Self-dual Yang–Mills Theory and One-Loop Maximally Helicity Violating Multi-Gluon Amplitudes

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Abstract

A scalar cubic action that classically reproduces the self-dual Yang–Mills equations is shown to generate one-loop QCD amplitudes for external gluon all with the same helicity. This result is related to the symmetries of the self-dual Yang–Mills equations.

Key words: Integrable, Yang–Mills, helicity, amplitude, self-dual

The study of multi-jet processes requires the calculation of QCD amplitudes with external quarks and gluons. The complexity of the calculations beyond tree level has motivated the development of ingenious and efficient methods that involve spinor helicity, color decomposition, supersymmetry, string theory, recursion relations, factorization and unitarity [1]. The easiest amplitudes to consider are the ones with all external gluons (conventionally taken as outgoing) in the same polarization state [2–4]. These so-called maximally helicity violating (MHV) processes vanish at tree level and take at one-loop a non-zero but very simple form. Zero tree amplitudes are a typical signature of a quantized integrable system [5,6] and it is tempting to look for an integrable model that reproduces these results. The relevance of two-dimensional current algebra for some MHV amplitudes was already noticed by Nair some time ago [7]. In a recent talk [8], Bardeen showed that some solutions of the self-dual Yang–Mills (SDYM) equations naturally appear in the calculation of tree MHV amplitudes (see also [9]). He also conjectured that the vanishing of the tree amplitudes is a consequence of the symmetries of the SDYM

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system and that an anomaly is responsible for the structure of the one-loop amplitudes.

In this paper, I extend Bardeen’s analysis and obtain the one-loop MHV QCD amplitudes from a scalar action whose Euler–Lagrange equations reproduce the SDYM ones. In the first section, I briefly review the calculation of QCD amplitudes. In section 2, I consider the SDYM equations and two actions associated to them. In section 3, I follow Bardeen’s idea and show the relation between SDYM solutions and MHV amplitudes. I extend this analysis to one-loop in section 4 and discuss in section 5 the symmetries of the SDYM system. I conclude in the last section with some remarks.

1 A brief review of QCD amplitudes

This section reviews some of the basic ingredients which simplify the calculation of QCD amplitudes. I only present the material relevant to my discussion and I refer the reader to Ref. [1] and references therein for a more complete presentation.

The spinor helicity notation [10] compactifies otherwise lengthy expressions. An on-shell momentum \( k \) (this always means \( k^2 = 0 \)) has three independent components that are usually expressed in terms of scalar products of massless Weyl spinors (see Appendix A). It is more natural in this paper to choose \( k_{0-z}, k_{x+iy} \) and the ratio

\[
Q = \frac{k_{0+z}}{k_{x+iy}} = \frac{k_{x-iy}}{k_{0-z}}. \tag{1}
\]

The combination

\[
X(k_1, k_2) = k_{1,x+iy}k_{2,0-z} - k_{1,0-z}k_{2,x+iy} = (Q_1 - Q_2)^* k_{1,0-z}k_{2,0-z} \tag{2}
\]

satisfies the identity

\[
2 k_1 \cdot k_2 = X(1, 2) (Q_1 - Q_2). \tag{3}
\]

One also introduces a set of polarization vectors in an arbitrary reference frame labelled by the null vector \( q \) \( (q \cdot k \neq 0) \). A change in \( q \) is analogous to a gauge transformation of the external legs and since the amplitudes are

\[\text{I use the conventions } k_{0\pm z} = k_0 \pm k_z \text{ and } k_{x\pm iy} = k_x \pm ik_y, \text{ with metric } g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \text{ and totally antisymmetric tensor } \epsilon_{0123} = 1.\]
gauge invariant, they do not depend on the reference frame chosen for the external particles. Nevertheless the calculations may drastically simplify for some reference vectors $q$. These polarization vectors are orthogonal to the momentum $k$ and normalized in the following way:

$$k \cdot \varepsilon^{\pm}(k; q) = 0, \quad \varepsilon^+(k; q) \cdot \varepsilon^-(k; q) = -1, \quad \varepsilon^\pm(k_1; q) \cdot \varepsilon^\pm(k_2; q) = 0. \quad (4)$$

Their general form is given in Appendix A. Comparison with the SDYM system is made later on in the “light-cone gauge” $q_\mu = (1, 0, 0, 1)$, in which the polarization vectors are denoted $\varepsilon^{(\pm)}_\mu(k)$ and have the non-zero components

$$\varepsilon^{(\pm)}_{0+z}(k) = -\sqrt{2} \frac{k_{0+z}}{k_{x+iy}}, \quad \varepsilon^{(\pm)}_{x-iy}(k) = -\sqrt{2} \frac{k_{x-iy}}{k_{x+iy}},$$

$$\varepsilon^{(\pm)}_{0+z}(k) = -\sqrt{2} \frac{k_{0+z}}{k_{x-iy}}, \quad \varepsilon^{(\pm)}_{x+iy}(k) = -\sqrt{2} \frac{k_{x+iy}}{k_{x-iy}}. \quad (5)$$

A positive helicity gauge field is parallel to $\varepsilon^{(+)}_\mu$:

$$\varepsilon^{(+)}(k) \cdot A(k) = -\frac{1}{\sqrt{2}} \frac{1}{k_{x+iy}} \left( k_{0+z} A_{0-z}(k) - k_{x-iy} A_{x+iy}(k) \right) = 0,$$

$$2 \ k \cdot A(k) = k_{0-z} A_{0+z} + k_{0+z} A_{0-z} - k_{x+iy} A_{x-iy} - k_{x-iy} A_{x+iy} = 0. \quad (6)$$

We will see in the next section that a self-dual gauge field in the light-cone gauge has positive helicity.

