Rigidity Theorem by the Minimal Point of the Bergman Kernel

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Abstract
We use the Suita conjecture (now a theorem) to prove that for any domain $\Omega \subset \mathbb{C}$ its Bergman kernel $K(\cdot, \cdot)$ satisfies $K(z_0, z_0) = \text{Volume}(\Omega)^{-1}$ for some $z_0 \in \Omega$ if and only if $\Omega$ is either a disk minus a (possibly empty) closed polar set or $\mathbb{C}$ minus a (possibly empty) closed polar set. When $\Omega$ is bounded with $C^\infty$-boundary, we provide a simple proof of this using the zero set of the Szegö kernel. Finally, we show that this theorem fails to hold in $\mathbb{C}^n$ for $n > 1$ by constructing a bounded complete Reinhardt domain (with algebraic boundary) which is strongly convex and not biholomorphic to the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

Keywords Bergman kernel · Minimal domain · Suita conjecture · Szegö kernel

Mathematics Subject Classification Primary 30C40; Secondary 30C35 · 30C85 · 30C20

1 Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$ and denote the Bergman space of $\Omega$ by $A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$ where $\mathcal{O}(\Omega)$ is the set of holomorphic functions on $\Omega$. The Bergman space is a separable Hilbert space under the $L^2$-inner product with Lebesgue volume measure. If $\{\phi_j\}_{j=0}^\infty$ is a complete orthonormal basis of $A^2(\Omega)$, then the Bergman kernel function $K : \Omega \times \Omega \to \mathbb{C}$ defined by
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\[ K(z, w) = \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)} \]  

(1.1)

is the unique function on \( \Omega \times \Omega \) which satisfies the properties

1. For all \( w \in \Omega \), \( K(., w) \in A^2(\Omega) \)
2. \( K(z, w) = \overline{K(w, z)} \)
3. For all \( f \in A^2(\Omega) \) and \( z \in \Omega \),
\[
f(z) = \int_{\Omega} f(w)K(z, w)dv(w).
\]

(1.2)

We note that the definition of \( K(z, w) \) is independent of the particular orthonormal basis chosen. We shall use the notation \( K_\Omega \) for the Bergman kernel of \( \Omega \) instead of \( K \) when we wish to emphasize that \( \Omega \) is the domain under consideration. If \( f : \Omega_1 \rightarrow \Omega_2 \) is a biholomorphic map, then the Bergman kernels for the respective domains are related by the transformation law of the Bergman kernel:

\[
K_{\Omega_1}(z, w) = f'(z)K_{\Omega_2}(f(z), f(w))f'(w).
\]

For further background on the Bergman kernel, we refer the readers to Krantz’s book [14].

If \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) and \( v \) is the Lebesgue \( \mathbb{R}^{2n} \)-measure, then \( v(\Omega)^{-1/2} \in A^2(\Omega) \) and \( \|v(\Omega)^{-1/2}\|_{L^2} = 1 \). Hence, if \( \{v(\Omega)^{-1/2}\} \cup \{\phi_j\}_{j=1}^{\infty} \) is a complete orthonormal basis for \( A^2(\Omega) \), then by (1.1),

\[
K(z, z) \geq \frac{1}{v(\Omega)}, \quad z \in \Omega.
\]

(1.3)

Equality in (1.3) is achieved for the unit ball \( \mathbb{B}^n \subset \mathbb{C}^n \), \( n \geq 1 \) with \( z = 0 \) because \( K_{\mathbb{B}^n}(z, w) = v(\mathbb{B}^n)^{-1}(1 - \overline{z}w)^{-n-1} \).

In this paper, we completely classify the domains in \( \mathbb{C} \) for which equality in (1.3) holds at some (minimal) point in the domain.

