DIRECT METHODS TO
LIEB–THIRRING KINETIC INEQUALITIES

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We review some recent progress on Lieb–Thirring inequalities, focusing on direct methods to kinetic estimates for orthonormal functions and applications for many-body quantum systems.

1. Introduction
The celebrated Weyl law states that the asymptotic behavior of negative eigenvalues

\[ E_1(-\Delta + \lambda V) \leq E_2(-\Delta + \lambda V) \leq \cdots < 0 \]

of the Schrödinger operator \(-\Delta + \lambda V(x)\) on \(L^2(\mathbb{R}^d)\) can be determined by the semiclassical approximation in the strong coupling limit \(\lambda \to \infty\), namely

\[
\sum_{n \geq 1} |E_n(-\Delta + \lambda V)|^\kappa \approx \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (2\pi k^2 + \lambda V(x))_+ \right|^\kappa \, dk \, dx \\
= L^{cl}_{\kappa,d} \int_{\mathbb{R}^d} |\lambda V(x)_-|^{\kappa + d/2} \, dx
\]

for all \(\kappa \geq 0\). Here \(t_- = \min\{t, 0\}\) is the negative part of \(t\) and

\[ L^{cl}_{\kappa,d} = \int_{\mathbb{R}^d} \left| (2\pi k^2 + 1) \right|^{\kappa} \, dk = (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + 1 + \frac{d}{2})}. \tag{1.1} \]

The case \(\kappa = 0\) corresponds to the number of negative eigenvalues. By formally taking the potential

\[
V(x) = \begin{cases} 
-1 & \text{if } x \in \Omega, \\
+\infty & \text{if } x \notin \Omega
\end{cases}
\]
with an open bounded set \( \Omega \subset \mathbb{R}^d \), we find that the number of eigenvalues below \( \lambda \) of the Dirichlet Laplacian on \( L^2(\Omega) \) is equal to

\[
L_{0,d}^\text{cl}|\Omega|\lambda^{d/2} + o\left(\lambda^{d/2}\right)_{\lambda \to \infty}.
\]

The latter formula was first proved by Weyl in 1911 [69,70]. We refer to [46, Chapter 12] for further discussion on Weyl’s law.

Lieb–Thirring inequalities are non-asymptotic estimates for eigenvalues of Schrödinger operators which agree with the semiclassical approximation, possibly up to a universal constant factor.

**Theorem 1.1:** (Lieb–Thirring inequalities) Let \( d \geq 1 \) and \( \kappa \geq 0 \). Let \( V : \mathbb{R}^d \to \mathbb{R} \) be a real-valued potential such that \( V^- \in L^{\kappa+d/2}(\mathbb{R}^d) \). Assume that the Schrödinger operator \(-\Delta + V(x)\) on \( L^2(\mathbb{R}^d) \) has negative eigenvalues \( \{E_n(-\Delta + V)\}_{n \geq 1} \). Then

\[
\sum_{n \geq 1} |E_n(-\Delta + V)|^\kappa \leq L_{\kappa,d} \int_{\mathbb{R}^d} |V(x)|^{-\kappa+\frac{d}{2}} \, dx \quad (1.2)
\]

for a constant \( L_{\kappa,d} \in (0, \infty) \) independent of \( V \), provided that

\[
\begin{cases}
\kappa \geq 0 & \text{if } d \geq 3, \\
\kappa > 0 & \text{if } d = 2, \\
\kappa \geq 1/2 & \text{if } d = 1.
\end{cases}
\]

This result was first proved by Lieb and Thirring in 1975 for \( \kappa = 1 \) and \( d = 3 \) [49]. Then they extended the inequality to all \( \kappa > 0 \) when \( d \geq 2 \) and all \( \kappa > 1/2 \) when \( d = 1 \) [50]. The case \( \kappa = 0 \) when \( d \geq 3 \), often referred to as the Cwikel–Lieb–Rozenblum inequality, was proved independently in [10, 11, 62]. The last critical case \( \kappa = 1/2 \) when \( d = 1 \) was solved by Weidl in 1996 [68]. The range of \( \kappa \) is optimal.

In general, the constant \( L_{\kappa,d} \) in (1.2) is not necessarily the same as the semiclassical constant \( L_{\kappa,d}^\text{cl} \) in (1.1). Determining the sharp Lieb–Thirring constant is an important topic in mathematical physics; see e.g. [20] for a recent study. So far the sharp value of \( L_{\kappa,d} \) is only known in two cases:

- \( L_{\kappa,d} = L_{\kappa,d}^\text{cl} \) for \( \kappa \geq 3/2, d \geq 1 \). It was proved for \( d = 1 \) by Lieb and Thirring in 1976 [50], and extended to all \( d \geq 1 \) by Laptev and Weidl in 2000 [34].
- \( L_{1/2,1} = 2L_{1/2,1}^\text{cl} \) for \( \kappa = 1/2, d = 1 \). It was proved by Hundertmark, Lieb and Thomas in 1998 [30].
We refer to [45, 46, 48, 66] for pedagogical introductions to the subject and [19] for a recent review of current research.

In this short note we will focus on the case $\kappa = 1$ (sum of eigenvalues) which is particularly interesting due to its application to the study of the ground state energy of Fermi gases. By a duality argument [49, 50], the bound on the sum of eigenvalues can be translated to a kinetic inequality for orthonormal functions in $L^2(\mathbb{R}^d)$.

**Theorem 1.2:** (Lieb–Thirring kinetic inequality) Let $d \geq 1$. For any $N \geq 1$, let $\{u_n\}_{n=1}^N$ be orthonormal functions in $L^2(\mathbb{R}^d)$ and define $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$. Then

$$\sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 \, dx \geq K_d \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2}{d}} \, dx. \quad (1.3)$$

The constant $K_d > 0$ is related to the sharp constant $L_{1,d}$ in (1.2) by

$$K_d \left( 1 + \frac{2}{d} \right) = \left[ L_{1,d} \left( 1 + \frac{d}{2} \right) \right]^{-2/d}. \quad (1.4)$$

Here the orthogonality of $\{u_n\}_{n=1}^N$ is crucial. Without that assumption, we only have at best

$$\sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 \, dx \geq \frac{C_{\text{GN}}(d)}{N^{2/d}} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2}{d}} \, dx \quad (1.5)$$

where $C_{\text{GN}}(d)$ is the optimal Gagliardo–Nirenberg constant

$$C_{\text{GN}}(d) := \inf_{u \in H^1(\mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx}{\int_{\mathbb{R}^d} |u(x)|^{2(1 + \frac{2}{d})} \, dx}. \quad (1.6)$$

In contrast, the constant $K_d$ in (1.3) is independent of $N$, making it very useful to study quantum systems with many particles. On the physical point of view, while the Sobolev and the Gagliardo–Nirenberg inequalities are quantitative versions of the uncertainty principle, Lieb–Thirring inequalities are deeper as they involve the exclusion principle as well.

Historically, the kinetic inequality (1.3) is a key ingredient in the short proof of the stability of matter in [49]. In [49], Lieb and Thirring derived (1.3) from the eigenvalue bound (1.2). The advantage of studying (1.2) is that it can be reduced to counting the number of eigenvalues $\geq 1$ of the Birman–Schwinger operator

$$K_E = \sqrt{|V_-|} \left( -\Delta + E \right)^{-1} \sqrt{|V_-|}, \quad E > 0.$$
This method has been at the heart of most semiclassical eigenvalue estimates since the 1970s.

Surprisingly, a direct proof of the kinetic bound (1.3) had not been available for a long time until the work of Eden and Foias in one dimension in 1991 [15], which was extended to all dimensions by Dolbeault, Laptev, and Loss in 2008 [12]. More recently, different proofs were found by Rumin in 2010 [63, 64] and Lundholm and Solovej in 2013 [56, 57]. These new approaches have immediately led to several interesting developments in the field.

In this note, we will review the latter two direct approaches and explain some new results originated from them in a rather self-consistent manner. Here is a summary of the next sections.

- In Section 2 we discuss Rumin’s method. In principle, this approach is based on a suitable decomposition of the Laplacian in the momentum space. As a warm-up, we represent a short proof of the standard Sobolev inequality by Chemin and Xu [9] in Section 2.1. In some sense, Rumin’s idea is to combine this technique with Bessel’s inequality. The kinetic inequality (1.3) and its extensions are proved in Section 2.2. We explain the implication to the eigenvalue bound (1.2) in Section 2.3. Interestingly, Rumin’s method can be modified to give the currently best bound for the constant in (1.3). This result from 21 will be reviewed in Section 2.4. Some further results obtained by related arguments are mentioned in Section 2.5.

- In Section 3 we discuss the Lundholm–Solovej method. This approach is based on a suitable decomposition of the Laplacian in the position space. The key observation is that Lieb–Thirring inequalities can be deduced from a local version of the exclusion principle. Interestingly, such a local exclusion bound was used also by Dyson and Lenard in their first proof of the stability of matter [13]. The approach in 56 was originally proposed to study anyons (particles with only fractional statistics), but it has led to several new Lieb–Thirring inequalities. We will explain the general strategy in Section 3.1 and review some new results in the next subsections.

Note that there are other direct approaches which are not covered here, see e.g. Sabin’s work [65]. Moreover, although the methods represented in this note seem fascinating, there are various extensions of Lieb–Thirring inequalities that have to be treated differently. We refer to Frank’s review [19] for further aspects of the subject.
To end the introduction, let us mention that the methods covered here are general enough to handle fractional Lieb–Thirring inequalities with rather little effort. Therefore, we will in most cases aim at this generalization. In particular, a more general version of Theorem 1.2 is

**Theorem 1.3:** (Fractional Lieb–Thirring kinetic inequality) Let \( d \geq 1 \) and \( s > 0 \). For any orthonormal functions \( \{u_n\}_{n=1}^N \) in \( L^2(\mathbb{R}^d) \) with density \( \rho(x) = \sum_{n=1}^N |u_n(x)|^2 \), we have

\[
\sum_{n=1}^N \left\| (\Delta)^{s/2} u_n \right\|_{L^2(\mathbb{R}^d)}^2 \geq K_{d,s} \int_{\mathbb{R}^d} \rho(x) (1 + |2\pi k|^{2s})^2 \, dx.
\]

(1.7)
The constant \( K_{d,s} > 0 \) is independent of \( N \) and \( \{u_n\}_{n=1}^N \).

