We consider in the unidirectional approximation the propagation of an ultra short electromagnetic pulse in a resonant medium consisting of molecules characterized by a transition operator with both diagonal and non-diagonal matrix elements. We find the zero-curvature representation of the reduced Maxwell-Bloch equations in the sharp line limit. This can be used to develop the inverse scattering transform method to solve these equations. Finally we obtain two types of exact traveling pulse solutions, one with the usual exponential decay and another with an algebraic decay.
Keywords: ultra-short pulses, two-level atoms model, high frequency Stark effect, inverse scattering transform method, steady state pulse, soliton.

1 Introduction

If the duration of an electromagnetic pulse is less than all the relaxation times in a medium, then the propagation of such a pulse is accompanied only by stimulated absorption and re-emission. These electromagnetic pulses we will refer to as ultra-short (USP). Usually the study of the propagation of USP within the framework of two-level atoms \[ \text{[1, 2]} \] assumes that the diagonal matrix elements of the dipole moment operator are zero (see also the reviews \[ \text{[3, 4, 5]} \]). This model could also describe the very short electromagnetic pulses containing a few optical cycles up to a half cycle \[ \text{[6, 7, 8, 9]} \].

An important approximation for the electromagnetic field called the slowly varying envelope approximation, assumes that the pulse envelope is a function slowly varying in space and time in comparison to the carrier monochromatic wave. In this approximation the Bloch equations describing the evolution of two-level atoms and the equation for the pulse envelope can be solved by the inverse scattering transform (IST) method \[ \text{[10, 11, 12]} \]. There are soliton and multi-solitons solutions, which describe the propagation and (elastic) interaction between USP in a resonant medium.

Second in importance is the unidirectional wave approximation. This corresponds to the propagation of the electromagnetic wave only in one of the two possible directions. The Maxwell and Bloch equations in this approxi-
mation allow to consider the USP propagation without imposing limitations on the pulse duration. As in the first case, the reduced Maxwell-Bloch equations can be solved via the IST method. Here the soliton corresponds to an extremely short pulse of the electromagnetic field, which contains half an optical cycle. The breather solutions correspond to pulses containing a few optical cycles. As already noted, two-level atoms media have been considered with zero diagonal matrix elements of the dipole moment operator. In a medium with a linear Stark shift, this assumption should be revised. The constant diagonal matrix elements of the dipole moment operator can be also induced by the external constant electric field. In an electromagnetic wave the resonant levels of this medium will be shifted. In analogy to a Kerr medium, for which a high frequency electromagnetic field results into a high frequency Kerr effect, a medium possessing a high frequency Stark effect may be referred to as a Stark medium. In Ref. [13] the interaction of a few-cycle electromagnetic pulses with a two-level Stark medium was considered in detail. Unlike [13] here we consider a simpler model, where we assume that all the matrix elements of the dipole moment operator are parallel to the linearly polarized electric field vector, neglect relaxation processes and the inhomogeneous broadening of the resonant lines, and assume unidirectional waves. This specific model differs from the standard two-level atoms model in a minimal degree. This results in a new kind of steady state propagation of the USP of extremely short duration. Furthermore, this model admits a zero-curvature representation, which can be used as a base to obtain exact solution by the IST method. After describing the reduced Maxwell-Bloch equations in section 2, we give a zero-curvature representation in section 3.
and find steady state pulse solutions in section 4.

2 The model and the principal equations

Following [11] we consider a plane electromagnetic wave propagating into a resonant medium consisting of molecules characterized by the operator of the dipole transition between resonant energy levels and let this operator have both non-diagonal and diagonal matrix elements. In the two-level approximation the Hamiltonian of the considered model can be written as

\[ \hat{H} = \frac{\hbar \omega_0}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} d_{11}E & d_{12}E \\ d_{21}E & d_{22}E \end{pmatrix}, \]

where \( E \) is the strength of electric field of the electromagnetic wave. The polarization of the medium is \( P = n_A p \), where \( n_A \) is the density of the molecules and the polarizability \( p \) is given by the expression \( p = \text{tr} \hat{\rho} \hat{d} = \rho_{11}d_{11} + \rho_{22}d_{22} + \rho_{12}d_{21} + \rho_{21}d_{12} \). We will consider the propagation of short electromagnetic pulses, for which all relaxation processes can be neglected. Hence, the density matrix \( \hat{\rho} \) obeys the constraint \( \rho_{11} + \rho_{22} = 1 \). Taking this relation into account, the polarizability can be written as

\[ p = \frac{1}{2} (d_{11} + d_{22}) + \frac{1}{2} (d_{11} - d_{22}) (\rho_{11} - \rho_{22}) + \rho_{12}d_{21} + \rho_{21}d_{12}. \]

The evolution of the elements of \( \hat{\rho} \) is given by the equation \( i\hbar \partial \hat{\rho} / \partial t = \hat{H} \hat{\rho} - \hat{\rho} \hat{H} \) and yields the Bloch equations.

