ON THE EXISTENCE OF AN APOLAR STAR CONFIGURATION

IMAN BAHMANI JAFARLOO

ABSTRACT. Our goal is to understand for which tuple \((d, l, n)\) there exists a star configuration \(X(l)\) apolar to the generic form \(F\) of degree \(d\) in \(n + 1\) variables.

1. INTRODUCTION

Let \(S = \mathbb{C}[x_0, \ldots, x_n]\) be the standard graded polynomial ring. For a given form \(F\) of degree \(d\), the Waring rank of \(F\) is the least value \(s\) for which there exist linear forms \(L_1, \ldots, L_s\) such that \(F = \lambda_1 L_1^d + \cdots + \lambda_s L_s^d\) where \(\lambda_i \in \mathbb{C}\) and it is denoted by \(\text{rk}(F)\). For example, the quadratic binary monomial \(M = x_0x_1\) can be written by sums of squares of two linear forms \(L_1 = x_0 + x_1\) and \(L_2 = x_0 - x_1\) with some coefficients. \(x_0x_1 = 1/4L_1^2 - 1/4L_2^2\). This shows that \(\text{rk}(M) \leq 2\). One can see that \(\text{rk}(M) > 1\) since \(M\) is not a perfect square. We conclude that \(\text{rk}(M) = 2\). The Waring rank of homogeneous forms has been studied widely [BBT13, BHMT18, One16, AH95, Ckov17, CCG12].

In [AH95], the authors computed the rank for any value of \(n\) and \(d\) that holds for all forms \(F\) in a dense Zariski open subset of the space of forms over the complex numbers and called the rank of a generic form in \(n\) variables of degree \(d\). Later, in [CCG12] a complete solution to the Waring problem was given in the case of monomials in any number of variables and sums of pairwise coprime monomials over \(\mathbb{C}\). The Variety of Sums of Powers of \(F\), \(\text{VSP}(F, d) \subset \text{Hilb}_s(\mathbb{P}^{n-1})^*\), is the closure in the Hilbert Scheme of \(\mathbb{P}^{n-1}\) of the set of \(s\)-tuples \([L_1], \ldots, [L_n]\), such that \(F = L_1^d + \cdots + L_n^d\).

We refer the reader to [BBT13, Car06] and references therein for better understanding the varieties of sums of powers. The authors in [BBT13, Theorem 1] prove that any Waring decomposition of a monomial is obtained from a complete intersection ideal. Let us consider the standard graded polynomial ring \(T = \mathbb{C}[y_0, \ldots, y_n]\). Suppose \(F \in S_d\) is a form such that \(F = \sum_{i=1}^r L_i^d\). Let \(I \subseteq T\) be the homogeneous ideal of forms vanishing on \(X = \{[L_1], \ldots, [L_r]\} \subset \mathbb{P}^{n}\). Then, we say that the ideal \(I\) is apolar to \(F\) or that \(X\) is apolar to \(F\). Recently, the authors in [MO18] by using Apolarity theory construct minimal decompositions of symmetric tensors with low rank.

Given any linear form \(L \in T_1\), we let \(\ell\) denote the corresponding line in \(\mathbb{P}^2\). Given a collection of \(l\) linear forms \(L_1, \ldots, L_l\) in \((k[y_0, y_1, y_2])_1\) such that \(\ell_i \cap \ell_j \cap \ell_k = \emptyset\) for all triples \(\{i, j, k\} \subseteq \{1, \ldots, l\}\), a star configuration is the set of \(\left(\begin{array}{c} l \\ 2 \end{array}\right)\) points formed by taking all possible intersections of lines. We will briefly refer to any such a set as \(X(l)\). These special configurations, and their generalizations in \(\mathbb{P}^n\) [Ghm13, AS12, Bh10, Shi11], have risen in prominence due, in part, to the fact that they have nice algebraic properties. The defining ideal of \(X(l)\) in the projective plane is then given by

\[
I_{X(l)} = \bigcap_{i \neq j} (L_i, L_j).
\]

2010 Mathematics Subject Classification. 13P05, 14N20, 14M05.

Key words and phrases. star configuration, Waring problem, apolar theory, ideals of points.
An easy calculation shows that the ideal associated to the star configuration $\mathbb{X}(l)$ is a homogeneous ideal generated in degree $l - 1$. (See, for details, [CGVT15, CVT11]).

$$I_{\mathbb{X}(l)} = \left( \hat{L}_1, \hat{L}_2, \ldots, \hat{L}_l \right), \quad \hat{L}_i = \prod_{i \neq j} L_j,$$

and $I_{\mathbb{X}(l)}$ is the ideal of $\{[L_1], \ldots, [L(l)]\} \subset \mathbb{P}^2$.

**Definition 1.1.** Let $\mathbb{X}$ be a set of points which is apolar to a form. We say that $\mathbb{X}$ is an apolar star configuration if the set $\mathbb{X}$ is a star configuration.

We consider the following natural question:

**Question 1.** For which tuples $(d, l, n)$ does the generic degree $d$ form have an apolar star configuration $\mathbb{X}(l)$?

In the following, we answer the above question. In Section 2, we review some known results concerned to the generic ranks of form. In Section 3, we prove that any generic ternary quadratic and cubic forms have an apolar star configuration $\mathbb{X}(l)$ for all $l \geq 3$ and $l \geq 4$, respectively. In Section 4, we recall the definition of star configuration in $\mathbb{P}^n$ and then find a necessary condition on the existence of an apolar star configuration $\mathbb{X}(l)$ for any generic form $F \in S_d$. We show that for $l \leq d$ with $d \geq 4$, the necessary condition fails and for $l > d + 1$ the question has an affirmative answer. In Conjecture 1, we state that any generic ternary form of degree $d \geq 5$ has an apolar star configuration $\mathbb{X}(d + 1)$. By the strategy we introduce in Remark 4.11, we are able to answer the conjecture only case by case. Computer experiments show that for any tuple $(d, d + 1, 2)$ there always exists an star configuration apolar to a generic ternary form of degree $d$ (see, Example 4.12 and Table 1). In Section 5, the question is given answer for $n \geq 3$ and the arguments are similar to Section 4. In Section 6, we look for apolar star configuration for some ternary cubic forms.

**Acknowledgments.** The author wants to thank his Ph.D. advisor Prof. Enrico Carlini for the patient guidance, encouragement, advice, and comments that greatly improved the manuscript.

