LINEAR POTENTIALS AND GALACTIC ROTATION CURVES - FORMALISM

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Abstract

In the first paper in this series we presented a typical set of galactic rotation curves associated with the conformal invariant fourth order theory of gravity which has recently been advanced by Mannheim and Kazanas as a candidate alternative to the standard second order Einstein theory. Reasonable agreement with data was obtained for four representative galaxies without the need for any non-luminous or dark matter. In this second paper we present the associated formalism. Additionally, we discuss the status of the Tully-Fisher relation in our theory, compare and contrast our theory with the dark matter theory, and make some general observations regarding the systematics of galactic rotation curve data.

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(I) Introduction

Despite the overwhelming consensus in the community favoring the existence of dark matter, as of today no explicit astrophysical or elementary particle dark matter candidate has yet been found, nor has any dark matter flux yet been measured. While this has not deterred the bulk of the community in any way, nonetheless a few authors have begun to question the validity of the underlying Newton-Einstein gravitational theory itself, taking the view that the need for dark matter in the standard theory may be symptomatic of an actual breakdown of the standard picture. Since this apparent need for dark matter is manifest on essentially every single distance scale from galactic all the way up to cosmological, while no such need is generally manifest on the much shorter distance scales where the standard theory was originally established in the first place, it is thus natural to consider the possibility that new physics (one might even refer to it as dark physics) may be opening up on these bigger distance scales. Indeed, rather than interpreting essentially every single current large distance scale gravitational observation as yet further evidence for the existence of dark matter (the common practice in both the learned and the popular literature), these selfsame data can just as equally be regarded as signaling the repeated failure of the standard theory; and definitively so if the only matter which actually exists in the Universe is that which is luminously observable. Thus the psychologically unwelcome empirical possibility suggested by the data is that Newton’s Law of Gravity may not be the correct weak gravitational theory, and that, accordingly, Einstein gravity may not then be the correct covariant one.

Now of course both the Newton and Einstein theories enjoy many successes (enough to convince most people that they are no longer even challengeable at all), and thus any alternate theory of gravity must be able to recover all their established features. To achieve this, one way to proceed is to begin with galactic rotation curve data (perhaps the most clear cut and well explored situation where the Newton-Einstein theory demands dark matter) and try to extract out a new weak gravity limit which encompasses Newton in an appropri-
ate limit (see e.g. Milgrom (1983) and Sanders (1990)) with a view to then subsequently working upwards to a covariant generalization (a program which is still in progress - see Sanders (1990) for a recent review). However, in order to ensure encompassing the Einstein successes from the outset, there is also much merit in beginning covariantly and then working downwards to a weak gravity limit. This latter approach has been advanced and explored by Mannheim and Kazanas in a recent series of papers (Mannheim and Kazanas (1989, 1991, 1992), Mannheim (1990, 1992, 1993a, 1993b), Kazanas (1991), Kazanas and Mannheim (1991a, 1991b)). Noting that there is currently no known theoretical reason which would select out the standard second order Einstein theory from amongst the infinite class of (all order) covariant, metric based theories of gravity that one could in principle at least consider, Mannheim and Kazanas reopened the question of what the correct covariant theory of gravity might be and developed an approach which works down from an additional fundamental principle above and beyond covariance, namely that of local scale or conformal invariance, i.e. invariance under any and all local conformal stretchings $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$ of the geometry, this being the invariance which is now believed to be possessed by the other three fundamental interactions, the strong, the electromagnetic and the weak. This invariance forces gravity to be described uniquely by the fourth order action

$$I_W = -\alpha \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha \int d^4x (-g)^{1/2} (R_{\lambda\mu} R^{\lambda\mu} - (R^\alpha_\alpha)^2/3)$$

(1)

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and $\alpha$ is a purely dimensionless coefficient.

In their original paper Mannheim and Kazanas (1989) obtained the complete and exact, non-perturbative exterior vacuum solution associated with a static, spherically symmetric gravitational source such as a star in this theory, viz.

$$-g_{00} = 1/g_{rr} = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - kr^2$$

(2)

where $\beta$, $\gamma$, and $k$ are three appropriate dimensionful integration constants. As can be seen, for small enough values of the linear and quadratic terms (i.e. on small enough distance
scales) the solution reduces to the familiar Schwarzschild solution of Einstein gravity, with
the conformal theory then enjoying the same successes as the Einstein theory on those
distance scales. On larger distance scales, however, the theory begins to differ from the
Einstein theory through the linear potential term, and (with the quadratic term only
possibly being important cosmologically, and with both the $\beta \gamma$ product terms being found
to be numerically negligible in the fits of Mannheim (1993b)) then yields a non-relativistic
gravitational potential

$$V(r) = -\beta/r + \gamma r/2$$

which may be fitted to data whenever the weak gravity limit is applicable.

The conformal theory thus not only generalizes Newton (Eq. (3)) it also generalizes
Schwarzschild (Eq. (2)), and even does so in way which is then able to naturally recover
both the Newton and Schwarzschild phenomenologies on the appropriate distance scales.
Since the conformal theory recovers the requisite solutions to Einstein gravity on small
enough distance scales (even while never recovering the Einstein Equations themselves -
observation only demands the recovery of the solutions not of the equations), that fact alone
makes the theory indistinguishable from and just as viable as the Einstein theory on those
distance scales, something recognized by Eddington (1922) as far back as the very early
days of Relativity. (Eddington was not aware of the full exact solution of Eq. (2) but was
aware that it was a solution to fourth order gravity in the restricted case where $\gamma = 0$. It
was only much later that the complete and exact solution of Eq. (2) was found and that its
consistency was established by successfully matching it on to the associated exact interior
solution (Mannheim and Kazanas (1992)). Thus in this sense conformal gravity should
always have been considered as a viable explanation of solar system physics. That it never
was so considered was in part due to the fact that strict conformal symmetry requires that
all particles be massless, something which would appear to immediately rule the symmetry
out. However, with the advent of modern spontaneously broken gauge theories manifest
in the other three fundamental interactions, it is now apparent that mass can still be
generated in the vacuum in otherwise dimensionless theories like the one associated with the action of Eq. (1). (And, interestingly, such dynamical mass generation is even found to still lead to geodesic motion (Mannheim (1993a)), despite the fact that the associated mass generating Higgs field which accompanies a test particle carries its own energy and momentum which the gravitational field also sees). Hence, it would appear that today the only non-relativistic way to distinguish between the two covariant theories is to explore their observational implications on larger distance scales where the linear potential term first makes itself manifest.

