Quantum Graphs which Optimize the Spectral Gap

Ram Band

Technion - Israel Institute of Technology

Joint work with Guillaume Lévy, Université Pierre et Marie Curie, Paris
(arXiv:1608.00520)

QMath 13, GeorgiaTech, Atlanta - October 2016
Eigenvalue optimization problems (for domains)

Fixing the topology, total volume and boundary conditions, we seek for the shape which maximizes\minimizes an eigenvalue.

Simply connected domains

Faber-Krahn [Dirichlet conditions]: the ball minimizes $\lambda_1$ (no sense maximizing).

Krahn-Szegö [Dirichlet conditions]: No minimizer for $\lambda_2$, but union of two balls serves as an infimizer.

Szegö-Weinberger [Neumann conditions]: the ball maximizes $\lambda_1$ (no sense minimizing).

Multi connected domains

Payne-Weinberger: Planar domains with a single hole, Dirichlet on outer boundary and Neumann on inner.
Fixing total area and length of outer boundary - annulus (concentric circles) maximizes $\lambda_1$.

More works by: Ashbaugh-Chatelain, Ashbaugh-Benguria, Exner-Mantile, Flucher, Harrell-Kröger-Kurata, Hersch, Kolokolnikov-Titcombe-Ward, and more...
Eigenvalue optimization problems (for domains)

Fixing the topology, total volume and boundary conditions, we seek for the shape which maximizes/minimizes an eigenvalue.

Simply connected domains

Faber-Krahn [Dirichlet conditions]: the ball minimizes $\lambda_1$ (*no sense maximizing*).

Krahn-Szegö [Dirichlet conditions]: No minimizer for $\lambda_2$, but union of two balls serves as an infimizer.

Szegö-Weinberger [Neumann conditions]: the ball maximizes $\lambda_1$ (*no sense minimizing*).

Multi connected domains

Payne-Weinberger: Planar domains with a single hole, Dirichlet on outer boundary and Neumann on inner.

Fixing total area and length of outer boundary - annulus (concentric circles) maximizes $\lambda_1$.

More works by: Ashbaugh-Chatelain, Ashbaugh-Benguria, Exner-Mantile, Flucher, Harrell-Kröger-Kurata, Hersch, Kolokolnikov-Titcombe-Ward, and more...
Eigenvalue optimization problems (for domains)

Fixing the topology, total volume and boundary conditions, we seek for the shape which maximizes\minimizes an eigenvalue.

**Simply connected domains**

Faber-Krahn [Dirichlet conditions]: the ball minimizes $\lambda_1$ (*no sense maximizing*).

Krahn-Szegö [Dirichlet conditions]: No minimizer for $\lambda_2$, but union of two balls serves as an *infimizer*.

Szegö-Weinberger [Neumann conditions]: the ball maximizes $\lambda_1$ (*no sense minimizing*).

**Multi connected domains**

Payne-Weinberger: Planar domains with a single hole, Dirichlet on outer boundary and Neumann on inner. Fixing total area and length of outer boundary - annulus (concentric circles) maximizes $\lambda_1$.

More works by: Ashbaugh-Chatelain, Ashbaugh-Benguria, Exner-Mantile, Flucher, Harrell-Kröger-Kurata, Hersch, Kolokolnikov-Titcombe-Ward, and more...
Eigenvalue optimization problems (for domains)

Fixing the topology, total volume and boundary conditions, we seek for the shape which maximizes\minimizes an eigenvalue.

Simply connected domains

Faber-Krahn [Dirichlet conditions]: the ball minimizes $\lambda_1$ (*no sense maximizing*).

Krahn-Szegö [Dirichlet conditions]: No minimizer for $\lambda_2$, but union of two balls serves as an *infimizer*.

Szegö-Weinberger [Neumann conditions]: the ball maximizes $\lambda_1$ (*no sense minimizing*).

Multi connected domains

Payne-Weinberger: Planar domains with a single hole, Dirichlet on outer boundary and Neumann on inner.

Fixing total area and length of outer boundary - annulus (concentric circles) maximizes $\lambda_1$.

More works by: Ashbaugh-Chatelain, Ashbaugh-Benguria, Exner-Mantile, Flucher, Harrell-Kröger-Kurata, Hersch, Kolokolnikov-Titcombe-Ward, and more...
Eigenvalue optimization problems (for domains)

Fixing the topology, total volume and boundary conditions, we seek for the shape which maximizes\minimizes an eigenvalue.

Simply connected domains

Faber-Krahn [Dirichlet conditions]: the ball minimizes $\lambda_1$ (*no sense maximizing*).

Krahn-Szegö [Dirichlet conditions]: No minimizer for $\lambda_2$,
but union of two balls serves as an infimizer.

Szegö-Weinberger [Neumann conditions]: the ball maximizes $\lambda_1$ (*no sense minimizing*).

Multi connected domains

Payne-Weinberger: Planar domains with a single hole,
Dirichlet on outer boundary and Neumann on inner.
Fixing total area and length of outer boundary - annulus (concentric circles) maximizes $\lambda_1$.

More works by: Ashbaugh-Chatelain, Ashbaugh-Benguria, Exner-Mantile, Flucher, Harrell-Kröger-Kurata, Hersch, Kolokolnikov-Titcombe-Ward, and more...
Outline

Introduction

Infimizers

Supremizers
  Upper bounds
  Spectral gap as a simple eigenvalue
  Gluing graphs

Summary & Conjectures
From a Discrete graph to a Quantum graph

$\mathcal{G}$ a discrete graph with $E < \infty$ edges and $V < \infty$ vertices. Space of edge lengths:

$$\mathcal{L}_G := \{(l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^{E} l_e = 1 \text{ and } \forall e, \ l_e > 0\}$$

$\Gamma(\mathcal{G}; l)$ denotes the metric graph obtained from $\mathcal{G}$ with edge lengths $l \in \mathcal{L}_G$.

Namely, the $e^{th}$ edge corresponds to an interval $[0, l_e]$.

