Adomian decomposition method for solving the population dynamics model of two species

Yulius Wahyu Putranto¹ and Sudi Mungkasi²

¹Postgraduate Program in Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University, Yogyakarta, Indonesia
²Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University, Yogyakarta, Indonesia

E-mail: juliogoza808@gmail.com, sudi@usd.ac.id

Abstract. Adomian decomposition method has been a powerful method to solve differential equations. In this paper, we propose the method to solve the population dynamics model of two species for mutualism, parasitism, and competition. These three scenarios are considered for the completion of our research. Adomian decomposition method uses initial values of the unknowns and provides series of approximate solutions to the problem. We obtain that the Adomian decomposition method provides fast computation for the solution.

1. Introduction

Population dynamics model has been developed in the field of mathematical biology. Populations of species increase or decrease over time depending on a number of factors. A population certainly interacts with other populations, causing a system dynamics [1].

Two populations interact with the properties of mutualism, parasitism, and competition. These interaction can be formed from a general model (system) of differential equations. Interaction type in both populations can vary depending on the value of each given parameter in the general model. Finding the solution to the model using the aid of computer nowadays is often desired [2-5]. This is because fast and accurate results are important to solve real problems.

In this paper, we solve the population dynamics model of two species using the Adomian decomposition method. The method is chosen because it is meshless, is an analytical approach, possess a fast convergence, and can be implemented on a computer easily [6-11]. It is often related to the variational iteration method [12-16].

The paper is organised as follows. Section 2 contains the problem formulation that we want to solve. Section 3 present the Adomian decomposition method for the population dynamics model. Computational results are provided in Section 4. We conclude the paper in Section 5.

2. Problem formulation

We consider the nonlinear system of the form [12]

\[
\begin{align*}
\frac{dx}{dt} &= x(a_1 + b_1x + c_1y), \\
\frac{dy}{dt} &= y(a_2 + b_2y + c_2x).
\end{align*}
\] (1)
Here $x = x(t)$ and $y = y(t)$ are populations of the first and second species at time $t$, respectively. In addition, $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$ are constants described as follows:

- $x$ is the population of the first species,
- $a_1$ denotes the growth rate of the population of the first species,
- $b_1$ denotes the carrying capacity of the population of the first species,
- $c_1$ denotes the interacting constant of the population of the first species with the second one,
- $y$ is the population of the second species,
- $a_2$ denotes the growth rate of the population of the second species,
- $b_2$ denotes the carrying capacity of the population of the second species,
- $c_2$ denotes the interacting constant of the population of the second species with the first one.

This model (1) governs mutualism, parasitism, and competition interactions.

3. Adomian decomposition method
We derive the Adomian decomposition method to solve the model following Batiha et al. [12].

Model (1) can be rewritten as

\[
\begin{align*}
\frac{dx}{dt} &= a_1 x + b_1 x^2 + c_1 xy, \\
\frac{dy}{dt} &= a_2 y + b_2 y^2 + c_2 xy.
\end{align*}
\] (2)

Introducing the derivative operator $L = \frac{d}{dt}$, we obtain

\[
\begin{align*}
L_x &= a_1 x + b_1 x^2 + c_1 xy, \\
L_y &= a_2 y + b_2 y^2 + c_2 xy.
\end{align*}
\] (3)

Applying $L^{-1} = \int_0^t \frac{d}{dt} \cdot dt$ to both sides of the nonlinear system (3) gives

\[
\begin{align*}
L^{-1}L_x &= L^{-1} a_1 x + L^{-1} b_1 x^2 + L^{-1} c_1 xy, \\
L^{-1}L_y &= L^{-1} a_2 y + L^{-1} b_2 y^2 + L^{-1} c_2 xy.
\end{align*}
\] (4)

The Adomian decomposition method admits the decomposition of $x$ and $y$ into an infinite series components

\[
x(t) = \sum_{n=0}^{\infty} x_n, \quad y(t) = \sum_{n=0}^{\infty} y_n
\] (5)

and the nonlinear terms $x^2, y^2$ and $xy$ are assumed to be in the following forms:

\[
x^2 = \sum_{n=0}^{\infty} A_n, \quad y^2 = \sum_{n=0}^{\infty} B_n, \quad xy = \sum_{n=0}^{\infty} D_n.
\] (6)

