On smooth rational threefolds of \( \mathbb{P}^5 \) with rational non-special hyperplane section

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Introduction

Linear systems of plane curves defining rational surfaces of \( \mathbb{P}^4 \) have been recently extensively studied. In particular: Alexander ([A], [A1]) gave theorems of existence and unicity for linear systems defining smooth surfaces in \( \mathbb{P}^4 \) with non-special or speciality one hyperplane sections, Catanese-Franciosi ([CF]) and Catanese-Hulek ([CH]) have performed a fine classification of base loci of linear systems defining smooth surfaces.

But nothing has been written in recent years about linear systems of surfaces of \( \mathbb{P}^3 \) giving rational threefolds in \( \mathbb{P}^r \). Nevertheless, something on the subject may be found in some classical papers by F. Jongmans and U. Morin ([J], [M], [M1]).

Here we study, from the above point of view, the smooth threefolds of \( \mathbb{P}^5 \) having a rational non-special surface of \( \mathbb{P}^4 \) as general hyperplane section. It is known ([I], [I1]) that there are exactly five of such threefolds: \( \mathbb{P}^2 \times \mathbb{P}^1 \), Del Pezzo, Castelnuovo and the scrolls of Bordiga and Palatini (of degrees 3,4,...,7 respectively). Among them we find all smooth threefolds which are scrolls over a rational surface ([O]). It has been in fact the wish of better understanding the geometry of the scrolls of Bordiga and Palatini which originally motivated our interest to this subject.

Our main result is that, for degree 3,4,5,6 the threefold \( X \) contains a line \( L \) such that the projection of center \( L \) gives a birational map between \( X \) and \( \mathbb{P}^3 \). In this situation the linear system \( |\Sigma| \) of surfaces in \( \mathbb{P}^3 \) defining the inverse map of the projection is particularly simple; in fact, if \( \deg X = d \), then \( \deg \Sigma = d-1 \); moreover the restriction of \( |\Sigma| \) to a general plane is a linear system giving a general hyperplane section of \( X \) containing \( L \).

We completely describe the base locus \( B \) of \( |\Sigma| \) in the four cases; precisely we determine the degree and the arithmetic genus for the irreducible components of \( B \), the infinitesimal structure we need to consider on them, and the way these components reciprocally intersect. An interesting consequence of this construction is the description of \( X \) as a suitable blowing-down of the blowing-up of \( \mathbb{P}^3 \) along \( B \).

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The situation is different for the Palatini scroll, because a similar line never exists. In a forthcoming paper ([MP]) we will give a different construction of a birational map $X \dashrightarrow \mathbb{P}^3$ in this last case.

The results on $|\Sigma|$ we obtain are collected in the following table (where “genus” always means “arithmetic genus”).

| Variety         | Degree | $\deg \Sigma$ | Base locus                                       |
|-----------------|--------|----------------|--------------------------------------------------|
| $\mathbb{P}^2 \times \mathbb{P}^1$ | 3      | 2              | • a point $P$                                    |
|                 |        |                | • a line                                         |
| Del Pezzo       | 4      | 3              | • a quintic curve of genus 2                     |
| Castelnuovo     | 5      | 4              | • a curve $B_1$ of degree 7 and genus 3 with a 5-secant line $B_2$ |
|                 |        |                | • the first infinitesimal nbh. of $B_2$          |
| Bordiga scroll  | 6      | 5              | • the first infinitesimal nbh. of                 |
|                 |        |                | • a curve $B_1$ of degree 3 and genus 0          |
|                 |        |                | • a curve $B_2$ of degree 7 and genus 0 with $\deg (B_1 \cap B_2) = 12$ |

It is natural to ask whether we can solve the inverse problem: characterize, among the linear systems satisfying the conditions of the above table, those defining Del Pezzo, Castelnuovo and Bordiga threefolds. We are able to completely solve this problem: the answer is that the base curve $B$ has to be locally Cohen–Macaulay and, roughly speaking, its general plane section has to satisfy the conditions given by Catanese–Franciosi.

Let us say a few words about the methods employed and the organization of the paper.

The study of the threefolds $X$ of degree 3 and 4, which is briefly reviewed in §1, is classical and easy. It motivates the analysis of the general properties of birational maps which are projections from an embedded line $L \subset X$, which is the content of §2.

In the cases of Castelnuovo 3-folds and Bordiga scrolls it is not a priori obvious that a similar projection exists. In the first case, a rather deep investigation of the geometry of $X$, inspired from [M], shows that in fact a suitable line always exists. We do this in §3.

We collect in §4 the geometric properties of the scrolls of Bordiga and Palatini we will need in the subsequent sections.

If $X$ is a Bordiga scroll, the existence of a suitable line is proved after a detailed study of the family of the adjunction maps for the hyperplane sections of $X$ (§5). Each of them contracts 10 lines to 10 points of $\mathbb{P}^2$; using a connectedness theorem of Debarre, we prove that for some hyperplane section $S$ these points are in special position, i.e. 7 of them are on a conic. The inverse image of such conic on $S$ is the desired line on $X$.

In §6 we show that for every Palatini scroll a line as above does not exist.

In §7 we reverse the point of view: we start from a linear system $|\Sigma|$ of surfaces in $\mathbb{P}^3$ and determine necessary and sufficient conditions on the base
locus in order that \(|\Sigma|\) defines a smooth threefold in \(\mathbb{P}^3\) of degree 3, 4, 5 and 6 respectively. One of the key tools here is the numerical character of a curve.

We will always work over an algebraically closed field \(k\) of characteristic zero. We will assume throughout this paper that \(X \subset \mathbb{P}^5\) is an irreducible, smooth, non-degenerate threefold. If \(E \subset \mathbb{P}^n\), we shall denote by \(\langle E \rangle\) the linear span of \(E\).

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1.- Preliminaries.

In this section we collect some classically known facts about birational maps and describe from our point of view the two first examples of rational threefolds, namely \(\mathbb{P}^2 \times \mathbb{P}^1\) and the Del Pezzo threefold, complete intersection of two quadric hypersurfaces. In both cases a birational map to \(\mathbb{P}^3\) is simply given by the projection from a suitable line contained in \(X\). At the end of the section, for the reader’s convenience, we will state a result of Catanese-Franciosi ([CF]) giving necessary and sufficient conditions on a zero-dimensional subscheme \(W \subset \mathbb{P}^2\) for the very ampleness on the blow-up of \(\mathbb{P}^2\) along \(W\) of the pull-back of a suitable linear system \(|H|\) of plane curves.

We recall first some classical facts on birational maps.

1.1 Fundamental points and exceptional divisors for a birational map.

Let \(f : X \dashrightarrow Y\) be a birational map between smooth varieties. A point \(x \in X\) is called fundamental for \(f\) in the case \(f\) is not regular at \(x\). Since \(X\) is smooth ("normal" is sufficient), the set \(F\) of the fundamental points for \(f\) is a closed subset of \(X\), of codimension at least 2. We will call \(F\) the fundamental locus for \(f\). An irreducible subvariety \(Z \subset X\) contained in the fundamental locus will be also called fundamental.

Van der Waerden’s Purity Theorem. Let \(f : X \dashrightarrow Y\) be a birational map of smooth varieties, and let \(W \subset Y\) be a fundamental variety for \(f^{-1}\). Assume that \(f(X) \cap W\) is dense in \(W\). Then any component \(E\) of \(f^{-1}(W)\) is of codimension 1 in \(X\).

For a proof we can refer to [EGA] (21.1), where the theorem is given under weaker assumptions.

We will call any \(E \subset X\) as above an exceptional divisor for \(f\).

Lemma 1.1 Let \(g : \mathbb{P}^n \dashrightarrow X\) be a birational map defined by the linear system \(|\Sigma|\) of hypersurfaces in \(\mathbb{P}^n\). Then any irreducible component of the fundamental locus for \(g\) is also a component of \(Bs(|\Sigma|)\), and conversely.

Proof. See [Z], Thm. 15.

Let us finally recall that a characteristic curve \(\Gamma\) of a linear system \(|\Sigma|\) of
surfaces of $\mathbb{P}^3$ is the free intersection of two general surfaces of $|\Sigma|$. If $E$ is an exceptional surface for $g^{-1}$ (notations are as in the lemma above), and $B$ is the corresponding base curve, then $\deg E = \deg (B \cap \Gamma)$.

1.2. The rational normal scroll $\mathbb{P}^2 \times \mathbb{P}^1$.

Let $X \subset \mathbb{P}^5$ be the image of $\mathbb{P}^2 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the Segre map $s$. For $x$ a point of $\mathbb{P}^1$, we will denote $F_1 = s(\mathbb{P}^2 \times x)$, a plane on $X$. Moreover, let $l$ be a line in $\mathbb{P}^2$ and set $F_2 = s(l \times \mathbb{P}^1)$. $F_2$ is a quadric surface on $X$. Finally, set $L := F_1 \cap F_2$.

The properties collected in the next proposition are classical ([SR]).

Proposition 1.2.1

(i) The projection map $\pi_L : X \dashrightarrow \mathbb{P}^3$ with center $L$ is birational;

(ii) the exceptional divisors of $\pi_L$ are $F_1$ and $F_2$, in particular $\pi_L(F_1)$ is a single point $P$ of $\mathbb{P}^3$, and $\pi_L(F_2)$ is a line $B \subset \mathbb{P}^3$ such that $P \notin B$;

(iii) the map $\pi_L^{-1}$ is defined by the linear system of quadrics $|\Sigma|$ with base locus $B \cup P$;

(iv) the only exceptional divisor for $\pi_L^{-1}$ is the plane $\Phi = \langle B \cup P \rangle$, which is contracted to the line $L$.

Proof. A general hyperplane through $L$ cuts $X$ into a rational normal scroll $S$. Another general hyperplane through $L$ cuts $S$ into the line $L$ plus a conic having only one point in common with $L$. Finally, a third general hyperplane through $L$ cuts this curve outside $L$ in exactly one point. Therefore, for a general point $A \in X$, the plane $\langle L \cup A \rangle$ cuts $X$ outside $L$ only in $A$, and (i) is proved.

The first part of (ii) is trivial and the second follows easily by remarking that $\pi_L$ is constant on any line of the ruling on $F_2$ which does not contain $L$.

A general hyperplane section $H$ of $X$ intersects $L$ at one point, hence it projects on a quadric $\Sigma$. $H$ meets $F_1$ and all lines of $F_2$, therefore the base locus of $|\Sigma|$ contains $B \cup P$. Conversely, the linear system of the quadric surfaces in $\mathbb{P}^3$ through $B \cup P$ has dimension 5, and (iii) follows.

(iv): let $R$ be a line of $\Phi$ through $P$; $R$ intersects any surface of $|\Sigma|$ in two base points, so it contracts to a point of $L$. $\square$

Remark 1.2.2. If we project from a line $s(y \times \mathbb{P}^1)$, $y \in \mathbb{P}^2$, of the ruling on $X$, then the projection is not birational, as it is easily seen.

