Dynamical Maps and Density Matrices

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Abstract.

The relations between dynamical maps and quantum states of bipartite systems are analyzed from the perspective of quantum conditional probability. In particular, we explore new interesting relations between completely positive maps, which correspond to quantum channels, and states of bipartite systems which correspond to correlations between the initial and final states. The new connection emerges in a natural way from the generalisation of the classical concept of conditional probability.

We develop applications of these relations which prove to be very useful in both directions, either for the classification of positive maps which are not completely positive, the classification of non-decomposable dynamical maps or for the classification of positive partial transpose and entangled states.

1. Introduction

The classic dichotomy between being and becoming has been translated into the description of physical systems as the dichotomy between bipartite states and dynamical maps. This dichotomy between statics and dynamics has been weakly questioned in both schemes, but persists as a key element of fundamental theories. In this respect quantum physics only differs from classical physics in the change of the nature of quantum states and quantum maps.

A surprising genuine feature of quantum physics is that it reinforces the existence of relations between these two different facets of quantum systems for the case of composite bipartite systems. The analysis of such relations has some additional interest because it permits to relate entanglement properties of quantum states of composite systems and the positivity properties of quantum maps. The aim of the present paper is to further analyse the relations between quantum states and dynamical maps from this perspective.

2. Quantum states and Unital Dynamical Maps

The basic elements for the description of quantum systems are the quantum states. Pure quantum states can be identified with rays of vectors |ψ⟩ in some Hilbert space ℋ. In the probabilistic interpretation, pure states are associated with the corresponding projectors ρψ = |ψ⟩⟨ψ|/⟨ψ|ψ⟩.

However, the knowledge of the state of a quantum system is not in general maximal and various pure states can be compatible with the experimental results. In this situation one has
to describe quantum states as statistical mixtures or density states [1].

If we assume that the results of our measurements are compatible with the possibility that our system may be found in the pure state \( \psi_1 \) with probability \( p_1 \), or in the state \( \psi_2 \) with probability \( p_2 \), probability \( p_n \) for the state \( \psi_n \) and so on, the state of our system is represented by the convex combination, usually called density matrix,

\[
\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| = \sum_j p_j \rho_j
\]

with

\[
0 \leq p_j \leq 1 \quad \sum_j p_j = 1.
\]

The family \((p_1, p_2, \ldots, p_n, \ldots)\) is usually said to represent a probability vector. Genuine quantum states involve therefore mixed states as well as pure states.

In general, for quantum system with only \( n \) level states \( H = \mathbb{C}^n \) any quantum state is described by an \( n \times n \) density matrix \( \rho \), i.e. a positive, selfadjoint matrix with unit trace,

\[
\begin{align*}
i) & \quad \rho^+ = \rho \\
ii) & \quad \rho \geq 0 \\
iii) & \quad \text{tr} \rho = 1
\end{align*}
\]

We shall denote by \( \mathcal{D}(H) \) the space of linear maps \( \rho \) associated to density matrices, i.e. matrices satisfying the properties (1)-(3). \( \mathcal{D}(H) \) is convex, i.e. for any \( \rho_1, \rho_2 \in \mathcal{D}(H) \) and \( 0 \leq s \leq 1 \)

\[
\rho = s\rho_1 + (1-s)\rho_2 \in \mathcal{D}(H).
\]

Pure states are extremal states in \( \mathcal{D}(H) \), i.e. states that cannot be decomposed as a convex sum of two different states (4).

Any evolution process or dynamical map has to transform quantum states into quantum states and preserve the superposition principle. Therefore a quantum dynamical map between two systems associated to the Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \) can be identified with a linear homomorphism \( \varphi \) mapping \( \mathcal{D}(\mathcal{H}_1) \) into \( \mathcal{D}(\mathcal{H}_2) \), which satisfies very intricate positivity conditions which complicate considerably its classification. If pure states are transformed into pure states these requirements imply that \( \varphi \) is a projective map \( \varphi : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \). However, in general a dynamical map can be any map \( \varphi : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_2) \) which preserves the basic properties of the quantum states [2]

\[
\begin{align*}
i) & \quad \text{Selfadjointness} \quad \varphi(\rho)^+ = \varphi(\rho) \\
ii) & \quad \text{Positivity} \quad \varphi(\rho) > 0 \\
iii) & \quad \text{Trace Condition} \quad \text{Tr} \varphi(\rho) = 1 \\
iv) & \quad \text{Linearity} \quad \varphi(s\rho_1 + (1-s)\rho_2) = s\varphi(\rho_1) + (1-s)\varphi(\rho_2)
\end{align*}
\]

