APPROACHING CENTRAL PROJECTIONS IN AF-ALGEBRAS

DANIELE MUNDICI

Abstract. Let $A$ be a unital AF-algebra whose Murray-von Neumann order of projections is a lattice. For any two equivalence classes $[p]$ and $[q]$ of projections we write $[p] \sqsubseteq [q]$ iff for every primitive ideal $p$ of $A$ either $p/p \preceq q/p \preceq (1 - q)/p$ or $p/p \succeq q/p \succeq (1 - q)/p$. We prove that $p$ is central iff $[p]$ is $\sqsubseteq$-minimal iff $[p]$ is a characteristic element in $K_0(A)$. If, in addition, $A$ is liminary, then each extremal state of $K_0(A)$ is discrete, $K_0(A)$ has general comparability, and $A$ comes equipped with a centripetal transformation $[p] \mapsto [p]$ that moves $p$ towards the center. The number $n(p)$ of $\sqsubseteq$-steps needed by $[p]$ to reach the center has the monotonicity property $[p] \sqsubseteq [q] \Rightarrow n(p) \leq n(q)$.

Our proofs combine the $K_0$-theoretic version of Elliott’s classification, the categorical equivalence $\Gamma$ between MV-algebras and unital $\ell$-groups, and Lőš ultraproduct theorem for first-order logic.

1. Introduction

Every $C^*$-algebra $A$ in this paper will be unital and separable. The ideals of $A$ will be closed and two-sided. We let $\text{proj}(A)$ be the set of projections of $A$, and $\text{prim}(A)$ be the space of primitive ideals of $A$ with the Jacobson topology, [14, §3.1].

Following [3], by an AF-algebra we mean the norm closure of the union of an ascending sequence of finite-dimensional $C^*$-algebras, all with the same unit.

Two projections $p, q$ of AF algebra $A$ are (Murray-von Neumann) equivalent, in symbols $p \sim q$, if there is an element $x \in A$ (necessarily a partial isometry) such that $p = x^*x$ and $q = xx^*$. We write $p \preceq q$ if $p$ is equivalent to a subprojection $r \preceq q$. The reflexive and transitive $\preceq$-relation is preserved under equivalence, and $\preceq$ has the antisymmetry property $p \preceq q \preceq p \Rightarrow p \sim q$, because $A$ is stably finite, [13, Theorem IV.2.3]. The resulting ordering on equivalence classes of projections in $A$ is called the Murray-von Neumann order of $A$.

Let $L(A)$ be the set of equivalence classes $[p]$ of projections $p$ of $A$. Elliott’s partial addition in $L(A)$ is defined by setting $[p] + [q] = [p + q]$ whenever $p$ and $q$ are orthogonal. One then obtains a countable partially ordered “local” semigroup, which by Elliott’s classification [17], is a complete classifier of AF-algebras. The adjective “local” means that the addition operation in $L(A)$ is not always defined. $L(A)$ inherits a partial order from the $\preceq$ relation, and Elliott’s partial addition is monotone with respect to this order.

When the Murray-von Neumann order of an AF-algebra $A$ is a lattice we say that $A$ is an AFℓ-algebra.

The theory of AFℓ-algebras is grounded in the following result, which will also be basic for the present paper:

Theorem 1.1. Let $A$ be an AF algebra and $L(A)$ the Elliott partially ordered local semigroup of $A$.

Date: September 6, 2018.

2000 Mathematics Subject Classification. Primary: 46L35. Secondary: 06D35, 06F20, 19A49, 46L80, 47L40.

Key words and phrases. AF algebra, Elliott classification, Elliott local semigroup, Murray-von Neumann order, Grothendieck $K_0$ group, MV algebra, liminary C*-algebra, Effros-Shen algebra, Behnke-Leptin algebras, Farey AF algebra.
(i) [27] Elliott’s partially defined addition $+_{L(A)}$ has at most one extension to an associative, commutative, monotone operation $\oplus: L(A)^2 \to L(A)$ satisfying the following condition: For each projection $p \in A_A \cap [1_A]$, the smallest element $[q] \in L(A)$ with $[p] \oplus [q] = [1_A]$. The unique semigroup $(S(A), \oplus)$ expanding the Elliott local semigroup $L(A)$ exists iff $A$ is an $\mathcal{A}F\ell$-algebra.

(ii) Let $A_1$ and $A_2$ be $\mathcal{A}F\ell$-algebras. For each $j = 1, 2$ let $\oplus_j$ be the extension of Elliott’s addition given by (i). Then the semigroups $(S(A_1), \oplus_1)$ and $(S(A_2), \oplus_2)$ are isomorphic iff so are $A_1$ and $A_2$.

(iii) [27] For any $\mathcal{A}F\ell$-algebra $A$ the semigroup $(S(A), \oplus)$ has the structure of a monoid $(E(A), 0, \neg, \oplus)$ with an involution operation $\neg[p] = [1_A - p]$. The Murray-von Neumann lattice order of equivalence classes of projections $[p], [q]$ is definable by the involutive monoidal operations of $E(A)$, upon setting $[p] \lor [q] = \neg(\neg[p] \oplus [q]) \oplus [q]$ and $[p] \land [q] = \neg(\neg[p] \lor \neg[q])$ for all $[p], [q] \in E(A)$.

(iv) [22] Theorem 3.9] Up to isomorphism, the map $A \to (E(A), 0, \neg, \oplus)$ is a one-one correspondence between $\mathcal{A}F\ell$-algebras and countable abelian monoids with a unary operation $\neg$ satisfying the equations:

$$\neg x = x, \quad \neg 0 \oplus x = -0, \quad \text{and} \quad \neg(\neg x \oplus y) \oplus y = \neg(y \oplus x) \oplus x.$$ 

These involutive monoids are known as MV-algebras. Let $\Gamma$ be the categorical equivalence between unital $\ell$-groups and MV-algebras. Then $(E(A), 0, \neg, \oplus)$ is isomorphic to $\Gamma(K_0(A))$.

(v) (From (ii)-(iv) via [10].) For any $\mathcal{A}F\ell$-algebra $A$ the dimension group $K_0(A)$ (which is short for $(K_0(A), K_0(A)^+, [1_A])$) is a countable lattice ordered abelian group with a distinguished strong order unit (for short, a unital $\ell$-group). All countable unital $\ell$-groups arise in this way. Let $A$ and $A'$ be $\mathcal{A}F\ell$-algebras. Then $K_0(A)$ and $K_0(A')$ are isomorphic as unital $\ell$-groups iff $A$ and $A'$ are isomorphic.

We refer to [10] and [29] for background on MV-algebras. The following characterization ([26] Proposition 4.13) will find repeated use throughout this paper:

**Proposition 1.2.** Let $J$ be an ideal (= kernel of a homomorphism) of an MV-algebra $B$. Then the following conditions are equivalent:

(i) The quotient $B/J$ is an MV-chain, meaning that the underlying order of $B/J$ is total.

(ii) Whenever $J$ coincides with the intersection of two ideals $H$ and $K$ of $B$, then either $J = H$ or $J = K$.

An ideal of $B$ will be said to be prime if it satisfies the two equivalent conditions above. For every MV-algebra $B$ we let

$$\text{Spec}(B)$$

(1)

denote the space of prime ideals of $B$ endowed with the Zariski (hull-kernel) topology, ([26] Definition 4.14).

**Corollary 1.3.** In any $\mathcal{A}F\ell$-algebra $A$ we have:

(i) $K_0$ induces an isomorphism

$$\eta: i \mapsto K_0(i) \cap E(A)$$

between the lattice of ideals of $A$ and the lattice of ideals of $E(A)$. Under this isomorphism, primitive ideals of $A$ correspond to prime ideals of $E(A)$.

