Improving the moment approach for astrometric binaries: possible application to Cygnus X-1

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Abstract

A moment approach for orbital determinations of astrometric binaries from astrometric observations alone has been recently studied for a low signal-to-noise ratio (Iwama et al. 2013, PASJ, 65, 2). Through avoiding a direct use of the time-consuming Kepler equation, temporal information is taken into account to increase the accuracy of statistical moments. As numerical tests, 100 realizations are done and the mean value and the standard deviation are also evaluated. For a semimajor axis, the difference between the mean of the recovered values and the true value decreases to less than a tenth in the case of 10000 observed points. Therefore, the present approach works better than the previous one for orbital determinations when one has a large number of observed points. The present approach is thus applicable to Cyg X-1.

Key words: astrometry — binaries: close — celestial mechanics — methods: analytical

1 Introduction

Space astrometry missions such as Gaia and JASMINE are expected to reach a few micro arcseconds (mas) (Mignard 2005; Perryman 2005; Gouda et al. 2007). Moreover, high-accuracy VLBI (Very Long Baseline Interferometry) is also available.

Orbit determinations for binaries have been considered for a long time. For visual binaries, formulations for orbit determinations have been well developed since the 19th century (Thiele 1883; Binnendijk 1960; Aitken 1964; Danby 1988; Roy 1988). At present, numerical methods are successfully used (Eichhorn & Xu 1990; Catovic & Olevic 1992; Olevic & Cvetkovic 2004). Furthermore, an analytic solution for an astrometric binary, where one object is unseen, has been found (Asada et al. 2004, 2007; Asada 2008). The solution requires that sufficiently accurate positions of a star (or a photocenter of the binary) are measured at more than four places for an orbital cycle of the binary system.

A moment approach for a low signal-to-noise (SN) ratio is proposed by Iwama, Asada, and Yamada (2013, hereafter the Iwama+ approach). For a close binary system with a short orbital period, we have a relatively large uncertainty in the position measurements. For instance, the orbital periods of Cyg X-1 and LS 5039 are nearly 6 d and 4 d, respectively, which are extremely shorter than those of normal binary stars, say a few months and several years. Although temporal information is not incorporated in the Iwama+ approach, this approach would be useful for obtaining recovered values of orbital parameters, when observational errors are much smaller than a binary apparent size. It would be convenient to use the recovered values as trial values of the steepest descent method for reaching the best-fitting parameter values.

On the other hand, if observational errors are comparable to, or larger than, a binary’s apparent size, the orbital parameters cannot be recovered well, because the expected values of the statistical moments are quite different from the...
true values. Hence, it is important to improve the *Iwama*+ approach in order to consider such a case of extremely low SN ratio. The main purpose of this paper is to improve the previous approach by using the temporal information of observed points. However, the use of the Kepler equation is still avoided the same as in the previous approach.

2 Moment formalism

We consider a Kepler orbit, whose semimajor axis, eccentricity, inclination, argument of periastron, and longitude of ascending node are \((a_K, e_K, i, \omega, \Omega)\) (see figure 1). Here, we focus on a binary whose orbital period \(P_K\) is known from other observations. Angular positions projected on to the celestial sphere are expressed by using the Thiele–Innes elements (Binnendijk 1960; Aitken 1964; Roy 1988).

Let us assume frequent observations of the angular position in the celestial sphere. Namely, we consider a large number of observed points. For such a case, the statistical average expressed as a summation is taken as the temporal average in an integral form; that is,

\[
\langle F \rangle = \frac{1}{T_{\text{obs}}} \int_0^{T_{\text{obs}}} F \, dt,
\]

where \(\langle \quad \rangle\) denotes the mean and \(T_{\text{obs}}\) the total time duration of observations.

In this paper, we focus on the periodic motion, so that the above expression becomes the integration over several orbital periods. We thus obtain

\[
\langle F \rangle = \frac{1}{P} \int_{t_0}^{t_0 + P} F \, dt
\]

\[
= \frac{1}{P} \int_{t_0}^{t_0 + P} F \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} F(1 - e_K \cos u) \, du,
\]

where \(j\) is an integer and we used the Kepler equation

\[
t = t_0 + \frac{P_K}{2\pi} (u - e_K \sin u),
\]

and \(dt = P_K(1 - e_K \cos u) \, du/2\pi\). Here, \(u\) and \(t_0\) denote the eccentric anomaly and the time of periastron passage, respectively.