A simple example is the unitary map defined by any unitary matrix \( U \) as

\[
\varphi(\rho) = U\rho U^\dagger.
\]

The set of quantum maps \( \text{Maps}_Q(\mathcal{H}_1, \mathcal{H}_2) \) is convex since for any pair of quantum maps \( \varphi_1, \varphi_2 \in \text{Maps}_Q(\mathcal{H}_1, \mathcal{H}_2) \) and \( 0 \leq s \leq 1 \)

\[
\varphi = (s\varphi_1 + (1-s)\varphi_2) \in \text{Maps}_Q(\mathcal{H}_1, \mathcal{H}_2)
\]

is a quantum map.

Sometimes it is convenient to consider a larger class of maps. Those which satisfy only the conditions (5) (6) and (8), but not the normalisation condition (7). In particular, maps with \( \varphi(\mathbb{I}) = \mathbb{I} \) are called unital maps and they differ by a normalisation factor from trace preserving maps \( \text{Tr} \varphi(\rho) = \text{Tr} \rho \) which satisfy the property (7).
3. Positive and Completely Positive Maps

A map \( \varphi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2) \) is said to be positive if \( \varphi(a) \geq 0 \) for any positive operator \( a \in \mathcal{L}(\mathcal{H}_1) \). Any dynamical map is positive, and because of the isomorphism \( \mathcal{L}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H} = \mathcal{H}^{\otimes 2} \) can be identified with a homomorphism \( \hat{\varphi} \) from \( \mathcal{H}^{\otimes 2} = \mathcal{H}_1 \otimes \mathcal{H}_1 \) into \( \mathcal{H}_2 \otimes \mathcal{H}_2 \). Moreover, by using the canonical Hilbert products of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), it can also be uniquely identified, with an endomorphism \( \hat{\varphi} \) of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). Thus, any dynamical map \( \varphi \) which is an homomorphism from \( \mathcal{L}(\mathcal{H}_1) \) into \( \mathcal{L}(\mathcal{H}_2) \) can also be also identified with an endomorphism \( \hat{\varphi} \) of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). The endomorphisms \( \hat{\varphi} \) corresponding to dynamical maps are selfadjoint endomorphisms of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). A dynamical map \( \varphi \) is said to be completely positive (CP) if the associated map \( \hat{\varphi} \) is a positive endomorphism of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

It is obvious that the class of completely positive maps is a subclass in the space of all positive maps. Indeed, any completely positive map is positive, but not any positive map needs to be completely positive. A simple counterexample is the transposition map

\[
\hat{\phi}_{\tau}(a) = a^T
\]

which is positive but not completely positive.

Thus, CP maps satisfy stronger conditions than simple positive maps. Simply positivity only requires that \( \varphi(a) > 0 \) for any linear positive operator \( a > 0 \), but completely positivity requires a more stringent condition \( \langle b, \hat{\varphi}(b) \rangle > 0 \) for any \( b \in \mathcal{H}_1 \otimes \mathcal{H}_2 \), which not any positive map satisfies.

Another structural difference between the two classes of maps is the behaviour under tensor product operations. Indeed, simple positivity is not preserved under tensorial products because in general \( \varphi = \varphi_1 \otimes \varphi_2 \) is not positive even if \( \varphi_1 \) and \( \varphi_2 \) are positive maps, whereas complete positivity is preserved under tensor products. For any completely positive maps \( \varphi_1 \) and \( \varphi_2 \) its tensor product \( \varphi = \varphi_1 \otimes \varphi_2 \) is always completely positive. In particular, it can be shown that a dynamical map \( \varphi \) is completely positive if and only if \( \varphi \otimes 1_n \) is a positive map for any identity matrix of arbitrary dimensions \( n \times n \) [3, 4].