Remark 1.2.3. The linear system $|\Sigma|$ contains a subsystem of dimension 3 formed by reducible quadrics, which are the union of the plane $\Phi$ and of a variable plane. The corresponding hyperplane sections of $X$ are precisely those which are cut out by hyperplanes containing $L$.

Construction 1.2.4. To better understand the projection $\pi_L$, let us perform the following construction. Let $\sigma : \widetilde{\mathbb{P}}^3 := \mathbb{P}^3(B \cup P) \dashrightarrow \mathbb{P}^3$ be the blowing-up of $\mathbb{P}^3$ along $B \cup P$. The linear system $|\Sigma|$ induces on $\widetilde{\mathbb{P}}^3$ the linear system $\Psi = |2H - E_B - E_P|$, where $H$ is the pull-back via $\sigma$ of the hyperplane divisor of $\mathbb{P}^3$, and $E_B, E_P$ are the exceptional divisors over $B$ and $P$ respectively. The
map defined by $\Psi$ is regular and fits in a commutative diagram:

\[\begin{array}{ccc}
\tilde{\mathbb{P}}^3 & \xrightarrow{\sigma} & \mathbb{P}^3 \\
& & \downarrow\pi_L^{-1} \\
\tilde{X} & \xrightarrow{\sim} & \pi_L^{-1}X
\end{array}\]

But $\Psi$ is not very ample, in fact on $X$ the images of $E_B$ and $E_P$ are the plane $F_1$ and the quadric $F_2$ which intersect along $L$.

Let us consider another linear system on $\tilde{\mathbb{P}}^3$: $\Psi' = |3H - E_B - E_P|$. It results to be very ample because the homogeneous ideal of $B \cup P$ in $\mathbb{P}^3$ is generated in degree 2 (see [Co], [BS]). Therefore $\Psi'$ defines an embedding $\tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^{14}$ with image $Y$, a smooth threefold of degree 23.

Let now $\tilde{X}$ be the blowing-up of $X$ along $L$. The homogeneous ideal of $L$ in $X$ is clearly generated in degree one; so by the results quoted above, the linear system $|2H_X - L|$ on $\tilde{X}$ is very ample and defines an embedding $\tilde{X} \rightarrow \mathbb{P}^{14}$. It is immediate to verify that the image is the variety $Y$. We have finally the following commutative diagram clarifying the geometry of the projection $\pi_L$, which results to be the composition of a blowing-up with the inverse of a blowing-up:

\[\begin{array}{ccc}
Y & \xleftarrow{\pi_L^{-1}} & \mathbb{P}^{14} \\
\downarrow & & \downarrow \\
\tilde{\mathbb{P}}^3 & \xrightarrow{\sim} & \tilde{X} \\
\downarrow & & \downarrow \\
\mathbb{P}^3 & \xrightarrow{\pi_L^{-1}} & X
\end{array}\]

**Remark 1.2.5.** The planes $\mathbb{P}^2 \times a$ on $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ correspond via $\pi_L$ to the planes in $\mathbb{P}^3$ containing the line $B$. The quadrics $R \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$, where $R \subset \mathbb{P}^2$ is a line, correspond via $\pi_L$ to the planes in $\mathbb{P}^3$ containing the point $P$.

### 1.3. Del Pezzo Threefold.

Let $X \subset \mathbb{P}^5$ be a Del Pezzo threefold, namely the complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$. Let $L \subset X$ be any line. The following properties, similar to those of Prop.1.2.1, hold ([SR]):

**Proposition 1.3.1**

(i) The projection map $\pi_L : X \rightarrow \mathbb{P}^3$ with center $L$ is birational;

(ii) the map $\pi_L^{-1}$ is defined by a linear system of cubic surfaces $|\Sigma|$ with base locus a general quintic curve $B$ of genus two;

(iii) the only exceptional divisor for $\pi_L^{-1}$ is the quadric $\Phi$ containing $B$, which is contracted by $\pi_L^{-1}$ to $L$;

(iv) the lines contained in $X$ and meeting $L$ generate a singular ruled surface $T$, of degree 8, having $L$ as triple locus, which is the only exceptional divisor of $\pi_L$. The surface $T$ is contracted by $\pi_L$ to $B$. 

Proof. (i) is similar to (i) of Proposition 1.2.1: now a general section of $X$ with a \( \mathbb{P}^3 \) containing $L$ is a quartic curve which splits in the union of $L$ with a rational cubic, having $L$ as a chord. Hence a general plane through $L$ intersects $X$ at a unique point outside $L$.

A general hyperplane section $H$ of $X$ intersects $L$ at one point, hence it projects on a cubic $\Sigma$. A general curve section $C$ of $X$ doesn’t intersect $L$, so the characteristic curves of $|\Sigma|$ are elliptic quartics. The linked curve to such a quartic $D$, in the complete intersection of two cubic surfaces containing it, is a quintic curve $B$ of genus 2, meeting $D$ in 8 points. Conversely, an easy calculation with cohomology shows that the linear system of the cubic surfaces in $\mathbb{P}^3$ through $B$ has dimension 5 and defines a rational map whose image is a threefold of degree 4.

$B$ is contained in a unique quadric $\Phi$, and is a divisor of type $(2, 3)$ on $\Phi$. Clearly the lines of $\Phi$ which meet $B$ at three points are contracted by $|\Sigma|$, those meeting $B$ at two points go to $L$.

The points of the base curve $B$ of $|\Sigma|$ come from lines in $X$ meeting $L$; the degree of the surface $T$ they generate is clearly $\deg (B \cap D) = 8$. Finally, $L$ is triple for $T$ because each point of $L$ comes from a trisecant line for $B$. \(\square\)

Remark 1.3.2. As in the case of $\mathbb{P}^2 \times \mathbb{P}^1$, the linear system $|\Sigma|$ contains a subsystem of dimension 3 formed by reducible cubic surfaces, which are the union of the quadric $\Phi$ and of a variable plane. This subsystem corresponds to hyperplane sections of $X$ containing $L$.

Remark 1.3.3. As in 1.2.4, we may factorize the rational map $\pi_L^{-1}$ through a variety $Y$ which is isomorphic to both the blowing-up of $\mathbb{P}^3$ along the quintic $B$ and to the blowing-up of $X$ along the line $L$. $Y$ is a threefold of degree 26 in $\mathbb{P}^{15}$ and can be seen as the image of the rational map defined by the linear system of quartic surfaces $|4H_{\mathbb{P}^3} - B|$ of $\mathbb{P}^3$; alternatively, it is defined by the linear system $|2H_X - L|$ on $X$. This follows as in 1.2.4 from the known fact that the homogeneous ideal of $B$ is generated in degree 3.

Remark 1.3.4. The lines on $X$ such that the projection with center one of them gives a birational map from $X$ to $\mathbb{P}^3$ (“good centers of projection”) form a family of dimension 3 and 2, respectively for $\mathbb{P}^2 \times \mathbb{P}^1$ and for a Del Pezzo threefold.

1.4. The Catanese-Franciosi conditions.

Let $W$ denote a zero-dimensional subscheme of $\mathbb{P}^2$ and let $|H|$ be a linear system of plane curves. We recall here under what conditions on $W$ the pull-back of $H$ on the blowing-up $\pi : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ along $W$ is very ample.

(CF1) $\deg W = 5$ (\(S\) is a Del Pezzo surface)

\[
W = \{x_1, \ldots, x_5\} \text{ and } |H| = |3H_{\mathbb{P}^2} - W| \quad \text{(here and in all other cases } H_{\mathbb{P}^2} \text{ denotes a line in } \mathbb{P}^2)\}; \text{ then } \pi^*H \text{ is very ample if and only if: }
\]

(0) $W$ has no infinitely near points ;

(1) $h^0(H_{\mathbb{P}^2} - x_i - x_j - x_k) = 0$, for any $i, j, k \in \{1, \ldots, 5\}$ .

(CF2) $\deg W = 8$ \(\quad (X\) is a Castelnuovo surface)
ample if and only if:

\[ W \text{ is very ample if and only if:} \]

1. \( h^0(H_x^2 - x_1 - y_i - y_j) = 0, \) for any \( i, j \in \{2, \ldots, 8\} \);
2. \( h^0(H_x^2 - \sum_{i \in \Delta} y_i) = 0 \) for \( \# \Delta \geq 4 \);
3. \( h^0(2H_x^2 - x_1 - \sum_{i \neq j} y_j) = 0, \) for any \( j \in \{2, \ldots, 8\} \).

(CF3) \( \deg W = 10 \) \((S \text{ is a Bordiga surface})\)

\[ W = \{y_1, \ldots, y_{10}\}, \text{ and } |H| = |4H_x^2 - W|; \text{ then } \pi^*H \text{ is very ample if and only if:} \]

1. \( W \text{ has no infinitely near points;} \)
2. \( h^0(H_x^2 - \sum_{i \in \Delta} y_i) = 0, \) for \( \# \Delta \geq 4 \);
3. \( h^0(2H_x^2 - \sum_{i \in \Delta} y_i) = 0, \) for \( \# \Delta \geq 8 \);
4. \( h^0(3H_x^2 - \sum_{1 \leq i \leq 10} y_i) = 0. \)

Actually, we will need to consider also another linear system \( |H'| \) of plane curves defining the Bordiga surface; precisely, \( |H'| \) is obtained from \( |H| \) by means of the standard quadratic transformation centered at three non collinear points among the \( y_i \)'s. Then the above conditions modify to (notations are fresh):

(CF3') \( \deg W = 10 \) \((S \text{ is again a Bordiga surface})\)

\[ W = \{x_1, x_2, x_3, y_1, \ldots, y_{10}\}, \text{ and } |H'| = |5H_x^2 - 2 \sum_{1 \leq i \leq 3} x_i - \sum_{4 \leq j \leq 10} y_j|; \text{ then } \pi^*H \text{ is very ample if and only if:} \]

1. \( \text{no two of the } y \text{'s are infinitely near each other; at most one among the } y \text{'s is infinitely near to one of the } x \text{'s;} \)
2. \( h^0(H_x^2 - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0, \) for \( 2\# \Delta + \# \Lambda \geq 5 \);
3. \( h^0(2H_x^2 - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0, \) for \( 2\# \Delta + \# \Lambda \geq 10 \);
4. \( h^0(3H_x^2 - \sum_{1 \leq i \leq 3} x_i - \sum_{4 \leq j \leq 10} y_j) = 0. \)

(CF4) \( \deg W = 11 \) \((S \text{ is a hyperplane section of a Palatini scroll})\)

\[ W = \{x_1, \ldots, x_6, y_7, \ldots, y_{11}\}, \text{ and } |6H_x^2 - \sum_{1 \leq i \leq 6} 2x_i - \sum_{7 \leq j \leq 11} y_j|; \text{ then } \pi^*H \text{ is very ample if and only if:} \]

1. \( \text{at most one } y_j \text{ is infinitely near to a point } x_i; \)
2. \( h^0(H_x^2 - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0, \) for \( 2\# \Delta + \# \Lambda \geq 6 \);
3. \( h^0(2H_x^2 - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0, \) for \( 2\# \Delta + \# \Lambda \geq 12 \);
4. \( h^0(3H_x^2 - \sum_{1 \leq i \leq 6} x_i - \sum_{j \neq h} y_j) = 0 \) for every \( h \in \{7, \ldots, 11\} \).

