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Geometry of Bi-Warped Product Submanifolds of Nearly Trans-Sasakian Manifolds

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Abstract: In the present work, we consider two types of bi-warped product submanifolds, $M = M_T \times f_i M_L \times f_k M_\phi$ and $M = M_\phi \times f_j M_T \times f_k M_1$, in nearly trans-Sasakian manifolds and construct inequalities for the squared norm of the second fundamental form. The main results here are a generalization of several previous results. We also design some applications, in view of mathematical physics, and obtain relations between the second fundamental form and the Dirichlet energy. The relationship between the eigenvalues and the second fundamental form is also established.

Keywords: geometric inequalities; nearly trans-sasakian; nearly sasakian; bi-warped products; dirichlet energy

1. Introduction

Let $M = M_1 \times M_2 \times M_3 \times \cdots \times M_k$ be the Cartesian product of Riemannian manifolds $M_1, M_2, \cdots, M_k$ and let $\pi_i : M \to M_k$ denote the canonical projection maps for $i = 1, \cdots, k$. If the positive-valued functions $f_1, \cdots, f_k$ are defined such that $f_1, \cdots, f_k : M_1 \to (0, \infty)$, then the Riemannian metric $g$ is defined as

$$g(X, Y) = g(\pi_1^* X, \pi_1^* Y) + \sum_{i=1}^{k} (f_i \circ \pi) g(\pi_i^* X, \pi_i^* Y),$$

where * is the symbol for tangent map, for any $X, Y$ tangent to $M$, then $M$ is called a multiple warped product manifold [1,2]. If we choose two fibers of a multiple warped product $M^1 \times f_1 M^2 \times \cdots \times f_k M^k$, such that $M = M_1 \times f_1 M_2 \times f_2 M_3$, then $M$ is defined as a bi-warped product submanifold, which satisfies the following:

$$\nabla_{U_1} Z = \sum_{i=2}^{3} (U_1 \ln f_i) Z_i,$$

where $U_1 \in \Gamma(TM_1), Z \in \Gamma(T(M_2 \times M_3))$ and $Z_i$ tangent to $M_i$, for each $i = 2, 3$. Moreover, $\nabla$ is the Levi–Civita connection and for more details, see [3–5]. An odd-dimensional analog of the nearly Kähler metric is the nearly Sasakian metric. The nearly Kähler cone over a Sasakian Einstein manifold has applications in physics. The Sasakian geometry has been extensively studied, due to its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics—for example, related to p-brane solutions in superstring theory and the Maldacena conjecture [6]. On the other hand, the vanishing of Dirichlet energies is equivalent to the Dirichlet condition with the unique solution of Poisson’s equation $\nabla^2 f = -4 f \rho$. This implies Neumann or Dirichlet boundary conditions, as classified by the electrostatic problem (see, e.g., [7]). In the present paper, we consider the bi-warped product submanifolds in a nearly trans-Sasakian manifold, inspired by the publication of Taştan’s seminal work [5], and obtained some inequalities for the Dirichlet energy and the second fundamental form. The study of bi-warped product...
submanifolds with two distinct fibers has been a topic of great interest; see, e.g., Naghi et al. [4], Ali et al. [8], Siraj et al. [9–11], and Awatif et al. [12]. It has been noted that the class of bi-warped product submanifolds is a generalization of several classes, such as CR-warped products, warped product semi-slant submanifolds, and warped product pseudo-slant submanifolds. On the other hand, as a generalization of nearly cosymplectic, nearly Sasakian [13], nearly Kenmotsu [8,14], nearly a-Sasakian, and nearly β-Kenmotsu manifolds, nearly trans-Sasakian manifolds have been studied on a large scale; see [15–19]. Therefore, our objective was to remove the gap in the nearly trans-Sasakian manifold literature, as they are an interesting structure of the almost contact manifolds that have generalized many others structures. The main goal of this paper was to discuss the geometry of bi-warped product submanifolds of the types $M_{T} \times_{f_{1}} M_{\perp} \times_{f_{2}} M_{\phi}$ and $M = M_{\phi} \times_{f_{1}} M_{T} \times_{f_{2}} M_{1}$ in a nearly trans-Sasakian manifold. Some inequalities for the length of the second fundamental form are obtained, including the length of warping functions and slant immersions. Various inequalities, which have been derived in [13,14,18–27], can be recovered from our inequalities under some conditions. Therefore, our results may find applications in mathematical physics.

2. Preliminaries

An odd-dimensional $C^\infty$-manifold $(\widetilde{M}, g)$ associated with an almost contact structure $(\psi, \zeta, \eta)$ is referred to as an almost contact metric manifold if there exist a $(1,1)$ tensor field $\psi$, a vector field $\xi$ (called a characteristic or Reeb vector field), and a 1-form $\eta$ satisfying the following conditions:

\begin{align*}
\psi^2 &= -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \psi(\zeta) = 0, \quad \eta \circ \psi = 0, \quad (2) \\
g(\psi U_1, \psi W_2) &= g(U_1, W_2) - \eta(U_1)\eta(W_2), \quad \eta(U_1) = g(U_1, \zeta), \quad (3) \\
\forall U_1, W_2 \in \Gamma(TM),
\end{align*}

where $\eta$ is the Riemannian connection associated with the metric $g$ on $\widetilde{M}$. According to the structure, we have the following classifications:

(i) A nearly trans-Sasakian $\widetilde{M}$ is a nearly cosymplectic if $\alpha = 0$ and $\beta = 0$ in (4).

(ii) If $\alpha = 1$ and $\beta = 0$ in (4), then $\widetilde{M}$ is a nearly Sasakian manifold under the condition

\begin{equation}
(\nabla_{U_1} \psi)V_1 + (\nabla_{V_1} \psi)U_1 = \alpha \left( 2g(U_1, V_1)\zeta - \eta(U_1)V_1 - \eta(V_1)U_1 \right) \\
- \beta \left( \eta(V_1)\psi U_1 + \eta(U_1)\psi V_1 \right), \quad (4)
\end{equation}

for any $U_1, V_1 \in \Gamma(TM)$, where is the Riemannian connection associated with the metric $g$ on $\widetilde{M}$. According to the structure, we have the following classifications:

(iii) A nearly trans-Sasakian $\widetilde{M}$ is a nearly Kenmotsu manifold if $\alpha = 0$ and $\beta = 1$ in (4).

(iv) Similarly, nearly $a$-Sasakian and nearly $\beta$-Kenmotsu manifolds can be defined from a nearly trans-Sasakian manifold by substituting $\beta = 0$ and $\alpha = 0$ in (4), respectively.

The Gauss and Weingarten formulas, which specify the relation between Levi–Civita connections $\nabla$ on a submanifold $M$ and $\nabla$ on an ambient manifold $\widetilde{M}$, are given by (for more detail, see [28]):

\begin{align*}
\nabla_{U_1} W_2 &= \nabla_{U_1} W_2 + B(U_1, W_2), \\
\nabla_{U_1} \zeta &= -A_{\zeta} U_1 + \nabla_{U_1} \zeta,
\end{align*}

for every $U_1, W_2 \in \Gamma(TM)$ and $\zeta \in \Gamma(T^{\perp}M)$. In addition, $B$ and $A_{\zeta}$ are the second fundamental form and shape operator, respectively, having the relation $g(B(U_1, W_2), \zeta) =$
\(g(A_\xi \mathcal{U}_1, \mathcal{V}_2)\). If we replace \(\mathcal{U}_1 = \xi, \mathcal{V}_1 = \xi\) in (4), we find that \((\tilde{\nabla}_\xi \psi) \xi = 0\), which implies that \(\psi \tilde{\nabla}_\xi \xi = 0\). Applying \(\psi\) and using (4), we get \(\tilde{\nabla}_\xi \xi = 0\). From the Gauss formula, we get \(\tilde{\nabla}_\xi \xi = 0\) and \(h(\xi, \xi) = 0\). For more classification, see [18,19]. We also have:

\[\psi \xi = \mathbf{T} \mathcal{U}_1 + \mathbf{F} \mathcal{U}_1, \tag{8}\]

in which \(\mathbf{F} \mathcal{U}_1\) and \(\mathbf{T} \mathcal{U}_1\) are normal and tangential elements of \(\psi \mathcal{U}_1\), respectively. If \(M\) is invariant and/or anti-invariant, then \(\mathbf{F} \mathcal{U}_1\) and/or \(\mathbf{T} \mathcal{U}_1\) are zero, respectively. Similarly, we have

\[\psi \xi = t_\xi + f_\xi, \tag{9}\]

where \(t_\xi\) (respectively \(f_\xi\)) are tangential (respectively normal) components of \(\psi \xi\). The covariant derivative of the endomorphism \(\psi\) is defined by

\[(\tilde{\nabla}_\xi \psi) \mathcal{X}_5 = \tilde{\nabla}_{\xi \mathcal{X}_5} \psi \mathcal{X}_5 - \psi \tilde{\nabla}_\xi \mathcal{X}_5, \quad \forall \mathcal{X}_5, \mathcal{X}_5 \in \Gamma(TM). \tag{10}\]

There is a motivating class of submanifolds, presented as slant submanifolds. For any non-zero vector \(U_1\) tangential to \(M\) at point \(p\), such that \(U_1\) is not proportional to \(\zeta_p\), \(0 \leq \phi(U_1) \leq \pi/2\) refers to the angle between \(\psi U_1\) and \(T_p M\), which is named the Wirtinger angle. If \(\phi(U_1)\) is constant for any \(U_1 \in T_p M\), \(\phi < \xi >\) at point \(p \in M\), then \(M\) is referred to as the slant submanifold [29] and \(\phi\) is the slant angle of \(M\). We consider the following necessary and sufficient for a submanifold \(M\) to be a slant submanifold [29,30]:

\[\mathbf{T}^2 = \cos^2 \phi (-I + \eta \otimes \zeta), \tag{11}\]

for \(0 \leq \phi \leq \frac{\pi}{2}\) and \(\mathbf{T}\) is an endomorphism defined in (8). The following result is from Equation (11):

\[g(\mathbf{T} \mathcal{U}_1, \mathbf{T} \mathcal{W}_2) = \cos^2 \phi \left\{ g(U_1, \mathcal{W}_2) - \eta(U_1) \eta(U_1) \right\}, \tag{12}\]

\[g(\mathbf{F} \mathcal{U}_1, \mathbf{F} \mathcal{W}_2) = \sin^2 \phi \left\{ g(U_1, \mathcal{W}_2) - \eta(U_1) \eta(\mathcal{W}_2) \right\}, \tag{13}\]

\(\forall U_1, \mathcal{W}_2 \in \Gamma(TM)\). The following are the definitions of semi-slant and psuedo-slant submanifolds.

