NUMERICAL SOLUTION OF TIME FRACTIONAL TIME REGULARIZED LONG WAVE EQUATION BY ADOMIAN DECOMPOSITION METHOD AND APPLICATIONS

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Abstract

In the paper, we develop the Adomian Decomposition Method for the fractional-order nonlinear Time Regularized Long Wave Equation (TRLW) equation. Caputo fractional derivatives are used to define fractional derivatives. We know that nonlinear physical phenomena can be explained with the help of nonlinear evolution equations. Therefore solving TRLW is very helpful to obtain the solution of many physical theories. In this paper, we will solve the time-fractional TRLW equation which may help researchers with their work. We solve some examples numerically, which will show the efficiency and convenience of the Adomian Decomposition Method.

Keywords: Time Regularized Long Wave equation, Fractional derivative, Adomian Decomposition Method, Convergence, Mathematica.

I. Introduction

In the present scenario, fractional calculus is useful in the various fields of science. In the past few years, the increase of interest in the subject is witnessed by series of conferences, research papers and several monographs. The dynamic models of a large number of phenomena can be modeled by fractional order partial differential equations which are characterized by fractional space and/or time derivatives [XIII],[XXII],[XXIII],[XXXI],[XXXII]. Fractional calculus is very useful to model many physical phenomena in viscoelastic materials, continuum mechanics, statistical mechanics, economics, etc [XIX], [XXI], [XXII], [XXXI]. But, many times it is difficult to obtain exact solutions, hence numerical methods must be used. Nowadays, Adomian Decomposition Method (ADM) is used to obtain the solution of fractional differential equations [I], [II], [III], [IV]. This method gives rapidly convergent series solutions by using a few iterations for both linear and nonlinear equations. This method
is very useful to avoid linearization, perturbation, massive computation and transformations \[\text{[X]}\]. In the last two decades, extensive work has been done using Adomian decomposition method \[\text{[VI],[VII],[VIII],[IX],[XI],[XII],[XIV],[XVII],[XVIII],[XX],[XXVI],[XXVII],[XXIX],[XX]}\]. The nonlinear physical phenomenon can be explained with the help of nonlinear evolution equations. Therefore solving these equations is very important. Time Regularized Long Wave (TRLW) equation was given by well-known mathematicians Joseph, Egri and Jaffery. Therefore it is also named as Joseph, Egri and Jaffery equation. This equation is also one of the alternative forms of the KdV equation. The equation can be expressed as:

$$r_t + r_x + \lambda r r_x + r_{xxt} = 0$$  \hspace{1cm} (1)

Where \(r\), \(t\) and \(x\) denote the amplitude, time and spatial co-ordinate respectively and \(\lambda\) is nonzero constant \[\text{[XXIII]}\]. The first term is the evolution term, the third term is nonlinear and the fourth term is the dispersion term. The nonlinear term accounts for steepening of the wave and the dispersion term represents the spreading of the wave. TRLW equation explains a large number of physical phenomena such as shallow-water waves and ion-acoustic plasma waves. This equation is studied as an application of various fields such as fluid mechanics, electrodynamics, chemical physics, chemical kinematics, plasma physics, optical fibers, solid-state physics, biology etc. In the paper \[\text{[XXIII]}\] researchers used the \text{exp} \((-\phi(\xi))-\exp\) expansion method to solve the TRLW equation and obtained hyperbolic, trigonometric, exponential and rational function solutions. In \[\text{[XXXIII]}\], the extended Jacobian elliptic function expansion method is used to find the exact traveling wave solution of shallow water wave equations and modified Liouville equation. They obtain the solitary wave solution of the shallow water wave equation and modified Liouville equation. Many methods such as nonlinear transform method, first integration method, homogeneous balance method, simplified Hirota’s method, complex hyperbolic function method, perturbation method \[\text{[XII]}\] etc. are used to solve the TRLW equation. In this paper, we will be using Adomian Decomposition Method to solve time-fractional TRLW. We organize the paper as follows: We have given some formulae and theorem in Section 2, which are useful for further developments. Section 3, is devoted to ADM to solve the time-fractional TRLW equation and prove convergence. In section 4, numerical problems are solved and presented their solutions graphically by using Mathematica software.

