Translationally invariant multipartite Bell inequalities involving only two-body correlators

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Abstract

Bell inequalities are natural tools that allow one to certify the presence of nonlocality in quantum systems. The known constructions of multipartite Bell inequalities contain, however, correlation functions involving all observers, making their experimental implementation difficult. The main purpose of this work is to explore the possibility of witnessing nonlocality in multipartite quantum states from the easiest-to-measure quantities, that is, the two-body correlations. In particular, we determine all three- and four-partite Bell inequalities constructed from one- and two-body expectation values that obey translational symmetry, and show that they reveal nonlocality in multipartite states. Also, by providing a particular example of a five-partite Bell inequality, we show that nonlocality can be detected from two-body correlators involving only nearest neighbours. Finally, we demonstrate that any translationally invariant Bell inequality can be maximally violated by a translationally invariant state and the same set of observables at all sites. We provide a numerical algorithm allowing one to seek for maximal violation of a translationally invariant Bell inequality.

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1. Historical background and introduction

Historical background. The first quantum mechanical revolution took place in the beginning of the XXth century. Quantum mechanics was then discovered and used to describe and explain the laws of micro-world, and to predict and calculate with great precision the properties of quantum systems. The fundamental nature of quantum mechanics and its paradoxes were, of course, discussed in those times [1], but they were treated as philosophical curiosities, rather than serious scientific challenges. All of that has changed due to the seminal work of John Bell [2]. As Alain Aspect terms it, Bell’s work has initiated the second quantum revolution, that has led to experimental confirmation of nonlocality of quantum mechanics, pioneered by Aspect himself [3]. The same developments triggered spectacular progress of ultraprecise studies of quantum single and few particle systems, that culminated with Nobel Prizes of Hans Dehmelt, Wolfgang Paul and Norman Ramsey in 1989, and Serge Haroche and Dave Wineland in 2012. Recent developments of quantum information theory and physics of ultracold matter mark the dawn of the third quantum mechanical revolution [4]. Quantum mechanics, after achieving maturity as a physical discipline describing quantitative and fundamental aspects of the micro-world, becomes nowadays the base for future quantum technologies. The paper, that we present below, belongs to these recent trends, and treats the challenging problem of feasible detection of nonlocality in many-body quantum systems.

Introduction. Nonlocality, that is, the existence of correlations that violate Bell inequalities, evidences that quantum phenomena cannot be explained by any local theory (see, e.g., [5]). Besides this fundamental significance, nonlocal correlations have also turned into a key resource for quantum information theory. In particular, nonlocal correlations perform better than classical resources at communication complexity problems [6] (see also [7]) and enable quantum key distribution [8–10]. Moreover, they can be used to certify the presence of true randomness in measurement statistics [11], and also to amplify it [12–15], in the sense that exploiting nonlocal correlations one can obtain perfectly random bits from partially random ones.

Certification of the presence of nonlocality in quantum states is therefore one of the central problems in quantum information theory. The most natural tool that serves the purpose are Bell inequalities [16]. These are linear inequalities, formulated in terms of expectation values involving products of local measurements performed by the parties. They are satisfied by all classical correlations, and possibly violated by some nonlocal ones. In principle, since classical correlations form a polytope, identifying all Bell inequalities corresponding to its facets would completely solve the above problem. Considerable effort has been devoted to achieve this goal and many Bell inequalities have been found (see, e.g., [17–21]). Usually, however, they contain expectation values involving all observers, and even if, intuitively, these are the strongest Bell inequalities, they are hardly applicable in experiments where the quantum system under consideration is large, in the sense that it contains a considerable number of particles. In fact, in these systems the measurement of expectation values involving all the observers is very challenging.

It is then an interesting question, both from fundamental and practical points of view, whether one can witness nonlocality relying solely on one- and two-body expectation values. Intuitively, they contain the least information about the correlations. However, they are also the easiest-to-measure quantities from which a Bell inequality can be constructed. It has recently been shown that such Bell inequalities are powerful enough to witness nonlocality in multipartite quantum states [22, 23]. In particular, in [22] we proposed classes of

4 Recall that one defines a polytope to be a bounded convex set with a finite number of extreme points.
permutationally invariant Bell inequalities involving only two-body correlators that detect nonlocality in multipartite quantum systems for any number of parties. The main aim of this paper is to follow the approach of [22], and search for all three- and four-partite Bell inequalities that obey a less restrictive symmetry: translational invariance. We group the Bell inequalities found within this approach into equivalence classes under certain symmetries, and check whether they are violated by quantum states. We then provide an example of a translationally invariant five-partite Bell inequality, constructed from two-body correlators, that involve only nearest neighbors, and check that it is violated by a genuinely multipartite entangled quantum state. We also show that any translationally invariant Bell inequality can be maximally violated by a translationally invariant state, when all parties measure the same set of observables.

2. Preliminaries

We begin by summarizing some known results on Bell inequalities and setting up the notation we will use throughout the paper.

Consider first $N$ spatially separated parties $A_1, \ldots, A_N$ sharing some $N$-partite quantum state $\rho$. Each party is allowed to perform measurements on its share of $\rho$. We restrict the study to the simplest scenario of party $A_i$ freely choosing between two observables $M_i^{(0)} (x_i = 0, 1)$, each having two outcomes $a_i = \pm 1$. The correlations that arise in such an experiment are described by a collection of conditional probabilities
\[
\{ p(a_1, \ldots, a_N | x_1, \ldots, x_N) \}_{a_1, \ldots, a_N; x_1, \ldots, x_N},
\]
with $p(a_1, \ldots, a_N | x_1, \ldots, x_N)$ denoting the probability of obtaining results $a_1, \ldots, a_N$ upon measuring $M_i^{(0)}, \ldots, M_N^{(0)}$. In our case, i.e., when each party chooses between two dichotomic observables, correlations can be equivalently described by a collection of expectation values
\[
\{ \langle M_i^{(0)} \cdots M_N^{(0)} \rangle \}_{i_1, \ldots, i_k; x_1, \ldots, x_k},
\]
where $x_i, \ldots, x_k = 0, 1$, $i_1 < \ldots < i_k = 1, \ldots, N$, and $k = 1, \ldots, N$. Notice that the set (2) contains all mean values of the local observables and mean values of their products involving up to $N$ parties. It follows that for a Bell experiment with two dichotomic observables per site, the set (2) has $3^N - 1$ elements, and it is convenient to think of them, after being ordered, as components of a vector $c$ belonging to $\mathbb{R}^D$ with $D = 3^N - 1$. In what follows, by a slight abuse of terminology, we will identify sets (2) with the corresponding vectors. Also, expectation values involving at least two parties will be called correlators.

Recall that in quantum theory
\[
\langle M_i^{(0)} \cdots M_N^{(0)} \rangle = \text{Tr} \left[ \rho_{A_1 \cdots A_N} \left( M_i^{(0)} \otimes \cdots \otimes M_N^{(0)} \right) \right]
\]
where $\rho_{A_1 \cdots A_N}$ stands for a subsystem of $\rho$ representing the quantum state held by the parties $A_1, \ldots, A_N$, i.e., the partial trace of $\rho$ over all the remaining parties and now $M_i^{(0)}$ with $x_i = 0, 1$ and $i = 1, \ldots, N$ denote Hermitian operators with eigenvalues $\pm 1$. It is known that the set of quantum correlations, denoted $Q$, which arise in the above experiment when the dimension of $\rho$ is unconstrained, is convex (cf [5]). As a proper subset it contains those correlations that, even if obtained from quantum states, the parties can simulate by using local strategies and some shared classical information represented by a random variable $\lambda$ with probability distribution $p(\lambda)$. As no quantum resources are needed to create them, we call...
such correlations classical or local. They form a polytope, denoted $P_N$, whose extremal points are those vectors (2) in which all correlators factorize, that is,

$$\langle M^{(i)}_{x_i} \cdots M^{(i)}_{x_i} \rangle = \langle M^{(i)}_{x_i} \rangle \cdots \langle M^{(i)}_{x_i} \rangle,$$

and each individual mean value $\langle M^{(i)}_{x_i} \rangle$ ($x_i = 0, 1, i = 1, \ldots, N$) equals either $-1$ or $1$. In other words, any vertex $v \in P_N$ represents correlations that the parties can produce by using local deterministic strategies, i.e., each local measurement has a perfectly determined outcome. Denoting then by $V_N$ the set of vertices of $P_N$, one finds that $|V_N| = 2^N$, while $\dim P_N = 3^N - 1$.

Classical correlations are thus represented by those vectors $c$ that can be written as a convex combination of vertices $v_j \in V_N$,

$$c = \sum \lambda_j n_j,$$

where the random variable $\lambda$ with probability distribution $p(\lambda)$ denotes the shared classical information among the parties. If $c$ does not admit such a decomposition, the corresponding correlations are called nonlocal. A multipartite state $\rho$ is then named nonlocal if one can generate nonlocal correlations from it.

