Nonlinear modeling and preliminary stabilization results for a class of piezoelectric smart composite beams

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ABSTRACT

Existing smart composite piezoelectric beam models in the literature mostly ignore the electro-magnetic interactions and adopt the linear elasticity theory. However, these interactions substantially change the controllability and stabilizability at the high frequencies, and linear models fail to represent and predict the governing dynamics since mechanical nonlinearities are pronounced in certain applications such as energy harvesting. In this paper, first, a consistent variational approach is used by considering nonlinear elasticity theory to derive equations of motion for a single-layer piezoelectric beam with and without the electromagnetic interactions (fully dynamic and electrostatic). This modeling strategy is extended for the three-layer piezoelectric smart composites by adopting the two widely-accepted sandwich beam theories. For both single-layer and three-layer models, the resulting infinite dimensional equations of motion can be formulated in the state-space form. It is observed that the fully dynamic nonlinear models are unbounded boundary control systems (same in linear theory) \( \dot{y}(t) = (A + N)y(t) + Bu(t) \), the electrostatic nonlinear models are unbounded bilinear control systems \( \dot{y}(t) = (A + N)y(t) + (B_1 + B_2y)u(t) \) in sharp contrast to the linear theory. Finally, we propose \( B^* \)-type feedback controllers to stabilize the single piezoelectric beam models. The filtered semi-discrete Finite Difference approximations is adopted to illustrate the findings.

Keywords: Piezoelectric smart composite beam, nonlinear piezoelectric beam, bilinear control, filtered semi-discrete Finite Difference Method, Mead-Marcus beam, Rao-Nakra beam, piezoelectric harvester.

1. INTRODUCTION

A perfectly-bonded three-layer piezoelectric smart composite beam is an elastic structure consisting of perfectly bonded piezoelectric and elastic layers \( \mathcal{D} \) and \( \mathcal{G} \), and a viscoelastic layer \( \mathcal{E} \), see Fig. 1a. The layer \( \mathcal{E} \) may be foam-based and it has the ability to passively suppress the vibrations on the device. These devices have become more and more promising in industrial applications in aeronautic, civil, defense, biomedical, and space structures due to their small size and high power density. Applications include cardiac pacemakers, course-changing bullets, structural health monitoring, nano-positioners, ultrasound imagers, ultrasonic welders and cleaners, energy harvesting due to the excellent advantages of the fast response time, large mechanical force, and extremely fine resolution. Accurate modeling and controlling of these structures are critical to achieve objectives for high-precision motion.

As the linear mechanical effects are considered, certain mechanical and electro-magnetic assumptions are required for each layer. The middle layer is not resistant to shear, and outer layers are relatively more stiff. Therefore, Mindlin-Timoshenko and Euler-Bernoulli-type linear elasticity theories are good fits for the middle and outer layers, respectively. For the electro-magnetic interactions on the piezoelectric layer, electrostatic assumption is almost standard through existing models. For the mechanical interaction of layers \( \mathcal{D} \), \( \mathcal{E} \) and \( \mathcal{G} \), the existing piezoelectric composite beam models use Mead-Marcus or Rao-Nakra sandwich beam theories. For instance, a Mead-Marcus type model is obtained by neglecting the rotational inertia terms for the longitudinal dynamics and rotational inertia for the bending dynamics. On the other hand, the model obtained through a consistent variational approach is more like a Rao-Nakra-type. All of these models reduce to the classical counterparts (sandwich laminates) once the piezoelectric strain is taken to be zero. Recently, Ahmet Özkan Özer is a faculty member in the Department of Mathematics of Western Kentucky University, Bowling Green, KY, 42101, USA ozkan.ober@wku.edu
A piezoelectric smart composite of length \( L \) with thicknesses \( h_1, h_2, h_3 \) for its layers \( \mathbb{1}, \mathbb{2}, \mathbb{3} \), respectively. Voltage control \( V(t) \) and strain controllers \( g_1^1(t), g_3^1(t) \) (in \( x \)-direction) control stretching motion of the layers \( \mathbb{1} \) and \( \mathbb{3} \). The whole device bends as a whole, and the bending motions are controlled by shear \( g(t) \) and moment \( M(t) \) controllers. These controllers are actuated, vibrational modes on the device can be suppressed in a fraction of a second.

A single piezoelectric beam of length \( L \) with thickness \( h \). Voltage control \( V(t) \) at the circuit and strain controller \( g(t) \) (in \( x \)-direction) control stretching motion. Since the nonlinear theory is used, the bending motions are controlled by shear \( g(t) \) and moment \( M(t) \) controllers.

Linear fully dynamic and electrostatic models are obtained through a consistent variational approach. The electrostatic models are simply derived by freeing the magnetic effects in the fully dynamic models. Keeping magnetic effects in the equations provide not only the natural control and observation operators but also exploring how the electrical and mechanical equations are coupled rigorously. In fact, the \( \mathbb{B}^* \)—observation is the total induced current at the electrodes. It is much easier and physical in terms of practical applications since measuring the total induced current at the electrodes is easier than measuring displacements or the velocity of the composite at one end of the beam in the electrostatic case, i.e. see. As well, there is no need of an external displacement or velocity sensor.

As the nonlinear mechanical effects are considered, the electro-magnetic behavior for a single piezoelectric layer, see Fig. 1b, is still modeled by the electrostatic assumption in the literature. In deep contrast to the linear elasticity theory, the voltage control actuating the piezoelectric layer can control both the stretching and bending. Moreover, the equations of motion form an unbounded bilinear system. This is first observed in. Even though it is known that the electrostatic theory is sufficient for many applications in piezoelectric acoustic wave devices, there are situations in which full electromagnetic coupling needs to be considered. To our knowledge, no work is reported yet on the fully-dynamic nonlinear vibrations of piezoelectric beams other than the port-Hamiltonian framework where it is shown that the quasi-static model lacks of asymptotic stability. For the nonlinear piezoelectric sandwich beams, an electrostatic model is obtained by adopting the Mead-Marcus sandwich theory. To our knowledge, the literature is scarce of rigorous nonlinear electrostatic and fully dynamic piezoelectric composite beam models and related stabilization results.

The paper is organized as the following. In Section 2, both dynamic and electrostatic nonlinear models for a single cantilevered piezoelectric beam are derived through a variational approach by using the Euler Bernoulli and Mindlin-Timoshenko theories. In Section 3, both dynamic and electrostatic nonlinear models for a three-layer piezoelectric smart cantilevered beam are derived by using Mead-Marcus and Rao-Nakra beam theories. In both sections, the electrostatic models are rigorously obtained by freeing the magnetic effects in the differential equations. All electrostatic models are “unbounded” bilinear (affine) control systems with the voltage input. These models form a solid foundation so that it is possible to analyze the well-posedness, stabilization, and approximation problems. In Section 4, we propose \( \mathbb{B}^* \)—type (nonlinear) stabilizing controllers for the closed-loop systems of electrostatic piezoelectric beam models. Finally, the filtered semi-discrete Finite Difference approximation is adopted to illustrate the findings. All of the models obtained in the paper can be also used for the dual problem “energy harvesting” by simply coupling the equations to a resistive circuit equation, see Remark 3.3.
2. FULLY DYNAMIC SINGLE LAYER PIEZOELECTRIC BEAM

Let $x, y$ and $z$ be the longitudinal and transverse directions, respectively, and the piezoelectric beam occupy the region $\Omega = [0, L] \times [-b, b] \times [-\frac{h}{2}, \frac{h}{2}]$ where $h << L$, see Fig. 1b. A very widely-used linear constitutive relationship for piezoelectric beams is

$$T = \begin{pmatrix} T_D \end{pmatrix} = \begin{bmatrix} \frac{\gamma \varepsilon}{\varepsilon} \end{bmatrix} \begin{pmatrix} S \ E \end{pmatrix}$$

where $T = (T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12})^T$, $S = (S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23})^T$, $D = (D_1, D_2, D_3)^T$, and $E = (E_1, E_2, E_3)^T$ are stress, strain, electric displacement, and electric field vectors, respectively. Moreover, the matrices $[\alpha]_{6 \times 6}, [\gamma]_{3 \times 6}, [\varepsilon]_{3 \times 3}$ are the matrices with elastic, electro-mechanic, and dielectric constant entries (for more details the reader can refer to 50). We assume the transverse isotropy and polarization in $x_3$-direction. Since $h << L$, assume that all forces acting in the $x_2$ direction are zero. We also assume that $T_{33}$ is zero, and therefore

$$T = (T_{11}, T_{13})^T, S = (S_{11}, S_{13})^T, D = (D_1, D_3)^T, E = (E_1, E_3)^T.$$ 

Let $v = v(x)$, $w = w(x)$ and $\psi = \psi(x)$ denote the longitudinal displacement of the center line, transverse displacement of the beam, and the rotation, respectively. We assume the Euler-Bernoulli (E-B) and Mindlin-Timoshenko (M-T) large-displacement assumptions related nonzero components of strains

$$S_{11} = \frac{\partial U_1}{\partial x} + \frac{1}{2} \left( \frac{\partial U_3}{\partial x} \right)^2, \quad S_{13} = \frac{\partial U_1}{\partial z} + \frac{\partial U_3}{\partial x} + \frac{1}{2} \frac{\partial U_3}{\partial x} \frac{\partial U_3}{\partial z}$$

where it is assumed that $\frac{\partial U_3}{\partial x}$ is small relative to $\frac{\partial U_1}{\partial x}$. Defining the physical variables

$$\gamma_1 = \gamma_{15}, \quad \gamma_3 = \gamma_{31}, \quad \beta_1 = \frac{1}{\varepsilon_1}, \quad \beta_3 = \frac{1}{\varepsilon_3}, \quad \alpha_1 = \alpha_{11} + \gamma_3^2 \beta_3, \quad \alpha_3 = \alpha_{55} + \gamma_3^2 \beta_1.$$  

