We derive the optical depth and photospheric radius of relativistic outflows using a model of a relativistic wind with a finite duration. We also discuss the role of radiative diffusion in such an outflow. We numerically solve the radiative transfer equation and obtain light curves and observed spectra of the photospheric emission. The spectra we obtain are nonthermal and in some cases have Band shapes.

**Key words:** opacity – plasmas – radiation mechanisms: thermal – X-rays: bursts

**Online-only material:** color figures

## 1. INTRODUCTION

Bulk motions of various outflows marked by high Lorentz factors are common in relativistic astrophysics. The best known examples are active galactic nuclei (see Maraschi 2003, and references therein), microquasars (Mirabel & Rodríguez 1999; Eikenberry et al. 2001), and gamma-ray bursts (GRBs; see, e.g., Piran 2004, and references therein). In the last case, outflows indeed reach ultra-relativistic velocities.

The fireshell model of GRBs (see, e.g., the review presented in Ruffini et al. 2009, and references therein) assumes that GRBs originate from an almost-instantaneous creation of an electron–positron plasma in a vacuum polarization process occurring during the formation of a black hole (Damour & Ruffini 1975; Preparata et al. 1998). For an earlier treatment of the instantaneous release of energy $E_0$ in a region of radius $R_0$ by an unspecified mechanism, see also Goodman (1986) and Shemi & Piran (1990). The optically thick electron–positron plasma formed in the source expands and reaches ultra-relativistic bulk motion velocities. This relativistically expanding plasma appears to a distant observer as a thin shell with a constant width $l = R_0$ measured in the laboratory frame (Ruffini et al. 1999). This result has been obtained previously assuming an instantaneous energy release (Piran et al. 1993). The interaction of the expanding plasma with a baryonic remnant of mass $M$ loads the plasma with baryons but does not change the width of the shell if the baryonic loading parameter $B = M
u^2/E_0$ satisfies the inequality $B < 10^{-2}$, where $\nu$ is the speed of light (Ruffini et al. 2000). When the relativistically expanding plasma becomes transparent to its own radiation, photons start to propagate freely toward the observer (Bianco et al. 2001). This first electromagnetic signal from the GRB is called the proper GRB (Ruffini et al. 2009, and references therein) assumes that the bulk Lorentz factor $\Gamma \gg 1$ (Paczynski 1986, 1990). This solution can be divided into two phases: accelerating and coasting. In the accelerating phase,

$$\Gamma \simeq r/R_0, \quad R_0 < r < \eta R_0, \quad (1)$$

while the coasting phase

$$\Gamma \simeq \eta = \text{const}, \quad r > \eta R_0, \quad (2)$$

where $r$ is the radial coordinate and $\eta = L/(\dot{M} c^2)$ is dimensionless entropy (Mészáros & Rees 2000). The continuity equation implies for a laboratory density of baryons

$$n = n_c \Gamma \simeq n_0 \left( \frac{r}{R_0} \right)^{-2}, \quad (3)$$

where $n_c$ is the comoving number density of baryons and $n_0$ is its value at the base of the wind.

The solutions to the relativistic hydrodynamic Equations (1)–(3) are also valid in the thin shell model of the fireball (Piran 1999; Mészáros 2006), provided that $\Gamma \gg 1$. In the case of the thin shell $l \simeq R_0$, $r$ should be interpreted as the radial position of the shell, which satisfies the equations of motion (1) and (2).

These equations are also valid as asymptotic regimes in the fireshell model. It should be noted, however, that the dynamics of the fireshell model are much richer and more complex and predict at least four different dynamical regimes (see Ruffini et al. 2000 for details): (1) the self-acceleration of the pure $e^+e^-$ plasma in a vacuum up to relativistic velocities, (2) the engulfment of the baryonic remnant, (3) the further self-acceleration of the newly formed $e^+e^-$-baryon plasma, and (4) the coasting phase. All these dynamical regimes are tightly interconnected; any attempt to make a theoretical model and interpret the observational data with a piecewise analysis of single regimes, without taking into due account the entire history of the system, is neither consistent nor viable (see, e.g.,
Ruffini et al. 2001b). Moreover, the dynamical equations of the fireshell model are not only the energy–momentum and baryon number conservation equations which asymptotically result in Equations (1)–(3); see, e.g., Ruffini & Vereshchagin (2013). The rate equation for electron–positron pairs is also added to this set of equations for self-consistency (Ruffini et al. 1999, 2000). The difference in dynamics of the relativistic shells in the fireball and fireshell models is discussed in detail in Bianco et al. (2006).

In the following, we present a treatment of the photospheric emission from relativistic outflows under the assumption that the dynamics of the outflow are given by either Equation (1) or (2). In this sense, the results we obtain are valid in limiting cases. These results are not directly applicable to GRB models such as the fireshell model where the complete dynamics of the relativistic shell are considered. We consider a spherically symmetric ultra-relativistic outflow with \( \Gamma \gg 1 \) as a wind with a finite duration \( \Delta t \) (Daigne & Mochkovitch 2002). The width of such an outflow is given by \( l \simeq c \Delta t \). When \( \Delta t = R_0/c \), the thin shell model is recovered with \( B = \frac{\eta}{c}, E_0 = L\Delta t, \) and \( M = M_0 \). For \( \Delta t \gg R_0/c \), the steady wind model is obtained.

The structure of the paper is as follows. In Section 2, we compute the optical depth and photospheric radius of the relativistic outflow. In Section 3, we discuss radiation diffusion in an expanding relativistic outflow. Observed light curves and spectra are computed in Sections 4 and 5. In Section 6, we discuss the main results obtained in this work and their implications for GRBs. Conclusions follow. Details of the derivations and applications of the radiative diffusion approximation are given in the Appendix.