(ii) The isomorphism $\eta$ is a homeomorphism of the space $\text{prim}(A)$ of primitive ideals of $A$ with the Jacobson topology, onto the space $\text{Spec}(E(A))$ of prime ideals of $E(A)$. 
(iii) Suppose \( I \) is an ideal of the countable MV-algebra \( B \). Let the AF\( \ell \)-algebra \( A \) be defined by \( E(A) = B \), in view of Theorem 1.1(iv). Let \( i \) be the ideal of \( A \) defined by \( \eta(i) = I \). Then \( B/I \) is isomorphic to \( E(A/i) \).

(iv) For every ideal \( i \) of \( A \), the map \( [p] \mapsto [p]_{\eta(i)} \), \( p \in \proj(A) \) is an isomorphism of \( E(A/i) \) onto \( E(A)/\eta(i) \). In particular, for every \( p \in \prim(A) \) the MV-algebra \( E(A/p) \) is totally ordered and \( A/p \) has comparability of projections in the sense of Murray-von Neumann.

(v) The map \( J \mapsto J \cap \Gamma(K_0(A)) \) is an isomorphism of the lattice of ideals of \( K_0(A) \) (i.e., kernels of unit preserving \( \ell \)-homomorphisms of \( K_0(A) \) into unital \( \ell \)-groups) onto the lattice of ideals of \( E(A) \). Further,

\[
\Gamma \left( \frac{K_0(A)}{J} \right) \cong \frac{\Gamma(K_0(A))}{J \cap \Gamma(K_0(A))}.
\]

Proof. (i) From [13, Proposition IV.5.1] and [19, p.196, 21H] one gets an isomorphism between the lattice of ideals of \( A \) and the lattice of ideals of the \( \ell \)-group \( K_0(A) \). The preservation properties of \( \Gamma \), [10, Theorems 7.2.2, 7.2.4] then yield the desired isomorphism. For the second statement, combine [3, Theorem 3.8] with the characterization given in Proposition 1.2 of prime ideals of an MV-algebra.

(ii) follows from (i), by definition of the topologies of \( \prim(A) \) and of \( \Spec(E(A)) \).

(iii) We have an exact sequence

\[
0 \to i \to A \to A/i \to 0.
\]

Correspondingly ([13, IV.15], [19] Corollary 9.2) we have an exact sequence

\[
0 \to K_0(i) \to K_0(A) \to K_0(A/i) \to 0,
\]

whence

\[
K_0 \left( \frac{A}{i} \right) \cong \frac{K_0(A)}{K_0(i)}.
\]

The preservation properties of \( \Gamma \) under quotients [10, Theorem 7.2.4], together with Theorem 1.1(iv)-(v) yield

\[
E \left( \frac{A}{i} \right) \cong \frac{\Gamma(K_0(A))}{\Gamma(K_0(i))} \cong \frac{E(A)}{\eta(i)} = B.
\]

(iv) Combine (i) and (iii) with the preservation properties of \( K_0 \) for exact sequences and the preservation properties of \( \Gamma \) under quotients. The MV-algebra \( E(A/p) \) is totally ordered by Proposition 1.2, because, by (ii), \( \eta(p) \in \Spec(E(A)) \) whenever \( p \in \prim(A) \). By Theorem 1.1(iv), \( A/p \) has comparability of projections.

(v) This follows by another application of [10, Theorems 7.2.2, 7.2.4]. \( \square \)

2. Central projections in AF\( \ell \)-algebras

MV-algebras were invented by C.C.Chang [8] to give an algebraic proof of the completeness of the Lukasiewicz axioms. For any MV-algebra \( D \) we let

\[
\text{Boole}(D) = \{ a \in D \mid a \oplus a = a \}.
\]

As observed by Chang in [8] Theorems 1.16-1.17, \( \text{Boole}(D) \) is a subalgebra of \( D \) which turns out to be a boolean algebra.
The following theorem and its extension Theorem 2.2 make precise the intuition that commutative AF-algebras stand to Boolean algebras as AF-algebras stand to MV-algebras:

**Theorem 2.1.** For every projection $p$ of an AF-algebra $A$ the following conditions are equivalent:

(i) $p/p \in \{0,1\} \subseteq A/p$ for all $p \in \text{prim}(A)$.

(ii) $[p] \in \text{Boole}(E(A))$.

(iii) $p$ is central in $A$.

(iv) $[p]$ is a characteristic element of $K_0(A)$, in the sense that $[p] \wedge [1_A - p]$ exists and equals 0, [20] Definition p.127.

**Proof.** (ii)$\Rightarrow$(i) From the assumption $[p] \oplus [p] = [p]$ it follows that $[p]/P \oplus [p]/P = [p]/P$, whence $[p]/P \in \text{Boole}(E(A)/P) = \{0,1\}$ for each $P \in \text{Spec}(E(A))$, because $E(A)/P$ is totally ordered (Proposition 1.2). Let $p$ be the primitive ideal of $A$ given by $\eta(p) = P$, with $\eta$ the isomorphism of Corollary 1.3(i)). Then $[p]/P \in \text{Boole}(E(A)/p) = \{0,1\}$, because by Corollary 1.3(iv), $E(A/p) \cong E(A)/P$ is totally ordered. Since $P$ is an arbitrary prime ideal of $E(A)$, then $p$ is an arbitrary primitive ideal of $A$. We conclude that $p/p \in \{0,1\}$ for all $p \in \text{prim}(A)$.

(i)$\Rightarrow$(iii) The hypothesis implies that $p/p$ is central in $A/p$ for each $p \in \text{prim}(A)$. Then $p$ is central in $A$.

(iii)$\Rightarrow$(ii) By way of contradiction assume $p$ central in $A$ but $[p] \notin \text{Boole}(E(A))$. By [10] Corollary 1.2.14,

$$\bigcap \{P \mid P \in \text{Spec}(E(A))\} = \{0\}.$$  

Therefore, for some $P \in \text{Spec}(E(A))$, $[p]/P$ does not belong to $\text{Boole}(E(A)/P)$. In view of Corollary 1.3(i), let $p \in \text{prim}(A)$ be defined by $\eta(p) = P$. Then

$$\frac{p}{p} \text{ is central and } \frac{p}{p} \notin \{0,1\} \subseteq \frac{A}{p}. \quad (2)$$

Claim. The projections $p/p$ and $(1_A - p)/p$ of $A/p$ are not Murray-von Neumann comparable.

Arguing by way of contradiction, let us suppose $p/p$ and $(1_A - p)/p$ are comparable, say,

$$u^* u = \frac{1_A - p}{p} \quad \text{and} \quad u u^* = \frac{q}{p} \leq \frac{p}{p},$$

for some partial isometry $u \in A/p$ and $q/p \in \text{proj}(A/p)$. From

$$\frac{q}{p} \frac{p}{p} = \frac{p}{p} \frac{q}{p} = \frac{q}{p}$$

it follows that

$$\frac{1_A - p}{p} \frac{1_A - p}{p} = \frac{u uu^* u}{p} = \frac{u^* u u}{p} \leq \frac{u^* q u}{p} = \frac{u^* p u}{p} = \frac{1_A - p}{p} \frac{p}{p} = 0,$$

against (2). Our claim is settled.

On the other hand, since $P$ is prime, then $E(A)/P$ is totally ordered, by Proposition 1.2. By Corollary 1.3(iii)-(iv), $A/p$ has comparability, which contradicts our claim.

(ii) $\Leftrightarrow$ (iv). By Theorem 1.1(iv)-(v) and definition of $\Gamma$ ([22] Definition 2.4), we can write $E(A) = \{x \in K_0(A) \mid 0 \leq x \leq u\}$, where $u$ is the order-unit of $K_0(A)$, coinciding with the unit element $1 \equiv [1_A]$ of $E(A)$. By [22] Theorem 2.5 the lattice order of $E(A)$ agrees with the restriction to $E(A)$ of the lattice order of $K_0(A)$. The desired conclusion now follows from [20] Theorem 8.7, p.130].
The ordering $\sqsubseteq$ and the map $\sigma^*$: $[0,1] \to [0,1]$. The rest of this section is devoted to proving the following extension of Theorem 2.1.

