Asymptotic Level Density in Heterotic String Theory and Rotating Black Holes

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ABSTRACT

We calculate the density of states with given mass and spin in string theory and obtain asymptotic formulas. We also compute the tree-level gyromagnetic couplings for arbitrary physical states in the heterotic string theory. These results are then applied to study whether fundamental strings can consistently describe the microphysics of the black hole horizon in the case of a general classical solution characterized by mass, charge and angular momentum.
1. Introduction

Massive excitations in the string spectrum constitute an interesting and nevertheless scarcely explored sector in string theory. Although the massless states suffice to describe all hitherto observed physics, in processes involving planckian distances, such as e.g. the big bang singularity or certain aspects of black hole physics, the entire string spectrum is expected to play a role. Highly-excited states are responsible of the ultraviolet finiteness of string perturbation theory, and they lead to a number of interesting effects. In particular, the level density increases so rapidly with mass that the thermodynamical partition function of a free string gas cannot be defined above a certain temperature [1]. A significant part of the highly-excited states lies within their Schwarzchild radii, and therefore they must be black holes. States with sufficiently high-angular momentum have an average radius larger than their Schwarzchild radii and thus they are not expected to be black holes.

A statistical mechanical explanation of the Bekenstein-Hawking entropy of black holes is unknown. The black hole horizon behaves as a system obeying the four thermodynamical laws, but their entropy was never explained in terms of counting of quantum states. ’t Hooft has stressed that such an explanation may provide a clue on the fundamental nature of matter [3]. Ordinary quantum field theory certainly does not give the correct result for the entropy. It gives an ultraviolet divergent answer [4]. A recent proposal developed in ref. [2] and further studied in ref. [5] entails a microscopic description of the horizon in terms of configurations of fundamental strings in a Rindler geometry. This approach has provided an interesting explanation of the Bekenstein-Hawking entropy, which is the essence of the information paradoxes associated with Hawking radiation.

In this paper various properties of classical black holes will be compared to a thermal average of fundamental strings in the heterotic string theory. In sect 2 we will calculate the density level for physical states in the string spectrum with a given angular momentum. This will establish a precise relation between the string
energy, the angular momentum, and the ADM mass of the configuration.

An intriguing property of rotating black holes in Einstein-Maxwell theory is that they have gyromagnetic ratio equal to two, i.e. the same as the tree-level $g$-factor of the electron [6]. This property holds true for rotating black holes in the heterotic string theory [7], but in other theories of gravity black holes may have $g \neq 2$, being, in general, a function of the black hole conserved quantum numbers (see, e.g., ref. [8]). We will derive the gyromagnetic couplings for arbitrary physical states in the heterotic string theory (in the case of the open string theory, this calculation was done in ref. [9]). Finally we will study the correspondence with rotating black hole in heterotic string theory.

Related results, but in a different direction, are investigated in ref. [10].

2. Density level of strings with given angular momentum

In this section we will calculate an asymptotic formula for the number of states with a given mass and angular momentum. We will consider the case of bosonic open strings, and then we shall generalize our results to other cases. To this purpose we modify the world-sheet Hamiltonian by adding a term containing the angular momentum in the $z$ direction with a Lagrange multiplier, i.e.

$$H = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \lambda J ,$$

(2.1)

where

$$J = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1) .$$

The Hamiltonian can be diagonalized by writing

$$\alpha_n^1 = \sqrt{n/2} (a_n + b_n) , \quad \alpha_n^2 = -i \sqrt{n/2} (a_n - b_n) ,$$
\[ \alpha_{-n}^1 = \sqrt{n/2}(a_n^\dagger + b_n^\dagger), \quad \alpha_{-n}^2 = i\sqrt{n/2}(a_n^\dagger - b_n^\dagger), \quad n = 1, \ldots \infty. \] (2.2)

One has \([a_n, a_m^\dagger] = \delta_{nm}, [b_n, b_m^\dagger] = \delta_{nm}\). The Hamiltonian takes the form

\[ H = \sum_{n=1}^{\infty} \left( \sum_{i=3}^{D-2} \alpha_{-n}^i \alpha_n^i + (n - \lambda)a_n^\dagger a_n + (n + \lambda)b_n^\dagger b_n \right). \] (2.3)

Let us now calculate the partition function,

\[ Z = \text{tr} \left[ e^{-\beta H} \right]. \] (2.4)

We obtain

\[ Z = \prod_{n=1}^{\infty} \left[ (1 - w^n)^{-D+4} (1 - cw^n)^{-1} (1 - \frac{w^n}{c})^{-1} \right], \] (2.5)

where \(w \equiv e^{-\beta}\) and \(c \equiv e^{\beta \lambda}\).

