RANDOMIZED POLYNOMIAL-TIME ROOT COUNTING IN PRIME POWER RINGS

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ABSTRACT. Suppose \( k, p \in \mathbb{N} \) with \( p \) prime and \( f \in \mathbb{Z}[x] \) is a univariate polynomial with degree \( d \) and all coefficients having absolute value less than \( p^k \). We give a Las Vegas randomized algorithm that computes the number of roots of \( f \) in \( \mathbb{Z}/(p^k) \) within time \( d^3(k \log p)^{2+o(1)} \). (We in fact prove a more intricate complexity bound that is slightly better.) The best previous general algorithm had (deterministic) complexity exponential in \( k \). We also present some experimental data evincing the potential practicality of our algorithm.

1. INTRODUCTION

Suppose \( k, p \in \mathbb{N} \) with \( p \) prime and \( f \in \mathbb{Z}[x] \) is a univariate polynomial with degree \( d \geq 1 \) and all coefficients having absolute value less than \( p^k \). Let \( N_{p,k}(f) \) denote the number of roots of \( f \) in \( \mathbb{Z}/(p^k) \) (see, e.g., \([24, 23, 2, 18, 14, 28]\) for further background on prime power rings). Computing \( N_{p,k}(f) \) is a fundamental problem occurring in polynomial factoring \([21, 9, 4, 25, 15]\), coding theory \([3]\), and cryptography \([19]\). The function \( N_{p,k}(f) \) is also a basic ingredient in the study of Igusa zeta functions and algorithms over \( \mathbb{Q}_p \) \([16, 11, 10, 4, 30, 5, 20, 29, 6, 1]\).

In spite of the fundamental nature of computing \( N_{p,k}(f) \), the fastest earlier general algorithms had complexity exponential in \( k \): \([8]\) gave a deterministic algorithm taking time \( (d \log(p) + 2^k)^{O(1)} \). While the \( O \)-constant was not stated in \([8]\), the proof of the main theorem there indicates that the dependence on \( k \) in their algorithm is linear in \( e^k \). Note that counting the roots via brute-force takes time \( dp^k(k \log p)^{1+o(1)} \), so the algorithm from \([8]\) is preferable, at least theoretically, for \( p \geq 3 \). Here, we present a simpler, dramatically faster randomized algorithm (Algorithm 2.3 of the next section) that appears practical for all \( p \).

**Theorem 1.1.** Following the notation above, there is a Las Vegas randomized algorithm that computes \( N_{p,k}(f) \) in time \( kd^3(k \log p)^{1+o(1)} + (dk \log^2 p)^{1+o(1)} \). In particular, the number of random bits needed is \( O(dk \log(dk)) \), and the space needed is \( O(dk \log^2 p) \) bits.

We prove Theorem 1.1 in Section 3 below. In our context, Las Vegas randomized means that, with a fixed error probability (which we can take to be, say, \( \frac{1}{4} \)), our algorithm under-estimates the number of roots. Our algorithm otherwise gives a correct root count, and always correctly announces whether the output count is correct or not. This type of randomization is standard in numerous number-theoretic algorithms, such as the fastest current algorithms for factoring polynomials over finite fields or primality checking (see, e.g., \([2, 17, 7]\)).

At a high level, our algorithm here and the algorithm from \([8]\) are similar in that they reduce the main problem to a collection of computations, mostly in the finite field \( \mathbb{Z}/(p^k) \), indexed by the nodes of a tree with size at worst exponential in \( k \). Also, both algorithms count by partitioning the roots in \( \mathbb{Z}/(p^k) \) into clusters having the same mod \( p \) reduction. Put another way, for each root \( \zeta_0 \in \mathbb{Z}/(p) \) of the mod \( p \) reduction of \( f \), we calculate the number of “lifts” \( \zeta_0 \) has to a root \( \zeta \in \mathbb{Z}/(p^k) \), paying special attention to those \( \zeta_0 \) that are degenerate: The latter kind of root might not lift to a root in \( \mathbb{Z}/(p^k) \), or might lift to as many as \( p^k-1 \) roots in \( \mathbb{Z}/(p^k) \) (see, e.g., Lemma 2.1 below). Another subtlety to be aware of is that we compute the number of roots in \( \mathbb{Z}/(p^k) \), without listing all of them: The number of roots

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in \( \mathbb{Z}/(p^k) \) can be as high as \( \max\{\lfloor d/k \rfloor p^{k-1}, p^{k-\lfloor k/d \rfloor}\} \): Indeed, consider the polynomials \((x^p - x)^k \) (when \( d \geq k \)) and \( x^d \) (when \( k \geq d \)). So we can’t attain a time or space bound sub-exponential in \( k \) unless we do something more clever than naively store every root (see Remark 1.2 below).

