Abstract

Fractional (or non-integer) differentiation is an important concept both from theoretical and applicational points of view. The study of problems of the calculus of variations with fractional derivatives is a rather recent subject, the main result being the fractional necessary optimality condition of Euler-Lagrange obtained in 2002. Here we use the notion of Euler-Lagrange fractional extremal to prove a Noether-type theorem. For that we propose a generalization of the classical concept of conservation law, introducing an appropriate fractional operator.

Keywords: calculus of variations; fractional derivatives; Noether’s theorem.

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1 Introduction

The notion of conservation law – first integral of the Euler-Lagrange equations – is well-known in Physics. One of the most important conservation laws is the integral of energy, discovered by Leonhard Euler in 1744: when a Lagrangian \( L(q, \dot{q}) \) corresponds to a system of conservative points, then

\[
-L(q, \dot{q}) + \partial_2 L(q, \dot{q}) \cdot \dot{q} \equiv \text{constant}
\]  

(1)

holds along all the solutions of the Euler-Lagrange equations (along the extremals of the autonomous variational problem), where \( \partial_2 L(\cdot, \cdot) \) denote the partial derivative of function \( L(\cdot, \cdot) \) with respect to its second argument. Many other examples appear in modern physics: in classic, quantum, and optical
mechanics; in the theory of relativity; etc. For instance, in classic mechanics, beside the conservation of energy (1), it may occur conservation of momentum or angular momentum. These conservation laws are very important: they can be used to reduce the order of the Euler-Lagrange differential equations, thus simplifying the resolution of the problems.

In 1918 Emmy Noether proved a general theorem of the calculus of variations, that permits to obtain, from the existence of variational symmetries, all the conservation laws that appear in applications. In the last decades, Noether’s principle has been formulated in various contexts (see [16, 17] and references therein). In this work we generalize Noether’s theorem for problems having fractional derivatives.

Fractional differentiation plays nowadays an important role in various seemingly diverse and widespread fields of science and engineering: physics (classical and quantum mechanics, thermodynamics, optics, etc), chemistry, biology, economics, geology, astrophysics, probability and statistics, signal and image processing, dynamics of earthquakes, control theory, and so on [3, 6, 8, 9]. Its origin goes back more than 300 years, when in 1695 L’Hopital asked Leibniz the meaning of \( \frac{d^n}{dx^n} \) for \( n = \frac{1}{2} \). After that, many famous mathematicians, like J. Fourier, N. H. Abel, J. Liouville, B. Riemann, among others, contributed to the development of the Fractional Calculus [6, 10, 14].

F. Riewe [12, 13] obtained a version of the Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives, that combines the conservative and non-conservative cases. More recently, O. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [1]. Then these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [4]. Here we use the results of [1] to generalize Noether’s theorem for the more general context of the Fractional Calculus of Variations.

The paper is organized in the following way. In Section 2 we recall the notions of right and left Riemann-Liouville fractional derivatives, that are needed for formulating the fractional problem of the calculus of variations. There are many different ways to approach classical Noether’s theorem. In Section 3 we review the only proof that we are able to extend, with success, to the fractional context. The method is based on a two-step procedure: it starts with an invariance notion of the integral functional under a one-parameter infinitesimal group of transformations, without changing the time variable; then it proceeds with a time-reparameterization to obtain Noether’s theorem in general form. The intended fractional Noether’s theorem is formulated and proved in Section 4. Two illustrative examples of application of our main result are given in Section 5. We finish with Section 6 of conclusions and some open questions.
2 Riemann-Liouville fractional derivatives

In this section we collect the definitions of right and left Riemann-Liouville fractional derivatives and their main properties [1][10][14].

**Definition 2.1** (Riemann-Liouville fractional derivatives). Let $f$ be a continuous and integrable function in the interval $[a, b]$. For all $t \in [a, b]$, the left Riemann-Liouville fractional derivative $a D^\alpha_t f(t)$, and the right Riemann-Liouville fractional derivative $t D^\alpha_b f(t)$, of order $\alpha$, are defined in the following way:

\[
a D^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t - \theta)^{n-\alpha-1} f(\theta) d\theta, \quad (2)
\]

\[
t D^\alpha_b f(t) = \frac{1}{\Gamma(n - \alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\theta - t)^{n-\alpha-1} f(\theta) d\theta, \quad (3)
\]

where $n \in \mathbb{N}$, $n - 1 \leq \alpha < n$, and $\Gamma$ is the Euler gamma function.