**Theorem 1** Let \( \Omega \subset \mathbb{C} \) be a domain. Suppose there exists a \( z_0 \in \Omega \) such that

\[
K(z_0, z_0) = \frac{1}{v(\Omega)},
\]

(1.4)

where we use the convention \( v(\Omega)^{-1} = 0 \) if \( v(\Omega) = \infty \).

(i) If \( v(\Omega) = \infty \), then \( \Omega = \mathbb{C} \setminus P \) where \( P \) is a possibly empty, closed polar set.
(ii) If \( v(\Omega) < \infty \), then \( \Omega = D(z_0, r) \setminus P \) where \( P \) is a possibly empty, polar set closed in the relative topology of \( D(z_0, r) \) with \( r = \sqrt{v(\Omega)\pi^{-1}} \).

We remark that
1. A set $P$ is said to be polar if there is a subharmonic function $u \neq -\infty$ on $\mathbb{C}$ such that $P \subset \{ z \in \mathbb{C} : u(z) = -\infty \}$. If $P$ is a closed polar subset of a domain $\Omega$, then $A^2(\Omega \setminus P) = A^2(\Omega)$ by [19]. It follows that if $\Omega_1 = D(z_0, r) \setminus P_1$ and $\Omega_2 = \mathbb{C} \setminus P_2$ where $P_i, i = 1, 2$, are relatively closed polar sets, then $K_{\Omega_i}$ extends to $\Omega_i \times \Omega_i$ and

$$K_{\Omega_1}(z_0, z_0) = \frac{1}{v(\Omega_1)}, \quad K_{\Omega_2}(0, 0) = 0.$$

2. Compact subsets of polar sets are totally disconnected. So the polar set $P$ will be empty if, for instance, the boundary of $\Omega$ is parametrized by non-trivial simple closed curves.

3. We can see that in Theorem 1, Statement (i) still holds in the Riemann surface setting, since on any non-planar open Riemann surface the Bergman kernel does not vanish. However, Statement (ii) cannot be generalized to an arbitrary open Riemann surface.

For example, let $X_{\tau,u} := X_{\tau} \setminus \{u\}$ be an open Riemann surface obtained by removing one single point $u$ from a compact complex torus $X_{\tau} := \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, for $\tau \in \mathbb{C}$ and $Im \tau > 0$. By the removable singularity theorem, the Bergman kernel on $X_{\tau,u}$ is the 2-form $K_{\tau,u} = (Im \tau)^{-1} dz \wedge d\bar{z}$, where $z$ is the local coordinate induced from the complex plane $\mathbb{C}$ (see [7,9]). Here $v(X_{\tau,u})$ is precisely $Im \tau$, the area of the fundamental parallelogram. So (1.4) holds true for $X_{\tau,u}$, which is not biholomorphic to $D(z_0, r) \setminus P$.

4. A bounded domain $\Omega \subset \mathbb{C}^n$ is called a minimal domain with a center $z_0 \in \Omega$ if $v(\Omega) \leq v(\Omega')$ for any biholomorphism $\varphi : \Omega \to \Omega'$ such that $det(J\varphi(z_0)) = 1$, where $J\varphi$ denotes the Jacobian matrix of $\varphi$. It is known [15] that equivalently a domain $\Omega$ is minimal with the center at $z_0$ if and only if

$$K(z, z_0) \equiv \frac{1}{v(\Omega)}, \quad z \in \Omega.$$

For more information about minimal and representative domains, see [12,13,15,22]. Theorem 1 classifies all minimal domains in $\mathbb{C}$. More precisely, we have the following corollary.

**Corollary 2** Let $\Omega \subset \mathbb{C}$ be a domain. Suppose there exists a $z_0 \in \Omega$ such that

$$K(z, z_0) \equiv C, \quad \text{for any } z \in \Omega.$$

Then $C = v(\Omega)^{-1}$ and

(i) If $C = 0$, then $\Omega = \mathbb{C} \setminus P$ where $P$ is a closed polar set.

