GRUNSKY OPERATOR, GRINSHPAN’S CONJECTURE AND UNIVERSAL TEICHMÜLLER SPACE

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Abstract. A. Grinshpan posed a deep conjecture on the norm of the Grunsky operator generated by univalent functions in the disk. It gives a quantitative answer in terms of the Grunsky coefficients, to which extent a univalent function determines the bound of dilatations of its quasiconformal extensions. We provide the proof of this conjecture and its various analytic, geometric and potential applications.

Another result concerns the model of universal Teichmüller space by Grunsky coefficients.

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1. PRELIMINARIES

1.1. The Grunsky operator on univalent functions. We consider the class $S_Q(\infty)$ of univalent functions $f(z) = z + a_2 z^2 + \ldots$ in the unit disk $D = \{ |z| < 1 \}$ admitting quasiconformal extensions to the whole Riemann sphere $\hat{C} = \mathbb{C} \cup \{ \infty \}$ and its completion $S(\infty)$ in the topology of locally uniform convergence in $D$.

The Beltrami coefficients of extensions are supported in the complementary disk $D^* = \hat{C} \setminus D = \{ z \in \hat{C} : |z| > 1 \}$ and run over the unit ball

$\text{Belt}(D^*)_1 = \{ \mu \in L_\infty(\mathbb{C}) : \mu(z)|D = 0, \|\mu\|_\infty < 1 \}$.

Each $\mu \in \text{Belt}(D^*)_1$ determines a unique homeomorphic solution to the Beltrami equation $\overline{\partial}w = \mu \partial w$ on $\mathbb{C}$ (quasiconformal automorphism of $\hat{C}$) normalized by $w^\mu(0) = 0$, $(w^\mu)'(0) = 1$, $w^\mu(\infty) = \infty$, whose restriction to $D$ belongs to $S_Q(\infty)$.

The Schwarzian derivatives of these functions

$S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2, \quad f(z) = w^\mu(z)|D$,

belong to the complex Banach space $B = B(D)$ of hyperbolically bounded holomorphic functions in the disk $D$ with norm

$\|\varphi\|_B = \sup_D (1 - |z|^2)^2 |\varphi(z)|$.
and run over a bounded domain in $\mathcal{B}$ modeling the universal Teichmüller space $\mathcal{T}$. Its origin (the base point) $\varphi = 0$ corresponds to the identity map $f(z) \equiv z$. The space $\mathcal{B}$ is dual to the Bergman space $A_1(\mathbb{D})$, a subspace of $L_1(\mathbb{D})$ formed by integrable holomorphic functions (quadratic differentials $\varphi(z)dz^2$ on $\mathbb{D}$).

One defines for any $f \in S_0(\infty)$ its *Grunsky coefficients* $\alpha_{mn}$ from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^m \zeta^n, \quad (z, \zeta) \in \mathbb{D}^2,$$

where the principal branch of the logarithmic function is chosen. These coefficients satisfy the inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f)x_m x_n \right| \leq 1 \quad (2)$$

for any sequence $x = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space $l^2$ with norm $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$; conversely, the inequality (1) also is sufficient for univalence of a locally univalent function in $\mathbb{D}$ (cf. [12]).

The minimum $k(f)$ of dilatations $k(f^\mu) = \|\mu\|_\infty$ among all quasiconformal extensions $w^\mu(z)$ of $f$ onto the whole plane $\hat{\mathbb{C}}$ (forming the equivalence class of $f$) is called the Teichmüller norm of this function. Hence,

$$k(f) = \tanh d_T(0, S_f),$$

where $d_T$ denotes the Teichmüller-Kobayashi metric on $\mathcal{T}$. This quantity dominates the Grunsky norm

$$\kappa(f) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f)x_m x_n \right| : x = (x_n) \in S(l^2) \right\}$$

by $\kappa(f) \leq k(f)$ (see, e.g., [23], [29]). These norms coincide only when any extremal Beltrami coefficient $\mu_0$ for $f$ (i.e., with $\|\mu_0\|_\infty = k(f)$) satisfies

$$\|\mu_0\|_\infty = \sup \left\{ \left| \int\int_{\mathbb{D}^*} \mu(z)\psi(z)dxdy \right| : \psi \in A_1^2(\mathbb{D}^*), \|\psi\|_{A_1} = 1 \right\} = \kappa(f) \quad (z = x + iy). \quad (3)$$

Here $A_1(\mathbb{D}^*)$ denotes the subspace in $L_1(\mathbb{D}^*)$ formed by integrable holomorphic functions (quadratic differentials) on $\mathbb{D}^*$ (hence, $\psi(z) = c_4 z^{-4} + c_5 z^{-5} + \ldots$, so $\psi(z) = O(z^{-4})$ as $z \to \infty$, and $A_1^2(\mathbb{D}^*)$ is its subset consisting of $\psi$ with zeros even order in $\mathbb{D}^*$, i.e., of the squares of holomorphic functions. Due to [16], every $\psi \in A_1^2(\mathbb{D}^*)$ has the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=4}^{\infty} \sqrt{mn} x_m x_n z^{-(m+n)}$$

and $\|\psi\|_{A_1(\mathbb{D}^*)} = \|x\|_2 = 1$, $x = (x_n)$.

In the general case, $\mu_0 \in \text{Belt}(\mathbb{D}^*)_1$ is extremal in its class if and only if

$$\|\mu_0\|_\infty = \sup \left\{ \left| \int\int_{\mathbb{D}^*} \mu(z)\psi(z)dxdy \right| : \psi \in A_1(\mathbb{D}^*), \|\psi\|_{A_1} = 1 \right\}.$$

Moreover, if $\kappa(f) = k(f)$ and the equivalence class of $f$ is a Strebel point of $\mathcal{T}$, which means that this class contains the Teichmüller extremal extension $f^{k(\psi_0)/\psi_0}$ with $\psi_0 \in A_1(\mathbb{D})$, then necessarily $\psi_0 = \omega^2 \in A_1^2$ (cf. [19], [23], [30], [37]).
An important fact is that the Strebel points are dense in any Teichmüller space (see [9]).

Every Grunsky coefficient $\alpha_{mn}(f)$ in (1) is represented as a polynomial of a finite number of the initial Taylor coefficients $a_2, \ldots, a_s$ and hence depends holomorphically on Beltrami coefficients $\mu_f \in \text{Belt}(\mathbb{D}^*)_1$ and on the Schwarzians $S_f \in \mathbf{T}$. This generates holomorphic maps

$$h_{\mathbf{x}}(S_f) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(S_f) \ x_m x_n : \mathbf{T} \to \mathbb{D}$$

with fixed $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ so that $\sup_{\mathbf{x} \in S(l^2)} h_{\mathbf{x}}(S_f) = \varkappa(f)$.

The holomorphy of these functions follows from the holomorphy of coefficients $\alpha_{mn}$ with respect to Beltrami coefficients $\mu \in \text{Belt}(D^*)_1$ and from the estimate

$$\left| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \right|^2 \leq \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2,$$

which holds for any finite $M, N$ and $1 \leq j \leq M, 1 \leq l \leq N$ (see [31], p. 61).

Due to [27], [23], the set of $f$ with $\varkappa(f) < k(f)$ is dense in $\Sigma$, and moreover, the Schwarzian derivatives of these functions form an open and dense set in the universal Teichmüller space $\mathbf{T}$. On the other hand, the functions with $\varkappa(f) = k(f)$ play a crucial role in applications of Grunsky inequalities to the Teichmüller space theory.

Both norms $\varkappa(f)$ and $k(f)$ are continuous plurisubharmonic functions of $S_f$ on the space $\mathbf{T}$ (in $\mathbf{B}$ norm); see, e.g. [23].