Theorem 2.2. Assume $A$ is an $\mathbb{A}F\ell$-algebra. For any $x,y \in E(A)$ let us write $x \sqsubseteq y$ iff for every prime ideal $P$ of $E(A)$

$$(y/P < -y/P \text{ implies } x/P \leq y/P) \quad \text{and} \quad (y/P > -y/P \text{ implies } x/P \geq y/P),$$

with $\leq$ the underlying total order of $E(A)/P$, (Corollary 3.2.8, Theorem 9.1.5). Then $\sqsubseteq$ endows $E(A)$ with a partial order (reflexive, transitive, antisymmetric) relation. Further, for every $p \in \text{proj}(A)$, the equivalent conditions (i)-(iv) in Theorem 2.1 are equivalent to $[p]$ being $\subseteq$-minimal in $E(A)$.

The following transformation will play a key role in the sequel: Let $\tau = \tau(X)$ be an MV-term in the variable $X$. [10] Definition 1.4.1]. For any MV-algebra $B$ and $a \in B$, by induction on the number of operation symbols in $\tau$ let us define

$$a_X = a, \quad a_{\tau_1 \oplus \tau_2} = a_{\tau_1} \oplus a_{\tau_2}, \quad a_{\neg} = \neg a_{\tau}.$$ (3)

This transforms $a$ into an element $a_{\tau} \in B$. The ambient algebra $B$ will always be clear from the context. For every ideal $I$ of $B$, induction on the number of operation symbols in $\tau$ yields

$$(a/I)_{\tau} = a_{\tau}/I.$$ (4)

Following [10] p.8, for any two MV-terms $\rho, \tau$ we let $\rho \odot \tau$ denote the MV-term $\neg(\neg \rho \oplus \neg \tau)$. Correspondingly, for any two elements $a, b$ of an MV-algebra $B$ we write

$$a \odot b \text{ as an abbreviation of } \neg(\neg a \oplus \neg b).$$ (5)

Let $\text{Free}_1$ denote the free one-generator MV-algebra. As a special case of McNaughton representation theorem, ([10] Corollary 3.2.8, Theorem 9.1.5)], $\text{Free}_1$ is the MV-algebra of all one-variable McNaughton functions, those continuous piecewise (affine) linear functions $f: [0,1] \to [0,1]$ whose linear pieces have integer coefficients. Further, the identity function $\pi_1: [0,1] \to [0,1]$ freely generates $\text{Free}_1$.

Proposition 2.3. With the notation of [4] and [5], let the MV-term $\sigma$ be defined by $\sigma = (X \odot (X \odot X)) \odot (X \odot X)$. Let us write $\sigma^*$ instead of $(\pi_1)_\sigma$.

(i) For all $x \in [0,1]$, $\sigma^*(x) = (x \odot (x \odot x)) \odot (x \odot x) = \min(1, \max(0,3x))$.

(ii) More generally, for any cardinal $\kappa > 0$, let $f$ belong to the free $\kappa$-generator MV-algebra $\text{Free}_\kappa$ (the algebra of McNaughton functions over the Tychonoff cube $[0,1]^\kappa$, [10] Theorem 9.1.5]). Then $f_\kappa = \sigma^* \circ f$, with $\circ$ denoting composition.

Proof. (i) A routine verification. See Figure 1 (ii) By induction on the number of operation symbols in an MV-term coding $f$. □

Following tradition, by the standard MV-algebra $[0,1]$ we mean the real unit real interval equipped by the operations $\neg y = 1 - y$ and $y \oplus z = \min(1, y + z)$. There will never be danger of confusion between the standard MV-algebra and the real unit interval.

Proposition 2.4. For any MV-algebra $D$ and $c \in D$, $c_{\sigma} \sqsubseteq c$.

Proof. Consider the conjunction $\psi$ of the following statements in the language of MV-algebras:

$$\forall z \ (\text{IF } (z < \neg z) \text{ THEN } (z \odot (z \odot z)) \odot (z \odot z) \leq z)$$

$$\forall z \ (\text{IF } (z > \neg z) \text{ THEN } (z \odot (z \odot z)) \odot (z \odot z) \geq z),$$
The graph of the function $\sigma^*(x) = (x \odot (x \oplus x)) \oplus (x \odot x) = \min(1, \max(0, 3x))$, and of some of its constituents. $\sigma^*: [0, 1] \to [0, 1]$ is a member of the free one-generator MV-algebra $\text{Free}_1$ consisting of all one-variable McNaughton functions. As usual, $x \odot x$ is an abbreviation of $\neg (\neg x \oplus \neg x)$.

in first-order logic with the usual connectives, quantifiers and identity. Here $\leq$ is the natural order of any MV-algebra: $a \leq b$ iff $\neg a \oplus b = 1$, [10, p.9]. $\psi$ is satisfied by the standard MV-algebra. This is so because $\psi$ says $\forall z \ (z \odot (z \oplus z)) \oplus (z \odot z) \leq z$. i.e., $\forall z \ z_\sigma \leq z$, which is easily verified in $[0, 1]$.

By Łoś ultraproduct theorem [9, Theorem 4.1.9, Corollary 4.1.10], $\psi$ is satisfied by every ultrapower $[0, 1]^*$ of the standard MV-algebra.

By Di Nola’s representation theorem, [10, 9.5.1], every MV-chain is embeddable in some ultrapower of $[0, 1]$. Since $\psi$ is a universal sentence, we have thus shown:

for each element $z$ of every MV-chain, $z_\sigma \leq z$. (6)

To conclude the proof, arguing by way of contradiction, suppose there is an MV-algebra $D$ and $c \in D$ such that $c_\sigma \not\leq c$ fails. By definition, there is $P \in \text{Spec}(D)$ such that in the quotient MV-chain $D/P$. (Proposition [12]), we either have $c/P < c_\sigma/P$ or $c_\sigma/P < c/P$. Say without loss of generality, $c/P < c_\sigma/P$ and $c_\sigma/P > c/P$. By [13], $c_\sigma = (c/P)_{\sigma}$. So $c/P$ is a counterexample of (5) in $D/P$, and the proof is complete.

\[ \text{Proosition 2.5.} \] Every MV-chain $C$ satisfies the conjunction of the following two sentences of first-order logic:

$\forall x \ (\text{IF } 0 < x < \neg x \ \text{THEN} \ (x \odot (x \oplus x)) \oplus (x \odot x) < x)$

$\forall x \ (\text{IF } \neg x < x < 1 \ \text{THEN} \ (x \odot (x \oplus x)) \oplus (x \odot x) > x)$.

\[ \text{Proof.} \] It is enough to deal with the first sentence, denoted $\chi$. First of all, observe that $\chi$ is a universal sentence of first-order logic in the language of MV-algebras: Thus, “0 < x” means “NOT (0 = x)”. Also, “x < ¬x” means “(x ≤
AND NOT $(x = \neg x)$, i.e., “$(\neg x \odot \neg x = 1)$ AND NOT $(x = \neg x)$”. Similarly, “$x_\sigma < x$” means “NOT$(x_\sigma = x)$ AND $(\neg x_\sigma \odot x = 1)$”. As we have seen, the MV-term $x_\sigma$ is definable from $x$ and the MV-algebraic operations. Arguing as in the proof of Proposition 2.4, $\chi$ is satisfied by the standard MV-algebra. By Los theorem, $\chi$ is satisfied by any ultrapower $[0, 1]^*$, whence it is satisfied by $C$, because $C$ can be embedded into some ultrapower of $[0, 1]$, by Di Nola’s theorem. □

**Proposition 2.6.** For any MV-algebra $B$, $\subset$ is a partial order relation on $B$.

**Proof.** Reflexivity is trivial. To verify transitivity, let us assume $x \subseteq y \subseteq z$ but $x \subseteq z$ fails (absurdum hypothesis). There is a prime ideal $P$ of $B$ such that, without loss of generality, $x/P < \neg x/P$ but $z/P \not\leq x/P$, whence

$$z/P > x/P,$$

because $B/P$ is totally ordered, by Proposition 1.2. From $x \subseteq y$ we have $y/P \leq x/P$. Thus, by the contrapositive property, $(10)$ Lemma 1.1.4(i)), $\neg y/P \geq \neg x/P > x/P \geq y/P$. From $y \subseteq z$ we now get $z/P \not\leq y/P \leq x/P$, a contradiction.

