Thermodynamic phase transition for Quintessence Dyonic Anti de Sitter Black Holes

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Abstract

We study thermodynamic of a dyonic AdS black hole surrounded by quintessence dark energy where negative cosmological constant of AdS space behaves as pressure of the black hole. We choose grand canonical ensemble of the black hole where its magnetic charge $Q_M$ and electric potential $\Phi_E$ are hold as constant. Our goal in this work is study physical effects of the magnetic charge and electric potential on the thermodynamic phase transition of the black hole in presence of quintessence dark energy. When barotropic index of the quintessence is $\omega = -\frac{7}{9}$ we obtained that compressibility factor of the black hole reduces to $Z_c = \frac{3}{8}$ which corresponds to the Van der Waals fluid. We obtained analogy between the small/large black hole phase transition and liquid/gas phase transition of the Van der Waals fluid. Numerical calculations predict that the black hole may born plasma phase which is fourth different state of the matter which does not appear in the Van der Waals fluid.

1 Introduction

Thermodynamic aspects of black holes have challenge when the argument of thermodynamic phase transition is under consideration. This is because of association of the phase transition to the thermodynamic variables for instance entropy, volume and etc. One of most controversial thermodynamic variables is volume of black holes undoubtedly. Consider a flat space-time including a black hole and an observer located at distances far from the black hole which can not observe small region of space-time hidden behind the black hole horizon. In fact this region which is equal with geometric volume of the black hole doesn’t have an straightforward definition due to use of different gravity models. It is well known that nonzero value of the vacuum energy

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density of the de Sitter space is relating to the cosmological constant. In the latter case a typical definition for the volume is there: That is related to total energy of the gravitational system but not to the black hole mass amount which is stand as hidden from point of view of an observer far from the black hole (ADM mass \[1,2,3\]). This means that the hidden energy from point of view of observer would be part of total energy which is equal to the ADM mass located on hidden volume. In the thermodynamic of the black holes the observed mass by an observer far from the black hole is considered to be the black hole thermal energy while the cosmological constant behaves as a fixed parameter. For an anti de-Sitter space the cosmological constant has negative values arising from an energy-momentum tensor which reads the vacuum equation of state \[P = -\rho = -\frac{\Lambda}{8\pi G N}\]. Regarding the latter concept one can write first law of black holes thermodynamic as follows.

\[dM = TdS + VdP \implies dU = TdS - PdV \tag{1.1}\]

where \(U = M - PV\) and \(H = U + PV\) are internal energy and enthalpy of the black holes respectively. This shows the black hole mass \(M\) can interpreted as enthalpy. In this view the cosmological constant is related to pressure of the AdS space-time \[4\]. Since the cosmological constant corresponds to space-time pressure thus its conjugate variable must have volume dimension. This is called thermodynamic volume for which we have \(V = (\frac{\partial M}{\partial P})_S [5]\). In general, this definition of the black hole volume is different with the geometric volume which we pointed previously. This contains some universal properties which they satisfy the condition of reverse iso-perimetric inequality \[6\]. Regarding the above definition for the black hole thermodynamic volume and the pressure, one can study critical behavior of the black holes in the P-V frame and in the extended phase space \[7\] (see also \[8\] for charged and \[9\] for rotating black holes). For a dyonic black hole, authors of ref. \[10\] studied magnetic charge effects on T-V critically of the black holes via the holographic approach. They observed that in constant electric potential and constant magnetic charge, the phase diagram of ensemble of dyonic black hole is similar to well known phase transition of Van der Waals fluid done in presence of its internal chemical potential. They obtained ferromagnetic like behavior of boundary theory of the dyonic black hole when the external magnetic field vanishes. Authors of the work \[11\] studied phase structure of the quintessence Reissner-Nordström-AdS black hole with the nonlocal observables such as holographic entanglement entropy and two point correlation functions. Results of their work show that, as the case of the thermal entropy,
both the observables exhibit the similar Van der Waals-like phase transition. They were check the equal area law and critical exponent of the heat capacity for the first and the second order phase transition respectively. Also they discussed the effect of the parameter of state equation on the phase structure of the nonlocal observables. Hartnoll et al studied AdS/CFT superconductor properties of a planar AdS Schwarzschild black hole which for temperatures less than the critical temperature, reduces to a charged condensate state. This is formed via a second order phase transition for which the conductivity reaches to infinite value \[12,13, 14,15,16\]. The study of the phase transition of the dyonic black hole surrounded by dark energy must be very complex and attractive because of dependence to the possibility existence of the dark energy. In fact There is a wide range of modern cosmological observations which confirm that our universe expands with positive acceleration. This leads us to infer that instead of the Newton’s gravity force there should exist other forces which is not still observed \[17,18,19\]. Dark energy is one of the most important candidates to explain this situation, which must occupied seventy percent of energy of our universe. Among diverse dark energy models, ‘quintessence’ as a canonical scalar field can be a good model to explain acceleration of the universe \[20,21,22\]. There are many works where various gravitational models are used to study thermodynamic aspect of black holes (see for instance \[23-26\]).

