric manner, see [8, 5]. Here we will adopt the notations from [8, 5]. Let \( B \) be a set of pieces with a symmetric and reflexive binary relation \( \mathcal{R} \), i.e., \( a R b \Leftrightarrow b R a \) and \( a R a \) for every \( a, b \in B \). A heap is a triple \( H = (P, \leq, \ell) \) where \( (P, \leq) \) is a poset and \( \ell \) is a map \( \ell : P \to B \) satisfying

1. For every \( x, y \in P \), if \( \ell(x) \mathcal{R} \ell(y) \), then \( x \) and \( y \) are comparable, i.e., \( x \leq y \) or \( y \leq x \).
2. For every \( x, y \in P \), if \( y \) covers \( x \) (i.e., \( x \leq y \) and for any \( z \) such that \( x \leq z \leq y \), \( z = x \) or \( z = y \)), then \( \ell(x) \mathcal{R} \ell(y) \).

Conditions (1) and (2) imply the relation \( \leq \) is the transitive closure of the relation \( \mathcal{R}^+ \) defined by: for \( x, y \in P \), \( x \mathcal{R}^+ y \) if and only if \( x \leq y \) and \( \ell(x) \mathcal{R} \ell(y) \). In view of this, a heap \( (P, \leq, \ell) \) can be identified as a graph \( G(P) \) with an acyclic orientation \( \gamma \) given as follows. Let \( G(P) \) be the graph whose vertex set is \( P \) and there is an edge between \( x \) and \( y \) where \( x \neq y \) if and only if \( x \) and \( y \) are comparable in the poset \( (P, \leq) \). Let \( \gamma(G(P)) \) be the graph \( G(P) \) with acyclic orientation \( \gamma \) that the edge \( \{x, y\} \) in \( G(P) \) has direction \( x \to y \) if \( x < y \) in the poset \( (P, \leq) \). In view of the transitivity and antisymmetry of \( \leq \) on the poset \( (P, \leq) \), the orientation \( \gamma \) is an acyclic orientation. Then a heap \( (P, \leq, \ell) \) defined by conditions (1) and (2) is equivalent to a directed graph \( (\gamma, G(P)) \) where \( \gamma \) is an acyclic orientation of \( G(P) \).

If \( \ell(x) = x \) for every \( x \in P \), a heap \( (P, \leq, \ell) \) can be regarded as the poset \( (P, \leq) \) satisfying conditions (1) and (2). Sometimes a heap \( (P, \leq, \ell) \) is simply denoted by \( P \). We denote by \( \emptyset \) the empty heap. A monoid \( S \) is a set that is closed under an associative binary operation \( \circ \) and has an identity element \( e \in S \) such that for all \( s \in S \), \( e \circ s = s \circ e = s \). Let \( \mathcal{H}(B, \mathcal{R}) \) be the set of heaps consisting of pieces from \( B \), including the empty heap \( \emptyset \). Then we define the product \( H_1 \circ H_2 \) of two heaps \( H_1, H_2 \) by the following: Suppose \( H_1 = (P_1, \leq_1, \ell_1) \), \( H_2 = (P_2, \leq_2, \ell_2) \), then \( H_1 \circ H_2 = (P_3, \leq_3, \ell_3) \) satisfies

1. \( P_3 \) is the disjoint union of \( P_1 \) and \( P_2 \).
2. \( \ell_3 \) is the unique map \( \ell_3 : P_3 \to B \) which restricts to \( P_1 \) (resp. \( P_2 \)) is \( \ell_1 \) (resp. \( \ell_2 \)).
3. The partial order \( \leq_3 \) is the transitive closure of the following relation \( \mathcal{R}^* \) for \( x, y \in P_3 \), \( x \mathcal{R}^* y \) if and only if one of (a), (b), (c) is satisfied.
   a. \( x \leq_1 y \) and \( x, y \in P_1 \)
   b. \( x \leq_2 y \) and \( x, y \in P_2 \)
   c. \( x \in P_1, y \in P_2 \) and \( \ell_1(x) \mathcal{R} \ell_2(y) \).

According to this, \( \mathcal{H}(B, \mathcal{R}) \) is closed under the associative product \( \circ \). If \( H_2 = \emptyset \) or \( H_1 = \emptyset \), then we have \( H_1 \circ \emptyset = \emptyset \circ H_1 = H_1 \). Therefore \( \mathcal{H}(B, \mathcal{R}) \) is a monoid with an identity element \( \emptyset \). We use \( |H| \) to denote the number of pieces in \( H \). A trivial heap \( T \) is a heap consisting of pieces that are pairwise unrelated, i.e., \( \ell(x) \mathcal{R} \ell(y) \) for any \( x, y \) in \( T \). Let \( \mathcal{T}(B, \mathcal{R}) \) be the set of trivial heaps contained in the set \( \mathcal{H}(B, \mathcal{R}) \). Let \( \mathbb{Z}[[\mathcal{H}(B, \mathcal{R})]] \) be the ring of all formal power series in \( \mathcal{H}(B, \mathcal{R}) \) with coefficients in the commutative ring \( \mathbb{Z} \), then the Cartier-Foata identity in \( \mathbb{Z}[[\mathcal{H}(B, \mathcal{R})]] \) states

\[
\sum_{H \in \mathcal{H}(B, \mathcal{R}) \atop T \in \mathcal{T}(B, \mathcal{R})} (-1)^{|T|} \cdot (T \circ H) = \emptyset.
\]

The Cartier-Foata monoid is a set \( \mathcal{M}(B, \mathcal{R}) \) of words \( w \) with letters from the set \( \{x_a : a \in B\} \) such that two adjacent letters \( x_a, x_b \) of \( w \) commute if \( \ell(a) \mathcal{R} \ell(b) \). Two words \( u, v \) are equivalent in \( \mathcal{M}(B, \mathcal{R}) \), i.e., \( u = v \) if and only if we can transform \( u \)
into $y$ by a sequence of transpositions of two adjacent letters $x_a, x_b$ if $\ell(a) \not\equiv \ell(b)$. The monoid $M(B, R)$ is isomorphic to the monoid $H(B, R)$. Here we only give the isomorphism $\varphi : H(B, R) \rightarrow M(B, R)$ and its inverse $\varphi^{-1}$ without the proof. We refer to Proposition 3.4 in [8] for a complete proof that $\varphi$ is an isomorphism.

Given a heap $H = (P, \leq, \ell) \in H(B, R)$, let $\eta : P \mapsto [n]$ be a bijection such that $\eta(x) \leq \eta(y)$ for every $x, y \in P$ and $x \leq y$, we define a word $w_\eta = x_{\eta^{-1}(1)} \cdots x_{\eta^{-1}(n)}$. Here we call the bijection $\eta$ as a natural labeling of the poset $(P, \leq)$. For any two different natural labelings $\eta, \theta$ of the poset $(P, \leq)$, it is true that $w_\eta = w_\theta$, which was proved in [8]. We set $\varphi(H) = w_\eta$ and $\varphi(H)$ is therefore well-defined.

Conversely, given a word $w = x_{a_1}x_{a_2} \cdots x_{a_n} \in M(B, R)$ such that two adjacent letters $x_{a_i}, x_{a_j}$ commute if $\ell(a_1) \not\equiv \ell(a_j)$, we can get a heap $\varphi^{-1}(w)$ as follows: For every $i \neq 1$, if $x_{a_i}x_{a_{i+1}} \neq x_{a_{i+1}}x_{a_i}$, then we have $a_i \leq a_{i+1}$ in the heap $\varphi^{-1}(w)$. Next we write $w = x_{a_1}w_1$ where $w_1 = x_{a_2} \cdots x_{a_n}$ and consider the word $w_1$. For every $i > 2$, if $x_{a_i}x_{a_{i-1}} \neq x_{a_{i-1}}x_{a_i}$, we have $a_i \leq a_{i-1}$ in the heap $\varphi^{-1}(w)$. We continue this process and get the heap $\varphi^{-1}(w) = (P, \leq, \ell)$ where $(P, \leq)$ is a poset of elements $\{a_1, \ldots, a_n\}$ and two elements $a_i, a_j$ are incomparable if $\ell(a_i) \equiv \ell(a_j)$.

We will use an example to illustrate the bijection $\varphi$. Let $B = \{a, b, c, d, e, z, g, h\}$ such that $a \not\equiv b, b \not\equiv d, a \not\equiv c, c \not\equiv d, e \not\equiv z, z \not\equiv g$ and $z \not\equiv h$. Let $H$ be a heap drawn as below where $\ell(x) = x$ for every $x \in H$.

![Diagram of a heap H](image)

Then we choose to label the elements of heap $H$ by the integers $1, 2, \ldots, 8$ from bottom-to-top and left-to-right, namely, $\varepsilon(a) = 1$, $\varepsilon(c) = 2$, $\varepsilon(b) = 3$, $\varepsilon(c) = 4$, $\varepsilon(z) = 5$, $\varepsilon(d) = 6$, $\varepsilon(g) = 7$, $\varepsilon(h) = 8$. It is easy to check that $\varepsilon$ is a natural labeling. The corresponding word $\varphi(H) = x_a x_d x_e x_g x_y x_b$ such that $x_a x_e = x_d x_b$.

In particular, $\varphi(\emptyset) = 1$ and for any two heaps $H_1, H_2 \in H(B, R)$, we have $\varphi(H_1 \circ H_2) = \varphi(H_1) \cdot \varphi(H_2)$, namely under $\varphi$ the empty heap $\emptyset$ corresponds to the identity word 1 and the composition of two heaps $H_1 \circ H_2$ corresponds to the product of two words $\varphi(H_1)$ and $\varphi(H_2)$ by juxtaposition. Let $|w|$ be the length of word $w \in M(B, R)$, then in particular $|\varphi(\emptyset)| = 0$. Let $\mathbb{Z}[[M(B, R)]]$ be the ring of all the formal power series in $M(B, R)$ with coefficients in the commutative ring $\mathbb{Z}$, then we can express eq. (2.1) in $\mathbb{Z}[[M(B, R)]]$ as below:

$$
(2.2) \sum_{H \in H(B, R)} \sum_{T \in T(B, R)} (-1)^{|\varphi(T)|} \cdot (\varphi(T) \cdot \varphi(H)) = 1.
$$

A pyramid is a heap with exactly one maximal element. Let $P(B, R)$ denote the set of pyramids consisting of pieces in $B$. Then according to the exponential formula for unlabeled combinatorial objects [7],

$$
(2.3) \sum_{H \in H(B, R)} \varphi(H) = \text{comm} \exp \left( \sum_{P \in P(B, R)} \frac{\varphi(P)}{|\varphi(P)|} \right),
$$

where $= \text{comm}$ means the identity holds in any commutative quotient of $\mathbb{Z}[[M(B, R)]]$. Together with eq. (2.2), it follows that

$$
(2.4) \sum_{P \in P(B, R)} \frac{\varphi(P)}{|\varphi(P)|} = \text{comm} - \log \left( \sum_{T \in T(B, R)} (-1)^{|\varphi(T)|} \varphi(T) \right).
$$
Recall that each heap $H = (W, \leq, \ell) \in \mathcal{H}(\mathcal{B}, \mathcal{R})$ can be identified as a directed graph $(\gamma, G(W))$ whose vertex set is $W$ and $\gamma$ is an acyclic orientation of $G(W)$ such that the edge $\{x, y\} \in G(W)$ has direction $x \rightarrow y$ if $x < y$ in the poset $(W, \leq)$. A source is a vertex having no outgoing arrows. In the same way, each pyramid $P = (R, \leq, \ell) \in \mathcal{P}(\mathcal{B}, \mathcal{R})$ can be identified as a directed graph $(\gamma_1, G(R))$ having a unique source where the vertex set of $G(R)$ is $R$ and $\gamma_1$ is an acyclic orientation of $G(R)$ such that the edge $\{x, y\} \in G(R)$ has direction $x \rightarrow y$ if $x < y$ in the poset $(R, \leq)$. For instance, consider the set $\mathcal{B} = \{a, b, c, d\}$ with relation $aRb, bRd, aRc, cRd$, let $P$ be a pyramid drawn as below where $\ell(x) = x$ for every $x$.

Then among the elements $a, b, c, d$, only $b$ and $c$ are not comparable, thus there is no edge between $b$ and $c$ in the graph $G(R)$. Since in the poset $P$, we have $a \leq b, b \leq d, a \leq d, a \leq c, c \leq d$. Then in the graph $G(R)$, we have acyclic orientation $\gamma$: $a \rightarrow b, b \rightarrow d, a \rightarrow d, a \rightarrow c, c \rightarrow d$. Jousuat-Vergés [4] had used this connection to interpret a new sequence in terms of heaps related to the perfect matchings. We will use this approach to interpret the numbers $r_n(\Pi_n^{(r)})$ and $r_n(Q_n^{(r)})$ defined in eq. (1.2). In the sequel, we simply use the set $\Pi_n^{(r)}$ (resp. $Q_n^{(r)}$) to represent the poset if we order the elements of $\Pi_n^{(r)}$ (resp. $Q_n^{(r)}$) by refinement.

3. On the poset $\Pi_n^{(r)}$

$\Pi_n^{(r)}$ is the poset of all the set partitions of $[rn]$ whose block sizes are divisible by $r$ if we order the elements of $\Pi_n^{(r)}$ by refinement. In particular $\Pi_n^{(1)} = \Pi_n$. Let $\Pi_{n,r}$ be the set of all the set partitions of $[rn]$ whose block sizes are exactly $r$. Then $\Pi_{n,r}$ is the set of minimal elements of the poset $\Pi_n^{(r)}$, consequently the number of minimal elements in $\Pi_n^{(r)}$ is equal to the size of $\Pi_{n,r}$, i.e., $M(n) = (rn)!/(n!r^n)!^{-1}$.

