Trace Formulas for Schrödinger Operators with Complex Potentials

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Abstract. We consider 3-dimensional Schrödinger operators with complex potential. We obtain new trace formulas with new terms, associated with singular measure. In order to prove these results, we study analytic properties of a modified Fredholm determinant as a function from Hardy spaces in the upper half-plane. In fact, we reformulate spectral theory problems as problems of analytic functions from Hardy spaces.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction

We consider the Schrödinger operator \( H = -\Delta + V \) on \( L^2(\mathbb{R}^3) \), where the potential \( V \) is complex and satisfies

\[
V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3).
\]

(1.1)

Here \( L^p(\mathbb{R}^d) \), \( p \geq 1 \), is the space equipped with the norm \( \|f\|_p = \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p} \) and let \( \|\cdot\| = \|\cdot\|_2 \). It is well-known that the operator \( H \) has essential spectrum \([0, \infty)\) plus \( N \in [0, \infty) \) eigenvalues (counted with multiplicity) in the cut domain \( \mathbb{C} \setminus [0, \infty) \). Denote by \( \lambda_j \in \mathbb{C} \setminus [0, \infty) \), \( j = 1, \ldots, N \) the eigenvalues (according multiplicity) of the operator \( H \). Note that the multiplicity of each eigenvalue is greater than or equal to 1, but we say the multiplicity of the eigenvalue is its \textit{algebraic multiplicity}.

Define an operator-valued functions \( Y_0(k) \) by

\[
Y_0(k) = V_1 R_0(k) V_2, \quad k \in \mathbb{C}_+,
\]

where \( R_0(k) = (-\Delta - k^2)^{-1} \) is the free resolvent and we have used the factorization

\[
V = V_1 V_2, \quad \text{where} \quad V_1 = |V|^{1/2}, \quad V_2 = |V|^{1/2} e^{i \arg V}.
\]

(1.3)

Let \( B \) denote the class of bounded operators. Let \( B_1 \) and \( B_2 \) be the trace and the Hilbert–Schmidt class equipped with the norm \( \|\cdot\|_{B_1} \) and \( \|\cdot\|_{B_2} \) correspondingly. It is well known that \( Y_0(k) \in B_2 \) but \( Y_0(k) \notin B_1 \) for all \( k \in \mathbb{C}_+ \). In this case, we cannot directly define the determinant \( \det(I + Y_0(k)) \) and we need some modification. It is well-known that the mapping \( Y_0 : \mathbb{C}_+ \to B_2 \) is analytic and continuous up to the real line. Then we can define the modified determinant

\[
\psi(k) = \det \left[ (I + Y_0(k)) e^{-Y_0(k)} \right], \quad k \in \mathbb{C}_+.
\]

(1.4)

The function \( \psi(k) \) is analytic in \( \mathbb{C}_+ \) and continuous up to the real line. It has \( N \leq \infty \) zeros (counted with multiplicity) \( k_1, \ldots, k_N \in \mathbb{C}_+ \) given by \( k_n = \sqrt{\lambda_n} \in \mathbb{C}_+ \) and labeled by

\[
\exists k_1 \geq \exists k_2 \geq \exists k_3 \geq \cdots \geq \exists k_N \geq \cdots
\]

(1.5)

and define the set \( \sigma_d = \{ k_1, \ldots, k_N \in \mathbb{C}_+ \} \). There are no other roots in \( \mathbb{C}_+ \). We describe the basic properties of the determinant \( \psi \) as an analytic function in \( \mathbb{C}_+ \).

The trace formulas for Schrödinger operators with real decaying potentials in three dimensional were proved by Buslaev [1], see also [3, 15, 34, 33].

Our main goal is to determine the trace formulas for Schrödinger operators with complex potentials. These formulas have a new term associated with singular measure, see (1.24) and (1.25). In order to describe the trace formulas, we need to discuss the properties of \( \psi \).
Theorem 1.1. Consider a potential \( V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \). Then the determinant \( \psi \) and the function \( \psi_2 = (1/2) \text{Tr} Y_0^2(k) \) are analytic in \( \mathbb{C}_+ \) and continuous up to the real line and \( \psi \) satisfies

\[ |\psi(k)| \leq \exp \left( C_* \| V \|_{3/2}^2 \right) \quad \forall \, k \in \mathbb{C}_+, \tag{1.6} \]

where \( C_* = 1/(8(4\pi)^{2/3}) \),

\[ \psi_2(k) = 1 + o(1), \quad \log |\psi(k)| - \psi_2(k) = O(k^{-1/2}) \tag{1.7} \]

as \( |k| \to \infty \), uniformly with respect to \( \arg k \in [0, \pi] \),

\[ |\psi_2(k)| \leq C_* \| V \|_{3/2}^2 \quad \forall \, k \in \mathbb{C}_+, \quad \| \psi_2(\cdot + iv) \| \leq (1/8(4\pi)^{1/3}) \| V \|_{3/2} \| V \| \quad \forall \, v \geq 0, \tag{1.8} \]

and the function \( h(k) = \log |\psi(k + i0)| \), \( k \in \mathbb{R} \), satisfies

\[ h \in L^1(-r, r), \quad h \in L^a(\mathbb{R} \setminus (-r, r)) \tag{1.9} \]

for some \( r > 0, \alpha \in (2, \infty) \) and the zeros \( \{ k_j \} \) of \( \psi \) in \( \mathbb{C}_+ \) satisfy

\[ \sum_{j=1}^N \Re k_j < \infty. \tag{1.10} \]

In fact, \( \psi \) is a function from the Hardy class. We now introduce some notation to make this precise. Let a function \( F(k), \, k = u + iv \in \mathbb{C}_+ \), be analytic on \( \mathbb{C}_+ \). For \( 0 < p \leq \infty \), we say that \( F \) belongs to the Hardy space \( \mathcal{H}_p = \mathcal{H}_p(C_+) \) if \( F \) satisfies \( \| F \|_{\mathcal{H}_p} < \infty \), where \( \| F \|_{\mathcal{H}_p} \) is given by

\[ \| F \|_{\mathcal{H}_p} = \begin{cases} \sup_{v > 0} (1/2\pi) \left( \int_{\mathbb{R}} |F(u + iv)|^p du \right)^{1/p} & \text{if } 0 < p < \infty, \\ \sup_{k \in \mathbb{C}_+} |F(k)| & \text{if } p = \infty. \end{cases} \]

Note that the definition of the Hardy space \( \mathcal{H}_p \) involves all \( v = \Re k > 0 \).

Thus from Theorem 1.1, we deduce that \( \psi, \psi_2 \in \mathcal{H}_\infty \) and \( \psi_2 \in \mathcal{H}_2 \).

1.2. Trace Formulas and Estimates

Theorem 1.1 gives the basic analytic properties of the function \( \psi \), in particular, \( \psi \in \mathcal{H}_\infty \). In order to study zeros of \( \psi(k) \) in the upper-half plane, we define the Blaschke product by

\[ B(k) = \prod_{j=1}^N \left( \frac{k - k_j}{k - k_j^\ast} \right), \quad k \in \mathbb{C}_+. \tag{1.11} \]

Here the Blaschke product \( B(k) \) converges absolutely for \( k \in \mathbb{C}_+ \), since, due to (1.7), all zeros are uniformly bounded, and the function \( B \in \mathcal{H}_\infty \) with \( \| B \|_{\mathcal{H}_\infty} \leq 1 \) (see, e.g., [13] or [21]). We describe the canonical factorization of \( \psi \).

Theorem 1.2. Let \( V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \). Then \( \psi \) has a canonical factorization in \( \mathbb{C}_+ \) given by

\[ \psi = \psi_{in}\psi_{out}, \quad \psi_{in}(k) = B(k)e^{-iK(k)}, \quad K(k) = 1/\pi \int_{\mathbb{R}} d\nu(t)/(k - t), \tag{1.12} \]

\[ \psi_{out}(k) = e^{iM(k)}, \quad M(k) = 1/\pi \int_{\mathbb{R}} (\log |\psi(t)|/(k - t)) dt. \]

(1) \( d\nu(t) \geq 0 \) is a singular compactly supported measure on \( \mathbb{R} \) and, for some \( r_+ > 0 \), it satisfies

\[ \int_{\mathbb{R}} d\nu(t) < \infty, \quad \text{supp} \, \nu \subset s\{z \in \mathbb{R} : \psi(z) = 0\} \subset s[-r_+, r_+]. \tag{1.13} \]
The function $K$ has an analytic continuation from $\mathbb{C}_+$ into the domain $\mathbb{C} \setminus [-r_*, r_*)$ and has the following Taylor series
\begin{equation}
K(k) = \sum_{j=0}^{\infty} K_j/k^{j+1}, \quad K_j = 1/\pi \int_{\mathbb{R}} t^j dv(t).
\end{equation}

The Blaschke product $B$ has an analytic continuation from $\mathbb{C}_+$ to the domain $\{|k| > r_0\}$, where $r_0 = \sup |k_n|$ and has the following Taylor series
\begin{equation}
\log B(z) = -iB_0/k - iB_1/2k^2 - iB_2/3k^3 - \cdots \text{ as } |k| > r_0,
\end{equation}
where each sum $B_n, n \geq 1$, is absolutely convergent and satisfies
\begin{equation}
|B_n| \leq 2 \sum_{j=1}^{N} |3k_j^2| \leq (\pi/2)(n+1)r_0^N B_0.
\end{equation}

Remark. (1) The function $\psi_{in}$ is the inner factor of $\psi$, and the function $\psi_{out}$ is the outer factor of $\psi$, see, e.g., [13] or [21]. These results will be used in the proof of trace formulas.

