THE VORTEX DYNAMICS OF A
GINZBURG-LANDAU SYSTEM UNDER PINNING
EFFECT *

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abstract

We study the vortex dynamical behaviour of a Ginzburg-Landau (G-L) system of related to inhomogeneous superconductors as well as to three-dimensional superconducting thin films having variable thickness. It is proved that the vortices are attracted by impurities or inhomogeneities in the superconducting materials. This rigorously verifies the fact predicted recently by a few authors using a method of formal asymptotics or approxiamate computation. Using this fact, furthermore, we prove the strong $H^1$-convergence of the solutions to the G-L system.

Keywords: Ginzburg-Landau system, vortex pinning, dynamics, elliptic estimate.

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1 Introduction and Main Results

Consider the solutions, \( V_\varepsilon = (V_\varepsilon^1, V_\varepsilon^2) : \Omega \rightarrow \mathbb{R}^2 \) for a smooth bounded domain \( \Omega \subset \mathbb{R}^N \) with \( N \geq 2 \), to the following initial-boundary value problem of Ginzburg-Landau system;

\[
\begin{cases}
\frac{\partial V_\varepsilon}{\partial t} = \Delta V_\varepsilon + \nabla \omega \nabla V_\varepsilon + A V_\varepsilon + \frac{B V_\varepsilon}{\varepsilon^2} (1 - |V_\varepsilon|^2) & \text{in } \Omega \times (0, \infty) \\
V_\varepsilon(x, t) = g(x) & \text{on } \partial \Omega \times (0, \infty) \\
V_\varepsilon(x, 0) = V_\varepsilon^0(x) & \text{in } \Omega,
\end{cases}
\]

where \( \omega, A \) and \( B \) are given smooth functions defined on \( \bar{\Omega} \). Equation (1.1) is a simple model which simulates inhomogeneous type II superconducting materials as well as three-dimensional superconducting thin films having variable thickness. In the inhomogeneous materials, the equilibrium density of superconducting electrons is not a constant, but a positive and smooth function on \( \bar{\Omega} \). Letting \( a(x) \) denote the density function and neglecting the magnetic field, one obtains that

\[
\frac{\partial u}{\partial t} - \Delta u = \frac{u}{\varepsilon^2} (a(x) - |u|^2). \tag{1.2}
\]

By setting \( V(x, t) = u(x, t)/\sqrt{a(x)} \), equation (1.2) reduces to the equation in (1.1) with

\[
\omega(x) = \ln a(x), \quad A(x) = \frac{\Delta \sqrt{a(x)}}{\sqrt{a(x)}} \text{ and } B(x) = a(x).
\]

We refer to [1], [2], [3] and the references therein for the detailed discussion of the motivation and physical background for equation (1.2) and its more general form involving magnetic field and electric field. There are several theoretical results on the static case of (1.2); see, for example, [3], [4], [5], [6] for the minimizers of the functional associated with the static case and [7] for the state-solutions.

Another model which also reduces to the equation in (1.1) is the following:

\[
\frac{\partial u}{\partial t} = \frac{1}{a(x)} \text{div}(a(x) \nabla u) + \frac{u}{\varepsilon^2} (1 - |u|^2). \tag{1.3}
\]
This equation is related to three-dimensional superconducting thin films having variable thickness. Let \( \Omega \times (-\delta a(x), \delta a(x)) \) be the domain occupied by these materials, where \( \Omega \subset R^2 \). Then this superconducting film was modeled as two-dimensional objects by equation (1.3) in [8], [9]. We refer the reader to [4], [8] and [9] for the study of the minimizers of a functional associated with the static case of (1.3). Obviously, (1.3) is the special form of the equation in (1.1) with

\[
\omega(x) = \ln a(x), \quad A \equiv 0 \text{ and } B \equiv 1.
\]

The connection between the steady solutions for (1.1) and the self-similar solutions for harmonic maps was studied in [10] recently.

Physically, the points at which a solution to problem (1.1) equals to zero are called vortices. In the case of \( N = 2 \), \( \omega(x) \equiv A(x) \equiv 0 \) and \( B(x) \equiv 1 \), the vortex dynamics was studied previously for the steady equations by Bethuel, Brezis and Hélein [11]. (For the minimum solution, see [12]). Furthermore, Lin [13], independently, Jerrard and Soner [14] and [15], studied the dynamical law for the vortices of \( u_\varepsilon(x,t) = V_\varepsilon(x,|\ln \varepsilon|t), \) where \( V_\varepsilon(x,t) \) solves the initial-boundary value problem (1.1) under the same case. Their dynamical law is described by an ODE, \( \frac{d}{dt} y(t) = -\nabla W(y(t)) \) where \( W \) is some known function related to the domain and the boundary condition and called as the renormalized energy functional associated with the steady problem [11], [13] or [15]. The results in [13] and in [15] were generalized to the Neumann boundary condition by Lin in [16].

However, there are few results for the vortex dynamics in the original time (not scaling by the time factor \( |\ln \varepsilon| \)), especially for equations (1.2) and (1.3), not to speak of for (1.1). Up to our limit knowledge, one can only locate one result for equation (1.3) by Lin in [16]. He proved that as \( \varepsilon \to 0 \), under some suitable assumptions on the initial and boundary data, the solutions, \( u_\varepsilon(x,t) \), of the Dirichlet initial-boundary valued problem for equation (1.3), subconverge in \( H^1_{loc}(\Omega \setminus \{y_1(t), y_2(t), \cdots, y_k(t)\}) \), where the functions \( y_j(t) : [0,T) \to \Omega \subset R^2 \) satisfy the following ODE:

\[
\begin{cases}
\frac{d}{dt} y_j(t) = -a^{-1}(y_j(t)))\nabla a(y_j(t)), & 0 \leq t \leq T, \\
y_j(0) = b_j
\end{cases}
\] (1.4)
Here $k, d_j, d$ are some constants related to the initial data, while $T$ is chosen so that $a_j(t)$ will stay inside $\Omega$ and $y_l(t) \neq y_j(t)$ for all $0 \leq t \leq T$ and for all $j \neq l, j, l = 1, 2, \ldots, k$. See Theorem 1.1 in [16] for the details. The first author in [17] proved that $T$, in fact, is $+\infty$ and each solution $y_j(t)$ of (1.4) converges to a critical point of $a(x)$ as $t \to \infty$ as long as $a(x)$ is analytic at its critical points and $b_j \in \Omega_j \subset \subset \Omega$ satisfies $\min_{x \in \partial \Omega_j} a(x) > a(b_j)$ for some domain $\Omega_j$.

In recent paper [18], the authors started studying the vortex dynamics of equation (1.2) with $N = 2$. They proved that the vortices are attracted by the local minimum points of $a(x)$ and the vortex dynamics is described by equation (1.4). Moreover, the authors conjectured that similar results should be true for equation (1.3).

In this paper, we will verify this conjecture. In fact, we will prove that all results for equation (1.2) in [18] are also true for problem (1.1) (see Theorem 1 and the first part of Theorem 2 below). In particular, we will prove that for most sufficient large $t$, under some suitable conditions, all the vortices of problem (1.1) are pinned together to the local minimum points of $\omega(x)$ in $\Omega$ as $\varepsilon \to 0$. This result was observed by Chapman and Richardson in [1] for equation (1.2) and Du and Gunzburger in [8] for equation (1.3). They used a matched asymptotic method or approximate computation method to predict that vortices for equation (1.2) or (1.3) (in fact, for a more complicated equation involving magnetic field and electric field), are attracted to the local minimum points of $a(x)$. Our results in this paper will show that their predictions are correct. See Remark 1 below.

As our second goal, we will study the strong $H^1$-convergence of solutions to problem (1.1). Although our strong convergence result, the second part of Theorem 2 below, can be viewed as a generalization of Theorem 1.1 in [16], our method to prove it is based on the vortex convergence, the first part of Theorem 2, and is completely different from the arguments in [16].

