CONTINUED FRACTIONS WITH $SL(2, \mathbb{Z})$-BRANCHES:
COMBINATORICS AND ENTROPY

CARLO CARMINATI, STEFANO ISOLA, GIULIO TIOZZO

Abstract. We study the dynamics of a family $K_\alpha$ of discontinuous interval maps whose (infinitely many) branches are Möbius transformations in $SL(2, \mathbb{Z})$, and which arise as the critical-line case of the family of $(a, b)$-continued fractions.

We provide an explicit construction of the bifurcation locus $E_{KU}$ for this family, showing it is parametrized by Farey words and it has Hausdorff dimension zero. As a consequence, we prove that the metric entropy of $K_\alpha$ is analytic outside the bifurcation set but not differentiable at points of $E_{KU}$, and that the entropy is monotone as a function of the parameter.

Finally, we prove that the bifurcation set is combinatorially isomorphic to the main cardioid in the Mandelbrot set, providing one more entry to the dictionary developed by the authors between continued fractions and complex dynamics.

1. Introduction

It is well-known that the usual continued fraction algorithm is encoded by the dynamics of the Gauss map $G(x) := \frac{1}{x} - [\frac{1}{x}]$; moreover, the Gauss map is known to be related, via a Poincaré section, to the geodesic flow on the modular surface $\mathbb{H}/SL(2, \mathbb{Z})$. In greater generality, the modular group $SL(2, \mathbb{Z})$ is generated by the transformations $Sx := -\frac{1}{x}$ and $Tx := x + 1$, and several different continued fraction algorithms have been constructed by applying the generators according to different rules.

In particular, for each $\alpha$ we can construct an interval map $K_\alpha$ by fixing a “fundamental interval” $[\alpha - 1, \alpha)$, and at each step applying the inversion $S$ followed by as many translations $T$ as are needed to come back to the fundamental domain. Thus, for each $\alpha \in (0, 1)$, we have the interval map $K_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ defined by $K_\alpha(0) = 0$ and

$$K_\alpha(x) = -\frac{1}{x} - c_\alpha(x)$$

where $c_\alpha(x) \in \mathbb{Z}$ is chosen so that the result lies in $[\alpha - 1, \alpha)$. Each $K_\alpha$ determines a continued fraction expansion of type

$$x = -\frac{1}{c_1 - \frac{1}{c_2 - \frac{1}{c_3 - \cdots}}}$$

with coefficients $c_\alpha := c_\alpha(K_\alpha^{-1}(x))$. Similarly to the Gauss map, each $K_\alpha$ has infinitely many expanding branches and a unique absolutely continuous invariant measure $\mu_\alpha$.

In recent years, S. Katok and I. Ugurcovici, following a suggestion of D. Zagier, defined the two-dimensional family $f_{a, b}$ of $(a, b)$-continued fraction transformations...
and studied their dynamics and natural extensions \cite{23,24}. The maps $K_\alpha$ are the first return maps of $f_{\alpha-1,\alpha}$ on the interval $[\alpha - 1, \alpha)$ and, as it will be explained, they capture all the essential dynamical features.

The family $K_\alpha$ interpolates between other well-known continued fraction algorithms: in particular, for $\alpha = 1/2$ one gets the continued fraction to the nearest integer going back to Hurwitz \cite{21}, while for $\alpha = 1$ one gets the backward continued fraction which is related to the reduction theory of quadratic forms \cite{35,22}.

The definition of $K_\alpha$ is very similar to the definition of the $\alpha$-continued fraction transformations $T_\alpha$ introduced by Nakada \cite{33} and subsequently studied by several authors \cite{30,26,34,10,6,27,12,1}. In this paper we shall use techniques similar to the ones in \cite{10} to study the $K_\alpha$: as we shall see in greater detail, this will also highlight the substantial differences in the combinatorial structures of the respective bifurcation sets. In particular, we shall see that the bifurcation set is canonically isomorphic to the set of external rays landing on the main cardioid of the Mandelbrot set (while the bifurcation set for the $\alpha$-continued fractions was shown to be isomorphic to the real slice of the Mandelbrot set \cite{6}).

From a dynamical systems perspective, we shall be interested in studying the variation of the dynamics of $K_\alpha$ as a function of the parameter. As we shall see, there exist infinitely many islands of “stability”, and each of them corresponds to a Farey word (see section 2). Namely, to each Farey word $w$ we shall associate an open interval $J_w \subseteq [0,1]$ called quadratic maximal interval, or qinterval for short (see section 3.2); the bifurcation set $E_{KU}$ is defined as the complement of all such intervals:

$$E_{KU} := [0,1] \setminus \bigcup_{w \in FW} J_w.$$  

The set $E_{KU}$ is homeomorphic to a Cantor set and has Hausdorff dimension zero (Proposition 3.4). We shall prove that on each $J_w$ we have the following matching between the orbits of $\alpha$ and $\alpha - 1$; namely, there exist integers $m_0$ and $m_1$ (which depend only on $J_w$) such that

$$K_\alpha^{m_0+1}(\alpha - 1) = K_\alpha^{m_1+1}(\alpha)$$

for all $\alpha \in J_w$.

One way to study the bifurcations of the family $K_\alpha$ is by considering its entropy, in the spirit of \cite{30}. Indeed, let us define $h(\alpha)$ to be the metric entropy of the map $K_\alpha$ with respect to the measure $\mu_\alpha$. We shall prove that the set $E_{KU}$ is precisely the set of parameters for which the entropy function is not smooth:

**Theorem 1.1.** The function $\alpha \mapsto h(\alpha)$

1. is analytic on $[0,1] \setminus E_{KU}$;
2. is not differentiable (and not locally Lipschitz) at any $\alpha \in E_{KU}$.

Thus, as the parameter $\alpha$ varies, the dynamics of $K_\alpha$ goes through infinitely many stable regimes, one for each connected component of the complement of $E_{KU}$. We shall prove, however, that the entropy function is globally monotone across the bifurcations. In order to state the theorem, let us note that the graph of the entropy function is symmetric with respect to the transformation $\alpha \mapsto 1 - \alpha$, because $K_\alpha$ and $K_{1-\alpha}$ are measurably conjugate (see equation 26). Moreover, it is not hard by an explicit computation to see that the entropy is constant (equal to $\frac{\pi^2}{6 \log(1+g)}$) on the interval $[g^2, g]$, where $g := \sqrt{\frac{\alpha-1}{2}}$ is the golden mean (so $g^2 = 1 - g = \frac{3 - \sqrt{5}}{2}$).


The main theorem is the following monotonicity result for $h$:

**Theorem 1.2.** The function $\alpha \mapsto h(\alpha)$ is strictly monotone increasing on $[0, g^2]$, constant on $[g^2, g]$ and strictly monotone decreasing on $[g, 1]$.

Note that Theorem 1.2 highlights a major difference with the $\alpha$-continued fraction case, where the entropy is not monotone in any neighbourhood of $\alpha = 0$, and actually the set of parameters where the entropy is locally non-monotone has Hausdorff dimension 1 

1.1. **Connection with the main cardioid in the Mandelbrot set.** The fact that each connected component of the complement of $\mathcal{E}_{KU}$ is naturally labelled by a Farey word can be used to draw an unexpected connection between the combinatorial structure of $\mathcal{E}_{KU}$ and the Mandelbrot set.

Recall the **main cardioid** of the Mandelbrot set is the set of parameters $c \in \mathbb{C}$ for which the map $f_c(z) := z^2 + c$ has an attractive or indifferent fixed point. The exterior of the Mandelbrot set admits a canonical uniformization map, and to each
angle $\theta \in \mathbb{R}/\mathbb{Z}$ there corresponds an associated external ray $R(\theta)$. Let us denote $\Omega$ to be the set of angles $\theta$ for which the ray $R(\theta)$ lands on the main cardioid.

Recall Minkowski’s question mark function $Q : [0, 1] \to [0, 1]$ is a homeomorphism of the interval which is defined by converting the continued fraction expansion of a number into a binary expansion. More precisely, if $x = [0; a_1, a_2, a_3, \ldots]$ is the usual continued fraction expansion of $x$, then we define

$$Q(x) := 0.0\ldots0 \underbrace{1}{a_1-1} 1 \ldots 1 \underbrace{0\ldots0}{a_2} \ldots \underbrace{0\ldots0}{a_3} \ldots$$

We shall prove that Minkowski’s function induces the following correspondence.

**Theorem 1.3.** Minkowski’s question mark function $Q(x)$ maps homeomorphically the bifurcation set $\mathcal{E}_{KU}$ onto the set $\Omega$ of external angles of rays landing on the main cardioid of the Mandelbrot set. In formulas, we have

$$Q(\mathcal{E}_{KU}) = \Omega.$$

The connection may seem incidental, but it is an instance of a more general correspondence discovered by the authors in recent years. Indeed, the Minkowski map provides an explicit dictionary between sets of numbers defined using continued fractions and sets of external angles for certain fractals arising in complex dynamics. More precisely, the question mark function:

1. maps homeomorphically the bifurcation set for $\alpha$-continued fractions onto the set of external rays landing on the real slice of the boundary of the Mandelbrot set (see [6] and [37], Theorem 1.1);
2. maps the sets of numbers of generalized bounded type defined in [11] to the sets of external rays landing on the real slice of the boundary of Julia sets for real quadratic polynomials ([37], Theorem 1.4);
3. conjugates the tuning operators defined by Douady and Hubbard for the real quadratic family to tuning operators corresponding to renormalization schemes for the $\alpha$-continued fractions [12].

For an introduction and more details about such correspondence we refer to one of the authors’ thesis [37]. The dictionary proves to be especially useful to derive results about families of continued fractions using the large body of information known about the combinatorics of the quadratic family; moreover, it can also be used to obtain new results about the quadratic family and the Mandelbrot set from the combinatorics of continued fractions (e.g. [37], Theorem 1.6).

Being intimately connected to the structure of $Q$, Farey words play a distinguished role in several other dynamical, combinatorial or algebraic problems. To list just a few, we mention: kneading sequences for Lorentz maps [20, 28], the coding of cutting sequences on the flat torus [15], as well as on the hyperbolic one-punctured torus ([23], pg. 726-727); the Markov spectrum (in particular the Cohn tree, see [5], pg. 201); primitive elements in rank two free groups [16]; the Burrows-Wheeler transform [31]; digital convexity [7]. For more information we also refer to the survey [2] or the books [15] and [3].

1.2. **Behaviour of $(a, b)$-continued fractions on the critical line.** We conclude the introduction by explaining in more detail the results of [23, 24] and how they relate to the present paper. For further details, see also section 6.
In [23], Katok and Ugarcovici consider the two parameter family of continued fraction algorithms induced by the maps

\[ f_{a,b}(x) := \begin{cases} 
  x + 1 & \text{if } x < a \\
  -1/x & \text{if } a \leq x < b \\
  x - 1 & \text{if } b \leq x 
\end{cases} \]

where the parameters \((a, b)\) range in a closed subset \(P\) of the plane. The segment

\[ C := \{(a, b) : b - a = 1, \ b \in [0, 1]\} \]

is a piece of the boundary of \(P\), and the first return map of \(f_{b-1,b}\) on the interval \([b-1, b)\) coincides with the map \(K_b\) we are going to study (these maps are also mentioned in [23] under the name “Gauss-like maps” and denoted \(\hat{f}_{b-1,b}\)).

Katok and Ugarcovici also consider a closely related family \((F_{a,b})(a,b)\in P\) of maps of the plane: each \(F_{a,b}\) has an attractor \(D_{a,b} \subset \mathbb{R}^2\) such that \(F_{a,b}\) restricted to \(D_{a,b}\) is invertible and it is a geometric realization of the natural extension of \(f_{a,b}\). They also show that for most parameters in \(P\) the attractor \(D_{a,b}\) has finite rectangular structure, meaning that it is a finite union of rectangles. Moreover, all exceptions to this property belong to a Cantor set \(\tilde{E}\) which is contained in the critical line \(C\) and whose 1-dimensional Lebesgue measure is zero.

It turns out that the set \(E_{KU}\) we are considering is just the projection of the set \(\tilde{E}\) onto the second coordinate, up to a countable set. Making explicit the structure of \(E_{KU}\) allows us to prove that it is not just a zero measure set, but it also has zero Hausdorff dimension.

**Structure of the paper.** In section 2 we shall start defining Farey words and establishing the properties which are needed to describe the combinatorial dynamics of the \((a, b)\)-continued fractions. Moreover, we define the binary bifurcation set \(E_B\) and show how it is related to the main cardioid of the Mandelbrot set.

In section 3 we then recall basic facts about continued fractions and then define the runlength map \(RL\) which passes from binary expansions to continued fraction expansions; then we shall see how the properties of Farey words translate into the properties of the qunervals which we shall define. Then, we shall apply all these properties to the case of \((a, b)\)-continued fractions; in section 4 we determine the combinatorial dynamics of the orbits of \(\alpha\) and \(\alpha - 1\), thus proving that the matching condition holds on each qunerval.

Then, in section 5 we shall draw consequences for the entropy function, proving Theorem 1.2: indeed, we prove that the Cantor set \(E_{KU}\) has Hausdorff dimension zero (Proposition 3.4) and \(h\) is Hölder continuous, so it can be extended to a monotone function across the Cantor set. Finally, in section 6 we shall combine the previous properties with the construction of the attractors given in [23] to prove Theorem 1.1. For the sake of readability, the proofs of some technical lemmas will be postponed to the appendix.

**Acknowledgements.** We wish to thank H. Bruin, P. Majer, C.G. Moreira and I. Ugarcovici for useful conversations. The pictures of the Mandelbrot and Julia sets are drawn with the software *mandel* of W. Jung.

2. Farey words and dynamics

We shall start by constructing the set of Farey words and establishing the properties which are needed in the rest of the paper. Many of these results appear in
various sources, for instance in the books [2, 3, 15, 29]. For the convenience of the reader, and in order to set up the notation for the rest of the paper, we shall give a fairly self-contained treatment.

2.1. **Alphabets and orderings.** An alphabet \( \mathcal{A} \) will be a finite set of symbols, which we shall call digits. Given an alphabet \( \mathcal{A} \), we shall denote as \( \mathcal{A}^n \) the set of words of length \( n \), as \( \mathcal{A}^\mathbb{N} \) the set of infinite words, and as \( \mathcal{A}^* := \bigcup_{n \geq 0} \mathcal{A}^n \) the set of **finite words** of arbitrary length. If \( w \) is a finite word, then the symbol \( \overline{w} \) will denote the infinite word given by infinite repetition of the word \( w \).

If \( w = (\epsilon_1, ..., \epsilon_\ell) \in \mathcal{A}^* \), we shall denote as \( |w| \) the length of the word \( w \), i.e. the number of digits; moreover, if we fix a digit \( a \in \mathcal{A} \), the symbol \( |w|_a \) will denote the number of digits in the word which are equal to \( a \). Moreover, given a word \( w = (\epsilon_1, ..., \epsilon_\ell) \in \mathcal{A}^* \), we define its transpose to be the word \( ^tw \) with

\[
^tw := (\epsilon_\ell, ..., \epsilon_1);
\]
a word which is equal to its transpose is called palindrome. Moreover, we define the cyclic permutation operator \( \tau \) to act on the word \( w = (\epsilon_1, ..., \epsilon_\ell) \) as

\[
\tau w := (\epsilon_2, ..., \epsilon_\ell, \epsilon_1).
\]

A total order \( < \) on the alphabet \( \mathcal{A} \) induces, for each \( n \), a total order on the set \( \mathcal{A}^n \) of words of length \( n \) by using the lexicographic order, and similarly it induces a total order on the set \( \mathcal{A}^\mathbb{N} \) of infinite words. We shall extend this order to a (partial) order on the set \( \mathcal{A}^* \) of finite words by defining that

\[
u < w \quad \text{if} \quad wu < vw.
\]

Note that it is not difficult to check that, if \( u, v \in \mathcal{A}^* \), then the inequality \( u < v \) is equivalent to \( \bar{u} < \bar{v} \) (this fact also proves that \( < \) is an order relation).

Finally, we shall also define the stronger partial order relation \( \ll \) on the set \( \mathcal{A}^* \) of finite strings by saying that

\[
u \ll w
\]
if there exist a prefix \( u_1 \) of \( u \) and a prefix \( v_1 \) of \( v \) with \( |u_1| = |v_1| \) and such that \( u_1 < v_1 \). Note that \( u \ll v \) implies \( u < v \), and moreover that any infinite word beginning with \( u \) is smaller than any infinite word beginning with \( v \).

In the following we will mainly be interested in the binary alphabet \( \mathcal{A} := \{0, 1\} \), with the natural order \( 0 < 1 \). For \( \epsilon \in \{0, 1\} \), we also define the negation operator \( \bar{\epsilon} := 1 - \epsilon \), which can be extended digit-wise to binary words: if \( w = (\epsilon_1, ..., \epsilon_\ell) \in \{0, 1\}^* \), we define \( \bar{w} := (\bar{\epsilon}_1, ..., \bar{\epsilon}_\ell) \).

Every infinite word \( w = (\epsilon_1, \epsilon_2, ...) \in \{0, 1\}^\mathbb{N} \) also corresponds to the unique real value in \([0, 1] \) which has \( w \) as its binary expansion, which will be denoted by \( .w := \sum_{k=1}^{\infty} \epsilon_k 2^{-k} \); the same is true for finite binary words in \( \{0, 1\}^* \), which correspond to dyadic rationals.

2.2. **Farey words.** We are now ready to define one of the main ingredients of the paper, namely the set of Farey words. As we shall see, several equivalent definitions can be given; we shall start with a recursive definition.

For each integer \( n \geq 0 \), we shall construct a list \( F_n \) of finite words in the alphabet \( \{0, 1\} \), called Farey list of level \( n \). Let us start with \( F_0 := (0, 1) \) the list consisting of the two one-digit words. For each \( n \), the next list \( F_{n+1} \) is obtained by inserting between two consecutive words \( v, w \) in the list \( F_n \) the concatenation \( vw \).
In formulas, if $F_n = (w_1, \ldots, w_k)$ with each $w_i$ a finite word, then the next list is $F_{n+1} = (v_1, \ldots, v_{2k-1})$ with
\[ v_{2i-1} := w_i \quad \text{for } 1 \leq i \leq k \]
\[ v_{2i} := w_i w_{i+1} \quad \text{for } 1 \leq i \leq k - 1. \]

**Definition 2.1.** The set of Farey words $FW$ is the union of all Farey lists:
\[ FW := \bigcup_{n \geq 0} F_n. \]

As an example, the first few Farey lists are
- $F_0 = (0, 1)$
- $F_1 = (0, 01, 1)$
- $F_2 = (0, 001, 01, 011, 1)$
- $F_3 = (0, 0001, 001, 00101, 01, 01011, 011, 0111, 1)$

and all their elements are Farey words. Note that each $F_n$ contains $2^n + 1$ elements. A Farey word will be called non-degenerate if it has more than one digit; we shall denote the set of non-degenerate Farey words as $FW^\ast = FW \setminus \{0, 1\}$.

