THE SCREW LINE OF THE RIEMANN ZETA-FUNCTION
AND ITS APPLICATIONS

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Abstract. We investigate the screw line corresponding to the screw function associated with the Riemann zeta-function under the Riemann hypothesis and derive three necessary and sufficient conditions for the Riemann hypothesis as applications. One of them explains the non-negativity of the Weil distribution by means of the norm.

1. Introduction

Let $g(t)$ be the even real-valued function on the real line defined by

$$g(t) := -4(e^{t/2} + e^{-t/2} - 2) - \frac{t}{2} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} \right) - \log \pi \right]$$

$$- \frac{1}{4} \left( \Phi(1, 2, 1/4) - e^{-t/2} \Phi(e^{-2t}, 2, 1/4) \right) + \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} (t - \log n)$$

(1.1)

for non-negative $t$, where $\Lambda(n)$ is the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^k$ with $k \in \mathbb{Z}_{>0}$ and $\Lambda(n) = 0$ otherwise, $\Gamma(s)$ is the gamma function, and $\Phi(z, s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} z^n$ is the Hurwitz–Lerch zeta-function.

We assume that the Riemann hypothesis (RH, for short) is true, that is, all nontrivial zeros of the Riemann zeta-function $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$. Then, $-g(t)$ is non-negative for all $t \in \mathbb{R}$ and vice versa ([4, Theorem 1.7]). This non-negativity can be understood by the Weil distribution ([6, Section 3.4]). In this paper, we demonstrate that the non-negativity can be explained by the norm of a Hilbert space, building on the same theoretical background used in [8] which explained the non-negativity of Li coefficients in terms of norms.

As stated in [6, Theorem 1.2], the even function $g(t)$ is a screw function on the real line in the sense of [4] if the RH is true, that is, the kernel defined by

$$G_g(t, u) := g(t - u) - g(t) - g(u) + g(0)$$

(1.2)

is non-negative definite on the real line. If $g(t)$ is a screw function, then there exists a Hilbert space $\mathcal{H}$ and a continuous mapping $t \mapsto x(t)$ from $\mathbb{R}$ into $\mathcal{H}$ such that $\langle x(t + v) - x(v), x(u + v) - x(v) \rangle_\mathcal{H}$ is independent of $v \in \mathbb{R}$ for all $t, u \in \mathbb{R}$ and the equality $\langle x(t) - x(0), x(u) - x(0) \rangle_\mathcal{H} = G_g(t, u)$ holds. Therefore, $\|x(t) - x(0)\|_\mathcal{H}^2 = -2g(t)$ if we note $g(0) = 0$. A mapping $x: \mathbb{R} \to \mathcal{H}$ endowed with the translation-invariance described above is called a screw line.

One of the screw lines corresponding to $g(t)$ can be constructed as follows. In general, an even real-valued function $\tilde{g}(t)$ on $\mathbb{R}$ with $\tilde{g}(0) = 0$ is a screw function if and only if it admits a representation

$$\tilde{g}(t) = \int_{-\infty}^{\infty} \frac{\cos(\gamma t) - 1}{\gamma^2} d\tilde{\tau}(\gamma)$$

with a non-negative measure $\tilde{\tau}$ such that $\int_{-\infty}^{\infty} d\tilde{\tau}(\gamma)/(1 + \gamma^2) < \infty$. Hence, there exists a non-negative measure $\tau$ representing $g(t)$ as above under the RH. Then the Hilbert
space $\mathcal{H} = L^2(\tau)$ and the mapping $t \mapsto x(t) := (e^{it\gamma} - 1)/\gamma$ provide a screw line satisfying $\|x(t) - x(0)\|^2_\mathcal{H} = -2g(t)$ ([12] §12). This spectral construction for a screw line is important and useful in analysis, but it is not very useful for studying the nontrivial zeros of $\zeta(s)$ without assuming the RH. Therefore, we provide a construction of a screw line that is at least superficially independent of the measure $\tau$, and apply it to the RH.