Simple $N = 1$ and $N = 2$ supersymmetry arguments [12,13] relate one-loop MHV amplitudes with either gluons, scalars or fermions running in the loop. For $SU(N_c)$ gauge fields, $n_s$ real scalar fields in the representation $R_s$ and $n_f$ Dirac fermions (four components) in the representation $R_f$, the amplitude has an overall factor of $^3$

$$\left( C_2(G) + \frac{n_s}{2} T_2(R_s) - 2 n_f T_2(R_f) \right). \quad (7)$$

Thus, it is sufficient to consider an (adjoint) scalar field $\chi$ coupled to a Yang–Mills field,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \text{tr} D_\mu \chi D^\mu \chi. \quad (8)$$

---

$^3$ I use the following group theoretic conventions for $SU(N_c)$: $C_2(G) = 2N_c \iff [T^a, T^b] = i\sqrt{2} f^{abc} T^c$, $T_2(F) = 1 \iff \text{tr}_F T^a T^b = \delta^{ab}$. 
where the covariant derivative is $D_\mu = \partial_\mu - igA_\mu/\sqrt{2}$ and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]/\sqrt{2}$, and to calculate one-loop amplitudes with classical gauge fields and scalar fields only in the loop. The scalar propagator is $\Delta = i/k^2$ with a color factor $\delta^{ab}/C_2(G)$.

The color structure of QCD amplitudes may be simplified by decomposing them in color-ordered partial amplitudes [14]: At one-loop, an $(n + 1)$ gluon amplitude with external momenta $k_1, \ldots, k_{n+1}$, helicities $\varepsilon_1, \ldots, \varepsilon_{n+1}$ and color indices $a_1, \ldots, a_{n+1}$ is written as

$$M_{n+1}^{\text{one-loop}}(\{k_i, \varepsilon_i, a_i\}) = \sum_{\text{non cyclic permutations of } (1 \cdots n+1)} \text{tr}(T^{a_1} \cdots T^{a_{n+1}}) m_{n+1}^{\text{one-loop}}(k_1, \varepsilon_1; \ldots; k_{n+1}, \varepsilon_{n+1}) + \ldots .$$

The triple dots at the end represent additional sub-leading color structures that can be calculated from the above leading-color partial amplitudes [15]. In other words, it is sufficient to calculate the fewer partial amplitudes using color-ordered Feynman rules where color indices are absent.

Amplitudes with all, or all but one, external gluons with the same helicity identically vanish at tree level:

$$m^{\text{tree}}(k_1, +; \ldots; k_n, +; k_{n+1}, \pm) = i k_1^2 \varepsilon^{+,\mu_1}(k_1; q_1) \cdots i k_n^2 \varepsilon^{+,\mu_n}(k_n; q_n) \times i k_{n+1}^2 \varepsilon^{+,\mu_{n+1}}(k_{n+1}; q_{n+1}) \left< A_{\mu_1}(k_1) \cdots A_{\mu_n}(k_n) A_{\mu_{n+1}}(k_{n+1}) \right>_c \bigg|_{k_1^2 = \cdots = k_n^2 = 0} \bigg|_{k_{n+1}^2 = 0} = 0 .$$

The subscript $c$ stands for connected Green functions. This can be proven using the freedom one has in the choice of the reference momenta $q_i$ for the external particle helicities [12]. Alternatively one can show that the current amplitude,

$$\left< A_\mu(k) \right>_{1+\ldots+n+}^{(+)} = \varepsilon^{(+),\mu_1}(k_1) \cdots \varepsilon^{(+),\mu_n}(k_n) \times (ik_1^2) \cdots (ik_n^2) \left< A_{\mu_1}(k_1) \cdots A_{\mu_n}(k_n) A_\mu(k) \right>_c \bigg|_{k_1^2 = \cdots = k_n^2 = 0} = 0 ,$$

has no multi-particle poles at tree level. We will later calculate the current amplitudes (10) using a self-dual system and demonstrate that they have no pole in $k^2$.

At one-loop, MHV amplitudes with four and more external legs do not vanish and are explicitly known. For example, the four particle amplitude is [16]
\[ m^{\text{one-loop}}(k_1, +; \cdots; k_4, +) = \]
\[ = -N_c g_4^4 \frac{i}{48\pi^2} \frac{k_{1,x+iy} k_{2,x+iy} k_{3,x+iy} k_{4,x+iy}}{Q_1 Q_2 Q_3 Q_4} X(k_1 + k_2, k_3)^2 (Q_1 - Q_2)^2, \] (11)

A more conventional expression written in spinor notations is given in Appendix A. It is the goal of this paper to reproduce all the MHV one-loop amplitudes in the framework of a SDYM system.

2 SDYM equations

The SDYM equations are most commonly studied in Euclidean space or (2 + 2)-dimensional spacetime so that the solutions are real. In Euclidean space, SDYM solutions describe instantons and in a (2 + 2) signature, they describe the gauge sector of an \( N = 2 \) heterotic or open string [5]. Here I explicitly work in Minkowski spacetime and consider the self-dual equations,

\[ F_{\mu\nu} = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \] (12)

whose solutions necessarily live in the complexification of the gauge group. Complex self-dual configurations have nevertheless a physical interpretation as waves of positive helicity. For Maxwell-like fields (\( g = 0, E^j = F_{ij}, B^j = \varepsilon^{ijkl} F_{kl}/2 \)), these equations simply state that electric and magnetic fields form a positive helicity wave, \( E^j = i B^j \). In components, Eqns. (12) read

\[ F_{0+z,x-iy} = 0, \quad F_{0-z,x+iy} = 0, \quad F_{0-z,0+z} = F_{x+iy,x-iy}. \] (13)

Considerable effort has been devoted to the study of these equations and various approaches have been proposed.

