Diffusive behavior from a quantum master equation
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Abstract

We study a general class of translation invariant quantum Markov evolutions for a particle on \( \mathbb{Z}^d \). We assume locality of the spatial jumps and exponentially fast relaxation in momentum space. It is shown that the particle position diffuses in the long time limit. We employ a fiber decomposition in momentum space of the evolution made possible by the translation invariance of the dynamics. A central limit theorem follows from perturbation theory around a single fiber whose evolution in the momentum representation is described by a Markov jump process on the \( d \)-dimensional torus.

1 Introduction

A classical problem in the study of dynamical systems is understanding diffusive behavior for some properly rescaled variable. When starting from a Hamiltonian dynamics, that often proceeds in two steps. First there is a the identification of relevant space-time scales under which certain variables obey a reduced autonomous description. That specifies the limit starting from a microscopic dynamics and leading to a master or Boltzmann-type equation, e.g. as the result of a weak coupling or a low density approximation, [1, 2, 3]. Already there some irreversible behavior may be exhibited. Additionally, a second step can further specify the irreversible properties of a more restricted set of degrees of freedom. The present paper deals with the second step, taking for granted a form of the master equation for the reduced description of a quantum particle hopping on the lattice. The specific derivation of that master equation, very much like a linear Boltzmann equation, starting from the unitary evolution of a particle in contact with a reservoir is not the subject of the paper, leaving background and further details to the literature, see e.g. [4].

We imagine a translation invariant law of motion wherein the free Hamiltonian evolution is interrupted by scattering events from interactions with the environment. The effective or resulting description is that of a Markovian open system. In quantum mechanics, Lindblad equations take the place of Langevin or Fokker-Planck equations in classical probability theory describing a dynamical system under the influence of an idealized noisy environment, cf. [12]. This Lindblad equation is a master equation for the evolution of the density matrix. The models we study in the present paper are translation invariant Markovian evolutions for a quantum particle on \( \mathbb{Z}^d \). Its state is described via a density matrix \( \rho_t \) for which, in position representation, the diagonal \( \rho_t(x, x) \) gives the probability density for its spatial location \( x \) at time \( t \). We show that under the right conditions it behaves diffusively, exactly the same result as for its classical counterpart: around the average position and as time \( t \) evolves, the spatial probability density \( \rho_t(x) \) becomes closer to a Gaussian, with parabolic space-time scaling \( x^2 \sim t \) after subtracting a possible systematic drift. We will also give some counterexamples to that result thereby making it less intuitive from a particular point of view. The questions and the applied techniques are

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however quite similar to what has been studied starting from classical Boltzmann-type equations, see e.g. [7, 8, 9]. A quantum example that is probably very related to ours is in [10].

The next section gives a heuristic discussion of our model and of its origin, together with a first description of the main result. Section 3 repeats the model in a more general mathematical setting and the main theorem 3.3 is being stated. The following section discusses the essentials of the situation. The rest of the paper contains the proof of the main theorem and in a final Appendix some further explanation is given about the form and the properties of the Lindblad generator.

2 Description of the result

We start directly from an open system dynamics, in terms of a quantum master equation. While we refer to [4, 5] for an introduction to the derivation of this quantum master equation, we give a heuristic description of the dynamics. We imagine the motion of our test particle to be steered by scattering events with a background fluid. The background fluid is homogeneous and the interaction with the particle is translation invariant. In a discrete time version we would say that the particle moves in the direction of its momentum where each time its energy and its momentum gets updated via the interaction with the background. In other words, first, if the momentum transfer is $\theta$, we get the shift in momentum or boost

$$
\rho \longrightarrow e^{-\frac{i}{\hbar} \theta X} \rho e^{\frac{i}{\hbar} \theta X}
$$

(2.1)

with $X$ the position operator. In the rest of the paper, we set $\hbar = 1$. Secondly, the state of the particle changes at fixed momentum. The result is given in momentum representation as a multiplication

$$
\rho(k_1, k_2) \longrightarrow M_\theta(k_1, k_2) \rho(k_1, k_2)
$$

with some positive kernel $M_\theta(k_1, k_2)$, that we do not yet specify further. Our mathematical assumptions are basically formulated on the kernel $M_\theta(k_1, k_2)$. These assumptions will express a certain locality in the jumps and they also allow a sufficiently smooth relaxation of the momentum. Specific expressions can be found in [5], and we come back to its spatial representation (for a particle hopping on the regular lattice $\mathbb{Z}^d$) for the examples in Section 4.3 and also in Appendix A.

Finally we also add the free evolution with some Hamiltonian $H$ that only depends on the particle’s momentum $\Omega$. The continuous time version gives us now an evolution of the density matrix $\rho_t$, $t \geq 0$, given by the master equation

$$
\frac{d}{dt} \rho_t = i[H, \rho_t] + \Psi(\rho_t) - \frac{1}{2} \{\Psi^*(I), \rho_t\}, \quad I = \text{identity}
$$

(2.2)

where $\Psi^* : \mathcal{B}(\ell^2(\mathbb{Z}^d)) \rightarrow \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is the adjoint of a completely positive map $\Psi : \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \rightarrow \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ of the form

$$
\Psi(\rho) = \int_{\mathbb{Z}^d} d\nu(\theta) e^{-i\theta X} M_\theta(\rho) e^{i\theta X},
$$

(2.3)
where $\nu$ is a positive $\sigma$-finite measure on the $d$–dimensional torus $\mathbb{T}^d$ and the $M_\theta's$ are completely positive maps that admit a Kraus decomposition of operators that are functions of $\Omega$, see also Appendix A.

The position distribution is found on the diagonal, $\rho_t(x, x)$, which is itself not Markovian. The main and mathematical result of the paper is the identification of natural assumptions under which the measure $\mu_t$, defined by

$$
\mu_t(R) = \sum_{x \in \sqrt{tr} - tv} \rho_t(x, x)
$$

for a drift velocity $v \in \mathbb{R}^d$ and for an arbitrary region $R \subset \mathbb{R}^d$, converges in distribution to a Gaussian law.

3 Model and results

3.1 A translation-covariant semigroup

We start with the kinematics. On the Hilbert space $\ell^2(\mathbb{Z}^d)$ we define the translation operators $\tau_y, y \in \mathbb{Z}^d$, and the (vector-valued) position operator as

$$(\tau_y f)(x) = f(x + y), \quad (X_j f)(x) = x_j f(x) \quad \text{for } f \in \ell^2(\mathbb{Z}^d), x \in \mathbb{Z}^d$$

Here, and in what follows, the subscript $j = 1, \ldots, d$ refers to the components in $\mathbb{Z}^d$ or $\mathbb{T}^d$. We will often consider the space $\mathcal{H}$ in its dual representation, i.e., as $L^2(\mathbb{T}^d)$ where $\mathbb{T}^d$ is identified with $[-\pi, \pi]^d$. For $g \in L^2(\mathbb{T}^d)$, we define the vector of ‘momentum’ operators $\Omega = (\Omega_j)$ as multiplication by $k \in \mathbb{T}^d$, i.e.,

$$
\Omega_j g(k) = k_j g(k), \quad k \in [-\pi, \pi]^d
$$

Although the $\Omega_j's$ are well-defined as bounded operators, they do not satisfy $[X_j, \Omega_j] = -i$, nor does $\Omega$ generate the translations $\tau_x$.

The space of trace-class operators is denoted by $\mathcal{B}_1(\ell^2(\mathbb{Z}^d))$. A density matrix $\rho \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ is positive and normalized, $\text{Tr}[\rho] = 1$. Through a Fourier transform, we can identify $\rho$ as an operator on $L^2(\mathbb{T}^d)$ with a well defined integral kernel $\rho(k_1, k_2)$ since $\rho$ is trace-class. We refer to this representation as the momentum representation. It is useful to change variables and to write

$$
[\rho]_p(k) := \rho(k - \frac{p}{2}, k + \frac{p}{2}), \quad k, p \in \mathbb{T}^d
$$

where we will think of $[\rho]_p$ as fibers of the density matrix $\rho$, indexed by $p \in \mathbb{T}^d$. The equation (3.2) defines a map

$$
[.]_p : \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \to L^1(\mathbb{T}^d).
$$

We will study this fiber decomposition with more care in Lemma 4.1. In particular, our conditions will ensure that the function $p \mapsto [\rho]_p$ can be chosen in $C^2(\mathbb{T}^d, L^1(\mathbb{T}^d))$. The zero fiber refers to taking $p = 0$. 

