3 Into 2 Doesn’t Go: 
(almost) chiral gauge theory 
on the lattice

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Kaplan recently proposed a novel lattice chiral gauge theory in which the bare theory is defined on $(2n+1)$-dimensions, but the continuum theory emerges in $2n$-dimensions. We explore whether the resulting theory reproduces all the features of continuum chiral gauge theory in the case of two-dimensional axial Schwinger model. We find that one can arrange for the two-dimensional perturbation expansion to be reproduced successfully. However, the theory fails to reproduce the 2-dimensional fermion number nonconservation.

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Research supported by NSF grants PHY90-21984, PHY89-04035, DOE grant DE-FG02-91ER40671, KOSEF-SRC and Ministry of Education.
1. Introduction

Under several reasonable assumptions to lattice gauge theory, such as translation-, gauge-, chiral-invariant, local and quadratic Hamiltonian, and an implicit assumption that the Higgs dynamics is unimportant, Nielsen and Ninomiya, and Karsten and Smit [1] proved that any attempt to put a chiral fermion on the lattice is afflicted with unremovable doubler fermions of opposite chirality.

Kaplan recently proposed a clever method [2] for defining chiral fermions on the lattice, which might evade the above Nielsen-Ninomiya No-Go theorem. His starting point is a $d = (2n+1)$-dimensional Wilson fermion with a Yukawa coupling to a $2n$-dimensional static topological defect. At low-energy below a threshold determined by Wilson mass and mass gap of the topological defect [3] the effective theory consists of massless chiral fermions trapped to the topological defect, while the ‘would-be’ doubling modes are naturally made heavy. The chiral zero mode currents are supplied by nontrivial $d$-dimensional lattice Chern-Simons currents [4] - a lattice version of the effect discovered by Callan and Harvey [5].

Kaplan’s proposal is not complete for defining a full-fledged chiral gauge theory on the lattice, however. One has to ensure that gauge field dynamics also reduces from the microscopic $d = (2n+1)$-dimensional theory to a low-energy $2n$-dimensional effective theory. Only if this ‘dynamical dimensional reduction’ is realized in a way compatible with the Kaplan’s lattice chiral fermions, a truly well-defined prescription of putting a chiral gauge theory on the lattice is accomplished. It is this issue we would like to address in this paper, in the simplest context of 3 into 2 dimensions and an abelian gauge theory.

This paper is organized as follows. In the next section, we briefly explain Kaplan’s proposal of lattice chiral fermion and anisotropic gauge theory in $2n + 1$ dimensions. In section 3, we recapitulate essential features of the two-dimensional axial Schwinger model. In section 4, the three-dimensional lattice gauge theory is shown to factorize, at the perturbative level, into a product of two two-dimensional chiral gauge theories at low energies: one on the wall and the other on the anti-wall. Infinite volume limit is then expected to isolate a chiral gauge theory induced on the wall or the anti-wall. Despite this success, in section 5, we argue that anomalous global symmetry such as fermion number violation is not reproduced and, in fact, completely suppressed in the infinite mass gap limit.
2. Kaplan’s Proposal for Lattice Chiral Fermion

Kaplan defines a lattice Dirac fermion starting from a one higher dimensions: \( d = (2n + 1) \)-dimensional Euclidean lattice \( x_\mu \equiv (x, x_{2n+1} \equiv s) \). In any concrete realization of his proposal, we must actually deal with a finite lattice with (periodic or antiperiodic) boundary conditions on the fields. The periodicity in the \( x \)-directions will not be of much concern to us, but the periodicity in the \( x_{2n+1} \)-direction does present several new features, which will be important to us. For concreteness, we will take \( s \in [-L, L] \), with periodic boundary conditions on the fields. The lattice action includes both a Wilson mass term and a \((2n+1)\)-th coordinate-dependent Dirac mass term:

\[
S_{\text{fermion}} = \sum_{x \in \mathbb{R}_d} \bar{\Psi}_x [K + M(s)] \Psi_x
\]

in which

\[
(K\Psi)_x \equiv \frac{1}{a} \sum_{\hat{\mu} = 1}^{2n+1} \frac{1}{2} \gamma^\mu (\Psi_{x+\hat{\mu}} - \Psi_{x-\hat{\mu}}) + ra(\Psi_{x+\hat{\mu}} - 2\Psi_x + \Psi_{x-\hat{\mu}}).
\]

