Universal recovery map for approximate Markov chains

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Abstract

A Markov chain is a tripartite quantum state $ρ_{ABC}$ whose $C$-part can be recovered from the $B$-part only, i.e., there exists a recovery map $R_{B→BC}$ such that $ρ_{ABC} = R_{B→BC}(ρ_{AB})$. More generally, an approximate Markov chain $ρ_{ABC}$ is a state whose distance to the closest recovered state $R_{B→BC}(ρ_{AB})$ is small. As proved recently in [Fawzi and Renner, arXiv:1410.0664], this distance can be bounded from above by the conditional mutual information $I(A:C|B)$ of the state. While the proof is non-constructive, one may expect that the recovery is universal, in the sense that the same map $R_{B→BC}$ works for all states with a given marginal $ρ_{BC}$. Here we show that this is indeed the case.

1 Introduction

A state $ρ_{ABC}$ on a tripartite quantum system $A \otimes B \otimes C$ forms a (quantum) Markov chain if it can be recovered from its marginal $ρ_{AB}$ on $A \otimes B$ by a quantum operation $R_{B→BC}$ from $B$ to $B \otimes C$, i.e.,

$$ρ_{ABC} = R_{B→BC}(ρ_{AB}) \quad (1)$$

An equivalent characterization of $ρ_{ABC}$ being a Markov chain is that the conditional mutual information $I(A:C|B) := H(AB)_ρ + H(BC)_ρ - H(B)_ρ - H(ABC)_ρ$ is zero [20, 23, 24] where $H(A)_ρ := -\text{tr}(ρ_A \log_2 ρ_A)$ is the von Neumann entropy. The structure of these states has been studied in various works. In particular, it has been shown that $A$ and $C$ can be viewed as independent conditioned on $B$, for a meaningful notion of conditioning [15]. Very recently it has been shown that Markov states can be alternatively characterized by having a generalized Rényi conditional mutual information that vanishes [9].

A natural question that is relevant for applications is whether the above statements are robust (see [19] for an example and [10] for a discussion). Specifically, one would like to have a characterization of the states that have a small (but not necessarily vanishing) conditional mutual information, i.e., $I(A:C|B) \leq ε$ for $ε > 0$. First results revealed that such states can have a large distance to Markov chains that is independent of $ε$ [8, 18], which has been taken as an indication that their characterization may be difficult. However, it has subsequently been realized that a more appropriate measure instead of the distance to a (perfect) Markov chain is to consider how well (1) is satisfied [33, 19, 34, 4]. This motivated the definition of approximate Markov chains as states where (1) approximately holds.

In recent work [10], it has been shown that the set of approximate Markov chains indeed coincides with the set of states with small conditional mutual information. In particular, the distance between the two terms in (1), which may be measured in terms of their fidelity $F$, is bounded by the conditional

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More precisely, for any state $\rho_{ABC}$ there exists a trace-preserving completely positive map $R_{B\rightarrow BC}$ (the recovery map) such that

$$I(A : C|B)_\rho \geq -2\log_2 F(\rho_{ABC}, R_{B\rightarrow BC}(\rho_{AB})) .$$

(2)

Furthermore, it can be shown that a converse inequality holds that is of the form $I(A : C|B)_\rho^2 \leq -c^2 \log_2 F(\rho_{ABC}, R_{B\rightarrow BC}(\rho_{AB}))$, where $c$ depends logarithmically on the dimension of $A$ [4, 10].

We also note that the fidelity term in (2), maximized over all recovery maps, i.e.,

$$F(A; C|B)_\rho := \sup_{R_{B\rightarrow BC}} F(\rho_{ABC}, R_{B\rightarrow BC}(\rho_{AB}))$$

(3)

is called fidelity of recovery, and has been introduced and studied in [26, 5]. With this quantity the main result of [10] can be written as

$$I(A : C|B)_\rho \geq -2\log_2 F(A; C|B)_\rho .$$

(4)

The fidelity of recovery has several natural properties, e.g., it is monotonous under local operations on $A$ and $C$, and it is multiplicative [5].

The result of [10] has been extended in various ways. Based on quantum state redistribution protocols, it has been shown in [7] that (2) still holds if the fidelity term is replaced by the measured relative entropy

$$D_M(\cdot, \cdot)$$

which is generally larger, i.e.,

$$I(A : C|B)_\rho \geq D_M(\rho_{ABC}\|R_{B\rightarrow BC}(\rho_{AB})) \geq -2\log_2 F(\rho_{ABC}, R_{B\rightarrow BC}(\rho_{AB})) .$$

(5)

Furthermore, in [5] an alternative proof of (2) has been derived that uses properties of the fidelity of recovery (in particular, multiplicativity). Another recent work [3] showed how to generalize ideas from [10] to prove a remainder term for the monotonicity of the relative entropy in terms of a recovery map that satisfies (2).

All known proofs of (2) are non-constructive, in the sense that the recovery map $R_{B\rightarrow BC}$ is not given explicitly. It is merely known [10] that if $A$, $B$, and $C$ are finite-dimensional then $R_{B\rightarrow BC}$ can always be chosen such that it has the form

$$X_B \mapsto V_{BC} p_{BC}^{1/2} U_B X_B U_B^\dagger p_{BC}^{-1/2} \otimes \text{id}_C \rho_{BC}^{1/2} V_{BC}^\dagger$$

(6)

on the support of $\rho_B$, where $U_B$ and $V_{BC}$ are unitaries on $B$ and $B \otimes C$, respectively. It would be natural to expect that the choice of the recovery map that satisfies (2) only depends on $\rho_{BC}$; however this is only known in special cases. One such special case are Markov chains $\rho_{ABC}$, i.e., states for which (1) holds perfectly. Here a map of the form (6) with $V_{BC} = \text{id}_BC$ and $U_B = \text{id}_B$ (sometimes referred to as transpose map or Petz recovery map) serves as a perfect recovery map [23, 24]. Another case where a recovery map that only depends on $\rho_{BC}$ is known explicitly are states with a classical $B$ system, i.e., qco-states of the form $\rho_{ABC} = \sum_b P_B(b)|b\rangle\langle b| \otimes \rho_{AC,b}$, where $P_B$ is a probability distribution, $\{|b\rangle\}_b$ an orthonormal basis on $B$ and $\{\rho_{AC,b}\}_b$ a family of states on $A \otimes C$. As discussed in [10], for such states (2) holds for the recovery map defined by $R_{B\rightarrow BC}(|b\rangle\langle b|) = |b\rangle\langle b| \otimes \rho_{C,b}$ for all $b$, where $\rho_{C,b} = \text{tr}_A(\rho_{AC,b})$. For general states, however, the previous results left open the possibility that the recovery map $R_{B\rightarrow BC}$ depends on the full state $\rho_{ABC}$ rather than the marginal $\rho_{BC}$ only. In particular, the unitaries $U_B$ and $V_{BC}$ in (6), although acting only on $B$ respectively $B \otimes C$, could have such a dependence.

In this work we show that for any state $\rho_{BC}$ on $B \otimes C$ there exists a recovery map $R_{B\rightarrow BC}$ that is universal—in the sense that the distance between any extension $\rho_{ABC}$ of $\rho_{BC}$ and $R_{B\rightarrow BC}(\rho_{AB})$ is bounded from above by the conditional mutual information $I(A : C|B)_\rho$. In other words we show that (2) remains valid if the recovery map is chosen depending on $\rho_{BC}$ only, rather than on $\rho_{ABC}$.

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1The fidelity of $\rho$ and $\sigma$ is defined as $F(\rho, \sigma) := \|\sqrt{\rho} \sigma^{1/2}\|_1$.

2The measured relative entropy is defined in Appendix B (Definition B.1).
2 Main result

**Theorem 2.1.** For any density operator \( \rho_{BC} \) on \( B \otimes C \) there exists a trace-preserving completely positive map \( \mathcal{R}_{B \to BC} \) such that for any extension \( \rho_{ABC} \) on \( A \otimes B \otimes C \)

\[
I(A : C|B)_\rho \geq -2 \log_2 F(\rho_{ABC}, \mathcal{R}_{B \to BC}(\rho_{AB})) ,
\]

where \( A, B, \) and \( C \) are separable Hilbert spaces.

**Remark 2.2.** If \( B \) and \( C \) are finite-dimensional Hilbert spaces, the statement of Theorem 2.1 can be tightened to

\[
I(A : C|B)_\rho \geq D_{\text{M}}(\rho_{ABC} \| \mathcal{R}_{B \to BC}(\rho_{AB})) .
\]

**Remark 2.3.** The recovery map \( \mathcal{R}_{B \to BC} \) predicted by Theorem 2.1 has the property that it maps \( \rho_B \) to \( \rho_{BC} \). To see this, note that \( I(A : C|B)_\beta = 0 \) for any density operator of the form \( \tilde{\rho}_{ABC} = \rho_A \otimes \rho_{BC} \). Theorem 2.1 thus asserts that \( \tilde{\rho}_{ABC} \) must be equal to \( \mathcal{R}_{B \to BC}(\rho_{AB}) \), which implies that \( \rho_{BC} = \mathcal{R}_{B \to BC}(\rho_B) \). We note that so far it was unknown whether recovery maps that satisfy (2) and have this property do exist.

We note that Theorem 2.1 does not reveal any information about the structure of the recovery map that satisfies (7). However, if we consider a linearized version of the bound (7), we can make more specific statements.

**Corollary 2.4.** For any density operator \( \rho_{BC} \) on \( B \otimes C \) there exists a trace-preserving completely positive map \( \mathcal{R}_{B \to BC} \) such that for any extension \( \rho_{ABC} \) on \( A \otimes B \otimes C \)

\[
F(\rho_{ABC}, \mathcal{R}_{B \to BC}(\rho_{AB})) \geq 1 - \frac{\ln(2)}{2} I(A : C|B)_\rho ,
\]

where \( A, B, \) and \( C \) are separable Hilbert spaces. Furthermore, if \( B \) and \( C \) are finite-dimensional then \( \mathcal{R}_{B \to BC} \) has the form

\[
X_B \mapsto \frac{1}{2} \rho_{BC}^{1/2} U_{BC \to BC}(\rho_B^{1/2} X_B \rho_B^{1/2} \otimes \text{id}_C) \rho_{BC}^{1/2}
\]

on the support of \( \rho_B \), where \( U_{BC \to BC} \) is a unital trace-preserving map from \( B \otimes C \) to \( B \otimes C \).

**Example 2.5.** For density operators with a marginal on \( B \otimes C \) of the form \( \rho_{BC} = \rho_B \otimes \rho_C \), a universal recovery map that satisfies (8) is uniquely defined on the support of \( \rho_B \) — it is the transpose map, which in this case simplifies to \( \mathcal{R}_{B \to BC} : X_B \mapsto X_B \otimes \rho_C \). It is straightforward to see that (8) holds. In fact, we even have equality if we consider the relative entropy (which is in general larger than the measured relative entropy), i.e.,

\[
I(A : C|B)_\rho = D(\rho_{ABC} \| \mathcal{R}_{B \to BC}(\rho_{AB})) .
\]

The uniqueness of \( \mathcal{R}_{B \to BC} \) on the support of \( \rho_B \) follows by using the fact that the universal recovery map should perfectly recover the Markov state \( \rho_{AB} \otimes \rho_C \) where \( \rho_{AB} \) is a purification of \( \rho_B \). This forces \( \mathcal{R}_{B \to BC} \) to agree with the transpose map on the support of \( \rho_B \) [23, 24].

The proof of Theorem 2.1 is structured into two parts. We first prove the statement for finite-dimensional Hilbert spaces \( B, \) and \( C \) in Section 3 and then show that this implies the statement for general separable Hilbert spaces in Section 4. The proof of Corollary 2.4 is given in Section 5.
3 Proof for finite dimensions

Throughout this section we assume that the Hilbert spaces $B$ and $C$ are finite-dimensional. In the proof Steps 1 - 3 below, we also make the same assumption for $A$, but then drop it in Step 4. We start by explaining why (8) is a tightened version of (7) which was noticed in [7]. Let $D_\alpha(\cdot||\cdot)$ be the $\alpha$-Quantum Rényi Divergence as defined in [21, 31] with $D_1(\rho||\sigma) = D(\rho||\sigma) := \text{tr}(\rho \log \rho - \log \sigma))$. By definition of the measured relative entropy (see Definition B.1) we find for any two states $\rho$ and $\sigma$

$$D_M(\rho||\sigma) = \sup_{M \in M} D(\mathcal{M}(\rho)||\mathcal{M}(\sigma)) \geq \sup_{M \in M} D_2 \big(\mathcal{M}(\rho)||\mathcal{M}(\sigma)\big) = -2 \log_2 \inf_{\mathcal{M} \in M} F(\mathcal{M}(\rho), \mathcal{M}(\sigma))$$

where $\mathcal{M} := \{\mathcal{M}: \mathcal{M}(\rho) = \sum_x \text{tr}(\rho M_x)|x\rangle\langle x| \text{ with } \sum_x M_x = \text{id}\}$ and $\{|x\rangle\}$ is a family of orthonormal vectors. The inequality step uses that $\alpha \mapsto D_\alpha(\rho||\sigma)$ is a monotonically non-decreasing function in $\alpha$ [21, Theorem 7] and the final step follows from the fact that for any two states there exists an optimal measurement that does not increase their fidelity [12, Section 3.3]. As a result, in order to prove Theorem 2.1 for finite-dimensional $B$ and $C$ it suffices to prove (8).

We first derive a proposition (Proposition 3.1 below) and then show how it can be used to prove (8) (and, hence, Theorem 2.1). The proposition refers to a family of functions

$$D(A \otimes B \otimes C) \ni \rho \mapsto \Delta_R(\rho) \in \mathbb{R} \cup \{-\infty\} ,$$

parameterized by recovery maps $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$, where $\text{TPCP}(B, B \otimes C)$ denotes the set of trace-preserving completely positive maps from $B$ to $B \otimes C$ and $D(A \otimes B \otimes C)$ denotes the set of density operators on $A \otimes B \otimes C$. Subsequently in the proof, the function family $\Delta_R(\cdot)$ will be constructed as the difference of the two terms in (8) (see Equation (49)) such that $\Delta_R(\rho) \geq 0$ corresponds to (8). The proposition asserts that if for any extension $\rho_{ABC}$ of $\rho_{BC}$ we have $\Delta_R(\rho) \geq 0$ for some $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$ and provided the function family $\Delta_R(\cdot)$ satisfies certain properties described below, then there exists a single recovery map $\mathcal{R}$ for which $\Delta_R(\rho) \geq 0$ for all extensions $\rho_{ABC}$ of $\rho_{BC}$ on a fixed $A$ system. We note that the precise form of the function family $\Delta_R(\cdot)$ is irrelevant for Proposition 3.1 as long as it satisfies a list of properties as stated below.

As described above, our goal is to prove that there exists a recovery map $\mathcal{R}_{B \rightarrow BC}$ such that $\Delta_R(\rho) \geq 0$ for all $\rho_{ABC} \in D(A \otimes B \otimes C)$ with a fixed marginal $\rho_{BC}$. Furthermore, for any fixed states $\rho_{0ABC}$ and $\rho_{ABC}$ on $A \otimes B \otimes C$ and $p \in [0,1]$, we define

$$\rho_{AABC}^p := (1-p)|00\rangle\langle 00| \otimes \rho_{0ABC}^p + p|11\rangle\langle 11| \otimes \rho_{ABC} ,$$

where $\hat{A}$ is an additional system with two orthogonal states $|0\rangle$ and $|1\rangle$. More generally, for any fixed state $\rho_{0ABC}$ and for any set $\mathcal{S}$ of density operators $\rho_{ABC}$ we set

$$\mathcal{S}^p := \{\rho_{AABC}^p : \rho_{ABC} \in \mathcal{S}\} .$$

Required properties of the $\Delta$-function.

1. For any $\rho_{0ABC}^p, \rho_{ABC} \in D(A \otimes B \otimes C)$ with identical marginals $\rho_{0BC}^0 = \rho_{BC}$ on $B \otimes C$, for any $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$, and for any $p \in [0,1]$ we have $\Delta_R(\rho^p) = (1-p)\Delta_R(\rho^0) + p\Delta_R(\rho)$.

2. For any $\mathcal{R}, \mathcal{R}' \in \text{TPCP}(B, B \otimes C)$, for any $\alpha \in [0,1]$, and $\mathcal{R} = \alpha\mathcal{R} + (1-\alpha)\mathcal{R}'$ we have $\Delta_{\mathcal{R}}(\rho) \geq \alpha\Delta_{\mathcal{R}}(\rho) + (1-\alpha)\Delta_{\mathcal{R}'}(\rho)$ for all $\rho \in D(A \otimes B \otimes C)$. 

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3. For any \( R \in \text{TPCP}(B, B \otimes C) \), the function \( \text{D}(A \otimes B \otimes C) \ni \rho \mapsto \Delta_R(\rho) \in \mathbb{R} \cup \{-\infty\} \) is upper semicontinuous.