Note that \(Z\) has poles at \(\lambda = \pm 1, \pm 2, \ldots\), which can be traced back to the fact that \(H\) will have negative eigenvalues.

The partition function, eq. (2.5), can be expressed in terms of the Jacobi \(\theta\)-function of the torus,

\[ \theta_1(z|\tau) = 2f(q^2)q^{1/4}\sin(\pi z) \prod_{n=1}^{\infty} \left( 1 - 2q^{2n}\cos(2\pi z) + q^{4n} \right), \] (2.6)

where

\[ f(q^2) = \prod_{n=1}^{\infty} (1 - q^{2n}) = \left( \frac{1}{2\pi q^{1/4}} \frac{d\theta_1(z|\tau)}{dz} \bigg|_{z=0} \right)^{1/3} = q^{-1/12}\eta(q^2), \quad q = e^{i\pi \tau}, \]

and \(\eta(q^2)\) is the Dedekind eta function. We find the following exact formula for
the partition function

\[ Z(w, \lambda) = 2 \frac{w^{\frac{1}{2}}}{f(w)^{D-5}} \sin(\pi z) \theta_1(z|\tau), \quad z = -\frac{i\beta\lambda}{2\pi}, \quad \tau = \frac{i\beta}{2\pi}. \quad (2.7) \]

In order to estimate the asymptotic density of states we use the modular transformation property

\[ \theta_1\left(-\frac{z}{\tau}| -\frac{1}{\tau}\right) = e^{i\pi s} \sqrt{\tau} \exp\left(i\pi z^2\right) \theta_1(z|\tau). \quad (2.8) \]

Applying eq.(2.8) to the partition function (2.7) we obtain

\[ Z = \left(\beta/2\pi\right)^{\frac{D-4}{2}} w^{\frac{D-2}{24}} e^{-\frac{\beta^2 z^4}{2}} e^{\frac{i\pi z^2}{\beta}} \frac{\sin(i\beta\lambda/2)f(e^{-\frac{4\pi^2}{\beta}})^{4-D}}{\sin(\pi\lambda)g(\beta, \lambda)}, \quad (2.9) \]

where

\[ a \equiv \sqrt{\frac{D-2}{6}} \pi, \]

and

\[ g(\beta, \lambda) \equiv \prod_{n=1}^{\infty} \left(1 - e^{i2\pi\lambda} e^{-\frac{4\pi^2}{\beta^2}n}\right)\left(1 - e^{-i2\pi\lambda} e^{-\frac{4\pi^2}{\beta^2}n}\right). \]

In the limit of high temperature \( Z \) reduces to

\[ Z(\beta, \lambda) = \text{const.} \, \beta^{\frac{D-2}{2}} e^{\frac{a^2}{\beta}} \frac{\lambda}{\sin(\pi\lambda)}. \quad (2.10) \]

In order to extract the level density \( d_{n, J} \) for states of level \( n \) and angular momentum \( J \), it is convenient to expand \( Z \) in the following way

\[ Z(w, k) = \sum_{n, J} d_{n, J} w^n e^{i k J}, \quad k = -i\beta\lambda. \quad (2.11) \]

Then \( d_{n, J} \) can be projected out by Fourier integrating over \( k \) and then integrating
\( w \) over a small circle around \( w = 0 \),
\[
d_{n,J} = \frac{1}{2\pi i} \int \frac{dw}{w^{n+1}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikJ} Z(w, k) .
\] (2.12)

Interestingly, the integral over \( k \) can be exactly carried out with the result
\[
\int_{-\infty}^{\infty} dk e^{-ikJ} \frac{k}{\sinh(\pi k/\beta)} = \frac{\beta^2}{2} \frac{1}{\cosh^2(\beta J/2)} .
\] (2.13)

The integral over \( w \) can be approximated by a saddle point evaluation. Indeed, for large \( n \) the integrand has a sharply defined saddle point at \( w \) given by the solution of the equation
\[
a^2 = n + 1 - J \tanh(J\beta/2) .
\] (2.14)

If \( J \ll n \), that is, away from the Regge trajectories, the second term on the right hand side can be ignored, and the solution is \( \beta \approx \frac{a}{\sqrt{n+1}} \). If, on the other hand, \(|J| = O(n)\), then one can see from eq. (2.14) that \(|J|\beta \gg 1\) and therefore the stationary point is at
\[
\beta \approx \frac{a}{\sqrt{n+1} - |J|} .
\] (2.15)

