Equivalence of non-minimally coupled cosmologies by Noether symmetries

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We discuss non-minimally coupled cosmologies involving different geometric invariants. Specifically, actions containing a non-minimally coupled scalar field to gravity described, in turn, by curvature, torsion and Gauss–Bonnet scalars are considered. We show that couplings, potentials and kinetic terms are determined by the existence of Noether symmetries which, moreover, allows to reduce and solve dynamics. The main finding of the paper is that different non-minimally coupled theories, presenting the same Noether symmetries, are dynamically equivalent. In other words, Noether symmetries are a selection criterion to compare different theories of gravity.

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I. INTRODUCTION

The Hilbert-Einstein action, linear in the Ricci curvature scalar $R$, gives rise to the field equations of General Relativity (GR) which is the theory of gravity capable of fitting a huge amount of phenomena ranging from gravitational waves, astrophysical compact objects, black holes up to cosmology. Despite of this large bulk of positive results, several shortcomings must be overcome by GR in order to match the whole budget of astrophysical and cosmological observations and to fix inconsistencies at quantum level. In other words, we lack a theory of gravity addressing the phenomenology, in a self-consistent way, from ultraviolet to infrared scales.

In particular, at large scales, discrepancies occur between the theoretical value of the Hubble constant predicted by GR and the measured one [1]; furthermore, without including dark matter or dark energy, the theory is unable to explain the today observed accelerated behavior of the universe [2–4], the missing matter in galaxies and the structure formation [5]. Furthermore, the “local” formulation of GR seems completely in disagreement with the intrinsic “non-locality” of quantum mechanics.

Besides the today observed cosmological acceleration, also the early epoch inflation cannot be framed considering only GR. Starting from 1980, A.A. Starobinsky [6], A. Guth [7], A. Linde [8], et al., proposed possible solutions for this problem: modifications of the gravity action containing minimal and non-minimal couplings between geometry and scalar fields or the introduction of higher-order curvature terms, behaving like scalar fields, e.g. the so called scalaron, have been taken into account. See [9] for a review.

The paradigm is that introducing a new field, a generic dilaton, shortcomings of Standard Cosmological Model can be solved. These modified actions, subsequently, have been studied also for late-time cosmology [10], for black holes [11], for string-dilaton cosmology considering different kinds of couplings and potentials [12, 13]. An important feature of these models is that, under suitable conformal transformations, they can be always reduced to GR plus scalar fields. The frame containing non-minimally coupled and higher-order terms is the Jordan Frame, while the standard GR plus scalar fields is restored in the Einstein Frame.

As a general remark, with the purpose of addressing or, at least, alleviating the above problems, alternative theories of gravity have been proposed. The approach is considering effective Lagrangians where further geometric invariants and/or scalar fields are included. For example, relaxing the hypothesis of second-order field equations, we may introduce into the action a function of the Ricci scalar like $f(R)$ [14, 15], of its derivatives [16, 17], or include other curvature invariants adopting a metric or metric-affine formulation of the theory [18]. Furthermore, in the context of GR extensions, non-minimal couplings between gravity and dynamical scalar fields are particularly useful for dynamical reasons. Among the second order curvature invariants, there are also models considering the Gauss–Bonnet topological invariant $\tilde{G}$ adopted to obtain ghost-free dynamics. For general reviews on modified gravity see, for example, [19–21].

Moreover, also affine connection can play a key role in modifications and extensions of standard GR. Levi-Civita connection, for instance, represents only the simplest case of torsionless affine connection and, without discarding the anti-symmetric part of $\Gamma^\alpha_{\mu\nu}$, torsion naturally arises. In this framework, the contribution of $\Gamma^\alpha_{\mu\nu}$ cannot be neglected and spacetime can be described both by curvature and torsion [22].

For the sake of completeness, it is worth mentioning that the most general connection can be written even without assuming metricity, i.e. $g_{\mu\nu,\alpha} = 0$, which, once relaxed, gives rise to a class of theories where non-metricity plays a crucial role in the description of spacetime. In [23], it is shown that actions constructed by curvature, torsion or non-metricity scalars are
equivalent at the level of field equations since they differ only for boundary terms. In this perspective, we can deal with the so-called geometric trinity of gravity [23]. The theory involving only the torsion scalar, with vanishing curvature in a purely affine formulation, is the Teleparallel Equivalent of General Relativity (TEGR); geometric foundations and applications of TEGR are discussed e.g. in [24–26]. It turns out that such a theory can be recast as a gauge theory for the translation group in a reference frame with vanishing spin connection. Despite the need of extending/modifying GR, self-interaction potentials, couplings and kinetic terms give rise to infinite choices which can lead to a frustrating indetermination in fitting observations and addressing conceptual problems. One can adjust models and parameters to match single datasets and phenomena but a theory in agreement with the whole phenomenology seems far to be achieved. In other words, any single theory loses its general predictive power and cannot be used to reproduce a self-consistent cosmic history starting from ultraviolet to infrared scales. Therefore, some selection criteria, based on physical requirements, are needed to discriminate among the plethora of modified gravities. These criteria can be based on symmetries, conservation laws and on general physical motivations.

Here, we want to consider scalar fields non-minimally coupled with different geometric invariants, in particular the Ricci scalar \( R \), the torsion scalar \( T \), and the Gauss-Bonnet scalar \( G \). The aim is to demonstrate that all these non-minimally coupled invariants can give rise to similar cosmological dynamics once we know how to transform each-other. Furthermore, these scalar-tensor theories can be dealt under the standard of Noether Symmetry Approach which allows to fix couplings, potentials and kinetic terms requiring the existence of symmetries and related conserved quantities. In other words, the purpose is to compare, by the Noether symmetries, the dynamics of three different actions, containing a scalar field non-minimally coupled to different geometric invariants pointing out the equivalence of the three representations of gravity. The final result of this study is that theories admitting the same Noether symmetries has the same dynamics and the same solutions. In this sense, they are equivalent.

The paper is organized as follows: in Sec. II we briefly argue some basic concepts about extended and modified theories of gravity. Sections III, IV, and V are devoted to the discussion of non-minimal coupling with \( R \), \( T \), and \( G \) invariants through the Noether symmetries. We derive the form of couplings, kinetic term and potential according to the existence of symmetries and related conserved quantities. After, the reduction of related dynamical systems, cosmological solutions are found out and compared in the three cases. In Sec. VI, thanks to the cyclic variables derived from the Noether method, we find the related Hamiltonians and compare the three theories also at this level. In Sec. VII, we draw our conclusions. Appendix A is a brief summary of the Noether Symmetry Approach.

II. EXTENDED AND MODIFIED GRAVITY

As already argued in the introduction, some of the shortcomings suffered by GR can be cured by extending the Einstein theory introducing in the action other curvature invariants like functions of \( R \), \( R^\mu_\nu R^\nu_\mu \), \( R^\alpha\beta\mu\nu R^{\alpha\beta}_\mu\nu \), \( W^{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu} \) where \( R_{\mu\nu} \), \( R^\alpha\beta\mu\nu \), and \( W_{\alpha\beta\mu\nu} \) are the Ricci, Riemann, and Weyl tensors respectively, or considering alternative geometrical representations of the gravitational field like the torsion scalar [25] or the non-metricity [23]. In the first case, we are dealing with Extended Gravity [15] because GR is a particular case of a large class of models while, in the other cases, we are dealing with Modified Gravity because dynamics is not given by \( R \) in the action. For a discussion see [27]. Also if conceptual foundations of these approaches are different (e.g. based on Equivalence Principle, affinities, non-metricity), phenomenology which they have to address is the same so the equivalence to GR and its extensions should be required for a self-consistent description of gravitational interaction.