A first step towards this phenomenological end was taken recently by Mannheim (1993b) with the above non-relativistic potential $V(r)$ being used in conjunction with observed surface brightness data to fit the rotation curves of four representative galaxies. The particular choice of galaxies was guided by the recent comprehensive survey of the $HI$ rotation curves of spiral galaxies made by Casertano and van Gorkom (1991) who found that those data fall into essentially four general groups characterized by specific correlations between the maximum rotation velocity and the luminosity. In order of increasing luminosity the four groups are dwarf, intermediate, compact bright, and large bright galaxies. Thus one representative galaxy from each group was studied, respectively the galaxies DDO154 (a gas dominated rather than star dominated galaxy), NGC3198, NGC2903, and NGC5907, with the fitting of Mannheim (1993b) being reproduced here as Fig. (1). (The reader is referred to the original paper for details). For NGC3198 the rotation curve of Begeman (1989) and the surface brightness data of Wevers et al. (1986) and Kent (1987) were used, for NGC 2903 the data were taken from Begeman (1987) and Wevers et al. (1986), for NGC 5907 from van Albada and Sancisi (1986) and Barnaby and Thronson (1992), and for DDO154 from Carignan and Freeman (1988) and Carignan and Beaulieu (1989). (While Carignan and his coworkers favor a distance of 4 Mpc to DDO154, Krumm and Burstein (1984) favor 10 Mpc. Since the gas contribution is extremely distance sensitive, for completeness we opted to fit this galaxy at both the candidate distances). As can be seen from Fig. (1), the conformal theory appears to be able to do justice to a data set
which involves a broad range of luminosities, and to even do so without the need for dark matter, a point we analyze further below.

In order to apply the linear potential to an extended object such as a disk it was found helpful to develop a general formalism, with the results of the formalism being used in Mannheim (1993b) to produce the fits of Fig. (1). In the present paper we present the actual details of the derivation of the formalism (something that will be useful for future studies), with the formalism actually even being of interest in its own right since it extends to linear potentials the earlier work of Toomre (1963), Freeman (1970), and Casertano (1983) on Newtonian disks. We present the general formalism in Sec. (2), and in Sec. (3) we present some general comments on our work, discuss the status of the Tully-Fisher (Tully and Fisher (1977)) relation in conformal gravity, and compare and contrast our theory with the standard dark matter theory.

(2) The Potential of an Extended Disk

In order to handle the weak gravity potential of an extended object such as a disk of stars each with gravitational potential \( V(r) = -\beta/r + \gamma r/2 \) many ways are possible with perhaps the most popular being due to Toomre (1963). Since the method he developed for the Newtonian case does not immediately appear to generalize to linear potentials, we have instead generalized his approach to non-thin disks (a step also taken by Casertano (1983)) and then to disks with linear potentials. To determine the Newtonian potential of an axially symmetric (but not yet necessarily thin) distribution of matter sources with luminosity density function \( \rho(R, z') \) we need to evaluate the quantity

\[
V_\beta(r, z) = -\beta \int_0^\infty dR \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{R\rho(R, z')}{(r^2 + R^2 - 2rr'\cos\phi' + (z - z')^2)^{1/2}}
\tag{4}
\]

where \( R, \phi', z' \) are cylindrical source coordinates and \( r \) and \( z \) are the only observation point coordinates of relevance. To evaluate Eq. (4) it is convenient to make use of the cylindrical Green’s function Bessel function expansion

\[
\frac{1}{|r - r'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(kr)J_m(kr')e^{im(\phi - \phi') - k|z - z'|}
\tag{5}
\]
whose validity can readily be checked by noting that use of the identity

$$\nabla^2 [J_m(kr)e^{im\phi-k|z-z'|}] = -2kJ_mE^{im\phi}\delta(z-z')$$  \hspace{1cm} (6)

leads to the relation

$$\nabla^2 \left( \frac{1}{|r-r'|} \right) = -4\pi\delta^3(r-r')$$  \hspace{1cm} (7)

(In his original study Toomre used a Bessel function discontinuity formula (essentially Eq. (6)) which only appears to be applicable to thin disks. Using the full completeness properties of the Bessel functions enables us to treat non-thin disks as well). While Eq. (5) is standard, it is not utilized as often as the more familiar modified Bessel function expansion

$$\frac{1}{|r-r'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk \cos[k(z-z')] \int_0^{\infty} J_m(kr)K_m(kR)e^{im(\phi-\phi')}$$  \hspace{1cm} (8)

since the product of the two modified Bessel functions has much better convergence properties at infinity than the product of the two ordinary Bessel functions. Nonetheless, the ordinary Bessel functions do actually vanish at infinity which is sufficient for our purposes here. A disadvantage of the expansion of Eq. (8) is that it involves oscillating $z$ modes rather than the bounded $z$ modes given in Eq. (5), with the bounded form of Eq. (5) actually being extremely convenient for a disk whose matter distribution is concentrated around $z = 0$. An additional shortcoming of the expansion of Eq. (8) is that when it is inserted into Eq. (4) it requires the $R$ integration range to be broken up into two separate pieces at the point of observation. However, inserting Eq. (5) into Eq. (4) leads to

$$V_\beta(r, z) = -2\pi\beta \int_0^\infty dk \int_0^{\infty} dR \int_{-\infty}^{\infty} dz' R\rho(R, z')J_0(kr)J_0(kR)e^{-k|z-z'|}$$  \hspace{1cm} (9)

which we see requires no such break up. Finally, taking the disk to be infinitesimally thin (viz. $\rho(R, z') = \Sigma(R)\delta(z')$) then yields for points with $z = 0$ the potential

$$V_\beta(r) = -2\pi\beta \int_0^\infty dk \int_0^{\infty} dRR\Sigma(R)J_0(kr)J_0(kR)$$  \hspace{1cm} (10)
which we immediately recognize as Toomre’s original result for an infinitesimally thin disk.

In passing we note that Eq. (9) also holds for points which do not lie in the $z = 0$ plane of the disk, and also applies to disks whose thickness may not in fact be negligible, with the form of Eq. (9) being particularly convenient if the fall-off of the matter distribution in the $z$ direction is itself exponential (see below).

