Time-evolution of entangled bipartite atomic systems in quantized radiation fields

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Abstract. We analyze the time-evolution of a pair of atoms that are initially entangled and are each of them in an independent isolated QED cavity. The cavities are assumed to contain a radiation bath of a given number of photons. The sudden death and recovering of entanglement are studied in terms of the concurrence introduced by Hill and Wootters. This last gives the value 1 for maximal entanglement and 0 for unentangled systems. We find that the entanglement of the bipartite atomic systems depends on the number of photons in the cavities when the radiation bath is assumed to be in a Fock state.

1. Introduction

Entanglement is a concept appearing in physics after the celebrated discussions deliberated by some of the quantum theory founder fathers (see e.g. [1–7] for a historical account on the matter). The term was introduced in 1935 by Schrödinger [8–10] to describe what occurs with our knowledge of two systems that are separated after they were interacting for a while, and from which we had maximal knowledge before the interaction [8]. According to Schrödinger, after the interaction, the systems can no longer be described in the same way as before. He accepted that, as a result of the interaction, instead of two isolated systems there is just a single composite system and therefore any change to one subsystem would affect the other, no matter the distance between them. In his own words “I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought” [9].

Considering the $\psi$ function as an expectation catalog, it results that one information is sharp while the other must be blurred (uncertainty principle) [11]. Wondering whether the blurring can affect also the macroscopic systems, Schrödinger illustrated a “ridiculous” situation in which a cat entangled to an unstable atom enters into a superposed state of “being alive” and “being dead” [8]. This humorous tale is nowadays known as the Schrödinger’s cat problem and has been the subject of polemic discussions over the years [5] (see also [7,11–13]). However, the results obtained from interaction-free measurements [14,15] are an indication that entanglement is indeed a fundamental property of the quantum systems. The quantum electrodynamics (QED) cavity experiments [16,17], on the other hand, illustrate the actuality of the superposed atomic energy states and the field quantization [16], as well as the reality of decisive though experiments just as the observation of a single photon in a cavity [17]. The latter results are a consequence of the entanglement between the energy states of the atom and the radiation bath in the cavity.

In general, the characterization of the differences between pure, mixed and maximally entangled states is a timely issue in contemporary physics; consider for example that the
entangled states represent a fundamental resource in quantum information, teleportation and computing [18–21]. Interestingly, the entangled mixed states could make a big difference in the usefulness of quantum correlations [22, 23]. The pairs of correlated photons produced from parametric down conversion, for instance, can be efficiently concentrated by simultaneously increasing both the purity and the degree of entanglement [22]. Besides, the mixed states can be more entangled in terms of the relative entropy of entanglement than the pure states if the appropriate measurements are done [23]. A fundamental question is then to what extent is a given quantum state entangled? The answer seems to be not unique [21, 24–26], though the concurrence of two qubits introduced by Hill and Wootters represents a good option [27, 28] (some generalizations can be found in e.g. [29–32]). Concurrence is a measure of the degree of entanglement that takes the value 1 for maximal entanglement and 0 for completely unentangled systems. This has been successfully applied in combination of the Jaynes-Cummings (JC) model [33] to the study of the entanglement of a pair of two-level atoms in QED cavities [34–39].

The interaction between a resonant cavity mode and a two-level atom can be analyzed in detail by using the JC model. Introduced in 1963, this model has been extended to be applicable in a diversity of situations (see e.g. [34–39] and references quoted therein). For instance, a “double JC” model consisting of a pair of two-level atoms, each one in its own perfect one-mode near-resonant cavity and isolated from the other atom and cavity, has been used to investigate the involved entanglement dynamics in [34]. By tracing over the cavity modes one arrives at a mixed state of the atoms, the entanglement of which can be conveniently measured by the Hill and Wooters concurrence. The results included in [34] assume that initially both cavities are prepared in the vacuum state and the atoms are prepared in a pure entangled state. Therefore, there is never more than one photon in the cavities. Using this model the so called sudden death effect is found as a result of the time-evolution of the system [34] (see also [35]). Although these results were immediately generalized to the case of six pairwise concurrences [36], the constraint of having no more than one photon in the cavities was preserved; similar results can be found in [37]. The case of an imperfect matching of the atoms to the cavity fields has been also analyzed in connection to the entanglement evolution [38]. A different situation has been reported in [39] where the case of a pair of two-level atoms are embedded in a common bath. There, it is found that using a ‘hierarchy equation method’ the rotating-wave approximation is not necessary to analyze the entanglement of the atoms. Indeed, the case of two photons in the cavities can be studied by including the counter-rotating-wave terms. It is also remarkable that the concurrence oscillates in time even though the initial state of the atoms is separable and the fields in cavities are in the vacuum state [39]. In almost all the previous works the vacuum is included as the initial condition of either one or both of the fields in the cavities as a manner of simplification in the calculations. The more elaborated initial condition includes no more than one photon in each of the cavities. A natural question arises: is the initial number of photons in each of the cavities playing a relevant role in the time-evolution of the atoms’ entanglement?

