Spacetime fluctuations and the spreading of wavepackets

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Received 22 July 2009, in final form 18 September 2009
Published 20 October 2009
Online at stacks.iop.org/CQG/26/225010

Abstract
Using a density matrix description in space we study the evolution of wavepackets in a fluctuating spacetime background. We assume that spacetime fluctuations manifest as classical fluctuations of the metric. From the non-relativistic limit of a non-minimally coupled Klein–Gordon equation, we derive a Schrödinger equation with an additive Gaussian random potential. This is transformed into an effective master equation for the density matrix. The solutions of this master equation allow us to study the dynamics of wavepackets in a fluctuating spacetime, depending on the fluctuation scenario. We show how different scenarios alter the diffusion properties of wavepackets.

PACS numbers: 02.50.Ey, 03.65.Yz, 04.60.Bc, 05.40.—a

1. Introduction

The search for a quantum theory of gravity is still a work in progress and leaves many open questions. Currently, there is no final version of a quantum gravity theory, but it is expected that one of its consequences is the appearance of some kind of spacetime foam. This can be thought of as spacetime fluctuations which could manifest classically in the low-energy limit as fluctuations of the metric or of the connection.

In the most simple version, one can think of a Minkowskian background on which small spacetime-dependent metrical fluctuations are imposed. In this context, it has been shown for quantum mechanical systems that such a scenario has the following consequences: (i) we obtain a modified inertial mass and, thus, an apparent violation of the weak equivalence principle [1–3]. (ii) Quantum systems will suffer decoherence induced by such quantum gravity-induced spacetime fluctuations which were discussed in [4–10]. In the context of the propagation of light (iii) fluctuating light cones and (iv) angular blurring were discussed in [11]. Since spacetime fluctuations can also lead effectively to non-localities they can manifest themselves as (v) modified dispersion relations [12–16]. (vi) Higher order time derivatives
can appear in the context of equations of motions of higher order \cite{17} leading to non-localities w.r.t. the time variable. It might be expected that these terms appear as a consequence of quantum gravity scenarios.

In this paper, we investigate a further consequence of the model worked out in \cite{3}. We consider the effect of spacetime fluctuations on the evolution of wavepackets and calculate their modified evolution in terms of the mean-squared displacement and higher order moments. The modification depends on the fluctuation scenario characterized by a spatial correlation function. Such kind of evolution equations have been discussed in \cite{18-20} in the context of quantum diffusion of particles in dynamical disordered continua. In this work, we will apply these methods to our model of spacetime fluctuations and show that also in our case one obtains modified wavepacket dynamics dependent on the properties of the stochastic model.

The paper is organized as follows. We start with an overview of a recently discussed model of spacetime fluctuations and generalize it by means of a non-minimal coupling scheme. From the modified Schrödinger equation, we will derive an effective master equation leading to modified wavepacket dynamics. Finally, we will analyze how different fluctuation scenarios influence the spreading of wavepackets.

2. Modified quantum dynamics

In our model, spacetime is regarded in the low energy limit as a fluctuating entity which consists of a classical fixed background on which Planck scale fluctuations are imposed. We assume that they appear as classical fluctuations of spacetime and model this as perturbations of the metric up to second order

\begin{equation}
  g_{\mu\nu}(x, t) = \eta_{\mu\nu} + h_{\mu\nu}(x, t),
\end{equation}

\begin{equation}
  g^{\mu\nu}(x, t) = \eta^{\mu\nu} - h^{\mu\nu}(x, t) + \tilde{h}^{\mu\nu}(x, t),
\end{equation}

where $|h_{\mu\nu}| \ll 1$. In this paper, Greek indices run from 0 to 3 and Latin indices from 1 to 3. The second-order perturbations are given by $\tilde{h}^{\mu\nu} = \eta_{\kappa\lambda}h^{\mu\kappa}h^{\nu\lambda}$, where indices are raised and lowered with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$.

We consider a scalar field $\phi(x)$ non-minimally coupled to gravity as given by the action

\begin{equation}
  S = \frac{1}{2} \int d^4x \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi(x)^* \partial_\nu \phi(x) - \left( \frac{m^2c^2}{\hbar^2} + \xi R(x) \right) \phi(x)^* \phi(x) \right),
\end{equation}

where $g = -\det g_{\mu\nu}$ is the determinant of the metric and $x = (x, t)$. Here $R(x)$ is the Ricci scalar and $\xi$ is a numerical factor. There are three values of $\xi$ which are of particular interest \cite{21}: (i) $\xi = 0$ is the minimally coupled case, (ii) $\xi = \frac{1}{6}$ is required by conformal coupling and can also be derived from the requirement that the equivalence principle is valid for the propagation of scalar waves in a curved spacetime \cite{22} and (iii) $\xi = \frac{1}{4}$ originates from the squaring the Dirac equation. The non-minimal coupling is introduced here for generality, but it will turn out that even only with minimal coupling the analysis will provide intriguing results.

