Uniform approximation of non-autonomous evolution equations *

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Abstract

We study $L^2$-maximal regularity in a Hilbert space $H$ for non-autonomous linear evolution equations of the form

$$\dot{u}(t) + A(t)u(t) = f(t) \quad t \in [0, T], \quad u(0) = u_0. \quad (1)$$

where $A(t), t \in [0, T]$ arise from a non-autonomous sesquilinear forms $a(t, \cdot, \cdot)$ with constant domain $V \subset H$. $L^2$-maximal regularity result is proved recently in [7] when $a$ is Hölder continuous of type $\alpha > 1/2$. In this paper we recover the same results by the approximation method developed in [19], [35] and [20]. The method uses an appropriate approximation $A_\Lambda(\cdot)$ of $A(\cdot)$ for which

$$\dot{u}_{\Lambda}(t) + A_\Lambda(t)u_{\Lambda}(t) = f(t) \quad t \in [0, T], \quad u_{\Lambda}(0) = u_0 \quad (2)$$

has $L^2$-maximal regularity where $\Lambda$ is a subdivision of $[0, T]$. Furthermore, we show that there exists a sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of subdivisions of $[0, T]$ depending on the modulus of continuity such that the sequence of the solutions $u_{\Lambda_n}$ of (2) converges in $L^2(0, T; V) \cap H^1(0, T; H) \cap C(0, T; V)$ uniformly on the initial datas $(u_0, f)$ to the solution $u$ of (1) as $n \to 0$. Moreover, we show that such an uniform converges with respect to initial datas holds for arbitrary subdivision of $[0, T]$ under a little more assumptions on the modulus of continuity.

keywords: Sesquilinear forms, non-autonomous evolution equations, maximal regularity, approximation.

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Introduction

Let $V, H$ be two separable Hilbert space such that $V$ is continuously and densely embedded into $H$. Consider a non-autonomous form

$$a : [0, T] \times V \times V \mapsto \mathbb{C}$$

such that $a(t, \cdot, \cdot)$ is sesquilinear for all $t \in [0, T]$, $a(\cdot, u, v)$ is measurable for all $u, v \in V$,

$$|a(t, u, v)| \leq M\|u\|_V \|v\|_V \quad t \in [0, T], u, v \in V, \quad \text{(boundedness)}$$

and

$$\Re a(t, u, u) \geq \alpha \|u\|_V^2 \quad t \in [0, T], v \in V, \quad \text{(coerciveness)}$$
for some $\alpha > 0$ and $M \geq 0$. For each $t \in [0, T]$ we associate a unique operator $\mathcal{A}(t) \in (V, V')$ such that

$$a(t, u, v) = \langle \mathcal{A}(t)u, v \rangle \quad \text{for all } u, v \in V.$$  

Then we say that the non-autonomous Cauchy problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0 \quad (3)$$

has $L^2$-maximal regularity in $H$ if for every $f \in L^2(0, T; H)$ and $u_0 \in V$ there exists a unique function $u$ belonging to $MR(V, H) := L^2(0, T; V) \cap H^1(0, T; H)$ such that $u$ satisfies (3).

Considering (3) on $V'$, Lions proved on 1961 (see [26] or [12, p. 620]) the following $L^2$-maximal regularity in $V'$ result:

**Theorem 0.1.** (Lions) For all $f \in L^2(0, T; V')$ and $u_0 \in H$, the problem (3) has a unique solution $u \in MR(V, V') := L^2(0, T; V) \cap H^1(0, T; V')$.

Theorem 0.1 requires only the measurability of $t \mapsto a(t, u, v)$ for all $u, v \in V$. However, in applications to boundary valued problems maximal regularity in $V'$ is not sufficient. Only the part $A(t)$ of $\mathcal{A}(t)$ in $H$ does realize the boundary conditions in question. One is more interested on $L^2$-maximal regularity in $H$:

**Problem 0.2.** If $f \in L^2(0, T; H)$ and $u_0 \in V$, does the solution $u$ of (3) belong to $H^1(0, T; H)$?

This problem is asked by Lions in [26, p. 68] for $u_0 = 0$ and $a(t, u, v) = a(t, v, u)$, i.e., $a$ is symmetric. A recent result by Dier [14], show that the answer of this question is negative in general. On the other hand, some positive results are due to Lions [26, p. 68, p. 94, ], [26, Theorem 1.1, p. 129] and [25, Theorem 5.1, p. 138] and to Bardos [9] under additional regularity assumptions on the form $a$, the initial value $u_0$ and the inhomogeneity $f$. More recently, this problem has been studied with some progress and different approaches by Arendt, Dier, Laasri and Ouhabaz [6], Arendt and Monniaux [7], Ouhabaz [27], Dier [15], Haak and Ouhabaz [29], Ouhabaz and Spina [30] and Dier and Zacher [16]. Results on multiplicative perturbation are established in [10] [13] [8].

In this paper we are interested with the following nice result due to Arendt and Monniaux [7]:

**Theorem 0.3.** Assume that $D(\mathcal{A}(0)^{1/2}) = V$ and that there exists $0 \leq \gamma < 1$ and a continuous function $\omega : [0, T] \rightarrow [0, +\infty)$ with

$$\sup_{t \in [0, T]} \frac{\omega(t)}{t^{1/2}} < \infty \quad \text{and} \quad \int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} < \infty \quad (4)$$

such that

$$|a(t, u, v) - a(s, u, v)| \leq \omega(|t - s|) \|u\|_V\|v\|_{V', \gamma} \quad (t, s \in [0, T], u, v \in V)$$

where $V, \gamma := [H, V]$ is the complex interpolation space. Then the Cauchy problem (3) has $L^2$-maximal regularity in $H$. Moreover, for each $f \in L^2(0, T, H)$ and $u_0 \in V$ the solution $u$ of (3) is continuous on $[0, T]$ with values in $V$.

The aim of this paper is to give an explicit approximation of the problem (3) under the assumption of Theorem 0.3 which is very useful to obtain qualitative properties of the unknown solution $u$ of (3). The method employs an approximation by discretisation of the function $\mathcal{A}(\cdot) : [0, T] \rightarrow (V, V')$ and then taking a suitable limit. Namely, let $\Lambda := \{0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T\}$ be a subdivision of $[0, T]$. Consider an approximation $\mathcal{A}_\Lambda : [0, T] \rightarrow \mathcal{L}(V, V')$ of $\mathcal{A}$ given by

$$\mathcal{A}_\Lambda(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \mathcal{A}_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} \mathcal{A}_{k+1} \quad \text{for } t \in [\lambda_k, \lambda_{k+1}]$$

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with

\[ A_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r) \, dr, \quad u \in V, k = 0, 1, \ldots, n. \]

The integral above makes sense since \( t \mapsto A(t)u \) is Bochner integrable on \([0, T]\) with values in \( V'\) for all \( u \in V\). Note that \( \|A(t)u\|_{V'} \leq M \|u\|_V \) for all \( u \in V\) and all \( t \in [0, T]\) and \( t \mapsto A(t)\) is strongly measurable by the Pettis' Theorem \([3]\) Theorem 1.1.1]. This is true since \( V\) and \( H\) are separable and \( t \mapsto A(t)\) is weakly measurable.

We prove that for all \( u_0 \in V\) and \( f \in L^2(0, T; H)\), the non-autonomous problem

\[ \dot{u}_\Lambda(t) + A_\Lambda(t)u_\Lambda(t) = f(t), \quad a.e. \text{ on } (0, T), \quad u_\Lambda(0) = u_0 \quad \tag{5} \]

has a unique solution \( u_\Lambda \in MR(V, H) \cap C(0, T, V)\), and \( (u_\Lambda)_\Lambda\) converges weakly in \( MR(V, H)\) as \( |\Lambda| \to 0\), and the weak limit \( u := \lim_{|\Lambda| \to 0} u_\Lambda\) solves uniquely \([3]\). This provides an alternative proof of Theorem [3] and an approximation of the solution. Moreover, we show that for each null sequence \((t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+\) such that

\[ \lim_{n \to \infty} \frac{\omega(t_n)}{t_n^{\gamma/2}} = 0 \]

and all uniforme subdivision \((\Lambda_n)_{n \in \mathbb{N}}\) of \([0, T]\) with \( |\Lambda_n| = \frac{t_n}{2}\), the sequence \((u_{\Lambda_n})_{n \in \mathbb{N}}\) converges (strongly) to \( u\) in \( MR(V, H) \cap C(0, T, V)\) as \( n \to 0\) uniformly on the initial datas \((x_0, f)\). Thanks to [4], such a null sequence exists. If, in addition, we assume that

\[ \lim_{t \to 0} \frac{\omega(t)}{t^{\gamma/2}} = 0, \]

then we show that \((u_\Lambda)_\Lambda\) converges to \( u\) uniformly on the initial datas \((u_0, f)\) in \( MR(V, H) \cap C(0, T, V)\) as \( |\Lambda| \to 0\) for arbitrary uniform subdivision \( \Lambda\) of \([0, T]\). More precisely, we obtain that

\[ \|u - u_\Lambda\|_{MR} \leq C \left[ \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} \, ds \right] \left[ \|u_0\|_V + \|f\|_{L^2(0, T; H)} \right], \]

and

\[ \|u - u_\Lambda\|_{C(0, T, V)} \leq C \left[ \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} \, ds \right] \left[ \|u_0\|_V + \|f\|_{L^2(0, T; H)} \right] \]

for some positive constant \( C > 0\) depending only on \( M, \alpha, \gamma\) and \( c_H\), where \( c_H\) is the continuous embedding constant of \( V\) into \( H\). For this we first prove that \((u_\Lambda)_\Lambda\) converges in \( MR(V', V')\) uniformly on the initial datas \((u_0, f)\) as \( |\Lambda| \to 0\). This will be proved in Section [4] in a more general situation.

It is well known that the solution of a non-autonomous linear evolution equation can be given by a strongly continuous evolution family \( \{U(t, s) : 0 \leq s \leq t \leq T\} \in (H)\). Our approximation approach will allows us to study whether or not this evolution family is eventually norm continuous. This will be the subject of a future work.