In this contribution we are interested in a double JC model consisting of a pair of atoms, each one in its own cavity containing a given initial number of photons and isolated from the other atom and cavity. The initial number of photons in the cavities can be either the same or substantially different and will be used as a tuning parameter. Our interest is focused on the time-evolution of the entanglement between the two atoms as a function of such a parameter. With this aim we are going to use the Hill and Wooters concurrence. As we shall see, the results show that the initial number of photons is fundamental in defining the concurrence. The paper is organized as follows. In Section 2 we first make a minimal account of the mathematics associated to the concepts of entanglement and concurrence. As we are interested in the interaction between atoms and photons, in Section 3 we use the Jaynes-Cummings model to determine the Hamiltonian of the atom+field system and solve the related Schrödinger equation (Section 3.1). These results are used in Section 3.2 to get the solution of the bipartite atomic
difference between the pure state and the ground state. The second result leads to the opposite distribution of atomic states. The main problem and to calculate the concurrence of the two entangled atoms when they are located in different isolated cavities. Some conclusions are given at the very end of this work.

2. Entanglement and Concurrence

Consider a bipartite system \( S = S_1 + S_2 \) which is a composition of a pair of two-level atoms \( S_1 \) and \( S_2 \). Let us label the excited and ground pure states of each of these atoms as \( |+\rangle_i \) and \( |-\rangle_i \), \( i = 1, 2 \), respectively. The related two-dimensional Hilbert spaces are then written as

\[
\mathcal{H}_1 = \text{Span}\{|+\rangle_1, |-\rangle_1\}, \quad \mathcal{H}_2 = \text{Span}\{|+\rangle_2, |-\rangle_2\}.
\]

(1)

The Hilbert space of the entire system atom+atom is then 4-dimensional and this is integrated by all the possible linear combinations of the Kronecker product between the basis elements of \( \mathcal{H}_1 \) and those of \( \mathcal{H}_2 \), that is

\[
\mathcal{H} = \text{Span}\{|+\rangle_1 \otimes |+\rangle_2, |+\rangle_1 \otimes |-\rangle_2, |-\rangle_1 \otimes |+\rangle_2, |-\rangle_1 \otimes |-\rangle_2\}.
\]

(2)

The general properties of the Kronecker product between vectors and/or operators can be consulted in [40] and references quoted therein. From now on we adhere to the simplified notation

\[
|+\rangle_1 \otimes |+\rangle_2 \equiv |\varphi_1\rangle, \quad |+\rangle_1 \otimes |-\rangle_2 \equiv |\varphi_2\rangle, \\
|-\rangle_1 \otimes |+\rangle_2 \equiv |\varphi_3\rangle, \quad |-\rangle_1 \otimes |-\rangle_2 \equiv |\varphi_4\rangle.
\]

(3)

Any pure state of the atom+atom system is represented as a linear combination of the basis (3). However, it is well known that not all these combinations can be factorized as the \( \otimes \)-product of a vector in \( \mathcal{H}_1 \) with a vector in \( \mathcal{H}_2 \). For instance, the Bell state

\[
|\beta\rangle = \frac{1}{\sqrt{2}} (|+\rangle_1 \otimes |+\rangle_2 + |-\rangle_1 \otimes |-\rangle_2) = \frac{1}{\sqrt{2}} (|\varphi_2\rangle + |\varphi_3\rangle),
\]

(4)

is such that there are not \( |\phi^{(1)}\rangle \in \mathcal{H}_1 \) and \( |\phi^{(2)}\rangle \in \mathcal{H}_2 \) fulfilling

\[
|\beta\rangle = |\phi^{(1)}\rangle \otimes |\phi^{(2)}\rangle.
\]

(5)

The statement is easily verified by taking \( \mathcal{H}_1 \ni |\phi^{(1)}\rangle = \alpha_1 |+\rangle + \alpha_2 |-\rangle \) and \( \mathcal{H}_2 \ni |\phi^{(2)}\rangle = \gamma_1 |+\rangle + \gamma_2 |-\rangle \), with the \( \alpha \)'s and \( \gamma \)'s to be determined such that (5) holds. Then, one arrives at the system

\[
\alpha_1 \gamma_1 = \alpha_2 \gamma_2 = 0, \quad \alpha_1 \gamma_2 = \alpha_2 \gamma_1 = \frac{1}{\sqrt{2}},
\]