Let us consider statistical moments. The second and the third moments of the projected position in \((x, y)\) coordinates are useful for determining orbital parameters. They are defined as follows:

\[
M_{xx} \equiv \langle (x - \langle x \rangle)^2 \rangle
\]

\[
= \frac{1}{2}(\alpha^2 + \beta^2) - \frac{1}{4} e_K^2 \alpha^2,
\]

\[
M_{yy} \equiv \langle (y - \langle y \rangle)^2 \rangle
\]

\[
= \frac{1}{2}(\gamma^2 + \delta^2) - \frac{1}{4} e_K^2 \gamma^2,
\]

\[
M_{xy} \equiv \langle (x - \langle x \rangle)(y - \langle y \rangle)\rangle
\]

\[
= \frac{1}{2}(\alpha \gamma + \beta \delta) - \frac{1}{4} e_K^2 \alpha \gamma,
\]

\[
M_{xx} \equiv \langle (x - \langle x \rangle)^3 \rangle
\]

\[
= \frac{3}{8} e_K \alpha (\alpha^2 + \beta^2) - \frac{3}{4} e_K^3 \alpha^3,
\]

\[
M_{yy} \equiv \langle (y - \langle y \rangle)^3 \rangle
\]

\[
= \frac{3}{8} e_K \gamma (\gamma^2 + \delta^2) - \frac{3}{4} e_K^3 \gamma^3,
\]

\[
M_{xy} \equiv \langle (x - \langle x \rangle)(y - \langle y \rangle)\rangle
\]

\[
= \frac{1}{8} e_K (3\alpha \gamma + \beta \delta) - \frac{1}{4} e_K^3 \alpha \gamma,
\]

\[
M_{yy} \equiv \langle (x - \langle x \rangle)(y - \langle y \rangle)\rangle
\]

\[
= \frac{1}{8} e_K (3\alpha \gamma^2 + \alpha \delta^2 + 2\beta \gamma \delta) - \frac{1}{4} e_K^3 \alpha \gamma^2.
\]

**Fig. 1.** Actual Keplerian orbit and apparent ellipse in three-dimensional space. We denote the inclination angle as \(i\), the argument of periastron as \(\omega\) and the longitude of ascending node as \(\Omega\). These angles relate two coordinates \((x', y')\) and \((x, y)\), both of which choose the origin as the common center of mass. Here, the \(x'\) axis is taken to lie along an arbitrary reference direction, while the \(x\)-axis is along the direction of the ascending node.
where observational errors are assumed to vanish at the last equal in each equation, and \( \alpha, \beta, \gamma, \) and \( \delta \) are the Thiele–Innes type elements defined by Iwama et al. (2013):

\[
\alpha \equiv a_K (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i), \quad (11)
\]

\[
\beta \equiv -b_K (\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i), \quad (12)
\]

\[
\gamma \equiv a_K (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i), \quad (13)
\]

\[
\delta \equiv -b_K (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i), \quad (14)
\]

where \( b_K = a_K \sqrt{1 - \varepsilon_K^2} \) is the semiminor axis. The moments \( M_{xx}, \ldots, M_{yy} \) are actually observable. For the moments’ calculation, temporal information of each observed position is smeared by averaging. If positions of a star are measured with sufficiently small observational errors, one can well recover the orbital parameters by using the \textit{Iwama+} approach (Iwama et al. 2013).

### 3 Improved moment approach

#### 3.1 Observational errors

In the above formalism, we assume that observed points are located on an apparent ellipse. However, position measurements are inevitably associated with observational errors. Therefore, it is very important to take into account observation noises. In this paper, we add Gaussian errors to position measurements as \( x \rightarrow x + \Delta x \) and \( y \rightarrow y + \Delta y \), where \( \Delta x \) and \( \Delta y \) have Gaussian distributions with a standard deviation of \( \sigma \). Then, the expected values of the moments are estimated to be

\[
E(M_{xx}^{(O)}) = M_{xx}^{(T)} + \frac{N - 1}{N} \sigma^2, \quad (15)
\]

\[
E(M_{yy}^{(O)}) = M_{yy}^{(T)} + \frac{N - 1}{N} \sigma^2, \quad (16)
\]