2.- Birational projections of a threefold from a line on it.

In the previous section we showed examples of threefolds \( X \subset \mathbb{P}^5 \) containing a line such that the projection from this line is a birational map \( X \dasharrow \mathbb{P}^3 \). Starting from this map it was very easy to find out a linear system of surfaces in \( \mathbb{P}^3 \) defining the birational inverse \( \mathbb{P}^3 \dasharrow X \) of the projection. In this section we want to formalize this procedure.
Let $X \subset \mathbb{P}^5$ be a threefold satisfying our general assumptions; we will denote by $d$ its degree. Moreover, we will assume that:

there is a line $L$ on $X$ such that the projection $\pi_L : X \to \mathbb{P}^3$ from the center $L$ is birational.

**Proposition 2.1** The surfaces in $\mathbb{P}^3$ of the linear system $|\Sigma|$ defining $\pi_L^{-1}$ have degree $d - 1$.

**Proof.** This follows easily from the fact that any hyperplane $H \subset \mathbb{P}^5$ intersects $L$ at a point. \qed

Let us denote by $M \simeq \mathbb{P}^3$ the target of the projection $\pi_L$. A key point here is the fact that, since $L$ and $M$ are disjoint, a hyperplane $H$ containing $L$ cannot contain also $M$, hence $H \cap M$ is a plane. From this remark we get at once the first part of

**Proposition 2.2** The set-theoretic image in $\pi_L$ of any hyperplane $H \subset \mathbb{P}^5$ such that $L \subset H$ is a plane in $M$. The elements of $|\Sigma|$ corresponding to the hyperplanes in $\mathbb{P}^5$ through $L$ break into a variable plane plus a fixed surface $\Phi$, of degree $d - 2$, which is the exceptional divisor for the map $\pi_L^{-1}$.

**Proof.** Let $u : \tilde{X} \to X$ be the blowing-up of $X$ along $L$, and let $v : \tilde{X} \to \mathbb{P}^3$ be the map which solves the singularities of $\pi_L : X \to \mathbb{P}^3$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} \\
\downarrow^u \\
X \xrightarrow{\pi_L} \mathbb{P}^3 \\
\end{array}
\]

Let $E$ denote the exceptional divisor of $u$. Then, clearly, a surface of $|\Sigma|$ corresponding to a hyperplane section $S \subset \mathbb{P}^5$ through $L$ breaks into the plane $\pi_L(S)$ and $\Phi := v(E)$. So, by definition, the map $\pi_L^{-1}$ contracts $\Phi$ to $L$. \qed

**Remark 2.3** In certain cases we can use the particular structure of the surfaces in $|\Sigma|$ corresponding to hyperplane sections through $L$ described above, to get informations on the base locus of $|\Sigma|$. Let $H \subset \mathbb{P}^5$ be a hyperplane such that $L \subset H$; set $S = X \cap H$ and $V = \pi_L(S) \subset M$. $V$ is a plane; the restriction of $|\Sigma|$ to $V$ is a linear system $|\Sigma'|$ of plane curves of degree $d - 1$. Of course, the dimension of $|\Sigma'|$ is 4, hence it defines a rational map whose image is a rational surface in $\mathbb{P}^4$, which has the same degree $d$ as $X$. The base locus of $|\Sigma'|$ is $V \cap B$, where $B$ is the 1-dimensional part of the base locus of $|\Sigma|$. In the case $|\Sigma'|$ is well understood and unique (see e.g. [A],[CF]) we get the desired informations about $B$.

We will give an example of the use of this remark in the next section (Theorem 3.7).
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**Proposition 2.4** The surface \( \Phi \) is irreducible, rational and ruled. Every general plane section of \( \Phi \) is a rational curve.

*Proof.* In fact, since \( E \) is irreducible, \( \Phi = v(E) \) is also irreducible.

If we think of the projection \( \pi_L \) as a rational map \( \mathbb{P}^5 \setminus L \dashrightarrow M \simeq \mathbb{P}^3 \), then \( \Phi \) is the union of the images in \( \pi_L \) of the (projectivized) tangent spaces to \( X \) at the points of \( L \). In particular, from this it follows that \( \Phi \) is ruled. In fact, let \( A \) be a general point of \( L \), and denote by \( T \) the tangent space to \( X \) at \( A \). Then \( \pi_L(T) \) is a line in \( M \), and \( \pi_L(T) \subset \Phi \).

The description of \( \Phi \) by means of \( \pi_L \) above shows that the map \( v \) restricted to \( E \) is still birational. Since \( E \) is a rational surface, \( \Phi \) is also rational.

Finally, let \( C \) be a general plane section of \( \Phi \). From \( \pi^{-1}_L(\Phi) = L \) it follows that \( \pi^{-1}_L \) induces a birational map between \( C \) and \( L \), hence \( C \) is a rational curve. \( \Box \)

**Remark 2.5** As the examples treated in § 1 show, we may have on \( \Phi \) more than one ruling of lines. In any case, we agree to consider as the ruling on \( \Phi \) the one considered in the above proof.

**Theorem 2.6** If there is a line \( L \subset X \) such that the projection \( \pi_L : X \dashrightarrow \mathbb{P}^3 \) from the center \( L \) is birational, then \( X \) is isomorphic to a blowing-down of \( \tilde{\mathbb{P}}^3 \), the blowing-up of \( \mathbb{P}^3 \) along \( B := Bs(|\Sigma|) \).

*Proof.* Let \( u : \tilde{X} \longrightarrow X \) be the blowing-up of \( X \) along \( L \). The homogeneous ideal of \( L \) in \( X \) is generated in degree one; then the linear system \( |2H_X - L| \) on \( X \) defines the rational map \( u^{-1} : X \dashrightarrow \tilde{X} \) ([Co],[BS]). Now, the linear system \( |H_X| \) on \( \mathbb{P}^3 \) is \( |(d - 1)H_{\mathbb{P}^3} - B| \). Finally, \( \pi^{-1}_L \) contracts \( \Phi \) to \( L \) and \( \Phi \subset |(d - 2)H_{\mathbb{P}^3} - L| \). Therefore, on \( \mathbb{P}^3 \) we have \( |2H_X - L| = |dH_{\mathbb{P}^3} - B| \). Let us recall that any smooth threefold of \( \mathbb{P}^5 \) is linearly normal, hence the linear system \( |(d - 1)H_{\mathbb{P}^3} - B| \) is complete. Therefore the coherent sheaf \( J_B(d - 1) \) is spanned: by [BS] it follows that \( \tilde{X} \) is isomorphic to the blow-up of \( \mathbb{P}^3 \) along \( B \), namely, we have the commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \overset{\sim}{\longrightarrow} & \tilde{\mathbb{P}}^3 \\
\downarrow u & & \downarrow \\
X \overset{\pi_L}{\longrightarrow} & \longrightarrow & \mathbb{P}^3
\end{array}
\]

We remark that the linear system \( |dH_{\mathbb{P}^3} - B| \) defines a rational map \( v : \mathbb{P}^3 \dashrightarrow \tilde{X} \). This yields the factorization of \( \pi^{-1}_L \)
where $w$ is the birational inverse of the map $v : \tilde{X} \to \mathbb{P}^3$ introduced in the proof of Proposition 2.2.

To prove the next proposition we need the following

**Lemma 2.7** Let $E \subset X$ be an irreducible exceptional divisor for the projection $\pi_L : X \dashrightarrow \mathbb{P}^3$, and set $B := \pi_L(E)$. Assume that the base locus of $|\Sigma|$ contains exactly the $r$-th infinitesimal neighbourhood of $B$. If $\dim(B) = 1$ and if $b \in B$ is a general point, then the fibre $\pi_L^{-1}(b)$ is a rational curve of degree $r + 1$, lying in the plane $\langle L \cup b \rangle$. If $B$ is a point, then $r = 0$ and $\pi_L^{-1}(B) = \langle L \cup B \rangle$.

**Proof.** Let $B$ be a curve. It is clear that $\pi_L^{-1}(b)$ is a curve lying in the plane $\langle L \cup b \rangle$. The preimage $D \subset \tilde{X}$ of $b \in \mathbb{P}^3$ in diagram (1) is clearly a rational curve in $\tilde{X}$, which is not contained into the exceptional divisor of $\tilde{X}$. Therefore, the map $u : D \to \pi_L^{-1}(b)$ is birational, hence $\pi_L^{-1}(b)$ is a rational curve. Finally, if $S$ denotes a hyperplane section of $X$, then $\deg \pi_L^{-1}(b) = \pi_L^{-1}(b) \cdot S = r + 1$, since the base locus of $|\Sigma|$ contains exactly the $r$-th infinitesimal neighbourhood of $B$.

Let $B$ be a point. Then, as a set, $\pi_L^{-1}(B) = \langle L \cup B \rangle$. Moreover, the general curve section of $X$ intersects $\langle L \cup B \rangle$ transversally at a single point. This shows that $r = 0$. \qed

**Proposition 2.8** Let $B$ be the base locus of $|\Sigma|$. Then, we have $B \subset \Phi$ scheme-theoretically. If $R$ is a line for which the scheme-theoretic intersection $R \cap B$ is a zero dimensional scheme of length $d - 1$ ("$R$ is $(d - 1)$-secant to $B$"), then $R \subset \Phi$. If $B$ is purely 1-dimensional, then the converse is also true ("the rational map $\pi_L^{-1} : B \dashrightarrow L$ has degree $d - 1$") and $H^0(\mathbb{P}^3, \mathcal{I}_B(d - 2))$ is generated by $\Phi$.

**Proof.** The subset $\{ \Phi + H \mid H \in \mathbb{P}^3 \} \subset |\Sigma|$ is still a linear system, whose base locus is exactly $\Phi$. Therefore we have $\Phi \subset H^0(\mathbb{P}^3, \mathcal{I}_B(d - 2))$.

From $B \subset \Phi$ and from the fact that the degree of $\Phi$ is $d - 2$, it follows immediately that any line which is $(d - 1)$-secant to $B$ lies on $\Phi$.