(6)

that admits not solution. In contraposition to the basis vectors (3) which are defined as the Kronecker products of the bases of the subsystems, the vectors that cannot be factorized just like (4) encode information about the correlation of the subsystems. Namely, \( |\beta\rangle \) represents a state of the atom+atom system that has \( \frac{1}{2} \) probability of being found in the configuration \( |+, -\rangle \) and \( \frac{1}{2} \) probability of being found in \( |-, +\rangle \). If the measurement device carries out the first result then we know with certainty that the atom \( S_1 \) is in the excited state while the atom \( S_2 \) is in the ground state. The second result leads to the opposite distribution of atomic states. The main difference between the pure state \( |+, -\rangle \) (equivalently \( |-, +\rangle \)) and the Bell vector (4) is that in the first case we know all about the system: \( S_1 \) is in \( |+\rangle \) and \( S_2 \) in \( |-\rangle \). (just because either we have ‘prepared them in’ or we have ‘measured them in’ those states). The Bell state \( |\beta\rangle \), on the other hand, represents the situation in which we know only that the subsystems are in orthogonal states with the configurations and probabilities indicated above, but we are not able to say any
more before the measurement. In both cases we deal with pure states which are represented by vectors in $\mathcal{H}$; the correlation between the subsystems is encoded in the orthogonality of their states. The usual mathematical definition of entangled pure states says that these are the states of a multipartite system that cannot be factorized as the Kronecker product of the subsystems’ factorized state. Here the parameter $
abla_i$ is obtained for pure states. All other pure states are named separable.

The bipartite atomic system $S = S_1 + S_2$, however, may not be in a pure state. In general, it would be in a statistical ensemble of pure states $\{p_k, \varphi_k\}_{k=1}^4$, where each pure state $\varphi_k$ occurs with probability $0 \leq p_k \leq 1$ and can be represented either by $|\varphi_k\rangle \in \mathcal{H}$ or by the orthogonal projector $p_k = |\varphi_k\rangle\langle\varphi_k|$. In this case, the mixed state $\{p_k, \varphi_k\}_{k=1}^4$ can be expressed as the convex combination

$$\rho = \sum_{k=1}^4 p_k \rho_k, \quad \sum_{k=1}^4 p_k = 1, \quad 0 \leq p_k \leq 1. \quad (7)$$

It is easy to verify that $\rho_k$ as well as $\rho$ fulfill the conditions to be density operators, that is they are (i) Hermitian (ii) positive definite and (iii) normalized operators (i.e., $\text{Tr} \rho = 1$). If $p_k = 1$ for a given $k \in \{1, 2, 3, 4\}$ then (7) is reduced to the orthogonal projector $\rho_k$ that fulfills $\text{Tr} \rho_k^2 = 1$, this last verifies that $|\varphi_k\rangle$ as well as $\rho_k$ represent the same pure state $\varphi_k$. In the case of a nontrivial combination (i.e., $0 < p_k < 1$ for at least two different values of $k$) one shows easily that $\text{Tr} \rho^2 < 1$. Thus, nontrivial convex combinations (7) represent mixed states of the bipartite atomic system. The intersection of all the possible convex combinations (7) that can be formed with the pure states $\rho_k$ as extreme points is the convex hull of the set $\{p_k\}_{k=1}^4$ and represents the state space of the bipartite atomic system we are dealing with. As a convex body, this state space can be visualized as a tetrahedron in which different geometric regions can be identified with different physical properties [21] (see also [41, 42]). Notice that the density operator $\rho$ so constructed is such that the probabilities $p_k$ and the pure states $|\varphi_k\rangle$ are the solutions of the eigenvalue equation $\rho |\varphi_k\rangle = p_k |\varphi_k\rangle$. That is, (7) is the spectral decomposition of $\rho$. Then one says that the mixed state $\rho$ is separable.

In a more general situation the density operator associated to the bipartite atomic state is not necessarily in its canonical (diagonal) form but it reads [40]:

$$\rho = \sum_{i,j=1}^4 \lambda_{i,j} X^{i,j}_{4}, \quad \lambda_{i,j} \in \mathbb{C}, \quad (8)$$

where the operators $X^{i,j}_{4}$ produce the transition from the pure state $|\varphi_j\rangle$ to the pure state $|\varphi_i\rangle$. The simplest representation of $X^{i,j}_{4}$ is a square matrix of order 4 that has entry 1 in position $(i,j)$ and zero in all other entries. These operators are named after Hubbard; their properties and applications can be consulted in [40] and references quoted therein. Remark that only the diagonal Hubbard operators are connected to pure states: $X^{i,i}_{4} \equiv \rho_k$. The $\lambda$-coefficients in (8) are in general different from zero and are defined in correspondence to the external fields that are applied on the bipartite atomic system [41]. When (8) cannot be reduced to the spectral decomposition (7) it is said that the state $\rho$ is entangled.