An example of this statement can be given considering TEGR. The torsion tensor \( T^\alpha_{\mu\nu} \) is defined as
\[
\Gamma^\alpha_{\mu\nu} := T^\alpha_{\mu\nu} .
\]
The most general connection involving curvature and torsion can be written as
\[
\Gamma^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu} + \frac{1}{2} \left( T^\alpha_{\mu\nu} + T^\alpha_{\nu\mu} - T^\alpha_{\mu\nu} \right),
\]
being \( \hat{\Gamma}^\alpha_{\mu\nu} \) the Levi-Civita connection. Furthermore, by defining the contorsion tensor as
\[
K^\nu_{\rho\mu} = \frac{1}{2} \left( T^\nu_{\mu\rho} + T^\nu_{\rho\mu} - T^\nu_{\rho\mu} \right)
\]
and the superpotential as
\[
S^{\rho\mu\nu} = K^{\rho\mu\nu} - g^{\rho\mu} T^{\sigma\nu}_{\alpha} + g^{\rho\mu} T^{\sigma\nu}_{\alpha},
\]
we derive the torsion scalar:

\[ T = T_{\mu \nu} S^{\mu \nu} \tag{5} \]

Setting to zero the curvature, we obtain a theory of gravity, equivalent to GR, described by torsion. Affine transformations are described by the Weitzenböck connection. As stated above, it turns out that TEGR can be seen as a gauge theory for translation group in the tangent spacetime, where the fundamental field is given by the \textit{tetrads}. Tetrads \( e^\mu_\alpha \) link the flat spacetime to the curved spacetime by means of the relation \( g_{\mu \nu} = e^\mu_\alpha e^\nu_\beta g_{\alpha \beta} \), being \( g_{\alpha \beta} \) the Minkowski tensor. See [24, 26–32] and reference therein.

However, it turns out that GR and TEGR are linked through the relation \( \hat{R} = -T + B \), where \( B \) is a boundary term [32]. This relation has practical applications. For instance, in a spatially flat cosmological Friedmann-Lemaitre-Robertson-Walker (FLRW) metric of the form \( g_{\mu \nu} = \text{diag}(1, -a(t)^2, -a(t)^2, -a(t)^2) \), the torsion takes the form

\[ T = -6 \left( \frac{\dot{a}}{a} \right)^2. \tag{6} \]

The Ricci scalar is

\[ R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \tag{7} \]

then immediately the relation between \( R \) and \( T \) can be recovered through the boundary term \( B \):

\[ B = -6 \left( \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} \right). \tag{8} \]

This means that the description of gravity of TEGR and GR, at cosmological level, is substantially equivalent. Differences emerge for \( f(T) \) and \( f(R) \) because the degrees of freedom of these two theories are different [25]. This is just an example of how two different representations of gravity can be practically connected. Below, we will discuss non-minimal coupling with \( R, T \), and \( G \) scalars pointing out that analogue (or identical) features emerge if couplings, kinetic terms, and potentials are selected by Noether symmetries.

A general scalar-tensor action, written in terms of the scalar curvature, reads as:

\[ S = \int \sqrt{-g} \left( F(\phi) R + \omega(\phi) g^{\mu \nu} \phi,_{\mu} \phi,_{\nu} - V(\phi) \right) d^4 x, \tag{9} \]

where \( F(\phi) \) is the coupling, \( \omega(\phi) \) the coefficient of the kinetic term and \( V(\phi) \) the potential. According to the previous considerations, such a theory can be both an extended gravity and a modified gravity depending on how GR, that is \( R \) with minimal coupling is recovered, i.e. \( F(\phi) \rightarrow F_0 \), with \( F_0 \) the Newton constant and \( \omega(\phi), V(\phi) \rightarrow 0 \). The variation of the action with respect to the metric tensor \( g_{\mu \nu} \), provides the field equations [33]

\[ R_{\mu \nu} F(\phi) - \frac{1}{2} g_{\mu \nu} [RF(\phi) + \omega(\phi) \phi,_{\alpha} \phi,_{\alpha} - V(\phi)] - F(\phi) g_{\mu \nu} + g_{\mu \nu} \square F(\phi) + \omega(\phi) \phi,_{\mu} \phi,_{\nu} = 0, \tag{10} \]

that clearly reduce to the Einstein equations considering the above conditions. Action (9) is the paradigm for a very large class of theories. For example, \( f(R) \) gravity in metric formalism can be easily recovered from (9) if \( \omega(\phi) \) is set to zero and

\[ V(\phi) = \frac{1}{2} \frac{f(R) - RF}{|f_R|^2}, \tag{11} \]

where the lower index means the derivative with respect to \( R \). See [15, 18] for details.

In general, minimally coupled Einstein GR plus a scalar field can be recovered by a conformal transformation

\[ \tilde{g}_{\mu \nu} = e^{2\Omega} g_{\mu \nu}, \tag{12} \]

where

\[ \tilde{\Gamma}_\alpha^\nu_\mu = \Gamma_\alpha^\nu_\mu + g^{\nu \sigma} (\partial_\alpha \Omega g_{\mu \nu} + \partial_\mu \Omega g_{\nu \sigma} - \partial_\sigma \Omega g_{\nu \mu}) \]

\[ \tilde{R}_{\alpha \beta} = R_{\alpha \beta} - 2\Omega g_{\alpha \beta} + 2\Omega g_{\alpha \beta} \Omega,_{\gamma} \Omega,_{\gamma} - 2g_{\alpha \beta} \Omega,_{\gamma} \Omega,_{\gamma} \]

\[ \tilde{R} = e^{-2\Omega} (R - 6\square \Omega - 6\Omega,_{\gamma} \Omega,_{\gamma}) \tag{13} \]
are the conformally transformed connection, Ricci tensor and Ricci scalar, respectively. According to this transformation, the stress-energy tensor for the conformal field is defined as

\[ \tilde{T}_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2} \tilde{g}_{\alpha\beta}\phi_{,\gamma}\phi_{,\gamma} + \tilde{g}_{\alpha\beta}V(\phi). \]  \hspace{1cm} (14)

Depending on the form of the functions, the scalar tensor action provides several features useful to describe early and late time evolution of the universe. In particular, it can be used in string-dilaton cosmology (see for example \cite{34,35,36,37}).

In order to select the unknown functions \( F(\phi), \omega(\phi) \) and \( V(\phi) \), we can adopt the Noether Symmetry Approach, summarized in Appendix A. By Noether symmetries, non-minimal couplings with different geometric invariants can be dealt under the same standard.

### III. NON-MINIMALLY COUPLED CURVATURE SCALAR

Let us start our analysis considering action (9) where we will fix the form of unknown functions by Noether symmetries. After selecting these functions, we will find out analytic cosmological solutions thanks to the reduction of dynamics.

From the variation of the action with respect to the scalar field \( \phi \), we get the Klein-Gordon equation

\[ \omega(\phi)\phi^{\alpha}\phi_{,\alpha} + 2\omega(\phi)\phi - RF(\phi) + V_{\phi}(\phi) = 0, \]  \hspace{1cm} (15)

where the subscript \( \phi \) denotes the derivative with respect to \( \phi \). Together with Eqs. (10), it completes the set of field equations.

