Decay of correlations, Lyapunov exponents and anomalous diffusion in the Sinai billiard

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Abstract - We compute the decay of the velocity autocorrelation function, the Lyapunov exponent and the diffusion constant for the Sinai billiard within the framework of dynamical zeta functions. The asymptotic decay of the velocity autocorrelation function is found to be $C(t) \sim c(R)/t$. The Lyapunov exponent for the corresponding map agrees with the conjectured limit $\lambda_{\text{map}} \to -2 \log(R) + C$ as $R \to 0$ where $C = 1 - 4 \log 2 + 27/(2\pi^2) \cdot \zeta(3)$. The diffusion constant of the associated Lorentz gas is found to be divergent $D(t) \sim \log t$.

1 Theory

1.1 Chaotic averages

Consider a chaotic Hamiltonian system with two degrees of freedom. Assign a weight $w(x_0, t)$ to the trajectory starting at phase space point $x_0$ and evolving during time $t$ (to point $x(x_0, t)$) in such a way that it is multiplicative along the flow: $w(x_0, t_1 + t_2) = w(x_0, t_1)w(x(x_0, t_1), t_2)$. The phase space average of $w(x_0, t)$ may be expanded in terms of periodic orbits as

$$\lim_{t \to \infty} \langle w(x_0, t) \rangle = \lim_{t \to \infty} \sum_p T_p \sum_{r=1}^\infty w_p^r \frac{\delta(t - rT_p)}{(1 - \Lambda_p^r)(1 - 1/\Lambda_p^r)},$$

where $r$ is the number of repetitions of primitive orbit $p$, having period $T_p$, and $\Lambda_p$ is the expanding eigenvalue of the Jacobian (transverse to the flow). $w_p$ is the

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weight integrated along with cycle \( p \). Zeta functions are introduced by observing that the average (1) may be written as

\[
\lim_{t \to \infty} \langle w(x_0, t) \rangle = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{Z_w(s)}{Z_w(0)} ds ,
\]

with the zeta function (2)

\[
Z_w(s) = \prod_p \prod_{m=0}^\infty \left( 1 - w_p e^{-sT_p} |\Lambda_p|^{m+n_p} \right)^{m+1} .
\]

The expressions are valid also for an area preserving map, like a Poincaré map of the above system. The time variable \( t \) and period \( T_p \) are now discrete and usually denoted as \( n \) and \( n_p \). The zeta function (3) is usually expressed in the \( z = e^{s} \) plane. The Dirac delta in (1) should be replaced by a Kronecker delta and the integral (2) accordingly performed along a circle \( |z| = C \).

We will be interested in three weights of the required type. In the discussion of Lyapunov exponents we will be interested in

\[
w_\lambda(x_0, t) = \Lambda(x_0, t)^\beta ,
\]

where \( \Lambda(x_0, t) \) is the (leading) eigenvalue of the Jacobian of the map from \( x_0 \) to \( x(x_0, t) \). It is not exactly multiplicative but we will discuss that problem below.

We now express the Lyapunov exponent in terms of the associated zeta function

\[
\lambda \equiv \lim_{t \to \infty} \frac{1}{t} \langle \log |\Lambda(x_0, t)| \rangle = \lim_{t \to \infty} \frac{1}{t} \int_{-\pi+\epsilon}^{i\pi+\epsilon} e^{st} \frac{d}{d\beta} \log Z_\lambda(s) |_{\beta=0} ds .
\]