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Now to the dynamics. The time-evolution is given by a norm–continuous completely positive semigroup \( \Phi_t \) on \( B_1(\ell^2(\mathbb{Z}^d)) \). As a result, by \([20]\), \( \Phi_t \) satisfies
\[
\frac{d}{dt} \Phi_t(\rho) = L(\Phi_t(\rho))
\]
where \( L \) can be written in the Lindblad form
\[
L \rho = i[H, \rho] + \Psi(\rho) - 1/2 \{ \Psi^*(I), \rho \}, \quad \rho \in B_1(\ell^2(\mathbb{Z}^d))
\]
with \( H \) a self-adjoint operator and \( \Psi \) a completely positive map, acting on \( B_1(\ell^2(\mathbb{Z}^d)) \), and \( \Psi^* \) is its adjoint, acting on \( B(\ell^2(\mathbb{Z}^d)) \).

We want the semigroup \( \Phi_t \) to be translation covariant
\[
\Phi_t(\tau_x^* \rho \tau_x) = \tau_x^* \Phi_t(\rho) \tau_x, \quad x \in \mathbb{Z}^d
\]
This implies that one can choose \( H = H(\Omega) \), a bounded function of the vector of momentum operators \( \Omega \) and, by \([15]\), \( \Psi \) is of the Holevo-form \((2.3)\) further discussed in Appendix \( A \). A crucial consequence of this translation invariant form is the invariance of the fibers, i.e., the dynamics gives rise to an autonomous evolution on each fiber; see also below under \((4.5)\).

3.2 Assumptions

There are basically two sets of assumptions, one having to do with the spatial locality of the dynamics, and the other with the dissipativeness. They are closely related by Fourier-transform. We here formulate these conditions and we discuss them more closely in Section \( 4 \) after having stated the main result of the paper.

The locality can first be formulated in terms of the completely positive maps \( \mathcal{M}_\theta \), the measure \( d\nu(\cdot) \) and the dispersion law \( H \).

**Assumption 3.1.** [Locality] We assume that the completely positive maps \( \mathcal{M}_\theta \) are defined by the kernels \( \mathcal{M}_\theta(k_1, k_2) \) as
\[
(\mathcal{M}_\theta(\rho))(k_1, k_2) = M_\theta(k_1, k_2) \rho(k_1, k_2)
\]
where the function \( M_\theta(k_1, k_2) \) is twice continuously differentiable in both \( k_1, k_2 \) and once continuously differentiable in \( \theta \). The function \( H \) is assumed to be twice continuously differentiable. Moreover we assume that \( d\nu(\theta) \) in \((2.3)\) is a probability measure whose Fourier transform vanishes at infinity:
\[
\int_{\mathbb{T}^d} d\nu(\theta) e^{ix\theta} \rightarrow 0, \quad |x| / \infty, \quad x \in \mathbb{Z}^d
\]

It is instructive to translate this Assumption \( 3.1 \) to real space such that its link to locality becomes obvious. In position representation
\[
\Psi(\rho)(x_1, x_2) = \sum_{y_1, y_2 \in \mathbb{Z}^d} N(x_1, y_1, y_2, x_2) \rho(y_1, y_2),
\]
for some function \( N(\cdot, \cdot, \cdot, \cdot) \). Translation covariance of \( \Psi \) is equivalent to the property
\[
N(x_1, y_1, y_2, x_2) = N(x_1 + z, y_1 + z, y_2 + z, x_2 + z), \quad \text{for all } z \in \mathbb{Z}^d
\]
Our locality Assumption 3.1 can be satisfied if
\[
\sum_{y_1, y_2 \in \mathbb{Z}^d} \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right) |N(x_1, y_1, y_2, x_2)| \xrightarrow{x_1-x_2 \to \infty} 0
\]
This locality can be compared with asking a finite variance for the jumps in a random walk and it is supposed to exclude superdiffusive behavior.

We now come to the dissipativeness. To formulate that conveniently, we introduce the Markov generator \( A \) on \( L^1(\mathbb{T}^d) \)
\[
(Af)(k) = \int_{\mathbb{T}^d} d\nu(\theta) r(k - \theta, \theta) f(k - \theta) - \int_{\mathbb{T}^d} d\nu(\theta) r(k, k + \theta) f(k) \tag{3.8}
\]
with the transition rates \( r(k, k') \) defined by,
\[
 r(k, k + \theta) := M_\theta(k, k) \geq 0. \tag{3.9}
\]
The measure \( d\nu(\theta) r(k, k + \theta) \) gives the probability per unit time of jumping to the state \( k + \theta \), conditioned on being in the state \( k \). As in (2.1) the parameter \( \theta \) plays the role of momentum transfer. Remember that \( r(\cdot, \cdot) \) is a \( C^1 \)-function by Assumption 3.1. One has \( \|Af\|_1 \leq c\|f\|_1 \) for any \( f \in C(\mathbb{T}^d) \) and \( c := \|r(\cdot, \cdot)\|_\infty \), and hence \( A \) is the bounded generator of a contractive (and positive) semigroup on \( L^1(\mathbb{T}^d) \), see the connection with the Hille-Yosida theory in e.g. [19]). Dissipativeness here refers to ergodic properties.

**Assumption 3.2. [Dissipativeness]** We assume

1. \( A \) has a simple eigenvalue 0 with eigenvector \( \mathcal{P} \in L^1(\mathbb{T}^d) \).

2. The eigenvalue 0 is separated from the rest of the spectrum by a gap \( b_A \),
\[
 b_A := -\sup \text{Re} \left( \text{spec}(A) \setminus \{0\} \right) > 0
\]

The assumptions above guarantee that the semigroup generated by \( A \), i.e., \( e^{tA} \), relaxes exponentially fast to the stationary distribution \( \mathcal{P} \). For future use, we let \( Y_t \) stand for the Markov process on \( \mathbb{T}^d \) generated by \( A \) and started from \( \mathcal{P} \). Using standard techniques for Markov processes on compact spaces, constructive conditions are available for guaranteeing the Assumption 3.2 in terms of \( r(k, k') \) and \( \nu \). We say more on that in Appendix B. The basic idea is however that one verifies the second condition of Assumption 3.2 by requiring
\[
\inf_{k} \int_{\mathbb{T}^d} d\nu(\theta) r(k, k + \theta) > 0, \tag{3.10}
\]
Using the \( L^1(\mathbb{T}^d) \) space enables nice continuity properties of the fiber decomposition, to be discussed under Lemma 4.1.

While the above are natural ergodicity or gap-assumptions, in fact for our result we need less. In particular, the exponential relaxation is not strictly necessary. We will however not describe that in detail.

\footnote{Note that there is a slight arbitrariness in choosing \( \nu \) and \( M_\theta \) since only the measures \( d\nu(\theta) M_\theta(k_1, k_2) \) are defined by specifying \( N \).}
The following standard construction will be useful in the statement of one of our results. Consider the scalar product
\[ \langle f, g \rangle_P := \int \! dk \mathcal{P}(k) \overline{f(k)} g(k), \quad f, g \in C(\mathbb{T}^d) \]
and let \( \mathcal{H}_P \) stand for the Hilbert space which is the completion of \( C(\mathbb{T}^d) \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_P \). Define the quadratic form \( A_P \) by
\[ \langle f, A_P g \rangle_P = \int \! dk \mathcal{P}(k) \left( A^\ast f \right)(k) g(k), \quad f, g \in C(\mathbb{T}^d) \] (3.11)
where \( A^\ast \) is the adjoint of \( A \) acting on \( L^\infty(\mathbb{T}^d) \).

The operator \( A \) appears naturally in a perturbation set-up around the zero fiber. In fact, the evolution on the zero fiber is the Markov process generated by \( A \).