Let $L^2(\mathbb{R})$ be the usual $L^2$-space on the real line with respect to the Lebesgue measure. Let $\xi(s) = 2^{-1}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the Riemann xi-function. The nontrivial zeros of $\zeta(s)$ coincide with the zeros of $\xi(s)$ with multiplicity. We define

$$E(z) := \xi(1/2 - iz) + \xi'(1/2 - iz),$$

$$\Theta(z) := \frac{E(z)}{E(\bar{z})},$$

and

$$\mathfrak{S}_t(z) := \frac{i(1 + \Theta(z))}{2\sqrt{\pi}} \mathfrak{P}_t(z)$$

with

$$\mathfrak{P}_t(z) := \frac{4(e^{t/2} - 1)}{1 - 2it} + \frac{4(e^{-t/2} - 1)}{1 + 2it} - \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} \frac{e^{iz(t - \log n)} - 1}{iz}$$

$$+ \frac{e^{iz} - 1}{iz} \left[ \frac{\Gamma'((1/2 - iz))}{\Gamma((1/2 - iz))} - \frac{1}{2} \log \pi + \frac{1}{2} \Gamma' \left( \frac{1}{4} + \frac{iz}{2} \right) \right]$$

$$- \frac{1}{2iz} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iz}{2} \right) - \Gamma' \left( \frac{1}{4} \right) \right]$$

$$- \frac{1}{2iz} e^{-t/2} \left[ \Phi(e^{-2it}, 1, 1/4) - \Phi(e^{-2it}, 1, 1/2 + iz) \right]$$

for a non-negative real number $t$ and a complex number $z$, where $Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. For negative $t$, we define $\mathfrak{S}_t(z) := \mathfrak{S}_{-t}(z)$. For this $\mathfrak{S}_t$, we first obtain the following.

**Proposition 1.1.** For any fixed $t \in \mathbb{R}$, $\mathfrak{S}_t(z)$ belongs to $L^2(\mathbb{R})$ as a function of $z$.

From this result, the mapping $t \mapsto \mathfrak{S}_t(z)$ from $\mathbb{R}$ to $L^2(\mathbb{R})$ is defined. For this mapping, the following holds under the RH.

**Theorem 1.1.** Assuming that the RH is true, the mapping $t \mapsto \mathfrak{S}_t(z)$ from $\mathbb{R}$ to $L^2(\mathbb{R})$ is a screw line corresponding to $g(t)$.

The following immediately follows from Theorem 1.1.

**Corollary 1.1.** The RH is true if and only if the equality

$$\|\mathfrak{S}_t\|^2_{L^2(\mathbb{R})} = -2g(t)$$

(1.7)

holds for all $t \geq t_0$ for some $t_0 \geq 0$.

In [6] Theorem 1.7, it has been proven that the non-negativity of $-g(t)$ is equivalent to the RH being true. Corollary 1.1 explains this non-negativity through a set of equalities involving norms. On the other hand, a screw function defines a non-negative definite hermitian form. As a variant of Corollary 1.1 the non-negativity of the hermitian form associated with $g(t)$ is also explained through a set of equalities involving norms.

The kernel ([1.2]) defines a hermitian form on the space $C^\infty_c(\mathbb{R})$ of all smooth and compactly supported function on $\mathbb{R}$ by

$$\langle \phi_1, \phi_2 \rangle_{G_g} := \int_{-\infty}^\infty \int_{-\infty}^\infty G_g(t, u)\phi_1(u)\overline{\phi_2(t)} dt du$$

for $\phi_1, \phi_2 \in C^\infty_c(\mathbb{R})$. (1.8)

This hermitian form is non-negative definite if $g(t)$ is a screw function ([12] §5), that is, if the RH is true. By the uniformity of the $L^2$-norm of $\mathfrak{S}_t(z)$ on a compact set of $t$ obtained in the proof of Proposition 1.1 and Minkowski’s integral inequality, we obtain the following.
Proposition 1.2. For \( \phi \in C_c^\infty(\mathbb{R}) \), we define
\[
\hat{\mathcal{P}}_\phi(z) := \int_{-\infty}^\infty \mathcal{G}_t(z) \phi(t) \, dt
\]
using (1.3). Then \( \hat{\mathcal{P}}_\phi(z) \) belongs to \( L^2(\mathbb{R}) \).

Based on this proposition, the following holds.

Theorem 1.2. The RH is true if and only if the equality
\[
\| \hat{\mathcal{P}}_\phi \|_{L^2(\mathbb{R})}^2 = \langle \phi, \phi \rangle_{G_\phi}
\]
holds for all \( \phi \in C_c^\infty(\mathbb{R}) \) satisfying \( \int_{-\infty}^\infty \phi(t) \, dt = 0 \). If the RH is true, equality (1.9) holds for all \( \phi \in C_c^\infty(\mathbb{R}) \).

The advantage of Corollary 1.1 and Theorem 1.2 is that it has turned the criterion of the RH from a set of inequalities like Weil’s criterion into a set of equalities. Theorem 1.2 can be reformulated as follows.