Note that the Grunsky (matrix) operator

$$\mathcal{G}(f) = \left( \sqrt{mn} \alpha_{mn}(f) \right)_{m,n=1}^{\infty}$$

acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$; the norm of this operator equals $\varkappa(f)$ (cf. [12]).

A deep theorem of Pommerenke and Zhuravlev states that any univalent function $f \in S$, with $\varkappa(f) \leq \kappa < 1$ has $\kappa_1$-quasiconformal extensions to $\mathbb{C}$ with $\kappa_1 = \kappa_1(\kappa) \geq \kappa$ (see [31]; [KK1, pp. 82-84], [39]). Some explicit estimates for $\kappa_1(k)$ are established in [20], [32].

The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces $X$ with a finite number of boundary components. The case of arbitrary quasidisks is described in [23].

In particular, for univalent functions $f(z) = z + a_2 z^2 + \ldots$ in an arbitrary disk $D_r = \{|z| < r\}, \ 0 < r < \infty$, the corresponding function (1) on $(z, \zeta) \in D_r^2$ provides the Grunsky norm

$$\varkappa(f) = \sup \left\{ \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} r^{-m-n} x_m x_n \ : \ \mathbf{x} = (x_n) \in S(l^2) \right\},$$

and accordingly, the holomorphic maps

$$h_{\mathbf{x},r}(S_f) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(S_f) r^{-m-n} x_m x_n : \mathbf{T} \to \mathbb{D}$$

with $\sup_{\mathbf{x} \in S(l^2)} h_{\mathbf{x},r}(S_f) = \varkappa(f)$. 
1.2. The root transform. One can apply to \( f \in S_Q(\infty) \) the rotational conjugation

\[ \mathcal{R}_p : f(z) \mapsto f_p(z) := f(z^p)^{1/p} = z + \frac{a_2}{p} z^{p-1} + \ldots \]

with integer \( p \geq 2 \), which transforms \( f \in S(\infty) \) into \( p \)-symmetric univalent functions accordingly to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_p & \xrightarrow{\mathcal{R}_p f} & \mathcal{C}_p \\
\downarrow{\pi_p} & & \downarrow{\pi_p} \\
\hat{\mathcal{C}} & \xrightarrow{f} & \hat{\mathcal{C}}
\end{array}
\]

where \( \mathcal{C}_p \) denotes the \( p \)-sheeted sphere \( \hat{\mathcal{C}} \) branched over 0 and \( \infty \), and the projection \( \pi_p(z) = z^p \).

This transform acts on \( \mu \in \text{Belt}(\mathbb{D}^*)_1 \) and \( \psi \in L_1(\mathbb{D}^*) \) by

\[ \mathcal{R}_p^* \mu = \mu(z^p) \overline{z}^{p-1}/z^{p-1}, \quad \mathcal{R}_p^* \psi = \psi(z^{-p}) p^2 z^{2p-2}; \]

then

\[ k(\mathcal{R}_p f) = k(f), \quad \kappa(\mathcal{R}_p f) \geq \kappa(f). \]

It was established by Grinshpan in [10] that the root transform does not decrease the Grunsky norm, i.e.

\[ \kappa_p(f) := \kappa(f_p) \geq \kappa(f) \]

(this also follows from the Kühnau-Schiffer theorem on reciprocity of the Grunsky norm to the least positive Fredholm eigenvalue of the curve \( L = f(|z| = 1) \); see [29], [30].

Note that the sequence \( \kappa_p(f) \), \( p = 2, 3, \ldots \), is not necessarily nondecreasing. For example, for the function \( f(z) = z/(1 + tz)^2 \) with \( |t| \leq 1 \), we have

\[ f_p(t) = z/(1 + tz^p)^{2/p}, \]

and \( \kappa(f_p) = |t| = k(f_p) \) for even \( p \), while \( \kappa(f_p) < |t| = k(f_p) \) for any odd \( p \geq 3 \) (see, e.g. [10]).

The Grunsky coefficients of every function \( \mathcal{R}_p f \) also are polynomials of \( a_2, \ldots, a_l \), which implies, similar to (4), the holomorphy of maps

\[ h_{\mathcal{R}_p f}(\mu) = \sum_{m,n=1}^\infty \sqrt{mn} \alpha_{mn}(\mathcal{R}_p f^\mu) \, x_m x_n : \text{Belt}(D^*)_1 \to \mathbb{D} \]

for any fixed \( p \) and any \( \mathbf{x} = (x_n) \in S(l^2) \), and \( \sup_{\mathbf{x} \in S(l^2)} h_{\mathcal{R}_p f}(\mu) = \kappa(\mathcal{R}_p f^\mu) \).

Every function \( h_{\mathcal{R}_p f}(\mu) \) descends to a holomorphic functions on the space \( \mathbf{T} \), which implies that the Grunsky norms \( \kappa(\mathcal{R}_p f^\mu) \) are continuous and plurisubharmonic on \( \mathbf{T} \) [21].

1.3. Additional remarks on universal Teichmüller space. The universal Teichmüller space \( \mathbf{T} = \text{Teich}(\mathbb{D}) \) is the space of quasisymmetric homeomorphisms of the unit circle \( S^1 \) factorized by Möbius maps. Its topology and real geometry is determined by Teichmüller metric which naturally arises from extensions of those \( h \) to the unit disk.

This space also admits the complex structure of a complex Banach manifold. This structure is defined on \( \mathbf{T} \) by factorization of the ball \( \text{Belt}(\mathbb{D}^*)_1 \), letting \( \mu_1, \mu_2 \in \text{Belt}(\mathbb{D})_1 \) be equivalent if the corresponding quasiconformal maps \( w^{\mu_1}, w^{\mu_2} \) coincide on the unit circle \( S^1 = \partial \mathbb{D} \) (hence, on \( \overline{\mathbb{D}} \)). Such \( \mu \) and the corresponding maps \( w^\mu \) are called \( \mathbf{T} \)-equivalent.
The equivalence classes $[w^\mu]_T$ are in one-to-one correspondence with the Schwarzian derivatives $S_{w^\mu}(z)$, $z \in \mathbb{D}$. The factorizing projection

$$\phi_T(\mu) = S_{w^\mu} : \text{Belt}(\mathbb{D}^*)_1 \to T$$

is a holomorphic map from $L_\infty(\mathbb{D}^*)$ to $B$. This map is a split submersion, which means that $\phi_T$ has local holomorphic sections (see, e.g., [GL]).

The basic intrinsic metric on the space $T$ is its Teichmüller metric

$$\tau_T(\phi_T(\mu), \phi_T(\nu)) = \frac{1}{2} \inf \left\{ \log K\left( (w^\mu)^{-1} \right) : \mu \in \phi_T(\mu), \nu \in \phi_T(\nu) \right\} ;$$

it is generated by the canonical Finsler structure on $\tilde{T}$ (in fact on the tangent bundle $\mathcal{T}(T) = T \times B$ of $T$).

The Carathéodory and Kobayashi metrics on $T$ are, as usually, the smallest and the largest semi-metrics $d$ on $T$, which are contracted by holomorphic maps $h : \mathbb{D} \to T$. Denote these metrics by $c_T$ and $d_T$, respectively; then

$$c_T(\psi_1, \psi_2) = \sup \{ d_{\mathbb{D}}(h(\psi_1), h(\psi_2)) : h \in \text{Hol}(T, \mathbb{D}) \} ;$$

while $d_T(\psi_1, \psi_2)$ is the largest pseudometric $d$ on $T$ satisfying

$$d(\psi_1, \psi_2) \leq \inf \{ d_{\mathbb{D}}(0, t) : h(0) = \psi_1, \text{ and } h(t) = \psi_2, h \in \text{Hol}(\mathbb{D}, T) \} ;$$

where $d_{\mathbb{D}}$ is the hyperbolic Poincaré metric on $\mathbb{D}$ of Gaussian curvature $-4$.

