Secure Software Leasing

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Abstract

We introduce the notion of secure software leasing (SSL): this allows for an authority to lease software to a third party with the security guarantee that after the lease expires, the third party can no longer produce any functionally equivalent software (including the original software) that can be run on the same platform. While impossible to achieve classically, this opens up the possibility of using quantum tools to realize this notion.

In this work, we initiate a formal study of this notion and present both positive and negative results under quantum hardness of classical cryptographic assumptions:

• **Negative Result**: We prove that it is impossible to construct SSL schemes for an arbitrary class of quantum unlearnable functions. In particular, our impossibility result also rules out quantum copy-protection [Aaronson CCC’09] for any class of quantum unlearnable functions; resolving an open problem on the possibility of constructing copy-protection for arbitrary quantum unlearnable circuits. Our techniques also rule out the existence of quantum VBB for classical circuits, answering an open problem posed by [Alagic and Fefferman arXiv’16].

Along the way, we introduce a notion called de-quantizable circuits and present the first construction of this notion which may be of independent interest.

• **Positive Result**: On the other hand, we show that we can realize SSL for a subclass of evasive circuits (that includes natural implementations of point functions, conjunctions with wild cards, and affine testers).

1 Introduction

Almost all proprietary software requires a legal document, called software license, that governs the use against illegal distribution of software, also referred to as pirating. The main security requirement from such a license is that any malicious user no longer has access to the functionality of the software after the lease associated with the software license expires. While ad hoc solutions existed in the real world, for a long time, no theoretical treatment of this problem was known.

This was until Aaronson, who in his seminal work [3] introduced and formalized the notion of quantum software copy-protection, a quantum cryptographic primitive that uses quantum no-cloning techniques to prevent pirating of software by modeling software as boolean functions.
Roughly speaking, quantum copy-protection says\(^1\) that given a quantum state computing a function \(f\), the adversary cannot produce two quantum states (possibly entangled) such that each of the states individually computes \(f\). This prevents a pirate from being able to create a new software from his own copy and re-distribute it; of course it can circulate its own copy to others but then it will lose access to its own copy.

For this notion to be interesting, Aaronson observed that only unlearnable functions can be copy-protected, since a function that can be learned from its input-output behavior alone can be pirated by learning the function in the first place, and then redistributing a software with the same functionality as the original function. In the same work, Aaronson showed how to achieve this notion for arbitrary unlearnable boolean functions in the quantum oracle model, and also proposed two heuristic candidates to copy-protect point functions in the standard model.

This leaves us with an unsatisfactory state of affairs, with the following questions still open since 2009:

- **Is quantum software copy-protection possible at all for arbitrary unlearnable functions in the standard model (i.e, without using any oracle) even under computational assumptions?**
- **Are there simple functions, for which we can construct software copy-protection based on classical cryptographic assumptions?**

In a recent blog post, Aaronson\(^2\) even mentioned constructing quantum copy-protection from cryptographic assumptions as one of the five big questions he wishes to solve.

**Our Work: Secure Software Leasing.** Towards understanding the questions above, we formulate a simpler version that still captures the essence of quantum copy-protection. We term this notion as **secure software leasing** (SSL). Roughly speaking, an SSL scheme allows for an authority (the lessor\(^3\)) to lease a classical circuit \(C\) to a user (the lessee\(^4\)) by providing a corresponding quantum state \(\rho_C\). The user can execute \(\rho_C\) to compute \(C\) on any input it desires. But at a later point in time, specified by the lease agreement, the lessee is supposed to return back \(\rho_C\) to the lessor. After it returns the state, we require the security property that the lessee can no longer compute \(C\).

In more detail, a secure software leasing scheme (SSL) for a family of circuits \(C\) is a collection, \((\text{Gen}, \text{Lessor}, \text{Run}, \text{Check})\), of quantum polynomial-time algorithms (QPT) satisfying the following conditions. \(\text{Gen}(1^\lambda)\), on input a security parameter \(\lambda\), outputs a secret key \(sk\) that will be used by a lessor to validate the states being returned after the expiration of the lease. For any circuit \(C : \{0, 1\}^n \to \{0, 1\}^m\) in \(C\), \(\text{Lessor}(sk, C)\) outputs a quantum state \(\rho_C\), where \(\rho_C\) allows \(\text{Run}\) to evaluate \(C\). Specifically, for any \(x \in \{0, 1\}^n\), we want that \(\text{Run}(\rho_C, x) = C(x)\). Finally, \(\text{Check}(sk, \rho_C)\) checks if \(\rho_C\) is a valid leased state. Any state produced by the lessor is a valid state and will pass the verification check.

An SSL scheme is said to be **lessor secure** if for any (malicious) QPT user \(A\), it holds that \(A(\rho_C)\) cannot output a (possibly entangled) bipartite states \(\sigma^*\) such that \(\sigma_1^* := \text{Tr}_2[\sigma^*]\) \(^4\) passes the lessor’s

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\(^1\) More generally, Aaronson considers the setting where the adversary gets multiple copies computing \(f\) and not just one.

\(^2\) The person who leases the software to another.

\(^3\) The person to whom the software is being leased to.

\(^4\) This denotes tracing out the second register.
verification \((\text{Check}(sk, \sigma^*_1) = 1)\) and such that the the resulting state, after the first register has been verified by the lessor, on the second register, \(\sigma^*_2\), can also be used to evaluate \(C\) with the \(\text{Run}\) algorithm, \(\text{Run}(\sigma^*_2, x) = C(x)\). To see why this security notion protects against pirating, consider the simpler case where a QPT adversary \(A\), is given \(\rho_C\), and outputs the state \(\rho_C \otimes \sigma^*_2\). Then, the state in the first register will pass the lessor’s verification since it is the original leased state, but in that case, the above security definition guarantees that \(\text{Run}(\sigma^*_2, \cdot)\) does not compute \(C\).

We study the questions stated above but in the context of SSL:

- **Q1:** Is secure software leasing possible at all for arbitrary unlearnable functions in the standard model under computational assumptions?
- **Q2:** Are there simple functions, for which we can construct secure software leasing based on classical cryptographic assumptions?

**Comparison with Copy-Protection.** A comparison between SSL and copy-protection is in order. Informally, a secure quantum copy-protection scheme for \(C\) satisfies the property that given \(C \in C\), a software distributor can produce a state \(\rho_C\) along with its own evaluation \(\text{Run}\) algorithm that computes \(\text{Run}(\rho_C, x) = C(x)\). Copy-protection security guarantees that no user can prepart two states that also computes \(C\) in any possible way from \((\text{Run}, \rho_C)\). That is, given \((\text{Run}, \rho_C)\), no adversary can output QPT algorithms \(\text{Run}'_1\) and \(\text{Run}'_2\), along with corresponding states \(\sigma_1\) and \(\sigma_2\) such that both compute \(C\) correctly, i.e. \(\text{Run}'_1(\sigma_1, x) = C(x)\) and \(\text{Run}'_2(\sigma_2, x) = C(x)\).

The existence of a copy-protection scheme for \(C\) would imply the existence of an SSL scheme for \(C\), as any successful pirate \(A\) that can produce \(\sigma_1\) and \(\sigma_2\) given \((\text{Run}, \rho_C)\) such that \(\text{Run}(\sigma_1, x) = C(x)\) and \(\text{Run}(\sigma_2, x) = C(x)\) would have also broken copy-protection.

We can view copy-protection as protecting against pirating of software that can be evaluated on any computing platform while our scheme protects against pirating of software for fixed platforms (modeled as \(\text{Run}\) in our setting). Moreover, SSL requires that the lessee returns the state back whereas quantum copy-protection has no such requirement. While weaker, there are still a number of concrete scenarios in which the SSL security guarantee suffices, as we discuss below. Furthermore, we could hope to construct simpler schemes that highlight what type of computational assumptions and tools are needed to get copy-protection for different circuit classes.

We list some applications of SSL below.

**Limited-Time Software.** Any company that owns a proprietary computing platform and desires to release a special edition program \(C\) for its platform for a limited time only can use SSL to achieve this. Consider the following examples. (1) *S.S.L.Inc* owns a proprietary operating system, and they might want to release a program \(C\) for their operating system that requires renewing the lease on a yearly basis. They can then distribute \(\rho_C \leftarrow \text{Lessor}(sk, C)\), and ask for this state a year later. (2) Before releasing the full version of \(C\), they might want to allow users to run a beta version of it, \(C_\beta\), in order to test it and get user feedback. Naturally, they do not want people to pirate their beta versions in order to not have to buy the full version when it comes out. Again, they can lease the beta version \(C_\beta\), expecting the users to return it back when the beta test is over. At this point, they would know if a user did not return their beta version and they can penalize such a user according to their lease agreement.
**Recalling Buggy Software.** Consider the scenario where a company, say S.S.L.Inc, is actively selling a feature $C$ to their proprietary computing platform. They soon discover that a bug in $C$ is compromising the security of the whole platform. If they distributed $C$ by using an SSL scheme, they can now recall $\rho_C$, instead of having to stop running the whole platform altogether. Once the feature $C$ has been recalled, if everyone returned the software, the platform security is no longer compromised—since no one can run $C$ in the platform anymore. Anyone not returning the software could then face possible legal consequences stated in the license agreement that the proprietary computing platform had in the first place.

**Malicious Former Employees.** Any company with employees that use company-owned proprietary software might worry that a former employee with malicious intentions kept or sold copies of the software. To protect against malicious former employees, a company S.S.L.Inc gives its employees a leased copy $\rho_C$ of their proprietary software $C$, which is needed for them to perform their job. Any employee that quits (or is fired) is required to return $\rho_C$. The lessor security would then guarantee S.S.L.Inc that the employee did not sell a copy of $C$ to a competitor that also uses the same platform; there are instances where different companies in the same sector use the same platform (e.g. Bloomberg Terminals).

### 1.1 Our Results

Given the long history of unclonable quantum cryptographic primitives (see Section 1.3) along with the recent boom in quantum cryptographic techniques [56, 42, 43, 26, 24, 29, 22, 28, 16, 11], one might hope that existing techniques could lead us to construct SSL schemes. We show, rather surprisingly, that there exist unlearnable class of circuits such that no SSL exists for this class. Our impossibility is conditional, we assume the existence of quantum fully homomorphic encryption schemes and the quantum hardness of learning with errors. Under cryptographic assumptions, this provides an answer to the question $Q_1$ stated earlier. While this dashes our hopes of achieving SSL for any class of unlearnable functions, we can still aim to construct SSL for a restricted class of unlearnable circuits. We show how to construct SSL for a subclass of evasive circuits (and not functions), partly answering $Q_2$.

#### 1.1.1 Impossibility Result

To demonstrate our impossibility result, we identify a class of classical circuits $C$ that we call a de-quantizable circuit class. This class has the nice property that given any efficient quantum implementation of $C \in C$, we can efficiently ‘de-quantize’ it to obtain a classical circuit $C' \in C$ that has the same functionality as $C$. If $C$ is learnable then, from the definition of learnability, there could be a QPT algorithm that finds $C'$. To make the notion interesting and non-trivial, we add the additional requirement that this class of circuits is quantum unlearnable. A circuit class $C$ is quantum unlearnable if given black box access to $C \in C$, any QPT algorithm cannot find a quantum implementation of $C$.

We show the existence of de-quantizable circuit class from cryptographic assumptions.

**Proposition 1 (Informal).** *Assuming the quantum hardness of learning with errors (QLWE), and assuming the existence of quantum fully homomorphic encryption (QFHE), there exists a de-quantizable class of*
circuits.

We show how non-black box techniques introduced in seemingly different contexts – proving impossibility of obfuscation [15, 21, 9] and constructing zero-knowledge protocols [17, 19, 12] – are relevant to proving the above proposition. We give an overview, followed by a formal construction, in Section 4.

We then show that for certain de-quantizable class of circuits, there does not exist an SSL scheme for this class. Combining this with the above proposition, we have the following:

**Theorem 2 (Informal).** Assuming the quantum hardness of learning with errors (QLWE), and assuming the existence of quantum fully homomorphic encryption (QFHE), there exists a class of quantum unlearnable circuits $C$ such that there is no SSL for $C$.

On the Assumption of QFHE: There are lattice-based constructions of QFHE proposed by [42, 20] although we currently don’t know how to base them solely on the assumption of LWE secure against QPT adversaries (QLWE). Brakerski [20] shows that the security of QFHE can be based on QLWE and a circular security assumption.

**Impossibility of Copy-Protection.** Since copy-protection implies SSL, we have the following result.

**Corollary 3 (Informal).** Assuming the quantum hardness of learning with errors (QLWE), and assuming the existence of quantum fully homomorphic encryption (QFHE), there exists a class of quantum unlearnable circuits $C$ that cannot be copy-protected.

Assuming QFHE and QLWE, this rules out constructing copy-protection for arbitrary classes of unlearnable circuits, resolving one of the questions stated in the introduction.

1.1.2 Main Construction

Our impossibility result does not rule out the possibility of constructing SSL schemes for specific circuit classes. For example, it does not rule out being able to construct SSL for evasive functions; this is a class of functions with the property that given black box access, an efficient algorithm cannot find an accepting input; that is, an input on which the output of the function is 1.

We identify a subclass of evasive circuits for which we can construct SSL.

**Searchable Compute-and-Compare Circuits.** We consider the following circuit class $C$: every circuit in $C$, associated with a circuit $C$ and a lock $\alpha$, takes as input $x$ and outputs 1 iff $C(x) = \alpha$. This circuit class has been studied in the cryptography literature in the context of constructing program obfuscation [54, 38]. We require this circuit class to additionally satisfy a searchability condition: there is an efficient (classical) algorithm, denoted by $S$, such that given any $C \in C, S(C)$ outputs $x$ such that $C(x) = 1$.

There are natural and interesting sub-classes of compute-and-compare circuits:

- Point circuits $C(\alpha, \cdot)$: the circuit $C(\alpha, \cdot)$ is a point circuit if it takes as input $x$ and outputs 1 iff $x = \alpha$. If we define the class of point circuits suitably, we can find $\alpha$ directly from the description of $C(\alpha, \cdot)$; for instance, $\alpha$ is the value assigned to the input wires of $C$. 
• Conjunctions with wild cards $C(S, \alpha, \cdot)$: the circuit $C(S, \alpha, \cdot)$ is a conjunction with wild card if it takes as input $x$ and outputs $C(S, \alpha, x) = 1$ iff $y = \alpha$, where $y$ is such that $y_i = x_i$ for all $i \in S$ and $y_i = 0$ for all $i \notin S$. Again, if we define this class of circuits suitably, we can find $S$ and $\alpha$ directly from the description of $C(S, \alpha, \cdot)$.

We emphasize that the notion of searchability is associated with a particular class of circuits and not with a function family. For instance, there could be two circuit classes that implement the same class of functions but one could be searchable but not the other one. We note that Aaronson [3] proposed candidates of copy-protection in the plain model only for searchable point circuits, although Aaronson doesn’t use the terminology of searchability.

We prove the following result. Our construction is in the common reference string (CRS) model. In this model, we assume that both the lessor and the lessee will have access to the CRS produced by a trusted setup. We note that our impossibility result also holds in the CRS model.

**Theorem 4 (SSL for Searchable Compute-and-Compare Circuits; Informal).** *Assuming the existence of: (a) quantum-secure subspace obfuscators [56] and, (b) learning with errors secure against sub-exponential quantum algorithms, there exists an SSL scheme in the common reference string model for searchable compute-and-compare circuits.*

Notice that for applications in which the lessor is the creator of software, the lessor can dictate how the circuit class is defined and thus would choose an implementation of the circuit class that is searchable.

A discussion about the primitives described in the above theorem statement is in order. A subspace obfuscator takes as input a subspace $A$ and outputs a circuit that tests membership of $A$ while hiding $A$ even against quantum adversaries. This was recently constructed by [56] based on the quantum-security of indistinguishability obfuscation [32].

While the assumption of learning with errors against sub-exponential quantum algorithms is non-standard, we firstly note that classical sub-exponential security of learning with errors has been used to construction many cryptographic primitives and secondly, there are no known significant quantum speedups to solving this problem.

In the technical sections, we prove a more general theorem.

**Theorem 5 (SSL for General Evasive Circuits; Informal).** *Let $C$ be a searchable class of circuits. Assuming the existence of: (a) quantum-secure input-hiding obfuscators [14] for $C$, (b) quantum-secure subspace obfuscators [56] and, (c) learning with errors secure against sub-exponential quantum algorithms, there exists an SSL scheme in the setup model for $C$.*

An input-hiding obfuscator is a compiler that converts a circuit $C$ into another functionally equivalent circuit $\tilde{C}$ such that given $\tilde{C}$ it is computationally hard to find an accepting point. We later show how to instantiate input-hiding obfuscators for searchable compute-and-compare functions from quantum hardness of learning with errors. However, we can envision quantum-secure instantiations of input-hiding obfuscators for more general class of searchable evasive circuits; we leave this problem open. We also leave open the problem of removing searchability condition from our result.
We admittedly use heavy cryptographic hammers to prove our result, but as will be clear in the overview given in the next section, each of these hammers will be necessary to solve the different technical challenges we face.

1.2 Overview of Construction of SSL

To construct an SSL scheme in the setup model (Setup, Gen, Lessor, Run, Check) against arbitrary quantum poly-time (QPT) pirates, we first focus on two weaker class of adversaries, namely, duplicators and maulers. Duplicators are adversaries who, given $\rho_C$ generated by the lessor for a circuit $C$ sampled from a distribution $D_C$, produce $\rho_C^{\otimes 2}$; that is, all they do is replicate the state. Maulers, who given $\rho_C$, output $\rho_C \otimes \rho'_C$, where $\rho'_C$ is far from $\rho_C$ in trace distance and $\rho_C$ is the copy returned by the mauler back to the lessor; that is the second copy it produces is a modified version of the original copy.

While our construction is secure against arbitrary pirates, it will be helpful to first focus on these restricted type of adversaries. We propose two schemes: the first scheme is secure against QPT maulers and the second scheme against QPT duplicators. Once we discuss these schemes, we will then show how to combine the techniques from these two schemes to obtain a construction secure against arbitrary pirates.

SSL against Maulers. To protect SSL against a mauler, we attempt to construct a classical scheme. The reason why it could be possible to construct a classical scheme is because maulers never produce a pirated copy $\rho_C^*$ that is the same as the original copy $\rho_C$ they obtained.

A natural attempt to construct an SSL scheme is to use virtual black-box obfuscation [15] (VBB); this notion is a compiler that transforms a circuit $C$ into another functionally equivalent circuit $\tilde{C}$ such that $\tilde{C}$ only leaks the input-output behavior of $C$ and nothing more. This is a powerful notion and implies most of the cryptographic primitives that exist under the sun. We generate the leased state $\rho_C$ to be the VBB obfuscation of $C$, namely $\tilde{C}$. The hope is that a mauler will not output another leased state $\rho'_C$ that is different from $\tilde{C}$.

Unfortunately, this scheme is insecure. A mauler on input $\tilde{C}$, obfuscates $\tilde{C}$ once more to obtain $\tilde{\tilde{C}}$ and outputs this re-obfuscated circuit. Moreover, note that the resulting re-obfuscated circuit still computes $C$. This suggests that program obfuscation is insufficient for our purpose. In hindsight, this should be unsurprising: VBB guarantees that given an obfuscated circuit, an efficient adversary should not learn anything about the implementation of the circuit, but this doesn’t prevent the adversary from being able to re-produce modified copies of the obfuscated circuit. To rectify this issue, we recognize the following useful properties:

- **Explanability**: this property says that there does not exist any adversary who given $\tilde{C}$ can produce a *different* circuit $\tilde{C}'$ and simultaneously produce $f(C)$, for some fixed function $f$; it will be soon clear what $f$ is.

- **Hardness of Property-Finding**: Given $\tilde{C}$, no efficient adversary can find $f(C)$.

Before explaining how these two properties are useful in constructing an SSL scheme, we first identify the right cryptographic tools to achieve both the above properties.
Explanability via Simulation-Extractable NIZKs [52, 30]: Towards tackling explanability, we consider the primitive of simulation-extractable non-interactive zero-knowledge [52, 30] (seNIZKs). A seNIZK system is a non-interactive protocol between a prover and a verifier with the prover trying to convince the verifier that a statement belongs to the NP language. By non-interactive we mean that the prover only sends one message to the verifier and the verifier is supposed to output the decision bit: accept or reject. Moreover, this primitive is defined in the common reference string model. In this model, there is a trusted setup that produces a common reference string and both the prover and the verifier have access to this common reference string.

As in a traditional interactive protocol, we require a seNIZK to satisfy the completeness property. Simulation-extractability, a property that implies both zero-knowledge and soundness, guarantees that if there exists an efficient adversary $A$ who upon receiving a simulated proof$^5$ for an instance $x$, produces an accepting proof for a different instance $x'$ then there also exists an adversary $B$ that given the same simulated proof produces an accepting proof for $x'$ along with simultaneously producing a valid witness for $x'$.