Variation of the action w.r.t. $\phi(x)$ yields the Klein–Gordon equation non-minimally coupled to the gravitational field:

\begin{equation}
  g^{\mu\nu} D_\mu \partial_\nu \phi(x) - \left( \frac{m^2c^2}{\hbar^2} + \xi R(x) \right) \phi(x) = 0,
\end{equation}

where $D_\mu$ is the covariant derivative based on the metric $g_{\mu\nu}$.
2.1. Non-relativistic limit

In the next step, we calculate the non-relativistic approximation of this equation by expanding the wavefunction $\phi(x, t) = e^{iS(x, t)/\hbar}$ in powers of $c^2$

$$S(x, t) = S_0(x, t)c^2 + S_1(x, t) + S_2(x, t)c^{-2} + \cdots,$$

according to the scheme worked out by Kiefer and Singh [23]. To orders $c^4$ and $c^2$ in the series expansion of the Klein–Gordon equation, the results are identical to the case where minimal coupling is assumed [3]. For obtaining the Schrödinger equation without relativistic corrections in the case of non-minimally coupling, we give a sketch of the calculations which are done here analogously to [3]. First one has to derive the equation of motion to order $c^0$ and non-Hermitian terms appear in the Hamiltonian. These terms are non-Hermitian w.r.t. the standard scalar product. Indeed, one must either use the scalar product for curved backgrounds or alternatively keep the usual ‘flat’ Euclidean scalar product and transform the operators and the wavefunction accordingly [24]. We proceed with the latter possibility and finally arrive at

$$\hat{\mathcal{H}}_\psi \psi = -\frac{(\delta^{ij} + \tilde{\alpha}^{ij}(t))\partial_i\partial_j}{2m} - \frac{\hbar^2}{2m} R(x)\psi + \frac{m}{2}g^{00}\psi + \frac{1}{2}[\hat{\mathcal{H}}_\partial\partial, g^{(0)}\psi].$$

Here $\Delta_{\text{cov}}$ represents the Laplace–Beltrami operator, $[\cdot, \cdot]$ is the anticommutator, $(^3g)$ is the determinant of the 3-metric $g_{ij}$ and $g^{00}$ includes the Newtonian potential $U(x)$. The Hamiltonian now is manifest Hermitian w.r.t. the chosen ‘flat’ scalar product.

2.2. The effective Schrödinger equation

In [3] it is assumed that the particle described by the modified Schrödinger equation has its own finite spatial resolution scale. Consequently, only an averaged influence of spacetime fluctuations can be detected. This is quantified by means of the spatial average $\langle \cdot \cdot \cdot \rangle_V$ of the modified Hamiltonian over the particle scale leading to an effective Hamiltonian

$$H = -\frac{\hbar^2}{2m}((\delta^{ij} + \tilde{\alpha}^{ij}(t))\partial_i\partial_j) - mU(x),$$

where the tensorial function $\alpha^{ij}(t)$ consists of the spatial average of squares of the metrical fluctuations. The tensorial function $\alpha^{ij}(t)$ can be split into a time-average part and a fluctuating part

$$\alpha^{ij}(t) = \tilde{\alpha}^{ij} + \gamma^{ij}(t) \quad \text{with} \quad \langle \gamma^{ij}(t) \rangle_T = 0,$$

where $\langle \cdot \cdot \cdot \rangle_T$ is the temporal average.

It has been shown that the part $\tilde{\alpha}^{ij}$ leads to a renormalized inertial mass implying an apparent breakdown of the weak equivalence principle [3], whereas the fluctuating part leads to a decay of coherences in the energy representation [10]. If we include non-minimal coupling then a spatially averaged Ricci scalar $\langle R \rangle_V$ appears leading to an additional term in the perturbation Hamiltonian

$$H_p = -\frac{\hbar^2}{2m} (\gamma^{ij} \partial_i \partial_j + \xi \langle R \rangle_V),$$

which commutes with the density operator $\hat{\rho}$ of the effective master equation in [10]. Consequently, this extra term does not change the decoherence properties of quantum systems.

In the following sections, we will derive the modifications of the dynamics of wave packets as induced by the fluctuating spacetime metric. This requires an appropriate approximation of the kinetic term.
2.3. Approximation of the kinetic term

Now we consider the full Hamiltonian (6) and analyze the kinetic term which is composed of the Laplace–Beltrami operator and the fluctuating metric quantities. It leads to the expression

\[ \frac{(3g^{1/4}) \Delta_{\text{cov}} (3g^{-1/4})}{g_{ij}(3g^{1/4}) \partial_i \partial_j (3g^{-1/4})} = \frac{1}{2} \partial_i \ln \sqrt{(3g)} \partial_j \ln \sqrt{(3g)} \]

\[ - \frac{1}{2} \partial_i g^{ij} \partial_j \ln \sqrt{(3g)} + g^{ij} \partial_i \partial_j + g^{ij} \partial_i \partial_j \]  

(10)

All contributions of the kinetic part as well as those terms in the anticommutator of equation (6) which do not include derivatives acting on the wavefunction give a modified stochastic scalar interaction term

\[ V(x, t) = -\frac{\hbar^2}{2m} (\xi R + g^{ij}((3g^{1/4}) \partial_i \partial_j (3g^{-1/4}))) \]

\[ - \frac{\hbar^2}{2m} \left( - \frac{1}{2} \partial_i \ln \sqrt{(3g)} \partial_j \ln \sqrt{(3g)} - \frac{1}{2} \partial_i g^{ij} \partial_j \ln \sqrt{(3g)} \right) + \frac{i\hbar}{2} \partial_i g^{0} \]  

(11)

appearing in the Schrödinger equation

\[ i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} (\xi R + g^{ij} \partial_i \partial_j + \partial_i g^{ij} \partial_j) \psi + i\hbar g^{ij} \partial_i \psi + V(x, t) \psi, \]

(12)

where for simplicity the Newtonian potential \( g^{0} \) is not included henceforth.