1.4. **Truncation.** Fix $0 < \rho < 1$ and consider for $\mu \in \text{Belt}(\mathbb{D}^*)_1$ the maps

$$f_\rho^\mu(z) = \rho \mu^{-1} f^\mu(\rho z), \quad z \in \mathbb{C}$$

with Beltrami coefficients $\tilde{\mu}(z) = \mu(\rho z)$. Truncating these coefficients by

$$\mu_\rho(z) = \begin{cases} \mu(\rho z), & |z| > 1, \\ 0, & |z| < 1, \end{cases} \quad (6)$$

one obtains a linear (hence holomorphic) map

$$\tau_\rho : \mu \mapsto \mu_\rho : \text{Belt}(\mathbb{D}^*)_1 \to \text{Belt}(\mathbb{D}_{1/\rho}^*)_1.$$ 

We compare this map with the holomorphic homotopy $f_t(z) = \frac{t}{2} f(tz)$ with $|t| \leq 1$, which determines for $|t| < 1$ a holomorphic map of the space $T$ into itself by

$$S_f(z) = S_{f_t}(z) = t^2 S_f(tz).$$

This is obtained by applying, for example, the following lemma of Earle [6].

**Lemma 1.** Let $E, T$ be open subsets of complex Banach spaces $X, Y$ and $B(E)$ be a Banach space of holomorphic functions on $E$ with sup norm. If $\phi(x, t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \phi(x, t)$ is holomorphic for each $x \in E$, then the map $\phi$ is holomorphic.

The following important lemma is a special case of a more general result from [14] (also presented in [15], p. 179). It concerns quasiconformal homeomorphisms with $L_\infty$ bounded but integrally small dilatations.

Consider in the space $L_p(\mathbb{C})$ with $p > 2$ the well-known integral operators

$$T_\rho(z) = -\frac{1}{\pi} \int_\mathbb{C} \frac{\rho(\zeta)d\xi d\eta}{\zeta - z}, \quad \Pi_\rho(z) = -\frac{1}{\pi} \int_\mathbb{C} \frac{\rho(\zeta)d\xi d\eta}{(\zeta - z)^2} = \partial_z T_\rho(z)$$
assuming for that $\rho$ has a compact support in $\mathbb{C}$. Then the second integral exists as a Cauchy principal value, and the derivative $\partial_T T$ generically is understanding as distributional. One of the basic fact in the theory of quasiconformal maps is that every quasiconformal automorphism $w^{\mu}$ with $\|\mu\|_{\infty} = k < 1$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\|\mu\|_{\infty} = k < 1$ is represented in the form

$$w^{\mu}(z) = z + T \rho(z),$$

where $\rho$ is the solution in $L_p$ (for $2 < p < p_0(k)$) of the integral equation

$$\rho = \mu + \mu \Pi \rho,$$

given by the series

$$\rho = \mu + \mu \Pi \mu + \mu \Pi \mu (\mu (\mu )) + \ldots .$$

Let $B_{p,R}$ denote the space of functions $f(z)$ on the disk $\mathbb{D}_R$, $f(0) = 0$, with norm

$$\|f\|_{B_{p,R}} = \sup_{z_1, z_2 \in \mathbb{D}_R} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|\partial_z f\|_{L_p} + \|\partial_{\bar{z}} f\|_{L_p}.$$

**Lemma 2.** Let $f^{\mu}$ be a quasiconformal automorphism of $\hat{\mathbb{C}}$ conformal in the disk $\mathbb{D}_R = \{|z| > R\}$ normalized by $f^{\mu}(z) = z + \text{const} + O(1/z)$ as $z \to \infty$ and $f^{\mu}(0) = 0$. Suppose that $\mu$ satisfies

$$\|\mu\|_{\infty} = k < 1, \quad \|\mu\|_{L_p} < \varepsilon,$$

where $r \geq p_0 p (p_0 - p) = r_0(k, p)$ with $p, p_0 > 2$ indicated above, and $\varepsilon$ is small. Then $f^{\mu}$ is represented in the form

$$f^{\mu}(z) = z - \frac{1}{\pi} \int_{\mathbb{D}_R} \mu(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\xi d\eta + \omega(z, \mu),$$

where

$$\|\omega\|_{B_{p,R}(\mathbb{D}_R)} \leq M(k, p, R) \varepsilon^2,$$

and the constant $M(k, p, R)$ depends only on $k, p, R$.

In other words, the bounded Beltrami coefficients, which are integrally small, define quasiconformal variations of the same form, and their remainder terms are uniformly small remainders of order $\varepsilon^2$.

Since the Beltrami coefficients of maps $f^{\mu}$ and $f_\varepsilon$ differ only on the annulus $\{|1 < |z| < 1/r\}$, one can compare these maps by Lemma 2 when $r = |t|$ is close to 1.

Assume that $f(z)$ is asymptotically conformal on the unit circle $S^1$; in other words, for any pair of points $a, b$ of the curve $L = f(S^1)$,

$$\max_{z \in L} \frac{|a - z| + |z - b|}{|a - b|} \to 1 \quad \text{as} \quad |a - b| \to 0,$$

where $z$ lies between $a$ and $b$. Such curves are quasicircles without corners, but can be rather pathological (see, e.g., [35], p.249). In particular, all $C^1$-smooth curves are asymptotically conformal.

Then the Schwarzian derivative has the growth $S_f(z) = o((1 - |z|)^2)$ as $|z| \to 1$ (so the function $(1 - |z|^2)^{-2} S_f(z)$ remains continuous under crossing the unit circle $S^1$, hence on $\mathbb{D}$). This implies that for any small $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) < 1$ such that

$$(1 - |z|^2)^{-2} |S_{f_\varepsilon}(z)| < \varepsilon \quad \text{for all} \quad r \in (r_0, 1).$$
We fix such \( r_0 \) and take \( r > r_0 \) satisfying \( (1 + r_0)/2 < r < 1 \). For such \( r \), we have by Lemma 2 the uniform bound
\[
\| S_{f^{pr}} - S_{f^r} \|_B = \alpha_1 (1 - r), \tag{7}
\]
where \( \alpha_1 (1 - r) \to 0 \) as \( r \to 1 \).

Combining this with the continuity of Grunsky norm on \( T \) and holomorphy of functions (5) on \( S_f, S_{f_0} \) and \( S_{f^{pr}} \), one obtains the estimate
\[
\kappa_p (f^r) - \kappa_p (f^{pr}) = \alpha_2 (1 - r) \to 0 \quad \text{as} \quad r \to 1; \quad p = 1, 2, \ldots \tag{8}
\]
The remainder depends on \( p \) and also is uniform under the indicated bounds for \( r \) and fixed \( r_0 \).

1.5. Quasireflections. The quasiconformal reflections (or quasireflections) are the orientation reversing quasiconformal homeomorphisms of the sphere \( \hat{\mathbb{C}} \) which preserve pointwise some (oriented) quasicircle \( L \subset \hat{\mathbb{C}} \) and interchange its interior and exterior domains. which we denote by \( D_L \) and \( D^*_L \), respectively.

One defines for \( L \) its reflection coefficient
\[
q_L = \inf k(f) = \inf \| \partial_z f / \partial \bar{z} f \|_\infty,
\]
taking the infimum over all quasireflections across \( L \), and quasiconformal dilatation
\[
Q_L = (1 + q_L)/(1 - q_L) \geq 1.
\]
Due to [2], [31], this dilatation is equal to the quantity \( Q_L = (1 + k_L)/(1 - k_L)^2 \), where \( k_L \) is the minimal dilatation among all orientation preserving quasiconformal automorphisms \( f_* \) of \( \hat{\mathbb{C}} \) carrying the unit circle onto \( L \), and \( k(f_*) = \| \partial_z f_*/\partial \bar{z} f_* \|_\infty \). On the properties of quasireflections and obtained results see, e.g., [2], [19], [31].

