The Lax equation and weak regularity of asymptotic estimate
Lie groups

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Abstract
We investigate the Lax equation in the context of infinite-dimensional Lie algebras. Explicit solutions are discussed in the sequentially complete asymptotic estimate context, and an integral expansion (sums of iterated Riemann integrals over nested commutators with correction term) is derived for the situation that the Lie algebra is inherited by an infinite-dimensional Lie group in Milnor’s sense. In the context of Banach Lie groups (and Lie groups with suitable regularity properties), we generalize the Baker–Campbell–Dynkin–Hausdorff formula to the product integral (with additional nilpotency assumption in the non-Banach case). We combine this formula with the results obtained for the Lax equation to derive an explicit representation of the product integral in terms of the exponential map. An important ingredient in the non-Banach case is an integral transformation that we introduce. This transformation maps continuous Lie algebra-valued curves to smooth ones and leaves the product integral invariant. This transformation is also used to prove a regularity statement in the asymptotic estimate context.

Keywords Infinite-dimensional Lie groups · Regularity of Lie groups · Lax equation · Generalized Baker-Campbell-Dynkin-Hausdorff formula · Nilpotent Lie algebras · Holonomies

1 Introduction
We investigate regularity properties of Lie groups in Milnor’s sense [3, 19] and extend the Baker–Campbell–Dynkin–Hausdorff formula for the exponential map to the product integral. The product integral generalizes the Riemann integral (for curves in Hausdorff locally convex

1 Throughout this paper, we work in the slightly more general setting introduced by Glöckner in [3]. Specifically, this means that any completeness presumption on the modeling space is dropped. Moreover, Milnor’s definition of an infinite-dimensional manifold \( M \) involves the requirement that \( M \) is a regular topological space, i.e., fulfills the separation axioms \( T_2, T_3 \). Deviating from that, in [3] only the \( T_2 \) property of \( M \) is explicitly assumed. This, however, makes no difference in the Lie group case, because topological groups are automatically \( T_3 \). We also refer to [21, 22] for introductions into this topic.

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vector spaces) to Lie groups, so that Lie algebra-valued curves are “integrated” to Lie group elements. The results are based on a deeper analysis of the Lax equation in the context of asymptotic estimate and sequentially complete Lie algebras. A further ingredient is an integral transformation that we introduce. This transformation maps continuous Lie algebra-valued curves to smooth ones and leaves the product integral invariant.

Explicitly, let $G$ be a Lie group in Milnor’s sense\(^1\) [3, 19] that is modeled over the Hausdorff locally convex vector space $E$. We denote the Lie algebra of $G$ by $(g, [\cdot, \cdot])$, the adjoint action by $\text{Ad}: G \times g \to g$, and set $\text{Ad}_g := \text{Ad}(g, \cdot)$ for each $g \in G$. The left and the right translation by some $g \in G$ is denoted by $L_g$ and $R_g$, respectively. For $a < b$, we set

$$C^1_\mu([a, b], G) := \{ \mu \in C^1([a, b], G) \mid \mu(a) = e \}.$$  

The evolution map is the inverse of the right logarithmic derivative\(^2\)

$$\delta^r : C^1_\mu([a, b], G) \to C^0([a, b], g), \quad \mu \mapsto d\mu R_{\mu^{-1}}(\dot{\mu}),$$  

and is denoted by

$$\text{Evol} : \bigcup_{a < b} \delta^r(C^1_\mu([a, b], G)) \to \bigcup_{a < b} C^1_\mu([a, b], G).$$  

The product integral is defined by

$$\int_a^b \phi := \text{Evol}(\phi|_{[a, b]})(b) \quad \text{for all } a < b \text{ and } \phi \in \mathcal{D} \text{ with } [a, b] \subseteq \text{dom}[\phi];$$

and for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, we set

$$\text{evol}_k : \mathcal{D} \cap C^k([0, 1], g) \ni \phi \mapsto \int_0^1 \phi \in G.$$  

We equip $C^k([0, 1], g)$ for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$ with the $C^k$-topology, as well as $\mathcal{D} \cap C^k([0, 1], g)$ with the corresponding subspace topology\(^3\):

- We say that $G$ is $C^k$-semiregular if $C^k([0, 1], g) \subseteq \mathcal{D}$ holds.
- We say that $G$ is weakly $C^k$-regular if $G$ is $C^k$-semiregular and $\text{evol}_k$ is differentiable.
- We say that $G$ is $C^k$-regular if $G$ is $C^k$-semiregular and $\text{evol}_k$ is smooth.

We briefly want to report on the continuity problem (under which circumstances is $\text{evol}_k$ continuous w.r.t. the $C^k$-topology):

- In [8], this problem had been solved for the case $k = 0$. Specifically, it was shown that $C^0$-continuity of $\text{evol}_0$ on its domain (also if $C^0$-semiregularity is not assumed) is equivalent to local $\mu$-convexity of $G$ [4, 8]. Roughly speaking, local $\mu$-convexity is a generalized triangle inequality for the Lie group multiplication.
- In [9], the continuity problem was completely solved in the asymptotic estimate context that we also consider in this paper (simply put, Theorem 1 in [9] states that all continuity notions are equivalent in the asymptotic estimate context).

A Lie group is said to be asymptotic estimate if its Lie algebra is asymptotic estimate. More generally, we have the following definitions for an infinite-dimensional Lie algebra $(q, [\cdot, \cdot])$\(^4\)

\(^1\) Injectivity of $\delta^r$ is easily verified, confer, e.g., Lemma 9 in [8].
\(^2\) The $C^k$-topology is Hausdorff locally convex (the defining seminorms are recalled in Sect. 2.2.1). In particular, differentiability (smoothness) of the evolution map has to be understood w.r.t. Bastiani’s differential calculus (recalled in Appendix A.1) in the following.
\(^3\) Specifically, $q$ is a Hausdorff locally convex vector space; and $[\cdot, \cdot] : q \times q \to q$ is bilinear, antisymmetric, continuous, and fulfills the Jacobi identity $[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]]$ for all $X, Y, Z \in q$.\(^4\)
A subset \( M \subseteq \mathfrak{g} \) is said to be an AE-set if for each \( v \in \text{Sem}(\mathfrak{g}) \), there exists \( v \leq w \in \text{Sem}(\mathfrak{g}) \) with
\[
v([X_1, [X_2, \ldots, [X_n, Y] \ldots]]) \leq w(X_1) \cdots w(X_n) \cdot w(Y)
\]
for all \( X_1, \ldots, X_n, Y \in M, \) and \( n \geq 1 \).

The Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) is said to be asymptotic estimate if \( \mathfrak{g} \) is an AE-set.

For instance, abelian Lie groups are asymptotic estimate, and the same is true for Lie groups with nilpotent Lie algebras. Also Banach Lie groups are asymptotic estimate, because their Lie bracket is submultiplicative. Notably, the class of asymptotic estimate Lie algebras has good permanence properties, as it is closed under passage to subalgebras, Hausdorff quotient Lie algebras, as well as closed under arbitrary Cartesian products (hence, e.g., under projective limits).

As already indicated above, Theorem 1 in [9] states that if \( G \) is asymptotic estimate, then \( G \) is locally \( \mu \)-convex if and only if \( \text{evol}_k \) is \( C^k \)-continuous for some (and then each) \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \). Based on the semiregularity results obtained in [8], this statement was used in [9] to prove that \( C^\infty \)-regularity implies \( C^k \)-regularity for each \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \), whereby for \( k = 0 \) additional completeness assumptions (on the Lie group not the Lie algebra) are required. Complementary to that, we prove the following theorem in the weakly regular context (cf. Theorem 2):

**Theorem 1** Assume that \( (\mathfrak{g}, [\cdot, \cdot]) \) is asymptotic estimate and sequentially complete. If \( G \) is weakly \( C^\infty \)-regular, then \( G \) is weakly \( C^k \)-regular for each \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \).

This theorem is obtained by a comprehensive analysis of the Lax equation in the sequentially complete asymptotic estimate context, as well as by application of an integral transformation that we introduce. As a by-product of these investigations, we obtain a generalization of the Baker–Campbell–Dynkin–Hausdorff formula (BCDH formula) in the Banach case (Proposition 4) as well as in the nilpotent weakly \( C^k \)-regular case (Theorem 1). In particular, this yields an explicit formula for the product integral in terms of the exponential map that involves iterated Riemann integrals over nested commutators (see Point (a)). More explicitly:

Given an infinite-dimensional Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) (not necessarily the Lie algebra of a Lie group), we denote its completion by \( \widehat{\mathfrak{g}} \) and set
\[
\text{ad}_Z: \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [Z, Y]
\]
for each \( Z \in \mathfrak{g} \). For \( \psi \in C^0([a, b], \mathfrak{g}) \) and \( X \in \mathfrak{g} \), we define the maps (\( \ell \geq 1 \))
\[
\lambda_{0, \psi}[X]: [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto X
\]
\[
\lambda_{\ell, \psi}[X]: [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{\ell-1}} ds_\ell (\text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_\ell)})(X),
\]
and set
\[
\lambda_{\ell, \psi}[t]: \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad X \mapsto \lambda_{\ell, \psi}[X](t) \quad \forall t \in [a, b], \quad \ell \in \mathbb{N}. \tag{1}
\]

We have the following results:

(a) Let \( G \) be a Banach Lie group such that the norm on \( \mathfrak{g} \) fulfills \( \|[X, Y]\| \leq \|X\| \cdot \|Y\| \) for all \( X, Y \in \mathfrak{g} \). Then, there exists \( \tau > 0 \) such that (Corollary 9 in Sect. 4.1)
\[
\int_a^t \phi = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_a^t \left( \sum_{\ell=1}^{\infty} \lambda_{\ell, \phi}[s] \right)^{n-1}(\phi(s)) \, ds \right) \quad \forall t \in [a, b] \tag{2}
\]
holds for all \( a < b \) and \( \phi \in C^0([a, b], \mathfrak{g}) \) with \( \int_a^b \|\phi(s)\| \, ds < \tau \).

The same formula is proven in the nilpotent weakly \( C^k \)-regular context (see Corollary 10 in Sect. 4.2.1), whereby the occurring sums are finite. For instance, if \( G \) is abelian and weakly \( C^k \)-regular for some \( k \in \mathbb{N} \cup \{\infty\} \), then (2) specializes to the well-known formula [18]

\[
\int_a^t \phi = \exp\left( \int_a^t \phi(s) \, ds \right) \quad \forall \ a \leq t \leq b, \ \phi \in C^k([a, b], \mathfrak{g}).
\]

Note that (2) provides an explicit formula for holonomies in principal bundles, as such holonomies are locally given by product integrals of curves that are pairings of (smooth) connections with derivatives of (smooth) curves in the base manifold of the principal bundle.

(b) The maps (1) are the elementary building blocks of solutions to the Lax equation

\[
\dot{\alpha} = [\psi, \alpha] \quad \text{with initial condition} \quad \alpha(a) = X,
\]

where \( X \in \mathfrak{q} \), \( \psi \in C^0([a, b], \mathfrak{q}) \) \( (a < b) \) are fixed parameters, and \( \alpha \in C^1([a, b], \mathfrak{q}) \) is the solution.

- Assume that \( (\mathfrak{q}, [\cdot, \cdot]) = (\mathfrak{g}, [\cdot, \cdot]) \) is the Lie algebra of a Lie group \( G \):
  - For \( \psi \in \mathcal{D}_{[a,b]} \), the unique solution to (3) is given by \( \alpha = \text{Ad}_{t \psi}^\phi(X) \) (cf. Corollary 7).
  - For \( \psi \in \mathcal{D}_{[a,b]} \) and \( \ell \geq 1 \), define

\[
\mathfrak{R}_{\ell, \psi}[X] : [a, b] \ni t \mapsto \int_a^t ds_1 \int_a^{s_1} ds_2 \ldots \int_a^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(s_1)} \circ \ldots \circ \text{ad}_{\psi(s_\ell)} \right)(\text{Ad}_{t \psi}^\phi(X)) \in \mathfrak{g}.
\]

A straightforward induction shows (cf. Lemma 13)

\[
\text{Ad}_{t \psi}^\phi(X) = \sum_{n=0}^\infty \lambda_{\ell, \psi}[X](t) + \mathfrak{R}_{n+1, \psi}[X](t) \quad \forall n \in \mathbb{N}, \ t \in [a, b].
\]

In particular, if \( M \) is an AE-set with \( \text{im}[\psi] \subseteq M \), then we have (cf. Corollary 8)

\[
\text{Ad}_{t \psi}^\phi(X) = \sum_{\ell=0}^\infty \lambda_{\ell, \psi}[X](t) \quad \forall X \in M, \ t \in [a, b].
\]

Notably, for \( \psi : [0, 1] \ni t \mapsto Y \in \mathfrak{g} \) constant, (4) reproduces the well-known formula

\[
\text{Ad}_{\exp(t \cdot Y)}(X) \overset{(48)}{=} \text{Ad}_{t \psi}^\phi(X) = \sum_{\ell=0}^\infty \frac{t^\ell}{\ell!} \cdot (\text{ad}_Y)^\ell(X) \quad \forall X \in \mathfrak{g}, \ t \in [0, 1].
\]

- Assume that \( (\mathfrak{q}, [\cdot, \cdot]) \) is sequentially complete and asymptotic estimate:
  Mimicking (4), for \( \psi \in C^0([a, b], \mathfrak{q}) \) and \( X \in \mathfrak{q} \), we define

\[
\text{Ad}_\psi[X] : [a, b] \to \mathfrak{q}, \quad t \mapsto \sum_{\ell=0}^\infty \lambda_{\ell, \psi}[X](t).
\]

- It is straightforward to see that \( \alpha := \text{Ad}_\psi[X] \) solves the Lax equation (3)—and is of class \( C^{k+1} \) if \( \psi \) is of class \( C^k \) for \( k \in \mathbb{N} \cup \{\infty\} \).
- It is ad hoc not clear that this solution is unique. To prove uniqueness, one basically has to show that for each \( t \in [a, b] \) one has

\[
\text{Aut}(\mathfrak{q}) \ni \text{Ad}_\psi[t] : q \to q, \quad X \mapsto \text{Ad}_\psi[X](t).
\]

Indeed, it is not hard to see that \( \text{Ad}_\psi[t] \) admits a left inverse; but, to prove the existence of a right inverse some effort is necessary.
In this paper, we use the solutions (5) to prove Theorem I. We combine them with an integral transformation that we introduce, and that is also used to prove Formula (2) in the nilpotent weakly $C^k$-regular situation. It naturally arises in the context of semiregular Lie groups:

- Let $G$ be $C^k$-semiregular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Then, the integral transformation
  $$\mathcal{T}: C^k([a, b], g) \to C^\infty([0, 1], g)$$
  $\phi \mapsto \left[0, 1] \ni t \mapsto \int_a^b \text{Ad}_{[t, \phi]}(\phi(s)) \, ds \right] \quad (6)$
  is defined for all $a < b$ (cf. Proposition 1) and has the invariance property
  $$\int_a^b \phi = \int_0^1 \mathcal{T}(\phi) \quad \forall \phi \in C^k([a, b], g).$$

- Let $(\mathfrak{q}, [\cdot, \cdot])$ be sequentially complete and asymptotic estimate. Motivated by (6), we consider the integral transformation (defined by Lemma 24)
  $$\mathcal{T}: C^0([a, b], \mathfrak{q}) \to C^\infty([0, 1], \mathfrak{q})$$
  $\phi \mapsto \left[0, 1] \ni t \mapsto \int_a^b \text{Ad}_{t,\phi([s, b])[b]}(\phi(s)) \, ds \right], \quad (7)$
  for which we have (Corollary 19)
  $$\int_a^b \mathcal{T}(\psi) = \int_c^b \mathcal{T}(\psi|_{[c, b]}) \cdot \int_a^c \mathcal{T}(\psi|_{[a, c]}) \quad \forall a < c < b, \psi \in C^0([a, b], \mathfrak{q}). \quad (8)$$

If $G$ is weakly $C^\infty$-semiregular and $\phi \in C^0([0, 1], \mathfrak{q})$, then
  $$\mu: [a, b] \ni z \mapsto \int_0^1 \mathcal{T}(\phi|_{[a, z]}) \in G$$
is of class $C^1$, whereby (8) implies $\delta^r(\mu) = \phi$ which establishes Theorem I.

This paper is organized as follows. In Sect. 2, we fix the notations and provide some basic facts and definitions concerning locally convex vector spaces, Lie groups, Lie algebras, and mappings. We furthermore discuss some elementary properties of power series that we need in the main text. Section 3 contains some preliminary results. We discuss the integral transformation (6), and provide an integral expansion for the adjoint action. We furthermore prove a differentiation result for the product integral in the weakly regular context. In Sect. 4, we generalize the Baker–Campbell–Dynkin–Hausdorff formula to the product integral and give some applications to it. In particular, we prove formula (2). In Sect. 5, we discuss the Lax equation for asymptotic estimate and sequentially complete Lie algebras. We also investigate the elementary properties of the integral transformation (7) that we finally use to prove Theorem I.

2 Preliminaries

In this section, we fix the notations and provide definitions and elementary facts concerning Lie groups, Lie algebras, and locally convex vector spaces that we shall need in the main text.

2.1 Conventions

Given sets $X, Y$, the set of all mappings $X \to Y$ is denoted by $\text{Maps}(X, Y) \equiv Y^X$. Let $Z$ be a topological space, and $S \subseteq Z$ a subset:
• We denote the closure of \( S \) in \( Z \) by \( \text{clos}_Z(S) \), or simply by \( \text{clos}(S) \) if it is clear from the context which topological space \( Z \) is meant.

• We denote the interior of \( S \) in \( Z \) by \( \text{int}_Z(S) \), or simply by \( \text{int}(S) \) if it is clear from the context which topological space \( Z \) is meant.

The class of Hausdorff locally convex vector spaces is denoted by \( \text{hlcVect} \). Given \( F \in \text{hlcVect} \), the system of continuous seminorms on \( F \) is denoted by \( \text{Sem}(F) \). We define

\[
B_{q,\epsilon} := \{ x \in E \mid q(x) < \epsilon \} \quad \text{and} \quad \overline{B}_{q,\epsilon} := \{ x \in E \mid q(x) \leq \epsilon \}
\]

for each \( q \in \text{Sem}(F) \) and \( \epsilon > 0 \). The completion of \( F \) is denoted by \( \hat{F} \in \text{hlcVect} \). For each \( q \in \text{Sem}(F) \), we let \( \overline{q} : \hat{F} \to [0, \infty) \) denote the continuous extension of \( q \) to \( \hat{F} \).

Given \( F_1, \ldots, F_n \in \text{hlcVect} \), we have \( F_1 \times \ldots \times F_n \in \text{hlcVect} \) via the Tychonoff topology generated by the seminorms

\[
\max[q_1, \ldots, q_n] : F_1 \times \ldots \times F_n \ni (x_1, \ldots, x_n) \mapsto \max\{q_k(x_k) \mid k = 1, \ldots, n\} \in [0, \infty)
\]

with \( q_k \in \text{Sem}(F_k) \) for \( k = 1, \ldots, n \). If \( \Phi : F_1 \times \ldots \times F_n \to F \) is a continuous \( n \)-multilinear map, then \( \hat{\Phi} : \hat{F}_1 \times \ldots \times \hat{F}_n \to \hat{F} \) denotes its continuous \( n \)-multilinear extension.

In this paper, manifolds and Lie groups are always understood to be in the sense of \([3]\), i.e., smooth, Hausdorff, and modeled over a Hausdorff locally convex vector space.\(^5\)

If \( f : M \to N \) is a \( C^1 \)-map between the manifolds \( M \) and \( N \), then \( df : TM \to TN \) denotes the corresponding tangent map between their tangent manifolds, and we write \( d_x f = df(x, \cdot) \). An interval is a non-empty, non-singleton connected subset \( D \subseteq \mathbb{R} \). A curve is a continuous map \( \gamma : D \to M \) for an interval and a manifold \( M \). If \( D \equiv I \) is open, then \( \gamma \) is said to be of class \( C^k \) if \( k \in \mathbb{N} \cup \{ \infty \} \) if it is of class \( C^k \) when considered as a map between the manifolds \( I \) and \( M \). If \( D \) is an arbitrary interval, then \( \gamma \) is said to be of class \( C^k \) if \( k \in \mathbb{N} \cup \{ \infty \} \) (we write \( \gamma \in C^k(D, M) \)) if \( \gamma = \gamma'|_D \) holds for a \( C^k \)-curve \( \gamma' : I \to M \) that is defined on an open interval \( I \) containing \( D \). If \( \gamma' : D \to M \) is of class \( C^1 \), then we denote the corresponding tangent vector at \( \gamma(t) \in M \) by \( \gamma'(t) \in T_{\gamma(t)}M \).

Throughout this paper, \( G \) denotes an infinite-dimensional Lie group in the sense of \([3]\) that is modeled over \( E \in \text{hlcVect} \). We fix a chart \( \Xi : G \supseteq \mathbb{U} \to \mathbb{V} \subseteq E \), with \( \mathbb{V} \) convex, \( e \in \mathbb{U} \), and \( \Xi(e) = 0 \). We denote the Lie algebra of \( G \) by \( (g, [\cdot, \cdot]) \), and set

\[
p(X) := (p \circ d_e \Xi)(X) \quad \forall p \in \text{Sem}(E), \ X \in g.
\]

We let \( m : G \times G \to G \) denote the Lie group multiplication, \( R_g := m(\cdot, g) \) the right translation by \( g \in G \), \( \text{inv} : G \ni g \mapsto g^{-1} \in G \) the inversion, and \( \text{Ad} : G \times g \to g \) the adjoint action. We have

\[
\text{Ad}(g, X) \equiv \text{Ad}_g(X) := d_e \text{Conj}_g(X) \quad \text{for} \quad \text{Conj}_g : G \ni h \mapsto g \cdot h \cdot g^{-1} \in G
\]

for each \( g \in G \) and \( X \in g \), as well as

\[
[X, Y] := d_e \text{Ad}(Y)(X) \quad \forall X \in g \quad \text{for} \quad \text{Ad}(Y) : G \ni g \mapsto \text{Ad}_g(Y) \in g.
\]

\(^5\) We explicitly refer to Definition 3.1 and Definition 3.3 in \([3]\). A review of the corresponding differential calculus is provided in Appendix A.1.

\(^6\) In other words, we equip \( g \) with the Hausdorff locally convex topology that is generated by the system of seminorms \( \{ p \circ d_e \Xi \mid p \in \text{Sem}(E) \} \). It is then not hard to see that \( \text{Sem}(g) = \{ p \circ d_e \Xi \mid p \in \text{Sem}(E) \} \) holds, hence \( \text{Sem}(g) \equiv \text{Sem}(E) \).
We recall the Jacobi identity (cf. also Example 5)

\[ [Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] \quad \forall X, Y, Z \in g \]  
(10)

as well as the product rule

\[ d_{(g, h)}(v, w) = d_g R_h(v) + d_h L_g(w) \quad \forall g, h \in G, \ v \in T_g G, \ w \in T_h G. \]  
(11)

### 2.2 Locally convex vector spaces

In this section, we collect some elementary facts and definitions concerning locally convex vector spaces that we shall need in the main text.

#### 2.2.1 Sets of curves

Let \( F \in \text{hlcVect} \) be given. For each \( X \in F \), we let \( \mathcal{E}_X : \mathbb{R} \ni t \mapsto X \in F \) denote the constant curve \( X \). Let \( a < b \) be given. We set \( C^c([a, b], F) := \{ \mathcal{E}_X|_{[a, b]} | X \in F \}; \) and let \( C^{\text{lip}}([a, b], F) \) denote the set of all Lipschitz curves on \([a, b]\), i.e., all curves \( \gamma : [a, b] \to F \) with

\[ \mathcal{L}(q, \gamma) := \sup \left\{ \frac{|q(\gamma(t) - \gamma(t'))|}{t - t'} \bigg| t, t' \in [a, b], \ t \neq t' \right\} \in [0, \infty) \quad \forall q \in \text{Sem}(F). \]

We define \( c + 1 := \infty, \infty + 1 := \infty, \text{lip} + 1 := 1; \) as well as

\[
\begin{align*}
q^{s}_{\infty}(\gamma) &:= \sup \left\{ q(\gamma^{(t)}(t)) \bigg| 0 \leq \ell \leq s, \ t \in [a, b] \right\} \quad \forall \gamma \in C^c([a, b], F) \\
q^{\infty}(\gamma) &:= q^{0}_{\infty}(\gamma) \quad \forall \gamma \in C^c([a, b], F) \\
q^{\text{lip}}_{\infty}(\gamma) &:= \max(q^{s}_{\infty}(\gamma), \mathcal{L}(q, \gamma)) \quad \forall \gamma \in C^{\text{lip}}([a, b], F)
\end{align*}
\]

for \( q \in \text{Sem}(F), k \in \mathbb{N} \cup \{\infty, c\}, 0 \leq s \leq k \). Here, the notation \( s \leq k \) means

- \( s = \text{lip} \) for \( k = \text{lip} \),
- \( \mathbb{N} \ni s \leq k \) for \( k \in \mathbb{N} \),
- \( s \in \mathbb{N} \) for \( k = \infty \),
- \( s = 0 \) for \( k = c \).

Given \( k \in \mathbb{N} \cup \{\text{lip}, \infty, c\} \), the \( C^k \)-topology on \( C^k([a, b], F) \) is the Hausdorff locally convex topology that is generated by the seminorms \( q^{s}_{\infty} \) for \( q \in \text{Sem}(F) \) and \( 0 \leq s \leq k \).

By an element in \( C^{\text{p}0}([a, b], F) \) (piecewise continuous curves), we understand a decomposition \( a = t_0 < \ldots < t_n = b \) for \( n \geq 1 \) together with a collection \( \gamma \equiv \{ \gamma[p] \}_{0 \leq p \leq n-1} \) of curves \( \gamma[p] \in C^0([t_p, t_{p+1}], F) \) for \( 0 \leq p \leq n - 1 \). Linear combinations as well as restrictions to compact intervals of elements in \( C^{\text{p}0}([a, b], F) \) are defined in the obvious way. We equip \( C^{\text{p}0}([a, b], F) \) with the \( C^0 \)-topology, i.e., the Hausdorff locally convex topology that is generated by the seminorms

\[ q^{\infty}(\gamma) = \max\{ q^{\infty}(\gamma[p]) \big| 0 \leq p \leq n - 1 \} \]

for \( q \in \text{Sem}(F) \), and \( \gamma \equiv \{ \gamma[p] \}_{0 \leq p \leq n-1} \) as above.

#### 2.2.2 Mackey convergence

Let \( F \in \text{hlcVect} \). A subsystem \( \mathcal{F} \subseteq \text{Sem}(F) \) is said to be a fundamental system if \( \{ B_{h, \varepsilon}(0) \}_{h \in \mathcal{F}, \varepsilon > 0} \) is a local base of zero in \( F \). We recall the following standard result \[ 17\]:
Lemma 1 Let $\mathcal{S} \subseteq \text{Sem}(F)$ be a fundamental system, and $\mathcal{G} \subseteq \text{Sem}(F)$ a subsystem. Then, $\mathcal{G}$ is a fundamental system if and only if to each $h \in \mathcal{G}$ there exist $c > 0$ and $s \in \mathcal{G}$ with $h \leq c \cdot s$.

Proof Confer, e.g., Lemma 20 in [11].

Let $\mathcal{S} \subseteq \text{Sem}(F)$ be a fundamental system. We write $\{X_n\}_{n \in \mathbb{N}} \rightarrow_X X$ for $\{X_n\}_{n \in \mathbb{N}} \subseteq F$, $X \in F$ if

$$h(X - X_n) \leq c_h \cdot \lambda_n \quad \forall n \in \mathbb{N}, \ h \in \mathcal{S}$$

holds for certain $\{c_h\}_{h \in \mathcal{S}} \subseteq \mathbb{R}_{\geq 0}$, and $\mathbb{R}_{> 0} \supseteq \{\lambda_n\}_{n \in \mathbb{N}} \rightarrow 0$. In this case, $\{X_n\}_{n \in \mathbb{N}}$ is said to be Mackey convergent to $X$.

Remark 1 • It is immediate from Lemma 1 that the definition made in (12) does not depend on the explicit choice of the fundamental system $\mathcal{S} \subseteq \text{Sem}(F)$.
• Since we will make use of the differentiability results obtained in [8], we explicitly mention that in [8] an equivalent definition of Mackey convergence (more suitable for the technical argumentation there) was used. Specifically, (12) is equivalent to require that

$$h(X - X_n) \leq c_h \cdot \lambda_n \quad \forall n \geq l_h, \ h \in \mathcal{S}$$

holds for certain $\{c_h\}_{h \in \mathcal{S}} \subseteq \mathbb{R}_{\geq 0}$, and $\mathbb{R}_{> 0} \supseteq \{\lambda_n\}_{n \in \mathbb{N}} \rightarrow 0$.
In other words, the indices $\{l_h\}_{h \in \mathcal{S}}$ in (13) can be circumvented (can be set equal to zero) if $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ instead of $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_{> 0}$ is required. □

Assume now $F = C^k([a, b], g)$ for $k \in \mathbb{N} \cup \{\text{lip}, \infty, c\}$ and $a < b$. We write

$$\{\phi_n\}_{n \in \mathbb{N}} \rightarrow_m C^k([a, b], g)$$

if $\{\phi_n\}_{n \in \mathbb{N}}$ is Mackey convergent to $\phi$ w.r.t. to the $C^k$-topology.

2.2.3 The Riemann integral and completeness

Let $F \in \text{hlcVect}$. The Riemann integral of $\gamma \in C^0([a, b], F)$ for $a < b$ is denoted by $\int \gamma(s) \, ds \in \widehat{F}$.

Remark 2 The Riemann integral in complete Hausdorff locally convex vector spaces can be defined exactly as in the finite-dimensional case; namely, as the limit of Riemann sums (which form a Cauchy sequence in $F$), confer Sect. 2 in [16]. In particular, the following assertions hold:
• If $F$ is sequentially complete, then $\int \gamma(s) \, ds \in F$ holds for $\gamma \in C^0([a, b], F)$.
• For $\gamma \in C^0([a, b], F)$ and $\gamma_- : [a, b] \ni t \mapsto \gamma(a + b - t)$, we have $\int \gamma_-(s) \, ds = \int \gamma(s) \, ds$.

Given $x < y$ and $\gamma \in C^0([x, y], F)$, then for $x \leq a < b \leq y$ and $x \leq c \leq y$, we set

$$\int_a^b \gamma(s) \, ds := \int \gamma|_{[a,b]}(s) \, ds, \quad \int_a^d \gamma(s) \, ds := -\int_a^b \gamma(s) \, ds, \quad \int_a^c \gamma(s) \, ds := 0.$$ (15)

Clearly, the Riemann integral is linear with

$$\int_a^c \gamma(s) \, ds = \int_a^b \gamma(s) \, ds + \int_b^c \gamma(s) \, ds \quad \forall x \leq a < b < c \leq y.$$ (16)

The Riemann integral admits the following additional properties:
For $a < b$, we have
\begin{align}
\gamma - \gamma(a) &= \int_a^b \gamma(s) \, ds \quad \forall \gamma \in C^1([a, b], F), \quad (17) \\
q(\gamma - \gamma(a)) &\leq \int_a^b q(\gamma(s)) \, ds \quad \forall \gamma \in C^1([a, b], F), \quad q \in \text{Sem}(F), \quad (18) \\
\hat{q}\left(\int_a^b \gamma(s) \, ds\right) &\leq \int_a^b q(\gamma(s)) \, ds \quad \forall \gamma \in C^0([a, b], F), \quad q \in \text{Sem}(F). \quad (19)
\end{align}

For $a < b$ and $\gamma \in C^0([a, b], F)$, we have
\begin{equation}
C^1([a, b], \hat{F}) \ni \Gamma: [a, b] \ni t \mapsto \int_a^t \gamma(s) \, ds \in \hat{F} \quad \text{with} \quad \hat{\gamma} = \gamma.
\end{equation}

For $a < b$, $a' < b'$, $\gamma \in C^1([a, b], F)$, $\varphi: [a', b'] \rightarrow [a, b]$ of class $C^1$, and $t \in [a', b']$, we have the substitution formula
\begin{equation}
\int_{a'}^{b'} \gamma(s) \, ds = \int_a^{\varphi(b')} \hat{\varphi}(s) \cdot (\gamma \circ \varphi)(s) \, ds + \int_a^{\varphi(a')} \hat{\varphi}(s) \cdot (\gamma \circ \varphi)(s) \, ds.
\end{equation}

Let $\tilde{F} \in hlc\text{Vect}$, $a < b$, and $L: F \rightarrow \tilde{F}$ a continuous and linear map. Then, the following implication holds:
\begin{equation}
\int_a^b \gamma(s) \, ds \in F \quad \text{for} \quad \gamma \in C^0([a, b], F) \\
\quad \Rightarrow \quad L\left(\int_a^b \gamma(s) \, ds\right) = \int_a^b L(\gamma(s)) \, ds.
\end{equation}

Next, let us recall the following definitions:
\begin{itemize}
\item We say that $F$ is Mackey complete if $\int_0^1 \gamma(s) \, ds \in F$ exists for each $\gamma \in C^\infty([0, 1], F)$. This is equivalent to require that $\int_a^b \gamma(s) \, ds \in F$ exists for each $\gamma \in C^\text{lip}([a, b], F)$ and $a < b$, confer Theorem 2.14 in [16] or Corollary 6 in [8].
\item We say that $F$ is integral complete if $\int_a^b \gamma(s) \, ds \in F$ exists for all $a < b$ and $\gamma \in C^0([a, b], F)$.
\end{itemize}

Evidently, $F$ is Mackey complete if $F$ is integral complete. Moreover, Remark 2 implies that $F$ is integral complete (hence, Mackey complete) if $F$ is sequentially complete.

Finally, for $\gamma \equiv \{\gamma[p]\}_{0 < p \leq n-1} \in \text{CP}^0([a, b], F)$ ($n \geq 1$), we define
\begin{equation}
\int_a^b \gamma(s) \, ds := \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} \gamma[p](s) \, ds.
\end{equation}

Clearly, the so-extended Riemann integral is linear, and $C^0$-continuous—specifically, (19) yields
\begin{equation}
\hat{q}\left(\int_a^t \gamma(s) \, ds\right) \leq (t - a) \cdot q_\infty(\gamma) \quad \forall a < t \leq b, \quad q \in \text{Sem}(F).
\end{equation}

For $x < y$ and $\gamma \in \text{CP}^0([x, y], F)$, we define $\int_a^b \gamma(s) \, ds$, $\int_a^b \gamma(s) \, ds$, $\int_c^b \gamma(s) \, ds$ for $x \leq a < b \leq y$ and $x \leq c \leq y$ as in (15). Clearly, then (16) also holds for the extended Riemann integral. We furthermore note that $[a, b] \ni t \mapsto \int_a^t \gamma(s) \, ds \in F$ is continuous for each $x \leq a < b \leq y$, as well as of class $C^1$ if $[a, b] \subseteq [t_p, t_{p+1}]$ holds for some $0 \leq p \leq n - 1$.

### 2.2.4 Extensions and completeness

Let $F \in hlc\text{Vect}$ be given. In this subsection, we collect some statements that we shall need in Sect. 5. We recall the following result.\(^7\)

\(^7\) The statement is due to a construction that goes back to Seeley [25]. We also refer to Proposition 24.10 in [16] for the case $k = \infty$ in the convenient setting.
Lemma 2 Let $D \subseteq \mathbb{R}$ be an interval, $k \in \mathbb{N} \cup \{\infty\}$, and $\gamma \in C^k(J, F)$ for $J := \text{int}_\mathbb{R}(D)$. Let $\{\gamma\}_{0 \leq \ell \leq k} \subseteq C^0(D, F)$ be given such that $\gamma^{(\ell)} = \gamma^{(\ell)}|_J$ holds for all $0 \leq \ell \leq k$. Then, there exists an open interval $I \subseteq \mathbb{R}$ with $D \subseteq I$ as well as $\tilde{\gamma} \in C^k(I, F)$ such that $\tilde{\gamma}|_D = \gamma$ holds.

Proof This is immediate from Theorem 1 in [12]. □

Lemma 3 If $F$ is sequentially complete, then $C^k([a, b], F)$ is sequentially complete w.r.t. the $C^k$-topology for each $k \in \mathbb{N} \cup \{\infty\}$ and $a < b$.