For $r \geq 2$, the sequence $(r_n(\Pi_n^{(r)}))_{n \geq 1}$ defined by eq. (1.2) satisfies

$$\sum_{n \geq 1} \frac{r_n(\Pi_n^{(r)}) z^{rn}}{(rn)!} = - \log \left( \sum_{n \geq 0} (-1)^n \frac{z^{rn}}{(rn)!} \right).$$

A permutation of the set $\{b_1, b_2, \ldots, b_n\}$ is a bijection $\pi : [n] \rightarrow \{b_1, b_2, \ldots, b_n\}$ that we can write $\pi = a_1a_2 \cdots a_n$ if $\pi(j) = a_j$ for $1 \leq j \leq n$. We denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. The descent set of permutation $\pi = a_1a_2 \cdots a_n \in \mathfrak{S}_n$ is the set $\text{Des } \pi = \{i : 1 \leq i < n \text{ and } a_i > a_{i+1}\}$. The generalized Euler number $E_{rn-1}$ counts the number of permutations $\pi \in \mathfrak{S}_{rn-1}$ such that $\text{Des } \pi = \{r, 2r, \ldots, rn-r\}$. One of our main results is a new proof of the following result due to Welker [10]:

**Theorem 1.** The number $r_n(\Pi_n^{(r)})$ defined in eq. (3.1) is equal to the generalized Euler number $E_{rn-1}$.

We will prove Theorem 1 by first interpreting the number $r_n(\Pi_n^{(r)})$ in terms of pyramids in Lemma 2, and then building a bijection between the set of the pyramids counted by $r_n(\Pi_n^{(r)})$ and the set of permutations of $[rn-1]$ with descent set $\{r, 2r, \ldots, rn-r\}$ in Lemma 3.

Before we proceed, we introduce some definitions and notations. We say $\pi = a_1a_2 \cdots a_{2n-1}$ is an alternating permutation of the set $\{b_1, b_2, \ldots, b_{2n-1}\}$ if $a_1 < a_2 > \cdots < a_{2n-2} > a_{2n-1}$, namely $\text{Des } \pi = \{2, 4, \ldots, 2n-2\}$. It is well-known that
the number of alternating permutations of \([2n - 1]\) is counted by the Euler number \(E_{2n-1}\) (also called as tangent number), whose exponential generating function is

\[
\sum_{n \geq 1} E_{2n-1} \frac{z^{2n-1}}{(2n-1)!} = \tan(z).
\]

There is a one-to-one correspondence \(T\) between the set of alternating permutations of \([2n - 1]\) and the set of decreasing binary trees of size \(2n - 1\). A binary tree is a tree whose vertices have either 2 or 0 children. We call the vertex having 0 children as a leaf, otherwise we call the vertex as an internal node. A decreasing binary tree of size \(n\) is a binary tree having \(n\) vertices labeled 1, 2, \ldots, \(n\) such that the labels along any path from the root are decreasing. The bijection extended to the set of alternating permutations of \(\{b_1, b_2, \ldots, b_n\}\) is given in [7]. Namely, let \(\pi = a_1 a_2 \cdots a_n\) be a permutation of \(\{b_1, b_2, \ldots, b_n\}\) such that \(\text{Des} \ \pi = \{2, 4, \ldots, 2n - 2\}\), the corresponding decreasing binary tree \(T(\pi)\) is defined inductively. If \(\pi = \varnothing\), then \(T(\pi) = \varnothing\). If \(\pi \neq \varnothing\), then let \(m\) be the maximal element of the set \(\{a_1, a_2, \ldots, a_n\}\), i.e., \(m = \max\{a_i : 1 \leq i \leq n\}\), therefore we can write \(\pi = \sigma m\tau\). We set \(m\) be the root of \(T(\pi)\), \(T(\sigma)\) and \(T(\tau)\) be the left and right subtree of the binary tree \(T(\pi)\) rooted at \(m\). This gives an inductive process to construct the decreasing binary tree \(T(\pi)\) that is labeled by \(a_1, a_2, \ldots, a_n\). We will need the bijection \(\pi \mapsto \tau(\pi)\) in the proof of Theorem 1.

3.1. Combinatorial interpretation of \(r_n(\Pi_n^{(2)})\). Recall that \(\Pi_{n,r}\) is the set of set partitions of \([rn]\) where each block has size \(r\). We say \(\{i_1, \ldots, i_r\}\) is a block of partition \(\pi \in \Pi_{n,r}\), denoted by \(\{i_1, \ldots, i_r\} \in \pi\), if \(1 \leq i_1 < \cdots < i_r \leq rn\). The diagram representation of \(\pi \in \Pi_{n,r}\) is given as follows: We draw \(rn\) dots in a line labeled with 1, 2, \ldots, \(rn\). For each block \(\{i_1, \ldots, i_r\} \in \pi\), we connect \(i_j\) and \(i_{j+1}\) by an edge for all \(1 \leq j < r\). We say two blocks \(\{i_1, \ldots, i_r\}\) and \(\{j_1, \ldots, j_r\}\) of \(\pi\) are not crossing if \(i_r < j_1\) or \(j_r < i_1\). Otherwise two blocks \(\{i_1, \ldots, i_r\}\) and \(\{j_1, \ldots, j_r\}\) are crossing. Let \(G(\pi)\) be the crossing graph of \(\pi\) where the vertex set consists of all the blocks of \(\pi\), and there is an edge between \(\{i_1, \ldots, i_r\}\) and \(\{j_1, \ldots, j_r\}\) if they are crossing. For instance, consider \(\pi = \{\{1, 2, 4\}, \{3, 5, 7\}, \{6, 8, 11\}, \{9, 10, 12\}\} \in \Pi_{4,3}\). The diagram representation of \(\pi\) and \(G(\pi)\) are drawn as follows.

\[
\pi: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12
\]

\[
G(\pi): \quad \{1, 2, 4\} \quad \{9, 10, 12\} \quad \{3, 5, 7\} \quad \{6, 8, 11\}
\]

Stanley [7] proved \(r_n(\Pi_n^{(2)}) = E_{2n-1}\) by using the exponential generating function of \(E_{2n-1}\) given in eq. (3.2). More precisely, the hyperbolic tangent function \(\tanh(z) = \frac{(e^z - e^{-z})(e^z + e^{-z})}{e^z + e^{-z}}\) satisfies \(\tanh(z) = -i \tan(i z)\). By setting \(r = 2\) to eq. (3.1) and replacing \(z^2\) by \(-z^2\), we get

\[
\sum_{n \geq 1} r_n(\Pi_n^{(2)}) (-1)^n \frac{z^{2n}}{(2n)!} = - \log \sum_{n \geq 0} \frac{z^{2n}}{(2n)!}.
\]

By differentiating both sides with respect to \(z\), we have

\[
\sum_{n \geq 1} (-1)^n r_n(\Pi_n^{(2)}) \frac{z^{2n-1}}{(2n-1)!} = -\left(\sum_{n \geq 0} \frac{z^{2n}}{(2n)!}\right)^{-1} \sum_{n \geq 1} \frac{z^{2n-1}}{(2n-1)!} = -\frac{e^z - e^{-z}}{e^z + e^{-z}} = -\tanh(z).
\]

In view of eq. (3.2) and \(\tanh(z) = -i \tan(i z)\), we obtain

\[
\sum_{n \geq 1} (-1)^n r_n(\Pi_n^{(2)}) \frac{z^{2n-1}}{(2n-1)!} = -\tanh(z) = \sum_{n \geq 1} (-1)^n E_{2n-1} \frac{z^{2n-1}}{(2n-1)!}.
\]
That yields \( r_n(\Pi_n^{(2)}) = E_{2n-1} \). For general \( r \), we shall show \( r_n(\Pi_n^{(r)}) \) counts the number of pyramids related to the set \( \Pi_{n,r} \).

**Lemma 2.** The number \( r_n(\Pi_n^{(r)}) \) counts the directed graphs \( (\gamma, G(\pi)) \) where \( \pi \in \Pi_{n,r} \), \( G(\pi) \) is connected and \( \gamma \) is an acyclic orientation of \( G(\pi) \) whose unique source is the block of \( \pi \) containing \( r_n \).

**Proof.** We will give a combinatorial interpretation of \( r_n(\Pi_n^{(r)}) \) in terms of pyramids. Consider the set \( \mathcal{B} = \{ \{i_1, i_2, \ldots, i_r\} : 1 \leq i_1 < i_2 < \cdots < i_r \leq r_n \} \) with a symmetric and reflexive binary relation \( \mathcal{R} \) defined by \( \{i_1, \ldots, i_r\} \mathcal{R} \{j_1, \ldots, j_r\} \) if and only if two blocks \( \{i_1, \ldots, i_r\} \) and \( \{j_1, \ldots, j_r\} \) are crossing. Let \( (P, \leq) \) be a poset where each element is labeled by \( \{i_1, \ldots, i_r\} \) such that

1. Two elements \( \{i_1, \ldots, i_r\} \) and \( \{j_1, \ldots, j_r\} \) are comparable if \( \{i_1, \ldots, i_r\} \) and \( \{j_1, \ldots, j_r\} \) are crossing.
2. If \( \{j_1, \ldots, j_r\} \) covers \( \{i_1, \ldots, i_r\} \) in the poset \( (P, \leq) \), then \( \{j_1, \ldots, j_r\} \) and \( \{i_1, \ldots, i_r\} \) are crossing.

Then by definition the poset \( (P, \leq) \) is a heap \( H = (P, \leq) \in \mathcal{H}(\mathcal{B}, \mathcal{R}) \). In a geometric way, we can represent each block \( \{i_1, i_2, \ldots, i_r\} \) by a line \( i_1--i_2--i_3--\cdots--i_r \) and put \( i_1--i_2--\cdots--i_r \) above \( j_1--j_2--\cdots--j_r \) if \( \{i_1, i_2, \ldots, i_r\} \leq \{j_1, j_2, \ldots, j_r\} \) in the heap \( H = (P, \leq) \). In this way, the line \( i_1--i_2--\cdots--i_r \) cannot move downwards without touching the line \( j_1--j_2--\cdots--j_r \). The heap \( H = (P, \leq) \) is therefore represented by a collection of lines \( i_1--i_2--\cdots--i_r \) in the diagram if \( \{i_1, i_2, \ldots, i_r\} \in P \), see the examples \( H_1, H_2 \) below when \( r = 2 \) and \( n = 3 \).

For any two heaps \( H_1, H_2 \), its composition \( H_1 \circ H_2 \) is the heap obtained by placing the pieces in \( H_2 \) on top of the pieces in \( H_1 \), see the example \( H_1 \circ H_2 \) below.

Consider the monoid \( \mathcal{M}(\mathcal{B}, \mathcal{R}) \), which is a set of words \( w \) with letters from the set \( \{x_{i_1}, \ldots, i_r : \{i_1, \ldots, i_r\} \in \mathcal{B}\} \) such that two adjacent letters \( x_{i_1} \ldots, i_r, x_{j_1} \ldots, j_r \) of \( w \) commute if \( \{i_1, \ldots, i_r\} \mathcal{R} \{j_1, \ldots, j_r\} \), or equivalently \( \{i_1, \ldots, i_r\} \) and \( \{j_1, \ldots, j_r\} \) are not crossing. We continue using the examples \( H_1, H_2 \) as before to show the words \( \varphi(H_1), \varphi(H_2) \in \mathcal{M}(\mathcal{B}, \mathcal{R}) \). By reading the labels of the elements in the poset \( H_1 \) (resp. \( H_2 \)) from bottom-to-top, left-to-right, we have \( \varphi(H_1) = x_{2,5}x_{1,3}x_{4,6}x_{3,4} \) where \( x_{1,3}x_{4,6} = x_{4,6}x_{1,3} \) (resp. \( \varphi(H_2) = x_{3,5}x_{2,6}x_{2,3}x_{1,4} \)). It is easy to check that \( \varphi(H_1 \circ H_2) = \varphi(H_1) \cdot \varphi(H_2) = x_{2,5}x_{1,3}x_{4,6}x_{3,4}x_{3,5}x_{2,6}x_{2,3}x_{1,4} \) subject to the condition \( x_{1,3}x_{4,6} = x_{4,6}x_{1,3} \).

Let \( \{a_i\}_{i \geq 1} \) be a sequence of variables that \( a_i^2 = 0 \) for every \( i \), we consider a ring homomorphism \( f : \mathbb{Z}[[\mathcal{M}(\mathcal{B}, \mathcal{R})]] \to \mathbb{Z}[[a_1, a_2, \ldots]] \) such that \( f(x_{i_1}, \ldots, i_r) = \prod_{j=1}^r a_i \),
and \( f(1) = 1 \). Then \( f \) induces a natural ring isomorphism
\[
 f_0 : \mathbb{Z}[[\mathcal{M}(\mathcal{B}, \mathcal{R})]]/\ker(f) \to \text{im}(f)
\]
via \( m + \ker(f) \mapsto f(m) \). After applying \( f_0 \) on both sides of eq. (2.4), we obtain an identity in the commutative ring \( \text{im}(f) \).

\[
(3.3) \quad \sum_{P \in \mathcal{P}(\mathcal{B}, \mathcal{R})} \frac{f_0(\varphi(P))}{|\varphi(P)|} = -\log \left( \sum_{T \in \mathcal{I}(\mathcal{B}, \mathcal{R})} (-1)^{|\varphi(T)|} f_0(\varphi(T)) \right).
\]

