A change of variables theorem for the multidimensional Riemann integral

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Abstract

The most general change of variables theorem for the Riemann integral of functions of a single variable has been published in 1961 (see [5]). In this theorem, the substitution is made by an ‘indefinite integral’, that is, by a function of the form $t \mapsto c + \int_a^t g =: G(t)$ where $g$ is Riemann integrable on $[a, b]$ and $c$ is any constant. We prove a multidimensional generalization of this theorem for the case where $G$ is injective – using the fact that the Riemann primitives are the same as those Lipschitz functions which are almost everywhere strongly differentiable in $(a, b)$. We prove a generalization of Sard’s lemma for Lipschitz functions of several variables that are almost everywhere strongly differentiable, which enables us to keep all our proofs within the framework of the Riemannian theory which was our aim.

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1 Introduction

As far as we know, the following theorem appeared first in ([5]).

**Theorem 1** If \( g : [a, b] \rightarrow \mathbb{R} \) is Riemann integrable, \( c \in \mathbb{R}, \forall t \in [a, b] G(t) := c + \int_a^t g \), and the function \( f \) is Riemann integrable on the range of \( G \), then \((f \circ G) \cdot g \) is Riemann integrable, and

\[
\int_{G(a)}^{G(b)} f = \int_a^b (f \circ G) \cdot g.
\]

Notice that the first statement of Theorem 1 is somewhat surprising because the composition \( f \circ G \) need not be Riemann integrable even if \( G \) is \( C^\infty \) (see [3, Example 34 in Chapter 8.]). Some years later, D. Preiss and J. Uher in [11] proved the converse: boundedness of \( f \) and integrability of \((f \circ G)g \) implies integrability of \( f \). The aim of the present paper is to formulate and prove a multidimensional version of these two theorems for the case where \( G \) is injective on the interior of its domain — with a proof that remains within the framework of the Riemann theory. In textbooks, the usual assumption on \( G \) is that it has a continuously differentiable extension to an open set that covers the closure of the original — Jordan measurable — domain. Observe that this assumption implies Lipschitz continuity (see Theorem 10). The starting point to the corresponding generalization is the fact (which seems not to be well-known) that a function \( G : [a, b] \rightarrow \mathbb{R} \) is a Riemann primitive if and only if it is Lipschitz and almost everywhere strongly differentiable.

In the next section, after introducing some notations and terminology, we summarize some well-known facts about Riemann integrability, and give a basic theorem about the change of variables with easy proof and 'hard-to-check' assumptions. In section 3, we investigate the notion 'strong differentiability' and other auxiliary tools, then in section 4 we prove that for injective functions \( G \) that are Lipschitz and almost everywhere strongly differentiable, and for properly chosen \( g \), conditions a), b) and c) of Theorem 7 are fulfilled (injectivity will be assumed only on the difference of \( D(G) \) and a set of Lebesgue measure zero).

2 Terminology and some basic facts about Riemann integrability

For any \( H \subset \mathbb{R}^m \), the set of Jordan measurable subsets of \( H \) will be denoted by \( \mathcal{J}_H \), in the case \( H := \mathbb{R}^m \) the subscript will be omitted. The volume or Jordan content of a Jordan measurable set \( X \subset \mathbb{R}^m \) will be denoted by \( V(X) \) and the outer Jordan content of a bounded set \( Y \) by \( V^*(Y) \).

By a Jordan partition of \( X \in \mathcal{J} \) we mean a finite collection of pairwise non-overlapping sets in \( \mathcal{J}_X \) the union of which is \( X \). The set of all Jordan partitions of \( X \in \mathcal{J} \) will be denoted by \( \Pi(X) \). By the norm of a Jordan partition \( \Phi \in \Pi(X) \)
we mean the number $|\Phi| := \max\{\text{diam}(H) : H \in \Phi\}$. The lower sum, upper sum and oscillation sum of a bounded function $f : X \to \mathbb{R}$ corresponding to the partition $\Phi \in \Pi(X)$ is defined by $s_f(\Phi) := \sum_{H \in \Phi} \inf f|_H V(H)$, $S_f(\Phi) := \sum_{H \in \Phi} \sup f|_H V(H)$, resp.

$$O_f(\Phi) := S_f(\Phi) - s_f(\Phi) = \sum \text{osc}_f(H)V(H),$$

where $\text{osc}_f(H) := \sup f|_H - \inf f|_H = \sup \{|f(y) - f(x)| : x \in H, y \in H\}$.

The lower and upper Darboux integral of $f$ is

$$\int_X f := \sup s_f, \quad \text{resp.} \quad \int_X f := \inf S_f.$$

The bounded function $f : X \to \mathbb{R}$ is integrable (with integral $\alpha \in \mathbb{R}$) if its lower and upper Darboux integrals agree (and are equal to $\alpha$).

By a dotted Jordan partition of $X \in \mathcal{J}$ we mean a finite set of ordered pairs $\eta := \{(H_1, y_1), \ldots, (H_n, y_n)\}$ such that $D(\eta) := \{H_1, \ldots, H_n\} \in \Pi(X)$, and $y_i \in H_i$ for $i = 1, \ldots, n$. The Riemann sum of the function $f : X \to \mathbb{R}$ corresponding to the dotted Jordan partition $\eta$ is $\sigma_f(\eta) := \sum_{i=1}^n f(y_i)V(H_i)$.

We will make use of the following well-known statements:

**Theorem 2 (Generalized Darboux Theorem)** For each $X \in \mathcal{J}$ and for each bounded $f : X \to \mathbb{R}$,

$$\lim_{|\Phi| \to 0} s_f(\Phi) = \int_X f, \quad \lim_{|\Phi| \to 0} S_f(\Phi) = \int_X f.$$

**Theorem 3 (modified Riemann’s condition)** For each $X \in \mathcal{J}$ and for each bounded $f : X \to \mathbb{R}$, integrability of $f$ is equivalent to the condition $\lim_{|D(\eta)| \to 0} O(\eta) = 0$.

**Theorem 4** For each $X \in \mathcal{J}$, $\alpha \in \mathbb{R}$ and $f : X \to \mathbb{R}$, the following two statements are equivalent: 1. $f$ is integrable with integral $\alpha$, 2. $\lim_{|D(\eta)| \to 0} \sigma_f(\eta) = \alpha$.