In finer detail, the algorithm from [8] solves a “small” polynomial system at each node of a recursion tree (using a specially tailored Gröbner basis computation [12]), while our algorithm performs a univariate factorization in \( (\mathbb{Z}/(p^k))[x] \) at each node of a smaller recursion tree. Our use of fast factorization (as in [17]) is why we avail to randomness, but this pays off: Gaining access to individual roots in \( \mathbb{Z}/(p) \) (as suggested in [8]) enables us to give a more streamlined algorithm.

**Remark 1.2.** von zur Gathen and Hartlieb presented in [14] a randomized polynomial-time algorithm to compute all factorizations of certain \( f \in (\mathbb{Z}/(p^k))[x] \). (Examples like \( x^2 = (x-p)(x+p) \in (\mathbb{Z}/(p^2))[x] \) show that unique factorization fails badly for \( k \geq 2 \), and the number of possible factorizations can be exponential in \( k \).) Their algorithm is particularly interesting since it uses a compact data structure to encode all the (possibly exponentially many) factorizations of \( f \). Unfortunately, their algorithm has the restriction that \( p^k \) not divide the discriminant of \( f \). Their complexity bound, in our notation, is the sum of \( d^k \log(p)(k \log(p) + \log d)^2 \) and a term involving the complexity of finding the mod \( p^k \) reduction of a factorization over \( \mathbb{Z}_p[x] \) (see Remarks 4.10–4.12 from [14]). The complexity of just counting the number of possible factorizations (or just the number of possible linear factors) of \( f \) from their data structure does not appear to be stated. ♦

Creating an efficient classification of the roots of \( f \) in \( \mathbb{Z}/(p^k) \) (also improving the data structure from [14] by removing all restrictions on \( f \)), within time polynomial in \( d + k \log p \), is a problem we hope to address in future work.

For the reader interested in implementations, we have a preliminary Maple implementation of Algorithm 2.3 freely downloadable from [www.math.tamu.edu/~rojas/countpk.map](http://www.math.tamu.edu/~rojas/countpk.map). A few timings (all done on a Dell XPS13 Laptop with 8Gb RAM and a 256Gb ssd, running Ubuntu Linux 14.04) are listed below:

| \( f(x) \) | \( p^k \) | Brute-force | Our Maple code |
|------------|-----------|-------------|----------------|
| \( x^{15} \) | 2^{250} | \( > 2 \times 10^{62} \) years | 0.077sec. |
| Random degree 75 | 10009^{15} | \( > 5 \times 10^{47} \) years | 0.116sec. |
| \( (x - 1234)^3(x - 7193)^4(x - 2030)^{12} \) | 123456791^{1} | 9min. 18sec. | 20.075sec |
| \( (x - 1234)^4(x - 7193)^4(x - 2030)^{12} \) | 123456791^{2} | \( > 2 \times 10^{11} \) years | 40.019sec. |
| \( (x - 1234)^3(x - 7193)^4(x - 2030)^{12} \) | 123456791^{10} | \( > 10^{68} \) years | 45.988sec. |
| \( (x - 1234)^3(x - 7193)^4(x - 2030)^{12} \) | 123456791^{23} | \( > 10^{173} \) years | 1min. 50.323sec. |

Our Maple implementations of brute-force and Algorithm 2.3 here are 5 lines long and 16 lines long, respectively. In particular, our random \( f \) above were generated by taking uniformly random integer coefficients in \( \{0, \ldots, p^k\} \) and then multiplying 5 (or 25) random cubic examples together: This results in longer timings for our code than directly picking a single random polynomial of high degree. The actual numbers of roots in the last 4 examples are respectively 3, 45724737732053043, 6662463731107084597239930491383079081613573366742531838643898960928425583, and

\(^{1}\)The timings in years were based on extrapolating (without counting the necessary expansion of laptop memory beyond 8Gb) from examples with much smaller \( k \) already taking over an hour.
1.1. A Recurrence from Partial Factorizations. Throughout this paper, we will use the integers \(\{0, \ldots, p^k - 1\}\) to represent elements of \(\mathbb{Z}/(p^k)\), unless otherwise specified. With this understanding, we will use the following notation:

**Definition 1.3.** For any \(f \in \mathbb{Z}[x]\) we let \(\bar{f}\) denote the mod \(p\) reduction of \(f\) and, for any root \(\zeta_0 \in \{0, \ldots, p-1\}\) of \(\bar{f}\), we call \(\zeta_0\) degenerate if and only if \(f'(\zeta_0) = 0 \mod p\). Letting \(\text{ord}_p : \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\}\) denote the usual \(p\)-adic valuation with \(\text{ord}_p(p) = 1\), we then define \(s(f, \zeta_0) := \min\left\{i + \text{ord}_p\left(\frac{f(i)(\zeta_0)}{i!}\right)\right\}\). Finally, fixing \(k \in \mathbb{N}\), let us inductively define a set \(T(f, p, k)\) of \(\{f(i, \zeta, k, \zeta)\} \subseteq \mathbb{Z}[x] \times \mathbb{N}\) as follows: We set \((f_{0,0}, k_{0,0}) := (f, k)\). Then, for any \(i \geq 1\) with \((f_{i-1, \mu}, k_{i-1, \mu}) \in T(f, p, k)\) and any degenerate root \(\zeta_{i-1} \in \mathbb{Z}/(p)\) of \(\bar{f}_{i-1, \mu}\) with \(s(f_{i-1, \mu}, \zeta_{i-1}) \in \{2, \ldots, k_{i-1, \mu} - 1\}\), we define \(\zeta := \mu + p^{i-1} \zeta_{i-1}\), \(k_{i, \zeta} := k_{i-1, \mu} - s(f_{i-1, \mu}, \zeta_{i-1})\) and \(f_{i, \zeta}(x) := \left[\frac{1}{p^{s(f_{i-1, \mu}, \zeta_{i-1})}} f(\zeta_{i-1} + px)\right] \mod p^{k_{i, \zeta}}\).

The “perturbations” \(f_{i, \zeta}\) of \(f\) will help us keep track of how the roots of \(f\) in \(\mathbb{Z}/(p^k)\) cluster (in a \(p\)-adic metric sense) about the roots of \(\bar{f}\). Since \(\frac{f(i)(\zeta_0)}{i!}\) is merely the coefficient of \(x^i\) in the Taylor expansion of \(f(x + \zeta_0)\) about 0, it is clear that \(\frac{f(i)(\zeta_0)}{i!}\) is always an integer (under the assumptions above).

We will see in the next section how \(T(f, p, k)\) can be identified with a finite rooted directed tree. In particular, it is easy to see that the set \(T(f, p, k)\) is always finite since, by construction, only \(f_{i, \zeta}\) with \(i \leq \lfloor (k - 1)/2 \rfloor\) and \(\zeta \in \mathbb{Z}/(p)\) are possible (see also Lemma 3.3 of the next section).

**Example 1.4.** Let us take \(p = 3\), \(k = 7\), and \(f(x) := x^{10} - 10x + 38\). Setting \((f_{0,0}, k_{0,0}) := (f, 7), a simple calculation then shows that \(\bar{f}_{0,0}(x) = x(x^9 - 1)\), which has roots \(\{0, 1\}\) in \(\mathbb{Z}/(3). The root 0 is non-degenerate so the only possible \(f_{1, \zeta}\) would be an \(f_{1,1} = f_{1,0} + 1\).

In particular, \(s(f_{0,0}, 1) = 4\) and thus \(k_{1,1} = 3\) and \(f_{1,1}(x) = 21x^4 + 13x^3 + 5x^2 + 9 \mod 3^3\).

Since \(f_{1,1}(x) = x^2(x - 1)\) and 1 is a non-degenerate root of \(\bar{f}_{1,1}\), we see that the only possible \(f_{2, \zeta}\) would be an \(f_{2,1} = f_{2,0}\).

In particular, \(s(f_{1,1}, 0) = 2\), so \(k_{2,1} = 1\) and \(f_{2,1}(x) = 2(x - 1)(x - 2) \mod 3\), which only has non-degenerate roots. So by Definition 1.3, there can be no \(f_{3, \zeta}\) and thus our collection of pairs \(T(f, p, k)\) consists of just 3 pairs.

Using base-\(p\) expansion, there is an obvious bijection between the ring \(\mathbb{Z}_p\) of \(p\)-adic integers and the set of root-based paths in an infinite \(p\)-ary tree. It is then natural to build a (finite) tree to store the roots of \(f\) in \(\mathbb{Z}/(p^k)\). This type of tree structure was studied earlier by Schmidt and Stewart in [26, 27], from the point of view of classification and (in our notation) upper bounds on \(N_{p,k}(f)\). However, it will be more algorithmically efficient to endow our set \(T(f, p, k)\) with a tree structure. The following fundamental lemma relates \(N_{p,k}(f)\) to a recursion tree structure on \(T(f, p, k)\).
Lemma 1.5. Following the notation above, let \( n_p(f_{0,0}) \) denote the number of non-degenerate roots of \( f_{0,0} \) in \( \mathbb{Z}/(p) \). Then, provided \( k \geq 2 \) and \( f \) is not identically 0 mod \( p \), we have

\[
N_{p,k}(f_{0,0}) = n_p(f_{0,0}) + \left( \sum_{\zeta_0 \in \mathbb{Z}/(p)} p^{k-1} \binom{n_{p,0}}{k-1} \right) + \sum_{\zeta_0 \in \mathbb{Z}/(p)} p^{s(f_{0,0},\zeta_0)-1} N_{p,k-s(f_{0,0},\zeta_0)}(f_{1,\zeta_0}).
\]