One approach comes from the original work of Yang [17] and views the two first equations as zero curvature conditions for the fields \((A_{0+z}, A_{x-iy})\) and \((A_{0-z}, A_{x+iy})\). They are solved using two elements \( h, \bar{h} \) of the (complexified) gauge group:

\[ -i g \sqrt{2} A_{0+z} = h^{-1} \partial_{0+z} h, \quad -i g \sqrt{2} A_{x-iy} = h^{-1} \partial_{x-iy} h, \]
\[ -i g \sqrt{2} A_{0-z} = \bar{h}^{-1} \partial_{0-z} \bar{h}, \quad -i g \sqrt{2} A_{x+iy} = \bar{h}^{-1} \partial_{x+iy} \bar{h}. \] (14)

The last self-dual equation involves the product \( H = h\bar{h}^{-1}, \)

\[ \partial_{0-z}(H^{-1} \partial_{0+z} H) - \partial_{x+iy}(H^{-1} \partial_{x-iy} H) = 0, \] (15)
and is known as the Yang equation. This is the Euler–Lagrange equation of an action proposed by Donaldson [18] and by Nair and Schiff [19]:

\[
S_{DNS}(H) = \frac{f_\pi^2}{2} \int d^4x \text{ tr} \left( \partial_{0+z} H \partial_{0-z} H^{-1} - \partial_{x-iy} H \partial_{x+iy} H^{-1} \right) \\
+ \frac{f_\pi^2}{2} \int d^4x \, dt \, \text{ tr} \left( [H^{-1} \partial_{0+z} H, H^{-1} \partial_{0-z} H] - [H^{-1} \partial_{x-iy} H, H^{-1} \partial_{x+iy} H] \right) H^{-1} \partial_t H .
\] (16)

The constant \( f_\pi \) has mass dimension one. This action is expected to be the prototype of a four-dimensional conformal theory that generalizes the two-dimensional Wess–Zumino–Novikov–Witten model [20]. Namely, a direct calculation shows that its beta function vanishes at one-loop order. (It is even argued that it vanishes at all orders in some circumstances [21].)

We will follow another approach [22], that takes advantage of the light-cone gauge \( A_{0-z} = 0 \). In this gauge, the two last equations in (13) are equivalent to the positive helicity conditions (6). The second equation implies \( A_{x+iy} = 0 \) and the third one is an integrability condition for the fields \( A_{0+z}, A_{x-iy} \),

\[
A_{x+iy} = 0 , \quad A_{0+z} = \sqrt{2} \, \partial_{x+iy} \Phi , \quad A_{x-iy} = \sqrt{2} \, \partial_{0-z} \Phi ,
\] (17)

or, in momentum space,

\[
A_{0+z}(k) = -i \sqrt{2} k_{x+iy} \, \Phi(k) \quad k^2 = 0 \implies A_\mu(k) = \varepsilon_\mu^{(+)}(k) \, \frac{k_{x+iy}}{Q} \, \Phi(k) .
\] (18)

A self-dual gauge field has positive helicity. The scalar field \( \Phi \) is constrained by the remaining first equation in (13):

\[
\partial^2 \Phi - i g \left[ \partial_{x+iy} \Phi, \partial_{0-z} \Phi \right] = 0 .
\] (19)

These are the equations of motion of the scalar action [23]:

\[
S_{\text{scalar}}(\phi) = f_\pi^2 \int d^4x \text{ tr} \left( \frac{1}{2} \partial \phi \cdot \partial \phi + \frac{ig}{3} \phi \left[ \partial_{x+iy} \phi, \partial_{0-z} \phi \right] \right) .
\] (20)

As before, \( f_\pi \) has mass dimension one so that the complex field \( \Phi \) is dimensionless, in agreement with (17). This action has three unconventional properties: it is not real,\(^4\) it explicitly breaks Lorentz invariance and it is not renormal-
izable by power counting since the interaction term contains two derivatives. Nevertheless, it turns out to reproduce correctly the MHV amplitudes up to one loop.

3 Tree amplitudes

I first recall the relation between a classical solution and a tree amplitude. An amplitude is obtained by truncating on-shell a connected Green function. A tree Green function is generated by the Legendre transform $W(J)$ of the classical action $S(\phi)$:

$$\frac{\delta S(\phi)}{\delta \phi(x)}|_{\phi=\Phi_J} + J(x) = 0,$$

$$W(J) = S(\Phi_J) + \int dx \, \Phi_J(x) J(x),$$

$$\langle \phi(x_1) \cdots \phi(x_{n+1}) \rangle_c = \frac{i^n \delta^{n+1} W(J)}{i \delta J(x_1) \cdots i \delta J(x_{n+1})} \bigg|_{J=0}.\ (23)$$

Since the classical solution $\Phi_J$ in presence of a source $J$ is given by the first variation of the generating functional $W(J)$, we have

$$\langle \phi(x_1) \cdots \phi(x_{n+1}) \rangle_c = \frac{\delta^n \Phi_J(n+1)}{i \delta J(x_1) \cdots i \delta J(x_{n})} \bigg|_{J=0}.\ (24)$$

The classical solution $\Phi_J$ is an infinite series in $J$ whose coefficients are the connected Green functions.