4. For any \( \rho \in \text{D}(A \otimes B \otimes C) \), the function \( \text{TPCP}(B, B \otimes C) \ni R \mapsto \Delta_R(\rho) \in \mathbb{R} \cup \{-\infty\} \) is upper semicontinuous.

Property 1 implies that for any state \( \rho^0_{ABC} \), for any set \( S \) of operators \( \rho_{ABC} \) with \( \rho_{BC} = \rho^0_{BC} \), and for any \( p \in [0,1] \) we have

\[
\Delta_R(S^p) = \inf_{\rho \in S} \Delta_R(\rho^p) = (1-p)\Delta_R(\rho^0) + p \inf_{\rho \in S} \Delta_R(\rho) = (1-p)\Delta_R(\rho^0) + p\Delta_R(S). \tag{17}
\]

Similarly, Property 2 implies

\[
\Delta_R(S) = \inf_{\rho \in S} \Delta_R(\rho) \geq \inf_{\rho \in S} \{ \alpha \Delta_R(\rho) + (1-\alpha)\Delta_R'(\rho) \} \\
\geq \alpha \inf_{\rho \in S} \Delta_R(\rho) + (1-\alpha)\inf_{\rho \in S} \Delta_R'(\rho) = \alpha \Delta_R(S) + (1-\alpha)\Delta_R'(S). \tag{18}
\]

**Proposition 3.1.** Let \( A, B, \) and \( C \) be finite-dimensional Hilbert spaces, \( \mathcal{P} \subseteq \text{TPCP}(B, B \otimes C) \) be compact and convex, \( S \) be a set of density operators on \( A \otimes B \otimes C \) with identical marginals on \( B \otimes C \), and \( \Delta_R(\cdot) \) be a family of functions of the form (13) that satisfies Properties 1-4. Then

\[
\forall \rho \in S \ \exists R \in \mathcal{P} : \Delta_R(\rho) \geq 0 \quad \implies \quad \exists \mathcal{R} \in \mathcal{P} : \Delta_R(S) \geq 0. \tag{19}
\]

We now proceed in four steps. In the first, we prove Proposition 3.1 for finite sets \( S \). This is done by induction over the cardinality of the set \( S \). We show that if the statement of Proposition 3.1 is true for all sets \( S \) with \( |S| = n \), this implies that it remains valid for all sets \( S \) with \( |S| = n + 1 \). In Step 2, we use an approximation step to extend this to infinite sets \( S \) which then completes the proof of Proposition 3.1. In the final two steps, we show how to conclude the statement of Theorem 2.1 for the finite-dimensional case from that. In Step 3 we prove (8) for the case where the recovery map that satisfies (8) could still depend on the dimension of the system \( A \). In Step 4 we show how this dependency can be removed.

**Step 1: Proof of Proposition 3.1 for finite size sets \( S \)**

We proceed by induction over the cardinality \( n := |S| \) of the set \( S \) of density operators. More precisely, the induction hypothesis is that for any finite-dimensional Hilbert space \( A \) and any set \( S \) of size \( n \) consisting of density operators on \( A \otimes B \otimes C \) with fixed marginal \( \rho_{BC} \) on \( B \otimes C \), the statement (19) holds. For \( n = 1 \), this hypothesis holds trivially for \( \mathcal{R} = \mathcal{R} \).

We now prove the induction step. Suppose that the induction hypothesis holds for some \( n \). Let \( A \) be a finite-dimensional Hilbert space and let \( S \cup \{ \rho_{ABC}^0 \} \) be a set of cardinality \( n + 1 \) where \( S \) is a set of states on \( A \otimes B \otimes C \) with fixed marginal \( \rho_{BC} \) on \( B \otimes C \) of cardinality \( n \) and \( \rho_{ABC}^0 \) is another state with \( \rho_{BC}^0 = \rho_{BC} \). We need to prove that there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \in \mathcal{P} \) such that

\[
\Delta_{\mathcal{R}}(S \cup \{ \rho_{ABC}^0 \}) \geq 0. \tag{20}
\]

Let \( p \in [0,1] \) and consider the set \( S^p \) as defined in (16). In the following we view the states \( \rho^p \) (see Equation (15)) in this set as tripartite states on \( (A \otimes A) \otimes B \otimes C \), i.e., we regard the system \( A \otimes A \) as one (larger) system. The induction hypothesis applied to the extension space \( A \otimes A \) and the set \( S^p \) (of size \( n \)) of states on \( (A \otimes A) \otimes B \otimes C \) implies the existence of a map \( \mathcal{R}_{B \rightarrow BC}^p \in \mathcal{P} \) such that

\[
\Delta_{\mathcal{R}^p}(S^p) \geq 0. \tag{21}
\]

\[\text{For } n = 0, \text{ we have } \Delta_{\mathcal{R}}(S) = \infty \geq 0 \text{ for any } \mathcal{R} \in \mathcal{P} \text{ since the infimum of an empty set is infinity.} \]
As by assumption the function $D(A \otimes B \otimes C) \ni \rho \mapsto \Delta_{R^p}(\rho) \in \mathbb{R} \cup \{-\infty\}$ satisfies Property 1 (and hence also Equation (17)) we obtain

$$(1 - p)\Delta_{R^p}(\rho^0) + p\Delta_{R^p}(S) \geq 0 .$$

(22)

This implies that

$$\Delta_{R^p}(\rho^0) \geq 0 \quad \text{or} \quad \Delta_{R^p}(S) \geq 0 .$$

(23)

Furthermore, for $p = 0$ the left inequality holds and for $p = 1$ the right inequality holds. By choosing $K_0 = \{p \in [0, 1] : \Delta_{R^p}(\rho^0) \geq 0\}$ and $K_1 = \{p \in [0, 1] : \Delta_{R^p}(S) \geq 0\}$, Lemma D.1 implies that for any $\delta > 0$ there exist $u, v \in [0, 1]$ with $0 \leq v - u \leq \delta$ such that

$$\Delta_{R^u}(\rho^0) \geq 0 \quad \text{and} \quad \Delta_{R^v}(S) \geq 0 .$$

(24)

Note also that $R_u B \rightarrow BC, R_v B \rightarrow BC \in \mathcal{P}$, since by the induction hypothesis $R_p B \rightarrow BC \in \mathcal{P}$ for any $p \in [0, 1]$.

We will use this to prove that the recovery map $\tilde{R} \in \mathcal{P}$ defined by

$$\tilde{R} := \alpha R_u + (1 - \alpha) R_v ,$$

(25)

for an appropriately chosen $\alpha \in [0, 1]$, satisfies

$$\Delta_{\tilde{R}}(\rho^0) \geq -c\delta \quad \text{and} \quad \Delta_{\tilde{R}}(S) \geq -c\delta ,$$

(26)

where $c$ is a constant defined by

$$c := 4 \max_{R \in \text{TPCP}(B, B \otimes C)} \max_{\rho \in D(A \otimes B \otimes C)} \Delta_{R}(\rho) < \infty .$$

(27)

Properties 3 and 4 together with Lemma C.1 and Remark C.3 ensure that the two maxima in (27) are attained which implies by the definition of the codomain of $\Delta_{\tilde{R}}(\cdot)$ (see Equation (13)) that $c$ is finite. In other words, for any $\delta > 0$ there exists a recovery map $\hat{R} \in \mathcal{P}$ such that

$$\Delta_{\hat{R}}(S \cup \{\rho^0\}) \geq -c\delta .$$

(28)

The compactness of $\mathcal{P}$ ensures that there exists a recovery map $\hat{R} \in \mathcal{P}$ and a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \delta_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \hat{R}^{\delta_n} = \hat{R} .$$

(29)

Because of (28) we have

$$\limsup_{n \to \infty} \Delta_{\hat{R}^{\delta_n}}(S \cup \{\rho^0\}) \geq \lim_{n \to \infty} -c\delta_n = 0 ,$$

(30)

which together with Property 4 implies that

$$\Delta_{\hat{R}}(S \cup \{\rho^0\}) = \min_{\rho \in S \cup \{\rho^0\}} \Delta_{\hat{R}}(\rho) \geq \min_{\rho \in S \cup \{\rho^0\}} \limsup_{n \to \infty} \Delta_{\hat{R}^{\delta_n}}(\rho) \geq \limsup_{n \to \infty} \min_{\rho \in S \cup \{\rho^0\}} \Delta_{\hat{R}^{\delta_n}}(\rho) = \limsup_{n \to \infty} \Delta_{\hat{R}^{\delta_n}}(S \cup \{\rho^0\}) \geq 0 ,$$

(31)

and thus proves (20).

It thus remains to show (26). To simplify the notation let us define

$$\Lambda^0 := \Delta_{R^u}(\rho^0) \quad \text{and} \quad \Lambda^1 := \Delta_{R^v}(S)$$

(32)

as well as

$$\bar{\Lambda}^0 := \Delta_{R^u}(\rho^0) \quad \text{and} \quad \bar{\Lambda}^1 := \Delta_{R^v}(S) .$$

(33)
It follows from (22) that 
\[
(1 - u)\Lambda^0 + u\bar{\Lambda}^1 \geq 0 .
\]
(34)
Similarly, we have 
\[
(1 - v)\bar{\Lambda}^0 + v\Lambda^1 \geq 0 .
\]
(35)
As by assumption the function \(\Delta_R(\cdot)\) satisfies Property 2 we find together with (35) that for any \(\alpha \in [0, 1]\) and \(\bar{R} = \alpha R^\ast + (1 - \alpha)R^\ast\), 
\[
\Delta_R(\rho^0) \geq \alpha \Delta_{R^\ast}(\rho^0) + (1 - \alpha)\Delta_{R^\ast}(\rho^0) = \alpha\Lambda^0 + (1 - \alpha)\bar{\Lambda}^0 \geq \alpha\Lambda^0 - (1 - \alpha)\frac{v}{1 - v}\Lambda^1 .
\]
(36)
(If \(v = 1\) it suffices to consider the case \(\alpha = 1\) so that the last term can be omitted; cf. Equation (40) below.) Analogously, using (18) and (34), we find 
\[
\Delta_R(S) \geq \alpha\Delta_{R^\ast}(S) + (1 - \alpha)\Delta_{R^\ast}(S) = \alpha\bar{\Lambda}^1 + (1 - \alpha)\Lambda^1 \geq -\alpha\frac{1 - u}{u}\Lambda^0 + (1 - \alpha)\Lambda^1 .
\]
(37)
(If \(u = 0\) it suffices to consider the case \(\alpha = 0\); cf. Equation (43) below.)
To conclude the proof of (26), it suffices to choose \(\alpha \in [0, 1]\) such that the terms on the right hand side of (36) and (37) satisfy 
\[
\alpha\Lambda^0 - (1 - \alpha)\frac{v}{1 - v}\Lambda^1 \geq -c\delta
\]
(38)
and 
\[
-\alpha\frac{1 - u}{u}\Lambda^0 + (1 - \alpha)\Lambda^1 \geq -c\delta .
\]
(39)
Let us first assume that \(u \geq \frac{1}{2}\). Since \(\Lambda^0\) and \(\Lambda^1\) are non-negative (see Equation (24)), we may choose \(\alpha \in [0, 1]\) such that 
\[
\alpha(1 - v)\Lambda^0 = (1 - \alpha)v\Lambda^1 .
\]
(40)
This immediately implies that the left hand side of (38) equals 0, so that the inequality holds. As \(\frac{1}{2} \leq u \leq v \leq 1\) and \(v - u \leq \delta\) we have 
\[
\left|\frac{1 - u}{u} - \frac{1 - v}{v}\right| \leq 4\delta .
\]
(41)
Combining this with (40) we find 
\[
-\alpha\frac{1 - u}{u}\Lambda^0 + (1 - \alpha)\Lambda^1 \geq -\alpha\Lambda^0\left(\frac{1 - v}{v} + 4\delta\right) + (1 - \alpha)\Lambda^1 = -4\alpha\Lambda^0\delta \geq -4\Lambda^0\delta ,
\]
which proves (39) because by (27) we have \(\Lambda^0 = \frac{\delta}{4}\).
Analogously, if \(u < \frac{1}{2}\), choose \(\alpha \in [0, 1]\) such that 
\[
\alpha(1 - u)\Lambda^0 = (1 - \alpha)u\Lambda^1 .
\]
(43)
This immediately implies that the left hand side of (39) equals 0, so that the inequality holds. Furthermore, for \(\delta > 0\) sufficiently small such that \(v \leq \frac{1}{2}\), we obtain 
\[
\left|\frac{v}{1 - v} - \frac{u}{1 - u}\right| < 4\delta .
\]
(44)
Together with (43) this implies 
\[
\alpha\Lambda^0 - (1 - \alpha)\frac{v}{1 - v}\Lambda^1 \geq \alpha\Lambda^0 - (1 - \alpha)\Lambda^1\left(\frac{u}{1 - u} + 4\delta\right) = -4(1 - \alpha)\Lambda^1\delta \geq -4\Lambda^1\delta ,
\]
(45)
which establishes (38). This concludes the proof of Proposition 3.1 for sets \(S\) of finite size.
Step 2: Extension to infinite sets $S$

All that remains to be done to prove Proposition 3.1 is to generalize the statement to arbitrarily large sets $S$. In fact, we show that there exists a recovery map $R_{B \rightarrow BC} \in \mathcal{P}$ such that $\Delta_R(S) \geq 0$, where $S$ is the set of all density operators on $A \otimes B \otimes C$ for a fixed finite-dimensional Hilbert space $A$ and a fixed marginal $\rho_{BC}$.

Note first that this set $S$ of all density operators on $A \otimes B \otimes C$ with fixed marginal $\rho_{BC}$ on $B \otimes C$ is compact (see Lemma C.2). This implies that for any $\varepsilon > 0$ there exists a finite set $S^\varepsilon$ of density operators on $A \otimes B \otimes C$ such that any $\rho \in S^\varepsilon$ is $\varepsilon$-close to an element of $S$. We further assume without loss of generality that $S^\varepsilon \subset S$ for $\varepsilon \geq \varepsilon$. Let $R^\varepsilon \in \mathcal{TPC}(B, B \otimes C)$ be a map such that $\Delta_{R^\varepsilon}(S^\varepsilon) \geq 0$, whose existence follows from the validity of Proposition 3.1 for sets of finite size (which we proved in Step 1). Since the set $\mathcal{TPC}(B, B \otimes C)$ is compact (see Remark C.3) there exists a decreasing sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ and $R \in \mathcal{TPC}(B, B \otimes C)$ such that

$$\lim_{n \to \infty} \varepsilon_n = 0 \quad \text{and} \quad \bar{R} = \lim_{n \to \infty} R^\varepsilon_n.$$ (46)

Combining this with Property 4 gives for all $n \in \mathbb{N}$

$$\Delta_{\bar{R}}(S^n) = \inf_{\rho \in S^n} \Delta_{\bar{R}}(\rho) \geq \inf_{\rho \in S^n} \limsup_{m \to \infty} \Delta_{R^m}(\rho) \geq \limsup_{m \to \infty} \inf_{\rho \in S^n} \Delta_{R^m}(\rho) \geq \limsup_{m \to \infty} \inf_{\rho \in S^n} \Delta_{R^m}(\rho) = \limsup_{m \to \infty} \Delta_{R^m}(\rho) \geq 0, \quad (47)$$

where the third inequality holds since $S^\varepsilon_n \subset S^\varepsilon_m$ for $\varepsilon_n \geq \varepsilon_m$, respectively $n \leq m$. The final inequality follows from the defining property of $R^\varepsilon$. For any fixed $\rho \in S$ and for all $n \in \mathbb{N}$, let $\rho^n \in S^n$ be such that $\lim_{n \to \infty} \rho^n = \rho \in S$. (By definition of $S^n$ it follows that such a sequence $(\rho^n)_{n \in \mathbb{N}}$ with $\rho^n \in S^n$ always exists.) Property 3 together with (47) yields

$$\Delta_{\bar{R}}(\rho) = \Delta_{\bar{R}}(\lim_{n \to \infty} \rho^n) \geq \limsup_{n \to \infty} \Delta_{\bar{R}}(\rho^n) \geq \limsup_{n \to \infty} \Delta_{\bar{R}}(S^n) \geq 0.$$ (48)

Since (48) holds for any $\rho \in S$, we obtain $\Delta_{\bar{R}}(S) \geq 0$, which completes the proof of Proposition 3.1.

Step 3: From Proposition 3.1 to Theorem 2.1 for fixed system $A$

We next show that Theorem 2.1, for the case where $A$ is a fixed finite-dimensional system, follows from Proposition 3.1. For this we use Proposition 3.1 for the function family

$$\Delta_{\mathcal{R}} : \mathcal{D}(A \otimes B \otimes C) \to \mathbb{R} \cup \{-\infty\}$$

$$\rho_{ABC} \mapsto I(A : C|B)_\rho - D_M(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})), \quad (49)$$

with $\mathcal{R}_{B \rightarrow BC} \in \mathcal{TPC}(B, B \otimes C)$. We note that since $C$ is finite-dimensional this implies that $\Delta_{\mathcal{R}}(\rho) < \infty$ for all $\rho \in \mathcal{D}(A \otimes B \otimes C)$. To apply Proposition 3.1, we have to verify that the function family $I(A : C|B \rho) \equiv \rho \mapsto \Delta_R(\rho) \in \mathbb{R} \cup \{-\infty\}$ of the form (49) satisfies the assumptions of the proposition. This is ensured by the following lemma.

