Abstract: The first estimate of the upper bound $\mu(\pi) \leq 42$ of the irrationality measure of the number $\pi$ was computed by Mahler in 1953, and more recently it was reduced to $\mu(\pi) \leq 7.6063$ by Salikhov in 2008. Here, it is shown that $\pi$ has the same irrationality measure $\mu(\pi) = \mu(\alpha) = 2$ as almost every irrational number $\alpha > 0$.

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1 Introduction

Let $\alpha \in \mathbb{R}$ be a real number. The irrationality measure $\mu(\alpha)$ of the real number $\alpha$ is the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the Diophantine inequality

$$\left| \alpha - \frac{p}{q} \right| \ll \frac{1}{q^{\mu(\alpha)}} \quad (1)$$

has finitely many rational solutions $p$ and $q$. The analysis of the irrationality measure $\mu(\pi) \geq 2$ was initiated by Mahler in 1953, who proved that

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{42}} \quad (2)$$

for all rational solutions $p$ and $q$, see [19], et alii. This inequality has an effective constant for all rational approximations $p/q$. Over the last seven decades, the efforts of several authors have improved this estimate significantly, see Table 1. More recently, it was reduced to $\mu(\pi) \leq 7.6063$, see [27]. This note has the followings result.

**Theorem 1.1.** For any number $\varepsilon > 0$, the Diophantine inequality

$$\left| \pi - \frac{p}{q} \right| \ll \frac{1}{q^{2+\varepsilon}} \quad (3)$$

has finitely many rational solutions $p$ and $q$. In particular, the irrationality measure $\mu(\pi) = 2$.

After some preliminary preparations, the proof of Theorem 1.1 is assembled in Section 3. Three distinct proofs from three different perspective are presented. Section 8 explores the relationship between the irrationality measure of $\pi = [a_0; a_1, a_2, \ldots]$ and the magnitude of the partial quotients $a_n$. In Section 12 there is an application to the convergence of the Flint Hills series. A second independent proof of the convergence of this series also appears in Section 10. Some numerical data for the ratio $\sin p_n / \sin 1/p_n$ is included in the last Section. These data match the theoretical result.

| Irrationality Measure Upper Bound | Reference   | Year       |
|----------------------------------|-------------|------------|
| $\mu(\pi) \leq 42$              | Mahler, [19]| 1953       |
| $\mu(\pi) \leq 20.6$            | Mignotte, [20]| 1974      |
| $\mu(\pi) \leq 14.65$           | Chudnovsky, [8]| 1982      |
| $\mu(\pi) \leq 13.398$          | Hata, [14]  | 1993       |
| $\mu(\pi) \leq 7.6063$          | Salikhov, [27]| 2008      |
2 Notation

The set of natural numbers is denoted by \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \), the set of integers is denoted by \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \), the set of rational numbers is denoted by \( \mathbb{Q} = \{a/b : a, b \in \mathbb{Z}\} \), the set of real numbers is denoted by \( \mathbb{R} = (-\infty, \infty) \), and the set of complex numbers is denoted by \( \mathbb{C} = \{x + iy : x, y \in \mathbb{R}\} \). For a pair of real valued or complex valued functions, \( f, g : \mathbb{C} \to \mathbb{C} \), the proportional symbol \( f \asymp g \) is defined by \( c_0 g \leq f \leq c_1 g \), where \( c_0, c_1 \in \mathbb{R} \) are constants. In addition, the symbol \( f \ll g \) is defined by \( |f| \leq c|g| \) for some constant \( c > 0 \).

3 Main Result

The analysis of the irrationality measure \( \mu(\pi) \) of the number \( \pi \) provided here is completely independent of the prime number theorem, and it is not related to the earlier analysis used by many authors, based on the Laplace integral of a factorial like function

\[
\frac{1}{i2\pi} \int_C \left( \frac{n!}{z(z-1)(z-2)(z-3)\cdots(z-n)} \right)^{k+1} e^{-t z} dz,
\]

where \( C \) is a curve around the simple poles of the integrand, and \( k \geq 0 \), see [19], [20], [8], [14], [27], and [6] for an introduction to the rational approximations of \( \pi \) and the various proofs. The improvements made using rational integrals and the prime number theorem, as the new estimate in [30], are limited to small incremental improvements, and no where near the numerical data, see Table 3. Continuing at this pace, it will take another 30 or more years to prove the true irrationality measure of \( \pi \), which is \( \mu(\pi) = 2 \) as approximated by the numerical data. These numerical approximations are not examples of the Strong law of small number, confer [13] for details. Here, the numerical data consists of very large numbers, including random evaluations.

3.1 First Proof

The first proof is based on elementary properties of the sine function and basic Diophantine analysis.

**Proof.** (Theorem 1.1) Assume \( \mu = \mu(\pi) > 2 \). Let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents of the number \( \pi \). Setting \( z = |\pi q_n - p_n| \) in the inequality

\[
z - \frac{z^3}{6} \leq \sin z \leq z,
\]

where \( |z| < 1 \), yields the inequality

\[
|\pi q_n - p_n| \ll |\sin(\pi q_n - p_n)| \ll |\pi q_n - p_n|
\]

for all large integers \( q_n \geq 1 \). Observe, that line (6) clearly shows that the lower bound of the sine function is independent of the irrationality measure \( \mu = \mu(\pi) \) of
the irrational number \( \pi \neq 0 \).

In addition, the Dirichlet approximation inequality
\[
\frac{1}{q_{n-1}^{\mu}} \ll |\pi q_n - p_n| \ll \frac{1}{q_n}
\]
holds for all large integers \( q_n \geq 1 \).

Combining (6) and (7) yield
\[
|\pi q_n - p_n| \ll \frac{1}{q_n} \ll |\sin(\pi q_n - p_n)| \ll |\pi q_n - p_n| \ll \frac{1}{q_n}
\]
(8)
for all large integers \( n \geq 1 \). On the contrary,
\[
|\pi q_n - p_n| \ll \frac{1}{q_{n-1}^{\mu}} \ll |\sin(\pi q_n - p_n)| \ll |\pi q_n - p_n| \ll \frac{1}{q_n}
\]
(9)
for all large integers \( q_n \geq 1 \). But, this contradicts the hypothesis \( \mu(\pi) > 2 \). Therefore, \( \mu(\pi) = 2 \).

3.2 Second Proof

The second proof is based on a result for the upper bound of the reciprocal sine function over the sequence of \( \{p_n : n \geq 1\} \) derived in Section 4. This is equivalent to a result in Section 6.

**Proof.** (Theorem 1.1) Let \( \varepsilon > 0 \) be an arbitrary small number, and let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents of the irrational number \( \pi \). By Theorem 5.1, the reciprocal sine function has the upper bound
\[
\left|\frac{1}{\sin \pi^2 q_n}\right| \ll q_n^{1+\varepsilon}.
\]
(10)
Moreover, \( \sin(\pi^2 q_n) = \sin(\pi^2 q_n - \alpha p) \) if and only if \( \alpha p = \pi p_n \), where \( p \) and \( p_n \) are integers. These information lead to the following relation.
\[
\frac{1}{q_n^{1+\varepsilon}} \ll \left|\sin \left(\pi^2 q_n\right)\right|
\]
(11)
\[
\ll \left|\sin \left(\pi^2 q_n - \pi p_n\right)\right|
\]
\[
\ll |\pi q_n - p_n|
\]
for all sufficiently large \( n \). Therefore,
\[
\left|\pi \frac{p_n}{q_n}\right| \gg \frac{1}{q_n^{2+\varepsilon}}
\]
(12)
\[
= \frac{1}{q_n^{\mu(\pi)+\varepsilon}}.
\]
Clearly, this implies that the irrationality measure of the real number \( \pi \) is \( \mu(\pi) = 2 \), see Definition 17.1. Quod erat demonstrandum.

This theory is consistent with the numerical data in Table 3, which shows the measure approaching 2 as the rational approximation \( p_n/q_n \to \pi \).
3.3 Third Proof

The third proof is based on the asymptotic expansion of the sine function in Section 7. Related results based on the cosine and sine functions appears in [1] and [10].

**Proof.** (Theorem 1.1) Given a small number $\varepsilon > 0$, it will be shown that $\psi(q) \leq q^{\mu(\pi)+\varepsilon} \leq q^{2+\varepsilon}$ is the irrationality measure of the real number $\pi$.

Observe that in Theorem 7.1, the asymptotic expansion of the sine function satisfies

$$|\sin (p_n)| \gg \left| \sin \left( \frac{1}{p_n} \right) \right|.$$

This inequality leads to the irrationality measures $\psi(q) = q^{2+\varepsilon}$ for real number $\pi$. Specifically,

$$|p_n - \pi q_n| \gg |\sin (p_n - \pi q_n)| \quad (13)$$

$$\geq |\sin (p_n)| \quad (14)$$

$$\gg \left| \sin \left( \frac{1}{p_n} \right) \right| \quad (15)$$

$$\gg \frac{1}{p_n} \quad (16)$$

$$\gg \frac{1}{q_n} \quad (17)$$

since $p_n = \pi q_n + O(q^{-1})$. Therefore,

$$\psi(q) \leq q^{\mu(\pi)+\varepsilon} \leq q^{2+\varepsilon}.$$

Quod erat demonstrandum.