In our case, the equations of motion with source are

$$\partial^2 \Phi_J - ig \left[ \partial_x + i g \Phi_J, \partial_{\theta_x} \Phi_J \right] + J = 0.\ (25)$$

These equations are solved with a Bethe Ansatz [8]. The solution is iteratively obtained as a series in the coupling constant $g$:

$$\Phi_J(x) = \sum_{m=1}^{\infty} \Phi_J^{(m)}(x),\quad \Phi_J^{(m)} \propto g^{m-1}.\ (26)$$

In momentum space, the first term is

$$\Phi_J^{(1)}(k) = J(k)/k^2 \equiv j(k).\ (27)$$
It is possible to add to $\Phi_J^{(1)}(k)$ a function with support on the light-cone $k^2 = 0$. This would shift the solution with source by a solution of Eq. (19) and lead to Green functions in the presence of a non-vanishing background field. Solutions of the SDYM equations (19) without source are obtained from $\Phi_J$ by taking the support of $j(k) = J(k)/k^2$ on the light-cone. By iteration we get

$$
\Phi_J^{(m)}(k) = ig \sum_{j=1}^{m-1} \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - k)
\times \frac{X(p_1, p_2)}{(p_1 + p_2)^2} \Phi^{(j)}(p_1) \Phi^{(m-j)}(p_2). 
$$

(28)

Each term $\Phi_J^{(m)}(k)$ is a sum of trees with cubic vertices, attached to a leg with momentum $k$ and ending up on $m$ sources $J(p)/p^2 = j(p)$. Since the current amplitude

$$
\langle \phi(k) \rangle_{1\ldots n} \equiv (-ik_1^2) f(k_1) \cdots (-ik_n^2) f(k_n) \langle \phi(k_1) \cdots \phi(k_n) \phi(k) \rangle \bigg|_{k_1^2 = \cdots = k_n^2 = 0}
$$

(29)

($f(k_j)$ is a function with support on the light-cone $k_j^2 = 0$) is obtained by differentiating the classical solution $n$ times with respect to the source $J$ and by truncating on-shell $n$ external legs (see (24)), we can restrict the support of $j(k)$ to be on the light-cone and then take the derivatives with respect to $j(k)$. Then Eqns. (28) are equivalent to the Berends and Giele recursion relations [3,24,11] and one shows by induction:

$$
\Phi_J^{(m)}(k) = (ig)^{m-1} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_m}{(2\pi)^4} (2\pi)^4 \delta(p_1 + \cdots + p_m - k)
\times j(p_1) \cdots j(p_m) (Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1} \cdots (Q_{m-1} - Q_m)^{-1}.
$$

(30)

If one takes $j$ to be a sum of $n$ independent (on-shell) plane waves,

$$
j(x) = -i \sum_{j=1}^n T^{a_j} e^{-ik_j x} f(k_j),
$$

(31)

one gets the Bethe Ansatz solution [8] of the SDYM equations. The relevant part in the calculation of the current amplitude (29) involves $n$ $j$'s and so is contained in $\Phi^{(n)}$:

$$
\Phi^{(n)}(x) = -ig^{n-1} \sum_{\text{permutations of } (1\ldots n)} T^{a_1} \cdots T^{a_n} e^{-i(k_1 + \cdots + k_n)x} f(k_1) \cdots f(k_n)
$$
\[ \times (Q_1 - Q_2)^{-1} (Q_2 - Q_3)^{-1} \cdots (Q_{n-1} - Q_n)^{-1} + \cdots . \]  

(32)

The piece given here explicitly involves all momenta \( k_j \) \((j = 1, \ldots, n)\) and is equal to the current amplitude (29). The additional terms denoted by the dots (as well as the other \( \Phi^{(m)}, \ m \neq n \)) correspond to tree diagrams with two or more legs having the same momentum. They are described in Appendix B.

The function \( f(k) \) has not yet been specified. One remarks that for \( f(k) = Q/k \), \( \Phi^{(1)}(k) = j(k) \) corresponds (see (18)) to a positive helicity gauge field: \( A_\mu^{(1)}(k) = \sum_{j=1}^n (2\pi)^4 \delta(k_j - k) \epsilon^{(+)\mu}(k_j). \) For this same function one finds a remarkable identity [8] between the tree SDYM current amplitude (29) and the tree MHV current amplitudes (10) in the light-cone gauge and in the reference frame (5) as calculated for example in [11]:

\[
\begin{align*}
\langle A_{0+z}(k) \rangle_{1+\ldots+n+}^{\text{tree}} &= -i\sqrt{2} k_{x+iy} \langle \phi(k) \rangle_{1\ldots n}, \\
\langle A_{x-iy}(k) \rangle_{1+\ldots+n+}^{\text{tree}} &= -i\sqrt{2} k_{0-z} \langle \phi(k) \rangle_{1\ldots n}.
\end{align*}
\]  

(33)

( compare with the self-dual \( \text{Ansatz} \) (18) ). The MHV amplitudes are obtained by contracting (33) with \( ik^2 \epsilon^{\pm \mu}(k; q) \) and taking the limit \( k^2 \to 0 \). A remarkable property of the solution (32) is that the multiparticle poles appearing in the branches of the tree completely disappear in the final solution. In particular, there is no pole in \( k^2 = (k_1 + \cdots + k_n)^2 \) and thus the tree amplitudes (9) vanish.

An analysis similar to the one done in this section can be carried [9] with a third \( \text{Ansatz} \) for the self-dual gauge field, originally proposed by ‘t Hooft for instantons solutions.

4 One-loop amplitudes

The color-ordered Feynman rules for the scalar action (20) are very simple. In momentum space the propagator is \( i/k^2 \) and the cubic vertex is

\[
\phi \overset{3}{\overbrace{\begin{array}{c} 2 \\ 1 \end{array}}} \phi = g \left\{ \frac{3}{2} X(k_3, k_1 - k_2) = g X(k_1, k_2) \right\}.
\]  

(34)

Minkowski kinematics is such that this vertex is zero on-shell in agreement with the vanishing of the three particle MHV amplitudes. This on-shell vertex is not zero for Euclidean and \((2+2)\) signature. Although two and three particle
amplitudes at one-loop involve IR and UV divergent integrals [23], they vanish for similar kinematics reasons. I have computed the color-ordered four particle amplitude using a symbolic algebra computer program and performed the resulting integrals in dimensional regularization [16]:

$$\left. (-i k_1^2) \cdots (-i k_4^2) \left\langle \phi(k_1) \cdots \phi(k_4) \right\rangle \right|_{k_1^2 = \cdots = k_4^2 = 0} = -\frac{N_c g^4 i}{48\pi^2} \frac{X(k_1 + k_2, k_3)^2}{(Q_1 - Q_2)^2}.$$  \hspace{1cm} (35)$$

If one multiplies each leg by $-iQ/k_{x+i_y}$, like at tree level, one gets a full agreement with QCD, see Eq. (11) (remember the factor of two in (7) between complex scalars and gluons contributions). In this section, I show that this result generalizes to one-loop amplitudes with an arbitrary number of external gluons.

