Estimation of the order parameter exponent of critical cellular automata using the enhanced coherent anomaly method

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Abstract

The stochastic cellular automaton of Rule 18 defined by Wolfram [Rev. Mod. Phys. 55, 601 (1983)] has been investigated by the enhanced coherent anomaly method. Reliable estimate was found for the $\beta$ critical exponent, based on moderate sized ($n \leq 7$) clusters.

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Calculating critical exponents of second order phase transitions is a challenging problem. For nonequilibrium systems, generalization of equilibrium statistical physics methods is under development. Among the most notable analytical tools are series expansion [1], transfer matrix diagonalization [2] and the mean-field renormalization-group method [3].

In a series of earlier papers [4–6], we have shown how the generalization of the mean-field technique with appropriate extrapolation can be used to describe the critical properties of cellular automata (CA) phase transitions.

The generalized mean-field approximation (GMF) first proposed for dynamical systems by Gutowitz et al. [7] and Dickman [8] is shown to converge slowly at criticality. In this method we set up equations for the steady state of the system based on \( n \)-point block probabilities. Correlations with a range greater than \( n \) are neglected. By increasing \( n \) from 1 (traditional mean-field) step by step we take into account more and more correlations and get better approximations. The GMF approximation can be used as the basis of the coherent anomaly method (CAM) calculation, and it gives a reasonably good \( \beta \) exponent for a dynamical system with large \( (n > 10) \) [9]. In this letter I show how an improved version of the CAM proposed very recently [10] works on cellular automata.

The essence of the CAM [11] is that the solution for singular quantities at a given \( (n) \) level of approximation \( (Q_n(p)) \) in the vicinity of the critical point is the product of the classical singular behavior multiplied by an anomaly factor \( (a(n)) \), which becomes anomalously large as \( n \to \infty \) (and \( p_n \to p_c \)):

\[
Q_n \sim a(n)(p/p_n^c - 1)^{\omega_{cl}}
\]

(1)

where \( p \) is the control parameter and \( \omega_{cl} \) is the classical critical index. The divergence of this anomaly factor scales as

\[
a(n) \sim (p_n^c - p_c)^{\omega - \omega_{cl}}
\]

(2)

thereby permitting the estimation of the true critical exponent \( \omega \), given a set of GMF approximation solutions. However such an estimation depends to some extent on the choice of the independent parameter \( (p \leftrightarrow 1/p) \). To avoid this a corrected CAM was proposed [10], based on a new parameter:

\[
\delta_n = (p_c/p_n^c)^{1/2} - (p_c/p_n^c)^{1/2}
\]

(3)

such that Eq. (3) is invariant under \( p \leftrightarrow p^{-1} \). This parametrization gives better estimates for the critical exponents of the 3-dimensional Ising model [10].

My target system for this kind of calculation was the one-dimensional, stochastic Rule 18 CA [12]. This range-one cellular automaton rule generates a '1' at time \( t \) only when the right or the left neighbor was '1' at \( t - 1 \):

\[
\begin{array}{ccc}
t-1 & : & 100 & 001 \\
t & : & 1 & 1
\end{array}
\]

with probability \( p \). In any other case the cell becomes '0' at time \( t \). The order parameter is the concentration \( (c) \) of '1'-s. For \( p < p_c \), the system evolves to an absorbing state \( (c = 0) \).
For \( p \geq p_c \) a finite concentration steady state appears with a continuous phase transition. This transition is known to belong to the universality class of directed percolation (DP) \([13]\). At \( t \to \infty \) the steady state can be built up from '00' and '01' blocks \([14]\). This permits one to map it onto stochastic Rule 6/16 CA with the new variables '01' \( \rightarrow \) 1' and '00' \( \rightarrow \) 0'.

\[
\begin{align*}
  t-1 : & \quad 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
  t : & \quad 0 & 1 & 1 & 0 & & & \\
\end{align*}
\]

and the GMF equations can be set up by means of pair variables. In an earlier work \([4]\) this was performed up to the order \( n = 6 \), and Padé extrapolation was applied to the results. Our best estimate for critical data was \( p_c = 0.7986 \) and \( \beta = 0.29 \).