2. OPTICAL DEPTH AND PHOTOSPHERIC RADIUS

The optical depth along the light-like worldline \( \mathcal{L} \) is defined as (Ehlers 1973)

\[
\tau = \int_\mathcal{L} \sigma j_\nu dx^\mu, \quad (4)
\]

where \( \sigma \) is the cross section, \( j_\nu \) is the 4-current of particles, and \( dx^\mu \) is the element of the worldline.

Consider the light-like worldline directed outward starting at time \( t \) at the interior boundary \( r = R \) of the outflow. The optical depth given by Equation (4) is then (see, e.g., Abramowicz et al. 1991)

\[
\tau = \int_{R}^{R+\Delta R} \frac{\sigma n (1-\beta \cos \theta)}{\cos \theta} \frac{dr}{\cos \theta}, \quad (5)
\]

where \( \beta \simeq 1 - 1/(2\Gamma^2) \), \( r \) is used as a parameter along the worldline, \( R+\Delta R \) is the radial coordinate at which the worldline crosses the outer boundary of the outflow, \( \theta \) is the angle between the worldline and the velocity vector of the outflow, and \( n \) is the laboratory number density of electrons.\(^3\) The quantity \( \Delta R \) is found from the equation of motion of the outflow. In particular, for an outflow expanding with a constant velocity (coasting phase), one has from Equation (2) \( r(t) = \beta ct + l \) for its outer boundary. For a radially directed light–light worldline, one has \( r(t) = ct \). Equating these expressions, we find that \( ct = \Delta R = \sqrt{l/(1-\beta)} \simeq 2\Gamma^2l \). The geometry of the outflow and the variables used here are illustrated in Figure 1.

When the contribution of positrons to Equation (5) can be neglected, the density of electrons is equal to the density of baryons given by Equation (3). Hence, using Equations (1) and (2) and assuming \( \sigma = \text{const} \), three asymptotic expressions for the optical depth along the line of sight are recovered from Equation (5): one for the accelerating phase and two for the coasting phase:

\[
\tau = \left\{ \begin{array}{ll}
\frac{1}{6} \frac{t_0}{R_0} \left( \frac{R_0}{R} \right)^3, & R_0 \ll R \ll \eta R_0, \\
\frac{1}{2\eta^2} \frac{t_0}{R_0} \left( \frac{R_0}{R} \right), & \eta R_0 \ll R \ll 2\eta^2 l, \\
\frac{t_0}{R_0}, & R \gg 2\eta^2 l,
\end{array} \right. \quad (6a, b, c)
\]

where

\[
\tau_0 = \sigma n_0 R_0 = \frac{\sigma E_0}{4\pi m_p c^2 R_0 \eta} = \frac{\sigma L}{4\pi m_p c^2 R_0 \eta}, \quad (7)
\]

and \( m_p \) is the proton mass.

The photospheric radius \( R_{\text{ph}} \) is defined by equating expression (6) to unity

\[
R_{\text{ph}} = \left\{ \begin{array}{ll}
R_0 \left( \frac{t_0}{6} \right)^{1/3}, & t_0 \ll \eta^3, \\
R_0 \frac{t_0}{2\eta}, & \eta^3 \ll t_0 \ll \eta^4 l/R_0, \\
(t_0 R_0 l)^{1/2}, & t_0 \gg \eta^4 l/R_0.
\end{array} \right. \quad (8)
\]

It is worth noting that the lower limit in the integral in Equation (5) corresponds to \( R_{\text{ph}} \).

Let us discuss for completeness the case when the number density of positrons exceeds the number density of baryons. Note that a pure electron–positron plasma reaches thermal equilibrium before expansion (Aksenov et al. 2007, 2010) and it keeps accelerating until it becomes transparent to radiation. Due to the exponential dependence of the thermal pair density on the comoving temperature \( T_c \), and hence on the radial coordinate, transparency is reached, rather independent of initial conditions, at

\[
kT_c^{\pm} \simeq 0.04 m_e c^2
\]

where \( k \) is the Boltzmann constant and \( m_e \) is the electron mass; both are independent of initial conditions. Given an initial temperature \( T_0 = (16\pi \sigma_{\text{SB}} R_0^2/L)^{-1/4} \), where \( \sigma_{\text{SB}} \) is the Stefan-Boltzmann constant, and an adiabaticity of expansion given by \( T_c = T_0 R_0/r \) (see, e.g., Ruffini & Vereshchagin 2013), we find

\[
R_{\text{ph}}^{\pm} = \frac{1}{T_c^{\pm}} \left( \frac{LR_0^2}{16\pi \sigma_{\text{SB}}} \right)^{1/4}. \quad (10)
\]
For typical parameters of GRBs, we have $R_{ph}^2 < 10^3 R_0$. Note that the optical depth for the expanding electron–positron–photon shell computed by Shemi & Piran (1990) is incorrect since it uses the asymptotic expression Equation (6c), which never apply to the pure $e^+ e^−$ outflows. The correct result (Equation (9)) can be found in, for example, Grimus & Wasserman (1998), Li & Sari (2008), and Toma et al. (2011).

Let us interpret Equation (6). On the one hand, Equations (6a) and (6b) imply that the radial distance $ΔR$ is sufficiently large so that the number density substantially decreases between $R$ and $R + ΔR$. In this respect, the outflow is “a thick wind,” even if the laboratory thickness of the outflow is small, $l \ll R$. We refer to this case as a photon-thick outflow. On the other hand, Equation (6c) implies that the number density of the outflow does not change substantially along the radial segment $ΔR$. In this respect, the outflow is “a thin shell” even if the duration may be long, $Δt \gg R_0/c$. We refer to this latter case as a photon-thin outflow. In particular, a geometrically thin ultra-relativistically expanding shell may be both photon thin or photon thick.