We prove Lemma 1.5 in the next section, where it will also easily follow that Lemma 1.5 applies recursively, i.e., our root counting formula still holds if one replaces \((f_{0,0}, k, f_{1,\zeta_0}, \zeta_0)\) with \((f_{i-1,\mu, k_{i-1,\mu}, f_{i,\mu+p^{-1}e_{i-1,\zeta_0}}, \zeta_{i-1})\). There we also show how Lemma 1.5 leads to our recursive algorithm (Algorithm 2.3) for computing \( N_{p,k}(f) \). Note that by construction, \( s(f, \zeta_0) \geq 2 \) implies that \( \zeta_0 \) is a degenerate root of \( \tilde{f} \). So the two sums above range over certain degenerate roots of \( \tilde{f} \). Note also that \( N_{p,k}(f) \) depends only on the residue class of \( f \) mod \( p^k \), so we will often abuse the notations \( N_{p,k}(f) \) and \( s(f, \zeta_0) \) by allowing \( f \in (\mathbb{Z}/(p^k))[x] \) as well. The following example illustrates how \( N_{p,k}(f) \) can be computed recursively.

Example 1.6. Revisiting Example 1.4, let \( p = 3 \), \( k = 7 \), and let us count the roots \( \zeta = \zeta_0 + 3\zeta_1 + \cdots + 3^8 \zeta_0 \in \mathbb{Z}/(3^7) \) of \( f_{0,0}(x) := x^{10} - 10x + 738 \). Lemma 1.5 and our earlier computation of \( T(f_{0,0}, p, k) \) then tell us that \( N_{3,7}(f_{0,0}) = 1 + 3^3 N_{3,3}(f_{1,1}) = 1 + 3^1 N_{3,1}(f_{2,1}) \). So we obtain \( N_{3,7}(f_{0,0}) = 1 + 3^7(1 + 3^1 \cdot 2) = 190 \). (Our Maple implementation confirmed this count in under 4 milliseconds.) ♦

While our \( T(f, p, k) \) from our running example has just 3 elements, the earlier tree structure from [26, 27] would have resulted in 190 nodes. We will now fully detail how to efficiently reduce root counting over \( \mathbb{Z}/(p^k) \) to computing \( p \)-adic valuations and factoring in \((\mathbb{Z}/(p))[x]\).

2. Algebraic Preliminaries and Our Algorithm

Let us first recall the following version of Hensel’s Lemma:

Lemma 2.1. (See, e.g., [22] Thm. 2.3, pg. 87, Sec. 2.6) Suppose \( f \in \mathbb{Z}[x] \) is not identically zero mod \( p \), \( k \in \mathbb{N} \), and \( \zeta_0 \in \mathbb{Z}/(p) \) is a non-degenerate root of \( \tilde{f} \). Then there is a unique \( \zeta \in \mathbb{Z}/(p^k) \) with \( \zeta = \zeta_0 \mod p \) and \( f(\zeta) = 0 \mod p^k \).

The following lemma enables us to understand the lifts of degenerate roots of \( \tilde{f} \).

Lemma 2.2. Following the notation of Lemma 2.1, suppose instead that \( \zeta_0 \in \mathbb{Z}/(p) \) is a root of \( \tilde{f} \) of (finite) multiplicity \( j \geq 2 \). Suppose also that \( k \geq 2 \) and there is a \( \zeta \in \mathbb{Z}/(p^k) \) with \( \zeta = \zeta_0 \mod p \) and \( f(\zeta) = 0 \mod p^k \). Then \( s(f, \zeta_0) \in \{2, \ldots, j\} \).

Proof of Lemma 2.2: We may assume, by base-\( p \) expansion that \( \zeta = \zeta_0 + p^i \zeta_1 + \cdots + p^{k-1} \zeta_{k-1} \) for some \( \zeta_1, \ldots, \zeta_{k-1} \in \{0, \ldots, p-1\} \). Note that \( f'(\zeta_0) = 0 \mod p \) since \( \zeta_0 \) is a degenerate root. Note also that \( i + \text{ord}_p \frac{f^{(j)}(\zeta_0)}{j!} \geq 2 \) for all \( i \geq 2 \). Letting \( \sigma := \zeta_1 + p^2 \zeta_2 + \cdots + p^{k-2} \zeta_{k-1} \) we then see that \( f(\zeta) = f(\zeta_0) + f'(\zeta_0) p^i + \cdots + f^{(k-1)}(\zeta_0) p^{k-1} \sigma^{k-1} \mod p^k \). So \( f(\zeta) = 0 \mod p^k \) implies that \( f(\zeta_0) = 0 \mod p^2 \) and thus \( s(f, \zeta_0) \geq 2 \).