Simulation-extractability is quite useful to handle explanability. In addition to giving the obfuscation of a circuit $C$, if we attach a proof that proves the knowledge of $f(C)$ (interpreted as an NP witness) then, if the adversary produces a different obfuscated circuit and a corresponding proof, we can use the simulation-extractability property to convert this into a different adversary that also simultaneously produces $f(C)$, as desired.

Hardness of Property-Finding via Input-Hiding Obfuscators [14]: To handle the hardness of property-finding, we use the primitive of input-hiding obfuscators [14]. An input-hiding obfuscator guarantees that given an obfuscated circuit $\tilde{C}$, any efficient adversary cannot find an accepting input $x$, i.e., an input $x$ such that $\tilde{C}(x) = 1$. Of course this notion is only meaningful for an evasive class of functions: a function is evasive if given oracle access to this function, any efficient adversary cannot output an accepting point. The work of Barak et al. [14] propose candidates for input-hiding obfuscators.

To see why this is a relevant tool, let us fix $f$ to be a function that on input $C$ produces an accepting input. Now, if an adversary given an obfuscation of $C$, produces $f(C)$ then this would violate the input-hiding property of the underlying obfuscation scheme.

Combining Simulation-Extractable NIZKs and Input-Hiding Obfuscators: We identified two different tools that solve explanability and hardness of property-finding separately. We now combine these two different techniques to obtain an SSL scheme secure against maulers. Our SSL scheme will be associated with searchable circuits; given a description of searchable circuit $C$, an input $x$ can be efficiently found such that $C(x) = 1$.

To lease a circuit $C$, do the following:

- Compute an input-hiding obfuscation of $C$, denoted by $\tilde{C}$,

$^5$A simulated proof is one that is generated by an efficient algorithm, called a simulator, who has access to some private coins that was used to generate the common reference string. Moreover, a simulated proof is indistinguishable from an honestly generated proof. A simulator has the capability to generate simulated proofs for YES instances even without knowing the corresponding witness for these instances.
• Produce a seNIZK proof \( \pi \) that proves knowledge of an input \( x \) such that \( C(x) = 1 \). Note that we can find this input using the searchability property.

Output \( (\tilde{C}, \pi) \) as the leased circuit. To evaluate on any input \( x \), we first check if \( \pi \) is a valid proof and if so, we compute \( \tilde{C} \) on \( x \) to obtain \( C(x) \).

To see why this scheme is secure against maulers, suppose an adversary \( A \) given \( (\tilde{C}, \pi) \) produces \( (\tilde{C}^*, \pi^*) \), where \( \tilde{C}^* \neq \tilde{C} \). Since \( A \) is a valid mauler we are guaranteed that \( \tilde{C}^* \) is functionally equivalent to \( C \). We first run the seNIZK simulator to simulate \( \pi \) (at this point, we don’t need \( x \) to generate \( \pi \)) instead; the adversary cannot distinguish simulated versus honestly generated proofs. Now, we invoke the simulation-extractability property to convert \( A \) into one who not only produces \( (\tilde{C}^*, \pi^*) \) but also simultaneously produces \( x \) such that \( \tilde{C}^*(x) = 1 \). Since \( \tilde{C}^* \) is functionally equivalent to \( C \), it follows that \( C(x) = 1 \) as well. But this violates the input-hiding property which says that no efficient adversary given \( \tilde{C} \) can produce an accepting input.

**Necessity of sub-exponential security:** There is a subtlety we skipped in the proof above. The maulers that we consider have multi-bit output which is atypical in the cryptographic setting where the focus is mainly on boolean adversaries. This causes an issue when we switch from the honestly generated proof to a simulated proof. Upon receiving the honestly generated proof, \( A \) outputs \( (\tilde{C}^*, \pi^*) \) such that \( \tilde{C}^* \) is functionally equivalent to \( C \) but upon receiving the simulated proof, the adversary outputs \( (\tilde{C}^*, \pi^*) \) where \( \tilde{C}^* \) differs from \( C \) on one point. From \( A \), we need to extract one bit that would help distinguish the real and simulated proofs. To extract this bit, we rely upon sub-exponential security. Given \( \tilde{C}^* \), we run in time \( 2^n \), where \( n \) is the input length, and check if \( \tilde{C}^* \) is still functionally equivalent to \( C \); if indeed \( \tilde{C}^* \) is not functionally equivalent to \( C \) then we know for a fact that the adversary was given a simulated proof, otherwise it received an honestly generated proof. We set the security parameter in the seNIZK system to be sufficiently large (for eg, \( 2^{n+\omega(\log(n))} \)) such that the seNIZK is still secure against adversaries running in time \( 2^n \). This is by now a fairly standard trick employed to make primitives sub-exponentially secure.

**SSL against Duplicators.** Next we focus on constructing SSL secure against duplicators. If our only goal was to protect against duplicators, we could achieve this with a simple scheme. The lessor, in order to lease \( C \), will output \( (|\psi\rangle, C) \) where \( |\psi\rangle \) is a random quantum state generated by applying a random polynomial sized quantum circuit \( U \) on input \( |0^{\lambda}\rangle \). Run on input \( (|\psi\rangle, C, x) \) ignores the quantum state \( |\psi\rangle \), and outputs \( C(x) \). By quantum no-cloning, an attacker cannot output two copies of \( (|\psi\rangle, C) \), which means that this scheme is already secure against duplicators.

Recall that we focused on designing SSL for duplicators in the hope that it will be later helpful for designing SSL for arbitrary pirates. But any SSL scheme in which \( \text{Run} \) ignores the quantum part would not be useful for obtaining SSL secure against arbitrary pirates; an attacker can simply replace the quantum state as part of the leased state with its own quantum state and copy the classical part. To overcome this insufficiency, we need to design SSL schemes where the Run algorithm only computes correctly when the input leased state belongs to a sparse set of quantum states. This suggests that the Run algorithm implicitly satisfies a verifiability property; it should
be able to verify that the input quantum state lies in this sparse set.

**Publicly Verifiable Unclonable States.** We wish to construct a family of efficiently preparable states \(|\psi_s\rangle\) with the following verifiability property. For any state \(|\psi_s\rangle\) in the family, there is a way to sample a classical description \(d_s\) for \(|\psi_s\rangle\) in such a way that it can be verified that \(d_s\) is a corresponding description of \(|\psi_s\rangle\). To be more precise, there should be a verification algorithm \(\text{Ver}(|\psi_s\rangle, d)\) that accepts if \(d\) is a valid description for \(|\psi_s\rangle\). Furthermore, we want the guarantee that given a valid pair \((|\psi_s\rangle, d_s)\), no QPT adversary can produce \(|\psi_s\rangle^{\otimes 2}\).

Our requirement has the same flavor as public-key quantum money, but a key difference is that we are not requiring any secret or public keys, as well as the fact that anyone should be able to generate such tuples \((|\psi_s\rangle, d_s)\), not only a minting authority (bank).

Given such verifiable family, we can define the Run algorithm as follows,

\[
\text{Run}(C, (|\psi_s\rangle, d), x):
\]

- If \(\text{Ver}(|\psi_s\rangle, d) = 0\), output \(\bot\).
- Otherwise, output \(C(x)\).

Any lessor can now lease a state \((|\psi_s\rangle, d_s, C)\), which would allow anyone to compute \(C\) using \(\text{Run}\). Of course, any pirate that is given \((|\psi_s\rangle, d_s, C)\) can prepare their own \((|\psi'_s\rangle, d'_s, C)\) and then input \((|\psi'_s\rangle, d'_s, C)\) into \(\text{Run}\). But recall that we are interested in ruling out duplicators. If the verifiable tuples \((|\psi_s\rangle, d_s)\) have the desired security guarantee, no such pirate could prepare \(|\psi_s\rangle^{\otimes 2}\) from \((|\psi_s\rangle, d_s)\), then no pirate can duplicate the leased state.

**Verifiable Unclonable States from Subspace Hiding Obfuscation.** Zhandry [56], in the context of constructing publicly-verifiable quantum money, shows how to achieve the notion we are looking for from subspace hiding obfuscation, a notion he introduced and constructed from quantum-secure indistinguishability obfuscation [15, 33] (qIO). Roughly speaking, a subspace hiding obfuscator (shO) takes as input a description of a linear subspace \(A\), and outputs a circuit that computes the membership function for \(A\), i.e. \(\text{shO}(A)(x) = 1\) iff \(x \in A\). Zhandry shows that for a uniformly random \(\frac{1}{2}\)-dimensional subspace \(A \subset \mathbb{Z}_q^4\), given \(|A\rangle := \frac{1}{\sqrt{q^{\frac{1}{2}}}} \sum a |a\rangle\) along with \(\tilde{g} \leftarrow \text{shO}(A), \tilde{g}_\perp \leftarrow \text{shO}(A^\perp)\), no QPT algorithm can prepare \(|A\rangle^{\otimes 2}\) with non-negligible probability. Nevertheless, because \(\tilde{g}\) and \(\tilde{g}_\perp\) compute membership for \(A\) and \(A^\perp\) respectively, it is possible to project onto \(|A\rangle\langle A|\) using \((\tilde{g}, \tilde{g}_\perp)\). This lets anyone check the tuple \((|\psi\rangle, (\tilde{g}, \tilde{g}_\perp))\) by measuring \(|\psi\rangle\) with the projectors \(|A\rangle\langle A|, I - |A\rangle\langle A|\).

**Main Template: SSL against Pirates.** Our goal is to construct SSL against arbitrary QPT pirates and not just duplicators or maulers. To achieve this goal, we combine the techniques we have developed so far.

To lease a circuit \(C\), do the following:

1. First prepare the state the state \(|A\rangle = \frac{1}{\sqrt{q^{\frac{1}{2}}}} \sum a |a\rangle\), along with \(\tilde{g} \leftarrow \text{shO}(A)\) and \(\tilde{g}_\perp \leftarrow \text{shO}(A^\perp)\).
2. Compute an input-hiding obfuscation of \(C\), namely \(\tilde{C}\).
3. Let \( x = S(C) \); that is, \( x \) is an accepting point of \( C \).

4. Compute a seNIZK proof \( \pi \) that (1) the obfuscations \((\tilde{g}, \tilde{g}_\perp, \tilde{C})\) were computed correctly (as they should have been) from \((A, A^\perp, C)\), and, (2) \( C(x) = 1 \).

5. Output \(|\psi_C\rangle = (|A\rangle, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi)\).

The Run algorithm on input \((\sigma, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi)\) and \( x \), first checks the proof \( \pi \), and outputs \( \perp \) if it does not accept the proof. If it accepts the proof, it knows that \( \tilde{g} \) and \( \tilde{g}_\perp \) are subspace obfuscators for some subspaces \( A \) and \( A^\perp \) respectively; it can use them to measure \(|A\rangle\langle A|\) on \( \sigma \). This way it checks whether \( \sigma \) is \(|A\rangle \) or not. If it is not, then it outputs \( \perp \). If it has not output \( \perp \) so far, then it computes \( \tilde{C} \) on \( x \) to obtain \( C(x) \).

Intuitively, the above construction would satisfy lessor security because a pirate cannot generate \(|A\rangle^{\otimes 2} \), so if it wants to have two valid copies to evaluate Run successfully, it is forced to come up with a new quantum state \(|B\rangle\). But if it does this, then it also has to change the classical part – specifically, the subspace obfuscated circuits. Otherwise, the Run algorithm will detect that \(|A\rangle\) was not given, and output \( \perp \). However, if the new copy has a different subspace obfuscators, the pirate would have to produce a proof a valid seNIZK proof for a different instance. This is something that the pirate cannot do, because then we could extract an accepting input to \( \tilde{C} \) which violates the simulation-extractability security of seNIZK.

To prove the lessor security of the above scheme, we consider two cases depending on the behavior of the pirate: (since the pirate has to return the original copy, we only focus on the copy he keeps after returning back the original copy)

- **Duplicator**: in this case, the pirate produces a new copy that is of the form \((\sigma^*, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi)\); that is, it has the same classical part as before. Now, we argue that either \( \sigma^* \) is close to \(|A\rangle\langle A|\), in which case it violates the no-cloning theorem or \( \sigma^* \) is far from \(|A\rangle\langle A|\) in which case, the new copy is not going to be functionally equivalent to \( C \). The Run algorithm will detect \((\sigma^*, \tilde{g}, \tilde{g}_\perp)\), and it will output \( \perp \).

- **Mauler**: If the pirate produces a new copy that is of the form \((\sigma^*, \tilde{g}^*, \tilde{g}_\perp^*, \tilde{C}^*, \pi^*)\) such that \((\tilde{g}^*, \tilde{g}_\perp^*, \tilde{C}^*) \neq (\tilde{g}, \tilde{g}_\perp, \tilde{C})\), we can invoke the simulation-extractability property to find an input \( x \) such that \( \tilde{C}^*(x) = 1 \). Since \( \tilde{C}^* \) is assumed to have the same functionality as \( C \), this means that \( C(x) = 1 \). This would contradict the security of input-hiding obfuscation, since any QPT adversary even given \( \tilde{C} \) should not be able to find an accepting input \( x \) such that \( C(x) = 1 \).

### 1.3 Related Work

SSL is an addition to the rapidly growing list of quantum cryptographic primitives with the desirable property of unclonability, and hence impossible to achieve classically. Besides the aforementioned connections to software copy-protection, our work on SSL is related to the following previous works.
Quantum Money and Quantum Lightning. Using quantum mechanics to achieve unforgeability has a history that predates quantum computing itself. Wiesner [55] informally introduced the notion of unforgeable quantum money – unclonable quantum states that can also be (either publicly or privately) verified to be valid states. A few constructions [3, 41, 34, 31, 6] achieved quantum money with various features and very recently, in a breakthrough work, Zhandry [56] shows how to construct publicly-verifiable quantum money from cryptographic assumptions. Zhandry also introduced a stronger notion of quantum money, which he coined quantum lightning, and constructed it from cryptographic assumptions.

Certifiable Deletion and Unclonable Encryption. Unclonability has also been studied in the context of encryption schemes. The work of [36] studies the problem of quantum tamper detection. Alice can use a quantum state to send Bob an encryption of a classical message \( m \) with the guarantee that any eavesdropper could not have cloned the ciphertext. After Bob receives the ciphertext, he can check if the state has been tampered with, and if this is not the case, he would know that a potential eavesdropper did not keep a copy of the ciphertext. In recent work, Broadbent and Lord [26] introduced the notion of unclonable encryption. Roughly speaking, an unclonable encryption allows Alice to give Bob and Charlie an encryption of a classical message \( m \), in the form of a quantum state \( \sigma(m) \), such that Bob and Charlie cannot ‘split’ the state among them.

In a follow-up work, Broadbent and Islam [24], construct a one-time use encryption scheme with certifiable deletion. An encryption scheme has certifiable deletion, if there is an algorithm to check that a ciphertext was deleted. The security guarantee is that if an adversary is in possession of the ciphertext, and it then passes the certification of deletion, the issuer of the encryption can now give the secret key to the adversary. At this point, the adversary still can’t distinguish which plaintext correspond to the ciphertext it was given.

Quantum Obfuscation. Our proof of the impossibility of SSL is inspired by the proof of Barak et al. [13] on the impossibility of VBB for arbitrary functions. Alagic and Fefferman[9] formalized the notion of program obfuscation via quantum tools, defining quantum virtual black-box obfuscation (qVBB) and quantum indistinguishability obfuscation (qiO), as the natural quantum analogues to the respective classical notions (VBB and iO). They also proved quantum analogues of some of the previous impossibility results from [13], as well as provided quantum cryptographic applications from qVBB and qiO.

Quantum One-Time Programs and One-Time Tokens. One natural question to ask is if quantum mechanics alone allows the existence of ‘one-time’ use cryptographic primitives. Quantum One-Time programs, that use only quantum information, are not possible even under computational assumptions [23]. This rules out the possibility of having a copy-protection scheme where a single copy of the software is consumed by the evaluation procedure. Despite the lack of quantum one-time programs, there are constructions of secure ‘one-time’ signature tokens in the oracle models [16] [11]. A quantum token for signatures is a quantum state that would let anyone in possession of it to sign an arbitrary document, but only once. The token is destroyed in the signing process.
Quantum Tomography. Quantum tomography is the task of learning a description of a mixed state $\rho$ given multiple copies of it [39] [46]. One possible way to break SSL (or copy-protection) would be to learn a valid description of the state $\rho_C$ directly from having access to multiple copies of the leased program, $\rho_C^\otimes k$. Indeed, in recent work in this area, Aaronson [5] showed that in order for copy-protection to be possible at all it must be based on computational assumptions.

Recent Work on Copy-Protection. While finishing this manuscript, we became aware of very recent work on copy-protection. Aaronson et al. [8] constructed copy-protection for unlearnable functions relative to a classical oracle. Our work complements their results, since we show that obtaining copy-protection in the standard model (i.e., without oracles) is not possible.

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2 Preliminaries

We assume that the reader is familiar with basic cryptographic notions such as negligible functions and computational indistinguishability (see [35]).

The security parameter is denoted by $\lambda$ and we denote $\text{negl}(\lambda)$ to be a negligible function in $\lambda$. We denote (classical) computational indistinguishability of two distributions $D_0$ and $D_1$ by $D_0 \approx_{\epsilon} D_1$. In the case when $\epsilon$ is negligible, we drop $\epsilon$ from this notation.

2.1 Quantum

For completeness, we present some of the basic quantum definitions, for more details see [45].

Quantum states and channels. Let $\mathcal{H}$ be any finite Hilbert space, and let $L(\mathcal{H}) := \{ \mathcal{E} : \mathcal{H} \to \mathcal{H} \}$ be the set of all linear operators from $\mathcal{H}$ to itself (or endomorphism). Quantum states over $\mathcal{H}$ are the positive semidefinite operators in $L(\mathcal{H})$ that have unit trace, we call these density matrices, and use the notation $\rho$ or $\sigma$ to stand for density matrices when possible. Quantum channels or quantum operations acting on quantum states over $\mathcal{H}$ are completely positive trace preserving (CPTP) linear maps from $L(\mathcal{H})$ to $L(\mathcal{H}')$ where $\mathcal{H}'$ is any other finite dimensional Hilbert space. We use the trace distance, denoted by $\| \rho - \sigma \|_{\text{tr}}$, as our distance measure on quantum states,

$$\| \rho - \sigma \|_{\text{tr}} = \frac{1}{2} \text{Tr} \left[ \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} \right]$$

A state over $\mathcal{H} = \mathbb{C}^2$ is called a qubit. For any $n \in \mathbb{N}$, we refer to the quantum states over $\mathcal{H} = (\mathbb{C}^2)^\otimes n$ as $n$-qubit quantum states. To perform a standard basis measurement on a qubit means projecting the qubit into $\{|0\rangle, |1\rangle\}$. A quantum register is a collection of qubits. A classical register is a quantum register that is only able to store qubits in the computational basis.

A unitary quantum circuit is a sequence of unitary operations (unitary gates) acting on a fixed number of qubits. Measurements in the standard basis can be performed at the end of the unitary circuit. A (general) quantum circuit is a unitary quantum circuit with 2 additional operations: (1) a
gate that adds an ancilla qubit to the system, and (2) a gate that discards (trace-out) a qubit from the system. A quantum polynomial-time algorithm (QPT) is a uniform collection of quantum circuits \( \{C_n\}_{n \in \mathbb{N}} \). We always assume that the QPT adversaries are non-uniform – a QPT adversary \( \mathcal{A} \) acting on \( n \) qubits could be given a quantum auxiliary state with \( \text{poly}(n) \) qubits.

**Quantum Computational Indistinguishability.** When we talk about quantum distinguishers, we need the following definitions, which we take from [53].

**Definition 6 (Indistinguishable collections of states).** Let \( I \) be an infinite subset \( I \subset \{0,1\}^* \), let \( p : \mathbb{N} \to \mathbb{N} \) be a polynomially bounded function, and let \( \rho_x \) and \( \sigma_x \) be \( p(|x|) \)-qubit states. We say that \( \{\rho_x\}_{x \in I} \) and \( \{\sigma_x\}_{x \in I} \) are quantum computationally indistinguishable collections of quantum states if for every QPT \( \mathcal{E} \) that outputs a single bit, any polynomially bounded \( q : \mathbb{N} \to \mathbb{N} \), and any auxiliary \( q(|x|) \)-qubits state \( \nu \), and for all \( x \in I \), we have that

\[
\left| \Pr[\mathcal{E}(\rho_x \otimes \nu) = 1] - \Pr[\mathcal{E}(\sigma_x \otimes \nu) = 1] \right| \leq \epsilon(|x|)
\]

for some function \( \epsilon : \mathbb{N} \to [0,1] \). We use the following notation

\[
\rho_x \approx_{Q,E} \sigma_x
\]

and we ignore the \( \epsilon \) when it is understood that it is a negligible function.

