An incidence Hopf Algebra of Convex Geometries

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A lattice \( L \) is “meet-distributive” if for each element of \( L \), the meets of the elements directly below it form a Boolean lattice. These objects are in bijection with “convex geometries”, which are an abstract model of convexity. Do they give rise to an incidence Hopf algebra of convex geometries?

1 Preliminaries

We will define some basic concepts in order to understand the question posed.

Definition 1.1. A finite lattice \( L \) is called meet distributive if for every \( x \in L - \{0\} \), the interval \([i(x), x]\) is a boolean lattice, where \( i(x) \) is the meet of all elements \( y \in L \) covered by \( x \) (i.e. \( y < x \) and there is no \( z \) such that \( y < x < z \)).

Definition 1.2. A closure operator on a set \( S \) is a map of subsets of \( S \), \( \text{cl} : P(S) \to P(S) \) such that for any \( X, Y \subseteq S \)

i) \( X \subseteq \text{cl}(X) \)

ii) \( X \subseteq Y \Rightarrow \text{cl}(X) \subseteq \text{cl}(Y) \)

iii) \( \text{cl}(\text{cl}(X)) = \text{cl}(X) \)

Definition 1.3. A closure operator \( \text{cl} \) on a set \( S \) has the antiexchange property if \( x \neq y \), \( y \in \text{cl}(A \cup x) \), \( y \notin \text{cl}(A) \) then \( x \notin \text{cl}(A \cup y) \).

Definition 1.4. A convex geometry is a pair \((Z, \text{cl})\) where \( Z \) is a finite set and \( \text{cl} \) is a closure operator on \( Z \) with the antiexchange property.

Definition 1.5. On a convex geometry \( Z, X \subseteq Z \) is called closed if \( \text{cl}(X) = X \). Closed sets of a convex geometry ordered by containment form a lattice \( L(Z, \text{cl}) \).
We have an equivalent definition of convex geometry

**Definition 1.6.** Let $Z$ be a finite nonempty set and $E \subseteq 2^Z$ a family of subsets of $Z$, then $E$ is called a convex geometry if

1. $\emptyset \in E, Z \in E$
2. $X, Y \in E \Rightarrow X \cap Y \in E$
3. $X \in E \\{Z\} \Rightarrow \exists z \in Z \setminus X : X \cup \{z\} \in E$

The elements of $E$ are called closed subsets of $Z$ and they induce a closure operator on $Z$ where

$$\text{cl}(A) = \bigcap_{X \in E, A \subseteq X} X$$

This closure operator has the antiexchange property. Conversely, given $(Z, \text{cl})$ with the antiexchange property, the family of closed subsets form a family $E$ with the properties defined previously and the closure operator defined as before coincides with cl.

The following theorem is due to Edelman [1].

**Theorem 1.7.** A finite lattice is meet distributive if and only if it is isomorphic to the lattice of closed sets of a convex geometry.

In order to construct an incidence Hopf Algebra of meet distributive lattices we must show that the family of such lattices is a hereditary family.

**Proposition 1.8.** The family of finite meet distributive lattices is closed under taking subintervals.

**Proof.** Let $L$ be a meet distributive lattice and $I = [a, b] \subseteq L$. $I$ is clearly a lattice. Let $x \in I - \{a\}$. There are fewer or an equal number of elements covered by $x$ in $I$ than in $L$ hence their meet $i_I(x) \geq i(x)$, the meet of all elements covered by $x$ in $L$, so $[i_I(x), x] \subseteq [i(x), x]$ which is a boolean lattice and subintervals of boolean lattices are boolean lattices.

**Proposition 1.9.** The family of meet distributive lattices is closed under direct product.
Proof. Let $L_1, L_2$ be meet distributive lattices. Let $(x_1, x_2) \in L_1 \times L_2 - \{\hat{0}_1, \hat{0}_2\}$. Clearly $i(x_1, x_2) = (i(x_1), i(x_2))$ and $[(i(x_1), i(x_2)), (x_1, x_2)] = [i(x_1), x_1] \times [i(x_2), x_2]$ which are boolean lattices and the product of boolean lattices is boolean.

\[\square\]

Corollary 1.10. The family of meet distributive lattices is a hereditary family. We denote such family as $D$. Under isomorphism it forms the incidence Hopf Algebra of meet distributive lattices, which we denote as $M$.