**Notation.** We will often denote by \( C \) a general positive constant whose value may change from line to line. In some cases, the dependence on key parameters will be included in the notation. However, it is important that all constants are always independent of the number of variables \( N \). Sometimes we write \( \int_{\Omega} f \) instead of \( \int_{\Omega} f(x) \, dx \).

**2. Rumin method**

### 2.1. A simple proof of Sobolev inequality

Recall the definition of the Sobolev space

\( H^s(\mathbb{R}^d) := \{ u \in L^2(\mathbb{R}^d) \mid |k|^s \hat{u}(k) \in L^2(\mathbb{R}^d) \} \)

with an arbitrary power \( s \geq 0 \) (not necessarily an integer). Here we use the following convention of the Fourier transform [46]

\[
\hat{u}(k) = \int_{\mathbb{R}^d} e^{-2\pi i k \cdot x} u(x) \, dx
\]

with \( i \) the imaginary unit (\( i^2 = -1 \)). Thus \( H^s(\mathbb{R}^d) \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{\hat{u}(k)} \hat{v}(k) (1 + |2\pi k|^{2s}) \, dk.
\]

On \( H^s(\mathbb{R}^d) \), the weak derivatives can be defined via the Fourier transform

\[
\hat{D^\alpha u}(k) = (2\pi k)^\alpha \hat{u}(k) \in L^2(\mathbb{R}^d)
\]
for any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots\}^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq s$. In particular,
\[ \langle u, (-\Delta)^{s/2} u \rangle_{L^2(\mathbb{R}^d)} = \left\| (-\Delta)^{s/2} u \right\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |2\pi k|^{2s} |\hat{u}(k)|^2 \, dk, \quad \forall u \in H^s(\mathbb{R}^d). \]

As a warm-up, let us consider
\[ K := \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^3} |2\pi k|^{2s} |\hat{u}(k)|^2 \, dk \]
\[ = \int_{\mathbb{R}^3} \left( \int_0^\infty \|\{2\pi k|^{2s} > E\}|\hat{u}(k)|^2 \, dE \right) \, dk \]
\[ = \int_0^\infty \left( \int_{\mathbb{R}^3} \|\{2\pi k|^{2s} > E\}|\hat{u}(k)|^2 \, dk \right) \, dE \]
\[ = \int_0^\infty \left( \int_{\mathbb{R}^3} |\hat{u}^E(k)|^2 \, dk \right) \, dE = \int_{\mathbb{R}^3} \left( \int_0^\infty |u^E(x)|^2 \, dE \right) \, dx \quad (2.1) \]
where the function $u^E$ is defined via the Fourier transform
\[ \hat{u}^E(k) = \mathbb{1}\{2\pi k|^{2s} > E\}\hat{u}(k). \]

When $d > 2s$, by Hölder’s inequality we have the uniform bound
\[ |u(x) - u^E(x)| = \left| \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \left( \hat{u}(k) - \hat{u}^E(k) \right) \, dk \right| \]
\[ = \left| \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \mathbb{1}\{2\pi k|^{2s} \leq E\} \hat{u}(k) \, dk \right| \]
\[ \leq \left( \int_{\mathbb{R}^d} |2\pi k|^{2s} |\hat{u}(k)|^2 \, dk \right)^{1/2} \left( \int_{\mathbb{R}^d} \mathbb{1}\{2\pi k|^{2s} \leq E\} \, dk \right)^{1/2} \]
\[ = C \sqrt{KE^{d-2s}} \]
with a finite constant $C = C(d, s) > 0$. By the triangle inequality,
\[
|u^{E+}(x)| \geq |u(x)| - |u(x) - u^{E+}(x)| \geq \left[|u(x)| - C \sqrt{KE} \frac{d-2s}{2s}\right]_+
\]
where $t_+ = \max\{t, 0\}$ is the positive part of $t$. Integrating over $E$ we get
\[
\int_0^\infty |u^{E+}(x)|^2 \, dE \geq \int_0^\infty \left[|u(x)| - C \sqrt{K}E \frac{d-2s}{2s}\right]^2_+ \, dE
\]
\[
\geq C |u(x)|^{\frac{d}{2s}} K^{-\frac{2s}{d}}.
\]
Inserting the latter bound in (2.1), we arrive at
\[
K \geq CK^{-\frac{2s}{d}} \int_{\mathbb{R}^d} |u(x)|^{\frac{d}{2s}} \, dx,
\]
which is equivalent to the desired inequality.

2.2. Lieb–Thirring kinetic inequality

The previous proof can be extended to bound the kinetic energy of a family of orthonormal functions.

**Theorem 2.2:** (Lieb–Thirring kinetic inequality) Let $d > 2\kappa \geq 0$ and $s \geq 0$ ($\kappa$ and $s$ are not necessarily integers). For any $N \geq 1$, let \{$(\Delta)^{\kappa/2} u_n\}_{n=1}^N$ be orthonormal functions in $L^2(\mathbb{R}^d)$ and denote $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$. Then
\[
\sum_{n=1}^N \|(-\Delta)^{\kappa/2} u_n\|_{L^2(\mathbb{R}^d)}^2 \geq K_{d,s,\kappa} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2s}{d-2s}} \, dx. \tag{2.2}
\]

The constant $K_{d,s,\kappa}$ is independent of $N$ and $\{u_n\}_{n=1}^N$.

The case $\kappa = 0$ is Theorem 1.3 in the Introduction. The case $\kappa = s$ is Lieb’s inequality [32], which implies the Cwikel–Lieb–Rozenblum inequality as we will see. The following proof for $\kappa = s$ is due to Rumin [63] (see also Frank [17]). The result for $\kappa \not\in \{0, s\}$ appears here for the first time.

**Proof:** By Plancherel’s and Fubini’s theorems we can write
\[
\sum_{n=1}^N \|(-\Delta)^{\kappa/2} u_n\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(\int_0^\infty \sum_{n=1}^N |u_n^{E+}(x)|^2 \, dE\right) \, dx \tag{2.3}
\]
where the function $u_n^{E+}$ is defined via the Fourier transform
\[
\hat{u}_n^{E+}(k) = 1(|2\pi k|^{2s} > E) \hat{u}_n(k).
\]
Now we use the assumption that \( \{(-\Delta)^{\kappa/2} u_n\}_{n=1}^N \) are orthonormal functions in \( L^2(\mathbb{R}^d) \). This implies that \( \{(2\pi k)^{\kappa} \hat{u}_n(k)\}_{n=1}^N \) are orthonormal functions in \( L^2(\mathbb{R}^d, dk) \). Hence, by Bessel’s inequality we have the uniform bound
\[
\sum_{n=1}^N |u_n(x) - u_n^{E+}(x)|^2 = \sum_{n=1}^N \left| \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \mathbf{1}\{ |2\pi k|^{2\kappa} \leq E \} \hat{u}_n(k) \, dk \right|^2 \\
= \sum_{n=1}^N \left| \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \mathbf{1}\{ |2\pi k|^{2\kappa} \leq E \} \frac{|2\pi k|^\kappa \hat{u}_n(k)}{|2\pi k|^\kappa} \, dk \right|^2 \\
\leq \left| e^{2\pi i k \cdot x} \mathbf{1}\{ |2\pi k|^{2\kappa} \leq E \} \right|^2 \left( \frac{2\pi}{|2\pi k|^\kappa} \right)^\frac{d-2\kappa}{2} \frac{2\pi}{|2\pi k|^{2\kappa}} \\
\leq C E^{\frac{d-2\kappa}{2}}.
\]

Here the constant \( C = C(d, \kappa) > 0 \) is finite when \( d > 2\kappa \).

Next, by the triangle inequality for vectors in \( \mathbb{C}^N \), we have
\[
\left( \sum_{n=1}^N |u_n^{E+}(x)|^2 \right)^{\frac{1}{2}} \geq \left| \sum_{n=1}^N |u_n(x)|^2 \right|^{\frac{1}{2}} - \left( \sum_{n=1}^N |u_n(x) - u_n^{E+}(x)|^2 \right)^{\frac{1}{2}} \\
\geq \sqrt{\rho(x) - CE^{\frac{d-2\kappa}{2}}}.
\]

Consequently,
\[
\int_0^\infty \sum_{n=1}^N |u_n^{E+}(x)|^2 \, dE \geq \int_0^\infty \left( \sqrt{\rho(x) - CE^{\frac{d-2\kappa}{2}}} \right)^2 \, dE \\
\geq C \rho(x)^{1+\frac{2\kappa}{d-2\kappa}}.
\]

Inserting the latter bound in (2.3), we obtain the desired inequality (2.2). \( \square \)

The extension of the above result to the case \( d \leq 2\kappa \) requires some modification. Here let us focus only on the case \( \kappa = s \).

**Theorem 2.3:** (Kinetic inequality in low dimensions) Let \( 2s \geq d \geq 1 \). For any \( N \geq 1 \) and \( E > 0 \), let \( \{\sqrt{(-\Delta)^s} + E u_n\}_{n=1}^N \) be orthonormal functions in \( L^2(\mathbb{R}^d) \) and denote \( \rho(x) = \sum_{n=1}^N |u_n(x)|^2 \).

- If \( d = 2s \), then there exist constants \( C_d, \alpha_d > 0 \) such that
  \[
  N \geq E \int_{\mathbb{R}^d} f(\rho(x)) \, dx, \quad f(t) = C_d t^{1+\alpha_d}.
  \]

- If \( d < 2s \), then \( \rho(x) \leq C_{s, \alpha} E^{\frac{d}{2s}-1} \) for a.e. \( x \in \mathbb{R}^d \).
Proof: By Plancherel’s and Fubini’s theorems we can write

\[ N = \sum_{n=1}^{N} \left\| \sqrt{(-\Delta)^s + E} u_n \right\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left( \int_{0}^{\infty} \sum_{n=1}^{N} \left| u_n^{L^+}(x) \right|^2 \, dL \right) \, dx \]

(2.4)

where the function \( u_n^{L^+} \) is defined via the Fourier transform

\[ \hat{u}_n^{L^+}(k) = \frac{1}{2\pi} \left( |2\pi k|^{2s} + E \right) \hat{u}_n(k). \]

Now we use the assumption that \( \{ \sqrt{(-\Delta)^s + E} u_n \}_{n=1}^{N} \) are orthonormal functions in \( L^2(\mathbb{R}^d) \). This implies that \( \{ \sqrt{|2\pi k|^{2s} + E} \hat{u}_n(k) \}_{n=1}^{N} \) are orthonormal functions in \( L^2(\mathbb{R}^d, dk) \). Hence, by Bessel’s inequality we have the uniform bound

\[ \sum_{n=1}^{N} \left| u_n(x) - u_n^{E^+}(x) \right|^2 \leq \int_{\mathbb{R}^d} \left( \frac{1}{|2\pi k|^{2s} + E} \right) \hat{u}_n(k) \, dk \]

\[ \leq C_d E^{\frac{d}{2s}-1} \int_{0}^{\infty} \frac{r^{d-1}}{r^{2s} + 1} \, dr. \]

Case \( d = 2s \). When \( L \geq E \) we get

\[ \sum_{n=1}^{N} \left| u_n(x) - u_n^{E^+}(x) \right|^2 \leq C_d \int_{0}^{\infty} \frac{r^{d-1}}{r^{2s} + 1} \, dr \leq C_d \log(L/E). \]

Hence, by the triangle inequality

\[ \int_{E}^{\infty} \sum_{n=1}^{N} \left| u_n^{L^+}(x) \right|^2 \, dL \geq \int_{E}^{\infty} \left( \left| \sqrt{\rho(x)} - \sqrt{C_d \log(L/E)} \right|_{+} \right)^2 \, dL \]

\[ = E \int_{1}^{\infty} \left( \left| \sqrt{\rho(x)} - \sqrt{C_d \log(L)} \right|_{+} \right)^2 \, dL \]

\[ \geq E \rho(x) e^{\alpha \rho(x)}. \]

Inserting the latter bound in (2.4), we obtain the desired inequality.