For our purposes here, the expansion of Eq. (5) can immediately be applied to the linear potential case too, and this leads directly (on setting $|\mathbf{r} - \mathbf{r}'| = (\mathbf{r} - \mathbf{r}')^2/|\mathbf{r} - \mathbf{r}'|$) to the potential

$$V_\gamma(r, z) = \frac{\gamma}{2} \int_0^\infty \frac{dR}{2\pi} \int_{-\infty}^\infty d\phi' \int_{-\infty}^\infty dz' \rho(R, z')[r^2 + R^2 - 2rR\cos\phi' + (z - z')^2]^{1/2}$$

$$= \pi\gamma \int_0^\infty \frac{dk}{2\pi} \int_0^\infty \frac{dR}{2\pi} \int_{-\infty}^\infty dz' \rho(R, z')[r^2 + R^2 + (z - z')^2]J_0(kr)J_0(kR)$$

$$- 2rRJ_1(kr)J_1(kR)e^{-k|z-z'|}$$

Equation (11) then reduces at $z = 0$ for infinitesimally thin disks to the compact expression

$$V_\gamma(r) = \pi\gamma \int_0^\infty \frac{dk}{2\pi} \int_0^\infty \frac{dR}{2\pi} RR\Sigma(R)[(r^2 + R^2)J_0(kr)J_0(kR) - 2rRJ_1(kr)J_1(kR)]$$

If the $k$ integrations are performed first in Eqs. (10) and (12) they lead to singular hypergeometric functions whose subsequent $R$ integrations contain infinities which, even while they are in fact mild enough to be integrable (as long as $\Sigma(R)$ is sufficiently damped at infinity), nonetheless require a little care when being carried out numerically. Thus unlike the sphere whose potential is manifestly finite at every step of the calculation, the disk, because of its lower dimensionality, actually encounters infinities at any interior point of observation on the way to a final finite answer. However, since the final answer is finite, it should be possible to obtain this answer without ever encountering any infinities at any stage of the calculation at all; and indeed, if the distribution function $\Sigma(R)$ is available in a closed form, then performing the $R$ integration before the $k$ integration can yield a calculation which is finite at every stage. Thus, for the exponential disk

$$\Sigma(R) = \Sigma_0 e^{-\alpha R}$$

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where $1/\alpha = R_0$ is the scale length of the disk and $N = 2\pi\Sigma_0 R_0^2$ is the number of stars in the disk, use of the standard Bessel function integral formulas

$$\int_0^\infty dRRJ_0(kR)e^{-\alpha R} = \frac{\alpha}{(\alpha^2 + k^2)^{3/2}}$$

(14)

$$\int_0^\infty dk \frac{J_0(kr)}{(\alpha^2 + k^2)^{3/2}} = (r/2\alpha)[I_0(\alpha r/2)K_1(\alpha r/2) - I_1(\alpha r/2)K_0(\alpha r/2)]$$

(15)

then leads directly to Freeman’s original result, viz.

$$V_\beta(r) = -2\pi\beta\Sigma_0 \int_0^\infty dk \frac{\alpha J_0(kr)}{(\alpha^2 + k^2)^{3/2}} = -\pi\beta\Sigma_0 r[I_0(\alpha r/2)K_1(\alpha r/2) - I_1(\alpha r/2)K_0(\alpha r/2)]$$

(16)

for the Newtonian potential of an exponential disk. The use of the additional integral formula

$$\int_0^\infty dRR^2 J_1(kR)e^{-\alpha R} = \frac{3\alpha k}{(\alpha^2 + k^2)^{5/2}}$$

(17)

and a little algebra (involving eliminating $J_1(kr) = -\left(dJ_0(kr)/dk\right)/r$ via an integration by parts) enable us to obtain for the linear potential contribution the expression

$$V_\gamma(r) = \pi\gamma\Sigma_0 \int_0^\infty dk \left(\frac{\alpha r^2 J_0(kr)}{(\alpha^2 + k^2)^{3/2}} - \frac{9\alpha J_0(kr)}{(\alpha^2 + k^2)^{5/2}} + \frac{15\alpha^3 J_0(kr)}{(\alpha^2 + k^2)^{7/2}} - \frac{6\alpha k r J_1(kr)}{(\alpha^2 + k^2)^{5/2}}\right)$$

$$= \pi\gamma\Sigma_0 \int_0^\infty dk J_0(kr) \left(\frac{\alpha r^2}{(\alpha^2 + k^2)^{3/2}} + \frac{15\alpha}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^3}{(\alpha^2 + k^2)^{7/2}}\right)$$

(18)

Equation (18) is readily evaluated through use of the modified Bessel function recurrence relations

$$I'_0(z) = I_1(z) \quad I'_1(z) = I_0(z) - I_1(z)/z$$

$$K'_0(z) = -K_1(z) \quad K'_1(z) = -K_0(z) - K_1(z)/z$$

(19)

in conjunction with Eq. (15), and yields

$$V_\gamma(r) = \pi\gamma\Sigma_0 \{(r/\alpha^2)[I_0(\alpha r/2)K_1(\alpha r/2) - I_1(\alpha r/2)K_0(\alpha r/2)]$$

$$+ (r^2/2\alpha)[I_0(\alpha r/2)K_0(\alpha r/2) + I_1(\alpha r/2)K_1(\alpha r/2)]\}$$

(20)
To obtain test particle rotational velocities we need only differentiate Eqs. (16) and (20) with respect to \( r \). This is readily achieved via repeated use of the recurrence relations of Eqs. (19) which form a closed set under differentiation so that higher modified Bessel functions such as \( I_2(\alpha r/2) \) and \( K_2(\alpha r/2) \) are not encountered; and the procedure is found to yield

\[
r V'(r) = (N \beta \alpha^3 r^2/2)[I_0(\alpha r/2)K_0(\alpha r/2) - I_1(\alpha r/2)K_1(\alpha r/2)]
+ (N \gamma r^2 \alpha/2)I_1(\alpha r/2)K_1(\alpha r/2)
\]

(21)

Using the asymptotic properties of the modified Bessel functions we find that at distances much larger than the scale length \( R_0 \) Eq. (21) yields

\[
r V'(r) \rightarrow \frac{N \beta}{r} + \frac{N \gamma r}{2} - \frac{3N \gamma R_0^2}{4r}
\]

(22)

as would be expected. We recognize the asymptotic Newtonian term to be just \( N \beta/r \) where \( N \) is the number of stars in the disk. The quantity \( N \beta \) is usually identified as \( MG/c^2 \) with \( M \) being taken to be the mass of the disk. For normalization purposes it is convenient to use this coefficient to define the velocity \( v_0 = (N \beta/R_0)^{1/2} \), the velocity that a test particle would have if orbiting a Newtonian point galaxy with the same total mass at a distance of one scale length. In terms of the convenient dimensionless parameter \( \eta = \gamma R_0^2/\beta \) Eq. (21) then yields for the rotational velocity \( v(r) \) of a circular orbit in the plane of a thin exponential disk the exact expression

\[
v^2(r)/v_0^2 = (r^2 \alpha^2/2)[I_0(\alpha r/2)K_0(\alpha r/2) + (\eta - 1)I_1(\alpha r/2)K_1(\alpha r/2)]
\]

(23)

an expression which is surprisingly compact. For thin disks then all departures from the standard Freeman result are thus embodied in the parameter \( \eta \) in the simple manner indicated.