In a scalar form the Maxwell equations lead to

\[ \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{4\pi n_A}{c^2} \frac{\partial^2}{\partial t^2} \langle p \rangle, \]

(1)
where the angular brackets stand for averaging over all molecules.

It is convenient to introduce the Bloch vector by defining its components

\[ r_1 = \rho_{12} + \rho_{21}, \quad r_2 = -i(\rho_{12} - \rho_{21}), \quad r_3 = \rho_{22} - \rho_{11}. \]

On can choose a constant phase for the matrix elements of the density matrix \( \hat{\rho} \) and the dipole operator \( \hat{d} \) so that \( d_{12} = d_{21} \). The polarizability \( p \) of the resonant medium can be written in terms of the Bloch vector components as

\[ p = \frac{1}{2} (d_{11} + d_{22}) + \frac{1}{2} (d_{22} - d_{11}) r_3 + d_{12} r_1. \] (2)

In this expression the first term corresponds to the constant polarizability of the molecules. As it gives no contribution to the radiation, it may be omitted. Furthermore, after averaging over all atoms it must be \( \langle d_{11} + d_{22} \rangle = 0 \) if one assumes that there is no constant polarization of the medium in the absence of an electromagnetic field.

So the total system of equations describing the model under consideration can be written in the following form

\[
\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{4\pi n_A}{c^2} \frac{\partial^2}{\partial t^2} \left< \frac{1}{2} (d_{22} - d_{11}) r_3 + d_{12} r_1 \right>,
\]

\[
\frac{\partial r_1}{\partial t} = -[\omega_0 + (d_{11} - d_{22}) E/\hbar] r_2,
\]

\[
\frac{\partial r_2}{\partial t} = -[\omega_0 + (d_{11} - d_{22}) E/\hbar] r_1 + 2(dE/\hbar) r_3,
\]

\[
\frac{\partial r_3}{\partial t} = -2(dE/\hbar) r_2.
\]

This system differs from the well-known Maxwell-Bloch equations [2]- [5] by terms containing the parameter \( (d_{11} - d_{22}) \). If \( n_A |d_{11} - d_{22}| \) and \( n_A |d_{12}| \) are small, then one can neglect the reflected wave and consider the unidirectional
propagation of the electromagnetic wave \[4, 9\]. The system of the Maxwell-Bloch equations is then reduced to

\[
\frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} = -\frac{2\pi n}{\hbar} A \left( \frac{1}{2} (d_{22} - d_{11}) r_3 + d_{12} r_1 \right),
\]

(3)

\[
\frac{\partial r_1}{\partial t} = -[\omega_0 + (d_{11} - d_{22}) E/\hbar] r_2,
\]

(4)

\[
\frac{\partial r_2}{\partial t} = -[\omega_0 + (d_{11} - d_{22}) E/\hbar] r_1 + 2(dE/\hbar) r_3,
\]

(5)

\[
\frac{\partial r_3}{\partial t} = -2(dE/\hbar) r_2.
\]

(6)

Let us consider here only the sharp line limit of the model. Introduce new dimensionless variables and field

\[
\tau = \omega_0 (t - z/c), \quad \zeta = z/L_{ab}, \quad q = 2dE/\hbar \omega_0,
\]

where \( L_{ab}^{-1} = 4\pi n A d_{12}^2 (\hbar c)^{-1} \) and the parameter \( \mu = (d_{11} - d_{22}) / 2d_{12} \). The reduced Maxwell-Bloch equations (2.3)-(2.6) take the form:

\[
r_{1,\tau} = -(1 + \mu q) r_2, \quad r_{2,\tau} = (1 + \mu q) r_1 + qr_3, \quad r_{3,\tau} = -qr_2,
\]

(7)

\[
q,\zeta = -(r_1 - \mu r_3),\tau.
\]

(8)

The last equation can be rewritten as

\[
q,\zeta = r_2.
\]

(9)

We will assume the general initial conditions where the medium is initially at rest so that \( q = 0, r_1 = r_2 = 0, r_3 = -1 \), at \( \tau \rightarrow -\infty \).