Consider the following eigenvalue equation on each $[0, l_e]$: 

$$-\frac{d^2}{dx_e^2} f|_e = k^2 f|_e,$$

with the Neumann (Kirchhoff) vertex conditions:

Continuity $\forall e_1, e_2 \sim v; \ f|_{e_1} (v) = f|_{e_2} (v)$

Vanishing sum of derivatives $\sum_{e \sim v} \frac{d}{dx_e} f|_e (v) = 0$

The spectrum, $\{k_n^2\}_{n=1}^{\infty}$ is discrete and bounded from below:

$$0 = k_0 < k_1 \leq k_2 \leq \ldots$$

We call $k_1$ the spectral gap of the graph.
From a Discrete graph to a Quantum graph

$\mathcal{G}$ a discrete graph with $E < \infty$ edges and $V < \infty$ vertices. Space of edge lengths:

\[ \mathcal{L}_{\mathcal{G}} := \left\{ (l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^E l_e = 1 \text{ and } \forall e, \ l_e > 0 \right\} \]

$\Gamma(\mathcal{G}; \ l)$ denotes the metric graph obtained from $\mathcal{G}$ with edge lengths $l \in \mathcal{L}_{\mathcal{G}}$.

Namely, the $e^{th}$ edge corresponds to an interval $[0, l_e]$

Consider the following eigenvalue equation on each $[0, l_e]$: 

\[ -\frac{d^2}{dx_e^2} f \big|_e = k^2 f \big|_e, \]

with the Neumann (Kirchhoff) vertex conditions:

- Continuity $\forall e_1, e_2 \sim v; \ f \big|_{e_1} (v) = f \big|_{e_2} (v)$
- Vanishing sum of derivatives $\sum_{e \sim v} \frac{d}{dx_e} f \big|_e (v) = 0$

The spectrum, $\{k_n^2\}_{n=1}^{\infty}$ is discrete and bounded from below:

\[ 0 = k_0 < k_1 \leq k_2 \leq \ldots \]

We call $k_1$ the spectral gap of the graph.
From a Discrete graph to a Quantum graph

\( \mathcal{G} \) a discrete graph with \( E < \infty \) edges and \( V < \infty \) vertices. Space of edge lengths:

\[
\mathcal{L}_G := \left\{(l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^E l_e = 1 \text{ and } \forall e, l_e > 0 \right\}
\]

\( \Gamma(G; l) \) denotes the metric graph obtained from \( \mathcal{G} \) with edge lengths \( l \in \mathcal{L}_G \).

Namely, the \( e \)th edge corresponds to an interval \([0, l_e]\)

Consider the following eigenvalue equation on each \([0, l_e]\):

\[
-\frac{d^2}{dx_e^2} f\big|_e = k^2 f\big|_e,
\]

with the Neumann (Kirchhoff) vertex conditions:

Continuity \( \forall e_1, e_2 \sim v; \ f\big|_{e_1}(v) = f\big|_{e_2}(v) \)

Vanishing sum of derivatives \( \sum_{e \sim v} \frac{d}{dx_e} f \big|_e(v) = 0 \)

The spectrum, \( \{k_n^2\}_{n=1}^{\infty} \) is discrete and bounded from below:

\[
0 = k_0 < k_1 \leq k_2 \leq \ldots
\]

We call \( k_1 \) the spectral gap of the graph.
Spectral gap dependence on edge lengths

\[ \mathcal{L}_G := \left\{ (l_1, \ldots, l_E) \in \mathbb{R}^E \bigg| \sum_{e=1}^E l_e = 1 \text{ and } \forall e, \ l_e > 0 \right\}. \]

\( \Gamma(G; \ l) \) denotes the metric graph obtained from \( G \) with edge lengths \( l \in \mathcal{L}_G \). Spectral gap is denoted \( k_1[\Gamma(G; \ l)] \).

Note: \( k_1[\Gamma(G; \ l)] \) is continuous in \( l \), which leads to consider also \( l \in \partial \mathcal{L}_G \) (some edge lengths vanish), possibly changing the topology of \( \Gamma(G; \ l) \).

**Definition 1.**

- \( \Gamma(G; \ l^*) \) a maximizer of \( G \) if \( l^* \in \mathcal{L}_G \) and \( k_1[\Gamma(G; \ l^*)] \geq k_1[\Gamma(G; \ l)], \ \forall l \in \mathcal{L}_G \).

- \( \Gamma(G; \ l^*) \) a supremizer of \( G \) if \( l^* \in \overline{\mathcal{L}_G} \) and \( k_1[\Gamma(G; \ l^*)] \geq k_1[\Gamma(G; \ l)], \ \forall l \in \overline{\mathcal{L}_G} \).

- Same definitions for minimizer and infimizer.

- Supremizer and infimizer always exist.

What about maximizer\/minimizer?

- Which graphs are spectral gap optimizers?
Spectral gap dependence on edge lengths

\[ \mathcal{L}_G := \left\{ (l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^{E} l_e = 1 \text{ and } \forall e, \; l_e > 0 \right\}. \]

\( \Gamma(G; \ l) \) denotes the metric graph obtained from \( G \) with edge lengths \( l \in \mathcal{L}_G \).
Spectral gap is denoted \( k_1[\Gamma(G; \ l)] \).
Note: \( k_1[\Gamma(G; \ l)] \) is continuous in \( l \),
which leads to consider also \( l \in \partial \mathcal{L}_G \) (some edge lengths vanish),
possibly changing the topology of \( \Gamma(G; \ l) \).

**Definition 1.**

- \( \Gamma(G; \ l^*) \) a maximizer of \( G \) if \( l^* \in \mathcal{L}_G \) and \( k_1[\Gamma(G; \ l^*)] \geq k_1[\Gamma(G; \ l)], \; \forall l \in \mathcal{L}_G \).
- \( \Gamma(G; \ l^*) \) a supremizer of \( G \) if \( l^* \in \overline{\mathcal{L}_G} \) and \( k_1[\Gamma(G; \ l^*)] \geq k_1[\Gamma(G; \ l)], \; \forall l \in \overline{\mathcal{L}_G} \).
- Same definitions for minimizer and infimizer.