For further purposes it is worth recalling that quantum correlations are not the only nonlocal ones. Indeed, there exists a larger set of correlations called nonsignalling that admit this same feature. It contains those correlations that satisfy the no-signalling principle: information cannot be transmitted instantaneously. In terms of a conditional probability distribution generated in the above experiment, this means that any of its marginals observed by any group of $N - 1$ parties does not depend on the choice of measurement made by the remaining party, i.e.,

$$\sum a_i \sum a_{i'} \cdots = \sum a_i \sum a_{i'} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
\(\mathcal{N}\), whose facets are given by inequalities (8). It follows that \(\mathcal{P}_\mathcal{N} \subset \mathcal{Q} \subset \mathcal{N}\) and it is known that in general \(\mathcal{P}_\mathcal{N} \neq \mathcal{Q}\) [16] and \(\mathcal{Q} \neq \mathcal{N}\) [24].

The usual tool to detect nonlocality of quantum states are Bell inequalities [16]. These are inequalities satisfied by all elements of \(\mathcal{P}_\mathcal{N}\), while possibly being violated by some quantum correlations. Generally, they can be written as

\[ I := \alpha \cdot c = \sum_{k=1}^{N} \sum_{i<j<k} \sum_{x_k=0}^{1} \alpha^{i,j,k} \cdot x_k \cdot \left\langle M^{(i)} \cdots M^{(k)} \right\rangle \geq -\beta_c, \tag{9} \]

where \(\alpha^{i,j,k}\) are the components of a vector \(\alpha \in \mathbb{R}^n\) and \(\beta_c = -\min_{c \in \mathcal{P}_\mathcal{N}} \alpha \cdot c\) is the so-called classical bound of (9). Notice that since \(\mathcal{P}_\mathcal{N}\) is a polytope, it is enough to optimize \(I\) over \(c \in \mathcal{P}_\mathcal{N}\) in order to determine \(\beta_c\). Accordingly, by \(\beta_v = \min_{c \in \mathcal{P}_\mathcal{N}} I\) and \(\beta_c = -\min_{c \in \mathcal{P}_\mathcal{N}} I\) we will be denoting the maximal violations of \(I\) by quantum and nonsignalling correlations, respectively. Clearly, \(\beta_v \leq \beta_y \leq \beta_c\) and, in particular, if \(\beta_c = \beta_v\), a Bell inequality does not have quantum violation, while if \(\beta_c = \beta_y\), it cannot be violated by any nonsignalling correlations. In the latter case we call such a Bell inequality trivial. It is worth mentioning that \(\beta_c\) can be efficiently determined via linear programming and therefore to prove that a Bell inequality lacks quantum violation it is in some cases easier to show that \(\beta_y = \beta_c\). We will use this fact later in sections 4.3 and 4.4.

Geometrically, Bell inequalities are half-spaces that constrain \(\mathcal{P}_\mathcal{N}\). In fact, since the latter is a polytope, a finite number of Bell inequalities is necessary to fully determine it. That is, the set \(\mathcal{P}_\mathcal{N}\) is fully described once all Bell inequalities corresponding to its facets—often referred to as tight—are known. This problem can be fully solved for the simplest scenarios using computer algorithms such as the cdd algorithm [25]; it was, for instance, solved for \(N = 3\) [20], and by imposing the permutational symmetry also for \(N = 4,5\) [21]. However, since the dimension of \(\mathcal{P}_\mathcal{N}\) and the number of its vertices grow exponentially with \(N\), the problem quickly becomes computationally intractable for larger \(N\).

### 3. Translationally invariant two-body Bell inequalities

#### 3.1. Bell inequalities from two-body correlators

We now study the problem of detecting nonlocality in a multipartite quantum state having access only to one and two-body expectation values

\[ \left\langle M^{(i)} \right\rangle, \quad \left\langle M^{(i)} M^{(j)} \right\rangle \tag{10} \]

with \(x_i, x_j = 0, 1\) and \(i < j = 1, \ldots, N\). As in the general case, the most straightforward way of tackling this task is to construct the corresponding polytope of local correlations, denoted \(\mathcal{P}_\mathcal{N}\). This is achieved by getting rid of all correlators of order larger than two in all elements of \(\mathcal{P}_\mathcal{N}\). That is, all correlations \(c\) can be written \(c = c_1 \oplus c_2\), where \(c_1 \in \mathbb{R}^{N^2}\) and \(c_2 \in \mathbb{R}^{-2N^2}\) stand for parts of \(c\) containing only one and two-body expectation values, and correlators of order higher than two, respectively. Denoting by \(P: \mathbb{R}^n \mapsto \mathbb{R}^{N^2}\) the projection \(Pc = c_1\) for any \(c\), one then formally has that \(\mathcal{P}_\mathcal{N} = \{ Pe \mid c \in \mathcal{P}_\mathcal{N}\}\). In particular, every vertex of \(\mathcal{P}_\mathcal{N}\) is uniquely mapped onto a vertex of \(\mathcal{P}_\mathcal{N}\), i.e., a vector \(v_2\) whose all two-body correlators factorize \((\langle M^{(i)} \rangle) (\langle M^{(j)} \rangle)\) for all \(x_i, x_j = 0, 1\) and \(i < j = 1, \ldots, N\) and every one-body expectation value equals \(\pm 1\). To see explicitly this one-to-one correspondence, let us denote by \(V_2^\mathcal{N}\) the set of extremal elements of \(\mathcal{P}_\mathcal{N}\) and notice that for any vertex \(v_2 \in V_2^\mathcal{N}\), \(v_2 = Pu\) with \(u \in V_1^\mathcal{N}\) (cf [21] for a proof). On the other hand, \(Pu \in V_2^\mathcal{N}\) for any \(u \in \mathcal{P}_\mathcal{N}\).
Indeed, assume that for some vertex \( \nu \in V_N \), \( \nu = \sum p_i \nu_i \), where \( p_i \in (0, 1) \) and \( \nu_i \) are some vertices (distinct) of \( V_N^2 \). This means that at least one of the components of \( \nu \) is different than \( \pm 1 \), meaning that \( \nu \) is not a vertex of \( V_N^2 \). As a consequence, \( \nu_i \neq \nu_j \) whenever \( p_i 
eq p_j \) for any \( \nu_i, \nu_j \in V_N \). In conclusion \( |V_N^2| = |V_N| = 2^{2N} \).

Having characterized the vertices of the polytope \( \mathcal{P}_N^2 \), the problem of witnessing non-locality from one- and two-body mean values can then be addressed by determining all its facets. That is, one has to find all tight Bell inequalities with two-body correlators, which most generally can be written as

\[
I := \sum_{i=1}^{N} \left( \alpha_i \left( M_0^{(i)} \right) + \beta_i \left( M_1^{(i)} \right) \right) + \sum_{i<j}^{N} \left( \gamma_{ij} \left( M_0^{(i)} M_0^{(j)} \right) + \omega_{ij} \left( M_1^{(i)} M_1^{(j)} \right) \right) \geq -\beta_c,
\]

where \( \alpha_i, \beta_i, \gamma_{ij}, \omega_{ij}, \) and \( \epsilon_{ij} \) are some real parameters, and \( \beta_c \) is the classical bound of \( I \): \( \beta_c = -\min_{\nu\in V_N} I \). In what follows we will shortly call any Bell inequality of the form (11) a two-body Bell inequality.

Although the dimension of the local polytope \( \mathcal{P}_N^2 \) is \( 2N^2 \) and, unlike \( \dim \mathcal{P}_N \), grows only polynomially in \( N \), the number of vertices is preserved \( |V_N^2| = |V_N| = 2^{2N} \), meaning that the problem of determining facets of \( \mathcal{P}_N^2 \) remains intractable for larger \( N \). One can, nevertheless, further simplify the classical polytope by demanding that the Bell inequalities (11) obey some symmetries. For instance, [22] focuses on Bell inequalities that are invariant under any permutation of the parties (see also [21] for symmetric Bell inequalities with full correlators). In that case the dimension of the local polytope is five irrespective of the number of parties, and the number of its vertices grows quadratically in \( N \). In this work we consider a less restrictive translational symmetry, i.e., the one where a Bell inequality is not changed if all parties are shifted by one to the right modulo \( N \). Here, the dimension of the local polytope is not constant with \( N \) as in the permutationally invariant case. However, it is reduced from quadratic to linear in \( N \).

### 3.2. Imposing the translational symmetry

We now move on to imposing translational invariance to the Bell inequalities (11), that is, we demand that they remain the same if the transformations

\[
M_j^{(i)} \rightarrow M_j^{(i+1)} \quad (j = 0, 1),
\]

are simultaneously applied to all parties with the convention that \( M_j^{(N+n)} = M_j^{(n)} \) for any \( n = 1, \ldots, N \) and \( j = 0,1 \). This directly translates to certain conditions on the parameters appearing in (11), namely,

\[
\alpha_i = \alpha_{i+1}, \quad \beta_i = \beta_{i+1} \quad (i = 1, \ldots, N-1)
\]

which imply that all \( \alpha_i \) and \( \beta_i \) must be equal. Then, \( \gamma_{ij}, \omega_{ij}, \) and \( \epsilon_{ij} \) must satisfy the following cycles of equalities:

\[
\gamma_{1,i+1} = \gamma_{2,i+2} = \cdots = \gamma_{N-k,N} = \gamma_{k+1,N} = \gamma_{k+2,N} = \cdots = \gamma_{N-1,N}
\]

In conclusion, |\( V_N^2 \)| = |\( V_N \)| = 2^{2N}.
and

$$
\begin{align*}
\epsilon_{1,1+4} = \epsilon_{2,2+4} = \ldots = \epsilon_{N-k,N} \\
\epsilon_{1,N-k+1} = \epsilon_{2,N-k+2} = \ldots = \epsilon_{N,N}
\end{align*}
$$

(15)

with \( k = 1, \ldots, \lfloor N/2 \rfloor \), and finally

$$
\begin{align*}
\omega_{1,1+4} = \omega_{2,2+4} = \ldots = \omega_{N-k,N} \\
\omega_{N-k+1,1} = \omega_{N-k+2,2} = \ldots = \omega_{N,N}
\end{align*}
$$

(16)

with \( k = 1, \ldots, N - 1 \).