To state our main results, we need the following assumptions:

(A$_1$) $g : \partial \Omega \to R^2$ is smooth, $|g(x)| \equiv 1$ on $\partial \Omega$;
(A$_2$) $\omega \in C^{2,\alpha}(\bar{\Omega}), A, B \in C^{\alpha}(\Omega)$ $(\alpha > 0)$, $\omega(x) > 0$ and $B(x) > 0$
for all \( x \in \Omega; \)

\[(A_3) \quad \text{the initial data } V_\varepsilon^0 \in C^2(\Omega; R^2) \quad (\varepsilon > 0) \text{ satisfies } V_\varepsilon^0(x) = g(x) \]
on \( \partial \Omega \) and

\[
\|V_\varepsilon^0\|_{C(\Omega)} \leq K, \quad \int_{\Omega} \rho^2(x)[|\nabla V_\varepsilon^0|^2 + \frac{1}{2\varepsilon^2}(|V_\varepsilon^0|^2 - 1)^2]dx \leq K
\]

for a constant \( K \) (independent of \( \varepsilon \)) and some \( m \) distinct points \( b_1, b_2, \cdots, b_m \) in \( \Omega \), where \( \rho(x) = \min\{|x - b_j|, j = 1, 2, \cdots, m\} \).

\[(A_4) \quad \text{For each } j, \text{ there exists some Lipschitz domain } G_j \text{ such that } b_j \in G_j \subset \subset \Omega, \min_{x \in \partial G_j} \omega(x) > \omega(b_j), j = 1, \cdots, m. \]

As a start point, consider the ordinary differential system

\[
\begin{cases}
\frac{d}{dt} y_j(t) = -\nabla \omega(y_j(t)), & 0 \leq t < \infty, \\
y_j(0) = b_j
\end{cases}
\]

(1.5)

where \( j = 1, 2, \cdots, m \), and \( \nabla \omega \) is the gradient of the function \( \omega \) with respect to \( x = (x_1, \cdots, x_N) \in R^N \). It is this system that will play an important rule in the course of the proof of our main results. As preliminary, we will generalize the results for (1.4) in [17] to system (1.5) in next section under an extra condition

\[(A_5) \quad \text{either } \omega \text{ has only nondegenerate critical points in } \Omega \text{ or it is analytic in some neighborhood of its critical points.} \]

Particularly, the existence and uniqueness of global solutions to (1.5) will be guaranteed by conditions \((A_2)\) and \((A_4)\)(see Lemmas 4 and 5 below).

As the main results, we will first prove the following compactness for the solutions to problem (1.1) for all \( N \geq 2 \) which is the generalization of Theorem 1.1 in [16] and Theorem 1.2 in [18] for \( N = 2 \).

**Theorem 1** Suppose that the hypotheses \((A_1), (A_2), (A_3)\) and \((A_4)\) are satisfied. Let \( y_j(t) \) be solutions to problem (1.5) \( (1 \leq j \leq m) \) and set

\[
\Omega(\omega) = \bar{\Omega} \times (0, \infty) \setminus \cup_{j=1}^m \{(x, t) : x = y_j(t), 0 < t < \infty\}. 
\]

Then there is a positive constant \( \varepsilon_0 \) (depending only on the infimum of \( B \) and the superemum of \(|A|\)) such that the set \( \{V_\varepsilon : \varepsilon \in (0, \varepsilon_0)\} \) of the classical solutions to problem (1.1) is bounded in \( H^1_{\text{loc}}(\Omega(\omega)) \). Moreover, given any sequence \( \varepsilon_n \downarrow 0 \), there exists a subsequence (still denoted by itself) such that \( V_{\varepsilon_n} \rightarrow V \)
weakly in $H^1_{loc}(\Omega(\omega))$ with $V$ satisfying $|V(x,t)| = 1$ a.e. in $\Omega(\omega)$, $V = g$ on $\partial\Omega \times (0,\infty)$, and the equation

$$\frac{\partial V}{\partial t} = \Delta V + V|\nabla V|^2 + \nabla \omega \nabla V \text{ in } D'(\Omega(\omega)).$$

We will use this compactness result, covering arguments and elliptic estimates to prove the following vortex convergence.

**Theorem 2** Suppose that $N = 2$ and all the hypotheses in Theorem 1 under this case are fulfilled. Let $V_\varepsilon$ be a classical solution to problem (1.1) for each $\varepsilon > 0$. Then for any $\delta \in (0,\frac{1}{4})$ and any interval $I \subset (0,\infty)$ with $|I| > 0$, one can find $t \in I$ and $\varepsilon_1 > 0$ such that the following two conclusions hold true for all $\varepsilon \in (0,\varepsilon_1)$:

(i)

$$\{ x \in \bar{\Omega} : |V_\varepsilon(x,t)| \leq \frac{1}{2} \} \subset \bigcup_{j=1}^{m} B_\delta(y_j(t))$$

and

$$||V_\varepsilon(x,t)| - 1| \leq C(\delta,t)\varepsilon^{\frac{1}{4}}, \forall x \in \bar{\Omega} \setminus \bigcup_{j=1}^{m} B_\delta(y_j(t)) := \Omega(\omega_\delta^\varepsilon);$$

(ii) if, furthermore, $A(x) \leq B(x)$ for all $x \in \bar{\Omega}$, then

$$|||V_\varepsilon(\cdot,t)| - 1||_{H^1(\Omega(\omega_\delta^\varepsilon))} + \varepsilon^{-1}|||V_\varepsilon(\cdot,t)| - 1||_{L^2(\Omega(\omega_\delta^\varepsilon))} \leq C(\delta,t)\varepsilon^{\frac{1}{4}}$$

and the convergence, $V_\varepsilon(\cdot,t)_{\varepsilon_n} \rightarrow V(\cdot,t)$, in Theorem 1, is strong convergence in $H^1(\Omega(\omega_\delta^\varepsilon))$ as $n \rightarrow \infty$.

**Corollary 3** Besides the assumptions in Theorem ??, we further assume that for some $t$, the set

$$\{ \partial_t V_\varepsilon(x,t), \varepsilon \in (0,\varepsilon_0) \}$$

is a bounded set in $L^2_{loc}(\bar{\Omega}\setminus\{y_1(t), \cdots, y_m(t)\})$. Then for any $\delta \in (0,\frac{1}{4})$, there is a constant $\varepsilon_1 > 0$ such that conclusions (i) and (ii) of Theorem 2 hold true for all $\varepsilon \in (0,\varepsilon_1)$.

The organization of this paper is as follows. As we have mentioned before, we will follow the arguments in [17] to study the system (1.5) and generalize the main result in [17] for our use later in next section. Section three will be
devoted to the proof of Theorem 1. Finally in section four, we will finish the proof of Theorem 2 and its corollary 3.

Before we are going to the detail proof of our results, we would like to make the following remarks.

**Remark 1** It is worth to point out that Theorem 2 (Corollary 3), together with Lemmas 4 and 5 below, imply that for most large $t$, all the vortices of $V_\varepsilon(x,t)$ for (1.1), as $\varepsilon \to 0$, move toward and eventually pin together at the critical points of $\omega(x)$ in $\Omega$ if $(A_1) - (A_5)$ are satisfied. In particular, if each $G_j$ contains no other critical points of $\omega(x)$ than its local minimum points, then all the vortices of $V_\varepsilon(x,t)$ are pinned to the minimum points.

**Remark 2** Obviously, the unique solution of (1.5) is $y_j(t) \equiv p_j$ ($j = 1, 2, \cdots, m$) for all $t \in [0, +\infty)$ if $\nabla \omega(p_j) = 0$. This shows that if $\omega \equiv$ constant, the vortices for (1.1) do not move in any finite time interval. In fact, in this case, formal analyses indicate that, if initial $V_0 \varepsilon$ has isolated vortices, then these vortices move with velocities of the order of $|\ln \varepsilon|^{-1}$. This prediction was proved rigorously in [13] and [15].