**Definition 7 (Indistinguishability of channels).** Let \( I \) be an infinite subset \( I \subset \{0,1\}^* \), let \( p, q : \mathbb{N} \to \mathbb{N} \) be polynomially bounded functions, and let \( \mathcal{D}_x, \mathcal{F}_x \) be quantum channels mapping \( p(|x|) \)-qubit states to \( q(|x|) \)-qubit states. We say that \( \{\mathcal{D}_x\}_{x \in I} \) and \( \{\mathcal{F}_x\}_{x \in I} \) are quantum computationally indistinguishable collection of channels if for every QPT \( \mathcal{E} \) that outputs a single bit, any polynomially bounded \( t : \mathbb{N} \to \mathbb{N} \), any \( p(|x|) + t(|x|) \)-qubit quantum state \( \rho \), and for all \( x \in I \), we have that

\[
\left| \Pr[\mathcal{E}(\mathcal{D}_x \otimes \text{Id})(\rho) = 1] - \Pr[\mathcal{E}(\mathcal{F}_x \otimes \text{Id})(\rho) = 1] \right| \leq \epsilon(|x|)
\]

for some function \( \epsilon : \mathbb{N} \to [0,1] \). We will use the following notation

\[
\mathcal{D}_x(\cdot) \approx_{Q,E} \mathcal{F}_x(\cdot)
\]

and we ignore the \( \epsilon \) when it is understood that it is a negligible function.

**Quantum Fourier Transform and Subspaces.** Our main construction uses the same type of quantum states (superpositions over linear subspaces) considered by [7, 56] in the context of constructing quantum money.

We recall some key facts from these works relevant to our construction. Consider the field \( \mathbb{Z}_q^\lambda \) where \( q \geq 2 \), and let FT denote the quantum fourier transform over \( \mathbb{Z}_q^\lambda \).

For any linear subspace \( A \), let \( A^\perp \) denote its orthogonal (dual) subspace,

\[
A^\perp = \{ v \in \mathbb{Z}_q^\lambda | \langle v, a \rangle = 0 \}.
\]

Let \( |A\rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} |a\rangle \). The quantum fourier Transform, FT, does the following:

\[
\text{FT}|A\rangle = |A^\perp\rangle.
\]
Since $(A^\perp)^\perp = A$, we also have $\text{FT}|A^\perp⟩ = |A⟩$.

Let $\Pi_A = \sum_{a \in A} |a⟩⟨a|$, then as shown in Lemma 21 of [7],

$$\text{FT}(\Pi_A^\perp)\text{FT}|A⟩ = |A⟩⟨A|.$$  

**Almost As Good As New Lemma.** We use the Almost As Good As New Lemma [2], restated here verbatim from [4].

**Lemma 8** (Almost As Good As New). Let $\rho$ be a mixed state acting on $C^d$. Let $U$ be a unitary and $(\Pi_0, \Pi_1 = 1-\Pi_0)$ be projectors all acting on $C^d \otimes C^d$. We interpret $(U, \Pi_0, \Pi_1)$ as a measurement performed by appending an ancillary system of dimension $d'$ in the state $|0⟩⟨0|$, applying $U$ and then performing the projective measurement $\{\Pi_0, \Pi_1\}$ on the larger system. Assuming that the outcome corresponding to $\Pi_0$ has probability $1 - \varepsilon$, i.e., $\text{Tr}[\Pi_0(U\rho \otimes |0⟩⟨0|U^\dagger)] = 1 - \varepsilon$, we have

$$\|\rho - \tilde{\rho}\|_F \leq \sqrt{\varepsilon},$$

where $\tilde{\rho}$ is state after performing the measurement and then undoing the unitary $U$ and tracing out the ancillary system:

$$\tilde{\rho} = \text{Tr}_{d'} \left(U^\dagger \left(\Pi_0 U (\rho \otimes |0⟩⟨0|) U^\dagger \Pi_0 + \Pi_1 U (\rho \otimes |0⟩⟨0|) U^\dagger \Pi_1\right) U\right)$$

We use this Lemma to argue that whenever a QPT algorithm $\mathcal{A}$ on input $\rho$, outputs a particular bit string $z$ with probability $1 - \varepsilon$, then $\mathcal{A}$ can be performed in a way that also lets us recover the initial state. In particular, given the QPT description for $\mathcal{A}$, we can implement $\mathcal{A}$ with an ancillary system, a unitary, and only measuring in the computational basis after the unitary has been applied, similarly to Lemma 8. Then, it is possible to uncompute in order to also obtain $\tilde{\rho}$.

**Notation about Quantum-Secure Classical Primitives.** For a classical primitive $X$, we use the notation $q$-$X$ to denote the fact that we assume $X$ to be secure against QPT adversaries.

### 2.2 Learning with Errors

We consider the decisional learning with errors (LWE) problem, introduced by Regev [51]. We define this problem formally below.

The problem $(n, m, q, \chi)$-LWE, where $n, m, q \in \mathbb{N}$ and $\chi$ is a distribution supported over $\mathbb{Z}$, is to distinguish between the distributions $(A, As + e)$ and $(A, u)$, where $A \sim \mathbb{Z}_q^{m \times n}, s \sim \mathbb{Z}_q^{n \times 1}, e \sim \chi^{m \times 1}$ and $u \sim \mathbb{Z}_q^{m \times 1}$.

The above problem has been believed to be hard against classical PPT algorithms – also referred to as LWE assumption – has had many powerful applications in cryptography. In this work, we conjecture the above problem to be hard even against QPT algorithms; this conjecture referred to as QLWE assumption has been useful in the constructions of interesting primitives such as quantum fully-homomorphic encryption [42, 20]. We refer to this assumption as QLWE assumption.
QLWE assumption: This assumption is parameterized by $\lambda$. Let $n = \text{poly}(\lambda)$, $m = \text{poly}(n \cdot \log(q))$ and $\chi$ be a discrete Gaussian distribution\(^6\) with parameter $\alpha q > 0$, where $\alpha$ can set to be any non-negative number.

Any QPT distinguisher (even given access to polynomial-sized advice state) can solve $(n, m, q, \chi)$-LWE only with probability $\text{negl}(\lambda)$, for some negligible function $\text{negl}$.

Remark 9. We drop the notation $\lambda$ from the description of the assumption when it is clear.

$(n, m, q, \chi)$-LWE is shown [51, 47] to be as hard as approximating shortest independent vector problem (SIVP) to within a factor of $\gamma = \tilde{O}(n/\alpha)$ (where $\alpha$ is defined above). The best known quantum algorithms for this problem run in time $2^{\tilde{O}(n/\log(\gamma))}$.

For our construction of SSL, we require a stronger version of QLWE that is secure even against sub-exponential quantum adversaries. We state this assumption formally below.

**T-Sub-exponential QLWE Assumption:** This assumption is parameterized by $\lambda$ and time $T$. Let $n = T + \text{poly}(\lambda)$, $m = \text{poly}(n \cdot \log(q))$ and $\chi$ be a discrete Gaussian distribution with parameter $\alpha q > 0$, where $\alpha$ can set to be any non-negative number.

Any quantum distinguisher (even given access to polynomial-sized advice state) running in time $2^{\tilde{O}(T)}$ can solve $(n, m, q, \chi)$-LWE only with probability $\text{negl}(\lambda)$, for some negligible function $\text{negl}$.

2.3 Quantum Fully Homomorphic Encryption

A fully homomorphic encryption scheme allows for publicly evaluating an encryption of $x$ using a function $f$ to obtain an encryption of $f(x)$. Traditionally $f$ has been modeled as classical circuits but in this work, we consider the setting when $f$ is modeled as quantum circuits and when the messages are quantum states. This notion is referred to as quantum fully homomorphic encryption (QFHE). We state our definition verbatim from [25].

**Definition 10.** Let $\mathcal{M}$ be the Hilbert space associated with the message space (plaintexts), $\mathcal{C}$ be the Hilbert space associated with the ciphertexts, and $\mathcal{R}_{evk}$ be the Hilbert space associated with the evaluation key. A quantum fully homomorphic encryption scheme is a tuple of QPT algorithms $\text{QFHE} = (\text{Gen}, \text{Enc}, \text{Dec}, \text{Eval})$ satisfying

- **QFHE.Gen($1^\lambda$):** outputs a a public and a secret key, $(pk, sk)$, as well as a quantum state $\rho_{evk}$, which can serve as an evaluation key.
- **QFHE.Enc($pk, \cdot$) ($L(\mathcal{M}) \rightarrow L(\mathcal{C})$):** takes as input a state $\rho$ and outputs a ciphertext $\sigma$
- **QFHE.Dec($sk, \cdot$) ($L(\mathcal{C}) \rightarrow L(\mathcal{M})$):** takes a quantum ciphertext $\sigma$, and outputs a qubit $\rho$ in the message space $L(\mathcal{M})$.
- **QFHE.Eval($E, \cdot$) ($L(\mathcal{R}_{evk} \otimes C^\otimes n) \rightarrow L(C^\otimes m)$):** takes as input a quantum circuit $E : L(M^\otimes n) \rightarrow L(M^\otimes m)$, and a ciphertext in $L(C^\otimes n)$ and outputs a ciphertext in $L(C^\otimes m)$, possibly consuming the evaluation key $\rho_{evk}$ in the process.

\(^6\)Refer [20] for a definition of discrete Gaussian distribution.
Semantic security and compactness are defined analogously to the classical setting, and we defer to \[\text{two.prop/five.prop}\] for a definition. For the impossibility result, we require a \(\text{QFHE}\) scheme where ciphertexts of classical plaintexts are also classical. Given any \(x \in \{0, 1\}\), we want \(\text{QFHE} \cdot \text{Enc}_{pk}(|x\rangle\langle x|)\) to be a computational basis state \(|z\rangle\langle z|\) for some \(z \in \{0, 1\}\) (here, \(l\) is the length of ciphertexts for 1-bit messages). In this case, we write \(\text{QFHE} \cdot \text{Enc}_{pk}(x)\). We also want the same to be true for evaluated ciphertexts, i.e. if \(E(|x\rangle\langle x|) = |y\rangle\langle y|\) for some \(x \in \{0, 1\}^n\) and \(y \in \{0, 1\}^m\), then

\[
\text{QFHE} \cdot \text{Enc}_{pk}(y) \leftarrow \text{QFHE} \cdot \text{Eval}(\rho_{v k}, E, \text{QFHE} \cdot \text{Enc}_{pk}(x))
\]
is a classical ciphertext of \(y\).

**Instantiation.** The works of \[\text{four.prop/two.prop}, \text{two.prop/zero.prop}\] give lattice-based candidates for quantum fully homomorphic encryption schemes; we currently do not know how to base this on learning with errors alone\(^7\). The desirable property required from the quantum FHE schemes, that classical messages have classical ciphertexts, is satisfied by both candidates \[\text{four.prop/two.prop}, \text{two.prop/zero.prop}\].

### 2.4 Circuit Class of Interest: Evasive Circuits

The circuit class we consider in our construction of SSL is a subclass of evasive circuits. We recall the definition of evasive circuits below.

**Evasive Circuits.** Informally, a class of circuits is said to be evasive if a circuit drawn from a suitable distribution outputs 1 on a fixed point with negligible probability.

**Definition 11 (Evasive Circuits).** A class of circuits \(C = \{C_\lambda\}_{\lambda \in \mathbb{N}}\), associated with a distribution \(D_C\), is said to be **evasive** if the following holds: for every \(\lambda \in \mathbb{N}\), every \(x \in \{0, 1\}^{\text{Poly}(\lambda)}\),

\[
\Pr_{C \leftarrow D_C} [C(x) = 1] \leq \text{negl}(\lambda),
\]

**Compute-and-compare Circuits.** The subclass of circuits that we are interested in is called compute-and-compare circuits, denoted by \(C_{\text{cnc}}\). A compute-and-compare circuit is of the following form: \(C[C, \alpha]\), where \(\alpha\) is called a lock and \(C\) has output length \(|\alpha|\), is defined as follows:

\[
C[C, \alpha](x) = \begin{cases} 1, & \text{if } C(x) = \alpha, \\ 0, & \text{otherwise} \end{cases}
\]

**Multi-bit compute-and-compare circuits.** We can correspondingly define the notion of multi-bit compute-and-compare circuits. A multi-bit compute-and-compare circuit is of the following form:

\[
C[C, \alpha, \text{msg}](x) = \begin{cases} \text{msg}, & \text{if } C(x) = \alpha, \\ 0, & \text{otherwise} \end{cases}
\]

where \(\text{msg}\) is a binary string.

We consider two types of distributions as defined by \[\text{five.prop/four.prop}\].

\(^7\)Brakerski \[\text{two.prop/zero.prop}\] remarks that the security of their candidate can be based on a circular security assumption that is also used to argue the security of existing constructions of unbounded depth multi-key FHE \[\text{two.prop/seven.prop}, \text{four.prop/four.prop}, \text{four.prop/eight.prop}, \text{two.prop/one.prop}\].
**Definition 12** (Distributions for Compute-and-Compare Circuits). We consider the following distributions on $C_{cnc}$:

- $D_{\text{unpred}}(\lambda)$: For any $(C[\alpha])$ along with aux sampled from this unpredictable distribution, it holds that $\alpha$ is computationally unpredictable given $(C, \text{aux})$.

- $D_{\text{pseud}}(\lambda)$: For any $C[\alpha]$ along with aux sampled from this distribution, it holds that $H_{\text{HILL}}(\alpha|(C, \text{aux})) \geq \lambda^\epsilon$, for some constant $\epsilon > 0$, where $H_{\text{HILL}}(\cdot)$ is the HILL entropy [40].

Note that with respect to the above distributions, the compute-and-compare class of circuits $C_{cnc}$ is evasive.

**Searchability.** For our construction of SSL for $C$, we crucially use the fact that given a circuit $C \in C$, we can read off an input $x$ from the description of $C$ such that $C(x) = 1$. We formalize this by defining a search algorithm $S$ that on input a circuit $C$ outputs an accepting input for $C$. For many interesting class of functions, there do exist a corresponding efficiently implementable class of circuits associated with a search algorithm $S$.

**Definition 13** (Searchability). A class of circuits $C = \{C_i\}_{i \in \mathbb{N}}$ is said to be $S$-searchable, with respect to a PPT algorithm $S$, if the following holds: on input $C$, $S(C)$ outputs $x$ such that $C(x) = 1$.

**Searchable Compute-and-Compare Circuits: Examples.** As mentioned in the introduction, there are natural and interesting classes of searchable compute-and-compare circuits. For completeness, we state them again below with additional examples [54].

- Point circuits $C(\alpha, \cdot)$: the circuit $C(\alpha, \cdot)$ is a point circuit if it takes as input $x$ and outputs $C(\alpha, x) = 1$ iff $x = \alpha$. If we define the class of point circuits suitably, we can find $\alpha$ directly from $C_\alpha$; for instance, $\alpha$ can be the value assigned to the input wires of $C$.

- Conjunctions with wild cards $C(S, \alpha, \cdot)$: the circuit $C(S, \alpha, \cdot)$ is a conjunction with wild cards if it takes as input $x$ and outputs $C(S, \alpha, x) = 1$ iff $y = \alpha$, where $y$ is such that $y_i = x_i$ for all $i \in S$. Again, if we define this class of circuits suitably, we can find $S$ and $\alpha$ directly from the description of $C(S, \alpha, \cdot)$. Once we find $S$ and $\alpha$, we can find the accepting input.

- Affine Tester: the circuit $C(A, \alpha, \cdot)$ is an affine tester, with $A, y$ where $A$ has a non-trivial kernel space, if it takes as input $x$ and outputs $C(A, \alpha, x) = 1$ iff $A \cdot x = \alpha$. By reading off $A$ and $\alpha$ and using Gaussian elimination we can find $x$ such that $A \cdot x = \alpha$.

- Plaintext equality checker $C(sk, \alpha, \cdot)$: the circuit $C(sk, \alpha, \cdot)$, with hardwired values decryption key $sk$ associated with a private key encryption scheme, message $\alpha$, is a plaintext equality checker if it takes as input a ciphertext $ct$ and outputs $C(sk, \alpha, ct) = 1$ iff the decryption of $ct$ with respect to $sk$ is $\alpha$. By reading off $\alpha$ and $sk$, we can find a ciphertext such that $ct$ is an encryption of $\alpha$.

**Remark 14.** We note that both the candidate constructions of copy-protection for point functions by Aaronson [3] use the fact that the accepting point of the point function is known by whoever is generating the copy-protected circuit.
2.5 Obfuscation

In this work, we use different notions of cryptographic obfuscation. We review all the required notions below, but first we recall the functionality of obfuscation.

**Definition 15 (Functionality of Obfuscation).** Consider a class of circuits $C$. An obfuscator $O$ consists of two PPT algorithms $\text{Obf}$ and $\text{Eval}$ such that the following holds: for every $\lambda \in \mathbb{N}$, circuit $C \in C$, $x \in \{0, 1\}^{\text{poly}(\lambda)}$, we have $C(x) \leftarrow \text{Eval}(\tilde{C}, x)$ where $\tilde{C} \leftarrow \text{Obf}(1^\lambda, C)$.

2.5.1 Lockable Obfuscation

In the impossibility result, we will make use of program obfuscation schemes that are (i) defined for compute-and-compare circuits and, (ii) satisfy distributional virtual black box security notion [15]. Such obfuscation schemes were first introduced by [54, 38] and are called lockable obfuscation schemes. We recall their definition, adapted to quantum security, below.

**Definition 16 (Quantum-Secure Lockable Obfuscation).** An obfuscation scheme $(\text{LO.Obf}, \text{LO.Eval})$ for a class of circuits $C$ is said to be a quantum-secure lockable obfuscation scheme if the following properties are satisfied:

- It satisfies the functionality of obfuscation.
- **Compute-and-compare circuits:** Each circuit $C$ in $C$ is parameterized by strings $\alpha \in \{0, 1\}^{\text{poly}(\lambda)}$, $\beta \in \{0, 1\}^{\text{poly}(\lambda)}$ and a poly-sized circuit $\tilde{C}$ such that on every input $x$, $C(x)$ outputs $\beta$ if and only if $C(x) = \alpha$.
- **Security:** For every polynomial-sized circuit $C$, string $\beta \in \{0, 1\}^{\text{poly}(\lambda)}$ for every QPT adversary $\mathcal{A}$ there exists a QPT simulator $\text{Sim}$ such that the following holds: sample $\alpha \leftarrow \{0, 1\}^{\text{poly}(\lambda)}$,

$$\left\{\text{LO.Obf}(1^\lambda, C)\right\} \approx_{Q, \varepsilon} \left\{\text{Sim}(1^\lambda, 1^{\text{poly}(\lambda)})\right\},$$

where $C$ is a circuit parameterized by $C, \alpha, \beta$ with $\varepsilon \leq \frac{1}{2^m}$.

**Instantiation.** The works of [54, 38, 37] construct a lockable obfuscation scheme based on polynomial-security of learning with errors (see Section 2.2). Since learning with errors is conjectured to be hard against QPT algorithms, the obfuscation schemes of [54, 38, 37] are also secure against QPT algorithms. Thus, we have the following theorem.

**Theorem 17 ([38, 54, 37]).** Assuming quantum hardness of learning with errors, there exists a quantum-secure lockable obfuscation scheme.

2.5.2 q-Input-Hiding Obfuscators

One of the main tools used in our construction is q-input-hiding obfuscators. The notion of input-hiding obfuscators was first defined in the classical setting by Barak et al. [14]. We adopt the same notion except that we require the security of the primitive to hold against QPT adversaries.
The notion of q-input-hiding obfuscators states that given an obfuscated circuit, it should be infeasible for a QPT adversary to find an accepting input; that is, an input on which the circuit outputs 1. Note that this notion is only meaningful for the class of evasive circuits.

The definition below is suitably adapted from Barak et al. [14]; in particular, our security should hold against QPT adversaries.

**Definition 18 (q-Input-Hiding Obfuscators [14]).** An obfuscator $qIHO = (Obf, Eval)$ for a class of circuits associated with distribution $D_C$ is **q-input-hiding** if for every non-uniform QPT adversary $A$, for every sufficiently large $\lambda \in \mathbb{N}$,

$$\Pr \left[ \begin{array}{c}
C(x) = 1 : \tilde{C} \leftarrow Obf(1^\lambda, C), \\
x \leftarrow A(1^\lambda, \tilde{C})
\end{array} \right] \leq \text{negl}(\lambda).$$

### 2.5.3 Subspace Hiding Obfuscators

Another ingredient in our construction is subspace hiding obfuscation. Subspace hiding obfuscation is a notion of obfuscation introduced by Zhandry [56], as a tool to build public-key quantum money schemes. This notion allows for obfuscating a circuit, associated with subspace $A$, that checks if an input vector belongs to this subspace $A$ or not. In terms of security, we require that the obfuscation of this circuit is indistinguishable from obfuscation of another circuit that tests membership of a larger random (and hidden) subspace containing $A$.

