Asymptotic behaviour of the logistic nonlinear diffusion equation

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Abstract

In this work we describe a hyperbolic model of repulsive chemotaxis with a dynamics in the population of cells. More precisely, we consider a population of cells producing a particular chemical which induces a motion of the cells towards lower concentrations of this chemical. The chemical serves as a proxy for the local population density and we assume that cells try to avoid crowded areas and prefer locally empty spaces which are further from the carrying capacity. We analyse the well-posed property of the associated Cauchy problem on a line, starting from essentially bounded initial conditions, and the preservation of properties of the initial condition such as the continuity, smoothness and monotonicity. We also describe in detail the behaviour of the level sets near the propagating boundary of the solution and find that an asymptotic jump is formed on the solution for a natural class of initial conditions. Finally, we prove the existence of sharp traveling waves for this model, which are particular solutions traveling at constant speed, and argue that sharp traveling waves are necessarily discontinuous. This analysis is confirmed by numerical simulations of the PDE problem.

1 Introduction

In this article we are concerned with the following diffusion equation with logistic source:

\[
\begin{aligned}
\frac{\partial u(t, x)}{\partial t} - \chi \frac{\partial}{\partial x}(u(t, x) \partial_x p(t, x)) &= u(t, x)(1 - u(t, x)), \quad t > 0, \ x \in \mathbb{R}, \\
u(t = 0, x) &= u_0(x),
\end{aligned}
\]

where \(u(t = 0, x) = u_0(x) \in L^\infty(\mathbb{R})\), \(\chi > 0\) is a sensing coefficient and \(p \in C^2(\mathbb{R})\) is an external pressure. Model (1.1) describes the behaviour of a population of cells \(u(t, x)\) living in a one-dimensional habitat \(x \in \mathbb{R}\), which undergo a logistic birth and death population dynamics, and in which individual cells follow the gradient of concentration of a chemical \(p\). The constant \(\chi\) characterizes the response of the cells to the effective gradient \(p_x\). In this work we will consider the case where \(p\) is itself determined by the state of the population \(u(t, x)\) as

\[
-\sigma^2 \partial_{xx} p(t, x) + p(t, x) = u(t, x), \quad t > 0, \ x \in \mathbb{R}.
\]

This corresponds to a scenario in which the chemical \(p(t, x)\) is produced by the cells (at rate one), diffuses to the whole space with diffusivity \(\sigma^2\) (for \(\sigma > 0\)), and is deteriorated at rate one. The fact that the production and degradation rates are set to one is a mathematical simplifications which could be lifted by considering different scalings in \(x\) and \(p\). \(p\) can be thought as a proxy for the local density of the population and individual cells follow the inverse gradient of \(p\) in order to avoid crowded areas. A similar model has been used in a recent work of the authors [16] to describe the motion of cancer cells in a Petri dish in the context of the cell co-culture experiments of Pasquier et al [29].

We are particularly interested in determining the spatial propagation dynamics of the solutions. To this end it can be noticed that, in the limit of slow diffusivity \(\sigma \to 0\), we get \(u(t, x) \equiv p(t, x)\) and (1.1) is equivalent to an equation with porous medium-type diffusion and logistic reaction

\[
u_t - \frac{1}{2}(u^2)_{xx} = u(1 - u).
\]

(1.3)
The propagation dynamics for this kind of equation was first studied, to the extent of our knowledge, by Aronson [2], Atkinson, Reuter and Ridler-Rowe [3], and later by de Pablo and Vazquez [10], in the more general context of nonlinear diffusion

\[ u_t = (u^m)_{xx} + u(1-u), \quad \text{with } m > 1. \]  

(1.4)

We refer to the monograph of Vazquez [32] for a detailed study of solutions to porous medium equations. In particular, it is known that solutions to (1.4) which start from compactly supported initial conditions eventually behave like a sharp traveling wave which propagates at a constant speed, i.e. a non-trivial self-similar solution traveling at constant speed \( u(t,x) = U(x - ct) \) for some \( c > 0 \) (traveling wave) which is sharp in the sense that \( U(x) \equiv 0 \) for \( x \geq 0 \). The qualitative behavior of the nonlinear diffusion model (1.4) is therefore comparable to the case of linear diffusion \( m = 1 \) studied in the seminal works of Fisher [15] and Kolmogorov, Petrovskii and Piskunov [23] (1937) (see also [1]). More precisely, the sharp traveling wave for (1.4) corresponds to the minimal value of the speed \( c^* > 0 \) of traveling waves (no traveling wave traveling at a speed \( 0 \leq c < c^* \) exist) and for each \( c > c^* \) there exists a traveling wave \( U_c \) which is smooth, positive and connects 1 near \(-\infty\) to 0 near \(+\infty\)

\[ U_c(-\infty) = 1, \quad U_c(+\infty) = 0, \quad U'_c \leq 0. \]

The minimal speed of traveling waves \( c^* \) also corresponds to the spreading speed of compactly supported initial conditions, see [1, 34] for details about this notion. Note finally that, for the particular case of equation (1.3), the critical speed and the corresponding wavefront are both explicit, namely \( c_* = 1/\sqrt{2} \) and \( U(z) = (1 - e^{-z/\sqrt{2}})_+ \) (see [28]).

As will be shown in the present paper, with the type of nonlinear diffusion considered in (1.1), compactly supported initial conditions are expected to reach an asymptotic propagation regime which is driven by traveling waves which are not only sharp but also discontinuous. This is quite different from the porous medium diffusion case (1.4) in which the sharp traveling wave is always continuous. The reason is that the cells are not sufficiently motile to compensate the growth of the population near the propagation front. Understanding and rigorously showing the existence of this discontinuity was one of the major mathematical difficulties of the present study, and is the reason why we develop a weak notion of solution integrated along the characteristics, which allows us to solve (1.1)-(1.2) with very weak assumptions on the initial condition \( u(t=0,x) \); in fact, we require nothing more than an essential bound on \( u_0(x) \).

Discontinuous traveling waves in hyperbolic partial differential equations have raised a lot of interest in the recent years. Travelling wave solutions with a shock or jump discontinuity have appeared e.g. in models of malignant tumor cells (Marchant, Norbury and Perumpanani [25], Harley et al. [19]) where the existence of discontinuous waves is proved by means of geometric singular perturbation theory for ODEs) or chemotaxis (Landman, Pettet and Newgreen [24] where both smooth and discontinuous travelling waves are found using phase plane analysis). Bouin, Calvez and Nadin [5] considered the following hyperbolic model

\[ \varepsilon^2 \partial_t \rho_{\varepsilon} + (1 - \varepsilon^2 F'(\rho_{\varepsilon})) \partial_x \rho_{\varepsilon} - \partial_{xx} \rho(t,x) = F(\rho_{\varepsilon}), \]

where the reaction term \( F \) is monostable. They identified two different regimes for the propagating behavior of solutions. In the first regime \( \varepsilon^2 F'(0) < 1 \), there exists a smooth travelling front (as in the Fisher-KPP case), whereas in the second regime \( \varepsilon^2 F'(0) > 1 \) the traveling wave is discontinuous. In the critical case when \( \varepsilon^2 F'(0) = 1 \), there exists a continuous travelling front with minimal speed \( \sqrt{F'(0)} \) which may present a jump in the derivative.

The particular relation between the pressure \( p(t,x) \) and the density \( u(t,x) \) in (1.2) strongly reminds the celebrated model of chemotaxis studied by Patlak (1953) and Keller and Segel (1970) [30, 21, 22] (parabolic-parabolic Keller-Segel model) and, more specifically, the parabolic-elliptic Keller-Segel model which is derived from the former by a quasi-stationary assumption on the diffusion of the chemical [20]. One of the difficulties in Keller-Segel-type models is that two opposite forces compete to drive the behavior of the equations: the diffusion due to the random motion of cells, on the one hand, and on the other hand the nonlocal advection due to the attractive chemotaxis; the former tends to regularize and homogenize the solution, while the latter promotes cell
aggregation and may lead to the blow-up of the solution in finite time \([8, 20]\). At this point let us mention that our study concerns repulsive chemotaxis with no diffusion, therefore no such blow-up phenomenon is expected in our study; however the absence of diffusion adds to the mathematical complexity of the study, because standard method of reaction-diffusion equations cannot be employed here. Traveling waves for the (attractive) parabolic-elliptic Keller-Segel model were studied by Nadin, Perthame and Ryzhik \([27]\), who constructed these traveling wave by a bounded interval approximation of the 1D system

\[
\begin{align*}
\frac{u_t + \chi (u p_x)_x}{-d p_{xx} + p} &= u_{xx} + u(1 - u), \\
- d p_{xx} + p &= u,
\end{align*}
\]

(1.5)

set on the real line \(x \in \mathbb{R}\), when the strength of the advection is not too strong \(0 < \chi < \min(1, d)\), and gave estimates on the speed of such a traveling wave: \(2 \leq c_* \leq 2 + \chi \sqrt{d/(d - \chi)}\).

Since the pressure \(p(t,x)\) is a nonlocal function of the density \(u(t,x)\) in (1.2), the spatial derivative appears as a nonlocal advection term in (1.1). In fact, our problem (1.1)–(1.2) can be rewritten as a transport equation in which the speed of particles is non-local in the density,

\[
\begin{align*}
\frac{\partial_t u(t,x) - \chi \partial_x (u(t,x) \partial_x (\rho \ast u)(t,x))}{u(t = 0, x) = u_0(x),}
\end{align*}
\]

(1.6)

where

\[
(\rho \ast u)(x) = \int_\mathbb{R} \rho(x - y) u(t, y) dy, \quad \rho(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}.
\]

(1.7)

Traveling waves for a similar diffusive equation with logistic reaction have been investigated for quite general nonlocal kernels by Hamel and Henderson \([18]\), who considered the model

\[
\begin{align*}
\frac{u_t + (u (K \ast u)_x)}{u(t, x)} &= u_{xx} + u(1 - u),
\end{align*}
\]

(1.8)

where \(K \in L^p(\mathbb{R})\) is odd and \(p \in [1, \infty]\). Notice that the attractive parabolic-elliptic Keller-Segel model (1.5) is included in this framework by the particular choice

\[
K(x) = -\chi \text{sign}(x) e^{-|x|/\sqrt{d}/(2\sqrt{d})}.
\]

They proved a spreading result for this equation (initially compactly supported solutions to the Cauchy problem propagate to the whole space with constant speed) and explicit bounds on the speed of propagation. Diffusive non-local advection also appears in the context of swarm formation \([26]\). Pattern formation for a model similar to (1.8) by Ducrot, Fu and Magal \([11]\). Let us mention that the inviscid equation (1.6) has been studied in a periodic cell by Ducrot and Magal \([12]\). Other methods have been established for conservative systems of interacting particles and there kinetic limit \([\text{Balagué et al} \ [4], \text{Carrillo et al} \ [6]]\) based on gradient flows set on measure spaces; those are difficult to adapt here because of the logistic term. Finally we refer to \([9, 17, 33, 14, 13]\) for other examples of traveling waves in nonlocal reaction-diffusion equations.

In this paper we focus on the particular case of (1.1)–(1.2) with \(\sigma > 0\) and \(\chi > 0\). The paper is organized as follows. In Section 2, we present our main results. In Section 3 we present numerical simulations to illustrate our theoretical results. In Section 4, we prove the propagation properties of the solution and describe the local behaviour near the propagating boundary (see Proposition 2.4 for definition), including the formation of a discontinuity for time-dependent solutions. In Section 5 we prove the existence of sharp traveling waves. We also prove that smooth traveling waves are necessarily positive, which shows that sharp traveling waves are necessarily singular (in this case, discontinuous). In particular, a solution starting from a compactly supported initial condition with polynomial behaviour at the boundary can never catch such a smooth travelling wave. Section 6 is devoted to the well-posedness of the Cauchy problem for system (1.1).

2 Main results and comments

We begin by defining our notion of solution to equation (1.1).
Definition 2.1 (Integrated solutions). Let \( u_0 \in L^\infty(\mathbb{R}) \). A measurable function \( u(t,x) \in L^\infty([0,T] \times \mathbb{R}) \) is an integrated solution to (1.1) if the characteristic equation

\[
\begin{align*}
\frac{d}{dt} h(t,x) &= -\chi(\rho_x \ast u)(t, h(t,x)) \\
h(t = 0, x) &= x.
\end{align*}
\]  

(2.1)

has a classical solution \( h(t,x) \) (i.e. for each \( x \in \mathbb{R} \) fixed, the function \( t \mapsto h(t,x) \) is in \( C^1([0,T], \mathbb{R}) \) and satisfies (2.1)), and for a.e. \( x \in \mathbb{R} \), the function \( t \mapsto u(t,h(t,x)) \) is in \( C^1([0,T], \mathbb{R}) \) and satisfies

\[
\begin{align*}
\frac{d}{dt} u(t,h(t,x)) &= u(t,h(t,x))(1 + \chi(\rho \ast u)(t,h(t,x)) - (1 + \hat{\chi})u(t,h(t,x))), \\
u(t=0,x) &= u_0(x),
\end{align*}
\]  

(2.2)

where \( \hat{\chi} := \frac{1}{\rho} \).

We define weighted space \( L^1_\eta(\mathbb{R}) \) as follows

\[
L^1_\eta(\mathbb{R}) := \left\{ f: \mathbb{R} \rightarrow \mathbb{R} \text{ measurable} \mid \int_{\mathbb{R}} |f(x)| e^{-\eta|x|} \, dx < \infty \right\}.
\]

\( L^1_\eta(\mathbb{R}) \) is a Banach space endowed with the norm

\[
\|f\|_{L^1_\eta} := \frac{\eta}{2} \int_{\mathbb{R}} |f(y)| e^{-\eta|y|} \, dy.
\]

Our first result concerns the existence of integrated solutions to (1.1).

Theorem 2.1 (Well-posedness). Let \( u_0 \in L^\infty(\mathbb{R}) \) and fix \( \eta > 0 \). There exists \( \tau^*(u_0) \in (0,\infty] \) such that for all \( \tau \in (0,\tau^*(u_0)) \), there exists a unique integrated solution \( u \in C^0([0,\tau], L^1_\eta(\mathbb{R})) \) to (1.1) which satisfies \( u(t=0,x) = u_0(x) \). Moreover \( u(t,\cdot) \in L^\infty(\mathbb{R}) \) for each \( t \in [0,\tau^*(u_0)) \) and the map \( t \mapsto u(t,\cdot) \) is a semigroup which is continuous for the \( L^1_\eta(\mathbb{R}) \)-topology.

The map \( u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R}) \) is continuous.

Finally, if \( 0 \leq u_0(x) \leq 1 \), then \( \tau^*(u_0) = +\infty \) and \( 0 \leq u(t,\cdot) \leq 1 \) for all \( t > 0 \).

Proposition 2.2 (Regularity of solutions). Let \( u(t,x) \) be an integrated solution to (1.1).

1. if \( u_0(x) \) is continuous, then \( u(t,x) \) is continuous for each \( t > 0 \).
2. if \( u_0(x) \) is monotone, then \( u(t,x) \) has the same monotony for each \( t > 0 \).
3. if \( u_0(x) \in C^1(\mathbb{R}) \), then \( u \in C^1([0,T] \times \mathbb{R}) \) and \( u \) is then a classical solution to (1.1).

Next we show the long-time behaviour of the solutions to (1.1).

Theorem 2.3 (Long-time behaviour). Let \( 0 \leq u_0(x) \leq 1 \) be a nontrivial nonnegative initial condition and \( u(t,x) \) be the corresponding integrated solution. Then \( 0 \leq u(t,x) \leq 1 \) for all \( t > 0 \) and \( x \in \mathbb{R} \). If moreover there exists \( \delta > 0 \) such that \( \delta \leq u_0(x) \leq 1 \) then

\[
u(t,x) \rightarrow 1, \text{ as } t \rightarrow \infty
\]

and the convergence holds uniformly in \( x \in \mathbb{R} \).

To get insight about the asymptotic propagation properties of the solutions, we focus on initial conditions whose support is bounded towards \( +\infty \). If the behaviour of the initial condition in a neighbourhood of the boundary of the support is polynomial, we can establish a precise estimate of the location of the level sets relative to the position of the rightmost positive point. Our first assumption requires that the initial condition is supported in \( (-\infty,0] \).

Assumption 1 (Initial condition). We assume that \( u_0(x) \) is a continuous function satisfying

\[
\begin{align*}
0 &\leq u_0(x) \leq 1, & \forall x \in \mathbb{R}, \\
u_0(x) &\geq 0, & \forall x \in (-\delta_0,0), \\
u_0(x) &> 0, & \forall x \in (-\delta_0,0),
\end{align*}
\]

for some \( \delta_0 > 0 \).
Under this assumption we show that $u$ is propagating to the right.

**Proposition 2.4** (The separatrix). Let $u_0(x)$ satisfy Assumption 1, and $h^*(t) := h(t,0)$ be the separatrix. Then $h^*(t)$ stays at the rightmost boundary of the support of $u(t,\cdot)$, i.e.

(i) we have

$$u(t,x) = 0 \text{ for all } x \geq h^*(t),$$  \hspace{1cm} (2.3)

(ii) for each $t > 0$ there exists $\delta > 0$ such that

$$u(t,x) > 0 \text{ for all } x \in (h^*(t) - \delta, h^*(t)).$$  \hspace{1cm} (2.4)

Moreover, $u$ is propagating to the right i.e.

$$\frac{d}{dt} h^*(t) > 0 \text{ for all } t > 0.$$

We precise the behaviour of the initial condition in a neighbourhood of 0 and estimate the steepness of $u$ in positive time.

**Assumption 2 (Polynomial behaviour near 0).** In addition to Assumption 1, we require that there exists $\alpha \geq 1$ and $\gamma > 0$ such that

$$u_0(x) \geq \gamma |x|^{\alpha}, \quad \forall x \in (-\delta, 0).$$

**Theorem 2.5** (Formation of a discontinuity). Let $u_0(x)$ satisfy Assumptions 1 and 2 and $u(t,x)$ solve (1.1) with $u(t=0,x) = u_0(x)$. For all $\delta > 0$ we have

$$\limsup_{t \to +\infty} \sup_{x \in (h^*(t) - \delta, h^*(t))} u(t,x) \geq \frac{1}{1 + \hat{\chi} + \alpha \chi} > 0.$$  \hspace{1cm} (2.5)

More precisely, define the level set

$$\xi(t,\beta) := \sup\{|x| \in \mathbb{R} | u(t,x) = \beta\},$$

for all $t > 0$ and $0 < \beta < \frac{1}{1 + \chi + \alpha \chi}$. Then, for each $0 < \beta < \frac{1}{1 + \chi + \alpha \chi}$, the distance between $\xi(t,\beta)$ and the separatrix is decaying exponentially fast:

$$h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\gamma}} e^{-\eta t} \leq \xi(t,\beta) \leq h^*(t),$$  \hspace{1cm} (2.6)

where $\eta \in (0,1)$ is given in Proposition 4.5 and $\hat{\chi} = \frac{\chi}{\sigma^2}$.

![Figure 1: A cartoon for the formation of the discontinuity. Here we choose $t_1 < t_2$ and $\xi(t,\beta), t = t_1, t_2$ are the level sets. Theorem 2.5 proves that when Assumptions 1 and 2 are satisfied, then the distance $|\xi(t,\beta) - h^*(t)|$ converges to 0 exponentially fast.](image)

In particular, the profile $u(t,x)$ forms a discontinuity near the boundary point $h^*(t)$ as $t \to +\infty$. By considering discontinuous integrated solutions, we are able to estimate the size of the jump for nonincreasing profiles, which leads to an estimate of the asymptotic speed.
Proposition 2.6 (Asymptotic jump near the separatrix). Let $u_0$ be a nonincreasing function satisfying $u_0(-\infty) \leq 1$, $u_0(0) > 0$ and $u_0(x) = 0$ for $x > 0$. Then

$$\liminf_{t \to +\infty} u(t, h^*(t)) \geq \frac{2}{2 + \hat{\chi}},$$  \hspace{1cm} (2.7)

$$\liminf_{t \to +\infty} \frac{d}{dt} h^*(t) \geq \frac{\sigma \hat{\chi}}{2 + \hat{\chi}},$$  \hspace{1cm} (2.8)

where $\hat{\chi} = \frac{\chi}{\sigma^2}$.

We finally turn to traveling wave solutions $u(t, x) = U(x - ct)$, which are self-similar profiles traveling at a constant speed.

Definition 2.2 (Traveling wave solution). A traveling wave is a positive solution $u(t, x)$ to (1.1) such that there exists a function $U \in L^\infty(\mathbb{R})$ and a speed $c \in \mathbb{R}$ such that $u(t, x) = U(x - ct)$ for a.e. $(t, x) \in \mathbb{R}^2$. By convention, we also require that $U$ has the following behavior at $\pm \infty$:

$$\lim_{z \to -\infty} U(z) = 1, \quad \lim_{z \to \infty} U(z) = 0.$$  

The function $U$ is the profile of the traveling wave.

Theorem 2.7 (Existence of a sharp discontinuous traveling wave). Let Assumption 3 be satisfied. There exists a traveling wave $u(t, x) = U(x - ct)$ traveling at speed

$$c \in \left( \frac{\sigma \hat{\chi}}{2 + \hat{\chi}}, \frac{\sigma \hat{\chi}}{2} \right),$$

where $\hat{\chi} = \frac{\chi}{\sigma^2}$. Moreover, the profile $U$ satisfies the following properties (up to a shift in space):
(i) $U$ is sharp in the sense that $U(x) = 0$ for all $x \geq 0$; moreover, $U$ has a discontinuity at $x = 0$ with $U(0^+) \geq \frac{2}{\pi^2}$. 

(ii) $U$ is continuously differentiable and strictly decreasing on $(-\infty, 0]$, and satisfies 

$$-cU' - \chi(UP)' = U(1 - U)$$ 

pointwise on $(-\infty, 0)$, where $P(z) := (\rho \ast U)(z)$.

Finally, we show that continuous traveling waves cannot be sharp, i.e. are necessarily positive on $\mathbb{R}$.

**Proposition 2.8** (Smooth traveling waves). Let $U(x)$ be the profile of a traveling wave solution to (1.1) and assume that $U$ is continuous. Then $U \in C^1(\mathbb{R})$, $U$ is strictly positive and we have the estimate:

$$- \chi(\rho \ast U)(x) < c \text{ for all } x \in \mathbb{R}. \quad (2.9)$$

In particular, by Theorem 2.5, any solutions starting from an initial condition satisfying Assumption 2 may never catch such a traveling wave.

### 3 Numerical Simulations

We first describe the numerical framework of this study.

- The parameters $\sigma$ and $\chi$ are fixed to $1$, $\sigma = 1$ and $\chi = 1$.

- We are given a bounded interval $[-L, L]$ and an initial distribution of $\phi \in C([-L, L])$;

- We solve numerically the following PDE using the upwind scheme ($p$ being given)

$$
\begin{aligned}
\partial_t u(t, x) - \partial_x (u(t, x)\partial_x p(t, x)) &= u(t, x)(1 - u(t, x)), \\
\nabla p(t, x) \cdot \nu &= 0 \\
u(0, x) &= \phi(x),
\end{aligned}
\quad t > 0, \ x \in [-L, L]. \quad (3.1)
$$

- The pressure $p$ is defined as

$$p(t, x) = (I - \Delta)^{-1} u(t, x), \quad t > 0, \ x \in [-L, L], \quad (3.2)$$

where $(I - \Delta)^{-1}$ is the Laplacian operator with Neumann boundary condition. Due to the Neumann boundary condition of the pressure $p$, we do not need boundary condition on $u$ (see [27, 16]).

Our numerical scheme reads as follows

$$
\begin{aligned}
\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} + \frac{1}{\Delta x} \left( G(u_{i+1}^{n}, u_{i}^{n}) - G(u_{i}^{n}, u_{i-1}^{n}) \right) &= u_{i}^{n}(1 - u_{i}^{n}), \\
i &= 1, 2, \ldots, M, \ n = 0, 1, 2, \ldots \\
u_{0} &= 1, \ u_{M+1} = 0,
\end{aligned}
$$

with $G(u_{i+1}^{n}, u_{i}^{n})$ defined as

$$G(u_{i+1}^{n}, u_{i}^{n}) = (v_{i+\frac{1}{2}}^{n})^+ u_{i}^{n} - (v_{i+\frac{1}{2}}^{n})^- u_{i+1}^{n} = 
\begin{cases} v_{i+\frac{1}{2}}^{n} u_{i}^{n}, & v_{i+\frac{1}{2}}^{n} \geq 0, \\
v_{i+\frac{1}{2}}^{n} u_{i+1}^{n}, & v_{i+\frac{1}{2}}^{n} < 0,
\end{cases} \quad i = 1, \ldots, M.$$

Moreover the velocity $v$ is given by

$$v_{i+\frac{1}{2}}^{n} = -\frac{p_{i+\frac{1}{2}}^{n} - p_{i}^{n}}{\Delta x}, \ i = 0, 1, 2, \ldots, M,$$
where from (3.2) we define
\[ P^n := (I - A)^{-1}U^n, \quad P^n = (p^n_i)_{M \times 1}, \quad U^n = (u^n_i)_{M \times 1}. \]
where \( A = (a_{i,j})_{M \times M} \) is the usual linear diffusion matrix with Neumann boundary condition. Therefore, by Neumann boundary condition \( p_0 = p_1 \) and \( p_{M+1} = p_M \), when \( i = 1, M \) we have
\[ G(u^n_1, u^n_0) = 0, \quad G(u^n_M, u^n_{M+1}) = 0, \]
which gives
\[ u^{n+1}_1 = u^n_1 - d \frac{\Delta t}{\Delta x} G(u^n_2, u^n_1) + \Delta t f(u^n_1), \]
\[ u^{n+1}_M = u^n_M + d \frac{\Delta t}{\Delta x} G(u^n_M, u^n_{M-1}) + \Delta t f(u^n_M). \]

Owing to the boundary condition, we have the conservation of mass for Equation (3.1) when the reaction term equals zero.

### 3.1 Formation of a discontinuity

In this part, we use numerical simulations to verify the theoretical predictions in the previous sections. Firstly, we choose the initial value \( \phi \in C^1([-L, L]) \) as follows
\[ \phi(x) = \frac{(x - x_0)^2}{(L + x_0)^2} [x, -L, x_0], \quad L = 20, \quad x_0 = -15. \quad (3.3) \]
Notice that this initial condition satisfies Assumptions 1 and 2. Due to Theorem 2.5, we should observe the formation of a discontinuity in space for large time.

We plot the evolution of the solution \( u(t, x) \) starting from \( u(0, x) = \phi(x) \) in Figure 3.

![Figure 3](image-url)

We observe that the jump is formed for large time and the height of the jump is greater than \( \frac{2}{3} \) which is in accordance with Theorem 2.7.

Next, we study the propagation speed of different level sets, namely,
\[ t \mapsto \xi(t, \beta) + L, \]
where \( \xi(t, \beta) := \sup\{x \in \mathbb{R} \mid u(t, x) = \beta\} \) and \( \beta = 0, 0.2, 0.8, 0.8, 2/3, 0.8. \) Note that the case \( \beta = 0 \) corresponds to the rightmost characteristic.
We compute the propagation speed in the following way: for different $\beta \in [0, 1]$, we choose $t_1 = 15$ and $t_2 = 40$ where the propagation speed is almost stable after $t = t_1$. Thus we can compute the mean propagation speed as follows

$$\text{Propagation speed at level } \beta = \frac{\xi(t_2, \beta) - \xi(t_1, \beta)}{t_2 - t_1}. \quad (3.4)$$

Figure 4: We plot the evolution of different level sets $t \mapsto \xi(t, \beta) + L$ under system (3.1). Our initial distribution is taken as (3.3). We plot the propagating speeds of the profile at $\beta = 0, 0.2, 2/3, 0.8$. The x-axis represents the time and the y-axis is the relative distance $\xi(t, \beta) + L$. The velocity is calculated by (3.4) for $t_1 = 15$ and $t_2 = 40$.

