Atmospheric Lamb waves transmit energy over great distances, and are useful indicators of volcanic eruptions, nuclear explosions or meteoritic impacts \cite{1, 2, 3}. They are trapped waves propagating along a solid boundary in density stratified, compressible fluids \cite{4}. When the density stratification exceeds a threshold, atmospheric Lamb waves suddenly disappear \cite{5}. Here we show that this abrupt transition has a topological origin, and we predict the existence of topologically trapped Lamb-like waves in the absence of a solid boundary, depending on the shape of the stratification profile. We relate the emergence of these atmospheric Lamb waves to two-band crossing points carrying opposite topological charges. The existence of these charges coincides with a restoration of the vertical mirror symmetry that is in general broken by gravity. In that case, the dispersion relation exhibits tilted Dirac cones of type III, that were up to now only observed in photonic resonator arrays \cite{6}. Atmospheric Lamb waves thus bear strong similarities with boundary modes encountered in quantum valley Hall effect \cite{7, 8, 9} and its classical analogues \cite{10, 11, 12}. Our work shows that such states can be observed in natural flows and could be manipulated in a density stratified fluid without the design of artificial crystals.

The simplest flow model supporting acoustic and internal gravity waves involves a compressible fluid in the presence of gravity $-g e_z$ in a vertical plane $(x, z)$. Owing to compressibility, fluids support propagation of acoustic waves with sound speed $c_s$. Gravity breaks the flow isotropy, adds an intrinsic frequency $g/c_s$ and allows for the propagation of internal gravity waves when the fluid is stratified, with density profile $\rho_0(z)$.

Due to stratification, the system admits another intrinsic frequency $N = \sqrt{-g \partial_z \rho_0/\rho_0 - g^2/c_s^2}$. This is the natural buoyancy frequency of fluid particles oscillating in the vertical direction, commonly called Brunt-Väisälä frequency, and known to rule a variety of phenomena in atmospheres and oceans \cite{4}. The density stratification is stable when $N$ is real.

The dynamics involves four fields: the horizontal and vertical velocity components $(u, w)$, the potential density $\theta$ and the pressure $p$. The temporal evolution of these fields is given by the conservation of momentum in both directions, mass and entropy. After proper rescaling of the different fields (see supplementary material 1), the linear dynamics around a state of rest is conveniently expressed as

$$
\partial_t \begin{pmatrix} u \\ w \\ \theta \\ p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -c_s \partial_x \\ 0 & 0 & -N & -S - c_s \partial_z \\ -c_s \partial_x & N & 0 & 0 \\ 0 & 0 & S - c_s \partial_z & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ \theta \\ p \end{pmatrix},
$$

(1)

where the stratification parameter

$$
S = \frac{1}{2} \left( \frac{N^2 c_s}{g} - \frac{g}{c_s} \right)
$$

(2)

plays a central role, as it breaks mirror symmetry in the $z$-direction $(z, w, \theta) \rightarrow -(z, w, \theta)$. In fact, the dynamics \cite{1} is left invariant by the transformation $(z, w, \theta, S) \rightarrow -(z, w, \theta, S)$ that we call stratification symmetry. When $S = 0$, mirror symmetry in the $z$-direction is restored. This occurs when $g = 0$, as one can expect, but also when $N^2 = g^2/c_s^2$ namely when the two intrinsic frequencies of the fluid match. By contrast, the system is always left invariant by two other discrete symmetries: mirror symmetry in the $x$ direction $(x, u) \rightarrow -(x, u)$, and time-reversal symmetry $(t, u, w) \rightarrow -(t, u, w)$.

Textbooks usually focus on the particular case of an ideal gas with an isothermal background stratification. In that case thermodynamical constraints impose $S < 0$ (see supplementary material). Here we will relax

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this simplifying assumption by considering the possibility for $S \geq 0$. This can for instance be achieved by considering non-isothermal atmospheres with sufficiently large buoyancy frequencies $N$. In the remaining of this paper, we adimensionalize the equations using $c_s^2/g$ and $c_s/g$ as a length and time unit, respectively, so that $S = \frac{1}{2}(N^2 - 1)$. The stability condition of the fluid $N^2 > 0$ then implies $S > -1/2$.