Conversely, assume that $B$ is purely 1-dimensional and let $R \subset \Phi$ be a line of the ruling. Therefore, there exists $\ell \in L$ such that $R = \pi_L(T_{X,\ell})$. We set $X \cap T_{X,\ell} = L \cup D$; then $\deg D = d - 1$. Assume that an irreducible component $D'$ of $D$ is not contained in the union of all exceptional divisors of $\pi_L$. Then $D'$ dominates $R$ through $\pi_L$ and the restriction $D' \dashrightarrow R$ of $\pi_L$ is birational. This contradicts $\pi_L^{-1}(R) = \ell$. Hence, by Lemma 2.7, $D$ consists of finitely many rational curves, each contained in a plane through $L$, and the sum of the degrees of these curves is $d - 1$. In view of Lemma 2.7 we conclude that the line $R$ is $(d - 1)$-secant to the base locus $B$. \hfill \qed
Finally, let $\Psi \in H^0(\mathbb{P}^3; \mathcal{J}_B(d-2))$. Any $(d-1)$-secant line to $B$ is contained in $\Psi$ and since these lines fill $\Phi$ we conclude $\Phi \subseteq \Psi$. But $\Phi$ and $\Psi$ have the same degree, hence $\Phi = \Psi$. $\square$

The assumption that $\pi_L : X \dashrightarrow \mathbb{P}^3$ is birational means that the general plane $U$ through $L$ cuts $X$ outside $L$ in a single point. Let $U = H_1 \cap H_2 \cap H_3$, where the $H_i$'s are hyperplanes through $L$. Now, $H_1 \cap H_2 \simeq \mathbb{P}^3$ and we can think of $U$ as a plane inside this $\mathbb{P}^3$. Moreover, $X \cap H_1 \cap H_2 = L \cup \Delta$. Therefore, $U$ cuts $\Delta$ outside $L$ in exactly one point. Since the degree of $\Delta$ is $d - 1$, we conclude

**Proposition 2.9** The projection $\pi_L : X \dashrightarrow \mathbb{P}^3$ is birational if and only if the general curve $\Delta$ defined above meets $L$ in $d - 2$ points.

Moreover, concerning the curves $\Delta$ we can say a little bit more:

**Proposition 2.10** The general curve $\Delta$ is rational.

**Proof.** The linear system on $X$ which defines the projection $\pi_L : X \dashrightarrow \mathbb{P}^3$ is $|H_X - L|$. Now, for any $H_1, H_2 \in |H_X - L|$, the curve $\Delta$ is the “variable part” of $H_1 \cap H_2$. Therefore, $\pi_L(\Delta)$ is the line, intersection of the two planes $H_1 \cap M$ and $H_2 \cap M$. $\square$

This can be refined as follows.

**Proposition 2.11** Let $\pi$ denote the sectional genus of $X$. Then the projection $\pi_L$ is birational if and only if for the general curve $\Delta$ it holds $p_a(\Delta) = \pi - d + 3$. In particular, if $X$ has non special, rational hyperplane sections, then $\pi_L$ is birational if and only if $p_a(\Delta) = 0$.

**Proof.** We have the relation

$$\pi = p_a(L \cup \Delta) = p_a(L) + p_a(\Delta) + \deg (L \cap \Delta) - 1$$

and the first part of the statement follows from Proposition 2.6. The last part follows from the fact that for such threefolds $\pi = d - 3$. $\square$

We conclude this section by reversing the point of view.

**Theorem 2.12** Let $B \subset \mathbb{P}^3$ be a closed subscheme. We assume that for some positive integer $d$:

(i) $\mathcal{J}_B(d-1)$ is spanned and $h^0(\mathcal{J}_B(d-1)) = 6$;

(ii) $h^0(\mathcal{J}_B(d-2)) = 1$;

(iii) for some plane $H \subset \mathbb{P}^3$, $B \cap H$ is the base locus of a linear system of curves in $H$, of degree $d - 1$, which is very ample on the blow-up of $H$ in $B \cap H$.

Then, denoted by $X$ the image of the rational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ defined by the linear system $|(d-1)H_{p3} - B|$, it follows that $X$ is a smooth threefold of degree $d$, the map $f$ is birational onto the image and there is a line $L \subset X$ such that $f^{-1}$ is given by the projection from $L$.

**Proof.** By (ii) we have $H^0(\mathcal{J}_B(d-2)) = k \cdot \Phi$ and the linear system $|(d-1)H_{p3} - B|$ contains the linear sub-system $\{\Phi + H \mid H \in \mathbb{P}^3\}$ of dimension 3. Therefore,
$f(\Phi)$ is contained in any hyperplane of a 3-dimensional linear family, hence $L := f(\Phi)$ is a line on $X$.

The restriction $\mathbb{P}^3 \setminus \Phi \to X \setminus L$ of $f$ is clearly a biregular map whose inverse is the projection from $L$. In particular, $\deg X = d$ and $X$ is smooth outside $L$. Finally, by (iii) there is a smooth hyperplane section of $X$ containing $L$, hence $X$ is smooth also along $L$. $\square$

3.- The Castelnuovo threefold.

Let $X \subset \mathbb{P}^5$ be a Castelnuovo threefold; it has degree 5 and sectional genus 2. $X$ can be defined by the $2 \times 2$ minors of a suitable $2 \times 3$ matrix of forms; then it is arithmetically Cohen-Macaulay. The hyperplane sections of $X$ are Castelnuovo surfaces, namely $S \subset \mathbb{P}^4$ is the image of a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$, birational onto $S$, defined by the linear system $|4H_2 - 2x_1 - \sum_{i \leq 3} y_i|$ (see [A]). The images in $\phi$ of the lines through $x_1$ are conics on $S$, which becomes in this way a conic bundle over $\mathbb{P}^1$. It is easily seen that the bundle map is just the adjunction map. The adjunction map $\varphi : X \to \mathbb{P}^1$ for $X$ realizes the Castelnuovo threefold as a quadric bundle over $\mathbb{P}^1$. (For more informations above the Castelnuovo threefold see [BOSS].)

In this section we will show the existence on any Castelnuovo threefold $X$ of a line $L$ such that $\pi_L : X \dashrightarrow \mathbb{P}^3$ is birational. Then we will apply the results of the previous section to construct the linear system $|\Sigma|$.

**Proposition 3.1** Let $L \subset X$ be a line. The projection $\pi_L : X \dashrightarrow \mathbb{P}^3$ is birational if and only if $L$ is unisecant the quadrics on $X$, i.e. $L$ is the image of a section of $\varphi$. In particular $\varphi(L) = \mathbb{P}^1$.

**Proof.** Assume that $\pi_L : X \dashrightarrow \mathbb{P}^3$ is birational. Let $S \subset X$ be a hyperplane section containing $L$. Since the projection of $S$ from $L$ is still birational, we have $|H_2| \subset |4H_2 - 2x_1 - \sum_i y_i|$, hence there is an effective divisor $\Gamma \in |3H_2 - 2x_1 - \sum_i y_i|$, whose image in $\phi : \mathbb{P}^2 \dashrightarrow S$ is the line $L$.

Note, in particular, that from this it follows that $L$ meets all the exceptional divisors on $S$.

Now, since the cubic $\Gamma$ has a node in $x_1$ it intersect a general line through $x_1$ at exactly one point away from $x_1$. This is no more true for the lines $x_1y_i$, $i > 1$, since $y_i \in \Gamma$. But $\phi$ maps $x_1y_i$ onto a line on $S$ and the conic is given by this line together with the exceptional divisor over $y_i$. As remarked above, $L$ intersects (in a unique point) also these conics.

Conversely, let $L \subset X$ be a line which is unisecant the quadrics on $X$. Consider a pencil of hyperplanes whose center contains $L$ and the corresponding hyperplane sections they cut on $X$. To this pencil it corresponds in the target of the projection a pencil of planes with center a fixed line. Then the desired conclusion follows from next Lemma. $\square$

**Lemma 3.2** Let $S$ be a Castelnuovo surface of $\mathbb{P}^4$. If there exists a line $L$ contained in $S$ and unisecant its conics, then the linear system on $S$ of
hyperplane sections containing \( L \) defines a birational map to the plane.

**Proof.** The condition “\( S \) smooth” ensures that the base points \( x_1, y_2, \ldots, y_8 \) satisfy the condition (CF2), see 1.4.

\( L \) being contained in a hyperplane section of \( S \), it is the image under \( \phi \) of a plane curve \( D \) of degree at most 3. It is easy to see that only three cases are possible:

1. \( D \) is a line passing through three of the points \( y_2, \ldots, y_8 \), or
2. \( D \) is a conic passing through \( x_1 \) and five of the points \( y_2, \ldots, y_8 \), or
3. \( D \in |3H_S - 2x_1 - \sum_i y_i| \).

In all the three cases the residual system to \( D \) is a homaloidal net in the plane; precisely, in case (a) it is the linear system of nodal cubics with node at \( x_1 \) and passing through 5 fixed points, in case (b) it is the net of conics through three fixed points, in case (c) it is the net of all lines of the plane. \( \square \)

The equations for \( X \) are the \( 2 \times 2 \) minors of a matrix

\[
M = \begin{pmatrix} L_1 & L_2 & F_1 \\ L_3 & L_4 & F_2 \end{pmatrix}
\]

where the \( L_i \) are linear forms and the \( F_j \) are quadratic forms.

**Lemma 3.3** Let \( Q \) be the quadric hypersurface defined by \( L_1L_4 - L_2L_3 = 0 \); then \( \text{rk}(Q) = 4 \).

**Proof.** Clearly \( \text{rk}(Q) \leq 4 \). Assume \( \text{rk}(Q) \leq 3 \). Then \( Q \) is a cone whose vertex contains a plane \( V \); we can assume that \( V \) is defined by the equations \( x_0 = x_1 = x_2 = 0 \). Then, after a change of coordinates, \( M \) has the form

\[
M = \begin{pmatrix} x_0 & x_1 & F_1 \\ x_1 & x_2 & F_2 \end{pmatrix}.
\]

Therefore, by evaluating the jacobian matrix at a point \( P \in V \) it follows that the points \( P \in V \) such that \( F_1(P) = F_2(P) = 0 \) are singular for \( X \), a contradiction. \( \square \)

Therefore \( \text{rk}(Q) = 4 \) and the vertex of \( Q \) is a line \( L' \); assume that \( x_0 = x_1 = x_2 = x_3 = 0 \) are equations for \( L' \). Then

\[
M = \begin{pmatrix} x_0 & x_1 & F_1 \\ x_2 & x_3 & F_2 \end{pmatrix}
\]

Note, in particular, that \( L' \subset X \). The quadric \( Q \) has equation \( x_0x_3 - x_1x_2 = 0 \). Then it is obtained by projecting from \( L' \) the smooth quadric defined by the same equation in some \( \mathbb{P}^3 \) disjoint from \( L' \). Hence on \( Q \) we have two 1-dimensional families of \( \mathbb{P}^3 \)'s \( \{M_\mu\}_{\mu \in \mathbb{P}^1}, \{N_\nu\}_{\nu \in \mathbb{P}^1} \) such that

- each \( M_\mu \) and \( N_\nu \) contains \( L' \);
- \( M_\mu \cap M_\mu' = L' \) and \( N_\nu \cap N_\nu' = L' \);
- \( M_\mu \cap N_\nu \) is a plane.
Lemma 3.4
(i) The quadrics on \( X \) are obtained as \( G_\mu := X \cap M_\mu \);
(ii) \( \{ F_\nu := X \cap N_\nu \} \) is a linear family of cubic surfaces, each containing \( L' \).