2. THE GRINSHSPAN CONJECTURE. MAIN THEOREM

2.1 The Grinshpan conjecture. A. Grinshpan posed in [11] the following deep

Conjecture. For every \( f \in S(\infty) \), we have the equality
\[
\limsup_{p \to \infty} \kappa_p (f) = k(f).
\]

Though this conjecture arose from the theory of univalent functions with quasiconformal extensions, it intrinsically relates to many important problems of geometric complex analysis and of Teichmüller space theory.

First of all, geometrically this conjecture implies the equality of the Carathéodory and Kobayashi metrics on the universal Teichmüller space, which leads to many important analytic, geometric and potential features of this space as well as of the space of univalent functions.
Somewhat modified (weakened) version of this conjecture is proven in [21]. We shall use here some results from this paper.

2.2. Main theorem. The aim of this paper is to prove the following theorem giving with its corollaries the answers to above problems.

Theorem 1. Every univalent function \( f(z) \in S(\infty) \) with Grunsky norm \( \kappa(f) < 1 \) admits quasiconformal extensions \( f^\mu \) to the whole sphere \( \hat{\mathbb{C}} \) with dilatations

\[
   k(f^\mu) \geq \hat{\kappa}(f) := \lim_{m \to \infty} \kappa_{2m}(f),
\]

and

\[
   k(f) = \hat{\kappa}(f) = \limsup_{p \to \infty} \kappa_p(f). \tag{9}
\]

The quantity \( \hat{\kappa}(f) \) can be regarded as the limit Grunsky norm of \( f \).

The fact that any \( f \in S \) with \( \kappa(f) < 1 \) belongs to \( S_Q \) follows from the Pommerenke-Zhuravlev theorem mentioned above. Theorem 1 strengthens this theorem giving explicitly the extremal dilatation of admissible quasiconformal extensions (the Teichmüller norm of \( f \)) and proves positively the Grinshpan conjecture. This theorem also has many other important consequences related to the complex and potential geometries of the universal Teichmüller space and to the well-known Ahlfors question on the intrinsic characterization of conformal maps of the disk onto domains with quasiconformal boundaries. The results are briefly presented in Section 5.

2.3. Example. Let us first mention that one cannot replace in the statement of Theorem 1 the assumption \( f(z) \in S(\infty) \) by \( f(z) \in S_Q \) (i.e., drop the third normalization condition), though it does not appear in the Pommerenke-Zhuravlev theorem. The point is that in this case the root transform \( \mathcal{R}_p \) can increase the Teichmüller norm.

For example, consider the extremal map \( g_r \) in Teichmüller’s Verschiebungssatz with minimal dilatation among quasiconformal automorphisms of the unit disk, which are identical on the boundary circle and move the origin into a given point \(-r \in (-1,0)\). Its Beltrami coefficient \( \mu_0 = k|\psi_0|/\psi_0 \) is defined by \( \psi_0 \), which is holomorphic and does not vanish on \( \mathbb{D} \setminus \{0\} \) and has simple pole at 0. This \( \psi_0 \) is orthogonal to all holomorphic quadratic differentials on \( \mathbb{D} \) with respect to pairing

\[
   \langle \varphi, \psi \rangle = \iint_{\mathbb{D}} (1 - |z|^2)^2 \varphi(z) \overline{\psi(z)} dx dy.
\]

For small \( r \), the dilatation \( k(g_r) = r/2 + O(r^2) \) (the corresponding formula in [38], p. 343, for extremal dilatation contains an error).

This map \( g_r \) extends trivially to a quasiconformal map of \( \hat{\mathbb{C}} \) by \( g_r(z) = z \) for \(|z| \geq 1\). Consider the translated map \( f_r(z) = g_r(z) + r \). Its restriction to \( \mathbb{D}^* \) has dilatation \( k(f_r) = k(g_r) = 0 \).

In contrast, the dilatation of the squared map \( f_{r,2} := \mathcal{R}_2^* f_r \) equals \( r \), since the differential \( \mathcal{R}_2^* \psi_0 \) is holomorphic on \( \mathbb{D} \) and therefore the Beltrami coefficient \( \mathcal{R}_2^* \mu_0 = \mu_{f_{r,2}} \) is extremal for the boundary values \( f_{r,2}|S^1 \) (note that \( f_{r,2}(z) = z + r/(2z) + \ldots \) for \(|z| > 1\)), and

\[
   \kappa(f_{r,2}) = k(f_{r,2}) = r + O(r^2), \quad r \to 0.
\]
Thus,
\[ \limsup_{p \to \infty} \mathcal{R}_p(f_r) = \mathcal{R}_2(f_r) > k(f_r). \]
Another example was constructed by R. Kühnau (private communication).

3. PROOF OF THEOREM 1

Step 1: Preliminary lemmas.
Given a function \( f \in S_Q(\infty) \), consider its extremal quasiconformal extension \( f^{\mu_0} \) to \( \mathbb{D}^* \) with Beltrami coefficient \( \mu_0 \in L_\infty(\mathbb{D}^*) \) (hence, \( k(f) = \|\mu_0\|_\infty \)) and assign to this function the quantity
\[
\alpha_\mathbb{D}^* = \sup \left\{ \left| \int \int_{\mathbb{D}^*} \mu_0(z) \psi(z) dx dy \right| : \psi \in A^2_1(D^*), \|\psi\|_{A^2_1} = 1 \right\}, \tag{10} \]

**Lemma 3.** \([15], [23] \) The Grunsky norm \( \mathcal{R}(f) \) of every function \( f \in S_Q(\infty) \) is estimated by its Teichmüller norm \( k = k(f) \) and the quantity (10) via
\[
\mathcal{R}(f) \leq k \frac{k + \alpha_\mathbb{D}^*(f)}{1 + \alpha_\mathbb{D}^*(f)k}, \tag{11} \]
and \( \mathcal{R}(f) < k \) unless \( \alpha_\mathbb{D}^*(f) = \|\mu_0\|_\infty \).

The last equality occurs if and only if \( \mathcal{R}(f) = k(f) \), and if in addition the equivalence class of \( f \) (the collection of maps equal to \( f \) on \( \partial D \)) is a Strebel point, then \( \mu_0 \) is necessarily of the form
\[
\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z), \quad \psi_0 \in A^2_1(\mathbb{D}^*). \]

The following lemma plays a crucial role in the proof of Theorem 1.

**Lemma 4.** Let \( f \in S_Q(\infty) \), and let \( f^{\mu_0} \) be an extremal extension of \( f \) to \( \hat{\mathbb{C}} \). Then
\[
k(f^{\mu_0}) = \mathcal{R}(f^{\mu_0}) := \sup_{\mu \in [f^{\mu_0}]} \limsup_{\rho \to 1} \sup_{p} \sup_{\psi \in A^2_1(\mathbb{D}^*), \|\psi\|_{A^2_1} = 1} \left| \int \int_{\mathbb{D}^*} \mathcal{R}_p^{\mu}(\psi(z)) dx dy \right|. \tag{12} \]

This quantity \( \mathcal{R}(f) \) cannot be replaced by a smaller lower bound for the dilatations of quasiconformal extensions of \( f \); we call it the **outer limit Grunsky norm** of \( f \).