Next we want to check whether the solutions of system (3.2) starting from two different initial values converge to the same discontinuous traveling wave solution. To that aim, given two different initial profiles $\phi_1$ and $\phi_2$ with $\phi_1 \leq \phi_2$ on $[-L, L]$, 

$$\phi_1(x) = -\frac{x + 15}{5} \mathbb{I}_{[-20, -15]}(x), \quad \phi_2(x) = \frac{x + 15}{10} \mathbb{I}_{[-17.5, -15]}(x) \quad (3.5)$$

We simulate the propagation of these two profiles in Figure 5.

Figure 5: We plot the propagation of two profiles under system (3.1) with initial distributions are taken as (3.5). The blue curves represent the profile with initial distribution $\phi_1$ while the red curves represent the profile with initial distribution $\phi_2$. We plot the propagation profiles at $t = 0, 15$ and $30$ (resp. dashed lines, dotted-dashed lines and solid lines). The simulation shows that the two profiles converge to the same discontinuous traveling wave solution.

3.2 Large speed traveling waves

As we know for porous medium equation, the existence of large speed $c > c_*$ traveling wave solutions is proved in [10] and it can be observed numerically by taking the exponentially decreasing function
as initial value. In this part, instead of taking a compactly supported initial value, we set the initial value
\[ \phi_\alpha(x) = \frac{1}{1 + e^{\alpha(x - x_0)}}, \quad x_0 = -15, \] (3.6)
where \( \alpha \geq 1 \) is a parameter introduced to describe the decaying rate of the initial value.

We compare the following three different scenarios with different parameters \( \alpha = 1, 2, 5 \) in the initial value (3.6).

Figure 6: We plot the propagation of the traveling waves under system (3.1) with the initial values (3.6) and the corresponding evolution of different level sets \( t \mapsto \xi(t, \beta) + L \). Figure (a) and (d) represent the evolution of the traveling wave and its level sets when \( \alpha = 1 \). Figure (b) and (e) correspond to the case when \( \alpha = 2 \). Figure (c) and (f) correspond to the case when \( \alpha = 5 \).

We observe the large speed traveling waves in Figure 6 when \( \alpha = 1, 2 \). We note that as the parameter \( \alpha \) in (3.6) is increasing, the propagation speed is decreasing and \( c \approx 1/\alpha \). When \( \alpha = 5 \), the propagation of the traveling waves is similar to the case in Figure 3 in which we started from the compactly supported initial value. In other word, we can observe the formation of discontinuity and the critical speed \( c_* \approx 0.414 \) is reached.

### 3.3 Comparison with porous medium equations: the vanishing jump

In this part, we compare the nonlocal advection model with the porous medium equation by introducing a new parameter \( \sigma \)
\[ p(t, x) = (I - \sigma^2 \Delta)^{-1} u(t, x) \] (3.7)
Thus if \( \sigma \to 0 \) then formally we have \( p(t, x) \to u(t, x) \). Thus, the first equation of (3.1) becomes
\[ u_t - \frac{1}{2} (u^2)_{xx} = u(1 - u), \]
which is the classical porous medium equation. It is well-known that this equation has the explicit traveling wave solution $U(z) = (1 - e^{z/\sqrt{2}})$ with critical speed $c_* = 1/\sqrt{2}$.

We consider the transition from the discontinuous traveling wave solution to the continuous sharp-type traveling wave solution by letting $\sigma \to 0$. Moreover, we want to see if the critical traveling speed of the discontinuous wavefront $c(\sigma)$ converges to $c_* = 1/\sqrt{2} \approx 0.707$ as $\sigma \to 0$. Our initial value is taken as $1/(1 + \exp(5 \ast (x + 15)))$, $x \in [-20, 20]$ in (3.6). We compare the following three different scenarios with different parameters $\sigma^2 = 0.5, 0.1, 0.01$ in kernel (3.7).

![Figure 7](image)

Figure 7: We plot the propagation of the traveling waves for system (3.1) with the kernel (3.7) and the corresponding evolution of different level sets $t \mapsto -\xi(t, \beta) + L$. Figure (a) and (d) represent the evolution of the traveling wave and its level sets when $\sigma^2 = 0.5$. Figure (b) and (e) correspond to the case when $\sigma^2 = 0.1$. Figure (c) and (f) correspond to the case when $\sigma^2 = 0.01$. Our initial value is taken as in (3.6) with $\alpha = 5$.

In Figure 7 we can observe that as $\sigma \to 0$ in the kernel, the discontinuous jump is gradually vanishing from (a) to (c). Moreover, the critical speed $c(\sigma)$ is increasing as $\sigma \to 0$ and is approaching the critical speed $c_* = 1/\sqrt{2} \approx 0.707$ for the porous medium case.

### 4 Properties of the time-dependent solutions

#### 4.1 The separatrix

In this section we study the qualitative properties of solutions to (1.1) starting from an initial condition supported in $(-\infty, 0]$.

**Proposition 4.1** (The separatrix). Let $u$ be a solution integrated along the characteristics to (1.1), starting from $u_0(x)$ satisfying Assumption 1. Let $h^*(t) := h(t, 0)$ be the separatrix (as in Proposition 2.4). Then $h^*(t)$ stays at the rightmost boundary of the support of $u(t, \cdot)$, i.e.

1. we have $u(t, x) = 0$ for all $x \geq h^*(t)$.

   \begin{equation}
   u(t, x) = 0 \quad \text{for all} \quad x \geq h^*(t). \tag{4.1}
   \end{equation}
(ii) for each $t > 0$ there exists $\delta > 0$ such that

$$u(t, x) > 0 \text{ for all } x \in (h^*(t) - \delta, h^*(t)).$$

(4.2)

Proof. By definition the characteristics are well-defined by (2.1) as the flow of an ODE. In particular, if $x \geq h^*(t) = h(t, 0)$ there exists $x_0 \geq 0$ such that $x = h(t, x_0)$. Since $u_0(x_0) = 0$ and in view of (2.2), we have indeed $u(t, x) = 0$. This proves Item (i).

By Assumption 1, there exists $\delta_0 > 0$ such that $u_0(x) > 0$ for $x \in (-\delta_0, 0)$. We remark that

$$\frac{d}{dt} u(t, h(t, x)) = \hat{\chi} u(t, h(t, x)) \left( (\rho \ast u)(t, h(t, x)) - u(t, h(t, x)) \right) + u(t, h(t, x)) (1 - u(t, h(t, x))) \geq u(t, h(t, x)) (1 - (1 + \hat{\chi}) u(t, h(t, x))).$$

By comparison with the solution to the ODE $v'(t) = v(t) (1 - (1 + \hat{\chi}) v(t))$ starting from $v(t = 0) = u_0(x) > 0$, we deduce that $u(t, x) \geq v(t) > 0$ for each $x \in (h(t, -\delta_0), h^*(t))$. Since $h(t, -\delta_0) < h(t, 0) = h^*(t)$, this proves Item (ii).

Next we investigate the propagation of $u$.

**Proposition 4.2 (u is propagating).** Let $u_0$ satisfy Assumption 1 and let $u$ be the solution integrated along the characteristics to (1.1) starting from $u(t = 0, x) = u_0(x)$. Then $u$ is propagating to the right, i.e.

$$\frac{d}{dt} h^*(t) > 0.$$  

Moreover, we have the estimate:

$$\frac{d}{dt} h^*(t) \leq \frac{\hat{\chi}}{2\sigma}.$$  

(4.4)

Proof. We have the following estimates:

$$\frac{d}{dt} h^*(t) = -\chi (\rho_x * u)(t, h^*(t))$$

$$= -\chi \int_{-\infty}^{+\infty} \rho_x(y) u(t, h^*(t) - y) dy$$

$$= \chi \int_{-\infty}^{+\infty} \text{sign}(y) \frac{e^{-\frac{|y|}{\sigma}}}{2\sigma^2} u(t, h^*(t) - y) dy$$

$$= \frac{\hat{\chi}}{\sigma} \int_{0}^{+\infty} \rho(y) u(t, h^*(t) - y) dy$$

$$> 0,$$

since $u(t, x) = 0$ for all $x > h^*(t)$. (4.3) is proved.

Then, since $0 \leq u \leq 1$, we have

$$\frac{d}{dt} h^*(t) = \frac{\hat{\chi}}{\sigma} \int_{0}^{+\infty} \rho(y) u(t, h^*(t) - y) dy \leq \frac{\hat{\chi}}{\sigma} \int_{0}^{+\infty} \rho(y) dy = \frac{\hat{\chi}}{2\sigma},$$

which proves (4.4).

These first two propositions together yield a proof of Proposition 2.4.

**Proof of Proposition 2.4.** Items (i) and (ii) have been proved in Proposition 4.1, and the propagating property follows from Proposition 4.2.

We continue with a technical lemma that will be used in the proof of Theorem 2.5.
Lemma 4.3 (Divergence speed near the separatrix). Let \( u_0(x) \) satisfy Assumptions 1 and 2 and \( u(t, x) \) be the corresponding solution to (1.1). Let \( h(t, x) \) be the characteristic flow of \( u \) and \( h^*(t) \) be the separatrix of \( u \), as defined in Proposition 2.4. For all \( t \geq 0 \) and \( x < 0 \) we have

\[
\frac{d}{dt}(h^*(t) - h(t, x)) \leq \chi(h^*(t) - h(t, x)) \sup_{y \in (h(t, x), h^*(t))} u(t, y). \tag{4.5}
\]

Proof. Recall that, by Proposition 4.1, \( u(t, x) = 0 \) for each \( x \geq h^*(t) \). For \( x < 0 \), we notice that:

\[
\frac{d}{dt}(h^*(t) - h(t, x)) = -\chi(p_x \ast u)(t, h^*(t)) + \chi(p_x \ast u)(h(t, x))
\]

\[
= \chi \int_{\mathbb{R}} \left(p_x(h(t, x) - y) - p_x(h^*(t) - y)\right)u(t, y)dy
\]

\[
= \chi \int_{-\infty}^{h(t, x)} \left(p_x(h(t, x) - y) - p_x(h^*(t) - y)\right)u(t, y)dy
\]

\[
+ \chi \int_{h(t, x)}^{h^*(t)} \left(p_x(h(t, x) - y) - p_x(h^*(t) - y)\right)u(t, y)dy.
\]

Therefore,

\[
\frac{d}{dt}(h^*(t) - h(t, x)) \leq \chi \int_{-\infty}^{h(t, x)} \left(p_x(h(t, x) - y) - p_x(h^*(t) - y)\right)u(t, y)dy
\]

\[
+ \chi \int_{h(t, x)}^{h^*(t)} \left(p_x(h(t, x) - y) - p_x(h^*(t) - y)\right)u(t, y)dy.
\]

Since \( p_x(y) = -\frac{1}{2\pi} \text{sign}(y)e^{-\frac{|y|^2}{2}} \) is increasing on \((0, +\infty)\), we have

\[
p_x(h(t, x) - y) - p_x(h^*(t) - y) \leq 0
\]

for each \( y \leq h(t, x) \), which shows (4.5). Lemma 4.3 is proved. \( \Box \)

Proposition 4.4 (Formation of a discontinuity). Let \( u_0(x) \) satisfy Assumptions 1 and 2 and \( u(t, x) \) be the corresponding solution to (1.1). For all \( \delta > 0 \) we have

\[
\limsup_{t \to +\infty} \sup_{x \in (h^*(t) - \delta, h^*(t))} u(t, x) \geq \frac{1}{1 + \hat{\chi} + \alpha \chi} > 0. \tag{4.6}
\]

Proof. We divide the proof in 2 steps.

Step 1: We show that for all \( \delta > 0 \),

\[
\sup_{t > 0} \sup_{x \in (h^*(t) - \delta, h^*(t))} u(t, x) \geq \frac{1}{1 + \hat{\chi} + \alpha \chi}. \tag{4.7}
\]

Assume by contradiction that there exists \( \delta > 0 \) such that

\[
\forall t > 0, \sup_{x \in (h^*(t) - \delta, h^*(t))} u(t, x) \leq \eta < \frac{1}{1 + \hat{\chi} + \alpha \chi}, \tag{4.8}
\]

where \( \alpha \geq 1 \) is the constant from Assumption 2.

We remark that the following inequality holds for \( x \in (h^*(t) - \delta, h^*(t)) \).

\[
\frac{d}{dt} u(t, h(t, x)) = \hat{\chi} u(t, h(t, x))(\rho \ast u)(t, h(t, x)) + u(t, h(t, x))(1 - (1 + \hat{\chi})u(t, h(t, x)))
\]

\[
\geq u(t, h(t, x))(1 - (1 + \hat{\chi})u(t, h(t, x))) \geq u(t, h(t, x)) \left(1 - \frac{1 + \hat{\chi}}{1 + \hat{\chi} + \alpha \chi}\right), \tag{4.9}
\]

therefore

\[
u(t, h(t, x)) \geq u(0, x) \exp \left((1 - (1 + \hat{\chi})\eta)t\right),
\]

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provided the characteristic \( h(t, x) \) does not leave the cylinder \((h^*(s) - \delta, h^*(s))\) for any \(0 \leq s \leq t\).

Next by (4.5) and (4.8), we have
\[
\frac{d}{dt}(h^*(t) - h(t, x)) \leq \chi(h^*(t) - h(t, x)) \times \eta,
\]
for each \(x \in (h^*(t) - \delta, h^*(t)) \). Hence by Grönwall’s Lemma
\[
(h^*(t) - h(t, x)) \leq -xe^{\eta t},
\]
provided the characteristic \( h(t, x) \) does not leave the cylinder \((h^*(s) - \delta, h^*(s))\) for any \(0 \leq s \leq t\).

In particular for \(0 > -\frac{1}{2} \delta e^{-\eta t} \geq x \geq -\delta e^{-\eta t} \), we find
\[
u(t, h(t, x)) \geq \nu(0, x) \exp \left( \left( 1 - \frac{1 + \hat{\chi}}{1 + \hat{\chi} + \alpha \chi} \right) t \right) \geq \gamma(-x) \alpha \exp \left( \left( 1 - \frac{1 + \hat{\chi}}{1 + \hat{\chi} + \alpha \chi} \right) t \right) \geq \frac{1}{2^\alpha} \gamma \alpha \exp \left( \left( 1 - (1 + \hat{\chi} + \alpha \chi) \eta \right) t \right) \xrightarrow{t \to +\infty} +\infty,
\]
by our assumption that \( \eta < \frac{1}{1 + \hat{\chi} + \alpha \chi} \). This is a contradiction.

**Step 2**: We show (4.6).

Assume by contradiction that there exists \(T > 0\) and \(\delta > 0\) such that
\[
\sup_{t \geq T} \sup_{x \in [h^*(t) - \delta, h^*(t)]} \nu(t, x) < \frac{1}{1 + \hat{\chi} + \alpha \chi}.
\]
Since the function \(\nu(t, x + h^*(t))\) is continuous on the compact set \([0, T] \times [-\delta, 0]\), it is uniformly continuous on this set and hence (recall that \(\nu(t, h^*(t)) = 0\)) there exists \(0 < \delta_0 \leq \delta\) such that
\[
\sup_{t \in [0, T], x \in [-\delta_0, 0]} \nu(t, x + h^*(t)) = \sup_{t \in [0, T], x \in [-\delta_0, 0]} \left( \nu(t, x + h^*(t)) - \nu(t, h^*(t)) \right) \leq \frac{1}{1 + \hat{\chi} + \alpha \chi}.
\]
Hence we conclude
\[
\sup_{t > 0, x \in [-\delta_0, 0]} \nu(t, x - h^*(t)) \leq \frac{1}{1 + \hat{\chi} + \alpha \chi}.
\]
This is in contradiction with Step 1. Proposition 4.4 is proved.

**Proposition 4.5** (Refined estimate on the level sets). Let \(u_0(x)\) satisfy Assumption 1 and 2. Define
\[
\xi(t, \beta) := \sup \{ x \in \mathbb{R} \mid \nu(t, x) = \beta \}
\]
for any \(0 < \beta < \frac{1}{1 + \hat{\chi} + \alpha \chi} \). Then, the level set function \(\xi(t, \beta)\) converges exponentially fast to \(h^*(t)\)
\[
h^*(t) - \left( \frac{\beta}{\gamma} \right)^\frac{1}{\nu} \leq \xi(t, \beta) \leq h^*(t), \quad (4.10)
\]
for each \(0 < \beta < \frac{1}{1 + \hat{\chi} + \alpha \chi} \), where \(\eta\) is given by
\[
\eta := 1 - \frac{1 + \hat{\chi} + \alpha \chi}{\beta} \in (0, 1).
\]
Proof. Let \(\eta \in (0, 1)\) be given and set \(\beta^* := \frac{1 - \eta}{1 + \hat{\chi} + \alpha \chi} \). Let us first remark that for any \(\beta \in (0, \beta^*)\), \(\xi(t, \beta)\) is well-defined by the continuity of \(x \mapsto \nu(t, x)\) and Assumption 2, that \(\nu(t, \xi(t, \beta)) = \beta\) and that \(\sup_{x \in (\xi(t, \beta), h^*(t))} \nu(t, x) \leq \beta\). Moreover \(\xi(0, \beta) < 0\) and \(u_0(\xi(0, \beta)) = \beta \geq \gamma |\xi(0, \beta)|\)^\alpha, therefore
\[
\xi(0, \beta) \geq - \left( \frac{\beta}{\gamma} \right)^\frac{1}{\nu} \quad (4.11)
\]
for each \(0 < \beta \leq \beta^* = \frac{1 - \eta}{1 + \hat{\chi} + \alpha \chi} \).
**Step 1:** We show that if \(u_0\) satisfies Assumption 1 and (4.11), then

\[
\xi(t, \beta) \geq h^*(t) - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{n}} e^{\frac{n}{\gamma} t},
\]

for all \(0 \leq t \leq t^* := \frac{1}{1+\eta} \ln \left( 1 + \frac{\eta}{2(1-\eta)} \right).

Let \(0 < \beta \leq \beta^*\). We remark that, by Assumption 1, we have \(0 \leq u(t, x) \leq 1\) hence \(0 \leq (\rho \ast u)(t, x) \leq 1\). It follows that, for all \(t \geq 0\),

\[
\frac{d}{dt} u(t, h(t, x)) = u(t, h(t, x)) \left( 1 + \chi \rho \ast u - (1 + \tilde{\chi})u(t, h(t, x)) \right) \leq (1 + \tilde{\chi})u(t, h(t, x)).
\]

In the remaining part of Step 1 we consider \(t \in [0, t^*]\). Using (4.5) from Lemma 4.3 we establish the following estimates on \(u\) and \(h\) for \(0 \leq t \leq t^*\) and \(\xi(0, \beta^*) \leq x \leq 0\):

- Since \(\frac{d}{dt} u(t, h(t, x)) \leq (1 + \tilde{\chi})u(t, h(t, x))\) we have \(u(t, h(t, x)) \leq u_0(x)e^{(1+\tilde{\chi})t}\) for all \(t \leq t^*\) and hence if \(x \geq \xi(0, \beta^*)\),

\[
u(t, h(t, x)) \leq \beta^* e^{\ln \left( 1 + \frac{\eta}{2(1-\eta)} \right)} \left( 1 + \frac{\eta}{2(1-\eta)} \right) = \frac{1 - \frac{\eta}{2}}{1 + \tilde{\chi} + \alpha \chi}.
\]

- Using (4.13) in the equation along the characteristic (2.2):

\[
\frac{d}{dt} u(t, h(t, x)) = u(t, h(t, x)) \left( 1 + \tilde{\chi}(\rho \ast u)(t, h(t, x)) - (1 + \tilde{\chi})u(t, h(t, x)) \right) \geq \left( 1 - \frac{\eta}{1 + \tilde{\chi} + \alpha \chi} \right) u(t, h(t, x)),
\]

we get

\[
u(t, h(t, x)) \geq u_0(x) e^{\left[ 1 - \frac{(1 + \tilde{\chi})(1 - \frac{\eta}{2})}{1 + \tilde{\chi} + \alpha \chi} \right] t}\]

(4.14)

- For all \(x \in (\xi(0, \beta^*), 0)\), since

\[
\sup_{y \in (h(t, x), h^*(t))} u(t, y) \leq \sup_{y \in (h(t, \xi(0, \beta^*)), h^*(t))} u(t, y) \leq \frac{1 - \frac{\eta}{2}}{1 + \tilde{\chi} + \alpha \chi},
\]

we have by (4.5):

\[
h^*(t) - h(t, x) \leq \exp \left( \frac{(1 - \frac{\eta}{2}) \chi}{1 + \tilde{\chi} + \alpha \chi} t \right) (h^*(0) - h(0, x)),
\]

hence

\[
h(t, x) \geq h^*(t) + x \exp \left( \frac{(1 - \frac{\eta}{2}) \chi}{1 + \tilde{\chi} + \alpha \chi} t \right).
\]

(4.15)

Since \(\beta \leq \beta^*\), we have \(\xi(0, \beta) \geq \xi(0, \beta^*)\). Using (4.14) with \(x = \xi(0, \beta)\) we find that

\[
u(t, h(t, \xi(0, \beta))) \geq \beta \exp \left[ 1 - \frac{(1 + \tilde{\chi})(1 - \frac{\eta}{2})}{1 + \tilde{\chi} + \alpha \chi} \right] t,\]

which implies

\[
\xi \left( t, \beta \exp \left[ 1 - \frac{(1 + \tilde{\chi})(1 - \frac{\eta}{2})}{1 + \tilde{\chi} + \alpha \chi} \right] t \right) \geq h(t, \xi(0, \beta)).
\]

Now by using \(x = \xi(0, \beta)\) in (4.15), we obtain

\[
h(t, \xi(0, \beta)) \geq h^*(t) + \xi(0, \beta) \exp \left( \frac{(1 - \frac{\eta}{2}) \chi}{1 + \tilde{\chi} + \alpha \chi} t \right).
\]
Using (4.11) we find that
\[
\xi(0, \beta \exp \left[ - \left( 1 - \frac{(1 + \hat{\chi})(1 - \frac{\gamma}{\alpha})}{1 + \hat{\chi} + \alpha \chi} \right) t \right]) \geq - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}} \exp \left[ - \frac{1}{\alpha} \left( 1 - \frac{(1 + \hat{\chi})(1 - \frac{\gamma}{\alpha})}{1 + \hat{\chi} + \alpha \chi} \right) t \right]
\]
which leads to
\[
\xi(t, \beta) \geq \tilde{h}^{*}(t) - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}} \exp \left[ - \frac{1}{\alpha} \left( 1 - \frac{(1 + \hat{\chi})(1 - \frac{\gamma}{\alpha})}{1 + \hat{\chi} + \alpha \chi} \right) t \right]
\]
and this estimate holds for each \( 0 \leq t \leq t^{*} \) and \( 0 < \beta \leq \beta^{*} \).

**Step 2:** We show that the estimate (4.12) can be extended by induction.

Define \( \tilde{u}_{0}(x) \equiv u(t^{*}, x + h(t^{*})) \) and \( \tilde{\xi}(t, \beta) = \xi(t + t^{*}, \beta) - \tilde{h}^{*}(t^{*}) \). We have for each \( 0 < \beta \leq \beta^{*} \)
\[
\tilde{\xi}(0, \beta) \geq - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}},
\]
where \( \bar{\gamma} = \gamma e^{\frac{1}{2} t^{*}} \). In particular the inequality (4.11) is satisfied by \( \tilde{u}_{0}(x) \), as well as Assumption 1. We can apply Step 1 and (4.12) gives
\[
\tilde{\xi}(t, \beta) \geq \tilde{h}^{*}(t) - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}} e^{-\frac{\beta}{\gamma} t^{*}} = h(t, h^{*}(t)) - \tilde{h}^{*}(t^{*}) - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}} e^{-\frac{\beta}{\gamma} (t + t^{*})}
\]
which yields
\[
\xi(t + t^{*}, \beta) \geq \tilde{h}^{*}(t + t^{*}) - \left( \frac{\beta}{\gamma} \right)^{\frac{1}{2}} e^{-\frac{\beta}{\gamma} (t + t^{*})}.
\]
The proof is completed. \( \square \)

We are now in the position to prove Theorem 2.5.

**Proof of Theorem 2.5.** The first part, equation 2.5, has been shown in Proposition 4.4, while the second part (equation 2.6) has been shown in Proposition 4.5. \( \square \)

We conclude this section by the proof of Proposition 2.6.

**Proof of Proposition 2.6.** Since \( x \mapsto u(t, x) \) is nonincreasing, we have \( u(t, x) \geq u(t, h^{*}(t)) \) for each \( x \leq h^{*}(t) \). Hence \( (\rho * u)(t, h^{*}(t)) \geq \frac{1}{2} u(t, h^{*}(t)) \) and
\[
\frac{d}{dt} u(t, h^{*}(t)) = u(t, h^{*}(t))(1 + \hat{\chi} \rho * u - (1 + \hat{\chi}) u(t, h^{*}(t))) \geq u(t, h^{*}(t)) \left( 1 - \left( 1 + \frac{\hat{\chi}}{2} \right) u(t, h^{*}(t)) \right).
\]
This yields
\[
u(t, h^{*}(t)) \geq \frac{u_{0}(0)}{\left( 1 + \frac{\hat{\chi}}{2} \right) u_{0}(0) + e^{-t} \left( 1 - \left( 1 + \frac{\hat{\chi}}{2} \right) u_{0}(0) \right)} \xrightarrow{t \to \infty} \frac{1}{1 + \frac{\hat{\chi}}{2}} = \frac{2}{2 + \hat{\chi}}.
\]
(2.7) is shown. Next, we have \( \frac{d}{dt} h^{*}(t) = -(\rho_{x} * u)(t, h^{*}(t)) \) which gives
\[
\frac{d}{dt} h^{*}(t) = \frac{\chi}{\sigma} \int_{0}^{\infty} \rho(y) u(t, h^{*}(t) - y) dy \geq u(t, h^{*}(t)) \times \frac{\chi}{2 \sigma} \xrightarrow{t \to \infty} \frac{\sigma \hat{\chi}}{2 + \hat{\chi}}.
\]
This proves (2.8) and finishes the proof of Proposition 2.6. \( \square \)
5 Traveling wave solutions

In this section we investigate the existence of particular solutions which consist in a fixed profile traveling at a constant speed $c$ (traveling waves). We are particularly interested in profiles which connect the stationary state $1$ near $-\infty$ to the stationary solution $0$ at a finite point of space, say, for any $x \geq 0$.

5.1 Existence of sharp traveling waves

We study the traveling wave solutions of equation (1.1):

$$
\begin{cases}
\partial_t u(t,x) - \chi \partial_x (u(t,x) \partial_x p(t,x)) = u(t,x)(1 - u(t,x)) \\
-\sigma^2 \partial_{xx}^2 p(t,x) + p(t,x) = u(t,x)
\end{cases}
$$

$t > 0$, $x \in \mathbb{R}$.

Let us formally derive an equation for the traveling wave solutions to (1.1). We consider the traveling wave solution $U(x-ct) = u(t,x)$. By using the resolvent formula of the second equation of (1.1) formula we deduce that

$$
p(t,x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-|x-y|} u(t,y) dy = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-|x-ct-y|} U(t) dt = P(x-ct)
$$

and the first equation in (1.1) becomes

$$
-cU''(x-ct) - \chi \partial_x \left( U(x-ct) \partial_x P(x-ct) \right) = U(x-ct)(1 - U(x-ct)), \quad t > 0, \quad x \in \mathbb{R}. \tag{5.1}
$$

By developing the derivative in (5.1) we obtain

$$
\left( -c - \chi P'(x-ct) \right) U''(x-ct) = U(x-ct)(1 + \hat{\chi} P(x-ct) - (1 + \hat{\chi}) U(x-ct)), \quad t > 0, \quad x \in \mathbb{R},
$$

where $\hat{\chi} = \frac{\chi}{2}$. Therefore, by letting $z = x - ct$, the traveling wave solutions of system (1.1) satisfy the following equation

$$
\begin{cases}
(-c - \chi P'(z)) U''(z) = U(z)(1 + \hat{\chi} P(z) - (1 + \hat{\chi}) U(z)), \\
-\sigma^2 P''(z) + P(z) = U(z).
\end{cases} \tag{5.2}
$$

Let us finally remark that

$$
P(z) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-|z-y|} U(z-y) dy = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-|z-ct-y|} U(t) dt. \tag{5.3}
$$

In particular if $U$ is non-constant and nonincreasing, then $z \mapsto P(z)$ is strictly decreasing.

