BOUNDS ON THE MULTIPLICITY OF THE HECKE EIGENVALUES

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Abstract. Fix an integer $N$ and a prime $p \nmid N$ where $p \geq 5$. We show that the number of newforms $f$ (up to a scalar multiple) of level $N$ and even weight $k$ such that $T_p(f) = 0$ is bounded independently of $k$, where $T_p(f)$ is the Hecke operator.

1. Introduction

1.1. Motivation. We begin by describing the spectral multiplicity hypothesis introduced in the work of Phillips and Sarnak [PS94], and motivating our results from the analytic point of view. Let $\mathbb{H}$ be the upper half-plane and consider the Hilbert space $L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H})$ equipped with the Laplace-Belteram operator $\Delta$ induced from the hyperbolic metric $\frac{1}{y^2}(dx^2 + dy^2)$ on $\mathbb{H}$. Selberg established the spectral decomposition of $\Delta$ acting on $L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H})$:

$$L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H}) = L^2_{\text{cusp}}(SL_2(\mathbb{Z}) \backslash \mathbb{H}) + \text{Eis}$$

where $\text{Eis}$ is the contribution of the Eisenstein series (form the continuous part of the spectrum filling out $[1/4, \infty)$) and its possible residues (the constant function which is the pole at $s = 1$), and $L^2_{\text{cusp}}(SL_2(\mathbb{Z})$ is the space of Maass cusp forms; see [Sel14]. It is widely believed that the entire cusp spectrum is simple; and this is very difficult to prove. The spectral multiplicity hypothesis is the assumption that a positive density of these cusp eigenvalues have a uniformly bounded multiplicity.

This assumption and its variations are essential in the deformation theory of hyperbolic surfaces that is initiated by Phillips, Sarnak and later developed by Wolpert. It is proved that, under this assumption, a generic hyperbolic curve with cusp, has only finitely many cusp eigenvalues unlike $L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H})$ or the arithmetic lattices where the cuspidal spectrum dominates the continuous spectrum.

The best bounds on the multiplicity of the eigenvalues are far from this conjecture. More precisely, let $m(t)$ be the multiplicity of the eigenvalue $1/4 + t^2$ on $L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H})$. The best known upper bound is

$$m(t) \leq \frac{\pi t}{24 \log(t)} (1 + o(1)),$$

where $o(1) \to 0$ as $t \to \infty$; see [Sar02]. As pointed out by Sarnak [Sar02 Page 2], even showing $m(t) = o(t/\log(t))$ is very difficult. The problem is that the Laplacian eigenvalues for large $t$ are not isolated (the expected consecutive distance is $1/t$) and this makes the problem inaccessible by analytic methods.

Bounding the multiplicity of the laplacian eigenvalues also appears in the quantum unique ergodicity. It is known that the existence of a sequence of eigenvalues
with large multiplicity violates the QUE conjecture \cite{Sar11}. It is expected that for every compact hyperbolic manifold $m(t) = t^\epsilon$, for any $\epsilon > 0$ and again the best known bound is $m(t) = O(t^{d-1}/\log(t))$ \cite{B77}, where $d$ is the dimension of the manifold.

Similarly, for the zeros of the Riemann zeta function, it is conjectured that the zeros are simple. Currently, there is no approach to prove simplicity of zeros even by assuming the Riemann hypothesis. However, in this case the average distance between two consecutive zeros in the interval $[1/2 + iT, 1/2 + i(T + 1)]$ is $1/\log(T)$ and this makes the problem easier from the analytic point of view. In fact, it is proven that a positive proportion of the zeros of the Riemann zeta function are simple; see \cite{HB79} and \cite{Mon73}.

In this paper, we consider the problem of bounding the multiplicity of Hecke operator’s eigenvalues for the family of the holomorphic modular forms with the fixed level and varying weight (as we note later the multiplicity is not bounded in the level aspect by the existence of CM modular forms).

Let $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, N|c \right\}$ be the Hecke congruence subgroup of level $N$. Let $S_{k,\chi}(N)$ be the space of holomorphic cusp forms of even weight $k \in \mathbb{Z}$, level $N$, and nebentypus character $\chi$. It is the space of the holomorphic functions $f$ on the upper half plane $\mathbb{H}$ such that

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k \chi(d) f(z) \tag{1.2}$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and $f$ converges to zero as it approaches each cusp (we have finitely many cusps for $\Gamma_0(N)$ that are associated to the orbits of $\Gamma_0(N)$ acting by Möbius transformations on $\mathbb{P}^1(\mathbb{Q})$). It is well-known that $S_{k,\chi}(N)$ is a finite dimensional vector space over $\mathbb{C}$, and is equipped with the Petersson inner product $\langle f, g \rangle := \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z)\overline{g(z)}y^k dx dy / y^2$ which makes it into a Hilbert space. Assume that $p$ is a fixed prime number where $p \nmid N$. Then one can define a self-adjoint Hecke operator $T_p$ on $S_k(N)$:

$$T_p(f)(z) := \sum_{n=1}^\infty a_n e(nz) + p^{k-1} \chi(p) \sum_{n=1}^\infty a_n e(pnz), \tag{1.3}$$

where $f(z) = \sum_{n=1}^\infty a_n e(nz)$ is the Fourier expansion of $f$ at the cusp $\infty$. In particular, if $f$ is an eigenfunction of $T_p$ with eigenvalue $\lambda_p(f)$ then $a_p = a_1 \lambda_p(f)$.