We denote the set of all zeros of \( \xi(1/2 - iz) \) without counting multiplicity by \( \Gamma \) and the multiplicity of \( \gamma \in \Gamma \) by \( m_\gamma \). Weil’s hermitian form on \( C_c^\infty(\mathbb{R}) \) is defined by
\[
\langle \psi_1, \psi_2 \rangle_W = \sum_{\gamma \in \Gamma} m_\gamma \int_{-\infty}^\infty \psi_1(t) e^{it\gamma t} \, dt \int_{-\infty}^\infty \psi_2(u) e^{iu\gamma u} \, du
\]
for \( \psi_1, \psi_2 \in C_c^\infty(\mathbb{R}) \). The relation
\[
\langle D\psi_1, D\psi_2 \rangle_{G_\phi} = \langle \psi_1, \psi_2 \rangle_W, \quad (D\psi)(t) := \psi'(t)
\]
of the hermitian forms in [6] Proposition 3.1 immediately leads to the following equivalence condition for the RH via Weil’s criterion, since the differential operator \( D \) gives a bijection from \( C_c^\infty(\mathbb{R}) \) to the subspace of \( C_c^\infty(\mathbb{R}) \) consisting of \( \phi \) with \( \int_{-\infty}^\infty \phi(t) \, dt = 0 \).

Corollary 1.2. The RH is true if and only if the equality
\[
\| \hat{\mathcal{P}}_{D\psi} \|_{L^2(\mathbb{R})}^2 = \langle \psi, \psi \rangle_W
\]
holds for all \( \psi \in C_c^\infty(\mathbb{R}) \).

A similar result was obtained in [7] Theorem 1.3], but Corollary 1.2 is simpler and more efficient as a statement.

In the following, we first prove one proposition (Proposition 2.1) used in the proof of Proposition 1.2 and Theorem 1.1 in Section 2. Then, we prove Proposition 1.1 in the same section. After that, we prove Theorems 1.1 and 1.2 after preparing a result (Proposition 2.1) on the theory of model spaces in Section 3. Finally, we mention two special values of \( \mathcal{G}_t(z) \) in Section 4. The strategy of the proof of Theorem 1.1 is similar to [8], however the computational details change. In [8], the analytic or geometric meaning of the functions that give the norms is unknown, but in this paper the functions that give the norms have the meaning as a screw line. Furthermore, as an advantage of using the screw line \( \mathcal{G}_t \), we obtain Theorem 1.2 whose analogue was not obtained in [8].

2. UNCONDITIONAL RESULTS FOR \( \mathcal{G}_t \)

2.1. Expansion of \( \mathcal{G}_t(z) \) over the zeros. For the basic properties of the Riemann zeta-function, we refer to [9]. Let \( \Gamma \) be the set of all zeros of \( \xi(1/2 - iz) \) without counting multiplicity. By two functional equations \( \xi(s) = \xi(1-s) \) and \( \xi(s) = \xi^*(s) \), if \( \gamma \in \Gamma \), then both \( -\gamma \) and \( \overline{\gamma} \) belong to \( \Gamma \) with the same multiplicity. Also, \( |\Im(\gamma)| < 1/2 \) for every \( \gamma \in \Gamma \), since all zeros of \( \xi(s) \) lie in the strip \( 0 < \Re(s) < 1 \). The RH is equivalent to all \( \gamma \in \Gamma \) are real. For \( E(z) \) of (1.4), we define
\[
A(z) = (E(z) + \overline{E(z)})/2.
\]
Then $A(z) = \xi(1/2 - iz)$, because $E(z) = \xi(1/2 - iz) - \xi'(1/2 - iz)$ by functional equations of $\xi(s)$. Therefore, $\Gamma$ coincides with the set of all zeros of both $A(z)$ and $1 + \Theta(z)$. We define

$$P_1(z) := \sum_{\gamma \in \Gamma} m_\gamma \frac{e^{iz\gamma} - 1}{\gamma} \cdot \frac{1}{z - \gamma} \quad (2.2)$$

for $t \in \mathbb{R}_{\geq 0}$, where $m_\gamma$ is the multiplicity of $\gamma \in \Gamma$. For negative $t$, we set $P_t(z) := P_{-t}(z)$. The series on the right-hand side (2.2) converges absolutely and uniformly on every compact subset of $\mathbb{C} \setminus \Gamma$, since $\sum_{\gamma \in \Gamma} m_\gamma |\gamma|^{-1-\delta} < \infty$ for any $\delta > 0$, because $A(z)$ is an entire function of order one. Therefore, $P_t(z)$ is a meromorphic function on $\mathbb{C}$ with $\Gamma$ as the set of all poles.

**Proposition 2.1.** Let $\Psi_t(z)$ and $P_t(z)$ be meromorphic functions defined by (1.6) and (2.2), respectively. Then, both coincide.