Case \( d < 2s \). When \( L \geq E \) we get

\[ \sum_{n=1}^{N} \left| u_n(x) - u_n^{E^+}(x) \right|^2 \leq C_d E^{\frac{d}{2s}-1} \int_{0}^{\infty} \frac{r^{d-1}}{r^{2s} + 1} \, dr \leq C_d E^{\frac{d}{2s}-1}. \]
Hence, if \( \rho(x) > C_d E^{\frac{d}{2s}-1} \), then by the triangle inequality

\[
\int_E \sum_{n=1}^{N} |u_n^+(x)|^2 \, dL \geq \int_E \left( \left[ \sqrt{\rho(x)} - \sqrt{C_d E^{\frac{d}{2s}-1}} \right]_+^2 \right) \, dL = \infty.
\]

Thus from (2.4) we conclude that \( \rho(x) \leq C_d E^{\frac{d}{2s}-1} \) for a.e. \( x \in \mathbb{R}^d \).

Note that in the above result, the case \( d = 2s \) is related to the question discussed in [19, Section 5.8]. Moreover, in the case \( d < 2s \), instead of using Rumin’s method we can also proceed as follows. Denote \( v_n = \sqrt{(-\Delta)^s + E} u_n \), then

\[
u_n = ((-\Delta)^s + E)^{-1/2} v_n = G_E \ast v_n, \quad \widehat{G}_E(k) = (|2\pi k|^{2s} + E)^{-1/2}.
\]

Since \( \{v_n\}_{n=1}^N \) are orthonormal functions in \( L^2(\mathbb{R}^d) \), by Bessel’s inequality and Plancherel’s theorem we have the uniform bound

\[
\rho(x) = \sum_{n=1}^{N} |u_n(x)|^2 = \sum_{n=1}^{N} \left| \int_{\mathbb{R}^d} G(x-y)v_n(y) \, dy \right|^2 \leq \|G\|_{L^2}^2 = C_d E^{\frac{d}{2s}-1}.
\]

### 2.3. Eigenvalue bounds for Schrödinger operators

Now let us consider an extension of Theorem 1.1 for the fractional Laplacian.

**Theorem 2.4:** (Eigenvalue bounds for fractional Laplacian) Let \( d \geq 1 \), \( s > 0 \) and \( \kappa \geq 0 \). Let \( V : \mathbb{R}^d \rightarrow \mathbb{R} \) be a real-valued potential such that \( V_\ast \in L^{\kappa + \frac{d}{2s}}(\mathbb{R}^d) \). Assume that the Schrödinger operator \((-\Delta)^s + V(x)\) on \( L^2(\mathbb{R}^d) \) has negative eigenvalues \( \{E_n((-\Delta)^s + V)\}_{n \geq 1} \). Then

\[
\sum_{n \geq 1} |E_n((-\Delta)^s + V)|^\kappa \leq L_{\kappa,d,s} \int_{\mathbb{R}^d} |V(x)|^{\kappa + \frac{d}{2s}} \, dx \quad (2.5)
\]

for a finite constant \( L_{\kappa,d,s} \) independent of \( V \), provided that

\[
\begin{align*}
\kappa &\geq 0 \quad \text{if} \ d > 2s, \\
\kappa &> 0 \quad \text{if} \ d = 2s, \\
\kappa &\geq 1 - \frac{d}{2s} \quad \text{if} \ d < 2s.
\end{align*}
\]

The Lieb–Thirring inequality for a generalized kinetic operator \( f(i\nabla) \) was studied in [11]. In the form of (2.5), all non-critical cases can be treated using the method in [50]. The critical case \( \kappa = 0 \) follows from the proofs in [10,62]. For the other critical case \( \kappa = 1 - d/(2s) \) see [15,61].
In the following we prove Theorem 2.4 using the kinetic inequalities in Section 2.2. The proof covers all cases except the critical case \( \kappa = 1 - d/(2s) \).

**Proof: Case 1:** \( d > 2s, \kappa = 0 \). The following duality argument is due to Frank [17]. Let \( W \) be the space spanned by eigenfunctions of negative eigenvalues of \( (-\Delta)^s + V \). Assume that \( \dim W \geq N \). Since the operator \( (-\Delta)^{s/2} \) is strictly positive on \( L^2(\mathbb{R}^d) \), we get

\[
\dim \left( (-\Delta)^{s/2} W \right) \geq N.
\]

Thus we can choose \( \{ u_n \}_{n=1}^N \subset W \) such that \( \{ (-\Delta)^{s/2} u_n \}_{n=1}^N \) are orthonormal in \( L^2(\mathbb{R}^d) \). By the kinetic inequality (2.2) with \( \kappa = s \), we have

\[
N = \sum_{n=1}^N \left\| (-\Delta)^{s/2} u_n \right\|_{L^2(\mathbb{R}^d)}^2 \geq K_{d,s} \int_{\mathbb{R}^d} \rho^{2s/(2s-d)}(x) \, dx
\]

with \( \rho(x) := \sum_{n=1}^N |u_n(x)|^2 \).

On the other hand, since \( \{ u_n \}_{n=1}^N \subset W \) we have

\[
0 \geq \sum_{n=1}^N \langle u_n, (-\Delta)^s + V \rangle_{L^2(\mathbb{R}^d)} = N + \int_{\mathbb{R}^d} V(x) \rho(x) \, dx.
\]

Putting together these inequalities, we find that

\[
N \leq - \int_{\mathbb{R}^d} V(x) \rho(x) \, dx \leq \int_{\mathbb{R}^d} |V_-(x)| \rho(x) \, dx
\]

\[
\leq \| V_- \|_{L^{d/(d-2s)}} \| \rho \|_{L^{2s/(d-2s)}} \leq \| V_- \|_{L^{d/(d-2s)}} \left( \frac{N}{K_{d,s}} \right)^{\frac{d-2s}{d}}.
\]

which is equivalent to

\[
N \leq K_{d,s}^{1-\frac{d}{d-2s}} \int_{\mathbb{R}^d} |V_-(x)|^{\frac{d}{d-2s}} \, dx.
\]

Then we conclude by taking \( N \to \dim W \).

**Case 2:** \( d > 2s, \kappa > 0 \). The result for \( \kappa > 0 \) follows from the case \( \kappa = 0 \), thanks to a general argument of Aizenman and Lieb [2]. We use the layer-cake representation

\[
|E_n|^\kappa = \kappa \int_0^\infty \mathbf{1}(E_n < -E) E^{\kappa-1} \, dE.
\]
Using the bound in Case 1 for the number of negative eigenvalues of $(-\Delta)^s + V + E$, we have

$$\sum_{n \geq 1} 1(E_n < -E) \leq C \int_{\mathbb{R}^d} |(V(x) + E)_-|^{\frac{s}{d}} \, dx$$

for a constant $C = C(d, s) > 0$. Thus

$$\sum_{n \geq 1} |E_n|^\kappa = \kappa \int_0^\infty \sum_{n \geq 1} 1(E_n < -E) E^{s-1} \, dE$$

$$\leq C\kappa \int_0^\infty \left( \int_{\mathbb{R}^d} |(V(x) + E)_-|^{\frac{s}{d}} E^{s-1} \, dx \right) \, dE$$

$$\leq C\kappa \int_{\mathbb{R}^d} \left( \int_0^\infty |(V(x) + E)_-|^{\frac{s}{d}} E^{s-1} \, dE \right) \, dx$$

$$= C' \int_{\mathbb{R}^d} |V(x)_-|^{\kappa + \frac{s}{d}} \, dx,$$

where

$$C' = C\kappa \int_0^\infty |(1 + E)_-|^{\frac{s}{d}} E^{s-1} \, dE < \infty.$$

**Case 3:** $d = 2s$, $\kappa > 0$. Let $N_E$ be the number of negative eigenvalues of $(-\Delta)^s + V + E$. We can bound $N_E$ by arguing as in Case 1, but replacing \(2.6\) by Theorem 2.3, namely

$$N_E \geq E \int_{\mathbb{R}^2} f(\rho(x)) \, dx, \quad f(t) = C_d t e^{\alpha_d t}$$

with constants $C_d, \alpha_d > 0$. Therefore,

$$N_E \leq 2 \int_{\mathbb{R}^d} |V_-(x)| \rho(x) \, dx - N_E \leq \int_{\mathbb{R}^d} \left( 2|V_-(x)| \rho(x) - E f(\rho(x)) \right) \, dx$$

$$\leq \int_{\mathbb{R}^d} E f^* \left( \frac{2|V_-(x)|}{E} \right) \, dx$$

(2.7)

where $f^*: [0, \infty) \to [0, \infty]$ is the Legendre transform of $f$, namely

$$f^*(y) = \sup_{t \geq 0} \{ yt - f(t) \}, \quad \forall y \geq 0.$$

Since $V+E = (V+E/2)+E/2$, we can also replace $(V, E)$ by $(V+E/2, E/2)$ and deduce from (2.7) that

$$N_E \leq \frac{E}{2} \int_{\mathbb{R}^d} f^* \left( \frac{4|(V(x) + E)_-|}{E} \right) \, dx.$$
Then we argue as in case 2 and obtain, by the layer-cake representation,

$$\sum_{n \geq 1} |E_n|^\kappa = \kappa \int_0^\infty N_E E^{\kappa-1} \, dE$$

$$\leq \kappa \int_0^\infty \left( \int_{\mathbb{R}^d} f^* \left( \frac{2|V(x) + E/2|}{E/2} \right) \frac{E^\kappa}{2} \, dx \right) \, dE$$

$$= C_{d,\kappa} \left( \int_{\mathbb{R}^d} |V(x) - E|^{\kappa+1} \, dx \right) \left( \int_0^1 f^* \left( \frac{2}{y} - 2 \right) y^\kappa \, dy \right).$$