These words are also sometimes called Christoffel words, as in the book [3], or standard words as in [31].

**Lemma 2.2.** Each Farey list $F_n$ is strictly increasing.

**Proof.** Since $0 < 1$, then the list $F_0$ is ordered increasingly. By induction, we just need to prove that given two words $u < v$, then we have $u < uv < v$.

Indeed, by definition $u < v$ means that
\[ uv < vu. \]

By prefixing $u$ on both sides of the inequality, we get $uv < uuv$, that we can interpret as $u( uv ) < (vu)u$, hence by definition of the order we have $u < uv$. Similarly, by adding $v$ on both sides at the end of (3) we get $uvv < vuv$, which by definition implies $uv < v$.

Note moreover that each Farey word is naturally equipped with a standard factorization; indeed, if $w$ is a Farey word, let $n$ be the smallest integer for which $w$ belongs to $F_n$; by definition, the word $w$ is generated in the iterative construction as a concatenation $w = w_1 w_2$ where $w_1$ and $w_2$ belong to the level $F_{n-1}$. Thus, the decomposition $w = w_1 w_2$ will be called the standard factorization of $w$. In the appendix we shall prove the following characterization:

**Proposition 2.3.** Given $w \in FW$, let us consider a decomposition $w = w' w''$ where $w', w''$ are non-empty words; then the following conditions are equivalent:

1. $w'$ and $w''$ are Farey words;
2. $w = w' w''$ is the standard factorization of $w$.

We shall now construct a natural correspondence between the set of Farey words and the set of rational numbers between 0 and 1. Given a word $w \in \{0, 1\}^*$, let us
Proposition 2.4. The map

\[ \rho(w) := \frac{|w|_1}{|w|} \]

Clearly, \(0 \leq \rho(w) \leq 1\). Moreover, we have the following correspondence:

**Proposition 2.4.** The map \( \rho : FW \to \mathbb{Q} \cap [0,1] \) is a bijection between the set of Farey words and the set of rational numbers between 0 and 1.

In the rest of this section we shall prove the Proposition 2.4, and meanwhile establish more properties of Farey words. In particular, we shall see how to construct an inverse of \( \rho \), i.e. to produce a Farey word given a rational number.

Let \( r := \frac{p}{q} \in [0,1] \) be a rational number, with \((p,q) = 1\), and consider the one-dimensional torus \( \mathbb{R}/\mathbb{Z} \), with the marked point \( x_0 = 0 \). Let \( C_q := \{ x \in \mathbb{R}/\mathbb{Z} : qx \not\equiv 0 \mod 1 \} \). For each \( x \in C_q \), we shall define the binary word \( \Phi_r(x) \in \{0,1\}^q \) by using the dynamics of the circle rotation

\[ R_r(x) := x + r \mod 1. \]

The word \( \Phi_r(x) \) will be constructed as follows: starting at \( x \), we successively apply the rotation \( R = R_r \) and each time we write down 1 if we cross the \( x_0 \) mark, and 0 otherwise. More precisely, we define \( \Phi_r(x) := (\epsilon_1, \ldots, \epsilon_q) \), where, for each \( k \) between 1 and \( q \), the \( k^{th} \) digit \( \epsilon_k \) is given by

\[ \epsilon_k := \begin{cases} 
0 & \text{if } x_0 \notin (R^{k-1}(x), R^k(x)] \\
1 & \text{if } x_0 \in (R^{k-1}(x), R^k(x)]. 
\end{cases} \]

It is immediate to check that one can also write the formula

\[ \epsilon_k = [x + kr] - [x + (k-1)r] \quad \text{for } 1 \leq k \leq q. \]

Note that the map \( \Phi_r \) intertwines the rotation with the cyclic permutation \( \tau \), i.e.

\[ \Phi_r \circ R_r = \tau \circ \Phi_r. \]

**Lemma 2.5.** The map \( \Phi_r \) is (weakly) increasing, in the sense that if \( 0 < x < y < 1 \), then \( \Phi_r(x) \leq \Phi_r(y) \).

**Proof.** If \( x < y \), then for each \( k \) we have \( [x + kr] \leq [y + kr] \). Thus, either for each \( 0 \leq k \leq q \) we have \( [x + kr] = [y + kr] \) (in which case \( \Phi_r(x) = \Phi_r(y) \)) or there exists a minimum \( k \leq q \) such that \( [x + kr] < [y + kr] \). In the latter case, \( \Phi_r(x) < \Phi_r(y) \) in lexicographical order.

Moreover, the map \( \Phi_r \) is constant on connected components of \( C_q \); we will be particularly interested in the word \( W_r \) defined as

\[ W_r := \Phi_r(0^+) = \lim_{x \to 0^+} \Phi_r(x). \]

**Lemma 2.6.** The map \( W : \mathbb{Q} \cap [0,1] \to \{0,1\}^* \) is a right inverse of \( \rho \); that is, for each \( r \in \mathbb{Q} \cap [0,1] \) we have

\[ \rho(W_r) = r. \]

**Proof.** Since all digits of \( W_r \) are either 0 or 1, then the number of 1 digits of \( W_r \) is just the sum of the digits, so by using equation (4) we get the telescoping sum:

\[ |W_r|_1 = \epsilon_1 + \cdots + \epsilon_q = \sum_{k=1}^{q} ([kr] - [(k-1)r]) = |qr| = p \]
so \( \rho(W_r) = |W_r|_1/|W_r| = p/q = r. \)

A pair \((r, r')\) of rational numbers \(r := \frac{p}{q}\) and \(r' := \frac{p'}{q'}\) with \((p, q) = (p', q') = 1\) and \(pq' - p'q = 1\) is called a Farey pair; the Farey sum of a Farey pair is defined as

\[
  r \oplus r' := \frac{p + p'}{q + q'}.
\]

It is easy to check that \(r \oplus r'\) lies in between \(r\) and \(r'\), that is if \(r < r'\) we have

\[
  r < r \oplus r' < r'.
\]

**Lemma 2.7.** Let \(r, r'\) be a Farey pair with \(r < r'\); then we have the identity

\[
  W_{r \oplus r'} = W_r W_r'
\]

where on the right-hand side we mean the concatenation of \(W_r\) and \(W_r'\).

**Proof of Proposition 2.4.** By Lemma 2.6, the function \(\rho\) is surjective, and moreover its restriction to the set

\[
  \text{Im } W := \{W_r : r \in \mathbb{Q} \cap [0, 1]\}
\]

is a bijection between \(\text{Im } W\) and \(\mathbb{Q} \cap [0, 1]\). Therefore, we just need to show that the set \(\text{Im } W\) coincides with the set \(\text{FW}\) of all Farey words. Now, since \(W_0 = 0\) and \(W_1 = 1\), the elements of the Farey list \(F_0 = (0, 1)\) belong to \(\text{Im } W\), and note that \((0, 1)\) is a Farey pair. Thus, by induction using Lemmas 2.7 for each \(n\) the elements of the list \(F_n\) belong to \(\text{Im } W\), so all Farey words belong to \(\text{Im } W\). Since it is well-known that every rational number can be obtained from 0 and 1 by taking successive Farey sums of Farey pairs, then \(W_r\) is a Farey word for any rational numbers \(r \in [0, 1]\), and the claim is proven. \(\square\)

Note moreover that the above \(\rho\) is a bijection between the tree of Farey words and the tree of Farey fractions; on one side, the operation is the concatenation of strings, while on the other side it is the Farey sum. For \(w = (\epsilon_1, \ldots, \epsilon_\ell) \in \{0, 1\}^*\) we set

\[
  \check{\check{w}} := (\check{\epsilon}_1, \epsilon_2, \ldots, \epsilon_\ell), \quad w^\vee := (\epsilon_1, ..., \epsilon_{\ell-1}, \check{\epsilon}_\ell).
\]

We shall now see Farey words have many symmetries, arising from the symmetries of the dynamical system \(R_r\).

**Proposition 2.8.** If \(w = W_r\) is a Farey word, then:

(a) the word \(^t\check{w}\) is still a Farey word: in particular,

\[
  W_{1-r} = ^t\check{w};
\]

(b) moreover, we have the identity

\[
  \Phi_{1-r}(0^-) = \check{w}
\]

(c) and

\[
  \Phi_r(0^-) = w^\vee = ^t\check{w};
\]

(d) both \(\check{w}\) and \(w^\vee\) are palindromes;

(e) finally, we have

\[
  ^t\check{w} < \check{w}.
\]

As an example, let us pick \(w = W_{2/5} = 00101\). One can check that \(\check{w} = 10101\) and \(w^\vee = 00100\) are both palindromes, and \(\check{w}^\vee = 10100\) equals the transpose of \(w\). Finally, the word \(^t\check{w} = 01011\) is also a Farey word (= \(W_{3/5}\)).
Proof. (a) Let us note that considering the rotation $R_{1-r}$ instead of $R_r$ is equivalent to inverting the direction (clockwise or counterclockwise) of the rotation. Thus, for each $x \in C_q$, the first $q + 1$ elements of the orbit of $x$ under $R_r$ are the same as the first $q + 1$ elements of the orbit of $x$ under $R_{1-r}$, but the order of visit is reversed (in symbols, $R^k_r(x) \equiv R^q_{1-r}(-x) \mod 1$ for $0 \leq k \leq q$), which proves the claim.

(b) This identity relies on the fact that the circle is symmetric under reflection $\sigma(x) := -x \mod 1$; indeed, for each $x$ the orbit of $x$ under $R_r$ is the reflection of the orbit of $1-x$ under $R_{1-r}$ (in symbols, $R^k_r(x) \equiv -R^q_{1-r}(-x) \mod 1$), while the marked point $x_0 = 0$ is fixed by $\sigma$.

(c) The first equality follows by noting that the iterates $R^k_r(0)$ encounter a discontinuity of the function $\lfloor \cdot \rfloor$ if and only if $k \equiv 0 \mod q$; thus, changing the starting point $x$ from $0^+$ to $0^-$ only affects the first and last digits of $\Phi_r(x)$. For the second equality, denote $v := \Phi_r(0^+) = t \Tv$; we have by (b) and then (a)
$$\Phi_r(0^-) = \Tv = t_w.$$ 
(d) follows immediately from (c): indeed we have 
$$t(\Tv) = (t^w)_\vee = (\Tv^w)_\vee = \Tv$$
where the first and third equalities are elementary, and the second one uses (c); a completely analogous proof works for $w^\vee$.

(e) Applying (c), and using the fact that the last digit of each (non-zero) Farey word is 1, we get 
$$t_w = \Tv < \Tv.$$ 
\[\square\]

It will be crucial in the following to study the ordering of the set of cyclic permutations of a given Farey word. The essential properties are contained in the following lemma.

Lemma 2.9. Let $w = W_r$ be a Farey word, and consider the set 
$$\Sigma(w) := \{\tau^kw : k \in \mathbb{N}\}$$
of its cyclic permutations. Moreover, let $w = w_1w_2$ be the standard factorization of $w$, and let $q_1 := |w_1|$, $q_2 := |w_2|$. Then the following are true:

1. the smallest cyclic permutation of $w$ is $w$ itself (i.e., $\min \Sigma(w) = w$); 
2. the second smallest cyclic permutation of $w$ is 
$$\tau^{q_1}w = w_2w_1;$$
3. the largest cyclic permutation of $w$ is 
$$\tau^{q_2}w = t_w.$$ 

Proof. Let us start by noting that, if $w = \Phi_r(x)$, then the set of cyclic permutations of $w$ is given by 
$$\{\tau^kw : 0 \leq k < q\} = \{\Phi_r(R^k_r(x)) : 0 \leq k < q\};$$
moreover, by Lemma 2.5, the order in the above set is the same as the order in the set 
$$S_r(x) := \{\{x + kr\} : 0 \leq k < q\}.$$ 
Thus, the smallest cyclic permutation of $w = W_r$ corresponds to the smallest possible value of $\{kr\}$, which is attained for $k = 0$, hence by $w = \Phi_r(0^+)$ itself, proving (1).
Moreover, let \( w = w_1w_2 \) be the standard factorization of \( w \). Then by definition we have \( w = W_r \), while \( w_1 = W_{r_1} \) and \( w_2 = W_{r_2} \), in such a way that \((r_1, r_2)\) is a Farey pair, with \( r_1 < r_2 \) and \( r := r_1 \oplus r_2 \). Note now that, writing \( r_1 = \frac{p_1}{q_1} \) and \( r_2 = \frac{p_2}{q_2} \), we have \( p_2q_1 - p_1q_2 = 1 \) by the definition of Farey pair, hence we can write
\[
q_1(p_1 + p_2) \equiv 1 \mod (q_1 + q_2).
\]
Thus, the second smallest element of \( S_r(0) \) is attained for \( k = q_1 \), hence the second smallest element of \( \Sigma(w) \) is \( \tau r_1w = w_2w_1 \), proving (2).

Finally, the largest element of the set \( \Sigma(w) \) is \( \tau k w \), where \( k \) is such that \( \{ kr \} = 1 - \frac{1}{q} \); thus, the corresponding word is \( \Phi_r(0^r) \), which equals \('w\) by Proposition 2.8 (c). Moreover, from equation (6) one also gets
\[
q_2(p_1 + p_2) \equiv -1 \mod (q_1 + q_2)
\]
hence \( \{ q_2 r \} = 1 - \frac{1}{q} \), so the largest element of the set \( \Sigma(w) \) is \( \tau q_2 w \).

Let us now state one more consequence of the previous lemma, in terms of ordering of subsets of the circle. Recall the doubling map \( D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) is defined as \( D(x) := 2x \mod 1 \). We say that a finite set \( X \subseteq S^1 \) has rotation number \( r = \frac{p}{q} \in \mathbb{Q} \) if it is invariant for the doubling map, and the restriction of \( D \) to \( X \) is conjugate to the circle rotation \( R_r \) via an orientation-preserving homeomorphism of \( S^1 \). More concretely, this means that if we write the elements of \( X \) in cyclic order as \( X = (\theta_0, \theta_1, \ldots, \theta_{q-1}) \) with \( 0 \leq \theta_0 < \theta_1 < \cdots < \theta_{q-1} < 1 \), then we have for each index \( i \)
\[
D(\theta_i) = \theta_{i+q} \mod q.
\]
The proof of the previous lemma also yields the following (uniqueness follows from [17], Corollary 8):

**Lemma 2.10.** Let \( w = W_r \) a Farey word. Then the set
\[
C(w) = \{0.\tau_k w : 0 \leq k \leq q - 1\} \subseteq S^1
\]
is the unique subset of \( S^1 \) which has rotation number \( r \) for the doubling map.

For an example, if \( w = 00101 \), then \( C(w) = (\frac{2}{21}, \frac{9}{21}, \frac{10}{21}, \frac{18}{21}, \frac{20}{21}) \) (see also Figure 2).

Recall that a word \( w \in \{0, 1\}^* \) which is minimal (with respect to lexicographic order) among all its cyclic permutations is also called a Lyndon word, hence property (e) of Lemma 2.9 can be paraphrased as saying that every Farey word is a Lyndon word (but not vice versa: e.g., \( 0011 \) is a Lyndon word but not a Farey word). Let us recall that all Lyndon words of length greater than 1 begin with the digit 0 and end with the digit 1; moreover one has the following (see [29])

**Proposition 2.11.** If \( w = ps \) is a Lyndon word (in particular, if \( w \) is a Farey word), then \( w \ll s \).

2.3. **Substitutions.** Another way to generate Farey words is by substitutions. Given a pair of words \( U = \begin{bmatrix} u_0 & u_1 \end{bmatrix} \in \{0, 1\}^* \times \{0, 1\}^* \) we can define the substitution operator associated to \( U \) to be the operator acting on \( \{0, 1\}^* \) (or on \( \{0, 1\}^N \)) as
\[
w = (\epsilon_1, \epsilon_2, \ldots) \mapsto (u_{\epsilon_1}, u_{\epsilon_2}, \ldots);
\]
the action of \( U \) on \( w \) will be denoted by \( w * U \). Let us note that if \( u_0 > u_1 \) then the operator is order preserving while if \( u_0 < u_1 \) it is order reversing; moreover, the
negation operator can be obtained as the substitution associated to \( V := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

We can also extend the substitution operator to pairs of words: if \( U = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \) and \( W = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \) let us define \( U \star W := \begin{bmatrix} u_0 \star W \\ u_1 \star W \end{bmatrix} \); in this way we get the following associativity property, that for each word \( w \) we have

\[
(w \star U) \star W = w \star (U \star W).
\]

Finally, the substitution and transposition operators are compatible, in the sense that

\[
\ell (w \star U) = \ell w \star \ell U \quad \text{where} \quad \ell U := \begin{bmatrix} \ell u_0 \\ \ell u_1 \end{bmatrix}.
\]

It turns out that one can produce all (non-degenerate) Farey words by successive iteration of two substitution operators, starting with the word \( w_0 = (01) \). Namely, let us define the two substitution operators

\[
U_0 : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \quad U_1 : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1. \end{cases}
\]

It is not difficult to realize that the action of \( U_0 \) and \( U_1 \) preserves the set of Farey words. More precisely, let us set \( F^*_n := F_n \setminus \{(0), (1)\} \),

\[
FW_0 := \{ w \in FW^* : |w_0| > |w_1| \} \quad FW_1 := \{ w \in FW^* : |w_0| < |w_1| \},
\]

and \( F^*_n := F_n \cap FW_\epsilon \) (note that, by Proposition 2.4) for each \( n \geq 2 \) one has \( F^*_n = F^*_n \cup \{(01)\} \cup F_1^* \). We can now formulate the following lemma.

**Lemma 2.12.** For each \( \epsilon = 0, 1 \), the operator \( U_\epsilon : F^*_n \to F^*_{n+1} \) is a bijection.

**Proof.** By induction, using the fact that substitution operators \( U_0, U_1 \) preserve the lexicographical order and respect concatenation, in the sense that

\[
(vw) \star U_\epsilon = (v \star U_\epsilon)(w \star U_\epsilon).
\]

\[\square\]

**Proposition 2.13.** For all \( n \geq 1 \), the following characterization holds:

\[
F^*_n = \{(01) \star U_{\epsilon_1} \star \ldots \star U_{\epsilon_\ell} : \epsilon_k \in \{0, 1\}, \ 0 \leq \ell < n\}.
\]

**Proof.** Again by induction: the base of the induction \( n = 1 \) is true, and if the claim holds at level \( n \) then we get, by Lemma 2.12 for each \( \epsilon = 0, 1 \)

\[
F^*_{n+1} = U_\epsilon (F^*_n) = \{(01) \star U_{\epsilon_1} \star \ldots \star U_{\epsilon_\ell} \star U_\epsilon : \epsilon_k \in \{0, 1\}, \ 0 \leq \ell < n\}.
\]

Thus, using the fact

\[
F^*_n = F^*_0 \cup F^*_1 \cup \{(01)\}
\]

the claim follows. \[\square\]
2.4. Farey words, kneading theory and external angles. Denote \([a,b]\) the closed interval of the circle from \(a\) to \(b\), with positive orientation. Let us define the binary bifurcation set \(E_B\) as

\[
E_B := \{ x \in [0, 1/2] : D^k(x) \in [x, x + 1/2] \ \forall k \in \mathbb{N} \}.
\]

The set \(E_B\) is a closed subset of the interval \([0, 1/2]\) and it has no interior as we will see. Let us point out that the only dyadic rationals which belong to \(E_B\) are 0 and \(1/2\). Moreover, we shall see that the connected components of the complement of \(E_B\) are canonically labelled by Farey words. Indeed, if \(w \in \{0, 1\}^*\) is a Farey word we set \(I_w := (a^-, a^+)\) with

\[
a^+ := \frac{1}{2}w = 0.\overline{w},
a^- := 0.\overline{w} - 1/2.
\]

For instance, if \(w = 00101\) then \(a^+ = 0.00101 = \frac{5}{31}\) and \(a^- = \frac{9}{62}\). We have the following properties.