**Proof.** For $t \geq 0$ and $z \in \mathbb{C}$ with $\Im(z) > 0$, we define

$$\phi_{z,t}(x) = (iz)^{-1} e^{ixz} (e^{izt} - e^{-iz \min(0,x)}) 1_{[-t, \infty)}(x),$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. The main tool for the proof is Weil’s explicit formula

$$\lim_{X \to \infty} \sum_{\gamma \in \Gamma, \Im(\gamma) \leq X} m_\gamma \int_{-\infty}^{\infty} \phi(x) e^{-ix\gamma} \, dx =\int_{-\infty}^{\infty} \phi(x)(e^{x/2} + e^{-x/2}) \, dx - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(\log n) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(-\log n)
$$

$$- (\log 4\pi + \gamma_0) \phi(0) - \int_{0}^{\infty} \{ \phi(x) + \phi(-x) - 2e^{-x^2/2} \phi(0) \} \frac{e^{x/2} \, dx}{e^x - e^{-x}},$$

which is obtained from the explicit formula in [1, p. 186] by taking $\phi(t) = e^{t/2} f(e^t)$ for test functions $f(x)$ in that formula with the conditions for $f(x)$ in [2, Section 3], where $\gamma_0$ is the Euler–Mascheroni constant. (Note that the formula in [2] has two typographical errors in the second line of the right-hand side.)

As is easily seen, Weil’s explicit formula can be applied to $\phi(x) = \phi_{z,t}(x)$. We have

$$\int_{-\infty}^{\infty} \phi_{z,t}(x) e^{-ix\gamma} \, dx = \frac{e^{izt} - 1}{\gamma} \cdot \frac{1}{z - \gamma} \quad \text{when } \Im(z) > \Im(\gamma).$$

Therefore, the left-hand side of Weil’s explicit formula for $\phi_{z,t}(x)$ gives $P_t(z)$ of (2.2) when $\Im(z) > 1/2$. Hence, if it is shown that the right-hand side is equal to $\Psi_t(z)$ for $\Im(z) > 1/2$, then the conclusion of the proposition follows by analytic continuation.

It is easy to verify

$$\int_{-\infty}^{\infty} \phi_{z,t}(x)(e^{x/2} + e^{-x/2}) \, dx = \frac{4(e^{t/2} - 1)}{1 - 2iz} + \frac{4(e^{-t/2} - 1)}{1 + 2iz}$$

and

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi_{z,t}(\log n) = \frac{e^{izt} - 1}{iz} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2 + iz}} = -\frac{e^{izt} - 1}{iz} \zeta\left( 1/2 - iz \right),$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi_{z,t}(-\log n) = \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} e^{iz(t - \log n) - 1} \cdot \frac{1}{iz}$$

for $\Im(z) > 1/2$ by direct calculation.
Therefore, the remaining task is to calculate the fifth term on the right-hand side. We split it into $\int_t^\infty$ and $\int_0^t$ and calculate each integral. For the first part,

$$\int_t^\infty \left\{ \phi_{z, t}(x) + \phi_{z, t}(-x) - 2e^{-x^2/2}\phi_{z, t}(0) \right\} \frac{e^{x^2/2}dx}{e^x - e^{-x}}$$

$$= e^{izt} - \frac{1}{iz} \int_t^\infty (e^{izx} - 2e^{-x^2/2}) \frac{e^{x^2/2}dx}{e^x - e^{-x}}$$

$$= e^{izt} - \frac{1}{iz} \int_t^\infty e^{izx} - 2e^{-x^2/2} \sum_{n=0}^{\infty} e^{-2nx} dx - \frac{1}{iz} \log e^t + 1$$

$$= e^{izt} - \frac{1}{iz} \left[ \frac{1}{2} e^{-t(z/2 + iz)} - \sum_{n=0}^{\infty} e^{-2nt} \log \coth(e^{t/2}) \right]$$

$$= e^{izt} - \frac{1}{iz} \left[ \frac{1}{2} e^{-t(z/2 + iz)} \Phi(-2t, 1, \frac{1}{2}(1 + iz)) \right].$$

For the second part,

$$\int_0^t \left\{ \phi_{z, t}(x) + \phi_{z, t}(-x) - 2e^{-x^2/2}\phi_{z, t}(0) \right\} \frac{e^{x^2/2}dx}{e^x - e^{-x}}$$

$$= \frac{1}{iz} \int_0^t \left\{ (e^{iz(t-x)} - 1) + (e^{izx} - 2e^{-x^2/2})(e^{izt} - 1) \right\} e^{-x^2/2} \sum_{n=0}^{\infty} e^{-2nx} dx.$$