The definition of the integral based on Riemann sums can be used in the matrix-valued case, too. In $\mathbb{R}^{m \times n}$, any metric induced by a norm can be used. In particular, for each $X \in \mathcal{J}$ and integer $m > 1$, a matrix-valued function $h : X \to \mathbb{R}^{m \times m}$ is integrable if and only if all the entries $h_{ik} : X \to \mathbb{R}$ ($i, k = 1, \ldots, m$) are integrable. This fact will be used in order to simplify the formulation of our last theorem.
\textbf{Theorem 5} For each $X \in \mathcal{J}$, $H \in \mathcal{J}_X$ and integrable $f : X \to \mathbb{R}$, the restriction $f|_H$ is integrable, for each $\Phi \in \Pi(X)$, $\int_X f = \sum_{H \in \Phi} \int_H f$.

\textbf{Theorem 6} For each $X \in \mathcal{J}$ and integrable $f : X \to \mathbb{R}$, the function $|f|$ is integrable, and the inequality $|\int_X f| \leq \int_X |f|$ holds.

\textbf{Definition 1} If $X \subset \mathbb{R}^m$, $D \subset \mathcal{J}_X$, $\cup D = X$ and $\Psi : D \to \mathbb{R}$, then by a density function of $\Psi$ we mean a function $g : X \to \mathbb{R}$ for which integrability of $g|_H$ and $\Psi(H) = \int_H g$ hold for each $H \in D$.

\textbf{Theorem 7} Let $X \in \mathcal{J}$, $G : X \to \mathbb{R}^m$ a Lipschitz function, $Y := G(X)$, $f : Y \to \mathbb{R}$ bounded and $g : X \to \mathbb{R}$ integrable. Suppose that

\begin{itemize}
\item[a)] for each $H \in \mathcal{J}_X$, $G(H) \in \mathcal{J}$,
\item[b)] for each pair of non-overlapping sets $A \in \mathcal{J}_X$, $B \in \mathcal{J}_X$, $G(A)$ and $G(B)$ are non-overlapping,
\item[c)] $g$ is a density function of the function $\mathcal{J}_X \ni H \mapsto V(G(H))$,
\item[d)] $f$ or $(f \circ G)g$ is integrable.
\end{itemize}

Then both $f$ and $(f \circ G)g$ are integrable and $\int_X f \circ G g = \int_Y f$.

\textbf{Proof.} Let $L > 0$ be a Lipschitz constant for $G$, $K > 0$ such that for every $x \in Y$, $|f(x)| \leq K$, and use the notation $\psi := (f \circ G)g$.

First, suppose that $f$ is integrable and let $\varepsilon$ be a positive number. We will show that for some $\delta > 0$, and for each dotted partition $\eta$ of $X$ with $|D(\eta)| < \delta$, we have $|\sigma_\psi(\eta) - \int_Y f| < \varepsilon$ (see Theorem 3). Choose positive numbers $\delta_f$ and $\delta_g$ such that $\sigma_f(\Phi) < \varepsilon / 2$ whenever $\Phi \in \Pi(Y)$ and $|\Psi| < \delta_f$, resp. $\sigma_g(\Phi) < \varepsilon / 2K$ whenever $\Phi \in \Pi(X)$ and $|\Phi| < \delta_g$ (see Theorem 4). Let $\{(H_k, \eta_k) : k = 1, \ldots, n\}$ a dotted partition of $X$ such that the norm of $\Phi := \{H_1, \ldots, H_n\}$ is less then $\min\{\delta_g, \delta_f / L\} =: \delta$. Conditions a) and b) imply that $\Psi := \{G(H_1), \ldots, G(H_n)\}$ is a Jordan partition of $Y$, Lipschitz condition and the definition of $L$ imply that the norm of this latter partition is smaller then $\delta_f$.

\begin{align*}
\left| \sum_{k=1}^n f(G(y_k))g(y_k)V(H_k) - \int_Y f \right| & \leq \left| \sum_{k=1}^n (f(G(y_k))g(y_k)V(H_k) - \int_{G(H_k)} f) \right| \\
& = \left| \sum_{k=1}^n \left[ f(G(y_k))[g(y_k)V(H_k) - V(G(H_k))] + f(G(y_k))V(G(H_k)) - \int_{G(H_k)} f \right] \right| \\
& \leq \sum_{k=1}^n \left| f(G(y_k)) \right| \left| g(y_k)V(H_k) - \int_{H_k} g \right| + \sum_{k=1}^n \left| f(G(y_k))V(G(H_k)) - \int_{G(H_k)} f \right| \\
& = \sum_{k=1}^n \left| f(G(y_k)) \right| \int_{H_k} |g(y_k) - g(x)| \, dx + \sum_{k=1}^n \int_{G(H_k)} |f(G(y_k)) - f(y)| \, dy \\
& \leq K \sum_{k=1}^n \int_{H_k} |g(y_k) - g(x)| \, dx + \sum_{k=1}^n \int_{G(H_k)} |f(G(y_k)) - f(y)| \, dy
\end{align*}
Denoting the collection of sets $G$, integral of $f$, upper Darboux integral of $f$, lower integral of $a$ positive number, we show that $\delta$ according to Theorem 3 – a that $S\sum_{k=1}^{n} \delta$ $(k=1, \ldots, n)$, we have the norm of $\Phi \in k$ and for each $\epsilon$ $\norm{\Psi}$ belongs less than the value of the first, second, third, respectively the fourth member on the right side is less than \(\epsilon/4\). As for the second member, this is seen from the following estimate:

\[
\left| \sum_{k=1}^{n} f(G(y_k)) [g(y_k) V(H_k) - V(G(H_k))] \right| < \frac{\epsilon}{4(V(Y) + 1)}
\]

Denoting the collection of sets $G(H_k)$ by $\Psi$ and the set of pairs $(H_k, y_k)$ by $\eta$ $(k=1, \ldots, n)$, we have

\[
\int_X \psi - \int_Y f = \left[ \int_X \psi - \sigma_\psi(\eta) \right] + \sum_{k=1}^{n} f(G(y_k))[g(y_k) V(H_k) - V(G(H_k))]
\]

\[
+ \sum_{k=1}^{n} [f(G(y_k)) - \sup_{G(H_k)} f] V(G(H_k)) + \left[ S_f(\Psi) - \int_Y f \right].
\]