A similar consideration may be applied to a part of the outflow with thickness $ξ$, see Figure 1. It is clear then that even in a photon-thick outflow, there is always a photon-thin layer located near the outer boundary. During the acceleration phase, such a photon-thin part accounts for a fraction no larger than $Γ^{-1}$ of the entire width of the outflow.4

Note that within the relativistic wind model, the asymptotic expressions (6a) and (6b) for the optical depth were obtained by Mészáros & Rees (2000). Their formulae agree with ours up to a numerical factor that comes from integration over the time. Expressions (6b) and (6c) were obtained by Daigne & Mochkovitch (2002), who did not consider an accelerating phase and ignored the asymptotic expression Equation (6a).

The optical depth within the relativistic wind model was computed by Shemi & Piran (1990), Mészáros et al. (1993), and Ruffini et al. (1999, 2000). In all these works, only the asymptotic expression (6c) is used. It is possible, however, that initial conditions satisfy the inequality $τ_0 \ll 4η^2$ which for the coating phase results in the asymptotic expression Equation (6b), as used in Nakar et al. (2005).

All asymptotic solutions for the optical depth (Equation (6)) have been considered by Mészáros et al. (2002) and Toma et al. (2011).

3. PHOTON DIFFUSION

In the previous discussion, we did not consider the diffusion of radiation out of the outflow by photon random walks. The effect of radiative diffusion in GRBs was discussed for the first time by Preparata et al. (2000), who determined the energy flux from an optically thick plasma considering the temperature gradient near its boundary. The diffusion crossing time over which photon is expected to cross an outflow with a comoving radial thickness of $l_c = Γ l$ is $t_{D,c} = l^2_c/D_c$. The diffusion coefficient is $D_c = (c/3\sigma_{nc})$, where $\lambda_c$ and $n_c$ are the comoving mean free path of photons and the comoving electron number density, respectively.

One may solve for the radial position $R_D$ at which the outflow arrives by a time $t_{D,c}$. At this time, measured from the beginning of expansion, photons actually cross the entire width of the outflow by diffusion. Neglecting the initial brief acceleration phase, when diffusion is irrelevant, we can use the equation of motion of the outflow $R = βct \simeq T_c t$, where $t_c$ is time measured in the comoving frame, to obtain

$$R_D = (τ_0 η^2 R_{ph}^2)^{1/3}. \quad (11)$$

This equation takes into account Equations (2) and (7).

This diffusion radius should be compared with the photospheric radius (Equation (8)). It follows that the photospheric radius of the photon-thick (thin) outflow is smaller (larger) than its diffusion radius $R_{ph} \ll R_D (R_{ph} \gg R_D)$. This result means that diffusion is irrelevant for the description of the photon-thick outflow. It also implies that in photon-thin outflows, radiation decouples not at the photospheric radius, as defined by Equation (8), but rather at the diffusion radius (when the expanding plasma is still opaque according to Equation (6)). In that case, photon decoupling occurs at the photosphere also, which is located near the outflow boundaries. Photons are transferred to this photosphere by diffusion. In this sense, the characteristic radius of the photospheric emission from the photon-thin outflows is not the photospheric radius (Equation (8)), but rather the radius of diffusion given by Equation (11). In what follows, we consider the decoupling of photons from the photon-thick and photon-thin outflows separately.

4. PHOTOSPHERIC EMISSION FROM PHOTON-THICK OUTFLOWS

4.1. Geometry and Dynamics of the Photosphere

The photosphere of an accelerating steady wind is shown in Figure 2. This surface has a concave shape. The photosphere of
the coasting steady relativistic wind with $\Gamma = \text{const}$ is analyzed by Abramowicz et al. (1991). The form of this surface, which also has a concave shape, is (Pe’er 2008):

$$\frac{r}{R_0} = \tau_0 \left( \frac{\theta}{\sin \theta} - \beta \right);$$

(12)

see Figure 3. The curvature of this surface is smaller than that of the accelerating steady wind. In both cases, these photospheres appear, for a distant observer, as static spots with radius

$$\rho = \pi \tau_0 R_0$$

(13)

and a temperature decreasing from the center to the edge.

Now consider the dynamic properties of the photosphere of a photon-thick outflow as seen by a distant observer. The arrival time of the radiation is defined as

$$t_a = t - r \cos \theta/c;$$

(14)

see, e.g., Bianco et al. (2001), Ruffini et al. (2001a). Therefore, photons emitted at the same time but from points with different $\theta$ are detected at different arrival times. On the other hand, photons detected with the same arrival time were emitted at different times from points with different $\theta$ values. The “equtemporal surfaces” (EQTS; see, e.g., Bianco et al. 2001, 2011; Bianco & Ruffini 2004, 2005a, 2005b) are the surface locus of points emitting photons with the same arrival time $t_a$ (i.e., they are the shapes which we would see at each fixed value of the arrival time $t_a$ if we were able to spatially resolve the emission coming from the outflow). In the following, we will use the acronym PhE (“photospheric EQTS”) to refer to the EQTS of the photospheric emission we are discussing.

In this section, we consider two different cases. In the accelerating photon-thick case, the photospheric radius is reached at the acceleration phase. In the coasting photon-thick case, the photospheric radius is reached at the coasting phase. Clearly, the coasting photon-thick outflow had a previous acceleration episode that is not considered here.