(1)

the displacement fields for (E-B) and (M-T) assumptions, and the corresponding constitutive relationship between $(T, E)$ and $(S, D)$ are obtained in Table 1.

| Euler-Bernoulli (E-B) | Mindlin-Timoshenko (M-T) |
|----------------------|-------------------------|
| $U_1 = v(x) - zw_x(x)$ | $U_1 = v(x) + z\psi(x)$ |
| $U_3 = w(x)$ | $U_3 = w(x)$ |
| $S_{11} = v_x - zw_{xx} + \frac{1}{2}w_x^2$ | $S_{11} = v_x + zw_{xx} + \frac{1}{2}w_x^2$ |
| $S_{13} = 0$ | $S_{13} = \frac{1}{2}(w_x + \psi)$ |
| $T_{11} = \alpha_1 S_{11} - \gamma_3 \beta_3 D_3$ | $T_{11} = \alpha_1 S_{11} - \gamma_3 \beta_3 D_3$ |
| $T_{13} = -\gamma_1 \beta_1 D_1$ | $T_{13} = \alpha_3 S_{13} - \gamma_1 \beta_1 D_1$ |
| $E_1 = \beta_1 D_1$ | $E_1 = -\gamma_1 \beta_1 S_{13} + 1 D_1$ |
| $E_3 = -\gamma_3 \beta_3 S_{11} + \beta_3 D_3$ | $E_3 = -\gamma_3 \beta_3 S_{11} + \beta_3 D_3$ |

Table 1: Nonlinear constitutive relationships for (E-B) and (M-T) assumptions.

The following Lagrangian is used for voltage-driven piezoelectric beams so that the applied voltage appears in the work term:

$$L = \int_0^T [K - (P + E) + B + W] \ dt$$

(2)

where $K$, $P$, $E$ and $B$ denote kinetic, potential, electrical, magnetic energies of the beam, respectively, and $W$ is the work done by the external forces. Moreover, $P + E$ is the total stored energy of the beam.32

The magnetic energy $B$ (as a function of $D$) is added to the Lagrangian $L$ through Maxwell’s equations. Let $B$ denote magnetic field vector In this paper, we first use the dynamic approach for the modeling of a piezoelectric beam. Maxwell’s equations including the effects of $B$ are $\nabla \cdot B = 0$, $\dot{B} = -\nabla \times E$, and $\dot{D} = \frac{1}{\mu}(\nabla \times B)$ where $\mu$ is the magnetic permeability. It follows from the last equation that $\frac{dB_2}{dx} = -\mu \dot{D}_3$, and so $B_2 = -\mu \int_0^x \dot{D}_3(\xi, x_3, t) \, d\xi$. 


Define \( p = \int_0^x D_3(\xi, t) \, d\xi \) to be the total charge of the beam at \( x \in (0, L) \) so that \( p_x = D_3 \). By using Table 1 (with the assumption \( D_1 = 0 \)), magnetic (electrical kinetic), kinetic, and stored energies of the beam are given respectively by

\[
B = \frac{\mu h}{2} \int_0^L \tilde{p}^2 \, dx, \quad (E-B), \quad (M-T)
\]

\[
K = \frac{\rho}{2} \int_\Omega (\tilde{U}^2 + \tilde{W}^2) \, dX = \frac{\rho h}{2} \int_0^L \left( \dot{\psi}^2 + \frac{h^2}{12} \psi_x^2 + \tilde{\psi}^2 \right) \, dx, \quad (E-B)
\]

\[
P + E = \frac{1}{2} \int_\Omega \left( T_{11} S_{11} + T_{33} S_{13} + D_3 E_3 \right) \, dX = \begin{cases}
\frac{h}{2} \int_0^L \left( \alpha_1 \left( v_x + \frac{1}{2} w_x^2 \right)^2 + \frac{h^2}{12} w_{xx}^2 \right), & (E-B) \\
-\frac{2\gamma_3 \beta_3 \left( v_x + \frac{1}{2} w_x^2 \right) p' + \beta_3 p_x^2}{\rho} \right) \, dx, & (M-T) \\
\frac{h}{2} \int_0^L \left( \alpha_1 \left( v_x + \frac{1}{2} w_x^2 \right)^2 + \frac{h^2}{12} \psi_x^2 \right) + \beta_3 p_x^2 + \alpha_3 (w_x + \psi)^2 - 2\gamma_3 \beta_3 \left( v_x + \frac{1}{2} w_x^2 \right) p_x \right) \, dx, & (M-T)
\end{cases}
\]

where \( \rho, \mu \) denote the volume density and magnetic permeability of the beam, respectively. Assume that the beam is subject to a distribution of boundary forces \( \tilde{g}^1, \tilde{g} \) along its edge \( x = L \). Now define

\[
g^1(x, t) = \int_{x-h/2}^{x+1/2} \tilde{g}(x, z, t) \, dz, \quad g(x, t) = \int_{x-h/2}^{x+1/2} \tilde{g}(x, z, t) \, dz, \quad m(x, t) = \int_{x-h/2}^{x+1/2} \tilde{g}^1(x, z, t) \, dz
\]

to be the external force resultants defined as in.\(^{25}\) For our model, it is appropriate to assume that \( \tilde{g}^1 \) is independent of \( z \). Let \( V(t) \) be the voltage applied at the electrodes of the piezoelectric layer. The total work done by the external forces is \( W = W_x + W_m \) where \( W_x = \int_0^L [-p_x V(t)] \, dx \) denotes the electrical force generated by applying voltage \( V(t) \) to the electrodes of the piezoelectric layer, and \( W_m \) denotes the external mechanical forces and defined by

\[
W_m = \begin{cases}
g^1 v^1(L) + g w(L) - m w_x(L), & (E-B) \\
g^1 v^1(L) + g w(L) - m \psi(L), & (M-T)
\end{cases}
\]

The application of Hamilton’s principle, using clamped-free boundary conditions and setting the variation of admissible displacements \( \{v, w, p\} \) for (E-B) and \( \{v, w, \dot{\psi}, \dot{p}\} \) for (M-T) of \( L \) to zero, respectively yields

\[
(E-B) \quad \begin{align}
\rho \frac{\partial \dot{v}}{\partial t} - \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right)_x + \gamma_3 \beta_3 h p_{xx} &= 0, \\
\mu h \ddot{w} - \beta_3 h p_{xx} + \gamma_3 \beta_3 h \left( v_x + \frac{1}{2} w_x^2 \right)_x &= 0, \\
\rho h \ddot{w} - \frac{1}{12} h^2 \psi_{xxx} + \frac{1}{12} h^2 w_{xxx} + \left( -\alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right) + \gamma_3 \beta_3 h p_x \right) w_x &= 0,
\end{align}
\]

\[
\begin{align}
\{v, p, w, \dot{w}, \dot{p}\} & \{0, 0, \alpha_1 k^1 w_x(L) = -(m(t)), \quad \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right) - \gamma_3 \beta_3 h p_x \} \{L = g^1(t), \quad (3)\}
\end{align}
\]

\[
(M-T) \quad \begin{align}
\rho \ddot{v} - \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right)_x + \gamma_3 \beta_3 h p_{xx} &= 0, \\
\frac{h^3}{12} \psi_{xx} + \alpha_1 h w_x + \alpha_1 h (w_x + \psi) &= 0, \\
\rho \ddot{w} - \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right)_x + \gamma_3 \beta_3 h (w_x p_x) - \alpha_3 h (w_x + \psi) &= 0, \\
\mu h \ddot{p} - \beta_3 h p_{xx} + \gamma_3 \beta_3 h \left( v_x + \frac{1}{2} w_x^2 \right)_x &= 0,
\end{align}
\]

\[
\begin{align}
\{v, p, w, \psi\} & \{0, 0, \alpha_1 k^1 \psi(L) = m(t), \quad \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right) - \gamma_3 \beta_3 h p_x \} \{L = g^1(t), \quad (4)\}
\end{align}
\]

\[
\begin{align}
\{v, p, w, \psi\} & \{0, 0, \alpha_1 k^1 \psi(L) = m(t), \quad \alpha_1 h \left( v_x + \frac{1}{2} w_x^2 \right) - \gamma_3 \beta_3 h p_x \} \{L = g^1(t), \quad (4)\}
\end{align}
\]
Note that these nonlinear equations are novel and they were never studied before. If one considers the linearization of (3) and (4) around \( v = w = p = \dot{v} = \dot{w} = \dot{p} = 0 \) for (E-B) and \( v = \psi = w = p = \dot{v} = \dot{w} = \dot{\psi} = \dot{p} = 0 \) for (M-T), or if we simply consider the linear stress-strain relationship in the beginning, we still obtain a boundary control system\(^3\).