The goal of this Section is to show that equation (5.2) can solved on the half-line $(-\infty, 0)$ which, as we will see later, will give a proof of Theorem 2.7. We begin by defining a set of admissible profiles, which is the set of function on which an appropriate fixed-point theorem will be used. The properties we impose are those who we suspect will be satisfied by the real profile of the traveling wave.

Definition 5.1. We say that the profile $U: \mathbb{R} \to [0,1]$ is admissible if

(i) $U \in C((-\infty, 0), \mathbb{R})$ and $\lim_{z \to 0^-} U(z)$ exists and belongs to $\left[\frac{2}{2 + \hat{\chi}}, 1\right]$;

(ii) $0 \leq U(z) \leq 1$ for any $z \in \mathbb{R}$;

(iii) the map $z \mapsto U(z)$ is non-increasing on $\mathbb{R}$;

(iv) $U(z) \equiv 0$ for any $z \geq 0$.

We denote $\mathcal{A}$ the set of all admissible functions.
Lemma 5.1. Let Assumption 3 hold and suppose that $U$ is admissible (as in Definition 5.1). Then the function $P$ defined by $P = (\rho \ast U)$ satisfies
\[ P'(0) < P'(z) \leq 0, \, \text{for all } z \in \mathbb{R} \setminus \{0\}. \]

Moreover, this estimate is locally uniform in $U$ on $(-\infty, 0)$ in the sense that for each $L > 1$ there is $\epsilon > 0$ independent of $U \in A$ such that
\[ P'(z) - P'(0) \geq \epsilon > 0, \, \text{for all } z \in \left[-L, -\frac{1}{L}\right]. \]

Proof. We divide the proof in five steps.

**Step 1.** We prove $P'(0) < P'(z)$ for any $z > 0$. Notice that, for $z > 0$, we have
\[ P(z) = \frac{1}{2\sigma} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma} \int_{z}^{\infty} e^{-\frac{y^2}{2\sigma^2}} U(y)dy = \frac{1}{2\sigma} e^{-\frac{z^2}{2\sigma}} \int_{-\infty}^{0} e^{\frac{x^2}{2\sigma}} U(y)dy. \]

Thus, taking derivative gives
\[ P'(z) = -\frac{1}{\sigma} e^{-\frac{z^2}{2\sigma}} \int_{-\infty}^{0} e^{\frac{x^2}{2\sigma}} U(y)dy = e^{-z} P'(0), \]

and since $U$ is strictly positive for negative values of $z$, we deduce that $P'(0) < P'(z)$ for any $z > 0$.

**Step 2.** We prove that $P'(0) < P'(z)$ for any $-\sigma \ln (\hat{\xi}) < z < 0$. In fact, we prove the stronger result
\[ P'(z) < 0 \text{ if } \sigma \ln \left(\frac{\hat{\chi}}{2}\right) < z < 0. \]

For any $z < 0$, we have
\[ P''(z) = \frac{1}{2\sigma^3} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma^3} \int_{z}^{\infty} e^{-\frac{y^2}{2\sigma^2}} U(y)dy - \frac{1}{\sigma^2} U(z) \]
\[ = \frac{1}{2\sigma^3} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma^3} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} U(y)dy - \frac{1}{\sigma^2} U(z). \]

Due to the assumption $U \leq 1$ and the fact that $U$ is decreasing we have
\[ \sigma^2 P''(z) \leq -\frac{1}{2\sigma} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} U(y)dy - U(z) \]
\[ = \frac{1}{2} + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} U(y)dy - U(z) \leq \frac{1}{2} + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} dy U(z) - U(z) \]
\[ = \frac{1}{2} - \frac{1}{2} \left(1 + e^{\hat{\xi}}\right) U(z) \leq \frac{1}{2} \frac{2 + \hat{\chi} - 2(1 + e^{\hat{\xi}})}{2 + \hat{\chi}} = \frac{-\hat{\chi} - 2e^{\hat{\xi}}}{2(2 + \hat{\chi})} < 0, \]

provided $z \in (\sigma \ln (\hat{\chi}/2), 0)$. In particular
\[ P'(z) - P'(0) = -\int_{z}^{0} P''(y)dy \geq \frac{1}{\sigma(2 + \hat{\chi})} \left(\frac{-\hat{\chi} - 2e^{\hat{\xi}}}{2(2 + \hat{\chi})}\right) > 0. \quad (5.4) \]

**Step 3.** We prove that $P'(0) < P'(z)$ for any $z < \sigma \ln \left(1 - \frac{\hat{\xi}}{2}\right)$. For any $z < 0$, we have
\[ \sigma P'(z) = -\frac{1}{2\sigma} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} U(y)dy, \quad \sigma P'(0) = -\frac{1}{2\sigma} \int_{-\infty}^{0} e^{\hat{\xi}U(y)dy}, \]

and
\[ \sigma(P'(z) - P'(0)) = \frac{1}{2\sigma} \int_{-\infty}^{0} e^{\hat{\xi}U(y)dy} - \frac{1}{2\sigma} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma^2}} U(y)dy + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y^2}{2\sigma^2}} U(y)dy. \]
Since for any $z \leq 0$, $-\frac{z}{2 + \chi} \leq U(z) \leq 1$, we have the following estimate

$$
\sigma(P'(z) - P'(0)) \geq \frac{1}{2\sigma} \int_{-\infty}^{0} e^{\frac{y}{2}} \times \frac{2}{2 + \chi} dy - \frac{1}{2\sigma} \int_{-\infty}^{z} e^{-\frac{y}{2+\chi}} dy + \frac{1}{2\sigma} \int_{z}^{0} e^{-\frac{y}{2+\chi}} \frac{2}{2 + \chi} dy
$$

$$
= \frac{1}{2 + \chi} \left( \frac{1}{2} + \frac{1}{2 + \chi} \left( 1 - e^{\frac{z}{\chi}} \right) \right)
= \frac{1}{2 + \chi} \left( 2 - e^{\frac{z}{\chi}} - \frac{1}{2} \left( 2 + \chi \right) \right) = \frac{1}{2 + \chi} \left( 1 - \frac{z}{\chi} - e^{\frac{z}{\chi}} \right). 
$$

(5.5)

By our assumption $z < \sigma \ln \left( 1 - \frac{\tilde{\chi}}{2} \right)$, we deduce that $P'(z) - P'(0) > 0$.

Notice that, if $\tilde{\chi} < 1$, we have $\sigma \ln \left( \frac{\tilde{\chi}}{2} \right) < \sigma \ln \left( 1 - \frac{\tilde{\chi}}{2} \right)$ and the estimate is done. If $1 \leq \tilde{\chi} < 2$ we still need to fill a gap between the two bounds.

**Step 4.** We assume that $\tilde{\chi} \geq 1$ and we prove that

$$
P'(z) - P'(0) \geq -\int_{z}^{0} P''(y) dy \geq \frac{z}{2\sigma} - \frac{1}{2\sigma} \ln \left( \frac{\tilde{\chi}}{2} \right) + \frac{\tilde{\chi}}{2\sigma(2 + \chi)} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right) > 0
$$

(5.6)

for any $z \in \left[ \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) - \frac{\sigma}{2 + \chi} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right), \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) \right]$. Notice that

$$
\frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} > 0,
$$

because $x \mapsto x \ln(x)$ is strictly convex.

By Step 2 we have for all $z \leq 0$:

$$
P''(z) \leq \frac{1}{2\sigma^2},
$$

therefore if $z \in \left[ \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) - \frac{\sigma}{2 + \chi} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right), \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) \right]$ we have

$$
P'(z) - P'(0) = P'(z) - P' \left( \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) \right) + P' \left( \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) \right) - P'(0)
$$

$$
\geq - \int_{z}^{\sigma \ln \left( \frac{\tilde{\chi}}{2} \right)} P''(y) dy + \frac{\tilde{\chi}}{2\sigma(2 + \chi)} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right)
$$

$$
\geq - \frac{1}{2\sigma^2} \left( \sigma \ln \left( \frac{\tilde{\chi}}{2} \right) - z \right) + \frac{\tilde{\chi}}{2\sigma(2 + \chi)} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right)
$$

$$
\geq \frac{z}{2\sigma^2} - \frac{\ln \left( \frac{\tilde{\chi}}{2} \right)}{2\sigma} + \frac{\tilde{\chi}}{2\sigma(2 + \chi)} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right) > 0.
$$

We have proved the desired estimate.

**Step 5.** We show the local uniformity. If $\tilde{\chi} < 1$ the local uniformity follows from Step 2 and Step 3 because $1 - \frac{\tilde{\chi}}{2} < \frac{\tilde{\chi}}{2}$. If $1 \leq \tilde{\chi} < 2$, then

$$
\ln \left( \frac{\tilde{\chi}}{2} \right) - \frac{2}{2 + \chi} \left( \frac{\tilde{\chi}}{2} \ln \left( \frac{\tilde{\chi}}{2} \right) + 1 - \frac{\tilde{\chi}}{2} \right) < \ln \left( 1 - \frac{\tilde{\chi}}{2} \right),
$$

(5.7)

because of Assumption 3 and Lemma B.1 (notice that (5.7) is equivalent to $f(\tilde{\chi}) < 0$, where $f$ is as defined in Lemma B.1). By the estimates (5.4), (5.5) and (5.6) from Step 2, Step 3 and Step 4, we find that $P'(z) - P'(0) > 0$ on every compact subset of $(-\infty, 0)$ and is bounded from below by a constant independent of $U$. This finishes the proof of Lemma 5.1.

Before resuming to the proof, let us define the mapping $T$ to which we want to apply a fixed-point theorem. Fix $U \in \mathcal{A}$, we define $T(U)$ as

$$
T(U)(z) := U(\sigma^{-1}(z)) \text{ for all } z < 0
$$

(5.8)
and $\mathcal{T}(U)(z) \equiv 0$ for all $z \geq 0$, where $\tau : \mathbb{R} \mapsto (-\infty, 0)$ is the solution of the following scalar ordinary differential equation

$$\begin{aligned}
\tau'(t) &= \chi(P'(0) - P'(\tau(t))), \\
\tau(0) &= -1,
\end{aligned} \tag{5.9}$$

and

$$U(t) = \left[(1 + \hat{\chi}) \int_{-\infty}^{t} \exp \left(-\int_{t}^{s} 1 + \hat{\chi} P_r(s) ds \right) dl \right]^{-1}, \forall t \in \mathbb{R}.$$  

**Lemma 5.2** (Stability of $A$). Let Assumption 3 be satisfied, let $U$ be admissible in the sense of Definition 5.1 and $\mathcal{T}$ be the map defined by (5.8). Then the image of $U$ by $\mathcal{T}$ has the following properties:

(i) $\frac{2}{2 + \hat{\chi}} \leq \mathcal{T}(U)(z) \leq 1$ for all $z \leq 0$;

(ii) $\mathcal{T}(U)$ is strictly decreasing on $(-\infty, 0]$;

(iii) $\mathcal{T}(U) \in C^1((-\infty, 0), \mathbb{R})$ and $\mathcal{T}(U)(0^-) = \lim_{z \to 0^-} \mathcal{T}(U)(z) = \frac{1 + \hat{\chi} P'(0)}{1 + \hat{\chi}}$.

In particular, $A$ is left stable by $\mathcal{T}$

$$\mathcal{T}(A) \subset A.$$  

**Proof.** We divide the proof in three steps.

**Step 1.** We prove that $\frac{2}{2 + \hat{\chi}} \leq \mathcal{T}(U)(z) \leq 1$ for all $z < 0$. For any $z \in \mathbb{R}$ we have

$$P(z) = \int_{-\infty}^{\infty} \rho(y) U(z - y) dy \leq \int_{-\infty}^{+\infty} \rho(y) dy = 1,

P(z) = \int_{-\infty}^{\infty} \rho(y) U(z - y) dy \geq 0.$$  

Thus, for any $z < 0$, we have for $z < 0$

$$P(z) \geq \frac{1}{2\sigma} \int_{z}^{+\infty} \exp \left(-\frac{|y|}{\sigma}\right) \times \frac{2}{2 + \hat{\chi}} dy = \frac{2}{2 + \hat{\chi}} \left(1 - \frac{e^{\frac{\hat{\chi} z}{2}}}{2}\right) \geq \frac{1}{2 + \hat{\chi}}.$$  

Thus, for any $z \leq 0$, we have $\frac{1}{2 + \hat{\chi}} \leq P(z) \leq 1$. Since $\tau(t)$ is the solution of

$$\begin{cases}
\tau'(t) &= \chi(P'(0) - P'(\tau(t))), \\
\tau(0) &= -1,
\end{cases}$$

and due to Lemma 5.1, $t \mapsto \tau(t)$ is strictly decreasing, continuous and

$$\lim_{t \to -\infty} \tau(t) = 0, \quad \lim_{t \to +\infty} \tau(t) = -\infty.$$  

Therefore,

$$\frac{1}{2 + \hat{\chi}} \leq P(\tau(t)) \leq 1, \quad t \in \mathbb{R}.$$  

Since by definition $U(t) = \left[(1 + \hat{\chi}) \int_{-\infty}^{t} e^{-\int_{t}^{s} 1 + \hat{\chi} P_r(s) ds} dl \right]^{-1}$, $U$ is monotone with respect to $P$, and we compute on the one hand

$$U(t) \leq \left[(1 + \hat{\chi}) \int_{-\infty}^{t} e^{-\int_{t}^{s} 1 + \hat{\chi} ds} dl \right]^{-1}.$$  


Moreover, for any Furthermore, we can see that is strictly decreasing, therefore the composition of two mappings deduce that

\[ \mathcal{U}(t) \geq \left(1 + \hat{\chi}\right) \int_{-\infty}^{t} \exp\left(-\int_{t}^{l} \frac{1}{2 + \hat{\chi}} dl\right) \exp\left(-\int_{t}^{l} \frac{\hat{\chi}}{2 + \hat{\chi}} (l - l) dl\right) \leq \frac{2}{2 + \hat{\chi}}. \]

This implies \( \frac{2}{2 + \hat{\chi}} \leq \mathcal{U}(t) \leq 1, \quad \forall t \in \mathbb{R} \). Since \( \tau^{-1} \) maps \((-\infty, 0)\) to \( \mathbb{R} \), for any \( z < 0 \) we have indeed

\[ \frac{2}{2 + \hat{\chi}} \leq \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z)) \leq 1. \]

Item (i) is proved.

**Step 2.** We prove that \( z \mapsto \mathcal{T}(U)(z) \) is strictly decreasing on \((-\infty, 0)\). First, we prove that \( t \mapsto \mathcal{U}(t) \) is strictly increasing. Indeed \( \mathcal{U} \) is differentiable and we have

\[ \mathcal{U}'(t) = -\frac{1}{1 + \hat{\chi}} \times \left(1 + \int_{-\infty}^{t} -\left(1 + \hat{\chi} P(\tau(t))\right) e^{-\int_{t}^{l} \hat{\chi} P(\tau(s)) ds} dl \right) \]

Moreover, for any \( l < t \), we have \( \tau(t) < \tau(l) \). Since \( P \) is strictly decreasing, \( P(\tau(l)) < P(\tau(t)) \). We deduce

\[ \int_{-\infty}^{t} e^{-\int_{t}^{l} \hat{\chi} P(\tau(s)) ds} \left(1 + \hat{\chi} P(\tau(t))\right) dl > \int_{-\infty}^{t} e^{-\int_{t}^{l} \hat{\chi} P(\tau(s)) ds} \left(1 + \hat{\chi} P(\tau(l))\right) dl \]

\[ = \int_{-\infty}^{t} \frac{d}{dl} \left(e^{-\int_{t}^{l} \hat{\chi} P(\tau(s)) ds}\right) = 1. \]

This implies \( \mathcal{U}'(t) > 0 \) and \( t \mapsto \mathcal{U}(t) \) is strictly increasing. Note that the inverse map \( z \mapsto \tau^{-1}(z) \) is strictly decreasing, therefore the composition of two mappings

\[ z \mapsto \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z)) \]

is also strictly decreasing on \((-\infty, 0)\). Item (ii) is proved.

**Step 3.** We prove that \( \mathcal{T}(U) \in C^{1}((-\infty, 0), \mathbb{R}) \) and compute the limit of \( \mathcal{T}(U) \) as \( z \to 0^{-} \).

Since for any \( z < 0 \)

\[ \sigma^{2} P''(z) = -\mathcal{U}(z) + P(z) \in C((-\infty, 0), \mathbb{R}), \]

\( P \) belongs to \( C^{2}((-\infty, 0), \mathbb{R}) \), which implies that \( t \mapsto \tau(t) \) belongs to \( C^{1}(\mathbb{R}, (-\infty, 0)) \). By (5.10), the function \( t \mapsto \mathcal{U}(t) \) is continuous and the inverse map \( z \to \tau^{-1}(z) \) is also of class \( C^{1} \) from \((-\infty, 0)\) to \( \mathbb{R} \). Thus, the function

\[ z \mapsto \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z)) \]

is of class \( C^{1} \) from \((-\infty, 0)\) to \( \mathbb{R} \). Moreover, the map \( t \mapsto \mathcal{U}(t) \) is strictly decreasing and is bounded from below by \( \frac{2}{2 + \hat{\chi}} > 0 \), thus \( \lim_{t \to -\infty} \mathcal{U}(t) \) exists. In particular

\[ \mathcal{T}(U)(0^{-}) := \lim_{z \to 0^{-}} \mathcal{U}(\tau^{-1}(z)) = \lim_{t \to -\infty} \mathcal{U}(t). \]

By the definition of \( \mathcal{U} \)

\[ \mathcal{T}(U)(0^{-}) = \lim_{t \to -\infty} \mathcal{U}(t) \]
\[
\begin{align*}
&= \lim_{t \to -\infty} \left[ (1 + \hat{\chi}) \int_{-\infty}^{t} e^{-\int_{s}^{t} 1 + \hat{\chi} P(\tau(s)) ds} dl \right]^{-1} \\
&= \lim_{t \to -\infty} \frac{e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds}}{(1 + \hat{\chi}) \int_{-\infty}^{t} e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds} dl}.
\end{align*}
\]

By employing L'Hôpital rule
\[
T(U)(0^-) = \lim_{t \to -\infty} \frac{e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds}}{e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds} dl}
= \lim_{t \to -\infty} \frac{(1 + \hat{\chi} P(\tau(t))) e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds}}{(1 + \hat{\chi}) e\int_{0}^{1 + \hat{\chi} P(\tau(s)) ds} dl}
= \frac{1 + \hat{\chi} P(0)}{1 + \hat{\chi}}.
\]

Therefore, \( T(U) \in C^1((-\infty, 0], \mathbb{R}) \cap C((-\infty, 0], \mathbb{R}) \) and \( T(U)(0) = (1 + \hat{\chi} P(0))/(1 + \hat{\chi}) \). This proves Item (iii) and concludes the proof of Lemma 5.2.

Next we focus on the continuity of \( T \) for a particular topology.

**Lemma 5.3 (Continuity of \( T \)).** Define the weighted norm
\[
\|U\|_{\eta} := \sup_{z \in (-\infty, 0)} \alpha(z) |U(z)|,
\]
where
\[
\alpha(z) := \sqrt{-z} e^{\eta z} \leq \frac{1}{\sqrt{2\pi\eta}}, \text{ for all } z \leq 0,
\]
with \( 0 < \eta < \sigma^{-1} \). If Assumption 3 is satisfied, then the map \( T \) is continuous on \( \mathcal{A} \) for the distance induced by \( \| \cdot \|_\eta \).

**Proof.** Let \( U \in \mathcal{A} \) and \( \varepsilon > 0 \) be given. Let \( \hat{U} \in \mathcal{A} \) be given and define the corresponding pressure and rescaled variable \( \hat{P} := \rho \star \hat{U} \) and \( \hat{\tau} \) as the solution to (5.9) with \( U \) replaced by \( \hat{U} \). We remark that:
\[
\begin{align*}
|T(U)(z) - T(\hat{U})(z)| & \leq \int_{-\infty}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z)} 1 + \hat{\chi} \hat{P}(\hat{\tau}(s)) ds} dl - \int_{-\infty}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z)} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl \\
& \leq \int_{-\infty}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z)} 1 + \hat{\chi} \hat{P}(\hat{\tau}(s)) ds} dl - \int_{-\infty}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z)} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl.
\end{align*}
\]

by Lemma 5.2. Define \( T_{-L}(U) := \int_{-L}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z)} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl \). We have \( T(U) = T_{-\infty}(U) \) and
\[
|T_{-\infty}(U) - T_{-\infty}(\hat{U})| \leq \int_{-\infty}^{\hat{\tau}^{-1}(z) - L} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\hat{\tau}(s)) ds} dl - \int_{-\infty}^{\hat{\tau}^{-1}(z) - L} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl \\
+ \int_{\hat{\tau}^{-1}(z) - L}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\hat{\tau}(s)) ds} dl - \int_{\hat{\tau}^{-1}(z) - L}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl
\leq e^{-L} + e^{-L} + \int_{\hat{\tau}^{-1}(z) - L}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\hat{\tau}(s)) ds} dl - \int_{\hat{\tau}^{-1}(z) - L}^{\hat{\tau}^{-1}(z)} e^{-\int_{s}^{\hat{\tau}^{-1}(z) - L} 1 + \hat{\chi} \hat{P}(\tau(s)) ds} dl
\leq e^{-L} + \frac{\varepsilon}{2\sqrt{2\pi\eta}} + \frac{\varepsilon}{2\sqrt{2\pi\eta}}.
\]
\[= \frac{\varepsilon}{2} \sqrt{2\eta e} + |T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)|,\]

for \(L := - \ln \left( \frac{\varepsilon}{2} \sqrt{\frac{T}{2}} \right) > 0.\)

Let \(z_0\) and \(z_1\) be respectively the smallest and the biggest negative root of the equation

\[\eta z + \frac{1}{2} \ln(-z) = \ln \left( \frac{\varepsilon}{4} \right).\]

Then if \(z \notin [z_0, z_1]\) we have \(\sqrt{-z}e^{\eta z} \leq \frac{\varepsilon}{4}\) and, since \(|T_{-L}(U)| \leq 1\) we have

\[\sqrt{-z}e^{\eta z}|T_{-L}(U)(z)| = \sqrt{-z}e^{\eta z} \left| \int_{\tau_{-1}(z)-L}^{\tau_{-1}(z)} e^{-\int_{\tau_{-1}(z)-L}^{t} \chi P(\tau(s))ds} dt \right| \leq \frac{\varepsilon}{4} \int_{\tau_{-1}(z)-L}^{\tau_{-1}(z)} e^{-\int_{\tau_{-1}(z)-L}^{t} \chi P(\tau(s))ds} dt = \frac{\varepsilon}{4} (1 - e^{-L}) \leq \frac{\varepsilon}{4}.
\]

Similarly, we have

\[\sqrt{-z}e^{\eta z}|T_{-L}(\tilde{U})(z)| \leq \frac{\varepsilon}{4}.\]

We have shown

\[\sup_{z \notin [z_0, z_1]} \sqrt{-z}e^{\eta z}|T(U)(z) - T(\tilde{U})(z)| \leq \varepsilon.
\]

There remains to estimate \(\sqrt{-z}e^{\eta z}|T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)|\) when \(z \in [z_0, z_1]\). We have

\[|T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)| = \left| \int_{\tau_{-1}(z)-L}^{\tau_{-1}(z)} e^{-\int_{\tau_{-1}(z)-L}^{t} \chi \tilde{P}(\tau(s))ds} dt - \int_{\tau_{-1}(z)-L}^{\tau_{-1}(z)} e^{-\int_{\tau_{-1}(z)-L}^{t} \chi P(\tau(s))ds} dt \right| \leq 2|\tilde{\tau}^{-1}(z) - \tau^{-1}(z)|
\]

\[+ \left| \int_{\tau_{-1}(z)-L}^{\tau_{-1}(z)} e^{-\int_{\tau_{-1}(z)-L}^{t} \chi \tilde{P}(\tau(s))ds} dt - e^{-\int_{\tau_{-1}(z)-L}^{t} \chi \tilde{P}(\tau(s))ds} dt \right| \leq 2|\tilde{\tau}^{-1}(z) - \tau^{-1}(z)|
\]

\[+ L \sup_{t \in (\tau_{-1}(z)-L, \tau_{-1}(z))} \left| e^{\int_{t}^{\tau_{-1}(z)} \chi \tilde{P}(\tau(s))ds - \int_{t}^{\tau_{-1}(z)} \chi \tilde{P}(\tau(s))ds - 1} \right|,
\]

and we remark that

\[\left| \int_{t}^{\tau_{-1}(z)} 1 + \chi \tilde{P}(\tau(s))ds - \int_{t}^{\tau_{-1}(z)} 1 + \chi \tilde{P}(\tau(s))ds \right| \leq 2|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| + \chi \left| \int_{t}^{\tau_{-1}(z)} P(\tau(s)) - \tilde{P}(\tau(s))ds \right| \leq 2|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| + \chi L \sup_{s \in (\tau^{-1}(z)-L, \tau^{-1}(z))} |P(\tau(s)) - \tilde{P}(\tau(s))| + \chi L \sup_{s \in (\tau^{-1}(z)-L, \tau^{-1}(z))} |P(\tau(s)) - \tilde{P}(\tau(s))|.
\]

To conclude the proof of the continuity of \(T\), we show that each of those three terms can be made arbitrarily small (uniformly on \([z_0, z_1]\)) by choosing \(\tilde{U}\) sufficiently close to \(U\) in the \(\| \cdot \|_q\) norm. We start with the second one. We have for all \(z \leq 0\):

\[|P(z) - \tilde{P}(z)| = \frac{1}{2\pi} \left| \int_{-\infty}^{0} e^{-\frac{i\pi}{\sigma}(U(y) - \tilde{U}(y))} dy \right| \leq \frac{1}{2\pi} \int_{-\infty}^{0} e^{\frac{i\pi}{\sigma} |U(y) - \tilde{U}(y)|} |U(y) - \tilde{U}(y)| dy + \frac{1}{2\pi} \int_{0}^{0} e^{\frac{i\pi}{\sigma} |U(y) - \tilde{U}(y)|} dy.
\]
In particular for $5.1.$ and therefore $P$ recalling that we have a uniform lower bound for $\|U - \tilde{U}\|_\eta$, we have
$\leq \frac{1}{2\sigma} \sqrt{\frac{2\eta}{e}} e^{-\frac{z}{2}} \int_{-\infty}^{z} \frac{e^{\frac{1}{2}(1-\sigma)y}}{\sqrt{-y}} dy + \frac{1}{2} \sqrt{\frac{2\eta}{e}} \int_{-\infty}^{0} e^{-\frac{1}{2}(1+\sigma)y} \frac{1}{\sqrt{-y}} dy \|U - \tilde{U}\|_\eta$

$= \sigma^{-1} \sqrt{\frac{\eta}{2e}} \left[ e^{-\frac{z}{2}} \int_{-\infty}^{z} \frac{e^{\frac{1}{2}(1-\sigma)y}}{\sqrt{-y}} dy + e^{\frac{z}{2}} \int_{z}^{0} e^{-\frac{1}{2}(1+\sigma)y} \frac{1}{\sqrt{-y}} dy \right] \|U - \tilde{U}\|_\eta$

$= : C_P(z)\|U - \tilde{U}\|_\eta.$

A similar computation shows that, for all $z \leq 0$,

$$|P'(z) - \tilde{P}'(z)| \leq \sigma^{-2} \sqrt{\frac{\eta}{2e}} \int_{-\infty}^{0} e^{\frac{1}{2}(1-\sigma)y} \frac{1}{\sqrt{-y}} dy \|U - \tilde{U}\|_\eta,$$

and therefore $P'(0)$ and $\tilde{P}'(0)$ can be chosen arbitrarily small. Next we show that $\tau(t)$ and $\tilde{\tau}(t)$ are uniformly close for $t \in [\tau^{-1}(z_0) - \nu, \tau^{-1}(z_1)]$. Indeed, we compute:

$|t| \leq C(0) + \max_{0 \leq s \leq \tilde{t}} C_P(\tau(s))[\|U - \tilde{U}\|_\eta + \chi \int_{0}^{t} |\tilde{\tau}(s) - \tau(s)| ds],$

where we have used the fact that $\sigma^2|P''(z)| = |P(z) - U(z)| \leq 1$. By Grönwall’s Lemma, we have therefore

$$|\tau(t) - \tilde{\tau}(t)| \leq \chi t[C_P(0) + \max_{0 \leq s \leq \tilde{t}} C_P(\tau(s))]\|U - \tilde{U}\|_\eta e^{\tilde{t}},$$

and we have shown that $\tau$ and $\tilde{\tau}$ can be made arbitrarily close by choosing $\|U - \tilde{U}\|_\eta$ sufficiently small. This gives an arbitrary control on the term

$$\sup_{s \in (\tau^{-1}(z) - L, \tau^{-1}(z))} |P(\tau(s)) - P(\tilde{\tau}(s))| \leq |P'(0)||\tau(s) - \tilde{\tau}(s)|,$$

since $P'(0) < P'(z) \leq 0$ by Lemma 5.1, and on the term

$$\sup_{s \in (\tau^{-1}(z) - L, \tau^{-1}(z))} |P(\tilde{\tau}(s)) - \tilde{P}(\tilde{\tau}(s))| \leq \sup_{s \in (\tau^{-1}(z) - L, \tau^{-1}(z))} C_P(\tilde{\tau}(s)) \|U - \tilde{U}\|_\eta.$$

Finally, we estimate $\tau^{-1}(z) - \tilde{\tau}^{-1}(z)$ by the remark:

$$|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| = \left| \int_{-1}^{z} \frac{1}{\tau^{-1}(\gamma)} d\gamma - \int_{-1}^{z} \frac{1}{\tilde{\tau}^{-1}(\gamma)} d\gamma \right|$$

$$= \frac{1}{\chi} \left| \int_{-1}^{z} \frac{1}{P'(0) - P'(y)} - \frac{1}{\tilde{P}'(0) - \tilde{P}'(y)} d\gamma \right|$$

$$\leq \frac{1}{\chi} \int_{-1}^{z} \left| P'(0) - P'(y) \right| + \left| \tilde{P}'(0) - \tilde{P}'(y) \right| d\gamma,$$

recalling that we have a uniform lower bound for $|P'(0) - P'(y)|$ and $|\tilde{P}'(0) - \tilde{P}'(y)|$ by Lemma 5.1.