(2) Due to (1.9), the integral $M(k)$, $k \in \mathbb{C}_+$, in (1.12) converges absolutely.

Corollary 1.3. Let $V \in L^3(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then the following trace formula
\begin{equation}
-2k \text{Tr} \left( R(k) - R_0(k) + R_0(k) V R_0(k) \right) = \sum_{j=1}^{N} 2i\Im k_j/((k - k_j)(k - \overline{k}_j)) + i/\pi \int_{\mathbb{R}} d\mu(t)/(t - k)^2
\end{equation}
holds for any $k \in \mathbb{C}_+ \setminus \sigma_d$, where the measure is $d\mu(t) = \log |\psi(t)| dt - d\nu(t)$ and the series converges uniformly in every bounded disk in $\mathbb{C}_+ \setminus \sigma_d$. If, in addition, $V \in L^1(\mathbb{R}^3)$, then
\begin{equation}
-2k \text{Tr} \left( R(k) - R_0(k) \right) + (i/4\pi) \int_{\mathbb{R}^3} V(x) dx = \sum_{j=1}^{N} 2i\Im k_j/((k - k_j)(k - \overline{k}_j)) + (i/\pi) \int_{\mathbb{R}} d\mu(t)/(t - k)^2.
\end{equation}

In order to prove the trace formulas, we introduce the space $W_m$ by
\begin{equation}
W_m = \{ f \in L^2(\mathbb{R}^3) : |\partial^\alpha f(x)| \leq C_V/(1 + |x|)^{\beta+|\alpha|}, \quad \forall \ |\alpha| \leq 2m + 1, \quad m \geq 1, \}
\end{equation}
where $\beta > 3$ and $m \geq 1$ is an integer. If $V \in W_m$, then the function $\psi(\cdot)$ is analytic in $\mathbb{C}_+$ and continuous up to the real line and satisfies
\begin{equation}
\log \psi(k) = Q_0/(ik) + Q_2/(ik^3) + Q_4/(ik^5) + \cdots + Q_{2m}/(ik^{2m+1}) + o(1)/(ik^{2m+1}),
\end{equation}
as $|k| \to \infty$ uniformly in $\arg k \in [0, \pi]$ (see [2], and also [3, 15, 33, 34]), where
\begin{equation}
Q_0 = (1/16\pi) \int_{\mathbb{R}^3} V^2(x) dx, \quad Q_2 = (1/3\pi^4)^3 \int_{\mathbb{R}^3} \left( (\nabla V(x))^2 + 2V^3(x) \right) dx, \ldots
\end{equation}
In Theorem 1.4, we show that the function $M(k)$, $k \in \mathbb{C}_+$, defined by (1.12) satisfies
\begin{equation}
M(k) = (1/\pi) \int_{\mathbb{R}} h(t)/k - t dt = (J_0 - iI_0)/t + J_1/t^2 + \cdots + (J_{2m} - iI_{2m})/t^{2m+1} + o(1)/t^{2m+1},
\end{equation}
as $\Im k \to \infty$, where $h := h_{-1} := \log |\psi(\cdot)|$, and
\begin{equation}
I_j = \Im Q_j, \quad h_j = t^{j+1}(h - P_j), \quad P_j = I_0/t + I_2/t^3 + \cdots + I_j/t^{j+1}, \quad J_0 = \text{v.s.}(1/\pi) \int_{\mathbb{R}} h(t) dt, \quad J_1 = \text{v.s.}(1/\pi) \int_{\mathbb{R}} (th(t) - I_0) dt, \quad J_j = \text{v.s.}(1/\pi) \int_{\mathbb{R}} h_{j-1}(t) dt, \quad j = 0, 1, \ldots, 2m. \text{ Here } I_{2j+1} = 0 \text{ and all integrals in (1.23) converge, since } \psi \text{ satisfies (1.20).}
Theorem 1.4. (Trace formulas) Let \( V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \). Then
\[
B_0 + \nu(\mathbb{R})/\pi = (1/\pi) \int_{\mathbb{R}} \log |D_4(k)| dk,
\]
where \( D_4 \) is the modified determinant (2.5). Let, in addition, \( V \in W_{m+1} \) for some \( m \geq 1 \). Then the function \( M \) defined by (1.12) satisfies (1.22). Moreover, the following identities hold:
\[
B_1/2 + K_1 = \mathcal{J}_1, \quad B_j/(j + 1) + K_j = \Re Q_j + \mathcal{J}_j, \quad j = 2, 3, \ldots, 2m.
\]

Remark. (1) Recall that trace formulas similar to (1.24), (1.25) were proved by Buslaev [2] for real potentials. In the case of complex potentials, there is an additional term associated with the singular measure \( \nu \), see (1.24), (1.25).

(2) Trace formulas for Schrödinger operators on the lattice are considered in [18] and for the case of complex potentials in [22, 25]. In our proof, we use methods from [22, 25].

(3) The higher regularity of \( V \) implies more trace formulas.

(4) Buslaev mainly considered the phase \( \phi_{sc} = \arg \psi \). In the present paper, the trace formulas for the conjugate function \( \log |\psi(k)| \) are used.

Theorem 1.5. (Estimates) Let \( V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \). Then the following bound
\[
\nu(\mathbb{R}) + \sum_{j=1}^N \Im k_j \leq \|V\|^2 F(\|V\|_{3/2})
\]
holds, where \( F(t) = a_1 t^{1/2} + a_2 t + a_3 t^{3/2} \), \( t > 0 \)
\[
a_1 = 8^{1/4} C_2/(4\pi)^{7/6}, \quad a_2 = C_2^2/(32)^{1/2}/(4\pi)^{4/3}, \quad a_3 = C_2^2/(6\pi^{3/2}),
\]
and \( C_2 \) is the constant in (2.14).

Remark. (1) We estimate the singular measure \( \nu(\mathbb{R}) \) plus eigenvalues \( \sum_{j=1}^N \Im k_j \) in terms of \( L^p \)-norms of the potential. It is a bound of a new type, since here we have a singular measure. If we use other identities in (1.25) for smoother potentials, we obtain a series of bounds. Note that singular measure is a big problem even for functions in Hardy spaces.

(2) In [18], there is a bound for the sum of \( \delta(\lambda_j) \) in terms of \( \|V\| \), where \( \delta(z) = \text{dist}(z, \mathbb{R}_+) \) for \( z \in \mathbb{C} \setminus [0, \infty) \).

There are a lot of results about the trace formulas for self-adjoint operators, see [2, 3, 15, 33, 34, 27, 26] and the references therein. Unfortunately, we know only few papers [22, 25, 32] for Schrödinger operators with complex-valued potentials decaying at infinity on the lattice only. We note that, in [22, 25], for Schrödinger operators with complex potentials on a lattice, the Hardy space on the unit disk was used. In the present paper, for the continuous case, we use the technique of [22, 25], but it is more natural to use the Hardy space in the upper half-plane. This gives some additional problems. Finally, note that the results of the present paper are used to discussed trace formulas for 1-dimensional Schrödinger operators with complex-valued potentials.

We need to say that, recently, uniform bounds on eigenvalues of Schrödinger operators in \( \mathbb{R}^d \) with complex-valued potentials decaying at infinity have attracted the attention of many specialists. In particular, bounds for sums of powers of eigenvalues were found in [10, 11, 29, 5, 6, 9, 14, 36, 35]. These bounds generalize the Lieb–Thirring bounds [31] to the nonself-adjoint setting. Note that, in [9], the author estimates the sum of the distances between the complex eigenvalues and the continuous spectrum \([0, \infty)\) in terms of \( L^p \)-norms of the potentials.