\[
\begin{align*}
\begin{cases}
\rho \ddot{v} - \alpha_1 hv_{xx} + \gamma_3 \beta_3 hp_{xx} &= 0, \\
\mu \ddot{p} - \beta_3 hp_{xx} + \gamma_3 \beta_3 hv_{xx} &= 0, \\
\rho \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xxx} + \frac{\alpha h^3}{12} w_{xxxx} &= 0,
\end{cases}
\end{align*}
\]

\[(E - B)\]

\[
\begin{align*}
\begin{cases}
\left[ v, p, w, w_x \right](0) = 0, & \left[ \alpha h^3 \ddot{w}_{xx} \right](L) = -m(t), & \left[ \alpha_1 hv_{xx} - \gamma_3 \beta_3 hp_{xx} \right](L) = g^1(t), \\
\left[ \beta_3 hp_{xx} - \gamma_3 \beta_3 hv_{xx} \right](L) = -V(t), & \left[ \frac{\rho h^3}{12} \ddot{w}_x - \frac{\alpha h^3}{12} w_{xxxx} \right](L) = g(t), \\
(v, w, p, v, w, \dot{v}, \dot{p})(x, 0) = (v_0, w_0, p_0, v_1, w_1, p_1),
\end{cases}
\end{align*}
\]

\[(M - T)\]

The bending equation in (E-B) and the bending and rotation angle equations in (M-T) are completely decoupled from the rest. If one considers the voltage control \( V(t) \) only, the bending and shear motions can never be controlled. Only certain longitudinal motions are controlled\(^1\) by measuring the total induced current accumulated at the electrodes.\(^35\) In practical applications, these uncontrolled longitudinal motions corresponding to the high-frequency solutions.

### 2.1 Electrostatic models

By the electrostatic assumption, all magnetic effects are discarded (i.e., take \( \ddot{p} \equiv 0 \)) and \( g^1(t) \equiv 0 \). Therefore (3) and (4) respectively reduce to

\[
\begin{align*}
\begin{cases}
\rho \ddot{v} - \alpha_{11} h \left( v_x + \frac{1}{2} w_x^2 \right)_x &= 0, \\
\rho \ddot{w} - \frac{\rho h^3}{12} w_{xx} + \frac{\alpha h^3}{12} w_{xxxx} - \left[ \left( \alpha_{11} h \left( v_x + \frac{1}{2} w_x^2 \right) + \gamma_3 H_L V(t) \right) w_x \right]_x &= 0,
\end{cases}
\end{align*}
\]

\[(E - B)\]

\[
\begin{align*}
\begin{cases}
\left[ v, w, w_x \right](0) = 0, & \alpha h^3 \ddot{w}_{xx}(L) = -m(t), \\
\left[ \alpha_{11} h \left( v_x + \frac{1}{2} w_x^2 \right) \right](L) = -\gamma_3 H_L V(t), & \left[ \frac{\rho h^3}{12} \ddot{w}_x - \frac{\alpha h^3}{12} w_{xxxx} \right](L) = g(t), \\
(v, w, v, \ddot{w})(x, 0) = (v_0, w_0, v_1, w_1),
\end{cases}
\end{align*}
\]

\[(M - T)\]

where we used (1), and \( H_L = H(x) - H(x - L) \) is the characteristic function of the interval \((0, L)\). Note that the fully elastic (E-B) and (M-T) beam models are derived and studied in\(^6\) respectively by taking \( V(t) \equiv 0 \). Observe that as the longitudinal inertia term \( \rho \ddot{w} \equiv 0 \) in (7)-(8) is neglected, the voltage control \( V(t) \) does not affect the bending equation anymore. This is the difference between elastic beam models\(^26,26\) and electrostatic
Terms in Table 2, the (E-B) systems (3) and (7) can be re-written by replacing the terms 
\( \alpha_1 v_x + \tilde{\alpha}_1 \dot{v}_x \) and \( \frac{\tilde{\alpha}_3}{12} w_{xxxx} + \frac{\tilde{\alpha}_3}{12} \dot{w}_{xxxx} \), respectively, where \( \tilde{\alpha}_1 \) is the damping coefficient. Note also that the 

| (E-B) with (K-V) damping | (M-T) with (K-V) damping |
|--------------------------|--------------------------|
| \( T_{11} = \alpha_1 S_{11} + \tilde{\alpha}_1 S_{11} - \gamma_3 \beta_3 D_3 \) | \( T_{11} = \alpha_1 S_{11} + \tilde{\alpha}_1 S_{11} - \gamma_3 \beta_3 D_3 \) |
| \( T_{13} = -\gamma_1 \beta_1 D_1 \) | \( T_{13} = \alpha_3 S_{13} + \tilde{\alpha}_3 S_{13} - \gamma_1 \beta_1 D_1 \) |

Table 2: Adding Kelvin-Voigt type damping to the constitutive equations.

The main challenge in the nonlinear electrostatic models (7) and (8) is the voltage control \( V(t) \) appearing at both the axial strain boundary condition at \( x = L \) and the \( w \)-equation. This makes the boundary control system not only unbounded but also bilinear and unbounded. To show this, consider (7) with the state \( x = [v, w, \dot{v}, \dot{w}]^T \).

Let \( D_x = \frac{d}{dx} D^n_x = \frac{d^n}{dx^n} \) for \( n > 1 \). Defining the operator \( \mathcal{P} = (I - \frac{k^2}{12} D_x^2)^{-1} \) where \( I \) is the identity operator, an unbounded bilinear control system for (7) is obtained as the following

\[
\dot{y} = (A + N)y + (B_1 + B_2 y) u(t) \quad \text{where} \\
A = \begin{pmatrix} \frac{\alpha_1}{\rho} D_x^2 & 0 & \cdots & 0 \\ 0 & \frac{\alpha_1}{\rho} \mathcal{P} \left(-\frac{k^2}{12} D_x^4 + D_x^2\right) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\alpha_1}{\rho} \mathcal{P} D_x (H_L D_x) \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_1}{\rho} \mathcal{P} D_x (H_L D_x) & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_1}{\rho} \mathcal{P} D_x (H_L D_x) & 0 & 0 \end{pmatrix},
\]

\( u(t) = \begin{pmatrix} V(t) \\ m(t) \end{pmatrix}, \quad N y = \begin{pmatrix} 0_{2 \times 1} \\ \mathcal{P} D_x \left(\frac{1}{2} (D_x (\cdot))^2\right) \end{pmatrix} \)

is a nonlinear operator, and \( \delta_L = \delta(x) - \delta(x - L) \). Define the natural energy space as

\[
H = H^1_L(0, L) \times H^2_L(0, L) \times L^2(0, L) \times H^1_L(0, L)
\]

where

\[
H^1_L(0, L) = \{ z \in H^1(0, L) : z(0) = 0 \}, \quad H^2_L(0, L) = \{ z \in H^2(0, L) : z(0) = z(0) = 0 \}.
\]

It can be shown that \( A \) is an infinitesimal generator of a unitary semigroup on \( H, N : H \to H \) is locally Lipschitz. As well, \( B_1 : \mathbb{C} \to H, B_2 : \mathbb{C} \to H \) are unbounded and bounded operators, respectively. The local controllability or the uniform (or asymptotic) stabilization of systems like (9) is still an open problem. A similar framework is recently studied in \(^{5,8,8}\) In Section 4, we take a stab of this hard problem in a numerical sense.

