Central limit theorem for eigenvalue statistics of sample covariance matrix with random population

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Abstract

Consider the sample covariance matrix
\[
\Sigma^{1/2} X X^T \Sigma^{1/2}
\]
(1)
where \(X\) is an \(M \times N\) random matrix with independent entries and \(\Sigma\) is an \(M \times M\) diagonal matrix. It is known that if \(\Sigma\) is deterministic, then the fluctuation of
\[
\sum_i f(\lambda_i)
\]
converges in distribution to a Gaussian distribution. Here \(\{\lambda_i\}\) are eigenvalues of (1) and \(f\) is a good enough test function. In this paper we consider the case that \(\Sigma\) is random and show that the fluctuation of
\[
\frac{1}{\sqrt{N}} \sum_i f(\lambda_i)
\]
converges in distribution to a Gaussian distribution. This phenomenon implies that the randomness of \(\Sigma\) decreases the correlation among \(\{\lambda_i\}\).

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1 Background

Consider the sample covariance matrix

\[ \Sigma^{1/2}X X^T \Sigma^{1/2} \tag{2} \]

where \( X \) is an \( M \times N \) random matrix with iid centered entries and the population \( \Sigma \) is an \( M \times M \) diagonal matrix with nonnegative entries. Assume that the empirical distribution of \( \Sigma \) converges to a deterministic probability measure \( \sigma \) and also assume that

\[ \frac{M}{N} \to \gamma_0 \in (0, \infty). \]

For the case \( \Sigma = I \), Marchenko and Pastur [19] proved that the empirical distribution of (2) converges to a deterministic probability measure with parameter \( \gamma_0 \). This measure is called the Marchenko-Pastur distribution and denoted by \( \mu_{MP,\gamma_0} \). For the case \( \Sigma \neq I \), it is known that empirical distribution of (2) converges to the so called multiplicative free convolution of \( \sigma \) and \( \mu_{MP,\gamma_0} \), denoted by \( \sigma \boxtimes \mu_{MP,\gamma_0} \). See [2, 24, 25].

To know how fast the empirical distribution of (2) converges to \( \sigma \boxtimes \mu_{MP,\gamma_0} \), people study the asymptotic behavior of the fluctuation of the linear statistics:

\[ \sum_i f(\lambda_i) - M \int f(t) d(\sigma \boxtimes \mu_{MP,\gamma_0})(t) \tag{3} \]

where \( \{\lambda_i\} \) are eigenvalues of (2) and \( f \) is a test function. The central limit theorem (CLT) of (3) was first proved by Johansson [11] for Wishart matrices where \( \Sigma = I \) and \( X \) have Gaussian entries. He proved that (3) converges to a Gaussian distribution. Note that this is different from the classic CLT for random variables which has the coefficient \( \frac{1}{\sqrt{N}} \). This phenomenon showed that the eigenvalues \( \{\lambda_i\} \) have very strong correlation. Bai and Silverstein [1] proved the CLT of (3) for general \( X \) and \( \Sigma \) with a condition on the fourth moment of \( X_{ij} \). This condition was later removed by the work of Pan and Zhou [21]. In a series of work [3, 18, 20, 22] the regularity condition of the test function was weakened. In [17] the CLT was proved in the case when the entries of \( \Sigma \) do not have a uniform bound. In [16] the mesoscopic CLT was obtained.

All the work cited above assume that \( \Sigma \) is a deterministic matrix. Kwak, the first author and Park [13] considered the largest eigenvalue of (2) for both deterministic and random \( \Sigma \) and proved that the fluctuation of the largest eigenvalue converges to a deterministic probability measure.

In this paper, we consider the sample covariance matrix with random population and prove the CLT of (3). In fact, for a good enough test function \( f \) we prove that the rescaled fluctuation

\[ \frac{1}{\sqrt{N}} \left( \sum_i f(\lambda_i) - M \int f(t) d(\sigma \boxtimes \mu_{MP,\gamma_0})(t) \right) \tag{4} \]

converges in distribution to a centered Gaussian distribution. Different from the case of deterministic \( \Sigma \), we have the coefficient \( \frac{1}{\sqrt{N}} \) in (4). This implies that the randomness of \( \Sigma \) decreases the correlation among the eigenvalues.

The similar phenomenon has been observed for the deformed Wigner matrix \( W + V \) where \( W \) is a Wigner matrix and \( V \) is a diagonal matrix. Suppose \( \{\mu_i\} \) are eigenvalues of \( W + V \). Ji and the first author [10] proved that if \( V \) is deterministic, then the fluctuation of

\[ \sum_i f(\mu_i) \tag{5} \]
converges to a Gaussian distribution and on the other hand, if \( V \) is random, then the fluctuation of (5) multiplied by \( \frac{1}{\sqrt{N}} \) converges to a Gaussian distribution.

## 2 Model and main results

Define

\[
C_+ = \{ x + iy | x \in \mathbb{R}, y > 0 \}.
\]

For a probability measure \( \pi \), define its Stieltjes transform by

\[
m_\pi(z) = \int \frac{d\pi(t)}{t - z}, \quad \forall z \notin \text{supp}(\pi).
\]

If \( \pi \) has density \( \rho \pi(t) \) on an open interval \( I \) and \( \rho \pi(t) \) is continuous on \( I \), then by the Sokhotski–Plemelj theorem,

\[
\rho(x) = \frac{1}{\pi} \lim_{y \to 0^+} \text{Im} m_\pi(x + iy) \quad \forall x \in I.
\]

**Lemma 1.** Let \( \pi \) be a compactly supported probability measure on \( \mathbb{R} \), and let \( r > 0 \).

- For each \( z \in C_+ \) there is a unique \( m = m(z) \in C_+ \) satisfying

  \[
  \frac{1}{m} = -z + r \int \frac{t}{1 + mt} d\pi(t).
  \]

  Moreover, \( m(z) \) is the Stieltjes transform of a probability measure with compact support in \([0, \infty)\). This probability measure is the multiplicative free convolution of \( \pi \) and the Marchenko–Pastur law with dimensional ratio \( r \):

  \[
  \pi \boxtimes \mu_{MP,r}
  \]

  where \( \mu_{MP,r} \) denotes the Marchenko–Pastur law with dimensional ratio \( r \).

- There is a continuous nonnegative function \( w(x) \) defined on \((0, \infty)\) such that

  \[
  \pi \boxtimes \mu_{MP,r} = (1 - r)^+ \delta_0 + w(x)dx.
  \]

  Moreover, \( w(x) \) is analytic wherever it is positive.

**Proof.** The first result is Lemma 2.2 in [12]. The second result can be found on Page 2271 of [9].\qed

**Definition 1.** Suppose \( X = (X_{ij}) \) is an \( M \times N \) matrix where \( M = M(N) \) and

- \( \{X_{ij}\} \) are iid real-valued random variables such that \( \mathbb{E}[X_{ij}] = 0 \), \( \mathbb{E}[X_{ij}^2] = \frac{1}{N} \)

- for any \( p \in \{1, 2, \ldots\} \) there exists \( C_p > 0 \) such that

  \[
  \mathbb{E}[|\sqrt{N}X_{ij}|^p] \leq C_p, \quad \forall i, j, N
  \]

- there exist constants \( \gamma_0 \in (0, \infty) \setminus \{1\} \) and \( c_0 > 0 \) such that

  \[
  |\frac{M}{N} - \gamma_0| \leq N^{-\frac{1}{2} - c_0}
  \]

Without loss of generality we assume \( c_0 < 0.01 \).
It is easy to see that if $\gamma_0 \in (0, 1)$, then $M < N$ when $N$ is large enough; if $\gamma_0 > 1$, then $M > N$ when $N$ is large enough. By (8), for any (small) $\epsilon > 0$ and (large) $D > 0$, if $N > N_0 = N_0(\epsilon, D)$ then
\[
P(|X_{ij}| \leq N^{-\epsilon - \frac{1}{2}}, \forall i, j) > 1 - N^{-D}.
\] (10)

**Definition 2.** Suppose $l \in (0, 1)$ is a constant and $\nu$ is a probability measure with $\text{supp}(\nu) = [l, 1]$. Let $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_M)$ be an $M \times M$ diagonal matrix such that the entries $\sigma_1, \ldots, \sigma_M$ are iid with distribution $\nu$. Moreover we assume that $\Sigma$ is independent of $X$.

By linear algebra, the set of eigenvalues of $X^T \Sigma X$ differs from the set of eigenvalues of $\Sigma^{1/2} X X^T \Sigma^{1/2}$ by $|M - N|$ zeros.

Suppose $\mu_{\Sigma}$ is the empirical measure of $\Sigma$:
\[
\mu_{\Sigma} = \frac{1}{M} \sum_{i=1}^{M} \delta_{\sigma_i}.
\]

**Lemma 2.** • Almost surely, $\mu_{\Sigma}$ converges weakly to $\nu$. In other words, if $f(t)$ is bounded and continuous, then almost surely we have
\[
\frac{1}{M} \sum_{i=1}^{M} f(\sigma_i) \to \int f(t) d\nu(t).
\] (11)

• Suppose $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$ are the eigenvalues of $X^T \Sigma X$. Let $\lambda_i = 0$ for $M \land N < i \leq M \lor N$. Almost surely, the empirical measure
\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}
\]
converge weakly to
\[
\mu_{\Sigma} := \nu \boxtimes \mu_{MP, \gamma_0}.
\]

**Proof.** The first conclusion is from the Glivenko–Cantelli Theorem. See Theorem 2.4.7 of [7]. The second conclusion can be induced from the first paragraph and the Equation (1.3) of [23].

**Definition 3.** • Let $\rho_{fc}(x)$ be the density function of the absolute continuous part of $\mu_{fc}$. According to Lemma [4], $\rho_{fc}(x)$ is continuous and
\[
\mu_{fc} = (1 - \gamma_0)^+ \delta_0 + \rho_{fc}(x)dx
\]

• Similarly, denote
\[
\hat{\mu}_{fc} := \mu_{\Sigma} \boxtimes \mu_{MP, M/N}
\]
and let $\hat{\rho}_{fc(t)}$ be the continuous function defined on $(0, \infty)$ such that
\[
\hat{\mu}_{fc} = (1 - \frac{M}{N})^+ \delta_0 + \hat{\rho}_{fc}(x)dx.
\]
Let $[L_-, L_+]$ be the smallest interval containing the support of $\rho_{fc}$. Let $[\hat{L}_-, \hat{L}_+]$ be the smallest interval containing the support of $\hat{\rho}_{fc}(x)$.

By Proposition 2.4 of [9] and the assumption $\gamma_0 \neq 1$, we have $L_- > 0$ and $\hat{L}_- > 0$ when $N$ is large enough.

**Theorem 1.** Suppose

$$\lim_{N \to \infty} \mathbb{P}\left(|\hat{L}_- - L_-| < \epsilon \quad \text{and} \quad |\hat{L}_+ - L_+| < \epsilon\right) \to 1 \quad \forall \epsilon > 0. \tag{12}$$

If $f(x)$ is a function analytic in a neighborhood of $[L_-, L_+]$, then as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(\lambda_i) - \sqrt{N} \int f(t) d\mu_{fc}(t)$$

converges in distribution to a centered Gaussian distribution whose variance is

$$-\frac{1}{4\pi^2} \oint_{\Gamma} \oint_{\Gamma} \frac{i^2 m_{fc}'(\xi_1)m_{fc}'(\xi_2)f(\xi_1)f(\xi_2)}{(1 + tm_{fc}(\xi_1))(1 + tm_{fc}(\xi_2))} dv(t) d\xi_1 d\xi_2 + \frac{1}{4\pi^2\gamma_0^2} \left( \oint_{\Gamma} f(\xi)m_{fc}(\xi)(\xi + \frac{1}{m_{fc}(\xi)}) d\xi \right)^2$$

where $\Gamma$ is a counterclockwise oriented path enclosing $[L_-, L_+]$ but not enclosing 0 such that $f$ is analytic in a neighborhood of the domain bounded by $\Gamma$.

Since the eigenvalues of $X^T \Sigma X$ differ from the eigenvalues of $\Sigma^{1/2} X X^T \Sigma^{1/2}$ by $|M - N|$ zeros, Theorem 1 immediately imply the central limit theorem of the linear statistics for eigenvalues of $\Sigma^{1/2} X X^T \Sigma^{1/2}$.

The next proposition provides a criteria of (12):

**Proposition 1.** If $m_{fc}(L_+) := \int \frac{d\mu_{fc}(t)}{t^2 L_+} \neq -1$ or $\gamma_0 \neq (\int \frac{1}{1 + t^2} dv(t))^{-1}$, then

$$\mathbb{P}(|L_+ - \hat{L}_+| < \epsilon) \to 1 \quad \forall \epsilon > 0. \tag{13}$$

If $m_{fc}(L_-) := \int \frac{d\mu_{fc}(t)}{t^2 L_-} \neq -1$ or $\gamma_0 \neq (\int \frac{1}{1 + t^2} dv(t))^{-1}$, then

$$\mathbb{P}(|L_- - \hat{L}_-| < \epsilon) \to 1 \quad \forall \epsilon > 0. \tag{14}$$

Obviously, (13) and (14) yield (12).

Proposition 1 is proved in Section 8. Now we give two explicit examples satisfying (12).

**Example 1.** Suppose $\nu_1$ is a probability measure supported on $[l, 1]$ such that its density function is bounded above and below:

$$0 < \inf_{x \in [l, 1]} \frac{d\nu_1(x)}{dx} \leq \sup_{x \in [l, 1]} \frac{d\nu_1(x)}{dx} < \infty. \tag{15}$$

(15) implies that $m_{fc}(L_+) = -\infty$ and $m_{fc}(L_-) = \infty$, so by Proposition 1 (12) must be true when $\nu = \nu_1$. 


Example 2. Suppose $\nu_2$ is the Jacobi measure on $[l, 1]$:

$$
\frac{d\nu_2}{dt} = \frac{1}{Z} \cdot d(x) \cdot (1 - t)^b
$$

where

- $b > 1$ is a constant
- $d(x) \in C^1([l, 1])$ and $d(x) > 0$ on $[l, 1]$
- $Z$ is the normalization constant: $Z = \int_l^1 d(t)(1 - t)^b dt$.

If

$$
\gamma_0 \in (0, (\int \frac{t^2}{(1 - t)^2} d\nu_2(t))^{-1}) \quad \text{and} \quad \gamma_0 \notin \{1, (\int \frac{t^2}{(t - l)^2} d\nu_2(t))^{-1}\}
$$

then by Proposition [1.12] is true. Moreover, according to Theorem 2.7 of [13], if $\nu = \nu_2$ and (16) holds, then $\rho_{fc}(t) \sim (1 - t)^b$ when $t \uparrow L^+$. This means (12) and our main theorem do not require $\rho_{fc}$ to decay like the square root near edges (which is a common requirement for sample covariance matrix).

Definition 4.

- Suppose $\tau \in (0, C_0^{-1})$ where $C_0 > 1$ is a constant defined in Lemma 3. Let

$$
D_\tau = \left\{ z = E + i \eta \in \mathbb{C}+ \left| \tau \leq E \leq \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1}, \text{dist}(E, [\hat{L} -, \hat{L} +]) \geq \tau \right. \right\}
$$

and

$$
D'_\tau = \left\{ z = E + i \eta \in \mathbb{C}+ \left| E \in [\tau, \tau^{-1}], \tau \leq \eta \leq \tau^{-1} \right. \right\}
$$

- Let $m_N$, $m_{fc}$ and $\hat{m}_{fc}$ denote the Stieltjes transform of $\sum_{i=1}^N \delta_{\lambda_i}$, $\mu_{fc}$ and $\hat{\mu}_{fc}$ respectively.
- For any $z \in \mathbb{C}\setminus \mathbb{R}$, let

$$
\Psi(z) = \frac{1}{N} \text{Im}z + \frac{1}{\sqrt{N}}.
$$

Notice that $D_\tau$ is a random subset of $\mathbb{C}$ but is independent of $X$. For any $z \in \mathbb{C}\setminus \mathbb{R}$ we have by (6):

$$
m_N(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}
$$

$$
\frac{1}{m_{fc}(z)} = -z + \gamma_0 \int \frac{t}{1 + tm_{fc}(z)} dt, \quad \frac{1}{\hat{m}_{fc}(z)} = -z + \frac{1}{N} \sum_{i=1}^M \frac{\sigma_i}{1 + \sigma_i \hat{m}_{fc}(z)}
$$

An important tool we use to prove the main theorem is the following local law.

Proposition 2. Suppose $\tau \in (0, C_0^{-1})$ where $C_0$ is the constant defined in Lemma 3. For any (small) $\epsilon > 0$ and (big) $D > 0$, if $N > N_0 = N_0(\epsilon, D, \tau)$, then

$$
\mathbb{P}(|m_N - \hat{m}_{fc}| \leq N^\epsilon \Psi^2 \quad \forall z \in D_\tau \cup D'_\tau) > 1 - N^{-D}
$$
Proposition 2 is proved in Section 6. It is similar to some results in [12]. See Theorem 3.14–3.16 and Lemma 5.6 of [12]. But the results in [12] have the following two differences from Proposition 2. First, the population matrix in [12] is deterministic. Because of the randomness of Σ, we have to define the conditional expectation $E_k$ (see 29) in a way different from that in [12]. This is to ensure (37) and is also important in the “binary tree” argument, see (167) and (168).

Second, for the spectral domain, three subdomains are considered in [12]: i) the subdomain in which the real part of the points is far away from supp($\hat{\rho}_{fc}$), ii) the subdomain in which the real part of the points is near edges of supp($\hat{\rho}_{fc}$), iii) the subdomain in which the real part of the points is in the bulk of supp($\hat{\rho}_{fc}$). For the second and third cases, some conditions on the regularity of the edges and bulks of supp($\hat{\rho}_{fc}$) are assumed. In this paper, when the real part of the points in the spectral domain is near supp($\hat{\rho}_{fc}$), we do not need the regularity of the edges or bulks (as we saw in Example 2), but we require the imaginary part of the points to be far away from zero (as we saw in the definition of $D'_\tau$).

It turns out that the differences mentioned above are comparatively minor and we can basically follow the method in [12] to prove Proposition 2. For the convenience of the readers we write down all details.

Proposition 2 yields the following corollary.

**Corollary 1.** Suppose $\tau \in (0, C_0^{-1})$ where $C_0$ is the constant defined in Lemma 3. For any (big) $D > 0$ we have

$$P\left(\lambda_1, \ldots, \lambda_{M \wedge N} \text{ are all in } [\hat{L}_- - \tau, \hat{L}_+ + \tau]\right) > 1 - N^{-D}$$

when $N > N_0 = N_0(\tau, D)$.

Lemma 10.1 of [12] proved same conclusion as Corollary 1 for a model with slight difference. In particular, Lemma 10.1 of [12] assumes some regularity conditions for the edges and bulks of supp($\hat{\rho}_{fc}$), as we mentioned above. Although our model does not necessarily satisfy the regularity conditions, the method in the proof of Lemma 10.1 of [12] still works. For the convenience of readers we write down the proof of Corollary 1 in Section 6.

### 3 Preliminaries

**Lemma 3.** There exist $C_0 > 1$ such that

1. if $N$ is large enough then

$$\hat{L}_+ < C_0 \quad \text{and} \quad \hat{L}_- > C_0^{-1}$$

2. for any (big) $D > 0$, if $N > N_0 = N_0(D)$ then

$$P(\lambda_i \in [C_0^{-1}, C_0] \ \forall i \in \{1, \ldots, M \wedge N\}) > 1 - N^{-D} \quad (21)$$

**Proof.** See Appendix A

**Definition 5.** For any $z \in \mathbb{C} \setminus \mathbb{R}$, define the $(M + N) \times (M + N)$ matrices:

$$H(z) := \begin{pmatrix} -\Sigma^{-1} & X \\ X^T & -z \end{pmatrix}, \quad G(z) := H(z)^{-1} \quad (22)$$
as well as the $M \times M$ matrix:
\[
G_M(z) := (-\Sigma^{-1} + z^{-1}XX^T)^{-1} = z\Sigma^{1/2}(\Sigma^{1/2}XX^T\Sigma^{1/2} - z)^{-1}\Sigma^{1/2}
\]
and the $N \times N$ matrix:
\[
G_N(z) := (X^T\Sigma XX - z)^{-1}.
\]

**Definition 6.** • Suppose $T$ is a subset of $\{1, \ldots, M+N\}$. For $z \in \mathbb{C}\setminus\mathbb{R}$, we use $H^{(T)}(z)$ to denote the $(M+N-|T|) \times (M+N-|T|)$ matrix:
\[
(H_{ij}(z))_{i,j \in \{1, \ldots, M+N\}\setminus T}
\]
• Let
\[
G^{(T)}(z) = (H^{(T)}(z))^{-1}
\]
• We also set
\[
\sum_{i}^{(T)} = \sum_{i \in T}^{(T)}, \quad \sum_{i,j}^{(T)} = \sum_{i \in T}^{(T)} \sum_{j \in T}^{(T)}
\]

**Remark 1.** 1. When $T = \{i\}$, we use $(i)$ instead of $(\{i\})$ in the above definitions. Similarly, we write $(ij)$ instead of $(\{i,j\})$. We use $(T)$ to denote $(T \cup \{i\})$.
2. In $H^{(T)}(z)$ and $G^{(T)}(z)$ we use the original values of matrix indices. For example, the indices for the rows and columns of $H^{(2)}(z)$ are 1, 3, 4, \ldots, $M+N$.
3. We often write $H(z)$ as $H$ for convenience. For $G(z)$, $G_M(z)$, $G_N(z)$, $H^{(T)}(z)$ and $G^{(T)}(z)$, we usually omit the variable $z$ too.
4. It is easy to see that $H^{(T)}(z)$ and $G^{(T)}(z)$ are symmetric matrices.