By Deligne’s result \cite{Del71} the Ramanujan-Petersson conjecture holds for $f$ and we have $|\lambda_p(f)| \leq 2p^{\frac{k-1}{2}}$. Under Langlands’ philosophy, the Hecke operator $T_p$ is the $p$-adic analogue of the Laplace operator (the eigenvalues of $T_p$ determine the Satake parameters of the associated local representation $\pi_p$ of $GL_2(\mathbb{Q}_p)$ just as the Laplace eigenvalue of the Maass form determines the associated local representation $\pi_\infty$ of $GL_2(\mathbb{R})$).

Let $m_p(\lambda, k, \chi, N)$ be the multiplicity of $\lambda$ as an eigenvalue of $T_p$ acting on $S_{k,\chi}(N)$. We fix the level $N$ in our paper and the implicit constants in the $O$ notations depend on $N$. The trivial upper bound is $m_p(\lambda, k, \chi, N) = O(k)$ which is the dimension of $S_{k,\chi}(N)$. By Eichler-Selberg trace formula Serre proved that $m_p(\lambda, k, \chi, N) = o(k)$ \cite{Ser97} and it is not hard to derive from his work that $m_p(\lambda, k, \chi, N) = O(k/\log(k))$; see \cite{MS09}. This is the analogue of (1.1) in the
weight aspect. Serre derived a number of striking consequences from this bound on the multiplicities. For example, he proved that if \(N_i\) is any sequence of positive integers and if \(d_i\) denotes the maximum of the dimensions of the simple abelian variety quotients of \(\text{Jac}(X_0(N_i))\), then \(d_i \to \infty\) as \(N_i \to \infty\). In particular, there are only finitely many positive integers \(N\) for which \(\text{Jac}(X_0(N_i))\) is isogenous to a product of elliptic curves.

For \(\lambda \neq 0\), Frank Calegari in his blog post [Cal15] proved that
\[
(1.4) \quad m_p(\lambda, k, \chi, N) = O(v_p(\lambda)^2),
\]
where the implicit constant in \(O(\cdot)\) is independent of \(k\) and only depends on the fixed numbers \(N\) and \(p\). We briefly explain his method. Let \(f\) be a modular form with \(p\)-th Hecke eigenvalue \(\lambda \neq 0\). Then the slope of the modular form \(f\) is finite. Since the level is fixed, there are only finitely many Coleman families of modular forms \([Col96, Col95]\) that cover these modular forms. (1.4) follows from Wan’s explicit quadratic bound [Wan98] on Gouvea and Mazur conjecture [Maz89a]. This method does not work for bounding \(m_p(0, k, \chi, N)\).

In this paper we use methods in the deformation theory of Galois representations and the Taylor-Wiles method to give a quantitative bound on \(m_p(0, k, \chi, N)\) which are independent of \(k\).

**Theorem 1.1.** We have
\[
(1.5) \quad m_p(0, k, \chi, N) \leq \frac{\alpha(N^2) \alpha(N)(p + 3)^2}{24^2},
\]
where \(\alpha(N) = N \prod_{p|N} (1 + 1/p)\).

We give an outline of our proof for bounding \(m_p(0, k, \chi, N)\) below.

1.2. **Outline of the proof.** We fix a semi-simple modular absolutely irreducible residual representation \(\bar{\rho}: G_\mathbb{Q} \to \text{GL}_2(\overline{\mathbb{F}}_p)\) with prime to \(p\) conductor dividing \(N\) and consider the problem only for \(f \in S_{k, \chi}(N)\) whose associated mod \(p\) representation is isomorphic to \(\bar{\rho}\). Note that by (part of) Serre’s conjecture [Ser87], there are only a bounded number (only depends on prime \(p\) and \(N\) and not on \(k\)) of such \(\bar{\rho}\); see Lemma [2.3]. Moreover, since \(\lambda_p(f) = 0\), the restriction of the \(p\)-adic representation \(r_f: G_\mathbb{Q} \to \text{GL}_2(\mathbb{Z}_p)\) associated to \(f\) to the decomposition group at \(p\) is a dihedral representation (induced from a Lubin-Tate character of \(G_\mathbb{Q}^\text{red}\)); see Theorem [2.1]. Moreover, it has irreducible reduction if \(k\) is not congruent to 1 mod \(p + 1\) and in particular has irreducible reduction when \(p > 2\) and \(k\) is even; see Theorem [2.1].

So, we study the deformation ring of deformations of \(\bar{\rho}\) unramified outside \(Np\) and whose restriction to the inertia subgroup of the decomposition group at \(p\) is dihedral. We construct a deformation ring \(R_D\) such that any \(r_f\) factors through it. By the class field theory \(R_D\) will be a \(\Lambda_p = \mathbb{Z}_p[[1 + p\mathbb{Z}_p^2] \times \mathbb{Z}_p]\)-algebra (the deformation ring associated to the characters of \(G_\mathbb{Q}^\text{red}\)). Note that \(\Lambda_p\) is isomorphic to the power series ring \(\mathbb{Z}_p[[S_1, S_2, S_3]]\) (at least if \(p > 2\)). To an integer \(k > 1\) there is a dimension 1 prime ideal \(P_k\) of \(\Lambda_p\) corresponding to the character of \(1 + p\mathbb{Z}_p^2\) sending \(a\) to \(a^{k-1}\) and the unmarried character sending Frobenius element to \(\sqrt{1}\); see Theorem [2.1]. If \(f\) has weight \(k\) then \(r_f\) corresponds to a point of \((R_D/P_k R_D)^\text{red}[1/p]\).