We call two convex geometries $(Z_1, \text{cl}_1), (Z_2, \text{cl}_2)$ isomorphic if they have isomorphic lattices $L(E_1, \text{cl}_1), L(E_2, \text{cl}_2)$. Then Edelman’s theorem gives rise to a bijection between isomorphism classes of meet-distributive lattices and isomorphism classes of convex geometries.

2 Some examples

Example: (Convex Shelling) Let $P$ be a finite set of points in $\mathbb{R}^n$. Define

$$E := \{X \subseteq P : \text{conv}(X) \cap P = X\}$$

Where $\text{conv}(X)$ denotes the convex hull of $X$. Then $(P, E)$ is a convex geometry. For example let $P_1 = \{a, b, c, d, e\} \subseteq \mathbb{R}^2$ where $a = (0, -1), b = (-1, 0), c = (0, 0), d = (1, 0), e = (0, 1)$ (Figure 1). Then the closed sets of $P_1$ in the convex shelling, ordered by size, are $[ab, cd], [abcd], [abc], [ac], [cd], [ce], [de], [a], [b], [c], [d]$, and the corresponding meet distributive lattice, with the elements in this
order, from left to right, and ranked by size is given in figure (2).

Figure 2: The lattice of the convex shelling $P_1$

**Example (Poset shelling):** Let $P = (E, \leq)$ be a finite poset. Define

$$E = \{ X \subseteq E : e \in X, f \leq e \Rightarrow f \in X \}$$

This is the set of downsets of $P$. This is a convex geometry. For example take the power set of $\{1, 2\}$ ordered by inclusion. Then the Poset shelling convex geometry is described by the meet distributive lattice in figure (3).

Figure 3: Lattice of down sets of $\{1, 2\}$

### 3 Convex geometries as a Hopf Algebra

Now we recall the basic structure of an incidence Hopf algebra. Let $K$ be a field. $K(D)$ is the free $K$-vector space over $D$, then the maps

$$\Delta(L) = \sum_{x \in L} [0_L, x] \otimes [x, 1_L]$$
\[ \epsilon(L) = \begin{cases} 1 & \text{if } |L| = 1 \\ 0 & \text{else} \end{cases} \]

\[ m(L_1, L_2) = L_1 \times L_2 \]

\[ S(L) = \sum_{k \geq 0} \sum_{0 = x_0 < \cdots < x_k = 1} (-1)^k [x_0, x_1] \times \cdots \times [x_{k-1}, x_k] \]

turn \( K(\mathcal{D}) \) into a Hopf algebra. We define the following

**Definition 3.1.** Let \((Z, E)\) be a convex geometry over a finite set, and let \(A, B\) be closed subsets such that \(A \subseteq B\). The minor of \((Z, E)\) with respect to \(A, B\) is

\[ M[A, B] = \{X \setminus A : X \in L, A \subseteq X \subseteq B\} \]

**Remark 3.2.** A minor is always a convex geometry

**Lemma 3.3.** Under Edelman’s bijection, any minor \(M(A, B)\) corresponds to the lattice \([A, B] \subseteq L(Z, E)\), the interval from \(A\) to \(B\) in the lattice of \((Z, E)\).

**Proof.** The map \(\phi : M(A, B) \to [A, B], X \mapsto X \cup A\) is clearly a bijection. If \(X, Y \in M(A, B), X \subseteq Y\), then \(X \cup A \subseteq Y \cup A\) so \(\phi(X) \leq \phi(Y)\).

**Definition 3.4.** Denote the free \(K\)-vector space over the set of isomorphism classes of convex geometries as \(C\).

**Corollary 3.5.** The coproduct \(\Delta : C \to C \otimes C\),

\[ (Z, E) \mapsto \sum_{X \in E} M(\emptyset, X) \otimes M(X, Z) \]

together with the count \(\epsilon(Z, E) = 1\) if \(Z = \emptyset\), \(\epsilon(Z, E) = 0\) otherwise, turn \(C\) into a coalgebra.