In our paper, we use classical results of complex analysis about Hardy space in the upper half-plane. In particular, we use the so-called canonical factorization of analytic functions in Hardy spaces via its inner and outer factors. This gives us a new class of trace formulas for Schrödinger operators with complex-valued potentials in which there is an additional term associated with a singular measure.

Let us briefly describe the plan of the paper. In Section 2, we present the main properties of the Fredholm determinant. In Section 3, we prove the main theorems. In particular, we give trace formulas. Section 4 is a collection of necessary facts about Hardy spaces.
2. PRELIMINARIES

2.1. Determinants and Trace Class Operators

Let us recall some well-known facts.

1. Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_1$. Then
   \[
   \text{Tr} \ AB = \text{Tr} \ BA, \quad \text{det} (I + AB) = \text{det} (I + BA). \tag{2.1}
   \]

2. If $A, B \in \mathcal{B}_1$, then
   \[
   | \text{det} (I + A) | \leq e^{\|A\|_{S}}. \tag{2.2}
   \]

3. We introduce the space $\mathcal{B}_p$, $p \geq 1$, of compact operators $A$ equipped with the norm
   \[
   \|A\|_{\mathcal{B}_p}^p = \text{Tr} (A^* A)^{p/2} < \infty.
   \]

   In the case of $A \in \mathcal{B}_n$, $n \geq 2$, we have
   \[
   (I + A)e^{-A + \Gamma_n (A)} - I \in \mathcal{B}_1, \quad \text{where} \quad \Gamma_n (z) = \sum_{2}^{n-1} (1/j)(-z)^j.
   \]

   Thus, we define the modified determinant $\text{det}_n (I + A)$ by
   \[
   \text{det}_2 (I + A) = \text{det} ((I + A)e^{-A}), \quad \text{det}_n (I + A) = \text{det} ((I + A)e^{-A + \Gamma_n (A)}), \quad n \geq 3. \tag{2.4}
   \]

   Note that the modified determinant satisfies
   \[
   \text{det}_n (I + A) = e^{\text{Tr} \ \Gamma_n (A)} \text{det}_2 (I + A) \quad \text{if} \quad A \in \mathcal{B}_2 \tag{2.5}
   \]

   and $I + A$ is invertible if and only if $\text{det}_n (I + A) \neq 0$, see Ch. IV of [16].

4. If $X \in \mathcal{B}_n$ for some $n \geq 2$, then (see [1, 7, 17])
   \[
   | \text{det}_2 (I + X) | \leq e^{(1/2)\|X\|_{\mathcal{B}_2}^2}, \quad | \text{det}_n (I + X) | \leq e^{\|X\|_{\mathcal{B}_n}^n}. \tag{2.6}
   \]

5. Suppose that a function $A(\cdot) : \mathcal{D} \to \mathcal{B}_1$ is analytic for a domain $\mathcal{D} \subset \mathbb{C}$ and the operator $(I + A(z))^{-1}$ is bounded for any $z \in \mathcal{D}$. Then the function $F(z) = \text{det} (I + A(z))$ satisfies
   \[
   F'(z) = F(z) \text{Tr} \ (I + A(z))^{-1} A'(z) \quad \forall \ z \in \mathcal{D}.
   \]

2.2. Analytic Functions

The kernel of the free resolvent $R_0 (k) = (-\Delta - k^2)^{-1}$ has the form
\[
R_0 (x - y, k) = e^{ik|x-y|/(4\pi|x-y|)}, \quad x, y \in \mathbb{R}^3, \quad k \in \mathbb{C}_+ . \tag{2.7}
\]

Recall that $Y_0 (k)$ is defined in (1.2) and $\sigma_d = \{ k_j = \sqrt{j}, j = 1, \ldots, N \} \subset \mathbb{C}_+$. Define an operator $Y(k) = |V|^{1/2} R(k)V^{1/2}$ for $k \in \mathbb{C}_+$. Below we will use the standard identity
\[
(I + Y_0 (k))(I - Y(k)) = I \quad \forall \ k \in \mathbb{C}_+ \setminus \sigma_d. \tag{2.8}
\]

Below we also need the Hardy–Littlewood–Sobolev inequality for $f, g \in L^{3/2}(\mathbb{R}^3)$:
\[
\int_{\mathbb{R}^6} |f(x)||g(y)||x - y|^{-2} dxdy \leq \pi^{4/3} 4^{1/3} \|f\|_{3/2} \|g\|_{3/2}. \tag{2.9}
\]
Lemma 2.1. (i) Let $V \in L^{3/2}(\mathbb{R}^3)$. Then the function $Y_0 : \mathbb{C}_+ \to \mathcal{B}_2$ is analytic and continuous up to the real line and satisfies, for all $k \in \mathbb{C}_+$, the inequality

$$\|Y_0(k)\|_{\mathcal{B}_2} \leq \int_{\mathbb{R}^6} |V(x)||V(y)|(4\pi)^{-2}|x-y|^{-2}dxdy \leq 2C\|V\|_{3/2}^2, \quad C = 1/(8(4\pi)^{2/3}). \quad (2.10)$$

(ii) If $V \in L^2(\mathbb{R}^3)$ and $\exists k > 0$, then

$$\|Y_0(k)\|_{\mathcal{B}_2} \leq \|V\|_2/\sqrt{8\pi\Im k}, \quad VR_0(k) \in \mathcal{B}_2. \quad (2.11)$$

(iii) If $V \in L^1(\mathbb{R}^3)$, then the function $Y'_0 : \mathbb{C}_+ \to \mathcal{B}_1$ is analytic and continuous up to the real line and satisfies, for all $k \in \mathbb{C}_+$, the inequality

$$\text{Tr} \ Y'_0(k) = 2k\text{Tr} \ (VR_0^2(k)) = (i/4\pi) \int_{\mathbb{R}^3} V(x)dx. \quad (2.12)$$

Proof. (i) The Hardy–Littlewood–Sobolev inequality (2.9) and (2.7) yield

$$\|Y_0(k)\|_{\mathcal{B}_2}^2 = (4\pi)^{-2} \int_{\mathbb{R}^6} |V(x)||V(y)|e^{-2\pi|x-y|}|x-y|^{-2}dxdy \leq (4\pi)^{-2} \int_{\mathbb{R}^6} |V(x)||V(y)||x-y|^{-2}dxdy \leq \pi^{4/3}4^{1/3}(4\pi)^{-2}\|V\|_{3/2}^2 = 2C\|V\|_{3/2}^2.$$

Similar arguments give the following well-known results: the operator-valued function $Y_0 : \mathbb{C}_+ \to \mathcal{B}_2$ is analytic and is continuous up to the real line.

(ii) Using identity (2.7), the formula $v = \Im k > 0$, and the Schwarz inequality for the function $f(x, y) = |V(x)|e^{-v|x-y|}/|x-y|$, we obtain

$$\|Y_0(k)\|_{\mathcal{B}_2}^2 = 1/(4\pi)^2 \int_{\mathbb{R}^6} f(x, y)f(y, x)dxdy \leq (4\pi)^{-2} \int_{\mathbb{R}^6} f^2(x, y)dxdy = \|V\|_2^2(4\pi)^{-1} \int_{0}^{\infty} e^{-2\pi v}dv = \|V\|_2^2/(8\pi v).$$

(iii) If $V \in L^1(\mathbb{R}^3)$, then similar arguments imply that the function $Y'_0(k)$ is analytic in $\mathbb{C}_+$ and is continuous up to the real line. Using (2.7), we obtain (2.12).\]

Below we shall need the following estimates from [12].

Let $3/2 < q \leq 2$ and $p = 2q/(3 - q)$. Then the $\mathcal{B}_p$-norms of the operator $Y_0(\lambda)$ satisfy

$$\|Y_0(\lambda)\|_{\mathcal{B}_p} \leq C_q|\lambda|^{3/q-2}\|V\|_q, \quad (2.13)$$

$$\|Y_0(k)\|_{\mathcal{B}_1} \leq C_2|k|^{-1/2}\|V\|_2, \quad (2.14)$$

where the constant $C_q > 0$ depends on $q$ only.

If $V \in L^{3/2}(\mathbb{R}^3)$, then $\|Y_0(k)\| = o(1)$ as $|k| \to \infty$ uniformly with respect to $\arg k \in [0, \pi]$.