2. Basic Facts

We make $T$ act via differentiation on $S$, e.g. we think of $y_j = \partial/\partial x_j$ (see, for example, [Ger96] or [IK99]). For any form $F$ of degree $d$ in $S$, we define the ideal $F^\perp \subseteq T$ as follows:

$$F^\perp = \{ \partial \in T : \partial F = 0 \}.$$  

Given a homogeneous ideal $I \subseteq T$ we denote by

$$\text{HF}(T/I, i) = \dim_k T_i - \dim_k I_i$$

its Hilbert function in degree $i$. It is well known that for all $i >> 0$ the function $\text{HF}(T/I, i)$ is a polynomial function with rational coefficients, called the Hilbert polynomial of $T/I$. We say that an ideal $I \subseteq T$ is one dimensional if the Krull dimension of $T/I$ is one, equivalently the Hilbert polynomial of $T/I$ is some integer constant, say $s$. The integer $s$ is then called the multiplicity of $T/I$. If, in addition, $I$ is a radical ideal, then $I$ is the ideal of a set of $s$ distinct points in $\mathbb{P}(S_1)$. We will use the fact that if $I$ is a one dimensional saturated ideal of multiplicity $s$, then $\text{HF}(T/I, i)$ is always $\leq s$. 
The following lemma, which we will call the Apolarity Lemma, is a consequence of [IK99, Lemma 1.31].

**Lemma 2.1** (Apolarity Lemma). A homogeneous degree $d$ form $F \in S$ can be written as

$$F = \sum_{i=1}^{s} \alpha_i L_i^d, \ L_i \in S_1 \text{ pairwise linearly independent, } \alpha_i \in \mathbb{C}$$

if and only if there exists $I \subseteq F^\perp$ such that $I$ is the ideal of a set of $s$ distinct points in $\mathbb{P}(S_1)$.

The Waring rank of a given specific form is not known in general. However, we know the rank for a generic form, that is,

**Theorem 2.2** (On the rank of the generic form [AH95]). If $F$ is a generic degree $d$ form in $n + 1$ variables, then

$$\text{rk}(F) = \left\lceil \frac{(d+n)}{d} \right\rceil$$

Unless $(n,d) = (n,2), (2,4), (3,4), (4,3), (4,4)$, the generic rank for these cases are respectively, $n + 1, 6, 10, 8, 15$.

However, the rank for a given specific form can be bigger or smaller than the rank of the generic form. Moreover, it is trivial to show that every form of degree $d$ is a sum of at most $\binom{d+n}{d}$ $d^{th}$ powers of linear forms. But, in general, it is not known how big the rank of a degree $d$ form can actually be (see, [BHMT18]).

**Remark 2.3.** Let $F$ be any form of degree $d$. We say that the Apolarity Lemma (AL) condition is satisfied for a tuple $(d,l,n)$ if $|X(l)| \geq \text{rk}(F)$.

3. **CASE $d = 2, 3$**

In this section, we give a complete proof for the existence of a star configuration $X(l)$ apolar to the generic ternary quadratic and cubic forms.

**Lemma 3.1.** Let $l \geq 3$ be an integer. Then, for any rank three ternary quadratic form $F$, $I_{X(l)} \subset F^\perp$ for some star configuration $X(l)$.

**Proof.** First we deal with the $l = 3$ case. By the Apolarity Lemma, since the rank of the generic ternary quadratic form is three, there is a set of three distinct points in $\mathbb{P}(S_1)$ such that its defining ideal is contained in $F^\perp$. The points only have two possible configurations as follow:

1. three collinear points
2. three non-collinear points

Case (1) fails, since $F$ is a not binary form. Hence, the only possible case is (2) and since any set of three non-collinear points is a star configuration $X(3)$, it implies that $I_{X(3)} \subset F^\perp$.

Now if $l \geq 4$, since $(F^\perp)_{l-1} = T_{l-1}$ and $I_{X(l)}$ is generated in degree $l - 1$, then we conclude that there exists a star configuration $X(l)$ apolar to $F$.

**Lemma 3.2.** Let $l \geq 3$ be an integer. The generic ternary cubic form $F$ is such that $F^\perp \supset I_{X(l)}$ for some star configuration $X(l)$ if and only if $l > 3$.
Proof. The Apolarity Lemma yields that there is no set of less than four points apolar to the
generic ternary cubic form $F$, since $\text{rk}(F) = 4$. Thus, there exists no star configuration $X(3)$
apolar to $F$. Let $\mathcal{A} = \{p_1, p_2, p_3, p_4\}$ be a set of four distinct points in $\mathbb{P}(S_1)$ which is apolar to $F$.
We need to consider the following three cases:

(1) the four points are collinear

(2) there are exactly three collinear points

(3) there are no more than two points on a line

In (1) $F$ is a binary form and in (2) $F$ is a cusp (see, [CCO17, Theorem 5.1]). Thus in both
cases (1) and (2) $F$ is not generic. It suffices that we only consider the third case. Let $J_A$ be the
defining ideal of $A$ in the case (3). The ideal $J_A \subset F^\perp$ is the ideal of the complete intersection
set of points $A$. Therefore, $J_A = (L_1 L_2, L_3 L_4)$ for suitable linear forms $L_1, L_2, L_3, L_4 \in T_1$. Set
$I = (L_2 L_3 L_4, L_1 L_3 L_4, L_1 L_2 L_4, L_1 L_2 L_3)$ and note that $I = I_X(4)$ for some $X(4)$. Since $I \subset J_A$,
we conclude that $I_X(4) \subset F^\perp$, and so the star configuration $X(4)$ is apolar to $F$.

Given any $l \geq 5$ and $\partial \in T_{l-1}$ then $\partial F = 0$. Hence, $T_{l-1} = (F^\perp)_{l-1}$ and we know that $I_X(l)$
-generated in degree $l - 1$. Therefore, $I_X(l) \subset (F^\perp)_{l-1}$ and the proof is complete.  

Remark 3.3. The argument of Lemma 3.2 yields that any rank four ternary cubic, which is not
a cusps, has an apolar $X(4)$.

4. CASE $d \geq 4$ AND $n = 2$

In this section, we prove Proposition 4.3 which is a necessary condition on the tuple $(d, l, n)$ for
the existence of an apolar star configuration $X(l)$. Since this result will be needed in Section 5,
we prove it for any generic form in any number of variables. We first restate the definition of star
configuration of points in $\mathbb{P}^n$.