We now investigate the consequences of the symmetries on the dispersion relation of acoustic gravity waves in unbounded geometries. In that case the spectrum is conveniently computed by performing a Fourier transform of $\Psi$ on the basis $e^{i(\omega t - k_xx - k_zz)}$. For each triplet of parameters $(k_x, k_z, S)$, the eigenfrequency and eigenmodes are then obtained by solving $\omega \Psi = H\Psi$ where $\Psi = (u, w, \theta, p)$, and where $H$ is an hermitian operator depending linearly on the parameters $(k_x, k_z, S)$ (see methods). The dispersion relation $\omega^2 = (K^2/2) \left[ 1 \pm (1 - (2Nk_x/K^2)^2)^{1/2} \right]$, with $K^2 = k_x^2 + k_z^2 + N^2 + S^2$, is plotted in figure 1 for three different values of $S$. This dispersion relation exhibits four bands $\omega^{(n)}$. High-frequency solutions $\omega^{(4)} = \omega_+$ and $\omega^{(1)} = -\omega_+$ are called acoustic waves, as their dispersion relation approaches $\omega_2^2 = (k_x^2 + k_z^2)$ at sufficiently small scales (large wave numbers). Low-frequency solutions $\omega^{(3)} = \omega_-$ and $\omega^{(2)} = -\omega_-$ are called internal gravity waves, as their dispersion relation approach $\omega_2^2 = N^2$ for sufficiently small scales. As expected from the analysis of the symmetries, the dispersion relation is left invariant by mirror symmetry in the $x$ direction $(k_x \rightarrow -k_x)$ and time reversal symmetry $(\omega \rightarrow -\omega)$. Interestingly, it is also left invariant by $k_z \rightarrow -k_z$, even though mirror symmetry in $z$ is broken by the dynamics. This property results from the combination of the two previous symmetries, together with a third one that is analogous to “particle-hole” symmetry in condensed matter physics. This third symmetry stems from the fact that the initial dynamical system describes real fields: each solution $\Psi(k_x, k_z, S)$ with eigenvalue $\omega$ has a partner $\Psi^*(-k_x, -k_z, S)$ with eigenvalue $-\omega$.

A footprint of the broken mirror symmetry in $z$ direction is given by the presence of a frequency gap whenever $S \neq 0$ (see figure 1). The gap between gravity and acoustic waves closes only when $S = 0$ at $k_z = 0$, that is to say when the symmetry is recovered. The dispersion relation can the interpreted in that case as the result of an horizontal sound wave $(u, p)$ of frequency $\omega = \pm k_x$ coexisting with a stratified background media $(w, \theta)$ oscillating vertically at $\omega = \pm N$. This leads to four two-fold degeneracy points at $k_z^2 = N^2$. The coupling between the fields $(u, p)$ and $(w, \theta)$ through the stratification parameter $S$ and vertical derivative leads to hybrid states separated by a band gap. This can be seen as a continuum counterpart of acoustic polaritons engineered recently in arrays of Helmholtz resonators.

In geophysical context, the existence and the consequences of the degeneracy points have, up to now, been overlooked. First, the linear dispersion relation around the two-band crossing points in $(k_x, k_z)$-space, with one of the band being flat in the direction $k_x$, realizes a peculiar case of tilted Dirac cones (type III), that have so far only been observed in an optical metamaterial. Second, beyond this remarkable dispersion relation, degeneracy points are known to induce peculiar geometrical properties for the system eigenmodes $\Psi^{(n)}$ in parameter space $(k_x, k_z, S)$. The main point of this letter is to unveil these properties that are revealed by the Berry curvature $F^{(n)} = (F^{(n)}_{k_x, k_z}, F^{(n)}_{k_z, S}, F^{(n)}_{S, k_x})$, whose components are

$$F^{(n)}_{p, p'} = i(\partial_p \Psi_j^{(n)} \partial_{p'} \Psi_j^{(n)} - \partial_p \Psi_j^{(n)} \partial_{p'} \Psi_j^{(n)})$$