**Definition 3.6.** Let \((Z_1, E_1)\) and \((Z_2, E_2)\) be two convex geometries. Suppose \(Z_1\) and \(Z_2\) are disjoint. Consider the family of subsets of \(Z_1 \cup Z_2\), the disjoint union of \(Z_1, Z_2\) defined by

\[ E_{Z_1Z_2} = \{X_1 \cup X_2 : X_1 \in E_1, X_2 \in E_2\} \]
Lemma 3.7. \((Z_1 \cup Z_2, E_{Z_1Z_2})\) is a convex geometry, called the product convex geometry, for disjoint \(Z_1, Z_2\).

Proof.  

i) \(\emptyset \in E_1, \emptyset \in E_2 \Rightarrow \emptyset = \emptyset \cup \emptyset \in E_{Z_1Z_2}\). \(Z_1 \in E_1, Z_2 \in E_2 \Rightarrow Z_1 \cup Z_2 \in E_{Z_1Z_2}\).

ii) Suppose \(X_1 \cup X_2, Y_1 \cup Y_2 \in E_{Z_1Z_2}\) then \(X_1, Y_1 \in E_1\) and \(X_1 \cap Y_1 \in E_1\) as \((Z_1, E_1)\) is a convex geometry. Analogously \(X_2 \cap Y_2 \in E_2\) then \((X_1 \cup X_2) \cap (Y_1 \cup Y_2) = (X_1 \cap Y_1) \cup (X_2 \cap Y_2) \in E_{Z_1Z_2}\).

iii) Suppose \(X_1 \cup X_2 \in E_{Z_1Z_2}\)\(\{Z_1 \cup Z_2\}\) then either \(X_1 \in E_1\)\(\{Z_1\}\) or \(X_2 \in E_1\)\(\{Z_2\}\). Suppose \(X_1 \in E_1\)\(\{Z_1\}\), then there exists some \(z \in Z_1\) such that \(X_1 \cup \{z\} \in E_1\) so that \(X_1 \cup X_2 \cup \{z\} = (X_1 \cup \{z\}) \cup X_2 \in E_{Z_1Z_2}\).

The other case follows by symmetry.

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Lemma 3.8. For disjoint convex geometries \((Z_1, E_1), (Z_2, E_2)\) the lattice \(L(Z_1 \cup Z_2)\) is isomorphic to \(L(Z_1) \times L(Z_2)\).

Proof. Define \(\phi : L(Z_1) \times L(Z_2) \rightarrow L(Z_1 \cup Z_2)\) as
\[
\phi(X, Y) = X \cup Y
\]

This map is clearly a bijection. Now if \((X_1, Y_1) \leq (X_2, Y_2)\) in \(L(Z_1) \times L(Z_2)\), then \(X_1 \subseteq X_2\) and \(Y_1 \subseteq Y_2\). It follows that
\[
\phi(X_1, Y_1) = X_1 \cup Y_1 \subseteq X_2 \cup Y_2 = \phi(X_2, Y_2)
\]

\(\)\n
Corollary 3.9. The multiplication map \(m : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}\) given by
\[
m(Z_1 \otimes Z_2) = Z_1 \cup Z_2
\]

Turns \(\mathcal{C}\) into a bialgebra.

Remark 3.10. Although dealing with isomorphism classes, this multiplication map is well defined by the previous lemma and the fact that one can choose representatives with disjoint \(Z_1\) and \(Z_2\).
4 Some sub-Hopf Algebras and further considerations

It is a matter of interest if certain subclasses of convex geometries form a sub-Hopf algebra of the general incidence Hopf algebra of convex geometries. We cite the following theorems

Theorem 4.1. Nakamura-Okamoto (2003) The convex geometry class of convex shellings of finite point sets is not closed under taking minors.

Theorem 4.2. Nakamura-Okamoto (2003) The class of digraph point search convex geometries is closed under taking minors.

Theorem 4.3. Nakamura (2003) The class of poset shellings convex geometries is closed under taking minors.

Theorem 4.4. Nakamura (2003) \((Z, E)\) is a poset shelling if and only if it contains no minor isomorphic to the following:

![Forbidden minor for poset shelling convex geometries](image)

Figure 4: Forbidden minor for poset shelling convex geometries \(F_1\)

Theorem 4.5. Nakamura-Okamoto \((Z, E)\) is a digraph point search convex geometry if and only if it contains no minor isomorphic to the following:

![Forbidden minor for digraph point search convex geometries](image)

Figure 5: Forbidden minor for digraph point search convex geometries \(F_2\)

Corollary 4.6. By theorem (4.1), the class of convex shellings does not yield a sub-Hopf algebra of \(\mathcal{C}\).

