On the global generation of higher direct images of pluricanonical bundles

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Abstract

Given a fibration $f$ between two projective manifolds $X$ and $Y$, we discuss the effective generation of the higher direct images $R^i f_*(K^m_X)$, where $K^m_X$ is the $m$-th tensor power of the canonical bundle of $X$. In particular, we answer two questions posed by Popa–Schnell in [PS14].

1. Introduction

Assume that $f: X \rightarrow Y$ is a fibration, i.e. a surjective morphism with connected fibres between two projective manifolds $X$ and $Y$. Denote by $K^m_X$ the $m$-th tensor power of the canonical bundle $K_X$ of $X$. The positivity of the associated higher direct image $R^i f_*(K^m_X)$ is of significant importance for understanding the geometry of this fibration. Fruitful results have been obtained on this subject, such as [BP08, Ber09, Hor10, Kaw81, Kaw82, Kol86a, Kol86b, Kol87, Vie82b, Vie83].

Popa and Schnell [PS14] proved the following result inspired by the brilliant work of Viehweg [Vie82b, Vie83] and Kollár [Kol86a, Kol86b]:

**Theorem 1.1 (Popa–Schnell).** Let $f: X \rightarrow Y$ be a fibration between two projective manifolds with dim$Y = n$, and $A$ be an ample and globally generated line bundle on $Y$. If $m \geq 1$ is an integer, then the sheaf

$$f_*(K^m_X) \otimes A^l$$

is 0-regular, and therefore globally generated, for $l \geq m(n + 1)$.

Popa and Schnell then posed a question whether the similar result holds for higher direct images. See Question in [PS14] after Corollary 2.10. In this paper, we give a positive answer to this question in some sense. Indeed, since by Proposition 2.1

$$f_*(K^m_X \otimes \mathcal{J}(f, \|K^m_X\|)) = f_*(K^m_X),$$

it is reasonable to involve the asymptotic multiplier ideal when we consider higher direct images. So our main result is as follows, which implies Theorem 1.1 when $i = 0$.

**Theorem 1.2.** Let $f: X \rightarrow Y$ be a fibration between two projective manifolds with dim$Y = n$, and $A$ be an ample and globally generated line bundle on $Y$. If $m \geq 1$ is an integer, then the sheaf

$$R^i f_*(K^m_X \otimes \mathcal{J}(f, \|K^m_X\|)) \otimes A^l$$

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is 0-regular, and therefore globally generated, for $i \geq 0$ and $l \geq m(n+1)$.

We use the strategy in [Kol86a, Kol86b] to prove Theorem 1.2. The idea is expanded as follows. First we prove a Kollár-type vanishing theorem.

**Theorem 1.3.** Let $f : X \to Y$ be a fibration between two projective manifolds with $\dim Y = n$, and $A$ be an ample and globally generated line bundle on $Y$. If $m \geq 1$ is an integer, then for any $l \geq (m-1)n + m$, $i \geq 0$ and $q > 0$,

$$H^q(Y, R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|)) \otimes A^l) = 0.$$

The Kollár-type vanishing theorem has been fully studied. See, for example, [Eno93, Fuj12, FM16, GM17, Kol86a, Kol86b, Mat14, Mat16, Ohs84, Wu20]. However, we cannot directly apply any result among these papers to obtain Theorem 1.3. The reason is that in general there does not exist a metric $\varphi$ on $K_X$ such that

$$i\Theta_{K_X, \varphi} \geq 0 \quad \text{and} \quad \mathcal{I}(\varphi) = \mathcal{I}(f, \|K_X\|).$$

We will construct a suitable metric on $K_X$ such that

$$i\Theta_{K_X, \varphi} \geq 0 \quad \text{and} \quad \mathcal{I}(\varphi) = \mathcal{I}(f, \|K_X\|).$$

Now Theorem 1.2 is a direct consequence of Theorem 1.3 combined with the Castelnuovo–Mumford regularity [Mum66].

After that, we prove a generic vanishing theorem for the higher direct images, which answers another question of Popa and Schnell [PS14] in some sense. See Question in [PS14] after Corollary 5.4. Note that T. Shibata [Shi16] provided an example to show that $R^i f_* (K_X^m)$, $m \geq 2$, are not necessarily GV-sheaves. Hence it is quite natural to consider instead $R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|))$.

**Theorem 1.4.** Let $f : X \to Y$ be a morphism from a projective manifold $X$ to an abelian variety $Y$. Then the sheaf

$$R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|))$$

is a GV-sheaf [PP11] for every $i \geq 0$ and $m \geq 1$.

This result leads in turn to the following vanishing and generation results which are stronger than those for morphisms to arbitrary varieties.

**Corollary 1.1.** If $f : X \to Y$ is a morphism from a projective manifold to an abelian variety and $A$ is an ample line bundle on $Y$, then for every $i \geq 0$ and $m \geq 1$ one has:

1. $R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|))$ is a nef sheaf on $Y$ (see Sect. 2.3).
2. $H^q(Y, R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|)) \otimes A) = 0$ for all $q > 0$.
3. $R^i f_* (K_X^m \otimes \mathcal{I}(f, \|K_X^m\|)) \otimes A^2$ is globally generated.