1 Uniform convergence on Banach spaces

In this section we consider a more general setting. Namely, let \((D, \|\cdot\|_D)\) and \((X, \|\cdot\|)\) be two Banach spaces such that \( D\) is continuously and densely embedded into \( X\) (we write \( D \hookrightarrow_d X\)) and let \( A : [0, T] \to \mathcal{L}(D, X)\) be a strongly measurable and bounded function. Let \( p \in (1, \infty)\) be fixed.
**Definition 1.1.** We say that \( A \) has \( L^p \)-maximal regularity on the bounded interval \([0,T]\), and we write \( A \in \mathcal{MR}_p(0,T) \), if for each interval \([a,b]\) \( \subset [0,T] \) and every \( f \in L^p(a,b;X) \) there exists a unique function \( u \) belonging to \( L^p(a,b;D) \cap W^{1,p}(a,b;X) \) such that
\[
\dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [a,b], \quad u(a) = 0.
\tag{6}
\]

Note that \( W^{1,p}(a,b;X) \subset C([a,b];X) \), so that \( u(a) = 0 \) in \((6)\) is well defined. The maximal regularity space
\[ MR_p(D,X) := MR_p(a,b,D,X) := L^p(a,b;D) \cap W^{1,p}(a,b;X) \]
is a Banach space for the norm
\[ \| u \|_{MR} := \| u \|_{L^p(a,b;D)} + \| u \|_{W^{1,p}(a,b;X)}. \]

Definition 1.1 can be reformulate in terms of sum methods. For this, we denote by \( A + B \) the closed subspace of \( MR_p(a,b,D,X) \) consisting of all functions \( u \) that satisfies \( u(a) = 0 \). For each \([a,b] \subset [0,\tau]\) consider the two unbounded linear operators \( A = A_{a,b} \) and \( B = B_{a,b} \) with domains \( D(A) = L^p(a,b;D) \) and \( D(B) = \{ u \in W^{1,p}(a,b;X), u(a) = 0 \} \) defined by
\[ (A f)(t) = A(t)f(t) \quad \text{and} \quad (B u)(t) = \dot{u}(t) \quad \text{for almost every } t \in [a,b]. \]

Thus \( A : [0,T] \to \mathcal{L}(D,X) \) has \( L^p \)-maximal regularity if and only if the unbounded operator \( A + B \) with domain \( D(A + B) = MR_0(D, X) \) is invertible.

**Remark 1.2.** (i) Assume that \( A \in \mathcal{MR}_p(0,T) \). Then the uniqueness of solvability in each subinterval \([a,b]\) implies that \((A_{a,b} + B_{a,b})^{-1}\) is the restriction to \( L^p(a,b;X) \) of \((A_{0,T} + B_{0,T})^{-1}\).

(ii) Remark that \( A \in \mathcal{MR}_p(D,X) \) if and only if \( \rho + A \in \mathcal{MR}_p(D,X) \) for some (or all) \( \rho \in \mathbb{C} \).

In fact, if \( f \in L^p(a,b;X) \), \( \rho \in \mathbb{C} \) and \( g(t) := e^{\rho t}f(t) \). Then a function \( u \in MR_p(D,X) \) satisfies
\[ \dot{u}(t) + A(t)u(t) + \rho u(t) = f(t), \quad \text{a.e. on } [a,b], \quad u(a) = 0 \]
if and only if \( v(\cdot) := e^{\rho \cdot}u(\cdot) \in MR_p(D,X) \) satisfies
\[ \dot{v}(t) + A(t)v(t) = g(t), \quad \text{a.e. on } [a,b], \quad v(a) = 0. \]

If \( A \in \mathcal{MR}_p(0,T) \), then for all \( 0 \leq a \leq b \leq T \) the homogeneous problem
\[ \dot{u}(t) + A(t)u(t) = f(t) \quad \text{a.e. on } [a,b], \quad u(a) = x \tag{7} \]
has a unique solution \( u \in MR_p(D,X) \) for all \( f \in L^p(a,b;X) \) and for all \( x \) in the **trace space**
\[ Tr = Tr_p(a,b,D,X) := \{ u(a), \quad u \in MR_p(a,b,D,X) \}. \]

The trace space is a Banach space with the norm
\[ \| x \|_{Tr} := \inf \{ \| u \|_{MR} : u(a) = x \}. \]

Note that the trace space does not depend on the interval \([a,b]\). It is isomorphic to the real interpolation space \((X,D)_{\frac{1}{p^*},p^*}\), where \( \frac{1}{p^*} + \frac{1}{p} = 1 \). Moreover,
\[ MR_p(D,X) \hookrightarrow C([a,b]; Tr). \]
The reader may consult e.g., [5, 32, 31] and the references therein for further references.

For autonomous Cauchy problems, that is if \(A(\cdot) = A\) is constant, \(L^p\)-maximal regularity is independent of the bounded interval \([0, T]\) and of \(p \in (1, \infty)\) [24, 11, 34]. Further, if \(A\) has \(L^p\)-maximal regularity then \(A\) is closed as unbounded operator on \(X\) and \(-A\) generates a holomorphic \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\) [4, 17, 24]. In Hilbert spaces an operator \(A\) has \(L^p\)-maximal regularity if and only if \(-A\) generates a holomorphic \(C_0\)-semigroup [13]. This equivalence is restricted to Hilbert spaces [24], see also [24]. In this section we will denote by \(\mathcal{MR}\) the set of all operators \(A \in L(D, X)\) having \(L^p\)-maximal regularity.

Now we recall that a strongly measurable function \(A : [0, T] \rightarrow (D, X)\) is (uniformly) relatively continuous (in the sense of [5, Definition 2.5]) if for every \(\varepsilon > 0\) there exist \(\delta > 0\) and \(\eta \geq 0\) such that for all \(x \in D\) and for all \(t, s \in [0, T]\) one has

\[
\|A(t)x - A(s)x\| \leq \varepsilon \|x\|_D + \eta \|x\| \tag{8}
\]

whenever \(|t - s| \leq \delta\). Note that if \(A\) is relatively continuous then \(A\) is bounded. The notion of relative continuity was introduced in by Arendt, Chill, Fornaro and Poupaud, to establish \(L^p\)-maximal regularity [5, Theorem 2.7].

Next we assume that there exists an approximation \(A_n : [0, \tau] \rightarrow L(D, X)\) (strongly measurable) of \(A\) with the following properties:

\((H_1)\) there exists \(C > 0\) such that \(\|A_n(t)\|_{L(D, X)} \leq C\) for all \(t \in [0, \tau]\) and \(n \in \mathbb{N}\),

\((H_2)\) for each \(x \in D\) one has \(A_n(t)x \rightarrow A(t)x\) as \(n \rightarrow \infty\) in \(X\) t.a.e. on \([0, \tau]\),

\((H_3)\) for every \(\varepsilon > 0\) there exist \(\eta \geq 0\), \(n_0 \in \mathbb{N}\) such that for all \(x \in D, n \geq n_0, t \in [0, \tau]\) one has

\[
\|A_n(t)x - A(t)x\| \leq \varepsilon \|x\|_D + \eta \|x\| \tag{9}
\]

\((H_4)\) \(A_n \in \mathcal{MR}_p(0, T)\) for all \(n \in \mathbb{N}\).

Then the following stability result was proved by EL-Mennaoui and Laasri [19, Theorem 4.5].

**Theorem 1.3.** Let \(A : [0, T] \rightarrow L(D, X)\) be strongly measurable and relatively continuous. Assume that \(A(t) \in \mathcal{MR}\) for all \(t \in [0, T]\) and \(A_n\) satisfy the hypothesis \((H_1)-(H_4)\). Let \(u_n \in Tr\) and \(f_n \in L^p(0, T; X)\) such that \(u_n \rightarrow x \) in \(Tr\) and \(f_n \rightarrow f\) in \(L^p(0, T; X)\). Then the sequence \((u_n)\) of solutions of

\[
\dot{u}_n(t) + A_n(t)u_n(t) = f_n(t) \quad a.e. \quad [0, T], \quad u_n(0) = x_n \tag{10}
\]

converges in \(\mathcal{MR}_p(D, X)\) and \(u := \lim_{n \rightarrow \infty} u_n\) is the unique solution of

\[
\dot{u}(t) + A(t)u(t) = f(t) \quad a.e. \quad [0, T], \quad u(0) = x. \tag{11}
\]

The aim of this section is to show that for \(x = x_n = 0\) the convergence established in Theorem 1.3 is actually uniform with respect the the inhomogeneity \(f\). If \(\eta = 0\) in (9), then we obtain that such a convergence is uniform with respect to both initial data \(f\) and \(x\).

**Theorem 1.4.** Let \(A : [0, T] \rightarrow L(D, X)\) be strongly measurable and relatively continuous. Assume that \(A(t) \in \mathcal{MR}\) for all \(t \in [0, T]\) and \(A_n\) satisfy the hypothesis \((H_1)-(H_4)\). Then for every \(\varepsilon > 0\) there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) one has

\[
\|(3 + 2\mathbb{B})^{-1} - (3 + 2\mathbb{B}_n)^{-1}\|_{L^p(0,T; \mathcal{MR}_p(D, X))} \leq \varepsilon. \tag{12}
\]
Proof. We proceed in three steps and follow the same idea as in the proof of [19, Theorem 4.5].

Step 1. By [19, Lemma 4.1] there exists a constant $M(A) > 0$ and $\rho_1 \geq 0$ independent on $t \in [0,T]$ such that
\[
||(|\rho + \mathfrak{A}(t) + \mathfrak{B})^{-1}||_{L^p(a,b,X),MR_2(a,b,D,X)} \leq M(A)
\] (13)
and
\[
||(|\rho + \mathfrak{A}(t) + \mathfrak{B})^{-1}||_{L^p(a,b,X)} \leq \frac{M(A)}{1 + \rho}
\] (14)
for all $\rho \geq \rho_1$ and all $[a, b] \subset [0, T]$. On the other hand, we have from [19, Lemma 4.2] that there exists $\rho_2 \geq 0, \delta > 0$ and $n_0 \in \mathbb{N}$ such that for each $[a, b] \subset [0, T], |b - a| \leq \delta$ implies that
\[
||(|\mathfrak{A}_n - \mathfrak{A}(t)(\rho + \mathfrak{A}(t) + \mathfrak{B})^{-1}||_{L^p(a,b,X)} \leq 3/4,
\] (15)
for all $t \in [0,T], n \geq n_0$ and all $\rho \geq \rho_2$. Since $A$ satisfies the assumptions $(H1) - (H4)$, we also have that
\[
||(|\mathfrak{A} - \mathfrak{A}(t)(\rho + \mathfrak{A}_n(t) + \mathfrak{B})^{-1}||_{L^p(a,b,X)} \leq 3/4,
\] (16)
for all $t \in [0,T]$ and all $\rho \geq \rho_2$ provided that $|b - a| \leq \delta$.