- Supremizer and infimizer always exist.
  What about maximizer/minimizer?
- Which graphs are spectral gap optimizers?
Spectral gap dependence on edge lengths

\[ \mathcal{L}_G := \{ (l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^{E} l_e = 1 \text{ and } \forall e, l_e > 0 \}. \]

\( \Gamma (G; l) \) denotes the metric graph obtained from \( G \) with edge lengths \( l \in \mathcal{L}_G \). Spectral gap is denoted \( k_1 [\Gamma (G; l)] \). Note: \( k_1 [\Gamma (G; l)] \) is continuous in \( l \), which leads to consider also \( l \in \partial \mathcal{L}_G \) (some edge lengths vanish), possibly changing the topology of \( \Gamma (G; l) \).

**Definition 1.**

- \( \Gamma (G; l^*) \) a maximizer of \( G \) if \( l^* \in \mathcal{L}_G \) and \( k_1 [\Gamma (G; l^*)] \geq k_1 [\Gamma (G; l)], \ \forall l \in \mathcal{L}_G \).

- \( \Gamma (G; l^*) \) a supremizer of \( G \) if \( l^* \in \overline{\mathcal{L}_G} \) and \( k_1 [\Gamma (G; l^*)] \geq k_1 [\Gamma (G; l)], \ \forall l \in \overline{\mathcal{L}_G} \).

- Same definitions for minimizer and infimizer.

- Supremizer and infimizer always exist.

  What about maximizer\ minimizer?

- Which graphs are spectral gap optimizers?
Spectral gap dependence on edge lengths

\[ \mathcal{L}_G := \{(l_1, \ldots, l_E) \in \mathbb{R}^E \mid \sum_{e=1}^{E} l_e = 1 \text{ and } \forall e, \ l_e > 0 \}. \]

\( \Gamma(G; l) \) denotes the metric graph obtained from \( G \) with edge lengths \( l \in \mathcal{L}_G \).

Spectral gap is denoted \( k_1[\Gamma(G; l)] \). \textbf{Note:} \( k_1[\Gamma(G; l)] \) is continuous in \( l \), which leads to consider also \( l \in \partial \mathcal{L}_G \) (some edge lengths vanish), possibly changing the topology of \( \Gamma(G; l) \).

**Definition 1.**

- \( \Gamma(G; l^*) \) a maximizer of \( G \) if \( l^* \in \mathcal{L}_G \) and \( k_1[\Gamma(G; l^*)] \geq k_1[\Gamma(G; l)], \ \forall l \in \mathcal{L}_G \).

- \( \Gamma(G; l^*) \) a supremizer of \( G \) if \( l^* \in \overline{\mathcal{L}_G} \) and \( k_1[\Gamma(G; l^*)] \geq k_1[\Gamma(G; l)], \ \forall l \in \overline{\mathcal{L}_G} \).

- Same definitions for minimizer and supremizer.

- Supremizer and infimizer always exist.

What about maximizer/minimizer?

- Which graphs are spectral gap optimizers?
Quantum Graphs which Optimize the Spectral Gap

- Supremizer and infimizer always exist. What about maximizer\minimizer?
- Which graphs are spectral gap optimizers?

A few examples

**Star** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1, 0, \ldots 0) = \pi$,

Maximum: $k_1(1/E, \ldots, 1/E) = \frac{E}{2} \pi$ (equilateral star)

(Recall: total edge length = 1)

**Flower** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1, 0, \ldots 0) = 2\pi$,

Maximum: $k_1(1/E, \ldots, 1/E) = E\pi$ (equilateral flower)

[Kennedy, Kurasov, Malenová, Mugnolo ’16]
Quantum Graphs which Optimize the Spectral Gap

- Supremizer and infimizer always exist. What about maximizer\minimizer?
- Which graphs are spectral gap optimizers?

A few examples

**Star** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1,0,\ldots,0) = \pi$,
Maximum: $k_1(1/E,\ldots,1/E) = \frac{E}{2} \pi$ (equilateral star)
(Recall: total edge length $= 1$)

**Flower** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1,0,\ldots,0) = 2\pi$,
Maximum: $k_1(1/E,\ldots,1/E) = E \pi$ (equilateral flower)

[Kennedy, Kurasov, Malenová, Mugnolo ’16]
Quantum Graphs which Optimize the Spectral Gap

• Supremizer and infimizer always exist. What about maximizer/minimizer?
• Which graphs are spectral gap optimizers?

A few examples

**Star** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1,0,\ldots,0) = \pi$,
Maximum: $k_1(1/E,\ldots,1/E) = \frac{E}{2}\pi$ (equilateral star)
(Recall: total edge length = 1)

**Flower** graph with $E \geq 2$ edges

Infimum (no minimum): $k_1(1,0,\ldots,0) = 2\pi$,
Maximum: $k_1(1/E,\ldots,1/E) = E\pi$ (equilateral flower)

[Kennedy, Kurasov, Malenová, Mugnolo ’16]
Quantum Graphs which Optimize the Spectral Gap

A few examples (continued)

**Stower** (Flétoile) graph with $E_p$ petals, $E_l$ leaves

Infimum (no minimum): $k_1(0\ldots,0,1) = \pi$,
Maximum: $k_1(l) = (E_p + \frac{E_l}{2})\pi$,
where $l = \frac{1}{2E_p+E_l}(2,\ldots,2,1,\ldots,1)$ ("equilateral" stower),
assuming $E_p + E_l \geq 2$ and $(E_p, E_l) \notin (1,1)$. [Shown in future slide].
This generalizes stars and flowers results.
Quantum Graphs which Optimize the Spectral Gap

A few examples (continued)

**Stower** (Flétoile) graph with $E_p$ petals, $E_l$ leaves

Infimum (no minimum): $k_1(0, \ldots, 0, 1) = \pi$,

Maximum: $k_1(l) = (E_p + \frac{E_l}{2})\pi$,

where $l = \frac{1}{2E_p+E_l}(2, \ldots, 2, 1, \ldots, 1)$ ("equilateral" stower),

assuming $E_p + E_l \geq 2$ and $(E_p, E_l) \notin (1, 1)$. [Shown in future slide].