This finishes the proof of Lemma 5.3. \qed
Lemma 5.4. Suppose $U$ is admissible in the sense of Definition 5.1 and that Assumption 3 holds. Then $T(U) \in C^1((-\infty,0], \mathbb{R})$ and

$$T(U)'(z) = T(U)(z) \frac{1 + \hat{\chi}P(z) - (1 + \hat{\chi})T(U)(z)}{\chi(P'(0) - P'(z))}, \quad \forall z < 0. \quad (5.12)$$

Moreover

$$\lim_{z \to 0^-} T(U)'(z) = \frac{P'(0)}{1 + \hat{\chi}} \frac{1 + \hat{\chi}P(0)}{1 + \hat{\chi}U(0^+)}. \quad (5.13)$$

Proof. We divide the proof in two steps.

Step 1. We prove (5.12).

We observe that

$$\tau'(\tau^{-1}(z)) := \chi(P'(0) - P'(z)),$$

therefore $T(U)$ is differentiable for each $z < 0$ and

$$T(U)'(z) = U'(\tau^{-1}(z)) \frac{1}{\tau'(\tau^{-1}(z))} = U'(\tau^{-1}(z)) \frac{1}{\chi(P'(0) - P'(z))}. \quad (5.12)$$

By Equation (5.10) in Lemma 5.2 we have

$$U'(t) = \frac{1}{1 + \hat{\chi}} \left[ \int_{-\infty}^t e^{-\int_t^s 1 + \hat{\chi}P(t(s))ds} \, dl \right]^{-2}
\times \left( \int_{-\infty}^t e^{-\int_t^s 1 + \hat{\chi}P(t(s))ds} \, dl - 1 \right)
= \left[ (1 + \hat{\chi}) \int_{-\infty}^t e^{-\int_t^s 1 + \hat{\chi}P(t(s))ds} \, dl \right]^{-2}
\times \left( (1 + \hat{\chi}) \int_{-\infty}^t e^{-\int_t^s 1 + \hat{\chi}P(t(s))ds} \, dl \right)
= \frac{U(t)}{1 + \hat{\chi}} \left( (1 + \hat{\chi}) \int_{-\infty}^t \, dl \right)
= U(t) \left( 1 + \hat{\chi}P(t) \right) \left( 1 + \hat{\chi} \right) \frac{U(t)}{1 + \hat{\chi}}.
$$

Therefore, we can rewrite $T(U)'(z)$ as

$$T(U)'(z) = \frac{U'(\tau^{-1}(z))}{\chi(P'(0) - P'(z))}
= \frac{U'(\tau^{-1}(z))(1 + \hat{\chi}P(z) - (1 + \hat{\chi})U(\tau^{-1}(z))}{\chi(P'(0) - P'(z))}
= \frac{T(U)(z)(1 + \hat{\chi}P(z) - (1 + \hat{\chi})T(U)(z))}{\chi(P'(0) - P'(z)).}$$

Equation (5.12) follows.

Step 2. Next we prove

$$\lim_{z \to 0^-} T(U)'(z) = \frac{P'(0)}{1 + \hat{\chi}} \frac{1 + \hat{\chi}P(0)}{1 + \hat{\chi}U(0^+)}. \quad (5.13)$$

Recall that

$$T(U)(z) = U(\tau^{-1}(z)) = \frac{1}{(1 + \hat{\chi}) \int_{-\infty}^t \, dl}
\times \frac{e^{\int_{-\infty}^t 1 + \hat{\chi}P(t(s))ds}}{(1 + \hat{\chi}) \int_{-\infty}^t \, dl}
= \frac{e^{\int_{-\infty}^t 1 + \hat{\chi}P(t(s))ds}}{(1 + \hat{\chi}) \int_{-\infty}^t \, dl}.$$

We have shown in Step 1 that for any $z < 0$

$$T(U)'(z) = \frac{T(U)(z)(1 + \hat{\chi}P(z) - (1 + \hat{\chi})T(U)(z))}{\chi(P'(0) - P'(z))}, \quad (5.13)$$

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and by Lemma 5.2 we have
\[
\lim_{z \to 0^-} T(U)(z) = \frac{1 + \hat{\chi} P(0)}{1 + \hat{\chi}}.
\]

Moreover,
\[
\frac{1 + \hat{\chi} P(z) - (1 + \hat{\chi}) T(U)(z)}{\chi(P'(0) - P'(z))} = \frac{(1 + \hat{\chi} P(z)) \int_0^{\tau^{-1}(z)} e^{\int_0^l 1 + \hat{\chi} P(\tau(s)) ds} dl - \int_0^{\tau^{-1}(z)} 1 + \hat{\chi} P(\tau(s)) ds}{\chi(P'(0) - P'(z)) \int_0^{\tau^{-1}(z)} e^{\int_0^l 1 + \hat{\chi} P(\tau(s)) ds}} =: \frac{N(z)}{D(z)},
\]

and
\[
\frac{N'(z)}{D'(z)} = \frac{\hat{\chi} P'(z) \int_0^{\tau^{-1}(z)} e^{\int_0^l 1 + \hat{\chi} P(\tau(s)) ds} dl}{\chi(P'(0) - P'(z)) \int_0^{\tau^{-1}(z)} e^{\int_0^l 1 + \hat{\chi} P(\tau(s)) ds} dl - \chi(P'(0) - P'(z)) \int_0^{\tau^{-1}(z)} 1 + \hat{\chi} P(\tau(s)) ds}
\]
\[
= \frac{\hat{\chi} (U(z) - P(z)) + (1 + \hat{\chi}) T(U)(z)}{z \to -0} \to \hat{\chi} U(0^-) + 1.
\]

Therefore, by using L'Hôpital's rule, \( T(U)'(z) \) admits a limit when \( z \to 0^- \) and
\[
\lim_{z \to -0^+} T(U)'(z) = \frac{P'(0)}{1 + \hat{\chi}} + \hat{\chi} U(0^-).
\]

\[\square\]

**Lemma 5.5 (Compactness of \( T \)).** Let Assumption 3 hold. The metric space \( A \) equipped with the distance induced by the \( \| \cdot \|_\eta \) norm (defined in (5.11)) is a complete metric space on which the map \( T : A \to A \) is compact.

**Proof.** Let us first briefly recall that the space \( A \) is complete. Let \( B_\eta \) be the set of all continuous functions defined on \(( -\infty, 0 )\) with finite \( \| \cdot \|_\eta \) norm:
\[
B_\eta := \{ u \in C^0(( -\infty, 0 )) \mid \| u \|_\eta < +\infty \}.
\]

It is classical that \( B_\eta \) equipped with the norm \( \| \cdot \|_\eta \) is a Banach space. Therefore, in order to prove the completeness of \( A \), it suffices to show that \( A \) is closed in \( B_\eta \). Let \( U_n \in A \), \( U \in B_\eta \) be such that \( \lim \| U_n - U \|_\eta = 0 \). Then \( U_n \) converges to \( U \) locally uniformly on \(( -\infty, 0 )\), and in particular we have
\[
U(z) \in \left[ \frac{2}{2 + \hat{\chi}}, 1 \right] \text{ for all } z \leq 0,
\]
\( U \) is non-increasing.

Therefore \( u \in A \) and the completeness is proved.

Let us show that \( T \) is a compact map of the metric space \( A \). We have shown in Lemma 5.2 that \( T \) is continuous on \( A \) and leaves \( A \) stable. Let \( U_n \in A \), then combining Equation (5.12) and the local uniform lower bound of \( P'(z) - P'(0) \) from Lemma 5.1, the family \( (U_n)'(\cdot|_{-k, -1/k}) \) is uniformly Lipschitz continuous on \([ -k, -1/k ]\) for each \( k \in \mathbb{N} \). Therefore the Ascoli-Arzela applies and the set \( \{ T(U_n)(\cdot|_{-k, -1/k}) \}_{n \geq 0} \) is relatively compact for the uniform topology on \([ -k, -1/k ]\) for each \( k \in \mathbb{N} \). Using a diagonal extraction process, there exists a subsequence \( \varphi(n) \) and a continuous function \( U \) such that \( U_{\varphi(n)} \to U \) uniformly on every compact subset of \(( -\infty, 0 )\). Let us show that \( \| U_{\varphi(n)} - U \|_\eta \to 0 \) as \( n \to +\infty \). Let \( \varepsilon > 0 \) be given, and let \( z_0, z_1 \) be respectively the smallest and largest root of the equation:
\[
\eta z + \frac{1}{2} \ln(-z) = \ln\left( \frac{\varepsilon}{2} \right).
\]

Then, on the one hand, for any \( z \notin [z_0, z_1] \), we have \( \sqrt{-z} e^{\eta z} \leq \frac{\varepsilon}{2} \) and therefore
\[
\sqrt{-z} e^{\eta z} |T(U_{\varphi(n)})(z) - T(U)(z)| \leq \sqrt{-z} e^{\eta z} (|U_{\varphi(n)}(z)| + |U(z)|) \leq \varepsilon.
\]

On the other hand, since \( T(U_{\varphi(n)}) \) converges locally uniformly to \( T(U) \), there is \( n_0 \geq 0 \) such that
\[
\sup_{z \in [z_0, z_1]} \sqrt{-z} e^{\eta z} |T(U_{\varphi(n)})(z) - T(U)(z)| \leq \varepsilon, \text{ for all } n \geq n_0.
\]
We conclude that 
\[ \| T(U_{\psi(n)}) - T(U) \|_\eta \leq \varepsilon, \]
for all \( n \geq n_0 \). The convergence is proved. This ends the proof of Lemma 5.5

We are now in the position to prove Theorem 2.7.

**Proof of Theorem 2.7.** We remark that the set of admissible functions \( \mathcal{A} \) is a nonempty, closed, convex, bounded subset of the Banach space \( B_{\eta} \), and \( T \) is a continuous compact operator on \( \mathcal{A} \) (Lemma 5.5). Therefore, a direct application of the Schauder fixed-point Theorem (see e.g. [35, Theorem 2.A p. 57]) shows that \( T \) admits a fixed point \( U \) in \( \mathcal{A} \):

\[ T(U) = U. \]

Applying Lemma 5.2 and 5.4, \( U \) is strictly decreasing on \( (-\infty, 0) \), \( U((-\infty, 0)) \subset \left[ \frac{2}{c^4}, 1 \right] \), \( U \) is \( C^1 \) on \( (-\infty, 0) \) and

\[
\lim_{z \to 0^-} U(z) = \frac{1 + \hat{\chi}P(0)}{1 + \hat{\chi}} \quad \text{and} \quad \lim_{z \to 0^-} U'(z) = \frac{P'(0)}{1 + \hat{\chi}U(0)}. 
\]

Finally

\[
U'(z) = U(z) \frac{1 + \hat{\chi}P(z) - (1 + \hat{\chi})U(z)}{\chi(P'(0) - P'(z))}, \quad \text{for all } z < 0, \tag{5.14}
\]

therefore

\[
\chi P'(0)U'(z) - \chi P'(z)U'(z) - \chi U(z)P''(z) = U(z)(1 - U(z)), \quad \text{for all } z < 0,
\]

and finally

\[
\chi P'(0)U'(z) - \chi (P'(z)U(z))' = U(z)(1 - U(z)), \quad \text{for all } z < 0.
\]

We now prove that \( U(-\infty) \coloneqq \lim_{z \to -\infty} U(z) = 1 \). Since \( U \) is monotone decreasing on \( (-\infty, 0) \) and is bounded by 1 from above, \( U(-\infty) \) exists and, by a direct application of Lebesgue’s dominated convergence theorem, \( P \) also converges to a limit near \( -\infty \), \( P(-\infty) = U(-\infty) \). Therefore \( U'(z) \to 0 \), \( P'(z) \to 0 \) and \( P''(z) \to 0 \) as \( z \to -\infty \). We conclude that

\[
\lim_{z \to -\infty} U(z)(1 - U(z)) = 0,
\]

which implies that \( U(-\infty) = 1 \).

Let us define \( u(t, x) \coloneqq U(x - ct) \), with \( c \coloneqq -\chi P'(0) \). The characteristics associated with \( u(t, x) \) are

\[
\frac{d}{dt} h(t, x) = -\chi (\rho \ast u)(t, h(t, x)) = \chi (\rho \ast U)(h(t, x) - ct) = -\chi P'(h(t, x) - ct),
\]

and \( u(t, x) \) satisfies for all \( x \) such that \( h(t, x) - ct < 0 \):

\[
\partial_t u(t, h(t, x)) = \partial_t (U(h(t, x) - ct)) = \left( \frac{d}{dt} (h(t, x) - ct) \right) U'(h(t, x) - ct)
\]

\[
= \chi (-P'(h(t, x) - ct) + P'(0))U'(h(t, x) - ct)
\]

\[
= u(t, h(t, x))(1 + \hat{\chi}u(t, h(t, x)))(1 + \hat{\chi})(h(t, x) - ct).
\]

If \( h(t, x) - ct > 0 \) then \( u(t, h(t, x)) = U(h(t, x) - ct) = 0 \) (locally in \( t \)) and therefore

\[
\partial_t u(t, h(t, x)) = 0 = u(t, h(t, x))(1 + \hat{\chi}(\rho \ast u)(t, h(t, x)) - (1 + \hat{\chi})u(t, h(t, x))).
\]

Since \( \{0\} \) is a negligible set for the Lebesgue measure, we conclude that \( u(t, x) \) is a solution integrated along the characteristics to (1.1) and thus \( U \) is a traveling wave profile with speed \( c = -P'(0) > 0 \) as defined in Definition 2.2. Finally

\[
c = -\chi P'(0) = \frac{\chi}{2\sigma} \int_{-\infty}^{0} e^{y}U(y)dy \in \left( \frac{\chi}{\sigma(2 + \hat{\chi})}, \frac{\chi}{2\sigma} \right) = \left( \frac{\sigma \hat{\chi}}{2 + \hat{\chi}}, \frac{\sigma \hat{\chi}}{2} \right).
\]

This finishes the proof of Theorem 2.7
5.2 Non-existence of continuous sharp traveling waves

Remark 5.1. This result tells us if \( U \) is a sharp traveling wave solution to (1.1), then it must be discontinuous. This situation is very different from the porous medium case. However, it does not exclude the existence of positive continuous traveling wave solutions which decay to zero near \(+\infty\). In fact, as we will show in the numerical simulations in the later section, we can observe numerically large speed traveling wave solutions that are smooth and strictly positive.

Proof of Proposition 2.8. We divide the proof in 3 steps.

**Step 1:** We show the estimate (2.9).

Assume by contradiction that there exists \( x \in \mathbb{R} \) such that

\[
-\chi \int_{\mathbb{R}} \rho_x(x - y)U(y)dy = c. \tag{5.15}
\]

We let \( P(x) := (\rho \ast U)(x) = \int_{\mathbb{R}} \rho(x - y)U(y)dy \). Since \( U \in C^0(\mathbb{R}) \), we have that \( P \in C^2(\mathbb{R}) \). Differentiating, we find that

\[
P'(x) = \int_{\mathbb{R}} \rho_x(x - y)U(y)dy = (\rho' \ast U)(x),
\]

\[
\sigma^2 P''(x) = \int_{\mathbb{R}} \rho(x - y)U(y)dy - U(x) = P(x) - U(x).
\]

Letting \( Y(x) := -\chi(\rho_x \ast U)(x) - c = -\chi P'(x) - c \), then \( Y \in C^1(\mathbb{R}) \) and we have

\[
Y'(x) = -\chi P''(x) = \hat{\chi}(U(x) - (\rho \ast U)(x)). \tag{5.16}
\]

Since \( \lim_{x \to +\infty} U(x) = 0 \), we have \( \lim_{x \to +\infty} Y(x) = -c < 0 \). Remark that by our assumption (5.15), \( Y \) has at least one zero and therefore the largest root of \( Y \) is well-defined:

\[
x_* := \inf \{ x | \forall y > x, Y(y) < 0 \}.
\]

We first remark that

\[
\frac{d}{dt}(h(t,x) - ct) = \frac{d}{dt} h(t,x) - c = -\chi(\rho_x \ast u)(t,h(t,x)) - c = Y(h(t,x) - ct),
\]

where we recall that \( u(t,x) := U(x - ct) \) is a solution to (1.1). In particular since \( Y(x_*) = 0 \) by the continuity of \( Y \), we have \( h(t,x_*) - ct = x_* \). Next by using (2.2) we have

\[
\frac{d}{dt} u(t,h(t,x_*)) = u(t,h(t,x_*))(1 + \hat{\chi}(\rho \ast u)(h(t,x_*)) - (1 + \hat{\chi})u(t,h(t,x_*)))
\]

\[
= U(h(t,x_*)) \big( 1 + \hat{\chi}(\rho \ast U)(h(t,x_*)) - (1 + \hat{\chi})U(h(t,x_*)) \big)
\]

\[
= U(x_*) \big( 1 + \hat{\chi}P(x_*) - (1 + \hat{\chi})U(x_*) \big),
\]

and since \( u(t,h(t,x_*)) = U(h(t,x_*)) = U(x_*) \) does not depend on \( t \), this yields

\[
0 = U(x_*) \big( 1 + \hat{\chi}P(x_*) - (1 + \hat{\chi})U(x_*) \big).
\]

We conclude that either \( U(x_*) = 0 \) or \( U(x_*) = \frac{1 + \hat{\chi}P(x_*)}{1 + \hat{\chi}} > 0 \). In the remaining part of this step we will show that these two cases lead to contradiction.

**Case 1:** \( U(x_*) = \frac{1 + \hat{\chi}P(x_*)}{1 + \hat{\chi}} > 0 \). By (5.16) we have:

\[
Y'(x_*) = \hat{\chi}(U(x_*) - P(x_*)) = (1 - P(x_*)) \frac{\hat{\chi}}{1 + \hat{\chi}},
\]

however \( U(x) \in [0,1], U(x) \neq 1 \) and thus \( P(x_*) = (\rho \ast U)(x_*) < 1 \) which shows \( Y'(x_*) > 0 \). Yet by definition of \( x_* \) we have \( Y(x_*) = 0 \) and \( Y(x) < 0 \) for all \( x > x_* \), hence \( Y'(x_*) \leq 0 \), which is a contradiction.
Case 2: \( U(x_*) = 0 \). By (5.16) we have
\[
Y'(x_*) = 0 - \hat{\chi}P(x_*) = -\hat{\chi}(\rho \ast U)(x_*) < 0. 
\]
Hence by the continuity of \( Y \), there exists a \( x_0 < x_* \), such that
\[
Y(x) > 0, \quad \forall x \in [x_0, x_*).
\]
Since for any \( t > 0 \), we have
\[
\frac{d}{dt}(h(t, x_0) - ct) = Y(h(t, x_0) - ct) > 0,
\]
the function \( t \mapsto h(t, x_0) - ct \) is increasing and converges to \( x_* \) as \( t \to +\infty \). In particular as \( t \to +\infty \) we have \( u(t, h(t, x_0)) = U(h(t, x_0) - ct) \to U(x_*) = 0 \). Let \( T > 0 \) be such that
\[
0 < u(t, h(t, x_0)) \leq \frac{1}{T} \text{ for all } t \geq T.
\]
We have
\[
\frac{d}{dt}u(t, h(t, x_0)) = u(t, h(t, x_0))(1 + \hat{\chi}(\rho \ast u)(t, h(t, x_0)) - (1 + \hat{\chi})u(t, h(t, x_0))\big)
\geq \frac{1}{2} u(t, h(t, x_0)),
\]
and then \( u(t, h(t, x_0)) \geq u(T, h(T, x_0)e^{-\frac{t-T}{2}}. \) In particular letting
\[
t^* := T - 2 \ln \left(u(T, h(T, x_0))\right) > T,
\]
we have
\[
\frac{u(t^*, h(t^*, x_0))}{1} > \frac{1}{2(1 + \chi)^2},
\]
which is a contradiction. Since both Case 1 and Case 2 lead to contradiction, we have shown (2.9).

Step 2: Regularity of \( u \).

We have shown in Step 1 that for all \( x \in \mathbb{R} \) the strict inequality:
\[
Y(x) = -\chi P(x) - c < 0
\]
holds. Let \( x \in \mathbb{R} \) and \( t_0 > 0 \). Then, there exists \( y \in \mathbb{R} \) such that \( h(t_0, y) = x \), where \( h \) is the characteristic semiflow defined by (2.1). Since
\[
\frac{d}{dt}(h(t, y) - ct) = -\chi(\rho \ast u)(t, h(t, y)) - c = Y(h(t, y)) \neq 0,
\]
the mapping \( t \mapsto h(t, y) - ct \) has a \( C^1 \) inverse which we denote \( \varphi(z) \), i.e.
\[
\forall z \exists t > 0, z = h(t, y) - ct, \quad h(\varphi(z), y) - c\varphi(z) = z.
\]
Then we have
\[
U(h(t, y) - ct) = u(t, h(t, y)) \quad \Leftrightarrow \quad U(z) = u(\varphi(z), h(\varphi(z), y)),
\]
with \( z = h(t, y) \) in a neighbourhood of \( x \). Since \( \varphi \) is \( C^1 \) and the function \( t \mapsto u(t, h(t, y)) \) is \( C^1 \), we conclude that \( U \) is \( C^1 \) in a neighbourhood of \( x \). The regularity is proved.

Step 3: We show that \( u \) is positive.

Combining Step 1 and 2, we know that \( u \) is a classical solution to the equation:
\[
-cU_x - \chi(\rho \ast U)_x = U(1 - U) \quad (-c - \chi P)U_x = U(1 + \hat{\chi}P - (1 + \hat{\chi})U) \quad U_x = \frac{U}{Y}(1 + \hat{\chi}P - (1 + \hat{\chi})U),
\]
and since \( Y < 0 \), the right-hand side is a locally Lipschitz vector field in the variable \( U \). In particular, the classical Cauchy-Lipschitz Theorem applies and the only solution with \( U(x) = 0 \) for some \( x \in \mathbb{R} \) is \( U \equiv 0 \). Since \( U \) is non-trivial by assumption, \( U \) has to be positive. \( \square \)
6 Well-posedness of the Cauchy problem

In this section we investigate the existence and uniqueness of solutions for the system (2.1)-(2.2). The idea to construct a fixed problem is to consider two variables

\[ w(t, x) = u(t, h(t, x)) \quad \text{and} \quad p(t, x) = (\rho \ast u)(t, x). \]

Before we state the theorem, let us introduce some functional spaces and definitions. We introduce the following weighted \( L^1 \) space for any \( \eta > 0 \), as

\[ L^1_\eta(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} \bigg| \int_{\mathbb{R}} |f(x)|e^{-\eta|x|} dx < \infty \right\}, \]

endowed with the norm \( \|f\|_{L^1_\eta(\mathbb{R})} := \frac{1}{\eta} \int_{\mathbb{R}} |f(y)|e^{-\eta|y|}dy \). Then for any \( \eta > 0 \) the space \( L^1_\eta(\mathbb{R}) \) is a Banach space and for any \( 0 < \eta < \eta' < +\infty \) we have

\[ L^\infty(\mathbb{R}) \subset L^1_\eta(\mathbb{R}) \subset L^1_{\eta'}(\mathbb{R}) \subset L^1_{\infty}(\mathbb{R}). \]

We will say that a measurable set \( U \subset \mathbb{R} \) is conull if \( |\mathbb{R}\setminus U| = 0 \), where \( |A| \) is the Lebesgue measure of the set \( A \). In what follows we need to work in the space of regular bounded functions on a measurable set \( U \subset \mathbb{R} \). Let us recall that the space

\[ \mathcal{L}^\infty(U) := \left\{ f : U \to \mathbb{R} \bigg| \sup_{x \in U} |f(x)| < +\infty \right\} \]

endowed with the norm \( \| f \|_{\mathcal{L}^\infty(U)} := \sup_{x \in U} |f(x)| \), is a Banach space. If \( U \) is conull then \( \mathcal{L}^\infty(U) \) is continuously embedded in \( L^\infty(\mathbb{R}) \) since

\[ \| f \|_{L^\infty(\mathbb{R})} \leq \| f \|_{\mathcal{L}^\infty(U)}. \]

Finally we introduce the fixed point problem which is the key element of our proof of Theorem 2.1. Let \( \tau > 0 \) and \( U \subset \mathbb{R} \) be a conull set, we introduce the function spaces:

\[ X^\tau_U := C^0([0, \tau], \mathcal{L}^\infty(U)), \quad \hat{X}^\tau_U := C^\tau([0, \tau], \mathcal{L}^\infty(U)), \quad Y^\tau := C^0([0, \tau], W^{1,\infty}(\mathbb{R})) \]

\[ \hat{Y}^\tau := \{ p \in Y^\tau \mid p(t, \cdot) \in W^{2,\infty}(\mathbb{R}) \text{ for all } t \in [0, \tau] \text{ and } \sup_{t \in [0, \tau]} \| p_{xx}(t, \cdot) \|_{L^\infty(\mathbb{R})} < +\infty \} \quad (6.1) \]

\[ Z^\tau_U := X^\tau_U \times \hat{Y}^\tau, \quad \hat{Z}^\tau_U := \hat{X}^\tau_U \times \hat{Y}^\tau. \]

Clearly, \( \hat{X}^\tau_U \) is closed in the Banach space \( C^0([0, \tau], \mathcal{L}^\infty(U)) \). \( \hat{Y}^\tau \) is not closed in \( C^0([0, \tau], W^{1,\infty}(\mathbb{R})) \), however for each \( K > 0 \), the set

\[ \hat{Y}^\tau_K := \{ p \in \hat{Y}^\tau \mid \sup_{t \in [0, \tau]} \| p_{xx}(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq K \} \quad (6.2) \]

is closed in \( Y^\tau \). Indeed, let \( p^n(t, x) \to p(t, x) \) be a converging sequence in \( Y^\tau \). Since \( C^0([0, \tau], W^{1,\infty}(\mathbb{R})) \) is a Banach space we have \( p \in C^0([0, \tau], W^{1,\infty}(\mathbb{R})) \). Moreover for each \( t \in [0, \tau] \) there exists a measurable set \( E_t \subset \mathbb{R} \) such that \( \int_{E_t} 1 dx = 0 \), \( p^n_x(t, x) \) and \( p_x(t, x) \) are well-defined for any \( x \in E_t \) and \( \lim_{n \to +\infty} p^n_x(t, x) = p_x(t, x) \) for each \( x \in E_t \). Let \( x, y \in E_t \), we have:

\[ |p_x(t, x) - p_x(t, y)| \leq |p_x(t, x) - p^n_x(t, x)| + |p^n_x(t, x) - p^n_x(t, y)| + |p^n_x(t, y) - p_x(t, y)| \leq |p_x(t, x) - p^n_x(t, x)| + |p^n_x(t, y) - p_x(t, y)|. \]

Taking the limit \( n \to +\infty \), we obtain

\[ |p_x(t, x) - p_x(t, y)| \leq K|x - y| \]

hence \( \| p_{xx} \|_{L^\infty} \leq K \) and \( p \in \hat{Y}^\tau_K \).