We finally ask some properties that appear directly linked with the notion of diffusion. We have of course already that the particle must be sufficiently localized since it is described by a density matrix \( \rho \in B_1(\ell^2(\mathbb{Z}^d)) \), but we also ask that the first two moments are well-defined in the following way
\[ X_j \rho, \; X_i X_j \rho \quad \text{are in} \; B_1(\ell^2(\mathbb{Z}^d)), \quad \text{for} \; i, j = 1, \ldots, d. \] (3.12)
Products of bounded and unbounded operators such as in (3.12) are to be understood as kernels of sequilinear forms with densely defined domains. For example, \( X_j \rho \in B(\ell^2(\mathbb{Z}^d)) \) (which is implied since \( B_1 \subset B \)) means that the form \( b(f, g) := \langle X_j f, pg \rangle \) restricted to \( (f, g) \in \text{Dom}(X_j) \times \ell^2(\mathbb{Z}^d) \) is bounded, and thus is extendable to \( \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d) \). Note that by the boundedness of the form \( b(f, g) \) and by the definition of the domain of the self-adjoint operator \( X_j \), we have that
\[ \rho g \in \text{Dom}(X_j), \quad \text{for any} \; g \in \ell^2(\mathbb{Z}^d), \] (3.13)
so that the product \( X_j \rho \) makes sense.

We write in general \( \rho_t \) for the solution of the equation (3.3) with initial condition \( \rho_0 \in B_1 \). We will show that there is a \( v \in \mathbb{R}^d \) such that
\[ v = \lim_{t \to \infty} \frac{1}{t} \sum_{x \in \mathbb{Z}^d} x \rho_t(x, x) \] (3.14)
That can be strengthened to a weak law of large numbers. One can obviously force \( v = 0 \) by requiring some additional symmetries. Getting our results does however not depend on these extra requirements. In particular, equilibrium conditions such as detailed balance are mostly irrelevant for the diffusive behavior around the drift, except when one wants for example to relate the diffusion constant to the mobility in linear response relations.

### 3.3 Result

We define \( T^{(1)} \) and \( T^{(2)} \) as, respectively, a vector and \( d \times d \) matrix of operators on \( L^1(\mathbb{T}^d) \), by
\[ (T^{(1)} f)(k) = i(\nabla H)(k) f(k) + \int_{\mathbb{T}^d} \! d\nu(\theta) \, m_\theta^{(1)}(k + \theta) f(k + \theta), \] (3.15)
where $m_\theta^{(1)}(k) = -i \text{Im} \left( \nabla_1 M_\theta(k, k) \right)$ (and $\nabla_1$ and $\nabla_2$ are the gradients with respect to the first and second variables of $M_\theta(k_1, k_2)$), and

$$
(T^{(2)} f)(k) = \int_{\mathbb{T}^d} d\nu(\theta) \left( (m_\theta^{(3)} - m_\theta^{(2)})(k + \theta) f(k + \theta) - m_\theta^{(3)}(k) f(k) \right) \tag{3.16}
$$

where

$$
(m_\theta^{(2)}(k))_{(i,j)} = \frac{1}{4} \left[ (\nabla_1 \nabla_2 M_\theta(k, k))_{(i,j)} + (\nabla_1 \nabla_2 M_\theta(k, k))_{(j,i)} \right],
$$

$$
(m_\theta^{(3)}(k))_{(i,j)} = \frac{1}{8} \left[ (\nabla_1^2 + \nabla_2^2) M_\theta(k, k))_{(i,j)} + (\nabla_1^2 + \nabla_2^2) M_\theta(k, k))_{(j,i)} \right].
$$

We let $P_0$ be the projection corresponding to the 0 eigenvalue of $A$, the Markov generator defined in (3.8), and take $S$ the reduced resolvent of $A$ at the eigenvalue 0, i.e., the solution of

$$
S(0 - A) = (0 - A)S = 1 - P_0, \quad SP_0 = P_0 S = 0.
$$

Finally, recall that $\mathcal{P}$ is the eigenvector corresponding to the eigenvalue 0. Hence $\mathcal{P}(k)dk$ is the stationary probability measure on the torus. In general the projection $P_0$ is non-orthogonal and has the form $P_0 = |\mathcal{P}|(1_{\mathbb{T}^d}|$, where $1_{\mathbb{T}^d}$ is the constant function on $\mathbb{T}^d$ with value 1. We use the notation $\langle g, f \rangle := \int_{\mathbb{T}^d}dk g(k) f(k)$ for the pairing between $f \in L^1(\mathbb{T}^d)$ and $g \in L^\infty(\mathbb{T}^d)$. To the time-evolved density matrix $\rho_t$, we associate the measures $\mu_t$ on $\mathbb{R}^d$, defined by

$$
\mu_t(R) := \text{Tr} [1_{\sqrt{t}R+vt}(X) \rho_t], \quad \text{for a Borel set } R \subset \mathbb{R}^d \tag{3.17}
$$

where $1_{\sqrt{t}R+vt}(X)$ is the spectral projection of the vector of position operators $X$ on the set $\sqrt{t}R + vt \subset \mathbb{R}^d$.

**Theorem 3.3.** Assume Assumptions 3.1 and 3.2 and let the initial density matrix $\rho_0 \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ satisfy (3.12). The limiting velocity exists and equals

$$
v := \lim_{t \to \infty} \frac{1}{t} \sum_{x \in \mathbb{Z}^d} x \rho_t(x, x) = -i \langle 1_{\mathbb{T}^d}, T^{(1)} \mathcal{P} \rangle \tag{3.18}
$$

The measure $\mu_t$, defined as in (3.17) with $v$ as in (3.18), converges, as $t \to \infty$, in distribution to a Gaussian with covariance matrix $\sigma = \beta + \beta^\dagger$, where $\beta^\dagger$ is the transpose of the matrix $\beta$, given by

$$
\beta := -\langle 1_{\mathbb{T}^d}, (T^{(2)} - T^{(1)}ST^{(1)}) \mathcal{P} \rangle. \tag{3.19}
$$

where the operators $T^2$ and $T^1$ as defined above.

The truncated second moments converge to the covariance matrix $\sigma$, i.e.

$$
\lim_{t \to \infty} \sum_{x \in \mathbb{Z}^d} \rho_t(x, x)(x_i - tv_i)(x_j - tv_j) = \sigma(i, j) \tag{3.20}
$$

Equation (3.20) tells us that the covariance matrix $\sigma$ can be interpreted as the diffusion tensor. In the case that $T^{(1)} = i \nabla H$ and $T^{(2)} = 0$, it reduces to another matrix that below we call $\alpha$. In the proposition below, the matrices $\sigma$ and $\alpha$ are “non-negative” in the sense of real valued vectors: $\forall v \in \mathbb{R}^d(v, \sigma v), (v, \alpha v) \geq 0$. 

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Proposition 3.4. Consider the diffusion matrix $\sigma$ appearing in (3.19).

1. $\sigma$ is non-negative.

2. Define the (vector) function
   \[ \zeta := \nabla H - \langle \nabla H, \mathcal{P} \rangle \] (3.21)
on $\mathbb{T}^d$. The real-valued $d \times d$-matrix $\alpha$ with entries
   \[ \alpha(i, j) = \frac{1}{2} \int_0^\infty dt \mathbb{E}_{\mathcal{P}}[(\zeta(Y_t))_i(\zeta(Y_0))_j + (\zeta(Y_t))_j(\zeta(Y_0))_i], \] (3.22)
is non-negative. (As in Section 3.2, $Y_t$ is the stationary Markov process on $\mathbb{T}^d$ with stationary density $\mathcal{P}$ and evolving according to the Markov evolution generated by $A$, cf. (3.8)).

3. Assume that for all $w \in \mathbb{R}^d$, the function $k \mapsto (w, \zeta(k))$ does not vanish identically (we have written $(\cdot, \cdot)$ for the scalar product on $\mathbb{R}^d$). That is, all components of the velocity $\nabla H$ can fluctuate. Assume in addition that $A_{\mathcal{P}}$, defined as a quadratic form in (3.11), extends to a bounded and sectorial operator on $\mathcal{H}_{\mathcal{P}}$. Sectoriality means that there is $0 < \gamma < \infty$ such that
   \[ |\langle f, \text{Im}(A_{\mathcal{P}})f \rangle_{\mathcal{P}}| \leq -\gamma \langle f, \text{Re}(A_{\mathcal{P}})f \rangle_{\mathcal{P}}, \quad \text{for all } f \in \mathcal{H}_{\mathcal{P}}. \] (3.23)
Then the matrix $\alpha$ is strictly positive (i.e., it has strictly positive eigenvalues).