To conclude, our multiplicity assumption implies that \( f^{(j)}(\zeta_0) \neq 0 \mod p \). So then \( j + \text{ord}_p \frac{f^{(j)}(\zeta_0)}{j!} \leq j \) and thus \( s(f, \zeta_0) \leq j \).}

We are now ready to state our main algorithm.
Before proving the correctness of Algorithm [2.3] it will be important to prove our earlier key lemma.

**Proof of Lemma [1.5]:** Proving our formula clearly reduces to determining how many lifts each possible root \( \zeta_0 \in \mathbb{Z}/(p) \) of \( \tilde{f}_{0,0} \) has to a root of \( f_{0,0} \) in \( \mathbb{Z}/(p^k) \). Toward this end, note that Lemma [2.1] implies that each non-degenerate \( \zeta_0 \) lifts to a unique root of \( f_{0,0} \) in \( \mathbb{Z}/(p^k) \).

In particular, this accounts for the summand \( n_p(f_{0,0}) \) in our formula. So now we merely need to count the lifts of the degenerate roots.

Assume \( \zeta_0 \in \mathbb{Z}/(p) \) is a degenerate root of \( \tilde{f}_{0,0} \), write \( \zeta = \zeta_0 + p \zeta_1 + \cdots + p^{k-1} \zeta_{k-1} \in \mathbb{Z}/(p^k) \) via base-\( p \) expansion as before, set \( \sigma := \zeta_1 + p \zeta_2 + \cdots + p^{k-2} \zeta_{k-1} \), and let \( s := s(f_{0,0}, \zeta_0) \).

Clearly then, \( f_{0,0}(\zeta) = p^s f_{1,\zeta_0}(\sigma) \mod p^k \) and, by construction, \( f_{1,\zeta_0} \in \mathbb{Z}[x] \) and is not identically 0 mod \( p \).

If \( s \geq k \) then \( f_{0,0}(\zeta) \equiv 0 \mod p^k \) independent of \( \sigma \). So there are exactly \( p^{k-1} \) values of \( \zeta \in \mathbb{Z}/(p^k) \) with \( \zeta = \zeta_0 \mod p \). This accounts for the second summand in our formula.

If \( s \leq k - 1 \) then \( \zeta \) is a root of \( f_{0,0} \) with \( \zeta = \zeta_0 \mod p \) if and only if \( f_{1,\zeta_0}(\sigma) \equiv 0 \mod p^{k-s} \). Also, \( s \geq 2 \) (thanks to Lemma [2.2]) because \( \zeta_0 \) is a degenerate root. Since the base-\( p \) digits \( \zeta_{k-s+1}, \ldots, \zeta_{k-1} \) do not appear in the last equality, the number of possible lifts \( \zeta \) of \( \zeta_0 \) is thus

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1 Here we use the fastest available Las Vegas factoring algorithm over \( (\mathbb{Z}/(p))[x] \) (currently \([17]\)) to isolate the degenerate roots of \( \tilde{f} \). Such factoring algorithms enable us to correctly announce failure to find all the degenerate roots, should this occur. We describe in the next section how to efficiently control the error probability.
exactly $p^{r-1}$ times the number of roots $\zeta_1 + p\zeta_2 + \cdots + p^{k-s-1}\zeta_{k-s} \in \mathbb{Z}/(p^{k-s})$ of $f_{1,0}$. So this accounts for the third summand in our formula and we are done. ■

We are at last ready to prove the correctness of Algorithm 2.3.

**Proof of Correctness of Algorithm 2.3.** Assume temporarily that Algorithm 2.3 is correct when $f_{0,0}$ is not identically 0 mod $p$. Since (for any integers $a, x, y$ with $a \leq k$) $p^a x = p^a y$ mod $p^k \iff x = y$ mod $p^{k-a}$, Steps 1–6 of our algorithm then clearly correctly dispose of the case where $f$ is identically 0 mod $p$. So let us now prove correctness when $f$ is not identically 0 mod $p$. Applying Lemma 1.5 we then see that it is enough to prove that the value of $M$ is the value of our formula for $N_{p,k}(f)$ when the For loops of Algorithm 2.3 runs correctly.

Step 7 ensures that the value of $M$ is initialized as $n_p(f)$. Steps 8–15 (once the For loop is completed) then simply add the second and third summands of our formula to $M$ thus ensuring that $M = N_{p,k}(f)$, provided the For loop has run correctly, along with all the For loops in the recursive calls to RandomizedPrimePowerRootCounting. Should any of these For loops run incorrectly, Steps 16–20 ensure that our algorithm correctly announces an under-count. So we are done. ■

3. **Our Complexity Bound: Proving Theorem 1.1**

Let us now introduce a tree structure on $T(f,p,k)$ that will simplify our complexity analysis.