From the Bloch equations (7) we obtain \( (r_1^2 + r_2^2 + r_3^2),\tau = 0 \). and using the initial conditions we get the following integral of motion

\[
r_1^2 + r_2^2 + r_3^2 = 1.
\]

(10)
We now proceed to give the zero-curvature representation of the system (7)-(9). We will see that the $\mu q$ term which makes the difference from the classical reduced Maxwell-Bloch equations [4, 5, 14] will generate complications of traditional IST method. We will also find two types of traveling pulse solutions which differ from the ones of the two classical reduced Maxwell-Bloch systems.

3 The zero-curvature representation

The inverse scattering transform method is based on considering the pair of linear equations [15]:

$$
\begin{align*}
\psi_{,\tau} &= \hat{U} \psi, \\
\psi_{,\zeta} &= \hat{V} \psi,
\end{align*}
$$

(11)

where

$$
\hat{V} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.
$$

The compatibility condition for these equations is $\hat{U}_{,\zeta} = \hat{V}_{,\tau} + \hat{V} \hat{U} - \hat{U} \hat{V}$, which yields the following system of equations

$$
\begin{align*}
A_{,\tau} + U_{21} B - U_{12} C &= U_{11,\zeta}, \\
B_{,\tau} + (U_{22} - U_{11}) B + 2U_{12} A &= U_{12,\zeta}, \\
C_{,\tau} - (U_{22} - U_{11}) C - 2U_{21} A &= U_{21,\zeta}.
\end{align*}
$$

(12) (13) (14)

It should be pointed that the reduced Maxwell-Bloch equations (7)-(9) can be represented in another form:

$$
\begin{align*}
i\psi_{1,\tau} &= -(1/2)(1 + \mu q)\psi_1 - (1/2)q\psi_2, \\
\end{align*}
$$

(15)
\[
i\psi_{2,\tau} = -(1/2)q\psi_1 + (1/2)(1 - \mu q)\psi_1, \quad (16)
\]

\[
q,\zeta = i(\psi_1^*\psi_2 - \psi_1\psi_2^*), \quad (17)
\]

where the components of the Bloch vector are constructed from the functions \(\psi\) according to the rule:

\[
r_1 = (\psi_1^*\psi_2 + \psi_1\psi_2^*), \quad r_2 = i(\psi_1^*\psi_2 - \psi_1\psi_2^*), \quad r_3 = |\psi_2|^2 - |\psi_1|^2.
\]

This suggests that the U-matrix of the zero-curvature representation for the reduced Maxwell-Bloch equations (7)-(9) could be written as

\[
\hat{U} = \begin{pmatrix}
-i\lambda + f_1(\lambda)q & g_1(\lambda)q \\
g_1(\lambda)q & i\lambda - f_2(\lambda)q
\end{pmatrix}, \quad (18)
\]

where \(f_{1,2}(\lambda)\) and \(g_{1,2}(\lambda)\) are unknown functions of the spectral parameter \(\lambda\). Taking into account this U-matrix, the compatibility conditions (12)-(14) can be written as

\[
A,\tau + g_2(\lambda)qB - g_1(\lambda)qC = f_1(\lambda)q, \quad (19)
\]

\[
B,\tau + (2\eta - f(\lambda)q)B + 2g_1(\lambda)qA = g_1(\lambda)q, \quad (20)
\]

\[
C,\tau - (2\eta - f(\lambda)q)C - 2g_2(\lambda)qA = g_2(\lambda)q, \quad (21)
\]

where \(f(\lambda) = f_1(\lambda) + f_2(\lambda)\), and \(\eta = i\lambda\). The system of equations (19)-(21) must coincide with the reduced Maxwell-Bloch equations (7)-(9).

It should be noted that the system of equations (7)-(9) contains the partial derivatives only of the Bloch vector components with respect to \(\tau\). Hence one can find \(B\) and \(C\) from (20) and (21) using the following expansions

\[
B = b_1r_1 + b_2r_2 + b_3r_3, \quad C = c_1r_1 + c_2r_2 + c_3r_3,
\]
where \( b_{1,2,3} \) and \( c_{1,2,3} \) are unknown functions of \( \lambda \). Substituting these expansions into equations (20) and (21), and equating the coefficients of \( r_1, r_2, r_3 \) and \( q \) in the resulting equations, we obtain the following relations between the \( b_{1,2,3} \) and \( c_{1,2,3} \)

\[
\begin{align*}
  b_2 + 2\eta b_1 &= 0, \quad -b_1 + 2\eta b_2 = g_1, \quad b_3 = 0, \\
  c_2 - 2\eta c_1 &= 0, \quad -c_1 - 2\eta c_2 = g_2, \quad c_3 = 0,
\end{align*}
\]