3.4 Four Proof

The fourth proof is based on the Diophantine inequality

$$\frac{1}{2q_{n+1}q_n} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^{\mu(\alpha)}}$$

for irrational numbers $\alpha \in \mathbb{R}$ of irrationality measure $\mu(\alpha) \geq 2$, confer Lemma 17.4, Definition 17.1, and a result for the partial quotients in Theorem 8.1.

**Proof.** (Theorem 1.1) Take the logarithm of the Diophantine inequality associated to the real number $\pi$. Specifically,

$$\frac{1}{2q_{n+1}q_n} \leq \left| \pi - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^{\mu(\pi)}}$$

to reach
\[ \mu(\pi) \leq \frac{\log 2q_{n+1}q_n}{\log q_n} = 1 + \frac{\log 2q_{n+1}}{\log q_n}. \] (18)

By Theorem 8.1, there is a constant \( c > 0 \) for which \( a_n \leq c \). Hence,
\[ q_{n+1} = a_{n+1}q_n + q_{n+1} \leq 2a_{n+1}q_n \leq 2cq_n. \] (19)

Substitute the last estimate (19) into (18) obtain
\[ \mu(\pi) \leq 1 + \frac{\log 4cq_n}{\log q_n} \leq 2 + \frac{\log 4c}{\log q_n} \] (20)
for all sufficiently large \( n \geq 1 \). Taking the limit yields
\[ \mu(\pi) = \lim_{n \to \infty} \left( 2 + \frac{\log 4c}{\log q_n} \right) = 2. \] (21)
Quod erat demonstrandum.

4 Harmonic Summation Kernels

The harmonic summation kernels naturally arise in the partial sums of Fourier series and in the studies of convergences of continuous functions.

**Definition 4.1.** The Dirichlet kernel is defined by
\[ D_x(z) = \sum_{-x \leq n \leq x} e^{inz} = \frac{\sin((2x + 1)z)}{\sin(z)}, \] (22)
where \( x \in \mathbb{N} \) is an integer and \( z \in \mathbb{R} - \pi\mathbb{Z} \) is a real number.

**Definition 4.2.** The Fejer kernel is defined by
\[ F_x(z) = \sum_{0 \leq n \leq x} \sum_{-n \leq k \leq n} e^{ikz} = \frac{1}{2} \frac{\sin((x + 1)z)^2}{\sin(z)^2}, \] (23)
where \( x \in \mathbb{N} \) is an integer and \( z \in \mathbb{R} - \pi\mathbb{Z} \) is a real number.

These formulas are well known, see [17] and similar references. For \( z \neq k\pi \), the harmonic summation kernels have the upper bounds \( |K_x(z)| = |D_x(z)| \ll |x| \), and \( |K_x(z)| = |F_x(z)| \ll |x^2| \).

An important property is the that a proper choice of the parameter \( x \geq 1 \) can shifts the sporadic large value of the reciprocal sine function \( 1/\sin z \) to \( K_x(z) \), and the
term $1/\sin(2x + 1)z$ remains bounded. This principle will be applied to the lacunary sequence $\{p_n : n \geq 1\}$, which maximize the reciprocal sine function $1/\sin z$, to obtain an effective upper bound of the function $1/\sin z$.

There are many different ways to prove an upper bound based on the harmonic summation kernels $D_x(z)$ and $F_x(z)$. An elementary approach is provided below.

The Dirichlet kernel in Definition 4.1 is a well defined continued function of two variables $x, z \in \mathbb{R}$. Hence, for fixed $z$, it has an analytic continuation to all the real numbers $x \in \mathbb{R}$.

**Lemma 4.1.** Let $k \geq 1$ be a small fixed integer, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number $\pi^k$, and $0 \neq z \in \mathbb{Z}$. Then

$$\frac{1}{|\sin(\pi^{k+1}z)|} \ll \frac{1}{|\sin(\pi^{k+1}q_n)|}. \quad (24)$$

**Proof.** By the best approximation principle, see Lemma 17.5,

$$|m - \pi^k z| \geq |p_n - \pi^k q_n| \quad (25)$$

for any integer $z \leq q_n$. Hence,

$$\frac{1}{|\sin(\pi^{k+1}z)|} = \frac{1}{|\sin(\pi m - \pi^{k+1}z)|} \leq \frac{1}{|\sin(\pi p_n - \pi^{k+1}q_n)|} = \frac{1}{|\sin(\pi^{k+1}q_n)|}, \quad (26)$$

as $n \to \infty$. \hfill \blacksquare

## 5 Upper Bound For $|1/\sin \pi^{k+1}z|$ 

As shown in Lemma 4.1, to estimate the upper bound of the function $1/|\sin \pi^{k+1}z|$ over the real numbers $z \in \mathbb{R}$, it is sufficient to fix $z = q_n$, and select a real number $x \in \mathbb{R}$ such that $q_n \approx x$. This idea is demonstrated below for small integer parameter $k \geq 1$.

**Lemma 5.1.** Let $k \geq 1$ be a small fixed integer, let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number $\pi^k$, and define the associated sequence $x_n = \left(\frac{2^{2+2v_2} + 1}{2^{2+2v_2}}\right) \frac{q_n}{\pi^k}$, \quad (27)

where $v_2 = v_2(q_n) = \max\{v : 2^v | q_n\}$ is the 2-adic valuation, and $n \geq 1$. Then

(i) $\sin \left(2(x_n - 1/2) + 1\right)\pi^{k+1}q_n = \pm 1$. 

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(ii) \( \sin (2(x_n + 1/2) + 1)\pi^{k+1}q_n) = \pm \cos 2\pi^{k+1}q_n. \)

(iii) \( \left| \sin (2x_n + 1/2)\pi^{k+1}q_n) \right| = 1 + O \left( \frac{1}{q_n^2} \right), \) as \( n \to \infty. \)

**Proof.** Observe that the value \( x_n \) in (27) yields

\[
\sin(2^{k+1}q_n x_n) = \sin \left( 2^{k+1}q_n \left( \frac{2^{2+2v} + 1}{2^{2+2v}} \right) \right) = \sin \left( \frac{\pi}{2} \cdot w_n \right) = \pm 1, \tag{28}
\]

and

\[
\cos (2^{k+1}q_n x_n) = \cos \left( 2^{k+1}q_n \left( \frac{2^{2+2v} + 1}{2^{2+2v}} \right) \right) = \cos \left( \frac{\pi}{2} \cdot w_n \right) = 0, \tag{29}
\]

where

\[
w_n = \left( \frac{2^{2+2v} + 1}{2^{2v}} \right) q_n^2 \tag{30}
\]

is an odd integer. (i) Routine calculations yield this:

\[
\sin((2(x_n - 1/2) + 1)\pi^{k+1}q_n) = \sin \left( 2^{k+1}q_n x_n \right) \tag{31}
\]

\[
= \sin \left( 2^{k+1}q_n \left( \frac{2^{2+2v} + 1}{2^{2+2v}} \right) \right) \frac{q_n}{\pi^k}
\]

\[
= \sin \left( \frac{\pi}{2} \cdot w_n \right)
\]

\[
= \pm 1,
\]

(ii) Routine calculations yield this:

\[
\sin \left( (2(x_n + 1/2) + 1)\pi^{k+1}q_n \right) = \sin(2^{k+1}q_n x_n + 2^{k+1}q_n) \tag{32}
\]

\[
= \sin(2^{k+1}q_n x_n) \cos(2^{k+1}q_n)
\]

\[
+ \cos(2^{k+1}q_n x_n) \sin(2^{k+1}q_n).
\]

Substituting (28) and (29) into (31) return

\[
\sin \left( (2(x_n + 1/2) + 1)\pi^{k+1}q_n \right) = \pm \cos \left( 2^{k+1}q_n \right). \tag{33}
\]

(iii) This follows from the previous result:

\[
\left| \sin \left( (2(x_n + 1/2) + 1)\pi^{k+1}q_n \right) \right| = \left| \pm \cos \left( 2^{k+1}q_n \right) \right| \tag{34}
\]

\[
= \left| \pm \cos \left( 2\pi p_n - 2^{k+1}q_n \right) \right|
\]

\[
= \left| \pm \cos \left( 2\pi \left( p_n - \pi^k q_n \right) \right) \right|
\]

\[
= 1 + O \left( \frac{1}{q_n^2} \right),
\]

since the sequence of convergents satisfies \( |p_n - \pi^k q_n| \leq 1/q_n \) as \( n \to \infty. \)
Lemma 5.2. Let \( k \geq 1 \) be a small fixed integer, let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents of the real number \( \pi^k \), and define the associated sequence

\[
x_n = \left( \frac{2^{2+2v_2} + 1}{2^{2+2v_2}} \right) \frac{q_n}{\pi^k},
\]

where \( v_2 = v_2(q_n) = \max\{v : 2^v | q_n\} \) is the 2-adic valuation, and \( n \geq 1 \). Then

\[
|\sin((2x^* + 1)\pi^{k+1}q_n)| \asymp 1,
\]

where \( x^* \in [x_n - 1/2, x_n + 1/2] \) is an integer.