Given \( p \in \hat{Y}^\tau \), let \( h \) be the solution of the following equation

\[
\begin{align*}
\frac{\partial}{\partial t} h(t, s; x) &= -\chi p_x(t, h(t, s; x)), \\
h(s, s; x) &= x.
\end{align*}
\]  

The existence of the solution \( h \) is ensured by \( p \in \hat{Y}^\tau \). Moreover,
Proof. \(h\) given (recall that by definition of \(\partial\) we have characteristic flows defined in \([0, \tau]\). In particular the image of \(U\) by \(h(t, s, \cdot)\) is still conull for any \(t, s \in [0, \tau]\).\

We are now in the position to define the mapping \(T_U^t[u_0]\) to which we aim at applying a fixed-point theorem:

\[
T_U^t[u_0](w, p)(t, x) = \left( u_0(x) \exp \left( \int_0^t 1 + \hat{\chi} p(l, h(l, 0; x)) - (1 + \hat{\chi} w(l; x))dl \right) \right)^T \int_\mathbb{R} \rho(r - h(l, 0; z)) u_0(z)e^{\int_{l}^{t} 1 - w(t, z)dl}dz, \tag{6.4}
\]

where

\[
(w, p) \in Z_U := X_U \times Y_U.
\]

Remark 6.1. In formula (6.4), the function \(h\) must be understood as the solution of (6.3) where \(p\) the argument of the function \(T_U^t[u_0](w, p)\).

Remark 6.2. Since we only impose \(u_0\) to be in \(L^\infty\) the time of local existence will depend on each value \(u_0(x)\). That is we are not considering the class of functions \(L^\infty\) for \(w(t, \cdot)\). Instead we consider some \(L^\infty(U)\) for \(w(t, \cdot)\).

Our first result is the well-definition of \(T_U^t[u_0]\). We start with a series technical Lemma.

**Lemma 6.1** (Lipschitz continuity of the characteristic flow). Let \(\tau > 0\), \(K > 0\) and \(p \in \tilde{Y}_K\) be given (recall that by definition of \(\tilde{Y}_K\), \(\sup_{t \in [0, \tau]} \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq K < +\infty\)). Then, the solution \(h(t, s, x)\) to (6.3) satisfies

\[
|h(t, s; x) - h(t, s; y)| \leq e^{K\chi|t-s|}|x - y|. \tag{6.5}
\]

**Proof.** The integrated form of (6.3) is

\[
h(t, s; x) = x + \int_s^t -\chi\rho_x(l, h(l, x; x))dl,
\]

therefore

\[
|h(t, s; x) - h(t, s; y)| \leq |x - y| + \chi \int_s^t |\rho_x(l, h(l, x; x)) - \rho_x(l, h(l, y; y))|dy
\]

\[
\leq |x - y| + \chi \sup_{t \in [0, \tau]} \|\rho_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_s^t |h(l, s; x) - h(l, s; y)|dy
\]

\[
\leq |x - y| + K\chi \int_s^t |h(l, s; x) - h(l, s; y)|dy,
\]

since \(p \in \tilde{Y}_K\). Grönwall’s inequality [7, Lemma 4.2.1] implies:

\[
|h(t, s; x) - h(t, s; y)| \leq e^{K\chi|t-s|}|x - y|.
\]

Lemma 6.1 is proved. \(\square\)

**Lemma 6.2.** Let \(\tilde{p}, p \in \tilde{Y}_K\) (where \(\tilde{Y}_K\) is defined as in (6.2)) and \(\tilde{h}, h\) be the corresponding characteristic flows defined in (6.3) with \(p\) and \(\tilde{p}\) respectively. Then for any \(\tau > 0\) and \(t, s \in [0, \tau]\) we have

\[
\|\tilde{h}(t, s; \cdot) - h(t, s; \cdot)\|_{L^\infty(\mathbb{R})} \leq |t - s|\chi \sup_{t \in [0, \tau]} \|\tilde{p}_x(t, \cdot) - p_x(t, \cdot)\|_{L^\infty(\mathbb{R})} e^{K\chi|t-s|}
\]

**Proof.** Without loss of generality we suppose \(t \geq s\), then

\[
\partial_s(\tilde{h}(t, s; x) - h(t, s; x)) = -\chi\tilde{p}_x(t, \tilde{h}(t, s; x)) + \chi\rho_x(t, h(t, s; x))
\]

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By the density of compactly supported smooth function in $L^\infty_0$, we have

$$
h(t,s) - h(t,s) \|_{L^\infty_0(\mathbb{R})} \leq |t-s| \chi \sup_{t \in [s,t]} \| p_x(t, \cdot) - p_x(t, \cdot) \|_{L^\infty_0(\mathbb{R})}
$$

$$
= - \chi \tilde{p}_x(t, h(t,s;x)) + \chi p_x(t, h(t,s;x)) - \chi p_x(t, h(t,s;x)) + \chi p_x(t, h(t,s;x)).
$$

Therefore, we have

$$
\| \tilde{h}(t,s) - h(t,s) \|_{L^\infty_0(\mathbb{R})} \leq |t-s| \chi \sup_{t \in [s,t]} \| p_x(t, \cdot) - p_x(t, \cdot) \|_{L^\infty_0(\mathbb{R})}
$$

$$
+ \chi \sup_{t \in [0,\tau]} \| p_{xx}(t, \cdot) \|_{L^\infty(\mathbb{R})} \int_s^t \| \tilde{h}(s, \cdot) - h(s, \cdot) \|_{L^\infty(\mathbb{R})} ds.
$$

The result follows from Grönwall’s inequality and the definition of $\tilde{Y}_r$.

**Lemma 6.3 (Continuity properties).** Let $(w,p) \in \tilde{Z}_\mathcal{U}^r$. Then, the function $u(t,x) := w(t,h(0,t,x))$, defined for each $t \in [0,\tau]$ and a.e. $x \in \mathbb{R}$, is a continuous function of time for the $L^\infty_0(\mathbb{R})$ topology (i.e., the map $t \mapsto u(t, \cdot)$ is continuous in $L^\infty_0(\mathbb{R})$). The maps $t \mapsto (\rho * u)(t, \cdot)$ and $t \mapsto (\rho * u)(t, \cdot)$ are continuous for the $C^0_0(\mathbb{R})$ topology and moreover $(\rho * u)(t, \cdot) \in W^{2,\infty}(\mathbb{R})$ for all $t \in [0,\tau]$.

**Proof.** Let $(w,p) \in \tilde{Z}_\mathcal{U}^r$ be given. We first remark that, since $p_x$ is Lipschitz continuous, the function $h(t,s;\cdot)$ is locally Lipschitz continuous for every $t,s \in [0,\tau]$ and therefore $h(t,0;\mathcal{U})$ is conull. In particular, $u(t,x)$ is well-defined for every $x \in h(t,0;\mathcal{U})$, therefore almost everywhere, for each $t \in [0,\tau]$.

We divide the rest of the proof in two steps.

**Step 1.** We show the continuity of $t \mapsto u(t, \cdot)$.

Let $t \in [0,\tau]$ and $\varepsilon > 0$ be given. For $s \in [0,\tau]$, we have:

$$
\| u(t, \cdot) - u(s, \cdot) \|_{L^\infty_0(\mathcal{U})} = \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - w(s,h(0,s;x))| e^{-\eta |x|} dx
$$

$$
\leq \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta |x|} dx
$$

$$
+ \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,s;x)) - w(s,h(0,s;x))| e^{-\eta |x|} dx.
$$

By the continuity of $t \mapsto w(t, \cdot)$ in $L^\infty(\mathcal{U})$, there is $\delta_0 > 0$ such that if $|t-s| \leq \delta_0$, then $\| w(t, \cdot) - w(s, \cdot) \|_{L^\infty(\mathcal{U})} \leq \frac{\varepsilon}{2}$. Therefore if $|t-s| \leq \delta_0$,

$$
\| u(t, \cdot) - u(s, \cdot) \|_{L^\infty_0(\mathcal{U})} \leq \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta |x|} dx + \| w(t, \cdot) - w(s, \cdot) \|_{L^\infty(\mathcal{U})}
$$

$$
\leq \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta |x|} dx + \frac{\varepsilon}{2}.
$$

Next we select $R > 0$ sufficiently large, so that

$$
\min(h(s,0;R),-h(s,0;-R)) \geq \frac{-1}{\eta} \ln \left( \frac{\varepsilon}{18 \sup_{t \in [0,\tau]} \| w \|_{L^\infty(\mathcal{U})}^2} \right)
$$

for all $s \in [t - \delta_0, t + \delta_0]$.

By the density of compactly supported smooth function in $L^1(-R,R)$, there is $\varphi \in C^0_0([-R,R])$ such that

$$
\| w - \varphi \|_{L^1(-R,R)} \leq \frac{\varepsilon}{18 \eta} e^{-K \chi_{\delta_0}(t+\delta_0)}.
$$

Then, we have:

$$
\| u(t, \cdot) - u(s, \cdot) \|_{L^\infty_0(\mathcal{U})} \leq \frac{\varepsilon}{2} + \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta |x|} dx
$$

$$
\leq \frac{\varepsilon}{2} + \frac{\eta}{2} \int_{\mathbb{R}} |w(t,h(0,t;x)) - \varphi(h(0,t;x))| e^{-\eta |x|} dx
$$

$$
+ \frac{\eta}{2} \int_{\mathbb{R}} |\varphi(h(0,t;x)) - \varphi(h(0,s;x))| e^{-\eta |x|} dx
$$

(6.6)

(6.7)

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Next we estimate (6.7) and (6.8) (remark that (6.6) is a particular case of (6.8), for \( s = t \), starting with (6.8). We have

\[
\frac{\eta}{2} \int_{\mathbb{R}} |\varphi(h(0, s; x)) - w(t, h(0, s; x))| e^{-\eta |x|} \, dx = \frac{\eta}{2} \int_{-\infty}^{h(s,0; -R)} |w(t, h(0, s; x))| e^{-\eta |x|} \, dx + \frac{\eta}{2} \int_{h(s,0; -R)}^{h(s,0; R)} |w(t, h(0, s; x)) - \varphi(h(0, s; x))| e^{-\eta |x|} \, dx + \frac{\eta}{2} \int_{h(s,0; R)}^{+\infty} |w(t, h(0, s; x))| e^{-\eta |x|} \, dx,
\]

then:

\[
\frac{\eta}{2} \int_{h(s,0; -R)}^{+\infty} |w(t, h(0, s; x))| e^{-\eta |x|} \, dx \leq \sup_{t \in [0, \tau]} \|w\|_{L^\infty} \frac{\int_{h(s,0; -R)}^{+\infty} e^{-\eta x} \, dx}{\eta} \leq \frac{\varepsilon}{36}.
\]

Similarly, we have

\[
\frac{\eta}{2} \int_{-\infty}^{h(s,0; -R)} |w(t, h(0, s; x))| e^{-\eta |x|} \, dx \leq \frac{\varepsilon}{36}.
\]

Moreover, changing the variable in the integral, we have

\[
\frac{\eta}{2} \int_{h(s,0; -R)}^{h(s,0; R)} |w(t, h(0, s; x)) - \varphi(h(0, s; x))| e^{-\eta |x|} \, dx \leq \frac{\eta}{2} \int_{-R}^{R} |w(t, y) - \varphi(y)| e^{-\eta |y|} \, dy \leq \frac{\eta}{2} \|\phi\|_{L^1(-R, R)} e^{-\eta K h(s,0; y)} |h_x(s,0; y)| \, dy \leq \frac{\eta}{2} e^{K y} \|w - \varphi\|_{L^1(-R, R)} \leq \frac{\eta}{2} e^{K y} \frac{\varepsilon}{18\eta} \leq \frac{\varepsilon}{36},
\]

where we recall that \(|h_x| \leq e^{K x} |t - s| \) by (6.5) and \( s \leq t + \delta_0 \). We have shown that

\[
\frac{\eta}{2} \int_{\mathbb{R}} |\varphi(h(0, s; x)) - w(t, h(0, s; x))| e^{-\eta |x|} \, dx \leq \frac{\varepsilon}{12},
\]

for each \( s \in (t - \delta_0, t + \delta_0) \), which is our desired estimate for (6.8) (and therefore for (6.6)).

Next we estimate (6.7). Let

\[
R' := \sup_{s \in (t - \delta_0, t + \delta_0)} \max \{h(s,0; R), -h(s,0; -R)\},
\]

which is well-defined by the continuity of \( s \mapsto h(s,0; \pm R) \) on \([t - \delta_0, t + \delta_0]\). Then the functions \( x \mapsto \varphi(h(0, s; x)) \) have their support in \((-R', R')\) for any \( s \in (t - \delta_0, t + \delta_0)\). In particular,

\[
\frac{\eta}{2} \int_{\mathbb{R}} |\varphi(h(0, t; x)) - \varphi(h(0,0; x))| e^{-\eta |x|} \, dx \leq \frac{\eta}{2} \|\varphi\|_{C^0([-R', R'])} \int_{-R'}^{R'} |h(0, t; x) - h(0, s; x)| e^{-\eta |x|} \, dx \leq \|\varphi\|_{C^0([-R', R'])} \sup_{x \in [-R', R']} |h(0, t; x) - h(0, s; x)|.
\]

Since \((s, x) \mapsto h(s, 0; x)\) is continuous on the compact set \([t - \delta_0, t + \delta_0] \times [-R', R']\), it is uniformly continuous on this set and there exists \( \delta_1 > 0 \) such that

\[
\sup_{x \in [-R', R']} |h(t, 0; x) - h(s, 0; x)| \leq \frac{\varepsilon}{6\|\varphi\|_{C^0([-R', R'])}}
\]

whenever \(|t - s| \leq \delta_1\). This finishes our estimate of (6.7).

Summarizing, we have found \( \delta_1 > 0 \) such that for all \( s \in [t - \delta_1, t + \delta_1] \), the inequality

\[
\|u(t, \cdot) - u(s, \cdot)\|_{L^1(R)} \leq \varepsilon
\]

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holds. This finishes the proof of the continuity of $u(t, \cdot)$ in $L^1_0(\mathbb{R})$.

**Step 2.** Define $p(t, x) := (\rho \ast u)(t, x) = \int_\mathbb{R} \rho(x-y)u(t,y)dy$ in the scope of this Step. We first show that for any $t \in [0, T]$ we have $p(t, \cdot) \in W^{2, \infty}(\mathbb{R})$. Indeed, since $\rho \in W^{1, \infty}(\mathbb{R})$ it is classical that $p_x(t, x)$ exists for each $t \in [0, T]$ and $x \in \mathbb{R}$ and

$$p_x(t, x) = \int_\mathbb{R} \rho_x(x-y)u(t,y)dy.$$

Next we remark that for $x \leq y$ we have

$$|p_x(t, x) - p_x(t, y)| = \left| \int_\mathbb{R} (\rho_x(x-z) - \rho_x(y-z))u(t,z)dz \right|$$

$$\leq \int_\mathbb{R} |\rho_x(x-z) - \rho_x(y-z)|d\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}dz$$

$$\leq \int_\mathbb{R} |\rho_x(z) - \rho_x(y-x+z)|dzd\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}dz$$

$$= \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \times \frac{1}{2\sigma^2} \int_{-\infty}^{x-y} -e^{z/\sigma} + e^{(y-x-z)/\sigma}dz$$

$$+ \int_{x-y}^{\infty} e^{z/\sigma} dz + \int_{0}^{\infty} e^{-z/\sigma} dz - e^{(x-y-z)/\sigma}dz$$

$$= \frac{\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}}{2} \times 4 \left( 1 - e^{-|x-y|/\sigma} \right) \leq \frac{2}{\sigma^2} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}|x-y|.$$

We deduce that

$$|p_x(t, x) - p_x(t, y)| \leq \frac{2}{\sigma^2} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})}|x-y|, \text{ for all } t \in [0, T].$$

In particular $p_x(t, \cdot)$ is globally Lipschitz continuous and thus $p(t, \cdot) \in W^{2, \infty}(\mathbb{R})$.

Next we prove that $p_x(t, x) = (\rho_x \ast u)(t, x) \in C^0([0, T] \times \mathbb{R})$. Let $\varepsilon > 0$ and $R := \ln \left( \frac{6\|u\|_{L^\infty([0, T] \times \mathbb{R})}}{\varepsilon} \right)$, then we have $\|p_x\|_{L^1(\mathbb{R} \setminus (-R, R))} = \varepsilon / (6\|u\|_{L^\infty([0, T] \times \mathbb{R})})$. Let $0 \leq t < s$, we have

$$|p_x(t, x) - p_x(s, y)| \leq |p_x(t, x) - p_x(t, y)| + |p_x(t, y) - p_x(s, y)|$$

$$\leq \frac{2}{\sigma^2} \|u\|_{L^\infty([0, T] \times \mathbb{R})}|x-y| + \int_{(-R, R)} |p_x(y-z)| |u(t,z) - u(s,z)|dz$$

$$+ \int_{\mathbb{R} \setminus (-R, R)} |p_x(y-z)u(t,z) - p_x(y-z)u(s,z)|dz$$

$$\leq \frac{2}{\sigma^2} \|u\|_{L^\infty([0, T] \times \mathbb{R})}|x-y| + \|\rho\|_{L^\infty} \|u(t, \cdot) - u(s, \cdot)\|_{L^1((-R, R))}$$

$$+ \|p_x\|_{L^1(\mathbb{R} \setminus (-R, R))} \times 2\|u\|_{L^\infty([0, T] \times \mathbb{R})}$$

$$\leq \frac{2}{\sigma^2} \|u\|_{L^\infty([0, T] \times \mathbb{R})}|x-y| + \|\rho\|_{L^\infty} \|u(t, \cdot) - u(s, \cdot)\|_{L^1((-R, R))} + \frac{\varepsilon}{3}.$$  

Hence, choosing $|x-y| \leq \frac{\alpha \varepsilon}{6\|u\|_{L^\infty([0, T] \times \mathbb{R})}}$ and $|t-s|$ sufficiently small so that $\|u(t, \cdot) - u(s, \cdot)\|_{L^1((-R, R))} \leq \frac{\varepsilon}{3\|\rho\|_{L^\infty}}$ we have

$$|p_x(t, x) - p_x(s, y)| \leq \varepsilon.$$

Hence $p_x$ is continuous. The continuity of $t \mapsto p(t, \cdot)$ in $L^\infty(\mathbb{R})$ can be shown similarly. \hfill $\square$

**Theorem 6.4 (Local existence and uniqueness of solutions).** Let $U$ be connul and $u_0 \in L^\infty(U)$ be given. There exists $\tau > 0$ such that $T_\tau[u_0]$ has a unique fixed point in $\mathcal{Z}^\tau$. Moreover $\tau$ can be chosen as a continuous function $\tau(\|u_0\|_{L^\infty(U)})$ of $\|u_0\|_{L^\infty(U)}$ and the mapping $u_0 \in C^0(\mathbb{R}) \mapsto \tau^\tau(w(t, x), p(t, x)) \in \mathcal{Z}^\tau$ is continuous in a neighborhood of $u_0$. 

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Proof. We divide the proof in three steps.

**Step 1. Stability of $\tilde{Z}_U^\tau$ by $T_U^\tau[u_0]$.** We show that $T_U^\tau[u_0](\tilde{Z}_U^\tau) \subset \tilde{Z}_U^\tau$. Define $(w^1, p^1) := T_U^\tau[u_0](w, p)$. We first prove $w^1 \in X^\tau = C([0, \tau], L^\infty(\mathcal{U}))$. By definition we have

$$w^1(t, \cdot) - w^1(s, \cdot) = u_0(\cdot) \exp \left( \int_0^t 1 + \tilde{\chi} p(l, h(l, 0; \cdot)) - (1 + \tilde{\chi}) w(l, \cdot) dt \right) - u_0(\cdot) \exp \left( \int_0^s 1 + \tilde{\chi} p(l, h(l, 0; \cdot)) - (1 + \tilde{\chi}) w(l, \cdot) dt \right).$$

Let us denote $\Theta[u] := |u| e^{\|u\|}$, $u \in \mathbb{R}$ and recall the inequality $e^u - 1 - u = \Theta[u]$ for all $u \in \mathbb{R}$. We have

$$\left\| u_0(\cdot) \exp \left( \int_0^t 1 + \tilde{\chi} p(l, h(l, 0; \cdot)) - (1 + \tilde{\chi}) w(l, \cdot) dt \right) - u_0(\cdot) \exp \left( \int_0^s 1 + \tilde{\chi} p(l, h(l, 0; \cdot)) - (1 + \tilde{\chi}) w(l, \cdot) dt \right) \right\|_{L^\infty(\mathcal{U})}$$

$$= \| u_0 \|_{L^\infty(\mathcal{U})} e^{s(1 + \tilde{\chi} \|p\|_{L^\infty([0, \tau] \times \mathbb{R})})} \exp \left( \int_s^t 1 + \tilde{\chi} p(l, h(l, 0; \cdot)) - (1 + \tilde{\chi}) w(l, \cdot) dt \right) - 1 \right\|_{L^\infty(\mathcal{U})}$$

$$\leq \| u_0 \|_{L^\infty(\mathcal{U})} e^{s(1 + \tilde{\chi} \|p\|_{L^\infty([0, \tau] \times \mathbb{R})})} \Theta \left( (t - s)(1 + \tilde{\chi} \|p\|_{L^\infty([0, \tau] \times \mathbb{R})}) + (1 + \tilde{\chi}) \sup_{t \in [0, \tau]} \| w(t, \cdot) \|_{L^\infty(\mathcal{U})} \right).$$

This implies

$$\| w^1(t, \cdot) - w^1(s, \cdot) \|_{L^\infty(\mathcal{U})} \leq \| u_0 \|_{L^\infty(\mathcal{U})} e^{s(1 + \tilde{\chi} \|p\|_{L^\infty([0, \tau] \times \mathbb{R})})} \Theta \left( (t - s)(1 + \tilde{\chi} \|p\|_{L^\infty([0, \tau] \times \mathbb{R})}) + (1 + \tilde{\chi}) \sup_{t \in [0, \tau]} \| w(t, \cdot) \|_{L^\infty(\mathcal{U})} \right).$$

(6.10)

Since $\chi[u] \to 0$ as $u \to 0$, the continuity of $w^1$ is proved.

Next we prove $p^1 \in \tilde{Y}^\tau$. Recall that, by definition of $\tilde{Y}^\tau$ (see (6.1)), the second derivative of $p$ in space is uniformly bounded: $\sup_{t \in [0, \tau]} \| p_{xx}(t, \cdot) \|_{L^\infty(\mathbb{R})} := K < +\infty$. For any $t, s \in [0, \tau]$ and $x \in \mathbb{R}$, we have

$$|p^1(t, x) - p^1(s, x)| = \left| \int_{\mathbb{R}} \left( \rho(x - h(t, 0; z)) e^{\int_0^t 1 - w(t, z) dt} - \rho(x - h(s, 0; z)) e^{\int_0^s 1 - w(t, z) dt} \right) u_0(z) \right|$$

$$\leq \| u_0 \|_{L^\infty(\mathbb{R})} \left( \left\| e^{\int_0^t 1 - w(t, \cdot) dt} - e^{\int_0^s 1 - w(t, \cdot) dt} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho(x - h(t, 0; z)) - \rho(x - h(s, 0; z))| dz \right).$$

(6.11)

Since $p \in \tilde{Y}^\tau$ we have $\| p_{xx} \|_{L^\infty([0, \tau] \times \mathbb{R})} \leq K$ and thus, recalling the Lipschitz property of $h$ (6.5),

$$\left\| e^{\int_0^t 1 - w(t, \cdot) dt} - e^{\int_0^s 1 - w(t, \cdot) dt} \right\|_{L^\infty(\mathbb{R})} \leq |t - s| (e^x + e^s) \left( 1 + \sup_{t \in [0, \tau]} \| w(t, \cdot) \|_{L^\infty(\mathcal{U})} \right)$$

$$\leq |t - s| 2e^x \left( 1 + \|x\|_{X^\tau_U} \right),$$

(6.12)

where we have used the classical inequality

$$|e^x - e^y| \leq (e^x + e^y) |x - y|$$

for all $x, y \in \mathbb{R}$.