Lemma 3.2. Let $A$ be a separable and $B$ and $C$ finite-dimensional Hilbert spaces. The function family $\Delta_{\mathcal{R}}(\cdot)$ defined by (49) satisfies Properties 1-4.

Proof. We first verify that the function $\Delta_{\mathcal{R}}(\cdot)$ satisfies Property 1. For any state $\rho^p$ of the form (15), we have by the definition of the mutual information

$$I(\hat{A}A : C|B)_{\rho^p} = H(C|B)_{\rho^p} - H(C|BA\hat{A})_{\rho^p}. \quad (50)$$

Because $\rho^p_{BC} = \rho_{BC}$, the first term, $H(C|B)_{\rho^p}$, is independent of $p$, i.e., $H(C|B)_{\rho^p} = H(C|B)_{\rho^p} = H(C|B)_{\rho^p}$. The second term can be written as an expectation over $A$, i.e.,

$$H(C|BA\hat{A})_{\rho^p} = (1 - p)H(C|BA)_{\rho^p} + pH(C|BA)_{\rho^p}. \quad (51)$$

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As a result we find
\[ I(\hat{A}A : C|B)_{\rho} = (1-p)I(A : C|B)_{\rho} + pI(A : C|B)_{\rho}. \] (52)

The density operator \( \mathcal{R}_{B \rightarrow BC}(\rho^p_{ABC}) \) can be written as
\[ \mathcal{R}_{B \rightarrow BC}(\rho^p_{ABC}) = (1-p)|0\rangle|0\rangle_{A} \otimes \mathcal{R}_{B \rightarrow BC}(\rho^0_{AB}) + p|1\rangle|1\rangle_{A} \otimes \mathcal{R}_{B \rightarrow BC}(\rho_{AB}). \] (53)

We can thus apply Lemma B.3, from which we obtain
\[ D_M(\rho^p_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho^p_{ABC})) = (1-p)D_M(\rho^0_{AB}||\mathcal{R}_{B \rightarrow BC}(\rho^0_{AB})) + pD_M(\rho^0_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho^0_{AB})). \] (54)

Equations (52) and (54) imply that
\[ \Delta_\mathcal{R}(\rho^p) = (1-p)\Delta_\mathcal{R}(\rho^0) + p\Delta_\mathcal{R}(\rho), \] (55)

which concludes the proof of Property 1.

That \( \Delta_\mathcal{R}(\cdot) \) satisfies Property 2 can be seen as follows. Let \( \mathcal{R}_{B \rightarrow BC}, \mathcal{R}'_{B \rightarrow BC} \in \text{TPCP}(B, B \otimes C), \) \( \alpha \in [0, 1] \) and \( \mathcal{R}_{B \rightarrow BC} = \alpha \mathcal{R}_{B \rightarrow BC} + (1-\alpha)\mathcal{R}'_{B \rightarrow BC}. \) Lemma B.4 implies that for any state \( \rho_{ABC} \) on \( A \otimes B \otimes C \) we have
\[ D_M(\rho_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho_{AB})) = D_M(\rho_{ABC}||\alpha \mathcal{R}_{B \rightarrow BC}(\rho_{AB}) + (1-\alpha)\mathcal{R}'_{B \rightarrow BC}(\rho_{AB})) \leq \alpha D_M(\rho_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho_{AB})) + (1-\alpha)D_M(\rho_{ABC}||\mathcal{R}'_{B \rightarrow BC}(\rho_{AB})) \] (56)
and hence
\[ \Delta_\mathcal{R}(\rho) \geq \alpha \Delta_\mathcal{R}(\rho) + (1-\alpha)\Delta_{\mathcal{R}'}(\rho). \] (57)

We next verify that the function \( \Delta_{\mathcal{R}}(\cdot) \) satisfies Property 3. The Alicki-Fannes inequality ensures that \( D(\alpha \otimes B \otimes C) \equiv \rho \mapsto I(A : C|B)_{\rho} \in \mathbb{R}^+ \) is continuous since \( C \) is finite-dimensional [1]. By the definition of \( \Delta_{\mathcal{R}}(\cdot) \) it thus suffices to show that \( D(\alpha \otimes B \otimes C) \equiv \rho_{ABC} \mapsto D_M(\rho_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \in \mathbb{R}^+ \) is lower semicontinuous. Let \( \{\rho^n_{ABC}\}_{n \in \mathbb{N}} \) be a sequence of states on \( A \otimes B \otimes C \) such that \( \lim_{n \to \infty} \rho^n_{ABC} = \rho_{ABC} \in D(A \otimes B \otimes C). \) By definition of the measured relative entropy (see Definition B.1), we find for \( M := \{M : M(\rho) = \sum_x \text{tr}(M_x)|x\rangle\langle x| \text{ with } \sum_x M_x = \text{id}\}, \)
\[ \liminf_{n \to \infty} D_M(\rho^n_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho^n_{AB})) = \liminf_{n \to \infty} \sup_{M \in M} D\left(M(\rho^n_{ABC})||M(\mathcal{R}_{B \rightarrow BC}(\rho^n_{AB}))\right) \geq \sup_{M \in M} \liminf_{n \to \infty} D\left(M(\rho^n_{ABC})||M(\mathcal{R}_{B \rightarrow BC}(\rho^n_{AB}))\right) \geq \sup_{M \in M} D\left(M(\rho_{ABC})||M(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}))\right) \]
\[ = D_M(\rho_{ABC}||\mathcal{R}_{B \rightarrow BC}(\rho_{AB})). \] (58)

In the penultimate step, we use that the relative entropy is lower semicontinuous [17, Exercise 7.22] and that \( M \) as well as \( \mathcal{R}_{B \rightarrow BC} \) are linear and bounded operators and hence continuous.

We finally show that \( \Delta_{\mathcal{R}}(\cdot) \) fulfills Property 4. It suffices to verify that \( \text{TPCP}(B, B \otimes C) \equiv \mathcal{R} \mapsto D_M(\rho_{ABC}||\mathcal{R}(\rho_{AB})) \in \mathbb{R}^+ \) is lower semicontinuous where by definition we have \( D_M(\rho_{ABC}||\mathcal{R}(\rho_{AB})) = \sup_{M \in M} D\left(M(\rho_{ABC})||M(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}))\right) \). Note that since \( \mathcal{R} \) and \( M \) are linear bounded operators and hence continuous for two states \( \sigma_1 \) and \( \sigma_2 \) defined by \( D(\sigma_1||\sigma_2) := \text{tr}(\sigma_1 \log \sigma_1 - \log \sigma_2) \) we find that \( \mathcal{R} \mapsto D(\mathcal{M}(\rho_{ABC})||\mathcal{M}(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}))) \) is continuous as the logarithm \( \mathbb{R}^+ \ni x \mapsto \log x \in \mathbb{R} \) is continuous. Since the supremum of continuous functions is lower semicontinuous [6, Chapter IV, Section 6.2, Theorem 4], the assertion follows.
What remains to be shown in order to apply Proposition 3.1 is that for any \( \rho \in \mathcal{S} \) where \( \mathcal{S} \) is the set of states on \( A \otimes B \otimes C \) with a fixed marginal \( \rho_{BC} \) on \( B \otimes C \), there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \in \mathcal{P} \) such that \( \Delta_{\mathcal{R}}(\rho) \geq 0 \). By choosing \( \mathcal{P} = TP_{C}(B, B \otimes C) \), the main result of [7] however precisely proves this. We have thus shown that \( \Delta_{\mathcal{R}}(\rho) \geq 0 \) holds for a universal recovery map \( \mathcal{R}_{B \rightarrow BC} \in \mathcal{P} \), so that (8) follows for any fixed dimension of the \( A \) system. This proves the statement of Remark 2.2 (and, hence, Theorem 2.1) for the case where \( A \) is a fixed finite-dimensional Hilbert space.

**Step 4: Independence from the \( A \) system**

Let \( \mathcal{S} \) be the set of all density operators on \( \bar{A} \otimes B \otimes C \) with a fixed marginal \( \rho_{BC} \) on \( B \otimes C \), where \( B \) and \( C \) are finite-dimensional Hilbert spaces and \( \bar{A} \) is the infinite-dimensional Hilbert space \( \ell^{2} \) of square summable sequences. We now show that there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \) such that \( \Delta_{\mathcal{R}}(\mathcal{S}) \geq 0 \).

Let \( \{\Pi_{a}^{A}\}_{a \in \mathbb{N}} \) be a sequence of finite-rank projectors on \( \bar{A} \) that converges to \( \text{id}_{\bar{A}} \) with respect to the weak operator topology. Let \( \mathcal{S}^{a} \) denote the set of states whose marginal on \( \bar{A} \) is contained in the support of \( \Pi_{a}^{A} \) and with the same fixed marginal \( \rho_{BC} \) on \( B \otimes C \) as the elements of \( \mathcal{S} \). For all \( a \in \mathbb{N} \), let \( \mathcal{R}_{B \rightarrow BC}^{a} \) denote a recovery map that satisfies \( \Delta_{\mathcal{R}}(\mathcal{S}^{a}) \geq 0 \). Note that the existence of such maps is already established by the proof of Theorem 2.1 for the finite-dimensional case. As the set of trace-preserving completely positive maps on finite-dimensional systems is compact (see Remark C.3) there exists a subsequence \( \{a_{i}\}_{i \in \mathbb{N}} \) such that \( \lim_{i \to \infty} a_{i} = \infty \) and \( \lim_{i \to \infty} \mathcal{R}_{BC}^{a_{i}} = \mathcal{R} \in TP_{C}(B, B \otimes C) \). For every \( \rho \in \mathcal{S} \) there exists a sequence of states \( \{\rho^{a}\}_{a \in \mathbb{N}} \) with \( \rho^{a} \in \mathcal{S}^{a} \) that converges to \( \rho \) in the trace norm (see Lemma E.3). Lemma 3.2 (in particular Properties 3 and 4), yields for any \( \rho \in \mathcal{S} \)

\[
\Delta_{\mathcal{R}}(\rho) \geq \limsup_{a \to \infty} \Delta_{\mathcal{R}}(\rho^{a}) \geq \limsup_{a \to \infty} \limsup_{i \to \infty} \Delta_{\mathcal{R}}(\rho^{a}) \geq \limsup_{a \to \infty} \inf_{\rho^{a} \in \mathcal{S}^{a}} \limsup_{i \to \infty} \Delta_{\mathcal{R}}(\rho^{a}) \geq \limsup_{i \to \infty} \inf_{\rho^{a} \in \mathcal{S}^{a}} \Delta_{\mathcal{R}}(\rho^{a}) = \limsup_{i \to \infty} \Delta_{\mathcal{R}}^{a_{i}}(\rho^{a_{i}}) \geq 0. \tag{59}
\]

The fourth inequality follows since \( a_{i} \geq a \) for large enough \( i \) and since this implies that \( \mathcal{S}^{a_{i}} \supset \mathcal{S}^{a} \), and the final inequality follows by definition of \( \mathcal{R}^{a_{i}} \). This shows that \( \Delta_{\mathcal{R}}(\mathcal{S}) \geq 0 \).

To retrieve the statement of Remark 2.2 (and hence Theorem 2.1 for finite-dimensional \( B \) and \( C \)), we need to argue that this same map \( \mathcal{R} \) remains valid when we consider any separable space \( A \). In order to do this, observe that any separable Hilbert space \( A \) can be isometrically embedded into \( \bar{A} \) [25, Theorem II.7]. To conclude, it suffices to remark that \( \Delta_{\mathcal{R}} \) is invariant under isometries applied on the space \( A \).

## 4 Extension to infinite dimensions

In this section we show how to obtain the statement of Theorem 2.1 for separable (not necessarily finite-dimensional) Hilbert spaces \( A, B, C \) from the finite-dimensional case that has been proven in Section 3. For trace non-increasing completely positive maps \( \mathcal{R}_{B \rightarrow BC} \) we define the function family

\[
\bar{\Delta}_{\mathcal{R}} : \mathcal{D}(A \otimes B \otimes C) \to \mathbb{R} \cup \{-\infty\},
\]

\[
\rho_{AB} \mapsto F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) - 2^{-\frac{1}{2}I(A;C|B)_{\rho}}, \tag{60}
\]

where \( \mathcal{D}(A \otimes B \otimes C) \) denotes the set of states on \( A \otimes B \otimes C \). We will use the same notation as introduced at the beginning of Section 3. In addition, we take \( \mathcal{S} \) to be the set of all states on \( A \otimes B \otimes C \) with a fixed marginal \( \rho_{BC} \) on \( B \otimes C \). The proof proceeds in two steps where we first show that there exists a sequence of recovery maps \( \{\mathcal{R}_{B \rightarrow BC}^{a}\}_{a \in \mathbb{N}} \) such that \( \lim_{a \to \infty} \bar{\Delta}_{\mathcal{R}}(\mathcal{S}) \geq 0 \), where the property that all elements of \( \mathcal{S} \) have the same marginal on the \( B \otimes C \) system will be important. In the second step we conclude by an approximation argument that there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \) such that \( \bar{\Delta}_{\mathcal{R}}(\mathcal{S}) \geq 0 \).
Step 1: Existence of a sequence of recovery maps

We start by introducing some notation that is used within this step. Let \( \{\Pi_B^b\}_{b \in \mathbb{N}} \) and \( \{\Pi_C^c\}_{c \in \mathbb{N}} \) be sequences of finite-rank projectors on \( B \) and \( C \) which converge to \( \text{id}_B \) and \( \text{id}_C \) with respect to the weak operator topology. For any given \( \rho_{ABC} \in \mathcal{D}(A \otimes B \otimes C) \) consider the normalized projected states

\[
\rho_{ABC}^{b,c} := \frac{(\text{id}_A \otimes \Pi_B^b \otimes \Pi_C^c)\rho_{ABC}(\text{id}_A \otimes \Pi_B^b \otimes \Pi_C^c)}{\text{tr}((\text{id}_A \otimes \Pi_B^b \otimes \Pi_C^c)\rho_{ABC})}
\]

and

\[
\rho_{ABC}^c := \frac{(\text{id}_A \otimes \text{id}_B \otimes \Pi_C^c)\rho_{ABC}(\text{id}_A \otimes \text{id}_B \otimes \Pi_C^c)}{\text{tr}((\text{id}_A \otimes \text{id}_B \otimes \Pi_C^c)\rho_{ABC})},
\]

where for any \( c \in \mathbb{N} \), the sequence \( \{\rho_{ABC}^{b,c}\}_{b \in \mathbb{N}} \) converges to \( \rho_{ABC}^c \) in the trace norm (see, e.g., Corollary 2 of [13]) and the sequence \( \{\rho_{ABC}^c\}_{c \in \mathbb{N}} \) converges to \( \rho_{ABC}^c \) also in the trace norm. Let \( \mathcal{S}^{b,c} \) be the set of states that is generated by (61) for all \( \rho_{ABC} \in \mathcal{S} \). We note that for any given \( b, c \) all elements of \( \mathcal{S}^{b,c} \) have an identical marginal on \( B \otimes C \). Let \( \mathcal{R}^{b,c}_{B \rightarrow BC} \) denote a recovery map that satisfies \( \Delta_{\mathcal{R}^{b,c}_{B \rightarrow BC}}(\mathcal{S}^{b,c}) \geq 0 \) whose existence is established in the proof of Theorem 2.1 for finite-dimensional systems \( B \) and \( C \) (see Section 3). We next state a lemma that explains how \( \Delta_{\mathcal{R}}(\rho) \) changes when we replace \( \rho \) by a projected state \( \rho_{b,c}^{b,c} \).