To handle the first half of this right-hand side, we calculate as

$$\int_0^t (e^{iz(t-x)} - 1) e^{-x^2/2} \sum_{n=0}^{N} e^{-2nx} dx$$

$$= \frac{1}{2} \sum_{n=0}^{N} e^{itz} - e^{-t/2}e^{-2nt} - 1 \sum_{n=0}^{N} 1 - e^{-t/2}e^{-2nt} \frac{1}{n + \frac{1}{2} + iz}$$

$$= \frac{1}{2} e^{itz} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2}(1 + iz)} - \frac{1}{2} e^{-t/2} \Phi(-2t, 1, \frac{1}{2}(1 + iz))$$

$$- \frac{1}{2} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2}} + \frac{1}{2} e^{-t/2} \Phi(-2t, 1, 1/4) + O(e^{-2Nt}),$$

where the implied constant depends on $t$ and $z$. To handle the second half of the right-hand side, we calculate as

$$\int_0^t (e^{izx} - 2e^{-x^2/2}) e^{-x^2/2} \sum_{n=0}^{N} e^{-2nx} dx$$

$$= \frac{1}{2} \sum_{n=0}^{N} 1 - e^{-t(1/2 + iz)} e^{-2nt} - \sum_{n=0}^{N} 1 - e^{-t} e^{-2nt}$$

$$= \frac{1}{2} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2}(1 + iz)} - \frac{1}{2} e^{-t(1/2 + iz)} \Phi(-2t, 1, \frac{1}{2}(1 + iz))$$

$$- \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2}} + \log \coth(e^{t/2}) + O(e^{-2Nt}),$$

where we used the series expansion of $\arctanh(e^{-t}) = 2^{-1} \log \coth(e^{t/2})$ and the implied constant depends on $t$ and $z.$
By the above preliminary calculations and the well-known series expansion
\[ \frac{\Gamma'(w)}{\Gamma(w)} = -\gamma_0 - \sum_{n=0}^{\infty} \left( \frac{1}{w+n} - \frac{1}{n+1} \right), \quad (2.3) \]
we obtain
\[
iz \int_0^t \left\{ \phi_{\pi,t}(x) + \phi_{\pi,t}(-x) - 2e^{-x/2}\phi_{\pi,t}(0) \right\} \frac{e^{x/2}dx}{e^x - e^{-x}} \\
= \frac{1}{2} e^{-t/2} \Phi(e^{-2t}, 1, 1/4) - \frac{1}{2} e^{-t/2} \Phi(e^{-2t}, 1, \frac{1}{2} + i\pi) \\
- (e^{itz} - 1) \frac{1}{2} e^{-t(\frac{1}{2} - iz)} \Phi(e^{-2t}, 1, \frac{1}{2} - iz) + (e^{itz} - 1) \log \coth(e^{t/2}) \\
+ (e^{itz} - 1) \lim_{N \to \infty} \left[ \frac{1}{2} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2} + iz} - \frac{N}{2} \right] - \frac{N}{2} \frac{1}{n + \frac{1}{2} + iz} \\
+ \frac{1}{2} \lim_{N \to \infty} \left[ \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2} + iz} - \frac{N}{2} \right] \\
= \frac{1}{2} e^{-t/2} \Phi(e^{-2t}, 1, 1/4) - \frac{1}{2} e^{-t/2} \Phi(e^{-2t}, 1, \frac{1}{2} + i\pi) \\
- (e^{itz} - 1) \frac{1}{2} e^{-t(\frac{1}{2} - iz)} \Phi(e^{-2t}, 1, \frac{1}{2} - iz) + (e^{itz} - 1) \log \coth(e^{t/2}) \\
+ (e^{itz} - 1) \frac{1}{2} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iz}{2} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{iz}{2} \right) \right] \\
+ \frac{1}{2} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iz}{2} \right) \right].
\]
Combining the results for \( \int_0^\infty \) and \( \int_0^t \),
\[
\int_0^\infty \left\{ \phi_{\pi,t}(x) + \phi_{\pi,t}(-x) - 2e^{-x/2}\phi_{\pi,t}(0) \right\} \frac{e^{x/2}dx}{e^x - e^{-x}} \\
= \frac{1}{2iz} e^{-t/2} \left[ \Phi(e^{-2t}, 1, 1/4) - \Phi(e^{-2t}, 1, \frac{1}{2} + i\pi) \right] \\
+ e^{itz} - 1 \frac{1}{iz} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iz}{2} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{iz}{2} \right) \right] \\
+ \frac{1}{2iz} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} \right) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{iz}{2} \right) \right].
\]
Finally, noting the special value \( (\Gamma'/\Gamma)(1/2) = -\gamma_0 - 2\log 2 \), we conclude that the right-hand side of Weil’s explicit formula for \( \phi_{\pi,t}(x) \) is equal to \( (1.6) \). \( \square \)