To check the antisymmetry property, suppose

$$x \subseteq y \subseteq x$$

but $x \not= y$, (absurdum hypothesis). Thus dist$(x, y) \not= 0$, where

$$\text{dist}(x, y) = (x \odot \neg y) \odot (y \odot \neg x),$$

is Chang’s distance function, [8] p.477, [10] Definition 1.2.4. By Proposition 1.2 there is a prime ideal $P$ of $B$ such that $\text{dist}(x/P, y/P) \not= 0$, i.e., $x/P \not= y/P$. We now argue by cases:

If $x/P < \neg x/P$ and $y/P < \neg y/P$ then from (7) we obtain $x/P \leq y/P \leq x/P$, whence $x/P = y/P$, a contradiction.

If $x/P > \neg x/P$ and $y/P > \neg y/P$ we similarly obtain a contradiction with $x/P \not= y/P$.

If $x/P < \neg x/P$ and $y/P > \neg y/P$, combining (7) with the contrapositive property $[10]$ Lemma 1.1.4(i)), we obtain $y/P \leq x/P < \neg x/P \leq \neg y/P$, whence $y/P < \neg y/P$, which is impossible.

If $x/P = \neg x/P$ and $y/P = \neg y/P$ then an easy verification similarly shows that $x/P = y/P$, another contradiction.

Without loss of generality the last possible case is $x/P = \neg x/P$ and $y/P < \neg y/P$. Then by (7), $x/P \leq y/P$, whence $\neg x/P = x/P \leq y/P < \neg y/P$. Again by the contrapositive property $[10]$ Lemma 1.1.4(i)], the two inequalities $x/P \leq y/P$ and $\neg x/P < \neg y/P$ are contradictory.

Having thus obtained a contradiction in all possible cases, we have completed the proof. □

**End of the proof of Theorem 2.2.** Trivially, every $b \in \text{Boole}(E(A))$ satisfies $b/P \in \{0, 1\} \subseteq E(A)/P$ for every prime ideal $P$ of $E(A)$. Thus $b$ is $\subset$-minimal. Conversely, for any element $b$ of $E(A)$ we will prove

$$\text{If } b \not\in \text{Boole}(E(A)) \text{ then } b \text{ is not } \subset\text{-minimal.}$$

(9) By way of contradiction assume $b \not\in \text{Boole}(E(A))$ and $b$ is $\subset$-minimal. Following [26] Definition 4.14, for any MV-algebra $B$ let

$$\mu(B)$$

(10) denote the maximal spectral space of $B$ equipped with the hull-kernel (Zariski) topology inherited from Spec$(B)$ by restriction. By [26] Proposition 4.15, $\mu(B)$ is a nonempty compact Hausdorff subspace of the prime spectral space Spec$(B)$. For
each $M \in \mathbf{m}(B)$ there is a unique embedding of $B/M$ into the standard MV-algebra $[0,1]$. \cite[Theorem 4.16]{20}. So for each $a \in B$ there is a unique $\alpha \in \mathbb{R}$ such that $a/M = \alpha$. We will throughout identify $a/M$ and $\alpha$ without fear of ambiguity.

**Claim 1:** For all $M \in \mathbf{m}(E(A))$ we have $b/M \in \{0,1/2,1\}$.

By way of contradiction, suppose $0 < b/M < 1/2$. (The case $1/2 < b/M < 1$ is similar.) By Proposition 2.5, in the MV-chain $E(A)/M \subseteq [0,1]$ we have $b_\sigma/M < b/M$, whence $b_\sigma \neq b$. On the other hand, by Proposition 2.4, $b_\sigma \subseteq b$. It follows that $b$ is not $\sqsubseteq$-minimal, a contradiction.

**Claim 2:** For all $M \in \mathbf{m}(E(A))$, if $b/M = 0$ then $b/P = 0$ for all $P \in \text{Spec}(E(A))$ contained in $M$.

Otherwise (absurdum hypothesis), there is a maximal ideal $M$ with $b/M = 0$, and a prime ideal $P \subseteq M$ with $b/P > 0$. Thus

$$0 < b/P < -b/P.$$  

(For otherwise $b/P \oplus b/P = 1$ whence a fortiori $b/M \oplus b/M = 1$, and $b/M \geq 1/2$, which is impossible.) By Proposition 2.5, in the MV-chain $E(A)/M \subseteq [0,1]$ we have $(b/P)_\sigma < b/P$, whence by \cite[4]{3}, $b_\sigma/P = (b/P)_\sigma \neq b/P$ whence $b_\sigma \neq b$. By Proposition 2.4, $b_\sigma \subseteq b$, thus contradicting the $\sqsubseteq$-minimality of $b$.

Similarly,

**Claim 3:** For all $M \in \mathbf{m}(E(A))$, if $b/M = 1$ then $b/P = 1$ for all $P \in \text{Spec}(E(A))$ contained in $M$.

**Claim 4:** For all $M \in \mathbf{m}(E(A))$, if $b/M = 1/2$ (i.e., $b/M = -b/M$) then $b/P = -b/P$, for all $P \in \text{Spec}(E(A))$ contained in $M$.

Otherwise (absurdum hypothesis), there is a maximal ideal $M$ and a prime ideal $P \subseteq M$ with $b/M = 1/2$ and $b/P \neq -b/P$, say without loss of generality $b/P < -b/P$ in the MV-chain $E(A)/P$. If $b/P = 0$, i.e., if $b \in P$, then $b \in M$ whence $M = 0$, which is impossible. So $b/P > 0$. By Proposition 2.5 in the MV-chain $E(A)/P$ we have $(b/P)_\sigma < b/P$, whence $b_\sigma/P = (b/P)_\sigma \neq b/P$ and $b_\sigma \neq b$. By Proposition 2.4, $b_\sigma \subseteq b$, again contradicting the $\sqsubseteq$-minimality of $b$.

We have thus shown that every prime ideal $P$ of $E(A)$ belongs to precisely one of the following three sets:

$$Y_0 = \{P \in \text{Spec}(E(A)) \mid b \subseteq P\}, \quad Y_1 = \{P \in \text{Spec}(E(A)) \mid \neg b \subseteq P\},$$

$$Y_{1/2} = \{P \in \text{Spec}(E(A)) \mid b/P = \neg b/P, \text{ i.e., dist}(b/P, \neg b/P) \subseteq P\},$$

where $\text{dist}(x,y) = (x \circ \neg y) \oplus (y \circ \neg x)$ is Chang’s distance function, (see \cite{8}).

To conclude, let $c \in E(A)$ be defined by $c = b \circ b = \neg(b \circ \neg b)$. For each $P \in \text{Spec}(E(A))$ the element $c/P = b/P \circ b/P$ equals 1 if $b/P = 1$, equals 0 if $b/P = 0$, and equals 0 if $b/P = -b/P$. It follows that $c \subseteq b$, because, as we have just seen, the prime quotients of $b/P$ have no other possibilities. Our hypothesis $b \notin \text{Boole}(E(A))$ implies $Y_{1/2} = \emptyset$, whence there is prime ideal $R$ of $E(A)$ such that $b/R = \neg b/R$. Since $c/R = 0$, then $c \neq b$, a contradiction with the $\sqsubseteq$-minimality of $b$. The proof of Theorem 2.2 is complete. \hfill \Box

Figure 2 is an illustration of the equivalence classes $[p]$ and $[p]_\sigma$ for $p$ a projection in the AF-algebra $\mathfrak{M}_2$ defined by $E(\mathfrak{M}_2) = \text{the free two-generator MV-algebra}$ $Free_2$ consisting of all McNaughton functions over the unit real square $[0,1]^2$. $\mathfrak{M}_2$ is well defined by Theorem 1.1(iv). As $n$ tends to $\infty$, letting $c(n) = \sigma \circ \cdots \circ \sigma$ ($n$ times) the grey zone in $[p]_{c(n)}$ gets thinner and thinner, and the density plot of $[p]_{c(n)}$ is almost everywhere white or black.
3. The special case of liminary C*-algebras with boolean spectrum

In this section we consider a class of C*-algebras whose central projections have a particularly simple realization.

