Scaling analysis of the quasiparticle tunneling in the $\mathbb{Z}_k$ parafermion states

Qi Li,1 Na Jiang,1 Xin Wan,2,3 and Zi-Xiang Hu1,4

1Department of Physics, Chongqing University, Chongqing, 401331, P.R. China
2Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, P.R. China
3Collaborative Innovation Center of Advanced Microstructures, Nanjing 210093, P.R. China

(Dated: December 22, 2015)

Quasiparticle tunneling between two counter-propagating edges through point contacts could provide information on the statistics of the quasiparticles. Previous study on a disk found a scaling behavior by varying the tunneling distance. It was found that in the limit with zero tunneling distance, the Abelian quasiparticles tunneling obey the scaling analysis while the non-Abelian quasiparticles exhibit some non-trivial behaviors on the scaling exponents. Because of the limitation of disk geometry, we put the fractional quantum Hall (FQH) state on the surface of a cylinder which has a larger tunable tunneling distance than that on disk by varying the aspect ratio $\gamma$. We analyze the scaling behavior of the quasiholes, especially the non-Abelian quasiholes in the Read-Rezayi $\mathbb{Z}_k$ parafermion states. We aim to address the existence of the anomalous correction of the scaling parameter in the long tunneling distance.

PACS numbers: 73.43.Cd, 73.43.Jn

I. INTRODUCTION

According to the experimentally realizable phases that support topological objects, the fractional quantum Hall (FQH) effect, since its discovery,1 has appealed to tremendous theoretical and experimental attractions and achieved us a collection of methods to study the strongly correlated electron systems. The quasiparticle excitations in the FQH liquids can have fractional charges and obey fractional statistics.2–4 Within a serious of filling factors of FQH states, some of them may support more exotic excitations with non-Abelian statistics, which have potential applications in the topological protected quantum computation.5–8 The FQH with an even denominator on the first Landau level at filling factor $\nu = 5/2$ is the most studied state that belongs to the family of non-Abelian FQH states. Since the seminal work of Moore and Read9,10, a connection between the wavefunction of the FQH state and the conformal field theory (CFT) has been established. Thereafter, a series of non-Abelian FQH states have been proposed which are described by the SU(2)$_k$ topological quantum field theory.11 The index $k$ describes the clustering properties in the model wavefunction and has a connection with its filling factor $\nu = \frac{8}{4k+2}$. They are addressed Read-Rezayi $\mathbb{Z}_k$ quantum Hall states since the corresponding wavefunctions can be calculated from the correlation functions in the $\mathbb{Z}_k$ parafermionic conformal field theory.12 Notably, it was found that the $\mathbb{Z}_k$ FQH states are the exact ground state of certain Hamiltonian with $k + 1$-body interaction for all integers $k \geq 1$. Further more, Bernevig et al.14–16 recently found that the homogeneous polynomial part of the $\mathbb{Z}_k$ FQH wavefunction can be obtained recursively from the Jack polynomials which is one of the polynomial solutions for Calogero-Sutherland Hamiltonian.13 The $\mathbb{Z}_k$ FQH wavefunctions in this language are labelled by a negative parameter $\alpha$ and a root configuration (or partition). Because of its computation advantages, we hereafter use the Jacks to produce the model wavefunctions for the ground state and their quasiparticle excitations in the following calculation.

The measurement of the transport properties of the quasiparticles propagating along the edge of the FQH states is crucial for identification of the topological nature of the systems. As standard practice in the noise and interference experiments,17–27, quantum point contacts are introduced to allow quasiparticles propagating on one edge to tunnel to another. This motivated us to study the quasiparticle tunneling amplitudes in FQH liquids in the disk geometry.28–30. On the disk, we considered a tunneling potential along a specific direction $V_{\text{tunnel}} = V(\theta)$. The tunneling amplitudes exhibit interesting scaling behavior, whose exponent is related to the conformal dimension and the charge of the tunneling quasiparticles. Specifically, from an effective field theory analysis, we found that the bare quasiparticle tunneling matrix element satisfies a scaling function

$$\Gamma_a = \langle 0 | H_T | \Psi_{\alpha}^a \rangle \propto N^{1-2\Delta_a} K_a(d) = N^\alpha K_a(d),$$