In the following, it will be shown that derivatives of fluctuating quantities dominate, provided that the Newtonian potential is a slowly varying function which can be regarded as a constant over the typical fluctuation scales. Therefore, our conclusions will not be affected by taking a Newtonian gravitational field into account.

Now we will employ some approximations which allow us to use established methods for the calculation of wavepacket dynamics. We assume that spacetime fluctuations act on a spacetime scale which is much smaller than typical lengthscales and timescales on which the wavefunction \( \psi(x, t) \) varies. Therefore, the amplitude of the terms containing derivatives of the fluctuating quantities (\( h^{ij} \) and the trace \( h \)) dominates in the Schrödinger equation (12) compared with the magnitude of the other fluctuating quantities. This allows us to state that the main contributions from spacetime noise stem from terms containing products of first derivatives of the fluctuation quantities. In short notation

\[ |\partial h \partial h| > (h, \bar{h}) \partial \partial, \quad |\partial h \partial h| > \hbar \partial \partial \quad \text{and} \quad |\partial h \partial h| > \partial \bar{h} \partial, \]

(13)

where for simplicity we omitted indices. Therefore, we can neglect the terms (\( \bar{h}^{ij} - h^{ij}) \partial_i \partial_j, \partial_i g^{ij} \partial_j \), and \( \partial_i g^{0} \). In addition, we note that in the following derivation of the averaged master equation only second moments of \( V \) will appear, see equation (22). This would lead to third- and fourth-order terms if we incorporate second-order fluctuation terms (third-order terms vanish in the average because of the Gaussian property which we will specify later). Therefore, we can neglect second-order terms already on the level of the Schrödinger equation leading to

\[ i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + V(x, t) \psi, \]

(14)

where \( \Delta \) is the Laplace operator in Cartesian coordinates and where the random scalar interaction (11) now reads

\[ V(x, t) = -\frac{\hbar^2}{2m} (\xi R + \delta^{ij}(3g^{1/4}) \partial_i \partial_j (3g^{-1/4})). \]

(15)

To first order in the perturbation metric this reduces to

\[ V(x, t) = -\frac{\hbar^2}{2m} \left( \xi R - \frac{1}{4} \Delta h \right), \]

(16)
where $h$ is the first-order spatial part of the trace $\text{tr}(g_{\mu\nu})$. To first order, the Ricci scalar is given by

$$R(x, t) = \partial_{\mu}\partial_{\nu}h^{\mu\nu}(x, t) - \Delta h(x, t).$$

(17)

Note that only spatial components are present which is an effect of the non-relativistic approximation (5). Compared to the first term in the fluctuating potential (16), the non-minimal coupling term $\xi R$ has the same structure and the same order of magnitude. Thus, this contribution to the Schrödinger Hamiltonian and to the analysis of spacetime fluctuations cannot be neglected in the context of our model and must be taken into consideration in the following calculations.

2.4. Stochastic properties

Now we have to specify the statistical properties of the fluctuating potential $V(x, t)$ as we like to derive an effective equation. Owing to the fluctuating metric, this term will be interpreted as a Gaussian random function $V(x, t)$ dependent on the quantities $h_{\mu\nu}(x, t)$. The main reason why we chose Gaussian fluctuations is that we want to calculate the main part of the modifications of quantum dynamics. This is given by Gaussian fluctuations while non-Gaussian fluctuations, which are certainly worth being studied, would induce higher order corrections to the main part. Furthermore, Gaussian terms are easier to handle—a fact that we will take advantage of when applying the Novikov theorem.

We will now continue the calculations for the one-dimensional case, as in the following section the effective master equation is derived for this specific case, and from now on the spatial coordinate is denoted by $x$. This yields for the fluctuating potential

$$V(x, t) = -\frac{\hbar^2}{2m} \left( \xi \partial_{x}^2 h_{11}(x, t) - \partial_{x}^2 h_{11}(x, t) \left( \xi + \frac{1}{4} \right) \right).$$

(18)

$$= \frac{\hbar^2}{8m} \partial_{x}^2 h_{11}(x, t).$$

(19)

As the quantities of interest are the metric perturbation terms, we choose the following statistical conditions:

$$\langle h_{11}(x, t) \rangle = 0$$

(20)

$$C(x, x', t, t') \equiv \langle h_{11}(x, t) h_{11}(x', t') \rangle = h_0^2 \delta(t - t')C(x - x'),$$

(21)

where $h_0$ is the strength of the fluctuations of $h(x, t)$ and $\langle \cdots \rangle$ symbolizes the average over the fluctuations. It has been shown by Heinrichs [18, 19] that the dynamics of wavepackets are unaffected by temporal correlations of the random function $h(x, t)$ if they are sufficiently small. This also applies to our model of spacetime fluctuations because it is commonly believed that quantum gravitational induced fluctuations appear on typical scales given by the Planck time $\tau_p$ and Planck length $l_p$. Therefore, we have chosen a $\delta$-correlation w.r.t. time. However, at the moment the spatial correlation function $C(x - x')$ will be left unspecified. For the following calculations, it is convenient to introduce the correlation function

$$\langle V(x, t) V(x', t') \rangle = V_0^2 \delta(t - t')g(x - x'),$$

(22)

where $g(x - x') = \partial_{x}^2 \partial_{x'}^2 C(x - x')$ and we defined $V_0 = \frac{\hbar^2}{8m} h_0$.