4 Discussion

4.1 Symmetries

We can erase drift terms and simplify the diffusion matrix $\sigma$ by imposing further symmetries. Define the linear and antilinear maps $U$ and $V$ acting on $f \in \ell^2(\mathbb{Z}^d)$ as
   \[ (Uf)(x) = f(-x) \quad \text{and} \quad (Vf)(x) = \overline{f(x)} \]
By assuming the symmetry
   \[ U\Phi_t(\rho)U^{-1} = \Phi_t(U\rho U^{-1}), \quad \rho \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \] (4.1)
(or on the generator $L$ of the semigroup $\Phi_t$), we have no drift i.e.
   \[ 0 = v = \lim_{t/\infty} \frac{1}{t} \sum_x x\rho_t(x, x) \] (4.2)
If rather than the symmetry $U$, we assume that the noise term $\Psi$ satisfies the symmetry
   \[ V\Psi(\rho)V^{-1} = \Psi(V\rho V^{-1}), \] (4.3)
then the drift depends on the stationary state $\mathcal{P}$ and the kinetic term $H$ as
   \[ v = \lim_{t/\infty} \frac{1}{t} \sum_x x\rho_t(x, x) = \langle \nabla H, \mathcal{P} \rangle. \]
When we have the symmetry \((4.3)\), then \(T^2 = 0\) and the diffusion matrix \(\sigma\) simplifies to a matrix that can be written in the form

\[
\sigma(i, j) = \int_{\mathbb{T}^d} dk \int_{\mathbb{T}^d} d\nu(\theta) \mathcal{P}(k) \left( [\nabla_1 \nabla_2 \rho_{\theta}]_{(i,j)}(k) + [\nabla_1 \rho_{\theta}]_{(j,i)}(k) \right) + \int_0^\infty dt \mathbb{E}_p[(\zeta(Y_t))_i(\zeta(Y_0))_j + (\zeta(Y_t))_j(\zeta(Y_0))_i], \tag{4.4}
\]

where the rightmost expression is defined as in \((3.22)\), and the first expression on the right is a non-negative matrix. We sketch the reasoning for this as follows. Since \(M_\theta\) is a completely positive map on \(B_1(\ell^2(\mathbb{Z}^d))\), for each \(\theta\), it follows that \(M_\theta(k_1, k_2)\) is the integral kernel for a positive operator on \(L^2(\mathbb{T}^d)\). This moreover implies that for any \(v \in \mathbb{R}^d\), \((v, [\nabla_1 \nabla_2 \rho_{\theta}]_{(k_1, k_2)}v)\) defines an integral kernel for a positive operator. The expression

\[
(v, \int_{\mathbb{T}^d} dk d\nu(\theta) \mathcal{P}(k) [\nabla_1 \nabla_2 \rho_{\theta}]_{(k,k)}v)
\]

is thus non-negative, since the diagonal entries in the integral kernel for a positive operator are non-negative. Symmetrizing \([\nabla_1 \nabla_2 \rho_{\theta}]_{(i,j)}(k, k)\) in the above expression makes no difference in the evaluation of expressions \((v, \sigma v)\) for \(v \in \mathbb{R}^d\) but ensures that we have a symmetric and real-valued diffusion matrix.

### 4.2 Idea of proof: perturbation theory

The main feature of our translation invariant models is that the evolution generated by \((2.2)\) can be decomposed along the fibers \((3.2)\), i.e., one can write \([L \rho]_p = L_p[\rho]_p\) for some operators \(L_p\) and the fibers of the density matrix, \([\rho_k]_p\), obey the differential equation

\[
\frac{d}{dt}[\rho_k]_p = L_p[\rho]_p, \tag{4.5}
\]

The expression for \(L_p\) can be determined as a quadratic form through a trace formula: for \(F \in L^\infty(\mathbb{T}^d)\),

\[
\text{Tr}[L^* (e^{ip/2\lambda} F(\Omega) e^{ip/2\lambda}) \rho] = \langle F, L_p[\rho]_p \rangle = \int_{\mathbb{T}^d} dk F(k) L_p[\rho]_p(k). \tag{4.6}
\]

A simple computation shows that \(L_p = ih_p + \Psi_p\) with

\[
(h_p f)(k) := \left( H(k - \frac{p}{2}) - H(k + \frac{p}{2}) \right) f(k)
\]

\[
(\Psi_p f)(k) := \int_{\mathbb{T}^d} d\nu(\theta) (r_p(k - \theta, k) f(k - \theta) - \bar{r}_p(k, k + \theta) f(k)), \tag{4.8}
\]

where \(r_p(k, k') = M_{k-k'}(k - \frac{p}{2}, k + \frac{p}{2})\). Note that \(r_{p=0}(k, k')\) was simply called \(r(k, k')\) in \((3.9)\).

The notation \(\Psi_p\) is consistent with the translation covariance of \(\Psi\), which implies that \([\Psi_p]_p = \Psi_p[\rho]_p\).

Since only the fibers around \(p = 0\) determine the diffusive behavior, this suggests using a perturbation argument to capture the essential dynamical properties of those fibers. Under our
assumptions on $A = L_0$, it has a gap between the zero eigenvalue corresponding to the sta-
tionary distribution $P$ and the rest of the spectrum which has strictly negative real part. Hence
small and sufficiently bounded perturbations of $L_0$ also have a gap.

The following lemma gives a condition on $\rho$ such that the function $p \mapsto [\rho]_p$ is twice differen-
tiable. This assures the existence of the first two moments.

**Proposition 4.1.** Assume that a density matrix $\rho \in B_1(\ell^2(\mathbb{Z}^d))$ satisfies

$$
X_j \rho, \quad X_i \rho X_j, \quad X_i X_j \rho \in B_1(\ell^2(\mathbb{Z}^d)) \quad \text{for } i, j = 1, \ldots, d.
$$

(as in (3.12)), then the function $\mathbb{T}^d \to L^1(\mathbb{T}^d) : p \mapsto [\rho]_p$ can be chosen to be twice continuously
differentiable.

**4.3 Examples**

We mention a certain subclass of models satisfying the conditions for (3.3) and another class of
models that do not satisfy our conditions and that possibly show a non-diffusive (e.g. super-
diffusive) behavior.

We consider first the case when the completely positive maps $M_\theta$ in (2.3) reduce to $M_\theta \rho = \rho$,
such that $\Psi$ becomes

$$
\Psi(\rho) = \int_{\mathbb{T}^d} d\nu(\theta) e^{i\theta X} \rho e^{-i\theta X}
$$

The map $\Psi$ then acts as multiplication in the position representation

$$
\Psi(\rho)(x, y) = \varphi(x - y) \rho(x, y)
$$

where $\varphi$ is the Fourier transform of the measure $\nu$. This is similar to the noise term for Gallis-
Flemming dynamics [13] for a quantum particle in $\mathbb{R}$ that has been frequently discussed in the
decoherence literature [11, 24]. The simplest case is when $\nu$ is a multiple of Lebesgue measure,
for which $\varphi(x, y) = c \delta_{x,y}$ and the particle changes position only by being carried through the
kinetic term.

Like in [13], the particle can be thought of as in contact with a reservoir at infinite temperature
and the stationary density $P$ of the zero-fiber Markov process is the uniform distribution. Since
$\nabla_1 \nabla_2 M_\theta(k_1, k_2) = 0$, the diffusion constant $\sigma$ takes the familiar form of the second term in (4.4).
That is, the diffusion matrix reduces to $\alpha$, as defined in Proposition 3.4.

