A Spectral Theorem for Zeon Matrices

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Abstract
In this paper, a spectral theorem for matrices with (complex) zeon entries is established. In particular, it is shown that when \( A \) is an \( m \times m \) self-adjoint matrix whose characteristic polynomial \( \chi_A(u) \) splits over the zeon algebra \( \mathbb{C}Z_n \), there exist \( m \) spectrally simple eigenvalues \( \lambda_1, \ldots, \lambda_m \) and \( m \) linearly independent normalized zeon eigenvectors \( v_1, \ldots, v_m \) such that
\[
A = \bigoplus_{j=1}^{m} \lambda_j \pi_j,
\]
where \( \pi_j = v_j v_j^\dagger \) is a rank-one projection onto the zeon submodule span\{\( v_j \)\} for \( j = 1, \ldots, m \).

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1 Introduction
Letting \( Z_n \) denote the multiplicative semigroup generated by a collection of commuting null-square variables \( \{\zeta_{(i)} : 1 \leq i \leq n\} \) and identity \( 1 = \zeta_\emptyset \), the resulting \( \mathbb{R} \)-algebra, denoted here by \( \mathfrak{Z}_n \) has appeared in multiple guises over many years. In recent years, \( \mathfrak{Z}_n \) has come to be known as the \( n \)-particle zeon\(^\dagger \) algebra. They naturally arise as commutative subalgebras of fermions.

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\(^\dagger\)The name “zeon algebra” was coined by Feinsilver \( \# \), stressing their relationship to both bosons (commuting generators) and fermions (null-square generators).
Combinatorial properties of zeons have been applied to graph enumeration problems, partition-dependent stochastic integrals, and routing problems in communication networks, as summarized in [11]. More recent combinatorial applications include graph colorings [12] and Boolean satisfiability [1]. Combinatorial identities involving zeons have also been developed in papers by Neto [6, 7, 8, 9].

A permanent trace formula analogous to MacMahon’s Master Theorem was presented and applied by Feinsilver and McSorley in [4], where the connections of zeons with permutation groups acting on sets and the Johnson association scheme were illustrated.

Power series of elementary zeon functions are reduced to zeon polynomials by virtue of the nilpotent properties of zeons [18]. The zeon extension of any analytic function is reduced to a polynomial of degree not exceeding the number of generators in the algebra. Finding zeros of zeon polynomials is of immediate interest for zeon differential calculus based on the power series approach [14].

From a different perspective, differential equations of first and second order in zeon algebras have recently been studied by Mansour and Schork [5]. Combinatorial properties of zeons are prominent throughout their work.

In the current paper, a spectral theorem for matrices with (complex) zeon entries is established. In particular, it is shown that when $A$ is an $m \times m$ self-adjoint matrix whose characteristic polynomial $\chi_A(u)$ splits over the zeon algebra $\mathbb{C}Z_n$, there exist $m$ spectrally simple eigenvalues $\lambda_1, \ldots, \lambda_m$ and $m$ linearly independent normalized zeon eigenvectors $v_1, \ldots, v_m$ such that $A = \bigoplus_{j=1}^m \lambda_j \pi_j$, where $\pi_j = v_j v_j^\dagger$ is a rank-one projection onto the zeon submodule span$\{v_j\}$ for $j = 1, \ldots, m$.

The paper is structured as follows. Terminology and essential properties of the complex zeon algebra $\mathbb{C}Z_n$ are found in Sections 1.1 and 1.2. Viewing the Cartesian product $\mathbb{C}Z_n^m$ of complex zeon algebras as a module, the zeon inner product is defined in Section 2. Zeon matrices are viewed as $\mathbb{C}Z_n$-linear transformations of the module $\mathbb{C}Z_n^m$ in Section 3, where their properties are investigated.

Essential results on complex zeon polynomials are reviewed in Section 4 before turning to zeon eigenvalues, eigenvectors, and the characteristic polynomial of a zeon matrix in Section 5. The zeon spectral theorem is established in Section 6. The paper concludes with a brief summary and discussion of avenues for future work in Section 7.

Examples appearing throughout the paper were computed using Mathe-
matica with the CliffMath [15] package, which is freely available through the Research link on the author’s web page: 
[http://www.siue.edu/~sstaple](http://www.siue.edu/~sstaple)

For a broader perspective of combinatorial and algebraic properties and applications of zeons, the interested reader is further directed to the books [11] and [13]

1.1 Preliminaries

For \( n \in \mathbb{N} \), let \( \mathbb{C}_3^n \) denote the complex abelian algebra generated by the collection \( \{ \zeta(i) : 1 \leq i \leq n \} \) along with the scalar \( 1 = \zeta_\emptyset \) subject to the following multiplication rules:

\[
\zeta(i) \zeta(j) = \zeta(i,j) = \zeta(j) \zeta(i) \quad \text{for } i \neq j, \quad \text{and} \\
\zeta(i)^2 = 0 \quad \text{for } 1 \leq i \leq n.
\]

It is evident that a general element \( u \in \mathbb{C}_3^n \) can be expanded as \( u = \sum_{I \in 2^{[n]}} u_I \zeta_I \), or more simply as \( \sum_I u_I \zeta_I \), where \( I \in 2^{[n]} \) is a subset of the \( n \)-set, \( [n] := \{1,2,\ldots,n\} \), used as a multi-index, \( u_I \in \mathbb{C} \), and \( \zeta_I = \prod_{i \in I} \zeta_i \).

The algebra \( \mathbb{C}_3^n \) is called the \((n\text{-particle})\ complex zeon algebra\footnote{The \( n \)-particle \( (\text{real})\ zeon algebra has been denoted by \( \mathcal{C}l_n^{\text{nil}} \) in a number of papers because it can be constructed as a subalgebra of the Clifford algebra \( \mathcal{C}l_{n,n} \).} \). As a vector space, this \( 2^n \)-dimensional algebra has a canonical basis of basis blades of the form \( \{ \zeta_I : I \subseteq [n] \} \). The null-square property of the generators \( \{ \zeta_j : 1 \leq j \leq n \} \) guarantees that the product of two basis blades satisfies the following:

\[
\zeta_I \zeta_J = \begin{cases} 
\zeta_{I \cup J} & I \cap J = \emptyset, \\
0 & \text{otherwise.}
\end{cases} \tag{1.1}
\]

It should be clear that \( \mathbb{C}_3^n \) is graded. For non-negative integer \( k \), the grade-\( k \) part of element \( u = \sum_I u_I \zeta_I \) is defined as

\[
\langle u \rangle_k = \sum_{\{I : |I|=k\}} u_I \zeta_I.
\]

Given a zeon element \( u = \sum_I u_I \zeta_I \), the complex conjugate of \( u \) is defined by

\[
\overline{u} = \sum_I \overline{u_I} \zeta_I.
\]
The real zeon algebra $\mathfrak{Z}_n$ is the subspace of $\mathbb{C}\mathfrak{Z}_n$ defined by

$$\mathfrak{Z}_n = \{ u \in \mathbb{C}\mathfrak{Z}_n : \overline{u} = u \}.$$

The maximal ideal consisting of nilpotent zeon elements will be denoted by

$$\mathbb{C}\mathfrak{Z}_n^\circ = \{ u \in \mathbb{C}\mathfrak{Z}_n : \mathfrak{C}u = 0 \}.$$

The multiplicative abelian group of invertible zeon elements is denoted by

$$\mathbb{C}\mathfrak{Z}_n^\times = \mathbb{C}\mathfrak{Z}_n \setminus \mathbb{C}\mathfrak{Z}_n^\circ = \{ u \in \mathbb{C}\mathfrak{Z}_n : \mathfrak{C}u \neq 0 \}.$$

For convenience, arbitrary elements of $\mathbb{C}\mathfrak{Z}_n$ will be referred to simply as “zeons.” In what follows, it will be convenient to separate the scalar part of a zeon from the rest of it. To this end, for $z \in \mathbb{C}\mathfrak{Z}_n$ we write $\mathfrak{C}z = \langle z \rangle_0$, the complex (scalar) part of $z$, and $\mathfrak{D}z = z - \mathfrak{C}z$, the dual part of $z$.