(3)

where $\Psi_j^{(n)}$ is the $j$th component of the (normalized) $n$th eigenvector, $i^2 = -1$, and $p$ and $p'$ are directions.
in parameter space \((k_x, k_z, S)\). The curvature \(\mathbf{F}\) is gauge-invariant: it is independent from the choice of the eigenmode’s phase. When the flux of this curvature is integrated over a surface \(\Sigma\) in parameter space, it yields the geometrical Berry phase gained by an eigenmode continuously deformed along the contour \(\partial \Sigma\) supporting the surface \(\Sigma\). The \(F_{k_x,k_z}^{(n)}\) component of the Berry curvature is shown for each band \(n\) on the dispersion relation in figure 2. Its amplitude is concentrated where the gaps are the smallest, as expected \([14]\). More importantly, the amplitude changes sign with \(S\). This reflects a topological property of the two-fold degeneracy points in \((k_x, k_z, S)\) space \([15]\). This topological property is captured by a quantized Berry flux through a closed surface \(\Sigma\) enclosing the degeneracy in parameter space: \(\int_{\Sigma} \mathbf{F}^{(n)} \cdot d\Sigma = 2\pi c^{(n)}\) where \(c^{(n)} \in \mathbb{Z}\) is the first Chern number. For this reason, the degeneracies are said to carry topological charges \(c^{(n)}\) that are the source of the Berry curvatures.

![Component of the Berry curvature](image)

**Figure 2:** Component \(F_{k_x,k_z}^{(n)}\) of the Berry curvature projected on each band for a given value of \(S \neq 0\). The extrema values \(\pm F_{k_x,k_z}^{(n),\text{max}}\) depend on the one of \(S\).

We find two degeneracy points in parameter space, at \((k_x, k_z, S) = (\pm N, 0, 0)\), leading to two sets of Chern numbers \(c^{(n)}_{\pm}\), that are related through mirror symmetry in \(x\)-direction: \(c^{(n)}_+ = -c^{(n)}_-\). Time reversal symmetry imposes that the Chern numbers of bands with opposite frequencies are the same. Since the sum of Chern numbers at a given degeneracy point must vanish, the computation of the topological charges is eventually reduced down to the computation of a single Chern number (see supplementary material). At the degeneracy point \(k_x = N\), we obtain \(c^{(n)} = -1\) for acoustic wave bands \((n \in \{1, 4\})\), and \(c^{(n)} = 1\) for the internal-gravity wave bands \((n \in \{2, 3\})\).

The existence of these topological charges is crucial as it is related to the existence of a spectral flow in the dispersion relation of the operator \(H_{\text{op}}\) that describes acoustic-gravity waves with varying stratification \(S(z)\). This operator is obtained from (1) after performing a Fourier transform on the basis \(e^{i(k_x x + k_z z)}\), and can be formally deduced from the matrix \(H(k_x, k_z, S)\), by identifying \((k_x, k_z, S) \rightarrow (k_x, i\partial_z, S(z))\). The spectral flow means that some eigenvalues of \(H_{\text{op}}(k_x, i\partial_z, S(z))\) transit from the internal-gravity wave band to the acoustic wave band (or the other way around) when varying \(k_z\), thus crossing a frequency gap in the vicinity of each degeneracy point. For this spectral flow to be non-zero – or equivalently for the Chern number to be non-zero – it is essential that \(S\) changes sign. When \(S\) is an increasing function of \(z\), and given our choice of orientation for parameter space \((k_x, k_z, S)\), the algebraic number of modes \(N\) that flows to the band \(n\) is given by \(N = c^{(n)}\) \([16, 15, 17]\). The sign of \(N\) is reversed when \(S\) is a decreasing function of \(z\), and this can be generalized to non-monotonic profiles. Moreover, the modes that transit are trapped around the altitude where \(S\) vanishes. This is a common feature of topological domain-wall states encountered in condensed matter, from 1D chains \([18]\) to massless Dirac and Weyl fermions subjected to magnetic field in 2D and 3D \([19, 20]\). The relation between the spectral flow of the operator \(H_{\text{op}}\) and the Chern numbers inferred from \(H(k_x, k_z, S)\) is rooted in Atiyah-Singer index theorem \([16]\). This deep connection has been successfully used to explain for instance the shape of molecular spectra \([21]\), the chiral boundary modes for generalized Dirac-like Hamiltonians in condensed matter \([22]\), and the existence of two eastward propagating equatorial waves \([23, 24]\). Based on this correspondence, we predict here a transition in the shape of the acoustic-gravity spectrum, controlled by the stratification profile \(S(z)\).