As the reader will recall, a totally disconnected compact Hausdorff space is said to be boolean.

Theorem 3.1. For any (always unital and separable) liminary C*-algebra $A$ the following conditions are equivalent:

(i) $A$ has a boolean (primitive) spectrum.

(ii) $A$ is an AF$\ell$-algebra.

Proof. (i)$\Rightarrow$(ii) From [18, Step(i), p.80] it follows that $A$ is an AF-algebra. Now by [18, Theorem 1], $K_0(A)$ is lattice-ordered. Finally, by Theorem [11,iv)-(v), $A$ is an AF$\ell$-algebra.

(ii)$\Rightarrow$(i) Since $A$ is liminary, all its primitive ideals are maximal, [14, 4.1.11(ii), 4.2.3]. Since $A$ is an AF$\ell$-algebra, by Corollary 1.3(i) every prime ideal of $E(A)$ is maximal. In symbols, by (1) and (10),

$$\text{Spec}(E(A)) = \mu(E(A)).$$

This is a necessary and sufficient condition for $E(A)$ to be hyperarchimedean, [10, Theorem 6.3.2]. Since the intersection of all prime ideals of $E(A)$ is zero, ([10, Corollary 1.2.14]), then $E(A)$ is semisimple, [10, p.72]. By [10, Proposition 1.2.10],
for every prime ideal $P$ of $E(A)$, the quotient $E(A)/P$ has no nonzero ideals, because $P$ is maximal. Equivalently, $E(A)/P$ is simple, [10] Theorem 3.5.1. By [25] Proposition 4.15, $\mu(E(A))$ is a nonempty compact Hausdorff space. By [25] Theorem 4.16, $E(A)$ is isomorphic to a separating MV-algebra of continuous $[0,1]$-valued functions on $\mu(E(A))$. Since $E(A)$ is hyperarchimedean, from [10] Corollary 6.3.5 it follows that $\mu(E(A))$ is a boolean space. By Corollary [13] ii), $\text{prim}(A)$ is boolean.

The following theorem provides a useful representation of $E(A)$ as an MV-algebra of continuous rational-valued functions over $\text{prim}(A)$:

**Theorem 3.2.** Suppose the liminary $C^*$-algebra $A$ satisfies the two equivalent conditions of Theorem 3.1. For every projection $q$ of $A$ let the dimension map $d_q : \text{prim}(A) \to \mathbb{Q} \cap [0,1]$ be defined by

$$d_q(p) = \frac{\dim \text{ range } \pi(q)}{\dim \pi} \quad (p \in \text{prim}(A)), \quad (12)$$

where $\pi$ is an arbitrary irreducible representation of $A$ such that $\text{ker}(\pi) = p$.

(I) The map $[q] \in E(A) \mapsto d_q \in [0,1]^{\text{prim}(A)}$ is an isomorphism of $E(A)$ onto the MV-algebra of dimension maps of $A$, with the pointwise operations of the standard MV-algebra $[0,1]$.

(II) Each dimension map is continuous and has a finite range.

(III) (Separation) For any two distinct $p,q \in \text{prim}(A)$ there is $p \in \text{proj}(A)$ such that $d_p(p) = 0$ and $d_p(q) = 1$. (Equivalently, there is $r \in \text{proj}(A)$ with $d_r(p) \neq d_r(q)$.)

**Proof.** Any two irreducible representations of $A$ with the same kernel are equivalent ([14] Theorem 4.3.7(ii)), and finite-dimensional ([14] 4.7.14(b)). Thus the actual choice of the representation $\pi$ with $\text{ker}(\pi) = p$ is immaterial in (12), and the dimension map $d_q$ is well defined. For each $p \in \text{prim}(A)$ and irreducible representation $\pi$ of $A$ with $\text{ker}(\pi) = p$, the quotient $A/p$ is simple, because $p$ is maximal. So upon setting $d = \dim \pi$ we have

$$A/p \cong M_d, \quad \text{the C*-algebra of } d \times d \text{ complex matrices.} \quad (13)$$

(I) We first show that $d_q$ depends on $q$ only via its Murray-von Neumann equivalence class $[q]$. For the proof we prepare:

**Claim 1:** For any $p,q \in \text{proj}(A)$,

$$p \sim q \quad \text{iff} \quad p/p \sim q/p \quad \text{for each } p \in \text{prim}(A).$$

Trivially, $p \sim q$ implies $p/p \sim q/p$ for all $p \in \text{prim}(A)$. Conversely, assuming $p/p \sim q/p$ for all $p \in \text{prim}(A)$, the continuity of the norm ensures that a partial isometry connecting $p$ and $q$ at a primitive ideal of $A$ can be lifted to a neighbourhood $\mathcal{N}$, which we may safely suppose to be clopen, because the topology of $\text{prim}(A)$ is boolean. Let $S$ be the set of ideals $\mathfrak{I} \in \text{prim}(A)$ such that both $p/\mathfrak{I}$ and $q/\mathfrak{I}$ are nonzero. Since $S$ is compact, a finite number of such clopen neighbourhoods $\mathcal{N}_1, \ldots, \mathcal{N}_k$ covers $S$. Without loss of generality, $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ whenever $i \neq j$. Adding up the associated partial isometries, we obtain $p \sim q$. Our first claim is settled.

**Claim 2:** For any $p,q \in \text{proj}(A)$ and $p \in \text{prim}(A)$,

$$d_p(p) = d_q(p) \quad \text{iff} \quad p/p \sim q/p.$$

As we already know, all primitive ideals of $A/p$ are maximal. Moreover, by [13], the finite-dimensional $C^*$-algebra $A/p$ is an isomorphic copy of the $C^*$-algebra $M_d$ of $d \times d$ complex matrices, with $d = \dim \pi$, and $\pi$ any irreducible representation with
kernel p. Thus \( p/p \sim q/p \) iff \( \dim \operatorname{range} \pi(p) = \dim \operatorname{range} \pi(q) \) iff \( d_p(p) = d_q(p) \).

Our second claim is settled.

Claims 1 and 2 jointly show that the map

\[
\theta : [q] \in E(A) \mapsto d_q \in [0, 1]^{\operatorname{prim}(A)}
\]

is well defined. A direct inspection shows that \( \theta \) is an isomorphism of \( E(A) \) onto the MV-algebra \( \theta(E(A)) \) of dimension maps, with the pointwise operations of the standard MV-algebra \([0, 1]\),

\[
\theta : E(A) \cong \{ \text{MV-algebra of dimension maps on } \operatorname{prim}(A) \}.
\]

(II) From [25] Theorem 4.16 we have an isomorphism

\[
\ast : [q] \in E(A) \mapsto [q]^* \in [0, 1]^\mu(E(A))
\]

of the semisimple MV-algebra \( E(A) \) onto a separating MV-algebra of continuous \([0, 1]\)-valued functions over the maximal spectral space \( \mu(E(A)) \). For every \( q \in \operatorname{proj}(A) \), the continuous function \([q]^*\) is defined by the following stipulation: For every maximal ideal \( N \in \mu(E(A)) \),

\[
[q]^*(N) = \text{the only real number corresponding to } \frac{[q]}{N}
\]

in the unique embedding of the simple MV-algebra \( E(A)/N \) into the standard MV-algebra \([0, 1]\].