(1)

where $|0\rangle$ and $|\Psi_{\alpha}^a\rangle$ are the ground state and quasihole wavefunction respectively and $\Delta_a$ is the conformal dimension of the quasiparticle operators. $H_T = \sum_a t_a | \Psi_{\alpha,1}^a (0) \Psi_{\alpha,2}^a (0) + h.c. \rangle$ is the edge-edge tunneling Hamiltonian. The function $K_a(d)$ reveals the dependence on the tunneling distance $d$. We analyzed the tunneling amplitude of the Abelian quasiparticle, such as the $e/3$ and $2e/3$ quasiholes in the Laughlin state at $\nu = 1/3$ and $e/2$ quasihole in the Moore-Read state at $\nu = 5/2$. As shown in Table 1 of Ref. 28, an excellent agreement with the above relation $\alpha = 1 - 2\Delta_a$ was found. However, for non-Abelian excitations, such as $e/4$ quasihole excitation in the Moore-Read state, it is likely that there is a correction on the scaling parameter, i.e.,

$$\alpha^{e/(k+2)} = 1 - 2\Delta_a - \frac{k - 1}{2k}.$$  

(2)
Incidentally, the anomalous term can be written as $-(k + 2)\Delta_a$ where $\Delta_a$ is the conformal dimension for the neutral part of the non-Abelian quasihole operator. We therefore, in the case of disk geometry, had an argument that this anomalous term may origin from the unconstrained tunneling of the neutral parafermions. However, because of the curvature the disk geometry, for a finite size system, the tunneling distance is limited by the radius of the system which is proportional to $\sqrt{N}$. A natural way to overcome this problem is putting the electrons on cylinder or torus. Comparing to the disk, the cylinder geometry has advantages that there is no curvature difference between Landau orbitals and the edge-edge distance is linearly proportional to the system size $N$. Moreover, the edge-edge distance can be easily tuned via varying the aspect ratio, by comparison, in a longer range. Our recent work on the quasihole tunneling and entanglement entropy in Laughlin state on cylinders found that a critical length scale of the edge-edge distance exists. It can be explained as a threshold value that the two edges of the cylinder can be treated independently, or the effects of the edge-edge interaction can only be neglected while $d > L_x$ where $L_x \sim 5B$. Therefore if we want to consider the scaling behavior of the quasiparticle tunneling between two independent edges with a longer tunneling distance, the cylinder geometry is a better choice than disk. In this paper, we study the physical properties of the FQH liquids and reconsider the quasiparticle tunneling amplitudes scaling on cylinder, especially focus on the scaling of the non-Abelian quasiparticles tunneling. We aim to address the question of the existence of the anomalous term in the scaling parameter for non-Abelian quasiholes in the whole region of tunneling distance.

The remainder of this paper is organized as follows. Section II gives a brief review of the mode and previous results in the disk geometry. The model of a quasihole tunneling on cylinder is introduced in section III. Section IV is devoted to the scaling analysis for the Abelian and non-Abelian quasiholes in $Z_k$ state. In section V we focus on the scaling analysis in other $Z_k$ states as the $k = 3$ and $k = 4$ cases. Section VI provides a conclusion and discussion of the paper.

