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Coherent states in Quantum Information: an example of experimental manipulations

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Abstract. A part of difficulties in implementing communication in Quantum Information stems from the fragility of Shroedinger cat-like superpositions. We describe here a recent experiment in Quantum Optics proving the feasibility of a feedback-mediated quantum measurement for discriminating between optical coherent states under photodetection. The measurements validate theoretical prediction by Helstrom (“Helstrom bound”), Dolinar and Geremia. This contribution gives an account of the implementation achieved by Cook, Martin and Geremia (2007) and explains the theoretical approaches to the subject.

1. Introduction
Quantum Information Processing is about exploiting quantum mechanical features in all facets of information processing (data communication, computing). The states act as information carriers, while the communication channels are the quantum operations [Audenaert 2007]. The sender encodes information by preparing the channel into a well-defined quantum state \( \rho \) belonging to an alphabet \( \mathcal{A} = \{ \rho_0, \rho_1, \ldots, \rho_M \} \). The receiver, following any relevant signal propagation, performs a measurement on the channel to ascertain which state was transmitted by the sender.

Quantum information theory is mainly based on superposition-basis and entanglement measurements. This requires high-fidelity implementation to be effective in the laboratory. Unfortunately, quantum measurements are “invasive” in the sense that little or no refinement is achieved by further observation of an already measured system. If the states in the sender alphabet are not orthogonal, no measurement can distinguish between overlapping quantum states without some ambiguity [von Neumann 1955, Holevo 2001, Fuchs 1996, Peres 1995, Helstrom 1976]. Then errors seem unavoidable: there exists a nonzero probability that the receiver will misinterpret the transmitted codeword.\(^1\)

However, this impossibility of discriminating non-orthogonal quantum states might represent an advantage for quantum key distribution [Bennett-Brassard 1984]. Indeed, nonorthogonality prevents an eavesdropper from acquiring information without disturbing the state. Also, in some cases it has been shown by Fuchs that the classical information capacity of a noisy channel is actually maximized by a nonorthogonal alphabet [Fuchs 1996].

Mathematically, the question of distinguishing between nonorthogonal states [Peres 1995, Fuchs 1996] is addressed by optimizing a state-determining measurement over all positive operator valued measures (POVM) [Helstrom 1976]. But arbitrary POVM’s are not easy to manipulate!

\(^1\) Even with transmission of orthogonal codewords, decoherence, energy dissipation and other imperfections deteriorate orthogonality.
The optical field produced by a laser provides a convenient quantum system for carrying information. Since optical coherent states \[|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \] are not orthogonal, one would be attempted to minimize the overlap \[\langle \alpha'|\alpha \rangle = e^{i\alpha^* (\alpha')^*} e^{-|\alpha - \alpha'|^2/2} \] by using large amplitude regimes. However, one faces to power limitation and the appearance of non-linear effects. So one is more inclined to develop optimization methods for communication processes based on small-amplitude optical coherent states and photodetection. Now, when one tries to distinguish between two non-orthogonal states through some receiver device, there exists a quantum error probability. The latter is bounded below by some minimum, named the quantum limit or Helstrom bound [Helstrom 1976] in this context.

In a recent work Cook, Martin, and Geremia (CMG) [Cook-Martin-Geremia 2007] have demonstrated that real-time quantum feedback can be used in place of a quantum superposition of the type “Schrödinger cat state”, to implement an optimal quantum measurement for discriminating between optical coherent states. The aim of the present contribution is precisely to give most of the elements needed to understand the CMG experiment, both on experimental and theoretical levels. The content is mainly extracted from the seminal paper by Geremia [Geremia 2004], the CMG Nature [Cook-Martin-Geremia 2007], and the chapter 4 of the textbook [Gazeau 2009]. We give in the next section some necessary definitions concerning the use of optical coherent states in Quantum Information. The Helstrom bound for binary communication with perfect and imperfect detection is then explained. We describe in Section 3 the principles, the experimental device and the results of the CMG experiment, which proved the ability of these authors to reach the quantum limit. In Section 4 we develop the theoretical background of the CGM experiment, namely the description of the Dolinar receiver that is at the basis of the CMG experiment.

