Para-Hopf algebroids and their cyclic cohomology

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Abstract. We introduce the concept of para-Hopf algebroid and define their cyclic cohomology in the spirit of Connes-Moscovici cyclic cohomology for Hopf algebras. Para-Hopf algebroids are closely related to, but different from, Hopf algebroids. Their definition is motivated by attempting to define a cyclic cohomology theory for Hopf algebroids in general. We show that many of Hopf algebraic structures, including the Connes-Moscovici algebra $H_{FM}$, are para-Hopf algebroids.

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1 Introduction

Broadly speaking, Hopf algebroids are quantizations of groupoids. More precisely, they are the noncommutative and non-cocommutative algebraic analogues of Lie groupoids and Lie algebroids. In their study of the index theory of transversely elliptic operators and in order to obtain more canonical formulas which work for non-flat transversals, Connes and Moscovici [4] had to replace their Hopf algebra $H_n$ by a so called extended Hopf algebra of transverse differential operators $H_{FM}$. Here $FM$ denotes the frame bundle of an $n$-dimensional manifold $M$. In fact $H_{FM}$ is a bialgebroid in the sense of [7, 11] and has a so called twisted antipode $\tilde{S}$ with $\tilde{S}^2 = 1$; precise definitions will be given in Section 2. To define a cyclic cohomology theory for $H_{FM}$, Connes and Moscovici used the natural action of $H_{FM}$ on the algebra $A_{FM}$ associated to the frame bundle of $M$ and defined the cocyclic module of $H_{FM}$ as a
certain submodule of the cocyclic module of the algebra $A_{FM}$. It is clear that in general one needs a theory, parallel to cyclic cohomology theory for Hopf algebras, that works for all bialgebroids endowed with a suitable additional structure related to an antipode. The goal of this paper is to isolate an appropriate class of bialgebroids, called here para-Hopf algebroids, for which one can define a cocyclic module and hence a cyclic cohomology theory extending the cyclic cohomology for Hopf algebras (Definition 2.1 and Theorem 3.1).

Among the properties of Hopf algebras that helped Connes and Moscovici in [2] (cf. also [3] for a survey) to prove that the structure discovered in [1] is actually a cocyclic module for any Hopf algebra endowed with a modular pair in involution are the following: the antipode $S$ is an anti-coalgebra map; the twisted antipode $\tilde{S}$ is a twisted anti-coalgebra map; $\tilde{S}^2 = 1$. The first two properties unfortunately are not well defined in the world of bialgebroids. Thus, our first task is to find good, i.e. well defined for bialgebroids, equivalents of these properties.

The main challenge is to find necessary and sufficient conditions, in terms of a single operator called here a para-antipode, for the cyclic operator of the Connes-Moscovici module to be well defined and to form a cocyclic module. One first realizes that the complex defined in [4] is defined for all bialgebroids endowed with an antialgebra map $T$ satisfying conditions (PH1) and (PH2) of Definition 2.1 and in fact it is always a cosimplicial module. After many attempts, we realized that if we just assume that the second and third powers of the cyclic operator satisfy

$$\tau_1^2(h) = h, \quad \tau_3^3(1_H \otimes_R h) = 1_H \otimes_R h,$$

for all $h \in H$, then we have a cocyclic module. These last two conditions are obviously necessary as well. We find it remarkable that the third power of the cyclic operator comes into the picture and, together with its second power, gives necessary and sufficient conditions to have a cocyclic module. Note that for Hopf algebras it is the second power of the cyclic operator that gives a necessary and sufficient condition to have a cocyclic module [2].

Our main theorem, coupled with the fact that the Connes-Moscovici algebra $\mathcal{H}_{FM}$ admits a cocyclic module, implies that $\mathcal{H}_{FM}$ is a para-Hopf algebroid. We also provide a few other examples of para-Hopf algebroids, including the algebra $A_\theta$ of noncommutative torus, and the bialgebroid of Example 2.4 below defined by Connes and Moscovici in their study of Rankin-Cohen brackets [5].

In an earlier version of this paper, of which a very brief sketch appeared in our survey article [6], a different definition was proposed for a class of bialgebroids that admit a cocyclic module. The axioms for this structure, called an extended Hopf algebra there, are unfortunately very difficult to verify.
In particular our claim there that the Connes-Moscovici algebra is an extended Hopf algebra seems to be wrong or at least our proof is not adequate. Our present notion of para-Hopf algebroid on the other hand seems to be more workable and useful.

We would like to thank Alain Connes and Henri Moscovici for their interest and for valuable comments and suggestions that played a crucial role in the development of our ideas specially with regard to the Connes-Moscovici algebra $\mathcal{H}_{FM}$. It is a pleasure to thank Daniel Sternheimer for several valuable suggestions that improved our exposition and style. We are also much obliged to a referee who suggested that we look into Example 2.4, as well as suggesting a new title, and a new name, para-Hopf algebroids, for the main object of study in this paper.