Step 2. Let $\delta > 0, \rho_0 := \max\{\rho_1, \rho_2\} \geq 0$ and $n_0 \in \mathbb{N}$ be as in the first step and assume that $T \leq \delta$. Let $t_0 \in [0,T]$ and $\rho > \rho_0$ be fixed. Let $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that
\[
\sum_{k=k_0+1}^{\infty} \|(\mathfrak{A}_n - \mathfrak{A}(t_0))(\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}\|^k_{L^p(0,T,X)} \leq \frac{\varepsilon}{3M(A)}
\] (17)
and
\[
\sum_{k=k_0+1}^{\infty} \|(\mathfrak{A} - \mathfrak{A}(t_0))(\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}\|^k_{L^p(0,T,X)} \leq \frac{\varepsilon}{3M(A)}.
\] (18)
For each $k \in \{1, \ldots, k_0\}$ and $n \in \mathbb{N}$ with $n \geq n_0$ we set
\[I_{k,n} := (\mathfrak{A}_n - \mathfrak{A}(t_0))(\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}k^k\] and $I_k := (\mathfrak{A} - \mathfrak{A}(t_0))(\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}k^k$.

According to $(H3)$, (15) and (16), there exists $n_1 \in \mathbb{N}$ and $\eta \geq 0$ such that for each $n \geq N_0 := \max\{n_0, n_1\}$
\[
\|I_{1,n}f - I_1f\|_{LP(0,T,X)} = \|(\mathfrak{A}_n - \mathfrak{A})(\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}f\|_{LP(0,T,X)}
\leq \frac{\varepsilon'}{2M(A)}\|(|\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}f\|_{MR_2(a,d,X)} + \eta\|(|\rho + \mathfrak{A}(t_0) + \mathfrak{B})^{-1}f\|_{LP(0,T,X)}
\leq \frac{\varepsilon'}{2}\|f\|_{LP(0,T,X)} + \frac{\eta M(A)}{1 + \rho}\|f\|_{LP(0,T,X)}
\]
where $\varepsilon' := \frac{4\varepsilon}{9M(A)^2}$. Thus choosing $\rho \geq \rho_0$ large enough we obtain
\[
\|I_{1,n}f - I_1f\|_{LP(0,T,X)} \leq \varepsilon'\|f\|_{LP(0,T,X)}
\]
for all $n \geq N_0$. This estimate together with (15) and (16), yield
\[
\|I_{2,n}f - I_2f\|_{LP(0,T,X)} = \|I_{1,n}I_{1,n}f - I_1I_1f\|_{LP(0,T,X)}
\leq \|I_{1,n}(I_{1,n} - I_1)f\|_{LP(0,T,X)} + \|(I_{1,n} - I_1)I_1f\|_{LP(0,T,X)}
\leq \frac{3}{4}\varepsilon'\|f\|_{LP(0,T,X)} + \frac{3}{4}\varepsilon'\|f\|_{LP(0,T,X)} = \frac{3}{2}\varepsilon'\|f\|_{LP(0,T,X)},
\]
and thus
\[ \|I_{k,n}f - I_k f\|_{L^p(0,T,X)} \leq \frac{3}{4} k_0^\varepsilon \|f\|_{L^p(0,T,X)} = \frac{\varepsilon}{3M(A)} \|f\|_{L^p(0,T,X)} \]
holds for all \( n \geq N_0 \) and every \( k = 1, 2, \ldots, k_0 \). Combining (17), (18) and (19) we deduce
\[ \|(\mathfrak{A}_n + \mathfrak{B})^{-1}f - (\mathfrak{A} + \mathfrak{B})^{-1}f\|_{\mathcal{M}R} \]
\[ \leq \|(\mathfrak{A}_n + \mathfrak{B})^{-1}(I + (\mathfrak{A}_n - \mathfrak{A})(\mathfrak{A}_n + \mathfrak{B})^{-1})^{-1}f \]
\[ - (\mathfrak{A}_n + \mathfrak{B})^{-1}(I + (\mathfrak{A} - \mathfrak{A}_n)(\mathfrak{A}_n + \mathfrak{B})^{-1})^{-1}f\|_{\mathcal{M}R_p(D,X)} \]
\[ < \frac{\varepsilon}{3M(A)} \|f\|_{L^p(0,T,X)} \]
\[ \leq M(A) \left( \frac{\varepsilon}{3M(A)} + \frac{\varepsilon}{3M(A)} \right) \|f\|_{L^p(0,T,X)} = \varepsilon \|f\|_{L^p(0,T,X)} \]
for all \( n \geq N_0 \).

**Step 2.** Let now \([0,T]\) be an arbitrary closed and bounded interval and set
\[ \tau := \max\{0 \leq \tau' \leq T \text{ such that (12) holds on } [0, \tau']\}. \]
Then \( \tau \geq \delta \). We show that \( \tau = T \). Assume by contradiction that \( \tau < T \) and let \( \tau' < \tau \) such that \( \tau - \tau' \leq \delta/2 \). Then (12) holds if we consider the Cauchy problems (10) and (11) on \([0, \tau'_0]\) or on \([\tau'_0, (\tau'_0 + \delta) \wedge T]\), and thus on \([0, (\tau'_0 + \delta) \wedge T]\) by taking into account Remark 1.2(i). Thus \( (\tau'_0 + \delta) \wedge T \leq \tau \), which is a contradiction. This completes the proof.

The main result of this section is the following.

**Theorem 1.5.** Let \( A : [0,T] \to \mathcal{L}(D,X) \) be strongly measurable and relatively continuous. Assume that \( A(t) \in \mathcal{M}R \) for all \( t \in [0,T] \) and \( A_n \) satisfy the hypothesis \( (H_1) - (H_4) \). Let \( x \in Tr \) and \( f \in L^p(0,T;X) \). Let \( u_n, u \in \mathcal{M}R_p(D,X) \) be, respectively, the solution of
\[ \dot{u}_n(t) + A_n(t)u_n(t) = f_n(t) \text{ a.e on } [0,T], \quad u_n(0) = x \]

and
\[ \dot{u}(t) + A(t)u(t) = f(t) \text{ a.e on } [0,T], \quad u(0) = x. \]

Then for each \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) and \( \tilde{\vartheta} > 0 \) such that for all \( n \geq n_0 \)
\[ \|u_n - u\|_{\mathcal{M}R} \leq \varepsilon \left[ \|x\|_{Tr} + \|f\|_{L^p(0,T,X)} \right] + \tilde{\vartheta} \|x\|_{Tr}. \]

**Proof.** Let \( \varepsilon > 0 \). Choose \( \vartheta \in \mathcal{M}R_p(D,X) \) such that \( \vartheta(0) = x \) and \( \|\vartheta\|_{\mathcal{M}R} \leq 2\|x\|_{Tr} \). Set \( g_n := -\vartheta - A_n(\cdot)\vartheta(\cdot) + f(\cdot) \) and \( g := -\vartheta - A(\cdot)\vartheta(\cdot) + f(\cdot) \in L^p(0,T;X) \). Then there exist \( v_n, u \in \mathcal{M}R_p(D,X) \) such that
\[ \dot{v}_n(t) + A_n(t)v_n(t) = g_n(t) \text{ a.e on } [0,T], \quad v_n(0) = 0 \]

and
\[ \dot{v}(t) + A(t)v(t) = g(t) \text{ a.e on } [0,T], \quad v(0) = 0. \]

By the uniqueness of solvability, \( u_n = v_n + \vartheta \) and \( u = v + \vartheta \). It follows,
\[ \|u_n - u\|_{\mathcal{M}R} = \|v_n - v\|_{\mathcal{M}R} = \|(\mathfrak{A}_n + \mathfrak{B})^{-1}g_n - (\mathfrak{A} + \mathfrak{B})^{-1}g\|_{\mathcal{M}R} \]
\[ \leq \|\mathfrak{A}_n + \mathfrak{B}\|^{-1}\|g_n - g\|_{\mathcal{M}R} + \|(\mathfrak{A}_n + \mathfrak{B})^{-1}g - (\mathfrak{A} + \mathfrak{B})^{-1}g\|_{\mathcal{M}R} \]
\[ \leq M(A,\vartheta - A\vartheta)_{L^p(0,T;X)} + \|(\mathfrak{A}_n + \mathfrak{B})^{-1}g - (\mathfrak{A} + \mathfrak{B})^{-1}g\|_{\mathcal{M}R} \]

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that there exists $\alpha, M > 0$ such that there exists $n_0 \in \mathbb{N}$ and $\eta \geq 0$ such that

$$\|A_n \theta - A\theta\|_{L^p(0,T;X)} \leq \frac{\varepsilon}{2M} \|\theta\|_{L^p(0,T;X)} + 2\eta \|\theta\|_{T^r}$$

and

$$\|(G_n + G) - G\|_{L^p(0,T;X)} \leq \frac{\varepsilon}{4C + 2} \|G\|_{L^p(0,T;X)}$$

$$\leq \frac{\varepsilon}{4C + 2} (\|\hat{\theta} + A\theta\|_{L^p(0,T;X)} + \|f\|_{L^p(0,T;X)})$$

$$\leq \frac{\varepsilon}{2} \|\theta\|_{T^r} + \varepsilon \|f\|_{L^p(0,T;X)}$$

for all $n \geq n_0$, where $c = \max\{1, \sup_{t \in [0,T]} \|A(t)\|_{L(D,X)}\}$. This shows the claims. □