We define $A_n$, $B_n$ and $D_n$ as

\[
A_n = \sum_{k=0}^{n} x_k x_{n-k}, \quad B_n = \sum_{k=0}^{n} y_k y_{n-k}, \quad D_n = \sum_{k=0}^{n} x_k y_{n-k}.
\] (7)

We have Adomian polynomials for $A_n, B_n$ and $D_n$:

\[
\begin{align*}
A_0 &= x_0 x_0, \\
A_1 &= x_0 x_1 + x_1 x_0, \\
A_2 &= x_0 x_2 + x_1 x_1 + x_2 x_0, \\
A_3 &= x_0 x_3 + x_1 x_2 + x_2 x_1 + x_3 x_0, \\
&\vdots
\end{align*}
\] (8)
\begin{align*}
B_0 &= y_0y_0 \\
B_1 &= y_0y_1 + y_1y_0 \\
B_2 &= y_0y_2 + y_1y_1 + y_2y_0 \\
B_3 &= y_0y_3 + y_1y_2 + y_2y_1 + y_3y_0 \\
&\vdots
\end{align*}
\begin{align*}
D_0 &= x_0y_0 \\
D_1 &= x_0y_1 + x_1y_0 \\
D_2 &= x_0y_2 + x_1y_1 + x_2y_0 \\
D_3 &= x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0 \\
&\vdots
\end{align*}
\begin{align}
&\frac{dx(t)}{dt} = x(t) - x(0) = L^{-1}a_1 \sum_{n=0}^{\infty} x_n + L^{-1}b_1 \sum_{n=0}^{\infty} A_n + L^{-1}c_1 \sum_{n=0}^{\infty} D_n \\
&\frac{dy(t)}{dt} = y(t) - y(0) = L^{-1}a_2 \sum_{n=0}^{\infty} y_n + L^{-1}b_2 \sum_{n=0}^{\infty} B_n + L^{-1}c_2 \sum_{n=0}^{\infty} D_n
\end{align}
\begin{align}
\sum_{n=0}^{\infty} x_n &= x(0) + L^{-1}a_1 x_n + L^{-1}b_1 A_n + L^{-1}c_1 D_n \\
\sum_{n=0}^{\infty} y_n &= y(0) + L^{-1}a_2 y_n + L^{-1}b_2 B_n + L^{-1}c_2 D_n
\end{align}

With initial values \( x(0) = x_0 \), \( y(0) = y_0 \), we can find the solution of the system. The iterations are determined by the following recursive formulas:
\begin{align}
x_{n+1} &= L^{-1}a_1 x_n + L^{-1}b_1 A_n + L^{-1}c_1 D_n, \\
y_{n+1} &= L^{-1}a_2 y_n + L^{-1}b_2 B_n + L^{-1}c_2 D_n.
\end{align}

In this paper, we use 7-term approximations to find the solution to the system. The solution is defined by \( X \) and \( Y \) as
\begin{align}
X &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_7, \\
Y &= y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_7.
\end{align}

4. Numerical results

For computational experiments in this section, we assume to have initial values \( x(0) = x_0 = 4 \) and \( y(0) = y_0 = 10 \). We can determine the type of interaction between both populations from the signs of parameter values. We compute components of 7-term approximations using the computer algebra package Maple. The interaction between populations can be mutualism, parasitism, and competition.

4.1. Mutualism interaction

Given that \( a_1 = 0.1; a_2 = 0.08; b_1 = -0.0014; b_2 = -0.001; c_1 = 0.0012; c_2 = 0.0009; \) and \( x_0 = 4 \) and \( y_0 = 10 \).

We obtain:
\begin{align}
x_1 &= 0.4256000000 t, \\
y_1 &= 0.7360000000 t, \\
x_2 &= 0.02321664000 t^2, \\
y_2 &= 0.02532000000 t^2, \\
x_3 &= 0.0008613579093 t^3, \\
&\ldots
\end{align}
Given that $a_4 = 0.1; a_2 = 0.08; b_1 = -0.0014; b_2 = -0.001; c_1 = 0.0012; c_2 = -0.0009$; and $x_0 = 4$ and $y_0 = 10$.