Since this solution also applies for \(|J| \ll n \) we can adopt it in the general case. After performing the Gaussian integration, what remains is
\[
d_{n,J} \approx \text{const} \cdot (n + 1 - |J|)^{-(D+3)/4} \exp \left[ \frac{a(2(n + 1) - |J|)}{\sqrt{n + 1} - |J|} \right] \frac{1}{\cosh^2 \left( \frac{aJ}{2\sqrt{n+1}-|J|} \right)} .
\] (2.16)

On the Regge trajectories \( J = \pm n \) \( d_{n,J} \rightarrow \text{const} \) which can be normalized to 1. Note that the number of states of level \( n \) with zero angular momentum \( d_{n,0} \) differs from the total number of states of level \( n \), \( d_n = \sum_J d_{n,J} \) only by a subleading factor of \( 1/\sqrt{n} \). Integrating over \( J \) we recover the familiar formula for \( d_n \) (this can be done by integrating \( J \) from \(-n^{1-\epsilon} \) to \( n^{1-\epsilon} \), \( \epsilon > 0 \), where \( J \) can be neglected as compared to \( n \)).
The density of level as a function of the mass is \((J << n)\)

\[
\rho(m,J) = \text{const. } m^{-(D+1)/2} e^{m/T_H} \frac{1}{\cosh^2(aJ/2\sqrt{\alpha'})},
\]

(2.17)

where \(T_H\) is the Hagedorn temperature \(T_H = 1/(2a\sqrt{\alpha'}) = 1/4\pi\sqrt{\alpha'}\).

3. Strings in a magnetic field and gyromagnetic coupling

In this section we will obtain the tree-level gyromagnetic coupling in three different string theories, namely the bosonic open string theory with U(1) Chan-Paton charges, a bosonic string theory where U(1) charges arise from Kaluza-Klein compactification, and the heterotic string theory.

A. Bosonic open string theory

The world-sheet action in presence of an electromagnetic background field is given by (to simplify the discussion we put a charge only at one end of the string)

\[
S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \partial_{\alpha} X^{\mu} \partial^{\alpha} X_{\mu} + \int d\tau d\sigma q \delta(\sigma) A_{\mu}(X) \dot{X}^{\mu}.
\]

(3.1)

In eq. (3.1) we have taken the sigma-model metric to be the Minkowski metric. This is a valid string background (with vanishing beta-functions) only up to terms of \(O(F_{\mu\nu}^2)\), which can be ignored for the purpose of deriving the gyromagnetic coupling which is linear in \(F_{\mu\nu}\). In what follows all terms of \(O(F_{\mu\nu}^2)\) will be dropped.

The world-sheet Hamiltonian and the canonical momenta are given by

\[
H = \frac{1}{4\pi\alpha'} \int_{0}^{\pi} d\sigma \left[ (2\pi\alpha')^2 (\Pi_{\mu} - q\delta(\sigma) A_{\mu}(X))^2 + X'^2 \right],
\]

(3.2)

\[
\Pi_{\mu} = \frac{1}{2\pi\alpha'} \partial_{\tau} X_{\mu} + q\delta(\sigma) A_{\mu}(X).
\]

(3.3)

Now let us consider the case in which only the component \(F_{12} = -F_{21}\) is different from zero, which for simplicity can be contemplated as a constant magnetic
field $F_{12} = \text{const.}$ The vector potential can be written as

$$A_\mu = -\frac{1}{2} F_{\mu\nu} X^\nu.$$ \hspace{1cm} (3.4)

In the case the magnetic field is not constant eq.(3.4) provides the first terms in an $\alpha'$ expansion. Successive terms are of higher order in derivatives and will not affect the gyromagnetic coupling.

The boundary conditions at $\sigma = 0$ are

$$X'_1(\tau, 0) = 2\alpha' \pi q F_{12} \dot{X}^2(\tau, 0), \quad X'_2(\tau, 0) = 2\alpha' \pi q F_{21} \dot{X}^1(\tau, 0),$$

$$X'_\mu(\tau, 0) = 0, \quad \mu \neq 1, 2.$$ \hspace{1cm} (3.5)