In the discussion of diffusion we will consider

\[
w_D(x_0, t) = e^{\beta\cdot(\tilde{x}(x_0, t) - \tilde{x}_0)} ,
\]

where \( \tilde{x} \) is the configuration space part of the phase space vector \( x \). The diffusion constant may now be expressed in terms of the associated zeta function

\[
D = \lim_{t \to \infty} \frac{1}{2t} \langle (\tilde{x}(x_0, t) - \tilde{x}_0)^2 \rangle = \lim_{t \to \infty} \frac{1}{2t} \left( \frac{d^2}{d\beta_1^2} + \frac{d^2}{d\beta_2^2} \right) e^{\beta\cdot(\tilde{x}(x_0, t) - \tilde{x}_0)} |_{\beta=0}
\]

\[
= \lim_{t \to \infty} \frac{1}{2t} \int_{-\pi+\epsilon}^{i\pi+\epsilon} e^{st} \frac{d^2}{d\beta_1^2} + \frac{d^2}{d\beta_2^2} \frac{d}{ds} \log Z_D(s) |_{\beta=0} ds .
\]

Next assume that we have a Poincaré map defined by a surface of section \( \Omega_{s.o.s.} \). Then, using weight

\[
e^{-sT(x_0, n)} w(x_0, n) ,
\]

(8)
where \( T(x_0, n) \) and \( w(x_0, n) \) is the integrated time and weight along a trajectory starting at \( x_0 \in \Omega_{s.o.s.} \) and evolving during \( n \) iterates of the map. We thus get a zeta function which combines the description of the map and the flow.

\[
Z(s, z) = \prod_p \prod_{m=0}^{\infty} \left( 1 - w_p z^n e^{-s T_p} \frac{1}{|\Lambda_p| \Lambda_p^m} \right)^{m+1}.
\] (9)

A more lucid introduction to chaotic averages may be found in [3] with proper references.

### 1.2 Approximate zeta functions

Given a time continuous system, suppose we have defined a surface of section in such a way that the corresponding map destroys all information from one iterate to another. Or more precisely, the weight associated with consecutive iterates are uncorrelated: \( \langle w_i w_{i+1} \rangle = \langle w_i \rangle \langle w_{i+1} \rangle \). This is called Assumption A in ref. [6]. This can of course not be exactly fulfilled in practice. But suppose our system is intermittent and we put the s.o.s. in the border between the laminar and the chaotic phase. Then the chaotic phase will do its best to destroy the memory from one laminar phase to the next. We will now average the the weight over the surface of section.

\[
\langle e^{-s T(x_0, n)} w(x_0, n) \rangle_{s.o.s.} = \langle e^{-s T(x_0, 1)} w(x_0, 1) \rangle^n_{s.o.s.},
\] (10)

which follows from assumption A. This type of behaviour is equivalent to their being only one zero \( z_0(s) = \langle \exp(-s T(x_0, 1)) w(x_0, 1) \rangle_{s.o.s.} \) of \( Z(s, z) \) in the \( z \)-plane. In the language of Ruelle resonances this means that there is an infinite gap to the first resonance. This in turn implies that the \( Z(s, z) \) may be written.

\[
Z(s, z) = e^{h(z, s)} \left( z - \langle e^{-s T(x_0, 1)} w(x_0, 1) \rangle_{s.o.s.} \right),
\] (11)

where \( h(z, s) \) is entire in \( z \) and \( s \). The zeta function for the flow is obtained as \( Z_{\text{flow}}(s) = Z(z = 1, s) \), see eq. (9), and we get

\[
Z_{\text{flow}}(s) = 1 - \langle e^{-s T(x_0, 1)} w(x_0, 1) \rangle_{s.o.s.}.
\] (12)

We had to put \( h = (z = 1, s) = 0 \) in order for \( Z(s) \) to obtain the correct limit at infinity. Normally, the best one can hope for is a finite gap to the leading resonance of the Poincaré map. The zeta function above is then only approximate, indicated by putting a hat on it: \( \hat{Z} \), we also suppress the clumsy subscript \( \text{flow} \).

The first two terms in an expansion around \( s = 0 \) for the special case \( w = 1 \) are however exact: \( \hat{Z}(s = 0) = Z(s = 0) = 0 \) and \( \hat{Z}'(s = 0) = Z'(s = 0) \) for linearity reasons.