(22)

\[
2g_1A + b_1 (\mu r_2 - fr_1) + b_2 (\mu r_1 + r_3 - fr_2) = 0,
\]

(24)

\[-2g_2A + c_1 (\mu r_2 + fr_1) + c_2 (\mu r_1 + r_3 + fr_2) = 0.
\]

(25)

From (22) and (23) it follows that

\[
\begin{align*}
  b_1 &= \frac{-g_1}{1 + 4\eta^2}, \quad b_2 = \frac{2\eta g_1}{1 + 4\eta^2}, \quad c_1 = \frac{-g_2}{1 + 4\eta^2}, \quad c_2 = \frac{-2\eta g_2}{1 + 4\eta^2}.
\end{align*}
\]

Substitution of these expressions into (24) and (25) leads to two equations for the matrix element \( A \). From (3.14) one gets

\[
2A + (1 + 4\eta^2)^{-1} \left[ 2\eta r_3 + (2\eta \mu + f) r_1 + (\mu - 2\eta f) r_2 \right] = 0,
\]

and from (3.15) one gets

\[
2A + (1 + 4\eta^2)^{-1} \left[ 2\eta r_3 + (2\eta \mu + f) r_1 - (\mu - 2\eta f) r_2 \right] = 0.
\]

Both expressions agree if \( \mu - 2\eta f = 0 \). Thus the auxiliary function \( f(\lambda) \) is specified: \( f(\lambda) = \mu/2\eta \), and the expressions for \( B, C \) and \( A \) are determined:

\[
\begin{align*}
  B &= \frac{g_1}{1 + 4\eta^2} (-r_1 + 2\eta r_2), \quad C = \frac{g_2}{1 + 4\eta^2} (-r_1 - 2\eta r_2), \\
  A &= \frac{-1}{1 + 4\eta^2} \left[ \eta r_3 + \mu \left( \eta + \frac{1}{4\eta} \right) r_1 \right].
\end{align*}
\]

(26)

(27)
To determine the auxiliary function \( g_{1,2}(\lambda) \), equation (3.9) has to be used. We assume the equality \( f_1(\lambda) = f_2(\lambda) \) so that \( f_1(\lambda) = \mu/4\eta \). Substitution of the equations (26) and (27) into (19) leads to the following relation

\[
\eta r_{3,\tau} + \mu \left( \eta + \frac{1}{4\eta} \right) r_{1,\tau} + \frac{\mu}{4\eta} \left( 1 + 4\eta^2 \right) q_{\zeta} = g_1 g_2 4\eta^2 r_2.
\]

Using equations (7) and (9) this becomes

\[
\eta + \frac{\mu^2}{4\eta} \left( 1 + 4\eta^2 \right) + 4\eta g_1 g_2 = 0. \tag{28}
\]

Now, we can require any correlation between \( g_1 \) and \( g_2 \), for instance, \( g_1(\lambda) = -g_2(\lambda) = g(\lambda) \). From (28) it then follows immediately that

\[
g(\lambda) = \frac{1}{2} \left[ 1 + \frac{\mu^2}{4\eta^2} \left( 1 + 4\eta^2 \right) \right]^{1/2}. \tag{29}
\]

Sometimes it may be more convenient to take \( g_1(\lambda) = g_2(\lambda) = ig(\lambda) \).

Thus, we have the U-V-matrices for a zero-curvature representation of the reduced Maxwell-\ Bloch equations (7) and (9) describing the USP propagation in Stark medium assuming a model of two-level atoms. They are

\[
\hat{U} = \begin{pmatrix}
-\eta + (\mu/4\eta) q & \frac{1}{2} \left[ 1 + (\mu/2\eta)^2 (1 + 4\eta^2) \right]^{1/2} q \\
-\frac{1}{2} \left[ 1 + (\mu/2\eta)^2 (1 + 4\eta^2) \right]^{1/2} q & \eta - (\mu/4\eta) q
\end{pmatrix},
\]

\[
\hat{V} = \frac{1}{2(1 + 4\eta^2)} \times
\begin{pmatrix}
-2[\eta r_3 + \mu(\eta + (1/4\eta)) r_1] & \left[ 1 + (\mu/2\eta)^2 (1 + 4\eta^2) \right]^{1/2} (2\eta r_2 - r_1) \\
[1 + (\mu/2\eta)^2 (1 + 4\eta^2)]^{1/2} (2\eta r_2 + r_1) & 2[\eta r_3 + \mu(\eta + (\mu/4\eta)) r_1]
\end{pmatrix},
\]

\[
10
\]
where \( \eta = i \lambda \). It should be noted that if \( \mu \rightarrow 0 \), then we obtain the well-known U-V representation for the classical reduced Maxwell-Bloch equations (see \([11, 12]\)).