In particular, we see that the magnetic field does not affect the boundary conditions for the light-cone coordinates $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$, so we can set, as usual, $X^+ = x^+ + 2\alpha' p^+ \tau$. By using eqs.(3.3) and (3.2) we find for the light-cone hamiltonian,

$$2\alpha' p^+ p^- = \frac{1}{4\pi \alpha'} \int_0^\pi d\sigma \left( (2\pi \alpha')^2 \Pi_i^2 + X'_i^2 + (2\pi \alpha')^2 q \delta(\sigma) F_{\mu\nu} \Pi^\mu X^\nu \right) - 1.$$ \hspace{1cm} (3.6)

Now it is easy to identify the gyromagnetic coupling for a physical state $|\Phi\rangle$ with mass $M$. We can measure the shift in the energy due to the interaction between the spin and the magnetic field. We use $2p^- p^+ = E^2 - p^{D-1}$ and expand the energy in powers of $1/M$ (in the presence of a magnetic field we cannot choose the rest frame because a combination of $p_1$ and $p_2$ does not commute with the Hamiltonian. In any case, terms containing $\vec{p}$ do not provide any contribution to the gyromagnetic coupling and are ignored). We obtain

$$\langle \Phi | H_{\text{mag}} | \Phi \rangle = \langle \Phi | -\frac{q \pi}{2M} F_{\mu\nu} X^\mu(0, \tau) \Pi^\nu(0, \tau) | \Phi \rangle.$$ \hspace{1cm} (3.7)

We can insert the free expansion for the string coordinates since the $O(F_{\mu\nu})$ cor-
rections will only give $O(F^2)$ contributions. We have
\begin{equation}
X^\mu(0, \tau) = x^\mu + 2\alpha' p^\mu + i\sqrt{2}\alpha' \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau}, \quad \dot{X}^\mu(0, \tau) = 2\alpha' p^\mu + \sqrt{2}\alpha' \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau}.
\end{equation}

$\mathcal{H}_{\text{mag}}$ can be written as the sum of an orbital angular momentum piece plus a purely spin part, $\mathcal{H}^O_{\text{mag}} + \mathcal{H}^s_{\text{mag}}$. By inserting eqs. (3.8) in eq. (3.7) and keeping the spin contribution $\mathcal{H}^s_{\text{mag}}$ only we find
\begin{equation}
\langle \Phi | \mathcal{H}^s_{\text{mag}} | \Phi \rangle = \langle \Phi \rangle - \frac{q}{2M} S^{\mu\nu} F_{\mu\nu} | \Phi \rangle,
\end{equation}

where

\begin{equation}
S^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^\mu \alpha_n^\nu - \alpha_n^\nu \alpha_n^\mu).
\end{equation}

From the standard form of the magnetic coupling, $\mathcal{H}_{\text{mag}} = -\vec{\mu} \cdot \vec{B}$ we obtain that the magnetic dipole moment is given by $\vec{\mu} = \frac{q}{M} \vec{J}$, whereby we deduce that $g = 2$ for all charged, spinning physical states in the bosonic open string theory. This is in agreement with a previous calculation by Ferrara et al [9].

**B. Bosonic string with U(1) Kaluza-Klein charges**

We begin by considering the sigma-model action for the bosonic string in presence of a metric and antisymmetric field:
\begin{equation}
S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ G_{AB} \eta^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B + B_{AB} \epsilon^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B \right],
\end{equation}

where $A, B = 0, 1, \ldots, D-1$, $\eta_{\alpha\beta}$ has signature $(-, +)$, $\epsilon^{12} = -\epsilon^{21} = 1$, and $D = 26$. Let us assume that one dimension, $X^{D-1} \equiv \varphi$, is compactified on $S^1$ and the metric
and antisymmetric fields take the following form:

$$S = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \left[ \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu + \partial_\alpha \varphi \partial^\alpha \varphi + 2A^\nu_\mu \partial_\alpha X^\mu \partial^\alpha \varphi + 2A^a_\mu \epsilon^{\alpha\beta} \partial_\beta X^\mu \partial_\alpha \varphi \right],$$

(3.11)

$$\mu, \nu = 0, 1, ..., D - 2.$$  

Let us introduce $x^\pm = \tau \pm \sigma$. The action (3.11) becomes

$$S = \frac{1}{\pi\alpha'} \int d^2 \sigma \left[ \partial_+ X^\mu \partial_- X_\mu + \partial_+ \varphi \partial_- \varphi + (A^\nu_\mu + A^a_\mu) \partial_- X^\mu \partial_+ \varphi + (A^\nu_\mu - A^a_\mu) \partial_+ X^\mu \partial_- \varphi \right].$$

(3.12)

In the particular case $A^a_\mu = A^\nu_\mu$, the model will provide a bosonic analog of heterotic string theory insofar as all the gauge quantum numbers will originate from the left sector.

