On the partition dimension of trees

Juan A. Rodríguez-Velázquez, Ismael G. Yero and Magdalena Lemańska

1Departament d’Enginyeria Informàtica i Matemàtiques
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.
ismael.gonzalez@urv.cat, juanalberto.rodriguez@urv.cat

2Department of Technical Physics and Applied Mathematics
Gdansk University of Technology, ul. Narutowicza 11/12 80-233 Gdansk, Poland
magda@mifgate.mif.pg.gda.pl

October 25, 2011

Abstract

Given an ordered partition Π = {P₁, P₂, ..., P_t} of the vertex set V of a connected graph G = (V, E), the partition representation of a vertex v ∈ V with respect to the partition Π is the vector r(v|Π) = (d(v, P₁), d(v, P₂), ..., d(v, P_t)), where d(v, P_i) represents the distance between the vertex v and the set P_i. A partition Π of V is a resolving partition of G if different vertices of G have different partition representations, i.e., for every pair of vertices u, v ∈ V, r(u|Π) ≠ r(v|Π). The partition dimension of G is the minimum number of sets in any resolving partition of G. In this paper we obtain several tight bounds on the partition dimension of trees.

Keywords: Resolving sets, resolving partition, partition dimension.
AMS Subject Classification numbers: 05C12
1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [2, 3, 4, 5, 6, 7, 8, 9, 10, 14]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds [12, 13] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in [3, 11, 14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions [4, 7, 8].

Given a graph $G = (V, E)$ and a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of $G$, the metric representation of a vertex $v \in V$ with respect to $S$ is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$ denotes the distance between the vertices $v$ and $v_i$, $1 \leq i \leq k$. We say that $S$ is a resolving set of $G$ if different vertices of $G$ have different metric representations, i.e., for every pair of vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The metric dimension of $G$ is the minimum cardinality of any resolving set of $G$, and it is denoted by $\text{dim}(G)$. The metric dimension of graphs is studied in [2, 3, 4, 5, 6, 18].

Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of the vertices of $G$, the partition representation of a vertex $v \in V$ with respect to the partition $\Pi$ is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, with $1 \leq i \leq t$, represents the distance between the vertex $v$ and the set $P_i$, i.e., $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. We say that $\Pi$ is a resolving partition of $G$ if different vertices of $G$ have different partition representations, i.e., for every pair of vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension of $G$ is the minimum number of sets in any resolving partition of $G$ and it is denoted by $\text{pd}(G)$. The partition dimension of graphs is studied in [4, 7, 8, 18].

\(^1\)Also called locating number.
2 The partition dimension of trees

It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph $G$ we have

$$pd(G) \leq dim(G) + 1.$$  \hspace{1cm} (1)

We know that the partition dimension of any path is two. That is, for any path graph $P$, it follows $pd(P) = dim(P) + 1 = 2$. A formula for the dimension of trees that are not paths has been established in [5, 9, 17]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree $T$ will be called a major vertex of $T$. Any leaf $u$ of $T$ is said to be a terminal vertex of a major vertex $v$ of $T$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $T$. The terminal degree of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $T$ is an exterior major vertex of $T$ if it has positive terminal degree.

Let $n_1(T)$ denote the number of leaves of $T$, and let $ex(T)$ denote the number of exterior major vertices of $T$. We can now state the formula for the dimension of a tree [5, 9, 17]: if $T$ is a tree that is not a path, then

$$dim(T) = n_1(T) - ex(T).$$  \hspace{1cm} (2)

As a consequence, if $T$ is a tree that is not a path, then

$$pd(T) \leq n_1(T) - ex(T) + 1.$$  \hspace{1cm} (3)

The above bound is tight, it is achieved for the graph in Figure 1 where $\Pi = \{\{8\}, \{4,9\}, \{1,2,3,5,6,7\}\}$ is a resolving partition and $pd(T) = 3$. 

Figure 1: In this tree the vertex 3 is an exterior major vertex of terminal degree two: 1 and 4 are terminal vertices of 3.
Figure 2: \( \Pi = \{\{1, 4, 9, 12\}, \{3, 5, 8, 11\}, \{2, 6, 7, 10\}\} \) is a resolving partition.