### 2 Non-autonomous forms: assumptions and preliminary results

Throughout the following sections $H, V$ are two separable Hilbert spaces over $\mathbb{C}$ such that $V \hookrightarrow H$; i.e., $V$ is densely embedded into $H$ and

$$\|u\| \leq c_H \|u\|_V \quad (u \in V)$$

for some constant $c_H > 0$. Let $V'$ denote the antidual of $V$. The duality between $V'$ and $V$ is denoted by $\langle \cdot, \cdot \rangle$. As usual, by identifying $H$ with $H'$, we have $V \hookrightarrow H \cong H' \hookrightarrow V'$. These embeddings are continuous and

$$\|f\|_{V'} \leq c_H \|f\| \quad (f \in V')$$

see e.g., [10]. We denote by $(\cdot, \cdot)_V$ the scalar product and $\|\cdot\|_V$ the norm on $V$ and by $(\cdot, \cdot), \|\cdot\|$ the corresponding quantities in $H$. Let $\alpha : [0,T] \times V \times V \to \mathbb{C}$ be a non-autonomous sesquilinear form satisfying

$$|\alpha(t, u, v)| \leq M \|u\|_V \|v\|_V \quad (t \in [0,T], u, v \in V) \tag{22}$$

and

$$\text{Re} \ \alpha(t, u, u) \geq \alpha \|u\|^2_\gamma \quad (t \in [0,T], u \in V) \tag{23}$$

for some constants $\alpha, M > 0$ and $\alpha(., u, v)$ is measurable for all $u, v \in V$. We assume in addition, that there exists $0 \leq \gamma < 1$ and a non-decreasing continuous function $\omega : [0,T] \to [0, +\infty)$ with

$$\sup_{t \in [0,T]} \frac{\omega(t)}{t^{\gamma/2}} < \infty, \tag{24}$$

$$\int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} dt < \infty \tag{25}$$

and

$$|\alpha(t, u, v) - \alpha(s, u, v)| \leq \omega(|t-s|) \|u\|_V \|v\|_V \tag{26}$$

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for all \( t, s \in [0, T] \) and for all \( u, v \in V \) where \( V_\gamma := [H, V]_\gamma \) is the complex interpolation space. Note that

\[
V \hookrightarrow V_\gamma \hookrightarrow H \hookrightarrow V'_\gamma \hookrightarrow V'
\]

with continuous embeddings. By the Lax-Milgram theorem, for each \( t \in [0, T] \) there exists an isomorphism \( A(t) : V \to V' \) such that \( \langle A(t)u, v \rangle = a(t, u, v) \) for all \( u, v \in V \). It is well known that \(-A(t)\), regarding as unbounded operator with domain \( V \), generates a bounded holomorphic semigroup \( e^{-\cdot A(t)} \) of angle \( \theta := \frac{\pi}{2} - \arctan\left(\frac{M}{\alpha}\right) \) on \( V' \). We call \( A(t) \) the operator associated with \( a(t, \cdot, \cdot) \) on \( V' \). We have also to consider the operator \( A(t) \) associated with \( a(t, \cdot, \cdot) \) on \( H \):

\[
D(A(t)) := \{ u \in V : A(t)u \in H \}
\]

Then \(-A(t)\) generates a holomorphic \( C_0 \)-semigroup (of angle \( \theta \)) \( e^{-sA(t)} \) on \( H \) which is the restriction to \( H \) of \( e^{-\cdot A(t)} \), and we have

\[
e^{-\cdot A(t)} = \frac{1}{2\pi i} \int_\Gamma e^{\mu}(\mu + A(t))^{-1}d\mu \quad (27)
\]

where \( \Gamma := \{ re^{i\varphi} : r > 0 \} \) for some fixed \( \varphi \in (\theta, \frac{\pi}{2}) \) (see e.g. [2, Lecture 7],[22],[28, Chapter 1] or [36, Chap. 2]).

The following proposition is of great interest for this paper.

**Proposition 2.1.** [7, Section 2] Let \( b \) be any sesquilinear form that satisfies assumptions [22]-[24] with the same constants \( M \) and \( \alpha \) and let \( \gamma \in [0, 1] \). Let \( B \) and \( B' \) be the associated operators on \( V' \) and \( H \), respectively. Then there exists a constant \( c > 0 \) which depends only on \( M, \alpha, \gamma \) and \( c_H \) such that

1. \( \| (\lambda - B)^{-1} \|_{(V'_\gamma, H)} \leq \frac{c}{(1 + |\lambda|)^{1 - \frac{\gamma}{2}}} \),
2. \( \| (\lambda - B)^{-1} \|_{(V)} \leq \frac{c}{1 + |\lambda|} \),
3. \( \| (\lambda - B)^{-1} \|_{(H, V)} \leq \frac{c}{(1 + |\lambda|)^{\frac{\gamma}{2}}} \),
4. \( \| (\lambda - B)^{-1} \|_{(V'_\gamma, H)} \leq \frac{c}{(1 + |\lambda|)^{\frac{\gamma}{2}}} \),
5. \( \| (\lambda - B)^{-1} \|_{(V'_\gamma, V)} \leq \frac{c}{(1 + |\lambda|)^{1 - \frac{\gamma}{2}}} \),
6. \( \| e^{-sB} \|_{(V'_\gamma, H)} \leq \frac{c}{s^{\gamma/2}} \),
7. \( \| e^{-sB} \|_{(V'_\gamma, V)} \leq \frac{c}{s^{\frac{\gamma}{2}}} \),
8. \( \| e^{-sB} \|_{(V', V)} \leq \frac{c}{s^{\frac{\gamma}{2}}} \),
9. \( \| B e^{-sB} \|_{(H)} \leq \frac{c}{s} \).
Remark 2.2. All estimates in Proposition 2.1 holds for the Kato square root property if

\[ |\Lambda| := \sup_t |\lambda_{t+1} - \lambda_t| = |\lambda_{k+1} - \lambda_k| \text{ for each } k = 0, 1, \ldots, n. \]

Consider a family of sesquilinear forms \( a_k : V \times V \to \mathbb{C} \) given by

\[ a_k(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(r; u, v)dr, \quad u, v \in V \]  

(28)

for each \( k = 0, 1, \ldots, n \). Remark that \( a_k \) satisfies (22)-(23) with the same constants \( \alpha \) and \( M \). The associated time dependent operator is denoted by

\[ A_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r)u dr, \quad u \in V, \quad k = 0, 1, \ldots, n. \]  

(29)

The function \( a_\Lambda : [0, T] \times V \times V \to \mathbb{C} \) defined for \( t \in [\lambda_k, \lambda_{k+1}] \) by

\[ a_\Lambda(t; u, v) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} a_k(u, v) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} a_{k+1}(u, v), \quad u, v \in V, \]  

(30)

is a non-autonomous sesquilinear form which satisfies (22)-(23) with the same constants \( \alpha \) and \( M \). The associated time dependent operator is denoted by

\[ A_\Lambda(\cdot) : [0, T] \to (V, V') \]  

(31)

and is given for \( t \in [\lambda_k, \lambda_{k+1}] \) by

\[ A_\Lambda(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} A_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} A_{k+1}. \]  

(32)

Then \( A_\Lambda \) converges strongly and almost everywhere to \( A \) and also on \( (L^2(0, T, V), L^2(0, T, V')) \) as \( |\Lambda| \to 0 \) [20] Lemma 2.1.

Remark 2.2. All estimates in Proposition 2.1 holds for \( A_\Lambda(t) \) with constant independent of \( \Lambda \) and \( t \in [0, T] \), since \( a_\Lambda \) satisfies (22)-(23) with the same constants \( M \) and \( \alpha \), also \( \gamma \) and \( c_H \) does not depend on \( \Lambda \) and \( t \in [0, T] \).

Recall that a coercive and bounded form \( b : V \times V \to \mathbb{C} \) associated with the operator \( B \) on \( H \) has the Kato square root property if

\[ D(B^{1/2}) = V. \]  

(33)

We prove below that \( a_\Lambda(\cdot; \cdot) \) has the square root property for all \( t \in [0, T] \) if \( a_\Lambda(0; \cdot, \cdot) \) has it. This is essentially based on the abstract result due to Arendt and Monniaux [7] Proposition 2.5]. They proved that for two sesquilinear forms \( a_1, a_2 : V \times V \to \mathbb{C} \) which satisfies (22)-(23), the form \( a_1 \) has the square root property if and only if \( a_2 \) has it provided that

\[ |a_1(u, v) - a_2(u, v)| \leq c\|u\|_V\|v\|_V, \quad u, v \in V \]

for some constant \( c > 0 \).
Proposition 2.3. Assume \( a(0, \ldots) \) has the square root property. Then \( a_\Lambda(t, \ldots) \) has the square root properties for all \( t \in [0, T] \), too.

Proof. Let \( t \in [0, T] \) and let \( k \in \{0, 1, \ldots, n\} \) be such that \( t \in [\lambda_k, \lambda_{k+1}] \). Then assumption \( \omega \) implies that

\[
|a_\Lambda(t, u, v) - a(0, u, v)| \leq \frac{1}{\lambda_{k+1} - \lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} |a(r; u, v) - a(0, u, v)| \, dr
\]

The following results will play an important role latter in the study of the convergence. We first prove that \( a_\Lambda \) has also a modulus of continuity of the same art as for \( a \). In what follows we extend \( \omega \) to \([0, 2T]\) by setting \( \omega(t) = \omega(T) \) for \( T \leq t \leq 2T \).

Proposition 2.4. For all \( u, v \in V \), \( t, s \in [0, T] \)

\[
|a_\Lambda(t, u, v) - a\Lambda(s, u, v)| \leq \omega_\Lambda(|t - s|)\|u\|_V\|v\|_V
\]

where \( \omega_\Lambda : [0, T] \rightarrow [0, +\infty] \) is defined by

\[
\omega_\Lambda(t) := \begin{cases} \frac{\omega(4|\Lambda|)}{|\Lambda|} & \text{for } 0 \leq t \leq 2|\Lambda|, \\ 2\omega(2t) & \text{for } 2|\Lambda| < t \leq T. \end{cases}
\]

Moreover,

\[
\int_0^T \frac{\omega_\Lambda(s)}{s^{1+\gamma/2}} \, ds \leq \frac{4}{1 - \frac{\gamma}{2}} \sup_{t \in [0, T]} \frac{\omega(t)}{t^{\gamma/2}} + 2t^{\gamma/2}\int_0^{2T} \frac{\omega(s)}{s^{1+\gamma/2}} \, ds < \infty,
\]

and

\[
\sup_{t \in [0, T]} \frac{\omega_\Lambda(t)}{t^{\gamma/2}} \leq 2^{1+\gamma/2}\sup_{t \in [0, T]} \frac{\omega(t)}{t^{\gamma/2}} < \infty.
\]

Proof. Let \( u, v \in V \) and \( t, s \in [0, T] \). For the proof of \( 35 \) we distinguish three cases