In [CGVT14], the authors studied for a generic form $F \in S$, when the hypersurface defined by $F$
in $\mathbb{P}^n$ contains a star configuration. Indeed, they determine particular polynomial decomposition
of $F$.

Definition 4.1. Let $L_1, \ldots, L_l$ be $l$ general linear forms in $T = k[y_0, \ldots, y_n]$. We define a star
configuration of points in $\mathbb{P}^n$ to be the set of $\binom{n}{l}$ points obtained by intersecting $n$ of the hyperplanes
$\{L_i = 0\}$ in all possible ways. A star configuration $X(l)$ is the algebraic variety in $\mathbb{P}^n$ defined by
the homogeneous ideal

$$J = \bigcap_{\tau = \{j_1, \ldots, j_n\} \subseteq [l]} (L_{j_1}, \ldots, L_{j_n}).$$

Theorem 4.2 ([CGVT14]). Let $X(l) \subset \mathbb{P}^n$ be a star configuration of points. Then $X(l)$ has a
generic Hilbert function, that is,

$$\text{HF}(X(l), t) = \dim \mathbb{C}(S/I_X(l))_t = \min\left\{\binom{n+t}{t}, \binom{l}{n}\right\}.$$

Furthermore, the ideal $I_X(l)$ is generated by $\binom{l}{n-1}$ forms of degree $l - n + 1$.

Now, we are ready to state the necessary condition for any 3-tuple $(d, l, n)$.
Proposition 4.3. Let \( l \geq 3 \) and \( n \geq 2 \) be integers. If \( F \) is a generic degree \( d \) form such that \( I_{X(l)} \subset F \), then \( \rho(d, l, n) \geq 0 \) where,
\[
\rho(d, l, n) = \binom{l}{n} + nl - \binom{d + n}{d}.
\]

Proof. We describe all star configurations \( X(l) \) in \( \mathbb{P}^n \). Let \( \mathbb{P}^n \) be the dual projective space of \( \mathbb{P}^n \). We consider the quasi-projective variety
\[
D_t \subseteq \prod_{t \text{-times}} \mathbb{P}^n = (\mathbb{P}^n)^l
\]
where \( (\ell_1, \ldots, \ell_t) \in D_t \) if and only if no \( n + 1 \) of the hyperplanes \( \ell_i \) pass through the same point. Set \( X(l) = \{ p_i : i = 1, \ldots, \binom{l}{n} \} \subseteq \mathbb{P}^n \). Since \( \mathbb{P}^n \cong \mathbb{P}(S_1) \) we have that \( p_i = [L_i], L_i \in S_1 \) for all \( i \). Hence, \( X(l) = \{ [L_1], \ldots, [L_{\binom{l}{n}}] \} \). We consider the following Veronese map
\[
\nu_d : \mathbb{P}(S_1) \cong \mathbb{P}^n \rightarrow \mathbb{P}^{N_{d,n}} \cong \mathbb{P}(S_d) \quad \text{where} \quad N_{d,n} = \binom{d+n}{d} - 1.
\]

Let \( H \) be the projectivization of the linear span of the set \( \{ \nu([L_1]), \ldots, \nu([L_{\binom{l}{n}}]) \} \), that is, \( H = \mathbb{P}(<[L_1^d], \ldots, [L_{\binom{l}{n}}^d]>). \) Since \( \text{HF}(X(l), d) = \dim_{\mathbb{C}}(T/I_{X(l)}d) = \dim_{\mathbb{C}} <[L_1^d], \ldots, [L_{\binom{l}{n}}^d] > \), we have that \( \dim H = \binom{l}{n} - 1 \), (see, [CGVT15, Lemma 2.1.]). We also define
\[
\Psi : D_t \rightarrow \text{Gr}(\mathbb{P}^{\binom{l}{n}}|_{\mathbb{P}^{N_{d,n}}}),
\]
that maps \( (\ell_1, \ldots, \ell_t) \) to \( <[L_1^d], \ldots, [L_{\binom{l}{n}}^d]> \).

For a generic point \( [F] \in H \), clearly \( F = \lambda_1 L_1^d + \ldots + \lambda_{\binom{l}{n}} L_{\binom{l}{n}}^d \) and we define the following incidence correspondence:
\[
\Sigma(d, l, n) = \{ ((\ell_1, \ldots, \ell_t), [F]) : <[L_1^d], \ldots, [L_{\binom{l}{n}}^d] \supseteq [F] \} \subseteq D_t \times \mathbb{P}^{N_{d,n}}.
\]

We also consider the natural projection maps \( \pi_1 : \Sigma(d, l, n) \rightarrow D_t \) and \( \pi_2 : \Sigma(d, l, n) \rightarrow \mathbb{P}^{N_{d,n}} \). Using standard fiber dimension argument for a generic \( (\ell_1, \ldots, \ell_t) \in D_t \), we see that
\[
\dim (\Sigma(d, l, n)) \leq \dim \pi^{-1}_1 ((\ell_1, \ldots, \ell_t)) + \dim D_t = \binom{l}{n} - 1 + nl \tag{*}
\]

We have that \( \pi_2 \) is dominant if and only if Question 1 has an affirmative answer. The map \( \pi_2 \) is dominant only if \( \dim (\Sigma(d, l, n)) - \dim (\mathbb{P}^{N_{d,n}}) \geq 0 \) and this is equivalent to
\[
\binom{l}{n} + nl - \binom{d + n}{d} \geq 0.
\]

\[\square\]

Corollary 4.4. Consider the previous proposition. If \( n = 2 \), then
\[
\rho(d, l, 2) = l(l-1) + 4l - (d+2)(d+1).
\]
Remark 4.5. If the AL condition is not satisfied for a tuple \((d, l, n)\) then \(\rho(d, l, n) < 0\). Assume that the AL condition does not hold. Hence, we have \(\left\lfloor \frac{d}{n} \right\rfloor = |X(l)| < \text{rk}(F) = \left\lfloor \frac{d+n}{n+1} \right\rfloor\). We know that \(\frac{(d+n)}{n+1} \leq \left\lfloor \frac{(d+n)}{n+1} \right\rfloor \leq \frac{(d+n)}{n+1} + 1\). Let \(\left\lfloor \frac{d}{n} \right\rfloor \). Then,

\[
\rho(d, l, n) = \left( \frac{l}{n} \right) + nl - \left( \frac{d+n}{d} \right) < 2 \left( \frac{l}{n} \right) - \left( \frac{d+n}{d} \right) < 2 \frac{(d+n)}{n+1} - \left( \frac{d+n}{d} \right)
\]

For any \(d \geq 2\), we have that \(\rho(d, l, n) < -\frac{(d+2)}{3} < 0\). Same argument if \(\left\lfloor \frac{d}{n} \right\rfloor < \frac{(d+n)}{n+1} + 1\).