**Proof.** First assume that \( \mu_0 \) is of Teichmüller form, which means
\[
\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z),
\]
with \( \hat{\mathbb{C}} \)-holomorphic quadratic differential
\[
\psi_0(z) = c_3 z^{-3} + c_4 z^{-4} + \ldots, \quad |z| > 1, \tag{13} \]
having at most simple pole at the infinite point.
If \( c_3 \neq 0, c_4 \neq 0 \), then, noting that \( \mathcal{R}(\mathcal{R}_2 f^{\mu_0}) = \mathcal{R}(f^{\mu_0}) \), one can start with the squared map \( \mathcal{R}_2 f^{\mu_0} \) whose defining quadratic differential is of the form
\[
\mathcal{R}_2^{*} \psi_0(z^2) = 4(c_3 z^{-4} + c_4 z^{-6} + \ldots)
and has at $z = \infty$ zero of even order. To avoid a complication of notations, we assume that this holds for $\psi_0$ (hence in (13) $c_3 = 0$). We only need to consider $\psi_0$ with at least two zeros of odd order.

After applying to $f^{\mu_0}$ the root transform, we get the Teichmüller map $R_\rho^* f = f^{k|R_\rho^* \psi_0|/R_\rho^* \psi_0}$ determined by quadratic differential

$$R_\rho^* \psi_0 = \psi_0(z^p)p^2 z^{2p-2} = \hat{\psi}_0(\zeta).$$

Note that if $\psi_0(z)$ has a zero at $z_0 \neq 0$, $z_0 \in \mathbb{D}$, then $\hat{\psi}_0(\zeta) = 0$ at the points $\zeta = z_0^{1/p}$, and $|z_0^{1/p}| \to 1$ as $p \to \infty$.

Assume that $f^{\mu_0}$ is asymptotically conformal on $S^1$ and fix $\rho_j < 1$ arbitrarily close to 1. Pick $p_j$ so large that all zeros of odd order of $R_\rho^* \psi_0$ are placed in the annulus $\{1 < |z| < 1/\rho_j\}$. Then, taking the truncated Beltrami coefficients $(R_\rho^* \mu_0)_{\rho_j}$ for

$$R_\rho^* \mu_0 = k|R_\rho^* \psi_0|/R_\rho^* \psi_0,$$

vanishing in the disks $\mathbb{D}_{1/\rho_j}$, and applying to these coefficients Lemma 3, one obtains that on each disk $\mathbb{D}_{1/\rho_j}$ the corresponding extremal map $f^{(R_\rho^* \mu_0)_{\rho_j}}$ is determined by holomorphic quadratic differential with zeros of even order. Therefore,

$$\varpi(f^{(R_\rho^* \mu_0)_{\rho_j}}) = \sup_{(z_n) \in S^2} \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} (f^{(R_\rho^* \mu_0)_{\rho_j}}) \rho_j^{m+n} x_m x_n \right|. \tag{14}$$

Using this equality, one can find the appropriate sequences $\{p_n\} \to 1$, $\{p_n\} \to \infty$ and $\{\psi_n\} \in A_1^0(\mathbb{D}^*)$ with $\|\psi_n\|_{A_1} = 1$ such that in the limit as $n \to \infty$ the above relations result in the desired equality (12). This proves the assertion of Lemma 4 for functions $f \in S_Q(\infty)$ admitting Teichmüller extensions and such that the curve $f(S^1)$ is asymptotically conformal.

In the case of the generic $f(z)$ having Teichmüller extension to $\mathbb{D}^*$, we pass to its homotopy stretching $f_\rho(z) = \frac{1}{\rho} f(r z)$ with $\rho < 1$ and apply the above arguments giving the prescribed sequences $\{p_n\}$, $\{p_n\}$, $\{\psi_n\}$ giving (13) for each $r$.

In view of properties of norms $\varpi_\rho(f_r)$ and $k(f_r)$ indicated in Section 1.3, one can select the appropriate subsequences $\{p_{n_j}\}$, $\{p_{n_j}\}$, $\{\psi_{n_j}\}$ so that

$$k(f) = \tilde{\varpi}(f) = \lim_{\rho_{n_j} \to 1} \sup_{p_{n_j}} \sup_{\psi_{n_j} \in A_1^0(\mathbb{D}^*)} \left| \int_{\mathbb{D}^*} R_{\rho \mu_0 p_{n_j}}(z) \psi_{n_j}(z) dxdy \right|, \tag{15}$$

which implies the equality (12) for $f$. So, Lemma 4 is valid for all Strebel points, hence, for a dense subset of $f \in S_Q(\infty)$.

Now consider an arbitrary function $f \in S_Q(\infty)$, and let $\mu_0$ be one of its extremal Beltrami coefficients in $\mathbb{D}^*$ (i.e., with minimal $L_\infty$ norm).

Take a neighborhood $U_0$ of $S_{f_{\mu_0}}$ in $\mathbf{T}$, in which the defining projection $\phi_{\mathbf{T}} : \text{Belt}(\mathbb{D}^*) \to \mathbf{T}$ has a local holomorphic section $s$, and a sequence of the Strebel points $S_{f_{\mu_n}} \to S_{f_{\mu_0}}$ in $\mathbf{B}$ norm. On this neighborhood, the upper semicontinuous normalization of the function $\tilde{\varpi}(f)$ is plurisubharmonic (as a function of the Schwarzians $\varphi = S_{f_0}$). In view of subharmonicity, we have

$$\tilde{\varpi}(\varphi_0) \geq \lim \sup \tilde{\varpi}(\varphi_m) \quad \text{for all sequences} \quad \varphi_m \to \varphi_0 = S_{f_{\mu_0}}.$$
The equalities (15) mean geometrically that the functions
\[ h_{x,1/\rho_{j}}(S_{f}^{p}x_{p_{j}}) = \sum_{m,n=1}^{\infty} \sqrt{m+n} \alpha_{mn}(f^{(R_{p_{j}}^{*})}) \rho_{j}^{m+n}x_{m}x_{n} \] (16)
form a maximizing sequence for the Carathéodory distance
\[ c_{\text{Belt}(\mathbb{D}^{*})}(0, \mu_{0}) = \tanh^{-1}(\|\mu_{0}\|_{\infty}), \quad \mu_{0} = s(\varphi_{0}), \]
on the ball \( \text{Belt}(\mathbb{D}^{*}) \) (which coincide with the Kobayashi and Teichmüller distances on this ball). The continuity of these metrics implies
\[ \tilde{\kappa}(f^{\mu_{n}}) \geq c_{\text{Belt}(\mathbb{D}^{*})}(0, S_{f_{0}^{\mu}}^{\mu_{n}}) - \varepsilon_{n} = \|\mu_{0}\|_{\infty} - \varepsilon_{n}, \quad \varepsilon_{n} \to 0, \]
and therefore, \( k(f^{\mu_{n}}) = \tilde{\kappa}(f^{\mu_{0}}) \), and these values coincide with right-hand part of (12). This completes the proof of Lemma 4.

The following lemma strengthens essentially the estimate (8).

**Lemma 5.** For any function (5), we have
\[ h_{x,p}(S_{f^{\rho_{r}}}) - h_{x,p}(S_{f_{r}}) = \alpha(1 - r), \]
where the function \( \alpha(t) \to 0 \) as \( t \to 0 \) and depends only on \( k(f) \).

**Proof.** Any function \( h_{x,p} \) maps the space \( T \) into the unit disk, and since the Kobayashi distance does not increase under holomorphic maps, both points \( h_{x,p}(S_{f^{\rho_{r}}}) \) and \( h_{x,p}(S_{f_{r}}) \) must lie in the disk \( \{ |w| \leq k \} \). In view of (7), the hyperbolic distance between these points satisfies
\[ d_{\mathbb{D}}(h_{x,p}(S_{f_{r}}), h_{x,p}(S_{f^{\rho_{r}}})) \leq d_{T}(S_{f^{\rho_{r}}}, S_{f_{r}}) = \alpha_{1}(1 - r). \]
For small \( 1 - r \), this implies that the Euclidean distance between these points also is estimated by a similar function
\[ |h_{x,p}(S_{f^{\rho_{r}}}) - h_{x,p}(S_{f_{r}})| = \alpha(1 - r), \quad \alpha(r) \to 0 \quad \text{as} \quad r \to 1, \]
and this function \( \alpha(t) \) depends only on \( k(t) \) and \( t \). This provides the assertion of Lemma 5.