More generally, by [10] Corollary 7.2.6, for any MV-algebra \( B \) there is at most one embedding \( \xi \) of \( B \) into the standard MV-algebra \([0, 1]\). Thus whenever such embedding \( \xi \) exists, we may identify any \( b \in B \) with the real \( \xi(b) \in [0, 1] \) without fear of confusion. By [10] Corollary 3.5.4, for all \( p \in \operatorname{prim}(A) \), letting \( \eta(p) \) be the prime (automatically maximal) ideal of \( E(A) \) corresponding to \( p \) by Corollary [1,3](i), we can write

\[
[q]^*(\eta(p)) = \text{the only real number corresponding to } \frac{[q]}{\eta(p)}.
\]

For each \( p \in \operatorname{prim}(A) \) and \( q \in \operatorname{proj}(A) \), recalling the definition of the isomorphism \( E(A)/p \cong E(A)/\eta(p) \), we can write

\[
d_q(p) = \frac{\dim \operatorname{range} \pi(q)}{\dim \pi} \in \mathbb{L}_d, \text{ with } d = \dim \pi \text{ and ker } \pi = p
\]

\[
= \text{the unique rational in } \mathbb{L}_d \text{ corresponding to } [\pi(q)] \in E(M_d) \text{ by } [13]
\]

\[
= \text{the unique image in } \mathbb{L}_d \text{ of } \frac{q}{p} \in E(A/p), \text{ by } [19]
\]

\[
= \text{the unique image in } \mathbb{L}_d \text{ of } \frac{[q]}{\eta(p)} \in E(A) / \eta(p) \text{ in } \mathbb{L}_d, \text{ by } [19]
\]

\[
= [q]^*(\eta(p)), \text{ by } [18].
\]
The composite function $d_q = [q]^* \circ \eta$: $\text{prim}(A) \rightarrow [0,1] \cap \mathbb{Q}$ is continuous. Since $E(A)^*$ is hyperarchimedean, the range of $d_q$ is finite by [11, Lemma 4.6]. For all $q \in \text{proj}(A)$, from [14, 15] we get

$$\theta([q]) = d_q = [q]^* \circ \eta.$$  \hfill (20)

(III) First of all, the two separation properties are equivalent: for the nontrivial direction, let us write $d_r(p) < d_r(q)$. Let $O$ be an open interval such that $d_r(p) < b < d_r(q)$ for all $b \in O$. By [10, Lemma 3.1.9], the free MV-algebra $F r e e^1$ contains a McNaughton function $\tau^*$ whose graph has three linear pieces as the graph of $\sigma^*$, with the additional property that the non-constant linear piece of $\tau^*$ is nonzero only over a nonempty open segment contained in $O$. Then the composite function $\tau^* \circ d_r$ has value 0 at $p$ and value 1 at $q$. Since $\tau^* \circ d_r$ is obtainable from $d_r$ by finitely many applications of the pointwise operations $\wedge$ and $\oplus$, then $\tau^* \circ d_r$ is a dimension map of $A$.

Having thus proved the equivalence of the two separation properties, the isomorphism $\ast$ in [16] maps $E(A)$ onto the separating MV-algebra of continuous $[0,1]$-valued functions over the maximal spectral space $\mu(E(A))$. By [16] and [20], the MV-algebra of dimension maps separates points.

In the light of Theorem 3.2 if $A$ satisfies the two equivalent conditions of Theorem 3.1, identifying via $\eta$ the primitive ideal space $\text{prim}(A)$ with the maximal spectral space $\mu(E(A))$, we will henceforth realize $E(A)$ as the MV-algebra of dimension maps

$$E(A) = \theta(E(A)) = E(A)^*.$$  \hfill (21)

In particular, any $f \in \text{Boole}(E(A))$ will be identified with a $\{0,1\}$-valued dimension map.

**Theorem 3.3.** Suppose the liminary C*-algebra $A$ satisfies the two equivalent conditions of Theorem 3.1. We then have:

(i) Every finite subset of $E(A)$ generates a finite subalgebra of $E(A)$. In other words, $E(A)$ is locally finite.

(ii) Every clopen $W \subseteq \text{prim}(A)$ is the zeroset of some $\{0,1\}$-valued dimension map.

(iii) For each dimension map $d_p$ and rational $\rho \in [0,1]$ there is a $\{0,1\}$-valued dimension map $b$ such that $d_p^{-1}(r) = b^{-1}(0)$.

(iv) Each extremal state $s$ of $K_0(A)$ is discrete, in the sense that $s(K_0(A))$ is a cyclic subgroup of $\mathbb{R}$, [20, p.70].

(v) $K_0(A)$ has general comparability, [20, p.131].

**Proof.** (i) From [11, Theorem 5.1(i) implies (ii)], in view of [19] and Corollary [13, i]).

(ii) Arguing as in the proof of Theorem 3.2 (III), for every $x \in W$ some dimension map $r \in E(A)$ vanishes precisely over a clopen neighbourhood of $x$ contained in $W$. Since 0 is isolated in the range of $r$, replacing if necessary $r$ by

$$r_W = r \oplus \cdots \oplus r$$

suitably many summands

we may assume $r_W$ to be $\{0,1\}$-valued. By compactness, $W$ is covered by finitely many pairwise disjoint clopens $W_1, \ldots, W_m$ and corresponding $\{0,1\}$-valued dimension maps $r_{W_1}, \ldots, r_{W_m}$, where for each $i = 1, \ldots, m$, the function $r_{W_i}$ vanishes precisely over $W_i$. The zeroset of the dimension map $r_{W_1} \wedge \cdots \wedge r_{W_m}$ coincides with $W$. 

(iii) By Theorem 3.2, the range of \(d_p\) is finite and \(d_p\) is continuous. It follows that \(d_p^{-1}(\rho)\) is a clopen subset of \(\text{prim}(A)\). Now apply (ii).

(iv) By Theorems 3.1 and Corollary 1.3(iv)-(v), \(K_0(A)\) is a lattice-ordered abelian group and \(E(A) = \Gamma(K_0(A))\). By [22] Theorem 12.18, the extremal states of \(K_0(A)\) coincide with the unit preserving \(\ell\)-homomorphisms of \(K_0(A)\) into the additive group \(\mathbb{R}\) of real numbers endowed with the usual order. So let \(s: K_0(A) \to \mathbb{R}\) be an extremal state. The kernel of \(s\) is a maximal ideal of \(K_0(A)\). Corollary 1.3(v) yields a unique maximal ideal \(s\) of \(A\) such that \(\ker s = K_0(s)\). As a matter of fact, recalling the notational stipulation (5), by [10, Lemma 1.1.2], \(K_0(A/s) \cong K_0(A)/\ker s\). Again, Corollary 1.3(v) yields a unique maximal ideal \(M\) of \(E(A)\) such that \(M = \eta(s) = K_0(A) \cap \Gamma(K_0(A))\). We then have isomorphisms

\[
E \left( \frac{A}{s} \right) \cong \frac{E(A)}{\eta(s)} \cong \frac{\Gamma(K_0(A))}{K_0(A) \cap E(A)} \cong \frac{\Gamma(K_0(A))}{K_0(s)} \cong \Gamma \left( \frac{K_0(A)}{\ker s} \right).
\]

By (22) Definition 2.4), \(E(A) = \Gamma(K_0(A))\) coincides with the unit interval of \(K_0(A)\) equipped with the order-unit 1 = \([1_A]\), and with the operations

\[
x \otimes y = u \wedge (x + y) \quad \text{and} \quad \neg y = 1 - y.
\]

Since \(M\) is a maximal ideal of \(E(A)\), by [22] Theorem 4.16, \(E(A)/M \cong E(A/s)\) are uniquely isomorphic to the same finite MV-subalgebra \(L\) of \([0, 1]\). By [10] Corollary 3.5.4, \(L = \{0, 1/m, \ldots, (m-1)/m, 1\}\) for a uniquely determined integer \(m \geq 1\). Since \(\Gamma\) is a categorical equivalence, from \(\Gamma(\mathbb{Z}, m) = L\) it follows that \(K_0(A/s) \cong (\mathbb{Z}, m)\), showing that the state \(s\) is discrete.