In what follows we assume we can prepare entangled bipartite atomic states by producing an interaction (somehow) between them. After the interaction, the entire system is in the state

$$|\psi_\alpha\rangle = \cos \alpha |+, -\rangle + \sin \alpha |-, +\rangle, \quad \alpha \in [0, \pi/2]. \quad (9)$$

Here the parameter $\alpha$ is a primary manifestation of entanglement since $\alpha = 0$ produces the factorized state $|+, -\rangle$ while $\alpha = \pi/2$ leads to $|-, +\rangle$. The equally weighted entangled state (4) is obtained for $\alpha = \pi/4$. Then $|\psi_{\alpha=0}\rangle = |+, -\rangle$ and $|\psi_{\alpha=\pi/2}\rangle = |-, +\rangle$ are not entangled.
Figure 1. Schematic representation of the preparation of entangled bipartite atomic states. In a first step a pair of two-level atoms collide in such a manner that their energy states become correlated. The state of the atom+atom system is given by Equation (9). Then each of the atoms is placed in isolated cavities where a radiation bath is present. We assume that the radiation in the cavities is quantized and described by the Fock states $|r⟩$ and $|s⟩$ respectively.

states while $|ψ_{α=π/4}⟩ = |β⟩$ is a maximally entangled state of the bipartite atomic system. Other intermediary states can be prepared. For example, the linear combinations

$$|ψ_{α=1π/20}⟩ = \sin \left( \frac{9}{20}π \right) |+,−⟩ + \cos \left( \frac{9}{20}π \right) |−,+⟩ \equiv |φ₊⟩,$$

$$|ψ_{α=9π/20}⟩ = \cos \left( \frac{9}{20}π \right) |+,−⟩ + \sin \left( \frac{9}{20}π \right) |−,+⟩ \equiv |φ₋⟩,$$

(10)

correspond to situations in which the system is prepared to be found in a specific configuration after the measuring of the state in the majority of the times. Namely, $|φ₊⟩$ represents a system having a probability of $\sin^2 \left( \frac{9}{20}π \right) ≈ 0.976$ to be found in the state $|+,−⟩$, and a probability of $\cos^2 \left( \frac{9}{20}π \right) ≈ 0.024$ to be found in the state $|−,+⟩$. If the system is in the linear combination $|φ₋⟩$ then the probabilities are interchanged. Once we have the atoms in the state (9) we place each of them in isolated cavities where a radiation bath is present. The radiation in cavities is quantized and prepared in the Fock states $|r⟩$ and $|s⟩$, so that the atoms interact with $r$ and $s$ photons respectively. The situation is depicted in Figure 1. Our problem is to determine the time-evolution of the atoms in the cavities and to evaluate the degree of entanglement as a function of the time.

Following [27, 28], given a two-qubit density operator $ρ$, the concurrence $C(ρ)$ is calculated from the expression

$$C(ρ) = \max \left\{ 0, \sqrt{λ₁} - \sqrt{λ₂} - \sqrt{λ₃} - \sqrt{λ₄} \right\},$$

(11)

where the $λ$’s are sorted in decreasing order and correspond to the eigenvalues of the matrix $\bar{ρ}(σ_2 ⊗ σ_2)ρ(σ_2 ⊗ σ_2)$, with $\bar{ρ}$ the complex conjugation of the matrix elements of $ρ$. The spin flip operator $σ_2 ⊗ σ_2$ preserves entangled states while cancels unentangled states$^1$. As a manner of example, the concurrence associated to the pure state

$$|φ⟩ = \sum_{k=1}^{4} a_k |φ_k⟩, \quad a_k ∈ C,$$

(12)

$^1$ For instance, the action of the spin flip on the factorized vector $|+,−⟩$ gives $⟨+,−| (σ_2 ⊗ σ_2)|+,−⟩ = 0$. 


reads as follows [24]:

\[ C (|\varphi\rangle\langle\varphi|) = 2|a_1a_4 - a_2a_3|. \] (13)

Setting \( a_1 = a_4 = 0, \) and \( a_2 = a_3 = 1/\sqrt{2}, \) the Bell state (4) is recovered and the concurrence yields the maximum value \( C (|\varphi\rangle\langle\varphi|) = 1. \) If the state (12) is separable, the coefficients satisfy \( a_1a_4 - a_2a_3 = 0 \) and the concurrence is in this case equal to zero. To get the state (9) we make \( a_2 = \cos \alpha, \) \( a_3 = \sin \alpha, \) and \( a_1 = a_4 = 0. \) Then the concurrence is \( C (|\psi_{\alpha}\rangle\langle\psi_{\alpha}|) = 2|\sin 2\alpha|, \) which is maximum for the equally weighted entangled state \( |\psi_{\alpha=\pi/4}\rangle = |\beta\rangle. \)