2. Binary Coherent State Communication and the Helstrom bound

2.1. Binary Coherent State Communication

Let us consider an alphabet consisting of two pure coherent states, \(\rho_0 = |\Psi_0\rangle\langle\Psi_0|\), \(\rho_1 = |\Psi_1\rangle\langle\Psi_1|\), corresponding to the logical “0” and “1” respectively. Without loss of generality, \(|\Psi_0(t)\rangle = |0\rangle\) can be chosen as the vacuum while

\[\Psi_1(t) = \psi_1(t) \exp \left[-i(\omega t + \varphi)\right] + c.c., \tag{1}\]

where \(\omega\) is the frequency of the optical carrier and \(\varphi\) is (ideally) a fixed phase.

The envelope function, \(\psi_1(t)\), is normalized such that

\[\int_0^\tau |\psi_1(t)|^2 dt = \bar{n}, \tag{2}\]

where \(\bar{n}\) is the mean number of photons to arrive at the receiver during the measurement interval, \(0 \leq t \leq \tau\). That is, \(\hbar\omega|\psi_1(t)|^2\) is the instantaneous average power of the optical signal for logical “1”. By combining the incoming signal with an appropriate local oscillator, the amplitude keying with the alphabet of two coherent states \(\mathcal{A} = \{|0\rangle, |\alpha\rangle\}\) can always be transformed to the phase-shift keyed alphabet, \(\{|-\frac{1}{2} \alpha\rangle, |\frac{1}{2} \alpha\rangle\}\), via the unitary displacement \(D(-\frac{1}{2} \alpha) = \exp(-\frac{1}{2} (\alpha a^\dagger - \bar{a}a))\). Similarly, if \(|\Psi_0\rangle \neq |0\rangle\), a simple displacement can be used to restore \(|\Psi_0\rangle\) to the vacuum state.

In the case of non-orthogonal quantum states as codewords, the receiver attempts to ascertain which state was transmitted by performing a quantum measurement, \(\Upsilon\), on the channel. \(\Upsilon\) is described by an appropriate POVM represented by a complete (here countable) set of positive operators [Peres 1995] resolving the identity,

\[\sum_i \Upsilon_i = I_d \quad \Upsilon_i \geq 0, \tag{3}\]

where \(i\) indexes the possible measurement outcomes.

\(^2\) We recall that \(D(\beta) |\alpha\rangle = e^{i\alpha (a^\dagger \beta)} |\alpha + \beta\rangle\)
An example of POVM in the euclidean plane is given by the following cyclotomic polygonal resolution of the unity:

\[
\frac{2}{n} \sum_{q=0}^{n-1} \Upsilon_{2q} = I_d, \quad \Upsilon_\theta = |\theta\rangle\langle\theta| = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}.
\]

For binary communication, for which the POVM resolution of the unity reads \( \Upsilon_0 + \Upsilon_1 = I_d \), the measurement by the receiver amounts to a decision between two hypotheses: \( H_0 \), that the transmitted state is \( \rho_0 \), selected when the measurement outcome corresponds to \( \Upsilon_0 \), and \( H_1 \), that the transmitted state is \( \rho_1 \), selected when the measurement outcome corresponds to \( \Upsilon_1 \).

### 2.2. The Quantum Error Probability or Helstrom bound

Now possibilities of errors mean that there is some chance that the receiver will select the null hypothesis, \( H_0 \) (resp. \( H_1 \)), when \( \rho_1 \) is actually present (resp. \( \rho_0 \)). Thus, we have in terms of conditional probabilities:

\[
p(H_0|\rho_1) = \text{tr}[\Upsilon_0 \rho_1] = \text{tr}[(I_d - \Upsilon_1) \rho_1], \quad p(H_1|\rho_0) = \text{tr}[\Upsilon_1 \rho_0].
\]

The total receiver error probability is then given by

\[
p[\Upsilon_0, \Upsilon_1] = \xi_0 p(H_1|\rho_0) + \xi_1 p(H_0|\rho_1), \quad \xi_0 + \xi_1 = 1,
\]

where \( \xi_0 = p_0(\rho_0) \) and \( \xi_1 = p_0(\rho_1) \) are the classical probabilities that the sender will transmit \( \rho_0 \) and \( \rho_1 \) respectively; they reflect the prior knowledge that enters into the hypothesis testing process implemented by the receiver, and in many cases \( \xi_0 = \xi_1 = 1/2 \).