**II. Basic Preliminaries and Properties of Fractional Derivatives**

Some basic concepts, which we will be using are as follows:-

**Definition 1.** The Caputo fractional derivative \[\text{[XX]}\] of the function \(f(x)\) is defined as

$$D^\alpha_x f(x) = J^{m-\mu} D^m f(x) = \frac{\Gamma(m-\beta)}{\Gamma(m-\beta)} \int_0^x \frac{f^{(m-\beta)}(t)}{(x-t)^{\beta}} dt,$$  \hspace{1cm} (2)

for \(-1 < \beta \leq m, m \in \mathbb{N}, x > 0, f \in C^m_{\beta+1}\).

**Properties:**

For \(f(x) \in C^\mu, \mu \geq -1, \alpha, \beta \geq 0 \text{ and } \gamma > -1\), we have

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(i) \( J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \),
(ii) \( J^\alpha f(x) = J^\beta J^\alpha f(x) \)
(iii) \( J^\alpha x^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{(\alpha+\gamma)} \).

Lemma 1. If \( m - 1 < \alpha \leq m \), \( m \in \mathbb{N} \) and \( f \in C^m_\mu, \mu \geq -1 \), then
\[
D_{\alpha} f(x) = f(x)
\]
\[
J^\alpha D_{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)} \left(0^+\right) \frac{x^k}{k!}, \quad x > 0.
\]

III. Fractional Adomian Decomposition Method (FADM)

We consider the following time-fractional TRLW equation to develop the time Fractional ADM [XX] for solving the TRLW equation,
\[
\begin{align*}
\frac{t^\alpha}{\Gamma(\alpha + 1)} r_x + r_x + \lambda r_x + r_{ttt} &= 0, \quad 0 < \alpha \leq 1, t > 0 \\
\text{initial condition : } r(x, 0) &= f(x)
\end{align*}
\]
We will operate \( J^\alpha \) on R.H.S. and L.H.S. of the equation,
\[
J^\alpha [r_t^\alpha + r_x + \lambda r_x + r_{xxxx}] = 0, \quad 0 < \alpha \leq 1, t > 0
\]
Now, consider the following decomposition series:
\[
r(x, t) = \sum_{n=0}^{\infty} r_n(x, t)
\]
The decomposed series of nonlinear terms \( N_r(x, t) \) are:
\[
r(x, t) = \sum_{n=0}^{\infty} A_n
\]
where the formula for Adomian polynomial is as follows:
\[
A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left( \sum_{k=0}^{n} \lambda^k r_k \right), \lambda = 0
\]
From (5) and using lemma (1), we get
\[
\sum_{k=0}^{\infty} r_n(x, t) = -J^\alpha \left[ \sum_{n=0}^{\infty} D_x r_n(x, t) + \sum_{n=0}^{\infty} D_{xxt} r_n(x, t) + \lambda \sum_{n=0}^{\infty} A_n \right], \quad x > 0
\]
The value of \( w_n(x, t), n \geq 0 \) can be obtained as follows:
\[
r_0(x, t) = r(x, 0) = f(x)
\]
\[
r_{n+1}(x, t) = -J^\alpha [D_x r_n(x, t) + D_{xxt} r_n(x, t) + \lambda A_n], \quad x > 0
\]
Now we can obtain solution by calculating value of each component.
\[
r_{n+1}(x, t) = -J^\alpha [D_x r_n(x, t) + D_{xxt} r_n(x, t) + \lambda A_n], \quad x > 0
\]
Now we can obtain solution by calculating value of each component