Let \( \mathcal{T}_n(\mathcal{B}, \mathcal{R}) \) be a set of trivial heaps of size \( n \) contained in the set \( \mathcal{T}(\mathcal{B}, \mathcal{R}) \), suppose \( T_n \in \mathcal{T}_n(\mathcal{B}, \mathcal{R}) \) is a trivial heap having \( n \) pieces \( \{i_1, \ldots, i_r\}, \{i_{r+1}, \ldots, i_{2r}\}, \ldots, \{i_{rn-r+1}, \ldots, i_{rn}\} \). Then \( i_1, i_2, \ldots, i_{rn} \) is a strictly increasing sequence. Let \( I_{T_n} = i_1, i_2, \ldots, i_{rn} \) denote this strictly increasing sequence. Thus the map \( T_n \mapsto I_{T_n} \) is a bijection between the set of trivial heaps of size \( n \) and the set of strictly increasing sequences of length \( rn \). By applying \( f_0 \) on the word \( \varphi(T_n) \in \mathcal{M}(\mathcal{B}, \mathcal{R}) \), we get \( f_0(\varphi(T_n)) = f(\varphi(T_n)) \neq 0 \) and consequently
\[
\sum_{T_n \in \mathcal{T}_n(\mathcal{B}, \mathcal{R})} (-1)^n f(\varphi(T_n)) = \sum_{I_{T_n}} (-1)^n \prod_{j=1}^{rn} a_{i_j} = (-1)^n \frac{(a_1 + a_2 + \cdots)^{rn}}{(rn)!}
\]
where the second summation runs over all the strictly increasing sequences \( I_{T_n} \) of length \( rn \) and the last equation holds because \( a_i^2 = 0 \) for every \( i \). It follows that the right hand side of eq. (3.3) is
\[
-\log \left( \sum_{T \in \mathcal{I}(\mathcal{B}, \mathcal{R})} (-1)^{|\varphi(T)|} f_0(\varphi(T)) \right) = -\log \left( \sum_{n \geq 0} \sum_{T_n \in \mathcal{T}_n(\mathcal{B}, \mathcal{R})} (-1)^n f(\varphi(T_n)) \right) = -\log \left( \sum_{n \geq 0} (-1)^n \frac{(a_1 + a_2 + \cdots)^{rn}}{(rn)!} \right).
\]

On the other hand, let \( \mathcal{P}_n(\mathcal{B}, \mathcal{R}) \) be the set of pyramids of size \( n \) contained in the set \( \mathcal{P}(\mathcal{B}, \mathcal{R}) \). Suppose \( P_n = (\pi, \leq) \in \mathcal{P}_n(\mathcal{B}, \mathcal{R}) \) be a pyramid of size \( n \) such that \( f(\varphi(P_n)) \neq 0 \), since the elements of the poset \( P_n = (\pi, \leq) \) are the blocks of \( \pi \) and \( f(\varphi(P_n)) \neq 0 \), \( \pi \) must be a partition of a set having \( rn \) elements that each block of \( \pi \) has size \( r \). We continue using \( I_{T_n} \) to represent any strictly increasing sequence \( i_1, i_2, \ldots, i_{rn} \). A strictly increasing sequence \( I_{T_n} \) is uniquely corresponding to a set \( \{i_1, i_2, \ldots, i_{rn}\} \). For a given set \( \{i_1, i_2, \ldots, i_{rn}\} \), let \( p_{n,r} \) count the number of pyramids \( P = (\pi, \leq) \) that \( \pi \) is a set partition of \( \{i_1, i_2, \ldots, i_{rn}\} \) and each block of \( \pi \) has size \( r \). Then we have
\[
\sum_{P_n \in \mathcal{P}_n(\mathcal{B}, \mathcal{R})} f(\varphi(P_n)) = \sum_{I_{T_n}} p_{n,r} \prod_{j=1}^{rn} a_{i_j}.
\]
Noting that the number \( p_{n,r} \) is independent of the choice of the set \( \{i_1, i_2, \ldots, i_{rn}\} \), we choose the set \( \{1, 2, \ldots, rn\} \) and \( p_{n,r} \) also counts the number of pyramids \( P_n = (\pi, \leq) \) where \( \pi \in \Pi_n^r \). It turns out the left hand side of eq. (3.3) is
\[
\sum_{P \in \mathcal{P}(\mathcal{B}, \mathcal{R})} \frac{f_0(\varphi(P))}{|\varphi(P)|} = \sum_{n \geq 0} \sum_{P_n \in \mathcal{P}_n(\mathcal{B}, \mathcal{R})} \frac{f(\varphi(P_n))}{|\varphi(P_n)|} = \sum_{n \geq 0} \frac{p_{n,r}}{n} \sum_{I_{T_n}} \prod_{j=1}^{rn} a_{i_j}
\]
\[
= \sum_{n \geq 0} \frac{p_{n,r}}{n} \frac{(a_1 + a_2 + \cdots)^{rn}}{(rn)!}
\]
In view of eq. (3.1), we can conclude \( r_n(\Pi_n^r) = n^{-1} p_{n,r} \). The number \( p_{n,r} \) is equal to the number of directed graphs \( (\delta, G(\pi)) \) that \( G(\pi) \) is connected with an acyclic
where $m$ is connected, map $\{T_\sigma\}$ blocks of $n$ ing sequence descent set $\{\leq\}$ ing for any $1 \leq a \leq n$, the left-to-right order is labeled by $\pi$ be labeled binary tree a $\pi$ $x$. The correspondence $\pi \mapsto \gamma$ is an acyclic orientation of $G(\pi)$ whose unique source is the block containing $rn$.

In the sequel we sometimes use $\{u_i\}_{i=1}^n$ to denote the sequence $u_1, u_2, \ldots, u_n$ and $\{u_i : 1 \leq i \leq n\}$ to denote the set $\{u_1, u_2, \ldots, u_n\}$.

**Lemma 3.** There is a bijection between the set of directed graphs $(\gamma, G(\pi))$ where $\sigma \in \Pi_{n,r}$, $G(\sigma)$ is connected and $\gamma$ is an acyclic orientation of $G(\sigma)$ whose unique source is the block of $\sigma$ containing $rn$, and the set of permutations of $[rn-1]$ with descent set $\{r, 2r, \ldots, rn-r\}$.

**Proof.** Let $\pi = a_1 \cdots a_{r-1} a_r a_{r+1} \cdots a_{rn-r} a_{rn-r+1} \cdots a_{rn-1}$ be a permutation of $\{a_1, \ldots, a_{rn-1}\}$ with descent set $\{r, 2r, \ldots, rn-r\}$, then we will first construct a bijection $\pi \mapsto f(\pi)$ that maps $\pi$ into a labeled binary tree $T(\pi)$. Since the descent set of $\pi$ is $\{r, 2r, \ldots, rn-r\}$, the sequence $a_{ri-r+1}, \ldots, a_{ri-1}$ is increasing for any $1 \leq i \leq n$. Now we introduce a map $h$ to reduce $\pi$ into $h(\pi) = a_1 a_{r+1} a_{2r+1} \cdots a_{rn-r+1} a_{rn-r} a_{rn-r+1}$, which is an alternating permutation since $a_1 < a_2 > a_{r+1} < a_{2r+1} < a_{3r+1} < \cdots < a_{rn-r} < a_{rn-r+1}$. By applying the bijection $T$ from the set of alternating permutations to the set of decreasing binary trees, we get a decreasing binary tree $T(h(\pi))$ labeled by $a_1, a_r, a_{r+1}, \ldots, a_{rn-r+1}$. We observe that the $i$-th leaf of the tree $T(h(\pi))$ in the left-to-right order is labeled by $a_{ri-r+1}$ for every $i$. For $1 \leq i \leq n$, let $x_i = a_{ri-r+1}, \ldots, a_{ri-1}$ be a strictly increasing sequence. Then we obtain a labeled binary tree $f(\pi)$ from $T(h(\pi))$ by replacing the label $a_{ri-r+1}$ by an increasing sequence $x_i$. The correspondence $\pi \mapsto f(\pi)$ is a bijection since the correspondences $\pi \mapsto h(\pi), \{x_i\}_{i=1}^n \mapsto (T(h(\pi)), \{x_i\}_{i=1}^n)$ are all bijections. For instance, for $n = 4, r = 3$ and $\pi = 12345678910 \in S_{11}$, we get $h(\pi), \{x_i\}_{i=1}^4 = (14376119, 1, 2, 3, 5, 6, 8, 9, 10)$ and the corresponding decreasing binary tree $T(h(\pi))$, labeled binary tree $f(\pi)$ are shown as below (left and middle).

Let $m^* = \max\{a_i : 1 \leq i < rn\}$ and $m$ be any integer that $m > m^*$, we first give a map $g$ from the set of pairs $(f(\pi), m)$ where $\pi$ is a permutation of $\{a_1, \ldots, a_{rn-1}\}$ with descent set $\{r, 2r, \ldots, rn-r\}$ to the set of directed graphs $(\gamma_{rn}, G(\sigma_f(\pi, m)))$ where $\sigma_f(\pi, m)$ is a partition of $\{a_1, \ldots, a_{rn-1}, m\}$ whose block sizes are $r$, $G(\sigma_f(\pi, m))$ is connected, $\gamma_{rn}$ is an acyclic orientation of $G(\sigma_f(\pi, m))$ whose unique source is the block of $\sigma_f(\pi, m)$ containing $m$, i.e., $g(\pi, m) = (\gamma_{rn}, G(\sigma_f(\pi, m)))$. We will next show the map $(\pi, m) \mapsto (f(\pi), m) \mapsto (\gamma_{rn}, G(\sigma_f(\pi, m)))$ is a bijection.

We observe that the internal nodes of tree $f(\pi)$ are labeled by $a_r, a_{2r}, \ldots, a_{rn-r}$ and the leaves of tree $f(\pi)$ are labeled by $x_1, x_2, \ldots, x_n$ in the left-to-right order. We will use the labels to represent the vertices in a tree. The map $g$ is defined as follows. Given a labeled binary tree $f(\pi)$ and an integer $m$, let $f(\pi)^m$ be the tree obtained from $f(\pi)$ by adding a new vertex labeled by $m$ to $m^*$ such that $m$ is the parent of $m^*$. Accordingly the root of $f(\pi)^m$ is $m$. Now we first give a correspondence $(f(\pi), m) \mapsto (f(\pi)^m, \sigma_f(\pi, m))$ that maps a pair $(f(\pi), m)$ into a partition $\sigma_f(\pi, m)$ of $\{a_1, \ldots, a_{rn-1}, m\}$ whose all the block sizes are $r$. Notice that
\( x_i \) is the label on the \( i \)-th leaf of \( f(\pi)^m \) in the left-to-right order, and we will use the labels \( a_1, a_2, \ldots, a_{r-1}, x_1, \ldots, x_n, m \) to represent the vertices of \( f(\pi)^m \). Let \( p^i(x) \) be the \((i+1)\)-th vertex on the path from \( x \) to the root of \( f(\pi)^m \), in particular \( i = 1 \), \( p(x) \) is the parent of \( x \) in the tree \( f(\pi)^m \). Then we will get a partition \( \sigma_{f(\pi),m} \) from \( f(\pi)^m \) as follows: Given a labeled binary tree \( f(\pi)^m \), we read the labels of the leaves of \( f(\pi)^m \) in the left-to-right order, and get the leaves \( x_1, x_2, \ldots, x_n \). In the first step, let \( B_1 = \{ \{ x_1, p(x_1) \} \} \) and \( A_1 = \{ x_1, p(x_1) \} \). Then in the second step, if \( p(x_2) \notin A_1 \), we set \( B_2 = B_1 \cup \{ \{ x_2, p(x_2) \} \} \) and \( A_2 = A_1 \cup \{ x_2, p(x_2) \} \). Otherwise, if \( p(x_2) = p(x_1) \in A_1 \), we set \( B_2 = B_1 \cup \{ \{ x_2, p^2(x_2) \} \} \) and \( A_2 = A_1 \cup \{ x_2, p^2(x_2) \} \). In the \( i \)-th step, let \( j_i \) be the minimal integer such that \( p^{j_i}(x_i) \notin A_{i-1} \). Then we set \( B_i = B_{i-1} \cup \{ \{ x_i, p^{j_i}(x_i) \} \} \) and \( A_i = A_{i-1} \cup \{ x_i, p^{j_i}(x_i) \} \). This process ends at the \( n \)-th step and we have the set \( B_n = \bigcup_{i=1}^{n} \{ \{ x_i, p^{j_i}(x_i) \} \} \) where \( x_i \) is the increasing sequence \( a_{r-1}, \ldots, a_{r-1} \) of length \( r-1 \). We further observe that \( a_{r-1} < p^{j_i}(x_i) \) for any \( 1 \leq i \leq n \). It follows that \( \sigma_{f(\pi),m} = \bigcup_{i=1}^{n} \{ \{ a_{r-1-1}, \ldots, a_{r-1}, p^{j_i}(x_i) \} \} \) is a partition of \( \{ a_1, \ldots, a_{r-1}, m \} \) whose block sizes are exactly \( r \). Particularly if \( \pi \in \mathcal{S}_{r-1} \) with descent set \( \{ r, 2r, \ldots, rn-r \} \), we choose \( m = rn \) and from the correspondence \( f(\pi)^r \mapsto \sigma_{f(\pi),rn} \) we have \( \sigma_{f(\pi),rn} \in \Pi_{1,3} \).

We continue using \( \pi = 12435768910 \in \mathcal{S}_{11} \) as an example. \( f(\pi) \) is drawn as before. \( f(\pi)^{12} \) is a tree obtained from \( f(\pi) \) by connecting 12 to 11 and making 12 as the new root. See the picture above. We will show the correspondence \( f(\pi)^{12} \mapsto \sigma_{f(\pi),12} \) as follows. We read the leaves in the left-to-right order that \( x_1 = 1, 2, x_2 = 3, 5, x_3 = 6, 8 \) and \( x_4 = 9, 10 \). In the first step, \( p(x_1) = 4 \), thus \( B_1 = \{ \{ 1, 2, 4 \} \} \) and \( A_1 = \{ 1, 2, 4 \} \). In the second step, since \( p(x_2) = p(x_1) \in A_1 \), we choose \( (x_2, p^2(x_2)) = (3, 5, 7) \) and set \( B_2 = \{ \{ 1, 2, 4 \}, \{3, 5, 7\} \} \) and \( A_2 = \{1, 2, 4, 3, 5, 7\} \). We continue this process until the 4-th step when we get \( B_4 = \{ \{ 1, 2, 4 \}, \{3, 5, 7\}, \{6, 8, 11\}, \{9, 10, 12\} \} \) and therefore \( \sigma_{f(\pi),12} = \{ \{ 1, 2, 4 \}, \{3, 5, 7\}, \{6, 8, 11\}, \{9, 10, 12\} \} \in \Pi_{1,3} \).