II. MODEL AND PREVIOUS RESULTS

In the disk geometry, we considered a single-particle tunneling potential $V_{\text{tunnel}} = V_0 \delta(\theta)$, which breaks the rotational symmetry. It defines a tunneling path for FQH quasiparticles under the gate influence at a quantum translation symmetry. It defines a tunneling path for FQH variate polynomials of the complex particle coordinates. Potentially, they can be the bosonic version of FQH wavefunction (appending the ubiquitous Gaussian factor $e^{-\sum_i |z_i|^2/4}$ on disk), or fermionic version with an extra Vandermonde determinant $\prod_{i<j}(z_i - z_j)$. A Jack $J_N^\alpha(z_1, z_2, \cdots, z_N)$ is parameterized by a negative rational number $\alpha$, which is related to the clustering properties of the wavefunction, and a root configuration $\lambda$, which satisfies a generalized Pauli exclusion principle and from which one can derive a set of monomials that form a basis for the wavefunction. For the Read-Rezayi $Z_k$ parafermion state, the $\alpha = -k - 1$ and the corresponding root configuration can be expressed by a binary format “$1^k\bar{0}^k\bar{1}^k\cdots$” in the occupation representation of Landau orbitals. Taking the Moore-Read state with $k = 2$ as an example, the root configuration for ground state is “11001100...” where the leftmost orbital represents the innermost Landau orbital with the symmetric gauge. In order to vary the tunneling distance, a large number of Abelian quasihole with charge $e/2$ were inserted at the center, namely a bunch of zeros are attached in front of the root configurations. Therefore, the tunneling distance for a $N$-electron $Z_k$ parafermion state is

$$d(n, N, k)/l_B = \sqrt{2n + \frac{2N(k + 2)}{k} - 4 - \sqrt{2n}},$$

in which $n$ is the number of quasiholes inserted at the center. The system evolves from disk to annulus and finally to a ring shape while increasing $n$. The tunneling distance $d \to 0$ in the one dimensional ring limit while $n \to \infty$. In this limit, we had a conjecture that the tunneling amplitudes for Abelian quasihole with charge $ke/k+2$ for $Z_k$ parafermion state is

$$2\pi\Gamma_k^{ke/(k+2)}(N) = \frac{N}{k+2}B \left( \frac{N}{k} - \frac{k}{k+2} \right),$$

where we introduce the $\beta$ function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Unfortunately, we did not find a universal formula for the non-Abelian quasihole tunneling amplitudes for $Z_k$ FQH states except for the $e/4$ quasihole in Moore-Read state at $k = 2$.

$$2\pi\Gamma^{e/4}(N) = \frac{N^2/4}{\sqrt{B} \left( \frac{N}{2} - \frac{\sqrt{3}}{4} \right) \sqrt{B} \left( \frac{N}{2} - \frac{\sqrt{3}}{4} \right)}.$$

Before discussing the scaling behavior of the tunneling amplitudes of the $Z_k$ parafermion states, we firstly write down the formula of the scaling dimension for the quasiholes. In $Z_k$ parafermionic CFT, the Abelian quasihole operator is $\psi_{\text{Abelian}}^{qh} = e^{i\phi\sqrt{k}/(k+2)}$ where $\phi$ is the charge bosonic field. The Abelian quasihole has charge $ke/k+2$ and scaling dimension $\Delta_{\text{Abelian}} = \frac{ke}{k+2}$. On the other hand, the operator for the smallest charged non-Abelian quasihole is $\psi_{\text{non-Abelian}}^{qh} = \sigma_1 e^{i\phi\sqrt{k}/(k+2)}$ in which the $\sigma_1$ is the neutral spin fields which has scaling dimension $\Delta_{\sigma_1} = \Delta_n = \frac{1}{2(k+2)}$. Therefore, the smallest charged...
The non-Abelian quasihole has charge \( \frac{\pi}{k+2} \) and scaling dimension \( \Delta_{\text{non-Abelian}} = \Delta_\alpha + \frac{1}{2(k+2)} \), where \( \alpha^q = 1 - 2\Delta^q \) is the scaling parameter. Fig. 1 presents the unrescaled and rescaled tunneling amplitudes for the \( e/2 \) quasihole in the MR state as a function of the tunneling distance \( d \) on a disk. With the scaling parameter \( \alpha^e/2 = 0.47 \simeq 1 - 2\Delta^e/2 = 0.5 \), the data for systems from \( 4 \) to \( 14 \) electrons collapse to the same value while \( d \rightarrow 0 \) which are in agreement with the scaling conjecture of Eq. (5).