As this is a model for spacetime fluctuations and because of the lack of a complete understanding of the microscopic structure of spacetime one has the freedom to characterize the statistical properties of the terms $h$ with a variety of Stochastic models given here by the correlator $C(x - x')$.

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3. Dynamics of the density matrix

3.1. Effective master equation

Most physical quantities of interest—in our case the mean square displacement—can be conveniently calculated in terms of the reduced density matrix $\langle \rho(x', x, t) \rangle$ and its evolution equation. For the sake of simplicity, we restrict all the calculations to the one-dimensional case. In the following, we pursue closely the strategy and calculation techniques realized in [20, 25], which lead to an exact solution of the master equation and thus for the mean-squared displacement of a wavepacket. In addition to this, we clarify some derivations and assumptions made in [20] and apply the results to our model of spacetime fluctuations.

We define the density operator

$$\rho(x', x, t) = \psi^*(x', t) \psi(x, t)$$

(23)

and set up the master equation according to

$$\frac{\partial_t \rho(x', x, t)}{\partial t} = \frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho(x', x, t) + i\hbar (V(x', t) - V(x, t)) \rho(x', x, t).$$

(24)

The formal solution of this master equation reads

$$\rho(x', x, t) = \rho(x', x, t = 0) + \frac{i\hbar}{2m} \int_0^t \! dt' \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho(x', x, t')$$

$$+ \frac{i}{\hbar} \int_0^t \! dt' (V(x', t') - V(x, t')) \rho(x', x, t').$$

(25)

In the next step, we calculate the effective equation of motion by applying the average over the fluctuations. Special care must be taken for the terms which are products of the fluctuating potential $V(x, t)$ and the density operator. This is due to the fact that the average of the product $\langle V(x, t) \rho(x', x, t) \rangle$ generally does not factorize because the density operator $\rho(x', x, t)$ is a functional of the fluctuation potential $V(x, t)$, hence $\rho[V]$.

We can handle this by using the fact that we have chosen Gaussian fluctuations (see the statements preceding equations (20) and (21)). A Gaussian process is characterized in a functional form by the Novikov theorem [26] valid for processes obeying $\langle V(x, t) \rangle = 0$ (here: $\partial^2_x [h_{11}(x, t)] = 0$), which yields in our case

$$\langle V(x, t) \rho(x', x, t) \rangle = \int \! dx'' \int \! dt'' \delta(t - t'') g(x - x'') \left\{ \frac{\delta \rho(x', x, t)}{\delta V(x', t'')} \right\},$$

(27)

where $g$ is the correlation function introduced in equation (22).

The Novikov theorem follows from a functional Taylor series expansion of $\rho[V]$ and utilizing the Gaussian property of $h_{11}(x, t)$ which leads to a factorization of moments [27].

The functional derivative can be calculated using the formal solution of the master equation (26) and yields

$$\left\{ \frac{\delta \rho(x', x, t)}{\delta V(x'', t'')} \right\} \equiv \frac{i}{\hbar} \int_0^t \! dt' \rho(x', x, t') \delta(x - x'') \delta(t' - t'') - \delta(x' - x'') \delta(t' - t'').$$

(28)

The Novikov theorem states that

$$\left\{ \frac{\delta \rho(x', x, t)}{\delta V(x'', t'')} \right\} = \frac{i}{2\hbar} \rho(x', x, t') \delta(x - x'') \delta(t' - t'').$$

(29)
This result used in (27) yields
\[ \langle V(x', t) \rho(x', x, t) \rangle = \frac{iV_0^2}{2\hbar} \langle \rho(x', x, t) \rangle (g(0) - g(x - x')) \]
and analogously, from equation (28),
\[ \langle V(x', t) \rho(x', x, t) \rangle = \frac{iV_0^2}{2\hbar} \langle \rho(x', x, t) \rangle (g(0) - g(x - x')). \]

Combining both results and inserting this into the averaged equation (25), we obtain the effective master equation for the reduced density matrix
\[ \partial_t \langle \rho(x', x, t) \rangle = -\frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x^2} \right) \langle \rho(x', x, t) \rangle + \frac{V_0^2}{\hbar} (g(0) - g(x - x')) \]

3.2. Solution of the effective master equation

In order to get rid of the time dependence, we perform a Laplace transformation according to
\[ r(x', x, s) = \mathcal{L}[\langle \rho(x', x, t) \rangle] = \int_0^\infty dt \langle \rho(x', x, t) \rangle e^{-st}, \]
where the Laplace variable \( s \) satisfies \( \text{Re}(s) > 0 \). The time derivative becomes
\[ \mathcal{L}[\partial_t \langle \rho(x', x, t) \rangle] = s \mathcal{L}[\langle \rho(x', x, t) \rangle] - \langle \rho(x', x, t = 0) \rangle \]
and the Laplace transformed master equation reads
\[ \frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x^2} \right) \mathcal{L}[\langle \rho(x', x, t) \rangle] + \left( s + \frac{V_0^2}{\hbar} (g(0) - g(x - x')) \right) \mathcal{L}[\langle \rho(x', x, t) \rangle] = \mathcal{L}[\langle \rho(x', x, t = 0) \rangle], \]
where \( \langle \rho(x', x, t = 0) \rangle \) is subject to initial conditions. For practical reasons, we introduce new coordinates according to
\[ X = x + x' \quad \text{and} \quad Y = x - x' \]
which yields
\[ \frac{2i\hbar}{m} \frac{\partial^2}{\partial X \partial Y} r(X, Y, s) + \left( s + \frac{V_0^2}{\hbar} (g(0) - g(Y)) \right) r(X, Y, s) = \langle \rho(X, Y, t = 0) \rangle. \]

This is the partial differential equation (PDE) which has to be solved.