We will prove the crossing graph \( G(\sigma_{f(\pi),m}) \) is connected by induction on the size of \( f(\pi)^m \). Consider the induction basis that \( f(\pi)^m \) has only two leaves \( x_1, x_2 \) and two internal nodes (including the root), then \( \sigma_{f(\pi),m} = \{ \{ x_1, m^* \}, \{ x_2, m \} \} \).

See the left figure below.

Since the minimal integer \( a_{r-1} \) of the sequence \( x_2 \) satisfies \( a_{r-1} < m^* < m \), we have the blocks \( \{ x_1, m^* \} \) and \( \{ x_2, m \} \) are crossing, i.e., \( G(\sigma_{f(\pi),m}) \) is connected. Assume that \( f(\pi)^m \) has more than 2 leaves, consider the subtree \( f(\pi)^m \) of \( f(\pi)^m \), we decompose \( f(\pi) \) into a left subtree \( T_L \) and a right subtree \( T_R \). Let \( (T_L)^m, (T_R)^m \) (resp. \( T_R \) resp. \( T_R \)) be the tree obtained from \( T_L \) (resp. \( T_R \)) by adding a vertex \( m^* \) (resp. \( m \)) to the root of \( T_L \) (resp. \( T_R \)) and making \( m^* \) (resp. \( m \)) as the new root. See the right figure above for the decomposition of \( f(\pi)^m \) into trees \( (T_L)^m \) and \( (T_R)^m \). Let \( x_k \) be the rightmost leaf of \( T_L \), let \( x_k \) be the rightmost leaf of \( T_R \). Then by induction hypothesis on the subtrees \( (T_L)^m \) and \( (T_R)^m \), the correspondence \( (T_L, m^*) \mapsto (T_L)^m \mapsto \sigma_{T_L,m^*} \) (resp. \( (T_R, m) \mapsto (T_R)^m \mapsto \sigma_{T_R,m} \)) gives a connected crossing graph \( G(\sigma_{T_L,m^*}) \) (resp. \( G(\sigma_{T_R,m}) \)).

From the correspondence \( (f(\pi), m) \mapsto f(\pi)^m \mapsto \sigma_{f(\pi),m} \) we have \( \sigma_{f(\pi),m} = \sigma_{T_L,m^*} \cup \sigma_{T_R,m} \). It follows that the vertex set of \( G(\sigma_{f(\pi),m}) \) is the union of the vertex set of \( G(\sigma_{T_L,m^*}) \) and \( G(\sigma_{T_R,m}) \). The edge set of \( G(\sigma_{f(\pi),m}) \) is the union of the edge set of \( G(\sigma_{T_L,m^*}) \) and \( G(\sigma_{T_R,m}) \), together with the new edges \{ \( u, v \) \} if \( u \) is a block of \( \sigma_{T_L,m^*}, v \) is a block of \( \sigma_{T_R,m} \) and \( u, v \) are crossing. We further observe that \( \{ \{ x_k, m^* \}, \{ x_k, m \} \} \subseteq \sigma_{f(\pi),m} \). Since the minimal integer \( a_{r-1}+1 \) of sequence
$x_t$ satisfies $a_{t,r} = 0 < m^* < m$, two blocks $\{x_k, m^*\}$ and $\{x_t, m\}$ are crossing, which means there is an edge between one vertex $\{x_k, m^*\}$ in $G(\sigma_{T_k,m^*})$ and one vertex $\{x_t, m\}$ in $G(\sigma_{T_k,m})$. In combination of the fact that both $G(\sigma_{T_k,m^*})$ and $G(\sigma_{T_k,m})$ are connected, we can conclude the graph $G(\sigma_{T_k,m})$ is connected. Therefore the correspondence $(f(\pi), m) \rightarrow f(\pi)^m \rightarrow (\gamma_{r,n}, G(\sigma_{f(\pi),m}))$ gives a connected crossing graph $G(\sigma_{f(\pi),m})$.

We next inductively define the correspondence $f(\pi)^m \rightarrow (\gamma_{r,n}, G(\sigma_{f(\pi),m}))$ that gives an acyclic orientation $\gamma_{r,n}$ of $G(\sigma_{f(\pi),m})$ whose unique source is the block containing the maximal element $m$. If $f(\pi)^m$ has only two leaves $x_1, x_2$ and two internal nodes (including the root), then $\gamma_{f(\pi),m} = \{\{x_1, m^*\}, \{x_2, m\}\}$. Since the unique source in the directed graph $(\gamma_{r,n}, G(\sigma_{f(\pi),m}))$ is the block of $\pi$ containing $m$, we choose $\{x_2, m\} \rightarrow \{x_1, m^*\}$ to be the single directed edge of the directed graph $(\gamma_{r,n}, G(\sigma_{f(\pi),m}))$.

Assume that $f(\pi)^m$ has more than 2 leaves, again we decompose $f(\pi)$ into a left subtree $T_L$ and a right subtree $T_R$. Suppose that on the trees $(T_L)^m$, $(T_R)^m$, we have defined an acyclic orientation $\gamma_{L,m^*}$ (resp. $\gamma_{R,m}$) on the connected $G(\sigma_{T_L,m^*})$ (resp. $G(\sigma_{T_R,m})$) whose unique source is the block $\{x_k, m^*\}$ (resp. $\{x_t, m\}$). Namely, $g(T_L, m^*) = (\gamma_{L,m^*}, G(\sigma_{T_L,m^*}))$ and $g(T_R, m) = (\gamma_{R,m}, G(\sigma_{T_R,m}))$. We will define the orientation $\gamma_{r,n}$ of $G(\sigma_{f(\pi),m})$ as follows: For any two crossing blocks $a, b$ contained in $\sigma_{f(\pi),m}$, if both $a, b$ are also crossing blocks contained in $\sigma_{T_L,m^*}$ (resp. $\sigma_{T_R,m}$), then $\gamma_{L,m^*}$ (resp. $\gamma_{R,m}$) determines the direction on the edge between $a$ and $b$. Otherwise if $a \in \sigma_{T_L,m^*}$, $b \in \sigma_{T_R,m}$ and $a, b$ are crossing blocks of $\sigma_{f(\pi),m}$. Then we add a directed edge $a \rightarrow b$. Consequently the orientation $\gamma_{r,n}$ of the graph $G(\sigma_{f(\pi),m})$ is acyclic. We further observe that in the directed graph $(\gamma_{r,n}, G(\sigma_{f(\pi),m}))$ there is an edge $\{x_k, m^*\} \rightarrow \{x_t, m\}$. That implies the source of directed graph $(\gamma_{r,n}, G(\sigma_{f(\pi),m}))$ is unique, which is the block $\{x_t, m\}$. Thus we have given the map $g(f(\pi), m) = (\gamma_{r,n}, G(\sigma_{f(\pi),m}))$. In particular if $\pi \in \mathcal{G}_{r,n-1}$ with descent set $\{r, 2r, \ldots, rn-r\}$ and $m = rn$, we have $g(f(\pi), rn) = (\gamma_{r,n}, G(\sigma_{f(\pi),rn}))$ where $\sigma_{f(\pi),rn} \in \Pi_{n,r}$.

We will use the example $\pi = 1 2 4 3 5 7 6 8 11 9 10 \in \mathcal{G}_{11}$ to show the correspondence $(f(\pi), 12) \rightarrow (\gamma_{3,4}, G(\sigma_{f(\pi),12}))$. First consider a subtree rooted at 4, denoted by $T_4$, its image $g(T_4, 7)$ is a digraph $\{1, 2, 4\} \rightarrow \{3, 5, 7\}$. Second, consider a subtree rooted at 7, denoted by $T_7$, we decompose $T_7$ into a left subtree $T_4$ and a right subtree $T_{6,8}$ having a single leaf 6, 8. Since $g(T_4, 7)$ is a digraph $\{1, 2, 4\} \rightarrow \{3, 5, 7\}$, $g(T_{6,8}, 11)$ is a graph with a single vertex $\{6, 8, 11\}$, and $\{3, 5, 7\}$ is crossing with $\{6, 8, 11\}$, thus by adding a directed edge $\{6, 8, 11\} \leftarrow \{3, 5, 7\}$ between two graphs $g(T_4, 7), g(T_{6,8}, 11)$, we get the graph $g(T_7, 11)$. 

\[
(T_4, 7) = \begin{pmatrix} 1, 2 & 3, 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1, 2 \end{pmatrix} \rightarrow \{1, 2, 4\} \\
(T_7, 11) = \begin{pmatrix} 1, 2 & 3, 5 & 6, 8 & 11 \end{pmatrix} \rightarrow \begin{pmatrix} 4 \end{pmatrix} \rightarrow \{3, 5, 7\}
\]
Finally consider the tree \( f(\pi) \), by decomposing \( f(\pi) \) into a left subtree \( T_7 \) (see the figure above) and a right subtree \( T_{9,10} \) which has a single leaf 9,10, we have \( g(T_7,11) \) is a digraph \( \{1,2,4\} \rightarrow \{3,5,7\} \rightarrow \{6,8,11\} \) and \( g(T_{9,10},12) \) is a graph with a single vertex \( \{9,10,12\} \). Since the blocks \( \{6,8,11\}, \{9,10,12\} \) are crossing, we add an additional edge \( \{6,8,11\} \rightarrow \{9,10,12\} \) between two graphs \( g(T_7,11), g(T_{9,10},12) \). Thus \( g(f(\pi),12) \) is a digraph \( \{1,2,4\} \rightarrow \{3,5,7\} \rightarrow \{6,8,11\} \rightarrow \{9,10,12\} \).

We will prove the given map \( g \) is a bijection by showing its inverse \( g^{-1} \) inductively. Given a directed graph \((\gamma,G(\sigma))\) where \( \sigma \) is a partition of \( \{a_1, \ldots , a_{rn}\} \) whose block sizes are \( r \), \( G(\sigma) \) is connected and the unique source is the block of \( \sigma \) containing the maximal element \( m \) of the set \( \{a_i : 1 \leq i \leq rn\} \). We will show the preimage \( g^{-1}(\gamma,G(\sigma)) \) is a pair \( (f(\pi),m) \) where \( \pi \) is a permutation of \( \{a_1, \ldots , a_{rn}\} \) with descent set \( \{r, \ldots , rn-r\} \). Let \( m^* \) be the second largest integer of the set \( \{a_i : 1 \leq i \leq rn\} \). \( G(\sigma) \) is a graph whose vertex set consisting of all the blocks of \( \sigma \).

We use \( V_G(\sigma) \) to denote the vertex set of graph \( G(\sigma) \). So \( y \in V_G(\sigma) \) means \( y \) is a block of partition \( \sigma \). In the sequel for any two vertices \( a, b \in V_G(\sigma) \), we say \( a < b \) if there is a directed path from \( a \) to \( b \) in the digraph \((\gamma,G(\sigma))\) and \( a \leq a \) for every \( a \) in the graph \( G(\sigma) \). Let \( c_x \) be the block of \( \sigma \) that contains \( x \), then we consider two blocks \( c_m \) and \( c_m^* \), there is an edge \( c_m \leftrightarrow c_m^* \) in the graph \( G(\sigma) \) with orientation \( \gamma \) and we consider a vertex-induced subgraph of \( G(\sigma) \), denoted by \( G_L(\sigma) \) whose vertex set is \( \{y : y \leq c_m^*, y \in V_G(\sigma)\} \). Let \( G_R(\sigma) \) be a vertex induced subgraph of \( G(\sigma) \) whose vertex set is the set \( \{y : y \geq c_m^*, y \in V_G(\sigma)\} \). Then \( c_m^* \) is a vertex of \( G_L(\sigma) \) and \( c_m \) is a vertex of \( G_R(\sigma) \). For any vertex \( c \in G_L(\sigma) \) and any vertex \( d \in G_R(\sigma) \), we have that \( c \rightarrow d \). If otherwise, \( c \leftarrow d \), namely \( d \leq c \). Since \( c \) is contained in the graph \( G_L(\sigma) \), then \( c \leq c_m^* \). It follows that \( d < c_m^* \) by the transitivity of \( \leq \) and therefore \( d \) is a vertex of \( G_L(\sigma) \), which contradicts to the assumption \( d \) is a vertex of \( G_R(\sigma) \).

Suppose \( \gamma_{r,n} \) (resp. \( \gamma_R \)) is the acyclic orientation \( \gamma_{r,n} \) that restricted to the graph \( G_L(\sigma) \) (resp. \( G_R(\sigma) \)), then in this way we split the digraph \((\gamma_{r,n},G(\sigma))\) into two independent digraphs \((\gamma_{L},G_L(\sigma))\) and \((\gamma_{R},G_R(\sigma))\). The unique source of \((\gamma_{L},G_L(\sigma))\) (resp. \((\gamma_{R},G_R(\sigma))\)) is the block \( c_m^* \) (resp. \( c_m \)). Based on this decomposition (see the left figure below) we will inductively prove \( g^{-1}(\gamma,G(\sigma)) \) is a pair \( (f(\pi),m) \) where \( \pi \) is a permutation of \( \{a_1, \ldots , a_{rn-1}\} \) with descent set \( \{r, \ldots , rn-r\} \). Consider the induction basis that \( G(\sigma) \) is a graph of size 2 that \( \sigma = \{c_m, \{a_1, \ldots , a_r\}\} \in \Pi_{2,r} \), suppose \( c_m = \{a_{r+1}, \ldots , a_{2r-1}, m\} \) and the orientation \( \gamma \) gives \( c_m \leftarrow \{a_1, \ldots , a_r\} \). Then \( g^{-1}(\gamma,G(\sigma)) \) is a pair \( (f(\pi),m) \) where \( f(\pi) \) is drawn as below.