**Remark 2.2.** If $\alpha$ is an integer, then from (2) and (3) one obtains the standard derivatives, that is,

\[
a D^\alpha_t f(t) = \left(\frac{d}{dt}\right)^\alpha f(t), \quad t D^\alpha_b f(t) = \left(-\frac{d}{dt}\right)^\alpha f(t).
\]

**Theorem 2.3.** Let $f$ and $g$ be two continuous functions on $[a, b]$. Then, for all $t \in [a, b]$, the following properties hold:

1. for $p > 0$, $a D^p_t (f(t) + g(t)) = a D^p_t f(t) + a D^p_t g(t)$;
2. for $p \geq q \geq 0$, $a D^p_t (a D^{-q}_t f(t)) = a D^{p-q}_t f(t)$;
3. for $p > 0$, $a D^p_t (a D^{-p}_t f(t)) = f(t)$ (fundamental property of the Riemann-Liouville fractional derivatives);
4. for $p > 0$, $\int_a^t (a D^p_t f(t)) g(t) dt = \int_a^t f(t) (a D^p_t g(t)) dt$ (we are assuming that $a D^p_t f(t)$ and $a D^p_t g(t)$ exist at every point $t \in [a, b]$ and are continuous).

**Remark 2.4.** In general, the fractional derivative of a constant is not equal to zero.

**Remark 2.5.** The fractional derivative of order $p > 0$ of function $(t-a)^\nu, \nu > -1$, is given by

\[
a D^p_t (t-a)^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(-p + \nu + 1)} (t-a)^{\nu-p}.
\]

**Remark 2.6.** In the literature, when one reads “Riemann-Liouville fractional derivative”, one usually means the “left Riemann-Liouville fractional derivative”. In Physics, if $t$ denotes the time-variable, the right Riemann-Liouville fractional derivative of $f(t)$ is interpreted as a future state of the process $f(t)$. For this reason, the right-derivative is usually neglected in applications, when

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the present state of the process does not depend on the results of the future development. From a mathematical point of view, both derivatives appear naturally in the fractional calculus of variations [1].

We refer the reader interested in additional background on fractional theory, to the comprehensive book of Samko et al. [14].

3 Review of the classical Noether’s theorem

There exist several ways to prove the classical theorem of Emmy Noether. In this section we review one of those proofs [7]. The proof is done in two steps: we begin by proving Noether’s theorem without transformation of the time (without transformation of the independent variable); then, using a technique of time-reparameterization, we obtain Noether’s theorem in its general form. This technique is not so popular while proving Noether’s theorem, but it turns out to be the only one we succeed to extend for the more general context of the Fractional Calculus of Variations.

We begin by formulating the fundamental problem of the calculus of variations:

\[
I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \, dt \rightarrow \min \quad (P)
\]

under the boundary conditions \( q(a) = q_a \) and \( q(b) = q_b \), and where \( \dot{q} = \frac{dq}{dt} \).

The Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is assumed to be a \( C^2 \)-function with respect to all its arguments.

**Definition 3.1** (invariance without transforming the time). Functional \( (P) \) is said to be invariant under a \( \varepsilon \)-parameter group of infinitesimal transformations

\[
\bar{q}(t) = q(t) + \varepsilon \xi(t, q) + o(\varepsilon)
\]

if, and only if,

\[
\int_{t_a}^{t_b} L(t, q(t), \dot{q}(t)) \, dt = \int_{t_a}^{t_b} L(t, \bar{q}(t), \dot{\bar{q}}(t)) \, dt \quad (5)
\]

for any subinterval \([t_a, t_b] \subseteq [a, b]\).

Along the work, we denote by \( \partial_i L \) the partial derivative of \( L \) with respect to its \( i \)-th argument.

**Theorem 3.2** (necessary condition of invariance). *If functional \((P)\) is invariant under transformations \((4)\), then*

\[
\partial_2 L(t, q, \dot{q}) \cdot \xi + \partial_3 L(t, q, \dot{q}) \cdot \dot{\xi} = 0 .
\]

**Proof.** Equation \((5)\) is equivalent to

\[
L(t, q, \dot{q}) = L(t, q + \varepsilon \xi + o(\varepsilon), \dot{q} + \varepsilon \dot{\xi} + o(\varepsilon)) .
\]

Differentiating both sides of equation \((7)\) with respect to \( \varepsilon \), then substituting \( \varepsilon = 0 \), we obtain equality \((6)\). \( \square \)
Definition 3.3 (conserved quantity). Quantity $C(t, q(t), \dot{q}(t))$ is said to be conserved if, and only if, $\frac{d}{dt}C(t, q(t), \dot{q}(t)) = 0$ along all the solutions of the Euler-Lagrange equations

$$\frac{d}{dt} \partial_3 L(t, q, \dot{q}) = \partial_2 L(t, q, \dot{q}).$$

(8)

Theorem 3.4 (Noether’s theorem without transforming time). If functional $\mathcal{P}$ is invariant under the one-parameter group of transformations \( \{ \bar{t} = t + \varepsilon \tau(t, q), \bar{q}(t) = q(t) + \varepsilon \xi(t, q) \} \), then

$$C(t, q, \dot{q}) = \partial_3 L(t, q, \dot{q}) \cdot \xi(t, q)$$

(9)

is conserved.