Beyond making actual applications to galaxies, a further advantage of having an exact solution in a particular case is that it can be used to test a direct numerical evaluation of the galactic potential (which involves integrable infinities) by also running the program
for a model exponential disk. Also, it is possible to perform the calculation analytically in various other specific cases. For a thin axisymmetric disk with a Gaussian surface matter distribution \( \Sigma(R) = \Sigma_0 \exp(-\alpha^2 R^2) \) and \( N = \pi \Sigma_0 / \alpha^2 \) stars (this being a possible model for the sometimes steeper central region of a galaxy in cases where there may be no spherical bulge) we find for the complete rotational velocity the expression

\[
V'(r) = \pi^{1/2} N \beta \alpha^3 r^2 [I_0(\alpha^2 r^2/2) - I_1(\alpha^2 r^2/2)] e^{-\alpha^2 r^2/2} \\
+ (\pi^{1/2} N \gamma \alpha r^2/4)[I_0(\alpha^2 r^2/2) + I_1(\alpha^2 r^2/2)] e^{-\alpha^2 r^2/2}
\]

(24)

Similarly, for a spherically symmetric matter distribution (the central bulge region of a galaxy) with radial matter density \( \sigma(r) \) and \( N = 4 \pi \int dr' r'^2 \sigma(r') \) stars we obtain the general expression

\[
V'(r) = \frac{4 \pi \beta}{r} \int_0^r dr' \sigma(r') r'^2 \\
+ \frac{2 \pi \gamma}{3r} \int_0^r dr' \sigma(r')(3r'^2 r^2 - r'^4) + \frac{4 \pi \gamma r^2}{3} \int_r^\infty dr' \sigma(r') r'
\]

(25)

which can readily be integrated once a particular \( \sigma(r) \) is specified.

Beyond the exact expressions obtained above there is one other case of practical interest namely that of non-thin but separable disks, a case which can also be greatly simplified by our formalism. For such separable disks we set \( \rho(R, z') = \Sigma(R) f(z') \) where the usually symmetric thickness function \( f(z') = f(-z') \) is normalized according to

\[
\int_{-\infty}^{\infty} dz' f(z') = 2 \int_0^{\infty} dz' f(z') = 1
\]

(26)

Recalling that

\[
e^{-k|z-z'|} = \theta(z - z') e^{-k(z-z')} + \theta(z' - z) e^{+k(z-z')}
\]

(27)

we find that Eqs. (9) and (11) then yield for points with \( z = 0 \)

\[
V_\beta(r) = -4 \pi \beta \int_0^\infty dk \int_0^\infty dR \int_0^{\infty} dz' R \Sigma(R) f(z') J_0(kr) J_0(kR) e^{-kz'}
\]

(28)

and

\[
V_\gamma(r) = 2 \pi \gamma \int_0^\infty dk \int_0^\infty dR \int_0^{\infty} dz' R \Sigma(R) f(z')
\]

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\[(r^2 + R^2 + z'^2)J_0(kr)J_0(kR) - 2rRJ_1(kr)J_1(kR)\]e^{-kz'} \tag{29}\]

in the separable case. Further simplification is possible if the radial dependence is again exponential (viz. \(\Sigma(R) = \Sigma_0 \exp(-\alpha R)\)) and yields, following some algebra involving the use of the recurrence relation \(J'_1(z) = J_0(z) - J_1(z)/z\), the expressions

\[rV'_\beta(r) = 2N\beta^3 r \int_0^\infty dk \int_0^\infty dz' f(z')e^{-kz'} \frac{kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} \tag{30}\]

and

\[rV'_\gamma(r) = N\gamma^3 r \int_0^\infty dk \int_0^\infty dz' f(z')e^{-kz'} \times \left( -\frac{4rJ_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{6\alpha^2 rJ_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{(r^2 + z'^2)kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{9kJ_1(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2 kJ_1(kr)}{(\alpha^2 + k^2)^{7/2}} \right) \tag{31}\]

As regards actual specific forms for \(f(z')\), two particular ones have been identified via the surface photometry of edge on galaxies, one by van der Kruit and Searle (1981), and the other by Barnaby and Thronson (1992). Respectively they are

\[f(z') = \text{sech}^2(z'/z_0)/2z_0 \tag{32}\]

and

\[f(z') = \text{sech}(z'/z_0)/\pi z_0 \tag{33}\]

each with appropriate scale height \(z_0\). We note that both of these thickness functions are falling off very rapidly in the \(z'\) direction just like the Bessel function expansion itself of Eq. (5). Consequently, Eqs. (28) - (31) will now have very good convergence properties. The thickness function of Eq. (32) is found to lead to rotational velocities of the form

\[rV'_\beta(r) = (N\beta^3 r^2/2)[I_0(\alpha r/2)K_0(\alpha r/2) - I_1(\alpha r/2)K_1(\alpha r/2)] \]

\[\times -N\beta^3 r \int_0^\infty dk \frac{k^2 J_1(kr)z_0 \beta(1 + k z_0/2)}{(\alpha^2 + k^2)^{3/2}} \tag{34}\]

where

\[\beta(x) = \int_0^1 \frac{t^{x-1}}{(1 + t)} \tag{35}\]

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and

\[ rV'_\gamma(r) = N\gamma\alpha^3 r \int_0^\infty dk (1 - kz_0\beta(1 + kz_0/2)) \]

\[ \times \left( -\frac{2rJ_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{3\alpha^2 rJ_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{r^2 kJ_1(kr)}{2(\alpha^2 + k^2)^{3/2}} + \frac{9kJ_1(kr)}{2(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2 kJ_1(kr)}{2(\alpha^2 + k^2)^{7/2}} \right) \]

\[ + N\gamma\alpha^3 r \int_0^\infty dk \frac{kJ_1(kr)}{2(\alpha^2 + k^2)^{3/2}} \frac{d^2}{dk^2} \left( kz_0\beta(1 + \frac{kz_0}{2}) \right) \]  

(36)

Similarly, the thickness function of Eq. (33) leads to

\[ rV'_\beta(r) = \frac{2N\beta\alpha^3 r}{\pi} \int_0^\infty dk \frac{kJ_1(kr)\beta(1/2 + kz_0/2)}{2(\alpha^2 + k^2)^{3/2}} \]  

(37)

and

\[ rV'_\gamma(r) = \frac{N\gamma\alpha^3 r}{\pi} \int_0^\infty dk \beta(1/2 + kz_0/2) \]

\[ \times \left( -\frac{4rJ_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{6\alpha^2 rJ_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{r^2 kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{9kJ_1(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2kJ_1(kr)}{(\alpha^2 + k^2)^{7/2}} \right) \]

\[ - \frac{N\gamma\alpha^3 r}{\pi} \int_0^\infty dk \frac{kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} \frac{d^2}{dk^2} \left( \beta\left(1 + \frac{kz_0}{2}\right) \right) \]  

(38)

The great utility of these expressions is that all of the functions of \(\beta(x)\) and their derivatives which appear in Eqs. (34) - (38) converge very rapidly to their asymptotic values as their arguments increase. Consequently the \(k\) integrations in Eqs. (34) - (38) converge very rapidly numerically while encountering no singularities at all.