Proof. This follows easily from the equations for \( X \), since
\[
x_0 - \mu x_2 = x_1 - \mu x_3 = 0 \quad \text{are equations for} \ M_\mu
\]
and
\[
x_0 - \nu x_1 = x_2 - \nu x_3 = 0 \quad \text{are equations for} \ N_\nu.
\]
\[\square\]

Lemma 3.5 A general quadric \( G_\mu \) and a general cubic \( F_\nu \) intersect in an irreducible conic.

Proof. We have \( G_\mu \cap F_\nu = X \cap (M_\mu \cap N_\nu) \). Therefore, by using equations it is easily seen that \( G_\mu \cap F_\nu \) is the intersection of the plane \( M_\mu \cap N_\nu \) with the irreducible quadric of equation \( \mu F_2 - F_1 = 0 \). \[\square\]

Proposition 3.6 There exists on \( X \) a line which is unisecant the quadrics.

Proof. We fix a general cubic surface \( F_\nu \) and we denote \( C_\mu := G_\mu \cap F_\nu \) the conics mentioned in the previous lemma; finally, we recall that \( L' \subset F_\nu \). The surface \( F_\nu \) cannot be singular along \( L' \) because otherwise the conic \( C_\mu \) would be reducible. On the other hand, the base locus of the linear system \( |F_\nu| \) is the line \( L' \), hence by Bertini the general \( F_\nu \) is smooth outside \( L' \). From this it follows that on \( F_\nu \) there exists a line \( L \) which is disjoint from \( L' \) (see [BL]).

The line \( L \) does not lie on any \( M_\mu \). Assume the contrary, and let \( L \subset M_\mu \). Then \( L \) is contained in the plane \( M_\mu \cap N_\nu \), which already contains \( L' \), a contradiction to \( L \cap L' = \emptyset \).

Therefore, for every \( \mu \in \mathbb{P}^1 \) the line \( L \) intersects \( M_\mu \) at a single point \( U \). Since \( L \subset X \), we have \( U \in X \cap M_\mu = Q_\mu \) and the proof is complete. \[\square\]

Theorem 3.7 Let \( X \) be a Castelnuovo threefold.

(i) There exists a line \( L \subset X \) such that the projection \( \pi_L \) of centre \( L \) from \( X \) to \( \mathbb{P}^3 \) is birational.

(ii) The birational map \( \pi_L^{-1} \) is defined by a linear system of quartic surfaces of \( \mathbb{P}^3 \) whose base locus is the union of a hyperelliptic curve \( B_2 \) of degree 7 and arithmetic genus 3, having a 5-secant line \( B_1 \), with the first infinitesimal neighbourhood of \( B_1 \). Moreover, the exceptional divisors of \( \pi_L \) and \( \pi_L^{-1} \) are respectively \( F_\nu \cup F' \), where \( F' \) is a ruled surface of degree 9, and a rational ruled cubic surface \( \Phi \).

Proof. The first assertion follows directly from 3.1 and 3.6.

To prove (ii) we may apply the results of §2, in particular Remark 2.3. The linear system \( |\Sigma| \) of surfaces of \( \mathbb{P}^3 \) which defines \( \pi_L^{-1} \) is formed by quartics; the base locus \( Bs(|\Sigma|) \) intersects a general plane in 7 simple points and one double point, or, to be more precise, the first infinitesimal neighbourhood of a point. Therefore, \( Bs(|\Sigma|) \) is the union of the first infinitesimal neighbourhood of a line \( B_1 \) with a curve \( B_2 \) of degree 7. A characteristic curve \( \Gamma \) of \( |\Sigma| \) is a quintic of genus 2.
To compute \( p_a(B_2) \) and \( \deg (B_1 \cap B_2) \) we will use the following degeneration argument. Assume that \( |\Sigma| \) contains two reducible surfaces of the form: \( \Sigma_1 = Q_1 \cup Q_2, \Sigma_2 = M \cup D \) where \( Q_1 \) and \( Q_2 \) are quadrics, \( M \) is a plane, \( D \) is a cubic, all containing \( B_1 \), and \( Q_1 \cap D \) contains a characteristic curve \( \Gamma \). Then

\[
\Sigma_1 \cap \Sigma_2 = (Q_1 \cap M) \cup (Q_1 \cap D) \cup (Q_2 \cap M) \cup (Q_2 \cap D) = \\
= (B_1 \cup s) \cup (\Gamma \cup B_1) \cup (B_1 \cup s') \cup (B_1 \cup \Gamma').
\]

In this case \( B_2 \) splits as \( B_2 = \Gamma' \cup s \cup s' \). From this we compute \( p_a(B_2) = 3 \) and \( \deg (B_1 \cap B_2) = 5 \), moreover \( \deg (\Gamma \cap B_1) = 3 \) and \( \deg (\Gamma \cap B_2) = 9 \). In particular, \( B_2 \) is hyperelliptic, because it has degree 7 and possesses the 5-secant line \( B_1 \). Hence \( B_2 \) is contained in two cubic surfaces \( G_1, G_2 \) (see for instance [dA]), and is linked, in the complete intersection of them, to a curve of degree 2 and genus \(-2\): clearly this is a double structure on \( B_1 \), therefore \( G_1 \) and \( G_2 \) are tangent along \( B_1 \).

The surface \( \Phi \) is in this case a rational ruled cubic generated by the chords of \( B_2 \) meeting \( B_1 \). The points of the double line of the base locus come via \( \pi_L \) from conics of planes containing \( L \): they form a surface ruled by conics over \( \mathbb{P}^1 \), whose degree is equal to \( \deg (\Gamma \cap B_1) = 3 \): it is precisely the surface \( F_\nu \) used in 3.6 to construct \( L \). The points of \( B_2 \) come from lines contained in \( X \) and meeting \( L \): they form a ruled surface \( F' \) of degree \( 9 = \deg (\Gamma \cap B_2) \), having \( L \) as a double line. \( \square \)

**Remark 3.8.** The lines on a Castelnuovo threefold \( X \) which are “good centers of projection” form a family of dimension 1.

In fact, the lines we exhibit in the proof of Proposition 3.6 lie on a cubic surface \( F_\mu \subset X \). The surfaces \( F_\mu \) are not ruled since they have at most isolated singularities. Hence, the dimension of the family of lines we are interested in is at least 1.

On the other hand, let us remark first that the only quadrics on \( X \) are the fibres of the adjunction map \( \varphi : X \to \mathbb{P}^1 \) (if \( Q \subset X \) is a quadric such that \( \varphi(Q) = \mathbb{P}^1 \), then \( Q \) intersects the general fibre of \( \varphi \) in a line, and we get a contradiction by a simple computation in \( \text{Pic}(X) \)). A similar argument shows that if \( V \) is a cubic surface on \( X \), then \( \langle V \rangle = \mathbb{P}^3 \). Therefore, the hyperplanes of the pencil of center \( \langle G_\mu \rangle \), where \( G_\mu \) is a general quadric, intersect \( X \) residually to \( G_\mu \) exactly in the cubic surfaces \( F_\nu \).

Now, if \( L \subset X \) is a line which is unisecant the quadrics on \( X \), for a fixed \( G_\mu \), the linear span \( \langle L \cup G_\mu \rangle \) is a hyperplane in \( \mathbb{P}^5 \) and \( L \) lies on the corresponding cubic \( F_\nu \).

**4.- Geometric properties of Bordiga and Palatini scrolls.**

Let \( X \subset \mathbb{P}^5 \) be a Bordiga scroll; it has degree 6 and sectional genus 3. \( X \) can be defined as the degeneracy locus of a generic map of vector bundles \( 3O_{\mathbb{P}^5} \to 4O_{\mathbb{P}^5}(1) \); therefore it is arithmetically Cohen-Macaulay. A Bordiga scroll is a scroll over \( \mathbb{P}^2 \), more precisely \( X = \mathbb{P}(\mathcal{E}) \) where \( \mathcal{E} \) is a rank two bundle over \( \mathbb{P}^2 \) with \( c_1(\mathcal{E}) = 4 \). The scroll map \( f : X \to \mathbb{P}^2 \) is the adjunction map. The hyperplane sections of \( X \) are Bordiga surfaces; in particular, \( S \subset \mathbb{P}^4 \) is the
image of a rational map (birational onto $S$) $\phi : \mathbb{P}^2 \to \mathbb{P}^4$ defined by the linear system $|4H - \sum_{1 \leq i \leq 10} y_i| ([A])$. The inverse of $\phi$ is defined on the whole $S$ and is the adjunction map.

Let $X \subset \mathbb{P}^5$ be a Palatini scroll; it has degree 7 and sectional genus 4. $X$ is arithmetically Buchsbaum; its coherent ideal has a $\Omega$-resolution

\[
\begin{array}{cccccc}
O & \to & 4\mathcal{O}_{\mathbb{P}^5} & \xrightarrow{\alpha} & \Omega^1_{\mathbb{P}^5}(2) & \to & \mathcal{I}_X(4) & \to & 0
\end{array}
\]

where $\alpha$ is generic. A Palatini scroll is a scroll over a smooth cubic surface $V \subset \mathbb{P}^3$, and the scroll map $f : X \to V$ is still the adjunction map. If $S$ denotes a general hyperplane section of $X$ we have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & V \\
\downarrow & & \downarrow \\
S & \xrightarrow{\psi} & \mathbb{P}^2
\end{array}
\]

where $\psi : S \to V$ is the adjunction map for $S$ and $g : V \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at the points $x_1, \ldots, x_6 \in \mathbb{P}^2$. The map $g \circ \psi : S \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at eleven points $x_1, \ldots, x_6, y_1, \ldots, y_5$; more precisely, the linear system of curves in $\mathbb{P}^2$ defining the birational map $(g \circ \psi)^{-1}$ is $|6H - \sum_{1 \leq i \leq 6} 2x_i - \sum_{1 \leq j \leq 5} y_j|$(see [A]).

**Proposition 4.1** If $S$ is a Bordiga surface then $S$ is a hyperplane section of a (smooth) Bordiga scroll of $\mathbb{P}^5$. If $S$ is a smooth, rational non special surface of degree 7 in $\mathbb{P}^4$, then $S$ is a hyperplane section of a unique arithmetically Buchsbaum threefold $X \subset \mathbb{P}^5$, whose ideal has a $\Omega$-resolution like (1). If $S$ is general, then $X$ is smooth, i.e. it is a Palatini scroll.