Two blocks \( \{a_1, \ldots , a_r\} \) and \( c_m \) are crossing since \( G(\sigma) \) is connected. Thus we have the minimal integer of the set \( \{a_{r+1}, \ldots , a_{2r-1}\} \) is strictly less that \( a_r \), i.e., \( a_{r+1} < a_r < m \). Consequently the inverse bijection \( f^{-1} \) on \( f(\pi) \) gives \( \pi = a_1 \ldots a_r a_{r+1} \ldots a_{2r-1} \) is a permutation with descent set \( \{r\} \).
Assume that $G(\sigma)$ has more than 2 vertices, we decompose $G(\sigma)$ into two independent graphs $G_L(\sigma), G_R(\sigma)$. Then by induction hypothesis, 
$g^{-1}(\gamma_L, G_L(\sigma)) = (f(\pi_1), m^*)$, $g^{-1}(\gamma_R, G_R(\sigma)) = (f(\pi_2), m)$ and $f(\pi_1) = T_L, f(\pi_2) = T_R$ as before. Suppose $\pi_1$ is a permutation of length $rs - 1$ whose descent set is \{r, \ldots, r(s - 1)\}, and $\pi_2$ is a permutation of length $r(n - s) - 1$ whose descent set is \{r, \ldots, rn - r\}. Then we choose $\pi = \pi_1 m^* \pi_2$ which is a permutation. Since $m^*$ is the largest integer of the set \{a_1, \ldots, a_{rn-1}\} - \{m\}, the descent set of $\pi$ is \{r, \ldots, rn - r\}. The corresponding tree $f(\pi)$ has root $m^*$ whose left subtree is $f(\pi_1)$ and right subtree is $f(\pi_2)$. So inductively we have shown the preimage $g^{-1}(\gamma, G(\sigma))$ is a pair $(f(\pi), m)$ where $\pi$ is a permutation of \{a_1, \ldots, a_{rn-1}\} - \{m\} with descent set \{r, \ldots, rn - r\} and $m = \max\{a_i : 1 \leq i \leq rn\}$. From this we can conclude for $\pi = a_1 \cdots a_{rn-1}$ with descent set \{r, \ldots, rn - r\} and any $m > \max\{a_i : 1 \leq i < rn\}$, the correspondence $(\pi, m) \mapsto (f(\pi), m) \mapsto f(\pi)^m \mapsto g(f(\pi), m)$ is a bijection. In particular, consider $\pi \in \mathfrak{S}_{rn-1}$ with descent set \{r, \ldots, rn - r\}, the correspondence $(\pi, rn) \mapsto (f(\pi), rn) \mapsto f(\pi)^rn \mapsto g(f(\pi), rn)$ is a bijection. The proof is complete. 

\[ \square \]

4. ON THE POSET $Q_n^{(r)}$

An $r$-partition of $[n]$ is a set 
\[
\pi = \{(B_{11}, \ldots, B_{1r}), (B_{21}, \ldots, B_{2r}), \ldots, (B_{kr}, \ldots, B_{kr})\}
\]
satisfying the following two conditions

1. For each $j \in [r]$, the set $\pi_j = \{B_{ij}, B_{2j}, \ldots, B_{kj}\}$ forms a partition of $S$ (into $k$ blocks),
2. For fixed $i$, $|B_{1i}| = |B_{2i}| = \cdots = |B_{ri}|.$

The set of all the $r$-partitions of set $[n]$, denoted by $Q_n^{(r)}$, has a partial ordering by refinement, namely, let 
\[
\pi = \{(B_{11}, \ldots, B_{1r}), (B_{21}, \ldots, B_{2r}), \ldots, (B_{kr}, \ldots, B_{kr})\},
\]
\[
\sigma = \{(A_{11}, \ldots, A_{1r}), (A_{21}, \ldots, A_{2r}), \ldots, (A_{kr}, \ldots, A_{kr})\}
\]
be two $r$-partitions of $[n]$ where we set for $1 \leq j \leq r$, 
\[
\pi_j = \{B_{1j}, B_{2j}, \ldots, B_{kj}\}, \sigma_j = \{A_{1j}, A_{2j}, \ldots, A_{lj}\}.
\]
Then $\pi \leq \sigma$ if for every $1 \leq j \leq r, \pi_j \leq \sigma_j$, i.e., every block of partition $\pi_j$ is contained in a block of partition $\sigma_j$. For instance, $\pi = \{(\{1\}, \{2\}), (\{2\}, \{3\}), (\{1\}, \{3\})\}$ and $\sigma = \{(\{1\}, \{2\}, \{3\}), (\{2\}, \{3\}, \{1\})\}$ be two 2-partitions of $[3]$. Then we have $\pi_1 = \pi_2 = \{(1, 2), (3), \}, \sigma_1 = \{(1, 2), (3)\}, \sigma_2 = \{(2, 3), (1)\}$ and $\pi_1 \leq \sigma_1, \pi_2 \leq \sigma_2$ by refinement. That implies $\pi \leq \sigma$ in the poset $Q_n^{(2)}$. From the definition of $r$-partition, a minimal element $\rho$ of $Q_n^{(r)}$ can be identified as an $(r - 1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})$ of permutations $\sigma_i \in \mathfrak{S}_n$ by
\[
\rho = \{(\{1\}, \{1\}), \ldots, \{\sigma_{r-1}(1)\}\}, \ldots, \{(n)\}, \{\sigma_1(n)\}, \ldots, \{\sigma_{r-1}(n)\} \}
\]
As a result, the number of minimal elements of $Q_n^{(r)}$ is equal to the number of ordered permutations on $[n]$, i.e., $\sigma_1, \sigma_2, \ldots, \sigma_{r-1}$ where $\sigma_i \in \mathfrak{S}_n$. It follows that $M(n) = n!^{r-1}$ and the sequence $\{r_n(Q_n^{(r)})\}_{n \geq 1}$ where $r \geq 2$ defined by eq. (1.2) satisfies
\[
\sum_{n \geq 1} r_n(Q_n^{(r)})z^n = -\log \left( \sum_{n \geq 0} (-1)^n \frac{z^n}{n!} \right).
\]
Furthermore, non-ambiguous trees were introduced by Aval et al. because of its connection to the tree-like tableaux. The non-ambiguous trees are embedded in a 2-dimensional grid $\mathbb{N} \times \mathbb{N}$. Let each vertex $v$ have coordinate $x(v) = \ldots$
(x_1(v), x_2(v)), then a 2-dimensional non-ambiguous tree of size n is a set A of n points (x_1(v), x_2(v)) ∈ \mathbb{N} \times \mathbb{N} such that

1. (0, 0) ∈ A, we call this point the root of A.
2. For a given non-root point p ∈ A, there exists one point q ∈ A such that x_2(q) < x_2(p) and x_1(q) = x_1(p), or one point s ∈ A such that x_1(s) < x_1(p) and x_2(s) = x_2(p), but not both.
3. There is no empty line between two given points: if there exists a point p ∈ A such that x_1(p) = x (resp. x_2(p) = y), then for every x' < x (resp. y' < y) there exists q ∈ A such that x_1(q) = x' (resp. x_2(q) = y').

A complete non-ambiguous tree is a non-ambiguous tree whose vertices have either 0 or 2 children. The non-ambiguous tree A has a unique tree structure since, except for the root, every point p ∈ A has a unique parent, which is the nearest point q described in the condition (2). That implies the set A of points determines the tree structure. Let T_A be the unique underlying tree associated to the vertex set A. For instance, there are four complete non-ambiguous trees of size 5 whose underlying trees are

With the notations and definitions given in the subsection 4.1, one of our main results is stated as follows:

**Theorem 4.** The number r_n(Q^{(r)}_n) defined in eq. (4.1) counts the directed graphs (δ, G(σ)) where σ is an (r − 1)-tuple (σ_1, . . . , σ_{r−1}) of permutations σ_i ∈ \mathcal{S}_n such that the crossing graph G(σ) is connected, and δ is an acyclic orientation of G(σ) whose unique source is the σ-path starting at 1. In particular, r_n(Q^{(2)}_n) is equal to the number of complete non-ambiguous trees of size 2n − 1.

We will prove Theorem 4 by giving a combinatorial meaning of r_n(Q^{(r)}_n) in terms of pyramids in Lemma 5, and then constructing a bijection between the set of pyramids counted by r_n(Q^{(2)}_n) and the set of complete non-ambiguous trees of size 2n − 1 in Lemma 6.

4.1. **Combinatorial interpretation of r_n(Q^{(r)}_n).** The diagram representation of permutation π ∈ \mathcal{S}_n is described as follows: we represent each permutation π ∈ \mathcal{S}_n by drawing 2n dots in two rows with each row having n dots labeled by 1, 2, . . . , n. Then we connect i from the 1st row with j from the 2nd row if and only if π(i) = j. Two edges (i, π(i)) and (j, π(j)) where i < j are crossing if π(i) > π(j), therefore the number of crossings is equal to the inversion number of π.

Given r sequences \{p_{mn−r+1}\}_{m=1}^{n}, \{p_{mn−r+2}\}_{m=1}^{n}, . . . , \{p_{mr}\}_{m=1}^{n}, if σ = (σ_1, σ_2, . . . , σ_{r−1}) be an (r − 1)-tuple of bijections σ_j satisfying σ_j(p_{mn−r+j}) = p_{mr−r+j+1} for every j, m. Then we write for every j,

\[ σ_j = \begin{pmatrix} \pi_j & p_{r+j} & \cdots & p_{mr−r+j} \\ p_{j+1} & pr+j+1 & \cdots & p_{nr−r+j+1} \end{pmatrix}. \]

We can represent σ by drawing rn dots in r rows with each row having n dots labeled by 1, 2, . . . , n. For any m, j we connect p_{mn−r+j} with p_{mr−r+j+1} by an edge. For each m ∈ [n], we call (p_{mn−r+1}, p_{mn−r+2}, . . . , p_{mr}) as a σ-path starting at p_{mn−r+1}. Two σ-paths (p_{mr−r+1}, p_{mr−r+2}, . . . , p_{mr}) and (p_{tr−r+1}, p_{tr−r+2}, . . . , p_{tr}) are crossing if there exists s such that p_{mn−r+s} < p_{tr−r+s} and p_{mr−r+s+1} > p_{mr−r+s+1}. Let G(σ) be the crossing graph of σ where the vertex set V_G(σ) consists of all the σ-paths, and two vertices are connected by an edge if the corresponding σ-paths
are crossing. In particular, if the set \( \{ p_{mr-r+j} : 1 \leq m \leq n \} = \{ 1, \ldots, n \} \) for every \( j \in [r] \), then \( \sigma \in \mathfrak{S}_n \) and \( \sigma = (\sigma_1, \ldots, \sigma_{r-1}) \) is an \((r - 1)\)-tuple of permutations. For instance, given 3 sequences, the first one is 1, 2, 3, 4, 5, the second one is 2, 3, 1, 4, 5 and the third one is 4, 1, 3, 2, 5. Then \( \sigma = (23145, 34125) \), i.e., \( \sigma_1 = 23145 = (1 \ 2 \ 3 \ 4 \ 5) \) and \( \sigma_2 = 34125 = (2 \ 3 \ 1 \ 4 \ 5) \). The paths (1, 2, 4), (2, 3, 1), (3, 1, 3), (4, 2), (5, 5, 5) are \( \sigma \)-paths. The diagram representation of \( \sigma \) and the crossing graph \( G(\sigma) \) are shown as below.

\[
\sigma: \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 2 & 4 & 1 \\
2 & 1 & 4 & 3 \\
1 & 4 & 2 & 3 \\
5 & 3 & 1 & 2
\end{array} \quad \text{G(\sigma):} \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 1 & 4 & 5 \\
4 & 1 & 3 & 2
\end{array}
\]

In general, we call \((i_1, \ldots, i_r)\) as a path if we connect \(i_j\) from the \(j\)-th row to \(i_{j+1}\) from the \((j+1)\)-th row for any \(1 \leq j < r\). Two paths \((i_1, \ldots, i_r)\) and \((j_1, \ldots, j_r)\) are crossing if there exists \(s\) such that \(i_s < j_s\) and \(i_{s+1} > j_{s+1}\).

The combinatorial interpretation of \(r_n(Q_n^{(r)})\) is stated as below:

**Lemma 5.** The number \(r_n(Q_n^{(r)})\) counts the directed graphs \((\delta, G(\sigma))\) where \(\sigma\) is an \((r - 1)\)-tuple \((\sigma_1, \ldots, \sigma_{r-1})\) of permutations \(\sigma_i \in \mathfrak{S}_n\) such that the crossing graph \(G(\sigma)\) is connected, and \(\delta\) is an acyclic orientation of \(G(\sigma)\) whose unique source is the \(\sigma\)-path starting at 1.

**Proof.** We consider the set of paths \(\mathcal{B} = \{(i_1, \ldots, i_r) : 1 \leq i_j \leq n, 1 \leq j \leq r\}\) with a symmetric and reflexive binary relation \(\mathcal{R}\). We say \((i_1, \ldots, i_r) \mathcal{R} (j_1, \ldots, j_r)\) if and only if two paths \((i_1, \ldots, i_r)\) and \((j_1, \ldots, j_r)\) are crossing. Let \((R, \leq)\) be a poset where each element is labeled by \((i_1, \ldots, i_r)\) such that

1. Two elements \((i_1, \ldots, i_r)\) and \((j_1, \ldots, j_r)\) are comparable if \((i_1, \ldots, i_r)\) and \((j_1, \ldots, j_r)\) are crossing.
2. If \((j_1, \ldots, j_r)\) covers \((i_1, \ldots, i_r)\) in the poset \((R, \leq)\), then \((j_1, \ldots, j_r)\) and \((i_1, \ldots, i_r)\) are crossing.