**Definition 1.** A submanifold \(M\) of an almost contact metric manifold \(\tilde{M}\) is said to be a semi-slant submanifold if there exists a pair of orthogonal distributions, \(\mathbf{D}\) and \(\mathbf{D}^\phi\), on \(M\) such that

(i) the tangent bundle \(TM\) admits the orthogonal direct decomposition \(TM = \mathbf{D} \oplus \mathbf{D}^\phi \oplus \zeta\);

(ii) the distribution \(\mathbf{D}\) is invariant under \(\psi\) (i.e., \(\psi \mathbf{D} = \mathbf{D}\)); and

(iii) the distribution \(\mathbf{D}^\phi\) is slant, with slant angle \(\phi\).

**Definition 2.** A submanifold \(M\) of an almost contact metric manifold \(\tilde{M}\) is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions, \(\mathbf{D}\) and \(\mathbf{D}^\phi\), on \(M\) such that

(i) the tangent bundle \(TM\) admits the orthogonal direct decomposition \(TM = \mathbf{D}^\perp \oplus \mathbf{D}^\phi \oplus \zeta\);

(ii) the distribution \(\mathbf{D}^\perp\) is totally real under \(\psi\) (i.e., \(\psi \mathbf{D}^\perp \subset T^\perp M\)); and

(iii) the distribution \(\mathbf{D}^\phi\) is slant, with slant angle \(\phi\).

The mean curvature vector, \(\mathcal{H}\), for an orthonormal frame \(\{e_1, e_2, \ldots, e_n\}\) of the tangent space \(TM\) on \(M^n\) is defined by

\[\mathcal{H} = \frac{1}{n} \text{trace}(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{B}(e_i, e_i), \tag{14}\]
where $n = \dim M$. In addition, we set
\[
||B||^2 = \sum_{\alpha, \beta = 1}^n g(B(e_{\alpha}, e_{\beta}), B(e_{\alpha}, e_{\beta})),
\] (15)
where $||B||^2$ is the length of the second fundamental form.

For a nearly trans-Sasakian manifold $\tilde{M}$, a bi-warped product submanifold $M = M_T \times f_1 M_{\perp} \times f_2 M_{\phi}$ in $\tilde{M}$, where $M_T, M_{\perp}, M_{\phi}$ refer to holomorphic, totally real, and proper slant submanifolds of $\tilde{M}$, respectively. Suppose that $TM = D \oplus D_{\perp} \oplus D_{\phi}$ and $T_{\perp}M = \phi D_{\perp} \oplus FD_{\phi} \oplus \mu$, where $\mu$ is classified as a $\psi$-invariant normal sub-bundle of $T_{\perp}M$.

The next practical consequence will be used later in this paper.

**Proposition 1.** Let $M = M_1 \times f_1 M_2 \times f_2 M_3$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then, $M$ is only a single warped product if the structure vector field $\zeta$ is tangent to the fiber; that is, either $\zeta \in \Gamma(D_2)$ or $\zeta \in \Gamma(D_3)$.

**Proof.** If we take $\zeta \in \Gamma(D_2)$, for any $U \in \Gamma(D_2)$, we then get
\[
\zeta(\ln f_a) = \beta,
\] \[\forall a = 1, 2, \] (16)
for each structure vector field $\zeta$ is tangent to $M_1$.

**Proposition 2.** Let $M = M_1 \times f_1 M_2 \times f_2 M_3$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then,
\[
\zeta(\ln f_a) = \beta,
\] \[\forall a = 1, 2, \] (16)
for each structure vector field $\zeta$ is tangent to $M_1$.

**Proof.** For any $X_0 \in \Gamma(D_1)$, we expand the following:
\[
\nabla_{X_0} \zeta = \beta X_0.
\]
Using (1) and (8), we obtain
\[
\zeta(\ln f_1) X_0 = \beta X_0.
\]
Taking the inner product with $X_0$, we obtain
\[
\zeta(\ln f_1) = \beta.
\]
Similarly, we can find that
\[
\zeta(\ln f_2) = \beta.
\]
Therefore, the statement is proved. 

**Remark 1.** It can be noticed that, if the structure vector field $\zeta$ is tangent to any fiber, then the warped product will be trivial due to Proposition 1. On the other hand, Proposition 2 assures that the warped product is always non-trivial if the structure vector field $\zeta$ is tangent to any base manifold of a nearly trans-Sasakian manifold.

**Lemma 1.** Let $M = M_T \times_f M_1 \times_f M_\varphi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Therefore, we have

$$g(B(X_0, X_1), \psi X_2) = 0, \quad g(B(X_0, X_1), \phi X_4) = 0,$$

$$g(B(X_0, X_2), \psi X_3) = -\left(\psi X_0(\ln f_1) + \alpha \eta(X_0)\right)g(X_2, X_3),$$

$$g(B(\psi X_0, X_2), \psi X_3) = \left(X_0(\ln f_1) - \eta(X_0)\right)g(X_2, X_3),$$

$$g(B(\zeta, X_2), \psi X_3) = -\alpha g(X_2, X_3),$$

for any $X_0, X_1 \in \Gamma(D), X_2, X_3 \in \Gamma(D^\perp)$ and $X_4 \in \Gamma(D^\varphi)$.

**Proof.** For all $X_0, X_1 \in \Gamma(D)$ and $X_2 \in \Gamma(D^\perp)$, we have

$$g(B(X_0, X_1), \psi X_2) = g(\nabla_{X_0} X_1, \psi X_2) = g((\nabla_{X_0} \psi)X_1, X_2) - g(\nabla_{X_0} \psi X_1, X_2).$$

Equation (1) gives the following:

$$g(B(X_0, X_1), \psi X_2) = g((\nabla_{X_0} \psi)X_1, X_2) + X_0(\ln f_1)g(\psi X_1, X_2).$$

The normality of vector fields implies that

$$g(B(X_0, X_1), \psi X_2) = g((\nabla_{X_0} \psi)X_1, X_2).$$

Replacing $X_0$ with $X_1$ in (22) gives

$$g(B(X_0, X_1), \psi X_2) = g((\nabla_{X_1} \psi)X_0, X_2).$$

Combining Equations (22) and (23) results in

$$2g(B(X_0, X_1), \psi X_2) = g((\nabla_{X_0} \psi)X_1 + (\nabla_{X_1} \psi)X_0, X_2).$$

Then, using Equation (4), we arrive at the first part (17). The second part, (18), can be obtained through a similar process as the first part. For the next part, we have:

$$g(B(X_0, X_2), \psi X_3) = g(\nabla_{X_2} X_0, \psi X_3) = g((\nabla_{X_2} \psi)X_0, X_3) - g(\nabla_{X_2} \psi X_0, X_3)$$
for all $\mathcal{X}_0 \in \Gamma(D)$ and $\mathcal{X}_2, \mathcal{X}_3 \in \Gamma(D^\perp)$. From (4), (6), and (1), we obtain

\[ g(B(\mathcal{X}_0, \mathcal{X}_2), \psi W) = -g((\nabla_{\mathcal{X}_2} \psi) \mathcal{X}_2, \mathcal{X}_3) + a \left\{ g(\frac{\partial}{\partial t}, \mathcal{X}_3) g(\mathcal{X}_0, \mathcal{X}_2)
\right.
\]

\[ - \eta(\mathcal{X}_2) g(\mathcal{X}_0, \mathcal{X}_3) - \eta(\mathcal{X}_0) g(\mathcal{X}_2, \mathcal{X}_3) \right\}
\]

\[ - \beta \left\{ g(\mathcal{X}_0) g(\psi \mathcal{X}_2, \mathcal{X}_3) + \eta(\mathcal{X}_2) g(\psi \mathcal{X}_0, \mathcal{X}_3) \right\} - \psi \mathcal{X}_0 (\ln f_1) g(\mathcal{X}_2, \mathcal{X}_3)
\]

\[ = -g(\nabla_{\mathcal{X}_0} \psi \mathcal{X}_2, \mathcal{X}_3) + g(\nabla_{\mathcal{X}_0} \mathcal{X}_2, \mathcal{X}_3) - \psi \mathcal{X}_0 (\ln f_1) g(\mathcal{X}_2, \mathcal{X}_3)
\]

\[ - \alpha \eta(\mathcal{X}_0) g(\mathcal{X}_2, \mathcal{X}_3). \]

This implies the following:

\[ 2g(B(\mathcal{X}_0, \mathcal{X}_2), \psi \mathcal{X}_3) = g(B(\mathcal{X}_0, \mathcal{X}_3), \psi \mathcal{X}_2) - \psi \mathcal{X}_0 (\ln f_1) g(\mathcal{X}_2, \mathcal{X}_3)
\]

\[ - \alpha \eta(\mathcal{X}_0) g(\mathcal{X}_2, \mathcal{X}_3). \]

Replacing $\mathcal{X}_2$ with $\mathcal{X}_3$ in (24) results in

\[ 2g(B(\mathcal{X}_0, \mathcal{X}_3), \psi \mathcal{X}_2) = g(B(\mathcal{X}_0, \mathcal{X}_2), \psi \mathcal{X}_3) - \psi \mathcal{X}_0 (\ln f_1) g(\mathcal{X}_2, \mathcal{X}_3)
\]

\[ - \alpha \eta(\mathcal{X}_0) g(\mathcal{X}_2, \mathcal{X}_3), \]

which implies Equation (19) from (24) and (25). Now, replacing $\mathcal{X}_0$ with $\psi \mathcal{X}_0$ in (19) and using (2), we obtain (20). If we substitute $\mathcal{X}_0 = \zeta$ in (19), we finally reach (21). This completes the proof of the lemma. □