**Definition 19 ([56]).** A subspace hiding obfuscator for a field $F$ and dimensions $d_0, d_1, \lambda$ is a tuple $(shO.Obf, shO.Eval)$ satisfying:

- **shO.Obf($A$):** on input an efficient description of a linear subspace $A \subseteq F^\lambda$ of dimensions $d \in \{d_0, d_1\}$ outputs an obfuscator $shO(A)$.

- **Correctness:** For any $A$ of dimension $d \in \{d_0, d_1\}$, it holds that

  $$\Pr[\forall x, shO.Eval(shO.Obf(A), x) = 1_A(x) : shO(A) \leftarrow shO.Obf(A)] \geq 1 - \text{negl}(\lambda),$$

  where: $1_A(x) = 1$ if $x \in A$ and 0, otherwise.

- **Quantum-Security:** Any QPT adversary $A$ can win the following challenge with probability at most negligibly greater than $\frac{1}{2}$.

  1. $A$ chooses a $d_0$-dimensional subspace $A \subseteq F^\lambda$.
  2. Challenger chooses uniformly at random a $d_1$-dimensional subspace $S \supseteq A$. It samples a random bit $b$. If $b = 0$, it sends $\tilde{g}_0 \leftarrow shO.Obf(A)$. Otherwise, it sends $\tilde{g}_1 \leftarrow shO.Obf(S)$
  3. $A$ receives $\tilde{g}_b$ and outputs $b'$. It wins if $b' = b$.

**Instantiation.** Zhandry presented a construction of subspace obfuscators from indistinguishability obfuscation [15, 33] secure against QPT adversaries.
2.6 q-Simulation-Extractable Non-Interactive Zero-Knowledge

We also use the tool of non-interactive zero-knowledge (NIZK) systems for NP for our construction. A NIZK is defined between a classical PPT prover \( P \) and a verifier \( V \). The goal of the prover is to convince the verifier \( V \) to accept an instance \( x \) using a witness \( w \) while at the same time, not revealing any information about \( w \). Moreover, any malicious prover should not be able to falsely convince the verifier to accept a NO instance. Since we allow the malicious parties to be QPT, we term this NIZK as qNIZK.

We require the qNIZKs to satisfy a stronger property called simulation extractability and we call a qNIZK satisfying this stronger property to be q-simulation-extractable NIZK (qseNIZK).

We describe the PPT algorithms of qseNIZK below.

- \( \text{CRSGen}(1^\lambda) \): On input security parameter \( \lambda \), it outputs the common reference string \( \text{crs} \).
- \( P(\text{crs}, x, w) \): On input common reference string \( \text{crs} \), NP instance \( x \), witness \( w \), it outputs the proof \( \pi \).
- \( V(\text{crs}, x, \pi) \): On input common reference string \( \text{crs} \), instance \( x \), proof \( \pi \), it outputs accept or reject. This is a deterministic algorithm.

This notion is associated with the following properties. We start with the standard notion of completeness.

**Definition 20 (Completeness).** A non-interactive protocol qseNIZK for a NP language \( L \) is said to be complete if the following holds: for every \( (x, w) \in \mathcal{R}(L) \), we have the following:

\[
\Pr \left[ V(\text{crs}, x, \pi) \text{ accepts } : \begin{array}{c}
\text{crs} \leftarrow \text{CRSGen}(1^\lambda) \\
\pi \leftarrow P(\text{crs}, x, w)
\end{array} \right] = 1
\]

**q-Simulation-Extractability.** We now describe the simulation-extractability property. Suppose there exists an adversary who upon receiving many proofs \( \pi_1, \ldots, \pi_q \) on all YES instances \( x_1, \ldots, x_q \), can produce a proof \( \pi' \) on instance \( x' \) such that: (a) \( x' \) is different from all the instances \( x_1, \ldots, x_q \) and, (b) \( \pi' \) is accepting with probability \( \epsilon \). Then, this notion guarantees the existence of two efficient algorithms Sim\(_1\) and Sim\(_2\) such that all the proofs \( \pi_1, \ldots, \pi_q \), are now simulated by Sim\(_1\), and Sim\(_2\) can extract a valid witness for \( x' \) from \( (x', \pi') \) produced by the adversary with probability negligibly close to \( \epsilon \).

**Definition 21 (q-Simulation-Extractability).** A non-interactive protocol qseNIZK for a language \( L \) is said to satisfy q-simulation-extractability if there exists a non-uniform QPT adversary \( A = (\mathcal{A}_1, \mathcal{A}_2) \) such that the following holds:

\[
\Pr \left[ \begin{array}{c}
V(\text{crs}, x', \pi') \text{ accepts} \\
\wedge \\
(V \in \{q\}, (x_i, w_i) \in \mathcal{R}(L)) : \\
\wedge \\
(V \in \{q\}, x' \neq x_i)
\end{array} : \\
\begin{array}{c}
\text{crs} \leftarrow \text{CRSGen}(1^\lambda), \\
\pi_i \leftarrow \mathcal{A}_1(\text{crs}) \\
V \in \{q\}, \pi_i \leftarrow P(\text{crs}, td, x_i) \\
(x', \pi') \leftarrow \mathcal{A}_2(\text{crs}, \pi_1, \ldots, \pi_q)
\end{array} \right] = \epsilon
\]
Then there exists QPT algorithms $\text{FkGen}$ and $\text{Sim} = (\text{Sim}_1, \text{Sim}_2)$ such that the following holds:

$$\Pr \left[ \begin{array}{c}
\forall (\text{crs}, x') \in \mathcal{R}(L), \exists (\text{ts}, \pi') \in \mathcal{R}(L) \setminus \mathcal{R}(L) \\
\forall i \in \{q\} \\
\forall (i, w) \in \mathcal{R}(L), \forall i \in \{q\}, \forall \pi \in \mathcal{P}(\text{crs}, x, \pi) \\
\forall (i, w) \in \mathcal{R}(L), \forall i \in \{q\}, \exists \pi \end{array} \right] \leq \text{negl}(\lambda)$$

We call a non-interactive argument system satisfying q-simulation-extractability property to be a qseNIZK system.

If q-simulation-extractability property holds against quantum adversaries running in time $2^{\tilde{O}(T)}$ ($\tilde{O}()$ notation suppresses additive factors in $O(\log(\lambda))$) then we say that $(\text{CRSGen}, \mathcal{P}, \mathcal{V})$ is a $T$-sub-exponential qseNIZK system.

**Remark 22.** The definition as stated above is weaker compared to other definitions of simulation-extractability considered in the literature. For instance, we can consider general adversaries who also can obtain simulated proofs for false statements which is disallowed in the above setting. Nonetheless, the definition considered above is sufficient for our application.

**Instantiation of qseNIZKs.** In the classical setting, simulation-extractable NIZKs can be obtained by generically [52, 30] combining a traditional NIZK (satisfying completeness, soundness and zero-knowledge) with a public-key encryption scheme satisfying CCA2 security. We observe that the same transformation can be ported to the quantum setting as well, by suitably instantiating the underlying primitives to be quantum-secure. These primitives in turn can be instantiated from QLWE. Thus, we can obtain a q-simulation-extractable NIZK from QLWE.

For our construction of SSL, it turns out that we need a q-simulation-extractable NIZK that is secure against quantum adversaries running in sub-exponential time. Fortunately, we can still adapt the same transformation but instead instantiating the underlying primitives to be sub-exponentially secure.

Before we formalize this theorem, we first state the necessary preliminary background.

**Definition 23 (q-Non-Interactive Zero-Knowledge).** A non-interactive system $(\text{CRSGen}, \mathcal{P}, \mathcal{V})$ defined for a NP language $L$ is said to be **q-non-interactive zero-knowledge (qNIZK)** if it satisfies Definition 20 and additionally, satisfies the following properties:

- **Adaptive Soundness:** For any malicious QPT prover $\mathcal{P}^*$, the following holds:

$$\Pr \left[ \begin{array}{c}
\forall (\text{crs}, x, \pi) \in \mathcal{R}(L), \exists (\text{ts}, \pi') \in \mathcal{R}(L) \\
\forall i \in \{q\}, \forall (i, w) \in \mathcal{R}(L) \\
\forall i \in \{q\}, \forall \pi \in \mathcal{P}(\text{crs}, x, \pi) \\
\forall (i, w) \in \mathcal{R}(L), \forall i \in \{q\}, \exists \pi \end{array} \right] \leq \text{negl}(\lambda)$$

- **Adaptive (Multi-Theorem) Zero-knowledge:** For any QPT verifier $\mathcal{V}^*$, there exists two QPT algorithms $\text{FkGen}$ and simulator $\text{Sim}$, such that the following holds:

$$\Pr \left[ \begin{array}{c}
\forall (\text{crs}, x, \pi) \in \mathcal{R}(L), \exists (\text{ts}, \pi') \in \mathcal{R}(L) \\
\forall i \in \{q\}, \forall (i, w) \in \mathcal{R}(L) \\
\forall i \in \{q\}, \forall \pi \in \mathcal{P}(\text{crs}, x, \pi) \\
\forall (i, w) \in \mathcal{R}(L), \forall i \in \{q\}, \exists \pi \end{array} \right] \leq \text{negl}(\lambda)$$
If both adaptive soundness and adaptive multi-theorem zero-knowledge holds against quantum adversaries running in time \(2^{\tilde{O}(T)}\) then we say that \((\text{CRSGen}, \mathcal{P}, \mathcal{V})\) is a T-sub-exponential qNIZK.

Remark 24. q-simulation-extractable NIZKs imply qNIZKs since simulation-extractability implies both soundness and zero-knowledge properties.

Definition 25 (q-CCA-secure PKE). A public-encryption scheme \((\text{Setup}, \text{Enc}, \text{Dec})\) (defined below) is said to satisfy q-CCA-security if every QPT adversary \(A\) wins in \(\text{Expt}_A\) (defined below) only with negligible probability.

- **Setup**\((1^\lambda)\): On input security parameter \(\lambda\), output a public key \(pk\) and a decryption key \(sk\).
- **Enc**\((pk, x)\): On input public-key \(pk\), message \(x\), output a ciphertext \(ct\).
- **Dec**\((sk, ct)\): On input decryption key \(sk\), ciphertext \(ct\), output \(y\).

For any \(x \in \{0, 1\}^{\text{poly}(\lambda)}\), we have \(\text{Dec}(sk, \text{Enc}(pk, x)) = x\).

\(\text{Expt}_A(1^\lambda, b)\):

- **Challenger** generates \(\text{Setup}(1^\lambda)\) to obtain \((pk, sk)\). It sends \(pk\) to \(A\).
- \(A\) has (classical) access to a decryption oracle that on input \(ct\), outputs \(\text{Dec}(sk, ct)\). It can make polynomially many queries.
- \(A\) then submits \((x_0, x_1)\) to the challenger which then returns \(ct^* \leftarrow \text{Enc}(pk, x_b)\).
- \(A\) is then given access to the same oracle as before. The only restriction on \(A\) is that it cannot query \(ct^*\).
- Output \(b'\) where the output of \(A\) is \(b'\).

\(A\) wins in \(\text{Expt}_A\) with probability \(\mu(\lambda)\) if \(\Pr[b = b' : b \xleftarrow{\{0,1\}} \text{Expt}_A(1^\lambda)] = \frac{1}{2} + \mu(\lambda)\).

If the above q-CCA security holds against quantum adversaries running in time \(2^{\tilde{O}(T)}\) then we say that \((\text{Setup}, \text{Enc}, \text{Dec})\) is a T-sub-exponential q-CCA-secure PKE scheme.

Remark 26. One could also consider the setting when the CCA2 adversary has superposition access to the oracle. However, for our construction, it suffices to consider the setting when the adversary only has classical access to the oracle.

Consider the following lemma.

Lemma 27. Consider a language \(L_\ell \in \mathsf{NP}\) such that every \(x \in L_\ell\) is such that \(|x| = \ell\).

Under the \(\ell\)-sub-exponential QLWE assumption, there exists a q-simulation-extractable NIZKs for \(L_\ell\) satisfying perfect completeness.

Proof: We first state the following proposition that shows how to generically construct a q-simulation-extractable NIZK from qNIZK and a CCA2-secure public-key encryption scheme.
**Proposition 28.** Consider a language $\mathcal{L}_\ell \in \text{NP}$ such that every $x \in \mathcal{L}_\ell$ is such that $|x| = \ell$.

Assuming $\ell$-sub-exponential qNIZKs for NP and $\ell$-sub-exponential q-CCA2-secure PKE schemes, there exists a $\ell$-sub-exponential qseNIZK system for $\mathcal{L}_\ell$.

**Proof.** Let qPKE be a $\ell$-sub-exponential qCCA2-secure PKE scheme. Let qNIZK be a $\ell$-sub-exponential qNIZK for the following relation.

\[ \mathcal{R}_{\text{qNIZK}} = \{((pk, ct_w, x), (w, r_w)) : (x, w) \in \mathcal{R}(\mathcal{L}_\ell) \land ct_w = \text{Enc}(pk, (x, w); r_w)\} \]

We present the construction (quantum analogue of algorithms $F_{k\text{Gen}}$) of q-simulation-extractable NIZK for $\mathcal{L}_\ell$ below.

- **CRSGen$(1^\lambda)$:** On input security parameter $\lambda$,
  - Compute qNIZK.CRSGen$(1^{\lambda_1})$ to obtain qNIZK.crs, where $\lambda_1 = \text{poly}(\lambda, \ell)$ is chosen such that qNIZK is a $\ell$-sub-exponential q-non-interactive zero-knowledge argument system.
  - Compute qPKE.Setup$(1^{\lambda_2})$ to obtain $(pk, sk)$, where $\lambda_2 = \text{poly}(\lambda, \ell)$ is chosen such that qPKE is a $\ell$-sub-exponential q-CCA2-secure PKE scheme.

Output $\text{crs} = (pk, \text{qNIZK.crs})$.

- **$P$(crs, $x$, $w$):** On input common reference string crs, instance $x$, witness $w$,
  - Parse crs as $(pk, qNIZK.crs)$.
  - Compute $ct_w \leftarrow qPKE.Enc(pk, (x, w); r_w)$, where $r_w \leftarrow \{0, 1\}^{\text{poly}(\lambda)}$.
  - Compute $qNIZK\.\pi \leftarrow qNIZK.P(qNIZK.crs, (pk, ct_w, x), (w, r_w))$.

Output $\pi = (qNIZK\.\pi, ct_w)$.

- **$V$(crs, $x$, $\pi$):** On input common reference string crs, NP instance $x$, proof $\pi$,
  - Parse crs as $(pk, ct, qNIZK.crs)$.
  - Output qNIZK.$V(qNIZK.crs, (pk, ct_w, x), \pi)$.

We prove that the above argument system satisfies q-simulation-extractability. We describe the algorithms FkGen and Sim = (Sim$_1$.Sim$_2$) below. Let qNIZK.FkGen and qNIZK.Sim be the QPT algorithms associated with the zero-knowledge property of qNIZK.

- **FkGen$(1^\lambda)$:** Compute $(qNIZK.crs, \tau) \leftarrow qNIZK.FkGen(1^\lambda)$. Compute $(pk, sk) \leftarrow qPKE.Setup(1^\lambda)$.
  Output $\text{crs} = (qNIZK.crs, pk, ct)$ and td = $(\tau, sk)$.

- **Sim$_1$ (crs, td, $\{x_i\}_{i \in [q]}$):** Compute qNIZK.Sim(qNIZK.crs, $\tau$, $(pk, ct, x_i)$) to obtain qNIZK.$\pi_i$, for every $i \in [q]$. Output $\{qNIZK.\pi_1, \ldots, qNIZK.\pi_q\}$ and $st = (td, crs, \{x_i\}_{i \in [q]})$. 

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Sim₂(st, x', π'): On input st = (td = (τ, sk), crs, \(\{x_i\}_{i \in [q]}\)), instance x', proof π' = (qNIZK, π', ct'_w), compute Dec(sk, ct'_w) to obtain w'. Output w'.

Suppose \(\mathcal{A}\) be a quantum adversary running in time \(2\tilde{O}(t)\) such that the following holds:

\[
\Pr \left[ \begin{array}{l}
\forall i \in [q], (x_i, w_i) \in R(L) \\
\forall i \in [q], \pi_i \leftarrow \mathcal{P}(\text{crs, } td, x_i) \\
(x_i, w_i') \in R(L) \\
(x_i, w_i^*) \in R(L)
\end{array} \right] = \delta
\]

Let \(\delta\) be such that the following holds:

\[
\Pr \left[ \begin{array}{l}
\forall i \in [q], (x_i, w_i) \in R(L) \\
\forall i \in [q], \pi_i \leftarrow \mathcal{P}(\text{crs, } td, x_i) \\
(x_i, w_i^*) \in R(L)
\end{array} \right] = \varepsilon
\]

We prove using a standard hybrid argument that \(|\delta - \varepsilon| \leq \text{negl}(\lambda)\).

Hybrid₁: \(\mathcal{A}\) is given \(\pi_1, \ldots, \pi_q\), where \(\pi_i \leftarrow \mathcal{P}(\text{crs, } x_i, w_i)\). Let \((x', \pi')\) be the output of \(\mathcal{A}\) and parse \(\pi' = (\text{qNIZK}, \pi', ct'_w)\). Decrypt \(ct'_w\) using sk to obtain \((x', w')\).

From the adaptive soundness of qNIZK, the probability that \((x', w') \in R(L_\ell)\) and \(x^* = x'\) is negligibly close to \(\varepsilon\).

Hybrid₂: \(\mathcal{A}\) is given \(\pi_1, \ldots, \pi_q\), where the proofs are generated as follows: first compute \((\text{qNIZK}, \pi_1, \ldots, \text{qNIZK}, \pi_q) \leftarrow \text{qNIZK.Sim}(\text{crs, } td, \{x_i\}_{i \in [q]})\), where \((\text{crs, } td) \leftarrow \text{qNIZK.FkGen}(1^\lambda)\). Then compute \(ct_{w_i} \leftarrow \text{Enc}(pk, (x_i, w_i))\) for every \(i \in [q]\). Set \(\pi_i = (\text{qNIZK}, \pi_i, ct_{w_i})\). The rest of this hybrid is defined as in Hybrid₁.

From the adaptive zero-knowledge property of qNIZK, the probability that \((x', w') \in R(L_\ell)\) and \(x^* = x'\) in the hybrid Hybrid₂ is still negligibly close to \(\varepsilon\).

Hybrid₃: This hybrid is defined similar to the previous hybrid except that \(ct_{w_i} \leftarrow \text{Enc}(pk, 0)\), for every \(i \in [q]\).

From the previous hybrids, it follows that \(ct'_{w_i} \neq ct_{w_i}\) for all \(i \in [q]\) with probability negligibly close to \(\varepsilon\); this follows from the fact that qPKE is perfectly correct and the fact that \(x^* = x'\) holds with probability negligibly close to \(\varepsilon\). Thus, we can invoke q-CCA2-security of qPKE, the probability that \((x', w') \in R(L_\ell)\) is still negligibly close to \(\varepsilon\).

But note that Hybrid₃ corresponds to the simulated experiment and thus we just showed that the probability that we can recover \(w'\) such that \((x', w') \in R(L_\ell)\) is negligibly close to \(\varepsilon\).
The primitives in the above proposition can be instantiated from sub-exponential QLWE by starting with existing LWE-based constructions of the above primitive and suitably setting the parameters of the underlying LWE assumption. We state the following propositions without proof.

**Proposition 29 ([49]).** Assuming \( \ell \)-sub-exponential QLWE (Section 2.2), there exists a \( \ell \)-sub-exponential \( q \)-NIZK for NP.

**Remark 30.** To be precise, the work of [49] constructs a NIZK system satisfying adaptive multi-theorem zero-knowledge and non-adaptive soundness. However, non-adaptive soundness implies adaptive soundness using complexity leveraging; the reduction incurs a security loss of \( 2^{\ell} \).

**Proposition 31 ([50]).** Assuming \( \ell \)-sub-exponential QLWE (Section 2.2), there exists a \( \ell \)-sub-exponential \( q \)-CCA2-secure PKE scheme.

\[ \square \]

## 3 Secure Software Leasing (SSL)

We present the definition of secure software leasing schemes. A secure software leasing (SSL) scheme for a class of circuits \( C = \{ C_\lambda \}_{\lambda \in \mathbb{N}} \) consists of the following QPT algorithms.