In order to characterise the difference between these two classes of dynamical maps it is convenient to introduce some structures in the space of density matrices.

Let us restrict ourselves for simplicity from now on to finite-dimensional Hilbert spaces. For finite dimensions the space of linear maps \( \mathcal{L}(\mathbb{C}^n) \), which can be identified with the algebra of \( n \times n \) complex matrices \( M_n \), has an hermitian product defined by

\[
(a, b) = \text{Tr}(a^\dagger b) \quad \text{for any } a, b \in M_n.
\]

This product induces another product \( \langle \cdot, \cdot \rangle \) in the space of dynamical maps \( \text{Maps}_Q(\mathbb{C}^n, \mathbb{C}^n) \) by the following mechanism. Let \( \{f_{\alpha}; \alpha = 1, 2 \cdots n^2\} \) be an orthonormal basis in \( M_n \)

\[
(f_{\alpha}, f_{\beta}) = \delta_{\alpha\beta}.
\]

Then, we define

\[
\langle \varphi_1, \varphi_2 \rangle = \sum_{\alpha=1}^{n^2} \text{Tr} \varphi_1(f_{\alpha})^\dagger \varphi_2(f_{\alpha}).
\]

Using this correspondence it is possible to introduce two different bases associated to the basis \( f_{\alpha} \) in the space of linear maps:

1. A basis: defined by

\[
e_{\alpha\beta}(a) = f_{\alpha}(f_{\beta}^\dagger, a)
\]
and,

2. B basis:

\[ \varphi_{\alpha\beta}(a) = f_{\alpha} a f_{\beta}^\dagger. \]

Any map can be expressed as

\[ \varphi(a) = \sum_{\alpha, \beta} A_{\alpha\beta} e_{\alpha\beta}(a) \quad \text{A matrix form} \]

or

\[ \varphi(a) = \sum_{\alpha, \beta} B_{\alpha\beta} \varphi_{\alpha\beta}(a) \quad \text{B matrix form} \]

By definition B matrix is hermitian: \( B_{\beta\alpha}^* = B_{\alpha\beta} \) [2]

Any orthonormal basis \( |i>; i = 1, 2, \cdots n \) in \( \mathbb{C}^n \) induces a basis in \( M_n \) by

\[ f_{\alpha} = e_{ij} = |i><j|. \]

In this basis the matrices A and B satisfy the following interesting properties,

1. A Matrix:

\[
\begin{align*}
A_{\alpha\alpha'} &= A_{ij,ij'}' \\
A_{ij,i'j'}' &= A_{ji,j'i'}' \\
A_{ij,i'j'}' x_i x_j^* y_{i'} y_{j'}^* &> 0 \\
A_{ii',jj'}' &= \delta_{i'j'}
\end{align*}
\]

(10)

2. B Matrix:

\[
\begin{align*}
B_{\alpha\alpha'} &= B_{i'i',jj'} \\
B_{ii',jj'}' &= B_{jj',ii'}' \\
B_{ii',jj'}' x_i x_j^* y_{i'} y_{j'}^* &> 0 \\
B_{ii',jj'}' &= \delta_{i'j'}
\end{align*}
\]

(11)

In terms of the hermitian matrix B it is very easy to characterize a map as completely positive or simply positive.

For any selfadjoint endomorphism \( \varphi \) of \( \mathcal{L}(\mathbb{C}^n) = M_n = \mathbb{C}^{n^2} \) it follows that

a) \( \varphi \) is completely positive iff the matrix \( B_{\alpha\beta} \) is positive

b) \( \varphi \) is completely positive iff

\[ \overline{\varphi \otimes I_k} : \mathbb{C}^n \otimes \mathbb{C}^k \rightarrow \mathbb{C}^n \otimes \mathbb{C}^k \]

is positive for any \( k \), where \( I_k \) denotes the identity matrix in \( \mathbb{C}^k \).