**Lemma 2.** Let $M = M_T \times f_1 M_{\perp} \times f_2 M_{\phi}$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then, we have

\[ g(B(\mathcal{X}_0, \mathcal{X}_2), FX_4) = \frac{1}{5} g(B(\mathcal{X}_0, \mathcal{X}_4), \psi \mathcal{X}_2) = 0, \]

\[ g(B(\mathcal{X}_0, \mathcal{X}_4), FX_5) = \frac{1}{3} \left\{ (\mathcal{X}_0 \ln f_2) - \beta \eta(\mathcal{X}_0) \right\} g(TX_4, \mathcal{X}_5)
\]

\[ - \psi \mathcal{X}_0 (\ln f_2) + \alpha \eta(\mathcal{X}_0) \right\} g(\mathcal{X}_4, \mathcal{X}_5), \]

\[ g(B(\psi \mathcal{X}_0, \mathcal{X}_4), FX_5) = \frac{1}{3} (\mathcal{X}_0 \ln f_2) g(TX_4, \mathcal{X}_5) + \{ \mathcal{X}_0 (\ln f_2) - \eta(\mathcal{X}_0) \} g(\mathcal{X}_4, \mathcal{X}_5), \]

\[ g(B(\mathcal{X}_0, \mathcal{X}_4), FTX_5) = \frac{1}{3} \left\{ (\mathcal{X}_0 \ln f_2) - \beta \eta(\mathcal{X}_0) \right\} \cos^2 \phi g(X_4, X_5)
\]

\[ - \psi \mathcal{X}_0 (\ln f_2) + \alpha \eta(\mathcal{X}_0) \right\} g(\mathcal{X}_4, TX_5), \]

\[ g(B(\psi \mathcal{X}_0, \mathcal{X}_4), FTX_5) = \frac{1}{3} (\mathcal{X}_0 \ln f_2) \cos^2 \phi g(X_4, X_5)
\]

\[ + \{ \mathcal{X}_0 (\ln f_2) - \eta(\mathcal{X}_0) \} g(TX_4, X_5), \]

\[ g(B(\mathcal{X}_0, TX_4), FX_5) = \frac{1}{3} \left\{ (\mathcal{X}_0 \ln f_2) - \beta \eta(\mathcal{X}_0) \right\} \cos^2 \phi g(X_4, X_5)
\]

\[ - \psi \mathcal{X}_0 (\ln f_2) + \alpha \eta(\mathcal{X}_0) \right\} g(TX_4, X_5), \]

\[ g(B(\psi \mathcal{X}_0, TX_4), FX_5) = -\frac{1}{3} (\mathcal{X}_0 \ln f_2) \cos^2 \phi g(X_4, X_5)
\]

\[ + \{ \mathcal{X}_0 (\ln f_2) - \eta(\mathcal{X}_0) \} g(TX_4, X_5), \]
the following:

Therefore, it follows from (6), (7), and (1) that

\[ g(B(x_0, x_4), \psi x_2) = (x_0 \ln f_2)g(T x_4, x_2) + g(A x_4 x_0, x_2) - g(B(x_0, x_4), \psi x_2). \]

Once more, by the use of (10)–(11), we construct

\[ g(B(x_0, x_4), \psi x_2) = \frac{1}{2} g(B(x_0, x_2), F x_4). \]

From (36) and (37), we obtain the second part of (i). For the third part, we calculate the following:
\[ g(B(x_0, x_4), Fx_5) = g(\tilde{\nabla}_{x_0} x_0, \psi x_5) - g(\tilde{\nabla}_{x_0} x_0, T x_5) \]
\[ = -g(\psi \tilde{\nabla}_{x_0} x_0, x_5) - (x_0 \ln f_2)g(x_4, T x_5) \]
\[ = g((\tilde{\nabla}_{x_0} \psi) x_0, x_5) - g(\tilde{\nabla}_{x_0} x_0, x_5) - (x_0 \ln f_2)g(x_4, T x_5) \]
\[ = -g((\tilde{\nabla}_{x_0} \psi) x_4, x_5) + \alpha \left\{ 2g(x_0, x_4)g(\xi, x_4) \right. \]
\[ \left. - \eta(x_4)g(x_0, x_5) - \eta(x_5)g(x_4, x_5) \right\} \]
\[ - \beta \left\{ \eta(x_0)g(x_4, x_5) - \eta(x_4)g(x_0, x_5) \right\} \]
\[ - (\psi x_0 \ln f_2)g(x_4, x_5) - (x_0 \ln f_2)g(x_4, T x_5) \]
\[ = g(\psi(\tilde{\nabla}_{x_0} x_4, x_5) - g(\tilde{\nabla}_{x_0} \psi x_4, x_5) - \beta \eta(x_4)g(T x_4, x_5) \]
\[ - (\psi x_0 \ln f_2)g(x_4, x_5) - (x_0 \ln f_2)g(x_4, T x_5) - \alpha \eta(x_0)g(x_4, x_5) \]
\[ = -g(\tilde{\nabla}_{x_0} x_5, x_5) - g(\tilde{\nabla}_{x_0} x_4, x_5) - g(\tilde{\nabla}_{x_0} x_4, T x_5) - \beta \eta(x_0)g(T x_4, x_5) \]
\[ - g(\tilde{\nabla}_{x_0} x_4, F x_5) - (\psi x_0 \ln f_2)g(x_4, x_5) - (x_0 \ln f_2)g(x_4, T x_5) \]
\[ - \alpha \eta(x_0)g(x_4, x_5). \]

By the use of (6)–(7) and (8), we have
\[ 2g(B(x_0, x_4), Fx_5) = g(B(x_0, x_5), Fx_4) - (x_0 \ln f_2)g(x_4, T x_5) \]
\[ - (\psi x_0 \ln f_2)g(x_4, x_5) - \beta \eta(x_0)g(T x_4, x_5) \]
\[ - \alpha \eta(x_0)g(x_4, x_5). \]

Equations (38) and (39) give the second part. The rest of the terms are derived by interchanging \(x_0\) with \(\psi x_0\), as well as \(x_4\) and \(x_5\) with \(T x_4\) and \(T x_5\), respectively. Thus, the proof is completed. \(\square\)

**Remark 2.** If \(B(U_1, x_2) = 0 \quad \forall U_1 \in \Gamma(D), \) and \(\forall x_2 \in \Gamma(D^\perp), \) \(B(U_2, x_3) = 0 \quad \forall U_2 \in \Gamma(D^\perp), \) and \(\forall x_3 \in \Gamma(D^\phi), \) then the bi-warped product submanifold \(M = M_T \times f_1 M_\perp \times f_2 M_\phi\) in a nearly trans-Sasakian manifold \(M\) refers to a \(D \oplus D^\perp\)-mixed totally geodesic (respectively, \(D \oplus D^\phi\))-mixed totally geodesic.

**Remark 3.** It easily proved that a proper bi-warped product submanifold \(M = M_T \times f_1 M_\perp \times f_2 M_\phi\) of a nearly trans-Sasakian manifold \(M\) is trivial, by using the \(D \oplus D^\phi\)-mixed totally geodesic and \(D \oplus D^\perp\)-mixed totally geodesic in (19) and (26).

Now, we are in the position to give the proof of our main result. More precisely, we give the following inequality theorem for bi-warped product submanifolds of the type \(M_T \times f_1 M_\perp \times f_2 M_\phi\).
Theorem 1. Let $M = M_T \times f_1 M_\perp \times f_2 M_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then, the following inequality is satisfied for the second fundamental form $B$:

$$
\|B\|^2 \geq 2n_2 \left\{ \|\nabla (\ln f_1)\|^2 + \alpha^2 - \beta^2 \right\} + n_3 \left\{ \left( \frac{2}{3} \cot^2 \phi + 2 \csc^2 \phi \right) \left( \|\nabla (\ln f_2)\|^2 - \beta^2 \right) + \alpha^2 \right\},
$$

(40)

where $n_2 = \dim M_\perp$, $n_3 = \dim M_\phi$. Moreover, $\nabla (\ln f_a)$ is the gradient of $\ln f_a$ for $a = 1, 2$, while $\xi \in M_T$, $M_\phi$, and $M_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\tilde{M}$, respectively. If the inequality (40) becomes an equality, then $M_\perp$ and $M_\phi$ are totally umbilical and $M_T$ is geodesic totally in $\tilde{M}$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $\tilde{M}$.