Unlike the electro-static models, the fully dynamic models (3), (4) with the choice of states \( y = [v, p, w, \dot{v}, \dot{p}, \dot{w}]^T \) can be written as

\[
\dot{y} = (A + N)y + Bu(t)
\]

where \( B \) is a boundary control operator with the input \( u(t) = [g(t), V(t), m(t), \dot{g}(t)]^T \). Then observation \( B^* y \) for only the voltage input corresponds to the total induced current \( \dot{\rho}(L) = \int_0^L D_3 \dot{w} \) accumulated at the electrodes.\(^{35}\) This is completely electro-magnetic, and is more practical than measuring tip velocity at the tip of the beam. Hence, it is more promising in energy harvesting applications.\(^{15}\)
If we linearize (7) along the equilibrium \( v = w = \dot{v} = \dot{w} = 0 \) for (E-B) and \( v = \psi = w = \dot{v} = \dot{w} = 0 \) for (M-T), or if we simply consider the linear stress-strain relationship in the beginning, we obtain a “degenerate” (strongly decoupled) system:

\[
\begin{align*}
\rho \ddot{v} - \alpha_{11} h \ddot{v}_{xx} &= 0, \\
\rho \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xx} + \frac{\alpha_3 h^3}{12} w_{xxxx} &= 0, \\
[v, w, w_x](0) &= 0, \\
\alpha_{11} h v_x(L) &= -\gamma_3 V(t), \\
\frac{\alpha_3 h^3}{12} \ddot{w}_x - \frac{\alpha_3 h^3}{12} w_{xxxx}(L) &= g(t), \\
(v, w, \dot{w}, \dot{w})(x, 0) &= (v_0, w_0, v_1, w_1),
\end{align*}
\]

where the unbounded boundary control \( V(t) \) only affects the stretching motion, not bending anymore. Including all three controllers \( V(t), m(t), g(t) \), both systems can be shown to be exponentially stabilizable.²⁶

**Remark 2.1.** For energy harvesting applications, the nonlinear models (7,8) and the linear models (10,11) are coupled to the extra circuit equation

\[
\frac{2 \alpha_{11} \varepsilon_3 b L}{\alpha_1 h} \dot{V}(t) + i(t) = 0
\]

where \( i(t) = \frac{1}{\pi} V(t) \) is the current generated by the piezoelectric beam, \( R \) is the resistance of the attached circuit.

### 3. Fully Dynamic Three-Layer Piezoelectric Smart Beam

The three-layer piezoelectric smart composite beam considered in this paper consists of a stiff layer, a compliant layer, and a piezoelectric layer, see Fig. 1a. The composite occupies the region \( \Omega = \Omega_{xy} \times (0, h) = [0, L] \times [-b, b] \times (0, h) \) at equilibrium. The total thickness \( h \) is assumed to be small in comparison to the dimensions of \( \Omega_{xy} \). The layers are indexed from 1 to 3 from the stiff layer to the piezoelectric layer, respectively.

Let \( 0 = z_0 < z_1 < z_2 < z_3 = h \), with \( h_i = z_i - z_{i-1}, \ i = 1, 2, 3 \). We use the rectangular coordinates \( X = (x, y) \) to denote points in \( \Omega_{xy} \), and \( (X, z) \) to denote points in \( \Omega = \Omega^s \cup \Omega^{ve} \cup \Omega^p \), where \( \Omega^s, \Omega^{ve} \), and \( \Omega^p \) are the reference configurations of the stiff, viscoelastic, and piezoelectric layers, respectively, and they are given by

\[
\Omega^s = \Omega_{xy} \times (z_0, z_1), \quad \Omega^{ve} = \Omega_{xy} \times (z_1, z_2), \quad \Omega^p = \Omega_{xy} \times (z_2, z_3).
\]

For \((X, z) \in \Omega\), let \( U(X, z) = (U_1, U_2, U_3)(X, z) \) denote the displacement vector of the point (from reference configuration). For the beam theory, all displacements are assumed to be independent of \( y \)-coordinate, and \( U_2 \equiv 0 \). The transverse displacements is \( w(x, y, z) = U_3(x) = w^i(x) \) for any \( i \) and \( x \in [0, L] \). Define \( u^i(x, y, z) = U_1(x, 0, z_i) = u^i(x) \) for \( i = 0, 1, 2, 3 \) and for all \( x \in (0, L) \).

Define the vectors \( \vec{v} = [\psi^1, \psi^2, \psi^3]^T \), \( \vec{\phi} = [\phi^1, \phi^2, \phi^3]^T \), \( \vec{v} = [v^1, v^2, v^3]^T \) where

\[
\psi^i = \frac{w^i - w^{i-1}}{h_i}, \quad \phi^i = \psi^i + w_x, \quad v^i = \frac{w^{i-1} + w^i}{2}, \quad i = 1, 2, 3,
\]

and \( \psi^i \) is the total rotation angles (with negative orientation) of the deformed filament within the \( i \)th layer in the \( x \) - \( z \) plane, \( \phi^i \) is the (small angle approximation for the) shear angles within each layer, \( v^i \) is the longitudinal displacement of the center line of the \( i \)th layer. For the middle layer, we apply Mindlin-Timoshenko constitutive assumptions, while for the outer layers Kirchhoff displacement assumptions. Therefore,

\[
\phi^1 = \phi^3 = 0, \quad \psi^1 = \psi^3 = -w_x, \quad \phi^2 = \psi^2 + w_x.
\]
Let $G_2$ be the shear modulus of the viscoelastic layer. Defining $\tilde{z}_i = \frac{i-1+i^*}{2}$, and

$$\alpha^3 = \alpha^3_1 + \gamma^2 \beta, \quad \alpha^3_1 = \alpha^3_1_1, \quad \alpha^2 = \alpha^2_1_1, \quad \alpha^1 = \alpha^1_1_1$$

(15)

where we use (1), the displacement fields, strains, and the constitutive relationships for each layer are given in Table 3.

| Layers | Displacements, Stresses, Strains, Electric fields, Electric displacements |
|--------|-------------------------------------------------------------------------|
| Layer $\Phi$ - Elastic | $U_i^\Phi(x, z) = v^\Phi(x) - (z - \tilde{z}_1)w_x, \quad U_3(x, z) = w(x)$ |
| $S_{11} = \frac{\partial v^\Phi}{\partial x} + \frac{1}{2} (w_x)^2 - (z - \tilde{z}_1) \frac{\partial w}{\partial x^2}, \quad S_{13} = 0$ |
| $T_{11} = \alpha_1 S_{11}, \quad T_{13} = 0$ |
| Layer $\Psi$ - Viscoelastic | $U_i^\Psi(x, z) = v^\Psi(x) + (z - \tilde{z}_2)\psi^\Psi(x), \quad U_3(x, z) = w(x)$ |
| $S_{11} = \frac{\partial v^\Psi}{\partial x} - (z - \tilde{z}_2) \frac{\partial \psi^\Psi}{\partial x^2}, \quad S_{13} = \frac{1}{2} \psi^2$ |
| $T_{11} = \alpha_2 S_{11}, \quad T_{13} = 2 G_2 S_{13}$ |
| Layer $\Omega$ - Piezoelectric | $U_i^\Omega(x, z) = v^\Omega(x) - (z - \tilde{z}_3)w_x, \quad U_3(x, z) = w(x)$ |
| $S_{11} = \frac{\partial v^\Omega}{\partial x} + \frac{1}{2} (w_x)^2 - (z - \tilde{z}_3) \frac{\partial w}{\partial x^2}, \quad S_{13} = 0$ |
| $T_{11} = \alpha_3 S_{11} - \gamma \beta D_3, \quad T_{13} = 0$ |
| $E_1 = \beta_1 D_1, \quad E_3 = -\gamma \beta S_{11} + \beta D_3$ |

Table 3: Nonlinear constitutive relationships for each layer.

Assume that the beam is subject to a distribution of boundary forces $(\tilde{g}^1, \tilde{g}^3, \tilde{g})$ along its edge $x = L$. Now define

$$g^i(x, t) = \int_{z_{i-1}}^{z_i} \tilde{g}^i(x, z, t) \, dz, \quad g(x, t) = \int_0^L \tilde{g}(x, z, t) \, dz, \quad m_i = \int_{z_{i-1}}^{z_i} (z - \tilde{z}_i)\tilde{g}^i(x, z, t) \, dz, \quad i = 1, 3$$

to be the external force resultants. The total work done by the external forces is $W = W_e + W_m$ where $W_e = \int_0^L [-p_x V(t)] \, dx$ denotes the electrical force generated by applying voltage $V(t)$ to the electrodes of the piezoelectric layer, and $W_m$ denotes the external mechanical forces and defined by

$$W_m = g^1 v^1(L) + g^3 v^3(L) + gw(L) - M w_x(L), \quad M = m_1 + m_3.$$  

(16)