This generalizes stars and flowers results.

Infimum: $k_1(0, 0, 1) = \pi$,

Maximum: $k_1(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}) = 2\frac{1}{2}\pi$
Quantum Graphs which Optimize the Spectral Gap

A few examples (continued)

**Stower** (Flétoile) graph with $E_p$ petals, $E_l$ leaves

Infimum (no minimum): $k_1(0, \ldots, 0, 1) = \pi$,

Maximum: $k_1(l) = (E_p + \frac{E_l}{2})\pi$,

where $l = \frac{1}{2E_p+E_l}(2, \ldots, 2, 1, \ldots, 1)$ ("equilateral" stower),

assuming $E_p + E_l \geq 2$ and $(E_p, E_l) \notin (1, 1)$. [Shown in future slide].

This generalizes stars and flowers results.

Infimum: $k_1(0, 0, 1) = \pi$,

Maximum: $k_1(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}) = 2\frac{1}{2}\pi$

Continuous family of infima: $k_1(0, t, 1 - t) = \pi$,

Continuous family of maxima: $k_1(1 - 2t, t, t) = 2\pi$
Quantum Graphs which Optimize the Spectral Gap

A few examples (continued)

Mandarin graph with $E$ edges

Infimum (no minimum): $k_1(1,0,\ldots,0) = 2\pi$,  
Maximum: $k_1(1/E,\ldots,1/E) = E\pi$.  
[Kennedy, Kurasov, Malenová, Mugnolo ’16]

Length dependence figures - courtesy of Lior Alon

- Which graphs have not only supremizer\/infimizer, but also maximizer\/minimizer?
- Which graphs are spectral gap optimizers?
Quantum Graphs which Optimize the Spectral Gap

A few examples (continued)

**Mandarin** graph with $E$ edges

Infimum (no minimum): $k_1(1, 0, \ldots, 0) = 2\pi$,  
Maximum: $k_1(1/E, \ldots, 1/E) = E\pi$.

[Kennedy, Kurasov, Malenová, Mugnolo ’16]

Length dependence figures - courtesy of Lior Alon

- Which graphs have not only supremizer\infimizer, but also maximizer\minimizer?
- Which graphs are spectral gap optimizers?
Lower bounds - Known results

\[ k_1[\Gamma] \geq \pi \]

with equality iff \( \Gamma \) is a single edge [Nicase ’87; Friedlander ’05; Kurasov, Naboko ’14].

If \( \Gamma \) has all vertex degrees even then

\[ k_1[\Gamma] \geq 2\pi, \quad [\text{Kurasov, Naboko ’14}] \]

with a single loop achieving equality (for example).

Remaining questions:

- What about other topologies?
- What are all possible minimizers?
Lower bounds - Known results

$$k_1[\Gamma] \geq \pi$$

with equality iff $\Gamma$ is a single edge [Nicaise ’87; Friedlander ’05; Kurasov, Naboko ’14].

If $\Gamma$ has all vertex degrees even then

$$k_1[\Gamma] \geq 2\pi, \quad [\text{Kurasov, Naboko ’14}]$$

with a single loop achieving equality (for example).

Remaining questions:

- What about other topologies?
- What are all possible minimizers\(\text{infimizers}\)?
Lower bounds - Known results

\[ k_1[\Gamma] \geq \pi \]

with equality iff \( \Gamma \) is a single edge [Nicase ’87; Friedlander ’05; Kurasov, Naboko ’14].

If \( \Gamma \) has all vertex degrees even then

\[ k_1[\Gamma] \geq 2\pi, \quad [\text{Kurasov, Naboko ’14}] \]

with a single loop achieving equality (for example).

Remaining questions:

- What about other topologies?
- What are all possible minimizers?
Inmizers - Solution

A **bridge** is an edge whose removal disconnects the graph.

**Theorem 2 (Band, Lévy).**

1. Let $G$ be a graph with a bridge. Then
   1.1 The infimal spectral gap of $G$ equals $\pi$.
   1.2 The unique infimizer is the unit interval.

2. Let $G$ be a bridgeless graph. Then
   2.1 The infimal spectral gap of $G$ equals $2\pi$.
   2.2 Any infimizer is a symmetric necklace graph.
A bridge is an edge whose removal disconnects the graph.

Theorem 2 (Band, Lévy).

1. Let $G$ be a graph with a bridge. Then
   1.1 The infimal spectral gap of $G$ equals $\pi$.
   1.2 The unique infimizer is the unit interval.
2. Let $G$ be a bridgeless graph. Then
   2.1 The infimal spectral gap of $G$ equals $2\pi$.
   2.2 Any infimizer is a symmetric necklace graph.

Figure: symmetric necklace graph
Inmizers - Solution

A bridge is an edge whose removal disconnects the graph.

**Theorem 2 (Band, Lévy).**

1. Let $G$ be a graph with a bridge. Then
   1.1 The infimal spectral gap of $G$ equals $\pi$.
   1.2 The unique infimizer is the unit interval.

2. Let $G$ be a bridgeless graph. Then
   2.1 The infimal spectral gap of $G$ equals $2\pi$.
   2.2 Any infimizer is a symmetric necklace graph.

- When is there a minimum?

*Figure: symmetric necklace graph*
Infimizers - Solution

A bridge is an edge whose removal disconnects the graph.

**Theorem 2 (Band, Lévy).**

1. Let $G$ be a graph with a bridge. Then
   1.1 The infimal spectral gap of $G$ equals $\pi$.
   1.2 The unique infimizer is the unit interval.