**Definition 1.1.** For a zeon $u \neq 0$, it is useful to define the minimal grade of $u$ by

$$\sharp u = \begin{cases} \min \{ k \in \mathbb{N} : \langle \mathfrak{D}u \rangle_k \neq 0 \} & \mathfrak{D}u \neq 0, \\ 0 & u = \mathfrak{C}u. \end{cases}$$

(1.2)

Note that $\sharp u = 0$ if and only if $u$ is a scalar. In this case, $u$ is said to be trivial.

### 1.2 Multiplicative Properties of Zeons

Since $\mathbb{C}\mathfrak{Z}_n$ is an algebra, its elements form a multiplicative semigroup. It is not difficult to establish convenient formulas for expanding products of zeons. As shown in [2], $u \in \mathbb{C}\mathfrak{Z}_n$ is invertible if and only if $\mathfrak{C}u \neq 0$. Moreover, the multiplicative inverse of $u$ is unique. The result is paraphrased without proof in Proposition 1.3 for review.

**Lemma 1.2** (Cancellation). Let $u, v \in \mathbb{C}\mathfrak{Z}_n$. If $uv = 0$, where $u \neq 0$, then $u, v \in \mathbb{C}\mathfrak{Z}_n^\circ$; i.e., $u$ and $v$ are both nilpotent. In particular, if $\mathfrak{C}u \neq 0$ and $uv = 0$, then $v = 0$.

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3 The term “dual” here is motivated by regarding zeons as higher-dimensional dual numbers.
Proof. Suppose \( uv = 0 \) and that \( Cu \neq 0 \). It follows from properties of zeon multiplication that \( \natural(\natural u) = \natural v \). Hence, \( v \) must be a scalar. The rest follows from the fact that \( C(uv) = (Cu)(Cv) \).

Proposition 1.3 (Existence and Uniqueness of Inverses). Let \( u \in \mathbb{C}Z_n \), and let \( \kappa \) denote the index of nilpotency\(^4\) of \( D_u \). It follows that \( u \) is uniquely invertible if and only if \( Cu \neq 0 \), and the inverse is given by

\[
 u^{-1} = \frac{1}{Cu} \sum_{j=0}^{\kappa-1} (-1)^j(Cu)^{-j}(Du)^j.
\]

For \( n \in \mathbb{N} \), \( \mathbb{C}Z_n^\times \) is defined to be the collection of invertible elements in \( \mathbb{C}Z_n \). That is,

\[
 \mathbb{C}Z_n^\times = \{ u \in \mathbb{C}Z_n : Cu \neq 0 \}.
\]

Since \( \mathbb{C}Z_n^\times \) is closed under (commutative) zeon multiplication, has multiplicative identity \( 1 \), and every element has a multiplicative inverse, the invertible zeons form an abelian group under multiplication.

1.3 Complex Zeon Roots: Existence and Recursive Formulations

Invertible zeons have roots of all orders; a recursive algorithm establishes their existence and provides a convenient method for their computation. The material below, taken from [17] was generalized from the original (real) context seen in [2].

Theorem 1.4. Let \( w \in \mathbb{C}Z_n^\times \), and let \( k \in \mathbb{N} \). Then, \( \exists u \in \mathbb{C}Z_n^\times \) such that \( u^k = w \), provided \( Cw \neq 0 \). Further, writing \( w = \varphi + \zeta(n)\psi \), where \( \varphi, \psi \in \mathbb{C}Z_{n-1} \), \( u \) is computed recursively by

\[
 u = w^{1/k} = \varphi^{1/k} + \zeta(n)\frac{1}{k} \varphi^{-(k-1)/k}\psi.
\]

As expected, any invertible complex zeon has \( k \) distinct complex \( k \)th roots.

Corollary 1.5. Let \( \alpha \in \mathbb{C}Z_n^\times \), and let \( k \in \mathbb{N} \). Then, \( \alpha \) has exactly \( k \) distinct \( k \)th roots; i.e., \( \sharp \{ u : u^k = \alpha \} = k \).

\(^4\)In particular, \( \kappa \) is the least positive integer such that \( (Du)^\kappa = 0 \).
Given an invertible zeon $u$ and positive integer $k$, the principal $k$th root of $u$ is defined to be the zeon $k$th root of $u$ whose scalar part is the principal complex root of $Cu \in \mathbb{C}$.

**Remark 1.6.** Following the approach of [18], an analytic complex-valued function $f : \mathbb{C}_n \to \mathbb{C}_n$ may be extended to a zeon-valued function $\varphi : \mathbb{C}_n \to \mathbb{C}_n$ by

$$\varphi(u) = \sum_{k=0}^{n} \frac{f^{(k)}(Cu)}{k!} (Du)^k.$$ 

A monomial function of the form $f(z) = z^t$ $(t \geq 0)$ then becomes

$$\varphi(u) = \sum_{k=0}^{n} \frac{(t)_k}{k!} (Cu)^{t-k}(Du)^k,$$

where $(t)_k = t(t-1) \cdots (t-k+1)$ denotes the falling factorial with $k$ factors. Consequently, one sees that values of $u^t$ exist for all $t \geq 0$.

## 2 The Zeon Inner Product

Let $x, y \in \mathbb{C}_n^m$. Writing $x$ and $y$ as column matrices $y = (y_1, \ldots, y_m)^\top$ and $x = (x_1, \ldots, x_m)^\top$, the zeon inner product of $x$ and $y$ is defined by

$$\langle x, y \rangle = y^\dagger x,$$  \hspace{1cm} (2.1)

where $y^\dagger = \overline{y}^\top$ denotes the complex conjugate transpose of $y$.

Some basic properties are established in the next lemma.

**Lemma 2.1.** Let $x, y, z \in \mathbb{C}_n^m$ and let $\alpha \in \mathbb{C}_n$. Then,

$$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$$ \hspace{1cm} (2.2)

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$ \hspace{1cm} (2.3)

$$\mathcal{C} \langle x, y \rangle = \langle \mathcal{C} x, \mathcal{C} y \rangle$$ \hspace{1cm} (2.4)

$$\mathcal{C} \langle x, x \rangle \geq 0$$ \hspace{1cm} (2.5)

$$\mathcal{C} \langle x, x \rangle = 0 \text{ iff } x \in (\mathbb{C}_n^o)^m.$$ \hspace{1cm} (2.6)

**Proof.** Proof is straightforward. Beginning with (2.2),

$$\langle \alpha x + y, z \rangle = z^\dagger (\alpha x + y) = \alpha z^\dagger x + z^\dagger y = \alpha \langle x, z \rangle + \langle y, z \rangle,$$

where $z^\dagger = \overline{z}^\top$ is the complex conjugate transpose of $z$.
while (2.3) is established as follows:

\[
\langle x, y \rangle = y^\dagger x = \sum_{\ell=1}^{m} \overline{y_\ell} x_\ell = \sum_{\ell=1}^{m} \overline{x_\ell} y_\ell = \langle y, x \rangle.
\]

Turning to (2.4),

\[
\mathcal{C} \langle x, y \rangle = \mathcal{C} \left( \sum_{\ell=1}^{m} \overline{y_\ell} x_\ell \right) = \sum_{\ell=1}^{m} (\mathcal{C} \overline{y_\ell})(\mathcal{C} x_\ell) = \langle \mathcal{C} x, \mathcal{C} y \rangle.
\]

Properties (2.5) and (2.6) follow immediately from (2.4).

The quantity \( \langle x, x \rangle \) is generally not scalar-valued, so it does not define a norm. However, the scalar part of \( \langle x, x \rangle \) defines a seminorm. Since \( \mathcal{C} \langle x, x \rangle \geq 0 \) for all \( x \in \mathbb{C}Z_3^n \), we define the spectral seminorm of \( x \) by

\[
|x|_* = \mathcal{C} \langle x, x \rangle^{1/2}.
\]

**Definition 2.2.** A zeon vector \( v \in \mathbb{C}Z_3^m \) is said to be null if its spectral seminorm is zero, i.e., \( \mathcal{C} \langle v, v \rangle = 0 \), or equivalently, \( \langle v, v \rangle \in \mathbb{C}Z_3^\circ \). A non-null zeon vector \( v \) is said to be normalized if and only if \( \langle v, v \rangle = 1 \).