We follow the same approach in Section 2 to include the magnetic effects. The magnetic field $B$ is perpendicular to the $x - z$ plane, and therefore, $B_2(x)$ is the only non-zero component. Assuming $E_1 = D_1 = 0$, the Lagrangian for the ACL beam is the same as (2) with

$$K = \frac{1}{2} \int_0^L \left[ \sum_{i=1}^3 \left( n_i h_i (v^i) \right)^2 + \left( \sum_{i=1}^3 n_i h_i \right) \left( \dot{w} \right)^2 + (\rho_1 h_1 + \rho_3 h_3) \dot{w}_x^2 + \rho_2 h_2 (\dot{\psi}^2)^2 \right] \, dx,$$

$$P + E = \frac{1}{2} \int_0^L \left[ \sum_{i=1,3} \alpha h_i \left( (v_x^i + \frac{h_i}{12} w_x^2) \right)^2 + \alpha h_2 \left( (v_x^i + h_2 w_x^2) \right)^2 - \gamma \beta h_3 \left( v_x^3 + \frac{h_3}{12} w_x^2 \right) \right] \, dx,$$

$$B = \frac{\mu b h_3}{2} \int_0^L \dot{\phi}^2 \, dx,$$

where $\rho_i$ is the volume density of the $i$th layer. By using (13)-(14), the variables $v^2, \phi^2$ and $\psi^2$ can be written as the functions of state variables as the following

$$v^2 = \frac{1}{2} (v_1 + v^3) + \frac{h_3 - h_1}{4} w_x, \quad \phi^2 = \frac{1}{h_2} (-v^1 + v^3) + \frac{h_1}{h_2} \dot{w}_x, \quad \psi^2 = \frac{1}{h_2} (-v^1 + v^3) + \frac{h_1}{h_2} \dot{w}_x.$$

Thus, we choose $w, v^1, v^3$ as the state variables. Let $H = h_1 + 2 h_2 + h_3$. Application of Hamilton’s principle, by using forced boundary conditions, i.e. clamped at $x = 0$, by setting the variation of admissible displacements
\{v^i, v^3, p, w\} of \mathbf{L} in (2) to zero yields the following coupled equations of stretching in odd layers, dynamic change in the piezoelectric layer, and the bending of the whole composite:

\[
\begin{align*}
\begin{cases}
\rho_1h_1\ddot{v}^1 + \frac{\rho_2h_2}{12}v^3 - \frac{1}{3}\alpha_2h_2v^3_{xx} - \left[ (\alpha_1h_1v^1_x + \frac{1}{2}(w_x)^2) + \frac{1}{2}\alpha_2h_2v^3_{xx} \right]_x - G_2\phi^2 \\
- \frac{\rho_2h_2}{12}(2h_1 - h_3)\ddot{w}_x + \alpha_2^2h_2(2h_1 - h_3)w_{xxx} = 0, \\
\rho_3h_3\ddot{v}^3 + \frac{\rho_2h_2}{12}v^1 - \frac{1}{2}\alpha_2h_2v^1_{xx} - \left[ (\alpha_3h_3v^3_x + \frac{1}{2}(w_x)^2) + \frac{1}{2}\alpha_2h_2v^3_{xx} \right]_x + G_2\phi^2 \\
+ \frac{\rho_2h_2}{12}(2h_3 - h_1)\ddot{w}_x - \alpha_2^2h_2(2h_3 - h_1)w_{xxx} + \gamma\beta h_3p_{xx} = 0,
\end{cases}
\end{align*}
\]

with clamped boundary conditions at the left end \(v^i, v^3, w, w_x, p\) \(|_{x=0} = 0\) and the free boundary conditions at \(x = L\):

\[
\begin{align*}
\begin{cases}
\frac{\rho_2h_2}{12}v^3 + \frac{\rho_2h_2}{12}(2h_1 - h_3)v^1 - \alpha_2^2h_2(2h_1 - h_3)w = g^1(t), \\
\frac{1}{3}\alpha_2^2h_2v^3 + \alpha_1h_1v^3_x + \frac{1}{2}(w_x)^2 - \alpha_2^2h_2(2h_1 - h_3)w = g^3(t), \\
\beta h_3p_{xx} - \gamma\beta h_3\left( v^3_x + \frac{1}{2}(w_x)^2 \right)_x = -\frac{1}{12}G(t), \\
\left( v^1, v^3, p, w, \dot{v}^1, \dot{v}^3, \dot{p}, \dot{w} \right)(x, 0) = (v^1_0, v^3_0, w_0, p_0, v^1_1, v^3_1, p_1, w_1).
\end{cases}
\end{align*}
\]

3.1 Fully dynamic Rao-Nakra (R-N) beam model

The model obtained above is still highly coupled. We assume that the viscoelastic layer is very thin and its stiffness is negligible, i.e. \(\rho_2, \alpha^2 \to 0\) as in.22 This approximation retains the potential energy of shear and transverse kinetic energy so that the model above reduces to

\[
\begin{align*}
\begin{cases}
\rho_1h_1\ddot{v}^1 - \alpha_1h_1\left( v^1_x + \frac{1}{2}(w_x)^2 \right)_x - G_2\phi^2 = 0, \\
\beta h_3p_x - \gamma\beta h_3\left( v^3_x + \frac{1}{2}(w_x)^2 \right)_x = 0,
\end{cases}
\end{align*}
\]

with initial conditions, and clamped boundary conditions at the left end \(v^1, v^3, p, w, w_x(0) = 0\) and the free boundary conditions at \(x = L\):

\[
\begin{align*}
\begin{cases}
\alpha^1h_1\left( v^1_x + \frac{1}{2}(w_x)^2 \right) = g^1(t), \\
\beta h_3p_x - \gamma\beta h_3\left( v^3_x + \frac{1}{2}(w_x)^2 \right)^2 = -V(t), \\
K_1\ddot{w} - K_2w_{xxx} + G_2H\phi^2 - \gamma\beta h_3p_x - \left[ \sum_{i=1,3} \alpha^i h_i \left( v^i_x + \frac{1}{2}(w_x)^2 \right)_x \right] = 0,
\end{cases}
\end{align*}
\]

where \(\rho = \rho_1h_1 + \rho_2h_2 + \rho_3h_3, K_1 = \frac{\rho_1h_1}{12} + \frac{\rho_2h_2}{12},\) and \(K_2 = \frac{\rho_1h_1}{12} + \frac{\rho_2h_2}{12} + \frac{\rho_2h_2}{12}\).

This model accounts for not only the coupling between the stretching and bending motions but also the dynamic behavior of the stretching of the piezoelectric layers, i.e. \(\ddot{v}^3 \neq 0\). Note that the linearized model is first derived in\(^{14}\) and it is shown in\(^{35}\) that certain vibration modes can not be controlled in the absence of \(g^1(t)\).
3.2 Electrostatic (R-N) beam model

By the electrostatic assumption, all magnetic effects are discarded, i.e. $\mu \to 0$ or $\mu \bar{p} = 0$. Since the $p-$equation in (19) can be solved for $p$, the system (19)-(20) are simplified as the following

\[ \begin{align*}
\rho_1 h_1 \ddot{v}_1 - \alpha_1^3 h_1 (v_1^2 + \frac{1}{2} (w_x)^2)_x - G_2 \phi^2 &= 0, \\
\rho_3 h_3 \dddot{v}_3 - \alpha_1^3 h_3 (v_3^2 + \frac{1}{2} (w_x)^2)_x + G_2 \phi^2 &= 0, \\
\rho \ddot{w} - K_1 \dddot{w}_x + K_2 w_{xxx} - G_2 H \phi_x^2 - \left( \sum_{i=1,3} \alpha^i h_i (v_i^2 + \frac{1}{2} (w_x)^2) \right) w_x = - (\gamma H \bar{w}_x)_x V(t) &= 0,
\end{align*} \]

(21)

with the initial conditions, clamped boundary condition at the left end $[v^1, v^3, w, w_x](0) = 0$, and free boundary conditions at $x = L$:

\[ \begin{align*}
\alpha_1^3 h_1 (v_1^2 + \frac{1}{2} (w_x)^2)(L) &= g^1(t), \\
\alpha_3^3 h_3 (v_3^2 + \frac{1}{2} (w_x)^2)(L) &= - \gamma V(t), \\
K_2 w_{xx}(L) &= - M(t), \\
K_1 \ddot{w}_x - K_2 w_{xxx} + G_2 H \phi_x^2 + \left( \sum_{i=1,3} \alpha^i h_i (v_i^2 + \frac{1}{2} (w_x)^2) \right) w_x (L) &= g(t), \\
(v^1, v^3, w, \ddot{v}, \dddot{w})(x, 0) &= (v_0^1, v_0^3, w_0, v_1, v_1, w_1).
\end{align*} \]

(22)

where we use (15), and $g^3 \equiv 0$. Note that the voltage controller is the only controller for controlling the strains in the piezoelectric layer.