Remark 4.6. Assume that \(F\) is a generic ternary form of degree four. A star configuration \(X(3)\) in projective plane consists of three points. By the Apolarity Lemma, there is no ideal \(I\) of a set of three points such that \(F\) is six. Therefore, there is no \(X(3)\) apolar to \(F\). By previous remark, since the AL condition is not held, so the necessary condition is not satisfied for the tuple \((4, 3, 2)\). An easy calculation verifies it, \(\rho(4, 3, 2) = -12 < 0\).

Remark 4.7. Let \(l = 4\) and \(F\) be a generic ternary quartic form. The AL condition satisfies for this case since \(|X(4)| = \binom{4}{3} = 6 = \text{rk}(F)\). But since \(\rho(4, 4, 2) = -2 < 0\), we conclude that the generic ternary quartic form does not have an apolar set of six distinct points which is a star configuration \(X(4)\).

Remark 4.8. Let \(F\) be a generic ternary form of degree five. Since \(\text{rk}(F) = 7\), the Apolarity Lemma yields that there is no set of less than seven points apolar to \(F^\perp\). Therefore, \(I_{X(l)} \not\subset F^\perp\) for \(l = 3, 4\). Suppose that \(l = 5\), then \(\rho(5, 5, 2) = -2 < 0\) and we conclude that no \(X(5)\) is apolar to \(F\).

Corollary 4.9. Let \(l \geq 3\) and \(d \geq 4\) be integers. If \(l \leq d\), then there is no star configuration \(X(l)\) apolar to a ternary generic degree \(d\) form \(F\).

Proof. For \(l - d \leq 0\), we have \(\rho(d, l, 2) = l(l - 1) + 4l - (d+2)(d+1) = (l - d)(3 + l + d) - 2 < 0\). Thus we conclude that there is no \(X(l)\) such that \(I_{X(l)} \subset F^\perp\). \(\square\)

Corollary 4.10. Let \(d \geq 4\) be an integer. If \(l \geq d + 2\), then there exists a star configuration \(X(l)\) apolar to a ternary degree \(d\) form \(F\).

Proof. For any \(\partial \in T_{l-1}\), we have \(\partial F = 0\). Since, \(T_{l-1} \subset (F^\perp)_{l-1}\) and \(I_{X(l)}\) is generated in degree \(l - 1\) form, we conclude that \(I_{X(l)} \subset (F^\perp)_{l-1}\). \(\square\)

Remark 4.11. Let us recall the natural projection map \(\pi_2 : \Sigma(d, l, n) \rightarrow \mathbb{P}^{N_d,n}\) from Proposition 4.3. Let \(d \geq 3\) and \(n \geq 2\) be integers. The closure of the image of \(\pi_2\) is the closure of the union of linear span of all possible \(X(l) \subset \mathbb{P}^n\), and we denote by

\[
U(d, l, n) := \overline{\text{Im} \pi_2}.
\]
We only consider tuples \((d, l, n)\) such that \(\rho(d, l, n) \geq 0\) since otherwise, Question 1 has negative answer. To compute the \(\dim \mathcal{U}(d, l, n)\), it is enough to find the dimension of the tangent space to \(\text{Im} \pi_2\) at a generic point \(p \in \text{Im} \pi_2\).

\[
\text{Im} \pi_2 = \bigcup_{\mathcal{X}(l) \subset \mathbb{P}^n} < [L^d_1], \ldots, [L^d_{(n)}] >.
\]

In order to compute algorithmically the dimension of the tangent space, we proceed as follows.

We construct \(l\) linear forms \(L_1, \ldots, L_l\) using \((n + 1)l\) variables.

\[
L_1 = a_{0,1}y_0 + a_{1,1}y_1 + \cdots + a_{n,1}y_n, \quad L_l = a_{0,l}y_0 + a_{1,l}y_1 + \cdots + a_{n,l}y_n.
\]

Let \(\mathcal{X}(l) = \{p_1, p_2, \ldots, p_{(l)}\}\) be the set of points that is obtained by constructing the star configuration \(\mathcal{X}(l)\) using \(L_1, \ldots, L_l\). Note that any \(p_i = [b_{0,i}, b_{1,i}, \cdots, b_{n,i}]\) for \(i = 1, \ldots, (l)\) is a point where,

\[
b_{j,i} = f_{j,i}(a_{0,1}, \ldots, a_{0,l}; \ldots; a_{j,1}, \ldots, a_{j,l}; \ldots; a_{n,1}, \ldots, a_{n,l}), \quad j = 0, \ldots, n
\]

is a polynomial and \(a_{j,1}, \ldots, a_{j,l}\) means the variables \(a_{j,1}, \ldots, a_{j,l}\) don’t appear in \(b_{j,i}\). For a pair \(((\ell_1, \ldots, \ell_l), [F]) \in \Sigma(d, l, n)\), \(F\) is a form of degree \(d\) with \(m\) variables where \(m = (n + 1)l + (l)\).

\[
F = \lambda_1 L_1^d + \cdots + \lambda_{(l)} L_{(l)}^d; \quad L_i = b_{0,i}x_0 + b_{1,i}x_1 + \cdots + b_{n,i}x_n.
\]

Let \(g_i := \text{coeff}_{m_i}(F)\), where \(m_i\) is the \(i\)-th element of the standard monomial basis of \(S_d\) respect to the lexicographic order, for \(i = 1, \ldots, \binom{d + n}{d}\). We define the map

\[
\Gamma : \mathbb{A}^m \longrightarrow \mathbb{A}^{N_{d,n} + 1}
\]

which maps every \(F\) to \(I = (g_1, g_1, \ldots, g_{N_{d,n} + 1})\). We find the rank of the Jacobian matrix \(m \times (N_{d,n} + 1)\) of the map evaluated at a generic point \(p\).