Then by definition the poset \((R, \leq)\) is a heap \((R, \leq) \in \mathcal{H}(\mathcal{B}, \mathcal{R})\). In a geometric way, we can represent each piece \((i_1, \ldots, i_r)\) as a path in the diagram, and put the path \((j_1, \ldots, j_r)\) on top of the path \((i_1, \ldots, i_r)\) if \((i_1, \ldots, i_r) \leq (j_1, \ldots, j_r)\) in the heap \(H = (R, \leq)\). Each heap \(H = (R, \leq)\) is therefore represented by a collection of paths \((i_1, \ldots, i_r)\) if \((i_1, \ldots, i_r) \in R\). We take the heaps \(H_1, H_2\) as examples, see the picture below.

\[
H_1: \quad \begin{array}{cccc}
(1, 1) & (1, 3) & (3, 1) & (2, 2) \\
2 & 3 & 1 & 4 \\
1 & 3 & 2 & 4 \\
1 & 2 & 3 & 4
\end{array} \quad \text{H_2:} \quad \begin{array}{cccc}
(2, 2) & (2, 3) & (1, 2) & (3, 1) \\
2 & 3 & 1 & 4 \\
1 & 3 & 2 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

We first put \((1, 1)\) at the bottom, then put \((1, 3)\) on top of \((1, 1)\), and then put \((3, 3)\) on top of \((1, 3)\). The path \((1, 1)\) (resp. \((3, 3)\)) cannot move vertically upwards (resp. downwards) without touching \((1, 3)\). But \((2, 2)\) and \((3, 3)\) are not crossing that the path \((2, 2)\) can move vertically without touching the path \((3, 3)\). That is equivalent to say \((2, 2), (3, 3)\) are incomparable and \((1, 1) \leq (1, 3) \leq (3, 3), (1, 3) \leq (2, 2)\) in the poset \(H_1\). See the examples \(H_1, H_2\) above. For any two heaps \(H_1, H_2\), the
composition of heaps $H_1, H_2, H_1 \circ H_2$, is the heap obtained by placing the pieces in $H_2$ on top of pieces in $H_1$, i.e.,

$$H_1 \circ H_2:$$

Consider the monoid $M(B, R)$, which is a set of words $w$ with letters from the set \{ $x_{1,\ldots,i_r}$ : $(i_1, \ldots, i_r) \in B$ \} such that two adjacent letters $x_{i_1,\ldots,i_r}, x_{j_1,\ldots,j_r}$ of $w$ commute if $i_m < j_m$ for any $1 \leq m \leq r$, or equivalently, the path $(i_1, \ldots, i_r)$ is not crossing with $(j_1, \ldots, j_r)$. We continue using the example $H_1, H_2$ as before to show the words $\varphi(H_1), \varphi(H_2) \in M(B, R)$. By reading the labels of the elements in the poset $H_1$ (resp. $H_2$) from bottom-to-top, left-to-right, we have $\varphi(H_1) = x_{1,1}x_{1,3}x_{2,2}x_{3,3}$ where $x_{2,2}x_{3,3} = x_{3,3}x_{2,2}$ (resp. $\varphi(H_2) = x_{3,1}x_{1,2}x_{2,3}$ where $x_{1,2}x_{2,3} = x_{2,3}x_{1,2}$).

It is easy to check that $\varphi(H_1 \circ H_2) = \varphi(H_1) \cdot \varphi(H_2) = x_{1,1}x_{1,3}x_{2,2}x_{3,3}x_{3,1}x_{1,2}x_{2,3}$ subject to the conditions $x_{2,2}x_{3,3} = x_{3,3}x_{2,2}$ and $x_{1,2}x_{2,3} = x_{2,3}x_{1,2}$.

Let $\{a_{i,j}\}_{i \geq 1,j \geq 1}$ be a sequence of variables that $a_{i,j}^2 = 0$ for every $i, j$, we consider a ring homomorphism $f : \mathbb{Z}[M(B, R)] \to \mathbb{Z}[\prod_{i,j} a_{i,j}]$ such that $f(x_{i_1,\ldots,i_r}) = \prod_{j=1}^{r} a_{i,j}$ and $f(1) = 1$. Then $f$ induces a natural isomorphism

$$f_0 : \mathbb{Z}[M(B, R)] / \ker(f) \to \text{im}(f)$$

via $m + \ker(f) \mapsto f(m)$. After applying $f_0$ on both sides of eq. (2.4), we obtain an identity in the commutative ring $\text{im}(f)$.

$$\sum_{P \in \mathcal{P}(B, R)} \frac{f_0(\varphi(P))}{|\varphi(P)|} = -\log \left( \sum_{T \in \mathcal{T}(B, R)} (-1)^{|\varphi(T)|} f_0(\varphi(T)) \right).$$

Let $\mathcal{T}_n(B, R)$ be a set of trivial heaps of size $n$ contained in the set $\mathcal{T}(B, R)$, suppose $T_n \in \mathcal{T}_n(B, R)$ is a trivial heap having $n$ pieces $(i_1, \ldots, i_r), (i_{r+1}, \ldots, i_{2r}), \ldots, (i_{rn-r+1}, \ldots, i_{rn})$ that the sequence $\{i_{rn-r+j}\}_{m=1}^{n}$ is a strictly increasing sequence for every $1 \leq j \leq r$. That is to say, $T_n \mapsto I^*_n$, where $I^*_n = (I_{T_n,1}, \ldots, I_{T_n,r})$ and $I_{T_n,j} = \{i_{rn-r+j}\}_{m=1}^{n}$, is a bijection between the set of trivial heaps of size $n$ and the set of $r$-tuple of strictly increasing sequences of length $n$. By applying $f_0$ on the word $\varphi(T_n) \in M(B, R)$, we get $f_0(\varphi(T_n)) = f(\varphi(T_n)) \neq 0$ and consequently

$$\sum_{T_n \in \mathcal{T}_n(B, R)} (-1)^n f(\varphi(T_n)) = \sum_{I^*_n} (-1)^n \prod_{m=1}^{r} \prod_{j=1}^{n} a_{i_{rn-r+j}, j}$$

$$= (-1)^n \prod_{j=1}^{r} \frac{(a_{1,j} + a_{2,j} + \cdots)^n}{n!}$$

$$= (-1)^n \frac{(\prod_{j=1}^{r} (\sum_{i=1}^{n} a_{i,j}))^n}{n!^r},$$

where the second summation runs over all the $r$-tuple $I^*_n$ of strictly increasing sequences of length $n$ and the last two equations hold because $a_{i,j}^2 = 0$ for every
It follows that the right hand side of eq. (4.2) is
\[- \log \left( \sum_{T \in \mathcal{T}(B, R)} (-1)^{|\varphi(T)|} f_0(\varphi(T)) \right) = - \log \left( \sum_{n \geq 0, T_n \in \mathcal{T}(R \cup B, R)} (-1)^n f(\varphi(T_n)) \right) = - \log \left( \sum_{n \geq 0} (-1)^n \left( \prod_{j=1}^{\infty} \left( \sum_{i=1}^{n} a_{i,j} \right) / n! \right)^n \right).\]

On the other hand, let \( \mathcal{P}_n(B, R) \) be the set of pyramids of size \( n \) contained in the set \( \mathcal{P}(B, R) \). Suppose \( P_n = (\sigma, \leq) \in \mathcal{P}_n(B, R) \) be a pyramid of size \( n \) such that \( f(\varphi(P_n)) \neq 0 \), since the elements of the poset \( P_n = (\sigma, \leq) \) are the \( \sigma \)-paths and \( f(\varphi(P_n)) \neq 0 \), \( \sigma \) must be an \( (r-1) \)-tuple \( (\sigma_1, \ldots, \sigma_{r-1}) \) of bijections that \( \sigma_j \) is a bijection between two sets of size \( n \) for every \( j \). We continue using \( I_{T_n} = (I_{T_n,1}, \ldots, I_{T_n,r}) \) to represent any \( r \)-tuple of strictly increasing sequences of length \( n \).

A strictly increasing sequence \( I_{T_n,j} = \{i_{m,r+j}\}_{m=1}^{n} \) is uniquely corresponding to a set \( \{i_{m,r+j}, 1 \leq m \leq n\} \). For a given \( r \)-tuple \( I_{T_n} \) of strictly increasing sequences, let \( \bar{p}_{n,r} \) count the number of pyramids \( P_n = (\sigma, \leq) \) that \( \sigma = (\sigma_1, \ldots, \sigma_{r-1}) \) is an \( (r-1) \)-tuple of bijections \( \sigma_j \) between the set \( \{i_{m,r+j}, 1 \leq m \leq n\} \) and the set \( \{i_{m,r+j+1}, 1 \leq m \leq n\} \) for every \( j, m \). Then we have

\[ \sum_{P_n \in \mathcal{P}_n(B, R), \sigma(\varphi(P_n)) \neq 0} f(\varphi(P_n)) = \sum_{I_{T_n}} \bar{p}_{n,r} \prod_{m=1}^{n} \prod_{j=1}^{r} a_{i_{m,r+j}, j} \]

Noting that the number \( \bar{p}_{n,r} \) is independent of the choice of the \( r \)-tuple \( I_{T_n} \) of strictly increasing sequences, we choose \( \{i_{m,r+j}, 1 \leq m \leq n\} = \{1, \ldots, n\} \) for every \( j \) and \( \bar{p}_{n,r} \) also counts the number of pyramids \( P_n = (\sigma, \leq) \) where \( \sigma = (\sigma_1, \ldots, \sigma_{r-1}) \) is an \( (r-1) \)-tuple of permutations \( \sigma_i \in \mathfrak{S}_n \). It turns out the left hand side of eq. (4.2) is

\[ \sum_{P \in \mathcal{P}(B, R), \sigma(\varphi(P)) \neq 0} f_0(\varphi(P)) \left| \varphi(P) \right| = \sum_{n \geq 0} \sum_{P_n \in \mathcal{P}_n(B, R), \sigma(\varphi(P_n)) \neq 0} \frac{f(\varphi(P_n))}{\left| \varphi(P_n) \right|} = \sum_{n \geq 0} \sum_{I_{T_n}} \bar{p}_{n,r} \prod_{m=1}^{n} \prod_{j=1}^{r} a_{i_{m,r+j}, j} = \sum_{n \geq 0} \bar{p}_{n,r} \left( \prod_{j=1}^{\infty} \left( \sum_{i=1}^{n} a_{i,j} \right) / n! \right)^n.\]

In view of eq. (4.1), we can conclude \( r_n(Q_n^{(r)}) = n^{-1} \bar{p}_{n,r} \). The number \( \bar{p}_{n,r} \) is equal to the number of directed graphs \( (\delta, G(\sigma)) \) that \( \sigma \) is an \( (r-1) \)-tuple of permutations of \( [n] \) and \( G(\sigma) \) is connected with an acyclic orientation \( \delta \) whose source is unique. Since there are \( n \) vertices in the graph \( G(\sigma) \), the number \( r_n(Q_n^{(r)}) = n^{-1} \bar{p}_{n,r} \) counts the number of directed graphs \( (\delta, G(\sigma)) \) that \( \sigma = (\sigma_1, \ldots, \sigma_{r-1}) \) is an \( (r-1) \)-tuple of permutations \( \sigma_i \in \mathfrak{S}_n \) and \( G(\sigma) \) is connected with an acyclic orientation \( \delta \) whose unique source is the \( \sigma \)-path starting with 1.

Let \( b_n \) be the number of complete non-ambiguous trees of size \( 2n - 1 \), Aval et al. proved the integers \( b_2 \) satisfy eq. (4.1) when \( r = 2 \). This implies \( r_n(Q_n^{(2)}) = b_n \). We will prove \( r_n(Q_n^{(2)}) = b_n \) by a bijection in Lemma 6.

**Lemma 6.** There is a bijection between the set of directed graphs \( (\delta, G(\sigma)) \) where \( \sigma \in \mathfrak{S}_n \), \( G(\sigma) \) is connected with an acyclic orientation \( \delta \) whose unique source is the path \( (1, \sigma(1)) \), and the set of complete non-ambiguous trees of size \( 2n - 1 \).

**Proof.** Given a complete non-ambiguous tree \( A \) of size \( 2n - 1 \), we first do a coordinate translation. Let \( A \mapsto \mu(A) \) be a bijection that for any \( v \in A \) having coordinates \( x(v) = (x_1(v), x_2(v)) \), we set \( \mu(A) = \{(x_1(v) + 1, x_2(v) + 1) : v \in A\} \). Recall that \( T_A \) is the unique underlying tree associated to \( A \), then via bijection \( \mu \) we shift the
tree $T_A$ from rooted at $(0,0)$ to the tree $T_{\mu(A)}$ rooted at $(1,1)$. Next we consider the coordinates of the leaves of $T_{\mu(A)}$. From the definition of complete non-ambiguous tree, we can let the coordinates of the leaves of $T_{\mu(A)}$ be $(1,x_1),\ldots,(n,x_n)$ where $x_1\cdots x_n \in \mathbb{S}_n$.

Consider the correspondence $T_{\mu(A)} \mapsto \sigma_{T_{\mu(A)}}$ that $\sigma_{T_{\mu(A)}} = x_1\cdots x_n \in \mathbb{S}_n$. The graph $G(\sigma_{T_{\mu(A)}})$ has vertex set $\{(1,x_1),\ldots,(n,x_n)\}$ and there is an edge between $(i,x_i)$ and $(j,x_j)$ if they are crossing. In order to prove $G(\sigma_{T_{\mu(A)}})$ is connected by induction, we need to introduce the definition of half non-ambiguous tree and complete half non-ambiguous tree. Let each vertex $v$ have coordinate $(x_1(v),x_2(v))$, then a half non-ambiguous tree of size $n$ is a set $A$ of $n$ points $(x_1(v),x_2(v)) \in \mathbb{N} \times \mathbb{N}$ such that

1. Let $a = \min\{x_1(v) : v \in A\}$ and $b = \min\{x_2(v) : v \in A\}$, then $(a,b) \in A$, we call this point the root of $A$.
2. For a given non-root point $p \in A$, there exists one point $q \in A$ such that $x_1(q) = x_1(p)$ and $x_2(q) < x_2(p)$, or one point $s \in A$ such that $x_2(s) = x_2(p)$ and $x_1(s) < x_1(p)$, but not both.

A complete half non-ambiguous tree is a half non-ambiguous tree whose vertices have either 0 or 2 children. For a half non-ambiguous tree $B$, from condition (2) we know the underlying tree $T_B$ associated to the vertex set $B$ is unique. Suppose $B$ is a complete half non-ambiguous tree and $T_B$ is a binary tree having $n$ leaves with coordinates $(\mu_1, \nu_1),\ldots,(\mu_n, \nu_n)$ where $\mu_1 < \cdots < \mu_n$, we will first show the map $h_B$ from the set of complete half non-ambiguous trees $B$ to the set of directed graphs $(\delta_{r,n,B},G(\sigma_{T_B}))$. Then we will prove $h_B$ is a bijection.