So it would suffice to show that \(\text{dim}_{\mathbb{Q}_p^\text{red}}(R_D/P_k R_D)^\text{red}[1/p]\) is bounded independent of \(k\). Equally it would suffice to show that \(R_D\) is finitely generated as a \(\Lambda_p\)-module. By the topological form of Nakayama’s lemma (see [Eis95, Exercise 7.2]) it suffices
to show that for one prime ideal $P$, $R_D/PR_D$ is finitely generated as a $\mathbb{Z}_p$-module. If $p \geq 5$ then we show that there exists a prime ideal $P_{\text{min}}$ so that the Hodge-Tate weights are of moderate Hodge-Tate type $[\text{FM93}]$ (in the Fontaine-Laffaille range, we use strong Serre’s conjecture for this step). By a modularity result in the work of Khare and Wintenberger, we show that there exists a finite $\mathbb{Z}_p$ algebra $\bar{R}_S^{k'-1}$ with a surjection to $R_D/P_{\text{min}}R_D$. We show that $\bar{R}_S^{k'-1}$ is isomorphic to an appropriate Hecke algebra which gives explicit bound on the number of generators of $\bar{R}_S^{k'-1}$ and $R_D/P_{\text{min}}R_D$ as finite $\mathbb{Z}_p$-modules.

1.3. Notations. We assume that $N$ is a fixed integer and $p$ is a fixed prime number where $\gcd(p,N) = 1$. We denote the space of modular forms of level $N$ and weight $k$ with nebentypus character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^*$ by $S_{k,\chi}(N)$ which is a finite dimensional vector space of dimension $\frac{\alpha(N)k}{12} + O(N^{1/2+\epsilon})$, where $\alpha(N) = N \prod_{p^i | N} (1 + 1/p)$. Define $S_k(N) := \bigoplus S_{k,\chi}(N)$. For a newform $f \in S_k(N)$, we write $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ for the Fourier expansion of $f$ at cusp $\infty$ and write $a_p(f)$ for the $p$-th Fourier coefficient of $f$. Let $G_{\mathbb{Q}} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $G_{\mathbb{Q}_p} := \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, $I_p$ be the inertia subgroup at $p$ and $\mathbb{Q}_{p^n}$ be the unique unramified extension of degree $n$ of $\mathbb{Q}_p$. We write $\varepsilon : G_{\mathbb{Q}} \to \mathbb{Z}_p^*$ for the cyclotomic character. We denote the local reciprocity map of the local field $\mathbb{Q}_{p^n}$ by $\text{rec}_{p^n} : \mathbb{Q}_{p^n}^* \to \mathbb{G}_a^{\text{ab}}_{\mathbb{Q}_{p^n}}$ which is well-defined up an embedding $\mathbb{Q}_{p^n} \to \bar{\mathbb{Q}}_p$. We fix an embedding $i : \mathbb{Q} \to \bar{\mathbb{Q}}_p$ which defines a $p$-adic valuation $\varepsilon_p$. Given a newform $f \in S_{k,\chi}(N)$, we denote the associated $p$-adic representation by $\rho_f : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}_p)$, its restriction to the decomposition group at $p$ by $\rho_{f,p} : G_{\mathbb{Q}_p} \to \text{GL}_2(\mathbb{Z}_p)$ and its residual representation by $\bar{\rho}_f : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F}_p)$ which are well-defined up to conjugation; see [Del71]. Define the two Lubin-Tate characters $\varepsilon_2, \varepsilon'_2 : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^*$ by requiring $\varepsilon_2, \varepsilon'_2 \circ \text{rec}_{p^n} = 1$ and $\bar{\varepsilon}_2, \bar{\varepsilon}'_2 \circ \text{rec}_{p^n} = \mathbb{Z}_p^2 \to \bar{\mathbb{Q}}_p$ given by two natural embedding of $\mathbb{Z}_p^2 \embed \bar{\mathbb{Q}}_p$.

Let $\mu_a : G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p$ be the unique unramified character which sends Frobenius to $a \in \bar{\mathbb{Q}}_p$.

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2. Modular forms with $a_p(f) = 0$

Let $f \in S_{k,\chi}(N)$ be a newform with $a_p(f) = 0$ and the residual representation $\bar{\rho}_f$, where $\gcd(p,N) = 1$. We cite a theorem which shows that the restriction of the Galois representation $\rho_f$ to the decomposition group at $p$ is a dihedral representation.