Minimizing the error in receiver measurement over all possible POVM’s \( (\Upsilon_0, \Upsilon_1) \) leads to the so-called quantum error probability or Helstrom bound,

\[
P_H = \min_{\Upsilon_0:\Upsilon_1} p[\Upsilon_0, \Upsilon_1],
\]

\( P_H \) is the smallest physically allowable error probability, given the overlap between \( \rho_0 \) and \( \rho_1 \).

The receiver error probability \( (5) \) can be written

\[
p[\Upsilon_0, \Upsilon_1] = \xi_0 \text{tr}[\Upsilon_1 \rho_0] + \xi_1 \text{tr}[(I_d - \Upsilon_1) \rho_1] = \xi_1 + \text{tr}[\Upsilon_1 (\xi_0 \rho_0 - \xi_1 \rho_1)],
\]

and is minimized by optimizing \( \min_{\Upsilon_1} \text{tr}[\Upsilon_1 \Gamma] \), \( \Gamma \overset{\text{def}}{=} \xi_0 \rho_0 - \xi_1 \rho_1 \), over \( \Upsilon_1 \) subject to \( 0 \leq \Upsilon_1 \leq I_d \).

Let \( \Gamma = \sum_n \lambda_n |\gamma_n\rangle\langle\gamma_n| \) be the spectral decomposition of the operator \( \Gamma \). One can write \( \text{tr}[\Upsilon_1 \Gamma] = \sum_n \lambda_n \langle \gamma_n | \Upsilon_1 | \gamma_n \rangle \). Then the Helstrom bound can be expressed as \( P_H = \xi_1 + \sum_{\lambda_n<0} \lambda_n \), which corresponds to the case in which \( \Upsilon_1 \) is the projector on all eigenstates \( |\gamma_n\rangle \) with negative \( \lambda_n \).

For pure states, where \( \rho_0 = |\Psi_0\rangle\langle\Psi_0| \) and \( \rho_1 = |\Psi_1\rangle\langle\Psi_1| \), \( \Gamma \) has two eigenvalues of which only one is negative,

\[
\lambda_- = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0 \xi_1 (|\Psi_0\rangle\langle\Psi_1|)^2} \right) - \xi_1 < 0,
\]

and the quantum error probability is therefore \( \text{[Helstrom 1976]} \)

\[
P_H = \xi_1 + \lambda_- = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0 \xi_1 (|\Psi_1\rangle\langle\Psi_0|)^2} \right).
\]

Consistently, this quantity vanishes for orthogonal states.
2.3. Helstrom bound for coherent states
From the expansion of coherent states over the number states, 
\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \]
the overlap between \( |\Psi_1\rangle \) and \( |\Psi_0\rangle \) is just given in terms of the expected number of photons or average value or lower symbol of the number operator in the coherent state \( |\alpha\rangle \):
\[ \langle \Psi_1 | \Psi_0 \rangle = \langle \alpha | 0 \rangle = e^{-|\alpha|^2/2} = e^{-\bar{n}/2}. \] (9)
So, the Helstrom bound is given by
\[ P_H = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \xi_0 \xi_1 e^{-\bar{n}}} \right). \] (10)

2.4. Helstrom bound with imperfect detection
It is further possible to evaluate the Helstrom bound for imperfect detection. Non-unit efficiency of a photodetector leads to a photon-count which is related to the ideal (efficiency \( \eta = 1 \)) pre-measured photon distribution by a Bernoulli transformation [Loudon 1973]. Accordingly, the probability \( p_n(\eta) \) to detect \( n \)-photons using a non-ideal photodetector (\( \eta < 1 \)) is given in terms of the probability \( p_m(\eta = 1) \) (using an ideal one) by
\[ p_n(\eta) = \sum_{m=n}^{\infty} \frac{m!}{n!} \eta^n (1 - \eta)^{m-n} p_m(\eta = 1) \] (11)

Coherent states have the convenient property that sub-unity quantum efficiency is equivalent to an ideal detector masked by a beam-splitter with transmission coefficient, \( \eta \leq 1 \). Indeed, in the case of coherent states, we have the Poisson distribution \( p_m(\eta = 1) = e^{-|\alpha|^2} |\alpha|^m / m! \), and so, by changing \( m \) into \( s = m - n \) in the summation (11) gives
\[ p_n(\eta) = \frac{\eta^n |\alpha|^2}{n!} e^{-|\alpha|^2} e^{-n|\alpha|^2} \] (12)
which amounts to replace \( \alpha \) by \( \sqrt{\eta \alpha} \) in the expression of coherent states. Accordingly, the Helstrom bound becomes
\[ P_H(\eta) = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \xi_0 \xi_1 e^{-\bar{n}\eta}} \right). \] (13)
This result and Eq. (10) indicate that there is a finite quantum error probability for all choices of \( |\Psi_1\rangle \), even when an optimal measurement is performed.