Proof Let $k \in \mathbb{N} \cup \{\infty\}$ be fixed; and let $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq C^k([a, b], F)$ be a Cauchy sequence w.r.t. the $C^k$-topology. Then, $\{(\gamma^{(\ell)}_n)\}_{n \in \mathbb{N}} \subseteq C^0([a, b], F)$ is a Cauchy sequence w.r.t. the $C^0$-topology for $0 \leq \ell \leq k$, and a standard $\varepsilon$-$\delta$-argument shows that it converges w.r.t. the $C^0$-topology to some $\gamma^{(\ell)} \in C^0([a, b], F)$. Let $\gamma := \gamma_0|_{(a, b)}$. By Lemma 2, it suffices to show

$$\gamma \in C^k((a, b), F) \quad \text{with} \quad \gamma^{(\ell)} = \gamma^{(\ell)}|_{(a, b)} \quad \text{for} \quad 0 \leq \ell \leq k. \quad (24)$$

Since the Riemann integral is $C^0$-continuous by (19), we have for $0 \leq \ell < k$ and $t \in [a, b]$ $\gamma^{(\ell)}(t) = \lim_n (\gamma_n^{(\ell)}(t)) = \lim_n ((\gamma_n^{(\ell)}(a) + \int_a^t (\gamma_n^{(\ell+1)}(s)) \, ds) = \gamma^{(\ell)}(a) + \int_a^t \gamma^{(\ell+1)}(s) \, ds.$

Then, (20) shows $\gamma^{(\ell)}_{\ell+1}|(a, b) \in C^1((a, b), F)$ with $\gamma^{(\ell+1)}_{\ell+1}|(a, b)$ for $0 \leq \ell < k$. Now, since $\gamma = \gamma_0|_{(a, b)}$ holds by definition, we can assume that there exists some $0 \leq q < k$ such that (24) holds for $k = q$. We obtain

$$\gamma^{(q+1)}|_{(a, b)} = (\gamma^{(q)}|_{(a, b)})^{(1)} = (\gamma^{(q+1)})^{(1)} = (\gamma^{(q)})^{(1)} = \gamma^{(q+1)},$$

hence $\gamma \in C^{q+1}((a, b), F)$. □

Corollary 1 Assume that $F$ is sequentially complete, and let $\mathcal{S} \subseteq \text{Sem}(F)$ be a fundamental system. Let $k \in \mathbb{N} \cup \{\infty\}$, $a < b$, and $\{\gamma_p\}_{p \in \mathbb{N}} \subseteq C^k([a, b], F)$ be given with

$$\sum_{p=0}^\infty q^k_p \gamma_p < \infty \quad \forall \ q \in \mathcal{S}, \quad 0 \leq s \leq k.$$ 

Then, $\gamma := \sum_{p=0}^\infty \gamma_p$ converges w.r.t. the $C^0$-topology, i.e., $\gamma^{(\ell)} = \sum_{p=0}^\infty (\gamma_p)^{(\ell)}$ converges w.r.t. the $C^0$-topology for each $0 \leq \ell \leq k$.

Proof The assumptions imply that $\{(\sum_{p=0}^n \gamma_p)\}_{n \in \mathbb{N}} \subseteq C^k([a, b], F)$ is Cauchy w.r.t. the $C^k$-topology. The rest is clear from Lemma 3. □

Corollary 2 Assume that $F$ is sequentially complete, and let $\mathcal{S} \subseteq \text{Sem}(F)$ be a fundamental system. Let $\{X_p\}_{p \in \mathbb{N}} \subseteq F$ and $\delta > 0$ be given, such that

$$\gamma[q]: (-\delta, \delta) \ni t \mapsto \sum_{p=0}^\infty t^p \cdot q(X_p) \in [0, \infty)$$

is defined for each $q \in \mathcal{S}$. Define $\gamma_p: (-\delta, \delta) \ni t \mapsto t^p \cdot X_p \in F$ for each $p \in \mathbb{N}$.

(a) Let $-\delta < a < b < \delta$. Then, $\gamma[a, b] := \sum_{p=0}^\infty \gamma_p|_{[a, b]} \in C^\infty([a, b], F)$ converges w.r.t. the $C^\infty$-topology, with

$$\gamma[a, b]^{(\ell)} = \sum_{p=0}^\infty (\gamma_p|_{[a, b]})^{(\ell)}: [a, b] \ni t \mapsto \sum_{p=0}^\infty \frac{p!}{(p-\ell)!} \cdot t^{p-\ell} \cdot X_p \in F \quad \forall \ \ell \in \mathbb{N}.$$ 

(b) We have $\gamma := \sum_{p=0}^\infty \gamma_p \in C^\infty((-\delta, \delta), F)$ with

$$\gamma^{(\ell)} = \sum_{p=0}^\infty (\gamma_p)^{(\ell)}: (-\delta, \delta) \ni t \mapsto \sum_{p=0}^\infty \frac{p!}{(p-\ell)!} \cdot t^{p-\ell} \cdot X_p \in F \quad \forall \ \ell \in \mathbb{N}.$$

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Proof Point b) is clear from Point a). To prove Point a), we observe that \( \gamma[q] \) is smooth for each \( q \in \mathcal{F} \), with
\[
\gamma[q]^{(\ell)}(t) = \sum_{p=\ell}^{\infty} \frac{p!}{(p-\ell)!} \cdot t^{p-\ell} \cdot q(X_p) \in [0, \infty) \quad \forall t \in (-\delta, \delta), \; \ell \in \mathbb{N}.
\]
This implies \( \sum_{p=\ell}^{\infty} q^s(\gamma_p|_{[a,b]}) < \infty \) for all \( q \in \mathcal{F} \), \( s \in \mathbb{N} \), \( a < b \), so that the claim is clear from Corollary 1. \( \square \)

2.2.5 Lie Algebras

Let \( (q, [\cdot, \cdot]) \) be a fixed Lie algebra (not necessarily the Lie algebra of a Lie group), i.e.,
- \( q \in \text{hlcVect} \),
- \( [\cdot, \cdot] : q \times q \to q \) is bilinear, antisymmetric, and continuous, with (Jacobi identity)
\[
[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] \quad \forall X, Y, Z \in q. \tag{25}
\]
We shall need the following definitions:
- Given \( X \in q \), we denote \( \text{ad}^1_X \equiv \text{ad}_X : q \ni Y \mapsto [X, Y] \in q \), \(^8\) and define inductively
  \[
  \text{ad}^0_X := \text{id}_q \quad \text{as well as} \quad \text{ad}^n_X := \text{ad}_X \circ \text{ad}^{n-1}_X \quad \forall n \geq 1.
  \]
- Given a subset \( S \subseteq q \), then \( (S) \subseteq q \) denotes the linear subspace that is generated by \( S \). We define inductively
  \[
  \mathcal{V}_1(S) := (S) \quad \text{as well as} \quad \mathcal{V}_{n+1}(S) := ([\mathcal{V}_n(S), \mathcal{V}_n(S)]) \quad \forall n \geq 1;
  \]
  and set \( \mathcal{G}_n(S) := (\bigcup_{\ell=0}^{\infty} \mathcal{V}_\ell(S)) \) for each \( n \geq 1 \).

A straightforward induction involving (25) shows (confer Appendix A.2 or Exercise 5.2.7 in [13])
\[
([\mathcal{V}_m(S), \mathcal{V}_n(S)]) \subseteq \mathcal{V}_{m+n}(S) \quad \forall m, n \geq 1. \tag{26}
\]
For \( n \geq 1 \), we set \( \overline{\mathcal{V}}_n := \text{clos}_q(\mathcal{V}_n) \) as well as \( \overline{\mathcal{G}}_n := \text{clos}_q(\mathcal{G}_n) \). Since \( [\cdot, \cdot] \) is continuous, (26) implies
\[
([\overline{\mathcal{V}}_m(S), \overline{\mathcal{V}}_n(S)]) \subseteq \overline{\mathcal{V}}_{m+n}(S) \quad \forall m, n \geq 1. \tag{27}
\]
We observe that \( \overline{\mathcal{G}}_{n+\ell}(S) \subseteq \overline{\mathcal{G}}_n(S) \) holds for all \( n, \ell \in \mathbb{N} \). We shall need the following definitions:
- A subset \( M \subseteq q \) is said to be an AE-set if to each \( v \in \text{Sem}(q) \), there exist \( v \leq w \in \text{Sem}(q) \) with
  \[
  v([X_1, [X_2, \ldots, [X_n, Y], \ldots]]) \leq w(X_1) \cdot \ldots \cdot w(X_n) \cdot w(Y) \quad \tag{28}
  \]
  for all \( X_1, \ldots, X_n, Y \in M \), and \( n \geq 1 \). We say that \( (q, [\cdot, \cdot]) \) is asymptotic estimate if \( q \) is an AE-set.

---

\(^8\) We note that if \( (q, [\cdot, \cdot]) \) is the Lie algebra of a Lie group \( Q \), then one usually defines \( \text{ad}_X : q \ni Y \mapsto d_0\text{Ad}(Y)(X) \in q \) for \( X \in g \). According to our definition (9) of the commutator, this is consistent with our notation in the general case (i.e., where \( (q, [\cdot, \cdot]) \) is not necessarily the Lie algebra of a Lie group).

\(^9\) Note that the first definition is recalled from Sect. 1.
- A subset \( N \subseteq q \) is said to be a Nil\(_q\)-set for \( q \geq 2 \) if

\[
[X_1, [X_2, \ldots, [X_{q-1}, X_q] \ldots]] = 0 \quad \forall X_1, \ldots, X_q \in N.
\]

Evidently, each Nil\(_q\)-set (for \( q \geq 2 \)) is an AE-set. We say that \( (q, [\cdot, \cdot]) \) is nilpotent if \( q \) is a Nil\(_q\)-set for some \( q \geq 2 \).

In view of Sect. 4.2 (the proof of Lemma 16), we observe the following.

**Remark 3** Let \( N \subseteq q \) be a Nil\(_q\)-set for some \( q \geq 2 \), then,

1. \( \overline{N}_{q+n}(N) = \{0\} \) holds for each \( n \in \mathbb{N} \).
2. \([\overline{N}_m(N), \overline{N}_n(N)] \subseteq \overline{N}_{m+n}(N)\) holds for \( m, n \geq 1 \), by (27).
3. \( \overline{N}_n(N) \) is a Nil\(_q\)-set for each \( n \geq 1 \), by the previous points 1) and 2).
4. Let \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \) be given; and assume that \( q \) is Mackey complete for \( k \in \mathbb{N}_{\geq 1} \cup \{\text{lip}, \infty\} \), as well as integral complete for \( k \equiv 0 \). Then, for \( n \in \{1, \ldots, q\} \) and \( \phi \in C^k([a, b], q) \) with \( \text{im}[\phi] \subseteq \overline{N}_n(N) \), we have

\[
\int_s^t \phi(s) \, ds \in \overline{N}_n(N) \quad \forall a \leq s < t \leq b.
\]

\[\Box\]

### 2.2.6 Some properties of maps

Let \( F_1, \ldots, F_n, F, E \in \text{hlcVect} \) be given.

**Lemma 4** Let \( X \) be a topological space; and let \( \Phi : X \times F_1 \times \ldots \times F_n \to F \) be continuous, such that \( \Phi(x, \cdot) \) is \( n \)-multilinear for each \( x \in X \). Then, to each compact \( K \subseteq X \) and each \( q \in \text{Sem}(F) \), there exist \( q_1 \in \text{Sem}(F_1), \ldots, q_n \in \text{Sem}(F_n) \) as well as \( O \subseteq X \) open with \( K \subseteq O \), such that

\[
(q \circ \Phi)(y, X_1, \ldots, X_n) \leq q_1(X_1) \cdots q_n(X_n) \quad \forall y \in O, \; X_1 \in F_1, \ldots, X_n \in F_n.
\]

**Proof** Confer, e.g., Corollary 1 in [8]. \[\Box\]

**Lemma 5** Let \( V \subseteq F \) be open with \( 0 \in V \). Let furthermore \( \Psi : V \times E \to E \) be smooth with \( \Psi(0, \cdot) = \text{id}_E \), such that \( \Psi(x, \cdot) \) is linear for each \( x \in V \). Then, to each \( p \in \text{Sem}(E) \), there exist \( q \in \text{Sem}(F) \) and \( w \in \text{Sem}(E) \) with

\[
p(\Psi(x, Y) - Y) \leq q(x) \cdot w(Y) \quad \forall x \in B_{q, 1} \subseteq V, \; Y \in E.
\]

**Proof** Confer Appendix A.3. \[\Box\]

**Lemma 6** Let \( F_1, F_2 \in \text{hlcVect} \), and \( f : F_1 \supseteq U \to F_2 \) of class \( C^2 \). Assume that \( \gamma : D \to F_1 \) is continuous at \( t \in D \), such that \( \lim_{h \to 0} 1/h \cdot (\gamma(t + h) - \gamma(t)) =: X \in \overline{F}_1 \) exists. Then, we have

\[
\lim_{h \to 0} 1/h \cdot (f(\gamma(t + h)) - f(\gamma(t))) = d_{\gamma(t)}f(X).
\]

**Proof** Confer, e.g., Lemma 7 in [8]. \[\Box\]

We close this subsection with the following convention concerning differentiable maps with values in Lie groups.

**Convention 1** Let \( F \in \text{hlcVect} \), \( U \subseteq F \), \( G \) a Lie group modeled over \( E \in \text{hlcVect} \). A map \( f : U \to G \) is said to be
differentiable at \( x \in U \) if there exists a chart \((\Xi', \mathcal{U}')\) of \( G \) with \( f(x) \in \mathcal{U}' \), such that
\[
(D_v^{\mathcal{U}'})f)(x) := \lim_{h \to 0} 1/h \cdot ((\Xi' \circ f)(x + h \cdot v) - (\Xi' \circ f)(x)) \in E \tag{30}
\]
events for all \( v \in F \). Lemma 6 applied to coordinate changes shows that (30) holds for one chart around \( f(x) \) if and only if it holds for each chart around \( f(x) \), and that
\[
d_x f(v) := (d_{\mathcal{U}'(f(x))}^{\mathcal{U}'})^{-1} \circ (D_v^{\mathcal{U}'})f)(x) \in T_{f(x)}G \quad \forall v \in F
\]
is independent of the explicit choice of \((\Xi', \mathcal{U}')\).

2.3 Power series

In this subsection, we collect some statements concerning power series in (Banach) algebras that we shall need to work informally in Sect. 4.1. We set
\[
U_{\varepsilon}(z) := \{ w \in \mathbb{C} \mid |w - z| < \varepsilon \} \quad \forall \varepsilon > 0, z \in \mathbb{C},
\]
and let \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) be fixed.

- Let \((N, *, N, 1_N)\) be a unital \(\mathbb{K}\)-algebra, and let \(N_q := \{ n \in N \mid n^q + 1 = 0 \}\) for \( q \geq 1 \). Set furthermore \( n^0 := 1_N \) and \( n^1 := n \) for each \( n \in N \).
- Let \((A, *_A, 1_A, \| \cdot \|_A)\) be a unital submultiplicative Banach algebra over \( \mathbb{K} \), and set \( a^0 := 1_A \) as well as \( a^1 := a \) for each \( a \in A \).

Let \( f : U_R(0) \ni z \mapsto \sum_{n=0}^{\infty} a_n \cdot z^n \in \mathbb{C} \) for \( \{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{K} \) be a power series with radius of convergence \( R > 0 \). We define
\[
|f|_r := \sum_{n=0}^{\infty} |a_n| \cdot r^n \in [0, \infty) \quad \forall 0 \leq r < R,
\]
and set \( f_p : U_R(0) \ni z \mapsto \sum_{n=0}^{p} a_n \cdot z^n \in \mathbb{C} \) for each \( p \in \mathbb{N} \). We furthermore let
\[
f(n) := \sum_{n=0}^{q} a_n \cdot n^q \in N \quad \forall n \in N_q \quad \text{with} \quad q \geq 1
\]
\[
f(a) := \sum_{n=0}^{\infty} a_n \cdot a^n \in A \quad \forall a \in A \quad \text{with} \quad \|a\|_A < R.
\]
Let \( g : U_S(0) \ni z \mapsto \sum_{n=0}^{\infty} b_n \cdot z^n \in \mathbb{C} \) be a power series with radius of convergence \( S > 0 \).

- Assume \( S = R \). The Cauchy product formula yields
\[
f * g : U_R \ni z \mapsto \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} a_{\ell} \cdot b_{n-\ell} \right) \cdot z^n = f(z) \cdot g(z) \in \mathbb{C}.
\]
\( \circ \) For \( n \in N_q \) with \( q \geq 1 \), we evidently have
\[
f(n) *_N g(n) = \sum_{n=0}^{q} \left( \sum_{\ell=0}^{n} a_{\ell} \cdot b_{n-\ell} \right) \cdot n^q = (f * g)(n) \in N.
\]
\( \circ \) For \( a \in A \) with \( \|a\|_A < R \), we obtain
\[
f(a) *_A g(a) = \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} a_{\ell} \cdot b_{n-\ell} \right) \cdot a^n = (f * g)(a) \in A.
\]
(Apply, e.g., Exercise 3.1.3 in [13], with \( X, Y, Z \equiv A \) and \( \beta \equiv *_A \) there.)
- Assume \( |g|_s = \sum_{n=0}^{\infty} |b_n| \cdot s^n < R \) for all \( 0 \leq s < S \), thus \( g(U_S) \subseteq U_R(0) \). By analyticity, we have
\[
f \circ g : U_S \ni z \mapsto \sum_{n=0}^{\infty} \frac{(f \circ g)^{(n)}(0)}{n!} \cdot z^n = f(g(z)) \in \mathbb{C}.
\]
The following statements are verified in Appendix A.4:
Assume $b_0 = 0$, and let $n \in \mathbb{N}_q$ with $q \geq 1$. We have $g(n) \in \mathbb{N}_q$ with

$$f(g(n)) = \sum_{n=0}^{q} c_n \cdot n^n = (f \circ g)(n) \in \mathbb{N}. \quad (31)$$

For $a \in A$ with $\|a\|_A < S$, we have

$$f(g(a)) = \sum_{n=0}^{\infty} c_n \cdot a^n = (f \circ g)(a) \in A. \quad (32)$$

**Example 1** Let $\mathbb{K} = \mathbb{R}$, and define the power series

$$f : (-1, 1) \ni t \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot t^{n-1} \in \mathbb{R} \quad \left(\frac{\ln(1+t)}{t}\right)$$

$$g : (-\ln(2), \ln(2)) \ni t \mapsto \sum_{n=1}^{\infty} \frac{1}{n!} \cdot t^{n} \in (-1, 1) \quad \left(\frac{e^t - 1}{t}\right)$$

$$h : (-1, 1) \ni t \mapsto \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \cdot t^{n} \in \mathbb{R}. \quad \left(\frac{e^t - 1}{t}\right)$$

In the context of the above notations, we have $b_0 = 0$, $R = 1$, $S = \ln(2)$, with $|g|_s < e^s - 1 = R$ for $0 \leq s < S$ as well as

$$((f \circ g) \ast h)(t) = f(g(t)) \cdot h(t) = 1 \quad \forall t \in (-\ln(2), \ln(2)). \quad (33)$$

We consider the power series

$$\tilde{f} : (-1, 1) \ni t \mapsto (1 + t) \cdot f(t) \in \mathbb{R}$$

$$\tilde{g} : (-\ln(2), \ln(2)) \ni t \mapsto g(-t) \in (-1, 1),$$

and obtain for $t \in (-\ln(2), \ln(2))$ that

$$\tilde{f} \circ \tilde{g}(t) = \tilde{f}(\tilde{g}(t)) = \tilde{f}(e^{-t} - 1) = e^{-t} \cdot \frac{e^{-t} - 1}{e^{-t} - 1} = f(g(t)) = (f \circ g)(t). \quad (34)$$

Let now $Z \in \mathfrak{g}$ be fixed, and assume that one of the following situations hold:

A) Let $\mathfrak{N} \subseteq \mathfrak{g}$ be a $\text{Nil}_q$-set for $q \geq 1$. We define $V := \mathfrak{G}_1(\mathfrak{N})$, and let $\text{End}(V)$ denote the set of all linear maps $V \rightarrow V$. Then, $(\mathfrak{N}, \ast_{\mathfrak{N}}, 1_{\mathfrak{N}}) \equiv (\text{End}(V), \circ, \text{id}_V)$ is a unital $\mathbb{R}$-algebra, with $\text{ad}_Z \in \text{End}_{\mathfrak{g}}(\mathfrak{N})$ for each $Z \in \mathfrak{N}$ by Remark 3.3.

B) $G$ is a Banach Lie group with

$$\|[X, Y]\| \leq \|X\| \cdot \|Y\| \quad \forall X, Y \in \mathfrak{g},$$

and we have $\|Z\| < \ln(2)$. We define $V := \mathfrak{g}$, and let $\text{End}^c(V)$ denote the set of all continuous linear maps $V \rightarrow V$. Then, $(\mathfrak{A}, \ast_{\mathfrak{A}}, 1_{\mathfrak{A}}, \|\|_{\mathfrak{A}}) \equiv (\text{End}^c(V), \circ, \text{id}_V, \|\|_{\text{op}})$ is a unital submultiplicative Banach algebra over $\mathbb{R}$, and we have $\text{ad}_Z \in \text{End}^c(V)$ with $\|\text{ad}_Z\|_{\text{op}} < \ln(2)$.

In both situations A) and B), the above discussions together with (33) and (34) show

$$f(g(\text{ad}_Z)) \circ h(\text{ad}_Z) = ((f \circ g) \ast h)(\text{ad}_Z) = \text{id}_V$$

$$\tilde{f}(\tilde{g}(\text{ad}_Z)) = f(g(\text{ad}_Z))$$

for $Z \in V$. This can be rewritten as

$$\tilde{f}(\tilde{g}(\text{ad}_Z)(Y)) = \tilde{f}(\Phi(\text{ad}_Z)(Y)) = Y \quad (35)$$

More specifically, $\tilde{f} = p \ast f$ for $p : (-1, 1) \ni t \mapsto (1 + t) \in \mathbb{R}$, and $\tilde{g} : (-\ln(2), \ln(2)) \ni t \mapsto \sum_{n=1}^{\infty} \frac{1}{n!} \cdot (-t)^n \in (-1, 1)$. 

\[ \square \] Springer
for each $Y \in V$, whereby for maps $\xi, \zeta : g \to g$ and $X \in g$ we set (convergence presumed)

$$
\Psi(\xi)(X) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (\xi - \text{id}_g)^{n-1}(X)
$$

$$
\widetilde{\Psi}(\xi)(X) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (\xi \circ (\xi - \text{id}_g)^{n-1})(X)
$$

$$
\Phi(\xi)(X) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \cdot \xi^n(X).
$$

Equation (35) will be relevant for our discussions in Sect. 4.1. $\square$

### 2.4 The evolution map

The subject of this section is the evolution map. We recall its elementary properties (confer also [8]), as well as the differentiability results obtained in [11]. We furthermore introduce the notion of weak $C^k$-regularity (confer Definition 1).

#### 2.4.1 Elementary definitions

The right logarithmic derivative is given by

$$
\delta^r : C^1(D, G) \to C^0(D, g), \quad \mu \mapsto d_{\mu}R_{\mu^{-1}}(\dot{\mu})
$$

for each interval $D \subseteq \mathbb{R}$. Notably, for $\mu \in C^1(D, G)$, $g \in G$, an interval $D' \subseteq D$, and $\varrho : D'' \to D$ of class $C^1$ for $D'' \subseteq \mathbb{R}$ an interval, we have

$$
\delta^r(\mu \cdot g) = \delta^r(\mu) \quad \delta^r(\mu|_{D'}) = \delta^r(\mu)|_{D'} \quad \delta^r(\mu \circ \varrho) = \dot{\varrho} \cdot (\delta^r(\mu) \circ \varrho). \quad (36)
$$

Moreover, for $\mu, v \in C^1(D, G)$, it follows from the product rule (11) that

$$
\delta^r(\mu \cdot v) = \delta^r(\mu) + \text{Ad}_{\mu}(\delta^r(v))
$$

holds. For $a < b$ and $k \in \mathbb{N} \cup \{\text{lip}, \infty, c\}$, we define

$$
\mathcal{D}_{[a,b]} := \delta^r(C^1([a,b], G)) \quad \text{as well as} \quad \mathcal{D}^k_{[a,b]} := \mathcal{D}_{[a,b]} \cap C^k([a,b], g).
$$

Now, $\delta^r$ restricted to the set

$$
C_*^1([a, b], G) := \{ \mu \in C^1([a, b], G) \mid \mu(a) = e \}
$$

is injective for $a < b$ (confer, e.g., Lemma 9 in [8]). We thus obtain a map

$$
\text{Evol} : \mathcal{D} := \bigcup_{a < b} \mathcal{D}_{[a,b]} \to \bigcup_{a < b} C_*^1([a, b], G),
$$

if for $a < b$ we define

$$
\text{Evol} : \mathcal{D}_{[a,b]} \to C_*^1([a, b], G), \quad \delta^r(\mu) \mapsto \mu \cdot \mu(a)^{-1}.
$$

Notably, for $a < b$ and $k \in \mathbb{N} \cup \{\text{lip}, \infty, c\}$, we have (confer, e.g., Lemma 10 in [8])

$$
\text{Evol} |_{\mathcal{D}^k_{[a,b]}} : \mathcal{D}^k_{[a,b]} \to C^{k+1}([a, b], G). \quad (37)
$$

Moreover, for $a < b$ and $\phi \in \mathcal{D}^k_{[a,b]}$, we have

$$
\phi|_{[a', b']} \in \mathcal{D}^k_{[a', b']} \quad \forall a \leq a' < b' \leq b
$$

by the second equality in (36) as well as Lemma 10 in [8].
Remark 4 Given $X \in \mathfrak{g}$, there exist $\varepsilon > 0$ and $\phi_X \in \mathcal{D} \mathcal{D}_{0, \varepsilon}^\infty$ with $\phi_X(0) = X$. In fact, fix $\varepsilon > 0$ with $(-2\varepsilon, 2\varepsilon) \cdot d_e \Xi(X) \subseteq \mathcal{V}$, and define

$$
\mu_X : (-\varepsilon, \varepsilon) \ni t \mapsto \Xi^{-1}(t \cdot d_e \Xi(X)) \in \mathcal{U}.
$$

Then, $\mu_X$ is of class $C^\infty$, and $\phi_X := \mathcal{D}^r(\mu_X|_{0, \varepsilon})$ has the desired properties. \hfill \Box

Example 2 (The Riemann Integral) Assume $(G, \cdot) \equiv (F, +)$ equals the additive group of some $F \in \text{hlcVect}$. We have $\mathcal{D}^r : C^1([a, b], F) \ni \gamma \mapsto \dot{\gamma} \in C^0([a, b], F)$ for each $a < b$, hence

$$
\mathcal{D}_{[a, b]} = \{ \gamma \in C^0([a, b], g) \mid \int_a^b \gamma(s) \, ds \in g \text{ for each } t \in [a, b] \}
$$

as well as Evol$(\gamma) : [a, b] \ni t \mapsto \int_a^t \gamma(s) \, ds$ for each $\gamma \in \mathcal{D}_{[a, b]}$. \hfill \Box

2.4.2 The product integral

The product integral is defined by

$$
\int_a^b \phi := \text{Evol}(\phi|_{[a, b]})(t) \in G \quad \forall [s, t] \subseteq \text{dom}(\phi), \phi \in \mathcal{D}.
$$

We set $\int \phi = \int_a^b \phi$ as well as $\int_c^a \phi := e$ for $a < b$, $\phi \in \mathcal{D}_{[a, b]}, c \in [a, b]$, and define

$$
ev_{\mathcal{D}_{[a, b]}}^k \equiv \int \mathcal{D}_{[a, b]}^k \quad \forall k \in \mathbb{N} \cup \{\text{lip}, \infty, c\}.
$$

We let $\text{evol}_k \equiv \text{evol}_{\mathcal{D}_{[0, 1]}^k}$ as well as $\mathcal{D}_k \equiv \mathcal{D}_{[0, 1]}^k$ for each $k \in \mathbb{N} \cup \{\text{lip}, \infty, c\}$. We furthermore let

$$
ev \equiv \text{evol}_0 : \mathcal{D} \equiv \mathcal{D}_0 \rightarrow G.
$$

The following elementary identities hold for $a < b$, confer [4, 15] or Sect. 3.5.2 in [8]:

(a) For each $\phi, \psi \in \mathcal{D}_{[a, b]}$, we have $\phi + \text{Ad}_{\mathcal{D}_{[a, b]}}(\psi) \in \mathcal{D}_{[a, b]}$ with

$$
\int_a^t \phi \cdot \int_a^t \psi = \int_a^t \phi + \text{Ad}_{\mathcal{D}_{[a, b]}}(\psi) \quad t \in [a, b].
$$

(b) For each $\phi, \psi \in \mathcal{D}_{[a, b]}$, we have $\text{Ad}_{\mathcal{D}_{[a, b]}}^{-1}(\psi - \phi) \in \mathcal{D}_{[a, b]}$ with

$$
[\int_a^t \phi]^{-1}\int_a^t \psi = \int_a^t \text{Ad}_{\mathcal{D}_{[a, b]}}^{-1}(\psi - \phi) \quad t \in [a, b].
$$

(c) For each $\phi \in \mathcal{D}_{[a, b]}$, we have $-\text{Ad}_{\mathcal{D}_{[a, b]}}^{-1}(\phi) \in \mathcal{D}_{[a, b]}$ with

$$
[\int_a^t \phi]^{-1} = \int_a^t -\text{Ad}_{\mathcal{D}_{[a, b]}}^{-1}(\phi) \quad t \in [a, b].
$$

(d) For $a = t_0 < \ldots < t_n = b$ and $\phi \in \mathcal{D}_{[a, b]}$, we have

$$
\int_{t_p}^{t_{p+1}} \phi = \int_{t_p}^{t_{p+1}} \phi \cdot \int_{t_p}^{t_{p-1}} \phi \cdot \ldots \cdot \int_{t_1}^a \phi \quad \forall t \in (t_p, t_{p+1}], \quad p = 0, \ldots, n - 1.
$$

(e) For $\varrho : [a', b'] \rightarrow [a, b]$ of class $C^1$ and $\phi \in \mathcal{D}_{[a, b]}$, we have $\dot{\varrho} \cdot (\phi \circ \varrho) \in \mathcal{D}_{[a', b']}$ with

$$
\int_{a'}^b \varrho(\dot{\varrho} \cdot (\phi \circ \varrho)) = \int_a^b \varrho(\dot{\varrho} \cdot (\phi \circ \varrho)) \cdot [\int_{a'}^b \varrho(\phi)] \phi.
$$

(f) For each homomorphism $\Psi : G \rightarrow H$ between Lie groups $G$ and $H$ that is of class $C^1$, we have

$$
\Psi \circ \int_a^b \phi = \int_a^b d_e \Psi \circ \phi \quad \forall \phi \in \mathcal{D}_{[a, b]}.
$$
Remark 5 Let $k \in \mathbb{N} \cup \{\operatorname{lip}, \infty\}$ and $a < b$ be given. We have by (a), (c), and Lemma 13 in [8] (confer also Lemma 8) that

\[
\psi^{-1} := -\operatorname{Ad}_{[a, b]}^k(\psi^{-1}) \in \mathcal{D}[a, b] \quad \text{with} \quad \int_a^b \psi^{-1} = [\int_a^b \psi]^{-1} \\
\phi \ast \psi := \phi + \operatorname{Ad}_{[a, b]}^k(\phi) \in \mathcal{D}[a, b] \quad \text{with} \quad \int_a^b \phi \ast \psi = [\int_a^b \phi - \int_a^b \psi]
\]

holds for each $\phi, \psi \in \mathcal{D}[a, b]$. It is then not hard to see that $(\mathcal{D}[a, b], \ast, \cdot, \cdot, \cdot, 0)$ is a group:

- We have $(\psi^{-1})^{-1} = \psi$ for each $\psi \in \mathcal{D}[a, b]$. In fact, applying (c) twice, we obtain

\[
\int_a^b (\psi^{-1})^{-1} = [\int_a^b \psi]^{-1} = [\int_a^b \psi]^{-1} = \int_a^b \psi.
\]

The claim is thus clear from injectivity of $\delta^r |_{C^1([a, b], G)}$.

- We have $\psi \ast \psi^{-1} = 0 = \psi^{-1} \ast \psi$ for each $\psi \in \mathcal{D}[a, b]$. In fact, it is clear that

\[
\psi \ast \psi^{-1} = 0 \quad \forall \psi \in \mathcal{D}[a, b].
\]

Then, we obtain from the previous point that

\[
\psi^{-1} \ast \psi = \psi^{-1} \ast (\psi^{-1})^{-1} = 0 \quad \forall \psi \in \mathcal{D}[a, b].
\]

- We have $\phi \ast (\psi \ast \chi) = (\phi \ast \psi) \ast \chi$ for all $\phi, \psi, \chi \in \mathcal{D}[a, b]$. In fact, we obtain from (a) that

\[
\int_a^b \phi \ast (\psi \ast \chi) = \int_a^b \phi \cdot \int_a^b (\psi \ast \chi) = \int_a^b \phi \cdot \int_a^b \psi \cdot \int_a^b \chi = \int_a^b (\phi \ast \psi) \cdot \int_a^b \chi = \int_a^b (\phi \ast \psi) \ast \chi.
\]

The claim is thus clear from injectivity of $\delta^r |_{C^1([a, b], G)}$.

We will reconsider this group structure in Sects. 4.2.1 and 5. □

Example 3 (The Riemann Integral) Assume we are in the situation of Example 2. Then,

\[
\int_a^b \phi = \int_a^t \phi(s) \, ds \quad \forall \phi \in \mathcal{D}[a, b], \quad t \in [a, b]
\]

holds; and, the identities (a), (d), (e), and (f) encode (in the given order) the additivity of the Riemann integral, (16), (21), and (22) (for if in (f), $(H, \cdot) \equiv (\tilde{F}, +)$ is the additive group of some further $\tilde{F} \in \operatorname{hlCVect}$), respectively. □

Example 4 For $a < b$, we define

\[
\operatorname{inv}: C^0([a, b], g) \to C^0([a, b], g) \quad \phi \mapsto [t \mapsto -\phi(a + b - t)]. 
\]

(38)

Let $a < b$ be fixed. Then, $\operatorname{inv}|_{C^0([a, b], g)}$ is linear. Moreover, for $\phi \in C^0([a, b], g)$ and $\varrho: [a, b] \ni t \mapsto a + b - t \in [a, b]$, we have $\operatorname{inv}(\phi) = \varrho \cdot \phi \circ \varrho$. Then, (e) shows that $\operatorname{inv}(\phi) \in \mathcal{D}[a, b]$ holds for each $\phi \in \mathcal{D}[a, b]$ and $k \in \mathbb{N} \cup \{\operatorname{lip}, \infty\}$, with

\[
eq \int_a^b \phi \cdot \int_a^b \operatorname{inv}(\phi) = \int_a^b \phi \cdot \int_a^b \operatorname{inv}(\phi) \quad \text{hence} \quad [\int_a^b \phi]^{-1} = \int_a^b \operatorname{inv}(\phi). 
\]

(39)

For instance, in the situation of Example 3, the right side of (39) reads

\[
-\int_a^b \phi(s) \, ds = \int_a^b -\phi(a + b - s) \, ds,
\]

which is in line with the second point in Remark 2. The relation $[\int_a^b \phi]^{-1} = \int_a^b \operatorname{inv}(\phi)$ will be useful for our argumentation in Sect. 3.2. □

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\textbf{Example 5} (The Lie bracket and Homomorphisms) Assume that we are in the situation of (f), and let \( \mathfrak{h}, [\cdot, \cdot]_\mathfrak{h} \) denote the Lie algebra of \( H \). Then, we have

\[
d_e \phi([X, Y]) = [d_e \phi(X), d_e \phi(Y)]_{\mathfrak{h}} \quad \forall X, Y \in \mathfrak{g}.
\]

\textbf{Proof of Equation (40)} Let \( X, Y \in \mathfrak{g} \) be fixed, and choose \( \phi_X : [0, \varepsilon_1] \to \mathfrak{g} \) as well as \( \phi_Y : [0, \varepsilon_2] \to \mathfrak{g} \) as in Remark 4. We obtain

\[
d_e \phi([X, Y]) = \left. \frac{d}{dt} \right|_{t=0} d_e \phi(\text{Ad}_{j_0^t} \phi_X(Y))
= \left. \frac{d}{dt} \right|_{t=0} \frac{d}{dy} \left|_{y=0} \phi(\text{Conj}_{j_0^t} \phi_X(j_0^y \phi_Y))
= \left. \frac{d}{dt} \right|_{t=0} \frac{d}{dy} \left|_{y=0} \text{Conj}_{j_0^t} \phi_X(j_0^y \phi_Y)
= \left. \frac{d}{dt} \right|_{t=0} \frac{d}{dy} \left|_{y=0} \text{Conj}_{j_0^t} d_e \phi_X(j_0^y d_e \phi_Y)
= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{j_0^t} d_e \phi_X(d_e \phi(Y))
= [d_e \phi(X), d_e \phi(Y)]_{\mathfrak{h}},
\]

which shows (40). \( \square \)

For instance, let \( \Psi \equiv \text{Conj}_g : G \to H = G \) with \( g \in G \). We obtain

\[
\text{Ad}_g([X, Y]) = [\text{Ad}_g(X), \text{Ad}_g(Y)] \quad \forall g \in G, X, Y \in \mathfrak{g}.
\]

Then, given \( X, Y, Z \in \mathfrak{g} \), fix \( \mu : (-\varepsilon, \varepsilon) \to G \) (\( \varepsilon > 0 \)) of class \( C^1 \), with \( \mu(0) = e \) and \( \mu(0) = Z \). We obtain from (41) and the parts b), d), e) of Proposition A.1 that

\[
[Z, [X, Y]] = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\mu(t)}([X, Y]) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\mu(t)}([X, Y])
= [[Z, X], Y] + [X, [Z, Y]]
\]

holds, which is the Jacobi identity (10). \( \square \)

### 2.4.3 Weak regularity

In this section, we recall certain differentiation results from [11] and introduce the notion of weak \( C^k \)-regularity for \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \) (cf. Definition 1). We say that \( G \) is \( C^k \)-semiregular for \( k \in \mathbb{N} \cup \{\text{lip}, \infty, c\} \) if \( D_k = C^k([0, 1], \mathfrak{g}) \) holds.