c) \( \varphi \) is completely positive if there exist a family of operators \( D_i; i = 1, 2 \cdots r \) in \( M_n \) such that \( \varphi \) can be decomposed as [5, 6]

\[ \varphi(a) = \sum_{i=1}^{N} D_i a D_i^\dagger \]

for any \( a \in M_n \). If \( N = 1, \varphi \) is extremal.
The most relevant property of completely positive maps arises from the following result. If \( \rho = \rho_1 \otimes \rho_2 \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n) \) is a separable state the unitary evolution map

\[
\text{Tr}_2 U \rho U^\dagger = \varphi_{\mathbb{C}}(\rho_1)
\]

is completely positive. This means, that the evolution of a system due to the coupling to any kind of environment is given always by a positive map. In particular, this implies that if the dynamical map has to be embedded into a map for composition of our system with any kind of environment the map has to be completely positive. But this is not a necessary physical requirement, therefore, dynamical maps which are not completely positive can have a physical meaning although their classification is more involved than that of completely positive maps (CP).

For positive maps in \( \mathcal{L}(\mathbb{C}^n) = M_n \) we have the following properties

i) \( \varphi \) is not completely positive if the matrix \( B_{\alpha\beta} \) has at least one negative eigenvalue

ii) \( \varphi \) is not completely positive if there is a \( k \in \mathbb{N} \) such that

\[
\varphi \otimes I_k : \mathbb{C}^n \otimes \mathbb{C}^k \rightarrow \mathbb{C}^n \otimes \mathbb{C}^k
\]

is not positive.

iii) \( \varphi \) is not completely positive if there are two families of operators \( D_i ; i = 1, 2 \cdots N \) and \( C_j ; j = 1, 2 \cdots M \) in \( M_n \) such that \( \varphi \) can be decomposed as

\[
\varphi(a) = \sum_{i=1}^{N} D_i a D_i^\dagger - \sum_{j=1}^{M} C_j a C_j^\dagger
\]

The possibility that the transposition operator might be useful to distinguish between positive and completely positive suggests the utility of consider a new class of maps: a map is said to be copositive if under composition with transposition becomes positive. More generally,

A map \( \varphi : M_n \rightarrow M_n \) is k-positive if

\[
\varphi \otimes I_k(a) \geq 0 \quad \text{for any} \quad a \in M_n,
\]

\( \varphi \) is k-copositive if under transposition

\[
\varphi^T(a) = \varphi(a^T)
\]

becomes k-positive, i.e. \( \varphi^T \otimes I_k(a) > 0 \) for any \( a > 0 \). \( \varphi \) is k-decomposable if both

\[
\varphi \otimes I_k \quad \text{and} \quad \varphi^T \otimes I_k
\]

are positive, and completely positive if it is k-positive for any k. Similarly, \( \varphi \) is decomposable if it is k-decomposable for any k.

A positive map \( \varphi \) is decomposable if can be expressed as the sum of positive and copositive maps [7]

\[
\varphi(a) = \sum_{i=1}^{N} D_i a D_i^\dagger + \sum_{i=1}^{M} C_i a^T C_i^\dagger
\]
In $n=2$ dimensions every positive map is decomposable [8, 3, 9, 10]. In higher dimensions positive maps which are not decomposable are called atomic [11]. A first example is provided by Choi map [9]

$$
\varphi_{\text{Choi}} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} = \begin{pmatrix}
a_{11} + a_{22} & -a_{12} & -a_{13} \\
-a_{21} & a_{22} + a_{33} & -a_{23} \\
-a_{31} & -a_{32} & a_{33} + a_{11}
\end{pmatrix}
$$

(12)

The physical relevance of positive maps which are not completely positive was thought to be meaningless for a long time because they could not appear from unitary evolution of separable states of system and environment. However, this property turned out to be crucial for pointing out the relevant role of those maps in the detection of entanglement. Non CP positive maps can arise in the unitary evolution of bipartite systems only from entangled states. In particular, they can appear from the interaction with the environment only if the system and the environment are entangled.

The relevance of positive maps for the entanglement detection is mainly due to the following result. A mixed state $\rho$ is separable if and only if

$$(\varphi \otimes \mathbb{I})\rho \geq 0$$

for any positive map $\varphi$ [12].

If there are negative modes in the associated map $\varphi \otimes \mathbb{I}_m$ they correspond to entangled states. Thus, positive but not CP maps can be used as detectors of entanglement. In particular, they can be used as good discriminators in the classification of entangled states.