Denote \( \alpha := \alpha_i \) and \( \beta := \beta_i \) \((i = 1, \ldots, N)\), and by \( \gamma_k \), \( \epsilon_k \) \((k = 1, \ldots, \lfloor N/2 \rfloor)\), and \( \omega_k \) \((k = 1, \ldots, N - 1)\) those parameters that form cycles in (14), (15), and (16) enumerated by \( k \).

Then, a general translationally invariant Bell inequality with one and two-body mean values, in what follows referred to as \( \text{translationally invariant} \) \( \text{two-body Bell inequality} \) reads

$$
\alpha S_0 + \beta S_1 + \sum_{k=1}^{|N/2|} \left( \gamma_k T_{00}^{(k)} + \epsilon_k T_{11}^{(k)} \right) + \sum_{k=1}^{N-1} \omega_k T_{01}^{(k)} \geq -\beta_c.
$$

(17)

Here, \( S_j \) stand for symmetrized local expectation values, i.e.,

$$
S_j = \sum_{n=1}^{N} \left\{ M_j^{(m)} \right\} \quad (j = 0, 1)
$$

(18)

and \( T_{ij}^{(k)} \) for all translationally invariant two-body correlators, given explicitly by

$$
T_{ij}^{(k)} = \sum_{n=1}^{N} \left\{ M_i^{(m)} M_j^{(m+k)} \right\} \quad (i \leq j = 0, 1),
$$

(19)

with \( k = 1, \ldots, \lfloor N/2 \rfloor \) for \( i = j \) and \( k = 1, \ldots, N - 1 \) for \( i < j \).

Again, the most efficient translationally invariant Bell inequalities are the facets of the corresponding \( \text{translationally invariant} \) \( \text{local polytope} \). To determine the latter, let \( P_{\mathcal{T}} \) be a map projecting every \( \epsilon_c \in \mathbb{R}^{2^{N}} \) onto a vector consisting of translationally invariant correlators (18) and (19), i.e., a vector of the form

$$
\left( S_0, S_1, T_{00}^{(1)}, \ldots, T_{00}^{(\lfloor N/2 \rfloor)}, T_{01}^{(1)}, \ldots, T_{01}^{(N-1)}, T_{11}^{(1)}, \ldots, T_{11}^{(\lfloor N/2 \rfloor)} \right).
$$

(20)

Then, the polytope of all translationally invariant local correlations is defined as \( P_N^{T} = P_{\mathcal{T}} P_N = \{ P_{\mathcal{T}} \epsilon_c \mid \epsilon_c \in P_N \} \). The dimension of \( P_N^{T} \) is \( N + 1 + 2 \lfloor N/2 \rfloor \), i.e., it grows linearly with the number of parties \( N \). Then, as in the previous case (cf section 3.1), vertices of \( P_N^{T} \), whose set we denote by \( V_N^{T} \), must arise as projections of vertices of the polytope \( P_N \). Precisely, for any \( \omega_{ij} \in V_N^{T} \) there is \( \epsilon_c \in V_N^{T} \) such that \( \omega_{ij} = P_{\mathcal{T}} \epsilon_c \). Although we are unable to determine \( |V_N^{T}| \), it is at most the number of vertices of \( P_N^{T} \) modulo translational symmetry (notice that \( P_{\mathcal{T}} \) can map two different vertices of \( P_N^{T} \) onto the same vertex of \( P_N^{T} \)), which can be counted by using Pólya enumeration theorem [26].

To be more precise, observe that every deterministic local strategy can be thought of as a function \( f : X \rightarrow Y \), where \( X = \{ A_1, \ldots, A_N \} \) is the set of parties and \( Y = \{(1,1), (1,-1), (-1,1), (-1,-1)\} \) is the set of possible pairs of predetermined outcomes assigned to each party’s pair of observables. Let us now denote by \( Y^X \) the set of all such functions, which is isomorphic to \( V_N^{T} \), and let \( G \) be a subgroup of the group of permutations of \( N \) parties. Let us then denote by \( Y^X/G \) the set of deterministic local strategies \( \text{modulo} \ G \), that is, two strategies \( f_1 \) and \( f_2 \) are equivalent if \( f_2 \) can be obtained from \( f_1 \) by permuting the parties with an element of \( G \). Then, Pólya’s enumeration theorem [26] states that
\[ \sum_{\sigma \in \mathcal{G}} Y_{\sigma}^X = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} |Y|^{c(\sigma)}, \]  

(21)

where the sum runs over all elements of \( \mathcal{G} \) and \( c(\sigma) \) counts the number of disjoint cycles in the representation of \( \sigma \) as a permutation.

For the considered translational symmetry, \( \mathcal{G} \) is the group generated by the full cycle \( 1 \mapsto 2 \mapsto \cdots \mapsto N \mapsto 1 \), denoted \( \tau \). The group \( \mathcal{G} \) has \( N \) elements of the form \( \tau^k \) (\( k = 1, \ldots, N \)), with \( \tau^N \) being the identity element of \( \mathcal{G} \). Due to the fact that the number of cycles of \( \tau^k \) as a permutation of an \( N \)-element set is precisely the greatest common divisor of \( N \) and \( k \), denoted \( \gcd(N, k) \), equation (21) can be rewritten as

\[ \sum_{\sigma \in \mathcal{G}} Y_{\sigma}^X = \frac{1}{N} \sum_{k=1}^{N} |Y|^{\gcd(N,k)}. \]  

(22)

The above expression can be further simplified by noting that for any \( k, d' = \gcd(k, N) \) is in particular a divisor of \( N \), i.e., \( d'|N \), and also by taking into account the fact that some \( k \)'s have the same greatest common divisor with \( N \). Let then for any divisor \( d' \) of \( N \), \( \{ 1 \leq k \leq N | \gcd(N, k) = d' \} \) be the set of all those \( k \)'s whose greatest common divisor with \( N \) is \( d' \). One finds that this set has the same number of elements as the set \( \{ 1 \leq l \leq N/d' | \gcd(N/d', l) = 1 \} \), whose cardinality is, by definition, the Euler totient function \( \phi(N/d') \). All this means that

\[ \sum_{\sigma \in \mathcal{G}} Y_{\sigma}^X = \frac{1}{N} \sum_{d' \mid N} \phi(N/d') |Y|^{d'}. \]  

(23)

Using the fact that if \( d'|N \), then \( d = N/d' \) is also a divisor of \( N \) and substituting \( |Y| = 4 \), we finally arrive at the upper bound on the number of vertices of \( V_N^{2,T} \):

\[ |V_N^{2,T}| \leq \frac{1}{N} \sum_{d \mid N} \phi(d) 4^{N/d}. \]  

(24)

As we shall see in the following sections, the above bound is not always tight: for \( N = 3, 5 \), (24) is tight indeed, but not for \( N = 4 \).

To conclude, let us mention that by repeating the arguments used in [21] to prove theorem 1, one can show that if a two-body Bell inequality is a facet of \( \mathcal{P}_N \), it must also be a facet of \( \mathcal{P}_N^2 \). Analogously, if a two-body translationally invariant Bell inequality is a facet of \( \mathcal{P}_N^2 \), then it is a facet of \( \mathcal{P}_N^{2,T} \). The opposite implications, however, are in general not true. Nevertheless, while it is rather hard to expect that a two-body Bell inequality can be a facet of \( \mathcal{P}_N \), some of translationally invariant two-body Bell inequalities are indeed facets of \( \mathcal{P}_N^2 \) (see section 4.4).

4. All three- and four-partite translationally invariant two-body Bell inequalities

In this section we eventually seek all three- and four-partite translationally invariant two-body Bell inequalities and characterize their properties. Using the symmetries described in section 4.1, we group the found Bell inequalities into equivalence classes that are listed in sections 4.3 and 4.4. We also test them with respect to the possibility of revealing nonlocality in quantum systems and the way in which we do this is discussed in section 4.2. Finally, in section 4.5 we discuss the case of \( N = 5 \) and give an example of a nontrivial five-partite translationally invariant Bell inequality in which two-body correlators involve only nearest neighbours.
4.1. Symmetries

According to the symmetries inherent to $\mathcal{P}_{N}$, multipartite Bell inequalities can be grouped into equivalence classes. Naming of parties, observables and outcomes is arbitrary and this is reflected in the structure of $\mathcal{P}_{N}$. In other words, permutation of any parties, measurements or outcomes in a given Bell inequality results in another Bell inequality.