In the end, we make some further discussions. Note that after [PS14], there are several references such as [Den21, DM19, Dut20, Iwa20] which aims to improve Theorem 1.1. It is remarkable that there is no more global generation required for $A$ in order to give the effective lower bound of $l$ such that $f_* (K_X^m) \otimes A^l$ is (generically) globally generated. Hence it is natural to try to remove the global generation condition in our theorems.

However, since Theorem 1.2 depends highly on the Castelnuovo–Mumford regularity, it is not easy to make such an extension. Currently, we can only make the following generalisation of Theorem 1.3 and also prove a Kollár-type injectivity theorem. These results seem to be of independent interest.
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Theorem 1.5. Let \( f : X \to Y \) be a smooth fibration between two projective manifolds with \( \dim Y = n \), and \( A \) be an ample line bundle on \( Y \). If \( m \geq 1 \) is an integer, then the following results hold.

1. For any \( q > 0, i \geq 0 \) and \( l > (m - 1)(n + 2) \),
   \[ H^q(Y, R^i f_* (K_X^m \otimes \mathcal{I}(f, \| K_X^{m-1} \|) \otimes A^l)) = 0. \]

2. For any integer \( j \geq 0 \) and a (non-zero) section \( s \) of \( f_* A^j \), the multiplication map induced by the tensor product with \( s \)
   \[ \Phi : H^q(X, K_X^m \otimes f^* A^l \otimes \mathcal{I}(f, \| K_X^{m-1} \|)) \to H^q(X, K_X^m \otimes f^* A^{l+j} \otimes \mathcal{I}(f, \| K_X^{m-1} \|)) \]
   is (well-defined and) injective for any \( q \geq 0 \) and \( l > (m - 1)(n + 2) \).

This paper is organised as follows. We first recall some background materials in Section 2, including the asymptotic multiplier ideal sheaf, the definition of \( GV \)-sheaves in the sense of Pareschi and Popa [PP11] and so on. Then, we prove Theorems 1.2 and 1.3 in Section 3, Theorem 1.4 in Section 4, and Theorem 1.5 in Section 5.

2. Preliminary

In this section we introduce some basic materials. Assume that \( f : X \to Y \) is a fibration between two projective manifolds, and \( L \) is a holomorphic line bundle on \( X \). Moreover, \( L^k \) refers to the \( k \)-th tensor power with the convention that \( L^0 = \mathcal{O}_X \) and \( L^k = (L^*)^{-k} \) for \( k < 0 \).

2.1 The asymptotic multiplier ideal sheaf

This part is mostly collected from [Laz04b].

First recall the definition of the multiplier ideal sheaf associated to an ideal sheaf \( a \subset \mathcal{O}_X \) and a positive real number \( c \). Let \( \mu : \tilde{X} \to X \) be a smooth modification such that \( \mu^* a = \mathcal{O}_{\tilde{X}}(-E) \), where \( E \) has the simple normal crossing support. Then the multiplier ideal sheaf is defined as
\[ \mathcal{I}(c \cdot a) := \mu_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor cE \rfloor). \]

Here \( \lfloor E \rfloor \) means the round-down.

Now suppose that \( L \) is a line bundle on \( X \) whose restriction to a general fibre of \( f \) has non-negative Iitaka dimension. For a positive integer \( k \), there is a naturally defined homomorphism
\[ \rho_k : f^* f_* L^k \to L^k. \]

The relative base-ideal \( a_{k,f} \) of \( |L^k| \) is then defined as the image of the induced homomorphism
\[ f^* f_* L^k \otimes L^{-k} \to \mathcal{O}_X. \]

Hence for a given positive real number \( c \), we have the multiplier ideal sheaf \( \mathcal{I}(f, c \cdot a_{k,f}) \) which is also denoted by \( \mathcal{I}(f, c \cdot \frac{1}{k}|L^k|) \). Hence
\[ \mathcal{I}(f, \frac{c}{k}|L^k|) \subset \mathcal{O}_X. \]

It is not hard to verify that for every integer \( p \geq 1 \) one has the inclusion
\[ \mathcal{I}(f, \frac{c}{k}|L^k|) \subset \mathcal{I}(f, \frac{c}{p k}|L^{pk}|). \]
Therefore the family of ideals
\[ \{ \mathcal{I}(f, \frac{c}{k}|L^k) \}_{(k \geq 0)} \]
has a unique maximal element from the ascending chain condition on ideals.

**Definition 2.1.** The relative asymptotic multiplier ideal sheaf associated to \( f, c \) and \(|L|\),
\[ \mathcal{I}(f, c||L||) \]
is defined to be the unique maximal member among the family of ideals \( \{ \mathcal{I}(f, \frac{c}{k}|L^k)|\}_{(k \geq 0)} \).

