Marshall-Olkin Extended Inverse Weibull Distribution and Its Application

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Abstract. Weibull distribution is the most popular distribution in wind speed energy literature. However, in real life, the wind speed data may not always be modelled by Weibull distribution. An alternative in modeling wind speed data is the inverse Weibull distribution. Inverse Weibull distribution is modifications from the Weibull distribution with the transformation variables. Marshall and Olkin (1997) introduced an interesting method of adding a parameter to a well established distribution so we extend the Invers Weibull distribution by the Marshall-Olkin method (IWMO). The probability density function (pdf), cumulative distribution function (cdf), hazard rate, survival function, moment and quantiles of IWMO are derived. We also discuss the estimation of the model parameters by maximum likelihood. The IWMO distribution was applied to wind speed data. The results were given which illustrate the IWMO distribution and were compared to Weibull distribution and Inverse Weibull distribution. Model comparison using the log likelihood, AIC, and BIC showed that IWMO fit the data better than the other distributions.

1. Introduction

Weibull distribution has attracted the attention of statisticians working on theory and methods as well as in various fields of applied statistics [1]. Two-parameter Weibull Distribution is commonly-used and is an accepted distribution in the wind energy literature [2]; see Soulouknga et al [3] using Weibull distribution for wind speed in Faya-Largeu region and Ozay et al [4] in Alaçatı region. However, in real life, the wind speed data may not always be modelled by using the Weibull distribution. In the other words, it may not represent all wind speed characteristic encountered in nature. Therefore, the alternative distributions are used in such cases like Inverse Weibull (IW) Distribution, as introduced by Keller et al see[5]. Akgül et al.[6] used IW distribution for modelled wind speed in two different stations of Turkey, i.e. Bursa and Sakarya, and show that Invers Weibull better than Weibull distribution.

IW distribution is modification from the Weibull Distribution with the transformation variables. Let Y is a continuous random variable with Weibull distribution with parameters $\theta, \tau$. Let $F_w(y)$ is the cumulative distribution function (cdf) of Weibull distribution, and $f_w(y)$ the probability distribution function (pdf) of Weibull distribution, then [7]

$$F_w(y) = 1 - e^{-\left(\frac{y}{\theta}\right)^\tau},$$

(1)

$$f_w(y) = \frac{\tau \left(\frac{y}{\theta}\right)^{\tau - 1} e^{-\left(\frac{y}{\theta}\right)^\tau}}{y},$$

(2)
With $\theta$ is scale parameters and $\tau$ is shape parameters.
The transformed variable $Y = \frac{1}{Y}$, has the cdf $F_{iw}(z)$ and pdf $f_{iw}(z)$

$$F_{iw}(z) = e^{-(\tau \theta z)^{-\tau}} \tag{3}$$
$$f_{iw}(z) = \tau \theta z^{-(\tau + 1)} e^{-(\tau \theta z)^{-\tau}} \tag{4}$$

Where $y, x, \theta, \tau > 0$
The equation (3) and (4) is cdf and pdf of IW distribution respectively. The pdf of IW distribution generally exhibits a long right [1].

![Figure 1. Plots of the pdf of the Weibull for $\theta = 1$](image1)

![Figure 2. Plots of the pdf of the Weibull for $\theta = 2$](image2)

![Figure 3. Plots of the pdf of the IW for $\theta = 1$](image3)

![Figure 4. Plots of the pdf of the IW for $\tau = 1$](image4)

Based on figures 1 and 2, the pdf of Weibull distribution is unimodal and decreasing function and has a light tail. Based on figures 3 and 4, the pdf IW distribution is unimodal and has a heavy or long tail.

In 1995 Marshall and Olkin [8] proposed a method to expand families of distributions based on the survival function of a distribution by adding a new parameter. If $\bar{F}(x)$ denote the survival function of a continuous random variable $X$, then the usual device of adding a new parameter results in another survival function $\bar{G}(x)$ defined by [9]

$$\bar{G}(x, \alpha) = \frac{\alpha \bar{F}(x)}{1 - \alpha \bar{F}(x)} ; -\infty < x < \infty ; \alpha > 0, \bar{\alpha} = 1 - \alpha \tag{5}$$

If $G(x)$, $g(x)$ and $r(x)$ are the cdf, pdf and hazard rate function corresponding to $\bar{G}$, then

$$G(x, \alpha) = \frac{F(x)}{1 - \alpha (1 - F(x))} ; -\infty < x < \infty ; \alpha > 0, \bar{\alpha} = 1 - \alpha \tag{6}$$
\[ g(x, \alpha) = \frac{af(x)}{1-\alpha F(x)}; \quad -\infty < x < \infty; \quad \alpha > 0, \quad \alpha = 1 - \alpha \quad (7) \]

\[ r(x, \alpha) = \frac{h(x)}{1-\alpha F(x)}; \quad -\infty < x < \infty; \quad \alpha > 0, \quad \alpha = 1 - \alpha \quad (8) \]

Where \( h(x) \) is the hazard rate function corresponding to \( f(x) \) and \( F(x) \), \( F(x) \) is the pdf and cdf corresponding to \( F(x) \).