\textbf{Remark 6} It follows from (e) when applied to affine transformations that \( G \) is \( C^k \)-semiregular if and only if \( D^k_{[a, b]} = C^k([a, b], \mathfrak{g}) \) holds for all \( a < b \) (confer, e.g., Lemma 12 in [8]). \( \square \)

We write \( \lim_n^{\infty} \mu_n = \mu \) for \( \{\mu_n\}_{n \in \mathbb{N}} \subseteq C^0([a, b], G) \) and \( \mu \in C^0([a, b], G) \) if \( \{\mu_n\}_{n \in \mathbb{N}} \) converges uniformly to \( \mu \), i.e., if to each open \( U \subseteq G \) with \( e \in U \), there exists \( N_U \in \mathbb{N} \) with

\[
\mu_n(t) \in \mu(t) \cdot U \quad \forall t \in [a, b].
\]

We recall (14), and say that \( G \) is Mackey k-continuous for \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \) if

\[
D_k \ni \{\phi_n\}_{n \in \mathbb{N}} \mapsto m.k. \phi \in D_k \quad \implies \quad \lim_n^{\infty} j_0^\ast \phi_n = j_0^\ast \phi.
\]

\textbf{Lemma A} \( G \) is Mackey k-continuous for \( k \in \mathbb{N} \cup \{\text{lip}, \infty\} \) if and only if for each \( a < b \) the following implication holds:

\[
D^k_{[a, b]} \ni \{\phi_n\}_{n \in \mathbb{N}} \mapsto m.k. \phi \in D^k_{[a, b]} \quad \implies \quad \lim_n^{\infty} j_a^\ast \phi_n = j_a^\ast \phi.
\]
Theorem 1 in [11] states:

**Theorem B** If $G$ is $C^k$-semiregular for $k \in \mathbb{N} \cup \{\text{lip, } \infty\}$, then $G$ is Mackey $k$-continuous.

Theorem 3 in [11] (in particular) states:

**Theorem C** Assume that $G$ is Mackey $k$-continuous for $k \in \mathbb{N} \cup \{\text{lip, } \infty\}$. Let $\Phi : I \times [a, b] \rightarrow \mathfrak{g}$ ($I \subseteq \mathbb{R}$ open) be given with $\Phi(z, \cdot) \in \mathcal{D}^k_{[a, b]}$ for each $z \in I$. Then, for $x \in I$ we have

$$\left. \frac{d}{dt} \right|_{t=0} \Xi \left( [f_a^b \Phi(x, \cdot)]^{-1} [f_a^b \Phi(x + h, \cdot)] \right) = \int_a^b \left( d_e \Xi \circ \text{Ad}_{[f_a^b \Phi(x, \cdot)]^{-1}} (\partial_t \Phi(x, s)) \right) ds \in \hat{E},$$

provided that the following conditions hold:

1. We have $(\partial_t \Phi)(x, \cdot) \in C^k([a, b], \mathfrak{g})$.
2. To $p \in \text{Sem}(E)$ and $s \leq k$, there exists $L_{p,s} \geq 0$ as well as $I_{p,s} \subseteq I$ open with $x \in I_{p,s}$, such that

$$p_{\infty}^{-1}(\Phi(x + h, \cdot) - \Phi(x, \cdot)) \leq |h| \cdot L_{p,s} \quad \forall h \in \mathbb{R} \neq 0 \text{ with } x + h \in I_{p,s}.$$ 

In particular, we have (recall Convention 1)

$$\left. \frac{d}{dt} \right|_{t=0} \int_a^b \Phi(x + h, \cdot) = d_e L_{p,s} \phi(x, \cdot) \left( \int_a^b \text{Ad}_{[f_a^b \Phi(x, \cdot)]^{-1}} (\partial_t \Phi(x, s)) \right)$$

if and only if the Riemann integral on the right side exists in $\mathfrak{g}$.

Recall from the end of Sect. 2.2.3 that the last condition in Theorem C concerning the Riemann integral is always fulfilled

- for $k \in \mathbb{N}_{\geq 1} \cup \{\text{lip, } \infty\}$ if $\mathfrak{g}$ is Mackey complete,
- for $k = 0$ if $\mathfrak{g}$ integral complete for $k = 0$.

In particular, this implies the following statement (cf. Theorem 2 in [11] and Corollary 3 in [11]):

**Proposition D** Assume that $G$ is $C^k$-semiregular for $k \in \mathbb{N} \cup \{\text{lip, } \infty\}$. Then, $\text{evol}_k$ is differentiable if and only if $\mathfrak{g}$ is Mackey complete for $k \in \mathbb{N}_{\geq 1} \cup \{\text{lip, } \infty\}$ as well as integral complete for $k = 0$. In this case, $\text{evol}^k_{[a, b]}$ is differentiable for each $a < b$, with

$$\text{d}_\phi \text{evol}^k_{[a, b]}(\psi) = d_e L_{p,s} \phi(x, \cdot) \left( \int_a^b \text{Ad}_{[f_a^b \Phi(x, \cdot)]^{-1}} (\psi(s)) \right) ds \quad \forall \phi, \psi \in C^k([a, b], \mathfrak{g}).$$

Moreover, for $a < b$ and $\phi, \psi \in C^k([a, b], \mathfrak{g})$, we have

$$C^1(\mathbb{R}, G) \ni \mu : \mathbb{R} \ni t \mapsto \int \phi + t \cdot \psi \in G.$$

**Proof** Clear from Theorem 2 in [11] and Corollary 3 in [11].

Proposition D motivates the following definition\(^{11}\):

**Definition 1** $G$ is said to be weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip, } \infty\}$ if $G$ is $C^k$-semiregular, and $\mathfrak{g}$ is Mackey complete for $k \in \mathbb{N}_{\geq 1} \cup \{\text{lip, } \infty\}$ as well as integral complete for $k = 0$. □

**Remark 7** Apart from $C^k$-semiregularity, the (standard) notion of $C^k$-regularity involves smoothness (and continuity) of the evolution map w.r.t. the $C^k$-topology. These additional assumptions, however, are unnecessarily strong for our purposes. This is because, due to the results stated above, the usual differentiability properties of the evolution map are already available in the weakly $C^k$-regular context. □

\(^{11}\) According to Proposition D, Definition 1 is equivalent to the definition given in Sect. 1.
Theorem C will in particular be applied to the situation in Example 2, i.e., where $(G, \cdot) \equiv (F, +)$ equals the additive group of some given $F \in \text{hlcVect}$. For this observe that $\int = \int$ is $C^0$-continuous, hence Mackey 0-continuous. Moreover, $G \equiv F$ is

- $C^0$-semiregular if $F$ is integral complete,
- $C^{\text{lip}}$-semiregular if $F$ is Mackey complete. □

For instance, we obtain the following statements that we shall need in Sect. 3.1.

Corollary 3 Let $F \in \text{hlcVect}$, $k \in \mathbb{N}$, $a < b$, and $I \subseteq \mathbb{R}$ an open interval. Let furthermore $\Theta^0 : I \times [a, b] \to F$ be a map with $\alpha_x := \Theta^0(\cdot, x) \in C^k(I, F)$ for each $x \in [a, b]$, and assume that for $0 \leq \ell \leq k$ the map

$$
\Theta^\ell : I \times [a, b] \ni (t, x) \mapsto \alpha_x^{(\ell)}(t) \in F
$$

is continuous such that

$$
\gamma^\ell : I \ni t \mapsto \int_a^b \Theta^\ell(t, x) \, dx \in F
$$

is defined. Then, $\gamma := \gamma^0 \in C^k(I, F)$ holds with $\gamma^{(\ell)} = \gamma^\ell$ for $0 \leq \ell \leq k$.

Proof For each $0 \leq \ell \leq k$, we have by (19), compactness of $[a, b]$, and continuity of $\Theta^\ell$ that $\gamma^{(\ell)} \in C^0(I, F)$ holds. Let now $0 \leq \ell < k$ be given, and set $\Phi \equiv \Theta^\ell$. We observe the following:

- For $t \in I$, we have $\Phi(t, \cdot) = \Theta^\ell(t, \cdot) \in C^0([a, b], F)$.
- For $t \in I$, we have $\partial_1 \Phi(t, \cdot) \in C^0([a, b], F)$, as
  $$
  \partial_1 \Phi(t, x) = \partial_1 \Theta^\ell(t, x) = \alpha_x^{(\ell+1)}(t) = \Theta^{\ell+1}(t, x) \quad \forall t \in I, x \in [a, b].
  $$
- For $q \in \text{Sem}(F)$, $x \in [a, b], t \in I$, and $\tau > 0$ with $t + [-\tau, \tau] \subseteq I$, we have by (18)
  $$
  1/|h| \cdot q(\Phi(t + h, x) - \Phi(t, x)) = 1/|h| \cdot q(\alpha_x^{(\ell)}(t + h) - \alpha_x^{(\ell)}(t)) \leq \sup \left\{ -\tau \leq s \leq \tau \mid q(\alpha_x^{(\ell+1)}(t + s)) \right\} \leq \sup \left\{ -\tau \leq s \leq \tau, a \leq y \leq b \mid q(\Theta^{\ell+1}(t + s, y)) \right\}
  $$
  for each $-\tau \leq h \leq \tau$.

Theorem C (Remark 8) shows that for each $t \in I$, we have

$$
\frac{d}{dt} \big|_{h=0} \gamma^{(\ell)}(t + h) = \frac{d}{dt} \big|_{h=0} \int_a^b \Theta^\ell(t + h, x) \, dx = \frac{d}{dt} \big|_{h=0} \int_a^b \Phi(t + h, x) \, dx = \int_a^b \partial_1 \Phi(t, x) \, dx = \int_a^b \partial_1 \Theta^\ell(t, x) \, dx = \int_a^b \alpha_x^{(\ell+1)}(t) \, dx = \int_a^b \Theta^{\ell+1}(t, x) \, dx = \gamma^{\ell+1}(t).
$$

Since this holds for each $0 \leq \ell < k$, the claim follows from $\gamma^{(0)} = \gamma = \gamma^0$ by induction. □
Lemma 7 Let $F \in hlcVect$ be Mackey complete, $a < b$, $I \subseteq \mathbb{R}$ an open interval, $\Theta^0 : I \times [a, b] \to F$ continuous, and $\Omega : F \times F \to F$ smooth. Assume that the following two conditions are fulfilled:

(a) $\Theta^0(t, \cdot) \in C^1([a, b], F)$ holds for each $t \in I$.

(b) $\alpha_x := \Theta^0(t, x) \in C^1(I, F)$ holds for each $x \in [a, b]$, with

$$\hat{\alpha}_x(t) = \Omega\left(\int_x^b \alpha_y(t) \ dy, \alpha_x(t)\right) \quad \forall t \in I.$$

Then, $\gamma[z] : I \ni t \mapsto \int_z^b \Theta^0(t, x) \ dx \in F$ is smooth for each $z \in [a, b]$.

Proof It suffices to prove the following statement.

Statement 1 Let $k \in \mathbb{N}$ be given. Then, $\alpha_x \in C^k(I, F)$ holds for each $x \in [a, b]$. Moreover,

$$\Theta^\ell : I \times [a, b] \ni (t, x) \mapsto \alpha_x^{(\ell)}(t) \in F$$

is continuous for $0 \leq \ell \leq k$, with $\Theta^\ell(t, \cdot) \in C^1([a, b], F)$ for each $t \in I$.

In fact, let $k \in \mathbb{N}$ be given. Since $F$ is Mackey complete, it follows from Statement 1 that

$$\gamma[z]^\ell : I \ni t \mapsto \int_z^b \Theta^\ell(t, x) \ dx \in F$$

exists for each $z \in [a, b]$ and $0 \leq \ell \leq k$. Moreover, Corollary 3 shows that $\gamma[z] \in C^k(I, F)$ holds for each $z \in [a, b]$, with

$$\gamma[z]^{(\ell)} = \gamma[z]^{\ell} = \int_z^b \Theta^\ell(\cdot, x) \ dx \quad \forall 0 \leq \ell \leq k. \quad (42)$$

Since $k \in \mathbb{N}$ was arbitrary, Lemma 7 follows.

It thus remains to prove Statement 1:

Proof of Statement 1 We first discuss the cases $k \in \{0, 1\}$ and then argue by induction:

- $k = 0$: Clear from the assumptions.
- $k = 1$: By (b), we have $\alpha_x \in C^1(I, F)$ for each $x \in [a, b]$. Since $\Theta^0$ is continuous,

$$\Gamma^0 : I \times [a, b] \ni (t, z) \mapsto \int_z^b \Theta^0(t, x) \ dx \in F$$

is continuous. It is clear that $\Gamma^0(t, \cdot) \in C^1([a, b], F)$ holds for each $t \in I$; and we have $\Theta^0(t, \cdot) \in C^1([a, b], F)$ for each $t \in I$ by (a). Now, (b) yields

$$\Theta^1(t, x) = \hat{\alpha}_x(t) = \Omega(\Gamma^0(t, x), \Theta^0(t, x)) \quad \forall t \in I, \ x \in [a, b].$$

Since $\Omega$ is smooth, $\Theta^1$ is continuous with $\Theta^1(t, \cdot) \in C^1([a, b], F)$ for each $t \in I$.

Assume now that Statement 1 holds for some $k \geq 1$. We observe the following:

(i) $\Theta^\ell$ is continuous for $0 \leq \ell \leq k$, hence

$$\Gamma^\ell : I \times [a, b] \ni (t, z) \mapsto \int_z^b \Theta^\ell(t, x) \ dx \in F$$

is continuous for $0 \leq \ell \leq k$.

(ii) For $z \in [a, b]$, we have $\alpha_z, \gamma[z] \in C^k(I, F)$, with

$$\alpha_z^{(\ell)} = \Theta^\ell(\cdot, z) \quad \text{and} \quad \gamma[z]^{(\ell)} = \Gamma^\ell(\cdot, z) \quad \forall 0 \leq \ell \leq k. \quad (43)$$

Then, (i) implies that the following maps are continuous:

$$I \times [a, b] \ni (t, z) \mapsto \alpha_z^{(\ell)}(t) \in F$$

$$I \times [a, b] \ni (t, z) \mapsto \gamma[z]^{(\ell)}(t) \in F \quad (44)$$

for $0 \leq \ell \leq k$. 

\[ \square \] Springer
(iii) For \( t \in I \) and \( 0 \leq \ell \leq k \), the maps

\[
[a, b] \ni z \mapsto \alpha^{(\ell)}_z(t) \in F
\]

\[
[a, b] \ni z \mapsto \gamma[z]^{(\ell)}(t) \in F
\]

are of class \( C^1 \) (use (43)).

Then, by (b) we have

\[
\dot{\alpha}_t(t) = \Omega(\gamma[x](t), \alpha_x(t)) \quad \forall t \in I, \ x \in [a, b].
\]

Together with ii), this shows \( \alpha_x \in C^{k+1}(I, F) \) for all \( x \in [a, b] \). Moreover, set \( \gamma[x]_1 := \alpha_x \) and \( \gamma[x]_2 := \gamma[x] \) for each \( x \in [a, b] \). It follows from (46) that

\[
\Theta^{k+1}: I \times [a, b] \ni (t, x) = \alpha_x^{(k+1)}(t) \in F
\]

is a sum of maps of the form

\[
I \times [a, b] \ni (t, x) \mapsto \Psi(\gamma[x]^{(\ell_1)}(t), \ldots, \gamma[x]^{(\ell_m)}(t)) \in F
\]

for certain \( 0 \leq \ell_1, \ldots, \ell_m \leq k \), \( 1 \leq i_1, \ldots, i_m \leq 2 \), \( m \geq 2 \), where \( \Psi: F^m \to F \) is smooth.

Then, (44) implies that \( \Theta^{k+1} \) is continuous, and (45) implies that \( \Theta^{k+1}(t, \cdot) \in C^1([a, b], F) \) holds for each \( t \in I \). This establishes the claim for \( k + 1 \), so that Statement 1 follows by induction.

This proves the Lemma 7.

### 2.4.4 The exponential map

We let \( i: g \ni X \rightarrow \mathbb{C}_X|_{[0,1]} \in C^c([0,1], g) \), hence

\[
i^{-1}: C^c([0,1], g) \rightarrow g, \quad \chi \mapsto \chi(0).
\]

The exponential map is given by

\[
\exp: \text{dom}[\exp] = i^{-1}(D_c) \rightarrow G, \quad X \mapsto \int \mathbb{C}_X|_{[0,1]} = (\text{evol}_c \circ i)(X).
\]

- Instead of saying that \( G \) is \( C^c \)-semiregular, in the following we rather say that \( G \) admits an exponential map.
- The relation (e) implies \( \mathbb{R} \cdot \text{dom}[\exp] \subseteq \text{dom}[\exp] \), as well as that \( t \mapsto \exp(t \cdot X) \) is a 1-parameter group for each \( X \in \text{dom}[\exp] \) with

\[
\exp(t \cdot X) = \int t \cdot \mathbb{C}_X|_{[0,1]} = \int_0^t \mathbb{C}_X|_{[0,1]} \quad \forall t \geq 0,
\]

confer, e.g., Remark 2.1) in [8].

Finally, Theorem 1 in [11] (cf. Theorem B), Corollary 6 in [11], and Remark 9.1 in [11] provide the following statement.

**Corollary 4** Let \( \tilde{X}: I \rightarrow \text{dom}[\exp] \subseteq g \) (\( I \subseteq \mathbb{R} \) an open interval) be of class \( C^1 \), and set \( \alpha := \exp \circ \tilde{X} \). Assume that \( G \) is weakly \( C^\infty \)-regular, or that \( \exp: g \rightarrow G \) is defined and of class \( C^1 \). Then, \( \alpha \) is of class \( C^1 \), with

\[
\dot{\alpha}(t) = d_\nu L_{\exp(\tilde{x}(t))} \left( \int_0^1 \text{Ad}_{\exp(t \cdot \tilde{x}(s))}(\dot{\tilde{x}}(s)) \, ds \right), \quad \forall t \in I.
\]

\( ^{12} \) The statement is obtained by a straightforward induction involving the differentiation rules in Proposition A.1, confer also the proof of Lemma 4 in [8].
Proof If $G$ is weakly $C^\infty$-regular, then the claim is clear from Theorem 1 in [11] and Corollary 6 in [11]. Assume now that exp is of class $C^1$. Then, Remark 9.1 in [11] shows that
\[ d_X \exp(Y) = d_L \exp(\int_0^1 \text{Ad} \exp(-s \cdot X)(Y) \, ds) \]
holds, so that the claim is clear from Part d of Proposition A.1. □

3 Preliminary results

In this section, we derive some elementary results from Theorem C (Proposition 1 in Sect. 3.1 and Proposition 2 in Sect. 3.2) and provide an integral expansion for the adjoint action (Lemma 13 in Sect. 3.3). Proposition 1 and Lemma 13 will be used in Sect. 4.2 to investigate the product integral of nilpotent curves. Also, Proposition 2 was supposed to be applied in this paper, but eventually turned out not to be necessary for our argumentation. We kept this result for academic reasons, and because it certainly will play a role in future applications. Our considerations in Sects. 3.1 and 3.2 furthermore serve as motivations for the constructions made in Sect. 5.

3.1 An integral transformation

Let $\mathcal{R} := \{ \phi \in \mathcal{D} \mid [0, 1] \cdot \phi \subseteq \mathcal{D} \}$. We define $\mathcal{T} : \mathcal{R} \to \text{Maps}([0, 1], \hat{g})$ by
\[
\mathcal{T} : \mathcal{R} \cap \mathcal{D}_{[a,b]} \to \text{Maps}([0, 1], \hat{g})
\phi \mapsto \left[ [0, 1] \ni t \mapsto \int_a^b \text{Ad}_{[\int_0^1 \hat{\phi} \cdot t \cdot \phi(s))} ds \right]
\]
for each $a < b$. In view of Sect. 4.2 (the proof of Lemma 16), for $a < b$ and $\phi \in \mathcal{R} \cap \mathcal{D}_{[a,b]}$, we define
\[
\mathcal{T}(\phi|\langle a,a \rangle) : [0, 1] \ni t \mapsto 0 \in g.
\]
In this section, we proof the following proposition.

**Proposition 1** Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Then, for $a < b$ and $\phi \in C^k([a, b], g)$, we have $\mathcal{T}(\phi) \in C^\infty([0, 1], g)$ with
\[
\int_a^b t \cdot \phi = \int_0^1 \mathcal{T}(\phi) \quad \forall t \in [0, 1].
\]
In particular, $\int_a^b \phi = \int_0^1 \mathcal{T}(\phi)$ holds, and $\mu : [0, 1] \ni t \mapsto \int_a^b t \cdot \phi$ is smooth by (37).

**Example 6** Assume that $G$ is abelian, as well as weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Then, Proposition 1 recovers the well-known formula [18]
\[
\int_a^x \phi = \exp \left( \int_a^x \phi(s) \, ds \right) \quad \forall x \in [a, b],
\]
for each $a < b$ and $\phi \in C^k([a, b], g)$. □

**Remark 9** Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$; and recall the group structure discussed in Remark 5, as well as the map (38) introduced in Example 4. It is straightforward from Proposition 1 and the properties of the product integral that
\[
\mathcal{T}(\psi)^{-1} = \mathcal{T}(\text{inv}(\psi))
\]
\[
\mathcal{T}(\psi) = \mathcal{T}(\psi|\langle c,b \rangle) \circ \mathcal{T}(\psi|\langle a,c \rangle)
\]
\[
\mathcal{T}(\psi) = \mathcal{T}(\hat{\phi} \circ (\psi \circ \hat{\phi}))
\]

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holds for $a < c < b$, $\psi \in C^k([a, b], g)$, as well as $\varrho: [a', b') \to [a, b]$ $(a' < b')$ of class $C^1$ with $\dot{\varrho}|_{(a', b')} > 0$, $\varrho(a') = a$, $\varrho(b') = b$.

- By (a) (first step), Proposition 1 (second step), and (39) (fourth step), we have
  \[ f^t_0 \Sigma(\psi) \star \Sigma(\inv(\psi)) = f^t_a \Sigma(\psi) \cdot f^t_0 \Sigma(\inv(\psi)) = (f^b_a t \cdot \psi) \cdot (f^c_0 t \cdot \inv(\psi)) = (f^b_a t \cdot \psi) \cdot (f^c_0 \inv(t \cdot \psi)) = (f^b_a t \cdot \psi) \cdot (f^c_0 t \cdot \psi)^{-1} = e \]
  for each $t \in [0, 1]$. Then, injectivity of $\delta^r|_{C^1([0, 1], G)}$ implies $\Sigma(\psi) \star \Sigma(\inv(\psi)) = 0$, which proves the first line in (52).

- By Proposition 1 (first and third step), (d) (second step), and (a) (fourth step), we have
  \[ f^t_0 \Sigma(\psi) = f^b_a t \cdot \psi = (f^c_0 t \cdot \psi) = f^t_0 \Sigma(\psi|_{[c, b]}) \cdot f^t_0 \Sigma(\psi|_{[a, c]}) \]
  for each $t \in [0, 1]$. The second line in (52) now follows from injectivity of $\delta^r$.

- Since $\dot{\varrho}|_{(a', b')} > 0$ holds, the substitution formula (e) applies to $\varrho|_{[x', y']}$ and $t \cdot \psi|_{[\varrho(x'), \varrho(y')]}/a' \leq x' < y' \leq b'$ and $t \in [0, 1]$. We obtain
  \[ \Sigma(\dot{\varrho} \cdot (\psi \circ \varrho)) = f^b_{a'} \Ad_{\varrho|_{[x', y']}}(\dot{\varrho}(t \cdot \varrho)) \cdot (\dot{\varrho}(s) \cdot (\psi \circ \varrho)(s)) \, ds = f^b_{a'} \dot{\varrho}(s) \cdot (\Ad_{\varrho|_{[x', y']}}(\dot{\varrho}(s))) \, ds = f^b_{a'} \Ad_{\varrho|_{[x', y']}}(\dot{\varrho}(s)) \, ds = \Sigma(\psi), \]
  where we have applied (21) in the third step. This proves the third line in (52). \hfill \Box

**Remark 10** The smoothness statement in Proposition 1, in particular, ensures that $\Sigma$ can be applied iteratively. We will use this fact in Sect. 4.2 to prove an identity for the product integral in the nilpotent context. \hfill \Box

For the proof of Proposition 1, we shall need the following statements.

**Lemma 8** Let $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, $a < b$, $\mu \in C^{k+1}([a, b], G)$, and $\phi \in C^{k}([a, b], g)$. Then, we have $\Ad_\mu(\phi) \in C^{k}([a, b], g)$.

**Proof** Confer, e.g., Lemma 13 in [8]. \hfill \Box

**Lemma 9** Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, and let $\phi \in C^{k}([a, b])$ $(a < b)$ be given. Then, $\kappa: \mathbb{R} \times [a, b] \ni (t, x) \mapsto f^b_0 t \cdot \phi \in G$ is continuous.

**Proof** Confer Appendix A.5. \hfill \Box

**Lemma 10** Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, and let $a < b$ as well as $\phi \in C^{k}([a, b])$ be given. Then, $\Sigma(\phi): \mathbb{R} \ni t \mapsto f^b_a \Ad_{\phi|_{[t, b]}}(\phi(s)) \, ds \in g$ is smooth.
Proof  We consider the map
\[ \Theta^0 : \mathbb{R} \times [a, b] \to \mathfrak{g}, \quad (t, x) \mapsto \text{Ad}_{\int_a^t r \cdot \phi} \phi(x), \]
and observe the following:

- Lemma 9 shows that \( \Theta^0 \) is continuous, as we have (with \( \kappa \) as in Lemma 9)
  \[ \Theta^0(t, x) = \text{Ad}_{\kappa(t, x)}(\phi(x)) \quad \forall t \in \mathbb{R}, \ x \in [a, b]. \]

- Given \( t \in \mathbb{R} \), we have \( \Theta^0(t, \cdot) \in C^k([a, b], \mathfrak{g}) \subseteq C^1([a, b], \mathfrak{g}) \) by Lemma 8, (37), and
  \[ \Theta^0(t, x) = \text{Ad}_{\int_a^t r \cdot \phi} \text{Ad}_{\int_a^t r \cdot \phi}^{-1} \phi(x)) \quad \forall t \in \mathbb{R}, \ x \in [a, b]. \]

- Given \( x \in [a, b] \), Proposition D yields
  \[ C^1(\mathbb{R}, G) \ni \mu_x : \mathbb{R} \ni t \mapsto \int_a^t \cdot \phi \in G. \]

Then, Lemma 8 shows
\[ C^1(\mathbb{R}, \mathfrak{g}) \ni \alpha_x := \Theta^0(\cdot, x) : \mathbb{R} \ni t \mapsto \text{Ad}_{\alpha_x(t)}(\phi(x)) \in \mathfrak{g}. \] (53)

Moreover, for \( t, h \in \mathbb{R} \) we have
\[ \mu_x(t + h) \overset{(a)}{=} \mu_x(t) \cdot \int_a^h \cdot \text{Ad}_{\int_a^{t+h} r \cdot \phi}^{-1}(\phi) = \text{Conj}_{\mu_x(t)}(\int_a^h \cdot \text{Ad}_{\int_a^{t+h} r \cdot \phi}^{-1}(\phi)) \cdot \mu_x(t) \]
\[ \overset{(f)}{=} [\int_a^h \cdot \text{Ad}_{\int_a^{t+h} r \cdot \phi}^{-1}(\phi)] \cdot \mu_x(t) \]
\[ = [\int_a^h \cdot \Theta^0(t, \cdot)] \cdot \mu_x(t). \] (54)

By Proposition D, the map \( \mathbb{R} \ni h \mapsto \int_a^h \cdot \Theta^0(t, \cdot) \in G \) is of class \( C^1 \) for \( t \in \mathbb{R} \) (recall that \( \Theta^0(t, \cdot) \in C^k([a, b], \mathfrak{g}) \) holds by the second point above), with
\[ \frac{d}{dt} \bigg|_{t=0} \int_a^h \cdot \Theta^0(t, \cdot) = d_0 \text{evol}_{[a,b]}^k(\Theta^0(t, \cdot)) = \int_a^b \Theta^0(t, y) dy \in \mathfrak{g}. \] (55)

We obtain for \( t \in \mathbb{R} \) that
\[ \hat{\alpha}_x(t) = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\alpha_x(t+h)}(\phi(x)) \]
\[ \overset{(54)}{=} \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\int_a^{t+h} r \cdot \Theta^0(t, \cdot)}(\text{Ad}_{\alpha_x(t)}(\phi(x))) \]
\[ \overset{(55)}{=} \int_a^b \Theta^0(t, y) dy, \alpha_x(t)) \]
\[ \overset{(53)}{=} \int_a^b \alpha_x(t) dy, \alpha_x(t)). \] (56)

The claim is now clear from Lemma 7 (with \( l = \mathbb{R}, \Omega = [\cdot, \cdot], \) and \( z = b \) there). \( \square \)

We are ready for the proof of Proposition 1.

Proof of Proposition 1 By Lemma 10, we have \( \Xi(\phi) \in C^\infty([0, 1], \mathfrak{g}) \). By Proposition D, we have \( C^1([0, 1], \mathfrak{g}) \ni \mu : [0, 1] \ni t \mapsto \int_a^b t \cdot \phi \) with
\[ \hat{\mu}(t) = d_{t \cdot \phi} \text{evol}_{[a,b]}^k(\phi) = d_{t \cdot L_{\mu(t)}}(\int_a^b \text{Ad}_{\int_a^{t+h} r \cdot \phi}^{-1}(\phi(s)) ds) \]
\[ = d_{t \cdot R_{\mu(t)}}(\int_a^b \text{Ad}_{\int_a^{t+h} r \cdot \phi}^{-1}(\phi(s)) ds). \]
In the third step, we have applied (22) with $L \equiv \text{Ad}_\mu(t)$. This shows that $\delta^r(\mu) = \Xi(\phi)$ holds, with $\mu(0) = e$. We obtain

$$ \int_a^b t \cdot \phi = \mu(t) = \int_0^t \delta^r(\mu) = \int_0^t \Xi(\phi), $$

which proves the claim.

The following corollary verifies the guess made in Remark 8.2 in [11].

**Corollary 5** Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, and let $\phi, \psi \in C^k([a, b], \mathfrak{g})$ be given. Then, $\mu : \mathbb{R} \ni t \mapsto \int_a^b \phi + t \cdot \psi \in G$ is smooth.

**Proof** We have

$$ \int_a^b (\phi + t \cdot \psi) \overset{(a)}{=} \left[ \int_a^b \phi \right] \cdot \left[ \int_a^b t \cdot \text{Ad}_{\mu(a)}^{-1}(\psi) \right] \quad \forall \phi, \psi \in C^k([a, b], \mathfrak{g}). $$

By Lemma 8, it thus suffices to prove the claim for $\phi = 0$ and arbitrary $\psi \in C^k([a, b], \mathfrak{g})$. Let thus $\psi \in C^k([a, b], \mathfrak{g})$ be given. Then, we have

$$ \int_a^b (t + h) \cdot \psi \overset{(a)}{=} \left[ \int_a^b t \cdot \psi \right] \cdot \left[ \int_a^b h \cdot \text{Ad}_{\mu(a)}^{-1}(\psi) \right] \quad \forall t, h \in \mathbb{R}. $$

By Lemma 8, it thus suffices to prove smoothness of the map $[0, 1] \ni h \mapsto \int_a^b h \cdot \chi \in G$ for each $\chi \in C^k([a, b], \mathfrak{g})$, so that the claim is clear from (the last statement in) Proposition 1. □

### 3.2 Differentiation of parameter-dependent integrals

In this section, we prove the following differentiation result.

**Proposition 2** Assume that $G$ is Mackey $k$-continuous for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Let $a < b$, $\sigma > 0$, $I \subseteq \mathbb{R}$ an open interval with $[a - \sigma, b + \sigma] \subseteq I$, as well as $\Phi : I \times [a - \sigma, b + \sigma] \rightarrow \mathfrak{g}$ a map with $\Phi(z, \cdot) \in \mathcal{D}^k_{[a - \sigma, b + \sigma]}$ for each $z \in I$. Let $x \in [a, b]$ be given, such that the following conditions hold:

(i) We have $(\partial_1 \Phi)(x, \cdot) \in C^k([a, b], \mathfrak{g})$ with $\int_a^x \text{Ad}_{\mu[a]}^{-1}(\partial_1 \Phi(x, s)) \, ds \in \mathfrak{g}$.

(ii) To $p \in \text{Sem}(E)$ and $s \leq k$, there exists $L_{p,s} \geq 0$ as well as $I_{p,s} \subseteq I$ open with $x \in I_{p,s}$, such that

$$ p^s_{\infty}(\Phi(x + h, \cdot) - \Phi(x, \cdot)) \leq |h| \cdot L_{p,s} \quad \forall h \in \mathbb{R}_{\neq 0} \text{ with } x + h \in I_{p,s}. $$

Then, we have

$$ \frac{d}{dt} \bigm|_{t=0} \int_a^x t \cdot \Phi(x + h, \cdot) = \text{d}_e R_{\mu[a]}^{-1}(\Phi(x, x)) + \text{d}_e L_{\mu[a]}^{-1}(\Phi(x, x)) \left( \int_a^x \text{Ad}_{\mu[a]}^{-1}(\partial_1 \Phi(x, s)) \, ds \right). $$

For the proof of Proposition 2, we shall need the following statements.

**Lemma 11** To $p \in \text{Sem}(E)$, there exist $p \leq q \in \text{Sem}(E)$ and $U \subseteq G$ open with $e \in U$, such that for each $a < b$ and $\chi \in \mathcal{D}_{[a, b]}$ with $\int_a^b \chi \in U$ we have

$$ (p \circ \Xi)(\int_a^b \chi) \leq \int_q^s q(\chi(s)) \, ds. $$

**Proof** Confer Lemma 4 in [11]. □

**Corollary 6** For each compact $C \subseteq G$ and each $q \in \text{Sem}(E)$, there exists some $q \leq m \in \text{Sem}(E)$, such that $q \circ \text{Ad}_g \leq m$ holds for each $g \in C$. □
Proof This is clear from Lemma 4, because \( \text{Ad} : G \times \mathfrak{g} \ni (g, X) \mapsto \text{Ad}_g(X) \in \mathfrak{g} \) is smooth as well as linear in the second argument.

**Proof of Proposition 2** We consider the maps

\[
\begin{align*}
\mu : [-\sigma, \sigma] & \ni h \mapsto \int_a^x \Phi(x + h, \cdot) \\
\nu_+ : [0, \sigma] & \ni h \mapsto [\int_x^{x+h} \Phi(x, \cdot)]^{-1} [\int_a^{x+h} \Phi(x + h, \cdot)] \\
\eta_+ : [0, \sigma] & \ni h \mapsto \int_a^{x+h} \Phi(x, \cdot) \\
\eta_- : [-\sigma, 0] & \ni h \mapsto [\int_a^{x+h} \Phi(x + h, \cdot)]^{-1} [\int_x^{x+h} \Phi(x, \cdot)],
\end{align*}
\]

and obtain

\[
\begin{align*}
\int_a^{x+h} \Phi(x + h, \cdot) = \eta_+(h) \cdot \nu_+(h) \cdot \mu(h) & \quad \forall h \in [0, \sigma] \\
\int_a^{x+h} \Phi(x + h, \cdot) = \eta_-(h) \cdot \nu_-(h) \cdot \mu(h) & \quad \forall h \in [-\sigma, 0].
\end{align*}
\]

We furthermore observe the following:

1. The maps \(\nu_+, \eta_-\) are continuous at \( h = 0 \), with \( \nu_+(0) = e = \eta_-(0) \). To show this, it suffices to prove the same property for the maps

\[
\begin{align*}
\alpha_+ : [0, \sigma] & \ni h \mapsto \int_x^{x+h} \Phi(x, \cdot) \\
\beta_+ : [0, \sigma] & \ni h \mapsto \int_a^{x+h} \Phi(x + h, \cdot) \\
\alpha_- : [-\sigma, 0] & \ni h \mapsto \int_x^{x+h} \Phi(x, \cdot) \\
\beta_- : [-\sigma, 0] & \ni h \mapsto \int_a^{x+h} \Phi(x + h, \cdot).
\end{align*}
\]

First, it is clear that \(\alpha_\pm, \beta_\pm\) is continuous (in particular at \( h = 0 \)) with \(\alpha_\pm(0) = e\), as we have

\[
\alpha_-(h) = [\int_a^{x-h} \Phi(x, \cdot)] \cdot [\int_a^{x+h} \Phi(x, \cdot)]^{-1} \quad \forall h \in [-\sigma, 0].
\]

Second, Condition ii) implies that for each sequence \([-\sigma, 0] \cup (0, \sigma] \ni (h_n)_{n \in \mathbb{N}} \to 0\), we have \(\Phi(x + h_n, \cdot))_{n \in \mathbb{N}} \to_m k \Phi(x, \cdot)\). Since \(G\) is Mackey k-continuous, Lemma A yields the following:

- For each sequence \((0, \sigma] \ni (h_n)_{n \in \mathbb{N}} \to 0\), we have

\[
\lim_{n \to \infty} \int_x^{x-h_n} \Phi(x + h_n, \cdot) = \int_x^{x} \Phi(x, \cdot) \quad \implies \lim_{n \to \infty} \int_x^{x+h_n} \Phi(x + h_n, \cdot) = e.
\]

This implies \(\lim_{h \to 0} \beta_+(h) = e\).