The action can be simplified by focusing on a spatially-flat FLRW metric and integrating out second order derivatives, we get

\[ S = 2\pi^2 \int a^3 \left[ RF(\phi) + \omega(\phi)\dot{\phi}^2 - V(\phi) \right] dt. \]  \hspace{1cm} (16)

Finally, replacing the cosmological expression of the Ricci scalar (7) into the action and integrating out second order derivatives, we get the cosmological point-like Lagrangian

\[ \mathcal{L} = 6F(\phi)a\ddot{a} + 6F_{\phi}(\phi)a^2\dot{a}\dot{\phi} + a^3\omega(\phi)\dot{\phi}^2 - a^3V(\phi). \]  \hspace{1cm} (17)

Field equations result in the Euler-Lagrange equations of (17). The energy condition \( E_{\mathcal{L}} = 0 \), that is the 00 equation, has to be also included for consistency. Finally, we have the dynamical system:

\[
\begin{align*}
\alpha & : 2F\ddot{a} + 4a (F_{\phi}\dot{a} + F\ddot{\phi}) + a^2 \left( V - \omega\dot{\phi}^2 + 2F_{\phi\phi}\dot{\phi}^2 + 2F_{\phi}\dot{\phi} \right) = 0 \\
\phi & : 6\dot{a}^2 F_{\phi} + 6\omega\dot{a}\dot{\phi} + a \left[ 6F_{\phi}\dot{a} + a \left( V_{\phi} + \dot{\phi}^2 w_{\phi} + 2w_{\phi} \right) \right] = 0 \\
E_{\mathcal{L}} & = 0 : 6\dot{a}^2 F_{\phi\phi} + 6aF_{\phi\phi}\dot{\phi} + a^2 \left( V + w_{\phi}^2 \right) = 0.
\end{align*}
\]  \hspace{1cm} (18)

We notice that the system is completely equivalent to that coming from the field equations derived from the variation of the action with respect to the metric and the scalar field, once the FLRW metric is imposed. It can be solved after the three functions of \( \phi \) are selected through Noether symmetries. The approach can be developed in the two-dimensional minisuperspace \( S = \{a, \phi\} \) whose corresponding symmetry generator is

\[ \mathcal{X} = \xi(a, \phi, t)\partial_t + \alpha(a, \phi, t)\partial_a + \beta(a, \phi, t)\partial_{\phi}. \]  \hspace{1cm} (19)

After equating to zero terms containing same time derivatives of variables, the application of Noether’s identity (A2) (see Appendix A) to Lagrangian (17) provides a system of 10 differential equations. Nevertheless, by imposing \textit{a priori} the condition \( \xi = \xi(t) \) (holding for Lagrangians in canonical forms) and neglecting redundant equations, the system reduces to 4 differential equations plus the condition on the infinitesimal generators, namely:

\[
\begin{align*}
3\alpha + \beta V_{\phi} + V\partial_t \xi &= 0 \\
\beta \left[ 2F_{\phi}V_{\phi}^2 + V^2 \omega_{\phi} - V(\omega V_{\phi} + 2F_{\phi}V_{\phi\phi}) \right] - 2V [V \omega (\partial_t \xi - \partial_{\phi} \beta) + F_{\phi}V_{\phi}\partial_a \beta] &= 0 \\
\alpha \left[ 3FV_{\phi}^2 + 3FV_{\phi\phi} - V(3F_{\phi}V_{\phi} + 2F_{\phi\phi}) \right] + V^2 \left[ F_{\phi} \left( -6\dot{\phi} \xi + 3\partial_{\phi} \beta \right) + a\omega_{\phi}\partial_a \beta \right] - VV_{\phi} (2F\partial_{\phi} \beta + aF_{\phi}\partial_a \beta) &= 0 \\
3\beta (VF_{\phi} - FV_{\phi}) + 3aF_{\phi}\partial_a \beta - 2F (3\partial_t \xi + aV_{\phi}\partial_a \beta) &= 0 \\
\alpha &= \alpha(a, \phi) \quad \beta = \beta(a, \phi) \quad \xi = \xi(t).
\end{align*}
\]  \hspace{1cm} (20)
The above system is clearly overdetermined and cannot provide any explicit form without imposing some constraint (see [38] for details). Since we want to investigate functions with physical meaning for cosmology, we replace into the system both power-law and exponential potentials, so that it provides the following solutions:

\[
\begin{align*}
\mathcal{X} &= (\xi_0 t + \xi_1) \partial_t - \frac{\xi_0}{3a} \left( \frac{k + c}{k - c} \right) \partial_a + \frac{2\xi_0}{k - c} \phi \partial_\phi, \\
F(\phi) &= F_0 e^{k\phi} \omega(\phi) = \omega_0 e^{k\phi} V(\phi) = V_0 e^{c\phi} k \neq c
\end{align*}
\]

with \( \alpha_0, \beta_0, \xi_0, k, c, \omega_0, F_0, V_0 \) real constants. We neglect trivial solutions, such as those with constant coupling or vanishing potential. Inserting the above functions into dynamics, it is reduced and equations of motion can be analytically solved.

Furthermore, there is one further solution of constant coefficient of the kinetic term \( \omega(\phi) = 1 \), namely

\[
\mathcal{X} = -\frac{2(s + 1)}{2s + 3} \beta_0 \alpha^{s+1} \phi^{2s+4/3} \partial_a + \beta_0 \alpha^s \phi^{2s+6s+3} \partial_\phi, \quad F(\phi) = \ell(s) \phi^2, \quad V(\phi) = V_0 e^{(2s+1)\phi} \ell(s) = \frac{(2s + 3)^2}{48(s + 1)(s + 2)}.
\]

with \( \alpha(s), \beta_0, \xi_0, k, c, \omega_0, F_0, V_0 \) real constants. We neglect trivial solutions, such as those with constant coupling or vanishing potential. Inserting the above functions into dynamics, it is reduced and equations of motion can be analytically solved.

Starting from the two main sets of functions selected above we are going to obtain exact cosmological solutions. It is worth noticing that the choice of exponential potential also leads to exponential coupling and exponential kinetic term like in string-dilaton cosmology [12, 13, 39]. This means that the string-dilaton Lagrangian can be naturally obtained from Noether symmetries [40]. From this point of view, solutions occurring in Eq. (21) can be considered more general than those provided in [12], since both the solutions outlined there by the authors are contained in the last two of Eq. (21). In particular, the exponential potential of string-dilaton cosmology is recovered for \( k = -2 \) and arbitrary \( c \), while the constant potential is recovered for for \( k = -2 \) and \( c = 0 \).

With these considerations in mind, let us solve the Euler-Lagrange Eqs. (18) for cases corresponding to the first and the third solution of Eq. (21). Let us start with the former one. In this case, the Lagrangian (17) takes the form:

\[
\mathcal{L} = 6F_0 \phi^2 a\dot{a}^2 + 6F_0 k \phi^{-1} a^2 \dot{\phi}^2 + a^3 \omega_0 \phi^{-2} \dot{\phi}^2 - a^3 V_0 e^c.
\]

The Euler-Lagrange equations can be analytically solved with the constraint \( k = c = 2 \) providing a de Sitter-like expansion of the form:

\[
a(t) = a_0 e^{2t}, \quad \phi(t) = \phi_0 \exp \left\{ \frac{1}{2} \left[ -3q \pm \sqrt{-48F_0 q^2 - 4V_0 + 9q^2 \omega_0} \right] \right\}, \quad q, a_0, \phi_0 \in \mathbb{R}.
\]

Considering the third case of (21), the point-like Lagrangian (17) can be written as

\[
\mathcal{L} = 6F_0 e^{k\phi} a^2 \dot{a}^2 + 6F_0 k e^{k\phi} a^2 \dot{\phi}^2 + a^3 \omega_0 e^{-2k\phi} \dot{\phi}^2 - a^3 V_0 e^{c\phi},
\]

and, even in this case, the equations of motion set the value of the parameter \( k \) and \( c \), introducing the further constraint \( k = c \). Therefore, discarding the solutions with minimal coupling, we find

\[
a(t) = a_0 e^{q t}, \quad \phi(t) = \frac{3F_0 k q \pm \sqrt{(3F_0 k q)^2 - 6F_0 \omega_0 q^2 - V_0 \omega_0}}{\omega_0}, \quad q \in \mathbb{R}.
\]

The values of the constants \( F_0, \omega_0, V_0 \) can be fixed according to cosmological observations [41]). In summary, de Sitter-like expansions are provided by symmetries. For the other cases, the analysis is similar.
IV. NON-MINIMALLY COUPLED TORSION SCALAR

With the above results in mind, let us develop similar considerations for non-minimally coupled teleparallel gravity. We will show that dynamics and solutions, derived from Noether symmetries, are equivalent to those obtained in Sec. III. In this sense, symmetries can be a criterion capable of comparing theories coming from different representations of gravity.