When the \( x_{2n+1} \)-direction is uncompactified, we can simply take the position-dependent mass term to be of the form of a “kink” centered at \( s = 0 \)

\[
M(s) \equiv M_0(s) = \frac{\sinh(aM_d)}{a}(\theta(s) - \theta(-s)).
\]

However, in a finite volume, the mass term must be a periodic function of \( s \), \( M(s) = M(s + 2L) \), so we will take it to consist of a kink at \( s = 0 \), and an “anti-kink” at \( s = L \)

\[
M(s) = \frac{1}{2} (M_0(s) - M_0(s-L))
\]

It is, perhaps, most elegant to imagine that the position-dependent mass term \( M(s) \) arises dynamically from a Yukawa coupling to a scalar field whose vacuum expectation value is that of a (pair of) \( 2n \)-dimensional topological defect(s). However, adding a scalar field to the theory only complicates the dynamics without adding any new desirable features. It is simpler, and this is the path we will follow, to just put in the position-dependent mass by hand.

As in the continuum theory [3], the defects trap fermion zero modes. The expressions for the zero modes are easiest to write down in the absence of the Wilson term \((r = 0)\). On a periodic lattice,

\[
\Psi_{LH}(x, s) = Ce^{-M_d f(s)} \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix}, \quad \Psi_{RH}(x, s) = Ce^{-M_d f(s-L)} \begin{pmatrix} 0 \\ \chi(x) \end{pmatrix}
\]
where \( f(s) \) is the periodic sawtooth function, \( f(s) = (|s| - |s+L| + |s+2L| - L) \equiv f(s+2L) \) and
\[
C = \sqrt{M_d \left( 1 - e^{-2M_d L} \right)^{-1/2}}.
\]
On the infinite lattice, only one of these, \( \Psi_{LH}(x, s) \), is a normalizable zero mode, and \( f(s) = |s| \). We have written the fermions in a two-component notation in anticipation that these components will correspond to the chiral components of \( 2n \)-dimensional spinors. Note that the chirality of the zero modes is correlated with the sign of the mass term \( M(s) \).

There are also zero modes of the opposite chirality, with momenta near \( \pi/a \), but turning on the Wilson term, \( r \neq 0 \), gives them large masses of order \( M_w \approx ar \left( \frac{\pi}{a} \right)^2 \sim r\pi^2/a \). Thus, in Kaplan’s proposal, one arranges that the doubler modes get large Wilson masses \([4]\), while leaving a chiral zero mode at the wall and a zero mode of the opposite chirality at the anti-wall. Of course, the finite volume theory is vector-like, as it must be. But we expect that in the continuum and infinite volume (\( L \to \infty \)) limits, the modes that are concentrated at the two walls will decouple from each other, leaving an effective \( 2n \)-dimensional chiral theory on each wall.

We would like to see if this scenario is still realized when one turns on dynamical gauge fields. Clearly, this is not going to be a trivial matter. We must ensure that

1) \( 2n \)-dimensional perturbation theory is successfully reproduced in the \( 2n+1 \)-dimensional theory.

2) The physics on the two walls is uncorrelated, \( i.e. \) the decoupling of the \( \psi \)’s and the \( \chi \)’s must persist even in the presence of dynamical gauge fields.

3) The nonperturbative physics of the \( 2n \)-dimensional chiral gauge theory is reproduced successfully, such as fermion number nonconservation.