Proof. Using the Euler-Lagrange equations \( \mathcal{P} \) and the necessary condition of invariance \( \mathcal{P} \), we obtain:

$$\frac{d}{dt} (\partial_3 L(t, q, \dot{q}) \cdot \xi(t, q))$$

$$= \frac{d}{dt} \partial_3 L(t, q, \dot{q}) \cdot \xi(t, q) + \partial_3 L(t, q, \dot{q}) \cdot \dot{\xi}(t, q)$$

$$= \partial_2 L(t, q, \dot{q}) \cdot \xi(t, q) + \partial_3 L(t, q, \dot{q}) \cdot \dot{\xi}(t, q)$$

$$= 0.$$

Remark 3.5. In classical mechanics, $\partial_3 L(t, q, \dot{q})$ is interpreted as the generalized momentum.

Definition 3.6 (invariance of $\mathcal{P}$). Functional $\mathcal{P}$ is said to be invariant under the one-parameter group of infinitesimal transformations

$$\begin{cases}
\bar{t} = t + \varepsilon \tau(t, q) + o(\varepsilon), \\
\bar{q}(t) = q(t) + \varepsilon \xi(t, q) + o(\varepsilon),
\end{cases}$$

(10)

if, and only if,

$$\int_{t_a}^{t_b} L(t, q(t), \dot{q}(t)) \, dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L(\bar{t}, \bar{q}(\bar{t}), \dot{\bar{q}}(\bar{t})) \, d\bar{t}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

Theorem 3.7 (Noether’s theorem). If functional $\mathcal{P}$ is invariant, in the sense of Definition 3.6, then

$$C(t, q, \dot{q}) = \partial_3 L(t, q, \dot{q}) \cdot \xi(t, q) + (L(t, q, \dot{q}) - \partial_3 L(t, q, \dot{q}) \cdot \dot{q}) \tau(t, q)$$

(11)

is conserved.
Proof. Every non-autonomous problem is equivalent to an autonomous one, considering as a dependent variable. For that we consider a Lipschitzian one-to-one transformation \([a, b] \ni t \mapsto \sigma \in [\sigma_a, \sigma_b]\) such that

\[ I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \, dt \]

\[ = \int_{\sigma_a}^{\sigma_b} L(t(\sigma), q(t(\sigma)), \frac{dq(t(\sigma))}{d\sigma}) \, d\sigma \]

\[ = \int_{\sigma_a}^{\sigma_b} L(t(\sigma), q(t(\sigma)), \frac{t'(\sigma)}{t''(\sigma)}) \, t' d\sigma \]

\[ = \int_{\sigma_a}^{\sigma_b} L(t(\sigma), q(t(\sigma)), \dot{t}^{'}(\sigma), \dot{q}^{'}(\sigma)) \, d\sigma \]

\[ = \int_{\sigma_a}^{\sigma_b} \bar{L}(t(\sigma), q(t(\sigma)), \dot{t}^{'}(\sigma), \dot{q}^{'}(\sigma)) \, d\sigma \]

where \(t(\sigma_a) = a, \ t(\sigma_b) = b, \ t'_{\sigma} = \frac{dt(\sigma)}{d\sigma}, \) and \(q'_{\sigma} = \frac{dq(t(\sigma))}{d\sigma}.\) If functional \(I[q(\cdot)]\) is invariant in the sense of Definition 3.6, then functional \(\bar{I}[t(\cdot), q(t(\cdot))]\) is invariant in the sense of Definition 3.1. Applying Theorem 3.4, we obtain that

\[ C(t, q, t', q') = \partial_4 \bar{L} \cdot \xi + \partial_3 \bar{L} \tau \] (12)

is a conserved quantity. Since

\[ \partial_4 \bar{L} = \partial_3 L \cdot (t, q, \dot{q}), \]

\[ \partial_3 \bar{L} = -\partial_3 L \cdot (t, q, \dot{q}) \cdot \frac{q''}{t''} + L(t, q, \dot{q}) \] (13)

substituting (13) into (12), we arrive to the intended conclusion (11). \(\square\)

4 Main Results

In 2002 [1], a formulation of the Euler-Lagrange equations was given for problems of the calculus of variations with fractional derivatives. In this section we prove a Noether’s theorem for the fractional Euler-Lagrange extremals.