As a practical matter, the observed scale heights \(z_0\) are usually much smaller than any observed scale lengths \(R_0\). Consequently the thickness corrections of Eqs. (34) - (38) usually only modify the thin disk formula of Eq. (21) in the central galactic region, and thus have essentially no effect on the linear potential contribution. For the Newtonian term the corrections of Eqs. (34) and (37) to the Freeman formula tend to reduce the overall Newtonian contribution (c.f. the second term in Eq. (34) and Casertano (1983)) and serve to ensure that the inner rotation curves of Fig. (1) are well described (see Mannheim (1993b)) by the luminous Newtonian contribution, to thus clear the way to explore the effect of the linear term on the outer region of the rotation curve, a region
where its presence is significant and where the thin disk formula of Eq. (21) provides a very good approximation to the dynamics.

(3) General Comments

In order to understand the general features of the rotation curves of Fig. (1) it is instructive to consider the generic implications of the thin disk formula of Eq. (23), a two parameter formula with \( v_0 \) fixing the overall normalization and \( \eta \) the relative contributions of the Newtonian and linear pieces. Moreover, if this overall normalization is fixed by the peak in the rise of the inner rotation curve (the so called maximum disk fit in which the Newtonian disk contribution gets to be as large as it possibly can be), then essentially the entire shape of the rest of the curve is fixed by just the one parameter (per galaxy) \( \eta \). As regards this maximum disk contribution, we note that the Newtonian term in Eq. (23) peaks at \( 2.15R_0 \) with \( v^2/v_0^2 \) receiving a Newtonian contribution of 0.387. This Newtonian contribution comes down to half of this value (i.e. 0.194) at \( 6.03R_0 \). Since the linear contribution is essentially negligible at \( 2.15R_0 \) (especially after we take the square root to get the velocity itself), if we choose the linear contribution at \( 6.03R_0 \) to be equal to the Newtonian contribution at that same distance (i.e. if numerically we set \( \eta \) equal to a critical value of 0.067), we will then have essentially achieved flatness over the entire 2 to 6 scale length region. Now at 6 scale lengths both the Newtonian and linear terms are quickly approaching their asymptotic values exhibited in Eq. (22). Consequently at close to 12 scale lengths (precisely at \( 11.62R_0 \)) the linear term contribution is just 0.387, the original maximum disk value at \( 2.15R_0 \). Hence between 6 and 12 scale lengths the curve again shows little deviation from flatness. However since the Newtonian contribution at 12 scale lengths is slightly bigger than the linear contribution at 2 scale lengths, the net outcome is that by 12 scale lengths the curve is actually beginning to show a slight rise, with flatness only being achieved out to about 10 scale lengths. Thus in general we see that by varying just one parameter we can naturally achieve flatness over the entire 2 to 10 scale length region, this intriguingly being about as large a range of scale lengths as has
up till now been observed in any rotation curve. In order to see just how flat a curve it is in principle possible to obtain, we have varied $\eta$ as a free parameter. Our most favored generic case is then obtained when $\eta$ takes the value 0.069 (i.e. essentially the critical value), with the resulting generic rotational velocity curve being plotted in Fig. (2). Over the range from 3 to 10 scale lengths the ratio $v(r)/v_0$ is found to take the values (0.666, 0.648, 0.632, 0.626, 0.628, 0.637, 0.651, 0.667) in unit step increases. Thus it has a spread of only $\pm 3\%$ about a central value of 0.647 in this region. Additionally, we find that even at 15 scale lengths the ratio $v(r)/v_0$ has still only increased to 0.763, a 14% increase over its value at 10 scale lengths. In the upper diagram in Fig. (2) we have plotted the generic $\eta = 0.069$ rotation curve out to 10 scale lengths to show just how flat it can be. In the lower diagram in Fig. (2) we have shown the continuation out to 15 scale lengths where the ultimate asymptotic rise is becoming apparent. We have deliberately juxtaposed the two diagrams in Fig. (2) since the flatness out to 10 scale lengths is usually taken as indicative of asymptotic flatness as well, with such ultimate flatness being characteristic (and even a primary motivation) of both isothermal gas sphere dark matter models and the MOND alternative (Milgrom (1983)). The possibility that flatness is only an intermediate and not an asymptotic phenomenon is one of the most unusual and distinctive features of the conformal gravity theory. (Of course it is always possible to build dark matter models with non flat asymptotic properties (see e.g. van Albada et al. (1985)) since the dark matter theory is currently so unconstrained. However, our point here is that the conformal theory is the first theory in which rotation curves are actually required to ultimately rise, even being predicted to do so in advance of any data). As regards other possible values for $\eta$, if $\eta$ exceeds the critical value of 0.067, then the curve will be flat for fewer scale lengths with the rise setting in earlier, while if $\eta$ is less than the critical value, the curve will drop perceptibly and come back to its maximum disk value at a greater distance.

As regards the generic critical value for $\eta$, we note that for a typical galaxy with a mass of $10^{11}$ solar masses and a 3 kpc scale length, the required value for the galactic $\gamma_{galaxy} (= N\gamma_{star}$ where $\gamma_{star}$ is the typical $\gamma$ used in the stellar potential $V(r)$
of Eq. (3)) then turns out to be of order $10^{-29}/\text{cm}$, which, intriguingly, is of order the inverse Hubble radius. Moreover, this characteristic value is in fact numerically attained in the fits of Fig. (1) for the stellar disk contribution in all of our four chosen galaxies (viz. $\gamma(154)=2.5\times10^{-30}/\text{cm}$, $\gamma(3198)=3.5\times10^{-30}/\text{cm}$, $\gamma(2903)=7.6\times10^{-30}/\text{cm}$, $\gamma(5907)=5.7\times10^{-30}/\text{cm}$). (While this same cosmological value is also found for DDO154, in some other aspects (such as possessing a rotation curve which has no observed flat region at all) the fitting to this dwarf irregular is found to be anomalous (see Mannheim (1993b)) presumably because the galaxy is gas rather than star dominated. Hence we shall only regard the three other galaxies, all regular spirals, as typical for the purposes of our discussion here). Thus not only is $\gamma_{galaxy}$ making the observed representative curves flat, and not only is it doing so with an effectively universal value, it is doing so with a value which is already known to be of astrophysical significance; thereby suggesting that $\gamma_{galaxy}$ may be of cosmological origin, perhaps being related to the scale at which galaxies fluctuate out of the cosmological background.