**Theorem 2.1** (Breuil). We have

\[
(2.1) \quad \rho_{f|G_{\mathbb{Q}_p}} = \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}}^{\varepsilon_2-1}} \otimes \mu_{\sqrt{-1}}^T.
\]

Moreover, $\text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}}^{\varepsilon_2-1}} \otimes \mu_{\sqrt{-1}}^T$ is a crystalline representation of $G_{\mathbb{Q}_p}$ for any integer $k \geq 2$, and its mod $p$ reduction is absolutely irreducible if $k$ is not congruent to $1$ mod $p + 1$. In particular $\bar{\rho}_f$ is absolutely irreducible when $p > 2$ and $k$ is even.
Proof. See [Bre93 Proposition 3.1.2].

2.1. Mod $p$ representations. We cite a theorem which shows that every modular form with $p$-th Fourier coefficient zero is congruent up to a twist with a modular form of weight at most $(p+3)/2$ mod $p$. Note that for $p \geq 5$, $(p+3)/2 \leq p - 1$ which is in the Fontaine-Laffaille range.

**Theorem 2.2** (Ash and Stevens - Edixhoven). Let $h \in S_{k,\chi}(N)$ be a newform then there exist integers $i$ and $k'$ with $0 \leq i \leq p - 1, k' \leq p + 1$ and a modular form $g \in S_{k',\chi}(N)$ such that $\tilde{\rho}_h \sim \varepsilon^i \otimes \tilde{\rho}_g$. Moreover, assume that $a_p(h) = 0, p$ is an odd prime number and $k$ is an even integer. Let $0 \leq a < b \leq p - 1$ be uniquely determined by the congruence condition $k - 1 = b + pa$ or $a + bp$ mod $p^2 - 1$. Then $\tilde{\rho}_g$ is absolutely irreducible and the only possible values for $(i, k')$ which all occur are:

\[
\begin{cases}
(a, 1 + b - a), (b - 1, p + 2 + a - b) & \text{if } b - a \neq 1 \\
(a, 2) & \text{if } b - a = 1.
\end{cases}
\]

Note that we can choose $k' \leq (p+3)/2$ and since $p \geq 5$ then $k'$ is in the Fontaine-Laffaille range.

**Proof.** See [Edi92 Theorem 3.4] and [Edi92 Theorem 4.5].

In the following Lemma, we use the above theorems and give an upper bound on the number of mod $p$ reduction of the newforms with level $N$ and $a_p(f) = 0$.

**Lemma 2.3.** Given $N$, even integer $k$ and a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}^*$, then the number of mod $p$ modular forms that has a lift to a newform $f \in S_{k,\chi}(N)$ with $a_g(f) = 0$ where $p$ is an odd prime number is bounded (independent of $k$) by $\frac{\varphi(N)}{24}$.

**Proof.** By Theorem 2.2 there exists a modular form $g \in S_{k',\chi}(N)$ such that $\varepsilon^i \otimes \tilde{\rho}_g = \tilde{\rho}_f$ and $k' \leq (p+3)/2$ where $k'$ and $i$ are uniquely determined by $k$ and $p$. The dimension of the vector space $S_{k',\chi}(N)$ is bounded by $\frac{\varphi(N)}{12}$. This concludes the proof of our lemma.

3. Deformation rings of Galois representations

In this section, we define several deformation rings which are complete Noetherian local rings with fixed residue field of characteristic $p$. We cite a result of Khare and Wintenberger [KW09 Theorem 10.1] on finiteness of deformation rings with some conditions. We keep our notations consistent with their work in this section and work with framed deformation rings. We refer the reader to the notes of Gebhard [Boe11] and the work of Mazur [Maz89] for an introduction on the deformations of Galois representations.

3.1. Galois deformation rings with local conditions. Let $S$ be the union of the set of primes in $pN$. Let $\tilde{\rho} : G_\mathbb{Q} \rightarrow GL_2(\mathbb{F}_p)$ be an absolutely irreducible modular representation of conductor dividing $N$ which is odd, i.e. $\det(\tilde{\rho}(c)) = -1$, where $c$ is complex conjugation. For $v \in S$, let $R^\otimes_v$ denote the local universal framed deformation ring of $\tilde{\rho}|_{G_{\mathbb{Q}_v}}$, with a distinguished basis lifting the standard basis of $\mathbb{F}_p^2$. Let $\psi : G_\mathbb{Q} \rightarrow \mathcal{O}_E^\times$ be a continuous homomorphism such that $\psi = \det(\tilde{\rho})$ mod $p$. Let $R^\otimes_{v,\psi}$ be the local framed deformation ring of $\tilde{\rho}|_{G_{\mathbb{Q}_v}}$ with the determinant
and only if $\psi$ the property that a continuous homomorphism $\psi : R^\square_p \to \mathbb{Q}_p$ factors through $R^\square_v,\psi$ if and only if $\psi \circ \rho^{\mathrm{univ}}$ is crystalline, and $HT(\psi \circ \rho^{\mathrm{univ}}) = (r, s)$, where $HT(\psi \circ \rho^{\mathrm{univ}})$ is the Hodge-Tate weights of $\psi \circ \rho^{\mathrm{univ}}$. Furthermore, $\tilde{R}^\square_v,\psi$ is isomorphic to a power series ring in 4 variables over $\mathbb{Z}_p$.