By using the averaging techniques to derive the effective equation (36), we converted the stochastic PDE (25) to an ‘ordinary’ PDE. Note that the information about the fluctuation scenario is encoded in the correlation function \( g(Y) \) and the amplitude \( V_0 \).

We rewrite the master equation (36) as an ordinary, inhomogeneous first-order differential equation by applying a Fourier transformation w.r.t. the variable \( X \):
\[ (\mathcal{F}_X r)(K, Y, s) = R(K, Y, s) = \int_{-\infty}^{\infty} dX e^{-iKX} r(X, Y, s). \]
This gives
\[ \partial_y R(K, Y, s) = -\frac{m}{2\hbar K} \left( s + \frac{V_0^2}{\hbar^2} (g(0) - g(Y)) \right) R(K, Y, s) = -\frac{m}{\hbar K} R(K, Y, t = 0), \]
where \( R(K, Y, t = 0) = (\mathcal{F}_X \langle \rho \rangle)(K, Y, t = 0) \) is the Fourier transform of the inhomogeneous part of the ordinary differential equation.

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This equation can easily be integrated and yields
\[
R(K, Y, s) = \exp \left( \frac{m s Y}{2 \hbar K} + G(Y) \right) R(K, Y_0, s) - \exp \left( \frac{m s Y}{2 \hbar K} + G(Y) \right) \times \frac{m}{\hbar K} \int_0^Y d^\prime \exp \left( - \frac{m s Y}{2 \hbar K} \right) R(K, Y', t = 0) \exp \left( - \frac{m G(Y')}{2 \hbar K} \right),
\]
where the term \( R(K, Y_0, s) = R(K, Y = 0, s) \) represents the initial value of the solution \( R(K, Y, s) \), and we define \( G(Y) = \frac{V_0}{2} \int_0^Y d^\prime (g(0) - g(Y')) \).

3. Mean-squared displacement

The mean-squared displacement of the particle is given by
\[
\sigma^2(t) = \int_{-\infty}^{\infty} dx x^2 \lim_{K \to 0} \frac{\partial^2}{\partial K^2} R(K, Y_0, t)
\]
and can be expressed in Fourier space as
\[
\sigma^2(t) = -\frac{1}{8} \frac{\partial^2}{\partial K^2} R(K, Y_0, t) \bigg|_{K=0}.
\]
Therefore, the quantity of interest which has to be calculated is the initial value of the solution \( R(K, Y, s) \), and we define \( G(Y) = \frac{V_0}{2} \int_0^Y d^\prime (g(0) - g(Y')) \).

The solution \( R(K, Y, s) \) is a quadratically integrable function and must vanish for \( Y \to \infty \).

Therefore, the rhs of equation (39) must be zero in this limit, leading to
\[
R(K, Y_0, s) = \frac{m}{\hbar K} \int_0^\infty d^\prime \exp \left( - \frac{m s Y}{2 \hbar K} \right) R(K, Y', t = 0) \exp \left( - \frac{m G(Y')}{2 \hbar K} \right).
\]
We make the substitution \( \tau = \frac{m}{h K} Y \) (the dimension of this quantity is in fact that of time) and note that the rhs represents a Laplace transformation
\[
R(K, Y_0, s) = 2 \int_0^\infty d^\prime \exp \left( -s \tau \right) R \left( K, \frac{2 h K}{m} \tau, t = 0 \right) \exp \left( -G(\tau) \right),
\]
if \( \tau \) is positive giving \( K \to |K| \). This equation can be inverted leading to the solution
\[
R(K, Y_0, \tau) = 2 R \left( K, \frac{2 h K}{m} \tau, t = 0 \right) \exp \left( - \frac{V_0}{2 h^2} \int_0^\tau d^\prime \left[ g(0) - g \left( \frac{2 h K}{m} \right) \right] \right) \bigg|_{K=0}.
\]
Inserting this expression into equation (41), we get for the mean-squared displacement
\[
\sigma^2(\tau) = \frac{1}{4} \exp \left( -G(Y) \right) R_0(Y) \left( \frac{\partial^2}{\partial t^2} G(Y) - (\partial_t G(Y))^2 + 2 \partial_t G(Y) \partial_t \ln R_0(Y) - \partial^2 \ln R_0(Y) - (\partial_t \ln R_0(Y))^2 \right) \bigg|_{K=0},
\]
where \( Y = \frac{2 h K}{m} \tau \) and \( R_0(Y) = R \left( K, \frac{2 h K}{m} \tau, t = 0 \right) \).