In this work we study thermodynamics of the AdS dyonic black hole which is surrounded by the quintessence dark energy. We calculate its equation of state on the event horizon hypersurface where the cosmological constant behaves as the black hole pressure. Then for different values of barotropic index of the quintessence dark energy \(\omega\) we obtained parametric forms of critical points \(\{T_c, V_c, P_c\}\) defined in terms of the constant electric potential and magnetic charge and \(\omega\). Then we set \(\omega = -\frac{7}{9}\) to obtain numerical values of the critical points. Because for this particular value of the quintessence barotropic index the black hole compressibility factor reaches to one which is obtained for the Van der Waals fluid as \(Z = \frac{3}{8}\). This restriction helps us to improve statement which extract from the work. The organization of the work is as follows.

In section 2 we review briefly phase transition of a Van der Waals fluid. In section 3 we define a dyonic AdS black hole in presence of quintessence dark energy counterpart. Then we obtain its equation of state by solving equation of the event horizon. Enthalpy, heat capacity at constant pressure and Gibbs free energy are other thermodynamic variables of the black hole which we
calculated in this section. At last we plot variations of the these variables versus the volume and the temperature. Also we obtained all possible numerical values for the critical points at zero electric potential $\Phi_E = 0$ versus different values of the magnetic charge $q_M$. Section 4 denotes to results of the this work and outlooks.

2 Phase transition of Van der Waals fluid

Equation of state for an ideal gas has simple form as $PV = Nk_B T$ in which for a constant temperature (isotherm curves) the pressure decreases absolutely by increasing the volume. While this is not ready for a real 'Van der Waals' fluid. In the latter case, the equation of state given by the following form behaves complex [27].

$$P(T, V) = \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2},$$  \hspace{1cm} (2.1)

where $N$ is number of particles in the fluid and the constant $b$ is intended to correct for the volume occupied by the molecules and the term $\frac{aN^2}{V^2}$ is a correction that accounts for the intermolecular forces of attraction. In fact, these constants are evaluated by noting that the critical isotherm passes through a point of inflection at the critical point and that the slope is zero at this point. The critical point $\{T_c, V_c, P_c\}$ is determined by solving

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$$  \hspace{1cm} (2.2)

which read

$$V_c = 3bN, \quad P_c = \frac{a}{27b^2}, \quad k_BT_c = \frac{8a}{27b}$$  \hspace{1cm} (2.3)

and one can calculate compressibility dimensionless factor $Z_c$ as follows.

$$Z_c = \frac{P_cv_c}{Nk_BT_c} = \frac{3}{8}.$$  \hspace{1cm} (2.4)

Substituting the dimensionless thermodynamic variables $p = \frac{P}{P_c}, \quad v = \frac{V}{V_c}, \quad t = \frac{T}{T_c}$ one can infer the equation of state (2.1) reads to a dimensionless form as follows (see ref. [27]).