Thinking about adding spatial jumps we arrive at models where the noise $\Psi$ resembles a
simple symmetric random walk. Yet, that easily breaks Assumption 3.2 Statement 2), see also
(3.10), and the model can become super-diffusive when the kinetic term is included. As an
illustration, we consider a one-dimensional model. Let $\Psi$ have the form

$$
\Psi(\rho)(x_1, x_2) = \sum_{y_1, y_2 \in \mathbb{Z}} N(x_1, y_1, y_2, x_2) \rho(y_1, y_2),
$$

where $N(x_1, y_1, y_2, x_2)$ is determined by a function $r(x), x \in \mathbb{Z}^d$, with a positive Fourier trans-
form through the equation

$$
N(x_1, y_1, y_2, x_2) = r(x_1 - x_2) \chi[|x_1 - y_1| = 1] \chi[|x_2 - y_2| = 1]
$$
where \( \chi[\cdot] \) is the indicator (1 or 0). In that case, by the formal derivation in Appendix A, the measure \( \nu(\theta) \) can be taken to be Lebesgue measure and the functions \( M_\theta(k_1, k_2) \) are determined by

\[
M_\theta(k_1, k_2) = 4 \hat{r}(\theta) \cos(k_1) \cos(k_2)
\]

\[
\Psi(\rho)(k_1, k_2) = 4 \int_\mathbb{T} d\theta \hat{r}(\theta) \cos(k_1 + \theta) \cos(k_2 + \theta) \rho(k_1 + \theta, k_2 + \theta)
\]

(4.10)

The smoothness of \( \hat{r}(\theta) = \frac{1}{(2\pi)^d} \sum_n e^{in\theta} r(n) \) depends on the decay of \( c \). Notice that the Markov process of the zero fiber is generated by

\[
(Af)(k) = 4 \int_\mathbb{T} d\theta \hat{r}(\theta) \left( \cos(k + \theta)^2 f(k + \theta) - \cos(k)^2 f(k) \right).
\]

That is describing a random walk on the torus \( \mathbb{T} \) where an escape from the point \( k \) occurs with rate \( 4 \cos^2(k) \left( \int_\mathbb{T} d\theta \hat{r}(\theta) \right) \) and the jump size \( \theta \) is independent of \( k \) and has probability density

\[
\frac{\hat{r}(\theta)}{\int_\mathbb{T} d\theta \hat{r}(\theta)}
\]

Assumption 3.2 is now violated in that, for \( k = \pm \frac{\pi}{2} \),

\[
\int_\mathbb{T} d\theta M_\theta(k, k) = 4 \left( \int_\mathbb{T} d\theta \hat{r}(\theta) \right) \cos^2(k) = 0
\]

This implies that there are degenerate stationary distributions of the form

\[
\mathcal{P}_\lambda(k) = \lambda \delta \left( \frac{\pi}{2} - k \right) + (1 - \lambda) \delta \left( - \frac{\pi}{2} - k \right)
\]

(4.11)

for \( 0 \leq \lambda \leq 1 \). The process is thus slow to leave the regions around \( k = \pm \frac{\pi}{2} \). We conjecture that for an arbitrary smooth probability distribution \( \mathcal{V} \) on \( \mathbb{T} \), \( e^{tA} \mathcal{V} \) converges in distribution to \( \mathcal{P}_{1/2} \), i.e., to (4.11) with \( \lambda = \frac{1}{2} \).

When the kinetic term \( H \) is zero, we observe that still the diffusion constant as such keeps making mathematical sense as

\[
\sigma = 2 \int_\mathbb{T} dk \mathcal{P}_{1/2}(k)(\partial_1 \partial_2 M)(k, k) = 8 \int_\mathbb{T} d\theta \hat{r}(\theta)
\]

When the kinetic term \( H \) is non-zero and \( H'(\pm \frac{\pi}{2}) \neq 0 \), we conjecture that the model exhibits a behavior closer to being ballistic with a width in position that grows on the order of \( t^{\frac{1}{2}} \) rather than \( t^{\frac{1}{4}} \). The basic idea is that when the particle attains a momentum close to \( k = \pm \frac{\pi}{2} \), then it tends to stay in or around that momentum for long intervals with infinite mean residence times. We are not sure about the physical relevance or importance of these models.

4.4 A classical analogue

There is a sense in which our present quantum problem differs little from an analogous classical problem that starts from a linear Boltzmann equation. We explain that analogy here. Among
other things it relates our results to the recent interest and work on diffusive behavior in systems of coupled oscillators where energy transport can be understood as a wave scattered by anharmonicities, [8].

Consider a stochastic dynamics with state space $S = \mathbb{Z}^d \times \mathbb{T}^d$ such that the probability density $P_t(x, k) = \Gamma_t(P)(x, k)$ evolves as

$$
\frac{d}{dt} P_t(x, k) = h(k)(P_t(x + 1, k) - P_t(x - 1, k)) + \int_{(x', k') \in S} (T(x', k'; x, k)P_t(x', k') - T(x, k; x', k')P_t(x, k)),
$$

for an initial $P_0(x, k) = P(x, k)$, where $T(x, k; x', k') \geq 0$ is a transition matrix describing the rates of Poisson timed jumps from $(x, k)$ to $(x', k')$ and the symbol $\int_{(x', k') \in S}$ stands for $\int_{\mathbb{T}^d} \sum_{x \in \mathbb{Z}^d}$. When $T$ satisfies $T(x + z, k; x' + z, k') = T(x, k; x', k')$, then the master equation (4.12) describes a formally translation-invariant evolution. From a classical physics perspective the evolution (4.12) is still somewhat strange as it involves both jumps in position $(x)$ and in momentum $(k)$ for the single particle’s state evolution. Still one can ask the classical questions about its diffusive behavior. We must then show that the centered position ($x - vt$) converges in distribution to a Gaussian with non-degenerate covariance matrix. We show here how this can proceed along the very same lines as for the (indeed, more general) quantum case.

By translation-invariance the dynamics does not “feel” the location of the particle and it follows that the “momentum” dynamics is Markovian, or, the marginal probability, $\tilde{P}_t(k) = \sum_{x \in \mathbb{Z}^d} P_t(x, k)$, for the $k$ variable obeys an autonomous first order equation:

$$
\frac{d}{dt} \tilde{P}_t(k) = \int_{\mathbb{T}^d} dk' (\tilde{T}(k', k) \tilde{P}_t(k') - \tilde{T}(k, k') \tilde{P}_t(k)),
$$

where $\tilde{T}(k', k) = \sum_{x} T(x + z, k'; x, k)$. This is analogous to the zero fiber of our decomposition for the quantum dynamics. Defining $[P_t]_p(k) = \sum_{x \in \mathbb{Z}^d} e^{ipx} P_t(x, k)$, the dynamics (4.12) operates as $[\Gamma_t]_p[P] = [\Gamma_t(P)]_p$ for some semigroup $[\Gamma_t]_p : L^1(\mathbb{T}^d) \to L^1(\mathbb{T}^d)$.

The closest thing to a joint probability density for position and momentum in the quantum case is the Wigner function. Here we go in the opposite direction and we define a “classical density matrix” from a joint distribution function $P(x, k)$. That is, for a momentum representation kernel, we formally define:

$$
\rho(k_1, k_2) = \sum_{x \in \mathbb{Z}^d} e^{-ix(k_1 - k_2)} P(x, \frac{k_1 + k_2}{2}).
$$

In this case, the positivity of $\rho$ is lost, since it is not in general the kernel of an operator with positive spectrum. The invariant fibers again correspond to $p = k_1 - k_2$ in the momentum representation. The dynamics can thus be written in its fibers

$$
\frac{d}{dt} [P_t]_p(k) = i \sin(p) h(k)[P_t]_p(k) + \int_{\mathbb{T}^d} dk' (\tilde{T}_p(k', k)[P_t]_p(k') - \tilde{T}_0(k, k')[P_t]_p(k))
$$
where $\tilde{T}_p(k', k) = \sum_{z \in \mathbb{Z}^d} e^{ipz}T(x, k'; x + z, k)$. One notices similarities with the fiber decomposition and with (4.7).

5 Proofs

We need a technical lemma to deal conveniently with trace-class operators. We write $B_1 = B_1(\mathcal{H})$, for some Hilbert space $\mathcal{H}$; in the application to our result $\mathcal{H} = \ell^2(\mathbb{Z}^d)$.

Lemma 5.1. Let $C \in B_1$ and let $X$ be a self-adjoint operator. If $XC \in B_1$, then

$$\lim_{\gamma \searrow 0} \frac{1}{\gamma} (e^{i\gamma X} - I)C = XC$$

(5.1)

where the convergence is meant in the sense of $B_1$.