**Proposition 2.14.** With the notation above we have

1. \(a^\pm \in E_B\);
2. if \(x \in I_w\) then \(x \notin E_B\);
3. for each Farey word \(w\), the length of \(I_w\) is

\[
|I_w| = \frac{1}{2(2^n - 1)}
\]

with \(n = |w|\);
4. each \(I_w\) is a connected component of \([0, 1/2] \setminus E_B\); moreover, we have

\[
[0, 1/2] \setminus E_B = \bigcup_{w \in FW} I_w;
\]
5. the Hausdorff dimension of \(E_B\) is zero.

Various similar constructions of this set appear in the literature; in particular, the above properties are proven by Bullett and Sentenac [8]; thus, for the convenience of the reader we still give a complete proof of the Proposition, but we postpone it to the appendix. The set \(E_B\) also appears in the kneading theory for Lorentz maps: indeed, it is the one-dimensional projection of the two-dimensional set of all kneading invariants for Lorentz maps (see [20, 28]).

Finally, the following lemma will be needed in the last section.

**Lemma 2.15.** If \(x \in E_B \cap [0, 1/6)\) then the following limit is infinite:

\[
\liminf_{\delta \to 0} \{|w|_0 - |w|_1 : I_w \subset [x - \delta, x + \delta]\} = +\infty.
\]

**Proof.** Indeed, note that setting \(\rho := \rho(w)\) we can rewrite \(|w|_0 - |w|_1 = |w|(1 - 2\rho)\); thus, if \(w_n\) is a sequence of Farey words such that \(I_{w_n}\) converge to \(x \in E_B\), then \(\rho(w_n)\) tends to a finite number, which is \(< \frac{1}{2}\) if \(x \in [0, 1/6)\), while \(|w|\) tends to infinity, hence the liminf of the product is infinite.

Let us now highlight a connection between the combinatorics of Farey words and the symbolic coding of rays landing on the main cardioid of the Mandelbrot set. We shall start by recalling few standard facts in complex dynamics: for an account, we refer to [32] and references therein.
Let us consider the family of quadratic polynomials \( f_c(z) := z^2 + c \) with \( c \in \mathbb{C} \). Recall the filled Julia set \( K(f) \) of a polynomial \( f(z) \) is the set of points with bounded orbits:

\[
K(f) := \{ z \in \mathbb{C} : \sup_n |f^n(z)| < \infty \}.
\]

If \( K(f) \) is connected, then its exterior is conformally isomorphic to a disk, hence it can be uniformized by a unique map \( \Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K(f) \) with \( \lim_{z \to \infty} |\Phi(z)| = \infty \) and \( \lim_{z \to \infty} \Phi(z)/z > 0 \). For each \( \theta \in \mathbb{R}/\mathbb{Z} \), the external ray at angle \( \theta \) is the set

\[
R(\theta) := \{ \Phi(\rho e^{2\pi i \theta}) \mid \rho > 1 \}.
\]

The ray \( R(\theta) \) is said to land if \( \lim_{\rho \to 1^+} \Phi(\rho e^{2\pi i \theta}) \) exists (and then it is a point on the boundary of \( K(f) \)). The Julia set \( J(f) \) is the topological boundary of \( K(f) \).

By Carathéodory’s theorem, if the Julia set is locally connected, then all rays land. The map \( f_c \) has two fixed points (which coalesce if \( c = \frac{1}{4} \)); we shall call \( \beta \)-fixed point the fixed point where the external ray of angle \( \theta = 0 \) lands, and \( \alpha \)-fixed point the other fixed point. A fixed point \( z_0 \) is called indifferent when the derivative \( f_c'(z_0) \) has modulus 1 (the derivative \( f_c'(z_0) \) is usually called the multiplier of the fixed point).

In parameter space, let us recall the Mandelbrot set \( \mathcal{M} \) is the set of parameters \( c \) for which the orbit under \( f_c \) of the critical point \( z = 0 \) is bounded:

\[
\mathcal{M} := \{ c \in \mathbb{C} : \sup_n |f_c^n(0)| < \infty \}.
\]

The set \( \mathcal{M} \) also equals the set of parameters \( c \in \mathbb{C} \) for which the Julia set of \( f_c \) is connected. Just as the Julia sets, the Mandelbrot set admits a unique uniformizing map \( \Phi_M : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M} \) such that \( \lim_{z \to \infty} |\Phi_M(z)| = \infty \) and \( \lim_{z \to \infty} \Phi_M(z)/z > 0 \).

Let us define the main cardioid \( \bigcirc \) of the Mandelbrot set as

\[
\bigcirc := \{ c \in \mathbb{C} : f_c(z) \text{ has an indifferent fixed point} \}.
\]

A simple computation shows \( \bigcirc \) can be parametrized as \( c = \frac{1}{2} e^{2\pi i \theta} - \frac{1}{4} e^{4\pi i \theta} \) where \( \theta \in [0, 1] \); in this parametrization, for each \( c \in \bigcirc \) the map \( f_c \) has multiplier \( e^{2\pi i r} \) at the \( \alpha \)-fixed point. Let \( \Omega \) denote the set of angles of external rays landing on the main cardioid of the Mandelbrot set:

\[
\Omega := \{ \theta \in \mathbb{R}/\mathbb{Z} : R_M(\theta) \text{ lands on } \bigcirc \}.
\]

The following proposition makes precise the connection between Farey words and the set of rays landing on the cardioid.

**Proposition 2.16.** Let \( r = \frac{p}{q} \in \mathbb{Q} \cap (0, 1) \), and \( w = W_r \) the corresponding Farey word. Let us now define the pair of angles \( (\theta^-, \theta^+) \) as

\[
\begin{align*}
\theta^- &= 0.\tau(w) \\
\theta^+ &= 0.\tau(w)
\end{align*}
\]

and let \( c \in \bigcirc \) denote the parameter on the main cardioid for which the \( \alpha \)-fixed point of \( f_c \) has multiplier \( e^{2\pi i r} \). Then we have the following properties:

1. in the Julia set of \( f_c \), the set of external rays landing at the \( \alpha \)-fixed point is the set

\[
C(w) = \{ 0.\tau^k w : 0 \leq k \leq q - 1 \}
\]

whose binary expansions are all cyclic permutations of \( w \);

2. in parameter space, the pair of angles of external rays \( (\theta^-, \theta^+) \) lands on the main cardioid at the parameter \( c \);
Figure 2. Left: the set $\Omega$ of external rays landing on the main cardioid of the Mandelbrot set. Right: the set $C(w)$ of external rays landing on the $\alpha$-fixed point of a Julia set for the center of a hyperbolic component tangent to the main cardioid (here, the rotation number is $r = 2/5$ and the corresponding Farey word is $w = 00101$.)

(3) we have the identity

$$\Omega = 2E_B.$$ 

As an example, if $r = \frac{2}{5}$ then $w = 00101$, and $\theta^+ = 0.01001 = \frac{9}{31}$, while $\theta^- = 0.10010 = \frac{10}{31}$. In the dynamical plane, the set of rays landing at the $\alpha$-fixed point is $C(w) = \left(\frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{18}{31}, \frac{20}{31}\right)$.

Proof. (1) Let $c \in \mathcal{O}$ be the parameter for which the map $f_c$ has an indifferent fixed point of multiplier $e^{2\pi ir}$. It is known that its Julia set $J(f_c)$ is locally connected, hence all external rays land in the dynamical plane of $f_c$, and the landing map $L(\theta) : \mathbb{R}/\mathbb{Z} \to J(f_c)$ defined as $L(\theta) := \lim_{\rho \to 1} \Phi(\rho e^{2\pi i\theta})$ is a continuous semiconjugacy between the doubling map and $f_c$, that is we have the commutative diagram (see also [32]):

$$
\begin{array}{ccc}
D & \xrightarrow{f_c} & J(f_c) \\
\mathbb{R}/\mathbb{Z} & \xrightarrow{L} & J(f_c)
\end{array}
$$

Let $S := L^{-1}(\alpha)$ the set of angles of external rays landing at the $\alpha$-fixed point. The map $f_c$ permutes the rays landing at $\alpha$, and preserves their cyclic orientation in the plane. Moreover, since the multiplier of $f_c$ at $\alpha$ is $e^{2\pi ir}$, then the set $S$ has rotation number $r$ under the doubling map, hence by Lemma 2.10 the set $S$ equals $C(w)$.

To pass to the statement in parameter space, note that it is known that the pair of rays landing on $c$ in parameter space corresponds to the elements of $S$ which delimit a sector which (in the dynamical plane) contains the critical value. Now, the set $S^1 \setminus S$ is the union of $q$ (connected) arcs, and the doubling map permutes their endpoints: hence, $D$ maps each arc of length smaller than $1/2$ homeomorphically to its image, and there is a unique arc $\ell_0$ of length at least $1/2$. Since the map $f_c$ is a local homeomorphism away from its critical point, then the component in the dynamical plane corresponding to $\ell_0$ must contain the critical point. Now note
that by Lemma 2.9 (1) and (3), the arc \( \ell = (0, \overline{w}, 0, \overline{v}) \) is a connected component of \( S^1 \setminus S \), and by Proposition 2.14 and the definition of \( \mathcal{E}_B \), the length of \( \ell \) is more than 1/2, so it must be \( \ell = \ell_0 \) the one which contains the critical point. As a consequence, the arc which contains the critical value is delimited by taking the forward image of the endpoints of \( \ell_0 \); thus, it is the arc \( \ell_1 = (0, \overline{\tau(w)}, 0, \overline{v}) = (\theta^-, \theta^+) \) and claim (2) is proven.

As for the last statement, the previous construction implies the correspondence \( \Omega \cap Q = 2(\mathcal{E}_B \cap Q) \); claim (3) follows by taking closures, as it is known that the set of angles of rays landing on the main cardioid is the closure of the set of rational angles of rays landing on the main cardioid (see [19], Corollary 4.4).

\[ \square \]

3. Regular continued fraction expansions

Let us first fix some notation regarding the classical continued fractions expansions. Any irrational number admits a unique infinite continued fraction expansion, which will be denoted as

\[ x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = [a_0; a_1, a_2, \ldots] \]

with \( a_k \in \mathbb{Z} \) \( \forall k \), and \( a_k \geq 1 \) \( \forall k \geq 1 \). Moreover, any rational value \( r \) admits exactly two finite expansions; indeed, we can write

\[ r = [a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_n - 1, 1] \]

with \( a_n \geq 2 \). Any nonempty string of positive integers \( S = (a_1, \ldots, a_n) \) defines a rational value \( r = [0; a_1, \ldots, a_n] \in (0, 1] \), which we will sometimes denote as \( r = [0; S] \).

We then define the right conjugate of \( S \) to be the only string \( S' \) which defines the same rational value as \( S \), i.e. such that \( [0; S'] = [0; S] \). For instance \((3, 1, 2, 1)' = (3, 1, 2, 1)\) and vice versa (conjugation is involutive, and affects only the last one or two digits). We also define the left conjugate \( S' \) of a (finite or infinite) string \( S \) in a similar way, just acting on the leftmost digits: that is, if \( S = (a_1, a_2, \ldots) \) we define

\[ S' := \begin{cases} (1, a_1 - 1, a_2, \ldots) & \text{if } a_1 \geq 2 \\ (1 + a_2, a_3, \ldots) & \text{if } a_1 = 1. \end{cases} \]

Thus, the left conjugate of \((3, 1, 3)\) will be \((3, 1, 3)' = (1, 2, 1, 3)\). It is not difficult to check that this manipulation on strings translates into the map \( \sigma : [0, 1] \to [0, 1] \) defined as \( \sigma(x) := 1 - x \) on the side of continued fraction expansions, namely for any string of positive integers we have

\[ \sigma([0; S]) = [0; S']. \]

Another operation on strings we shall often use in the following is the operator \( \partial \) defined on (finite or infinite) strings as

\[ \partial(a_1, a_2, \ldots) := \begin{cases} (a_1 - 1, a_2, \ldots) & \text{if } a_1 > 1 \\ (a_2, \ldots) & \text{if } a_1 = 1. \end{cases} \]

We shall sometimes also use the transposition: the transpose string of \( S = (a_1, \ldots, a_\ell) \) is the string \( \overline{S} = (a_\ell, \ldots, a_1) \). Finally, if \( S \) is a finite string of positive integers we will denote by \( q(S) \) the denominator of the rational number whose c.f. expansion is \( S \), i.e. such that \( \frac{p(S)}{q(S)} = [0; S] \) with \( (p(S), q(S)) = 1, q(S) > 0. \)
Let us also recall the well-known estimate

\[ q(S)q(T) \leq q(ST) \leq 2q(S)q(T). \] (10)

Moreover, we define the map \( f_S : x \mapsto S \cdot x \), which corresponds to appending the string \( S \) at the beginning of the continued fraction expansion of \( x \). That is, if \( S = (a_1, \ldots, a_n) \) we can write, by identifying matrices with Möbius transformations,

\[ S \cdot x := \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \cdot x. \] (11)

It is easy to realize that concatenation of strings corresponds to composition, namely \((ST) \cdot x = S \cdot (T \cdot x)\); moreover the map \( f_S \) is increasing if \(|S|\) is even, decreasing if \(|S|\) is odd. The image of \( f_S \) is a cylinder set

\[ I(S) := \{ x = S \cdot y, \ y \in [0,1] \} \]

which is a closed interval with endpoints \([0; a_1, \ldots, a_n]\) and \([0; a_1, \ldots, a_n + 1]\). The map \( f_S \) is a contraction of the unit interval, and it is easy to see that

\[ \frac{1}{4q(S)^2} \leq |f_S'(x)| \leq \frac{1}{q(S)^2} \quad \forall x \in [0,1] \] (12)

and the length of \( I(S) \) is bounded by

\[ \frac{1}{2q(S)^2} \leq |I(S)| \leq \frac{1}{q(S)^2}. \] (13)

Given two strings of positive integers \( S = (a_1, \ldots, a_n) \) and \( T = (b_1, \ldots, b_n) \) of equal length, let us define the alternate lexicographic order as

\[ S < T \text{ if } \exists k \leq n \text{ s.t. } a_i = b_i \ \forall 1 \leq i \leq k - 1 \text{ and } \begin{cases} a_n < b_n & \text{if } n \text{ even} \\ a_n > b_n & \text{if } n \text{ odd}. \end{cases} \]

The importance of such order lies in the fact that given two strings of equal length \( S < T \iff [0; S] < [0; T] \). In order to compare quadratic irrationals with periodic expansion, the following string lemma (\cite{10}, Lemma 2.12) is useful: for any pair of strings \( S, T \) of positive integers, we have the equivalence

\[ ST < TS \Leftrightarrow [0; S] < [0; T]. \] (14)

The order \(<\) is a total order on the strings of positive integers of fixed length; to be able to compare strings of different lengths we define the partial order

\[ S \ll T \text{ if } \exists i \leq \min\{|S|, |T|\} \text{ s.t. } S_1^i < T_1^i \]

where \( S_1^i = (a_1, \ldots, a_i) \) denotes the truncation of \( S \) to the first \( i \) characters. Let us note the following basic properties:

1. if \(|S| = |T|\), then \( S < T \iff S \ll T \);
2. if \( S, T, U \) are any strings, \( S \ll T \Rightarrow SU \ll T, S \ll TU \);
3. if \( S \ll T \), then \( S \cdot z < T \cdot w \) for any \( z, w \in (0,1) \).
3.1. Farey legacy. We shall now see how to construct, using continued fractions, an irrational number given a binary word; this way, starting from the set of Farey words we shall define the fractal subset $E_{KU}$ of the interval, and establish its properties from the properties of Farey words we obtained in the previous sections.

Indeed, let us define the runlength map $RL$ to be the map which associates to a (finite or infinite) binary word $w$ the string of positive integers which records the size of blocks of consecutive equal digits: namely, if

$$w = \underbrace{0 \ldots 0}_{a_1} \underbrace{1 \ldots 1}_{a_2} \ldots,$$

we set

$$RL(w) := (a_1, a_2, \ldots).$$

For instance, $RL(0001001001) = RL(1110110110) = (3, 1, 2, 1, 2, 1)$; note that $RL$ is a two-to-one map ($RL(w) = RL(\overline{w})$), but it is strictly increasing when restricted to words beginning with the digit 0. If $S = RL(w)$ for some $|w| > 1$ then

$$RL(\overline{w}) = S, \quad RL(w\overline{w}) = S' \quad (15)$$

Note also that, if $w = w_0w_1$ and the last digit of $w_0$ is different from the first of $w_1$, then one has

$$RL(w) = RL(w_0)RL(w_1)$$

(note this is always the case when $w_i$ are non-degenerate Farey words). For the runlength string of Farey words some more nice properties hold:

**Lemma 3.1.** Let $w \in FW^*$ be a Farey word and $S := RL(w)$. Then the length $|S|$ is even and

(i) there exists an integer $a \geq 1$ and a Farey word $f = (\epsilon_1, \ldots, \epsilon_n)$ such that one can write

$$S = B_\epsilon \ldots B_n,$$

with

$$B_0 = (a + 1, 1), \quad B_1 = (a, 1) \quad \text{if } w \in FW_0$$

or

$$B_0 = (1, a), \quad B_1 = (1, a + 1) \quad \text{if } w \in FW_1.$$  

The Farey word $f$ is unique as long as $w \neq (01)$; since it plays a central role in the following, it will be referred to as the Farey structure of the string $S$.