Analogously, translationally invariant Bell inequalities obey certain symmetries and we will use them below to group the facets of $\mathcal{P}_{N}^{T}$ and $\mathcal{P}_{N}^{3T}$ into equivalence classes. These are the following:

- Renaming of parties in a cyclical way $\mathcal{M}_{ij}^{(i)} \rightarrow \mathcal{M}_{ij}^{(i+1)}$ for all $i = 1,\ldots,N$ and $j = 0, 1$. By construction, this symmetry leaves $\mathcal{P}_{N}^{T}$ invariant.
- Renaming of observables $\mathcal{M}_{ij}^{(i)} \leftrightarrow \mathcal{M}_{ij}^{(j)}$ for all parties. On the level of the Bell inequalities (17) this corresponds to the changes $\alpha \leftrightarrow \beta$, $\gamma \leftrightarrow \epsilon_{i}$ ($k = 1, \ldots,[N/2]$) and $\omega_{k} \leftrightarrow \omega_{N-k}$ ($k = 1, \ldots,[N/2]$).
- Renaming of outcomes of the $j$th observable at all sites, i.e., $\mathcal{M}_{ij}^{(i)} \leftrightarrow -\mathcal{M}_{ij}^{(i)}$ for all $i = 1,\ldots,N$. In (17) this symmetry changes $\alpha \leftrightarrow -\alpha$ and $\omega_{k} \leftrightarrow -\omega_{k}$ ($k = 1, \ldots,N-1$) if $j = 0$, and $\beta \leftrightarrow -\beta$ and $\omega_{k} \leftrightarrow -\omega_{k}$ ($k = 1, \ldots,N-1$) if $j = 1$.
- Renaming of parties $\mathcal{M}_{ij}^{(i)} \leftrightarrow \mathcal{M}_{ij}^{(j)}$ for all $i, j$. This symmetry changes in (17) $\omega_{k} \leftrightarrow \omega_{N-k}$ ($k = 1, \ldots,[N/2]$).

It should be stressed that these symmetries form only a proper subset of all the aforementioned symmetries of the global polytope $\mathcal{P}_{N}$ that are usually used to classify Bell inequalities (cf [20]). This restricted choice of symmetries is, however, dictated by the fact that in our classification we want to preserve the translational invariance of Bell inequalities; a transformation that is not of the above form if applied to a translationally invariant Bell inequality may result in another Bell inequality that is no longer translationally invariant. Notice, however, that the above symmetries preserve translational invariance of the found Bell inequalities in the most general case, i.e., when the coefficients in (17) are unconstrained. However, if some of them are zero, then translational invariance does not need to be preserved for the corresponding correlator and one can exploit further symmetries. For example, if $\alpha = \beta = 0$ and $N$ is even, applying $\mathcal{M}_{ij}^{(i)} \leftrightarrow -\mathcal{M}_{ij}^{(i)}$ for any even $i$ and any $j$ leads to the symmetry changes in (17): $\gamma \leftrightarrow -\gamma$, $\omega_{k} \leftrightarrow -\omega_{k}$, $\epsilon_{i} \leftrightarrow -\epsilon_{i}$ for all odd $k$.

4.2. Quantum violation

To search for quantum violation of the Bell inequalities presented in next sections we use the fact that every multipartite Bell inequality with two dichotomic measurements per site can be maximally violated with a multiqubit state and real one-qubit local measurements [27, 28]. Consequently, we can assume that

$$
\mathcal{M}_{ij}^{(i)} = \cos \phi_{ij}^{(i)} \sigma_{i} + \sin \phi_{ij}^{(i)} \sigma_{e},
$$

(25)

where $0 \leq \phi_{ij}^{(i)} \leq \pi/2$ for $x = 0, 1$ and $i = 1,\ldots,N$, and $\sigma_{i}$ and $\sigma_{e}$ stand for the real Pauli matrices. A given Bell inequality is then violated by quantum theory if the corresponding Bell operator (including also the term $\beta_{1}$ with $I$ being the identity matrix) has a negative eigenvalue. Then, to find its maximal quantum violation we optimize the lowest eigenvalue of the corresponding Bell operator over the parameters $\phi_{ij}^{(i)}$.

Notice that if we further impose that $\phi_{ij}^{(i)} = \phi$, for all $i = 1,\ldots,N$ and $j = 0, 1$, then the Bell operator is also translationally invariant. One could naively think that in such case the Bell inequality is maximally violated by a pure translationally invariant state. This is
generally not the case (a particular example is inequality #25 presented in section 4.4), however, one can always construct a multiqubit translationally invariant mixed state achieving the maximal violation. To be more explicit, let $|\psi\rangle \in (\mathbb{C}^2)^\otimes N$ be a pure state maximally violating a given Bell inequality with the same pair of observables at each site. Let also $V_i$ be a shift operator defined through

$$V_i|\psi_1\rangle \cdots |\psi_N\rangle = |\psi_N\rangle |\psi_1\rangle \cdots |\psi_{N-1}\rangle$$

(26)

with $|\psi_i\rangle \in \mathbb{C}^d$ for every $i$. Then, one checks by hand that the following translationally invariant mixed state

$$\rho = \frac{1}{N} \sum_{k=0}^{N-1} V_k^2 |\psi\rangle \langle \psi| V_k^2$$

(27)

violates the Bell inequality maximally with the same observables. Later, in section 5 we will show that any translationally invariant Bell inequality with $M$ dichotomic observables per site (not necessarily the two-body one) can be maximally violated by a translationally invariant mixed state and the same set of observables at all sites.

4.3. $N = 3$

We now consider the case $N = 3$. The general formula (17) reduces to

$$\alpha S_0 + \beta S_1 + \gamma T_{00} + \epsilon T_{11} + \omega_1 T_{01}^{(1)} + \omega_2 T_{01}^{(2)} + \beta_c \geq 0,$$

(28)

where we denoted $\gamma_i$ and $\epsilon_i$ by $\gamma$ and $\epsilon$, respectively, and also skipped the superscripts in $T_{00}^{(1)}$ and $T_{11}^{(1)}$. The latter, as well as $T_{01}^{(1)} + T_{01}^{(2)}$, are permutationally invariant, and therefore, for $\omega_1 = \omega_2$, one obtains a symmetric Bell inequality (cf [22]).

In this case $\dim P^3_{N,T} = 6$. By using the cdd algorithm [25] solving the convex hull problem, we find that $P^3_{3,T}$ has 38 facets that under the symmetries discussed in the preceding section are grouped into 6 classes presented in table 1. Moreover, the polytope has 24 vertices, meaning that in this case the bound (24) is saturated.

Noticably, only the last Bell inequality in table 1 is violated by quantum states. By construction, it is a facet of $P^3_{3,T}$, but not of $P^3_{N,T}$. The rest of the classes in table 1 are trivial in the sense that they are not violated by any nonsignalling correlations.

The maximal quantum violation of this only nontrivial inequality is $\beta_q = 10.02$ (while $\beta_q = 13$) and it can be realized with the following pure state

$$|\psi_3\rangle = \frac{1}{N} \sum_{k=0}^{N-1} V_k^2 |\psi\rangle \langle \psi| V_k^2$$

(29)

and the measurements given by $\phi^{(i)}_{0} = -1.1946$ and $\phi^{(i)}_{1} = 0.0957$ for all $i = 1, 2, 3$; both chosen so that the coefficients in front of $|000\rangle$ and $|111\rangle$ are equal. Let us now shortly discuss the properties of the state $|\psi_3\rangle$. First, it is symmetric and therefore genuinely multipartite entangled [29]. Second, all its bipartite reductions are local in the sense that they do not violate the CHSH Bell inequality [31]. Then, one notices that the state $|\psi_3\rangle$ is quite close to the three-qubit W state $|W_3\rangle = (1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$ suggesting that the latter also violates this Bell inequality. Indeed, one finds that the maximal violation of our Bell inequality by $|W_3\rangle$ is 9.85 with the measurements given by $\phi^{(i)}_{0} = 5.2556$ and $\phi^{(i)}_{1} = 0.2285$ ($i = 1, 2, 3$).

Let us finally check how entangled is the state $|\psi_3\rangle$ by computing its geometric measure of entanglement. Recall that for pure $N$-partite states from $(\mathbb{C}^d)^\otimes N$ the latter is defined as [32]:

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where the maximum is taken over all fully product $N$-partite states $|\phi_{\text{prod}}\rangle = |e_1\rangle \otimes \cdots \otimes |e_N\rangle \in \mathbb{C}^d^\otimes N$ with $|e_i\rangle \in \mathbb{C}^d$ being local single-party states. By exploiting the fact that for pure symmetric states the maximum in equation (30) is always realized by a symmetric product vector $|\phi\rangle = |\phi\rangle \otimes \cdots \otimes |\phi\rangle \in \mathbb{C}^d^\otimes N$, $E_G$ of our state $|\psi\rangle$ can be computed almost by hand giving $E_G(|\psi\rangle) = 0.2726$. It is worth pointing out that $|\psi\rangle = E_W(1/3)$ and therefore although the state $|\psi\rangle$ is less entangled than the $W$ state with respect to $E_G$, it gives a stronger violation of the above Bell inequality than $|W\rangle$.