Next, we explain the analytic counterpart of the relative multiple ideal sheaf. By definition,
\[ \mathcal{I}(f, c||L||) = \mathcal{I}(f, \frac{c}{k}|L^k|) = \mathcal{I}(\frac{c}{k} \cdot a_{k,f}) \]
for some \( k \). In this case, we will say that \( k \) computes \( \mathcal{I}(f, c||L||) \). Let \( U \) be a local coordinate ball of \( Y \). By definition, we can pick \( \{ u_1, ..., u_m \} \) in \( \Gamma(f^{-1}(U), L^k) \) which generate \( a_{k,f} \) on \( f^{-1}(U) \). Let \( \varphi_U = \log(|u_1|^2 + \cdots + |u_m|^2) \) which is a singular metric on \( L^k|_{f^{-1}(U)} \). We verify that
\[ \mathcal{I}(f, \frac{c}{k}|L^k|) = \mathcal{I}(\frac{c}{k} \varphi_U) \text{ on } f^{-1}(U). \]

Indeed, let \( \mu : X \to X \) be a smooth modification of \( a_{k,f} \). Then \( \mu^*a_{k,f} = \mathcal{O}_X(-E) \) such that \( E + \text{except}(\mu) \) has the simple normal crossing support. Here \text{except}(\mu) \) is the exceptional divisor of \( \mu \). Now it is computed in [Dem12] that
\[ \mathcal{I}(\frac{c}{k} \varphi_U) = \mu_* \mathcal{O}_X(K_{X/X} - [\frac{c}{k} E]) \text{ on } f^{-1}(U) \]
which coincides with the definition of \( \mathcal{I}(f, \frac{c}{k}|L^k|) \). Furthermore, if \( v_1, ..., v_m \) are alternative generators and \( \psi_U = \log(|v_1|^2 + \cdots + |v_m|^2) \), obviously we have \( \mathcal{I}(\frac{c}{k} \varphi_U) = \mathcal{I}(\frac{c}{k} \psi_U) \). Hence all the \( \mathcal{I}(\frac{c}{k} \varphi_U) \) patch together to give a globally defined multiplier ideal sheaf \( \mathcal{I}(\frac{c}{k} \varphi) \) such that
\[ \mathcal{I}(\frac{c}{k} \varphi) = \mathcal{I}(f, \frac{c}{k}|L^k|) = \mathcal{I}(f, c||L||). \]

Note that \( \{ f^{-1}(U), \frac{1}{k} \varphi_U \} \) does not give a globally defined metric on \( L \) in general. The \( \frac{1}{k} \varphi \) is interpreted as the collection of functions \( \{ f^{-1}(U), \frac{1}{k} \varphi_U \} \) by abusing the notation, which is called the collection of (local) singular metrics on \( L \) associated to \( \mathcal{I}(f, c||L||) \). Certainly it depends on the choice of \( k \) and is not unique.

The following elementary property is collected from [Laz04b].

**Proposition 2.1** (c.f. Proposition 11.2.15, [Laz04b]). Let \( f : X \to Y \) be a fibration between two projective manifolds, and \( L \) be a line bundle on \( X \). Given a positive integer \( k \), let
\[ a_{k,f} = a(f, |L^k|) \]
be the relative base-ideal of \(|L^k| \) relative to \( f \). Then the canonical map \( \rho_k : f^*f_*L^k \to L^k \) factors through the inclusion \( L^k \otimes \mathcal{I}(f, |L^k|) \), i.e.
\[ a_{k,f} \subseteq \mathcal{I}(f, |L^k|). \]

Equivalently, the natural map
\[ f_*(L^k \otimes \mathcal{I}(f, |L^k|)) \to f_*(L^k) \]
is an isomorphism.
2.2 GV-sheaves in the sense of Pareschi and Popa

2.2.1 Definition We concentrate now on the case of more specific morphisms \( f : X \to Y \), where \( X \) is a projective manifold and \( Y \) is an abelian variety. We denote by \( P \) the normalised Poincaré bundle on the product \( Y \times \text{Pic}^0(Y) \), and by \( P_\alpha \) its restriction to the slice \( Y \times \{\alpha\} \); this is of course just a different name for the point \( \alpha \in \text{Pic}^0(Y) \).

**Definition 2.2 (c.f. [PP11])** A coherent sheaf \( \mathcal{F} \) on \( X \) is said to be a GV-sheaf if

\[
\text{codim}_{\text{Pic}^0(Y)} \{\alpha \in \text{Pic}^0(Y) \mid H^q(X, \mathcal{F} \otimes f^*P_\alpha) \neq 0\} \geq q
\]

for every \( q \geq 0 \).

2.2.2 A brief review of the former results If \( f \) is generically finite, then \( K_X \) is a GV-sheaf by a special case of the generic vanishing theorem of Green and Lazarsfeld [GL87]. This result was generalised by Hacon [Hac04], to the effect that for an arbitrary \( f \) the higher direct images \( R^if_*K_X \) are GV-sheaves on \( Y \) for all \( i \geq 0 \). On the other hand, there exist simple examples showing that even when \( f \) is generically finite, the powers \( K_X^m \) with \( m \geq 2 \) are not necessarily GV-sheaves; see [PP11], Example 5.6. Therefore it is quite surprising that Popa and Schnell [PS14] showed that \( f_*K_X^m \) are GV-sheaves on \( Y \) for all \( m \geq 1 \). They then asked whether the higher direct images are also GV-sheaves. We positively answer this question in some sense in Theorem 1.4, i.e., we prove that \( R^if_*K_X^m \otimes \mathcal{I}(f, \|K_X^{m-1}\|) \) are GV-sheaves.