(v) We prepare:

**Claim 1:** For any \(p, q \in \text{proj}(A)\) there are clopens \(X, Y \subseteq \text{prim}(A)\) = maximal ideal space of \(A\), such that for every \(m \in \text{prim}(A)\)

\[
m \in X \iff d_p(m) \leq d_q(m) \quad \text{and} \quad m \in Y \iff d_p(m) \geq d_q(m).
\]

As a matter of fact, recalling the notational stipulation (5), by [10] Lemma 1.1.2, \(d_p(m) \leq d_q(m)\) iff \((d_p \circ \neg d_q)(m) = 0\). Similarly, and \(d_p(m) \geq d_q(m)\) iff \((d_q \circ \neg d_p)(m) = 0\). Now the zeroset \(f^{-1}(0) \subseteq \text{prim}(A)\) of any dimension map \(f\) is clopen, because the range of \(f\) is finite and \(f\) is continuous. Conversely, by (iii), every clopen subset of \(\text{prim}(A)\) is the zeroset of some \([0, 1]\)-valued dimension map \(f\), i.e., (Theorem 2.11 ii)\(\iff (iv)\)), the zeroset of some characteristic element of \(K_0(A)\). Our first claim is settled.

A routine variant of the proof of Claim 1 yields:

**Claim 2:** For any \(p, q \in \text{proj}(A)\) there are clopens \(X, Y \subseteq \text{prim}(A)\) such that for every \(m \in \text{prim}(A)\),

\[
m \in X \iff d_p(m) < d_q(m) \quad \text{and} \quad m \in Y \iff d_p(m) \geq d_q(m).
\]

Next let

\[
\text{maxspec}(K_0(A))
\]

denote the maximal spectral space of \(K_0(A)\), By [10] Theorems 7.2.2, 7.2.4 and Corollary 1.3(ii),

\[
\text{maxspec}(K_0(A)) \cong \mu(E(A)) \cong \text{prim}(A)
\]

(23)

and \(\text{maxspec}(K_0(A))\) can be safely identified with \(\text{prim}(A)\) and with \(\mu(E(A))\). By [22] Theorems 3.8-3.9, \(K_0(A)\) is (isomorphic to) the unital \(\ell\)-group of functions on \(\text{prim}(A)\) generated by the dimension maps, with the constant \(u = 1 = [1_A]\) as
the unit, and with the pointwise \( \ell \)-group operations of \( \mathbb{R} \). By Theorem 3.2, each function in \( K_0(A) \) is continuous, rational-valued, and has a finite range.

As explained in [22, p.126], to prove that \( K_0(A) \) has general comparability, for all \( h, k \in K_0(A) \) we must find a direct product decomposition

\[
K_0(A) = G_1 \times G_2
\]

such that the \( G_1 \)-components of \( h \) and \( k \) satisfy \( h_1 \leq k_1 \) while the \( G_2 \)-components of \( h \) and \( k \) satisfy \( h_2 \geq k_2 \). By the translation invariance of the lattice order of \( K_0(A) \) and the defining property of the unit \( u \) of \( K_0(A) \), replacing, if necessary, \( h, k \) by \( h + mu, k + mu \) (for a suitably large integer \( m \)), we may assume \( h, k \geq 0 \).

Claim 3: There is a clopen \( X_1 \subseteq \text{maxspec}(K_0(A)) \) coinciding with the set \( X_1 \) of maximal ideals of \( N \) of \( K_0(A) \) such that \( h/N \leq k/N \).

As a matter of fact, let

\[
x_1, x_2, \ldots, x_{n_1} \quad \text{and} \quad y_1, y_2, \ldots, y_{n_2}
\]

be elements of \( \Gamma(K_0(A)) = E(A) \) having the following properties:

\[
x_i \oplus x_{i+1} = x_i, \quad \sum_{i=1}^{n_1} x_i = h, \quad y_i \oplus y_{i+1} = y_i, \quad \sum_{i=1}^{n_2} y_i = k.
\]

Their existence is ensured by [22, Proposition 3.1(i)]. (Actually, these sequences are uniquely determined by \( h \) and \( k \), up to a tail of zeros.) Adding a finite tail of zeros to the shortest sequence, we may assume \( n_1 = n_2 = n \) without loss of generality. Recalling the notational stipulation (5), for each \( i = 1, \ldots, n \) let \( X_{1,i} \) be the zeroset of the dimension map \( x_i \circ \neg y_i \). The identification (23) yields

\[
X_{1,i} = \{ N \in \text{maxspec}(K_0(A)) \mid x_i/N \leq y_i/N \text{ for all } i = 1, \ldots, n \},
\]

because ([10, Lemma 1.1.2]),

\[
x_i/N \circ \neg y_i/N = 0 \iff x_i/N \leq y_i/N.
\]

Now by [22, Proposition 3.1(ii)], for any \( N \in \text{maxspec}(K_0(A)) \) the inequality \( h/N \leq k/N \) is equivalent to the simultaneous occurrence of the inequalities

\[
x_1/N \leq y_1/N, \ldots, x_n/N \leq y_n/N.
\]

As a consequence, the set \( X_1 = \bigcap_{i=1}^n X_{1,i} \) satisfies

\[
N \in X_1 \iff h/N \leq k/N.
\]

Since the range of every dimension map \( f \) is finite and \( f \) is continuous, each \( X_{1,i} \) is a clopen subset of \( \text{maxspec}(K_0(A)) \), and so is \( X_1 \). Thus \( X_1 \) has the desired properties, and our third claim is settled.

The complementary clopen \( X_2 = \text{maxspec}(K_0(A)) \setminus X_1 \) has the property that for every maximal ideal \( N \) of \( K_0(A) \), \( N \in X_2 \Leftrightarrow h/N > k/N \).

In view of (iii), for each \( j = 1, 2 \) let \( e_j \) be the uniquely determined \( \{0,1\} \)-valued dimension map satisfying \( e_j^{-1}(0) = X_j \). Each \( e_j \) is a characteristic element of \( K_0(A) \), (Theorem 2.1). Let \( I_j \) be the ideal of \( K_0(A) \) generated by \( e_j \). The \( \ell \)-homomorphisms of \( K_0(A) \) into itself induced by the two ideals \( I_1, I_2 \) provide the desired direct product decomposition \( K_0(A) \cong K_0(A)/I_1 \times K_0(A)/I_2 \). Up to isomorphism, every \( g \in K_0(A) \) splits into its restrictions \( g_1 = g|X_1 \) and \( g_2 = g|X_2 \). The \( K_0(A)/I_1 \) components of \( h \) and \( k \) satisfy \( h_1 \leq k_1 \). The \( K_0(A)/I_2 \) components satisfy \( h_2 > k_2 \). A fortiori, \( K_0(A) \) has general comparability.
Central projections as fixpoints.

**Corollary 3.4.** Suppose the liminary C*-algebra \( A \) satisfies the two equivalent conditions of Theorem 3.1. For all \( p, q \in \text{proj}(A) \) we have:

1. (Fixpoint) The sequence \([p] \supseteq [p]_\sigma \supseteq [p]_{\sigma \circ \sigma} \supseteq [p]_{\sigma \circ \sigma \circ \sigma} \supseteq \ldots\) is eventually constant.