Two stratification profiles \(S(z) = S_0 e^{-z/\ell_0} + S_\infty\) differing by the sign of \(S_\infty\) are considered in figure 3. When \(S_\infty > 0\), then the stratification is always positive, and the spectrum of \(H_{\text{op}}\) is gapped. In contrast, when
When the e-folding depth $z_0$ tends to zero, with $S_0$ sufficiently large, the profile of $S(z)$ tends to a constant $S_\infty$ in the bulk. For finite wave number, the spectra are found to be identical to the classical ones computed for $S$ constant together with an impermeability condition $w = 0$ at $z = 0$ [4]. The similarity between the bottom intensified profile of $S(z)$ and the impermeability constraint is due to the fact that large stratification prevents vertical motion, thus mimicking the condition $w = 0$ at $z = 0$. The modes that transit in this case correspond to the celebrated atmospheric Lamb waves [1], that should not be confused with other Lamb waves encountered in elastic plates [25], and in particular in mechanical metamaterials [26]. Atmospheric Lamb waves only exist when $S < 0$ and may disappear whenever an other boundary condition than the impermeability one is imposed [5]. In contrast, the topological spectral flow described here only depends on the zeros in the profile of $S$.

Our study shows that Lamb-like waves filling a frequency gap between acoustic and internal gravity wave bands can emerge even in the absence of a solid boundary, provided that the profile $S(z)$ changes sign. It will be interesting to search for such configurations in planetary atmospheres, oceans, and even star tachoclines (stably stratified). The topological transition we describe could also be observed in experiments on compressible stratified fluids, by probing the spectrum of excitations with different stratification profiles. Up to now, topological waves in classical systems with time-reversal symmetry have been designed in lattices enforcing specific symmetries to engineer either Kramers degeneracies [27, 28] (with $Z_2$ invariants), or Dirac points.
Methods

Expressions of the symbol and the operator. The symbol $\mathcal{H}$ is obtained after performing a Fourier transform of (1) on the basis $e^{i(\omega t-k_x x-k_z z)}$, and considering $S$ (or equivalently $N$, assuming $g$ prescribed) as an external parameter. The dual operator $\mathcal{H}_{op}$ is obtained after performing a Fourier transform of (1) on the basis $e^{i(\omega t-k_x x)}$, considering now $S(z)$ as a given function of $z$. Their expression is

$$
\mathcal{H}(k_x, k_z, S) = \begin{pmatrix}
0 & 0 & 0 & k_x \\
0 & 0 & -iN & k_z - iS \\
k_x & iN & 0 & 0 \\
k_z & k_z + iS & 0 & 0
\end{pmatrix}, \quad \mathcal{H}_{op}(k_x) = \begin{pmatrix}
0 & 0 & 0 & k_x \\
0 & 0 & -iN(z) & i\partial_z - iS(z) \\
k_x & iN(z) & 0 & 0 \\
k_z & i\partial_z + iS(z) & 0 & 0
\end{pmatrix} \quad (4)
$$

Numerical computation of the spectra with arbitrary $S(z)$ profiles. The spectra shown in figure 3 are computed with dedalus code [29]. In practice, for a given profile $S(z)$ in a domain $[0, L_z]$, the operator is projected on a Chebytchev basis, and the linear system is solved using the tau method. Boundary conditions are required at $z = 0$ and $z = L_z$. We used $w = 0$ at $z = 0$, but we checked that other choice of the boundary conditions at this point did not affect the spectrum, given the choice of an exponential stratification profile. Indeed, this profile is equivalent to an effective condition $w = 0$ around $z = 0$. We follow the Iga prescription to avoid spurious eigenmodes at $z = L_z$ [5]: this amounts to impose the condition $p = 0$ when $S > 0$ and $w = 0$ when $S < 0$. This procedure allows us to identify the boundary $z = L_z$ with $z = \infty$, which can be checked by comparing the numerical spectra with the analytical spectra obtained in the case of a flow taking place in the upper half place $z > 0$, with $S = S_{\infty}$. For a given normalized eigenmode $\Psi(z)$, the localisation of the mode is computed as $z^* = 1/L_z \int_0^{L_z} \Psi^* \cdot \Psi dz$. We provide the python scripts in supplementary material.