Note that \( R^if_*K_X^m \) are not necessarily GV-sheaves. In fact, [Shi16] constructed such a counterexample; see Example 4.5 there. After that, T. Shibata [Shi16] (Proposition 4.8) proved that \( R^if_*K_X^m \) are GV-sheaves with the assumptions that \( \dim X = 2 \) and \( \kappa(X) \geq 0 \). We will make a brief illustration (see Remark 4.1) that our Theorem 1.4 actually implies this result. Therefore, it is quite natural to consider the twist by the asymptotic multiplier ideal.

2.3 Nef coherent sheaves

To any coherent sheaf \( \mathcal{F} \) on a projective manifold \( Y \), one associates the scheme [Har77]

\[
P(\mathcal{F}) := \text{Proj}(\oplus_{m \geq 0}\text{Sym}^m \mathcal{F})
\]

and an inevitable sheaf \( \mathcal{O}_{P(\mathcal{F})}(1) \) on \( P(\mathcal{F}) \). Then we have the following definition.

**Definition 2.3.** A coherent sheaf \( \mathcal{F} \) is said to be nef if \( \mathcal{O}_{P(\mathcal{F})}(1) \) is.

3. Main theorem

3.1 The vanishing theorem

We first prove Theorem 1.3. We will apply the following Kollár-type vanishing theorem.

**Theorem 3.1 (c.f. [FM16], Theorem D).** Let \( f : X \to Y \) be a surjective morphism from a compact Kähler manifold \( X \) onto a projective variety \( Y \). Let \( F \) be a holomorphic line bundle on \( X \) with a (singular) metric \( \varphi_F \) such that \( i\Theta_{F,\varphi_F} \geq 0 \). Let \( N \) be a holomorphic line bundle on \( X \). Assume that there exist two positive integers \( a \) and \( b \) and an ample line bundle \( A \) on \( Y \) such that \( N^a = f^*A^b \). Then

\[
H^q(Y, R^if_*K_X \otimes F \otimes \mathcal{I}(\varphi_F) \otimes N) = 0
\]

for every \( q > 0 \) and \( i \geq 0 \).
We can not pick $K_{X}^{m-1}$ as $F$ in Theorem 3.1 since in general we can not obtain a metric $\psi$ on $K_{X}$ such that

$$i\Theta_{K_{X},\psi} \geq 0$$

and

$$\mathcal{K}((m-1)\psi) = \mathcal{K}(f, \|K_{X}^{m-1}\|).$$

Instead we consider $F = K_{X}^{m-1} \otimes f^*A^{(m-1)(n+1)}$, where $n = \dim Y$.

**Proof of Theorem 1.3.** Let $\{U_{\alpha}\}$ be a finite local coordinate chart of $Y$. Let $p$ be a divisible and large enough integer which computes $\mathcal{K}(f, \|K_{X}^{m-1}\|)$, and let $\varphi = \{\varphi_{\alpha}\}$ be the associated metrics. So $\varphi_{\alpha}$ is a singular metric on $K_{X}|_{f^{-1}(U_{\alpha})}$ of the form that

$$\varphi_{\alpha} = \frac{1}{p} \log \sum_{i} |u_{i,\alpha}|^2,$$

where $u_{i,\alpha} \in \Gamma(f^{-1}(U_{\alpha}), K_{X}^{p})$. In particular, $\{u_{i,\alpha}\}$ are local generators of the relative base-ideal $a_{p,f}$ of $|K_{X}^{p}|$ and

$$\mathcal{K}((m-1)\varphi) = \mathcal{K}(f, \|K_{X}^{m-1}\|).$$

Now $f_{*}(K_{X}^{p}) \otimes A^{p(n+1)}$ is globally generated by Theorem 1.1. Then there exist sections

$$\{v_{ij,\alpha}\} \subseteq H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)})$$

such that $\log \sum |u_{i,\alpha}|^2$ and $\log \sum |v_{ij,\alpha}|^2$ are equivalent with respect to the singularities on $f^{-1}(U_{\alpha})$. In fact, if we denote, by abusing the notation,

$$H^{0}(Y, f_{*}(K_{X}^{p}) \otimes A^{p(n+1)})$$

and

$$H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)})$$

the trivial vector bundles on $Y$ and $X$ respectively, the morphism

$$H^{0}(Y, f_{*}(K_{X}^{p}) \otimes A^{p(n+1)}) \to f_{*}(K_{X}^{p}) \otimes A^{p(n+1)}$$

as well as

$$H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)}) \to f^*f_{*}(K_{X}^{p}) \otimes f^*A^{p(n+1)}$$

is surjective by definition. Therefore

$$H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)}) \otimes K_{X}^{-p} \to f^*f_{*}(K_{X}^{p}) \otimes K_{X}^{-p} \otimes f^*A^{p(n+1)}$$

is also surjective. In particular, we have the following surjection:

$$H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)}) \otimes K_{X}^{-p} \to a_{p,f} \otimes f^*A^{p(n+1)},$$

where $a_{p,f} \otimes f^*A^{p(n+1)}$ by definition is the image of the natural morphism:

$$f^*f_{*}(K_{X}^{p}) \otimes K_{X}^{-p} \otimes f^*A^{p(n+1)} \to f^*A^{p(n+1)}.$$