We can reformulate the formula above for the special case where \( w(x_0, 1) \) is a function of \( T(x_0, 1) \equiv \Delta(x_0) \) denoted \( w(\Delta) \). Then the probability distribution of recurrence times is

\[
p(\Delta) = \langle \delta(\Delta - \Delta(x_0)) \rangle_{s.o.s.},
\] (13)
and
\[ \hat{Z}(s) = 1 - \int_0^\infty w(\Delta)p(\Delta)e^{-s\Delta}d\Delta \, . \] (14)

Approximate zeta functions were introduced in refs. [4, 5], to a large extent inspired by ref. [6], the above derivation is though much more easy to follow than the one in [4].

## 2 The Sinai Billiard

We will apply the method of approximate zeta functions to the Sinai billiard [7], a particle with unit speed bouncing inside a unit square with a scattering circle of radius \( R \) in the middle. The only sensible choice of \( \Omega_{s.o.s.} \) is the disk itself. The disk to disk map will be hyperbolic but without a finite Markov partition, most likely it will have a finite gap and exhibit exponential mixing [8, 9]. In the small scatterer limit one may derive the following exact expression for the distribution of recurrence times [10].

\[
p_{R \to 0}(\Delta) = \begin{cases} 12R & \xi < 1 \\ \frac{12R}{\pi^2} \log(\xi) + 4(\xi-1)^2 \log(\xi-1) - (2-\xi)^2 \log(2-\xi) + 4(\xi-1)^2 \log(\xi-1) - (2-\xi)^2 \log(2-\xi) & \xi > 1 \end{cases} \, ,
\] (15)

where \( \xi = \Delta^2 R \) In the following we will e.g. need the mean \( <\Delta> \) in this distribution, and the large \( \Delta \) limit: \( p \sim 1/(\pi^2 R^2 \Delta^3) \).

### 2.1 The Lyapunov exponent

Consider a trajectory with \( n \) intersections with \( \Omega_{s.o.s.} \). The consecutive disk to disk time of flight and scattering angles will be denoted as \( \Delta_i \) and \( \alpha_i \) being functions of the s.o.s coordinate \( x_i \). It may be shown [10] that the (by construction) multiplicative weight
\[
w(x_0, n) = \prod_{i=1}^n \frac{2\Delta_i}{R \cos \alpha_i} \, ,
\] (16)
give the correct \( R \to 0 \) limit of the Lyapunov exponent
\[
\lambda_{map} = \lim_{n \to \infty} \frac{1}{n} \left( \log w(x_0, n)_{s.o.s} + O(R) \right) = \left( \log \frac{2\Delta(x)}{R \cos \alpha(x)} \right)_{s.o.s} + O(R) \, ,
\] (17)

\[ = \log(2/R) + \langle \log \Delta(x) \rangle_{s.o.s} - \langle \log \cos \alpha(x) \rangle_{s.o.s} + O(R) \, . \]

The problem is then reduced to computing the average of \( \log \Delta \) (the average of \( \cos \alpha \) is trivial). To that end we use eq. (15). The result is [10]
\[
\lambda_{map} \to -2 \log R + C \quad R \to \infty \, ,
\] (18)
with \( C = 1 - 4 \log 2 + 27/(2\pi^2) \cdot \zeta(3) \), confirming previous conjectures \( 11, 12, 13 \) but with \( C \) computed exactly for the first time.