\[
E(M_{xy}^{(O)}) = M_{xy}^{(T)}, \quad (17)
\]

\[
E(M_{xx}^{(O)}) = M_{xx}^{(T)}, \quad (18)
\]

\[
E(M_{yy}^{(O)}) = M_{yy}^{(T)} + \frac{N - 1}{N} \sigma^2, \quad (19)
\]

\[
E(M_{xy}^{(O)}) = M_{xy}^{(T)}, \quad (20)
\]

\[
E(M_{yy}^{(O)}) = M_{yy}^{(T)} + \frac{N - 1}{N} \sigma^2, \quad (21)
\]

where \( N \) is the total number of observed points, and the superscripts \( (O) \) and \( (T) \) denote observable values including observational errors and true values corresponding to equations (4)–(10), respectively. Since \( N \) is a large number, \( (N - 1)/N \simeq 1 \). Equations (15) and (16) suggest that orbital parameters are not recovered well in the case that \( \sigma^2 \) is comparable to, or larger than, \( M_{xx} \) and \( M_{yy} \), even if \( N \) approaches the infinity. In this section, we improve the \textit{Iwama+} approach to obtain the moments with higher accuracy for such large observational errors produced by incorporating temporal information.

#### 3.2 Averaging operation

By incorporating temporal information, we average the coordinates of observed points which are neighboring positions. Let us assume that an orbital period of a binary, \( P_k \), is known with high accuracy by another observation, such as observations of absorption lines (e.g., Brocksopp et al. 1999 for Cyg X-1; Sarty et al. 2011 for LS 5039). If observational errors are very large, a neighboring position on the orbit can be considered as the same position with some error. In other words, one can identify a point observed at a time \( t_1 \) with another one at a time \( t_2 \) when

\[
\Delta t \ll \sigma, \quad (22)
\]

in units of \( a_K = 1 \), where

\[
\Delta t = |t_1 - t_2| (\text{mod } P_k). \quad (23)
\]

Let us divide the apparent ellipse into small bins, each of which corresponds to an equal short time interval, e.g., \([t_0, t_0 + P_K/n_m]\), where \( n_m \) is the number of bins. If the same star is observed at fixed intervals, then every bin has the equal number of observed points, \( n_a = N/n_m \), and one obtains more bins near the apastron than near the periastron. Namely, every bin will contain the same number of points if and only if the interval between the observations is not a multiple of the bin duration. Note that we can use the data over several orbital periods, so that each bin may include observed points of different orbital periods.

In order to reduce statistical errors, we calculate the average positions of \( n_a \) observed points for each bin and obtain \( n_m \) averaged points (see figure 2). Through this averaging operation, the expected values of the moments are given as follows:

\[
E(\overline{M}_{xx}^{(O)}) = M_{xx}^{(T)} + \frac{N - 1}{N} \sigma^2 \frac{1}{n_a}, \quad (24)
\]

\[
E(\overline{M}_{yy}^{(O)}) = M_{yy}^{(T)} + \frac{N - 1}{N} \sigma^2 \frac{1}{n_a}, \quad (25)
\]

where the bar denotes the value obtained from \( n_m \) averaged points. Hence, if \( n_a \) is sufficiently large, the errors of \( M_{xx}^{(O)} \) and \( M_{yy}^{(O)} \) could be neglected safely. Therefore, the \textit{Iwama+}
approach is improved with regard to the accuracy of the moments by the averaging operation.

4 Results

4.1 Numerical test

In equations (1) and (2), we assume that one can integrate observed quantities. In practice, however, observations are so discrete that the integration should be replaced by a summation. The integration and the summation could agree on the condition that \( n_m \) approaches infinity. In addition, it is necessary that \( n_a \) is so large as to reduce errors. According to the numerical calculations, the present approach recovers orbital parameters for \( n_m = 100 \) [see the discussion in Iwama et al. (2013) on the absence of the averaging operation, i.e., \( n_a = 1 \) & \( n_m = N \)]. Therefore, one can use \( N/100 \) points for the averaging operation on each bin.