In this case, we can define the radius $r_0 > 0$ by the condition

$$\sup_{\exists k \geq 0, |k| \geq r_0} \|Y_0(k)\| \leq 1/2. \quad (2.15)$$

Let $\psi_2 = (1/2) \text{Tr} \ Y_0^2$. Define the Fourier transformation

$$V(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i(p,x)}\hat{V}(p)dp,$$

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^3$. 

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 27 No. 1 2020
Lemma 2.2. Let $V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then the function $\psi_2 = (1/2) \text{Tr} Y_0^2$ satisfies
\[ \psi_2(k) = \int_0^\infty e^{i2kt} \gamma(t) dt = -C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)/(i(2k - (p,\omega))) dp, \tag{2.16} \]
where $C = 1/(2(4\pi)^2)$ and
\[ \gamma(t) = C \int_{|\omega|=1} d\omega \int_{\mathbb{R}^3} V(x-t\omega)V(x) dx = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)e^{-it(p,\omega)} dp, \tag{2.17} \]
\[ |\gamma|_1 \leq C \cdot |V|^2_{3/2} \quad \text{if} \quad V \in L^{3/2}(\mathbb{R}^3), \]
\[ |\gamma|_\infty \leq |V|^2/(8\pi) \quad \text{if} \quad V \in L^2(\mathbb{R}^3), \]
\[ |\psi_2| \leq 1/(8(4\pi)^{1/3})|V|^2_{3/2} |V| \quad \text{if} \quad V \in L^2(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3), \]
\[ \psi_2(k) = -(Q_0 + o(1))/(ik) \quad \text{as} \quad \Im k \to \infty, \tag{2.19} \]
\[ \psi_2 \in \mathcal{H}_2, \tag{2.20} \]
\[ \psi_2(k) = o(1) \quad \text{as} \quad |k| \to \infty, \tag{2.21} \]
uniformly with respect to $k$. and here $Q_0 = (1/16)\pi \int_{\mathbb{R}^3} V^2(x) dx$.

Proof. Due to (2.7), we have
\[ \psi_2(k) = (1/2) \int_{\mathbb{R}^6} V(x)R_0^2(x-y,k)V(y)dxdy = C \int_{\mathbb{R}^6} V(x)V(y)e^{i2k|x-y|}|x-y|^{-2}dxdy. \]
If we set $x-y = tw$, where $\omega$ belongs to the unit sphere $\mathbb{S}^2$ and $t = |x-y| > 0$, then we obtain
\[ \psi_2(k) = \int_0^\infty e^{i2kt} \gamma(t) dt, \quad \gamma(t) = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} V(x-t\omega)V(x) dx. \tag{2.22} \]
Using the Fourier transformation, we rewrite the function $\gamma$ in the form
\[ \gamma(t) = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} V(x-t\omega)V(x) dx = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)e^{-it(p,\omega)} dp. \tag{2.23} \]
Thus we have for $k \in \mathbb{C}_+$:
\[ \psi_2(k) = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)dp \int_0^\infty e^{it(2k-(p,\omega))} dt = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)/(i((p,\omega)-2k)) dp. \tag{2.24} \]
From this identity and the Lebesgue Theorem, we obtain
\[ \psi_2(k) + Q_0/(ik) = C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)(1/(i2k) - 1/(i(2k-(p,\omega)))) dp \]
\[ = C/(i2k) \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} \hat{V}(p)\hat{V}(-p)p\omega/(2k-p\omega) dp = o(1)/k, \]
since $p\omega/(2k-p\omega) \to 0$ as $\Im k \to \infty$ and $V \in L^2(\mathbb{R}^3)$.

Let us show (2.18). Let $V \in L^{3/2}(\mathbb{R}^3)$. Then, using (2.10), we can write
\[ \int_0^\infty |\gamma(t)| dt \leq C \int_0^\infty dt d\omega \int_{\mathbb{R}^3} |V(x-t\omega)V(x)| dx = C \int_{\mathbb{R}^6} |V(x)||V(y)||x-y|^{-2}dxdy < C \cdot |V|^2_{3/2}. \]
If $V \in L^2(\mathbb{R}^3)$, then, for any $t \geq 0$, we obtain
\[ |\gamma(t)| \leq C \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} |V(x-t\omega)V(x)| dx \leq C |\mathbb{S}^2| \cdot \|V\|^2 = \|V\|^2/(8\pi). \tag{2.25} \]
Thus $\psi_2 \in L^2(\mathbb{R})$ and $\psi_2 \in \mathcal{H}_2$, and we have
\[ 1/\pi\|\psi_2\|^2 = \|\gamma\|^2 \leq \|\gamma\|_\infty \int_0^\infty |\gamma(t)| dt \leq (\|V\|^2/(8\pi))C \cdot |V|^2_{3/2}, \]
which gives (2.18). Using identity (2.16), where $\gamma \in L^1(0, \infty)$, we obtain (2.21).
Lemma 2.3. Let $V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then the function $\psi_3 = (1/3) \text{Tr} Y_0^3$ is analytic in $\mathbb{C}_+$ and continuous up to the real line and satisfies the inequalities

$$|\psi_3(k)| \leq (1/(96\pi))\|V\|_{3/2}^3,$$

$$|\psi_3(k)| \leq (1/3)\|Y_0^3(k)\|_{B_1} \leq C|k|^{-1/2}\|V\|_{3/2}^2\|V\| \quad \forall \ k \in \mathbb{C}_+ \setminus \{0\},$$

$$\psi_3(i\tau) = O(\tau^{-3/2}) \quad \text{as} \ quad \tau \to +\infty.$$  \hspace{1cm} (2.26)

Proof. It follows from Lemma 2.1 that the function $\psi_3 = (1/3) \text{Tr} Y_0^3$ is analytic in $\mathbb{C}_+$ and is continuous up to the real line. From (2.10), we get

$$|\psi_3(k)| = (1/3)|\text{Tr} Y_0^3(k)| \leq (1/3)|Y_0(k)|_{B_2}^3 \leq (1/(96\pi))\|V\|_{3/2}^3 \quad \forall \ k \in \mathbb{C}_+.$$  \hspace{1cm} (2.27)

Let $k \in \mathbb{C}_+$ and $k \neq 0$. Then from (2.10) and (2.14), we obtain

$$|\psi_3(k)| \leq (1/3)\|Y_0^3(k)\|_{B_1} \leq (1/3)\|Y_0(k)\|_{B_2}^2 \|Y_0(k)\|_{B_4} \leq C|k|^{-1/2}\|V\|_{3/2}^2V\|$$

for some absolute constant $C$. From (2.11), we have, as $k = i\tau$, $\tau \to \infty$,

$$|\psi_3(k)| \leq (1/3)|Y_0(k)|_{B_2}^3 = O(k^{-3/2}).$$  \hspace{1cm} (2.28)

Note some properties of the determinants $\psi = D_2$ and $D_4 = \det ((I + Y_0)e^{-Y_0 + (1/2)Y_0^2 - (1/3)Y_0^3})$.

Lemma 2.4. Let $V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and let $k \in \mathbb{C}_+$. Then $D_m \in \mathcal{H}_\infty$, $m \geq 2$, and it is continuous up to the real line and satisfies the inequalities

$$|\psi(k)| \leq e^{C\|V\|_{3/2}^3},$$

$$\log |D_m(\cdot + i0)| \in L^1_{loc}(\mathbb{R}),$$

and if, in addition $|k| \geq r_0$, where $r_0$ is given by (2.15), then

$$\log \psi(k) = \text{Tr} \Gamma_m(Y_0(k)) + \log D_m(k),$$

$$\log D_m(k) = \sum_{n=m}^{\infty} ((-1)^{n+1}/n) \text{Tr} Y_0^n(k),$$

where the series converges absolutely and uniformly, and

$$|\log D_m(k)| \leq (2/m)\|Y_0^m(k)\|_{B_1},$$

$$\psi(k) = 1 + o(1)$$

as $|k| \to \infty$ uniform with respect to $\arg k \in [0, \pi]$.

Proof. Lemma 2.1 shows that the operator-valued function $Y_0 : \mathbb{C}_+ \to \mathcal{B}_2$ is analytic on $\mathbb{C}_+$ and continuous up to the real line. Then $\psi(k)$ is analytic on $\mathbb{C}_+$ and continuous up to the real line. The bound (2.28) follows from (2.10) and (2.6). Moreover, (2.29) holds, since we have $D_m \in \mathcal{H}_\infty$, see, e.g., [21].