Marshall and Olkin [8] have noted that the method has a stability property, if the method is applied twice, nothing new is obtained. Marshall-Olkin extended distributions offer a wide range of behavior than the basic distributions from which they are derived. The property that the extended form of \( F \) distribution with the parameters \( u \) and \( v \) can have an interesting hazard function depending on the value of the added parameter \( \alpha \) and therefore can be used to model real situation in a better manner than the basic distribution, resulted in the detailed study of Marshall-Olkin extended family of distributions by some researchers like Okasha et al [10], Ghitany et al [11], Josse et al [12], Jayakumar and Mathew [13], Pérez-Casany, M., and Casellas [14], Krishna et al [15], Gui [16], and Lemonte[17].

2. IWMO Distribution and its properties

An extended distribution with marshall-olkin is commonly used for reliability applications, such as on the Inverse Weibull extension with Marshall-Olkin and its application in reliability, see [10]. But in this paper we use slightly different constructions parameters on its Inverse Weibull distribution.

2.1. IWMO distribution

In this section we will extend the inverse weibull distribution with pdf and cdf in equations (3) and (4) using the Marshall-Olkin method. Let \( X \) be the random continuous variable that have IWMO distribution with the parameters \( \theta, \tau > 0 \). And \( \alpha \) are the real constants, \( \alpha > 0 \), by substituting equations (3) and (4) into equation (5) - (8) we get cdf, pdf, hazard rate function and survival function of IWMO respectively as follows

\[ G_{iwmo}(x|\theta, \tau, \alpha) = \frac{e^{-(\alpha\theta)^{-\tau}}}{(a-(a-1)e^{-(\alpha\theta)^{-\tau}})} \quad (9) \]

\[ g_{iwmo}(x|\theta, \tau, \alpha) = \frac{a\tau\theta^{-\tau}x^{-\tau}e^{-(\alpha\theta)^{-\tau}}}{[a-(a-1)e^{-(\alpha\theta)^{-\tau}}]} \quad (10) \]

\[ r_{iwmo}(x|\theta, \tau, \alpha) = \frac{\tau\theta^{-\tau}x^{-\tau}e^{-(\alpha\theta)^{-\tau}}}{(1-e^{-(\alpha\theta)^{-\tau}})(a-(a-1)e^{-(\alpha\theta)^{-\tau}})} \quad (11) \]

\[ G_{iwmo}(x|\theta, \tau, \alpha) = \frac{\alpha(1-e^{-(\alpha\theta)^{-\tau}})}{a-(a-1)e^{-(\alpha\theta)^{-\tau}}} \quad (12) \]
From figures 5 and 6 we can see pdf of IWMO is decreasing and unimodal function.

2.2. Moment

In this section will discuss the \( k \)th moments of IWMO distribution. If \( X \) be the random variable that IWMO distribution with parameters \( \theta, \tau, \alpha \) then \( r \)th moments of IWMO is

\[
E(X^k) = \int_0^{\infty} x^k g_{IWMO}(x) \, dx
\]

then

\[
E(X^k) = \sum_{j=0}^{\infty} \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right)^j \left( \frac{1}{\theta \tau} (j+1) \right)^k \tau \left( 1 - \frac{k}{\tau} \right), \quad \tau > k
\]

We can get mean and variance by substituting \( k = 1 \) and \( k = 2 \) respectively.

2.3. Quantiles

In this section will discuss the quantiles of IWMO distribution. For a constant \( p \in (0,1) \), the quantile \( p \) of a value \( \omega_p \) is solution for

\[
G_{IWMO}(\omega_p) = p
\]

then the quantile \( p \) of a value \( \omega_p \)

\[
\omega_p = \left[ \theta^\tau \ln \left( 1 - \frac{p-1}{\alpha \tau} \right) \right] ^{-\frac{1}{\tau}}
\]

We can get median by substituting \( p = 0.5 \).