$E_G(|\psi\rangle) = 1 - \max_{|\phi_{\text{prod}}\rangle} |\langle \phi_{\text{prod}} | \phi \rangle|^2,$

(30)

Here $\dim P_{4,2}^T = 9$, while the number of different local deterministic strategies amounts to $|V_{4,2}^T| = 2^{4} \times 2^{4} = 256$. Modulo translational invariance, this number reduces to $(1 \times 4^4 + 1 \times 4^4 + 2 \times 4^4)/4 = 70$, which, through (24), upper bounds the actual number of elements of $V_{4,2}^T$. By using the cdd algorithm [25] we find that 68 out of these 70 ‘translationally invariant’ local strategies correspond to extremal vertices of $P_{4,2}^T$, i.e., the bound (24) is not tight in this case. This is because the deterministic local strategies $\{(1, 1), (-1, 1), (1, -1), (-1, -1)\}$ and $\{(1, 1), (-1, -1), (1, -1), (-1, 1)\}$ which are translationally inequivalent give exactly the same one and two-body translationally invariant correlators, as well as $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ and $\{(1, 1), (-1, -1), (-1, 1), (1, -1)\}$ do. Hence, they correspond to the same vertex of $P_{4,2}^T$.

Furthermore, the cdd algorithm [25] shows that there are 1038 tight Bell inequalities in this scenario, which we group into 103 classes, collected in table 2. Inequalities #1 to #20 are trivial because $\beta_0 = \beta_1$. Inequalities #21 to #24 are not violated by quantum correlations, although they are violated by nonsignalling correlations. Noticeably, these inequalities are a bit in the spirit of the Guess-Your-Neighbour’s-Input (GYNI) Bell inequalities [30]. That is, they represent distributed tasks at which quantum theory does not provide any advantage over classical correlations, while there exist supra-quantum nonsignalling correlations outperforming them. Recall that GYNI Bell inequalities are facets of $P_N$, whereas our inequalities are not; still inequality #21 is a facet of $P_N^*$. Then, all the remaining Bell inequalities (#25 to #103) are violated by quantum correlations. Inequalities #25 to #63 can be violated maximally by using the same pair of qubit
Table 2. The list of all classes of four-partite translationally invariant two-body Bell inequalities. As before, by $\beta_N$, $\beta_Q$, $\beta_C$ we denote the maximal value of the Bell inequality for nonsignalling, quantum and classical correlations, respectively. Then, $\beta_Q^{\text{TI}}$ stands for maximal quantum violation with the same single-qubit observables per site (recall that in this case the corresponding Bell operator is permutationally invariant).

| #  | $\beta_{NS}$ | $\beta_Q$ | $\beta_Q^{\text{TI}}$ | $\beta_C$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\epsilon_1$ | $\gamma_2$ | $\alpha_3$ | $\epsilon_1$ |
|----|--------------|-----------|------------------------|-----------|----------|--------|--------|----------|----------|----------|-------------|--------|----------|-------------|
| 1  | 4            | 4.00      | 4.00                   | 4         | 0        | 2      | 0      | 0        | 0        | 0        | 1           | 0      | 0        | 1            |
| 2  | 4            | 4.00      | 4.00                   | 4         | 1        | 1      | 0      | 0        | 1        | 0        | 0           | 0      | 0        | 0            |
| 3  | 4            | 4.00      | 4.00                   | 4         | 1        | 1      | 0      | 0        | 0        | 0        | 1           | 0      | 0        | 0            |
| 4  | 4            | 4.00      | 4.00                   | 4         | 0        | 2      | 0      | 0        | 0        | 2        | 0           | 0      | 1        | 0            |
| 5  | 4            | 4.00      | 4.00                   | 4         | 0        | 0      | 0      | 0        | 1        | 0        | 0           | 1      | 0        | 0            |
| 6  | 4            | 4.00      | 4.00                   | 4         | 0        | 2      | 0      | 0        | 0        | 1        | 0           | 0      | 0        | 0            |
| 7  | 4            | 4.00      | 4.00                   | 4         | 0        | 0      | 0      | 0        | 1        | 0        | 0           | 0      | 0        | 1            |
| 8  | 4            | 4.00      | 4.00                   | 4         | 0        | 0      | 0      | 0        | 0        | 2        | 0           | 0      | 1        | 0            |
| 9  | 8            | 8.00      | 8.00                   | 8         | −1       | −1     | −1     | 1        | 1        | 1        | 0           | 1      | 0        | 1            |
| 10 | 8            | 8.00      | 8.00                   | 8         | −1       | 1      | 1      | 1        | 1        | 0        | 0           | 1      | 1        | 0            |
| 11 | 8            | 8.00      | 8.00                   | 8         | 0        | 2      | −1     | −1       | 1        | 1        | 0           | 0      | 0        | 0            |
| 12 | 8            | 8.00      | 8.00                   | 8         | 1        | 1      | 0      | 1        | 1        | −2       | 0           | 0      | −1       | 1            |
| 13 | 8            | 8.00      | 8.00                   | 8         | −1       | 3      | 0      | −1       | −1       | 2        | 0           | 0      | −1       | 1            |
| 14 | 8            | 8.00      | 8.00                   | 8         | −2       | −2     | 1      | 1        | 1        | 1        | 0           | 0      | 0        | 2            |
| 15 | 8            | 8.00      | 8.00                   | 8         | −2       | −2     | 0      | 2        | 2        | 0        | 1           | 0      | 0        | 1            |
| 16 | 16           | 16.00     | 16.00                  | 16        | 0        | 4      | 0      | 2        | 2        | 4        | 1           | 2      | 1        | 1            |
| 17 | 16           | 16.00     | 16.00                  | 16        | −4       | 0      | 2      | 2        | 2        | 2        | 1           | 2      | 1        | 2            |
| 18 | 16           | 16.00     | 16.00                  | 16        | 2        | 2      | −2     | 3        | 1        | −2       | 1           | −2     | 1        | 2            |
| 19 | 20           | 20.00     | 20.00                  | 20        | −3       | 5      | 0      | −3       | −3       | 4        | 0           | −3     | 2        | 1            |
| 20 | 20           | 20.00     | 20.00                  | 20        | −1       | 5      | −1     | −2       | −2       | 5        | 1           | −3     | 1        | 1            |
| 21 | 44/3         | 12.00     | 12.00                  | 12        | 0        | 2      | −2     | 0        | 2        | 2        | 1           | 0      | 0        | 0            |
| 22 | 116/5        | 20.00     | 20.00                  | 20        | 0        | 4      | −1     | −3       | 3        | 3        | −1          | 2      | 0        | 2            |
| 23 | 32           | 28.00     | 28.00                  | 28        | −2       | −8     | −2     | −2       | 4        | 4        | −1          | 2      | 2        | 2            |
| 24 | 32           | 28.00     | 28.00                  | 28        | −2       | −8     | −4     | 0        | 4        | 4        | 1           | 0      | 2        | 0            |
| 25 | 48/5         | 8.42      | 8.42                   | 8         | −2       | −2     | 1      | 0        | 1        | 1        | 0           | 1      | 0        | 0            |
| 26 | 12           | 9.27      | 9.27                   | 8         | 0        | 0      | −2     | −1       | −1       | 2        | 1           | 0      | 1        | 1            |
Table 2. (Continued).