The fundamental functional of the fractional calculus of variations is defined as follows:

\[ I[q(\cdot)] = \int_a^b L(t, q(t), aD^\alpha q(t), bD^\beta q(t)) \, dt \rightarrow \min, \] (PF)

where the Lagrangian \(L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) is a \(C^2\) function with respect to all its arguments, and \(0 < \alpha, \beta \leq 1.\)
Remark 4.1. In the case $\alpha = \beta = 1$, problem $[P_f]$ is reduced to problem $[P]$:

$$I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \, dt \longrightarrow \min$$

with

$$L(t, q, \dot{q}) = L(t, q, -\dot{q}).$$

(14)

Theorem 4.2 summarizes the main result of [1].

Theorem 4.2 ([1]). If $q$ is a minimizer of problem $[P_f]$, then it satisfies the fractional Euler-Lagrange equations:

$$\partial_2 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) + tD_\beta^\beta \partial_3 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) + aD_\alpha^\alpha \partial_4 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) = 0. \quad (15)$$

Definition 4.3 (cf. Definition 3.1). We say that functional $[P_f]$ is invariant under the transformation (4) if, and only if,

$$\int_{t_a}^{t_b} L \left( t, q(t), aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) \, dt = \int_{t_a}^{t_b} L \left( t, \bar{q}(t), aD_\alpha^\alpha \bar{q}(t), tD_\beta^\beta \bar{q}(t) \right) \, dt \quad (16)$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

The next theorem establishes a necessary condition of invariance, of extreme importance for our objectives.

Theorem 4.4 (cf. Theorem 3.2). If functional $[P_f]$ is invariant under transformations (4), then

$$\partial_2 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) \cdot \xi(t, q) + \partial_3 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) \cdot aD_\alpha^\alpha \xi(t, q) + \partial_4 L \left( t, q, aD_\alpha^\alpha q(t), tD_\beta^\beta q(t) \right) \cdot tD_\beta^\beta \xi(t, q) = 0. \quad (17)$$

Remark 4.5. In the particular case $\alpha = \beta = 1$, we obtain from (17) the necessary condition $[9]$ applied to $L$ [14].

Proof. Having in mind that condition (16) is valid for any subinterval $[t_a, t_b] \subseteq [a, b]$, we can get rid off the integral signs in (16). Differentiating this condition with respect to $\varepsilon$, substituting $\varepsilon = 0$, and using the definitions and properties...
of the Riemann-Liouville fractional derivatives given in Section 2 we arrive to

\[ 0 = \partial_2 L \left( t, q, a D_t^\alpha q, t D_b^\beta q \right) \cdot \xi(t, q) \]

\[ + \partial_3 L \left( t, q, a D_t^\alpha q, t D_b^\beta q \right) \cdot \frac{d}{dz} \left[ \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-\theta)^{n-\alpha-1} q(\theta) d\theta \right] \]  

\[ + \varepsilon \frac{\Gamma(n-\alpha)}{\Gamma(n-\beta)} \left( \frac{d}{dt} \right)^n \int_a^t (t-\theta)^{n-\alpha-1} \xi(\theta, q) d\theta \right]_{\varepsilon=0} \]

\[ + \partial_4 L \left( t, q, a D_t^\alpha q, t D_b^\beta q \right) \cdot \frac{d}{dz} \left[ \frac{1}{\Gamma(n-\beta)} \left( -\frac{d}{dt} \right)^n \int_t^b (\theta-t)^{n-\beta-1} q(\theta) d\theta \right] 

\[ + \varepsilon \frac{\Gamma(n-\alpha)}{\Gamma(n-\beta)} \left( -\frac{d}{dt} \right)^n \int_t^b (\theta-t)^{n-\beta-1} \xi(\theta, q) d\theta \right]_{\varepsilon=0} \cdot \]  

Expression (18) is equivalent to (17). The following definition is useful in order to introduce an appropriate concept of fractional conserved quantity.

**Definition 4.6.** Given two functions \( f \) and \( g \) of class \( C^1 \) in the interval \([a, b]\), we introduce the following notation:

\[ \mathcal{D}_t^\gamma (f, g) = -g \cdot D_t^\alpha f + f \cdot D_t^\alpha g, \]

where \( t \in [a, b] \) and \( \gamma \in \mathbb{R}_0^+ \).

**Remark 4.7.** If \( \gamma = 1 \), the operator \( \mathcal{D}_t^1 \) is reduced to

\[ \mathcal{D}_t^1 (f, g) = -g \cdot D_t^\alpha f + f \cdot D_t^\alpha g = \hat{f} g + f \hat{g} = \frac{d}{dt}(fg). \]

In particular, \( \mathcal{D}_t^1 (f, g) = \mathcal{D}_t^1 (g, f) \).

**Remark 4.8.** In the fractional case (\( \gamma \neq 1 \)), functions \( f \) and \( g \) do not commute: in general \( \mathcal{D}_t^\gamma (f, g) \neq \mathcal{D}_t^\gamma (g, f) \).