The following lemma is Lemma 4.4 of [12].

**Lemma 4.** For any $z \in \mathbb{C}\setminus\mathbb{R}$ we have the following results.

1. \[
G(z) = 
\begin{bmatrix}
\Sigma XG_NX^T\Sigma - \Sigma \\
G_NX^T\Sigma \\
\end{bmatrix}
\begin{bmatrix}
\Sigma XG_N \\
G_N \\
\end{bmatrix}
= 
\begin{bmatrix}
G_M & z^{-1}G_MX \\
z^{-2}X^TG_MX - z^{-1} \\
\end{bmatrix}
\]
\quad (23)

2. If $M+1 \leq p \leq M + N$ then
\[
1 \frac{G_{pp}}{G_{pp}} = -z - \sum_{i,j=1}^{M} X_{i(p-M)}X_{j(p-M)}G_{ij}^{(p)}.
\]
\quad (24)

If $p$ and $q$ are distinct elements in $\{M+1, \ldots, M+N\}$ then
\[
G_{pq} = -G_{pp} \sum_{i=1}^{M} X_{i(p-M)}G_{iq}^{(p)} = -G_{qq} \sum_{i=1}^{M} G_{pi}^{(q)}X_{i(q-M)} = G_{pp}G_{qq} \sum_{i,j=1}^{M} X_{i(p-M)}G_{ij}^{(pq)}X_{j(q-M)}
\]
3. If \(i \in \{1, \ldots, M\}\) then
\[
\frac{1}{G_{ii}} = -\frac{1}{\sigma_i} - \sum_{p=1}^{M+N} X_{i(p-M)}G_{ip(i)}^{(i)}.
\]
(25)

If \(i\) and \(j\) are distinct elements in \(\{1, \ldots, M\}\), then
\[
G_{ij} = -G_{ii} \sum_{p=M+1}^{M+N} X_{i(p-M)}G_{ip(j)}^{(i)} = -G_{jj} \sum_{p=M+1}^{M+N} G_{ip(j)}^{(j)} X_{j(p-M)} = G_{ii}G_{jj} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{j(q-M)}G_{pq(i)}^{(j)}.
\]
(26)

4. If \(i \in \{1, \ldots, M\}\) and \(p \in \{M+1, \ldots, M+N\}\), then
\[
G_{ip} = -G_{pp} \sum_{j=1}^{M} G_{ij}^{(p)} X_{j(p-M)} = -G_{ii} \sum_{q=M+1}^{M+N} X_{i(q-M)}G_{qip}^{(i)}
\]
\[
= G_{ii}G_{pp}^\prime \left(-X_{i(p-M)} + \sum_{q \in \{1, \ldots, M\} \setminus \{i\}} X_{i(q-M)}X_{j(p-M)}G_{qj}^{(ip)}\right)
\]
\[
= G_{pp}G_{ii} \left(-X_{i(p-M)} + \sum_{q \in \{1, \ldots, M\} \setminus \{i\}} X_{i(q-M)}X_{j(p-M)}G_{qj}^{(ip)}\right)
\]
(27)

5. Suppose \(r \in \{1, \ldots, M+N\}\) and \(s, t\) are elements in \(\{1, \ldots, M+N\} \setminus \{r\}\), then
\[
G_{st(r)} = G_{st} - \frac{G_{sr}G_{rt}}{G_{rr}}.
\]
(28)

Definition 7. For any \(k \in \{1, \ldots, M+N\}\) we define the conditional expectation
\[
\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot| \sigma_k] \text{ sigma algebra generated by } \{\text{entries of } H^{(k)}\} \cup \{\text{entries of } \Sigma\}.
\]
(29)

For any \(z \in \mathbb{C} \setminus \mathbb{R}\) we define
\[
h(z) = -\frac{1}{z} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 + z\sigma_i}
\]
\[
Z_p = Z_p(z) = (1 - \mathbb{E}_p) \sum_{i,j=1}^{M} X_{i(p-M)}X_{j(p-M)}G_{ij}^{(p)}, \quad \forall p \in \{M+1, \ldots, M+N\}
\]
\[
Z_i = Z_i(z) = (1 - \mathbb{E}_i) \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq(i)}, \quad \forall i \in \{1, \ldots, M\}
\]
\[
A_i = A_i(z) = \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(G_{pp})^2}{G_{ii}}, \quad \forall i \in \{1, \ldots, M\}
\]
Remark 2. Because of the randomness of Lemma 5. way as those in [12]. See (5.6), (5.7) and (5.8) of [12].

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Suppose \( i, j \in \{1, \ldots, M\} \) and \( p, q \in \{M + 1, \ldots, M + N\} \) then
\[
|G_{ij}(z)| = |(G_M)_{ij}(z)| \leq \frac{|z|}{|Imz|}, \quad |G_{pq}(z)| = |(G_N)_{(p-M)(q-M)}(z)| \leq \frac{1}{|Imz|}, \tag{30}
\]
\[
|G_{ip}(z)| \leq \frac{N}{|Imz|} \max_{a,b} |X_{ab}|, \tag{31}
\]

Similarly, if \( T \subset \{1, \ldots, M + N\}, \{i', j'\} \subset \{1, \ldots, M\} \setminus T \) and \( \{p', q'\} \subset \{M + 1, \ldots, M + N\} \setminus T \), then
\[
|G_{i'j'}^{(T)}(z)| \leq \frac{|z|}{|Imz|}, \quad |G_{p'q'}^{(T)}(z)| \leq \frac{1}{|Imz|} \quad \text{and} \quad |G_{i'j'}^{(T)}(z)| \leq \frac{N}{|Imz|} \max_{a', b', \{a' \in \{1, \ldots, M\} \setminus T, b' \in \{M + 1, \ldots, M + N\} \setminus T\}} |X_{a'b'}|. \tag{32}
\]

2. Suppose \( T \subset \{1, \ldots, M + N\} \) and \( p \in \{M + 1, \ldots, M + N\} \setminus T \) then
\[
\sum_{q \in \{M + 1, \ldots, M + N\} \setminus T} |G_{pq}^{(T)}(z)|^2 \leq \frac{ImG_{pp}^{(T)}}{|Imz|} \tag{33}
\]

Suppose \( S \subset \{1, \ldots, M + N\} \) and \( i \in \{1, \ldots, M\} \setminus S \) then
\[
\sum_{j \in \{1, \ldots, M\} \setminus S} |G_{ij}^{(S)}(z)|^2 \leq 2 \frac{\|\tilde{X}\tilde{X}^T\|_{Imz}}{|Imz|^2} + 2 \leq 2 \|\tilde{X}\tilde{X}^T\| \cdot \frac{|z|}{|Imz|^2} + 2 \tag{34}
\]

where \( \| \cdot \| \) denotes the operator norm and \( \tilde{X} \) is a matrix constructed from \( X \) by removing the rows with indices in \( S \cap \{1, \ldots, M\} \) and removing the columns with indices in \( \{p - M | p \in S \cap \{1, \ldots, M\} \setminus \{M + 1, \ldots, M + N\}\} \).

3. There exists a constant \( C_w > 0 \) depending only on \( \gamma_0 \) such that the following holds. For any \( k > 0 \) and (big) \( D > 0 \), if \( N > N_0 = N_0(D, k) \), then
\[
\mathbb{P}(\|XX^T\| < C_w) > 1 - N^{-D}, \quad \mathbb{P}(\|\tilde{X}\tilde{X}^T\| < C_w) > 1 - N^{-D} \tag{35}
\]

where \( \tilde{X} \) is constructed from \( X \) by removing no more than \( k \) columns or rows of \( X \).
4. For any \( i \in \{1, \ldots, M\} \) and \( p \in \{M + 1, \ldots, M + N\} \)

\[
\frac{1}{G_{ii}} = -\frac{1}{\sigma_i} - z_i - m_N + A_i, \quad \frac{1}{G_{pp}} = -z - Z_p - \frac{1}{N} \sum_{i=1}^{M} G_{ii} + A_p \tag{36}
\]

\[
Z_i = -(1 - E_i) \frac{1}{G_{ii}}, \quad Z_p = -(1 - E_p) \frac{1}{G_{pp}} \tag{37}
\]

5.

\[
m_N(z)\hat{m}_{fc}(z) (h(m_N(z)) - z) = \alpha_1 (m_N(z) - \hat{m}_{fc}(z))^2 + \alpha_2 (m_N(z) - \hat{m}_{fc}(z)) \tag{38}
\]

where

\[
\alpha_1 := -\frac{1}{N} \sum_{i=1}^{M} \frac{\hat{m}_{fc}(z)\sigma_i^2}{(1 + \sigma_i m_N(z))(1 + \hat{m}_{fc}(z)\sigma_i)^2}, \quad \alpha_2 := 1 - \frac{1}{N} \sum_{i=1}^{M} \frac{\hat{m}_{fc}(z)\sigma_i^2}{(1 + \hat{m}_{fc}(z)\sigma_i)^2} = \frac{\hat{m}_{fc}^2(z)}{m_{fc}^2(z)}. \tag{39}
\]

6.

\[
h(m_N) - h(\hat{m}_{fc}) = h(m_N) - z = B_N - \frac{1}{N} \sum_{p=M+1}^{M+N} A_p + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(m_N - G_{pp})^2}{m_N^2 G_{pp}} + \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2}{(1 + \sigma_i m_N)^2} Z_i \tag{40}
\]

4  A weak local law on \( D_\tau \)

Definition 8. For any \( \tau \in (0, 1) \), let

\[
D_\tau = \left\{ z = E + i\eta \in \mathbb{C} \mid \tau \leq E \leq \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1}, \text{dist}(E, [\hat{L}_-, \hat{L}_+]) \geq \tau \right\} \tag{41}
\]

Since \([\hat{L}_-, \hat{L}_+]\) depends on \( \Sigma \), \( D_\tau \) is a random subset of \( \mathbb{C} \), but it is independent of \( X \).

Theorem 2. For any \( \tau \in (0, 1) \), (small) \( \epsilon > 0 \) and (big) \( D > 0 \), if \( N > N_0 = N_0(\epsilon, D, \tau) \) then

\[
\mathbb{P}\left(|m_N(z) - \hat{m}_{fc}(z)| \leq \frac{N^\epsilon}{N^{1+m}} \quad \forall z \in D_\tau\right) > 1 - N^{-D} \tag{42}
\]

\[
\mathbb{P}\left(|G_{ij}(z)| \leq N^\epsilon\left(\frac{1}{N^{1+m}} + \sqrt{\frac{\text{Im}\hat{m}_{fc}(z)}{N^{1+m}}}, \quad \forall i \neq j \in \{1, \ldots, M + N\} \text{ and } z \in D_\tau\right) \right) > 1 - N^{-D} \tag{43}
\]

\[
\mathbb{P}\left(|G_{ii}(z) + \frac{\sigma_i}{1 + \sigma_i \hat{m}_{fc}(z)}| \leq N^\epsilon\left(\frac{1}{N^{1+m}} + \sqrt{\frac{\text{Im}\hat{m}_{fc}(z)}{N^{1+m}}}, \quad \forall i \in \{1, \ldots, M\} \text{ and } z \in D_\tau\right) \right) > 1 - N^{-D} \tag{44}
\]

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\[ \mathbb{P}\left( |G_{pp}(z) - \hat{m}_{fc}(z)| \leq N^\epsilon \left( \frac{1}{N \text{Im} z} + \sqrt{\frac{\text{Im} m_{fc}(z)}{N \text{Im} z}} \right) \right) > 1 - N^{-D} \quad (45) \]

**Proof.** Consider the random variable:

\[ A(X, \Sigma, z) := \mathbb{1}_{z \in D_\tau} \cdot |m_N(z) - \hat{m}_{fc}(z)| \frac{N \text{Im} z}{N^\epsilon}. \]

Let \( \sigma(\Sigma) \) be the sigma algebra generated by entries of \( \Sigma \) and let \( \omega \) denote a sample point in the probability space. Then

\[
\mathbb{P}\left( \mathbb{1}_{z \in D_\tau} \cdot |m_N(z) - \hat{m}_{fc}(z)| \leq \frac{N^\epsilon}{N \text{Im} z} \right) = \mathbb{P}\left( A(X, \Sigma, z) \leq 1 \right) = \mathbb{E}[\mathbb{E}[\mathbb{1}_{A(X,\Sigma,z) \leq 1} | \sigma(\Sigma)]]
\]

\[
= \int \mathbb{E}[\mathbb{1}_{A(X,\Sigma,z) \leq 1} | \sigma(\Sigma)](\omega) d\mathbb{P}(\omega) = \int \mathbb{E}[\mathbb{1}_{A(X,\Sigma(\omega),z) \leq 1} | \sigma(\Sigma)](\omega) d\mathbb{P}(\omega) = \int \mathbb{P}(A(X, \Sigma(\omega), z) \leq 1) d\mathbb{P}(\omega)
\]

where we used the independence of \( X \) and \( \Sigma \) in the fourth identity.

Note that \( \Sigma(\omega) \) satisfies the conditions on the population matrix in Theorem 3.16 of [12]. Thus, according to Theorem 3.16 of [12], if \( N > N_0(\epsilon, D, \tau) \) then

\[ \mathbb{P}(A(X, \Sigma(\omega), z) \leq 1) \geq 1 - N^{-D} \]

and

\[ \mathbb{P}\left( \mathbb{1}_{z \in D_\tau} \cdot |m_N(z) - \hat{m}_{fc}(z)| \leq \frac{N^\epsilon}{N \text{Im} z} \right) \geq 1 - N^{-D} \]

thus by a classical “lattice” argument we have

\[ \mathbb{P}\left( \mathbb{1}_{z \in D_\tau} \cdot |m_N(z) - \hat{m}_{fc}(z)| \leq \frac{N^\epsilon}{N \text{Im} z}, \forall z \in \left\{ |E + i\eta| \leq \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1} \right\} \right) \geq 1 - N^{-D} \]

which yields (42). The other conclusions can be proved similarly using results from [12]. \( \Box \)

## 5 Weak local law on \( D' \)

**Lemma 6.** Suppose \( p \) and \( q \) are distinct numbers in \( \{M + 1, \ldots, M + N\} \). Suppose \( i \) and \( j \) are distinct numbers in \( \{1, \ldots, M\} \). For any (small) \( \epsilon' > 0 \) and (big) \( D' > 0 \), there exists \( N_0 = N_0(\epsilon', D') > 0 \) such that if \( N > N_0 \) and \( z \in \mathbb{C} \setminus \mathbb{R} \) then

\[ \mathbb{P}\left( \left| \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} G_{ks}^{(pq)} \right| \leq N^\epsilon' \sqrt{\frac{1}{N^2} \sum_{k,s=1}^{M} |G_{ks}^{(pq)}|^2} \right) > 1 - N^{-D'} \quad (46) \]

\[ \mathbb{P}\left( \left| \sum_{s=1}^{M} X_{s(p-M)} G_{ks}^{(p)} \right| \leq N^\epsilon' \sqrt{\frac{1}{N} \sum_{s=1}^{M} |G_{ks}^{(p)}|^2} \right) > 1 - N^{-D'}, \forall k \in \{1, \ldots, M\} \quad (47) \]
\[ P\left( \left| \sum_{k,s \in [1,M]} X_{k(p-M)} X_{s(p-M)} G_{ks}^{(p)} \right| \leq N^{r} \sqrt{\frac{1}{N^2} \sum_{k,s \in [1,M]} |G_{ks}^{(p)}|^2} > 1 - N^{-D'} \right) = \frac{C}{N^2} \]  

(48)

\[ P\left( \left| \sum_{k=1}^{M} (X_{k(p-M)})^2 - \frac{1}{N} |G_{kk}| \right| \leq N^{r} \sqrt{\frac{1}{N^2} \sum_{k=1}^{M} |G_{kk}|^2} > 1 - N^{-D'} \right) \]

(49)

\[ P\left( \left| \sum_{k,s=M+1}^{M+N} X_{i(k-M)} X_{j(s-M)} G_{ks}^{(i)} \right| \leq N^{r} \sqrt{\frac{1}{N^2} \sum_{k,s=M+1}^{M+N} |G_{ks}^{(i)}|^2} > 1 - N^{-D'} \right) \]

(50)

\[ P\left( \left| \sum_{k,s \in [M+1,M+N]} X_{i(k-M)} X_{j(s-M)} G_{ks}^{(i)} \right| \leq N^{r} \sqrt{\frac{1}{N^2} \sum_{k,s \in [M+1,M+N]} |G_{ks}^{(i)}|^2} > 1 - N^{-D'} \right) \]

(51)

\[ P\left( \left| \sum_{k=1}^{M+N} (X_{i(k-M)})^2 - \frac{1}{N} |G_{kk}| \right| \leq N^{r} \sqrt{\frac{1}{N^2} \sum_{k=1}^{M+N} |G_{kk}|^2} > 1 - N^{-D'} \right) \]

(52)

Proof. Suppose \( \mathcal{G}_1 \) is the sigma algebra generated by entries of \( H^{(pq)} \). Then \( X_{k(p-M)} \) and \( X_{s(q-M)} \) are independent of \( \mathcal{G}_1 \). Let

\[ B_{ks} = \frac{\sqrt{\sum_{k,s=1}^{M} |G_{ks}^{(pq)}|^2}}{N} \]

For any natural number \( r \) and any sample point \( \omega \) in the probability space, we have

\[ E\left[ \left( \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} B_{ks} \right)^{2r} \right| \mathcal{G}_1 (\omega) \right) = E\left[ \left( \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} B_{ks} \right)^{2r} \right| \mathcal{G}_1 \right] (\omega) \]

\[ = E\left[ \left( \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} B_{ks}(\omega) \right)^{2r} \right| \mathcal{G}_1 \right] \]

\[ \leq C \left( \sum_{k,s=1}^{M} \left| \frac{B_{ks}(\omega)}{N} \right|^2 \right)^{r} = CN^{-2r} \]

where \( C > 0 \) depends only on \( r \). We used \([8]\) and the Marcinkiewicz-Zygmund inequality (see, for example, Lemma 9 of \([13]\)) in the inequality. So,

\[ P\left( \left| \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} G_{ks}^{(pq)} \right| > N^{r} \sqrt{\frac{1}{N^2} \sum_{k,s=1}^{M} |G_{ks}^{(pq)}|^2} \right) = P\left( \left| \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} B_{ks} \right| > N^{r} \right) \]
\[\leq N^{-2r' \epsilon} \mathbb{E} \left[ N \sum_{k,s=1}^{M} X_{k(p-M)} X_{s(q-M)} B_{ks}^{2r'} \right] \leq C N^{-2r' \epsilon}.\]

Choose \( r \) large enough such that \( 2r' \epsilon > D' \), then we complete the proof of the first conclusion.

Other conclusions can be proved in the same way by using different variants of the Marcinkiewicz-Zygmund inequality. These variants of the Marcinkiewicz-Zygmund inequality can be found in Lemma 9 of [14]. Also see Lemma 13 of [14] for similar conclusions.

In this section we provide the weak local law on \( D'_s \). The weak local law on \( D'_s \) is similar as Theorem 3.14 and Theorem 3.15 of [12].