| #  | $\beta_0$ | $\beta_0$ | $\beta_0 T$ | $\alpha$ | $\beta$ | $\gamma_1$ | $\omega_0$ | $\omega_1$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|----|----------|----------|-------------|----------|--------|-----------|------------|------------|------------|------------|------------|
| 27 | 16       | 11.31    | 11.31       | 8        | 0      | 0         | 0          | 0          | 0          | -1         | 2          |
| 28 | 76/5     | 12.26    | 12.26       | 12       | 0      | 2         | -2         | 0          | 0          | 2          | 1          | 2          |
| 29 | 76/5     | 12.97    | 12.97       | 12       | 0      | 2         | -3         | -1         | -1         | 1          | 1          | 2          |
| 30 | 52/3     | 13.60    | 13.60       | 12       | 0      | 0         | -1         | -1         | -2         | 4          | 0          | -1         |
| 31 | 20       | 14.42    | 14.42       | 12       | 0      | 1         | 1          | 2          | 2          | -1         | 3          |
| 32 | 20       | 14.77    | 14.77       | 12       | 0      | 0         | -1         | -1         | 4          | -1         | -2         |
| 33 | 96/5     | 16.60    | 16.60       | 16       | -2     | -2        | 1          | 2          | -4         | 0          | 0          | 3          |
| 34 | 96/5     | 16.72    | 16.72       | 16       | -4     | -4        | 1          | 2          | 0          | 0          | 0          | 2          |
| 35 | 64/3     | 17.25    | 17.25       | 16       | -1     | -1        | -5         | 2          | 2          | 1          | 3          | -1         |
| 36 | 24       | 17.50    | 17.50       | 16       | 1      | 3         | 1          | 2          | 3          | -1         | 3          |
| 37 | 24       | 18.02    | 18.02       | 16       | 0      | 0         | -2         | -3         | -3         | 4          | 1          | 0          |
| 38 | 24       | 18.37    | 18.37       | 16       | -2     | -2        | 2          | 2          | 2          | 2          | -1         | 4          |
| 39 | 116/5    | 20.36    | 20.36       | 20       | -2     | -4        | 0          | 2          | 4          | 1          | 4          |
| 40 | 24       | 20.77    | 20.77       | 20       | -2     | -8        | -1         | 2          | 2          | 4          | 0          | 0          |
| 41 | 24       | 20.84    | 20.84       | 20       | -2     | -8        | -1         | 1          | 1          | 4          | 0          | 2          |
| 42 | 28       | 21.18    | 21.18       | 20       | -2     | 4         | 0          | -2         | -2         | 4          | -1         | -4         |
| 43 | 28       | 21.89    | 21.89       | 20       | 0      | 2         | -6         | 2          | 2          | 3          | -2         |
| 44 | 28       | 21.93    | 21.93       | 20       | 0      | 2         | -4         | -2         | -2         | 0          | 2          | -4         |
| 45 | 32       | 25.01    | 25.01       | 24       | 0      | 4         | -2         | 2          | 4          | 0          | -1         | -4         |
| 46 | 32       | 25.30    | 25.30       | 24       | -2     | -2        | -6         | 4          | 2          | 5          | 0          | 1          |
| 47 | 32       | 28.40    | 28.40       | 28       | -2     | -8        | -4         | 0          | 0          | 4          | 1          | 4          |
| 48 | 156/5    | 28.41    | 28.41       | 28       | -6     | 8         | 1          | -4         | -4         | 4          | 0          | -4         |
| 49 | 156/5    | 28.48    | 28.48       | 28       | -6     | 8         | 0          | -4         | -4         | 4          | 1          | -4         |
| 50 | 36       | 29.20    | 29.20       | 28       | -1     | -7        | -2         | 4          | 4          | 2          | 0          | -3         |
| 51 | 36       | 29.25    | 29.25       | 28       | -2     | 4         | -2         | -6         | -4         | 6          | 1          | -2         |
| 52 | 36       | 29.29    | 29.29       | 28       | -4     | -4        | 0          | 3          | 3          | 4          | 2          | -6         |
| 53 | 44       | 31.73    | 31.73       | 28       | -2     | 4         | 1          | -3         | -3         | 7          | -2         | -6         |
| 54 | 44       | 31.84    | 31.84       | 28       | -2     | 4         | 0          | -2         | -2         | 6          | -2         | -6         |
| 55 | 40       | 33.64    | 33.64       | 32       | -4     | 0         | 0          | 4          | 4          | -4         | -1         | -6         |
| #  | $\beta_\alpha$ | $\beta_\beta$ | $\beta_\gamma$ | $\alpha$ | $\beta_1$ | $\gamma_1$ | $\omega_1$ | $\omega_3$ | $\epsilon_1$ | $\epsilon_2$ | $\omega_2$ | $\epsilon_3$ |
|----|----------------|----------------|----------------|--------|----------|-----------|----------|----------|------------|------------|----------|------------|
| 56 | 124/3          | 36.57          | 36.57          | 36     | 4        | 8         | -1       | -5       | -6         | 8          | 0        | -5         |
| 57 | 52             | 39.11          | 39.11          | 36     | 8        | 4         | -4       | -4       | -4         | 8          | 3        | 0          |
| 58 | 60             | 42.82          | 42.82          | 36     | 4        | 8         | -2       | -3       | -3         | 8          | 5        | -3         |
| 59 | 48             | 40.92          | 40.92          | 40     | -4       | 8         | 1        | -5       | -5         | 9          | -3       | 0          |
| 60 | 56             | 42.32          | 42.32          | 40     | -4       | 8         | 1        | -5       | -5         | 9          | -2       | -8         |
| 61 | 64             | 50.49          | 50.49          | 48     | -4       | 8         | -4       | -10      | -10        | 8          | 3        | -2         |
| 62 | 500/7          | 62.89          | 62.89          | 60     | -14      | 16        | -4       | -8       | -8         | 4          | 5        | -4         |
| 63 | 104            | 74.50          | 74.50          | 64     | -4       | 8         | 4        | -8       | -8         | 12         | -5       | -14        |
| 64 | 10             | 8.83           | 8.00           | 8      | -2       | 0         | 1        | 1        | -1         | 1          | 0        | 0          |
| 65 | 18             | 16.56          | 16.00          | 16     | -3       | -1        | 1        | -2       | 3          | 1          | 0        | 2          |
| 66 | 56/3           | 16.59          | 16.00          | 16     | 2        | 2         | -2       | 1        | -1         | -2         | 1        | 2          |
| 67 | 68/3           | 20.46          | 20.00          | 20     | -2       | 4         | -1       | -2       | -4         | 4          | 0        | -2         |
| 68 | 24             | 21.24          | 20.00          | 20     | -2       | 4         | -2       | -2       | -4         | 4          | 1        | -2         |
| 69 | 100/3          | 29.15          | 28.00          | 28     | -2       | 4         | -2       | -2       | -6         | 6          | 0        | -2         |
| 70 | 44/3           | 12.52          | 12.05          | 12     | -1       | 3         | -2       | -1       | -2         | 2          | 1        | 0          |
| 71 | 24             | 21.31          | 20.06          | 20     | -2       | 6         | 1        | -3       | 0          | 4          | -1       | -3         |
| 72 | 192/5          | 32.69          | 32.10          | 32     | -1       | -7        | -1       | -4       | 5          | 5          | -2       | 4          |
| 73 | 96/5           | 16.45          | 16.18          | 16     | -2       | -2        | 1        | -2       | 4          | 1          | -1       | 2          |
| 74 | 44/3           | 12.60          | 12.21          | 12     | -1       | -1        | -3       | 1        | 2          | 1          | 2        | 0          |
| 75 | 96/5           | 16.60          | 16.28          | 16     | -4       | -2        | 2        | -1       | 3          | 1          | 0        | 2          |
| 76 | 20             | 17.66          | 16.36          | 16     | 0        | 4         | 0        | -2       | 2          | 2          | -1       | 2          |
| 77 | 20             | 17.66          | 16.43          | 16     | -1       | -5        | 0        | -1       | 2          | 3          | -1       | 2          |
| 78 | 28             | 24.47          | 24.44          | 24     | -4       | 8         | 1        | -2       | -2         | 4          | -1       | -4         |
| 79 | 16             | 13.66          | 12.58          | 12     | 0        | 2         | 1        | 0        | 2          | 2          | -1       | 2          |
| 80 | 24             | 21.43          | 20.63          | 20     | -2       | -8        | 0        | 0        | 2          | 4          | -1       | 2          |
| 81 | 24             | 21.43          | 20.68          | 20     | -4       | 2         | 2        | -4       | 2          | 2          | 0        | -2         |
| 82 | 500/7          | 61.83          | 60.69          | 60     | -10      | 12        | -12      | -8       | -8         | 4          | 9        | 0          |
| 83 | 32             | 25.00          | 24.70          | 24     | 0        | 4         | 2        | 2        | 4          | 4          | -1       | 4          |

Table 2. (Continued.)
Table 2. (Continued).

| #  | $\beta_{33}$ | $\beta_{Q}$ | $\beta_{Q}^T$ | $\delta_{C}$ | $\alpha$ | $\beta$ | $\gamma$ | $\omega_1$ | $\omega_2$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ | $\gamma_1$ | $\omega_3$ | $\epsilon_1$ | $\epsilon_2$ |
|----|--------------|--------------|--------------|-------------|----------|--------|---------|----------|----------|-------------|-------------|-------------|-----------|----------|-------------|----------|
| 84 | 16           | 13.66        | 12.75       | 12          | 0        | 2      | -2      | 2         | 2         | 2           | 2           | 2           | 0         | 1         |              |          |
| 85 | 36           | 31.31        | 28.82       | 28          | -2       | 8      | 1       | -4        | 0         | 6           | -2          | -4          | 2         |          |              |          |
| 86 | 44           | 36.89        | 36.85       | 36          | -4       | 8      | 0       | -5        | -5        | 8           | -1          | -6         | 4         |          |              |          |
| 87 | 44           | 39.31        | 36.90       | 36          | -6       | -12    | 1       | 0         | 4         | 4           | -2          | 4          | 2         |          |              |          |
| 88 | 28           | 25.31        | 24.90       | 24          | -8       | -4     | 3       | 3         | 0         | 1           | 1           | 3          | -1        |          |              |          |
| 89 | 24           | 21.43        | 20.93       | 20          | -2       | -8     | -2      | 2         | 2         | 4           | 1           | 0          | 2         |          |              |          |
| 90 | 16           | 13.66        | 12.93       | 12          | 0        | 2      | -1      | 0         | 2         | 0           | -1          | -2         | 1         |          |              |          |
| 91 | 16           | 13.66        | 13.14       | 12          | -2       | -2     | 1       | 1         | 1         | 0           | -1          | 2         | 1         |          |              |          |
| 92 | 36           | 31.31        | 29.15       | 28          | -2       | -8     | -1      | -2        | 2         | 4           | -2          | 4         | 2         |          |              |          |
| 93 | 32           | 27.31        | 25.19       | 24          | -2       | 6      | -4      | -4        | -4        | 4           | 3          | 0         | 3         |          |              |          |
| 94 | 20           | 17.66        | 17.21       | 16          | 0        | 4      | -2      | 2         | 2         | 2           | 1          | -2        | 1         |          |              |          |
| 95 | 40           | 30.62        | 29.27       | 28          | -2       | 4      | -4      | -6        | -6        | 4           | 3          | 0         | 4         |          |              |          |
| 96 | 44           | 39.31        | 37.82       | 36          | -6       | -12    | -4      | 4         | 4         | 4           | 3          | 0         | 2         |          |              |          |
| 97 | 36           | 31.31        | 29.87       | 28          | -2       | -8     | -4      | 4         | 4         | 4           | 4          | 2         | -2        | 3         |          |              |          |
| 98 | 44           | 38.70        | 37.98       | 36          | -2       | -8     | -8      | 4         | 4         | 4           | 3          | -4        | 2         |          |              |          |
| 99 | 40           | 30.85        | 30.06       | 28          | -2       | 4      | -8      | -4        | -4        | 4           | 5          | 0         | 2         |          |              |          |
| 100| 268/5        | 46.34        | 46.15       | 44          | -10      | 12     | 3       | -4        | -4        | 4           | -2         | -8        | 2         |          |              |          |
| 101| 32           | 23.31        | 22.77       | 20          | -2       | 4      | 1       | -2        | -2        | 4           | -2         | -4        | 2         |          |              |          |
| 102| 88           | 68.91        | 67.63       | 64          | -4       | 8      | -8      | -14       | -14       | 8           | 5          | 2         | 9         |          |              |          |
| 103| 80           | 69.75        | 67.97       | 64          | -4       | -16    | -8      | 10        | 10        | 8           | 3          | -6        | 7         |          |              |          |
measurements at all sites ($\beta' = \beta'^{TI}_Q$), and also, except for inequality #27, they are maximally violated by pure states that are genuinely multipartite entangled. In particular, inequalities #26 to #63 are maximally violated by translationally invariant pure states, whereas inequality #25 by a state orthogonal to the subspace spanned by translationally invariant states; still it can be maximally violated by a translationally invariant mixed state (cf section 4.2).