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Acknowledgements
Author Contributions
Author Information The authors declare no competing financial interests.
Supplementary Material for:
Topological transition in compressible stratified fluids

1 Hermitian operator for acoustic-gravity waves

The dynamics of an isothermal compressible stratified fluids around a state of rest can be written on the form a Hermitian linear operator \[30\]. The general case, without hypothesis on the nature of the fluid, was discussed by \[5\], in order to address the possible existence of Lamb waves in oceans. However, \[5\] did not write down the linear operator on the Hermitian form \[S14\]. Here we show for completeness how to write down the linear dynamics in the form of an Hermitian operator.

We consider a compressible stratified fluid with a prescribed density profile \(\rho_0(z)\) at rest. We restrict ourself to the case of a non-rotating flow taking place in the plane \((x, z)\), with \(z\) the vertical axis, in the direction of the (constant) gravity field. Momentum and mass conservation equations write

\[
\rho D_t \mathbf{u} = -\nabla p - \rho g e_z \tag{S1}
\]

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{S2}
\]

where \(\mathbf{u} = (u, w)\) is the velocity field, \(\rho\) the fluid density, \(p\) the pressure and \(D_t = \partial_t + (\mathbf{u} \cdot \nabla)\) the Lagrangian derivative. Denoting \(s\) the entropy field, pressure variation \(p(s, \rho)\) is given by

\[
dp = \left(\frac{\partial p}{\partial s}\right)_\rho ds + \left(\frac{\partial p}{\partial \rho}\right)_s d\rho \tag{S3}
\]

Assuming adiabatic particle displacements \((ds = 0)\) leads to

\[
D_t \rho = \frac{1}{c_s^2} D_t p \tag{S4}
\]

where \(c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s\) is sound speed.

In basic state \((\rho_0(z), p_0(z))\), the fluid is at rest and hydrostatically balanced

\[
\frac{dp_0}{dz} = -\rho_0 g. \tag{S5}
\]

We will consider small disturbances \(\mathbf{u}', \rho', p'\) of this basic state

\[
\mathbf{u} = 0 + \mathbf{u}'(x, z, t), \quad \rho = \rho_0(z) + \rho'(x, z, t), \quad p = p_0(z) + p'(x, z, t) \tag{S6}
\]

\[
\mathbf{u}', \mathbf{w}' \ll c_s, \quad \rho' \ll \rho_0, \quad p' \ll p_0. \tag{S7}
\]

Using \[S5\] to express advection of pression leads to

\[
\begin{cases}
\partial_t \rho_0 \partial_t \mathbf{u}' = -\rho_0 \nabla p' - \rho' g e_z \\
\partial_t \rho' + w' \partial_z \rho_0 + \rho_0 \nabla \cdot \mathbf{u}' = 0 \\
\partial_t \rho_0 + w' \partial_z \rho_0 = \frac{1}{c_s^2} (\partial_t \rho' - \rho' g)
\end{cases} \tag{S8a,b,c}
\]

We use the following change of variables

\[
\tilde{u} = u' \rho_0^{1/2}, \quad \tilde{w} = w' \rho_0^{1/2}, \quad \tilde{p} = p' \rho_0^{-1/2}, \quad \tilde{\rho} = \rho' \rho_0^{-1/2} \tag{S9}
\]

that leads to

\[
\begin{cases}
\partial_t \tilde{u} = -\partial_z \tilde{p} \\
\partial_t \tilde{w} = -\partial_z \tilde{p} - \frac{1}{2} \frac{\partial_z \rho_0}{\rho_0} \tilde{p} - \tilde{g} \\
\partial_t \tilde{p} = g \tilde{w} - c_s^2 \nabla \cdot \mathbf{\tilde{u}} + \frac{1}{2} c_s^2 \frac{\partial_z \rho_0}{\rho_0} \tilde{w} \\
\partial_t \tilde{\rho} = -\tilde{w} \frac{\partial_z \rho_0}{\rho_0} + \frac{1}{c_s^2} (\partial_t \tilde{p} - \tilde{w} g)
\end{cases} \tag{S10a,b,c,d}
\]
The density field is substituted by \( \tilde{\theta} = \tilde{\rho} - \frac{1}{c_s^2} \rho \). This leads the following set of equations:

\[
\begin{aligned}
\partial_t \tilde{u} &= -\partial_x \tilde{p} \\
\partial_t \tilde{w} &= -\partial_z \tilde{p} - \left( \frac{1}{2} \frac{\partial_x \rho_0}{\rho_0} + \frac{g}{c_s^2} \right) \tilde{p} - \tilde{\theta} g \\
\partial_t \tilde{p} &= \left( g + \frac{1}{2} c_s^2 \frac{\partial_z \rho_0}{\rho_0} \right) \tilde{w} - c_s^2 \nabla \cdot \tilde{u} \\
\partial_t \tilde{\theta} &= -\left( \frac{\partial_x \rho_0}{\rho_0} + \frac{g}{c_s^2} \right) \tilde{w}
\end{aligned}
\]  

(S11a) \hspace{1cm} (S11b) \hspace{1cm} (S11c) \hspace{1cm} (S11d)

Let us introduce the square of buoyancy frequency

\[ N^2 = -g \frac{\partial_z \rho_0}{\rho_0} - \frac{g^2}{c_s^2}. \]  

(S12)

Stable configurations correspond to \( N^2 > 0 \). In that case \( N \) is interpreted as an intrinsic frequency of the system, referred to as the Brunt-Väisälä frequency, or the buoyancy frequency. We also introduce another frequency of the system

\[ S = \frac{1}{2} \left( \frac{N^2 c_s}{g} - \frac{g}{c_s} \right). \]  

(S13)

Assuming \( c_s \) constant, and denoting \( \theta = -\tilde{\theta} N/g, \tilde{p} = \tilde{p} / c_s \), the linear dynamics is expressed as

\[
\begin{pmatrix}
\partial_t \tilde{u} \\
\partial_t \tilde{w} \\
\partial_t \tilde{\theta} \\
\partial_t \tilde{p}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & -c_s \partial_x \\
0 & 0 & -N & -S - c_s \partial_z \\
0 & N & 0 & 0 \\
-c_s \partial_x & S - c_s \partial_z & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{w} \\
\tilde{\theta} \\
\tilde{p}
\end{pmatrix}
\]  

(S14)

This is the operator studied in the main text, where the symbols "~" and "\bar{}" are dropped

### 2 Stratification parameter in isothermal atmospheres

In non-isothermal atmospheres, non-constant stratification profiles with \( S < 0 \) and \( S > 0 \) can occur. However in most textbooks, Lamb waves are derived from the case of an isothermal atmosphere. Here we show that in that case thermodynamical constraints impose \( S < 0 \).

Let’s consider that the atmosphere is described as an ideal gas at a constant temperature \( T_0 \). The state equation \( p_0(z) = \rho_0(z) R^* T_0 \) holds, where \( R \) is the specific gas constant. At rest, the atmosphere is hydrostatically balanced, so \( \frac{dp_0}{dz} = -\rho_0(z) g \). Combining the two last equations yields to the density profile of the isothermal atmosphere:

\[ \rho_0(z) = \rho_0(z = 0) e^{-\frac{gz}{RT_0}} \]  

(S15)

Now recall that the stratification parameter is given by \( S = 1/2(-c_s \partial_z \rho_0/\rho_0 - 2g/c_s) \) and that sound speed in ideal gas is \( c_s = \sqrt{\gamma RT_0} \), where \( \gamma \) is the heat capacity ratio. This leads to

\[ S = \frac{g}{2c_s} (\gamma - 2) \]  

(S16)

which is always negative. Indeed in ideal gases, \( \gamma = 1 + 2/f \), where \( f \geq 3 \) is the number of degrees of freedom for a single gas molecule.