Case 1: If \( \lambda_k \leq s < t \leq \lambda_{k+1} \) for some fixed \( k \in \{0, 1, \ldots, n\} \). Then we obtain, using \( 26 \) and the fact that \( \omega \) is non-decreasing, that

\[
|a_\Lambda(t, u, v) - a\Lambda(s, u, v)| = \left|\frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k}a_k(u, v) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k}a_{k+1}(u, v) - \frac{\lambda_{k+1} - s}{\lambda_{k+1} - \lambda_k}a_k(u, v) - \frac{s - \lambda_k}{\lambda_{k+1} - \lambda_k}a_{k+1}(u, v)\right|
\]

\[
= \frac{(t - s)}{|\Lambda|} \left|a_k(u, v) - a_{k+1}(u, v)\right|
\]

\[
\leq \frac{(t - s)}{|\Lambda|} \frac{1}{|\Lambda|} \int_0^{|\Lambda|} |a(r + \lambda_k, u, v) - a(r + \lambda_{k+1}, u, v)| \, dr
\]

\[
\leq \frac{(t - s)}{|\Lambda|} \frac{1}{|\Lambda|} \int_0^{|\Lambda|} \omega(\lambda_{k+1} - \lambda_k)|u||V||v|_V \, dr \leq \frac{(t - s)}{|\Lambda|} \omega(|\Lambda|)\|u\|_V\|v\|_V
\]

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Case 2: If \( \lambda_k \leq s \leq \lambda_{k+1} \leq t \leq \lambda_{k+2} \), then we deduce from Step 1 that

\[
|a_\Lambda(t, u, v) - a_\Lambda(s, u, v)| \leq |a_\Lambda(t, u, v) - a_\Lambda(\lambda_{k+1}, u, v)| + |a_\Lambda(\lambda_{k+1}, u, v) - a_\Lambda(s, u, v)| \\
\leq \frac{t - \lambda_{k+1}}{|\Lambda|} |\omega(|\Lambda|)||u||v||v_\gamma| + \frac{\lambda_{k+1} - s}{|\Lambda|} |\omega(|\Lambda|)||u||v||v_\gamma| \\
= \frac{t - s}{|\Lambda|} |\omega(|\Lambda|)||u||v||v_\gamma|.
\]

Case 3: If now \( \lambda_k \leq s \leq \lambda_{k+1} < \cdots < \lambda_l \leq t \leq \lambda_{l+1} \). Then \( \lambda_l - \lambda_{k+1} \leq t - s \leq \lambda_{l+1} - \lambda_k \) and thus

\[
|t - s + \lambda_{k+1} - \lambda_{l+1}| \leq |\Lambda|, \tag{37}
\]

It follows that

\[
a_\Lambda(t, u, v) - a_\Lambda(s, u, v)
= \frac{\lambda_{l+1} - t}{\lambda_l - \lambda_{l+1}} a_\Lambda(u, v) + \frac{t - \lambda_l}{\lambda_{l+1} - \lambda_l} a_\Lambda(\lambda_{l+1}, u, v) - \frac{\lambda_{k+1} - s}{\lambda_{l+1} - \lambda_l} a_\Lambda(u, v) - \frac{s - \lambda_k}{\lambda_{l+1} - \lambda_k} a_\Lambda(\lambda_{k+1}, u, v) \\
= \frac{\lambda_{l+1} - t}{|\Lambda|} [a_\Lambda(u, v) - a_\Lambda(\lambda_{l+1}, u, v)] + \frac{t - \lambda_l}{|\Lambda|} [a_\Lambda(\lambda_{l+1}, u, v) - a_\Lambda(\lambda_{k+1}, u, v)] \\
+ \frac{\lambda_{k+1} - \lambda_k + t - s}{|\Lambda|} a_\Lambda(u, v) + \frac{\lambda_k - \lambda_l + t - s}{|\Lambda|} a_\Lambda(\lambda_{k+1}, u, v)
\]

Because of (37) and since \( \lambda_k - \lambda_l = \lambda_{k+1} - \lambda_{l+1} \), we deduce that

\[
|a_\Lambda(t, u, v) - a_\Lambda(s, u, v)| \leq \frac{\lambda_{k+1} - t}{|\Lambda|} |\omega(\lambda_l - \lambda_k)| + \frac{t - \lambda_k}{|\Lambda|} |\omega(\lambda_{l+1} - \lambda_{k+1})| \\
+ \frac{|t - s + \lambda_{k+1} - \lambda_{l+1}|}{|\Lambda|} |\omega(\lambda_{l+1} - \lambda_l)| \\
\leq \omega(\lambda_k - \lambda_l) + \omega(\lambda_{l+1} - \lambda_l) \\
\leq 2\omega(2(t - s)).
\]

This completes the proof of (34). Let now prove (35). By construction we have

\[
\int_0^T \frac{\omega_\Lambda(t)}{t^{1+\gamma/2}} dt = \int_0^{2|\Lambda|} \frac{\omega(4|\Lambda|)}{|\Lambda|} t^{-\gamma/2} dt + \int_0^T \frac{\omega(2t)}{t^{1+\gamma/2}} dt \\
\leq \frac{2^{\gamma+1}}{1 - \frac{\gamma}{2}} \frac{\omega(4|\Lambda|)}{|\Lambda|} + 2^{\gamma/2} \int_0^{2T} \frac{\omega(t)}{t^{1+\gamma/2}} dt \\
\leq \frac{4}{1 - \frac{\gamma}{2}} \sup_{t \in [0,T]} \frac{\omega(t)}{t^{1+\gamma/2}} + 2^{\gamma/2} \int_0^{2T} \frac{\omega(t)}{t^{1+\gamma/2}} dt
\]

which is finite by (24). The inequality (36) is easy to prove. \(\square\)

Note that condition (26) implies that \( A(t) - A(s) \in (V, V'_\gamma) \) for each \( t, s \in [0, T] \) and

\[
||A(t) - A(s)||_{(V, V'_\gamma)} \leq \omega(|t - s|). \tag{38}
\]

According to Proposition 2.4 similar estimates hold for \( A_\Lambda(\cdot) : \)

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Lemma 2.5. For each \( t, s \in [0, T] \) we have \( \|A_{\lambda}(t) - A_{\lambda}(s)\|_{(V, V')} \leq \omega_{\lambda}(|t - s|) \) \( (39) \) and

\[ \|A_{\lambda}(t) - A(t)\|_{(V, V')} \leq 2\omega(2|\lambda|). \] \( (40) \)

Proof. The estimate \( (39) \) follows from \( (34) \). For the second statement, let \( t \in [0, T] \) and let \( k \in \{0, 1, \ldots, n\} \) be such that \( t \in [\lambda_k, \lambda_{k+1}] \). Then

\[ A_{\lambda}(t) - A(t) = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} [A_{\lambda_k} - A(t)] + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} [A_{\lambda_{k+1}} - A(t)] \]

\[ = \frac{\lambda_{k+1} - t}{(\lambda_{k+1} - \lambda_k)^2} \int_{\lambda_k}^{\lambda_{k+1}} [A(r) - A(t)]dr + \frac{t - \lambda_k}{(\lambda_{k+1} - \lambda_k)^2} \int_{\lambda_{k+1}}^{\lambda_{k+2}} [A(r) - A(t)]dr. \]

Then using \( (33) \) and the fact that \( \omega \) is non-decreasing we obtain

\[ \|A_{\lambda}(t) - A(t)\|_{(V, V')} \leq \frac{\lambda_{k+1} - t}{(\lambda_{k+1} - \lambda_k)^2} \int_{\lambda_k}^{\lambda_{k+1}} \omega(t - r)dr + \frac{t - \lambda_k}{(\lambda_{k+1} - \lambda_k)^2} \int_{\lambda_{k+1}}^{\lambda_{k+2}} \omega(t - r)dr \]

\[ \leq \omega(|\lambda|) + \omega(2|\lambda|) \leq 2\omega(2|\lambda|), \]

which proves the claim. \( \square \)

3 \( L^2 \)-maximal regularity in \( H \) : a weak approximation

Recall that \( V, H \) denote two separable Hilbert spaces and \( a : [0, T] \times V \times V \to \mathbb{C} \) is a non-autonomous form satisfying \( (22)-(26) \) such that \( D(A(0)^{1/2}) = V \). Let \( A(t) \) the operator associated with \( a(t, \cdot, \cdot) \) on \( V' \) for each \( t \in [0, T] \) and consider the non-autonomous Cauchy problem

\[ \dot{u}(t) + A(t)u(t) = f(t) \quad a.e. \text{ on } [0, T], \quad u(0) = u_0 \quad (41) \]

Let \( \Lambda \) be an uniform subdivision of \([0, T]\),

\[ A_{\lambda} : [0, T] \to (V, V') \quad \text{and} \quad a_{\lambda} : [0, T] \times V \times V \to \mathbb{C} \]

be given by \( (31)-(32) \) and \( (30) \), respectively, and consider the Cauchy problem

\[ \dot{u}_{\lambda}(t) + A_{\lambda}(t)u_{\lambda}(t) = f(t) \quad a.e. \text{ on } [0, T], \quad u_{\lambda}(0) = u_0. \quad (42) \]

Clearly, \( t \mapsto a_{\lambda}(\cdot, u, v) \) is piecewise \( C^1 \) for all \( u, v \in V \). Moreover, \( a_{\lambda} \) has the Kato square property by Lemma 2.3. Then the Cauchy problem \( (42) \) has \( L^2 \)-maximal regularity in \( H \) by \( [9] \) Theorem 1.1 or \( [6] \) Theorem 4.2. On the other hand, we known by Lions' theorem that for a given \( f \in L^2(0, T; H) \) and \( u_0 \in V \) the Cauchy problem \( (41) \) has a unique solution \( u \in MR(V, V') \). Furthermore, it is known that the sequence \( (u_{\lambda}) \) of solutions of \( (42) \) converges (strongly) in \( MR(V, V') \) to \( u \) as \( |\lambda| \) goes to \( 0 \) \( [20] \) Proposition 3.2. The main result of this section show that \( (u_{\lambda}) \) converges weakly in \( MR(V, H) \) to \( u \) as \( |\lambda| \) goes to \( 0 \). This in particular gives an alternative proof of Theorem 0.3.