This relation extends to a more general parallelism in the approaches to the classification the degree of entanglement of bipartite systems and the classification of positive but not CP maps, which might be useful in both directions [13, 15, 14].

Indeed in the space of physical states of bipartite systems we have three main classes of states: Separable (non-entangled states), partial positive transpose (PPT) states and entangled states. The class of PPT states overlaps with the class of separable states but in higher dimensions also with that of entangled states. The entangled states can be further subdivided into those of zero discord or non-zero discord$^1$. On the space of dynamical maps we have: CP maps, decomposable maps and indecomposable maps.

4. Unital Quantum Maps and States of Bipartite Systems

Let us now explore further the properties of the afore mentioned one-to-one correspondence$^7$ between elements of $\mathbb{C}^{n^2} \otimes \mathbb{C}^{m^2}$ and linear maps from $\mathbb{C}^{n^2}$ into $\mathbb{C}^{m^2}$ which permits to establish a correspondence between quantum maps $\varphi$ from $\mathbb{C}^{n^2}$ into $\mathbb{C}^{m^2}$ and quantum states $\omega$ of the composite system $\mathbb{C}^{n^2} \otimes \mathbb{C}^{m^2}$ [4, 20].

$$
\varphi : \mathbb{C}^{n^2} \otimes \mathbb{C}^{m^2} \leftrightarrow \varphi \in \mathcal{L}(\mathbb{C}^{n^2}, \mathbb{C}^{m^2})
$$

4.1. Bipartite Mixed states

If the quantum system under consideration is a bipartite compound, density states will be described by selfadjoint positive matrices $R$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ which have unit trace, i.e

$$
R^+ = R, \quad R \geq 0 \quad \text{and} \quad \text{tr} \ R = 1.
$$

(13)

$^1$ The concept of quantum discord was introduced by W. Zurek [16]-[18]

$^7$ Similar results hold for infinite dimensional systems thanks to nuclear theorems [19].
As the Hilbert space is a tensor product we may introduce partial traces with respect to both factor spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \),

\[
\text{tr}_1 R = \rho^{(2)}, \quad \text{tr}_2 R = \rho^{(1)}
\]
giving rise to density states for the subsystems.

We can concatenate any number of subsystems to construct a multipartite density state [2] whose partial traces (all the traces except one) will give density states for the remaining subsystem. Here we shall restrict our interest to bipartite systems where we have

\[
\text{tr}_1 \text{tr}_2 R = \text{tr}_1 \rho^{(2)} = \text{tr}_2 \rho^{(1)} = 1.
\]

The fact that \( R \) is a density matrix guarantees that all the partial traces are also density matrices. In particular,

\[
R^+ = R \quad \text{implies} \quad \rho^{(1)+} = \rho^{(1)}, \quad \rho^{(2)+} = \rho^{(2)}
\]
\[
R \geq 0 \quad \text{implies} \quad \rho^{(1)} \geq 0, \quad \rho^{(2)} \geq 0
\]
\[
\text{tr} R = 1 \quad \text{implies} \quad \text{tr} \rho^{(1)} = \text{tr} \rho^{(2)} = 1.
\]

If we introduce suitable bases \( e_a \) in \( \mathcal{H}_1 \) and \( e_r \) in \( \mathcal{H}_2 \), we could write the matrix for \( R \) in component form \( R_{ar,rb} = (e_a \otimes e_b, R e_a \otimes e_r) \). This matrix, satisfies the following identity

\[
R_{ar,rb} = R^*_{bs,ra},
\]
\[
z_{ar} R_{ar,rb} z_{br} \geq 0
\]
\[
R_{ar,ra} = 1,
\]

for any matrix \((z_{ab}) \in M_{mn}\)

The canonical connection between bipartite systems and positive maps between the two subsystems follows from a very general principle. Indeed, in any Hilbert space \( \mathcal{H} \) there is an one-to-one correspondence between the set \( \mathcal{P}_q^p \) of \( p \)-contravariant \( q \)-covariant tensors and the set \( \mathcal{P}_{p+q} \) of \( p + q \) covariant tensors. The equivalence is due to the identification of \( \mathcal{H} \) and its dual space \( \mathcal{H}^* \) by means of the hermitian product. Consequently, endomorphisms of \( \mathcal{H} \) which are in \( \mathcal{P}_1^1 \) are in one-to-one correspondence with \( 2 \)-covariant tensors of \( \mathcal{P}_2 \).