**Convention:** Throughout this paper, we use the letter $C$ to denote various constants independent of $\varepsilon$ but maybe depending on $\Omega$, $\omega(x)$, $A(x)$, $B(x)$, $g$, $K$ and other known constants.

## 2 Preliminaries

Observing that ordinary differential system (1.5) is nothing but (1.4) with $\ln a(x)$ replaced by $\omega(x)$, we can apply the result and method for (1.4) in [17] to (1.5).

First, following the arguments from (2.1) to (2.3) in [17], we have

**Lemma 4** Suppose $\omega$ satisfies $(A_2)$ and $(A_4)$. Then (1.5) has a unique $C^3$ solution $(y_1, y_2, \cdots, y_m) : [0, \infty) \to R^{mN}$. Furthermore, if $b_j \neq b_l$ for all $j \neq l$, then

a. for all $t \in [0, \infty)$ and $1 \leq j \neq l \leq m$, $y_j(t) \neq y_l(t)$;

b. for all $t \in [0, \infty)$ and $1 \leq j \leq m$, $y_j(t) \in G_j$. 

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Lemma 5 Assume that \( \omega \) satisfies \((A_2)\) and \((A_4)\) as well as \((A_5)\). Then for each \( j = 1, 2, \ldots, m \), there exists \( b_j \in \bar{G}_j \) which are critical point of \( \omega \) such that \( y_j(t) \to b_j \) as \( t \to +\infty \).

Proof. Repeating the proof of Theorem 1.2 in [17] with \( lna(x) \) replaced by \( \omega(x) \), one easily obtains the desired results. Here we would like to point out two facts in order for the reader easy to follow the arguments.

1. To prove \( y(t) \to b \) as \( t \to \infty \) under the conditions that \( \nabla \omega(b) = 0 \) and \( \omega(y(t)) \downarrow \omega(b) \) for all \( t \in (0, \infty) \). Otherwise, it is easy to find a \( t_0 > 0 \) such that \( \omega(y(t)) = \omega(b) \) for all \( t \in (t_0, \infty) \). This yields
\[
0 = \nabla \omega(y(t)) \cdot y'(t) = -|y'(t)|^2
\]
by (1.5). Hence \( y(t) = b \) for all \( t \in (t_0, \infty) \).

2. The hypothesis that \( \omega \) has only nondegenerate critical points \( b \) implies that
\[
|\nabla \omega(x)| \geq \theta_1 |\omega(x) - \omega(b)|^{\frac{1}{2}} \quad \forall x \in B_{\theta_2}(b)
\]
for some positive constants \( \theta_1 \) and \( \theta_2 \). In fact, Since \( \nabla \omega(b) = 0 \) and \( \text{det}(\nabla^2 \omega(b)) \neq 0 \), we have
\[
|\nabla \omega(x)| = |\nabla^2 \omega(b) \cdot (x - b)| + o(|x - b|)
\]
\[
= \{|(x - b)^{\top} \cdot (\nabla^2 \omega(b))^\top \nabla^2 \omega(b) \cdot (x - b)|^\frac{1}{2} + o(|x - b|)
\]
\[
\geq \lambda_0 |x - b| + o(|x - b|)
\]
and
\[
|\omega(x) - \omega(b)| = |(x - b)^{\top} \cdot \nabla^2 \omega(b) \cdot (x - b)| + o(|x - b|^2)
\]
\[
\leq \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} |x - b|^2 + o(|x - b|^2),
\]
where \( \lambda_0 \) is the minimum eigenvalue of \( (\nabla^2 \omega(b))^\top \nabla^2 \omega(b) \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( \nabla^2 \omega(b) \). Thus the hypothesis \( \text{det}(\nabla^2 \omega(b)) \neq 0 \) guarantees that (2.1) is true.

Corollary 6 In Lemma 5, if \( \Omega \) is replaced by \( \mathbb{R}^N \) and \((A_4)\) replaced by the following assumption:
\[
|x| \leq \Psi(\omega(x))
\]
(2.2)
and
\[ |\nabla \omega(x)| \leq \phi(|x|), \tag{2.3} \]
where \( \Psi : \mathbb{R} \rightarrow \mathbb{R}^+ \) is an increasing function and \( \phi \) is a positive function satisfying
\[ \int_\alpha^{+\infty} \phi^{-1}(t)dt = +\infty \tag{2.4} \]
for some \( \alpha > 0 \), then all the conclusion of Lemma 5 is also true.

**Proof.** Using (2.3) and (2.4) and applying Wintner’s theorem (Theorem 2.5 of Chapter 1 in [19]) to system (1.5), one can conclude that (1.5) has a unique solution in \([0, +\infty)\). Moreover, by (2.2), one can see that the solution is bounded in \( \mathbb{R}^N \) uniformly for \( t \in [0, \infty) \). The remaining part of the proof is the same as in [17].

### 3 Proof of Theorem 1

Throughtout this section, we assume \((A_1), (A_2), (A_3)\) and \((A_4)\), although some conclusions below need only parts of these assumptions.

**Lemma 8.** Let \( V_\varepsilon \) be classical solutions to problem (1.1). Then there exists an \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \)
\[ |V_\varepsilon(x,t)|^2 \leq C, \quad \forall (x,t) \in \bar{\Omega} \times [0, \infty) \tag{3.1} \]
and
\[ |\nabla V_\varepsilon(x,t)|^2 + \left| \frac{\partial V_\varepsilon}{\partial t} \right| \leq \frac{C}{\varepsilon^2}, \quad \forall (x,t) \in \bar{\Omega} \times [\varepsilon^2, \infty). \tag{3.2} \]

**Proof.** Denote
\[ \varepsilon_0 = \left[ \frac{\inf_{x \in \Omega} B(x)}{1 + \sup_{x \in \Omega} |A(x)|} \right]^\frac{1}{2}. \tag{3.3} \]
Let \( W = |V_\varepsilon|^2 \). Dropping the subscript \( \varepsilon \), we see that the equation in (1.1) reads as
\[ \partial_t W = \Delta W + \nabla \omega \nabla W - 2|\nabla V|^2 + \frac{2BW}{\varepsilon^2} (1 - W) + 2AW. \tag{3.4} \]

If (3.1) were not true for some \( \varepsilon \in (0, \varepsilon_0) \), we could use \((A_1)\) and \((A_3)\), and employ the usual arguments for maximum principle to find a point \((x_\varepsilon, t_\varepsilon) \in \]
\[ \Omega \times (0, \infty) \] (for each \( \varepsilon \)) at which

\[ W > 2, \ \nabla W = 0, \ \partial_t W \geq 0 \text{ and } \Delta W \leq 0. \quad (3.5) \]

Moreover, (3.4) gives us \( \partial_t W \leq 2(A - B)W \) at \((x_\varepsilon, t_\varepsilon)\). This yields a contradiction as \( \varepsilon \in (0, \varepsilon_0) \).

By a scaling argument, considering the equation for \( U_\varepsilon(x, t) = V_\varepsilon(\varepsilon x, \varepsilon^2 t) \) and using (1.1) and standard local parabolic estimates, we immediately obtain (3.2).

**Corollary 8** Let \( V_\varepsilon \) be classical solutions to problem (1.1). If \( A(x) \leq B(x) \) for all \( x \in \Omega \), then \( |V_\varepsilon(x, t)|^2 \leq 1 + \varepsilon^2 \) for all \((x, t) \in \Omega \times [0, \infty) \) and for all \( \varepsilon > 0 \).

**Proof.** It is obvious from the proof of Lemma 7, observing that \( W > 1 + \varepsilon^2 \) and so \( \partial_t W \leq 2(A - B)W \) at \((x_\varepsilon, t_\varepsilon)\).