Note that the unperturbed solution is given by the last two logarithmic terms, where for \( K = 0 \) we get \( \exp \left( -G(Y) \right) R_0(Y) = 1 \), if the initial state \( G_0 \) is given by a Gaussian wavepacket. The information about the fluctuation scenario is given by the correlation function appearing in \( G(Y) \). In our case, we regard even correlation functions \( G(Y) = G(-Y) \) excluding a preferred direction in space. For our purposes, it is assumed to be an exponential function having the form \( G(-Y^{2n+1}) \), for \( n \in \mathbb{N} \) avoiding the introduction of the modulus of the argument \( Y \) as the correlation function must be differentiable and analytic at the point \( Y = 0 \). In this case, the first derivative \( \partial_t G(Y) \) vanishes in the limit \( K = 0 \), therefore modifications to the free (unperturbed) mean-squared displacement come from the second derivative \( \partial^2_t G(Y) \).

\[3\] We assume that the frame of reference (lab frame) is comoving with the center of mass of the wavepacket and in the case of exponential functions exhibiting the form \( G(-Y^{2n+1}) \) this would lead to a directional dependent asymmetric spreading of wavepackets.
3.4. Gaussian correlator

With this result (45), we can now calculate the mean-squared displacement of a wavepacket. In the preceding discussion, we imposed several restrictions and assumptions for the argument of the correlation function \( C(Y) \). For this reason and as a first starting point, we choose a Gaussian correlator

\[
C(Y) = \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{Y^2}{a^2}\right),
\]

(46)

where \( a \) represents a finite correlation length, and we choose for the initial conditions a Gaussian wavepacket

\[
R(K, Y, t = 0) = \mathcal{F}_X \left[ \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{X^2}{8\sigma^2}\right) \right]
\]

\[
= 2 \exp\left(-\left(\frac{Y^2}{8\sigma^2} + 2K^2\sigma^2\right)\right).
\]

(47)

We set \( Y = 2\hbar|K|\tau \) and the mean-squared displacement yields

\[
\sigma_x^2(t) = \sigma_x^2(0) + \frac{\hbar^2}{4m^2\sigma^2_x(0)}t^2 + \frac{5V_0^2}{\sqrt{2\pi m^2a^3}}t^3.
\]

(49)

The first two terms correspond to the expression that is obtained from the free Schrödinger equation, whereas the last term accounts for superdiffusive\(^4\) behavior. The asymptotic behavior for \( t \to \infty \) is dominated by the \( t^3 \) dependence

\[
\sigma_x^2(t) = \sigma_x^2(0) + \frac{5V_0^2}{\sqrt{2\pi m^2a^3}}t^3, \quad \text{for} \quad t \to \infty.
\]

(50)

We can state that for long evolution times the wavepacket undergoes superdiffusion if the spatial correlation properties of spacetime fluctuations are given by (46). The effect of this special model of spacetime fluctuations is such that the evolution of quantum particles is superdiffusive.

The inclusion of a white-noise scenario by setting \( C(x - x') = \delta(x - x') \) is not feasible in this framework. The reason is that our equations are only valid for analytical correlation functions (analytic at \( K = 0 \)) but singular quantities like \( \delta(0) \) would appear in the expressions which were derived here. These expressions would render infinite momentum contributions which must be limited by a momentum cutoff. In our framework, this is motivated by a fundamental minimal lengthscale which is given by the Planck length, since we consider possible effects emerging from quantum gravity. This leads to the desired momentum cutoff and hence the Dirac \( \delta \)-function is not the appropriate choice in this context.

However, a modified behavior for wavepackets was already sketched in [10] where a white-noise scenario for the fluctuating phase of the wavefunctions was considered yielding

\[
\sigma_x^2(t) = \sigma_x^2(0) + \frac{\sigma_{px}(0)}{m}t + \frac{\sigma_{p}^2}{m^2}t^2 + \frac{\sigma_{p}^2}{m^2}t \tau_p t.
\]

(51)

The quantities appearing here are given by \( \sigma_{px}(t) = \langle px + xp \rangle_t - 2\langle p \rangle_t \langle x \rangle_t \) and \( \sigma_{p}^2(t) = \langle p^2 \rangle_t - \langle p \rangle^2_t \) which is constant in time. The constant \( \tau_p \) appearing in the dissipative term is the Planck time and given by the amplitude of the white-noise scenario, whereas the average is defined by the trace over the effective density operator \( \langle A \rangle_t = \text{tr}(A(\rho(t))) \). However, for

\(^4\) The nomenclature is as follows. For the mean-squared displacement, \( \langle x^2 \rangle \propto t^\nu \) the exponent \( \nu = 1 \) renders diffusive motion, \( \nu = 2 \) is a ballistic motion and \( \nu = 3 \) denotes superdiffusion.
large times $t \gg \tau_p$, the quadratically increasing ballistic term dominates and the dissipative linear term can be neglected, thus reproducing the behavior of a wavepacket described by a free Schrödinger equation.

Therefore, the result (50) derived here by introducing a finite correlation length (Gaussian correlator) represents a new effect emerging from our model for spacetime fluctuations.

3.5. Other correlators and higher order moments

In general, it is not possible to distinguish different fluctuation scenarios on the level of wavepacket dynamics only by the second moment (41). This does not suffice to fully characterize the modified wavepacket evolution what we demonstrate here by selecting the correlation function

$$C(Y) = C_4 \exp \left( -\frac{Y^4}{4a^4} \right),$$

where $C_4$ is a normalization factor. We arrive at

$$\sigma_x^2(t) = \sigma_x^2(0) + \frac{\hbar^2 t^2}{4m^2\sigma_x^2(0)},$$

showing no superdiffusion and no deviation from the usual wavepacket dynamics.