- For each sequence \((-\sigma, 0] \ni (h_n)_{n \in \mathbb{N}} \to 0\), we have

\[
\lim_{n \to \infty} \int_a^{x-h_n} \Phi(x + h_n, \cdot) = \int_a^{x} \Phi(x, \cdot) \quad \implies \lim_{n \to \infty} \int_a^{x+h_n} \Phi(x + h_n, \cdot) = \int_a^{x} \Phi(x, \cdot).
\]

This implies \(\lim_{h \to 0} \beta_-(h) = e\), as we have

\[
\beta_-(h) = [\int_a^{x-h} \Phi(x + h, \cdot)] \cdot [\int_a^{x+h} \Phi(x + h, \cdot)]^{-1} \quad \forall h \in [-\sigma, 0].
\]

2. We have

\[
\dot{\mu}(0) = d_e L_{\int_a^x \Phi(x, \cdot)} \left( \int_a^x \text{Ad}[\int_a^x \Phi(x, \cdot)]^{-1} (\partial_t \Phi(x, s)) \, ds \right) \\
\dot{\nu}_-(0) = \Phi(x, x)
\]
\[ \dot{\eta}_\pm(0) = \Phi(x, x). \] (60)

In fact, (60) is evident, (58) is clear from Theorem C, and (59) is obtained as follows:

\[ \square \]

**Proof of Equation (59).** Given \(-\sigma \leq h < 0\), we define \( q_h : [x, x-h] \ni t \mapsto t+h \in [x+h, x], \) and obtain

\[ \int_{x+h}^x \psi = \int_{x+h}^{x-h} (\varepsilon) \psi = \int_x^{x-h} \psi(\cdot + h) \quad \forall \psi \in \mathcal{D}_{[x-\sigma, x]} . \] (61)

According to Example 4 (with \( \varphi : [x+h, x] \ni t \mapsto 2x + h - t \in [x+h, x] \) there), we have

\[ \nu_-(h) = [\int_{x+h}^x \Phi(x, \cdot)]^{-1} = \int_{x+h}^x \Phi(x, 2x + h - \cdot) = \int_{x}^{x-h} \Phi(x, 2x - \cdot) . \]

We obtain

\[ \lim_{0 < h \to 0} \frac{1}{h} \cdot (\Xi(\nu_-(h)) - \Xi(\nu_-(0))) = - \lim_{0 < h \to 0} \frac{1}{h} \cdot \Xi(\nu_-(h)^{-1}) = - \lim_{0 < h \to 0} \frac{1}{h} \cdot \Xi(\int_{x}^{x-h} \Phi(x, 2x - \cdot)) = d_\varepsilon \Xi(\Phi(x, x)) , \]

which proves (59). \[ \square \]

(3) By 1), there exists \( 0 < \delta \leq \sigma \), such that the following maps are defined:

\[ \Delta_+ : (0, \delta] \ni h \mapsto \frac{1}{h} \cdot \Xi(\nu_+(h)) \quad \text{and} \quad \Delta_- : [-\delta, 0) \ni h \mapsto \frac{1}{h} \cdot \Xi(\eta_-(h)) . \]

Then, to establish the proof, it suffices to show

\[ \lim_{0 < h \to 0} \Delta_+(h) = 0 \quad \text{as well as} \quad \lim_{0 < h \to 0} \Delta_-(h) = 0 . \] (62)

In fact, let \((\Xi', \mathcal{U}')\) be a chart around \( \mu(0) \). Let furthermore \( O \subseteq \mathcal{U} \) and \( O' \subseteq \mathcal{U}' \) be open with \( e \in O \) and \( \mu(0) \in O' \), such that \( O \cdot O' \subseteq \mathcal{U} \) holds. We set \( W := \Xi(O), \ W' := \Xi'(O'), \ V' := \Xi'(\mathcal{U}') \). We furthermore define

\[ f : W \times W \times W' \to V' \]

\[ (X, Y, Z) \mapsto \Xi'(\Xi^{-1}(X) \cdot \Xi^{-1}(Y) \cdot \Xi'^{-1}(Z)) , \]

as well as (shrink \( \delta > 0 \) if necessary, to ensure \( \text{im}[\eta_{\pm}] \subseteq O, \text{im}[v_{\pm}] \subseteq O, \text{im}[\mu] \subseteq O' \))

\[ \gamma_+ : [0, \delta] \ni h \mapsto (\Xi(\eta_+(h)), \Xi(\nu_+(h)), \Xi'(\mu(h))) \in E \times E \times E \]

\[ \gamma_- : [-\delta, 0] \ni h \mapsto (\Xi(\eta_-(h)), \Xi(\nu_-(h)), \Xi'(\mu(h))) \in E \times E \times E . \]

Then, Lemma 6 (with \( F_1 \equiv E \times E \times E, F_2 \equiv E, \) and \( \gamma \equiv \gamma_\pm \) there) together with Part c) of Proposition A.1 (cf. also (11), (57), (58), (59), (60), (62)) implies the claim.

Finally, to prove (62), let \( p \in \text{Sem}(E) \) be fixed, and choose \( p \leq q \in \text{Sem}(E) \) as well as \( U \subseteq G \) as in Lemma 11. Since both \( v_+, \nu_- \) are continuous by 1), there exists \( 0 < \delta_U \leq \delta (\delta \text{ as in 3}) \) with \( v_+(0, \delta_U] \subseteq U \) and \( \eta_-([-\delta_U, 0]) \subseteq U \). We observe that

\[ v_+(h) = \int_{x}^{x+h} \text{Ad}_{\alpha_+}^{-1}(\Phi(x + h, \cdot) - \Phi(x, \cdot)) \quad \forall h \in (0, \sigma] \]

\[ \eta_-(h) = \int_{x}^{x+h} \text{Ad}_{\beta_-}^{-1}(\Phi(x, \cdot) - \Phi(x + h, \cdot)) \quad \forall h \in [-\sigma, 0) \]

holds by (b), and obtain from Lemma 11 that

\[ p(\Delta_+(h)) \leq \frac{1}{|h|} \cdot \int_{x}^{x+h} q(\text{Ad}_{\alpha_+}^{-1}(\Phi(x + h, \cdot) - \Phi(x, \cdot))) \, ds \quad \forall h \in (0, \delta_U] \]

\[ \square \]
\[ p(\Delta_-(h)) \leq \frac{1}{|h|} \int_0^{x+h} q(Ad_{\beta_-(\cdot)}^{-1}(\Phi(x, s) - \Phi(x + h, s))) ds \quad \forall \ h \in [-\delta_U, 0). \]

Let \( q \leq m \in \text{Sem}(E) \) be as in Corollary 6, for \( C \equiv \text{inv(im}[\alpha_] + \text{inv(im}[\beta_-)] \) there. We shrink \( 0 < \delta_U \leq \delta \) such that \( x + (-\delta_U, \delta_U) \subseteq I_{m,0} \) holds, for \( I_{m,0} \) as in Condition ii) (with \( s = 0 \) there). Then, Condition ii) yields the following:

- For \( 0 < h < \delta_U \), we have
  \[ p(\Delta_+(h)) \leq \frac{1}{|h|} \int_0^{x+h} q(Ad_{\alpha_+(-)}^{-1}(\Phi(x, s) - \Phi(x, s))) ds \leq \sup_{s \leq x \leq x + h} m(\Phi(x, s) - \Phi(x, s)) \leq |h| \cdot L_{m,0}. \]

- For \( -\delta_U < h < 0 \), we have
  \[ p(\Delta_-(h)) \leq \frac{1}{|h|} \int_0^{x+h} q(Ad_{\beta_-(\cdot)}^{-1}(\Phi(x, s) - \Phi(x + h, s))) ds \leq \sup_{s \leq x + h \leq x + h} m(\Phi(x, s) - \Phi(x + h, s)) \leq |h| \cdot L_{m,0}. \]

This proves (62), hence the claim. \( \square \)

### 3.3 An integral expansion for the adjoint action

Let \( a < b \) and \( \psi \in \mathcal{D}_{[a,b]} \) be given. For \( X \in \mathfrak{g} \), we define
\[
\text{Ad}_{\psi}[X]: [a, b] \rightarrow \mathfrak{g}, \quad t \mapsto \text{Ad}_{[a, \psi]^{\pm 1}}(X).
\]

We furthermore define
\[
\text{Ad}_{\psi}^\pm[t]: \mathfrak{g} \ni X \mapsto \text{Ad}_{\psi}[X](t) \in \mathfrak{g} \quad \forall \ t \in [a, b],
\]
\[
\text{Ad}_{\psi}^\pm := \text{Ad}_{\psi}[b].
\]

For \( \chi \in C^0([a, b], \mathfrak{g}) \), we set
\[
\eta^\pm(\psi, \chi): [a, b] \ni t \mapsto \text{Ad}_{\psi}^\pm[\chi(t)](t) \in \mathfrak{g}.
\]

The following assertions are immediate from the definitions:

- We have \( \eta^\pm(\psi, \chi) \in C^{k+1}([a, b], \mathfrak{g}) \) for \( k \in \mathbb{N} \cup \{\infty\} \), for each \( \psi \in \mathcal{D}_{[a,b]}^k \) and \( \chi \in C^{k+1}([a, b], \mathfrak{g}) \) (by (37) and smoothness of the group operations).

- We have \( \eta^\pm(\psi, \eta^\pm(\psi, \chi)) = \chi \) for each \( \psi \in \mathcal{D}_{[a,b]} \) and \( \chi \in C^1([a, b], \mathfrak{g}) \).

We furthermore observe the following:

**Lemma 12** Let \( a < b, \psi \in \mathcal{D}_{[a,b]}, \) and \( X \in \mathfrak{g} \) be given. Then, \( \text{Ad}_{\psi}^\pm[X] \in C^1([a, b], \mathfrak{g}) \) holds, with \( \text{Ad}_{\psi}^\pm[X](a) = X \) as well as
\[
\partial_t \text{Ad}_{\psi}^\pm[X](t) = [\psi(t), \text{Ad}_{\psi}^\pm[X](t)]
\]
and
\[
\partial_t \text{Ad}_{\psi}^-[X](t) = -\text{Ad}_{\psi}^-[[\psi(t), X]](t) \tag{64}
\]
for each \( t \in [a, b] \). In particular, for \( \chi \in C^1([a, b], \mathfrak{g}) \), we have
\[
\eta^+(\psi, \chi) = \eta^+(\psi, \chi) + [\psi, \eta^+(\psi, \chi)]
\]
\[
\eta^-(\psi, \chi) = \eta^-(\psi, \chi) - \eta^-(\psi, [\psi, \chi]). \tag{65}
\]
Lemma 13
We obtain the following statement.

Proof
Equation (64) is verified in Appendix A.6; and (65) is clear from (64) as well as the parts b), d), e) of Proposition A.1.

For the sake of completeness, we want to mention the following well-known result:

Corollary 7
Let \( a < b, \psi \in \mathcal{D}_{[a,b]}, X \in \mathfrak{g} \). Then, \( \text{Ad}_\psi^+ [X] \) is the unique solution \( \alpha \in C^1([a,b], \mathfrak{g}) \) to the differential equation (Lax equation) \( \dot{\alpha} = [\psi, \alpha] \), with the initial condition \( \alpha(a) = X \).

Proof
By Lemma 12, it remains to show uniqueness. Let thus \( \alpha \in C^1([a,b], \mathfrak{g}) \) be given, with \( \dot{\alpha} = [\psi, \alpha] \) and \( \alpha(a) = X \). Then, \( \dot{\eta}^-(\psi, \alpha) = 0 \) holds by (65), with \( \eta^-(\psi, \alpha)(a) = X \). Then, (17) yields \( \eta^-(\psi, \alpha) = X \), hence

\[
\text{Ad}_\psi^+ [X] = \eta^+(\psi, C X |_{[a,b]}) = \eta^+(\psi, \eta^-(\psi, \alpha)) = \alpha.
\]

This proves the claim.

Let \( \widehat{\mathfrak{g}} \) denote the completion of \( \mathfrak{g} \). Let \( \psi \in C^0([a,b], \mathfrak{g}), X \in \mathfrak{g}, n \in \mathbb{N} \) be given. We set

\[
\lambda_{0,\psi}^\pm [X] : [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto X,
\]

and define for \( \ell \geq 1 \)

\[
\lambda_{\ell,\psi}^+ [X] : [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto \int_a^t ds_1 \int_a^{s_1} ds_2 \ldots \int_a^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(s_1)} \circ \ldots \circ \text{ad}_{\psi(s_\ell)} \right) (X)(s_\ell) \nabla - 1 \quad \ell = 1,
\]

\[
\lambda_{\ell,\psi}^- [X] : [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto (-1)^\ell \cdot \int_a^t ds_1 \int_a^{s_1} ds_2 \ldots \int_a^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(s_1)} \circ \ldots \circ \text{ad}_{\psi(s_\ell)} \right) (X)(s_\ell).
\]

For \( \ell \geq 1 \), we set

\[
\mathcal{N}_{\ell,\psi}^+ [X] : [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto \int_a^t ds_1 \int_a^{s_1} ds_2 \ldots \int_a^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(s_1)} \circ \ldots \circ \text{ad}_{\psi(s_\ell)} \right) (\text{Ad}_\psi^+ [X](s_\ell))
\]

\[
\mathcal{N}_{\ell,\psi}^- [X] : [a, b] \to \widehat{\mathfrak{g}}, \quad t \mapsto (-1)^\ell \cdot \int_a^t ds_1 \int_a^{s_1} ds_2 \ldots \int_a^{s_{\ell-1}} ds_\ell \text{Ad}_\psi^- ((\text{ad}_{\psi(s_1)} \circ \ldots \circ \text{ad}_{\psi(s_\ell)})(X))(s_\ell).
\]

To simplify the notations, we define

\[
\lambda_{\ell,\psi}^\pm [t] : g \ni X \mapsto \lambda_{\ell,\psi}^\pm [X](t) \in \widehat{\mathfrak{g}} \quad \forall t \in [a, b], \ell \in \mathbb{N}
\]

\[
\lambda_{\ell,\psi}^\pm := \lambda_{\ell,\psi}^\pm [b] \quad \forall \ell \in \mathbb{N}.
\]

We obtain the following statement.

Lemma 13
Let \( \psi \in \mathcal{D}_{[a,b]}, X \in \mathfrak{g}, \) and \( n \in \mathbb{N} \) be given. Then, for \( t \in [a, b] \) we have

\[
\text{Ad}_\psi^+ [X](t) = \sum_{\ell=0}^n \lambda_{\ell,\psi}^+ [X](t) + \mathcal{N}_{n+1,\psi}^+ [X](t)
\]

\[
\text{Ad}_\psi^- [X](t) = \sum_{\ell=0}^n \lambda_{\ell,\psi}^- [X](t) + \mathcal{N}_{n+1,\psi}^- [X](t)
\]

Proof
By Lemma 12 and (17), we have

\[
\text{Ad}_\psi^+ [X](t) = X + \int_a^t \psi(s), \text{Ad}_\psi^+ [X](s) ds = \lambda_{0,\psi}^+ [X](t) + \mathcal{N}_{1,\psi}^+ [X](t)
\]

\[
\text{Ad}_\psi^- [X](t) = X - \int_a^t \text{Ad}_\psi^- [[\psi(s), X](s)] ds = \lambda_{0,\psi}^- [X](t) + \mathcal{N}_{1,\psi}^- [X](t).
\]

We thus can assume that the claim holds for some \( n \in \mathbb{N} \). We obtain from (22) (third steps) that

\[
\text{Ad}_\psi^+ [X](t) = \sum_{\ell=0}^n \lambda_{\ell,\psi}^+ [X](t) + \mathcal{N}_{n+1,\psi}^+ [X](t)
\]

\[
= \sum_{\ell=0}^n \lambda_{\ell,\psi}^+ [X](t).
\]
Then, as well as

\[ \sum_{t=0}^{n} \lambda_{t,\psi}^{+} [X](t) + (\lambda_{n+1,\psi}^{+} [X] + 2\mathfrak{R}_{n+2,\psi}^{+} [X](t)) \]

as well as

\[ \mathbf{Ad}_{\psi}^{-} [X](t) = \sum_{t=0}^{n} \lambda_{t,\psi}^{-} [X](t) + 2\mathfrak{R}_{n+1,\psi}^{-} [X](t) \]

\[ \mathbf{Ad}_{\psi}^{\pm} [X](t) = \lim_{t \to \infty} \sum_{t=0}^{\infty} \lambda_{t,\psi}^{\pm} [X] = \sum_{t=0}^{\infty} \lambda_{t,\psi}^{\pm} [X] \] (68)

The claim now follows by induction. \( \square \)

We recall the definition of an AE-set in (28), and obtain the following corollary.

**Corollary 8** Let \( M \subseteq \mathfrak{g} \) be an AE-set, \( a < b \), and \( \psi \in D_{[a,b]} \) with \( \text{im}[\psi] \subseteq M \) be given. Then,

\[ \mathbf{Ad}_{\psi}^{\pm} [X] = \lim_{n} \sum_{t=0}^{n} \lambda_{t,\psi}^{\pm} [X] = \sum_{t=0}^{\infty} \lambda_{t,\psi}^{\pm} [X] \] (68)

converges uniformly for each fixed \( X \in M \). In particular, the following assertions hold:

1. For \( \nu \leq w \) as in (28), we have

\[ \nu \circ \mathbf{Ad}_{\psi}^{\pm} [X] \leq w(X) \cdot e^{\int_{a}^{b} \nu(s) ds} \quad \forall X \in M. \]

2. By (48), for \( Z \in \text{dom}[\exp] \cap M \) we have

\[ \mathbf{Ad}_{\exp(t, Z)^{\pm 1}} [X](t) = \mathbf{Ad}_{CZ_{[0,1]}}^{\pm} [X](t) = \sum_{t=0}^{\infty} \frac{t^{n}}{n!} \cdot \mathbf{ad}_{\pm Z}^{X} (X) \quad \forall X \in M, \ t \in [0, 1]. \]

**Proof** Equation (68) is clear from the following two estimates:

- Let \( \nu \leq w \) be as in (28), as well as \( w \leq m \) be as in Corollary 6 for \( q \equiv w \) and \( C \equiv \text{im}[\nu^{*}, \psi] \) there. We obtain from Lemma 13 that

\[ \nu(\mathbf{Ad}_{\psi}^{\pm} [X](t)) - \sum_{t=0}^{n} \lambda_{t,\psi}^{\pm} [X](t)) = \nu(\mathfrak{R}_{n+1,\psi}^{\pm} [X](t)) \leq \frac{(b-a)^{n+1}}{(n+1)!} \cdot w_{\infty}(\psi)^{n+1} \cdot w_{\infty}(\mathbf{Ad}_{\psi}^{\pm} [X]) \]

holds for each \( t \in [a, b] \), \( X \in \mathfrak{g} \), and \( n \in \mathbb{N} \).

- Let \( q \leq m \) be as in Corollary 6 for \( C \equiv \text{im}[\text{inv} \circ f_{a}^{*}, \psi] \) there. Let furthermore \( m \leq w \) be as in (28) for \( \nu \equiv m \) there. We obtain from Lemma 13 that

\[ q(\mathbf{Ad}_{\psi}^{\pm} [X](t)) - \sum_{t=0}^{n} \lambda_{t,\psi}^{\pm} [X](t)) = q(\mathfrak{R}_{n+1,\psi}^{\pm} [X](t)) \leq \frac{(b-a)^{n+1}}{(n+1)!} \cdot w_{\infty}(\psi)^{n+1} \cdot w(X) \]

holds for each \( t \in [a, b] \), \( X \in \mathfrak{g} \), and \( n \in \mathbb{N} \).
Now, Part 2) is just clear from (68), and Part 1) is clear from (68) as well as
\[ v(\lambda_{\ell, \psi}^{\pm}[X](t)) \leq \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{\ell-1}} ds_\ell \ w(\psi(s_1)) \cdots w(\psi(s_\ell)) \cdot w(X) = \frac{1}{\ell!} \cdot (\int_a^t w(\psi(s)) \ ds)^\ell \cdot w(X) \]
for each \( \ell \geq 1 \). (The equality in the second step just follows by induction over \( \ell \geq 1 \), via taking the derivative of both expressions.)

**Example 7** Let \( N \subseteq g \) be a Nil\(_q\)-set for some \( q \geq 2 \). Then, \( N \) is an AE-set, and Corollary 8 shows
\[ \text{Ad}_{\psi}^{\pm}[X] = \sum_{\ell=0}^{q-2} \lambda_{\ell, \psi}^{\pm}[X] \]
for each \( X \in N \) and \( \psi \in \mathcal{D} \) with \( \text{im}[\psi] \subseteq N \). In particular, we have
\[ \text{Ad}_{\exp(t, Z)^{\pm}}(X) = \sum_{\ell=0}^{q-2} \ell! \cdot \text{ad}_Z^\ell(X) \quad \forall \ t \in [0, 1] \]
for each \( X \in N \) and \( Z \in \text{dom}[\exp] \cap N \).

### 4 A generalized BCDH formula

In the first part of this section, we generalize the Baker–Campbell–Dynkin–Hausdorff formula (for the exponential map) to the product integral (cf. Proposition 3). In the second part, we apply this formula (together with the integral transformation introduced in Sect. 3.1) to express the product integral of nilpotent curves\(^{13}\) (cf. Theorem 1) via the exponential map. Various applications of the derived formula are discussed in Sect. 4.2.1.

#### 4.1 A BCDH formula for the product integral

Let \( \mathcal{X} \in C^1([a, b], g) \) for \( a < b \) be given, with \( \text{im}[\mathcal{X}] \subseteq \text{dom}[\exp] \). In this section, we consider the following two situations:

A) \( G \) is a Banach Lie group with \( \| [X, Y] \| \leq \| X \| \cdot \| Y \| \) for all \( X, Y \in g \), and we have \( \| \mathcal{X} \|_{\infty} < \ln(2) \). We set \( V := g \) as well as \( I, J := \infty \), and observe the following:

- (\( g, [\cdot, \cdot] \)) is asymptotic estimate.
- \( \exp : g \to g \) is defined and smooth, and a local diffeomorphism. In particular, there exists an identity neighborhood \( O \subseteq G \) on which \( \exp \) is a diffeomorphism, such that \( \| \exp^{-1}(g) \| < \ln(2) \) holds for each \( g \in O \).
- \( G \) is \( C^0 \)-regular (even \( L^1 \)-regular by Theorem C in [5]). In particular, there exists some \( \tau > 0 \), such that for all \( \phi \in C^0([a, b], g) \) (\( a < b \)) and \( \psi \in C^0([a', b'], g) \) (\( a' < b' \)) with \( \int_a^b ||\phi(s)|| \ ds + \int_a^{b'} ||\psi(s)|| \ ds < \tau \), we have\(^{14}\)
\[ \int_a^t \phi \cdot \int_a^t \psi \in O \quad \forall \ t \in [a, b]. \quad \tau \in [a', b'] \]  \[ (69) \]

B) \( G \) is weakly \( C^\infty \)-regular, or \( \exp : g \to G \) is defined (\( \text{dom}[\exp] = g \)) and of class \( C^1 \). Moreover, \( N := \text{im}[\mathcal{X}] \cup \text{im}[\mathcal{Y}] \) is a Nil\(_q\)-set for some \( q \geq 2 \). We set \( V := \mathcal{G}_1(N) \) as well as \( I := q - 1 \) and \( J := q - 2 \), and recall that \( V \) is a Nil\(_q\)-set by Remark 3.3).

---

\(^{13}\) Specifically, such \( g \)-valued curves whose image is a Nil\(_q\)-set for some \( q \geq 2 \), recall (29).

\(^{14}\) Alternatively, combine Proposition 14.6 in [4] (or Example 2 in [8]) with Proposition 2 in [8].
In both situations, the presumptions in Corollary 4 are fulfilled. Moreover, by Example 1, for \( t \in [a, b] \) and \( Y \in V \), we have

\[
\Psi \left( \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \text{ad}^n_{\tilde{X}(t)} \right)(\Phi(\text{ad}\tilde{X}(t))(Y)) = Y
\]  
(70)

\[
\tilde{\Psi} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \text{ad}^n_{-\tilde{X}(t)} \right)(Y) = \Psi \left( \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \text{ad}^n_{\tilde{X}(t)} \right)(Y)
\]  
(71)

with \( \Phi(\text{ad}\tilde{X}(t))(Y) \in V \) (by Remark 3 in the situation of B)). Here, for maps \( \xi, \zeta : g \to g \) as well as \( X \in g \), we set (convergence presumed)

\[
\Psi(\xi)(X) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (\xi - \text{id}_g)^{n-1}(X)
\]

\[
\tilde{\Psi}(\xi)(X) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (\xi \circ (\xi - \text{id}_g)^{n-1})(X)
\]

\[
\Phi(\xi)(X) := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \cdot \xi^n(X).
\]

We obtain the following proposition:

**Proposition 3** Let \( \phi \in \mathcal{D}_{[a,b]}, \ g \in G, \tilde{X} \in C^1([a,b], g) \) be given, with \( \text{im}[\tilde{X}] \subseteq \text{dom}[\exp] \) as well as

\[
(\int_a^t \phi) \cdot g = \exp(\tilde{X}(t)) \quad \forall t \in [a,b].
\]  
(72)

Assume furthermore that we are in the situation of A) or B). Then, we have \( \text{im}[\phi] \subseteq V \), and the following identities hold for each \( t \in [a,b] \):

\[
\tilde{X}(t) - \tilde{X}(a) = \int_a^t \Psi(\text{Ad}_{[a,b]_g} \circ \text{Ad}_g)(\phi(s)) \, ds
\]  
(73)

\[
\tilde{X}(t) - \tilde{X}(a) = \int_a^t \tilde{\Psi}(\text{Ad}_{g^{-1}} \circ \text{Ad}_{[a,b]_g^{-1}})(\phi(s)) \, ds
\]  
(74)

\[
\tilde{X}(t) - \tilde{X}(a) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \int_a^t (\text{Ad}_{[a,b]_g} \circ \text{Ad}_g - \text{id}_g)^{n-1}(\phi(s)) \, ds.
\]  
(75)

**Proof** We observe the following:

- For \( t \in [a,b] \), we have by Corollary 8.2 (and Example 7)

\[
\text{Ad}_{\exp(\phi_{\tilde{X}(t)})} | V = \sum_{n=0}^{\infty} \frac{j^n}{n!} \cdot \text{ad}_{\tilde{X}(t)} | V.
\]  
(76)

- For \( t \in [a,b] \), we have

\[
\Phi(\text{ad}\tilde{X}(t))(\tilde{X}(t)) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \cdot \text{ad}_{\tilde{X}(t)}(\tilde{X}(t)) \in V.
\]  
(77)

- For \( t \in [a,b] \), we have by (70), (71), and (77) that

\[
\dot{\tilde{X}}(t) = \Psi(\sum_{n=0}^{\infty} \frac{1}{n!} \cdot \text{ad}^n_{\tilde{X}(t)})(\Phi(\text{ad}\tilde{X}(t)))(\dot{\tilde{X}}(t))
\]

\[
= \tilde{\Psi}(\sum_{n=0}^{\infty} \frac{1}{n!} \cdot \text{ad}^n_{\tilde{X}(t)})(\Phi(\text{ad}\tilde{X}(t)))(\dot{\tilde{X}}(t)).
\]  
(78)

Set \( \alpha := \exp \circ \tilde{X} \in C^1([a,b], g) \). Then, given \( t \in [a,b] \), we obtain from (72) (first step), Corollary 4 (second step), (22) (third step), (76) (fourth step), \( C^0 \)-continuity of the Riemann integral in the situation of A) (fifth step), as well as (77) (fifth step) that

\[
d_t R_{\alpha(t)}(\phi(t)) = \dot{\alpha}(t)
\]

\[
= d_t L_{\alpha(t)}(\int_0^1 \text{Ad}_{\exp(-s, \tilde{X}(t))}(\dot{\tilde{X}}(t)) \, ds)
\]

\[
= d_t R_{\alpha(t)}(\int_0^1 \text{Ad}_{\exp((1-s) \cdot \tilde{X}(t))}(\dot{\tilde{X}}(t)) \, ds)
\]
We obtain from (17), (78), (79), and (76) the following identities for each $t$:

$$= d_x R_{a(t)} \left( \int_0^1 \sum_{n=0}^J (1-s)^n/n! \cdot \text{ad}_{X(t)}^n (\dot{X}(t)) \, ds \right)$$

Together with (77), this shows

$$= d_x R_{a(t)} \left( \sum_{n=0}^J \frac{1}{(n+1)!} \cdot \text{ad}_{X(t)}^n (\dot{X}(t)) \right).$$

We now finally have to prove (75):

Together with (72), this implies (73) and (74). We now finally have to prove (75):

- Assume that we are in the situation of A). Then, (76) (first step) together with (79), we obtain from (17), (78), (79) (first step), linearity of the Riemann integral (third step), (76) (fourth step), and (72) (fifth step) that

$$X(t) - X(a) = \int_a^t \Psi \left( \sum_{\ell=0}^\infty \frac{1}{\ell!} \cdot \text{ad}_{X(t)}^\ell (\dot{X}(t)) \right) \, ds$$

holds for each $t$. Let $\|X\|_\infty < \ln(2)$ (last step) implies

$$\|(\text{Ad}_{\exp(X(t))} - \text{id}_g)^k (\phi(t))\| \leq (\| \sum_{\ell=1}^\infty \frac{1}{\ell!} \cdot \text{ad}_{X(t)}^\ell \|_{\text{op}})^k \cdot \|\phi\|_\infty$$

for each $t \in [a, b]$ and $k \geq 1$. We obtain from (80) (first step), $C^0$-continuity of the Riemann integral and (81) (third step), as well as (72) (fourth step) that

$$X(t) - X(a) = \int_a^t \Psi \left( \sum_{\ell=1}^\infty \frac{1}{\ell!} \cdot \text{ad}_{X(t)}^\ell (\dot{X}(t)) \right) \, ds$$

holds for each $t \in [a, b]$. □

**Proposition 4** Assume that $G$ is a Banach Lie group, and let $\tau > 0$ be as in A). Let $\phi \in C^0([a, b], g)$ $(a < b)$ and $\psi \in C^0([a', b'])$ $(a' < b')$ be given, with $\int_a^b \|\phi(s)\| \, ds + \int_{a'}^{b'} \|\psi(s)\| \, ds < \tau$. Then, we have

$$\int_a^t \phi \cdot \int_{a'}^{b'} \psi = \exp \left( X_{\psi} (b') + X_{\phi} (t) \right) \quad \forall t \in [a, b].$$
provided that for \( t' \in [a', b'] \) and \( t \in [a, b] \) we set
\[
\mathcal{X}_\psi(t') := \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right)^{n-1} \cdot \int_{a'}^{t'} \left( \sum_{\ell=1}^{\infty} \lambda_{\ell, \psi}^+(s) \right)^{n-1} (\psi(s)) \, ds
\]
\[
\mathcal{X}_{\phi, \psi}(t) := \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right)^{n-1} \cdot \int_{a}^{t} \left( \sum_{\ell=1}^{\infty} \lambda_{\ell, \phi}^+(s) \right) \circ \left( \sum_{\ell=1}^{\infty} \lambda_{\ell, \psi}^+(b') \right) - \text{id}_g)^{n-1} (\phi(s)) \, ds.
\]

**Proof** By definition, there exist open intervals \( I, I' \subseteq \mathbb{R} \) with \([a, b] \subseteq I\) and \([a', b'] \subseteq I'\), as well as \( \mu \in C^1(I, G) \) and \( \nu \in C^1(I', G) \), with \( \delta'^I(\mu)|_{[a,b]} = \phi \) and \( \delta^I(\nu)|_{[a',b']} = \psi \). We make the following modifications to these curves:

- We replace \( \mu \) by \( \mu \cdot \mu(a)^{-1} \) to ensure \( \mu|_{[a,b]} = \int_{a}^{b} \phi. \)
- We replace \( \nu \) by \( \nu \cdot \nu(a')^{-1} \) to ensure \( \nu|_{[a',b']} = \int_{a'}^{b'} \psi. \)
- We define \( \alpha := \mu \cdot \nu(b') \), and observe \( \alpha|_{[a,b]} = \int_{a}^{b} \phi \cdot \int_{a'}^{b'} \psi. \)
- We shrink \( I \) around \([a, b]\) and \( I' \) around \([a', b']\), to ensure \( \mu(I) \cup \nu(I') \subseteq O \) (recall (69)).

Then, the maps
\[
\mathcal{X} : [a, b] \ni t \mapsto \exp^{-1}(\alpha(t)) \in g \quad \text{and} \quad \mathcal{X}' : [a', b'] \ni t' \mapsto \exp^{-1}(\nu(t')) \in g
\]
are defined and of class \( C^1 \), with
\[
\mathcal{X}(a) = \mathcal{X}'(b'), \quad \mathcal{X}'(a') = 0, \quad \int_{a}^{b} \phi \circ \mathcal{X}' = \exp \circ \mathcal{X}, \quad \int_{a'}^{b'} \psi \circ \mathcal{X}' = \exp \circ \mathcal{X}.
\]
We conclude the following:

- Equation (75) in Proposition 3 for \( g \equiv e \) there, together with Corollary 8 shows \( \mathcal{X}' = \mathcal{X}_\psi \).
- Equation (75) in Proposition 3 for \( g \equiv \int_{a'}^{b'} \psi \) there, together with Corollary 8 and the previous point shows
\[
\mathcal{X} - \mathcal{X}_\psi(b') = \mathcal{X} - \mathcal{X}'(b') = \mathcal{X} - \mathcal{X}(a) = \mathcal{X}_{\phi, \psi},
\]
which proves the claim. □

**Corollary 9** Assume that \( G \) is a Banach Lie group, and let \( \tau > 0 \) be as in \( A \). Then, for \( \phi \in C^0([a,b], g) \) \((a < b)\) with \( \int_{a}^{b} \|\phi(s)\| \, ds < \tau \), we have
\[
\exp^{-1} \left( \int_{a}^{b} \phi \right) = \sum_{n=1}^{\infty} \left( \frac{-1}{n} \right)^{n-1} \cdot \int_{a}^{t} \left( \sum_{\ell=1}^{\infty} \lambda_{\ell, \phi}^+(s) \right)^{n-1} (\phi(s)) \, ds \quad \forall \ t \in [a, b].
\]

**Proof** Apply Proposition 4 with \( \psi = 0 \). □

**Example 8** (The BCDH Formula) Assume that we are in the situation of Proposition 3, with \([a, b] = [0, 1], \quad g = \exp(-X), \quad \mathcal{X}(0) = -X \) for certain \( X, Y \in g \) and \( \mathcal{X} \in C^1([0,1], g) \). We obtain the well-known \(^{15}\) integral representation of the Baker–Campbell–Dynkin–Hausdorff formula:
\[
\exp(X) \cdot \exp(t \cdot Y) = \exp \left( X + \int_{0}^{t} \tilde{\Psi}(\text{Adexp}(X) \circ \text{Adexp}(s \cdot Y))(Y) \, ds \right) \quad \forall \ t \in [0,1].
\]

**Proof of Equation (83)** By (48), we have \( \int_{0}^{t} \phi = \exp(-t \cdot Y) \) for each \( t \in [0, 1] \), hence
\[
\exp(X) \cdot \exp(t \cdot Y) = \left( \int_{0}^{t} \phi \cdot g \right)^{-1} \overset{(72)}{=} \exp(\mathcal{X}(t))^{-1} = \exp(-\mathcal{X}(t)).
\]

\(^{15}\) For the finite-dimensional case, confer, e.g., Proposition 3.4.4 in \([13]\), with \( \Psi \equiv \tilde{\Psi} \).
Then, (74) in Proposition 3 shows
\[-(X(t) - X(0)) = -\int_0^t \tilde{\Psi} (\text{Ad}_{g^{-1}} \circ \text{Ad}_{[0_\phi]} (-Y)) \, ds\]
for each \( t \in [0, 1] \), from which the claim follows. \( \square \)

For instance, the assumptions in (82) are fulfilled in the following situations:

- \( G \) is a Banach Lie group, with \( \|X\| + \|Y\| < t \). Indeed, by A), then there exists \( X \in C^1([0, 1], g) \) with \( X(0) = -X \), such that (72) holds for \( \phi \) and \( g \) as in (82).

- \( G \) is weakly \( C^k \)-regular for \( k \in \{ \text{lip}, \infty \}, \) \( \{X, Y\} \) is a \( N_q \)-set for some \( q \geq 2 \). Indeed, we will reconsider this situation in Corollary 11 in Sect. 4.2. There, we construct some \( X \in C^1([0, 1], g) \) with \( X(0) = -X \), such that (72) holds for \( \phi \) and \( g \) as in (82). \( \square \)

### 4.2 The product integral of nilpotent curves

Assume that \( G \) is weakly \( C^k \)-regular for \( k \in \{ \text{lip}, \infty \}, \) \( \text{and} \) \( \phi \in C^k([a, b], g) \) for \( a < b \) as well as \( \psi \in C^k([a', b'], g) \) for \( a' < b' \) be given.