Suppose we have \( n_L \) fermions with one sign of \( M(s) \) and \( n_R \) fermions with the other sign of \( M(s) \) in (2.1). The free action (2.1) then has a \( U(n_L) \times U(n_R) \) symmetry, and we can gauge a subgroups \( G \in U(n_L) \times U(n_R) \) by replacing
\[
(\mathbf{K}\Psi)_x \rightarrow (\hat{\mathbf{K}}\Psi)_x \equiv \frac{1}{a} \sum_{\mu=1}^{2n+1} \frac{1}{2} \gamma^\mu (U_\mu \Psi_{x+\hat{\mu}} - U_{-\hat{\mu}} \Psi_{x-\hat{\mu}}) + ra(U_\mu \Psi_{x+\hat{\mu}} - 2\Psi_x + U_{-\hat{\mu}} \Psi_{x-\hat{\mu}})
\]
where we have introduced the gauge field through the link variables \( U_\mu(x, s) \equiv e^{iaA_\mu(x, s)} \in G \).

Since the fermion action explicitly breaks the \( (2n+1) \)-dimensional (discrete) Euclidean symmetry down to a \( 2n \)-dimensional symmetry, there is no reason to assume that the
gauge action possesses the full \((2n + 1)\)-dimensional symmetry. Rather, we introduce an anisotropic generalization of the Wilson action:

\[
S_{\text{gauge}} = \sum_{x \in \mathbb{R}^d} \left[ \beta_{||}(a) \operatorname{Tr} \prod_{P_{x||}} U_\mu + 2\beta_{\perp}(a) \operatorname{Tr} \prod_{P_{x\perp}} U_\mu + \text{h.c.} \right].
\quad (2.7)
\]

with independent couplings for plaquettes which lie in \(2n\)-dimensional sub-lattices \((\beta_{||})\) and for plaquettes lying across \((2n + 1)\)-th dimension \((\beta_{\perp})\).

### 3. Two-Dimensional Axial Schwinger Model

As a concrete example, in the remainder of this paper, we consider a two-dimensional Pythagorean Schwinger model as the simplest chiral gauge theory. In Euclidean spacetime, the light cone coordinates are rotated into a complex coordinate \(z \equiv \tau + ix\) and its complex conjugate. The model consists of three chiral fermions: two left-handed \(\psi_r, \psi_s\) and one right-handed \(\psi_t\), and an abelian gauge field \(A_z, \bar{A}_\bar{z}\) with a field strength \(\mathcal{E} \equiv -2i(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)\). Subscripts to the chiral fermions denote electric charge assignments \(Q = e(r, s, t)\). We will take the charges to be commensurate, which while not essential to our arguments, is the situation which most resembles nonabelian gauge theory in 4 dimensions. We may then, without loss of generality, take the charges \(r, s, t\) to be integer-valued and ordered as \(0 < r < s < t\). The action is

\[
L = \frac{1}{2e^2} \mathcal{E}^2 + \bar{\psi}_r (\partial_z + rA_z) \psi_r + \bar{\psi}_s (\partial_z + sA_z) \psi_s + \bar{\psi}_t (\bar{\partial}_{\bar{z}} + t\bar{A}_{\bar{z}}) \psi_t.
\quad (3.1)
\]

The theory has a \(U(1)^3\) symmetry corresponding to independent phase rotations on the fermions. Denote the corresponding currents by

\[
J^r_z = \bar{\psi}_r \psi_r, \quad J^s_z = \bar{\psi}_s \psi_s, \quad J^t_{\bar{z}} = 0, \quad J^t_z = 0, \quad J^r_{\bar{z}} = 0, \quad J^s_{\bar{z}} = \bar{\psi}_t \psi_t.
\quad (3.2)
\]

One linear combination of these currents is simply the gauge current

\[
J_z = rJ^r_z + sJ^s_z, \quad J_{\bar{z}} = tJ^t_{\bar{z}}.
\]