Theorem 3.1. Let \( f \in L^2(0, T; H) \) and \( u_0 \in V \) and let \( u_{\lambda} \in MR(V, H) \) be the solution of \( (42) \). Then \( u_{\lambda} \) converges weakly in \( MR(V, H) \) as \( |\lambda| \to 0 \) and \( u := \lim_{|\lambda| \to 0} u_{\lambda} \) satisfies \( (41) \).
For the proof we need first some preliminary lemmas. Let \( f \in L^2(0, T; H) \) and \( u_0 \in V \), then the solution \( u_\Lambda \) of (42) satisfies the following key formula

\[
u_\Lambda(t) = e^{-tA_\Lambda(t)}u_0 + \int_0^t e^{-(t-s)A_\Lambda(t)}f(s)ds + \int_0^t e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s))u_\Lambda(s)ds \tag{43}\]

for all \( t \in [0, T] \). This formula is due to Acquistapace and Terreni [1] and was proved in a more general setting in [7, Proposition 3.5]. For the operator valued function \( A_\Lambda \), this formula can be derived in a more classical way. In the sequel we will use the following notations:

\[
u_{\Lambda,1}(t) := e^{-tA_\Lambda(t)}u_0, \quad \nu_{\Lambda,2}(t) := \int_0^t e^{-(t-s)A_\Lambda(t)}f(s)ds \tag{44}\]

and

\[
u_{\Lambda,3}(t) := \int_0^t e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s))u_\Lambda(s)ds. \tag{45}\]

The next two lemmas follow, thanks to Proposition 2.4, Lemma 2.5 and Remark 2.2, by using the same argument as in the proof of Arendt and Monniaux [7, Theorem 4.1].

**Lemma 3.2.** Let \( Q_\Lambda^\mu : L^2(0, T, H) \rightarrow L^2(0, T, H) \) denotes the linear operator defined for all \( g \in L^2(0, T, H) \) and \( \mu \geq 0 \) by

\[
(Q_\Lambda^\mu g)(t) := \int_0^t (A_\Lambda(t) + \mu)e^{-(t-s)(A_\Lambda(t)+\mu)}(A_\Lambda(t) - A_\Lambda(s))(A_\Lambda(s) + \mu)^{-1}g(s)ds \quad t\text{-a.e.} \tag{46}\]

Then \( \lim_{\mu \to \infty} \|Q_\Lambda^\mu\|_{L^2(0, T, H)} = 0 \) uniformly on \( \Lambda \) and thus \( I - Q_\Lambda^\mu \) is invertible on \( L^2(0, T, H) \) for \( \mu \) large enough and for all \( \Lambda \).

**Lemma 3.3.** There exists constant \( c > 0 \) depending only on \( \alpha, M, \gamma \) and \( c_H \) such that

\[
\|A_\Lambda u_{\Lambda,1}\|_{L^2(0, T, H)} \leq c\|u_0\|_V, \tag{47}\]

\[
\|A_\Lambda u_{\Lambda,2}\|_{L^2(0, T, H)} \leq c\|f\|_{L^2(0, T, H)}. \tag{48}\]

According to Lemma 3.2 and replacing \( A_\Lambda(t) \) with \( A(t) + \mu \), we may assume without loss of generality that \( Q_\Lambda = Q_\Lambda^1 \) satisfies \( \|Q_\Lambda\|_{L^2(0, T, H)} < 1 \), and then \( I - Q_\Lambda \) is invertible by the Neumann series. Now we can give the proof of Theorem 3.1.

**Proof.** (of Theorem 3.1) Since \((I - Q_\Lambda)\) is invertible in \( L^2(0, T, H) \), we deduce from (43) that

\[
\dot{u}_\Lambda \Lambda = A_\Lambda u_\Lambda = (I - Q_\Lambda)^{-1}(A_\Lambda u_\Lambda^1 + A_\Lambda u_\Lambda^2). \]

This equality and Lemma 3.3 yield the estimate

\[
\|\dot{u}_\Lambda\|_{L^2(0, T, H)} \leq c\|u_0\|_V + \|f\|_{L^2(0, T, H)} \tag{49}\]

for a constant \( c > 0 \) independent of the subdivision \( \Lambda \). Since for all \( t \in [0, T] \) one has \( u_\Lambda(t) = u_\Lambda(0) + \int_0^t \dot{u}_\Lambda(s)ds \), we conclude that

\[
\|u_\Lambda\|_{H^1(0, T, H)} \leq c\|u_0\|_V + \|f\|_{L^2(0, T, H)} \tag{50}\]

for some constant \( c > 0 \) independent of the subdivision \( \Lambda \). Then there exists a subsequence of \((u_\Lambda)\), still denoted by \((u_\Lambda)\) that converges weakly to some \( v \in H^1(0, T, H) \) as \( |\Lambda| \to 0 \).
We known that the Cauchy problem (11) has a unique solution $u \in MR(V, V')$ by Lions’ theorem. On the other hand, $(u_\Lambda)$ converges strongly to $u$ on $MR(V, V')$ by [20, Proposition 3.2]. In particular, $u_\Lambda \to u$ in $L^2(0, T, V)$. Thus $A_\Lambda u_\Lambda \to Au$ in $L^2(0, T, V')$ as $|\Lambda| \to 0$ by [20, Lemma 2.1]. It follows, by the uniqueness of the limits, that $u = v \in H^1(0, T, H)$ since

$$
\dot{u}_\Lambda = f - A_\Lambda u_\Lambda \to f - Au = \dot{u} \quad \text{in} \quad L^2(0, T, V').
$$

This completes the proof. 

4 $L^2$-maximal regularity in $H$: uniform approximation

Assume that $H, V$ and $a : [0, T] \times V \times V \to \mathbb{C}$ are as in Section 2. Let $(f, u_0) \in L^2(0, T) \times H \times V$, $\Lambda$ be a uniform subdivision of $[0, T]$ and $u, u_\Lambda \in MR(V, H)$ be the solutions of (41) and (42) respectively. In the previous section we have seen that $(u_\Lambda)$ converges weakly to $u$ in $MR(V, H)$ as $|\Lambda| \to 0$. The aim of this section is to prove that this convergence holds for the strong topology of $MR(V, H)$ and uniformly on the initial data $u_0$ and $f$.

The following result is the key idea of this section.

**Theorem 4.1.** There exists a positive constant $c > 0$ depending only on $M, \alpha, \gamma$ and $c_H$ such that

$$
\|u - u_\Lambda\|_{H^1(0, T, H)} \leq c \left[ \omega(2|\Lambda|) + \frac{\omega(2|\Lambda|)}{|\Lambda|^{1/2}} + \int_0^{2|\Lambda|} \frac{\omega(t)}{t^{1+\gamma/2}} dt \right] \left[ \|f\|_{L^2(0, T, H)} + \|u_0\|_V \right], \quad (51)
$$

With this estimate theorem in hand, the study of the uniform convergence, with respect to initial data, of $u_\Lambda \to u$ in $MR(V, H)$ becomes easy. Due to the results of Section 1 and hypothesis that $\omega$ satisfies all we need is to look when

$$
\lim_{|\Lambda| \to 0} \frac{\omega(2|\Lambda|)}{|\Lambda|^{1/2}} = 0 \quad (52)
$$

holds. Endee, clearly (55) implies that $A : [0, T] \to (V, V')$ is, in particular, continuous. Moreover, $A_\Lambda : [0, T] \to (V, V')$ satisfies conditions $(H_1)$-$(H_4)$ (see Section 1) by taking $D = V$ and $X = V'$. Thus one can apply Theorem 1.1 and conclude that $u_\Lambda \to u$ in $L^2(0, T, V)$ uniformly on the initial data $u_0 \in V \subset H = Tr(V, V')$ and the homogeneity $f \in L(0, T; H)$.

**Corollary 4.2.** There exists a null sequence $(t_n)_{n \in \mathbb{N}} \subset [0, T]$ depending on $\omega$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has

$$
\|u - u_\Lambda_n\|_{MR} \leq \varepsilon \left[ \|u_0\|_V + \|f\|_{L^2(0, T, H)} \right].
$$

for all subdivisions $\Lambda_n$ of $[0, T]$ with $2|\Lambda_n| = t_n$.

**Proof.** We claim that

$$
\liminf_{t \to 0} \frac{\omega(t)}{t^{1/2}} = 0.
$$

Other wise the integral

$$
\int_0^T \frac{\omega(s)}{s^{1+\gamma/2}} ds = \infty
$$

which contradict the assumption (25). This and Theorem 4.1 completes the proof. 

}
Finally, if we assume that \( \omega \) satisfies the following addition condition

\[
\lim_{t \to 0} \frac{\omega(t)}{t^{3/2}} = 0,
\]

then the statement of Corollary 4.2 holds for all uniform subdivision \( \Lambda \) of \([0, T]\).

**Corollary 4.3.** For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for each subdivision \( \Lambda \) of \([0, T]\)

\[
|\Lambda| \leq \delta \implies \|u - u_\Lambda\|_{MR} \leq \varepsilon \left[ \|x_0\|_V + \|f\|_{L^2(0, T, H)} \right].
\]

Now we give the proof of Theorem 4.1.

**Proof of Theorem 4.1** We will use the representation formula (43), notations (44)-(45) and the corresponding quantities for the solution \( u \) of (11). We proceed by several steps.

a) First, we estimate \( A_\Lambda u_{\Lambda, 1} - A u_1 \) in \( L^2(0, T, H) \). Let \( t \neq 0 \). We obtain using the second estimate in Proposition 2.5 and the estimates (3) and (10) in Proposition 2.1 that

\[
\|A_\Lambda(t)u_{\Lambda, 1}(t) - A(t)u_1(t)\|_H = \|A_\Lambda(t)e^{-tA_\Lambda(t)}u_0 - A(t)e^{-tA(t)}u_0\|_H \\
\leq \|e^{-tA_\Lambda(t)}[A_\Lambda(t)u_0 - A(t)u_0]\|_H + \|e^{-tA_\Lambda(t)} - e^{-tA(t)}\|A_\Lambda(t)u_0\|_H \\
= \|e^{-tA_\Lambda(t)}[A_\Lambda(t)u_0 - A(t)u_0]\|_H + \int_0^t \|e^{-(t-s)A_\Lambda(t)}(A\Lambda(t) - A(t))e^{-sA(t)}u_0\|_H \\
\leq 2c_\omega(2|\Lambda|) \left( \frac{1}{t^{3/2}} + c \int_0^t \frac{1}{s^{3/2}} \, ds \right) \|u_0\|_V.
\]