$$p(v, t) = \frac{8t}{3v-1} - \frac{3}{v^2}.$$  \hspace{1cm} (2.5)
Gibbs free energy difference for a Van der Waals fluid is given by

\[ dG = -SdT + VdP \]  \hspace{1cm} (2.6)

where \( S \) is entropy. For a constant temperature the first term in right hand side of the above equation is eliminated and so we obtain

\[ G = \int VdP = PV - \int P(V)dV \]  \hspace{1cm} (2.7)

which for the dimensionless equation of state (2.5) reads

\[ g(t, v) = \frac{8t}{3} \left[ \frac{3v}{3v-1} - \ln(3v-1) \right] - \frac{6}{v} \]  \hspace{1cm} (2.8)

Holding the temperature as constant, we plot diagrams for the equation of state (2.5) and for the Gibbs free energy (2.8) in figures 1-c and 1-b respectively. Here \( t = 0.5 < t_c, t = t_c = 1 \) and \( t = 1.5 > t_c \) correspond to dash, solid and dash-dot lines respectively. (1-c) shows for \( t < t_c \) there is a phase transition for the fluid which coexistence of two phase occurs at the minimum pressure. This point has minimum Gibbs free energy which is shown in figure (1-b). Figure (1-a) shows variation of the Gibbs free energy versus the temperature. It shows at all, the Van der Waals fluid has different phases which they cross each other at the phase transition point. This point corresponds to coexistence regime of both the gas and the fluid phases of the Van der Waals matter which occurs at small scale of thermodynamic volume. We obtained by numerical calculations that for \( p \geq p_c = 1 \) the diagrams of \( p - v, g - t \) and or \( g - v \) have not a minimum point in phases space (not shown) and so the phase transition dose not appeared in the latter case.

Our aim in this work is to study thermodynamic phase transition of a quintessence AdS dyonic black hole and obtain situations for which the black hole behaves thermodynamically as a Van der Waals fluid.

3 Quintessence Dyonic AdS black hole thermodynamic

According to the work presented by Kiselev in [23] for metric of a spherically symmetric static Reissner Nordström black hole surrounded by quintessence
Figure 1: Diagrams of Gibbs free energy and pressure of a Van der Waals fluid are plotted versus the temperature $g - t$ at (a) and against the volume $g - v$ and $p - v$ at (b) and (c) respectively by setting $p_c = 1$.

dark energy we use an extension of his metric solution where an extra magnetic charge $Q_M$ is added and so it is called as dyonic AdS quintessence black hole with the following metric form.

\[ ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \]

in which for quintessence regimes $-1 < \omega < -\frac{1}{3}$ we have

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q_M^2}{r^2} + \frac{Q_E^2}{r^2} - \left(\frac{r_q}{r}\right)^{1+3\omega} - \frac{\Lambda r^2}{3}, \]

where $r_q$ is the dimensional normalization constant and the negative cosmological constant $\Lambda < 0$ of the AdS space behaves as thermodynamic pressure of the black hole. Quintessence counterpart of the above metric solution $(r_q/r)^{1+3\omega}$ comes from leading order terms of a convergent series solution which is obtained originally by the Kiselev by applying the condition of linearity and additivity of the Einstein metric equation [23].

As we know the phase structure depends on the statistical ensemble which we choose to study the phase transition. There is no new interesting results if the black hole phase transition is studied by particular ensemble in which both of the electric and the magnetic charges are hold as constant.
In the contrary if we hold magnetic charge together with the electric potential as constant parameters then there is obtained possibly new physics from study of the phase transition. Former case is a canonical ensemble where a small/large black hole phase transition dose not appeared in absence of the quintessence while the latter setting become grand (mixed) canonical ensemble for which a small/large black hole phase transition occurs [10, 28]. However the former situation is similar to case where the electric charge is shifted by $Q_E^2 \rightarrow Q_E^2 + Q_M^2$ while in the latter case the first law of thermodynamics is described in extended phase space (see for instance [29]). To approach our goal in this work we extend the first law of the black hole thermodynamic given by the equation (2.12) in ref. [30] as follows.

$$dM = T dS + \Phi_E dQ_E + \Phi_M dQ_M + V dP + A d r_q$$

(3.3)

where $A$ is conjugate variable for the normalization factor $r_q$ of the quintessence dark energy. It is defined by

$$A = \left( \frac{\partial M}{\partial r_q} \right)_{S, Q_E, Q_M, r_q}$$

(3.4)

