Rotational Viscosity in Linear Irreversible Thermodynamics and its Application to Neutron Stars

Alfredo Sandoval-Villalbazo¹*, Ana Laura García-Perciante¹ and L.S García Colín²†
1. Departamento de Ciencias, Universidad Iberoamericana, Santa Fé, México D.F. México
2. Departamento de Física, Universidad Autónoma Metropolitana, Iztapalapa, México D.F. México

A generalized analysis of the local entropy production of a simple fluid is used to show that, if intrinsic angular momentum is taken into account, rotational viscosity must arise in the linear non-equilibrium regime. As a consequence, the stress tensor of dense rotating matter, such as the one present in neutron stars, possesses a significant non-vanishing antisymmetrical part. A simple argument suggests that, due to the extreme magnetic fields present in neutron stars, the relaxation time associated to rotational viscosity is large ($\approx 10^{21}$ s). The formalism leads to generalized Navier-Stokes equations useful in neutron star physics which involve vorticity in the linear regime.

1. INTRODUCTION

In a recent paper [1], Rezania and Maartens have shown that relevant effects due to vorticity are present in neutron star physics. Their communication was primarily based on results derived from kinetic theory which show that, if Boltzmann equation is treated by means of Grad’s method, a coupling between vorticity and shear necessarily arises [2]. This paper is intended to present a study of an alternative role that vorticity may play in neutron stars. This approach is based on a purely phenomenological formalism capable of describing the thermodynamical role of vorticity and rotational viscosity, without introducing any assumption foreign to the concept of local equilibrium and the validity of the main conservation laws.

A word of caution is here pertinent. In principle one could require the use of relativistic irreversible thermodynamics to correctly cope with this question as it has been developed in previous work [3]. Nevertheless, in order to compare our results with those obtained in Ref. [1] we shall overlook this fact and perform a non-relativistic analysis to emphasize on the role played by the rotational viscosity.

The paper is divided as follows: Section 2 gives a review of the Meixner-Prigogine approach to non-equilibrium thermodynamics [4]. Attention should be paid to the fact that in this scheme, nothing forbids the existence of a non-vanishing antisymmetric part of the stress-energy tensor. Section 3 is devoted to the analysis of the constitutive equations relating the thermodynamical forces and fluxes, and its immediate effects in neutron star dynamics. Section 4 includes a brief comparison between the vorticity coupling suggested in Ref. [1] and the one here presented. An outline of how it is possible to support the existence of a non-vanishing antisymmetric part of the stress tensor with kinetic theory is also included.

2. MEIXNER-PRIGOGINE FORMALISM INCLUDING INTRINSIC ANGULAR MOMENTUM

2.1 Conservation laws

The Meixner-Prigogine approach to irreversible thermodynamics is based in the consideration of the conservation laws of mass, linear momentum, angular momentum and total energy, plus the assumption of local thermodynamical equilibrium. The balance equations for mass and linear momentum, written in differential form, are respectively given by:

$$\frac{\partial \rho}{\partial t} + \left( \rho u^\ell \right)_\ell = \frac{D\rho}{Dt} + \rho \left( \dot{u}^\ell \right)_\ell = 0$$

$$\frac{\partial \left( \rho u^k \right)}{\partial t} + \left( \rho u^\ell u^k + \tau^{\ell k} \right)_\ell = \rho \frac{Du^k}{Dt} + \tau^{\ell k} = f^k$$

*Present address: Relativity & Cosmology Group, University of Portsmouth, Portsmouth PO1 2EG, Britain. Email: alfredo.sandoval@port.ac.uk
†E-mail: lgc@xanum.uam.mx
where the summation is taken for repeated indices, $\rho$ is the mass density, $u^\ell$ the local hydrodynamic velocity, $\tau^{\ell k}$ the stress tensor and $f^k$ the external force density term. The forces of interest in our formalism correspond to gravitational and magnetic fields. Latin indices will run from 1 to 3, and ; indicates a covariant derivative.

Total angular momentum per unit of mass in any local element is properly described by an antisymmetric tensor $J^{\ell k}$. This tensor possesses an orbital part $L^{\ell k}$ and an intrinsic contribution $S^{\ell k}$, so that

$$J^{\ell k} = L^{\ell k} + S^{\ell k}$$

Since total momentum is conserved in purely rotating systems, we may write

$$\rho \frac{D J^{\ell k}}{D t} + \left( x^k \tau^{a \ell} - x^\ell \tau^{a k} \right)_{; a} = 0$$

where $x^k$ stands for the position vector. It is important to stress at this stage that the external force $f^k$ does not contribute to equation (4) for two reasons. The first obvious one is that the gravitational force being radial generates no torques. The second one is a little bit more subtle. Assuming that the external magnetic field $\vec{B}$ is constant and points along the rotation axis, the Lorentz force generates only a radial component whence again generates no torque. On the other hand internal magnetic fields are accounted for in the intrinsic angular momentum equation as it will be pointed out below. Now, orbital angular momentum per unit of mass is given by