The gauge current is nonanomalous, and the theory consistent, if and only if the charges satisfy the Pythagorean relation

\[
r^2 + s^2 = t^2.
\quad (3.3)
\]
The other two linear combinations correspond to global $U(1)$ symmetries and can be taken to be

\begin{align}
J_L^z &= s J_r^z - r J_s^z & J_F^z &= J_r^z + J_s^z \\
J_L^\bar{z} &= 0 & J_F^\bar{z} &= J_t^z.
\end{align}

(3.4)

The first is a symmetry of the quantum theory, for the current is nonanomalous

$$\partial_\bar{z} J_L^z = 0$$

The latter current, which is simply fermion number in this two-dimensional theory, is anomalous

$$2(\partial_\bar{z} J_F^z + \partial_z J_F^{\bar{z}}) = \frac{1}{\pi} (r + s - t) E$$

(3.5)

Anomalous nonconservation of the fermion number is a notable feature of the Pythagorean Schwinger model. It is the precise analog of the nonconservation of $B + L$ fermion number in the $SU(2)_L \times U(1)_Y$ standard model, as discovered by ‘t Hooft. The fermion number anomaly (3.5) is the two-dimensional analog of the $B + L$-number nonconservation by the weak instantons [6]. Fermion number nonconservation has proven to be a sensitive test of whether a lattice theory correctly reproduces the nonperturbative physics of chiral gauge theories like the standard model [7,8] and, as we shall see, the same is true here.

4. Three Dimensional Lattice Gauge Theory: Continuum & Infinite Volume Limits

Having discussed the two-dimensional chiral gauge theory we hope to reproduce, let us return to the corresponding 3-dimensional lattice theory. We are interested in exploring the continuum limit of

$$S = S_{\text{fermion}} + S_{\text{gauge}}$$

$$S_{\text{fermion}} = \sum_{x \in \mathbb{R}^d} \bar{\Psi}_x^r (\hat{K} + M(s)) \Psi_x^r + \bar{\Psi}_x^s (\hat{K} + M(s)) \Psi_x^s + \bar{\Psi}_x^t (\hat{K} - M(s)) \Psi_x^t$$

(4.1)

where $S_{\text{gauge}}$ is given by (2.7). This may seem somewhat formidable, but, at least in perturbation theory, there is an obvious simplification which we would like to exploit. One can view the 3-dimensional fermi field $\Psi$ as an infinite collection (on a finite lattice, actually a finite collection) of 2-dimensional fermi fields. Usually, this is not a very profitable point of view, but here we note that almost all of these fermions have masses of order $M_d$, so
if we are interested in physics well below this scale, we can integrate them out, retaining only the zero modes (2.3). In order to do this in a clean way in the interacting theory (4.1), it behooves us to partially gauge-fix (4.1), choosing \( A_3 = 0 \) gauge\(^1\). In this case, the separation of the light from the heavy fermion modes can be made cleanly, since the equation determining the zero modes is independent of the gauge field. The light fermions are again given by (2.5). Of course, by working in a particular gauge, our formulæ will not be manifestly invariant under the full 3-dimensional gauge symmetry, but we will demand manifest invariance under the group of residual gauge transformations – the group of 2-dimensional gauge transformations. So we will really be considering an effective theory of 2-dimensional fermions coupled to a 3-dimensional gauge field in \( A_3 = 0 \) gauge.

Actually, there is one more term in the gauge action which plays a crucial role in realizing the anomalies of the theory \([2]\), namely the 3-dimensional Chern-Simons term, which in \( A_3 = 0 \) gauge is simply

\[
\frac{c}{\pi} \int d^3 x \, \epsilon^{ij} A_i \partial_3 A_j
\]  

(4.2)