### 2.2. Proof of Proposition 2.1

We have \( |\Theta(z)| = 1 \) for every \( z \in \mathbb{R} \) by definition. In fact, zeros of \( E(z) \) in the denominator cancel out in the numerator \( E(z) \), even if they exist. Further, \( \Psi(z) \) has poles of order one at \( \gamma \in \Gamma \), but \( \mathcal{S}_t(z) \) is holomorphic there, since \( (1 + \Theta(z))/2 = A(z)/E(z) = A(z)/(A(z) + iA'(z)) = (z - \gamma)(-1/m_\gamma + o(1)) \) near \( z = \gamma \) by direct calculation. Hence, \( \mathcal{S}_t(z) \) is bounded and holomorphic on the real line by \( (1.5), (2.2) \), and Proposition 2.1. On the other hand, in the horizontal strip \( |\Im(z)| \leq 1/2 \), we have the well-known estimate \( (\Gamma'/\Gamma)(1/4 + iz/2) \ll \log |z| \) and
\[
\frac{\gamma'}{\zeta} \left( \frac{1}{2} - iz \right) = \sum_{|\Re(z) - \gamma| \leq 1/2} \frac{i}{z - \gamma} + O(\log |z|)
\]
by \( \delta \) Theorem 9.6 (A). In both estimates, implied constants are uniform in \( |\Im(z)| \leq 1/2 \). The number of zeros \( \gamma \in \Gamma \) satisfying \( |\Re(z) - \gamma| \leq 1 \) is \( O(\log |z|) \) counting with multiplicity by \( \delta \) Theorem 9.2. Therefore, \( \mathcal{S}_t(z) \ll |z|^{-1} \log |z| \) as \( |z| \to \infty \) with an
implied constant depending on a compact set of \( t \) by (1.6). Hence \( \mathcal{G}_t(z) \) belongs to \( L^2(\mathbb{R}) \) and the norm is uniformly bounded on a compact set of \( t \).

\[ \square \]

### 3. Proof of the main results

#### 3.1. Preparation on the theory of the model spaces.

For this part, we refer to [8, Section 3.1], including precise definitions of notions. Let \( \mathbb{H}^2 \) be the Hardy space on the upper half-plane. As usual, we identify \( \mathbb{H}^2 \) with a closed subspace of \( L^2(\mathbb{R}) \) via boundary values. Then, the inner product of \( \mathbb{H}^2 \) coincides with the standard inner product of \( L^2(\mathbb{R}) \).

Assuming the RH is true, \( E(z) \) of (1.3) is an entire function satisfying \( |E(z)| < |E(z)| \) if \( \Im(z) > 0 \) [8, Theorem 1]). Therefore, it generates the de Branges space \( \mathcal{H}(E) \), which is a Hilbert space of entire functions isomorphic to the model subspace \( \mathcal{K}(\Theta) := \mathbb{H}^2 \ominus \Theta \mathbb{H}^2 \) by the mapping \( F(z) \mapsto F(z)/E(z) \) from \( \mathcal{H}(E) \) into \( \mathbb{H}^2 \), where \( \Theta(z) \) is the meromorphic function defined in (1.4). The model subspace \( \mathcal{K}(\Theta) \) is a subspace of \( L^2(\mathbb{R}) \) as a Hilbert space. In particular, the inner product of \( \mathcal{K}(\Theta) \) matches that of \( L^2(\mathbb{R}) \).

**Proposition 3.1.** Assuming the RH is true, the family

\[ F_\gamma(z) := \sqrt{m_\gamma} \frac{i(1 + \Theta(z))}{2(z - \gamma)}, \quad \gamma \in \Gamma \]  

forms an orthonormal basis of the Hilbert space \( \mathcal{K}(\Theta) \).

**Proof.** See [8, Proposition 3.1]. \( \square \)

#### 3.2. Proof of Theorem 1.1

By Proposition 2.1, we have

\[ \mathcal{G}_t(z) = \sum_{\gamma \in \Gamma} \sqrt{m_\gamma} e^{\gamma t} - \frac{1}{\gamma} F_\gamma(z) \]  

unconditionally. Further, the coefficients on the right-hand side converge in \( L^2 \)-sense:

\[ \sum_{\gamma \in \Gamma} \left| \sqrt{m_\gamma} e^{\gamma t} - \frac{1}{\gamma} \right|^2 \leq \sum_{\gamma \in \Gamma} m_\gamma < \infty. \]