(ii) The runlength of the Farey word $\overline{w}$ is

$$RL(\overline{w}) = 'S' = 'S;$$

(iii) if $S = (a_1, \ldots, a_\ell)$ and $1 \leq k < \ell/2$, we set $P_k := (a_1, \ldots, a_{2k})$, $S_k := (a_{2k+1}, \ldots, a_\ell)$, (so that $S = P_kS_k$), then

$$(16) \quad S << S_k;$$

(iv) using the same notation as above, if $w \in FW_0$ then

$$(17) \quad S_kP_k << \partial S.$$
Proof. (i). Recall that \(FW^* = FW_0 \cup (01) \cup FW_1\). Clearly, for \(w = (01)\) we have \(S = RL(w) = (1, 1)\), so \(a = 1\) and we can choose \(f = (0)\) or \(f = (1)\). Let us now assume \(w \in FW_0\). Then by Proposition 2.13 and Lemma 2.12 we can write
\[
(18) \quad w = f \ast U_1 \ast U_0^a
\]
for some \(f = (\epsilon_1, \ldots, \epsilon_l) \in FW\) and \(a \geq 1\); on the other hand \(U_1 \ast U_0^a = \left[\begin{array}{c}
(0^a+1) \\
(0^a1)
\end{array}\right]\) and \(RL(0^a1) = (a, 1)\) so, calling \(B_0 := (a + 1, 1)\) and \(B_1 := (a, 1)\) we get that \(S\) is the concatenation \(B_{\epsilon_1} \ldots B_{\epsilon_l}\). Note that, since the image of \(U_0\) is contained in \(FW_0\) and the image of \(U_1\) is contained in \(FW_1\) (Lemma 2.12), then the factorization of equation (18) is unique, hence also the Farey structure of \(S\) is unique. The case \(w \in FW_1\) is analogous.

(ii). The second claim is an immediate consequence of equation (15) and the fact that \(\mathcal{V}w^* = t^* w\), together with the fact that \(RL(w^*) = RL(w)\).

(iii). It follows from Proposition 2.11 and the fact that the runlength map preserves the strong order \(<\subset\subset\) when restricted to words which begin with 0.

(iv). By unwinding the definitions it is not hard to see that, if \(w \in FW_0\), we can write the identities
\[
S = RL(f \ast U_1 \ast U_0^a)
\]
\[
\partial S = RL(\langle f \rangle \ast U_1 \ast U_0^a)
\]
\[
S_k P_k = RL(\langle \tau^k f \rangle \ast U_1 \ast U_0^a).
\]
Moreover, by Proposition 2.8 (c) and Lemma 2.9 (3) we have
\[
\langle f \rangle \triangleright \ association \langle t^* f \rangle \quad \forall k
\]
hence the claim follows from the fact that both the substitution operator \(f \mapsto f \ast U_1 \ast U_0^a\) and the runlength map (when restricted to words beginning with 0) are order-preserving.

3.2. Quantervals. For \(x \in [0, 1/2]\) we shall consider the map \(\phi : [0, 1/2] \to [0, 1]\) induced by runlength as follows: if \(x = \sum_{j \geq 1} \epsilon_j 2^{-j}\) is the binary expansion of \(x\), with \(\epsilon_j \in \{0, 1\}\), then we define \(\phi(x)\) to be the number with continued fraction
\[
\phi(x) := [0; RL(\epsilon)]
\]
where \(RL(\epsilon)\) is the runlength of the sequence \((\epsilon_j)_{j \geq 1}\). This map is certainly well-defined for those values of \(x\) which admit a unique (and infinite) binary expansion; in fact, it also extends continuously to dyadic rationals, since the two binary expansions of a dyadic rational are mapped onto two continued fraction expansions of the same rational. For instance, if \(x = \frac{3}{8} = 0.011\) then \(\phi(x) = [0; 1, 2] = \frac{2}{3}\); on the other hand, we can write \(\frac{3}{8} = 0.0101\), which maps to \([0; 1, 1, 1, \infty]\) = \(\frac{2}{3}\). It is not difficult to check that this map is a homeomorphism between \([0, 1/2]\) and \([0, 1]\).

The inverse of \(\phi\) is essentially Minkowski’s question mark function \(Q : [0, 1] \to [0, 1]\) sending \(x = [0; a_1, a_2, \ldots]\) to
\[
Q(x) := 0.0 \ldots 0 \overline{1} \ldots 1 \ldots
\]
In fact, one has for each \(x \in [0, 1/2]\)
\[
(19) \quad Q(\phi(x)) = 2x.
\]

For properties of Minkowski’s question mark function, we refer to [35].
Definition 3.2. Given $w \in FW^*$ we will call qumterval of label $w$ the interval $J_w$ which is the image under $\phi$ of the interval $I_w$ appearing in Lemma 2.14, that is

$$J_w := \phi(I_w).$$

Note that, if $S := RL(w)$, then by Lemma 2.14 the endpoints of $J_w$ are given by

$$J_w = (\alpha^-, \alpha^+)$$

with

$$\alpha^+ := \phi(a^+) = [0; S]$$
$$\alpha^- := \phi(a^-) = [0; S']$$

As an example, the Farey word $w = 001$ yields $S = (2, 1)$, hence $\alpha^+ = [0; 2, 1] = \sqrt{3} - 1$ and $\alpha^- = [0; 3, 1, 2] = 2 - \sqrt{3}$. The rational value $r := \phi(w) = [0; S]$ is the (unique!) rational value in $J_w$ with least denominator, and will be called the pseudocenter of $J_w$ (see [10] for more general properties of the pseudocenter of an interval). Note that, by using equation (9) and Lemma 3.1 (ii), the left endpoint $\alpha^-$ can also be described by the property

$$1 - \alpha^- = [0; \alpha^]$$

We also define $\mathcal{E}_{KU} := \phi(\mathcal{E}_B)$. In this way we get

$$[0, 1] \setminus \mathcal{E}_{KU} = \bigcup_{w \in FW^*} J_w.$$

Let us point out that, since $\phi$ is a bijection between dyadic rationals and rationals in $[0, 1]$, we have $\mathcal{E}_{KU} \cap \mathbb{Q} = \{0, 1\}$. Theorem 1.3 now follows immediately from our setup.

We now have the tools to prove one of the results stated in the introduction.

Proof of Theorem 1.3. Since by definition $\mathcal{E}_{KU} = \phi(\mathcal{E}_B)$, then by using equation (19) and Proposition 2.16 (3) we get

$$Q(\mathcal{E}_{KU}) = Q(\phi(\mathcal{E}_B)) = 2 \mathcal{E}_B = \Omega.$$

3.3. Thickening $\mathbb{Q}$. We shall now perform an alternative construction of $\mathcal{E}_{KU}$ which is not essential for the main results of this paper, but it is useful for a comparison with the results in [10]. Given any rational value $r \in (0, 1)$, let us consider its continued fraction expansion of even length $r = [0; S]$; then set $\beta(r) := [0; S]$ and

$$\tilde{J}_r := (\sigma \beta(\sigma r), \beta(r)).$$

Since $\beta(r) > r$ and $\sigma$ is order reversing, we can easily see that the $\tilde{J}_r$ is an open interval containing $r$, and in fact $r$ is the pseudocenter of $\tilde{J}_r$. For all $w \in FW^*$ we have that $J_w = \tilde{J}_r$ for $r = \phi(w)$. Indeed qumtervals have the following maximality property (which will be proven in the appendix):

Proposition 3.3. For any $r' \in \mathbb{Q} \cap (0, 1)$ there is a Farey word $w \in FW^*$ such that $\tilde{J}_{r'} \subset J_w$.

As a consequence of the above proposition one gets the identity

$$\mathcal{E}_{KU} = [0, 1] \setminus \bigcup_{r \in \mathbb{Q} \cap (0, 1)} \tilde{J}_r.$$

Let us now compute the dimension of $\mathcal{E}_{KU}$. 
Figure 3. The quadratic intervals $\tilde{J}_r$. Each interval is represented by a half-circle with the same endpoints. The intervals which are maximal with respect to inclusion are precisely the connected components of the complement of the bifurcation set $\mathcal{E}_{KU}$.

Proposition 3.4. The Hausdorff dimension of $\mathcal{E}_{KU}$ is zero:

$$\text{H.dim } \mathcal{E}_{KU} = 0.$$ 

Proof. We shall actually prove the stronger statement that for each $N \geq 2$ the set $\mathcal{E}_{KU} \cap [\frac{1}{N+1}, \frac{1}{N}]$ has zero box-counting dimension: the claim then follows since the box-counting dimension is an upper bound for the Hausdorff dimension. Fix $N \geq 2$, set

$$C_N := \left\{ w \in FW_0 : J_w \cap \left[ \frac{1}{N+1}, \frac{1}{N} \right] \neq \emptyset \right\}$$

and consider the geometric $\zeta$-function defined by

$$\zeta_N(s) := \sum_{w \in C_N} |J_w|^s.$$ 

Since the abscissa of convergence of the series $\zeta_N$ coincides with the upper box dimension of $\mathcal{E}_{KU} \cap [\frac{1}{N+1}, \frac{1}{N}]$ (see [14], pg. 54), it is enough to prove that the above series converges for any $s > 0$. Now, it is not hard to prove that, for all $N \geq 2$, one has

$$|J_w| < 2 b^{\|w\|} \quad \forall w \in C_N, \quad \text{where } b := N^{-\frac{1}{N+1}}.$$ 

Indeed, it is easy to check that

$$|r - \beta(r)| = |S \cdot 0 - S \cdot \beta(r)| \leq \sup |f_S'| \beta(r) \leq \frac{1}{q(S)^2}$$

where the last inequality is a consequence of equation (12); since an analogous estimate holds for the distance between $r$ and the left endpoint, one gets

$$|J_w| < \frac{2}{q(S)^2}.$$ 

On the other hand, if $w \in C_N$ then $S = RL(w)$ is a concatenation of $n$ blocks of the type $B_0 := (N, 1)$ or $B_1 := (N - 1, 1)$, where $n(N + 1) < |w|$. Thus we get that

$$q(S) = q(B_{e_1}...B_{e_n}) \geq q(B_1^n) \geq q(B_1)^n$$

and since $q(B_1) \geq N$ we get $q(S) \geq N^{\frac{1}{N+1}}$; thus (23) follows from this last estimate and equation (24). Since $\#\{w \in FW : |w| = k\} \leq k$, the estimate (23) implies that $\zeta_N$ is dominated by the sum $2^s \sum_{k=1}^{\infty} k b^{sk}$, therefore it converges for all $s > 0$, proving the claim. 

$\square$
Let us conclude this section by comparing the bifurcation set $E_{KU}$ with the bifurcation set (or exceptional set) $E_N$ for Nakada’s $\alpha$-continued fraction transformations (see [10]). By comparing equation (22) with the definition of $E_N$ from [10], one can easily check the inclusion
\[(25)\quad E_{KU} \cap [0, 1/2] \subset E_N.\]
Note that the inclusion is strict (and actually, the Hausdorff dimension of $E_N$ is 1, while the dimension of $E_{KU}$ is 0).

Since both sets in (25) are related to sets of rays landing in the Mandelbrot set, it is interesting to see what our dictionary tells us when we transport the previous inclusion to the world of complex dynamics. First, using Theorem 1.3, the Minkowski question mark map $Q$ maps $E_{KU}$ homeomorphically to the set $\Omega$ of external angles of rays landing on the main cardioid. Meanwhile, by the main theorem of [6], the set $E_N$ is related to the set of rays landing on the real slice of the Mandelbrot set. Indeed, if we let $R$ be the set of external angles of rays whose impression intersects the real slice of the Mandelbrot set, then we have the homeomorphism (6, Theorem 1.1)
\[\psi(E_N) = R \cap [1/2, 1)\]
where $\psi(x) := \frac{1}{2} + \frac{Q(x)}{4}$. Thus we have the following commutative diagram, where $i$ is the inclusion map:

\[
\begin{array}{ccc}
E_{KU} \cap [0, 1/2] & \overset{i}{\rightarrow} & E_N \\
\downarrow Q & & \downarrow \psi \\
\Omega \cap [0, 1/2] & \overset{T}{\rightarrow} & \mathcal{R}
\end{array}
\]
As a consequence, the map $T(\theta) := \psi(Q^{-1}(\theta))$, which can be just expressed as
\[T(\theta) = \frac{1}{2} + \frac{\theta}{4}\]
maps the set of rays landing on the upper half of the main cardioid into the set of real rays, i.e.
\[T(\theta) \subseteq \mathcal{R}\]
for each $\theta \in \Omega \cap [0, 1/2]$. This fact is known in the folklore as “Douady’s magic formula” (see [4], Theorem 1.1).

4. Matching intervals for continued fractions with $SL(2, \mathbb{Z})$-branches.

Let us return to the maps $K_\alpha : [\alpha - 1, \alpha] \rightarrow [\alpha - 1, \alpha]$ which are defined by $K_\alpha(0) = 0$ and
\[K_\alpha(x) = -\frac{1}{x} - c_\alpha(x), \quad c_\alpha(x) := \left\lfloor -\frac{1}{x} + 1 - \alpha \right\rfloor \in \mathbb{Z}.
\]
The goal of this section is to prove that a matching condition between the orbits of the endpoints $\alpha$ and $\alpha - 1$ is achieved for any parameter which belongs to some quantum (Theorem 4.1 and Corollary 4.2). In order to formulate the result precisely, we need some notation.

\footnote{In [10], the bifurcation set is simply denoted by $E$, and its complement is denoted by $M$. See also [12], section 3.}
Recall the group $PSL(2, \mathbb{Z})$ acts on the real projective line by Möbius transformations. Indeed, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $x \in \mathbb{R} \cup \{\infty\}$ then we shall write $Ax := \frac{ax + b}{cx + d}$. The group $PSL(2, \mathbb{Z})$ is generated by the two elements $S$ and $T$, which are represented by the matrices

$$
S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
$$

and act respectively as the inversion $Sx := -1/x$ and the translation $Tx := x + 1$. For any fixed $\alpha \in (0, 1)$, the map $K_\alpha$ is just given by the inversion followed by an integer power of the translation which brings the point back to the interval $[\alpha - 1, \alpha]$. Thus, each branch of $K_\alpha(x)$ is represented by the map $x \mapsto T^{-c_\alpha(x)}Sx$. Now, in order to keep track of the inverse branches of the powers of $K_\alpha$, we shall now use the notation $c_{j,\alpha}(x) := c_\alpha(K_j^{-1}(x))$ for each positive integer $j$, and define the matrices

$$
M_{\alpha,x,\ell} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Note that these matrices represent the inverses of $K_\alpha$, in the sense that

$$
M_{\alpha,x,\ell}(K_\alpha^\ell(x)) = x
$$

for each $\alpha \in [0, 1], x \in [\alpha - 1, \alpha], \ell \in \mathbb{N}$. Finally, note that the family $K_\alpha$ possesses the following fundamental symmetry: the maps $K_\alpha$ and $K_{1-\alpha}$ are measurably conjugate, namely one has

$$
K_\alpha(x) = -K_{1-\alpha}(-x)
$$

for all $x \in [\alpha - 1, \alpha] \setminus \bigcup_{k \in \mathbb{Z}} \frac{1}{k - \alpha}$ (the countable set of exceptions is due to the convention about the floor function). As a consequence, it is sufficient to study the dynamics for $\alpha \in [0, 1/2]$.

We are now ready to formulate the main result of this section.

**Theorem 4.1.** Let $w \in FW^*$ a Farey word, $m_0 := |w|_0, m_1 := |w|_1$ and $J_w$ the corresponding quaternial; let moreover $S := RL(w)$ denote the runlength of $S$ and $r := [0; S]$ be the pseudocenter of $J_w$. Then there exist two elements $M, M' \in PSL(2, \mathbb{Z})$ such that, for all $\alpha \in J_w$ we have the equalities

$$
M_{\alpha,\alpha-1,m_0} = M,
M_{\alpha,\alpha,m_1} = M'.
$$

Moreover, the following matching condition holds:

$$
TM = M'ST^{-1}S.
$$

The matching condition (28) implies the following identification between of the orbits of the two endpoints $\alpha$ and $\alpha - 1$.

**Corollary 4.2.** For each parameter $\alpha \in J_w$ we have the identity

$$
K_\alpha^{m_0}(\alpha - 1) = K_\alpha^{m_1}(\alpha).
$$

Note that (27) implies that the first $m_0$ steps of the (symbolic) itinerary of $\alpha$ is constant for all $\alpha$ in the same quaternial, and the same is true for the first $m_1$ steps of the orbit of $m_1$. A condition of this kind is called **strong cycle condition** in [23]; see Section 6 for a more detailed comparison.
As an illustration of Theorem 4.1, let us consider the case $\alpha \in J_w$ with $w = 01$: it turns out that for every $\alpha \in J_{01} = (g^2, g)$ the following identity holds:

$$K_\alpha^2(\alpha) = K_\alpha^2(\alpha - 1) \quad \forall \alpha \in (g^2, g).$$

Indeed, this is due to the fact that the analytic expression of $K_\alpha$ at the endpoints does not change as $\alpha \in J_{01}$; in this simple case in fact we can work out the explicit form of $K_\alpha$, and we get

$$K_\alpha(\alpha) = T^2 S \alpha = \frac{2\alpha - 1}{\alpha} \quad K_\alpha(\alpha - 1) = T^{-2} S (\alpha - 1) = \frac{2\alpha - 1}{1 - \alpha}.$$

whence we have $M = ST^2$ and $M' = ST^{-2}$, and we can check that

$$TM = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = M'ST^{-1}S$$

which is an instance of equation (28). Note that the essential point is that the matrices $M$ and $M'$ do not depend on the particular $\alpha$ as long as $\alpha$ belongs to $J_w$; thus, the matching condition is just an identity between elements of the group $PSL(2, \mathbb{Z})$. However, to different matching intervals $J_w$ there correspond different identities in the group.

The proof of Theorem 4.1 follows from an explicit description of the symbolic orbits of $\alpha$ and $\alpha - 1$ in terms of the regular continued fraction expansion of the pseudocenter of $J_w$, as stated in the following proposition.

**Proposition 4.3.** Let $w \in FW_0$ (hence $J_w \subset (0, 1/2)$), denote $m := |w|_0$ and $n := |w|_1$ and let $RL(w) = (a_1, 1, ..., a_n, 1)$ be the associated string of positive integers. Then for each $\alpha \in J_w$ we have the identities

$$M_{\alpha,\alpha-1,m} = M$$
$$M_{\alpha,\alpha,n} = M'$$

where the above matrices are constructed as

$$M := (ST^2)^{a_1} T (ST^2)^{a_2} T ... (ST^2)^{a_{n-1}} T (ST^2)^{a_n}$$
$$M' := ST^{-a_1-1} ST^{-a_2-2} ... T^{-a_{n-1}-2} ST^{-a_n-2}.$$

Let us point out that in the special case $n = 1$ we have $RL(w) = (N, 1)$ for some $N \geq 1$, and the above equations must be interpreted as yielding $M = (ST^2)^N$, $M' = ST^{-N-1}$. Moreover, as a consequence of the Proposition, the matrices determining matching conditions behave well under concatenation; indeed, if the Farey word $w$ is the concatenation of two Farey words $w'$ and $w''$, then the left-hand side of the matching condition on $J_w$ is the concatenation of the left-hand sides of the matching conditions of $J_{w'}$ and $J_{w''}$.