As mentioned in the first section, one-loop QCD amplitudes may be calculated from the scalar QCD action (8) with a classical gauge field $A_\mu$ and a quantized scalar field $\chi$ running in the loop. In light-cone gauge, (8) reads

$$S_{\text{QCD}} = \int d^4x \text{tr} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial \chi \cdot \partial \chi - \frac{ig}{2\sqrt{2}} [A_{0+}, \chi] \partial_{0-} \chi + \frac{ig}{2\sqrt{2}} [A_{x-i y}, \chi] \partial_{x+i y} \chi + \frac{ig}{2\sqrt{2}} [A_{x+i y}, \chi] \partial_{x-i y} \chi + \frac{g^2}{4} [A_{x+i y}, \chi][A_{x-i y}, \chi] \right),$$  \hspace{1cm} (36)$$

and generates five color-ordered vertices coupling gauge fields and scalar fields:

$$A_{0+} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \\ \chi \end{array} \rightarrow \begin{array}{c} 2 \chi \\ 1 \\ \chi \end{array} = \frac{ig}{2\sqrt{2}} (k_1 - k_2)_{0-} ,$$

$$A_{x-i y} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \\ \chi \end{array} \rightarrow \begin{array}{c} 2 \chi \\ 1 \\ \chi \end{array} = -\frac{ig}{2\sqrt{2}} (k_1 - k_2)_{x+i y} ,$$

$$A_{x+i y} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \\ \chi \end{array} \rightarrow \begin{array}{c} 2 \chi \\ 1 \\ \chi \end{array} = -\frac{ig}{2\sqrt{2}} (k_1 - k_2)_{x-i y} ,$$

$$A_{x+i y} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \\ \chi \end{array} \rightarrow \begin{array}{c} 2 \chi \\ 1 \\ \chi \end{array} = -\frac{ig^2}{4} .$$

Gluon vertices are not shown here since they have already been correctly taken into account in the previous section. One observes that this action looks very similar to the SDYM action (20) in a background field $\Phi$,

$$S_{\text{scalar}}(\Phi + \phi) = S(\Phi) + \int d^4 x \, \text{tr} \left( \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{ig}{2} [\partial_{x+iy} \Phi, \phi] \partial_{0-z} \phi ight)$$

$$+ \frac{ig}{2} [\partial_{0-z} \Phi, \phi] \partial_{x+iy} \phi - \phi \left( \partial^2 \Phi + ig [\partial_{x+iy} \Phi, \partial_{0-z} \Phi] \right)$$

$$+ \frac{ig}{3} \phi [\partial_{x+iy} \phi, \partial_{0-z} \phi] \right),$$

(38)

when one uses the relation (17) or (33) between $A_\mu$ and $\Phi$.

A general one-loop partial amplitude is obtained by gluing with a scalar propagator the two extremities of the QCD dressed vertex

$$(-ik_1^2)(-ik_2^2) \langle \chi(k_1) \chi(k_2) \rangle_{1+...n+} = \chi \begin{array}{c} \text{tree} \\ \text{tree} \end{array} \chi = \chi \begin{array}{c} \text{tree} \\ \text{tree} \end{array} \chi,$$

(39)

or the SDYM dressed vertex $(-ik_1^2)(-ik_2^2) \langle \phi(k_1) \phi(k_2) \rangle_{1+...n}$. We found in (33) that there are no current amplitudes $\langle A_{x+iy} \rangle$, so the two quartic vertices and the third cubic vertex in (37) do not appear. For each segment carrying one tree in the above diagram, we have (remember that a current amplitude has a propagator attached to its off-shell leg):
\[
= -i\sqrt{2}\langle \phi(k_3) \rangle_{1, \ldots, n} k_{3, x+i y} A_{0+z}^{3, 2, 1, x} \]
\[
= -i\sqrt{2}\langle \phi(k_3) \rangle_{1, \ldots, n} k_{3, 0-z} A_{x-i y}^{3, 2, 1, x} \]
\[
= \langle \phi(k_3) \rangle_{1, \ldots, n} \phi_{3, 2, 1} = \phi_{1, 2, 3} \cdot \phi \cdot \phi \cdot \phi . \tag{40}
\]

For the second equality, we use the tree level identity (33) relating \( \langle A(k) \rangle_{1, \ldots, n} \) and \( \langle \phi(k) \rangle_{1, \ldots, n} \). For the third equality, we replace a linear combination of two QCD vertices (37) with a \( \phi \)-vertex (34). The dressed vertices of scalar QCD and of the SDYM action coincide and their one-loop amplitudes are therefore equal.