- If \( \text{im} [\phi] \) is a \( N_q \)-set for some \( q \geq 2 \), then for each \( t \in [a', b'] \) we set
  \[ X_{\psi}(t) := \sum_{n=1}^{q-1} (-1)^{n-1} \cdot \int_a^t (\text{Ad}_{\phi} \cdot \psi - \text{id}_g)^{n-1}(\psi(s)) \, ds \]
  \[ = \sum_{n=1}^{q-1} (-1)^{n-1} \cdot \int_a^t (\sum_{\ell=1}^{q-2} \lambda_{\ell, \psi}^+ [s])^{n-1}(\psi(s)) \, ds. \]
  The second line is due to (22) and Example 7.

- If \( \text{im}[\phi] \cup \text{im}[\psi] \) is a \( N_q \)-set for some \( q \geq 2 \), then for each \( t \in [a, b] \) we set
  \[ X_{\phi, \psi}(t) := \sum_{n=1}^{q-1} (-1)^{n-1} \cdot \int_a^t (\text{Ad}_{\phi} \cdot \psi - \text{id}_g)^{n-1}(\phi(s)) \, ds \]
  \[ = \sum_{n=1}^{q-1} (-1)^{n-1} \cdot \int_a^t ((\sum_{\ell=0}^{q-2} \lambda_{\ell, \phi}^+ [s]) \circ (\sum_{\ell=0}^{q-2} \lambda_{\ell, \psi}^+ [b']) - \text{id}_g)^{n-1}(\phi(s)) \, ds. \]
  The second line is due to (22) and Example 7.

Notably, both maps take values in \( g \) by Lemma 8 and are thus of class \( C^1 \) by (20). In this section, we combine Proposition 3 with the integral transformation introduced in Sect. 3.1 to prove the following statement.

**Theorem 1** Assume that \( G \) is weakly \( C^k \)-regular for \( k \in \{ \text{lip}, \infty \}, \) \( \phi \in C^k([a, b], g), \) \( \psi \in C^k([a', b'], g) \) be given, such that \( N := \text{im}[\phi] \cup \text{im}[\psi] \) is a \( N_q \)-set for some \( q \geq 2 \). Then,
\[ \int_a^t \phi \cdot \int_a^t \psi = \exp (X_{\psi}(b') + X_{\phi, \psi}(t)) \quad \forall t \in [a, b]. \]

The proof of Theorem 1 is given in Sect. 4.2.2. We now first want to discuss some consequences.

### 4.2.1 Some consequences

Setting \( \psi = 0 \) in Theorem 1, we immediately obtain the following statement.
Corollary 10  Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$, and let $\phi \in C^k([a, b], \mathfrak{g})$ be given such that $\text{im}[\phi]$ is a $\text{Nil}_q$-set for some $q \geq 2$. Then, we have

$$
\int_a^t \phi = \exp \left( \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_a^t \left( \sum_{\ell=1}^{q-2} \lambda_{\ell, \phi}[s] \right)^{n-1} (\phi(s)) \, ds \right) \quad \forall t \in [a, b].
$$

Example 9  Let $G$ be abelian, as well as weakly $C^k$-regular for some $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Then, $\mathfrak{g}$ is a $\text{Nil}_2$-set, and Corollary 10 recovers formula (51) in Example 6.

Moreover, combining Theorem 1 with Example 8, we obtain the BCDH formula:

Corollary 11  Assume that $G$ is weakly $C^k$-regular for $k \in \mathbb{N} \cup \{\text{lip}, \infty\}$. Let furthermore $X, Y \in \mathfrak{g}$ be given, such that $[X, Y]$ is a $\text{Nil}_q$-set for some $q \geq 2$. Then, we have

$$
\exp(X) \cdot \exp(t \cdot Y) = \exp \left( X + \int_0^t \tilde{\Psi}(\text{Ad}_{\exp(X)} \circ \text{Ad}_{\exp(s \cdot Y)}) (Y) \, ds \right) \quad \forall t \in [0, 1].
$$

Proof  Set $\phi := -\mathcal{E}_Y|_{[0, 1]}$ as well as $\psi := -\mathcal{E}_X|_{[0, 1]}$, and let $\mathcal{X}_\psi, \mathcal{X}_{\phi, \psi}$ be as in (84),(85). Then, $\mathcal{X} : [0, 1] \ni t \mapsto \mathcal{X}_\psi(t) + \mathcal{X}_{\phi, \psi}(t) \in \mathfrak{g}$ is of class $C^1$, with $\mathcal{X}(0) = \mathcal{X}_\psi(0) = -X$, as well as

$$
\int_0^t \phi \cdot \int_0^t \psi = \exp(\mathcal{X}(t)) \quad \forall t \in [0, 1]
$$

by Theorem 1. The claim is thus clear from Example 8.

Next, we define

$$
\Gamma_{\text{Nil}}^\infty := \{ \phi \in \mathcal{D}_0^\infty \mid \text{im}[\phi] \text{ is a } \text{Nil}_q\text{-set for some } q \geq 2 \}
$$

and set $G_{\text{Nil}}^\infty := \text{evol}_\infty(\Gamma_{\text{Nil}}^\infty)$.

We observe the following:

Lemma 14  If $G$ is weakly $C^\infty$-regular, then $G_{\text{Nil}}^\infty = \text{im[exp]}$ holds.

Proof  Obviously, we have $\text{im[exp]}((\text{dom}[\exp])) \subseteq \Gamma_{\text{Nil}}^\infty$, hence $\text{im[exp]} \subseteq G_{\text{Nil}}^\infty$. Moreover, Corollary 10 shows $G_{\text{Nil}}^\infty \subseteq \text{im[exp]}$, which proves the claim.

Lemma 15  Assume that $G$ is connected. Then, $\text{evol}_\infty$ is surjective.

Proof  Let $\mathcal{A} \subseteq G$ denote subgroup of all $g \in G$, such that there exist $\mu_1, \ldots, \mu_n \in C^\infty([0, 1], G)$ $(n \in \mathbb{N})$, with $g = \mu_n(1) \cdot \ldots \cdot \mu_1(1)$ as well as $\mu_\ell(0) = e$ for $\ell = 1, \ldots, n$. Set furthermore $\mathcal{O} := \mathcal{U} \cap \text{inv}(\mathcal{U})$. Since $G$ is connected, we have $G = \bigcup_{n \geq 1} \mathcal{O}^n$. Since $\mathcal{O} \subseteq \mathcal{U} \subseteq \mathcal{A}$ holds, we obtain $G = \mathcal{A}$. Let now $g \in G$ be fixed, and choose $\mu_1, \ldots, \mu_n \in C^\infty([0, 1], G)$ with $\mu_1(0), \ldots, \mu_n(0) = e$ and $g = \mu_n(1) \cdot \ldots \cdot \mu_1(1)$. We set $\phi_\ell := \delta^\ell(\mu_\ell) \in \mathcal{D}_0^\infty$ for $\ell = 1, \ldots, n$, and observe

$$
g = \mu_n(1) \cdot \ldots \cdot \mu_1(1) = \int_0^1 \phi_n \cdot \ldots \int_0^1 \phi_1.
$$

Applying (a) inductively, we obtain (recall Remark 5)

$$
g = \int_0^1 \phi_n \cdot \ldots \int_0^1 \phi_1 = \int_0^1 \phi_n^{\cdot*(\phi_{n-1}^{\cdot*(\ldots(\phi_2^{\cdot*(\phi_1)}))})} \in \mathcal{D}_0^\infty
$$

which proves the claim.

\footnote{For $g \in \mathcal{U}$, consider $C^\infty([0, 1], G) \ni \mu_1 : [0, 1] \ni t \mapsto \mathcal{E}^{-1}(t \cdot \mathcal{E}(g)) \in G.$}
Combining Lemma 14 with Lemma 15, we obtain the following proposition.

**Proposition 5** Assume that $G$ is weakly $C^\infty$-regular and connected. If $\Gamma^\infty_{\text{Nil}} = C^\infty([0, 1], g)$ holds, then $\exp: g \to G$ is surjective.

**Proof** Lemma 14 shows $\text{im}(\exp) = G^\infty_{\text{Nil}} = \text{evol}_{\infty}(\Gamma^\infty_{\text{Nil}})$, with $\text{evol}_{\infty}(\Gamma^\infty_{\text{Nil}}) = \text{evol}_{\infty}(C^\infty([0, 1], g))$ by assumption. Since $\text{evol}_{\infty}$ is surjective by Lemma 15, the claim follows. \hfill \Box

Proposition 5 immediately implies the following statement.

**Corollary 12** Assume that $G$ is weakly $C^\infty$-regular and connected. If $(g, [\cdot, \cdot])$ is nilpotent, then $\exp$ is surjective.

**Remark 11** (1) The statement in Corollary 12 follows from Theorem IV.2.6 in [21] provided we replace “weakly $C^\infty$-regular” by “$g$ is Mackey complete and $\exp$ is smooth”.

(2) Let $g$ be finite-dimensional. Then, the hypothesis $\Gamma^\infty_{\text{Nil}} = C^\infty([0, 1], g)$ in Proposition 5 is equivalent to nilpotency of $g$.

In fact, the one implication is evident. Assume thus that $\Gamma^\infty_{\text{Nil}} = C^\infty([0, 1], g)$ holds, and let $\{X_0, \ldots, X_{n-1}\} \subseteq g$ be a base of $g$. Let furthermore $\rho: [0, 1/n] \to [0, \infty)$ be a bump function, i.e., $\rho$ is smooth with

$$\rho((0,1/n)) > 0 \quad \text{as well as} \quad \rho(1)(0) = 0 = \rho(1/n) \quad \forall \ell \in \mathbb{N}.$$

We define $\phi: [0, 1] \to g$ by $\phi(0) := 0$ and $\phi(\ell/n, (\ell + 1)/n) \ni t \mapsto \rho(t - \ell/n) \cdot X_{\ell} \in g \quad \forall \ell = 0, \ldots, n - 1$.

Then, $\phi$ is easily seen to be smooth (for details confer, e.g., Lemma 24 and Appendix B.2 in [8]). By assumption, there exists some $q \geq 2$, such that $N := \text{im}(\phi)$ is a Nil$_q$-set. Then, $\mathcal{V}_1(N)$ is a Nil$_q$-set by Remark 3, with $\{X_0, \ldots, X_{n-1}\} \subseteq \mathcal{V}_1(N)$, hence $g = \mathcal{V}_1(N)$. \hfill \Box

We finally want to mention that Proposition 5 also provides an easy proof of the following well-known result in the finite-dimensional context.

**Corollary 13** If $G$ is compact and connected, then $\exp$ is surjective.

**Proof** Let $T \subseteq G$ be a maximal torus with Lie algebra $t$. Since $T$ is abelian, Corollary 12 shows that $\exp|_t: t \to T$ is surjective. Since $G$ is covered by maximal tori (confer, e.g., Theorem 12.2.2.(iii) in [13]), the claim follows. \hfill \Box

**Remark 12** The proof of Corollary 13 can be simplified. Specifically, surjectivity of $\exp|_t: t \to T$ also follows from Lemma 15 and formula (51) (as $T$ is abelian). Notably, in finite dimensions, (51) follows easily from smoothness of the exponential map:

**Proof of (51) in the finite-dimensional case** Let $G$ be abelian and finite-dimensional, and let $\phi \in C^0([a, b], g)$ be given. Let $\psi \in C^0([a, b + \varepsilon), g)$ for $\varepsilon > 0$ be a continuous extension of $\phi$. Then,

$$\mu: [a, b + \varepsilon) \ni t \mapsto \exp\left(\int_a^t \psi(s) \, ds\right) \in G$$

is of class $C^1$ as $\exp$ is smooth. Since $G$ is abelian, we have

$$(\mu(t + h) \cdot \mu(t))^{-1} = \exp\left(\int_t^{t+h} \psi(s) \, ds\right) \quad \forall t \in [a, b], \ 0 < h < \varepsilon.$$

Since $d_e \exp = \text{id}_g$ holds, we obtain $\delta^r(\mu|_{[a, b]}) = \phi$, hence

$$\int_a^b \phi = \mu|_{[a, b]} = \exp(\int_a^b \phi(s) \, ds)$$

by injectivity of $\delta^r|_{C^1([a, b], G)}$. \hfill \Box
4.2.2 The proof of Theorem 1

In this subsection, we prove Theorem 1. Our argumentation is based on the following (straightforward but technical) lemma that is proven in Sect. 4.2.3.

**Lemma 16** Let \( G \) be weakly \( C^k \)-regular for \( k \in \mathbb{N} \cup \{ \text{lip}, \infty \} \), and let \( \phi \in C^k([a, b], g) \) be given such that \( \text{im}[\phi] \) is a \( \text{Nil}_q \)-set for some \( q \geq 2 \). Then, the following assertions hold:

- We have \( \mathcal{T}^{-1}(\phi|[a, \tau]) \in C^\infty([0, 1], g) \) for each \( \tau \in [a, b] \), hence the map (recall (47))
  \[ \chi: [a, b] \ni \tau \mapsto i^{-1}(\mathcal{T}^{-1}(\phi|[a, \tau])) \in g \]
  is defined.
- We have \( \chi \in C^1([a, b], g) \), and \( \text{im}[\chi] \cup \text{im}[\tilde{\chi}] \) is a \( \text{Nil}_q \)-set.

In particular, iterated application of Proposition 1 yields

\[ \int_0^\tau \phi = \int_0^1 \mathcal{T}^{-1}(\phi|[a, \tau]) = \exp(\chi(\tau)) \quad \forall \tau \in [a, b]. \]

**Proof** Confer Sect. 4.2.3. \( \square \)

**Remark 13** Let \( \psi \in C^0([x, y], g) \) with \( x < y \) be given. Let \( \varphi: [x', y'] \to [x, y] \) \((x' < y')\) be of class \( C^1 \), with \( \varphi(x') = x \) and \( \varphi(y') = y \). We set \( \phi := \dot{\varphi} \cdot (\psi \varphi) \) and obtain inductively from (21) that

\[ \int_x^{y'} \lambda^{+}_{\ell, \varphi}[s](\psi(s)) \, ds = \int_x^{y} \lambda^{+}_{\ell, \psi}[s](\psi(s)) \, ds \]

holds for each \( \ell \in \mathbb{N} \). \( \square \)

We are ready for the proof of Theorem 1.

**Proof of Theorem 1** Let \( \varrho_1: [0, 1] \to [a', b'] \) and \( \varrho_2: [1, 2] \to [a, b] \) be smooth, such that \( \varrho_j := \dot{\varrho}_j \) is a bump function for \( j = 1, 2 \), i.e., we have

\[ \varrho_1|_{(0, 1)}, \varrho_2|_{(1, 2)} > 0 \quad \text{as well as} \quad \varrho_1^{(\ell)}(0), \varrho_1^{(\ell)}(1), \varrho_2^{(\ell)}(1), \varrho_2^{(\ell)}(2) = 0 \quad \forall \ell \in \mathbb{N}. \]

Define \( \chi: [0, 1] \to g \) by

\[ \chi|_{[0, 1]} := \varrho_1 \cdot (\psi \circ \varrho_1) \quad \text{as well as} \quad \chi|_{(1, 2]} := (\varrho_2 \cdot (\phi \circ \varrho_2))|_{(1, 2]}. \]

Then, \( \chi \) is easily seen to be of class \( C^k \) (confer, e.g., Lemma 24 and Appendix B.2 in [8]), with \( \text{im}[\chi] \subseteq \mathcal{V}_1(N) \subseteq \mathcal{F}_1(N) \) (a \( \text{Nil}_q \)-set by Remark 3). Then, for \( \tau \in [0, 1] \) and \( t \in [1, 2] \), we have

\[ \int_0^\tau \chi = \int_0^\tau \varrho_1 \cdot (\psi \circ \varrho_1) = \int_0^{\varrho_1(\tau)} \psi, \]

\[ \int_1^t \chi = \int_1^t \varrho_2 \cdot (\phi \circ \varrho_2) = \int_1^{\varrho_2(t)} \phi, \]

\[ \int_0^t \chi = \left[ \int_1^t \chi |_{(1, 2]} \right] \cdot \left[ \int_0^1 \chi |_{[0, 1]} \right] = \left[ \int_1^t \varrho_2 \cdot (\phi \circ \varrho_2) \right] \cdot \left[ \int_0^1 \varrho_1 \cdot (\psi \circ \varrho_1) \right] = \left[ \int_1^{\varrho_2(t)} \phi \right] \cdot \left[ \int_0^{\varrho_1(\tau)} \psi \right]. \]

Now, Lemma 16 provides some \( \tilde{\chi} \in C^1([0, 2], g) \) with \( \int_0^t \chi = \exp \circ \tilde{\chi} \), such that \( \text{im}[\tilde{\chi}] \cup \text{im}[\tilde{\chi}] \) is a \( \text{Nil}_q \)-set. In particular, we have

\[ \int_0^t \chi = \exp(\tilde{\chi}|_{[0, 1]}(t)) \quad \forall t \in [0, 1]; \]

\[ \text{Confer, e.g., Sect. 4.3 in [8] for explicit constructions of such bump functions.} \]
and obtain from Proposition 3 (specifically from (75) with $g \equiv e$ and $\mathcal{X} \equiv \tilde{\mathcal{X}}|_{[0,1]}$ there) and (68) in Corollary 8 (first step), as well as Remark 13 (second step) that
\[
\tilde{\mathcal{X}}|_{[0,1]}(1) = \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_0^1 \left( \sum_{z=1}^{q-2} \lambda^+_{z,\mathcal{X}}(s) \right)^{n-1}(\chi(s)) \, ds \\
= \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_0^1 \left( \sum_{z=1}^{q-2} \lambda^+_{z,\psi}(s) \right)^{n-1}(\psi(s)) \, ds \\
= \mathcal{X}_\psi(b')
\]
holds. Moreover, (75) in Proposition 3 applied with $g \equiv \int_0^1 \chi, \phi \equiv \chi|_{[1,2]}, \mathcal{X} \equiv \tilde{\mathcal{X}}|_{[1,2]}$ there (first step) yields for $t \in [1,2]$ that
\[
\tilde{\mathcal{X}}(t) = \tilde{\mathcal{X}}(1) + \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_1^t \left( \text{Ad}_{\tilde{\mathcal{X}}(s)} \circ \text{Ad}_{\tilde{\mathcal{X}}(0)} ight)^{n-1}(\chi(s)) \, ds \\
\equiv \mathcal{X}_\psi(b') + \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_1^t \left( \text{Ad}_{\chi(t)} \circ \text{Ad}_{\chi(0)} - \text{id}_{\mathfrak{g}} \right)^{n-1}(\chi(s)) \, ds \\
\equiv \mathcal{X}_\psi(b') + \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_1^t \rho_2(s) \cdot \left( \text{Ad}_{\chi(t)} \circ \text{Ad}_{\chi(0)} - \text{id}_{\mathfrak{g}} \right)^{n-1}(\phi(s)) \, ds \\
\equiv \mathcal{X}_\psi(b') + \sum_{n=1}^{q-1} \frac{(-1)^{n-1}}{n} \cdot \int_1^t \left( \text{Ad}_{\phi(t)} \circ \text{Ad}_{\phi(0)} - \text{id}_{\mathfrak{g}} \right)^{n-1}(\phi(s)) \, ds \\
\equiv \mathcal{X}_\psi(b') + \mathcal{X}_\phi,\psi(\varphi_2(t)).
\]

We obtain
\[
\int_0^t \phi \cdot \int_{t}^1 \psi \overset{(86)}{=} \int_0^1 \chi \overset{(87)}{=} \exp(\tilde{\mathcal{X}}(t)) = \exp(\mathcal{X}_\psi(b') + \mathcal{X}_\phi,\psi(\varphi_2(t))) \quad \forall t \in [1,2].
\]

From this, the claim follows, because $\varphi_2$ is strictly increasing on $(1,2)$, thus strictly increasing on $[1,2]$, hence bijective. \hfill \Box

### 4.2.3 The proof of Lemma 16

Let $\mathcal{N} \subseteq \mathfrak{g}$ and $\ell \geq 1$ be given:

We let $\mathcal{K}_\ell(\mathcal{N})$ denote the set of all maps $\alpha : [a, b] \times [0, 1] \rightarrow \mathfrak{g}$ that admit the following properties:

1. We have $\text{im}[\alpha] \subseteq \overline{\mathfrak{g}}_{\ell+1}(\mathcal{N})$.
2. There exists a continuous map $L : [a, b] \times [0, 1] \rightarrow \overline{\mathfrak{g}}_1(\mathcal{N}) \subseteq \mathfrak{g}$, as well as a map $\tilde{\alpha} : I \times [0, 1] \rightarrow \mathfrak{g}$ for $I \subseteq \mathbb{R}$ an open interval with $[a, b] \subseteq I$, such that the following conditions hold:

   a. $\tilde{\alpha}$ is continuous, with $\tilde{\alpha}|_{[a,b] \times [0,1]} = \alpha$.
   b. To each $\tau \in [a, b]$, there exist $\delta_\tau > 0$ as well as $\epsilon_\tau : (-\delta_\tau, \delta_\tau) \times [0, 1] \rightarrow \mathfrak{g}$ continuous with $\lim_{t \to 0} p_{\infty}(\epsilon_\tau(h, \cdot)) = 0$ for each $p \in \text{Sem}(E)$, such that
   \[
   \tilde{\alpha}(\tau + h, t) = \tilde{\alpha}(\tau, t) + h \cdot L(\tau, t) + h \cdot \epsilon_\tau(h, t) \quad (h, t) \in (-\delta_\tau, \delta_\tau) \times [0, 1].
   \]
   In particular, $\Phi \equiv \tilde{\alpha}$ fulfills the assumptions in Theorem C for $(G, \cdot) \equiv (\mathfrak{g}, +)$ there (recall Remark 8), i.e., we have
   \[
   \frac{d}{dt}|_{t=0} \int_0^1 \alpha(\tau + h, s) \, ds = \int_0^1 L(\tau, s) \, ds \quad \forall \tau \in [a, b].
   \]

We let $\mathcal{K}_\ell(\mathcal{N})$ denote the set of all (continuous) maps of the form
\[
\eta : [a, b] \times [0, 1] \ni (\tau, t) \mapsto \tilde{\eta}(\tau) + \alpha(\tau, t) \in \mathfrak{g}.
\]
for $j \in C^1([a, b], g)$ with $\im[3] \cup \im[j] \subseteq \overline{G}_1(N)$, and $\alpha \in \mathcal{H}_\ell(N)$ (in particular, $\im[\eta] \subseteq \overline{G}_1(N)$).

We recall Convention (50), and are ready for the proof of Lemma 16.

**Proof of Lemma 16** Set $N := \im[\phi]$. We prove by induction that

$$\mathcal{H}_\ell(N) \ni \eta_\ell : [a, b] \times [0, 1] \ni (\tau, t) \mapsto \mathfrak{X}_\ell^\ell(\phi_{(\tau, 1)})(t) \in g \quad \forall \ell = 1, \ldots, q - 1 \quad (90)$$

holds. Evaluating this for $\ell = q - 1$, the claim follows because:

- For $\alpha \in \mathcal{H}_{q-1}(N)$, we have $\im[\alpha] \subseteq \overline{G}_q(N) = \{0\}$ by Condition 1 and Remark 3.1.
- $\overline{G}_1(N)$ is a Nil$_q$-set by Remark 3.3.

To perform the induction, we fix $\varepsilon > 0$ and $\tilde{\phi} \in C^k([a - \varepsilon, b + \varepsilon], g)$ with $\tilde{\phi}_{|[a, b]} = \phi$. We observe that for each $\psi \in \mathcal{D}_{[a, b]}$ with $\im[\psi]$ a Nil$_q$-set, and $\lambda, t \in [a, b]$ we have (confer Example 7)

$$\mathfrak{X}(\psi_{|[\alpha, \lambda]}(t)) = \int_a^\lambda \Ad_{\tilde{J}_{t, \lambda}^\alpha, \psi}(\psi(s)) \, ds$$

$$= \int_a^\lambda \Ad_{\tilde{J}_{t, \lambda}^\alpha, \psi}(\psi(s))(\lambda) \, ds$$

$$= \int_a^\lambda \psi(s) \, ds + \sum_{p=1}^{q-2} \int_a^\lambda \lambda_{p, t, \psi_{|[\alpha, \lambda]}}(\psi(s))(\lambda) \, ds$$

$$= \int_a^\lambda \psi(s) \, ds$$

$$+ \sum_{p=1}^{q-2} t^p \cdot \int_a^\lambda \, ds \int_s^{s_1} \, ds_1 \int_s^{s_2} \, ds_2 \ldots \int_s^{s_{p-1}} \, ds_p \ (\Ad_{\psi(s_1)} \circ \ldots \circ \Ad_{\psi(s_p)})(\psi(s)).$$

**Induction Basis:** Let $\ell = 1$. By (91), for $t \in [a, b]$ and $t \in [0, 1]$ we have

$$\eta_1(\tau, t) = \int_a^\tau \phi(s) \, ds$$

$$= \int_a^\tau \phi(s) \, ds$$

$$+ \sum_{p=1}^{q-2} t^p \cdot \int_a^\tau \, ds \int_s^{s_1} \, ds_1 \int_s^{s_2} \, ds_2 \ldots \int_s^{s_{p-1}} \, ds_p \ (\Ad_{\psi(s_1)} \circ \ldots \circ \Ad_{\psi(s_p)})(\psi(s)),$n

with

- $j \in C^1([a, b], g)$ by (20), and $\im[3] \cup \im[j] \subseteq \overline{G}_1(N)$ by Remark 3.4.
- $\im[\alpha] \subseteq \overline{G}_2(N)$ (hence Condition 1) for $\ell = 1$ there) by Remark 3.4.

To verify Condition 2, for $\tau_1, \tau_2 \in I := (a - \varepsilon, b + \varepsilon)$ and $t \in [0, 1]$, we set

$$\tilde{\beta}(\tau_1, \tau_2) := \sum_{p=1}^{q-2} t^p \cdot \int_{a-\varepsilon}^{\tau_1} \, ds \int_s^{s_1} \, ds_1 \int_s^{s_2} \, ds_2 \ldots \int_s^{s_{p-1}} \, ds_p \ (\Ad_{\psi(s_1)} \circ \ldots \circ \Ad_{\psi(s_p)})(\psi(s)),$n

and define $\tilde{\alpha}(\tau, t) := \tilde{\beta}(\tau, t) - \tilde{\beta}(a, \tau, t)$ for $\tau \in I$ and $t \in [0, 1]$. Then, $\tilde{\alpha}$ is continuous, with $\tilde{\alpha}_{|[a, b] \times [0, 1]} = \alpha$, hence fulfills Condition 2.a). Moreover, for $\tau \in I$, $t \in [0, 1]$, and $|h| > 0$ suitably small, we have

$$\tilde{\alpha}(\tau + h, t) - \tilde{\alpha}(\tau, t)$$

$$= \sum_{p=1}^{q-2} t^p \cdot \int_{a-\varepsilon}^{\tau+h} \, ds \int_s^{\tau+\varepsilon} \, ds_1 \int_s^{s_1} \, ds_2 \ldots$$

$$\int_s^{s_{p-1}} \, ds_p \ (\Ad_{\psi(s_1)} \circ \ldots \circ \Ad_{\psi(s_p)})(\psi(s))$$

$$+ \sum_{p=1}^{q-2} t^p \cdot \int_{a-\varepsilon}^{\tau+\varepsilon+h} \, ds \int_s^{\tau+\varepsilon+h} \, ds_1 \int_s^{s_1} \, ds_2 \ldots$$

$$\int_s^{s_{p-1}} \, ds_p \ (\Ad_{\psi(s_1)} \circ \ldots \circ \Ad_{\psi(s_p)})(\psi(s)).$$
\[ \tilde{\alpha}(\tau, t) - \tilde{\alpha}(\tau - h, t) = \sum_{p=1}^{q-2} t^p \cdot \int_{\tau}^{\tau-h} ds \int_{\tau-h}^{s} ds_1 \int_{s}^{s_1} ds_2 \ldots \int_{s}^{s_{p-1}} ds_p \rho((\text{ad}_{\tilde{\phi}(s_1)} \circ \ldots \circ \text{ad}_{\tilde{\phi}(s_p)})(\tilde{\phi}(s))) \]

\[ + \sum_{p=1}^{q-2} t^p \cdot \int_{\tau}^{\tau-h} ds \int_{\tau-h}^{s} ds_1 \int_{s}^{s_1} ds_2 \ldots \int_{s}^{s_{p-1}} ds_p \rho((\text{ad}_{\tilde{\phi}(s_1)} \circ \ldots \circ \text{ad}_{\tilde{\phi}(s_p)})(\tilde{\phi}(s))). \]

We define \( L : [a, b] \times [0, 1] \to \overline{G}_1(N) \subseteq g \) (recall Remark 3.4) by

\[ L(\tau, t) := \sum_{p=1}^{q-2} t^p \cdot \int_{\tau}^{\tau-h} ds \int_{\tau-h}^{s} ds_1 \int_{s}^{s_1} ds_2 \ldots \int_{s}^{s_{p-1}} ds_p \rho((\text{ad}_{\tilde{\phi}(s_1)} \circ \ldots \circ \text{ad}_{\tilde{\phi}(s_p)})(\tilde{\phi}(s))) \]

for all \( \tau \in [a, b] \) and \( t \in [0, 1] \). Clearly, \( L \) is continuous; and it follows from (19) as well as continuity of the involved maps that

\[ \lim_{h \to 0} 1/|h| \cdot p_\infty(\tilde{\alpha}(\tau + h, \cdot) - \tilde{\alpha}(\tau, \cdot) - h \cdot L(\tau, \cdot)) = 0 \quad \forall \, p \in \text{Sem}(E), \, \tau \in [a, b] \]

holds, which verifies Condition 2.b). We thus have shown (90) for \( \ell = 1 \). In particular, this proves the claim for \( q = 2 \).

**Induction Step:** Assume that (90) holds for some \( 1 \leq \ell \leq q - 1 \) (with \( q \geq 3 \)), hence we have

\[ \mathcal{K}_\ell(N) \ni \eta_\ell : [a, b] \times [0, 1] \ni (\tau, t) \mapsto \tilde{\eta}_\ell(\tau) + \alpha_\ell(\tau, t) \in \overline{G}_1(N) \subseteq g \quad (92) \]

for some \( \tilde{\eta}_\ell \in C^1([a, b], g) \) with \( \text{im}[\tilde{\eta}_\ell] \cup \text{im}[\tilde{\eta}_\ell] \subseteq \overline{G}_1(N) \), as well as \( \alpha_\ell \in \mathcal{H}_\ell(N) \). Let \( I \subseteq \mathbb{R} \) be as in Condition 2 for \( \alpha \equiv \alpha_\ell \) there. We can shrink \( I \) around \([a, b]\) in such a way that \( \tilde{\eta}_\ell \) extends to a \( C^1 \)-curve \( \tilde{\eta}_\ell \in C^1(I, g) \), and set

\[ \tilde{\eta}_\ell : I \times [0, 1] \ni (\tau, t) \mapsto \tilde{\eta}_\ell(\tau) + \tilde{\alpha}_\ell(\tau, t) \in g. \quad (93) \]

Now, (91) for \( \psi \equiv \eta_\ell(\tau, \cdot) \) and \( \lambda \equiv 1 \) yields for \( \tau \in [a, b] \) and \( t \in [0, 1] \) that

\[ \eta_{\ell+1}(\tau, t) = \mathcal{T}^{\ell+1}(\phi|_{[a, \tau]})(t) \]

\[ = \mathcal{T}(\eta_\ell(\tau, \cdot))(t) \]

\[ = \int_{0}^{1} \eta_\ell(\tau, s) \, ds \]

\[ =: \tilde{\alpha}_{\ell+1}(\tau, t) \]

We obtain from (92) that \( \tilde{\eta}_{\ell+1} = \tilde{\eta}_\ell + \int_{0}^{1} \alpha_\ell(\cdot, s) \, ds \) holds, and conclude the following:

- Let \( L : [a, b] \times [0, 1] \to \overline{G}_1(N) \subseteq g \) be as in Condition 2.b) for \( \alpha \equiv \alpha_\ell \) there (induction hypothesis). Then, we have

\[ \tilde{\eta}_{\ell+1}(\tau) = \tilde{\eta}_\ell(\tau) + \int_{0}^{1} L(\tau, s) \, ds \in \overline{G}_1(N) \quad \forall \, \tau \in [a, b]. \]

Since \( L \) is continuous, this implies \( \tilde{\eta}_{\ell+1} \in C^1([a, b], g) \) with \( \text{im}[\tilde{\eta}_{\ell+1}] \cup \text{im}[\tilde{\eta}_{\ell+1}] \subseteq \overline{G}_1(N) \).

- We have \( \text{im}[\alpha_{\ell+1}] \subseteq \overline{G}_{\ell+2}(N) \) by Remark 3.2, Remark 3.4, bilinearity of [·, ·], and

\[ - (\text{ad}_{\tilde{\alpha}_\ell(\tau)} \circ \ldots \circ \text{ad}_{\tilde{\alpha}_\ell(\tau)})(\tilde{\alpha}_\ell(\tau)) = 0 \quad \forall \, \tau \in [a, b], \]

\[ - \text{im}[\tilde{\alpha}_\ell] \subseteq \overline{G}_1(N) \]

\[ \text{Springer} \]
where for \( \tau \leq \tau \)

It follows from continuity and bilinearity of \( \tau \) as well as for \( \tau \)

with \( \lim_{h \to 0} p \) \( \) for each \( p \in \mathrm{Sem}(E) \), such that

holds. We define for \( \tau \in I \) and \( t \in [0,1] \)

where for \( 1 \leq p \leq q - 2 \) and \( 0 \leq s_1, \ldots, s_p, s \leq 1 \), we set

as well as for \( 1 \leq k \leq p \)


It follows from continuity and bilinearity of \( [\cdot, \cdot] \), (94), and (19) that

holds, which verifies Condition 2).b) for \( \alpha \equiv \alpha_{\ell+1} \) there.

5 Asymptotic estimate Lie algebras

Throughout this section, \((q, [\cdot, \cdot])\) denotes a fixed sequentially complete and asymptotic
estimate Lie algebra, i.e., the following conditions are fulfilled:

- \( q \in \mathrm{hlcVect} \) is sequentially complete.
- \([\cdot, \cdot] : q \times q \to q \) is bilinear, antisymmetric, continuous, and fulfills the Jacobi identity (25). Moreover, to each \( v \in \mathrm{Sem}(q) \), there exist \( v \leq w \in \mathrm{Sem}(q) \) with

for all \( x_1, \ldots, x_n, y \in q \) and \( n \geq 1 \).

The first part of this section (Sect. 5.1) is dedicated to a comprehensive analysis of the Lax equation

for \( x \in q \) and \( \psi \in C^0([a, b], q) \) \( a < b \) fixed, and \( \alpha \in C^1([a, b], q) \). We first show the existence and uniqueness of solutions and discuss their elementary properties (Sect. 5.1.1
and Sect. 5.1.2). More specifically, let $\text{Ad}_\psi^+ [X]$ be defined by the right hand side of (68). We show that the unique solution to the above equation is given by $\alpha = \text{Ad}_\psi^+ [X]$. We also show\footnote{Although it might be obvious, we explicitly mention at this point that in contrast to the finite-dimensional case, in the context considered it not suffices to show the existence of a left- or right inverse of $\text{Ad}_\psi^+ [t]$ but both. Basically, this is the reason for the necessity of the investigations made in Sect. 5.1.2.}

$$\text{Aut}(q) \ni \text{Ad}_\psi^+ [t] : q \in X \mapsto \text{Ad}_\psi^+ [X](t) \in q \quad \forall t \in [a, b],$$

with $\text{Ad}_\psi^+ [t]^{-1} = \text{Ad}_\psi^+ [t]$ (both statements are proven in Proposition 6). This allows to define a group structure on $C^k([a, b], q)$ for $k \in \mathbb{N} \cup \{\infty\}$ (in analogy to Remark 5 and Sect. 1.14 in [2]) by

$$\psi^{-1} : [a, b] \ni t \mapsto -\text{Ad}_\psi^+ [t](\psi(t)) \in q$$

$$\phi \star \psi : [a, b] \ni t \mapsto \phi(t) + \text{Ad}_\psi^+ [t](\psi(t)) \in q$$

for $\phi, \psi \in C^k([a, b], q)$. The group axioms are verified in Sect. 5.1.3.