Set

\[ E_\varepsilon(V(x, t)) = \frac{e^{\omega(x)}}{2} \left[ |\nabla V(x, t)|^2 + \frac{B(x)}{2\varepsilon^2} (1 - |V(x, t)|^2)^2 \right]. \quad (3.6) \]

**Lemma 9** Let \( V_\varepsilon \) be solutions to (1.1). Then for any \( T > 0 \), there exist two positive constants \( C(T) \) and \( \sigma(T) \) (both depending on \( T \)) such that for all \( \varepsilon \in (0, \varepsilon_0) \), all \( \delta \in (0, \sigma(T)) \) and all \( t \in [0, T] \), one has

\[ B_\delta(y_l(t)) \subset \Omega, \ B_\delta(y_l(t)) \cap B_\delta(y_j(t)) = \emptyset \quad \text{for } l \neq j \]

and

\[ \int_0^T \int_{\Omega \setminus \bigcup_{j=1}^m B_\delta(y_j(t))} |\frac{\partial V_\varepsilon}{\partial t}|^2 dxdt + \sup_{0 \leq t \leq T} \int_{\Omega \setminus \bigcup_{j=1}^m B_\delta(y_j(t))} E_\varepsilon(V_\varepsilon) dx \leq \delta^{-2} C(T). \]

**Proof.** For each \( T > 0 \), by Lemma 4 we can find a \( \sigma = \sigma(T) > 0 \) such that

\[ \sigma \leq \frac{2}{1 + \sup_{x \in \Omega} |\nabla \ln B|} \quad (3.7) \]

and for all \( t \in [0, T] \),

\[ \min_{1 \leq i,j \leq m} \{ \text{dist}(y_j(t), \partial \Omega), \ |y_j(t) - y_i(t)| \} \text{ for } l \neq j \geq 4\sigma. \quad (3.8) \]
Motivated by a method in [15], we choose a smooth monotone function \( \phi : [0, \infty) \rightarrow [0, \infty) \) such that

\[
\phi(r) = \begin{cases} 
  r^2, & \text{if } r \leq \sigma \\
  \sigma^2, & \text{if } r \geq 2\sigma.
\end{cases}
\]  

(3.9)

Let

\[
\rho(x, t) = \min_{1 \leq j \leq m} |x - y_j(t)|.
\]

It follows easily from (3.8) that \( \phi(\rho(x, t)) \) is smooth in \( x \) as well as in \( t \) for all \( (x, t) \in \bar{\Omega} \times [0, T] \). Dropping the subscript \( \varepsilon \), applying integration by parts, noting \( \partial_t V = \partial_t g = 0 \) on \( \partial \Omega \times (0, \infty) \) and using (3.6) and the equation in (1.1), we obtain

\[
\frac{d}{dt} \int_{\Omega} \phi(\rho(x, t)) E(V) dx \leq \int_{\Omega} \frac{d\phi(\rho)}{dt} E(V) + \int_{\Omega} \phi(\rho) e^\omega [\nabla V \nabla V_i - \varepsilon^{-2} BV(1 - |V|^2) V_i]
\]

\[
= \int_{\Omega} \frac{d\phi(\rho)}{dt} E(V) - \int_{\Omega} V_i \nabla V \nabla \phi(\rho) e^\omega - \int_{\Omega} \phi(\rho) [\nabla (e^\omega \nabla V) + \varepsilon^{-2} Be^\omega V(1 - |V|^2)] V_i
\]

\[
= \int_{\Omega} \frac{d\phi(\rho)}{dt} E(V) - \int_{\Omega} V_i \nabla V \nabla \phi(\rho) + \int_{\Omega} e^\omega V_i AV \phi(\rho) - \int_{\Omega} e^\omega V_i \nabla V \phi(\rho)
\]

\[
\leq \int_{\Omega} \frac{d\phi(\rho)}{dt} E(V) - \frac{1}{2} \int_{\Omega} \phi(\rho) e^\omega |V_i|^2 + \frac{1}{2} \int_{\Omega} \phi(\rho) e^\omega |AV|^2 - \int_{\Omega} e^\omega V_i \nabla V \phi(\rho)
\]

\[
\leq C - \frac{1}{2} \int_{\Omega} \phi(\rho) e^\omega |V_i|^2 + \int_{\Omega} \frac{d\phi(\rho)}{dt} E(V) - e^\omega V_i \nabla V \phi(\rho) dx,
\]  

(3.10)

where we have used (3.1) and (3.9) and \( C \) depends only on \( K, \sigma, ||A||_{C(\Omega)} \) and \( ||\omega||_{C(\Omega)} \).

One can use the notation \( U_i = \frac{\partial U}{\partial x_i} \) and the summation convention to compute

\[
e^\omega V_i \nabla V \nabla \phi(\rho) = e^\omega \nabla V \nabla \phi[\Delta V + \nabla \omega \nabla V + AV + \varepsilon^{-2} BV(1 - |V|^2)]
\]

\[
= e^\omega \phi_i \{-(4\varepsilon^2)^{-1}[B(1 - |V|^2)]_i + (4\varepsilon^2)^{-1}(1 - |V|^2)^2 B_i - 2^{-1}(|\nabla V|^2) + [V_i V_j] + V_i \omega_j V_j + AV V_i \}.
\]

By virtue of this equality, integration by parts and the fact of \( \nabla^k \phi(\rho) = 0 \) on \( \partial \Omega \) (see (3.8) and (3.9)), one gets that

\[
\int_{\Omega} e^\omega V_i \nabla V \nabla \phi(\rho) dx
\]
\[ \int_{\Omega} [\Delta \phi E(V) + (4\varepsilon^2)^{-1} e^{\omega} (1 - |V|^2)^2 \nabla B \nabla \phi + \nabla \phi \nabla \omega E(V) - V_i V_j (e^{\omega} \phi_i)_j + e^{\omega} (V_i V_j \omega + A V_i \phi)] \, dx. \]  

(3.11)

Combing (3.10) and (3.11) yields

\[
\frac{d}{dt} \int_{\Omega} \phi(\rho(x,t)) E(V) \, dx \leq C - \frac{1}{2} \int_{\Omega} \phi(\rho)e^{\omega}|V_i|^2 + \int_{\Omega} \{E(V)[(\phi(\rho)_t - \nabla \omega \nabla \phi] + I_1(V)\} \, dx + I_2(V),
\]

(3.12)

where

\[
I_1(V) = e^{\omega} V_i V_j \phi_{ij} - \Delta \phi E(V) - (4\varepsilon^2)^{-1} e^{\omega} (1 - |V|^2)^2 \nabla B \nabla \phi
\]

(3.13)

and

\[
I_2(V) = - \int_{\Omega} e^{\omega} A V \nabla V \nabla \phi \\
\leq \int_{\Omega} |A V|^2 \frac{\nabla \phi^2}{\phi} + \phi |\nabla V|^2 e^{\omega} \\
\leq C[1 + \int_{\Omega} \phi |\nabla V|^2].
\]

(3.14)

Here we have used (3.1) and (3.9).