The Smarr relation is obtained from the mass function of the black hole because the black hole mass is equal to the enthalpy. Applying the black hole metric (3.2) and solving the horizon equation $f(r_h) = 0$ we obtain mass of the AdS quintessence dyonic black hole versus the thermodynamics variables as follows.

$$M = \frac{r_h^2}{2} + \frac{Q_M^2}{2r_h} + \frac{Q_E^2}{2r_h} - \frac{r_q^{1+3\omega}}{2r_h^{3\omega}} + \frac{4\pi r_h^3}{3} P$$

(3.5)

where we substitute pressure $P$ of the AdS space with corresponding radius $r_{AdS}$ as follows.

$$P = -\frac{\Lambda}{8\pi}, \quad r_{AdS} = \sqrt{-\frac{3}{\Lambda}}.$$  

(3.6)

Using (3.3) one can calculate thermodynamic volume of the quintessence dyonic AdS black hole as

$$V = \left( \frac{\partial M}{\partial P} \right)_{A, Q_E, Q_M, r_q} = \frac{4\pi r_h^3}{3}$$

(3.7)

which its form is similar to the geometric volume of the horizon with surface area $4\pi r_h^2$. We should notice that the latter statement is not correct for metric
of an arbitrary curved space time and in general the black hole thermodynamic volume is differ with its geometric one. From (3.5) we obtain

$$\Phi_E = \left( \frac{\partial M}{\partial Q_E} \right)_{S,P,Q,M,r_q} = \frac{Q_E}{r_h}$$  \hspace{1cm} (3.8)

which is called as the electric potential of the black hole on the horizon. Substituting (3.5) and calculating (3.4) we obtain

$$A = -\frac{(1 + 3\omega)}{2} \left( \frac{r_q}{r_h} \right)^{2\omega}$$ \hspace{1cm} (3.9)

The Hawking temperature of the above black hole can be derived as follows.

$$T = \frac{f'(r)}{4\pi} \bigg|_{r=r_h} = \frac{1}{4\pi} \left( \frac{1}{r_h} - \frac{Q^2_M}{2r_h} + \frac{Q^2_E}{2r_h} - \frac{3\omega r_q^{1+3\omega}}{2r_h^{3\omega+2}} + 8\pi r_h P \right)$$ \hspace{1cm} (3.10)

where we substitute (3.5) to remove $M$. Since the black hole mass $M$ is interpreted as enthalpy as $H = M$ thus one can infer that the equation (3.5) reads to the following form.

$$H = U + PV$$ \hspace{1cm} (3.11)

where

$$U = \frac{r_h^2}{2} + \frac{Q^2_M}{2r_h} + \frac{Q^2_E}{2r_h} - \frac{3\omega r_q^{1+3\omega}}{2r_h^{3\omega+2}}$$ \hspace{1cm} (3.12)

is internal energy. To study location of critical point on the P-V plan it is appropriate to use dimensionless forms of the thermodynamic functions as follows.

$$p = 8\pi r_q^2 P, \hspace{0.5cm} t = 4\pi r_q T, \hspace{0.5cm} v = \frac{r_h}{r_q}, \hspace{0.5cm} q_M = \frac{Q_M}{r_q}$$ \hspace{1cm} (3.13)

Substituting the above definitions and (3.8) into the equation (3.10) we obtain dimensionless equation of state for the quintessence dyonic AdS black hole such that

$$p(v,t) = t - \frac{(1 - \Phi^2_E)}{v^2} + \frac{q^2_M}{v^4} - \frac{3\omega}{v^{3(1+\omega)}}.$$ \hspace{1cm} (3.14)