Therefore, assuming the RH and applying Proposition 3.1 to \( \mathcal{G}_t(z) \) via formula (3.2), we find that \( \mathcal{G}_t(z) \) belongs to the subspace \( \mathcal{K}(\Theta) \) of \( L^2(\mathbb{R}) \) and

\[ \langle \mathcal{G}_{t+u} - \mathcal{G}_u, \mathcal{G}_{s+u} - \mathcal{G}_u \rangle_{L^2(\mathbb{R})} = \sum_{\gamma \in \Gamma} m_\gamma e^{\gamma t} - \frac{1}{\gamma} e^{-\gamma s} - \frac{1}{\gamma} \]  

holds. The right-hand side is equal to \( G_g(t, s) \) by [6, (1.9)]. Hence, \( \mathcal{G}_t : \mathbb{R} \to L^2(\mathbb{R}) \) is a screw line of \( g(t) \) under the RH.

We find that \( \mathcal{G}_0(z) \) is identically zero by (1.5) and (1.6), since

\[ \lim_{t \to 0} \Phi(e^{-2it}, 1/4) - \Phi(e^{-2it}, (1/2 + iz)/2) = -\frac{\Gamma'}{\Gamma}(1/4) + \frac{\Gamma'}{\Gamma}((1/2 + iz)/2) \]

by (2.3). Therefore, by taking \( u = 0 \) in (3.3),

\[ \| \mathcal{G}_t \|^2_{L^2(\mathbb{R})} = \sum_{\gamma \in \Gamma} m_\gamma e^{\gamma t} - \frac{1}{\gamma} = 2 \sum_{\gamma \in \Gamma} m_\gamma \frac{1 - \cos(\gamma t)}{\gamma^2}. \]

On the other hand,

\[ -g(t) = \sum_{\gamma \in \Gamma} m_\gamma \frac{1 - \cos(\gamma t)}{\gamma^2} \]  

by [6, Theorem 1.1 (2)]. Hence equality (1.7) follows from (3.4) and (3.5). \( \square \)
3.3. **Proof of Corollary 1.1.** Theorem 1.1 states that (1.7) is a necessary condition for the RH. Therefore, it suffices to show that (1.7) is a sufficient condition for the RH.

We suppose that equality (1.7) holds for all \( t \geq t_0 \). Then \(-g(t)\) is nonnegative on \([t_0, \infty)\), which implies that the RH is true by [6] Theorems 1.7 and 11.1.

3.4. **Proof of Theorem 1.2.** First, we show equation (1.9) assuming that the RH is true. For any \( \phi \in C_c^\infty(\mathbb{R}) \), we have

\[
\mathcal{P}_\phi(z) = \sum_{\gamma \in \Gamma} \sqrt{m_\gamma} \frac{\widehat{\phi}(\gamma) - \widehat{\phi}(0)}{\gamma} F_\gamma(z)
\]

by (3.2), where \( \widehat{\phi}(z) := \int_{-\infty}^{\infty} \phi(t) e^{izt} \, dt \). Therefore,

\[
\|\mathcal{P}_\phi\|_{L^2(\mathbb{R})}^2 = \sum_{\gamma \in \Gamma} m_\gamma \left| \frac{\widehat{\phi}(\gamma) - \widehat{\phi}(0)}{\gamma} \right|^2
\]

by Proposition 3.1. Applying

\[
G_g(t, u) = \sum_{\gamma} \frac{(e^{\gamma t} - 1)(e^{-i\gamma u} - 1)}{\gamma^2}
\]

in [6] (1.9) to (1.8) and noting the symmetry \( \gamma \mapsto -\gamma \), we find that the right-hand side of (3.6) is equal to \( \langle \phi, \phi \rangle_{G_g} \).

Conversely, we show that the RH is true assuming equality (1.9). We show that a contradiction arises if the RH is false. We take a non-real \( \gamma_0 \in \Gamma \). For any \( \epsilon > 0 \), there exists \( \psi_1, \psi_2 \in C_c^\infty(\mathbb{R}) \) such that \( \widehat{\psi}_1(\gamma_0) = i, \widehat{\psi}_2(\gamma_0) = -i \), \( |\widehat{\psi}_1(\gamma)| \leq \epsilon |\gamma_0 - \gamma|^{-1-\delta} \) for every \( \gamma \in \Gamma \setminus \{\gamma_0\} \), and \( |\widehat{\psi}_2(\gamma)| \leq \epsilon |\gamma_0 - \gamma|^{-1-\delta} \) for every \( \gamma \in \Gamma \setminus \{\gamma_0\} \) by [10] Lemma 1.