If we linearize (21) along the equilibrium $v^1 = v^3 = w = \dot{v} = \ddot{v} = \dddot{w} = 0$, or if we simply consider the linear stress-strain relationship in the beginning, we obtain the coupled system

\[ \begin{align*}
\rho_1 h_1 \ddot{v}_1 - \alpha_1^3 h_1 v_{xx}^1 - G_2 \phi^2 &= 0, \\
\rho_3 h_3 \ddot{v}_3 - \alpha_1^3 h_3 v_{xx}^3 + G_2 \phi^2 &= 0, \\
\rho \dddot{w} - K_1 \dddot{w}_x + K_2 w_{xxx} - G_2 H \phi_x^2 &= 0, \\
v^1(0) = v^3(0) = w(0) = w_x(0) = 0, \\
\alpha_1^3 h_1 (v_1^2) = g^1(t), \\
\alpha_3^3 h_3 (v_3^2) = - \gamma V(t), \\
K_2 w_{xx}(L) &= - M(t), \\
K_1 \ddot{w}_x - K_2 w_{xxx} + G_2 H \phi_x^2 &= g(t), \\
(v^1, v^3, w, \ddot{v}, \dddot{w})(x, 0) &= (v_0^1, v_0^3, w_0, v_1, v_1, w_1).
\end{align*} \]

(23)

(24)

where the only coupling is due to the shear $\phi^2 = \frac{1}{r^2} (- v^1 + v^3 + H w_x)$ of the middle layer. This model is first obtained in.\(^\text{34}\) Later, it is shown to be exponentially stabilizable by the appropriate choice of controllers.\(^\text{35}\) For elastic-viscoelastic-elastic model, there is currently a large literature for the controllability and stabilization\(^\text{38,39}\) with various other boundary conditions.

3.3 Fully dynamic Mead-Marcus (M-M) beam model

By the Mead-Marcus sandwich beam theory, we assume that the longitudinal and rotational inertia terms, i.e. $\ddot{v}_1, \dddot{w}, \dddot{v}_3$, are negligible in (17)-(18). Therefore, we obtain

\[ \begin{align*}
- \frac{1}{6} \alpha^2 h_2 v_{xx}^3 - \left[ (\alpha^1 h_1 (v_1^4 + \frac{1}{2} (w_x)^2)) + \frac{1}{3} \alpha^2 h_2 v_2^4 \right]_{xx} + \frac{\alpha^3 h_3 (2h_3 - h_3) w_{xx} - G_2 \phi^2}{12} &= 0, \\
- \frac{1}{6} \alpha^2 h_2 v_{xx}^3 - \left[ (\alpha^3 h_3 (v_3^4 + \frac{1}{2} (w_x)^2)) + \frac{1}{3} \alpha^2 h_2 v_2^4 \right]_{xx} - \frac{\alpha^3 h_3 (2h_3 - h_3) w_{xx} + G_2 \phi^2 + \gamma h_3 p_{xx}}{12} &= 0, \\
\mu h_3 \dddot{p} - \gamma h_3 \phi_x^2 + \gamma h_3 (v_2^2 + \frac{1}{2} (w_x)^2)_x &= 0, \\
\rho \ddot{w} - H \phi_x^2 + \frac{1}{12} (\alpha^1 h_1^3 + \alpha^3 h_3^3 + \alpha^2 h_2 (h_3 + h_3 - h_1 h_3)) w_{xxx} &= 0, \\
- \frac{\alpha^3 h_3 (2h_3 - h_3) v_{xxx}}{12} + \frac{\alpha^3 h_3 (2h_3 - h_3) v_{xxx} - \left( \sum_{i=1,3} \alpha^i h_i (v_i^2 + \frac{1}{2} (w_x)^2) - \gamma h_3 p_x \right) w_x}{12} &= 0,
\end{align*} \]

(25)
with clamped boundary conditions at the left end $v^1, v^3, w, w_x, p \mid _{x=0} = 0$ and the free boundary conditions at $x = L$:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{6} \alpha^2 h^2 v^3 + (\alpha^1 h^1 (v^1 + \frac{1}{2} (w_x)^2) + \frac{1}{3} \alpha^2 h^2 v^4 + \frac{\alpha^2 h^2 (2h_1 - h_3)}{12} w_{xx} = 0, \\
\frac{1}{6} \alpha^2 h^2 v^3 + \alpha^3 h^3 (v^2 + \frac{1}{2} (w_x)^2) + \frac{1}{3} \alpha^2 h^2 v^4 + \frac{\alpha^2 h^2 (2h_3 - h_1)}{12} w_{xx} = 0, \\
\beta h_3 x - \gamma h_3 (v^2 + \frac{1}{2} (w_x)^2) = -V(t), \\
\frac{\alpha^1 (h_1^2) + \alpha^3 (h_3^2) + \alpha^2 h^2 (h_1^2 + h_3^2 - h_1 h_3)}{12} w_{xx} - \frac{\alpha^2 h^2 (2h_1 - h_3)}{12} v^1 + \frac{\alpha^2 h^2 (2h_3 - h_1)}{12} v^3 = 0,
\end{array} \right.
\end{align*}
$$

G_2 h^2 \phi^2 + \frac{\alpha^2 h^2}{12} (2h_1 - h_3) v^1_x - \frac{\alpha^2 h^2}{12} (2h_3 - h_1) v^3_x + \left( \sum_{i=1,3} \{ \alpha^1 h_1 (v^i_x + \frac{1}{2} (w_x)^2) \} + \gamma h_3 p_x \right) w_x + \left( -\alpha^1 (h_1^2) - \alpha^1 (h_3^2) + \alpha^2 h^2 ((h_1^2 + h_3^2) - h_1 h_3) \right) w_{xxx} = g(t),

(v^1, v^3, w, p, \dot{v}^1, \dot{v}^3, \dot{w}, \dot{p})(x, 0) = (v^1_0, v^3_0, w_0, p_0, v^1_1, v^3_1, w_1, p_1).
$$

Define the constants by

$$
D := \alpha^2 h^2 \left( 4 \alpha^1 h_1 + 4 \alpha^2 h_2 + 4 \alpha^3 h_3 \right) + 12 \alpha^1 \alpha^3 h_1 h_2 h_3,
$$

$$
A = \frac{1}{12} \left( \alpha^1 h_1^3 + \alpha^3 h_3^3 + \alpha^2 h^2 (5 \alpha^2 h^2 (\alpha^1 h_1^2 + \alpha^3 h_3^2) + 12 \alpha^1 \alpha^3 h_1 h_2 h_3 (h_1^2 + h_3^2 - h_1 h_3)) \right),
$$

$$
B_1 = \frac{1}{12} \left( \alpha^2 (\alpha^1 h_1^2 + \alpha^3 h_3^2) + 2 \alpha^2 h^2 (h_1^2 + h_3^2 + 3 \alpha^2 h^2 (h_1^2 + h_3^2)) \right) + \frac{1}{12} \alpha^1 \alpha^3 h_1 h_2 h_3,
$$

$$
B_2 = \frac{6 \alpha^2 h^2 + 12 \alpha^1 h_1}{D},
$$

$$
B_3 = \frac{1}{12} \left( \alpha^2 (\alpha^1 h_1^2 + \alpha^3 h_3^2) + 2 \alpha^2 h^2 (h_1^2 + h_3^2 + 3 \alpha^2 h^2 (h_1^2 + h_3^2)) \right) + \frac{1}{12} \alpha^1 \alpha^3 h_1 h_2 h_3,
$$

$$
B_4 = \frac{6 \alpha^2 h^2 + 12 \alpha^1 h_1}{D},
$$

$$
C = \frac{12 \alpha^1 h_1 + 12 \alpha^3 h_3}{D},
$$

$$
E_1 = \frac{6 \alpha^2 h^2 (\alpha^1 h_1 - \alpha^3 h_3)}{D},
$$

$$
E_2 = \frac{12 \alpha^2 h^2}{D},
$$

Consider $g(t) \equiv 0$. Now we multiply the first and second equations in (25) by $(\frac{1}{21} \alpha^2 h_2 + \frac{1}{12} \alpha^3 h_3)$ and $-\left(\frac{1}{21} \alpha^2 h_2 + \frac{1}{12} \alpha^3 h_1 \right)$, respectively, and add these two equations. An alternate formulation is obtained as the following

$$
\begin{align*}
\left\{ \begin{array}{l}
\rho \ddot{w} + Aw_{xxxx} - B_1 G_2 \gamma h_2 h_3 \phi^2 + \gamma \beta B_3 p_{xxx} + \frac{1}{2} (B_1 E_1 w_x w_{xx} + E_1 \phi^2_x w_x + \gamma \beta E_2 p_x w_x) \\
+ 3E_3 \left( w_{xx} \right)^2 w_{xx} = 0, \\
C G_2 \phi^2 - \phi^2_{xx} + B_1 w_{xxx} + B_2 p_{xx} + E_1 w_x w_{xx} = 0, \\
\mu \ddot{p} + \beta (-B_3 p_{xx} + \gamma \beta h_3 h_2 G_2 B_2 \phi^2 - \gamma B_3 w_{xxxx} + \gamma B_3 p_{xx} w_x) = 0, \\
\left[ \begin{array}{l}
w, w_x, \phi^2, p \end{array} \right] (0) = 0, \\
\left[ Aw_{xx} + \gamma \beta B_3 P_x + \frac{B_2 E_1}{2} (w_x)^2 \right] (L) = 0, \\
\left[ \begin{array}{l}
-B_4 p_x - \gamma B_2 p_{xx} + 2B_2 \frac{E_2}{2} (w_x)^2 \end{array} \right] (L) = -\frac{\nu V(t)}{h_3 B_4}, \\
\left[ Aw_{xx} - B_1 G_2 \gamma h_2 h_3 \phi^2 + \gamma \beta B_3 p_{xx} + \frac{1}{2} (B_1 E_1 w_x w_{xx} + E_1 \phi^2_x w_x + \gamma \beta E_2 p_x w_x) + E_3 \left( w_x \right)^3 \right] (L) = 0 \\
(w, p, \dot{w}, \dot{p})(x, 0) = (w_0, p_0, w_1, p_1).
\end{array} \right.
\end{align*}
$$

### 3.4 Electrostatic (M-M) beam model

Notice that if the magnetic effects in (27) are neglected, i.e. $\mu \ddot{p} \equiv 0$, the last equation can be solved for $p_{xx}$.