The dynamic PhE of the photon-thick outflow initially have convex shapes. The PhE progressively become concave, however; see Figures 2 and 3 for the accelerating and coasting cases, respectively. The PhE in both cases approach the photosphere of the steady wind. In the coasting case, the approach to that surface is only asymptotic, while in the accelerating case the photosphere actually reaches it at a finite arrival time. The external boundary of the PhE for a given $t_a$ shown in Figure 3 is defined by the condition that the optical depth for photons emitted from the outermost layer of the outflow equals unity. Note that this boundary is wider than the relativistic beaming surface (these are tube and cone for accelerating and coasting outflows, respectively). As soon as the inner boundary of the outflow reaches the photospheric radius, the inner boundary of the PhE expands with $t_a$, and the observer sees the wind switching off.

4.2. Observed Flux and Spectrum

The basis for calculating spectra and fluxes is the radiative transfer equation for a specific intensity $I_\nu$ along a ray (see, e.g., Rybicki & Lightman 1979, p. 11)

$$\frac{dI_\nu}{ds} = j_\nu - \kappa_\nu I_\nu,$$

(15)

where $j_\nu$ is the monochromatic emission coefficient for the frequency $\nu$, $\kappa_\nu$ is the absorption coefficient, and $s$ is the distance, measured along the ray. We refer the reader to Figure 1.

The spectral intensity of radiation at infinity on a ray approaching the observer at some arrival time $t_a$ is given by the formal solution (Beloborodov 2011)

$$I_\nu(v, \rho, t_a) = \int I_\nu(v, r, \theta, t) \frac{d}{ds} \{\exp[-\tau(v, r, \theta, t)]\} \, ds$$

$$= \int I_\nu(v, r, \theta, t) \exp[-\tau(v, r, \theta, t)] \, d\tau,$$

(16)

where $I_\nu(r, \theta, t)$ is the source function equal to the ratio of the emission and absorption coefficients $I_\nu = j_\nu/\kappa_\nu$. The optical depth $\tau$ is given by

$$\tau = \int_s^\infty \kappa_\nu ds,$$

(17)

consistent with Equation (4). The variables $(r, \theta, t)$ are connected by $t_a = t - (r/c) \cos \theta$, and $\rho = r \sin \theta$; see Figure 1.

The total observed flux is an integral over all rays:

$$F_\nu(v, t_a) = 2\pi \Delta \Omega \int I_\nu(v, \rho, t_a) \rho \, d\rho,$$

(18)

where $\Delta \Omega$ is the solid angle of the observer’s detector as seen from the outflow in the laboratory frame and $2\pi \rho d\rho$ is an element of area in the plane of the sky.

We assume that the emissivity $j_\nu$ is thermal and isotropic in the comoving frame of the outflow and comoving $\kappa_{\nu, c} = \text{const}$. The laboratory source function is then

$$I_\nu(v, r, \theta, t) = \frac{2\hbar}{c^2} \frac{v^3}{\exp \left( \frac{k_0 T_c (1 - \beta \cos \theta)}{k_{\nu, c} r(v, r)} \right) - 1},$$

(19)

where $\hbar$ is the Planck constant. This approximation is justified when the radiation field is tightly coupled to the matter. The photospheric emission comes from the entire volume of the
outflow (see also Bianco et al. 2001) and the computational method sketched above is closely related to that used by Beloborodov (2011). This latter author introduced the concept of the “fuzzy photosphere.” Our method will then be referred to as the fuzzy photosphere approximation.

Most of energy reaching the observer is emitted from the region near the PhE, where the probability density function along the ray is maximized:

$$P(r, \theta, t) = P_0 \frac{d}{ds} \exp[-\tau(r, \theta, t)].$$

$P_0$ is a normalizing constant. For this reason, the dynamics of PhE studied in the previous section determine both the light curves and the spectra of the observed photospheric emission. When the time dependence in this equation is discarded $P(r, \theta)$ becomes the probability density function of the last scattering defined in Pe’er (2008). Assuming that all the energy comes from the PhE only, i.e., a surface instead of the volume discussed above, the computation may be reduced to one-dimensional integration by the substitution of the function $P$ with a Dirac delta function. Such a crude approximation, in contrast to the fuzzy photosphere approximation, will be referred to as the sharp photosphere approximation.

For a photon-thick outflow, the optical depth becomes a function of $r$ and $\theta$ only and the comoving temperature also depends only on radius. In this respect, the photon-thick case is similar to the steady wind. Then, the integrand in Equation (18) does not depend on time and only the limits of integration provide a time dependence due to the motion of the outflow boundaries. An interesting consequence of the ultra-relativistic motion of the outflow is that even a geometrically thin outflow is that even a geometrically thin outflow.

$$\delta t = \frac{R_0^2}{(R_{\text{ph}} c)^2} = \frac{R_0}{\Gamma_{\text{ph}} c},$$

where $\Gamma_{\text{ph}}$ is the Lorentz factor at the photospheric radius $R_{\text{ph}}$. There is no simple analytic expression describing the full light curve though; however, its decreasing part is close to a power law with index $-4.7$ and $-6.5$ within the fuzzy and sharp photosphere approximations, respectively. As the minimum duration of the photon-thick outflow $\Delta t$ is of order $R_0/c$, $\Delta t \gg \delta t$ and the light curve has an almost rectangular shape.

Such an accelerating outflow appears to a distant observer as a spot with size $\rho = (R_0^2 - (t_a/c)^2)^{1/2}$, for $-R_0/c \leq t_a \leq 0$. As soon as the PhE reaches the corresponding accelerating steady wind photosphere at $t_a = 0$, the spot size starts to increase almost linearly with time as $\rho \simeq R_0 + c t_a$. Finally, as the innermost part of the outflow reaches the photospheric radius, the spot transforms to a ring with rapidly decreasing width and brightness; see Figure 2.