Let us derive the Hamiltonian corresponding to eq. (3.11). The canonical momenta are given by

$$\Pi_\mu = \frac{1}{2\pi\alpha'} \left( \partial_\tau X_\mu + A^\nu_\mu \partial_\tau \varphi + A^a_\mu \partial_\sigma \varphi \right),$$

(3.13)

$$\Pi_\varphi = \frac{1}{2\pi\alpha'} \left( \partial_\tau \varphi + A^\nu_\mu \partial_\tau X^\mu - A^a_\mu \partial_\sigma X^\mu \right).$$

(3.14)

The Hamiltonian is (as before we ignore $O(A^2)$ terms)

$$H = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left[ (2\pi\alpha')^2 \Pi_\mu^2 + X'^2 + (2\pi\alpha')^2 \Pi_\varphi^2 + \varphi'^2 - 2A^\nu_\mu \left( (2\pi\alpha')^2 \Pi_\varphi X^\mu - \varphi' X^\mu \right) \right] - 2A^a_\mu \left( (2\pi\alpha')^2 \Pi_\varphi - \varphi' X^\mu \right)$$

(3.15)

Let us first discuss the particular case $A^a_\mu = 0$. As in the previous subsection, let us consider the case in which only the $F^\nu_{12} = -F^\nu_{21}$ is non-vanishing. Then the free-field equations of motion for the light-cone coordinates $X^\pm$ are not affected and we can set $X^+ = x^+ + 2\alpha' p^+ \tau$. 

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The term in the Hamiltonian containing the interaction with the electromagnetic field is
\[ H_{\text{int}} = \frac{1}{4\pi \alpha'} F_{\mu \nu}^v \int_{0}^{\pi} d\sigma X^\nu \left( (2\pi \alpha')^2 \Pi_\varphi \Pi^\mu - X^\mu \varphi' \right). \] (3.16)

We note that there is an unusual term \[ \varphi' F_{\mu \nu}^v X^\nu X^\mu \] for states with non-vanishing winding number.

In order to identify the gyromagnetic coupling for a physical state \[ |\Phi\rangle \] with mass \[ M \], again we use \[ 2p^- p^+ = E^2 - p^D - 1 \] and expand the energy in powers of \[ 1/M \]. We obtain
\[ \langle \Phi | H_{\text{mag}} | \Phi \rangle = \langle \Phi \frac{1}{2\alpha'} M | H_{\text{int}} | \Phi \rangle. \] (3.17)

We can insert the free mode expansion for the string coordinates since the \[ O(F_{\mu \nu}^v) \] corrections will only give \[ O(F^2) \] contributions. Only the zero-mode part of \( \Pi_\varphi \) and \( \varphi' \), representing charge and winding number, respectively, will contribute to this linear order in \( F_{\mu \nu}^v \). Thus we find
\[ \langle \Phi | H_{\text{mag}}^s | \Phi \rangle = -\frac{Q}{4M} F_{\mu \nu}^v S_{\mu \nu} + \frac{W}{4M} F_{\mu \nu}^v (S_{L}^{\mu \nu} - S_{R}^{\mu \nu}), \] \[ \equiv -\frac{Q}{2M} B_{z} B_{z} \] (3.18)

where \( Q \) and \( W \) are respectively the \( U(1) \) charge and the winding number given by
\[ Q = \frac{1}{2\pi \alpha'} \int_{0}^{\pi} d\sigma \varphi, \quad W = \frac{1}{2\pi \alpha'} \int_{0}^{\pi} d\sigma \varphi', \] (3.19)

and \( S_{\mu \nu} = S_{R}^{\mu \nu} + S_{L}^{\mu \nu} \),
\[ S_{R}^{\mu \nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^{\mu} \alpha_n^{\nu} - \alpha_n^{\mu} \alpha_{-n}^{\nu} ) , \quad S_{L}^{\mu \nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_n^{\nu} - \tilde{\alpha}_n^{\mu} \tilde{\alpha}_{-n}^{\nu} ). \] (3.20)

From eq. (3.18) one can see that the magnetic moment operator in general does not commute with \( S^2 \) and therefore they cannot be simultaneously diagonalized.
The gyromagnetic factor must be defined as the ratio between expectation values. For states labelled by $S^2, S_z$ we have

$$\langle \Phi | \mu_z | \Phi \rangle = g \frac{Q}{2M} \langle \Phi | S_z | \Phi \rangle .$$  \hfill (3.21)

Thus in this theory physical states will have

$$\langle g \rangle = 1 + \frac{W Q}{\langle S \rangle} \langle S_R - S_L \rangle .$$  \hfill (3.22)

It is interesting to note the physical state $\alpha_{-1} \mu |Q,W\rangle$ (and similarly for the state $\tilde{\alpha}_{-1} \mu |Q,W\rangle$), because it is subject to the Virasoro condition $N_R - N_L = QW = 1$, it will have $g = 2$ only at the self-dual point where $Q = W$.