If \( \rho(x,t) \geq \sigma \), by (3.9) one has

\[
E(V)|\phi(\rho)_t - \nabla \omega \nabla \phi| + |I_1(V)| \leq C \phi(\rho) E(V).
\]

(3.15)

If \( \rho(x,t) \leq \sigma \), on the other hand, then \( \phi(\rho(x,t)) = |x - y_l(t)|^2 \) for some \( l \). Hence \( \phi_{ij} = \delta_{ij} \) and

\[
I_1(V) = e^{\omega} (1 - \frac{N}{2})|\nabla V|^2 - \frac{e^{\omega} B(1 - |V|^2)^2}{4\varepsilon^2} [N + 2(x - y_l(t)) \cdot \nabla \ln B] \leq 0
\]

(3.16)

by (3.7). Moreover, using (1.5) we have

\[
\phi(\rho)_t - \nabla \omega \nabla \phi = 2(x - y_l(t))(\nabla \omega(y_l(t)) - \nabla \omega(x)) \\
\leq 2|x - y_l(t)|^2 \|\omega\|_{C^2(\Omega)} \\
= 2\|\omega\|_{C^2(\Omega)} \phi(\rho(x,t)).
\]

(3.17)
Combing (3.12)-(3.17), we obtain
\[
\frac{d}{dt} \int_{\Omega} \phi(\rho(x,t)) E_{\epsilon}(V_\epsilon) \, dx + \frac{1}{2} \int_{\Omega} \phi(\rho(x,t)) e^\omega |V_\epsilon|^2 \, dx \\
\leq C [1 + \int_{\Omega} \phi(\rho(x,t)) E_{\epsilon}(V_\epsilon) \, dx]
\]
for all \( t \in [0, T] \). Hence, by Gronwall’s inequality and \((A_3)\), we deduce that
\[
\int_{\Omega} \phi(\rho(x,t)) E_{\epsilon}(V_\epsilon) \, dx + \frac{1}{2} \int_0^t \int_{\Omega} E_{c(t-s)} \phi(\rho(x,t)) e^\omega |\partial_s V_\epsilon|^2 \, dxds \leq C.
\]
This result, together with the fact that \( \phi(\rho(x,t)) \geq \delta^2 \) for all \( t \in [0, T], \) all \( x \in (\bar{\Omega} \setminus \cup_{j=1}^m B_\delta(y_j(t))) \) and any \( \delta \in (0, \sigma(T)) \), immediately implies the conclusion of Lemma 9.

**Proof of Theorem 1:** Recall the \( G_j \) in \((A_4)\) and let
\[
\delta_0 = \min_{1 \leq j \leq m} \text{dist}(G_j, \partial \Omega).
\]
Then by Lemma 4, we have
\[
\min_{1 \leq j \leq m} \text{dist}(y_j(t), \partial \Omega) \geq \delta_0, \forall t \in [0, \infty).
\]
Set
\[
\Omega_r(t) = \Omega \setminus \cup_{j=1}^m B_r(y_j(t)).
\]
For any \( T > 0 \) and any \( \delta \in (0, \delta_0) \), it follows from Lemma 9 and \((A_2)\) that
\[
\int_0^T \int_{\Omega_{\delta/4}(t)} |\partial_t V_\epsilon|^2 \, dxdt + \sup_{0 \leq s \leq T} \int_{\Omega_{\delta/4}(t)} |\nabla V_\epsilon|^2 + \frac{1}{\epsilon^2} (1 - |V_\epsilon|^2)^2 \, dx \leq C(\delta, T)
\]
for all \( \epsilon > 0 \). This shows that the set \( \{ V_\epsilon : \epsilon > 0 \} \) is bounded in \( H^1_{\text{loc}}(\Omega(\omega)) \).

Using (3.21) and applying a diagonal method for \( \delta \downarrow 0 \) and \( T \uparrow \infty \), we see that, for any sequence \( \epsilon_n \to 0 \), there is a subsequence \( V_{\epsilon_n} \) (denoted still by itself) such that \( V_{\epsilon_n} \rightharpoonup V \) weakly in \( H^1_{\text{loc}}(\Omega(\omega)) \). Moreover, (3.21), (3.2) and Lebesgue’s dominated convergence theorem imply that
\[
|V(x,t)| = 1 \quad \text{a.e. in} \quad \Omega(\omega).
\]
By taking the wedge product of \( V_{\epsilon_n} \) with the equation in (1.1), we have
\[
V_{\epsilon_n} \wedge \frac{\partial V_{\epsilon_n}}{\partial t} = \text{div}(V_{\epsilon_n} \wedge \nabla V_{\epsilon_n}) + V_{\epsilon_n} \wedge (\nabla \omega \nabla V_{\epsilon_n}).
\]
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Passing to the limit, we conclude that
\[ V \land \frac{\partial V}{\partial t} = \text{div}(V \land \nabla V) + V \land (\nabla \omega \nabla V) \quad \text{in} \quad D'(\Omega(\omega)). \]

But (3.22) yields
\[ V \frac{\partial V}{\partial t} = 0, \quad \text{div}(V \nabla V) = 0 \quad \text{in} \quad D'(\Omega(\omega)). \]

Combing the last three equations with (3.22), one gets
\[
\begin{cases}
V_t = \Delta V + V|\nabla V|^2 + \nabla \omega \nabla V & \text{in} \quad D'(\Omega(\omega)) \\
|V| = 1 & \text{in} \quad \Omega(\omega) \\
u = g & \text{on} \quad \partial \Omega.
\end{cases}
\]

This completes the proof of Theorem 1.

4 Proof of Theorem 2

We assume \( N = 2 \) as well as \((A_1), (A_2), (A_3)\) and \((A_4)\) throughout this section.

Let \( V_\varepsilon \) be classical solution to problem (1.1). Then all conclusions in the last section hold true. We will use covering arguments and ellitic estimates to prove Theorem 2 by the coming lemmas.

**Lemma 10** For any \( T_0 > 0 \), there exist constants \( C(T_0) > 0 \) and \( \sigma_1 = \sigma_1(T_0), \sigma_1 \in (0, \sigma(T_0)/4) \) with the same \( \sigma(T_0) \) as in Lemma 9 such that for all \( t \in [0, T_0] \), for all \( x_0 \in \Omega \setminus \bigcup_{j=1}^m B_{\sigma_1}(y_j(t)) \), for all \( r \in (0, \sigma_1) \) and for all \( \varepsilon \in (0, \varepsilon_0) \), one has
\[
\int_{\partial \Omega_r(x_0)} |x - x_0| |\frac{\partial V_\varepsilon}{\partial \nu}|^2 ds + \int_{\Omega_r(x_0)} \frac{1}{\varepsilon^2} (1 - |V_\varepsilon|^2)^2 dx
\leq C(T_0) \left\{ \int_{\partial \Omega_r(x_0)} |x - x_0| \left( |\frac{\partial V_\varepsilon}{\partial \nu}|^2 + \frac{1}{4 \varepsilon^2} (1 - |V_\varepsilon|^2)^2 \right) ds \\
+ \int_{\Omega_r(x_0)} |x - x_0| \left( 1 + |\nabla V_\varepsilon|^2 + |\nabla V_\varepsilon| |\partial_\nu V_\varepsilon| \right) dx \right\}, \quad (4.1)
\]
where \( \Omega_r(x_0) = B_r(x_0) \cap \Omega, \nu \) and \( T \) are, respectively, the exterior unit normal vector and tangent vector of \( \partial \Omega_r \) such that \((\nu, T)\) is direct.