Let $\{s_{j}\}$ be the set of global sections that generates $f^*A^{p(n+1)}$. Due to (3.1), all of the sections $\{u_{i,\alpha} \otimes s_{j}\}$ extend over $X$ as the global sections $\{v_{ij,\alpha}\}$ of

$$H^{0}(X, K_{X}^{p} \otimes f^*A^{p(n+1)}).$$

Since $\{s_{j}\}$ generates $f^*A^{p(n+1)}$, $\log \sum |u_{i,\alpha}|^2$ and $\log \sum |v_{ij,\alpha}|^2$ are equivalent with respect to the singularities on $f^{-1}(U_{\alpha})$.

Now the sections $\{v_{ij,\alpha}\}$, as $i, j, \alpha$ vary, together define a (singular) metric $\chi$ on

$$K_{X}^{p} \otimes f^*A^{p(n+1)}$$
with positive curvature current. Next we show that
\[ \mathcal{I}(p^{-1}(m-1)\chi) = \mathcal{I}(f, \|K_X^{m-1}\|). \]

Let \( a \) be the ideal sheaf defined by \( \{v_{ij,\alpha}\} \) (as \( i, j, \alpha \) vary) and let \( a_\alpha \) be the ideal sheaf (on \( f^{-1}(U_\alpha) \)) defined by \( \{u_{i,\alpha}\} \) (as \( i \) varies). Then by the choice of \( \{u_{i,\alpha}\} \), for every \( \alpha \) we have
\[ \mathcal{I}(p^{-1}(m-1) \cdot a) = \mathcal{I}(p^{-1}(m-1) \cdot a_\alpha) \text{ on } f^{-1}(U_\alpha). \tag{3.2} \]

In fact, notice that \( a_\alpha \) is just the relative base-ideal of \( |K_X^p| \) restricted on \( f^{-1}(U_\alpha) \). Accordingly, \( \mathcal{I}(p^{-1}(m-1) \cdot a_\alpha) \) is the restriction of \( \mathcal{I}(f, p^{-1}(m-1)|K_X^p|) \) on \( f^{-1}(U_\alpha) \). However,
\[ \mathcal{I}(f, p^{-1}(m-1)|K_X^p|) = \mathcal{I}(f, \|K_X^{m-1}\|) \]
is now the unique maximal element in
\[ \{ \mathcal{I}(f, \frac{1}{k}|K_X^{(m-1)}|) \}_{k \geq 0}. \]

So \( \mathcal{I}(f, p^{-1}(m-1)|K_X^p|) \), as well as \( \mathcal{I}(p^{-1}(m-1) \cdot a_\alpha) \) should be stable. In other words, if \( u \) is a section of \( \Gamma(f^{-1}(U_\alpha), K_X^p) \), we must have
\[ \mathcal{I}(p^{-1}(m-1) \cdot a_\alpha) = \mathcal{I}(f, p^{-1}(m-1)|K_X^p|)_{f^{-1}(U_\alpha)} = \mathcal{I}(p^{-1}(m-1) \cdot (a_\alpha \cup (u))). \]

Here \( a_\alpha \cup (u) \) refers to the ideal sheaf generated by both of \( a_\alpha \) and \( u \).

On the other hand, by construction the sections \( \{v_{ij,\alpha}\} \), as \( i, j \) vary, also generate \( a_\alpha \) on \( f^{-1}(U_\alpha) \). By stability, the sections \( \{v_{ij,\alpha}\} \), as \( i, j, \alpha \) vary, will lead to the same multiplier ideal sheaf here. It implies (3.2). Equivalently,
\[ \mathcal{I}(p^{-1}(m-1)\chi) = \mathcal{I}(p^{-1}(m-1) \cdot a) = \mathcal{I}(f, \|K_X^{m-1}\|) \]
on \( f^{-1}(U_\alpha) \) hence everywhere.

Now let
\[ (F, \varphi_F) = (K_X^{m-1} \otimes f^*A^{(m-1)(n+1)}, p^{-1}(m-1)\chi). \]
Then as is shown before,
\[ \mathcal{I}(\varphi_F) = \mathcal{I}(f, \|K_X^{m-1}\|). \]

Furthermore, since \( l \geq (m-1)n + m \) by hypothesis, \( l - (m-1)(n+1) \geq 1 \). Now let
\[ N = f^*A^{l-(m-1)(n+1)}, \]
we have
\[ R^lf_*((K_X \otimes F \otimes \mathcal{I}(\varphi_F) \otimes N) = R^lf_*((K_X^{m-1} \otimes \mathcal{I}(f, \|K_X^{m-1}\|) \otimes A^l). \]

Applying Theorem 3.1 (with the same notation there), we then obtain the desired vanishing result. \( \square \)

3.2 Global generation
Using Theorem 1.3 we can prove the global generation of higher direct images, namely Theorem 1.2. We first review the definition and a basic result of the Castelnuovo–Mumford regularity [Mum66].