**Remark 14** (A Lie Group Construction) Let $k \in \mathbb{N} \cup \{\infty\}$ and $a < b$ be given; and set

$$G_k = E_k = g_k := C^k([a, b], q), \quad e_k := 0 \in C^k([a, b], q), \quad \Xi_k := \text{id}_{C^k([a, b], q)}.$$

As already mentioned, we will show that the maps

$$\text{inv}_k : G_k \to G_k, \quad \psi \mapsto \psi^{-1}$$

$$m_k : G_k \times G_k \to G_k, \quad (\phi, \psi) \mapsto \phi \star \psi$$

are defined (confer Lemma 17), and that $(G_k, m_k, \text{inv}_k, e_k)$ is a group (confer Lemma 22). Expectably, the group operations (96) are smooth w.r.t. the $C^k$-topology; hence give rise to a Lie group $G_k$ with global chart $\Xi_k$, and sequentially complete Lie algebra $g_k$ (recall Lemma 3). It is furthermore to be expected that the corresponding Lie bracket $[\cdot, \cdot]_k : g_k \times g_k \to g_k$ is given by

$$[\phi, \psi]_k(t) = [\int_a^t \phi(s) \, ds, \psi(t)] + [\phi(t), \int_a^t \psi(s) \, ds] \quad \forall t \in [a, b]$$

for $\phi, \psi \in g_k$, just as in Sect. 1.14 in [2] (confer Proposition 1.14.1 in [2]). Presumably, $(g_k, [\cdot, \cdot], \Xi_k)$ is asymptotic estimate, and $G_k$ is $C^0$-regular. The technical details will be worked out in a separate paper. This serves as a preparation for a possible extension of Lie’s third theorem to the infinite-dimensional asymptotic estimate case—just by performing the same (a similar) construction made in Sect. 1.14 in [2] to prove this theorem (Theorem 1.14.3 in [2]) in the finite-dimensional context. Notably, this construction had already been used in [1], to prove Lie’s third theorem in the context of Lie algebroids. \hfill \square

In the last part of Sect. 5.1, we use the proven statements to investigate the properties of the following integral transformation that mimics (49) in Sect. 3.1:

$$\Xi : C^0([a, b], q) \to C^\infty([0, 1], q)$$

$$\phi \mapsto \left[ [0, 1] \ni t \mapsto \int_a^b \text{Ad}_r^+ \phi([s, b])[b](\phi(s)) \, ds \right].$$

The investigations serve as a preparation for the discussions in Sect. 5.2. There, the above transformation is applied to the situation where $(q, [\cdot, \cdot]) \equiv (g, [\cdot, \cdot])$ equals the Lie algebra of a given Lie group $G$. Specifically, we prove the following regularity result (cf. Theorem 1):
Theorem 2 Assume that \((g, [\cdot, \cdot])\) is asymptotic estimate and sequentially complete. If \(G\) is weakly \(C^\infty\)-regular, then \(G\) is weakly \(C^k\)-regular for each \(k \in \mathbb{N} \cup \{\text{lip, } \infty\}\).

Remark 15 Note that in the situation of Theorem 2, \(G\) is weakly \(C^k\)-regular for \(k \in \mathbb{N} \cup \{\text{lip, } \infty\}\) if and only if \(G\) is \(C^k\)-semiregular, just because \(g\) is assumed to be sequentially complete.

This result complements Theorem 2 in [9] that essentially states that \(C^\infty\)-regularity implies \(C^k\)-regularity for each \(k \in \mathbb{N} \cup \{\text{lip, } \infty\}\), where for \(k = 0\) additionally sequentially completeness of the Lie group has to be assumed. Hence, in a certain sense, the assumption in Theorem 2 in [9] that \(G\) is sequentially complete, had been replaced in Theorem 2 by the assumption that \(g\) is sequentially complete. The key result proven in [9] (Theorem 1 in [9]) states that in the asymptotic estimate context, \(C^\infty\)-continuity of the product integral is equivalent to its \(C^0\)-continuity (local \(\mu\)-convexity of \(G\)), which makes the semiregularity results obtained in [8] applicable—Basically, in analogy to the definition of the Riemann integral as a limit over Riemann sums, in [8] the product integral is obtained as a limit over product integrals of piecewise integrable curves (under certain assumptions on the Lie group that are automatically fulfilled in the asymptotic estimate context). The proof of Theorem 2 works differently: For some given \(\phi \in C^0([a, b], g)\), we show that (this is defined by \(C^\infty\)-semiregularity of \(G\), as well as \(\text{im}[\Sigma] \subseteq C^\infty([0, 1], g)\))

\[
\mu: [a, b] \ni z \mapsto \int_0^1 \Sigma(\phi|_{[a, z]}) \in G
\]

is of class \(C^1\), with \(\delta^r(\mu) = \phi\).

Remark 16 We will tacitly use throughout this section that sequentially completeness of \(\mu\) implies (recall Remark 2)

\[
\int \phi(s) \, ds \in \mu \quad \forall \phi \in C^0([a, b], \mu), \quad a < b.
\]

We will also use that \([\cdot, \cdot]\) is smooth (apply, e.g., the parts b), e) of Proposition A.1). □

Theorem 2, together with Theorem 4 in [8] and Theorem 1 in [9], yields the following statement:

Corollary 14 If \((g, [\cdot, \cdot])\) is asymptotic estimate and sequentially complete, then the following assertions are equivalent:

(a) \(G\) is \(C^k\)-semiregular for some \(k \in \mathbb{N} \cup \{\infty\}\), and \(\text{evol}_k\) is of class \(C^\ell\) for some \(\ell \in \mathbb{N} \cup \{\infty\}\).

(b) \(G\) is \(C^0\)-semiregular, and \(\text{evol}_0\) is \(C^0\)-continuous (\(G\) is locally \(\mu\)-convex).

(c) \(G\) is \(C^k\)-regular for all \(k \in \mathbb{N} \cup \{\infty\}\).

Proof The implication \((c) \Rightarrow (a)\) is evident. Moreover:

(a) \(\Rightarrow (b)\): If the assertion \((b)\) holds, then

- \(G\) is \(C^\infty\)-semiregular, as \(C^\infty([0, 1], g) \subseteq C^k([0, 1], g)\) holds. Hence, \(G\) is weakly \(C^\infty\)-regular by Remark 15; thus, \(G\) is \(C^0\)-semiregular by Theorem 2.

- \(\text{evol}_k\) is \(C^k\)-continuous, as of class \(C^\ell\) (with \(\ell \geq 0\)). Hence, \(\text{evol}_0\) is \(C^0\)-continuous (\(G\) is locally \(\mu\)-convex) by Theorem 1 in [9].

\[19\] We recall that in contrast to the definition of weak \(C^k\)-regularity for \(k \in \mathbb{N} \cup \{\text{lip, } \infty\}\), the definition of \(C^k\)-regularity additionally involves continuity of the product integral w.r.t. the \(C^k\)-topology.

\[20\] Sequentially completeness of a Lie group and sequentially completeness of its Lie algebra are ad hoc different properties; details can be found in [8].
b) \Rightarrow c): Assume that the assertion b) holds, and let \( k \in \mathbb{N} \cup \{\infty\} \) be given. Then,

- \( G \) is \( C^k \)-semiregular, as \( C^k([0, 1], \mathbb{g}) \subseteq C^0([0, 1], \mathbb{g}) \) holds.
- \( \text{evol}_k = \text{evol}_0|_{C^k([0,1],\mathbb{g})} \) is \( C^k \)-continuous, as even \( C^0 \)-continuous.

Since \( \mathbb{g} \) is sequentially complete (hence, integral complete and Mackey complete), Theorem 4 in [8] shows that \( \text{evol}_k \) is smooth. \( \square \)

### 5.1 The Lax equation

Let \( a < b \) and \( \psi \in \mathcal{CP}^0([a, b], \mathbb{q}) \) be given.\(^{21}\) Motivated by Sect. 3.3, for \( X \in \mathbb{q} \) we set

\[
\lambda_{0, \psi}^\pm [X]: [a, b] \rightarrow \mathbb{q}, \quad t \mapsto X,
\]

as well as (recall (23))

\[
\lambda_{\ell, \psi}^\pm [X]: [a, b] \rightarrow \mathbb{q}, \quad t \mapsto \int_a^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(\delta s_1)} \circ \cdots \circ \text{ad}_{\psi(\delta s_\ell)} \right) (X),
\]

\[
\lambda_{-\ell, \psi}^\pm [X]: [a, b] \rightarrow \mathbb{q}, \quad t \mapsto (-1)^{\ell} \int_a^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{\ell-1}} ds_\ell \left( \text{ad}_{\psi(\delta s_1)} \circ \cdots \circ \text{ad}_{\psi(\delta s_\ell)} \right) (X)
\]

for each \( \ell \geq 1 \). We furthermore define (definedness is covered by Lemma 17)

\[
\Lambda_{\psi}^\pm [X](t) := \sum_{\ell=0}^\infty \lambda_{\ell, \psi}^\pm [X](t) \quad \forall X \in \mathbb{q}, \ t \in [a, b],
\]

and let

\[
\Lambda_{\psi}^\pm: [a, b] \times \mathbb{q} \rightarrow \mathbb{q}, \quad (t, X) \mapsto \Lambda_{\psi}^\pm [X](t).
\]

To simplify the notations, we define

\[
\text{Ad}_{\psi}^\pm [X](t) := \sum_{\ell=0}^\infty \lambda_{\ell, \psi}^\pm [X](t) \quad \forall X \in \mathbb{q}, \ t \in [a, b],
\]

and let

\[
\Lambda_{\psi}^\pm: [a, b] \times \mathbb{q} \rightarrow \mathbb{q}, \quad (t, X) \mapsto \text{Ad}_{\psi}^\pm [X](t).
\]

### 5.1.1 Elementary properties

In this section, we discuss the elementary properties of the objects defined in the beginning of Sect. 5.1. We start with the following lemma.

**Lemma 17** Let \( \psi \in C^0([a, b], \mathbb{q}) \) be given. Then, the following assertions hold:

1. The maps \( \Lambda_{\psi}^\pm \) are defined and of class \( C^1 \), and we have

\[
\partial_t \Lambda_{\psi}^\pm (t, X) = \partial_t \text{Ad}_{\psi}^\pm [X](t) = \left[ \psi(t), \text{Ad}_{\psi}^\pm [X](t) \right]
\]

for all \( t \in [a, b] \) and \( X \in \mathbb{q} \).

2. \( \Lambda_{\psi}^\pm (t, \cdot) = \text{Ad}_{\psi}^\pm [t] \) is linear and continuous for each \( t \in [a, b] \).

3. If \( \psi \in C^k([a, b], \mathbb{q}) \) holds for \( k \in \mathbb{N} \cup \{\infty\} \), then \( \Lambda_{\psi}^\pm \) is of class \( C^{k+1} \).

\(^{21}\) We consider \( C^0([a, b], \mathbb{q}) \subseteq \mathcal{CP}^0([a, b], \mathbb{q}) \) as a subset in the obvious way.
Proof The proof is straightforward but technical. It is provided in Appendix A.7.

Remark 17 The substitution formula (21) yields the following:

- Let \( a < b, a' < b', \psi \in C^0([a, b], q) \), as well as \( \varphi : [a', b'] \rightarrow [a, b] \) be of class \( C^1 \) with \( \varphi(a') = a \) and \( \varphi(b') = b \). We set \( \phi := \varphi \circ \psi \) and obtain inductively from (21) that

\[
\lambda_{\ell, \phi}^\pm [t] = \lambda_{\ell, \psi}^\pm [\varphi(t)] \quad \forall t \in [a', b']
\]

holds for all \( \ell \in \mathbb{N} \). Hence, we have \( \text{Ad}_{\varphi \circ \psi}^\pm [\cdot] = \text{Ad}_\psi^\pm [\varphi(\cdot)] \). In particular, if additionally \( \dot{\varphi}|_{(a, b)} > 0 \) holds, then we have

\[
\text{Ad}_{\varphi \circ \psi}^\pm [\cdot]|_{(a', b')} = \text{Ad}_\psi^\pm [\varphi(\cdot)] \quad \forall a' \leq x' < y \leq b' \quad (97)
\]

- The previous point applied to affine transformations yields

\[
\text{Ad}_\chi^\pm [b] = \text{Ad}_{\chi(-\chi)}^\pm [b + x]
\]

for \( a < b, \chi \in C^0([a, b], q) \), and \( x \in \mathbb{R} \).

Corollary 15 For \( \psi \in C^0([a, b], q) \) and \( t \in [a, b] \), we have \( \text{Ad}_\psi^\pm [t] \circ \text{Ad}_\psi^\pm [t] = \text{id}_q \).

Proof Let \( Z \in q \) be given, and define

\[
\alpha_Z : [a, b] \ni t \mapsto \Lambda_{\psi}^+(t, \Lambda_{\psi}^+(t, Z)) = (\text{Ad}_\psi^+[t] \circ \text{Ad}_\psi^+[t])(Z) \in q.
\]

Then, Lemma 17 shows \( \alpha_Z \in C^1([a, b], q) \); and the parts b), b), c) of Proposition A.1 yield

\[
\dot{\alpha}_Z(t) = -\Lambda_{\psi}^-(t, [\psi(t), \Lambda_{\psi}^+(t, Z)]) + \Lambda_{\psi}^+(t, [\psi(t), \Lambda_{\psi}^+(t, Z)]) = 0 \quad \forall t \in [a, b],
\]

dence \( \alpha_Z(t) = Z + \int_a^t \dot{\alpha}_Z(s) \, ds = Z \) for each 1 \in [a, b] by (17).

Lemma 18 For \( \psi \in CP^0([a, b], q) \) and \( a < c < b \), we have \( \text{Ad}_\psi^+ = \text{Ad}_{\psi|_{[c, b]}}^+ \circ \text{Ad}_{\psi|_{[a, c]}}^+ \).

Proof Let \( X \in q \) be fixed. We thus prove by induction that

\[
\lambda_{\ell, \psi}^+ [X](s) = \sum_{m=0}^\ell \lambda_{m, \psi|_{[c, b]}}^+ [\lambda_{\ell-m, \psi|_{[a, c]}}^+ [X](c)](s) \quad \forall c < s \leq b \quad (98)
\]

holds for each \( \ell \in \mathbb{N} \). It is clear from the definitions that

\[
\lambda_{0, \psi}^+ [X](s) = X = \lambda_{0, \psi|_{[c, b]}}^+ [\lambda_{0, \psi|_{[a, c]}}^+ [X](c)](s) \quad \forall c < s \leq b.
\]

We thus can assume that (98) holds for some \( \ell \in \mathbb{N} \), and obtain for \( c < s \leq b \) that

\[
\lambda_{\ell+1, \psi}^+ [X](s) = \int_a^s \left[ \lambda_{\ell, \psi}^+ [X](s_0) \right] ds_0
\]

\[
= \int_a^s \left[ \psi(s_0), \lambda_{\ell, \psi}^+ [X](s_0) \right] ds_0 + \int_a^s \left[ \lambda_{\ell, \psi}^+ [X](s_0) \right] ds_0
\]

\[
= \lambda_{\ell+1, \psi}^+ [X](c) + \sum_{m=0}^\ell \int_a^s \left[ \lambda_{\ell-m, \psi|_{[c, b]}}^+ [X](c) \right] ds_0
\]

\[
= \lambda_{\ell+1, \psi}^+ [X](c) + \sum_{m=0}^\ell \lambda_{m, \psi|_{[c, b]}}^+ \left[ \lambda_{\ell+1-m, \psi|_{[a, c]}}^+ [X](c) \right] ds_0
\]

\[
= \sum_{m=0}^{\ell+1} \lambda_{m, \psi|_{[c, b]}}^+ [\lambda_{\ell+1-m, \psi|_{[a, c]}}^+ [X](c)](s)
\]

\[\text{\textsuperscript{22} Specifically, in the previous point set } \psi := \chi(-x) : [a + x, b + x] \rightarrow q \text{ as well as } \varphi : [a, b] \ni t \mapsto t + x \in [a + x, b + x].\]
holds, so that (98) follows by induction. Then, for $n \geq 1$ we have

\begin{equation}
\Delta_n := \sum_{\ell=0}^{2n} \lambda_{\ell, \psi}^+(X) - \sum_{p=0}^{n} \lambda_{p, \psi, \varepsilon}^+(X) (\sum_{q=0}^{n} \lambda_{q, \psi, \delta}^+(X))
\end{equation}

(99)

\begin{equation}
= \sum_{\ell=0}^{2n} \sum_{m=0}^{\ell} \lambda_{m, \psi, \varepsilon}^+(X) (\lambda_{\ell-m, \psi, \delta}^+ (X)) - \sum_{p=0}^{n} \lambda_{p, \psi, \varepsilon}^+(X) (\lambda_{q, \psi, \delta}^+ (X)).
\end{equation}

(100)

For $v \leq w$ as in (95), we obtain from (100) and the binomial theorem that

\begin{equation}
v(\Delta_n) \leq \sum_{n+1}^{2n} \sum_{m=0}^{\ell} v(\lambda_{m, \psi, \varepsilon}^+(X) (\lambda_{\ell-m, \psi, \delta}^+ (X)))
\end{equation}

\begin{equation}
\leq w(X) \cdot \sum_{n+1}^{\ell} \sum_{m=0}^{\ell-m} \frac{1}{m!} (b-c)^m \cdot (c-a)^{\ell-m} \cdot w_\infty (\psi)\!
\end{equation}

\begin{equation}
= w(X) \cdot \sum_{n+1}^{\ell} \frac{1}{m!} (b-a)^m \cdot w_\infty (\psi)\!
\end{equation}

holds, which yields

\begin{equation}
0 = \lim_n \Delta_n \overset{(99)}{=} \lim_n \sqrt[n]{\lambda_{p, \psi, \varepsilon}^+(X) (\lambda_{q, \psi, \delta}^+ (X))}.
\end{equation}

It thus remains to show $\lim_n \beta_n = \text{Ad}_{\psi, \varepsilon}^+(X)$. For this, we observe that

\begin{equation}
\tilde{\Delta}_n := \text{Ad}_{\psi, \varepsilon}^+(X) - \beta_n
\end{equation}

\begin{equation}
= \sum_{n=0}^{\ell} \lambda_{p, \psi, \varepsilon}^+(X) (\lambda_{\ell-n, \psi, \delta}^+ (X)) + \sum_{p=n+1}^{\ell} \lambda_{p, \psi, \varepsilon}^+(X) (\lambda_{q, \psi, \delta}^+ (X))
\end{equation}

holds for each $n \in \mathbb{N}$ and choose $w \leq m$ as in (95) for $v \equiv w$ and $w \equiv m$ there. We obtain

\begin{equation}
v(T_1(n)) \leq e^{(b-c) \cdot w_\infty (\psi)} \cdot w(\sum_{n=0}^{\ell} \lambda_{p, \psi, \varepsilon}^+(X))
\end{equation}

\begin{equation}
\leq e^{(b-c) \cdot w_\infty (\psi)} \cdot \sum_{n=0}^{\ell} \frac{1}{m!} (b-c)^m \cdot w_\infty (\psi)\!
\end{equation}

\begin{equation}
\text{hence } \lim_n T_1(n) = 0. \text{ We furthermore obtain}
\end{equation}

\begin{equation}
v(T_2(n)) \leq (\sum_{n=0}^{\ell} \frac{1}{m!} (b-c)^m \cdot w_\infty (\psi)\!) \cdot w(\text{Ad}_{\psi, \varepsilon}^+(X))
\end{equation}

\begin{equation}
\leq (\sum_{n=0}^{\ell} \frac{1}{m!} (b-c)^m \cdot w_\infty (\psi)\!) \cdot e^{(c-a) \cdot w_\infty (\psi)} \cdot m(X),
\end{equation}

\begin{equation}
\text{hence } \lim_n T_2(n) = 0. \text{ It follows that } \lim_n \tilde{\Delta}_n = 0 \text{ holds, which proves the claim.}
\end{equation}

\begin{remark}
Lemma 18, together with the second point in Remark 17 shows

\begin{equation}
\text{Ad}_{\psi, \varepsilon}^+[b] = \text{Ad}_{\psi, \varepsilon}^+[-a] \cdot [b + x]
\end{equation}

for $a < b, \psi \in C^0([a, b], q)$, and $x \in \mathbb{R}$.\hfill\Box
\end{remark}

\begin{lemma}
Let $\psi \in C^0([a, b], q)$ and $\{\psi_n\}_{n \in \mathbb{N}} \subseteq C^0([a, b], q)$ be given, with $\{\psi_n\}_{n \in \mathbb{N}} \to \psi$ w.r.t. the $C^0$-topology. Then, for each $X \in q$, we have $\{\text{Ad}_{\psi_n}^+[X]\}_{n \in \mathbb{N}} \to \text{Ad}_{\psi}^+[X]$ w.r.t. the $C^0$-topology.
\end{lemma}

\begin{proof}
For each $\ell \geq 1$ and $X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, X \in q$, we have

\begin{equation}
\text{ad}_{X_1} \circ \ldots \circ \text{ad}_{X_\ell}(X) - (\text{ad}_{Y_1} \circ \ldots \circ \text{ad}_{Y_\ell})(X)
\end{equation}

\begin{equation}
= \sum_{p=1}^{\ell} \text{ad}_{Y_1} \circ \ldots \circ \text{ad}_{Y_{p-1}} \circ \text{ad}_{X_p} \circ \ldots \circ \text{ad}_{X_{\ell}}(X).
\end{equation}

(101)

Let $v \leq w$ be as in (95). For $N \in \mathbb{N}$ suitably large, we have $w_\infty (\psi_n) \leq 2 \cdot w_\infty (\psi)$ for each $n \geq N$. We obtain for $n \geq N$ from (101) (second step) that

\begin{equation}
v(\text{Ad}_{\psi}^+[X](t) - \text{Ad}_{\psi}^+[X](t))
\end{equation}

\begin{equation}
\leq \sum_{p=1}^{\ell} \text{ad}_{Y_1} \circ \ldots \circ \text{ad}_{Y_{p-1}} \circ \text{ad}_{X_p} \circ \ldots \circ \text{ad}_{X_{\ell}}(X).
\end{equation}

(101)

\begin{equation}
\leq (\sum_{p=0}^{\ell} \frac{1}{m!} (b-c)^m \cdot w_\infty (\psi)\!) \cdot e^{(c-a) \cdot w_\infty (\psi)} \cdot m(X),
\end{equation}

\begin{equation}
\text{hence } \lim_n T_2(n) = 0. \text{ It follows that } \lim_n \tilde{\Delta}_n = 0 \text{ holds, which proves the claim.}
\end{equation}

\begin{remark}
Lemma 18, together with the second point in Remark 17 shows

\begin{equation}
\text{Ad}_{\psi, \varepsilon}^+[b] = \text{Ad}_{\psi, \varepsilon}^+[-a] \cdot [b + x]
\end{equation}

for $a < b, \psi \in C^0([a, b], q)$, and $x \in \mathbb{R}$.\hfill\Box
\end{remark}

\begin{lemma}
Let $\psi \in C^0([a, b], q)$ and $\{\psi_n\}_{n \in \mathbb{N}} \subseteq C^0([a, b], q)$ be given, with $\{\psi_n\}_{n \in \mathbb{N}} \to \psi$ w.r.t. the $C^0$-topology. Then, for each $X \in q$, we have $\{\text{Ad}_{\psi_n}^+[X]\}_{n \in \mathbb{N}} \to \text{Ad}_{\psi}^+[X]$ w.r.t. the $C^0$-topology.
\end{lemma}

\begin{proof}
For each $\ell \geq 1$ and $X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, X \in q$, we have

\begin{equation}
(\text{ad}_{X_1} \circ \ldots \circ \text{ad}_{X_\ell}(X) - (\text{ad}_{Y_1} \circ \ldots \circ \text{ad}_{Y_\ell})(X)
\end{equation}

\begin{equation}
= \sum_{p=1}^{\ell} \text{ad}_{Y_1} \circ \ldots \circ \text{ad}_{Y_{p-1}} \circ \text{ad}_{X_p} \circ \ldots \circ \text{ad}_{X_{\ell}}(X).
\end{equation}

(101)

\begin{equation}
\leq (\sum_{p=0}^{\ell} \frac{1}{m!} (b-c)^m \cdot w_\infty (\psi)\!) \cdot e^{(c-a) \cdot w_\infty (\psi)} \cdot m(X),
\end{equation}

\begin{equation}
\text{hence } \lim_n T_2(n) = 0. \text{ It follows that } \lim_n \tilde{\Delta}_n = 0 \text{ holds, which proves the claim.}
\end{equation}

\begin{remark}
Lemma 18, together with the second point in Remark 17 shows

\begin{equation}
\text{Ad}_{\psi, \varepsilon}^+[b] = \text{Ad}_{\psi, \varepsilon}^+[-a] \cdot [b + x]
\end{equation}

for $a < b, \psi \in C^0([a, b], q)$, and $x \in \mathbb{R}$.\hfill\Box
\end{remark}
\[ \leq \sum_{\ell=0}^{\infty} \nu(\ell, \psi_n \{X\}(t) - \chi_{\ell, \psi}[X](t)) \]
\[ \leq (b - a) \cdot w_{\infty}(\psi_n - \psi) \cdot w(X) \cdot \sum_{\ell=1}^{\infty} \frac{(b-a)^{\ell-1}}{(\ell-1)!} \cdot 2^{\ell-1} \cdot w_{\infty}(\psi)^{\ell-1} \]
\[ = (b - a) \cdot w_{\infty}(\psi_n - \psi) \cdot w(X) \cdot e^{2(b - a) \cdot w_{\infty}(\psi)} \]

holds for each \( t \in [a, b] \), which proves the claim. \( \square \)

### 5.1.2 Uniqueness of the solution

For \( \psi \in C^0([a, b], q) \) with \( a < b \), we define \( \text{inv}(\psi) \in C^0([a, b], q) \) as in Example 4 by
\[
\text{inv}(\psi): [a, b] \ni t \mapsto -\psi(a + b - t) \in q.
\]

Notably, we have \( \text{inv}(\text{inv}(\psi)) = \psi \). In this section, we prove the following proposition.\(^{23}\)

**Proposition 6** Let \( \psi \in C^0([a, b], q) \) be given. Then, the following assertions hold:

1. We have \( \text{Ad}_\psi^{\pm}[t] \in \text{Aut}(q) \) for each \( t \in [a, b] \), with \( \text{Ad}_\psi^{\pm}[t]^{-1} = \text{Ad}_\psi^{\mp}[t] = \text{Ad}_\psi^{\pm}(\text{inv}(\psi)[a, t]) \).
2. Let \( X \in q \) be given. Then, \( \text{Ad}_\psi^{+}[X] \) is the unique solution \( \alpha \in C^1([a, b], q) \) to the differential equation (Lax equation)
\[
\dot{\alpha} = [\psi, \alpha] \quad \text{with initial condition} \quad \alpha(a) = X.
\]

We obtain the following corollaries.

**Corollary 16** For each \( \psi \in C^0([a, b], q) \), we have
\[
\text{Ad}_\psi^{\pm}[X, Y] = [\text{Ad}_\psi^{\pm}[X], \text{Ad}_\psi^{\pm}[Y]] \quad \forall X, Y \in q.
\]

**Proof** Let \( X, Y \in q \) and \( \psi \in C^0([a, b], q) \) be given. Lemma 17 shows
\[
C^1([a, b], q) \ni \alpha: [a, b] \ni t \mapsto [\text{Ad}_\psi^{+}[X](t), \text{Ad}_\psi^{+}[Y](t)] \in q,
\]

with \( \alpha(a) = [X, Y] \). The parts b), e) of Proposition A.1 (first step), and the Jacobi identity (25) (second step) show that
\[
\dot{\alpha}(t) = [[\psi(t), \text{Ad}_\psi^{+}[X](t)], \text{Ad}_\psi^{+}[Y](t)(t)] + [\text{Ad}_\psi^{+}[X](t), [\psi(t), \text{Ad}_\psi^{+}[Y](t)](t)]
\]
\[
= [\psi(t), [\text{Ad}_\psi^{+}[X](t), \text{Ad}_\psi^{+}[Y](t)](t)]
\]
\[
= [\psi(t), \alpha(t)]
\]

holds for each \( t \in [a, b] \). Proposition 6.2 yields \( \alpha = \text{Ad}_\psi^{+}[X, Y] \), which proves the one part of the statement. Then, Proposition 6.1) shows
\[
\text{Ad}_\psi^{+}[X, Y](t)(t) = \text{Ad}_\psi^{+}[r]([\text{Ad}_\psi^{+}[r] \circ \text{Ad}_\psi^{+}[r])(X), (\text{Ad}_\psi^{+}[r] \circ \text{Ad}_\psi^{+}[r])(Y)])
\]
\[
= (\text{Ad}_\psi^{+}[r] \circ \text{Ad}_\psi^{+}[r])([\text{Ad}_\psi^{+}[X](t), \text{Ad}_\psi^{+}[Y](t)](t))
\]

for each \( t \in [a, b] \) and \( X, Y \in q \), which establishes the claim. \( \square \)

**Corollary 17** For \( \psi \in C^0([a, b], q) \) and \( \varphi: [a', b'] \to [a, b] \) of class \( C^1 \), we have
\[
\text{Ad}_\psi^{+}[\varphi(t)] = \text{Ad}_\psi^{+}[(\psi \circ \varphi)[r] \circ \text{Ad}_\psi^{+}[\varphi(a')] \quad \forall t \in [a', b'].
\]

\(^{23}\) For \( a < b \) and \( \chi \in C^0([a, b], q) \), set \( \text{Ad}_{\chi|[a, a]} := \text{id}_q \).
Proof For $Z \in q$, define

$$\alpha_Z(t) := \text{Ad}^+_{\psi}(\varphi(t))(\text{Ad}^-_{\psi}(\varphi(a'))(Z)) \quad \forall t \in [a', b'].$$

Proposition 6 shows $\alpha_Z \in C^1([a', b'], q)$, with $\alpha_Z(a') = Z$ and (additionally apply b) in Proposition A.1)

$$\dot{\alpha}_Z(t) = \dot{\varphi}(t) \cdot (\psi \circ \varphi)(t), \alpha_Z(t) \quad \forall t \in [a', b'].$$

The uniqueness statement in Proposition 6 (second identity) yields

$$\text{Ad}^+_{\psi}(\varphi(t))(\text{Ad}^-_{\psi}(\varphi(a'))(Z)) = \alpha_Z(t) = \text{Ad}^+_{\varphi(\psi \circ \varphi)}[r](Z) \quad \forall t \in [a, b], Z \in q.$$

Given $X \in q$, we set $Z := \text{Ad}^+_{\varphi(\psi)}[\varphi(a')](X)$ and obtain from Proposition 6.1) that

$$\text{Ad}^+_{\varphi(\psi)}[\varphi(t)](X) = \text{Ad}^+_{\varphi(\psi \circ \varphi)}[r](\text{Ad}^+_{\varphi(\psi)}[\varphi(a')](X)) \quad \forall t \in [a', b']$$

holds, from which the claim is clear. □

Remark 19 Let $\mathcal{L}: q \to q$ be linear and continuous, with

$$\mathcal{L}([X, Y]) = [\mathcal{L}(X), \mathcal{L}(Y)] \quad \forall X, Y \in q. \quad (102)$$

It follows by iterated application of (102) and (22) that

$$\mathcal{L} \circ \text{Ad}^+_{\psi}[r] = \text{Ad}^+_{\mathcal{L} \circ \psi} [r] \circ \mathcal{L} \quad \forall a < b, \psi \in C^0([a, b], q), t \in [a, b] \quad (103)$$

holds. This equality alternatively follows from Proposition 6:

Proof of Equation (103) Given $\psi \in C^0([a, b], q)$ and $Z \in q$, define $\alpha := \mathcal{L} \circ \text{Ad}^+_{\psi}[Z] \in C^1([a, b], q)$. Then, we have $\alpha(a) = \mathcal{L}(Z)$, with

$$\dot{\alpha} = \mathcal{L}([\psi, \text{Ad}^+_{\psi}[Z]]) = [\mathcal{L} \circ \psi, \alpha].$$

Proposition 6.2) shows $\alpha = \text{Ad}^+_{\mathcal{L} \circ \psi}[\mathcal{L}(Z)]$, which establishes the “+”-case in (103). Together with Proposition 6.1), we obtain

$$\mathcal{L} \circ \text{Ad}^+_{\psi}[r](Z) = \mathcal{L} \circ \text{Ad}^+_{\text{inv}(\psi)[b,a]}(Z) = \text{Ad}^+_{\mathcal{L} \circ \text{inv}(\psi)[a,b]}(\mathcal{L}(Z))$$

for $a < t \leq b$ and $\psi \in C^0([a, b], q)$, which establishes the “−”-case in (103). □

For the proof of Proposition 6, we shall need the following facts and definitions:

• For $\phi \equiv \{\phi[p]\}_{0 \leq p \leq n-1} \in C^0([a, b], q)$, we define $\text{inv}(\phi) \in C^0([a, b], q)$ by

$$\text{inv}(\phi) := \text{inv}(\phi[(n - 1) - p])_{0 \leq p \leq n-1}.$$
Let \( \phi = \{ \phi[p] \}_{0 \leq p \leq n-1} \in \text{CP}^0([a, b], q) \) (\( a < b, n \geq 1 \)) and \( \phi' = \{ \phi'[q] \}_{0 \leq q \leq n'-1} \in \text{CP}^0([a', b'], q) \) (\( a' < b', n' \geq 1 \)) be given. We set
\[
\psi[\ell] := \begin{cases} 
\phi[\ell] & \text{for } 0 \leq \ell \leq n - 1 \\
\phi'[\ell - n] & \text{for } n \leq \ell \leq n + n' - 1,
\end{cases}
\]
and define \( \phi \triangleright \phi' \in \text{CP}^0([a, b + (b' - a')] , q) \) by
\[
\phi \triangleright \phi' := \{ \psi[\ell] \}_{0 \leq \ell \leq n + n' - 1}.
\]

Lemma 18 together with Remark 18 shows
\[
\text{Ad}_{\phi \triangleright \phi'}^{+} = \text{Ad}_{\phi \triangleright \phi'}^{+} \circ \text{Ad}_{\phi \triangleright \phi'}^{+}[0,b,b'+(b'-a')] \circ \text{Ad}_{\phi \triangleright \phi'}^{+}[a,b] = \text{Ad}_{\phi \triangleright \phi'}^{+} \circ \text{Ad}_{\phi \triangleright \phi'}^{+}.
\]

We have \( \text{inv}(\text{inv}(\psi)) = \psi \) for each \( \psi \in \text{C}^0([a, b], q) \).

For \( a < b \) and \( Z \in q \), we have (observe \( \text{inv}(C_Z|_{[a,b]}) = -C_Z|_{[a,b]} \))
\[
\text{Ad}_{C_Z|_{[a,b]}}^{+}[b] = \sum_{\ell=0}^{\infty} \frac{(a-b)\ell}{\ell!} \cdot \text{ad}_Z = \text{Ad}_{\text{inv}(C_Z|_{[a,b]})}^{+}[b].
\]

Corollary 15 (second step) yields
\[
\text{Ad}_{C_Z|_{[a,b]}}^{+}[b] \circ \text{Ad}_{\text{inv}(C_Z|_{[a,b]})}^{+}[b] \equiv \text{Ad}_{\text{inv}(C_Z|_{[a,b]})}^{+}[b] \circ \text{Ad}_{\text{inv}(C_Z|_{[a,b]})}^{+}[b] = \text{id}_q.
\]

We obtain the following statement.

**Lemma 20** For \( \psi \in \text{C}^0([a, b], q) \), we have \( \text{Ad}_{\psi}^{+} \circ \text{Ad}_{\text{inv}(\psi)}^{+} = \text{id}_q \).

**Proof** For \( n \geq 1 \) and \( p = 0, \ldots, n \), we define \( t_{n,p} := a + \frac{p}{n} \cdot (b - a) \) and \( Z_{n,p} := \psi(t_{n,p}) \).

We set
\[
\chi_{n.p} := C_{Z_{n,p}}|_{[t_{n,p},t_{n,p+1}]} \quad \forall n \geq 1, \ p = 0, \ldots, n - 1.
\]

For \( n \geq 1 \), we define \( \psi_n^{+} \equiv \{ \psi_n^{+}[p] \}_{0 \leq p \leq n-1} \in \text{CP}^0([a, b], q) \) by
\[
\psi_n^{+}[p] := \chi_{n,p}^{+} \quad \text{as well as} \quad \psi_n^{-}[p] := \text{inv}(\chi_{n,(n-1)-p}) \quad \text{for} \quad p = 0, \ldots, n - 1.
\]

(hence \( \psi_n^{-} = \text{inv}(\psi_n^{+}) \)). Obviously, we have \( \{ \psi_n^{+} \}_{n \geq 1} \rightarrow \psi \) as well as \( \{ \psi_n^{-} \}_{n \geq 1} \rightarrow \text{inv}(\psi) \) w.r.t. the \( \text{C}^0 \)-topology. Moreover, for \( n \geq 1 \), we have by Lemma 18 and (106) that
\[
\text{Ad}_{\psi_n^{+}}^{+} \circ \text{Ad}_{\psi_n^{-}}^{+} = \text{Ad}_{\chi_{n,0}}^{+} \circ \cdots \circ (\text{Ad}_{\chi_{n,n-1}}^{+} \circ \text{Ad}_{\text{inv}(\chi_{n,n-1})}^{+}) \circ \cdots \circ \text{Ad}_{\text{inv}(\chi_{n,0})}^{+} = \text{id}_q
\]
holds. We define \( \phi := \text{inv}(\psi) \triangleright \psi \), as well as \( \phi_n := \psi_n^{-} \triangleright \psi_n^{+} \) for each \( n \geq 1 \). Then, \( \{ \phi_n \}_{n \geq 1} \rightarrow \phi \) converges w.r.t. the \( \text{C}^0 \)-topology, and we obtain from Lemma 19 (second step) that
\[
(\text{Ad}_{\psi}^{+} \circ \text{Ad}_{\text{inv}(\psi)}^{+})(X) \equiv \text{Ad}_{\phi}^{+}(X) = \lim_n \text{Ad}_{\phi_n}^{+}(X) \equiv \lim_n (\text{Ad}_{\psi_n^{-}}^{+} \circ \text{Ad}_{\psi_n^{+}}^{+})(X) \equiv X
\]
holds for each \( X \in q \), which proves the claim.