4 Steady state solutions of the Maxwell-Bloch equations

To obtain the equations describing the propagation of stationary USP one should assume that the components of the Bloch vector and the normalized pulse envelope depend on only one variable \( \eta \equiv \tau - \beta \zeta \equiv \omega_0(t - z/V) \). Under this assumption the system (7) and (9) transforms into the system of ordinary differential equations

\[
\begin{align*}
r_{1,\eta} &= -(1 + \mu q)r_2, & r_{2,\eta} &= (1 + \mu q)r_1 + qr_3, & r_{3,\eta} &= -qr_2, \quad (32) \\
\beta q,\eta &= -r_2. \quad (33)
\end{align*}
\]

Let us consider the solution of equations (32) and (33) describing a solitary steady state wave with the boundary conditions \( q = 0, r_1 = r_2 = 0, r_3 = -1 \) at \( |\eta| \rightarrow \infty \).

From (32) it follows that

\[
r_2 = -(r_1 - \mu r_3),\eta \quad (34)
\]

We now take (33) and the boundary conditions into account to get

\[
r_1 = \beta q + \mu(1 + r_3). \quad (35)
\]

From the last equation of (32) and equation (33) we obtain

\[
r_3 = -1 + \beta q^2/2. \quad (36)
\]
We define \( q_0^2 = 4/\beta \) to renormalize \( q \) as \( w = q/q_0 \) and write all components of the Bloch vector as \( r_1 = 4w/q_0 + 2\mu w, r_2 = -4w, r_3 = -1 + 2w^2 \). The substitution of the Bloch vector components into expression (10) leads at once to the equation for \( w \):

\[
\left( \frac{dw}{d\eta} \right)^2 = \left( \frac{q_0^2}{4} - 1 \right) w^2 - \mu q_0 w^3 - \frac{q_0^2}{4} \left( 1 + \mu^2 \right) w^4.
\]  

This equation has a solution in the form of a solitary wave if the factor in the first term is positive. Let us denote it as \( q_0^2/4 - 1 = \theta^{-2} \) and introduce the new variable

\[
y = 2 \left( q_0 w \sqrt{1 + \mu^2} \right)^{-1}.
\]

In terms of this variable equation (17) takes the form

\[
\left( \frac{dy}{d\eta} \right)^2 = \frac{1}{\theta^2} \left( y^2 - \frac{2\mu \theta^2}{\sqrt{1 + \mu^2}} y - \theta^2 \right).
\]  

Let be \( u = y - \mu \theta^2/(1 + \mu^2)^{1/2} \). Then equation (38) can be rewritten as

\[
\theta \left( \frac{du}{d\eta} \right) = \left( u^2 - B^2 \right)^{1/2},
\]

where

\[
B^2 = \theta^2 \left( 1 + \frac{\mu^2 \theta^2}{1 + \mu^2} \right).
\]

The solution of this equation is \( u(\eta) = B \cosh \left[ (\eta - \eta_0) / \theta \right] \), with \( \eta_0 \) as the integrating constant. Taking this result into account one can obtain that

\[
q(\eta) = \frac{2}{\theta \left\{ \sqrt{1 + \mu^2 (1 + \theta^2)} \cosh \left[ (\eta - \eta_0) / \theta \right] + \mu \theta \right\}}.
\]  

This expression represents a one-parameter family of solutions of equations (32) and (33), with \( \theta \) as parameter. It is convenient to introduce the duration
of a stationary USP \( t_s \) by the relation \( \theta = t_s \omega_0 \), so that \( q(\eta) \) may be written as

\[
q(\eta) = \frac{(2/t_s \omega_0)}{\left\{ \sqrt{1 + \mu^2 + (\mu t_s \omega_0)^2} \right\}} \cosh \left[ (t - z/V - t_0) / t_s \right] + \mu t_s \omega_0 .
\] (40)