**Lemma 4.1.** For any \( \rho_{BC} \in \mathcal{D}(B \otimes C) \) there exists a sequence of reals \( \{\xi_{b,c}^{b,c}\}_{b,c \in \mathbb{N}} \) that satisfies \( 4 \lim_{b,c \rightarrow \infty} \lim_{b \rightarrow \infty} \xi_{b,c}^{b,c} = 0 \), such that for any \( R \in \text{TPCP}(B,B \otimes C) \), any extension \( \rho_{ABC} \) of \( \rho_{BC} \), and \( \rho_{ABC}^c \) as given in (61) we have

\[
\Delta_{\mathcal{R}}(\rho_{b,c}^{b,c}) - \Delta_{\mathcal{R}}(\rho) \leq \xi_{b,c}^{b,c} \quad \text{for all} \quad b,c \in \mathbb{N}.
\]

**Proof.** We note that local projections applied to the subsystem \( C \) can only decrease the mutual information, i.e.,

\[
\text{tr}(\Pi_C^c \rho_{BC})I(A : C|B)_\rho \leq I(A : C|B)_\rho.
\]

The Alicki-Fannes inequality [1] ensures that for a fixed finite-dimensional system \( C \) the conditional mutual information \( I(A : C|B)_\rho = H(C|B)_\rho - H(C|AB)_\rho \) is continuous in \( \rho \) with respect to the trace norm, i.e.,

\[
I(A : C|B)_{\rho_{b,c}} \leq I(A : C|B)_{\rho} + 8 \varepsilon_{b,c}^B \log(\text{rank} \Pi_C^c) + 4h(\varepsilon_{b,c}^B),
\]

where \( \varepsilon_{b,c} = \|\rho_{ABC}^{b,c} - \rho_{ABC}^c\|_1 \) and \( h(\cdot) \) denotes the binary Shannon entropy function defined by \( h(p) := -p \log_2(p) - (1 - p) \log_2(1 - p) \) for \( 0 \leq p \leq 1 \). Using the Fuchs-van de Graaf inequality [11] and Lemma E.1, we find

\[
\varepsilon_{b,c}^B \leq 2\sqrt{1 - F(\rho_{ABC}^{b,c}, \rho_{ABC}^c)^2} \leq 2\sqrt{1 - \text{tr}(\Pi_B^b \otimes \Pi_C^c \rho_{BC})/\text{tr}(\Pi_C^c \rho_{BC})}.
\]

Combining (64) and (65) yields

\[
I(A : C|B)_{\rho_{b,c}} \leq \frac{1}{\text{tr}(\Pi_C^c \rho_{BC})} I(A : C|B)_{\rho} + 8 \varepsilon_{b,c}^B \log(\text{rank} \Pi_C^c) + 4h(\varepsilon_{b,c}^B).
\]

Since \( x^y \leq x - y + 1 \) for \( x, y \in [0,1] \),

\[
2^{-\frac{1}{2}I(A:C|B)_\rho} - 2^{-\frac{1}{2}I(A:C|B)_{\rho_{b,c}}} \leq 2^{-\frac{1}{2}I(A:C|B)_\rho} - 2^{-\frac{1}{2}\text{tr}(\Pi_C^c \rho_{BC})I(A:C|B)_{\rho_{b,c}}} - \text{tr}(\Pi_C^c \rho_{BC}) + 1.
\]

\footnote{The precise form of the sequence \( \{\xi_{b,c}^{b,c}\}_{b,c \in \mathbb{N}} \) is given in the proof (see Equation (75)).}

\footnote{For \( x = 0 \) the statement clearly holds. For \([0,1] \times [0,1] \ni (x, y) \mapsto f(x, y) := x^y - x + y - 1 \in \mathbb{R} \) we find by using the convexity of \( y \mapsto f(x, y) \) that \( \max_{x \in [0,1]} \max_{y \in [0,1]} f(x, y) = 0. \)
According to (67) and since $2^{-x} \geq 1 - \ln(2)x$ for $x \in \mathbb{R}$, we have

$$2^{-\frac{1}{2}} \text{Tr} (\Pi_C \rho) I(A:C|B)_{a,b,c} \geq 2^{-\frac{1}{2}} I(A:C|B)_{a,b,c} - 2^{-\frac{1}{2}} \text{Tr} (\Pi_C \rho) (8e^{b,c} \log(\text{rank} \Pi_C) + 4h(e^{b,c}))$$

$$\geq 2^{-\frac{1}{2}} I(A:C|B)_{a,b,c} - \frac{\ln(2)}{2} \text{Tr} (\Pi_C \rho) (8e^{b,c} \log(\text{rank} \Pi_C) + 4h(e^{b,c})) . \quad (69)$$

Combining (68) and (69) yields

$$2^{-\frac{1}{2}} I(A:C|B)_{a,b,c} - \frac{\ln(2)}{2} \text{Tr} (\Pi_C \rho) (8e^{b,c} \log(\text{rank} \Pi_C) + 4h(e^{b,c})) \leq 1 - \text{tr} (\Pi_B \otimes \Pi_C \rho_{BC}) . \quad (70)$$

For two states $\sigma_1$ and $\sigma_2$ let $P(\sigma_1, \sigma_2) := \sqrt{1 - F(\sigma_1, \sigma_2)^2}$ denote the purified distance. Applying the Fuchs-van de Graaf inequality [11] and Lemma E.1 gives

$$P(\rho_{ABC}, \rho_{ABC}^{b,c})^2 = 1 - F(\rho_{ABC}, \rho_{ABC}^{b,c})^2 \leq 1 - \text{tr} (\Pi_B^b \otimes \Pi_C \rho_{BC}) . \quad (71)$$

Since the purified distance is a metric [28] that is monotonous under trace-preserving completely positive maps [27, Theorem 3.4], (71) gives

$$P(\rho_{ABC}, R_{B \to BC}(\rho_{AB}))$$

$$\leq P(\rho_{ABC}, \rho_{ABC}^{b,c}) + P(\rho_{ABC}^{b,c}, R_{B \to BC}(\rho_{AB}^{b,c})) + P(R_{B \to BC}(\rho_{AB}^{b,c}), R_{B \to BC}(\rho_{AB}))$$

$$\leq 2P(\rho_{ABC}, \rho_{ABC}^{b,c}) + P(\rho_{ABC}^{b,c}, R_{B \to BC}(\rho_{AB}^{b,c}))$$

$$\leq P(\rho_{ABC}^{b,c}, R_{B \to BC}(\rho_{AB}^{b,c})) + 2\sqrt{1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})} . \quad (72)$$

As the fidelity for states lies between zero and one, (72) implies

$$F(\rho_{ABC}^{b,c}, R_{B \to BC}(\rho_{AB}^{b,c}))^2$$

$$\leq F(\rho_{ABC}, R_{B \to BC}(\rho_{AB}))^2 + 4(1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})) + 4\sqrt{1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})}$$

$$\leq F(\rho_{ABC}, R_{B \to BC}(\rho_{AB}))^2 + 8\sqrt{1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})}$$

$$\leq \left( F(\rho_{ABC}, R_{B \to BC}(\rho_{AB})) + 2\sqrt{2}(1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})) \right)^{\frac{1}{2}} . \quad (73)$$

This implies that

$$F(\rho_{ABC}^{b,c}, R_{B \to BC}(\rho_{AB}^{b,c})) \leq F(\rho_{ABC}, R_{B \to BC}(\rho_{AB})) + 2\sqrt{2}(1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})) \frac{1}{2} . \quad (74)$$

By definition of the quantity $\Delta_R(\cdot)$ (see Equation (60)) the combination of (70) and (74) yields

$$\Delta_R(\rho^{b,c}) - \Delta_R(\rho)$$

$$\leq \frac{\ln(2)}{2} (8e^{b,c} \log(\text{rank} \Pi_C) + 4h(e^{b,c})) + (1 - \text{tr}(\Pi_C \rho_{BC})) + 2\sqrt{2}(1 - \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC})) \frac{1}{2}$$

$$= \xi^{b,c} , \quad (75)$$

where $e^{b,c}$ is bounded by (66). By Lemma E.2, we find $\lim_{b \to 0^+} \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC}) = \text{tr}(\Pi_C \rho_{BC})$ for all $c \in \mathbb{N}$ and hence $\lim_{b \to 0^+} e^{b,c} = 0$ for any $c \in \mathbb{N}$. Furthermore, we have $\lim_{c \to 0^+} \text{tr}(\Pi_C \rho_{PC}) = 1$ and $\lim_{c \to 0^+} \lim_{b \to 0^+} \text{tr}(\Pi_B^b \otimes \Pi_C \rho_{BC}) = 1$ which implies that $\lim_{c \to 0^+} \lim_{b \to 0^+} \xi^{b,c} = 0$. This proves the assertion.
By Lemma 4.1, using the notation defined at the beginning of Step 1, we find
\[
\lim_{c \to \infty} \lim_{b \to \infty} \sup \Delta_{\mathcal{R}^{b,c}}(\mathcal{S}) = \limsup_{c \to \infty} \limsup_{b \to \infty} \inf_{\rho \in \mathcal{S}} \Delta_{\mathcal{R}^{b,c}}(\rho) = \limsup_{c \to \infty} \limsup_{b \to \infty} \inf_{\rho \in \mathcal{S}} \left\{ \Delta_{\mathcal{R}^{b,c}}(\rho^{b,c}) - \xi^{b,c} \right\} \\
= \limsup_{c \to \infty} \limsup_{b \to \infty} \inf_{\rho^{b,c} \in \mathcal{S}^{b,c}} \Delta_{\mathcal{R}^{b,c}}(\rho^{b,c}) - \xi^{b,c} = \limsup_{c \to \infty} \limsup_{b \to \infty} \sup \Delta_{\mathcal{R}^{b,c}}(\mathcal{S}) \geq 0, 
\]
where the second equality step is valid since all states in $\mathcal{S}$ have the same fixed marginal on $B \otimes C$ and since the sequence $\{\xi^{b,c}\}_{b,c \in \mathbb{N}}$ only depends on this marginal. The penultimate step uses that $\lim_{c \to \infty} \lim_{b \to \infty} \xi^{b,c} = 0$. The final inequality follows by definition of $\mathcal{R}^{b,c}_{B \to BC}$. Inequality (76) implies that there exist sequences $\{b_k\}_{k \in \mathbb{N}}$ and $\{c_k\}_{k \in \mathbb{N}}$ such that $\limsup_{k \to \infty} \Delta_{\mathcal{R}^{b_k,c_k}}(\mathcal{S}) \geq 0$. Setting $\mathcal{R}^{k}_{B \to BC} = \mathcal{R}^{b_k,c_k}_{B \to BC}$ then implies that there exists a sequence $\{\mathcal{R}^{k}_{B \to BC}\}_{k \in \mathbb{N}}$ of recovery maps that satisfies
\[
\lim_{k \to \infty} \sup \Delta_{\mathcal{R}^{k}}(\mathcal{S}) \geq 0. 
\]

**Step 2: Existence of a limit**

Recall that $\mathcal{S}$ is the set of density operators on $A \otimes B \otimes C$ with a fixed marginal $\rho_{BC}$ on $B \otimes C$. The goal of this step is to use (77) to prove that there exists a recovery map $\mathcal{R}_{B \to BC}$ such that
\[
\Delta_{\mathcal{R}}(\mathcal{S}) \geq 0. 
\]

Let $\{\Pi^m_A\}_{m \in \mathbb{N}}$ and $\{\Pi^m_B\}_{m \in \mathbb{N}}$ be sequences of projectors with rank $m$ that weakly converge to $\id_B$ and $\id_C$, respectively. Furthermore, for any $m$ and any $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$ let $[\mathcal{R}]^m$ be the trace non-increasing map obtained from $\mathcal{R}$ by projecting the input and output with $\Pi^m_A$ and $\Pi^m_B \otimes \Pi^m_C$, respectively. We start with a preparatory lemma that proves a relation between $\Delta_{[\mathcal{R}]^m}(\mathcal{S})$ and $\Delta_{\mathcal{R}}(\mathcal{S})$.

**Lemma 4.2.** For any $\rho_{BC} \in \mathcal{D}(B \otimes C)$ there exists a sequence of reals $\{\delta^m\}_{m \in \mathbb{N}}$ with $\lim_{m \to \infty} \delta^m = 0$, such that for any $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$ we have
\[
\Delta_{[\mathcal{R}]^m}(\mathcal{S}) \geq \Delta_{\mathcal{R}}(\mathcal{S}) - \delta^m - 4\varepsilon^\perp, 
\]
where $\|\mathcal{R}(\rho_B) - \rho_{BC}\|_1 \leq \varepsilon$.

**Proof.** For any $\rho_{ABC} \in \mathcal{S}$ and any $m \in \mathbb{N}$ let us define the non-negative operator $\hat{\rho}^m_{AB} := (\id_A \otimes \Pi^m_B)\rho_{AB}$ and $\id_A \otimes \Pi^m_B$. By definition of $\Delta_{\mathcal{R}}(\cdot)$ (see Equation (60)), it suffices to show that for any $\rho_{ABC} \in \mathcal{S}$, any $\mathcal{R} \in \text{TPCP}(B, B \otimes C)$, $\varepsilon \in [0, 2]$ such that $\|\mathcal{R}(\rho_B) - \rho_{BC}\|_1 \leq \varepsilon$ and
\[
\hat{\rho}^m_{ABC} := (\id_A \otimes \Pi^m_B \otimes \Pi^m_C)\mathcal{R}_{B \to BC}(\hat{\rho}^m_{AB})(\id_A \otimes \Pi^m_B \otimes \Pi^m_C) 
\]
we have $F(\rho_{ABC}, \hat{\rho}^m_{ABC}) \geq F(\rho_{ABC}, \mathcal{R}_{B \to BC}(\rho_{AB})) - \delta^m - 4\varepsilon^\perp$. As in Step 1 let $P(\cdot, \cdot)$ denote the purified distance. Lemma E.1 implies that
\[
P(\rho_{AB}, \hat{\rho}^m_{AB})^2 = 1 - F(\rho_{AB}, \hat{\rho}^m_{AB})^2 \leq 1 - \text{tr}(\rho_B \Pi^m_B)^2. 
\]
Similarly, we obtain
\[
P(\mathcal{R}_{B \to BC}(\hat{\rho}^m_{AB}), \hat{\rho}^m_{AB})^2 \leq 1 - \text{tr}(\mathcal{R}_{B \to BC}(\hat{\rho}^m_{AB})\Pi^m_B \otimes \Pi^m_C)^2 = 1 - \text{tr}(\mathcal{R}_{B \to BC}(\hat{\rho}^m_{AB})\Pi^m_B)^2 \cdot \Pi^m_C)^2. 
\]

By Hölder’s inequality, monotonicity of the trace norm for trace-preserving completely positive maps [30, Example 9.1.8 and Corollary 9.1.10] and (81) together with the Fuchs-van de Graaf inequality [11] and Lemma E.1 we find

\[\text{The precise form of the sequence } \{\delta^m\}_{m \in \mathbb{N}} \text{ can be found in the proof (see Equation (86)).}\]
\[
\left| \text{tr} \left( (\mathcal{R}_{B \rightarrow BC}(\hat{\rho}_B^m) - \mathcal{R}_{B \rightarrow BC}(\rho_B))\Pi_B^m \otimes \Pi_C^m \right) \right| \leq \| \mathcal{R}_{B \rightarrow BC}(\rho_B) - \mathcal{R}_{B \rightarrow BC}(\hat{\rho}_B^m) \|_1 \| \Pi_B^m \otimes \Pi_C^m \|_{\infty} \\
= \| \mathcal{R}_{B \rightarrow BC}(\rho_B) - \mathcal{R}_{B \rightarrow BC}(\hat{\rho}_B^m) \|_1 \leq \| \rho_B - \hat{\rho}_B^m \|_1 \leq \| \rho_{AB} - \hat{\rho}_{AB}^m \|_1 \leq 2\sqrt{1 - \text{tr}(\rho_B \Pi_B^m)^2} . \quad (83)
\]

Combining (82), (83) and Hölder’s inequality together with the assumption \( \| \mathcal{R}(\rho_B) - \rho_{BC} \|_1 \leq \varepsilon \) gives
\[
P(\mathcal{R}_{B \rightarrow BC}(\hat{\rho}_{AB}^m), \rho_{AB}^m) \leq 1 - \text{tr}(\mathcal{R}_{B \rightarrow BC}(\rho_{AB})\Pi_B^m \otimes \Pi_C^m) + 4\sqrt{1 - \text{tr}(\rho_B \Pi_B^m)^2} + 2\varepsilon . \quad (84)
\]

Inequalities (81), (84) and the monotonicity of the purified distance under trace-preserving and completely positive maps [27, Theorem 3.4] show that
\[
P(\rho_{ABC}, \rho_{ABC}^m) \leq P(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) + P(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}), \mathcal{R}_{B \rightarrow BC}(\rho_{AB}^m)) + P(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}), \rho_{ABC}) \leq P(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) + P(\rho_{AB}, \rho_{AB})^m + P(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}), \rho_{ABC})^m \leq P(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) + (\delta^m)^2 / 8 + 2\varepsilon , \quad (85)
\]

for
\[
\delta^m := \sqrt{8 \left( \sqrt{1 - \text{tr}(\rho_B \Pi_B^m)^2} + \sqrt{\text{tr}(\rho_{BC} \Pi_B^m \otimes \Pi_C^m)^2} + 4\sqrt{1 - \text{tr}(\rho_B \Pi_B^m)^2} \right)^{- \frac{1}{2}}} . \quad (86)
\]

As the purified distance between two states lies inside the interval \([0, 1]\) and since \((\delta^m)^2 / 8 + 2\varepsilon \in [0, 6]\), (85) implies that whenever \(F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB}))^2 \geq (\delta^m)^2 + 8\sqrt{2\varepsilon}\), we have
\[
F(\rho_{ABC}, \rho_{ABC}^m)^2 \geq F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB}))^2 - (\delta^m)^2 - 8\sqrt{2\varepsilon} \geq F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) - \sqrt{(\delta^m)^2 + 8\sqrt{2\varepsilon}}^2 . \quad (87)
\]

As a result, we find
\[
F(\rho_{ABC}, \rho_{ABC}^m) \geq F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) - \delta^m - \sqrt{8(2\varepsilon)^{\frac{1}{2}}} , \quad (88)
\]

which proves (79) since \(\sqrt{5} 2^{\frac{1}{2}} \leq 4\).