2. Let $G$ be a bridgeless graph. Then
   2.1 The infimal spectral gap of $G$ equals $2\pi$.
   2.2 Any infimizer is a symmetric necklace graph.

- When is there a minimum?
- Proof idea - rearrangement method on graphs.
Upper bounds - Known results

- **Global bound**
  
  $$k_1[\Gamma] \leq E\pi,$$

  equality if and only if $\Gamma$ is an equilateral mandarin or equilateral flower [Kennedy, Kurasov, Malenová, Mugnolo ’16].

  This fully answers optimization for flowers and mandarins: supremizers (also maximizers) are equilateral.

- If $\Gamma$ is a tree then

  $$k_1[\Gamma] \leq \frac{E}{2}\pi,$$

  equality if and only if $\Gamma$ is an equilateral star [Rohleder ’16].

  This fully answers optimization for trees: supremizers are stars.
Upper bounds - Known results

• Global bound

\[ k_1 [\Gamma] \leq E\pi, \]

equality if and only if \( \Gamma \) is an equilateral mandarin or equilateral flower [Kennedy, Kurasov, Malenová, Mugnolo ’16].

This fully answers optimization for flowers and mandarins: supremizers (also maximizers) are equilateral.

• If \( \Gamma \) is a tree then

\[ k_1 [\Gamma] \leq \frac{E}{2} \pi, \]

equality if and only if \( \Gamma \) is an equilateral star [Rohleder ’16].

This fully answers optimization for trees: supremizers are stars.
Upper bounds - Further progress

**Proposition 3 (Band, Lévy).**

If $\Gamma$ is a tree with $E_l$ leaves then $k_1[\Gamma] \leq \frac{E_l}{2}\pi$.

**Proof idea.**

$d(\Gamma) := \max\{d(x,y) | x, y \in \Gamma\}$ graph diameter.

Combine $k_1[\Gamma] \leq \frac{\pi}{d(\Gamma)}$ with $d(\Gamma) \geq \frac{2}{E_l}$ (the latter true for trees).

**Proposition 4 (Band, Lévy).**

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \ l \in \mathcal{L}_G$, $k_1[\Gamma(G; l)] \leq \pi \left(E - \frac{E_l}{2}\right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma(G; l)$ is either an equilateral mandarin ($E_l = 0$) or an equilateral stower ($E_l \geq 0$).
Upper bounds - Further progress

**Proposition 3 (Band, Lévy).**

If $\Gamma$ is a tree with $E_l$ leaves then $k_1[\Gamma] \leq \frac{E_l}{2}\pi$.

**Proposition 4 (Band, Lévy).**

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \; l \in \mathcal{L}_G, \quad k_1[\Gamma(G; l)] \leq \pi \left(E - \frac{E_l}{2}\right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma(G; l)$ is either an equilateral mandarin ($E_l = 0$) or an equilateral stower ($E_l \geq 0$).

Proof idea.

Take $\Gamma$ and attach two vertices to obtain $\Gamma'$ (illegal move in our game). Get $k_1(\Gamma) \leq k_1(\Gamma')$.

Repeatedly attach all inner vertices to obtain a stower with $E_l$ leaves and $E - E_l$ petals.

Use bound on stowers: $k_1[\Gamma] \leq \pi \left(E - \frac{E_l}{2}\right)$ [to appear in a future slide]
Upper bounds - Further progress

Proposition 3 (Band, Lévy).

If $\Gamma$ is a tree with $E_l$ leaves then $k_1[\Gamma] \leq \frac{E_l}{2}\pi$.

Proposition 4 (Band, Lévy).

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \: l \in \mathcal{L}_G, \quad k_1[\Gamma(G; \: l)] \leq \pi\left( E - \frac{E_l}{2} \right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma(G; \: l)$ is either an equilateral mandarin ($E_l = 0$) or an equilateral stower ($E_l \geq 0$).

Proof idea.

Take $\Gamma$ and attach two vertices to obtain $\Gamma'$ (illegal move in our game). Get $k_1(\Gamma) \leq k_1(\Gamma')$.

Repeatedly attach all inner vertices to obtain a stower with $E_l$ leaves and $E - E_l$ petals.

Use bound on stowers: $k_1[\Gamma] \leq \pi \left( E - \frac{E_l}{2} \right)$ [to appear in a future slide]
Spectral gap as a simple eigenvalue - Critical points

Try to find supremizers by seeking for local critical points in $\mathcal{L}_G$.

Derivatives with respect to edge lengths may be calculated for simple eigenvalues.

Theorem 5 (Band, Lévy).

Let $\mathcal{G}$ be a discrete graph and $l \in \mathcal{L}_G$.

Assume that $\Gamma(\mathcal{G}; l)$ is a supremizer of $\mathcal{G}$ with simple spectral gap $k_1[\Gamma(\mathcal{G}; l)]$.

Then $\Gamma(\mathcal{G}; l)$ is not a unique supremizer:

there exists $l^* \in \overline{\mathcal{L}_G}$ s.t. $\Gamma(\mathcal{G}; l^*)$ is an equilateral mandarin and

$$k_1[\Gamma(\mathcal{G}; l)] = k_1[\Gamma(\mathcal{G}; l^*)].$$

Proof ingredients.

- A supremizer is a critical point of some $\mathcal{L}_{\hat{G}}$ ($\hat{G}$ maybe different than $\mathcal{G}$).

- $\forall e \frac{\partial}{\partial l_e} (k^2) = -(f'^2 + k^2 f^2)|_e$ where $f$ eigenfunction which corresponds to $k$.

- This implies restrictions on eigenfunction derivatives.

- Courant nodal domain theorem. $f$ has exactly two nodal domains.
Spectral gap as a simple eigenvalue - Critical points

Try to find supremizers by seeking for local critical points in $\mathcal{L}_G$.

Derivatives with respect to edge lengths may be calculated for simple eigenvalues.