Given non-null \( x \in \mathbb{C}Z_3^m \), one normalizes \( x \) via the mapping \( x \mapsto \hat{x} \) where

\[
\hat{x} = (\langle x, x \rangle^{-1})^{1/2} x.
\]

It is not difficult to verify that \( \langle \hat{x}, \hat{x} \rangle = 1 \) whenever \( \mathcal{C} \langle x, x \rangle \neq 0 \).

**Example 2.3.** Consider the following elements of \( \mathbb{C}Z_3^3 \):

\[
v_1 = \begin{pmatrix} i + \zeta_{(1)} \\ \zeta_{(2)} - \zeta_{(2,3)} \\ 2 - \zeta_{(1,2,3)} \end{pmatrix}; \quad v_2 = \begin{pmatrix} \zeta_{(1,3)} \\ \zeta_{(2)} \\ \frac{i}{2} \zeta_{(1,3)} \end{pmatrix}.
\]

Direct calculation shows that

\[
\langle v_1, v_1 \rangle = 5 - 4\zeta_{(1,2,3)}
\]

\[
\langle v_1, v_2 \rangle = 0
\]

\[
\langle v_2, v_2 \rangle = 0.
\]
Although $v_2$ is null, $v_1$ can be normalized by mapping 

$$v_1 \mapsto w_1 = \langle v_1, v_1 \rangle^{-1/2} v_1,$$

where 

$$\langle v_1, v_1 \rangle^{-1/2} = \frac{2\zeta_{(1,2,3)}}{\sqrt{5}} \, + \, \frac{1}{\sqrt{5}}.$$

In this case, the normalized vector $w_1$ is

$$w_1 = \left( \begin{array}{c}
\frac{2i\zeta_{(1,2,3)}}{5\sqrt{5}} + \frac{\zeta_{(1)}}{\sqrt{5}} + \frac{i}{\sqrt{5}} \\
\frac{\zeta_{(2)}}{\sqrt{5}} - \frac{\zeta_{(2,3)}}{\sqrt{5}} \\
\frac{2}{\sqrt{5}} - \frac{\zeta_{(1,2,3)}}{5\sqrt{5}}
\end{array} \right).$$

Finally, direct computation verifies that $\langle w_1, w_1 \rangle = 1$.

3 Zeon Matrices as $\mathbb{C}3_n$-Linear Transformations

Given positive integer $m$, the algebra of square zeon matrices $\text{Mat}(m, \mathbb{C}3_n)$ will be regarded as the algebra of linear transformations on the $\mathbb{C}3_n$-module 

$$\mathbb{C}3_n^m = \underbrace{\mathbb{C}3_n \times \cdots \times \mathbb{C}3_n}_{m \text{ times}}$$

$$= \{(\alpha_1, \alpha_2, \ldots, \alpha_m) : \alpha_1, \ldots, \alpha_m \in \mathbb{C}3_n\}.$$

As a complex vector space, $\mathbb{C}3_n^m$ has dimension $\dim_{\mathbb{C}}(\mathbb{C}3_n^m) = m 2^n$. As a $\mathbb{C}3_n$-module, the dimension is $\dim(\mathbb{C}3_n^m) = m$.

Lemma 3.1. Let $X$ be an $r \times c$ matrix having entries in $\mathbb{C}3_n^\circ$, and let $U$ be a $c \times t$ matrix having entries in $\mathbb{C}3_n$. Then, the entries of $XU$ are in $\mathbb{C}3_n^\circ$, and $XU$ is therefore nilpotent.

Proof. Write $X = (x_{jk})$, $U = (u_{jk})$, and $XU = (y_{jk})$. Fixing indices $j, k$ such that $1 \leq j \leq r$ and $1 \leq k \leq t$, one has $y_{jk} = \sum_{\ell=1}^{c} x_{j\ell} u_{\ell k}$. Since $\mathbb{C}3_n^\circ$ is an ideal, it is clear from the properties of zeon multiplication that $x_{j\ell} u_{\ell k} \in \mathbb{C}3_n^\circ$ for all $j, k, \ell$. Hence, the result. \qed
For \( m, n \in \mathbb{N} \), let \( \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) denote the algebra of \( m \times m \) matrices having entries from the \( n \)-particle zeon algebra \( \mathbb{C}\mathcal{Z}_n \). A matrix \( A \in \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) can naturally be written as a sum of the form \( A = A_\varnothing + A \) of a complex-valued matrix \( A_\varnothing = \mathcal{C}A \) and a nilpotent, zeon-valued matrix \( A = \mathcal{D}A \).

Naturally, matrices in \( \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) can be viewed as linear operators on the vector space \( \mathbb{C}\mathcal{Z}_n^m \). As a vector space over \( \mathbb{C} \), the dimension of \( \mathbb{C}\mathcal{Z}_n^m \) is \( m2^n \). In terms of unit basis vectors of \( \mathbb{C}^m \), i.e., \( e_j = \begin{pmatrix} 0, \ldots, 0, 1 \text{ at } j^{\text{th}} \text{ pos.}, 0, \ldots \end{pmatrix} \), the canonical basis of \( \mathbb{C}\mathcal{Z}_n^m \) is defined to be
\[
B = \{ \zeta_I e_j : I \in 2^{[n]}, 1 \leq j \leq m \}.
\]

The mapping \( 1 \mapsto 1, \zeta_I \mapsto 0 \) for all \( I \neq \varnothing \) is denoted by \( \langle \cdot \rangle_0 \). Its kernel is the maximal ideal of nilpotent zeon elements, denoted by \( \mathbb{C}\mathcal{Z}_n^\circ \). The algebras of \( m \times m \) matrices over \( \mathbb{C}\mathcal{Z}_n \) and \( \mathbb{C}\mathcal{Z}_n^\circ \) are then conveniently denoted by \( \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) and \( \text{Mat}(m, \mathbb{C}\mathcal{Z}_n^\circ) \), respectively.

Note that the invertible elements of \( \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) constitute an abelian multiplicative group but do not constitute a submodule, as they are not closed under addition.

Let \( x \in \mathbb{C}\mathcal{Z}_n^m \), let \( A \in \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \), and define \( y \in \mathbb{C}\mathcal{Z}_n^m \) by
\[
y = A x.
\]

It is not difficult to verify that \( A \) is a \( \mathbb{C}\mathcal{Z}_n \)-linear operator on \( \mathbb{C}\mathcal{Z}_n^m \); i.e.,
\[
A(\alpha x_1 + x_2) = \alpha Ax_1 + Ax_2
\]
for \( x_1, x_2 \in \mathbb{C}\mathcal{Z}_n^m \) and \( \alpha \in \mathbb{C}\mathcal{Z}_n \).

The Determinant

The determinant of \( A \in \text{Mat}(m, \mathbb{C}\mathcal{Z}_n) \) is defined in the usual way by
\[
|A| = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{j=1}^m a_{j\sigma(j)},
\]
where \( S_m \) is the symmetric group of order \( m! \) and \( \text{sgn}(\sigma) \) is the signature of the permutation \( \sigma \).
Lemma 3.2. For all $A \in \text{Mat}(m, \mathbb{CZ}_n)$, $\mathcal{C}[A] = |\mathcal{C}A|$.  

Proof. Given arbitrary $u, v \in \mathbb{CZ}_n$, one observes that 
\[
\mathcal{C}(uv) = \mathcal{C}((\mathcal{C}u + \mathcal{D}u)(\mathcal{C}v + \mathcal{D}v)) = \mathcal{C}(\mathcal{C}u\mathcal{C}v + \mathcal{C}u\mathcal{D}v + \mathcal{C}v\mathcal{D}u + \mathcal{D}u\mathcal{D}v) = \mathcal{C}u\mathcal{C}v.
\]
Extending to arbitrary products in the determinant, one obtains 
\[
\mathcal{C}[A] = \mathcal{C}\left(\sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{j=1}^{m} (\mathcal{C}a_{\sigma(j)} + \mathcal{D}a_{\sigma(j)})\right) = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \mathcal{C}\left(\prod_{j=1}^{m} (\mathcal{C}a_{\sigma(j)} + \mathcal{D}a_{\sigma(j)})\right) = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{j=1}^{m} \mathcal{C}a_{\sigma(j)}.
\]

The next result, first established for real zeon matrices in [16], is now generalized to matrices over $\mathbb{CZ}_n$.