**Theorem 3.** Suppose \( \tau \in (0, C^{-1} \epsilon) \). For any (small) \( \epsilon' > 0 \) and (big) \( D' > 0 \), if \( N > N_0 = N_0(\epsilon', D', \tau) \) and \( z \in D'_s \), then

1. if \( p, q \) are both in \( \{M+1, \ldots, M+N\} \), then
\[
\mathbb{P} \left( \left| G_{pq}(z) - \delta_{pq} m_N(z) \right| \leq N^{\epsilon' - \frac{1}{2}} \right) > 1 - N^{-D'}
\] (53)

2. if \( i \in \{1, \ldots, M\} \) and \( p \in \{M+1, \ldots, M+N\} \), then
\[
\mathbb{P} (|G_{ip}(z)| \leq N^{\epsilon' - \frac{1}{2}}) > 1 - N^{-D'}.
\] (54)

3. if \( i \) and \( j \) are distinct numbers in \( \{1, \ldots, M\} \), then
\[
\mathbb{P} (|G_{ij}(z)| \leq N^{\epsilon' - \frac{1}{2}}) \geq 1 - N^{-D'}
\] (55)

4. if \( i \in \{1, \ldots, M\} \), then
\[
\mathbb{P} \left( \left| G_{ii}(z) + \frac{\sigma_i}{1 + \sigma_i m_N(z)} \right| \leq N^{\epsilon' - \frac{1}{2}} \right) > 1 - N^{-D'}
\] (56)

**Proof.**

1. Suppose \( p \) and \( q \) are distinct numbers in \( \{M+1, \ldots, M+N\} \). By (20) and (32)
\[
|G_{pq}| = |G_{pp} G_{qq}^{(p)} \sum_{i,j=1}^{M} X_{i(p-M)} X_{j(q-M)} G_{ij}^{(pq)}| \leq \frac{1}{|Imz|^2} \sum_{i,j=1}^{M} X_{i(p-M)} X_{j(q-M)} G_{ij}^{(pq)}|
\]

Note \( |z| \) is bounded and \( \text{Im} z \geq \tau \) when \( z \in D'_s \). By (10), (34) and (35) we obtain (53) in the case \( p \neq q \).

To consider the case \( p = q \), we let \( u \) and \( v \) be two indices in \( \{M+1, \ldots, M+N\} \), not necessarily distinct. By (21) and (23),
\[
\frac{1}{G_{uu}} = -z - \sum_{i,j=1}^{M} X_{i(u-M)} X_{j(u-M)} G_{ij}^{(u)}
\]
\[
= -z - \frac{1}{N} \sum_{k=1}^{M} G_{kk}^{(u)} + \sum_{k=1}^{M} G_{kk} \left( \frac{1}{N} - (X_{k(u-M)})^2 \right) - \sum_{i,j \in [1, M]} X_{i(u-M)} X_{j(u-M)} G_{ij}^{(u)}
\]

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\[
= -z - \frac{1}{N} \sum_{k=1}^{M} G_{kk} + \frac{1}{N} \sum_{k=1}^{M} \left( \frac{G_{ku}}{G_{uu}} \right)^2 + \sum_{k=1}^{M} G_{ku} \left( \frac{1}{N} - (X_{k(u-M)})^2 \right) - \sum_{i,j \in [1,M], i \neq j} X_{i(u-M)} X_{j(u-M)} G_{ij}^{(u)}
\]

which implies:

\[
G_{uu} - G_{vv} = G_{uu} G_{uv} \left( \frac{1}{G_{uv}} - \frac{1}{G_{uu}} \right) = G_{uu} \sum_{k=1}^{M} \left( \frac{G_{kv}}{G_{uu}} \right)^2 - G_{vv} \sum_{k=1}^{M} \left( \frac{G_{ku}}{G_{uu}} \right)^2
\]

\[
+ G_{uu} G_{vv} \sum_{k=1}^{M} G_{kk}^{(v)} \left( \frac{1}{N} - (X_{k(v-M)})^2 \right) - G_{uu} G_{vv} \sum_{k=1}^{M} G_{kk}^{(u)} \left( \frac{1}{N} - (X_{k(u-M)})^2 \right)
\]

\[
- G_{uu} G_{vv} \sum_{i,j \in [1,M], i \neq j} X_{i(v-M)} X_{j(v-M)} G_{ij}^{(v)} + G_{uu} G_{vv} \sum_{i,j \in [1,M], i \neq j} X_{i(u-M)} X_{j(u-M)} G_{ij}^{(u)}
\]

Next, we estimate each term on the right hand side of (57).

**Estimation of the first two terms in (57).** By (27) and (32),

\[
\left| \frac{G_{uu}}{N} \sum_{k=1}^{M} \left( G_{kv} \right)^2 \right| \leq \frac{1}{N |\text{Im}z|^2} \sum_{k=1}^{M} \left| G_{vv} \sum_{r=1}^{M} G_{rk}^{(v)} X_{r(v-M)} \right|^2 \leq \frac{1}{N |\text{Im}z|^3} \sum_{k=1}^{M} \left| G_{rk}^{(v)} X_{r(v-M)} \right|^2
\]

Note that \( \text{Im}z \geq \tau \) since \( z \in D'_s \). By (32), (41) and (56), if \( N > N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P} \left( \left| \frac{G_{uu}}{N} \sum_{k=1}^{M} \left( G_{kv} \right)^2 \right| \leq N^{\epsilon'-1} \right) \geq 1 - N^{-D'}
\]

Similarly we can obtain same bound for the second term in (57).

**Estimation of the third term and the fourth term in (57).** By (32) and (49), if \( N > N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P} \left( \left| \text{sum of the third and fourth terms in (57)} \right| \leq N^{\epsilon'-\frac{1}{2}} \right) \geq 1 - N^{-D'}
\]

**Estimation of the last two terms in (57).** By (48), (41), (59) and (52), if \( N > N_0(\epsilon', D', \tau) \)

\[
\mathbb{P} \left( \left| \text{sum of the last two terms in (57)} \right| \leq N^{\epsilon'-\frac{1}{2}} \right) \geq 1 - N^{-D'}
\]

From the above estimations of the terms in (57), if \( N > N_0(\epsilon', D', \tau) \), then

\[
\mathbb{P}(|G_{uu} - G_{vv}| \leq N^{\epsilon'-\frac{1}{2}}) > 1 - N^{-D'}
\]

(58)

For any \( p \in \{M + 1, \ldots, M + N\} \), by (28):

\[
|G_{pp} - m_N| = |G_{pp} - \frac{1}{N} \sum_{s=M+1}^{M+N} G_{ss}| \leq \frac{1}{N} \sum_{s=M+1}^{M+N} |G_{pp} - G_{ss}|
\]

which together with (58) prove (53) in the case \( p = q \).
Lemma 7. Suppose 
\[ P \leq 0. \]
By (25),
\[ G_{ip} = -G_{pp} \sum_{k=1}^{M} X_{k(p-M)}G_{ki}^{(p)} \]
which together with (32), (47), (34) and (35) complete the proof of (54).

3. By (25),
\[ G_{ij} = G_{ii}G_{jj}^{(i)} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{j(q-M)}G_{pq}^{(i)} \]
Then using (50), (33) and (32), we proved (55).

4. By (25) and (28),
\[ G_{ii} \left( \frac{1}{G_{ii}} + \frac{1}{\sigma_i} + m_N \right) = G_{ii} \left( \frac{1}{N} \sum_{p=M+1}^{M+N} G_{pp} - \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \right) \]
\[ = G_{ii} \left( \frac{1}{N} \sum_{p=M+1}^{M+N} \left[ G_{pp} - G_{ii} \right] - \sum_{p=M+1}^{M+N} \left( \frac{1}{N} - (X_{i(p-M)})^2 \right) \right) G_{ii}^{(i)} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \]
\[ = \frac{1}{N} \sum_{p=M+1}^{M+N} (G_{pi})^2 + G_{ii} \sum_{p=M+1}^{M+N} \left( \frac{1}{N} - (X_{i(p-M)})^2 \right) \]
\[ \geq 1 \sum_{p=M+1}^{M+N} \left( \frac{1}{G_{ii}} \right) G_{ii}^{(i)} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \]
(60)

Then using (50), (33) and (32), we proved (55).

4. By (25) and (28),
\[ G_{ii} \left( \frac{1}{G_{ii}} + \frac{1}{\sigma_i} + m_N \right) = G_{ii} \left( \frac{1}{N} \sum_{p=M+1}^{M+N} G_{pp} - \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \right) \]
\[ = G_{ii} \left( \frac{1}{N} \sum_{p=M+1}^{M+N} \left[ G_{pp} - G_{ii} \right] - \sum_{p,q=M+1}^{M+N} \left( \frac{1}{N} - (X_{i(p-M)})^2 \right) \right) G_{ii}^{(i)} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \]
\[ = \frac{1}{N} \sum_{p=M+1}^{M+N} (G_{pi})^2 + G_{ii} \sum_{p=M+1}^{M+N} \left( \frac{1}{N} - (X_{i(p-M)})^2 \right) \]
\[ \geq 1 \sum_{p=M+1}^{M+N} \left( \frac{1}{G_{ii}} \right) G_{ii}^{(i)} \sum_{p,q=M+1}^{M+N} X_{i(p-M)}X_{i(q-M)}G_{pq}^{(i)} \]
(60)

For the terms on the right hand side of (60), using (51) to estimate the first term, using (52) and (32) to estimate the second term, using (61), (32) and (33) to estimate the third term, we conclude that if \( N > N_0(\epsilon', D', \tau) \), then
\[ P \left( \left| \frac{G_{ii}}{G_{ii}} + \frac{1}{\sigma_i} + m_N \right| \leq N^{\epsilon' - \frac{1}{2}} \right) \geq 1 - N^{-D'}. \]
(61)

Notice \( |1 + \sigma_i m_N| \geq l \text{Im} m_N = \frac{l \sigma_i}{|\lambda|} \geq \frac{l \sigma_i}{\max(|\lambda|, |\lambda_i|)} \), so
\[ \left| G_{ii} + \frac{\sigma_i}{1 + \sigma_i m_N} \right| = \left| \frac{\sigma_i}{1 + \sigma_i m_N} G_{ii} \left( \frac{1}{G_{ii}} \right) + \frac{1}{\sigma_i} + m_N \right| \leq \frac{\max(|\lambda|, |\lambda_i|)^2}{l \tau} \left| G_{ii} \left( \frac{1}{G_{ii}} \right) + \frac{1}{\sigma_i} + m_N \right|. \]
This together with (61) and (25) complete the proof of (54).

\[ \square \]

Lemma 7. Suppose \( \tau \in (0, C_0^{-1}) \). For any (small) \( \epsilon' > 0 \) and (big) \( D' > 0 \), if \( N > N_0 = N_0(\epsilon', D', \tau) \) then
1. \( P(|A_{kk}| \leq N^{\epsilon'} \| \forall k \in \{1, \ldots, M + N\} \);\n2. \( P(|A_k| \leq N^{\epsilon' - 1} \| \forall k \in \{1, \ldots, M + N\} \);
3. \( \mathbb{P}(|Z_k| \leq N^\epsilon - \epsilon') > 1 - N^{-D'} \) for all \( k \in \{1, \ldots, M + N\} \);

4. \( \mathbb{P}(|B_N| \leq N^\epsilon - \epsilon') > 1 - N^{-D'} \).

\textit{Proof.} 1. Suppose \( z \in D'_r, i \in \{1, \ldots, M\} \) and \( p \in \{M + 1, \ldots, M + N\} \). By (21) and (25)

\[
\frac{1}{G_{ri}} = - \sum_{p' = M + 1}^{M + N} (X_i(p' - M))^2 G_{i,p'}^{(i)} - \sum_{p', q' \in [M + 1, M + N]} X_i(p' - M) X_i(q' - M) G_{i,p'}^{(i)} G_{i,q'}^{(i)} - \frac{1}{\sigma_i} \tag{62}
\]

\[
\frac{1}{G_{pp}} = - \sum_{i' = 1}^{M} (X_{i'}(p - M))^2 G_{i',p}^{(p)} - \sum_{i', j' \in [1, M]} X_{i'}(p - M) X_{j'}(p - M) G_{i',j'}^{(p)} - z \tag{63}
\]

Using (10) and (32) to estimate the first term of the RHS of (63) and the first term of the RHS of (62), using (18) and (32) the second term of the RHS of (63), using (51) and (32) the second term of the RHS of (62), we have that if \( N > N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P}(|\frac{1}{G_{kk}}| \leq N^\epsilon' \quad \forall k \in \{1, \ldots, M + N\}) > 1 - N^{-D'}. \tag{64}
\]

Using a classic “lattice” argument we obtain the first statement.

2. The first conclusion together with (53), (54) and (55) and a “lattice” argument prove the second conclusion.

3. Suppose \( z \in D'_r, i \in \{1, \ldots, M\} \) and \( p \in \{M + 1, \ldots, M + N\} \). By definition,

\[
Z_p = \sum_{i' = 1}^{M} \left( (X_{i'}(p - M))^2 - \frac{1}{N} \right) G_{i',p}^{(p)} + \sum_{i', j' \in [1, M]} X_{i'}(p - M) X_{j'}(p - M) G_{i',j'}^{(p)} \tag{65}
\]

\[
Z_i = \sum_{p' = M + 1}^{M + N} \left( (X_i(p' - M))^2 - \frac{1}{N} \right) G_{p',i}^{(i)} + \sum_{p', q' \in [M + 1, M + N]} X_i(p' - M) X_i(q' - M) G_{p',q'}^{(i)} \tag{66}
\]

Using (10) and (32) to estimate the first term of the RHS of (65), using (18), (32) and (33) to estimate the second term of the RHS of (65), using (52) and (32) to estimate the first term of the RHS of (66), using (51), (33) and (32) the second term of the RHS of (66), we have for \( N > N_0(\epsilon', D', \tau) \) and \( k \in \{1, \ldots, M + N\} \):

\[
\mathbb{P}(|Z_k| \leq N^\epsilon' - \epsilon') > 1 - N^{-D'}
\]

Using a “lattice” argument we obtain the third statement.
4. Suppose \( z \in D'_\tau \). Note \(|m_N| \leq \frac{1}{\ln z} \leq \tau^{-1}\) and \( \text{Im} m_N = \frac{1}{N} \sum_{i=1}^{N} \frac{\text{Im} z_{\lambda_i - z}}{|\lambda_i - z|^2} \geq \frac{\max(|\lambda_i|, |\lambda_j|)}{\max((|z| + |\lambda_i|)^2, |z| + |\lambda_j|)^2} \). So we have from (21) and the first two conclusions of this lemma that if \( N > N_0(\epsilon', D', \tau) \) and \( 1 \leq i \leq M \) then

\[
\mathbb{P}\left(|1 + \sigma_i m_N| \geq l|\text{Im} m_N| \geq \frac{l\tau}{(|z| + C_0)^2}\right) > 1 - N^{-D'} \tag{67}
\]

where \( C_0 > 0 \) is defined in Lemma 3. By (67), the definition of \( B_N \) and the first two conclusions of this lemma, we have for \( N > N_0(\epsilon', D', \tau) \):

\[
\mathbb{P}(\|B_N\| \leq N'^{-1}) > 1 - N^{-D'}
\]

Using a “lattice” argument we obtain the last statement.

\[ \square \]

**Theorem 4.** Suppose \( \tau \in (0, C_0^{-1}) \). For any (small) \( \epsilon' > 0 \) and (big) \( D' > 0 \), if \( N > N_0 = N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P}\left(|m_N - \hat{m}_{fc}| \leq N'^{-\frac{3}{4}} \quad \forall z \in D'_\tau\right) > 1 - N^{-D'}
\]

**Proof.** By (40), Lemma 6, Theorem 3 and (67), if \( N > N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P}\left(|h(m_N) - z| \leq N'^{-\frac{3}{4}}, \forall z \in D'_\tau\right) > 1 - N^{-D'} \tag{68}
\]

According to (68) and (21), if \( N > N_0(\epsilon', D', \tau) \) then

\[
\mathbb{P}(E_N) > 1 - N^{-D'} \tag{69}
\]

where

\[
E_N = \{ h(m_N) - z \leq N'^{-\frac{3}{4}}, \forall z \in D'_\tau \} \cap \{ \lambda_i \in [0, C_0], \forall 1 \leq i \leq N \}.
\]

By (38) we have the quadratic equation:

\[
\beta_1 \cdot (m_N(z) - \hat{m}_{fc}(z))^2 + (m_N(z) - \hat{m}_{fc}(z)) = \beta_2
\]

where

\[
\beta_1 = -\frac{1}{N} \sum_{i=1}^{M} \frac{\hat{m}_{fc}(z) \sigma_i^2}{\hat{m}_{fc}(1 + \sigma_i m_N(z))(1 + \hat{m}_{fc}(z) \sigma_i)^2}
\]

\[
\beta_2 = \frac{m_N \hat{m}_{fc}(h(m_N) - z)}{\hat{m}_{fc}}
\]

Suppose \( E_N \) holds and \( z \in D'_\tau \). By Lemma 3

\[
\text{Im} \hat{m}_{fc}(z) = \text{Im} z \int \frac{d\hat{m}_{fc}(t)}{|t - z|^2} \geq \frac{\text{Im} z}{(|z| + \hat{L})^2} \geq \frac{\tau}{(|z| + C_0)^2}
\]

\[
\text{Im} m_N(z) = \frac{\text{Im} z}{N} \sum_{i=1}^{N} \frac{1}{|\lambda_i - z|^2} \geq \frac{\tau}{(|z| + C_0)^2}
\]

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and therefore

$$|\beta_1| \leq C_1, \quad |\beta_2| \leq C_1 N^{\epsilon' - \frac{1}{2}}$$  \hspace{1cm} (70)

where $C_1 > 0$ is a constant determined by $\tau$.

According to the above quadratic equation, we have

$$m_N - \hat{m}_{fc} = m_{(1)} := \frac{-1 + x_1}{2\beta_1}$$ or $$m_N - \hat{m}_{fc} = m_{(2)} := \frac{-1 + x_2}{2\beta_1}$$

where $x_{1,2}$ are square roots of $1 + 4\beta_1 \beta_2$ such that $x_1$ is close to $-1$ and $x_2 = -x_1$ is close to $1$. So if $N > N_0(\epsilon', D', \tau)$ then we have:

$$|1 + x_1| \leq 4|\beta_1| \beta_2 | \leq 4C_1^2 N^{\epsilon' - \frac{1}{2}}, \quad |1 - x_2| \leq 4|\beta_1| \beta_2 | \leq 4C_1^2 N^{\epsilon' - \frac{1}{2}},$$

$$|m_{(1)} - m_{(2)}| = \frac{|-1 + x_1 - (-1 + x_2)|}{2|\beta_1|} = \frac{|x_2|}{2|\beta_1|} \geq \frac{1}{2|\beta_1|} \geq \frac{1}{2C_1}$$

so by continuity, we have

$$m_N - \hat{m}_{fc} = m_{(1)} = \frac{-1 + x_1}{2\beta_1} \quad \forall z \in D'_r$$  \hspace{1cm} (71)

or

$$m_N - \hat{m}_{fc} = m_{(2)} = \frac{-1 + x_2}{2\beta_1} \quad \forall z \in D'_r$$  \hspace{1cm} (72)

By definition we have $D_r \cap D'_r \neq \emptyset$, so Theorem 2 implies that (71) cannot happen. In summary, if $N > N_0(\epsilon', D', \tau)$ and $E_N$ holds, then (72) holds and

$$|m_N - \hat{m}_{fc}| = |m_{(2)}| \leq 4C_1^2 N^{\epsilon' - \frac{1}{2}}, \quad \forall z \in D'_r.$$

Note that $\epsilon'$ can be arbitrarily small. So we use (69) and complete the proof. \hfill \Box

6 A strong local law on $D_r \cup D'_r$: proof of Proposition 2

According to Theorem 2, Theorem 4 and the fact that $\operatorname{Im} \hat{m}_{fc} \sim \frac{1}{N \operatorname{Im} z}$ on $D_r \cup D'_r$ (which will be proved in Lemma 8), for any (small) $\epsilon > 0$ and (big) $D > 0$, if $N > N_0 = N_0(\epsilon, D, \tau)$, then

$$\mathbb{P}(|m_N - \hat{m}_{fc}| \leq N^{\epsilon} \Psi(z) \quad \forall z \in D_r \cup D'_r) > 1 - N^{-D}$$  \hspace{1cm} (73)

where

$$\Psi(z) = \frac{1}{N \operatorname{Im} z} + \frac{1}{\sqrt{N}}$$  \hspace{1cm} (74)

as we defined in (18). In this section we prove that the $\Psi(z)$ in (73) can be improved to $\Psi^2(z)$.
Lemma 8. Suppose \( \tau \in (0, C_0^{-1}) \). There exists a constant \( C_\tau > 1 \) such that if \( z \) is in
\[
\{ x + iy | 0 \leq y \leq \tau^{-1}, x \in [\tau, \hat{L}_- - \tau] \cup [\hat{L}_+ + \tau, \tau^{-1}] \} \cup D'_\tau
\]
then
1. 
\[
C_\tau^{-1} \leq \frac{\text{Im} \hat{m}_{fc}(z)}{\text{Im} z} \leq C_\tau
\]
\[
|\hat{m}_{fc}(z)| \leq C_\tau
\]
2. for any \( i \in \{1, \ldots, M\} \) we have
\[
C_\tau^{-1} \leq |1 + \sigma_i \hat{m}_{fc}(z)| \leq C_\tau
\]
3. for any (big) \( D > 0 \), if \( N > N_0 = N_0(D, \tau) \) then
\[
\mathbb{P}(|G_{ii}| \leq C_\tau, \ \forall i \in \{1, \ldots, M + N\} \text{ and } z \in D_\tau \cup D'_\tau) > 1 - N^{-D}
\]
Proof. 1. Notice that
\[
\frac{\text{Im} \hat{m}_{fc}(z)}{\text{Im} z} = \int \frac{1}{|t - z|^2} d\hat{\mu}_{fc}(t)
\]
Since \( z \in (75) \), we have \( |t - z| \geq \tau \) for \( t \) in the support of \( \hat{\mu}_{fc} \), so by (80) we know the right inequality in (76) is correct. On the other hand, \( |t - z| \leq |t| + |z| \) is bounded by a constant for all \( t \) in the support of \( \hat{\mu}_{fc} \), so by (80) we know the left inequality in (76) is correct. (77) can be proved by noticing
\[
|\hat{m}_{fc}(z)| \leq \int \frac{1}{|z - t|} d\hat{\mu}_{fc}(t)
\]
and the fact that \( |t - z| \geq \tau \) for \( t \) in the support of \( \hat{\mu}_{fc} \).