**Definition 3.1.** Let us identify the elements of $T(f,p,k)$ with nodes of a labelled rooted directed tree $T(f,p,k)$ defined inductively as follows:

1. The root node of $T(f,p,k)$ is labelled $(f_{0,0}, k_{0,0})$.
2. The non-root nodes of $T(f,p,k)$ are uniquely labelled by each $(f_{i,\zeta}, k_{i,\zeta}) \in T(f,p,k)$ with $i \geq 1$.
3. There is an edge from node $(f_{i',\zeta'}, k_{i',\zeta'})$ to node $(f_{i,\zeta}, k_{i,\zeta})$ if and only if $i' = i - 1$ and there is a degenerate root $\zeta_{i-1} \in \mathbb{Z}/(p)$ of $\tilde{T}_{i',\zeta'}$ with $s(f_{i',\zeta'}, \zeta_{i-1}) \in \{2, \ldots, k_{i',\zeta'} - 1\}$ and $\zeta = \zeta' + p^{i-1} \zeta_{i-1}$.
4. The label of a directed edge from node $(f_{i,\zeta}, k_{i,\zeta})$ to node $(f_{i,\zeta}, k_{i,\zeta})$ is $p^{s(f_{i',\zeta'}, \zeta_{i-1})}$.

In particular, the edges are labelled by powers of $p$ in $\{p^1, \ldots, p^{k-2}\}$, and the labels of the nodes lie in $\mathbb{Z}[x] \times \mathbb{N}$. ◦

**Example 3.2.** Letting $f_{0,0}(x) := x^{10} - 10x + 738$ (from Example 1.4) and $g_{0,0}(x) := x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8$, the trees $T(f_{0,0}, 3, 7)$ and $T(g_{0,0}, 17, 100)$ are drawn below, on the left and right respectively.
Suppose Lemma 3.3.

\[ \cdot \]

Lemma 1.5 and \( N \) for all \( i \) no polynomials have \( g \)

\[ \lfloor 1 + \frac{1}{k, p} \rfloor = \frac{100 - x}{2} = 49 \text{ and exactly} \]

1 + \[ 2 \frac{1}{2} \] \( \lfloor (100 - 1)/2 \rfloor = 83 \) nodes. \]

Recall, from Example 1.6, that \( g \)

To count the roots of \( g \), let us first observe that \( g \).

Also, \( N_{17,2}(g_{49,1}) = 17 \) by

\[ \begin{align*}
N_{17,1}(g_{33,2}) &= 1 \text{ trivially. So by Lemma 1.3 once more, } g_{0,0} \text{ has exactly } \\
&= 17 \cdot 17^{100} + 17^{33} = 17^{50} + 17^{66} \text{ roots in } \mathbb{Z}/(17^{100}). \text{ Expanded in base-10, this count is } \\
&= 1620424537653706124196923258781575759359875675913436470380245486276378993995166018. 
\end{align*} \]

The following lemma will be central in our complexity analysis.

**Lemma 3.3.** Suppose \( k, p \in \mathbb{N} \) with \( p \) prime, and \( f \in \mathbb{Z}[x] \) has degree \( d \). Then:

1. The depth of \( T(f, p, k) \) is at most \( \lfloor (k - 1)/2 \rfloor \).
2. The degree of the root node of \( T(f, p, k) \) is at most \( \lfloor d/2 \rfloor \).
3. The degree of any non-root node of \( T(f, p, k) \) labelled \( (f_i, j_i, k_i, \zeta_i) \), with parent \( (f_{i-1}, j_{i-1}, k_{i-1}, \zeta_{i-1}) \) and \( \zeta_{i-1} := (\zeta - \mu)/p^{i-1} \), is at most \( \lfloor s(f_{i-1}, j_{i-1}, k_{i-1}, \zeta_{i-1})/2 \rfloor \). In particular, \( \deg f_i, j_i \leq s(f_{i-1}, j_{i-1}, k_{i-1}) \leq k_{i-1} \) and \( \sum_{(f_{i, j_i}, \zeta_{i-1}) \text{ a child of } (f_{i-1}, j_{i-1}, k_{i-1})} s(f_{i-1}, j_{i-1}, k_{i-1}, \zeta_{i-1}) \leq \deg f_{i-1}, j_{i-1} \).
4. \( T(f, p, k) \) has at most \( \lfloor \frac{d}{2} \rfloor \) nodes at depth \( i \geq 1 \), and thus a total of no greater than \( 1 + \lfloor \frac{d}{2} \rfloor \lfloor \frac{k-1}{2} \rfloor \) nodes.
Proof of Lemma 3.3

Assertion (1): By Definitions 1.3 and 3.1, the labels \((f_{i,\zeta}, k_{i,\zeta})\) satisfy 
\[2 \leq k_{i-1,\mu} - k_{i,\zeta} \leq k_{i-1,\mu} - 1\] 
for any child \((f_{i,\zeta}, k_{i,\zeta})\) of \((f_{i-1,\mu}, k_{i-1,\mu})\), and \(1 \leq k_{i,\zeta} \leq k - 2\) for all \(i \geq 1\). So considering any root to leaf path in \(T(f, p, k)\), it is clear that the depth of \(T(f, p, k)\) can be no greater than \(1 + \lceil(k - 2)/2\rceil\). 