3. Parameters Estimation

We use the Maximum Likelihood estimation method for parameters estimation. Let \( X_1, X_2, \ldots, X_n \) be a random sample from IWMO distribution, then the likelihood function is given by

\[
L(x|\theta, \tau, \alpha) = \frac{\tau^\alpha e^{-\tau e^{-\left( \frac{x}{\theta} \right) \tau}}}{\alpha^n \Gamma(\alpha) \prod_{i=1}^{n} x_i^{-(\tau+1)}}.
\]

Then the logarithm of the likelihood function

\[
\ln L(x|\theta, \tau, \alpha) = n \ln(\theta^\tau) + n \ln \tau - n \ln \alpha - \left( \tau + 1 \right) \sum_{i=1}^{n} \ln(x_i) - \theta^{-\tau} \sum_{i=1}^{n} (x_i)^{-\tau} - 2 \sum_{i=1}^{n} \ln \left( 1 - \left( \frac{1}{\alpha} \right) e^{-\left( \frac{x_i}{\theta} \right)^\tau} \right).
\]
The maximum equation (18) can be obtained by taking the first partial derivatives of the log-likelihood function with respect to the three parameters, the results are

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \theta^{-1} - \tau \sum_{i=1}^{n} (x_i)^{-\tau} + \frac{\tau e^{-\frac{(x_i)^{-\tau}}{\theta}} (1 - \frac{1}{\theta})}{1 - e^{-\frac{(x_i)^{-\tau}}{\theta}}} = 0,$$

$$\frac{\partial \ln L}{\partial \tau} = \frac{n}{\tau} - n \ln \theta - \sum_{i=1}^{n} x_i + \theta^{-\tau} \ln \theta \sum_{i=1}^{n} (x_i)^{-\tau} - \left( \theta^{-\tau} \sum_{i=1}^{n} \ln x_i (x_i)^{-\tau} - 2 \left( \sum_{i=1}^{n} e^{-\frac{(x_i)^{-\tau}}{\theta}} (1 - \frac{1}{\theta}) \left( \frac{1}{\theta} \right) \right) \right),$$

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n}{\alpha} - 2 \left( \sum_{i=1}^{n} e^{-\frac{(x_i)^{-\tau}}{\theta}} (1 - \frac{1}{\theta}) \right).$$

The maximum likelihood estimates of $\theta, \tau$, and $\alpha$ are obtained by solving the nonlinear equations. These equations are not in closed form and the values of the parameters $\alpha, \beta$, and $\theta$ must be found by numerical methods.

4. Wind Speed data

In this section, the wind speed data obtained from Central Java, Indonesia, namely Jragung station [18] is used to determine the best fitting distributions. The data is the monthly average wind speed miles per day from 2011 to 2015, the data consists 60 observations. The location of the jragung station is within 07 ° 09' 17.16” south latitude and 110 ° 33' 40.26” east longitude. Statistical descriptions of the data are given in Table 1. From figure 7, we see that the data has extreme values. Table 2 is an estimate of the parameters using the maximum likelihood method.

| TABLE 1. Descriptive statistics for wind speed data |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Median | Mean | Variance | Skewness | Kurtosis | Min | Max |
| 22.115 | 30.79 | 2423.225 | 6.130 | 42.893 | 2.74 | 377.09 |

![Figure 7. The histogram wind speed data](image)

| TABLE 2. Maximum Likelihood Estimation for wind speed data |
|-----------------|-----------------|-----------------|-----------------|
| Distribution | $\hat{\theta}$ | $\hat{\tau}$ | $\hat{\alpha}$ |
| IWMO | 0.30276 | 1.8707 | 27.36221 |
| IW | 0.08166 | 1.12 | - |
4.1. Fitted models distributions

For the comparison for fitted models distribution IWMO with the IW and Weibull distribution, we used the log-likelihood values (log L), the Akaike information criteria (AIC) defined by -2log L + 2q, and the Bayesian information criterion (BIC) defined by -2log L + q log(n), where q is the number of estimated parameters and n is the sample size. The best model would give by the high value of log L and the lowest values from AIC and BIC. From Table 3 we have IWMO better than the others distributions.

| Distribution | log L   | AIC     | BIC     |
|--------------|---------|---------|---------|
| IWMO         | -256.766| 519.533 | 525.816 |
| IW           | -261.565| 527.129 | 531.318 |
| Weibull      | -265.63 | 535.259 | 539.448 |

4.2. Goodness of fit distribution models

For based on Kolgomorov-Smirnov test with level of significance \( \alpha = 0.05 \), the value of \( w_{1-\alpha} = 0.1755 \). From table 4, all value of T is smallest than 0.1755, so that all models fit in explaining the data.

| Distribution | T   |
|--------------|-----|
| IWMO         | 0.06975 |
| IW           | 0.1252  |
| Weibull      | 0.0978  |

Figure 8 gives The empirical and fitted cumulative functions, and from Figure 9 the graph of pdf IWMO is not too sharp like IW but also not decreased like Weibull.

5. Conclusion

In this paper we extend the Inverse Weibull distribution by the Marshall-Olkin method (IWMO) corresponding to the Weibull distribution [7]. Some properties of our proposed model IWMO were derived. The estimation of parameters was obtained by maximum likelihood method. The results of simulation on wind speed data in Jragung Station, Indonesia show that IWMO is better than Weibull and IW distributions.
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