Proposition 3.3. The mapping $A \mapsto \mathcal{C}A$ is an algebra homomorphism $\text{Mat}(m, \mathbb{CZ}_n) \to \text{End(}\mathbb{C}^m\text{)}$.

Proof. Recognizing that $\text{End}(\mathbb{C}^m)$ is isomorphic to the algebra of $m \times m$ complex matrices, the result is established by verifying that $\mathcal{C}(A+B) = \mathcal{C}A + \mathcal{C}B$ and that $\mathcal{C}(AB) = (\mathcal{C}A)(\mathcal{C}B)$ for matrices $A, B \in \text{Mat}(m, \mathbb{CZ}_n)$.  

By Proposition 3.3, $\mathcal{C}(A^k) = (\mathcal{C}A)^k$. Hence, the following corollary.

Corollary 3.4. A matrix $A \in \text{Mat}(m, \mathbb{CZ}_n)$ is singular if and only if $\mathcal{C}A$ is singular.

Lemma 3.5 (Properties of the determinant). Let $A$ and $B$ be $m \times m$ matrices over $\mathbb{CZ}_n$, and let $\alpha \in \mathbb{CZ}_n$. Then the following hold:
\[
|AB| = |A||B|,
\]
\[
|\alpha A| = \alpha^m|A|.
\]
In particular, $|A^{-1}| = |A|^{-1}$ when $A$ is invertible.
Proof. The result follows directly from the determinant definition.

The following result was initially established for matrices over \( \mathbb{Z}_n \). Its proof is straightforward, but the interested reader can find details in [16].

Lemma 3.6. Let \( A \in \text{Mat}(m, \mathbb{C}_n) \). Then, \( A \) is nilpotent if and only if \( \mathcal{C}A \) is nilpotent.

The Matrix Inverse

With Lemma 3.1 established, conditions for invertibility of zeon matrices can be discussed. The next proposition was established for real zeon matrices in [15]. The proof for the complex generalization differs only by substituting \( \mathcal{C}A \) for \( \mathcal{R}A \).

Proposition 3.7. Let \( A = (a_{ij}) \) be a square matrix having entries from \( \mathbb{C}_n \), and write \( A = \mathcal{C}A + \mathcal{D}A \), where \( \mathcal{C}A = (\mathcal{C}a_{ij}) \). It follows that \( A \) is invertible if and only if \( \mathcal{C}A \) is invertible. In this case, the inverse is given by

\[
A^{-1} = (\mathcal{C}A)^{-1} \sum_{\ell=0}^{\kappa(\mathcal{D}A(\mathcal{C}A)^{-1})-1} (-1)^\ell (\mathcal{D}A(\mathcal{C}A)^{-1})^\ell.
\]

In light of Lemma 3.2, the next corollary follows immediately.

Corollary 3.8. A matrix \( A \in \text{Mat}(m, \mathbb{C}_n) \) is singular if and only if \( |A| \) is nilpotent.

Zeon Gaussian Elimination

It is easily verified that the following elementary (row) operations have the familiar effects on the matrix determinant. If \( A \mapsto A' \) via an elementary operation, then the effects are as follows:

i. exchanging two rows changes the sign, i.e., \( |A'| = -|A| \);

ii. multiplying a row by an invertible zeon constant \( u \) implies \( |A'| = u|A| \);

iii. adding any zeon multiple of one row to another gives \( |A| = |A'| \).
Much as in the real case, elementary zeon matrices are defined as matrices obtained from the identity matrix by elementary zeon row operations. Moreover, the standard method of computing inverses via Gaussian elimination can be applied, and every zeon matrix can be placed in upper triangular form via a sequence of elementary operations.

Example 3.9. Consider the matrix $A$ given by

$$A = \begin{pmatrix} 2 + \zeta_{(1)} & \zeta_{(2)} & 0 \\ \zeta_{(2)} & 2 - \zeta_{(2)} & 3\zeta_{(1,2,3)} \\ 0 & 3\zeta_{(1,2,3)} & 1 - \zeta_{(1,2,3)} + \zeta_{(1)} \end{pmatrix},$$

along with elementary matrix $E_1$ exchanging rows 1 and 2, and the elementary matrix $E_2$ that multiplies row 3 by the constant $2 + 3\zeta_{(1,2)}$, as given here

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 + 3\zeta_{(1,2)} \end{pmatrix}.$$  

Then,

$$E_1 A = \begin{pmatrix} \zeta_{(2)} & 2 - \zeta_{(2)} & 3\zeta_{(1,2,3)} \\ 2 + \zeta_{(1)} & \zeta_{(2)} & 0 \\ 0 & 3\zeta_{(1,2,3)} & 1 - \zeta_{(1,2,3)} + \zeta_{(1)} \end{pmatrix},$$

and

$$E_2 A = \begin{pmatrix} 2 + \zeta_{(1)} & \zeta_{(2)} & 0 \\ \zeta_{(2)} & 2 - \zeta_{(2)} & 3\zeta_{(1,2,3)} \\ 0 & 6\zeta_{(1,2,3)} & 2 + 3\zeta_{(1,2)} - 2\zeta_{(1,2,3)} + 2\zeta_{(1)} \end{pmatrix}.$$  

Further,

$$|A| = 4 - 3\zeta_{(1,2)} - 4\zeta_{(1,2,3)} + 6\zeta_{(1)} - 2\zeta_{(2)};$$

$$|E_1 A| = -4 + 3\zeta_{(1,2)} + 4\zeta_{(1,2,3)} - 6\zeta_{(1)} + 2\zeta_{(2)} = -|A|;$$

$$|E_2 A| = 8 + 6\zeta_{(1,2)} - 8\zeta_{(1,2,3)} + 12\zeta_{(1)} - 4\zeta_{(2)} = (2 + 3\zeta_{(1,2)})|A|.$$  

The Adjugate and Cramer’s Rule

When $A$ is a square matrix with entries in a commutative ring $R$ (and thus in $\mathbb{C}\zeta_n$), the **adjugate** of $A$ is the transpose of the cofactor matrix of $A$.  

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When $A$ is invertible, the adjugate satisfies $\text{adj}(A) = |A|A^{-1}$ and hence, $\text{adj}(A)A = |A|I$.

Consequently, Cramer’s Rule holds for invertible complex zeon matrices. If $A \in \text{Mat}(m, \mathbb{C}Z_n)$ is invertible and $b = (b_1, \ldots, b_m)\top$ is a column vector with complex zeon entries, then solutions of $Ax = b$ satisfy

$$\text{adj}(A)Ax = |A|x = \text{adj}(A)b,$$

so that $x = \frac{\text{adj}(A)}{|A|}b$.

## 4 Complex Zeon Polynomials

The interested reader is directed to the recent paper [17] for a detailed study of zeon polynomials and their zeros. For purposes of computing eigenvalues and eigenvectors of zeon matrices, only essential results are recalled here.

When the leading coefficient of a zeon polynomial is invertible, polynomial division is possible in the familiar sense. This is formalized as follows.

**Theorem 4.1 (Zeon polynomial division algorithm).** Let $\varphi(u)$ and $\psi(u)$ be polynomials with zeon coefficients such that the leading coefficient of $\psi(u)$ is invertible. Then, there exist unique zeon polynomials $q(u)$ and $r(u)$ such that

$$0 \leq \deg r(u) < \deg \psi(u) \quad \text{and} \quad \varphi(u) = \psi(u)q(u) + r(u). \quad (4.1)$$

The following corollary follows immediately from Theorem 4.1.

**Corollary 4.2 (Zeon remainder theorem).** Let $\varphi(u)$ be a polynomial with zeon coefficients and let $z \in \mathbb{C}Z_n$. Then, $\varphi(z)$ is the remainder after dividing $\varphi(u)$ by the monic polynomial $u - z$.

Generally, the Fundamental Theorem of Algebra does not hold for zeon polynomials. That is, if $\varphi(u)$ is a nonconstant complex zeon polynomial, then $\varphi(u) = 0$ does not necessarily have a complex zeon root. For example, it is straightforward to verify that $\varphi(u) = (u - 1)^2 + \zeta_{(1)}$ has no zeon zeros.