**Theorem 5 (Band, Lévy).**

Let $\mathcal{G}$ be a discrete graph and $l \in \mathcal{L}_G$.
Assume that $\Gamma(\mathcal{G}; l)$ is a supremizer of $\mathcal{G}$ with simple spectral gap $k_1[\Gamma(\mathcal{G}; l)]$.

Then $\Gamma(\mathcal{G}; l)$ is not a unique supremizer:
there exists $l^* \in \overline{\mathcal{L}_G}$ s.t. $\Gamma(\mathcal{G}; l^*)$ is an equilateral mandarin and

$$k_1[\Gamma(\mathcal{G}; l)] = k_1[\Gamma(\mathcal{G}; l^*)].$$

**Proof ingredients.**

- A supremizer is a critical point of some $\mathcal{L}_{\hat{\mathcal{G}}}$ ($\hat{\mathcal{G}}$ maybe different than $\mathcal{G}$).
- $\forall e \quad \frac{\partial}{\partial l_e} (k^2) = - (f'^2 + k^2 f^2) \big|_e$ where $f$ eigenfunction which corresponds to $k$.
- This implies restrictions on eigenfunction derivatives.
- Courant nodal domain theorem - $f$ has exactly two nodal domains.
Spectral gap as a simple eigenvalue - Critical points

**Theorem 5 (Band, Lévy).**

Let $\mathcal{G}$ be a discrete graph and $l \in \mathcal{L}_\mathcal{G}$.
Assume that $\Gamma(\mathcal{G}; l)$ is a supremizer of $\mathcal{G}$ with simple spectral gap $k_1[\Gamma(\mathcal{G}; l)]$.
Then $\Gamma(\mathcal{G}; l)$ is not a unique supremizer:
there exists $l^* \in \overline{\mathcal{L}_\mathcal{G}}$ s.t. $\Gamma(\mathcal{G}; l^*)$ is an equilateral mandarin and

$$k_1[\Gamma(\mathcal{G}; l)] = k_1[\Gamma(\mathcal{G}; l^*)].$$

**Proof ingredients.**

- A supremizer is a critical point of some $\mathcal{L}_{\hat{\mathcal{G}}}$ ($\hat{\mathcal{G}}$ maybe different than $\mathcal{G}$).
- $\forall e \quad \frac{\partial}{\partial l_e} (k^2) = - (f'^2 + k^2 f^2) \bigg|_e$ where $f$ eigenfunction which corresponds to $k$.
- This implies restrictions on eigenfunction derivatives.
- Courant nodal domain theorem - $f$ has exactly two nodal domains.
Gluing graphs - Vertex connectivity one

Let $G_1, G_2$ be discrete graphs, and $v_i \ (i = 1, 2)$ be a vertex of $G_i$. Let $G$ be the graph obtained by identifying (gluing) $v_1$ and $v_2$. If we know the supremizers $\Gamma_1, \Gamma_2$ of $G_1, G_2$, can we tell the supremizer of $G$?

Corollary 6.

Let $G_1, G_2$ be discrete graphs. Let $G$ obtained by identifying two non-leaf vertices $v_1$ and $v_2$. If the (unique) supremizer of $G_i$ is the “equilateral” stower with $E_p^{(i)}$ petals and $E_l^{(i)}$ leaves, such that $E_p^{(i)} + E_l^{(i)} \geq 2$, then the (unique) supremizer of $G$ is an “equilateral” stower with $E_p^{(1)} + E_p^{(2)}$ petals and $E_l^{(1)} + E_l^{(2)}$ leaves.
Let $\mathcal{G}_1, \mathcal{G}_2$ be discrete graphs, and $v_i$ ($i = 1, 2$) be a vertex of $\mathcal{G}_i$.
Let $\mathcal{G}$ be the graph obtained by identifying (gluing) $v_1$ and $v_2$.
If we know the supremizers $\Gamma_1$, $\Gamma_2$ of $\mathcal{G}_1$, $\mathcal{G}_2$,
can we tell the supremizer of $\mathcal{G}$?

Yes

---

**Corollary 6.**

Let $\mathcal{G}_1, \mathcal{G}_2$ be discrete graphs.
Let $\mathcal{G}$ obtained by identifying two non-leaf vertices $v_1$ and $v_2$.
If the (unique) supremizer of $\mathcal{G}_i$ is the “equilateral” stower
with $E_p^{(i)}$ petals and $E_l^{(i)}$ leaves, such that $E_p^{(i)} + E_l^{(i)} \geq 2$,
then the (unique) supremizer of $\mathcal{G}$ is an “equilateral” stower
with $E_p^{(1)} + E_p^{(2)}$ petals and $E_l^{(1)} + E_l^{(2)}$ leaves.
Gluing graphs - Vertex connectivity one

Let $G_1, G_2$ be discrete graphs, and $v_i$ ($i = 1, 2$) be a vertex of $G_i$. Let $G$ be the graph obtained by identifying (gluing) $v_1$ and $v_2$. If we know the supremizers $\Gamma_1, \Gamma_2$ of $G_1, G_2$, can we tell the supremizer of $G$?

Yes (under some conditions on $k_1(\Gamma_1), k_1(\Gamma_2)$)

For brevity, skip here the theorem and move on to its corollaries.

**Corollary 6.**

Let $G_1, G_2$ be discrete graphs. Let $G$ obtained by identifying two non-leaf vertices $v_1$ and $v_2$. If the (unique) supremizer of $G_i$ is the “equilaterial” stower with $E_p^{(i)}$ petals and $E_l^{(i)}$ leaves, such that $E_p^{(i)} + E_l^{(i)} \geq 2$, then the (unique) supremizer of $G$ is an “equilateral” stower with $E_p^{(1)} + E_p^{(2)}$ petals and $E_l^{(1)} + E_l^{(2)}$ leaves.
Let $G_1, G_2$ be discrete graphs, and $v_i \ (i = 1, 2)$ be a vertex of $G_i$. Let $G$ be the graph obtained by identifying (gluing) $v_1$ and $v_2$. If we know the supremizers $\Gamma_1, \Gamma_2$ of $G_1, G_2$, can we tell the supremizer of $G$?