$\delta \leq \delta_\psi$, $\delta \leq \delta_\eta$, the choice of the points $y_k$ and $\delta \leq \delta_f$ imply that the absolute value of the first, second, third, respectively the fourth member on the right hand side is less than $\epsilon/4$. As for the second member, this is seen from the following estimate:

\[
\left| \sum_{k=1}^{n} f(G(y_k)) [g(y_k) V(H_k) - V(G(H_k))] \right| \leq \sum_{k=1}^{n} f(G(y_k)) \int_{H_k} [g(y_k) - g(x)] \, dx
\]

\[
\leq \sum_{k=1}^{n} \left| f(G(y_k)) \right| \int_{H_k} \osc_g(H_k) \leq K\circ_g(\Phi).
\]
3 Auxiliary tools

3.1 Strong differentiability

Definition 2 Let \( m \) and \( n \) be positive integers, \( U \subset \mathbb{R}^m \), and \( u \) an interior point of \( U \). The function \( f : U \to \mathbb{R}^n \) is strongly differentiable at the point \( u \), if there exists a linear map \( A : \mathbb{R}^m \to \mathbb{R}^n \) such that

\[
\lim_{(x,y) \to (u,u)} \frac{1}{\|x - y\|} [f(x) - f(y) - A(x - y)] = 0,
\]

where on \( \mathbb{R}^m \times \mathbb{R}^m \) one can use any metric induced by a norm, e.g. \( d((x,y),(z,w)) := \max\{\|x - z\|,\|y - w\|\} \).

We make some remarks about this notion.

Some authors use the term ‘strict differentiability’ instead of ‘strong differentiability.’

In the definition the spaces \( \mathbb{R}^m \) and \( \mathbb{R}^n \) could be replaced by any normed spaces, but in this case (if the first space is infinite dimensional) one says ‘continuous linear’ instead of ‘linear’.

If \( f \) is strongly differentiable at \( u \) then it is differentiable there and \( f'(u) = A \) must hold.

Strong differentiability of \( f \) at \( u \) implies the existence of a neighborhood of \( u \) on which \( f \) is a Lipschitz function.

If one replaces the assumption on continuous differentiability of \( f \) at the point \( u \) by strong differentiability at \( u \) in the local inverse function theorem (see [4]), then one can state existence of a neighborhood \( U \) of \( u \) such that the restriction \( f|_U \) is injective, its range is a neighborhood of \( f(u) \), the local inverse (the inverse of this restriction) is strongly differentiable at the point \( f(u) \), and the derivative of the local inverse at \( f(u) \) is equal to the inverse of \( f'(u) \). (see for example [6] or [7]). The most difficult part of the proof is essentially contained in the proof of the next theorem (the proof of the fact that \( f(u) \) is an interior point of the range of the injective restriction).

Theorem 8 Let \( u \) be an element of the open set \( \Omega \subset \mathbb{R}^m \), suppose that the function \( G : \Omega \to \mathbb{R}^m \) is strongly differentiable at \( u \) and \( M := G'(u) \) is regular. Then for each \( \varepsilon \in (0,1) \), there exists a \( \delta > 0 \) such that \( \forall (x,r) \in \Omega \times (0,\infty) \),

\[
\mathcal{B}(x,r) \subset \overline{\mathcal{B}}(u,\delta) \implies \ell_x \mathcal{B}(x,(1-\varepsilon)r) \subset G(\mathcal{B}(x,r)) \subset \ell_x \mathcal{B}(x,(1+\varepsilon)r),
\]

where \( \ell_x : \mathbb{R}^m \to \mathbb{R}^m \) denotes the affine function \( z \mapsto G(x) + M(z - x) \). In particular, \( G(u) \in \text{int} R(G) \).

Proof. Define the function \( \varrho : \Omega \times \Omega \to \mathbb{R}^m \) as follows: if \( x,z \in \Omega \) and \( x = z \) then \( \varrho(x,z) := 0 \), otherwise

\[
\varrho(x,z) := \frac{1}{\|z - x\|}[G(z) - G(x) - M(z - x)].
\]
The strong differentiability condition implies that for each $\varepsilon \in (0, 1)$, one can find a $\delta > 0$ such that
\[
\| q(x, z) \| < \frac{\varepsilon}{\| M^{-1} \|}, \quad \text{whenever} \quad x, z \in \overline{B}(u, \delta).
\]

Fix a pair $(x, r)$ satisfying the condition $\overline{B}(x, r) \subset \overline{B}(u, \delta)$, and, in order to prove the first inclusion, fix an element $y = \ell_x(v)$ with $\| v - x \| \leq (1 - \varepsilon)r$ as well. One can apply Banach’s fixed point theorem on the metric subspace $X := \overline{B}(x, r)$ of $\mathbb{R}^m$ to the function
\[
f : X \to \mathbb{R}^m, \quad z \mapsto z - M^{-1}[G(z) - y] = M^{-1}[y - G(z) + Mz]
\]
(of course, the fixed point is a point $z \in \overline{B}(x, r)$ with $G(z) = y$). Indeed, $f$ maps $X$ into $X$, because for each $z \in X$ we have
\[
\| f(z) - x \| = \| M^{-1}[y - G(x) + G(x) - G(z) - M(x - z)] \| \leq \| M^{-1} \| \| M^{-1} \| \| G(z) - M(x - z) \| \| \leq \| M^{-1} \| \| \| G(z) - M(x - z) \| \|.
\]
and it is a contraction with Lipschitz constant $\varepsilon$, because for each pair $(z, w) \in X \times X$ we have
\[
\| f(z) - f(w) \| = \| M^{-1}[G(w) - G(z) - M(w - z)] \| \leq \| M^{-1} \| \| \varepsilon \| \| w - z \|.
\]
To prove the second inclusion, fix an element $v \in \overline{B}(x, r)$ and set
\[
w := M^{-1}[v - x]q(x, v).
\]
Then $\| w \| \leq \| M^{-1} \| \cdot \| q(x, v) \| \cdot \| v - x \|$, hence $\| v + w - x \| \leq r + \varepsilon r$, and
\[
G(v) = G(x) + M(v - x) + \| v - x \|q(x, v) = G(x) + M(v + w - x) = \ell_x(v + w).
\]
To show that $G(u)$ is an interior point of the range, apply the first inclusion with $\varepsilon := 1/2$, $x := u$, $r := \delta$:
\[
G(\overline{B}(u, \delta)) \supset G(u) + MB(0, \delta/2) \supset G(u) + B(0, \delta/(2\| M^{-1} \|)).
\]

**Remark 1** Let $m, n, U$ and $u$ be the same as in Definition 2. The function $f : U \to \mathbb{R}^n$ is strongly differentiable at $u$ if and only if each component of $f$ has this property.