Before delving into the core of the proofs of Proposition 4.3 and Theorem 4.1 let us make some elementary observations and define some more notation. The action of both $S, T$ can be easily expressed in terms of regular continued fraction expansion, thus the action of $K_\alpha$ on the regular continued fraction expansion of $x$ follows some simple rules. Namely, if $x = [-1; a_1, a_2, a_3, ...] \in (\alpha - 1, 0)$ then one gets the formulas

$$K_\alpha(x) = \begin{cases} T^{-2} S x = [-1; a_1 - 1, a_2, a_3, a_4, ...] & a_1 > 1 \\ T^{-(a_2+1-\epsilon)} S x = [\epsilon; a_3, a_4, ...] & a_1 = 1 \end{cases} \quad \epsilon \in \{-1, 0\}$$
where to decide whether \( \epsilon \) is \(-1\) or \(0\) one has to check which of these choices returns an element in \([\alpha - 1, \alpha]\). On the other hand, if \( x = [0; a_1, a_2, a_3, \ldots] \in (0, \alpha] \) then

\[
K_\alpha(x) = T^{a_1+1}\alpha Sx = \begin{cases} 
[\epsilon; 1, a_2 - 1, a_3, \ldots] & a_2 > 1 \\
[\epsilon; 1 + a_3, a_4, \ldots] & a_2 = 1
\end{cases} \quad \epsilon \in \{-1, 0\};
\]

again, the choice between \( \epsilon = -1 \) and \( \epsilon = 0 \) is forced by the condition that the range of \( K_\alpha \) must be \([\alpha - 1, \alpha]\). To write some of the above branches of \( K_\alpha \) in compact form we shall also use the following fractional transformations:

\[
\partial^- := S T^{-1} S, \quad \partial := S T S.
\]

Note that, if \( 0 < x < 1 \), one has \( \partial^- x < x \); in fact in terms of regular continued fractions we get \( \partial^- ([0; a_1, a_2, a_3, \ldots]) = [0; 1 + a_1, a_2, a_3, \ldots] \), while \( \partial \) is the inverse of \( \partial^- \) (and is consistent with the previous definition in section 3). Finally, if \( S = (a_1, \ldots, a_n) \) and \( x \in [0, 1] \), we shall use the string action notation \( S \cdot x \) to denote the number whose continued fraction expansion is obtained by appending \( S \) at the beginning of the continued fraction expansion of \( x \); in terms of \( S \) and \( T \), this can be defined as

\[
S \cdot x := ST^{-a_1}ST^{-a_2} \ldots ST^{-a_{n-1}}ST^{a_n}x.
\]

**Proof of Proposition 4.3.** Let \( w \in FW_0 \), and \( \alpha \in J_w \). Let us denote \( S := RL(w) \) the runlength of \( S \) and \( r := [0; S] \) the pseudocenter of \( J_w \). Now, by Lemma 3.1 we have that \( S \) is of the form

\[
S = (a_1, 1, a_2, 1, \ldots, a_m, 1)
\]

with \( a_j \in \{a, a+1\} \), \( m_0 = \sum_{j=1}^{m} a_j \), and \( m_1 = m \). Moreover for \( 1 \leq k \leq m \) we define the even prefixes and suffixes of \( S \) as

\[
P_k := (a_1, 1, a_2, 1, \ldots, a_k, 1), \quad S_k := (a_{k+1}, 1, a_{k+2}, 1, \ldots, a_m, 1).
\]

Recall also that by definition the endpoints of \( J_w = (\alpha^-, \alpha^+) \) are

\[
\alpha^- = [0; S T S] \quad \alpha^+ = [0; S].
\]

**Case A.** Let us first take into account the case \( \alpha \in [r, \alpha^+] \). Then we can write \( \alpha := S \cdot y \) for some \( y \in [0, \alpha^+] \). In this case we claim that the orbits of the endpoints \( \alpha \) and \( \alpha - 1 \) under \( K_\alpha \) eventually match, and before getting to the matching point the orbits (and symbolic orbits, on the right column) of \( \alpha - 1 \) and \( \alpha \) are given by table 1.

One can go from one line to the following just using the rules (33) (for the upper part) or (34) (for the lower part). So we only have to check that at each stage we actually get a value which lies in the interval \([\alpha - 1, \alpha]\).

As far as the orbit of \( \alpha - 1 \) is concerned, all items in the list except for the last one are negative, so we just have to check that we never drop below \( \alpha - 1 \); since the operator \( \partial \) increases the value of its argument (i.e. \( \partial x \geq x \)), it is sufficient to check the iterates of \( K_\alpha \) of order \( a_1 + \ldots + a_k \) (with \( k \in \{1, \ldots, m\} \)): the corresponding lines are marked by the symbol \( \checkmark \). That is, we need to check the following:

1. \( -1 + S_k \cdot y \geq \alpha - 1 \) for all \( k \in \{1, \ldots, m - 1\} \)
2. \( y \leq \alpha \).

(1) is true by Lemma 3.1 (iii): indeed, we have the inequality \( S_k \gg S \), from which follows that

\[
-1 + S_k \cdot y \geq -1 + S \cdot y = -1 + \alpha
\]
as needed. Now, since by construction $S \cdot y = \alpha < \alpha^+$ and the map $x \mapsto S \cdot x$ is increasing with a fixed point at $\alpha^+$, we have that $y \leq S \cdot y = \alpha$, which proves (2).

Checking that the values in the lower part of table 1 are actually in $[\alpha - 1, \alpha]$ is slightly more tricky. We need to prove that $\partial^- S_k \cdot y \leq S \cdot y = \alpha$ for $k \in \{1, \ldots, m-1\}$; as a matter of fact by Lemma 3.1-(iv) one has

$$S_k P_k \ll \partial S,$$

which implies, since $P_k$ is a prefix of the continued fraction expansion of $\alpha^+$,

$$S_k \cdot \alpha^+ \leq \partial S \cdot y.$$

Since the map $x \mapsto S_k \cdot x$ is increasing we then get $S_k \cdot y \leq S_k \cdot \alpha^+ < \partial S \cdot y$, which implies, by applying $\partial^-$ to both sides of the equation, $\partial^- S_k \cdot y < S \cdot y = \alpha$. Since $0 < \partial^- y < y < S \cdot y = \alpha$, we get the last step for free.

**Case B.** We must now settle the case $\alpha \in (\alpha^- , r]$. Let us recall that $S' = \frac{1}{r} S$, so that we must have $\alpha = S' \cdot y$ for $0 \leq y \leq \lfloor \frac{1}{r} S \rfloor$, which is equivalent, $\sigma \alpha = S' \cdot y$, $0 \leq y \leq \lfloor \frac{1}{r} S \rfloor$. In this case we claim that the orbits of the endpoints, before reaching the matching point, are given by table 2.

Again, one can go from one line to the following just using the rules (33) for the upper list or (34) for the lower; we just have to check that at each stage we actually get values inside the interval $[\alpha - 1, \alpha]$.

As far as the orbit of $\alpha - 1$ is concerned, all items of the list are negative, and we just have to check that we never drop below $\alpha - 1$, therefore the only steps which need some comment are those corresponding to iterates of $K_\alpha$ of order $a_1 + \ldots + a_k$.

### Table 1. Orbits of $\alpha$ and $\alpha - 1$, for $\alpha \in [r, \alpha^+)$.

| $\alpha - 1$ | $-1 + S \cdot y$ |
| $K_\alpha(\alpha - 1)$ | $-1 + \partial S \cdot y$ |
| $K_\alpha^2(\alpha - 1)$ | $-1 + \partial^2 S \cdot y$ |
| $\ldots$ | $\ldots$ |
| $K_\alpha^{a_1-1}(\alpha - 1)$ | $-1 + \partial^{a_1-1} S \cdot y$ |
| $\alpha^+$ | $S \cdot y$ |
| $K_\alpha(\alpha)$ | $\partial^- S_1 \cdot y$ |
| $K_\alpha^2(\alpha)$ | $\partial^- S_2 \cdot y$ |
| $\ldots$ | $\ldots$ |
| $K_\alpha^{m-1}(\alpha)$ | $\partial^- S_{m-1} \cdot y$ |
| $K_\alpha^m(\alpha)$ | $\partial^- y = \frac{y}{y+1}$ |
| $c_{1,\alpha}$ | $\leq a_1 - 1$ |
| $c_{2,\alpha}$ | $\leq a_2 - 2$ |
| $\ldots$ | $\ldots$ |
| $c_{m-1,\alpha}$ | $\leq a_{m-1} - 2$ |
| $c_{m,\alpha}$ | $\leq a_m - 2$ |
proving the required inequality. The last step is immediate, since 

\(-1 + S' \cdot y\)

and the last inequality holds because we can apply Lemma 3.1 (iii) to \(t\) since \(y < S\) is order reversing and \(\sigma\)

\(\alpha = S' \cdot y\)

\(K_0(\alpha) = \partial^{-} S'_1 \cdot y\)

\(K_0(\alpha) = \partial^{-} S'_2 \cdot y\)

\(K_{m-1}(\alpha) = \partial^{-} S'_{m-1} \cdot y\)

\(K_{m}(\alpha) = -y\)

proving the required inequality. The last step is immediate, since 

\(-1/2 > \alpha - 1\) (note that \(\alpha < 1/2\) since \(w \in FW_0\)).

To check that the values in the lower part of the list of table 2 are actually in \([\alpha - 1, \alpha]\) we need to prove that \(\partial^{-} S'_k \cdot y < S' \cdot y\) for \(k \in \{1, ..., m-1\}\); indeed, since \(\sigma\) is order reversing and \(S' = \iota S\), we get

\(\partial^{-} S'_k \cdot y < S' \cdot y\) \iff \(\sigma(\partial^{-} S'_k \cdot y) > \sigma(S' \cdot y) \iff (\partial^{-} S'_k) \cdot y = (\iota S)_k \cdot y > S' \cdot y\)

and the last inequality holds because we can apply Lemma 3.1 (iii) to \(\iota S\), yielding \((\iota S)_k \gg S\).

Finally, to check that \(K^m(\alpha) = -y\) we have to show that \(-y > \alpha - 1\): indeed, since \(y < [0; \lfloor S \rfloor]\) we get \(\iota S \cdot y > y\), and \(1 - \alpha = \iota S' \cdot y = \iota S \cdot y > y\).

If we now keep track of the symbolic orbit of \(\alpha - 1\) and \(\alpha\) as we described (see the right column of the tables 1, 2), we realize that the values of the coefficients
Thus, by writing $TST$ (36). Thus, in general for each $1 \leq n \leq m_0$ we have

$$\begin{align*}
(c_{j, \alpha}(\alpha - 1))_{1 \leq j \leq m_0} &= (\frac{2, \ldots, 2, 3, 2, \ldots, 3, 2, \ldots, 2}{a_1 - 1, a_2, \ldots, a_m}),
\end{align*}$$

$$\begin{align*}
(c_{j, \alpha}(\alpha))_{1 \leq j \leq m_1} &= (-a_1 - 1, -a_2 - 2, \ldots, -a_m - 2),
\end{align*}$$

thus we can compute the matrices $M_{\alpha, \alpha - 1, m_0}$ and $M_{\alpha, \alpha, m_1}$, recovering formula (32).

\textbf{Proof of Theorem 4.1.} By symmetry (eq. (26)), we need only to check what happens for $\alpha \leq \frac{1}{2}$. Now, the previous proposition gives us formulas for $M$ and $M'$, so we just need to check that equation (28) holds given these formulas; this is a simple algebraic manipulation as follows. Let us first prove the case $n = 1$, for which equation (28) becomes

$$T(ST^2)^N = ST^{-N-1}ST^{-1}S$$

(with $N = a_1$). It is well-known and easy to check that $(ST)^{3} = \text{identity}$, from which it follows

$$\text{TST} = ST^{-1}S.$$ 

Thus, by writing $TST = T(ST^2)T^{-1}$ and raising both sides to the $N^{th}$ power, we have

$$T(ST^2)^N T^{-1} = ST^{-N}S$$

from which $T(ST^2)^N = ST^{-N}ST = ST^{-N-1}TST = ST^{-N-1}ST^{-1}S$, proving (36). Thus, in general for each $1 \leq k \leq n$ we have the identity $T(ST^2)^{ak} = ST^{-ak}ST^{-1}S$, and concatenating all pieces we get precisely equation (28).

\textbf{Proof of Corollary 4.2}. By taking the inverses of both sides of equation (28), and acting on $\alpha$ we get the equality

$$M_{\alpha, \alpha - 1, m_0} T^{-1} \alpha = STSM_{\alpha, \alpha, m_1}^{-1}(\alpha)$$

hence using that $K_{\alpha}^{m_1}(\alpha) = M_{\alpha, \alpha, m_0}^{-1}(\alpha)$ and $K_{\alpha}^{m_0}(x) = M_{\alpha, \alpha - 1, m_0}^{-1}(\alpha - 1)$ we get that

$$\frac{1}{K_{\alpha}^{m_1}(\alpha)} + 1 = -\frac{1}{K_{\alpha}^{m_0}(\alpha - 1)}$$

hence for any $k \in \mathbb{Z}$ we have

$$-\frac{1}{K_{\alpha}^{m_1}(\alpha)} - k \in [\alpha - 1, \alpha] \Leftrightarrow -\frac{1}{K_{\alpha}^{m_0}(\alpha - 1)} - (k + 1) \in [\alpha - 1, \alpha]$$

thus $c_{m_0 + 1, \alpha}(\alpha - 1) = c_{m_1, \alpha}(\alpha) + 1$ and the claim follows.

Let us conclude this section by studying the ordering between the iterates of $K_{\alpha}$, which will be needed in the last section. As we shall see, this also follows from the combinatorics of the underlying Farey words: in particular, the ordering is the same as the ordering between the cyclic permutations of their Farey structure.

\textbf{Lemma 4.4.} Let $w \in FW_{0}$; then for any $\alpha \in J_{w}$ the ordering of the set

$$\{K_{\alpha}^{j}(\alpha - 1) : 0 \leq j \leq m_0\}$$

of the first $m_0 + 1$ iterates of $\alpha - 1$ under $K_{\alpha}$ is independent of $\alpha$. Similarly, the ordering of the set of the first $m_1 + 1$ iterates of $\alpha$ is also independent of $\alpha$. 

Proof. The first part of the claim can be rephrased as saying that for each \( j, j' \in \{0, \ldots, m_0\} \) and for each \( \alpha, \alpha' \in J_w \)
\[
K^i_\alpha(\alpha - 1) < K^j_\alpha(\alpha - 1) \text{ if and only if } K^i_\alpha(\alpha - 1) < K^j_\alpha(\alpha - 1).
\]
Let now \( 0 \leq j < j' \leq m_0 \) be fixed; to check that \( K^j_\alpha(\alpha - 1) \) and \( K^{j'}_\alpha(\alpha - 1) \) are ordered in the same way for all \( \alpha \in J_w \) it is enough to prove that they are always different, and to prove this latter statement it is enough to prove that \( K^j_\alpha(\alpha - 1) < K^{m_0}_\alpha(\alpha - 1) \) for all \( \alpha \in J_w \) and all \( j \in \{0, \ldots, m_0 - 1\} \). This follows from the explicit description of the orbits given in the proof of Proposition 4.3 of which we will keep the notation (see tables 1 and 2). Indeed, in case A the claim holds because of the inequality
\[
K^j_\alpha(\alpha - 1) < 0 \leq y = K^{m_0}_\alpha(\alpha - 1);
\]
in case B we have
\[
K^{m_0}_\alpha(\alpha - 1) = \frac{-y}{y + 1} = -1 + \frac{1}{1 + y} = [-1; 1, a, ...] \quad \text{for some } a > 1,
\]
while the largest of the previous iterates has a continued fraction expansion beginning with \([-1; 1, 1, ...]\).

The corresponding claim about iterates of \( \alpha \) is proven in the same way. Indeed, it is enough to check that if \( j < m_1 \) then \( K^{m_1}_\alpha(\alpha - 1) < K^j_\alpha(\alpha) \): this is obvious in case B, while in case A we just have to check that \( \partial^{-} y < \partial^{-} S_j \cdot y \); now, since \( y < S \cdot y \) (see also proof of Proposition 4.3 case A) and \( S < S_j \) (Lemma 3.1(iii)), one gets \( \partial^{-} y < \partial^{-} S \cdot y \leq \partial^{-} S_j \cdot y \) as claimed. \(\square\)

**Proposition 4.5.** Let \( w \in FW_0 \) a Farey word, and \( w = w'w'' \) be its standard factorization, and denote \( j_0 := |w'|_0 \) and \( j_1 = |w''|_1 \). Then for each \( \alpha \in J_w \) the following holds:
\[
K^{j_0}_\alpha(\alpha - 1) = \min\{K^j_\alpha(\alpha - 1) : 1 \leq j \leq m_0\}
\]
\[
K^{j_1}_\alpha(\alpha) = \max\{K^j_\alpha(\alpha) : 1 \leq j \leq m_1\}.
\]

**Proof.** By Lemma 4.4 and since all the maps \( \alpha \to K^j_\alpha(\alpha - 1) \) for \( 0 \leq j \leq m_0 \) are continuous on the closure of \( J_w \) (they are given by equation (52), it is sufficient to verify the statement for \( \alpha = \alpha^+ \) the right endpoint of \( J_w \). In this case, by Proposition 4.3 the first iterates of \( \alpha - 1 \) are given by
\[
K^{a_1, \ldots, a_k + \delta}(\alpha - 1) = -1 + \partial^h S_k \cdot \alpha^+
\]
with \( 0 \leq h \leq a_{k+1} - 1, 0 \leq k \leq m - 1 \). Note that the homeomorphism \( \phi \) defined in section 3.2 semiconjugates the shift in the binary expansion with the map \( \partial \), hence for each prefix \( v \) of \( w \) we have
\[
\phi(0.\tau^l w) = K^{l_0}_\alpha(\alpha - 1) + 1
\]
where \( l := |v| \) and \( l_0 := |v|_0 \). Now, in order to find the smallest nontrivial iterate, recall that by Lemma 2.9 the smallest cyclic permutation of \( w \) is \( w \) itself, while the second smallest is \( \tau^lw \), i.e.
\[
w < \tau^lw \leq \tau^k w \quad \text{for all } 1 \leq k \leq m_0 + m_1.
\]
Thus, since the homeomorphism \( \phi \) is increasing on the interval \([0, 1/2]\), we get by equation (37) that for each \( j \in \{1, \ldots, m_0\} \)
\[
K^{j_0}_\alpha(\alpha - 1) \leq K^j_\alpha(\alpha - 1)
\]
where \( j_0 = |w'_0| \) as claimed. Similarly, for the orbit of \( \alpha \), we know by Proposition 4.3 that the iterates in case \( \alpha = \alpha^+ \) are given by
\[
K^j_\alpha(\alpha) = \partial^- S_j \cdot \alpha^+ \quad 1 \leq j \leq m_1.
\]
Once again from Lemma 2.9, the largest cyclic permutation of \( w \) is \( \tau_l \) with \( l = |w''| \), hence the largest value of \( S_j \cdot \alpha^+ \) is attained for \( j = |w''|_1 = j_1 \) and the claim follows. \( \square \)

5. Entropy

We shall now use the combinatorial description of the orbits of \( K_\alpha \) we have obtained in the previous section to derive consequences about the entropy of the maps, proving Theorem 1.2.