The observed photospheric emission of the coasting photon-thick outflow results in the flux changing as

$$F = F_{\text{max}}[1 - (t_p/t_a)^2],$$

with

$$t_p = \frac{R_{\text{ph}}}{2 \Gamma_{\text{ph}} c}.$$

Up to the saturation value $F_{\text{max}} \propto L$, observers see the rising part of the light curve in Figure 5, both in the sharp and the fuzzy photosphere approximations. The radius of the visible spot then reaches its maximum size (Equation (13)). As the arrival time exceeds $t_p + \Delta t$, the innermost part of the outflow approaches the wind photosphere, (Equation (12)), along the line of sight. The spot transforms to a ring and the flux decreases rapidly in both approximations,

$$F \propto t_p^{-3} \left[ \frac{1}{(t_a - \Delta t)^2} - \frac{1}{t_a^2} \right].$$

For $t_a \gg \Delta t$, it behaves as $F \propto t_a^{-3}$; see the decreasing part of the light curve in Figure 5. Similar to the accelerating photon-thick outflow, the light curve for $\Delta t \gg t_p$ has an almost rectangular shape due to the fact that its increase and decay times are much shorter than $\Delta t$; see Figure 3.

Accelerating photon-thick outflows exhibit photospheric spectra close to thermal ones; see Figure 6. In the ultra-relativistic case, spectra computed using both the sharp and
fuzzy photosphere approximations are very similar to each other. Both have small deviations from a thermal spectrum in the low-energy part with corresponding Band low energy indices of $\alpha = 0.82$ and $\alpha = 0.71$, respectively.

In contrast, the spectrum of photospheric emission for the coasting photon-thick outflow is significantly wider than the thermal spectrum; see Figure 7. This spectrum is composed of various Planck spectra (multicolor black body of Pe’er & Ryde 2011), as illustrated by Figure 3. The low-energy part is described by a power law with Band indices $\alpha = 0.34$ and $\alpha = 0$, respectively, for the sharp and fuzzy photosphere approximations.

After the initial phase of evolution, namely the rising of the low-energy part, the spectra do not evolve until the observer detects emission from the innermost part of the outflow. At that time, there is a transition to another phase characterized by the fast decrease of both temperature and flux. Considering a time-integrated spectrum, we find that the characteristic times of the first and third phases are much shorter than those of the second phase. The spectrum is close to the instantaneous one described above.

4.3. Comparison with Previous Work

Now we compare our results with ones obtained using other methods. Beloborodov (2011) considered the photospheric emission from a steady wind in both the acceleration and coasting phases. He solved the corresponding steady radiative transfer equation. His main conclusion was that in addition to usual relativistic beaming leading to anisotropy of radiation in the laboratory frame, in the coasting wind another anisotropy in the comoving frame of the outflow is developing. This comoving anisotropy results from the fraction of photons that already underwent their last scattering in the bulk photon field of the outflow. The anisotropy of such photons grows with increasing radius for geometrical reasons. Since the amount of such photons increases with radius, the entire photon field becomes increasingly anisotropic.

For the finite photon-thick outflow, the radiative transfer problem becomes explicitly time dependent. The expanding outflow at a given laboratory time spans only a finite part of the probability density distribution shown in Figures 2 and 3. As such, differences are observed in the spectra for the finite and infinite cases.

The effect of additional comoving anisotropy on the source function found by Beloborodov (2011) is actually small. The difference between the probability of last scattering given by Equation (20) and the corresponding function in a steady wind does not exceed several percent.

Our method is also similar to the one used by Pe’er (2008) and Pe’er & Ryde (2011). These authors described the late-time photospheric emission of switching off a relativistic wind considering the probability density function for the last scattering of photons (Equation (20)). An additional approximation adopted by these authors is the possibility to split radial and angular dependencies. Actually, Pe’er (2008) does not compute the traditional energy flux understood as the energy crossing a unit area in a unit time, but rather the photon flux as the number of photons crossing a unit area in a unit time. For this reason, his decay law for the photon flux at late times is $F^{\text{out}}(t_{\text{w}}) \propto E_{\text{w}}^{-3}$. A Lorentz transformation of the photon energy from the comoving frame to the laboratory one results in an additional multiplier $(1 - \beta \cos \theta)^{-1}$ in the energy flux that leads to the observed flux $F \propto E_{\text{w}}^{-3}$. This result agrees with our result in Equation (24); see also Pe’er & Ryde (2011).

We conclude that the fuzzy photosphere approximation in fact closely follows the methods of Pe’er & Ryde (2011) and Beloborodov (2011). In fact, we obtained similar results for the probability of last scattering as did more sophisticated treatments of radiative transfer (Beloborodov 2011). The sharp photosphere approximation provides a good description of light curves, including their rising and decaying parts. The observed spectrum from an accelerating outflow is also well described in this approximation, while there are some differences for the coasting case. The advantage of the sharp photosphere approximation for computing observed light curves and spectra is evident for intrinsically variable and dynamic outflows. The relations between our results and the ones of Pe’er & Ryde (2011) and Beloborodov (2011), discussed in Sections 4.2 and 4.3, are currently under further consideration.

5. PHOTOSPHERIC EMISSION FROM PHOTON-THIN OUTFLOWS

Now we turn to photon-thin outflows. In Section 3, we pointed out that most of the radiation leaves the outflow not at its photospheric radius, but earlier, before the diffusion radius. Given that the opacity of the outflow is still large, the emission escapes only from a very narrow region near the outer boundary...
of the outflow. The probability density function, Equation (20), is strongly peaked there and the photospheric emission for a given arrival time originates from this narrow region. The sharp photospheric approximation is thus completely justified in this case.