We denote the series in (2.30) by $F(k)$. It follows from (2.15) that

$$|\text{Tr} Y_0^n(k)| \leq \|Y_0^m(k)\|_{B_1}\|Y_0(k)\|^m \leq \|Y_0^m(k)\|_{B_1}\varepsilon_k^{n-m}, \quad \varepsilon_k = \|Y_0(k)\| \leq 1/2$$

(2.33)
for \(|k| > r_0\). Then the function \(F(k)\) converges absolutely and uniformly, and each term is analytic for \(|k| > r_0\). Then \(F(k)\) is analytic for \(|k| > r_0\). Differentiating (2.30) and using (2.8), we obtain

\[
F'(k) = -i \sum_{n=2}^{\infty} \text{Tr} \left( -Y_0(k) \right)^n Y_0'(k) = i \text{Tr} Y(k) Y_0'(k), \quad |k| > C_0.
\]

Then \(F(k) = -i \log \psi(k)\), since \(F(i\tau) = o(1)\) as \(\tau \to \infty\). Using (2.30) and (2.33), we obtain (2.31).

Let us prove (2.32). From (2.30), we have \(\log \psi(k) = \psi_2(k) + \log D_3(k)\) for \(|k| > r_0\). The asymptotics (2.31) and (2.26) give \(\log D_3(k) = O(k^{-1/2})\) and, using (2.21), we obtain (2.32).

Define the function \(\psi_j = (1/j) \text{Tr} Y_0^j(k),\) \(j \geq 2\). Using (2.5), we have

\[
D_4 = D_2 e^{\psi_2 - \psi_3},
\]

which yields

\[
h(k) = \log |D_2(k)| = \log |D_4(k)| - \Re(\psi_2(k) - \psi_3(k)), \quad k \in \mathbb{C}_+.
\]

**Lemma 2.5.** Let \(V \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)\). Then the following identity holds true:

\[
(1/\pi) \int_{\mathbb{R}} (|\psi(t)|/(k - t)) dt = (1/\pi) \int_{\mathbb{R}} (|D_4(t)|/(k - t)) dt + i\psi_2(k) - i\psi_3(k) \quad \forall k \in \mathbb{C}_+,
\]

and, if \(k = i\tau, \tau \to +\infty,\) then

\[
(1/\pi) \int_{\mathbb{R}} (|\psi(t)|/(k - t)) dt = (C_M + o(1))/k, \quad C_M = (1/\pi) \int_{\mathbb{R}} |D_4(t)| dt - Q_0.
\]

**Proof.** Let \(f(t) = \log |D_4(t + i0)|,\) \(t \in \mathbb{R}\). From Lemma 2.4, we have \(f \in L^1_{loc}(\mathbb{R})\). Moreover, from (2.31) and (2.14), we can deduce that

\[
\left| \log D_4(t + i0) \right| \leq \|Y_0^4(t + i0)\|_{B_1} = O(t^{-2}) \quad \text{as} \quad t \to \pm \infty,
\]

which gives (2.37). Using (2.35) and (4.12) and Lemmas 2.2 and 2.3, we obtain, for all \(k \in \mathbb{C}_+\),

\[
\frac{1}{\pi} \int_{\mathbb{R}} (h(t)/(k - t)) dt = \frac{1}{\pi} \int_{\mathbb{R}} ((f(t) - \Re \psi_2(t) + \Re \psi_3(t))/(k - t)) dt = \frac{1}{\pi} \int_{\mathbb{R}} (f(t)/(k - t)) dt + i\psi_2(k) - i\psi_3(k).
\]

Let us prove (2.38). Let \(k = i\tau, \tau \to \infty\). The Lebesgue Theorem and (2.37) give

\[
(1/\pi) \int_{\mathbb{R}} (f(t)/(k - t)) dt = (1/\pi) \int_{\mathbb{R}} (f(t)/k) dt = (1/(\pi k)) \int_{\mathbb{R}} (tf(t)/(k - t)) dt = o(1)/k,
\]

since \(t/(i\tau - t) \to 0\) as \(\tau \to \infty\) for each \(t \in \mathbb{R}\). Then, substituting the asymptotics (2.40), (2.19), and (2.27) into (2.36) yields (2.38).

**Proof of Theorem 1.1.** Due to Lemma 2.4, the function \(\psi(k)\) is analytic on \(\mathbb{C}_+\) and satisfies (1.6), since we have (2.10).

From (2.21), we have the first asymptotics in (1.7). From (2.30), we have

\[
\log \psi(k) - \psi_2(k) = \log D_3(k), \quad k \in \overline{\mathbb{C}_+},
\]

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 27 No. 1 2020
and from (2.31), (2.26), we get the other asymptotics in (1.7).

From (2.10), we have the first estimate in (1.8), and from (2.18), we have the other one.

We have \( \psi \in \mathcal{H}_\infty \), and then (4.13) and the asymptotics (1.7) yield (1.9).

The asymptotics (1.7) shows that all zeros of \( \psi \in \mathcal{H}_\infty \) are uniformly bounded. Then these zeros satisfy (1.10), since this is the standard fact for such functions, see Theorem 2.3 in [14]. We note that (see page 53 of [14]), in general, condition (1.10) is replaced in the upper half plane by the larger condition:

\[
\sum \Im z_j/(1 + |z_j|^2) < \infty,
\]

and the Blaschke product \( B \) with zeros \( z_j \) has the form

\[
B(z) = ((z - i)^m/(z + i)^m) \prod_{z_j \neq 0} ((1 + z_j^2)/(1 + z_j^2))((z - z_j)/(z - \overline{z}_j)), \quad z \in \mathbb{C}_+.
\]

If all moduli \( |z_n| \) are uniformly bounded, then the bound (2.41) becomes \( \sum \Im z_j < \infty \), and the convergence factors in (2.42) are not needed, since \( \prod_{z_j \neq 0} (z - \overline{z}_j) \) already converges.\[\blacksquare\]

3. PROOF OF THE MAIN THEOREMS

We describe the determinant \( \psi(k), k \in \mathbb{C}_+ \), in terms of a canonical factorization.

**Proof of Theorem 1.2.** From Theorem 1.1, we deduce that \( \psi \in \mathcal{H}_\infty \). Recall that, due to (2.32), each zero of \( \psi \) in \( \mathbb{C}_+ \) belongs to the half-disk \( \{k \in \mathbb{C}_+ : |k| \leq r_*\} \) for some \( r_* > 0 \). From Theorem 1.1, we deduce that the function \( \psi \in \mathcal{H}_\infty \) satisfies all conditions of Theorem 4.3, and so we obtain all results of Theorem 1.2. Note that all needed results about the Blaschke product \( B \) are proved in Lemma 4.1.

We can now prove the first main result about trace formulas.

**Proof of Corollary 1.3.** It follows from (1.12) that, for all \( k \in \mathbb{C}_+ \),

\[
\psi'(k)/\psi(k) = B'(k)/B(k) - (i/\pi) \int_{\mathbb{R}} (k - t)^{-2} d\mu(t), \quad B'(k)/B(k) = \sum 2i \Im k_j/((k - k_j)(k - \overline{k}_j)),
\]

where \( d\mu(t) = h(t)dt - dv(t) \). Differentiating the modified determinant \( \psi = D_2 \), we obtain

\[
\psi'(k)/\psi(k) = -2k \text{Tr} \left( R(k) - R_0(k) + R_0(k)VR_0(k) \right).
\]

Combining (3.1) and (3.2), we deduce (1.17). If, in addition, \( V \in L^1(\mathbb{R}^3) \), then (2.12) yields (1.18).\[\blacksquare\]

Let us now prove the second main result about the trace formulas.

**Proof of Theorem 1.4.** We show (1.24). Since \( \psi \in \mathcal{H}_\infty \), using Theorem 4.3 and (2.38), the following asymptotic identity

\[
\psi(k) = \exp \left[ -i(Q_0 + o(1))/k \right] = \exp \left[ -i(B_0 + K_0 + o(1))/k \right] \exp \left[ i(C_M + o(1))/k \right]
\]
as \( \Re k \to \infty \), which yields (1.24). We show (1.25). Let \( V \in W_{m+1}, m \geq 1 \). From Lemma 4.2 and from asymptotics (1.20), we obtain

\[
\psi(k) = \exp \left[ -iQ_0/k - iQ_2/k^3 - iQ_4/k^5 - \cdots - i(Q_{2m} + o(1))/k^{2m+1} \right]
= \exp \left[ -iB_0/k - iB_1/2k^2 - iB_2/3k^3 - \cdots \right] \exp \left[ -iK_0/k - iK_1/k^2 \cdots \right]
\times \exp i \left[ (J_0 - iI_0)/k + (J_1 - iI_1)/k^2 + \cdots + (J_{2m} - iI_{2m})/k^{2m+1} + M_{2m}(k)/k^{2m+1} \right],
\]

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 27 No. 1 2020
which yields \( Q_j = B_j/(j + 1) + K_j - J_j + iI_j, \ j = 0, 1, \ldots \) Thus we have (1.25), since \( I_j = 3Q_j \).