However, there are graphs for which the following bound gives better result than bound (3), for instance, the graph in Figure 2.

Let \( S = \{s_1, s_2, ..., s_\kappa\} \) be the set of exterior major vertices of \( T = (V, E) \) with terminal degree greater than one, let \( \{s_{i1}, s_{i2}, ..., s_{il_i}\} \) be the set of terminal vertices of \( s_i \) and let \( \tau = \max_{1 \leq i \leq \kappa}\{l_i\} \). With the above notation we have the following result.

**Theorem 1.** For any tree \( T \) which is not a path,

\[
\text{pd}(T) \leq \kappa + \tau - 1.
\]

**Proof.** For a terminal vertex \( s_{ij} \) of a major vertex \( s_i \in S \) we denote by \( S_{ij} \) the set of vertices of \( T \), different from \( s_i \), belonging to the \( s_i - s_{ij} \) path. If \( l_i < \tau - 1 \), we assume \( S_{ij} = \emptyset \) for every \( j \in \{l_i + 1, ..., \tau - 1\} \). Now for every \( j \in \{2, ..., \tau - 1\} \), let \( B_j = \bigcup_{i=1}^{\kappa} S_{ij} \) and, for every \( i \in \{1, ..., \kappa\} \), let \( A_i = S_{i1} \). Let us show that \( \Pi = \{A, A_1, A_2, ..., A_\kappa, B_2, ..., B_{\tau-1}\} \) is a resolving partition of \( T \), where \( A = V - \left( \bigcup_{i=1}^{\kappa} A_i \cup \bigcup_{j=2}^{\tau-1} B_j \right) \). We consider two different vertices \( x, y \in V \). Note that if \( x \) and \( y \) belong to different sets of \( \Pi \), we have \( r(x|\Pi) \neq r(y|\Pi) \).

Case 1: \( x, y \in S_{ij} \). In this case, \( d(x, A) = d(x, s_i) \neq d(y, s_i) = d(y, A) \).

Case 2: \( x \in S_{ij} \) and \( y \in S_{kl}, i \neq k \). If \( j = 1 \) or \( l = 1 \), then \( x \) and \( y \) belong to different sets of \( \Pi \). So we suppose \( j \neq 1 \) and \( l \neq 1 \). Hence, if
\[ d(x, A_i) = d(y, A_i), \text{ then} \]
\[
d(x, A_k) = d(x, s_i) + d(s_i, s_k) + 1
\]
\[
= d(x, A_i) + d(s_i, s_k)
\]
\[
= d(y, A_i) + d(s_i, s_k)
\]
\[
= d(y, s_k) + 2d(s_k, s_i) + 1
\]
\[
= d(y, A_k) + 2d(s_k, s_i)
\]
\[> d(y, A_k). \]

Case 3: \( x \in S_{i\tau} \) and \( y \in A - \bigcup_{l=1}^\kappa S_{l\tau} \). If \( d(x, A_i) = d(y, A_i) \), then \( d(x, s_i) = d(y, s_i) \). Since \( y \notin S_{l\tau}, l \in \{1, \ldots, \kappa\} \), there exists \( A_j \in \Pi, j \neq i \), such that \( s_i \) does not belong to the \( y-s_j \) path. Now let \( Y \) be the set of vertices belonging to the \( y-s_j \) path, and let \( v \in Y \) such that \( d(s_i, v) = \min_{u \in Y} \{d(s_i, u)\} \).

