Maximum matching on random graphs

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Abstract. – The maximum matching problem on random graphs is studied analytically by the cavity method of statistical physics. When the average vertex degree \( c \) is larger than 2.7183, groups of max-matching patterns which differ greatly from each other gradually emerge. An analytical expression for the max-matching size is also obtained, which agrees well with computer simulations. Discussion is made on this continuous glassy phase transition and the absence of such a glassy phase in the related minimum vertex covering problem.

Introduction. – Studies on spin glasses focus on systems with random frustrations \cite{1,2}. The energy landscape of a spin glass is very rough. When environmental temperature is lower than certain critical value, the system gets trapped in one of many local regions of the whole configurational space (ergodicity breaking). The deep connection between frustrations in spin glasses and constraints in combinatorial optimization problems was noticed by many authors, and the replica method developed during the study of spin glass physics \cite{4} has been applied to hard combinatorial optimization problems including the k-satisfiability \cite{4}, the number partitioning \cite{5}, the Euclidean matching \cite{6}, the vertex covering \cite{7}, and many others. However, sophistication of replica method renders analytical discussion to be limited usually to the replica-symmetric (RS) level.

Recently, considerable success has been attained in applying the cavity method \cite{8,9} to combinatorial optimization problems \cite{10,11,12,13}. The cavity formalism enables analytical calculations to be carried out to first-step replica-symmetry-breaking (RSB). For random 3-satisfiability (3-Sat) problems it was discovered \cite{10} that, between the Easy-SAT and UNSAT phase there is a Hard-SAT phase. The Hard-SAT phase is a glassy phase, with great many states of the same ground-state energy density which are separated by very high energy barriers. The Easy-SAT to Hard-SAT phase-transition is abrupt. Similar behavior was observed in random 3-XOR-Sat \cite{12}. On the other hand, work performed on minimum vertex covering (min-covering) of random graphs \cite{13} suggested there is no proliferation of ground-states in this model system even when replica symmetry is broken.
The following questions arise: (1) Why in the RSB domain, the structural entropy density (complexity) of min-covering is zero? (2) If there is a glassy phase-transition in a combinatorial optimization problem, is it always discontinuous? In this article, we looked into these questions by studying the maximum matching (max-matching) problem of random graphs. We chose to work with max-matching because (a) as will be shown later, max-matching is equivalent to min-covering in the RS parameter range and, (b) there exist polynomial algorithms to find a max-matching, so that theory can be checked by experiment.

We found that, when the average number of nearest-neighboring vertices of each vertex in a random graph, called the average vertex degree $c$, is larger than $c_{cr} = e$ the system is in a glassy phase with many max-matchings that have very large Hamming distance between each other. Contrast to the situations in random 3-Sat and 3-XOR-Sat, these patterns appear gradually; their number increases exponentially with $c$. When $c < e$, max-matching and min-covering are equivalent (there is no edge-redundancy). When $c \geq e$, the appearance of redundant edges, while adding great freedom to max-matching, cause severe constraints to min-covering patterns, since each edge needs to be covered in the later case.

We also obtained an analytical expression for the average max-match size, which is in agreement with computer simulations in the whole range of parameter $c$.

**Model and cavity calculations.** - Consider a random graph $G(n, c/(n-1))$. There are $n$ vertices in the vertex set $V = \{v_1, v_2, \ldots, v_n\}$; between any pair of vertices $v_i$ and $v_j$, an edge is present in the edge set $E$ with probability $c/(n-1)$ and absent with probability $1 - c/(n-1)$. The average number of edges incident to a randomly chosen vertex is $c$; and for large graph size $n$, the number $k$ of edges associated with a given vertex obeys the Poisson distribution $P_P(c, k) = e^{-c}c^k/k!$. A matching $M$ is a subset of the edge set $E$ such that no two edges in $M$ share a common vertex. We are interested in the max-matching $M^*$, a matching with the largest number of edges. What is the size $|M^*|$ of a max-matching for a random graph $G(n, c/(n-1))$ constructed by the above mentioned procedure? This question could be answered analytically and algorithmically. First, we investigate random graph max-matching by zero-temperature statistical physics, and an analytical formula will be given. The max-matching patterns could be classified into different groups based on their mutual similarity. It turns out that the total number of such groups grow exponentially with graph size $n$ when the average vertex degree $c > e$.