III. QUASIHOLE TUNNELING ON CYLINDER

As modelled in Fig. 3 for a cylinder with circumference \( L_y \) in \( y \) direction, the lowest Landau level wave function for single electron in a magnetic field with Landau gauge is:

\[
\psi_j(r) = \frac{1}{\sqrt{\pi l_B^2 L_y}} e^{ik_y y} e^{-\frac{1}{2}(x+k_y)^2},
\]

in which \( k_y = \frac{2\pi j}{L_y} \), \( j = 0, \pm 1, \pm 2 \cdots \) are the equilibrium positions for each Landau orbital. Here the magnetic length \( l_B = \sqrt{\hbar c/eB} \) has been set to one. The magnetic field is perpendicular to the cylinder surface and the number of orbits \( N_{\text{orb}} \) equals to the number of magnetic flux quantum penetrating from the surface. As each state or each orbit occupies a constant area \( 2\pi l_B^2 \), the total area is \( A = 2\pi l_B^2 N_{\text{orb}} \) for a finite size system. The aspect ratio is defined as \( \gamma = L_y/L_x \).

To study the quasiparticle tunneling on cylinder, we use a simple delta tunneling potential \( V_{\text{tunnel}} = V_\delta(y) \) for rescaled data for \( e/4 \) quasihole in Moore-Read state. It is shown that the data from different system sizes in the ring limit collapse well at \( \alpha = 0.5 \) instead of at \( \alpha = 0.75 = 1 - 2\Delta^e/4 \). Since the correction of the scaling parameter for \( Z_k \) with \( k = 2, 3, 4, 5 \) can be approximated by \( -(k+2)\Delta_\alpha \), we speculated that the anomalous scaling behavior for the non-Abelian quasihole may originate from the effect of non-independent edges for neutral component, or the edge-edge interactions while in small \( d \). On the other hand, if we look carefully at the Fig. 2(b), for the range \( d > 5l_B \), although few data points limited by the geometry and system size, there is still a hint that the scaling behavior works well in large \( d \) regime without the neutral part correction. Therefore, a natural question is whether the scaling conjecture of Eq. (5) works for large tunneling distance? With this motivation, we reconsider the scaling behavior of the tunneling amplitudes for \( Z_k \) parafermion states on cylinder in the following section.

![FIG. 1: The unrescaled (a) and rescaled tunneling amplitude \( N^{-\alpha} e^{(d/2l_B)^2} \) with \( \alpha = 0.47 \) for \( e/2 \) quasihole in the MR state as a function of the tunneling distance \( d \) on disk.](image1)

![FIG. 2: The rescaled tunneling amplitude \( N^{-\alpha} e^{(d/2l_B)^2} \) for \( e/4 \) quasihole in the \( Z_2 \) parafermion, or Moore-Read state as a function of the tunneling distance \( d \) on disk.](image2)

![FIG. 3: The sketch of a cylinder we used to make the quasihole tunnel from the left edge to the right under a tunneling potential \( V_{\text{tunnel}} = V_\delta(y) \). \( L_y \) is the circumference of the edge and \( L_x \) is the length of the finite cylinder.](image3)
a single particle similar to the disk geometry. This potential allows a quasiparticle to tunnel along the $x$-direction while the $y$-direction doesn’t have translational symmetry. The matrix element $\langle k|V_{\text{tunnel}}|m \rangle$ describes a particle tunneling from one single particle state $|m\rangle$ to another state $|k\rangle$. If we set $V_f = 1$ for convenience, it is easy to get the matrix element form as follows:

$$v_p(k, m) = \langle k|V_{\text{tunnel}}|m \rangle = e^{-\frac{\delta^2}{4d^2}},$$

(8)

where $d$ is the distance between two single states. In many-body case, the tunneling operator is just the summation of the single particle tunneling potential $H_T = V_f \sum \delta(y_i)$. Then the tunneling amplitude is obtained by calculating

$$\Gamma = \langle \psi_{qh}|\tau|\psi_0\rangle = \sum_i \langle k_1 k_2 \cdots k_n|\delta(y_i)C_k^+ C_m|m_1 m_2 \cdots m_n\rangle,$$