**Step 2: Equality of the generalized Grunsky norms.**

First assume that \( f = f_{0} \) is holomorphic in the closed unit disk \( \overline{\mathbb{D}} \) (hence, in some disk \( \mathbb{D}_{d} \) of radius \( d > 1 \), and \( f'(z) \neq 0 \) in \( \mathbb{D}_{d} \)). Then this function admits the Teichmüller extremal extension across any circle \( \{ |z| = d' \} \), \( d' < d \), so the Schwarzian \( S_{f_{0}} \) is a Strebel point of the space \( T \) (in other words, the equivalence class of \( f_{0} \) contains a Teichmüller extremal map).

For such \( f_{0} \), the arguments applied in the proof of Lemma 4 provides the sequences \( \{ r_{n} \} \to 1 \), \( \{ p_{n} \} \to \infty \), \( \{ x_{n} \} \subset S(\mathbb{D}) \) and \( \{ \psi_{n} \} \in A_{1}^{2} \) defining the extremal extensions \( f_{0}^{(R_{p_{n}}^{*})1/r_{n}} \) such that
\[ \kappa_{p_{n}}(f_{0}^{(R_{p_{n}}^{*})1/r_{n}}) \geq \tilde{\kappa}(f_{0}^{(R_{p_{n}}^{*})1/r_{n}}) - \varepsilon_{n} = k(f_{0}^{\rho_{r_{n}}}) - \varepsilon_{n} \]
with \( \varepsilon_{n}, \varepsilon'_{n} > 0 \) monotone decreasing to zero as \( n \to \infty \). Hence, the corresponding holomorphic functions \( h_{n}(S_{f_{0}^{\rho_{r_{n}}}}) : T \to \mathbb{D} \) of type (4) or (16) also satisfy
\[ |h_{x,n}(S_{f_{0}^{(R_{p_{n}}^{*})1/r_{n}}})| \geq \tilde{\kappa}(f_{0}) - \varepsilon_{n} = k(f_{0}) - \varepsilon_{n}. \] (17)
By Lemmas 2 and the estimate (7) the functions (17) are approximated by the corresponding functions $h_n(S_{fl,r_n})$ defining $\tilde{z}(f_0)$. Then Lemma 5 implies

$$|h_n(S_{fl,r_n})| \geq \tilde{z}(f_0) - \varepsilon'_n, \quad \varepsilon'_n \to 0,$$

which results in the limit equalities

$$\tilde{z}(f_0) = \tilde{z}(f_0) = k(f_0). \quad (18)$$

These relations easily extend to all functions $f \in S_{Q}(\infty)$ having Teichmüller extremal extensions by preliminary using their homotopy functions $f_r(z)$. By the previous step, the equalities (18) are valid for all these functions, so $\tilde{z}(f_r) = k(f_r) \to k(f)$ as $r \to 1$, which implies the assertion of the theorem for the indicated $f$.

In fact, we have established somewhat more: since the function $S_f \to S_{fl}$ is holomorphic in $t$ as a map from $T$ into $T \times D$, the relation (12) already established for the Strebel points implies the equality $c_T(0, \varphi) = \tau_T(0, \varphi)$ on all Teichmüller disks

$$\{\phi_T(t|\psi_0)/\psi_0 : t \in D, \ \psi_0 \in A_1(D^*)\} \subset T.$$

Finally, consider the remained case, when $S_{fl}$ does not be a Strebel point (hence, generically the curve $f^\mu(S^1)$ is fractal). Then one can repeat the arguments from the last step in the proof of Lemma 3 taking an extremal quasiconformal extension $f^\mu$ of $f$ and its homotopy functions $f^\mu_r(z)$, which obey (18). This $S_{fl}$ is approximated by Strebel points $S_{f_{p_n}}$, and in view of the above remark, one can now deal with the Carathéodory distances $c_T(0, S_{f_{p_n}})$ (i.e., consider the corresponding maps (16) as the functions of the Schwarzians $S_f \in T$).

The desired equalities (18), equivalent to (9), again follow by applying the enveloping function (15) and taking the limit as $r \to 1$. This completes the proof of Theorem 1.

4. SECOND PROOF OF LEMMA 4

Lemma 4 plays a crucial role in the proof of Theorem 1. Thus we provide also alternate proof of the last step, which does not involve the properties of invariant distances and of the defining projection $\phi_T$.

Consider an arbitrary function $f \in S_{Q}(\infty)$, and let $\mu$ be one of its extremal Beltrami coefficients in $\mathbb{D}^*$ (i.e., with minimal $L_\infty$ norm).

Truncate this $\mu$ by (6) close to $\rho$. Restricting the obtained Beltrami coefficient $\mu_\rho$ to the disk $\mathbb{D}_{1/\rho'} = \{|z| > 1/\rho'\}$, one obtains, in view of conformality on the annulus

$$U_{1/\rho,1/\rho'} = \{1/\rho < |z| < 1/\rho'\},$$

that the equivalence class of $f^\mu|_{\rho'}$ on the disk $\mathbb{D}_{1/\rho'}$, i.e., among the maps conformal in the disk $\mathbb{D}_{1/\rho'}$ and with the fixed values on the circle $\{|z| = 1/\rho'\}$ is a Strebel point. It admits Teichmüller extension onto the disk $\mathbb{D}_{1/\rho'}^*$ with Beltrami coefficient $\mu_{\rho,\rho'}$ satisfying

$$|\mu_{\rho,\rho'}(z)| = \|\mu_{\rho,\rho'}\|_\infty = \|\mu\|_\infty - o(1) < \|\mu\|_\infty,$$

where, in view of the properties of extremal Beltrami coefficients, $o(1) = \beta_1(\rho' - \rho) \to 0$ as $\rho' \to \rho$ and both $\rho, \rho'$ approach 1.

The arguments from the second step in the above proof of the lemma applied to $f^\mu_{\rho,\rho'}$ provide the equality

$$\kappa_p(f^{R_{\rho,\rho'}}) = k(f^{R_{\rho,\rho'}}) \quad (19)$$

for all $p \geq p_0(\rho)$ such that $R_{\rho,\rho'}$ have no zeros of odd order in the disk $\mathbb{D}_{1/\rho'}^*$. 


We also have for any fixed natural $p \geq 1$,
\[
\lim_{\rho \to 1} \kappa_p(f^{\mu,\mu'}) = \kappa_p(f^\mu)
\]
and that for a fixed $\rho$ the functions $\kappa_p(f^{\mu,\mu'})$ are plurisubharmonic with respect to the Schwarzians $S_{f^\mu}$ and $S_{f^{\mu'}}$ in $\mathcal{T}$ and bounded by $k(f^\mu)$. This yields that the upper semi-continuous regularization of the function
\[
\kappa_0(f^\mu) = \limsup_{\rho \to 1} \sup_p \kappa_p(f^{\mu,\mu'})
\]
also is plurisubharmonic on the space $\mathcal{T}$. It admits the circular symmetry on each homotopy disk \\{\{S_{f^\mu}\}\} (inherited from the symmetry of $f^\mu|D$) and thus $\kappa_0(f^\mu_t)$ is continuous in $|t|$ on $[0, 1]$.

Applying the relations (19), (20), one derives the desired equality (12), which completes the proof of the lemma.

5. APPLICATIONS OF THEOREM 1

The aim of the section is to illustrate the importance of the limit Grunsky norm, which leads to various sharp bounds for univalent functions with quasiconformal extensions and for the basic curvilinear functionals.

5.1. Ahlfors’ problem. In 1963, Ahlfors posed in [1] (and repeated in his book [2]) the following question which gave rise to various investigations of quasiconformal extendibility of univalent functions.