**Proof.** Consider the continuous function \( f(x) = |\sin((2x + 1)\pi^{k+1}q_n)| \) over the interval \([x_n-1/2, x_n+1/2]\). By Lemma 5.1, it has a local maximal at \( x = x_n-1/2 \in \mathbb{R} \):

\[
|\sin((2x + 1)\pi^{k+1}z)| = |\sin((2(x_n - 1/2) + 1)\pi^{k+1}q_n)|
\]

\[
= 1,
\]

and it has a local minimal at \( x = x_n + 1/2 \in \mathbb{R} \):

\[
|\sin((2x + 1)\pi^{k+1}z)| = |\sin((2(x_n + 1/2) + 1)\pi^{k+1}q_n)|
\]

\[
= 1 + O\left(\frac{1}{q_n^2}\right).
\]

Since \( f(x) \) is continuous over the interval \([x_n - 1/2, x_n + 1/2]\), it follows that

\[
1 + O\left(\frac{1}{q_n^2}\right) \leq |\sin((2x^* + 1)\pi^{k+1}z)| \leq 1
\]

for any integer \( x^* \in [x_n - 1/2, x_n + 1/2] \)

**Theorem 5.1.** If \( k \geq 1 \) is a small fixed integer, and \( z \in \mathbb{N} \) is a large integer, then,

\[
\frac{1}{|\sin\pi^{k+1}z|} \ll |z|.
\]

**Proof.** Let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents of the real number \( \pi^k \). Since the denominators sequence \( \{q_n : n \geq 1\} \) maximize the reciprocal sine function \( 1/|\sin\pi^{k+1}z| \), see Lemma 4.1, it is sufficient to prove it for \( z = q_n \). Define the associated sequence

\[
x_n = \left( \frac{2^{2+2v_2} + 1}{2^{2+2v_2}} \right) \frac{q_n}{\pi^k},
\]

where \( v_2 = v_2(q_n) = \max\{v : 2^v | q_n\} \) is the 2-adic valuation, and \( n \geq 1 \). Let \( f(x) = |\sin((2x + 1)\pi^{k+1}z)| \), and let \( z = q_n \). The function \( f(x) \) is bounded over the interval \([x_n - 1/2, x_n + 1/2]\), see Lemma 5.1. Replacing the integer parameters \( x^* \in [x_n - 1/2, x_n + 1/2] \), \( z = q_n \), and applying Lemma 5.1 return

\[
|\sin((2x + 1)\pi^{k+1}z)| = |\sin((2x^* + 1)\pi^{k+1}q_n)|
\]

\[
\asymp 1.
\]
Rewrite the reciprocal sine function in terms of the harmonic kernel in Definition 4.1, and splice all these information together, to obtain

\[
\left| \frac{1}{\sin \pi k^{+1}z} \right| = \left| \frac{D_x(\pi k^{+1}z)}{\sin((2x + 1)\pi k^{+1}z)} \right| \\
\ll |D_x| \left| \frac{1}{\sin((2x^* + 1)\pi k^{+1}q_n)} \right| \\
\ll |x^*| \cdot 1 \\
\ll |z|
\]

since \(|z| \asymp x^* \asymp p_n \asymp q_n\), and the trivial estimate \(|D_x(z)| \ll |x|\). \hfill \blacksquare

### 6 Elementary Techniques

A similar case arises for the sine function \(\sin(\alpha \pi n)\) as \(n \to \infty\). This is handled by the observing that \(1/q_n^{a-1} < |p_n - \alpha q_n| \leq 1/q_n\) is small as \(n \to \infty\). Thus, the denominators sequence \(\{q_n : n \geq 1\}\) maximize the reciprocal sine function \(1/\sin z\). Hence, it is sufficient to consider the infinite series over the denominators sequence \(\{q_n : n \geq 1\}\).

**Theorem 6.1.** Let \(\alpha \in \mathbb{R}\) be an irrational number of irrationality measure \(\mu(\alpha) = a\). Then

\[
\frac{1}{|\sin \alpha \pi q_n|} \ll q_n^{a-1}. \tag{43}
\]

**Proof.** Let \(\alpha = [a_0, a_1, a_2, \ldots]\) be the continued fraction of the number \(\alpha\), and let \(\{p_n/q_n : n \geq 1\}\) be the sequence of convergents. Then

\[
\frac{1}{|\sin \alpha \pi q_n|} = \frac{1}{|\sin(\pi p_n - \alpha \pi q_n)|} \geq \frac{1}{\pi |p_n - \alpha q_n|} > q_n^{a-1}.
\]

as \(n \to \infty\). \hfill \blacksquare

### 7 Asymptotic Expansions Of The Sine Function

**Theorem 7.1.** Let \(p_n/q_n\) be the sequence of convergents of the irrational number \(\pi\). Then,

\[
\sin \left(\frac{1}{p_n}\right) \asymp \sin p_n \tag{45}
\]

as \(p_n \to \infty\).

**Proof.** The Taylor series of the sine function leads to the inequality

\[
z - \frac{z^3}{6} \leq \sin z \leq z, \tag{46}
\]
where \(|z| < 1\). Replacing \(z = |\pi q_n - p_n|\) yields the symmetric inequality
\[
|\pi q_n - p_n| \ll |\sin(\pi q_n - p_n)| \ll |\pi q_n - p_n|.
\] (47)

Moreover, \(|\pi q_n - p_n| \leq 1/q_n\), implies the associated symmetric inequality
\[
|\pi q_n - p_n| \ll \frac{1}{q_n} \ll |\sin(\pi q_n - p_n)| \ll |\pi q_n - p_n| \ll \frac{1}{q_n}.
\] (48)

Since \(p_n \sim q_n\), the last inequality leads to the relation
\[
|\sin p_n| = |\pi q_n - p_n| \gg \frac{1}{p_n} \gg \sin \left(\frac{1}{p_n}\right).
\] (49)

On the other direction,
\[
\sin \left(\frac{1}{p_n}\right) \gg \frac{1}{p_n} \gg \frac{1}{q_n} \gg |p_n - \pi q_n| = |\sin (p_n - \pi q_n)| = |\sin (p_n)|,
\] (50)

These prove that
\[
\sin p_n \gg \sin \left(\frac{1}{p_n}\right) \quad \text{and} \quad \sin \left(\frac{1}{p_n}\right) \gg \sin p_n,
\] (51)
as \(n \to \infty\).

\[\blacksquare\]

**Lemma 7.1.** Let \(p_n/q_n\) be the sequence of convergents of the irrational number \(\pi = [a_0; a_1, a_2, \ldots]\). Then, the followings hold.

(i) \(\sin p_n = \sin (p_n - \pi q_n)\), for all \(p_n\) and \(q_n\).

(ii) \(\sin \left(\frac{1}{p_n}\right) = \frac{1}{p_n} \left(1 - \frac{1}{3! p_n^2} + \frac{1}{5! p_n^4} - \cdots \right)\), as \(p_n \to \infty\).

(iii) \(\cos 2p_n = \cos (2(p_n - \pi q_n))\), for all \(p_n\) and \(q_n\).

### 8 Bounded Partial Quotients

A result for the partial quotients based on the properties of continued fractions and the asymptotic expansions of the sine function is derived below.
Theorem 8.1. Let $\pi = [a_0; a_1, a_2, \ldots]$ be the continued fraction of the real number $\pi \in \mathbb{R}$. Then, the $n$th partial quotients $a_n \in \mathbb{N}$ are bounded. Specifically, $a_n = O(1)$ for all $n \geq 1$.

Proof. Consider the real numbers

$$C = p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \quad \text{and} \quad D = \frac{1}{q_n}.$$  \hspace{1cm} (52)

For all sufficiently large integers $n \geq 1$, the Dirichlet approximation theorem and the addition formula $\sin(C + D) = \cos C \sin D + \cos D \sin C$ lead to

$$\frac{1}{q_{n+1}} \gg |p_{n+1} - \pi q_{n+1}| \quad \gg |\sin (p_{n+1} - \pi q_{n+1})| \quad = |\sin (p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} + \frac{1}{q_n})| \quad = |\cos C \sin \left(\frac{1}{q_n}\right) + \cos \left(\frac{1}{q_n}\right) \sin C|.$$  \hspace{1cm} (53)

The sequence $\{p_n : n \geq 1\}$ maximizes the cosine function and minimizes the sine function. Hence, by Lemma 8.2,

$$1 - \frac{2}{q_n^2} \leq 1 - \frac{C^2}{2} \leq \cos C \leq 1,$$  \hspace{1cm} (54)

and

$$\frac{1}{2q_n} - \frac{1}{48q_n^3} \leq C - \frac{C^3}{6} \leq \sin C \leq \frac{2}{q_n}.$$  \hspace{1cm} (55)

Substituting these estimates into the reverse triangle inequality $|X + Y| \geq ||X| - |Y||$, produces

$$\frac{1}{q_{n+1}} \gg |\cos C \sin \left(\frac{1}{q_n}\right) + \cos \left(\frac{1}{q_n}\right) \sin C| \quad \geq \left|\left(1 \left(\frac{1}{q_n} - \frac{1}{6q_n^3}\right)\right) - \left(1 \left(\frac{1}{2q_n} - \frac{1}{48q_n^3}\right)\right)\right| \quad \geq \frac{c_0}{q_n},$$

where $c_0 > 0$ is a constant. These show that

$$\frac{1}{q_{n+1}} \gg \frac{1}{q_n}$$  \hspace{1cm} (57)

as $n \to \infty$. Furthermore, (57) implies that $q_n \gg q_{n+1} = a_n q_n + q_{n-1}$ as claimed. ■