(9)

where $|k_1 k_2 \cdots k_n\rangle \in \psi_{qh}$ and $|m_1 m_2 \cdots m_n\rangle \in \psi_0$. The matrix elements have non-zero components coming from the identical sets $|k_1 k_2 \cdots k_n\rangle$ and $|m_1 m_2 \cdots m_n\rangle$ except a single pair $m'$ and $k'$ which have constant value of momentum difference. Taking Moore-Read state as an example, the $e/2$ case need the $k' - m' = N$ while the $e/4$ case need the $k' - m' = N/2$ where $N$ is the number of electrons. Thus, we easily get $v_p^{e/2} = e^{-\frac{\pi^2}{2d^2}N^2}$ and $v_p^{e/4} = e^{-\frac{\pi^2}{2d^2}N^2/2}$, or the charge of the quasihole should appear in the Gaussian factor.

IV. SCALING ANALYSIS FOR THE $\mathbb{Z}_2$ STATE

In this section, we systematically study the scaling behavior of the tunneling amplitude in the case of $\mathbb{Z}_2$, or Moore-Read state on a cylinder. In the language of the Jack polynomial description, the Moore-Read state and its quasihole state are labelled by root configurations $|\psi_{MR}\rangle = |11001100 \cdots 110011\rangle$, $|\psi_{e/2}^{\psi_{qh}}\rangle = |011001100 \cdots 110110\rangle$ and $|\psi_{e/4}^{\psi_{qh}}\rangle = |010101010 \cdots 0101\rangle$ respectively. Here the extra zero in the $|\psi_{e/4}^{\psi_{qh}}\rangle$ on the left means a flux quantum, or an Abelian $e/2$ quasihole is created on the left edge of the cylinder. The pattern in the $|\psi_{e/4}^{\psi_{qh}}\rangle$ means there are two $e/4$ quasiholes, one on each edge of the cylinder since the non-Abelian Majorana fermion modes are embedded in the $e/4$ quasihole excitation which must appear in pair. As was introduced in above section, the tunneling path of the quasihole in this case is the length of the finite cylinder in $x$-direction, namely $d = L_x = \sqrt{2\pi N_{\text{orb}}/\gamma}$. In this case, we have a parameter $\gamma$ that can smoothly tune the tunneling distance from zero to infinity. As a comparison, in the disk geometry, the tunneling distance has a upper limit which is the radius of the system $R = \sqrt{2N_{\text{orb}}}$ and $d \sim N_{\text{orb}}/\sqrt{2N_{\text{qh}}}$ for $N_{\text{qh}} \gg N_{\text{orb}}$, thus the disk geometry is not suitable to large $d$ physics. Interestingly, in the limit of $L_x \to \infty$ or $\gamma \to 0$, the two adjacent Landau orbitals have practically zero overlap and thus the Hamiltonian is dominated by the electrostatic repulsion. This thin cylinder limit has ground state which is called a charge density wave state, or Tao-Thouless state on torus with occupation pattern 1001001001 $\cdots$. However previous studies in this limit shows that the topological properties of the FQH state does not change as varying $\gamma$, or there is no phase transition while varying the aspect ratio. Therefore, we feel more comfortable to say that the results calculated from Eq.(9) in the whole range of $L_x$ are really the tunneling amplitudes for FQH quasiholes. Since the Landau wavefunctions both on the disk and cylinder have the similar feature to Gaussian wave, following the work on disk, we believe that the quasiparticle tunneling amplitude on cylinder has the same scaling behavior as that on disk in Eq.(4). Fig.4 (a) shows that the tunneling amplitudes for $e/2$ quasihole as a function of the tunneling distance $L_x$. By the help of Jacks, the largest system size we have reached is 18 electrons. The data is the same as that on disk in Fig.3(a) except that there are more data for large $L_x$ on cylinder. In the limit $L_x \to 0$, or the CFT limit, with the same argument as that on disk, the value of the tunneling amplitude are exactly the same as that in Eq. (4). As $L_x$ increasing, the tunneling amplitudes dramatically drop to zero which dominated by the gaussian factor in Eq.(6).