This term has a calculable coefficient in the low-energy effective theory \([2]\): \( c = r^2 + s^2 - t^2 \). It vanishes precisely when the 2-dimensional theory is anomaly-free (3.3). When the theory is anomalous, the nonzero Chern-Simons current

\[
\partial_3 J^{CS}_3 = -\frac{c}{\pi} \mathcal{E}
\]

compensates for the anomaly in the 2-dimensional gauge current, so that the full 3-dimensional current is conserved. Note that it is the Chern-Simons current, not the \( x_3 \)-component of the fermion current that cancels the anomaly. The fermions are massive off the wall, and massive degrees of freedom cannot carry off the charge (remember, the anomaly is a low-energy effect). Rather, it is the gauge-field, which can propagate off the wall, that, through the Chern-Simons term, carries off the charge. In a sense, the effective theory is not really 2-dimensional at all; massless degrees of freedom can propagate in the \( x_3 \)-direction, carrying off the charge. We can only hope to recover a truly 2-dimensional theory in the anomaly-free theory, in which the Pythagorean relation (3.3) is satisfied and the coefficient of the Chern-Simons vanishes.

\(^1\) This leaves a residual gauge-invariance – the ability to make \( x_3 \)-independent gauge transformations – which we will exploit in the following.
We will, henceforth, assume that (3.3) is satisfied. Also, since we are doing perturbation theory, we will adopt a continuum language to discuss our calculations. This is, of course, completely appropriate, since we are interested in probing the theory at length scales \( \gg 1/M_d \gg a \). Finally, we will first consider the case of a single wall \((L = \infty)\), returning to discuss the finite \( L \) corrections later.

Consider sticking a left-handed fermion zero mode

\[
\Psi(x, s) = \sqrt{M_d} \begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} e^{-M_d |s|}
\]

into the fermion action. We get

\[
\int ds \int d^2x \; M_d e^{-2M_d |s|} \bar{\psi}(x) (\partial_z + A_z(x, s)) \psi(x)
\]

Clearly the first term just gives the standard 2-dimensional free fermion action and leads to the standard 2-dimensional fermion propagator . The second term represents the coupling of the 2-dimensional fermion to the 3-dimensional gauge field. In momentum space, it is

\[
\int \frac{d^2q d^2k}{(2\pi)^4} \int \frac{dk_3}{2\pi} \bar{\psi}(q) A_z(k, k_3) \psi(-q - k) F(k_3)
\]

where

\[
F(k_3) = \left(1 + \frac{k_3}{2M_d} \right)^2
\]

This interaction is, of course, invariant under the residual gauge symmetry group of \( x_3 \)-independent (i.e. 2-dimensional) gauge transformations.

The action for the gauge field is simply

\[
\int ds \int d^2x \frac{1}{4e^2} F_{ij}^2 + \frac{1}{2e^2} (\partial_s A_i)^2
\]

which leads to the propagator (in covariant gauge)\(^2\)

\[
D_{ij}(k, k_3) = \left(\frac{1}{e^2} k^2 + \frac{1}{e^2} k_3^2 \right)^{-1} \begin{pmatrix} g_{ij} - (1 - \xi) \frac{k_i k_j}{k^2 + \xi \frac{e^2}{e_\perp^2} k_3^2} \end{pmatrix}
\]

\(^2\) Recall that we need to fix the residual 2-dimensional gauge invariance.
In a generic Feynman diagram for this theory, we encounter for every internal photon line a momentum integral of the form

\[
\ldots \int \frac{d^2k}{(2\pi)^2} \int \frac{dk_3}{2\pi} F(k_3)^2 \gamma^i D_{ij}(k, k_3) \gamma^j \ldots
\]

where \( F(k_3) \) is the form-factor \((4.7)\) in the interaction vertex \((4.4)\). If we had been doing the same Feynman diagram in the 2-dimensional theory, we would have encountered the momentum integral

\[
\ldots \int \frac{d^2k}{(2\pi)^2} \gamma^i \frac{e^2}{k^2} \left[ g_{ij} - (1 - \xi) \frac{k_i k_j}{k^2} \right] \gamma^j \ldots
\]

We can obtain agreement between these two expressions (up to corrections of order \( k^2/M_d^2 \) or \( e^2/M_d^2 \)) provided we take

\[
e^2 \parallel \ll M_d \ll e^2 \perp \ll 1/a \quad (4.8)
\]

and identify the 2-dimensional gauge coupling to be

\[
e^2 = e^2 \parallel M_d/2. \quad (4.9)
\]

With this hierarchy of mass scales we obtain complete agreement between the perturbation theory of this exotic effective field theory (probed at energies \( \ll M \)) and the genuinely 2-dimensional Pythagorean Schwinger model.