**Remark 2** Let $m, n, U$ and $u$ be the same as in Definition 2. If the function $f : U \to \mathbb{R}^n$ is strongly differentiable at $u$, then, without any assumption on the domain of $f'$, the function $x \mapsto f'(x)$ is continuous at $u$. For a proof, see [3].

**Remark 3** Let $m, n, U$ and $u$ be the same as in Definition 2. If the function $f : U \to \mathbb{R}^n$ is differentiable in a neighborhood of $u$ and $f'$ is continuous at $u$ then $f$ is strongly differentiable at $u$. As for the proof: apply the mean value inequality to the function $z \mapsto f(z) - f'(u)z$ on the line segment $[x, y]$. 

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Before our last remark we introduce a definition which is a slight modification of Nijenhuis’ definition (see [9]).

**Definition 3** Let \( m \) and \( j \) be positive integers, \( j \leq m \), \( U \subset \mathbb{R}^m \), and \( u \) an interior point of \( U \). The function \( f : U \to \mathbb{R} \) is strongly partially differentiable with respect to the \( j \)-th variable at the point \( u \), if there exists a real number \( D_j f(u) \) such that for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) with the following property: if \( x, y \in B(u, \delta) \), \( y_j \neq x_j \), but for all \( i \neq j \) \( y_i = x_i \), then

\[
\left| \frac{f(x) - f(y)}{x_j - y_j} - D_j f(u) \right| < \varepsilon.
\]

**Remark 4** Let \( U \) be a subset of \( \mathbb{R}^m \), and \( u \) an interior point of \( U \). The function \( f : U \to \mathbb{R} \) is strongly differentiable at \( u \) if and only if \( f \) is strongly partially differentiable at \( u \) with respect to all its variables. (The proof is an easy exercise, for the case \( m = 2 \) see [9].)

### 3.2 Cubes, set-functions

By a (closed) cube we mean a Cartesian product of \( m \) number of closed one dimensional intervals of equal length, by a cube-partition of a cube \( Q \) we mean a finite set of pairwise non-overlapping cubes, the union of which is \( Q \). Analogously, by a dotted cube-partition of a cube \( Q \) we mean a finite set

\[
\{(Q_1, y_1), \ldots, (Q_n, y^n)\}
\]

of ordered pairs, where \( \{Q_1, \ldots, Q_n\} \) forms a cube-partition of \( Q \) and \( y^i \in Q_i \) for \( i = 1, \ldots, n \). Equivalently, dotted cube-partitions of a cube \( Q \) can be viewed as functions: a function \( \eta : \mathcal{A} \to Q \) is a dotted cube-partition of the cube \( Q \), if \( \mathcal{A} \) is a cube-partition of \( Q \) and for each cube \( I \in \mathcal{A} \), \( \eta_I := \eta(I) \in I \). In the following, (dotted) partition of a cube will mean always a (dotted) cube-partition.

In the space \( \mathbb{R}^m \) we use the norm \( x \mapsto \max |x_i| =: \|x\| \), therefore the closed balls \( \overline{B}(y, r) \) are cubes and the open balls \( B(y, r) \) are open cubes.

Let \( \delta \) be a positive valued function defined on a cube \( Q \). A dotted partition \( \eta \) of \( Q \) is said to be \( \delta \)-fine, if for each \( I \in D(\eta) \), \( I \subset B(\eta_I, \delta(\eta_I)) \). The following statement will be called ‘Cousin’s lemma’:

**Lemma 1 (Cousin’s lemma)** For each cube \( Q \) and each function \( \delta : Q \to (0, +\infty) \), \( Q \) has a \( \delta \)-fine dotted partition.

For a proof of this assertion, see for example the proof of [10] Lemma 7.3.2.

**Definition 4** A real valued function \( \Phi \) defined on a set \( C \subset \mathbb{J} \) will be called

1. additive, if for each \( H \in C \) and each Jordan partition \( \mathcal{A} \subset C \) of \( H \),

\[
\Phi(H) = \sum_{J \in \mathcal{A}} \Phi(J),
\]
2. Lipschitz, if there exists a nonnegative number $L$ such that for each $H \in C$, 
$\Phi(H) \leq L \cdot V(H)$.

**Definition 5** Suppose that for some $r > 0$ and $u \in \mathbb{R}^m$, each closed subcube of 
$B(u, r)$ belongs to the domain of the real valued set-function $\Phi$. Then $\Phi$ is called 
differentiable (resp. strongly differentiable) at $u$, if for some real number $\Phi'(u)$ 
and for each $\omega > 0$ there exists a $\delta > 0$ for which $u \in I \in D(\Phi)$ (resp. $I \in D(\Phi)$) 
and $I \subset B(u, \delta)$ imply 
$$\left| \frac{\Phi(I)}{V(I)} - \Phi'(u) \right| < \omega.$$ 

Of course, if a cube-function $\Phi$ is differentiable at $u$ then the number $\Phi'(u)$ 
in the definition of differentiability is unique. Note that if $m = 1$, $f : [a, b] \to \mathbb{R}$ 
and $\Phi$ is defined on the set of closed subintervals of $[a, b]$ by $[\alpha, \beta] \mapsto f(\beta) - f(\alpha)$, 
then $\Phi$ is additive, $\Phi$ is Lipschitz if and only if $f$ is Lipschitz, $\Phi$ is (strongly) 
differentiable at $u \in (a, b)$ if and only if $f$ is (strongly) differentiable there.

**Remark 5** Another important and well-known example: if $X \in \mathcal{J}$ and $g : X \to \mathbb{R}$ 
is integrable, then the set-function $\mathcal{J}X \ni H \mapsto \int_H g = : \Psi(H)$ is additive, 
Lipschitz, and strongly differentiable at the continuity points $u$ of $g$ with $\Psi'(u) = g(u)$.

### 3.3 Lipschitz functions and Lipschitz set-functions

**Lemma 2** If $(M, d)$ is a metric space, $\emptyset \neq X \subset M$ and $G : X \to \mathbb{R}^m$ is a 
Lipschitz function with Lipschitz constant $L$, then $G$ has a Lipschitz extension $F : M \to \mathbb{R}^m$ 
with the same Lipschitz constant $L$.