**Proof of Theorem 1.5.** Let \( r > 0 \). Represent (1.24) in the form:

\[
B_0 + \nu(\mathbb{R})/\pi = X_1 + X_2, \quad X_1 = (1/\pi) \int_{-r}^{r} \xi(k)dk, \quad X_2 = (1/\pi) \int_{\mathbb{R}\setminus[-r,r]} \xi(k)dk, \quad (3.3)
\]

where \( \xi(k) = \log|D_4(k + i0)|, \ k \in \mathbb{R} \). Let us estimate \( X_2 \). From (2.6), (2.14), we have

\[
\xi(k) \leq \|Y_0(k)\|^4_{\mathcal{B}_4} \leq C_2^4 \|V\|^4/|k|^2 \quad \forall k \in \mathbb{R}, \quad (3.4)
\]

which yields

\[
X_2 = (1/\pi) \int_{\mathbb{R}\setminus[-r,r]} \xi(k)dk \leq (1/\pi) \int_{\mathbb{R}\setminus[-r,r]} \|Y_0(k)\|^4_{\mathcal{B}_4} dk \leq A \int_{k > r} \|V\|^4/|k|^2 dk = (A/r)\|V\|^4, \quad (3.5)
\]

where \( A = (2/\pi)C_2^4 \). Let us estimate \( X_1 \). Using identity (2.35), represent \( X_1 \) in the form

\[
X_1 = X_{11} + X_{12} - X_{13}, \quad X_{11} = (1/\pi) \int_{-r}^{r} \log |\psi(k)|dk, \quad X_{1j} = (1/\pi) \int_{-r}^{r} \Re \psi_j(k)dk, \quad (3.6)
\]

where \( \psi_j = (1/j) \text{Tr} Y_0^j(k), j = 1, 2 \). Estimate (1.6) gives

\[
X_{11} = (1/\pi) \int_{-r}^{r} \log |\psi(k)|dk \leq B r \|V\|_{3/2}^2, \quad B = (2/\pi)C_\bullet. \quad (3.7)
\]

Estimates (1.8) gives

\[
X_{12} = (1/\pi) \int_{-r}^{r} \Re \psi_2(k)dk \leq (2r^{1/2}/\pi) \|\psi_2\| \leq (r^{1/2}/(4\pi)^{1/3}) \|V\|_{3/2} \|V\|. \quad (3.8)
\]

Using the estimate \( |\psi_3(k)| \leq \|Y_0(k)\|_2^2 \|Y_0(k)\|_{\mathcal{B}_4} \) and (2.14) and (2.10), we obtain

\[
X_{13} \leq (1/(3\pi)) \int_{-r}^{r} \|Y_0(k)\|^2_{\mathcal{B}_2} \|Y_0(k)\|_{\mathcal{B}_4} dk \leq 2K \|V\|_{3/2}^2 \|V\| \int_0^r k^{1/2} dk = K \|V\|_{3/2}^2 \|V\| r^{1/2}, \quad (3.9)
\]

where \( K = 8C_\bullet C_2/(3\pi) \). Collecting (3.6)–(3.9) at \( \alpha = \|V\|, \beta = \|V\|_{3/2} \), we obtain for all \( r > 0 \):

\[
X_1 + X_2 \leq A\alpha^4/r + B\beta^2 r + \alpha \beta r^{1/2} E, \quad E = (4\pi)^{-4/3} + K \beta. \quad (3.10)
\]

The function \( A\alpha^4/r + B\beta^2 r \) has the minimum at \( r_o = (A/B)^{1/2} \alpha^2/\beta \). This implies

\[
X_1 + X_2 \leq 2(A/B)^{1/2} \alpha^2 \beta + (A/B)^{1/4} \alpha^2 \beta^{1/2} E = \alpha^2 [2(A/B)^{1/2} \beta + (4\pi)^{-4/3}(A/B)^{1/4} \beta^{1/2} + (A/B)^{1/4} \beta^{3/2} K] \quad (3.11)
\]

and, using the definitions of \( A, B, K, \alpha, \) and \( \beta \), we obtain (1.26).
4. APPENDIX. ANALYTIC FUNCTIONS ON THE UPPER HALF-PLANE

Let us describe functions in the Hardy spaces. If \( f \in \mathcal{H}_\infty \), then

\[ \lim_{v \to +0} B(u + i v) = B(u + i 0), \quad |B(u + i 0)| = 1 \text{ a.e., } u \in \mathbb{R}, \]

\[ \lim_{v \to 0} \int \log |B(u + i v)| du = 0. \]

**Lemma 4.1.** Let \( f \in \mathcal{H}_\infty \) and let all its zeros \( \{z_j\} \) in \( \mathbb{C}_+ \) be uniformly bounded by \( r_0 \). Then

(i) the coefficient \( B_n = 2 \sum_j \Im z_j^{n+1}, n \geq 0 \), satisfies

\[ |B_n| \leq 2 \sum \Im z_j^{n+1} < \infty \quad \forall \ n \geq 0, \]

\[ |B_n| \leq (\pi/2)(n + 1)r_0^n B_0 \quad \forall \ n \geq 1. \]

(ii) The function \( \log B(z) \) has an analytic continuation from \( \mathbb{C}_+ \setminus \{|z| < r_0\} \) to the domain \( \{|z| > r_0\} \), where \( r_0 = \sup |z_n| \), and has the following Taylor expansion:

\[ \log B(z) = -IB_0/z - iB_1/(2z^2) - iB_2/(3z^3) - \cdots - iB_{n-1}/(nz^n) - \cdots \]

**Proof.** (i) Consider the function \( F(w) = f(z(w)) \), where the conformal mapping \( w: \mathbb{C}_+ \to \mathbb{D} \) is given by

\[ w = w(z) = (i - z)/(i + z), \quad z \in \mathbb{C}_+, \quad iw + zw = i - z, \quad z = z(w) = i(1 - w)/(1 + w). \]

Thus \( F \in \mathcal{H}_\infty(\mathbb{D}) \), and the zeros of \( F \) have the form \( w_j = w(z_j) \). We have the identity

\[ 1 - |w|^2 = 1 - |i - z|^2/|i + z|^2 = 1 - ((1 - y)^2 + x^2)/|i + z|^2 = 4y/|i + z|^2, \]

which yields

\[ \sum (1 - |w_j|^2) = \sum 4y_j|i + z_j|^{-2}. \]

Consider the function \( f_1(l) = f(\sqrt{l}), l \in \mathbb{C}_+ \) and let \( w(\lambda) \) be conformal mapping. The function \( f_1 \) on \( \mathbb{C}_+ \) is analytic and uniformly bounded, i.e., \( f_1 \in \mathcal{H}_\infty(\mathbb{C}_+) \). The function \( f_1 \) has the zeros \( \lambda_j = z_j^2 \in \mathbb{C}_+ \) which satisfy

\[ \Im \lambda_j = \sum \Im z_j^2 < \infty \]

If we apply similar arguments to the function \( f(\sqrt{\lambda}), \lambda \in \mathbb{C}_- \), we obtain

\[ \sum \Im \lambda_j < 0 \quad \sum \Im z_j^2 < \infty. \]

Thus we have proved (4.3) in the case \( n = 2 \). The proof for the case \( n \geq 3 \) is similar.