For any fixed \( \alpha \in (0, 1) \), the map \( K_\alpha \) is a uniformly expanding map of the interval and many general facts about its measurable dynamics are known (see [24]). Indeed, each \( K_\alpha \) has a unique absolutely continuous invariant probability measure (a.c.i.p. for short) which we will denote \( d\mu_\alpha(x) = \rho_\alpha(x) \, dx \), and the dynamical system \( (K_\alpha, \mu_\alpha) \) is ergodic. In fact, it is even exact and isomorphic to a Bernoulli shift; moreover, its ergodic properties can also be derived from the properties of the geodesic flow on the modular surface.

Let \( h(\alpha) \) be the metric entropy of \( K_\alpha \) with respect to the measure \( \mu_\alpha \): we shall be interested in studying the properties of the function \( \alpha \mapsto h(\alpha) \). Recall that for an expanding map of the interval the entropy can also be given by Rohlin’s formula
\[
h(\alpha) = \int_{\alpha-1}^{\alpha} \log |K'_\alpha| \, d\mu_\alpha.
\]
Moreover, for maps generating continued fraction algorithms such as \( K_\alpha \), the entropy is also related to the growth rate of denominators of convergents to a “typical” point. More precisely, we can define the \( \alpha \)-convergents to \( x \) to be the sequence \((p_{n,\alpha}(x)/q_{n,\alpha}(x))_{n\in\mathbb{N}}\) where
\[
\begin{pmatrix} p_{n,\alpha}(x) \\ q_{n,\alpha}(x) \end{pmatrix} := M_{\alpha,x,n} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
For each \( x \), the sequence \( p_{n,\alpha}(x)/q_{n,\alpha}(x) \) tends to \( x \). Then, for \( \mu_\alpha \)-almost every \( x \in [\alpha - 1, \alpha] \) we have
\[
h(\alpha) = 2 \lim_{x \to +\infty} \frac{1}{n} \log |q_{n,\alpha}(x)|.
\]

As far as the global regularity of the entropy function is concerned, one can easily adapt the strategy of [36] to prove that \( h(\alpha) \) is Hölder continuous in \( \alpha \):

**Theorem 5.1.** For any \( \alpha \in (0, 1/2) \) and any \( \eta \in (0, 1/2] \), the function \( \alpha \mapsto h(\alpha) \) is Hölder continuous of exponent \( \eta \) on \([\alpha, 1 - \alpha] \).

On the complement of \( \mathcal{E}_{KU} \) we can exploit the rigidity due to the matching to gain much more regularity. The key tool will be the following proposition.

**Proposition 5.2.** Let \( \alpha, \alpha' \in J_w \) be nearby points which lie both on the same side with respect to the pseudocenter, with \( \alpha' < \alpha \). Then the following formulas hold:
\[
h(\alpha) = [1 + (|w_0| - |w_1|)\mu_\alpha([\alpha', \alpha])]h(\alpha') \tag{38}
\]
\[
h(\alpha') = [1 - (|w_0| - |w_1|)\mu_{\alpha'}([\alpha' - 1, \alpha - 1])]h(\alpha) \tag{39}
\]
The proof of this proposition follows very closely the proof of the corresponding statement for $\alpha$-continued fractions in ([34], Theorem 2); a sketch of the argument is included in the appendix. As a first straightforward consequence of Proposition 5.2 we get the local monotonicity of $h$ on the complement of $E_{KU}$.

**Corollary 5.3.** The entropy is locally monotone on $[0,1] \setminus E_{KU}$. More precisely:

1. the entropy is strictly increasing on $J_w$ if $w \in FW_0$;
2. the entropy is constant on $J_w$ if $w = (01)$;
3. the entropy is strictly decreasing on $J_w$ if $w \in FW_1$.

Let us point out that Corollary 5.3 alone is still not enough to deduce the monotonicity on $[0, g^2]$ stated in Theorem 1.2. Indeed, $h$ is strictly increasing on each open interval $J_w$ for all $w \in FW_0$ and the union of all such opens is dense in $[0, g^2]$, but to conclude that $h$ is monotone on $[0, g^2]$ we must exclude that $h$ displays pathological behaviour like the “devil’s staircase” function. We shall take care of this issue proving that $h$ is absolutely continuous.

Let us consider the decomposition $h(t) = h_r(t) + h_s(t)$ where

\[(40)\]

$h_r(t) := \sum_{w \in FW} \text{Var}_{J_w \cap [0,t]} h$, \quad $h_s(t) := h(t) - h_r(t)$.

Recall the notation $\text{Var}_I f$ means the total variation of the function $f$ on the interval $I$. Intuitively, $h_r$ is the “regular part” which takes into account the behaviour of $h$ on the (open and dense) union of the $J_w$, while $h_s$ is the remaining “singular part”, which we will actually prove to be zero.

**Lemma 5.4.** The function $h_s$ is locally constant on $[0, g^2] \setminus E_{KU}$.

**Proof.** If $w \in FW_0$ and $t_1, t_2 \in J_w$, $t_1 < t_2$, then the monotonicity of $h_{|J_w}$ implies that

$h(t_2) - h(t_1) = \text{Var}_{J_w \cap [t_1,t_2]} h = h_r(t_2) - h_r(t_1)$

which implies $h_s(t_2) - h_s(t_1) = 0$, whence the claim. \qed

We shall now need the following lemma in fractal geometry, whose proof we postpone to the appendix.

**Lemma 5.5.** Let $I_1, I_2, \ldots$ a countable family of disjoint subintervals of some close interval $I \subseteq \mathbb{R}$ and denote

$G := I \setminus \bigcup_{i=1}^{\infty} I_i$.

Let the upper box-dimension of $G$ be $\delta_0$. Given any $\eta$ and $\delta$ with $\eta > \delta > \delta_0$, there exists a constant $C$ such that for any choice of a subsequence $J_1, J_2, \ldots$ of the family $\{I_i\}$, one gets the following inequality:

$\sum_{i=1}^{\infty} |J_i|^{\eta} \leq C \left( \sum_{i=1}^{\infty} |J_i| \right)^{\eta - \delta}$.

**Lemma 5.6.** For every $a$ and $\eta$ in $(0, 1/2)$ both functions $h_s$ and $h_r$ are Hölder continuous of exponent $\eta$ on the interval $[a, 1/2]$. 
Proof. By Theorem 5.1 there is $C = C(a, \eta)$ such that
\begin{equation}
|h(t) - h(t')| \leq C|t - t'|^{\eta} \quad \forall t, t' \in [a, 1/2].
\end{equation}
Note that in order to prove Hölder continuity of $h_r$ on the whole interval $[a, 1/2]$ it is sufficient to show that $h_r$ is Hölder continuous on $[a, 1/2] \cap \mathcal{E}_{KU}$. If $\beta, \beta' \in \mathcal{E}_{KU} \cap [a, 1/2]$, $\beta < \beta'$, then
\begin{equation}
|h_r(\beta') - h_r(\beta)| = \sum_{J_w \subset [\beta, \beta']} \text{Var}_J \varrho.
\end{equation}
By equation (41), for each $J_w$ we have that $	ext{Var}_J \varrho \leq C|J_w|^{\eta}$; then we get
\begin{equation}
|h_r(\beta') - h_r(\beta)| \leq C \sum_{J_w \subset [\beta, \beta']} |J_w|^\eta \leq
\end{equation}
\begin{equation}
\leq CC' \left( \sum_{J_w \subset [\beta, \beta']} |J_w| \right)^{\eta - \delta} = CC'|\beta' - \beta|^{\eta - \delta}
\end{equation}
for any $\delta \in (0, \eta)$; the last line is a direct consequence of Lemma 5.5 and the fact that the box-dimension of $\mathcal{E}_{KU}$ is 0 (Proposition 3.4). Finally $h_s$ is Hölder continuous since it is a difference of two Hölder continuous functions (note the domain has unit length).

\begin{lemma}
Let $f : I \to I$ be a map of a closed real interval which is Hölder continuous of exponent $\eta > 0$. Suppose $E \subseteq I$ is a closed, measurable subset of Hausdorff dimension less than $\eta$, and that $f$ is locally constant on the complement on $E$. Then $f$ is constant on all $I$.
\end{lemma}

Proof. If $f$ is Hölder continuous of exponent $\eta$, it is easy to check using the definition of Hausdorff dimension that for any measurable subset $E$ of the interval one has the estimate
\begin{equation}
\text{H.dim } f(E) \leq \frac{1}{\eta} \text{H.dim } E.
\end{equation}
Now, if $f$ is locally constant on $I \setminus E$, then the image of any connected component of the complement of $E$ is a point, hence we have
\begin{equation}
\text{H.dim } f(I) = \text{H.dim } f(E).
\end{equation}
Now, if $f$ is not constant, then by continuity the image $f(I)$ is a non-degenerate interval, hence it has dimension $1$. Thus one has by using equation (42)
\begin{equation}
\text{H.dim } E \geq \eta \text{H.dim } f(I) = \eta
\end{equation}
which is a contradiction.

\begin{proof}
By virtue of the symmetry of $h$, the third statement is a consequence of the first one, so we just need to prove that the function $h$ is strictly increasing on $[0, g^2]$.

Since the function $h$ is strictly increasing on each $J_w$ which intersects $[0, g^2]$ (Corollary 5.3 (1)) and the union of the $J_w$ is dense, then the function $h_r$ is strictly increasing on $[0, g^2]$. The claim then follows if we prove that the function $h_s$ is identically zero, so $h = h_r$. Now, by Lemma 5.4, $h_s$ is locally constant on the complement of $\mathcal{E}_{KU}$, it is $\eta$-Hölder continuous for any positive $\eta < 1/2$ by Lemma 5.6, so since $\text{H.dim } \mathcal{E}_{KU} = 0 < \eta$ the function $h_s$ is globally constant by Lemma 5.7.
\end{proof}
Figure 4. The parameter space of \((a, b)\)-continued fractions. We only consider the critical line case \(b - a = 1\).

6. Natural extension and regularity properties of the entropy

Finally, in this section we shall analyze the regularity properties of the entropy function \(h(\alpha)\), proving Theorem 1.1. In order to do so, we need some results from the theory of \((a, b)\)-continued fractions due to S. Katok and I. Ugarcovici. Therefore we will outline here some of the results contained in [23, 24]; meanwhile, we shall also explain to the reader how our constructions relate to the work of Katok and Ugarcovici, and translate between the different notations.

The starting point are the “slow” maps \(f_{a, b} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}\) defined as

\[
 f_{a, b}(y) := \begin{cases} 
 T_y & \text{if } y < a \\
 S_y & \text{if } a \leq y < b \\
 T^{-1}y & \text{if } b \leq y 
\end{cases}
\]

where the parameters \((a, b)\) range in the closed region

\[
 P := \{(a, b) \in \mathbb{R}^2 : a \leq 0 \leq b, \ b - a \geq 1, \ -ab \leq 1\}
\]

which is plotted in Figure [4].

An essential role in the theory is played by a condition called cycle property, which we recall briefly. If \(f\) is a real map and \(x\) is a point, we call upper orbit (resp. lower orbit) of \(x\) the countable set of elements \(f^k_t(x) := \lim_{t \to x^+} f^k(t)\) (resp. \(f^k_{-1}(x) := \lim_{t \to x^{-}} f^k(t)\)), with \(k \in \mathbb{N}\). We say that the map \(f_{a, b}\) satisfies the cycle property at the discontinuity points \(a, b\) if the upper and lower orbit of \(a\) eventually collide, and the same is true for \(b\). As a matter of fact for most parameters the cycle property is strong, meaning that it is the consequence of an identity in \(SL(2, \mathbb{Z})\), which is stable on an open set.
To build a geometrical realization of the natural extension one defines the family of maps of the plane

\[
F_{a,b}(x, y) := \begin{cases}
(Tx, Ty) & \text{if } y < a \\
(Sx, Sy) & \text{if } a \leq y < b \\
(T^{-1}x, T^{-1}y) & \text{if } b \leq y.
\end{cases}
\]

Katok and Ugarcovici prove that each \(F_{a,b}\) has an attractor \(D_{a,b} \subset \mathbb{R}^2\) such that \(F_{a,b}\) restricted to \(D_{a,b}\) is invertible and it is a geometric realization of the natural extension of \(f_{a,b}\). In fact, for most values of \((a, b)\) the attractor \(D_{a,b}\) has a simple structure:

**Theorem 6.1** ([23]). There exists an uncountable set \(\hat{E}\) of one-dimensional Lebesgue measure zero that lies on the diagonal boundary \(b - a = 1\) of \(\mathcal{P}\) such that:

1. for all \((a, b) \in \mathcal{P} \setminus \hat{E}\) the map \(F_{a,b}\) has an attractor \(D_{a,b}\) (which is disjoint from the diagonal \(x = y\)) on which \(F_{a,b}\) is essentially bijective;
2. the set \(D_{a,b}\) consists of two (or one, in the “degenerate” case \(ab = 0\)) connected components each having finite rectangular structure, i.e. bounded by non-decreasing step functions with a finite number of steps;
3. almost every point \((x, y)\) off the diagonal \(x = y\) is mapped to \(D_{a,b}\) after finitely many iterations of \(F_{a,b}\).

The above result shows that exceptions to the finiteness condition dwell on the critical line \(b - a = 1\). For this reason, we only consider these cases. With a slight abuse of notation we shall always write \(b\) critical line of \(\alpha\) and by Theorem 6.1 the extension of \(f_{b-1,b}\) restricted to \([-b, b]\) is precisely the first return map of \(f_{b-1,b}\) on the interval \([b - 1, b]\).

Note also that in the symmetric case \(a + b = 0\) (with \(b \in [1/2, 1]\)) the system determined by the first return on \([-b, b]\) is equivalent to a twofold cover of the \(\alpha\)-continued fraction transformation \(T_b\).

Let us note that along the critical line \(b - a = 1\) the cycle property is a bit easier to state. Indeed, in this case the upper orbit of \(a\) coincides with the orbit of \(a\), while the lower orbit of \(a\) coincides with the orbit of \(b\); on the other hand the upper orbit of \(b\) coincides with the orbit of \(a\), while the lower orbit of \(b\) is just the orbit of \(b\). Thus, in this case the cycle property for \(f_{b-1,b}\) is essentially equivalent to the condition [1] for the fast map \(K_b\). Indeed, the set \(\mathcal{E}_{KU}\) that we explicitly described coincides, up to a countable set of points, with the projection on the \(x\)-axis of the set \(\tilde{\mathcal{E}}\) mentioned in Theorem 6.1.

Now, for \(\alpha \in J_w\) the map \(K_\alpha\) satisfies the algebraic matching condition (28), hence \(f_\alpha\) satisfies the strong cycle property (see [23]) and by Theorem 6.1 the extension \(F_\alpha\) has an attractor \(D_\alpha\) with finite rectangular structure. We may also consider the “first return map” map \(\tilde{F}_\alpha : \mathbb{R} \times [\alpha - 1, \alpha) \to \mathbb{R} \times [\alpha - 1, \alpha)\) defined as

\[
\tilde{F}_\alpha(x, y) := (T^{-c_\alpha(y)}Sx, T^{-c_\alpha(y)}Sy).
\]

Note that the map \(K_\alpha\) is a factor of \(\tilde{F}_\alpha\). Obviously the set \(D_\alpha \cap \mathbb{R} \times [\alpha - 1, \alpha)\) is an attractor for \(\tilde{F}_\alpha\). In order to compactify the attractor it is convenient to make the change of coordinates \(\xi = Sx\); then, the natural extension map becomes

\[
\Phi_\alpha(\xi, y) := (ST^{-c_\alpha(y)}\xi, T^{-c_\alpha(y)}Sy),
\]

Precisely, the projection of \(\hat{E}\) equals \(\mathcal{E}_{KU} \setminus \bigcup_{w \in F^* \setminus J_w} \partial J_w\).
which is just the map $\tilde{F}_\alpha$ in the new coordinates: $\tilde{F}_\alpha \circ (S \times id) = (S \times id) \circ \Phi_\alpha$.

The map $\Phi_\alpha$ will have the attractor $\Delta_\alpha := SD_\alpha = (S \times id)(D_\alpha \cap \mathbb{R} \times [\alpha - 1, \alpha])$, which is bounded and has finite rectangular structure.

6.1. Structure of the attractor $\Delta_\alpha$ and entropy formula. In [23], Section 5) the authors prove that the attractor $D_\alpha$ has finite rectangular structure providing an explicit recipe to build it; one can easily translate this recipe in order to obtain the following analogue description for $\Delta_\alpha$ for $\alpha \in J_w$.

Let us fix $w \in FW$ a Farey word, with $m_i := |w|_i$, and pick $\alpha \in J_w$. Then the attractor $\Delta_\alpha$ is the union of finitely many rectangles with sides parallel to the coordinate axes, and the sides of these rectangles are determined by the dynamics of $K_\alpha$ prior to the matching, as we now describe.

The horizontal segments which delimit $\Delta_\alpha$ are of precisely two types, corresponding to the orbits of $\alpha$ and $\alpha - 1$, respectively; in particular, the set of levels (ordinates) of the horizontal segments on the “lower-right” part of $\Delta_\alpha$ is precisely the set

$$\{\alpha - 1, K_\alpha(\alpha - 1), ..., K_\alpha^{m_0}(\alpha - 1)\}$$

of iterates of $\alpha - 1$ up to the matching, while the set of levels of the horizontal segments on the “upper-right” side is the set

$$\{\alpha, K_\alpha(\alpha), ..., K_\alpha^{m_1}(\alpha)\}.$$ 

The coordinates of the vertical sides of the boundary of $\Delta_\alpha$ (hence the abscissae of its corners) can instead be found in a slightly indirect way, also described in [23]: in order to explain it, let $(x, \alpha)$ and $(y, \alpha - 1)$ denote, respectively, the upper-right and lower-left corners of the attractor $\Delta_\alpha$.