By applying the above dimensionless parameters the enthalpy (3.11) reads

$$m = (1 + \Phi^2_E)v + \frac{q^2_M}{v} - \frac{1}{v^{3\omega}} + \frac{p t^3}{3}$$ \hspace{1cm} (3.15)
in which
\[ m = \frac{M}{2r_q} \]  \hspace{1cm} (3.16)
is assumed to be dimensionless mass of the black hole. Solving the equations
\[ \left( \frac{\partial p}{\partial v} \right)_{\text{other parameters}} = \left( \frac{\partial^2 p}{\partial v^2} \right)_{\text{other parameters}} = 0 \]  \hspace{1cm} (3.17)
we can obtain parametric form of the critical point \( \{ p_c, t_c, v_c \} \) which satisfy the following equations.
\[ 2(1 - \Phi_E^2)v_c^{3\omega+1} - 12q_M^2v_c^{3\omega-1} + 9\omega(1 + \omega)(2 + 3\omega) = 0 \]  \hspace{1cm} (3.18)
\[ t_c = \frac{2(1 - \Phi_E^2)}{v_c} - \frac{4q_M^2}{v_c^3} + \frac{9\omega(1 + \omega)}{v_c^{2+3\omega}} \]  \hspace{1cm} (3.19)
\[ p_c = \frac{(1 - \Phi_E^2)}{v_c^2} - \frac{3q_M^2}{v_c^4} + \frac{3\omega(2 + 3\omega)}{v_c^{3(1+\omega)}}. \]  \hspace{1cm} (3.20)
The above algebraic equations have several roots due to different values of the black hole parameters \( \{ \omega, q_M, \Phi_E \} \) which are into account where \(-1 < \omega < -\frac{1}{3}\) corresponds to the quintessence dark energy regime. To compare compressibility factor of this black hole with which one obtained for the Van de Waals fluid as \( Z = \frac{3}{8} \) given by the equation (2.4), it is appropriate to substitute \( 1 - \Phi_E^2 \) from (3.18) into the equations (3.19) and (3.20). In the latter case (3.19) and (3.20) reduce to the following forms respectively.
\[ t_c = \frac{8q_M^2}{v_c^3} - \frac{9\omega(1 + \omega)^2}{v_c^{2+3\omega}} \]  \hspace{1cm} (3.21)
and
\[ Z = \frac{p_cv_c}{t_c} = \frac{3}{8} - \frac{3\omega(1 + 3\omega)(7 + 9\omega)}{8t_c v_c^{2+3\omega}}. \]  \hspace{1cm} (3.22)
The above relation shows that for a quintessence regime of the dark energy where \(-1 < \omega < -\frac{1}{3}\) the compressibility of the dyonic AdS black hole behaves as a Van der Waals fluid if we set
\[ \omega_{Van} = -\frac{7}{9}. \]  \hspace{1cm} (3.23)
Substituting (3.23) into the equations (3.14), (3.18), (3.21) and (3.22) we will have respectively

\[ p = \frac{t}{v} - \frac{(1 - \Phi_E^2)}{v^2} + \frac{q_M^2}{v^4} + \frac{7}{3} \frac{1}{v^2} \quad (3.24) \]

\[ 1 - \Phi_E^2 = \frac{6q_M^2}{v_c^2} - \frac{7}{27} \frac{4}{v_c^2}, \quad (3.25) \]

\[ t_c = \frac{8q_M^2}{v_c^3} + \frac{28}{81} \frac{4}{v_c^2} \quad (3.26) \]

and

\[ p_c = \frac{3q_M^2}{v_c^4} + \frac{7}{54} \frac{1}{v_c^2}. \quad (3.27) \]

The equations (3.26) and (3.27) show that the critical temperature \( t_c \) and the critical pressure \( p_c \) are sensitive to the magnetic charge of the black hole \( q_M \) just at small values of the critical volume \( v_c \ll 1 \) but not for its large amounts.

For positive temperatures \( t \geq 0 \) (here we exclude negative temperatures because they are in usual way un-physical) one can infer that \( p - v \) diagram plotted by the equation (3.24) possess inflection points under condition

\[ 0 \leq \Phi_E^2 < 1 \quad (3.28) \]

and these inflection points disappear for electric potentials \( \Phi_E^2 \geq 1 \). Because for (3.28) coefficient of the term \( \frac{1}{v_c^2} \) given by (3.24) remains negative. Substituting (3.28) into the equation (3.25) we can obtain admissible numerical values for the critical volume as

\[ 0 < \frac{6q_M^2}{v_c^2} - \frac{7}{27} \frac{4}{v_c^3} \leq 1. \quad (3.29) \]