3.1.2. $\tilde{R}^\square_v,\psi$ for $v \neq p$ : Deformations of fixed inertial type. We follow closely [Gee14 Section 3.30]. Given a local representation $\rho : G_{\mathbb{Q}_v} \to GL_2(\mathbb{Q}_p)$ there is a Weil-Deligne representation $WD(\rho)$ associated to $\rho$. If $WD = (r, N)$ is a Weil-Deligne representation, then we call $(r|I_{Q_v}, N)$ an inertial WD-type, where $r|I_{Q_v}$ is the restriction of $r$ to the inertia subgroup $I_{Q_v}$. In particular, if $\rho$ is unramified then its inertial WD-type is $(Id, 0)$. Let $\tau$ be an inertial WD-type of conductor $|v|_{\mathrm{ord}, (N)}$.

Theorem 3.1. There is a unique reduced, p-torsion free quotient $R^\square_v,\psi$ of $R^\square_v$ with the property that a continuous homomorphism $\psi : R^\square_v \to \mathbb{Q}_p$ factors through $R^\square_v,\psi$ if and only if $\psi \circ \rho^{\mathrm{univ}}$ has inertial WD-type $\tau$. Moreover, for all but finitely many $\tau$, we have $\tilde{R}^\square_v,\psi = 0$ and if $\tilde{R}^\square_v,\psi$ is nonzero then it is Cohen-Macaulay, has Krull dimension 4, and the generic fibre $\tilde{R}^\square_v,\psi[1/p]$ is irreducible and formally smooth.

3.2. Finiteness of deformation rings. We follow the same notations as in [KW09 Section 10]. Let $R^\square_S,\psi$ be the global framed deformation ring of deformations of $\tilde{\rho} : G_{\mathbb{Q}_v} \to GL_2(\mathbb{F}_p)$ with fixed determinant $\psi$ which are unramified outside $S$. Let $R^\square_v,\psi$ be the local framed deformation ring of deformations of $\tilde{\rho}|_{G_{Q_v}}$ with fixed determinant $\psi$. Define $R^\square,\psi := \hat{\otimes}_{v \in S} \hat{R}^\square_v,\psi$ which is the completed tensor product of $R^\square_v,\psi$ over $\mathbb{Z}_p$. Similarly, define $\hat{R}^\square,\psi := \hat{\otimes}_{v \in S} \hat{R}^\square_v,\psi$, where $\hat{R}^\square_v,\psi$ were defined in sections 3 and 3.1.2.

(3.1) $R^\square_S,\psi := R^\square_S,\psi \otimes_{\hat{R}^\square,\psi} \hat{R}^\square,\psi$,

and $\tilde{R}^\square_S,\psi$ the image of the universal unframed deformation ring $R_S$ in $\hat{R}^\square_S,\psi$. We cite the following theorem from [KW09 Theorem 10.1]

Theorem 3.3. The ring $\tilde{R}^\square_S,\psi$ is finite as a $\mathbb{Z}_p$-module.

3.3. $\tilde{R}^\square_S,\psi$ is Cohen-Macaulay. We cite a result of Snowden [Sho18 Proposition 5.0.6] which implies that $\tilde{R}^\square_S$ is flat over $\mathbb{Z}_p$ and is Cohen-Macaulay. This result is motivated by the suggestion in the paragraph before Corollary 4.7 in [KW09]. This shows that $\tilde{R}^\square_S$ (rather than $\tilde{R}^\square_S[1/p]_{\mathrm{red}}$) is isomorphic to an appropriate completed
We introduce a deformation ring associated to modular forms with $p$-adic $G$-representation theory. For simplicity, we present our proof for $p = 5$. By fixing some inertial WD-type with conductor dividing $N$, we introduce a deformation ring $\bar{R}_S^\psi$ isomorphic to $Z_p[[t_1, \ldots, t_d]]$. Hence, it has Krull dimension 5 and it is Cohen-Macaulay, has Krull dimension 4. So, all the conditions of Proposition 3.4 are satisfied and this concludes our Theorem.

Proof. By Theorem 2.1, it follows that $\bar{\rho}$ is even. Since $\bar{\rho}$ is a deformation of $\tilde{\rho}$, it is a deformation of a $p$-adic $G$-representation. By Theorem 3.3, $\bar{\rho}$ is a deformation of $\tilde{\rho}$, where $\tilde{\rho}$ is a $p$-adic $G$-representation. Then we give explicit bounds on the number of generators of $\bar{R}_S^\psi$ is flat over $Z_p$ and Cohen-Macaulay.

Corollary 3.5. $\bar{R}_S^\psi$ is finite and Cohen-Macaulay over $Z_p$.

Proof. By Theorem 3.3, $\bar{R}_S^\psi$ is finite over $Z_p$. By Theorem 3.1, $\bar{R}_S^\psi$ is isomorphic to $Z_p[[t_1, \ldots, t_d]]$. Hence, it has Krull dimension 5 and it is Cohen-Macaulay, has Krull dimension 4. So, all the conditions of Proposition 3.4 are satisfied and this concludes our Theorem.

4. Proof of Theorem 1.1

For simplicity, we present our proof for $\chi = id$, our method generalizes for every character $\chi$ by fixing some inertial WD-type with conductor dividing $N$. We introduce a deformation ring associated to modular forms with $p$-th Fourier coefficient zero and fixed mod $p$ residual representation $\tilde{\rho}$. We show that this deformation ring is a finite $\Lambda_p$ module, and the number of its generators gives an upper bound on the multiplicity of 0 as an eigenvalue of $\mathcal{T}_p$.