- **Private-key Generation**, \( \text{Gen}(1^\lambda) \): On input security parameter \( \lambda \), it outputs a private key \( sk \).
- **Software Lessor**, \( \text{Lessor}(sk, C) \): On input the private key \( sk \) and a poly\((n)\)-sized classical circuit \( C \in C_\lambda \), with input length \( n \) and output length \( m \), it outputs a quantum state \( \rho_C \).
- **Evaluation**, \( \text{Run}(\rho_C, x) \): On input the quantum state \( \rho_C \) and an input \( x \in \{0, 1\}^n \), it outputs \( y \), and some state \( \rho_C^{x} \).
- **Check of Returned Software**, \( \text{Check}(sk, \rho_C^x) \): On input the private key \( sk \) and the state \( \rho_C^x \), it checks if \( \rho_C^x \) is a valid leased state and if so it outputs 1, else it outputs 0.

**Setup.** In this work, we only consider SSL schemes in the setup model. In this model, all the lessors in the world have access to a common reference string generated using a PPT algorithm \( \text{Setup} \). The difference between \( \text{Setup} \) and \( \text{Gen} \) is that \( \text{Setup} \) is run by a trusted third party whose output is used by all the lessors while \( \text{Gen} \) is executed by each lessor separately. We note that our impossibility result rules out SSL schemes for all quantum unlearnable class of circuits even in the setup model.

We define this notion below.

**Definition 32 (SSL with Setup).** A secure software leasing scheme \( (\text{Gen}, \text{Lessor}, \text{Run}, \text{Check}) \) is said to be in the common reference string (CRS) model if additionally, it has an algorithm \( \text{Setup} \) that on input \( 1^\lambda \) outputs a string \( \text{crs} \).

Moreover, the algorithms \( \text{Gen} \) now takes as input \( \text{crs} \) instead of \( 1^\lambda \) and \( \text{Run} \) additionally takes as input \( \text{crs} \).

We require that an SSL scheme, in the setup model, satisfies the following properties.
Definition 33 (Correctness). An SSL scheme (Setup, Gen, Lessor, Run, Check) for \( C = \{C_\lambda\}_{\lambda \in \mathbb{N}} \) is \( \varepsilon \)-correct if for all \( C \in C_\lambda \), with input length \( n \), the following two properties holds for some negligible function \( \varepsilon \):

- Correctness of Run:

\[
\Pr \left[ \forall x \in \{0, 1\}^n, y = C(x) : \begin{aligned}
&\text{crs} \leftarrow \text{Setup}(1^\lambda), \\
&\text{sk} \leftarrow \text{Gen}(\text{crs}), \\
&\rho_\lambda \leftarrow \text{Lessor}(\text{sk}, C) \\
&\rho_{\lambda,x} \leftarrow \text{Run}(\text{crs}, \rho_\lambda, x)
\end{aligned} \right] \geq 1 - \varepsilon
\]

- Correctness of Check:

\[
\Pr \left[ \text{Check} (\text{sk}, \rho_\lambda) = 1 : \begin{aligned}
&\text{crs} \leftarrow \text{Setup}(1^\lambda), \\
&\text{sk} \leftarrow \text{Gen}(\text{crs}), \\
&\rho_\lambda \leftarrow \text{Lessor}(\text{sk}, C)
\end{aligned} \right] \geq 1 - \varepsilon
\]

Reusability. A desirable property of an SSL scheme is reusability: the lessee should be able to repeatedly execute Run on multiple inputs. An SSL scheme does not necessarily guarantee reusability; for instance Run could destroy the state after executing it once. But fortunately, we can transform this scheme into another scheme that satisfies reusability.

We define reusability formally.

Definition 34. (Reusability) An SSL scheme (Setup, Gen, Lessor, Run, Check) for \( C = \{C_\lambda\}_{\lambda \in \mathbb{N}} \) is said to be reusable if for all \( C \in C \) and for all \( x \in \{0, 1\}^n \),

\[
\|\rho_{\lambda,x} - \rho_\lambda\|_{tr} \leq \text{negl}(\lambda).
\]

Note that the above requirement \(\|\rho_{\lambda,x} - \rho_\lambda\|_{tr} \leq \text{negl}(\lambda)\) would guarantee that an evaluator can evaluate the leased state on multiple inputs; on each input, the original leased state is only disturbed a little which means that the resulting state can be reused for evaluation on other inputs.

The following proposition states that there is a way to generically transform any SSL scheme into one that is reusable.

Proposition 35. Let (Setup, Gen, Lessor, Run, Check) be any SSL scheme (not necessarily satisfying the reusability condition). Then, there is a QPT algorithm Run’ such that (Setup, Gen, Lessor, Run’, Check) is a reusable SSL scheme.

Proof. For any \( C \in C \) and for any \( x \in \{0, 1\}^n \), we have that Run(crs, \( \rho_\lambda, x \)) outputs \( C(x) \) with probability \( 1 - \varepsilon \). By the Almost As Good As New Lemma (Lemma 8), there is a way to implement Run such that it is possible to obtain \( C(x) \), and then recover a state \( \tilde{\rho}_\lambda \) satisfying \( \|\tilde{\rho}_\lambda - \rho_\lambda\|_{tr} \leq \sqrt{\varepsilon} \).

We let Run’ be this operation.

Thus, it suffices to just focus on the correctness property when constructing an SSL scheme.
3.1 Security

We require the following security guarantee: a QPT adversary (pirate) after receiving a leased copy of C generated using Lessor, denoted by $\rho_C$, cannot produce a bipartite state $\sigma^*$ on registers $R_1$ and $R_2$, such that $\sigma^*_1 := \text{Tr}_2[\sigma^*]$ can be checked (verified using Check), and the resulting post-measurement state on $R_2$ (after the check on $R_1$), which we will denote by $P_2(\sigma^*)$, still computes $C$ by $\text{Run}(P_2(\sigma^*), x) = C(x)$. We call this perfect lessor security and the reason why we use the word perfect here is because we require $\text{Run}(P_2(\sigma^*), x) = C(x)$ with overwhelming probability on every input $x$. Note that $\text{Run}$ is not necessarily deterministic (for instance, it could perform measurements) and thus we allow it to output the incorrect value with some probability. In the case that the adversary only gets one leased copy, we term this security notion as perfect lessor security and the more general notion as generalized perfect lessor security.

Before formally stating the definition, let us fix some notation. We will use the following notation for the state that the pirate keeps after the initial copy has been returned and verified. If the pirate outputs the bipartite state $\sigma^*$, then we will write

$$P_2(\text{sk}, \sigma^*) \propto \text{Tr}_1[\Pi_1[\text{Check}(\text{sk}, \cdot)_1 \otimes I_2(\sigma^*)]]$$

for the state that the pirate keeps after the first register has been returned and verified. Here, $\Pi_1$ denotes projecting the output of Check onto 1, and where $\text{Check}(\text{sk}, \cdot)_1 \otimes I_2(\sigma^*)$ denotes applying the Check QPT onto the first register, and the identity on the second register of $\sigma^*$. In other words, $P_2(\text{sk}, \sigma^*)$ is used to denote the post-measurement state on $R_2$ conditioned on $\text{Check}(\text{sk}, \cdot)$ accepting on $R_1$.

**Definition 36** (Perfect Lessor Security). *We say that a SSL scheme $(\text{Setup}, \text{Gen}, \text{Lessor}, \text{Run}, \text{Check})$ for a class of circuits $C = \{C_\lambda\}_{\lambda \in \mathbb{N}}$ is said to satisfy $(\beta, \gamma, D_C)$-perfect lessor security, with respect to a distribution $D_C$ on $C$, if for every QPT adversary $A$ (pirate) that outputs a bipartite (possibly entangled) quantum state on two registers, $R_1$ and $R_2$, the following holds:

$$\Pr \left[ \begin{array}{c}
\text{Check}(\text{sk}, \sigma^*_1) = 1 \\
\forall x, \Pr[\text{Run}(\text{crs}, P_2(\text{sk}, \sigma^*), x) = C(x)] \geq \beta
\end{array} \right] \leq \gamma$$

**Generalized Perfect Lessor Security.** It is desirable to have a stronger security guarantee where the adversary will not be able to create a pirated copy even after receiving many leased copies of the same circuit (and not just one as we considered before). We formalize this security notion, termed as generalized perfect lessor security, below.

Similar to the perfect lessor security setting, we will use the notation $P_{q+1}(\text{sk}, \sigma^*)$ to denote the post-measurement state on register $R_{q+1}$ conditioned on $\text{Check}(\text{sk}, \cdot)$ accepting on all registers $R_1$ through $R_q$.

**Definition 37** (Generalized Perfect Lessor Security). *We say that a SSL scheme $(\text{Setup}, \text{Gen}, \text{Lessor}, \text{Run}, \text{Check})$ for a class of circuits $C = \{C_\lambda\}_{\lambda \in \mathbb{N}}$ is said to satisfy generalized $(\beta, \gamma, D_C)$-perfect lessor security...*
security, with respect to a distribution $D_C$ on $C$, if for every security parameter $\lambda \in \mathbb{N}$, every $q = \text{poly}(\lambda)$, every QPT adversary $A$ (pirate) that outputs a multipartite (possibly entangled) quantum state on registers, $R_1, \ldots, R_{q+1}$, the following holds:

$$\Pr \left[ \exists (sk, \cdot)^{\otimes q}(a_{1,\ldots,q}^\ast) \mid \forall x, \Pr[\text{Run}(crs, P_{q+1}(sk, a^\ast), x) = C(x)] \geq \beta \right] \leq \gamma,$$

where $\text{Check}(sk, \cdot)^{\otimes q}(a_{1,\ldots,q}^\ast)$ denotes applying $\text{Check}(sk, \cdot)$ to each register from $R_1$ through $R_q$.

Remark 38. In the general (multiple-copies) setting, we can consider schemes in which correctness is not negligibly close to 1. However, similarly to [3], an SSL scheme with generalized perfect lessor security can always be amplified to have $\varepsilon \leq \text{negl}(\lambda)$ by providing $k = \Omega(\lambda)$ number of copies, $\rho_C^{\otimes k}$ instead of a single one, $\rho_C$.

4 Impossibility of SSL

To prove the impossibility of SSL, we first construct de-quantizable class of circuits.

4.1 De-Quantizable Circuits: Definition

A de-quantizable class of circuits $C$ is a class of circuits for which there is a QPT algorithm that given any quantum circuit with the same functionality as $C \in C$, it finds a (possibly different) classical circuit $C' \in C$ with the same functionality as $C$. Of course if $C$ is learnable, then it could be possible to just observe the input-output behavior of the quantum circuit to find such a $C'$. To make this notion meaningful, we additionally impose the requirement that $C$ needs to be quantum unlearnable; given only oracle access to $C$, any quantum algorithm can find a circuit (possibly a quantum circuit and an auxiliary input state $\rho$) with the same functionality as $C$ with only negligible probability.

Definition 39. We say that a collection of QPT algorithms, $\{U_C, \rho_C\}_{C \in C}$, computes $C$ if for any $C \in C$, with input length $n$ and output length $m$, $\rho_C$ is a $\text{poly}(n)$-qubits auxiliary state, and $U_C$ a QPT algorithm satisfying that for all $x \in \{0, 1\}^n$,

$$\Pr[U_C(\rho_C, x) = C(x)] \geq 1 - \text{negl}(\lambda),$$

where the probability is over the measurement outcomes of $U_C$. We also refer to $(U_C, \rho_C)$ as an efficient quantum implementation of $C$. A class of classical circuits $C$, associated with a distribution $D_C$, is said to be de-quantizable if the following holds:

- **Efficient De-quantization**: There is a QPT algorithm $B$ such that, for any $\{U_C, \rho_C\}_{C \in C}$ that computes $C$, the following holds:
\[
\Pr \left[ \bigwedge_{x \in \{0,1\}^n, C(x) = c''(x)} \forall C' \in \mathcal{D}_C : C' \leftarrow \mathcal{D}_C \left| C \leftarrow \mathcal{D}_C \left( x \leftarrow \mathcal{B}(U_C, \rho_C) \right) \right. \right] \geq 1 - \negl(\lambda)
\]

- \(\nu\)-Quantum Unlearnability: For any QPT adversary \(A\), the following holds:

\[
\Pr \left[ \forall x, \Pr[U'(\rho^*, x) = C(x)] \geq \nu : C \leftarrow \mathcal{D}_C \left( U^*, \rho' \leftarrow A^{\{1^n\}} \right) \right] \leq \negl(\lambda)
\]

Remark 40. By the Almost As Good As New Lemma (Lemma 8), we can assume that the QPT algorithm \(U_C\) also output a state \(\rho'_{C,x}\) that is negligibly close in trace distance to \(\rho_C\), i.e. for all \(C \in \mathcal{C}\) and \(x \in \{0,1\}^n\) it holds that

\[
\Pr[U_C(\rho_C, x) = (\rho'_{C,x}, C(x))] \geq 1 - \negl(\lambda)
\]

and \(\|\rho'_{C,x} - \rho_C\|_r \leq \negl(\lambda)\).

Remark 41. We emphasize that the efficient de-quantization property requires that the circuit \(C'\) output by the adversary should be in the same circuit class \(C\).

Remark 42. We can relax the unlearnability condition in the above definition to instead have a distribution over the inputs and have the guarantee that the adversary has to output a circuit \((U^*, \rho^*)\) such that it agrees with \(C\) only on inputs drawn from this distribution. Our impossibility result will also rule out this relaxed unlearnability condition; however, for simplicity of exposition, we consider the unlearnability condition stated in the above definition.

From the above definition, we can see why a de-quantizable class \(C\) cannot be copy-protected, as there is a QPT \(B\) that takes any \((U_C, \rho_C)\) efficiently computing \(C\), and outputs a functionally equivalent classical circuit \(C'\), which can be copied. In the following theorem we will show that if every circuit \(C \in \mathcal{C}\) have a unique representation in \(\mathcal{C}\), then it is also not possible to have SSL for this circuit class. To see why we need an additional condition, let's consider a QPT pirate \(A\) that wants to break SSL given \((\text{Run}, \rho_C)\) computing \(C \in \mathcal{C}\). Then, \(A\) can run \(B\) to obtain a circuit \(C' \in \mathcal{C}\), but in the process it could have destroyed \(\rho_C\), hence it wouldn't be able to return the initial copy. If \(B\) takes as input \((\text{Run}, \rho_C)\) and outputs a fixed \(C'\) with probability negligibly close to 1, then by the Almost As Good As New Lemma, it could uncompute and recover \(\rho_C\). The definition of de-quantizable class does not guarantee that \(B\) will output a fixed circuit \(C'\), unless each circuit in the family has a unique representation in \(\mathcal{C}\). If each circuit has a unique representation, the pirate would obtain \(C' = C\) with probability negligibly close to 1, and uncompute to recover \(\rho_C\). At this point, the pirate can generate its own leasing keys \(sk'\), and run \(\text{Lessor}(sk', C')\) to obtain a valid leased state \(\rho'_{C'}\). The pirate was able to generate a new valid leased state for \(C\), while preserving the initial copy \(\rho_C\), which it can later return to the lessor.

Theorem 43. Let \((C, \mathcal{D}_C)\) be a de-quantizable class of circuits in which every circuit in the support of \(\mathcal{D}_C\) has a unique representation in \(C\). Then there is no SSL scheme \((\text{Setup}, \text{Gen}, \text{Lessor}, \text{Run}, \text{Check})\) (in CRS model) for \(C\) satisfying \(\varepsilon\)-correctness and \((\beta, \gamma, \mathcal{D}_C)\)-perfect lessor security for any negligible \(\gamma\), and any \(\beta \leq (1 - \varepsilon)\).
Proof. Consider the QPT algorithm $\mathcal{A}$ (pirate) that is given $\rho_C \leftarrow \text{Lessor}(sk, C)$ for some $C \leftarrow D_C$. The pirate will run $B$, the QPT that de-quantizes $(C, D_C)$, on input $(\text{Run}, \rho_C)$ to obtain a functionally equivalent circuit $C' \in C$. Because $C$ has a unique representation in $C$, we have $C' = C$. Since this succeeds with probability negligibly close to 1, by the Almost As Good As New Lemma 8, it can all be done in a way such that it is possible to obtain $C$ and to recover a state $\overline{\rho_C}$ satisfying $\|\overline{\rho_C} - \rho_C\|_{\text{tr}} \leq \text{negl}(\lambda)$. At this point, the pirate generates its own key $sk' \leftarrow \text{Gen}(crs)$, and prepares $\rho'_C \leftarrow \text{Lessor}(sk', C)$. It outputs $\overline{\rho_C} \otimes \rho'_C$.

This means that $\rho'_C$ is a valid leased state and by correctness of the SSL scheme,

$$\Pr \left[ \forall x \in \{0,1\}^n, \text{Run}(crs, \rho'_C, x) = C(x) : \text{crs} \leftarrow \text{Setup}(1^1), sk' \leftarrow \text{Gen}(crs), \rho'_C \leftarrow \text{Lessor}(sk', C) \right] \geq 1 - \varepsilon$$

Furthermore, since $\|\overline{\rho_C} - \rho_C\|_{\text{tr}} \leq \text{negl}(\lambda)$, the probability that $\overline{\rho_C}$ passes the return check is negligibly close to 1. Putting these together, we have

$$\Pr \left[ \text{Check}(sk, \overline{\rho_C}) = 1 \land \forall x, \Pr[\text{Run}(crs, \rho'_C, x) = C(x)] \geq 1 - \varepsilon \right. \left. \iff \text{crs} \leftarrow \text{Setup}(1^1), C \leftarrow D_C(\lambda), \text{sk} \leftarrow \text{Gen}(crs), \rho \leftarrow \text{Lessor}(sk, C), \rho'_C \leftarrow \text{Lessor}(sk', C), \right] \geq 1 - \text{negl}(\lambda)$$

4.2 De-quantizable Circuit Class: Construction

All that remains in the proof of impossibility of SSL is the construction of a de-quantizable circuits class $(C, D_C)$ in which every circuit in the support of $D_C$ has a unique representation in $C$. We begin with an overview of the construction.

Constructing De-Quantizable Circuits: Challenges. The starting point is the seminal work of Barak et al. [15], who demonstrated a class of functions, where each function is associated with a secret key $k$, such that: (a) Non black box secret extraction: given non black box access to any classical circuit implementation of this function, the key can be efficiently recovered, (b) Classical Unlearnability of secrets: but given black box access to this circuit, any classical adversary who can only make polynomially many queries to the oracle cannot recover the key.

While the result of Barak et al. has the ingredients suitable for us, it falls short in many respects:

- The proof of non black box secret extraction crucially relies upon the fact that we are only given a classical obfuscated circuit. In fact there are inherent difficulties that we face in adapting Barak et al. to the quantum setting; see [9].

- As is the case with many non black box extraction techniques, the proof of Barak et al. involves evaluating the obfuscated circuit multiple times in order to recover the secret. As is typically the case with quantum settings, evaluating the same circuit again and again is not always easy – the reason being that evaluating a circuit once could potentially destroy the state thus rendering it impossible to run it again.
• Barak et al. only guarantees extraction of secrets given non black box access to the classical circuit implementation of the function. However, our requirement is qualitatively different: given a quantum implementation of the classical circuit, we need to find a (possible different) classical circuit with the same functionality.

• Barak et al.’s unlearnability result only ruled out adversaries who make classical queries to the oracle. On the other hand, we need to argue unlearnability against QPT adversaries who can perform superposition queries to the oracle.

Nonetheless, we show that the techniques introduced in a simplified version of Barak⁸ can be suitably adapted for our purpose by using two tools: quantum fully homomorphic encryption (QFHE) and lockable obfuscation. Combining QFHE and lockable obfuscation for the purpose of secret extraction has been recently used in a completely different context, that of building zero-knowledge protocols [19, 12] (and in classical setting was first studied by [17]).

**Construction.** We present the construction of de-quantizable circuits.

**Theorem 44.** Assuming the quantum hardness of learning with errors (QLWE), and assuming that there is a QFHE that supports evaluation of arbitrary polynomial-sized quantum circuits (see 2.3), and has the property that ciphertexts of classical poly-sized messages have poly-sized classical descriptions, there is a de-quantizable class of circuits \(\mathcal{C}, \mathcal{D}_C\).

**Proof.** We define a de-quantizable class of circuits \(\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \mathbb{N}}\), where every circuit in \(\mathcal{C}_\lambda\) is defined as follows:

\[
\mathcal{C}_{a,b,r,pk,\mathcal{O}}(x):
\]

1. If \(x = 0 \cdots 0\), output QFHE.Enc\((pk, a; r)\) |\(\mathcal{O}\)|\(pk\).
2. Else if \(x = a\), output \(b\).
3. Otherwise, output \(0 \cdots 0\)

We will suitably pad with zeroes such that all the inputs (resp., outputs) are of the same length \(n\) (resp., of the same length \(m\)).