Proof. Assume that $M = M_T \times f_1 M_\perp \times f_2 M_\phi$ is an $n$-dimensional proper bi-warped product submanifold of the nearly trans-Sasakian manifold $\tilde{M}_{2n+1}$. In addition, let the local orthonormal vector fields $v_1, \ldots, v_n$ of $TM$ be as follows:

$$
\begin{align*}
D &= \text{span}\{v_1, \ldots, v_r, v_{r+1} = \psi v_1, \ldots, v_{2l} = \psi v_l\} \\
D^\perp &= \text{span}\{v_{2l+1} = \tilde{v}_1, \ldots, v_{2l+2l} = \tilde{v}_l\} \\
D^\phi &= \text{span}\{v_{2l+1} = v^1, \ldots, v_{2l+2l+k} = v^k, v_{2l+2l+k+1} = \sec \phi T v^1, \\
&\ldots v_n = \sec \phi T v^k\}.
\end{align*}
$$

Therefore, $\dim M_T = n_1 = 2l + 1$, $\dim M_\perp = n_2 = l$, and $\dim M_\phi = n_3 = 2k$. Furthermore, the orthonormal frame fields $v_1, \ldots, v_{2m+1-n-l-2k}$ of the normal sub-bundle $T^\perp M$ are as follows:

$$
\begin{align*}
\psi D^\perp &= \{v_1 = \psi \tilde{v}_1, \ldots, v_1 = \psi \tilde{v}_l\} \\
FD^\phi &= \text{span}\{v_{l+1} = \tilde{v}_1 = \csc \phi F v^1, \ldots, v_{l+k} = \csc \phi F v_k, \\
v_{l+k+1} = \csc \phi \sec \phi FT v^1, \ldots, v_{l+2k+1} = \csc \phi \sec \phi FT v_k\}
\end{align*}
$$

$$
\mu = \text{span}\{v_{l+2k+1}, \ldots, v_{2m+1-n-l-2k}\}.
$$

According the definition $B$, we have

$$
\|B\|^2 = \sum_{a,b=1}^n g\left(B(v_a, v_b), B(v_a, v_b)\right).
$$

The above equation can be broken into the components of submanifolds of $M_T$, $M_\phi$, and $M_\perp$ as follows:

$$
\begin{align*}
\|B\|^2 &= \sum_{r=1}^{l} \sum_{a,b=1}^n g^2(B(v_a, v_b), \psi \tilde{v}_r) + \sum_{r=1}^{l+2k} \sum_{a,b=1}^n g^2(B(v_a, v_b), v_r) \\
&\quad + \sum_{r=l+2k+1}^{2m+1-n-l-2k} \sum_{a,b=1}^n g^2(B(v_a, v_b), v_r),
\end{align*}
$$

(41)
where clearly $\dim \mu = 2m + 1 - n - l - 2k$. Expanding the above equation, according to the orthonormal bases of $D^\theta, D$, and $D^\perp$ (except for the last term), we arrive at

$$
\|B\|^2 \geq \sum_{r=1}^{l} \sum_{a,b=1}^{2k+1} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g(B(\varphi_a, \varphi_b), \psi \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k+1} g^2(B(v_a, v_b^*), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k+1} g^2(B(v_a^*, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a^*, v_b^*), \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k+1} g(B(v_a, v_b^*), \varphi_r) + 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b^*), \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k+1} g^2(B(v_a, v_b^*), \varphi_r) + 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b^*), \varphi_r).
$$

Utilizing Equations (17), (18), and (26) in the preceding equation, we obtain

$$
\|B\|^2 \geq \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g(B(\varphi_a, \varphi_b), \psi \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(\varphi_a, \varphi_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \varphi_r).
$$

Leaving out all terms except for the last two relations in the above equation, we find

$$
\|B\|^2 \geq 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{2k} g^2(B(v_a, v_b), \varphi_r).
$$

From the frame fields of tangent and normal sub-bundles of $M$, after ignoring the $\mu$-components part in (41), we derive

$$
\|B\|^2 \geq 2 \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \psi \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \varphi_r)
$$

$$
+ 2 \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \psi \varphi_r) + \sum_{r=1}^{l} \sum_{a,b=1}^{k} g^2(B(v_a, v_b), \varphi_r).
$$

$$
+ 2 \csc^2 \phi \sum_{r,b=1}^{k} \sum_{a,d=1}^{l} \left\{ g^2(B(v_a, v_b^*), F_v^r) + g^2(B(\psi v_a, v_b^*), F_v^r) \right\}
$$

$$
+ 2 \csc^2 \phi \sec^2 \phi \sum_{r,b=1}^{k} \sum_{a,d=1}^{l} \left\{ g^2(B(v_a, T v_b^*), F_v^r) + g^2(B(\psi v_a, T v_b^*), F_v^r) \right\}.
$$
in the remaining terms, we arrive at
\begin{equation}
\ln \zeta \parallel f \B_1 \parallel \parallel \nabla f \parallel^2
\end{equation}

For a nearly trans-Sasakian manifold, two conditions are satisfied:
\begin{equation}
\zeta \ln f_1 = \beta \quad \text{and} \quad \zeta \ln f_2 = \beta
\end{equation}

Utilizing Equations (19), (20), and (21) in the first terms, we substitute (26)–(35) to obtain
\begin{equation}
\| B \|^2 \geq 2l \sum_{a=1}^{l} \left\{ -\psi v_a \ln f_1 + a \eta(v_a) \right\}^2 + (v_a \ln f_1 - \eta(v_a))^2 \right\}
\end{equation}

+ 2k \csc^2 \phi \sec^2 \phi \sum_{a=1}^{l} \left\{ \left( \psi v_a \ln f_2 - a \eta(v_a) \right)^2 + (v_a \ln f_2 - \eta(v_a))^2 \right\}
\end{equation}

+ \frac{2k}{g} \csc^2 \phi \sec^2 \phi \cos^2 \phi \sum_{a=1}^{l} \left\{ \left( v_a \ln f_2 - \beta \eta(v_a) \right)^2 + (\psi v_a \ln f_2)^2 \right\}
\end{equation}

+ \frac{2k}{g} \csc^2 \phi \sec^2 \phi \cos^2 \phi \sum_{a=1}^{l} \left\{ \left( -\psi v_a \ln f_2 - a \eta(v_a) \right)^2 + (v_a \ln f_2 - \eta(v_a))^2 \right\}
\end{equation}

+ 2k \csc^2 \phi \sec^2 \phi \cos^2 \phi \sum_{a=1}^{l} \left\{ \left( -\psi v_a \ln f_2 - a \eta(v_a) \right)^2 + (v_a \ln f_2 - \eta(v_a))^2 \right\}
\end{equation}

+ 2l \alpha^2 + 4k \alpha^2.

From the orthonormal frame \( \eta(v_a) = 0 \) for \( 1 \leq i \leq 2l \), and with some rearrangement in the remaining terms, we arrive at
\begin{equation}
\| B \|^2 \geq 2l \sum_{a=1}^{l} (v_a \ln f_1)^2 + 2l \alpha^2 + 4k \csc^2 \phi \sum_{a=1}^{l} (v_a \ln f_1)^2 + \frac{4k}{g} \cot^2 \phi \sum_{a=1}^{l} (v_a \ln f_1)^2 + 4k \alpha^2.
\end{equation}

Using trigonometric identities and adding and subtracting some terms, we obtain
\begin{equation}
\| B \|^2 \geq 2l \| \nabla \ln f_1 \|^2 - 2l (v_{2l+1} \ln f_1)^2 + 2l \alpha^2 + \frac{4k}{g} \left( 10 \csc^2 \phi - 1 \right) \| \nabla \ln f_2 \|^2 - \frac{4k}{g} \left( 10 \csc^2 \phi - 1 \right) (v_{2l+1} \ln f_2)^2 + 4k \alpha^2.
\end{equation}

For a nearly trans-Sasakian manifold, two conditions are satisfied: \( \xi \ln f_1 = \beta \) and \( \xi \ln f_2 = \beta \) for the structure vector field \( \xi \) tangent to the base \( M_T \). The following inequality is obtained by substituting into the proceeding equation:
\begin{equation}
\| B \|^2 \geq 2l \| \nabla \ln f_1 \|^2 - 2l \beta^2 + 2l \alpha^2
\end{equation}

where
\begin{equation}
\| B \|^2 \geq 2l \| \nabla \ln f_1 \|^2 - 2l \beta^2 + 2l \alpha^2 + \frac{4k}{g} \left( 10 \csc^2 \phi - 1 \right) \| \nabla \ln f_2 \|^2 - \frac{4k}{g} \left( 10 \csc^2 \phi - 1 \right) \beta^2 + 4k \alpha^2,
\end{equation}

from which we reach the final result, (40), which we wanted to prove.

If the inequality (40) becomes an equality, the missing third part in (41) gives:
\begin{equation}
B (TM, TM) \perp \mu.
\end{equation}

(43)
Using terms (i) and (ii), which were not considered in (17), and (42), we can derive $\mathbf{B}(\mathbf{D}, \mathbf{D}) \perp \mathbf{q}\mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}, \mathbf{D}) \perp \mathbf{F}\mathbf{D}^\phi$. It is obtained that $\mathbf{B}(\mathbf{D}, \mathbf{D}) = 0$. The missing first and fourth terms in (42) lead to the following:

$$\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \perp \mathbf{q}\mathbf{D}^\perp \ & \ & \mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) \perp \mathbf{F}\mathbf{D}^\phi. \quad (44)$$

It can easily be seen that $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, by using the missing terms in (43) and (44). In addition, the second and third terms that were left in (42) give $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp \mathbf{q}\mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$. This leads to $\mathbf{B}(\mathbf{D}^\phi, \mathbf{D}^\phi) = 0$. In addition, terms numbers five and six that were missed in (42) imply that $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{q}\mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$. These give us $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$. We obtain $\mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) \subset \mathbf{q}\mathbf{D}^\perp$ by considering the second term in (26), along with (43). From the first term in (26) along with (43), we obtain $\mathbf{B}(\mathbf{D}, \mathbf{D}^\perp) \subset \mathbf{F}\mathbf{D}^\phi$. As $\mathbf{M}_T$ is totally geodesic in $\mathbf{M}$, by the use of this point along with $\mathbf{B}(\mathbf{D}, \mathbf{D}) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$, we find that $\mathbf{M}_T$ is totally geodesic in $\mathbf{M}$. As $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical in $\mathbf{M}$ and we have $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\perp) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) = 0$, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{q}\mathbf{D}^\perp$, and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$, it can be concluded that $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ together are totally umbilical in $\mathbf{M}$. Furthermore, $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{q}\mathbf{D}^\perp$ and $\mathbf{B}(\mathbf{D}^\perp, \mathbf{D}^\phi) \perp \mathbf{F}\mathbf{D}^\phi$ imply that $\mathbf{M}$ is neither $\mathbf{D} \oplus \mathbf{D}^\perp$-mixed totally geodesic nor $\mathbf{D} \oplus \mathbf{D}^\phi$-mixed totally geodesic at $\mathbf{M}$, in light of Remark 2. This completes the proof of the main theorem.