4.1. The dihedral property at $p$. Let $\prod_{p,D}$ (the subscript $D$ is for “dihedral”) be the maximal profinite quotient group of $G_{\mathbb{Q}_p}$ in which the image of $G_{\mathbb{Q}_p}$ is abelian and $\pi_{D,p} : G_{\mathbb{Q}_p} \to \prod_{p,D}$ be the natural projection. Let

$$\bar{\rho}_{p,k} = \text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} \varepsilon_2 k-1 \otimes \mu_{\sqrt{-1}}.$$ 

By Theorem 2.1 it follows that $\bar{\rho}_{p,k}$ is absolutely irreducible when $p > 2$ and $k$ is even. Since $\bar{\rho}_{p,k}$ is a dihedral representation then there exists $\bar{\rho}_{p,k} : \prod_{p,D} \to GL_2(\mathbb{F}_p)$ such that $\bar{\rho}_{p,k} = \bar{\rho}_{p,k} \circ \pi_{D,p}$. Let $W_{\mathbb{Q}_p}$ be the Weil subgroup of $G_{\mathbb{Q}_p}$ and $\text{rec} : \mathbb{Q}_p^\times \to W_{\mathbb{Q}_p}$ be the Artin reciprocity isomorphism defined by the local class field theory [Ser77]. Let $\Lambda_p$ be the universal deformation ring of the deformations of $\bar{\rho}_{p,k}$; see [Maz89a, Proposition 1].

Proposition 4.1. $\Lambda_p$ is isomorphic to $Z_p[[t_1, t_2, t_3]]$.

Proof. By the class field theory, it follows that $\Lambda_p \cong Z_p[[\{1 + p\mathbb{Z}_p^2\} \times \mathbb{Z}]$. The theorem follows from the well-known isomorphism $Z_p[[1 + p\mathbb{Z}_p^2]] \cong Z_p[[t_1, t_2]]$; see [Maz89a, Proposition 11].

Let $(i, k')$ be the pair of integers associated to the weight $k$ and the prime $p$ that are defined in Theorem 2.2. It follows from Theorem 2.2 that $\text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} k'-1 \otimes \varepsilon^i \mu_{\sqrt{-1}}$ is a deformation of $\bar{\rho}_{p,k}$. Note that the difference of its associated Hodge-Tate weights is $k' - 1$ smaller than $\frac{p+1}{2}$. Let $r_{k', i} : \Lambda_p \to Z_p$ be the unique homomorphism associated to $\text{Ind}_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_p}} k'-1 \otimes \varepsilon^i \mu_{\sqrt{-1}}$. Let $P_{\text{min}} := \text{ker}(r_{k', i})$ which is a prime ideal of $\Lambda_p$. 

Let $f \in S_k(N)$ be a Hecke eigenform with $a_p(f) = 0$ and fixed mod $p$ residual representation $\bar{\rho}$. By Theorem 2.1 it follows that $\rho_{f|G_D} = \text{Ind}_{G_{\mathbb{Q}_p}^2}^{G_D} \varepsilon_2^{k-1} \otimes \mu_{\sqrt{-1}}$ is a dihedral representation, and $\bar{\rho}_{G_D}$ is absolutely irreducible, so does $\bar{\rho}$. Let $R_S$ be the global deformation ring of deformations $\rho$ of $\bar{\rho}$ which are unramified outside $pN$ and $\text{det}(\rho)$ is unramified outside only $p$. Let $R_p$ be the universal local deformation ring of $\bar{\rho}_p$; see [Maz89a, Proposition 1]. By the universal property of $R_S$, $R_p$, and $\Lambda_p$, we have the following natural maps induced from the inclusions $i_p : G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$

and the projection $\pi_{D,p} : G_{\mathbb{Q}_p} \to \prod_{p,D}$

We define

(4.1) $R_D := R_S \hat{\otimes}_{R_p} \Lambda_p$.

and

$$\rho_D := G_{\mathbb{Q}} \to GL_2(R_S) \to GL_2(R_D).$$

By the universal property of the tensor product, there is a unique map $\rho_{f,D} : R_D \to \mathbb{Q}_p$ such that the following diagram commutes

(4.2)

Finally we give a proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 2.3, $\bar{\rho}_f$ has only $\frac{(p+3)\alpha(N)}{24}$ possibilities. Fix $\bar{\rho}$ one of those $\frac{(p+3)\alpha(N)}{24}$ possible residual representations where $\bar{\rho}_{f,p} = \bar{\rho}_{p,k}$ is absolutely irreducible. Let $m_p(0, k, N, \bar{\rho})$ be the number of newforms $g \in S_k(N)$ with $a_p(g) = 0$ and $\bar{\rho}_g = \bar{\rho}$. It is enough to prove that $m_p(0, k, \bar{\rho}) \leq \frac{\alpha(N^2)(p+3)}{24}$.