Now, I have extended the GMF calculations up to \( n = 7 \) (see Table \([\) with the help of the symbolic Mathematica software. This required the setting up and solution of a set of nonlinear equations of 72 variables. I obtained \( p_c^7 = 0.7729 \), which is still 5% off the result obtained by steady state simulation: \( p_c = 0.8086(2) \) \([15]\) or from the more accurate time dependent simulation data: \( p_c = 0.8094(2) \) \([16]\).

The CAM analysis of \((a(n), \delta_n)\) data was done, taking into account the correction term

\[
a(n) = b \delta_n^{\beta - \beta_{cl}} + c \delta_n^{\beta - \beta_{cl} + 1}
\]

and examining the stability of the solution by omitting different points from the \((n = 1, ..., 7)\) data set. For the fitting \( \beta_{cl} = 1 \) and \( p_c = 0.8094 \) were used. As was pointed out in Ref. \([10]\) the CAM data may contain departures from ideal scaling; moreover there is no clear dependence on the order of the approximations. I found relatively stable estimates using the correction formula \([4]\) on the data set with the omission of the \( n = 3 \) point. The \( n = 3 \) approximation result does not fit into the \((\log(\delta_n) - \log(a(n)))\) curve either (see Fig. ). Table \([4]\) shows the stability of the results, with the mean \( \beta = 0.2796(2) \) calculated from them. This compares very well with the value \( \beta = 0.2769(2) \) obtained by Dickman and Jensen \([4]\) from series expansion. If the CAM calculation based on \( p \) or \( 1/p \) independent variables the results differ by \( \pm 0.005 \) from the present enhanced version \( \beta \) estimates.

Another critical model with non-DP universality, the nonequilibrium kinetic Ising model, has been examined with the enhanced CAM method, and the \( \beta \) exponent estimate is in agreement with the simulation results \([17]\).

The conclusion of this study is that the enhanced version CAM method with careful data analysis gives good estimates for the critical exponent for moderate \( n < 10 \) level GMF approximations. Calculation of the \( n = 5, 6, 7, ... \) GMF approximations is possible on moderate sized workstations. The solution of the \( n = 7 \) level approximation took about 10 hours CPU time on a SUN Sparc-10. This provides an efficient analytical tool for exploring universalities of nonequilibrium systems.

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FIGURES

Result obtaining by applying the improved CAM method on $n$-pair ($n = 1, ..., 7$) approximation data. The logarithm of the anomaly coefficient $a(n)$ is plotted versus the logarithm of the improved independent variable $\delta_n$. Fitting was done according to Eq. (4).
### TABLE I. GMF calculation results for pair approximation data

| $n$ | $p_0^n$ | $a(n)$ |
|-----|---------|--------|
| 1   | 0.5000  | 0.5000 |
| 2   | 0.6666  | 1.5000 |
| 3   | 0.7094  | 2.3484 |
| 4   | 0.7413  | 2.8816 |
| 5   | 0.7543  | 3.5345 |
| 6   | 0.7656  | 4.2545 |
| 7   | 0.7729  | 4.8463 |

### TABLE II. CAM calculation results for pair approximation data

| data | $\beta$ |
|------|---------|
| 1 − 2 − 4 − 5 | 0.273 |
| 1 − 2 − 4 − 5 − 6 | 0.271 |
| 1 − 2 − 4 − 5 − 6 − 7 | 0.282 |
| 1 − 2 − 4 − 5 − 7 | 0.285 |
| 1 − 2 − 4 − 6 − 7 | 0.275 |
| 1 − 2 − 5 − 6 − 7 | 0.310 |
| 1 − 4 − 5 − 6 − 7 | 0.275 |
| 2 − 4 − 5 − 6 − 7 | 0.266 |
| mean | 0.2796(2) |

Padé extrapolation, ref. [4] 0.29

Simulation, ref. [15] 0.285(5)

Series expansion for DP ref. [1] 0.2769(2)