(6.13)

There remains to estimate the second term in the right-hand side of (6.11). Using (6.13) we have

$$\int_{\mathbb{R}} |\rho(x - h(t, 0; z)) - \rho(x - h(s, 0; z))| dz \leq \int_{\mathbb{R}} |\rho(x - y)\partial_x h(t, 0; y) dy|$$

$$\leq e^{Kx^t}.$$
\[ = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|x-h(t,0,z)|}{\sigma}} - e^{-\frac{|x-h(s,0,z)|}{\sigma}} \, dz \]

Moreover, since
\[ \leq \frac{1}{2\sigma^2} \int_{\mathbb{R}} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) \sigma^{-1} |h(t,0;z) - h(s,0;z)| \, dz \]

\[ \leq \frac{1}{2\sigma^2} \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})} \left( \int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} h_x(0,t;y) \, dy + \int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} h_x(0,s;y) \, dy \right) \]

\[ \leq \sigma^{-1} \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})} (e^{K\tau} + e^{K\sigma^2}) \]

\[ \leq 2\sigma^{-1} e^{K\tau} \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})}. \]  

Moreover, since
\[ h(t,0;x) - h(s,0;x) = - \int_s^t \chi p_x(l,h(l,0;x)) \, dl, \]  

we have \( \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})} \leq |t-s| \sup_{t \in [0,\tau]} ||p_x(t,:)||_{L^\infty(\mathbb{R})}. \) Combining (6.11) and (6.14) we have
\[ \| p_1(t,:) - p_1(s,:) \|_{L^\infty(\mathbb{R})} \leq |t-s| \times 2e^{(K\tau+1)\sigma} \| u_0 \|_{L^\infty(U)} \left( 1 + \| u \|_{X^\sigma} + \sigma^{-1} \chi \| p \|_{Y^\tau}. \right) \]  

This proves \( p_1 \in C([0,\tau], L^\infty(\mathbb{R})). \)

Similarly, we compute for any \( t, s \in [0,\tau] \) and \( x \in \mathbb{R} : \)
\[ \| p_2(t,x) - p_2(s,x) \| \leq |t-s| \times 2\sigma^{-1} e^{(K\tau+1)\sigma} \| u \|_{L^\infty(U)} \left( 1 + \| u \|_{X^\sigma} \right) \]  

\[ + \| u \|_{L^\infty(U)} e^{\sigma} \int_{\mathbb{R}} |p_x(x - h(t,0;z)) - p_x(x - h(s,0;z))| \, dz. \]  

In order to estimate the last term in (6.17), suppose first that \( h(0,0;x) \leq h(0,s;x). \) We have
\[ \int_{\mathbb{R}} |p_x(x - h(t,0;z)) - p_x(x - h(s,0;z))| \, dz \]

\[ \leq \frac{1}{2\sigma^2} \int_{-\infty}^{h(0,t;x)} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) |h(t,0;z) - h(s,0;z)| \, dz \]

\[ + \frac{1}{2\sigma^2} \int_{h(0,s;x)}^{\infty} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) |h(t,0;z) - h(s,0;z)| \, dz \]

\[ + \frac{1}{2\sigma^2} \int_{h(0,t;x)}^{h(0,s;x)} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) \, dz. \]  

Using (6.13) and (6.12) we have
\[ \int_{\mathbb{R}} |p_x(x - h(t,0;z)) - p_x(x - h(s,0;z))| \, dz \]

\[ \leq \frac{1}{2\sigma^2} \int_{-\infty}^{h(0,t;x)} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) |h(t,0;z) - h(s,0;z)| \, dz \]

\[ + \frac{1}{2\sigma^2} \int_{h(0,s;x)}^{\infty} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) |h(t,0;z) - h(s,0;z)| \, dz \]

\[ + \frac{1}{2\sigma^2} \int_{h(0,t;x)}^{h(0,s;x)} \left( e^{-\frac{|x-h(t,0,z)|}{\sigma}} + e^{-\frac{|x-h(s,0,z)|}{\sigma}} \right) \, dz \]

\[ \leq \frac{1}{2\sigma^2} \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})} \left( \int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} \, dy + e^{-\frac{|x-y|}{\sigma}} \right) \, dz \]

\[ + \frac{1}{2\sigma^2} \int_{h(0,t;x)}^{h(0,s;x)} 2\, dz \]

\[ \leq 2\sigma^{-1} e^{K\tau} \| h(t,0,:) - h(s,0,:) \|_{L^\infty(\mathbb{R})} + \sigma^{-2} \| h(0,0,:) - h(0,s,:) \|_{L^\infty(\mathbb{R})}. \]  

Moreover by (6.15) we have \( \| h(0,0,:) - h(0,s,:) \|_{L^\infty(\mathbb{R})} \leq |t-s| \chi \| p \|_{Y^\tau}. \) Combining (6.17) and (6.18) we have
\[ \| p_2(t,:) - p_2(s,:) \|_{L^\infty(\mathbb{R})} \leq |t-s| \times \| u_0 \|_{L^\infty(U)} \sigma^{-1} (2e^{(K\tau+1)\sigma} \chi \| p \|_{X^\tau} + \chi \| p \|_{Y^\tau} + \sigma^{-1} \| p \|_{Y^\tau}). \]  

(6.19)
This proves $p^r_\tau \in C([0,\tau],L^\infty(\mathbb{R}))$. According to (6.16) and (6.19) we have
\[
\|p^r(t,\cdot) - p^1(s,\cdot)\|_{W^1,\infty(\mathbb{R})} \leq C|t-s| \times \|u_0\|_{L^\infty(U)} e^{(K\chi+1)\tau},
\] (6.20)
where $C$ is a constant depending on $\sigma$, $\chi$, $\|w\|_{X^r}$ and $\|p\|_{Y^r}$. Therefore $p^r \in Y^r$.

There remains to show that sup$_{t \in [0,\tau]} \|p^r_\tau(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty$. Let $t, s \in [0, \tau]$ and $x \in \mathbb{R}$. We have
\[
\|p^r_\tau(t,x) - p^r_\tau(t,y)\| = \left|\int_\mathbb{R} \rho_x(x - h(t,0;z)) - \rho_x(y - h(t,0;z)) u_0(z) e^{\int_0^t -w(t,z)dt} dz\right|
\leq \|u_0\|_{L^\infty(\mathbb{R})} e^{t} \int_\mathbb{R} |\rho_x(x - z) - \rho_x(y - z)| b_x(0,t;z) dz
\leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{L^\infty(\mathbb{R})} |x - y|.
\]
Therefore
\[
\sup_{t \in [0,\tau]} \|p^r_\tau(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{L^\infty(U)} < +\infty.
\] (6.21)
We have shown the stability of $\tilde{Z}^r_\tau$.

**Step 2. Local stability of a vicinity.** We show the stability of the set
\[
\mathcal{B}_r := \{(w, p) \in \tilde{Z}^r_\tau \mid \sup_{t \in [0,\tau]} \|u_0 - w(t,\cdot)\|_{L^\infty(U)} \leq r \text{ and } p \in \hat{Y}_r \text{ and } \|p - (\rho \ast u_0)\|_{Y^r} \leq r\},
\] (6.22)
for any $r > 0$ and $\tau > 0$ sufficiently small, where $K := 4\sigma^{-1} \|u_0\|_{L^\infty(U)}$. Note that $\mathcal{B}_r$ is closed in $\tilde{Z}^r_\tau$ for any $r > 0$.

Let $(w, p) \in \mathcal{B}_r$, and define $\kappa := \|(u_0, \rho \ast u_0)\|_{Z^r} + r$. By definition, we have
\[
\|(w, p)\|_{\hat{Z}^r} \leq \|u_0, \rho \ast u_0\|_{Z^r} + r = \kappa.
\]
On the one hand by (6.10) (with $s = 0$) we find that
\[
\sup_{t \in [0,\tau]} \|w^1(t,\cdot) - u_0(\cdot)\|_{L^\infty(U)} = \sup_{t \in [0,\tau]} \|w^1(t,\cdot) - w^1(0,\cdot)\|_{L^\infty(U)}
\leq \|u_0\|_{L^\infty(U)} \Theta \left[\tau(1 + \hat{\chi})\|p\|_{Y^r} + (1 + \hat{\chi})\|w\|_{X^r}\right]
\leq \kappa \chi \left[\tau(1 + (1 + 2\hat{\chi})\kappa)\right] \xrightarrow{\tau \to 0} 0 < r,
\]
where $\Theta[u] := |u| e^{u}$. On the other hand, by (6.20) (with $s = 0$), for all $t \in [0, \tau]$,
\[
\|p^1(t,x) - (\rho \ast u_0)(x)\|_{Y^r} = \sup_{t \in [0,\tau]} \|p^1(t,\cdot) - p^1(0,\cdot)\|_{W^1,\infty(\mathbb{R})}
\leq C \tau \times \|u_0\|_{L^\infty(U)} e^{(K\chi+1)\tau},
\leq C \tau \kappa e^{(K\chi+1)\tau} \xrightarrow{\tau \to 0} 0 < r.
\]
Finally by (6.21),
\[
\sup_{t \in [0,\tau]} \|p^r_\tau(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{L^\infty(U)} \xrightarrow{\tau \to 0} 2\sigma^{-1} \|u_0\|_{L^\infty(U)} < 4\sigma^{-1} \|u_0\|_{L^\infty(U)} = K.
\]
We conclude that for any $r > 0$ there is $\tau > 0$ sufficiently small so that $T^r_\tau[u_0](\mathcal{B}_r) \subset \mathcal{B}_r$.

**Step 3. $T^r_\tau[u_0]$ is a contraction.** More precisely, we show that $T^r_\tau[u_0]$ is contracting for $\tau$ sufficiently small.

Let $r > 0$ be given and $\tau > 0$ be sufficiently small so that $\mathcal{B}_r$ is left stable by $T^r_\tau[u_0]$, and define $\kappa := \|(u_0, \rho \ast u_0)\|_{Z^r} + r$ as in Step 2. Let $(w, p) \in \mathcal{B}_r$ and $(\hat{w}, \hat{p}) \in \mathcal{B}_r$ be given, we observe that for any $t, s \in [0, \tau]$ and $x \in U$,
\[
|\tilde{w}^1(t,x) - w^1(t,x)| \leq \|u_0\|_{L^\infty([t])} \left| e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \omega(l,x) \, dt} - e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \hat{\omega}(l,x) \, dt} \right| \\
\leq \|u_0\|_{L^\infty([t])} e^{(1+\bar{\chi})\|p\|_{\mathcal{V}}} \left| 1 - e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \hat{\omega}(l,x) - \omega(l,x) \, dt} \right| \\
\leq \kappa e^{(1+\bar{\chi})} \left| 1 - e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \hat{\omega}(l,x) - \omega(l,x) \, dt} \right| \\
\leq \kappa e^{(1+\bar{\chi})} \Theta \left( \tau \left( \bar{\chi} \sup_{t \in [0,\tau]} |\tilde{p}(l,\tilde{h}(l,0;v;x)) - p(l,h(l,0;v;x))| + (1 + \bar{\chi}) \|\tilde{w} - w\|_{X^\tau} \right) \right),
\]
where we have used the inequality \(|e^u - 1| \leq |u|e^{|u|} =: \Theta[u], \forall u \in \mathbb{R}\). Moreover, we have
\[
\sup_{t \in [0,\tau]} |\tilde{p}(l,\tilde{h}(l,0;v;x)) - p(l,h(l,0;v;x))| \\
\leq \sup_{t \in [0,\tau]} ||\tilde{p}(l,\cdot) - p(l,\cdot)||_{L^\infty(\mathbb{R})} + \sup_{t \in [0,\tau]} |p(l,\tilde{h}(l,0;v;x)) - p(l,h(l,0;v;x))| \\
\leq ||\tilde{p} - p||_{Y^\tau} + \sup_{t \in [0,\tau]} ||\tilde{p}_x(l,\cdot)||_{L^\infty(\mathbb{R})} \sup_{t \in [0,\tau]} ||\tilde{h}(l,0;\cdot) - h(l,0;\cdot)||_{L^\infty(\mathbb{R})} \\
\leq ||\tilde{p} - p||_{Y^\tau} + \kappa \sup_{t \in [0,\tau]} ||\tilde{h}(l,0;\cdot) - h(l,0;\cdot)||_{L^\infty(\mathbb{R})}.
\]
According to Lemma 6.2 we have
\[
||\tilde{h}(l,0;\cdot) - h(l,0;\cdot)||_{L^\infty(\mathbb{R})} \leq \tau \chi \sup_{t \in [0,\tau]} ||\tilde{p}_x(l,\cdot) - p_x(l,\cdot)||_{L^\infty(\mathbb{R})} e^{K\chi \tau},
\]
which yields
\[
\sup_{t \in [0,\tau]} |\tilde{p}(l,\tilde{h}(l,0;v;x)) - p(l,h(l,0;v;x))| \leq ||\tilde{p} - p||_{Y^\tau} (1 + \kappa \chi \tau e^{K\chi \tau}).
\]
This implies
\[
||\tilde{w}^1 - w^1||_{X^\tau} \leq \epsilon e^{(1+\kappa \chi)} \Theta \left( \tau \left( \bar{\chi} ||\tilde{p} - p||_{Y^\tau} (1 + \kappa \chi \tau e^{K\chi \tau}) + (1 + \bar{\chi}) \|\tilde{w} - w\|_{X^\tau} \right) \right).
\] (6.23)
On the other hand, we have
\[
|\tilde{p}^1(t,x) - p^1(t,x)| \\
= \left| \int_{\mathbb{R}} \left( \rho(x - \tilde{h}(t,0;v;z)) e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \omega(l,x) \, dt} - \rho(x - h(t,0;v;z)) e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt} \right) u_0(z) \, dz \right| \\
= \left| \int_{\mathbb{R}} \left( \rho(x - \tilde{h}(t,0;v;z)) e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \omega(l,x) \, dt} - \rho(x - h(t,0;v;z)) e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt} \right) u_0(z) \, dz \right| \\
\leq \|u_0\|_{L^\infty(\mathbb{R})} \left| \left( e^{\int_0^t \bar{\gamma}(\rho(l,h(l,t;v;x)) - 1 + \bar{\chi}) \omega(l,x) \, dt} - e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt} \right) \right| u_0(z) \, dz \\
+ \|e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt}\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho(x - \tilde{h}(t,0;v;z)) - \rho(x - h(t,0;v;z))| \, dz.
\]
In order to estimate the term \(\|e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt}\|_{L^\infty(\mathbb{R})}\), we write
\[
\|e^{\int_0^t \gamma(l,h(l,t;v;x)) - \omega(l,x) \, dt}\|_{L^\infty(\mathbb{R})} \leq 2\epsilon \tau \left| \int_0^t \tilde{w}(l,\cdot) - w(l,\cdot) \, dt \right|_{L^\infty(\mathbb{R})} \leq 2\epsilon \tau \|\tilde{w} - w\|_{X^\tau},
\]
where we have used (6.13). Next we notice that \(\tilde{p} \in Y^\tau\) implies \(\tilde{p}_x \|_{L^\infty((0,\tau) \times \mathbb{R})} \leq K\), thus we obtain by a change of variable (recall the Lipschitz continuity of \(\tilde{h}\) by Lemma 6.1)
\[
\int_{\mathbb{R}} |\rho(x - \tilde{h}(t,0;v;z))| \, dz = \int_{\mathbb{R}} |\rho(x - z)\partial_v \tilde{h}(0,t;v;z)| \, dz \leq e^{K\chi \tau}.
\]
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Finally we have
\[
\int \rho(x - \tilde{h}(t, 0; z)) - \rho(x - h(t, 0; z)) \, dz \\
= \frac{1}{2\sigma} \int R \left[ e^{-|x - \tilde{h}(t, 0; z)|^2} - e^{-|x - h(t, 0; z)|^2} \right] \, dz \\
\leq \frac{1}{2\sigma} \int R \left( e^{-|x - \tilde{h}(t, 0; z)|^2} + e^{-|x - h(t, 0; z)|^2} \right) \, dz \\
\leq \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{2\sigma} \int e^{-|x - \tilde{h}(t, 0; z)|^2} + e^{-|x - h(t, 0; z)|^2} \, dz \\
\leq \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} (e^{K\tau t} + e^{K\tau t}) \\
\leq 2e^{K\tau t} \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})}.
\]
Applying Lemma 6.2 yields
\[
\int \rho(x - \tilde{h}(t, 0; z)) - \rho(x - h(t, 0; z)) \, dz \leq 2e^{K\tau} \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} \\
\leq 2\chi \tau e^{2K\tau t} \|\tilde{p} - \rho\|_{\mathcal{Y}^t}
\]
We have shown the following estimate on \(p\):
\[
\sup_{t \in [0, T]} \|p^1(t, \cdot) - p^1(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 2\kappa \tau e^{(K\chi + 1)\tau} \|\tilde{w} - w\|_{\mathcal{X}^t} + 2\kappa \chi \tau e^{2(K\chi + 1)\tau} \|p - \rho\|_{\mathcal{Y}^t}.
\]
Next we estimate the gradient of \(p\). We have:
\[
|\dot{p}_x^1(t, x) - p_x^1(t, x)| \\
= \left| \int R \left( \rho_x(x - \tilde{h}(t, 0; z)) e^{\int_0^t 1-\tilde{w}(t,z) \, dt} - \rho_x(x - h(t, 0; z)) e^{\int_0^t 1-w(t,z) \, dt} \right) u_0(z) \right| \\
\leq \|u_0\|_{L^\infty(\mathbb{R})} \left( \left\| e^{\int_0^t 1-\tilde{w}(t,z) \, dt} - e^{\int_0^t 1-w(t,z) \, dt} \right\|_{L^\infty(\mathbb{R})} \right) \int R |\rho_x(x - \tilde{h}(t, 0; z)) - \rho_x(x - h(t, 0; z))| \, dz \\
+ \left| e^{\int_0^t 1-w(t,z) \, dt} \right|_{L^\infty(\mathbb{R})} \int R |\rho_x(x - \tilde{h}(t, 0; z)) - \rho_x(x - h(t, 0; z))| \, dz \\
\leq 2\sigma^{-1} K \tau e^{(K\chi + 1)\tau} \|\tilde{w} - w\|_{\mathcal{X}^t} + \kappa \tau e^{(K\chi + 1)\tau} \int R |\rho_x(x - \tilde{h}(t, 0; z)) - \rho_x(x - h(t, 0; z))| \, dz.
\]
For the need of this computation, let us introduce the quantities \(h^- := \min(\tilde{h}(0, t; x), h(0, t; x))\) and \(h^+ := \max(\tilde{h}(0, t; x), h(0, t; x))\). We have:
\[
\int R |\rho_x(x - \tilde{h}(t, 0; z)) - \rho_x(x - h(t, 0; z))| \, dz \\
\leq \frac{1}{2\sigma^2} \int_{-\infty}^{h^-} \left( e^{-\frac{|x - \tilde{h}(t, 0; z)|^2}{2\sigma^2}} + e^{-\frac{|x - h(t, 0; z)|^2}{2\sigma^2}} \right) \tilde{h}(t, 0; z) - h(t, 0; z) \, dz \\
+ \frac{1}{2\sigma^2} \int_{h^+}^{\infty} \left( e^{-\frac{|x - \tilde{h}(t, 0; z)|^2}{2\sigma^2}} + e^{-\frac{|x - h(t, 0; z)|^2}{2\sigma^2}} \right) \tilde{h}(t, 0; z) - h(t, 0; z) \, dz \\
+ \frac{1}{2\sigma^2} \int_{h^-}^{h^+} \left( e^{-\frac{|x - \tilde{h}(t, 0; z)|^2}{2\sigma^2}} + e^{-\frac{|x - h(t, 0; z)|^2}{2\sigma^2}} \right) \tilde{h}(t, 0; z) - h(t, 0; z) \, dz \\
\leq \frac{1}{2\sigma^2} \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} \left( e^{-\frac{|x - h(t, 0; z)|^2}{2\sigma^2}} + e^{-\frac{|x - h(t, 0; z)|^2}{2\sigma^2}} \right) \, dz \\
+ \frac{1}{2\sigma^2} \int_{h^-}^{h^+} 2 \, dz \\
\leq 2\sigma^{-1} e^{K\tau t} \|\tilde{h}(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} + \sigma^{-2} \|\tilde{h}(0, t; \cdot) - h(0, t; \cdot)\|_{L^\infty(\mathbb{R})}.
\]
According to Lemma 6.2 we have then

\[
\int_{\mathbb{R}} |\rho_x(x - \bar{h}(t, 0; z)) - \rho_x(x - h(t, 0; z))| dz \\
\leq 2\sigma^{-1} e^{K_\tau} \|h(t, 0; \cdot) - h(t, 0; \cdot)\|_{L^\infty(\mathbb{R})} + \sigma^{-2} \|\bar{h}(0, 0; \cdot) - h(0, 0; \cdot)\|_{L^\infty(\mathbb{R})} \\
\leq (2\tau\sigma^{-1} e^{2K_\tau} + \sigma^{-2} \tau e^{K_\tau}) \|\tilde{p} - p\|_{Y^\tau}.
\]

This implies

\[
\sup_{t \in [0, \tau]} \left\| p^1_x(t, \cdot) - p^1_x(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq 2\sigma^{-1} \kappa \tau e^{(K_\chi + 1)\tau} \left\| \tilde{w} - w \right\|_{X^\tau} \\
+ (2\kappa \chi \sigma^{-1} e^{(2K_\chi + 1)\tau} + \kappa \chi \sigma^{-2} \tau e^{(K_\chi + 1)\tau}) \|\tilde{p} - p\|_{Y^\tau}.
\] (6.25)

Combining (6.23), (6.24) and (6.25), there exists a mapping \( \tau \mapsto L(\tau) \) with \( L(\tau) \to 0 \) as \( \tau \to 0 \) such that

\[
\| T_{U}^{\tau}[u_0](\tilde{w}, \tilde{p}) - T_{U}^{\tau}[u_0](w, p) \|_{Z^\tau} \leq L(\tau) \| (\tilde{w}, \tilde{p}) - (w, p) \|_{Z^\tau}.
\] (6.26)

Thus for \( \tau > 0 \) sufficiently small we have \( L(\tau) < 1 \) in which case \( T_{U}^{\tau}[u_0] \) is a contraction on the complete metric space \( \hat{Z} \). By the Banach contraction principle, there exists then a unique fixed point to \( T_{U}^{\tau}[u_0] \). Moreover \( \tau \) can be chosen as a continuous function of \( \|u_0\|_{C^\infty(\mathbb{R})} \).

Finally, the continuous dependency of \( (w, p) \) with respect to \( u_0 \) is a direct application of the continuous dependency of the fixed point with respect to a parameter [35, Proposition 1.2].

In order to show the semigroup property satisfied by \( (w, p) \) and to make the link with the integrated solutions to (1.1), we need the following technical Lemma.

**Lemma 6.5** (The derivatives of \( p \) and \( h \)). Let \( U \subset \mathbb{R} \) be conull and \( \tau > 0 \) be given. Let \( (w, p) \in \hat{Z} \) be a fixed point of \( T_{U}^{\tau}[u_0] \). Then there exists a conull set \( U' \) such that

(i) for any \( t, s \in [0, \tau] \), the solution \( h(t, s; x) \) to (6.3) is differentiable for each \( x \in h(s, 0; U') \) (therefore for almost every \( x \in \mathbb{R} \)) and we have

\[
h_x(t, s; x) = \exp \left( \tilde{\chi} \int_{s}^{t} w(l, x) - p(l, h(l, s; x)) dl \right) \text{ for a.e. } x \in U.
\] (6.27)

(ii) for every \( t \in [0, \tau] \) and \( x \in \mathbb{R} \) we have

\[
p(t, x) = \int_{\mathbb{R}} \rho(x - y) w(t, h(0, t; y)) dy \text{ and } p_x(t, x) = \int_{\mathbb{R}} \rho_x(x - y) w(t, h(0, t; y)) dy.
\]

(iii) for every \( x \in U' \), the function \( p_x(t, \cdot) \) is differentiable at \( h(t, 0; x) \) and we have

\[
\sigma^2 p_{xx}(t, h(t, 0; x)) = p(t, h(t, 0; x)) - w(t, x).
\]

**Proof.** We divide the proof in three steps.

**Step 1.** We prove item (i).

Let \( x \leq y \) and \( t, s \in [0, \tau] \) be given, we first remark that

\[
p_s(t, h(t, 0; y)) - p_s(t, h(t, 0; x))
\]
Next we remark that, with those functions differentiable at every point.

Applying Lemma A.1, we conclude that there exists a conull set

\[ f(t,0; y) - h(t,0; x) - g(t, x) \]

where

\[ f(t; y) := \left( \int_{-\infty}^{t} + \int_{t}^{+\infty} \right) \left( \frac{\rho_x(h(t,0; y) - h(t,0; z))}{h(t,0; y) - h(t,0; x)} \right) u_0(z) e^\int_{t}^{t} 1-w(t,z) dl \]

and

\[ g(t; x, y) := \frac{1}{2 \sigma^2} \left( \int_{x}^{y} \right) \left( e^\frac{h(t,0; y) - h(t,0; z)}{\sigma} + e^\frac{-h(t,0; z) + h(t,0; z)}{\sigma} \right) u_0(z) e^\int_{t}^{t} 1-w(t,x) dl \]

Next we remark that, with those functions \( f \) and \( g \), we have

\[ h(t,0; y) - h(t,0; x) = y - x - \chi \int_{0}^{t} p_x(l, h(l,0; y)) - p_x(l, h(l,0; x)) dl \]

\[ = y - x - \chi \int_{0}^{t} f(t; x, y)(h(l,0; y) - h(l,0; x)) - g(t; x, y) dl \]

\[ = (y - x) e^{-\chi \int_{t}^{t} f(t; x, y) dl} + \chi \int_{0}^{t} g(\sigma; x, y) e^{-\chi \int_{t}^{t} f(t; x, y) dl} d\sigma \]

For a given \( x \in \mathbb{R} \), we have

\[ f(t; x, y) \xrightarrow{y \to x} \frac{1}{\sigma^2} p(t, h(t,0; x)) \]

uniformly in \( t \), because of Lebesgue’s dominated convergence theorem.

Next we remark that, given \( t \in [0, \tau] \), if \( x \) is a Lebesgue point of the function \( z \mapsto u_0(z) e^\int_{t}^{t} 1-w(t,z) dl \in C^0([0, \tau], L^\infty(\mathcal{U})) \), then \( \frac{2(t,y)}{y-x} \) has a limit as \( y \to x \) and

\[ \lim_{y \to x} \frac{g(t; x, y)}{y-x} = \frac{1}{\sigma^2} u_0(x) e^\int_{t}^{t} 1-w(t,x) dx \]

Applying Lemma A.1, we conclude that there exists a conull set \( \mathcal{U} \subset \mathcal{U} \) on which \( h(t,0; \cdot) \) is differentiable at every point \( x \in \mathcal{U} \) for all \( t > 0 \) and we have

\[ h_x(t,0; x) = e^{-\hat{x} \int_{0}^{t} p(l, h(l,0; x)) dl} + \sigma^{-2} \int_{0}^{t} u_0(x) e^\int_{t}^{t} 1-w(t,x) dl e^{-\hat{x} \int_{t}^{t} p(l, h(l,0; x)) dl} d\sigma \]

\[ = e^{-\hat{x} \int_{0}^{t} p(l, h(l,0; x)) dl} \left( 1 + \hat{x} \int_{0}^{t} u_0(x) e^\int_{t}^{t} 1+\hat{x} p(l, h(l,0; x)) - w(t,x) dl d\sigma \right) \]

\[ = e^{-\hat{x} \int_{0}^{t} p(l, h(l,0; x)) dl} \left( 1 + \int_{t}^{t} \hat{x} w(\sigma, x) e^{\hat{x} \int_{t}^{t} w(t,x) dx d\sigma} \right) \]

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\[ e^{-\int_a^t p(l,h(t,0;x))dt} \left( 1 + \int_0^t \left( e^{\int_0^s \hat{\chi}w(l,x)dx} \right) \, d\sigma \right) \]

\[ = \exp \left( \hat{\chi} \int_0^t w(l, x) - p(l, h(0,0;x)) \, dt \right). \]

Since \( h(0,t;x) = [h(t,0;\cdot)]^{-1}(x) \), the function \( h(0,t;\cdot) \) is differentiable at each \( x \in h(t,0;\mathcal{U}') \) and

\[ h_x(0,t;x) = \frac{1}{h_x(0,0;x)} = \exp \left( -\hat{\chi} \int_0^t w(l, h(0,t;x)) - p(l, h(l,t;x)) \, dt \right). \]

The formula (6.27) can be deduced from the remark \( h(t,s;x) = h(t,0;h(0,s;x)) \), where the right-hand side is differentiable for all \( x \in h(s,0;\mathcal{U}') \).

**Step 2.** We show item (ii).

We have, by definition,

\[ p(t,x) = \int_\mathbb{R} \rho(x - h(t,0;z))u_0(z)e^{\int_0^1 -w(l,z) \, dz} \, dz \]

and item (i) allows a change of variables which yields

\[ p(t,x) = \int_\mathbb{R} \rho(x - y)u_0(h(0,0;y))e^{\int_0^1 -w(l,h(0,t;z)) \, dz} h_x(0,t;z) \, dz \]

\[ = \int_\mathbb{R} \rho(x - y)u_0(h(0,0;y))e^{\int_0^1 -w(l,h(0,t;z)) \, dz} - \hat{\chi} \int_0^t w(l,h(0,t;y)) - p(l,h(t,t;y)) \, dt \, dz \]

\[ = \int_\mathbb{R} \rho(x - y)u_0(h(0,0;y))e^{\int_0^1 \hat{\chi}p(l,h(t,t;x)) \, dz} \, dz \]

\[ = \int_\mathbb{R} \rho(x - y)w(t,h(0,0;y)) \, dy. \]

The formula for \( p_x \) is proven similarly. Item (ii) is proved.

**Step 3.** We show item (iii).

Using the formula for \( p_x \) established in item (ii), we have

\[ p_x(t,y) - p_x(t,x) = \int_{-\infty}^\infty (\rho_x(y - z) - \rho_x(x - z))w(t,h(0,t;z)) \, dz \]

\[ = \left( \int_{-\infty}^y + \int_y^{\infty} \right) (\rho_x(y - z) - \rho_x(x - z))w(t,h(0,t;z)) \, dz \]

\[ = \frac{1}{2\sigma^2} \int_x^y (e^{\frac{-z^2}{\sigma^2}} + e^{\frac{-x^2}{\sigma^2}})w(t,h(0,0;z)) \, dz, \]

therefore \( p_x(t,\cdot) \) is differentiable each time \( x \) is a Lebesgue point of \( z \mapsto w(t,h(0,0;z)) \) and we have

\[ p_{xx}(t,x) = p(t,x) - w(t,h(0,0;x)). \]

To finish our statement, we show that there exists \( \mathcal{U}'' \subset \mathcal{U}' \) (see the definition of \( \mathcal{U}' \) given in item (i)) such that every \( x = h(t,0;x_0) \) with \( x_0 \in \mathcal{U}'' \) is a Lebesgue point of \( z \mapsto w(t,h(0,0;z)) \).