In this context, three receivers have been described and compared by Geremia [Geremia 2004].

Kennedy [Kennedy 1972] proposed in 1972 a receiver based on simple photon counting to distinguish between two different coherent states. However, the Kennedy receiver error probability lies above the quantum mechanical minimum, i.e. the Helstrom bound.

Then, Dolinar [Dolinar 1973] proposed a measurement scheme capable of achieving the quantum limit. Dolinar’s receiver, while still based on photon counting, approximates an optimal POVM by superposing a local feedback signal to the channel. A serious experimental drawback was that real-time adjustment of the local signal following each photon was considered (at that time) as quite impracticable.

As a result, Sasaki and Hirota [Sasaki Hirota 1996, Takeoka Sasaki 2008] later proposed an alternative receiver that applies an open-loop unitary transformation to the incoming coherent state signals to render them more distinguishable by simple photon counting.

However, the 2007 Cook-Martin-Geremia experiment has validated the Dolinar receiver.
3. Cook-Martin-Geremia experiment

In [Cook-Martin-Geremia 2007] Cook, Martin and Geremia demonstrate that shot noise can be surpassed and even the quantum limit can be approached by using real-time quantum feedback in place of the cat-state measurement. They exploit the finite duration of any real measurement and quantum states $|0\rangle$ and $|\alpha\rangle$ are realized as optical wavepackets with spatiotemporal extent.

Measurements on an optical pulse inherently persist for a time set by the pulse length $\tau$. Photon counting generates a measurement record $\Xi_{[0,\tau]} \equiv (t_1, t_2, \ldots, t_n)$ that consists of the observed photon arrival times even if is modelled using standard quantum measurement theory by viewing the total number of photon arrivals in the counting interval $[0, \tau]$ as one aggregate “instantaneous” measurement of the number operator.

![Figure 1](image_url)

**Figure 1.** The CMG Measurement combines photon counting with feedback-mediated optical displacements to enact quantum-limited state discrimination between the coherent states $|0\rangle$ and $|\alpha\rangle$. Reprinted by permission from Macmillan Publishers Ltd: [Nature] (Cook, R.L., Martin, P.J., and Geremia, J.M. 446, p.774, 2007), copyright (2007)

In the closed-loop measurement as is sketched in Figure[1] and detailed in Figure[2] photon counting is combined with feedback-mediated optical displacements applied during the photon counting interval. The amplitude of the displacement $u_t$ applied at each time $t$ during the measurement is conditional on the accumulated measurement record $\Xi_{[0,t]}$ and based on an evolving bayesian estimate of the incoming wavepacket state.

Discrimination is performed by selecting the state $|\psi\rangle \in \{ |0\rangle, |\alpha\rangle \}$ that maximizes the conditional probability $P (\Xi_{[0,t]} | \psi, u_{[0,t]} )$ that the measurement record $\Xi_{[0,t]}$ would be observed given the state $\psi$ and the history of applied displacements denoted by $u_{[0,t]}$.

The feedback controller determines which state is most consistent with the accumulating record $\Xi_{[0,t]}$ and chooses the feedback amplitude at each point in time to minimize the probability of error over the remainder of the measurement interval $(t, \tau]$.

The premise behind the closed-loop measurement is **to displace the field to the vacuum in each shot and decide which state is present based on the displacement applied to cancel the field**. As the controller gains increased confidence in its guess, it is better able to perform the correct nulling displacement. The displacement magnitude $|u^*_t|$ is inversely proportional to the time-dependent decision uncertainty. The statistical optimality for the closed-loop measurement is reached by reversing state hypothesis with each detector click during the counting interval. As the measurement record accumulates, the controller eventually settles on its final (correct) decision. In Figure[3] are shown