**Definition 3.1.** Let \( X \) be a projective manifold and \( L \) an ample and globally generated line bundle on \( X \). Given an integer \( m \), a coherent sheaf \( F \) on \( X \) is \( m \)-regular with respect to \( L \) if for all \( i \geq 1 \)
\[ H^i(X, F \otimes L^{m-i}) = 0. \]
Theorem 3.2. (c.f. [Mum66]) Let $X$ be a projective manifold and $L$ an ample and globally generated line bundle on $X$. If $F$ is a coherent sheaf on $X$ that is $m$-regular with respect to $L$, then the sheaf $F \otimes L^m$ is globally generated.

After this, we can prove Theorem 1.2.

**Proof of Theorem 1.2.** Since $l \geq m(n + 1)$, $l - q \geq (m - 1)n + m$ when $q \leq n$. It then follows from Theorem 1.3 that for every $q \geq 1$,

$$H^q(Y, R^i f_*(K^m_X \otimes \mathcal{J}(f, \|K^{m-1}_X\|)) \otimes A^{l-q}) = 0$$

Hence the sheaf $R^i f_*(K^m_X \otimes \mathcal{J}(f, \|K^{m-1}_X\|)) \otimes A^l$ is 0-regular with respect to $A$. So it is globally generated by Theorem 3.2.

4. Generic vanishing

We first prove that the higher direct images are $GV$-sheaves.

**Proof of Theorem 1.4.** In view of [PP11, PS14], it is enough to show that for every finite étale morphism $\beta : Z \to Y$ of abelian varieties and an ample and globally generated line bundle $H$ on $Z$, we have

$$H^q(Z, H^l \otimes \beta^* R^i f_*(K^m_X \otimes \mathcal{J}(f, \|K^{m-1}_X\|))) = 0$$

for every $q > 0$ with $l$ large enough. Here large enough means that there exists a bound $d$ depending only on $n$ and $m$ such that vanishing (4.1) holds for any $Z$ and $H$ as long as $l \geq d$. In particular, we cannot apply the Serre asymptotic vanishing theorem [Har77] here.

Now we prove this vanishing result. Put $W := Z \times_Y X$ the fibre product [Har77]. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
W & \xrightarrow{\alpha} & X \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{\beta} & Y
\end{array}
$$

By construction, $\alpha$ is also finite and étale. Hence we have $\alpha^* K_X = K_W$ and

$$\alpha^* \mathcal{J}(f, \|K^{m-1}_X\|) = \mathcal{J}(g, \|K^{m-1}_W\|)$$

by the behaviour of asymptotic multiplier ideals under étale covers (c.f. [Laz04b], Theorem 11.2.16).

By the flat base change theorem [Har77],

$$\beta^* R^i f_*(K^m_X \otimes \mathcal{J}(f, \|K^{m-1}_X\|)) \simeq R^i g_*(K^m_W \otimes \mathcal{J}(f, \|K^{m-1}_W\|)).$$

Hence we obtain vanishing (4.1) by Theorem 1.3.

**Remark 4.1.** A variant of Theorem 1.4 can actually recover the following proposition, which is originally obtained in [Shi16].

**Proposition 4.1** (c.f. [Shi16], Proposition 4.8). Let $f : X \to Y$ be a morphism from a smooth projective surface $X$ to an abelian variety $Y$. Assume that $\kappa(X) \geq 0$. Then $R^i f_*(K^m_X)$ are $GV$-sheaves for every $i \geq 0$ and $m \geq 1$. 

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Reproof via Theorem 1.3. The proof is similar to the one in [Shi16], hence we only sketch it.

We may assume without loss of generality that dim f(X) ≥ 1. Then it is enough to show that R1f*(KmX) is a GV-sheaf. Since κ(X) ≥ 0, we can take a series of contractions of (−1)-curves ε : X → X′ such that KX′ is semi-ample, and obtain a natural morphism f′ : X′ → Y such that f′ ◦ ε = f. Combined with the Leray spectral sequence [Har77], it is left to prove that

\[ H^q(Y, f'^* R^1 \varepsilon_* (K^m_X) \otimes H^i) = 0 \]  \tag{4.2} 

and

\[ H^q(Y, R^1 f'^* (K^m_{X'}) \otimes H^i) = 0 \]  \tag{4.3} 

for every q > 0 and any ample and globally generated line bundle H on Y with l large enough. The vanishing (4.2) is quite obvious for the reason of dimension, while (4.3) is a direct consequence of Theorem 1.3 since the semi-ampleness implies that \( \mathcal{I}(f', \|K^{m-1}_X\|) = \mathcal{O}_{X'} \). □

Once we know generic vanishing the situation is in fact much better than what we obtained for morphisms to arbitrary varieties.

Proof of Corollary 1.1. In view of [PS14], Corollary 5.4, everything is immediately verified once we know that Rifi*(KmX ⊗ I(f, \|K^{m-1}_X\|)) is a GV-sheaf. Hence we omit the proof here. □

5. Further discussions

In this section, we prove Theorem 1.5 based on results in [Den21, DM19, Dut20, Iwa20]. More precisely, we will apply the following theorem.