Considering the case $C(Y) = C_6 \exp \left( -\frac{Y^6}{6a^6} \right)$ leads us to

$$\sigma_x^2(t) = \sigma_x^2(0) + \frac{\hbar^2}{4m^2\sigma_x^2(0)} t^2 + \frac{45V_0^2\Gamma(\frac{5}{6})}{\sqrt{2}\pi m^2 a^7} t^3,$$

where $\Gamma$ is the Gamma function. This expression has the same structure and asymptotic behavior like equation (49).

Generalizing these results to correlation functions $C(Y) = C_n \exp \left( -\left( \frac{Y}{\sqrt{2a}} \right)^{2n} \right)$ [20], one yields again expression (53) for all $n \in \mathbb{N}_+ \setminus \{3\}$. In order to deal with this fact, one must consider higher order moments $\langle x^{2n}(t) \rangle$ with $n \in \mathbb{N}$ and $n > 1$.

For correlation functions having exponents of order $2n$, one must include moments up to the order $2n$ to fully characterize the response of the wavepacket to spacetime fluctuations, where the higher order moments are given by

$$\langle x^{2n}(t) \rangle \propto \frac{\partial^{2n}}{\partial K^{2n}} R(K, Y_0, \tau) |_{K=0}. \quad (55)$$

For example, we get for the correlator (52) the asymptotic ($t \to \infty$) fourth moment

$$\langle x^4(t) \rangle \approx 96\sigma_x^4(0) + \frac{V_0^2\hbar^2}{\pi m^2 a^5} t^5. \quad (56)$$

Thus, in order to be able to distinguish between different Stochastic scenarios for spacetime fluctuations, one must characterize the wavepacket spreading by the inclusion of higher order moments.

3.6. Connection to holographic noise

As a side remark, we want to argue whether and in which way holographic noise would affect the wavepacket propagation. Holographic noise is currently gaining attention because it may serve as a possible explanation for a so-called ‘mystery noise’ at the gravitational wave detector GEO600 [28]. In that framework holographic noise leads to a square-root dependence $\sqrt{\tau_p}$ of the displacement of the beam splitter in the gravitational wave detector.
displacement noise with a similar behavior has been discussed in [29] where the square-root noise scenario is a special case of general displacement noise spectra taken into account as a possible effect of spacetime fluctuations. For our case, we adopt the holographic noise scenario as discussed by Ng and coworkers in [30] and [31]. In contrast to the scenario of Hogan [28], this leads to a different scaling law for the displacement noise but still exhibits an amplification of spacetime noise compared with simple estimates leading to a linear scaling of the measurement uncertainty with the Planck length.

The holographic principle states that the number of degrees of freedom of a region of spacetime is bounded by the area of the region in Planck units $l_p^2$. Alternatively, we can state that although the world appears to have three spatial dimensions, its contents can be encoded on a two-dimensional surface, like a hologram. According to this principle length, measurements of a distance $l$ exhibit an intrinsic uncertainty given by
\[
\delta l \geq (l l_p^2)^{1/3}
\]
where the exponent $2/3$ for the Planck length $l_p$ is characteristic for holographic noise. It can be shown [30, 31] that this translates to fluctuations of the metric
\[
\delta g_{\mu\nu} \geq (l_p/l)^{2/3} a_{\mu\nu},
\]
where $a_{\mu\nu}$ is a tensor of order $O(1)$. This corresponds to a power spectral density (PSD) of
\[
S(f) = f^{-5/6} (c l_p^2)^{1/3},
\]
where $f$ is the frequency and $c$ is the speed of light. We write this in terms of a PSD in momentum space obtaining
\[
S(k) = k^{-5/6} l_p^{2/3},
\]
where $k$ is the wavenumber. It can be readily checked that this expression leads to $\delta l = (l l_p^2)^{1/3}$ by calculating $(\delta l)^2 = \int_{k_{\text{max}}} d\mathbf{k} (S(k))^2$, where $k_{\text{max}}$ is subject to a cut-off given by the Planck length $l_p$ and the length $l$ is given by the experiment. This shows that—by means of a correlation function—a finite, non-vanishing correlation length is involved where the correlator scales with $l_p^{2/3}$ modifying the wavepacket evolution accordingly. The $l_p^{2/3}$ dependence distinguishes the holographic noise scenario from the other fluctuations scenarios discussed in the previous sections.

3.7. Experimental suggestions

For the experimental observation of the predicted effects on wavepacket dynamics, it is necessary that long evolution times of quantum particles can be achieved. So far, it seems that no dedicated experiment concerning the spreading of quantum mechanical wavepackets has been performed. We suggest to test this effect. In this context, it is feasible that the gravitational field of earth is suppressed as best as possible to facilitate the free evolution of wavepackets. This goal can be reached by performing experiments in microgravity environments which is realized for example by the QUANTUS [32] and PRIMUS projects. The first already demonstrated that a Bose–Einstein condensate (BEC) in free fall can be realized with propagation times of about 1 s. In the PRIMUS project, BECs will be used to realize a matter wave interferometer under microgravity conditions. We suggest that in both projects cold atom gases could in principle be used to study the long time behavior of wavepackets with high sensitivity.