1. The highest horizontal segment which delimits $\Delta_\alpha$ has endpoints $(y_{y - 1}, \alpha)$ and $(x, \alpha)$, while the lowest one is the segment of endpoints $(y, \alpha - 1)$ and $(x_{y - 1}, \alpha - 1)$;
2. the horizontal segments which form the upper boundary of $\Delta_\alpha$ are images under $\Phi_\alpha$ of the segment of endpoints $(y_{y - 1}, \alpha)$ and $(x, \alpha)$, and similarly the
Figure 6. The attractor $\Delta_\alpha$ for $\alpha = 4/15$: the numbers and arrows indicate the dynamics of the horizontal boundary segments. Note that the vertical ordering of the horizontal segments follows the ordering of the cyclic translates of the corresponding Farey word (in this case, $w = 00101$).

horizontal segments which bound $\Delta_\alpha$ from below are images under $\Phi_\alpha$ of
the segment of endpoints $(y, \alpha - 1)$ and $(\frac{1}{\alpha}, \alpha - 1)$;

(3) the values of $x$ and $y$ are determined by asking that the projection of the
horizontal segments bounding $\Delta_\alpha$ from above (resp. below) project to
adjacent segments; it turns out that it is enough to check this condition on
a couple of adjacent levels on the top and on the bottom, and this boils
down to an algebraic relation which only depends on the symbolic orbit of
$\alpha$ and $\alpha - 1$. In particular, it is enough to ask that the right endpoint of the
lowest level matches with the left endpoint of the level immediately above
it. Similarly, one needs to ask that the left endpoint of the highest level
matches with the right endpoint of the level immediately below it. That
is, if we let $\pi_1$ be the projection on the $x$-coordinate and $j_0, j_1$ be chosen
such that

$$K_{\alpha}^{j_1}(\alpha) = \max\{K_{\alpha}^j(\alpha) : 1 \leq j \leq m_1\}, \quad K_{\alpha}^{j_0}(\alpha - 1) = \min\{K_{\alpha}^j(\alpha - 1) : 1 \leq j \leq m_0\}$$
then, as a consequence of this discussion, the values \( x, y \) are determined by the following system:

\[
\begin{align*}
\text{ST}^{-1}\text{S}x &= \pi_1(\Phi_\alpha^0(y, \alpha)) \\
\text{ST}^{-1}\text{S}y &= \pi_1(\Phi_\alpha^j(x, \alpha))
\end{align*}
\]

Once \( x \) and \( y \) are known, then the other vertical levels are obtained by iterating \( \Phi_\alpha \) on \( x \) and \( y \); note that, as a consequence, the abscissae of the vertical segments depend only on the qumterval \( J_w \) and do not depend on the particular \( \alpha \) inside \( J_w \).

We shall now combine this recipe with the results of section 4 and find the following explicit formulas for \( x \) and \( y \).

**Proposition 6.2.** Let \( J_w \subset [0, 1/2] \) be a qumterval and \( RL(w) = (a_1, \ldots, a_n, 1); \) then \( x = [0; 1, a_n, \ldots, 1, a_1] \) and \(-y = [0; a_1, 1, \ldots, a_n, 1].\)

**Proof.** In general if \((a_1, 1, \ldots, a_1)\) and \((a_{\ell+1}, 1, \ldots, a_n, 1)\) is the splitting of \((a_1, 1, \ldots, a_n, 1)\) which corresponds to the standard factorization of \( w \) then by Proposition 4.5 \( j_0 = \sum_{i=1}^n a_i, \) \( j_1 = n - \ell \) and (46) becomes

\[
\begin{align*}
\text{ST}^{-1}\text{S}x &= \text{ST}^{-3} (\text{ST}^{-2})^{a_{\ell-1}} \ldots \text{ST}^{-3} (\text{ST}^{-2})^{a_1-1} y \\
\text{ST}^{-1}\text{S}y &= \text{ST}^{-3} (\text{ST}^{-2})^{a_{\ell-1}+2} \ldots \text{ST}^{-3} (\text{ST}^{-2})^{a_1+1} x
\end{align*}
\]

Let us point out that, by Lemma 3.1(ii), \((a_1 - 1, a_2, \ldots, a_n)\) is palindrome, thus \((a_{n-\ell}, a_{n-\ell-1}, \ldots, a_2, a_1 - 1) = (a_{\ell+1}, \ldots, a_n); \) on the other hand, since \((a_{\ell+1}, \ldots, a_n)\) has Farey structure as well, Lemma 3.1 implies that

\((a_{n-\ell}, a_{n-\ell-1}, \ldots, a_2, a_1 - 1) = (a_{\ell+1}, \ldots, a_n) = (a_n + 1, a_{n-1}, \ldots, a_{\ell+2}, a_{\ell+1} - 1); \)

therefore the first equation of (47) can be written as

\[
\text{ST}^{-1}\text{S}y = \text{ST}^{a_n+3} \ldots \text{ST}^{a_{\ell+2}+2} \text{ST}^{a_{\ell+1}+1} x.
\]

Note that by applying the equality \( \text{TST} = \text{ST}^{-1}\text{S} \) one gets for each \( k \)

\[
\text{T}^{k+1}\text{ST} = \text{T}^{k}\text{TST} = \text{T}^{k}\text{ST}^{-1}\text{S}
\]

hence by applying this identity to each block on the right-hand side of (48) we get

\[
y = \text{ST}^{a_n+1} \ldots \text{ST}^{a_{\ell+1}} x.
\]

Similarly, in order to modify the second equation of (47), we note that by leveraging the elementary identity \( \text{T}^{-1}\text{ST}^{-1} = \text{ST} \) we get for each \( k \) the equality

\[
\text{T}^{-1}(\text{ST}^{-2})^k = (\text{T}^{-1}\text{ST}^{-1})^k \text{T}^{-1} = (\text{ST})^k \text{T}^{-1} = \text{ST}^k \text{ST}^{-1}.
\]

Now, if we apply it to each block on the right-hand side of the second line of (47), we get the equation

\[
x = \text{ST}^{-1}\text{ST}^{a_{\ell+1}} \ldots \text{ST}^{-1}\text{ST}^{a_1-1} y.
\]

We will just prove the claim for \( y, \) the other case following in the same way. By putting together (49) and (50), one finds that \( y \) satisfies the fixed point equation

\[
y = G y
\]

where

\[
G = \text{ST}^{a_n+1} \ldots \text{ST}^{a_{\ell+1}} \ldots \text{ST}^{a_2} \text{ST}^{-1}\text{ST}^{a_1-1} \text{ST}^{-1}
\]

Again, since \((a_1, \ldots, a_n)\) has Farey structure, we can use Lemma 3.1(iii) to infer that \((a_n + 1, a_{n-1}, \ldots, a_2, a_1 - 1) = (a_1, \ldots, a_n),\) hence \( G \) can be expressed as

\[
G = \text{ST}^{a_1} \ldots \text{ST}^{-1}\text{ST}^{a_2} \ldots \text{ST}^{-1}\text{ST}^{a_n} \text{ST}^{-1}
\]
Recalling properties (35) we can check that setting
\[ \hat{G} := ST^{-a_1} ST ST^{-a_2} ST \ldots ST^{-a_{n-1}} ST ST^{-a_n} ST \]
one gets that \( -y = \hat{G}(-y) \); on the other hand it is immediate to check that \( \hat{G} \)
coincides with the string action induced by \( Z = (a_1, 1, \ldots, a_n, 1) \), and since \(-y > 0\) we get \(-y = [0; a_1, 1, \ldots, a_n, 1] \). □

For instance, in the case \( \alpha = \frac{1}{N+1} \) we get \( j_0 = N, j_1 = 1 \), so (46) reads
\[ STS y = ST^{N+1} x, \quad ST^{-1} S x = (ST^{-2})^N y, \]
and a simple computation yields \( x = [0; 1, N], -y = [0; N, 1] \).

The map \( \Phi_\alpha \) admits the invariant density \( (1 + xy)^{-2} dx dy \); and it is then easy to check that the a.c.i.p. for \( K_\alpha \) is \( d\mu_\alpha = \rho_\alpha(t) dt \) with invariant density
\[ \rho_\alpha(t) := \left( \frac{\int_{\Delta_\alpha \cap \{ y = t \}} (1 + xy)^{-2} dx}{\int_{\Delta_\alpha} (1 + xy)^{-2} dx dy} \right). \]

Moreover, for each \( \alpha \) the following formula holds (see [24]):
\[ h(\alpha) \int_{\Delta_\alpha} \frac{dx dy}{(1 + xy)^2} = \frac{\pi^2}{3}. \]

6.2. Consequences. Formula (52) says that, instead of studying the behaviour of the entropy, we may just study the function \( \alpha \mapsto A_\alpha := \int_{\Delta_\alpha} \frac{dx dy}{(1 + xy)^2} \), and the explicit description of \( \Delta_\alpha \) provides us with an effective tool to do it.

Proof of Theorem 1.1. The function \( \alpha \mapsto A_\alpha \) is smooth on \( J_w \), since the levels of the vertical segments which bound \( \Delta_\alpha \) are the same for all \( \alpha \in J_w \), while the levels of the horizontal segments vary analytically with \( \alpha \). Thus, by equation (52) the function \( \alpha \mapsto h(\alpha) \) is smooth as well on each qunterval.

In order to prove the second claim, let us prove that the invariant densities \( \rho_\alpha \) are locally bounded from below. In order to do so, let \( \alpha \notin \mathcal{E}_{KU}, \) and \( J_w = (\alpha^-, \alpha^+) \) be the qunterval to which \( \alpha \) belongs. Now, by formula (21) and Proposition 6.2 we have that
\[ 1 - \alpha^- = [0; 7] = [0; 1, a_n, \ldots, 1, a_1] = x; \]
on the other hand, recall that by the discussion in section 6.1 the right endpoint of the lowest horizontal boundary in \( \Delta_\alpha \) has absissa
\[ x_0 = \frac{x}{x + 1} = \frac{1 - \alpha^-}{2 - \alpha^-} \geq \frac{1}{3}, \]
since \( \alpha^- \leq \alpha \in [0, 1/2] \), therefore the following inclusion holds:
\[ \Delta_\alpha \supset [0, 1/3] \times [\alpha - 1, \alpha]. \]

As a consequence, we can bound the invariant density \( \rho_\alpha \) by writing for each \( t \in [\alpha - 1, \alpha] \)
\[ \int_{\Delta_\alpha \cap \{ y = t \}} (1 + xy)^{-2} dx \geq \int_0^{1/3} (1 + xt)^{-2} dx \geq \frac{1}{3} \cdot 2^{-2} \]
from which, using (51) and (52), it immediately follows that \( \rho_\alpha(t) \geq \frac{1}{|w|_0} = \frac{h(\alpha)}{2\pi t} \).

Now, by Proposition 5.2 the difference quotient of the entropy function \( h(\alpha) \) on quaternals is given in terms of \( \rho_\alpha \) and the difference \( |w|_0 - |w|_1 \):
\[
\frac{h(\alpha) - h(\alpha')}{\alpha - \alpha'} = (|w|_0 - |w|_1) h(\alpha') \frac{1}{\alpha - \alpha'} \int_0^t \rho_\alpha.
\]

Thus, by combining it with the previous lower bound we get for each \( \alpha \in J_w \)
\[
|h'(\alpha)| \geq C_\alpha |w|_0 - |w|_1
\]
where \( C_\alpha \) is bounded away from zero as long as \( \alpha \) is bounded away from 0 or 1.

Now, let us pick \( \alpha \in E_{KU}, \alpha \neq 0, 1 \): for any \( \alpha' \) sufficiently close to \( \alpha \) we have
\[
|h(\alpha') - h(\alpha)| = \sum_{J_w \subseteq [\alpha, \alpha']} \text{Var} h \geq \sum_{J_w \subseteq [\alpha, \alpha']} C ||w|_0 - |w|_1||J_w|
\]
hence, since \( E_{KU} \) has measure zero,
\[
|h(\alpha') - h(\alpha)| \geq C \inf_{J_w \subseteq [\alpha, \alpha']} ||w|_0 - |w|_1||\alpha - \alpha'|
\]

Let us first assume \( \alpha \neq g^2 \). Then by Lemma 2.13 the difference \( ||w|_0 - |w|_1|| \) tends to \( \infty \) as soon as \( \alpha' \) tends to some \( \alpha \in E_{KU} \), thus \( h \) is not differentiable (and not even Lipschitz continuous) at \( \alpha \).

Suppose instead \( \alpha = g^2 \) (the other case is analogous by symmetry). Then we know \( h \) is constant to the right of \( \alpha \); on the other hand, by equation (54) and the fact that \( ||w|_0 - |w|_1|| \geq 1 \) to the left of \( \alpha \), we get by the same reasoning as before that
\[
\liminf_{\alpha' \rightarrow \alpha^-} \frac{h(\alpha) - h(\alpha')}{\alpha - \alpha'} \geq C > 0
\]
hence the function \( h \) is not differentiable at \( \alpha \). Finally, since \( g^2 \) is an accumulation point of parameters in \( E_{KU} \) for which the derivative is unbounded, then \( h \) is also not locally Lipschitz at \( \alpha = g^2 \).

\[
\square
\]

In the same way, one can also use formula (52) to prove the following asymptotic estimate, which is analogous to the result obtained in [34] for the family of Nakada’s \( \alpha \)-continued fractions.

**Proposition 6.3.** The asymptotic behaviour of \( h \) at 0 is
\[
h(t) \sim \frac{\pi^2}{3 \log(1/t)} \quad \text{as} \quad t \to 0^+,
\]
hence \( \lim_{t \to 0^+} h(t) = 0 \), and \( h \) is not locally Hölder continuous at 0.

**Proof.** We shall use formula (52) and prove the asymptotic estimate (55) simply checking that
\[
A_\alpha = \log(1/\alpha) + O(1) \quad \text{as} \quad \alpha \to 0^+.
\]
By Theorem 1.2 we know that \( \alpha \rightarrow A_\alpha \) is decreasing on \([0, g^2]\), therefore it is enough to prove (56) for \( \alpha = \frac{1}{N+1} \) with \( N \) a positive, even integer.

Following the recipe of [23] described earlier in this section we see that, for \( \alpha = \frac{1}{N+1} \), the attractor \( \Delta_\alpha \) has a very simple structure, which can be completely described. In particular, it is not difficult to check that the left endpoint of the lowest horizontal boundary of \( \Delta_\alpha \) has coordinates \((0; 2, N+1), -\frac{N}{N+1}\); the other
Figure 7. The attractor $\Delta_\alpha$ for $\alpha = 1/5$: the inner and outer rectangles give the lower and upper bounds for the entropy as in the proof of Proposition 6.3.

lower boundaries are obtained from the lowest applying the function $\Phi_\alpha$, so that the lower-right corners of the attractor are the points $(x_k, y_k) := \Phi_k^\alpha(x_0, y_0)$ with

$$x_k = [0; 1, k, 1, N]$$
$$y_k = -\frac{N-k+1}{N-k+1}$$

for $1 < k < N$. Now, if we pick an even value $N = 2h$, we get that $-y_h < x_h$, hence the attractor contains the square of coordinates $[0, -y_h] \times [y_h, 0]$. Integrating the invariant density $(1 + xy)^{-2}dxdy$ on this square we get the lower bound for the measure of the attractor

$$A_{1/(N+1)} \geq \log(N) - \log(4).$$

On the other hand, for the upper bound we note that, using the notation of section 6.1 we have for each $\alpha \notin E_{KU}$ the inclusion $\Delta_\alpha \subseteq [y, x] \times [\alpha - 1, \alpha]$: then by taking $\alpha = 1/N$ and using Proposition 6.2 one gets that the attractor $\Delta_{1/N}$ is contained in the rectangle $[-1/(N-1), 1-1/(N+2)] \times [1/N - 1, 1/N]$, which leads to the upper bound $A_{1/N} \leq \log(N) + O(1)$ as $N \to +\infty$. This, together with the previous inequality, proves (56). □

6.3. Comparison with Nakada’s $\alpha$-continued fractions and open questions. Let us remark that the study of the entropy $h_N$ in the case of the family $(T_\alpha)$ of $\alpha$-continued fractions of Nakada is indeed much more complicated than the case examined in this paper. Actually many statements that we proved before should hold also for the family $(T_\alpha)$, but proofs are missing.

The structure of the matching set for $\alpha$-continued fractions is quite well understood ([10], [6]), but in this case matching intervals with different monotonic behaviours are mixed up in a complicated way ([12]), so even the fact that the entropy $h_N$ attains its maximum value at $1/2$ is still conjectural.

Another feature which is still unproved is the smoothness of entropy on matching intervals. This is due to the fact that the natural extension has no finite rectangular
structure when \( \alpha \) ranges in a matching interval (see [27]). We conjecture that, as in the case we examined in this paper, on a matching interval densities are piecewise continuous, with discontinuity points located on the forward images of the endpoints (before matching occurs), while the branches of these densities are fractional transformation which move smoothly with the parameter (see also [9], Conj. 5.3).

**Appendix**

We shall now give the proofs of a few technical lemmas we postponed in the main body of the article.

**Proof of Lemma 2.7** Let us denote \( r = \frac{p}{q} \), \( r' = \frac{p'}{q} \) with \( (p, q) = (p', q') = 1 \), and \( r'' := r \oplus r' = \frac{p+p'}{q+q'} \). The claim follows immediately from the two facts that for each \( 0 \leq k \leq q \) we have
\[ [kr''] = [kr] \]
and for \( 0 \leq k \leq q' \) we have
\[ [(k+q)r''] = p + [kr']. \]

To prove the first fact, let us note that, using the fact that \( r \) and \( r' \) form a Farey pair, we have for each \( 0 \leq k \leq q \)
\[ 0 \leq kr'' - kr \leq \frac{1}{q+q'} \]
thus, from the definition of integer part and the fact that \( r \) has denominator \( q \), we get
\[ [kr] \leq kr'' \leq kr + \frac{1}{q+q'} \leq [kr] + \frac{q-1}{q} + \frac{1}{q+q'} < [kr] + 1 \]
hence \( [kr''] = [kr] \). To prove the second fact, let us note that for each \( 0 \leq k \leq q' \) we have
\[
0 \leq (k + q)r'' - (kr' + p) \leq \frac{1}{q + q'}
\]
thus, similarly as before, we get
\[
[kr''] + p \leq (k + q)r'' \leq kr' + p + \frac{1}{q + q'} \leq [kr'] + p + \frac{1}{q + q'} \leq [kr''] + p + 1
\]
so \( [(k + q)r''] = [kr'] + p \).

**Proof of Proposition 2.14.**

(1) Let \( a^+ = 0.\overline{w} \), where \( w \) is a Farey word. Then by Proposition 2.8(e), (f) and (h) one has for each \( k \geq 0 \)
\[
w \leq \tau^k w \leq t^k w < \vee w
\]
which, passing from binary words to real numbers, yields \( a^+ \in \mathcal{E}_B \). Similarly, by Proposition 2.8(e), (f) and (h) we have
\[
\vee (\overline{t^k w}) < w \leq \tau^k w \leq t^k w
\]
which passing to real numbers yields
\[
b^- - \frac{1}{2} \leq D^k(b^-) \leq b^- \quad \forall k \in \mathbb{N}
\]
and using that \( a^- = b^- - \frac{1}{2} \) yields the claim.

(2) Let \( x \in I_w \) with \( n := |w| \), and define \( a_0 := 0.\overline{w} \) the dyadic rational with \( w \) as its (finite) binary expansion. Then the map \( D^n \) is an orientation-preserving homeomorphism between the intervals \([a_0, a^+]\) and \([0, a^+]\), hence if \( x \in [a_0, a^+] \) we have
\[
0 \leq \{2^nx\} \leq a^+
\]
and using that \( a^- = a^+ - \frac{1}{2} \) yields the claim.