Given a complex zeon polynomial $\varphi(u) = \alpha_m u^m + \cdots + \alpha_1 u + \alpha_0$, a complex polynomial $f_\varphi : \mathbb{C} \to \mathbb{C}$ is induced by

$$f_\varphi(z) = \sum_{\ell=1}^m (\mathbb{C}\alpha_\ell)z^\ell.$$
It follows that

\[ f_\varphi(Cu) = \sum_{\ell=1}^{m}(C\alpha_\ell)(Cu)\ell = C(\varphi(u)), \]

so that \( f_\varphi \circ C = C \circ \varphi. \)

Considering \( \varphi(u) = (u - 1)^2 + \zeta_{(1)} \), one sees that the induced complex polynomial \( f_\varphi(z) = (z - 1)^2 \), which has a zero of multiplicity 2. Thus, \( \varphi \) need not have zeros when \( f_\varphi \) has complex zeros.

On the other hand, it was shown in [2] that every invertible zeon element with positive scalar part has a real square root. Extending to complex zeons, \((u - 1)^2 + (a + b\zeta_{(1)})\) has zeros for all \( a, b \in \mathbb{C} \) where \( a \neq 0 \). In this particular example, one sees that \( u = i\sqrt{a} + \frac{ib}{2\sqrt{a}}\zeta_{(1)} \) is a zero.

When \( f_\varphi(z) \) has a multiple root \( w_0 \in \mathbb{C}, \varphi(u) \) may or may not have a zero \( w \) satisfying \( Cw = w_0 \). For this reason, discussion is now restricted to simple zeros of polynomials with zeon coefficients.

### 4.1 Spectrally Simple Zeros of Complex Zeon Polynomials

Recalled in this section is a useful version of the Fundamental Theorem of Algebra for zeon polynomials. It not only guarantees the existence of a simple zeon zero of \( \varphi(u) \) when the complex polynomial \( f_\varphi(z) \) has a simple complex zero, but it provides an algorithm for computing such a zero.

Letting \( \varphi(u) \) be a nonconstant monic zeon polynomial, we consider \( \lambda \in \mathbb{C}Z_n \) to be a simple zero of \( \varphi \) if \( \varphi(u) = (u - \lambda)g(u) \) for some zeon polynomial \( g \) satisfying \( g(\lambda) \neq 0 \). Recalling that the spectrum of an element \( u \) in a unital algebra is the collection of scalars \( \lambda \) for which \( u - \lambda \) is not invertible, it follows that when \( u \in \mathbb{C}Z_n \), the spectrum of \( u \) is the singleton \( \{\lambda = Cu\} \).

**Definition 4.3.** A simple zero \( \lambda_0 \in \mathbb{C}Z_n \) of \( \varphi(u) \) is said to be a spectrally simple if \( C\lambda_0 \) is a simple zero of the complex polynomial \( f_\varphi(z) \).

With the notion of spectrally simple zeros in hand, the fundamental theorem of algebra for zeon polynomials was presented in [17].
Theorem 4.4 (Fundamental Theorem of Zeon Algebra). Let \( \varphi(u) \) be a monic polynomial of degree \( m \) over \( \mathbb{C}_3^n \), and let \( f_\varphi(z) \) be the complex polynomial induced by \( \varphi \). If \( \lambda_0 \in \mathbb{C} \) is a simple zero of \( f_\varphi(z) \), let \( g \) be the unique complex polynomial satisfying \( f_\varphi(Cu) = (Cu - \lambda_0)g(Cu) \). It follows that \( \varphi(u) \) has a simple zero \( \lambda \) such that \( C\lambda = \lambda_0 \). In particular, for \( 1 \leq k \leq n \), the grade-\( k \) part of \( \lambda \) (denoted \( \lambda_k \)) is given by

\[
\lambda_k = -\frac{1}{g(\lambda_0)} \left\langle \varphi \left( \sum_{j=0}^{k-1} \lambda_j \right) \right\rangle_k.
\]

Moreover, such a zero \( \lambda \) is unique.

Remark 4.5. The requirement that \( \varphi(u) \) be monic can be relaxed by simply requiring the leading coefficient \( \alpha_m \) of \( \varphi \) to be invertible. In that case, the proposition is applied to the monic polynomial \( \alpha_m^{-1}\varphi(u) \).

Corollary 4.6. Let \( \varphi(u) \) be a complex zeon polynomial of degree \( m \geq 1 \). If \( f_\varphi(z) \) is a nonconstant complex polynomial whose zeros are all simple, then \( \varphi(u) \) has exactly \( m \) complex zeon zeros. In this case, we say \( \varphi \) splits over \( \mathbb{C}_3^n \).

Proof. By Theorem 4.4, \( \varphi \) has a zero of the form \( w = w_0 + Dw \), where \( w_0 \neq 0 \) is a zero of the complex polynomial \( f_\varphi(z) \). If the zeros of \( f_\varphi(z) \) are simple, then there exist \( m \) such zeros. \( \square \)

4.1.1 Computing spectrally simple zeon zeros

Presented here is an algorithm for finding a spectrally simple zeon zero \( \lambda \) of polynomial \( \varphi(u) \).

Example 4.7. Consider the following zeon polynomial \( \varphi(u) \) over \( \mathbb{C}_3^4 \):

\[
\varphi(u) = u^4 - 6u^3 + (-\zeta_{1,2} - \zeta_{1,3} - \zeta_{1,4} + 12)u^2 + (2\zeta_{1,2} + 2\zeta_{1,3} + 2\zeta_{1,4} - 10)u + 3 - \zeta_{1,2} - \zeta_{1,3} - \zeta_{1,4}.
\]

The associated complex polynomial is \( f_\varphi(z) = z^4 - 6z^3 + 12z^2 - 10z + 3 \), which has zeros 3 (of multiplicity 1) and 1 (of multiplicity 3). The unique complex zeon zero \( \lambda \) such that \( \lambda_0 = C(\lambda) = 3 \) is

\[
\lambda = 3 + \frac{1}{2} \left( \zeta_{1,2} + \zeta_{1,3} + \zeta_{1,4} \right).
\]
input: Zeon polynomial \( \varphi(u) \) over \( \mathbb{C} \mathcal{Z}_n \) and a simple nonzero root \( \lambda_0 \) of the associated complex polynomial \( \mathcal{C}(\varphi(u)) \).

output: Zeon zero \( \lambda \) of \( \varphi(u) \) with \( \mathcal{C}\lambda = \lambda_0 \).

Initialize complex polynomial \( g(\mathcal{C}u) \).

\[
g(\mathcal{C}u) \leftarrow \frac{\mathcal{C}(\varphi(u)) \mathcal{C}u - \lambda_0}{\mathcal{C}u - \lambda_0};
\]

Note \( g(\mathcal{C}u) \) satisfies \( \mathcal{C}(\varphi(u)) = (\mathcal{C}u - \lambda_0)g(\mathcal{C}u) \), where \( g(\lambda_0) \neq 0 \).

\[
\xi \leftarrow \frac{\varphi(\lambda_0)\xi}{g(\lambda_0)};
\]

\[
\lambda \leftarrow \lambda_0 - \xi;
\]

while \( 0 < \xi \leq n \) do

\[
\xi \leftarrow \varphi(\lambda)\xi/g(\lambda_0);
\]

\[
\lambda \leftarrow (\lambda - \xi);
\]

end

return \( \lambda \);

Algorithm 1: Compute spectrally simple zeon zero.

### Multiple and Non-spectrally Simple Zeon Zeros

Given any multiple or non-spectrally simple zeon zero \( w \) of \( \varphi \), there are infinitely many zeon zeros \( v \) such that \( \mathcal{C}v = \mathcal{C}w \).

**Theorem 4.8.** Let \( \varphi(u) \) be a monic polynomial over \( \mathbb{C} \mathcal{Z}_n \) with zeon zero \( w \in \mathbb{C} \mathcal{Z}_n \) such that \( \mathcal{C}w \) is a multiple zero of \( f_{\varphi} \). Then, \( \varphi \) has infinitely many zeros \( v \) satisfying \( \mathcal{C}v = \mathcal{C}w \).