In the theory of quantum information, completely positive maps correspond to quantum channels, and its relation with states of bipartite systems correspond to the existence of correlations between the initial and final states. In the classical information theory the relation is based on the concept of conditional probability. A generalisation of that idea for quantum systems is possible. The analogy requires the introduction of quantum conditional probability operators.

Let us consider two systems with physical states in \( \mathcal{D}(\mathbb{C}^n) \) and \( \mathcal{D}(\mathbb{C}^m) \), respectively. The first one describes an initial (input) states and the second one a final (output) states. Let us consider a map \( \varphi : \mathcal{D}(\mathbb{C}^n) \rightarrow \mathcal{D}(\mathbb{C}^m) \), such that its dual map \( \varphi^\dagger : \mathcal{D}(\mathbb{C}^m) \rightarrow \mathcal{D}(\mathbb{C}^n) \) is completely positive and unital, i.e. \( \varphi(I_m) = I_n \).

For any pair of initial \( \rho \in \mathcal{D}(\mathbb{C}^n) \) and final states \( \varphi(\rho) \in \mathcal{D}(\mathbb{C}^m) \), the interesting composite states \( \omega_\rho^\varphi \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m) \) encoding the correlation between the input and output of \( \varphi \) should satisfy the following two conditions:

\[
\text{i)} \quad \text{Tr} \omega_\rho^\varphi a \otimes I_m = \text{Tr} \rho^\dagger a, \quad \text{for all} \ a \in \mathcal{D}(\mathbb{C}^n)
\]
\[
\text{ii)} \quad \text{Tr} \omega_\rho^\varphi I_n \otimes b = \text{Tr} \varphi(\rho)^\dagger b, \quad \text{for all} \ b \in \mathcal{D}(\mathbb{C}^m).
\]

It is well known that conditional probability does not generally exist for quantum systems, therefore it is difficult to define a compound state \( \omega_\rho^\varphi \) satisfying the above conditions.
The first construction of a composite state $\omega_\phi$ satisfying the above two conditions is due to Ohya [21, 22] and proceeds as follows. Let $\rho \in D(\mathbb{C}^m)$ be density matrix with the following spectral decomposition

$$\rho = \sum_k \lambda_k m_k \rho_k$$

with

$$\rho_k = \frac{1}{m_k} P_k, m_k = \text{Tr} P_k,$$

where $\lambda_k$ are the eigenvalues of $\rho$, and $P_k$ are eigenprojectors of $\rho$, respectively.

Then, for any $\phi : D(\mathbb{C}^n) \rightarrow D(\mathbb{C}^m)$ the Ohya composite state $\omega_\phi \in D(\mathbb{C}^n \otimes \mathbb{C}^m)$ is defined by

$$\omega_\phi = \sum_k \lambda_k m_k \rho_k \otimes \phi(\rho_k).$$

Let us observe that by construction the Ohya state $\omega_\phi$ is non-linear with respect to $\rho$, but since $\phi$ is a positive unital map, $\omega_\phi$ is a separable state in $D(\mathbb{C}^n \otimes \mathbb{C}^m)$.

The construction of the Ohya map is very similar to that of the Jamiolkowski map but it satisfies the properties (18) (19) which are not fulfilled in the later case. We recall that Jamiolkowski map [4, 3] is nothing but the image under $\phi^\dagger \otimes \mathbb{1}_m$ of the maximally entangled element

$$\sum_{\alpha, \beta} e_{\alpha\beta} \otimes e_{\alpha\beta}$$

of the tensor product $M_n \otimes M_m$ defined by any orthonormal basis $e_{\alpha\beta}$ of $M_m$, i.e.

$$\omega_\phi = \sum_{\alpha, \beta=1}^n \phi^\dagger(e_{\alpha\beta}) \otimes e_{\alpha\beta}$$

4.2. Quantum Conditional Probability

There are other ways of generating composite states from dynamical maps satisfying all the conditions (18) (19). One interesting family is provided by quantum conditional probability operators [20].