Recall that \(B\) and \(C\) are separable Hilbert spaces and that \(\{\Pi_B^m\}_{m \in \mathbb{N}}\) and \(\{\Pi_B^m \otimes \Pi_C^m\}_{m \in \mathbb{N}}\) converge weakly to \(\text{id}_B\) and \(\text{id}_B \otimes \text{id}_C\) respectively. Lemma E.2 thus shows that \(\lim_{m \rightarrow \infty} \text{tr}(\rho_B \Pi_B^m) = 1\) and \(\lim_{m \rightarrow \infty} \text{tr}(\rho_{BC} \Pi_B^m \otimes \Pi_C^m) = 1\), which implies \(\lim_{m \rightarrow \infty} \delta^m = 0\).

The following lemma proves that for sufficiently large \(m\) and a recovery map \(\mathcal{R}_{B \rightarrow BC}\) that maps \(\rho_B\) to density operators that are close to \(\rho_{BC}\), the operator \([\mathcal{R}]^m(\rho_{AB})\) has a trace that is bounded from below by essentially one.

**Lemma 4.3.** Let \(A, B,\) and \(C\) be separable Hilbert spaces. For any density operator \(\rho_{AB} \in \mathcal{D}(A \otimes B)\) and any \(\mathcal{R} \in \text{TPCP}(B, B \otimes C)\) we have
\[
\text{tr}([\mathcal{R}]^m(\rho_{AB})) \geq \text{tr}(\Pi_B^m \otimes \Pi_C^m) - 2\sqrt{1 - \text{tr}(\Pi_B^m \rho_B)} - \| \mathcal{R}(\rho_B) - \rho_{BC} \|_1 . \quad (89)
\]

**Proof.** We first note that by Hölder’s inequality and monotonicity of the trace norm for trace-preserving completely positive maps [30, Example 9.1.8 and Corollary 9.1.10] we have
\[
|\text{tr} (\Pi_B^m \otimes \Pi_C^m (\mathcal{R}(\rho_B) - \mathcal{R}(\Pi_B^m \rho_B \Pi_B^m)))| \leq \| \mathcal{R}(\rho_B) - \mathcal{R}(\Pi_B^m \rho_B \Pi_B^m) \|_1 \leq \| \rho_B - \Pi_B^m \rho_B \Pi_B^m \|_1 . \quad (90)
\]
Together with Hölter’s inequality this implies
\[
\text{tr}([R]^m(\rho_{AB})) = \text{tr}(\Pi_B^m \otimes \Pi_C^m \mathcal{R}(\Pi_B^m \rho_{AB} \Pi_B^m)) = \text{tr}(\Pi_B^m \otimes \Pi_C^m \mathcal{R}(\Pi_B^m \rho_B \Pi_B^m)) \\
\geq \text{tr}(\Pi_B^m \otimes \Pi_C^m \mathcal{R}(\rho_B)) - \|\rho_B - \Pi_B^m \rho_B \Pi_B^m\|_1 \\
\geq \text{tr}(\Pi_B^m \otimes \Pi_C^m \rho_{BC}) - \|\rho_B - \Pi_B^m \rho_B \Pi_B^m\|_1 - \|\mathcal{R}(\rho_B) - \rho_{BC}\|_1 .
\] (91)
Combining Lemma A.2 and Lemma E.1 gives
\[
\|\rho_B - \Pi_B^m \rho_B \Pi_B^m\|_1 \leq 2 \sqrt{1 - F(\rho_B, \Pi_B^m \rho_B \Pi_B^m)^2} = 2 \sqrt{1 - \text{tr}(\Pi_B^m \rho_B) F(\rho_B, \Pi_B^m \rho_B \Pi_B^m/\text{tr}(\Pi_B^m \rho_B))}^2 \\
\leq 2 \sqrt{1 - \text{tr}(\Pi_B^m \rho_B)} ,
\] (92)
which together with (91) proves the assertion.

According to (77) the mappings $\mathcal{R}^k$ satisfy
\[
\bar{\Delta}_{\mathcal{R}^k}(S) \geq -\bar{\epsilon}^k ,
\] (93)
with $\bar{\epsilon}^k \geq 0$ such that $\liminf_{k \to \infty} \bar{\epsilon}^k = 0$. As explained in Remark 2.3, by considering a state $\bar{\rho}_{ABC} = \rho_A \otimes \rho_{BC} \in S$, (93) implies $F(\rho_{BC}, \mathcal{R}^k(\rho_B)) \geq -\bar{\epsilon}^k + 1$. Applying the Fuchs-van de Graaf inequality [11] gives
\[
\|\rho_{BC} - \mathcal{R}^k(\rho_B)\|_1 \leq 2 \sqrt{\bar{\epsilon}^k(2 - \bar{\epsilon}^k)} =: \varepsilon^k ,
\] (94)
where $\liminf_{k \to \infty} \varepsilon^k = 0$ because $\liminf_{k \to \infty} \bar{\epsilon}^k = 0$.

By Lemma 4.2 we have
\[
\bar{\Delta}_{[\mathcal{R}^k]^m}(S) \geq \bar{\Delta}_{\mathcal{R}^k}(S) - 4(\bar{\epsilon}^k)^{1/2} - \delta^m .
\] (95)
Hence, using our starting point (77) above,
\[
\limsup_{k \to \infty} \bar{\Delta}_{[\mathcal{R}^k]^m}(S) \geq \limsup_{k \to \infty} \bar{\Delta}_{\mathcal{R}^k}(S) - 4(\bar{\epsilon}^k)^{1/2} - \delta^m \geq -\delta^m .
\] (96)
Because, for any fixed $m \in \mathbb{N}$, the mappings $[\mathcal{R}^k]^m$, for $k \in \mathbb{N}$, are all contained in the same finite-dimensional subspace (i.e., the set of trace non-increasing maps from operators on the support of $\Pi_B^m$ to operators on the support of $\Pi_B^m \otimes \Pi_C^m$), and because the space of all such mappings is compact (see Remark C.3), for any fixed $m \in \mathbb{N}$ there exists a subsequence of the sequence $\{[\mathcal{R}^k]^m\}_{k \in \mathbb{N}}$ that converges. Specifically for any fixed $m \in \mathbb{N}$ there exists a sequence $\{k^m_i\}_{i \in \mathbb{N}}$ such that
\[
\bar{\mathcal{R}}^m := \lim_{i \to \infty} [\mathcal{R}^{k^m_i}]^m
\] (97)
is well defined. Furthermore, because of the continuity of $\mathcal{R} \mapsto \bar{\Delta}_{\mathcal{R}^k}(\rho_{ABC})$ on the set of maps from operators on the support of $\Pi_B^m$ to operators on the support of $\Pi_B^m \otimes \Pi_C^m$ (see Lemma C.4), we have
\[
\bar{\Delta}_{\mathcal{R}^m}(S) = \inf_{\rho \in S} \bar{\Delta}_{\mathcal{R}^m}(\rho) = \inf_{\rho \in S} \lim_{i \to \infty} \bar{\Delta}_{[\mathcal{R}^{k^m_i}]^m}(\rho) \geq \limsup_{i \to \infty} \inf_{\rho \in S} \bar{\Delta}_{[\mathcal{R}^{k^m_i}]^m}(\rho) = \limsup_{i \to \infty} \bar{\Delta}_{[\mathcal{R}^{k^m_i}]^m}(S) \geq -\delta^m ,
\] (98)
and, hence,
\[
\liminf_{m \to \infty} \bar{\Delta}_{\mathcal{R}^m}(S) \geq 0 .
\] (99)
For any $m \in \mathbb{N}$, let $\rho_{B:C:B}^m$ be the operator obtained by applying $\tilde{R}^m$ to a purification $\rho_{B:B}$ of $\rho_B$. Without loss of generality we can assume that the projector $\Pi_B^m$ is in the eigenbasis of $\rho_B$. Let $\{k_i^m\}_{i \in \mathbb{N}}$ be a subsequence of $\{k_i^m\}_{i \in \mathbb{N}}$. Using the definition of $\tilde{R}^m$ and that $\Pi_B^m \leq \Pi_B^m'$, $\Pi_C^m \leq \Pi_C^m'$, and $\Pi_B^m \leq \Pi_B^m$ for $m \leq m'$, we obtain

$$\rho_{BC:B}^m = \tilde{R}^m(\rho_{B:B}) = \lim_{i \to \infty} \langle \tilde{R}^{k_i^m}\rangle^m(\rho_{B:B}) = \lim_{i \to \infty} (\Pi_B^m \otimes \Pi_C^m)[\tilde{R}^{k_i^m+1}](\Pi_B^m \rho_{B:B} \Pi_B^m)(\Pi_B^m \otimes \Pi_C^m)$$

$$= \lim_{i \to \infty} (\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)[\tilde{R}^{k_i^m+1}](\Pi_B^m \rho_{B:B} \Pi_B^m)(\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)$$

$$= (\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)^{m+1}(\Pi_B^m \rho_{B:B} \Pi_B^m)(\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)$$

$$= (\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)^{m+1}(\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m).$$

(100)

As a result, since $\Pi_B^m \leq \Pi_B^m$, $\Pi_C^m \leq \Pi_C^m'$, and $\Pi_B^m \leq \Pi_B^m$ for $m \leq m'$, we have for any $m \leq m'$

$$\rho_{BC:B}^m = (\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)^{m+1}(\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m).$$

(101)

Lemma E.1 together with (101) implies

$$F(\rho_{BC:B}^m, \rho_{BC:B}^{m'}) = F(\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m, \Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m, \Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m)$$

$$\geq \text{tr}(\rho_{BC:B}^{m'} \Pi_B^2 \otimes \Pi_C^2 \otimes \Pi_B^2) = \text{tr}(\rho_{BC:B}^{m'}).$$

(102)

Lemma A.2 yields for $m' \geq m$

$$\|\rho_{BC:B}^m - \rho_{BC:B}^{m'}\|_1 \leq 2 \sqrt{\text{tr}(\rho_{BC:B}^{m'})^2 - F(\rho_{BC:B}^m, \rho_{BC:B}^{m'})^2} \leq 2 \sqrt{\text{tr}(\rho_{BC:B}^{m'})^2 - \text{tr}(\rho_{BC:B}^m)^2}.$$

(103)

We now prove that as $m \to \infty$, $\text{tr}(\rho_{BC:B}^m)$ goes to 1. Note that since $B$ is a separable Hilbert space and $\rho_{B:B}$ is normalized it can be written as $\rho_{B:B} = |\psi \rangle \langle \psi|$, where $|\psi \rangle$ is a state on $B \otimes B$. Furthermore as $\Pi_B^m \otimes \Pi_C^m \otimes \Pi_B^m \leq \text{id}_{BC:B}$, (101) implies that

$$\text{tr}(\rho_{BC:B}^m) \leq \text{tr}(\rho_{BC:B}^{m'}) \leq 1 \quad \text{for } m' \geq m.$$

(104)

By definition of $\rho_{BC:B}^m$, Lemma 4.3 together with (94) implies that

$$\lim_{m \to \infty} \text{tr}(\rho_{BC:B}^m) = \lim_{m \to \infty} \text{tr}(\rho_{BC:B}^m) = 1.$$
that this does not uniquely define the recovery map $R_{B \rightarrow BC}$, which is not a problem as Theorem 2.1 proves the existence of a recovery map that satisfies (7) and does not claim that this map is unique. It remains to show that $R_{B \rightarrow BC}$ has the property (78). This follows from the observation that any density operator $\rho_{AB}$ can be obtained from the purification $\rho_{B;\tilde{B}}$ by applying a trace-preserving completely positive map $T_{\tilde{B} \rightarrow A}$ from $\tilde{B}$ to $A$. By Lemma C.5 and because $T_{\tilde{B} \rightarrow A}$ commutes with any recovery map $R_{B \rightarrow BC}$ from $B$ to $B \otimes C$, we have

$$R_{B \rightarrow BC}(\rho_{AB}) = (R_{B \rightarrow BC} \circ T_{\tilde{B} \rightarrow A})(\rho_{\tilde{B};B}) = (T_{\tilde{B} \rightarrow A} \circ R_{B \rightarrow BC})(\rho_{B;\tilde{B}}) = T_{\tilde{B} \rightarrow A}(\rho_{B;\tilde{B}}) = \lim_{m \rightarrow \infty} (R_{B \rightarrow BC} \circ T_{\tilde{B} \rightarrow A})(\rho_{B;B}) = \lim_{m \rightarrow \infty} R_{B \rightarrow BC}^{m}(\rho_{AB}).$$

(107)

Using the continuity of the fidelity (see, e.g., Lemma B.9 in [10]), this implies that

$$\Delta_{R}(\rho) = \lim_{m \rightarrow \infty} \Delta_{R}^{m}(\rho),$$

(108)

for any $\rho \in S$. Combining this with (99) gives

$$\Delta_{R}(S) = \inf_{\rho \in S} \Delta_{R}(\rho) = \inf_{\rho \in S} \lim_{m \rightarrow \infty} \Delta_{R}^{m}(\rho) = \lim_{m \rightarrow \infty} \inf_{\rho \in S} \Delta_{R}^{m}(\rho) = \lim_{m \rightarrow \infty} \Delta_{R}^{m}(S) \geq 0,$$

(109)

which concludes Step 2 and thus completes the proof of Theorem 2.1 in the general case where $B$ and $C$ are no longer finite-dimensional.

## 5 Proof of Corollary 2.4

The first statement of Corollary 2.4 that holds for separable Hilbert spaces follows immediately from Theorem 2.1, since $2^{-\frac{1}{2}I(A:C|B)_{\rho}} \geq 1 - \frac{\ln(2)}{2} I(A:C|B)_{\rho}$. The proof of the second statement of Corollary 2.4 is partitioned into three steps.\(^8\) We first show that a similar method as used in Section 3 can be used to reveal certain insights about the structure of the recovery map $R_{B \rightarrow BC}$ (which is not universal) that satisfies

$$F(\rho_{ABC},R_{B \rightarrow BC}(\rho_{AB})) \geq 1 - \frac{\ln(2)}{2} I(A:C|B)_{\rho}.$$  

(110)

In a second step, by invoking Proposition 3.1, we use this knowledge to prove that for a fixed $A$ system there exists a recovery map that satisfies (110) which is universal and preserves the structure of the non-universal recovery map from before. Finally, in Step 3 we show how the dependency on the fixed $A$ system can be removed.

### Step 1: Structure of a non-universal recovery map

We will show that for any density operator $\rho_{ABC}$ on $A \otimes B \otimes C$, where $A$, $B$, and $C$ are finite-dimensional Hilbert spaces there exists a trace-preserving completely positive map $R_{B \rightarrow BC}$ that satisfies (110) and is of the form

$$X_{B} \mapsto \rho_{BC}^{\frac{1}{2}}W_{BC}(\rho_{B}^{\frac{1}{2}}X_{B}\rho_{B}^{\frac{1}{2}} \otimes \text{id}_{C})W_{BC}^{\dagger}\rho_{BC}^{\frac{1}{2}},$$

(111)

on the support of $\rho_{B}$, where $W_{BC}$ is a unitary on $B \otimes C$. We start by proving the following preparatory lemma.