Indeed, let \( \mathcal{U}'' \) be the set given by Lemma A.1 applied to the function \( w \in C^0([0,\tau], L^\infty(\mathcal{U}')) \). If \( x = h(t,0;x_0) \) we have:

\[ \frac{1}{y-x} \int_x^y |w(t,h(0,t;z)) - w(t,h(0,t;x))| \, dz = \frac{1}{y-x} \int_{h(0,t;x_0)}^{h(0,t,y)} |w(t,z) - w(t,x_0)| h_x(0,t;y) \, dz \]

\[ \leq \frac{h(0,t;y) - h(0,t;x)}{y-x} \frac{1}{h(0,t;y) - h(0,t;x)} \int_{h(0,t;x_0)}^{h(0,t,y)} |w(t,z) - w(t,x_0)| h_x(0,t;\cdot) \, dz = \frac{1}{L^\infty(\mathcal{U})}. \]

Since \( h(0,t;x) \) is differentiable for each \( x \in h(t,0;\mathcal{U}') \supset h(t,0;\mathcal{U}'') \), the right-hand side converges to 0 as \( y \to x \) when \( x_0 \in \mathcal{U}'' \) is a Lebesgue point of \( w(t,\cdot) \). Lemma 6.5 is proved. \( \square \)
Since the existence time given by Theorem 6.4 depends only on \( \tau \), we concentrate on the most important point which is the well-definition of \( \mathcal{T}_t u_0 \). The reason is that, for a semigroup property to hold, the property \( p(t,x) = \int_S p(x-y)w(t,y)dy \) would have to hold so that the vector \((w(t,\cdot),p(t,\cdot))\) can be taken as an initial condition; however, this is very unlikely in view of Lemma 6.5. In order to continue our construction of the integrated solutions, we first show that the solution can be defined in \( L^\infty \) with little modification.

Given \( u_0 \in L^\infty(\mathbb{R}) \), we define the operator induced by the family \( \mathcal{T}_t[u_0] : Z^+ \to Z^+ \) (for \( U \subset \mathbb{R} \) conull) as

\[
T^+_t[u_0](w, p) = \mathcal{T}_t[u_0](w, p)
\]

where \( \mathcal{T}_t[u_0] \) is obtained by (6.4) with an initial condition equal to \( u_0 \) a.e. and \( Z^+ := C^0([0, \tau], L^\infty(\mathbb{R})) \times Y^+, \quad \tau := C^0([0, \tau], L^\infty(\mathbb{R})) \times Y^+ \). The fact that \( T^+_t[u_0] \) is well-defined is shown in the following Corollary.

**Corollary 6.6 (Well-posedness in \( L^\infty(\mathbb{R}) \)).** Let \( u_0 \in L^\infty(\mathbb{R}) \) be given. Let \( U \) and \( U' \) be two conull set and \( u^U_0 \in L^\infty(U) \) and \( u^{U'}_0 \in L^\infty(U') \) be such that \( u_0 = u^U_0 = u^{U'}_0 \) almost everywhere. There exists \( \tau = \tau(\|u_0\|_{L^\infty(\mathbb{R})}) > 0 \) and a conull set \( \mathcal{U} \subset U \cap U' \) such that the solutions \( w^U \in C^0([0, \tau], L^\infty(U)) \) and \( w^{U'} \in C^0([0, \tau], L^\infty(U')) \) given by Theorem 6.4 coincide for all \( t \in [0, \tau^U] \cup [0, \tau^{U'}] \) and \( x \in \mathcal{U} \). Moreover we have \( \tau \geq \max(\tau^U, \tau^{U'}) \).

In particular, let \( u_0 \in L^\infty(\mathbb{R}) \) be such that \( u_0 = u_0 \) almost everywhere and \( \|u_0\|_{L^\infty(\mathbb{R})} = \|\tilde{u}_0\|_{L^\infty(\mathbb{R})} \) and define \( w(t, \cdot) \) as the \( L^\infty \) class of the solution \( \tilde{w} \in C^0([0, \tau], L^\infty(\mathbb{R})) \) given by Theorem 6.4. Then \( w \in C^0([0, \tau], L^\infty(\mathbb{R})) \) and \( w \) is the unique fixed point on the operator \( T^+_t[u_0] \) induced by the operator \( \mathcal{T}_t[u_0] \) defined in (6.4).

**Proof.** Most of the arguments involved in the proof of Corollary 6.6 are very classical therefore we concentrate on the most important point which is the well-definition of \( w \) in \( L^\infty \). The set \( U^\prime \subset U \cap U' \) mentioned in the corollary can be defined as

\[
U^\prime = U \cap U' \cap \{ u^U_0(x) \leq \|u_0\|_{L^\infty} \}.
\]

Since the existence time given by Theorem 6.4 depends only on \( \|u^U_0\|_{L^\infty(U')} \), we have \( \tau^U \geq \max(\tau^U, \tau^{U'}) \). Moreover since \( U^\prime \subset U \) it follows from the uniqueness of the fixed point of \( \mathcal{T}_t[u_0] \) that \( w^U \) and \( w^{U'} \) coincide on \( U^\prime \), and similarly \( \tilde{w}^U = \tilde{w}^{U'} \) on \( U^\prime \). The remaining statements are classical. 

We are now equipped with a family of operators \( T_t \) defined for \( u \in L^\infty(\mathbb{R}) \) and \( t \in [0, \tau(\|u_0\|_{L^\infty})] \) as

\[
T_t u_0(x) := w(t, h(0, t; x)) \in L^\infty(\mathbb{R}),
\]

where \( w \) and \( \tau(\|u_0\|_{L^\infty}) \) are given by Corollary 6.6. Next we show that the family \( T_t \) satisfies a semigroup property. We deduce the existence of a maximal solution for each \( u_0 \in L^\infty(\mathbb{R}) \).

**Theorem 6.7 (Maximal solutions).** Let \( u_0 \in L^\infty(\mathbb{R}) \) be given. The number

\[
\tau^*(u_0) := \sup\{ \tau > 0 | T^+_t[u_0] \text{ has a unique fixed point} \}
\]

is well-defined and belongs to \( (0, +\infty] \), where \( T^+_t[u_0] \) is the operator defined in (6.28). Moreover, there exists a conull set \( \mathcal{U} \subset \mathbb{R} \) and \( \tilde{u}_0 \in L^\infty(U) \) such that the operator \( \mathcal{T}_t[u_0] \) has a unique fixed point \( \tilde{w} \in C^0([0, \tau], L^\infty(\mathbb{U})) \) for each \( \tau \in (0, \tau^*(u_0)) \) and

\[
\tilde{w}(t,x) = w(t,x) \text{ for a.e. } x \in \mathbb{R}.
\]

The map \( u_0 \in L^\infty(\mathbb{R}) \mapsto (\tilde{w}, p) \in Z^+_U \) (and therefore \( u_0 \in L^\infty(\mathbb{R}) \mapsto (w, p) \in Z^+ \)) is continuous for each \( \tau \in (0, \tau^*(u_0)) \).

Finally, the map \( t \in [0, \tau^*(u_0)) \mapsto T_t u_0 \in L^\infty(\mathbb{R}) \) is a semigroup which is continuous for the \( L^\infty(\mathbb{R}) \) topology for any \( \eta \in (0, 1) \), where \( T_t \) is defined by (6.29), and if \( \tau^*(u_0) < +\infty \) then we have

\[
\lim_{t \to \tau^*(u_0)^-} \| T_t u_0 \|_{L^\infty(\mathbb{R})} = +\infty.
\]

The map \( u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^\infty(\mathbb{R}) \) is continuous for each \( t \in (0, \tau^*(u_0)) \).
The positiveness of \( \tau^*(u_0) \) is a consequence of Corollary 6.6. We show the existence of \( \mathcal{U} \) as defined in the Theorem. Let \( \mathcal{U}^0 := \mathbb{R} \) and let \( \tilde{u}_0 \in L^\infty(\mathbb{R}) \) be a bounded measurable function on \( \mathbb{R} \) such that \( \|\tilde{u}_0\|_{L^\infty(\mathbb{R})} = \|u_0\|_{L^\infty(\mathbb{R})} \). In the rest of the proof we identify \( u_0 \) and \( \tilde{u}_0 \) and consequently drop the tilde. We recursively construct a sequence of connect sets \( \mathcal{U}^n, n \in \mathbb{R}, \) such that \( \mathcal{U}^{n+1} \subset \mathcal{U}^n \), and a sequence of functions \( u_{0}^{n} \in L^\infty(\mathcal{U}^n) \), such that:

(i) \( u_{0}^{n+1}(x) := w^n(\tau_n, h^n(0, \tau_n; x)) \) where \( \tau_n := \tau(\|u_{0}^{n}\|_{L^\infty}) \), \((w^n, p^n)\) is the unique fixed point of the operator \( T^n \) (given by Theorem 6.4) with initial condition \( u_{0}^{n} \) and \( h^n \) is the solution of (6.3) corresponding to \( p^n \).

(ii) \( \mathcal{U}^{n+1} = \mathcal{U}^n \cap h^n(0, \tau_n; \mathcal{U}^n) \cap \{ x \mid u_{0}^{n+1}(x) \leq \|u_{0}^{n+1}\|_{L^\infty} \} \).

We let \( \mathcal{U} := \bigcap_{n \in \mathbb{N}} \mathcal{U}^n \). Remark that, since each \( \mathcal{U}^n \) is connect, the set \( \mathcal{U} \) is still connect. Next we show that \( T^n[u_0] \) has a unique fixed point for each \( \tau \in [0, \sum_{n \in \mathbb{N}} \tau_n] \).

Let \( T_0 = 0 \) and \( T_n := \sum_{k=0}^{n-1} \tau_{n+1} \), for all \( t \in [T_n, T_n + 1] \) we define

\[
w(t, x) := w^n(t - T_n, h_{n-1}(\tau_n, 0; x)) \quad \text{for all } x \in \mathcal{U},
\]

\[
p(t, x) := p^n(t - T_n, h_{n-1}(\tau_n, 0; x)) \quad \text{for all } x \in \mathbb{R}.
\]

We show that \((w, p)\) is the unique fixed point of \( T^n \) for all \( \tau \in [0, T_\infty) \) by induction. Indeed, the property is a consequence of Theorem 6.4 for all \( \tau \leq T_1 \). Suppose that \((w, p)\) is the unique fixed point of \( T^n \) for all \( \tau \leq T_n, n \geq 1 \). The formula

\[
w(t, x) = u_0(x) \exp \left( \int_0^t 1 + \hat{\chi} p(l, h(l, 0; x)) - (1 + \hat{\chi}) w(l, x) dl \right)
\]

is valid for all \( t \leq T_n \). For \( t \in [T_n, T_{n+1}] \) we have

\[
w^n(t - T_n, x) = u_{0}^{n}(x) \exp \left( \int_0^{t-T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) dl \right)
\]

\[
= w(T_n, h(T_n, 0; x)) \exp \left( \int_0^{T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) dl \right)
\]

\[
= u_0(h(0, T_n; x)) \exp \left( \int_0^{T_n} 1 + \hat{\chi} p^n(l, h(l, 0; h(0, T_n; x))) - (1 + \hat{\chi}) w(l, h(0, T_n; x)) dl \right)
\]

\[
\times \exp \left( \int_0^{T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) dl \right),
\]

so that

\[
w^n(t - T_n, h(T_n, 0; x)) = u_0(x) \exp \left( \int_0^{T_n} 1 + \hat{\chi} p(l, h(l, 0; x)) - (1 + \hat{\chi}) w(l, x) dl \right) \tag{6.30}
\]

\[
\times \exp \left( \int_{T_n}^t 1 + \hat{\chi} p^n(l - T_n, h^n(l - T_n, 0; h(T_n, 0; x))) - (1 + \hat{\chi}) w^n(l - T_n, h(T_n, 0; x)) dl \right).
\]

Next we remark that, by Lemma 6.5, the formula

\[
p(T_n, x) = \int_{\mathbb{R}} p(x - y) w(T_n, h(0, T_n; y)) dy = \int_{\mathbb{R}} p(x - y) u_{0}^{n}(y) dy = p^n(0, x)
\]

\[
p_x(T_n, x) = \int_{\mathbb{R}} p_x(x - y) w(T_n, h(0, T_n; y)) dy = \int_{\mathbb{R}} p_x(x - y) u_{0}^{n}(y) dy = p^n(0, x)
\]

hold, therefore \( p(t, x) \) can be extended to a function \( p \in C^0([0, T_{n+1}], W^{1, \infty}(\mathbb{R})) \) by defining \( p(t, x) = p^n(t - T_n, x) \) when \( t \geq T_n \), and moreover the extended function \( h(t, s; x) \) defined on
solves (6.3). Therefore (6.30) can be rewritten as:

\[ w^n(t - T_n, h(0, T_n; x)) = u_0(x) \exp \left( \int_0^{T_n} 1 + \chi p(l, h(l, 0; x)) - (1 + \chi)w(l, x)dl \right) + \int_0^t 1 + \chi p(l, h^n(l - T_n, 0; h(T_n, 0; x)) - (1 + \chi)w^n(l - T_n, h(T_n, 0; x))dl \]

\[ = u_0(x) \exp \left( \int_0^t 1 + \chi p(l, h(l, 0; x)) - (1 + \chi)w(l, x)dl \right), \]

where we have extended \( w \in C^0([0, T_{n+1}], L^\infty(\mathcal{U})) \) by defining \( w(t, x) := w^n(t - T_n, h(0, T_n; x)) \) when \( t \geq T_n \). Finally

\[ p(t, x) = \int_{\mathbb{R}} \rho(x - y)w(t, h(0, t; y))dy = \int_{\mathbb{R}} \rho(x - h(t, 0; x))w(t, z)h_z(t, 0; x)dz = \int_{\mathbb{R}} \rho(x - h(t, 0; z))u_0(z)e^{\int_0^t -w(t, z)dt}dz. \]

We have shown that \((w, p)\) is a fixed point of \( T_{n+1}^t[u_0] \), for all \( t \leq T_{n+1} \). Uniqueness follows from the remark: let \( w, \tilde{w} \) of \( T_{n+1}^t[u_0] \) be two fixed points of \( T_{n+1}^t \). Then \( w \) and \( \tilde{w} \) coincide in \([0, T_n]\) (by uniqueness of the fixed point) therefore \( w(T_n, x) = \tilde{w}(T_n, x) \). We have shown that \( w(T_n, x) = \tilde{w}(T_n, x) \) and by the uniqueness of the fixed point in the interval \([T_n, T_{n+1}] \) we conclude \( w(t, \cdot) = \tilde{w}(t, \cdot) \). The uniqueness is proved. We have shown by induction that \( T_{n+1}^t[u_0] \) has a unique fixed point for all \( \tau \in [0, T_\infty] \). As a by-product, this is also true for \( T^\tau[u_0] \) and therefore \( T_\infty \leq \tau^*(u_0) \).

Next we remark that \( \tau_n = \tau(\|u_0^n\|_{L^\infty}) \) is a positive continuous function of \( u_0^n \) and therefore \( T_\infty = \sum \tau_n < +\infty \) implies \( \|w(T_n, \cdot)\|_{L^\infty} = \|u_0^n\|_{L^\infty} \rightarrow +\infty \). This shows that \( \tau^*(u_0) \leq T_\infty \) and therefore

\[ \tau^*(u_0) = T_\infty. \]

Obviously if \( T_\infty = +\infty \) then we have \( \tau^*(u_0) \geq T_\infty = +\infty \). We have shown the equality between the quantities.

Finally, the continuity of \( u_0 \in L^\infty(\mathcal{U}) \rightarrow (w, p) \in Z_{t_0}^\tau \) is a consequence of the continuity of the map \( u_0^n \rightarrow (w^n, p^n) \in Z_{t_0}^\tau \) given by Theorem 6.4.

Next we prove the semigroup property of \( t \mapsto T_t u_0 \). This follows from a direct computation: let \( 0 \leq t \leq s < \tau^*(u_0) \), then for almost all \( x \in \mathbb{R} \) we have

\[ T_{t+s} u_0(x) = u_0(h(0, t + s; x)) \exp \left( \int_0^{t+s} 1 + \chi p(l, h(l, t + s; x)) - (1 + \chi)w(l, h(0, t + s; x))dl \right) \]

\[ = \left[ u_0(h(0, t; h(t, t + s))) \exp \left( \int_0^t 1 + \chi p(l, h(t, t + s; x)) - (1 + \chi)w(l, h(0, t + s; x))dl \right) \right. \]

\[ \times e^{\int_t^{t+s} 1 + \chi p(l, h(l, t + s; x)) - (1 + \chi)w(l, h(0, t + s; x))dl} \]

\[ = T_t u_0(h(t, t + s; x)) \exp \left( \int_0^s 1 + \chi p(l, h(t + l, t + s; x)) - (1 + \chi)w(l, h(0, t + s; x))dl \right). \]

Let \( \tilde{p}(t, x), \tilde{b}(t, s; x), \tilde{w}(t, x) \) be the quantities corresponding to the initial condition \( \tilde{u}_0 = T_t u_0(x) \). By Lemma 6.5 we have

\[ p(t, x) = \int_{\mathbb{R}} \rho(x - y)w(t, h(0, t; y))dy = \int_{\mathbb{R}} \rho(x - y)T_t(u_0)(y)dy, \]
therefore by the uniqueness of the fixed point we have
\[ \tilde{p}(l, y) = p(l + l, y), \quad \tilde{h}(l, \sigma; x) = h(l + l, \sigma; x), \quad \tilde{w}(l, x) = w(l + l, h(0, t; x)). \]

We conclude that
\[ T_{t+s}u_0(x) = T_{t}u_0(\tilde{h}(0, s; x)) \exp \left( \int_0^s 1 + \tilde{\chi}\tilde{p}(l, \tilde{h}(l, s; x)) - (1 + \tilde{\chi})\tilde{w}(l, \tilde{h}(0, s; x))dl \right). \]

The continuity of \( t \mapsto T_t u_0 \) in the \( L^1_\eta \) topology follows directly from Lemma 6.5 and 6.3.

What remains to show is the continuity of \( u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^\infty_t(\mathbb{R}) \). We use the sequential characterization of continuity. Let \( u_0, u^n_0 \in L^\infty(\mathbb{R}) \) be such that
\[ \|u^n_0 - u_0\|_{L^\infty(\mathbb{R})} \xrightarrow{n \to \infty} 0, \]
and let \( 0 < t < \tau^*(u_0) \). Let us recall that the map \( u_0 \in L^\infty \mapsto (w, p) \in Z^\tau \) is continuous, therefore we have \( \tau^*(u_n) > t \) for \( n \) sufficiently large and
\[ \|w^n(t, \cdot) - w(t, \cdot)\|_{L^\infty(\mathbb{R})} \xrightarrow{n \to \infty} 0, \]
where \((w^n, p_n)\) is the fixed point of \( T^t[u^n_0] \). Define \( h^n \) as the solution to (6.3) associated with \( u^n \), then we have
\[ \|u(t, \cdot) - u^n(t, \cdot)\|_{L^1_t(\mathbb{R})} = \frac{\eta}{2} \int_{\mathbb{R}} e^{-\eta|z|}|u(t, x) - u^n(t, x)|dx \]
\[ = \frac{\eta}{2} \int_{\mathbb{R}} e^{-\eta|z|}|w(t, h(t, 0; x)) - w^n(t, h^n(t, 0; x))|dx \]
\[ \leq \frac{\eta}{2} \int_{\mathbb{R}} e^{-\eta|z|}|w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))|dx \]
\[ + \frac{\eta}{2} \int_{\mathbb{R}} e^{-\eta|z|}|w(t, h^n(t, 0; x)) - w(t, h^n(t, 0; x))|dx \]
\[ \leq \frac{\eta}{2} \int_{\mathbb{R}} e^{-\eta|z|}|w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))|dx + \|w(t, \cdot) - w^n(t, \cdot)\|_{L^\infty(\mathbb{R})}. \]

Next we remark that the function \( w(t, h^n(t, 0; x)) \) converges to \( w(t, h(t, 0; x)) \) for a.e. \( x \in \mathbb{R} \). Indeed, let \( x \in \mathbb{R} \) be a Lebesgue point of \( w(t, h(t, 0; x)) \), then we have
\[ \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; z)) - w(t, h^n(t, 0; z))|dz \leq \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; z)) - w(t, h(t, 0; x))|dz \]
\[ + \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; z))|dz \]
\[ = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; z)) - w(t, h(t, 0; x))|dz \]
\[ + \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(0, t, h^n(t, 0, x+\varepsilon))) - w(t, h(0, t, 0; y)))h_x(t, 0; y)dy \]
\[ \times h^n(t, 0; h(0, t, 0; y))h_x(t, 0; y)dy \]
\[ \leq \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; z)) - w(t, h(t, 0; x))|dz \]
\[ + \frac{C}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; x)) - w(t, h(t, 0; y)))h_x(t, 0; y)dy |dy, \]
where \( C := \|h^n(t, 0; \cdot)\|_{L^\infty} \|h_x(t, 0; \cdot)\|_{L^\infty} \), so that
\[ \limsup_{n \to +\infty} \int_{x-\varepsilon}^{x+\varepsilon} |w(t, h(t, 0; z)) - w(t, h^n(t, 0; z))|dz = o(\varepsilon). \]
Define
\[ E_{\delta} := \{ x \in \mathbb{R} \mid \limsup_{n \to \infty} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))| \geq \delta \}, \]
and take a compact set \( K \subset E_{\delta} \) which is contained in a open set \( O \) with finite Lebesgue measure. Then \( K \) can be covered by a finite union of the interval in the family \( \Omega_{\mu} \) of intervals \( I_{x, \varepsilon, \mu} := (x - \varepsilon, x + \varepsilon) \) such that \( x \) is a Lebesgue point of \( w(t, h(t, 0; \cdot)) \), \( I \subset O \) and

\[
\limsup_{n \to +\infty} \int_{I_{x, \varepsilon, \mu}} |w(t, h(t, 0; z)) - w(t, h^n(t, 0; z))|dz \leq 2\mu \epsilon.
\]

Applying the Vitali covering lemma [31, Theorem 8.3 p. 154], there is a finite disjoint subcollection \( I_{x_k, \varepsilon_k, \mu} = (x_k, \varepsilon_k) \) \( (1 \leq k \leq n < +\infty) \) such that \( |K \setminus \bigcup I_{x_k, \varepsilon_k, \mu}| = 0 \) and therefore

\[
\delta |K| \leq \int_{K} \limsup_{n \to +\infty} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))|dx
\leq \sum_{k=1}^{n} \int_{I_{x_k, \varepsilon_k, \mu}} \limsup_{n \to +\infty} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))|dx
\leq \sum_{k=1}^{n} \limsup_{n \to +\infty} \int_{I_{x_k, \varepsilon_k, \mu}} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))|dx
\leq \sum_{k=1}^{n} 2\mu \varepsilon_k = \mu \sum_{k=1}^{n} |I_{x_k, \varepsilon_k, \mu}| \leq \mu |O|,
\]

Since \( O \) is independent of \( \mu \) we take the limit \( \mu \to 0 \) to find \( |K| = 0 \) and therefore

\[ |E_\delta| = \sup_{K \text{ compact}, K \subset E_\delta} |K| = 0. \]

Since \( \delta > 0 \) arbitrary we have shown that the set of where \( w(t, h^n(t, 0; x)) \) does not converge to \( w(t, h(t, 0; x)) \) is included in \( \bigcup_{n \geq 0} E_{1/n} \), which is still negligible for the Lebesgue measure. We have shown the convergence of \( w(t, h^n(t, 0; \cdot)) \) to \( w(t, h(t, 0; \cdot)) \) almost everywhere in \( \mathbb{R} \). The convergence of \( u^n(t, \cdot) \) to \( u(t, \cdot) \) in \( L^p_0(\mathbb{R}) \) is then a consequence of Lebesgue’s dominated convergence Theorem.

We are now in the position to link the constructed maximal solution with the integrated solutions to (1.1).

**Proposition 6.8** (Integrated solutions). Let \( \tau > 0 \) and \( u_0 \in L^\infty(\mathbb{R}) \).

(i) If \( u \in C^0([0, \tau], L^1_{\text{loc}}(\mathbb{R})) \) is an integrated solution to (1.1), then \( \tau^*(u_0) \geq \tau \) and \( u(t, \cdot) = T_t u_0 \) for all \( t \in [0, \tau] \).

(ii) Conversely, if \( u(t, x) := T_t u_0(x) \) for all \( t \in [0, \tau] \), then \( u(t, x) \) is an integrated solution to (1.1).

**Proof.** We first prove Item (i). Assume \( u(t, x) \in C^0([0, \tau], L^1_{\text{loc}}(\mathbb{R})) \) is an integrated solution. Define \( p(t, x) := \int_{\mathbb{R}} \rho(x-y) u(t, y) dy \). We first show that \( p \in C^0([0, \tau], L^1(\mathbb{R})) \). We have:

\[
|p(t, x) - p(s, x)| \leq \int_{\mathbb{R}} \rho(x-y)|u(t, y) - u(s, y)|dy,
\]
and since \( t \mapsto u(t, \cdot) \) is bounded in \( L^\infty \) and continuous in \( L^1_{\text{loc}} \), both right-hand sides can be made arbitrarily small (recall \( \rho \) and \( \rho_x \)). This shows \( p \in C^0([0, \tau], L^1(\mathbb{R})) \).

Next we show that \( p(t, \cdot) \in W^{2, \infty}(\mathbb{R}) \) for all \( t \in [0, \tau] \) and that \( \sup \rho_{x|0,\tau} \|p_{x^2}(t, \cdot)\|_{L^\infty} < +\infty. \)

Indeed, take \( x \leq y \), we have

\[
p_x(t, x) - p_x(t, y) = \int_{\mathbb{R}} (p_x(x-z) - p_x(y-z)) u(t, z) dz
\]
and we have
\[ p_x(t, x) = p(t, x) - w(t, x) \quad \text{for a.e. } x \in \mathbb{R}. \]

Next, define the solution \( h \) to (6.3). According to Definition 2.1, there exists a conull set \( \mathcal{U} \) on which \( t \mapsto u(t, h(t, 0; x)) \) is a classical solution to (2.2). Therefore, by a direct integration, we have
\[
w(t, x) = \varphi_0(x) \exp \left( \int_0^t 1 + \chi p(t, h(t, 0; x)) - (1 + \chi) w(t, x) \, dt \right),
\]
where \( w(t, x) := u(t, h(t, 0; x)) \). In particular, \( w(t, x) \in C^0([0, \tau], \mathcal{C}^\infty(\mathcal{U})) \). By Lemma A.1, there exists a subset \( \mathcal{U}' \subset \mathcal{U} \) such that for each \( x \in \mathcal{U}' \) and all \( t \in [0, \tau] \), \( x \) is a Lebesgue point of \( w(t, x) \). Since \( h(t, s; \cdot) \) is Lipschitz continuous for all \( t, s \in [0, \tau] \), we have
\[
\int_{-1}^1 |u(t, x + \varepsilon y) - u(t, x)| \, dy = \int_{h(0,t,x+c)}^{h(0,t,x-c)} |u(t, h(t, 0; z)) - u(t, x)| h_x(t, 0; z) \, dz
\]
\[
\leq \int_{h(0,t,x-c)}^{h(0,t,x+c)} |w(t, z) - w(t, h(0, t; x))| h_x(t, 0; z) \, dz
\]
\[
\leq K \int_{h(0,t,x)-K\varepsilon}^{h(0,t,x)+K\varepsilon} |w(t, z) - w(t, h(0, t; x))| \, dz,
\]
where \( K \) is the Lipschitz constant of \( h(t, 0; \cdot) \). Therefore \( x \) is a Lebesgue point of \( u \) whenever \( h(0, t; x) \) is a Lebesgue point of \( u \). In particular, for \( x \in \mathcal{U}' \), \( p_{xx}(t, h(t, 0; x)) \) is the derivative of \( p_x \) and we have
\[
\sigma^2 p_{xx}(t, h(t, 0; x)) = p(t, h(t, 0; x)) - w(t, x).
\]
In particular, writing
\[
h(t, 0; x) - h(t, 0; y) = x - y - \chi \int_0^t p_x(t, h(t, x))) \, dt
\]
\[
= x - y - \chi \int_0^t \frac{p_x(t, h(t, 0; x)) - p_x(t, h(t, 0; y))}{h(t, 0; x) - h(t, 0; y)} \, dt
\]
\[
= (x - y) \exp \left( -\chi \int_0^t \frac{p_x(t, h(t, 0; x)) - p_x(t, h(t, 0; y))}{h(t, 0; x) - h(t, 0; y)} \, dt \right),
\]
we find that the formula
\[
h_x(t, 0; x) = e^{\chi t} \int_0^t w(t, x) - p(t, h(t, 0; x)) \, dt
\]
holds for all \( x \in \mathcal{U}' \). Therefore
\[
p(t, x) = \int_{\mathbb{R}} \rho(x - y) u(t, y) \, dy = \int_{\mathbb{R}} \rho(x - h(t, 0; z)) u(t, h(t, 0; z)) h_x(t, 0; z) \, dz
\]
\[
= \int_{\mathbb{R}} \rho(x - h(t, 0; z)) w(t, x) e^{\chi t} \int_t^0 w(t, z) - p(t, h(t, 0; z)) \, dt \, dz
\]
\[
= \int_{\mathbb{R}} \rho(x - h(t, 0; z)) \varphi_0(x) e^{\chi t} \int_0^1 w(t, z) \, dz \, dz.
\]
Therefore \((w, p)\) is a fixed point of \( T^*_{\mathcal{U}}[\varphi_0] \).