This shows upper and lower bounds of the critical volume \( v_c \) which are become restricted by the magnetic charge \( q_M \). For simplicity we choose equal sign in the above inequality identity for which we will have

\[ q_M(\Phi_E = 0) = \pm \frac{v_c}{18} \sqrt{14v_c^4 + 54}. \quad (3.30) \]

Substituting (3.30) into the relations (3.26) and (3.27) we obtain

\[ t_c = \frac{81 + 77v_c^4}{162v_c} \quad (3.31) \]
and

\[ p_c = \frac{14v_c^2 + 27v_c^2}{54v_c^2} \]  

(3.32)

for zero electric potential \( \Phi_E = 0 \). Substituting (3.30) and \( \Phi = 0 \) into the equation (3.24) we obtain

\[
p = \frac{t}{v} + \frac{v^2}{v^4} \left( 1 + \frac{7}{27}v_c^2 \right) \left[ \frac{1}{6} - \left( \frac{v}{v_c} \right)^2 \right] + \frac{7}{3} \frac{1}{v^4} \left[ 9 + \left( \frac{v}{v_c} \right)^2 \right].
\]  

(3.33)

This shows that the for some positive constant temperatures \( p - v \) diagrams will have inflection points only for

\[
\frac{v}{v_c} > \frac{1}{\sqrt{6}}
\]  

(3.34)

because of negativity sign of the term \( \left[ \frac{1}{6} - \left( \frac{v}{v_c} \right)^2 \right] \). Now we talk about the enthalpy function (3.15) which under the condition (3.30) reads

\[
m = \left( 2 + \frac{7}{27}v_c^2 \right) v + \frac{v^2}{v} \left( 1 + \frac{7}{27}v_c^2 \right) \left[ \frac{1}{6} - \left( \frac{v}{v_c} \right)^2 \right] - \frac{1}{v^\frac{1}{3}} + pv^3.
\]  

(3.35)

Other quantity which can be used to study thermal stability of black holes is heat capacity at constant pressure defined by

\[ C_P = \left( \frac{\partial M}{\partial T} \right)_p \]  

(3.36)

which for \( C_p < 0 \) the black hole is unstable while for \( C_p > 0 \) is stable. Applying (3.33) and (3.35) the equation (3.36) reads

\[ c_p = \left( \frac{\partial m}{\partial t} \right)_p = \frac{W(v)}{O(v)} \]  

(3.37)

in which \( W(v) \) and \( O(v) \) are obtained respectively as follows.

\[
W(v) = \left( \frac{\partial m}{\partial v} \right)_p = 2 + pv^2 + \frac{7}{3} \frac{1}{v^\frac{1}{3}} - \frac{v_c^2}{v^2} \left( 1 + \frac{7}{27}v_c^2 \right) \left[ \frac{1}{6} + \left( \frac{v}{v_c} \right)^2 \right]
\]  

(3.38)

and

\[
O(v) = \left( \frac{\partial t}{\partial v} \right)_p = p + \frac{v_c^2}{v^4} \left( 1 + \frac{7}{27}v_c^2 \right) \left[ \frac{1}{2} - \left( \frac{v}{v_c} \right)^2 \right] + \frac{7}{3} \frac{1}{v^\frac{1}{3}} \left[ \left( \frac{v}{v_c} \right)^2 - 3 \right]
\]  

(3.39)
with
\[ c_p = \frac{C_P}{8\pi r_q^2}. \]  
(3.40)

Gibbs free energy of the quintessence dyonic AdS Black hole is given by (see Eq. (3.2) in ref. [10])
\[ G = M - TS - \Phi_E Q_E - \Phi_M Q_M = \mu N \]  
(3.41)

where \( S \) is entropy of the black hole which by according to the Bekenstein-Hawking entropy formula is quarter of the black hole horizon area [31] (see also [32]).
\[ S = \int \frac{dM}{T} = \int_0^{r_h} dr_h \left( \frac{\partial M}{\partial r_h} \right) = \frac{4\pi r_h^2}{4} = \frac{\pi r_h^2}{4} = \text{horizon area}. \]  
(3.42)