On the other hand, inequalities #64 to #103 and inequality #27 are maximally violated by states that are product across some bipartition. In particular inequality #27 is a sum of two CHSH Bell inequalities [31]: one between $A_1$ and $A_3$ and the other one between $A_2$ and $A_4$. It is then maximally violated by a product of two two-qubit maximally entangled states.

Accordingly, in all the cases from #64 to #103, different pairs of measurement settings are required to achieve maximal violation ($\beta' > \beta'^{TI}_Q$). Noticeably, for inequalities #64 to #69 the same settings at all sites do not suffice to achieve quantum violation ($\beta'^{TI}_Q = \beta_c$).

It should also be stressed that in the case of inequalities #25, #73, #81, #87, and #88, $\beta'^{TI}_Q$ is obtained with multipartite states whose all bipartite subsystems are local. Consequently, nonlocality of these states cannot be revealed by any bipartite Bell inequality. In this sense, some of the above violations are purely multipartite, even if obtained from only bipartite correlations.

Let us finally mention that inequalities 4, 10, 17, 20, 21, 25, 28, 36, 38, 43, 51, 54, 57, 69, 81, 84, 89, 94 are also facets of the two-body local polytope $\mathcal{P}^2_N$.  

4.5. $N = 5$

Let us finally consider the five-partite case. Here, $\dim \mathcal{P}^2_{5, T} = 10$ and the local polytope $\mathcal{P}^2_N$ has $2^{5^5} = 1024$ vertices, which modulo translational invariance reduce to $(1 \times 4^4 + 4 \times 4)/5 = 208$. Interestingly, all the resulting translationally invariant correlations uniquely correspond to vertices of $\mathcal{P}^2_{5, T}$, thus, like for $N = 3$, the bound (24) is saturated in this case. Then, using the algorithm [25] one finds that the polytope $\mathcal{P}^2_{5, T}$ has 34484 facets, which, after applying the symmetries from section 4.1 can be grouped into 4198 different classes. This number is already too large to list explicitly all these Bell inequalities. The case $N = 6$ is already intractable.

Let us finally mention that the complexity of the local polytope can further be simplified by imposing additional constraints on the Bell inequalities. For instance, one can require that a TI Bell inequality contains only correlators between nearest neighbors. A general form of such an $N$-partite Bell inequality is

$$\alpha S_0 + \beta S_1 + \gamma T^{(1)}_{00} + \alpha_1 T^{(1)}_{01} + \alpha_2 T^{(1)}_{01} + \epsilon T^{(1)}_{11} + \beta_c \geq 0$$

and the corresponding local polytope has dimension six for any number of parties. As an illustrative example we consider a five-partite Bell inequality of this form with $\alpha = -2$, $\beta = -6$, $\gamma = -2$, $\alpha_1 = 2$, $\alpha_2 = 4$, and $\epsilon = 5$. For this choice of parameters one finds that $\beta_c = 35$.

Interestingly, the resulting Bell inequality is capable of revealing nonlocality in multipartite quantum states. First, assuming that all parties measure the same pair of single-qubit observables, the maximal quantum violation of this Bell inequality amounts to 35.29 and it is for instance realized by the following five-qubit translationally invariant state

$$\psi' = -0.3710(\lvert 00000 \rangle + \lvert 11111 \rangle) - 0.1817 \lvert T_{0000} \rangle + 0.1260 \lvert T_{0001} \rangle - 0.1418 \lvert T_{0010} \rangle + 0.2645 \lvert T_{0011} \rangle - 0.0603 \lvert T_{0101} \rangle + 0.0486 \lvert T_{0111} \rangle,$$

$$
(33)$$
where the states $|T_{abcd,e}\rangle$ are defined through the shift operator (26) as
\[
|T_{abcd,e}\rangle = \sum_{i=0}^{4} V_{i}^{2} |abcde\rangle.
\] (34)

The corresponding measurements are given by $\phi_{0}^{(i)} = 1.2967$ and $\phi_{1}^{(i)} = 1.9866$ with $i = 1, \ldots, 5$. Again, both the state and the measurements are chosen so that the coefficients in front of $|0\rangle^{\otimes 5}$ and $|1\rangle^{\otimes 5}$ are the same. As one checks, the state $|\psi\rangle$ is genuinely multipartite entangled and all its bipartite subsystems are local in the sense that they do not violate the CHSH Bell inequality. Moreover, its geometric measure of entanglement (30) amounts to $E_{G}(\psi) = 0.4980$.

If one then considers any possible real qubit measurements at all sites, then the maximal quantum violation of this Bell inequality is $\beta_{Q} \approx 36.21$, which is realized by the state $|\psi\rangle = |001 \rangle \otimes |00 \rangle - |01 \rangle + |10 \rangle + |11 \rangle$, entangled only at the last two qubits. The corresponding measurements are given by $\phi_{0}^{(i)} = 0$ for all $i$ and $\phi_{1}^{(i)} = 0$ with $i = 1, 2, 3$, and $\phi_{1}^{(4)} = 4.7378$ and $\phi_{1}^{(5)} = 1.2083$.

5. Bell violation with translationally invariant multiqudit states

Following the results of [34], we now show that the maximal quantum violation $\beta_{Q}$ of any translationally invariant Bell inequality with $M$ dichotomic measurements per site can always be attained with multipartite quantum states obeying translational symmetry and the same set of observables at all sites (below referred to as symmetric measurement settings). The local dimension sufficient to do so is $dN$, where $d$ is the local dimension of a pure state (not necessarily translationally invariant) violating the Bell inequality maximally. We then provide a numerical method seeking both the translationally invariant multipartite quantum states and observables (the same at all sites) that realize $\beta_{Q}$ for a given Bell inequality. This algorithm is further tested on Bell inequalities presented in table 2, on which it shows good performance. In particular, all these Bell inequalities are violated maximally by translationally invariant states of local dimensions smaller than $dN$.

5.1. The construction

Let us go back to the general Bell inequality (9) and generalize it to the case of $M$ observables at each site, i.e., $x_{i} = 0, \ldots, M - 1$ for all $i$. Assume then that it is translationally invariant, i.e., elements of $\alpha$ obey the following equalities
\[
\alpha_{k_{1}, \ldots, k_{i}} = \alpha_{k_{1} + 1, \ldots, k_{i} + 1}
\] (35)
for all $i_{1} < \ldots < i_{k} = 1, \ldots, N$, $x_{i_{1}}, \ldots, x_{i_{k}} = 0, \ldots, M - 1$ and $k = 1, \ldots, N$, and with the convention that if $i_{k} = N$, then
\[
\alpha_{k_{1}, \ldots, k_{i} + 1} = \alpha_{k_{1}, \ldots, k_{i}, i_{k} + 1}
\] (36)
Assume then that this inequality is violated maximally by a pure $N$-partite quantum state $|\psi\rangle$ belonging to $(\mathbb{C}^{d})^{\otimes N}$ and local measurements $\mathcal{M}_{i}^{(j)}$. At each site we extend the Hilbert space to $(\mathbb{C}^{d} \otimes \mathbb{C}^{N})^{\otimes N}$ and construct the following translationally invariant state acting on $(\mathbb{C}^{d} \otimes \mathbb{C}^{N})^{\otimes N}$:
\[
q^{\otimes N} = \frac{1}{N} \sum_{j=0}^{N-1} V_{j}^{2} |\psi\rangle\langle \psi| (V_{j}^{2})^{\dagger} \otimes V_{N,0}^{2} |0, 1, \ldots, N-1\rangle\langle 0, 1, \ldots, N-1|_{N-1}, (37)
\]
where \( V_d \) stands for the shift operator defined in (26). Then, the corresponding dichotomic observables acting on \( \mathbb{C}^d \otimes \mathbb{C}^N \) are taken to be the same at all sites and defined as

\[
\widehat{M}_j^{(1)} = \widehat{M}_j^{(2)} = \cdots = \widehat{M}_j^{(N)} = \sum_{i=0}^{N-1} M_{ij}^{(i+1)} \otimes |j\rangle \langle i| \tag{38}
\]

with \( j = 0, \ldots, M - 1 \). Now, exploiting (35) it is fairly straightforward to check that the given Bell inequality is maximally violated by the state (37) and settings (38).