Conversely if \( u(t, x) = T_t \varphi_0(x) \) for all \( t \in [0, \tau] \) then by definition \( u \) is a fixed point of \( T^*[\varphi_0] \) and we have see in Theorem 6.7 that there exists \( \mathcal{U} \subset \mathbb{R} \) conull such that \( T^*_t[u_0](w, p) = (w, p) \) for a \( p \in \mathcal{Y}^* \), with \( w(t, x) = u(t, h(t, 0; x)) \). It then follows from Lemma 6.5 that \( p = \rho \cdot u \) and elementary computation then show that \( u \) is indeed a classical solution to (2.2) for all \( x \in \mathcal{U} \). This proves Item (ii).

This finishes the proof of Proposition 6.8. \( \square \)
Now we prove Lemma 6.2 which is used in the proof of Lemma 6.4. Next we prove that the solutions remain bounded by 0 and 1.

**Lemma 6.9** (Boundedness of the solutions). Let \( \tau > 0 \) be given and let \( u_0 \in L^\infty(\mathbb{R}) \) satisfy \( 0 \leq u_0(x) \leq 1 \). Let \( u(t,x) \) be the corresponding integrated solution to (1.1). Then

\[
0 \leq u(t,x) \leq 1.
\]

**Proof.** Let \( w(t,x) := u(t,h(t,0;x)) \in C^0([0,T]; L^\infty(\mathcal{U})) \) for some \( T > 0 \) and a conull set \( \mathcal{U} \subset \mathbb{R} \) (the continuity of \( t \mapsto w(t,\cdot) \) follows from Theorem 6.7) be such that \( t \mapsto w(t,x) \) is a classical solution to (2.2) for each \( x \in \mathcal{U} \). We prove the uniform bound:

\[
\|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \leq 1. \tag{6.31}
\]

Let \( \varepsilon > 0 \) and assume by contradiction that there exists \( t \in [0,T) \) with

\[
\|w(t,\cdot)\|_{L^\infty(\mathcal{U})} > 1 + \varepsilon.
\]

Define

\[
t^* := \inf \{ t > 0 \mid \|w(t,\cdot)\|_{L^\infty} > 1 + \varepsilon \} < T.
\]

Then by the continuity of \( t \mapsto \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \) we have \( \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})} = 1 + \varepsilon \). In particular there exists a sequence \((t_n,x_n)\) with \( t_n > t^*, t_n \to t^* \) as \( n \to +\infty \) and \( x_n \in \mathcal{U} \) which satisfies

\[
w(t_n,x_n) \to \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})}, \quad \text{as } n \to \infty,
\]

\[
w(t_n,x_n) > 1 + \varepsilon \quad \forall n \in \mathbb{N}. \tag{6.32}
\]

We claim that there exists a \( N \) such that for any \( n \geq N \) and \( t \in [t^*,t_n] \), we have

\[
w(t,x_n) > \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} = \frac{\varepsilon}{2(1 + \chi)} \quad \text{and} \quad \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \geq \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})} - \frac{\varepsilon}{2} \tag{6.33}
\]

Indeed, for \( t \in [t^*,t_n] \) we have

\[
|w(t,x_n) - \|w(t,\cdot)\|_{L^\infty(\mathcal{U})}| \leq \frac{\varepsilon}{2} \quad \text{if } |t - t^*| \leq \delta_0 \quad \text{and by the continuity of } t \mapsto \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \text{ there exists } \delta_1 > 0 \text{ such that } |\|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})} - \|w(t,\cdot)\|_{L^\infty(\mathcal{U})}| \leq \frac{\varepsilon}{2(1 + \chi)} \text{ if } |t - t^*| \leq \delta_1. \]

Since \( t_n \to t^* \) as \( n \to +\infty \) and \( w(t_n,x_n) \to \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})} \) we can choose \( N > 0 \) such that for all \( n \geq N \), we have \( |t_n - t^*| \leq \min(\delta_0,\delta_1) \) and

\[
|w(t_n,x_n) - \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})}| \leq \frac{\varepsilon}{2(1 + \chi)} \quad \text{and}
\]

\[
|w(t_n,x_n) - \|w(t,\cdot)\|_{L^\infty(\mathcal{U})}| \leq \frac{\varepsilon}{2(1 + \chi)}, \quad \text{for all } t \in [t^*,t_n].
\]

Finally, using (6.33) we have for all \( t \in [t^*,t_n] \):

\[
\frac{d}{dt} w(t,x_n) = w(t,x_n) (1 + \tilde{\chi}(\rho * w)(t,h(t,0;x_n)) - (1 + \tilde{\chi})w(t,x_n)) \leq \frac{\varepsilon}{2} + \tilde{\chi} \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} - (1 + \tilde{\chi}) \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \leq w(t,x_n) \left( 1 + \tilde{\chi} \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} - (1 + \tilde{\chi}) \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} + \frac{\varepsilon}{2} \right)
\]

\[
\leq w(t,x_n) \left( 1 + \frac{\varepsilon}{2} - \|w(t,\cdot)\|_{L^\infty(\mathcal{U})} \right)
\]

\[
\leq w(t,x_n) \left( 1 + \frac{\varepsilon}{2} - \|w(t^*,\cdot)\|_{L^\infty(\mathcal{U})} + \frac{\varepsilon}{2} \right) \leq 0.
\]
This implies
\[ w(t, x_n) \leq w(t^*, x_n) \leq 1 + \varepsilon, \quad \forall t \in [t^*, t_n]. \]

On the other hand, due to (6.32) we have
\[ w(t_n, x_n) > 1 + \varepsilon. \]

This is a contradiction. Thus for any \( t > 0, \|w(t, \cdot)\|_{L^\infty(I)} \leq 1 + \varepsilon. \) Since \( \varepsilon \) is arbitrary, (6.31) holds.

In particular, the solution constructed in Step 1 and 2 can be extended up to \( T = +\infty. \) We are now in the position to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( u_0 \in L^\infty(\mathbb{R}). \)

**Existence and uniqueness.** The existence and uniqueness of the integrated solution follows directly from Theorem 6.7 (existence and uniqueness of a fixed-point problem) and Proposition 6.8 (consistency between the fixed-point problem and the integrated solutions).

**Continuity.** The continuity in the space \( L^1_0(\mathbb{R}) \) and the continuity of \( u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_0(\mathbb{R}) \) have been shown in Theorem 6.7.

**Other properties.** The semigroup property follows directly from the form of the operator has been shown in Theorem 6.7. The uniform bound when \( 0 \leq u_0(x) \leq 1 \) has been shown in Lemma 6.9 and the fact that \( \tau^*(u_0) = +\infty \) from the fact that the \( L^\infty \) norm of \( u(t, \cdot) \) cannot blow-up in finite time.

This ends the proof of Theorem 2.1. \( \square \)

Next we show that our model preserves certain properties of the initial condition.

**Proposition 6.10 (Properties of the solutions).** Let \( u(t, x) \) be an integrated solution to (1.1) and suppose \( u_0 \in L^\infty(\mathbb{R}) \) with \( 0 \leq u_0 \leq 1. \) Then

(i) if \( u_0(x) \) is continuous, then \( u \in C^0([0, T] \times \mathbb{R}). \)

(ii) if \( u_0(x) \in C^1(\mathbb{R}), \) then \( u \in C^1([0, T] \times \mathbb{R}) \) and \( u \) is then a classical solution to (1.1).

(iii) if \( u_0(x) \) is monotone, then \( u(t, x) \) has the same monotony for each \( t > 0. \)

**Proof.** From (2.2) we can directly solve the solution \( w(t, x) = u(t, h(t, 0; x)) \) as
\[
 w(t, x) = \frac{u_0(x) \exp \left( \int_0^t (1 + \tilde{x}(\rho \ast u)(l, h(l, 0; x)) dl \right)}{1 + (1 + \tilde{x}) u_0(x) \exp \left( \int_0^t (1 + \tilde{x}(\rho \ast u)(\sigma, h(\sigma, 0; x)) d\sigma \right)}.
\]

for all \( t > 0 \) and almost all \( x \in \mathbb{R}, \) which is equivalent to
\[
 u(t, x) = \frac{u_0(h(0, 0; t)) \exp \left( \int_0^t (1 + \tilde{x}(\rho \ast u)(l, h(l, 0; x)) dl \right)}{1 + (1 + \tilde{x}) u_0(h(0, 0; t)) \exp \left( \int_0^t (1 + \tilde{x}(\rho \ast u)(\sigma, h(\sigma, 0; x)) d\sigma \right)}.
\]

Since \( (t, x) \to h(t, s; x) \) is continuous, the right-hand side is a continuous function. This shows (i).

Let us show (ii). By (i) we have \( u \in C^0([0, T] \times \mathbb{R}). \) Thus, the spatial derivative of the vector field of (2.1) satisfies
\[
 -\sigma^2(\rho \ast u)_x(t, x) = u(t, x) - (\rho \ast u)(t, x) \in C^0([0, T] \times \mathbb{R}).
\]

Therefore, the characteristic flow \( (t, s, x) \to h(t, s; x) \in C^1([0, T] \times [0, T] \times \mathbb{R}). \) If we denote
\[
 \phi(t, x) := e^{\int_0^t (1 + \tilde{x}(\rho \ast u)(l, h(l, 0, x)) dl}, \quad (6.34)
\]
then \((t, x) \mapsto \phi(t, x)\) is \(C^1\), which implies \(w \in C^1([0, T] \times \mathbb{R})\). Since \(u(t, x) = w(t, h(t; t; x))\) we have \(u \in C^1([0, T] \times \mathbb{R})\).

Finally we show \((iii)\). We will assume that \(u_0(x)\) is decreasing (the increasing case can be treated with a similar argument). We let \(w(t, x) := u(t, h(t, x))\) where \(u\) is the solution to (1.1) starting from \(u(t = 0, x) \equiv u_0(x)\), and \(h(t, s; x)\) be the corresponding characteristic flow, i.e. the solution to (6.3) with \(p(t, x) := \int_0^1 \rho(x - z) w(t, h(0, t; z)) dz\). Our aim is to show that \(w\) is a fixed point of the map

\[
\tilde{T}_r : C^0([0, \tau], L^\infty(\mathbb{R})) \rightarrow C^0([0, \tau], L^\infty(\mathbb{R}))
\]

\[
\tilde{w} 
\]

\[
\left. \begin{array}{c}
\tilde{w} \\
\end{array} \right\} \mapsto \frac{u_0(x) \exp \left( \int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds \right)}{1 + (1 + \tilde{\chi})u_0(x) \int_0^t \exp \left( \int_0^t 1 + \tilde{\rho}(s, h(s, 0; x)) ds \right) dl}
\]

where \(\tilde{\rho}(t, x)\) is defined in the above formula by

\[
\tilde{\rho}(t, x) := \int_0^1 \rho(x - z) \tilde{w}(t, h(0, t; z)) dz
\]

we stress that \(h\) is the characteristic flow corresponding to the “real” solution to (1.1) and is independent of \(\tilde{w}\).

As the proof is more involved, we subdivide it in four steps.

**Step 1:** Let \(r > 0\), we show that there exists \(\tau_0\) such that the ball

\[
B_r := \left\{ w \in C^0([0, \tau], L^\infty(\mathbb{R})) \mid \|w(t, x) - u_0(x)\|_{C^0([0, \tau], L^\infty(\mathbb{R}))} \leq r \right\}
\]

is left stable by \(\tilde{T}_r\) for \(0 < \tau \leq \tau_0\).

Let \(u_0 \in B_r\). We compute:

\[
\|\tilde{T}_r(\tilde{w}) - u_0(x)\| = \left| \frac{u_0(x)e^{\int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds} - u_0(x)}{1 + (1 + \tilde{\chi})u_0(x) \int_0^t e^{\int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds} dl} \right|
\]

\[
\leq u_0(x) \left| \frac{e^{\int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds} - 1 - (1 + \tilde{\chi})u_0(x)}{1 + (1 + \tilde{\chi})u_0(x) \int_0^t e^{\int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds} dl} \right|
\]

\[
\leq \left\| u_0 \right\|_{L^\infty(\mathbb{R})} \left( e^{1 + \tilde{\chi}\|u_0\|_{L^\infty(\mathbb{R})} + \tilde{\chi} r} \left| \int_0^t 1 + \tilde{\chi}(s, h(s, 0; x)) ds \right| + (1 + \tilde{\chi}\|u_0\|_{L^\infty(\mathbb{R})} + \tilde{\chi} r) \right)
\]

\[
\leq C r,
\]

where \(C\) depends on \(\|u_0\|_{L^\infty(\mathbb{R})}\), \(r\), and \(\tilde{\chi}\). The existence of \(\tau_0\) is proved.

**Step 2:** Let \(r > 0\), we show that there exists \(\tau_1 > 0\) such that \(\tilde{T}_r\) is contracting on \(B_r\) for \(0 < \tau < \tau_1\).

Let \(\tilde{w}_1, \tilde{w}_2 \in B_r\), and let \(\kappa := 1 + r\) so that \(\|\tilde{w}_1\|_{L^\infty(\mathbb{R})} \leq \kappa\) and \(\|\tilde{w}_2\|_{L^\infty(\mathbb{R})} \leq \kappa\). For notational compactness we define in advance

\[
\tilde{p}_i(t, x) := \int_0^1 \rho(x - z) \tilde{w}_i(t, h(0, t; z)) dz,
\]

\[
D_i(t, x) := 1 + (1 + \tilde{\chi})u_0(x) \int_0^t \exp \left( \int_0^t 1 + \tilde{p}_i(s, h(s, 0; x)) ds \right) dl,
\]

\(i \in \{1, 2\}\).

We compute:

\[
\|\tilde{T}_r(\tilde{w}_1(t, x) - \tilde{T}_r(\tilde{w}_2(t, x))\|\]

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Moreover, the contraction principle, Step 4:

\[ \tilde{w}(t, x) = u_0(x) e^{\int_0^t 1+\tilde{\chi}p(s,h(s,x))ds} D_2(t, x) \]

where we have used the fact that \( u_0 \) is nonincreasing. This shows the stability of \( \tilde{w} \).

**Step 3:** We show that the map \( \tilde{T} \) preserves the monotony of \( u_0 \), i.e. the set

\[ \mathcal{D} := \{ w \in C^0([0, \tau], L^\infty(\mathbb{R})) | w(t, \cdot) \text{ is nonincreasing} \} \]

is left stable by \( \tilde{T} \).

Indeed, let \( \tilde{w} \) be nonincreasing with respect to \( x \). Let \( \tilde{w}^1(t, x) := \tilde{T}(w)(t, x) \). We first show that \( \tilde{P} \) is nonincreasing:

\[ \tilde{p}(t, x) - \tilde{p}(t, y) = \int_0^t \rho(z) (\tilde{w}(t, h(0, t; x - z)) - \tilde{w}(t, h(0, t; y - z))) dz \leq 0, \]

since the characteristic flow \( h(t, s; \cdot) \) is increasing. Next we let

\[ D(t, x) := 1 + (1 + \tilde{\chi}u_0(x)) \int_0^t \exp \left( \int_0^t 1 + \tilde{\chi}p(s, h(s, x))ds \right) dl. \]

We compute:

\[ \tilde{w}^1(t, x) - \tilde{w}^1(t, y) = u_0(x) e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,x))ds} D(t, y) - u_0(y) e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,y))ds} D(t, x) \]

\[ = \frac{u_0(x) e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,x))ds} D(t, y) - u_0(y) e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,y))ds} D(t, x)}{D(t, x)D(t, y)} \]

\[ + u_0(x) u_0(y) \int_0^t e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,x))ds} D(t, y) \int_0^t 1+\tilde{\chi}p(s,h(s,0,x))ds D(t, x) \]

\[ - e^{\int_0^t 1+\tilde{\chi}p(s,h(s,0,x))ds} D(t, y) \int_0^t 1+\tilde{\chi}p(s,h(s,0,y))ds D(t, x) \]

\[ \leq \frac{u_0(x) u_0(y)}{D(t, x)D(t, y)} \int_0^t e^{\tilde{\chi} \int_0^t \tilde{p}(s,h(s,0,x)) - \tilde{p}(s,h(s,0,y)) ds} \int_0^t 1+\tilde{\chi}p(s,h(s,0,z))ds D(t, y) \]

\[ \times \left( e^{\tilde{\chi} \int_0^t \tilde{p}(s,h(s,0,x)) - \tilde{p}(s,h(s,0,y)) ds} - 1 \right) dl \leq 0, \]

since \( \tilde{P} \) is nonincreasing. This shows the stability of \( \mathcal{D} \).

**Step 4:** We conclude.

Let \( \tau := \min(\tau_0, \tau_1) \) where \( \tau_0, \tau_1 \) are as in Step 1 and 2. By a direct application of the Banach contraction principle, \( \tilde{T} \) has a unique fixed point in \( \tilde{B}_\tau \), which is \( w \) (since \( w \) happens to be a fixed point). Moreover \( w \) can be obtained as the limit of the iteration scheme:

\[ w^0(t, x) := u_0(x), \quad w^{n+1}(t, x) := \tilde{T}(w^n)(t, x). \]
Since \( u_0 \) is nonincreasing and \( \tilde{T}_\tau \) preserves the monotony, it follows that \( w \) is nonincreasing (\( \mathcal{D} \) is closed for the considered topology).

Since \( \tau \) does not depend on \( u_0 \), the monotony of \( u(t, \cdot) \) for all \( t > 0 \) follows from an induction argument.

\[ \] \[ \] \[ \]

**Theorem 6.11** (Long-time behaviour). Let \( \delta \in (0,1) \) and \( u_0(x) \) be such that \( \delta \leq u_0(x) \leq 1 \). Let \( u(t,x) \) be the corresponding integrated solution to (1.1). Then

\[
\lim_{t \to \infty} \|1 - u(t,\cdot)\|_{L^\infty(\mathbb{R})} = 0.
\]

**Proof.** Let \( \theta \) be defined as

\[
\theta := \liminf_{t \to +\infty} \inf_{x \in \mathbb{R}} u(t,x),
\]

and assume by contradiction that \( \theta < 1 \). We first remark that for any \( x \in \mathbb{R} \) we have

\[
\left\{ \begin{array}{ll}
\partial_t w(t,x) = w(t,x) \left( 1 + \chi (\rho \ast u)(t,h(t,0;x)) - (1 + \hat{\chi}) w(t,x) \right) & t > 0, \\
 w(0,x) & \geq \delta.
\end{array} \right.
\]

Thus, for each \( x \in \mathbb{R} \),

\[
w(t,x) \geq \delta, \quad x \in \mathbb{R}, t > 0.
\]

In particular \( (\rho \ast u)(t,h(t,0;x)) = \int_{\mathbb{R}} \rho(h(t,0;x) - y) u(t,y) dy \geq \delta \int_{\mathbb{R}} \rho(h(t,0;x) - y) dy = \delta \). We deduce that

\[
\left\{ \begin{array}{ll}
\partial_t w(t,x) = w(t,x) \left( 1 + \chi (\rho \ast u)(t,h(t,0;x)) - (1 + \hat{\chi}) w(t,x) \right) & t > 0, \\
 w(0,x) & \geq \delta.
\end{array} \right.
\]

This implies for any \( t > 0, x \in \mathbb{R} \)

\[
w(t,x) \geq \frac{\delta e^{t(1+\hat{\chi})}}{1 + \frac{(1+\chi)\delta}{1+\hat{\chi}} e^{t(1+\hat{\chi})} - 1} \xrightarrow{t \to \infty} \frac{1 + \hat{\chi}\delta}{1 + \hat{\chi}}.
\]

In particular

\[
\theta \geq \frac{1 + \hat{\chi}\delta}{1 + \delta} \geq \frac{1}{1 + \hat{\chi}}.
\]

(6.35)

It is not difficult to see that for each \( \alpha \in (0,1) \) there exists \( T_\alpha \) such that, for all \( t \geq T_\alpha \), we have

\[
\inf_{x \in \mathbb{R}} w(t,x) \geq \alpha \theta.
\]

Therefore for all \( t \geq T_\alpha \),

\[
(\rho \ast u)(t,h(t,0;x)) \geq \alpha \theta \int_{\mathbb{R}} \rho(h(t,0;x) - y) dy = \alpha \theta,
\]

which yields

\[
\left\{ \begin{array}{ll}
\partial_t w(t,x) = w(t,x) \left( 1 + (\rho \ast u)(t,h(t,0;x)) - (1 + \hat{\chi}) w(t,x) \right) & t > T_1, x \in \mathbb{R} \\
 w(T_1,x) & \geq \frac{1 + \hat{\chi} \alpha \theta}{1 + \hat{\chi}}
\end{array} \right.
\]

and finally

\[
\theta = \liminf_{t \to +\infty} \inf_{x \in \mathbb{R}} w(t,x) \geq \frac{1 + \hat{\chi} \alpha \theta}{1 + \hat{\chi}}.
\]

This is a contradiction if \( \alpha \) is chosen as

\[
\alpha = 1 - \frac{1}{\hat{\chi}} \left( 1 + \frac{1}{\theta} - 1 \right).
\]

and this choice is admissible because

\[
\frac{1}{\hat{\chi}} \left( 1 + \frac{1}{\theta} - 1 \right) < \frac{1}{\hat{\chi}} (1 + \hat{\chi} - 1) = 1
\]

by (6.35). This concludes the proof of Theorem 6.11. \( \square \)

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Appendix

A Lebesgue points along continuous trajectories

Here we show that the space $\mathcal{L}^\infty(\mathcal{U})$ is well-behaved with respect to Lebesgue points when $\mathcal{U}$ is a subset of $\mathbb{R}$.

**Lemma A.1** (Lebesgue points along continuous trajectories). Let $\mathcal{U} \subset \mathbb{R}$ be conull. Let $w \in C^0([0, \tau], \mathcal{L}^\infty(\mathcal{U}))$ be given, then there exists a conull set $\mathcal{U}' \subset \mathcal{U}$ such that each $x \in \mathcal{U}'$ is a Lebesgue points of $w(t, \cdot)$ for all $t \in [0, \tau]$.

**Proof.** Recall that a Lebesgue point of a measurable function $f : \mathcal{U} \to \mathbb{R}$ is characterized by the property

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |f(z) - f(x)|dz = 0$$

or, equivalently,

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_{-1}^{1} |f(x + \varepsilon y) - f(x)|dy = 0.$$

Let $w \in C^0([0, \tau], \mathcal{L}^\infty(\mathcal{U}))$ be given. Given $q \in \mathbb{Q} \cap [0, \tau]$ we define the failure set

$$\mathcal{F}_q := \{ x \in \mathcal{U} | x \text{ is not a Lebesgue point of } w(q, \cdot) \}.$$

It is classical that for each $q$ the set $\mathcal{F}_q$ is negligible for the Lebesgue measure $\lambda$, i.e. $\lambda(\mathcal{F}_q) = 0$. Since the family $(\mathcal{F}_q)_{q \in \mathbb{Q} \cap [0, \tau]}$ is countable, we have

$$\lambda \left( \bigcup_{q \in \mathbb{Q} \cap [0, \tau]} \mathcal{F}_q \right) = 0$$

therefore the set $\mathcal{U}' := \mathcal{U} \setminus \bigcup_{q \in \mathbb{Q} \cap [0, \tau]} \mathcal{F}_q$ is conull.

Let us show that $\mathcal{U}'$ is composed of Lebesgue points of $w(t, \cdot)$. Let $x \in \mathcal{U}'$ and $t \in [0, \tau]$, then there exists a sequence of rational numbers $t_n \in \mathbb{Q}$ such that $t_n \to t$. By definition of $\mathcal{U}'$, $x$ is not in any $\mathcal{F}_{t_n}$ and therefore $x$ is a Lebesgue point of the functions $w(t_n, \cdot)$ for all $n \in \mathbb{N}$. We have:

$$\int_{-1}^{1} |w(t, x + \varepsilon y) - w(t, x)|dy \leq \int_{-1}^{1} |w(t, x + \varepsilon y) - w(t_n, x + \varepsilon y)|dy + \int_{-1}^{1} |w(t_n, x + \varepsilon y) - w(t_n, x)|dy$$

$$+ \int_{-1}^{1} |w(t_n, x) - w(t, x)|dy$$

$$\leq \int_{-1}^{1} |w(t_n, x + \varepsilon y) - w(t_n, x)|dy + 2\|w(t, \cdot) - w(t_n, \cdot)\|_{L^\infty(\mathcal{U})},$$

therefore the right-hand side is arbitrarily small when $\varepsilon \to 0$. We conclude that $x$ is a Lebesgue point of $w(t, \cdot)$. Lemma A.1 is proved.

B An nonlinear function

We study a function used in the proof of Lemma 5.1 and Assumption 3.

**Lemma B.1.** The function

$$f(x) := \ln \left( \frac{2 - x}{x} \right) + \frac{2}{2 + x} \left( \frac{x}{2} \ln \left( \frac{x}{2} \right) + 1 - \frac{x}{2} \right)$$

defined for $x \in (0, 2)$ is strictly decreasing and satisfies

$$\lim_{x \to 0^+} f(x) = +\infty, \quad \lim_{x \to 2^-} f(x) = -\infty.$$  

In particular $f(x)$ has a unique root in $(0, 2)$.

Finally, we have $f(1) > 0$.  

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Proof. The behaviour of $f$ at the boundary is standard. The strict monotonicity requires the computation of the derivative:

\[ f'(x) = \left( -\frac{x - (2 - x)}{x^2} \right) \times \frac{x}{2 - x} + \frac{-2}{(2 + x)^2} \left( \frac{x}{2} \ln \frac{x}{2} + 1 - \frac{x}{2} \right) + \frac{2}{2 + x} \ln \frac{x}{2}. \]

Recalling that

\[ \frac{\hat{\chi}}{2} \ln \left( \frac{\hat{\chi}}{2} \right) + 1 - \frac{\hat{\chi}}{2} > 0, \]

because $x \mapsto x \ln(x)$ is strictly convex, all three terms in the expression of $f'(x)$ are negative, therefore

\[ f'(x) < 0 \]

for all $x \in (0, 2)$. The fact that $f(1) > 0$ can also be deduced from (B.1). Lemma B.1 is proved.}

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