Let us now consider the case $A^a_\mu = A^\nu_\mu, A^L_\mu = \frac{1}{2} (A^\nu_\mu + A^a_\mu) \neq 0$, which, as mentioned above, is a bosonic analog of the heterotic string theory. The term in the light-cone Hamiltonian containing the interaction with the electromagnetic field is

$$H_{\text{int}} = \frac{1}{4\pi\alpha'} F_{\mu\nu}^L \int_0^\pi d\sigma X^\nu (2\pi\alpha' \Pi^\mu - X^\mu) (\varphi' + 2\pi\alpha' \Pi \varphi) .$$  \hfill (3.23)

By following similar steps as before we find

$$\langle \Phi | H^s_{\text{mag}} | \Phi \rangle = - \frac{Q}{2M} F_{\mu\nu}^L S^\mu_{\nu R} = - \frac{Q}{M} B_z S^z_{\nu R} ,$$  \hfill (3.24)

where

$$Q_L = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma (\varphi' + \varphi) ,$$  \hfill (3.25)

and

$$S^\mu_{\nu R} = -i \sum_{n=1}^\infty \frac{1}{n} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n) .$$  \hfill (3.26)

From eq. (3.24) we conclude that in this model physical states have a magnetic dipole moment $\vec{\mu} = \frac{Q}{M} \vec{S}_R$. The same result will appear in the case of the heterotic string theory considered below.
C. Heterotic string theory

The action of the heterotic string in presence of background gauge fields is given by [11]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[ \partial_\alpha X^\mu \partial_\alpha X_\mu - 2i\psi_-^{\mu} \partial_+ \psi_- - 2i\lambda_\pm \partial_- \lambda_+^r - 2\lambda_+^r (T^M)_{rs} \lambda_+^s A^M_{\mu} \partial_- X_\mu + \frac{i}{2} \lambda_+^r (T^M)_{rs} \lambda_+^s F^M_{\mu\nu} \psi_-^{\mu} \psi_-^{\nu} \right].$$  (3.27)

Again, $O(F^2)$ corrections to metric and other sigma-model backgrounds will be ignored since they do not affect the gyromagnetic coupling. The canonical momenta are given by

$$2\pi\alpha' \Pi_\mu = \dot{X}_\mu + \dot{O}^M A^M_{\mu} \partial_- X_\mu + \frac{i}{2} \lambda_+^r (T^M)_{rs} \lambda_+^s F^M_{\mu\nu} \psi_-^{\mu} \psi_-^{\nu}$$

and

$$2\pi\alpha' \Pi_\psi = \frac{i}{2} \psi_-^\mu, \quad 2\pi\alpha' \Pi_\lambda = \frac{i}{2} \lambda_+^r. \quad (3.28)$$

One finds for the Hamiltonian

$$H = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left[ (2\pi\alpha')^2 \dot{\Pi}_\mu^2 + X_\mu^2 - i\psi_-^{\mu} \partial_\sigma \psi_- - i\lambda_+^r \partial_\sigma \lambda_+^r \right]$$

$$-2\dot{O}^M A^M_{\mu} (2\pi\alpha' \Pi^\mu - X^\mu) + i\dot{O}^M F^M_{\mu\nu} \psi_-^{\mu} \psi_-^{\nu}. \quad (3.29)$$

Now we consider the particular background where only one of the components, $F_{12}^1 = -F_{21}^1$, is different from zero. The light-cone gauge can then be fixed in the standard way by setting $X^+ = x^+ + 2\alpha' p^+ \tau$ and $\psi^+ = 0$. Proceeding as in the previous model, we obtain that the magnetic dipole moment for a physical state
with charge $q$ and mass $M$ is given by

$$\mu^z = \frac{q}{M} S^z_R,$$

(3.30)

$$q = \langle \Phi | \frac{1}{2\pi \alpha'} \int_0^\pi d\sigma \frac{1}{2} \lambda^r_+ (T^1)_rs \lambda^s_+ | \Phi \rangle,$$

(3.31)

where $S^z_R$ is the right contribution to the angular momentum, that is,

$$S^z_R = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1) - \frac{i}{2} [d_0^1, d_0^2] - i \sum_{n=1}^{\infty} (d_{-n}^1 d_n^2 - d_{-n}^2 d_n^1),$$

(3.32)

in the Ramond sector, and the analog expression in the Neveu-Schwarz sector. This is the result anticipated above.