Lemma 8.1. Let $p_n/q_n$ be the sequence of convergents of the real number $\pi = [a_0; a_1, a_2, \ldots]$. Then, as $n \to \infty$, 

...
\[(i) \frac{1}{2q_n} \leq \left| p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \right| \leq \frac{2}{q_n}, \quad (ii) \frac{1}{2p_n} \leq \left| p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \right| \leq \frac{13}{p_n}. \]

**Proof.** (i) Using the triangle inequality and Dirichlet approximation theorem yield the upper bound

\[
\left| p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \right| \leq \frac{c_0}{q_n} + \frac{1}{q_n}
\]

\[
\leq \frac{2}{q_n},
\]

since \(q_{n+1} = a_{n+1}q_n + q_{n-1}\). The lower bound

\[
\frac{1}{q_n} - \left| p_{n+1} - \pi q_{n+1} \right| \geq \frac{1}{q_n} - \left| p_{n+1} - \pi q_{n+1} \right|
\]

\[
\geq \frac{1}{q_n} - \frac{c_1}{q_n + 1}
\]

\[
\geq \frac{1}{2q_n},
\]

where \(c_0 > 0\) and \(c_1 > 0\) are small constants. (ii) Use the Hurwitz approximation theorem

\[
\pi q_n - \frac{1}{\sqrt{5}q_n^2} \leq p_n \leq \pi q_n + \frac{1}{\sqrt{5}q_n^2}
\]

(60)

to convert it to the upper bound in term of \(p_n\):

\[
\left| p_{n+1} - \pi q_{n+1} - \frac{1}{q_n} \right| \leq \frac{2}{q_n} \leq \frac{13}{p_n}.
\]

as \(n \to \infty\) as claimed.

**Lemma 8.2.** Let \(\{p_n/q_n : n \geq 1\}\) be the sequence of convergents of \(\pi\), and let \(C = p_{n+1} - \pi q_{n+1} - q_n\). Then,

\[(i) \ 1 - \frac{2}{q_n^2} \leq \cos C \leq 1 - \frac{2}{q_n^2} + \frac{1}{6q_n^4}, \]

\[(ii) \ \frac{1}{2q_n} - \frac{1}{48q_n^3} \leq \sin C \leq \frac{2}{q_n}. \]

**Proof.** Use Lemma 8.1.

9 Statistics And Example

The result of Theorem 8.1 indicates that the real number \(\pi\) fits the profile of a random irrational number. The geometric mean value of the partial quotients of a random irrational number has the value

\[
K_0 = \lim_{n \to \infty} \left( \prod_{k \leq n} a_k \right)^{1/n} = 2.6854520010\ldots,
\]

(61)
which is known as the Khinchin constant. In addition, the Gauss-Kuzmin distribution specifies the frequency of each value by

\[ p(a_k = k) = -\log_2 \left( 1 - \frac{1}{(k+1)^2} \right). \]  

\[ (62) \]

For a random irrational number, the proportion of partial quotients \( a_k = 1, 2 \) is about \( p(a_1) + p(a_2) \leq 0.5897 \). A large value \( a_k \geq 10^6 \) is very rare. In fact,

\[ p(a_k = 10^6) = -\log_2 \left( 1 - \frac{1}{(10^6+1)^2} \right) = 0.00000000000144, \]

and the sum of probabilities is

\[ \sum_{k \geq 10^6} p(a_k = k) = \sum_{k \geq 10^6} -\log_2 \left( 1 - \frac{1}{(k+1)^2} \right) \leq 0.000001. \]  

\[ (63) \]

\[ (64) \]

**Example 9.1.** A short survey of the partial quotients \( \pi = [a_0; a_1, a_2, \ldots] \) is provided here to sample this phenomenon, most computer algebra system can generate a few thousand terms within minutes. This limited numerical experiment established that \( a_{432} = 20776 \) is the only unusual partial quotient. This seems to be an instance of the Strong Law of Small Number.

\[ \pi = [3, 7, 15, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, \ldots]. \]  

\[ (65) \]

**10 Convergence Of The Flint Hills Series I**

The first analysis of the the convergence of the Flint Hills series is based on the upper bound of the reciprocal sine function \( 1/\sin n \) as \( n \to \infty \), in Theorem 5.1. This is equivalent to the upper bound of the harmonic summation kernel

\[ D_x(z) = \sum_{-x \leq n \leq x} e^{i2nz} = \frac{\sin((2x + 1)z)}{\sin(z)}, \]  

\[ (66) \]

consult Definition 4.1. Let \( \pi = [a_0, a_1, a_2, \ldots] \) be the continued fraction of the number \( \pi \), and let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents. The difficulty in proving the convergence of the series \((69)\) arises from the sporadic maximal values of the function

\[ \frac{1}{|\sin p_n|} = \frac{1}{|\sin(p_n - \pi q_n)|} \geq \frac{1}{|p_n - \pi q_n|} > q_n^{42}. \]  

\[ (67) \]

at the integers \( z = p_n \) as \( n \to \infty \). The irrationality exponent 42 in the Diophantine inequality

\[ \frac{1}{q_n^{42}} \leq |\pi - p_n/q_n| \leq \frac{1}{q_n} \]  

\[ (68) \]
was determined in [19]. Earlier questions on the convergence the series (69) appears in [24, p. 59], [28, p. 583], [1], et alii.

**Theorem 10.1.** If the real numbers \( u > 0 \) and \( v > 0 \) satisfy the relation \( u - v > 0 \), then the Flint Hills series
\[
\sum_{n \geq 1} \frac{1}{n^u \sin^v n}
\]  
(69)
is absolutely convergent.

**Proof.** As \( 1/q_1^{41} < |p_n - \pi q_n| \leq 1/q_n \) is small as \( n \to \infty \), the numerators sequence \( \{p_n : n \geq 1\} \) maximize the reciprocal sine function \( 1/\sin z \). Hence, it is sufficient to consider the infinite series over the numerators sequence \( \{p_n : n \geq 1\} \). To accomplish this, write the series as a sum of a convergent infinite series and the lacunary infinite series (over the sequence of numerators):
\[
\sum_{n \geq 1} \frac{1}{n^u \sin^v n} = \sum_{m \geq 1} \frac{1}{m^u \sin^v m} + \sum_{n \geq 1} \frac{1}{p_n^u \sin (p_n)^v}
\]  
(70)

where \( c_0, c_1, c_2, c_3, c_4 > 0 \) are constants. By Theorem 5.1, the reciprocal of the sine function is bounded
\[
\left| \frac{1}{\sin (p_n)} \right| \leq c_1 p_n.
\]  
(71)

Applying this bound yields
\[
\sum_{n \geq 1} \frac{1}{p_n^u (\sin p_n)^v} \leq \sum_{n \geq 1} \frac{(c_1 p_n)^v}{p_n^u}
\]  
\leq c_2 \sum_{n \geq 1} \frac{1}{p_n^{u-v}}.
\]  
(72)

By the Binet formula for quadratic recurrent sequences, the sequence \( p_n = a_n p_{n-1} + p_{n-2}, a_n \geq 1 \), has exponential growth, namely,
\[
p_n \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n
\]  
(73)

for \( n \geq 2 \). Now, replacing (73) into (72) returns
\[
c_2 \sum_{n \geq 1} \frac{1}{p_n^{u-v}} \leq c_3 \sum_{n \geq 1} \left( \frac{2}{1 + \sqrt{5}} \right)^{(u-v)n}
\]  
\leq c_4 \sum_{n \geq 1} \left( \frac{1}{2} \right)^{(u-v)n}.
\]  
(74)

Hence, it immediately follows that the infinite series converges whenever \( u - v > 0 \). \[\blacksquare\]
Example 10.1. The infinite series

\[ \sum_{n \geq 1} \frac{1}{n^3 \sin^2 n} \]  

has \( u - v = 3 - 2 = 1 > 0 \). Hence, by Theorem 10.1, it is convergent.

Example 10.2. Let \( \varepsilon > 0 \) be an arbitrary small number. The infinite series

\[ \sum_{n \geq 1} \frac{1}{n^{1+\varepsilon} \sin n} \]  

has \( u - v = 1 + \varepsilon - 1 = \varepsilon > 0 \). Hence, by Theorem 10.1, it is absolutely convergent.

The Partial Sum \( P_x = \sum_{n \leq x} \frac{1}{n \sin^2 n} \)

11 The Flint Hills Series And The Lacunary Sine Series

A comparison of the partial sum of the Flint Hills series

\[ P_x = \sum_{n \leq x} \frac{1}{n^u \sin^v n} \]  

and the partial sum of the Lacunary Sine series

\[ Q_x = \sum_{p_n \leq x} \frac{1}{p_n^u \sin (p_n)^v} \]  

respectively, is tabulated in Table 2. It demonstrates that the Lacunary Sine series infuses an overwhelming contribution to the complete sum.
Table 2: Comparison Of The Partial Sums $P_x$ and $Q_x$ For $(u, v) = (3, 2)$.

| $x$ | $P_x$     | $Q_x$  |
|-----|-----------|--------|
| 1   | 1.422829  | 1.422829 |
| 3   | 3.423233  | 1.887049 |
| 22  | 4.754112  | 3.085767 |
| 355 | 29.405625 | 27.683949 |

12 Convergence Of The Flint Hills Series II

The second analysis of the convergence of the Flint Hills series is based on the asymptotic relation

$$\sin(p_n) \asymp \sin \left( \frac{1}{p_n} \right), \quad (79)$$

where $p_n/q_n \in \mathbb{Q}$ is the sequence of convergents of the real number $\pi \in \mathbb{R}$. In some way, the representation (79) removes any reference to the difficult problem of estimating the maximal value of the function $1/\sin n$ as $n \to \infty$. Earlier study of the convergence the series (80) appears in [28, p. 583], [1], et alii.