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\[ \phi_N(x,t) = \sum_{n=0}^{N-1} r_n(x,t) \quad (9) \]
\[ \lim_{N \to \infty} \phi_N = r(x,t) \quad (10) \]

**Theorem 1 Uniqueness Theorem [XXVIII]:**

Consider time-fractional TRLW equation for \( \lambda = 1 \), as follows:

\[ r_t^\alpha + r_x + \lambda r_{rx} + r_{xxt} = 0, \quad 0 < \alpha \leq 1, > 0 \quad (11) \]

Initial condition: \( r(x,0) = f(x) \)

The equation has a unique solution whenever \( 0 < \eta < 1 \) where \( \eta = \frac{(M_1 + M_2 + M_3)\Gamma\alpha}{\Gamma\alpha + 1} \)

**Proof:** Let \( X = (C(I), \| \cdot \|) \) be the Banach space of all continuous functions on \( I = [0,T] \) with norm

\[ \| r(t) \| = \max_{t \in I} | r(t) | \]

We define a mapping \( S : X \to X \), such that

\[ S(r(t)) = f(x) - J^\alpha N(r(t)) - J^\alpha F(r(t)) - J^\alpha P(r(t)) \]

Now, \( N(r(t)) \) denotes nonlinear term and \( F(r(t)) \) denotes first order spatial term and \( P(r(t)) \) denotes third order time - spatial term. Also nonlinear term \( N(r(t)) \) is Lipschitzian that is \( | N(r) - N(p) | \leq M_1 | r - p | \)

where \( M_1 \) is Lipschitz constant. Let \( r, r' \in X \), we have

\[ \| S(r) - S(r') \| = \max_{t \in I} | - J^\alpha N(r(t)) - J^\alpha F(r(t)) - P(r(t)) + J^\alpha N(r'(t)) + J^\alpha F(r'(t)) + J^\alpha P(r'(t)) | \]
\[ = \max_{t \in I} | J^\alpha (Nr - Nr') + J^\alpha (Fr - Fr') - J^\alpha (Pr - P r') | \]
\[ = \max_{t \in I} | J^\alpha (Nr - Nr') + J^\alpha (Fr - Fr') + (Pr - P r') | \]
\[ \leq \max_{t \in I} | J^\alpha (Nr - Nr') | + | J^\alpha (Fr - Fr') | + | J^\alpha (Pr - P r') | \]

Now suppose \( F(r(t)) \) and \( P(r(t)) \) are also Lipschitzian that is

\[ | F(r) - F(p) | \leq M_2 | r - p | \]

and

\[ | P(r) - P(p) | \leq M_3 | r - p | \]

where \( M_2 \) and \( M_3 \) are Lipschitz constants.

Therefore

\[ \| S(r) - S(r') \| \leq \max_{t \in I} (M_1 J^\alpha |r - r'| + M_2 J^\alpha |r - r'| + (M_3 J^\alpha |r - r'|) \]

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\[ \| S(r) - S(r') \| \leq \gamma \| r - r' \|, \text{where} \gamma = \frac{(M_1 + M_2 + M_3)t^\alpha}{\Gamma\alpha + 1} \]

Therefore, whenever \( 0 < \eta < 1 \), the mapping is contraction. Hence with the referen Banach fixed point theorem for contraction, we proved that the equation has unique solution.

**Theorem 2. Convergence Theorem [XXVIII]**

Let \( V_n \) be the \( n^{th} \) partial sum, that is

\[ V_n = \sum_{i=0}^{n} \eta (x, t) \quad (13) \]

Then we shall prove that \( \{ V_n \} \) is a Cauchy sequence in Banach space \( X \).