Note that the massless particles with gauge quantum numbers in the heterotic string theory get all their angular momentum from the right sector, that is $S^\mu_\nu = S^\mu_\nu$. As a result, $\mu^z = q S^z / M$, and thus they have $g = 2$ (a small mass $M$ is given by sitting infinitesimally away the self-dual point).

In the general case $g$ will be given by $g = 2 \langle S^z_R \rangle / S^z$. In the next section we will be interested in a thermal ensemble of heterotic string states with macroscopic masses ($M^2 >> 1/\alpha'$). In this case we obtain the average gyromagnetic factor of a thermal average of heterotic strings with a given mass, charge and angular momentum.
4. Correspondence with rotating black holes

In ref. [2] (see also [5]) a statistical mechanical explanation of the Bekenstein-Hawking entropy as counting of quantum states was suggested. The proposal entails a description of the black hole horizon in terms of strings moving in a Rindler geometry. This study was carried out for the Schwarzschild black hole and a remarkable correspondence was obtained. It is important to understand to what extent a string description is feasible. So let us consider the most general solution given by the rotating, charged black hole solution, that is, the heterotic string analog of the Kerr-Newman black hole (see ref. [7]). We shall compare our formulas for the level density derived for a free string with the entropy of the rotating black hole solution. It should be noted that the angular momentum is quantized and it is also an adiabatic invariant, and therefore it should not change if the gravitational coupling were gradually increased in such a way the initial free string gravitational collapses.

In addition to the entropy, if a charged and rotating black hole is to be described in terms of a fundamental string, it should have the same gyromagnetic factor. The gyromagnetic factor of a black hole varies according to the theory (see e.g. ref. [8]). In the case of the heterotic string theory one finds $g = 2$ for all values of the charge $Q$ [7]. The same result holds in Einstein-Maxwell theory, but, generically, $g$ is different than two; it is a function of the parameters characterizing the black hole.

It is straightforward to generalize the result of section 2 to the case of the heterotic string theory. Eq. (2.16) can be applied to obtain expressions for the left and right sectors of the bosonic string separately. In addition we have the Virasoro constraint which sets $n_L = n_R = n$ (more precisely, $n_L = n_R + \tilde{a}$, the value of $\tilde{a}$ according to the sector, but for large the normal ordering constant $\tilde{a}$ can be neglected). The discussion can be simplified by setting $J << n$ in eq. (2.16), which means that $J$ is not on a Regge trajectory. It turns out that in the large $n$
case we are considering this does not affect integrals over $J$. We have

$$d_{n,J_L,J_R} \approx \text{const. } n^{-23/2} \frac{e^{2(a_L+a_R)\sqrt{n}}}{\cosh^2(a_LJ_L/2\sqrt{n}) \cosh^2(a_RJ_R/2\sqrt{n})}, \quad (4.1)$$

where $a_L = 2\pi$ and $a_R = \sqrt{2}\pi$. Thus

$$d_{n,J} = \int_{-n}^{n} dJ_L d_{n,J_L,J-J_L} = \int_{-n}^{n} dJ_R d_{n,J-R,J_R}. \quad (4.2)$$

So let us consider a macroscopic four-dimensional rotating and electrically charged black hole in the heterotic string theory and let us choose the $z$-direction along the angular momentum, so that $J_z = |J|$, $J_x = J_y = 0$. Now let us study a thermal ensemble of strings with given mass, charge and angular momentum. The charge $Q$ is the quantum number of a zero-mode operator and characterizes the Fock space vacuum. As long as $Q \ll m$, it does not play any role in the following discussion. If $Q = O(m)$ then the condition $n_L \approx n_R$ must be modified by the addition of a term proportional to $Q^2$. Let us compute the average magnetic dipole moment $\langle \vec{\mu} \rangle \propto \langle \vec{J}_R \rangle$ in the thermal ensemble. Clearly $\langle J^x_R \rangle = \langle J^y_R \rangle = 0$, since for any configuration with $J^x_R$ there is a configuration with $-J^x_R$. The average gyromagnetic factor will be given by

$$\langle g \rangle = 2 \frac{\langle J^z_R \rangle}{\langle J \rangle} = \frac{\langle J_R \rangle}{\langle J_R \rangle + \langle J_L \rangle}, \quad (4.3)$$

with

$$\langle J_L \rangle = \frac{1}{d_{n,J}} \int_{-n}^{n} dJ_L d_{n,J,L} = \langle J_R \rangle = \frac{1}{d_{n,J}} \int_{-n}^{n} dJ_R d_{n,J,R}. \quad (4.4)$$