Then we obtain the following model with simplified boundary conditions

$$
\begin{align*}
\rho \ddot{w} + \hat{A} w_{xxxx} - G_2 \gamma \beta h_2 h_3 \hat{B} \phi^2 + (N_1 w_x w_{xx} + N_2 w_x \phi^2_x + N_3 (w_x)^3 + \frac{\gamma E_2}{\nu C B_4} V(t) H(w_x) w_x = 0, \\
G_2 C \phi^2 - \phi^2_{xx} + \hat{B} w_{xxx} + N_4 w_x w_{xx} = 0,
\end{align*}
$$

$$
\begin{align*}
\left[ \begin{array}{l}
w, w_x, \phi^2 \end{array} \right] (0) = 0, \\
\left[ \begin{array}{l}
-\phi^2_x + \hat{B} w_{xx} + \frac{1}{2} N_4 (w_x)^2 \end{array} \right] (L) = -\frac{B_2 V(t)}{h_3 C B_4}, \\
\hat{A} w_{xx} + \frac{1}{2} \left( N_1 + \frac{\gamma \beta B_3 E_2}{C B_4} \right) \left( w_x \right)^2 (L) = -\frac{2B_3 V(t)}{h_3 C B_4}, \\
\hat{A} w_{xx} - G_2 \gamma \beta h_2 h_3 \hat{B} \phi^2 + N_1 w_x w_{xx} + N_2 w_x \phi^2_x + N_3 (w_x)^3 + \frac{\gamma E_2}{h_3 C B_4} V(t) w_x \right) (L) = 0, \\
(w, \dot{w})(x, 0) = (w_0, w_1)
\end{align*}
$$
where the coefficients are functions of material parameters

\[
\begin{align*}
\tilde{A} &= A - \gamma \beta B^2_0, & \tilde{B} &= B_1 - \gamma \beta B_0 B_2, & \tilde{B}_3 &= B_3 + \gamma \beta h_3 B_0 B_2, & \tilde{B}_4 &= B_4 + \gamma \beta h_3 B^2_0, & \tilde{B}_5 &= B_5 - \beta h_3 B_0 E_1, \\
\tilde{C} &= \frac{C B_4}{B_3}, & N_1 &= \frac{B_1 E_1}{C} + \gamma \beta B_3 \left( \frac{B_3}{B_4} - \frac{E_2}{C B_4} \right), & N_2 &= \frac{E_2}{C} + \gamma ^2 \beta h_3 B^2_0 E_2, & N_3 &= E_3 + \frac{\gamma ^2 \beta E_2 B_5}{6 C B_4}, \\
N_4 &= E_1 + \gamma ^2 B_6 B_3.
\end{align*}
\]

Unlike the electrostatic Rao-Nakta model (21), the voltage control simultaneously controls the shear of the middle layer and the bending motion of the composite. This model is similar to the model obtained in (9). Remark 3.1. for the linearized (M-M) beam equations (27) with PID controllers.

Remark 3.2. The observation \( B^* y \) corresponds to the total induced current accumulated at the electrodes for the voltage input.\(^{35}\) This is completely electromagnetic. Remark 3.3. The semi-discretized Finite Difference technique is performing well for many de-coupled equations, see \(^{15}\) multigrid techniques for high frequencies in finite differences,\(^{1,2,29}\) or mixed finite element methods.\(^{3,6,29}\) Among these, the semi-discretized Finite Difference technique with the introduction of an artificial (numerical) viscosity term is applicable for many de-coupled equations, see \(^{49}\) The main idea to suppress the spurious high frequency oscillations is either (i) to control the projected high-frequency solutions on corresponding subspace so these spurious solutions are filtered, or (ii) we add an extra boundary control. In fact, in a recent paper,\(^{36}\) the filtered Finite Difference Method has powerful results for the linearized (M-M) beam equations (27) with PID controllers.

4. PRELIMINARY STABILIZATION RESULTS FOR THE SEMI-DISCRETE APPROXIMATIONS OF A SINGLE PIEZOELECTRIC BEAM

For the numerical analysis of the piezoelectric beam models, the Galerkin-based Finite Element Method (FEM) is a very standard way to see the effectiveness of the control design on the first several vibrational modes of the composite. However, our models are highly coupled and require more careful treatment for the spurious high frequency modes which may cause the so-called spillover effect. There are recent attempts to mathematically address these issues such as, filtering techniques for the high frequencies in finite differences,\(^{1,2,29}\) multigrid techniques\(^{33}\) or mixed finite element methods.\(^{3,6,29}\) Among these, the semi-discretized Finite Difference technique with the introduction of an artificial (numerical) viscosity term is performing well for many de-coupled equations, see \(^{49}\)
The aim of this section is to present some simple numerical experiments in order to show that $\mathcal{B}^*$-type stabilizing feedback controller can be designed in a similar fashion as for the linearized models. The fully dynamic models are currently left out to save space. We only provide preliminary stabilization results for the electrostatic (E-B) and (M-T) single beam cases. Furthermore, we skip the convergency and consistency analyses. These will be comprehensively discussed in.\textsuperscript{40} The total energy of a single beam for (7) and (8) is

$$E(t) = \frac{h}{2} \left\{ \int_0^L \rho \left( \dot{v}^2 + \frac{h^2}{12} \ddot{w}_x^2 + \ddot{w}^2 \right) + \left( \alpha_1 (v_x + \frac{1}{2} \ddot{w}_x)^2 + \frac{\alpha_3}{12} h^2 \ddot{w}_x^2 \right) \right\} dx, \quad \text{(E-B)}$$

$$\left\{ \int_0^L \rho \left( \ddot{v}^2 + \frac{h^2}{12} \ddot{w}_x^2 + \ddot{w}^2 \right) + \left( \alpha_1 (v_x + \frac{1}{2} \ddot{w}_x)^2 + \frac{\alpha_3}{12} h^2 \ddot{w}_x^2 \right) \right\} dx, \quad \text{(M-T)}$$

For $c_i > 0$, we design a $\mathcal{B}^*$-type stabilizing control law as in Table 4. Notice that, $\mathcal{B}^*$-feedback has a nonlinear integral controller $\int_0^L w_x \dot{w}_x \, dx$, which is the contribution of the nonlinearity. The main advantage of an integral controller is that it provides high-gain feedback at low frequencies, and therefore, integral controllers can overcome creep and hysteresis effects and lead to precision positioning (since the vibrational dynamics is not dominant at low frequencies).\textsuperscript{14} With this choice, the energy for each model is dissipative

$$\frac{dE(t)}{dt} = \begin{cases} -c_1 \dot{v}(L,t) + \int_0^L w_x \dot{w}_x \, dx \left( 2 - c_2 |w_x(L,t)|^2 - c_3 |\dot{w}(L,t)|^2 \right) & \text{(E-B)} \\ -c_1 \dot{v}(L,t) + \int_0^L w_x \dot{w}_x \, dx \left( 2 - c_2 |\dot{w}(L,t)|^2 - c_3 |\dot{w}(L,t)|^2 \right) & \text{(M-T)} \end{cases} \leq 0.$$

Now consider the discretization of the interval [0, L] with the fictitious points $x_{-1}$ and $x_{N+1}$ as the following

$x_{-1} < 0 = x_0 < x_1 < x_2, \ldots < x_N = L < x_{N+1}$, \quad $x_i = i \cdot dx, \quad i = -1, 0, 1, \ldots, N, N + 1$, \quad $dx = \frac{L}{N+1}$.

The following are the second order finite difference approximations for different order derivatives:

$$z_x = \frac{z(x_{i+1}) - z(x_{i-1})}{2dx} + O(dx^2), \quad \text{or} \quad \frac{3z(x_i) - 4z(x_{i-1}) + z(x_{i-2})}{2dx} + O(dx^2), \quad z_{xx} = \frac{z(x_{i+2}) - 2z(x_{i+1}) + 2z(x_{i-1}) - z(x_{i-2})}{dx^2} + O(dx^2).$$

(33)

Henceforth, to simplify the notation, we use $z(x_i) = z_i$. We also non-dimensionalize the time variable $t = A_1 t^*$ with $t^* \to t$.