We have

\[ z_j = |z_j|e^{i\phi_j}, \quad \phi_j = \begin{cases} \phi_j^+ \quad \text{for } \phi_j \in (0, \pi/2], \\ \pi - \phi_j^- \quad \text{for } \phi_j \in (\pi, \pi/2). \end{cases} \]

This yields

\[ B_0 = 2 \sum \Im z_j = 2 \sum |z_j| \sin \phi_j < 2 \sum |z_j|\phi_j^+ = A \]
and
\[ B_0 = 2 \sum z_j = 2 \sum |z_j| \sin \phi_j > (4/\pi) \sum |z_j| \phi_j^\pm = (2/\pi) A. \] (4.9)

These estimates give us
\[ \sum \Re z_j^n = \sum |z_j|^n \sin n\phi_j, \] (4.10)
and so
\[ \left| \sum \Re z_j^n \right| \leq \sum |z_j|^n \sin n\phi_j^\pm \leq nr_0^{n-1} \sum |z_j| \phi_j^\pm = nr_0^{n-1} A \leq (\pi/2)nr_0^{n-1} B_0, \] (4.11)
which yields (4.4).

ii) Let \( c = z_j/z \) and \( \bar{c} = \bar{z}_j/z \). The function \( b_j(z) = \log ((z - z_j)/(z - \bar{z}_j)) \) has an analytic continuation from \( \mathbb{C}_+ \setminus \{|z| < r_0\} \) into the domain \( \{|z| > r_0\} \) and has the following Taylor series:
\[ b_j(z) = \log ((z - z_j)/(z - \bar{z}_j)) = \log(1 - c) - \log(1 - \bar{c}) = -\sum_{n=1}^\infty 2i\Re z_j^n/(nz^n). \]

Let \( b_{jm}(z) = -\sum_{n=1}^m 2i\Re z_j^n/(nz^n) \). We have the following simple estimates for \( |z| > r_0 \):
\[ \left| \log B(z) \right| \leq \sum_j |\log ((z - z_j)/(z - \bar{z}_j))| \leq \sum_j \sum_{n \geq 1} 2|\Re z_j^n|/(|n||z|^n), \]
and
\[ \left| \log B(z) - \sum_{j=1}^\infty b_{jm}(z) \right| \leq \sum_{j=1}^\infty \sum_{n \geq m} 2|\Re z_j^n|/(|n||z|^n) \leq \sum_{n \geq m} r_0^{n-1} A|z|^{-n} = A/r_0 (r_0/|z|)^{m+1}/(1 - r_0/|z|), \]
which yields (ii) and, in particular, (4.5). \[ \square \]

We need the following results (see [21, p. 128]):
- If \( u \in L^p(\mathbb{R}) \) and \( p \in (1, \infty) \), then
\[ F(k) = (i/\pi) \int_{\mathbb{R}} (u(t)/(k - t))dt \in \mathcal{H}_p \] (4.12)
and \( RF(k) \rightarrow u(t) \) as \( k \rightarrow t \) a.e in \( t \) (as \( k \rightarrow t \) non-tangentially asusual).
- If \( f \in \mathcal{H}_p, p \geq 1 \), then
\[ \log |f(t + i0)| \in L^p_{\text{loc}}(\mathbb{R}). \] (4.13)

We need some results about functions in Hardy spaces. We begin with asymptotics.

**Definition.** A function \( h \) belongs to the class \( \mathfrak{X}_m = \mathfrak{X}_m(\mathbb{R}) \) if \( h \in L^1_{\text{real,loc}}(\mathbb{R}) \) and
\[ h(t) = P_m(t) + h_m(t)/t^{m+1}, \quad P_m(t) = I_0/t + I_1/t^2 + \cdots + I_m/t^{m+1}, \]
\[ h_m(\cdot)/(1 + |\cdot|^a) \in L^1(\mathbb{R}), \quad h_m(t) = o(1) \quad \text{as} \quad t \rightarrow \pm \infty \] (4.14)
for some real constants \( I_0, I_1, \ldots, I_m \), where \( m \geq 0 \), and \( a < 1 \).

Note that, if \( h \in \mathfrak{X}_m \) for some \( m \geq 0 \), then there exist finite integrals (the principal values):
\[ J_j = \text{v.p.}(1/\pi) \lim_{s \rightarrow \infty} \int_{-s}^s h_{j-1}(t)dt, \quad h_j = t^{j+1}(h(t) - P_j(t)), \quad h_{-1} = h, \] (4.15)
for all \( j = 0, 1, 2, \ldots, m - 1 \). For \( h \in \mathfrak{X}_m \), we define the integrals
\[ M(k) = (1/\pi) \int_{\mathbb{R}} (h(t)/(k - t))dt, \quad M_m(k) = (1/\pi) \int_{\mathbb{R}} (h_m(t)/(k - t))dt, \quad k \in \mathbb{C}_+. \] (4.16)
In order to obtain the trace formulas in Theorem 1.4, we need to determine the asymptotics of \( M_m \).
Lemma 4.2. Let \( M(k) = (1/\pi) \int_{\mathbb{R}} (h(t)/(k - t))dt \), \( k \in \mathbb{C}_+ \), for some \( h \in \mathcal{X}_m, m \geq 0 \). Then the following identity holds:

\[
M(k) = (\mathcal{J}_0 - iI_0)/(k + \mathcal{J}_1 - iI_1)/k^2 + \cdots + (\mathcal{J}_m - iI_m + M_m(k))/k^{m+1} \quad \forall k \in \mathbb{C}_+, \tag{4.17}
\]

where the real constants \( \mathcal{J}_0, \ldots, \mathcal{J}_m \) and the functions \( M_m \) are given by (4.16) and satisfy

\[
M_m(i\tau) = o(1) \quad \text{as} \quad \tau \to \infty. \tag{4.18}
\]

Proof. Consider the case \( m = 0 \); the proof for the case \( m \geq 1 \) is similar. By (4.14), we have

\[
h \in L^1_{\mathrm{loc}}(\mathbb{R}), \quad h(t) = (I_0 + h_0(t))/t, \quad h_0(t) = o(1) \quad \text{as} \quad t \to \pm \infty, \tag{4.19}
\]

for some real constant \( I_0 \). Introduce the functions

\[
f(t) = h(t) - f_0(t), \quad f_0(t) = I_0 t/(t^2 + 1), \quad t \in \mathbb{R},
\]

and note that \( f(t) = h(t) - f_0(t) = o(1)/t \) as \( t \to \pm \infty \). Let \( k \in \mathbb{C}_+ \). Then we have

\[
F_0(k) = (1/\pi) \int_{\mathbb{R}} (f(t)/(k - t))dt = (I_0/k + M_0(k))/k + I_0/(k + i), \tag{4.20}
\]

Thus the function \( M(k) = (1/\pi) \int_{\mathbb{R}} (h(t)/(k - t))dt \) is well defined and has the form

\[
M(k) = (1/\pi) \int_{\mathbb{R}} (h(t)/(k - t))dt = F_0(k) + F(k) = -iI_0/k + i + F(k),
\]

\[
F(k) = (1/\pi) \int_{\mathbb{R}} (f(t)/(k - t))dt = (1/\pi) \lim_{s \to \infty} \int_{-s}^{s} (f(t)/(k - t))dt = \mathcal{J}_0/k + M_0(k)/k + I_0/(k + i),
\]

since, using \( 1/(k - t) = 1/k + t/(k(k - t)) \) and \( h_0 = th - I_0 \), we have

\[
\int_{-s}^{s} (f(t)/(k - t))dt = \frac{1}{k} \int_{-s}^{s} f(t)dt + \frac{1}{k} \int_{-s}^{s} tf(t)/(k - t)dt = \frac{1}{k} \int_{-s}^{s} h(t)dt + \frac{1}{k} \int_{-s}^{s} (h(t) + I_0/((t^2 + 1)(k - t)))dt,
\]

\[
\left(\frac{1}{\pi}\right) \lim_{s \to \infty} \int_{-s}^{s} h(t)dt = \mathcal{J}_0, \quad \left(\frac{1}{\pi}\right) \lim_{s \to \infty} \int_{-s}^{s} (h(t)/k - t)dt = \int_{\mathbb{R}} (h(t)/k - t)dt = M_0(k), \tag{4.23}
\]

\[
\left(\frac{1}{\pi}\right) \lim_{s \to \infty} \int_{-s}^{s} (I_0/((t^2 + 1)(k - t)))dt = \left(\frac{1}{\pi}\right) \int_{\mathbb{R}} (I_0/((t^2 + 1)(k - t)))dt = I_0/(k + i). \tag{4.24}
\]

Collecting (4.20)–(4.21), we obtain

\[
M(k) = -iI_0/(k + i) + F(k) = -iI_0/(k + i) + \mathcal{J}_0/k + M_0(k)/k + I_0/(k(k + i)) = (\mathcal{J}_0 - iI_0 + M_0(k))/k, \tag{4.25}
\]

where

\[
M_0(k) = (1/\pi) \int_{\mathbb{R}} (h_0(t)/(k - t))dt, \quad h_0(t) = th(t) - I_0, \quad \mathcal{J}_0 = \text{v.p.}(1/\pi) \int_{\mathbb{R}} h(t)dt.
\]

In order to show (4.18), define a function \( g_k(t) = (1 + |t|)/k - t \), \( t \in \mathbb{R} \), and note that \( \|g_k\|_{\infty} = o(1) \) as \( k = i\tau, \tau \to \infty \). Then we have

\[
|M_0(k)| = (1/\pi) \left| \int_{\mathbb{R}} (h_0(t)/(k - t))dt \right| \leq (\|g_k\|_{\infty}/\pi) \int_{\mathbb{R}} |h_0(t)|/(1 + |t|)^{\alpha}dt = o(1)
\]

as \( k = i\tau, \tau \to \infty \), which yields (4.18). \( \blacksquare \)