We have computed the integrals in eqs. (4.2) and (4.4) by numerical integration for different values of $J$ and $n$. Three cases can be distinguished:
i) \( \lim_{n \to \infty} \frac{J}{\sqrt{n}} = 0 \). In this case we obtain \( \langle g \rangle \approx 1.27 \).

ii) \( \lim_{n \to \infty} \frac{J}{\sqrt{n}} = \infty \), for which we find \( \langle g \rangle = 2 \).

iii) \( \lim_{n \to \infty} \frac{J}{\sqrt{n}} = \bar{J} = \text{finite} \). Then \( \langle g \rangle \) is a monotonically increasing function of \( \bar{J} \) which interpolates between 1.27 and 2.

As we will see, the correct scaling arises automatically from the correspondence between level densities.

Because a string in the vicinity of the horizon is described in terms of a Rindler geometry, in ref. [2] it was argued that the black hole ADM mass \( M \) must be related to the string energy by \( m \sim M^2 \). The Bekenstein-Hawking entropy for a rotating black hole is given by \( S = A/4G \), with \( (Q^2 << M) \) [7]

\[
A = 8\pi M^2 G^2 \left[ 1 + \left( 1 - \frac{J^2}{G^2 M^4} \right)^{1/2} \right].
\]

(4.5)

On the other hand from eqs. (4.1), (4.2) we see that, for large \( n \), \( d_{n,J} \) can be written as

\[
d_{n,J} = e^{2(a_L + a_R)\sqrt{n}} f(J/\sqrt{n}).
\]

(4.6)

The function \( f(J/\sqrt{n}) \) can be explicitly obtained from eq. (4.2) for large \( \bar{J} = J/\sqrt{n} \). It is of the form \( f(J) = c_0 e^{-a_R \bar{J}} + c_1 e^{-a_L \bar{J}} + \ldots \).

From eqs. (4.5) and (4.6) we see that correspondence between heterotic string and black hole density level requires

\[
2(a_L + a_R)\sqrt{n} + \log f(J/\sqrt{n}) = M^2 GF(J/GM^2)
\]

(4.7)

where

\[
F(J/GM^2) = 2\pi \left[ 1 + \left( 1 - \frac{J^2}{G^2 M^4} \right)^{1/2} \right] \]

Eq. (4.7) establishes a relationship between the string energy and the black hole ADM mass as a function of the angular momentum. Remarkably, this relation
implies that the only consistent scaling is \( m \sim M^2 \). Indeed, let us take the limit \( J \to \infty \) and \( M \to \infty \) at \( j \equiv J/M^2 \) fixed. In case i), we have \( \log f(0) = \text{finite} \), and we get from eq. (4.7) \( m = \text{const.} \ M^2 + \text{finite} \). This is in contradiction with the assumption that \( J/\sqrt{n} \to 0 \) at \( J/M^2 \) fixed, and therefore this case is not consistent with eq. (4.7). Similarly, in case ii) we get \( m \sim M^2 + M^2/m \). Again, this contradicts the assumption that \( J/\sqrt{n} \to \infty \) at \( J/M^2 \) fixed. Finally, the case iii) leads to an equation of the form

\[
\frac{m}{T_H} + \log f\left(2J/m\sqrt{\alpha'}\right) = M^2 GF\left(J/M^2 G\right), \tag{4.8}
\]

where \( T_H = \frac{1}{\pi \sqrt{\alpha'}} \left(1 - \frac{1}{\sqrt{2}}\right) \). Eq. (4.8) generalizes the relation \( m = 4\pi M^2 GT_H \) to the case \( J \neq 0 \).

Thus the correspondence between the level density requires that \( m \sim M^2 \) and hence \( J \) scales with \( \sqrt{n} \). As mentioned above, in this case the average gyromagnetic factor is a function of \( J/M^2 G \) varying between \( \sim 1.27 \) and 2. However, it should be noted that the calculation of \( g \) is extremely sensitive to the scaling, since logarithmic corrections to the scaling, such as \( J \sim n^{1/2+\epsilon}, \ \epsilon > 0 \) would lead to \( g = 2 \), which is the value for rotating black holes in heterotic string theory.

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