2. According to Lemma 2.5 of [12],
\[
x_1 = \hat{m}_{fc}(\hat{L}_+) := \lim_{\eta \to 0^+} \hat{m}_{fc}(\hat{L}_+ + i\eta)
\]
where \( x_1 \) is the unique critical point of the function
\[
r \mapsto \frac{1}{r} + \frac{1}{N} \sum_{j=1}^{M} \frac{\sigma_j}{1 + r\sigma_j}
\]
on \((- (\max \sigma_j)^{-1}, 0)\). It’s easy to see \( \hat{m}_{fc} \) is negative and is increasing on \([\hat{L}_+, \infty)\), so
\[
|\hat{m}_{fc}(L_+ + \tau) - \hat{m}_{fc}(\hat{L}_+)| \geq \frac{\tau}{(\hat{L}_+ + \tau)^2}
\]
which implies

\[ |\hat{m}_{fc}(L_+ + \tau)| \leq |x_1| - \frac{\tau}{(L_+ + \tau)^2} < \frac{1}{\max \sigma_j} - \frac{\tau}{(L_+ + \tau)^2}. \]

Since \( \hat{m}_{fc}(\cdot) \) is increasing on \((\hat{L}_+, \infty)\), we have for all \( x \geq \hat{L}_+ + \tau \):

\[ 1 + \sigma_i \hat{m}_{fc}(x) \geq 1 + \sigma_i \hat{m}_{fc}(L_+ + \tau) > 1 - \sigma_i \left( \frac{1}{\max \sigma_j} - \frac{\tau}{(L_+ + \tau)^2} \right) > \frac{l\tau}{(L_+ + \tau)^2} \quad (83) \]

If \( \gamma_0 > 1 \) then \( \hat{\mu}_{fc} \) does not have an atom at 0, so \( \hat{m}_{fc} > 0 \) on \((-\infty, \hat{L}_-)\) and we have

\[ \|1 + \sigma_i \hat{m}_{fc}(x)\| > 1, \quad \forall x \in (-\infty, \hat{L}_-) \quad (84) \]

Suppose \( \gamma \in (0, 1) \). According to Lemma 2.5 of [12],

\[ x'_1 = \hat{m}_{fc}(\hat{L}_-) := \lim_{\eta \to 0^+} \hat{m}_{fc}(\hat{L}_- + i\eta) \]

where \( x'_1 \) is the unique critical point of (81) on \((\hat{L}_-, \hat{L}_+ - \tau)^2\). It’s easy to see \( \hat{m}_{fc} \) is increasing on \((0, \hat{L}_-)\), so

\[ |\hat{m}_{fc}(L_- - \tau) - \hat{m}_{fc}(\hat{L}_-)| > \frac{\tau}{(L_+ + \tau)^2} \]

which implies

\[ \hat{m}_{fc}(L_- - \tau) < x'_1 - \frac{\tau}{(L_+ + \tau)^2} < -\frac{1}{\max \sigma_j} - \frac{\tau}{(L_+ + \tau)^2}. \]

Since \( \hat{m}_{fc}(\cdot) \) is increasing on \((0, \hat{L}_-)\), we have for all \( x \in (0, \hat{L}_-)\):

\[ 1 + \sigma_i \hat{m}_{fc}(x) \leq 1 + \sigma_i \hat{m}_{fc}(L_- - \tau) < -\frac{l\tau}{(L_+ + \tau)^2} \quad (85) \]

Then, by (83), (84) and (85), if \( x \in [\tau, \hat{L}_- - \tau] \cup [\hat{L}_+ + \tau, \tau^{-1}] \), then

\[ |1 + \sigma_i \hat{m}_{fc}(x)| \geq C_1 := 1 \wedge \frac{l\tau}{(L_+ + \tau)^2} \]

and for \( 0 < y < C_1 \tau^2 / 2 \) we have

\[ |\hat{m}_{fc}(x + iy) - \hat{m}_{fc}(x)| = \left| \int \frac{iy}{(t - x)(t - x - iy)} d\hat{\mu}_{fc}(t) \right| \leq \frac{y}{\tau^2} \]

thus

\[ |1 + \sigma_i \hat{m}_{fc}(x + iy)| \geq |1 + \sigma_i \hat{m}_{fc}(x)| - \frac{y}{\tau^2} > C_1 / 2. \quad (86) \]
On the other hand, if $z \in (75)$ and $\text{Im} z \geq C_1 \tau^2 / 2$, then

$$|1 + \sigma_i \hat{m}_{f_e}(z)| \geq t \cdot \text{Im} \hat{m}_{f_e}(z) = t \cdot \text{Im} \int \frac{d\hat{p}_{f_e}(t)}{|t-z|^2} \geq t \cdot \text{Im} z \cdot \frac{1}{(|z| + L_+)^2} \geq \frac{l C_1 \tau^2 / 2}{(\sqrt{2} \tau^{-1} + C_0)^2} \tag{87}$$

where we used the facts $|z| \leq \sqrt{2} \tau^{-1}$ and $\hat{L}_+ \leq C_0$ (see Lemma 3) in the last inequality. \ref{80} and \ref{87} complete the proof of the first inequality in \ref{78}. The second inequality in \ref{78} is directly from \ref{77}.

3. Notice that $D^*_+ \cup D^*_r$ is contained in \ref{75}. So \ref{70} comes from Theorem 2, Theorem 3, Theorem 4 and the first two conclusions of this lemma.

\[\square\]

**Lemma 9.** Suppose $\tau \in (0, C_0^{-1})$. For any (small) $\epsilon' > 0$ and (big) $D' > 0$, if $N > N_0 = N_0(\epsilon', D', \tau)$ then

1. $$\mathbb{P}(|\frac{1}{G_{kk}}| \leq N^{-\epsilon'}, \ \forall z \in D_{\tau} \cup D'_{\tau}) \text{ for all } k \in \{1, \ldots, M+N\} \tag{88}$$

$$\mathbb{P}(|m_N| \geq N^{-\epsilon'}, \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'} \tag{89}$$

$$\mathbb{P}(|G^{(r)}_{kk}| \leq N^{-\epsilon'} \Psi(z), \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'} \text{ for all } k \neq s \in \{1, \ldots, M+N\} \setminus \{r\} \tag{90}$$

$$\mathbb{P}(|G^{(r)}_{ss}| \leq 2C_{\tau}, \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'} \text{ for all } s \in \{1, \ldots, M+N\} \setminus \{r\} \tag{91}$$

where $C_{\tau}$ was defined in Lemma 8.

2. $\mathbb{P}(|A_k| \leq N^{-\epsilon'} \Psi^2(z), \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'} \text{ for all } k \in \{1, \ldots, M+N\}$;

3. $\mathbb{P}(|Z_k| \leq N^{-\epsilon'} \Psi^2(z), \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'} \text{ for all } k \in \{1, \ldots, M+N\}$;

4. $\mathbb{P}(|B_N| \leq N^{-\epsilon'} \Psi^2(z), \ \forall z \in D_{\tau} \cup D'_{\tau}) > 1 - N^{-D'}$

where $\Psi(z) = \frac{1}{N m_z} + \frac{1}{\sqrt{N}}$ (as defined in \ref{74}).

**Proof.** Thanks to Lemma 7 it suffices to prove each conclusion with $D_{\tau} \cup D'_{\tau}$ replaced by $D_{\tau}$.

1. By \ref{41}, \ref{76} and \ref{77} we obtain \ref{88} for $1 \leq k \leq M$. Now suppose $M+1 \leq p \leq M+N$ and $z$ is in

$$\left\{ x + iy | \tau \leq x \leq \tau^{-1}, N^{-1+\tau} \leq y \leq \tau^{-1} \right\}. \tag{92}$$

By \ref{21} and \ref{28} we have:

$$\frac{1}{G_{pp}} = -z - \sum_{i=1}^{M} (X_{i(p-M)})^2 \left[ G_{ii} - \frac{(G_{ip})^2}{G_{pp}} \right] - \sum_{i,j \in [1, M] \setminus \{p\}} X_{i(p-M)} X_{j(p-M)} G_{ij}^{(p)} \tag{93}$$
and
\[
\frac{1}{G_{pp}} \left[ 1 - \sum_{i=1}^{M} (X_{i(p-M)} G_{ip})^2 \right] = -z - \sum_{i=1}^{M} (X_{i(p-M)})^2 G_{ii} - \sum_{i,j \in [1, M], \ i \neq j} X_{i(p-M)} X_{j(p-M)} G_{ij}. 
\]

(94)

By (28) and (48), if \( N > N_0(\epsilon', D', \tau) \), then we have with at least \( 1 - N^{-D'} \) probability that:
\[
\left| \sum_{i,j \in [1, M], \ i \neq j} X_{i(p-M)} X_{j(p-M)} G_{ij} \right| \leq N^\epsilon' \sqrt{2N^\epsilon' - 1} \left( \frac{1}{|G_{pp}|} \left| \sum_{i,j \in [1, M], \ i \neq j} G_{ij} \right|^2 \right) \leq \left| \sum_{i,j \in [1, M], \ i \neq j} (G_{ip} G_{jp})^2 \right|
\]
and by (34):
\[
\mathbb{1}_{z \in D, \tau} \cdot \frac{1}{|G_{pp}|} \left[ 1 - \sum_{i=1}^{M} (X_{i(p-M)} G_{ip})^2 \right] - \sqrt{2N^\epsilon' - 1} \left( \frac{1}{|G_{pp}|} \left| \sum_{i,j \in [1, M], \ i \neq j} G_{ij} \right|^2 \right) \leq \mathbb{1}_{z \in D, \tau} \left( |z| + \sum_{i=1}^{M} (X_{i(p-M)})^2 G_{ii} \right) + \sqrt{2N^\epsilon' - 1} \left( \frac{1}{|G_{pp}|} \left| \sum_{i,j \in [1, M], \ i \neq j} G_{ij} \right|^2 \right)
\]
where
- \( \mathbb{1}_{z \in D, \tau} \sum_{i=1}^{M} (X_{i(p-M)} G_{ip})^2 \leq N^\epsilon' - 2\tau \) by (10), (43) and (94)
- \( \mathbb{1}_{z \in D, \tau} \sum_{i,j \in [1, M], \ i \neq j} |G_{ip} G_{jp}|^2 \leq N^2 + \epsilon' - 2\tau \) and \( \mathbb{1}_{z \in D, \tau} \sum_{i,j \in [1, M], \ i \neq j} |G_{ij}|^2 \leq N^2 + \epsilon' - 2\tau \) by (48)
- \( \mathbb{1}_{z \in D, \tau} \sum_{i=1}^{M} (X_{i(p-M)})^2 G_{ii} \leq N^\epsilon' \) by (10) and (79).

Since \( \epsilon' \) can be arbitrarily small, we have for \( N > N_0(\epsilon', D', \tau) \):
\[
\mathbb{P}(\mathbb{1}_{z \in D, \tau} \cdot \frac{1}{|G_{pp}|} \leq N^\epsilon') > 1 - N^{-D'}.
\]

Using a “lattice” argument on (92) we prove (88) for \( k \in \{M + 1, \ldots, M + N\} \), thus complete the proof of (88). (89) is directly from (88), (42) and (45). (91) comes directly from (79), (43), (88) and (28). (90) is from (43), (88) and (28).

2. The second conclusion is directly from (43) and the first conclusion.
3. Suppose \( i \in \{1, \ldots, M\}, p \in \{M + 1, \ldots, M + N\} \) and \( z \) is in (92). By definition,
\[
Z_p = \sum_{i' \in [1, M]} \left( (X_{i'(p-M)})^2 - \frac{1}{N} \right) G_{i'i'}(p) + \sum_{i', j' \in [1, M], \ i' \neq j'} X_{i'(p-M)} X_{j'(p-M)} G_{i'j'}(p),
\]
(95)
\[
Z_i = \sum_{p' = M + 1}^{M+N} \left( (X_{i(p'-M)})^2 - \frac{1}{N} \right) G_{p'p'} + \sum_{p', q' \in \{M+1, M+N\}, p' \neq q'} X_{i(p'-M)} X_{i(q'-M)} G_{p'q'}. \tag{96}
\]

Using (49) and (91) to estimate the first term of the RHS of (95), using (48) and (90) to estimate the second term of the RHS of (95), using (52) and (91) to estimate the first term of the RHS of (96), using (51) and (90) to estimate the second term of the RHS of (96), we have for \( N > N_0(\epsilon', D', \tau) \) and \( k \in \{1, \ldots, M + N\} \):

\[
P\left( 1 \leq |Z_k| \leq N^{\epsilon'} \Psi(z) \right) > 1 - N^{-D'}
\]

Using a “lattice” argument we obtain the third statement.

4. According to (42), (78) and the second and third conclusions of this lemma, if \( N > N_0(\epsilon', D', \tau) \), then we have with probability at least 1 - \( N^{-D'} \) that:

\[
|1 + \sigma_i m_N(z)| \geq \frac{1}{2C_\tau}, \quad \forall z \in D_\tau,
\]

\[
|1 + \sigma_i m_N(z) + \sigma_i Z_i + \sigma_i A_i| \geq \frac{1}{2C_\tau}, \quad \forall z \in D_\tau.
\]

(97), (98) and the second and third conclusions of this lemma complete the proof of the fourth conclusion.

\[ \square \]

**Lemma 10.** Suppose \( \tau \in (0, C_0^{-1}) \) and \( p_0 \in \{0, 1, 2, \ldots\} \). For any (small) \( \epsilon > 0 \) and (big) \( D > 0 \), if \( N > N_0(\epsilon, D, \tau, p_0) \), then we have

\[
P(|G_{ij}^{(T)}| \leq N^{\epsilon} \Psi(z), \quad \forall z \in D_\tau \cup D'_\tau) > 1 - N^{-D} \tag{99}
\]

\[
P(|G_{ii}^{(T)}| \leq (p_0 + 1)C_\tau, \quad \forall i \in \{1, \ldots, M + N\} \text{ and } z \in D_\tau \cup D'_\tau) > 1 - N^{-D} \tag{100}
\]

\[
P\left( \frac{1}{G_{ii}^{(T)}} \leq N^\epsilon, \quad \forall z \in D_\tau \cup D'_\tau \right) > 1 - N^{-D} \tag{101}
\]

\[
P\left( \left| 1 - E_i \right| \frac{1}{G_{ii}^{(T)}} \leq N^\epsilon \Psi(z), \quad \forall z \in D_\tau \cup D'_\tau \right) > 1 - N^{-D} \tag{102}
\]

for any \( i \neq j \in \{1, \ldots, M + N\} \setminus T \) and \( T \subset \{1, \ldots, M + N\} \) satisfying \(|T| \leq p_0\).

**Proof.** We first prove (99), (100) and (101). Suppose \(|T| = 0\). In this case, Theorem 2 and Theorem 3 imply (99); (88) gives (101); (79) gives (100). If \(|T| = 1\), then using (23) and the conclusion for \(|T| = 0\), we obtain (99), (100) and (101). Repeating this procedure for finite many times we proved (99), (100) and (101) for all \(|T| \leq p_0\).

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Now we prove (102). Suppose \( i \in \{1, \ldots, M\} \setminus T, p \in [M + 1, M + N] \setminus T \) and \( z \) is in \( G \). Using (24) and (25) (with \( G \) replaced by \( G^{(T)} \)) and the definition of \( E_k \) we get:

\[
(1 - E_p) \frac{1}{G_{ii}^{(T)}} = - \sum_{p', q' \in [M+1, M+N] \setminus T} X_{i(p' - M)} X_{i(q' - M)} G_{i'i'}^{(T)} - \sum_{p' \in [M+1, M+N] \setminus T} \left( (X_{i(p' - M)})^2 - \frac{1}{N} \right) G_{i'i'}^{(p'T)}
\]

(103)

\[
(1 - E_p) \frac{1}{G_{pp}^{(T)}} = - \sum_{i', j' \in [1, M]}, T X_{i'(p' - M)} X_{j'(p' - M)} G_{ij'}^{(p'T)} - \sum_{i' \in [1, M] \setminus T} \left( (X_{i'(p' - M)})^2 - \frac{1}{N} \right) G_{ij'}^{(p'T)}.
\]

(104)

Using (51) (with \( G \) replaced by \( G^{(T)} \)) and (99) to estimate the first term on RHS of (103), using (52) (with \( G \) replaced by \( G^{(T)} \)) and (100) to estimate the second term on RHS of (103), using (48) (with \( G \) replaced by \( G^{(T)} \)) and (99) to estimate the first term on RHS of (104), using (49) (with \( G \) replaced by \( G^{(T)} \)) and (100) to estimate the second term on RHS of (104), we have for \( N > N_0(\epsilon, D, \tau, p_0) \):

\[
\mathbb{P} \left( \mathbf{1}_{\bar{z} \in D_r \cup D'_r} \left| (1 - E_i) \frac{1}{G_{ii}^{(T)}} \right| \leq N^\epsilon \Psi(z) \right) > 1 - N^{-D}
\]

\[
\mathbb{P} \left( \mathbf{1}_{\bar{z} \in D_r \cup D'_r} \left| (1 - E_p) \frac{1}{G_{pp}^{(T)}} \right| \leq N^\epsilon \Psi(z) \right) > 1 - N^{-D}
\]

Using the “lattice” argument we complete the proof of (102).

Lemma 11. Suppose \( \tau \in (0, C_0^{-1}) \). For any (small) \( \epsilon > 0 \) and (big) \( D > 0 \), if \( N > N_0(\epsilon, D, \tau) \) then we have:

\[
\mathbb{P} \left( \frac{1}{N} \sum_{p = M + 1}^{M + N} Z_p \right) \leq N^\epsilon \Psi^2, \quad \forall \bar{z} \in D_r \cup D'_r > 1 - N^{-D}
\]

\[
\mathbb{P} \left( \frac{1}{N} \sum_{i = 1}^{M} \frac{\sigma_i^2}{(1 + \sigma_i m_N)^2} Z_i \right) \leq N^\epsilon \Psi^2, \quad \forall \bar{z} \in D_r \cup D'_r > 1 - N^{-D}
\]

The proof of Lemma 11 follows the standard “binary tree” argument. This argument was first introduced to study the Wigner matrix. See [5]. Later it was used for sample covariance matrix with deterministic population. See [6] and [12]. In our model, although the population is random, the “binary tree” argument still works. The key point is to define \( E_k \) as (24). For the convenience of the readers we write down all details of the binary tree argument under the settings of this paper. See Appendix [3] Its proof basically follows the steps in the proof of Proposition 6.1 in [3].