**Remark 4.9.** The linearity of the operators \( a D_t^\gamma \) and \( t D_b^\beta \) imply the linearity of the operator \( \mathcal{D}_t^\gamma \).

**Definition 4.10** (fractional-conserved quantity – cf. Definition 3.3). We say that \( C_f \left( t, q, a D_t^\alpha q, t D_b^\beta q \right) \) is fractional-conserved if (and only if) it is possible to write \( C_f \) in the form

\[ C_f (t, q, d_t, d_r) = \sum_{i=1}^m C_i^1 (t, q, d_t, d_r) \cdot C_i^2 (t, q, d_t, d_r) \]  

(19)

for some \( m \in \mathbb{N} \) and some functions \( C_i^1 \) and \( C_i^2 \), \( i = 1, \ldots, m \), where each pair \( C_i^1 \) and \( C_i^2 \), \( i = 1, \ldots, m \), satisfy

\[ \mathcal{D}_t^\gamma \left( C_i^1 \left( t, q, a D_t^\alpha q, t D_b^\beta q \right), C_i^2 \left( t, q, a D_t^\alpha q, t D_b^\beta q \right) \right) = 0 \]  

(20)
Remark 4.11. Noether conserved quantities are a sum of products, like the structure we are assuming in Definition 4.10. For \( \alpha = \beta = 1 \) is equivalent to the standard definition \( \frac{d}{dt} |C_f(t, q(t), \dot{q}(t), -\dot{q}(t))| = 0 \).

Example 4.12. Let \( C_f \) be a fractional-conserved quantity written in the form with \( m = 1 \), that is, let \( C_f = C_{i}^1 \cdot C_{j}^2 \) be a fractional-conserved quantity for some given functions \( C_{i}^1 \) and \( C_{j}^2 \). Condition (20) of Definition 4.10 means one of four things: or \( D_t^\alpha (C_{i}^1, C_{j}^2) = 0 \), or \( D_t^\alpha (C_{i}^1, C_{j}^2) = 0 \), or \( D_t^\alpha (C_{i}^1, C_{j}^2) = 0 \), or \( D_t^\alpha (C_{i}^1, C_{j}^2) = 0 \).

Remark 4.13. Given a fractional-conserved quantity \( C_f \), the definition of \( C_{i}^1 \) and \( C_{j}^2 \), \( i = 1, \ldots, m \), is never unique. In particular, one can always choose \( C_{i}^1 \) to be \( C_{j}^2 \) and \( C_{j}^2 \) to be \( C_{i}^1 \). Definition 4.10 is immune to the arbitrariness in defining the \( C_{i}^1, i = 1, \ldots, m, j = 1, 2 \).

Remark 4.14. Due to the simple fact that the same function can be written in several different but equivalent ways, to a given fractional-conserved quantity \( C_f \) it corresponds an integer value \( m \) in which is, in general, also not unique (see Example 4.15).

Example 4.15. Let \( f, g \) and \( h \) be functions satisfying \( D_t^\alpha (g, f) = 0 \), \( D_t^\alpha (f, g) \neq 0 \), \( D_t^\alpha (h, f) = 0 \), \( D_t^\alpha (f, h) \neq 0 \) along all the fractional Euler-Lagrange extremals of a given fractional variational problem. One can provide different proofs to the fact that \( C = f(g + h) \) is a fractional-conserved quantity: (i) \( C \) is fractional-conserved because we can write \( C \) in the form with \( m = 2 \), \( C_{i}^1 = g, C_{i}^2 = f, C_{j}^1 = h, C_{j}^2 = f, \) and \( C_{j}^2 = f, \) satisfying (20) \( D_t^\alpha (C_{i}^1, C_{j}^2) = 0 \) and \( D_t^\alpha (C_{i}^2, C_{j}^2) = 0 \); (ii) \( C \) is fractional-conserved because we can write \( C \) in the form with \( m = 1 \), \( C_{i}^1 = g + h, \) and \( C_{j}^2 = f, \) satisfying (20) \( D_t^\alpha (C_{i}^1, C_{j}^2) = D_t^\alpha (g + h, f) = D_t^\alpha (g, f) + D_t^\alpha (h, f) = 0 \).

Theorem 4.16 (cf. Theorem 3.4). If the functional \( D_f \) is invariant under the transformations \( \xi \), in the sense of Definition 4.3 then

\[
C_f \left( t, q, a D_t^\alpha q, b D_b^\beta q \right) = \left[ \partial_3 L \left( t, q, a D_t^\alpha q, b D_b^\beta q \right) - \partial_4 L \left( t, q, a D_t^\alpha q, b D_b^\beta q \right) \right] \cdot \xi(t, q) \tag{21}
\]

is fractional-conserved.

Remark 4.17. In the particular case \( \alpha = \beta = 1 \), we obtain from (21) the conserved quantity \( \xi \) applied to \( L \) (14).