Hence,
\[
d(x, A_j) = d(x, s_i) + d(s_i, v) + d(v, s_j) + 1
\]
\[
= d(y, s_i) + d(s_i, v) + d(v, s_j) + 1
\]
\[
= d(y, v) + 2d(v, s_i) + d(v, s_j) + 1
\]
\[
= d(y, A_j) + 2d(v, s_i)
\]
\[> d(y, A_j). \]

Case 4: \( x, y \in A' = A - \bigcup_{l=1}^\kappa S_{l\tau} \). If for some exterior major vertex \( s_i \in S \), the vertex \( x \) belongs to the \( y-s_i \) path or the vertex \( y \) belongs to the \( x-s_i \) path, then \( d(x, A_i) \neq d(y, A_i) \). Otherwise, there exist at least two exterior major vertices \( s_i, s_j \) such that the \( x-y \) path and the \( s_i-s_j \) path share more than one vertex (if not, then \( x, y \notin A' \)). Let \( W \) be the set of vertices belonging to the \( s_i-s_j \) path. Let \( u, v \in W \) such that \( d(x, u) = \min_{z \in W} \{d(x, z)\} \) and \( d(y, v) = \min_{z \in W} \{d(y, z)\} \). We suppose, without loss of generality, that \( d(s_i, u) > d(v, s_i) \). Hence, if \( d(x, v) = d(y, v) \), then \( d(x, u) \neq d(y, u) \), and if \( d(x, u) = d(y, u) \), then \( d(x, v) \neq d(y, v) \). We have,
\[
d(x, A_j) = d(x, u) + d(u, s_j) + 1
\]
\[
\neq d(y, u) + d(u, s_j) + 1
\]
\[
= d(y, A_j) \]
or

\[ d(x, A_i) = d(x, v) + d(v, s_i) + 1 \]
\[ \neq d(y, v) + d(v, s_i) + 1 \]
\[ = d(y, A_i). \]

Therefore, for different vertices \( x, y \in V \), we have \( r(x|\Pi) \neq r(y|\Pi) \). \( \square \)

One example where \( pd(T) = \kappa + \tau - 1 \) is the tree in Figure 1.

Any vertex adjacent to a leaf of a tree \( T \) is called a support vertex. In the following result \( \xi \) denotes the number of support vertices of \( T \) and \( \theta \) denotes the maximum number of leaves adjacent to a support vertex of \( T \).

**Corollary 2.** For any tree \( T \) of order \( n \geq 2 \), \( pd(T) \leq \xi + \theta - 1 \).

**Proof.** If \( T \) is a path, then \( \xi = 2 \) and \( \theta = 1 \), so the result follows. Now we suppose \( T \) is not a path. Let \( v \) be an exterior major vertex of terminal degree \( \tau \). Let \( x \) be the number of leaves adjacent to \( v \) and let \( y = \tau - x \). Since \( \kappa + y \leq \xi \) and \( x \leq \theta \), we deduce \( \kappa + \tau \leq \xi + \theta \). \( \square \)

The above bound is achieved, for instance, for the graph of order six composed by two support vertices \( a \) and \( b \), where \( a \) is adjacent to \( b \), and four leaves; two of them are adjacent to \( a \) and the other two leaves are adjacent to \( b \). One example of graph for which Theorem 1 gives better result than Corollary 2 is the graph in Figure 1.

Since the number of leaves, \( n_1(T) \), of a tree \( T \) is bounded below by \( \xi + \theta - 1 \), Corollary 2 leads to the following bound.

**Remark 3.** For any tree \( T \) of order \( n \geq 2 \), \( pd(T) \leq n_1(T) \).

Now we are going to characterize all the trees for which \( pd(T) = n_1(T) \).

It was shown in [7] that \( pd(G) = 2 \) if and only if the graph \( G \) is a path. So by the above remark we obtain the following result.

**Remark 4.** Let \( T \) be a tree of order \( n \geq 4 \). If \( n_1(T) = 3 \), then \( pd(T) = 3 \).

**Theorem 5.** Let \( T \) be a tree with \( n_1(T) \geq 4 \). Then \( pd(T) = n_1(T) \) if and only if \( T \) is the star graph.
Proof. If \( T = S_n \) is a star graph, it is clear that \( pd(T) = n_1(T) \). Now, let \( T = (V, E) \neq S_n \), such that \( pd(T) = n_1(T) \geq 4 \). Note that by (3) we have \( ex(T) = 1 \). Let \( t = n_1(T) \) and let \( \Omega = \{u_1, u_2, \ldots, u_t\} \) be the set of leaves of \( T \).