In Fig. 4(b), we plot the rescaled data $\tilde{\Gamma} = N^{-\alpha} e^{(L_x/2\pi\Delta^2)^\alpha/2}$ as a function of the tunneling distance. While $\alpha = 1 - 2\Delta^2/2 = 0.5$, the scaling behavior is obviously better than that on disk as shown in Fig.(1b). The data from 8–18 electrons not only in the CFT limit $L_x \to 0$, but also in the large $L_x$ region collapse onto

![Fig. 4: The bare (a) and rescaled tunneling amplitude $N^{-\alpha} e^{(L_x/2\pi\Delta^2)^\alpha/2}$ with $\alpha = 0.5$ (b) for $e/2$ quasihole in the Moore-Read state as a function of the tunneling distance $L_x$ on cylinder. The system size ranges from 8 to 18 electrons.](image-url)
each other. Therefore we see that the scaling function still works, for charged $e/2$ Abelian quasihole, in the region far away from the CFT limit. The next question is whether it is workable for the non-Abelian quasihole tunneling amplitude, and whether the anomalous term in the scaling parameter still exist?

In Fig. 5(a), we plot the tunneling amplitude for $e/4$ quasihole as a function of $L_x$ for systems ranges from 8 to 18 electrons. The values in the CFT limit are still consistent to the Eq. (3). Comparing with the Abelian case as shown in Fig. 4(a), the bare tunneling amplitudes for non-Abelian $e/4$ quasihole do not decay monotonically as increasing tunneling distance. There is a bump around $L_x \simeq L_x^c \simeq 5L_B$ which is consistent to the crirical threshold value at which the interaction between two edges of FQH states can not be neglected. The more interesting is that, as shown in Fig. 4(b), while the tunneling amplitudes are rescaled by Eq. (4), with a modified scaling parameter $\alpha e^4/4 = 0.5$ as expressed in Eq. (2), the data approaching to the CFT limit still scales very well. This result has consistency with that in the disk geometry. However, with this scaling parameter, the tunneling amplitudes for large $L_x$ do not collapse onto each other. Since the two edges of the cylinder in this case are getting more and more independent while increasing the edge-edge distance, it motivates us to look the scaling behavior without the correction term in Eq. (2). The results are shown in Fig. 5(c). Here we set the scaling parameter $\alpha = 1 - 2\Delta_{qh} = 0.75$. It is shown that the rescaled data for all systems scales very well in the range $L_x > L_x^c$ and obviously, the scaling behavior in the CFT limit in this case is broken in this case.

As a conclusion for this section, we find that in the cylinder geometry, the tunneling amplitude for the Abelian $e/2$ quasihole in the Moore-Read state obeys the scaling behavior of Eq. (4) with $\alpha^q = 1 - 2\Delta_{qh}$ both in the region with short tunneling distance (CFT limit) and in the region of long tunneling distance. However, for the non-Abelian $e/4$ quasihole, the above scaling parameter is only workable for $L_x > L_x^c$ where the $L_x^c$ is the critical distance above which the two edges of the cylinder can be treated as two independent ones. As the previous study on disk, the data for the tunneling amplitudes in $L_x < L_x^c$ obey the similar scaling function with a non-trivial modification on the scaling parameter $\alpha$ as expressed in Eq. (2).

V. SCALING ANALYSIS IN OTHER $Z_k$ STATES

To furtherly confirm that our conclusions for the case of $k = 2$ are suitable for all the $Z_k$ parafermion FQH states. In this section, we look at the scaling behavior and the related scaling parameter in the cases of $k = 3$ and $k = 4$. First of all, since we need to consider the relation between the scaling parameter $\alpha$ and the scaling dimension of the quasiholes in FQH states, we list the quasihole charge, scaling dimension and its corresponding scaling parameters for Abelian and non-Abelian quasiholes in the following two tables.