**Proof.** (The following argument was inspired by [Ch].)

Both the Bordiga and Palatini scrolls are defined as degeneracy loci of a suitable map $\phi : \mathcal{E} \to \mathcal{F}$ of vector bundles over $\mathbb{P}^5$. Let $H \subset \mathbb{P}^5$ be a general hyperplane. From the exact sequence

\[
0 \to \tilde{\mathcal{E}} \otimes \mathcal{F}(-1) \to \tilde{\mathcal{E}} \otimes \mathcal{F} \to \tilde{\mathcal{E}} \otimes \mathcal{F}|_H \to 0
\]

we get the long exact cohomology sequence

\[
\begin{array}{cccccc}
H^0(\tilde{\mathcal{E}} \otimes \mathcal{F}(-1)) & \to & H^0(\tilde{\mathcal{E}} \otimes \mathcal{F}) & \to & H^0(\tilde{\mathcal{E}} \otimes \mathcal{F}|_H) & \to & H^1(\tilde{\mathcal{E}} \otimes \mathcal{F}(-1))
\end{array}
\]

We recall that $H^0(\tilde{\mathcal{E}} \otimes \mathcal{F}) \simeq \text{Hom}(\mathcal{E}, \mathcal{F})$ and $H^0(\tilde{\mathcal{E}} \otimes \mathcal{F}|_H) \simeq \text{Hom}(\mathcal{E}|_H, \mathcal{F}|_H)$.

If $S$ is a Bordiga surface the last term in (3) is zero. Since $S$ is arithmetically Cohen-Macaulay, we conclude that it is a hyperplane section of some smooth Bordiga scroll $X$ ([HTV]). Note, however, that $h^0(\tilde{\mathcal{E}} \otimes \mathcal{F}(-1)) = 12$, hence $X$ is not uniquely determined (this should be compared with the case of the Palatini scroll below).
If $S$ is a smooth, rational non special surface of degree 7 in $\mathbb{P}^4$, by Bott's periodicity formulas, the first and the last term in (3) are zero. Then any map $\overline{\phi} : E|_H \to \mathcal{F}|_H$ is the restriction to $H$ of exactly one map $\phi : \mathcal{E} \to \mathcal{F}$. This can also be seen in down to earth terms as follows. Any global section of $\Omega^1_{\mathbb{P}^n}(2)$ can be identified with a $(n+1) \times (n+1)$ skew-symmetric matrix with entries in the base field $k([O])$. Then $\phi : E \to F$ corresponds to four $6 \times 6$ skew-symmetric matrices. Let $x_5 = 0$ be the equation of $H$ in $\mathbb{P}^5$. Then a section of $\mathcal{F}|_H$ corresponds to a $5 \times 5$ skew-symmetric matrix, plus a linear form $f = a_0 x_0 + a_1 x_1 + \ldots + a_4 x_4$. Starting from $A$ and $f$ we can construct the skew-symmetric $6 \times 6$ matrix

$$
\begin{pmatrix}
  & a_{05} \\
A & \\
-a_{05} & \ldots & -a_{45} & 0
\end{pmatrix},
$$

hence a section of $\Omega^1_{\mathbb{P}^5}(2)$. It is clear how to reverse this procedure and that this yields the identification $\text{Hom}(\mathcal{E}, \mathcal{F}) = \text{Hom}(\mathcal{E}|_H, \mathcal{F}|_H)$.

A general map $\overline{\phi} : \mathcal{E}|_H \to \mathcal{F}|_H$ is the restriction to $H$ of a general map $\phi : \mathcal{E} \to \mathcal{F}$, and we have the desired conclusion that $S$ is a hyperplane section of a (smooth) Palatini scroll. \qed

We collect here three Lemmas, to be used in next Sections. The first one is due to F.L.Zak and S.L.L’vovsky, as well as the part of Lemma 4.3 concerning the Palatini scroll ([ZL]).

**Lemma 4.2** Let $X \subset \mathbb{P}^r$ be an irreducible, non degenerate threefold which is a scroll over the surface $V$, and let $f : X \to V$ be the scroll map. Let $Z \subset X$ be an irreducible surface such that $\dim \langle Z \rangle < r$. Then either $f(Z) \subset V$ is a curve, or $f$ maps $Z$ birationally onto $V$.

**Proof.** If $f(Z)$ is not a curve, then $Z$ is mapped dominantly onto $V$. If this map is not birational, then the intersection of $Z$ with a fibre of $f$ is a zero dimensional scheme of length greater than one. Hence the fibre of $f$, which is a line, is contained in $\langle Z \rangle$. This yields $X \subset \langle Z \rangle$, a contradiction since $X$ is non degenerate. \qed

**Lemma 4.3** There are no planes on the scrolls of Bordiga and Palatini. The Palatini scroll contains exactly 27 quadrics; each of them is smooth and has the form $f^{-1}(R)$, where $R$ is a line on $V$, and $f$ denotes the scroll map.

**Proof.** If $M \subset X$ is a plane, then obviously $f(M)$ cannot be a curve. By Lemma 4.2 we get in any case a birational map $f|_M : \mathbb{P}^2 \to \mathbb{P}^2$, whence $f|_M$ is an isomorphism. In the case of the Palatini scroll we reach immediately a contradiction because $f|_M$ factorizes through the blow-up $V \to \mathbb{P}^2$. In the case of the Bordiga scroll, the plane $M$ cuts on a general hyperplane section $S \subset X$ one of the 10 lines on $S$ (see Lemma 4.5 below). We have the commutative diagram
where $\psi : S \to \mathbb{P}^2$ is the adjunction map for $S$, which is the blow-up of $\mathbb{P}^2$ at ten points. Therefore, any line on $S$ is contracted to a point, namely it is a fibre of $f$. Therefore, $f$ contracts the plane $M$ to a point, a contradiction.

Let $X$ be a Palatini scroll, and let $Q \subset X$ be an irreducible quadric. Since a quadric cannot be mapped birationally onto a smooth cubic surface of $\mathbb{P}^3$, from Lemma 4.2 it follows that $D = f(Q)$ is a rational curve on $V$, and $Q = f^{-1}(D)$.

Now, let $E$ denote the $rk$ 2 vector bundle on $V$ such that $X = \mathbb{P}(E)$; we have $c_1(E) = \mathcal{O}_V(2) ([O])$. Therefore, for any curve $Y \subset V$:

$$deg f^{-1}(Y) = deg \mathbb{P}(E|_Y) = c_1(E) \cdot Y = 2 deg Y.$$  

This shows that $D$ is a line and, conversely, if $D \subset V$ is a line, then $f^{-1}(D)$ is a smooth quadric surface on $X$.  \hfill $\Box$

Let $S$ be a hyperplane section of a Bordiga scroll, and let $u : S \to \mathbb{P}^2$ be the blow-up whose inverse is defined by the linear system of plane curves $|4H_p - \sum_{i=1}^{6} x_i|$. If $R \subset S$ is a line, assume $R = au^*H_p - \sum_{i=1}^{10} b_iE_i$ in the Picard group of $S$, where $E_i = u^{-1}(x_i)$, $a \geq 0$ and $b_i \geq 0$ for any $i$. Since $R$ is properly contained in a hyperplane section of $S$, we can assume, moreover, that $a < 4$. Then $1 = R \cdot H_S = 4a - \sum b_i$. If $a = 1$, three of the points $x_i$ should lie on a line, whereas if $a = 2$ seven points $x_i$ should lie on a conic. Finally, $a = 3$ implies $\sum b_i = 11$; but this is ruled out by (CF3), since $S$ is smooth.

If $C \subset S$ is a conic, then $C$ is also contained in a hyperplane section $H_S$ of $S$ and it is different from $H_S$. Then, if $C = au^*H_p - \sum_{i=1}^{10} b_iE_i$ in $Pic(S)$, we can assume $0 < a < 4$. From $2 = C \cdot H_S = 4a - \sum b_i$ it follows that for $a = 2$ six of the points $x_i$ should lie on a conic. For $a = 3$ all the points $x_i$ should lie on a cubic, which is impossible by (CF3).

A similar argument can be applied to determine the conics lying on a general hyperplane section of a Palatini scroll. We conclude:

**Lemma 4.4**

(i) Let $S$ be a smooth hyperplane section of a Bordiga scroll. If $h^0(H_p - \sum_{i\in\Delta} x_i) = 0$ for $\# \Delta \geq 3$ and $h^0(2H_p - \sum_{i\in\Delta} x_i) = 0$ for $\# \Delta \geq 7$, then the lines on $S$ are only the exceptional divisors of $u$. Moreover, if $h^0(2H_p - \sum_{i\in\Delta} x_i) = 0$ for $\# \Delta \geq 6$, then the conics on $S$ correspond exactly to the 45 lines $x_ix_j \subset \mathbb{P}^2$.

(ii) Let $S$ be a smooth hyperplane section of a Palatini scroll $X$. Assume that the inverse of the blow-up $u : S \to \mathbb{P}^2$ is defined by the linear system $|6H_p - \sum_{i=1}^{6} 2x_i - \sum_{1\leq j \leq 5} y_j|$. If $h^0(H_p - \sum_{i\in\Delta} x_i - \sum_{j\in\Lambda} y_j) = 0$ for
2#Δ + #Λ ⩾ 4 (where Λ ≠ ∅), and if $h^0(2H - \sum_{i\in\Delta} x_i - \sum_{j\in\Lambda} y_j) = 0$ for 2#Δ + #Λ ⩾ 10, then the conics on $S$ are only: the 6 exceptional divisors $u^{-1}(x_i)$, the strict transforms of the 15 lines $x_i x_h$, and the strict transforms of the 6 conics through five of the points $x_i$. In particular, on $S$ we have exactly 27 conics.

By Lemma 4.1 the hypotheses of the above lemma are fulfilled for a general hyperplane section of a general Palatini scroll. Then, from Lemma 4.3 it follows at once

**Corollary 4.5** The only conics on a general Palatini scroll $X$ are those contained in the 27 quadrics on $X$. Hence they form a (reducible) 3-dimensional family of plane curves which do not invade $X$. Moreover, the planes of these conics do not invade $\mathbb{P}^5$.

5.- Existence of a “good” line on the Bordiga scroll.

In this section $X$ will denote a smooth Bordiga scroll and $\phi$ the adjunction map for $X$. As in §3 for the Castelnuovo threefold, we will show the existence on $X$ of a line $L$ such that $\pi_L : X \dasharrow \mathbb{P}^3$ is birational. Then we will construct the linear system $|\Sigma|$.

**Proposition 5.1** Let $L \subset X$ be a line. Let $S$ be a smooth hyperplane section of $X$ containing $L$ and $\phi_S : S \to \mathbb{P}^2$ be the adjunction map. The projection $\pi_L : X \dasharrow \mathbb{P}^3$ is birational if and only if $\phi_S(L)$ is a conic $\gamma$. In this case, $\gamma$ passes through 7 of the points $p_1, \ldots, p_{10}$ which are the images of the ten lines contracted by $\phi_S$.