Proof. Let

$$C = \sum_{n \in \mathbb{N}} \lambda_n |f_n\rangle\langle g_n|, \quad f_n, g_n \in \mathcal{H}, \quad \sum_n |\lambda_n| < \infty$$

(5.2)

be the singular value decomposition of $C \in B_1$. Both families $f_n$ and $g_n$ are an orthonormal set. By the comment (3.13) and $Cg_n = \lambda_n f_n$, it follows that when $\lambda_n \neq 0$, then $f_n \in \text{Dom} X$.

If there are only finitely many terms in the singular value decomposition (5.2), then the convergence (5.1) is guaranteed by an application of Stone’s theorem for each $n$. When there are an infinite number of terms in (5.2), an extra estimate is required to bound the tail of the sum.

Defining the projection $P_n = \sum_{n=1}^{N} |g_n\rangle\langle g_n|$, then we have the following bound

$$\| \frac{1}{\gamma}(e^{i\gamma X} - I)C - XC \|_1 \leq \sum_{n=1}^{N} \| (\frac{1}{\gamma}(e^{i\gamma X} - I) - X)f_n \| \langle g_n|$$

$$+ \| XC(I - P_N) \|_1 + \| \frac{1}{\gamma}(e^{i\gamma X} - I)C(I - P_N) \|_1.$$  

(5.3)

Our strategy will be to pick, for each $\epsilon > 0$ a number $N_\epsilon$ such that the last two terms on the right are bounded by $\epsilon$, for $N \geq N_\epsilon$ and arbitrary $\gamma$. Since the first term, with $N = N_\epsilon$ vanishes as $\gamma \searrow 0$ by the reasoning above, the claim will follow:

Since $XC \in B_1$, we have $\| XC(I - P_N) \|_1 \to 0$, as $N \to \infty$. To bound the third term on the right-hand-side, we note that for $A = \frac{1}{\gamma}(e^{i\gamma X} - I)$ and $B = 2X$, we have $0 \leq |A|^2 \leq |B|^2$, and hence also $D^*|A|^2D \leq D^*|B|^2D$ for $D = C(1 - P_N)$. Since $\cdot \mapsto \sqrt{\cdot}$ is an operator monotone function, we have $\sqrt{D^*|A|^2D} \leq \sqrt{D^*|B|^2D}$. Hence

$$\| AC \|_1 \leq \text{Tr} \sqrt{C^*|A|^2C} \leq \text{Tr} \sqrt{C^*|B|^2C} = \| BC \|_1.$$ 

\[\square\]

We continue with the proof of Proposition 4.1.
Step 1
Assume the singular value decomposition for $C$, as in (5.2). Then

$$k \mapsto \Upsilon_p(k) := \sum_{n \in \mathbb{N}} \lambda_n (e^{i \frac{p}{2} X} f_n)(k) (e^{-i \frac{p}{2} X} g_n)(k)$$

is a $L^1$-function (since it is the product of two $L^2$-functions), which depends continuously on $p$ (since $p \mapsto e^{i \frac{p}{2} X}$ is a strongly continuous group on $L^2(\mathbb{T}^d)$). It is straightforward to verify that $[C]_p := \Upsilon_p$ satisfies our definition of the fiber decomposition (3.2). In other words,

$$C(k_1, k_2) := \Upsilon_{k_2-k_1}(\frac{k_1 + k_2}{2})$$

is a kernel for the operator $C$.

Step 2
We show that, under assumption (4.9), $[C]_p$ is actually in $C_1$. If both $C$ and $XC, CX$ are in $B_1$, then

$$\frac{i}{2} \{ [X_j, C] \}_p = \frac{\partial}{\partial p_j} [C]_p.$$  \hspace{1cm} (5.6)

because of (5.1). The continuity of $\frac{\partial}{\partial p_j} [C]_p$ in $p$, follows from (5.6) and from the general argument in Step 1, with $C$ replaced by $\frac{i}{2} \{ C, X \}$.

Step 3
The existence and continuity of the second derivative follows by repeating Step 2, since $\{ C, X \} X$ and $X \{ C, X \}$ are in $B_1$.

The following lemma lays out the standard perturbation theory [18] that we make use of.

**Lemma 5.2.** Consider a family of bounded operators $L_p$ on a Banach space for $p \in \mathbb{T}^d$. Assume that the $\frac{\partial^2}{\partial p_i \partial p_j} L_p$ are bounded and continuous as a functions of $p$ in some neighborhood of $0 \in \mathbb{T}^d$. Finally, assume that $\text{spec}(L_0)$, the spectrum of $L_0$, contains $0$ as an isolated point, corresponding to a simple eigenvalue $0$. Then, for $|p|$ small enough, the operator $L_p$ has a unique eigenvalue $D_p$ such that

$$D_p = (p, \text{tr}[T^{(1)} P_0]) + (p, \text{tr}[T^{(2)} P_0]) - (p, \text{tr}[T^{(1)} S T^{(1)} P_0] + o(p^2)), \hspace{1cm} (5.7)$$

where $T^{(1)}$ and $T^{(2)}$ are, respectively, a vector of operators and a self-transpose matrix of operators (i.e. $T^{(2)}_{i,j} = T^{(2)}_{j,i}$), defined by the expansion

$$L_p - L_0 = (p, T^{(1)}) + (p, T^{(2)} p) + o(p^2),$$

$P_0$ is the projection corresponding to the eigenvalue $0$ of $L_0$, and $S$ is the reduced resolvent of $L_0$, at the eigenvalue $0$, i.e. the solution of

$$S(0 - L_0) = (0 - L_0) S = 1 - P_0, \hspace{1cm} SP_0 = P_0 S = 0.$$

Moreover, $\text{spec}(L_p) \setminus \{ D_p \}$ lies at a distance $o(p)$ from $\text{spec}(L_0) \setminus \{ 0 \}$. 

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For our model the projection \( P_0 \) has the form \( P_0 = |\mathcal{P}\rangle\langle 1_{\mathbb{T}^d}| \), where \( \mathcal{P} \) is the stationary density for the Markov process of the zero fiber and \( 1_{\mathbb{T}^d} \) is the indicator function over \( \mathbb{T}^d \). The first-order perturbation term \( T^{(1)} \) is given in (3.15). The second-order perturbation \( T^{(2)} \) is given by (3.16) and it only depends on the map \( \Psi \), not on \( H \). With the assumption of (4.3), so that the second term on the righthand-side of (3.15) is zero, we have the following explicit expressions

\[
\text{tr}[T^{(1)}P_0] = i \int_{\mathbb{T}^d} dk \mathcal{P}(k)(\nabla H)(k), \quad (5.8)
\]

\[
\text{tr}[T^{(2)}P_0] = - \int_{\mathbb{T}^d} dk \int_{\mathbb{T}^d} d\nu(\theta) \mathcal{P}(k)m^{(2)}_{\theta}(k), \quad (5.9)
\]

\[
\text{tr}[T^{(1)}ST^{(1)}P_0] = - \int_0^\infty dt \int_{\mathbb{T}^d} dk \zeta(k)(e^{tL_0}\zeta)(k), \quad (5.10)
\]

with the function \( \zeta \) as defined in Proposition 3.4. We have used that on the range of \( S \), the integral \( \int_0^\infty dt e^{tL_0} \) is well defined and equal to \( S \).

The expression (5.9) is non-positive since the matrices \( \nabla_1\nabla_2 M_{\theta}(k, k) \), are non-negative, as explained in Section 4.1. Seeing that the expression (5.10) is non-positive is a little more tricky, but it helps to rewrite the righthand-side as the following:

\[
\int_0^\infty dt \int_{\mathbb{T}^d} dk (e^{tL_0}\zeta)(k) \zeta(k)\mathcal{P}(k) = \int_0^\infty dt \mathbb{E}_\mathcal{P}[(\zeta(Y_t)\zeta(Y_0)].
\]

The evolution has been shifted to operate on the observables which can then be rewritten in terms of the expectation \( \mathbb{E}_\mathcal{P} \) of the Markov process \( Y_t \) started from the stationary distribution \( \mathcal{P} \). We show the non-negativity of this expression in the proof of Proposition 3.4.