### TABLE I: The scaling exponents for charge $ke/(k+2)$ Abelian quasiholes tunneling amplitudes scaling in the Read-Rezayi states. $\Delta_{qh}$ is the scaling dimension for quasihole.

| $k$ | $Q$ | $\Delta_{qh}$ | $\alpha$ |
|-----|-----|---------------|---------|
| 3   | $3e/5$ | 3/10          | 2/5     |
| 4   | $4e/6$ | 1/3           | 1/3     |

| $k$ | $Q$ | $\Delta_{ch}$ | $\Delta_{qh}$ | $\alpha = 1 - 2\Delta_{qh}$ |
|-----|-----|---------------|---------------|------------------------------|
| 3   | $e/5$ | 1/30         | 1/15          | 7/15(0.4667)                |
| 4   | $e/6$ | 1/48         | 1/16          | 11/24(0.4583)               |

Due to the limitation of storing the basis of the wavefunction, the system sizes for $Z_3$ and $Z_4$ parafermion states we considered are up to 21 and 28 electrons respectively. As shown in Fig. 5, like in the $Z_2$ case, the tunneling amplitudes for Abelian quasiholes in the $Z_3$ and $Z_4$ states monotonically decay exponentially as increasing the tunneling distance. After being rescaled by Eq. (4), all the data locate on the same curve except for that of the smallest system size. The scaling parameters $\alpha^q$ we used in the plot are exact the expect values...
the interaction of the two edges of the cylinder brings the anomalous correction to the scaling parameter. On the other hand, if this anomalous term is neglected, as shown in Fig. 7(c) and (f), the data for the long distance tunneling scales better than that with the correction, although there are strong finite size effects.

VI. SUMMARY AND DISCUSSION

In this work, we systematically study the scaling behavior of the quasihole tunneling amplitudes for the $Z_k$ parafermion states on the cylinder geometry. Comparing with the disk geometry we previously studied, the Landau orbitals on cylinder do not have curvature difference and the tunneling distance can be smoothly tuned from zero to infinity via varying the aspect ratio. While the length of the finite cylinder $L_x$ decreasing from a thin cylinder limit to the CFT limit, there is a critical length scale $L_x^c \simeq 5l_B$ at which the two independent edges of the cylinder become interacting with each other. Therefore, the length scale of the tunneling distance is automatically separated into two regions. Our calculations reveals that the scaling behaviors for the overall region are in good agreement with the scaling conjecture in Eq. (4). For the Abelian quasihole with charge $e/(k+2)$, in all $Z_k$ states, by using the scaling parameter $\alpha^{qh} = 1 - 2\Delta^{qh}$ as expected from the analysis of the effective field theory, the data in two regions collapses onto each other very well. However, the things get more complicated in the non-Abelian case. When $L_x > L_x^c$, or with two independent edges, the scaling parameter $\alpha$ is the same as that in the Abelian case instead of substituting the quantum dimension $\Delta^{qh}$ by the one for non-Abelian quasiholes. And in the case of $L_x < L_x^c$, or approaching to the CFT limit, similar to the result on disk geometry, the scaling parameter need a modification which is shown in Eq. (2).

Based on the following reasons: (1) the tunneling amplitudes of the Abelian quasiholes do not have this term in this CFT limit; (2) this anomalous term can be rewritten as $-(k+2)\Delta_q$, where $\Delta_q$ is the conformal dimension of the neutral component; (3) our recent work elsewhere on the density difference between the bosonic and fermionic edge states shows that the width of the fermionic edge states in $Z_k$ states is larger than that of bosonic edge states. Or in other words, the neutral fermionic component in the edge state or quasihole state is more sensitive to the length scale of the edge-edge distance. We conclude that the anomalous correction term is contributed from the neutral component of the non-Abelian quasiholes which can be treated as another feature of the non-Abelian quasiholes. This characteristic may be detected in the realistical systems, such as in the shot noise or the point contact interference experiments.