\[
\text{Jac} \mathcal{X} = \begin{bmatrix}
\frac{\partial g_1}{\partial a_{0,1}} & \cdots & \frac{\partial g_1}{\partial a_{0,l}} & \cdots & \frac{\partial g_1}{\partial a_{n,1}} & \cdots & \frac{\partial g_1}{\partial \lambda_1} & \cdots & \frac{\partial g_1}{\partial \lambda_{(l)}} \\
\vdots & & \frac{\partial g_1}{\partial a_{0,1}} & \cdots & \frac{\partial g_1}{\partial a_{0,l}} & \cdots & \frac{\partial g_1}{\partial a_{n,1}} & \cdots & \frac{\partial g_1}{\partial \lambda_1} & \cdots & \frac{\partial g_1}{\partial \lambda_{(l)}} \\
\vdots & & \vdots & & \ddots & & \ddots & & \ddots & & \ddots & \cdots & \frac{\partial g_1}{\partial \lambda_{(l)}} \\
\vdots & & \vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial a_{0,1}} & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial a_{0,l}} & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial a_{n,1}} & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial a_{n,l}} & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial \lambda_1} & \cdots & \frac{\partial g_{N_{d,n} + 1}}{\partial \lambda_{(l)}}
\end{bmatrix}
\]

Using this fact \(\text{rank}(\text{Jac} I)_p = \dim T_p \text{Im} \pi_2\) ([CK16, Dimension]), the map \(\pi_2\) is dominant if the Jacobian matrix at a generic point \(p\) is full rank.

\[
\dim \mathcal{U}(d, l, n) = \dim \text{Im} \pi_2 = \text{rank}(\text{Jac} I)_p - 1.
\]

**Example 4.12.** As an application of the above remark we take the tuple \((4, 5, 2)\). We see that the necessary condition is satisfied \(\rho(4, 5, 2) = 10 \geq 0\). We used the following code in the computer algebra system \texttt{Macaulay2} [GS] to compute \(\text{rank}(\text{Jac} I)_p\). We obtained \(\text{rank}(\text{Jac} I)_p = 15\). Hence, \(\dim \mathcal{U}(4, 5, 2) = 14\) and the map \(\pi_2\) is dominant. This generic form of degree \(d\) has an apolar star configuration \(\mathcal{X}(5)\).
Macaulay2, version 1.11
i1 : needsPackage "NumericalImplicitization"

i2 : rkJacI=(d,l,n)->(    S=CC[x_(0,1) .. x_(n,l), c_0 .. c_(binomial(l,n)-1)];
R= S[X_0 .. X_n];
M=for i from 1 to l list matrix{toList (x_(0,i) .. x_(n,i))};
H=for b in (for i from 0 to #subsets(M,n)-1 list
    for a in (subsets(M,n))_i list(a)) list matrix b;
P=for t from 0 to #H-1 list
    for j from 0 to n list(-1)^(j)*(minors(n,H_t))_(n-j);
F=sum for i from 0 to #P-1 list
    c_(i)*(sum for j from 0 to n list P_i_j*X_j)^d;
I=transpose substitute((coefficients F)#1,S);
     p=point{apply(toList(0..<#gens ring I),i->
        sub(random(-100,100), coefficientRing ring I))};
     numericalImageDim(I, ideal 0_(ring I), p)
)
o2 = rkJacI
o2 : FunctionClosure

i3 : rkJacI(4,5,2)
o3 = 15

Remark 4.13. As we showed in Remark 4.7, for the tuple (4, 4, 2), the AL condition is satisfied, but the necessary condition is not satisfied \( \rho(4, 4, 2) = -2 \). Using the above code for the tuple \((d, l, n) = (4, 4, 2)\), we verify that the map \( \pi_2 \) is not dominant and its image is codim one in \( \mathbb{P}^{14} \).

\[
\text{codim} \mathcal{U}(4, 4, 2) = 14 - \dim \mathcal{U}(4, 4, 2) = 14 - 13 = 1.
\]

i4 : rkJacI(4,4,2)
o4 = 14

We are interested to know the degree of the image. One can check that degree \( \mathcal{U}(4, 4, 2) \) by definition is the cardinality of \( \mathcal{L} \cap \mathcal{U}(4, 4, 2) \) where the intersection is a finite set of reduced points and \( \mathcal{L} \in \text{Gr}(\mathbb{P}^1, \mathbb{P}^{14}) \) is a general linear space. In order to find the degree of the image we proceeded as follows:

i5 : numericalImageDegree(I, ideal 0_(ring I), maxThreads => allowableThreads)
o5 = a pseudo-witness set, indicating the degree of the image is 15
o5 : PseudoWitnessSet

The degree of the image is 15 and Bezout’s Theorem for higher dimension yields that the degree of the hypersurface \( \mathcal{U}(4, 4, 2) = V(\mathcal{F}) \) is 15.

By Corollaries 4.9, 4.10 and Example 4.12, we can answer Question 1 unless \( l = d + 1 \) where \( d \geq 5 \).

Conjecture 1. Let \( d \geq 5 \) be an integer. For a generic ternary degree \( d \) form \( F \) there exists a star configuration \( \mathcal{X}(d+1) \) apolar to it.
Let us first check the necessary condition for our claim.

\[ \rho(d, l, 2) = \rho(d, d + 1, 2) = 2(d + 1) \geq 0, \]

for any \( d \geq 5 \) and the necessary condition is satisfied. Computer experiments suggest that the Jacobian matrix is full rank for \( d \geq 5 \).

| \( d \) | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | \cdots | \( t \) |
|-----|----|----|----|----|----|----|----|----|----|--------|--------|
| \( \rho(d, d + 1, 2) \) | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | \cdots | 2(t + 1) |
| \( \dim U(d, d + 1, 2) \) | 20 | 27 | 35 | 44 | 54 | 65 | 77 | 90 | 104 | \cdots | \( N_{t,2} \) |

Table 1. Computation in Macaulay2

Table 1 verifies that the conjecture holds for \( d \leq 13 \). We expect that the strategy used to show that the map \( \pi_2 \) is dominant will show that any generic ternary form of degree \( d \geq 5 \) has an apolar star configuration \( X(d+1) \); the difficulty is finding the rank of just only one maximal minor of the Jacobian matrix for any \( d \geq 5 \).