**Proof:** For proving this theorem, we consider

\[ \| V_{n+p} - V_n \| = \max_{t \epsilon I} | V_{n+p} - V_n | \]

\[ = \max_{t \epsilon I} \left| \sum_{i=n+1}^{n+p} r_{i-1}(x, t) \right| \]

\[ = \max_{t \epsilon I} \left| - J^\alpha \sum_{i=n+1}^{n+p} N r_{i-1} - J^\alpha \right| \]

\[ = \max_{t \epsilon I} \left| \frac{1}{n} \sum_{i=n+1}^{n+p} \right| \]

\[ = \max_{t \epsilon I} \left| J^\alpha \left( |NV_{n+p} - NV_{n-1}| \right) \right| \]

\[ + \max_{t \epsilon I} \left| J^\alpha \left( |PV_{n+p} - PV_{n-1}| \right) \right| \]

\[ \leq (M_1 + M_2 + M_3) \frac{t^\alpha}{\Gamma\alpha + 1} \| V_{n+p} - V_n \| \]

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\[ \| V_{n+p} - V_n \| \leq \eta \| V_{n+p+1} - V_{n-1} \|, \text{ where } \eta = (M_1 + M_2 + M_3) \frac{t^\alpha}{\Gamma(\alpha+1)} \]
\[ \| V_{n+p} - V_n \| \leq \eta \| V_{n+p-1} - V_{n-1} \| \]

Similarly, we have
\[ \| V_{n+p} - V_n \| \leq \eta^2 \| V_{n+p-2} - V_{n-2} \| \]
\[ \vdots \]
\[ \leq \eta^n \| V_p - V_0 \| \]
\[ \leq \eta^n \| V_1 - V_0 \|. \text{ for } p = 1 \]
\[ \leq \eta^n \| r_1 \| \]

Now, for \( n > m \), where \( n, m \in \mathbb{N} \),
\[ \| V_{n-m} \| \leq \| V_{m+1} - V_m \| + \| V_{m+2} - V_{m+1} \| + \cdots + \| V_n - V_{n-1} \| \]
\[ \leq (\eta^m + \eta^{m+1} + \cdots + \eta^{n-1}) \| r_1 \| \]
\[ \leq \eta^m \left[ \frac{1 - \gamma^{n-m}}{1 - \gamma} \right] \| r_1 \| \]

Since, \( 0 < \eta < 1 \), then \( 1 - \eta^n < 1 \), so we have,
\[ \| V_n - V_m \| \leq \frac{\eta^m}{1 - \eta} \| r_1 \| \]

Since, \( r(t) \) is bounded, therefore \( \| r_1 \| < \infty \)
\[ \lim_{n \to \infty} \| V_n - V_m \| \to 0 \]

Hence, we proved that solution is convergent because \( \{V_n\} \) is a Cauchy sequence in \( X \).

**IV. Numerical Examples**

**Example 1:** We will consider the following time-fractional TRLW equation

\[ r_t^\alpha + r_x - 6rr_x + r_{xxt} = 0, 0 < \alpha \leq 1, t > 0 \] \hspace{1cm} (14)
\[ \text{initial condition: } r(x, 0) = \frac{1}{6}(x - 1) \] \hspace{1cm} (15)

Now, using equations (7) and (8), we have
\[ r_0(x, t) = r(x, 0) = f(x) \]
\[ r_{k+1}(x, t) = -f^\alpha(D_xr_k)(x, t) + D_{xxt}r_k(x, t) - 6A_k, x > 0 \]
\[ r_0(x, t) = r(x, 0) = \frac{1}{6}(x - 1) \]

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\[ r_1(x, t) = -J^\alpha[D_x r_0(x, t) + D_{xxt} r_0(x, t) - 6A_0] \]

\[ A_0 = r_0 D_x(r_0) = \frac{1}{6^2}(x - 1) \]

\[ r_1(x, t) = -J^\alpha\left[\frac{1}{6} - \frac{1}{6}(x - 1)\right] \]

\[ r_1(x, t) = \frac{1}{6}(x - 1) \frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{1}{6} \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ r_1(x, t) = \frac{1}{6}(x - 2) \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ r_2(x, t) = -J^\alpha[D_x r_1(x, t) + D_{xxt} r_1(x, t) - 6A_1] \]