Now we prove that both the poset shellings and the digraph point search classes are closed under products.
**Theorem 4.7.** The Convex geometry class of poset shellings and the class of digraph point search are closed under products.

*Proof.* Let \((Z_1, E_1), (Z_2, E_2)\) be two convex geometries of the same class. Let \(L_1\) and \(L_2\) be their corresponding closed subsets lattice. By lemma (), the minors of such convex geometries correspond to subintervals in the corresponding lattice. By theorems (4.4), (4.5) we then know that both \(L_1\) and \(L_2\) contain no subinterval isomorphic to either \(F_1\), in the case of poset shellings, or \(F_2\), in the case of digraph point search. Now by lemma (3.8) we know the lattice of closed sets of \(Z_1 \cup Z_2\) corresponds to the product of the lattices \(L_1 \times L_2\).

Any subinterval \(I\) of \(L_1 \times L_2\) is isomorphic to a product \(I_1 \times I_2\) where \(I_1\) is a subinterval of \(L_1\) and \(I_2\) is a subinterval of \(L_2\). Also we have the equality \(|I| = |I_1||I_2|\). Now suppose \(L_1 \times L_2\) contains an interval \(I\) isomorphic to either \(F_1\) or \(F_2\) then we have a decomposition \(|F_i| = |I_1||I_2|\), but the size of \(F_i\) is prime in both cases so either \(|I_1| = 1\) or \(|I_2| = 1\). Without loss of generality \(|I_2| = 1\). As \(I_2\) is a one element lattice we have that \(F_i \simeq I_1 \times I_2 \simeq I_1\) which is a contradiction as \(L_1\) contains no subinterval isomorphic to \(F_i\).

\[\square\]

**Corollary 4.8.** The class of poset shellings and the class of digraph point search convex geometries form two sub-Hopf Algebras of the general incidence Hopf Algebra of convex geometries \(C\).

Now consider the set \(Z_n := [n]\) and the family of subsets \(E = \{[k] : 1 \leq k \leq n\} \cup \{\emptyset\}\). It is straightforward to check that \((Z, E)\) is a convex geometry, moreover, it forms a finite linear order and any finite linear order is isomorphic to some \(Z_n\). Therefore we also have

**Lemma 4.9.** The Hopf algebra of symmetric functions is a sub Hopf algebra of \(C\).

*Proof.* Consider the map

\[\phi(\text{finite linear order of size } n) = Z_{n-1}\]

This is extends to an injective Hopf algebra map from \(Sym\) to \(C\).

\[\square\]

The set of isomorphism classes of convex geometries is really large. We don’t expect a simplified formula for the antipode, however we may find simplified formulas for the antipode on certain families of convex geometries.
Example Consider the family of convex shellings of regular $n$-gons $\{P_n\}$ in two dimensional space. We call the convex hull of two different points the 2-gon and a single point the 1-gon. For the regular $n$-gon, we see that any subset of the vertices belongs to the convex shelling, so we can distinguish an individual vertex $v$ and separate the elements of the convex shelling as those containing $v$ and those not containing $v$. The elements of the convex shelling that do not contain $v$ is the convex shelling of a $(n - 1)$-gon.

We see at once that the convex shelling of the $n$-gon is the same as the product of the convex shelling of the $(n - 1)$-gon and the convex shelling of a vertex, $P_1$. Hence by induction we have that $P_n = (P_1)^n$. Then $S(P_n) = S(P_1)^n$.

Now $P_1$ corresponds to the lattice of a linear order of size 2, $L_1$. Using the formula of the antipode we have $S(P_1) = S(L_1) = -L_1 = -P_1$ then

$$S(P_n) = (-1)^n P_n$$

An analogous argument with some other considerations would yield a formula for the antipode of the convex shellings of the vertices of simplicial polytopes (every facet is a simplex) on higher dimension.
References

[1] Edelman, Paul H, (1980). Meet-distributive lattices and the anti-exchange closure, Algebra Universalis 10 (1): 290-299.

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[3] Masataka Nakamura, (2003). Excluded-minor characterizations of antimatroids arisen from posets and graph searches. Discrete Applied Mathematics 129 (23): 487-498