5. THE \( n \geq 3 \) CASE

In this section, we present a complete solution to the existence of a star configuration \( X(l) \) apolar to the generic form of degree \( d \geq 2 \) in \( n + 1 \) variables, where \( n \geq 3 \).

**Remark 5.1.** By Theorem 4.2, we know that the ideal \( I_{X(l)} \) is generated by \( \left( \frac{l}{n-1} \right) \) forms of degree \( l - n + 1 \). Let \( F \) be any form of degree \( d \leq l - n \). For any \( \partial \in T_{l-n+1} \), we have that \( \partial F = 0 \). Hence, \( T_{l-n+1} = (F^\perp)_{l-n+1} \) and it follows that \( I_{X(l)} \subseteq (F^\perp)_{l-n+1} \). Therefore, for \( l \geq d + n \) there exists a star configuration \( X(l) \) apolar to \( F \).

**Lemma 5.2.** Let \( F \) be a generic quadratic form and \( n \geq 3 \) be an integer. If \( l = n + 1 \), then there exists a star configuration \( X(l) \) apolar to \( F \). If \( l < n + 1 \), then there is no star configuration \( X(l) \) apolar to \( F \).

**Proof.** The rank of a generic quadratic form \( F \) in \( n + 1 \) variables is exactly the number of variables which is equal to

\[ \text{rk}(F) = n + 1 = \binom{n+1}{n} = \binom{l}{n} = |X(l)|. \]

We claim that any \( n + 1 \) points in \( \mathbb{P}^n \) are a star configuration if there exits no hyperplane contains all the points. We give a proof by induction on \( n \). The rank for a generic ternary quadratic form is three, \( \text{rk}(F) = 3 \). Any set of three non-collinear points in \( \mathbb{P}^2 \) is a star configuration. Consider
In this case \( \text{rk}(F) = 4 \). Let \( X = \{p_1, p_2, p_3, p_4\} \) be the set of four points such that all the points are not on a hyperplane. Consider \( H_1, \ldots, H_4 \) be the general hyperplanes such that

\[
H_1 = \langle p_2, p_3, p_4 \rangle, \quad H_2 = \langle p_1, p_3, p_4 \rangle, \quad H_3 = \langle p_1, p_2, p_4 \rangle, \quad H_4 = \langle p_1, p_2, p_3 \rangle,
\]

where \( H_2 \cap H_3 \cap H_4 = \{p_1\}, \ldots, H_1 \cap H_2 \cap H_3 = \{p_4\} \). By definition of star configuration in \( \mathbb{P}^n \), we immediately conclude that \( X \) is a star configuration \( X(4) \). Now, let \( F \) be a generic quadratic form in \( n + 1 \) variables. In this case \( \text{rk}(F) = n + 1 \). Let \( X = \{p_1, p_2, p_3, \ldots, p_{n+1}\} \) be the set of \( n + 1 \) points which all are not on a hyperplane. Consider \( H_1, \ldots, H_{n+1} \) be the general hyperplanes such that

\[
H_i = \langle p_1, \ldots, \hat{p}_i, \ldots, p_{n+1} \rangle, \quad \bigcap_{i \neq j} H_j = \{p_i\}, \quad i, j \in \{1, 2, \ldots, n + 1\}.
\]

Hence, \( X \) is a star configuration \( X(n+1) \) which is apolar to \( F \).

For \( l < n + 1 \), we have that \( X(l) \) is set of only one point or \( \emptyset \). Hence, the AL condition is not satisfied and the is no \( X(l) \) apolar to \( F \).

**Theorem 5.3.** Let \( d \geq 3 \) be an integer and \( n = 3, 4, 5 \). Let \( F \) be a generic form of degree \( d \). If \( l \geq d + n \), then there exists a star configuration \( X(l) \) apolar to \( F \). If \( l < d + n \), then there does not exist a star configuration \( X(l) \) apolar to \( F \) unless,

\[
(d, l, n) = (3, 5, 3), (4, 6, 3), (5, 7, 3), (3, 6, 4), (3, 7, 5).
\]

**Proof.** By Remark 5.1, we conclude that for any \( l \geq d + n \) there exits a star configuration \( X(l) \) apolar to \( F \). Now, assume that \( l < d + n \) and consider the following possibilities:

**(a)** If \( l = d + n - 1 \), then

\[
\rho(d, d + n - 1, n) = \binom{d + n - 1}{n} + n(d + n - 1) - \binom{d + n}{d} = n(d + n - 1) - \binom{d + n - 1}{d},
\]

and we have the following cases:

1. **case \( n = 3 \)**

\[
\rho(d, d + 2, 3) = 3(d + 2) - \binom{d + 2}{d} = (d + 2)(5 - d)/2.
\]

Therefore, for \( d \geq 3 \) we have \( (d + 2)(5 - d)/2 < 0 \) unless \( d = 3, 4, 5 \).

2. **case \( n = 4 \)**

\[
\rho(d, d + 3, 4) = 4(d + 3) - \binom{d + 3}{d} = ((d + 6)(3 - d) + 4)(d + 3)/6.
\]

Hence, for all \( d \geq 3 \), \( \rho(d, d + 3, 4) < 0 \) except \( d = 3 \).

3. **case \( n = 5 \)**

\[
\rho(d, d + 4, 5) = 5(d + 4) - \binom{d + 4}{d} = (d + 4)(3 - d)(d^2 + 9d + 38)/24.
\]

We conclude that \( \rho(d, d + 4, 5) < 0 \) for all \( d \geq 3 \) unless \( d = 3 \).
(b) If \( l \leq d + n - 2 \), then we have \( \binom{l}{n} \leq \binom{d+n-2}{n} \) and \( nl \leq n(d + n - 2) \). Hence,
\[
\rho(d, l, n) = \binom{l}{n} + nl - \binom{d+n}{d} \leq \binom{d+n-2}{n} + n(d + n - 2) - \binom{d+n}{d}.
\]

Same as part (a), we consider the following cases:

1. case \( n = 3 \)
   \[
   \rho(d, l, 3) \leq \frac{d+1}{3} + 3(d+1) - \binom{d+3}{d} = -(d+1)(d-2).
   \]
   It is obvious to see that \(-(d+1)(d-2)\) \( < 0 \) for any \( d \geq 3 \).