(ii) If $C > 0$, then $\Omega = D(z_0, r) \setminus P$ where $P$ is a possibly empty, polar set closed in the relative topology of $D(z_0, r)$ with $r = \sqrt{v(\Omega)}\pi^{-1}$.

Consequently, all minimal domains with the center $z_0$ in $\mathbb{C}$ are disks centered at $z_0$ possibly minus closed polar sets.
In $\mathbb{C}^n$, $n \geq 1$, Equality (1.4) is achieved for the unit ball and more generally for complete Reinhardt domains. A domain $\Omega$ in $\mathbb{C}^n$ is said to be complete Reinhardt if for all $z = (z_1, \ldots, z_n) \in \Omega$

$$(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega, \quad |\lambda_i| \leq 1.$$  

For a bounded complete Reinhardt domain, $\{z^\alpha\}_{\alpha \in \mathbb{N}^n}$ is a complete orthogonal system of $A^2(\Omega)$. It follows from (1.1) that $K_\Omega(0, 0) = v(\Omega)^{-1}$. Equality (1.4) also holds for complete circular domains (cf. [6]).

When $n > 1$, the unit polydisk $D(0, 1)^n$ is an example of a complete Reinhardt domain which is not biholomorphic to $\mathbb{B}^n$. Thus, to generalize Theorem 1 to $\mathbb{C}^n$, $n \geq 1$, $D(z_0, r)$ cannot simply be replaced by a translation and rescaling of $\mathbb{B}^n$. However, the polydisk does not have smooth boundary, whereas the unit ball is strongly convex with algebraic boundary. So, we also consider whether Theorem 1 generalizes to $\mathbb{C}^n$ if $\Omega$ is required to be complete Reinhardt, strongly convex with algebraic boundary. The answer is no as the next theorem shows.

**Theorem 3** Let $\Omega = \{z \in \mathbb{C}^2 : |z_1|^4 + |z_1|^2 + |z_2|^2 < 1\}$ be a domain with algebraic boundary. Then $\Omega$ is complete Reinhardt, strongly convex and not biholomorphic to $\mathbb{B}^2$.

The paper is organized as follows. In Sect. 2, we give a proof of Theorem 1 using only complex analysis of one variable for the case where $\Omega$ has $C^\infty$-boundary. In Sect. 3, Theorem 1 is proved in full generality using the Suita conjecture and Corollary 2 is proved. In Sect. 4, Theorem 3 is proved.

**2 Proof of $C^\infty$ Boundary Case of Theorem 1**

The main ingredient in the proof of Theorem 1 will be (2.1). The theory that follows can be found in Bell’s book [1]. We begin by recalling the Szegö kernel.

Let $\Omega$ be a bounded domain with $C^\infty$-smooth boundary and denote its boundary by $b\Omega$. Then $\Omega$ is $n$-connected with $n < \infty$ and the boundary consists of $n$ simple closed curves parametrized by $C^\infty$ functions $z_j : [0, 1] \to \mathbb{C}$. Without loss of generality, let $z_n$ parametrize the outermost boundary curve; that is, $z_n$ parametrizes the boundary component which bounds the unbounded component of the complement of the domain. Additionally, the boundary component parametrized by $z_j$, $j = 1, \ldots, n$, is denoted by $b\Omega_j$.

Let $T(z)$ denote the unit tangent vector to the boundary and $ds$ denote the arc-length measure of the boundary. Define $L^2(b\Omega) = \{f : b\Omega \to \mathbb{C} : \|f\|_{L^2(b\Omega)} < \infty\}$ where the norm $\|\cdot\|_{L^2(b\Omega)}$ is induced by the inner product

$$\langle f, g \rangle = \int_{b\Omega} f \overline{g} \, ds.$$  

Let $A^\infty(b\Omega)$ denote the boundary values of functions in $O(\Omega) \cap C^\infty(\overline{\Omega})$. The Hardy space of $b\Omega$ denoted $H^2(b\Omega)$ is the $L^2(b\Omega)$ closure of $A^\infty(b\Omega)$. If $P : L^2(b\Omega) \to$
\( H^2(b\Omega) \) is the orthogonal projection, then the Szegö kernel for \( \Omega \), \( S(z, a) \), is defined by

\[
P(C_a(\cdot))(z) = S(z, a), \quad a, z \in \Omega, \quad C_a(z) = \frac{1}{2\pi i} \frac{T(z)}{z-a}
\]