The 3-dimensional theory \((1.1)\) contained several dimensionful parameters: the physical size of extra dimension \( L \), the three-dimensional gauge couplings \( e_{\perp}, e_{\parallel} \), the domain wall mass scale \( M_d \), and the lattice spacing \( a \). We might have enquired at the outset, what combination of these dimensionful quantities is supposed to be identified as the 2-dimensional gauge coupling \( e^2 \)? Also, since the 2-dimensional theory has no dimensionless parameters (which could be formed by ratios of the above dimensionful ones), what is the hierarchy among these dimensionful quantities which is necessary to reproduce the 2-dimensional theory? Equations \((4.8)\) and \((4.9)\) provide the answers to these questions.

At this point, the attentive reader might object to our use of \((1.6)\) as our gauge action. After all, what we have done in obtaining \((1.4)\) and \((1.6)\), is truncate the original 3-dimensional theory by dropping the heavy fermion degrees of freedom. This is not the same as integrating them out. Integrating them out introduces corrections to the effective action for the gauge field \((1.6)\). We should show that these corrections do not (qualitatively) change our results.
The corrections result from doing the 1-loop diagram with 3-dimensional fermions (with the position-dependent mass term) running around the loop. This is slightly formidable, but some of the qualitative features can be seen from doing the same calculation for fermions with a constant mass. Each term in the effective action comes with a power of $1/M_d$ given by dimensional analysis. Terms in the effective action with more powers of the photon field $A_i$, therefore end up having their effects suppressed by powers of $k^2/M_d^2$. The only terms we actually have to worry about for our low-energy calculation involve two external photon lines, which correct the photon propagator. We have arranged our fermion charges so that the parity-violating part of the vacuum polarization graph cancels, and so the corrections to the photon propagator are all expressed in terms of a single scalar function $\pi((k^2 + k_3^2)/M_d^2)$. The point is that, for the low-energy processes we are considering, we always have $k^2 \ll M_d^2$. (Note that it is crucial here that we are working with a super-renormalizable theory, so that, even in internal lines, the photon momenta are “soft”.) Thus we can always neglect the $k^2$-dependence of the vacuum polarization and consider the function $\pi(k_3^2/M_d^2)$. We, of course, cannot neglect the $k_3$-dependence, because the typical photon $k_3$ momenta may be of order $M_d$. However, again, up to effects of order $k^2/M_d^2$, this correction to the photon propagator shifts the constant of proportionality between $e^2$ and $c_{11} M_d$ in (4.9) but does not qualitatively change our conclusions.

What changes when we consider finite $L$, and zero modes $\psi$ concentrated on the wall and $\chi$ concentrated on the anti-wall? Clearly now, a photon line can couple the fermions $\psi$ on one wall with the fermions $\chi$ on the other. The strength of the coupling is given by the overlap of the two zero mode wave functions (2.5)

$$\int_{-L}^{L} ds \ C e^{-M_d f(s)} \cdot C e^{-M_d f(s+L)}$$

(Previously, both of the wave functions were $\sqrt{M_d} e^{-M_d |s|}$, and the integral extended from $-\infty$ to $\infty$.) Otherwise, the computation is as before. The coupling between $\psi$’s and $\chi$’s is exponentially small for large $L$; it goes as $e^{-M_d L}$. Thus if we take care to make $L \gg 1/M_d$, the fermions on the two walls decouple, and, at least in perturbation theory, we have succeeded in reproducing the 2-dimensional chiral gauge theory.
5. Nonperturbative Fermion Number Nonconservation

Finally, we need to return to discuss the fate of the fermion-number current in the effective theory. We expect it not to be conserved. The anomaly equation (3.5) means that in the theory with dynamical gauge fields, there are real physical processes which violate fermion number.