In the last equality we have changed the variable $E = 2|V(x)|y$. It remains to show that:

$$\int_0^1 f^* \left( \frac{2}{y} - 2 \right) y^\kappa \, dy < \infty.$$

Note that if $f \geq g$, then $f^* \leq g^*$. Moreover, $(t^p/p)^* = t^q/q$ with $1/p + 1/q = 1$ by Young’s inequality. Since $f(t)$ grows faster than any polynomial, we find that

$$f^*(t) \leq C_q t^q$$

for any $q \in (1, 1 + \kappa)$. Hence

$$\int_0^1 f^* \left( \frac{2}{y} - 2 \right) y^\kappa \, dy \leq C_q \int_0^1 \left( \frac{2}{y} - 2 \right) y^q \, dy \leq C_q \int_0^1 y^{\kappa-q} \, dy < \infty.$$

**Case 4:** $d < 2s$, $\kappa > 1 - \frac{d}{2s}$. Again we bound $N_E$, the number of negative eigenvalues of $(-\Delta)^s + V + E$, as in case 1 but replacing (2.6) by the uniform bound

$$\rho(x) \leq C_d E^{\frac{d}{2s} - 1}, \quad \text{a.e. } x \in \mathbb{R}^d,$$

from Theorem 2.3. Thus

$$N_E \leq \int_{\mathbb{R}^d} |V-| \rho \leq C_d E^{\frac{d}{2s} - 1} \int_{\mathbb{R}^d} |V-(x)| \, dx.$$

Again, we can write $V + E = (V + E/2) + E/2$ and obtain

$$N_E \leq C_d (E/2)^{\frac{d}{2s} - 1} \int_{\mathbb{R}^d} |(V(x) + E/2)| \, dx.$$
Thus by the layer-cake representation,
\[
\begin{align*}
\sum_{n \geq 1} |E_n|^\kappa &= \kappa \int_0^\infty N_E E^{\kappa-1} \, dE \\
&\leq \kappa C_d \int_0^\infty E^{\kappa-1} (E/2)^{\frac{d}{2}} \left( \int_{\mathbb{R}^d} \left| (V(x) + E/2)_- \right| \, dx \right) \, dE \\
&= C_{d,\kappa,s} \left( \int_{\mathbb{R}^d} |V_-(x)|^{\kappa+\frac{d}{2}} \, dx \right) \left( \int_0^\infty E^{\kappa+\frac{d}{2}} (1 - E) \, dE \right).
\end{align*}
\]
Finally, when \( \kappa > 1 - d/(2s) \) we have
\[
\int_0^\infty E^{\kappa+\frac{d}{2}} (1 - E) \, dE = \int_0^1 E^{\kappa+\frac{d}{2}} (1 - E) \, dE < \infty.
\]
This completes the proof of Theorem 2.4 (except for the critical case \( \kappa = 1 - d/(2s) \)).

2.4. Best known constant for kinetic inequality

In the non-relativistic case, Lieb and Thirring \([49, 50]\) conjectured that the optimal constant in the kinetic inequality (1.3) is
\[
K_d = \min \{ K_d^{cl}, C_{GN}(d) \} = \begin{cases} 
K_d^{cl} & \text{if } d \geq 3, \\
C_{GN}(d) & \text{if } d = 1, 2,
\end{cases} \quad (2.8)
\]
where \( C_{GN}(d) \) is defined in (1.6). Thanks to the relation (1.4), the Lieb–Thirring conjecture is equivalent to
\[
L_{1,d} = \max \{ L_{1,d}^{cl}, L_{1,d}^{So} \} = \begin{cases} 
L_{1,d}^{cl} & \text{if } d \geq 3, \\
L_{1,d}^{So} & \text{if } d = 1, 2,
\end{cases} \quad (2.9)
\]
where \( L_{1,d}^{So} \) is the optimal constant in the one-body bound
\[
\int_{\mathbb{R}^d} \left( \nabla u(x)^2 + V(x) |u(x)|^2 \right) \, dx \geq -L_{1,d}^{So} \int_{\mathbb{R}^d} |V(x)^-|^{1+d/2} \, dx. \quad (2.10)
\]
The original proof of Lieb and Thirring \([49]\) gave \( L_{1,d}/L_{1,d}^{cl} \leq 4\pi \) in \( d = 3 \). This bound has been improved further in \([7, 12, 15, 29, 44]\). The latest improvement in \([21]\) is

**Theorem 2.5:** For all \( d \geq 1 \) we have \((K_d^{cl}/K_d)^{d/2} = L_{1,d}/L_{1,d}^{cl} \leq 1.456\).

In one dimension, this bound is about 26% bigger than the expected value \( L_{1,1}^{So}/L_{1,1}^{cl} = 2/\sqrt{3} = 1.155 \ldots \) in \([50]\). For any \( d \geq 1 \), by the Aizenman–Lieb monotonicity \([2]\), Theorem 2.5 implies that \( L_{\kappa,d}/L_{\kappa,d}^{cl} \leq 1.456 \) for all \( \kappa \geq 1 \),
which is also the best known result for all \(1 \leq \kappa < 3/2\) (when \(\kappa \geq 3/2\), we know that \(L_{\kappa,d} = L_{\kappa,d}^{cl}\) \([34,50]\)).

The proof of Theorem 2.5 uses crucially the technique of optimizing momentum decompositions. More precisely, it contains two main steps.

- First, we improve the kinetic inequality using a modification of Rumin’s method. This gives \(L_{1,d}/L_{1,d}^{cl} \leq 1.456\) in \(d = 1\) (and worse bounds in higher dimensions).
- Second, we use the Laptev–Weidl lifting argument \([34]\) to extend the bound to higher dimensions.

The first step can be extended to fractional Laplacian to bound the constant in Theorem 1.3. For every \(s > 0\), the corresponding semiclassical constant is

\[
K_{d,s}^{cl} = \frac{d}{d+2s} \left( \frac{(2\pi)^d}{|B_1|} \right)^{\frac{2}{d}}.
\]

We have

**Theorem 2.6:** For all \(d \geq 1\) and \(s > 0\), the best constant in (1.7) satisfies

\[
\frac{K_{d,s}}{K_{d,s}^{cl}} \gtrsim \frac{d}{d+2s} \left( \frac{2s}{d+2s} \right)^{\frac{2}{d}} C_{d,s}^{\frac{2}{d}}
\]

where

\[
C_{d,s} := \inf \left\{ \left( \int_0^\infty \varphi(r)^2 \, dr \right)^{\frac{d}{2s}} \frac{d}{2s} \int_0^\infty \left| 1 - \int_0^\infty \varphi(E) f(E) \, dE \right|^2 \frac{dt}{t^{1+\frac{d}{2s}}} \right\}
\]

(2.11)

with the infimum taken over all functions \(f, \varphi : [0,\infty) \to [0,\infty)\) satisfying \(\int_0^\infty f(r)^2 \, dr = 1\).

When \(d = s = 1\), we have \(C_{1,1} \leq 0.373556\) by taking in (2.11)

\[
f(t) = (1 + \mu_0 t^{4.5})^{-0.25}, \quad \varphi(t) = \frac{(1 - t^{0.36})^{2.1}}{1 + t} 1(t \leq 1)
\]

with \(\mu_0\) determined by \(\int_0^\infty f^2 = 1\). This implies \(L_{1,1}/L_{1,1}^{cl} \leq 1.456\).

**Proof:** Using \(\int_0^\infty f(r)^2 \, dr = 1\) we can write

\[
|2\pi k|^{2s} = \int_0^\infty f(E)|2\pi k|^{-2s} \, dE.
\]
Thus
\[
\sum_{n=1}^{N} \|(-\Delta)^{s/2} u_n\|^2_{L^2} = \int_{\mathbb{R}^d} \left( \sum_{n=1}^{N} \int_{0}^{\infty} |u_n^{E^+}(x)|^2 \, dE \right) \, dx \tag{2.12}
\]
where
\[
\hat{u}_n^{E^+}(k) = f(E|2\pi k|^{-2s})\hat{u}_n(k).
\]

Next, for every function \( \varphi : [0, \infty) \to [0, \infty) \), by the Cauchy–Schwarz inequality and the triangle inequality in \( \mathbb{C}^d \), we have
\[
\sum_{n=1}^{N} \left( \int_{0}^{\infty} \varphi(E)^2 \, dE \right) \left( \int_{0}^{\infty} |u_n^{E^+}(x)|^2 \, dE \right) \geq \sum_{n=1}^{N} \left| \int_{0}^{\infty} \varphi(E) u_n^{E^+}(x) \, dE \right|^2
\]
\[
\geq \left( \sum_{n=1}^{N} |u_n(x)|^2 \right)^{1/2} - \left( \sum_{n=1}^{N} |u_n(x)|^2 - \int_{0}^{\infty} \varphi(E) u_n^{E^+}(x) \, dE \right)^{1/2}.
\]

Next, using again the fact that \( \{u_n\}_{n=1}^{N} \) are orthonormal functions in \( L^2(\mathbb{R}^d) \) and Bessel’s inequality, we have the uniform bound
\[
\sum_{n=1}^{N} \left| u_n(x) - \int_{0}^{\infty} \varphi(E) u_n^{E^+}(x) \, dE \right|^2
\]
\[
= \sum_{n=1}^{N} \left| \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \hat{u}_n(k) \left( 1 - \int_{0}^{\infty} \varphi(E) f(E|2\pi k|^{-2s}) \, dE \right) \, dk \right|^2
\]
\[
\leq \int_{\mathbb{R}^d} \left| 1 - \int_{0}^{\infty} \varphi(E) f(E|2\pi k|^{-2s}) \, dE \right|^2 \, dk
\]
\[
= \frac{|dB_1|}{2s(2\pi)^d} \int_{0}^{\infty} \left| 1 - \int_{0}^{\infty} \varphi(E) f(Et) \, dE \right|^2 \left( \frac{dt}{t^{1+\frac{s}{d}}} \right).
\]