We obtain:

\begin{align*}
y_3 &= 0.0005198410667t^3, \\
x_4 &= 0.00002377241892t^4, \\
y_4 &= 0.00000715508705t^4, \\
x_5 &= 4.762365706 \times 10^4(-7)t^5, \\
y_5 &= 1.122902100 \times 10^4(-7)t^5, \\
x_6 &= 4.92081361 \times 10^4(-9)t^6, \\
y_6 &= 3.924828700 \times 10^4(-9)t^6, \\
x_7 &= -9.99735783 \times 10^4(-11)t^7, \\
y_7 &= 1.355344556 \times 10^4(-10)t^7. \\
\end{align*}

4.2. Parasitism interaction

Given that $a_4 = 0.1; a_2 = 0.08; b_1 = -0.0014; b_2 = -0.001; c_1 = 0.0012; c_2 = -0.0009$; and $x_0 = 4$ and $y_0 = 10$.

We obtain:

\begin{align*}
x_1 &= 0.4256000000t, \\
y_1 &= 0.6640000000t, \\
x_2 &= 0.02304384000t^2, \\
y_2 &= 0.01680960000t^2, \\
x_3 &= 0.0008296779093t^3, \\
y_3 &= 0.0000151441067t^3, \\
x_4 &= 0.00002079741808t^4, \\
y_4 &= -0.00001228646758t^4, \\
x_5 &= 2.877838352 \times 10^4(-7)t^5, \\
y_5 &= -4.066091675 \times 10^4(-7)t^5, \\
x_6 &= -3.968167659 \times 10^4(-9)t^6, \\
y_6 &= -5.05637136 \times 10^4(-9)t^6, \\
x_7 &= -4.223308270 \times 10^4(-10)t^7, \\
y_7 &= 8.804344521 \times 10^4(-11)t^7. \\
\end{align*}

4.3. Competition interaction

Given that $a_1 = 0.1; a_2 = 0.08; b_1 = -0.0014; b_2 = -0.001; c_1 = -0.0012; c_2 = -0.0009$; and $x_0 = 4$ and $y_0 = 10$.

We obtain:

\begin{align*}
x_1 &= 0.3296000000t, \\
y_1 &= 0.6640000000t, \\
x_2 &= 0.01106304000t^2, \\
y_2 &= 0.01724160000t^2, \\
x_3 &= 0.0001173886293t^3, \\
y_3 &= 0.0000783313067t^3, \\
x_4 &= -0.00004301207450t^4, \\
y_4 &= -0.00007815319423t^4, \\
x_5 &= -1.851824568 \times 10^4(-7)t^5, \\
y_5 &= -2.13685195 \times 10^4(-7)t^5, \\
x_6 &= -1.635576185 \times 10^4(-9)t^6, \\
y_6 &= -7.0058057 \times 10^4(-9)t^6, \\
x_7 &= 8.190722735 \times 10^4(-11)t^7, \\
y_7 &= 1.240343622 \times 10^4(-10)t^7. \\
\end{align*}

We find the solution to the system for each initials conditions. Approximate solutions for both populations are given by:
Approximate solutions for mutualism interaction are given by:
\[
X = x_0 + x_1 + x_2 + x_3 + x_4 + x_6 + x_7,
Y = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7.
\]

Approximate solutions for the first population
\[
X_1 = 4 + 0.4256000000 \times t + 0.023216640000 \times t^2 + 0.0008613579093 \times t^3
+ 0.00002377241892 \times t^4 + 4.762365706 \times 10^{-7} \times t^5 + 4.92081361 \times 10^{-9} \times t^6
- 9.99735783 \times 10^{-11} \times t^7,
\]
\[
Y_1 = 10 + 0.7360000000 \times t + 0.02532000000 \times t^2 + 0.0005198410667 \times t^3
+ 0.0000715508705 \times t^4 + 1.122902100 \times 10^{-7} \times t^5 + 3.924828700 \times 10^{-9}
+ 1.355344556 \times 10^{-10} \times t^7.
\]