3. The Jaynes-Cummings Model

In this section we consider the interaction of a two-level atom with a single mode of quantized radiation through the dipolar interaction term. The approach is a milestone introduced in quantum optics by Jaynes and Cummings [33]. We start with the atomic Hamiltonian given by

\[ \tilde{H}_a = \frac{\hbar \omega_a}{2} \sigma_3, \] (14)

where \( \omega_a \) is the transition frequency and \( \sigma_3 \) is the Pauli matrix

\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (15)

This last together with the spin-1/2 ladder operators

\[ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \] (16)

close the \( su(2) \) algebra. The operators (15) and (16) are in their simplest representation for which

\[ |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (17)

are orthonormal eigenvectors of \( \sigma_3 \) and span the state space of the atom \( \mathcal{H}_a = \text{Span}\{|+\rangle, |-\rangle\}. \) It is straightforward to verify that the following properties hold

\[ \sigma_+^2 = \sigma_-^2 = 0, \quad \sigma_+^\dagger = \sigma_-, \quad \sigma_\pm |\pm\rangle = 0, \quad \sigma_\pm |\mp\rangle = |\pm\rangle. \] (18)

On the other hand, a single mode radiation field can be written in terms of the boson ladder operators

\[ \bar{E}(t) = e \left(a e^{-i\omega_f t} + a^\dagger e^{i\omega_f t}\right) e_f, \] (19)

where \( \omega_f \) and \( e_f \) are respectively the frequency and polarization of the field. The observable of the energy is the oscillator’s Hamiltonian

\[ \bar{H}_f = \hbar \omega_f (N + 1/2). \] (20)

Hereafter we use the Fock’s representation for the operators associated to the field if there is no confusion. That is, the number 1/2 in (20) should read (1/2)\( \mathbb{1}_f \), with \( \mathbb{1}_f \) the identity operator defined on the Hilbert space \( \mathcal{H}_f = \text{Span}\{|n\rangle\}_{n \geq 0}, \) where the nth Fock state \( |n\rangle \) is given by

\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle, \quad n \in \mathbb{Z}^+. \] (21)
The ladder operators $a$, $a\dagger$, and the number operator $N$ satisfy the boson algebra
\[
[a, a\dagger] = 1, \quad [N, a] = -a, \quad [N, a\dagger] = a\dagger,
\]
and are defined in the Fock’s representation. That is
\[
a|n\rangle = \sqrt{n}|n-1\rangle, \quad a\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle.
\]
The energy of the entire system for no interaction between the atom and the field reads:
\[
\tilde{H}_0 = \hbar\omega_f I_a \otimes \left( N + \frac{1}{2} \right) + \frac{\hbar\omega_a}{2}\sigma_3 \otimes I_f = \begin{pmatrix}
\hbar\omega_f(N + 1) + \frac{\hbar\Delta}{2} & 0 \\
0 & \hbar\omega_f N - \frac{\hbar\Delta}{2}
\end{pmatrix},
\]
with $I_a$ the identity operator in the state space of the atom, $\Delta := \omega_a - \omega_f$ the detuning between the radiation field and the atomic transition. For each $n \geq 1$ the Hamiltonian (24) has two eigenvectors
\[
|+, n\rangle := |\rangle \otimes |n\rangle = \left( |\rangle \right)_n, \quad |-, n + 1\rangle := |\rangle \otimes |n + 1\rangle = \left( |\rangle \right)_{n+1},
\]
where $|\rangle$ is the null vector in $H_f$. The corresponding eigenvalues are given by
\[
\tilde{H}_0|+, n\rangle = \hbar \left[ \omega_f(n + 1) + \frac{\Delta}{2} \right]|+, n\rangle,
\quad \tilde{H}_0|-, n + 1\rangle = \hbar \left[ \omega_f(n + 1) - \frac{\Delta}{2} \right]|-, n + 1\rangle.
\]
The vector $|-, 0\rangle := |\rangle \otimes |0\rangle$ is the eigenstate belonging to the eigenvalue 0. Therefore, the Hilbert space $\mathcal{H}$ of the atom+field system splits into a direct sum of subspaces $\Xi_n = \text{Span}\{|+, n\rangle, |-, n+1\rangle\}$, $n = 0, 1, 2 \ldots$. In addition, the one-dimensional space $\Xi_\ast = \text{Span}\{|-, 0\rangle\}$ is required to make the Hilbert space $\mathcal{H}$ complete [41, 42]. Thus, we have
\[
\mathcal{H} = \Xi_\ast \oplus \Xi_0 \oplus \Xi_1 \oplus \Xi_2 \oplus \ldots = \bigoplus_{n=\ast,0}^{\infty} \Xi_n.
\]
Two vectors belonging to different subspaces $\Xi_m$ and $\Xi_n$ ($m, n = 0, 1, 2, \ldots$) are orthogonal
\[
\langle +, m |+, n \rangle = \langle -, m + 1 |-, n + 1 \rangle = \delta_{m,n}, \quad \langle +, m |-, n + 1 \rangle = \langle -, m + 1 |+, n \rangle = 0.
\]
Remark as well that $|-, 0\rangle$ is orthogonal to any $\Xi_n$, $n = 0, 1, 2, \ldots$.