[1, Sect. 7]. It can be shown that \( S(z, a) = \overline{S(a, z)} \), and from the proof of the Ahlfors Mapping Theorem, for each \( a \), \( S(\cdot, a) \) has \( n-1 \) zeros counting multiplicity [1, Theorem 13.1]. We note that the proof of the Ahlfors Mapping Theorem just cited requires \( C^\infty \)-boundary regularity. Since we will need the fact about the \( n-1 \) zeros of \( S(\cdot, a) \), we have imposed a \( C^\infty \) boundary regularity assumption on \( \Omega \) in this section.

Let \( \omega_j \) be the (unique) solution to the Dirichlet boundary-value problem

\[
\begin{cases}
\Delta u(z) = 0 & z \in \Omega \\
u(z) = 1 & z \in b\Omega_j \\
u(z) = 0 & z \in b\Omega_k, \ k \neq j
\end{cases}
\]

and define \( F_j : \Omega \rightarrow \mathbb{C} \) by \( F_j(z) = 2\partial \omega_j / \partial z \). Then the Bergman kernel and Szegö kernel are related by

\[
K(z, a) = 4\pi S(z, a)^2 + \sum_{j=1}^{n-1} \lambda_j F_j(z),
\]

where \( \lambda_j \) are constants in \( z \) and depend on \( a \) [1, Theorem 23.2]. Since \( \omega_j \in C^\infty(\overline{\Omega}) \) is harmonic, \( F_j \in \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega}) \subset A^2(\Omega) \). We now prove Theorem 1 when \( \Omega \) is bounded with \( C^\infty \) boundary.

**Proof** After a translation we may assume that \( z_0 = 0 \). Let \( \{v(\Omega)^{-1/2}\} \cup \{\phi_j\}_{j=1}^\infty \) be a complete orthonormal basis for \( A^2(\Omega) \). Then

\[
\frac{1}{v(\Omega)} = K(0, 0) = \frac{1}{v(\Omega)} + \sum_{j=1}^\infty \phi_j(0) \overline{\phi_j(0)},
\]

which implies that \( \phi_j(0) = 0 \), for all \( j \). It follows that \( K(0, a) = v(\Omega)^{-1} \), and for any \( f \in A^2(\Omega) \) by the reproducing property (1.2),

\[
f(0) = \frac{1}{v(\Omega)} \int_\Omega f(w) dv(w).
\]
In particular for $F_j, j = 1, \ldots, n-1$,
\[
F_j(0) = \frac{1}{2i v(\Omega)} \int_{\Omega} 2 \frac{\partial \omega_j}{\partial w} d\bar{w} \wedge dw = -\frac{1}{i v(\Omega)} \int_{\partial \Omega} \omega_j d\bar{w} = -\frac{1}{i v(\Omega)} \int_{\partial \Omega_j} 1 d\bar{w} = 0.
\]

Hence setting $z = 0$ in (2.1),
\[
\frac{1}{v(\Omega)} = K(0, a) = 4\pi S^2(0, a).
\]

Since $S(0, \cdot) = \overline{S(\cdot, 0)}$ has $n-1$ zeros counting multiplicity, $n = 1$; that is, $\Omega$ is simply connected.