These amplitudes have been explicitly calculated in the context of QCD for an arbitrary number of legs: their form has first been conjectured [25] using gluons in the loop and then derived [26] using fermions in the loop. The result is amazingly simple:

\[
(-ik_1^2) \cdots (-ik_n^2) \langle \phi(k_1) \cdots \phi(k_n) \phi(k) \rangle_{k_1^2 = \cdots = k_n^2 = 0} \]
\[
= \frac{\frac{\pi}{2} N_c g^n f^{n+1}}{48\pi^2} \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} k_{i, \mu} (k_1 + \cdots + k_i)_\nu k_{j, \lambda} (k_1 + \cdots + k_j)_\rho (Q_1 - Q_2)(Q_2 - Q_3) \cdots (Q_n - Q_1) \]
\[
\times \text{tr} \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\lambda \sigma^\kappa , \tag{41}
\]

with the Pauli matrices, \( \sigma^\mu = (1, \sigma^i), \bar{\sigma}^\mu = (1, -\sigma^i) \).

I end this section with some comments on the alternative SDYM action (16). Since its equations of motion are also the SDYM ones, this action reproduces correctly the tree amplitudes. At one-loop, the proliferation of vertices makes the analysis more difficult. I checked that the four particle amplitude gives again (11). With \( J = \exp i\pi^a T^a / f_\pi \), the cubic vertex is a slight modification of (34):

\[
\pi_{3, 2, 1} \pi = -\frac{1}{6f_\pi} \left( X(k_3, k_1 - k_2) + k_1^2 - k_2^2 \right) . \tag{42}
\]
The additional term \((k_1^2 - k_2^2)\) is irrelevant for the four particle amplitude, but does contribute to higher amplitudes, as do the quartic, quintic, etc. vertices. This also happens at tree level but does not change the final result. It is plausible that the same cancellations work at one-loop and that the Donaldson–Nair–Schiff action reproduces also all the one-loop MHV amplitudes.

5 Symmetries of the SDYM system

In fact, the dressed vertices (39) have an interesting interpretation in terms of the symmetries of the SDYM equations (see also [9]). Consider a small perturbation \(\Lambda(x)\) around a solution \(\Phi\) of the SDYM equations. It satisfies the linearization of Eq. (19),

\[
\partial^2 \Lambda - ig [\partial_{x+iy} \Lambda, \partial_{0-z} \Phi] - ig [\partial_{x+iy} \Phi, \partial_{0-z} \Lambda] = 0. \quad (43)
\]

A similar equation is obtained by differentiating (25) with respect to \(J(y)\):

\[
\partial^2 \frac{\delta \Phi f(x)}{\delta J(y)} - ig [\partial_{x+iy} \frac{\delta \Phi f(x)}{\delta J(y)}, \partial_{0-z} \Phi] - ig [\partial_{x+iy} \Phi, \partial_{0-z} \frac{\delta \Phi f(x)}{\delta J(y)}] + \delta(x-y) = 0. \quad (44)
\]

Actually, \(\Lambda(q) \equiv (-iq^2) \lambda \frac{\delta \Phi f(x)}{i \delta J(q)}\) is nothing else than the generating functional of the dressed vertex \((-ik^2)(-iq^2)\lambda \langle \phi(k)\phi(q) \rangle_{1\cdots n}\). The constant \(\lambda\) will be fixed later. We can solve for \(\Lambda(q)\) as we did in section 3. Take \(\Phi\) to be the Bethe Ansatz solution but with the plane wave \(j = j^*\) missing (\(j^*\) is some index between 1 and \(n\)) and expand \(\Lambda_{q=k^*_j}\) in powers of the coupling constant \(g\):

\[
\Phi(x) = \sum_{m=1}^{\infty} \Phi^{(m)}(x) \quad \text{with} \quad \Phi^{(1)}(x) = -i \sum_{j \neq j^*}^{n} T^{a_j} e^{-ik_j x} f(k_j),
\]

\[
\Lambda_{j^*}(x) = \sum_{m=1}^{\infty} \Lambda_{j^*}^{(m)}(x) \quad \text{with} \quad \Lambda_{j^*}^{(1)}(x) = -iT^{a_j^*} e^{-ik_{j^*} x}. \quad (45)
\]

The function \(f(k_j)\) has support on the light-cone \(k_j^2 = 0\). However \(k_{j^*}\) is unrestricted and may be off-shell. At successive orders in the coupling constant \(g\), we have diagrammatically

\[\text{Fig.}\]

5 A solution of the homogeneous equation (43) is obtained by restricting \(k_{j^*}\) to be on-shell.
The trees in \( \Lambda_{j*}^{(n)}(k) \) have \((n-1)\) on-shell legs and two off-shell legs with momenta \( k \) and \( k_{j*} \). The sum of the trees ending with \( n \) different momenta \( k_{j*} \) \((j = 1, \ldots, n)\) reproduces the dressed vertex \((-i k_{j*}^2) \lambda \langle \phi(k) \phi(k_{j*}) \rangle_{1 \cdots n} \) with two off-shell momenta \( k \) and \( k_{j*} \). One can even find a closed analytic expression for it; for \( f(k) = Q/k_{x+i y} \) as before and \( \lambda = -2 \), equations (46) are precisely the equations that Dunn, Mahlon and Yan have for two off-shell current amplitudes in QCD [11]. Explicit expressions can be found in their article and will not be reproduced here.