Thus
\[
\sum_{n=1}^{N} \int_{0}^{\infty} |u_n^{E^+}(x)|^2 \, dE \geq \left( \int_{0}^{1} \varphi(r)^2 \, dr \right)^{-1}
\]
\[
\times \left[ \sqrt{\rho(x)} - \left( \frac{|dB_1|}{2s(2\pi)^d} \int_{0}^{\infty} \left| 1 - \int_{0}^{\infty} \varphi(E) f(Et) \, dE \right|^2 \left( \frac{dt}{t^{1+\frac{s}{d}}} \right) \right)^{1/2} \right]^{2}.
\]
Replacing $\varphi(E) \mapsto \ell \varphi(\ell E)$ and optimizing over $\ell > 0$ we get
\[
\sum_{n=1}^{N} \int_{0}^{\infty} |u_{n}^{E+}(x)|^{2} \, dE \geq \rho(x)^{1 + \frac{2d}{d + 2s}} \left( \frac{d}{d + 2s} \right)^{2} \left( \frac{2s}{d + 2s} \right)^{\frac{d}{s}} \left( \int_{0}^{1} \varphi(r)^{2} \, dr \right)^{-1} \left( \frac{d|B_{1}|}{2s(2\pi)^{d}} \int_{0}^{\infty} |1 - \int_{0}^{\infty} \varphi(E)f(Et) \, dE|^{2} \frac{dt}{t^{1 + \frac{s}{2}}} \right)^{-\frac{2}{d}}.
\]
Finally, optimizing over $f$ and $\varphi$ we conclude that
\[
\sum_{n=1}^{N} \int_{0}^{\infty} |u_{n}^{E+}(x)|^{2} \, dE \geq \rho(x)^{1 + \frac{2d}{d + 2s}} \left( \frac{d}{d + 2s} \right)^{2} \left( \frac{2s}{d + 2s} \right)^{\frac{d}{s}} \left( |B_{1}| \left( \frac{d}{d + 2s} \right) \right)^{-\frac{2d}{s}} C_{d,s}^{-\frac{2d}{s}}.
\]
Inserting this bound in (2.12) we get the desired inequality.

2.5. Further results

The idea of optimizing momentum decompositions is also useful to improve the Lieb–Thirring kinetic constant on the sphere and on the torus in [32], and to improve the constant in the Cwikel–Lieb–Rozenblum inequality in [31]. This technique can be developed to derive new semiclassical inequalities; see [23,24] for a positive density analogue of the Lieb–Thirring inequality.

3. Lundholm-Solovej method

3.1. Kinetic inequality via local exclusion principle

The Lieb–Thirring inequality [49, 50] was originally invented to give an energy lower bound for fermionic particles. From first principles of quantum mechanics, a system of $N$ identical (spinless) fermions in $\mathbb{R}^{d}$ can be described by a normalized wave function $\Psi_{N} \in L^{2}(\mathbb{R}^{d})^{N}$ satisfying
\[
\Psi_{N}(x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{N}) = -\Psi_{N}(x_{1}, \ldots, x_{j}, \ldots, x_{i}, \ldots, x_{N}), \quad \forall i \neq j.
\]
(3.1)
Here $x_{i} \in \mathbb{R}^{d}$ is the position of the $i$-th particle (we ignore the spin for simplicity) and $|\Psi_{N}|^{2}$ is interpreted as the probability density of $N$ particles.
The anti-symmetry condition (3.1), also called Pauli’s exclusion principle, implies that two fermionic particles cannot occupy the same position. Clearly, \( \Psi_N = 0 \) if \( x_i = x_j \) for \( i \neq j \). Moreover, if we define the one-body density matrix \( \gamma_{\Psi_N} \) as an operator on \( L^2(\mathbb{R}^d) \) with kernel

\[
\gamma_{\Psi_N}(x, y) := \sum_{j=1}^{N} \int_{\mathbb{R}^d(N-1)} \frac{1}{(N-1)} \Psi(x_1, \ldots, x_{j-1}, x, x, x_{j+1}, \ldots, x_N) \prod_{i \neq j} dx_i,
\]

then we have the operator inequality \[0 \leq \gamma_{\Psi_N} \leq 1 \] on \( L^2(\mathbb{R}^d) \). (3.2)

This is nontrivial since \( \text{Tr} \gamma_{\Psi_N} = N \). Without the anti-symmetry condition (3.1), \( \gamma_{\Psi_N} \) may have an eigenvalue as large as \( N \) (in fact, \( \gamma_{\Psi_N} = N|u\rangle\langle u| \) if \( \Psi_N = u \otimes u \otimes \ldots \otimes u \)). The Lieb–Thirring inequality allows us to bound the kinetic energy of \( \Psi_N \) in terms of its one-body density

\[
\rho_{\Psi}(x) := \gamma_{\Psi_N}(x, x) = \sum_{j=1}^{N} \int_{\mathbb{R}^d(N-1)} |\Psi(x_1, \ldots, x_{j-1}, x, x, x_{j+1}, \ldots, x_N)|^2 \prod_{i \neq j} dx_i.
\]

**Theorem 3.1:** (Lieb–Thirring kinetic inequality for fermions) Let \( d \geq 1 \) and \( s > 0 \). Let \( \Psi_N \in L^2((\mathbb{R}^d)^N) \) be a normalized wave function satisfying the anti-symmetry (3.1). Then

\[
\left\langle \Psi_N, \sum_{i=1}^{N} (-\Delta)^s \Psi_N \right\rangle \geq K_{d,s} \int_{\mathbb{R}^d} \rho_{\Psi_N}^{1+2/s}(x) dx.
\]

with the same constant \( K_{d,s} \) in (1.7).

Note that the left side of (3.3) is nothing but

\[ \text{Tr}(\Psi_N (-\Delta)^s) = \text{Tr}(\gamma_{\Psi_N} (-\Delta)^s/2). \]

Hence, applying (3.3) to the Slater determinant \( \Psi_N = u_1 \wedge u_2 \wedge \cdots \wedge u_N \) we recover the kinetic inequality for orthonormal functions in (1.7). On the other hand, we can deduce (3.3) from (1.7) and the operator bound (3.2) by a convexity argument (see [19, Lemma 3]). Moreover, the operator bound (3.2) also allow us to deduce (3.3) from the eigenvalue bound in Theorem 2.4. Nevertheless, we will discuss an alternative approach to (3.3) below as it will open the way to further developments.
In 2013, Lundholm and Solovej [56, 57] realized that one can deduce the Lieb–Thirring inequality (3.3) using only a weaker version of Pauli’s exclusion principle (3.1). More precisely, they need only a rather simple consequence of the operator inequality (3.2), which holds for a larger class of quantum systems than just Fermi gases. We refer to Lundholm’s lecture notes [51] for a pedagogical introduction to the theory. In the following we will follow the simplified representation in [52].

Decomposition in position space. If \( \mathbb{R}^d \) is covered by disjoint domains \( \{\Omega\} \), then

\[
(-\Delta)^s = \sum_{\Omega} (-\Delta)^s_{|\Omega} \quad \text{on} \quad L^2(\mathbb{R}^d)
\]

where the Neumann Laplacian \((-\Delta)^s_{|\Omega}\) is defined via the quadratic form

\[
\langle u, (-\Delta)^s_{|\Omega} u \rangle_{L^2(\mathbb{R}^d)} = \|u\|^2_{\dot{H}^s(\Omega)}.
\]

Here the seminorm \(\|u\|^2_{\dot{H}^s(\Omega)}\) is defined as follows,

\[
\|u\|^2_{\dot{H}^s(\Omega)} = \begin{cases} 
\sum_{|\alpha|=s} \frac{s!}{\alpha!} \int_{\Omega} |D^\alpha u(x)|^2 \, dx & \text{if } s \in \mathbb{N}, \\
\mathbf{c}_{d,\sigma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\Omega} \int_{\Omega} |D^\alpha u(x) - D^\alpha u(y)|^2 \frac{dx \, dy}{|x-y|^{d+2\sigma}} & \text{if } s \not\in \mathbb{N}.
\end{cases}
\]

In the case \( s \not\in \mathbb{N} \), we have used the notation \( s = m + \sigma \) with \( m \in \mathbb{N} \), \( \sigma \in (0,1) \) and

\[
\mathbf{c}_{d,\sigma} := \frac{2^{2\sigma-1} \Gamma(d/2 + \sigma)}{\pi^{d/2} |\Gamma(-\sigma)|}.
\]

The coefficient \( \mathbf{c}_{d,\sigma} \) comes from the well-known formula (see e.g. [22, Lemma 3.1])

\[
\langle u, (-\Delta)^\sigma u \rangle = \mathbf{c}_{d,\sigma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^2 \frac{dx \, dy}{|x-y|^{d+2\sigma}}.
\]

Consequently, for any \( N \)-body wave function \( \Psi_N \in H^s(\mathbb{R}^{dN}) \) we can decompose

\[
\mathcal{E}_{\mathbb{R}^d}[\Psi_N] := \left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i})^s \Psi_N \right\rangle = \sum_{\Omega} \mathcal{E}_\Omega[\Psi_N] \quad \text{(3.4)}
\]

where the local energy on \( \Omega \) is defined by

\[
\mathcal{E}_\Omega[\Psi_N] := \left\langle \Psi_N, \sum_{j=1}^N (-\Delta_{x_j})^s_{|\Omega} \Psi_N \right\rangle. \quad \text{(3.5)}
\]
So far we have defined the Neumann Laplacian \((-\Delta)_\Omega^s\) on \(L^2(\mathbb{R}^d)\) for convenience, but it can be restricted naturally to an operator on \(L^2(\Omega)\). It is important that the kernel of this restriction has only finite dimensions.

**Lemma 3.2:** (Lower bound on Neumann Laplacian) Let \(d \geq 1\) and \(s > 0\). Then for any cube \(Q \subset \mathbb{R}^d\) we have the operator inequality on \(L^2(Q)\):

\[
(-\Delta)^s_Q \geq \frac{C_{d,s}}{|Q|^{2s/d}} (1 - P_q)
\]

for a constant \(C_{d,s} > 0\) independent of \(Q\) and a rank-\(q\) projection \(P_q\), where

\[q := \#\{\text{multi-indices } \alpha \in \{0,1,\ldots\}^d : 0 \leq |\alpha| < s\}.
\]

**Proof:** If \(s = 1\), then the result is obvious since the eigenvalues of the Neumann Laplacian on \(L^2(Q)\) are given explicitly by \(\{|Q|^{-2/d}|2\pi k|^2 | k \in \{0,1,\ldots\}^d\}\); in particular the eigenvalue 0 is single. We refer to [52, Lemma 11] for the general case.