**Proof.** Suppose \( \varphi \) has a complex zeon zeros \( w = \rho + \mathcal{D}w \) such that \( \varphi(u) = (u - w)g(u) \), where \( g(u) \) is a nonzero zeon polynomial satisfying \( \mathcal{C}g(\rho) = 0 \).

It follows immediately that \( g(\rho + \lambda) \in \mathbb{C} \mathcal{Z}_n^\circ \) is nilpotent for any nilpotent \( \lambda \in \mathbb{C} \mathcal{Z}_n^\circ \). Recalling that \( \zeta_{[n]} = \zeta_{\{1,2,\ldots,n\}} = \zeta_1 \cdots \zeta_n \), it thereby follows that for any nonzero scalar \( a \),

\[
\varphi(w + a\zeta_{[n]}) = (w + a\zeta_{[n]} - w)g(w + a\zeta_{[n]})
\]

\[
= a\zeta_{[n]}g(\rho + \mathcal{D}w + a\zeta_{[n]})
\]

\[
= 0.
\]

\[\square\]
Remark 4.9. It is worth noting that Theorem 4.8 does not characterize all zeon zeros of a polynomial with multiple or non-spectrally simple zeon zeros.

5 Eigenvalues, Eigenvectors, and the Characteristic Polynomial

Given a square matrix $A \in \text{Mat}(m, \mathbb{C}Z^n)$, the characteristic polynomial of $A$ is defined in the usual way as the zeon polynomial $\chi_A(t) = |tI - A|$. 

Definition 5.1. Let $A \in \text{Mat}(m, \mathbb{C}Z^n)$. The zeon eigenvalues of $A$ are defined to be the spectrally simple zeros of the characteristic polynomial $\chi_A(t)$.

The fact that the Cayley-Hamilton Theorem holds for matrices over arbitrary commutative rings is well known and can be deduced from the “usual” formulation for matrices over fields.

Theorem 5.2 (Cayley-Hamilton Theorem). Let $A \in \text{Mat}(m, \mathbb{C}Z^n)$. Then, $A$ satisfies its own characteristic equation; i.e.,

$$\chi_A(A) = 0.$$ 

It is easy to verify that if $\lambda$ is a zeon eigenvalue of $A$, then $C\lambda$ is a complex eigenvalue of $CA$. The next lemma is an immediate consequence.

Lemma 5.3. If $A$ is invertible, then all eigenvalues of $A$ are invertible.

Proof. If $A$ is invertible, then all eigenvalues of $CA$ are nonzero. Hence, the scalar part of any zero of $\chi_{CA}(u)$ must be nonzero. \qed

The following theorem is an immediate corollary of Theorem 4.4.

Theorem 5.4. Let $A \in \text{Mat}(m, \mathbb{C}Z^n)$. If $\chi_{CA}(u)$ has simple complex zero $u_0$, then $A$ has a unique zeon eigenvalue $\lambda$ satisfying $\lambda u_0 = u_0$.

5.1 Zeon eigenvectors

With zeon eigenvalues in hand, attention now turns to eigenvectors. To begin, a few remarks concerning linear independence of vectors in $\mathbb{C}Z^n$ are
in order. A subset \( \{v_1, \ldots, v_k\} \) of \( \mathbb{C}^m_n \) is said to be \textit{linearly independent} if and only if for all coefficients \( \alpha_1, \ldots, \alpha_k \in \mathbb{C}^m_n, \)

\[
\alpha_1 v_1 + \cdots + \alpha_k v_k = 0
\]

implies \( \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0. \)

Note that for any nonzero \( v \in \mathbb{C}^m_n \), the singleton \( \{v\} \) can only be linearly independent if it has at least one invertible component. That is, \( \{v\} \) linearly independent implies \( v \notin (\mathbb{C}^m_n)^\circ \); equivalently, \( \{v\} \) is linearly independent if and only if \( \zeta[n] v \neq 0. \)

**Definition 5.5.** Given a matrix \( A \in \text{Mat}(m, \mathbb{C}^m_n) \), a nonzero \( \xi \in \mathbb{C}^m_n \) is said to be a \textit{zeon eigenvector} of \( A \) if the following are satisfied:

i. there exists a zeon eigenvalue \( \lambda \in \mathbb{C}^m_n \) such that \( A \xi = \lambda \xi \), and

ii. \( \xi \notin (\mathbb{C}^m_n)^\circ \), i.e., at least one component of \( \xi \) is invertible.

Note that the requirement \( \xi \notin (\mathbb{C}^m_n)^\circ \) is necessary for linear independence of zeon eigenvectors.

Next, suppose \( \xi \) is a zeon eigenvector associated with zeon eigenvalue \( \lambda \) and let \( f(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0 \) be an arbitrary zeon polynomial. The zeon matrix evaluation of \( f \) then satisfies

\[
f(A)\xi = (a_k A^k + a_{k-1} A^{k-1} + \cdots + a_1 A + a_0)\xi = (a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_1 \lambda + a_0)\xi = f(\lambda)\xi.
\]

**Theorem 5.6.** Let \( A \in \text{Mat}(m, \mathbb{C}^m_n) \). If \( \lambda \in \mathbb{C}^m_n \) is a spectrally simple zero of \( \chi_A(u) \), then there exists \( \xi \in \mathbb{C}^m_n \setminus (\mathbb{C}^m_n)^\circ \) such that \( A \xi = \lambda \xi. \)

**Proof.** Since \( \lambda \) is a spectrally simple zero of the characteristic polynomial \( \chi_A(u) \), it follows immediately that reducing \( \lambda I - A \) to upper triangular form (via Gaussian elimination), one finds \( m - 1 \) invertible elements along the main diagonal. It follows immediately that the last row consists of all zeros, giving one free variable in a system of equations. It thereby follows that for each value of this free variable, the solutions of the remaining equations can be uniquely determined. Any such resulting vector serves as a basis for the corresponding eigenspace. \( \square \)
Corollary 5.7. Let $A \in \text{Mat}(m, \mathbb{C}\mathfrak{Z}_n)$. If $\lambda$ is a zeon eigenvalue of $A$ associated with zeon eigenvector $\xi$, then $C\lambda$ is an eigenvalue of $CA$ associated with eigenvector $C\xi$.

Proof. Expanding $A\xi = \lambda\xi$ as

$$(CA + D)(C\xi + D\xi) = (C\lambda + D\lambda)(C\xi + D\xi),$$

one finds that

$$CA\xi + DAE\xi + CADE\xi = C\lambda C\xi + D\lambda C\xi + C\lambda D\xi + D\lambda D\xi.$$  

Since $\mathbb{C}\mathfrak{Z}_n^\circ$ is an ideal, it is easy to see that $CA\xi$ and $C\lambda C\xi$ are the only terms of the equation that exist in $\mathbb{C}^m$. All other terms are elements of $(\mathbb{C}\mathfrak{Z}_n^\circ)^m$. Hence, the result: $CA\xi = C\lambda C\xi$.

6 The Zeon Spectral Theorem

In this section, linear independence of eigenvectors associated with distinct zeon eigenvalues is established and utilized to construct rank-one zeon projection operators. These operators are used to construct a resolution of the identity and, ultimately, to establish a spectral theorem for zeon matrices.

Proposition 6.1. Let $A \in \text{Mat}(m, \mathbb{C}\mathfrak{Z}_n)$. Let $v_1, v_2 \in \mathbb{C}\mathfrak{Z}_n^m$ be zeon eigenvectors of $A$ associated with zeon eigenvalues $\lambda_1, \lambda_2$, respectively, where $C\lambda_1 \neq C\lambda_2$. Then, $v_1$ and $v_2$ are linearly independent.