Let \( u \in V \) be the unique exterior major vertex of \( T \). Let us suppose, without loss of generality, \( u_t \) is a leaf of \( T \) such that \( d(u_t, u) = \max_{u_i \in \Omega} \{d(u_i, u)\} \).

For the leaves \( u_1, u_2, u_t \in \Omega \) let the paths \( P = uu_1u_2\ldots u_{t-1}u_t \), \( Q = uu_1u_2\ldots u_{t-1}u_1 \) and \( R = uu_2u_3\ldots u_{t-1}u_t \). Let us form the partition \( \Pi = \{A_1, A_2, \ldots, A_{t-2}, A_t\} \), such that \( A_1 = \{u_1, u_2, \ldots, u_{t-1}, u_1, u_t, u_{t-3}, \ldots, u_{t-1}, u_t\} \), \( A_2 = \{u_2, u_3, \ldots, u_{t-2}, u_2, u_1\} \), \( A_i = \{u_i\}, i \in \{3, \ldots, t-2\} \) and \( A = V - \bigcup_{i=1}^{t-2} A_i \). Let us consider two different vertices \( x, y \in V \). Hence, we have the following cases,

Case 1: \( x, y \in A_1 \). Let us suppose \( x \in P \) and \( y \in Q \). If \( d(x, A_2) = d(y, A_2) \), then we have

\[
\begin{align*}
d(x, A) &= d(x, u_1) + 1 \\
&= d(x, A_2) + 1 \\
&= d(y, A_2) + 1 \\
&= d(y, A) + 2 \\
&> d(y, A).
\end{align*}
\]

Now, if \( x, y \in P \) or \( x, y \in Q \), then \( d(x, A) \neq d(y, A) \).

Case 2: \( x, y \in A_2 \). If \( x = u_1 \) or \( y = u_1 \), then let us suppose for instance, \( x = u_1 \), so we have \( d(x, A_1) = 1 < 2 \leq d(y, A_1) \). On the contrary, if \( x, y \in R \), then \( d(x, A) \neq d(y, A) \).

Case 3: \( x, y \in A \). If \( d(x, A_1) = d(y, A_1) \), then \( t \geq 5 \) and there exists a leaf \( u_i, i \neq 1, 2, t-1, t \), such that \( d(x, A_i) = d(x, u_i) \neq d(y, u_i) = d(y, A_i) \).

Therefore, for different vertices \( x, y \in V \) we have \( r(x|\Pi) \neq r(y|\Pi) \) and \( \Pi \) is a resolving partition in \( T \), a contradiction. 

Figure 3: A Comet graph where \( 3 = \theta = Pd(T) \).
Let $T$ be the Comet graph showed in Figure 3. A resolving partition for $T$ is $\Pi = \{A_1, A_2, A_3\}$, where $A_1 = \{x, t\}$, $A_2 = \{y, z\}$ and $A_3 = \{u, w\}$. In this case, $Pd(T) = 3\theta$.

**Remark 6.** For any tree $T$ of order $n \geq 2$, $pd(T) \geq \theta$.

**Proof.** Since different leaves adjacent to the same support vertex must belong to different sets of a resolving partition, the result follows.

Other examples where $pd(T) = \theta$ are the star graphs and the graph in Figure 2.