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\(^8\)Although Corollary 2.4 does not immediately follow from Theorem 2.1 it is justified to term it as such, as it follows by the same proof technique that is used to derive Theorem 2.1 (in particular it makes use of Proposition 3.1).
Lemma 5.1. For any density operator $\rho_{ABC}$ on $A \otimes B \otimes C$, where $A$, $B$, and $C$ are finite-dimensional Hilbert spaces there exists a trace-preserving completely positive map $R_{B \rightarrow BC}$ of the form

$$X_B \mapsto V_{BC} \rho_{BC}^{\frac{1}{2}} (\rho_B^{\frac{1}{2}} U_B X_B U_B^\dagger \rho_B^{-\frac{1}{2}} \otimes id_C) \rho_{BC}^{\frac{1}{2}} V_{BC}^\dagger,$$

(112)

where $U_B$ is diagonal with respect to some orthonormal basis $\{|b\rangle\}_b$ on $B$, $U_B U_B^\dagger \leq id_B$, and $V_{BC}$ is a unitary on $B \otimes C$ such that $R_{B \rightarrow BC}(\rho_B) = \rho_{BC}$ and

$$F(\rho_{ABC}, R_{B \rightarrow BC}(\rho_{AB})) \geq 1 - \frac{\ln(2)}{2} I(A : C|B)_{\rho}.$$  

(113)

Proof. Let $\rho_{ABC}$ be an arbitrary state on $A \otimes B \otimes C$ and let $\rho_{0}^{\rho}_{ABC}$ be a Markov chain with the same marginal on the $B \otimes C$ system, i.e., $\rho_{BC}^{0} = p_{BC}$. For $p \in (0, 1]$, define the state

$$\rho_{AA'BC}^{p} := (1 - p)|0\rangle \langle 0|_{A} \otimes \rho_{ABC}^{0} + p|1\rangle \langle 1|_{A} \otimes \rho_{ABC}.$$  

(114)

The main result of [10] (see Theorem 5.1 and Remark 4.3 in [10]) implies that there exists a recovery map $R_{B \rightarrow BC}$ of the form

$$X_B \mapsto V_{BC} \rho_{BC}^{\frac{1}{2}} (\rho_B^{\frac{1}{2}} U_B X_B U_B^\dagger \rho_B^{-\frac{1}{2}} \otimes id_C) \rho_{BC}^{\frac{1}{2}} V_{BC}^\dagger,$$

(115)

where $U_B$ is diagonal with respect to some orthonormal basis $\{|b\rangle\}_b$ on $B$, $U_B U_B^\dagger \leq id_B$ and $V_{BC}$ is a unitary on $B \otimes C$, such that

$$F(\rho_{AA'BC}^{p}, R_{B \rightarrow BC}(\rho_{AB})) \geq 1 - \frac{\ln(2)}{2} I(\tilde{A} : C|B)_{\rho_{AA'BC}}.$$  

(116)

By Lemma 5.2, using that $I(A : C|B)_{\rho} = 0$ since $\rho_{ABC}^{0}$ is a Markov chain, this may be rewritten as

$$p (1 - F(\rho_{ABC}, R_{B \rightarrow BC}(\rho_{AB}))) + (1 - p) (1 - F(\rho_{ABC}^{0}, R_{B \rightarrow BC}(\rho_{ABC}^{0}))) \leq p \frac{\ln(2)}{2} I(A : C|B)_{\rho}.$$  

(117)

Let us assume by contradiction that any recovery map $R_{B \rightarrow BC}$ that satisfies (117) does not leave $\rho_{ABC}^{0}$ invariant, i.e., $\rho_{ABC}^{0} \neq R_{B \rightarrow BC}(\rho_{ABC}^{0})$. This implies that there exists a $\delta \in (0, 1]$, which may depend on the recovery map $R_{B \rightarrow BC}$, such that $1 - F(\rho_{ABC}^{0}, R_{B \rightarrow BC}(\rho_{ABC}^{0})) = \delta$. In the following we argue that there exists a universal (i.e., independent of $R_{B \rightarrow BC}$) constant $\delta \in (0, 1]$ such that $1 - F(\rho_{ABC}^{0}, R_{B \rightarrow BC}(\rho_{ABC}^{0})) \geq \delta$ for all recovery maps $R_{B \rightarrow BC}$ that satisfy (117). Since the set of trace-preserving completely positive maps from $B$ to $B \otimes C$ that satisfy (117) is compact and the function $f : TP(CP(B, B \otimes C) \ni R_{B \rightarrow BC} \mapsto 1 - F(\rho_{ABC}^{0}, R_{B \rightarrow BC}(\rho_{ABC}^{0})) \in [0, 1]$ is continuous (see Lemma C.4), Weierstrass’ theorem ensures that $\delta := \min_{R_{B \rightarrow BC}} f(\rho_{ABC}^{0}, R_{B \rightarrow BC}(\rho_{ABC}^{0})) \in [0, 1]$ is continuous (see Lemma C.4). Weierstrass’ theorem then implies compactness.

---

\footnote{This set is bounded as the set of trace-preserving completely positive maps from $B$ to $B \otimes C$ is bounded (see Remark C.3). Furthermore, this set is closed since the set of trace-preserving completely positive maps from $B$ to $B \otimes C$ is closed (see Remark C.3) and the mapping $R_{B \rightarrow BC} \mapsto F(\rho_{ABC}, R_{B \rightarrow BC}(\rho_{AB}))$ is continuous for all states $\rho_{ABC}$ (see Lemma C.4). The Heine-Borel theorem then implies compactness.}
since $C$ is assumed to be a finite-dimensional system and as such $I(A : C|B)_{\rho} < \infty$. Hence for $p < \gamma$, inequality \((118)\) is violated. Since by \([10]\) for any $p \in (0, 1]$ there exists a recovery map $\mathcal{R}_{B \to BC}$ of the form \((112)\) that satisfies \((117)\) we conclude that for sufficiently small $p$ there exists a recovery map $\mathcal{R}_{B \to BC}$ of the form \((112)\) that satisfies \((117)\) and leaves $\rho_{ABC}^0$ invariant. However for recovery maps that leave $\rho_{ABC}^0$ invariant, \((117)\) simplifies to \((113)\) for all $p$. Thus, there exists a recovery map $\mathcal{R}_{B \to BC}$ of the form \((112)\) satisfying \((113)\) that leaves $\rho_{ABC}^0$ invariant, i.e., $\mathcal{R}_{B \to BC}(\rho_{AB}^0) = \rho_{ABC}^0$. Since $\rho_{ABC}^0 := \rho_A \otimes \rho_{BC}$ is a Markov chain with marginal $\rho_{BC}^0 = \rho_{BC}$, the condition $\mathcal{R}_{B \to BC}(\rho_{AB}^0) = \rho_{ABC}^0$ implies that $\mathcal{R}_{B \to BC}(\rho_B) = \rho_{BC}$ which proves the assertion.

Lemma 5.1 implies that there exists a recovery map $\mathcal{R}_{B \to BC}$ that satisfies \((113)\) and fulfills

$$\mathcal{R}_{B \to BC}(\rho_B) = V_{BC} \rho_{BC}^0(U_B U_B^\dagger \otimes \text{id}_C) \rho_{BC}^0 \! V_{BC}^\dagger = \rho_{BC} \ . \tag{120}$$

Using the fact that $\mathcal{R}_{B \to BC}$ is trace preserving and the invariance of the trace under unitaries we find

$$1 = \text{tr}(V_{BC} \rho_{BC}^0(U_B U_B^\dagger \otimes \text{id}_C) \rho_{BC}^0 \! V_{BC}^\dagger) = \text{tr}(\rho_{BC}(U_B U_B^\dagger \otimes \text{id}_C)) = \text{tr}(\rho_B U_B U_B^\dagger) \ . \tag{121}$$

This implies that $\text{tr}(\rho_B(\text{id}_B - U_B U_B^\dagger)) = 0$. Using the fact that $U_B U_B^\dagger \leq \text{id}_B$, we conclude that $U_B U_B^\dagger = \text{id}_B$ on the support of $\rho_B$. This simplifies \((120)\) to $V_{BC}\rho_{BC}V_{BC}^\dagger = \rho_{BC}$, i.e., $V_{BC}$ and $\rho_{BC}$ commute, which implies that the mapping \((112)\) can be written as

$$X_B \mapsto \rho_{BC}^0 W_{BC} \rho_{BC}^0 X_B \rho_{BC}^0 \otimes \text{id}_C W_{BC}^\dagger \rho_{BC}^0 \ , \tag{122}$$

with $W_{BC} = V_{BC} U_B \otimes \text{id}_C$ which is a unitary as $V_{BC}$ and $U_B$ are unitaries. Furthermore, $W_{BC}$ is such that \((122)\) is trace-preserving.

**Step 2: Structure of a universal recovery map for fixed $A$ system**

In this step we show that the recovery map satisfying \((110)\) of the form \((111)\), whose existence has been established in Step 1, can be made universal without sacrificing the (partial) knowledge about its structure. The idea is to apply Proposition 3.1 for the function family

$$\tilde{\Delta}_{\mathcal{R}}(\rho) : D(A \otimes B \otimes C) \to \mathbb{R} \cup \{- \infty\}$$

$$\rho_{ABC} \mapsto F(\rho_{ABC}, \mathcal{R}_{B \to BC}(\rho_{AB})) - 1 + \frac{\ln(2)}{2} I(A : C|B)_{\rho} \ . \tag{123}$$

We therefore need to verify that the assumptions of Proposition 3.1 are fulfilled. This is done by the following lemma. We first note that since $C$ is finite-dimensional this implies that $\tilde{\Delta}_{\mathcal{R}}(\rho) < \infty$ for all $\rho \in D(A \otimes B \otimes C)$.

**Lemma 5.2.** Let $A$ be a separable and $B$ and $C$ finite-dimensional Hilbert spaces. The function family $\tilde{\Delta}_{\mathcal{R}}(\cdot)$ defined by \((123)\) satisfies Properties 1-4.

**Proof.** We start by showing that $\tilde{\Delta}_{\mathcal{R}}(\cdot)$ satisfies Property 1. For $\rho_{AABC}^p$, as defined in \((15)\), we have

$$F(\rho_{AABC}^p, \mathcal{R}_{B \to BC}(\rho_{AB}^p)) = (1 - p) F(\rho_{ABC}^0, \mathcal{R}_{B \to BC}(\rho_{AB}^0)) + p F(\rho_{ABC}, \mathcal{R}_{B \to BC}(\rho_{AB})) \ . \tag{124}$$

The density operator $\mathcal{R}_{B \to BC}(\rho_{AAB}^p)$ can be written as

$$\mathcal{R}_{B \to BC}(\rho_{AAB}^p) = (1 - p) |0\rangle \langle 0|_\tilde{A} \otimes \mathcal{R}_{B \to BC}(\rho_{AB}^0) + p |1\rangle \langle 1|_\tilde{A} \otimes \mathcal{R}_{B \to BC}(\rho_{AB}) \ . \tag{125}$$

The relevant density operators thus satisfy the orthogonality conditions for equality in Lemma A.1, from which \((124)\) follows. Furthermore, as explained in the proof of Lemma 3.2 we have

$$I(\tilde{A} : C|B)_{\rho^p} = (1 - p) I(A : C|B)_{\rho} + p I(A : C|B)_{\rho} \ . \tag{126}$$
Equations (124) and (126) imply that
\[ \tilde{\Delta}_R(p) = (1 - p)\Delta_R(p) + p\tilde{\Delta}_R(p). \] (127)

We next verify that $\tilde{\Delta}_R(\cdot)$ fulfills Property 2. Let $R_{B \to BC}, R'_{B \to BC} \in \text{TPCP}(B \otimes C)$, $\alpha \in [0, 1]$ and $\tilde{R}_{B \to BC} = \alpha R_{B \to BC} + (1 - \alpha)R'_{B \to BC}$. Lemma A.1 implies that for any state $\rho_{ABC}$ on $A \otimes B \otimes C$
\[ F(\rho_{ABC}, \tilde{R}_{B \to BC}(\rho_{AB})) = F(\rho_{ABC}, \alpha R_{B \to BC}(\rho_{AB}) + (1 - \alpha)R'_{B \to BC}(\rho_{AB})) \geq \alpha F(\rho_{ABC}, R_{B \to BC}(\rho_{AB})) + (1 - \alpha)F(\rho_{ABC}, R'_{B \to BC}(\rho_{AB})) , \] (128)
and, hence by the definition of $\tilde{\Delta}_R(\cdot)$
\[ \tilde{\Delta}_R(\rho) \geq \alpha \Delta_R(\rho) + (1 - \alpha)\tilde{\Delta}_R(\rho). \] (129)

The function $\rho \mapsto \tilde{\Delta}_R(\rho)$ is continuous which clearly implies Property 3. To see this, recall that by the Alicki-Fannes inequality $\rho \mapsto I(A : C|B)_\rho$ is continuous for a finite-dimensional $C$ system [1]. Furthermore, since $\rho_{AB} \mapsto R_{BC}(\rho_{AB})$ is continuous (see Lemma C.5), Lemma B.9 of [10] implies that $\rho_{ABC} \mapsto F(\rho_{ABC}, R_{B \to BC}(\rho_{AB}))$ is continuous, which then establishes Property 3.

Finally it remains to show that $\Delta_R(\cdot)$ satisfies Property 4, which however follows directly by Lemma C.4.

Let $\mathcal{P} \subseteq \text{TPCP}(B, B \otimes C)$ be the convex hull of the set of trace-preserving completely positive mappings from the $B$ to the $B \otimes C$ system that are of the form (111). We note that the elements of $\mathcal{P}$ are mappings of the form (10), since a convex combination of unitary mappings are unital and a convex combination of trace-preserving maps remains trace-preserving. Proposition 3.1, which is applicable as shown in Lemma 5.2 together with Step 1 therefore proves the assertion for a fixed $A$ system.

**Step 3: Independence from the $A$ system**

Let $\mathcal{S}$ be the set of all density operators on $A \otimes B \otimes C$ with a fixed marginal $\rho_{BC}$ on $B \otimes C$, where $B$ and $C$ are finite-dimensional Hilbert spaces and $A$ is the infinite-dimensional Hilbert space $l^2$ of square summable sequences.

We note that the set of trace-preserving completely positive maps of the form (10) on finite-dimensional systems is compact, which follows by Remark C.3 together with the fact that the intersection of a compact set and a closed set is compact. Hence, using Lemma 5.2 (in particular Properties 3 and 4) and the result from Step 2 above, the same argument as in Step 4 of Section 3 can be applied to conclude the existence of a recovery map $R_{B \to BC}$ of the form (10) such that $\Delta_R(S) \geq 0$.

As every separable Hilbert space $A$ can isometrically embedded into $\tilde{A}$ [25, Theorem II.7] and since $\Delta_R$ is invariant under isometries applied on the extension space $A$, we can conclude that the recovery map $R_{B \to BC}$ remains valid for any separable extension space $A$. This proves the statement of Corollary 2.4 for finite-dimensional $B$ and $C$ systems.

**6 Discussion**

Our main result is that for any density operator $\rho_{BC}$ on $B \otimes C$ there exists a recovery map $R_{B \to BC}$ such that the distance between any extension $\rho_{ABC}$ of $\rho_{BC}$ acting on $A \otimes B \otimes C$ and $R_{B \to BC}(\rho_{AB})$ is bounded from above by the conditional mutual information $I(A : C|B)_\rho$. It is natural to ask whether such a map can be described as a simple and explicit function of $\rho_{BC}$. In fact, it was conjectured in [19, 4] that (2) holds for a very simple choice of map, namely
\[ T_{B \to BC} : X_B \mapsto \frac{1}{\sqrt{\|X_B\|^2 + 1}}(\rho_{BC} \otimes id_C)X_B \rho_{BC} \otimes \rho_{BC} , \] (130)
called the transpose map or Petz recovery map. This conjecture, if correct, would have important consequences in obtaining remainder terms for the monotonicity of the relative entropy [3]. As discussed in the introduction, if $\rho_{ABC}$ is such that it is a (perfect) quantum Markov chain or the $B$ system is classical, the claim of the conjecture is known to hold.

One possible approach to prove a result of this form would be to start from the result (2) for an unknown recovery map and then show that the transpose map $T_{B\to BC}$ cannot be much worse than any other recovery map. In fact, a theorem of Barnum and Knill [2] directly implies that when $\rho_{ABC}$ is pure, we have

$$F(\rho_{ABC}, T_{B\to BC}(\rho_{AB})) \leq F(A; C|B)_\rho \leq \sqrt{F(\rho_{ABC}, T_{B\to BC}(\rho_{AB}))}.$$  \hspace{1cm} (131)

This shows that, if $\rho_{ABC}$ is pure, an inequality of the form (2), with the fidelity replaced by its square root, holds for the transpose map. In order to generalize this to all states, one might hope that (131) also holds for mixed states $\rho_{ABC}$. However, this turns out to be wrong even when the state $\rho_{ABC}$ is completely classical (see Appendix F for an example).

Another interesting question is whether the lower bound in terms of the measured relative entropy (8) can be improved to a relative entropy. Such an inequality is known to be false if we restrict the recovery map to be the transpose map (130) [33], but it might be true when we optimize over all recovery maps. It is worth noting that in case such an inequality holds for any $\rho_{ABC}$ and a corresponding recovery map, then the argument presented in this work would imply that there exists a universal recovery map satisfying (8) with the relative entropy instead of the measured relative entropy. This can be seen by defining the function family $\rho \mapsto \Delta_R(\rho) := I(A : C|B)_\rho - D(\rho_{ABC}||\mathcal{R}_{B\to BC}(\rho_{AB}))$. Lemma B.2, the convexity of the relative entropy [22, Theorem 11.12] and the lower semicontinuity of the relative entropy [17, Example 7.22] imply that $\Delta_R(\cdot)$ satisfies Properties 1-4. As a result, Proposition 3.1 is applicable which can be used to prove the existence of a universal recovery map.

**Appendices**

**A  General facts about the fidelity**

The following lemma states a standard concavity property of the fidelity which is presented here for completeness and since we are interested in the case where equality holds.