Yes (under some conditions on $k_1(\Gamma_1), k_1(\Gamma_2)$)

For brevity, skip here the theorem and move on to its corollaries.

**Corollary 6.**

*Let $G_1, G_2$ be discrete graphs. Let $G$ obtained by identifying two non-leaf vertices $v_1$ and $v_2$. If the (unique) supremizer of $G_i$ is the “equilateral” stower with $E_p^{(i)}$ petals and $E_l^{(i)}$ leaves, such that $E_p^{(i)} + E_l^{(i)} \geq 2$, then the (unique) supremizer of $G$ is an “equilateral” stower with $E_p^{(1)} + E_p^{(2)}$ petals and $E_l^{(1)} + E_l^{(2)}$ leaves.*
**Gluing graphs - Corollaries**

**Corollary 7.**

Let $G$ be a stower with $E_p + E_l \geq 2$ and $(E_p, E_l) \neq (1, 1)$. Then a maximizer is the “equilateral” stower graph with spectral gap $\pi \left( E_p + \frac{E_l}{2} \right)$. This maximizer is unique for $(E_p, E_l) \notin \{(2, 0), (1, 2)\}$.

**Proof idea.**

Prove the statement for “small” stowers. Then glue them to construct any stower.

Recall

**Proposition 4:**

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \ l \in \mathcal{L}_G, \ k_1 [\Gamma (G; \ l)] \leq \pi \left( E - \frac{E_l}{2} \right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma (G; \ l)$ is either an equilateral mandarin ($E_l = 0$) or an equilateral stower ($E_l \geq 0$).

We use Corollary 7 in its proof.
**Corollary 7.**

Let $G$ be a stower with $E_p + E_l \geq 2$ and $(E_p, E_l) \neq (1, 1)$. Then a maximizer is the “equilaterial” stower graph with spectral gap $\pi \left( E_p + \frac{E_l}{2} \right)$. This maximizer is unique for $(E_p, E_l) \notin \{(2, 0), (1, 2)\}$.

**Proof idea.**

Prove the statement for “small” stowers. Then glue them to construct any stower.

Recall

**Proposition 4:**

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \ I \in \mathcal{L}_G$, $k_1 \Gamma (G; I) \leq \pi \left( E - \frac{E_l}{2} \right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma (G; I)$ is either an equilaterial mandarin ($E_l = 0$) or an equilaterial stower ($E_l \geq 0$).

We use Corollary 7 in its proof.
**Gluing graphs - Corollaries**

**Corollary 7.**

Let $G$ be a stower with $E_p + E_l \geq 2$ and $(E_p, E_l) \neq (1, 1)$. Then a maximizer is the “equilateral” stower graph with spectral gap $\pi \left( E_p + \frac{E_l}{2} \right)$. This maximizer is unique for $(E_p, E_l) \notin \{(2, 0), (1, 2)\}$.

**Proof idea.**

Prove the statement for “small” stowers. Then glue them to construct any stower.

Recall

**Proposition 4:**

Let $G$ be a graph with $E$ edges, out of which $E_l$ are leaves.

If $(E, E_l) \notin \{(1, 1), (1, 0), (2, 1)\}$ then $\forall \, l \in \mathcal{L}_G$, $k_1 [\Gamma \left( G; \, l \right)] \leq \pi \left( E - \frac{E_l}{2} \right)$.

Assuming $(E, E_l) \notin \{(2, 0), (3, 2)\}$ equality above implies $\Gamma \left( G; \, l \right)$ is either an equilateral mandarin ($E_l = 0$) or an equilateral stower ($E_l \geq 0$).

We use Corollary 7 in its proof.
Summary

- Optimization problem fully solved for infimizers/minimizers.

- Supremizers
  - Improved upper bounds by conditioning on number of leaves, $k_1 \leq \pi \left( E - \frac{E_t}{2} \right)$ (global) and $k_1 \leq \pi \frac{E_t}{2}$ (for trees).
  - Simple spectral gaps are never better than that of the mandarin.
  - Construct supremizer by gluing known supremizers.
Summary

- Optimization problem fully solved for infimizers/minimizers.

- Supremizers
  - Improved upper bounds by conditioning on number of leaves, $k_1 \leq \pi \left( E - \frac{E_l}{2} \right)$ (global) and $k_1 \leq \pi \frac{E_l}{2}$ (for trees).
  - Simple spectral gaps are never better than that of the mandarin.
  - Construct supremizer by gluing known supremizers.
Summary

- Optimization problem fully solved for infimizers/\minimizers.

- Supremizers
  - Improved upper bounds by conditioning on number of leaves,
    \[ k_1 \leq \pi \left( E - \frac{E_l}{2} \right) \] (global) and \[ k_1 \leq \pi \frac{E_l}{2} \] (for trees).
  - Simple spectral gaps are never better than that of the mandarin.
  - Construct supremizer by gluing known supremizers.
Summary

- Optimization problem fully solved for infimizers\minimizers.

- Supremizers
  
  ▶ Improved upper bounds by conditioning on number of leaves,
  \[ k_1 \leq \pi \left( E - \frac{E_l}{2} \right) \text{ (global)} \text{ and } k_1 \leq \pi \frac{E_l}{2} \text{ (for trees)}. \]
  
  ▶ Simple spectral gaps are never better than that of the mandarin.
  
  ▶ Construct supremizer by gluing known supremizers.
Summary

- Optimization problem fully solved for infimizers/\minimizers.