**Proof.** Because of our choice of the norm in $\mathbb{R}^m$, the lemma follows from the 
special case where $m = 1$, which can be applied to the component functions. 
But this special case is a well-known theorem, for a proof see for example [10, 
6.6.5 and 6.6.6].

**Theorem 9** Let $X \in \mathcal{J}$ and $G : X \to \mathbb{R}^m$ a Lipschitz function, then the set-
function $\mathcal{J}X \ni H \mapsto V^*(G(H)) = : \Psi(H)$ is again Lipschitz.

**Proof.** Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be an extension of $G$ satisfying the Lipschitz 
condition with Lipschitz constant $L$ (see Lemma 2), so for each cube $I = B(u, r) \subset \mathbb{R}^m$, 
Lipschitz condition yields $F(I) \subset \overline{B}(F(u), Lr)$, consequently 
$V^*(F(I)) \leq L^m V(I)$. Let $H \in \mathcal{J}X$, $\varepsilon > 0$ and $\{I_1, \ldots, I_n\}$ be a finite set of 
cubes such that 
$$H \subset \bigcup_{k=1}^n I_k \quad \text{and} \quad \sum_{k=1}^n V(I_k) < V(H) + \varepsilon.$$ 

$V^*$ is monotonic and subadditive, therefore 
$$V^*(G(H)) \leq V^* \left[ \bigcup_{k=1}^n F(I_k) \right] \leq \sum_{k=1}^n V^*(F(I_k)) \leq L^m \sum_{k=1}^n V(I_k) < L^m (V(H) + \varepsilon)$$
and this gives the inequality $\Psi \leq L^m V_{\beta_X}$.

**Theorem 10** If $(X_1, d_1)$ is a compact metric space, $(X_2, d_2)$ a metric space and $f : X_1 \to X_2$ is locally Lipschitz in each point of $X_1$ then $f$ is Lipschitz.

Proof. Using on $X_1 \times X_1$ – for example – the metric

$$((x, y), (u, v)) \mapsto \max\{d_i(x, u), d_i(y, v)\} \quad (i = 1, 2),$$

$X_1 \times X_1$ is compact, therefore (being a closed subset of this compact space) the diagonal $\Delta := \{(x, x) : x \in X_1\}$ is also compact. This fact and the local Lipschitz condition gives a positive integer $n$, elements $z_1, \ldots, z_n \in X_1$ and positive numbers $r_1, \ldots, r_n, L_1, \ldots, L_n$ such that for each $k = 1, \ldots, n$, $f|_{B(z_k, r_k)}$ is Lipschitz with Lipschitz constant $L_k$, and $\Delta \subset \bigcup_{k=1}^n B(z_k, r_k) \times B(z_k, r_k) =: \Gamma$.

$(X_1 \times X_1) \setminus \Gamma$ is again compact, the metrics and $G$ are continuous, thus the restriction to $(X_1 \times X_1) \setminus \Gamma$ of the function

$$(X_1 \times X_1) \setminus \Delta \ni (x, y) \mapsto \frac{d_2(f(x), f(y))}{d_1(x, y)} =: h(x, y)$$

has an upper bound $L_0$. This implies that $\max\{L_0, L_1, \ldots, L_n\}$ is an upper bound of $h$.

### 3.4 Some consequences of Cousin’s lemma

The following lemma is known from several proofs of Sard’s lemma (see for example [13, proof of Theorem 3.14.]).

**Lemma 3** Suppose that $A$ is a subset of $\mathbb{R}^m$, $G : A \to \mathbb{R}^m$ is strongly differentiable at the interior point $u$ of $A$ and let $G'(u)$ be singular. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B(u, \delta) \subset A$, and the inequality $V^*(G(I)) \leq \varepsilon V(I)$ holds for all cubes $I$ covered by $B(u, \delta)$.

Now we prove an interesting version of the so-called Sard’s lemma. Observe that as A. Sard himself writes in [12], the real valued $C^1$-case is due to A. P. Morse (see [3]). If $G$ is differentiable at an interior point $x$ of its domain then the Jacobi matrix of $G$ at $x$ will be denoted by $J_G(x)$.

**Theorem 11** Suppose that $\Omega$ is an open subset of $\mathbb{R}^m$, $K \subset \Omega$ is a set of Lebesgue measure 0 and $G : \Omega \to \mathbb{R}^m$ is a Lipschitz function which is strongly differentiable at every points of $\Omega \setminus K$. Then the image under $G$ of the set