**Theorem 7.** Let $T$ be a tree. If every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then

$$pd(T) \leq \max\{\kappa, \tau + 1\}.$$

**Proof.** If $T$ is a path, then $\tau = 2$ and $\kappa = 1$, so the result follows. We suppose $T = (V, E)$ is not a path. Let $S = \{s_1, s_2, ..., s_\kappa\}$ be the set of exterior major vertices of $T$ with terminal degree greater than one and let $B_i = \{s_i\}$, $i = 1, ..., \kappa$. If $\kappa < \tau + 1$, then for $i \in \{\kappa + 1, ..., \tau + 1\}$ we assume $B_i = \emptyset$. Let $l_i$ be the terminal degree of $s_i$, $i \in \{1, ..., \kappa\}$. If $l_i < i$, then we denote by $\{s_{i_1}, ..., s_{il_i}\}$ the set of terminal vertices of $s_i$. On the contrary, if $l_i \geq i$, then the set of terminal vertices of $s_i$ is denoted by $\{s_{i1}, ..., s_{il_i-1}, s_{il_i+1}, ..., s_{il_i+1}\}$. Also, for a terminal vertex $s_{ij}$ of a major vertex $s_i$ we denote by $S_{ij}$ the set of vertices of $T$, different from $s_i$, belonging to the $s_i - s_{ij}$ path. Moreover, we assume $S_{ij} = \emptyset$ for the following three cases: (1) $i = j$, (2) $i \leq l_i < \tau$ and $j \in \{l_i + 2, ..., \tau + 1\}$, and (3) $i > l_i$ and $j \in \{l_i + 1, ..., \tau + 1\}$. Now, let $t = \max\{\kappa, \tau + 1\}$ and let $\Pi = \{A_1, A_2, ..., A_t\}$ be composed by the sets $A_i = B_i \cup (\bigcup_{j=1}^{t} S_{ij})$, $i = 1, ..., t$. Since every vertex belonging to the path between two exterior major vertices of terminal degree greater than one, is an exterior major vertex of terminal degree greater than one, then $\Pi$ is a partition of $V$.

Let us show that $\Pi$ is a resolving partition. Let $x, y \in V$ be different vertices of $T$. If $x, y \in A_i$, we have the following three cases.

**Case 1:** $x, y \in S_{ji}$. In this case $d(x, A_j) = d(x, s_j) \neq d(y, s_j) = d(y, A_j)$.

**Case 2:** $x \in S_{ji}$ and $y \in S_{kj}$, $j \neq k$. If $d(x, A_k) = d(y, A_k)$ we have $d(y, A_j) > d(y, s_k) = d(y, A_k) = d(x, A_k) > d(x, s_j) = d(x, A_j)$.

**Case 3:** $x = s_i$ and $y \in S_{ji}$. As $s_i$ has at least two terminal vertices, there
exists a terminal vertex $s_i$ of $s_i$, $l \neq j$, such that $d(x, A_l) = d(x, S_{il}) = 1$. Hence, $d(y, A_l) > d(y, s_j) \geq 1 = d(x, A_l)$. Therefore, for different vertices $x, y \in V$, we have $r(x|\Pi) \neq r(y|\Pi)$.

The above bound is achieved, for instance, for the graph in Figure 4.

Figure 4: $\Pi = \{\{1, 8, 11, 14\}, \{2, 5, 12, 15\}, \{3, 6, 9, 16\}, \{4, 7, 10, 13\}\}$ is a resolving partition.

3 On the partition dimension of generalized trees

A cut vertex in a graph is a vertex whose removal increases the number of components of the graph and an extreme vertex is a vertex such that its closed neighborhood forms a complete graph. Also, a block is a maximal biconnected subgraph of the graph. Now, let $\mathcal{F}$ be the family of sequences of connected graphs $G_1, G_2, \ldots, G_k$, $k \geq 2$, such that $G_1$ is a complete graph $K_{n_1}$, $n_1 \geq 2$, and $G_i$, $i \geq 2$, is obtained recursively from $G_{i-1}$ by adding a complete graph $K_{n_i}$, $n_i \geq 2$, and identifying a vertex of $G_{i-1}$ with a vertex in $K_{n_i}$.

From this point we will say that a connected graph $G$ is a generalized tree if and only if there exists a sequence $\{G_1, G_2, \ldots, G_k\} \in \mathcal{F}$ such that $G_k = G$ for some $k \geq 2$. Notice that in these generalized trees every vertex is either, a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every $G_i$ is isomorphic to $K_2$, then $G_k$ is a tree, thus justifying the terminology used. In this section we will be centered in the study of partition dimension of generalized trees.