**Lemma A.1.** For any density operators $\rho$, $\rho'$, $\sigma$, and $\sigma'$, and for any $p \in [0,1]$ we have

$$F(p\rho + (1-p)\rho', p\sigma + (1-p)\sigma') \geq pF(\rho, \sigma) + (1-p)F(\rho', \sigma') ,$$  \hspace{1cm} (132)

with equality if both of $\rho$ and $\sigma$ are orthogonal to both of $\rho'$ and $\sigma'$.

**Proof.** Note first that for any two normalized and mutually orthogonal vectors $|0\rangle$ and $|1\rangle$ in an ancilla space, we have

$$F(p\rho + (1-p)\rho', p\sigma + (1-p)\sigma') \geq F(p\rho \otimes |0\rangle\langle 0| + (1-p)\rho' \otimes |1\rangle\langle 1|, p\sigma \otimes |0\rangle\langle 0| + (1-p)\sigma' \otimes |1\rangle\langle 1|) ,$$  \hspace{1cm} (133)

because of the monotonicity of the fidelity under the partial trace. Furthermore, if both of $\rho$ and $\sigma$ are orthogonal to both of $\rho'$ and $\sigma'$ then there exists a trace-preserving completely positive map that generates the corresponding state $|0\rangle$ or $|1\rangle$ of the ancilla system. This implies that, in this case, the inequality also holds in the other direction. It therefore suffices to prove (132) with $\rho$ and $\sigma$ replaced by $\rho \otimes |0\rangle\langle 0|$ and $\sigma \otimes |0\rangle\langle 0|$, and with $\rho'$ and $\sigma'$ replaced by $\rho' \otimes |1\rangle\langle 1|$ and $\sigma' \otimes |1\rangle\langle 1|$, respectively. In other words, it remains to show that, for the case where $\rho$ and $\sigma$ are orthogonal to $\rho'$ and $\sigma'$, (132) holds with equality, i.e.,

$$F(\bar{\rho}, \bar{\sigma}) = pF(\rho, \sigma) + (1-p)F(\rho', \sigma') ,$$  \hspace{1cm} (134)
where \( \bar{\rho} = p\rho + (1-p)\rho' \) and \( \bar{\sigma} = p\sigma + (1-p)\sigma' \).

For this, let \( |\phi\rangle, |\phi'\rangle, |\psi\rangle \), and \( |\psi'\rangle \) be purifications of \( \rho, \rho', \sigma, \) and \( \sigma' \), respectively, such that \( F(\rho, \sigma) = \langle \phi | \psi \rangle \) and \( F(\rho', \sigma') = \langle \phi' | \psi' \rangle \). It is easy to verify that
\[
|\bar{\phi}\rangle = \sqrt{p}|\phi\rangle \otimes |0\rangle + \sqrt{1-p}|\phi'\rangle \otimes |1\rangle \quad \text{and} \quad |\bar{\psi}\rangle = \sqrt{p}|\psi\rangle \otimes |0\rangle + \sqrt{1-p}|\psi'\rangle \otimes |1\rangle
\]
are purifications of \( \bar{\rho} \) and of \( \bar{\sigma} \), respectively. Hence,
\[
pF(\rho, \sigma) + (1-p)F(\rho', \sigma') = p\langle \phi | \psi \rangle + (1-p)\langle \phi' | \psi' \rangle = \langle \bar{\phi} | \bar{\psi} \rangle \leq F(\bar{\rho}, \bar{\sigma}) ,
\]
which proves one direction of (134).

To prove the other direction, let \( \pi \) be the projector onto the joint support of \( \rho \) and \( \sigma \), i.e., \( \pi\rho = \rho \) and \( \pi\sigma = \sigma \). Similarly, let \( \pi' \) be the projector onto the joint support of \( \rho' \) and \( \sigma' \), i.e., \( \pi\rho' = \rho' \) and \( \pi\sigma' = \sigma' \). By the condition that \( \rho \) and \( \sigma \) are orthogonal to \( \rho' \) and \( \sigma' \), the two projectors must be orthogonal, i.e., \( \pi\pi' = 0 \). Furthermore, let \( |\bar{\phi}\rangle \) be a purification of \( \bar{\rho} \) and let \( |\bar{\psi}\rangle \) be a purification of \( \bar{\sigma} \) such that \( F(\bar{\rho}, \bar{\sigma}) = \langle \bar{\phi} | \bar{\psi} \rangle \). Because
\[
p\rho = \pi \bar{\rho} \pi \quad \text{and} \quad (1-p)\rho' = \pi' \bar{\rho}' \pi'
\]
\( \pi |\bar{\phi}\rangle \) and \( \pi' |\bar{\phi}\rangle \) are purifications of \( p\rho \) and \( (1-p)\rho' \), respectively. Similarly, \( \pi |\bar{\psi}\rangle \) and \( \pi' |\bar{\psi}\rangle \) are purifications of \( p\sigma \) and \( (1-p)\sigma' \), respectively. Hence, we have
\[
F(\bar{\rho}, \bar{\sigma}) = \langle \bar{\phi} | \bar{\psi} \rangle = \langle \bar{\phi} | \pi |\bar{\psi}\rangle + \langle \bar{\phi} | \pi' |\bar{\psi}\rangle \leq F(p\rho, p\sigma) + F((1-p)\rho', (1-p)\sigma')
\]
\[
= pF(\rho, \sigma) + (1-p)F(\rho', \sigma') .
\]
This proves the other direction of (134) and thus concludes the proof.

The following lemma generalizes the Fuchs-van de Graaf inequality which has been proven for states to non-negative operators. The result is standard and stated here for completeness.

**Lemma A.2.** For any two non-negative operators \( \rho \) and \( \sigma \) with \( \text{tr}(\rho) \geq \text{tr}(\sigma) \), the trace norm of their difference is bounded from above by
\[
\|\rho - \sigma\|_1 \leq 2\sqrt{\text{tr}(\rho)^2 - F(\rho, \sigma)^2} .
\]

**Proof.** Let \( \omega \) be a non-negative operator with \( \text{tr}(\omega) = \text{tr}(\rho) - \text{tr}(\sigma) \), whose support is orthogonal to the support of both \( \rho \) and \( \sigma \), and define \( \sigma' = \sigma + \omega \). Then \( \text{tr}(\rho) = \text{tr}(\sigma') \) and
\[
\|\rho - \sigma\|_1 = \|\rho - \sigma'\|_1 \quad \text{and} \quad F(\rho, \sigma) = F(\rho, \sigma') .
\]
It therefore suffices to show that the claim holds for operators with \( \text{tr}(\rho) = \text{tr}(\sigma) = c \in \mathbb{R}^+ \). Furthermore for \( c > 0 \), defining \( \bar{\rho} = \rho/c \) and \( \bar{\sigma} = \sigma/c \) and noting that
\[
\|\rho - \sigma\|_1 = c\|\bar{\rho} - \bar{\sigma}\|_1 \quad \text{and} \quad F(\rho, \sigma) = cF(\bar{\rho}, \bar{\sigma}) ,
\]
it suffices to verify that the claim holds for \( \text{tr}(\rho) = \text{tr}(\sigma) = 1 \) which follows by the Fuchs-van de Graaf inequality [11].

**B General facts about the measured relative entropy**

**Definition B.1.** The **measured relative entropy** between density operators \( \rho \) and \( \sigma \) is defined as the supremum of the relative entropy with measured inputs over all POVMs \( \mathcal{M} = \{M_x\} \), i.e.,
\[
D_M(\rho|\sigma) = \sup\{D(\mathcal{M}(\rho)||\mathcal{M}(\sigma)) : \mathcal{M}(\rho) = \sum_x \text{tr}(\rho M_x|x\rangle\langle x|) \text{ with } \sum_x M_x = \text{id} \} ,
\]
where \( \{|x\rangle\} \) is a finite set of orthonormal vectors.
This quantity was studied in [16, 14] where it was shown that \( \frac{1}{n}D_{\text{M}}(\rho^\otimes n\|\sigma^\otimes n) \) converges to the relative entropy \( D(\rho\|\sigma) := \text{tr}(\rho(\log \rho - \log \sigma)) \).

**Lemma B.2.** Let \( \rho, \rho', \sigma, \) and \( \sigma' \) be density operators such that both \( \rho \) and \( \sigma \) are orthogonal to both \( \rho' \) and \( \sigma' \). For any \( p \in [0, 1] \) we have

\[
D(pp + (1-p)\rho' \| p\sigma + (1-p)\sigma') = pD(\rho\|\sigma) + (1-p)D(\rho'\|\sigma').
\] (143)

**Proof.** By the orthogonality of \( \rho \) and \( \rho' \) (respectively \( \sigma \) and \( \sigma' \)) we have

\[
\log(pp + (1-p)\rho') = \log(pp) + \log((1-p)\rho') = \log(p) + \log(1-p) + \log(\rho) + \log(\rho')
\] (144)

and \( \rho \log \rho' = 0 \). Thus by definition of the relative entropy we obtain the desired statement.

**Lemma B.3.** Let \( \rho, \rho', \sigma, \) and \( \sigma' \) be density operators such that both \( \rho \) and \( \sigma \) are orthogonal to both \( \rho' \) and \( \sigma' \). For any \( p \in [0, 1] \) we have

\[
D_{\text{M}}(pp + (1-p)\rho' \| p\sigma + (1-p)\sigma') = pD_{\text{M}}(\rho\|\sigma) + (1-p)D_{\text{M}}(\rho'\|\sigma').
\] (145)

**Proof.** Let \( \mathcal{M} = \{ M_x \} \), \( \mathcal{M}' = \{ M'_y \} \) be measurements and define the POVM on \( N \) whose elements are given by \( \{ M_x \}_x \cup \{ M'_y \}_y \). Then we can write

\[
\mathcal{N}(pp + (1-p)\rho') = p\sum_x \text{tr}(M_x\rho)|x\rangle \langle x| + (1-p)\sum_y \text{tr}(M'_y\rho')|y\rangle \langle y|.
\] (146)

As a result using Lemma B.2,

\[
D_{\text{M}}(pp + (1-p)\rho' \| p\sigma + (1-p)\sigma') \geq D\left(\mathcal{N}(pp + (1-p)\rho') \bigg\| \mathcal{N}(p\sigma + (1-p)\sigma')\right)
\]

\[
= pD\left(\sum_x \text{tr}(M_x\rho)|x\rangle \langle x| \bigg\| \sum_x \text{tr}(M_x\sigma)|x\rangle \langle x|\right) + (1-p)D\left(\sum_y \text{tr}(M'_y\rho')|y\rangle \langle y| \bigg\| \sum_y \text{tr}(M'_y\sigma')|y\rangle \langle y|\right).
\] (147)

As this inequality is valid for any measurements \( \mathcal{M} \) and \( \mathcal{M}' \), taking the supremum over such measurements gives

\[
D_{\text{M}}(pp + (1-p)\rho' \| p\sigma + (1-p)\sigma') \geq pD_{\text{M}}(\rho\|\sigma) + (1-p)D_{\text{M}}(\rho'\|\sigma').
\] (148)

For the other direction, consider a measurement \( \mathcal{M} = \{ M_x \} \). We can write

\[
\mathcal{M}(pp + (1-p)\rho') = \sum_x p\text{tr}(M_x\rho)|x\rangle \langle x| + (1-p)\text{tr}(M_x\rho')|x\rangle \langle x|.
\] (149)

Combining this with the joint convexity of the relative entropy [22, Theorem 11.12], we get

\[
D_{\text{M}}(pp + (1-p)\rho' \| p\sigma + (1-p)\sigma') = D\left(\mathcal{M}(pp + (1-p)\rho') \bigg\| \mathcal{M}(p\sigma + (1-p)\sigma')\right)
\]

\[
\leq pD\left(\sum_x \text{tr}(M_x\rho)|x\rangle \langle x| \bigg\| \sum_x \text{tr}(M_x\sigma)|x\rangle \langle x|\right) + (1-p)D\left(\sum_x \text{tr}(M_x\rho')|x\rangle \langle x| \bigg\| \sum_x \text{tr}(M_x\sigma')|x\rangle \langle x|\right)
\]

\[
\leq pD_{\text{M}}(\rho\|\sigma) + (1-p)D_{\text{M}}(\rho'\|\sigma').
\] (150)

\[\square\]

**Lemma B.4.** For density operators \( \rho, \sigma, \) and \( \sigma' \) and \( p \in [0, 1] \) the measured relative entropy satisfies

\[
D_{\text{M}}(\rho p\sigma + (1-p)\sigma') \leq pD_{\text{M}}(\rho\|\sigma) + (1-p)D_{\text{M}}(\rho\|\sigma').
\] (151)
Proof. For any measurement $\mathcal{M}$,

\[
D(\mathcal{M}(\rho)\|\mathcal{M}(p\sigma + (1-p)\sigma')) = D(\mathcal{M}(\rho)\| p\mathcal{M}(\sigma) + (1-p)\mathcal{M}(\sigma')) \\
\leq p D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) + (1-p) D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma')) \\
\leq p D_M(\rho\|\sigma) + (1-p) D_M(\rho\|\sigma') ,
\]

where the first inequality step uses the convexity of the relative entropy [22, Theorem 11.12]. Taking the supremum over $\mathcal{M}$, we get the desired result.

C Basic topological facts

For completeness we state here some standard topological facts about density operators and trace-preserving completely positive maps.

Lemma C.1. Let $\alpha \in \mathbb{R}^+$. The space of non-negative operators on a finite-dimensional Hilbert space $E$ with trace smaller or equal to $\alpha$ (respectively equal to $\alpha$) is compact.

Proof. Let $D'(E) := \{\rho \in \text{Pos}(E) : \text{tr}(\rho) \leq \alpha\}$ denote the set non-negative operators on $E$ with trace not larger than one, where $\text{Pos}(E)$ is the set of non-negative operators on $E$. Consider the ball $B := \{e \in E : \|e\| \leq \alpha\}$ which is compact. The function $B \ni e \mapsto f(e) = ee^\dagger \in D'(E)$ is continuous and thus the set $f(B) = \{ee^\dagger : e \in E, \|e\| \leq \alpha\}$ is compact, as continuous functions map compact sets to compact sets. By the spectral theorem it follows that $D'(E) = \text{conv}f(B)$. As the convex hull of every compact set is compact this proves the assertion. The same argumentation (by replacing the inequalities with equalities) proves that the set of non-negative operators on $E$ with trace $\alpha$ is compact.

Lemma C.2. Let $E$, $G$ be finite-dimensional Hilbert spaces and let $\sigma_G \in \text{Pos}(G)$. The space of non-negative operators on $E \otimes G$ with a marginal on $G$ smaller or equal to $\sigma_G$ (respectively equal to $\sigma_G$) is compact.

Proof. Let $\sigma_G \in \text{Pos}(G)$. By Lemma C.1, the set of non-negative operators on $E \otimes G$ with trace not larger than $\alpha \in \mathbb{R}^+$ is compact. The set $\{X \in E \otimes G : \text{tr}_E(X) \leq \rho_G\}$ is closed. The intersection of a compact set and a closed set is compact which implies that $\{X \in \text{Pos}(E \otimes G) : \text{tr}_E(X) \leq \rho_G\}$ is compact. Since the set $\{X \in E \otimes G : \text{tr}_E(X) = \rho_G\}$ is closed the same argumentation shows that $\{X \in \text{Pos}(E \otimes G) : \text{tr}_E(X) = \rho_G\}$ is compact.

Remark C.3. Let $E$ and $G$ be two finite-dimensional Hilbert spaces. The space of trace-non-increasing (respectively trace-preserving) completely positive maps from $E$ to $G$ is compact. To see this note that Lemma C.2 implies that the set $\mathcal{F} := \{X \in \text{Pos}(E \otimes G) : \text{tr}_E(X) \leq \text{id}_E\}$ is compact. By the Choi-Jamiołkowski representation $\mathcal{F}$ is however isomorphic to the set of all trace-non-increasing completely positive maps from $E$ to $G$. The same argumentation applied to the set $\mathcal{F} := \{X \in \text{Pos}(E \otimes G) : \text{tr}_E(X) = \text{id}_E\}$ shows that the set of trace-preserving completely positive maps from $E$ to $G$ is compact.

Lemma C.4. Let $G$ and $K$ be finite-dimensional Hilbert spaces and let $\sigma_{EGK} \in D(E \otimes G \otimes K)$. The mapping $\text{TPCP}(G,G \otimes K) \ni \mathcal{R} \mapsto F(\sigma_{EGK},\mathcal{R}_{G \rightarrow GK}(\sigma_{EGK})) \in [0,1]$ is continuous.

Proof. This follows directly from the continuity of $\mathcal{R} \mapsto \mathcal{R}_{G \rightarrow GK}(\sigma_{EG})$ and the continuity of the fidelity (see, e.g., Lemma B.9 of [10]).

Lemma C.5. Let $E$, $G$, and $K$ be separable Hilbert spaces and $\mathcal{R} \in \text{TPCP}(G,K)$. Then the mapping $D(E \otimes G) \ni X \mapsto \mathcal{I}_E \otimes \mathcal{R}_{G \rightarrow K}(X_{EG}) \in D(E \otimes K)$ is continuous.