### 3 Symmetries of bulk waves

The symmetries of the partial differential equation describing acoustic-gravity waves are given in the main text. These symmetries can now be translated in Fourier space \( (t, x, z) \rightarrow (\omega, k_x, k_z) \), as symmetries of the operator \( \mathcal{H}(k_x, k_z, S, N) \) defined in Eq. (4) of the method section. When doing so, several arbitrary choices are possible, each of them being compatible when all symmetries are taken into account.
As an example, consider time reversal symmetry \( t \rightarrow -t \) that imposes \( u \rightarrow -u \). Let us then decompose the velocity in Fourier space as

\[
\mathbf{u}(t, \mathbf{x}) = \int d\omega dk e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \mathbf{u}(\omega, \mathbf{k}).
\]  

(S17)

Since \( \mathbf{u}(t, \mathbf{x}) \) is a real-valued vector field, this decomposition imposes \( \mathbf{u}(\omega, \mathbf{k}) = \mathbf{u}^*(-\omega, -\mathbf{k}) \). At the level of the operator \( \mathcal{H}(k_x, k_z, S, N) \), this real field condition leads to \( \mathcal{H}(k_x, k_z, S, N) = -\mathcal{H}(-k_x, -k_z, S, N)^* \), that is the analog of particle-hole symmetry in condensed matter. As a consequence, the frequency spectrum has the symmetry \( \omega(k_x, k_z) \rightarrow -\omega(-k_x, -k_z) \).

Time-reversal symmetry \( u(t, \mathbf{x}) \rightarrow -u(-t, \mathbf{x}) \) can then be satisfied in two different ways:

- (i) \( \omega \rightarrow -\omega \) and \( \mathbf{u}(\omega, \mathbf{k}) \rightarrow \mathbf{u}(-\omega, \mathbf{k}) \),
- (ii) \( i \rightarrow -i \), \( \mathbf{k} \rightarrow -\mathbf{k} \) and \( \mathbf{u}(\omega, \mathbf{k}) \rightarrow \mathbf{u}(\omega, -\mathbf{k}) \).

Here we refer to choice (i) as time reversal symmetry for the Fourier modes, because \( \omega \) has the dimension of the inverse of a time. Choice (ii) coincides with the standard time reversal symmetry in quantum mechanics, where \( \mathbf{k} \) is the momentum. In that case there exists an anti-unitary operator that commutes with \( \mathcal{H}(k_x, k_z, S, N) \) for \( \mathbf{k} = 0 \). In contrast, choice (i) leads to the existence of a unitary operator that anti-commutes with \( \mathcal{H}(k_x, k_z, S, N) \). It follows that the frequency spectrum has the symmetry \( \omega(k_x, k_z) \rightarrow -\omega(k_x, k_z) \). This usually refers to chiral or sublattice or bi-partite symmetry in condensed matter. Choice (ii) is automatically satisfied by combining the real field condition with choice (i).

All the symmetries of the physical system can then be formally rephrased in Fourier space by introducing a unitary operator for each of them. This draws an interpretation of \( \mathcal{H}(k_x, k_z, S, N) \) as a classical counterpart of a quantum Hamiltonian:

| Symmetry                      | Operator                  | Commutation Relation |
|-------------------------------|---------------------------|----------------------|
| Mirror in \( x: (x, u) \rightarrow -(x, u) \) | \( \mathcal{M} = \text{diag}(-1, 1, 1) \) | \( \mathcal{M} \mathcal{H}(k_x, k_z, S, N) \mathcal{M}^{-1} = \mathcal{H}(-k_x, k_z, S, N) \) |
| Time reversal: \( (t, u, w) \rightarrow -(t, u, w) \) | \( \mathcal{T} = \text{diag}(1, 1, -1, -1) \) | \( \mathcal{T} \mathcal{H}(k_x, k_z, S, N) \mathcal{T}^{-1} = -\mathcal{H}(k_x, k_z, S, N) \) |
| Stratification: \( (z, w, \theta, S) \rightarrow -(z, w, \theta, S) \) | \( \mathcal{S} = \text{diag}(1, -1, -1, 1) \) | \( \mathcal{S} \mathcal{H}(k_x, k_z, S, N) \mathcal{S}^{-1} = \mathcal{H}(k_x, -k_z, -S, N) \) |
| Real fields \( (u, w, \theta, p) \in \mathbb{R}^3 \) | complex conjugation \( \mathcal{K} \) | \( \mathcal{K} \mathcal{H}(k_x, k_z, S, N) \mathcal{K}^{-1} = -\mathcal{H}(-k_x, -k_z, S, N) \) |

Table 1: Symmetries of the linear operator projected in Fourier space.