We consider a sample piezoelectric beam with height $L = 1$m, and thickness $h = 0.01$m. The material constants are $\rho = 7600$ kg/m$^3$, $\alpha_1 = 1.4 \times 10^5$ N/m$^2$, $\alpha_3 = 4.5 \times 10^5$ N/m$^2$, $\gamma = 10^{-3}$ C/m$^2$, $\beta = 10^6$ m/F, $G_2 = 100$ GN/m$^2$. We consider $N = 60$ with the initial data $w(x,0) = v(x,0) = \dot{v}(x,0) = 10^{-3} e^2 \epsilon_0^{0.5}$ and $\psi(x,0) = \psi(x,0) = \dot{\psi}(x,0) = 0$. The simulations are computed for (non-dimensionalized) the final time $T_{\text{final}} = 300$.

The filtering technique is successfully applied in\textsuperscript{16,49} so that high frequency solutions, causing artificial instability in the approximated solution, are filtered by adding a viscosity term $(dx)^2 u_{xxt}$ to the wave equation and $w_{xxt}$ to the fourth order beam equation. Both terms vanish uniformly as $dx \to 0$. We adopt the same idea to derive a filtered semi-discrete scheme in finite differences to simulate the effects of the stabilizing controller. For this purpose, additional viscosity terms (boxed terms in the following) $(dx)^2 v_{xxt}, \frac{1}{2} w_{xxt}, (dx)^2 \psi_{xxt}$ to the (E-B)
equations and \((dx)^2 v_{xx}, (dx)^2 w_{xx}, (dx)^2 \psi_{xx}\) to the \((M-T)\) equations are added with appropriate coefficients:

\[
\begin{align*}
(E-B) \quad 
\begin{cases}
\dot{v}_i = \frac{\dot{v}_{i+1} - 2\dot{v}_i + \dot{v}_{i-1}}{dx^2} - \frac{v_{i+1} - 2v_i + v_{i-1}}{dx^2} + \frac{1}{L} \left[ w_{i+1} - 2w_i + w_{i-1} \right] \\
\ddot{w} = -\frac{h^2}{12L^2} w_{i+1} - 2w_i + w_{i-1} + \frac{h^2}{12} \left[ w_{i+1} - 2w_i + w_{i-1} \right]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(M-T) \quad 
\begin{cases}
\dot{v}_i = \frac{\dot{v}_{i+1} - 2\dot{v}_i + \dot{v}_{i-1}}{dx^2} - \frac{v_{i+1} - 2v_i + v_{i-1}}{dx^2} + \frac{1}{L} \left[ w_{i+1} - 2w_i + w_{i-1} \right] \\
\ddot{w} = -\frac{h^2}{12L^2} w_{i+1} - 2w_i + w_{i-1} + \frac{h^2}{12} \left[ w_{i+1} - 2w_i + w_{i-1} \right]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\dot{v}_i = \frac{\dot{v}_{i+1} - 2\dot{v}_i + \dot{v}_{i-1}}{dx^2} - \frac{v_{i+1} - 2v_i + v_{i-1}}{dx^2} + \frac{1}{L} \left[ w_{i+1} - 2w_i + w_{i-1} \right] \\
\ddot{w} = -\frac{h^2}{12L^2} w_{i+1} - 2w_i + w_{i-1} + \frac{h^2}{12} \left[ w_{i+1} - 2w_i + w_{i-1} \right]
\end{cases}
\end{align*}
\]

where the approximated controllers are designed

\[
\begin{align*}
(E-B) \quad 
\begin{cases}
V(t) = c_1 \left( \dot{v}_N + \frac{dx}{3} \frac{d}{dt} \left( \frac{3w_{N-1} - 4w_{N-1} + w_{N-1}}{2dx} \right)^2 + 4 \sum_{i=1}^{N/2-1} \left( w_{2i+1} - w_{2i-1} \right)^2 \right), \\
m(t) = -c_2 \frac{dx}{dt} \frac{3w_{N-1} - 4w_{N-1} + w_{N-1}}{2dx}, \quad g(t) = c_3 \dot{w}_N
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(M-T) \quad 
\begin{cases}
V(t) = c_4 \left( \dot{v}_N + \frac{dx}{3} \frac{d}{dt} \left( \frac{3w_{N-1} - 4w_{N-1} + w_{N-1}}{2dx} \right)^2 + 4 \sum_{i=1}^{N/2-1} \left( w_{2i+1} - w_{2i-1} \right)^2 \right), \\
m(t) = -c_5 \dot{v}_N, \quad g(t) = -c_6 \dot{w}_N
\end{cases}
\end{align*}
\]

where the feedback gains \(c_1, \ldots, c_6 > 0\) are chosen appropriately in the numerical code. The Differential Algebraic system of equations are solved by the Mathematica’s NDSolve command with AccuracyGoal and PrecisionGoal options to be set to MachinePrecision/2. The simulations are run for three major cases:

- Fully controlled: \(V(t), m(t), g(t) \neq 0\)
• Partially controlled: $V(t) \equiv 0$ and $m(t), g(t) \neq 0$.

• No control: $V(t), m(t), g(t) \equiv 0$.

In the fully controlled case, the feedback controller corresponding to the voltage control $V(t)$ in (36) is very powerful to exponentially stabilize the stretching solutions $v(x, t)$ for both models, see Tables 5 and 6. In the partially controlled case for both (E-B) and (M-T) models, two controllers $m(t)$ and $g(t)$ are able to decay the bending and shear to zero polynomially. Since the voltage controller is freed in this case, the stretching solutions in both (E-B) and (M-T) cases are not controlled at all. All results are also compared to the uncontrolled models in Tables 5 and 6.

The tip velocities $\dot{v}_N, \dot{w}_N$ for the (E-B) model and $\dot{v}_N, \dot{w}_N, \dot{\psi}_N$ for the (M-T) model are simulated in Tables 8 and 9, respectively. It can be observed that the $\dot{v}_N$ decay to zero exponentially for both models. The polynomial decay pattern for the $\dot{v}_N$ and $\dot{w}_N$ is still observed. The energy of the fully controlled models, shown in Table 7, decay to zero in a few seconds. Unlike the fully elastic models, it is our observation that the voltage controller $V(t)$, controlling stretching equations, dominates the other controls in the approximated schemes (34) and (35). This implies that the boundary control design for the approximated solutions (34) and (35) should be improved by maybe adding additional viscosity terms (boundary type or distributed type). Adaptive controllers may also be considered to improve the decay rate of bending and shear solutions. Analytical work to find out the decay rates of the nonlinear models is currently under investigation.

5. DISCUSSION AND CONCLUSION

A large class of nonlinear piezoelectric smart beam models are obtained in the framework of a consistent variational approach. All models derived in this manuscript fit in the category of quasi-linear hyperbolic systems in, and therefore, the well-posedness of these models can be shown in appropriate Hilbert spaces. Since the spectral analysis for these models is very tedious, the Finite Difference Method is very suitable for stable approximations. Comparing our results obtained by the filtered Finite Differences with the ones obtained by the mixed-Finite Element method is essential. This together with the extended numerical analysis with fourth order approximations is currently under investigation.

It is also an ongoing project to find out the decay rate of the global stabilization in discrete/continuous of electrostatic and fully dynamic models with $B^*$-type feedback controllers.

Another ongoing work is the implementation of the Lyapunov’s direct method to design non-trivial boundary controllers for the electrostatic and fully dynamic models. The designed controller’s ability to stabilize the composite at its equilibrium position is proven analytically.

For energy harvesting applications, three-layer models obtained in Section 3 are attached to a circuit equation, see Remark 3.3. It is a future work to carry out a detailed spectral analysis for the corresponding linearized models since it may be helpful for deducing stability characteristics of the nonlinear models rigorously. This analysis is similar to the one carried out for a unimorph.

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Stretching - $v(x, t)$  
Bending - $w(x, t)$  
Rotation angle $\psi_N(x, t)$

Table 6: Graphs in each column show simulations for the approximated solutions $\{v, w, \psi\}$ of the (M-T) closed-loop system for the uncontrolled, controlled, and partially controlled ($V(t) = 0$) cases, respectively. The $t$–axis is real-time.

Table 7: Total approximated energy of the (E-B) and (M-T) controlled, partially controlled, uncontrolled models, respectively. The $t$–axis is non-dimensional with the non-dimensionality constant $A_1 = 0.023$. As $dx \to 0$ (larger $N$), the actual decay can be observed better.

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Table 9: Graphs in each column show simulations for the tip velocities \( \dot{v}_N, \dot{\psi}_N, \dot{w}_N \) of the (M-T) closed-loop system for the controlled, partially controlled, and uncontrolled models. The \( t \)–axis is non-dimensional with the non-dimensionality constant \( A_1 = 0.023 \).

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