We are ready for the proof of Proposition 6.
Proof of Proposition 6  (1) The claim is clear for $t = a$. Then, it suffices to prove the statement for $t = b$, as then we can apply it to the restriction $\psi|_{[a, t]}$ for each $a < t < b$. By Lemma 20 (first step), and Corollary 15 (third step), we have
\[
\text{Ad}_{\psi}^{-}[b] = \text{Ad}_{\psi}^{-}[b] \circ (\text{Ad}_{\psi}^{+}[b] \circ \text{Ad}_{\psi}^{+}_{\text{inv}(\psi)}[b])
\]
\[
= (\text{Ad}_{\psi}^{-}[b] \circ \text{Ad}_{\psi}^{+}[b]) \circ \text{Ad}_{\psi}^{+}_{\text{inv}(\psi)}[b] = \text{Ad}_{\psi}^{+}_{\text{inv}(\psi)}[b].
\]
Applying $\text{Ad}_{\psi}^{+}$ from the left, Lemma 20 yields (Corollary 15 in the second equality)
\[
\text{Ad}_{\psi}^{+}[b] \circ \text{Ad}_{\psi}^{-}[b] = \text{id}_{\mathcal{A}} = \text{Ad}_{\psi}^{-}[b] \circ \text{Ad}_{\psi}^{+}[b].
\]
Together, we have shown the following properties:
- $\text{Ad}_{\psi}^{+}[b] \in \text{Aut}(q)$ with $\text{Ad}_{\psi}^{+}[b]^{-1} = \text{Ad}_{\psi}^{-}[b]$,
- $\text{Ad}_{\psi}^{+}[b] = \text{Ad}_{\psi}^{+}_{\text{inv}(\psi)}[b]$.

Applying the third property to $\tilde{\psi} := \text{inv}(\psi)$ (second step), we obtain from $\text{inv}(\text{inv}(\psi)) = \psi$ (third step) that
\[
\text{Ad}_{\text{inv}(\psi)}^{-}[b] = \text{Ad}_{\psi}^{-}[b] = \text{Ad}_{\text{inv}(\psi)}^{+}[b] = \text{Ad}_{\psi}^{+}[b]
\]
holds, which completes the proof.

(2) By Lemma 17, for $\alpha \equiv \text{Ad}_{\psi}^{+}[X]$, we have $\alpha \in C^1([a, b], q)$ with $\alpha(a) = X$ and $\dot{\alpha} = [\psi, \alpha]$. This shows the solution property. For uniqueness, let $\alpha \in C^1((a, b), q)$ be given, such that $\alpha(a) = X$ and $\dot{\alpha} = [\psi, \alpha]$ holds. By Lemma 17, we have $\beta := \Lambda_{\psi}^{-}([\psi, \alpha]) \in C^1((a, b), q)$ with (use Part c) of Proposition A.1 on the left side, as well as (20) on the right side.
\[
\hat{\beta} = -\Lambda_{\psi}^{-}([\psi, \alpha]) + \Lambda_{\psi}^{-}([\psi, \alpha]) = 0
\]
\[
\implies \beta = \beta(a) + \int_{a}^{t} \dot{\beta}(s) \, ds = X.
\]
Applying Part I), we obtain
\[
\text{Ad}_{\psi}^{+}[X](t) = \text{Ad}_{\psi}^{+}[I](\beta(t)) = (\text{Ad}_{\psi}^{+}[I] \circ \text{Ad}_{\psi}^{-}[I])(\alpha(t)) = \alpha(t) \quad \forall t \in [a, b],
\]
which proves the claim. \hfill \Box

5.1.3 A group structure

For $\phi, \psi \in C^0([a, b], q)$ with $a < b$, we define
\[
C^0([a, b], q) \ni \psi^{-1} := -\Lambda_{\psi}^{-}([\psi, \psi]) : [a, b] \ni t \mapsto -\text{Ad}_{\psi}^{-}[t](\psi(t)) \in q
\]
\[
C^0([a, b], q) \ni \phi*\psi := \phi([\cdot, \psi]) + \Lambda_{\psi}^{+}([\cdot, \psi]) : [a, b] \ni t \mapsto \phi(t) + \text{Ad}_{\psi}^{+}[t](\psi(t)) \in q.
\]
We observe the following.

Lemma 21 Let $\phi, \psi \in C^0([a, b], q)$ be fixed. Then, the following assertions hold:

1. We have $\text{Ad}_{\psi*\psi}^{-}[t] = \text{Ad}_{\psi}^{-}[t]$ for each $t \in [a, b]$.
2. We have $\text{Ad}_{\psi*\psi}^{+}[t] = \text{Ad}_{\psi}^{+}[t] \circ \text{Ad}_{\psi}^{+}[t]$ for each $t \in [a, b]$.\hfill \Box
For \( X \in q \) be given, and set \( \alpha_X := \text{Ad}_\psi^{-1}[X] \in C^1([a, b], q) \). We have \( \alpha_X(a) = X \); and Lemma 17 shows \( \alpha_X := \text{Ad}_\psi^{-1}[X] \in C^1([a, b], q) \) with (for the second step use Corollary 16)

\[
\dot{\alpha}_X(t) = -\text{Ad}_\psi^{-1}[t](\dot{X}(t) + \text{Ad}_\psi^{-1}[X](t)) = [- \text{Ad}_\psi^{-1}[t](\dot{X}(t)), \text{Ad}_\psi^{-1}[X](t)] = [\dot{\psi}^{-1}(t), \alpha_X(t)]
\]

for each \( t \in [a, b] \). Then, Proposition 6.2) gives \( \alpha_X = \text{Ad}_\psi^{-1}[X] \).

(2) For \( X \in q \), we define

\[
\alpha_X : \{a, b\} \ni t \mapsto (\text{Ad}_\phi^{-1}[t] \circ \text{Ad}_\psi^{-1}[t])\circ \text{Ad}_\psi^{-1}[X](t) \in q.
\]

We have \( \alpha_X(a) = X \); and Lemma 17 shows \( \alpha_X \in C^1([a, b], q) \) with (for the first step additionally apply the parts b) and e) of Proposition A.1; and for the second step apply Corollary 16)

\[
\dot{\alpha}_X(t) = \text{Ad}_\phi^{-1}[t]([\phi(t), \alpha_X(t)] + \text{Ad}_\phi^{-1}[t](\text{Ad}_\psi^{-1}[t](X)))
\]

\[= [\phi(t), \alpha_X(t)] + [\text{Ad}_\phi^{-1}[t](\phi(t)), \alpha_X(t)]
\]

\[= \{(\phi \ast \psi)(t), \alpha_X(t)\}.
\]

Proposition 6.2) yields \( \alpha_X = \text{Ad}_{\phi \ast \psi}^{-1}[X] \), which proves the claim. \( \square \)

**Lemma 22** Given \( \phi, \psi, \chi \in C^0([a, b], q) \), then we have

\[
0 \ast \psi = \psi = \psi \ast 0,
\]

\[
\psi^{-1} \ast \psi = 0 = \psi \ast \psi^{-1},
\]

\[
(\phi \ast \psi) \ast \chi = \phi \ast (\psi \ast \chi).
\]

**Proof** We have \( \text{Ad}_\phi^{-1}[t] = \text{id}_{q} \) and \( \text{Ad}_\psi^{-1}[t](0) = 0 \) for each \( t \in [a, b] \); hence,

\[
(0 \ast \psi)(t) = 0 + \text{Ad}_\phi^{-1}[t](\psi(t)) = \psi(t) = \psi(t) + \text{Ad}_\psi^{-1}[t](0) = (\psi \ast 0)(t) \quad \forall t \in [a, b].
\]

Next, Proposition 6.1) yields

\[
(\psi \ast \psi^{-1})(t) = \psi(t) - (\text{Ad}_\psi^{-1}[t] \circ \text{Ad}_\psi^{-1}[t])(\psi(t)) = 0 \quad \forall t \in [a, b].
\]

Moreover, Lemma 21.1) shows

\[
(\psi^{-1} \ast \psi)(t) = \psi^{-1}(t) + \text{Ad}_\psi^{-1}[t](\psi(t)) = \psi^{-1}(t) + \text{Ad}_\psi^{-1}[t](\psi(t))
\]

\[= \psi^{-1}(t) - \psi^{-1}(t) = 0
\]

for each \( t \in [a, b] \). Finally, we obtain from Lemma 21.2) that

\[
((\phi \ast \psi) \ast \chi)(t) = (\phi \ast \psi)(t) + \text{Ad}_{\phi \ast \psi}^{-1}[t](\chi(t))
\]

\[= \phi(t) + \text{Ad}_\phi^{-1}[t](\psi(t)) + (\text{Ad}_\psi^{-1}[t] \circ \text{Ad}_\psi^{-1}[t])(\chi(t))
\]

\[= \phi(t) + \text{Ad}_\phi^{-1}[t]((\chi \ast \psi)(t))
\]

\[= (\phi \ast (\psi \ast \chi))(t)
\]

holds for each \( t \in [a, b] \), which establishes the proof. \( \square \)

**Corollary 18** For \( \phi, \psi \in C^0([a, b], q) \) and \( t \in [a, b] \), we have

\[
\text{Ad}_{\phi \ast \psi}^{-1}[t] = \text{Ad}_\psi^{-1}[t] \circ \text{Ad}_{\phi \ast \psi}^{-1}[t](\psi(t)).
\]
Proof By Lemma 22, we have
\[(\phi^{-1} \ast (\phi + \psi))(t) = \phi^{-1}(t) + \text{Ad}_{\phi^{-1}}^+[t](\phi(t)) + \text{Ad}_{\phi^{-1}}^+[t](\psi(t))\]
\[= (\phi^{-1} \ast \phi)(t) + \text{Ad}_{\phi^{-1}}^+[t](\phi(t))\]
\[= \text{Ad}_{\phi^{-1}}^+[t](\phi(t))\]
for each \(t \in [a, b]\). Lemma 21.2 (first step) and Lemma 21.1 (third step) yield
\[\text{Ad}_{\phi^{-1}}^+[t] \circ \text{Ad}_{\phi^{-1}}^+[t] = \text{Ad}_{\phi^{-1}}^+[t] \circ [\phi + \psi][t] = \text{Ad}_{\phi^{-1}}^+[t](\phi(t))\]
\[= \text{Ad}_{\phi^{-1}}^+[t](\psi(t))\]
for each \(t \in [a, b]\). Applying \(\text{Ad}_{\phi^{-1}}^+[t]\) to both hand sides, the claim follows from Lemma 21.2, Lemma 22, as well as \(\text{Ad}_{\phi^{-1}}^+[t] = \text{id}_{\mathfrak{g}} \text{ for each } t \in [a, b]\).
\[\square\]

5.1.4 An integral transformation

In analogy to (49), for \(a < b\) and \(\phi \in C^0([a, b], \mathfrak{g})\), we set
\[\Xi(\phi) : [0, 1] \ni t \mapsto \int_a^b \text{Ad}_{t \cdot \phi|_{[a,b]}}^+[\phi(s)] \, ds \in \mathfrak{g}\]
(108)
as well as \(\Xi(\phi|_{[a,b]}) : [0, 1] \ni t \mapsto 0 \in \mathfrak{g}\). Observe that (108) is defined as the integrand is continuous; because, by Lemma 18 and Proposition 6 we have
\[\text{Ad}_{t \cdot \phi|_{[a,b]}}^+[\phi(s)] = \text{Ad}_{t \cdot \phi|_{[a,b]}}^+[\phi(s)] \quad \forall s \in [a, b].\]
Continuity of the integrand also implies the following statement.

Remark 20 Let \(a < b\) and \(\psi \in C^0([a, b], \mathfrak{g})\). Let \(\varrho : [a', b') \to [a, b] (a' < b')\) be of class \(C^1\), with \(\varrho|_{(a', b')} > 0\) as well as \(\varrho(a') = a\) and \(\varrho(b') = b\). Then, (97) in Remark 17 (second step) shows
\[\Xi(\varrho \cdot (\psi \circ \varrho))(t) = \int_{a'}^{b'} \text{Ad}_{t \cdot \varrho|_{[a',b]}}^+[\varrho(s) \cdot (\psi \circ \varrho)(s)] \, ds\]
\[= \int_{a'}^{b'} \varrho(s) \cdot \text{Ad}_{t \cdot \varrho|_{[a',b]}}^-\varrho(s) \right) \, ds\]
\[= \int_{a'}^{b'} \text{Ad}_{t \cdot \varrho|_{[a',b]}}^+[\psi(\varrho(s))] \, ds\]
\[= \Xi(\psi)\]
where we have applied the substitution formula (21) in the third step.
\[\square\]

In this section, we investigate the elementary properties of this map, and provide some statements that we shall need in Sect. 5.2. We start with some technical properties:

Lemma 23 Let \(a < b\), \(\psi \in C^0([a, b], \mathfrak{g})\), and \(X \in \mathfrak{g}\) be given. Then, \(\eta_{\psi, X} : \mathbb{R} \ni t \mapsto \text{Ad}_{\psi(t)}^+[b](X) \in \mathfrak{g}\) is smooth, with
\[\eta_{\psi, X}^{(n)}(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \cdot \lambda_{\psi, b}(X) \quad \forall n \in \mathbb{N}.\]

Proof It follows from the definitions that \(\eta_{\psi, X}(t) = \sum_{p=0}^{\infty} t^p \cdot \lambda_{\psi, b}(X)\) holds for each \(t \in \mathbb{R}\). Now, for \(v \leq w\) as in (95), we have
\[\nu(\lambda_{\psi, b}(X)) \leq w(X) \cdot \frac{(b-a)^p}{p!} \cdot \nu(\psi)^p \quad \forall p \in \mathbb{N}.\]
Consequently, \(\gamma[q] : \mathbb{R} \ni t \mapsto \sum_{p=0}^{\infty} t^p \cdot q(\lambda_{\psi, b}(X)) \in [0, \infty)\) is defined for each \(q \in \text{Sem}(\mathfrak{g})\), so that the claim is clear from Corollary 2.
\[\square\]
Next, let $a < b$ and $\psi \in C^0([a, b], q)$ be given:

- For $a \leq s < z \leq b$, we define $\mathcal{Y}_0(\psi, s, z) := \psi(s)$, as well as
  \[\mathcal{Y}_p(\psi, s, z) := \int_s^z ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{p-1}} ds_p (\operatorname{ad}_{\psi(s_1)} \circ \cdots \circ \operatorname{ad}_{\psi(s_p)})(\psi(s)) \quad \forall \, p \geq 1.\]

For each $t \in [0, 1]$ and $a < z \leq b$, we have

\[\mathcal{T}(\psi|_{[a, z]})(t) = \int_a^z \sum_{p=0}^{\infty} \lambda^+_p(t, \psi|_{[z, z]})(\psi(s)) \, ds = \int_a^z \sum_{p=0}^{\infty} t^p \cdot \mathcal{Y}_p(\psi, s, z) \, ds. \quad (109)\]

- We define
  \[\mathcal{T}_p(\psi, z, t) := t^p \cdot \int_a^z \mathcal{Y}_p(\psi, s, z) \, ds \quad \forall \, p \in \mathbb{N}, \, t \in [0, 1], \, z \in [a, b]. \quad (110)\]

We have the following statement.

**Lemma 24** For each $a < b$, we have $\mathcal{T} : C^0([a, b], q) \to C^\infty([0, 1], q)$. Moreover,

\[\mathcal{T}(\psi|_{[a, z]}) = \sum_{p=0}^{\infty} \mathcal{T}_p(\psi, z, \cdot)\]

converges w.r.t. the $C^\infty$-topology for each $\psi \in C^0([a, b], q)$ and $a < z \leq b$; and we have

\[\sup\{a \leq z \leq b \mid q^\infty(\mathcal{T}(\psi|_{[a, z]})) < \infty\} \quad \forall \, \psi \in C^0([a, b], q), \, q \in \operatorname{Sem}(q), \, s \in \mathbb{N}. \quad (111)\]

**Proof** Let $\psi \in C^0([a, b], q)$ be given, and let $v \leq w$ be as in (95). Then, we have

\[v_\infty(\mathcal{Y}_p(\psi, \cdot, z)) \leq w_\infty(\psi) \cdot \frac{(b-a)^p}{p!} \cdot w_\infty(\psi)^p \quad \forall \, p \in \mathbb{N}, \, a < z \leq b.\]

It follows that $\{\sum_{p=0}^\ell t^p \cdot \mathcal{Y}_p(\psi, \cdot, z)\}_{\ell \in \mathbb{N}} \to \sum_{p=0}^{\infty} t^p \cdot \mathcal{Y}_p(\psi, \cdot, z)$ converges uniformly on $[a, b]$, for each fixed $t \in [0, 1]$ and $a < z \leq b$. Since the Riemann integral is $C^0$-continuous, we obtain

\[\mathcal{T}(\psi|_{[a, z]})(t) = \sum_{p=0}^{\infty} t^p \cdot \int_a^z \mathcal{Y}_p(\psi, s, z) \, ds =: X_p(z) \quad \forall \, t \in [0, 1], \, a < z \leq b.\]

Now, for each $p \in \mathbb{N}$ and $a < z \leq b$, we have

\[v(X_p(z)) \leq (b-a) \cdot w_\infty(\psi) \cdot \frac{(b-a)^p}{p!} \cdot w_\infty(\psi)^p.\]

Consequently, $\gamma[q] : \mathbb{R} \ni t \mapsto \sum_{p=0}^{\infty} t^p \cdot q(X_p(z)) \in [0, \infty)$ is defined for each $q \in \operatorname{Sem}(q)$ and $a < z \leq b$, so that the claim is clear from Corollary 2.

**Lemma 25** For $\psi \in C^0([a, b], q)$, we have

\[\operatorname{Ad}_{t^p \psi}[b] = \operatorname{Ad}_{t^p \psi}^+[t] \quad \forall \, t \in [0, 1].\]

**Proof** Let $X \in q$ be given. Lemma 23 yields the following:

- We have $C^\infty(\mathbb{R}, q) \ni \eta_{\psi, X} : \mathbb{R} \ni t \mapsto \operatorname{Ad}_{t^p \psi}^+[b](X) \in q$.

- By Lemma 17, we have
  \[\chi_t := \operatorname{Ad}_{t^p \psi}^-[t](\psi(\cdot)) \in C^0([a, b], q) \quad \forall \, t \in \mathbb{R}.\]

Lemma 23 shows that $\eta_{\chi_t, X} : \mathbb{R} \ni h \mapsto \operatorname{Ad}_{h \chi_t}^+[b](X) \in q$ is smooth for each $t \in \mathbb{R}$, with

\[\dot{\eta}_{\chi_t, X}(0) = \lambda^+_{1, \chi_t}[b](X) = \int_a^b \chi_t(s) \, ds, X. \quad (112)\]
By Lemma 18 (second step) and Proposition 6.1 (third step), we have
\[
\text{Ad}_{r,\psi}^+ [b] \circ \text{Ad}_{-r,\psi}^-[s] = (\text{Ad}_{r,\psi}^+[s] \circ \text{Ad}_{-r,\psi}^+)[b] = \text{Ad}_{-r,\psi}^+ \circ \text{Ad}_{r,\psi}^+ = \text{Ad}_{r,\psi}^+
\]
\[
\forall t \in \mathbb{R}, \ a \leq s \leq b.
\]
This yields (second step), together with Corollary 16 and (112) (first step) that
\[
\text{Ad}_{r,\psi}^+[b](\dot{\eta}_{h, t} X(0)) = [\int_a^b \text{Ad}_{r,\psi}^+[b](\chi_t(s)) \, ds, \text{Ad}_{r,\psi}^-[b](X)]
\]
\[
= [\int_a^b \text{Ad}_{r,\psi}^+[b](\psi(s)) \, ds, \text{Ad}_{r,\psi}^-[b](X)]
\]
holds for each \( t \in [0, 1] \). Now, Corollary 18 (second step) shows
\[
\eta_{\psi, X}(t + h) = \text{Ad}_{r,\psi + h, \psi}^+[b](X)
\]
\[
= \text{Ad}_{r,\psi}^+[b](\text{Ad}_{-h, \psi}^-[\eta_{\psi}](\psi(\cdot)))[b](X)
\]
for \( t, h \in \mathbb{R} \). Hence, (113), together with Lemma 17 and Part b) of Proposition A.1 yields
\[
\dot{\eta}_{\psi, X}(t) = [\Sigma(\psi)(t), \eta_{\psi, X}(t)] \quad \forall t \in [0, 1],
\]
with \( \eta_{\psi, X}(0) = X \). Then, Proposition 6.2 (second step) shows
\[
\text{Ad}_{r,\psi}^+[b](X) = \eta_{\psi, X}(t) = \text{Ad}_{r,\psi}^+[\Sigma(\psi)(t)](X) \quad \forall t \in [0, 1].
\]
Since \( X \in q \) was arbitrary, the claim follows.

We obtain the following statement.

\[\textbf{Lemma 26}\]

\textit{Let} \( \psi \in C^0([a, b + \varepsilon), q) \) \textit{with} \( a < b \) \textit{and} \( \varepsilon > 0 \) \textit{be given}. \textit{Then, we have}
\[
\Sigma(\psi|_{[a, z+h]}) = \Sigma(\psi|_{[z, z+h]})*\Sigma(\psi|_{[a, z]}) \quad \forall a < z \leq b, \ 0 < h < \varepsilon.
\]

\[\textbf{Proof}\]

\textit{Let} \( a < z \leq b \) \textit{and} \( 0 < h < \varepsilon \) \textit{be fixed}. \textit{For} \( t \in [0, 1] \), \textit{we have}
\[
\Sigma(\psi|_{[a, z+h]})(t) = \int_a^{z+h} \text{Ad}_{r,\psi|_{[z, z+h]}}^+[\psi(s)] \, ds
\]
\[
= \int_a^z \text{Ad}_{r,\psi|_{[z, z+h]}}^+[\psi(s)] \, ds + \int_a^h \text{Ad}_{r,\psi|_{[z, z+h]}}^+[\psi(s)] \, ds.
\]
Now, Lemma 18 and (22) (first step), as well as Lemma 25 (third step) yield
\[
\alpha(t) = \text{Ad}_{r,\psi|_{[z, z+h]}}^+[\int_a^z \text{Ad}_{r,\psi|_{[z, z+h]}}^+[\psi(s)] \, ds]
\]
\[
= \text{Ad}_{r,\psi|_{[z, z+h]}}^+[z+h](\Sigma(\psi|_{[a, z]})(t))
\]
\[
= \text{Ad}_{r,\psi}^+\Sigma(\psi|_{[z, z+h]})(t)(\Sigma(\psi|_{[a, z]})(t))
\]
for each \( t \in [0, 1] \). \textit{We obtain}
\[
\Sigma(\psi|_{[a, z+h]})(t) = \Sigma(\psi|_{[z, z+h]}) + \text{Ad}_{r,\psi}^+\Sigma(\psi|_{[z, z+h]})(t)(\Sigma(\psi|_{[a, z]})(t))
\]
\[
= (\Sigma(\psi|_{[z, z+h]})\Sigma(\psi|_{[a, z]}))(t)
\]
for each \( t \in [0, 1] \), which proves the claim. \( \Box \)
We obtain the following statement (in analogy to Remark 9).

**Proposition 7** Let \( a < c < b, \psi \in C^k([a, b], q) \), and \( \varrho : [a', b') \to [a, b] (a' < b') \) be of class \( C^1 \) with \( \dot{\varrho} \mid_{(a', b')} > 0 \), \( \varrho(a') = b \), \( \varrho(b') = b \). Then, we have

\[
\begin{align*}
\Xi(\psi)^{-1} &= \Xi(\mathrm{inv}(\psi)) \quad (114) \\
\Xi(\psi) &= \Xi(\psi\mid_{(c, b)}) \cdot \Xi(\psi\mid_{[a, c]}) \quad (115) \\
\Xi(\psi) &= \Xi(\dot{\varrho} \cdot (\psi \circ \varrho)). \quad (116)
\end{align*}
\]

**Proof** Lemma 26 shows (115), and Remark 20 shows (116). To prove (114), let \( \varrho_1 : [a, b] \to [a, b] \) with \( \varrho_1(a) = a \), \( \varrho_1(b) = b \), as well as \( \varrho_2 : [b, 2b + a] \to [a, b] \) with \( \varrho_2(b) = a \), \( \varrho_2(2b + a) = b \) both be of class \( C^1 \), such that \( \dot{\varrho}_1 |_{(a, b)} > 0 \) as well as \( \dot{\varrho}_2 |_{(b, 2b + a)} > 0 \) holds with

\[
\dot{\varrho}_1(a) = \dot{\varrho}_1(b) = \dot{\varrho}_2(2b + a) = 0.
\]

We set \( \chi_1 := \dot{\varrho} \cdot (\psi \circ \varrho_1) \in C^0([a, b], q) \) as well as \( \chi_2 := \dot{\varrho}_2 \cdot (\mathrm{inv}(\psi) \circ \varrho_2) \in C^0([b, 2b + a], q) \), and define \( \phi \in C^0([a, 2b + a], q) \) by

\[
\phi(t) := \begin{cases} 
\chi_1(t) & \text{for } t \in [a, b] \\
\chi_2(t) & \text{for } t \in (b, 2b + a)
\end{cases}
\]

for each \( t \in [a, 2b + a] \). Then, we have

\[
\Xi(\psi) \cdot \Xi(\mathrm{inv}(\psi)) \quad (116) \quad \Xi(\chi_1) \cdot \Xi(\chi_2) \quad (115) \quad \Xi(\phi). 
\]

Let now \( t \in [0, 1] \) and \( X \in q \) be given (and observe \( t \cdot \mathrm{inv}(\psi) = \mathrm{inv}(t \cdot \psi) \)).

- For \( s \in [a, b] \), we have by Lemma 18 (first step), (97) in Remark 17 (third step), as well as Lemma 18 and Proposition 6.1 (fourth step) that

\[
\begin{align*}
\text{Ad}^{+}_{r, \varrho_1 |_{[s, 2b + a]}}(X) &= \text{Ad}^{+}_{r, \varrho_1 |_{[b, 2b + a]}}\left(\text{Ad}^{+}_{r, \varrho_1 |_{[s, b]}}(X)\right) \\
&= \text{Ad}^{+}_{r, \chi_2}\left(\text{Ad}^{+}_{r, \chi_1 |_{[s, b]}}(X)\right) \\
&= \text{Ad}^{+}_{\mathrm{inv}(r, \psi)}\left(\text{Ad}^{+}_{r, \psi |_{[\varrho_2(s), b]}}(X)\right) \\
&= \text{Ad}^{+}_{r, \psi |_{[\varrho_2(s), b]}}(X).
\end{align*}
\]

- For \( s \in [b, 2b + a] \), we define

\[
\chi : \varrho_2(s), b] \ni z \mapsto \mathrm{inv}(t \cdot \psi |_{[a, a + b - \varrho_2(s)]})(z - (\varrho_2(s) - a)) \in q.
\]

We observe the following:

- (i) We have \( \chi = \mathrm{inv}(t \cdot \psi) |_{\varrho_2(s), b} \) as

\[
\chi(z) = \mathrm{inv}(t \cdot \psi |_{[a, a + b - \varrho_2(s)]})(z - (\varrho_2(s) - a)) = t \cdot \psi(a + b - z)
\]

holds for each \( z \in \varrho_2(s), b \).

- (ii) By the second point in Remark 17 (second step), we have

\[
\text{Ad}^{+}_{\chi} = \text{Ad}^{+}_{\chi |_{[-(a - \varrho_2(s)), b + (a - \varrho_2(s))]}}(b).
\]
\[ = \text{Ad}^+_{\text{inv}(t \cdot \psi \mid [a, b - \varrho_2(s)])} (b + (a - \varrho_2(s))) \]
\[ = \text{Ad}^+_{\text{inv}(t \cdot \psi \mid [a, a + b - \varrho_2(s)])}. \]

We obtain from (97) in Remark 17 (second step), Point i) (third step), Point ii) (fourth step), as well as Proposition 6.1 (fifth step) that
\[ \text{Ad}^+_{\phi \mid [z, 2b-a]} (X) = \text{Ad}^+_{\chi \cdot \psi \mid [z, 2b-a]} (X) \]
\[ = \text{Ad}^+_{\text{inv}(t \cdot \psi \mid [\varrho_2(s), b])} (X) \]
\[ = \text{Ad}^+_{\chi} (X) \]
\[ = \text{Ad}^+_{\text{inv}(t \cdot \psi \mid [a, a + b - \varrho_2(s)])} (X) \]
\[ = \text{Ad}^+_{\psi \mid [a, a + b - \varrho_2(s)]} (X). \]  

We obtain from (118) and (119) (third step), (21) (fourth step), as well as the second point in Remark 2 (sixth step) that
\[ \mathcal{T}(\phi)(t) = \int_a^{2b+a} \text{Ad}^+_{\phi \mid [z, 2b-a]} (\phi(s)) \, ds \]
\[ = \int_a^{b} \text{Ad}^+_{\phi \mid [z, 2b-a]} (\phi(s)) \, ds + \int_b^{2b+a} \text{Ad}^+_{\phi \mid [z, 2b-a]} (\phi(s)) \, ds \]
\[ = \int_a^{b} \dot{\varphi}_1(s) \cdot \text{Ad}^+_{\phi \mid [z, \varrho_1(s)]} (\dot{\varphi}_1(s)) \, ds \]
\[ + \int_b^{2b+a} \dot{\varphi}_2(s) \cdot \text{Ad}^+_{\phi \mid [z, a + b - \varrho_2(s)]} (\text{inv}(\psi)(\varrho_2(s))) \, ds \]
\[ = \int_a^{b} \text{Ad}^+_{\phi \mid [z, a]} (\phi(s)) \, ds + \int_a^{b} \text{Ad}^+_{\phi \mid [z, a + b - s]} (\text{inv}(\psi)(s)) \, ds \]
\[ = \int_a^{b} \text{Ad}^+_{\phi \mid [z, a]} (\phi(s)) \, ds - \int_a^{b} \text{Ad}^+_{\phi \mid [z, a + b - s]} (\psi(a + b - s)) \, ds \]
\[ = 0 \]
holds for each \( t \in [0, 1] \). Together with (117), this proves (116). \[ \square \]

Moreover, applying Lemma 26 to the situation where \( (q, [\cdot, \cdot]) \equiv (g, [\cdot, \cdot]) \) is an asymptotic estimate and sequentially complete Lie algebra that is inherited by a Lie group \( G \), we obtain the following statement.

**Corollary 19** Let \( G \) be weakly \( C^\infty \)-regular, with sequentially complete and asymptotic estimate Lie algebra \( (g, [\cdot, \cdot]) \). Then, for \( \psi \in C^0([a, b + \varepsilon]) \) with \( a < b \) and \( \varepsilon > 0 \), we have
\[ \int_0^1 \mathcal{T}(\psi \mid [a, z + h]) = \int_0^1 \mathcal{T}(\psi \mid [z, z + h]) \cdot \int_0^1 \mathcal{T}(\psi \mid [a, z]) \quad \forall a < z \leq b, \ 0 < h \leq \varepsilon. \]

**Proof** Let \( a < z \leq b \) and \( 0 < h \leq \varepsilon \) be fixed. Corollary 8 shows
\[ \text{Ad}^+_{\mathcal{T}(\psi \mid [z, z + h])} (X) = \text{Ad}^+_{\mathcal{T}(\psi \mid [z, z + h])} (\mathcal{T}(\psi \mid [a, z + h])) \quad \forall t \in [0, 1], \ X \in g. \]

We thus obtain from (a) (first step) as well as Lemma 26 (last step) that
\[ \int_0^1 \mathcal{T}(\psi \mid [z, z + h]) \cdot \int_0^1 \mathcal{T}(\psi \mid [a, z]) = \int_0^1 \mathcal{T}(\psi \mid [z, z + h]) + \text{Ad}^+_{\mathcal{T}(\psi \mid [z, z + h])} (\mathcal{T}(\psi \mid [a, z])) \]

\[ \text{Recall that in Sect. 3.3, } \text{Ad}^+_{\psi} [X] \text{ had been defined in Equation (63) by } \text{Ad}^+_{\psi} [X], \text{ for } \psi \in \mathcal{D}_{[0, 1]} \supset C^\infty([0, 1], g) \text{ and } X \in g. \text{ Furthermore, observe that the right side of (68) in Corollary 8 obviously equals the definition of } \text{Ad}^+_{\psi} [X] \text{ in the beginning of Sect. 5.1.} \]
Lemma 27. Let \( a \leq b, 0 < \varepsilon < 1 \), and \( \psi \in C^0([a, b + \varepsilon], g) \) be given.

- We define \( \Phi_\psi : (a - 1, b + \varepsilon) \times [0, 1] \to g \) by
  \[
  \Phi_\psi(z, t) := \begin{cases} 
  (z - a) \cdot \psi(a) & \text{for } z \in (a - 1, a) \\
  \mathcal{T}(\psi\big|_{a, z})(t) & \text{for } z \in [a, b + \varepsilon)
  \end{cases}
  \]
  for each \( t \in [0, 1] \). Lemma 24 yields the following:
  - For \( z \in (a - 1, b + \varepsilon) \), we have \( \Phi_\psi(z, \cdot) \in C^\infty([0, 1], g) \).
  - For \( z \in [a, b + \varepsilon) \), we have
    \[
    \Phi_\psi(z, t) = \int_0^t \text{Ad}_{\Phi_\psi(s)}^+ [\psi(s)] \, ds = \sum_{\ell=0}^\infty \mathcal{T}_\ell(\psi, z, t) \quad \forall t \in [0, 1],
    \]
    where \( \Phi_\psi(z, \cdot) = \sum_{\ell=0}^\infty \mathcal{T}_\ell(\psi, z, \cdot) \) converges w.r.t. the \( C^\infty \)-topology.
  - For \( p \in \text{Sem}(E) \) and \( s \in \mathbb{N} \), we have
    \[
    \sup\{a - 1 \leq z \leq b + \varepsilon \mid p_s^\infty(\Phi_\psi(z, \cdot))\} < \infty. \tag{121}
    \]

- For each \( t \in [0, 1] \), we define the map
  \[
  \mu_{t, \psi} : (a - 1, b + \varepsilon) \ni z \mapsto \int_0^t \Phi_\psi(z, \cdot) \in G. \tag{122}
  \]
  We observe the following:
  - We have
    \[
    \mu_{t, \psi}(z) = \int_0^t \mathcal{T}(\psi\big|_{a, z}) \quad \forall z \in [a, b + \varepsilon). \tag{123}
    \]
  - We have
    \[
    \Phi_\psi(a, \cdot) = \mathcal{T}(\psi\big|_{a, a})(\cdot) = 0 \quad \text{hence} \quad \mu_{t, \psi}(a) = e \quad \text{for each} \quad t \in [0, 1]. \tag{124}
    \]

We obtain the following lemma.

Lemma 27. Let \( a \leq b, 0 < \varepsilon < 1 \), and \( \psi \in C^0([a, b + \varepsilon], g) \) be given. We define \( \psi \in C^0((a - 1, b + \varepsilon), g) \) by

\[
\psi(z) := \begin{cases} 
  \psi(a) & \text{for } z \in (a - 1, a) \\
  \psi(z) & \text{for } z \in [a, b + \varepsilon),
  \end{cases}
\]

and set

\[
\mathcal{D}(z, t) := \psi(z) + t \cdot [\psi(z), \Phi_\psi(z, t)] \quad \forall z \in (a - 1, b + \varepsilon), t \in [0, 1).
\]

Then, the following assertions hold:
For $z \in (a - 1, b + \varepsilon)$, $p \in \text{Sem}(E)$, and $s \in \mathbb{N}$, we have
\[ \lim_{h \to 0} \frac{1}{|p|} \cdot p_{\infty}^s (\Phi_\psi(z + h, \cdot) - \Phi_\psi(z, \cdot) - h \cdot D(z, \cdot)) = 0. \tag{125} \]

(2) The following assertions hold:

(a) We have $\partial_t \Phi_\psi(z, \cdot) = D(z, \cdot) \in C^\infty([0, 1], \mathfrak{g})$ for each $z \in (a - 1, b + \varepsilon)$.

(b) To each $p \in \text{Sem}(E)$, $s \leq k$, and $z \in (a - 1, b + \varepsilon)$, there exists $L_{p,s} \geq 0$ as well as $I_{p,s} \subseteq (a - 1, b + \varepsilon)$ open with $z \in I_{p,s}$, such that
\[ p_{\infty}^s (\Phi_\psi(z + h, \cdot) - \Phi_\psi(z, \cdot)) \leq |h| \cdot L_{p,s} \quad \forall h \in \mathbb{R}_{\neq 0} \text{ with } z + h \in I_{p,s}. \]

(c) $\Phi_\psi$ is continuous.