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Note that the term “arrival time is not really appropriate from an experimental point of view. Time interval is more appropriate.
Figure 2. Diagram of the laboratory implementation of Fig. [1]. Light from an external-cavity grating-stabilized diode laser operating at 852nm is coupled into a polarization maintaining (PM) fibre-optic MachZehnder interferometer. The input beam splitter (FBS1) provides two optical fields with a well defined relative phase: the upper arm of the interferometer acts as the target quantum system for state discrimination and the lower arm provides an auxiliary field used to perform closed-loop displacements at the second beam splitter (FBS2). Photon counting on the outcoupled field is implemented using a gated silicon avalanche photodiode (APD). Feedback controller is constructed from a combination of programmable waveform generators and high-speed digital signal processing electronics (feedback bandwidth 30MHz). A digital counter records the number of photon counter clicks generated in each measurement interval $[0, \tau]$, during which the feedback controller determines the feedback amplitude $u^*_t (n_{[0,t]})$ via the accumulating count record $n_{[0,t]}$. Reprinted by permission from Macmillan Publishers Ltd: [Nature] (Cook, R.L., Martin, P.J., and Geremia, J.M. 446, p.774, 2007), copyright (2007)

The measured probability of error versus mean photon number for both direct photon counting (red squares) and the CMG closed-loop measurement interpreted using a bayesian estimator that assumes application of the optimal closed-loop control policy (blue circles) and one that accounts for experimental imperfections (green triangles). All data points were obtained from ensembles of 100,000 measurement trajectories, with error bars that reflect the sample standard deviation. The data in Fig. [3] have been adjusted to account for finite detector efficiency and optical losses. That is to say, the coherent state amplitudes in the experiment are normalized with respect to the average photon number $\bar{n}_\alpha$, measured by the APD over a $t = 20\mu s$ square-envelope pulse. Owing to the nature of coherent states, it has been shown that the detection efficiency $\eta$ (resulting from the combination of detector quantum efficiency $\eta_d$ and optical efficiency $\eta_e$) factors out of a comparison between the shot-noise and quantum limits. For comparison, the shot-noise error and quantum limits that would correspond to ideal detection ($\eta = 1$) have been plotted. The intrinsic efficiency of the apparatus has been independently determined to be approximately $\eta \approx 0.35$. 
4. The principles of the Dolinar receiver

The Dolinar receiver utilizes an adaptive strategy to implement a feedback approximation to the Helstrom POVM [Dolinar 1973]. It operates by combining the incoming signal, $\Psi(t)$, with a separate local signal,

$$U(t) = u(t) \exp\left[-i(\omega t + \phi)\right] + \text{c.c.} \quad (14)$$

Here $u(t)$ is the “displacement” or “feedback” amplitude.

The detector counts photons with total instantaneous mean rate,

$$\Phi(t) = |\psi(t) + u(t)|^2 \quad (15)$$

where we recall that $\psi(t) = 0$ (for logical “0”) when the channel is in the state $\rho_0$, and $\psi(t) = \psi_1(t)$ (for logical “1”) when the channel is in $\rho_1$.

4.1. Photon counting distributions

Given the alphabet $A = (\rho_0, \rho_1)$, the feedback amplitude $u(t)$, a transmission coefficient $\eta$, and some subdivision $(t_0 \equiv 0, t_1, \ldots, t_n, t_{n+1} \equiv \tau)$ of the measurement time interval (or “counting interval”) $[0, \tau]$, the conditional probability $w[t_k|\rho_i, u(t)]$ that a photon will arrive at time $t_k$ and that it will be the only click during the half-closed interval, $(t_{k-1}, t_k]$ [Glauber 1963-2] is called the exponential waiting
time distribution for optical coherent states. It is defined as

\[ w[t_k|\rho_i, u(t)] = \eta \Phi(t_k) \exp \left( -\eta \int_{t_{k-1}}^{t_k} \Phi(t') \, dt' \right) , \]

The corresponding exclusive counting densities for the measurement interval reflect the likelihood that \( n \) photon arrivals occur precisely at the times \( t_1, \ldots, t_n \), given that: the channel is in the state, \( \rho_i \), the feedback amplitude is \( u(t) \), and the detector quantum efficiency is \( \eta \):

\[ p_\eta[t_1, \ldots, t_n|\rho_i, u(t)] = \prod_{k=1}^{n+1} w_\eta[t_k|\rho_i, u(t)] . \]

They allow to evaluate, using the Bayes’ rule, the conditional arrival time probabilities

\[ p_\eta[t_1, \ldots, t_n|\rho_i, u(t)] = p_\eta[t_1, \ldots, t_n|\rho_i, u(t)] \, p_0(\rho_i) . \]

4.2. Decision criterion of the receiver

The receiver decides between hypotheses \( H_0 \) and \( H_1 \) by selecting the one that is more consistent with the record of photon arrival times observed by the detector given the choice of \( u(t) \). \( H_1 \) is selected when the ratio of conditional arrival time probabilities,

\[ \Lambda = \frac{p_\eta[H_1|\rho_1, u(t)]}{p_\eta[H_0|\rho_0, u(t)]} \]

is greater than one; otherwise it is assumed that \( \rho_0 \) was transmitted.