Since, $\bar{\rho}$ is absolutely irreducible and dihedral at $p$ there exists $R_D$ defined in (4.1). Note that $R_D$ is a $\Lambda_p$ algebra. In what follows, we show that in order to bound $m_p(0, k, N, \bar{\rho})$, it is enough to show that $R_D$ is a finite $\Lambda_p$ module. Assume that $g \in S_k(N)$ with $a_p(g) = 0$ and $\bar{\rho}_g = \bar{\rho}$. By the diagram (4.2), there is a unique map $\rho_{g,D} : R_D \to O_g$. Let $P_k$ be the prime ideal in $\Lambda_p$ associated to the representation $\text{Ind}_{G_{\mathbb{Q}_p}^2}^{G_D} \varepsilon_2^{k-1} \otimes \mu_{\sqrt{-1}}$. Therefore, $P_k \subset \ker(\rho_{g,D})$ and $\rho_{g,D} \in \text{Hom}(R_D/P_k R_D[1/p]^{\text{red}}, \mathbb{Q}_p)$. Hence, $m_p(0, k, N, \bar{\rho}) \leq \dim_{\mathbb{Q}_p} \left( \frac{R_D/P_k R_D[1/p]^{\text{red}}}{\mathbb{Q}_p} \right)$. Assume that $R_D$ is finitely generated as a $\Lambda_p$ module with $m$ generators. It follows that

$$m_p(0, k, N, \bar{\rho}) \leq \dim_{\mathbb{Q}_p} \left( \frac{R_D/P_k R_D[1/p]^{\text{red}}}{\mathbb{Q}_p} \right) \leq m.$$
By the topological from of the Nakayama’s lemma (see [Eis95, Exercise 7.2]) if for some prime ideal \( P \), \( R_D/PR_D \otimes_{\Lambda_p} \mathbb{F}_p \) is generated by the image of \( r_1, \ldots, r_s \in R_D \) then \( r_1, \ldots, r_s \) generates \( R_D \) as an \( \Lambda_p \) module. Therefore,

\[
m_p(0, k, N, \tilde{p}) \leq \dim_{\mathbb{F}_p} \left( R_D/PR_D \otimes \mathbb{F}_p \right),
\]

where \( P \) is any prime ideal of \( \Lambda_p \). Recall the prime \( P_{\min} \) of \( \Lambda_p \) that is associated to the dihedral representation \( \text{Ind}_{G_{2p}}^{G_{2p}} \varepsilon_2^{k'-1} \otimes \varepsilon^i \mu_\sqrt{-1} \). We give an upper bound on \( \dim_{\mathbb{F}_p} \left( R_D/P_{\min}R_D \otimes \mathbb{F}_p \right) \).

Let \( \rho_{\min} \) be the quotient of \( \rho_D \otimes \varepsilon^{-i} \) obtained by the following composition

\[
G_Q \to GL_2(R_S) \to GL_2(R_D) \to GL_2(R_D/P_{\min}R_D).
\]

Let \( r_{\min} : R_S \to R_D/P_{\min}R_D = R_S \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \) be the map associated to \( \rho_{\min} \), where we identified \( \Lambda_p/P_{\min} \Lambda_p \) with \( \mathbb{Z}_p \). Note that \( r_{\min} \) is surjective. In the following lemma, we show that the determinant of \( \rho_{\min} \) is a power of the cyclotomic character.

**Lemma 4.2.** We have

\[
\det(\rho_{\min}) = \varepsilon^{k'-1}.
\]

**Proof.** Recall that \( R_S \) is the global deformation ring of \( \tilde{\rho} \) with unramified determinant outside \( \{ p \} \), then \( \det(\rho_{\min}) \) is unramified every where except at prime \( p \). At \( p \), we know that the representation is isomorphic to \( \text{Ind}_{G_{2p}}^{G_{2p}} \varepsilon_2^{k'-1} \otimes \varepsilon^i \mu_\sqrt{-1} \) which has determinant \( \varepsilon^{k'-1} \). Therefore, the character \( \varepsilon^{k'-1} \det(\rho_{\min})^{-1} : G_Q \to R_D/P_{\min}R_D \), is unramified everywhere and the only unramified character on \( G_Q \) is the identity character. Hence

\[
\det(\rho_{\min}) = \varepsilon^{k'-1}.
\]

This concludes our lemma. 

Since \( k' - 1 \leq \frac{p-1}{p} \leq p - 2 \) for \( p \geq 5 \), by Theorem 3.1, there exists \( \tilde{R}_p^{\square, \varepsilon^{k'-1}} \) which is the unique reduced, \( p \)-torsion free quotient \( \tilde{R}_p^{\square, \psi} \) of \( R_p^{\square} \) with the property that a continuous homomorphism \( \psi : R_p^{\square} \to \hat{\mathbb{Q}}_p \) factors through \( \tilde{R}_p^{\square, \psi} \) if and only if \( \psi \circ \rho^{\text{univ}} \) is crystalline, and \( HT(\psi \circ \rho^{\text{univ}}) = (0, k' - 1) \). Note that the dihedral representation \( \text{Ind}_{G_{2p}}^{G_{2p}} \varepsilon_2^{k'-1} \otimes \varepsilon^i \mu_\sqrt{-1} \) is crystalline and its Hodge-Tate’s weights are \( (0, k' - 1) \).