We first consider the correspondence $T_B \mapsto \sigma_{T_B}$ where $\sigma_{T_B}$ is a bijection satisfying $\sigma_{T_B}(\mu_i) = \nu_i$ for any $i$. $G(\sigma_{T_B})$ is the crossing graph of $\sigma_{T_B}$ whose vertex set is $\{\{(\mu_1, \nu_1),\ldots,(\mu_n, \nu_n)\}$ and there is an edge between $(\mu_i, \nu_i)$ and $(\mu_j, \nu_j)$ if $(\mu_i, \nu_i)$ is crossing with $(\mu_j, \nu_j)$. We will prove the crossing graph $G(\sigma_{T_B})$ is connected by induction on the size of $T_B$. If $T_B$ is a binary tree with 2 leaves $(\mu_1, \nu_1), (\mu_2, \nu_2)$ where $\mu_1 < \mu_2$, then $\nu_1 > \nu_2$. It follows that there is an edge between $(\mu_1, \nu_1)$ and $(\mu_2, \nu_2)$ in the graph $G(\sigma_{T_B})$. If $T_B$ is a binary tree with more than 2 leaves, i.e., $(\mu_1, \nu_1), (\mu_2, \nu_2), \ldots, (\mu_n, \nu_n)$ are leaves of $T_B$ such that $\mu_1 < \mu_2 < \cdots < \mu_n$, then we decompose $T_B$ into a left subtree $T_{B_1}$ and a right subtree $T_{B_2}$. By induction hypothesis on $T_{B_1}$ and $T_{B_2}$, the correspondence $T_{B_1} \mapsto \sigma_{T_{B_1}}, T_{B_2} \mapsto \sigma_{T_{B_2}}$ gives two connected crossing graphs $G(\sigma_{T_{B_1}}), G(\sigma_{T_{B_2}})$. Furthermore, the leaf $(\mu_1, \nu_1)$ is contained in the left subtree $T_{B_1}$, and the leaf $(\mu_n, \nu_n)$ is contained in the right subtree $T_{B_2}$. By noticing that $\nu_n < \nu_1$ and $\mu_1 < \mu_n$, two paths $(\mu_1, \nu_1)$ and $(\mu_n, \nu_n)$ are crossing, i.e., there is an edge $\{(\mu_1, \nu_1), (\mu_n, \nu_n)\}$ in the graph $G(\sigma_{T_B})$ that connects a vertex $(\mu_1, \nu_1)$ from $G(\sigma_{T_{B_1}})$ and a vertex $(\mu_n, \nu_n)$ from $G(\sigma_{T_{B_2}})$. In combination with the fact both $G(\sigma_{T_{B_1}})$ and $G(\sigma_{T_{B_2}})$ are connected, we can conclude for any complete half non-ambiguous tree $B$, the correspondence $T_B \mapsto \sigma_{T_B}$ gives a connected graph $G(\sigma_{T_B})$. In particular, for a complete non-ambiguous tree $A$, from $T_{\mu(A)} \mapsto \sigma_{T_{\mu(A)}}$ we get a connected crossing graph $G(\sigma_{T_{\mu(A)}})$ where $\sigma_{T_{\mu(A)}} \in \mathbb{S}_n$.

We next inductively define the map $T_B \mapsto (\delta_{r,n,B},G(\sigma_{T_B}))$ where $\delta_{r,n,B}$ is an acyclic orientation of $G(\sigma_{T_B})$ whose unique source is $(\mu_1, \nu_1)$ and $\nu_1 = \sigma_{T_B}(\mu_1)$. If $T_B$ is a binary tree with 2 leaves $(\mu_1, \nu_1), (\mu_2, \nu_2)$ where $\mu_1 < \mu_2$, then we choose the orientation $\delta_{r,n,B}$ to be $(\mu_1, \nu_1) \leftarrow (\mu_2, \nu_2)$. If $T_B$ is a binary tree with more than 2 leaves, assume $T_B$ has $n$ leaves $(\mu_1, \nu_1), (\mu_2, \nu_2), \ldots, (\mu_n, \nu_n)$ where $\mu_1 < \mu_2 < \cdots < \mu_n$, any subtree rooted at an internal node of $T_B$ is a complete half non-ambiguous tree. We start from the subtrees of $T_B$ having 2 leaves. If $T_{B_1}$ is a subtree contained in $T_B$ with 2 leaves $(\mu_1, \nu_1), (\mu_2, \nu_2)$ where $\mu_1 < \mu_2$, then $\nu_1 > \nu_2$ and we choose the orientation $\delta_{r,n,B}$ on the edge $\{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ to be $(\mu_1, \nu_1) \leftarrow (\mu_2, \nu_2)$ (resp. $(\mu_1, \nu_1) \rightarrow (\mu_2, \nu_2)$) if $T_B$ is the left (resp. right) subtree of his parent. Next we consider the subtrees of $T_B$ with more than 2 leaves. If $T_{B_1}$ is a subtree contained
in $T_B$ with more than 2 leaves $(\mu_i, \nu_i), (\mu_{i2}, \nu_{i2}), \ldots, (\mu_{im}, \nu_{im})$ where $\mu_i < \mu_{i2} < \cdots < \mu_m$, then two paths $(\mu_i, \nu_i)$ and $(\mu_{im}, \nu_{im})$ are crossing in the diagram representation of $\sigma_{T_B^*}$ since $\mu_{i2} < \mu_{im}$ and $\nu_{i2} > \nu_{im}$. We decompose $T_B$ into a left subtree $T_{B_1}$ and a right subtree $T_{B_2}$. Assume that we have defined an acyclic orientation $\delta_{r,n,B_1}$ on the graph $G(\sigma_{T_{B_1}}^*)$ whose unique source is $(\mu_i, \nu_i)$ and an acyclic orientation $\delta_{r,n,B_2}$ on the graph $G(\sigma_{T_{B_2}}^*)$ whose unique source is $(\mu_{im}, \nu_{im})$.

We will define the orientation $\delta_{r,n,B^*}$ on the graph $G(\sigma_{T_B^*})$ as follows. For any two leaves $(\mu_i, \nu_i), (\mu_j, \nu_j)$ contained in the tree $T_B$, if both $(\mu_i, \nu_i), (\mu_j, \nu_j)$ are crossing in the diagram representation of $\sigma_{T_B^*}$ (resp. $\delta_{r,n,B_1}^*$), then $\delta_{r,n,B_2}$ (resp. $\delta_{r,n,B_2}$) determines the direction on the edge between $(\mu_i, \nu_i)$ and $(\mu_j, \nu_j)$. Otherwise if $(\mu_i, \nu_i)$ is a leaf contained in the tree $T_{B_1}$, $(\mu_j, \nu_j)$ is a leaf contained in the tree $T_{B_2}$ and $(\mu_i, \nu_i), (\mu_j, \nu_j)$ are crossing paths in the diagram representation of $\sigma_{T_B^*}$. Then we choose the direction $(\mu_i, \nu_i) \rightarrow (\mu_j, \nu_j)$ (resp. $(\mu_i, \nu_i) \rightarrow (\mu_j, \nu_j)$) if $T_B$ is the left (resp. right) subtree of his parent or $T_B = T_B$. Consequently the orientation $\delta_{r,n,B^*}$ is acyclic. Since two paths $(\mu_i, \nu_i)$ and $(\mu_{im}, \nu_{im})$ are crossing, the unique maximal element is $(\mu_i, \nu_i)$ (resp. $(\mu_{im}, \nu_{im})$) if $T_B$ is the left (resp. right) subtree of his parent or $T_B = T_B$. Thus we have inductively given the map $B \mapsto h_2(B) = (\delta_{r,n,B}, G(\sigma_{T_B^*}))$ where $\delta_{r,n,B}$ is an acyclic orientation on the graph $G(\sigma_{T_B^*})$ whose unique source is $(\mu_i, \nu_i)$. In particular, the map $\mu(A) \mapsto h_2(\mu(A)) = (\delta_{n,m,\mu(A)}, G(\sigma_{T_{\mu(A)}}))$ gives a connected graph $G(\sigma_{T_{\mu(A)}})$ with an acyclic orientation $\delta_{n,m,\mu(A)}$ whose unique source is $(1, x_1)$ and $x_1 = \sigma_{T_{\mu(A)}}(1)$.

For instance, consider the set $A = \{(0,0), (1,0), (2,0), (0,2), (1,1)\}$, then $T_{\mu(A)}$ has 3 leaves $(1, 3), (2, 2), (3, 1)$ in the left-to-right order, see the figure below. Thus $\sigma_{T_{\mu(A)}} = (1, 3, 2, 2, 3) \equiv 321 \in \mathcal{S}_3$. We decompose $T_{\mu(A)}$ into a left subtree $T_{\mu(A)_1}$, and a right subtree $T_{\mu(A)_2}$ and the image $h_2(\mu(A)_1)$ (resp. $h_2(\mu(A)_2)$) is a digraph whose unique source is $(1, 3)$ (resp. $(3, 1)$). $h_2(\mu(A))$ is obtained from $h_2(\mu(A)_1)$ and $h_2(\mu(A)_2)$ by adding $(1, 3) \leftarrow (3, 1)$ and $(1, 3) \leftarrow (2, 2)$ since $(1, 3)$ is crossing with $(3, 1)$ and $(2, 2)$.

We will show the given map $h_2$ is a bijection by showing its inverse $h_2^{-1}$ inductively. Given a pair $(\delta, G(\sigma))$ where $\sigma \in \mathcal{S}_n$, $G(\sigma)$ is connected with an acyclic orientation $\delta$ whose unique source is $(1, \sigma(1))$. Recall that the vertex set of $G(\sigma)$ consists of all the $\sigma$-paths and there is an edge between two $\sigma$-paths if they are crossing. We use $V_{G(\sigma)}$ to denote the vertex of graph $G(\sigma)$. So $y \in V_{G(\sigma)}$ means $y$ is a $\sigma$-path. Consider the induction basis $n = 2$, the digraph $(\delta, G(\sigma))$ is $(\mu_1, \nu_1) \leftarrow (\mu_2, \nu_2)$ where $\mu_1 < \mu_2$, then $\nu_1 > \nu_2$ and $h_2^{-1}(\delta, G(\sigma))$ is the set $\{(\mu_1, \nu_1), (\mu_2, \nu_2), (\nu_1, \nu_2)\}$ which is a complete half non-ambiguous tree rooted at $(\mu_1, \nu_1)$. Assume that $G(\sigma)$ has more than 2 leaves $(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)$ where $\mu_1 < \cdots < \mu_n$, then we have $\nu_n < \min\{\nu_i : 1 \leq i < n\}$, we say $c < d$ if there is a directed path from $c$ to $d$ in the directed graph $(\delta, G(\sigma))$ and $c \leq c$ for every $c$ in the graph $G(\sigma)$. We consider a vertex-induced subgraph of $G(\sigma)$, denoted by $G_R(\sigma)$ whose vertex set is $\{y : y \leq (\mu_n, \nu_n), y \in V_{G(\sigma)}\}$. Let $G_L(\sigma)$ be a vertex induced subgraph of $G(\sigma)$ whose vertex set is $\{y : y \not\leq (\mu_n, \nu_n), y \in V_{G(\sigma)}\}$. For any vertex $c$ in $G_L(\sigma)$ and $d$ in $G_R(\sigma)$, we have $c \leftarrow d$. Otherwise, it would contradict to the assumption that $c$ is
contained in the graph $G_L(\sigma)$. In this way we split graph $G(\sigma)$ into two independent graphs $G_L(\sigma)$ and $G_R(\sigma)$. Let $\delta_L$ (resp. $\delta_R$) be the acyclic orientation $\delta$ restricted to the graph $G_L(\sigma)$ (resp. $G_R(\sigma)$). Therefore the directed graph $(\delta_L, G_L(\sigma))$ (resp. $(\delta_R, G_R(\sigma))$) has unique source $(\mu_1, \nu_1)$ (resp. $(\mu_n, \nu_n)$). By induction hypothesis, $h_2^{-1}(\delta_L, G_L(\sigma)) = B_1$ gives a complete half non-ambiguous tree $B_1$ with root $(\mu_1, \nu_1)$ and $h_2^{-1}(\delta_R, G_R(\sigma)) = B_2$ gives a complete half non-ambiguous tree $B_2$ with root $(\mu_n, \nu_n)$. Thus $h_2^{-1}(\delta, G(\sigma))$ is the set $B_1 \cup B_2 \cup \{(\mu_1, \nu_1)\}$, which is a complete half non-ambiguous tree with root $(\mu_1, \nu_1)$. Consequently the proof that $h_2$ is a bijection is complete.

For instance, there are four complete non-ambiguous trees with 3 leaves whose underlying trees are

They are corresponding to the directed graphs

one by one from left to right. □

4.2. From heaps to pairs of permutations. A non-ambiguous forest introduced by Aval et al. [1] is a set of points $x(v) = (x_1(v), x_2(v)) \in \mathbb{N} \times \mathbb{N}$ satisfying the following conditions:

(1) For a given non-root point $p \in A$, the set $\{s \in A : x_1(s) < x_1(p), x_2(s) = x_2(p)\} \cup \{q \in A : x_2(q) < x_2(p), x_1(q) = x_1(p)\}$ has at most 1 element.

(2) There is no empty line between two given points: if there exists a point $p \in A$ such that $x_1(p) = x$ (resp. $x_2(p) = y$), then for every $x' < x$ (resp. $y' < y$) there exists $q \in A$ such that $x_1(q) = x'$ (resp. $x_2(q) = y'$).