\[ A_1 = r_1 D_x(r_0) + r_0 D_x(r_1) = \frac{1}{6^2}(2x - 3) \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ r_2(x, t) = -2 \frac{6^3 t^{2\alpha}}{\Gamma(2\alpha + 1)} + 2 \frac{6^5 x t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{6^3 \alpha(\alpha - 1)\Gamma(\alpha - 1)t^{2\alpha-2}}{\Gamma(2\alpha - 1)\Gamma(\alpha + 1)} \]

Fig. 1: Graphical presentation of time fractional TRLW equation with \( \alpha = 0.9 \)

Example 2: We will consider the following time-fractional TRLW equation

\[ r_t^\alpha + r_x - 6rr_x + r_{xxt} = 0, \ 0 < \alpha \leq 1, \ t > 0 \] (16)

initial condition : \( r(x, 0) = e^{-x} \) (17)

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Now, using equations (7) and (8), we have
\[
\begin{align*}
\dot{r}_0(x, t) &= r(x, 0) = f(x) \\
\dot{r}_k(x, t) &= -J^\alpha [D_x \dot{r}_k(x, t) + D_{xtt} \dot{r}_k(x, t) - 6A_k], x > 0 \\
\dot{r}_0(x, t) &= r(x, 0) = e^{-x} \\
\dot{r}_1(x, t) &= -J^\alpha [D_x \dot{r}_0(x, t) + D_{xtt} \dot{r}_0(x, t) - 6A_0] \\
A_0 &= r_0D_x(r_0) = -e^{-2x} \\
\dot{r}_1(x, t) &= -J^\alpha [-e^{-x} + 6e^{-2x}] \\
\dot{r}_1(x, t) &= [e^{-x} - 6e^{-2x}] \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
\dot{r}_2(x, t) &= -J^\alpha [D_x \dot{r}_1(x, t) + D_{xtt} \dot{r}_1(x, t) - 6A_1] \\
A_1 &= \dot{r}_1D_x(\dot{r}_0) + r_0D_x(\dot{r}_1) = [-2e^{-2x} + 18e^{-3x}] \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
\dot{r}_2(x, t) &= [e^{-x} - 24e^{-2x} + 108e^{-3x}] \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&+ [e^{-x} - 12e^{-2x}] \frac{\alpha(\alpha - 1)\Gamma(\alpha - 1) t^{2\alpha - 2}}{\Gamma(2\alpha - 1)\Gamma(\alpha + 1)}
\end{align*}
\]

After calculating and substituting values of various components, we have
\[
\begin{align*}
n(x, t) &= \dot{r}_0(x, t) + \dot{r}_1(x, t) + \cdots \\
n(x, t) &= e^{-x} + [e^{-x} - 6e^{-2x}] \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
&+ [e^{-x} - 24e^{-2x} + 108e^{-3x}] \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
&+ [e^{-x} - 12e^{-2x}] \frac{\alpha(\alpha - 1)\Gamma(\alpha - 1) t^{2\alpha - 2}}{\Gamma(2\alpha - 1)\Gamma(\alpha + 1)} \cdots
\end{align*}
\]

Fig. 2: Graphical presentation of time fractional TRLW equation with \( \alpha = 0.9 \)
Example 3: We will consider the following time-fractional TRLW equation

\[ r_t^\alpha + r_x - 6rr_x = 0, \quad 0 < \alpha \leq 1,1 > 0 \]

initial condition : \( r(x, 0) = \sin x \)