Since all the radiation is emitted from the PhE, we briefly discuss their geometry and dynamics. The PhE of the photon-thin outflow are similar to EQTS of infinitesimally thin, constantly-emitting relativistic shells first considered by Couderc (1939) and then by Rees (1966, 1967). The EQTS of these shells appear to a distant observer as ellipsoids with axes ratios equal to $\Gamma$. However, the PhE of photon-thin outflow are not the entire ellipsoid: they are only a part of that surface, limited by the beaming cone; see Figure 8. The external boundary of the PhE for a given $t_{\text{in}}$ is defined by the condition that photons emitted from the outermost layer of the outflow toward observer leave the outflow. In the photon-thin asymptotics, this surface coincides with the relativistic beaming cone.

We again start with the radiative transfer described by Equation (15). In contrast with the photon-thick case, here the source function $I$ in Equation (16) strongly depends on both $r$ and $t$. The main process by which photons are coupled to the matter is Compton scattering, which conserves the number of photons. Since the opacity is large, other processes that do not conserve the photon number lead to local thermodynamic equilibrium with a thermal comoving radiation intensity $\dot{I}_r$, number density, and spectrum of photons in the outflow. Hence, we use the Rosseland radiative diffusion approximation (see, e.g., Rybicki & Lightman 1979, pp. 39–42) that we derive from the radiative transfer equations (Equation 15) for expanding outflows. Details of the calculations are presented in the Appendix.

It is useful to introduce the function accounting for the adiabatic cooling of radiation in an expanding outflow, $L_c(\xi, t) = (t/t_0)^{8/3} L(\xi, t)$, where $t_0$ is a normalization parameter. Here, both $\xi$ and time $t$ are measured in the laboratory frame, while $L(\xi, t)$ is measured in the comoving frame. The diffusion equation with these variables is well behaved in the ultra-relativistic limit; see Beloborodov (2011). We then obtain

$$\frac{\partial L_c}{\partial t} + \frac{c^2 T^3}{3} \frac{\partial^2 L_c}{\partial \xi^2} = 0, \quad \Delta = \frac{1}{r^2}. \quad (25)$$

Note that the diffusion coefficient is explicitly time dependent due to the expansion of the outflow. In the Appendix, we also discuss initial and boundary conditions and obtain an approximate analytic solution for the radiation field inside the photon-thin outflow. The rising part of the corresponding flux of $L_c$ through the external boundary of the outflow scales as $t^{1/2}$, while its decaying part is quasi-exponential as shown in Figure 8. Consequently, while diffusion in a static object decreases the flux as $t^{-1/2}$, in our case the observed flux given by Equation (18) decreases even more slowly:

$$F \propto t_{\infty}^{-1/6}, \quad (26)$$

up to the arrival time of diffusion

$$t_{\infty} = \frac{R_D}{2n^2 c} \simeq 0.12 E_{54}^{1/3} \eta_2^{-5/3} l_0^{-1/3} s, \quad (27)$$

where a large part of the energy has left the outflow already ($E_0 = E_{54} 10^{54}$ erg, $\eta = \eta_2 10^3$, $l = l_0 10^8$ cm). At this moment, the energy decrease due to diffusion becomes substantial even in the deepest parts of the outflow. Later, the observed flux decreases quasi-exponentially with arrival time.

The comoving temperature of radiation on the photosphere is determined by the balance between the energy diffusion from the interior of the outflow and radiative losses. This temperature is much smaller than the temperature in the interior. The variation of the observed temperature across the PhE is small; see Figure 8. Hence, the observed instantaneous spectrum is very close to the thermal one and peaks near the observed temperature of the line of sight. We find that the latter decreases as $t_{\infty}^{-3/2}$, in contrast with the adiabatic law $t_{\infty}^{-2/3}$. However, at the diffusion radius, both temperatures agree to give the line of sight temperature

$$kT_{\text{LOS}} \simeq 162 \eta_2^{4/9} \text{keV}. \quad (28)$$

The time-integrated spectrum has a Band shape with a cutoff near the temperature $T_0$ see Figure 9. The low-energy part of the spectrum has a slope $\alpha = 1$, while the high-energy part has $\beta \simeq -3.5$.

6. DISCUSSION

Expression (27) gives an estimate for the duration of the photospheric emission for photon-thin outflows in the coasting phase. When the available observed spectra are integrated on time intervals comparable to $t_{\infty}^D$, the observed spectrum of photospheric emission is expected to have a Band shape. Thus, starting from a comoving thermal spectrum for the photospheric emission, we obtain, for the first time, an observed spectrum that may be well described by the Band function with a high-energy power-law index $\beta$ determined by the density profile of the outflow. We find this result quite remarkable, even though we assume strongly simplified dynamics.

Note that nonthermal spectra resulting from the convolution of thermal spectra over time have been introduced for afterglows of GRBs by Blinnikov et al. (1999). Double convolution of thermal or quasi-thermal spectra over EQTS and arrival time is
corresponding spectrum of the outflow. The diffusion length first and modifies the initial part of the light curve and the fusing out from this part of the outflow arrives at the observer (dashed green curve on the right) and the arrival time of diffusion (dashed red curve on the left).

(A color version of this figure is available in the online journal.)

also one of the key ideas in the fireshell model to reproduce the observed nonthermal spectra of GRB-extended afterglows (see, e.g., Ruffini et al. 2003; Bernardini et al. 2005; Patricelli et al. 2012).

Band spectra for photospheric emission have been obtained by now, in the context of GRB modeling, only assuming additional dissipation mechanisms (Rees & Mészáros 2005; Pe’er et al. 2006) such as magnetic reconnection (Giannios 2006), collisional heating (Beloborodov 2010), and internal shocks (Daigne & Mochkovitch 2002; Toma et al. 2011; Ryde et al. 2011). In our treatment, such additional assumptions are not required.