$$L^{\ell k} = x^k u^\ell - x^\ell u^k$$

From equations (2) and (5), it follows that, in the presence of the fields of interest,

$$\rho \frac{D S^{\ell k}}{D t} = \tau^{\ell k} - \tau^{k \ell}$$

so that, subtracting equation (6) from equation (4) we have, for the intrinsic angular momentum per unit of mass, that,

$$\rho \frac{D S^{\ell k}}{D t} = \tau^{\ell k} - \tau^{k \ell} \equiv -2 \tau^{[\ell k]}$$

where square brackets on indices indicate the skew part. Intrinsic angular momentum, i.e. spin, in any local element of a neutron star is assumed to be proportional to its local magnetization $M^{\ell k}$, per unit of area, per unit of charge, so that:

$$S^{\ell k} = \beta M^{\ell k}$$

where the coefficient $\beta$ has units of $(\text{Length})^2$ and is related to thermodynamical properties of the element. This coefficient will be estimated in Section 4. It follows then, from the corresponding balance of intrinsic rotational energy, per unit of mass, $e_{[rot]}$, that using equations (7) and (8)

$$\frac{\partial}{\partial t} \left( \rho e_{[rot]} \right) + \left( J^{[mec]} + J^{[Q]} \right)_{; k} = \frac{\rho D}{4} \left( \beta M^{\ell k} \right) = -M_k \tau^{[k,\ell]}$$

where the second equality is due to the fact that, the rotational contribution to the internal energy in a paramagnetic substance is proportional to $B^{\ell k}B_k$ and since the magnetization is proportional to the magnetic field $B^{\ell k}$, equation (10) follows directly.

Let us now consider the energy balance. Since total energy is conserved, one can write for this conservation law the expression,

$$\frac{\partial}{\partial t} \left( \rho e_{[mec]} + e_{[int]} \right) + \left( J_{[mec]} + J_{[Q]} \right)_{; k} = 0$$

where $J_{[Q]}$ is the heat flux density and $e_{[int]} u^k$ the drift transport of internal energy. The explicit form of the heat flux $J_{[Q]}$ will be given later.

Multiplying equation (2) by the covariant vector $u^\ell$ and making use of equations (7) and (8) we get that:

$$\frac{\partial}{\partial t} e_{[mec]} + \left( J_{[mec]} \right)_{; k} = u_{\ell;k} \tau^{\ell k} - M_k \tau^{[k,\ell]}$$
where
\[ e_{\text{mec}} = \frac{1}{2} \rho u^2 + \rho \varphi + \rho e_{\text{rot}} \]  
(12)
and
\[ J^k_{\text{mec}} = \frac{1}{2} \rho u^2 u^k + \rho \varphi u^k + \rho e_{\text{rot}} u^k + u_{\ell} \tau^{\ell k} \]  
(13)
where \( \varphi \) stands for the gravitational potential and use has been made of the fact that no dissipation arises from the magnetic field, since \( u_{\ell} e_{n}^{\ell} u^k B^n = 0 \). If one now combines equations (1) and (11), it is possible to obtain the balance equation for the internal energy,
\[ \frac{D e_{\text{int}}}{D t} = - \left( J^\ell_{[Q]};_{\ell} \right) - u_{\ell,k} \tau^{\ell k} + M_{\ell k} \tau^{\ell k} \]  
(14)
If the stress tensor is splitted into two parts, one related to the (scalar) equilibrium pressure \( p \) and one associated with the viscous pressure tensor \( \Pi^{\ell k} \), then
\[ \tau^{\ell k} = p \delta^{\ell k} + \Pi^{\ell k}, \]  
and equation (13) reads:
\[ \frac{D e_{\text{int}}}{D t} = - \left( J^\ell_{[Q]};_{\ell} \right) - u_{\ell,k} \Pi^{\ell k} - p (u^a_\alpha) + M_{\ell k} \Pi^{\ell k} \]  
(15)
It must be noticed that \( \tau^{[\ell,k]} = \Pi^{[\ell,k]} \).