This work was supported by NSFC Project No. 1127403, 11547305, Fundamental Research Funds for the Central Universities No. CQDXWL-2014-Z006. NJ was also supported by Chongqing Graduate Student Research.
Innovation Project No. CYB14033. XW was supported by the 973 Program Project No. 2012CB927404, NSF-China Grant No. 11174246.

* Electronic address: zxhu@cqu.edu.cn

1 D. C. Tsui, H. L Stomer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
2 F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982).
3 B. I. Halperin, Phys. Rev. Lett. 52, 1583 (1984).
4 D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).
5 M. H. Freedman, Proc. Natl Acad. Sci. U. S. A. 95, 98 (1998).
6 J. Preskill, in Introduction to Quantum Computation, edited by H. -K. Lo, S. Popescu, and T. P. Spiller (World Scientific, Singapore, 1998).
7 A. Kitaev, Ann. Phys. (N. Y.) 303, 2 (2003).
8 B. I. Halperin, Phys. Rev. Lett. 52, 1583 (1984).
9 D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).
10 M. H. Freedman, Proc. Natl Acad. Sci. U. S. A. 95, 98 (1998).
11 N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
12 V. A. Fateev and A. B. Zamalodchikov, Sov. Phys. JETP 62, 215 (1985).
13 B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Not. 2002, 1223 (2002); ibid, 2003, 1015 (2003).
14 B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 100, 246802 (2008).
15 B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 101, 246806 (2008).
16 B. A. Bernevig and N. Regnault, Phys. Rev. Lett. 103, 206801 (2009).
17 C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen, Phys. Rev. B 55, 2331 (1997).
18 E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, Nucl. Phys. B 516, 704 (1998).
19 S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. 94, 166802 (2005).
20 A. Stern and B. I. Halperin, Phys. Rev. Lett. 96, 016802 (2006).
21 P. Bonderson, A. Kitaev, and K. Shtengel, Phys. Rev. Lett. 96, 016803 (2006).
22 B. Rosenow, B. I. Halperin, S. H. Simon, and A. Stern, Phys. Rev. Lett. 100, 226803 (2008).
23 W. Bishara and C. Nayak, Phys. Rev. B 77, 165302 (2008).
24 P. Bonderson, K. Shtengel, and J. K. Slingerland, Phys. Rev. Lett. 97, 016401 (2006).
25 L. Fidkowski, [arXiv:0704.3291] (unpublished).
26 P. Bonderson, K. Shtengel, and J. K. Slingerland, Ann. Phys. (N.Y.) 323, 2709 (2008).
27 E. Ardonne and E. -A. Kim, J. Stat. Mech.: Theory Exp. L04001 (2008).
28 H. Chen, Z. -X. Hu, K. Yang, E. H. Rezayi, and X. Wan, Phys. Rev. B 80, 235305 (2009).
29 Z. -X. Hu, K. H. Lee, E. H. Rezayi, X. Wan, and K. Yang, New J. Phys. 13, 035020 (2011).
30 Z. -X. Hu, K. H. Lee, and X. Wan, Int. J. Mod. Phys. Conf. Ser. 11, 70 (2012).
31 Q. Li, N. Jiang, Z. Zhu, and Z.-X. Hu, New J. Phys. 17, 095006 (2015).
32 R. Tao and D. J. Thouless, Phys. Rev. B 28, 1142 (1983).
33 D. J. Thouless, Surf. Sci. 142, 147 (1984).
34 A. Seidel, H. Fu, D-H. Lee, J. M. Leinaas, and J. Moore, Phys. Rev. Lett. 95, 266405 (2005).
35 E. J. Bergholtz and A. Karlhede, Phys. Rev. B 77, 155308 (2008).
36 E. J. Bergholtz, T. H. Hansson, M. Hermanns, A. Karlhede, and S. Viefers, Phys. Rev. B 77, 165325 (2008).
37 N. Jiang and Z. -X. Hu, unpublished.