2. case \( n = 4 \)
   \[
   \rho(d, l, 4) \leq \frac{d+2}{4} + 4(d+2) - \binom{d+4}{d} = -(d+2)(2d^2 + 5d - 21)/6
   \]
   \[
   \leq -2(d+2).
   \]
   So, for all \( d \geq 3 \) we have that \(-2(d+2)\) \( < 0 \).

3. case \( n = 5 \)
   \[
   \rho(d, l, 5) \leq \frac{d+3}{5} + 5(d+3) - \binom{d+5}{d} = (d+3)(d^3 + 5d^2 + 8d - 56)/6
   \]
   \[
   \leq -10(d+3)/3.
   \]
   Therefore, \(-10(d+3)/3\) \( < 0 \) for all \( d \geq 3 \).

Hence, for \( l < d + n \) the necessary condition is not satisfied, \( \rho(d, l, n) < 0 \), except for five tuples which were appeared in (a)(1), (a)(2) and (a)(3). So, to complete the proof we only need to prove that the above five cases hold. By the strategy in Remark 4.11, if we show that \( \dim \mathcal{U}(d, l, n) = N_{d,n} \) for the above cases, then the proof is done. Here is the computation in Macaulay2.

\[
\begin{align*}
18 : & \text{toList} \{ \text{rkJacI}(3,5,3), \text{rkJacI}(3,6,4), \\
& \quad \text{rkJacI}(3,7,5), \text{rkJacI}(4,6,3), \text{rkJacI}(5,7,3) \} \\
o8 : & \{ 20, 35, 56, 35, 56 \} \\
o8 : & \text{List}
\end{align*}
\]

| \( (d, l, n) \) | \( (3,5,3) \) | \( (3,6,4) \) | \( (3,7,5) \) | \( (4,6,3) \) | \( (5,7,3) \) |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| \( N_{d,n} \)   | 19          | 34          | 55          | 34          | 55          |
| \( \rho(d, l, n) \) | 5           | 4           | 0           | 3           | 0           |
| \( \dim \mathcal{U}(d, l, n) \) | 19          | 34          | 55          | 34          | 55          |

Table 2. The exceptional cases

Table 2 confirms that the necessary condition \( \rho(d, l, n) \) is satisfied for the exceptional cases and shows that the map \( \pi_2 \) is dominant. Hence, we conclude that Question 1 has positive answers for those tuples.
There exists a star configuration \( \mathbb{X}(l) \) apolar to a generic form of degree \( d \). If \( l \geq d + n \), then there exists a star configuration \( \mathbb{X}(l) \) apolar to a generic form of degree \( d \).

**Proof.** If \( l \geq d + n \), then from Remark 5.1 there exists a star configuration \( \mathbb{X}(l) \) apolar to a generic form of degree \( d \). If \( l < d + n \), then \( \rho(d, l, n) < 0 \), and it follows that there is no star configuration \( \mathbb{X}(l) \) apolar to a generic form of degree \( d \). For \( l \leq d + n - 1 \), we have that \( \binom{l}{n} \leq \binom{d + n - 1}{n} \) and \( nl \leq n(d + n - 1) \). Therefore,

\[
\rho(d, l, n) = \left( \frac{l}{n} \right) + nl - \binom{d + n - 1}{d} \\
\leq \binom{d + n - 1}{n} + n(d + n - 1) - \binom{d + n - 1}{d} \\
= n(d + n - 1) - \binom{d + n - 1}{d} \\
= n(d + n - 1) - \frac{1}{d!} (d + n - 1) \cdots (n + 1)n \\
= n(d + n - 1) \left( 1 - \frac{1}{d(d - 1)} \binom{d + n - 2}{d - 2} \right) n(d + n - 1) \\
= n(d + n - 1) \left( 1 - \frac{1}{d(d - 1)} \binom{d + n - 2}{d - 2} \right).
\]

Since \( n(d + n - 1) > 0 \), then it suffices to prove that \( \binom{d + n - 2}{d - 2} > d(d - 1) \), which is true because

\[
\binom{d + n - 2}{d - 2} > \binom{d + 5 - 2}{d - 2} = \frac{1}{5!} (d + 3)(d + 2)(d + 1)d(d - 1) \\
\geq \frac{1}{5!} (3 + 3)(3 + 2)(3 + 1)d(d - 1) = d(d - 1).
\]

Hence, for \( l < d + n \) there is no \( \mathbb{X}(l) \) apolar to the generic form of degree \( d \). \( \square \)

6. NOT GENERIC CASE

In this section, we investigate the existence of an apolar star configuration to possibly not generic forms. As an application we show that there exists a star configuration \( \mathbb{X}(4) \) apolar to ternary cubic forms. We recall same non-generic cases for which the Waring rank is known.

**Theorem 6.1** ([CCG12]). If \( M = x_0^{d_0} \cdots x_n^{d_n}, \) with \( d_0 \leq \ldots \leq d_n \), then,

\[
\mathrm{rk}(M) = \frac{1}{d_0 + 1} \prod_{i=0}^{n} (d_i + 1).
\]

**Theorem 6.2** ([CCG12]). Consider the degree \( d \) form

\[
F = M_0 + \cdots + M_r = x_{i_0,1}^{a_{i_0,1}} \cdots x_{i_0,n_0}^{a_{i_0,n_0}} + \cdots + x_{i_r,1}^{a_{i_r,1}} \cdots x_{i_r,n_r}^{a_{i_r,n_r}},
\]

where \( a_{i,1} + \cdots + a_{i,n_i} = d, 1 \leq a_{i,1} \leq \cdots \leq a_{i,n_i} (0 \leq i \leq r). \) If \( d = 1 \), then \( \mathrm{rk}(F) = 1. \) If \( d \geq 2, \) then \( \mathrm{rk}(F) = \sum_{i=0}^{r} \mathrm{rk}(M_i). \)
Remark 6.3. If \( \mathcal{M} = x_0 x_1 x_2 \) we can not apply Lemma 3.2, to deduce the existence of \( \mathbb{X}(4) \). However, since \( \text{rk}(\mathcal{M}) = 4 \) and any minimal apolar set is a complete intersection we can conclude. It is also possible to follow an algorithmic approach which will be useful in what follows.