If now the interaction is allowed, assume that this is ruled by the dipolar term $\tilde{H}_I = \vec{p} \cdot \vec{E}$ which, in the rotating wave approximation is given by
\[
\tilde{H}_I = \hbar \lambda (\sigma_+ \otimes a + \sigma_- \otimes a\dagger),
\]
where $\lambda$ is the coupling constant. At resonance ($\Delta = 0$) the Hamiltonian of the whole system $\tilde{H} = \tilde{H}_0 + \tilde{H}_I$ can be rewritten in dimensionless form $H = \tilde{H}/(\hbar\omega)$ to read
\[
H = H_0 + H_I, \quad H_0 = \mathbb{I}_a \otimes \left( N + \frac{1}{2} \right) + \frac{1}{2}\sigma_3 \otimes I_f, \quad H_I = \gamma (\sigma_+ a + \sigma_- a\dagger), \quad \gamma = \lambda/\omega.
\]
Notice that $[H_0, H_I] = 0$, so the basis (25) is common to $H_0$ and $H_I$. 
3.1. Dynamical evolution

The Schrödinger equation

\[ i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \]  

(31)

with \( H \) defined in (30) and \( t \) the dimensionless time-parameter, can be solved by time-translations of the initial vector state \( |\psi(t)\rangle = U(t)|\psi(0)\rangle \). In this case the time evolution operator \( U(t) \) is given by

\[ U(t) = e^{iHt} = e^{i(H_0 + H_I)t} = e^{iH_0 t} e^{iH_I t}. \]  

(32)

We are considering the time-evolution of a given definite state of the energy, so that the operator \( e^{iH_0 t} \) leads to a global phase in the solution, which will be not considered. Then

\[ U(t) = e^{iH_I t} = \begin{pmatrix} \cos(t\gamma \sqrt{a a^\dagger}) & -i \frac{1}{\sqrt{a a^\dagger}} \sin(t\gamma \sqrt{a a^\dagger}) \rangle a^\dagger \\ -i \frac{1}{\sqrt{a a^\dagger}} \sin(t\gamma \sqrt{a a^\dagger}) \rangle a & \cos(t\gamma \sqrt{a a^\dagger}) \]  

(33)

Consider the initial situation in which the atom is in the excited state \( |+\rangle \) and the cavity is filled with \( k \) photons. The time-evolved state of the composite system is

\[ |\psi(t)\rangle = U(t)|+,k\rangle = \cos(\gamma t \sqrt{k + 1}) |+,k\rangle - i \sin(\gamma t \sqrt{k + 1}) |-,k + 1\rangle. \]  

(34)

The calculation of the corresponding atomic population inversion yields

\[ \langle \sigma_3(t) \rangle = \cos(2\gamma t \sqrt{k + 1}), \]  

(35)

so that whole system oscillates in time between the states \( |+,k\rangle \) and \( |-,k + 1\rangle \) according to the Rabi frequency \( 2\gamma \sqrt{k + 1} \).

3.2. Two atoms dynamics and entanglement

The Hamiltonian of two non-interacting atoms that are placed in separate cavities is of the form

\[ H = H_1 + H_2, \]

\[ H_1 = H_{0:1} + H_{I:1}, \quad H_{0:1} = N_1 + \frac{1}{2} + \frac{1}{2} \sigma_3, \quad H_{I:1} = \gamma_1 (\sigma_+ a + \sigma_- a^\dagger), \]  

\[ H_2 = H_{0:2} + H_{I:2}, \quad H_{0:2} = N_2 + \frac{1}{2} + \frac{1}{2} \sigma_3, \quad H_{I:2} = \gamma_2 (s_+ b + s_- b^\dagger), \]  

(36)

where \( N_i, i = 1, 2 \), is the number operator in the \( i \)th cavity. The boson ladder operators \( b, b^\dagger \), and the atomic \( su(2) \) generators \( \{s_3, s_\pm\} \) are associated to the second cavity. Using the Kronecker product properties \[41,42\] one arrives at the Hilbert space \( \mathcal{H} \) of the entire system

\[ \mathcal{H} = \bigoplus_{n,m} \Xi_n^{(1)} \otimes \Xi_m^{(2)}. \]  

(37)