Some Geometric Consequences

If we consider $\dim \mathbf{M}_\phi = 2k = 0$, then Theorem 1 gives the following:

**Theorem 2.** (Theorem 4.1 of [18]) Let $\mathbf{M} = \mathbf{M}_T \times f_\mathbf{M}_\perp$ be a CR-warped product submanifold in a nearly trans-Sasakian manifold. Then, we have:

$$\|\mathbf{B}\|^2 \geq 2n_2 \left\{ \|
abla (\ln f_1)\|^2 + \alpha^2 - \beta^2 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$.

If $\mathbf{M}_\perp$ vanishes, then Theorem 1 implies the following:

**Theorem 3.** (Theorem 4.1 of [19]) Let $\mathbf{M} = \mathbf{M}_T \times f_\mathbf{M}_\phi$ be a warped product semi-slant submanifold of a nearly trans-Sasakian $\mathbf{M}$, then inequality (40) implies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2n_3 \left\{ \left( \frac{2}{3} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|
abla (\ln f_2)\|^2 - \beta^2 \right\} + +\alpha^2 \right\},$$

where $n_3 = \dim \mathbf{M}_\phi$.

Therefore, Theorem 1 is an extension of Theorem 4.1 of [19]. Similarly, for $\alpha = 0$, we have:

**Theorem 4.** Let $\mathbf{M} = \mathbf{M}_T \times f_\mathbf{M}_\perp \times f_\mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly $\beta$-Kenmotsu manifold $\mathbf{M}$. Then, the second fundamental form $\mathbf{B}$ satisfies the following inequality:

$$\|\mathbf{B}\|^2 \geq 2l \left\{ \|
abla (\ln f_1)\|^2 - \beta^2 \right\} + 2k \left\{ \left( \frac{2}{3} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|
abla (\ln f_2)\|^2 - \beta^2 \right\} \right\},$$

where $l = \dim \mathbf{M}_\perp, k = \frac{1}{2} \dim \mathbf{M}_\phi$. In addition, $\nabla (\ln f_\zeta)$ is the gradient of $\ln f_\zeta$, while $\zeta \in \mathbf{M}_T, \mathbf{M}_\phi$, and $\mathbf{M}_\perp$ are holomorphic, proper slant, and totally real submanifolds of $\mathbf{M}$, respectively. If inequality (40) becomes an equality, then $\mathbf{M}_\perp$ and $\mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic.
totally in $\tilde{M}$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $\tilde{M}$.

Inserting $\alpha = 0$ and $\beta = 1$ into Theorem 1, we have

**Theorem 5.** (Theorem 1 of [8]) Let $M = M_T \times f_1 M_\perp \times f_2 M_\phi$ be a bi-warped product submanifold of a nearly Kenmotsu manifold $M$. Then, the second fundamental form $B$ satisfies the following inequality:

$$
\|B\|^2 \geq 2n_2 \left\{ \|\nabla (\ln f_1)\|^2 - 1 \right\} + n_3 \left\{ \left( \frac{2}{3} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla (\ln f_2)\|^2 - 1 \right\} \right\},
$$

where $n_2 = \dim M_\perp$, $n_3 = \frac{1}{2} \dim M_\phi$. In addition, $\nabla (\ln f_1)$ is the gradient of $\ln f_1$, while $\zeta \in M_T$, $M_\phi$, and $M_\perp$ are holomorphic, proper slant, and totally real submanifolds of $M$, respectively. If inequality (40) becomes an equality, then $M_\perp$ and $M_\phi$ are totally umbilical and $M_T$ is geodesic totally in $M$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $\tilde{M}$.

If we put $\alpha = 0$, $\beta = 0$ into Theorem 1, the following is obtained for the bi-warped product submanifold of a nearly cosymplectic manifold:

**Theorem 6.** Let $M = M_T \times f_1 M_\perp \times f_2 M_\phi$ be a bi-warped product submanifold of a nearly cosymplectic manifold $M$. Then, the second fundamental form $B$ satisfies the following inequality:

$$
\|B\|^2 \geq 2n_2 \left\{ \|\nabla (\ln f_1)\|^2 \right\} + n_3 \left\{ \left( \frac{2}{3} \cot^2 \phi + 2 \csc^2 \phi \right) \|\nabla (\ln f_2)\|^2 \right\},
$$

where $n_2 = \dim M_\perp$, $n_3 = \frac{1}{2} \dim M_\phi$. Moreover, $\nabla (\ln f_1)$ is the gradient of $\ln f_1$, while $\zeta \in M_T$, $M_\phi$, and $M_\perp$ are holomorphic, proper slant, and totally real submanifolds of $M$, respectively. If inequality (40) becomes an equality, then $M_\perp$ and $M_\phi$ are totally umbilical and $M_T$ is geodesic in $M$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $\tilde{M}$.

Again, we get the following by using $\alpha = 0$, $\beta = 0$, and $\dim M_\phi = 0$.

**Theorem 7.** (Theorem 3.2 of [31]) Let $M = M_T \times f M_\perp$ be a CR-warped product submanifold of a nearly cosymplectic manifold. Then, we have:

$$
\|B\|^2 \geq 2n_2 \|\nabla (\ln f_1)\|^2,
$$

where $n_2 = \dim M_\perp$.

Inserting $\alpha = 0$, $\beta = 0$ and $\dim M_\perp = 0$ into Theorem 1, we obtain the following:

**Theorem 8.** (Theorem 3.5 of [32]) Let $M = M_T \times f M_\phi$ be a warped product semi-slant submanifold of a nearly cosymplectic $M$. Then, inequality (40) implies the following inequality:

$$
\|B\|^2 \geq 4k \left\{ \left( \frac{1}{9} \cot^2 \phi + \csc^2 \phi \right) \|\nabla (\ln f_2)\|^2 \right\},
$$

where $2k = \dim M_\phi$. 

By inserting $\beta = 0$ and $\alpha = 1$, $\beta = 0$ into Theorem 1, the following results for the bi-warped product submanifolds of a nearly $\alpha$-Sasakian manifold and a nearly Sasakian manifold, respectively, can be obtained:

**Theorem 9.** Let $M = \mathbf{M}_T \times f_1 \mathbf{M}_\perp \times f_2 \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly $\alpha$-Sasakian manifold $M$. Then, the second fundamental form $B$ satisfies the following inequality:

$$\|B\|^2 \geq 2n_2 \left\{ \|\nabla (\ln f_1)\|^2 + \alpha^2 \right\}$$

$$+ n_3 \left\{ \left( \frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla (\ln f_2)\|^2 \right\} + \alpha^2 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. If the above inequality (40) becomes an equality, then $\mathbf{M}_\perp, \mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $M$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $M$.

**Theorem 10.** Let $M = \mathbf{M}_T \times f_1 \mathbf{M}_\perp \times f_2 \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly Sasakian manifold $M$. Then, the second fundamental form $B$ satisfies the following inequality:

$$\|B\|^2 \geq 2n_2 \left\{ \|\nabla (\ln f_1)\|^2 + 1 \right\} + n_3 \left\{ \left( \frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left\{ \|\nabla (\ln f_2)\|^2 \right\} + 1 \right\},$$

where $n_2 = \dim \mathbf{M}_\perp$, $n_3 = \frac{1}{2} \dim \mathbf{M}_\phi$. If inequality (40) becomes an equality, then $\mathbf{M}_\perp, \mathbf{M}_\phi$ are totally umbilical and $\mathbf{M}_T$ is geodesic totally in $M$. The $D \oplus D^\perp$-mixed totally geodesic and $D \oplus D^\phi$-mixed totally geodesic do not exist in $M$ of $M$.

**Remark 4.** It can be easily seen that Theorem 4.1 of [18], Theorem 4.1 of [19], Theorem 3.2 of [31], Theorem 3.5 of [32], and Theorem 1 of [8] are special cases of our main Theorem 1.

3. Inequality for Bi-Warped Product Submanifold of the Type $M = \mathbf{M}_\phi \times f_1 \mathbf{M}_\perp \times f_2 \mathbf{M}_T$

In this section, we consider the bi-warped product submanifold of type $M = \mathbf{M}_\phi \times f_1 \mathbf{M}_T \times f_2 \mathbf{M}_\perp$ in a nearly trans-Sasakian manifold $\mathbf{M}$, with respect to the tangent spaces of $\mathbf{M}_T, \mathbf{M}_\perp$ and $\mathbf{M}_\phi$, which are integral manifolds of $D, D^\perp$, and $D^\phi$, respectively.

**Lemma 3.** Let $M = \mathbf{M}_\phi \times f_1 \mathbf{M}_T \times f_2 \mathbf{M}_\perp$ be a bi-warped product submanifold of a trans-Sasakian manifold $M$. Then, we construct the following:

$$g(\mathbf{B}(X_0, X_1), \mathbf{X}_4) = \left\{ (\mathbf{T}X_4 \ln f_1) + a \eta(X_4) \right\} g(X_0, X_1)$$

(45)

$$2g(\mathbf{B}(X_2, X_3), \mathbf{X}_4) = g(\mathbf{B}(X_4, X_2), \psi X_3) + g(\mathbf{B}(X_4, X_3), \psi X_2)$$

$$+ 2 \left\{ \mathbf{T}X_4 \ln f_2 + a \eta(X_4) \right\} g(X_2, X_3)$$

(46)

$$g(\mathbf{B}(X_4, X_5), \psi X_2) = g(\mathbf{B}(X_4, X_2), \mathbf{X}_5),$$

$$g(\mathbf{B}(X_0, X_5), \mathbf{X}_4) = 0.$$

**Proof.** For any $X_0, X_1 \in \Gamma(D)$ and $X_4, X_5 \in \Gamma(D^\phi \oplus \xi)$, we have

$$g(\mathbf{B}(X_0, X_1), \mathbf{X}_4) = g(\nabla_{X_0} X_1, \mathbf{X}_4) = g(\nabla_{X_0} X_1, \psi X_4) - (\nabla_{X_0} X_1, \mathbf{T}X_4)$$

$$= g((\nabla_{X_0} \psi)X_1, X_4) - g(\nabla_{X_0} \psi X_1, X_4) + g(X_1, \nabla_{X_0} \mathbf{T}X_4).$$
Using (4), (8), (10), and (1), we get
\[
g(B(X_0, X_1), FX_4) = -g(\nabla_{X_1}X_0, X_4) + \{2g(X_0, X_1)g(\xi, X_4) - \eta(X_0)g(X_1, X_4) - \eta(X_1)g(X_0, X_4)\}
\]
\[
- \beta\{\eta(X_0)g(\xi, X_1, X_4) + \eta(X_1)(\xi, X_0, X_4)\} + TX_4[ln f_1]g(X_0, X_1)
\]
\[
= -g(\nabla_{X_1}X_0, X_4) - g(\nabla_{X_1}X_0, FX_4) - g(\nabla_{X_1}X_0, TX_4) + 2\eta f_4)g(X_0, X_1)
\]
\[
+ TX_4[ln f_1]g(X_0, X_1)
\]
\[
= g(X_4[ln f_1]g(\xi, X_1, X_4) - X_4[ln f_1]g(\xi, X_1, X_4) + 2TX_4[ln f_1]g(\xi, X_1, X_4).
\]