4 Computation of the Chern number

Here we show how to compute the Chern numbers associated with degeneracy points in the spectrum of the symbol \( \mathcal{H}(k_x, k_z, S) \) defined Eq. (4) in method section. This 4 by 4 matrix can be written as

\[
\mathcal{H} = \begin{pmatrix}
0 & h^1 \\
h^1 & 0
\end{pmatrix}, \quad \text{where } h = \begin{pmatrix}
0 & k_x \\
-1iN & k_z - iS
\end{pmatrix}.
\]  

(S18)

The study of the spectrum of \( \mathcal{H} \) is then conveniently reduced down to the study of the spectrum of \( hh^1 \), provided that the eigenvalues of \( \mathcal{H} \) are non zero. Indeed each solution of \( hh^1 \phi = \alpha \phi \) corresponds to two solutions of \( \omega \Psi = \mathcal{H} \Psi, \) with \( \omega = \sqrt{\alpha} \) and \( \Psi = (\phi, (h^1 \phi/\omega)) \). This reduction procedure is classically used in textbooks on compressible stratified fluids, when writing the dynamics as a second order in time linear PDE for \( u \) and \( w \) [1][5]. One can check that the Berry curvature generated by \( \phi \) is the same as the Berry curvature generated by \( \Psi \). The matrix \( hh^1 \) is expressed as

\[
hh^1 = \frac{k_x^2 + k_z^2 + N^2 + S^2}{2} I_2 + \mathbf{g} \cdot \sigma, \quad \mathbf{g} = \begin{pmatrix}
k_x k_z \\
-k_z S \\
\frac{k_x^2 - k_z^2 - N^2 - S^2}{2}
\end{pmatrix}
\]  

(S19)

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli matrices. In that form, eigenvalues of \( hh^1 \) are given by

\[
\alpha^{(n)} = \frac{k_x^2 + k_z^2 + N^2 + S^2}{2} + (-1)^n \| \mathbf{g} \|, \quad n = 1, 2.
\]  

(S20)

3
Note that in the 2 by 2 case, the wave bands \( n = 1 \) and \( n = 2 \) correspond to gravity waves and acoustic waves, respectively. From now on, we consider \( N^2 = (2S + 1) \). The degeneracy points \( p_\pm = (k_x = \pm 1, k_z = 0, S = 0) \) are easily identified as they correspond to nullification points of \( \|\mathbf{g}\| \). A classical calculation leads to

\[
\frac{1}{2\pi} E^{(n)}_{p, p'} = (-1)^{(n)} \frac{1}{4\pi \|\mathbf{g}\|^3} \cdot (\partial_p \mathbf{g} \times \partial_{p'} \mathbf{g}),
\]

(S21)

see e.g. [31]. This allows us to identify the topological charge of the lower (internal-gravity) band \( c_+^{(1)} \) carried by a degeneracy point \( p_\pm \) with the index of the vector field \( \mathbf{g} \) at \( p_\pm \). Finally this index equals \( \text{sign}(\det(d_{p_\pm} \mathbf{g})) \), where \( d_{p_\pm} \mathbf{g} \) is the Jacobian matrix (the matrix of partial differentials). We find

\[
d_{p_\pm} \mathbf{g} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}
\]

(S22)

so the topological charge of the gravity wave band is \( c_+^{(1)} = 1 \) at \( k_x = 1 \). Note that in the 4 by 4 problem, the gravity wave and acoustic wave bands are indexed by \( n \in \{2, 3\} \) and \( n \in \{1, 4\} \), respectively. The topological charges of the degeneracy points for these bands are deduced from the 2 by 2 case, using symmetries of the problem:

\[
(c_+^{(1)}, c_+^{(2)}, c_+^{(3)}, c_+^{(4)}) = (-1, 1, 1, -1) \text{ at } k_x = 1,
\]

(S23)

\[
(c_-^{(1)}, c_-^{(2)}, c_-^{(3)}, c_-^{(4)}) = (1, -1, -1, 1) \text{ at } k_x = -1.
\]

(S24)