Let us consider the teleparallel equivalent of action (9), i.e.:

\[ S = \int e [T F(\phi) + \omega(\phi) \phi, \alpha \phi^\alpha - V(\phi)] \, d^4 x, \]  

whose Klein-Gordon equation reads as

\[ \omega(\phi) \phi^\alpha \phi_{,\alpha} + 2 \omega(\phi) \Box \phi - T F_\phi(\phi) + V_\phi(\phi) = 0. \]  

Here \( e \) takes the place of \( \sqrt{-g} \) and stands for the determinant of tetrad fields. Unlike the previous case, the cosmological expression of torsion does not contain second derivatives which must be integrated out; therefore, the point-like Lagrangian can be easily found only by replacing the relation (6) into the action and by integrating the three-dimensional surface:

\[ \mathcal{L} = -6a F(\phi) \dot{a}^2 + a^3 \omega(\phi) \dot{\phi}^2 - a^3 V(\phi). \]  

Note that this Lagrangian is already canonical and the equations of motion are simplified with respect to Eqs. (18). They are:

\[
\begin{align*}
  a : & -2F \dot{a}^2 + a^2 (V - \omega \dot{\phi}^2) - 4a (F_\phi \dot{\phi} + F \ddot{a}) = 0 \\
  \phi : & 6a^2 F_\phi + 6a \omega \dot{a} \dot{\phi} + a^4 (V_\phi + \omega \dot{\phi}^2 + 2 \omega \ddot{\phi}) = 0 \\
  E_\mathcal{L} = & 0 : -6a F \dot{a}^2 + a^3 (V + \omega \dot{\phi}^2) = 0.
\end{align*}
\]

In order to solve the system (30), we select functions related to symmetries; the minisuperspace considered is two-dimensional as in the previous case \( (S = \{a, \phi\}) \) and the generator of the symmetry is, in turn,

\[ \mathcal{X} = \xi(t) \partial_t + \alpha(a, \phi, t) \partial_a + \beta(a, \phi, t) \partial_\phi, \]

where, being the Lagrangian in a canonical form, the condition \( \xi = \xi(t) \) immediately holds. The application of the extended Noether vector to the point-like Lagrangian (29) provides a system of 12 equations, which can be reduced to a system of 4 differential equations with the constraints on the infinitesimal generators, that is:

\[
\begin{align*}
  {6F \partial_\phi} \alpha & - 2 \omega a^2 \partial_\alpha \beta = 0 \\
  \alpha F + \beta a F_\phi + a F \partial_t \xi - 2a F \partial_\alpha \alpha = 0 \\
  3\alpha V + \beta a V_\phi - a V \partial_\xi = 0 \\
  3\alpha \omega + \beta a \omega_\phi - a \omega \partial_t \xi + 2a \omega \partial_\phi \beta = 0 \\
  \alpha = \alpha(a, \phi) & \quad \beta = \beta(a, \phi) \quad \xi = \xi(t).
\end{align*}
\]

After some manipulations, the system can be recast as a system of two differential equations containing the three functions \( F(\phi), \omega(\phi), V(\phi) \) and two unknown infinitesimal generators. It is therefore clear that the system cannot provide a unique solution, and an initial choice must be performed in order to fix the related dynamics. However, the assumption is not too much strict, since only the form of the potential is needed in order to exactly solve the system. Solutions containing power-law and exponential potentials are:

\[
\begin{align*}
  \mathcal{X} = (\xi_0 t + \xi_1) \partial_t - \frac{\xi_0}{3} a \left( \frac{k + c}{k - c} \right) \partial_a + \frac{2\xi_0}{k - c} \phi \partial_\phi \\
  F(\phi) = F_0 \phi^k \quad \omega(\phi) = \omega_0 \phi^{k-2} \quad V(\phi) = V_0 \phi^c \quad k \neq c
\end{align*}
\]

\[
\begin{align*}
  \mathcal{X} = (\xi_0 t + \xi_1) \partial_t - \frac{\xi_0}{3} a \left( \frac{k + c}{k - c} \right) \partial_a + \frac{2\xi_0}{k - c} \partial_\phi \\
  F(\phi) = F_0 e^{k\phi} \quad \omega(\phi) = \omega_0 e^{k \phi} \quad V(\phi) = V_0 e^{c \phi} \quad k \neq c
\end{align*}
\]

\[
\begin{align*}
  \mathcal{X} = \frac{k}{3} a \beta(\phi) \partial_a + \beta(\phi) \partial_\phi \\
  F(\phi) = F_0 e^{k \phi} \quad \omega(\phi) = \omega_0 e^{k \phi} \quad V(\phi) = V_0 e^{k \phi}
\end{align*}
\]
and those with unitary kinetic term are

\[
\begin{align*}
X &= \frac{2\beta_0}{2s+3}a^{s+1}\phi^{\frac{2s+4}{s+3}}\partial_\chi + \beta_0 a^s\phi^{\frac{2s}{s+3}}\partial_\phi \\
F(\phi) &= \frac{\phi^2}{48} V(\phi) = V_0\phi^{\frac{6}{s+3}} \\
X &= -\frac{2}{3}a^2(c_2 + 2c_3\phi)\partial_\chi + a^\frac{2}{3}(c_1 + c_2\phi + c_3\phi^2)\partial_\phi \\
F(\phi) &= \frac{3}{64c_3}(c_1 + c_2\phi + c_3\phi^2) V(\phi) = V_0(c_1 + c_2\phi + c_3\phi^2)^2.
\end{align*}
\] (34)

Also in this case, the exponential solutions of Noether system allow us to find out the teleparallel equivalent of string-dilaton cosmology, namely the string-dilaton action with torsion instead of curvature. Once finding the functions, it is possible to get the exact cosmological solutions.

Let us now solve the Euler-Lagrange equations (30) for two different set of couplings, potentials and kinetic terms. We choose the most general solutions among those in (33), namely the first and the second. In the former case the Lagrangian (29) turns out to be:

\[
\mathcal{L} = -6F_0a^k\dot{a}^2 + \omega_0 a^3\phi^2 e^{\frac{2}{3}} - V_0a^3 \phi^c.
\] (35)

Assuming the condition \( k = c \) and the de Sitter-like expansion for the scale factor, we have:

\[
a(t) = a_0e^{qt}, \quad \phi(t) = \phi_0 \exp\left\{ \pm \sqrt{\frac{6F_0q^2 - V_0}{\omega_0}} t \right\}, \quad q = \sqrt{\frac{V_0\omega_0}{6F_0\omega_0 - 4k^2F_0^2}}.
\] (36)

By taking into account the second set of functions, the Lagrangian takes the form

\[
\mathcal{L} = -6F_0ae^{k\phi}a^2 + \omega_0 a^3 e^{k\phi} \phi^2 - V_0a^3 e^{c\phi},
\] (37)

leading to the exponential solutions constrained by the relation \( k = c \):

\[
a(t) = a_0e^{qt}, \quad \phi(t) = \pm \sqrt{\frac{6F_0q^2 - V_0}{\omega_0}} t, \quad q = \sqrt{\frac{V_0\omega_0}{6F_0\omega_0 - 4k^2F_0^2}}.
\] (38)

It is worth stressing the difference between the scalar field coupled to the curvature scalar and the corresponding torsion one. The Noether approach performed in Sec. III allows to find exact expressions for the scalar field and for the scale factor, but the analytic relations between the free parameters cannot be obtained analytically. In the case treated here, instead, such a relation can be analytically found, so that an exact solution of Euler-Lagrange equations (30) occurs. This is due to the cosmological expression of \( T \) which, not containing second derivatives, leads immediately to a canonical Lagrangian.

Similar results occurs considering the Gauss–Bonnet topological term non minimally coupled to a scalar field, as we are going to discuss in the forthcoming section.