By employing Bayes’ rule, \( \Lambda \), can be reexpressed in terms of the photon counting distributions

\[ \Lambda = \frac{p_\eta[t_1, \ldots, t_n|\rho_1, u(t)] \, p_0(\rho_1)}{p_\eta[t_1, \ldots, t_n|\rho_0, u(t)] \, p_0(\rho_0)} = \frac{\xi_1 \, p_\eta[t_1, \ldots, t_n|\rho_1, u(t)]}{\xi_0 \, p_\eta[t_1, \ldots, t_n|\rho_0, u(t)]} , \]

In terms of error probabilities, the likelihood ratio is given by

\[ \Lambda = \frac{p_\eta[H_1|\rho_1, u(t)]}{p_\eta[H_0|\rho_0, u(t)]} = \frac{1 - p_\eta[H_0|\rho_1, u(t)]}{p_\eta[H_1|\rho_0, u(t)]} , \]

(i.e., the receiver definitely selects \( H_1 \)), and

\[ \Lambda = \frac{p_\eta[H_0|\rho_1, u(t)]}{p_\eta[H_1|\rho_0, u(t)]} = \frac{p_\eta[H_0|\rho_1, u(t)]}{1 - p_\eta[H_1|\rho_0, u(t)]} , \]

(i.e., the receiver definitely selects \( H_0 \)).

The minimization over \( u(t) \) of the Dolinar receiver error probability,

\[ P_D[u(t)] = \xi_0 \, p_\eta[H_1|\rho_0, u(t)] + \xi_1 \, p_\eta[H_0|\rho_1, u(t)] , \]

can be accomplished by employing the technique of dynamic programming [Bertsekas 2000], which leads to the control policy,

\[ u_*^1(t) = -\psi_1(t) \left( 1 + \frac{J[u_*^1(t)]}{1 - 2J[u_*^0(t)]} \right) . \]

Bayes’ theorem relates the conditional and marginal probabilities of events \( A \) and \( B \), where \( B \) has a non-vanishing probability

\[ P(A|B) = \frac{P(B|A) \, P(A)}{P(B)} . \]

More generally, let \( \Omega = \bigcup_i A_i \), \( A_i \cap A_j = \emptyset \) for \( i \neq j \), be a partition of the event space. Then we have

\[ P(A_i|B) = \frac{P(B|A_i) \, P(A_i)}{\sum_j P(B|A_j) \, P(A_j)} \text{ for any } A_i \text{ in the partition.} \]
for $\Lambda > 1$, where $p_{\eta}[H_0|\rho_0, u^*_1(t)] = 0$ and

$$\mathcal{J}[u^*_1(t)] = \xi_1 p_{\eta}[H_1|\rho_0, u^*_1(t)] = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\bar{n}(t)}} \right),$$

Here we recall that $\bar{n}(t) = \int_0^t |\psi_1(t')|^2 dt'$ is the average number of photons expected to arrive at the detector by time $t$ when the channel is in the state $\rho_1$. Conversely, the optimal control takes the form,

$$u^*_0(t) = \psi_1(t) \left( \frac{\mathcal{J}[u^*_0(t)]}{1 - 2\mathcal{J}[u^*_0(t)]} \right)$$

for $\Lambda < 1$, where $p_{\eta}[H_1|\rho_0, u^*_0(t)] = 0$ and

$$\mathcal{J}[u^*_0(t)] = \xi_0 p_{\eta}[H_0|\rho_1, u^*_0(t)] = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\bar{n}(t)}} \right).$$

4.3. Dolinar hypothesis testing procedure
The Hamilton-Jacobi-Bellman solution leads to a conceptually simple procedure for estimating the state of the channel. The receiver begins at $t = 0$ by favoring the hypothesis that is more likely based on the prior probability $p_0(0) = \bar{\xi}_0$ and $p_1(0) = \xi_1$. Assuming that $\xi_1 \geq \bar{\xi}_0$ (for $\bar{\xi}_0 > \xi_1$, the opposite reasoning applies), the Dolinar receiver always selects $H_1$ during the initial measurement segment. The probability of deciding on $H_0$ is exactly zero prior to the first photon arrival such that an error only occurs when the channel is actually in $\rho_0$.