3. LOCAL THERMODYNAMICAL EQUILIBRIUM AND ENTROPY BALANCE

We now turn ourselves towards the question of introducing the entropy production. For this purpose we invoke the local equilibrium assumption which is crucial to the formulation of several versions of irreversible thermodynamics [8]. This assumption states that the local entropy \( s \) of the system is a time-independent functional of the local scalar thermodynamic densities, in this case the mass density \( \rho (x^a, t) \) and the internal energy density \( e_{\text{int}} (x^a, t) \). Then,
\[ s = s \left[ \rho (x^a, t), e_{\text{int}} (x^a, t) \right] \]  
(16)
whence,
\[ \frac{D s}{D t} = \left( \frac{\partial s}{\partial \rho} \right)_{e_{\text{int}}} \frac{D \rho}{D t} + \left( \frac{\partial s}{\partial e_{\text{int}}} \right)_{\rho} \frac{D e_{\text{int}}}{D t} \]  
(17)
By the local equilibrium assumption per se, the thermodynamic coefficients are the local expressions of their equilibrium counterparts,
\[ \left( \frac{\partial s}{\partial \rho} \right)_{e_{\text{int}}} = - \frac{p}{\rho^2 T}, \quad \left( \frac{\partial s}{\partial e_{\text{int}}} \right)_{\rho} = \frac{1}{T} \]  
(18)
where \( p \) and \( T \) are the local pressure and temperature, respectively, and the time rates of change of \( \rho \) and \( e_{\text{int}} \) are given through equations (1) and (14) respectively. Putting all this information into equation (17) and after some algebraic manipulation, one arrives at an expression of the form
\[ \frac{D s}{D t} + \left( \frac{J^\ell_{[Q]}}{T} \right);_{\ell} = - \left( \frac{J^\ell_{[Q]}}{T^2} \frac{\partial T}{\partial x^\ell} \right) - \frac{1}{T} u_{\ell,k} \Pi^{\ell k} + \frac{1}{T} M_{\ell k} \Pi^{\ell k} \]  
(19)
From standard irreversible thermodynamics, the entropy production or Clausius’ uncompensated heat term \( \Sigma \) is precisely the right hand side of equation (19) namely,
\[ \Sigma = - \left( \frac{J^\ell_{[Q]}}{T^2} \frac{\partial T}{\partial x^\ell} \right) - \frac{1}{T} u_{\ell,k} \Pi^{\ell k} + \frac{1}{T} M_{\ell k} \Pi^{\ell k} \]  
(20)
Only when \( \Sigma > 0 \), which depends on the nature of the constitutive equations adopted, the theory will be consistent with the second law of thermodynamics. We will come back to this questions in the next section.
4. CONSTITUTIVE RELATIONS AND NEUTRON STAR DYNAMICS

4.1 Linear constitutive relations

We are now able to propose constitutive relations for thermodynamical fluxes and forces using Curie’s principle [4, 5]. This principle states that, given an expression such as (20) for the entropy production in an isotropic system, only tensors of equal rank may be coupled. According to this idea, we must decompose tensors \( \Pi_{\ell k} \) and \( u_{\ell k} \) in their irreducible forms, namely:

\[
\Pi_{\ell k} = \Pi_{\ell k}^{[\ell k]} + \Pi_{\ell k}^{\langle \ell k \rangle} + \Pi_{\ell k}^{\delta_{\ell k}} \tag{21}
\]

and

\[
u_{\ell k} = \nu_{\ell k}^{[\ell k]} + \nu_{\ell k}^{\langle \ell k \rangle} + (u_{\alpha}^{a}) \delta_{\ell k} \tag{22}
\]

where angular brackets denote the symmetric trace-free part and \( \Pi \) represents one third of the trace of \( \Pi_{\ell k} \).

Introducing these relations in equation (20) we obtain:

\[
\Sigma = -J_{[\ell]} \frac{\partial T}{\partial x^\ell} - \frac{1}{T} \Pi_{\ell k}^{[\ell k]} u_{\ell k} + \frac{1}{T} \Pi_{\ell k}^{\langle \ell k \rangle} u_{\langle \ell k \rangle} - \frac{1}{T} \Pi^{\delta_{\ell k}} + \frac{1}{T} M_{\ell k} \Pi^{[\ell k]} \tag{23}
\]

Since we wish to have \( \Sigma > 0 \), linear constitutive relations are admissible to relate thermodynamical fluxes and forces, so that

\[
J_{[\ell]} = -\kappa \frac{\partial T}{\partial x^\ell} \tag{24}
\]

\[
\Pi = -\zeta u_{a}^{a} \tag{25}
\]

\[
\Pi^{[\ell k]} = -2\eta R \left( u^{[\ell k]} - M^{[\ell k]} \right) \tag{26}
\]

and

\[
\Pi^{\langle \ell k \rangle} = -2\eta \left( u^{\langle \ell k \rangle} \right) \tag{27}
\]

The transport coefficients introduced here are \( \kappa \), the heat conductivity, \( \eta \), the bulk viscosity, \( \zeta \), the shear viscosity, and \( \eta R \) the rotational viscosity. The property sought for \( \Sigma \) will be satisfied if these coefficients are themselves positive. As is well known, this is borne out by experiment.