**Question.** Let $f$ be a conformal map of the disk (or half-plane) onto a domain with quasiconformal boundary (quasicircle). How can this map be characterized?

He conjectured that the characterization should be in analytic properties of the logarithmic derivative $\log f' = f''/f'$, and indeed, many results on quasiconformal extensions of holomorphic maps have been established using $f''/f'$ and other invariants (see, e.g., the survey [19] and the references there).

This question relates to another still not completely solved problem in geometric complex analysis:

*To what extent does the Riemann mapping function $f$ of a Jordan domain $D \subset \hat{\mathbb{C}}$ determine the geometric and conformal invariants (characteristics) of the boundary $\partial D$ and of complementary domain $D^* = \hat{\mathbb{C}} \setminus D$?*

Theorem 1 implies a natural qualitative answer to all these questions and shows how the inner features of conformality completely prescribe the admissible distortion under quasiconformal extensions of function $f$ and determine the hyperbolic features of the universal Teichmüller space. We present this important consequence from Theorem 1 as

**Corollary 1.** For any function $f$ mapping conformally the unit disk onto a domain with quasiconformal boundary $L = f(|z| = 1)$, the reflection coefficient $q_L$ of the curve $L$ is
determined by the limit Grunsky norm of this function via

\[
\frac{1+q_L}{1-q_L} = \left( \frac{1 + \hat{\kappa}(f)}{1 - \hat{\kappa}(f)} \right)^2.
\]  

(21)

Hence the right-hand side of (21) represents the minimal dilatation of quasiconformal reflections across \( L \).

5.2. Invariant metrics on universal Teichmüller space. We already mentioned in the proof of Lemma 4 the connection between the function (13) and the invariant distances on \( T \). Here we give their explicit representation generated by the original univalent functions. It is elementary that the Carathéodory, Kobayashi and Teichmüller metrics of any Teichmüller space \( \tilde{T} \) are related by

\[
c_{\tilde{T}}(\cdot, \cdot) \leq d_{\tilde{T}}(\cdot, \cdot) \leq \tau_{\tilde{T}}(\cdot, \cdot)
\]  

(22)

(and similarly for their infinitesimal forms).

The fundamental Royden-Gardiner theorem states that the metrics \( d_{\tilde{T}} \) generated and \( \tau_{\tilde{T}} \) coincide on any space \( \tilde{T} \) (see, e.g., [7], [9]). On the other hand, due to [24], the Carathéodory metric of the universal Teichmüller space \( T \) also coincides with its Teichmüller metric, but this does not hold, for example, for finite dimensional Teichmüller spaces of dimension greater than 1, see [8]. Theorem 1 gives a new proof of this fact and represents these metrics explicitly in terms of \( \hat{\kappa}(f) \).

Corollary 2. The Carathéodory metric of the universal Teichmüller space \( T \) coincides with its Teichmüller metric; hence all non-expansive holomorphically invariant metrics on the space \( T \) are equal, in particular, for any two point \( \varphi_1, \varphi_2 \in T \). In particular, for any pair \((\varphi_1, \varphi_2) \in T \times T\),

\[
c_T(\varphi_1, \varphi_2) = d_T(\varphi_1, \varphi_2) = \tau_T(\varphi_1, \varphi_2),
\]  

(23)

and similarly for the infinitesimal forms of these metrics.

Proof. It follows from Theorem 1 and (19) that for any point \( S_{f^\mu} \in T \), we have the equalities

\[
c_T(S_{f^\mu}, 0) = d_T(S_{f^\mu}, 0) = \tau_T(S_{f^\mu}, 0) = \tanh^{-1} \hat{\kappa}(f^\mu).
\]  

(24)

Now consider two arbitrary points \( \varphi_1 = S_{f_1} \) and \( \varphi_2 = S_{f_2} \) in \( T \). Since the universal Teichmüller space is a complex homogeneous domain in \( B \), this general case is reduced to (24) by moving one of these points to the origin \( \varphi = 0 \), applying a right translation of the space \( T \). Such translations preserve the invariant distances and the Teichmüller distance which, together with (24), yields the equalities (23).

In view of maximality of \( d_T \) and minimality of \( c_T \), all intermediate invariant metrics also obey (22). This completes the proof of Corollary 2.

This Corollary completely determines the complex geometry of the space \( T \). Note also that the equalities similar to (24) are also valid for the infinitesimal forms of metrics \( c_T, d_T, \tau_T \), which provides that all these metrics have holomorphic curvature \(-4\). We will not go here into details, because the proof involves essentially the results lying out of the framework of this paper.
5.3. Pluricomplex Green function of universal Teichmüller space. Corollary 2 also determines the potential features of the space $T$. We illustrate this by representation of its pluricomplex Green function.

Recall that the pluricomplex Green function $g_D(x, y)$ of a domain $D$ in a complex Banach space $X$ with pole $y$ is defined by $g_D(x, y) = \sup u_y(x)$ $(x, y \in D)$ followed by the upper semicontinuous regularization

$$v^*(x) = \lim_{\varepsilon \to 0} \sup_{\|x' - x\| < \varepsilon} v(x').$$

The supremum here is taken over all plurisubharmonic functions $u_y : D \to [-\infty, 0)$ such that $u_y(x) = \log \|x - y\|_X + O(1)$ in a neighborhood of the pole $y$. Here $\| \cdot \|_X$ denotes the norm on $X$, and the remainder term $O(1)$ is bounded from above. The Green function $g_D(x, y)$ is a maximal plurisubharmonic function on $D \setminus \{y\}$ (unless it is identically $-\infty$).

Corollary 3. For every point $\varphi = S_f \in T$, the pluricomplex Green function $g_T(0, S_f)$ with pole at the origin of $T$ is given by

$$g_T(0, S_f) = \log \hat{\kappa}(f), \quad (25)$$

and for any pair $(\varphi, \psi)$ of points in $T$, we have

$$g_T(\varphi, \psi) = \log \tanh d_T(\varphi, \psi) = \log \tanh c_T(\varphi, \psi) = \log k(\varphi, \psi),$$

where $k(\varphi, \psi)$ denotes the extremal dilatation of quasiconformal maps determining the Teichmüller distance between $\varphi$ and $\psi$.

The equality (25) is in fact a special case of the general equality

$$g_D(x, y) = \log \tanh d_D(x, y),$$

which holds for any hyperbolic Banach domain whose Kobayashi metric $d_D$ is logarithmically plurisubharmonic (cf. [5], [13], [18]).

6. MODELING UNIVERSAL TEICHMÜLLER SPACE BY GRUNSKY COEFFICIENTS

There are several models of the universal Teichmüller space. The most applicable is the Bers model via the domain $T$ in the Banach space $B$ of the Schwarzian derivatives; this model was applied above.

We mention here another model constructed in [25] by applying the Grunsky coefficients of univalent functions in the disk. In this model, the space $T$ is represented by a bounded domain in a subspace of $l_\infty$ determined by the Grunsky coefficients. This domain is biholomorphically equivalent to the Bers domain $T$.

Consider the univalent nonvanishing functions in the disk $D^* = \{|z| > 1\}$ with hydrodynamical normalization

$$f(z) = z + b_1 z^{-1} + \cdots : D^* \to \hat{\mathbb{C}} \setminus \{0\}, \quad (26)$$
and denote their collection by $\Sigma$. The Grunsky coefficients $c_{mn}$ of these functions defined from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{m,n=1}^{\infty} c_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2,$$

and satisfy (2). Note that $c_{mn} = c_{nm}$ for all $m, n \geq 1$ and $c_{m1} = b_m$ for any $m \geq 1$.

Denote by $\Sigma_k$ the subset of $\Sigma$ formed by the functions with $k$-quasiconformal extensions to $\mathbb{D}$, and let $\Sigma^0 = \bigcup_k \Sigma_k$.