Let us describe the canonical factorization.
Theorem 4.3. Let $f \in \mathcal{H}_p$ for some $p \geq 1$ and $f(k) = 1 + o(1)$ as $|k| \to \infty$ uniformly with respect to $\arg k \in [0, \pi]$. Then $f$ has a canonical factorization in $\mathbb{C}_+$ given by

$$f = f_{\text{in}} f_{\text{out}}, \quad f_{\text{in}}(k) = B(k) e^{-iK(k)}, \quad K(k) = (1/\pi) \int_{\mathbb{R}} (k - t)^{-1} d\nu(t).$$  \hspace{1cm} (4.26)

- $d\nu(t) \geq 0$ is some singular compactly supported measure on $\mathbb{R}$ satisfying
  $$\nu(\mathbb{R}) = \int_{\mathbb{R}} d\nu(t) < \infty, \quad \text{supp} \nu \to [-r_c, r_c],$$
  \hspace{1cm} (4.27)
  for some $r_c > 0$. In particular, if $f$ is continuous on $\overline{\mathbb{C}}_+$, then
  $$\text{supp} \nu \subset \{ k \in \mathbb{R} : f(k) = 0 \} \subset [-r_c, r_c].$$  \hspace{1cm} (4.28)

- The function $K(\cdot)$ has an analytic continuation from $\mathbb{C}_+$ into the domain $\mathbb{C} \setminus [-r_c, r_c]$ and has the following Taylor series:
  $$K(k) = \sum_{j=0}^{\infty} K_j/(k^{j+1}), \quad K_j = (1/\pi) \int_{\mathbb{R}} t^j d\nu(t).$$ \hspace{1cm} (4.29)

- $B$ is the Blaschke product for $\exists k > 0$ given by (1.11). Let, in addition, $\log |f(\cdot)|/(1 + |\cdot|)^a \in L^1(\mathbb{R})$ for some $a < 1$. Then the outer factor $f_{\text{out}}$ is
  $$f_{\text{out}}(k) = e^{iM(k)}, \quad M(k) = (1/\pi) \int_{\mathbb{R}} \log(|f(t)|/(k - t)) dt, \quad k \in \mathbb{C}_+.$$ \hspace{1cm} (4.30)

**Remark.** (1) These results are crucial to determine the trace formulas in Theorem 1.4.

(2) The integral $M(k), k \in \mathbb{C}_+$, in (4.30) converges absolutely since $f(t) = 1 + O(1)/t$ as $t \to \pm \infty$.

**Proof.** From (4.34), we deduce that each zero $\in \overline{\mathbb{C}}_+$ of $f \in \mathcal{H}_p^0$ belongs to the half-disk $\{ k \in \overline{\mathbb{C}}_+ : |k| \leq r_c \}$, for some $r_c > 0$. It is well known (see [21, p. 119]) that the function $f(k), k \in \mathbb{C}_+$, has a standard factorization $f = f_{\text{in}} f_{\text{out}}$, where $f_{\text{in}}$ is the inner factor given by

$$f_{\text{in}}(k) = e^{i\gamma + ia} B(k) e^{-iK(k)}, \quad K(k) = (1/\pi) \int_{\mathbb{R}} (1/(k - t) + t/(t^2 + 1)) d\nu(t)$$ \hspace{1cm} (4.31)

with $\gamma \in \mathbb{R}, \alpha > 0$, and $d\nu(t) \geq 0$ is a singular compactly supported measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} (1 + t^2)^{-1} d\nu(t) < \infty$ and $\text{supp} \nu \subset [-r_c, r_c]$. Here $B$ is the Blaschke product for $\exists z > 0$ given by (1.11), since all zeros of $\psi$ are uniformly bounded. Without loss of generality (since $\text{supp} \nu \subset [-r_c, r_c]$), we can write $K(k)$ in the form

$$K(k) = (1/\pi) \int_{-r_c}^{r_c} (k - t)^{-1} d\nu(t).$$ \hspace{1cm} (4.32)

Due to (4.32), we deduce that the function $K$ is real on the set $\mathbb{R} \setminus [-r_c, r_c]$. Thus the function $K$ has an analytic extension from $\mathbb{C}_+$ into the whole cut plane $\mathbb{C} \setminus [-r_c, r_c]$. Moreover, $K$ has the Taylor series in the domain $\{ |k| > r_c \}$. We determine the Taylor series. For large $|k|$, we have

$$K(k) = (1/k) \pi \int_{-r_c}^{r_c} d\nu(t)/(1 - (t/k)) = \sum_{n \geq 0} \int_{\mathbb{R}} t^n d\nu(t)/k^{n+1} = K_0/k + K_1/k^2 + K_2/k^3 + K_3/k^4 + \cdots$$

for $|k| > r_c$,

The function $f_{\text{out}}$ is the outer factor given by

$$f_{\text{out}}(k) = e^{iM(k)}, \quad M(k) = (1/\pi) \int_{\mathbb{R}} (1/(k - t) + t/(t^2 + 1)) \log |f(t)| dt.$$ \hspace{1cm} (4.33)

Consider $M(k)$. Due to (4.34), we have $f(t) = 1 + O(1/t)$ as $t \to \pm \infty$. Thus without loss of generality, we can rewrite $M(k)$ in the form (4.30).

As is well known (see of [21, p. 119]), if $f$ is continuous in $\overline{\mathbb{C}}_+$, then (4.28) holds true. \hfill \blacksquare

Let $f \in \mathcal{H}_p$ for some $0 < p \leq \infty$. For integer $m \geq 0$, we say that $f$ belongs, the class $\mathcal{H}^m_p = \mathcal{H}^m_p(\mathbb{C}_+)$ if $f$ satisfies

$$\log f(k) = Q_0/(ik) + Q_1/(ik^2) + Q_2/(ik^3) + \cdots + (Q_m + o(1))/(ik^{m+1}),$$ \hspace{1cm} (4.34)

as $|k| \to \infty$ uniformly in $\arg k \in [0, \pi]$, for some constants $Q_j \in \mathbb{C}$. We describe a canonical factorization of functions in $\mathcal{H}^m_p$. 

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 27 No. 1 2020
Theorem 4.4. Let a function $f$ belong to $\mathcal{H}_p^m$ for some $m \geq 0$ and $p \geq 1$ and let the function $h(t) = \log |f(t)|$, $t \in \mathbb{R}$, belong to $\mathcal{X}_m$. Then $f$ has a canonical factorization $f = f_{in} f_{out}$ in $\mathbb{C}_+$ given by Theorem 4.3, where the function $M$ satisfies the following identity for any $k \in \mathbb{C}_+$:

$$M(k) = (1/\pi) \int_{\mathbb{R}} (h(t)/(k - t))dt = (J_0 - iI_0)/k + (J_1 - iI_1)/k^2 + \cdots + (J_m - iI_m + M_m(k))/k^{m+1},$$

where

$$M_m(k) = (1/\pi) \int_{\mathbb{R}} (h_m(t)/(k - t))dt, \quad J_j = v.p.(1/\pi) \int_{\mathbb{R}} h_{j-1}(t)dt, \quad h_{j-1} = (I_j + h_j(t)0/t, \quad M_m(k) = o(1) \quad \exists k \to \infty, \quad j = 0, 1, \ldots, m, \text{ and } h_{-1} = h. \text{ Moreover, the following trace formulas holds:}$$

$$B_j + K_j = \Re Q_j + J_j, \quad j = 0, 1, \ldots, m.$$  

Proof. From Lemma 4.2, we deduce relations (4.35)–(4.37). From Theorem 4.3, we have $i \log f(k) = i \log B(k) + K(k) - M(k)$, where

- the function $K$ has the Taylor series (4.29),
- the function $B(k)$ has the Taylor series (1.15),
- the function $M$ has the asymptotics given by (4.35)–(4.37),
- the function $f$ has the asymptotics given by (4.34).

Substituting all these asymptotics into identity (4.39), we obtain

$$i \log f(k) = Q_0/k + Q_1/k^2 + Q_2/k^3 + \cdots + O(1)/k^{m+2}$$

$$= ((B_0 + K_0)/k + (B_1/2 + K_1)/k^2 + (B_2/3 + K_2)/k^3 + \cdots)$$

$$- ((J_0 - iI_0)/k + (J_1 - iI_1)/k^2 + (J_2 - iI_2)/k^3 + \cdots)$$

which gives (4.38).}

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