Proof. We use the fractional Euler-Lagrange equations

\[
\partial_2 L \left( t, q, a D_t^\alpha q, b D_b^\beta q \right) = -a D_t^\beta \partial_3 L \left( t, q, a D_t^\alpha q, b D_b^\beta q \right)
\]

\[
- a D_t^\beta \partial_4 L \left( t, q, a D_t^\alpha q, b D_b^\beta q \right)
\]
The proof is complete.

**Definition 4.18** (invariance of \(P_f\)) – cf. Definition 4.10. Functional \(P_f\) is said to be invariant under the one-parameter group of infinitesimal transformations \(\{P_f\}\) if, and only if,

\[
\int_{t_a}^{t_b} L \left( t, q(t), aD_t^\alpha q(t), tD_\sigma^\beta q(t) \right) dt = \int_{t_a}^{t_b} L \left( \tilde{t}, \tilde{q}(\tilde{t}), \tilde{a}D_{\tilde{t}}^\alpha \tilde{q}(\tilde{t}), \tilde{t}D_{\tilde{\sigma}}^\beta \tilde{q}(\tilde{t}) \right) d\tilde{t}
\]

for any subinterval \([t_a, t_b] \subseteq [a, b]\).

The next theorem provides the extension of Noether’s theorem for Fractional Problems of the Calculus of Variations.

**Theorem 4.19** (fractional Noether’s theorem). If functional \(P_f\) is invariant, in the sense of Definition 4.18 then

\[
\begin{align*}
C_f \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) &= \left[ \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \right] \\
&- \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \cdot \xi(t, q)
\end{align*}
\]

is fractional-conserved (Definition 4.10).

**Remark 4.20.** If \(\alpha = \beta = 1\), the fractional conserved quantity \(22\) gives \(\Pi\) applied to \(L\),

Proof. Our proof is an extension of the method used in the proof of Theorem 3.7. For that we reparameterize the time (the independent variable \(t\)) by the Lipschitzian transformation

\[
[a, b] \ni t \mapsto \sigma f(\lambda) \in [\sigma_a, \sigma_b]
\]

that satisfies \(t_\sigma = f(\lambda) = 1\) if \(\lambda = 0\). Functional \(P_f\) is reduced, in this way, to an autonomous functional:

\[
\bar{I} [t(\cdot), q(t(\cdot))] = \int_{\sigma_a}^{\sigma_b} L \left( t(\sigma), q(t(\sigma)), \sigma_a D_{t(\sigma)}^\alpha q(t(\sigma)), t(\sigma)D_{\sigma(\sigma)}^\beta q(t(\sigma)) \right) t'_\sigma d\sigma,
\]

\[\text{in (17), obtaining}
\]

\[\begin{align*}
&- \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \cdot \xi(t, q) + \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \cdot aD_t^\alpha \xi(t, q) \\
&- aD_t^\beta \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \cdot \xi(t, q) + \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \cdot tD_\sigma^\beta \xi(t, q)
\end{align*}
\]