This is implies the first part (45) of the lemma. For the next part, we have, from (8),
\[
g(B(X_2, X_3), FX_4) = g(\nabla_{X_2}X_3, \psi X_4) - g(\nabla_{X_2}X_3, TX_4).
\]

Using (10) and (1), we obtain
\[
g(B(X_2, X_3), FX_4) = g((\nabla_{X_2}\psi)X_3, X_4) - g(\nabla_{X_2}X_3, X_4) + TX_4[ln f_2]g(X_2, X_3).
\]

By use of (4), (8), and (10), we get
\[
g(B(X_2, X_3), FX_4) = -g((\nabla_{X_3}X_2)X_2, X_4) + a\{2\eta f_4)g(X_2, X_3)
\]
\[
- \eta(X_2)g(X_3, X_4) - \eta(X_3)g(X_2, X_4)\}
\]
\[
- \beta\{\eta(X_2)g(\xi, X_3, X_4) - \eta(X_3)g(\xi, X_2, X_4)\} + TX_4[ln f_2]g(X_2, X_3)
\]
\[
+ g(B(X_4, X_2), X_3)
\]
\[
= g(B(X_4, X_2), X_3) + g(B(X_4, X_3), X_2)
\]
\[
+ TX_4[ln f_2]g(X_2, X_3) + 2\eta f_4)g(X_2, X_3)
\]
\[
= g(\nabla_{X_3}X_2, X_4) - g(\nabla_{X_3}X_2, TX_4)
\]
\[
= g(B(X_4, X_2), X_3) + g(B(X_4, X_3), X_2)
\]
\[
+ 2TX_4[ln f_2]g(X_2, X_3) + 2\eta f_4)g(X_2, X_3).
\]

This is equivalent to (46). For any $X_4, X_5 \in \Gamma(D^\phi), X_2 \in \Gamma(D^\perp)$,
\[
g(B(X_4, X_5), X_2) = g(\nabla_{X_4}X_5, X_2) = g((\nabla_{X_4}X_5, X_2) \nabla_{X_4}X_5, X_2).
\]

Utilizing (4), we get the following:
\[
g(B(X_4, X_5), X_2) = g(\nabla_{X_4}X_2, X_5) = g(\nabla_{X_4}X_2, FX_5) + g(\nabla_{X_4}X_2, FX_5).
\]

In view of (6) and (7), we get
\[
g(B(X_4, X_5), X_2) = g(B(X_4, X_2), FX_5).
\]

For the last part, we have
\[
g(B(X_0, X_5), FX_4) = g(\nabla_{X_0}X_5, X_4) - g(\nabla_{X_0}X_0, TX_4)
\]
\[
= -g(\nabla_{X_0}X_0, X_4) + g((\nabla_{X_0}X_0, X_4) - X_5[ln f_1]g(X_0, TX_4),
\]
for any $X_0 \in \Gamma(D)$ and $X_4, X_5 \in \Gamma(D^\phi)$. Using (4), (7), and the normality of vector fields, we arrive at
\[
g(B(X_0, X_5), FX_4) = 0.
\]

Thus, the required results are obtained. $\square$
The following equalities can be derived by exchanging $X_0$ with $\psi X_0$ and $X_1$ with $\psi X_1$ in Lemma 3:

$$g(B(\psi X_0, X_1), F X_4) = \left\{ T X_4 (\ln f_1) + \alpha \eta(X_4) \right\} g(\psi X_0, X_1),$$  \hspace{1cm} (47)

and

$$g(B(X_0, \psi X_1), F X_4) = \left\{ T X_4 (\ln f_1) + \alpha \eta(X_4) \right\} g(X_0, \psi X_1),$$  \hspace{1cm} (48)

Similarly, if we interchange $X_4$ with $TX_4$ in Lemma 3, (47)–(49), and (16), we can derive

$$g(B(X_0, X_1), T X_4) = - \cos^2 \phi \left\{ (X_4 \ln f_1) - \eta(X_4) \right\} g(X_0, X_1),$$  \hspace{1cm} (50)

$$g(B(\psi X_0, X_1), T X_4) = - \cos^2 \phi \left\{ (X_4 \ln f_1) - \eta(X_4) \right\} g(\psi X_0, X_1)$$  \hspace{1cm} (51)

and

$$g(B(\psi X_0, \psi X_1), T X_4) = - \cos^2 \phi \left\{ (X_4 \ln f_1) - \eta(X_4) \right\} g(\psi X_0, \psi X_1),$$  \hspace{1cm} (52)

Again, interchanging $X_4$ with $TX_4$ in (46), Lemma 3, and (16), we obtain

$$2g(B(X_2, X_3), T X_4) = g(B(T X_4, X_2), \psi X_3) + g(B(T X_4, X_3), \psi X_2) - 2 \cos^2 \phi \left( - \eta(X_4) + (X_4 \ln f_2) \right) g(X_2, X_3).$$  \hspace{1cm} (54)

**Lemma 4.** Let $M = M_\phi \times f_1 M_T \times f_2 M_{\perp}$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then, we have

$$g(B(X_0, X_5), \psi X_2) = g(B(X_0, X_2), F X_5) = 0,$$  \hspace{1cm} (55)

for any $X_0 \in \Gamma(D)$, $X_5 \in \Gamma(D^\phi \oplus \zeta)$ and $X_2 \in \Gamma(D^\perp)$.

**Proof.** For any $X_0 \in \Gamma(D)$, $X_5 \in \Gamma(D^\phi \oplus \zeta)$, and $X_2 \in \Gamma(D^\perp)$, we have

$$g(B(X_0, X_2), F X_5) = g(\nabla_{X_2} X_0, \psi X_5) + g(\nabla_{X_2} X_5, X_0),$$

$$= g(\nabla_{X_2} \psi X_0, X_5) - g(\nabla_{X_2} \psi X_5, X_0) + T X_5 (\ln f_2) g(X_0, X_2).$$

Equations (4) and (10) imply $g(B(X_0, X_2), F X_5) = 0$. For the next equality, we compute

$$g(B(X_0, X_2), F X_5) = g(\nabla_{X_0} X_2, \psi X_5) + g(\nabla_{X_0} X_5, X_2),$$

$$= g(\nabla_{X_0} \psi X_2, X_5) - g(\nabla_{X_0} \psi X_5, X_2) + T X_5 (\ln f_1) g(X_0, X_2)$$

Again making use of (4) and (1), we find the following:

$$g(B(X_0, X_5), \psi X_2) = g(B(X_0, X_2), F X_5),$$
which is the first equality. Therefore, the result is proved. □

In this direction, we provide a relationship between the squared norm of the second fundamental form and the warping function for the bi-warped product. Before giving the next relationship, we define an orthonormal frame. Taking $\zeta$ tangent to the base manifold $M_\phi$ of an $n$-dimensional bi-warped product submanifold $M = M_\phi \times f_1 M_T \times f_2 M_{\perp}$ in a $(2n + 1)$-dimensional nearly trans-Sasakian manifold $\tilde{M}$, we consider the dimensions $\text{dim}(M_T) = n_1$, $\text{dim}(M_{\perp}) = n_2$, and $\text{dim}(M_\phi) = n_3$. We provide proof of the main theorem as follows.

For the second type of bi-warped product submanifold, $M_\phi \times f_1 M_T \times f_2 M_{\perp}$, we prove the following result:

**Theorem 11.** Assume that $M = M_\phi \times f_1 M_T \times f_2 M_{\perp}$ is a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. If $D^\phi - D^\perp$ is mixed totally geodesic and $\zeta$ is tangent to $M_\phi$, then the length of the second fundamental form $\beta$ is defined as

$$||\beta||^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left(||\tilde{\nabla} \ln f_1||^2 - \beta^2\right) + n_2 \cot^2 \phi \left(||\tilde{\nabla} \ln f_2||^2 - \beta^2\right), \quad (56)$$

where $n_1 = \text{dim}(M_T)$ and $n_2 = \text{dim}(M_{\perp})$. The gradients $\tilde{\nabla} (\ln f_1)$ and $\tilde{\nabla} (\ln f_2)$ of $f_1$ and $f_2$ are along $M_T$ and $M_{\perp}$, respectively. If the inequality (56) becomes an equality, then $M_T$ and $M_{\perp}$ are totally umbilical submanifolds, and $M_\phi$ is a totally geodesic submanifold in $M$. Furthermore, $M$ is a $D^\phi$-totally geodesic submanifold of $\tilde{M}$.