Let \(\mathcal{D}_C(\lambda)\) be the distribution that outputs a circuit from \(\mathcal{C}_\lambda\) by sampling \(a, b, r \leftarrow \{0, 1\}^\lambda\), then computing \((pk, sk) \leftarrow \text{QFHE.Gen}(1^\lambda)\), and finally computing an obfuscation \(\mathcal{O} \leftarrow \text{LO.Obf}(\mathcal{C}[\text{QFHE.Dec}(sk, \cdot), b, (sk|r)])\), where \(\mathcal{C}\) is a compute-and-compare circuit.

We show that with respect to this distribution: (a) \(\mathcal{C}\) is quantum unlearnable (Proposition 45) and, (b) \(\mathcal{C}\) is efficiently de-quantizable (Proposition 48).

**Proposition 45.** For any non-negligible \(\nu\), the circuit class \(\mathcal{C}\) is \(\nu\)-quantum unlearnable with respect to \(\mathcal{D}_C\).

**Proof.** We first rule out QPT adversaries, who given black box access to the circuit, can find the secret key \(sk\) with non-negligible probability. Once we rule out this type of adversaries, we then show how to reduce a QPT adversary who breaks the quantum unlearnability property of the de-quantizable class of circuits to one who finds the secret key \(sk\); thus completing the proof.

---

⁸See [18] for a description of this simplified version.
Claim 46. For all QPT $A$ with oracle access to $C_{a,b,r,pk,O}(\cdot)$ (where the adversary is allowed to make superposition queries), we have

$$\Pr_{(a, b, r, pk, O) \leftarrow \mathcal{D}_C} \left[ sk \leftarrow A^{C_{a,b,r,pk,O}(1^\lambda)} \right] \leq \text{negl}(\lambda)$$

Proof. Towards proving this, we make some simplifying assumptions; this is only for simplicity of exposition and they are without loss of generality.

Simplifying Assumptions. Consider the following oracle $O_{a,b,r,pk,O}$:

$$O_{a,b,r,pk,O}(x)z = \begin{cases} |x\rangle z \oplus C_{a,b,r,pk,O}(x), & \text{if } x \neq 0 \cdots 0 \\ |x\rangle z, & \text{if } x = 0 \cdots 0 \end{cases}$$

The first simplifying assumption is that the adversary $A$ is given access to the oracle $O_{a,b,r,pk,O}$, instead of the oracle $C_{a,b,r,pk,O}$. In addition, $A$ is given $\text{Enc}(pk, a; r), pk,$ and $O$ as auxiliary input.

The second simplifying assumption is that $A$ is given some auxiliary state $|\xi\rangle$, and that it only performs computational basis measurements right before outputting (i.e. $A$ works with purified states).

Proof Overview. Our proof follows the basic adversary method proof technique [10]. We prove this by induction on the number of queries. We show that after every query the following invariant is maintained: the state of the adversary has little amplitude over $a$. More precisely, we argue that the state of the adversary after the $t^{th}$ query, is negligibly close to the state just before the $t^{th}$ query, denoted by $|\psi^t\rangle$. After the adversary obtains the response to the $t^{th}$ query, it then applies a unitary operation to obtain the state $|\psi^{t+1}\rangle$, which is the state of the adversary just before the $(t+1)^{th}$ query. This observation implies that there is another state $|\phi^t\rangle$ that: (a) is close to $|\psi^{t+1}\rangle$ (here, we use the inductive hypothesis that $|\phi^t\rangle$ is close to $|\psi^t\rangle$) and, (b) can be prepared without querying the oracle at all.

Let $U_i$ denote the unitary that $A$ performs right before its $i^{th}$ query, and let $A, X,$ and $Y$ denote the private, oracle input, and oracle output registers of $A$, respectively.

Just before the $t^{th}$ query, we denote the state of the adversary to be:

$$|\psi^t\rangle := U_t O \cdots O U_1 |\psi^0\rangle$$

where $|\psi^0\rangle = |\xi\rangle |\text{Enc}(pk, a; r), O, pk\rangle |0\cdots 0\rangle_X |0\cdots 0\rangle_Y$ is the initial state of the adversary. Let $\Pi_a = (|a\rangle \langle a|)_X \otimes I_{Y_A}$.

Note that any $A$ that outputs $sk$ with non-negligible probability can also query the oracle on a state $|\psi\rangle$ satisfying $\text{Tr}[\Pi_a |\psi\rangle \langle \psi|] \geq \text{non-negl}(\lambda)$ with non-negligible probability. Since $A$ outputs $sk$ with non-negligible probability, it can decrypt $\text{Enc}(pk, a; r)$, to find $a$ and then query the oracle on $a$. In other words, if there is an adversary $A$ that finds $sk$ with non-negligible probability, then there is an adversary that at some point queries the oracle with a state $|\psi\rangle \text{Tr}[\Pi_a |\psi\rangle \langle \psi|] \geq \text{non-negl}(\lambda)$.
also with non-negligible probability.

Hence, it suffices to show that for any adversary $\mathcal{A}$ that makes at most $T = \text{poly}(\lambda)$ queries to the oracle, it holds that

$$\Pr[\forall j, \operatorname{Tr}[\Pi_a|\psi^j\rangle \langle \psi^j|] \leq \text{negl}(\lambda)] \geq 1 - \text{negl}(\lambda).$$

This would then imply that $\mathcal{A}$ cannot output $\text{sk}$ with non-negligible probability, thus proving Claim 46.

Towards proving the above statement, consider the following claim that states that if $\mathcal{A}$ has not queried the oracle with a state that has large overlap with $\Pi_a$, then its next query will also not have large overlap with $\Pi_a$.

**Claim 47 (No Good Progress).** Let $T$ be any polynomial in $\lambda$. Suppose for all $t < T$, the following holds:

$$\operatorname{Tr}[\Pi_a|\psi^t\rangle \langle \psi^t|] \leq \text{negl}(\lambda)$$

Then, $\Pr[\operatorname{Tr}[\Pi_a|\psi^T\rangle \langle \psi^T|] \leq \text{negl}(\lambda)] \geq 1 - \text{negl}(\lambda)$.

**Proof.** For all $j$, let $|\phi^j\rangle = U_j U_{j-1} \ldots U_1 |\psi^0\rangle$.

We will proceed by induction on $T$. Our base case is $T = 1$ (just before the first query to the oracle); that is, $|\psi^1\rangle = |\phi^1\rangle$. Suppose the following holds:

$$\Pr[\operatorname{Tr}[\Pi_a|\psi^j\rangle \langle \psi^j|] \geq \text{non-negl}(\lambda)] \geq \text{non-negl}(\lambda).$$

The first step is to argue that if $\mathcal{A}$ can prepare a state such that $\operatorname{Tr}[\Pi_a|\psi^1\rangle \langle \psi^1|] \geq \text{non-negl}(\lambda)$ given $\text{Enc}(pk, a; r)$, $pk$ and $O \leftarrow \text{LO.Obf}(\text{C[QFHE.\text{Dec}(sk, \cdot); b, (sk|r)])}$ without querying the oracle, then it can also prepare a state with large overlap with $\Pi_a$ if its given the simulator of the lockable obfuscation instead. We will use $\mathcal{A}$ (specifically, the first unitary that $\mathcal{A}$ applies, $U_1$) to construct an adversary $\mathcal{B}$ that breaks the security of lockable obfuscation. $\mathcal{B}$ is given $a, \text{Enc}(pk, a; r), pk$ and $O$ as well as auxiliary state $|\xi\rangle$. It the prepares $|\psi_{1,O}\rangle = U_1 |\xi\rangle |\text{Enc}(pk, a; r), O, pk|0 \cdots 0\rangle_X |0 \cdots 0\rangle_Y$, and measures in computational basis. If the output of this measurement is $a$, it outputs 1; otherwise, it outputs 0.

Consider the following hybrids.

- **Hyb$_1$** In this hybrid, $\mathcal{B}$ is given $a, \text{Enc}(pk, a; r), pk, O \leftarrow \text{LO.Obf}(\text{C[QFHE.\text{Dec}(sk, \cdot); b, (sk|r)])}$.

- **Hyb$_2$** In this hybrid, $\mathcal{B}$ is given $a, \text{Enc}(pk, a; r), pk$ and $O \leftarrow \text{Sim}(1^\lambda)$.

Since the lock $b$ is chosen uniformly at random, by security of lockable obfuscation, the probability that $\mathcal{B}$ outputs 1 in the first hybrid is negligibly close to the probability that $\mathcal{B}$ outputs 1 in the second hybrid. This means that if $\operatorname{Tr}[\Pi_a|\psi_{1,O}\rangle \langle \psi_{1,O}|] \geq \text{non-negl}(\lambda)$ with non-negligible probability when $O \leftarrow \text{LO.Obf}(\text{C[QFHE.\text{Dec}(sk, \cdot); b, (sk|r)])}$, then this still holds when $O \leftarrow \text{Sim}(1^\lambda)$.

But we show that if $\operatorname{Tr}[\Pi_a|\psi_{1,O}\rangle \langle \psi_{1,O}|] \geq \text{non-negl}(\lambda)$, when $O$ is generated as $O \leftarrow \text{Sim}(1^\lambda)$, then QFHE is insecure.

- Consider the following QFHE adversary who is given $|\xi\rangle$ as auxiliary information, and chooses two messages $m_0 = 0 \cdots 0$ and $m_1 = a$, where $a$ is sampled uniformly at random from $\{0, 1\}^\lambda$. It sends $(m_0, m_1)$ to the challenger.
The challenger of QFHE then generates $ct_d = \text{Enc}(pk, m_d)$, for some bit $d \in \{0, 1\}$ and sends it to the QFHE adversary.

The QFHE adversary computes $O \leftarrow \text{Sim}(1^\lambda)$.

The QFHE adversary computes $\text{Enc}(\xi)$, for some negligible $\delta_0$, and measures register $X$ in the computational basis.

If $d = 0$, the probability that the QFHE adversary obtains $a$ as outcome is negligible; since $a$ is independent of $U_1, pk, |\xi\rangle$, and $O$. But from our hypothesis ($\Pr[\text{Tr}[\Pi_0|\psi^1]\langle\psi^1|] \geq \text{negl}(\lambda)] \geq \text{non-negl}(\lambda)$), the probability that the QFHE adversary obtains $a$ as outcome is non-negligible for the case when $d = 1$. This contradicts the security of QFHE as the adversary can use $a$ to distinguish between these two cases.

To prove the induction hypothesis, suppose that for all $t < T$, the following two conditions hold:

1. $\Pr[\Pi_a|\psi^t]\langle\psi^t|] \leq \text{negl}(\lambda)$
2. $\langle\phi^t|\psi^t\rangle = 1 - \delta_t$

for some negligible $\delta_1, \ldots, \delta_{T-1}$. We can write

$$|\langle\phi^T|\psi^T\rangle| = |\langle\phi^{T-1}|O|\psi^{T-1}\rangle|$$

By hypothesis (2) above, we have $|\phi^{T-1}| = (1 - \delta_{T-1})e^{ia}|\psi^{T-1}\rangle + \sqrt{2\delta_{T-1} - \delta_{T-1}^2}|\tilde{\psi}^{T-1}\rangle$, here $\alpha$ is some phase, and $|\tilde{\psi}^{T-1}\rangle$ is some state orthogonal to $|\psi^{T-1}\rangle$. Then

$$|\langle\phi^T|\psi^T\rangle| = |(1 - \delta_{T-1})e^{ia}\langle\psi^{T-1}|O|\psi^{T-1}\rangle + \sqrt{2\delta_{T-1} - \delta_{T-1}^2}\langle\tilde{\psi}^{T-1}|O|\psi^{T-1}\rangle|$$

$$\geq |(1 - \delta_{T-1})e^{ia}\langle\psi^{T-1}|O|\psi^{T-1}\rangle| - \sqrt{2\delta_{T-1} - \delta_{T-1}^2}$$

By hypothesis (1) above, and since the oracle acts non-trivially only on $a$, we have $|\langle\psi^{T-1}|O|\psi^{T-1}\rangle| \geq 1 - \text{negl}(\lambda)$, which gives us

$$|\langle\phi^T|\psi^T\rangle| \geq 1 - \text{negl}(\lambda).$$

Now we want to show that $\Pr[\Pi_a|\psi^T]\langle\psi^T|] \leq \text{negl}(\lambda)$. This follows from the security of lockable obfuscation and QFHE similarly to $T = 1$ case. Since $|\langle\phi^T|\psi^T\rangle| \geq 1 - \text{negl}(\lambda)$, we have that

$$\Pr[\Pi_a|\phi^T]\langle\phi^T|] \leq \text{negl}(\lambda) \implies \Pr[\Pi_a|\psi^T]\langle\psi^T|] \leq \text{negl}(\lambda).$$

From a similar argument to the $T = 1$ case but using $U_T U_{T-1} \cdots U_1$ instead of just $U_1$, we have that $\Pr[\text{Tr}[\Pi_a|\phi^T]\langle\phi^T|] \leq \text{negl}(\lambda)] \geq 1 - \text{negl}(\lambda)$. \hfill $\square$

Let $E_t$ denote the event that $\Pr[\Pi_a|\psi^t]\langle\psi^t|] \leq \text{negl}(\lambda)$. Let $p_T$ be the probability that $\Pr[\Pi_a|\psi^t]\langle\psi^t|] \leq \text{negl}(\lambda)$ for all the queries $t \leq T$. Using the previous claim, we have that
\[
\begin{align*}
    p_T &= \prod_{t=1}^{T} \Pr[E_t | \forall j < t, E_j] \\
    &\geq (1 - \text{negl}(\lambda))^T \\
    &\geq (1 - T \cdot \text{negl}(\lambda))
\end{align*}
\]

Suppose that there is a QPT \( \mathcal{B} \) that can learn \( C \) with respect to \( \mathcal{D}_C \) with non-negligible probability \( \delta \). In other words, for all inputs \( x \),
\[
\Pr\left[U(\rho, x) = C_{a,b,r,\mathcal{B},\mathcal{O}}(x) : C_{a,b,r,\mathcal{B},\mathcal{O}} \leftarrow \mathcal{D}_C\right] = \delta
\]

We use \( \mathcal{B}^{C_{a,b,r,\mathcal{B},\mathcal{O}}} \) to construct a QPT \( \mathcal{A}^{C_{a,b,r,\mathcal{B},\mathcal{O}}} \) that can find \( \text{sk} \) with probability negligibly close to \( \delta \), contradicting Claim 46. To do this, \( \mathcal{A} \) first prepares \((U, \rho) \leftarrow \mathcal{B}^{C_{a,b,r,\mathcal{B},\mathcal{O}}(1^1)} \cdot\) Then, \( \mathcal{A}^{C_{a,b,r,\mathcal{B},\mathcal{O}}} \) queries the oracle on input \( 0 \cdots 0 \), obtaining \( c_{t_1} = \text{QFHE.Enc}(pk, a; r) \) along with \( pk \) and \( O = \text{LO.Obf}(\text{QFHE.Dec}sk, \cdot, b, (sk[r])) \). Finally, it homomorphically computes \( c_{t_2} \leftarrow \text{QFHE.Eval}(U(\rho, \cdot), c_{t_1}) \). Then it computes \( sk'|r' = O(ct_{t_2}) \), and outputs \( sk' \).

By the correctness of the QFHE and because \( U(\rho, a) = b \) holds with probability \( \delta \), we have that \( \text{QFHE.Dec}_{sk}(ct_{t_2}) = b \) with probability negligibly close to \( \delta \). By correctness of lockable obfuscation \( O(ct_{t_2}) \) will output the right message \( sk \). This means that output of \( \mathcal{A} \) is \( sk \) with probability negligibly close to \( \delta \).

\[\square\]

**Proposition 48.** \((C, \mathcal{D}_C)\) is efficiently de-quantizable.

**Proof.** We will start with an overview of the proof.

**Overview:** Given a quantum circuit \((U_C, \rho_C)\) that computes \( C_{a,b,r,\mathcal{O}}(\cdot) \), first compute on the input \( x = 0 \cdots 0 \) to obtain \( \text{QFHE.Enc}(pk, a; r)|\mathcal{O}|pk \). We then homomorphically evaluate the quantum circuit on \( \text{QFHE.Enc}(pk, a; r) \) to obtain \( \text{QFHE.Enc}(pk, b') \), where \( b' \) is the output of the quantum circuit on input \( a \); this is part where we crucially use the fact that we are given \((U_C, \rho_C)\) and not just black box access to the functionality computing \((U_C, \rho_C)\). But \( b' \) is nothing but \( b \)! Given QFHE encryption of \( b \), we can then use the lockable obfuscation to recover \( sk \); since the lockable obfuscation on input a valid encryption of \( b \) outputs \( sk \). Using \( sk \) we can then recover the original circuit \( C_{a,b,r,\mathcal{O}}(\cdot) \). Formal details follow.

For any \( C \in C \), let \((U_C, \rho_C)\) be any QPT algorithm (with auxiliary state \( \rho_C \)) satisfying that for all \( x \in \{0,1\}^n \),
\[
\Pr\left[U_C(\rho_C, x) = \left(\rho'_{C,x}, C(x)\right)\right] \geq 1 - \text{negl}(\lambda),
\]
where the probability is over the measurement outcomes of \( U_C \), and \( \rho'_{C,x} \) is negligibly close in trace distance to \( \rho_C \) (see Remark 40). We will show how to construct a QPT \( \mathcal{B} \) to de-quantize \((C, \mathcal{D}_C)\).
\( \mathcal{B} \) will perform a QFHE evaluation, which we describe here. Given \( \text{QFHE.Enc}(\pk, x) \), we want to homomorphically evaluate \( C(x) \) to obtain \( \text{QFHE.Enc}(\pk, C(x)) \). To do this, first prepare \( \text{QFHE.Enc}(\pk, \rho_C, x) \), then evaluate \( U_C \) homomorphically to obtain the following:

\[
\text{QFHE.Enc}(\pk, \rho'_{C,x}, C(x)) = \text{QFHE.Enc}(\pk, \rho'_{C,x})|\text{QFHE.Enc}(\pk, C(x))
\]

Consider the following QPT algorithm \( \mathcal{B} \) that is given \((U_C, \rho_C)\) for any \( C \in C \).

\( \mathcal{B}(U_C, \rho_C) \):

1. Compute \((\rho', \ct_1|O'|\pk') \leftarrow U_C(\rho_C, 0 \cdots 0) \).
2. Compute \( \sigma|\ct_2 \leftarrow \text{QFHE.Eval}(U_C(\rho', \cdot), \ct_1) \).
3. Compute \( \sk'|r' \leftarrow O(\ct_2) \).
4. Compute \( a' \leftarrow \text{QFHE.Dec}(\sk', \ct_1), b' \leftarrow \text{QFHE.Dec}(\sk', \ct_2) \).
5. Output \( C_{a',b',r',\pk',O'} \).

We claim that with probability negligibly close to \( 1 \), \( (a', b', r', \pk', O') = (a, b, r, \pk, O) \) when \( C := C_{a,b,r,\pk,O} \leftarrow D_C \). This would finish our proof.

Let's analyze the outputs of \( \mathcal{B} \) step-by-step.

- After Step (1), with probability negligibly close to \( 1 \), we have that \( \ct_1 = \text{QFHE.Enc}(\pk, a; r) \), \( \pk' = \pk \), and \( O' = O \leftarrow \text{LO.Obf}([\text{QFHE.Enc}(\sk, \cdot), b, (\sk|r)]) \). Furthermore, we have that \( \rho' \) is negligibly close in trace distance to \( \rho_C \).

- Conditioned on Step (1) computing \( C(0 \cdots 0) \) correctly, we have that \( \text{QFHE.Eval}(U_C(\rho', \cdot), \ct_1) \) computes correctly with probability negligibly close to 1. This is because \( \|\rho' - \rho_C\|_{\text{tr}} \leq \text{negl}(\lambda) \), and by correctness of both QFHE and \((U_C, \rho_C)\). Conditioned on \( \ct_1 = \text{QFHE.Enc}(\pk, a; r) \), when Step (2) evaluates correctly, we have \( \ct_2 = \text{QFHE.Enc}(\pk, C(a)) = \text{QFHE.Enc}(\pk, b) \).

- Conditioned on \( \ct_2 = \text{QFHE.Enc}(\pk, b) \), by correctness of lockable obfuscation, we have that \( O(\ct_2) \) outputs \( \sk|r \). Furthermore, by correctness of QFHE, decryption is correct: \( \text{QFHE.Dec}(\sk, \ct_1) \) outputs \( a \) with probability negligibly close to \( 1 \), and \( \text{QFHE.Dec}(\sk, \ct_2) \) outputs \( b \) with probability negligibly close to \( 1 \).