**Proof.** By Proposition 2.9, $\pi_L$ is birational if and only if deg $\big(\Delta \cap L\big) = 4$, where $\Delta$ is the residual of $L$ in a general curve section of $X$. Since clearly $L$ cannot be a line of the scroll, there are only two possibilities, i.e. either $\phi_S(L)$ is a line and $\phi_S(\Delta)$ is a cubic, or both $\phi_S(L)$ and $\phi_S(\Delta)$ are conics. Only in the second case deg $\big(\Delta \cap L\big) = 4$. Last assertion follows by remarking that the degree of $\phi_S^*(aH - b_1p_1 - \ldots - b_{10}p_{10})$ is $4a - b_1 - \ldots - b_{10}$. □

If $H$ is a hyperplane of $\mathbb{P}^5$, the adjunction map for $X \cap H$ will be denoted by $\phi_H$. Each $H$ determines 10 points of $\mathbb{P}^2$ (not necessarily distinct) are determined: the images of the ten lines contracted by $\phi_H$. Next lemma gives a “bound” on the set of hyperplanes for which these points are not distinct.

**Lemma 5.2** Let $X$ be a Bordiga scroll and $\check{X}$ denote its dual variety. Then a general point of $\check{X}$ represents a hyperplane which is tangent to $X$ at a unique point. The hyperplanes which are tangent to $X$ along a line form a subset of codimension at least 2 in $\check{X}$.

**Proof.** Let us recall that, if dim $\check{X} = 4 - h$, then the contact locus of a general tangent hyperplane is a linear space of dimension $h$. In our case dim $\check{X} = 4$ (see [E]), hence the first assertion follows. Let now $\check{X} \subset X \times \check{X}$ denote the conormal variety of $X$, and let $p, q$ its projections to $X$ and $\check{X}$. Denote by $A$ the subset of $\check{X}$ representing hyperplanes $H$ such that $p(q^{-1}(H))$ is a line, hence
\[ \dim q^{-1}(H) = 1. \] The general fibres of the restriction of \( q: q^{-1}(A) \to A \) have dimension 1, therefore if \( \dim A = 3 \), then \( \dim q^{-1}(A) = 4 \) and \( q^{-1}(A) = \tilde{X} \): a contradiction.

We introduce the regular map \( f: \mathbb{P}^5 \to S^{(10)}(\mathbb{P}^2) \), from the dual of \( \mathbb{P}^5 \) to the tenth symmetric power of the plane, which takes \( H \) to the images of the ten lines contracted by \( \phi_H \). Let \( V \) denote the image of \( f \): it is projective irreducible of dimension 5. Our aim is to show that \( V \) meets the codimension two irreducible subvariety \( W \) of \( S^{(10)}(\mathbb{P}^2) \) parametrizing 10-uples of points, 7 of them lying on a conic.

Let us consider the natural map \( p: (\mathbb{P}^2)^{10} \to S^{(10)}(\mathbb{P}^2) \) and set \( V' = p^{-1}(V), W' = p^{-1}(W) \).

**Lemma 5.3** With the above notation, \( V' \cap W' \neq \emptyset \). Hence \( \dim V' \cap W' \geq 3 \).

**Proof.** We shall use the following result of Debarre ([D]):

Let \( \mathbb{P} \) be a product of projective spaces: \( \mathbb{P} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r} \), and \( Z,Y \) be closed irreducible subvarieties of \( \mathbb{P} \). For any subset \( I = \{i_1, \ldots, i_n\} \neq \emptyset \) of \( \{1, \ldots, r\} \), denote by \( p_I \) the projection \( \mathbb{P} \to \mathbb{P}^I := \mathbb{P}^{n_{i_1}} \times \ldots \times \mathbb{P}^{n_{i_n}} \). If \( \dim p_I(Z) + \dim p_I(Y) \geq \dim \mathbb{P}^I \) for any \( I \), then \( Z \cap Y \neq \emptyset \).

We shall apply this result in the situation \( \mathbb{P} = (\mathbb{P}^2)^{10} \) to a pair of varieties \( Z,Y \) where \( Z \) (resp. \( Y \)) is an irreducible component of \( V' \) (resp. \( W' \)). The Debarre’s condition becomes: \( \text{codim} \ p_I(Y) \leq \dim p_I(Z) \). It is easy to see that \( \text{codim} \ p_I(Y) \leq 2 \) for any \( I \), so we have simply to exclude the possibility that \( \dim p_I(Z) \leq 1 \). Otherwise a general fiber of \( p_I: Z \to p_I(Z) \), corresponding to points \( q_1, \ldots, q_7 \), would have dimension at least 4; i.e. there would be a 4-dimensional family of hyperplane sections of the threefold \( X \) containing the 7 lines contracted to \( q_1, \ldots, q_7 \) by the adjunction map: this possibility is excluded because these lines are pairwise disjoint. \( \square \)

**Theorem 5.4** Let \( X \) be a Bordiga scroll.

(i) There exists a line \( L \subset X \) such that the projection \( \pi_L \) of centre \( L \) from \( X \) to \( \mathbb{P}^3 \) is birational.

(ii) The birational map \( \pi_L^{-1} \) is defined by a linear system of quintic surfaces of \( \mathbb{P}^3 \) whose base locus is the union of the first infinitesimal neighbourhood of a cubic curve \( B_1 \) of arithmetic genus 0 with a curve \( B_2 \) of degree 7 and \( p_\ast(B_2) = 0 \). Moreover, we have \( \deg (B_1 \cap B_2) = 12 \). The exceptional divisors are: for \( \pi_L \) a rational scroll \( F \) of degree 8, unbalanced of type \( (1,7) \), and a rational surface of degree 8; for \( \pi_L^{-1} \) a rational ruled quartic surface \( \Phi \).

**Proof.** By Lemma 5.3 it follows that the above introduced varieties \( V \) and \( W \) have non-empty intersection of dimension \( \geq 3 \). Lemma 5.2 allows to exclude the possibility that the intersection is formed only by 10-uples of non-distinct points. Hence a general 10-uple in \( V \cap W \) represents 10 distinct points, 7 lying on a conic \( \gamma \); it comes via \( f \) from a hyperplane \( H \) such that \( \phi_H^{-1}(\gamma) \) is a line \( L \) as required in (i) by Proposition 5.1.

By the results of §2, \( |\Sigma| \) is a linear system of quintic surfaces, having as
base locus the first infinitesimal neighbourhood of a cubic curve $B_1$ and a curve $B_2$ of degree 7. The base locus is contained in a rational surface $Φ$ of degree 4, with rational plane sections, which is the exceptional divisor for $π_{L}^{-1}$. A general surface like this is the projection of a rational normal surface of $ℙ^5$ from a line and has a cubic of arithmetic genus 0 as singular locus ([C]). Let us consider the surface $F$ defined as the union of the lines of the scroll $X$ intersecting $L$, i.e. $F = φ^{-1}(φ(L))$. It is a smooth rational scroll and $deg F = c_1(Ε)deg φ(L) = 8$.

Since $L$ is a unisecant line of $F$, the surface $F$ is of type $(1, 7)$. The projection centered in $L$ contracts $F$ to a rational curve of degree 7, which is precisely $B_2$. $B_1$ comes from the surface $F'$ generated by the conics contained in $X$ and having $L$ as a chord. If $Γ$ is a characteristic curve of $|Σ|$, of degree 6, the degree of $F'$ is equal to the number of intersections of $Γ$ with $B_1$, i.e. $8 (Γ · Σ = 30 = deg X + 2 deg (Γ ∩ B_1) + deg (Γ ∩ B_2))$.

Finally we get $deg (B_1 ∩ B_2) = 12$ by an easy computation in the Picard group of $Φ'$, regardless $Φ'$ is balanced or not. \[\square\]

**Remark 5.5.** There is only a finite number of lines on a Bordiga scroll $X$ which are “good centers of projection”.

In fact, a hyperplane section $S$ of $X$ contains at most one of such lines, because the adjunction map $φ : S → ℙ^2$ takes any such line to a conic, and any of these conics must contain seven of the points images of the lines contracted by $φ$. Since the hyperplanes in $ℙ^5$ containing a fixed line form a 3-dimensional family, the intersection $V' ∩ W'$ in Lemma 5.3 is purely of dimension 3, and each of its components corresponds to a “good center of projection” on $X$.

**6.- Non-existence of a “good” line on the Palatini scroll.**

The purpose of this section is to prove the following

**Theorem 6.1** Let $X ⊂ ℙ^5$ be a Palatini scroll. Then there does not exist any line $L ⊂ X$ such that the projection $π_L : X → ℙ^3$ is birational.

**Proof.** We shall argue by contradiction.

From the results of § 2, if such a line exists, then the surfaces of the linear system $|Σ|$ have degree 6. In particular, from Remark 2.3 and from (CF4), it follows that the 1-dimensional part $B$ of $Bs(|Σ|)$ decomposes into a curve $B_1$ of degree 5 and the first infinitesimal neighbourhood of a curve $B_2$ of degree 6. Let $S ⊂ X$ be a hyperplane section containing $L$, and let $H ⊂ ℙ^5$ be a hyperplane containing $L$, with $⟨S⟩ ≠ H$. Then $S ∩ H = L ∪ Δ$, where $Δ$ is a rational curve of degree 6, meeting $L$ at 5 points by Proposition 2.9. With the notations of diagram (2) (§4) we set $C := g ◦ ψ(L)$ and $D := g ◦ ψ(Δ)$; the curves $C$ and $D$ are both irreducible and rational and $C · D = 5$. Therefore, either $C$ or $D$ is a line. Since $Δ$ is the variable part of $S ∩ H$, in the case $deg C = 5$ and $deg D = 1$ all the base points $x_1, ..., x_6, y_1, ..., y_5$ lie on $C$. In the case $deg C = 1$ and $deg D = 5$, in order that the curves $D$ form a net, the double points $x_1, ..., x_5$ must lie on $D$ and the line $C$ contains $y_1, ..., y_5$ (actually, both cases are possible by (CF4)).

In the case $deg C = 5$ the curve $C$ has a node at any point $x_i$, then the strict transform of $C$ in $f$, which is nothing but $φ(L)$, is bisecant to any
exceptional divisor $E_i \subset V$ of $f$. Therefore, the line $L$ is bisecant to any of the six quadric surfaces $\varphi^{-1}(E_i)$, which are pairwise disjoint.

In the case $\deg C = 1$ the curve $\varphi(L)$ is bisecant to any line $R_i$ on $V$, which is the strict transform of one of the six conics on $\mathbb{P}^2$ through 5 of the points $x_1, \ldots, x_6$. Hence the line $L$ is bisecant to any of the six quadric surfaces $\varphi^{-1}(R_i)$, which are pairwise disjoint.

By Lemma 2.7, any point of the double curve $B_2$ comes from a conic on $X$ with two points in common with $L$. Therefore, the curve $B_2$ contains the six lines $L_i$ and, by degree, $B_2$ actually coincides with the union of the six lines $L_i$.

Now, the ruled surface $\Phi$ introduced in Proposition 2.2 has degree 5. Since the first infinitesimal neighbourhood of $B_2$ is contained in $\Phi$, we have that $\Phi$ contains the trisecant lines to $B_2$. If we take three lines among the $L_i$’s, there is a quadric surface $Q \subset \mathbb{P}^3$ containing them. Therefore $Q \subset \Phi$, a contradiction since $\Phi$ is irreducible by Proposition 2.4.