V. NON-MINIMALLY COUPLED GAUSS–BONNET SCALAR

Among extended theories of gravity, there is a particular combination of second-order curvature invariants which turns into a topological surface term which is the Gauss–Bonnet topological scalar:

\[
\mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}.
\] (39)

It represents the four-dimensional Euler class, whose integral over the manifold provides the Euler Characteristic, according to the Generalized Gauss–Bonnet Theorem [42–44]. The Gauss–Bonnet scalar is a topological boundary term even in less than four dimensions and it is non-trivial starting from five dimensions. It naturally arises, for instance, in the Lovelock four-dimensional Lagrangian [45] or in the Chern-Simons Lagrangian [46]. Nevertheless, a non-linear function of the Gauss–Bonnet scalar provides non-trivial contributions to the field equations even in four dimensions, while starts being a surface term in less than four; for this reason it is possible to get non-trivial equations of motion from the action \( S = \int \sqrt{-g}f(\mathcal{G})d^4x \), (with \( f(\mathcal{G}) \neq 0 \)). In several papers, the action containing the function \( f(\mathcal{G}) \) is considered [47, 48] and it turns out that it provides a possible explanation of inflation [49, 50]. Black Hole solutions [51] or generalizations of the ΛCDM model [52]. Other works consider
functions depending on both the Gauss–Bonnet term and the Ricci scalar, obtaining results in cosmology [53], alternatives to Dark Energy [54], corrections to ΛCDM model [55] and other achievements. Most of them add into the action also the scalar curvature $R$; even though this choice is useful to recover GR as a limit, the introduction of $R$ into the action is not the only way to recover GR, since the same can be obtained even considering a particular forms of function $f(\mathcal{G})$. As a matter of fact, it turns out that, in cosmology, the curvature invariants $R^\mu\nu R_{\mu\nu}$ and $R^{\mu\nu\rho\tau} R_{\mu\nu\rho\tau}$ are comparable with the term $R^2$. According to this statement, $R = C_0 \sqrt{|g|}$, with $C_0$ being a constant, can be consistently adopted in cosmology\(^1\) (see [56]). Similar analogies shows that a function of $\mathcal{G}$ also provides interesting solutions in spherical symmetry [57]. Therefore, non-linear functions of $\mathcal{G}$ can describe, in principle, phenomenology at any scales.

Here, we will consider functions of $\mathcal{G}$ non-minimally coupled with a scalar field in order to discuss solutions analogue to the above non-minimally coupled curvature and torsion cases. Let us start considering the action

$$S = \int \sqrt{-g} [\mathcal{G}^n F(\phi) + \omega(\phi) \phi_{,\alpha} \phi^{,\alpha} - V(\phi)] \, d^4x , \quad \text{with} \quad n \in \mathbb{R},$$

(40)

whose Klein-Gordon equation and field equations read respectively

$$\omega(\phi) \phi_{,\alpha} \phi^{,\alpha} + 2\omega(\phi) \square \phi - \mathcal{G}^n F_\phi(\phi) + V_\phi(\phi) = 0,$$

(41)

$$\frac{1}{2} g_{\mu\nu} \mathcal{G}^n F(\phi) - 2n F(\phi) \left( R R_{\mu\nu} - 2 R_{\mu\alpha} R^\alpha_{\nu} + R^{\alpha \beta \gamma} R_{\alpha \beta \gamma} - 2 R^{\alpha \beta} R_{\mu \nu \alpha \beta} \right) \mathcal{G}^{n-1} +$$

$$+ n F(\phi) \left[ 2 R \nabla_\mu \nabla_\nu + 4 G F_{\mu \nu} \square \phi - 4 R^\rho \nabla_{\mu} R^\nu_{\rho} \nabla_{\sigma} - 4 G F_{\mu \nu \alpha \beta} \nabla^\alpha \nabla^\beta \right] \mathcal{G}^{n-1} +$$

$$- \frac{1}{2} g_{\mu\nu} \omega(\phi) \phi_{,\alpha} \phi^{,\alpha} - F_\phi(\phi) \mu \nu + g_{\mu \nu} F(\phi) + \omega(\phi) \phi_{,\mu} \phi_{,\nu} + \frac{1}{2} g_{\mu \nu} V(\phi) = 0,$$

(42)

where $\nabla$ denotes the covariant derivative and, as above, $\square$ is the D’Alember operator defined as $\square = \nabla_\alpha \nabla^\alpha$. Note that, with respect to Secs. III and IV, we introduced into the action a new degree of freedom, hence the minisuperspace in no longer two-dimensional, but it contains one more variable, that is $\mathcal{G}$. This is linked to the term $\mathcal{G}^n$ which cannot be treated at the same level of $R$ and $T$ due to the power $n$. By replacing the cosmological expression of $\mathcal{G}$ into the action, we obtain second order derivatives which cannot be eliminated through a simple integration. In order to find out the point-like Lagrangian, we have to define a further Lagrange multiplier and introduce into the action a new Lagrange parameter $\lambda$ with the constraint

$$\mathcal{G} = \frac{24 \dot{a}^2}{a^3}.$$

(43)

After integrating the surface term, the action turns out to be:

$$S = 2 \pi^2 \int a^3 \left\{ \left[ F(\phi) \mathcal{G}^n + \omega(\phi) \phi^2 - V(\phi) \right] - \lambda \left( \mathcal{G} - \frac{24 \dot{a}^2}{a^3} \right) \right\} \, dt.$$

(44)

The Lagrange multiplier can be found by varying the action with respect to the Gauss–Bonnet invariant. It is:

$$\frac{\delta S}{\delta \mathcal{G}} = 0 \quad \rightarrow \quad \lambda = a^3 \mathcal{G}^{-n-1} F(\phi).$$

(45)

Replacing now the result into the action and integrating out the second derivatives, the point-like Lagrangian becomes

$$\mathcal{L} = (1 - n) a^3 \mathcal{G}^n F(\phi) - 8n a^2 \dot{a} F_\phi(\phi) \mathcal{G}^{n-1} + a^3 \omega(\phi) \phi^2 - a^3 V(\phi) - 8n(n - 1) \mathcal{G}^{n-2} \dot{a}^3 \mathcal{G} F(\phi).$$

(46)

Clearly the Gauss–Bonnet contribution disappears for $n = 1$. In this case we have three equations of motion and the energy condition; the further equation is the one for $\mathcal{G}$, which provides the cosmological expression of the Gauss–Bonnet surface term by construction. The Euler-Lagrange equations therefore read

$$a : 3 \dot{a}^2 \left[ (n - 1) \mathcal{G} F(\phi) + V - \omega \phi^2 \right] - 24 n \dot{\mathcal{G}} \left[ (n - 1) F \left\{ 2 \mathcal{G} \dot{\phi} + \mathcal{G} \dot{\phi} + 2 F_\phi (n - 1) \mathcal{G} \dot{\phi} + 2 F_\phi \mathcal{G} \right\} \right] +$$

$$+ 2 G^2 F_\phi \dot{\phi} \dot{\phi} + \alpha \mathcal{G} \left[ \mathcal{G} \dot{\phi} F_\phi + 2 F_\phi (n - 1) \mathcal{G} \dot{\phi} + \mathcal{G} F_\phi \right] = 0$$

$$\phi : 6 a^2 \omega \dot{\phi} - 24 n \mathcal{G}^{n-1} \dot{a}^2 F_\phi \dot{\phi} + a^3 \left[ (n - 1) \mathcal{G} F(\phi) + V + \omega \phi^2 + 2 \omega \phi \dot{\phi} \right] = 0$$

$$G : a^3 \dot{\mathcal{G}} - 24 \dot{a} \dot{a} \dot{a} = 0$$

(47)