\( M \) is the black hole mass which is equal to the black hole enthalpy energy. \( T \) is the black hole Hawking temperature and \( \Phi_{E,M} \) and \( Q_{E,M} \) are electromagnetic potential and electromagnetic charge of the black hole. \( N \) is number of micro particles and \( \mu \) is their mutual chemical potential. Substituting (3.5), (3.8), (3.10) and (3.39) into the Gibbs free energy (3.38) one can infer
\[ G = \frac{(1 - \Phi_E^2) r_h}{4} - \frac{1}{4} \frac{Q_M^2}{r_h} - \frac{2\pi r_h^3 P}{3} - \frac{(2 + 3\omega) r_q^{1+3\omega}}{4r_h^{3\omega}} \]  
(3.43)

which by substituting (3.13) leads to a dimensionless form as follows.
\[ g(p, v) = (1 - \Phi_E^2)v - \frac{q_M^2}{v} - \frac{(2 + 3\omega)}{v^{3\omega}} - \frac{pv^3}{3} \]  
(3.44)

where we defined dimensionless Gibbs free energy as follows.
\[ g = \frac{4G}{r_q}. \]  
(3.45)

Substituting (3.23), (3.25) and (3.30) with \( \Phi_E = 0 \) into the black hole Gibbs free energy (3.44) we obtain
\[ g(p, v) = \frac{v^2}{v} \left[ 1 + \frac{7}{27} \left( \frac{v}{v_c} \right)^{\frac{4}{3}} \right] \left[ \left( \frac{v}{v_c} \right)^{\frac{2}{3}} - \frac{1}{6} \right] + \frac{7}{27} v \left[ 9 - \left( \frac{v}{v_c} \right)^{\frac{4}{3}} \right] - \frac{pv^3}{3} \]  
(3.46)
Now we plot diagrams of state equation of the black hole $p(v, t)$, its Gibbs free energy $g(p, v)$ versus the volume and the temperature for different values of the pressure and then proceed to interpretation of these diagrams. Diagrams of the thermodynamic variables such as the pressure (3.33), the enthalpy (3.35), the heat capacity (3.37), and Gibbs free energy (3.46) together with all possible numerical values of the critical points of the quintessence AdS dyonic black hole are plotted in figures 2 and 3.

Diagram of figure (2-a) shows all possible values for critical volume obtained from the equation (3.25) by substituting $|q_M| > 0$ and $0 \leq \Phi^2_E < 1$. Setting $\Phi_E = 0$ we plot all possible values for $(q_M, v_c)$ by applying (3.30) in figure (2-b). The figure (2-c) denotes variation of the critical compressibility of the black hole $z_c = \frac{p_{c}}{v_{c}}$ which can be calculated from (3.31) and (3.32) versus the critical volume $v_c$. This figure shows that for large $v_c$ the black hole system takes a single phase, while for small $v_c$ the black hole system is made from to subsystem (phase) with different compressibility factor which they reach to a coexistence stable state with smallest value of $v_c$. Figure (2-d) describes variations of the critical temperature versus the critical volume containing a local minimum value. In other words by decreasing and increasing the critical volume the critical temperature raises. We show absolutely decreasing behavior of the critical pressure by raising the critical volume in the figure (2-e). Applying (3.31) and (3.32) we plot possible values of the critical pressure against the possible values of the critical temperature in the figure (2-f). This shows monotonically raising behavior for $p_c$ when the temperature increases $t_c$ for approximate values $t_c \geq 2300$ but not for lesser values. For small values of the critical temperature, $p_c$ increases rapidly. Diagram of the equation (3.33) is plotted in figure (2-g) at constant pressure. This figure shows a local maximum and a local minimum critical temperature for small and large volume of the black holes respectively. Looking to this figure one can infer that for temperatures $3.5 < t < 5.8$ there is three unstable state for the black holes under consideration. Minimum of the temperature diagram appears at $v \approx 0.18$ and all black holes with $v < 0.18$ is unstable and so they are hotter. They incline to reach a cooler larger stable state thermally with volume $\sim 0.18$. This means small/large black hole phase transition. Also all black holes with $v > 0.18$ incline to compress so that become smaller one with stable volume $\sim 0.18$. This is also means that unstable large black holes exhibit with a large/small phase transition. This phase transition can be interpreted in figure (2-h) at constant temperature for a given pressure $-20 < p < 60$. This domain is depended to some fixed temperatures which
we used to plot the diagram. One can extract more physics from behavior of
the black hole by looking the figure (2-i). In contrary the figure (1-a) for the
Van der Waals fluid which has a single crossing point the figure (2-i) predicts
a double crossing points. A single crossing point at (1-a) shows a coexistence
of a fluid containing two subsystems or phase (gas/liquid), while (2-i) predicts
that the black hole matter contains two coexistence state because of existence
two crossing point. This shows that the black hole can have 4 subsystems or
phase. Importance of the diagram (2-i) with respect to other diagrams (2-g)
and (2-h) is this: We saw that by setting $t = constant$ and $p = constant$
in the figures (2-g) and (2-h) there is obtained three particular points with
different volumes which predict three unstable state of the black hole matter.
Physically these 3 unstable state can be considered as gas/liquid/solid of phase
the black hole material. They do not predict forth plasma phase of the matter
which can be coexistence with other phases of the black hole matter. In
contrary the figure (2-i) namely variation of the black hole Gibbs free energy
versus the temperature can be predict the plasma phase of the black hole
matter which can be occurs under the particular conditions.