We associate with each edge $e_i$ of graph $G(n, c/(n-1))$ a spin variable $S = \{0, 1\}$. There are altogether $2^{|E|}$ possible microscopic spin configurations for the graph. We introduce the following energy functional for each microscopic spin configuration:

$$\mathcal{H}(\{S\}) = - \sum_{e_i \in E} S_{e_i} + \lambda \sum' S_{e_i} S_{e_j},$$  \hspace{1cm} (1)

where $\lambda > 1$ is a constant and the prime indicates the second summation is restricted to edges $e_i$ and $e_j$ that share a common vertex. For a max-matching $M^*$, we assign $S_{e_i} = 1$ to all the edges $e_i \in M^*$ and $S_{e_j} = 0$ to all the other edges $e_j$. The configurational energy is $-|M^*|$. Because $\lambda > 1$, no microscopic spin configurations could attain energy lower than $-|M^*|$: therefore, finding the max-matching size is converted to finding the ground energy states of eq. (1). Furthermore, each energy (local) minimum configuration corresponds to a matching of the graph $G$. Hereafter we consider only these energy minimum configurations. For very large systems ($n \gg 1$) we group them into different (macroscopic) states. A state

\[\text{(1)}\text{Notice that, different from ref. [13], spins are associated with edges of the graph and cavity fields (see below) are associated with vertices of the graph.}\]
of the system contains a set of microscopic spin configurations, all of which have the same (minimum) energy and only finite number (or number at most proportional to $n^\theta$ with $\theta < 1$) of spin flips is needed to transit from one to another of these configurations [8].

For each state, say $\alpha$, we define a quantity $h_{\alpha}(v_i)$ (called the cavity field) for each vertex $v_i$ according to the following rule: $h_{\alpha}(v_i) = -1$, if in each microscopic configuration of state $\alpha$ one of the edges that meets $v_i$ has spin value $S = 1$; and $h_{\alpha}(v_i) = 0$, if in one or more microscopic configurations of state $\alpha$ all the edges that meet $v_i$ have spin value $S = 0$.

A random graph $G(n, c/(n - 1))$ can be obtained by first generating a random graph $G(n - 1, c/(n - 1))$ and then adding a new vertex (say $v_0$) and setting up $k$ edges $e_1, e_2, \ldots, e_k$ to $k$ randomly chosen vertices (say $v_1, v_2, \ldots, v_k$) of the old graph. The value of $k$ is governed by $P_P(c, k)$ when $n$ is considerably large. If the original system is in state $\alpha$, the energy difference between the enlarged and the original system is

$$\Delta E_{\alpha} = \min_{S_1, \ldots, S_k} \left[-\sum_{i=1}^{k} [h_{\alpha}(v_i) + 1]S_{e_i} + \lambda \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} S_{e_i} S_{e_j}\right].$$

(2)

For the purpose of favoring microscopic spin configurations that lead to an energy decrease, we introduce an “effective inverse temperature” parameter $y$ [10, 9]. Denote $P_{v_i}^{(y)}(h)$ as the distribution of the cavity field $h$ of vertex $v_i$ over different states $\alpha$ of the graph at given effective inverse temperature $y$. The following recursive equation could be written down for this probability distribution:

$$P_{v_0}^{(y)}(h_0) = \delta(k)\delta(h_0) + [1 - \delta(k)]C \prod_{i=1}^{k} [\int dh_i P_{v_i}^{(y)}(h_i)]e^{-y\Delta E}\delta[h_0 + \theta(\sum_{i=1}^{k} h_i + k)],$$

(3)

where $\delta(x)$ is the Kronecker symbol; $C$ is a normalization constant; and $\theta(x) = 1$ if $x > 0$ and $= 0$ otherwise. Equation (3) is understood as follows: (1) if vertex $v_0$ is isolated, it feels no field (the first term); (2) if $k \geq 1$ but in state $\alpha$ each of $v_0$’s neighbors has already associated with an edge of $S = 1$, the edges of $v_0$ should all assign $S = 0$ to decrease energy ($h_0 = 0$); (3) otherwise, it is energetically favorable to assign $S = 1$ to one of the edges of vertex $v_0$ ($h_0 = 1$). Situation (2) and (3) correspond to the second term of eq. (3).