Since any $f \in \Sigma^0$ does not vanish in $\mathbb{D}^*$, its inversion $F_f(z) = 1/f(1/z)$ is holomorphic and univalent in the unit disk $\mathbb{D}$; both functions $f$ and $F_f$ have the same Grunsky coefficients $c_{mn}$.

These coefficients span a $\mathbb{C}$-linear space $L^0$ of sequences $c = (c_{mn})$ which satisfy the symmetry relation $c_{mn} = c_{nm}$ and

$$|c_{mn}| \leq C(c)/\sqrt{mn}, \quad C(c) = \text{const} < \infty \quad \text{for all} \quad m, n \geq 1,$$

with finite norm

$$\|c\| = \sup_{m,n} \sqrt{mn} \ |c_{mn}| + \sup_{x = (x_n) \in S(l^2)} \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \ c_{mn} x_m x_n \right|. \quad (27)$$

Denote the closure of span $L^0$ by $L$ and note that the limits of convergent sequences $\{c^{(p)} = (c_{mn}^{(p)})\} \subset L$ in the norm (27) also generate the double series

$$\sum_{m,n=1}^{\infty} c_{mn} z^{-m} \zeta^{-n}$$

convergent absolutely in the bidisk $\{(z, \zeta) \in \hat{\mathbb{C}}^2 : \ |z| > 1, \ |\zeta| > 1\}$.

It is established in [25] that the sequences $c$ corresponding to functions $f \in \Sigma^0$ having quasiconformal extensions to the disk $\mathbb{D}$ fill a bounded domain $\tilde{T}$ in the indicated Banach space $L$ containing the origin, and the correspondence

$$S_f \leftrightarrow c = (c_{mn})$$

determines a biholomorphism of the domain $\tilde{T}$ onto the space $T$.

In this model, the Grunsky norm of any function from $S_Q(\infty)$ arises as a canonical part of the Banach norm of its representative $c$ in $T$, and the hyperbolic length of the limit Grunsky norm of this function is equal by Corollary 2 to each of invariant distances in $\tilde{T}$ between the point $c$ and the origin. This determines the basic features of both Grunsky norms.

The corresponding holomorphic functions

$$h_{\mathcal{K}^0}(c) = \sum_{m,n=1}^{\infty} \sqrt{mn} \ c_{mn} x_m^0 x_n^0$$

generating the norm $\mathcal{K}(f)$ become linear on $\tilde{T}$, which provide some interesting applications.

We establish here the following property of this domain.

**Theorem 2.** The domain $\tilde{T}$ is not starlike (with respect to the origin of $L$).

The problem on starlikness of Teichmüller spaces in Bers’ embedding was stated in 1970 in the collection of problems in the book [4]. This problem still does not be solved completely.
Its negative solution for the universal Teichmüller space $T$ was given in [17]. This result also covers other models of $T$ and later has been extended to finite dimensional Teichmüller spaces $T(0, n)$ of sufficiently large dimensions. Another proof of non-starlikness of $T$ is given in [22]. Both proofs explicitly provide the functions violating this property.

**Proof.** We apply the same construction as in [22]. Pick unbounded convex rectilinear polygon $P_n$ with finite vertices $A_1, \ldots, A_{n-1}$ and $A_n = \infty$. Denote the exterior angles at $A_j$ by $\pi \alpha_j$ so that $\pi < \alpha_j < 2\pi$, $j = 1, \ldots, n-1$. The conformal map $f_n$ of the lower half-plane $H^* = \{ z : \text{Im} z < 0 \}$ onto the complementary polygon $P_n^* = \mathbb{C} \setminus \overline{P_n}$ is represented by the Schwarz-Christoffel integral

$$f_n(z) = d_1 \int_0^z (\xi - a_1)^{n-1}(\xi - a_2)^{n-1} \cdots (\xi - a_{n-1})^{n-1} d\xi + d_0,$$

with $a_j = f_n^{-1}(A_j) \in \mathbb{R}$ and complex constants $d_0, d_1$; here $f_n^{-1}(\infty) = \infty$. Its Schwarzian derivative is given by

$$S_{f_n}(z) = b_{f_n}'(z) - \frac{1}{2} b_{f_n}^2(z) = \sum_{j=1}^{n-1} \frac{C_j}{(z - a_j)^2} - \sum_{j,l=1}^{n-1} \frac{C_{jl}}{(z - a_j)(z - a_l)},$$

where $b_f = f''/f'$ and

$$C_j = - (\alpha_j - 1) - (\alpha_j - 1)^2/2 < 0, \quad C_{jl} = (\alpha_j - 1)(\alpha_l - 1) > 0.$$

It defines a point of the universal Teichmüller space $T$ modelled as a bounded domain in the space $B(H^*)$ of hyperbolically bounded holomorphic functions on $H^*$ with norm

$$\| \varphi \|_{B(H^*)} = \sup_{H^*} |z - \tau|^2 |\varphi(z)|.$$

By the Ahlfors-Weill theorem [3], every $\varphi \in B(H^*)$ with $\| \varphi \|_{B(H^*)} < 1/2$ is the Schwarzian derivative of a univalent function $f$ in $H^*$, and this function has quasiconformal extension onto the upper half-plane $H = \{ z : \text{Im} z > 0 \}$ with Beltrami coefficient of the form

$$\mu_\varphi(z) = -2y^2 \varphi(\overline{z}), \quad \varphi = S_f(z = x + iy \in H^*)$$

called harmonic.

Denote by $r_0$ the positive root of the equation

$$\frac{1}{2} \left[ \sum_{j=1}^{n-1} (\alpha_j - 1)^2 + \sum_{j=1}^{n-1} (\alpha_j - 1)(\alpha_l - 1) \right] r^2 - \sum_{j=1}^{n-1} (\alpha_j - 1) r - 2 = 0,$$

and put

$$S_{f_n, \tau} = t b_{f_n}' - b_{f_n}^2/2, \quad t > 0.$$

Then for appropriate vertices $\alpha_j$ and appropriate Moebius map $\sigma : \mathbb{D}^* \to H^*$, we have the following result from [22].

**Lemma 6.** For any convex polygon $P_n$, the Schwarzians $r S_{f_n, r_0}$ with $0 < r < r_0$ create the univalent function $w_r = f_n : H^* \to \mathbb{C}$ whose harmonic Beltrami coefficients $\nu_r(z) = -(r/2)y^2 S_{f_n, r_0}(\overline{z})$ in $H$ are extremal in their equivalence classes, and

$$k(f_n \circ \sigma) = \mathcal{K}(f_n \circ \sigma) = \frac{r}{2} \| S_{f_n, r_0} \|_{B(H^*)}. \quad (30)$$
This lemma yields that any function $w_r$ with $r < r_0$ does not admit extremal quasi-conformal extensions of Teichmüller type; the extremal extensions have harmonic Beltrami coefficients $\mu_{S_{w_r}}$ given by (29). Therefore, the Schwarzians $S_{w_r}$ for some $r$ between $r_0$ and 1 must lie outside of the space $T$; so this space is not a starlike domain in $B(H^*)$ and in $B(\mathbb{D}^*)$.

Consider the corresponding Grunsky coefficients and the interval

$$I = \{tc(f_n \circ \sigma) : 0 \leq t \leq 1\}.$$

Since $S_{f_n, \sigma}$ is an inner point of $T$, it has a neighborhood $U_0$ whose intersection with the image of the interval $I$ in $T$ contains a non-degenerate subinterval. Hence, all $tc(f_n \circ \sigma)$ with $t$ sufficiently close to 1 belong to $\overline{T}$ (correspond to univalent functions), and the same holds by for all $t \in [0, r_0]$, but the interval $I$ does not lie entirely in $T$. So this domain is not starlike.

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