Let $F : D(0, 1) \to \Omega$ be the inverse of the Riemann map with $F(0) = 0, F'(0) > 0$. By the transformation law of the Bergman kernel,
\[
\frac{1}{\pi} = K_{D(0, 1)}(z, 0) = F'(z)K_{\Omega}(F(z), 0)F'(0) = \frac{F'(z)F'(0)}{v(\Omega)}.
\]

So $F$ is linear; hence $\Omega = D(0, F'(0))$. \qed

3 Proof of Theorem 1

The proof of Theorem 1 in the previous section used methods specific to bounded domains with $C^{\infty}$-boundary and cannot be used to prove the general case where $\Omega \subset \mathbb{C}$ is a domain. So instead, we will use the Suita conjecture to prove the general case.

Let $SH(\Omega)$ be the set of subharmonic functions on $\Omega$. The (negative) Green’s function of a domain $\Omega \subset \mathbb{C}$ is defined by
\[
g(z, w) = \sup \left\{ u(z) : u < 0, u \in SH(\Omega), \limsup_{\zeta \to w} (u(\zeta) - \log |\zeta - w|) < \infty \right\}.
\]

A domain admits a Green’s function if and only if there exists a non-constant, negative subharmonic function on $\Omega$. In particular any bounded domain has a Green’s function.

**Definition 4** The Robin constant is defined by
\[
\lambda(z_0) = \lim_{z \to z_0} g(z, z_0) - \ln |z - z_0|.
\]
In 1972, Suita [20] made the following conjecture.

**Suita Conjecture.** If $\Omega$ is an open Riemann surface admitting a Green’s function, then

$$\pi K(z, z) \geq e^{2\lambda(z)}$$

and if equality holds at one point, then $\Omega$ is biholomorphic to $D(0, 1) \setminus P$, where $P$ is a possibly empty, closed (in the relative topology of $D(0, 1)$) polar set.

The inequality part of the Suita conjecture for planar domains was proved by Błocki [3]. See also [2]. The equality part of the Suita conjecture was proved by Guan and Zhou [10]. See also [8] for related work.

We will use the following result in Helm’s book [11, Theorem 5.6.1], which is attributed to Myrberg [16].

**Theorem 5** Let $\Omega$ be a non-empty open subset of $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus \Omega$ is not a polar set if and only if $\Omega$ admits a Green’s function.

**Proof of Theorem 1** First suppose, $v(\Omega) = \infty$. Since $e^{2\lambda(z_0)} > 0$, the inequality part of the Suita conjecture implies that $\Omega$ does not have a Green’s function. By Theorem 5, $\Omega = \mathbb{C} \setminus P$ where $P$ is a polar set. $P$ is closed because it is the complement in $\mathbb{C}$ of an open set.

Now suppose $v(\Omega) < \infty$. Since polar sets have two-dimensional Lebesgue measure 0, by Theorem 5, $\Omega$ admits a Green’s function. As in the proof of the $C^\infty$ boundary case, after a translation, $z_0 = 0$ and $K_\Omega(\cdot, 0) \equiv v(\Omega)^{-1}$. Let $\Omega_\tau = \{ z \in \Omega : g(z, 0) < \tau \}$. Let

$$r_0 := e^{\tau - \lambda(0) - \epsilon}, \quad r_1 := e^{\tau - \lambda(0) + \epsilon}.$$  

Then for $\tau < 0$ sufficiently negative,

$$D(0, r_0) \subset \Omega_\tau \subset D(0, r_1)$$

(cf. [2,4].) Hence,

$$\frac{e^{-2\epsilon} e^{2\lambda(0)}}{\pi} \leq \frac{e^{2\tau}}{v(\Omega_\tau)} \leq \frac{e^{2\epsilon} e^{2\lambda(0)}}{\pi}.$$  

Letting $\epsilon \to 0^+$,

$$\frac{e^{2\tau}}{v(\Omega_\tau)} \approx \frac{e^{2\lambda(0)}}{\pi}, \quad \text{as } \tau \to -\infty.$$  