Additionally, we note that this apparent universality for $\gamma_{galaxy}$ has implications for the status of the Tully-Fisher relation in our theory. Specifically, the average velocity $v_{ave}$ (the velocity dispersion) of the critical generic curve is equal to the maximum disk value since the curve is flat. Thus we can set (using $N = 2\pi \Sigma_0 R_0^2$ and letting $L$ denote the galactic luminosity)

$$v_{ave}^4 = \left(\frac{0.387 N \beta}{R_0}\right)^2 = 0.300 \pi \Sigma_0 \beta^2 L \left(\frac{N}{L}\right) \quad (39)$$

At the critical value for $\eta$ (the fits yield $\eta(3198)=0.044$, $\eta(2903)=0.038$, $\eta(5907)=0.057$) we also can set

$$\gamma_{galaxy} = \frac{0.067 N \beta}{R_0^2} \quad (40)$$

so that Eq. (39) may be rewritten as

$$v_{ave}^4 = 2.239 \gamma_{galaxy} \beta L \left(\frac{N}{L}\right) \quad (41)$$

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If we assume that all galaxies possess the same universal value for the mass to light ratio (our fits yield $M/L(3198)=4.2$, $M/L(2903)=3.5$, $M/L(5907)=6.1$ in units of $M_\odot/L_{B\odot}$), we then see that given a universal $\gamma_{\text{galaxy}}$, Eq. (41) then yields none other than the Tully-Fisher velocity-luminosity relation. (Observationally the Tully-Fisher relation is not thought to hold for the stellar component of the dwarf irregular DDO154, as may be anticipated since DDO154 is phenomenologically found to have an anomalously small $M/L$ ratio ($M/L(154)$ takes the value 1.4 in our fits and is essentially zero in the dark matter and MOND fits of Begeman et al. (1991)) - since the above discussion does not include any non-stellar component it is anyway not applicable to gas dominated galaxies). Additionally, according to Eq. (40) the universality of $\gamma_{\text{galaxy}}$ also entails the universality of $\Sigma_0$, the central surface brightness, a phenomenological feature first identified for spirals by Freeman (1970). (In turn the universality of $\Sigma_0$ entails a mass - radius squared relation for galaxies). The (near) universality of $\gamma_{\text{galaxy}}$ and $\eta$ thus correlates in one fell swoop the observed flatness of rotation curves, the universality of $\Sigma_0$, and the Tully-Fisher relation, and does so in a theory in which rotation curves must eventually rise. (In passing, we note that in his review article Sanders (1990) argues against the possibility of being able to do precisely this in a theory which possesses one new non-Newtonian scale (such as $\gamma_{\text{galaxy}}$). However, his arguments were made in the explicit context of asymptotically flat rotation curves, and are thus bypassed here since our curves only enjoy flatness as an intermediate phenomenon). Thus the establishing of a cosmological origin for $\gamma_{\text{galaxy}}$ and $\eta$ (which would establish a cosmological origin for $N\beta/R_0^2$ as well thereby making the galactic Newtonian and linear contributions comparable) would then lead naturally to flatness and the Tully-Fisher relation. The above given discussion provides a generalization to axially symmetric systems of an earlier discussion (Kazanas (1991), Mannheim and Kazanas (1991b)) based on the simplification of using Eq. (2) itself as the galactic metric. As we now see, the ideas developed in those two earlier papers carry over to the present more detailed treatment. (In passing we should point out the mass - radius squared relation which was also identified in those two previous papers was actually found to have phenomenological validity on many
other astrophysical scales as well, something which still awaits an explanation).

While we have categorized our fits as having two parameters per galaxy, the actual situation is slightly more constrained. Specifically, we note that the Newtonian and linear contributions are both proportional to $N$ according to Eq. (22). Thus if there existed universal average stellar parameters $\beta$ and $\gamma$ to serve as input for Eqs. (4) and (11), $\eta$ would then be fixed by the scale length $R_0$ of each galaxy, resulting in one parameter ($N$) per galaxy fits. Ordinarily, one thinks of $\beta$ as being the Schwarzschild radius of the Sun, and then in the fits the numerical value of the mass to light ratio of the galaxy is allowed to vary freely in the fitting, with $M/L$ ratios then being found which are actually remarkably close to each other (without such closeness there would be severe violations of the Tully-Fisher relation because of the $N/L = M/LM_\odot$ factor in Eqs. (39) and (41)).

However, in reality each galaxy comes with its own particular mix of stars, both in overall population and, even more significantly, in the spatial distribution of the mix. Now, of course ideally we should integrate Eq. (4) over the true stellar distribution allowing $\beta$ to vary with position according to where the light and heavy stars (stars whose luminosities do not simply scale linearly with their masses) are physically located within the stellar disk. Instead we use an average $\beta$ (which incidentally enables us to derive exact formulas such as Eq. (9)). However, two galaxies with the identical morphological mix of stars but with different spatial distributions of those stars should each be approximated by a different average $\beta$, since the Newtonian potential weights different distances unequally. Since we do not give two galaxies of this type different average $\beta$ parameters to begin with, we can then compensate later by giving them different mass to light ratios (even though for this particular example we gave them the same morphological mix). Hence we extract out a quantity $N\beta_\odot$ from the data which simulates $N_{ave}\beta_{ave}$ where $N_{ave}$ is the true average number of stars in the galaxy. Because of the difference between these two ways of defining the number of stars in a galaxy, it is not clear whether the currently quoted mass to light ratios as found in the fits (in essentially all theories of rotation curve systematics) are merely reflecting this difference or whether they are exploiting this uncertainty to come up
with possibly unwarrantable mass to light ratios. Thus a first principles determination of actual values or of a range of allowed values of galactic mass to light ratios prior to fitting would be extremely desirable.