It is even more remarkable that GRBs appear to be the only known objects in nature able to reach the photon-thin asymptotics in their ultra-relativistic expansion. For thermally accelerated relativistic plasmas discussed in connection with their possible synthesis in ground-based laboratories (e.g., Ruffini et al. 2010), these asymptotics are unreachable. The photon-thin asymptotics are reached for

$$\tau_0 \gg 4\Gamma_8^3 \frac{I}{R_0} = 4 \times 10^{8}\eta_2^4 \frac{l_{8}}{R_8},$$

where $R_0 = R_810^8$ cm. GRBs clearly can satisfy this constraint since the contribution of baryons only gives $\tau_0 \simeq 3.5 \times 10^{8}E_{54} \eta_2^{-1}R_8^{-1}l_{8}^{-1}$.

We obtained both time-integrated and instantaneous observed spectra of the accelerating photon-thick outflow. In each case, nearly thermal spectra are recovered. Small deviations in the Rayleigh–Jeans part are observed, in agreement with the results of Grimsrud & Wasserman (1998).

The time-integrated observed spectrum of the coasting photon-thick outflow is broader than the thermal one, and deviates from it both at low- and high-energies. This broadening has also been found by Beloborodov (2010) and Béguel et al. (2013) using Monte-Carlo simulations; our result is in agreement with these works for the isotropic scattering model.

As discussed earlier in Section 2, each photon-thick outflow always contains a photon-thin layer with depth $\xi_{\text{thin}} = (R + l)/(2\eta^2)$ located near its outer boundary. Radiation diffusing out from this part of the outflow arrives at the observer first and modifies the initial part of the light curve and the corresponding spectrum of the outflow. The diffusion length $\xi_D = [(R + l)^3/(\eta^2\tau_0 R_0)]^{1/2}$ always remains within this photon-thin layer $\xi_{\text{thin}}$ and our solution for the photon-thin outflow is applicable for the description of this early emission. The photospheric emission from the well-developed photon-thick outflow and its late-time behavior occurring when the outflow crosses the photospheric radius may be described by either the fuzzy or sharp photosphere approximations.

When the outflow becomes transparent in the transition from photon-thick to photon-thin conditions, the observed time-integrated spectrum will contain both a Band component produced by the early emission from the photon-thin layer and a superimposed thermal-like component coming from the photon-thick part. We recall that our treatment contains several simplified assumptions of the outflow dynamics and therefore our treatment cannot be directly applied to GRB modeling. Nevertheless, it is interesting to note that some GRBs analyzed by Ryde & Pe’er (2009) present both power-law and black body components.

We presented analytic expressions for the photon flux in the simple model of the relativistic wind with a finite duration. With a more complex density profile composed of presumably many shells, the light curve is expected to be variable. The minimum timescale of variability is given by Equation (27); it may be very small for small baryon loading. It is necessary to emphasize that the decaying part of the light curve follows $t_{\text{c}}^{-3}$ for photon-thick outflows. A steeper decay of the light curve of photospheric emission is a clear signature of the photon-thin outflow.

The photospheric emission should be additionally identified by spectral analysis. In particular, power-law spectra extending to high energies above 10 MeV cannot be produced by photospheric emission unless additional mechanisms are involved. What we have shown here, though, is that in the special case of assuming simplified dynamics of the outflow, it is possible to have photospheric emission that exhibits a nonthermal spectral shape to a distant observer.

7. CONCLUSIONS

In summary, we have presented a treatment of photospheric emission from relativistic outflows under the main assumption that the dynamics of the outflow are given by Equations (1) and (2). In this sense, the results we obtained are valid in limiting cases. These results are not directly applicable to GRB models where the complete dynamics of the relativistic shell are considered (such as the fireshell model).

We proposed a new physically motivated classification of relativistic outflows: “photon thin” and “photon thick.” This new classification may be useful for understanding, in particular, why a geometrically thin shell may appear as a thick wind with respect to the photospheric emission. We then re-examined the existing literature and pointed out the advantage of the proposed classification versus the existing ones.

We studied the geometry of dynamic photospheres in relativistic outflows, albeit with the simplifying assumptions recalled above. Since we are interested in the appearance of the photosphere to a distant observer, we described the dynamics of the corresponding photospheric EQTS.

We computed both the energy flux and the observed spectra of photon-thick outflows under two additional different approximations derived from the radiative transfer equation. In our fuzzy photosphere approximation, the effect of simultaneous emission from the entire volume of the outflow is taken into account. We also used another more computationally
simple sharp photosphere approximation that well reproduces both light curves and spectra. The validity of these approximations has recently been tested with Monte-Carlo simulations of photon scattering in expanding relativistic outflows, as described by Equation (2) in Béguel et al. (2013). These results generalize the corresponding results in the literature for steady relativistic winds.

In photon-thin outflows, under the simplified dynamical assumptions recalled above, most of the radiation is shown to originate not at its photospheric radius but rather at smaller radii (due to radiation diffusion). Starting from the radiative transfer equation for time-dependent outflows, we derived the diffusion equation and obtained an approximate analytic solution for the energy flux. We present both instantaneous and time-integrated spectra as they would appear to a distant observer. The latter are well described by the Band function. For our simple density profile, we find values for the low-energy power-law index $\alpha = 1$ and the high-energy power-law index $\beta \simeq -3.5$.

More work is needed and is currently ongoing to check if these results are still valid when more realistic dynamics of the outflow are considered. Investigating if these results can therefore be applied to real astrophysical sources like GRBs is of great importance.