Proof of Proposition [2] For any \( z \) in \( \mathbb{C} \setminus \mathbb{R} \), we have:

\[
\left| \frac{1}{N} \sum_{i = 1}^{M} \frac{\sigma_i^2}{(1 + \sigma_i m_N)^2} Z_i \right| \leq \frac{1}{N} \sum_{i = 1}^{M} \frac{\sigma_i^2}{(1 + \sigma_i m_N)^2} \left| Z_i \right| \leq \frac{1}{N} \sum_{i = 1}^{M} \left( \frac{|m_N - \bar{m}_f|}{1 + \sigma_i m_N |1 + \sigma_i m_N f^2|} + \frac{|m_N - \bar{m}_f|}{1 + \sigma_i m_N |1 + \sigma_i m_N f^2|} \right)
\]

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thus by (40),

$$|h(m_N) - z| \leq |B_N| + \frac{1}{N} \sum_{p=M+1}^{M+N} |A_p| + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{|m_N - G_{pp}|^2}{|m_N^2 G_{pp}|} + \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p$$

$$+ \frac{1}{N} \sum_{i=1}^{M} \sigma_i^2 \left(1 + \sigma_i \hat{m}_{fc} \right)^2 Z_i + \frac{1}{N} \sum_{i=1}^{M} \left( \frac{|m_N - \hat{m}_{fc}| |Z_i|}{1 + \sigma_i m_N^2 |1 + \sigma_i \hat{m}_{fc}|^2} + \frac{|m_N - \hat{m}_{fc}| |Z_i|}{1 + \sigma_i m_N^2 |1 + \sigma_i \hat{m}_{fc}|} \right)$$

(105)

Now we estimate the RHS of (105). Using Lemma 9 to control $|B_N|$, $|A_p|$ and $\frac{1}{N} \sum_{p=M+1}^{M+N} \frac{|m_N - G_{pp}|^2}{|m_N^2 G_{pp}|}$, using Theorem 2 and (53) (and a “lattice” argument) to control $|m_N - G_{pp}|^2$, using Lemma 11 to control $\frac{1}{N} \sum_{p=M+1}^{M+N} Z_p$ and $\frac{1}{N} \sum_{i=1}^{M} \sigma_i^2 \left(1 + \sigma_i \hat{m}_{fc} \right)^2 Z_i$, using (78), (42), Theorem 4 and Lemma 9 to control the last term, we have for $N > N_0(\epsilon, D, \tau)$:

$$\mathbb{P}(|h(m_N) - z| \leq N^\epsilon \Psi^2, \forall z \in D_\tau \cup D'_\tau) > 1 - N^{-D}$$

(106)

From (89), Theorem 2 and Theorem 4 we have for $N > N_0(\epsilon, D, \tau)$:

$$\mathbb{P}(\frac{1}{|\hat{m}_{fc}|} \leq N^\epsilon, \forall z \in D_\tau \cup D'_\tau) > 1 - N^{-D}.$$

(107)

Since the distance between $D_\tau$ and $D'_\tau$ and the support of $\hat{m}_{fc}$ is $\tau$, we have:

$$|\hat{m}_{fc}(z)| \leq \frac{\tau}{\tau} \text{ and } |\hat{m}_{fc}(z)| \leq \frac{\tau}{\tau} \forall z \in D_\tau \cup D'_\tau$$

(108)

thus by (78), (42) and Theorem 4 if $N > N_0(\epsilon, D, \tau)$ then

$$\mathbb{P}(|m_N(z)| \leq \frac{2}{\tau} \text{ and } |1 + \sigma, m_N(z)| \geq \frac{1}{2C_\tau}, \forall i \in \{1, \ldots, M\}, z \in D_\tau \cup D'_\tau) > 1 - N^{-D}.$$  

(109)

By (106), (107) and (109), if $N > N_0(\epsilon, D, \tau)$ then

$$\mathbb{P}(E_N) > 1 - N^{-D}$$

(110)

where

$$E_N = \{|h(m_N) - z| \leq N^\epsilon \Psi^2, \frac{1}{|\hat{m}_{fc}|} \leq N^\epsilon, |m_N| \leq \frac{2}{\tau} |1 + \sigma_i m_N(z)| \geq \frac{1}{2C_\tau}, \forall z \in D_\tau \cup D'_\tau, i \in [1, M]\}$$

By (88) we have the quadratic equation:

$$\beta_1 \cdot (m_N(z) - \hat{m}_{fc}(z))^2 + (m_N(z) - \hat{m}_{fc}(z)) = \beta_2$$

where

$$\beta_1 = -\frac{1}{N} \sum_{i=1}^{M} \frac{\hat{m}_{fc}(z)^2}{m_N(1 + \sigma_i m_N(z))(1 + \hat{m}_{fc}(z)(1 + \sigma_i \hat{m}_{fc}(z))^2)}$$

$$\beta_2 = \frac{m_N \hat{m}_{fc}(h(m_N) - z)}{\hat{m}_{fc}}$$

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Suppose \( E_N \) holds and \( z \in D_{\tau} \cup D_{\tau}' \). We have from (78), (108) and the definition of \( E_N \):
\[
|\beta_1| \leq C_1 N^\epsilon, \quad |\beta_2| \leq C_1 N^{2\epsilon} \Psi^2 \tag{111}
\]
where \( C_1 > 0 \) is a constant determined by \( \tau \).

According to the above quadratic equation, we have
\[
m_N - \hat{m}_fc = m_{(1)} := \frac{-1 + x_1}{2\beta_1} \quad \text{or} \quad m_N - \hat{m}_fc = m_{(2)} := \frac{-1 + x_2}{2\beta_1}
\]
where \( x_{1,2} \) are square roots of \( 1 + 4\beta_1\beta_2 \) such that \( x_1 \) is close to \(-1\) and \( x_2 = -x_1 \) is close to \(1\). So if \( N > N_0(\epsilon, D, \tau) \) then
\[
|1 + x_1| \leq |4\beta_1\beta_2| \leq 4C_1^2 N^{3\epsilon} \Psi^2, \quad |1 - x_2| \leq |4\beta_1\beta_2| \leq 4C_1^2 N^{3\epsilon} \Psi^2
\]
\[
|m_{(1)} - m_{(2)}| = \left| \frac{-1 + x_1}{2\beta_1} - \frac{-1 + x_2}{2\beta_1} \right| = \left| \frac{x_2 - x_1}{2\beta_1} \right| \geq \frac{1}{2C_1} N^{-\epsilon}
\]
\[
|m_{(2)}| = \left| \frac{-1 + x_2}{2\beta_1} \right| = \left| \frac{2\beta_2}{x_2 + 1} \right| \leq 4C_1 N^{2\epsilon} \Psi^2 \quad (\text{since} \ x_2^2 = 1 + 4\beta_1\beta_2)
\]
\[
|m_{(1)}| \geq |m_{(1)} - m_{(2)}| - |m_{(2)}| \geq \frac{1}{2C_1} N^{-\epsilon} - 4C_1 N^{2\epsilon} \Psi^2 \geq \frac{1}{4C_1} N^{-\epsilon} \tag{112}
\]
so by continuity, we have
\[
m_N - \hat{m}_fc = m_{(1)} = \frac{-1 + x_1}{2\beta_1} \quad \forall z \in D_{\tau} \cup D_{\tau}' \tag{113}
\]
or
\[
m_N - \hat{m}_fc = m_{(2)} = \frac{-1 + x_2}{2\beta_1} \quad \forall z \in D_{\tau} \cup D_{\tau}' \tag{114}
\]
However, Theorem 2 and (112) imply that (113) cannot happen. In summary, if \( N > N_0(\epsilon, D, \tau) \) and \( E_N \) holds, then (113) holds and
\[
|m_N - \hat{m}_fc| = |m_{(2)}| \leq 4C_1 N^{2\epsilon} \Psi^2, \quad \forall z \in D_{\tau} \cup D_{\tau}'.
\]
Note that \( \epsilon \) can be arbitrarily small, so we use (110) and complete the proof. \( \square \)

Proof of Corollary 7. Lemma 10.1 of [12] proved same conclusion for a model with slight difference. But the method also works for our model. For the convenience of readers we write down the details.

Because of (21), it suffices to show that if \( N > N_0(\tau, D) \) then
\[
\mathbb{P}\left( \text{none of} \; \lambda_1, \ldots, \lambda_N \; \text{is in} \; [\tau, \hat{L}_- - \tau] \cup [\hat{L}_+ + \tau, \tau^{-1}] \right) > 1 - N^{-D}. \tag{115}
\]

Without loss of generality assume \( \tau < \frac{1}{100} \). Set \( \epsilon = 0.01 \). If we have \( \lambda_i = x_0 \in [\tau, \hat{L}_- - \tau] \cup [\hat{L}_+ + \tau, \tau^{-1}] \), then for \( y_0 := N^{-\frac{\epsilon}{2}} \), we have \( x_0 + iy_0 \in D_\tau \) and:
\[
\text{Im} \; m_N(x_0 + iy_0) = \frac{1}{N} \sum_{j=1}^{N} \frac{y_0}{|\lambda_j - x_0 - iy_0|^2} \geq \frac{1}{Ny_0} = N^{-\frac{\epsilon}{2} + \epsilon}
\]

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\[ \text{Im}\hat{m}_{fc}(x_0 + iy_0) \geq C^{-1}y_0 = C^{-1}N^{-\frac{1}{2} - \epsilon} \quad \text{(by (76))} \]

thus for \( N > N_0(\tau) \):

\[ |m_N(x_0 + iy_0) - \hat{m}_{fc}(x_0 + iy_0)| \geq \frac{1}{2}N^{-\frac{1}{2} + \epsilon} > N^\epsilon(N^{-\frac{1}{2} + \epsilon} + \frac{1}{\sqrt{N}})^2 = N^\epsilon \Psi^2(x_0 + iy_0). \quad \text{(116)} \]

However, according to Proposition 2 when \( N > N_0(\tau, D) \), then with probability no less than \( 1 - N^{-D} \), the inequality (116) does not hold. So (115) is true and the corollary is proved.

7 Proof of the main theorem

In this section we prove Theorem 1. Suppose its assumptions hold. First of all we suppose

- \( d \in (0, \frac{1}{10} \wedge C_0^{-1}) \) is a small constant such that \( f \) is analytic on a neighborhood of the closed rectangle with vertices \( L_+ + 2d \pm 2di \) and \( L_- - 2d \pm 2di \);
- \( \Gamma \) is the boundary of the rectangle described above with counterclockwise orientation;
- \( \Omega_N = \{ |\hat{L}_-_L| < \frac{d}{2} \text{ and } |\hat{L}_+_L| < \frac{d}{2} \} \)
  \( \tilde{\Omega}_N = \{ \lambda_1, \ldots, \lambda_{M\wedge N} \text{ are all in } [\hat{L}_-_L - \frac{d}{2}, \hat{L}_+_L + \frac{d}{2}] \} \cap \Omega_N \)

By the assumption (12) and Corollary 1, we have

\[ \mathbb{P}(\tilde{\Omega}_N) \to 1. \]

Suppose \( \tilde{\Omega}_N \) holds, then we have:

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(\lambda_i) - \sqrt{N} \int f(t) d\mu_{fc}(t) \]

\[ = \frac{1}{\sqrt{N}} \left[ \frac{1}{2\pi i} \sum_{\Gamma} \frac{f(\xi) d\xi}{\xi - \lambda_i} - \frac{N}{2\pi i} \int_{\Gamma} f(\xi) d\xi \int_{\Gamma} f(\xi) d\xi dt + f(0)[(N - M)^+ - N(1 - \gamma_0)^+] \right] \]

\[ = \frac{1}{\sqrt{N}} \left[ \int_{\Gamma} f(\xi) \left( N \int_{\xi}^{\xi+1} \frac{\rho_{fc}(t) dt}{t - \xi} - \sum_{i=1}^{M\wedge N} \frac{1}{\lambda_i - \xi} \right) d\xi + f(0) \frac{N}{\sqrt{N}} [(N - M)^+ - (N - 1 - \gamma_0)^+] \right] \]

\[ = \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} f(\xi) (m_{fc}(\xi) - m_N(\xi)) d\xi + \frac{f(0)}{\sqrt{N}} [(N - M)^+ - (N - 1 - \gamma_0)^+] \quad \text{(117)} \]

where we used

\[ \int_{\Gamma} f(\xi) \int_{\xi}^{\xi+1} \frac{\rho_{fc}(t) dt}{t - \xi} d\xi = \int_{\Gamma} f(\xi) \left( \int \frac{d\mu_{fc}(t)}{t - \xi} + \frac{(1 - \gamma_0)^+}{\xi} \right) d\xi = \int_{\Gamma} f(\xi) \int \frac{d\mu_{fc}(t)}{t - \xi} d\xi \]
and
\[
\int_{\Gamma} f(\xi) \sum_{i=1}^{\mathsf{M} \wedge \mathsf{N}} \frac{1}{\lambda_i - \xi} d\xi = \int_{\Gamma} f(\xi) \left( \sum_{i=1}^{\mathsf{N}} \frac{1}{\lambda_i - \xi} + \frac{(\mathsf{N} - \mathsf{M})_+}{\xi} \right) d\xi = \int_{\Gamma} f(\xi) \sum_{i=1}^{\mathsf{N}} \frac{1}{\lambda_i - \xi} d\xi
\]
in the last step of (117).

Note that \( P(\mathcal{\Omega}_N) \to 1 \) and
\[
\left| \frac{f(0)}{\sqrt{N}} \left( (\mathsf{N} - \mathsf{M})_+ - \mathsf{N}(1 - \gamma_0)_+ \right) \right| \leq f(0)\sqrt{N}|\gamma_0 - \frac{\mathsf{M}}{\mathsf{N}}| \to 0 \quad \text{(by \( \theta \)).}
\]

So it suffices to prove that
\[
\mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} f(\xi)(\hat{m}_{f_c}(\xi) - m_N(\xi))d\xi = \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} f(\xi)(\hat{m}_{f_c}(\xi) - m_N(\xi))d\xi + \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} f(\xi)(\hat{m}_{f_c}(\xi) - \hat{m}_{f_c}(\xi))d\xi
\]
converges in distribution to the limiting distribution in the statement of Theorem 11.

When \( \mathcal{\Omega}_N \) holds, we have by definition:
\[
\text{dist}(\xi, \text{supp}(\hat{m}_{f_c})) \geq d \quad \text{and} \quad \text{dist}(\xi, \lambda_i) \geq d \quad \forall \xi \in \Gamma, i \in \{1, \ldots, \mathsf{N}\}
\]
and then
\[
|m_N(\xi)| \leq d^{-1}, \quad |\hat{m}_{f_c}(\xi)| \leq d^{-1}, \quad \forall \xi \in \Gamma.
\]

Choose \( \tau = 0.01 \wedge \frac{d}{4} \), so if \( \mathcal{\Omega}_N \) holds then \( \Gamma \cap \{ z | \text{Im} z > \mathsf{N}^{-1+\tau} \} \subset D_\tau \cup D'_\tau \).

By Proposition 2 with \( \epsilon = 0.1 \) and \( D = 1 \), if \( \mathsf{N} > \mathsf{N}_0(d) \), then with probability larger than \( 1 - \mathsf{N}^{-1} \) we have
\[
\left| \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} (\hat{m}_{f_c}(\xi) - m_N(\xi))f(\xi)d\xi \right|
\]
\[
\leq \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi} \int_{\Gamma \cap \{ |\text{Im}\xi| < \mathsf{N}^{-1+\tau} \}} (\hat{m}_{f_c}(\xi) - m_N(\xi))f(\xi)d\xi + \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi} \int_{\Gamma \cap \{ |\text{Im}\xi| > \mathsf{N}^{-1+\tau} \}} (\hat{m}_{f_c}(\xi) - m_N(\xi))f(\xi)d\xi
\]
\[
\leq \left( \frac{\sqrt{N}}{2\pi} 4\mathsf{N}^{-1+\tau} \cdot \frac{2}{d} + \frac{\sqrt{N}}{2\pi} \int_{\Gamma \cap \{ |\text{Im}\xi| > \mathsf{N}^{-1+\tau} \}} 2N^{0.1}(\frac{1}{N^2|\text{Im}\xi|^2} + \frac{1}{N})d\xi \right) \sup_{\xi \in \Gamma} |f(\xi)|
\]
\[
\leq \left( \frac{2}{d} \mathsf{N}^{-0.5+\tau} + \mathsf{N}^{-0.3} \right) \sup_{\xi \in \Gamma} |f(\xi)|
\]

So \( \mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} (\hat{m}_{f_c}(\xi) - m_N(\xi))f(\xi)d\xi \) converges in distribution to 0 and now we only need to show that
\[
\mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} (m_{f_c}(\xi) - \hat{m}_{f_c}(\xi))f(\xi)d\xi
\]
or
\[
\mathbb{1}_{\mathcal{\Omega}_N} \frac{\sqrt{N}}{2\pi i} \int_{\Gamma} (m_{f_c}(\xi) - \hat{m}_{f_c}(\xi))f(\xi)d\xi
\]
converges in distribution to the limiting distribution in the statement of Theorem 11.

**Lemma 12.** There exists a constant \( C_d > 0 \) depending on \( d \) such that
• \[ |1 + tm_{fc}(z)| > C_d \quad \forall z \in \Gamma, t \in [l, 1] \]

• if \( \Omega_N \) holds then
  \[ |1 + \sigma_i m_{fc}(z)| > C_d \quad \forall z \in \Gamma, 1 \leq i \leq M \]

**Proof.** The second conclusion is induced from Lemma 8. To prove the first conclusion, we cite the following result from Page 2271–2272 of [9]:

For any interval \( I \subset (-\infty, 0) \cup (0, 1) \cup (l^{-1}, \infty), \) \(-m_{fc}(x)\) is decreasing on \( I \) and the function

\[
r(x) := \frac{1}{x} + \gamma_0 \int \frac{t}{1-tx} d
\]

is the inverse of \(-m_{fc}\) on \(-m_{fc}(I)\), thus \(r(x)\) is decreasing on \(-m_{fc}(I)\).

(Note that the Stieltjes transform in [9] differs by a sign from the one in this paper.)

• To estimate \([1 + tm_{fc}(L_+ + 2d)]\), we take \( I = [L_+ + 2d, \infty) \). From the above result we see that \((0, -m_{fc}(L_+ + 2d))\) is contained in the domain of \( r \), so \(|m_{fc}(L_+ + 2d)| < 1 \) and \(|1 + tm_{fc}(L_+ + 2d)|\) is larger than a constant which depends only on \( d \).

• If \( \gamma_0 > 1 \), then \( \mu_{fc} \) does not have an atom at origin. So \( m_{fc}(L_- - 2d) > 0 \) and \(|1 + tm_{fc}(L_- - 2d)| > 1 \).

• If \( \gamma_0 \in (0, 1) \), we take \( I = (0, L_- - 2d) \). Then \(-m_{fc}(x)\) approaches \( \infty \) when \( x \to 0^+ \). From the above result we see that \((-m_{fc}(L_- - 2d), \infty)\) is contained in the domain of \( r \), so \( m_{fc}(L_- - 2d) < -l^{-1} \) and \(|1 + tm_{fc}(L_- - 2d)|\) is larger than a constant which depends only on \( d \).

By continuity there exists \( C_1 > 0 \) and \( y_0 > 0 \) both determined by \( d \) such that if \(|y| < y_0\) then

\[ |1 + tm_{fc}(L_+ + 2d + iy)| > C_1 \quad \text{and} \quad |1 + tm_{fc}(L_- - 2d + iy)| > C_1. \]

When \( z \in \Gamma \) but \( |\text{Im}z| \geq y_0 \), then

\[ |1 + tm_{fc}(z)| \geq l|\text{Im}m_{fc}(z)| \geq l y_0 \int \frac{d\mu_{fc}(t)}{|z - t|^2} \]

which is bounded below since \(|t - z|\) is bounded above uniformly. So the first conclusion is proved.

**Definition 9.** For \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( 1 \leq i \leq M \) we define

\[
g_i(z) = \frac{\sigma_i}{1 + \sigma_i m_{fc}(z)}, \quad \hat{g}_i(z) = \frac{\sigma_i}{1 + \sigma_i \tilde{m}_{fc}(z)}
\]

**Lemma 13.** Suppose \( a_1 > 0, a_2 > 0 \) are constants. Then

\[
P\left( \left| \frac{1}{N \frac{1}{2} + a_1} \sum_{i=1}^{N} (g_i(\xi) - \mathbb{E}[g_i(\xi)]) \right| \leq a_2, \quad \forall \xi \in \Gamma \right) \rightarrow 1 \quad \text{as} \ N \rightarrow \infty
\]

\[
P\left( \left| \frac{1}{N \frac{1}{2} + a_1} \sum_{i=1}^{N} (g_i^2(\xi) - \mathbb{E}[g_i^2(\xi)]) \right| \leq a_2, \quad \forall \xi \in \Gamma \right) \rightarrow 1 \quad \text{as} \ N \rightarrow \infty
\]
Proof. It can be proved in the same way as Lemma 19 in [13]. Note that $g_i$ are bounded on $\Gamma$ thanks to Lemma 12.