\(^1\) In the following, we are assuming the modulus of $\mathcal{G}$ when dealing with the square root because the invariant has to be real.
The generator of symmetry in three-dimensional minisuperspace contains one further infinitesimal generator related to the $\mathcal{G}$:

$$X = \xi(a, \phi, \mathcal{G}, t)\partial_t + \alpha(a, \phi, \mathcal{G}, t)\partial_a + \beta(a, \phi, \mathcal{G}, t)\partial_\phi + \gamma(t, a, \phi, \mathcal{G})\partial_\mathcal{G}$$  (48)

so that the application of $X^{[1]}$ to Lagrangian (46) provides the following system of 4 differential equations:

$$\begin{align*}
\left\{ \begin{array}{l}
(\mathcal{G} - 1) (\gamma F_\phi + F \partial_\phi \gamma) + \beta \mathcal{G} F_\phi + F_\mathcal{G} (3 \partial_t \xi + \partial_a \alpha) = 0 \\
\beta \mathcal{G} F_\phi + F [(n - 1) \mathcal{G} - 3 \partial_t \xi + \partial_a \alpha + 3 \partial_a \alpha] = 0 \\
3 \alpha \omega + \alpha [\beta \omega_\phi - \omega (\partial_t \xi - 2 \partial_a \beta)] = 0 \\
3 \alpha \mathcal{G} [(n - 1) \mathcal{G}^n + V] + a (n - 1) \mathcal{G}^n (n \gamma + \mathcal{G} \partial_\xi \xi) + a \mathcal{G} \beta [(n - 1) \mathcal{G}^n F_\phi + V_\phi] + a \mathcal{G} V \partial_t \xi = 0
\end{array} \right.
\end{align*}$$  (49)

As in the previous cases, the system is overdetermined and admits an infinite class of solutions depending on the form of the unknown coupling, namely

$$\begin{align*}
\alpha = \alpha_0 a, & \quad \beta = \frac{(3 \alpha_0 + \xi_0 - 4 n \xi_0)}{V} F(\phi) \quad \gamma = -4 \xi_0 \mathcal{G} \quad \xi = \xi_0 t + \xi_1 \\
\omega = \frac{F(\phi)}{(3 \alpha_0 + \xi_0 - 4 n \xi_0)^2} V = V_0 F(\phi) \quad \beta = \frac{3 \alpha_0 + \xi_0 - 4 n \xi_0}{k} \quad \gamma = -4 \xi_0 \mathcal{G} \quad \xi = \xi_0 t + \xi_1
\end{align*}$$  (50)

Therefore, by choosing exponential and power-law couplings, Eq. (50) can be split in two different solutions:

$$\begin{align*}
\alpha = \alpha_0 a, & \quad \beta = -\frac{3 \alpha_0 + \xi_0 - 4 n \xi_0}{k} \quad \gamma = -4 \xi_0 \mathcal{G} \quad \xi = \xi_0 t + \xi_1 \\
\omega = \frac{k^2}{(3 \alpha_0 + \xi_0 - 4 n \xi_0)^2} \quad V = V_0 F(\phi) \quad \beta = \frac{3 \alpha_0 + \xi_0 - 4 n \xi_0}{k} \quad \gamma = -4 \xi_0 \mathcal{G} \quad \xi = \xi_0 t + \xi_1
\end{align*}$$  (51)

The above hold as long as $n \neq 1$; otherwise we obtain the following:

$$\begin{align*}
\alpha = 0 & \quad \beta = \frac{3 \xi_0 \phi}{k} \quad \xi = \xi_0 t + \xi_1 \\
\omega(\phi) = \frac{1}{k^2} e^{k\phi} \quad V(\phi) = V_0 e^{k\phi} \\
\alpha = 0 & \quad \beta = \frac{3 \xi_0 \phi}{k} \quad \xi = \xi_0 t + \xi_1 \\
\omega(\phi) = \frac{1}{k^2} e^{k\phi} \quad V(\phi) = V_0 e^{k\phi}
\end{align*}$$  (53)

where, as before, $\xi_0, \alpha_0, \beta_0, \gamma_0, k, n$ are real constants. The $n = 1$ limit is topologically trivial only when $k = 1$, where the contribution of the geometry in the corresponding action turns into a topological surface term. For this reason, there is no interest in investigating cosmological solutions occurring for $k = n = 1$ and, in what follows, we will only focus on the $n = 1/2$ case, which represents the Gauss–Bonnet equivalent to GR in the cosmological framework.

Let us derive cosmological solutions for the Noether symmetry (50). We will solve the Euler-Lagrange equations (47) for $n = 1/2$ in order to compare the results with the above curvature case. For $f(\mathcal{G}) = \mathcal{G}^{1/2}$, Noether’s solutions (52) can be written as:

$$\begin{align*}
\alpha = \frac{\ell}{6} (z + k) a & \quad \beta = -\ell \quad \gamma = -2 \ell (z - k) \mathcal{G} \quad \xi = \frac{\ell}{2} (z - k) t + \xi_1 \\
\omega(\phi) = \frac{1}{k^2} e^{k\phi} \quad V(\phi) = V_0 e^{k\phi} \quad F(\phi) = F_0 e^{k\phi}
\end{align*}$$  (55)

$$\begin{align*}
\alpha = \frac{\ell}{6} (z + k) a & \quad \beta = -\ell \phi \quad \gamma = -2 \ell (z - k) \mathcal{G} \quad \xi = \frac{\ell}{2} (z - k) t + \xi_1 \\
\omega(\phi) = \frac{1}{k^2} e^{k\phi} \quad V(\phi) = V_0 e^{k\phi} \quad F(\phi) = F_0 e^{k\phi}
\end{align*}$$  (56)
where we have defined
\[ \ell \equiv \frac{3\alpha_0 - \xi_0}{k} \quad V_0 F_0^{1+\frac{2s}{3}} \equiv \tilde{V}_0 \quad z \equiv k \left( \frac{3\alpha_0 + \xi_0}{3\alpha_0 - \xi_0} \right). \] (57)

Let us start by analyzing the action containing a power-law coupling, namely:
\[ S = \int \sqrt{-g} \left[ F_0 \sqrt{G} \phi^k + \frac{1}{\ell^2} \phi^2 \phi^{k-2} + \tilde{V}_0 \phi^6 \right] d^4x, \] (58)
corresponding to the solution of Eq. (52). After solving the system (47), we obtain a de Sitter-like solution which fixes the
values of \( k \) and \( z \) to \( k = z = \frac{\sqrt{6}}{2\ell^2 F_0} \). It reads as:
\[ a(t) = a_0 e^{\frac{\sqrt{6} \xi_0}{2\ell^2 F_0} t}, \quad \phi(t) = \phi_0 e^{\frac{\sqrt{6} \xi_0}{2\ell^2 F_0} t}, \quad G(t) = \frac{54 \tilde{V}_0^2}{\ell^4 F_0^2}. \] (59)

This means that, by merging the result provided by the Euler-Lagrange equations with those coming from the Noether approach,
the only generator associated to this case is:
\[ \mathcal{X} = \frac{1}{\sqrt{6\ell^2 F_0}} a \partial_t - \ell \phi \partial_z, \] (60)
which describes an internal gauge symmetry. Let us now analyze the second solution with exponential coupling, potential and
kinetic term; the corresponding action takes the form:
\[ S = \int \sqrt{-g} \left[ F_0 \sqrt{G} \phi^k + \frac{1}{\ell^2} e^2 \phi^2 + \tilde{V}_0 e^6 \right] d^4x. \] (61)

By replacing Eq. (55) into the equations of motion (47), it turns out that these latter can be analytically solved by imposing the
constraint \( k = z \), so that the scale factor and the scalar field behave like
\[ a(t) = a_0 \exp \left\{ \frac{k \ell}{3} \sqrt{\tilde{V}_0 (1 + \sqrt{2})} t \right\}, \quad \phi(t) = -\ell \sqrt{\tilde{V}_0} t, \quad G(t) = \frac{8k^4 \ell^4}{27} \tilde{V}_0^2 (1 + \sqrt{2})^4, \quad k = \frac{3}{\ell^2} \sqrt{\frac{1}{F_0} \sqrt{21 - 12\sqrt{2}}} \] (62)

As final remark, it is worth noticing that the non-minimal couplings with the invariants \( R \), \( T \), and \( G \) all admit de Sitter solutions
which can be easily compared each-other. It is important to point out that the Gauss-Bonnet topological invariant can be defined
also in the case of teleparallel gravity [58, 59] so that the above representations of gravity can be made totally equivalent also at
this level.