[Proof of Theorem 3.3]

To prove convergence in distribution for \( \mu_t \), we show pointwise convergence of the characteristic functions.

\[
\varphi_{\mu_t}(\gamma) := \int_{\mathbb{R}^d} d\mu_t(x)e^{ix\gamma} = \text{Tr}[\rho te^{i\sqrt{\gamma}(X-xt)}] = e^{-i\sqrt{\gamma}t} \langle 1_{\mathbb{T}^d}, [e^{tL}(p)]_{X} \rangle \quad (5.11)
\]

where the third equality makes use of the fiber decomposition of our dynamics. In particular, we used the relation (cf. the proof of Lemma 4.1)

\[
[e^{i\sqrt{\gamma}X}Ce^{i\sqrt{\gamma}X}]_p = [C]_{p+\gamma}, \quad C \in \mathcal{B}_1.
\]

For a fixed \( \gamma \) the limit involves only small fibers \( p \propto t^{-\frac{1}{2}} \), which suggests using a perturbation argument around the zero fiber. By Lemma (5.2), for a small enough neighborhood of \( p = 0 \), say \( U \subset \mathbb{T}^d \), we have that

\[
[e^{tL}\rho]_p = P_p e^{tD_p} + (1 - P_p)e^{tV_p}[p], \quad V_p := L_p - D_pP_p
\]

and \( \|e^{tV_p}\| = O(e^{-tb}) \) as \( t \uparrow \infty \), for some \( b > 0 \) satisfying \( b - b_A \to 0 \) as \( p \searrow 0 \). The norm refers to the operator norm on \( \mathcal{B}(L^1(\mathbb{T}^d)) \) and \( b_A \) is the gap of the operator \( A \) (Assumption 3.2).
We now show that, in $L^1(\mathbb{T}^d)$,
\[
e^{-i\sqrt{tv}\gamma[e^{tL}p]}\mathcal{P} \xrightarrow{t/\infty} e^{-\frac{1}{2}(\gamma,\sigma\gamma)}\mathcal{P}
\]  
(5.14)

This follows by combining (5.13), the relation
\[
D_p = i(p,v) - \frac{1}{2}(p,\sigma p) + o(p^2)
\]  
(5.15)

(which follows from Lemma 5.2), and the fact that
\[
P_p \xrightarrow{t/\infty} P_0 = |P\rangle\langle 1|_{\mathbb{T}^d}, \quad [\rho]_p \xrightarrow{t/\infty} [\rho]_0
\]  
(5.16)

where the first convergence is in $B(L^1(\mathbb{T}^d))$ and the second in $L^1(\mathbb{T}^d)$. The first claim of (5.16) follows again from Lemma 5.2, the second is a consequence of Proposition 4.1. We have shown (5.14). Since norm convergence implies weak convergence, in particular $\int_{\mathbb{T}^d} e^{tL}[\rho]_p \, d\mathcal{T}$ integrated against the indicator $1_{\mathbb{T}^d}$ converges to the desired value. Hence $\mu_t$ converges in distribution.

We now prove the convergence of the first and second moments (3.19). By (5.11), and the usual connection between moments and derivatives of the characteristic function, we have
\[
\frac{1}{t} \sum_{x \in \mathbb{Z}^d} \rho_t(x,x)(x_i - tv_i)(x_j - tv_j) = \frac{1}{t} \left( - \frac{\partial^2}{\partial p_i \partial p_j} \langle 1, e^{tL}\rho_0 \rangle_p \bigg|_{p=0} + \frac{\partial}{\partial p_i} \langle 1, e^{tL}\rho_0 \rangle_p \bigg|_{p=0} \frac{\partial}{\partial p_j} \langle 1, e^{tL}\rho_0 \rangle_p \bigg|_{p=0} \right)
\]  
(5.17)

Note that since the operator $L_p$ has two continuous derivatives, the operators $P_p, V_p$ (defined above) and the eigenvalue $D_p$ do also. In particular $\| \left( \frac{\partial^2}{\partial p_i \partial p_j} \right) V_p \|$ is bounded for $p \in \mathcal{U}$. One can see that,
\[
\sup_{p \in \mathcal{U}} \| \left( \frac{\partial^2}{\partial p_i \partial p_j} \right) e^{tV_p} \| = O(t^2 e^{-bt}), \quad t \nearrow \infty.
\]

(and a similar bound for the first-order derivatives). By Lemma 4.1, the function $p \mapsto [\rho]_p$ is $C^2$ and we obtain
\[
\frac{\partial^2}{\partial p_i \partial p_j} (e^{tL_p}[\rho]_p) \bigg|_{p \in \mathcal{U}} \xrightarrow{t/\infty} 0.
\]

Hence, in (5.17) we can replace $e^{tL_p}$ with $P_p e^{tD_p}$, for large $t$, and we see the convergence to $\sigma$ using the expansion (5.15).

[Proof of Proposition 3.4]}

**Proof of Statement 1)** In the proof of Theorem 3.3, we showed the pointwise convergence of the characteristic function $\varphi_{\mu_t}$, i.e.,
\[
\varphi_{\mu_t}(\gamma) = \int_{\mathbb{R}^d} d\mu_t(x)e^{ix\gamma} \xrightarrow{t/\infty} e^{-\frac{1}{2}(\gamma,\sigma\gamma)}
\]  
(5.18)
(see e.g. (5.11) and (5.14)). Suppose there were a $\gamma \in \mathbb{R}^d$ such that $(\gamma, \sigma \gamma) < 0$, then, for large enough $t, |\varphi_{\mu}(\gamma)| > 1$, which is impossible since $\varphi_{\mu}$ is the characteristic function of a probability measure. This proves the non-negativity of the diffusion matrix $\sigma$. \hfill \square

**Proof of Statement 2)** Now we consider the non-negativity of the matrix $\alpha$. We show that $\alpha$ is a non-negative matrix by showing that the expression $(w, \alpha w)$ for $w \in \mathbb{R}^d$ is always non-negative. Using an unsymmetrized form for $\alpha$, we can rewrite our evaluation as

\[(w, \alpha w) = \int_0^\infty \text{d}t \, \mathbb{E}_P[f(Y_t)f(Y_0)], \tag{5.19}\]

where $f(k) := (w, \zeta(k))$ is real valued. Let the stationary Markov process $Y_t$ be extended to all negative values of $t$. Then $G(t) = \mathbb{E}_P[f(Y_t)f(Y_0)]$ is an even function and so (5.19) is twice the value of the Fourier transform $\tilde{G}(z)$ of $G(t)$ at $z = 0$. We will show that $\tilde{G}(z)$ is non-negative valued. Using Bochner’s theorem we just need to check that $G(t - s)$ defines a positive operator on $L^2(\mathbb{R})$. Let $\eta \in L^2(\mathbb{R})$, then

\[\int_{\mathbb{R}^2} \text{d}t \, \text{d}s \, \eta(t)G(t - s)\eta(s) = \mathbb{E}_P[|\int_{\mathbb{R}} \text{d}t \, \eta(t)f(Y_t)|^2]\]

where we have used the stationarity property $\mathbb{E}_P[f(Y_{t-s})f(Y_0)] = \mathbb{E}_P[f(Y_t)f(Y_s)]$. \hfill \square

**Proof of Statement 3)** The strict positivity of $\alpha$ is established as follows. Let $w \in \mathbb{R}^d$. By the assumption that the velocity fluctuates, $g(k) := (w, \zeta(k)) \in \mathcal{H}_P$ satisfies

\[g \neq 0, \quad \langle 1_{\mathbb{T}^d}, g \rangle_P = 0\]

where $1_{\mathbb{T}^d} \in \mathcal{H}_P$, the identity function on $\mathbb{T}^d$, is the 0-eigenvector of $A_P$. Note further that

\[(w, \alpha w) = \langle g, (A_P)^{-1}g \rangle_P, \tag{5.20}\]

since one can easily check that $h := (A_P)^{-1}g \in \mathcal{H}_P$ by using the fact that $A_P$ is bounded and $A$ has a gap. Assume that $\alpha$ is not strictly positive. Then there is a $w \in \mathbb{R}^d$ such that (recall that $g$ and hence $h$ depend on $w$)

\[\langle g, (A_P)^{-1}g \rangle_P = \langle A_P h, h \rangle = 0.\]

In particular, this implies that

\[\langle \text{Re}(A_P) h, h \rangle_P = 0. \tag{5.21}\]