We define \( \psi := \psi_1 + \psi_2 (\neq 0) \) and \( \phi := D\psi \). Then, \( \widehat{\phi}(0) = 0 \) by definition and \( \langle \phi, \phi \rangle_{G_g} = \langle \psi, \psi \rangle_{W} \) by (1.10). The right-hand side is equal to \( \sum_{\gamma \in \Gamma} m_\gamma \widehat{\psi}(\gamma) \psi(\gamma) = -m_{\gamma_0} + O(\epsilon) \), since \( \sum_{\gamma \in \Gamma} m_\gamma |\gamma|^{-1-\delta} < \infty \). Therefore, \( \langle \phi, \phi \rangle_{G_g} \) is negative for a sufficiently small \( \epsilon > 0 \), but it contradicts the non-negativity that follows from (1.9). Hence the RH is true.

4. Special values of the screw line \( \mathcal{S}_t(z) \)

The screw line \( \mathcal{S}_t(z) \) has the following unconditional relations with the screw function \( g(t) \). It is interesting that they are not a special case of equations obtained from the general theory of screw functions.

**Theorem 4.1.** Let \( g(t) \) and \( \mathfrak{P}_t(z) \) be functions of (1.1) and (1.6), respectively. Then the following equations hold independently of the truth of the RH:

\[
\mathfrak{P}_t(0) = -g(t),
\]

\[
\lim_{y \to +\infty} \left[ y \mathfrak{P}_t(\nu) - \frac{1}{2} \Gamma'(\frac{1}{4} - \frac{y}{2}) + \frac{1}{2} \log \pi \right] = -g'(t),
\]

where we assume \( t \neq \log n \) for any \( n \in \mathbb{N} \) in (1.2).

**Proof.** Equality (4.1) follows from (2.2), Proposition 2.1 and (3.5), but it also follows directly from (1.1) and (1.6). In fact, by \( \Phi(z, s, a) = \sum_{n=0}^{\infty} z^n (n + a)^{-s} \) and (2.3),

\[
\lim_{z \to 0} \frac{1}{iz} \left[ \Phi(e^{-2it}, 1, 1/4) - \Phi(e^{-2it}, 1, \frac{1}{4} + iz) \right] = \frac{1}{2} \Phi(e^{-2it}, 2, 1/4),
\]

\[
\lim_{z \to 0} \frac{1}{iz} \left[ \Gamma'\left(\frac{1}{4} - \frac{1}{4} + \frac{iz}{2}\right) \right] = \frac{1}{2} \psi_1\left(\frac{1}{4}\right),
\]

where \( \psi_1(z) \) is the polygamma function of order one. The expansion \( \psi_1(w) = \sum_{n=0}^{\infty} (w + n)^{-2} \) gives \( \psi_1(1/4) = \Phi(1, 2, 1/4) \). By \( Z(s) = Z(1-s) \), we have \( (Z'/Z)(1/2) = 0 \). Hence, by taking the limit \( z \to 0 \) in (1.6), we obtain the minus of (1.1).
To show (4.2), we multiply (1.6) by $\frac{y}{1 + 2y}$ and substitute $iy$ for $z$:

$$y \mathcal{P}_t(iy) = \frac{4y(e^{t/2} - 1)}{1 + 2y} + \frac{4y(e^{-t/2} - 1)}{1 - 2y} + \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} (e^{-y(t - \log n)} - 1)$$

$$- (e^{-yt} - 1) \left[ \frac{Z'}{Z} \left( \frac{1}{2} + y \right) - \frac{1}{2} \log \pi + \frac{1}{2} \Gamma' \left( \frac{1}{4} - \frac{y}{2} \right) \right]$$

$$+ \frac{1}{2} \left[ \Gamma' \left( \frac{1}{4} \right) - \Gamma' \left( \frac{1}{4} - \frac{y}{2} \right) \right]$$

$$+ \frac{1}{2} e^{-t/2} \left[ \Phi(e^{-2t}, 1, 1/4) - \Phi(e^{-2t}, 1, 1/2 - y) \right].$$

Therefore, for positive $t > 0$,

$$\lim_{y \to +\infty} \left[ y \mathcal{P}_t(iy) - \frac{1}{2} \Gamma' \left( \frac{1}{4} + \frac{y}{2} \right) + \frac{1}{2} \log \pi \right]$$

$$= 2(e^{t/2} - e^{-t/2}) - \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} + \frac{1}{2} \left[ \Gamma' \left( \frac{1}{4} \right) - \log \pi \right] + \frac{1}{2} e^{-t/2} \Phi(e^{-2t}, 1, 1/4).$$

The right-hand side equals to $-g'(t)$ if $t \neq \log n$ by (1.1) and $(d/dt)(e^{-t/2} \Phi(e^{-2t}, 2, 1/4)) = -2e^{-t/2} \Phi(e^{-2t}, 2, 1/4)$ follows from $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n (n + a)^{-s}$. □

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