VI. EQUIVALENCE OF HAMILTONIAN DYNAMICS

We want to show now that for internal symmetries, namely for \( \xi(t) = 0 \), the Noether Approach provides transformation laws
allowing to introduce cyclic variables into the point-like cosmological Lagrangian. In order to get internal symmetries, we have
to set the infinitesimal generator, related to the time variation, equal to zero. In the case of Ricci scalar coupled to the scalar
field, the only solution containing symmetries which, after setting \( \xi(t) = 0 \), does not lead to trivial results, is that written in Eq.
(22) (see also [38]). With regards to the torsional case, the only compatible solutions are (34), while, for the \( \sqrt{G} \) scalar, both
solutions can be equivalently considered. It is worth noticing that the generators in (22), the first in Eq. (34) and that in Eq.
(56), under appropriate conditions, are equivalent. For this reason, only the Hamiltonian dynamics provided by the following
generators will be investigated:

\[ R \rightarrow \begin{cases} \mathcal{X} = \frac{2(s + 1)}{2s + 3} \beta_0 a^{s+1} \phi^{\frac{2s^2 + 4s + 3}{2s + 1}} \partial_a + \beta_0 a^s \phi^{\frac{2s^2 + 6s + 3}{2s + 1}} \partial_a \\ F(\phi) = \frac{48(s + 1)(s + 2)}{(2s + 3)^2} \phi^2 V(\phi) = V_0 \phi^{\frac{6(s + 1)}{2s + 1}} \end{cases} \] (63)

\[ T \rightarrow \begin{cases} \mathcal{X} = \frac{2\beta_0}{2s + 3} \phi^{\frac{s + 1}{s + 3}} \partial_a + \beta_0 a^s \phi^{\frac{s}{s + 3}} \partial_a \\ F(\phi) = \frac{48}{(2s + 3)^2} \phi^2 V(\phi) = V_0 \phi^{\frac{s}{s + 3}} \end{cases} \] (64)
\[ G \rightarrow \begin{cases} X = \frac{k\ell}{3}a_\alpha a - \ell \phi a_\phi \\ F(\phi) = F_0 \phi^k \\ V = V_0 \phi^k \end{cases} \quad \omega(\phi) = \frac{1}{2} \phi^{k-2}. \quad (65) \]

In the last solution, to obtain the condition \( \xi = 0 \), we set \( k = z, \xi_1 = 0 \). In order to compare the solutions, we set the coefficient of the kinetic term in the Gauss–Bonnet case (65) to be constant, as naturally provided by the Noether Approach in the other two cases (63) and (64). Therefore, by setting \( s = 0 \) and \( k = 2 \), (63), (64) and (65) become

\[ R \rightarrow \begin{cases} X = -2\beta_0 a_\alpha a + \beta_0 \phi a_\phi \\ F(\phi) = \frac{3}{32} \phi^2 \\ V = V_0 \phi^2 \end{cases}, \quad (66) \]

\[ T \rightarrow \begin{cases} X = -2\beta_0 a_\alpha a + \beta_0 \phi a_\phi \\ F(\phi) = \frac{1}{16} \phi^2 \\ V = V_0 \phi^2 \end{cases}, \quad (67) \]

\[ G \rightarrow \begin{cases} X = 2\ell a_\alpha a - \ell \phi a_\phi \\ F(\phi) = F_0 \phi^2 \\ V = V_0 \phi^2 \end{cases}, \quad (68) \]

that is the symmetries fix the equivalence among the three representations of gravity when a scalar field is coupled with \( R, T, \) and \( \sqrt{G} \). This is the main result of this paper.

To finalize our approach, let us consider the generator (63). Thanks to the system (A8) in the Appendix A, we can perform the change of variables induced by the Noether symmetry which allows to introduce a cyclic variable into the Lagrangian [38]. The system (A8) takes the form:

\[ \begin{align*}
X z &= 2\beta_0 a_\alpha a z - \beta_0 \phi a_\phi z = 1 \\
X u &= 2\beta_0 a_\alpha a u - \phi a_\phi u = 0,
\end{align*} \quad (69) \]

where \( z \) represents the cyclic variable and the minisuperspace of configurations is transformed from \( S = \{a, \phi\} \) to \( S' = \{z, u\} \).

A possible solution of the above system is

\[ \begin{align*}
z &= -\frac{1}{\beta_0} \ln \phi \\
u &= a^\frac{2}{3} \phi
\end{align*} \rightarrow \begin{align*}
\phi &= e^{-\beta_0 z} \\
a &= u^\frac{2}{3} e^{\frac{2\phi}{3}}.
\end{align*} \quad (70) \]

Replacing the new variables \( u, z \) into the Lagrangian (17), we get

\[ \mathcal{L}_R = -V_0 u^2 + \frac{1}{2} \ell^2 u^2 z^2 - \frac{1}{4} \ell u \dot{u} z + \frac{1}{4} u^2, \quad (71) \]

where we set \( \ell \equiv \beta_0 \) in order to conform the notation to the other examples. Clearly this form of \( \mathcal{L}_R \) is cyclic in \( z \). After finding the time-derivatives of the variables as functions of the conjugate momenta, we can easily get the Hamiltonian:

\[ \mathcal{H}_R = \pi_u^2 + \frac{4}{\ell^2} \frac{\pi_z^2}{u^2} + V_0 u^2. \quad (72) \]

Classical trajectories (24) can be recovered by means of the Hamilton–Jacobi equations by going back to the old variables (70). It is worth noticing that, in order to provide a comparison among the three equivalent cases, the Hamiltonian dynamics has been studied for the solution (63) only; however, the change of variables coming from the Noether approach can be also found for the other solutions of Noether system, as shown, e.g., in [38, 60–64].

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2 In the case of the Gauss-Bonnet coupling, the vector \( X \) is equivalent to the previous cases multiplying by -1.
The next case is the torsion non-minimally coupled to the scalar field. Let us focus on the solution (64) with \( s = 0 \). With this assumption, the system providing the suitable change of variables is:

\[
\begin{cases}
-\frac{2\beta_0}{3} a \partial_a z + \beta_0 \phi \partial_\phi z = 1 \\
-\frac{2}{3} a \partial_a u + \phi \partial_\phi u = 0,
\end{cases}
\]  

(73)

whose possible solution is the same as before, namely

\[
\begin{cases}
z = -\frac{1}{\beta_0} \ln \phi \\
u = \frac{a}{\ell} \phi
\end{cases}
\rightarrow
\begin{cases}
\phi = e^{-\beta_0 z} \\
a = \frac{u}{\ell} e^{\frac{2\beta_0}{3} z}
\end{cases}
\]  

(74)

and, setting \( \beta_0 \equiv \ell \), the new Lagrangian reads

\[
\mathcal{L}_T = -\frac{1}{8} \frac{u^2}{u} + \frac{1}{2} \ell \dot{u} \dot{z} + \frac{1}{2} \ell^2 u \dot{z}^2 - V_0 u.
\]

(75)

Also here \( z \) is the cyclic variable which permits to write the Hamiltonian as

\[
\mathcal{H}_T = \frac{\pi_z^2}{2w} + 2\pi_z \pi_u + \frac{3}{16} u a^2 u + V_0 w,
\]

(76)

where \( \pi_z, \pi_u \) are the conjugate momenta.