$$\{x \in \Omega \setminus K : J_G(x) \text{ is singular}\}$$

is of Lebesgue measure 0.
Proof. \( \Omega \) is a countable union of cubes, so it is enough to prove that for any cube \( Q \subset \Omega \), the image under \( G \) of the set
\[
S := \{ x \in Q \setminus K : J_G(x) \text{ is singular} \}
\]
has Jordan content 0. Set
\[
R := \{ x \in Q \setminus K : J_G(x) \text{ is regular} \}, \quad \text{and} \quad T := Q \cap K.
\]
First, observe that \( R \subset \text{ext} \, S \). Indeed, \( J_G \) is continuous at every point of \( R \) (see Remark 2) and so is the function \( \det : \mathbb{R}^{m \times m} \to \mathbb{R} \), thus a neighborhood of a point of \( R \) in which for every \( x \) we have \( \det J_G(x) \neq 0 \) cannot intersect \( S \). Second, observe that if \( L \) is a Lipschitz constant for \( G \) then for any cube \( I := B(u, r) \subset \Omega \) we have \( G(I) \subset B(G(u), Lr) \), so \( V^*(G(I)) \leq L^m V(I) \). In order to apply Cousin’s lemma, we define a positive valued function \( \delta \) on \( Q \). Let \( \varepsilon \) be a positive number, fix a countable set \( \mathcal{I} \) of open intervals with the sum of volumes being less than \( \varepsilon / 2L^m \), the union of which covers the set \( T \). If \( u \in R \) then let \( \delta(u) > 0 \) be such that \( B(u, \delta(u)) \cap S = \emptyset \). For each \( u \in S \), using the previous lemma, select a \( \delta(u) > 0 \) such that \( V^*(G(I)) \leq \varepsilon V(I) / 2V(Q) \) holds for every cube \( I \) satisfying conditions \( u \in I \subset B(u, \delta(u)) \). For each \( u \in T \), select first a \( J_u \in \mathcal{I} \) that contains \( u \) and then a \( \delta(u) > 0 \) such that \( B(u, \delta(u)) \subset J_u \). Fix a \( \delta \)-fine dotted partition \( \eta \) of \( Q \) and write its domain in the form \( \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) where for a cube \( I \in D(\eta) \), \( I \in \mathcal{A} \) means \( \eta \in R \), \( I \in \mathcal{B} \) means \( \eta \in S \) and \( I \in \mathcal{C} \) means \( \eta \in T \). As \( \eta \) is \( \delta \)-fine, each \( I \in \mathcal{A} \) is disjoint from \( S \), hence
\[
\begin{align*}
G(S) \subset \bigcup_{I \in \mathcal{B} \cup \mathcal{C}} G(I).
\end{align*}
\]
Finally, using subadditivity of \( V^* \) and the definition of \( \delta \), we have
\[
\begin{align*}
V^* \left( \bigcup_{I \in \mathcal{B} \cup \mathcal{C}} G(I) \right) & \leq \sum_{I \in \mathcal{B} \cup \mathcal{C}} V^*(G(I)) = \sum_{I \in \mathcal{B}} V^*(G(I)) + \sum_{I \in \mathcal{C}} V^*(G(I)) \\
& \leq \sum_{I \in \mathcal{B}} \frac{\varepsilon}{2V(Q)} V(I) + L^m \sum_{I \in \mathcal{C}} V(I) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{align*}
\]
Lemma 4.1. Let \( Q \) be a cube, denote by \( \mathcal{C} \) the set of subcubes of \( Q \) and let \( \Phi : \mathcal{C} \to \mathbb{R} \) be an additive Lipschitz function such that \( \Phi'(u) = 0 \) holds for almost all interior points \( u \) of \( Q \). Then \( \Phi \) is the constant 0 function. 2. If \( X \in \mathcal{I} \) and \( \Psi : \mathcal{I}_X \to \mathbb{R} \) is an additive Lipschitz function such that \( \Psi'(u) = 0 \) holds for almost all interior points of \( X \), then \( \Psi \) is the constant 0 function.

Proof. 1. Suppose the contrary, then there exists a subcube \( K \) such that \( \varepsilon := |\Phi(K)| > 0 \). From the assumptions we have a positive \( L \) such that for each \( I \in \mathcal{C}, |\Phi(I)| \leq L \cdot V(I) \), and we have a subset \( H \) of \( K \) with Lebesgue measure 0, which contains all the boundary points of \( K \), such that for all points \( u \in K \setminus H \), \( \Phi'(u) = 0 \). Consequently, we have a countable set \( T \) of open intervals with
\[ \sum_{J \in \mathcal{T}} V(J) < \varepsilon/2L \] the union of which covers \( H \). To apply Cousin’s lemma, define a positive valued function \( \delta \) on \( K \). Assign to each \( u \in H \) a \( J_u \in \mathcal{T} \) that contains the point \( u \) and then a positive \( \delta(u) \) such that \( B(u, \delta(u)) \subset J_u \), while to each \( u \in K \setminus H \), a \( \delta(u) > 0 \) for which the following implication holds: if a cube \( I \in \mathcal{C} \) satisfies the condition \( u \in I \subset B(u, \delta(u)) \) then

\[ \left| \frac{\Phi(I)}{V(I)} \right| < \frac{\varepsilon}{2V(K)}. \]

Fix a \( \delta \)-fine dotted partition \( \eta \) of \( K \). The domain of \( \eta \) can be written as \( A \cup B \) where for \( I \in A \) and for \( I \in B \) we have \( \eta_I \in H \) and \( \eta_I \in K \setminus H \), respectively. We get a contradiction in the form \( \varepsilon < \varepsilon \):

\[ \varepsilon = |\Phi(K)| \leq \sum_{I \in A} |\Phi(I)| + \sum_{I \in B} |\Phi(I)| \leq \sum_{I \in A} L \cdot V(I) + \sum_{I \in B} \frac{\varepsilon \cdot V(I)}{2V(K)} \leq L \cdot \sum_{I \in A} V(I) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \varepsilon = \varepsilon. \] (1)

Inequality (\(*)\) can be proved as follows. Using \( \delta \)-fineness of \( \eta \), each \( I \in A \) is a subset of \( J_u \), so the sum of volumes \( V(I) \) for cubes \( I \) belonging to the same \( J_u \) can be majorized by the volume of this common \( J_u \), consequently, for some finite subset \( \mathcal{T}_0 \) of \( \mathcal{T} \) we have

\[ \sum_{I \in A} V(I) \leq \sum_{J \in \mathcal{T}_0} V(J) \leq \sum_{J \in \mathcal{T}} V(J) < \frac{\varepsilon}{2L}. \]

2. If \( Y \subset X \) is Jordan measurable, \( \varepsilon \) is a positive number and \( L > 0 \) is a Lipschitz constant for \( \Psi \), then there exists a set \( H \subset \text{int} Y \) which is a finite union of cubes with \( V(Y \setminus H) < \varepsilon/L \), therefore part 1. of the theorem yields

\[ |\Psi(Y)| = |\Psi(Y \setminus H) + \Psi(H)| = |\Psi(Y \setminus H)| \leq \varepsilon. \]

**Remark 6** Repeating a part of the proof of assertion 1. we can get an elementary proof of the fact that a bounded function \( f \) defined on a cube \( K \) which is continuous in almost all interior points of \( K \), is integrable (consequently the same is true for a bounded function defined on a Jordan measurable set (this is essentially the same proof that one can find in [B, Theorem 4] for the case \( m = 1 \) : the domain is again a union \( A \cup B \) where \( A \) is of small content and \( B \) is a finite union of cubes). Indeed, let \( \varepsilon \) be a positive number. For any subcube \( I \subset K \), define \( L := \text{osc}_f(K) \), \( H := \partial K \cup \text{dis} f \). For \( u \in H \), let the definition of \( \delta(u) \) be the same as in the previous proof, while for \( u \in K \setminus H \), let \( \delta(u) \) be any positive number satisfying the condition \( \text{osc}_f(B(u, \delta(u))) < \varepsilon/2V(K) \), and let the definitions of \( \eta, A \) and \( B \) be the same as before. Then a proof of the inequality

\[ \sum_{I \in A} \text{osc}_f(I)V(I) + \sum_{I \in B} \text{osc}_f(I)V(I) < \varepsilon, \]
which implies integrability of \( f \), can be formulated as follows. Write the left hand side of this inequality followed by a “\( \leq \)” sign and then switch to line (1) and copy the previous proof.