Thus one can obtain the expression for the electric field of this extremely short steady state pulse:

\[
E_s(t, z) = \frac{(\hbar / t_s d)}{\left\{ \sqrt{1 + \mu^2 + (\mu t_s \omega_0)^2} \right\}} \cosh \left[ (t - z/V - t_0) / t_s \right] + \mu t_s \omega_0 .
\] (41)

From the definition of \( \eta \) one can obtain the velocity of propagation \( V \) of the USP in the laboratory reference frame

\[
\frac{c}{V} = 1 + \frac{(t_s \omega_0)^2}{1 + (t_s \omega_0)^2} \left( \frac{4 \pi n_A d_{12}^2}{\hbar \omega_0} \right).
\] (42)

This expression for the velocity of the propagation of stationary pulse is identical to the one found in [6]. Hence the linear high frequency Stark effect does not influence the rate of propagation of the steady state USP.

The expression for the normalized electric field strength (33) has been obtained under the condition that \( \theta^2 = q_0^2 / 4 - 1 > 0 \). Let us consider the case \( \theta = 0 \), i.e., \( q_0 = 2 \). Equation (37) takes the form

\[
\left( \frac{dw}{d\eta} \right)^2 = -2 \mu w^3 - \left( 1 + \mu^2 \right) w^4 .
\] (43)

If \( \mu \) is positive we have only trivial solutions of this equation, \( w = 0 \). If \( \mu \) is negative there are nontrivial solutions. We rewrite (37) as

\[
\left( \frac{dw}{d\eta} \right)^2 = w^3 \left[ 2 |\mu| - (1 + \mu^2) w \right].
\] (44)
and introduce $u = 2 |\mu| / (1 + \mu^2) w$ so that the equation becomes

$$\left( \frac{du}{d\eta} \right)^2 = \frac{4\mu^2}{(1 + \mu^2)} (u - 1).$$

The solution of this simple equation is

$$u = 1 + \frac{\mu^2}{(1 + \mu^2)} (\eta - \eta_0)^2.$$

Hence

$$y = \frac{1 + \mu^2}{2 |\mu|} \left\{ 1 + \frac{\mu^2}{1 + \mu^2} (\eta - \eta_0)^2 \right\},$$

and

$$q_{al}(\eta) = \frac{4 |\mu|}{(1 + \mu^2)} \left\{ 1 + \frac{\mu^2}{1 + \mu^2} (\eta - \eta_0)^2 \right\}^{-1}. \quad (45)$$

In terms of physical values, this algebraic steady state USP can be represented as

$$E_{al}(t, z) = \frac{E_m}{1 + \left[ (t - z/V_{al} - t_0)/t_{al} \right]^2}, \quad (46)$$

where the amplitude of USP $E_m$, its duration $t_{al}$, and velocity $V_{al}$ are defined as

$$E_m = \frac{4\hbar \omega_0 (d_{22} - d_{11})}{\sqrt{4d_{12}^2 + (d_{22} - d_{11})^2}}, \quad t_{al} = \frac{\omega_0 (d_{22} - d_{11})}{\sqrt{4d_{12}^2 + (d_{22} - d_{11})^2}}, \quad \frac{c}{V_{al}} = 1 + \frac{4\pi n_A d_{12}^2}{\hbar \omega_0}. \quad (47)$$

The possibility of the propagation of such an algebraic solitary wave is the distinguishing feature of the Stark medium under consideration here.

5 Conclusion

We have introduced and analyzed a model for the propagation of ultra-short electromagnetic pulses moving in one direction in a Stark medium. This is
described by two-level atoms taking into account the high frequency linear Stark shift of the energy levels. Two families of exact analytical solutions of the reduced Maxwell-Bloch equations have been found, which correspond to steady state pulses propagating with different pulse widths. We find algebraic solitary waves apart from the usual exponentially decreasing ones.

It was found that the system of reduced Maxwell-Bloch equations admits a zero-curvature representation in the sharp line limit. Unlike the well-known Zakharov-Shabat spectral problem the one obtained here shows complicated analytical properties. Nevertheless we assume that it can be used to develop the inverse scattering transform method to solve these equations.

There is another class of stationary waves – cnoidal waves amongst the solutions of the Maxwell-Bloch equations (3)-(6), or (7) and (9). These are periodic continuous waves, different from the solitary waves described above. Since equations (7) and (9) are valid when the duration of the wave is less or much less than the relaxation times of the atomic subsystem, cnoidal waves seem to be mathematical objects lying beyond the physical meaning of the original equations.

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