What about the Lyapunov exponent of the flow. The zeta function is now, cf. eqs. (4) and (12),

\[
\hat{Z} = 1 - \langle \Lambda(x_0) e^{-s\Delta(x_0)} \rangle_{s, o.s} = s(\Delta) + O(s^2 \log s) - \beta(\lambda_{map} + O(s)) \ldots \quad (19)
\]

Putting this into eq. (5) we get

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \frac{1}{2\pi i} \int_{i\infty+\epsilon}^{i\infty-\epsilon} e^{st} \left( \frac{\lambda_{map}}{\langle \Delta \rangle} + O(s \log s) \right) ds
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \frac{\lambda_{map}}{\langle \Delta \rangle} t + O(\log t) = \frac{\lambda_{map}}{\langle \Delta \rangle}, \quad (20)
\]

which is just Abramovs formula \( 14 \). We got an exact result (for the weight \( 16 \)) since the relevant terms in the approximate zeta function were indeed exact. We saw that the leading zero also being a branch point did not do any harm to the Lyapunov exponent but it will in fact induce a phase transition among the generalized Lyapunov exponents \( 1 \).

### 2.2 Diffusion

Next we consider the associated Lorentz gas obtained by infinitely unfolding the billiard. The zeta function is, cf. eqs. (3) and (12),

\[
\hat{Z}(s) = 1 - \langle e^{-s\Delta} e^{x_1\beta_1 + x_2\beta_2} \rangle = 1 - \langle e^{-s\Delta} \frac{1}{2\pi} \int_0^{2\pi} e^{\Delta \beta \cos(\theta)} d\theta \rangle, \quad (21)
\]

where we used the asymptotic isotropy (which is not really necessary cf refs. \( 1, 15 \)), and \( \Delta = \sqrt{x_1^2 + x_2^2} \). We only need to expand to second order in \( \beta \)

\[
\hat{Z}(s) = 1 - \langle e^{-s\Delta} \rangle - \frac{\beta^2}{4} (\Delta^2 e^{-s\Delta}) \ldots
\]

\[
= 1 - \frac{s}{2R} + O(s^2 \log s) + \frac{\beta^2}{4\pi^2 R^2} (-\log s + O(1)) \ldots \quad . \quad (22)
\]

Putting it into eq. (6) we get

\[
D = \lim_{t \to \infty} \frac{1}{2t \pi^2 R^2} \int_{i\infty+\epsilon}^{i\infty-\epsilon} e^{st} \frac{-\log s + O(1)}{s^2} ds = \frac{1}{\pi^2 R} (\log(t) + O(1)) \quad , \quad (23)
\]

which is indeed the exact result \( 13 \). Exact result may also be obtained for finite \( R \). \( 1, 13 \). The exactness of the result indicates that the term \(-\frac{\beta^2}{4\pi^2 R^2} \log s \) in eq. (22) agrees with the corresponding one in the exact zeta function.
2.3 Correlation functions

As regarding correlation functions we will restrict ourselves to observables $A(x_0)$ that are changed only by bounces on the disk, such as $A = |v_x|$. The autocorrelation function is e.g. obtained as the time average

$$C(t) = \langle A(t_0 + t)A(t_0) \rangle_{t_0} - \langle A^2 \rangle .$$

(24)

Measuring $A$ at two points of time separated by $t$, then there is a probability $p_0(t)$ that the trajectory has not hit the disk in between and according to assumption $A$ we get [16]

$$C(t) = p_0(t)\langle A^2 \rangle + (1 - p_0(t))\langle A \rangle^2 - \langle A^2 \rangle = p_0(t)V(A) ,$$

(25)

where $V(A)$ is the variance of $A$. The function $p_0(t)$ may be expressed in terms of $p(\Delta)$ [16]

$$p_0(t) = \frac{1}{\langle \Delta \rangle} \int_{t}^{\infty} \int_{u}^{\infty} p(\Delta) d\Delta \, du .$$

(26)

The $1/\Delta^3$ decay of $p(\Delta)$ thus implies a $1/t$ of the correlation function. In fig 1 we compare the numerical correlation function with $A = |v_x|$ with eq (25) with $p_0(t)$ computed from eq. (15). The $1/t$ decay law has also been suggested in refs [17, 9].

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Figure captions

Figure 1: Numerical correlation function (full line), for $R = 0.106$. The dashed line represents eq. (25) with $p_0(t)$ computed from eqs. (26) and (15). From ref. [16]