Proof. Suppose, to the contrary, that for some nonzero constants $\alpha_1, \alpha_2 \in \mathbb{C}\mathfrak{Z}_n$, the following holds:

$$0 = \alpha_1 v_1 + \alpha_2 v_2 = (\alpha_1 v_1 + \alpha_2 v_2)\lambda_1.$$  

(6.1)

Further, linearity of $A$ implies

$$0 = A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \lambda_1 v_2 + \alpha_2 \lambda_2 v_2.$$  

(6.2)

Subtracting (6.2) from (6.1) gives

$$0 = \alpha_2(\lambda_2 - \lambda_1)v_2.$$
Note that since $\lambda_1$ and $\lambda_2$ are spectrally simple by Definition 5.1, it follows that $C(\lambda_2 - \lambda_1) \neq 0$. Hence, $\lambda_2 - \lambda_1$ is invertible and not a zero divisor. Hence, $\alpha_2 v_2 = 0$, which further implies $\alpha_1 v_1 = 0$ by (6.1) so that $v_1, v_2 \in (C3_n^\circ)^m$, a contradiction. 

**Remark 6.2.** Note that the only zeon eigenvalue that can be associated with the nullspace of matrix $A \in \text{Mat}(m, C3_n)$ is zero. If $\lambda v = 0$ for nonzero $\lambda$, then $\lambda$ is nilpotent and $v \in (C3_n^\circ)^m$, violating the requirement that an individual eigenvector be linearly independent.

**Remark 6.3.** In light of Theorem 4.8 and Corollary, the zeon eigenvalues of a matrix are restricted to spectrally simple zeros of the matrix’s characteristic polynomial. Suppose $v$ is an eigenvector of $X$ associated with distinct eigenvalues $\lambda_1, \lambda_2$ satisfying $C\lambda_1 = C\lambda_2$. Then, 

$$Xv = \lambda_1 v = \lambda_2 v$$

$$\Rightarrow (\lambda_2 - \lambda_1)v = 0,$$

contradicting linear independence of $v$.

**Resolution of the Identity**

Two vectors $v_1, v_2 \in C3_n^m$ are said to be orthogonal if and only if $\langle v_1, v_2 \rangle = 0$. Note that any collection of zeon vectors $\{v_1, \ldots, v_m\}$ spanning $C3_n^m$ can be orthogonalized via Gaussian elimination. Normalizing then yields an orthonormal basis for $C3_n^m$.

**Lemma 6.4.** Let $v \in C3_n^m$ be a normalized zeon vector. The matrix $vv^\dagger$ represents orthogonal projection onto $\text{span}\{v\}$.

**Proof.** Given normalized $v$, let $\{u_1, \ldots, u_{m-1}\}$ be an orthonormalized collection of zeon vectors orthogonal to $v$. Letting $x \in C3_n^m$ be arbitrary, there exist zeon coefficients $\alpha_0, \ldots, \alpha_{m-1}$ such that

$$x = \alpha_0 v + \alpha_1 u_1 + \cdots + \alpha_{m-1} u_{m-1}.$$ 

It follows that

$$(vv^\dagger)x = vv^\dagger(\alpha_0 v + \alpha_1 u_1 + \cdots + \alpha_{m-1} u_{m-1})$$

$$= v\alpha_0 v^\dagger v + \alpha_1 v^\dagger u_1 + \cdots + \alpha_{m-1} v^\dagger u_{m-1}$$

$$= \alpha_0 \langle v, v \rangle v + \alpha_1 \langle u_1, v \rangle v + \cdots + \alpha_{m-1} \langle u_{m-1}, v \rangle v$$

$$= \alpha_0 v.$$
It follows that when \( \{u_1, \ldots, u_m\} \) is an orthonormalized collection of zeon vectors, a \textit{resolution of the identity} is given by

\[
I = \bigoplus_{j=1}^{m} u_j u_j^\dagger.
\]

A matrix \( A \in \text{Mat}(m, \mathbb{C}Z_n) \) satisfying \( A^\dagger = A \) is clearly self-adjoint w.r.t. the inner product (2.1) because for arbitrary \( x, y \in \mathbb{C}Z_n^m \), the following holds:

\[
\langle Ax, y \rangle = y^\dagger (Ax) = y^\dagger (x^\dagger A^\dagger) = y^\dagger A^\dagger x = (Ax)^\dagger = \langle x, Ay \rangle.
\]

Eigenvalues of self-adjoint zeon matrices are elements of the real zeon algebra \( \mathfrak{Z}_n \); i.e., \( \chi_A(\lambda) = 0 \) implies \( \lambda = \overline{\lambda} \) when \( A \) is self-adjoint. Further, eigenvectors of self-adjoint zeon matrices associated with distinct eigenvalues are orthogonal. Given distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) associated with eigenvectors \( v_1 \) and \( v_2 \), respectively,

\[
\lambda_1 \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle
\]

implies \( \langle v_1, v_2 \rangle = 0 \).

\textbf{Remark 6.5.} Consider zeon eigenvalues \( \lambda \) of \( A \), where \( \mathcal{C}\lambda \) is a simple eigenvalue of \( \mathcal{C}A \). Simple zeros give spectrally simple zeon eigenvalues. An \( m \times m \) matrix with \( m \) distinct eigenvalues is diagonalizable.

\textbf{Theorem 6.6 (Zeon Spectral Theorem).} Let \( A \in \text{Mat}(m, \mathbb{C}Z_n) \) be a self-adjoint zeon matrix with \( m \) spectrally simple eigenvalues. Let \( v_1, \ldots, v_m \) denote normalized zeon eigenvectors associated with these eigenvalues and set \( \pi_j = v_j v_j^\dagger \) for \( j = 1, \ldots, m \). Then,

\[
A = \bigoplus_{j=1}^{m} \lambda_j \pi_j.
\]
Proof. Assuming that $A$ is self-adjoint with spectrally simple eigenvalues, it follows that the corresponding normalized eigenvectors $\{v_1, \ldots, v_m\}$ are orthogonal, resulting in an orthonormal zeon basis of $\mathbb{C}^m$. For arbitrary $z \in \mathbb{C}^m$, it follows that

$$Az = A \sum_{j=1}^{m} \langle z, v_j \rangle v_j$$

$$= \sum_{j=1}^{m} \lambda_j \langle z, v_j \rangle v_j$$

$$= \sum_{j=1}^{m} \lambda_j \pi_j z$$

$$= \left( \bigoplus_{j=1}^{m} \lambda_j \pi_j \right) z.$$

\[\square\]

Example 6.7. Consider the self-adjoint zeon matrix

$$A = \begin{pmatrix}
5 + \zeta_{\{2\}} & 2\zeta_{\{3\}} & -\zeta_{\{1\}} \\
2\zeta_{\{3\}} & 3 + \zeta_{\{1,2\}} & 1 \\
-\zeta_{\{1\}} & 1 & -1
\end{pmatrix}.$$  

The zeon eigenvalues of $A$ are

$$\lambda_1 = 5 - 0.363636\zeta_{\{1,3\}} + 0.264463\zeta_{\{1,2,3\}} + 1.\zeta_{\{2\}}$$

$$\lambda_2 = 3.23607 + 0.947214\zeta_{\{1,2\}} + 0.507064\zeta_{\{1,3\}} - 0.287463\zeta_{\{1,2,3\}}$$

$$\lambda_3 = -1.23607 + 0.0527864\zeta_{\{1,2\}} - 0.143428\zeta_{\{1,3\}} + 0.0229998\zeta_{\{1,2,3\}}.$$  