**Proof.** By Lemma 9 and the fact that \( \Omega \) is smooth, we can find constant \( \alpha = \alpha(T_0), \sigma_1 = \sigma_1(T_0) \in (0, \sigma(T_0)/4) \) such that for all \( t \in [0, T_0] \), all \( x_0 = \ldots \)
\(x_0(t) \in \Omega \setminus \bigcup_{j=1}^{m} B_{\sigma_1}(y_j(t))\) and all \(r \in (0, \sigma_1),\)

\[(x-x_0) \cdot \nu \geq \alpha |x-x_0|, \forall \in \partial \Omega_r(x_0). \quad (4.2)\]

Multiplying the equation in (1.1) by \(\nabla V \cdot (x-x_0)\) and integrate it over \(\Omega_r = \Omega_r(x_0)\). Neglecting the subscript \(\varepsilon\), we obtain that

\[
\frac{1}{4\varepsilon^2} \int_{\Omega_r} (1-|V|^2)^2 \text{div}(B(x) \cdot (x-x_0)) \nonumber \\
= \frac{1}{4\varepsilon^2} \int_{\partial \Omega_r} (1-|V|^2) B(x) \nu \cdot (x-x_0) \\
+ I_3 + I_4 + I_5, 
\]

where

\[
I_3 = \int_{\partial \Omega_r} \frac{\partial (-V)}{\partial \nu} \cdot (\nabla V \cdot (x-x_0)), \\
I_4 = \int_{\Omega_r} \nabla V \cdot (\nabla V(x-x_0)), \\
\]

and

\[
I_5 = \int_{\Omega_r} \partial_t V \cdot (\nabla V \cdot (x-x_0)) - (\nabla \omega \nabla V + AV)(\nabla V \cdot (x-x_0)) \\
\leq \int_{\Omega} C(|V|^2 + |\nabla V|^2 + |\partial_t V| |\nabla V|)|x-x_0|dx. 
\]

By virtue of \((A_2)\) and the smallness of \(r\) we may assume

\[
div(B \cdot (x-x_0)) \geq \lambda > 0 \quad (4.4)\]

for all \(x \in \Omega_r\) and some constant \(\lambda\) depending only on \(A\). On the other hand, the integrand in \(I_3\) can be written as

\[
- \frac{\partial V}{\partial \nu} \left[ (\frac{\partial V}{\partial \nu} \nu + \frac{\partial V}{\partial T}) (x-x_0) \right] \leq \frac{-4}{5} \frac{\partial V}{\partial \nu}^2 \nu \cdot (x-x_0) + C \frac{\partial V}{\partial T}^2 |x-x_0|. \quad (4.5)\]

The integrand in \(I_4\) is nothing but \(\text{div}(\frac{1}{2} |\nabla V|^2 (x-x_0))\). Hence, we have

\[
I_4 = \int_{\partial \Omega_r} \left( \frac{1}{2} |\nabla V|^2 \nu \cdot (x-x_0) \right) ds \quad (4.6)\]

Combing (4.2)-(4.6) and using (3.1), we have deduced the desired (4.1).

**Lemma 11** For any interval \(I \subset (0, \infty)\) with \(|I| > 0\) and any \(\delta \in (0, \frac{1}{4})\), there exist \(t \in I\) and \(\eta_0 > 0\) such that

\[
\int_{\Omega_{\frac{1}{4}\eta_0}(t)} |\frac{\partial V_\varepsilon}{\partial t}|^2 dx + \int_{\Omega_{\frac{1}{4}\eta_0}(t)} |\nabla V_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1-|V_\varepsilon|^2)^2 dx \leq C(t, \delta) < +\infty \quad (4.7)\]
for all $\varepsilon \in (0, \eta_0)$. Moreover, there exist a $\varepsilon_1 \in (0, \eta_0)$ such that

$$\{ x \in \Omega : |V_\varepsilon(x, t)| < \frac{1}{2} \} \subset \bigcup_{j=1}^m B_{\delta/4}(y_j(t)), \forall \varepsilon \in (0, \varepsilon_1).$$

(4.8)

**Proof.** The proof is almost the same as the one of Lemma 4.2 in [18]. Here we copy it just for the convenience.

First, Using (3.21) and standard methods in real analysis, we easily get (4.7). Thus, what we need do is only to prove (4.8) for sufficiently small $\delta > 0$ and the same $t$ as in (4.7). If the conclusion were not true, we could find $\delta_1 \in (0, \sigma(t))$ (with the same $\sigma(t)$ as in Lemma 9), a sequence $\varepsilon_k \searrow 0$, $\varepsilon_k \in (0, \delta_1)$, and $\{ x_k \} \subset \Omega \setminus \bigcup_{j=1}^m B_{\delta/4}(y_j(t))$ such that $|V_{\varepsilon_k}(x_k, t)| < \frac{1}{2}$ for all $k$. Hence, by virtue of (3.2) and the fact $|V_\varepsilon| = |g| = 1$ on $\partial \Omega$, we see that there is a ball $B_{C_1 \varepsilon_k}(x_k) \subset \Omega \setminus \bigcup_{j=1}^m B_{\delta/4}(y_j(t))$ for some constant $C_1 > 0$ with $|V_{\varepsilon_k}(x, t)| \leq \frac{3}{4}$ for all $x \in B_{C_1 \varepsilon_k}(x_k)$ and all sufficiently large $k \geq k_0$. Let $r_k = C_1 \varepsilon_k$, $B_k = B_{r_k}(x_k)$ and $V_k = V_{\varepsilon_k}$. Since $N = 2$, one has that

$$\varepsilon_k^{-2} \int_{B_k} (1 - |V_k(x, t)|^2)^2 dx \geq C_2 > 0$$

(4.9)

for all $k \geq k_0$ and some positive constant $C_2$ depending only on $C_1$.

On the other hand, as $r_k \to 0$, (4.7) implies that

$$\int_{\Omega \setminus \bigcap_{k=1}^\infty \{ x_k \}} \left| \frac{\partial V_k}{\partial t} \right|^2 dx + \int_{\Omega \setminus \bigcap_{k=1}^\infty \{ x_k \}} \left[ \left| \nabla V_k \right|^2 + \varepsilon_k^{-2} (1 - |V_k|^2)^2 \right] dx \leq C(t, \delta)$$

(4.10)

for all $k \geq k_0$. Thus, letting

$$f_k(r) = \int_{\partial B_r(x_k) \cap \Omega} \left[ \left| \nabla V_k \right|^2 + \varepsilon_k^{-2} (1 - |V_k|^2)^2 \right] ds$$

we have

$$C(t, \delta) \geq \int_{B_{C_1 \varepsilon_k}(x_k) \setminus B_r(x_k)} \left[ \left| \nabla V_k \right|^2 + \varepsilon_k^{-2} (1 - |V_k|^2)^2 \right] dx$$

$$= \int_{r_k}^{r_k \sqrt{\lambda_k}} f_k(r) dr$$

$$\geq \frac{1}{2} \ln r_k \min_{r_k \leq r \leq \sqrt{\lambda_k}} \{ r f_k(r) \}, \forall k \geq k_0.$$

Therefore, for each $k \geq k_0$, we can find a $\lambda_k \in (r_k, \sqrt{\lambda_k})$ such that

$$\lambda_k f_k(\lambda_k) \leq 2 \ln \lambda_k |^{-1} C(t, \delta).$$

(4.11)
Using Lemma 10 for $x_0 = x_k$ and $r = \lambda_k$, we obtain
\[\epsilon_k^{-2} \int_{B_k} (1 - |V_k|^2)^2 \leq \epsilon_k^{-2} \int_{\Omega_{\lambda_k}(x_k)} (1 - |V_k|^2)^2 \]
\[\leq C\lambda_k \left[ \int_{\Omega_{\lambda_k}(x_k)} (1 + |\nabla V_k|^2 + |\nabla V_k|)|\partial_t V_k| \right] dx \]
\[+ \int_{\partial\Omega} \left| \frac{\partial V_k}{\partial T} \right|^2 ds + \int_{\partial B_{\lambda_k} \cap \Omega} (|\frac{\partial V_k}{\partial T}|^2 + \frac{1 - |V_k|^2}{4\epsilon_k^2}) ds \]
\[\leq C[\lambda_k + 2|\ln \lambda_k|^{-1}] \quad (by \ (4.10) \ and \ (4.11)).\]

This contradicts with (4.9) because of the fact $\lambda_k \to 0$. In this way, we finish the proof of Lemma 11.