Given \( n \) and \( m \) each subspace is of the form

\[ \Xi_n^{(1)} \otimes \Xi_m^{(2)} = \text{Span}\{|+,n\rangle \otimes |+,m\rangle, |+,n\rangle \otimes |-,m+1\rangle, |-,n+1\rangle \otimes |+,m\rangle, |-,n+1\rangle \otimes |-,m+1\rangle\}. \]
We consider the $\alpha$-entangled state (9) as the initial condition for the bipartite atomic system with $r$ and $s$ photons in cavities 1 and 2, respectively. Thus, at time $t = 0$ the state of the whole system is

$$|\psi(0)\rangle = \cos \alpha |+, r\rangle \otimes |-, s\rangle + \sin \alpha |-, r\rangle \otimes |+, s\rangle.$$  \hspace{1cm} (38)

Since the two atoms do not interact, the time evolution operator is given as follows

$$U(t) = e^{iHt} = e^{iH_{\text{1,2}}t} \otimes e^{iH_{\text{1,2}}t}.$$  \hspace{1cm} (39)

Hence, the state at time $t$ reads

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

$$= \cos \alpha \left[ -i \cos(\gamma t \sqrt{r + 1}) \sin(\gamma t \sqrt{s}) |+, r\rangle \otimes |+, s - 1\rangle \\
+ \cos(\gamma t \sqrt{r + 1}) \cos(\gamma t \sqrt{s}) |+, r\rangle \otimes |-, s\rangle \\
- \sin(\gamma t \sqrt{r + 1}) \sin(\gamma t \sqrt{s}) |-, r + 1\rangle \otimes |+, s - 1\rangle \\
- i \sin(\gamma t \sqrt{r + 1}) \cos(\gamma t \sqrt{s}) |-, r + 1\rangle \otimes |-, s\rangle \right]$$  \hspace{1cm} (40)

$$+ \sin \alpha \left[ -i \cos(\gamma t \sqrt{s + 1}) \sin(\gamma t \sqrt{r}) |+, r - 1\rangle \otimes |+, s\rangle \\
- \sin(\gamma t \sqrt{r}) \sin(\gamma t \sqrt{s + 1}) |+, r - 1\rangle \otimes |-, s + 1\rangle \\
+ \cos(\gamma t \sqrt{r}) \cos(\gamma t \sqrt{s + 1}) |-, r\rangle \otimes |+, s\rangle \\
- i \cos(\gamma t \sqrt{r}) \sin(\gamma t \sqrt{s + 1}) |-, r\rangle \otimes |-, s + 1\rangle \right].$$

The density operator $\rho$ of the whole system is the pure state $\rho = |\psi(t)\rangle \langle \psi(t)|$. In order to obtain information about the bipartite atomic system we calculate $\rho_{\text{atoms}} = Tr_{\text{1,2}} \rho$ to get

$$\rho_{\text{atoms}} = \begin{pmatrix}
\Omega_{11} & 0 & 0 & 0 \\
0 & \Omega_{22} & \Omega_{23} & 0 \\
0 & \Omega_{32} & \Omega_{33} & 0 \\
0 & 0 & 0 & \Omega_{44}
\end{pmatrix},$$  \hspace{1cm} (41)

where

$$\Omega_{11} = \cos^2 \alpha \cos^2(\gamma t \sqrt{r + 1}) \sin^2(\gamma t \sqrt{s}) + \sin^2 \alpha \sin^2(\gamma t \sqrt{r}) \cos^2(\gamma t \sqrt{s + 1})$$

$$\Omega_{22} = \cos^2 \alpha \cos^2(\gamma t \sqrt{r + 1}) \cos^2(\gamma t \sqrt{s}) + \sin^2 \alpha \sin^2(\gamma t \sqrt{s + 1}) \sin^2(\gamma t \sqrt{r})$$

$$\Omega_{33} = \cos^2 \alpha \sin^2(\gamma t \sqrt{r + 1}) \sin^2(\gamma t \sqrt{s}) + \sin^2 \alpha \cos^2(\gamma t \sqrt{s + 1}) \cos^2(\gamma t \sqrt{r})$$

$$\Omega_{44} = \cos^2 \alpha \sin^2(\gamma t \sqrt{r + 1}) \cos^2(\gamma t \sqrt{s}) + \sin^2 \alpha \sin^2(\gamma t \sqrt{s + 1}) \cos^2(\gamma t \sqrt{r})$$

$$\Omega_{23} = \Omega_{32} = \cos \alpha \sin \alpha \cos(\gamma t \sqrt{r + 1}) \cos(\gamma t \sqrt{r}) \cos(\gamma t \sqrt{s + 1}) \cos(\gamma t \sqrt{s}).$$

The eigenvalues of the matrix $\rho_{\text{atoms}}(\sigma_2 \otimes \sigma_2) \rho_{\text{atoms}}(\sigma_2 \otimes \sigma_2)$ read as follows

$$\lambda_1 = \lambda_4 = \Omega_{11},$$

$$\lambda_2 = (\Omega_{23} + \sqrt{\Omega_{22} \Omega_{33}})^2,$$  \hspace{1cm} (42)

$$\lambda_3 = (\Omega_{23} - \sqrt{\Omega_{22} \Omega_{33}})^2.$$
Finally, we use (11) to calculate the concurrence of the bipartite atomic system.