A precisely similar situation also obtains for the \( \gamma \) dependent terms. Again we use an average stellar \( \gamma \) and compensate for its possible average variation from galaxy to galaxy by allowing the galactic gamma to light ratio \( (\gamma_{\text{galaxy}}/L = N\gamma/L) \) to vary phenomenologically (i.e. we use \( N\gamma \) to simulate \( N_{\text{ave}}\gamma_{\text{ave}} \) where \( N \) is determined once and for all by normalizing the data to \( N\beta_\odot \)). The fits to our representative galaxies are found to yield \( N\gamma/L_B(3198)=3.9, N\gamma/L_B(2903)=5.1, N\gamma/L_B(5907)=3.2 \) (in units of \( 10^{-40}/\text{cm}/L_B\odot \)), values which again are remarkably close to each other and which are of a par with the mass to light ratios \( M/L(3198)=4.2, M/L(2903)=3.5, M/L(5907)=6.1 \) found for the same galaxies. We would not expect the \( M/\gamma_{\text{galaxy}} \) ratio to be the same for the entire sample, simply because even if the stellar \( \beta \) and \( \gamma \) parameters were to change by the same proportion in going from one morphological type of star to another (a reasonable enough expectation), nonetheless, as the galactic spatial distributions change, the inferred average stellar \( \beta \) and \( \gamma \) parameters would then change in essentially unrelated ways, since the Newtonian and linear potentials weight the differing spatial regions of the galaxy quite differently to thus yield different average values. Nonetheless, it is intriguing to find that the variation in the average \( \beta \) and \( \gamma \) shows such mild dependence on specific galaxy within our sample; \( N \) and \( N_{\text{ave}} \) thus appear to be very close. To within this (mild) variation, our fits are thus effectively one parameter per galaxy fits. (In its pure form the MOND explanation of the systematics of galactic rotation curves is also a one parameter per galaxy theory. However, in its successful practical applications (Begeman et al. (1991)), it is generally found necessary to introduce at least one more fitting parameter per galaxy, such as by allowing a (generally quite mild) variation in the fundamental acceleration parameter \( a_0 \) over the galactic sample. Phenomenologically then MOND would thus appear to be on a par with our linear potential theory). For our linear potential theory we note that given the apparent uniformity of the average stellar \( \gamma/\beta \) ratio, we see that we really have to
normalize $N$ to the maximum disk mass and that we are really not free to vary the normalizations of the Newtonian and linear pieces separately, since they both are proportional to $N$. Specifically, if we make the Newtonian piece too small we would have to arbitrarily increase the linear contribution, something we are not able to do in a consistent manner. Thus the Newtonian contribution in our fit cannot be too small. Similarly, it can never be allowed to be too large (this would give too high a velocity); and, hence, the Newtonian contribution in our theory is bounded both above and below, and essentially forced to the maximum disk mass; and thus our theory is reduced to almost parameter free fitting. Since dark matter fits can generally adjust the relative strengths of the luminous and dark matter pieces at will, they are not so constrained, and often yield much smaller luminous Newtonian contributions, and thus large amounts of dark matter. Thus a first principles determination of galactic mass to light ratios might enable one to discriminate between rival theories. (Actually, the Tully-Fisher relation in the form of Eq. (39) is also a statement about the rotational velocity at the Newtonian inner region peak. Since the Tully-Fisher relation is generally found to hold phenomenologically for the maximum disk velocity (the relation was initially found for the inner region velocity maximum long before outer region flatness was ever established), that fact alone would seem to constrain the mass to light ratios to the values implied by Eq. (39); to thus potentially constrain the dark matter models, models which generally seem to be curiously silent regarding the whole issue of the validity or otherwise of the Tully-Fisher relation).

In order to compare our work with that of other approaches it is useful to clarify the significance of the term ‘flat rotation curve’. In the literature it is generally thought that rotation curves will be flat asymptotically (though of course the more significant issue here is the fact that the curves deviate from the luminous Newtonian prediction at all, rather than in what particular way); and of course since our model predicts that velocities will eventually grow as $r^{1/2}$, the initial expectation is that our model is immediately ruled out. However, the rotation curve fits that have so far been made are not in fact asymptotic ones. Firstly, the $HII$ optical studies pioneered by Rubin and coworkers (Rubin et al.
(1978, 1980, 1982, 1985)), even while they were indeed yielding flat rotation curves, were restricted to the somewhat closer in optical disk region since the $HII$ regions are only to be found in the vicinity of hot stars which ionize those regions. And eventually, after a concentrated effort to carefully measure the surface brightness of such galaxies, it was gradually realized (see e.g. Kaljnas (1983) and Kent (1986)) that the $HII$ curves could be described, albeit coincidentally, by a standard luminous Newtonian prediction after all; even in fact for galaxies such as UGC2885 for which the data go out to as much as 80 kpc, a distance which turns out to only be of order 4 scale lengths ($R_0=22$ kpc for UGC2885, an atypically high value - this galaxy is just very big). Thus, not only are the optical studies limited (by their very nature in fact) to the optical disk region where there is some detectable surface brightness, but it turns out, coincidentally, that they are also limited to the region where an extended Newtonian source is actually yielding flat rotation curves to a rather good degree. Thus this inner region flatness has nothing at all to do with any possible asymptotic flatness, though it will enable flatness to set in as early as 2 or 3 scale lengths in fits to any data which do go out to many more scale lengths.

While the $HII$ data do not show any substantive non-canonical behaviour, nonetheless, the pioneering work of Rubin and coworkers brought the whole issue of galactic rotation curves into prominence and stimulated a great deal of study in the field. Now it turns out that neutral hydrogen gas is distributed in galaxies out to much farther distances than the stars, thus making the $HI$ studies ideal probes of the outer reaches of the rotation curves and of the luminous Newtonian prediction. (That $HI$ studies might lead to a conflict with the luminous Newtonian prediction was noted very early by Freeman (1970) from an analysis of NGC300 and M33, by Roberts and Whitehurst (1975) from an analysis of M31, and by Bosma (1978) who made the first large 21 cm line survey of spiral galaxies). Thus with the $HI$ studies (there are now about 30 well studied cases) it became clear that there really was a problem with the interpretation of galactic rotation curve data, which the community immediately sought to explain by the introduction of galactic dark matter since the Newton-Einstein theory was presumed to be beyond question. Fits to
the HI data have been obtained using dark matter (Kent (1987) provides a very complete analysis), and while the fits are certainly phenomenologically acceptable, they nonetheless possess certain shortcomings. Far and away their most serious shortcoming is their ad hoc nature, with any found Newtonian shortfall then being retroactively fitted by an appropriately chosen dark matter distribution. In this sense dark matter is not a predictive theory at all but only a parametrization of the difference between observation and the luminous Newtonian expectation. As to possible dark matter distributions, no specific distribution, or explicit set of numerical parameters for a distribution, has convincingly been derived from first principles as a consequence, say, of galactic dynamics or formation theory (for a recent critical review see Sanders (1990)). (The general community would not appear to regard any specific derivation as being all that convincing since no distribution has been heralded as being so theoretically secure that any failure of the data to conform to it would necessitate the abandoning of the Newton-Einstein theory). Amongst the candidate dark matter distributions which have been considered in the literature the most popular is the distribution associated with a modified isothermal gas sphere (a two parameter spherical matter density distribution \( \rho(r) = \rho_0/(r^2 + r_0^2) \) with an overall scale \( \rho_0 \) and an arbitrarily introduced non-zero core radius \( r_0 \) which would cause dark matter to predominate in the outer rather than the inner region - even though a true isothermal sphere would have zero core radius). The appeal of the isothermal gas sphere is that it leads to an asymptotically logarithmic galactic potential and hence to asymptotically flat rotation curves, i.e. it is motivated by no less than the very data that it is trying to explain. However, careful analysis of the explicit dark matter fits is instructive. Recalling that the inner region (around, say, \( 2R_0 \) for definitiveness) is already flat for Newtonian reasons, the dark matter parameters are then adjusted so as to join on to this Newtonian piece (hence the ad hoc core radius \( r_0 \)) to give a continuously flat curve in the observed region. This matching of the luminous and dark matter pieces is for the moment completely fortuitous (van Albada and Sancisi (1986) have even referred to it as a conspiracy) and not yet explained by galactic dynamics. What is done in the fits is actually even a double conspiracy. Not only are
the outer \((10R_0)\) and the inner \((2R_0)\) regions given the same velocity (by adjusting \(\rho_0\)), the intermediate \((6R_0)\) region is adjusted through the core radius \(r_0\) to ensure that the curve does not fall and then rise again in that region. Hence flatness in the \(r_0\) dominated region has almost nothing at all to do with the presumed asymptotically flat isothermal gas sphere contribution. Even worse, in the actual fits the dark matter contributions are found to actually still be rising at the largest observed \((10R_0)\) distances, and thus not yet taking on their asymptotic values at all. Hence the curves are made flat not by a flat dark matter contribution but rather by an interplay between a rising dark matter piece and a falling Newtonian one, with the asymptotically flat expectation not yet actually having even been tested. (Prospects for pushing the data out to farther distances are not good because \(HI\) surface densities typically fall off exponentially fast at the edge of the explored region). Thus for the moment, even though both available \(HI\) and \(HII\) type data sets are flat in their respective domains, each data set is flat for its own coincidental reason, and it would appear to us that region of true galactic asymptotics has yet to be explored; with the observed flatness of the galactic rotation curves (just like the apparent flatness of total proton proton scattering cross sections over many energy decades before an eventual rise) perhaps only being an intermediate rather than an asymptotic phenomenon.