**Theorem 12.1.** If the real numbers $u > 0$ and $v > 0$ satisfy the relation $u - v > 0$, then the Flint Hills series

$$\sum_{n \geq 1} \frac{1}{n^u \sin^v n} \quad (80)$$

is absolutely convergent.

**Proof.** Let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents of the real number $\pi$. Since the numerators sequence $\{p_n : n \geq 1\}$ maximize the reciprocal sine function $1/\sin z$. Hence, it is sufficient to consider the infinite series over the lacunary numerators sequence $\{p_n : n \geq 1\}$. Substituting the numerators sequence returns

$$\sum_{n \geq 1} \frac{1}{n^u \sin^v n} \ll \sum_{n \geq 1} p_n^u (\sin p_n)^v \ll \sum_{n \geq 1} \frac{1}{p_n^u \sin \left( \frac{1}{p_n} \right)^v}, \quad (81)$$

see Theorem 7.1. Substituting the Taylor series at infinity, see Lemma 7.1, return

$$\sum_{n \geq 1} \frac{1}{p_n^u \sin \left( \frac{1}{p_n} \right)^v} = \sum_{n \geq 1} \frac{1}{p_n^u} \left( \frac{1}{p_n} \left( 1 - \frac{1}{3! p_n^2} + \frac{1}{5! p_n^4} - \cdots \right) \right)^v \ll \sum_{n \geq 1} \frac{1}{p_n^{u-v}}. \quad (82)$$

By the Binet formula for recurrent quadratic sequences, the sequence $p_n = a_n p_{n-1} + p_{n-2}$, $a_n \geq 1$, has exponential growth, namely,

$$p_n \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \quad (83)$$
for \( n \geq 2 \). Replacing (83) into (82) returns

\[
\sum_{n \geq 1} \frac{1}{p_n^{\mu - v}} \ll \sum_{n \geq 1} \left( \frac{2}{1 + \sqrt{5}} \right)^{(u-v)n}
\]

\[
\ll \sum_{n \geq 1} \left( \frac{1}{2} \right)^{(u-v)n}.
\]

(84)

Hence, it immediately follows that the infinite series converges whenever \( u - v > 0 \). ■

13 Convergence Of Some Flint Hills Type Series

The analysis for the convergence of some modified Flint Hills series as

\[
\sum_{n \geq 1} \frac{1}{n^u \sin^v \alpha \pi \phi}
\]

have many similarities to the of the Flint Hills series. Given an irrational number \( \alpha \in \mathbb{R} \) of irrationality measure \( \mu(\alpha) = a \), let \( \pi = [a_0, a_1, a_2, \ldots] \) be the continued fraction of the number \( \alpha \), and let \( \{p_n/q_n : n \geq 1\} \) be the sequence of convergents. The difficulty in proving the convergence of the series (69) arises from the sporadic maximal values of the function

\[
\frac{1}{|\sin \alpha \pi q_n|} = \frac{1}{|\sin(\pi p_n - \alpha \pi q_n)|}
\]

\[
\geq \frac{1}{\pi |p_n - \alpha q_n|}
\]

\[
> q_n^{a-1}.
\]

as \( n \to \infty \).

Theorem 13.1. If the real numbers \( u > 0 \) and \( v > 0 \) satisfy the relation \( u - (a - 1)v > 0 \), then the series

\[
\sum_{n \geq 1} \frac{1}{n^u \sin^v \alpha \pi \phi}
\]

is absolutely convergent.

Proof. As \( 1/q_n^{a-1} < |p_n - \alpha q_n| \leq 1/q_n \) is small as \( n \to \infty \), the numerators sequence \( \{p_n : n \geq 1\} \) maximize the reciprocal sine function \( 1/\sin z \). Hence, it is sufficient to consider the infinite series over the numerators sequence \( \{p_n : n \geq 1\} \). To accomplish this, write the series as a sum of a convergent infinite series and the lacunary infinite series (over the sequence of numerators):

\[
\sum_{n \geq 1} \frac{1}{n^u \sin^v n} = \sum_{m \geq 1} \frac{1}{m^u \sin^v m} + \sum_{n \geq 1} \frac{1}{q_n^u \sin(\alpha \pi q_n)^v} \]

\[
\leq c_0 \sum_{n \geq 1} \frac{1}{q_n^a \sin(\alpha \pi q_n)^v}.
\]

(88)
Irrationality Measure of Pi

where $c_0, c_1, c_2, c_3, c_4 > 0$ are constants. By hypothesis, the reciprocal of the sine function is bounded by

$$\left| \frac{1}{\sin(\alpha \pi q_n)} \right| \leq c_1 q_n^{a-1}.$$  \hfill (89)

see Theorem 6.1 for more details. Applying this bound yields

$$\sum_{n \geq 1} \frac{1}{q_n^u \sin(\alpha \pi q_n)^v} \leq \sum_{n \geq 1} \frac{(c_1 q_n)^{(a-1)v}}{q_n^u} \leq c_2 \sum_{n \geq 1} \frac{1}{q_n^{u-(a-1)v}}.$$  \hfill (90)

By the Binet formula for quadratic recurrent sequences, the sequence $q_n = a_n q_{n-1} + q_{n-2}$, $a_n \geq 1$, has exponential growth, namely,

$$q_n \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n$$  \hfill (91)

for $n \geq 2$. Now, replacing (91) into (72) returns

$$c_2 \sum_{n \geq 1} \frac{1}{q_n^{u-(a-1)v}} \leq c_3 \sum_{n \geq 1} \left( \frac{2}{1 + \sqrt{5}} \right)^{(u-(a-1)v)n} \leq c_4 \sum_{n \geq 1} \left( \frac{1}{2} \right)^{(u-(a-1)v)n}. \hfill (92)$$

Hence, it immediately follows that the infinite series converges whenever $u - (a - 1)v > 0$.

**Example 13.1.** The irrational number $\alpha = \sqrt{2}$ has irrationality measure $\mu(\sqrt{2}) = 2$, and the infinite series

$$\sum_{n \geq 1} \frac{1}{n^3 \sin^2(\sqrt{2} \pi n)}$$  \hfill (93)

has $u - (a-1)v = 3 - 2 = 1 > 0$. Hence, by Theorem 13.1, it is absolutely convergent.

**Example 13.2.** Let $\varepsilon > 0$ be an arbitrary small number. The irrational number $\alpha = \sqrt{3} \sqrt{2}$ has irrationality measure $\mu(\sqrt{3} \sqrt{2}) = 2$, and the infinite series

$$\sum_{n \geq 1} \frac{1}{n^{1+\varepsilon} \sin(\sqrt{2} \pi n)}$$  \hfill (94)

has $u - (a - 1)v = 1 + \varepsilon - 1 = \varepsilon > 0$. Hence, by Theorem 13.1, it is absolutely convergent.
14 Convergence Of The Flat Hills Series

Let \( \{x\} \) be the fractional part function and let \( \|\alpha\| = \min\{|n - \alpha| : n \in \mathbb{Z}\} \) be the least distance to the nearest integer. Another classes of problems arise from the other properties of the number \( \pi \) and other irrational numbers. One of these problems is the convergence of the Flat Hills series.

**Definition 14.1.** Let \( a > 1 \) and \( b \neq 0 \) be a pair of real parameters. A Flat Hills series is defined by an infinite sum of the following forms.

\[
\begin{align*}
(i) & \sum_{n \geq 1} \frac{1}{n^a \sin^b \|\pi^n\|}, \\
(ii) & \sum_{n \geq 1} \frac{1}{n^a \sin^b \|\pi 10^n\|}, \\
(iii) & \sum_{n \geq 1} \frac{1}{n^a \sin^b \{\pi^n\}}, \\
(iv) & \sum_{n \geq 1} \frac{1}{n^a \sin^b \{\pi 10^n\}}.
\end{align*}
\]

The known properties of some algebraic irrational numbers \( \alpha \in \mathbb{R} \) can be used to determine the convergence or divergence of the Flat Hills series

\[
\sum_{n \geq 1} \frac{1}{n^a \sin^b \|\alpha^n\|},
\]

see Exercise 19.26. In the case \( \alpha = \pi \), the analytic properties of sequences such as \( \{\|\pi^n\| : n \geq 1\} \) and \( \{\|\pi 10^n\| : n \geq 1\} \) are unknown. Accordingly, the convergence or divergence of the infinite series (95) are unknown. Likewise, for the case \( \alpha = e \), the analytic properties of sequences such as \( \{\|e^n\| : n \geq 1\} \) and \( \{\|e 10^n\| : n \geq 1\} \) are unknown. Accordingly, the convergence or divergence of the infinite series (95) are unknown, see Lemma 17.6 for some details.

15 Representations

The Flint Hills series and similar class of infinite series have representations in terms of other analytic functions. These reformulations could be useful in further analysis of these series.