A one-parameter family of symmetries \( \Lambda_s \) is constructed from the following pair of recursion relations [27]:

\[
\begin{align*}
\partial_{0-z} \Lambda_{s+1} &= \partial_{x-iy} \Lambda_s - ig \left[ \partial_{0-z} \Phi, \Lambda_s \right], \\
\partial_{x+iy} \Lambda_{s+1} &= \partial_{0+z} \Lambda_s - ig \left[ \partial_{x+iy} \Phi, \Lambda_s \right].
\end{align*}
\]

These equations are compatible if \( \Lambda_s \) is a solution of (43). Moreover, \( \Lambda_{s+1} \) is a symmetry if \( \Phi \) is a solution of the SDYM equations. It is known that these symmetries form an affine Lie algebra. This is at the basis of the conjecture [28] that all integrable models are some reduction of a SDYM system. Of course, the \( \Lambda \)'s are symmetries of the equations of motion and not necessarily of the action. In fact, in the hierarchy (47), only \( \Lambda_0 = T^a \) and \( \Lambda_1 = -ig \left[ \phi, T^a \right] \) are true symmetries of the action. Nevertheless, the hierarchy (47) defines an infinite set of conserved currents whose classical expressions are

\[
\begin{align*}
\mathcal{J}_{s,0+z} &= \partial_{0+z} \Lambda_s - ig \left[ \partial_{x+iy} \Phi, \Lambda_s \right], \\
\mathcal{J}_{s,x-iy} &= \partial_{x-iy} \Lambda_s - ig \left[ \partial_{0-z} \Phi, \Lambda_s \right], \\
\partial_{\mu} \mathcal{J}_{s}^{\mu} &= \frac{1}{2} ig \left[ \Lambda_{x-1}, \partial^{2} \Phi - ig \left[ \partial_{x+iy} \Phi, \partial_{0-z} \Phi \right] \right] = 0.
\end{align*}
\]

Since these currents are only known for classical solutions \( \Phi \), one can only derive tree level identities and not true Ward identities relating Green functions of different orders in \( \hbar \).

6 Conclusion

In this paper, I have investigated the idea that QCD computations may be simplified when considering only maximally helicity (MHV) violating processes.
Although this statement seems natural, its concrete realization is based upon special tools like supersymmetry identities and self-dual Yang–Mills (SDYM) equations. I showed that a scalar action reproduces exactly the MHV amplitudes up to one loop. The extension of this result beyond one-loop is not obvious, since I can no more invoke supersymmetric arguments to restrict myself to scalar QCD. It is unlikely that the SDYM action is sufficient to describe the MHV amplitudes at two-loop, even when all external helicities are set equal. Namely, any cut in a diagram that isolates a tree amplitude will be automatically zero for the SDYM action, whereas it will be non-zero in QCD owing to the presence of different helicities in the intermediate states.

The vanishing of tree amplitudes is the signature of a quantized integrable model [5]. We have further probed Bardeen’s idea that the $S$-matrix of QCD between positive helicity gluons is intimately related to the quantization of a SDYM system. The fact that one-loop amplitudes are calculated from a generator of the symmetry of the SDYM equations is one more piece of evidence in favour of an anomaly-type mechanism to generate the simple non-vanishing one-loop amplitudes. A more detailed investigation of this question is still needed.

A necessary step is to study the quantum symmetries of the system. It was noted at the end of the previous section that the SDYM action has global symmetries and that classically an infinite set of conserved currents can be found. It is still an open question whether they are responsible for the vanishing of tree amplitudes. The Donaldson–Nair–Schiff action (16) may be an interesting alternative, since it also possesses an infinite-dimensional current algebra symmetry. This model is moreover viewed [20] as an example of a four-dimensional conformal theory and it was already suggested in Ref. [5] that the infinite number of symmetries generates Ward identities that ensure the vanishing of tree amplitudes. We have checked that the Donaldson–Nair–Schiff action reproduces the correct QCD tree current amplitudes and one-loop four particle amplitude. However, due to a proliferation of vertices, we have been unable so far to show that it also reproduces the other one-loop amplitudes. If this proves to be true, then all the one-loop amplitudes of this conformal theory may be deduced from available QCD calculations [25,26].

There remains several other open questions. What can this approach bring in return to QCD? Can the mixing of helicities be described by some appropriate coupling between a self-dual and an anti-self-dual fields? These questions deserve further investigation.

---

6 Incidentally, there is a delicate issue with the signature of spacetime. The Donaldson–Nair–Schiff action and the SDYM equations are usually written using a $(2 + 2)$ signature. One must be careful with analytic continuation; for example, I already mentioned that the cubic vertex within this signature does not vanish.
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Note added

After completion of this work, two related papers appeared. Chalmers and Siegel [29] propose yet another SDYM action \( S = \int d^4x \Lambda (\partial^2 \phi - ig[\partial_{x+iy} \phi, \partial_{0-z} \phi]). \)

It is derived as a truncation of the \( N = 4 \) supersymmetric Yang–Mills action in the light-cone formalism. The equations of motion for the Lagrange multiplier \( \Lambda \) are identical to the equations (43) for the generators of the SDYM symmetries. Moreover, the authors prove my conjecture that the Donaldson-Nair-Schiff action gives the correct one-loop amplitudes. Korepin and Oota [30] derive a closed expression for the Bethe Ansatz solution described in Appendix B.

A Spinor notations

This appendix is intended for readers familiar with the spinor notation. Consider first a massless Weyl spinor

\[
\begin{align*}
\alpha(k) &= \begin{pmatrix} \frac{k_{x+iy}}{\sqrt{|k_{0+z}|}}, -\text{sign}(k_0) \frac{k_{0+z}}{\sqrt{|k_{0+z}|}} \end{pmatrix}.
\end{align*}
\]

(A.1)

Notice that on-shell we have either \( k_{0+z}, k_{0-z}, k_0 > 0 \) or \( k_{0+z}, k_{0-z}, k_0 < 0 \). The spinor products

\[
\begin{align*}
\langle 12 \rangle &= \alpha^\dagger(k_1)\alpha(k_2) = -(Q_1 - Q_2) \frac{k_{1,x+iy}}{\sqrt{|k_{1,0+z}|}} \frac{k_{2,x+iy}}{\sqrt{|k_{2,0+z}|}},
\end{align*}
\]

(A.2)

\[
\begin{align*}
[12] &= \text{sgn} \langle \theta_1 \theta_2 \rangle \langle 21 \rangle^* = X(k_1, k_2) \frac{\sqrt{|k_{1,0+z}|}}{k_{1,x+iy}} \frac{\sqrt{|k_{2,0+z}|}}{k_{2,x+iy}},
\end{align*}
\]

(A.3)
One defines polarization vectors in a reference frame $q$ by their projection on an arbitrary null vector $p$,

$$
 p \cdot \varepsilon^+(k; q) = \frac{\langle qp \rangle [pk]}{\sqrt{2} \langle qk \rangle} \text{sign}(k_0), 
 p \cdot \varepsilon^-(k; q) = \frac{\langle kp \rangle [pq]}{\sqrt{2} [kq]} \text{sign}(k_0).
$$

(Notice the opposite sign in Ref. [11].)