**Lemma 12** *With the same $\delta, t$ and $\epsilon_1$ as in Lemma 11, one has that for all $x \in \overline{\Omega} \setminus \bigcup_{j=1}^{m_j} B_{\delta_j}(y_j(t))$,*
\[1 - C(\delta, t)\epsilon^{\frac{1}{2}} \leq |V_\epsilon(x, t)| \leq 1 + C(\delta, t)\epsilon^{\frac{1}{2}}, \quad \forall \epsilon \in (0, \epsilon_1). \quad (4.12)\]

**Proof.** we follow the proof of Lemma 4.3 in [18]. Here we give the details just for the reader’s convenience. Fix $x_0 \in \partial \Omega$ and let $\sigma_2 = \frac{1}{4} \min\{1, [\sigma(T)]^2\}$. Using (4.7) and the arguments from (4.10) to (4.11) one can easily see that
\[\int_{\partial B_{\lambda_k}(x_0) \cap \Omega} \left[ |\nabla V_\epsilon(x, t)|^2 + \frac{1 - |V_\epsilon(x, t)|^2}{\epsilon^2} \right] dx \leq C(\delta, t, \sigma_2) \quad (4.13)\]
for some $\lambda_\epsilon \in [\sigma_2, \sqrt{\sigma_2}]$ and all $\epsilon \in (0, \epsilon_1)$. Moreover, (4.13), (4.7) and Lemma 10 imply that $\int_{\partial\Omega} |\frac{\partial V_\epsilon}{\partial N}|^2 \leq C$ independent of $\epsilon$. Therefore, we deduce that
\[\int_{\partial\Omega} |\nabla V_\epsilon|^2 ds + \int_{\Omega_{\delta/4}(t)} (|\nabla V_\epsilon|^2 + |\frac{\partial V_\epsilon}{\partial t}|^2 + \frac{1 - |V_\epsilon|^2}{\epsilon^2}) dx \leq C(\delta, t) \quad (4.14)\]
and
\[\alpha_1 \geq |V_\epsilon(x, t)|^2 \geq \alpha_2 > 0, \quad \forall x \in \Omega_{\delta/4}(t) \quad (by \ (3.2) \ and \ (4.8)) \quad (4.15)\]
for all $\epsilon \in (0, \epsilon_1)$. Let $R_0$ be a positive constant to be determined later. As we will see, it depends only on $\alpha_1, \alpha_2$ and $C(\delta, t)$ in (4.14). Fix a constant $r_0 \in (0, \min\{\frac{\delta}{4}, R_0\})$ which will be suitably small at last. For an arbitrary $y \in \Omega_{\delta}(t)$, choose a number $R \in (0, \min\{\frac{\delta}{8}, \frac{R_0 - r_0}{2}\})$ satisfying $B_{2R + r_0}(y) \subset \Omega_{\frac{\delta}{8}}(t)$. 

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Write
\[ V_\varepsilon(x, t) = \rho_\varepsilon(x, t)e^{i\psi_\varepsilon(x,t)} \]
on $B_{2R+r_0}(y) \times (t - C_\varepsilon, t + C_\varepsilon)$ for some $C_\varepsilon > 0$ (see (3.2)) so that the equation in (1.1) turns to be
\[ \text{div}(\rho_\varepsilon^2 \nabla \psi_\varepsilon) = \rho_\varepsilon^2 (\partial_t \psi_\varepsilon - \nabla \psi_\varepsilon \nabla \omega) \quad \text{in } B_{2R+r_0}(y) \tag{4.16} \]
and
\[ \Delta \rho_\varepsilon + B \frac{(1 - \rho_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon = |\nabla \psi_\varepsilon|^2 \rho_\varepsilon + \partial_t \rho_\varepsilon - \nabla \rho_\varepsilon \nabla \omega - A \rho_\varepsilon \quad \text{in } B_{2R+r_0}(y). \tag{4.17} \]
Moreover, using (4.14) and the Fubini's theorem (see the arguments from (4.10) to the (4.11)), one can find $R_\varepsilon \in (R, R + \frac{r_0}{2})$ such that
\[ \int_{\partial B_{R_\varepsilon}(y)} |\nabla V_\varepsilon|^2 + \frac{(1 - |V_\varepsilon|^2)^2}{\varepsilon^2} \leq 2r_0^{-1}C(\delta, t). \tag{4.18} \]
It easily follows from (4.18) and Lemma 7 that
\[ \max_{x \in \partial B_{R_\varepsilon}(y)} |1 - |V_\varepsilon(x, t)|^2| \leq C(r_0^{-1}, \delta)\varepsilon^{\frac{1}{2}} \]
which, together with (3.1), implies that
\[ |1 - |V_\varepsilon(x, t)|| \leq C_2 \varepsilon^{\frac{1}{2}} \tag{4.19} \]
for all $x \in \partial B_{R_\varepsilon}(y)$, all $\varepsilon \in (0, \varepsilon_1)$ and some constant $C_2$ depending only on $r_0^{-1}$, $\delta$ and $t$.

On one hand, applying Theorem 2.2 of Chapter V in [20] to the equation (4.16) for $\psi_\varepsilon$ with the coefficient $\rho_\varepsilon$ satisfying (4.15) and using the notation $\mathcal{E} = \frac{1}{\varepsilon} \int E$, we obtain that
\[ \left( \int_{B_{R_\varepsilon}(y)} |\nabla \psi_\varepsilon|^p dx \right)^{\frac{1}{p}} \leq C_3 \left( \int_{B_{2R+r_0}(y)} |\nabla \psi_\varepsilon|^2 dx \right)^{\frac{1}{2}} \]
\[ + \frac{R}{2} \left( \int_{B_{2R+r_0}(y)} (|\nabla \psi| |\nabla \omega| + |\partial_t \psi_\varepsilon|)^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \tag{4.20} \]
for some $p \in (2, 3]$ depending only on $\alpha_1$, $\alpha_2$ in (4.15), for some $R_0 > 0$ depending only on $\alpha_1$, $\alpha_2$ and $C$ in (4.14), for some $C_3 > 0$ depending only on $p$, $\alpha_1$ and $\alpha_2$, for all $R < \frac{R_0 - r_0}{2}$, and for all $\varepsilon \in (0, \varepsilon_1)$. 