In deriving eq. (3) it was assumed that, before the addition of vertex $v_0$, the cavity field distributions for the vertices $v_1, v_2, \ldots, v_k$ are mutually independent [5,9]. This assumption is based on the argument that, before $v_0$ is added these vertices are usually far apart. It certainly does not hold for regular networks.

A careful inspection of eqs. (2) and (3) leads to the following form for the cavity field distribution:

$$P^{(y)}(h) = \left\{ \begin{array}{ll} \delta(h + 1), & \text{probability } p_1 \\ \delta(h), & \text{probability } p_2 \\ \alpha \delta(h + 1) + (1 - \alpha)\delta(h), & \text{probability } p_3 \end{array} \right.$$  

(4)

where $0 < \alpha < 1$ is a random number obeying certain distribution; and

$$p_1 = 1 - e^{-c p_2}, \quad p_2 = e^{-c(1-p_1)}, \quad p_3 = 1 - p_1 - p_2.$$  

(5)

We introduce the following transformation for variable $\alpha$ in eq. (4): $\alpha = 1/[1 + \tau e^{-y/2}]$. It could be shown that $\tau > 0$ is governed by the population dynamics equation

$$\rho^{(y)}(\tau) = \sum_{m=1}^{\infty} \frac{P_{P^{(c P_3, m)}}}{1 - e^{-c P_3}} \prod_{i=1}^{m} \left[\int d\tau_i P^{(y)}(\tau_i)\right] \delta \left[ \tau - \tau_i \sum_{j=1}^{m} \frac{e^{-y/2}}{[1 + \tau_j e^{-y/2}] - 1} \right].$$  

(6)
\[
= \sum_{m=1}^{\infty} \frac{P_p(m, cp_3)}{1 - e^{-cp_3}} \prod_{i=1}^{m} \left[ \int d\tau_i \rho(y)(\tau_i) \delta \left( \tau - \frac{1}{\tau_1 + \ldots + \tau_m} \right) \right] (y \to +\infty). \tag{7}
\]

The free energy density at given value of the effective inverse temperature \(y\) is obtained following the procedure given in ref. [15] (see also ref. [13]). The expression reads

\[
\Phi(y) = \frac{1}{2} \left( -1 - p_1 + p_2 + cp_2 - cp_1 p_2 + (p_3/y) \left[ (1 + cp_2) \ln(\tau + e^{-y/2}) \right. \right.
\]
\[
+ \left. \frac{c}{2} p_3^2 \ln(1 + \tau_1 \tau_2 + \tau_1 e^{-y/2} + \tau_2 e^{-y/2}) - (1 + c - cp_1) \ln(1 + \tau e^{-y/2}) \right] \tag{8}
\]
\[
= \frac{1}{2} \left( -1 - p_1 + p_2 + cp_2 - cp_1 p_2 \right)
\]
\[
+ (p_3/y) \left( (1 + cp_2) \ln \tau + \frac{c}{2} p_3^2 \ln(1 + \tau_1 \tau_2) \right) \quad (y \gg 1). \tag{9}
\]

In the above equation and hereafter, an overline means performing an averaging.

We are interested in the average size of a max-matching for a random graph \(G(n, c/(n-1))\). This corresponds to the free energy of the system at \(y \to \infty\), provided that the complexity or structural entropy density, \(\Sigma(y) = d\Phi(y)/d(1/y)\), is non-negative (that is, if there exist states at \(y \to \infty\) [10]. When the average vertex degree \(c \leq e\), the only solution of eq. \(1\) is \(p_1 = 1 - W(e)/c\) and \(p_2 = W(e)/c\), where \(W(e)\) is the root of \(W e^W = c\). In this case the average size of the max-matching (in units of the graph size \(n\)) is

\[
x(c) = \lim_{n \to \infty} \frac{|M^*|}{n} = 1 - W(c)/c - W^2(c)/2c. \tag{10}
\]

We see the average max-matching size is identical to the average minimum vertex cover size obtained in refs. [7,10]. This should be the case. Actually, for \(c \leq e\) the leaf-removal algorithm of ref. [17] will at the same time report a min-covering and a max-matching. Expression (10) is exact for \(c \leq e\); while for \(c > e\) it is just an upper-bound (for \(c \geq 4\) it even exceeds 1/2).