Let $G = (V, E)$ be a generalized tree and let $R_1, R_2, \ldots, R_k$ be the blocks of $G$. A cut vertex $v \in V$ is a support cut vertex if there is at least a
Figure 5: $\Pi = \{\{4\}, \{7\}, \{10\}, \{5, 8, 11\}, \{1, 2, 3, 6, 9, 12\}\}$ is a resolving partition for the generalized tree.

block $R_i$ of $G$, in which $v$ is the unique cut vertex belonging to the block $R_i$. An extreme vertex is an exterior extreme vertex if it is adjacent to only one cut vertex. Let $S = \{s_1, s_2, ..., s_\zeta\}$ be the set of support cut vertices of $G$ and let $\{s_{i_1}, s_{i_2}, ..., s_{i_l}\}$ be the set of exterior extreme vertices adjacent to $s_i \in S$. Also, let $Q = \{Q_1, Q_2, ..., Q_\vartheta\}$ be the set of blocks of $G$ which contain more than one cut vertex and more than one extreme vertex and let $\{q_{i_1}, q_{i_2}, ..., q_{i_l}\}$ be the set of extreme vertices belonging to $Q_i \in Q$. Now, let $\phi = \max_{1 \leq i \leq \zeta, 1 \leq j \leq \vartheta} \{l_i, t_j\}$. With the above notation we have the following result.

**Theorem 8.** For any generalized tree $G$,

$$pd(G) \leq \begin{cases} \zeta + \vartheta + \phi - 1, & \text{if } \phi \geq 3; \\ \zeta + \vartheta + 1, & \text{if } \phi \leq 2. \end{cases}$$

**Proof.** For each support cut vertex $s_i \in S$, let $A_i = \{s_i\}$ and for each block $Q_j \in Q$, let $B_j = \{q_{i_1}\}$. Let us suppose $\phi \geq 3$. For every $j \in \{2, ..., l_i\}$ we take $M_{ij} = \{s_{ij}\}$ and, if $l_i < \phi - 1$, then for every $j \in \{l_i+1, ..., \phi - 1\}$ we consider $M_{ij} = \emptyset$. Analogously, for every $j \in \{2, ..., l_i\}$ we take $N_{ij} = \{q_{ij}\}$ and, if $l_i < \phi - 1$, then for every $j \in \{l_i+1, ..., \phi - 1\}$ we consider $N_{ij} = \emptyset$. Now, let $C_j = \bigcup_{i=1}^{\max\{\zeta, \vartheta\}} (M_{ij} \cup N_{ij})$, with $j \in \{2, ..., \phi - 1\}$.

Let us prove that $\Pi = \{A, A_1, A_2, ..., A_\zeta, B_1, B_2, ..., B_\vartheta, C_2, C_3, ..., C_{\phi-1}\}$ is a resolving partition of $G$, where $A = V - \bigcup_{i=1}^{\zeta} A_i - \bigcup_{i=1}^{\vartheta} B_i - \bigcup_{i=2}^{\phi-1} C_i$. To begin with, let $x, y$ be two different vertices of $G$. We have the following cases.

Case 1: $x$ is a cut vertex or $y$ is a cut vertex. Let us suppose, for instance, $x$ is a cut vertex. So there exists an extreme vertex $s_{i_1}$ such that $x$ belongs to a shortest $y - s_{i_1}$ path or $y$ belongs to a shortest $x - s_{i_1}$ path. Hence, we have $d(x, A_i) = d(x, s_{i_1}) \neq d(y, s_{i_1}) = d(y, A_i)$.
Case 2: $x, y$ are extreme vertices. If $x, y$ belong to the same block of $G$, then $x, y$ belong to different sets of $\Pi$. On the contrary, if $x, y$ belong to different blocks in $G$, then let us suppose there exists an extreme vertex $c$ such that $d(x, c) \leq 1$ or $d(y, c) \leq 1$. We can suppose $c \in A_i$, for some $i \in \{1, \ldots, \zeta\}$, or $c \in B_j$, for some $j \in \{1, \ldots, \vartheta\}$. Without loss of generality, we suppose that $d(x, c) \leq 1$. Since $x$ and $y$ belong to different blocks of $G$, we have $d(y, c) > 1$. So we obtain either $d(x, A_i) = d(x, c) \leq 1 < d(y, c) = d(y, B_j)$ or $d(x, B_j) = d(x, c) \leq 1 < d(y, c) = d(y, A_i)$.