To see what happens when a photon does arrive at the detector, it is necessary to investigate the behavior of $\Lambda(t)$ at the boundary between two measurement segments. Substituting the optimal control policy, $u^*(t)$, which alternates between $u^*_1(t)$ and $u^*_0(t)$, into the photon counting distribution leads to

$$p(t_1, \ldots, t_n|\rho_i) = \eta^n \prod_{k=0}^{n+1} \Phi_i[u_k|2(t_{k-1}, t_k)] \times \exp \left( -\eta \int_{t_{k-1}}^{t_k} \Phi_i[u_k|2(t'_{k-1}, t'_k)] dt' \right),$$

where $k|2$ stands for $k \mod 2$. This expression can be used to show that the limit of $\Lambda(t)$ approaching a photon arrival time, $t_k$, from the left is the reciprocal of the limit approaching from the right,

$$\lim_{t \rightarrow t_k^-} \Lambda(t) = \left[ \lim_{t \rightarrow t_k^+} \Lambda(t) \right]^{-1}.$$  (17)

That is, if $\Lambda > 1$ such that $H_1$ is favored during the measurement interval ending at $t_k$, the receiver immediately swaps its decision to favor $H_0$ when the photon arrives. Evidently, the optimal control policy, $u^*(t)$, engineers the feedback such that the photon counter is least likely to observe additional clicks if it is correct based on its best knowledge of the channel state at that time. Each photon arrival invalidates the current hypothesis and the receiver completely reverses its decision on every click. This result implies that $H_1$ is selected when the number of photons, $n_k$, is even (or zero) and $H_0$ when the number of photons is odd.

It is also asserted [Dolinar 1973] that, despite the discontinuities in the conditional probabilities, $p_{\eta}[H_1|\rho_0, u^*(t)]$ and $p_{\eta}[H_0|\rho_1, u^*(t)]$, at the measurement segment boundaries, the total Dolinar receiver error probability, $P_D(\eta, t) = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\bar{n}(t)}} \right)$, evolves smoothly, i.e. we have

$$\lim_{t \rightarrow t_k^-} \mathcal{J}[u^*(t)] = \lim_{t \rightarrow t_k^+} \mathcal{J}[u^*(t)]$$

at the boundaries.

Recognizing that $\bar{n}(\tau) = \bar{n}$ leads to the final Dolinar receiver error,

$$P_D(\eta) = \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\eta \bar{n}}} \right)$$

which is equal to the Helstrom bound for all values of the detector efficiency, $0 < \eta \leq 1$.

5 If $\xi_0 = \xi_1$, then neither hypothesis is a priori favored and the Dolinar receiver is singular with $P_D = \frac{1}{2}$. 
5. Conclusion
Quantum feedback can be viewed as manipulating the outcome statistics of the number operator $N$. In the absence of feedback, the detailed measurement record consisting of photon arrival times $\Xi_{[0,\tau]} = (t_1, t_2, \ldots, t_n)$ provides no more information than the total number $n$: Poisson processes are stationary in time, but with feedback, the significance of each click depends on when it occurs, even though the field is described by some coherent state at each point in time. The optimal feedback policy applies displacements in a manner that extracts as much information out of each photon arrival as possible. It is in this manner that shot noise can be surpassed to achieve the fundamental quantum limit over a non-trivial range of $|\alpha|$. Furthermore, this procedure appears as less demanding on the measurement resources needed to achieve optimal statistics than a direct implementation of a cat state. At no point in time has a superposition between optical coherent states been generated, yet the optimal result has been effectively achieved by exploiting the time-dependence of the measurement.

It should be added that recently Wittmann et al [Wittmann et al 2008] have experimentally realized a new quantum measurement that detects binary optical coherent states with fewer errors than the homodyne and the Kennedy receiver for all amplitudes of the coherent states. Although the scheme is not capable of achieving the Helstrom bound the implementation discriminates binary coherent states with an error probability lower than the optimal Gaussian receiver, namely the homodyne receiver. For more details on the theoretical background, see [Takeoka Sasaki 2008]

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