□

7.- The inverse problem: sufficient conditions on the base locus.

In the previous sections we showed that on $\mathbb{P}^2 \times \mathbb{P}^1$ and on the Del Pezzo, Castelnuovo and Bordiga threefolds it is always possible to find a line $L \subset X$ such that the projection $\pi_L : X -\to \mathbb{P}^3$ is birational, and we described the base locus $B$ of the linear system defining $\pi_L^{-1}$. In this section we will reverse the point of view by studying the sufficiency of the conditions on $B$. The case of $\mathbb{P}^2 \times \mathbb{P}^1$ is straightforward. It will turn out that in all the other cases curves $B$ with components of degree, arithmetic genus and reciprocal position as in Proposition 1.3.1, Theorem 3.7 and Theorem 5.4 respectively, and satisfying condition (iii) of Theorem 2.12 are the base locus of a linear system of surfaces (of degree 3, 4, 5 respectively) which define birational maps with smooth image; in other words for these curves all the hypotheses of Theorem 2.12 are fulfilled.

**Theorem 7.1.** Let $B \subset \mathbb{P}^3$ be a locally Cohen-Macaulay curve verifying one of the following conditions:

\begin{itemize}
  \item[(DP)] $B$ has degree 5, $p_a(B) = 2$ and the general plane section $Z$ of $B$ satisfies (CF1);
  \item[(C)] $B$ is the union of a curve $B_2$ of degree 7 and arithmetic genus 3, with the first infinitesimal neighbourhood of a line $B_1$ such that $\deg(B_1 \cap B_2) = 5$; the general plane section $Z$ of $B_1 \cup B_2$ satisfies (CF2);
  \item[(B)] $B$ is the union of the first infinitesimal neighbourhood of a cubic curve $B_1$ of arithmetic genus 0 with a curve $B_2$ of degree 7, such that $p_a(B_2) = 0$, and $\deg(B_1 \cap B_2) = 12$; the general plane section $Z$ of $B_1 \cup B_2$ satisfies (CF3').
\end{itemize}

Then $B$ fulfills the assumptions of Theorem 2.12, with $d = 4, 5, 6$ respectively.

**Proof.** The main lines of the proof are the same in all cases. More precisely, we investigate the numerical character of $B$ (see [GP] for the definition and first properties of the numerical character; see also [B]) and we prove that conditions (CF) determine it uniquely. Using the properties of this character we show that
B is arithmetically Cohen-Macaulay and that $H^1(O_B(d-2)) = 0$. Finally, we show that $J_B(d-1)$ is spanned by a liaison argument. We shall give the details only for the Bordiga scroll.

To start, let us compute the arithmetic genus of $B$. We denote the first infinitesimal neighbourhood of $B_1$ by $2B_1$. The arithmetic genus of $2B_1$ is 8; moreover, from $\deg(B_1 \cap B_2) = 12$ it follows $\deg(2B_1 \cap B_2) = 24$. Therefore, we have $p_a(B) = 31$.

Now, let $\Gamma$ be a general plane section of $B$ and $\sigma$ the minimum degree of a plane curve containing $\Gamma$. First of all, we observe that $\sigma = 4$. In fact: $\deg \Gamma = 16$, hence $\sigma \leq 5$; it is easily seen that there is only one possibility for the numerical character of $B$ if $\sigma = 5$, namely: $\chi = (6, 5, 5, 5, 5)$. But the genus of this character is $g(\chi) = 30$ which contradicts $31 = p_a(B) \leq g(\chi)$. On the other hand, $\sigma < 4$ is impossible because, otherwise, $Z$ would be contained in a plane cubic, hence contradicting (CF3').

For $\sigma = 4$ the possibilities for the numerical character $\chi = (n_0, n_1, n_2, n_3)$ of $B$ are collected in the following table:

|   |   |   |   |   |
|---|---|---|---|---|
| a | 10 - r | 4 + r | 4 | 4, $0 \leq r \leq 2$ |
| b | 7 | 7 | 4 | 4 |
| c | 8 | 5 | 5 | 4 |
| d | 7 | 6 | 5 | 4 |
| e | 6 | 6 | 6 | 4 |
| f | 7 | 5 | 5 | 5 |
| g | 6 | 6 | 5 | 5 |

In the cases a), b), c), e), f) the character is disconnected: this implies that $B$ is the union of two subcurves $C_1$ and $C_2$ whose characters depend on the position of the first gap. In the cases a), c), f) the first gap is after $n_0$: hence one of the subcurves is plane, of degree respectively $10 - r, 8, 7$; this contradicts (CF3').

In the case b) the gap is after $n_1$, then one of the subcurves is contained in a quadric; moreover we compute its degree to be 13: this also contradicts (CF3').

In the case c) the gap is after $n_2$: this implies that one of the two subcurves, say $C_1$, is a line, while $C_2$ is contained in a cubic surface and has degree 15. If $C_1 \subset B_1$ then $\Gamma$, and therefore also $Z$, is on a cubic, against (CF3'). If $C_1 \subset B_2$, then $C_2 \supset B_1$ and a general plane section of $C_2$ is contained in a plane cubic $D$, which necessarily has 3 singular points and therefore splits; by considering the various possibilities for the splitting type of $D$, we again get a contradiction to (CF3').

So the character $\chi$ is connected: we want to exclude the possibility d). In fact, if $\chi = (7, 6, 5, 4)$, then $\Gamma$ is contained in two quartics $F, G$. Since $B$ (and therefore also $\Gamma$) is certainly not a complete intersection, by genus reasons, we conclude that $F$ and $G$ have a common component, of degree $h \leq 3$. By [B], Proposition 1.5, $B$ is the union of two subcurves $C_1$ and $C_2$, and precisely: if
If \( h = 3 \), then one of them is a line, and we may argue as in case \( f \); if \( h = 2 \), then one of them has character \((7,6)\) and has therefore degree 12 and is contained in a quadric; if \( h = 1 \), then there is a plane component of degree 7: in both cases we reach a contradiction.

We have proved that \( \chi = (6,6,5,5) \). Now: the speciality of \( B \) is \( \leq n_{\sigma-1}-2 \), so \( h^1(O_B(4)) = 0 \). Moreover \( g(\chi) = 31 = p_a(B) \), which implies that \( B \) is arithmetically Cohen-Macaulay. We get \( h^0(J_B(4)) = 1 \) and \( h^0(J_B(5)) = 6 \). We may perform a liaison of type \((4,5)\), and find that the linked curve has degree 4 and arithmetic genus 1 and is therefore a complete intersection of two quadrics; by mapping cone we get the minimal free resolution of \( J_B \), showing that \( J_B(5) \) is globally generated. \( \square \)

References

[A] J. Alexander: “Surfaces rationnelles non-spéciales dans \( \mathbb{P}^4 \)”, Math. Z. 200 (1988), 87-110

[A1] J. Alexander: “Speciality one rational surfaces in \( \mathbb{P}^4 \)”, in Complex Projective Geometry (G. Ellingsrud, C. Peskine, G. Sacchiero, S. A. Stromme Eds.), Cambridge University Press (1992), 1-23

[dA] J. D’Almeida: “Courbes de l’espace projectif: Séries linéaires incomplètes et multisécantes”, J. reine angew. Math. 370 (1986), 30-51

[B] V. Beorchia: “On the arithmetic genus of locally Cohen-Macaulay space curves”, Intern. J. of Math. 6 (1995), 491-502

[BOSS] R. Braun - G. Ottaviani - M. Schneider - F.-O. Schreyer: “Classification of conic bundles in \( \mathbb{P}^5 \)”, Ann. Scuola Norm. Sup. Pisa 23 (1996), 69-97

[BL] M. Brundu - A. Logar: “Classification of cubic surfaces with computational methods”, Quaderno Matematico n.375, Università di Trieste, (1996)

[BS] M. C. Beltrametti - A. J. Sommese: “Notes on embeddings of blowups”, preprint 1996, to appear in J. of Algebra

[CF] F. Catanese - M. Franciosi: “Divisors of small genus on surfaces and projective embeddings”, in Israel Mathematical Conference Proceedings, Vol.9 (1996), 109-140

[CH] F. Catanese - K. Hulek: “Rational surfaces in \( \mathbb{P}^4 \) containing a plane curve”, preprint (1995)

[Ch] M. C. Chang: “On the hyperplane sections of certain codimension 2 subvarieties in \( \mathbb{P}^m \)”, Arch. Math. 58 (1992), 547-550

[C] F. Conforto: “Le superficie razionali”, Zanichelli, Bologna, 1939

[Co] M. Coppens: “Embeddings of blowing-ups”, in Seminari di Geometria 1991-93, Università di Bologna, 89-100

[D] O. Debarre: “Théorèmes de connexité pour les produits d’espaces projectifs et les Grassmanniennes”, Amer. J. Math. 118 (1996), 1347-1367

[E] L. Ein: “Varieties with small dual varieties, I”, Invent. math. 86 (1986), 63-74
[EGA] A. Grothendieck - J. Dieudonné: *Éléments de Géométrie Algébrique* Chap. IV (Quatrième Partie), Publ. Math. I.H.E.S. 32 1967

[GP] L. Gruson - Ch. Peskine: “Genre des courbes de l’espace projectif”, in Algebraic Geometry, Tromsø1977, LNM 687 (1978), 31-60

[HTV] J. Herzog - N.V. Trung - G. Valla: “On hyperplane sections of reduced irreducible varieties of low codimension”, J. Math. Kyoto Univ. 34 (1994), 47-72

[I] P. Ionescu: “Embedded projective varieties of small invariants, II”, Rev. Roumaine math. pures appl. 31 (1986), 539-544

[I1] P. Ionescu: “Generalized adjunction and applications”, Math.Proc.Camb. Phil.Soc. 99 (1986), 457-472

[J] F. Jongmans: “Les variétés algébriques à trois dimensions dont les courbes - sections ont le genre trois”, Acad. Roy. Belgique, Bull. Cl. Sci. (5), 30 (1943), 766-782, 823-835

[M] U. Morin: “Sui tipi di sistemi lineari di superficie algebriche a curva - caratteristica di genere due”, Ann. Mat. Pura Appl., Ser. IV 19 (1940), 257-288

[M1] U. Morin: “Sulle varietà algebriche a curve - sezioni di genere tre”, Ann. Mat. Pura Appl., Ser. IV 21 (1942), 1-43

[MP] E. Mezzetti - D. Portelli: “Linear systems representing threefolds which are scrolls on a rational surface” in preparation

[O] G. Ottaviani: “On 3-folds in $\mathbb{P}^5$ which are scrolls”, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. Ser. (4) 19 (1992), 451-471

[SR] J. G. Semple - L. Roth: *Introduction to Algebraic Geometry*, Clarendon Press, Oxford, 1949

[Z] O. Zariski: “Foundations of a general theory of birational correspondences”, Trans. Amer. Math. Soc. 53 (1943), 490-542

[ZL] F. L. Zak - S. M. L’vovsky: “Around Palatini variety”, unpublished notes