With probability negligibly close to \( 1 \), we have shown that \( (a', b', r', \pk', O') = (a, b, r, \pk, O) \).

Note that it is also possible to recover \( \rho'' \) that is negligibly close in trace distance to \( \rho_C \). This is because \( \sigma = \text{QFHE.Enc}(\pk, \rho'') \) for some \( \rho'' \) satisfying \( \|\rho'' - \rho_C\|_{\text{tr}} \). Once \( \sk' = \sk \) has been recovered, it is possible to also decrypt \( \sigma \) and obtain \( \rho'' \). To summarize, we have shown a QPT \( \mathcal{B} \) satisfying

\[
\Pr[\mathcal{B}(U_C, \rho_C) = (\rho'', C) : C \leftarrow D_C] \geq 1 - \text{negl}(\lambda)
\]

where \( \|\rho'' - \rho_C\|_{\text{tr}} \leq \text{negl}(\lambda) \).
Implications to Copy-Protection. We have constructed a class $C$ and an associated distribution $D_C$ that is efficient de-quantizable. In particular, this means that there is no copy-protection for $C$. If for all inputs $x$, there is a QPT $(U_C, \rho_C)$ to compute $U_C(\rho_C, x) = C(x)$ with probability $1 - \epsilon$ for some negligible $\epsilon$, then it is possible to find, with probability close to 1, a circuit $C'$ that computes the same functionality as $C$. We also proved that $(C, D_C)$ is quantum unlearnable. We summarize the result in the following corollary.

**Corollary 49.** There is $(C, D_C)$ that is quantum unlearnable, but $C$ cannot be copy-protected against $D_C$. Specifically, for any $C \leftarrow D_C$ with input length $n$, and for any QPT algorithm $(U_C, \rho_C)$ satisfying that for all $x \in \{0,1\}^n$,

$$\Pr[U_C(\rho_C, x) = C(x)] \geq 1 - \epsilon$$

for some negligible $\epsilon$, there is a QPT algorithm (pirate) that outputs a circuit $C'$, satisfying $C'(x) = C(x)$ for all $x \in \{0,1\}^n$, with probability negligibly close to 1.

**Further Discussion.** Notice that in our proof that $C$ is efficient de-quantizable, we just need to compute $U_C(\rho_C, x)$ at two different points $x_1 = 0 \cdot \cdots \cdot 0$ and $x_2 = a$, where the evaluation at $x_2$ is done homomorphically. This means that any scheme that lets a user evaluate a circuit $C$ at least 2 times (for 2 possibly different inputs) with non-negligible probability cannot be copy-protected. Such a user would be able to find all the parameters of the circuit, $(a, b, r, pk, O)$, successfully with non-negligible probability, hence it can prepare as many copies of a functionally equivalent circuit $C'$.

In our proof, we make use of the fact that $(U_C, \rho_C)$ evaluates correctly with probability close to 1. This is in order to ensure that the pirate can indeed evaluate at 2 points by uncomputing after it computes $C(0 \cdots 0)$. Since any copy-protection scheme can be amplified to have correctness negligibly close to 1 by providing multiple copies of the copy-protected states, our result also rules out copy-protection for non-negligible correctness parameter $\epsilon$, as long as the correctness of $(U_C, \rho_C)$ can be amplified to negligibly close to 1 by providing $\rho_C^{\otimes k}$ for some $k = \text{poly}(\lambda)$.

**Impossibility of Quantum VBB with single uncloneable state.** Our techniques also rule out the possibility of quantum VBB for classical circuits. In particular, this rules the possibility of quantum VBB for classical circuits with the obfuscated circuit being a single uncloneable state, thus resolving an open problem by Alagic and Fefferman [9].

**Proposition 50.** Assuming QFHE and QLWE, the following holds: there exists a circuit class $C$ such that there does not exist any quantum VBB for $C$.

**Proof.** We construct a circuit class $C = \{C_\lambda\}_{\lambda \in \mathbb{N}}$, where every circuit in $C_\lambda$ is of the form $C_{a,b,r,pk,O}$ defined in the proof of Theorem 44.

Given any quantum VBB of $C_{a,b,r,pk,O}$, there exists an adversary $A$ that recovers $b$ and outputs the first bit of $b$. The adversary $A$ follows steps 1-4 of $B$ defined in the proof of Proposition 48 and then outputs the first bit of $b'$. In the same proof, we showed that the probability that $b' = b$ is negligibly close to 1 and thus, the probability it outputs the first bit of $b$ is negligibly close to 1.

On the other hand, any QPT simulator $Sim$ with superposition access to $C_{a,b,r,pk,O}$ can recover $b$ with probability negligibly close to $1/2$. To prove this, we rely upon the proof of Claim 46. We
will start with the same simplifying assumptions as made in the proof of Claim 46. Suppose \( T \) is the number of superposition queries made by Sim to \( C_{a,b,r,\mathsf{pk},O} \). Let \( |\psi^0\rangle \) be the initial state of Sim and more generally, let \( |\psi^t\rangle \) be the state of Sim after \( t \) queries, for \( t \leq T \).

We define an alternate QPT simulator Sim’ which predicts the first bit of \( b \) with probability negligibly close to \( \text{Sim} \). Before we describe Sim’, we give the necessary preliminary background. Define \( |\phi^t\rangle = U_1 U_{t-1} \cdots U_1 |\psi^0\rangle \). We proved the following claim.

Claim 51. \( \langle \phi^t | \psi^t \rangle = 1 - \delta_t \) for every \( t \in [T] \).

Sim’ starts with the initial state \( |\psi^0\rangle \). It then computes \( |\phi^T\rangle \). If \( U \) is a unitary matrix Sim applies on \( |\psi^T\rangle \) followed by a measurement of a register \( D \) then Sim’ also performs \( U \) on \( |\phi^T\rangle \) followed by a measurement of \( D \). By the above claim, it then follows that the probability that Sim’ outputs 1 is negligibly close to the probability that Sim outputs 1. But the probability that Sim’ predicts the first bit of \( b \) is \( 1/2 \). Thus, the probability that Sim predicts the first bit of \( b \) is negligibly close to \( 1/2 \). □

5 qIHO for Compute-and-Compute Circuits

To complement the impossibility result, we present a construction of SSL for a subclass of evasive circuits. Specifically, the construction works for circuit classes that have q-Input-Hiding obfuscators. In the following section, we show that there are q-Input-Hiding obfuscators for Compute-and-Compare circuits.

Barak et al. [14] present a construction of input-hiding obfuscators secure against classical PPT adversaries; however, it is unclear whether their construction is secure against QPT adversaries. Instead we present a construction of input-hiding obfuscators (for a class of circuits different from the ones considered in [14]) from QLWE. Specifically, we show how to construct a q-input-hiding obfuscator for compute-and-compare circuits \( C_{cnc} \) with respect to a distribution \( D_C \) defined in Definition 12.

**Lemma 52** (qIHO for Compute-and-Compare Circuits). Consider a class of compute-and-compare circuits \( C_{cnc} \) associated with a distribution \( D_C \) (Definition 12). Assuming QLWE, there exists qIHO for \( C_{cnc} \).

**Proof.** We prove this in two steps: we first construct a qIHO for the class of point functions and then we use this to build qIHO for compute-and-compare class of circuits.

**qIHO for point functions:** To prove this, we use a theorem due to [14] that states that an average-case VBB for circuits with only polynomially many accepting points is already an input-hiding obfuscator for the same class of circuits; their same proof also holds in the quantum setting. Any q-average-case VBB for circuits with only polynomially many accepting points is already a qIHO. As a special case, we have a qIHO for point functions from q-average-case VBB for point functions. Moreover, we can instantiate q-average-case VBB for point functions from QLWE and thus, we have qIHO for point functions from QLWE.

We describe the formal details below. First, we recall the definition of average-case VBB.

**Definition 53** (q-Average-Case Virtual Black-Box Obfuscation (VBB)). Consider a class of circuits \( C = \{C_\lambda\}_{\lambda \in \mathbb{N}} \) associated with a distribution \( D_C \). We say that (Obf, Eval) is said to be a q-average-case
We consider a quantum analogue of a proposition proven in [14]. We omit the proof details since this is identical to the proof provided by [14] albeit in the quantum setting.

**Proposition 54.** Consider a class of evasive circuits \( C = \{ C_\lambda \}_{\lambda \in \mathbb{N}} \) associated with a distribution \( D_C \) such that each circuit \( C \in C_\lambda \) has polynomially many accepting points.

Assuming q-average-case virtual black-box obfuscation for point functions, there is a qIHO for compute-and-compare circuits below; we denote this by \( \text{qIHO} \) (see for example [54, 38]). Thus, we have the following proposition.

**Proposition 55 (q-Input-Hiding Obfuscator for Point Functions).** Consider the class of circuits \( C = \{ C_\lambda \}_{\lambda \in \mathbb{N}} \) defined as follows: every circuit \( C \in C \) is associated with \( x \) such that it outputs 1 on \( x \) and 0 on all other points.

Assuming QLWE, there is a qIHO for \( C \).

qIHO for compute-and-compare circuits from qIHO for point functions: We now show how to construct qIHO for compute-and-compare circuits \( C_{\text{cnc}} \) associated with distribution \( D_{\text{cnc}} \) (Definition 12), from qIHO for point functions. Denote \( \text{PO.qIHO} \) to be a qIHO for point functions \( G = \{ G_\lambda \}_{\lambda \in \mathbb{N}} \) associated with distribution \( D_{\text{po}} \), where \( D_{\text{po}} \) is a marginal distribution of \( D_{\text{cnc}} \) on \( \{ \alpha \} \). We construct qIHO for compute-and-compare circuits below; we denote this by cnc.qIHO.

\[ \text{cnc.qIHO.Obf}(1^\lambda, C[C, \alpha]) : \text{It takes as input security parameter } \lambda, \text{ compute-and-compare circuit } C[C, \alpha], \text{ associated with lock } \alpha. \text{ Compute } \text{PO.qIHO}(1^\lambda, G_\alpha \in G_\lambda) \text{ to obtain } \widetilde{G}_\alpha. \text{ Output } \widetilde{C} = (C, \widetilde{G}_\alpha). \]

\[ \text{cnc.qIHO.Eval}(\widetilde{C}, x) : \text{On input obfuscated circuit } \widetilde{C} = (C, \widetilde{G}_\alpha), \text{ input } x, \text{ do the following:} \]

- Compute \( C(x) \) to obtain \( \alpha' \).
- Compute \( \text{PO.Eval}(\widetilde{G}_\alpha, \alpha') \) to obtain \( b \).
- Output \( b \).

**Claim 56.** Assuming \( \text{PO.qIHO} \) is an input-hiding obfuscator for \( G \) associated with \( D_{\text{po}} \), cnc.qIHO is an input-hiding obfuscator for \( C \) associated with \( D_{\text{cnc}} \).

**Proof.** Suppose there exists a QPT adversary \( \mathcal{A} \) such that the following holds:

\[ \Pr \left[ \widetilde{C}(x) = 1 : \widetilde{C} \leftarrow \text{cnc.qIHO}(1^\lambda, C[C, \alpha]), \ x \leftarrow \mathcal{A}(1^\lambda, \widetilde{C}) \right] = \delta \]
Our first observation is that \( \Pr \left[ C(x) = \alpha \mid \tilde{C}(x) = 1 \right] = 1 \). Using this, we can construct another adversary \( \mathcal{A}' \) that violates the input-hiding property of \( \text{PO.qIHO} \). On input \( G\alpha(\cdot) \), \( \mathcal{A}' \) computes \( \tilde{C} = \left( C, G\alpha(\cdot) \right) \); denote the output to be \( x \). Finally, \( \mathcal{A}' \) outputs \( \alpha' = C(x) \).

From the above observations, it holds that \( \mathcal{A}' \) breaks the input-hiding property of \( \text{PO.qIHO} \) with probability \( \delta \). From the security of \( \text{PO.qIHO} \), we have that \( \delta = \text{negl}(\lambda) \) and thus the proof of the claim follows.

\( \square \)

**Conclusion**: Combining Claim 56 and Proposition 55, we have qIHO for compute-and-compare circuits from QLWE.

\( \square \)

## 6 Main Construction

In this section, we present the main construction of SSL; our SSL satisfies perfect lessor security (and not the generalized lessor security).

Let \( C = \{ C_\lambda \} \) be the class of \( S \)-searchable circuits associated with SSL. We denote \( s(\lambda) = \text{poly}(\lambda) \) to be the maximum size of all circuits in \( C_\lambda \). And let \( D_C \) be the distribution associated with \( C \).

**Ingredients.**

1. Input-hiding obfuscator \( \text{qIHO} = (\text{qIHO.Obf}, \text{qIHO.Eval}) \) for \( C \).
2. Subspace hiding obfuscation \( \text{shO} = (\text{shO.Obf}, \text{shO.Eval}) \). The field associated with \( \text{shO} \) is \( \mathbb{Z}_q \) and the dimensions will be clear below.
3. \( q \)-simulation-extractable non-interactive zero-knowledge system \( \text{qseNIZK} = (\text{CRSGen}, \mathcal{P}, \mathcal{V}) \) for NP with sub-exponential security as guaranteed in Lemma 27.

**Construction.** We describe the scheme of SSL below.

- **Setup(1^\lambda):** Compute \( \text{crs} \leftarrow \text{CRSGen}(1^{\lambda_1}) \), where \( \lambda_1 = \lambda + n \) and \( n \) is the input length of the circuit. Output \( \text{crs} \).
- **Gen(crs):** On input common reference string \( \text{crs} \), choose a random \( \frac{1}{2} \)-dimensional subspace \( A \subset \mathbb{Z}_q^{\lambda_1} \). Set \( \text{sk} = A \).

- **Lessor(sk = A, C):** On input secret key \( \text{sk} \), circuit \( C \in C_\lambda \), with input length \( n \),
  1. Prepare the state \( |A\rangle = \frac{1}{\sqrt{q^{\lambda_1/2}}} \sum_{a \in A} |a\rangle \).
  2. Compute \( \tilde{C} \leftarrow \text{qIHO.Obf}(C; r_\alpha) \)
  3. Compute \( \tilde{g} \leftarrow \text{shO}(A; r_A) \).
  4. Compute \( g_{\perp} \leftarrow \text{shO}(A_{\perp}; r_{A_{\perp}}) \).
5. Let \( x = S(C) \); that is, \( x \) is an accepting point of \( C \).

6. Let \( L \) be the NP language defined by the following NP relation.

\[
\mathcal{R}_L := \left\{ \left( \widetilde{g}, \widetilde{g}_\perp, \widetilde{C} \right), (A, r, r_A, r_{A^+}, C, x) \right\} \quad \text{if} \quad \widetilde{g} = \text{shO}(A; r_A) \quad \text{and} \quad \widetilde{g}_\perp = \text{shO}(A^+; r_{A^+}) \quad \text{and} \quad C = \text{qIHO.Ob}(C; r_0), \quad C(x) = 1
\]

Compute \( \pi \leftarrow \mathcal{P} \left( \text{crs}, (\widetilde{g}, \widetilde{g}_\perp, \widetilde{C}), (A, r, r_A, r_{A^+}, C, x) \right) \)

7. Output \( \rho_C = |\Phi_C\rangle\langle\Phi_C| = (|A\rangle\langle A|, \widetilde{g}, \widetilde{g}_\perp, \widetilde{C}, \pi) \).

- **Run(crs, \( \rho_C, x \):**
  1. Parse \( \rho_C \) as \( (\rho, \widetilde{g}, \widetilde{g}_\perp, \widetilde{C}, \pi) \). In particular, measure the last 4 registers.
     
     **Note:** This lets us assume that the input to those registers is just classical, since anyone about to perform **Run** might as well measure those registers themselves.

  2. Abusing notation, we denote the operation \( \text{shO.Eval}(\widetilde{g}, |x\rangle|y\rangle) = |x\rangle|y \oplus 1_A(x)\rangle \) by \( \widetilde{g}[|x\rangle|y\rangle] \). Compute \( \widetilde{g}[\rho \otimes |0\rangle\langle 0|] \) and measure the second register. Let \( a \) denote the outcome bit, and let \( \rho' \) be the post-measurement state.

  3. As above, we denote the operation \( \text{shO.Eval}(\widetilde{g}_\perp, |x\rangle|y\rangle) = |x\rangle|y \oplus 1_A(x)\rangle \) by \( \widetilde{g}_\perp[|x\rangle|y\rangle] \). Compute \( \widetilde{g}_\perp[\text{FT}\rho^{\text{FT}} \otimes |0\rangle\langle 0|] \) and measure the second register. Let \( b \) denote the outcome bit.

     **Note:** in Step 2 and 3, **Run** is projecting \( \rho \) onto \( |A\rangle\langle A| \) if \( a = 1 \) and \( b = 1 \).

  4. Afterwards, perform the Fourier Transform again on the first register of the post-measurement state, let \( \rho'' \) be the resulting state.

  5. Compute \( c \leftarrow \mathcal{V} \left( \text{crs}, (\widetilde{g}, \widetilde{g}_\perp, \widetilde{C}), \pi \right) \)

  6. If either \( a = 0 \) or \( b = 0 \) or \( c = 0 \), reject and output \( \perp \).

  7. Compute \( y \leftarrow \text{qIHO.Eval}(\widetilde{C}, x) \).

  8. Output \( (\rho'', \widetilde{g}, \widetilde{g}_\perp, \widetilde{C}, \pi) \) and \( y \).

- **Check(sk = A, \( \rho_C \):**
  1. Parse \( \rho_C \) as \( (\rho, \widetilde{g}, \widetilde{g}_\perp, \widetilde{C}, \pi) \).

  2. Perform the measurement \{\( |A\rangle\langle A|, I - |A\rangle\langle A| \)\} on \( \rho \). If the measurement outcome corresponds to \( |A\rangle\langle A| \), output 1. Otherwise, output 0.

**Lemma 57** (Overwhelming probability of perfect correctness.). The above scheme satisfies \( e = \text{negl}(\lambda) \) correctness.

**Proof.** Since qIHO is perfectly correct, it suffices to show that **Run** will not output \( \perp \). For this to happen, it has to be the case that \( a, b, c = 1 \). If \( \pi \) is a correct proof, then by perfect correctness of
qseNIZK, we have that $\Pr[c = 1] = 1$. Since $\widetilde{g} = \text{shO}(A)$ and $\widetilde{g}_\perp = \text{shO}(A^\perp)$, the $a$ and $b$ check Run just projects the input quantum state onto $|A\rangle\langle A|$ (with probability negligibly close to 1, due to correctness of shO). But the input is the state $|A\rangle$ in the first place, so $a = 1$ and $b = 1$ with negligibly probability close to 1.

To see that correctness of Check holds, note that the leased state is $\rho = |A\rangle\langle A|$, which will pass the check with probability 1.

\[ \square \]

**Lemma 58.** Fix $\beta = \mu(\lambda)$, where $\mu(\lambda)$ is any non-negligible function. Assuming the security of qIHO, qseNIZK and shO, the above scheme satisfies $(\beta, \gamma, D_C)$-perfect lessor security, where $\gamma$ is a negligible function.

**Proof.** For any QPT adversary $A$, define the following event.

**Process($1^\lambda$):**

- $\text{crs} \leftarrow \text{Setup}(1^\lambda)$,
- $\text{sk} \leftarrow \text{Gen}(\text{crs})$,
- $C \leftarrow D_C(\lambda)$,
- $(\rho_C = (|A\rangle\langle A|, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi)) \leftarrow \text{Lessor}(\text{sk}, r)$
- $(\tilde{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}^{(1)}_\perp, \tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{(2)}_\perp, \pi^{(2)}, \sigma^*) \leftarrow A(\text{crs}, \rho_C)$

That is, $A$ outputs two copies; the classical part in the first copy is $(\tilde{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}^{(1)}_\perp, \pi^{(1)})$ and the classical part in the second copy is $(\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{(2)}_\perp, \pi^{(2)})$. Moreover, it outputs a single density matrix $\sigma^*$ associated with two registers $R_1$ and $R_2$; the state in $R_1$ is associated with the first copy and the state in $R_2$ is associated with the second.