As three important corollaries, we give a characterization of the ‘indefinite integral’ of a given integrable function, a characterization of the density functions of the constant zero set-functions and a characterization of the set-functions \( \Psi : J_X \to \mathbb{R} \) possessing a density function.

**Theorem 12** If \( X \) is a Jordan measurable set, \( g : X \to \mathbb{R} \) an integrable function and \( \Psi : J_X \to \mathbb{R} \) then the following two statements are equivalent: 1. \( \Psi \) is an additive Lipschitz function such that \( \Psi'(u) = g(u) \) holds for almost all interior points \( u \) of \( X \), 2. \( \Psi(H) = \int_H g \) for each \( H \in J_X \) (in other words: \( g \) is a density function of \( \Psi \)).

**Proof.** 1.\( \Rightarrow \) 2. Assertion 2. of Lemma 4 can be applied to the set-function \( J_X \ni H \mapsto \Psi(H) - \int_H g \).

2.\( \Rightarrow \) 1. Additivity and Lipschitz condition are well-known, \( \Psi'(u) = g(u) \) holds in each continuity points \( u \in \text{int} \ X \) of \( g \).

**Theorem 13** If \( X \) is a Jordan measurable set, \( g : X \to \mathbb{R} \) an integrable function and \( \Psi : J_X \to \mathbb{R} \) is the constant zero set-function then the following three statements are equivalent: 1. \( g(u) = 0 \) holds for almost all points \( u \in \text{int} \ X \), 2. \( g \) is a density function of \( \Psi \), 3. \( g(u) = 0 \) holds for all continuity points \( u \in \text{int} \ X \) of \( g \).

**Proof.** 1.\( \Rightarrow \) 2. See assertion 1.\( \Rightarrow \) 2. of Theorem 12. 2.\( \Rightarrow \) 3. See Remark 6. 3.\( \Rightarrow \) 1. Integrability implies continuity in almost all interior points.

**Theorem 14** Given a Jordan measurable set \( X \subset \mathbb{R}^m \) and a set-function \( \Psi : J_X \to \mathbb{R} \), the following two assertions are equivalent: 1. \( \Psi \) is an additive Lipschitz function that is strongly differentiable in almost all interior points of \( X \), 2. \( \Psi \) has a density function.

**Proof.** 1.\( \Rightarrow \) 2. We prove that the function \( g : X \to \mathbb{R} \) defined by

\[
g(x) := \begin{cases} 
\inf \{ \sup \{ \frac{\Psi(I)}{V(I)} : I \text{ is a subcube of } B(x, r) \} : r > 0 \}, & \text{if } x \in \text{int} \ X, \\
0, & \text{if } x \in \partial \ X \cap \ X
\end{cases}
\]

is a density function of \( \Psi \). First we show that \( g \) is integrable, that is bounded, and continuous in almost all points of \( \text{int} \ X \). If \( L \) is a Lipschitz constant for \( \Psi \), then the range of \( g \) is contained in the interval \([-L, L]\), thus it suffices to show that if \( \Psi \) is strictly differentiable at an interior point \( u \) of \( X \), then \( g \) is continuous at \( u \). Let \( u \in \text{int} \ X \), from the definition of strong differentiability we have \( \Psi'(u) = g(u) \). Let \( \varepsilon \) be a positive number and \( \delta > 0 \) such that \( B(u, \delta) \subset \text{int} \ X \) and for each subcube \( I \) of \( B(u, \delta) \) \( |\Psi(I)/V(I) - g(u)| < \varepsilon \) holds. This implies
that if \( \|x - u\| < \delta \) and \( r < \delta - \|x - u\| \), then – beeing each subcube \( I \) of \( B(x, r) \) a subcube of \( B(u, \delta) \)

\[
\sup \left\{ \frac{\Psi(I)}{V(I)} : I \text{ is a subcube of } B(x, r) \right\} \in [g(u) - \varepsilon, g(u) + \varepsilon],
\]

therefore \( g(x) \in [g(u) - \varepsilon, g(u) + \varepsilon] \), whenever \( x \in B(u, \delta) \). Now, Theorem \[12\] implies that \( g \) is a density function of \( \Psi \).

2. \( \Rightarrow \) 1. It is well-known that \( \Psi \) is additive, Lipschitz, and strongly differentiable in the continuity points \( u \in \text{int } X \) of \( g \).

4 Back to the change of variables

**Theorem 15** If \( A \subset \mathbb{R}^m \) is Jordan measurable, \( G : A \to \mathbb{R}^m \) is a Lipschitz map and \( G \) is strongly differentiable at almost all interior points of \( A \), then \( G(A) \) is Jordan measurable.

**Proof.** Let \( L \) be a Lipschitz constant for \( G, T \) the set of those interior points of \( A \) where \( G \) is not strongly differentiable, \( R \) and \( S \) the set of those points \( x \in (\text{int } A) \setminus T \), for which \( J_G(x) \) is regular or singular, respectively. Finally, let \( F \) be the unique continuous extension of \( G \) defined on \( \overline{A} \) (which is again a Lipschitz function with Lipschitz constant \( L \)). \( G(A) \) is bounded, because it is a subset of the compact set \( F(\overline{A}) \). Continuity of \( F \) implies \( \overline{G(A)} = F(\overline{A}) = F(A) \), therefore

\[
\partial G(A) = \overline{G(A) \setminus \text{int } G(A)} = F(\overline{A}) \setminus \text{int } G(A)
\]

\[
= [G(R) \cup G(S) \cup G(T) \cup F(\partial A)] \setminus \text{int } G(A)
\]

\[
\subset [G(R) \setminus \text{int } G(A)] \cup G(S) \cup G(T) \cup F(\partial A) = G(S) \cup G(T) \cup F(\partial A).
\]

The last equality follows from the inclusion \( G(R) \subset \text{int } G(A) \) which is a consequence of Theorem \[8\]. Theorem \[11\] can be applied to the function \( G|_{\text{int } A} \), from this we get that \( G(S) \) is a Lebesgue-0-set. As both \( T \) and \( \partial A \) are Lebesgue-0-sets, to finish the proof it is enough to observe that the image under a Lipschitz map of a Lebesgue-0-set is a Lebesgue-0-set.