Finally, let us consider the Gauss–Bonnet equivalent Hamiltonian for \( \sqrt{\mathcal{G}} \). The symmetry generators for this case are in Eq. (56). For \( k = 2 \), where we have a constant kinetic term, the Lagrangian can be written as:

\[
\mathcal{L} = -V_0 a^3 \phi^2 + \frac{1}{2} F_0 a^3 \sqrt{G} \phi^2 + 2 F_0 (\phi^2 a^3 \dot{G}) G^{-\frac{3}{2}} - 8 F_0 \phi \dot{a}^3 \phi G^{-\frac{3}{2}} + \frac{1}{12} a^3 \phi^2.
\]

(77)

The condition (A8) permits to change the minisuperspace variables from \( \bar{S} = \{a, \phi, G\} \) to \( \bar{S}' = \{z, u, G\} \) and gives rise to the same system of differential equations as Eq. (69) and Eq. (73), i.e.

\[
\begin{cases}
X z = \frac{2\ell}{\beta} a \partial_a z - \ell \partial_\phi z = 1 \\
X u = \frac{3}{3} a \partial_a u - \partial_\phi u = 0,
\end{cases}
\]

(78)

with \( z \) being the cyclic variable. One possible solution is

\[
\begin{cases}
z = -\frac{1}{\ell} \ln \phi \\
u = \frac{a}{\ell} \phi
\end{cases}
\rightarrow
\begin{cases}
\phi = e^{-\ell z} \\
a = u e^{\frac{2\ell}{3} z}
\end{cases}
\]

(79)

Replacing the new variables \( u, z \) into the Lagrangian (77), we get

\[
\mathcal{L}_G = \frac{G^{-\frac{3}{2}}}{54u} \left[ 27 F_0 G^2 u^3 - 54 G^2 u^3 (V_0 - \ell^2 \dot{z}^2) + 32 F_0 \dot{G} (\ell u \dot{z} + \dot{u})^3 + 128 F_0 \ell \dot{G} \dot{z} (\ell u \dot{z} + \dot{u})^3 \right]
\]

(80)

where, as expected, \( z \) is cyclic. By a straightforward Legendre transformation, we find the Hamiltonian

\[
\mathcal{H}_G = \frac{1}{4 u^2} \left[ 16 G^2 \pi_z^2 + 8 u G \pi_G \pi_u - 2 F_0 \sqrt{G} u^4 + u^2 \pi_u^2 + 4 V_0 u^4 + 3 \cdot 2 \frac{G^2 u}{G_F} \right] \left( \pi_z + \ell \pi_u \right)
\]

(81)

As final remark, we can state that if the models have the same Noether symmetries, they are dynamically equivalent.
VII. CONCLUSIONS

We analyzed non-minimal coupling between a scalar field and gravity, taking into account different geometric invariants, namely $R$, $T$, and $G$, the curvature, torsion, and Gauss-Bonnet scalars respectively. In all cases, the action contains three functions of the scalar field, namely the coupling, the kinetic term and the potential. We showed that, by the Noether Symmetry Approach, it is possible to fix the form of the above functions of the scalar field and solve exactly dynamics. Furthermore, it is possible to demonstrate that if the symmetries coincide, cosmologies coming from curvature, torsion and Gauss-Bonnet gravity are equivalent. In particular, this statement holds as soon as exponential and power–law expansions of the scale factor of the universe are derived as exact solutions. Interestingly, GR can be recovered in all representations as soon as $R = -T + B$ and $f(G) = \sqrt{G}$. Here $B$ is the torsion boundary term.

As concluding remark we can say that Noether symmetries are a general paradigm by which deal with cosmologies coming from different theories of gravity. According to the present results, different theories showing the same symmetries are dynamically equivalent also if coming from different conceptual foundations. Specifically, GR and metric theories require the Equivalence Principle, the Lorentz Invariance and so on. On the other hand, TEGR and its generalization are gauge theories invariant under the translational group. Also if we start from these very different assumptions, if the related dynamics are governed by the same Noether symmetries, theories are equivalent.

In forthcoming studies, these results will be generalized to other classes of gravitational theories.

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Appendix A: The Noether Symmetry Approach

The Noether Symmetry Approach is widely used to deal with cosmologies coming from different theories of gravity. For example, in [65–71], the approach has been used to deal with $f(R)$ gravity. In [72–77], extended $f(T)$ TEGR models have been discussed in cosmology and spherical symmetry. In [56, 57, 78, 79], the Noether theorem has been used to study $f(G)$ and $f(R, G)$ dynamics. Scalar-tensor actions have been studied in [41, 61, 80–82], where the coupling and the potential are found by symmetries. The basic formulation of the Noether Theorem for dynamical systems and cosmology is presented in [38, 56, 83, 84].

For the purpose of this paper, the Noether theorem can be summarized as follows. Let us consider the set of transformations

$$
\begin{align*}
\mathcal{L}(t, q^i, \dot{q}^i) & \rightarrow \mathcal{L}(\tau, \pi^i, \dot{\pi}^i) \\
\tau & = t + \epsilon \xi(t, q^i) + O(\epsilon^2) \\
\pi^i & = q^i + \epsilon \eta^i(t, q^i) + O(\epsilon^2)
\end{align*}
$$

(A1)

where $\mathcal{L}$ is the Lagrangian of the system, $t$ an affine parameter (e.g. time) and $q^i$ the coordinates. If such a transformation leaves the equations of motion invariant, then the condition

$$
\left[ \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + (\dot{q}^i - \dot{q}^i \xi) \frac{\partial}{\partial \dot{q}^i} \right] \mathcal{L} = \dot{\xi} - \xi \frac{\partial \mathcal{L}}{\partial \dot{q}^i}
$$

(A2)

holds, and the quantity

$$
J(t, q^i, \dot{q}^i) = \xi \left( \dot{q}^i \frac{\partial \mathcal{L}}{\partial q^i} - \mathcal{L} \right) - \eta^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} + g(t, q^i)
$$

(A3)

is a constant of motion.

Some remarks are necessary at this point: i) the quantity in the LHS of Eq. (A2) is named the First Prolongation of Noether Vector. It is indicated as $X^{(1)}$. It is called first prolongation since the transformation (A1) involves the first derivative of the variables, discarding the possibility of higher-order Lagrangians. By setting $\xi = 0$, we get the non-extended Noether vector $X$, which provides symmetries which do not depend on the coordinates transformation. In such a case, the condition (A2) can be rewritten as:

$$
\left[ \eta^i \frac{\partial}{\partial q^i} + \eta^i \frac{\partial}{\partial q^i} \right] \mathcal{L} = 0 \quad \Rightarrow \quad J = \eta^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}
$$

(A4)
and it can be recast in terms of Lie derivative as

\[ L_X L = 0. \quad (A5) \]

It means that the Lie derivative of a Lagrangian, containing symmetries along the flux of vector \( X \), vanishes identically. The vector \( X \) provides the possibility to introduce a cyclic variable into the system. To this purpose, let us consider the coordinates transformation

\[ q^i \rightarrow Q^i(q^i), \quad (A6) \]

and the inner derivative of the new variables \( Q^i \), defined as:

\[ i_X dQ^i \equiv \delta q^j \frac{\partial Q^i}{\partial q^j}. \quad (A7) \]

According to these definitions, the non-extended Noether vector can be written in terms of the variables \( Q^i \) as

\[ X' = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \frac{\partial (i_X dQ^k)}{\partial t} \frac{\partial}{\partial \dot{Q}^k}. \]

Imposing

\[ i_X dQ^1 = 1, \quad \text{and} \quad i_X dQ^i = 0, \quad i \neq 1, \quad (A8) \]

the infinitesimal generator of the variable \( Q^1 \) is constant and the conserved quantity is

\[ J = \partial_{Q^1} L = \pi_{Q^1}, \quad (A9) \]

where \( \pi_{Q^1} \) is the conserved momentum. In this way, the conjugate momentum related to \( Q^1 \) is a constant of motion and, therefore, \( Q^1 \) is a cyclic variable. Summing up, the relations (A8) allow to replace a variable with the corresponding integral of motion. In summary, we search for symmetries. If they exist, the Lagrangian is invariant under certain transformations; we write down the generator of such transformations and, by imposing Noether’s identity (A2), we get the infinitesimal generators and the form of the unknown functions into the Lagrangian. The procedure allows to reduce dynamics and, eventually, to solve it finding out exact solutions.

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