By Theorem 3 of [5], $\frac{e^{2\tau}}{v(\Omega_\tau)}$ is a decreasing function on $(-\infty, 0]$; hence,

$$K(0, 0) = \frac{1}{v(\Omega)} \leq \frac{e^{2\lambda(0)}}{\pi} \leq K(0, 0).$$  

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By the equality part of the Suita conjecture, there exists a biholomorphic map \( f : D(0, 1) \setminus P \rightarrow \Omega \) where \( P \) is a closed polar set. After a Möbius transformation of the unit disk, we may assume \( 0 \notin P \), \( f(0) = 0 \) and \( f'(0) > 0 \). Since \( P \) is removable for functions in \( A^2(D(0, 1) \setminus P) \), \( K_{D(0,1)}(\cdot, \cdot) = K_{D(0,1)}(\cdot, \cdot) \) when both sides are well defined. As in the case where \( \Omega \) has \( C^\infty \)-boundary, by the transformation law of the Bergman kernel, \( f \) is linear. Hence \( \Omega = D(0, f'(0)) \setminus f(P) \) and \( f(P) \) is a closed polar set. □

**Proof of Corollary 2** Since \( 1 = \int_\Omega K(w, z_0)dv(w) \), it follows that \( C = v(\Omega)^{-1} \) and \( K(z_0, z_0) = v(\Omega)^{-1} \). The result now follows from Theorem 1. □

### 4 Proof of Theorem 3

In this section, we provide a short proof of Theorem 3. Historically, biholomorphic mappings between Reinhardt domains have been an active area of research since the earliest days of several complex variables (see [17,21]).

**Proof of Theorem 3** It is easy to see that \( \Omega \) is complete Reinhardt with algebraic boundary. To verify that \( \Omega \) is strongly convex, one lets \( \rho(z) = |z_1|^4 + |z_1|^2 + |z_2|^2 - 1 \) and verifies that the real-Hessian of \( \mathcal{H}(\rho(z)) \) satisfies
\[
w^T \mathcal{H}(\rho(z_0))w > 0, \quad z_0 \in \partial \Omega, \quad w \in \mathbb{R}^4 \setminus \{0\}.
\]
Suppose towards a contradiction that there exists an \( F : \mathbb{B}^2 \rightarrow \Omega \) which is biholomorphic. Since the holomorphic automorphism group of \( \mathbb{B}^2 \) is transitive, we may suppose that \( 0 \mapsto 0 \). By Henri Cartan’s theorem, [18, Theorem 2.1.3.], \( F \) is linear; that is
\[
F(z) = (a_1 z_1 + a_2 z_2, a_3 z_1 + a_4 z_2).
\]
Consequently, \( F : b \mathbb{B}^2 \rightarrow b \Omega \). After composing with a holomorphic rotation of \( \mathbb{B}^2 \), we may also suppose \( F((0, 1)) = (0, 1) \). Then, \( a_2 = 0 \), \( a_4 = 1 \). Since for all \( \theta \in [0, 2\pi] \),
\[
b \Omega \ni F \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} e^{i\theta} \right) = \left( \frac{a_1}{\sqrt{2}}, \frac{a_3}{\sqrt{2}} + \frac{e^{i\theta}}{\sqrt{2}} \right),
\]
we see that
\[
\frac{|a_1|^4}{4} + \frac{|a_1|^2}{2} + \frac{|a_3|^2}{2} + Re(a_3, e^{i\theta}) + \frac{1}{2} = 1,
\]
which implies that \( a_3 = 0 \). Thus,
\[
|a_1|^4 + 2|a_1|^2 = 2. \tag{4.1}
\]
Since \( (a_1, 0) = F((1, 0)) \in b \Omega \),
\[
|a_1|^4 + |a_1|^2 = 1. \tag{4.2}
\]

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Equations (4.1) and (4.2) do not have a simultaneous solution. Thus, $F$ does not exist.

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