By fixing a basis for \( \tilde{\rho} \) and lifting it to the local universal framed deformation ring \( R_p^{\square, \varepsilon^{k'-1}} \) and its image in the global framed deformation ring \( R_S^{\square, \varepsilon^{k'-1}} \), we obtain a surjective map from the framed deformation rings to the unframed deformation rings. By Lemma 4.2, \( \det(\rho_{\min}) = \varepsilon^{k'-1} \). Hence this lift of the frame gives a surjective map from \( R_S^{\square, \varepsilon^{-i}} \otimes_{R_p^{\square, \varepsilon^{k'-1}}} \tilde{R}_p^{\square, \varepsilon^{k'-1}} \) to \( R_D/P_{\min}R_D = R_S^{k'-1} \otimes_{\mathbb{Z}_p^{k'-1}} \mathbb{Z}_p \).

Hence, we have the following commutative diagram

\[
\begin{array}{ccc}
R_S & \longrightarrow & R_S^{\square, \varepsilon^{k'-1}} \otimes_{R_p^{\square, \varepsilon^{k'-1}}} \tilde{R}_p^{\square, \varepsilon^{k'-1}} \\
\downarrow & & \downarrow \\
R_D/P_{\min}R_D
\end{array}
\]
Recall that \( \overline{R}_S^{k'} \) is the image of \( R_S \) in \( \overline{R}_S^{k'-1} \otimes_{\mathbb{R}_p} \overline{R}_p^{k'-1} \). By the above diagram \( \overline{R}_S^{k'} \) surjects to \( R_D/P_{\text{min}}R_D \). By Theorem 3.3 \( \overline{R}_S^{k'} \) is finite as a \( \mathbb{Z}_p \) module. Hence \( \dim_{\mathbb{F}_p} \left( \overline{R}_S^{k'} \otimes \mathbb{F}_p \right) = O(1) \). Therefore, we have

\[
\dim_{\mathbb{F}_p} \left( R_D/P_{\text{min}}R_D \otimes \mathbb{F}_p \right) \leq \dim_{\mathbb{F}_p} \left( \overline{R}_S^{k'} \otimes \mathbb{F}_p \right).
\]

Finally, we apply a modularity result and give a quantitate bound on \( \dim_{\mathbb{F}_p} \left( \overline{R}_S^{k'} \otimes \mathbb{F}_p \right) \). By Corollary 3.5, \( \overline{R}_S^{k'} \) is finite and Cohen-Macaulay over \( \mathbb{Z}_p \). By the discussion in [Sno18, Section 5], it follows that \( \overline{R}_S^{k'} \) is isomorphic to a Hecke algebra \( T_m \) completed at a maximal ideal \( m \) associated to the residual representation \( \overline{\rho} \). Therefore, we have

\[
\dim(\overline{R}_S^{k'} \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \leq \dim(\prod_f \mathbb{F}_p)
\]

where \( f \in S_{k'}(\Gamma_0(N_f)) \) is a newform with some level \( N_f \), its associated local representation is crystalline at \( p \), \( \det(\rho_f) = \varepsilon^{k'-1} \), and \( \rho_f = \text{Frob}_i \circ \overline{\rho} \) for some \( i \in \mathbb{Z} \).

We give an upper bound on the number of such newforms \( f \). Since the local representation is crystalline at \( p \), by Saito’s result [Sai97], \( \gcd(N_f, p) = 1 \). By our deformation conditions, \( \rho \) is unramified at all primes outside \( pN \), hence the prime divisors of \( N_f \) are a subset of prime devisors of \( N \). By a Theorem of Carayol [Car89] we know that \( N(\overline{\rho})|N_f \). More precisely, Carayol [Car89] proved that

\[
\text{Ord}_f(N_f) - \text{Ord}_f(N(\overline{\rho})) = \dim((\overline{\rho})^T - \dim(\rho_f)^T)
\]

where \( \dim(\overline{\rho}^p) \) and \( \dim(\rho_f)^p \) are the dimension of the inertial invariant subspace as \( \mathbb{F}_p^\nu \) and \( E \) vector spaces. It follows from the above that

\[
\text{Ord}_f(N_f) - \text{Ord}_f(N(\overline{\rho})) \leq \begin{cases} 1 & \text{if } p | N(\overline{\rho}) \\ 2 & \text{otherwise,} \end{cases}
\]

Therefore, \( N_f|N^2 \). The set of all new forms \( f \in S_{k'}(\Gamma_0(N_f)) \), where \( N_f|N^2 \) lifts to a subspace of \( S_{k'}(\Gamma_0(N^2)) \). Therefore,

\[
\dim(\overline{R}_S^{k'} \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \leq \dim \left( S_{k'}(\Gamma_0(N^2)) \right) \leq \frac{\alpha(N^2)k'}{12} \leq \frac{\alpha(N^2)(p+3)}{24}
\]

where \( \alpha(N) = N \prod_{p|N}(1 + 1/p) \). Hence,

\[
\dim_{\mathbb{F}_p} \left( R_D/P_{\text{min}}R_D \otimes \mathbb{F}_p \right) \leq \frac{\alpha(N^2)(p+3)}{24}.
\]

By Lemma 2.3 there are at most \( \frac{(p+3)\alpha(N)}{24} \) possible residual representation \( \overline{\rho} \). Hence,

\[
m_p(0, k, N) = \frac{\alpha(N^2)\alpha(N)(p+3)^2}{24^2}.
\]

This concludes the proof of Theorem 1.1. ■
POST-CLASSICAL THEOREMS OF THE HAMMER — 11

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