Now, using equations (7) and (8), we have

\[ r_0(x, t) = r(x, 0) = f(x) \]

\[ r_{k+1}(x, t) = -J^\alpha[D_xr_k(x, t) + D_xttr_k(x, t) - 6A_k], \quad x > 0 \]

\[ r_0(x, t) = r(x, 0) = \sin x \]

\[ r_1(x, t) = -J^\alpha[D_xr_0(x, t) + D_xttr_0(x, t) - 6A_0] \]

\[ A_0 = r_0D_x(r_0) = \frac{\sin 2x}{2} \]

\[ r_1(x, t) = -J^\alpha[\cos x + 3\sin 2x] \]

\[ r_1(x, t) = [\cos x + 3\sin 2x] \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ r_2(x, t) = -J^\alpha[D_xr_1(x, t) + D_xttr_1(x, t) - 6A_1] \]

\[ A_1 = r_1D_x(r_0) + r_0D_x(r_1) \]

\[ = [\cos 2x + 3\sin 2x \cos x + 6\cos 2x \sin x] \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ r_2(x, t) = [-\sin x + 12\cos 2x + 18\sin 2x \cos x + 36\cos 2x \sin x] \frac{t^\alpha}{\Gamma(2\alpha + 1)} \]

\[ + [\sin x - 6\cos 2x] \frac{\alpha(\alpha - 1)\Gamma(\alpha - 1)t^{2\alpha - 2}}{\Gamma(2\alpha - 1)\Gamma(\alpha + 1)} \]

After calculating and substituting values of various components, we have

\[ r(x, t) = r_0(x, t) + r_1(x, t) + \cdots \]

\[ r(x, t) = \sin x + [\cos x + 3\sin 2x] \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

\[ + [-\sin x + 12\cos 2x + 18\sin 2x \cos x + 36\cos 2x \sin x] \frac{t^\alpha}{\Gamma(2\alpha + 1)} \]

\[ + [\sin x - 6\cos 2x] \frac{\alpha(\alpha - 1)\Gamma(\alpha - 1)t^{2\alpha - 2}}{\Gamma(2\alpha - 1)\Gamma(\alpha + 1)} + \cdots \]
Example 4: We will consider the following time-fractional TRLW equation

\[ r_t^\alpha + r_x - 6rr_x + r_{xxt} = 0, \quad 0 < \alpha \leq 1, \quad t > 0 \]  

*initial condition:* \( r(x, 0) = 3 \text{sech}^2 x \)

Now, using equations (7) and (8), we have

\[ r_0(x, t) = r(x, 0) = f(x) \]

\[ r_{k+1}(x, t) = -r^\alpha[D_xr_k(x, t) + D_{xxt}r_k(x, t) - 6A_k], \quad x > 0 \]

\[ r_0(x, t) = r(x, 0) = 3 \text{sech}^2 x \]

\[ r_1(x, t) = r^\alpha[D_xr_0(x, t) + D_{xxt}r_0(x, t) - 6A_0] \]

\[ A_0 = r_0D_x(r_0) = -18 \text{sech}^4 xtanhx \]

\[ r_1(x, t) = -r^\alpha[6 \text{sech}^2 xtanhx - 108 \text{sech}^4 xtanhx] \]

\[ r_1(x, t) = [6 \text{sech}^2 xtanhx - 108 \text{sech}^4 xtanhx] \frac{t^\alpha}{\Gamma(\alpha + 1)} \]

After calculating and substituting values of various components, we have

\[ r(x, t) = r_0(x, t) + r_1(x, t) + \ldots \]

\[ r_1(x, t) = 3 \text{sech}^2 x + [6 \text{sech}^2 xtanhx - 108 \text{sech}^4 xtanhx] \frac{t^\alpha}{\Gamma(\alpha + 1)} + \ldots \]
V. Conclusions:

The time-fractional TRLW equation is solved by using ADM and we can say the formula of ADM polynomials is powerful to obtain the solution of nonlinear fractional partial differential equation. The graphical presentation of solutions of time-fractional TRLW equation reveals the reliability of the mathematical procedure. We also prove the uniqueness and convergence theorem for the time-fractional TRLW equation.

Conflict of Interest:

Authors declared : No conflict of interest regarding this article

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