A map

$$\pi : L(\mathbb{C}^n \otimes \mathbb{C}^m) \rightarrow L(\mathbb{C}^n \otimes \mathbb{C}^m)$$

is said to be a quantum conditional probability operator (QCPO) if for any $\sigma \in D(\mathbb{C}^n \otimes \mathbb{C}^m)$ satisfies that

$$\pi(\sigma) > 0$$

$$\text{Tr}_2 \pi(\sigma) = \mathbb{1}_n.$$ (23)

Such an operator $\pi$ is the quantum analogue of the classical conditional probability. Another definition of quantum conditional probability has been given in Ref. [24].

One way of building QCPO is to consider the map $\pi : L(\mathbb{C}^n \otimes \mathbb{C}^m) \rightarrow L(\mathbb{C}^n \otimes \mathbb{C}^m)$ which transforms any $\sigma \in D(\mathbb{C}^n \otimes \mathbb{C}^m)$ into the operator $\pi(\sigma) : \mathbb{C}^n \otimes \mathbb{C}^m \rightarrow \mathbb{C}^n \otimes \mathbb{C}^m$, defined by

$$\pi(\sigma) = (\sigma_1^{-1/2} \otimes I_m) \sigma (\sigma_1^{-1/2} \otimes I_m),$$

where $\sigma_1$ is the first entry of $\sigma$.
where
\[ \text{tr}_2 \sigma = \sigma_1. \] (25)

Since \( \sigma \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m) \) is a state of the bipartite system \( \mathbb{C}^n \otimes \mathbb{C}^m \), \( \sigma_1 > 0 \) and \( \pi \) satisfies the properties (23) (24) required for QCPO.

Now, given a QCPO \( \pi \) one can define a composite state \( \omega^\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m) \) for any \( \rho \in \mathcal{D}(\mathbb{C}^n) \) by
\[ \omega^\rho = (\rho^{1/2} \otimes I_m) \pi (\rho^{1/2} \otimes I_m) \]
which verifies that
\[ \text{tr}_2 \omega^\rho = \rho \] (26)
and defines by tracing out the \( \mathbb{C}^n \) component
\[ \text{tr}_1 \omega^\rho = \text{tr}_1 \pi (\rho \otimes I_m) = \varphi(\rho) \] (27)
a dynamical \( \varphi \)
\[ \varphi : \mathcal{D}(\mathbb{C}^n) \rightarrow \mathcal{D}(\mathbb{C}^m). \]
This correspondence defines in an explicit way a relation between QCPO and dynamical maps.

The study of this relation between \( \varphi \) and \( \pi \) is based on duality between quantum maps and composite states which has been investigated Refs. [23, 24, 25, 26] (see also references therein). In the case of CP maps the correspondence is invertible. Let \( n = m \) and \( \varphi \) a CP map, by using the image of the Jamiolkowski map associated to the basis \( e_{ij} \) we can define
\[ \varphi(e_{ij}) \otimes e_{ij} = (\varphi \otimes \mathbb{I}_n) \sum_{ij=1}^n e_{ij} \otimes e_{ij} \in M_n \otimes M_n \]
and, since \( \varphi \) is CP, \( \sigma_1 = \text{Tr}_2 \sigma_\varphi > 0 \), by
\[ \pi_{\sigma_\varphi} = (\sigma_1^{-1/2} \otimes \mathbb{I}_n) \sigma_\varphi (\sigma_1^{-1/2} \otimes \mathbb{I}_n) \]
a quantum conditional probability operator. Finally, we define the bipartite state
\[ \omega_\varphi^\rho = (\rho^{1/2} \otimes \mathbb{I}_n) \pi_{\sigma_\varphi} (\rho^{1/2} \otimes \mathbb{I}_n) \]
which closes the relation.

We can apply this construction to obtain relations between of entanglement properties of \( \omega_\varphi^\rho \) and properties of the positive map \( \varphi \). Indeed, one can show that [20]

1. The bipartite state \( \omega_\varphi^\rho \) has a positive partial transpose (PPT) iff \( \varphi \) is k-completely positive and k'-completely copositive
2. The bipartite state \( \omega_\varphi^\rho \) has negative partial transpose (NPT) iff \( \varphi \) is k-completely positive and k'-copositive \( (k' < n) \) provided that rank \( \rho = n \)

Let us consider some applications to point out the utility of QPCO.