15.1 Gamma Function Representation

The gamma function reflection formula produces a new reformulation as

\[
\sum_{n \geq 1} \frac{1}{n^a \sin^b n} = \sum_{n \geq 1} \frac{\Gamma(1 - n/\pi)^v \Gamma(n/\pi)^v}{n^a},
\]

see Lemma 16.1.
15.2 Zeta Function Representation

The zeta function reflection relation

\[ \zeta(s) = 2^s \pi^{1-s} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) \]  

(97)
evaluated at \( s = \frac{2n}{\pi} \) produces a new reformulation as

\[ \sum_{n \geq 1} \frac{1}{n^u \sin^n n} = \sum_{n \geq 1} \frac{\left(\frac{2^{2n}/\pi^{1-2n}/\pi \Gamma(1-2n/\pi)}{\pi} \zeta(1-2n/\pi)\right)^v}{n^u \zeta(2n/\pi)^v}, \]  

(98)

15.3 Harmonic Kernel Representation

The harmonic summation kernel is defined by

\[ K_x(z) = \sum_{-x \leq n \leq x} e^{i2nz} = \frac{\sin((2x+1)z)}{\sin(z)}, \]  

(99)

where \( x, z \in \mathbb{C} \) are complex numbers. For \( z \neq k\pi \), the harmonic summation kernel has the upper bound \(|K_x(z)| \ll |x|\). Replacing it into the series produces a new reformulation as

\[ \sum_{n \geq 1} \frac{1}{n^u \sin^n n} = \sum_{n \geq 1} \frac{K_x(n)^v}{n^u \sin^n((2x+1)n)}. \]  

(100)

Here the choice of parameter \( x = \alpha n \), where \( \alpha > 0 \) is an algebraic irrational number, shifts the large value of the reciprocal sine function \( 1/\sin(n) \) to \( K_x(z) \), and the \( 1/\sin(2x+1)n = \sin(2\alpha n + 1)n \) remains bounded. Thus, it easier to prove the convergence of the series (100).

16 Beta and Gamma Functions

Certain properties of the beta function \( B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt \), and gamma function \( \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \) are useful in the proof of the main result.

**Lemma 16.1.** Let \( B(a, b) \) and \( \Gamma(z) \) be the beta function and gamma function of the complex numbers \( a, b, z \in \mathbb{C} - \{0, -1, -2, -3, \ldots\} \). Then,

(i) \( \Gamma(z + 1) = z \Gamma(z) \), the functional equation.

(ii) \( B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \), the multiplication formula.

(iii) \( \frac{B(1-z, z)}{\pi} = \frac{1}{\sin \pi z} \).

(iv) \( \frac{\Gamma(1-z)\Gamma(z)}{\pi} = \frac{1}{\sin \pi z} \), the reflection formula.

**Proof.** Standard analytic methods on the \( B(a, b) \), and gamma function \( \Gamma(z) \), see [11, Equation 5.12.1], [2, Chapter 1], et cetera.

\[ \blacksquare \]
Irrationality Measure of Pi

The current result on the irrationality measure $\mu(\pi) = 7.6063$, see Table 1, of the number $\pi$ implies that

$$|\Gamma(z + 1)\Gamma(z)| \ll |z|^{7.6063}. \quad (101)$$

A sharper upper bound is computed here.

**Lemma 16.2.** Let $\Gamma(z)$ be the gamma function of a complex number $z \in \mathbb{C}$, but not a negative integer $z \neq 0, -1, -2, \ldots$. Then,

$$|\Gamma(z + 1)\Gamma(z)| \ll |z|. \quad (102)$$

**Proof.** Assume $\Re(z) > 0$. Then, the functional equation

$$\Gamma(z + 1) = z\Gamma(z), \quad (103)$$

see Lemma 16.1, or [11, Equation 5.5.1], provides an analytic continuation expression

$$\frac{\Gamma(1 - z)\Gamma(z)}{\pi} = \frac{1}{\sin \pi z},$$

for any complex number $z \in \mathbb{C}$ such that $z \neq 0, -1, -2, -3, \ldots$. By Theorem 7.1,

$$\frac{1}{\sin |\pi z|} \asymp \frac{1}{\sin \left(\frac{1}{|\pi z|}\right)} \asymp |\pi z|$$

as $|\pi z| \to \infty$. \hfill \blacksquare

### 17 Basic Diophantine Approximations Results

All the materials covered in this section are standard results in the literature, see [15], [18], [21], [25], [26], [29], et alii.

#### 17.1 Rationals And Irrationals Numbers Criteria

A real number $\alpha \in \mathbb{R}$ is called *rational* if $\alpha = a/b$, where $a, b \in \mathbb{Z}$ are integers. Otherwise, the number is *irrational*. The irrational numbers are further classified as *algebraic* if $\alpha$ is the root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $\deg(f) > 1$, otherwise it is *transcendental*.

**Lemma 17.1.** If a real number $\alpha \in \mathbb{R}$ is a rational number, then there exists a constant $c = c(\alpha)$ such that

$$\frac{c}{q} \leq \left|\alpha - \frac{p}{q}\right| \quad (104)$$

holds for any rational fraction $p/q \neq \alpha$. Specifically, $c \geq 1/b$ if $\alpha = a/b$. 

This is a statement about the lack of effective or good approximations for any arbitrary rational number $\alpha \in \mathbb{Q}$ by other rational numbers. On the other hand, irrational numbers $\alpha \in \mathbb{R} - \mathbb{Q}$ have effective approximations by rational numbers. If the complementary inequality $|\alpha - p/q| < c/q$ holds for infinitely many rational approximations $p/q$, then it already shows that the real number $\alpha \in \mathbb{R}$ is irrational, so it is sufficient to prove the irrationality of real numbers.

**Lemma 17.2** (Dirichlet). Suppose $\alpha \in \mathbb{R}$ is an irrational number. Then there exists an infinite sequence of rational numbers $p_n/q_n$ satisfying

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$  \hspace{1cm} (105)

for all integers $n \in \mathbb{N}$.

**Lemma 17.3.** Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction of a real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}$$  \hspace{1cm} (106)

for all integers $n \in \mathbb{N}$.

This is standard in the literature, the proof appears in [15, Theorem 171], [26, Corollary 3.7], [16, Theorem 9], and similar references.

**Lemma 17.4.** Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction of a real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then

(i) $\frac{1}{2q_{n+1}q_n} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}$,  
(ii) $\frac{1}{2a_{n+1}q_n^2} \leq \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}$,

for all integers $n \in \mathbb{N}$.

The recursive relation $q_{n+1} = a_{n+1}q_n + q_{n-1}$ links the two inequalities. Confer [22, Theorem 3.8], [16, Theorems 9 and 13], et alii. The proof of the best rational approximation stated below, appears in [25, Theorem 2.1], and [26, Theorem 3.8].

**Lemma 17.5.** Let $\alpha \in \mathbb{R}$ be an irrational real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then, for any rational number $p/q \in \mathbb{Q}^\times$,

(i) $|\alpha q_n - p_n| \leq |\alpha q - p|$,  
(ii) $\left| \alpha - \frac{p_n}{q_n} \right| \leq \left| \alpha - \frac{p}{q} \right|$,

for all sufficiently large $n \in \mathbb{N}$ such that $q \leq q_n$. 

17.2 Irrationalities Measures

The concept of measures of irrationality of real numbers is discussed in [29, p. 556], [5, Chapter 11], et alii. This concept can be approached from several points of views.

**Definition 17.1.** The irrationality measure $\mu(\alpha)$ of a real number $\alpha \in \mathbb{R}$ is the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the Diophantine inequality

$$|\alpha - \frac{p}{q}| \ll \frac{1}{q^{\mu(\alpha)}}$$

has finitely many rational solutions $p$ and $q$. Equivalently, for any arbitrary small number $\varepsilon > 0$

$$|\alpha - \frac{p}{q}| \gg \frac{1}{q^{\mu(\alpha)+\varepsilon}}$$

for all large $q \geq 1$.

**Theorem 17.1.** ([7, Theorem 2]) The map $\mu : \mathbb{R} \rightarrow [2, \infty) \cup \{1\}$ is surjective function. Any number in the set $[2, \infty) \cup \{1\}$ is the irrationality measure of some irrational number.

**Example 17.1.** Some irrational numbers of various irrationality measures.

(1) A rational number has an irrationality measure of $\mu(\alpha) = 1$, see [15, Theorem 186].

(2) An algebraic irrational number has an irrationality measure of $\mu(\alpha) = 2$, an introduction to the earlier proofs of Roth Theorem appears in [25, p. 147].

(3) Any irrational number has an irrationality measure of $\mu(\alpha) \geq 2$.

(4) A Champernowne number $\kappa_b = 0.123\cdots b - 1 \cdot b \cdot b + 1 \cdot b + 2 \cdots$ in base $b \geq 2$, concatenation of the $b$-base integers, has an irrationality measure of $\mu(\kappa_b) = b$.

(5) A Mahler number $\psi_b = \sum_{n \geq 1} b^{-\lceil \tau \rceil^n}$ in base $b \geq 3$ has an irrationality measure of $\mu(\psi_b) = \tau$, for any real number $\tau \geq 2$, see [7, Theorem 2].