When the average degree \(c > e\), eq. [9] has another solution with \(p_1 + p_2 < 1\), and eq. [10] leads to the following max-matching size expression

\[
x(c) = \left( 1 + p_1 - p_2 - cp_2 + cp_1 p_2 \right)/2. \tag{11}
\]

The structural entropy density at effective inverse temperature \(y = \infty\) is

\[
\Sigma = p_3(1 + cp_2) \ln \tau + \left( c/2 \right) p_3^2 \ln(1 + \tau_1 \tau_2). \tag{12}
\]

Based on eqs. [10] and [12], the ground-state structural entropy density is calculated numerically for various values of \(c\) and is shown in fig. [4,10]. The structural entropy density is zero for \(c \leq e\) and it gradually increases from zero as \(c > e\). Therefore, for \(c > e\) there exist many degenerate ground states and the system is in a zero-temperature glassy phase. Figure [4] also reveals that, the larger the value of \(c\), the larger the value of the structural entropy density. This observation is quite different from the case of random 3-Sat and 3-XOR-Sat problem [10][19]. In random 3-Sat and 3-XOR-Sat problems, at the phase-transition point, the complexity jumps from zero to a finite value and then gradually decreases to zero as the average vertex degree \(c\) increases.

The “phase-diagram” fig. [1] is also quite different from that of the minimum vertex covering problem [13]. It is noticeable that, while the minimum vertex covering problem could be mapped to an energy functional very similar to eq. [10] in form, in that system there is no zero-temperature glassy phase [13]. We will discuss this in the last section.
Fig. 1 – The structural entropy density at $y = \infty$ [eq. (12)] as a function of average vertex degree $c$. The structural entropy density could exceed $\ln 2$, since the total number of configurations is $2^{cn/2}$ rather than $2^n$.

In fig. 2 the average size of max-matching eq. (11) is shown for various values of $c$. Also shown are the results obtained by an exact algorithm mentioned in the appendix. The analytical results are in complete agreement with the experimental result in the whole range of $c$. We suggest that the analytical expressions eq. (11) and (12) obtained by statistical mechanics method are exact in the limit $n \gg 1$. The scaling of max-matching size for $c \gg 1$ is

$$x(c) \sim \frac{1}{2} e^{-c/2} + c^2 e^{-2c/2}.$$ (13)

An interesting question is to ask the probability for a random graph $G(n, c/(n - 1))$ to

Fig. 2 – The average size of the max-matchings for a random graph of fixed average vertex degree (solid line) and its asymptotic form (dashed line, see eq. (13)). Filled circles are average max-matching sizes obtained by a matching algorithm (see appendix). At each given average vertex degree $c$, in the numerical simulation, 1000 samples of a random graph of 10000 vertices were generated and the max-matching size for each of them was obtained and then averaged. The errors in the experimental data are all smaller than the radius of the circles. Inset is the probability of a random graph of $n$ vertices to have a matching containing at least $0.49n$ edges: $n = 10000$, filled circles (2000 samples per point); $n = 5000$, empty circles (2000 samples per point); $n = 1000$, empty triangles (3000 samples per point).
have a perfect matching, a matching that has \( n/2 \) edges. If the random graph contains one or more isolated vertices, of cause there could be no perfect matching. The average number of isolated vertices in a random graph \( G(n, c/(n-1)) \) is \( \langle n_0 \rangle = ne^{-c} \). It must be less than unit for a perfect matching to be possible. Therefore, \( c \sim \ln(n) \). For \( n = 10^4 \), \( c \sim 9.21 \). Numerical experiment reveals that, around \( c \simeq 10 \) there is a sharp change in the probability of perfect matching for random graphs of size \( n = 10^4 \). This is in agreement with our above analysis. For \( c > 10 \) probability of perfect matching approaches unity while for \( c < 10 \) it approaches zero. Such a phenomenon was also observed in many other combinatorial optimization problems. In the inset of fig. 2 we demonstrate the experimental probability for a random graph \( G(n, c/(n-1)) \) of \( n = 10^4 \) to have a matching containing at least 4900 edges. We see a sharp transition occurs at \( c \simeq 4.1 \), as would have been expected according to eq. (11), which predicted that at \( c = 4.1 \) the average max-matching size is \( 0.49n \). Figure 2 also shows that the sharpness of this transition is related to system size \( n \), because the relative importance of fluctuations in the max-matching size scales as \( n^{-1/2} \).