Proof. As the map is linear it suffices to show that it is bounded. For that we can decompose $X = P - N$ with $P$ and $N$ orthogonal non-negative operators. Then we have

\[
\|\mathcal{I}_E \otimes \mathcal{R}_{G \rightarrow K}(X)\|_1 \leq \|\mathcal{I}_E \otimes \mathcal{R}_{G \rightarrow K}(P)\|_1 + \|\mathcal{I}_E \otimes \mathcal{R}_{G \rightarrow K}(N)\|_1 = \text{tr}(P) + \text{tr}(N) = \|X\|_1 .
\]
D Touching sets lemma

We prove here a basic fact that is used in the proof of Theorem 2.1.

**Lemma D.1.** Let $K_0$ and $K_1$ be two sets such that $K_0 \cup K_1 = [0, 1]$ and $0 \in K_0$, $1 \in K_1$. Then for any $\delta > 0$ there exists $u \in K_0$ and $v \in K_1$ such that $0 \leq v - u \leq \delta$.

**Proof.** We define $\mu := \inf K_1$ and distinguish between the two cases $\mu \in K_0$ and $\mu \not\in K_0$.

If $\mu \in K_0$, it suffices to show that for any $\delta > 0$ we have $[\mu, \mu + \delta] \cap K_1 \neq \emptyset$, since by choosing $u = \mu$ this implies that $u \in K_0$ and there exists a $v \in [\mu, \mu + \delta]$ such that $v \in K_1$. By contradiction, we assume that $[\mu, \mu + \delta] \cap K_1 = \emptyset$. This implies that either $\inf K_1 < \mu$ or $\inf K_1 \geq \mu + \delta$, which contradicts $\mu := \inf K_1$.

If $\mu \not\in K_0$ it suffices to show that for any $\delta > 0$ we have $[\mu - \delta, \mu] \cap K_0 \neq \emptyset$, since by choosing $v = \mu$ this ensures that $v \in K_1$ and that there exists a $u \in [\mu - \delta, \mu]$ such that $u \in K_0$. Assume by contradiction that $[\mu - \delta, \mu] \cap K_0 = \emptyset$, which implies that $[\mu - \delta, \mu] \subset K_1$. This however contradicts $\mu := \inf K_1$.

\[ \square \]

E Properties of projected states

We first prove variant of the gentle measurement lemma [32], which is used repeatedly in the proof of Theorem 2.1.

**Lemma E.1.** Let $E$ and $G$ be separable Hilbert spaces and let $\Pi_G$ be a finite-rank projector on $G$. For any non-negative operator $\sigma_{EG}$ on $E \otimes G$ we have

\[ F\left(\sigma_{EG}, \frac{(\id_E \otimes \Pi_G)\sigma_{EG}(\id_E \otimes \Pi_G)}{\text{tr}(\id_E \otimes \Pi_G)\sigma_{EG}}\right)^2 \geq \text{tr}(\Pi_G\sigma_{EG}) \]  \tag{154}

and

\[ F(\sigma_{EG}, (\id_E \otimes \Pi_G)\sigma_{EG}(\id_E \otimes \Pi_G)) \geq \text{tr}(\Pi_G\sigma_{EG}) \]  \tag{155}

**Proof.** Let $|\psi\rangle$ be a purification of $\sigma_{EG}$ then by Uhlmann’s theorem [29] we find

\[ F\left(\sigma_{EG}, \frac{(\id_E \otimes \Pi_G)\sigma_{EG}(\id_E \otimes \Pi_G)}{\text{tr}(\id_E \otimes \Pi_G)\sigma_{EG}}\right)^2 \geq \frac{(\langle \psi|\Pi_G|\psi\rangle)^2}{\text{tr}(\id_E \otimes \Pi_G)\sigma_{EG}} = \text{tr}(\Pi_G\sigma_{EG}) \]  \tag{156}

and

\[ F(\sigma_{EG}, (\id_E \otimes \Pi_G)\sigma_{EG}(\id_E \otimes \Pi_G))^2 \geq (\langle \psi|\Pi_G|\psi\rangle)^2 = \text{tr}(\Pi_G\sigma_{EG})^2 \]  \tag{157}

\[ \square \]

We next prove a basic statement about converging projectors that is used several times in the proof of Theorem 2.1.

**Lemma E.2.** Let $E$ be a separable Hilbert space and let $\{\Pi^E_\epsilon\}_{\epsilon \in E}$ be a sequence of finite-rank projectors on $E$ which converges to $\id_E$ with respect to the weak operator topology. Then for any density operator $\sigma_E$ on $E$ we have $\lim_{\epsilon \to \infty} \text{tr}(\Pi^E_\epsilon\sigma_E) = \text{tr}(\sigma_E)$.

**Proof.** By assumption the Hilbert space $E$ is separable which implies that any state $\sigma_E$ can be written as $\sigma_E = \sum_i p_i |x_i\rangle \langle x_i|$, where $p_i \geq 0$, $\sum_i p_i = 1$ and $\{|x_i\rangle\}$ is an orthonormal basis on $E$. As the sequence $\{\Pi^E_\epsilon\}_{\epsilon \in E}$ weakly converges to $\id_E$, we find

\[ \lim_{\epsilon \to \infty} \text{tr}(\Pi^E_\epsilon\sigma_E) = \lim_{\epsilon \to \infty} \sum_i p_i \langle x_i|\Pi^E_\epsilon|x_i\rangle = \sum_i p_i \lim_{\epsilon \to \infty} \langle x_i|\Pi^E_\epsilon|x_i\rangle = \sum_i p_i \langle x_i|\id_E|x_i\rangle = \text{tr}(\sigma_E), \]  \tag{158}
where the second step uses the dominated convergence theorem that is applicable since $|\langle x_i | \Pi^e_E | x_i \rangle| \leq |\langle x_i | I_{E} | x_i \rangle|$ for all $e \in \mathbb{N}$.

Let $E$ and $G$ be separable Hilbert spaces and let $\mathcal{S}$ denote the set of bipartite density operators on $E \otimes G$ with a fixed marginal $\sigma_G$ on $G$. Let $\{\Pi^e_E\}_{e \in \mathbb{N}}$ be a sequence of projectors with rank $e$ that weakly converge to $I_{E}$ and $\mathcal{S}^e$ be the set of bipartite states on $E \otimes G$ whose marginal on $E$ is contained in the support of $\Pi^e_E$ and whose marginal on $G$ is identical to $\sigma_G$.

**Lemma E.3.** For every $\sigma_{EG} \in \mathcal{S}$ there exists a sequence $\{\sigma^e_{EG}\}_{e \in \mathbb{N}}$ with $\sigma^e_{EG} \in \mathcal{S}^e$ that converges to $\sigma_{EG}$ with respect to the trace norm.

**Proof.** For $\sigma_{EG} \in \mathcal{S}$, let

$$\bar{\sigma}^e_{EG} := \frac{(\Pi^e_E \otimes I_G)\sigma_{EG}(\Pi^e_E \otimes I_G)}{\text{tr}((\Pi^e_E \otimes I_G)\sigma_{EG})},$$

which has the desired support on $E$, however, $\bar{\sigma}^e_{G} \neq \sigma_{G}$ in general. This is fixed by considering

$$\sigma^e_{EG} := \text{tr}((\Pi^e_E \otimes I_G)\sigma_{EG})\bar{\sigma}^e_{EG} + |0\rangle\langle 0|_E \otimes \text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}(\Pi^e_E \otimes I_G))_{G},$$

where $|0\rangle_E$ is a normalized state on $E$. Since the partial trace on $E$ is cyclic on $E$ we obtain

$$\sigma^e_{G} = \text{tr}_E(\sigma^e_{EG}) = \text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}(\Pi^e_E \otimes I_G)) + \text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}(\Pi^e_E \otimes I_G))$$

$$= \text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}) + \text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}) = \text{tr}_E(\sigma_{EG}) = \sigma_{G}. \quad (161)$$

By the multiplicativity of the trace norm under tensor products and since $\|A\|_1 = \text{tr}(\sqrt{AA^*})$, the triangle inequality implies that

$$\|\sigma^e_{EG} - \sigma^e_{EG}\|_1 \leq 1 - \text{tr}((\Pi^e_E \otimes I_G)\sigma_{EG}) + \|\text{tr}_E((\Pi^e_E \otimes I_G)\sigma_{EG}(\Pi^e_E \otimes I_G))\|_1$$

$$= 1 - \text{tr}((\Pi^e_E \otimes I_G)\sigma_{EG}) + \text{tr}(\Pi^e_E \sigma_{EG}) = 2(1 - \text{tr}(\Pi^e_E \sigma_{EG})). \quad (162)$$

Lemma E.2 now implies that $\lim_{e \to \infty} \text{tr}(\Pi^e_E \sigma_{EG}) = 1$. We note that the sequence $\{\sigma^e_{EG}\}_{e \in \mathbb{N}}$ converges to $\sigma_{EG}$ in the trace norm since by the Fuchs-van de Graaf inequality [11], Lemma E.1 and Lemma E.2

$$\lim_{e \to \infty} \|\sigma_{EG} - \sigma^e_{EG}\|_1 \leq \lim_{e \to \infty} 2\sqrt{1 - F(\sigma_{EG}, \sigma^e_{EG})} \leq \lim_{e \to \infty} 2\sqrt{1 - \text{tr}(\Pi^e_E \sigma_{EG})} = 0. \quad (163)$$

Combining this with (162) and the triangle inequality proves that $\{\sigma^e_{EG}\}_{e \in \mathbb{N}}$ converges to $\sigma_{EG}$ in the trace norm.

**F** The transpose map is not square-root optimal

As discussed in Section 6, for pure states $\rho_{ABC}$ it is known [2] that

$$F(A; C|B)_\rho \leq \sqrt{F(\rho_{ABC}; T_{B \to BC}(\rho_{AB}))} \quad (164)$$

holds for $T_{B \to BC}$ the transpose map. In this appendix we show that (164) does not hold for all mixed states. Let $\dim A = \dim B = \dim C = 2$ and consider the state

$$\rho_{ABC} = \frac{1}{2} |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B \otimes |0\rangle\langle 0|_C + \frac{1}{8} |1\rangle\langle 1|_A \otimes I_{BC}. \quad (165)$$

The transpose map satisfies

$$T_{B \to BC}(|0\rangle\langle 0|_B) = \frac{5}{6} |00\rangle\langle 00|_{BC} + \frac{1}{6} |01\rangle\langle 01|_{BC} \quad \text{and} \quad T_{B \to BC}(|1\rangle\langle 1|_B) = \frac{1}{2} |10\rangle\langle 10|_{BC} + \frac{1}{2} |11\rangle\langle 11|_{BC}. \quad (166)$$

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If we consider a recovery map $R_{B \to BC}$ that is defined by
\[ R_{B \to BC}(0\langle 0 | B) = |00\rangle \langle 00 | BC \quad \text{and} \quad R_{B \to BC}(1\langle 1 | B) = \frac{1}{3} (|01\rangle \langle 01 | BC + |10\rangle \langle 10 | BC + |11\rangle \langle 11 | BC) , \]
we find
\[ F(\rho_{ABC}, R_{B \to BC}(\rho_{AB})) > 0.9829 \quad \text{and} \quad \sqrt{F(\rho_{ABC}, T_{B \to BC}(\rho_{AB}))} < 0.9696 , \]
which shows that (164) cannot hold since $F(\rho_{ABC}, R_{B \to BC}(\rho_{AB})) \leq F(A; C | B)_{\rho}$.

This does not show that one cannot prove a non-trivial guarantee on the performance of the transpose map relative to the optimal recovery map, but it suggests that such a guarantee would have to be worse than the square root (and actually worse that the fourth root as well using another example), or perhaps it is more naturally expressed using a different distance measure (using similar examples, the trace distance does not seem to be a good candidate, either). We further note that this example does not show that (2) is wrong for the transpose map.

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References
[1] R. Alicki and M. Fannes. Continuity of quantum conditional information. Journal of Physics A: Mathematical and General, 37(5):55–57, 2004.
[2] H. Barnum and E. Knill. Reversing quantum dynamics with near-optimal quantum and classical fidelity. Journal of Mathematical Physics, 43(5):2097–2106, 2002.
[3] M. Berta, M. Lemm, and M. Wilde. Monotonicity of quantum relative entropy and recoverability, 2014. arXiv:1412.4067.
[4] M. Berta, K. P. Seshadreesan, and M. M. Wilde. Rényi generalizations of the conditional quantum mutual information. Journal of Mathematical Physics, 56(2), 2015.
[5] M. Berta and M. Tomamichel. The fidelity of recovery is multiplicative, 2015. arXiv:1502.07973.
[6] N. Bourbaki. Elements of Mathematics: General Topology. Hermann, Éditeures des Sciences et des Arts, 1966.
[7] F. G. S. L. Brandão, A. W. Harrow, J. Oppenheim, and S. Strelchuk. Quantum conditional mutual information, reconstructed states, and state redistribution, 2014. arXiv:1411.4921.
[8] M. Christandl, N. Schuch, and A. Winter. Entanglement of the antisymmetric state. Communications in Mathematical Physics, 311(2):397–422, 2012.
[9] N. Datta and M. Wilde. Quantum Markov chains, sufficiency of quantum channels, and Rényi information measures, 2015. arXiv:1501.05636.
[10] O. Fawzi and R. Renner. Quantum conditional mutual information and approximate Markov chains, 2014. arXiv:1410.0664v3.
[11] C. Fuchs and J. van de Graaf. Cryptographic distinguishability measures for quantum-mechanical states. *IEEE Transactions on Information Theory*, 45(4):1216–1227, May 1999.

[12] C. A. Fuchs. Distinguishability and accessible information in quantum theory. *PhD Thesis, University of New Mexico*, 1996. arXiv:quant-ph/9601020.

[13] F. Furrer, J. Åberg, and R. Renner. Min- and max-entropy in infinite dimensions. *Communications in Mathematical Physics*, 306(1):165–186, 2011.

[14] M. Hayashi. Asymptotics of quantum relative entropy from a representation theoretical viewpoint. *Journal of Physics A: Mathematical and General*, 34(16):3413, 2001.

[15] P. Hayden, R. Jozsa, D. Petz, and A. Winter. Structure of states which satisfy strong subadditivity of quantum entropy with equality. *Communications in Mathematical Physics*, 246(2):359–374, 2004.

[16] F. Hiai and D. Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143(1):99–114, 1991.

[17] A. S. Holevo. *Quantum Systems, Channels, Information*. De Gruyter Studies in Mathematical Physics 16, 2012.

[18] B. Ibinson, N. Linden, and A. Winter. Robustness of quantum Markov chains. *Communications in Mathematical Physics*, 277(2):289–304, 2008.

[19] I. Kim. Application of conditional independence to gapped quantum many-body systems, 2013. http://www.physics.usyd.edu.au/quantum/Coogee2013/Presentations/Kim.pdf.

[20] E. Lieb and M. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *Journal of Mathematical Physics*, 14(12):1938–1941, 1973.

[21] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel. On quantum Rényi entropies: A new generalization and some properties. *Journal of Mathematical Physics*, 54(12):–, 2013.

[22] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[23] D. Petz. Sufficient subalgebras and the relative entropy of states of a von Neumann algebra. *Communications in Mathematical Physics*, 105(1):123–131, 1986.

[24] D. Petz. Monotonicity of quantum relative entropy revisited. *Reviews in Mathematical Physics*, 15(01):79–91, 2003.

[25] M. Reed and B. Simon. *Functional Analysis*. Elsevier, Academic Press, 1980.

[26] K. P. Seshadreesan and M. Wilde. Fidelity of recovery, geometric squashed entanglement, and measurement recoverability, 2014. arXiv:1410.1441.

[27] M. Tomamichel. A framework for non-asymptotic quantum information theory. *PhD thesis, ETH Zurich*, 2012. arXiv:1203.2142.

[28] M. Tomamichel, R. Colbeck, and R. Renner. Duality between smooth min- and max-entropies. *IEEE Transactions on Information Theory*, 56(9):4674–4681, Sept 2010.

[29] A. Uhlmann. The “transition probability” in the state space of a *-algebra. *Reports on Mathematical Physics*, 9(2):273 – 279, 1976.

[30] M. Wilde. *Quantum Information Theory*. Cambridge University Press, June 2013.
[31] M. M. Wilde, A. Winter, and D. Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Communications in Mathematical Physics*, 331(2):593–622, 2014.

[32] A. Winter. Coding theorem and strong converse for quantum channels. *IEEE Transactions on Information Theory*, 45(7):2481–2485, Nov 1999.

[33] A. Winter and K. Li. A stronger subadditivity relation? With applications to squashed entanglement, sharability and separability, 2012. [http://www.maths.bris.ac.uk/~csajw/stronger_subadditivity.pdf](http://www.maths.bris.ac.uk/~csajw/stronger_subadditivity.pdf).

[34] L. Zhang. Conditional mutual information and commutator. *International Journal of Theoretical Physics*, 52(6):2112–2117, 2013.