The four point function in these notations takes the compact form

$$
 m_{\text{one-loop}}(k_1, +; \ldots k_4, +) = -N_c g^4 \frac{i}{48\pi^2} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle},
$$

which is the one usually quoted in the literature.

### B The Bethe Ansatz for a SDYM solution

In this appendix, I indicate why (30) is indeed a solution and I comment on the terms left over in the Bethe Ansatz solution (32).

It is easy to see that the recursion relation (28) is satisfied by (30) provided the following identity is valid on-shell:

$$
 (p_1 + \cdots + p_m)^2 = \sum_{j=1}^{m-1} X(p_1 + \cdots + p_j, p_{j+1} + \cdots + p_m) (Q_j - Q_{j+1})
$$

The proof goes as follow:

$$
 \sum_{j=1}^{m-1} X(p_1 + \cdots + p_j, p_{j+1} + \cdots + p_m) (Q_j - Q_{j+1}) \\
 = \sum_{j=1}^{m-1} \sum_{s=1}^j \sum_{t=j+1}^m X(p_s, p_t) (Q_j - Q_{j+1}) \\
 = \sum_{1 \leq s \leq j < t \leq m} 2(p_s \cdot p_t) (Q_s - Q_t)^{-1}(Q_j - Q_{j+1})
$$
After inserting the plane wave expansion (31) for \( j(x) \) into (30), the \( p_j \)'s momenta are replaced by the plane wave momenta. It may happen that two or more consecutive \( p_j, p_{j+1} \) are replaced by the same momentum \( k_\ell \). In that case one may worry about factors like \( (Q_j - Q_{j+1})^{-1} \). The way to deal with this problem is first to replace \( Q_j \) by \( Q_\ell + \epsilon_j \), then to symmetrize all the \( \epsilon_j \)'s and finally to take the limit \( \epsilon_j \to 0 \). For example, the limits

\[
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1} \xrightarrow{\quad p_1 = p_2 \neq p_3 \quad} \frac{1}{2} (Q_1 - Q_3)^{-2}, \\
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1} \xrightarrow{\quad p_1 = p_2 = p_3 \quad} 0, \\
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1}(Q_3 - Q_4)^{-1} \xrightarrow{\quad p_1 \neq p_4 \neq p_3 \quad} \frac{1}{2} (Q_1 - Q_4)(Q_1 - Q_2)^{-2} \times (Q_2 - Q_4)^{-2}, \\
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1}(Q_3 - Q_4)^{-1} \xrightarrow{\quad p_1 = p_2 \neq p_3 \neq p_4 \quad} \frac{1}{2} (Q_1 - Q_4)^{-3}, \\
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1}(Q_3 - Q_4)^{-1} \xrightarrow{\quad p_1 \neq p_2 \neq p_3 \quad} 0, \\
(Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1}(Q_3 - Q_4)^{-1} \xrightarrow{\quad p_1 = p_2 = p_3 \quad} \frac{1}{6} (Q_1 - Q_4)^{-3},
\]

enter in the expressions for the first few terms, \( \Phi^{(1)}(x), \Phi^{(2)}(x), \Phi^{(3)}(x) \) and \( \Phi^{(4)}(x) \), in the power expansion of the Bethe Ansatz solution:

\[
\Phi^{(1)}(x) = -i \sum_{j=1}^{n} T^{a_j} e^{-ik_j x} f(k_j), \quad (B.3)
\]

\[
\Phi^{(2)}(x) = -ig \sum_{\text{permutations of (12) in \( \{1 \ldots n\} \)}} T^{a_1} T^{a_2} e^{-i(k_1+k_2)x} f(k_1) f(k_2) (Q_1 - Q_2)^{-1} \quad (B.4)
\]

\[
\Phi^{(3)}(x) = -ig^2 \sum_{\text{permutations of (123) in \( \{1 \ldots n\} \)}} T^{a_1} T^{a_2} T^{a_3} e^{-i(k_1+k_2+k_3)x} f(k_1) f(k_2) f(k_3) \\
\times (Q_1 - Q_2)^{-1}(Q_2 - Q_3)^{-1} \\
-16g^2 \sum_{\text{permutations of (12) in \( \{1 \ldots n\} \)}} T^{a_1} T^{a_1} T^{a_2} e^{-i(2k_1+k_2)x} f(k_1)^2 f(k_2)(Q_1 - Q_2)^{-2}
\]

and is based on the algebraic relations (2) and (3).
\[
+ ig^2 \sum_{\text{permutations of } (12) \text{ in } \{1\ldots n\}} T^{a_1} T^{a_2} T^{a_1} e^{-i(2k_1+k_2)x} f(k_1)^2 f(k_2)(Q_1 - Q_2)^{-2}
\]

\[- \frac{i g^2}{2} \sum_{\text{permutations of } (12) \text{ in } \{1\ldots n\}} T^{a_1} T^{a_2} T^{a_2} e^{-i(k_1+2k_2)x} f(k_1)f(k_2)^2(Q_1 - Q_2)^{-2} \quad \text{(B.5)}
\]

It would be interesting to get a closed formula for all \(\Phi^{(m)}(x)\).

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