Now, if there exists no such a vertex $c$, then there exist two blocks $H, K \notin Q$ with $x \in H$ and $y \in K$, which contain more than one cut vertex and only one extreme vertex. So $x, y \in A$. Let $u \in H$ be a cut vertex such that $d(y, u) = \max_{v \in H} d(y, v)$. Hence, there exists an extreme vertex $s_i$ such that $u$ belongs to a shortest $x - s_i$ path and $d(y, s_i) = d(y, u) + d(u, s_i)$. As $x, y$ belong to different blocks and $d(y, u) = \max_{v \in H} d(y, v)$ we have $d(y, u) \geq 2$. Thus,

$$
\begin{align*}
 d(y, A_i) &= d(y, s_i) \\
 &= d(y, u) + d(u, s_i) \\
 &\geq 2 + d(u, s_i) \\
 &> 1 + d(u, s_i) \\
 &= d(x, u) + d(u, s_i) \\
 &= d(x, A_i).
\end{align*}
$$

Hence, we conclude that if $\phi \geq 3$, then for every $x, y \in V$, $r(x|\Pi) \neq r(y|\Pi)$. Therefore, $\Pi$ is a resolving partition.

On the other hand, if $\phi \leq 2$, then $\Pi' = \{A, A_1, A_2, \ldots, A_\zeta, B_1, B_2, \ldots, B_\vartheta\}$ is a partition of $V$. Proceeding as above we obtain that $\Pi'$ is a resolving partition. \qed

The above bound is achieved, for instance, for the graph in Figure 5, where $\zeta = 3$, $\vartheta = 0$ and $\phi = 3$. Also, notice that for the particular case of trees we have $\zeta = \xi$, $\phi = \theta$ and $\vartheta = 0$. So the above result leads to Corollary 2.

References

[1] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, *Mathematica Bohemica* 128 (1) (2003) 25–36.
[2] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of Cartesian product of graphs, *SIAM Journal of Discrete Mathematics* 21 (2) (2007) 273–302.

[3] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Mathematics* 22 (2005) 129–133.

[4] G. Chappell, J. Gimbel, C. Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Combinatoria* 88 (2008) 349–366.

[5] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Applied Mathematics* 105 (2000) 99–113.

[6] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, *Computers and Mathematics with Applications* 39 (2000) 19–28.

[7] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Mathematicae* (1-2) 59 (2000) 45–54.

[8] M. Fehr, S. Gosselin, O. R. Oellermann, The partition dimension of Cayley digraphs *Aequationes Mathematicae* 71 (2006) 1–18.

[9] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria* 2 (1976) 191–195.

[10] T. W. Haynes, M. Henning, J. Howard, Locating and total dominating sets in trees, *Discrete Applied Mathematics* 154 (2006) 1293–1300.

[11] B. L. Hulme, A. W. Shiver, P. J. Slater, A Boolean algebraic analysis of fire protection, *Algebraic and Combinatorial Methods in Operations Research* 95 (1984) 215–227.

[12] M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *Journal of Biopharmaceutical Statistics* 3 (1993) 203–236.

[13] M. A. Johnson, Browsable structure-activity datasets, *Advances in Molecular Similarity* (R. Carbó-Dorca and P. Mezey, eds.) JAI Press Connecticut (1998) 153–170.
[14] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Applied Mathematics* **70** (1996) 217–229.

[15] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision Graphics and Image Processing* **25** (1984) 113–121.

[16] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs: a survey, *International Journal of Mathematics and Mathematical Sciences* **38** (2004) 1997–2017.

[17] P. J. Slater, Leaves of trees, Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congressus Numerantium* **14** (1975) 549–559.

[18] I. Tomescu, Discrepancies between metric and partition dimension of a connected graph, *Discrete Mathematics* **308** (2008) 5026–5031.