For the true parameters \((a_K, e_K, i, \omega, \Omega) = (1.0, 0.1, 30^\circ, 30^\circ, 30^\circ)\), we consider two cases for \( N = 10000 \). Case 1: the observational error for each position measurement is equal to a binary size, namely, \( \sigma = 1 \) in units of \( a_K \). Case 2: \( \sigma = 5 \) in units of \( a_K \). For each case, \( \sigma/\sqrt{N} \) is fixed where we imagine an instrument, such as the Small-JASMINE. For each parameter set, 100 realizations are done and the mean value and the standard deviation are also evaluated.

Figure 3 shows the apparent orbits for the mean values of the recovered parameters using the Iwama+ approach and the present one. The present approach can recover the orbital parameters better than the Iwama+ approach. Especially, one can see that the true orbit and the recovered orbit from the present approach almost overlap each other for \( \sigma = 1 \).

Table 1 is a list of orbital parameters that are recovered through the Iwama+ approach and the present one for \( n_a = 100 \). In both cases, the difference between the true value of the semimajor axis and the mean of the recovered one decreases to less than a tenth. This can be seen in figure 3, and is consistent with an order-of-magnitude estimation from equations (24) and (25) (see the Appendix). On the other hand, the dispersion of recovered parameters is not improved by the averaging operation since the order of magnitude of the dispersion depends not on \( n_a \) but on \( \sigma/\sqrt{N} \).

In case 1, the longitude of ascending node is well recovered with an accuracy of less than 10% of the true value,
while the other recovered parameters from the *Iwama*+ approach are quite different from the true values. Through the averaging operation, all of the mean recovered values approach the true values. In case 2, all the recovered values except \( \omega \) are improved.

Our numerical tests suggest that \( \omega \) and \( \Omega \) are not always improved by the present approach. However, this point is not important, since the change of the differences between the recovered values of \( \omega \) and \( \Omega \) and their true values is smaller than the dispersion of the recovered values.

In order to confirm the reliability of the above results, we calculate the recovered values for \( 16(=2^4) \) parameter sets as \( \alpha_K = 0.1 \) and \( 0.5 \), \( i = 30^\circ \) and \( 60^\circ \), \( \omega = 30^\circ \) and \( 60^\circ \), and \( \Omega = 30^\circ \) and \( 60^\circ \). One example (\( \alpha_K, \omega, \Omega = (0.5, 30^\circ, 60^\circ, 60^\circ) \)) is added to the above results in table 1 to save space. Fourier analyses recover the orbital period from numerically simulated data of the above two cases with accuracies of \( \sim 1\% \) and \( 5\% \).

### 4.2 Possible application to Cyg X-1

Let us consider a possible application to Cyg X-1, whose angular radius is \( \sim 0.03 \) mas. The required precision of the Small-JASMINE is \( 0.01 \) mas, so that \( \sigma/\sqrt{N} \approx 0.3 \). For Cyg X-1, the Small-JASMINE is expected to measure the position of the star with an accuracy of 3 mas, which corresponds to \( \sigma = 100 \), for each imaging. Hence, position measurements of \( N \approx 10^5 \) times are required as one data-set for \( \sigma/\sqrt{N} \approx 0.3 \).

Since the Small-JASMINE is expected to measure for 3–4 orbital periods of Cyg X-1, \( \sim 10^6 \) observed points, which correspond to 10 data-sets, will be obtained. This means that \( \sigma/\sqrt{N} \approx 0.1 \) exceeds the required precision of the Small-JASMINE. However, every observed point has the systematic error of the Small-JASMINE as \( \sim 0.01 \) mas, so that the recovered parameters might also have an error of \( \sim 0.01 \) mas.

In this paper, we consider two cases as numerical tests where we fix \( \sigma/\sqrt{N} \) for each data-set. Case 1: \( N = 100, \sigma = 3, \) and \( n_s = 1 \). In this case, the present approach reduces to the *Iwama*+ one. Case 2: \( N = 1000, \sigma = 9.5, \) and \( n_s = 10 \). See table 2 for a comparison of these cases of one data-set. In case 2, the present approach recovers the semimajor axis and the inclination better than the *Iwama*+ one.