Theorem 5.1 (c.f. [Iwa20], Theorem 1.4). Let f : X → Y be a fibration between two projective manifolds with dim Y = n, and A be an ample line bundle on Y. For any integer m ≥ 1 and l ≥ \( \frac{1}{2} n(n-1) + m(n+1) \), the sheaf

\[ f_*(K^m_X \otimes A^l) \]

is generated by the global sections at a regular value y of f.

The proof of Theorem 1.5 then involves the same argument as Theorem 1.3 with slightly adjustment.

Proof of Theorem 1.5. (1) Let \{Uα\} be a finite local coordinate chart of Y. Let p be a divisible and large enough integer which computes \( \mathcal{I}(f, \|K^{m-1}_X\|) \). Moreover, \( f^* A^{p(n+2)} \) is globally generated and

\[ p \geq \frac{1}{2} n(n-1) \]  \tag{5.1} 

Let \( \varphi = \{\varphi_\alpha\} \) be the metrics associated with \( \mathcal{I}(f, \|K^{m-1}_X\|) \). Then \( \varphi_\alpha \) is a singular metric on \( K_X|_{f^{-1}(U_\alpha)} \) of the form that

\[ \varphi_\alpha = \frac{1}{p} \log \sum |u_{i,\alpha}|^2, \]

where \( u_{i,\alpha} \in \Gamma(f^{-1}(U_\alpha), K^p_X) \). Moreover,

\[ \mathcal{I}(m-1)\varphi = \mathcal{I}(f, \|K^{m-1}_X\|). \]

We write inequality (5.1) as

\[ p(n+2) \geq \frac{1}{2} n(n-1) + p(n+1). \]
So by Theorem 5.1, the sheaf
\[ f_*(K_X^p) \otimes A^{p(n+2)} \]
is globally generated on \( Y \). Now let \( \{s_j\} \) be the set of global sections that generates \( f^*A^{p(n+2)} \). Then as is shown in the proof of Theorem 3.3 all of the sections \( \{u_{i,\alpha} \otimes s_j\} \) extend over \( X \) as the sections \( \{v_{ij,\alpha}\} \) in
\[ H^0(X, K_X^p \otimes f^*A^{p(n+2)}). \]
The sections \( \{v_{ij,\alpha}\} \), as \( i, j, \alpha \) vary, together define a (singular) metric \( \chi \) on
\[ K_X^p \otimes f^*A^{p(n+2)} \]
with positive curvature current. Then by the same argument as Theorem 1.3, we obtain
\[ \mathcal{J}(p^{-1}(m-1)\chi) = \mathcal{J}(f, \|K_X^{m-1}\|). \]

Let
\[ (F, \varphi_F) = (K_X^{m-1} \otimes f^*A^{(m-1)(n+2)}, p^{-1}(m-1)\chi). \]
As is shown before,
\[ \mathcal{J}(\varphi_F) = \mathcal{J}(f, \|K_X^{m-1}\|). \]
Furthermore, since \( l > (m-1)(n+2) \) by hypothesis, \( l - (m-1)(n+2) \geq 1 \). So let
\[ \mathcal{N} = f^*A^{l-(m-1)(n+2)}. \]
We have
\[ R^1f_*(K_X \otimes F \otimes \mathcal{J}(\varphi_F) \otimes N) = R^1f_*(K_X^{m-1} \otimes \mathcal{J}(f, \|K_X^{m-1}\|)) \otimes A^l. \]
Then by Theorem 3.1 (with the same notation there), we obtain the desired vanishing result.

(2) The strategy is to apply the following injectivity theorem in [Mat14].

**Theorem 5.2** (c.f. [Mat14], Theorem 1.5). Let \((F, \varphi_F)\) and \((M, \varphi_M)\) be line bundles with (singular) metrics on a compact Kähler manifold \( X \). Assume the following conditions:

(a) There exists a subvariety \( Z \) on \( X \) such that \( \varphi_F \) and \( \varphi_M \) are smooth on \( X \setminus Z \);
(b) \( i\Theta_{F,\varphi_F} \geq \gamma \) and \( i\Theta_{M,\varphi_M} \geq \gamma \) on \( X \) for some smooth \((1,1)\)-form \( \gamma \) on \( X \);
(c) \( i\Theta_{F,\varphi_F} \geq \varepsilon i\Theta_{M,\varphi_M} \) for some positive number \( \varepsilon > 0 \).

Then for a (non-zero) section \( s \) of \( M \) with \( \sup_X |s|_{\varphi_M} < \infty \), the multiplication map induced by the tensor product with \( s \)
\[ \Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{J}(\varphi_F)) \to H^q(X, K_X \otimes F \otimes M \otimes \mathcal{J}(\varphi_F + \varphi_M)) \]
is (well-defined and) injective for any \( q \).