Beyond these fitting questions (two dark matter parameters per galaxy is, however, still fairly economical), the outcome of the fitting is that galaxies are then 90% or so non-luminous. Thus not only is the Universe to be dominated by this so far undetected material, the stars in a galaxy are demoted to being only minor players, an afterthought as it were. Since dark matter only interacts gravitationally it is extremely difficult to detect (its actual detection with just the requisite flux would of course be a discovery of the first magnitude), and since it can be freely reparametrized as galactic data change or as new data come on line, it hardly qualifies as even being a falsifiable idea, the sine qua non for a physical theory. While some possible dark matter candidate particle may eventually be detected, the issue is not whether the particle exists at all - it may exist for some wholly unrelated reason, but rather whether its associated flux is big enough to dominate galaxies.
Since the great appeal of Einstein gravity is its elegance and beauty, using an approach as ad hoc and contrived as dark matter for it almost defeats the whole purpose, and would even appear to be at odds with Einstein’s own view of the way nature works. Indeed, Einstein always referred to the Einstein Equations as being a bridge between the beautiful geometry of the Einstein tensor and the ugliness of the energy-momentum tensor. The dark matter idea only serves to make the energy-momentum tensor even more ugly. The great aesthetic appeal of the conformal theory is that it adds beauty to both sides of the gravitational equations of motion by both retaining covariance and by endowing both the sides of the bridge with the additional, highly restrictive, symmetry of conformal invariance; and, as we have seen, the theory can then even eliminate the need for dark matter altogether. (We do not assert that there is no dark matter in the Universe, only that there is no gravitational basis for assuming its existence - even with conformal gravity dark matter could still exist for reasons totally unrelated to gravity, albeit at a flux much lower than the standard one).

Given the success (so far) of the linear potential theory in fitting the rotation curve data without needing to invoke dark matter, it would thus appear to us that at the present time one cannot categorically assert that the sole gravitational potential on all distance scales is the Newtonian one; and that, in the linear potential, the standard $1/r$ potential would not only appear to have a companion but to have one which would even dominate over it asymptotically. Indeed, the very need for dark matter in the standard theory may simply be due to trying to apply just the straightforward Newtonian potential in a domain for which there is no prior (or even current for that matter) justification. Even though the observational confirmation on terrestrial to solar system distance scales of both the Newton theory and its general relativistic Einstein corrections technically only establishes the validity of the Newton-Einstein theory on those scales, nonetheless, for most workers in the field, it seems to have established the standard theory on all other distance scales too; despite the fact that many other theories could potentially have the same leading perturbative structure on a given distance scale and yet differ radically elsewhere. Since
we have shown that the conformal theory also appears to be able to meet the constraints of data, one has to conclude that at the present time the Newton-Einstein theory is only sufficient to describe data, but not yet necessary. Indeed, it is the very absence of some principle which would single out the Einstein theory from amongst all other possible covariant theories which one could in principle at least consider which prevents the Einstein theory from yet being a necessary theory of gravity. In fact, in a sense, it is the absence of some underlying principle which would ensure its uniqueness that is the major theoretical problem for the Einstein theory, rather than its phenomenological inability to fit data without invoking dark matter; with this very lack itself actually opening the door to other contenders (Mannheim (1993c)).

Finally, as regards our actual fitting, we see that is not in fact necessary to demand flatness in the asymptotic region in order to obtain flat rotation curves in the explored intermediate region. Thus, unlike the dark matter fits, we do not need to know the structure of the data prior to the fitting, or need to adapt the model to a presupposed asymptotic flatness. Further, not only is our linear potential theory more motivated than the dark matter models (Eq. (3) arises in a fundamental, fully covariant, uniquely specified theory), it possesses one fewer free parameter per galaxy (\(\gamma\) instead of \(\rho_0\) and \(r_0\)). Consequently, according to the usual criteria for evaluating rival theories, as long as conformal gravity continues to hold up, it is to be preferred.

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**Figure Captions**

Figure (1). The calculated rotational velocity curves associated with the metric of Eq. (2) for the four representative galaxies, the intermediate sized NGC3198, the compact bright NGC2903, the large bright NGC5907, and the dwarf irregular DDO154 (at two possible adopted distances). In each graph the bars show the data points with their quoted errors, the full curve shows the overall theoretical velocity prediction (in km/s) as a function of distance (in arc minutes) from the center of each galaxy, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials of Eq. (3) would produce when integrated over the luminous matter distribution of each galaxy. No dark matter is assumed.

Figure (2). The flattest possible rotation curve for a thin exponential disk of stars each with conformal gravity potential $V(r) = -\beta/r + \gamma r/2$ which is obtained when the dimensionless ratio $\eta$ takes the value 0.069. The full curve shows the overall theoretical velocity prediction (in units of $v/v_0$) as a function of distance (in units $R/R_0$), while the two indicated dotted curves show the rotation curves that separate Newtonian and linear potentials would produce. In the upper diagram the rotation curve is plotted out to 10 scale lengths to fully exhibit its flatness, while in the lower diagram it is plotted out to 15 scale lengths to exhibit its eventual asymptotic rise.