**Proof.** Suppose the orthogonal frames of the corresponding tangent spaces of $D, D^\perp$, and $D^\phi$, are as follows:

$$D = \text{span}\{v_1, \ldots, v_p, v_{p+1} = \psi v_1, \ldots, v_{n_1} = v_{2p} = \psi v_p\},$$

$$D^\perp = \text{span}\{v_{n_1+1} = v_1, \ldots, v_{n_1+n_2} = v_{n_2}\},$$

$$D^\phi = \text{span}\{v_{n_1+n_2+1} = v_1^\phi, \ldots, v_{n_1+n_2+q} = v_q^\phi, v_{n_1+n_2+q+1} = v_{q+1}^\perp = \sec \phi T v_1^\phi, \ldots, v_{n_1+n_2+2q} = v_{2q+1} = \sec \phi T v_q^\phi, v_m = v_n = v_{q+1} = \zeta\}.$$

Then, the orthonormal frame fields of the normal sub-bundles of $\psi D^\perp, FD^\phi$, and $\mu$, respectively, are as follows:

$$\psi D^\perp = \text{span}\{v_{m+1} = v_1 = \vartheta_1, \ldots, v_{m+n_2} = v_{n_2} = \vartheta_{n_2}\},$$

$$FD^\phi = \text{span}\{v_{m+n_2+1} = \vartheta_{n_2+1} = \csc \phi F v_1^\phi, \ldots, v_{m+n_2+q+1} = \vartheta_{n_2+q+1} = \csc \phi F v_q^\phi, \ldots, v_{m+n_2+2q} = v_{2q+1} = \vartheta_{n_2+2q+1} = \zeta\},$$

$$\mu = \text{span}\{v_{m+n_2+n_3} = \vartheta_{n_2+n_3+1}, \ldots, v_{2n+1} = \vartheta_{2(n-n_2-n_3+1)+1}\}.$$

From the definition of the second fundamental form, we have

$$||\beta||^2 = \sum_{a,b=1}^{m} g(B(v_a, v_b), B(v_a, v_b)) = \sum_{r=m+1}^{2n+1} \sum_{a,b=1}^{m} g(B(v_a, v_b), v_r)^2.$$

The above expression can be expanded, according the frame vector fields, as

$$||\beta||^2 = \sum_{r=1}^{n_2} \left(\sum_{a,b=1}^{m} g(B(v_a, v_b), \vartheta_r)^2\right) + \sum_{r=n_2+1}^{n_2+n_3} \left(\sum_{a,b=1}^{m} g(B(v_a, v_b), \vartheta_r)^2\right). \quad (57)$$
Similarly, we ignore the first, second, fourth, and ninth terms, due to the lack of connections. Inserting the equations from Lemma 3 with (47)–(54), we derive

\[
\|B\|^2 \geq \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} g(B(v_a, v_b), \psi \psi_r)^2 + \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} g(B(\varphi_a, \varphi_b), \psi \psi_r)^2 \\
+ \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} g(B(v_a^*, v_b^*), \psi \psi_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} g(B(v_a, v_b), \psi \psi_r)^2 \\
+ 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(B(v_a, v_b), \psi \psi_r)^2 + 2 \sum_{r=1}^{n_2} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(B(\varphi_a, \varphi_b), \psi \psi_r)^2 \\
+ \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} g(B(v_a, v_b), \varphi_r)^2 + \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} g(B(\varphi_a, \varphi_b), \varphi_r)^2 \\
+ \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} g(B(v_a^*, v_b^*), \varphi_r)^2 + 2 \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(B(v_a, v_b), \varphi_r)^2 \\
+ 2 \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(B(v_a^*, v_b^*), \varphi_r)^2 + 2 \sum_{r=n_2+1}^{n_2+n_1-1} \sum_{a=1}^{n_1} \sum_{b=1}^{n_3} g(B(\varphi_a, \varphi_b), \varphi_r)^2. 
\]

(58)

As we assumed that M is a $D^+ - D^\phi$ mixed totally geodesic, this forces the third, sixth, and twelfth terms to vanish. In view of Lemma 4, the fifth and tenth terms are zero. Similarly, we ignore the first, second, fourth, and ninth terms, due to the lack of connections for warping functions. Now, the rest of the terms (i.e., the seventh and eighth) can be written as

\[
\|B\|^2 \geq \sum_{r=1}^{q} \sum_{a=1}^{n_1} g(B(v_a, v_b), \csc \phi F \psi_r)^2 + \sum_{r=1}^{q} \sum_{a=1}^{n_1} g(B(\varphi_a, \varphi_b), \csc \phi \sec \phi FT \psi_r)^2 \\
+ \sum_{r=1}^{q} \sum_{a=1}^{n_1} g(B(v_a^*, v_b^*), \csc \phi F \psi_r)^2 + \sum_{r=1}^{q} \sum_{a=1}^{n_1} g(B(\varphi_a, \varphi_b), \csc \phi \sec \phi FT \psi_r)^2.
\]

Inserting the equations from Lemma 3 with (47)–(54), we derive

\[
\|B\|^2 \geq n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=1}^{q} \left( T \psi_r (\ln f_1) + a \eta(v_r^*) \right)^2 \\
+ n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^{q} \left( v_r^*(\ln f_1) - \eta(v_r^*) \right)^2 \\
+ n_2 \csc^2 \phi \sum_{r=1}^{q} \left( T \psi_r (\ln f_1) + a \eta(v_r^*) \right)^2 + n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^{q} \left( v_r^*(\ln f_2) - \eta(v_r^*) \right)^2.
\]

From the orthonormal frame $\eta(\psi_r^*) = 0$, for $1 \leq r \leq q$, we have

\[
\|B\|^2 \geq n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=1}^{2q+1} \left( T \psi_r (\ln f_1) \right)^2 - n_1 \csc^2 \phi (1 + \sec^2 \phi) \sum_{r=q+1}^{2q} g(v_r^*, T \nabla (\ln f_1))^2 \\
- n_1 \csc^2 \phi (1 + \sec^2 \phi) (T \psi_{2q+1} (\ln f_1))^2 + n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^{q} (v_r^*(\ln f_1))^2 \\
+ n_2 \csc^2 \phi \sum_{r=1}^{2q+1} \left( T \psi_r (\ln f_2) \right)^2 - n_2 \csc^2 \phi \sum_{r=q+1}^{2q} g(v_r^*, T \nabla (\ln f_2))^2 \\
- n_2 \csc^2 \phi (T \psi_{2q+1} (\ln f_2))^2 + n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^{q} (v_r^*(\ln f_2))^2.
\]
The third and seventh terms are equal to zero by \(e_{2q+1}^2 = \zeta\) and \(T\zeta = 0\). Thus, the preceding inequality takes the following form:

\[
||B||^2 \geq n_1 \csc^2 \phi (1 + \sec^2 \phi) ||T\nabla (\ln f_1)||^2 - n_1 \csc^2 \phi (1 + \sec^2 \phi) \sec^2 \phi \sum_{r=1}^{q} g(T\nu_r^*, T\nabla (\ln f_1))^2
\]

\[
+ n_1 \csc^2 \phi (1 + \cos^2 \phi) \sum_{r=1}^{q} (\nu_r^* (\ln f_1))^2 + n_2 \csc^2 \phi ||T\nabla (\ln f_2)||^2
\]

\[
- n_2 \csc^2 \phi \sec^2 \phi \sum_{r=1}^{q} g(T\nu_r^*, T\nabla (\ln f_2))^2
\]

\[
+ n_2 \csc^2 \phi \cos^2 \phi \sum_{r=1}^{q} (\nu_r^* (\ln f_2))^2.
\]

Using (11) and the fact that \(\zeta (\ln f_a) = \beta, \quad i = 1, 2\) from Proposition 2, we finally obtain

\[
||B||^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left(||\nabla (\ln f_1)||^2 - \beta^2\right) + n_2 \cot^2 \phi (||\nabla (\ln f_2)||^2 - \beta^2).
\]

For the equality case in (56), considering the third \(\mu\)-component term in (57), we derive

\[
B(TM, TM) \perp \mu.
\]  

By using the missing first, second, and fourth terms in (58), we find that

\[
B(D, D) \perp \psi D^\perp, \quad B(D^\perp, D^\perp) \perp \psi D^\perp, \quad B(D, D^\perp) \perp \psi D^\perp.
\]

Evaluating the ninth term of (58), we get

\[
B(D^\phi, D^\phi) \perp FD^\phi.
\]

Then, from (59) and (60), we obtain

\[
B(D, D) \in FD^\phi, \quad B(D^\perp, D^\perp) \in FD^\phi, \quad B(D, D^\perp) \in FD^\phi.
\]

As \(M\) is \(D^\perp - D^\phi\) mixed totally geodesic, we can conclude the following:

\[
B(D^\perp, D^\phi) = 0.
\]

From the vanishing third term of (58), we get

\[
B(D^\phi, D^\phi) \perp \psi D^\perp.
\]

From (59), (61), and (64), we get

\[
B(D^\phi, D^\phi) = 0.
\]

Evaluating the fifth, tenth, and eleventh terms of (58), we can derive

\[
B(D, D^\phi) \perp \psi D^\perp, \quad F(D, D^\perp) \perp FD^\phi, \quad B(D, D^\phi) \perp FD^\phi.
\]

Therefore, with (59), (60), and (66), we obtain

\[
B(D, D^\phi) = 0, \quad B(D, D^\perp) = 0.
\]

It is well-known that \(M^\phi\) is a totally geodesic submanifold in \(M\) and we can analyze that \(M^\phi\) is totally geodesic in \(\hat{M}\) by (63), (65), and (67). Additionally, \(M_T\) and \(M_\perp\) are totally umbilical submanifolds in \(M\) by (62), as \(M_T\) and \(M_\perp\) are totally umbilical in \(M\). Equations
(60)–(67) show that $\mathbf{M}$ is a $D^\phi$-totally geodesic submanifold in $\tilde{\mathbf{M}}$. This completes the proof of the theorem.

3.1. Geometric Applications

In this section, we find a particular case of our main results. Particularizing $\beta = 0$ and $\alpha = 1$ or $\alpha = 0$, along with $n_1 = \dim \mathbf{M}_T = 0$, in Theorem 11, we get the following:

**Theorem 12.** (Theorem 3.1 of [13] and Theorem 3.1 of [33]) Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a nearly Sasakian manifold or a nearly cosymplectic manifold. Then, we have

$$||\mathbf{B}||^2 \geq n_2 \cot^2 \phi ||\nabla \ln f_2||^2,$$

where $\dim \mathbf{M}_\perp = n_2$.