In the same way that a non-ambiguous tree has an underlying binary tree structure, a non-ambiguous forest has an underlying binary forest structure. A complete non-ambiguous forest is a non-ambiguous forest such that all trees in the underlying binary forest are complete. We denote by $\tau(n)$ the number of complete non-ambiguous forests with $n$ leaves. Then Aval et al. [1] proved that

$$\sum_{n \geq 0} \tau(n) \frac{x^n}{n!^2} = \left( \sum_{n \geq 0} \frac{(-1)^n x^n}{n!^2} \right)^{-1}. \quad (4.3)$$

In addition, Carlitz et al. [2] proved the same identity by enumerating the pairs of permutations with no common rise. Let $\omega(n)$ count the number of pairs $(\pi, \xi)$ of permutations of $[n]$ such that any two consecutive entries cannot be rising both in $\pi$ and $\xi$, i.e., there is no integer $i$ such that $\pi(i) < \pi(i+1)$ and $\xi(i) < \xi(i+1)$. Carlitz et al. [2] showed

$$\sum_{n \geq 0} \omega(n) \frac{x^n}{n!^2} = \left( \sum_{n \geq 0} \frac{(-1)^n x^n}{n!^2} \right)^{-1}. \quad (4.4)$$
This implies $\tau(n) = \omega(n)$ and Aval et al. asked for a bijective proof of $\tau(n) = \omega(n)$. Here we will give the bijection between the set of complete non-ambiguous forests with $n$ leaves and the set of pairs $((\pi, \xi))$ of permutations $\pi, \xi \in \mathfrak{S}_n$ with no common rise, by using the heaps as intermediate objects.

**Theorem 7.** There is a bijection between the set of complete non-ambiguous forests with $n$ leaves and the set of pairs $((\pi, \xi))$ of permutations of $[n]$ with no common rise.

**Proof.** We first define the number $h_{n,r}$ by the equation

$$
\sum_{n \geq 0} h_{n,r} x^n = \left( \sum_{n \geq 0} \frac{(-1)^n x^n}{n!} \right)^{-1}.
$$

Similar to the proof of Lemma 5, we can prove that $h_{n,r}$ counts the number of directed graphs $((\delta, G(\sigma)))$ where $\sigma = (\sigma_1, \ldots, \sigma_{r-1})$ is an $(r-1)$-tuple of permutations $\sigma_i \in \mathfrak{S}_n$ and $\delta$ is an acyclic orientation on the graph $G(\sigma)$. Here we omit the proof of this fact. In particular, $h_{n,2}$ counts the directed graphs $((\delta, G(\sigma)))$ that $\sigma \in \mathfrak{S}_n$ and $\delta$ is an acyclic orientation on the graph $G(\sigma)$. We will first show $\tau(n) = h_{n,r}$ by using bijection.

Given a complete non-ambiguous forest $A$ with $n$ leaves, let $v \in A$ have coordinate $(x_1(v), x_2(v)) \in \mathbb{N} \times \mathbb{N}$, we first obtain the set $\mu(A) = \{(x_1(v) + 1, x_2(v) + 1) : v \in A\}$ via the bijection $A \mapsto \mu(A)$ given in Lemma 6. Recall that $T_{\mu(A)}$ is the underlying forest associated with the vertex set $\mu(A)$, we consider the coordinates of the leaves in the forest $T_{\mu(A)}$. From the definition of complete non-ambiguous forest, we can let the coordinates of the leaves of $T_{\mu(A)}$ be $(1, x_1), (2, x_2), \ldots, (n, x_n)$ where $x_1 \cdots x_n \in \mathfrak{S}_n$. We will give the correspondence $T_{\mu(A)} \mapsto ((\delta, \sigma_{T_{\mu(A)}}))$ where $\sigma_{T_{\mu(A)}} = x_1 x_2 \cdots x_n$ and $\delta$ is an acyclic orientation of $G(\sigma_{T_{\mu(A)}})$. Suppose the forest $T_{\mu(A)}$ contains exactly $k$ binary trees $T_{B_1}, \ldots, T_{B_k}$ that are associated to the vertex sets $B_1, \ldots, B_k$ respectively, we observe every binary tree $T_{B_i}$ in the forest $T_{\mu(A)}$ is a complete half non-ambiguous tree. Let $(\mu_{1,i}, \nu_{1,i}), \ldots, (\mu_{m_i,i}, \nu_{m_i,i})$ be the leaves in the tree $T_{B_i}$ for every $i$ where $\mu_{1,i} < \cdots < \mu_{m_i,i}$, then by applying the bijection $B \mapsto h_2(B) = ((\delta_{r,n,B}, G(\sigma_{T_{B_i}})))$ given in Lemma 6 on every complete half non-ambiguous tree $B_i$, we have the corresponding directed graphs $((\delta_{r,n,B_i}, G(\sigma_{T_{B_i}})))$ where $\sigma_{T_{B_i}}(\mu_{j,i}) = \nu_{j,i}$ for every $j$, $1 \leq j \leq m_i$ and $G(\sigma_{T_{B_i}})$ is connected with an acyclic orientation $\delta_{r,n,B_i}$, whose unique source is $(\mu_{1,i}, \nu_{1,i})$. Without loss of generality, we assume $\mu_{1,1} < \mu_{1,2} < \cdots < \mu_{1,k}$. We will define the orientation $\delta$ on the graph $G(\sigma_{T_{\mu(A)}})$ as follows. Recall that the vertex set of $G(\sigma_{T_{\mu(A)}})$ is the set $\{(1, x_1), \ldots, (n, x_n)\}$. For any two $\sigma_{T_{\mu(A)}}$-paths $(i, x_i), (j, x_j)$ that are crossing, if they are contained in the diagram representation of $\sigma_{T_{B_i}}$, then the orientation $\delta_{r,n,B_i}$ determines the orientation $\delta_{r,n,\mu(A)}$ on the edge $\{(i, x_i), (j, x_j)\}$. Otherwise if $(i, x_i)$ is a $\sigma_{T_{B_i}}$-path, $(j, x_j)$ is a $\sigma_{T_{B_i}}$-path and they are crossing, then we choose the directed edge $(i, x_i) \rightarrow (j, x_j)$ if $s < t$ and choose $(i, x_i) \leftarrow (j, x_j)$ if $s > t$. In this way the orientation $\delta$ is acyclic. Thus we have given the map $\mu(A) \mapsto T_{\mu(A)} \mapsto ((\delta, G(\sigma_{T_{\mu(A)}})))$ where $\sigma_{T_{\mu(A)}} = x_1 \cdots x_n \in \mathfrak{S}_n$ and $\delta$ is an acyclic orientation on the graph $G(\sigma_{T_{\mu(A)}})$.

We next prove the map $\mu(A) \mapsto ((\delta, G(\sigma_{T_{\mu(A)}})))$ is a bijection by showing its inverse map. For a given directed graph $((\delta, G(\sigma)))$ where $\sigma \in \mathfrak{S}_n$ and $\delta$ is an acyclic orientation of $G(\sigma)$, we consider the vertex-induced subgraphs of $G(\sigma)$. We say $c < d$ is there is a directed path from $c$ to $d$ in the directed graph $((\delta, G(\sigma)))$ and $c \leq c$ for every $c$ in the graph $G(\sigma)$. Let $\{(1, x_1), \ldots, (n, x_n)\}$ be the vertex set of $G(\sigma)$ and $x_i = \sigma(i)$ for every $i$. We denote by $V_G$ the vertex set of graph $G$. Then we choose the subgraph $G_1$ induced by the vertex set $V_{G_1} = \{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \leq (1, x_1), (x_1(v), x_2(v)) \in V_G(\sigma)\}$. Next from the set $\{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \in V_G(\sigma) - V_{G_1}\}$ we select the element $(\mu_2, \sigma(\mu_2))$ that $\mu_2 = \min\{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \in V_G(\sigma) - V_{G_1}\}$ and choose the
subgraph $G_2$ induced by the vertex set $V_{G_2} = \{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \leq (\mu_2, \sigma(\mu_2)), (x_1(v), x_2(v)) \in V_{G_1} \}$. Then from the set $\{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \in V_{G_1} \}$ we select the element $(\mu_3, \sigma(\mu_3))$ that $\mu_3 = \min\{x_1(v) : (x_1(v), x_2(v)) \in V_{G_1} \}$ and choose the subgraph $G_3$ induced by the vertex set $V_{G_3} = \{(x_1(v), x_2(v)) : (x_1(v), x_2(v)) \leq (\mu_3, \sigma(\mu_3)), (x_1(v), x_2(v)) \in V_{G_1} \}$. By continuing this process until no vertices are left, we decompose the graph $G(\sigma)$ into independent graphs $G_1, G_2, \ldots, G_k$ and $1 < \mu_2 < \cdots < \mu_k$. Let $\delta_i$ be the orientation $\delta$ restricted to the subgraph $G_i$, then we split the directed graph $(\delta, G(\sigma))$ into independent graphs $(\delta_1, G_1), (\delta_2, G_2), \ldots, (\delta_k, G_k)$ whose unique sources are $(1, x_1), (\mu_2, \sigma(\mu_2)), \ldots, (\mu_k, \sigma(\mu_k))$ respectively and $1 < \mu_2 < \cdots < \mu_k$. For any two crossing paths $(i, x_i), (j, x_j)$ that $(i, x_i) \in V_{G_i}$ and $(j, x_j) \in V_{G_j}$, under the orientation $\delta$ we have $(i, x_i) \rightarrow (j, x_j)$ if $s < t$ and $(i, x_i) \leftarrow (j, x_j)$ if $s > t$. By applying the inverse bijection $(\delta, n, B, G(\sigma_{TB})) \rightarrow TB$ given in Lemma 6 on the graphs $(\delta_1, G_1), (\delta_2, G_2), \ldots, (\delta_k, G_k)$, we retrieve a forest of binary trees $T_{B_1}, T_{B_2}, \ldots, T_{B_k}$ whose vertex set is a complete non-ambiguous forest. Therefore the map $\mu(A) \mapsto (\delta, G(\sigma_{\mu(A)}))$ is a bijection.

For instance, consider the complete non-ambiguous forest that contains vertices $(0, 1), (1, 2), (2, 0)$.

They are corresponding to the directed graphs

one by one from left to right.

The next step we will show $h_{n,r} = \omega(n)$ by constructing a bijection between the set of directed graphs $(\delta, G(\sigma))$ where $\sigma \in \mathfrak{S}_n$, $\delta$ is an acyclic orientation of $G(\sigma)$, and the set of pairs $(\pi, \xi)$ of permutations of $[n]$ with no common rise. For a given pair $(\pi, \xi)$ of permutations $\pi, \xi \in \mathfrak{S}_n$ without common rise, suppose $\pi = a_1 a_2 \cdots a_n$ and $\xi = b_1 b_2 \cdots b_n$, we choose a permutation $\sigma \in \mathfrak{S}_n$ satisfying $\sigma(a_i) = b_i$ for every $i$. It remains to define the acyclic orientation $\delta$ on the graph $G(\sigma)$. The vertex set of graph $G(\sigma)$ is $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$. For every two crossing $\sigma$-paths $(a_i, b_i), (a_j, b_j)$, we define the orientation $\delta$ on the edge $\{(a_i, b_i), (a_j, b_j)\}$ to be $(a_i, b_i) \leftarrow (a_j, b_j)$ if $i < j$. Otherwise if $i > j$, we define the orientation $(a_i, b_i) \rightarrow (a_j, b_j)$. In this manner, the orientation $\delta$ is acyclic. This can be proved by using contradiction. Assume $(a_i, b_i) \rightarrow (a_j, b_j) \rightarrow \cdots \rightarrow (a_k, b_k) \rightarrow (a_i, b_i)$ is a cycle in the graph $(\delta, G(\sigma))$, then according to the orientation $\delta$, we must have $i < j < k < \cdots < i$ which contradicts with the assumption. Thus we have given the map $(\pi, \xi) \mapsto (\delta, G(\sigma))$.

We next prove this map is a bijection by showing its inverse map. Given a graph $G(\sigma)$ with an acyclic orientation $\delta$, we need to give a linear order on the vertex set of graph $G(\sigma)$. For any two $\sigma$-paths $(a_i, b_i), (a_j, b_j)$, we define $(a_i, b_i) < (a_j, b_j)$ if $(a_i, b_i) \leftarrow (a_j, b_j)$ in the directed graph $(\delta, G(\sigma))$. If there is no edge $\{(a_i, b_i), (a_j, b_j)\}$ in the graph $G(\sigma)$ and there is no directed path from $(a_i, b_i)$ to $(a_j, b_j)$, or from $(a_j, b_j)$ to $(a_i, b_i)$, we define $(a_i, b_i) > (a_j, b_j)$ if $a_i < a_j$. According
to this, any two $\sigma$-paths \((a_i, b_i), (a_j, b_j)\) are comparable, which allows us to rewrite the elements in the set \(\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}\) by this linear order. Suppose the sequence \((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\) satisfies \((a_i, b_i) < (a_i, b_i) < \cdots < (a_i, b_i)\) and \(\delta = b_1 \cdots b\in S_n\). There is no common rise for the pair \((\pi, \xi)\). If there exists \(i\) such that \(a_i < a_{i+1}\) and \(b_i < b_{i+1}\), then there is no edge \(\{(a_i, b_i), (a_{i+1}, b_{i+1})\}\) in the graph \(G(\pi)\) and there is no directed path from \((a_i, b_i)\) to \((a_{i+1}, b_{i+1})\) or from \((a_{i+1}, b_{i+1})\) to \((a_{i}, b_{i})\). In this case the linear order defines \((a_i, b_i) > (a_{i+1}, b_{i+1})\), contradicting to the assumption that \((a_i, b_i) < (a_{i+1}, b_{i+1})\). Consequently the map \((\pi, \xi) \mapsto (\delta, G(\sigma))\) is a bijection.

For instance, the directed graphs

\[
\begin{align*}
(1, 2) & \quad (1, 2) & \quad (1, 2) & \quad (1, 2) \\
(3, 1) & \quad (3, 1) & \quad (3, 1) & \quad (3, 1) \\
(2, 3) & \quad (2, 3) & \quad (2, 3) & \quad (2, 3)
\end{align*}
\]

are corresponding to the pairs of permutations, \((132, 213), (321, 132), (231, 312), (213, 321)\) one by one from left to right.

\[
\Box
\]

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