(3) We have $\mu_{t,\psi} \in C^1((a - 1, b + \varepsilon), G)$ for each $t \in [0, 1]$, with
\[ \dot{\mu}_{t,\psi}(z) = d_\psi L_{\mu_{t,\psi}}(z) \left( \int_0^1 \text{Ad}_{\mu_{t,\psi}(z)^{-1}} (\psi(z) + s \cdot [\psi(z), \Phi_\psi(z, s)]) \, ds \right) \]
\[ \forall z \in (a - 1, b + \varepsilon). \]

In particular, for each $t \in [0, 1]$ and $z \in [a, b + \varepsilon)$, we have
\[ \dot{\mu}_{t,\psi}(z) = d_\psi L_{\mu_{t,\psi}}(z) \left( \int_0^t \text{Ad}_{\mu_{t,\psi}(z)^{-1}} (\psi(z) + s \cdot [\psi(z), \mathfrak{T}(\psi)|_{[a,z]}(s)]) \, ds \right). \tag{126} \]

Proof (1) For $t \in [0, 1]$, $z \in (a - 1, a]$ and $h \neq 0$ with $z + h \in (a - 1, a]$, we have
\[ \Phi_\psi(z + h, t) - \Phi_\psi(z, t) - h \cdot D(z, h) \]
\[ = (z + h - a) \cdot \psi(a) - (z - a) \cdot \psi(a) \]
\[ - h \cdot (\psi(a) + t \cdot [\psi(a), (z - a) \cdot \psi(a)]) \]
\[ = 0. \]

Hence, (125) holds for each $z \in (a - 1, a)$, and we have
\[ \lim_{h \to 0} \frac{1}{|p|} \cdot p_{\infty}^s (\Phi_\psi(a + h, \cdot) - \Phi_\psi(a, \cdot) - h \cdot D(a, \cdot)) = 0 \]
\[ \forall \, p \in \text{Sem}(E), \, s \in \mathbb{N}. \tag{127} \]

For the remaining cases, we need the following observations:

- For $t \in [0, 1]$ and $z \in [a, b + \varepsilon)$, we define $L_0(z, t) := \psi(z)$ as well as (recall (110)))
\[ L_\ell(z, t) := t \cdot [\psi(z), \mathcal{T}_{\ell-1}(\psi, z, t)] \]
\[ = t \cdot [\psi(z), t^{\ell-1} \cdot \int_0^t \mathfrak{g}_{\ell-1}(\psi, s, z) \, ds] \]
\[ = t^\ell \cdot \int_a^z ds_1 \cdots \int_a^{s_{\ell-1}} ds_{\ell-1} \cdots \int_a^{s_0} \mathfrak{g}_{\ell}(ad_{\psi}(z) \circ ad_{\psi(s_0)} \circ \cdots \circ ad_{\psi(s_{\ell-1})})(\psi(s)) \]
for $\ell \geq 1$. Since $(\mathfrak{g}, [\cdot, \cdot])$ is asymptotic estimate and sequentially complete, Corollary 2 shows that the map
\[ L : [a, b + \varepsilon) \times [0, 1) \to \mathfrak{g}, \quad (z, t) \mapsto \sum_{\ell=0}^\infty L_\ell(z, t) \tag{128} \]

is defined, such that $C^\infty([0, 1], \mathfrak{g}) \ni L(z, \cdot) = \sum_{\ell=0}^\infty L_\ell(z, t)$ converges w.r.t. the $C^\infty$-topology for each $z \in [a, b + \varepsilon)$.

- Let $t \in [0, 1]$, $z \in [a, b + \varepsilon)$, and $h > 0$ be fixed.
- If $z + h < b$ holds, we have $T_0(\psi, z + h, t) - T_0(\psi, z, t) = \int_{z}^{z+h} \psi(s) \, ds$, as well as for $\ell \geq 1$

$$T_\ell(\psi, z + h, t) - T_\ell(\psi, z, t)$$

$$= t^\ell \cdot \int_a^z ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$= t^\ell \cdot \int_a^z ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$= t^\ell \cdot \int_a^z ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$(\text{In the second step, we have split the third integral in the first summand at } z.)$$

- If $a < z - h$ holds, we have $T_0(\psi, z, t) - T_0(\psi, z - h, t) = \int_{z-h}^{z} \psi(s) \, ds$, as well as for $\ell \geq 1$

$$T_\ell(\psi, z, t) - T_\ell(\psi, z - h, t)$$

$$= t^\ell \cdot \int_a^{z-h} ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$= t^\ell \cdot \int_a^{z-h} ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$= t^\ell \cdot \int_a^{z-h} ds \int_{s}^{s+h} ds_1 \int_{s_1}^{s_1+h} ds_2 \cdots \int_{s_{\ell-1}}^{s_{\ell-1}+h} ds_{\ell} \left( \text{ad}_{\psi(s_1)} \circ \cdots \circ \text{ad}_{\psi(s_{\ell})} \right)(\psi(s))$$

$$(\text{In the second step, we have split the third integral in the first summand at } z - h.)$$

Let $v \leq w$ be as in (95), and set

$$\Delta(w, z, h) := w_{\infty}(\psi(z) - \psi([z-h, z+h] \cap [a, b+\varepsilon])) \quad \forall z \in [a, b + \varepsilon], \ h \neq 0.$$ 

Let $z \in [a, b + \varepsilon]$ be fixed, and set

$$D := \{x - z \mid x \in [a, b + \varepsilon]\}.$$ 

For $s \in \mathbb{N}$, $\ell \geq 1$, and $h, L \in D$, we obtain

$$v_{\infty}(T_0(\psi, z + h, \cdot) - T_0(\psi, z, \cdot) - h \cdot L_0(z, \cdot))$$

$$= v(\int_{z}^{z+h} \psi(s) \, ds - \psi(z))$$

$$\leq |h| \cdot \Delta(w, z, h),$$

$$v_{\infty}(T_\ell(\psi, z + h, \cdot) - T_\ell(\psi, z, \cdot) - h \cdot L_\ell(z, \cdot))$$

$$\leq s! \cdot |h| \cdot \Delta(w, z, h) \cdot (b + \varepsilon - a) \cdot w_{\infty}(\psi) \cdot \frac{(b+\varepsilon-a)^{\ell-1}}{(\ell-1)!} \cdot w_{\infty}(\psi)^{\ell-1}$$

$$+ s! \cdot (b + \varepsilon - a) \cdot w_{\infty}(\psi) \cdot |h|^2 \cdot w_{\infty}(\psi)^2 \cdot \frac{(b+\varepsilon-a)^{\ell-2}}{(\ell-2)!} \cdot w_{\infty}(\psi)^{\ell-2}$$

$$+ s! \cdot |h|^2 \cdot w_{\infty}(\psi)^2 \cdot \frac{(b+\varepsilon-a)^{\ell-1}}{(\ell-1)!} \cdot w_{\infty}(\psi)^{\ell-1}.$$ 

Now, the maps (with $h \in D$)

$$\Phi(\psi, z + h, \cdot) = \sum_{\ell=0}^{\infty} T_\ell(\psi, z + h, \cdot), \quad \Phi(\psi, z, \cdot) = \sum_{\ell=0}^{\infty} T_\ell(\psi, z, \cdot),$$

$$L(z, \cdot) = \sum_{\ell=0}^{\infty} L_\ell(z, \cdot)$$
converge w.r.t. the $C^\infty$-topology; and continuity of $\psi$ implies $\lim_{h \to 0} \Delta(w, z, h) = 0$. The triangle inequality thus yields
\[
\lim_{D(h) \to 0} \frac{1}{|h|} \cdot p_\infty^\bullet(\Phi_\psi(z + h, \cdot) - \Phi_\psi(z, \cdot) - h \cdot L(z, \cdot)) = 0.
\]
Together with (127) this proves the claim, because for $z \in \{a, b + \varepsilon\}$ and $t \in [0, 1]$, we have
\[
L(z, t) = \psi(z) + t \cdot \sum_{t=1}^\infty [\psi(z), \mathcal{T}_t(\psi, z, t)] = \psi(z) + t \cdot [\psi(z), \Phi_\psi(z, t)] = \mathcal{D}(z, t)
\]
by continuity of $\cdot \cdot$.

(2) Point (a) is clear from Part 1). For the points (b) and (c), we observe the following:

- For $p \in \text{Sem}(E)$, $s \in \mathbb{N}$, $z \in (a - 1, b + \varepsilon)$, and $h \in \mathbb{R}$ with $z + h \in (a - 1, b + \varepsilon)$, we have
  \[
p_\infty^\bullet(\Phi_\psi(z + h, \cdot) - \Phi_\psi(z, \cdot)) \\
  \leq p_\infty^\bullet(\Phi_\psi(z + h, \cdot) - \Phi_\psi(z, \cdot) - h \cdot \mathcal{D}(z, \cdot)) + |h| \cdot p_\infty^\bullet(\mathcal{D}(z, \cdot)).
  \]

- Since $\cdot \cdot$ is continuous and bilinear, there exists $q \in \text{Sem}(E)$ with
  \[
p([X, Y]) \leq q(X) \cdot q(Y) \quad \forall X, Y \in g.
  \]

Consequently, (121) implies
\[
\sup\{|a - 1 \leq z \leq b + \varepsilon | p_\infty^\bullet(\mathcal{D}(z, \cdot))\} < \infty \quad \forall p \in \text{Sem}(E).
\]

Point (b) is now clear from Part 1), (129), and (130). For point (c), we set $s := 0$. Then, by Part 1) and (130), both summands in (129) tend to zero if $h$ tends to zero. Point (c) now follows from the triangle inequality, as well as continuity of $\Phi_\psi(z, \cdot)$ for each $z \in (a - 1, b + \varepsilon)$.

(3) Let $z \in (a - 1, b + \varepsilon)$ and $\mathbb{R} \ni \{h_n\}_{n \in \mathbb{N}} \to 0$ be given. Then, Part 2).b) implies that \{\Phi_\psi(z + h_n, \cdot)\}_{n \in \mathbb{N}} \to \mu_\infty \Phi_\psi(z, \cdot) holds (recall (14)). Theorem B and Lemma A yield
\[
\lim_{n \to \infty} \int_0^\bullet \Phi_\psi(z + h_n, \cdot) = \int_0^\bullet \Phi_\psi(z, \cdot) = \mu_{s, \psi}(z) \in G
\]
It follows that the map
\[
(a - 1, b + \varepsilon) \times [0, 1] \ni (z, s) \mapsto \int_0^\bullet \Phi_\psi(z, \cdot) = \mu_{s, \psi}(z) \in G
\]
is continuous. In particular, $\mu_{t, \psi}$ is continuous for each $t \in [0, 1]$.

Let now $t \in [0, 1]$ be fixed. The points (a) and (b) in Part 2) show that the map $\Phi_\psi |_{(a - 1, b + \varepsilon) \times [0, t]}$ fulfills the assumptions in Theorem C. We obtain (apply Part 2).a) in the third step
\[
\hat{\mu}_{t, \psi}(z) = \frac{d}{dh} \bigg|_{h=0} \int_0^t \Phi_\psi(z + h, \cdot) \\
= \frac{d}{dt} \bigg|_{t=0} \int_0^t \Phi_\psi(z, \cdot) \bigg( \int_0^t \Phi_\psi(z, \cdot) \bigg) = \int_0^t \mu_{t, \psi}(z) \bigg( \int_0^t \Phi_\psi(z, \cdot) \bigg) \bigg)
\]
for each $z \in (a - 1, b + \varepsilon)$. It follows from smoothness of the group operations, continuity of (131), and continuity of $\Phi_\psi$ (by Part 2).c) that $\hat{\mu}_{t, \psi}$ is continuous. This shows that $\mu_{t, \psi}$ is of class $C^1$, which proves the claim. \hfill \Box

We are ready for the proof of Theorem 2.
Proof of Theorem 2} Since \( g \) is sequentially complete, \( g \) is both Mackey complete and integral complete. To prove the claim, it thus suffices to show that \( G \) is \( C^0 \)-semiregular. For this, let \( \phi \in C^0([0, 1], g) \) be given, and find a continuous extension \( \psi : [0, 1 + \varepsilon] \to g \) for some \( \varepsilon > 0 \). We define \( \mu_{1, \psi} : (-1, 1 + \varepsilon) \to G \) as in (122) for \( a \equiv 0 \) and \( b \equiv 1 \) there. Lemma 27.2) shows that \( \mu_{1, \psi} \) is of class \( C^1 \), so that it remains to verify \( \delta'(\mu_{1, \psi}|_{[0, 1]}) = \phi \). For this, let \( z \in [0, 1] \) be fixed. Corollary 19 shows (\( \Phi_\psi \) is defined as in (120))

\[
\mu_{1, \psi}(z+h) = \int_0^1 \Phi_\psi(z+h, \cdot) = \int_0^1 \mathcal{X}(\psi|[0,z+h]) = \int_0^1 \mathcal{X}(\psi|[z,z+h]) \cdot \int_0^1 \mathcal{X}(\psi|[0,z])
\]  

(132)

for each \( h \in [0, \varepsilon] \). Set \( \hat{\psi} := \psi|[z,z+\varepsilon] \). Then,

\[
\mu_{1, \hat{\psi}}(z+h) = \int_0^1 \mathcal{X}(\hat{\psi}|[z,z+h]) = \int_0^1 \mathcal{X}(\psi|[z,z+h]) \quad \forall \ h \in [0, \varepsilon)
\]

holds by definition, so that we have

\[
\mu_{1, \psi}(z+h) \cdot \mu_{1, \psi}(z)^{-1} = \mu_{1, \hat{\psi}}(z+h) \quad \forall \ h \in [0, \varepsilon).
\]  

(133)

Lemma 27.2) shows \( \mu_{1, \hat{\psi}} \in C^1((z-1, z + \varepsilon), G) \), and Equation (126) in Lemma 27.2) together with both sides of (124) (first step) yields

\[
\dot{\mu}_{1, \hat{\psi}}(z) = \hat{\psi}(z) = \psi(z) = \phi(z).
\]

Then, (133) shows \( \delta'(\mu_{1, \psi})(z) = \phi(z) \), which proves the claim. \( \square \)

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Data availability The data that support the findings of this study are included in this paper.

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A Appendix

A.1 Bastiani’s Differential Calculus

In this appendix, we recall Bastiani’s differential calculus, confer also [3, 7, 19–22]. Let \( E, F \in \text{hlcVect} \) be given. A map \( f : U \to F \), with \( U \subseteq E \) open, is said to be differentiable if

\[
(D_v f)(x) := \lim_{t \to 0} 1/t \cdot (f(x + t \cdot v) - f(x)) \in F
\]
exists for each \( x \in U \) and \( v \in E \). The map \( f \) is said to be \( k \)-times differentiable for \( k \geq 1 \) if

\[
D_{v_k, \ldots, v_1} f := D_{v_k} (D_{v_{k-1}} (\ldots (D_{v_1} (f)) \ldots)) : U \to F
\]

is defined for all \( v_1, \ldots, v_k \in E \). Implicitly, this means that \( f \) is \( p \)-times differentiable for each \( 1 \leq p \leq k \), and we set

\[
d_x^p f (v_1, \ldots, v_p) \equiv d^p f (x, v_1, \ldots, v_p) := D_{v_p, \ldots, v_1} f (x) \quad \forall x \in U, \ v_1, \ldots, v_p \in E
\]

for \( p = 1, \ldots, k \). We furthermore define \( d f := d^1 f \), as well as \( d_x f := d_x^1 f \) for each \( x \in U \).

The map \( f : U \to F \) is said to

- be of class \( C^0 \) if it is continuous. In this case, we define \( d^0 f := f \).
- be of class \( C^k \) for \( k \geq 1 \) if it is \( k \)-times differentiable, such that

\[
d^p f : U \times E^p \to F, \quad (x, v_1, \ldots, v_p) \mapsto D_{v_p, \ldots, v_1} f (x)
\]

is continuous for \( p = 0, \ldots, k \). In this case, \( d^p f \) is symmetric and \( p \)-multilinear for each \( x \in U \) and \( p = 1, \ldots, k \), cf. [3].
- be of class \( C^\infty \) if it is of class \( C^k \) for each \( k \in \mathbb{N} \).

**Remark** Assume that \( E, F \) are normed spaces. Let \( U \subseteq E \) be non-empty open, as well as \( k \in \mathbb{N} \cup \{\infty\} \). Let \( \mathcal{F}^C_k(U, F) \) denote the set of all \( k \)-times Fréchet differentiable maps \( U \to F \). Then, \( C^{k+1}(U, F) \subseteq \mathcal{F}^C_k(U, F) \subseteq C^k(U, F) \) holds [20, 26], hence \( C^\infty(U, F) = \mathcal{F}^C\infty(U, F) \). □

We have the following differentiation rules [3].

**Proposition A.1** (a) A map \( f : E \supseteq U \to F \) is of class \( C^k \) for \( k \geq 1 \) if and only if \( d f \) is of class \( C^{k-1} \) when considered as a map \( E \times E \supseteq U \times U \to F \).

(b) Let \( f : E \to F \) be linear and continuous. Then, \( f \) is smooth, with \( d_x^1 f = f \) for each \( x \in E \), as well as \( d^p f = 0 \) for each \( p \geq 2 \).

(c) Let \( F_1, \ldots, F_m \) be Hausdorff locally convex vector spaces, and \( f_q : E \supseteq U \to F_q \) be of class \( C^k \) for \( k \geq 1 \) and \( q = 1, \ldots, m \). Then,

\[
f := f_1 \times \ldots \times f_m : U \to F_1 \times \ldots \times F_m, \quad x \mapsto (f_1 (x), \ldots, f_m (x))
\]

if of class \( C^k \), with \( d^p f = d^p f_1 \times \ldots \times d^p f_m \) for \( p = 1, \ldots, k \).

(d) Let \( F, \tilde{F}, \bar{F} \in \text{hlCVect}, \ 1 \leq k \leq \infty \), as well as \( f : F \supseteq U \to \tilde{U} \subseteq \tilde{F} \) and \( \tilde{f} : \bar{F} \to \bar{F} \supseteq \tilde{U} \subseteq \tilde{F} \) be of class \( C^k \). Then, \( \tilde{f} \circ f : U \to \bar{F} \) is of class \( C^k \), with

\[
d_x (\tilde{f} \circ f) = d_{f(x)} \tilde{f} \circ d_x f \quad \forall x \in U.
\]

(e) Let \( F_1, \ldots, F_m, E \in \text{hlCVect} \), and \( f : F_1 \times \ldots \times F_m \supseteq U \to E \) be of class \( C^0 \). Then, \( f \) is of class \( C^1 \) if and only if for \( p = 1, \ldots, m \), the partial derivatives

\[
\partial_p f : U \times F_p \ni (x_1, \ldots, x_m, v_p) \mapsto \lim_{t \to 0} 1/t \cdot (f(x_1, \ldots, x_p + t \cdot v_p, \ldots, x_m) - f(x_1, \ldots, x_m))
\]

exist in \( E \) and are continuous. In this case, we have

\[
d f ((x_1, \ldots, x_m), v_1, \ldots, v_m) = \sum_{p=1}^m \partial_p f ((x_1, \ldots, x_m), v_p)
\]

\[
= \sum_{p=1}^m \left( d f ((x_1, \ldots, x_m), (0, \ldots, 0, v_p, 0, \ldots, 0)) \right)
\]

for each \((x_1, \ldots, x_m) \in U\), and \( v_p \in F_p \) for \( p = 1, \ldots, m \).
A.2 Proof of Equation (26)

The claim is clear for \( m = 1 \) and \( n \geq 1 \). Assume now that (26) holds for some \( m \geq 1 \), as well as each \( n \geq 1 \). For \( X \in V_1(S), \alpha \in V_m(S), \) and \( \beta \in V_n(S) \) with \( n \geq 1 \), we obtain from (25) and antisymmetry of \([\cdot, \cdot]\) that

\[
[X, \alpha, \beta] = [X, [\alpha, \beta]] = [\alpha, [X, \beta]] \in V(S)_{(m+1)+n}
\]

holds. Since \([\cdot, \cdot]\) is bilinear, this implies \( \{V_{m+1}(S), V_n(S)\} \subseteq V_{(m+1)+n}(S) \) for each \( n \geq 1 \), so that the claim follows inductively.

A.3 Proof of Lemma 5

**Lemma 5** Let \( V \subseteq F \) be open with \( 0 \in V \). Let furthermore \( \Psi : V \times E \to E \) be smooth with \( \Psi(0, \cdot) = \text{id}_E \), such that \( \Psi(x, \cdot) \) is linear for each \( x \in V \). Then, to each \( p \in \text{Sem}(E) \), there exist \( q \in \text{Sem}(E) \) and \( w \in \text{Sem}(E) \) with

\[
p(\Psi(x, Y) - Y) \leq q(x) \cdot w(Y) \quad \forall x \in B_{q,1} \subseteq V, Y \in E.
\]

**Proof of Lemma 5** We consider the continuous map

\[
\Phi : V \times E \times F \to E, \quad (x, Y, Z) \mapsto \partial_1 \Psi((x, Y), Z)
\]

that is linear in the last two arguments. Then, for \( p \in \text{Sem}(E) \) fixed, by Lemma 4 there exist \( q \in \text{Sem}(E) \) and \( p \leq w \in \text{Sem}(E) \) with

\[
(p \circ \Phi)((x, Y), Z) \leq w(Y) \cdot q(Z) \quad \forall x \in B_{q,1} \subseteq V, Y \in E, Z \in F.
\] (134)

For \( x \in B_{q,1} \), we let \( \gamma : [0, 1] \ni t \mapsto t \cdot x \in B_{q,1} \), and define

\[
C^\infty([0, 1], E) \ni \alpha \gamma : [0, 1] \ni t \mapsto \Psi(\gamma(t), Y) \quad \forall Y \in E.
\]

Then, the parts b) and e) of Proposition A.1 yield \( \dot{\gamma} = \dot{\Phi}((\gamma, Y), \gamma) = \Phi((\gamma, Y), x) \), and we obtain

\[
p(\Psi(x, Y) - Y) = p(\alpha \gamma(1) - \alpha \gamma(0)) \leq \int_0^1 p(\dot{\alpha} \gamma(s)) \, ds
\]

\[
\leq \int_0^1 p(\Phi((\gamma(s), Y), x)) \, ds \leq p_\infty(\Phi((\gamma, Y), x)) \leq w(Y) \cdot q(x)
\] (134)

for all \( x \in B_{q,1} \) and \( Y \in E \). \( \square \)

A.4 Appendix to Sect. 2.3

For combinatorial reasons, there exist \( c[p]_0, \ldots, c[p]_{p^2} \in \mathbb{K} \) with

\[
f_p(g_p(z)) = (f_p \circ g_p)(z) = \sum_{n=0}^{p^2} c[p]_n \cdot z^n
\]

\[
f_p(g_p(n)) = (f_p \circ g_p)(n) = \sum_{n=0}^{p^2} c[p]_n \cdot n^n
\]

\[
f_p(g_p(a)) = (f_p \circ g_p)(a) = \sum_{n=0}^{p^2} c[p]_n \cdot a^n.
\]

Set \( c[p]_n := 0 \) for \( n \geq p^2 + 1 \). Since \( \{f_p \circ g_p\}_{p \in \mathbb{N}} \to f \circ g \) converges compactly on \( U_\delta(0) \), the Weierstrass convergence theorem yields

\[
\lim_{p \to \infty} c[p]_n = c_n \quad \forall n \in \mathbb{N}.
\] (135)
Proof of Equation (31) Since \( b_0 = 0 \) holds, we have \( c[q + \ell]_n = c[q]_n \) for \( n = 0, \ldots, q \) and \( \ell \in \mathbb{N} \), hence \( c[q]_n = c_n \) for \( n = 0, \ldots, q \) by (135). We obtain
\[
 f(g(n)) = \sum_{n=0}^q a_n \cdot (\sum_{\ell=1}^q b_\ell \cdot n^\ell)_n = f_q(g_q(n)) = \sum_{n=0}^q c[q]_n \cdot n^n
\]
which proves (31).

Proof of Equation (32) Fix \( \|a\|_A < s < S \), and set \( M := \|f\|_{L^s} \). Then, \( |(f_p \circ g_p)(z)| \leq M \) holds for all \( p \in \mathbb{N} \) and \( z \in C \) with \( |z| \leq s \). Cauchy’s estimate yields \( |c[p]|_n \leq \frac{M}{s^n} \) for all \( p, n \in \mathbb{N} \), hence
\[
 \sum_{n=N+1}^\infty |c[p]|_n \cdot (\|a\|_A)^n = \sum_{n=N+1}^\infty |c[p]|_n \cdot s^n \cdot \left( \frac{\|a\|_A}{s^n} \right)^n \leq M \cdot \sum_{n=N+1}^\infty \left( \frac{\|a\|_A}{s^n} \right)^n
\]
for all \( N, p \in \mathbb{N} \). Given \( \varepsilon > 0 \), there thus exists \( N_\varepsilon \in \mathbb{N} \) with
\[
 \|\sum_{n=N_\varepsilon+1}^\infty c_n \cdot a^n\|_A < \frac{\varepsilon}{4} \quad \text{as well as} \quad \|\sum_{n=N_\varepsilon+1}^\infty c[p]_n \cdot a^n\|_A < \frac{\varepsilon}{4} \quad \forall \ p \in \mathbb{N}.
\]
By (135), there exists \( m \in \mathbb{N} \) with
\[
 |c[p]_n - c_n| < \frac{\varepsilon}{4(N_\varepsilon+1):\max(\|a\|_A|0 \leq q \leq N_\varepsilon)} \quad \forall \ 0 \leq n \leq N_\varepsilon, \ p \geq m.
\]
Increasing \( m \) if necessary, we can additionally assume\(^{25}\)
\[
 \|f(g(a)) - f_p(g_p(a))\|_A < \frac{\varepsilon}{4} \quad \forall \ p \geq m.
\]
Set \( p := m \). Then, we have
\[
 \|f(g(a)) - (f \circ g)(a)\|_A \leq \|f(g(a)) - f_p(g_p(a))\|_A + \|f_p(g_p(a)) - \sum_{n=0}^\infty c_n \cdot a^n\|_A
\]
with
\[
 \begin{align*}
 A & \leq \sum_{n=0}^{N_\varepsilon} c[p]_n \cdot a^n - \sum_{n=0}^{N_\varepsilon} c_n \cdot a^n + \| \sum_{n=N_\varepsilon+1}^\infty c[p]_n \cdot a^n - \sum_{n=N_\varepsilon+1}^\infty c_n \cdot a^n \|_A \\
 & \leq \sum_{n=0}^{N_\varepsilon} |c[p]_n - c_n| \cdot \|a^n\|_A + \| \sum_{n=N_\varepsilon+1}^\infty c[p]_n \cdot a^n \|_A + \| \sum_{n=N_\varepsilon+1}^\infty c_n \cdot a^n \|_A \\
 & \leq \frac{3}{4} \cdot \varepsilon
\end{align*}
\]
by (136) and (137).

A.5 Proof of Lemma 9

Lemma 9 Assume that \( G \) is weakly \( C^k \)-regular for \( k \in \mathbb{N} \cup \{\infty\} \), and let \( \phi \in C^k([a, b]) \) \((a < b)\) be given. Then, \( \kappa : \mathbb{R} \times [a, b] \ni (t, x) \mapsto [\int_x^t \phi]_x \cdot \phi \in G \) is continuous.

Proof Let \( z \in [a, b] \), \( t \in \mathbb{R} \), and \( U \subseteq G \) be an open neighborhood of \( e \). We fix \( W \subseteq G \) open with \( e \in W \) and \( W \cdot W \subseteq U \), and observe that
\[
 \kappa(t, \cdot) : [a, b] \ni x \mapsto [f_x^b t \cdot \phi]_x \cdot [f_x^b t \cdot \phi]^{-1} \in G
\]
\(^{25}\) Apply the triangle inequality in the form \( \|f(g(a)) - f_p(g_p(a))\|_A \leq \|f(g(a)) - f(g_p(a))\|_A + \|f(g_p(a)) - f_p(g_p(a))\|_A \).

\(\square\) Springer
is continuous. Since \([a, b]\) is compact, there exists some open \(V \subseteq G\) with \(e \in V = V^{-1}\), such that

\[
(f^b x \cdot \phi)^{-1} \cdot V \cdot (f^b x \cdot \phi) \subseteq W \quad \forall x \in [a, b].
\]

Theorem B together with Lemma 15 in [11] implies that there exists \(\delta > 0\) with

\[
\int_a^\bullet (t + h) \cdot \phi \in V \cdot [f^b_a t \cdot \phi] \quad \forall |h| < \delta.
\]

For \(a \leq x \leq b\) and \(|h| < \delta\), we obtain

\[
\int_x^b (t + h) \cdot \phi = [f^b_a (t + h) \cdot \phi] \cdot [f^x_a (t + h) \cdot \phi]^{-1}
\]

\[
\in V \cdot ([f^b_a t \cdot \phi] \cdot [f^x_a t \cdot \phi]^{-1}) \cdot V^{-1}
\]

\[
= V \cdot (f^b_a t \cdot \phi) \cdot V
\]

\[
\subseteq W \cdot f^b_a t \cdot \phi.
\]

Since \(\kappa(t, \cdot)\) is continuous, we can shrink \(\delta > 0\) such that for each \(x \in (z - \delta, z + \delta) \cap [a, b]\), we have \([(f^b_x t \cdot \phi) \cdot (f^x_t t \cdot \phi)]^{-1} \in W\). We obtain

\[
[f^x_t (t + h) \cdot \phi] \cdot [f^b_x t \cdot \phi]^{-1} = ([f^x_t (t + h) \cdot \phi] \cdot [f^b_x t \cdot \phi]^{-1})
\]

\[
= [f^b_x t \cdot \phi] \cdot [f^x_t t \cdot \phi]^{-1} \in W \cdot W \subseteq U
\]

for \(|h| < \delta\) and \(x \in (z - \delta, z + \delta) \cap [a, b]\).

\(\square\)

**A.6 Proof of Equation (64)**

**Proof of Equation (64)** By definition, we have \(\delta^r(\mu)|_{[a, b]} = \psi\) for certain \(\mu \in C^1((a - \varepsilon, b + \varepsilon), G)\) and \(\varepsilon > 0\). Replacing \(\mu\) by \(\mu \cdot \mu(a)^{-1}\) if necessary, we can assume that \(\mu(a) = e\) holds, hence

\[
\mu|_{[a, b]} = \int_a^\bullet \phi = \int_a^\bullet \psi \quad \text{for} \quad \phi := \delta^r(\mu).
\]

Then, \(\alpha^\pm := \text{Ad}_{\mu^\pm}(X) \in C^1((a - \varepsilon, b + \varepsilon), G)\) holds by Lemma 8, with \(\alpha^\pm|_{[a, b]} = \text{Ad}_{\psi}[X]\). This shows \(\text{Ad}^\pm_{\psi}[X] \in C^1([a, b], g)\), with \(\text{Ad}^\pm_{\psi}[X](a) = \text{Ad}_e(X) = X\). Let now \(t \in [a, b]\) and \(0 < h < \varepsilon\) be given. We observe

\[
[f^t+h \phi]^{-1} \overset{(\text{c})}{=} f^t+h -\text{Ad}_{[f^t \phi]^{-1}}(\phi),
\]

and obtain from (d) that

\[
\alpha^+(t + h) = \text{Ad}_{f^{t+h} \phi} (\text{Ad}_{\psi}[X](t))
\]

\[
\alpha^-(t + h) = \text{Ad}_{\psi}^{-1}[\text{Ad}_{f^{t+h} \phi^{-1}}(X)](t)
\]

\[
= \text{Ad}_{\psi}^{-1}[\text{Ad}_{f^{t+h}}^{-1} -\text{Ad}_{[f^t \phi^{-1}]}^{-1}(\phi)](X)](t).
\]

Differentiating both hand sides and then applying (9), we obtain (64).

\(\square\)

**A.7 The Proof of Lemma 17**

**Lemma 17** Let \(\psi \in C^0([a, b], q)\) be given. Then, the following assertions hold:
(1) The maps $\Lambda^\pm_\psi$ are defined and of class $C^1$, and we have

$$
\partial_t \Lambda^+_\psi(t, X) = \partial_t \Lambda^+\psi[X](t) = [\psi(t), \Lambda^+\psi[X](t)] \\
\partial_t \Lambda^-\psi(t, X) = \partial_t \Lambda^-\psi[X](t) = -\Lambda^-\psi[[\psi(t), X]](t)
$$

for all $t \in [a, b]$ and $X \in \mathfrak{q}$.

(2) $\Lambda^\pm_\psi(t, \cdot) = \Lambda^\pm\psi[t]$ is linear and continuous for each $t \in [a, b]$.

(3) If $\psi \in C^k([a, b], \mathfrak{q})$ holds for $k \in \mathbb{N} \cup \{\infty\}$, then $\Lambda^\pm_\psi$ is of class $C^{k+1}$.

**Proof** Let $X \in \mathfrak{q}$ be fixed. Then, $\lambda^\pm_\ell,\psi[X]$ is of class $C^1$ for each $\ell \in \mathbb{N}$ by (20); with $\partial_t \lambda^\pm,\psi[X] = 0$ as well as

$$
\partial_t \lambda^+_\psi[X](t) = [\psi(t), \lambda^+_\psi[X](t)] \quad \text{and} \quad \partial_t \lambda^-\psi[X](t) = -\lambda^-\psi[[\psi(t), X]](t)
$$

for each $\ell \geq 1$ and $t \in [0, 1]$. For $v \leq w$ as in (95), we have

$$
\sum_{\ell=0}^{\infty} v(\lambda^+_\ell,\psi[X]) \leq w(X) \cdot e^{(b-a)\cdot w_\infty(\psi)} < \infty. \tag{138}
$$

Corollary 1 thus implies $\Lambda^\pm(\cdot, X) = \Lambda^\pm\psi[X] \in C^1([a, b], \mathfrak{q})$, with

$$
\partial_t \Lambda^\pm\psi[X](t) = \sum_{\ell=0}^{\infty} [\psi(t), \lambda^\pm_\ell,\psi[X](t)] = [\psi(t), \Lambda^\pm\psi[X](t)] \\
\partial_t \Lambda^\pm\psi[X](t) = -\sum_{\ell=0}^{\infty} \lambda^-_\ell,\psi[[\psi(t), X]](t) = -\Lambda^-\psi[[\psi(t), X]](t)
$$

for each $t \in [a, b]$, which establishes Part 1. Now, it is clear from the definitions that $\Lambda^\pm_\psi(t, \cdot)$ is linear for each $t \in [a, b]$; and we furthermore observe that

$$
v\left(\sum_{\ell=0}^{\infty} \lambda^\pm_\ell,\psi[X](t) - \sum_{\ell=0}^{\infty} \lambda^\pm_\ell,\psi[X](t)\right) \\
\leq |t - t'| \cdot w(X) \cdot w_\infty(\psi) \cdot \sum_{\ell=1}^{\infty} \frac{(b-a)^{\ell-1}}{\ell-1} \cdot w_\infty(\psi)^{\ell-1} \\
= |t - t'| \cdot w(X) \cdot w_\infty(\psi) \cdot e^{(b-a)\cdot w_\infty(\psi)} \tag{139}
$$

holds for all $t, t' \in [a, b]$ and $X \in \mathfrak{q}$. We obtain from (138) and (139) that

$$
v(\Lambda^\pm_\psi(t', X') - \Lambda^\pm_\psi(t, X)) \leq v(\Lambda^\pm_\psi(t', X') - \Lambda^\pm_\psi(t', X)) + v(\Lambda^\pm_\psi(t', X) - \Lambda^\pm_\psi(t, X)) \\
\leq (w(X' - X) + |t - t'| \cdot w(X) \cdot w_\infty(\psi)) \cdot e^{(b-a)\cdot w_\infty(\psi)}
$$

for all $t, t' \in [a, b]$ and $X, X' \in \mathfrak{q}$. This shows that $\Lambda^\pm_\psi$ is continuous, which establishes Part 2. We furthermore conclude that the partial derivatives

$$
\partial_t \Lambda^\pm_\psi((t, X), \lambda) = \lambda \cdot [\psi(t), \Lambda^\pm_\psi(t, X)] \quad \forall t \in [a, b], \lambda \in \mathbb{R}, X \in \mathfrak{q} \\
\partial_t \Lambda^\pm_\psi((t, X), \lambda) = -\lambda \cdot \Lambda^\pm_\psi(t, [\psi(t), X]) \quad \forall t \in [a, b], \lambda \in \mathbb{R}, X \in \mathfrak{q} \\
\partial_2 \Lambda^\pm_\psi((t, X), Y) = \Lambda^\pm_\psi(t, Y) \quad \forall t \in [a, b], X, Y \in \mathfrak{q}
$$

are continuous. Part e) of Proposition A.1 thus shows that the maps $\Lambda^\pm_\psi$ are of class $C^1$, with
\[
\begin{align*}
    \d\Lambda_+^\psi(t, X, (\lambda, Y)) &= \lambda \cdot [\psi(t), \Lambda_+^\psi(t, X)] + \Lambda_+^\psi(t, Y) \\
    \d\Lambda_-^\psi(t, X, (\lambda, Y)) &= -\lambda \cdot \Lambda_-^\psi(t, [\psi(t), X]) + \Lambda_-^\psi(t, Y).
\end{align*}
\]

Finally, assume that \( \psi \in C^k([a, b], q) \) holds for \( k \geq 1 \). Then, (140) together with the parts a), c), d) of Proposition A.1 imply that given \( 1 \leq \ell \leq k \), then \( \Lambda_+^\psi \) is of class \( C^{\ell+1} \) if \( \Lambda_-^\psi \) is of class \( C^\ell \), which establishes Part 3). It thus follows by induction that \( \Lambda_+^\psi \) is of class \( C^{k+1} \). \( \square \)

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