**Conclusion.** – To summarize, we have obtained an analytical expression for the average max-matching size of a random graph as a function of the average vertex degree \( c \), and this formula was verified by a numerical experiment. The analytical calculation was performed on a designed spin model using the cavity method of statistical physics. When \( c > e \) the energy landscape of this model system has many valleys of the same energy, separated by large energy barriers between them. The total number of such low-energy valleys is also estimated. Different from the discontinuous glassy phase transition in problems such as random 3-Sat, the transition in max-matching is continuous, with a continuously growing structural entropy.

In a random graph, why are there great many maximum matching patterns but only very few minimum vertex-covering patterns [13]? We think the reason is edge redundancy. When the average vertex degree \( c > e \), redundant edges give additional freedoms to the max-matching patterns, but it cause additional constraints to the min-covering patterns, as all edges should be covered. As was shown here, when \( c < e \) max-matching and min-covering are equivalent for random graphs (in this sense, there is no redundant edges). Beyond \( c = e \), there may still be deep connections between the solution spaces of these two problems. An interesting question is to investigate the possibility of constructing a solution to the min-covering problem with the guide of the solutions of the corresponding max-matching problem.

The excellent agreement between theory and experiment suggests that, for short-range spin-glass models on finite connectivity random graphs, the cavity field assumption that corresponds to the first-order replica-symmetry breaking might be accurate enough that no higher order RSBs is needed for many purposes. For infinite connectivity SK model [3] and finite connectivity models on random graphs [8, 9] we now enjoy satisfactory understandings. A challenge now is the solution of spin glass models on finite connectivity regular lattices.

**Appendix: matching algorithm.** – A max-matching pattern can be found with time proportional to polynomials of graph size \( n \) [15]. The algorithm mentioned below is inspired by the concept of matching-alternating chain [19]. Suppose \( M \) is a matching of a graph \( (V, E) \). An incomplete matching-alternating chain (IMAC) of length zero is composed of a vertex that does not meet \( M \); an IMAC of length \( p \geq 1 \) is composed of a sequence of distinct vertices \( v_0, v_1, \ldots, v_p \) such that (1) \( \{v_i, v_{i+1}\} \in E \) for \( i \in \{0, \ldots, p-1\} \); (2) \( v_0 \) does not meet \( M \) and
The algorithm works by inputting a matching \( M \). It either returns a new matching with size equaling to \(|M| + 1\) or stops if the input matching is a max-matching.

1. Find all the IMACs of length zero. Set all vertices as unlabeled.
2. If there is no IMACs, stop. Else, select one IMAC (say \( C_v \), which is specified by its last element \( v \)). Construct a set \( A = \{v_i : V_i \text{ unlabeled}, \{v_a, v_i\} \in (E - M)\} \).
3. If \(|A| \neq 0\), select one of its elements (say \( v_b \)) and delete \( v_b \) from \( A \). If \( v_b \) is already in chain \( J_v \), return to step (3). Else, mark \( v_b \) as labeled. If \( v_b \) does not meet \( M \), the chain formed by adding \( v_b \) to the end of \( C_v \) is a CMAC; jump to (3a). Else, jump to (3b).
4. Create a new IMAC by adding \( v_b \) and then \( v_c \) to chain \( C_v \). Return to step (3).
5. Delete chain \( C_v \), and return to step (2).

If the algorithm returns a new matching, the algorithm is run again with this new matching as input, till no new matching is returned. It could proved that the last returned matching is a max-matching.

For each given value of \( c \), we generate 1000–3000 instances for a random graph \( G(n, c/(n - 1))\) of order \( n \sim 10^4 \). The max-matching size for each of them is obtained by the above-mentioned algorithm. The average max-matching sizes are shown in fig. 2 (filled circles).

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