Lemma 14. Suppose $z \in \mathbb{C} \setminus \mathbb{R}$. Then we have
\[
\hat{m}_{fc} - m_{fc} = A_N(\hat{m}_{fc} - m_{fc})^2 + B_N
\]  
where
\[
A_N = \frac{m'_{fc} M}{m_{fc} N} \int \left( \frac{t}{1 + tm_{fc}} \right)^2 dt - \frac{m'_{fc} \hat{m}_{fc}}{N m_{fc}} \sum_{i=1}^{M} (\hat{g}_i)^2 \hat{g}_i
\]
\[
B_N = -\frac{m'_{fc}}{N} \sum_{i=1}^{M} (g_i - \mathbb{E} \hat{g}_i) + \frac{m'_{fc} \hat{m}_{fc}}{N m_{fc}} \sum_{i=1}^{M} (g_i^2 - \mathbb{E} [g_i^2]) - \frac{m'_{fc} \hat{m}_{fc}}{m_{fc}} \left( \frac{M}{N} - \gamma_0 \right) \int t dt
\]

Proof. By (119),
\[
\hat{m}_{fc} - m_{fc} = \hat{m}_{fc} m_{fc} \left[ \frac{1}{m_{fc}} - \frac{1}{\hat{m}_{fc}} \right] = -\hat{m}_{fc} m_{fc} \left[ \frac{1}{N} \sum_{i=1}^{M} (\hat{g}_i - \mathbb{E} \hat{g}_i) + \left( \frac{M}{N} - \gamma_0 \right) \int t dt \right]
\]  

Note that
\[
\sum_{i=1}^{M} (\hat{g}_i - \mathbb{E} \hat{g}_i) = \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + \sum_{i=1}^{M} (\hat{g}_i - g_i) = \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + (m_{fc} - \hat{m}_{fc}) \sum_{i=1}^{M} g_i \hat{g}_i
\]
\[
= \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + (m_{fc} - \hat{m}_{fc}) \left( \sum_{i=1}^{M} g_i^2 + (m_{fc} - \hat{m}_{fc}) \sum_{i=1}^{M} \hat{g}_i^2 \right)
\]
\[
= \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + (m_{fc} - \hat{m}_{fc}) \sum_{i=1}^{M} g_i^2 + (m_{fc} - \hat{m}_{fc}) \sum_{i=1}^{M} \hat{g}_i^2
\]
\[
= \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + (m_{fc} - \hat{m}_{fc}) \sum_{i=1}^{M} (g_i^2 - \mathbb{E} [g_i^2]) + (m_{fc} - \hat{m}_{fc})^2 \sum_{i=1}^{M} \hat{g}_i^2 - M(m_{fc} - m_{fc}) \mathbb{E} [g_i^2].
\]  

Plugging (120) into (119), we have
\[
\hat{m}_{fc} - m_{fc} = -\frac{m_{fc} \hat{m}_{fc}}{N} \sum_{i=1}^{M} (g_i - \mathbb{E} g_i) + \frac{m_{fc} \hat{m}_{fc}}{N} (\hat{m}_{fc} - m_{fc}) \sum_{i=1}^{M} (\hat{g}_i)^2 \hat{g}_i
\]
\[
- \frac{m_{fc} \hat{m}_{fc}}{N} (\hat{m}_{fc} - m_{fc})^2 \sum_{i=1}^{M} \hat{g}_i^2 - m_{fc} (\frac{M}{N} - \gamma_0) \int t dt + \frac{m_{fc} m_{fc} (\hat{m}_{fc} - m_{fc}) \mathbb{E} [g_i^2]}{N}
\]  

(121)
In (121), we replace the last term by
\[ m_f c (\hat{m}_{f c} - m_{f c}) \mathbb{E}[g_1^2] \left( \frac{M}{N} (\hat{m}_{f c} - m_{f c}) + m_{f c} \left( \frac{M}{N} - \gamma_0 \right) \right) + (\hat{m}_{f c} - m_{f c}) m_{f c}^2 \gamma_0 \mathbb{E}[g_1^2], \]
then move the term \((\hat{m}_{f c} - m_{f c}) m_{f c}^2 \gamma_0 \mathbb{E}[g_1^2]\) to the left hand side, then use the fact that
\[ 1 - m_{f c}^2 \gamma_0 \mathbb{E}[g_1^2] = 1 - \gamma_0 m_{f c}^2 \int \frac{t^2 d\nu(t)}{(1 + tm_{f c})^2} = \frac{m_{f c}^2}{m'_{f c}} \]
then we obtain (118). Here we used the definition of \(g_i\) in the first identity of (122) and used the self-consistent equation (6) to find the \(m'_{f c}\) for the second identity of (122).

Let
\[ \Omega'_N = \Omega_N \cap \left\{ \left| \frac{1}{N^{\frac{1}{2}} + 0.1c_0} \sum_{i=1}^{N} (g_i(\xi) - \mathbb{E}[g_i(\xi)]) \right| \leq 1, \left| \frac{1}{N^{\frac{1}{2}} + 0.1c_0} \sum_{i=1}^{N} (g_i^2(\xi) - \mathbb{E}[g_i^2(\xi)]) \right| \leq 1 \quad \forall \xi \in \Gamma \right\} \]
where \(c_0 \in (0, 0.01)\) is defined in Definition 1. Then \(\mathbb{P}(\Omega') \to 1.\) On \(\Omega'\) we have for all \(\xi \in \Gamma:\)
\[ |A_N| \leq C_1, \quad |B_N| \leq C_1 N^{-0.5+0.1c_0} \]
where \(C_1 > 0\) is a constant determined by \(d.\) By Lemma 13 on \(\Omega'_N\) we have for \(z \in \Gamma\)
\[ \hat{m}_{f c} - m_{f c} = \frac{1 + x_1}{2A_N} \quad \text{or} \quad \hat{m}_{f c} - m_{f c} = \frac{1 + x_2}{2A_N} \]
where \(x_{1,2}\) are square roots of \(1 - 4A_N B_N\) such that \(x_1\) is close to \(-1\) and \(x_2\) is close to \(1.\) So if \(N > N_0(d)\) and \(\Omega'_N\) holds, then
\[ |1 + x_1| \leq 4C_1^2 N^{-0.5+0.1c_0}, \quad |1 - x_2| \leq 4C_1^2 N^{-0.5+0.1c_0}, \quad \forall z \in \Gamma \]
\[ \left| \frac{1 + x_1}{2A_N} - \frac{1 + x_2}{2A_N} \right| \leq \frac{1}{2C_1}, \forall z \in \Gamma \]
so by continuity, we have
\[ \hat{m}_{f c} - m_{f c} = \frac{1 + x_1}{2A_N}, \forall z \in \Gamma \]
(123)
or
\[ \hat{m}_{f c} - m_{f c} = \frac{1 + x_2}{2A_N}, \forall z \in \Gamma \]
(124)
Suppose \(z_0 \in \Gamma\) and \(\text{Im}z_0 = 2d,\) by Lemma 2 and Theorem 2 we see \(m_{f c}(z_0) - \hat{m}_{f c}(z_0) = m_{f c}(z_0) - m_N(z_0) + m_N(z_0) - \hat{m}_{f c}(z_0)\) converges in probability to 0. Note that if \(N > N_0(d),\) then \(\frac{1 + x_{1,2}}{2A_N(z_0)} \geq \frac{1}{2C_1}\) holds on \(\Omega'_N,\) so (124) cannot happen, so we have (123). Plugging (123) into (118), we see that if \(N > N_0(d),\) \(\Omega'_N\) holds and \(z \in \Gamma\) then
\[ (1 + x_1)(1 - x_1) = 4A_N B_N \]
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that if $\Omega$ has the same limit in distribution as $C$ for some constant $C_2 = C_2(d) > 0$ such that if $\Omega_N$ holds and $N > N_0(d)$ then

$$B_N = -\frac{m'_{f_c}}{N} \sum (g_i - \mathbb{E}g_i) + R_N$$

where

$$|R_N| \leq C_2N^{-\frac{1}{4}-c_0}$$

and therefore by (118), (125) and estimation of $A_N$,

$$\hat{m}_{f_c} - m_{f_c} = -\frac{m'_{f_c}}{N} \sum (g_i - \mathbb{E}g_i) + W_N$$

where

$$|W_N| \leq 4C_1^N N^{-1+0.2c_0} + C_2N^{-\frac{1}{4}-c_0}.$$ 

So, for $N > N_0(d)$,

$$\left|1_{\Omega_N} \sqrt{\frac{N}{2\pi}} \int_{\Gamma} (m_{f_c}(\xi) - \hat{m}_{f_c}(\xi))f(\xi)d\xi - \left[ \frac{1}{2\pi i} \int_{\Gamma} m'_{f_c}(\xi) g_i(\xi) f(\xi)d\xi \right] \right| \leq C_3N^{-c_0} \sup_{\xi \in \Gamma} |f(\xi)|$$

for some constant $C_3$. Since both $\mathbb{P}(\Omega'_{N})$ and $\mathbb{P}(\Omega_{N})$ go to 1, we see that

$$\left|1_{\Omega_N} \sqrt{\frac{N}{2\pi}} \int_{\Gamma} (m_{f_c}(\xi) - \hat{m}_{f_c}(\xi))f(\xi)d\xi \right|$$

has the same limit in distribution as

$$\frac{\sqrt{N}}{2\pi i} \int_{\Gamma} m'_{f_c}(\xi) N \sum (g_i - \mathbb{E}g_i) f(\xi)d\xi = \frac{1}{\sqrt{N}} \sum_{i=1}^{M} \left\{ \frac{1}{2\pi i} \int_{\Gamma} m'_{f_c}(\xi) g_i(\xi) f(\xi)d\xi - \mathbb{E} \left[ \frac{1}{2\pi i} \int_{\Gamma} m'_{f_c}(\xi) g_i(\xi) f(\xi)d\xi \right] \right\}$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{M} \left\{ \frac{1}{2\pi i} \int_{\Gamma} \frac{\sigma_i m'_{f_c}(\xi)}{1 + \sigma_i m_{f_c}(\xi)} f(\xi)d\xi - \mathbb{E} \left[ \frac{1}{2\pi i} \int_{\Gamma} \frac{\sigma_i m'_{f_c}(\xi)}{1 + \sigma_i m_{f_c}(\xi)} f(\xi)d\xi \right] \right\}$$

which, by central limit theorem, converges in distribution to a centered Gaussian distribution with variance

$$\mathbb{E} \left[ \left( \frac{1}{2\pi i} \int_{\Gamma} \frac{\sigma_i m'_{f_c}(\xi)}{1 + \sigma_i m_{f_c}(\xi)} f(\xi)d\xi \right)^2 \right] - \left( \mathbb{E} \left[ \frac{1}{2\pi i} \int_{\Gamma} \frac{\sigma_i m'_{f_c}(\xi)}{1 + \sigma_i m_{f_c}(\xi)} f(\xi)d\xi \right] \right)^2$$

$$= -\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} \int_{\Gamma} (1 + tm_{f_c}(\xi))(1 + tm_{f_c}(\xi)) f(\xi) f(\xi_2) dv(u)d\xi_1d\xi_2 + \frac{1}{4\pi^2\gamma_0^2} \left( \int_{\Gamma} f(\xi) m'_{f_c}(\xi)(\xi + \frac{1}{m_{f_c}(\xi)})d\xi \right)^2$$

where we used the fact $\int_{\Gamma} \frac{dv(t)}{1 + tm_{f_c}(\xi)} = \gamma_0^{-1}(\xi + \frac{1}{m_{f_c}(\xi)})$ in the last identity. So the proof of the main theorem is complete.
8 Proof of Proposition 1

Definition 10. Define $D = (-\infty, 0) \cup (0, 1) \cup \left(\frac{1}{2}, \infty\right)$ and

$$g(x) = \frac{1}{x} + \gamma_0 \int_1^{x \gamma_0} \frac{t}{1 - xt} \, dt$$

and $x \in D$

$$h_0(x) := \int_1^x \left(\frac{xt}{1 - xt}\right)^2 \, dt$$

and $x \in D$

$$h_1(x) := \frac{1}{M} \sum_{i=1}^M \left(\frac{x \sigma_i}{1 - x \sigma_i}\right)^2$$

$x \in \mathbb{R}\{0, \sigma_1^{-1}, \ldots, \sigma_M^{-1}\}$.

According to Remark 2.2 and the first paragraph of Appendix A of [9], the support of $\rho_{fc}$ must be the union of a few finite intervals:

$$\text{supp}(\rho_{fc}) = [a_1, b_1] \cup \cdots \cup [a_k, b_k].$$

The following lemma is Proposition 2.3 of [9].

Lemma 15. A number $c$ satisfies $c = b_i$ for some $i \in \{1, \ldots, k\}$ (i.e., $c$ is the right endpoint of a connected component of $\text{supp}(\rho_{fc})$) if and only if one of the following two conditions is satisfied:

1. there exists $x \in D$ such that $c = g(x)$, $g'(x) = 0$ and $g''(x) > 0$

2. there exists $x \in \partial D$ and $\epsilon > 0$ such that: i) $(x - \epsilon, x) \subset D$, ii) $g(t)$ is decreasing on $(x - \epsilon, x)$ and iii) $c = \lim_{t \to x^+} g(t)$

A number $c'$ satisfies $c' = a_i$ for some $i \in \{1, \ldots, k\}$ (i.e., $c'$ is the left endpoint of a connected component of $\text{supp}(\rho_{fc})$) if and only if one of the following two conditions is satisfied:

3. there exists $x \in D$ such that $c' = g(x)$, $g'(x) = 0$ and $g''(x) < 0$

4. there exists $x \in \partial D$ and $\epsilon > 0$ such that: i) $(x, x + \epsilon) \subset D$, ii) $g(t)$ is decreasing on $(x, x + \epsilon)$ and iii) $c' = \lim_{t \to x^-} g(t)$

Remark 3. Proposition 2.3 of [9] requires the edges in this lemma to be “soft” (i.e., the density function $\rho_{fc}(t)$ goes to a finite value as $t$ approaches this edge). But this condition is automatically satisfied by our model because $\gamma_0 \neq 0$. It is known that only $a_1$ may fail to be soft and this happen only when $\gamma_0 = 1$.

Corollary 2. In our model, $[L_-, L_+]$ satisfy the following conditions.

- $L_+ = 1 + \gamma_0 \int \frac{1}{1 - x t} \, dt$ or $L_+ = g(x)$ for some $x \in (0, 1)$ satisfying $h_0(x) = \frac{1}{\gamma_0}$.

- $L_- = 1 - \gamma_0 \int \frac{1}{1 - x t} \, dt$ or $L_- = g(x)$ for some $x \in (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$ satisfying $h_0(x) = \frac{1}{\gamma_0}$.

Remark 4. Suppose $L_- = g(x)$ for some $x \in (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$. If $\gamma_0 > 1$, then $x \in (-\infty, 0)$. This is because the condition $g'(x) = 0$ implies $\int (\frac{1}{1 - tx})^2 \, dt = \gamma_0^{-1} < 1$, so $x$ cannot be in $(l^{-1}, \infty)$, otherwise the integrand in $\int (\frac{1}{1 - tx})^2 \, dt$ is larger than 1. Similarly, if $\gamma_0 \in (0, 1)$, then $x \in (l^{-1}, \infty)$.
Proof. Notice that \( \partial D = \{0, 1, \frac{1}{l}\} \). For \( L_+ \), we see that 1 is the only point in \( \partial D \) which may satisfy Condition 2 in Lemma 15 with a finite \( c \). So \( L_+ \), as a right endpoint of a connected component of \( \text{supp}(\rho_{f,c}) \), may be \( \lim_{t \to 1^-} g(t) = 1 + \gamma_0 \int 1_{t < x} \nu(t) \). On the other hand, if \( x \in D \) satisfies Condition 1 in Lemma 15 then we claim that \( x \) must be in \((0, 1)\). We prove this claim by contradiction.

- If \( x \in (-\infty, 0) \), by the condition \( g''(x) > 0 \),

  \[
  \int_1^1 \left( \frac{-tx}{1 - xt} \right)^3 \nu(t) > \frac{1}{\gamma_0}.
  \] (126)

  According to the condition \( g'(x) = 0 \), we have \( \frac{1}{\gamma_0} = \int_1^1 \left( \frac{tx}{1 - xt} \right)^2 \nu(t) \). Plug this into (126), we have

  \[
  \int_1^1 \left( \frac{tx}{1 - xt} \right)^3 \left[ \frac{-tx}{1 - xt} - 1 \right] \nu(t) > 0.
  \] (127)

  But (127) cannot hold since the integrand in it is negative because \( x \leq 0 \).

- If \( x \in (\frac{1}{l}, \infty) \), then similarly,

  \[
  \int_1^1 \left( \frac{tx}{xt - 1} \right)^3 \nu(t) < \frac{1}{\gamma_0} = \int_1^1 \left( \frac{tx}{xt - 1} \right)^2 \nu(t).
  \] (128)

  So we have

  \[
  \int_1^1 \left( \frac{tx}{xt - 1} \right)^2 \left[ \frac{tx}{xt - 1} - 1 \right] \nu(t) < 0.
  \] (129)

  But (129) cannot hold since the integrand in it is positive because \( x_0 > 1/l \).

So the claim is true and we proved the first conclusion.

For \( L_+ \), we see that \( \frac{1}{l} \) is the only point in \( \partial D \) which may satisfy Condition 4 in Lemma 15 with a finite \( c \). So \( L_+ \), as a left endpoint of a connected component of \( \text{supp}(\rho_{f,c}) \), may be \( \lim_{t \to l^-} g(t) = 1 - \gamma_0 \int 1_{x < l} \nu(t) \). On the other hand, if \( x \in D \) satisfies Condition 3 in Lemma 15 then we claim that \( x \) must be in \((\infty, 0) \cup (l, \infty)\). We prove this claim by contradiction. Suppose \( x \in (0, 1) \) satisfies Condition 3 in Lemma 15. Then by \( g''(x) < 0 \) we have

  \[
  \int_1^1 \left( \frac{tx}{xt - 1} \right)^3 \nu(t) > \frac{1}{\gamma_0}.
  \]

  But \( \frac{1}{\gamma_0} \) equals \( \int_1^1 \left( \frac{tx}{xt - 1} \right)^2 \nu(t) \), so

  \[
  \int_1^1 \left( \frac{tx}{xt - 1} \right)^2 \left[ \frac{tx}{xt - 1} - 1 \right] \nu(t) > 0
  \] (130)

  which cannot be true because \( x < 1 \). So the claim that \( x \in (\infty, 0) \cup (l, \infty) \) is true and we proved the second conclusion. \(\square\)
Lemma 16. If

\[ L_+ = g(x_0) \]  \hspace{1cm} (131)

for some \( x_0 \in (0, 1) \) with \( h_0(x_0) = \frac{1}{\gamma_0} \), then

\[ \mathbb{P}(|L_+ - \hat{L}_+| < \epsilon) \to 1 \quad \forall \epsilon > 0. \]

If

\[ L_- = g(x'_0) \]  \hspace{1cm} (132)

for some \( x'_0 \in (-\infty, 0) \cup (l^{-1}, \infty) \) with \( h_0(x'_0) = \frac{1}{\gamma_0} \), then

\[ \mathbb{P}(|L_- - \hat{L}_-| < \epsilon) \to 1 \quad \forall \epsilon > 0. \]

Proof. By Lemma 2.4–2.6 of [12],

\[ \hat{L}_+ = \frac{1}{x_1} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 - x_1 \sigma_i} \]  \hspace{1cm} (133)

where \( x_1 \in (0, \frac{1}{\max \sigma_i}) \) with \( h_1(x_1) = \frac{N}{M} \). Note that \( x_0 \) is deterministic and is independent of \( N \), but \( x_1 \) is random and depends on \( N \).

Suppose \( c > 0 \) is a small enough constant such that \([x_0 - c, x_0 + c] \subset (0, 1)\). We claim that

\[ \mathbb{P}(x_1 \in [x_0 - c, x_0 + c]) \to 1. \]  \hspace{1cm} (134)

In fact, since \( h_0 \) is increasing on \([0, 1]\), there exists \( a > 0 \) such that

\[ h_0(x_0 - c) + a < h_0(x_0) < h_0(x_0 + c) - a. \]

Note that both \( \sqrt{M}(h_0(x_0 - c) - h_1(x_0 - c)) \) and \( \sqrt{M}(h_0(x_0 + c) - h_1(x_0 + c)) \) converge in distribution to Gaussian distributions, so

\[ \mathbb{P}\left(|h_0(x_0 \pm c) - h_1(x_0 \pm c)| < a\right) \to 1. \]  \hspace{1cm} (135)

When \( |h_0(x_0 \pm c) - h_1(x_0 \pm c)| < a \) holds, we have

\[ h_1(x_0 - c) < h_0(x_0 - c) + a < h_0(x_0) < h_0(x_0 + c) - a < h_1(x_0 + c) \]

Notice \( h_1(x_1) - h_0(x_0) \to 0 \) since \( \left|\frac{1}{x_0} - \frac{N}{M}\right| \to 0 \). So if \( N \) is large enough and \( |h_0(x_0 \pm c) - h_1(x_0 \pm c)| < a \) holds,

\[ h_1(x_0 - c) < h_1(x_1) < h_1(x_0 + c) \]

thus by monotonicity of \( h_1 \):

\[ x_0 - c \leq x_1 \leq x_0 + c \quad \text{whenever } |h_0(x_0 \pm c) - h_1(x_0 \pm c)| < a \]  \hspace{1cm} (136)

(135) and (136) prove (134).
Now suppose $\epsilon > 0$. Choose $\delta$ small enough such that $[x_0 - \delta, x_0 + \delta] \in (0, 1)$ and
\[
\left| \int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t) - \int_{t}^{1} \frac{t}{1 - (x_0 \pm \delta) t} d\nu(t) \right| < \epsilon
\] (137)
and
\[
|\frac{1}{x_0} - \frac{1}{x_0 \pm \delta}| < \epsilon.
\]

Since $\frac{1}{\sqrt{M}} \sum_{i=1}^{M} \frac{\sigma_i}{1 - (x_0 \pm \delta) \sigma_i} - \sqrt{M} \int_{t}^{1} \frac{t}{1 - (x_0 \pm \delta) t} d\nu(t)$ converges in distribution to a normal distribution, we have
\[
\mathbb{P}\left(\frac{1}{M} \sum_{i=1}^{M} \frac{\sigma_i}{1 - (x_0 \pm \delta) \sigma_i} - \int_{t}^{1} \frac{t}{1 - (x_0 \pm \delta) t} d\nu(t) \right| < \epsilon \right) \to 1. \quad (138)
\]

By (131) and (133), the event
\[
\{|x_0 - x_1| < \delta\} \cap \left\{ \left| \frac{1}{M} \sum_{i=1}^{M} \frac{\sigma_i}{1 - (x_0 \pm \delta) \sigma_i} - \int_{t}^{1} \frac{t}{1 - (x_0 \pm \delta) t} d\nu(t) \right| < \epsilon \right\}
\] (139)
has a probability going to 1. On (139), we have by (137):
\[
\int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t) - 2\epsilon < \frac{1}{M} \sum_{i=1}^{M} \frac{\sigma_i}{1 - x_1 \sigma_i} < \int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t) + 2\epsilon
\]
and then for large enough $N$
\[
|L_+ - \hat{L}_+| \leq \left| \frac{1}{x_1} - \frac{1}{x_0} \right| + \left| \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 - x_1 \sigma_i} - \int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t) \right| \quad \text{(by (131) and (133))}
\]
\[
\leq \left| \frac{1}{x_0 + \delta} - \frac{1}{x_0} \right| + \left| \frac{1}{x_0 - \delta} \right| + \left| \frac{1}{M} \sum_{i=1}^{M} \frac{\sigma_i}{1 - x_0 \sigma_i} - \int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t) \right| + \left| \frac{M}{N} - \gamma_0 \right| \int_{t}^{1} \frac{t}{1 - x_0 t} d\nu(t)
\]
\[
\leq \epsilon + \frac{M}{N} \cdot 2\epsilon + \epsilon \quad \text{(since $\left| \frac{M}{N} - \gamma_0 \right|$ goes to 0)}
\]

Recall $\mathbb{P}(139) \to 1$, so we have proved
\[
\mathbb{P}(|L_+ - \hat{L}_+| \leq 2\epsilon(1 + \frac{M}{N})) \to 1.
\]

Since $\frac{M}{N}$ is bounded and $\epsilon$ can be arbitrarily small, the first conclusion of Lemma 10 is proved.