5 Here the Planck length is assumed to be the fundamental minimal length.
4. Summary and conclusion

We have calculated the mean-squared displacement for a quantum particle subject to spacetime fluctuations. As we have seen, the properties of wavepacket dynamics are strongly dependent on the spatial correlation function. For two special cases, we encounter superdiffusion of the wavepacket for the second moment on large timescales (characterized by the exponent of the time variable). For other correlators, the second moment is unaffected, and the unperturbed expression is recovered. Therefore, higher order moments must be included in the analysis showing modifications of the unperturbed case. This leads also to a superdiffusive behavior and—in addition—allows us to distinguish different stochastic models experimentally. For a white-noise scenario, we encounter a diffusive behavior for short timescales. On large timescales, compared to the typical evolution time of a wavepacket, the unperturbed case is recovered as the ballistic term dominates. Furthermore, we discussed the influence of holographic noise on the wavepacket evolution and argued that it should be dependent on the Planck length appearing with the exponent 2/3.

Therefore, interesting effects representing deviations from the usual wavepacket dynamics can only be obtained by inclusion of general correlation functions having finite correlation lengths. In our approach, the modifications on wavepacket dynamics depend only on the statistical fluctuation properties and the magnitude, whereas the latter is determined by the fluctuation strength independently of the statistical model.

Furthermore, we observe that in our model the strength of the modifications to the free Schrödinger equation for the minimally and the non-minimally coupled scenario have the same order of magnitude. However, the results for the one-dimensional case are independent of the non-minimal coupling term. For the three-dimensional case, we expect modifications coming from the non-minimally coupling terms but they will not affect our conclusions significantly. Since the structure of the equations remains the same and the form of the correlation functions are identical to those in one dimension, we should not encounter major differences for the expressions for the wavepacket dynamics. The magnitude and the temporal behavior of the modifications should remain the same. Nevertheless, possible effects from anisotropic spacetime fluctuations cannot be described in the one-dimensional case, of course, and therefore our conclusions do not include this possibility.

We conclude that a possible experimental verification of our model of spacetime foam is feasible by means of experiments with cold atoms with long propagation times—preferably in a microgravity environment. This gives the opportunity of setting bounds for different scenarios of spacetime fluctuations.

Acknowledgments

We would like to thank Ž. Marojević for fruitful discussions. All authors thank the German–Mexican DFG–CONACyT cooperative grants. EG gratefully acknowledges the support by the German Research Foundation (DFG) and the Centre for Quantum Engineering and SpaceTime Research (QUEST), and CL the support by the German Aerospace Center (DLR) grant no. 50WM0534.

References

[1] Jaekel M-T and Reynaud S 1994 Phys. Lett. A 185 143
[2] Camacho A 2003 Gen. Rel. Grav. 35 319
[3] Göklu E and Lämmerzahl C 2008 Class. Quantum Grav. 25 105012
[4] Karolyhazy F 1966 Il Nuovo Cimento A, 42 390
[5] Jack Ng Y and van Dam H 1994 Mod. Phys. Lett. A 9 335
[6] Amelino-Camelia G 1994 Mod. Phys. Lett. A 9 3415
[7] Power W L and Percival L C 2000 Proc. R. Soc. Lond. A 456 955–68
[8] Wang C H-T, Bingham R and Mendonca J-T 2006 Class. Quantum Grav. 23 L59–65
[9] Bonifacio Paolo M, Wang C H-T, Mendonca J-T and Bingham R 2009 Class. Quantum Grav. 26 145013
[10] Breuer H-P, Gökülü E and Lämmerzahl C 2009 Class. Quantum Grav. 26 105012
[11] Ford L H 2005 Int. J. Theor. Phys. 44 175368
[12] Bernadotte S and Klinkhamer F R 2007 Phys. Rev. D 75 024028
[13] Amelino-Camelia G, Ellis J, Mavromatos N E, Nanopoulos D V and Sarkar S 1998 Nature 393 763
[14] Amelino-Camelia G 2000 Phys. Rev. D 62 024015
[15] Alfaro J, Morales-Tecotl H A and Urrutia L F 2000 Phys. Rev. Lett. 84 2318
[16] Camacho A 2002 Gen. Rel. Grav. 34 1839
[17] Lämmerzahl C and Rademaker P 2009 arXiv:0904.4779v1
[18] Heinrichs J 1992 Z. Phys. B 89 115–21
[19] Heinrichs J 1996 Z. Phys. B 100 327–8
[20] Jayannavar A M 1993 Phys. Rev. E 48 837
[21] Birell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press)
[22] Sonego S and Faraoni V 1993 Class. Quantum Grav. 10 1185–7
[23] Kiefer C and Singh T P 1991 Phys. Rev. D 44 1067
[24] Lämmerzahl C 1995 Phys. Lett. A 203 12–7
[25] Jayannavar A M 1982 Phys. Rev. Lett. 48 553
[26] Novikov E A 1964 Zh. Eksp. Teor. Fiz. 47 1919
Novikov E A 1965 Sov. Phys.—JETP 20 1990
[27] Sancho J M, San Miguel M, Katz S L and Gunton J D 1982 Phys. Rev. A 26 1589
[28] Hogan C J 2009 arXiv:0905.4803v4 [gr-qc]
[29] Amelino-Camelia G 1999 Nature 398 216–8
[30] Jack Ng Y 2002 Int. J. Mod. Phys. D 11 1585–90
[31] Jack Ng Y 2003 Mod. Phys. Lett. A 18 1073–98
[32] Vogel A et al 2006 Appl. Phys. B 84 663