Corresponding (non-normalized) zeon eigenvectors are

$$v_1 = \begin{pmatrix}
-0.8999999 \zeta_{\{1\}} + 1.09091 \zeta_{\{3\}} - 0.0661157 \zeta_{\{1,2\}} - 0.61157 \zeta_{\{2,3\}} + 0.595041 \zeta_{\{1,2,3\}} \\
-0.181818 \zeta_{\{1\}} + 0.181818 \zeta_{\{3\}} - 0.0413223 \zeta_{\{1,2\}} + 0.132231 \zeta_{\{2,3\}} + 0.0991736 \zeta_{\{1,2,3\}} \\
0.566915 \zeta_{\{1\}} - 4.80298 \zeta_{\{3\}} - 0.321393 \zeta_{\{1,2\}} + 2.72288 \zeta_{\{2,3\}} - 3.65313 \zeta_{\{1,2,3\}} \\
4.23607 + 0.947214 \zeta_{\{1,2\}} - 4.29592 \zeta_{\{1,3\}} + 2.43542 \zeta_{\{1,2,3\}} \\
-0.236068 + 0.0527864 \zeta_{\{1,2\}} - 0.0229998 \zeta_{\{1,3\}} + 0.010859 \zeta_{\{1,2,3\}}
\end{pmatrix}.$$  

$$v_2 = \begin{pmatrix}
0.160357 \zeta_{\{1\}} + 0.0757105 \zeta_{\{3\}} - 0.0257145 \zeta_{\{1,2\}} - 0.0121407 \zeta_{\{2,3\}} - 0.0162885 \zeta_{\{1,2,3\}} \\
0.181818 \zeta_{\{1\}} + 0.181818 \zeta_{\{3\}} - 0.0413223 \zeta_{\{1,2\}} + 0.132231 \zeta_{\{2,3\}} + 0.0991736 \zeta_{\{1,2,3\}} \\
0.198815 \zeta_{\{1\}} - 0.480298 \zeta_{\{3\}} - 0.321393 \zeta_{\{1,2\}} + 2.72288 \zeta_{\{2,3\}} - 3.65313 \zeta_{\{1,2,3\}} \\
4.23607 + 0.947214 \zeta_{\{1,2\}} - 4.29592 \zeta_{\{1,3\}} + 2.43542 \zeta_{\{1,2,3\}} \\
-0.236068 + 0.0527864 \zeta_{\{1,2\}} - 0.0229998 \zeta_{\{1,3\}} + 0.010859 \zeta_{\{1,2,3\}}
\end{pmatrix}.$$  

$$v_3 = \begin{pmatrix}
0.160357 \zeta_{\{1\}} + 0.0757105 \zeta_{\{3\}} - 0.0257145 \zeta_{\{1,2\}} - 0.0121407 \zeta_{\{2,3\}} - 0.0162885 \zeta_{\{1,2,3\}} \\
0.181818 \zeta_{\{1\}} + 0.181818 \zeta_{\{3\}} - 0.0413223 \zeta_{\{1,2\}} + 0.132231 \zeta_{\{2,3\}} + 0.0991736 \zeta_{\{1,2,3\}} \\
0.198815 \zeta_{\{1\}} - 0.480298 \zeta_{\{3\}} - 0.321393 \zeta_{\{1,2\}} + 2.72288 \zeta_{\{2,3\}} - 3.65313 \zeta_{\{1,2,3\}} \\
4.23607 + 0.947214 \zeta_{\{1,2\}} - 4.29592 \zeta_{\{1,3\}} + 2.43542 \zeta_{\{1,2,3\}} \\
-0.236068 + 0.0527864 \zeta_{\{1,2\}} - 0.0229998 \zeta_{\{1,3\}} + 0.010859 \zeta_{\{1,2,3\}}
\end{pmatrix}.$$
Normalizing, the orthogonal zeon projections are determined by setting
\[ \pi_j = \langle v_j, v_j \rangle^{-1} v_j v_j^\dagger \quad (j = 1, 2, 3). \]

The resulting matrices are large. For exposition, the first one is presented here as columns \( \pi_1 = (p_1 \mid p_2 \mid p_3) \), where

\[
\begin{align*}
p_1 & = 1.049463 \zeta_{\{1,3\}} + 0.318557 \zeta_{\{1,2,3\}} - 0.0909091 \zeta_{\{1\}} + 1.09091 \zeta_{\{3\}} + 0.0661157 \zeta_{\{1,2\}} - 0.61157 \zeta_{\{2,3\}} + 0.595041 \zeta_{\{1,2,3\}} - 0.198347 \zeta_{\{1,3\}} + 0.255447 \zeta_{\{1,2,3\}} - 0.214876 \zeta_{\{1,3\}} + 0.180316 \zeta_{\{1,2,3\}} - 0.0661157 \zeta_{\{1,3\}} + 0.0631104 \zeta_{\{1,2,3\}} \\
p_2 & = -0.0909091 \zeta_{\{1\}} + 1.09091 \zeta_{\{3\}} + 0.0661157 \zeta_{\{1,2\}} - 0.61157 \zeta_{\{2,3\}} + 0.595041 \zeta_{\{1,2,3\}} - 0.198347 \zeta_{\{1,3\}} + 0.255447 \zeta_{\{1,2,3\}} - 0.214876 \zeta_{\{1,3\}} + 0.180316 \zeta_{\{1,2,3\}} - 0.0661157 \zeta_{\{1,3\}} + 0.0631104 \zeta_{\{1,2,3\}} \\
p_3 & = -0.181818 \zeta_{\{1\}} + 0.181818 \zeta_{\{3\}} + 0.0413223 \zeta_{\{1,2\}} - 0.132231 \zeta_{\{2,3\}} + 0.0991736 \zeta_{\{1,2,3\}} - 0.214876 \zeta_{\{1,3\}} + 0.180316 \zeta_{\{1,2,3\}} - 0.0661157 \zeta_{\{1,3\}} + 0.0631104 \zeta_{\{1,2,3\}}.
\end{align*}
\]

Using Mathematica, the following properties are quickly verified:

i. \( \pi_j^2 = \pi_j \) for \( j = 1, 2, 3 \) (idempotent);

ii. \( \pi_j \pi_k = \pi_k \pi_j = 0 \) for \( j \neq k \) (orthogonality);

iii. \( \pi_1 + \pi_2 + \pi_3 = I_3 \) (resolution of the identity); and

iv. \( A = \lambda_1 \pi_1 + \lambda_2 \pi_2 + \lambda_3 \pi_3 \) (spectral decomposition).

7 Conclusion and Avenues for Further Research

Viewing a matrix \( M \in \text{Mat}(m, \mathbb{C}3_n) \) as a linear operator on the \( \mathbb{C}3_n \)-module \( \mathbb{C}3_n^m \), zeon eigenvalues of \( M \) have been defined as the spectrally simple zeon zeros of the characteristic polynomial of \( M \). Further, essential notions of zeon eigenvectors and linear independence have been established and used to define orthogonal projections and a resolution of the identity.

Finally, it has been shown that given a self-adjoint zeon matrix \( A \in \text{Mat}(m, \mathbb{C}3_n) \) having \( m \) spectrally simple eigenvalues \( \lambda_1, \ldots, \lambda_m \) associated with normalized zeon eigenvectors \( v_1, \ldots, v_m \), the following holds:

\[
A = \bigoplus_{j=1}^m \lambda_j \pi_j.
\]
where $\pi_j = v_j v_j^\dagger$ is an orthogonal projection onto $\text{span}\{v_j\}$ for $j = 1, \ldots, m$.

Looking toward future work, a number of promising avenues become apparent. Two are briefly described here.

**Spectral properties of the zeon combinatorial Laplacian**

It was shown in [16] that the zeon combinatorial Laplacian of a simple graph with no isolated vertices is always invertible and that the entries of the inverse matrix represent enumeration of paths in the graph. Spectral properties of the zeon combinatorial Laplacian are therefore of particular interest due to their potential combinatorial applications in graph theory and computer science.

**Putzer’s method for evaluating matrix exponentials**

Putzer’s method for analytically evaluating matrix exponentials uses only eigenvalues and components of the solution of a relatively simple linear system. The method is characterized by the following theorem [10].

**Theorem 7.1 (Putzer).** Let $A$ be a square matrix of order $n$, and let $\lambda_1, \ldots, \lambda_n$ be the (not necessarily distinct) eigenvalues of $A$ in some arbitrary but specified order. Then

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j,$$

(7.1)

where $P_0 = I$ and $P_j$ is defined for $j = 2, \ldots, n$ by

$$P_j = \prod_{k=1}^{j} (A - \lambda_k I).$$

(7.2)

Here, $r_1(t), \ldots, r_n(t)$ is the solution of the triangular system

$$\dot{r}_1 = \lambda_1 r_1$$
$$\dot{r}_j = r_{j-1} + \lambda_j r_j \quad (j = 2, \ldots, n)$$

with initial values $r_1(0) = 1$ and $r_j(0) = 0$ for $j = 2, \ldots, n$.

It is not obvious that Putzer’s method extends to the case of zeon matrices, but it is worth noting that the number of terms contained in Putzer’s expansion corresponds to the degree of the minimal polynomial of the associated matrix, as opposed to an infinite series expansion.
Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of Interest

The author declares that he has no conflict of interest or competing interests.

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