\[= D_t^\alpha \left( \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right), \xi(t, q) \right) - D_t^\beta \left( \xi(t, q), \partial_t L \left( t, q, aD_t^\alpha q, tD_\sigma^\beta q \right) \right) = 0.
\]
where \( t(\sigma_a) = a, \ t(\sigma_b) = b, \)
\[
\sigma_a D^\alpha_a q(t(\sigma)) \]
\[
= \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt(\sigma)} \right)^n \int^{\sigma f(\lambda)}_{\lambda} (\sigma f(\lambda) - \theta)^{n - \alpha - 1} q(\theta f^{-1}(\lambda)) \, d\theta \\
= \frac{(t'_\sigma)^{-\alpha}}{\Gamma(n - \alpha)} \left( \frac{d}{d\sigma} \right)^n \int^{\sigma}_{\lambda} (\sigma - s)^{n - \alpha - 1} q(s) \, ds \\
= (t'_\sigma)^{-\alpha} D^{\alpha}_a q(\sigma),
\]
and, using the same reasoning,
\[
\sigma_b D^\beta_b q(t(\sigma)) = (t'_\sigma)^{-\beta} D^{\beta}_b q(\sigma).
\]
We have
\[
\bar{I}[t(\cdot), q(t(\cdot))] \\
= \int^{\sigma_b}_{\sigma_a} \tilde{L}_f \left( t(\sigma), q(t(\sigma)), (t'_\sigma)^{-\alpha} \frac{d}{(t'_\sigma)^2} D^\alpha_a q(\sigma), (t'_\sigma)^{-\beta} \frac{d}{(t'_\sigma)^2} q(\sigma) \right) t'_\sigma d\sigma \\
= \int^{\sigma_b}_{\sigma_a} \bar{L}_f \left( t(\sigma), q(t(\sigma)), (t'_\sigma)^{-\alpha} \frac{d}{(t'_\sigma)^2} D^\alpha_a q(t(\sigma)), (t'_\sigma)^{-\beta} \frac{d}{(t'_\sigma)^2} q(t(\sigma)) \right) d\sigma \\
= \int_a^b \tilde{L}_f \left( t, q(t), a D^\alpha_t q(t), t D^\beta_b q(t) \right) dt \\
= \bar{I}[q(\cdot)].
\]
By hypothesis, functional \( \text{23} \) is invariant under transformations \( \text{10} \), and it follows from Theorem \( \text{4.16} \) that
\[
C_f \left( t(\sigma), q(t(\sigma)), (t'_\sigma)^{-\alpha} \frac{d}{(t'_\sigma)^2} D^\alpha_a q(t(\sigma)), (t'_\sigma)^{-\beta} \frac{d}{(t'_\sigma)^2} q(t(\sigma)) \right) \\
= (\partial_4 \tilde{L}_f - \partial_3 \bar{L}_f) \cdot \xi + \frac{\partial}{\partial t'_\sigma} \tilde{L}_f \tau \quad \text{(24)}
\]
is a fractional conserved quantity. For \( \lambda = 0, \)
\[
\frac{d}{(t'_\sigma)^2} D^\alpha_a q(t(\sigma)) = a D^\alpha_t q(t), \\
\frac{d}{(t'_\sigma)^2} D^\beta_b q(t(\sigma)) = t D^\beta_b q(t),
\]
and we get
\[
\partial_4 \tilde{L}_f - \partial_3 \bar{L}_f = \partial_3 L - \partial_4 L, \quad \text{(25)}
\]
and
\[
\frac{\partial}{\partial \nu'_{\sigma}} \bar{L}_f = \partial_4 \bar{L}_f \cdot \frac{\partial}{\partial \nu'_{\sigma}} \left[ \frac{(t'_{\sigma})^{-\alpha}}{\Gamma(n-\alpha)} \left( \frac{d}{d\sigma} \right)^n \int_{\sigma}^{\sigma'} (\sigma-s)^{n-\alpha-1} q(s) ds \right] t'_{\sigma} \\
+ \partial_5 \bar{L}_f \cdot \frac{\partial}{\partial \nu'_{\sigma}} \left[ \frac{(t'_{\sigma})^{-\beta}}{\Gamma(n-\beta)} \left( \frac{d}{d\sigma} \right)^n \int_{\sigma}^{\sigma'} (s-\sigma)^{n-\beta-1} q(s) ds \right] t'_{\sigma} + L \\
= \partial_4 \bar{L}_f \cdot \left[ \frac{-\alpha(t'_{\sigma})^{-\alpha-1}}{\Gamma(n-\alpha)} \left( \frac{d}{d\sigma} \right)^n \int_{\sigma}^{\sigma'} (\sigma-s)^{n-\alpha-1} q(s) ds \right] \\
+ \partial_5 \bar{L}_f \cdot \left[ \frac{-\beta(t'_{\sigma})^{-\beta-1}}{\Gamma(n-\beta)} \left( \frac{d}{d\sigma} \right)^n \int_{\sigma}^{\sigma'} (s-\sigma)^{n-\beta-1} q(s) ds \right] + L \\
= -\alpha \partial_3 L \cdot D_{\alpha}^\alpha q - \beta \partial_4 L \cdot D_{\beta}^\beta q + L.
\]

Substituting (25) and (26) into equation (24), we obtain the fractional-conserved quantity (22).

5 Examples

In order to illustrate our results, we consider in this section two problems from [4, §3], with Lagrangians which are linear functions of the velocities. In both examples, we have used [5] to compute the symmetries (10).

Example 5.1. Let us consider the following fractional problem of the calculus of variations (Pf) with n = 3:
\[
I[q(\cdot)] = I[q_1(\cdot), q_2(\cdot), q_3(\cdot)] \\
= \int_0^b \left( (a D_{\alpha}^\alpha q_1) q_2 - (a D_{\beta}^\beta q_2) q_1 - (q_1 - q_2) q_3 \right) dt \rightarrow \min.
\]

The problem is invariant under the transformations (10) with
\[
(\tau, \xi_1, \xi_2, \xi_3) = (-ct, 0, 0, cq_3),
\]
where c is an arbitrary constant. We obtain from our fractional Noether’s theorem the following fractional-conserved quantity (22):
\[
C_f(t, q, a D_{\alpha}^\alpha q) = [ (1 - \alpha) ((a D_{\alpha}^\alpha q_1) q_2 - (a D_{\beta}^\beta q_2) q_1) - (q_1 - q_2) q_3 ] t. \quad (27)
\]