- Supremizers
  - Improved upper bounds by conditioning on number of leaves,
    \[ k_1 \leq \pi \left( E - \frac{E_l}{2} \right) \text{ (global)} \text{ and } k_1 \leq \pi \frac{E_l}{2} \text{ (for trees)}. \]
  - Simple spectral gaps are never better than that of the mandarin.
  - Construct supremizer by gluing known supremizers.
Summary

Supremizer candidates are stowers and mandarins (are there any others?)
⇒ lower bounds on supremal spectral gap

Getting to a stower gives $\pi \left( \beta + \frac{E_l}{2} \right)$,
where $\beta := E - V + 1$ is the graph’s first Betti number.

Getting to a mandarin:
Partition vertices $V = V_1 \cup V_2$.
$E(V_1, V_2) := \#$ of edges connecting $V_1$ to $V_2$.
Maximal spectral gap among all mandarins is
$\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$. (Cheeger-like constant)

Compare $\pi \left( \beta + \frac{E_l}{2} \right)$ (stower) with $\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$ (mandarin).

$E(V_1, V_2) = \beta + 1 - (\beta_1 + \beta_2)$, where $\beta_i$ is the Betti number of $V_i$ graph.

If $E_l \leq 1$ then mandarin wins if and only if we find $\beta_1 = \beta_2 = 0$.
If $E_l \geq 2$ then mandarin never wins (possibility for a tie).
Summary

Supremizer candidates are stowers and mandarins (are there any others?)
⇒ lower bounds on supremal spectral gap

Getting to a stower gives $\pi \left( \beta + \frac{E_l}{2} \right)$,
where $\beta := E - V + 1$ is the graph’s first Betti number.

Getting to a mandarin:
Partition vertices $V = V_1 \cup V_2$.
$E(V_1, V_2) := \# \text{ of edges connecting } V_1 \text{ to } V_2$.
Maximal spectral gap among all mandarins is
$\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$. (Cheeger-like constant)

Compare $\pi \left( \beta + \frac{E_l}{2} \right)$ (stower) with $\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$ (mandarin).

$E(V_1, V_2) = \beta + 1 - (\beta_1 + \beta_2)$, where $\beta_i$ is the Betti number of $V_i$ graph.
If $E_l \leq 1$ then mandarin wins if and only if we find $\beta_1 = \beta_2 = 0$.
If $E_l \geq 2$ then mandarin never wins (possibility for a tie).
Summary

Supremizer candidates are stowers and mandarins (are there any others?)
⇒ lower bounds on supremal spectral gap

Getting to a stower gives $\pi \left( \beta + \frac{E_l}{2} \right)$,
where $\beta := E - V + 1$ is the graph’s first Betti number.

Getting to a mandarin:
Partition vertices $V = V_1 \cup V_2$.
$E(V_1, V_2) := \#$ of edges connecting $V_1$ to $V_2$.
Maximal spectral gap among all mandarins is
$\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$. (Cheeger-like constant)

Compare $\pi \left( \beta + \frac{E_l}{2} \right)$ (stower) with $\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$ (mandarin).

$E(V_1, V_2) = \beta + 1 - (\beta_1 + \beta_2)$, where $\beta_i$ is the Betti number of $V_i$ graph.
If $E_l \leq 1$ then mandarin wins if and only if we find $\beta_1 = \beta_2 = 0$.
If $E_l \geq 2$ then mandarin never wins (possibility for a tie).
Summary

Supremizer candidates are stowers and mandarins (are there any others?)
⇒ lower bounds on supremal spectral gap

Getting to a stower gives $\pi \left( \beta + \frac{E_l}{2} \right)$,
where $\beta := E - V + 1$ is the graph’s first Betti number.

Getting to a mandarin:
Partition vertices $V = V_1 \cup V_2$.
$E(V_1, V_2) := \#$ of edges connecting $V_1$ to $V_2$.
Maximal spectral gap among all mandarins is
$\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$. (Cheeger-like constant)

Compare $\pi \left( \beta + \frac{E_l}{2} \right)$ (stower) with $\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$ (mandarin).

$E(V_1, V_2) = \beta + 1 - (\beta_1 + \beta_2)$, where $\beta_i$ is the Betti number of $V_i$ graph.

If $E_l \leq 1$ then mandarin wins if and only if we find $\beta_1 = \beta_2 = 0$.
If $E_l \geq 2$ then mandarin never wins (possibility for a tie).
Summary

Getting to a stower gives $\pi \left( \beta + \frac{E_l}{2} \right)$, where $\beta := E - V + 1$ is the graph’s first Betti number.

Getting to a mandarin:
Partition vertices $V = V_1 \cup V_2$.
$E(V_1, V_2) := \#$ of edges connecting $V_1$ to $V_2$.
Maximal spectral gap among all mandarins is
$\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$. (Cheeger-like constant)

Compare $\pi \left( \beta + \frac{E_l}{2} \right)$ (stower) with $\pi \cdot \max_{V_1, V_2} E(V_1, V_2)$ (mandarin).

$E(V_1, V_2) = \beta + 1 - (\beta_1 + \beta_2)$, where $\beta_i$ is the Betti number of $V_i$ graph.

If $E_l \leq 1$ then mandarin wins if and only if we find $\beta_1 = \beta_2 = 0$.
If $E_l \geq 2$ then mandarin never wins (possibility for a tie).

Leads to conjectures....
Conjectures

- Supremizer is either a mandarin or a stower.
- Supremum is obtained when order of symmetry group is maximized.
- Supremum is obtained when multiplicity of spectral gap is maximized.
Conjectures

• Supremizer is either a mandarin or a stower.

• Supremum is obtained when order of symmetry group is maximized.

• Supremum is obtained when multiplicity of spectral gap is maximized.
Conjectures

- Supremizer is either a mandarin or a stower.
- Supremum is obtained when order of symmetry group is maximized.
- Supremum is obtained when multiplicity of spectral gap is maximized.
Quantum Graphs which Optimize the Spectral Gap

Ram Band

Technion - Israel Institute of Technology

Joint work with Guillaume Lévy, Université Pierre et Marie Curie, Paris
(arXiv:1608.00520)

QMath 13, GeorgiaTech, Atlanta - October 2016