On the other hand, the recovered eccentricity from the *Iwama*+ approach is close to the true value, which is considered to be a chance coincidence. Numerical calculations of other parameter sets suggest that the recovered eccentricities from the *Iwama*+ approach and the present one are close to 0.1 and 0.25, respectively, for any true eccentricity in case 2. Hence, the recovered eccentricities from the two approaches may not be reliable when \( \sigma/\sqrt{N} = 0.3 \). For the reliability of the recovered eccentricity, the position measurement with an accuracy of...
\[ \sigma/\sqrt{N} = 0.01 \], which corresponds to case 1 in table 1, is required.

The recovered values from the present approach are comparable to those from the *Iwama* approach for the argument of periastron and the longitude of ascending node. The recovered parameters from the present approach in case 2 are comparable to case 1. These numerical results suggest that one can obtain the similar results in \( \sigma = 3 \) and \( \sigma = 9.5 \) through the averaging operation. Hence, the present approach works well to reduce \( \sigma \) effectively for \( n_m = 100 \).

Next, let us consider these two cases for 10 data-sets. Table 3 shows the recovered values from the *Iwama* approach and the present approach for 10 data-sets. In both cases, each bin has 10 data-sets of \( n_3 \) observed points, so that \( 10 \times n_3 \) observed points are effectively averaged in the present approach. On the other hand, every observed point is not averaged in the previous approach.

For the semimajor axis, the mean values of the recovered parameters from the previous approach of 10 data-sets in both cases are comparable to those of one data-set. On the other hand, the recovered semimajor axes from the present approach of 10 data-sets in both cases are much better than those of one data-set.

However, the recovered eccentricities from both approaches may not be reliable if \( \sigma/\sqrt{N} = 0.3 \) by the similar reason as the low \( S/N \) ratio case of one data-set. For the reliability of the recovered eccentricity, the position measurement with an accuracy of \( \sigma/\sqrt{N} = 0.01 \), which corresponds to case 1 in table 1, is required. For the inclination, the mean values of the recovered parameters from the present approach are better than those from the previous approach. The recovered values from the present approach are comparable to those from the previous approach for the argument of periastron and the longitude of ascending node. Note that the dispersion of the recovered semimajor axis for 10 data-sets corresponds to a random error of observations. Therefore, the actual observational errors, including the systematic errors of the Small-JASMINE, will be comparable to the dispersion in table 2. These results suggest that the semimajor axis of Cyg X-1 is recovered with an accuracy comparable to, or smaller than, the true value of the semimajor axis by the Small-JASMINE observations. In order

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**Table 2.** Reconstructing the parameters of numerical simulations for two different cases as \( (N = 100, \sigma = 3, n_3 = 1) \) and \( (N = 1000, \sigma = 9.5, n_3 = 10) \) of one data-set.*

| \( \sigma \) | \( N \) | Approach | \( a_K \) | \( e_K \) | \( \iota[^1] \) | \( \omega[^1] \) | \( \Omega[^1] \) |
|---|---|---|---|---|---|---|---|
| 3 | 100 | *Iwama* | 4.76 ± 0.41 | 0.285 ± 0.133 | 27.2 ± 8.3 | 51.9 ± 25.0 | 41.4 ± 23.5 |
| 9.5 | 1000 | *Iwama* | 13.77 ± 0.27 | 0.086 ± 0.045 | 15.7 ± 4.3 | 48.0 ± 25.1 | 41.4 ± 27.8 |
| 9.5 | 1000 | Present | 4.78 ± 0.39 | 0.254 ± 0.142 | 28.1 ± 7.3 | 55.3 ± 24.5 | 38.4 ± 24.6 |

*Standard deviation \( \sigma = 0 \) row indicates the true orbital parameters, whereas the \( \sigma = 3 \) row and the row \( \sigma = 9.5 \) provide the recovered values for adding Gaussian errors (3 and 9.5 in units of the true semimajor axis, respectively). For each parameter set, 100 realizations are done, and the mean value and the standard deviation are also evaluated.