Approximate solutions for parasitism interaction are given by:
\[
X_2 = 4 + 0.4256000000 \times t + 0.023043840000 \times t^2 + 0.0008296779093 \times t^3
+ 0.00002079741808 \times t^4 + 2.877838352 \times 10^{-7} \times t^5 - 3.968167659 \times 10^{-9}
+ 2.877838352 \times 10^{-10} \times t^7,
\]
\[
Y_2 = 10 + 0.6640000000 \times t + 0.01680960000 \times t^2 + 0.0001514441067 \times t^3
- 0.00001228646758 \times t^4 - 4.660091675 \times 10^{-7} \times t^5 + 5.050637136 \times 10^{-9}
+ 8.804344521 \times 10^{-11} \times t^7.
\]

Approximate solutions for competition interaction are given by:
\[
X_3 = 4 + 0.3296000000 \times t + 0.011063040000 \times t^2 + 0.0001173886293 \times t^3
- 0.000004301207450 \times t^4 - 1.851824568 \times 10^{-7} \times t^5 - 1.635576185 \times 10^{-9}
+ 8.190722735 \times 10^{-11} \times t^7,
\]
\[
Y_3 = 10 + 0.6640000000 \times t + 0.01724160000 \times t^2 + 0.0000783313067 \times t^3
- 0.00007815319423 \times t^4 - 2.136855195 \times 10^{-7} \times t^5 - 7.0058057 \times 10^{-11}
+ 1.240343622 \times 10^{-10} \times t^7.
\]
(a). Solution for the first population $X_2$.  
(b). Solution for the second population $Y_2$. 

**Figure 2.** Solutions for both populations with parasitism interaction.

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(a). Solution for the first population $X_3$.  
(b). Solution for the second population $Y_3$. 

**Figure 3.** Solutions for both populations with competition interaction.
Table 1. Population dynamics for both species obtained using the Adomian decomposition method.

| t  | Mutualism interaction | Parasitism interaction | Competition interaction |
|----|-----------------------|------------------------|-------------------------|
|    | $X_1$ | $Y_1$ | $X_2$ | $Y_2$ | $X_3$ | $Y_3$ |
| 0  | 4.000000 | 10.00000 | 4.000000 | 10.00000 | 4.000000 | 10.00000 |
| 0.1| 4.042793 | 10.07385 | 4.042791 | 10.06657 | 4.033071 | 10.06657 |
| 0.2| 4.086056 | 10.14822 | 4.086048 | 10.13347 | 4.066363 | 10.13349 |
| 0.3| 4.129793 | 10.22309 | 4.129777 | 10.20071 | 4.099879 | 10.20075 |
| 0.4| 4.174010 | 10.29848 | 4.173981 | 10.26829 | 4.133617 | 10.26836 |
| 0.5| 4.218713 | 10.37440 | 4.218666 | 10.33620 | 4.167580 | 10.33632 |
| 0.6| 4.263907 | 10.45083 | 4.263838 | 10.40445 | 4.201767 | 10.40462 |
| 0.7| 4.309597 | 10.52779 | 4.309501 | 10.47304 | 4.236180 | 10.47327 |
| 0.8| 4.355790 | 10.60527 | 4.355661 | 10.54196 | 4.270819 | 10.54227 |
| 0.9| 4.402489 | 10.68329 | 4.402324 | 10.61122 | 4.305684 | 10.61162 |
| 1  | 4.449702 | 10.76185 | 4.449495 | 10.68081 | 4.340776 | 10.68131 |

Illustration of the solutions for all three cases are shown in Figure 1, Figure 2, and Figure 3. Numerical results are representatively given in Table 1. From these results, the Adomian decomposition method is successful in solving the population dynamics model. These results compared with the Runge-Kutta numerical solutions lead to the discrepancy of order lower than $10^{-6}$. This means that solutions obtained using the Adomian decomposition method are very accurate.

5. Conclusion
We have solved the population dynamics model and for three different sets of parameters for mutualism, parasitism, and competition. The Adomian decomposition method is meshless, so we can obtain approximate solution simply from the explicit solutions. It gives approximate solutions at every time value without any discretisation of the time domain. This is an advantage of using the Adomian decomposition method to solve the problem.

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