4.2 Navier-Stokes equations in the presence of rotational viscosity and relaxation time.

An immediate consequence of the validity of the constitutive relations (24-27) is the effect of vorticity in the equation of motion (2), since

\[
\frac{\partial \tau^{\ell k}}{\partial x^\ell} = \frac{\partial}{\partial x^\ell} \left[ \Pi^{[\ell k]} + \Pi^{\langle \ell k \rangle} + (\Pi + p) \delta^{\ell k} \right] \tag{28}
\]

or

\[
\frac{\partial \tau^{\ell k}}{\partial x^\ell} = \frac{\partial}{\partial x^\ell} \left[ -2\eta R \left( u^{[\ell k]} - M^{[\ell k]} - 2\eta u^{\langle \ell k \rangle} - \zeta (u_{a}^{a}) \delta^{\ell k} \right) \right] + \frac{\partial p}{\partial x^k} \tag{29}
\]

Now, as de Groot and Mazur observe [3], this effect of vorticity involves a relaxation time obtainable from the assumption that \( u^{[\ell k]} \) does not change substantially in a time interval \( 0 < t < \Gamma \) where \( \Gamma \) is the relaxation time characterizing the local magnetization and defined below in equation (33). Thus, from equations (27) and (2), we have that,
\[2\Pi^{[\ell, k]} = -4\eta_R \left( u^{[\ell; k]} - M^{\ell k} \right) = -\rho D S^{k\ell} \frac{D t}{D t} \tag{30}\]

or, in terms of the local magnetization, equation (8)

\[4\eta_R \left( u^{[\ell; k]} - M^{\ell k} \right) = \rho \beta D M^{k\ell} \frac{D t}{D t} \tag{31}\]

which yields, under the uniformity assumption mentioned before, and assuming a vanishing magnetization at \(t = 0\):

\[M^{\ell k} = u^{[\ell; k]} \left[ 1 - \exp \left( -\frac{4\eta_R}{\rho \beta} t \right) \right] \tag{32}\]

This coupling between intrinsic angular momentum and vorticity involves a relaxation time given by:

\[\Gamma = \frac{\rho \beta}{4\eta_R} \tag{33}\]

Densities in neutron stars are extremely large, typically near to \(10^{18} \text{ kg m}^3\), so that \(\Gamma\) essentially depends on the ratio \(\beta/\eta_R\). Its magnitude will be explored below.

### 4.3 Order of magnitude for the relaxation time

The other parameter left in order to obtain a good idea about the relevance of this vorticity-magnetization coupling is \(\beta/\eta_R\) in equation (33). Strong magnetic fields in the star, close to \(B = 10^6\) Tesla, totally associated with local spin, suggest that the coefficient \(\beta\) may be estimated by a very simple argument, namely the relation between local magnetization, per unit of area, per unit of charge, and spin per kilogram of mass of a perfect paramagnetic solid, which is given by:

\[S^{k\ell} = \frac{1}{2\pi \mu} \frac{h}{\mathcal{L}(\mu B/kT)} M^{k\ell} \tag{34}\]

where \(h\) is Planck’s constant, \(\mu = 10^{-24} \text{ A.m}^2\) is Bohr’s magneton and \(\mathcal{L}(T)\) is the Langevin function at temperature \(T\). In our approximation, \(\mu B/kT \approx 10^{-5}\). Direct comparison with equation (8) yields,

\[\beta = \frac{1}{2\pi \mu} \frac{h}{\mathcal{L}(\mu B/kT)} \approx 10^{-6} \text{ m}^2 \tag{35}\]

and, thus

\[\Gamma \approx \frac{10^{11} \text{ Pa.s}^2}{\eta_R} \tag{36}\]

which may be quite significant if \(\eta_R\) is a relatively small number. Estimates of this coefficient are scarce in the literature, but even if we assume that it is of the order of \(10^3\) Pa s², \(\Gamma\) would be very large, if compared with relaxation times of the so-called “fast variables” associated with extended irreversible thermodynamics.

### 5. FINAL REMARKS

It must be stated that the formalism here developed is not in conflict with the one presented in Ref.[1], where vorticity appeared without taking into account intrinsic spin, but by means of a coupling between shear and vorticity. Whether the vorticity effects associated to intrinsic spin are relevant or not to stability of neutron stars is not clear. Boltzmann equation itself cannot predict values of rotational viscosity, since it leads to symmetric stress tensors and involves no information about internal degrees of freedom. A kinetic treatment involving the generalization suggested by Waldmann and Snider [9, 10] may be useful in order to compute precisely the value of \(\eta_R\) in equation (33) and so conclude about the real significance of this relaxation time.

On the other hand, if the dynamical effect of spin were negligible in dense systems like neutron stars, then, according to equation (33), rotational viscosity must have large values and, from this point of view, further studies related to astrophysical implications of this coefficient should developed.
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