- $\sigma^*_1 = \text{Tr}_2[\sigma^*]$
- $\rho_C^{(1)} = (\sigma^*_1, \tilde{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}^{(1)}_\perp, \pi^{(1)}) \wedge \rho_C^{(2)} = (\rho_2(\text{sk}, \sigma^*), \tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{(2)}_\perp, \pi^{(2)})$

To prove the lemma, we need to prove the following:

$$ \Pr \left[ \begin{array}{c} \text{Check}(\text{sk}, \rho_C^{(1)}) = 1 \\ \forall x, \Pr[\text{Run}(\text{crs}, \rho_C^{(2)}_x, x) = C(x)] \geq \beta \end{array} : \text{Process } (1^\lambda) \right] = \gamma $$

Consider the following:

- Define $\gamma_1$ as follows:

$$ \Pr \left[ \begin{array}{c} \text{Check}(\text{sk}, \rho_C^{(1)}) = 1 \\ \forall x, \Pr[\text{Run}(\text{crs}, \rho_C^{(2)}_x, x) = C(x)] \geq \beta \\ (\tilde{C}, \tilde{g}, \tilde{g}_\perp) = (\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{(2)}_\perp) \end{array} : \text{Process } (1^\lambda) \right] = \gamma_1 $$

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• Define $\gamma_2$ as follows:

$$
\gamma_2 = \begin{cases}
\text{Check}(sk, \rho_C^{(1)}) = 1 \\
\land \\
\Pr \left( \forall x, \Pr[\text{Run}(\text{crs}, \rho_C^{(2)}, x) = C(x)] \geq \beta : \text{Process}\left(1^\lambda\right) \right)
\end{cases} = \gamma_2
$$

Note that $\gamma = \gamma_1 + \gamma_2$. In the next two propositions, we prove that both $\gamma_1$ and $\gamma_2$ are negligible which will complete the proof of the lemma.

**Proposition 59.** $\gamma_1 \leq \text{negl}(\lambda)$

**Proof.** If $\langle A | \sigma^*_i | A \rangle \leq \text{negl}(\lambda)$, we would be done, since then the probability that Check$(sk, \sigma^*_i)$ passes in the first place would be negligible.

Suppose $\langle A | \sigma^*_i | A \rangle$ is non-negligible. We now prove the following claim.

**Claim 60.** $\langle A | P_2(sk, \sigma^*) | A \rangle \leq \text{negl}(\lambda)$

**Proof.** Suppose not. Then, we can use $\mathcal{A}$ to break quantum no-cloning. Specifically, Zhandry [56] showed that no QPT algorithm on input $\langle A \rangle$, $\tilde{g} := \text{shO}(A)$, $\tilde{g}_\perp := \text{shO}(A^\perp)$ can prepare the state $\langle A \rangle ^{\otimes 2}$ with non-negligible probability. We will show that $\mathcal{A}$ allows us to do exactly this if $\langle A | P_2(sk, \sigma^*) | A \rangle$ is non-negligible.

Consider the following adversary $\mathcal{B}'$. It runs $\mathcal{A}$ and then projects the output of $\mathcal{A}$ onto $\langle A \rangle \langle A \rangle ^{\otimes 2}$; the output of the projection is the output of $\mathcal{B}'$.

$\mathcal{B}'(C)$:

1. Compute crs, sk as in the construction

2. Compute $\rho_C \leftarrow \text{Lessor}(sk, C)$. Let $\rho_C = \langle A \rangle \langle A \rangle , \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi$.

3. Compute $\mathcal{A}(crs, \rho_C)$ to obtain $\langle \tilde{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}_\perp^{(1)}, \pi^{(1)}, \tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}_\perp^{(2)}, \pi^{(2)}, \sigma^* \rangle$.

4. Then, project $\sigma^*$ onto $\langle A \rangle \langle A \rangle ^{\otimes 2}$ by using $\tilde{g}$ and $\tilde{g}_\perp$. Let $m$ be the outcome of this projection, so $m = 1$ means that the post measured state is $\langle A \rangle \langle A \rangle ^{\otimes 2}$.

5. Output $m$.

The projection $\langle A \rangle \langle A \rangle ^{\otimes 2}$ can be done by first projecting the first register onto $\langle A \rangle \langle A \rangle$ and then the second register. Conditioned on the first register passing Check, means that $\sigma^*_i$ is successfully projected onto $\langle A \rangle \langle A \rangle$. By our assumption that $\langle A | \sigma^*_i | A \rangle$ is non-negligible, this will happen with non-negligible probability. Conditioned on this being the case, if $\langle A | P_2(sk, \sigma^*) | A \rangle$ is non-negligible, then projecting the second register onto $\langle A \rangle \langle A \rangle$ will also succeed with non-negligible probability. This means that $m = 1$ with non-negligible probability.

Consider the following adversary. It follows the same steps as $\mathcal{B}'$ except in preparing the states $\langle A \rangle$ and computing obfuscations $\tilde{g}, \tilde{g}_\perp$; it gets these quantities as input. Moreover, it simulates
the proof \( \pi \) instead of computing the proof using the honest prover. This is because unlike \( B' \), the adversary \( B \) does not have the randomness used in computing \( \tilde{g} \) and \( \tilde{g}_\perp \) and hence cannot compute the proof \( \pi \) honestly.

\[
\mathcal{B}(|A\rangle, \tilde{g}, \tilde{g}_\perp):
\]

1. Sample randomness \( r_o \) and compute \( \tilde{C} \leftarrow \text{qIHO.Obf}(C; r_o) \).
2. Let \( \text{FkGen} \) and \( \text{Sim} \) be associated with the simulation-extractability property of \text{qseNIZK}. Compute \((\tilde{\text{crs}}, \tilde{\text{td}}) \leftarrow \text{FkGen}(1^\lambda)\).
3. Compute \((\pi, \text{st}) \leftarrow \text{Sim}\left(\tilde{\text{crs}}, \tilde{\text{td}}, \left(\tilde{g}, \tilde{g}_\perp, \tilde{C}\right)\right)\)
4. Let \( \rho_C = \langle |A\rangle \langle A|, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi \rangle\)
5. Run \( \mathcal{A}(\tilde{\text{crs}}, \rho_C) \) to obtain \((\tilde{C}^{(1)}, \tilde{g}^{(1)}_\perp, \tilde{g}^{(1)}_\perp, \pi^{(1)}, \tilde{C}^{(2)}, \tilde{g}^{(2)}_\perp, \tilde{g}^{(2)}_\perp, \pi^{(2)}, \sigma^*)\).
6. Then, project \( \sigma^* \) onto \(|A\rangle \langle A| \otimes 2\) by using \( \tilde{g} \) and \( \tilde{g}_\perp \). Let \( m \) be the outcome of this projection, so \( m = 1 \) means that the post measured state is \(|A\rangle \langle A| \otimes 2\).
7. Output \( m \).

Note that from the q-simulation-extractability property\(^9\) of \text{qseNIZK}, it follows that the probability that \( B \) outputs 1 is negligibly close to the probability that \( B' \) outputs 1 because everything else is sampled from the same distribution. This implies that \( B \) on input \(|A\rangle, \tilde{g}, \tilde{g}_\perp \) outputs \(|A\rangle \otimes 2\) with non-negligible probability, contradicting \([56]\).

\[\Box\]

At this point, we want to show that if \( \left(\tilde{g}^{(2)}_\perp, \tilde{g}_\perp^{(2)}\right) = (\tilde{g}, \tilde{g}_\perp) \), and \( \langle A|P_2(\text{sk}, \sigma^*)|A\rangle \leq \text{negl}(\lambda) \), then the probability that \( \text{Run}(\text{crs}, P_2(\text{sk}, \sigma^*), x) \) evaluates \( C \) correctly is negligible.

By correctness of \( \text{shO} \), we have

\[
\Pr[\forall x \; \tilde{g}^{(2)}_\perp(x) = 1_A(x)] \geq 1 - \text{negl}(\lambda)
\]

\[
\Pr[\forall x \; \tilde{g}_\perp^{(2)}(x) = 1_{A^\perp}(x)] \geq 1 - \text{negl}(\lambda)
\]

This means that with probability negligibly close to 1, the first thing that the Run algorithm does on input \((P_2(\text{sk}, \sigma^*), \tilde{g}^{(2)}_\perp, \tilde{g}_\perp^{(2)}, \tilde{C}, \pi) \) is to measure \(|A\rangle \langle A|, I - |A\rangle \langle A|\) on \( P_2(\text{sk}, \sigma^*) \). If \( I - |A\rangle \langle A| \) is obtained, then the Run algorithm will output \( \perp \). By Claim \( 60 \), the probability that this happens is negligibly close to 1. Formally, when \( \tilde{g} \) and \( \tilde{g}_\perp \) are subspace obfuscations of \( A \) and \( A^\perp \) respectively, the check \( a = 1 \) and \( b = 1 \) performed by the Run algorithm is a projection onto \(|A\rangle \langle A| \).

\(^9\)We don’t need the full-fledged capability of q-simulation-extractability to argue this part; we only need q-zero-knowledge property which is implied by q-simulation-extractability.
\[ \Pr[a = 1, b = 1] = \text{Tr}[^4 \Pi_A \Pi \Pi A P_2(\text{sk}, \sigma^*)] \]
\[ = \text{Tr}[\langle A | P_2(\text{sk}, \sigma^*) \rangle] \]
\[ = \langle A | P_2(\text{sk}, \sigma^*) | A \rangle \]
\[ \leq \text{negl}(\lambda) \]

where \( \Pi_A = \sum_{a \in A} |a \rangle \langle a | \) and \( \Pi_{A^\perp} = \sum_{a \in A^\perp} |a \rangle \langle a | \). From this, we have that \( \Pr[\text{Run(crs, } \rho_C^{(2)}, x) = \perp] \geq 1 - \text{negl}(\lambda) \), and we have \( \Pr[\text{Run(crs, } \rho_C^{(2)}, x) = C(x)] \leq \text{negl}(\lambda) \) with probability negligibly close to 1.

This finishes our proof that if \( \beta \) is non-negligible, then \( \gamma_1 \leq \text{negl}(\lambda) \). \( \square \)

**Proposition 61.** \( \gamma_2 \leq \text{negl}(\lambda) \).

**Proof.** We consider the following hybrid process.

HybProcess\(_1(1^3)\):

- (crs, td) \( \leftarrow \) FkGen \((1^3)\),
- \( \text{sk} \leftarrow \text{Gen(crs)} \),
- \( C \leftarrow D_C(\lambda) \),
- Sample a random \( \frac{1}{2} \)-dimensionall sub-space \( A \subset Z_4^2 \). Prepare the state \( |A\rangle = \frac{1}{\sqrt{2^{1/2}}} \sum_{a \in A} |a\rangle \).
- Compute \( \tilde{g} \leftarrow \text{shO} (A; r_A) \),
- Compute \( \tilde{g}_{\perp} \leftarrow \text{shO} (A^\perp; r_{A^\perp}) \),
- Compute \( \bar{C} \leftarrow \text{qlHO.Obf} (C; r_\sigma) \),
- \( (\pi, st) \leftarrow \text{Sim}_1 \left( \text{crs, td, } \left( \tilde{g}, \tilde{g}_{\perp}, \bar{C} \right) \right) \)
- Set \( \rho_C = \left( |A\rangle \langle A|, \tilde{g}, \tilde{g}_{\perp}, \bar{C}, \pi \right) \).
- \( \left( \bar{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}_{\perp}^{(1)}, \pi^{(1)}, \bar{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}_{\perp}^{(2)}, \pi^{(2)}, \sigma^* \right) \leftarrow A (\text{crs, } \rho_C) \)
- Set \( \sigma_1^* = \text{Tr}_2[\sigma^*] \)
- Set \( \rho_C^{(1)} = \left( \sigma_1^*, \bar{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}_{\perp}^{(1)}, \pi^{(1)} \right) \) and \( \rho_C^{(2)} = \left( P_2(\text{sk}, \sigma^*), \bar{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}_{\perp}^{(2)}, \pi^{(2)} \right) \)
- \( \left( A^*, r_A^*, r_{A^*}, C^*, x^* \right) \leftarrow \text{Sim}_2 \left( \pi, \left( \tilde{g}_{\perp}^{(2)}, \bar{C}^{(2)} \right), \pi^{(2)} \right) \).

The proof of the following claim follows from the q-simulation-extractactability property of qseNIZK.
Claim 62. Assuming that qseNIZK satisfies q-simulation extractability property secure against QPT adversaries running in time $2^n$, we have:

$$\Pr\left[\begin{array}{c}
\left(\tilde{g}^{(2)}, \tilde{g}^{(1)}, \tilde{C}^{(2)}, \left(A, r_o, r_A, r^{A^\perp}, C, x\right)\right) \in \mathcal{R}(1^n)
\land
\text{Check}(\text{sk}, \rho^{(1)}) = 1
\land
\forall x, \Pr[\text{Run}(\text{crs}, \rho^{(2)}, x) = C(x)] \geq \beta
\land
\left(\tilde{C}, \tilde{g}, \tilde{g}^{\perp}\right) \neq \left(\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{\perp^{(2)}}\right)
\end{array}\right] = \delta_1$$

Then, $|\delta_1 - \gamma_2| \leq \text{negl}(\lambda)$.

Remark 63. Note that $n$ is smaller than the length of the NP instance and thus, we can invoke the sub-exponential security of the seNIZK system guaranteed in Lemma 27.

Proof of Claim 62. Consider the following qseNIZK adversary $B$:

- It gets as input crs.
- It samples and computes $(C, A, \tilde{g}, \tilde{g}^{\perp}, \tilde{C})$ as described in HybProcess$_1(1^n)$. It sends the following instance-witness pair to the challenger of seNIZK:

$$\left(\left(C, A, \tilde{g}, \tilde{g}^{\perp}, \tilde{C}\right), ((A, r_o, r_A, r^{A^\perp}, C, x)\right),$$

where $r_o, r_A, r^{A^\perp}$ is, respectively, the randomness used to compute obfuscations $\tilde{g}$, $\tilde{g}^{\perp}$ and $\tilde{C}$.
- The challenger returns back $\pi$.
- $B$ then sends $\left(|A\rangle, \tilde{g}, \tilde{g}^{\perp}, \tilde{C}, \pi\right)$ to $A$.
- $A$ then outputs $\left(\tilde{C}^{(1)}, \tilde{g}^{(1)}, \tilde{g}^{\perp^{(1)}}, \pi^{(1)}, \tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{\perp^{(2)}}, \pi^{(2)}, \sigma^*\right)$.
- $B$ sets $\sigma^*_1 = \text{Tr}_2[\sigma^*]$.
- Finally, $B$ performs the following checks:
  - Verify if the first copy passes the Check algorithm: Perform the measurement $\{|A\rangle\langle A|, I - |A\rangle\langle A|\}$ on $\sigma^*_1$. If the measurement outcome does not correspond to $|A\rangle\langle A|$, output $\perp$.
  - Verify if second copy computes C: If the measurement above does not output $\perp$, set $\rho^{(2)}_C = \left(P_2(\text{sk}, \sigma^*), \tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{\perp^{(2)}}, \pi^{(2)}\right)$. For every $x$, check if $\text{Run}(\text{crs}, \rho^{(2)}_C, x) = C(x)$. If for any $x$, the check fails, output $\perp$. // Note that this step takes time $2^{O(n + \log(n))}$.
  - Verify if the classical parts are different: Check if $\left(\tilde{C}, \tilde{g}, \tilde{g}^{\perp}\right) \neq \left(\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{g}^{\perp^{(2)}}\right)$, output $\perp$.
- Output $\left(\tilde{g}^{(2)}, \tilde{g}^{\perp^{(2)}}, \tilde{C}^{(2)}\right), \pi^{(2)}$. 

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Note that $\mathcal{B}$ is a valid qseNIZK adversary: it produces a proof on an instance different from one for which it obtained a proof (either real or simulated) and moreover, the proof produced by $\mathcal{B}$ (conditioned on not $\perp$) is an accepting proof.

If $\mathcal{B}$ gets as input honest CRS and honestly generated proof $\pi$ then this corresponds to $\text{Process}_1(1^\lambda)$ and if $\mathcal{B}$ gets as input simulated CRS and simulated proof $\pi$ then this corresponds to $\text{HybProcess}_1(1^\lambda)$.

Thus, from the security of q-simulation-extractable NIZKs, we have that $|\gamma - \delta_1| \leq \text{negl}(\lambda)$. □

We first prove the following claim.

**Claim 64.**

\[
\begin{array}{l}
\left(\left(\tilde{\gamma}^{(2)}(2), \tilde{g}^{(2)}(2), \tilde{C}^{(2)}(2)\right), (A^*, r^*_o, r^*_A, r^*_\lambda, C^*, x^*)\right) \in \mathcal{R}(L) \land \forall x, \text{Pr}\left[\text{Run}\left(\text{crs}, \rho^{(2)}_C, x\right) = C(x)\right] \geq \beta
\end{array}
\implies C(x^*) = 1,
\]

**Proof.** We first claim that $\forall x, \text{Pr}\left[\text{Run}\left(\text{crs}, \rho^{(2)}_C, x\right) = C(x)\right] \geq \beta$ implies that $\tilde{C}^{(2)} \equiv C$, where $\equiv$ denotes functional equivalence. Suppose not. Let $x'$ be an input such that $\tilde{C}^{(2)}(x') \neq C(x')$ then this means that $\text{Run}(\text{crs}, \rho^{(2)}_C, x')$ always outputs a value different from $C(x')$; follows from the description of $\text{Run}$. This means that $\text{Pr}[\text{Run}(\text{crs}, \rho^{(2)}_C, x') = C(x')] = 0$, contradicting the hypothesis.

Moreover, $\left(\left(\tilde{\gamma}^{(2)}(2), \tilde{g}^{(2)}(2), \tilde{C}^{(2)}(2)\right), (A^*, r^*_o, r^*_A, r^*_\lambda, C^*, x^*)\right) \in \mathcal{R}(L)$ implies that $\tilde{C}^{(2)} = \text{qIHO}(1^\lambda, C^*, r^*_o)$ and $C(x^*) = 1$. Furthermore, perfect correctness of $\text{qIHO}$ implies that $\tilde{C}^{(2)} \equiv C^*$.

So far we have concluded that $\tilde{C}^{(2)} \equiv C$, $\tilde{C}^{(2)} \equiv C^*$ and $C(x^*) = 1$. Combining all of them together, we have $C(x^*) = 1$.

\[\Box\]

Consider the following inequalities.

\[
\begin{align*}
\delta_1 &= \text{Pr}\left[\left(\left(\tilde{\gamma}^{(2)}(2), \tilde{g}^{(2)}(2), \tilde{C}^{(2)}(2)\right), (A^*, r^*_o, r^*_A, r^*_\lambda, C^*, x^*)\right) \in \mathcal{R}(L) \land \text{Check}(s, \rho^{(1)}_C) = 1 \land \forall x, \text{Pr}[\text{Run}(\text{crs}, \rho^{(2)}_C, x) = C(x)] \geq \beta \land \left(\tilde{C}, \tilde{g}, \tilde{\gamma}\right) \neq \left(\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{\gamma}^{(2)}\right)\right] & : \text{HybProcess}_1
\end{align*}
\]

\[
\begin{align*}
&= \text{Pr}\left[C(x^*) = 1 \land \text{Check}(s, \rho^{(1)}_C) = 1 \land \left(\tilde{C}, \tilde{g}, \tilde{\gamma}\right) \neq \left(\tilde{C}^{(2)}, \tilde{g}^{(2)}, \tilde{\gamma}^{(2)}\right)\right] & : \text{HybProcess}_1
\end{align*}
\]

\[
\leq \text{Pr}[C(x^*) = 1 : \text{HybProcess}_1]
\]

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Let $\Pr[C(x^*) = 1 : \text{HybProcess}_1] = \delta_2$.

**Claim 65.** Assuming the q-input-hiding property of $\text{qIHO}$, we have $\delta_2 \leq \text{negl}(\lambda)$

**Proof.** Suppose $\delta_2$ is not negligible. Then we construct a QPT adversary $B$ that violates the q-input-hiding property of $\text{qIHO}$, thus arriving at a contradiction.

$B$ now takes as input $\tilde{C}$ (an input-hiding obfuscator of $C$), computes $(\tilde{crs}, td) \leftarrow \text{FkGen}(1^\lambda)$ and then computes $\rho_C = (|A\rangle, \tilde{g}, \tilde{g}_\perp, \tilde{C}, \pi)$ as computed in $\text{HybProcess}_1$. It sends $(\tilde{crs}, \rho_C)$ to $A$ who responds with $(\tilde{C}(1), \tilde{c}^{(1)}, \tilde{g}_\perp^{(1)}, \pi^{(1)}, \tilde{C}^{(2)}, \tilde{g}_\perp^{(2)}, \pi^{(2)}, \sigma^*)$. Compute $(A^*, r_o^*, r_A^*, r_{A^\perp}^*, C^*, x^*)$ by generating $\text{Sim}_2(st, (\tilde{g}^{(2)}, \tilde{g}_\perp^{(2)}, \tilde{C}^{(2)}), \pi^{(2)})$, where $st$ is as defined in $\text{HybProcess}_1$. Output $x^*$.

Thus, $B$ violates the q-input-hiding property of $\text{qIHO}$ with probability $\delta_2$ and thus $\delta_2$ has to be negligible. □

Combining the above observations, we have that $\gamma_2 \leq \text{negl}(\lambda)$ for some negligible function $\text{negl}$. This completes the proof. □

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