Let us consider the following four linear forms

\[
\begin{align*}
L_1 &= a_1 y_0 + b_1 y_1 + c_1 y_2, \\
L_2 &= a_2 y_0 + b_2 y_1 + c_2 y_2, \\
L_3 &= a_3 y_0 + b_3 y_1 + c_3 y_2, \\
L_4 &= a_4 y_0 + b_4 y_1 + c_4 y_2.
\end{align*}
\]

The star configuration \( \mathbb{X}(4) \) constructed by the linear forms \( L_1, L_2, L_3, L_4 \) is apolar to \( \mathcal{M} \), or \( I_{\mathbb{X}(4)} \subset \mathcal{M}^\perp = (y_0^2, y_1^2, y_2^2) \) if and only if the monomial \( y_0 y_1 y_2 \) does not appear in the generators of the ideal \( I_{\mathbb{X}(4)} \), i.e.,

\[
\begin{align*}
C_1 &:= \text{coeff}_{y_0 y_1 y_2} L_1 = c_2 b_3 a_4 + b_2 c_3 a_4 + c_2 a_3 b_4 + a_2 c_3 b_4 + b_2 a_3 c_4 + a_2 b_3 c_4 = 0, \\
C_2 &:= \text{coeff}_{y_0 y_1 y_2} L_2 = c_1 b_3 a_4 + b_1 c_3 a_4 + c_1 a_3 b_4 + a_1 c_3 b_4 + b_1 a_3 c_4 + a_1 b_3 c_4 = 0, \\
C_3 &:= \text{coeff}_{y_0 y_1 y_2} L_3 = c_1 b_2 a_4 + b_1 c_2 a_4 + c_1 a_2 b_4 + a_1 c_2 b_4 + b_1 a_2 c_4 + a_1 b_2 c_4 = 0, \\
C_4 &:= \text{coeff}_{y_0 y_1 y_2} L_4 = c_1 b_2 a_3 + b_1 c_2 a_3 + c_1 a_2 b_3 + a_1 c_2 b_3 + b_1 a_2 c_3 + a_1 b_2 c_3 = 0.
\end{align*}
\]

Assume that

\[
P = [a_1 : b_1 : c_1; a_2 : b_2 : c_2; a_3 : b_3 : c_3; a_4 : b_4 : c_4]
\]

\[
= [-15/7 : 3 : 0; 5 : 7 : 0; -1/2 : -3/16 : 1; -1/3 : -1/8 : -2/3].
\]

Since the point \( P \in V(C_1, C_2, C_3, C_4) \) and any three minors of the matrix

\[
A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}
\]

are non zero (i.e., they are linearly independent), then we conclude that the star configuration \( \mathbb{X}(4) \) evaluated at \( P \) is apolar to \( \mathcal{M} \).

Remark 6.4. For a not generic ternary cubic form \( \mathcal{N} \) could exist an apolar star configuration \( \mathbb{X}(l) \) even for \( l \leq 3 \). Consider for example \( \mathcal{N} = x_0^3 + x_1^3 + x_2^3 \).

Lemma 6.5. Any ternary cuspidal cubic is projectively equivalent to \( V(x_0^3 - x_1^2 x_2) \).

Proof. See, [Gib98, Lemma 15.5]. \( \square \)

Proposition 6.6. Any ternary cuspidal cubic has an apolar star configuration \( \mathbb{X}(4) \).

Proof. By Lemma 6.5, it is enough to show that the normal form \( \mathcal{C} = x_0^3 - x_1^2 x_2 \) has an apolar star configuration \( \mathbb{X}(4) \). By Theorem 6.2, we know that \( \text{rk}(x_0^3 - x_1^2 x_2) = 4 \). Hence, the AL condition is satisfied for \( l = 4 \). With an elementary calculation on can see that

\[
\mathcal{C}^\perp = (y_2^2, y_0 y_2, y_0 y_1, y_1^2, y_0^3 + 3 y_1^2 y_2).
\]

Similarly in Remark 6.3, we impose the ideal \( I_{\mathbb{X}(4)} \) constructed by the four linear forms (\( \star \)) to the ideal \( \mathcal{C}^\perp \). With a little bit more work, the star configuration \( \mathbb{X}(4) \) constructed by the four linear forms \( L_1 = y_0, L_2 = y_1, L_3 = y_1 - y_2, L_4 = y_0 + y_1 + y_2 \) is apolar to \( \mathcal{C} \). Hence, there exists an apolar star configuration for any ternary cuspidal cubic. \( \square \)
In the following we consider an example to find an apolar star configuration $X(4)$ for a non-generic cubic of rank five in projective plane.

**Lemma 6.7.** Any rank five ternary cubic is projectively equivalent to $V(x_0(x_2^2 + x_0x_1))$.

**Proof.** See, [Gib98, Lemma 15.6].

**Proposition 6.8.** There exists an apolar star configuration $X(4)$ for any ternary cubic of rank five (conic plus tangent line).

**Proof.** Using Lemma 6.7, we only need to find an apolar star configuration $X(4)$ for the normal form of conic plus tangent type $G = x_0(x_2^2 + x_0x_1)$. Since the $rk(G) = 5$ and $|X(4)| = 6$ then the AL condition is satisfied when $l = 4$. We have that, $G^\perp = (y_1y_2, y_1^2, y_0y_1 - y_2^2, y_0^2y_2, y_0^3)$. Same as in Proposition 6.6, we force the ideal $I_{X(4)}$ constructed by the linear forms $(\star)$ to be in the ideal $G^\perp$. One can easily check that the four linear forms $L_1 = y_0 + (47/132)y_1 - 3y_2$, $L_2 = 4y_0 - (20/3)y_1 - 10y_2$, $L_3 = 2y_0 + (862/33)y_1 + 7y_2$ and $L_4 = 11y_0 - (421/12)y_1 + 6y_2$ are an apolar star configuration $X(4)$ for $G$. Hence, any ternary cubic curves of conic plus tangent type has an apolar star configuration $X(4)$ and the proof is done.

**Theorem 6.9.** Any ternary cubic form has an apolar star configuration $X(4)$.

**Proof.** By Proposition 6.8, the proof is followed.

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I. BAHMANI JAFARLOO,
DIPARTIMENTO DI SCIENZE MATEMATICHE, POLITECNICO DI TORINO (DISMA), ITALY,
DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, ITALY.

E-mail address: iman.bahmanijafarloo@polito.it
E-mail address: iman.bahmanijafarloo@unito.it