Similarly, combining the estimates (1) and (3) in Proposition 2.1 and the estimate (10) in Proposition 2.1 we obtain

\[
\|A_\Lambda(t)u_{\Lambda, 2}(t) - A(t)u_2(t)\|_H \\
\leq \int_0^t \|e^{-(t-s)A_\Lambda(t)} - A(t)e^{-(t-s)A(t)}\|f(s)\|_H \, ds \\
\leq \frac{1}{2\pi} \int_0^t \int \|\lambda\| e^{-(t-s)\Re \lambda}\|\lambda - A_\Lambda(t)\|^{-1}(A\Lambda(t) - A(t))(\lambda - A(t))^{-1} f(s)\|_H \, d\lambda \, ds \\
\leq \frac{1}{\pi} \int_0^t \|f(s)\|_H \int_0^\infty e^{-\left(\frac{t-s}{r}\right)\cos(\nu)} \nu^{2\omega(2|\Lambda|)} \, d\nu \\
= \frac{c_\omega(2|\Lambda|)}{2\pi} \int_0^t \|f(s)\|_H \int_0^\infty e^{-\left(\frac{t-s}{r}\right)\cos(\nu)} \nu^{\frac{3}{2} - \frac{2\omega(2|\Lambda|)}{\pi}} \, d\nu \\
= \frac{c_\omega(2|\Lambda|)}{\pi} \int_0^\infty \frac{e^{-\rho\cos(\nu)}}{\rho^{\frac{3}{2}} \left(\frac{\rho}{r}\right)^{\frac{3}{2} - \frac{2\omega(2|\Lambda|)}{\pi}}} \, d\rho \int_0^t \|f(s)\|_H (t - s)^{-\frac{1+\nu}{2}} \, ds.
\]

The last integral is well defined since the function \( h : \mathbb{R} \to \mathbb{R} \) given by \( h(t) = t^{-\frac{1+\nu}{2}} \) for \( t \in [0, T] \) and \( h(t) = 0 \) for \( t \in (-\infty, 0]\cap[T, +\infty) \) belongs to \( L^1(\mathbb{R}) \) because \( \frac{1+\nu}{2} < 1 \). The estimates (53) and (55) yield, respectively,

\[
\|A_\Lambda u_{\Lambda, 1} - A u_1\|_{L^2(0, T; H)} \leq c_\omega(2|\Lambda|) \|u_0\|_V
\]

\[
\|A_\Lambda u_{\Lambda, 2} - A u_2\|_{L^2(0, T; H)} \leq c_\omega(2|\Lambda|) \|u_0\|_V
\]
and
\[ \|A_3u_{\Lambda,2} - A_4u_2\|_{L^2(0,T,H)} \leq c\omega(2|\Lambda|)\|f\|_{L^2(0,T,H)} \] (57)
for a positive constant \(c > 0\) that depends only on \(M, \alpha, \gamma\) and \(\varepsilon_H\).

b) Next, we prove the following estimate
\[ \|Q_\Lambda - Q\|_{L^2(0,T,H)} \leq c\left[\omega(2|\Lambda|) + \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \int_0^{|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}}\right] \] (58)
where \(Q : L^2(0,T,H) \rightarrow L^2(0,T,H)\) is defined via formula which is analogous to (46). To this end, for \(g \in L^2(0,T,H)\) and \(t \in [0,T]\) we write

\[ \|(Qg)(t) - (Qg)(t)\|_H \]

\[ \leq \int_0^t \|A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s))(A_\Lambda^{-1}(s) - A^{-1}(s))g(s)\|_H ds \]

\[ + \int_0^t \|A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s) + A(s))A^{-1}(s)g(s)\|_H ds \]

\[ + \int_0^t \|g(A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s)))g(s)\|_H ds \]

\[ = I_{\Lambda,1}(t) + I_{\Lambda,2}(t) + I_{\Lambda,3}(t) \]

Replacing \(A(s)\) by \(A(s) + \mu\) and according to Proposition 2.1 we may assume \(\|A_\Lambda^{-1}(s)\|_{(V,\bar{V})} \leq c\) and \(\|A_\Lambda^{-1}(s)\|_{(H,\bar{V})} \leq c\). Next, by the estimates (56) and (59) in Proposition 2.1 together with (39) and (40), we have

\[ I_{\Lambda,1}(t) = \int_0^t \|A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A_\Lambda(s))(A_\Lambda^{-1}(s) - A^{-1}(s))g(s)\|_H ds \]

\[ \leq 2^{1+\gamma/2}c^2 \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}}\|(A_\Lambda^{-1}(s) - A^{-1}(s))g(s)\|_V ds \]

\[ = 2^{1+\gamma/2}c^2 \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}}\|(A_\Lambda^{-1}(s)(A_\Lambda(s) - A(s))A^{-1}(s))g(s)\|_V ds \]

\[ \leq 2^{1+\gamma/2}c^2\omega(2|\Lambda|) \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}}\|A_\Lambda^{-1}(s)(V,\bar{V})\|_{A^{-1}(s)(V,\bar{V})}g(s)\|_H ds \]

\[ \leq 2^{1+\gamma/2}c^2\omega(2|\Lambda|) \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}}\|g(s)\|_H ds \]

\[ = 2^{1+\gamma/2}c^2\omega(2|\Lambda|)h_\Lambda s\|g(s)\|_H(t), \]

where \(h_\Lambda(t) := \omega(t)\lambda^{-1-\gamma/2}\) for \(t \in [0,T]\) and \(h_\Lambda(t) := 0\) for \(t \in (-\infty, 0]\cap[T, +\infty)\). Proposition 2.4 implies that \(h_\Lambda \in L^2(\mathbb{R})\) and that \(h_\Lambda^2\) is bounded uniformly with respect to the subdivision \(\Lambda\). Therefore we obtain
\[ \int_0^T I_{\Lambda,1}(s)ds \leq c\omega(2|\Lambda|)^2 \int_0^T \|g(s)\|^2_H ds \] (59)
where the positive constant \(c > 0\) is independent of \(\Lambda\).
Again using as above the estimates (6) and (9) in Proposition 2.1, we obtain for the second term $I_{\Lambda,2}$

$$I_{\Lambda,2}(t) := \int_0^t \|A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A(t) - A_\Lambda(s) + A(s))A^{-1}(t)g(s)\|_H ds$$

$$\leq 2^{1+\gamma/2} c_\Lambda \int_0^t \|A_\Lambda(t) - A(t) - A_\Lambda(s) + A(s)\|_{(V_\gamma',H)} \frac{\|g(s)\|_H}{(t-s)^{1+\gamma/2}} ds$$

$$\leq 2^{1+\gamma/2} c_\Lambda \int_0^t \kappa_\Lambda(t-s)\|g(s)\|_H ds$$

where

$$\kappa_\Lambda(t) := \left\{ \begin{array}{ll}
|\omega(t) + \omega_\Lambda(t)|t^{-(1+\frac{2}{\gamma})} & \text{if } 0 \leq t < 2|\Lambda|, \\
4\omega(2|\Lambda|)t^{-(1+\frac{2}{\gamma})} & \text{if } 2|\Lambda| < t \leq 2T, \\
0 & \text{if } t \in ]-\infty, 0]\cap[2T, +\infty].
\end{array} \right.$$}

Here we have used simultaneously both estimates (6) and (9) from Lemma 2.5. Because of (25) and (33), the function $t \mapsto \kappa_\Lambda(t)$ belongs to $L^1(\mathbb{R})$, and by a simple calculation we obtain

$$\|\kappa_\Lambda\|_{L^1(\mathbb{R})} \leq c \left( \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} ds \right)$$

and therefore,

$$\int_0^T I_{\Lambda,2}(s) ds \leq c \left( \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} ds \right)^2 \int_0^T \|g(s)\|_H^2 ds$$

(60)

for a constant $c = c(M, \alpha, c_H, \gamma) > 0$ independent of $\Lambda$.

b) For the last term $I_{\Lambda,3}(t)$, we set $\tilde{g}(t, \cdot) := (A(t) - A(\cdot))A^{-1}(\cdot)g(\cdot)$. Again by Lemma 2.3 and 4.1 and 5. from Proposition 2.1 and we obtain

$$I_{\Lambda,3}(t) := \int_0^t \|A_\Lambda(t)e^{-(t-s)A_\Lambda(t)}(A_\Lambda(t) - A(t) - A_\Lambda(s) + A(s))A^{-1}(t)\tilde{g}(t,s)\|_H ds$$

$$\leq \frac{1}{2\pi} \int_0^t \int_{\Gamma} |\lambda| e^{-(t-s)Re \lambda} \|[(\lambda - A_\Lambda(t))^{-1}(A_\Lambda(t) - A(t))(\lambda - A(t))^{-1}\tilde{g}(t,s)]\|_H d\lambda ds$$

$$\leq \frac{1}{2\pi} \int_0^t \int_{\Gamma} |\lambda| e^{-(t-s)Re \lambda} \|[(\lambda - A_\Lambda(t))^{-1}\|_{(V_\gamma',H)}\|A_\Lambda(t) - A(t)\|_{(V_\gamma',H)}\|\tilde{g}(t,s)\|_{V_\gamma'} d\lambda ds$$

$$\leq \frac{C_{V_\gamma'}}{2\pi} \int_0^t \int_{\Gamma} |\lambda| e^{-(t-s)Re \lambda} \frac{C_\omega(2|\Lambda|)}{1 + |\lambda|}^{1/2} \|\tilde{g}(t,s)\|_{V_\gamma'} d\lambda ds$$

$$\leq \omega(2|\Lambda|) \frac{C_{V_\gamma'}}{\pi} \int_0^t \int_{\Gamma} e^{-(t-s)Re \lambda} \frac{\lambda}{1 + |\lambda|}^{1/2} \|\tilde{g}(t,s)\|_{V_\gamma'} d\lambda ds$$

$$\leq \omega(2|\Lambda|) \frac{C_{V_\gamma'}}{\pi} \int_0^t \int_{\Gamma} e^{-(t-s)Re \lambda} \|\tilde{g}(t,s)\|_{V_\gamma'} d\lambda ds$$

$$\leq \omega(2|\Lambda|) \frac{C_{V_\gamma'}}{\pi} \int_0^t \int_0^\infty r^{\frac{\gamma}{2}} e^{-(t-s)r|\cos(\nu)|} \|\tilde{g}(t,s)\|_{V_\gamma'} dr d\nu$$

where $C_{V_\gamma'}$ is the injection constant of $V_\gamma'$ into $V'$. Next, since

$$\|\tilde{g}(t,s)\|_{V_\gamma'} \leq \omega(t-s)\|A^{-1}(t)\|_{(H,V)} \|g(s)\|_H,$$

we conclude that
where $h : \mathbb{R} \rightarrow \mathbb{R}$ is defined analogously as $h_\Lambda$ above. Taking into account \([25]\), it follows
\[
\int_0^T I_{\lambda, \Lambda}^2(s)ds \leq c\omega(2|\Lambda|)^2 \int_0^T \|g(s)\|^2_H ds
\]
for a constant $c > 0$ independent of $\Lambda$, and thus the desired estimate \([58]\) is proved.