Diagrams of the figures (3-a) and (3-b) show the above situations via vari-
ations of the heat capacity of the black hole at constant pressure versus the
volume and the temperature respectively. We know that a thermodynamical
system become unstable when its heat capacity takes negative values and
stable state for its positive values. These diagrams show that this black hole
will be stable $C_p > 0$ thermally just when its volume is not more larger or
smaller than the critical volume $v_c = 0.1$. There is small volume $v_s < 0.1$
and large volume $v_L > 0.1$ where sign of the heat capacity is changed. This
is also predicts only a small/large black hole phase transition which can be
contain three phase of the black hole matter. Diagram of the figure (3-b)
shows a phase of the black hole matter which by raising the temperature the
heat capacity has zero value $C_v \approx 0$ while two other phases of the black hole
matter behave as heat ${-}$ giver ($C_p < 0$) and heat + receiver ($C_p > 0$).
There is a particular temperature $t \rightarrow 1.5$ where the heat capacity takes a
degenerate state. Last diagram (3-c) shows the enthalpy which has positive
and negative values for large and small black holes respectively. When we
say black hole is small then is should be comparable with the critical volume
$v_c = 0.1$.

Last point which should be presented is this: All diagrams are plotted easily
except (g-t). This has more sensitive to initial values. For instance with
$p = 30$ and $0 < v < 1$ the diagram shows the black hole have two cross-
ing points but for the number of these crossing points can be reduced by changing the pressure. Our numerical calculations predicts they can be even disappeared by changing the volume domain and the pressure. However as future work, it’s worth looking at finding more points of this type which predict different phases of the black hole material. If there is obtained more than 2 crossing point then one can result that the black hole exhibits with other phases of the matter called as the ‘Bose-Einstein condensate’ state and the ‘Fermionic condensate’ state which are fifth and sixth superfluid phase of the matter respectively [34,35].

4 Conclusion

In this work we studied quintessence dark energy effects on the phase transition of the AdS dyonic black holes. We determined regimes on the barotropic index of the quintessence dark energy and electric potential and magnetic charge of the black hole where the small/large black hole phase transition is occurred. Also diagram of the p-v curves predicts for large values of the thermodynamic volume of the black hole phase transition can be reach to the well known Hawking-Page phase transition where the black hole disappears and a vacuum AdS space remains. Our work other phases of the black hole matter is predicted such as plasma, Bose – Einstein and Fermionic condensate in addition to three well known states called as solid – fluid – gas of the matter. One of applicable work which we can be investigate it in the future is study quintessence dark energy effects on entanglement entropy of AdS dyonic black holes in relation to the Maxwell equal area law which does not satisfied for AdS dyonic black holes [28].

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Figure 2: Diagram of critical points and small/large phase transition of the quintessence AdS Dyonic black hole
Figure 3: Diagrams of the heat capacity are plotted versus the volume and the temperature at (a) and (b) respectively and the black hole enthalpy is plotted versus the volume at (c).