From (1) we know that
\[ (K_X^{m-1} \otimes f^*A^{(m-1)(n+2)}, p^{-1}(m-1)\chi) \]
is a Hermitian line bundle with positive curvature current and satisfies
\[ \mathcal{J}(p^{-1}(m-1)\chi) = \mathcal{J}(f, \|K_X^{m-1}\|). \]
In particular, \( \chi \) is smooth outside a subvariety. Let \( \psi \) be a smooth metric on \( f^*A \) with positive curvature. Let
\[ (F, \varphi_F) = (K_X^{m-1} \otimes f^*A^l, p^{-1}(m-1)\chi + (l - (m-1)(n+2))\psi) \]
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\[ (M, \varphi_M) = (f^* A^j, j\psi). \]

Since \( l > (m - 1)(n + 2) \) by hypothesis, \( l - (m - 1)(n + 2) \geq 1 \). Therefore let \( \varepsilon = \frac{1}{l} \), we have
\[ i\Theta_{F, \varphi_F} \geq \varepsilon i\Theta_{M, \varphi_M} \geq 0. \]

Moreover, since \( \psi \) is smooth,
\[ I(\varphi_F + \varphi_M) = I(\varphi_F) = I(f, \|K_X^m\|). \]

So the proof is finished by directly applying Theorem 5.2. \( \square \)

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References

BP08 B. Berndtsson, M. Păun, Bergman kernels and the pseudoeffectivity of relative canonical bundles, Duke Math. J. 145 (2008), 341-378.

Ber09 B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Ann. of Math. 169 (2009), 531-560.

Dem12 J.-P. Demailly, Analytic methods in algebraic geometry, Surveys of Modern Mathematics 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

Den21 Y. Deng, Applications of the Ohsawa-Takegoshi Extension Theorem to Direct Image Problems, Int. Math. Res. Notices. 23(2021), 17611-17633.

DM19 Y. Dutta, T. Murayama, Effective generation and twisted weak positivity of direct images, Algebra and Number theory. 13 (2019), 425-454.

Dut20 Y. Dutta, On the Effective Freeness of the Direct Images of Pluricanonical Bundles, Ann. de l’Institut Fourier. 70 (2020), 1545-1561.

Eno93 I. Enoki, Kawamata-Viehweg vanishing theorem for compact Kähler manifolds, Einstein metrics and Yang–Mills connections. Marcel Dekker, (1993), 59-68.

Fuj12 O. Fujino, A transcendental approach to Kollár’s injectivity theorem, Osaka J. Math. 49 (2012), 833-852.

FM16 O. Fujino, S. Matsumura, Injectivity theorem for pseudo-effective line bundles and its applications, Transactions of the AMS. 8, (2021), 849-884.

GM17 Y. Gongyo, S. Matsumura, Versions of injectivity and extension theorems, Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), 479-502.

GL87 M. Green, R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90 (1987), 389-407.

Har77 R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg, 1977.

Hac04 C. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math. 575 (2004), 173-187.

Hor10 A. Hötting, Positivity of direct image sheaves—a geometric point of view, Enseign. Math. (2) 56 (2010), 87-142.

Iwa20 M. Iwai, On the global generation of direct images of pluri-adjoint line bundles. Math. Z. 294 (2020), 201-208.
On the global generation of higher direct images of pluricanonical bundles

Kaw81 Y. Kawamata, Characterization of abelian varieties Compositio Math. 43 (1981), 253-276.
Kaw82 Y. Kawamata, A generalization of Kodaira-Ramanujam’s vanishing theorem, Math. Ann. 261 (1982), 43-46.
Kol86a J. Kollár, Higher direct images of dualizing sheaves. I, Ann. of Math. (2) 123 (1986), 11-42.
Kol86b J. Kollár, Higher direct images of dualizing sheaves. II, Ann. of Math. (2) 124 (1986), 171-202.
Kol87 J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, Algebraic Geometry, 361-398 (1985) Adv. Stud. Pure Math. 10. North-Holland, Amsterdam (1987)
Laz04a R. Lazarsfeld, Positivity in algebraic geometry. I. Classical setting: line bundles and linear series, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004. xviii+387 pp. ISBN: 3-540-22533-1.
Laz04b R. Lazarsfeld, Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004. xviii+385 pp. ISBN: 3-540-22534-X.
Mat14 S. Matsumura, A Nadel vanishing theorem via injective theorems, Math. Ann. 359 (2014), 785-802.
Mat16 S. Matsumura, A vanishing theorem of Kollár-Ohsawa type, Math. Ann. 366 (2016), 1451-1465.
Mum66 D. Mumford, Lectures on Curves on an Algebraic Surface, Annals of Mathematics Studies. 59, Princeton University Press (1966)
Ohs84 T. Ohsawa, Vanishing theorems on complete Kähler manifolds, Publ. Res. Inst. Math. Sci. 20 (1984), 21-38.
PP11 G. Pareschi, M. Popa, GV-sheaves, Fourier–Mukai transform, and generic vanishing, Amer. J. Math. 133 (2011), 235-271.
PS14 M. Popa, C. Schnell, On direct images of pluricanonical bundles, Algebra Number Theory 8 (2014), 2273-2295.
Shi16 T. Shibata, On generic vanishing for pluricanonical bundles, Michigan Math. J. 65 (2016), 873-888.
Vie82b E. Viehweg, Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs, J. Reine Angew. Math. 330 (1982), 132-142.
Vie83 E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, Algebraic Varieties and Analytic Varieties. 329-353 (1981) Adv. Stud. Pure Math. 1. North-Holland, Amsterdam, 1983.
Wu20 J. Wu, A Kollár-type vanishing theorem, Math. Z. 295 (2020), 331-340.

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