c) Finally, by using Lemma \([3.3]\) we conclude from \([a) - b)]\) that
\[
\|A_\Lambda u_\Lambda - Au\|_{L^2(0,T,H)} \leq \| (I - Q_\Lambda)^{-1}(A_\Lambda u_\Lambda - Au_1) \|_{L^2(0,T,H)}
+ \| (I - Q_\Lambda)^{-1} [A_\Lambda u_\Lambda - Au_2] \|_{L^2(0,T,H)}
+ \| (I - Q_\Lambda)^{-1} (Q - Q_\Lambda)(I - Q)^{-1}(Au_1 + Au_2) \|_{L^2(0,T,H)}
\leq c \left[ \omega(2|\Lambda|) + \omega(2|\Lambda|) \right] \int_0^T \|u_0\|_V + \|f\|_{L^2(0,T,H)}
\]
where $c > 0$ is independent of $\Lambda$. Further, since $u$ and $u_\Lambda$ satisfy \([42]\) and \([44]\), respectively, we have
\[
\|\tilde{u}_\Lambda - \tilde{u}\|_{L^2(0,T,H)} \leq c \left[ \omega(2|\Lambda|) + \omega(2|\Lambda|) \right] \int_0^T \|u_0\|_V + \|f\|_{L^2(0,T,H)}.
\]
Now since $u(t)$ and $u_\Lambda(t)$ belong to $V$ for almost every $t \in [0,T]$, we have
\[
u_\Lambda(t) = u_\Lambda(0) + \int_0^t \tilde{u}_\Lambda(s)ds \quad \text{and} \quad u(t) = u(0) + \int_0^t \tilde{u}(s)ds
\]
almost everywhere. This completes the proof. \(\Box\)

## 5 Continuity of solutions

Assume that $H, V$ and $\mathfrak{a} : [0,T] \times V \times V \rightarrow \mathbb{C}$ are as in the previous section. The aim of the this section is the prove that $(u_\Lambda)_\Lambda$ converges to $u$ in the space $C([0,T], V)$ uniformly on $(f, u_0)$ provided that \([55]\) holds. Note that $u_\Lambda \rightarrow u$ in $C([0,T], H)$ since $MR(V, V')$ is continuously embedded into $C(0,T, H)$.

**Theorem 5.1.** Assume that $\mathfrak{a} : [0,T] \times V \times V \rightarrow \mathbb{C}$ satisfies \([22] - [26]\) with $D(\Lambda(0)^{1/2}) = V$. Let $\Gamma$ be a another subdivision of $[0,T]$ that is finer than $\Lambda$. Then
\[
\|u_\Gamma - u_\Lambda\|_{C(0,T,V)} \leq c \left( \omega(2|\Lambda|) + \frac{\omega(2|\Gamma|)}{|\Gamma|^{1/2}} + \frac{\omega(2|\Lambda|)}{|\Lambda|^{1/2}} \right) \left[ \|u_0\|_V + \|f\|_{L^2(0,T,H)} \right]
\]
for some positive constant $c > 0$ depending only on $M, \alpha, \gamma$ and $c_H$. 

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Proof. We will use the notation of the the previous sections and we will proceed, as in the proof
of Theorem 4.1, in several steps. To this end, we will adapt the proof of [7, Theorem 4.4] to our
situation.

Step a: By using (2) and (5) in Proposition 2.1, for \((\lambda - \mathcal{A}_T(t))^{-1}\) and \((\lambda - \mathcal{A}_\Lambda(t))^{-1}\), respectively, and (10) we obtain for every \(t \in [0, T]\) that

\[
\|u_{1, \Lambda}(t) - u_{1, \Gamma}(t)\|_V \leq \frac{1}{2\pi} \int_{\Gamma} e^{-t\Re \lambda} \|\lambda - \mathcal{A}_\Lambda(t)\|^2 (\mathcal{A}_\Lambda(t) - \mathcal{A}_T(t)) (\lambda - \mathcal{A}_T(t))^{-1} u_0 \|_V d\lambda
\]

\[
\leq \frac{2c^2\omega(2|\Lambda|)}{\pi} \int_{\Gamma} \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1+r} f(f(s)) H ds d\lambda dr d\lambda
\]

\[
\leq \frac{2c^2\omega(2|\Lambda|)}{\pi} \left[ \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1+r} f(f(s)) H ds d\lambda dr \right]^{1/2}
\]

Step b: Again the estimates (4) and (5) in Proposition 2.1 and formula (10) imply that

\[
\|(\lambda - \mathcal{A}_\Lambda(t))^{-1}(\mathcal{A}_\Lambda(t) - \mathcal{A}_T(t)) (\lambda - \mathcal{A}_T(t))^{-1} f(s)\|_V \leq 2c^2\omega(2|\Lambda|) \frac{\|f(s)\|_H}{(1 + |\lambda|)^{1-\nu}}.
\]

Therefore, we obtain by using Fubini’s theorem that for all \(\lambda \in \Gamma \setminus \{0\}\)

\[
\|u_{2, \Lambda}(t) - u_{2, \Gamma}(t)\|_V = \| \int_{\Gamma} \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1+r} f(f(s)) H ds d\lambda dr d\lambda
\]

\[
\leq \frac{c^2\omega(2|\Lambda|)}{\pi} \left[ \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1+r} f(f(s)) H ds d\lambda dr \right]^{1/2}
\]

Step c: It remains to estimate \(\|u_{3, \Lambda}(\cdot) - u_{3, \Gamma}(\cdot)\|_V\). For this for each \(h \in C(0, T, V)\) we set

\[
(P_{\Lambda}h)(t) := \int_{0}^{t} e^{-(t-s)\mathcal{A}_\Lambda(t)} (\mathcal{A}_\Lambda(t) - \mathcal{A}_\Lambda(s)) h(s) ds.
\]

From [7, Lemma 4.5] we have \(P_{\Lambda}h \in C(0, T, V)\). Thanks to Proposition 2.4 and Lemma 2.5 one can prove in a similar way as in Step 3 of the proof of [7, Theorem 4.4] (see also Step 3 of the proof of Lemma 5.2 that \(\|P_{\Lambda}\|_{(C(0, T, V))} \leq 1/2\) and thus \(I - P_{\Lambda}\) is invertible on \((C(0, T, V))\). Therefore, we obtain by using the representation formula (13)

\[
u_{1,3} - u_{1, \Gamma, 3} = (I - P_{\Lambda})^{-1} (u_{1, \Lambda, 3} - u_{1, \Gamma, 1}) + (I - P_{\Gamma})^{-1} (u_{1, \Gamma, 2} - u_{1, \Gamma, 2})
\]

\[
u_{2,3} - u_{2, \Gamma, 3} = (I - P_{\Lambda})^{-1} (P_{\Lambda} - P_{\Gamma}) (I - P_{\Gamma})^{-1} (u_{2, \Gamma, 1} + u_{2, \Gamma, 2})
\]

The term on the right hand side of (64) is treated in Step a)-b). We need only to estimate the
we obtain

Because of (25) and (35), the function $t \mapsto k_{\Lambda, \Gamma}(t)$ belongs to $L^1(\mathbb{R})$, and by a simple calculation we obtain
for some positive constant $c > 0$ depending only on $M, \alpha, \gamma$ and $c_H$. We conclude then
\[
\|u_{3, \Lambda}(t) - u_{3, \Gamma}(t)\|_V \leq c \left( \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \frac{\omega(2|\Gamma|)}{|\Gamma|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} ds \right) \left[ \|u_0\|_V + \|f\|_{L^2(0,T,H)} \right],
\]
for some positive constant $c > 0$ (probably different from the previous one) depending only on $M, \alpha, \gamma$ and $c_H$, and the proof is complete.

\[ \square \]

We finish this section with our main result.

**Theorem 5.2.**
1. There exists a null sequence $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ depending on $\omega$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has
\[
\|u_{\Gamma_n} - u_{\Lambda_n}\|_{C([0,T],V)} \leq \varepsilon \left[ \|u_0\|_V + \|f\|_{L^2(0,T,H)} \right]
\]
for all subdivision $\Lambda_n$ and $\Gamma_n$ of $[0,T]$ with $|\Gamma_n| < |\Lambda_n| = \frac{T}{N}$.

2. $u \in C([0,T],V)$ and there exists a null sequence $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ depending on $\omega$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has
\[
\|u_{-} - u_{\Lambda_n}\|_{C([0,T],V)} \leq \varepsilon \left[ \|u_0\|_V + \|f\|_{L^2(0,T,H)} \right]
\]
for all subdivision $\Lambda_n$ of $[0,T]$ with $|\Lambda_n| = \frac{T}{N}$.

3. Assume moreover that (5.5) holds. Then $u_{\Lambda} \rightarrow u$ in $C([0,T],V)$ uniformly on $u_0$ and $f$ as $|\Lambda| \rightarrow 0$. More precisely,
\[
\|u - u_{\Lambda}\|_{C([0,T],V)} \leq c \left( \frac{\omega(2|\Lambda|)}{|\Lambda|^{\gamma/2}} + \frac{\omega(2|\Gamma|)}{|\Gamma|^{\gamma/2}} + \int_0^{2|\Lambda|} \frac{\omega(s)}{s^{1+\gamma/2}} ds \right) \left[ \|u_0\|_V + \|f\|_{L^2(0,T,H)} \right]
\]
for all subdivision $\Lambda$ and for some positive constant $c > 0$ independent of $\Lambda$.

**Proof.** Assertion 1) follows by similar argument as in the proof of Corollary 4.2. By Theorem 5.1 $u_{\Lambda}$ is a Cauchy sequence, and thus converges in $C([0,T],V)$. In other hand, we known that $u_{\Lambda} \rightarrow u$ strongly in $C([0,T],H)$. Therefore, assertion 2) is a direct consequence of 1) and the assertion 3) follows directly from the estimate (5.5) and the additional condition (5.5).

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