But 3-dimensional lattice theory (4.1) has fermion number as an exact symmetry of the lattice action. Thus the 3-dimensional fermion number current must be exactly conserved. Have we arrived at a paradox?

At first, it appears that the situation resembles the case when the 2-dimensional gauge current is anomalous \((r^2 + s^2 \neq t^2)\). Charge is not conserved in the would-be 2-dimensional theory. But the effective theory has a nonzero Chern-Simons term, and the Chern-Simons current built out of the gauge-field carries off the charge, which is exactly conserved in the 3-dimensional theory. We conclude that the theory which wanted to be an anomalous 2-dimensional gauge theory really isn’t 2-dimensional at all. There are light degrees of freedom (built out of the gauge field) which can carry the charge off into the third dimension.

But in the nonanomalous theory \((r^2 + s^2 = t^2)\), the coefficient of the Chern-Simons term vanishes and the theory really is supposed to be 2-dimensional. There are no light degrees of freedom to carry the fermion number off the wall. What is going on?

The answer is, simply, a failure of decoupling. The situation is very similar to that uncovered by Banks & Dabholkar [9]. The “heavy” fermions that we integrated out to obtain the effective theory of 2-dimensional chiral fermions have zero modes in an instanton background. Though the fields themselves are massive, it is a mistake to integrate them out because these zero modes fail to decouple in an instanton background. If we insist on integrating them out, they come back to haunt us by inducing nonlocal terms in the resulting effective action. These terms restore the \(U(1)\) symmetry to an exact symmetry of the 2-dimensional theory.

A more enlightened (and local!) point of view is to say that because of this failure of decoupling, the theory isn’t really 2-dimensional at all on the nonperturbative level (i.e. when we allow large fluctuations of the gauge field). Because of their zero modes, it is simply incorrect to integrate out the “heavy” fermions, and the theory really is 3-dimensional.

The theory turns out to be 3-dimensional, not just when the gauge symmetry of the would-be 2-dimensional theory is anomalous, but when any of the global symmetries
present in the 3-dimensional lattice theory would be anomalous in the 2-dimensional chiral theory. Whenever that happens, there is a failure of decoupling which prevents us from integrating out the “massive” fermion degrees of freedom to obtain an effective 2-dimensional theory. When the anomalous symmetry is a gauge symmetry, the failure is seen in perturbation theory – due to the presence of a nonvanishing Chern-Simons term. When it is a global symmetry, the failure manifests itself only nonperturbatively – when “large” fluctuations of the gauge field are taken into account.

The dilemma faced by this theory is a stark one. Fermion number is an exact symmetry of the lattice action. This means that either there are light degrees of freedom that carry the fermion number off of the wall – in which case, it is inconsistent to think of the effective theory as 2-dimensional – or the fermion number symmetry is also an exact symmetry of the 2-dimensional theory. Either way, we do not obtain the physics that we want: a 2-dimensional theory with fermion number violation.

The lesson that we should draw from this analysis is that for proposal for putting chiral fermions on the lattice must not only correctly reproduce the gauge anomalies of the would-be chiral gauge theory, it must also reproduce the anomalies in whatever global symmetries may be present in the model (like $B+L$ in the standard model). This is a tall order, and so far, neither Kaplan’s model nor any other proposal \cite{kaplan} seems to be up to the task.

Acknowledgment

We are grateful to D. Kaplan whose seminar stimulated our interest to this problem, and to T. Banks, C.G. Callan, J. March-Russell, J. Preskill and F. Wilczek for useful discussions. We would also like to gratefully acknowledge the hospitality of the ITP at UC Santa Barbara, where this manuscript was written.
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