(3) Note that every non-degenerate Farey word \( w \) is of the form \( w = 0.v \) for some (possibly empty) binary word \( v \). Then, using the fact that \( \overline{t^k w} = \vee w \overline{v} \) (Proposition 2.8(c)), we have that \( a^- \) has a binary expansion \( a^- = 0.\overline{v}1010 \), hence by comparing it with the expansion of \( a^+ = 0.\overline{w} \) we get
\[
a^+ - a^- = 0.0011 = \frac{1}{2(2^N + 1 - 1)}
\]
with \( N = |v| + 1 = n - 1 \), which proves the claim.

(4) Note that by summing up all the lengths of all the intervals \( I_w \) and using (3) we have:
\[
\sum_{0 < p < q} |I_{W_{p/q}}| = \frac{1}{2} \sum_{0 < p < q} \frac{1}{2p - 1} = \frac{1}{2} \sum_{0 < p < q} \sum_{k=1}^{\infty} 2^{-pk} = \frac{1}{2} \sum_{0 < p < q} 2^{-Q} = \frac{1}{2} \sum_{Q=2}^{\infty} (Q - 1)2^{-Q} = \frac{1}{2}
\]
and by setting \( P := pk \) and \( Q := qk \) (note that \( k = (P, Q) \)) we get
\[
= \frac{1}{2} \sum_{0 < P < Q} 2^{-Q} = \frac{1}{2} \sum_{Q=2}^{\infty} (Q - 1)2^{-Q} = \frac{1}{2}
\]

(5) As in the proof of Proposition 3.4 it is enough to check that the \( \zeta \)-function
\[
\sum_{w \in \mathcal{F}_w} |I_w|^s
\]
converges for each \( s > 0 \). From (3) we have that the above series equals
\[
\sum_{n=1}^{\infty} \phi(n)(2^n - 1)^{-s} \leq \sum_{n=1}^{\infty} n(2^n - 1)^{-s} < \infty
\]
(where \( \phi(n) \) is the Euler function) hence the claim is proven.

\[\square\]

**Proof of Proposition 2.3.** Since the claim is true for Farey words of low order, let us prove it by induction. Assume the claim is true for all elements of \( F_n \) for \( n \geq 1 \), and let us consider \( w \in F_{n+1} \setminus F_n \) such that \( w = w'w'' \) with \( w', w'' \in FW \). Without loss of generality we may assume that \( w \in FW_0 \). Now, note that by construction no element of \( FW_0 \) contains consecutive 1s, while all non-degenerate elements of \( FW_1 \) contain consecutive 1s: for this reason, we have \( w' \in FW_0 \) and \( w'' \in FW_0 \cup \{ 01 \} \). Therefore there are \( v', v'' \in FW \) such that \( w' = v' \star U_0 \) and \( w'' = v'' \star U_0 \), and so we can write \( w = (v'v'') \star U_0 \). On the other hand \( v = v'v'' \in F_n \), hence by inductive hypothesis \( v = v'v'' \) is the standard factorization of \( v \), hence \( w = w'w'' \) is the standard factorization of \( w \). \[\square\]

**Proof of Lemma 5.2.** Since the proof follows closely the same strategy as in [34] we give here just a sketch of the main steps. The idea is based on comparing the return times of \( K_\alpha \) on the intervals \([\alpha' - 1, \alpha - 1]\) and \([\alpha', \alpha]\): by the ergodic theorem, these give information on how the invariant measure changes.

Let \( J_w \) be the quintile interval labelled by the Farey word \( w \) and let \( \alpha, \alpha' \in J_w \) be such that either
\[
r \leq \alpha' \leq \alpha < \alpha^+, \quad S \cdot \alpha' > \alpha
\]
or
\[
\alpha^- < \alpha' < \alpha < r, \quad \forall S \cdot \sigma(\alpha) < \alpha'.
\]
Then, for every \( x \in [\alpha', \alpha] \) there exist two increasing sequences (visiting times) \( (n_0(k))_{k \in \mathbb{N}}, (n_1(k))_{k \in \mathbb{N}} \) such that
\[\begin{align*}
(1) & \quad n_0(k) \text{ and } n_1(k) \text{ are } k\text{-return times on } (\alpha' - 1, \alpha - 1) \text{ and } (\alpha', \alpha), \text{ respectively:} \\
& \quad K_{\alpha}^n(x - 1) \in (\alpha' - 1, \alpha - 1) \iff n = n_0(k) \text{ for some } k \in \mathbb{N} \\
& \quad K_{\alpha}^n(x) \in (\alpha', \alpha) \iff n = n_1(k) \text{ for some } k \in \mathbb{N}
\end{align*}\]
\[\text{(2) although the return times may depend on } x, \text{ their difference just depends on } w: \quad n_0(k) - n_1(k) = k(|w|_0 - |w|_1).
\]
\[\text{(3) the matching property induces a synchronization of } k\text{-returns:}
\]
\[\begin{align*}
K_{\alpha'}^{n_0(k)}(x - 1) &= K_{\alpha}^{n_1(k)}(x - 1) - 1 \\
\text{T}M_{\alpha', x - 1, n_0(k)} &= M_{\alpha, x, n_1(k)} T
\end{align*}\]
\[\text{(4) Just before the } k\text{-th return the two orbits are together:}
\]
\[K_{\alpha'}^{n_0(k)}(x - 1) = K_{\alpha}^{n_1(k) - 1}(x - 1), \quad \text{T}M_{\alpha', x - 1, n_0(k) - 1} = M_{\alpha, x, n_1(k) - 1}.
\]
It is now possible to choose \( x \in (\alpha', \alpha) \) such that both the following conditions hold:
\[\begin{align*}
(\text{a) } & \quad x \text{ is a typical point for } K_\alpha, \text{ namely} \\
& \quad \lim_{x \to +\infty} \frac{1}{n} \# \{ i < n : K_\alpha^i(x) \in (\alpha', \alpha) \} = \mu_\alpha([\alpha', \alpha]) \\
& \quad \lim_{x \to +\infty} \frac{1}{n} \log q_{n, \alpha}(x) = h(\alpha).
\end{align*}\]
(b) $x - 1$ is typical for $K_{\alpha'}$, that is:

\[
\lim_{x \to +\infty} \frac{1}{n} \# \{ j < n : K_{\alpha'}^j (x - 1) \in (\alpha' - 1, \alpha - 1) \} = \mu_{\alpha'}([\alpha' - 1, \alpha - 1]);
\]

\[
\lim_{x \to +\infty} \frac{1}{n} \log q_{n,\alpha'} (x - 1) = h(\alpha')
\]

Therefore, on one hand we have

\[
\lim_{k \to +\infty} \frac{k}{n_0 (k)} = \mu_{\alpha'}([\alpha' - 1, \alpha - 1]), \quad \lim_{k \to +\infty} \frac{k}{n_1 (k)} = \mu_\alpha([\alpha', \alpha]),
\]

which implies by taking the quotient and using (2)

\[
\lim_{k \to +\infty} \frac{n_0 (k)}{n_1 (k)} = \lim_{k \to +\infty} \frac{1}{n_1 (k)} \log q_{n_0 (k) - 1, \alpha} (x) = 1 + \left( |w|_0 - |w|_1 \right) \mu_\alpha([\alpha', \alpha])h(\alpha').
\]

Then, putting everything together we get

\[
h(\alpha) = \lim_{k \to +\infty} \frac{2}{n_1 (k)} \log q_{n_1 (k) - 1, \alpha} (x) = \lim_{k \to +\infty} \frac{n_0 (k) - 1}{n_1 (k) - 1} \log q_{n_0 (k) - 1, \alpha'} (x - 1) = (1 + \left( |w|_0 - |w|_1 \right) \mu_\alpha([\alpha', \alpha])) h(\alpha')
\]

which proves the claim. □

Proof of Lemma 5.5. We may assume, without loss of generality, that $I = [0, 1]$ and that the intervals are indexed in decreasing size: $|I_1| \geq |I_2| \geq \ldots$. By definition the upper box-dimension of $G$ is given by

\[
B \text{ dim } G := \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}
\]

where $N(\epsilon)$ is the minimum cardinality of a cover of $G$ with intervals of diameter less or equal than $\epsilon$. Let us now define

\[
M(\epsilon) := \sup \{ i : |I_i| \geq \epsilon \}.
\]

Notice now that any cover of $[0, 1] \setminus \bigcup_{i=1}^{M(\epsilon)} I_i$ with intervals of diameter less than $\epsilon$ necessarily must have cardinality at least $M(\epsilon) + 1$, because any such interval intersects at most one connected component. Hence

\[
M(\epsilon) \leq N(\epsilon).
\]

If we now fix $\delta \in (\delta_0, \eta)$, by (59) there is some $C > 0$ such that

\[
M(\epsilon) \leq N(\epsilon) \leq C \epsilon^{-\delta} \quad \forall \epsilon \leq 1.
\]

Now

\[
\sum_{M(\epsilon)/2^k + 1}^{M(\epsilon/2^{k+1})} |I_i|^\eta \leq \left( \frac{\epsilon}{2^k} \right)^\eta M(\epsilon/2^{k+1}) \leq C 2^k \epsilon^{\eta - \delta} 2^{k(\delta - \eta)}
\]

hence summing over $k \geq 0$

\[
\sum_{M(\epsilon)+1}^{\infty} |I_i|^\eta \leq \epsilon^{\eta - \delta} \frac{C 2^\delta}{1 - 2^{\delta - \eta}}.
\]
Now, given any subsequence \( J_1, J_2, \ldots \), if we set \( \epsilon := \sum_{i=1}^{\infty} |J_i| \), then all elements in the subsequence have length smaller than \( \epsilon \), hence
\[
\sum_{i=1}^{\infty} |J_i|^p \leq \sum_{M(\epsilon)+1}^{\infty} |I_i|^p \leq \left( \sum_{i=1}^{\infty} |J_i| \right)^{\eta-\delta} \frac{C^2 \delta}{1 - 2^{\delta-\eta}}.
\]

Proof of Proposition 3.3. Let us pick \( r' \in (0, 1) \cap \mathbb{Q} \), since \( \mathbb{Q} \cap E_{KU} = \{0, 1\} \) then \( r' \notin E_{KU} \). Therefore there is \( w \in FW^* \) such that \( r' \in J_w \), and Proposition 3.3 will be proved once we prove that
\[
(60) \quad r' \in J_w \Rightarrow \hat{J}_{r'} \subset J_w.
\]
Now, let \( S := RL(w) \) and \( r := [0; S] \) be the pseudocenter of \( J_w \) (so that, by virtue of equation (20), \( \hat{J}_r = J_w \)). Let us first assume that \( r' > r \): this means that \( r' = [0; ST] \) with \( |T| \) even. Since \( S' \ll S \), it is clear that the left endpoint \( \sigma \beta(\sigma r') \) of \( J_{r'} \) belongs to \( J_r \). To prove that the right endpoint \( \beta(r') \) satisfies \( \beta(r') \leq \beta(r) \), let us set \( m := \max \{ j : |S_j| \leq |T| \} \). Then, either (A) \( T \ll S^{m+1} \) or (B) \( T = S^m P \) with \( S = PZ \) (i.e. \( T \) is prefix of \( S^{m+1} \)): We claim that in both cases \( \beta(r') = [0; \overline{ST}] \leq [0; \overline{S}] = \beta(r) \). In case (A) this claim is trivial, and the same is true in case (B) if \( P = S \); on the other hand, in the case when \( P \) is a proper prefix of \( S \), by virtue of Lemma 3.1-(3) one gets that \( PZ < ZP \) so that using the Lemma of eq. (14)
\[
PPZ < PZP \iff S^{m+1} PS < SS^{m+1} P \iff [0; \overline{ST}] = [0; \overline{S^{m+1}P}] < [0; \overline{S}]
\]
completing the proof of the inclusion \( \hat{J}_{r'} \subset \hat{J}_r = J_w \).

On the other hand, if \( r' < r \), then \( \sigma(r') > \sigma(r) \) and \( \sigma(r') \in \hat{J}_{\sigma r} = J_{\bar{t}_w} \), so the previous case implies
\[
\hat{J}_{r'} = \sigma(\hat{J}_{\sigma r'}) \subset \sigma(\hat{J}_{\sigma r}) = \hat{J}_r
\]
and (60) is thus proven. \( \square \)

References

[1] P. Arnoux, T.A. Schmidt, Cross sections for geodesic flows and \( \alpha \)-continued fractions, Nonlinearity 26 (2013), no. 3, 711–726.
[2] J. Berstel, Sturmian and episturmian words (a survey of some recent results), in Algebraic informatics, Lecture Notes in Comput. Sci. 4728, Springer, Berlin, 2007.
[3] J. Berstel, A. Lauve, C. Reutenauer, F. Saliola, Combinatorics on Words: Christoffel Words and Repetition in Words, CRM monograph series 27, American Mathematical Society, Providence, 2008.
[4] G. Blé, External arguments and invariant measures for the quadratic family, Discrete Contin. Dyn. Syst. 11 (2004), no. 2-3, 241–260.
[5] E. Bombieri, Continued fractions and the Markoff tree, Expo. Math. 25 (2007), no. 3, 187–213.
[6] C. Bonanno, C. Carminati, S. Isola, G. Tiozzo, Dynamics of continued fractions and kneading sequences of unimodal maps, Discrete Contin. Dyn. Syst. 33 (2013), no. 4, 1313–1332.
[7] S. Bullett, J.-O. Lachaud, X. Provençal, C. Reutenauer, Lyndon + Christoffel = digitally convex, in Pattern Recognition 42 (2009), 2239–2246.
[8] S. Bullett, P. Sentenac, Ordered orbits of the shift, square roots, and the devil’s staircase, Math. Proc. Cambridge Philos. Soc. 115 (1994), no. 3, 451–481.
[9] C Carminati, S Marmi, A Profeti, G Tiozzo, The entropy of \( \alpha \)-continued fractions: numerical results, Nonlinearity 23 (2010) 2429–2456.
[10] C. Carminati, G. Tiozzo, A canonical thickening of Q and the entropy of α-continued fractions, Ergodic Theory Dynam. Systems 32 (2012), no. 4, 1249–1269.
[11] C. Carminati, G. Tiozzo, The bifurcation locus for the set of bounded type numbers, arXiv:1109.0516 [math.DS].
[12] C. Carminati, G. Tiozzo, Tuning and plateaux for the entropy of α-continued fractions, Nonlinearity 26 (2013), 1049–1070.
[13] R. Devaney, The Mandelbrot set, the Farey tree, and the Fibonacci sequence, Amer. Math. Monthly 106 (1999), no. 4, 289–302.
[14] K. Falconer, Fractal geometry: mathematical foundations and applications, Wiley, Chichester, 1990.
[15] N.P. Fogg, Substitutions in dynamics, arithmetics and combinatorics, edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel, Lecture Notes in Mathematics 1794, Springer-Verlag, Berlin, 2002.
[16] J. Gilman, L. Keen, Enumerating palindromes and primitives in rank two free groups., J. Algebra 332 (2011), 1–13.
[17] L.R. Goldberg, Fixed points of polynomial maps. I. Rotation subsets of the circles, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 6, 679–685.
[18] G.A. Hedlund, M. Morse, Symbolic dynamics II. Sturmian trajectories., Amer. J. Math. 62 (1940), 1-42.
[19] J.H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, in Topological methods in modern mathematics (Stony Brook, NY, 1991), 467–511, Publish or Perish, Houston, TX, 1993.
[20] J.H. Hubbard, C.T. Sparrow, The classification of topologically expansive Lorenz maps, Comm. Pure Appl. Math. 43 (1990), no. 4, 431–443.
[21] A. Hurwitz, Über eine besondere Art der Kettenbruch-Entwickelung reeller Grössen, Acta Math. 12 (1889), no. 1, 367–405.
[22] S. Katok, Coding of closed geodesics after Gauss and Morse, Geom. Dedicata 63 (1996), no. 2, 123–145.
[23] S. Katok, I. Ugarcovici, Structure of attractors for (a,b)-continued fraction transformations, J. Mod. Dyn. 4 (2010), no. 4, 637–691.
[24] S. Katok, I. Ugarcovici, Applications of (a,b)-continued fraction transformations, Ergodic Theory Dynam. Systems 32 (2012), no. 2, 755–777.
[25] L. Keen, C. Series, Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori, Topology 32 (1993), 719–749.
[26] C. Kraaikamp, A new class of continued fraction expansions, Acta Arith. 57 (1991), no. 1, 1–39.
[27] C. Kraaikamp, T. A. Schmidt, W. Steiner, Natural extensions and entropy of α-continued fractions, Nonlinearity 25 (2012), no. 8, 2207–2243.
[28] R. Labarca, C. G. Moreira, Essential dynamics for Lorenz maps on the real line and the lexicographical world, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 5, 683–694.
[29] M. Lothaire, Combinatorics on words, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1997.
[30] L. Luzzi, S. Marmi, On the entropy of Japanese continued fractions, Discrete Contin. Dyn. Syst. 20 (2008), 673–711.
[31] S. Mantaci, A. Restivo, M. Sciortino, Burrows-Wheeler transform and Sturmian words, Inform. Process. Lett. 86 (2003), 241–246.
[32] J. Milnor, Periodic orbits, external rays and the Mandelbrot set: an expository account, in Géométrie complexe et systèmes dynamiques (Orsay, 1995), Astérisque 261 (2000), xiii, 277–333.
[33] H. Nakada, Métrical theory for a class of continued fraction transformations and their natural extensions, Tokyo J. Math. 4 (1981), 399–426.
[34] H. Nakada, R. Natsu, The non-monotonicity of the entropy of α-continued fraction transformations, Nonlinearity 21 (2008), 1207–1225.
[35] R. Salem, On some singular monotonic functions which are strictly increasing, Trans. Amer. Math. Soc. 53 (1943), 427–439.
[36] G. Tiozzo, The entropy of α-continued fractions: analytical results, to appear on Ann. Sc. Norm. Super. Pisa Cl. Sci. (2014).
[37] G. Tiozzo, *Entropy, dimension and combinatorial moduli for one-dimensional dynamical systems*, Ph.D. thesis, Harvard University (2013), available at http://dash.harvard.edu/handle/1/11124847.

[38] D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer-Verlag, Berlin, 1981.

(Carlo Carminati) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
E-mail address, Carlo Carminati: carminat@dm.unipi.it

(Stefano Isola) Dipartimento di Matematica e Informatica, Università di Camerino, via Madonna delle Carceri, 62032 Camerino, Italy
E-mail address, Stefano Isola: stefano.isola@unicam.it

(Giulio Tiozzo) ICERM, Brown University, 121 South Main St, Providence RI 02903, USA
E-mail address, Giulio Tiozzo: Giulio_Tiozzo@brown.edu