A consequence of (3.6) and (3.2) is

**Lemma 3.3:** (Local exclusion for fermions) Let \(d \geq 1\) and \(s > 0\). Let \(\Psi_N\) be a normalized fermionic wave function in \(H^s(\mathbb{R}^{dN})\) satisfying (3.1). Then for any cube \(Q \subset \mathbb{R}^d\), we have

\[
E_Q[\Psi_N] \geq C_{d,s}|Q|^{-2s/d} \int_Q \rho_{\Psi_N}(x) \, dx - q,
\]

where \(E_Q[\Psi_N]\) is defined in (3.5) and \(q\) is given in Lemma 3.2.

In the non-relativistic case \(s = 1\), this weak formulation of the exclusion principle was used by Dyson and Lenard in their first proof of the stability of matter [13]. As realized in [56,57], this can be used as a key tool to derive Lieb–Thirring inequalities.

**Proof of Lemma 3.3:** We use the spectral decomposition

\[
\gamma_{\Psi_N} = \sum_{n \geq 1} \lambda_n \langle u_n | u_n \rangle \quad \text{on } L^2(\mathbb{R}^d)
\]

where \(\{u_n\}_{n \geq 1} \subset H^s(\mathbb{R}^d)\) are orthonormal in \(L^2(\mathbb{R}^d)\) and \(0 \leq \lambda_n \leq 1\) due...
Direct methods to Lieb–Thirring kinetic inequalities

Then using (3.6) we have
\[
E_Q[\Psi_N] = \text{Tr}((-\Delta)^s \gamma \Psi_N) = \sum_{n \geq 1} \lambda_n \langle u_n, (-\Delta_x)^s \gamma u_n \rangle_{L^2(\mathbb{R}^d)}
\]
\[
\geq \frac{C_{d,s}}{|Q|^{2s/d}} \sum_{n \geq 1} \lambda_n \langle u_n, 1 - P_q \rangle_{L^2(\mathbb{R}^d)}
\]
\[
\geq \frac{C_{d,s}}{|Q|^{2s/d}} \left[ \sum_{n \geq 1} \lambda_n \int_Q |u_n(x)|^2 \, dx - \text{Tr}(P_q) \right]
\]
\[
= \frac{C_{d,s}}{|Q|^{2s/d}} \left[ \int_Q \rho_{\Psi_N}(x) \, dx - q \right].
\]
Also obviously the left side of (3.7) is nonnegative. This completes the proof of (3.7).

The second key ingredient to prove the Lieb–Thirring inequality (3.3) is a Gagliardo–Nirenberg inequality on bounded domains (see [52, Lemma 8]). This part requires no symmetry condition on the wave functions.

**Lemma 3.4:** (Local uncertainty) Let \( d \geq 1 \) and \( s > 0 \). Let \( \Psi_N \) be a wave function in \( H^s(\mathbb{R}^{dN}) \) for arbitrary \( N \geq 1 \) and let \( Q \) be an arbitrary cube in \( \mathbb{R}^d \). Then
\[
E_Q[\Psi_N] \geq C_{d,s} \int_Q \rho_{\Psi_N}(x)^{1+2s/d} \, dx \left( \int_Q \rho_{\Psi_N}(x)^{2s/d} \, dx \right)^{2s/d} - \frac{1}{|Q|^{2s/d}} \int_Q \rho_{\Psi_N}(x) \, dx. \tag{3.9}
\]

**Sketch of proof:** By translation and dilation, it suffices to prove (3.9) for \( Q = [0,1]^d \). When \( N = 1 \), (3.9) is equivalent to the Gagliardo–Nirenberg inequality
\[
\|u\|_{H^s(\mathbb{R}^d)}^{1-\theta} \|u\|_{L^2(\mathbb{R}^d)} \geq C_{d,s} \|u\|_{L^q(\mathbb{R}^d)}, \quad q = 2 + \frac{4s}{d}, \quad \theta = \frac{d}{d + 2s}. \tag{3.10}
\]
By the extension theorem (see [11 Theorem 7.41]), it suffices to prove that
\[
\|U\|_{H^s(\mathbb{R}^d)} \|U\|_{L^2(\mathbb{R}^d)} \geq C_{d,s} \|U\|_{L^q(\mathbb{R}^d)}, \quad q = 2 + \frac{4s}{d}, \quad \theta = \frac{d}{d + 2s}
\]
which follows from Sobolev’s embedding theorem.

For \( N \geq 1 \), we can use the spectral decomposition (3.8) and write
\[
E_Q[\Psi_N] = \text{Tr}((-\Delta)^s \gamma \Psi_N) = \sum_{n \geq 1} \|v_n\|_{H^s(Q)}^2.
\]
with \( v_n = \lambda^{1/2} u_n \). Using Hölder’s inequality (for sums), the one-body inequality \((3.10)\) and the triangle inequality we get

\[
\left( \int_Q \rho \Psi(x) \, dx \right)^{\frac{2s}{d}} \left( E_Q[\Psi_N] + \int_Q \rho \Psi(x) \, dx \right)^{\frac{2s-d}{d}} 
\geq \sum_{n \geq 1} \| v_n \|_{L^4(Q)}^4 \geq \| \rho \Psi \|_{L^{1+2s/d}(Q)}^2. 
\]

This is equivalent to \((3.9)\).

The third key ingredient to prove the Lieb–Thirring inequality \((3.3)\) is a covering lemma, which allows to combine the local exclusion and local uncertainty principles in an efficient way. The following is a simplified version of \([52, \text{Lemma 12}]\).

**Lemma 3.5:** (Covering lemma) Let \( 0 \leq f \in L^1(\mathbb{R}^d) \) be a function with compact support. Take \( 0 \leq \Lambda < \int_{\mathbb{R}^d} f(x) \, dx \). Then we can cover \( \mathbb{R}^d \) by a collection of disjoint cubes \( \{Q\} \) such that

\[
\int_Q f(x) \, dx \leq \Lambda, \quad \forall Q \tag{3.11}
\]

and there exists \( C_{\alpha,q} > 0 \) for every \( \alpha > 0 \) and \( 0 < q \leq (1 - \varepsilon)A2^{-d} \) such that

\[
\sum_{Q} \frac{1}{|Q|^\alpha} \left( \int_Q f(x) \, dx - q \right) - \varepsilon(1 - 2^{-\alpha d})A^{-d} \int_Q f(x) \, dx \geq 0. \tag{3.12}
\]

**Proof:** First, we cover \( \text{supp } f \) by a big cube \( Q_0 \). Then we divide \( Q_0 \) into \( 2^d \) disjoint sub-cubes of half-length side. For every sub-cube \( Q \),

- If \( \int_Q f < \Lambda \), then we stop dividing \( Q \).
- If \( \int_Q f \geq \Lambda \), then we continue dividing \( Q \) into \( 2^d \) disjoint sub-cubes and iterate.

This procedure stops after finitely many steps (since \( f \) is integrable) and we obtain a division of \( Q_0 \) into finitely any cubes \( Q \)’s. We can distribute all these cubes into disjoint groups \( \{F\} \) such that in each group \( F \):

\[
\sum_{Q \in F} \frac{1}{|Q|^\alpha} \left( \int_Q f(x) \, dx - q \right) - \varepsilon(1 - 2^{-\alpha d})A^{-d} \int_Q f(x) \, dx \geq 0.
\]
• There exists a smallest cube in \( F \) such that \( \int_Q f \geq 2^{-d}\Lambda \).

• There are at most \( 2^d \) cubes of every volume level.

Now we consider each group \( F \). By the first property of \( F \), we can find a smallest cube \( Q_m \in F \) with \( |Q_m| = m \) and

\[
\sum_{Q \in F} \frac{1}{|Q|^\alpha} \left[ \int_Q f(x) \, dx - q \right]_+ \geq \frac{1}{|Q_m|^\alpha} \left[ \int_{Q_m} f(x) \, dx - q \right]_+ \geq \frac{1}{m^{\alpha}} (2^{-d}\Lambda - q) \geq \frac{\varepsilon}{2^d m^\alpha}.
\]

On the other hand, by the second property of \( F \), there are at most \( 2^d \) cubes in \( F \) of volume \( 2^{kd}m \) for each \( k = 0, 1, \ldots \). Moreover, \( \int_Q f < \Lambda \) for every cube \( Q \). Hence,

\[
\sum_{Q \in F} \frac{1}{|Q|^\alpha} \int_Q f(x) \, dx \leq \sum_{k \geq 0} \frac{2^d}{(2^{kd}m)^\alpha} \Lambda = \frac{2^d}{1 - 2^{-\alpha d}} \Lambda.
\]

Thus

\[
\sum_{Q \in F} \frac{1}{|Q|^\alpha} \left( \left[ \int_Q f(x) \, dx - q \right]_+ - \varepsilon (1 - 2^{-\alpha d}) 4^{-d} \int_Q f(x) \, dx \right) \geq 0.
\]

Summing over all groups \( F \)'s we get the desired inequality.

Now we are ready to give an alternative proof of the Lieb–Thirring inequality (3.3).

**Proof:** Let \( q \) be as in Lemma 3.3 and let \( \Lambda = 2^{d+1}q \). If \( N \leq \Lambda \), then the desired bound follows immediately from (3.4) by taking \( Q \to \mathbb{R}^d \). If \( N > \Lambda \), then by a standard density argument we can assume that \( \Psi_N \in \mathcal{C}_c^\infty(\mathbb{R}^d) \). Then we apply Lemma 3.5 with \( f = \rho_{\Psi_N}, \alpha = 2s/d \) and obtain a collection of disjoint cubes \( \{Q\} \) covering \( \rho_{\Psi_N} \).