**Theorem 16** If \( X \in \mathcal{J}, \ K \subset X \) is a set of Lebesgue measure 0 and \( G : X \to \mathbb{R} \) is a Lipschitz function which is injective on \( \text{int } X \setminus K \), then for any two non-overlapping \( A \in \mathcal{J} X, B \in \mathcal{J} X \), their images under \( G \) are also non-overlapping.

**Proof.** The inclusion

\[
G(A) \cap G(B) \subset G(K) \cup G(\partial A \cap A) \cup G(\partial B \cap B)
\]

follows from the fact that if \( y = G(a) = G(b), a \in A \setminus K \) and \( b \in B \setminus K \), then the relations \( a \in \text{int } A, b \in \text{int } B \) cannot hold at the same time: in the case \( a = b \) this would contradict to the fact that \( A \) and \( B \) are non-overlapping, in the case
Let Theorem 17 is strongly differentiable at $u\Phi$ already been settled in Lemma 3. Let $\omega Q$ set of subcubes of $A$ such that Lemma 5 yields a $\delta I$ each subcube if Theorem 18 is denoted by $g \Psi$ is strongly differentiable at $u$ and $\Phi'(u) = |\det J_G(u)|$.

Proof. Set $D := |\det J_G(u)|$. Suppose that $D \neq 0$ as the other case has already been settled in Lemma 3. Let $\omega$ be a positive number and $\varepsilon \in (0,1)$ such that

$$D - \omega < (1 - \varepsilon)^m D \leq (1 + \varepsilon)^m D < D + \omega.$$ 

Theorem 5 yields a $\delta$ for this $\varepsilon$; we may and do suppose that $\overline{B}(u, \delta) \subset Q$. For each subcube $I = \overline{B}(x, r) \subset \overline{B}(u, \delta)$ we use the notations

$$I_- := \overline{B}(x, (1 - \varepsilon)r), \quad I_+ := \overline{B}(x, (1 + \varepsilon)r)$$

and apply Theorem 8

$$D - \omega < (1 - \varepsilon)^m D = (1 - \varepsilon)^m \frac{V(\ell_x(I_-))}{V(I)} = \frac{V(\ell_x(I_-))}{V(I)} \leq \frac{V(\ell_x(I_+))}{V(I)} \leq (1 + \varepsilon)^m \frac{V(\ell_x(I_+))}{V(I)} \leq (1 + \varepsilon)^m D < D + \omega.$$ 

Theorem 18 If $X \subset \mathbb{R}^m$ is a Jordan measurable set, $K \subset X$ a Lebesgue-0 set, $G : X \to \mathbb{R}^m$ a Lipschitz map which is strongly differentiable in almost all points of $\text{int} X$, injective on $X \setminus K$ and the set-function $\mathcal{J}_X \ni H \mapsto V(G(H))$ is denoted by $\Psi$, then a density function $g$ of $\Psi$ can be constructed in this way: using the notation

$$B^j(x, r) := \{(y, z) \in B(x, r) \times B(x, r) : y_j \neq z_j, y_i = z_i \text{ if } i \neq j\} \quad (x \in \mathbb{R}^m, r > 0),$$

let $g : X \to \mathbb{R}$ be the function $x \mapsto |\det(\mathcal{g}_{ij}(x))|$, where for all pairs of integers $1 \leq i, j \leq m$, the definition of the functions $\mathcal{g}_{ij} : X \to \mathbb{R}$ is

$$\mathcal{g}_{ij}(x) := \begin{cases} \inf \{ \sup \{ \frac{G_i(z) - G_i(y)}{z_j - y_j} : (y, z) \in B^j(x, r) \} : r > 0 \}, & \text{if } x \in \text{int} X, \\ 0, & \text{if } x \in \partial X \cap X \end{cases}.$$
Theorem 16. As for condition c), 16 implies that condition a) of Theorem 7 is satisfied, while b) is implied by 2. Theorem 7 can be applied with \( \| \cdot \| \) holds. If \( J \) is a function of \( x \) on a set of Lebesgue measure 0, according to Theorem 18, the latter is a density function of \( \Psi \). These facts and Theorem 12 imply that there is integrable, and in the strong differentiability points of \( G \). Moreover, at the strong differentiability points \( u \in \text{int} X \) of \( G \), we get that \( \Psi \) is an additive Lipschitz function. These facts and Theorem 19 imply that \( g \) is a density function of \( \Psi \).

Theorem 19. If \( X \subset \mathbb{R}^m \) is a Jordan measurable set, \( K \subset X \) a Lebesgue-0-set, \( G : X \rightarrow \mathbb{R}^m \) a Lipschitz map which is strongly differentiable in almost all points of \( X \), and \( f : G(X) \rightarrow \mathbb{R} \) is any bounded function, then

1. there exists an integrable function \( \overline{g} : X \rightarrow \mathbb{R}^m \) such that for almost all \( x \in \text{int} X \), \( \overline{g}(x) = J_G(x) \).

2. for each integrable function \( \overline{h} : X \rightarrow \mathbb{R}^m \) with this property, the function

\[
\psi : X \rightarrow \mathbb{R}, \quad \psi(x) := f(G(x)) \cdot |\det \overline{h}(x)|
\]

is integrable if and only if \( f \) is integrable, and

\[
\int_{G(X)} f = \int_X \psi
\]

holds whenever one of \( f \) and \( \psi \) is (that is both of \( f \) and \( \psi \) are) integrable.

Proof. 1. From Theorem 18 we already know that the function \( \overline{g} \) defined there is integrable, and in the strong differentiability points \( x \) of \( G \), \( \overline{g}(x) = J_G(x) \) holds.

2. Theorem 7 can be applied with \( X \ni x \mapsto |\det \overline{h}(x)| =: g(x) \). Indeed, Theorem 13 implies that condition a) of Theorem 7 is satisfied, while b) is implied by Theorem 16. As for condition c), \( g \) differs from the function \( X \ni x \mapsto |\det \overline{g}(x)| \) on a set of Lebesgue measure 0; according to Theorem 18 the latter is a density function of \( \overline{g} \), so the same is true for \( g \) (see Theorem 14).
Remark 7 In Theorem 19, the injectivity assumption cannot be omitted, in particular, Theorem 24.26 in [1] is false. Counterexample: $m = 2$, $A := [1, 2] \times [0, 4\pi]$, $G(x, y) := (e^x \cos y, e^x \sin y)$, $f(x, y) := 1$.

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