Assertion (2): Since \(\tilde{f}_{0,0} = \tilde{f}\) has degree \(\leq d\), and the multiplicity of any degenerate root of \(\tilde{f}_{0,0}\) is at least 2, we see that \(\tilde{f}_{0,0}\) has no more than \([d/2]\) degenerate roots in \(\mathbb{Z}/(p)\). Every edge emanating from the root node of \(T(f, p, k)\) corresponds to a unique degenerate root of \(\tilde{f}_{0,0}\) (and not every degenerate root of \(\tilde{f}\) need yield a valid edge emanating from the root of \(T(f, p, k)\)), so we are done.

Assertion (3): The degree bound for non-root nodes follows similarly to the degree bound for the root node: Letting \(s := s(f_{i-1,\mu}, \zeta_{i-1})\), it suffices to prove that \(\deg \tilde{f}_{i,\zeta} \leq s\) for all \(i \geq 1\). Note that we must have
\[s = \min_{j \in \{0, \ldots, k_{i-1,\mu} - 1\}} \left\{ j + \ord_{p} \frac{f_{i-1,\mu}(\zeta_{i-1})}{f_{i,\zeta}} \right\},\]
since \(f_{i,\zeta} \in \mathbb{Z}/(p^{k_{i-1,\mu}})[x]\) for \(i \geq 1\). So then, the coefficient of \(x^{\ell}\) in \(f_{i-1,\mu}(\zeta_{i-1} + px)\) must be divisible by \(p^{\ell+1}\) for all \(\ell \geq s + 1\). In other words, the coefficient of \(x^{\ell}\) in \(f_{i,\zeta}(x)\) must be divisible by \(p\) for all \(\ell \geq s + 1\), and thus \(\deg \tilde{f}_{i,\zeta} \leq s\). That \(s \leq k_{i-1,\mu} - 1\) follows from the definition of \(s(f, \zeta)\), and \(k_{i-1,\mu} \leq k\) since \(k_{0,0} := k\) and \(k_{i-1,\mu}\) is a strictly decreasing function of \(i\).

To prove the final bound, note that Lemma 2.2 implies that the term \(s(f_{i-1,\mu}, \zeta_{i-1})\) in the sum is at most the multiplicity of the root \(\zeta_{i-1}\) of \(\tilde{f}_{i-1,\mu}\). Since the sum of the multiplicities of the degenerate roots of \(\tilde{f}_{i-1,\mu}\) is no greater than \(\deg \tilde{f}_{i-1,\mu}\), we are done.

Assertion (4): By Assertion (3), the sum of the degrees of the \(\tilde{f}_{i,\zeta}\) (as \((f_{i,\zeta}, k_{i,\zeta})\) ranges over all depth 1 node labels of \(T(f, p, k)\)) is no greater than \(\deg \tilde{f}_{0,0}\), which is at most \(d\).

By applying Assertion (3) to all nodes of depth \(i \geq 2\), the sum of the degrees of the \(\tilde{f}_{i,\zeta}\) (as \((f_{i,\zeta}, k_{i,\zeta})\) ranges over all depth \(i\) node labels of \(T(f, p, k)\)) is no greater than the sum of the degrees of the \(\tilde{f}_{i-1,\mu}\) (as \((f_{i-1,\mu}, k_{i-1,\mu})\) ranges over all depth \(i - 1\) node labels of \(T(f, p, k)\)).

Since \(\deg \tilde{f}_{0,0} \leq d\) we thus obtain that, for every depth \(i\), the sum of the degrees of the \(\tilde{f}_{i,\zeta}\) (as \((f_{i,\zeta}, k_{i,\zeta})\) ranges over all depth \(i\) node labels of \(T(f, p, k)\)) is no greater than \(d\). So by the final part of Assertion (3), our tree \(T(f, p, k)\) has no more than \([d/2]\) nodes at any fixed depth \(\geq 1\). So by Assertion (1) we are done.

We are at last ready to prove our main theorem.

Proof of Theorem 1.1: Since we already proved at the end of the last section that Algorithm 2.3 is correct, it suffices to prove the stated complexity bound for Algorithm 2.3.

Proving that Algorithm 2.3 runs as fast as stated will follow easily from (a) the fast randomized Kedlaya-Umans factoring algorithm from 17 and (b) applying Lemma 3.3 to show that the number of necessary factorizations and \(p\)-adic valuation calculations is well-bounded.