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APPENDIX

For large opacities, the distribution function of photons in the comoving reference frame is close to the isotropic distribution and the radiative diffusion approximation is accurate. Following Beloborodov (2011), we use the spectral intensity in the comoving frame $J_\nu(t, \xi, \mu)$. Starting from the radiative transfer equation (15) along the ray $s$ in the laboratory frame, we transform all variables except time $t$, depth $\xi$, and distance $s$ into the comoving reference frame

$$\nu^3 \frac{d}{ds} \left( \frac{J_\nu}{\nu^3} \right) = \frac{\kappa_\nu}{\mathcal{D}} (S_\nu - J_\nu). \tag{A1}$$

Integrating over the comoving frequency $\nu$, we have

$$\frac{1}{c} \frac{\partial J}{\partial t} - \mu \frac{\partial J}{\partial \xi} + \frac{1}{c} \frac{\partial J}{\partial \mu} + \frac{4}{c} \frac{\partial}{\partial \nu} \left( \frac{1}{c} - \frac{\partial J}{\partial \nu} \right) J = \frac{\kappa}{\mathcal{D}} (S - J), \tag{A2}$$

where $\mu = \cos \theta$, $\theta$ is the photon angle with respect to the radial direction in the comoving frame, $\mathcal{D} = \Gamma (1 + \beta \mu)$ is the Doppler factor, $J = \int J_\nu d\nu$ is the total intensity, $S = \int S_\nu d\nu$ is the total source function, $\kappa = \int \kappa_\nu J_\nu d\nu$ is the effective opacity, and $\kappa_\nu = \sigma n_\nu$ is the opacity in the comoving frame.

In the case of small deviations from isotropy, the decomposition $J = J_0(t, \xi) + \mu J_1(t, \xi)$ can be applied. Introducing it into Equation (A2) and integrating it over $\Delta \mu$ and over $\Delta \nu \Delta \mu \Delta \xi$ after some algebra for coherent scattering with $S = S_0 = J_0$, we have

$$\frac{\partial J_0}{\partial ct} + \frac{\beta J_0}{3} \frac{\partial J_0}{\partial \xi} - \frac{1}{3} \frac{\partial J_1}{\partial \mu} + \frac{2 J_1}{3(\nu - \xi)} + \frac{4 J_0 \beta}{3(\nu - \xi)} = 0, \tag{A3}$$

$$\frac{\partial J_1}{\partial ct} + \frac{\beta J_0}{3} \frac{\partial J_1}{\partial \xi} - \frac{1}{3} \frac{\partial J_0}{\partial \mu} + \frac{8 J_1 \beta}{5(\nu - \xi)} = -\frac{\kappa J_1}{\Gamma}. \tag{A4}$$

The diffusion approximation is based on the slow variation of the total flux through the entire sphere $L_1(t/t_0^2)$ over the mean free path, where $t_0$ is an initial time. Then, $\partial L_1/\partial ct = 0$ and it provides $J_1$ from Equation (A4). Inserting this into Equation (A3), after simple but tedious calculations in the ultra-relativistic $\beta \simeq 1$ photon-thin case $\Gamma^2 \xi \ll \nu$ for the function $L = J_0(t/t_0)^{\nu/3}$, we obtain the diffusion equation (25):

$$\frac{\partial L}{\partial ct} - \frac{c^2 t^2 \Delta^2 L}{3 R_0} \frac{\partial^2 L}{\partial \xi^2} = 0, \quad \Delta = \frac{1}{\Gamma^2 t_0}. \tag{A5}$$

This equation should be supplemented with boundary conditions. There are two types of boundary conditions used frequently: free-streaming, for example in two-stream approximation (Rybicki & Lightman 1979, pp. 42–45), and zero boundary conditions. The latter can be used at the “extrapolated boundary” as replacements for free-streaming conditions (Haskell et al. 1994). We find that the position of the “extrapolated boundary” $\xi = -\alpha c^2 t^2 \Delta/R_0$ ($\alpha$ is a constant of order unity, dependent on the approximation used for free-streaming description) for the main part of emission is very close to the real boundary, and in the case of zero boundary conditions $L|_{\xi=0} = L|_{\xi=\infty} = 0$, there is a series expansion of the solution. For initial conditions $L(\xi, t_0) = 1$, the solution is:

$$L(\xi, t) = \sum_{n=0}^{\infty} \frac{4}{(2n + 1)\pi} \exp \left[ -\frac{\Delta(2n + 1)^2 \pi^2 c^3 (t^3 - t_0^3)}{9 R_0^2} \right] \times \sin \left[ \frac{(2n + 1)\pi \xi}{l} \right]. \tag{A6}$$

This solution deviates from the numerical one with free-streaming boundary conditions by only a few percent.

The flux of $L$ is characterized by an initial burst and then tends to the asymptotic solution. This result corresponds to solution (A6) with $t_0 = 0$, with flux

$$F(t) = \frac{4\Delta c^2 t^2}{3 R_0 l^2} \frac{\partial}{\partial \xi} \left[ 0, \exp \left( -\frac{4\Delta \pi^2 c^3 t^3}{9 R_0 l^2} \right) \right], \quad \tag{A7}$$

where $\partial$ is the Jacobi elliptic theta function; see its graphical representation in Figure 8. The peak of the flux of $L$ is near the diffusion time

$$t_D = \frac{1}{c} \left( \frac{R_0}{\Delta} \right)^{1/3}, \tag{A8}$$

and the “extrapolated boundary” at that time is $\xi = -\alpha t_0 (\Delta/R_0)^{1/3} \ll l$. This boundary is very close to the real one as $\Delta \ll 1$, ensuring the accuracy of Equation (A6).

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