Figure 2 shows the behaviour of the concurrence as a function of time for the equally weighted entangled state (4), i.e., $|\psi_{\alpha=\pi/4}\rangle = |\beta\rangle$, and for the "−" weighted state (10), considered these last as different initial conditions for the bipartite atomic state. The state $|\phi_{-}\rangle$ corresponds to the situation in which the atom+atom system has the probability $\cos^2\left(\frac{9}{20}\pi\right) \approx 0.024$ to be in the state $|+, -\rangle$ and $\sin^2\left(\frac{9}{20}\pi\right) \approx 0.976$ to be in the state $|-, +\rangle$. That is, the repetition of measurement will give $|-, +\rangle$ in the majority of the occasions the system is asked about what state is it in. Remark that in this case the correlation obtained from the interaction process (see Figure 1) is present because the atoms are unavoidably in orthogonal states but we have now more information about the system: This is most probable to find it in the state $|-, +\rangle$ rather than in the state $|+, -\rangle$. In this context, we say that the state $|\phi_{-}\rangle$ encodes a weak entanglement between the components of the entire system as compared to the Bell state $|\beta\rangle$ which is the common reference of a maximally entangled state. In Figure 2 we can appreciate that the concurrence for the Bell state (4) departures from a maximally entangled condition (where concurrence is equal to one) and after a while this goes to the unentangled condition (i.e., this is equal to zero). The process is reverted after the same time-interval and the concurrence is again equal to 1. If $|\phi_{-}\rangle$ is the initial condition then we see that the process is quite similar except that this starts in a lower value of concurrence because the entanglement is not perfect here.

![Figure 2](image.png)

**Figure 2.** The time-evolution of the concurrence associated to the a bipartite atomic state $|\psi_{\alpha}\rangle$ defined in (9) when the radiation bath in both cavities includes 10 photons and $\alpha = \pi/4$ (left), $\alpha = 9\pi/20$ (right).

The Figure 3 shows the superposition of the concurrences discussed above. Here we can say that the time-evolution of entanglement is quite similar in both cases, no matter how perfect the entanglement was at the very beginning. Moreover, we can see that the maxima of both of these curves are centered around the same times. This is because in both cases the pair of cavities has exactly the same number of photons. Thus, it is the number of photons which determines the profile of the entanglement time-evolution. The statement is supported with the fact that the frequency $\Omega$ is a function of the number operator. Besides, it must be clear that the probability of atomic transitions increases with the number of photons that integrate the radiation bath. This is because the atom has at its disposal a bigger variety of options to capture a given photon in the cavity when the bath has more photons than having only a few of them. It is interesting that not only the frequency but also the manner in which the entanglement suffers the sudden death are dominated by the number of photons, as this is shown in Figure 4.

4. Conclusions

The time-evolution of the energy states of a pair of entangled atoms has been studied when the atoms are in isolated QED cavities each one. The concurrence of these systems has been calculated in order to measure entanglement. In contrast with other approaches already reported (see e.g. [34–39]), the present work provides an analysis of the bipartite atomic concurrence when
Figure 3. The curves representing concurrence in Figure 2 are superposed for comparison. The red curve is for $\alpha = 9\pi/20$ and the blue one for $\alpha = \pi/4$. Notice that the maxima are centered around the same times in both curves.

Figure 4. Time-evolution of the concurrence for a bipartite atomic entangled state which is initially in the Bell state $|\psi_{4\pi/20}\rangle$ defined in (4). The cavities include a radiation field of 100 photons each one. Compare the times of the sudden death and the recovering of entanglement with those reported in Figures 2 and 3.

the initial field state in each of the cavities is not longer the vacuum or the single photon state but an arbitrary Fock state. We have found that the entanglement of the atoms depends on the initial number of photons that integrate the radiation baths in the cavities. For the same initial number of photons in both cavities, the frequencies of sudden death and recovering of entanglement are well defined, no matter how perfect the entanglement between the atoms was at the very beginning. The intervals of time in which the entanglement is missing are shorter as bigger is the initial number of photons in the cavities. Some other questions as the dependence of the concurrence with the statistical distribution of the photons in the cavities are going to be discussed elsewhere.

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