(6) A Liouville number $\ell_b = \sum_{n \geq 1} b^{-n!}$ parameterized by $b \geq 2$ has an irrationality measure of $\mu(\ell_b) = \infty$, see [15, p. 208].

**Definition 17.2.** A measure of irrationality $\mu(\alpha) \geq 2$ of an irrational real number $\alpha \in \mathbb{R}^x$ is a map $\psi : \mathbb{N} \rightarrow \mathbb{R}$ such that for any $p, q \in \mathbb{N}$ with $q \geq q_0$,

$$|\alpha - \frac{p}{q}| \geq \frac{1}{\psi(q)}.$$ 

Furthermore, any measure of irrationality of an irrational real number satisfies $\psi(q) \geq \sqrt{5}q^{\mu(\alpha)} \geq \sqrt{5}q^2$. 
Theorem 17.2. For all integers \( p, q \in \mathbb{N} \), and \( q \geq q_0 \), the number \( \pi \) satisfies the rational approximation inequality

\[
|\pi - \frac{p}{q}| \geq \frac{1}{q^{7.6063}}.
\]  

(110)

**Proof.** Consult the original source [27, Theorem 1].

\[\blacksquare\]

17.3 Normal Numbers

The earliest study was centered on the distribution of the digits in the decimal expansion of the number \( \sqrt{2} = 1.414213562373\ldots \), which is known as the Borel conjecture.

**Definition 17.3.** An irrational number \( \alpha \in \mathbb{R} \) is a normal number in base 10 if any sequence of \( k \)-digits in the decimal expansion occurs with probability \( \frac{1}{10^k} \).

**Lemma 17.6.** (Wall) An irrational number \( \alpha \in \mathbb{R} \) is a normal number in base 10 if and only if the sequence \( \{\alpha 10^n : n \geq 1\} \) is uniformly distributed modulo 1.

18 Numerical Data

The maxima of the function \( 1/\sin x \) occur at the numerators \( x = p_n \) of the sequence of convergents \( p_n/q_n \rightarrow \pi \). The first few terms of the sequence \( p_n \), which is cataloged as A046947 in [23], are:

\[
N_{\pi} = \{1, 3, 22, 333, 355, 103993, 104348, 208341, 312689, 833719, 1146408, 4272943, 5419351, 80143957, 165707065, 245850922, 411557987, \ldots\}
\]  

(111)

18.1 Data For The Irrationality Measure

A few values were computed to illustrate the prediction in Theorem 1.1. The numerators \( p_n \) and the denominators \( q_n \) are listed in OEIS A002485 and A002486 respectively. The numerical data and Theorem 1.1 are very well matched. The values of the approximate irrationality measure \( \mu_n(\alpha) \geq 2 \) of the irrational number \( \alpha \neq 0 \) is defined by

\[
\mu_n(\alpha) = -\frac{\log|\alpha - p_n/q_n|}{\log q_n},
\]  

(112)

where \( n \geq 2 \).

**Example 18.1.** A large convergent is used here to illustrate the calculations, using 50 digits accuracy in the computer algebra system SAGE. The 80th convergent \( p_n/q_n \) is given by

(a) \( p_{80} = 32265750565715036834586769616835078002345262 \),

(b) \( q_{80} = 10270507390207332847445984588758042509963443 \),
(c) \( p_{90} = 306243329449682257532162387854374057879036650780 \),
(d) \( q_{90} = 974802793416785521474303406201616853353780695273 \).

Likewise, the corresponding 90th approximation of the irrationality measure is
\[
\mu_{90}(\pi) = -\frac{\log|\pi - p_{90}/q_{90}|}{\log q_{90}} = 2.00343624458326071981396152293441039867289426019.
\]

The range of values for \( n \leq 25 \) is plotted in Figure 1.

18.2 The Sine Asymptotic Expansions

A few values were computed in Table 4 to illustrate the prediction in Theorem 7.1. Note that (79) is the restriction to convergents, so never vanishes, it is bounded below by \( 1/p_n \). The numerical data in Table 4 confirms this result.

18.3 The Sine Reflection Formula

Observe that the substitution \( z \rightarrow p_n/\pi \) in \( \Gamma(1 - z)\Gamma(z) \) leads to
\[
\frac{\Gamma(1 - z)\Gamma(z)}{z} = \frac{\pi}{z \sin \pi z} = \frac{\pi^2}{p_n \sin p_n} = O(1).
\]
A few values were computed to illustrate the prediction in Lemma 16.2. The ratio (115) is tabulated in the third column of Table 5.
Table 3: Numerical Data For Irrationality Measure $|p_n/q_n - \pi| \geq q_n^{\mu_n(\pi)}$.

| $n$ | $p_n$ | $q_n$ | $\mu_n(\pi)$ |
|-----|-------|-------|----------------|
| 1   | 3     | 1     |                |
| 2   | 22    | 7     | 3.429288       |
| 3   | 333   | 106   | 2.014399       |
| 4   | 355   | 113   | 3.201958       |
| 5   | 103993| 33102 | 2.043905       |
| 6   | 104348| 33215 | 2.096582       |
| 7   | 208341| 66317 | 2.055815       |
| 8   | 312689| 99532 | 2.107950       |
| 9   | 833719| 265381| 2.039080       |
| 10  | 1146408| 364913| 2.120203       |
| 11  | 4272943| 1360120| 2.020606      |
| 12  | 5419351| 1725033| 2.189381       |
| 13  | 80143857| 25510582| 2.057220      |
| 14  | 165707065| 52746197| 2.044100      |
| 15  | 245850922| 78256779| 2.040522       |
| 16  | 411557987| 131002976| 2.058941      |
| 17  | 1068966896| 340262731| 2.052652      |
| 18  | 2549491779| 811528438| 2.049381      |
| 19  | 6167950454| 1963319607| 2.057213      |
| 20  | 14885392687| 473816765| 2.203610      |
| 21  | 21053343141| 6701487259| 2.196409      |
| 22  | 1783366216531| 567663097408| 2.034261     |
| 23  | 3587785776203| 1142027682075| 2.032066    |
| 24  | 5371151992734| 1709690779483| 2.019525     |
| 25  | 8958937768937| 2851718461558| 2.096513     |

19 Problems

19.1 Flint Hill Class Series

Exercise 19.1. The infinite series

$$S_1 = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \quad \text{and} \quad S_2 = \sum_{n \geq 1} \frac{1}{n^2 \sin n}$$

have many asymptotic similarities. The first is conditionally convergent. Is the series $S_2$ conditionally convergent?

Exercise 19.2. Show that the infinite series

$$S_3 = \sum_{n \geq 1} \frac{\cos^v n}{n} \quad \text{and} \quad S_4 = \sum_{n \geq 1} \frac{\sin^v n}{n}$$

are conditionally convergent for odd $v \geq 1$. Are these series absolutely conditionally convergent for any $v > 0$?
Exercise 19.3. Explain the convergence of the infinite series

\[ \frac{\pi}{\sin \pi z} = \frac{1}{z} + 2z \sum_{n \geq 1} \frac{(-1)^n}{z^2 - n^2} \]

at the real numbers \( z = n/\pi \), where \( n \geq 1 \) is an integer. Is this series convergent?

Exercise 19.4. Explain the convergence of the infinite series

\[ \pi \cot \pi z = \frac{1}{z} + 2z \sum_{n \geq 1} \frac{1}{z^2 - n^2} \]

at the real numbers \( z = n/\pi \), where \( n \geq 1 \) is an integer. Is this series convergent? The derivation of this series appears in [9, p. 122].

Exercise 19.5. Explain the convergence of the infinite series

\[ \frac{\pi}{\sin^2 \pi z} = \frac{1}{\pi^2} \sum_{-\infty < n < \infty} \frac{1}{(z - n)^2} \]
Table 5: Numerical Data For $\pi / \sin p_n$

| $n$ | $p_n$ | $\Gamma(1 - p_n/\pi)\Gamma(p_n/\pi)$ | $\pi^2/p_n \sin p_n$ |
|-----|-------|-------------------------------------|---------------------|
| 1   | 3     | 22.2619                             | 23.3126             |
| 2   | 22    | -354.93                             | -50.6838            |
| 3   | 333   | -356.143                            | -3.35992            |
| 4   | 355   | -104218.0                           | -922.286            |
| 5   | 103993| -164229.0                           | -4.9613             |
| 6   | 104348| -285210.0                           | -8.58678            |
| 7   | 208341| 387167.0                            | 5.83812             |
| 8   | 312689| 1.08305 $\times$ 10$^6$             | 10.8814             |
| 9   | 833719| 1.35828 $\times$ 10$^6$             | 5.11823             |
| 10  | 1146408| -5.34484 $\times$ 10$^6$         | -14.6469            |
| 11  | 4272943| -5.71636 $\times$ 10$^6$         | -4.20283            |
| 12  | 5419351| -8.22936 $\times$ 10$^7$         | -47.6742            |
| 13  | 80143857| -2.1266 $\times$ 10$^8$       | -8.33615            |
| 14  | 165707065| -3.62989 $\times$ 10$^8$     | -6.88181            |
| 15  | 245850922| 5.13494 $\times$ 10$^8$         | 6.56166             |
| 16  | 411557987| 1.23845 $\times$ 10$^9$         | 9.45359             |
| 17  | 1068966896| 3.00736 $\times$ 10$^9$        | 8.83836             |
| 18  | 2549491779| 7.02111 $\times$ 10$^9$        | 8.65171             |
| 19  | 6167950454| 2.09811 $\times$ 10$^{10}$     | 10.6866             |
| 20  | 14885392687| 2.12297 $\times$ 10$^{10}$     | 4.8058              |
| 21  | 2105343141| 1.79173 $\times$ 10$^{12}$     | 267.364             |
| 22  | 1783366216531| 4.50764 $\times$ 10$^{12}$    | 7.9407              |
| 23  | 3587785776203| 8.73917 $\times$ 10$^{12}$    | 7.65233             |
| 24  | 5371151992734| -9.30941 $\times$ 10$^{12}$  | -5.44508            |
| 25  | 8958937768937| 1.42671 $\times$ 10$^{14}$    | -50.0299            |

at the real numbers $z = n/\pi$, where $n \geq 1$ is an integer. Is this series convergent? The derivation of this series appears in [9, p. 122], [12, p. 276].