2. Let \( n(p) \) be the least integer \( m \) such that
   \[
   [p]_{\sigma \circ \cdots \circ \sigma} = [p]_{\sigma \circ \cdots \circ \sigma}.
   \]
   Then
   i. If \( p \) is central or \( p \sim 1_A - p \), \( n(p) = 0 \).
   ii. If \( n(p) = 0 \) then for every \( p \in \text{prim}(A) \), either \( p/p \in \{0, 1\} \) or \( p/p \sim (1_A - p)/p \).
   iii. Suppose for each \( p \in \text{prim}(A) \),
       \[
       [p]/p \sim (1_A - p)/p \Leftrightarrow [q]/p \sim (1 - q)/p.
       \]
       Then \([p] \sqsubseteq [q] \Rightarrow n(p) \leq n(q)\). (1)

3. Let the set \( C_p \subseteq E(A) \) be defined by
   \[
   C_p = \{[r] \in E(A) | r \text{ is a central projection in } A \text{ such that } [r] \sqsubseteq [p] \}.
   \]
   (1) \( C_p \) nonempty.
   (II) \( C_p \) is a singleton iff for no \( p \in \text{prim}(A) \) we have \( p/p \sim (1_A - p)/p \).
   (III) When \( C_p \) is a singleton, the unique element \([r] \in C_p \) equals the fixpoint
       \[
       [p]_{\sigma \circ \cdots \circ \sigma},
       \]
       \( n(p) \) times
   (IV) If \( p/p \sim (1_A - p)/p \) for some \( p \in \text{prim}(A) \) then
       \[
       [p]_{\sigma \circ \cdots \circ \sigma} \oplus [p]_{\sigma \circ \cdots \circ \sigma} \text{ and } [p]_{\sigma \circ \cdots \circ \sigma} \ominus [p]_{\sigma \circ \cdots \circ \sigma}
       \]
       \( n(p) \) times
   are two distinct elements of \( C_p \).

**Proof.** In view of Theorem 3.2 throughout we will identify \( E(A) \) with the separating MV-algebra of all dimension maps. Thus each \([p] \in E(A) \) is a continuous rational-valued function \( d_p \) with a finite range over the boolean space \( \mu(E(A)) = \text{Spec}(E(A)) \cong \text{prim}(A) \).

1. The value \( 1/2 \) is isolated in range\((d_p)\). The definition of the map \( \sigma^* : [0, 1] \rightarrow [0, 1] \) immediately yields the desired conclusion.

To prove (2) we argue as follows:

i. If \( p \) is central, range\((d_p) \subseteq \{0, 1\} \), whence \( n(p) = 0 \), because \( \sigma^*(0) = 0 \) and \( \sigma^*(1) = 1 \). If \( p \sim 1_A - p \) then \( d_p(p) = 1/2 \) for all \( p \in \text{prim}(A) \). From \( \sigma^*(1/2) = 1/2 \) it follows that \( n(p) = 0 \).

ii. If \( n(p) = 0 \) then for every \( p \in \text{prim}(A) \), \( p/p \in \{0, 1\} \) or \( p/p \sim (1_A - p)/p \), because \( \sigma^*(t) = t \) iff \( t \in \{0, 1/2, 1\} \).

iii. The hypothesis means \( d_p^{-1}(1/2) = d_q^{-1}(1/2) \). The conclusion then follows by definition of \( \sqsubseteq \) and \( n(p) \).

3. Let \( f_p = [p]_{\sigma \circ \cdots \circ \sigma} = (d_p)_{\sigma \circ \cdots \circ \sigma} \) \((n(p) \) times). Let \( C \subseteq \text{prim}(A) \) be defined by \( C = f_p^{-1}(1/2) \). By Theorem 3.2 \( C \) is clopen.
If $C = \emptyset$, then $f_p = (f_p)_p$ by definition of $v(p)$. For every $p \in \text{prim}(A)$, either $f_p(p) < 1/2$, in which case $f_p(p) = 0$, or else $f_p(p) > 1/2$, in which case $f_p(p) = 1$. Thus $f_p$ is the only element of $C_p$. This proves (I)-(IV) for the present case.

If $C \neq \emptyset$, then $f_p \circ f_p$ pushes the graph of $f_p \mid C$ down to $0$, leaving unaltered the rest of $f_p$; evidently $f_p \circ f_p$ is a $\{0, 1\}$-valued dimension map and $f_p \circ f_p = (f_p \circ f_p)_p$. Similarly, $f_p \oplus f_p$ pushes the graph of $f_p \mid C$ up to $1$, leaving the rest unaltered. So $f_p \oplus f_p = (f_p \oplus f_p)_p$. This completes the proof of (I)-(IV).

Intuitively, the map $[p] \mapsto [p]^{\bowtie} = [p]_{\sigma}$ is “centripetal” in the sense that $[p]^{\bowtie} \subseteq [p]$, and a finite number of iterations of the map leads to a unique $\bowtie$-fixpoint, in such a way that if $p$ is central then $[p] = [p]^{\bowtie}$. If $p$ is not central and $p/p \neq (1_A - p)/p$ for every primitive ideal $p$ of $A$, then the $\bowtie$-fixpoint $[q]$ of $[p]$ arises from some central projection $q$ of $A$. If $p/p = (1_A - p)/p$ for some primitive ideal $p$ of $A$, then the same holds (not for $[q]$, but) for $[q] \oplus [q]$, or for $[q] \circ [q]$.

4. Concluding remarks

AF-$\ell$-algebras include many interesting classes of AF algebras, well beyond the trivial examples of commutative AF algebras and finite-dimensional $C^*$-algebras. Nontrivial examples are given by the CAR algebra and, more generally, Glimm’s UHF algebras, \[16\] [2, 15, 23, 24, 25], the Effros-Shen $C^*$-algebras $\mathfrak{F}_\theta$ for irrational $\theta \in [0, 1]$, \[16\] p.65], which play an interesting role in topological dynamics, \[1, 4, 28\]. Further examples are provided by AF-$\ell$-algebras whose $K_0$-group has general comparability, \[20\] Proposition 8.9, p.131. Non-simple examples include the Behnke-Leptin $C^*$-algebras $\mathcal{A}_{m,n}$ with a two-point dual \[1\], and AF algebras with a directed set of finite dimensional $\ast$-subalgebras, \[21\].

The “universal” AF algebra $\mathfrak{M}$ of \[22\] \S 8] is an AF-$\ell$-algebra. It is defined by

\[ E(\mathfrak{M}) = \text{Free}_\omega = \text{the free countably generated MV-algebra}. \]

Every AF algebra with comparability of projections is a quotient of $\mathfrak{M}$ by a primitive essential ideal, \[22\] Corollary 8.7. Every (possibly non-unital) AF algebra may be embedded into a quotient of $\mathfrak{M}$, \[22\] Remark 8.9. One more example is given by the Farey AF-$\ell$-algebra $\mathfrak{M}_1$ introduced in \[23\]. It is defined by $E(\mathfrak{M}_1) = \text{Free}_1$. By \[28\], every irrational rotation $C^*$-algebra is embeddable into some (Effros-Shen) simple quotient of $\mathfrak{M}_1$, \[23\] Theorem 3.1 (ii)]. $\mathfrak{M}_1$, in turn, is embeddable into Glimm’s universal UHF algebra, \[25\] Theorem 1.5]. As shown in \[24\] \& \[25\], the AF algebra $\mathfrak{A}$ more recently considered by Boca \[2\] coincides with $\mathfrak{M}_1$. For an account of the interesting properties and applications of $\mathfrak{M}_1$ see \[2, 15, 23, 24\] \& \[25\].

Liminal $C^*$-algebras with boolean spectrum, and more generally with Hausdorff spectrum, are considered by Dixmier \[14\] passim. Liminal $C^*$-algebras with boolean spectrum per se are the main topic of \[12\]. Here the authors consider the analogue of Kaplansky’s problem for these algebras, and prove that the Murray von Neumann order of projections alone is sufficient to uniquely recover the $C^*$-algebraic structure. Thus If two liminary $C^*$-algebras with Boolean spectrum have order-isomorphic Murray von Neumann posets, then they are isomorphic.

**Problem 4.1.** Extend the characterization

\[ p \text{ central in } A \iff [p] \subseteq \text{-minimal } \iff [p] \text{ characteristic in } K_0(A) \iff [p] \text{ a fixpoint} \]

outside the domain of AF-$\ell$-algebras.
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(D. Mundici) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE “ULISSE DINI”, UNIVERSITY OF FLORENCE, VIALE MORAGNA 67/A, I-50134 FLORENCE, ITALY
E-mail address: mundici@math.unifi.it