If we assume that $\alpha = 0$, $\beta = 1$ with $\dim \mathbf{M}_\perp = 0$, we find another theorem:

**Theorem 13.** Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_T$ be a warped product semi-slant submanifold of nearly Kenmotsu manifold. Then, we have

$$||\mathbf{B}||^2 \geq n_1 \csc^2 \phi (1 + \cos^2 \phi) \left( ||\nabla \ln f_1||^2 - 1 \right).$$

**Remark 5.** It can be noted that only on a nearly Kenmotsu manifold does the warped product semi-slant submanifold of the type $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_T$ exist; in other structures, it becomes a trivial case (see [22]).

Similarly, if $\alpha = 0$, $\beta = 1$ with $\dim \mathbf{M}_T = 0$, then we have:

**Theorem 14.** (Theorem 4.2 of [14]) Let $\mathbf{M} = \mathbf{M}_\phi \times_f \mathbf{M}_\perp \times_f \mathbf{M}_\phi$ be a mixed totally geodesic warped product pseudo-slant submanifold of a nearly Kenmotsu manifold. Then, we have

$$||\mathbf{B}||^2 \geq n_2 \cot^2 \phi \left( ||\nabla \ln f_2||^2 - 1 \right),$$

where $\dim \mathbf{M}_\perp = n_2$.

3.2. Some Applications Related to Mathematical Physics

In this section, we investigate the Dirichlet energy, which satisfies the following for a compact submanifold $\mathbf{M}$ and differentiable function $\theta : \mathbf{M} \rightarrow \mathbb{R}$:

$$E(\theta) = \frac{1}{2} \int_{\mathbf{M}} ||\nabla \theta||^2 dV,$$

(68)

where $dV$ is a volume element. Considering this, we give the following Theorem by combining (40) and (68), where $\mathbf{M}_T$ is compact without boundary.

**Theorem 15.** Let $\mathbf{M} = \mathbf{M}_T \times_f \mathbf{M}_\perp \times_f \mathbf{M}_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{\mathbf{M}}$, such that $\mathbf{M}_T$ is compact without boundary. Then, we have

$$4n_2 E(\ln f_1) + 2n_3 \left( \frac{2k}{9} \cot^2 \phi + 2 \csc^2 \phi \right) E(\ln f_2) \leq \int_{\mathbf{M}_T \times \{n_2\} \times \{n_3\}} \left\{ ||\mathbf{B}||^2 + n_3 \beta^2 \left( \frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) + 2n_2 \left( \beta^2 - \alpha^2 \right) - n_3 \alpha^2 \right\} dV,$$
where \( E(\ln f_1) \) and \( E(\ln f_2) \) are the Dirichlet energies of the warping functions \( f_1 \) and \( f_2 \), respectively. The equality cases are the same as in Theorem 1.

Similarly, if \( M_\phi \) is a compact base without boundary, then Theorems 11 and (68) give the following:

**Theorem 16.** Let \( M = M_\phi \times f_1 M_T \times f_2 M_\perp \) be a bi-warped product submanifold of a nearly trans-Sasakian manifold \( \tilde{M} \), such that \( M_\phi \) is compact without boundary. Then, we have

\[
2n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) E(\ln f_1) + n_2 \cot^2 \phi E(\ln f_2)
\leq \int_{M_\phi \times \{n_1\} \times \{n_2\}} \left\{ ||B||^2 + \beta^2 \left(n_2 \cot^2 \phi + n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) \right) \right\} dV.
\]

As immediate applications of Theorem 15, we give corollaries in the following.

**Corollary 1.** Assume that \( M = M_T \times f_1 M_\perp \) is a CR-warped product submanifold of a nearly trans-Sasakian manifold \( \tilde{M} \), such that \( M_T \) is compact without boundary. Then, we have

\[
E(\ln f_1) \leq \frac{1}{4n_2} \int_{M_T \times \{n_2\}} \left( ||B||^2 + 2n_2 \left( \beta^2 - \alpha^2 \right) \right) dV.
\]

**Corollary 2.** Assume that \( M = M_T \times f_2 M_\phi \) is a warped product semi-slant submanifold of a nearly trans-Sasakian manifold \( \tilde{M} \), such that \( M_T \) is compact without boundary. Then, we have

\[
E(\ln f_2) \leq \left( \frac{9 \sin^2 \phi}{8k(10 - \sin^2 \phi)} \right) \int_{M_T \times \{n_1\}} \left( ||B||^2 + 2k\beta^2 \left( \frac{2}{5} \cot^2 \phi + 2 \csc^2 \theta \right) \right) dV.
\]

On the other hand, substituting \( n_1 = 0 \) and \( n_2 = 0 \) into Theorem 16, we can derive the following corollaries.

**Corollary 3.** Let \( M = M_\phi \times f_1 M_\perp \) be a warped product pseudo-slant submanifold of a nearly trans-Sasakian manifold \( \tilde{M} \), such that \( M_\phi \) is compact without boundary. Then, we have

\[
E(\ln f_2) \leq \frac{1}{n_2 \cot^2 \phi} \int_{M_T \times \{n_2\}} \left( ||B||^2 + \beta^2 n_2 \cot^2 \phi \right) dV.
\]

**Corollary 4.** Let \( M = M_\phi \times f_2 M_T \) be a warped product semi-slant submanifold of a nearly trans-Sasakian manifold \( \tilde{M} \), such that \( M_\phi \) is compact without boundary. Then, we have

\[
2n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) E(\ln f_1) \leq \int_{M_T \times \{n_1\}} \left( ||B||^2 + \beta^2 n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) \right) dV.
\]

For the Laplacian, many applications in mathematics as well as in physics can be found. This is possible due to the eigenvalue problem of \( \Delta \). The corresponding Laplace eigenvalue equation is defined as follows: A real number \( \lambda \) is called an eigenvalue if there exists a non-vanishing function \( \theta \) which satisfies the following equation:

\[ \Delta \theta = \lambda \theta, \text{ on } M. \]
with appropriate boundary conditions. Considering a Riemannian manifold $M^n$ with no boundary, the first non-zero eigenvalue of $\Delta$, defined as $\lambda_1$, includes variational properties (cf. [34]):

$$\lambda_1 = \inf \left\{ \frac{\int_M |\nabla \theta|^2 dV}{\int_M \|\theta\|^2 dV} \mid \theta \in W^{1,2}(M^n) \backslash \{0\}, \int_M \theta dV = 0 \right\}.$$ 

Inspired by the above characterization, using the first non-zero eigenvalue of the Laplace operator and the maximum principle for the first non-zero eigenvalue $\lambda_1$, we deduce the following:

**Theorem 17.** Let $M = M_T \times f_1 M_\perp \times f_2 M_\phi$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$ with $\zeta \in M_T$ and such that $M_T$ is a compact base without boundary. If $\lambda_1$ and $\mu_1$ are the first eigenvalues of eigenfunctions $f_1$ and $f_2$, respectively, then we have following inequality:

$$\int_{M_T \times \{n_1\} \times \{n_2\}} \|B\|^2 dV \geq 2 \mu_2 \int_{M_T \times \{n_1\} \times \{n_3\}} \left\{ \lambda_1 (\ln f_1)^2 + n^2 - \beta^2 \right\} dV + \mu_3 \int_{M_T \times \{n_2\} \times \{n_3\}} \left\{ \left( \frac{2}{9} \cot^2 \phi + 2 \csc^2 \phi \right) \left( \mu_1 (\ln f_2)^2 - \beta^2 \right) + n^2 \right\} dV. \quad (69)$$

**Proof.** Assuming that $\theta$ is a non-constant warping function, by use of the minimum principle for the first eigenvalue $\lambda_1$, one can obtain (p. 186, [34]):

$$\lambda_1 \int_{M^n} (\theta)^2 dV \leq \int_{M^n} |\nabla \theta|^2 dV. \quad (70)$$

The equality holds if and only if $\Delta \theta = \lambda_1 \theta$. Taking the integral in (40) and using (70), we get the required result (69). \qed

Similarly, using Theorem 11, we have

**Theorem 18.** Let $M = M_\phi \times f_1 M_T \times f_2 M_\perp$ be a bi-warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. If $D^\perp = D^\phi$ is mixed totally geodesic and $\zeta$ is tangent to the compact base $M_\phi$, then we get

$$n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) \lambda_1 \int_{M_\phi \times \{n_1\} \times \{n_2\}} (\ln f_1)^2 dV + n_2 \cot^2 \phi \mu_1 \int_{M_\phi \times \{n_1\} \times \{n_2\}} (\ln f_2)^2 dV \leq \int_{M_\phi \times \{n_1\} \times \{n_2\}} \left\{ \|B\|^2 + \beta^2 \left( n_2 \cot^2 \phi + n_1 \csc^2 \phi \left( 1 + \cos^2 \phi \right) \right) \right\} dV.$$

4. Conclusions

It is noted that the nearly trans-Sasakian structures generalize some remarkable geometric structures on manifolds, like nearly cosymplectic, nearly Sasakian, nearly Kenmotsu, nearly $\alpha$-Sasakian, and nearly $\beta$-Kenmotsu. The main target of this paper is to discuss the geometry of bi-warped product submanifolds of some special types in nearly trans-Sasakian manifolds. In particular, we derived some basic inequalities, which turn out to be generalization of various known results obtained by several mathematicians in the last decade. The eigenvalues inequalities are established and the Dirichlet energy inequalities are derived that give the new motivation of such studies.

**Author Contributions:** Writing and original draft, A.A.; funding acquisition, editing and draft, A.H.A.; review and editing, A.A.; methodology, project administration, A.A.; formal analysis, resources, A.A. All authors have read and agreed to the published version of the manuscript.
**Funding:** This research was funded by the Deanship of Scientific Research at King Khalid University under Grant No. R.G.P.1/186/41.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** This research was funded by the Deanship of Scientific Research at King Khalid University under Grant No. R.G.P.1/186/41.

**Conflicts of Interest:** The authors declare no conflict of interest.

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