Since $-\text{Re}(A_P)$ is a positive operator (as follows from the fact that $A_P$ is a Markov generator), we get $\text{Re}(A_P) h = 0$. By the sectoriality assumption, it follows that also $\text{Im}(A_P) h = 0$ and hence $A_P h = 0$. Indeed, for $\lambda > 0$ and $v \in \mathcal{H}_P$,

\[|\langle (h + \lambda v), \text{Im}(A_P)(h + \lambda v) \rangle_P| \leq \gamma |\langle (h + \lambda v), \text{Re}(A_P)(h + \lambda v) \rangle_P| = \lambda^2 \langle v, \text{Re}(A_P)v \rangle_P\]

and hence the $O(\lambda)$-term has to vanish on the lefthand-side for all $v$. Since the zero-eigenvector of $A_P$ is unique by assumption, we obtain $h = c 1_{\mathbb{T}^d}, c \in \mathbb{R}$. This leads to a contradiction with the fact that $h = (A_P)^{-1}g$ and $\langle 1_{\mathbb{T}^d}, P \rangle = 0$. Hence $\alpha$ is strictly positive. \hfill \square
Note that (5.19) is related to the familiar central limit theorem for Markov processes
\[
\lim_{T \to \infty} \frac{1}{\sqrt{T}} \int_0^T dt f(Y_t) \to \mathcal{N}(0, \sigma)
\]
where the convergence is in distribution, \(\sigma = \int_0^\infty dt \mathbb{E}_\rho[f(Y_t)f(Y_0)]\), and the function \(f\) satisfies \(\int_{\mathbb{R}^d} dk f(k)P(k)dk = 0\).

**APPENDIX**

**A  Holevo form**

The form of the Lindblad generator as a translation invariant particle dynamics has been analyzed by Holevo, [16]. We give here more details concerning the relation of the Holevo form with the intuition of a spatially defined walk. We start with some formal calculations for defining the maps \(M_\theta\).

We look back at (3.7) that we can also write as
\[
\Psi(\rho)(x_1, x_2) = \text{Tr}_1[N(I \otimes \rho^\dagger)](x_1, x_2)
\]
for some positive operator \(N \in B(\ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}^d))\) which is partial trace class and satisfies \(N(x_1 + z, y_1 + z, y_2 + z, x_2 + z) = N(x_1, y_1, y_2, x_2); \rho^\dagger\) is the transpose of \(\rho\) in the position basis. Since \(N\) is translation invariant, we can write \(N(x_1, y_1, y_2, x_2) = R(x_1 - x_2, x_1 - y_1, x_2 - y_2)\) for some function \(R : \mathbb{Z}^{3d} \to \mathbb{C}\). We can therefore define the map \(F_\theta\) as
\[
F_\theta(x_1 - y_1, x_2 - y_2) = \frac{1}{(2\pi)^d} \sum_{x_1 - x_2 \in \mathbb{Z}^d} e^{-i\theta(x_1 - x_2)} N(x_1, y_1, y_2, x_2),
\]
where in the above we have held \(x_1 - y_1\) and \(x_2 - y_2\) fixed. Then \(F_\theta(\ell_1, \ell_2)\) defines a positive operator kernel on \(\ell^2(\mathbb{Z}^d)\), which we will assume can be written as \(F_\theta(\ell_1, \ell_2) = \sum_\alpha \hat{f}_{\theta,\alpha}(\ell_1) \hat{f}_{\theta,\alpha}(\ell_2)\) for some functions \(f_{\theta,\alpha}\) on \(\mathbb{Z}^d\). Now looking at the action of the kernel \(F_\theta(x_1 - y_1, x_2 - y_2)\) as determining a map on \(B_1(\ell^2(\mathbb{Z}^d))\)
\[
\sum_{y_1, y_2} F_\theta(x_1 - y_1, x_2 - y_2) \rho(y_1, y_2) = \sum_{y_1, y_2} \sum_\alpha \hat{f}_{\theta,\alpha}(x_1 - y_1) \rho(y_1, y_2) \hat{f}_{\theta,\alpha}(x_2 - y_2)
\]
which is a Kraus decomposition. The \(\hat{f}_{\theta,\alpha}, f_{\theta,\alpha}\) act through convolution which means that they are multiplication operators in the momentum basis:
\[
\sum_\alpha \hat{f}_{\theta,\alpha}(k_1) \rho(k_1, k_2) \hat{f}_{\theta,\alpha}(k_2) = M_\theta(k_1, k_2) \rho(k_1, k_2)
\]
and we get back
\[ M_\theta(\rho)(k_1, k_2) = M_\theta(k_1, k_2) \rho(k_1, k_2) \]
\[ \Psi(\rho) = \int_{T^d} d\theta e^{i\theta X} M_\theta(\rho) e^{-i\theta X} \]  \hspace{1em} (A.1)

**B Dissipativeness of the zero fiber classical Markov process**

We give some further discussion on the Assumption 3.2. While that brings us to the general theory of Markov jump processes on compact spaces which is not the subject of the paper, we do not give full proofs here but we sketch the central arguments.

First of all, the conditions

1. Continuity and strict positivity of \( r(k, k + \theta) \),
2. Decay at infinity of the Fourier transform of the measure \( \nu \), i.e. Assumption (3.6)

appear sufficient for the irreducibility of the semigroup \( e^{tA} \), i.e. the property that for all \( g \geq 0, g \neq 0 \)
\[ \text{Supp}\{e^{tA}g\} \subset \text{Supp}\{g\} \Rightarrow \text{Supp}\{g\} = T^d \]

Since \( r(k, k + \theta) \) is strictly positive, the support of the measure \( \nu \) determines the possible jump increments for the Markov process. Recall that the dynamical system of repeated shifts by a vector \( \theta = (\theta_1, \cdots, \theta_d) \in T^d \) is ergodic if and only if \( \theta \in I \), the set of points satisfying
\[ n_1\theta_1 + \cdots + n_d\theta_d \neq 0 \hspace{1em} \text{on} \ T \]  \hspace{1em} (B.1)

for any combination of integers \( n_1, \cdots, n_d \in \mathbb{Z} \) for which not all \( n_j \) are zero, see [23]. We now argue that the measure \( \nu \) has its support on \( I \) due to (3.6), the decay of the Fourier transform \( \varphi_\nu \). Consider the subset \( T(s,s_1,\cdots,s_d) \subset T^d \) for \( s_1, \cdots, s_d \in \mathbb{Z}, s \in T \) consisting of all points in \( \theta \in T^d \) such that
\[ s_1\theta_1 + \cdots + s_d\theta_d = s \hspace{1em} \text{on} \ T \]

Notice that (3.6) requires that, for \( m \in \mathbb{Z} \) tending to infinity,
\[ \varphi_\nu(ms_1,\cdots,ms_d) = \int_{T^d} d\nu(\theta_1, \cdots, \theta_d)e^{im(s_1,\cdots,s_d)}(\theta_1, \cdots, \theta_d) = \int_{T} d\mu(s)e^{ims} \rightarrow 0 \]
where formally \( d\mu(s) = \int_{T(s,s_1,\cdots,s_d)} d\nu(\theta_1, \cdots, \theta_d) \). It follows that \( d\mu \) cannot have atoms and hence \( \nu(T(s,s_1,\cdots,s_d)) = 0 \). The complement of the set \( I \) is a countable union of the subsets \( T(0,n_1,\cdots,n_d) \) over \( n_1 \cdots n_d \in \mathbb{Z} \). Hence, \( I \) has full measure and the Markov process must be irreducible.

Next we exploit the inequality (3.10). We write \( A = K + R \) where \( K \) and \( R \) correspond to the two terms on the right side of (3.8). \( R \) is a multiplication operator whose spectrum lies on the negative real line and by condition (3.10) we have that \( \sup \sigma(R) < 0 \). Assumption (3.6) implies that the integral operator \( K \) is compact. Therefore, by Weyl’s theorem, the essential spectrum of \( A \) and \( R \) are equal and hence the eigenvalues above the continuous spectrum \( \sigma(R) \) form a discrete set (see e.g. p. 101 of [22]).

To end the argument, we must check that the only purely imaginary eigenvalue is at 0 and that it is simple. This follows from a Perron-Frobenius argument, e.g. Thm. 13.6.12 in [14], which yields the desired result upon using that \( A \) is bounded and that it preserves \( C(T^d) \).
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