From the local uncertainty (3.9) and (3.11) we have

\[
\mathcal{E}_{\mathbb{R}^d}[\Psi_N] = \sum_Q \mathcal{E}_Q[\Psi_N] \geq C_{d,s} \frac{1}{\Lambda^{2s/d}} \int_{\mathbb{R}^d} \rho_{\Psi_N}^{1+2s/d} - \sum_Q \frac{1}{|Q|^{2s/d}} \int_Q \rho_{\Psi_N}.
\]

On the other hand, by the local exclusion (3.7), for all \( L > 0 \) we have

\[
L \mathcal{E}_{\mathbb{R}^d}[\Psi_N] = L \sum_Q \mathcal{E}_Q[\Psi_N] \geq \sum_Q \frac{LC_{d,s}}{|Q|^{2s/d}} \left[ \int_Q \rho_{\Psi_N} - q \right]_+.
\]
If we choose \( L = L_{d,s} \) such that
\[
\frac{1}{L_{d,s}C_{d,s}} = \frac{1}{2}(1 - 2^{-\alpha_d})4^{-d},
\]
then (3.11) gives us
\[
\sum_{Q} \frac{1}{|Q|^{2s/d}} \left( L_{d,s}C_{d,s} \left[ \int_Q \rho \Psi_N - q \right] - \int_Q \rho \Psi_N \right) \geq 0.
\]
Thus we conclude that
\[
(1 + L_{d,s})\mathcal{E}_{\mathbb{R}^d}[\Psi_N] \geq \frac{C_{d,s}}{\Lambda^{2s/d}} \int_{\mathbb{R}^d} \rho \Psi_N^{1+2s/d}.
\]
This completes the proof of (3.3).

3.2. Kinetic inequality with semiclassical constant and error term

A natural impression from the proof in Section 3.2 is that it involves several non-optimal estimates and potentially gives a rather bad control on the constant. However, it turns out that this proof can be modified to achieve the semiclassical constant, up to an error which is normally small in applications. Here we focus only on the non-relativistic case \( s = 1 \). The following result is taken from [60].

**Theorem 3.6:** Let \( d \geq 1 \). For any \( N \geq 1 \), let \( \{u_n\}_{n=1}^N \subset H^1(\mathbb{R}^d) \) be orthonormal functions in \( L^2(\mathbb{R}^d) \) and define the density \( \rho(x) = \sum_{n=1}^N |u_n(x)|^2 \). Then for all \( \varepsilon > 0 \)
\[
\sum_{n=1}^N \int_{\mathbb{R}^d} \left| \nabla u_n(x) \right|^2 \, dx \geq (1 - \varepsilon)\mathcal{E}_{\mathbb{R}^d}[\Psi_N] \geq \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2}{d}} \, dx - \frac{C_d}{\varepsilon^{3+4/d}} \int_{\mathbb{R}^d} \left| \nabla \sqrt{\rho(x)} \right|^2 \, dx.
\] (3.13)

Note that our bound (3.13) implies the Lieb–Thirring inequality (1.4) with a non-sharp constant, thanks to the Hoffmann–Ostenhof inequality [28] (or the diamagnetic inequality)
\[
\sum_{n=1}^N \int_{\mathbb{R}^d} \left| \nabla u_n(x) \right|^2 \, dx \geq \int_{\mathbb{R}^d} \left| \nabla \sqrt{\rho(x)} \right|^2 \, dx.
\] (3.14)

Moreover, in many applications, the gradient term is much smaller than the kinetic energy. For example, at the ground state of \( N \) ideal (i.e. non-interacting) fermions in a fixed volume, the gradient term is proportional
to $N$ while the kinetic energy grows as $N^{1+2/d}$. The gradient terms have also appeared in recent improvements [6, 38] of the Lieb–Oxford estimate on Coulomb exchange energy [47].

Recall that when $d \leq 2$, we know that the optimal value of $K_d$ is strictly smaller than $K_d^{cl}$. On the other hand, our bound (3.13) holds for all $d \geq 1$, so the additional error term is unavoidable.

**Sketch of proof:**

**Step 1.** For every cube $Q \subset \mathbb{R}^d$ and every $\mu > 0$ we can write

$$\sum_{n \geq 1} \int_Q |\nabla u_n|^2 = \text{Tr}_{L^2(Q)}((-\Delta Q - \mu)\gamma_Q) + \mu \int_Q \rho \geq \text{Tr}_{L^2(Q)}(-\Delta Q - \mu) - + \mu \int_Q \rho$$

with $-\Delta Q$ the Neumann Laplacian on $L^2(Q)$ and $\gamma_Q = \sum_{n \geq 1} |\mathbb{I}_Q u_n\rangle\langle \mathbb{I}_Q u_n|$ satisfying $0 \leq \gamma_Q \leq 1$ on $L^2(\mathbb{R}^d)$. Using the explicit eigenvalues of the Neumann Laplacian, we have

$$\text{Tr}_{L^2(Q)}(-\Delta Q - \mu) = |Q|^{-2/d} \sum_{p \in \{0, 1, \ldots\}^d} [\pi^2 |p|^2 - \mu]_+ \geq |Q|^{-2/d} \left( -L_{1,d}^{cl} \mu^{1+2/d} - C_d(\mu^{1+1/d} + 1) \right).$$

Optimizing over $\mu$ we obtain

$$\sum_n \int_Q |\nabla u_n(x)|^2 \, dx \geq K_d^{cl} |Q|^{-2/d} \left[ \left( \int_Q \rho \right)^{1+2/d} - C \left( \int_Q \rho \right)^{1+1/d} \right]. \quad (3.15)$$

**Step 2.** By a density argument, we can assume $\{u_n\} \subset C^\infty_c(\mathbb{R}^d)$ and hence $\rho$ has compact support. As in the covering lemma, we can cover $\text{supp} \rho$ by disjoint cubes $\{Q\}$ such that

$$\int_Q \rho \leq \Lambda, \quad \forall Q,$$

and

$$\sum_Q |Q|^{-2/d} \left[ C \Lambda^{-1/d} \left( \int_Q \rho \right)^{1+2/d} - \left( \int_Q \rho \right)^{1+1/d} \right] \geq 0. \quad (3.17)$$
Step 3. By Poincaré’s inequality we can show that for every $\varepsilon > 0$,

\[
\frac{1}{|Q|^{2/d}} \left( \int_Q \rho \right)^{1+2/d} \geq \frac{1}{(1 + \varepsilon)^{(1+4)/d}} \int_Q \rho^{1+2/d} - \frac{C}{\varepsilon^{(1+4)/d}} \left( \int_Q |\nabla \sqrt{\rho}|^2 \right) \left( \int_Q \rho \right)^{2/d}.
\]

Combining the latter bound with (3.16), (3.17) and (3.15), we obtain

\[
\sum_Q \int_Q |\nabla u_n|^2 \geq K_d (1 - C\Lambda^{-1/d}) \sum_Q |Q|^{-2/d} \left( \int_Q \rho \right)^{1+2/d} \geq K_d (1 - C\Lambda^{-1/d}) \left( \int_{\mathbb{R}^d} \rho \right)^{1+2/d} - \frac{C\Lambda^2/d}{\varepsilon^{(1+4)/d}} \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2.
\]

Taking $\varepsilon = \Lambda^{-1/d}$ and optimizing over $\Lambda$ we get the desired result. $\square$

Theorem 3.6 can be seen as a first step towards the local density approximation for many-body quantum systems. More precisely, for any $N \geq 1$ and $\rho \geq 0$ with $\int_{\mathbb{R}^d} \rho = N$, we can define the Levy–Lieb energy functional [37, 43] for the kinetic operator

\[
K(\rho) = \inf_{\rho \in L^1_{\text{ferm}}(\mathbb{R}^d N)} \left\langle \Psi_N, \sum_{i=1}^N (-\Delta x_i) \Psi_N \right\rangle_{L^2(\mathbb{R}^d N)}.
\]

Here the infimum is taken over all (normalized) fermionic wave functions in $L^2(\mathbb{R}^d N)$ whose one-body density is exactly equal to $\rho$. Then (3.13) is equivalent to the lower bound

\[
K(\rho) \geq (1 - \varepsilon)K_d (1 - C\Lambda^{-1/d}) \int_{\mathbb{R}^d} \rho(x)^{1+2/d} dx - \frac{C_2}{\varepsilon^{(1+4)/d}} \int_{\mathbb{R}^d} |\nabla \sqrt{\rho(x)}|^2 dx, \quad \forall \varepsilon > 0.
\]

Of course, by the Lieb–Thirring conjecture one expects that both the gradient term and the $\varepsilon$ dependence can be removed when $d \geq 3$.

On the other hand, it is conjectured [43, 59] that $K(\rho)$ satisfies the upper bound

\[
K(\rho) \leq K_d (1 - C\Lambda^{-1/d}) \int_{\mathbb{R}^d} \rho(x)^{1+2/d} dx + \int_{\mathbb{R}^d} |\nabla \sqrt{\rho(x)}|^2 dx.
\]

The appearance of the gradient term on the right side of (3.19) is reasonable since the kinetic energy cannot be controlled by an integral of $\rho$ alone due...
Direct methods to Lieb–Thirring kinetic inequalities

Recently, Lewin, Lieb and Seiringer proved in [39] that, for a grand-canonical analogue \( \tilde{K}(\rho) \) of \( K(\rho) \),

\[
\tilde{K}(\rho) \leq (1 + \varepsilon) K_{cl}^{cl} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{d}{2}} \, dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^d} \left| \nabla \sqrt{\rho(x)} \right|^2 \, dx, \quad \forall \varepsilon > 0. \tag{3.20}
\]

A result weaker than (3.20) was used in [26] in the context of proving Gamma-convergence of the Levy–Lieb model to Thomas–Fermi theory. Removing \( \varepsilon \) in both (3.18) and (3.20) is interesting and difficult. Nevertheless, in the current form, they are already useful to justify the local density approximation in certain regimes; see [39, 40] for further details.

3.3. Kinetic inequality for functions vanishing on diagonal set

Recall that Pauli’s exclusion principle (3.1) implies that the wave function \( \Psi_N \) vanishes on the diagonal set of \((\mathbb{R}^d)^N\), namely

\[
\Psi(x_1, \ldots, x_N) = 0 \quad \text{if } x_i = x_j \text{ for some } i \neq j. \tag{3.21}
\]

Thus a natural question is whether the Lieb–Thirring inequality (3.3) remains valid if (3.1) is replaced by the weaker condition (3.21). This question is nontrivial since (3.21) is not sufficient to ensure the operator inequality (3.2). The following answer is taken from [36].

**Theorem 3.7:** (Lieb–Thirring inequality for wave functions vanishing on diagonals) Let \( d \geq 1 \) and \( s > 0 \). Let \( N \geq 1 \) and let \( \Psi_N \in C_c^\infty(\mathbb{R}^{dN}) \) be a normalized wave function satisfying (3.21). Then we have the Lieb–Thirring inequality

\[
\left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{x_i})^s \Psi_N \right\rangle \geq C_{d,s} \int_{\mathbb{R}^d} \rho_{\Psi_N}(x)^{1+2s/d} \, dx \tag{3.22}
\]

with a constant \( C_{d,s} > 0 \) independent of \( N \) and \( \Psi_N \) if and only if \( 2s > d \).