We remark that the fractional-conserved quantity (27) depends only on the fractional derivatives of q_1 and q_2 if \alpha \in [0, 1]. For \alpha = 1, we obtain the classical result:
\[
C(t, q) = (q_1 - q_2) q_3 t
\]
is conserved for the problem of the calculus of variations
\[
\int_a^b (\dot{q}_1 q_2 - \dot{q}_2 q_1 - (q_1 - q_2)q_3) \, dt \rightarrow \min.
\]

**Example 5.2.** We consider now a variational functional \(I[q(\cdot)]\) with \(n = 4\):
\[
I[q(\cdot)] = -\int_a^b \left( iD^\beta_0 q_1 \right) q_2 + \left( iD^\beta_0 q_3 \right) q_4 - \frac{1}{2} (q_2^2 - 2q_2q_3) \right) \, dt.
\]
The problem is invariant under (10) with
\[
(\tau, \xi_1, \xi_2, \xi_3, \xi_4) = \left( \frac{2c}{3} t, cq_1, -cq_2, \frac{c}{3} q_3, -\frac{c}{3} q_4 \right),
\]
where \(c\) is an arbitrary constant. We conclude from (22) that
\[
C_f (t, q, iD^\beta_0 q) = \left[ (\beta - 1) \left( iD^\beta_0 q_1 \right) q_2 + \left( iD^\beta_0 q_3 \right) q_4 \right] \frac{2}{3} t + q_1 q_2 + \frac{q_3 q_4}{3} \quad (28)
\]
is a fractional conserved quantity. In the particular case \(\beta = 1\), the classical result follows from (28):
\[
C(t, q) = q_1 q_2 + \frac{q_1 q_4}{3} + \frac{1}{3} (q_4^2 - 2q_2q_3) t
\]
is preserved along all the solutions of the Euler-Lagrange differential equations (8) of the problem
\[
\int_a^b \dot{q}_1 q_2 + \dot{q}_3 q_4 + \frac{1}{2} (q_2^2 - 2q_2q_3) \, dt \rightarrow \min.
\]

**6 Conclusions and Open Questions**

The fractional calculus is a mathematical area of a currently strong research, with numerous applications in physics and engineering. The theory of the calculus of variations for fractional systems was recently initiated in [1], with the proof of the fractional Euler-Lagrange equations. In this paper we go a step further: we prove a fractional Noether’s theorem.

The fractional variational theory is in its childhood so that much remains to be done. This is particularly true in the area of fractional optimal control, where the results are rare. A fractional Hamiltonian formulation is obtained in [11], but only for systems with linear velocities. A study of fractional optimal control problems with quadratic functionals can be found in [2]. To the best of the author’s knowledge, there is no general formulation of a fractional version of Pontryagin’s Maximum Principle. Then, with a fractional notion of Pontryagin extremal, one can try to use the techniques of [15] to extend the present results to the more general context of the fractional optimal control.
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References

[1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 272 (2002), no. 1, 368–379.

[2] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynam. 38 (2004), no. 1-4, 323–337.

[3] O. P. Agrawal, J. A. Tenreiro Machado and J. Sabatier, Introduction [Special issue on fractional derivatives and their applications], Nonlinear Dynam. 38 (2004), no. 1-4, 1–2.

[4] D. Baleanu and Tansel Avkar, Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, Nuovo Cimento, 119 (2004), 73–79.

[5] P. D. F. Gouveia and D. F. M. Torres, Automatic computation of conservation laws in the calculus of variations and optimal control, Comput. Methods Appl. Math., 5 (2005), no. 4, 387–409.

[6] R. Hilfer, Applications of fractional calculus in physics, World Sci. Publishing, River Edge, NJ, 2000.

[7] J. Jost and X. Li-Jost, Calculus of variations, Cambridge Univ. Press, Cambridge, 1998.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, Vol. 204, Elsevier, 2006.

[9] M. Klimek, Stationarity-conservation laws for fractional differential equations with variable coefficients, J. Phys. A 35 (2002), no. 31, 6675–6693.

[10] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[11] S. I. Muslih and D. Baleanu, Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives, J. Math. Anal. Appl. 304 (2005), no. 2, 599–606.

[12] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) 53 (1996), no. 2, 1890–1899.
[13] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) 55 (1997), no. 3, part B, 3581–3592.

[14] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives – theory and applications, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[15] D. F. M. Torres, On the Noether Theorem for Optimal Control, European Journal of Control 8 (2002), no. 1, 56–63.

[16] D. F. M. Torres, Quasi-invariant optimal control problems, Port. Math. (N.S.) 61 (2004), no. 1, 97–114.

[17] D. F. M. Torres, Proper extensions of Noether’s symmetry theorem for nonsmooth extremals of the calculus of variations, Commun. Pure Appl. Anal. 3 (2004), no. 3, 491–500.