Now we estimate $|L_- - \hat{L}_-|$. Suppose (132) holds for some $x_0' \in (-\infty, 0) \cup (l^{-1}, \infty)$ with $h_0(x_0') = \frac{1}{\gamma_0}$. By Remark 3, $x_0' \in (-\infty, 0)$ if $\gamma_0 > 1$; $x_0' \in (l^{-1}, \infty)$ if $\gamma_0 \in (0, 1)$.

By Lemma 2.4–2.6 of [12] we have the following. $\hat{L}_- = \frac{1}{x_i} + \frac{1}{M} \sum_{i=1}^{M} \frac{\sigma_i}{1 - x_i \sigma_i}$ where $x_i$ is a point in $(-\infty, 0) \cup (\frac{1}{\min \sigma_i}, \infty)$ such that $h_1(x_i') = \frac{N}{M}$. Similarly as what we explained in Remark 3, if $\gamma_0 > 1$, then $x_i' \in (-\infty, 0)$; if $\gamma_0 \in (0, 1)$, then $x_i' \in (\frac{1}{\min \sigma_i}, \infty)$.  

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Then using similar argument as above we can prove (134) with $x_0$ and $x_1$ replaced by $x'_0$ and $x'_1$ respectively. Then similarly as above we can prove
\[ P(|L_- - \hat{L}_-| \leq 2\epsilon(1 + \frac{M}{N})) \to 1. \]

Since $\frac{M}{N}$ is bounded and $\epsilon$ can be arbitrarily small, the second conclusion of Lemma 16 is proved. So we have proved Lemma 16. \hfill \Box

**Proof of Proposition 7.** Suppose $m_{fc}(L_+) \neq -1$. If $L_+ = 1 + \gamma_0 \int \frac{t}{1-t} dv(t)$, then by Lemma 15
\[ m_{fc}(L_+) = \lim_{t \uparrow 1} m_{fc}\left( \frac{1}{t} + \gamma_0 \int \frac{s}{1-ts} dv(s) \right). \]

By Proposition 2.1 of [9],
\[ \lim_{t \uparrow 1} m_{fc}(\frac{1}{t} + \gamma_0 \int \frac{s}{1-ts} dv(s)) = -\lim_{t \uparrow 1} t = -1 \]
which is contradictory to the assumption $m_{fc}(L_+) \neq -1$. Notice that the Stieltjes transform in [9] differs by a sign from the Stieltjes in this paper. So we proved $L_+ \neq 1 + \gamma_0 \int \frac{t}{1-t} dv(t)$. By Corollary 2, $L_+ = g(x)$ for some $x \in (0, 1)$ satisfying $h_0(x) = \frac{1}{\gamma_0}$. Then using Lemma 15 we obtain (13) under the assumption $m_{fc}(L_+) \neq -1$.

Now we prove (13) under the assumption that $\gamma_0 \neq (\int (\frac{t}{1-t})^2 dv(t))^{-1}$. By Corollary 2 and Lemma 16 without loss of generality, we can suppose
\[ L_+ = 1 + \gamma_0 \int \frac{t}{1-t} dv(t) \quad (140) \]

By the second condition in Lemma 15, $g(t)$ is decreasing when $t$ is close enough but smaller than 1. So
\[ \lim_{t \uparrow 1} \left( -\frac{1}{x^2} + \gamma_0 \int (\frac{t}{1-xt})^2 dv(t) \right) = \lim_{t \uparrow 1} g'(t) \leq 0 \]
which implies
\[ \int (\frac{t}{1-t})^2 dv(t) \leq \frac{1}{\gamma_0}. \]
This together with the assumption that $\gamma_0 \neq (\int (\frac{t}{1-t})^2 dv(t))^{-1}$ yields:
\[ \int (\frac{t}{1-t})^2 dv(t) < \frac{1}{\gamma_0}. \]

As we saw in (133),
\[ \hat{L}_+ = \frac{1}{x_1} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1-x_1\sigma_i} \quad (141) \]
where $x_1 \in (0, \frac{1}{\max \sigma_i})$ satisfying $h_1(x_1) = \frac{N}{M}$. 38
Since \(\frac{1}{\sqrt{M}} \sum (\frac{\sigma_i}{1-\sigma_i})^2 - \sqrt{M} f(\frac{1}{1-t})^2 dv(t)\) converges in distribution to a Gaussian distribution, we have

\[ P(Q_N) \rightarrow 1 \quad (142) \]

where

\[ Q_N = \left\{ \left| \frac{1}{M} \sum (\frac{\sigma_i}{1-\sigma_i}) - \int \frac{t}{1-t} dv(t) \right| \leq \frac{1}{\gamma_0} - \int \frac{t}{1-t} \right\} \]

When \(N\) is large enough and \(Q_N\) holds, we must have \(x_1 > 1\), because otherwise

\[ h_1(x_1) \leq h_1(1) \leq \int \frac{t}{1-t} dv(t) + \frac{1}{2} \left( \frac{1}{\gamma_0} - \int \frac{t}{1-t} dv(t) \right) = \frac{1}{2} \left( \frac{1}{\gamma_0} + \int \frac{t}{1-t} dv(t) \right) < \frac{1}{\gamma_0} \]

which is contradictory to \(h_1(x_1) = \frac{N}{M} \rightarrow \frac{1}{\gamma_0}\). Moreover, \(x_1 > 1\) implies \(x_1 \in (1, \frac{1}{\max \sigma_i})\) and then

\[
\left| \frac{1}{M} \sum \frac{\sigma_i}{1-\sigma_i} - \frac{1}{N} \sum \frac{\sigma_i}{1-\sigma_i} \right| \leq |x_1 - 1| \frac{1}{M} \sum \frac{\sigma_i^2}{(1-\sigma_i)(1-x_1 \sigma_i)} \\
\leq |1-x_1| \frac{1}{M} \sum \frac{\sigma_i^2}{(1-x_1 \sigma_i)^2} = |1-x_1| \cdot \frac{N}{Mx_1^2} \quad \text{(by definition of } x_1) \\
\leq \frac{1-\max \sigma_i}{\max \sigma_i} \cdot \frac{N}{M} \quad (143)
\]

Suppose \(\epsilon > 0\). Since \(\text{supp } \nu = [l, 1]\), we have

\[ P(\max \sigma_i \in (1-\epsilon, 1)) \rightarrow 1. \]

Similarly as \((142)\) we have

\[ P(Q_N') \rightarrow 1 \]

where

\[ Q_N' = \left\{ \left| \frac{1}{M} \sum \frac{\sigma_i}{1-\sigma_i} - \int \frac{t}{1-t} \right| \leq \epsilon \right\} \]

When \(N\) is large enough and \(Q_N \cap Q_N' \cap \{\max \sigma_i \in (1-\epsilon, 1)\}\) holds, we have by \((140), (141)\) and the fact \(x_1 \in (1, \frac{1}{\max \sigma_i}) \subset (1, \frac{1}{\gamma_0})\) that

\[
|\hat{L}_+ - L_+| \leq |1 - \frac{1}{x_1}| + |\gamma_0| \int \frac{t}{1-t} dv(t) - \frac{1}{N} \sum \frac{\sigma_i}{1-\sigma_i} \leq \epsilon + |\gamma_0| \left( \frac{M}{N} \int \frac{t}{1-t} dv(t) \right) \\
+ \frac{M}{N} \int \frac{t}{1-t} dv(t) - \frac{1}{M} \sum \frac{\sigma_i}{1-\sigma_i} \leq \epsilon + \frac{\gamma_0}{1-\epsilon} - \frac{1}{M} \sum \frac{\sigma_i}{1-\sigma_i} - \frac{1}{M} \sum \frac{\sigma_i}{1-x_1 \sigma_i} \\
\leq 2\epsilon + N^{-\frac{1}{2}-c_0} \int \frac{t}{1-t} dv(t) + \frac{M}{N} \epsilon \quad (144)
\]

where we used \((11)\) and \((123)\) in the last inequality. Note that the RHS of \((144)\) goes to \(\frac{2\epsilon}{1-\epsilon} + \gamma_0\epsilon\) and the probability of \(Q_N \cap Q_N' \cap \{\max \sigma_i \in (1-\epsilon, 1)\}\) goes to 1. So

\[ P(\hat{L}_+ - L_+ \leq 2\epsilon) \rightarrow \frac{2\epsilon}{1-\epsilon} + \gamma_0\epsilon \rightarrow 1. \]
Since $\epsilon$ can be arbitrarily small, we have for any (small) $\epsilon' > 0$,

$$P(|\hat{L}_+ - L_+| < \epsilon') \to 1$$  \hspace{1cm} (145)

and (144) is proved under the assumption that $\gamma_0 \neq \left( \int \left( \frac{1}{t} \right)^2 dt \right)^{-1}$.

So the first conclusion of Proposition 1 is proved. The second conclusion can be proved similarly. \hfill \square

A Proof of some auxiliary lemmas

*Proof of Lemma 3.*

1. $\hat{L}_+ < C_0$ is from Lemma 2.5 of [12]. By Lemma 2.4 and Lemma 2.5 of [12],

$$\hat{L}_- = f(x)$$

where

$$f(t) = -\frac{1}{t} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 + t \sigma_i}$$

and $x$ is the unique critical point of $f$ on $(-\infty, -(\min \sigma_i)^{-1}) \cup (0, \infty)$. By definition,

$$\frac{N}{M} = \frac{1}{M} \sum_{i=1}^{M} \left( \frac{x \sigma_i}{1 + x \sigma_i} \right)^2$$  \hspace{1cm} (146)

so,

$$\hat{L}_- = f(x) = -\frac{1}{x} + \frac{1}{M} \sum_{i=1}^{M} \frac{x \sigma_i}{1 + x \sigma_i} = \frac{\sum_{i=1}^{M} \frac{\sigma_i}{1 + x \sigma_i}}{\sum_{i=1}^{M} \frac{x^2 \sigma_i^2}{1 + x \sigma_i}}$$

So it is enough to show $|x|$ is bounded above by a constant when $N > N_0$. But this is directly from (146) because $\frac{N}{M} \to \gamma_0^{-1} \neq 1$.

2. Suppose $\lambda'_1 \geq \cdots \geq \lambda'_N \geq 0$ are eigenvalues of $XX^T$. By Theorem 2.10 of [6], there exists a constant $C_1 > 1$ such that

$$P(\lambda'_1 < C_1 \text{ and } \lambda'_{M \wedge N} > C_1^{-1}) > 1 - N^{-D}$$  \hspace{1cm} (147)

when $N$ is large enough. This together with the facts that $\lambda_1 = \| \Sigma^{1/2} XX^T \Sigma^{1/2} \| \leq \| XX^T \| = \lambda'_1$ yield:

$$P(\lambda_1 \leq C_0) > 1 - N^{-D}$$  \hspace{1cm} (148)

for $N > N_0(D)$.

To estimate $\lambda_{M \wedge N}$, we do singular value decomposition:

$$(O_1 X O_2)_{ij} = \delta_{ij} \sqrt{\lambda'_{ii}}$$
where $O_1$ and $O_2$ are orthogonal matrices. Recall that if $W$ is a real symmetric matrix with nonnegative eigenvalues, then its smallest eigenvalues equals

$$
\min_{||v||=1} \langle v, Wv^T \rangle.
$$

If $M \leq N$, then $\lambda_{M \wedge N} = \lambda_M$ is the smallest eigenvalue of $\Sigma^{1/2}X^T\Sigma^{1/2}$ and

$$
\lambda_{M \wedge N} = \min_{||v||=1} \langle v, \Sigma^{1/2}X^T\Sigma^{1/2}v^T \rangle = \min_{||v||=1} \langle v, O_1^T \text{Diag}(\lambda'_1, \ldots, \lambda'_M)O_1 \Sigma^{1/2}v^T \rangle
$$

$$
= \min_{||u||=1} \langle u, \text{Diag}(\lambda'_1, \ldots, \lambda'_M)u^T \rangle \quad \text{(where } u = v\Sigma^{1/2}O_1^T \text{)}
$$

$$
\geq \min_{||v||=1} ||u|| \cdot \lambda'_M \geq \sqrt{\lambda} \cdot \lambda'_M = \sqrt{\lambda} \cdot \lambda'_{M \wedge N}. \quad (149)
$$

If $M > N$, then $\lambda_{M \wedge N} = \lambda_N$ is the smallest eigenvalue of $X^T\Sigma X$ and

$$
\lambda_{M \wedge N} = \min_{||v||=1} \langle v, X^T\Sigma Xv^T \rangle \geq \lambda_N \cdot \min_{||v||=1} \langle v, X^T Xv^T \rangle = \lambda_N \cdot \lambda'_{M \wedge N} \quad (150)
$$

By (147), (149) and (150),

$$
P(\lambda_{M \wedge N} \geq C_0^{-1}) > 1 - N^{-D}
$$

for $N > N_0(D)$. This together with (148) complete the proof of (21).

\[\Box\]

**Proof of Lemma 5**  
1. (30) is from the definitions of $G_M$ and $G_N$. (31) is induced from (23) and (30). (32) can be proved similarly.

2. (33) is the Ward’s identity (see (3.6) of [5]). Now we use the argument on Page 283 of [12] to prove (33). For simplicity we let $S = \emptyset$. When $S \neq \emptyset$ the proof is the same. Consider the singular value decomposition:

$$
\Sigma X = U\Lambda V^T \quad (151)
$$

where $U$ is an $M \times M$ orthogonal matrix, $V$ is an $N \times N$ orthogonal matrix and $\Lambda$ is an $M \times N$ matrix with $A_{rs} = \delta_{rs}\eta_r\eta_s$ where $\eta_1 \geq \cdots \geq \eta_N \geq 0$ are eigenvalues of $X^T\Sigma^2 X$. By (23),

$$
\sum_{j=1}^{M} |G_{ij}|^2 = \sum_{j=1}^{M} |(\Sigma X G_N X^T \Sigma)_{ij} - \Sigma_{ij}|^2 \leq 2 \sum_{j=1}^{M} |(\Sigma X G_N X^T \Sigma)_{ij}|^2 + 2 \sum_{j=1}^{M} (\Sigma_{ij})^2
$$

$$
\leq 2(\Sigma X G_N X^T \Sigma^2 X G_N X^T \Sigma)_{ii} + 2(\Sigma^2)_{ii} \quad \text{(note: } G_N \text{ is symmetric)}
$$

$$
= 2(\Sigma X G_N V^T \text{Diag}(\eta_1, \ldots, \eta_N)V G_N X^T \Sigma)_{ii} + 2(\Sigma^2)_{ii} \quad \text{(by (151))}
$$

$$
= 2 \sum_{k=1}^{N} \eta_k |(\Sigma X G_N V^T)_{ik}|^2 + 2(\Sigma^2)_{ii} \leq 2 \eta_i \sum_{k=1}^{N} |(\Sigma X G_N)_{ik}|^2 + 2(\Sigma^2)_{ii}
$$

$$
= 2 \|X^T \Sigma^2 X \| \cdot (\Sigma X G_N \bar{G}_N X^T \Sigma)_{ii} + 2(\Sigma^2)_{ii} \quad (152)
$$

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where \( \| \cdot \| \) denotes the operator norm. Since there exists orthogonal matrix \( O \) such that 
\[ G_N = O \text{Diag}(\frac{1}{\lambda_1 - z}, \ldots, \frac{1}{\lambda_N - z})O^T, \]
we have 
\[
(\Sigma XG_N \bar{G}_N X^T \Sigma)_{ii} = \sum_{k=1}^{N} \frac{1}{|\lambda_k - z|^2} ((\Sigma XO)_{ik})^2 = \frac{1}{\text{Im}z} \sum_{k=1}^{N} \text{Im} \left( \frac{1}{\lambda_k - z} \right) ((\Sigma XO)_{ik})^2 
= \frac{\text{Im}(\Sigma XG_N X^T \Sigma)_{ii}}{\text{Im}z} = \frac{\text{Im}(G_M)_{ii}}{\text{Im}z} \quad \text{(by (23))} \quad (153)
\]
Plug (153) and the fact 
\[ \|X^T \Sigma^2 X\| = \|\Sigma X X^T \Sigma\| \leq \|X X^T\| \|\Sigma\|^2 \leq \|X X^T\| \] into (152), then we have
\[
\sum_{j=1}^{M} |G_{ij}|^2 \leq 2 \frac{\|X X^T\| \|\Sigma\|^2}{\text{Im}z} (\text{Im}(G_M)_{ii}) + 2 \quad (154)
\]
Use singular value decomposition for \( X \):
\[ X = O_1 R O_2 \]
where \( O_1 \) and \( O_2 \) are orthogonal matrices, \( R_{ij} = \delta_{ij} \sqrt{\mu_i} \) and \( \mu_1 \geq \cdots \geq \mu_N \geq 0 \) are eigenvalues of \( X^T X \). So
\[ z^{-1} X X^T - \Sigma^{-1} = O_1 \text{Diag}(z^{-1} \mu_i - \sigma_i^{-1}) O_1^T. \]
Then, by the definition of \( G_M \), we have
\[ |(G_M)_{ii}| = \left| \sum_{k=1}^{M} \frac{z \sigma_k}{\mu_k \sigma_k - z} ((O_1)_{ik})^2 \right| \leq \frac{|z|}{\text{Im}z} \sum_{k=1}^{M} ((O_1)_{ik})^2 = \frac{|z|}{\text{Im}z}. \]
This together with (154) complete the proof of (34).

3. (35) is a direct corollary of Theorem 2.10 in [6].

4. The first identity in (36) is from (25) and (28). The second identity in (36) is from (24) and (28). (37) is from (24), (25) and the definitions of \( Z_i \) and \( Z_p \).

5. (38) is (A.13) of [12]. The last identity in (39) can be shown by taking derivative on each term of 
\[ \frac{1}{\hat{m}_{fc}} = -z + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 + \sigma_i \hat{m}_{fc}} \]
which comes from (6).