**Exercise 19.6.** Explain the convergence of the infinite series

$$
\frac{\sin z}{\sin \pi z} = \frac{2}{\pi} \sum_{n \geq 1} (-1)^n \sin nz \frac{n}{z^2 - n^2}
$$

at the real numbers $z = n/\pi$, where $n \geq 1$ is an integer. This series has a different structure than the previous exercise, is this series is absolutely convergent? conditionally convergent? The derivation of this series appears in [12, p. 276].

### 19.2 Complex Analysis

**Exercise 19.7.** Assume that the infinite series is absolutely convergent. Use a Cauchy integral formula to evaluate the integral

$$
\frac{1}{i2\pi} \int_C \frac{1}{z^n \sin^n z} dz,
$$
where $C$ is a suitable curve.

**Exercise 19.8.** Determine whether or not there is a complex valued function $f(z)$ for which the Cauchy integral evaluate to
\[
\frac{1}{i2\pi} \oint_C f(z)dz = \sum_{n \geq 1} \frac{1}{n^u \sin^n n},
\]
where $C$ is a suitable curve, and the infinite series is absolutely convergent.

**Exercise 19.9.** Let $z \in \mathbb{C}$ be a large complex variable. Prove or disprove the asymptotic relation
\[
\frac{1}{\sin |\pi z|} \asymp \frac{1}{\sin \left(\frac{1}{|\pi z|}\right)} \asymp |\pi z|.
\]

**Exercise 19.10.** Consider the product
\[
\sin\left(\frac{1}{z}\right) \times \sin z = 1 + O(1/z^2)
\]
of a complex variable $z \in \mathbb{C}$. Explain its properties in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and at infinity.

**Exercise 19.11.** Let $z \in \mathbb{C}$ be a large complex variable. Use the properties of the gamma function to prove the asymptotic relation
\[
|\Gamma(1 - z)\Gamma(z)| = O(|z|).
\]

**Exercise 19.12.** Let $x \in \mathbb{R}$ be a large real variable. Use the trigamma reflection formula
\[
\psi_1(1 - x) + \psi_1(x) = \frac{\pi^2}{\sin^2 (\pi x)}
\]
to derive an asymptotic relation
\[
\frac{1}{\sin^2 |\pi x|} \asymp |\pi x|^2.
\]
Explain any restrictions on the real variable $x \in \mathbb{R}$.

### 19.3 Irrationality Measures

**Exercise 19.13.** Do the irrationality measures of the numbers $\pi$ and $\pi^2$ satisfy $\mu(\pi) = \mu(\pi^2) = 2$? This is supported by the similarity of $\sin(n) = \sin(n - m\pi)$ and
\[
\frac{\sin x}{x} = \prod_{m \geq 1} \left(1 - \frac{x^2}{\pi^2 m^2}\right)
\]
at $x = n$. More generally, $x = n^k \pi^{-k+1}$

**Exercise 19.14.** What is the relationship between the irrationality measures of the numbers $\pi$ and $\pi^k$, for example, do these measures satisfy $\mu(\pi) = \mu(\pi^k) = 2$ for $k \geq 2$?
19.4 Partial Quotients

Exercise 19.15. Determine an explicit bound $a_n \leq B$ for the continued fraction of the irrational number $\pi = [a_0; a_1, a_2, a_3, \ldots]$ for all partial quotients $a_n$ as $n \to \infty$.

Exercise 19.16. Does $\pi^2 = [a_0; a_1, a_2, a_3, \ldots]$ have bounded partial quotients $a_n$ as $n \to \infty$?

Exercise 19.17. Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and $\beta = [b_0; b_1, b_2, b_3, \ldots]$ be a pair of continued fractions. Is there an algorithm to determine whether or not $\alpha = \beta$ or $\alpha \neq \beta$ based on sequence of comparisons $a_n = b_0, a_1 = b_1, a_2 = b_2, \ldots, a_N = b_N$ for some $N \geq 1$?

Exercise 19.18. Let $e = [a_0; a_1, a_2, a_3, \ldots]$, where $a_{3k} = a_{3k+2} = 1$, and $a_{3k+1} = 2k$, be the of continued fraction of the natural base $e$. Compute the geometric mean value of the partial quotients:

$$K_e = \lim_{n \to \infty} \left( \prod_{k \leq n} a_k \right)^{1/n} = \infty?$$

Exercise 19.19. Let $\pi = [a_0; a_1, a_2, a_3, \ldots]$ be the of continued fraction of the natural base. A short calculation give

$$K_\pi = \lim_{n \leq 10} \left( \prod_{k \leq n} a_k \right)^{1/n} = 3.361 \ldots, K_\pi = \lim_{n \leq 20} \left( \prod_{k \leq n} a_k \right)^{1/n} = 2.628 \ldots.$$ 

Compute a numerical approximation for the geometric mean value of the partial quotients:

$$K_\pi = \lim_{n \leq 1000} \left( \prod_{k \leq n} a_k \right)^{1/n} = ?$$

Exercise 19.20. Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ be the of continued fraction of a real number. Assume it has a lacunary subsequence of unbounded partial quotients $a_{n_i} = O(\log \log n)$, such that $n_{i+1}/n_i > 1$, otherwise $a_n = O(1)$. Is the geometric mean value of the partial quotients unbounded

$$K_\alpha = \lim_{n \to \infty} \left( \prod_{k \leq n} a_k \right)^{1/n} = \infty?$$

19.5 Concatenated Sequences

Exercise 19.21. A Champernowne number $\kappa_b = 0.123 \ldots b - 1 \cdot b + 1 \cdot b + 2 \ldots$ in base $b \geq 2$ is formed by concatenating the sequence of consecutive integers in base $b$ is irrationality. Show that the number $0.F_0F_1F_2F_3\ldots = 1/F_{11}$ formed by concatenating the sequence of Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$ is rational.

Exercise 19.22. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial, and let $D_n = |f(n)|$. Show that the number $0.D_0D_1D_2D_3\ldots$ formed by concatenating the sequence of values is irrational.
Exercise 19.23. Let \( \{D_n \geq 0 : n \geq 0\} \) be an infinite sequence of integers, and let \( \alpha = 0.D_0D_1D_2D_3\ldots \) formed by concatenating the sequence of integers. Determine a sufficient condition on the sequence of integers to have an irrational number \( \alpha > 0 \).

19.6 Exact Evaluations Of Power Series

Exercise 19.24. The exact evaluation of the first series below is quite simple, but the next series requires some work:

\[
L_1(1/2) = \sum_{n \geq 1} \frac{1}{2^n n} = \log 2 \quad \text{and} \quad L_2(1/2) = \sum_{n \geq 1} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{1}{2} (\log 2)^2.
\]

Prove this, use the properties of the polylogarithm function \( L_k(z) = \sum_{n \geq 1} z^n/n^k \).

Exercise 19.25. Determine whether or not the series

\[
L_3(1/2) = \sum_{n \geq 1} \frac{1}{2^n n^3}
\]

has a closed form evaluation (exact).

19.7 Flat Hill Class Series

Exercise 19.26. Let \( \alpha = (1 + \sqrt{5})/2 \) be a Pisot number. Show that \( ||\alpha^n|| \to 0 \) as \( n \to \infty \). Explain the convergence of the first infinite series

\[
\sum_{n \geq 1} \sin ||\alpha^n|| \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{n^2 \sin ||\alpha^n||}
\]

the divergence of the second infinite series.

Exercise 19.27. Assume that \( \pi \) is a normal number base 10. Show that \( ||\pi 10^n|| \) is uniformly distributed. Explain the convergence or divergence of the infinite series

\[
\sum_{n \geq 1} \frac{1}{n^2 \sin ||\pi 10^n||}
\]

Exercise 19.28. Assume that \( \pi \) is a normal number base 10. Determine the least parameters \( a > 1 \) and \( b > 0 \) for which the infinite series

\[
\sum_{n \geq 1} \frac{1}{n^a \sin^b ||\pi^n||}
\]

converges. For example, is \( a - b > 0 \) sufficient?

Exercise 19.29. Assume that \( \pi \) is a normal number base 10. Determine the least parameters \( a > 1 \) and \( b > 0 \) for which the infinite series

\[
\sum_{n \geq 1} \frac{1}{n^a \sin^b ||\pi 10^n||}
\]

converges. For example, is \( a - b > 0 \) sufficient?
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