Thus, any translationally invariant Bell inequality can always be maximally violated by a state that obeys the same symmetry defined on a product Hilbert space whose local dimension is \( dN \). In particular, it follows that all the TI two-body Bell inequalities found here for \( N = 3 \) and \( N = 4 \) (cf section 4) can be maximally violated by TI states of local dimension 6 and 8, respectively.

Although the above construction is general, it remains unclear whether \( dN \) is the minimal local dimension necessary to produce \( \beta_Q \) with TI states and symmetric settings. For instance, in the three-partite case the Bell inequality \#6 is maximally violated by a three-qubit translationally invariant state and the same pair of measurements at all sites (see section 4.3). Below we show that all those Bell inequalities in table 2 that have quantum violation are violated maximally by TI states of local dimension lower than eight.

5.2. Numerical technique

Here we discuss an algorithm providing a lower bound on the maximal violation \( \beta_Q \) of a translationally invariant Bell inequality in fixed local dimensions \( D \), assuming that the underlying quantum state is translationally invariant and the measurement settings are the same at all sites. It also provides an upper bound on the minimal dimension \( d_{\text{min}} \) of the local Hilbert space required to attain \( \beta_Q \) with translationally invariant states and symmetric settings. In the case of small number of parties (say, \( N < 6 \)) and low dimensions (\( D < 7 \)), we conjecture that the algorithm finds the maximal quantum violation for a given local dimension \( D \). This would imply that the resulting upper bounds on \( d_{\text{min}} \) are the best possible ones, that is, the algorithm finds \( d_{\text{min}} \) with high confidence. In particular, \( d_{\text{min}} \)’s found by the algorithm for the Bell inequalities from table 2 improve the general upper bound \( dN \) discussed in section 5.1.

Let us now move on to the algorithm. It is a variant of the see-saw type iteration technique introduced in [35] (see also [36]) and it is formulated as follows:

(i) For a given translationally invariant Bell inequality with \( M \) dichotomic measurements per site represented by the vector \( \alpha \) (cf in equation (9)) fix the dimension of the local Hilbert space as \( D \) for all parties. Then, generate random unitary matrices \( U_j \) (\( j = 0, \ldots, M - 1 \)) according to a uniform distribution (see [37] for a simple method allowing to do so) and take the first party’s dichotomic observables as

\[
\mathcal{M}_{ij}^{(1)} = U_i \Lambda U_j^\dagger, \tag{39}
\]

where \( \Lambda \) is a \( D \)-dimensional diagonal matrix with randomly generated \( \pm 1 \) entries.

(ii) Identify the observables of the remaining \( N - 1 \) parties with the first party’s ones, i.e.,

\[
\mathcal{M}_{ij}^{(1)} = \mathcal{M}_{ij}^{(1)} \tag{40}
\]

for all \( i = 2, \ldots, N \) and \( j = 0, \ldots, M - 1 \).
(iii) Form the corresponding Bell operator

\[ B = \sum_{k=1}^{N} \sum_{i_1 < \cdots < i_N} \sum_{x_1, \ldots, x_N=0}^{M-1} \alpha_{x_1 \cdots x_N}^{i_1 \cdots i_N} M_{x_1}^{(i_1)} \otimes \cdots \otimes M_{x_N}^{(i_N)} \]  

(41)

and minimize Tr (Bρ) subject to ρ ≥ 0 and Tr ρ = 1. This amounts to finding the minimal eigenvalue of B with the corresponding eigenvector |ψ⟩, and the Bell inequality violation \( \tilde{\beta} = -\langle \psi|B|\psi \rangle \).

(iv) Generate the translationally invariant state

\[ q^{TI} = \frac{1}{N} \sum_{i=0}^{N-1} V_i |\psi⟩⟨\psi| V_i^†, \]  

(42)

where \( V_i \) is defined as in equation (26) with \( d \) replaced by \( D \). Notice that \( q^{TI} \) features the same violation of the Bell inequality as \( |ψ⟩ \), i.e., Tr \( (q^{TI} B) = (\langle \psi|B|\psi \rangle) = -\beta \).

(v) Then, for the Bell operator (41) and the state (42) one has

\[ T r \left( B q^{TI} \right) = \sum_{j=0}^{M-1} T r \left( M_j^{(1)} F_j \right), \]  

(43)

where (to get the expression below we have used the fact that the state \( q^{TI} \) is translationally invariant)

\[ F_j = \sum_{k=1}^{N} \sum_{1 \leq i_1 < \cdots < i_N \leq N} \sum_{x_1, \ldots, x_N=0}^{M-1} \alpha_{x_1 \cdots x_N}^{1, i_1 \cdots i_N} \times T r_{A_2 \cdots A_N} \left( 1_{A_1} \otimes M_{x_1}^{(i_1-1)} \otimes \cdots \otimes M_{x_N}^{(i_N-1)} q^{TI} \right). \]  

(44)

Now, minimize the expression

\[ \tilde{\beta} = \sum_{j=0}^{M-1} T r \left( M_j^{(1)} F_j \right) \]  

(45)

over all Hermitian matrices \( M_j^{(1)} \) such that \( -1 \leq M_j^{(1)} \leq 1 \) (\( j = 0, \ldots, M-1 \)). This can easily be done by introducing the eigenvalue decomposition of each \( F_j \),

\[ F_j = \sum_{i} \lambda_{j}^{(i)} |\phi_{i}^{(j)}⟩⟨\phi_{i}^{(j)}| \]  

(46)

Then, one directly finds that the optimal first party’s measurements are given by

\[ M_j^{(1)} = -\sum_{i} sgn \left( \lambda_{j}^{(i)} \right) |\phi_{i}^{(j)}⟩⟨\phi_{i}^{(j)}| \]  

(47)

and the resulting Bell violation \( \tilde{\beta} \geq \beta \) (however, now the measurement settings are no longer symmetric).

(vi) Go back to step (ii) and keep repeating the protocol until convergence of the objective value \( \beta \) is reached.

The above algorithm differs from the one proposed in [36] in that it contains two additional steps (ii) and (iv). The first one is to guarantee that the violation is obtained with the same pair of measurements at each site, while the second one to ensure that the state realizing this violation is translationally invariant. It should be noticed, however, that due to step (ii) it is unclear whether the violations \( \beta \) produced by our version of the algorithm are nondecreasing at each iteration step. This is because after requiring at step (ii) that all parties...
measure the same observables, the violation could in principle drop. However, our numerical studies show that this is not the case and at each iteration step of the algorithm the value of $\beta$ is nondecreasing. We then conjecture that this is always the case.

5.3. Observations

We applied the above procedure to inequalities $\#64 - \#103$ in table 2, i.e., those for which $\beta_Q > \beta_Q^{\text{ini}}$ with four-qubit states, leading to the following observations regarding values of $d_{\text{min}}$:

- $d_{\text{min}} \leq 6$ for each inequality. That is, a six-dimensional Hilbert space at each site is enough to obtain $\beta_Q$ with translationally invariant states in all cases. In fact, three-dimensional component spaces suffice in the majority of cases; the only exceptions are:
- $d_{\text{min}} \leq 4$ for inequalities $\#64, \#65, \#73, \#78, \#81, \#86, \#91, \#99, \#100$,
- $d_{\text{min}} \leq 5$ for inequalities $\#70, \#82$,
- $d_{\text{min}} \leq 6$ for inequalities $\#67, \#68, \#69, \#74, \#75$.

We conjecture that the above bounds are all tight, that is, we can replace the inequality with equality. Moreover, we found that in all the cases real-valued measurements suffice to saturate these bounds.

6. Conclusion

In this work we have explored the question if Bell inequalities involving only one and two-body expectation values are capable of witnessing nonlocality in multipartite quantum states, pursuing the research started in [22]. To simplify our considerations we have considered only those Bell inequalities that obey the translational symmetry. We have found all tight Bell inequalities (facets of the corresponding polytopes of classical correlations) for the three and four-partite cases and grouped them into equivalence classes under certain symmetries. We have then characterized their properties and checked whether they are violated by quantum theory. Noticeably, in some cases the states violating these inequalities have all bipartite subsystems local, meaning that their nonlocality cannot be revealed by any bipartite Bell inequality. We have then provided an example of a five-partite translationally invariant Bell inequality which contains correlators involving only the nearest neighbours and checked that it is powerful enough to detect nonlocality in multipartite states. Finally, we have shown that any translationally invariant Bell inequality with $M$ dichotomic observables per site can always be violated maximally by a translationally invariant state (mixed in general) and the same set of observables per site. We have also discussed an algorithm that finds a maximal quantum violation in the above setting for a fixed local dimension.

Clearly, our studies can be further developed. For instance, one could generalize our results to an arbitrary number of parties. The other possibility is to consider other symmetries than permutational or translational, and also more complicated scenarios, i.e., with more measurements and more outcomes per site, leading in both cases to stronger Bell inequalities. A bit more nontrivial direction is to construct nontrivial $N$-partite Bell inequalities consisting of correlators involving only the nearest neighbours. Such Bell inequalities, being in the spirit of entanglement witnesses constructed from two-body Hamiltonians [38], would facilitate nonlocality detection in many-body systems in which correlations between the nearest neighbours are the dominating ones.
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