6. The first identity of (40) is from (6). Now we prove the second identity of (40).
For any \( p \in \{M + 1, \ldots, M + N\} \),
\[
\frac{1}{G_{pp}} = \frac{1}{m_N} \left[ 1 + \left( \frac{m_N}{G_{pp}} - 1 \right) \right] = \frac{1}{m_N} + \frac{1}{m_N} \frac{m_N - G_{pp}}{G_{pp}} = \frac{1}{m_N} + \frac{m_N - G_{pp}}{m_N^2} = \frac{(m_N - G_{pp})^2}{m_N} + \frac{m_N - G_{pp}}{m_N^2} \]
which (together with $Nm_N = \sum_{p=M+1}^{M+N} G_{pp}$) implies:

$$\frac{1}{N} \sum_{p=M+1}^{M+N} \frac{1}{G_{pp}} = \frac{1}{m_N} + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(m_N - G_{pp})^2}{m_N^2 G_{pp}}$$  (155)

By (39),

$$\frac{1}{N} \sum_{i=1}^{M} \left( G_{ii} + \frac{\sigma_i}{1 + m_N \sigma_i} \right) = \frac{1}{N} \sum_{i=1}^{M} \left( \frac{-\sigma_i}{1 + \sigma_i m_N + \sigma_i Z_i - \sigma_i A_i} + \frac{\sigma_i}{1 + m_N \sigma_i} \right)$$

$$= \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2(Z_i - A_i)}{(1 + \sigma_i m_N + \sigma_i Z_i - \sigma_i A_i)(1 + m_N \sigma_i)}$$

$$= \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2 Z_i}{(1 + \sigma_i m_N)(1 + \sigma_i m_N + \sigma_i Z_i - \sigma_i A_i)} - \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2 A_i}{(1 + \sigma_i m_N)(1 + \sigma_i m_N + \sigma_i Z_i - \sigma_i A_i)} = \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2 Z_i}{(1 + \sigma_i m_N)^2} + B_N(z).$$  (156)

By (39), (155), (156) and the definition of $h_i$,

$$h(m_N) - z = -\frac{1}{N} \sum_{p=M+1}^{M+N} \frac{1}{G_{pp}} + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(m_N - G_{pp})^2}{m_N^2 G_{pp}} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 + m_N \sigma_i} - z$$

$$= \frac{1}{N} \sum_{i=1}^{M} G_{ii} - \frac{1}{N} \sum_{p=M+1}^{M+N} A_p + \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{m_N - G_{pp}}{m_N^2 G_{pp}} + \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i}{1 + m_N \sigma_i}$$

$$= \frac{1}{N} \sum_{i=1}^{M} \left( G_{ii} + \frac{\sigma_i}{1 + m_N \sigma_i} \right) - \frac{1}{N} \sum_{p=M+1}^{M+N} A_p + \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(m_N - G_{pp})^2}{m_N^2 G_{pp}}$$

$$= \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2 Z_i}{(1 + \sigma_i m_N)^2} + B_N(z) - \frac{1}{N} \sum_{p=M+1}^{M+N} A_p + \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p + \frac{1}{N} \sum_{p=M+1}^{M+N} \frac{(m_N - G_{pp})^2}{m_N^2 G_{pp}}$$

So we proved the second identity of (39).  

B **A binary tree argument: proof of Lemma 11**

**Proof.** This proof follows the steps in the proof of Proposition 6.1 in [5] with some slight modification. Suppose $k \in \{1, 2, \ldots\}$ and $z$ is in

$$\left\{ x + iy \mid \tau \leq x \leq \tau^{-1}, N^{-1+\tau} \leq y \leq \tau^{-1} \right\}.$$  (157)

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According to (37),

\[
\mathbb{E} \left[ \mathbf{1}_{z \in D \cup D'} \cdot \left( \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p \right)^{2k} \right] = \sum_{B \in \mathcal{B}_k} \frac{1}{N^{2k}} \sum_{p} \mathbb{E}[\mathbf{1}_{z \in D \cup D'} \cdot V(p)] \cdot \mathbb{1}(g_p = B) \tag{158}
\]

where

- \( \mathcal{B}_k \) is the set of partitions of \( \{1, \ldots, 2k\} \);
- \( p = (p_1, \ldots, p_{2k}) \) runs over \( \{M + 1, \ldots, M + N\}^{2k} \);
- \( g_p \) is the partition of \( \{1, \ldots, 2k\} \) induced by the coincidence of the coordinates of \( p \) (i.e., \( a \) and \( b \) are in the same block if and only if \( p_a = p_b \));
- \( V(p) = Z_{p_1} \cdots Z_{p_k} \bar{Z}_{p_{k+1}} \cdots \bar{Z}_{p_{2k}} \)
- \( (1 - E_{p_1}) \left[ \frac{1}{G_{p_1p_1}} \right] \cdots (1 - E_{p_k}) \left[ \frac{1}{G_{p_kp_k}} \right] (1 - E_{p_{k+1}}) \left[ \frac{1}{G_{p_{k+1}p_{k+1}}} \right] \cdots (1 - E_{p_{2k}}) \left[ \frac{1}{G_{p_{2k}p_{2k}}} \right] \)

Given \( B \in \mathcal{B}_k \), and \( p \in \{M + 1, \ldots, M + N\}^{2k} \) with \( g_p = B \),

- we let \( L(B) = \{ r \in \{1, \ldots, 2k\} \mid \text{the block in } B \text{ containing } r \text{ is } \{r\} \} \);
- we set \( p_L = \{p_r \mid r \in L(B)\} \);
- if \( p_L \subset T \cup \{x, y\} \), then we say \( G_{xy}^{(T)} \) is maximally expanded;
- set \( \mathcal{A} \) to be the set of monomials in elements of

\[
\left\{ G_{xy}^{(T)} \mid T \subset p_L, x \neq y, \{x, y\} \subset p \setminus T \right\} \cup \left\{ \frac{1}{G_{xx}^{(T)}} \mid T \subset p_L, x \subset p \setminus T \right\} \tag{159}
\]

- for each \( A \in \mathcal{A} \), let \( d(A) \) denote the number of off-diagonal entries (i.e., entries of form \( G_{xy}^{(T)} \)) in \( A \).

Given \( B \in \mathcal{B}_k \), and \( p \in \{M + 1, \ldots, M + N\}^{2k} \) with \( g_p = B \), suppose \( A \in \mathcal{A} \). We can do the following algorithm to expand \( A \).

1. If every factor of \( A \) is maximally expanded or \( d(A) \geq 2k + 1 \), then stop the algorithm.
2. Otherwise choose a factor of \( A \) that is not maximally expanded. If this entry is off-diagonal, \( G_{xy}^{(T)} \), write

\[
G_{xy}^{(T)} = G_{xy}^{(Tu)} + \frac{G_{xu}^{(T)} G_{uy}^{(T)}}{G_{uu}^{(T)}}. \tag{160}
\]
where \( u \) is the smallest element in \( p_L \setminus (T \cup \{ x, y \}) \). If the chosen factor is diagonal, \( \frac{1}{G_{xx}} \), then write
\[
\frac{1}{G_{xx}} = \frac{1}{G_{(T)^{u}}(T)} G_{ku}(T) G_{uu}(T) G_{xx}(T)
\]
for the smallest \( u \in p_L \setminus (T \cup \{ x, y \}) \). Here (160) and (161) are induced from (28). Now write \( A = w_0(A) + w_1(A) \) by replacing the chosen factor by the RHS of (160) or (161). Here \( w_0(A) \) (resp. \( w_1(A) \)) denotes the terms containing the first therm on RHS of (160) (resp. 161).

Obviously
\[
d(w_0(A)) = d(A), \quad d(w_1(A)) \geq \max(2, d(A) + 1).
\]

Given \( B \in B_k \), and \( p \in \{ M + 1, \ldots, M + N \}^{2k} \) with \( g_p = B \), we apply the above algorithm on each
\[
A^r := \frac{1}{G_{p_{p_r}}}, \quad r \in \{1, \ldots, 2k\}.
\]

We set \( A^r_{00} := w_0(A^r) \) and \( A^r_{11} := w_1(A^r) \). Then we apply the algorithm again and set
\[
A^r_{00} := w_0(A^r_{00}), \quad A^r_{11} := w_1(A^r_{11}), \quad A^r_{01} := w_1(A^r_{01}), \quad A^r_{10} := w_1(A^r_{10}), \quad A^r_{11} := w_1(A^r_{11})
\]
and so on. Notice that the lower indices are sequences of 0 and 1. So this algorithm generates a binary tree. After finite many steps, the algorithm stops and we have:
\[
A_r = \sum_{\sigma \in L_r} A^r_{\sigma} \quad \text{and} \quad V(p) = \sum_{\sigma_1 \in L_1} \cdots \sum_{\sigma_{2k} \in L_{2k}} (1 - E_{p_{1}})[A_{\sigma_1}^1] \cdots (1 - E_{p_{k}})[A_{\sigma_k}^k](1 - E_{p_{k+1}})[A_{\sigma_{k+1}}^{k+1}] \cdots (1 - E_{p_{2k}})[A_{\sigma_{2k}}^{2k}]
\]

where \( L_r \) is the set of leaves (i.e., vertices with no child) in the binary tree generated by the algorithm. It is easy to see:

- \( A^r_{\sigma} \in A \) such that \( d(A^r_{\sigma}) \geq 2k + 1 \) or every factor in \( A^r_{\sigma} \) is maximally expanded;
- by the stopping rule of the algorithm, we have \( d(A^r_{\sigma}) \leq 2k + 2 \) for each \( \sigma \in L_r \);
- each application of \( w_1 \) increases \( d(\cdot) \) by at least one, and in the first step by two, so each \( \sigma \in L_r \) has no more than \( 2k + 1 \) ones;
- the number of Green function entries increases by at most four through each application of \( w_1 \) and by zero through each application of \( w_0 \), so \( A^r_{r} \) has no more than \( 8k + 5 \) Green function entries;
- each Green function entry \( G_{xx}^{(T)} \) or \( \frac{1}{G_{xx}} \) in \( A^r_{r} \) satisfies \( T \subset L \), so by \( |p_L| \leq 2k \), the total number of upper indices in \( A^r_{r} \) is no more than \( 2k(8k + 5) \);
- each application of \( w_0 \) increases the total number of upper indices by one, so \( \sigma \) has no more then \( 2k(8k + 5) \) zeros;
- since each \( \sigma \in L_r \) has no more than \( 2k + 1 \) ones and no more than \( 2k(8k + 5) \) zeros, we have \( |L_r| < C_k \) where \( C_k > 0 \) depends only on \( k \).
Lemma 17. Suppose $k \in \{1, 2, \ldots\}$, $B \in \mathcal{B}_k$ and $z$ is in $(\bar{1}, \bar{157})$. For any (small) $\epsilon > 0$, if $N > N_0 = N_0(\epsilon', \tau, k)$, then:

$$|E[\mathbb{1}_{z \in D_\tau \cup D_{\tau}'} \cdot V(p)]| \leq N^{\epsilon'} |\Psi(z)|^{2k + |L(B)|}$$  

provided $p \in \{M + 1, \ldots, M + N\}^{2k}$ and $g_p = B$.

Proof. According to Lemma [10] for any $D' > 0$, if $N > N_0(\epsilon', D', \tau, k)$ then

$$P(A_N) > 1 - N^{-D'}$$  

(163)

where

$$A_N = \left\{ \frac{|G_{ij}^{(T)}|}{|G_{ij}|} \leq N^{\epsilon'} \forall i \neq j, |T| \leq 2k, z \in D_\tau \cup D_{\tau}' \right\}$$

and

$$\cap \left\{ \frac{1}{|G_{ij}|} \leq N^{\epsilon'} \forall |T| \leq 2k, z \in D_\tau \cup D_{\tau}' \right\}$$

Suppose $p \in \{M + 1, \ldots, M + N\}^{2k}$ and $g_p = B$. Recall that

$$V(p) = \sum_{\sigma_1 \in L_1} \cdots \sum_{\sigma_{2k} \in L_{2k}} (1 - E_{p_1})[A_{\sigma_1}^1] \cdots (1 - E_{p_2})[A_{\sigma_2}^k](1 - E_{p_{k+1}})[\tilde{A}_{\sigma_{k+1}}^{k+1}] \cdots (1 - E_{p_{2k}})[A_{\sigma_{2k}}^{2k}].$$

(164)

Recall that each $A_{\sigma_i}^i$ on RHS of (164) has no more than $8k + 5$ Green function entries. Also note that if $\sigma_i$ has no one, then $A_{\sigma_i}^i = \frac{1}{G_{ij}^{(T)}}$ for some $T$, thus $d(A_{\sigma_i}^i) = 0$. On the other hand, if $\sigma_i$ has ones, then the number of ones is at most $d(A_{\sigma_i}^i) - 1$ (since the first $w_1$ adds two off-diagonal entries). So we have on $A_N$:

$$\mathbb{1}_{z \in D_\tau \cup D_{\tau}'} \cdot |(1 - E_i)[A_{\sigma_i}^i]| \leq \begin{cases} N^{\epsilon'} |\Psi(z)| & \text{if } d(A_{\sigma_i}^i) = 0 \\ 2N^{\epsilon'}(8k + 5) \cdot |\Psi(z)|^{d(A_{\sigma_i}^i)} & \text{if } d(A_{\sigma_i}^i) > 0 \\ \end{cases}$$

$$\leq 2N^{\epsilon'}(8k + 5) \cdot |\Psi(z)|^{1 + (\text{number of ones in } \sigma_i)}$$

(165)

Now we consider two cases. First, if $A_{\sigma_i}^r$ $(1 \leq r \leq 2k)$ contains a factor which is not maximally expanded, then by the algorithm, $d(A_{\sigma_i}^r) \geq 2k + 1$. So if $N > N_0(\epsilon', D', \tau, k)$, then on $A_N$ we have

$$\mathbb{1}_{z \in D_\tau \cup D_{\tau}'} \cdot \left| (1 - E_{p_1})[A_{\sigma_1}^1] \cdots (1 - E_{p_2})[A_{\sigma_k}^k](1 - E_{p_{k+1}})[\tilde{A}_{\sigma_{k+1}}^{k+1}] \cdots (1 - E_{p_{2k}})[A_{\sigma_{2k}}^{2k}] \right|$$

$$\leq N^{C_1 \epsilon'} |\Psi(z)|^{4k} \leq N^{C_1 \epsilon'} |\Psi(z)|^{2k + |L(B)|}$$

(166)

where $C_1 > 0$ is a constant depending only on $k$.

Second, if every factor in each $A_{\sigma_i}^r$ $(1 \leq i \leq 2k)$ is maximally expanded and

$$E \left[ \mathbb{1}_{z \in D_\tau \cup D_{\tau}'} \cdot \left| (1 - E_{p_1})[A_{\sigma_1}^1] \cdots (1 - E_{p_2})[A_{\sigma_k}^k](1 - E_{p_{k+1}})[\tilde{A}_{\sigma_{k+1}}^{k+1}] \cdots (1 - E_{p_{2k}})[A_{\sigma_{2k}}^{2k}] \right| \right]$$

is nonzero, then we claim that for each $s \in L(B)$, there exists an element $a(s) \in \{1, \ldots, 2k\}\setminus\{s\}$ such that $A_{\sigma_{a(s)}}^a(s)$ contains a Green function entry with lower index $p_s$. This is because otherwise
there exists $s \in L(B)$ such that for each $i \in \{1, \ldots, 2k\}\setminus\{s\}$, $A^0_{\sigma_i}$ has no factor with lower index $p_s$, and therefore $A^0_{\sigma_i}$ must be $H(p_s)$-measurable (since all factors of $A^0_{\sigma_i}$ are maximally expanded), so (167) equals:

$$
\mathbb{E}\left[(1 - \mathbb{E}_{p_s})\left[\prod_{i \in [1, k]} (1 - \mathbb{E}_{p_i})[A^1_{\sigma_i}] \prod_{j \in [k+1, 2k]} (1 - \mathbb{E}_{p_j})[\bar{A}^2_{\sigma_j}]\right]\right] = 0 \tag{168}
$$

but this is contradictory to our assumption that (167) is nonzero. So the claim must be true. For each $i \in \{1, \ldots, 2k\}$, if $l \in a^{-1}(i)$ (i.e., $l \in L(B)$ and $a(l) = i$), then by the definition of $L(B)$, $p_l$ is not the lower index of $A' = \frac{1}{G_{p_l,p_l}}$, so $p_l$ must be added to lower index of $A_{\sigma_i}$ through an application of $w_1$ (because $w_0$ does not change lower index while $w_1$ add one element in lower index). So $\sigma_i$ should have at least $|a^{-1}(i)|$ ones. So by (105), if $N > N_0(\epsilon', D', \tau, k)$, then on $A_N$ we have:

$$
\prod_{i \in [1, k]} (1 - \mathbb{E}_{p_i})[A^1_{\sigma_i}] \prod_{j \in [k+1, 2k]} (1 - \mathbb{E}_{p_j})[\bar{A}^2_{\sigma_j}] \leq N^{C_2}\epsilon' \left|\Psi(\tau)\right|^{2k + \sum_{|a^{-1}(i)|} |a^{-1}(i)|} = N^{C_2}\epsilon' \left|\Psi(\tau)\right|^{2k + |L(B)|} \tag{169}
$$

where $C_2 > 0$ is a constant depending only on $k$. On $A_N$ we have the naive bound of the Green function entries:

$$
\left|G_{xy}^{(T)}\right| \leq N^2(1 + \max_{a,b} |X_{ab}|) \quad (\text{by } 32) \tag{170}
$$

Using (24) and (25) (with $G$ replaced by $G^{(T)}$) and (170) we have:

$$
\frac{1}{\left|G_{xy}^{(T)}\right|} \leq l^{-1} + \tau^{-1} + (M \lor N)^4(1 + \max_{a,b} |X_{ab}|) \max_{a,b} |X_{ab}|^2 \tag{171}
$$

Note that $\epsilon'$ can be arbitrarily small. Using (170) and (171) on $A_N$, using (166) and (169) on $A_N$, using (103) to control $\mathbb{P}(A_N)$, using (104), we complete the proof of Lemma 17.

Now we continue to prove Lemma 11. Note that if $B \in B_k$, then:

$$
\frac{1}{N^{2k}} \sum_{p \in (M+1, \ldots, M+N)^{2k}} \mathbb{I}(g_p = B) \leq N^{|B| - 2k} \leq N^{-k + \frac{1}{2}|L(B)|} = (N^{-1/2})^{2k - |L(B)|} \tag{172}
$$

where we used $2k - |B| > \frac{2k - |L(B)|}{2}$ in the last inequality.

By (158), Lemma 17 and (172), for any (small) $\epsilon' > 0$, if $N > N_0(\epsilon', \tau, k)$, then for any $z$ in (157) we have:

$$
\mathbb{E}\left[\mathbb{I}_{z \in D_{\tau} \cup D_{\tau}^C} \left|\frac{1}{N} \sum_{p = M+1}^{M+N} \mathbb{E}_{Z_p} \right|^{2k}\right] \leq \left|\mathbb{E}_k[\epsilon'] \left|\Psi(z)\right|^{2k + |L(B)|} \right| (N^{-1/2})^{2k - |L(B)|} \leq |\mathbb{E}_k| N^{\epsilon'} \left|\Psi(z)\right|^{4k}
$$

where we used $\Psi(z) \geq N^{-1/2}$ in the last inequality. For any $\epsilon > 0$ and $D > 0$, choose $k$ large enough such that $2k\epsilon > 2D$, then for $N > N_0(\epsilon, D, \tau)$ we have:
\[
\mathbb{P}(1 \in D \cup D' \mid \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p > N^2 |\Psi(z)|^2) \leq N^{-2k}\mathbb{E}[1 \in D \cup D' \mid \frac{1}{N} \sum_{p=M+1}^{M+N} Z_p^2] \\
\leq |B_k| N^e N^{-2k} < N^{-D}
\]

where we used the fact that \(|B_k|\) is a constant depending only on \(k\) in the last inequality. Finally, using the “lattice” argument we complete the proof of the first conclusion.

For the second conclusion, we notice:

\[
\mathbb{E}[1 \in D \cup D' \mid \frac{1}{N} \sum_{i=1}^{M} \frac{\sigma_i^2}{(1 + \sigma_i \bar{m}_{fc})^2} Z_i^{2k}] = \sum_{B \in B_k} \frac{1}{N^{2k}} \sum_p \mathbb{E}[1 \in D \cup D' \mid \check{V}(p)] \cdot 1(g_p = B) \quad (173)
\]

where \(p\) runs over \(\{1, \ldots, M\}^{2k}\) and

\[
\check{V}(p) = \left(\frac{\sigma_{p_1}^2}{(1 + \sigma_{p_1} \bar{m}_{fc})^2} \bar{Z}_{p_1}\right) \cdots \left(\frac{\sigma_{p_k}^2}{(1 + \sigma_{p_k} \bar{m}_{fc})^2} \bar{Z}_{p_k}\right) \left(\frac{\sigma_{p_{k+1}}^2}{(1 + \sigma_{p_{k+1}} \bar{m}_{fc})^2} \bar{Z}_{p_{k+1}}\right) \cdots \left(\frac{\sigma_{p_{2k}}^2}{(1 + \sigma_{p_{2k}} \bar{m}_{fc})^2} \bar{Z}_{p_{2k}}\right)
\]

By (173), \(|\frac{\sigma_i^2}{(1 + \sigma_i \bar{m}_{fc})^2}|\) is bound uniformly in \(i \in \{1, \ldots, M\}\) and in \(z \in D \cup D'\). We can use the same proof to show that the conclusion of Lemma 17 holds with \(V(p)\) replaced by \(\check{V}(p)\), provided \(p \in \{1, \ldots, M\}^{2k}\) and \(g_p = B\). The remaining part of the proof of the second conclusion is similar as that of the proof of the first conclusion. \(\square\)

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