**Table 3.** Reconstructing the parameters of numerical simulations for two different cases as \( (N = 100, \sigma = 3, n_3 = 1) \) and \( (N = 1000, \sigma = 9.5, n_3 = 10) \) of 10 data-sets.*

| \( \sigma \) | \( N \) | Approach | \( a_K \) | \( e_K \) | \( \iota[^1] \) | \( \omega[^1] \) | \( \Omega[^1] \) |
|---|---|---|---|---|---|---|---|
| 3 | 100 | *Iwama* | 4.43 ± 0.08 | 0.090 ± 0.046 | 15.0 ± 3.8 | 46.3 ± 23.9 | 45.4 ± 25.8 |
| 9.5 | 1000 | *Iwama* | 13.56 ± 0.08 | 0.031 ± 0.016 | 9.1 ± 2.5 | 46.9 ± 27.0 | 43.2 ± 25.7 |
| 9.5 | 1000 | Present | 1.79 ± 0.13 | 0.236 ± 0.137 | 28.0 ± 7.1 | 52.0 ± 23.0 | 35.1 ± 23.1 |

*Standard deviation \( \sigma = 0 \) row indicates the true orbital parameters, whereas the \( \sigma = 3 \) row and the row \( \sigma = 9.5 \) provide the recovered values for adding Gaussian errors (3 and 9.5 in units of the true semimajor axis, respectively). For each parameter set, 100 realizations are done, and the mean value and the standard deviation are also evaluated.
5 Conclusion

This paper improved the Iwama+ approach to consider the case with an extremely low S/N, where observational errors are comparable to, or larger than, a binary size. Through avoiding a direct use of the time-consuming Kepler equation, we take temporal information into account to increase the accuracy of statistical moments. For numerical tests, 100 realizations are done and the mean value and the standard deviation are also evaluated. For instance, the difference between the mean of the recovered values of the semimajor axis and its true value decreases to less than a tenth in the case of 10000 observed points. Therefore, when one has a large number of the observed points, the present approach significantly improves the previous one for the orbital determination. For Cyg X-1, the semimajor axis is expected to be recovered with an accuracy comparable to, or smaller than, the true value from astrometric observations alone. Although the inversion formula by Asada, Akasaka, and Kasai (2004) is also discussed, numerical calculations show that the averaging operation does not work well in the analytic method. It is more convenient to start with the values that are recovered from the present moment approach and next to use the steepest descent method for finally reaching the best-fitting parameter values. It is left for future work.

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Appendix. Estimation of the recovered semimajor axis

Let us estimate the difference between the recovered value and true one for the semimajor axis. By using equations (11)–(14), (4), and (5), equations (15) and (24) are rewritten as follows:

\[
E \left[ f \left( \hat{e}_k^{(O)} \right) \right] = f \left( \hat{e}_k^{(T)} \right) + \frac{N - 1}{N} \sigma^2, \quad (A1)
\]

\[
E \left[ f \left( \hat{a}_k^{(O)} \right) \right] = f \left( \hat{a}_k^{(T)} \right) + \frac{N - 1}{N} \frac{\sigma^2}{n_e}, \quad (A2)
\]
where
\[ f(e_K, i, \omega, \Omega) \equiv \frac{1}{2} \left[ (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i)^2 ight. \\
+ (1 - e_K^2)(\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i)^2 \\
\left. - \frac{1}{4} e_K^2 (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i)^2 \right]. \quad (A3) \]

In the order-of-magnitude estimation, one can assume that \( a_K \) and \( f(e_K, i, \omega, \Omega) \) are independent of each other. Hence,
\[ E\left[f(e_K, i, \omega, \Omega) a_K^2\right] \simeq E\left[f(e_K, i, \omega, \Omega)\right] E(a_K^2). \quad (A4) \]

In addition, because \( e_K \) and the trigonometric functions are from 0 to 1, one finds \( f(e_K, i, \omega, \Omega) = O(1) \). Therefore, the expected values of the recovered semimajor axis from the \textit{Iwama}+ approach and from the present one are expressed approximately as follows:
\[ E\left(a_K^{(O)}\right) \sim \sqrt{\left(a_K^{(T)}\right)^2 + \sigma^2} \geq a_K^{(T)} + \sigma \quad (A5) \]
and
\[ E\left(\bar{a}_K^{(O)}\right) \sim \sqrt{\left(a_K^{(T)}\right)^2 + \frac{\sigma^2}{n_a}} \geq a_K^{(T)} + \frac{\sigma}{\sqrt{n_a}} \quad (A6) \]
respectively. Equations (A5) and (A6) suggest that the difference between the true value of the semimajor axis and the mean value of the recovered one decreases to nearly \( 1/\sqrt{n_a} \) by conducting the averaging operation. If \( n_a \geq 100 \), this difference decreases to nearly a tenth.

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