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On the other hand, equation (4.17) implies that the function
\[ \bar{U}_\varepsilon(x,t) \equiv 1 - \rho_\varepsilon(x,t) \]
satisfies
\[ -\Delta \bar{U}_\varepsilon + \frac{C_\varepsilon(x)}{\varepsilon^2} \bar{U}_\varepsilon = f_\varepsilon \]
in \( B_{R_\varepsilon}(y) \) with \( 0 < C(\alpha_1) \leq C_\varepsilon(x) \equiv B(1 + \rho_\varepsilon)\rho_\varepsilon \leq C(\alpha_2) \) and
\[ f_\varepsilon \equiv |\nabla \psi_\varepsilon|^2 \rho_\varepsilon + \partial_t \rho_\varepsilon - \nabla \omega \nabla \rho_\varepsilon - A\rho_\varepsilon \in L^2(B_{R_\varepsilon}(y)). \]
By (4.20) and (4.14), we see that
\[ ||f_\varepsilon||_{L^2(B_{R_\varepsilon}(y))} \leq C(\alpha_1, \alpha_2, \delta), \quad \forall \varepsilon \in (0, \varepsilon_1). \]
Moreover, (4.19) yields \(-C_4 \varepsilon^{\frac{1}{2}} \leq \bar{U}_\varepsilon \leq C_4 \varepsilon^{\frac{1}{2}} \) on \( \partial B_{R_\varepsilon}(y) \). Therefore, a standard elliptic estimate (Theorem 8.16 in [21]) gives us
\[ |\bar{U}_\varepsilon| \leq C(C_2, \alpha_1, \alpha_2) \varepsilon^{\frac{1}{2}} \text{ in } B_{R_\varepsilon}(y) \times \{ t \}. \]
Particularly, we have
\[ 1 - C_4 \varepsilon^{\frac{1}{2}} \leq \rho_\varepsilon(x) = |V_\varepsilon(x,t)| \leq 1 + C_4 \varepsilon^{\frac{1}{2}}, \quad \forall x \in B_R(y), \quad \forall \varepsilon \in (0, \varepsilon_1). \quad (4.21) \]
Now for any \( G_0 \subset \subset \Omega \), choose \( r_0 = \frac{1}{4} \min\{R_0, \delta, \text{dist}(G_0, \partial\Omega)\} \) and fix
\[ R = \min\{\frac{\delta}{8}, \frac{R_0 - r_0}{2}, \frac{1}{4}\text{dist}(G_0, \partial\Omega)\}. \]
Then, by the arbitrariness of \( y \in \bar{G}_0 \setminus \bigcup_{j=1}^m B_\delta(y_j(t)) \), we can find finite balls, \( B_R(y_i), i = 1, 2, \cdots, n \), such that \( \bigcup_{i=1}^n B_R(y_i) \supset \bar{G}_0 \setminus \bigcup_{j=1}^m B_\delta(y_j(t)) \) and (4.21) holds true for all \( x \in B_R(y_i) \) and all \( i = 1, 2, \cdots, n \). In this way, we conclude
\[ 1 - C_4 \varepsilon^{\frac{1}{2}} \leq |V_\varepsilon(x,t)| \leq 1 + C_4 \varepsilon^{\frac{1}{2}}, \quad \forall x \in \bar{G}_0 \setminus \bigcup_{j=1}^m B_\delta(y_j(t)), \forall \varepsilon \in (0, \varepsilon_1) \quad (4.22) \]
for some constant \( C \) and \( \varepsilon_1 \) both independent of \( \varepsilon \). Moreover, using the fact
\[ \int_{\partial\Omega} |\nabla V_\varepsilon|^2 \leq C(\delta)(\text{see (4.14)}) \] and repeating the argument above for \( \Omega_R = B_R(y) \cap \Omega \) with \( y \in \partial\Omega \), we can find a domain \( G' \subset \subset \Omega \) such that (4.22) holds true for all \( x \in \bar{\Omega} \setminus G' \). Combining this result with (4.22), we have proved Lemma 12.
Now, combining Lemmas 11 and 12, we have completed the proofs of conclusion (i) of Theorem 2 as well as the corresponding part of Corollary 3.

Next, we are going to prove the second part of Theorem 2.

**Lemma 13** \( \{\psi_\varepsilon : \varepsilon \in (0, \varepsilon_1)\} \) is compact in \( H^1(B) \) with \( B = B_{R_1 + r_0}(y) \).

**Proof.** (4.14) and (4.15) imply that \( \{\nabla \rho_\varepsilon^2 : \varepsilon \in (0, \varepsilon_1)\} \) is bounded in \( L^2(B) \). Thus, by (4.20) and Holder inequality, we see that \( \{\nabla \rho_\varepsilon^2 \nabla \psi_\varepsilon : \varepsilon \in (0, \varepsilon_1)\} \) is bounded in \( L^q(B) \) with \( q = \frac{2p}{p+1} > 1 \). Hence \( \{\psi_\varepsilon : \varepsilon \in (0, \varepsilon_1)\} \) is bounded in \( W^{2,q}(B) \) by (4.7), (4.15), equation (4.16) and standard elliptic estimates. Furthermore, the Rellich-Kondrachov’s theorem tells us that the set \( \{\psi_\varepsilon : \varepsilon \in (0, \varepsilon_1)\} \) is compact in \( H^1(B) \).

To complete the proof of conclusion (ii) of Theorem 2, we need only to prove estimate (1.6). This is because the strong convergence of \( V_{\varepsilon n} \to V \) is a direct consequence of Theorem 1, Lemma 13, and (1.6).

In order to prove (1.6), we rewrite (4.17) as

\[
\Delta \rho_\varepsilon + \frac{(1 + \varepsilon^2 - \rho_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon = |\nabla \psi_\varepsilon|^2 \rho_\varepsilon + \partial_t \rho_\varepsilon - \nabla \rho_\varepsilon \nabla \omega - (A-1) \rho_\varepsilon.
\]

Multiplying this equality by \( \rho_\varepsilon \) and integrating it over \( B_{R_1}(y) \), we have

\[
\int_{B_{R_1}(y)} \frac{(1 + \varepsilon^2 - \rho_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon^2 dx = \int_{B_{R_1}(y)} [ |\nabla \psi_\varepsilon|^2 \rho_\varepsilon + |\nabla \psi_\varepsilon|^2 \rho_\varepsilon^2 + (\partial_t \rho_\varepsilon - \nabla \rho_\varepsilon \nabla \omega) - (A-1) \rho_\varepsilon] dx - \int_{\partial B_{R_1}(y)} \frac{\partial \rho_\varepsilon}{\partial \nu} \rho_\varepsilon dx.
\]

It is easy to see that the sum on the right side hand can be bounded by a constant \( C_3 = C(\alpha_1, \alpha_2, \delta, t, r_0^{-1}) \) due to (4.7), (4.15) and (4.18). Therefore, applying (4.15), corollary 8 and conclusion (i) of Theorem 2, we obtain that

\[
\int_{B_{R_1}(y)} \frac{(1 + \varepsilon^2 - \rho_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon^2 dx \leq \alpha_2^{-2} \int_{B_{R_1}(y)} \frac{(1 + \varepsilon^2 - \rho_\varepsilon^2)}{\varepsilon^2} \rho_\varepsilon^2 dx \\
\leq \max_{(x,t) \in \Omega(\delta/2)} (1 + \varepsilon^2 - \rho_\varepsilon^2) C_3 \\
\leq C(\alpha_1, \delta, t) \varepsilon^{-\frac{1}{2}}.
\]

Hence

\[
\varepsilon^{-2} \int_{B_{R_1}(y)} (1 - |V_\varepsilon|^2)^2 dx = \varepsilon^{-2} \int_{B_{R_1}(y)} [(1 + \varepsilon^2 - \rho_\varepsilon^2)^2 - \varepsilon^4 - 2 \varepsilon^2 (1 - \rho_\varepsilon^2)] dx
\]

20
On the other hand, setting $P_\varepsilon = 1 - |V_\varepsilon|^2$, we have, by equation (4.17), that

$$-\Delta P_\varepsilon + 2 \frac{\rho_\varepsilon^2 P_\varepsilon}{\varepsilon^2} = 2|\nabla \psi_\varepsilon|^2 \rho_\varepsilon^2 + \partial_t \rho_\varepsilon^2 + 2|\nabla \rho_\varepsilon|^2 - \nabla \rho_\varepsilon^2 \nabla \omega - 2A \rho_\varepsilon^2$$

in $B_{R_0}(y)$. Multiplying this equation by $P_\varepsilon$, integrating the resulted equality over $B_{R_0}(y)$, and using (4.7), (4.15), (4.18), (4.23), the boundedness of $|\psi_\varepsilon|^2$ in $L^2(B)$, and the estimate

$$|P_\varepsilon| = (1 + |V_\varepsilon|)(1 - |V_\varepsilon|) \leq C(\delta, t)\varepsilon^{\frac{1}{2}}$$

from the conclusion (i) of Theorem 2, we obtain that

$$\int_{B_{R_0}(y)} |\nabla P_\varepsilon|^2 dx \leq C(\delta, t)\varepsilon^{\frac{1}{2}}. \tag{4.24}$$

Observing $\nabla |V_\varepsilon| = \nabla \rho_\varepsilon = -(2\rho_\varepsilon)^{-1} \nabla P_\varepsilon$, we have, by (4.15) and (4.24), that

$$\int_{B_{R_0}(y)} |\nabla |V_\varepsilon||^2 dx \leq C(\delta, t)\varepsilon^{\frac{1}{2}}. \tag{4.25}$$

Finally, using (4.23), (4.25) and Lemma 13, and repeating the covering arguments in the end of the proof of (i) of Theorem 2, we have proved the (1.6) and thus (ii) of Theorem 2.

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