Actually, the condition \( 2s > d \) is related to the Sobolev embedding \( H^s(\mathbb{R}^d) \subset C(\mathbb{R}^d) \). Heuristically, this is the minimum condition for (3.21) to be “nontrivial,” otherwise (3.22) must fail. We refer to [36] for a detailed discussion on this negative direction, and below let us focus only on the derivation of (3.22) when \( 2s > d \).

From the general strategy in Section 3.1 it suffices to derive a local exclusion bound similar to (3.7).
Sketch of the local exclusion bound:

Step 1. First, note that (3.7) can be reduced to a simpler estimate: for scale-covariant systems, such a local exclusion simply boils down to the strict positivity of the local energy. This idea is inspired by Lundholm and Seiringer [58] and seems very helpful for future applications. The following abstract formulation is taken from [36, Lemma 4.1].

**Lemma 3.8:** (Covariant energy bound) Assume that to any \( n \in \mathbb{N}_0 \) and any cube \( Q \subset \mathbb{R}^d \) there is associated a non-negative number (‘energy’) \( E_n(Q) \) satisfying the following properties, for some constant \( s > 0 \):

(i) (scale-covariance) \( E_n(\lambda Q) = \lambda^{-2s} E_n(Q) \) for all \( \lambda > 0 \);

(ii) (translation-invariance) \( E_n(Q + x) = E_n(Q) \) for all \( x \in \mathbb{R}^d \);

(iii) (superadditivity) For any collection of disjoint cubes \( \{Q_j\}_{j=1}^J \) such that their union is a cube,

\[
E_n\left( \bigcup_{j=1}^J Q_j \right) \geq \min_{\{n_j\} \in \mathbb{N}_0^J \text{ s.t. } \sum_j n_j = n} \sum_{j=1}^J E_{n_j}(Q_j);
\]

(iv) (a priori positivity) There exists \( q \geq 0 \) such that \( E_n(Q) > 0 \) for all \( n \geq q \).

Then there exists a constant \( C > 0 \) independent of \( n \) and \( Q \) such that

\[
E_n(Q) \geq C |Q|^{-2s/d} n^{1+2s/d}, \quad \forall n \geq q. \tag{3.23}
\]

Step 2. The above abstract result applies to the local energy

\[
E_N(Q) := \inf \left\{ \int_{Q^N} \Psi_N, \sum_{j=1}^N (-\Delta x_j)\Psi_N \right\}_{L^2(\Omega^N)} \tag{3.24}
\]

where the infimum is taken over all wave functions \( \Psi_N \in C_c^\infty(\mathbb{R}^d) \) satisfying (3.21) and normalized \( \| \Psi_N \|_{L^2(Q^N)} = 1 \). Note that the right side of (3.24) is in different from \( \mathcal{E}_Q[\Psi_N] \) in (3.5) because only the “completely localized energy” in (3.24) satisfies the superadditivity in Lemma 3.8 (iii). Moreover, the conditions in (i) and (ii) obviously hold.

The key assumption (3.21) is used to derive the strict positivity in Lemma 3.8 (iv). This is nontrivial. The central facts used in the proof is that the kernel of the associated Neumann Laplacian must be a polynomial (of many variables), and that if a polynomial vanishes on too many diagonals then it must be zero. We refer to [36, Theorem 5.1] for details.
Step 3. Finally, using (3.23) and a many-body localization technique, we can deduce the desired local exclusion bound

$$E_Q[\Psi_N] \geq C|Q|^{-2s/d} \left[ \int_Q \rho_{\Psi_N} - q \right]_+$$

with the same constants $C, q$ in (3.24) and with $E_Q[\Psi_N]$ defined in (3.5). See [36, Lemma 4.4] for details.

3.4. Lieb–Thirring inequality for interacting systems

In order to obtain an exclusion bound, instead of putting a condition on wave functions, one can also add a repulsive term to the Hamiltonian. Given the kinetic operator $(-\Delta)^s$, it is natural to consider the interaction potential $w(x) = |x|^{-2s}$ which has the same scaling property. This leads to the following

**Theorem 3.9:** (Lieb–Thirring inequality for interacting systems)

Let $d \geq 1$, $s > 0$ and $\lambda > 0$. For any $\Psi_N \in H^s(\mathbb{R}^{dN})$ which is normalized in $L^2(\mathbb{R}^{dN})$, we have

$$\langle \Psi_N, \left( \sum_{i=1}^{N} (-\Delta_{x_i})^s + \sum_{1 \leq i < j \leq N} \frac{\lambda}{|x_i - x_j|^{2s}} \right) \Psi_N \rangle \geq C_{LT}(d, s, \lambda) \int_{\mathbb{R}^d} \rho_{\Psi_N}(x)^{1+\frac{2s}{d}} \, dx. \quad (3.25)$$

The constant $C_{LT}(d, s, \lambda) > 0$ is independent of $N$ and $\Psi_N$.

This result was first proved by Lundholm and Solovej for $s = 1$ and $d = 1$ in [53]. The extension to $s = 1, d > 1$ was done by Lundholm, Portmann, and Solovej in [53]. The general power $s > 0$ was treated in [52]. The proof in [52] is based on the strategy in Section 3.1 but now the local exclusion bound is derived from the interaction.

**Lemma 3.10:** (Local exclusion by interaction) For all $d \geq 1$, $s > 0$, for every normalized function $\Psi \in L^2(\mathbb{R}^{dN})$ and for an arbitrary collection of disjoint cubes $Q$’s in $\mathbb{R}^d$, one has

$$\langle \Psi, \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \Psi \rangle \geq \sum_Q \frac{1}{2d^s|Q|^{2s/d}} \left[ \left( \int_Q \rho_{\Psi} \right)^2 - \int_Q \rho_{\Psi} \right]_+.$$
Proof: This result follows from the operator estimate
\[
\sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \geq \sum_Q \sum_{1 \leq i < j \leq N} \frac{1}{d^s|Q|^{2s/d}} \int_Q \frac{1}{|x_i - x_j|^{2s}} \left[ \left( \sum_{i=1}^N 1_{Q}(x_i)(x_i) \right)^2 - \sum_{i=1}^N 1_{Q}(x_i) \right]
\]
and the Cauchy–Schwarz inequality.

As explained in [52], if \(2s < d\), then the Lieb–Thirring inequality (3.25) can be also derived from the one-body interpolation inequality
\[
\langle u, (-\Delta)^su \rangle_{L^2(\mathbb{R}^d)}^{1 - \frac{2s}{d}} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^{2s}} \, dx \, dy \right)^{\frac{2s}{d}} \geq C \int_{\mathbb{R}^d} |u(x)|^{2(1 + \frac{2s}{d})} \, dx.
\]
(3.26)
The inequality (3.26) was first proved in [4] for \(s = 1/2, d = 3\), and extended to all \(0 < s < d/2\) in [5] (see [3] for further results in this direction). The existence of optimizers for (3.26) is an interesting open problem; see [5] for related discussions.

It was conjectured in [52] that in the strong coupling limit \(\lambda \to \infty\), the optimal constant \(C_{LT}(s, d, \lambda)\) converges to the Gagliardo–Nirenberg constant
\[
C_{GN}(d, s) := \inf_{u \in H^s(\mathbb{R}^d)} \frac{\langle u, (-\Delta)^su \rangle}{\|u\|_{L^2}^{2(1 + \frac{2s}{d})}}.
\]
This was proved recently in [33].

Theorem 3.11: (Lieb–Thirring constant in the strong-coupling limit)
For any \(d \geq 1\) and \(s > 0\), we have
\[
\lim_{\lambda \to \infty} C_{LT}(s, d, \lambda) = C_{GN}(d, s).
\]
The heuristic idea behind Theorem 3.11 is that in the strong-coupling limit, each particle is forced to stay away from the others and the many-body interacting problem reduces to a one-body non-interacting system. However, proving this is nontrivial since we have to prove estimates uniformly in the number of particles.

The proof of Theorem 3.11 in [33] is based on a new construction of covering sub-cubes. In Lemma 3.5, the division into sub-cubes follows by a
standard “stopping time argument:” any cube $Q$ with the mass $\int_Q \rho \Psi$ bigger than a given quantity will be divided into $2^d$ sub-cubes. Consequently, the masses in final sub-cubes may differ up to a factor $2^d$, leading to a similar factor loss in the Lieb–Thirring constant. In [33], the stopping time argument is applied to “clusters of cubes” rather than to individual cubes. At the end, each cluster has essentially at most one particle, allowing us to recover the constant $C_{GN}(d, s)$ by using a refined version of the local uncertainty principle (3.9). The localization error is compensated by the interaction energy. This localization argument seems very flexible and may be useful in other contexts.

### 3.5. Further results

The method represented in Section 3.1 was originally invented to derive Lieb–Thirring inequalities for anyons (particles satisfying only some fractional statistics between bosons and fermions). See [33, 54–58] for various results in this direction. A similar method was used to prove a Lieb–Thirring inequality for fermionic particles with point interactions in [25].

This method is also useful to recover the Hardy–Lieb–Thirring inequality in [14, 16, 22], where the kinetic operator is replace by $(-\Delta)^d - C_{d,s}|x|^{-2s}$ with $C_{d,s}$ is the optimal constant in Hardy’s inequality [27]. This requires $2s < d$. The results in Theorems 3.9, 3.11 were also extended to the case of Hardy operator; see [33, 52]. The key additional ingredient is the refined Hardy inequality: for all $s > t > 0$ and $\ell > 0$,

$$(-\Delta)^s - \frac{C_{s,d}}{|x|^2} \geq \ell^{s-t}(-\Delta)^t - C_{d,s,t}\ell^s \text{ on } L^2(\mathbb{R}^d).$$

This bound was first proved for $s = 1/2$, $d = 3$ by Solovej, Sørensen, and Spitzer [67, Lemma 11] and then generalized to the full range $0 < s < d/2$ by Frank [16, Theorem 1.2].

Recently, the fermionic Hardy–Lieb–Thirring inequality has been extended to include fractional Pauli operators in [8]. It is unclear whether the approach in this section could be adapted to study this case.

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Direct methods to Lieb–Thirring kinetic inequalities

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