**Example 1:**

Indeed, the previous properties of QPCO can be used to show that
\[ \varphi(a) = \sum_{ij} c_{ij} e_{ij} a e_{ij}^T + \mu a \] (28)
is completely positive iff
\[ c_{ij} \geq 0, \quad i \neq j \quad \text{and} \quad |c_{ii}\delta_{ij} + \mu| \geq 0 \]
and completely copositive if
\[ c_{ii} + \mu \geq 0 : \quad c_{ij} + c_{ji} > 2|\mu|, \quad i \neq j \]
\[ c_{ij}c_{ji} \geq \mu^2, \quad i \neq j \]

The same techniques can be applied to the following two examples are based on SU(n) symmetry. Let \( \lambda_1, \lambda_2, \ldots, \lambda_{n^2-1} \) be the Gell-Mann-Tilma basis of \( M_n \).

**Example 2.** The map [27]
\[ \varphi(a) = \sum_{\alpha=1}^{n^2-1} x_\alpha \lambda_\alpha T a \lambda_\alpha + \frac{1}{n} I T r a \]
(29)
is unital \( \varphi(1) = I \), invariant under transposition
\[ \varphi^T(a) = \varphi(a) \]
and positive, provided that
\[ |x_\alpha| \leq 1 \quad \alpha = 1, 2, \ldots, n^2 - 1, \]
because
\[ T r |x><x| \varphi(|y><y|) = \sum_{\alpha=1}^{n^2-1} x_\alpha (x, \lambda_\alpha y)(y, \lambda_\alpha y) + \frac{1}{n} \geq 0 \]
(30)
Moreover, \( \varphi \) is also completely positive because in that case
\[ \sum_{\alpha=1}^{n^2-1} \varphi(e_{ij}) \otimes e_{ij} = \sum_{\alpha=1}^{n^2-1} x_\alpha \lambda_\alpha \otimes \lambda_\alpha^T + \frac{1}{n} I \otimes I_n \geq 0. \]
(31)

**Example 3.** Given a a mixed state \( \omega \) in \( C^n \): \( \omega \geq 0 \) with \( T r \omega = 1 \) we can consider two bases of \( M_n \) defined
\[ f_\alpha = \begin{cases} \lambda_\alpha & \text{for } \alpha = 1, 2, \ldots, n^2 - 1, \\ \omega & \text{for } \alpha = n^2. \end{cases} \]
(32)
and
\[ g_\alpha = \begin{cases} \lambda_\alpha - I T r (\omega^\dagger \lambda_\alpha); & \text{for } \alpha = 1, 2, \ldots, n^2 - 1, \\ I; & \text{for } \alpha = n^2. \end{cases} \]
(33)
with \( x_{n^2} = 1 \) and \( x_\alpha \) satisfying (30) for \( \alpha = 1, 2, \ldots, n^2 - 1 \). Notice that the bases (32)(33) are dual one to each other
\[ T r f_\alpha g_\beta = \delta_{\alpha\beta}. \]
Then, the unital maps $\varphi$ and $\varphi^T$ defined by

$$\varphi(a) = \sum_{\alpha=1}^{n^2} x_\alpha f_\alpha \text{Tr}(a g_\alpha)$$  \hspace{1cm} (34)$$

and

$$\varphi^T(a) = \sum_{\alpha=1}^{n^2} x_\alpha g_\alpha \text{Tr}(a f_\alpha)$$  \hspace{1cm} (35)$$

are both positive

$$\sum_{\alpha=1}^{n^2} x_\alpha (x_\alpha g_\alpha x)(y, f_\alpha y) \geq 0$$

and completely positive

$$\sum_{\alpha=1}^{n^2} x_\alpha g_\alpha \otimes f_\alpha^T \geq 0$$

provided that

$$|x_\alpha| \leq 1 \hspace{1cm} \alpha = 1, 2, \ldots n^2 - 1.$$ 

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