More precisely, the For loops and recursive calls of Algorithm 2.3 can be interpreted as a depth-first search of \(T(f, p, k)\), with \(T(f, p, k)\) being built along the way. In particular, we begin at the root node by factoring \(\tilde{f}_{0,0} = \tilde{f}\) in \((\mathbb{Z}/(p))[x]\) via 17, in order to find the degenerate roots of \(\tilde{f}\). (Factoring in fact dominates the complexity of the gcd computation that gives us \(n_{p}(f_{0,0})\) if we use a deterministic near linear-time gcd algorithm such as
that of Knuth and Schönhage (see, e.g., [BCS97, Ch. 3]). This factorization takes time $(d^{1.5} \log p)^{1+o(1)} + (d \log^2 p)^{1+o(1)}$ and requires $O(d)$ random bits.

Now, in order to continue the recursion, we need to compute $p$-adic valuations of polynomial coefficients in order to find the $s(f_{0,0}, \zeta_0)$ and determine the edges emanating from our root. Expanding each $f_{0,0}(\zeta_0 + px)$ can clearly be done mod $p^k$, so each such expansion takes time no worse than $d^2(k \log p)^{1+o(1)}$ via Horner’s method and fast finite ring arithmetic (see, e.g., [2, 13]). Computing $s(f_{0,0}, \zeta_0)$ then takes time no worse than $d(k \log p)^{1+o(1)}$ using, say, the standard binary method for evaluating powers of $p$. There are no more than $[d/2]$ possible $\zeta_0$ (thanks to Lemma 3.3), so the total work so far is

$$d^3(k \log p)^{1+o(1)} + (d \log^2 p)^{1+o(1)}.$$

(To simplify our bound, we are rolling multiplicative constants into the exponent, at the price of a negligible increase in the little-$o(\cdot)$ terms in the exponent.)

The remaining work can then be bounded similarly, but with one small twist: By Assertion (4) of Lemma 3.3, the number of nodes at depth $i$ of our tree is no more than $[d/2]$ and, by Assertion (3), the sum of the degrees of the $f_{i,\zeta}$ at level $i$ is no greater than $d$.

Now observe that (for $i \geq 2$) the amount of work needed to compute the $s(f_{i-1,\mu}, \zeta_{i-1})$ at level $i-1$ (which are used to define the polynomials at level $i$) is no greater than $d \cdot d(k \log p)^{1+o(1)}$, and this will be dominated by the subsequent computations of the expansions of the $f_{i,\zeta}$. In particular, by the basic calculus inequality $r_1^t + \cdots + r_t^t \leq (r_1 + \cdots + r_t)^t$ (valid for any $r, t \geq 1$), the total amount of work for the factorizations for each subsequent level of $T(f, p, k)$ will be

$$d^{1.5}(k \log p)^{1+o(1)} + (d \log^2 p)^{1+o(1)},$$

with a total of $O(d)$ random bits needed. The expansions of the $f_{i,\zeta}$ at level $i$ will take time no greater than $d^3(k \log p)^{1+o(1)}$ to compute. So our total work at each subsequent level is then

$$d^3(k \log p)^{1+o(1)} + (d \log^2 p)^{1+o(1)}.$$

So then, the total amount of work for our entire tree will be

$$kd^3(k \log p)^{1+o(1)} + k (d \log^2 p)^{1+o(1)},$$

and the number of random bits needed is $O(dk)$.

We are nearly done, but we must still ensure that our algorithm has the correct Las Vegas properties. In particular, while finite field factoring can be assumed to succeed with probability $\geq 2/3$, we use multiple calls to finite field factoring, each of which could fail. The simplest solution is to simply run our finite field factoring algorithm sufficiently many times to reduce the over-all error probability. In particular, thanks to Lemma 3.3 it is enough to enforce a success probability of $O(\frac{1}{dk})$ for each application of finite field factoring. This implies that we should run the algorithm from [17] $O(\log(dk))$ many times each time we need a factorization over $\mathbb{Z}/(p))[x]$. So, multiplying our last total by $\log(dk)$, this yields a final complexity bound of

$$kd^3(k \log p)^{1+o(1)} + (dk \log^2 p)^{1+o(1)}$$

(since computing the expansions of the $f_{i,\mu}(\zeta_{i-1} + x)$ dominates our complexity) and a total number of $O(dk \log(dk))$ random bits needed.
To conclude, note that as our algorithm proceeds with depth first search, we need only keep track of collections of $f_i \zeta$ occurring along a root-to-leaf path in $T(f, p, k)$. A polynomial of degree $d$ with integer coefficients all of absolute value less than $p^k$ requires $O(dk \log p)$ bits to store, and Lemma 3.3 tells us that the depth of $T(f, p, k)$ is $O(k)$. So we never need more than $O(dk^2 \log p)$ bits of memory. ■

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