2 Para-Hopf algebroids

In this section we first recall the definitions of bialgebroids and Hopf algebroids from [7, 11]. We then define a para-Hopf algebroid as a bialgebroid endowed with an extra structure that we call a para-antipode. Finally we show that several classes of bialgebroids are in fact para-Hopf algebroids. They include the Connes-Moscovici algebra $\mathcal{H}_{FM}$, the groupoid algebra of a groupoid with a finite number of objects, and the algebraic noncommutative torus $A_\theta$.

Hopf algebroids can be regarded as not necessarily commutative or cocommutative algebraic analogues of groupoids. Commutative Hopf algebroids were introduced in [9] as cogroupoid objects in the category of commutative algebras. Hence the total algebra and the base algebra are both commutative. Equivalently they can be defined as representable functors from the category of commutative algebras to the category of groupoids. The main examples of commutative Hopf algebroids are algebras of functions on an algebraic groupoid.

In [8] Hopf algebroids were defined where the total algebra need not be commutative but the base algebra is still commutative and the source and target maps land in the center of the total algebra. In the following we first recall a general definition of a bialgebroid and a Hopf algebroid, due to Lu, from [7, 11] and then define our para-Hopf algebroids. Note that a general definition of a bialgebroid, equivalent to Lu’s definition, was also given by Takeuchi in [10].

Let $k$ be a field of characteristic zero. A bialgebroid $(H, R, \alpha, \beta, \Delta, \epsilon)$ over $k$ consists of the following data:

- BA1: A $k$-algebra $H$, a $k$-algebra $R$, an algebra homomorphism $\alpha : R \to H$,
- and an anti algebra homomorphism $\beta : R \to H$ such that the images of $\alpha$ and $\beta$ agree on $R$. 

In particular our claim there that the Connes-Moscovici algebra is an extended Hopf algebra seems to be wrong or at least our proof is not adequate. Our present notion of para-Hopf algebroid on the other hand seems to be more workable and useful.

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α and β commute in H, i.e. for all a, b in R

\[ \alpha(a)\beta(b) = \beta(b)\alpha(a). \]

It follows that H has an R-bimodule structure defined by

\[ axb = \alpha(a)\beta(b)x \quad \forall a, b \in R, \ x \in H. \]

In particular the bimodule tensor product \( H \otimes_R H \) is defined and is an \((R, R)\)-bimodule. Similarly for \( H \otimes_R H \otimes_R H \) and higher bimodule tensor products. H is called the total algebra, R the base algebra, α the source map, and β the target map of the bialgebroid.

**BA2:** A coproduct, i.e. an \((R, R)\)-bimodule map \( \Delta : H \to H \otimes_R H \) which satisfies the following conditions:

- **cp1)** \( \Delta(1) = 1 \otimes_R 1 \)
- **cp2)** Coassociativity :

\[ (\Delta \otimes_R id_H)\Delta = (id_H \otimes_R \Delta)\Delta : H \to H \otimes_R H \otimes_R H, \]

- **cp3)** Compatibility with the product: for all \( a, b \in H \) and \( r \in R \),

\[ \Delta(a)(\beta(r) \otimes 1 - 1 \otimes \alpha(r)) = 0 \text{ in } H \otimes_R H, \]

\[ \Delta(ab) = \Delta(a)\Delta(b). \]

In the first relation the natural right action of \( H \otimes H \) on \( H \otimes_R H \) defined by \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \) is used. While \( H \otimes_R H \) need not be an algebra, it can be easily checked that the left annihilator of the image of \( \beta \otimes 1 - 1 \otimes \alpha \) is an algebra. Hence, by the first relation, the multiplicative property of \( \Delta \) makes sense.

**BA3:** A counit, i.e. an \((R, R)\)-bimodule map \( \epsilon : H \to R \) satisfying,

- **cu1)** \( \epsilon(1_H) = 1_R \)
- **cu2)** \( (\epsilon \otimes_R id_H)\Delta = (id_H \otimes_R \epsilon)\Delta = id_H : H \to H \)

In this paper we suppress the summation notation and write \( \Delta(h) = h^{(1)} \otimes_R h^{(2)} \) to denote the coproduct of a bialgebroid. Similarly, \( \Delta^n(h) = h^{(1)} \otimes_R \cdots \otimes_R h^{(n+1)} \) denotes the higher iterations of the coproduct.

Although it is not used in this paper, for reader’s convenience in comparing the definitions, we recall that a bialgebroid \((H, R, \alpha, \beta, \Delta, \epsilon)\) is called a Hopf algebroid if there is a bijective map \( S : H \to H \), called antipode, which is an antialgebra map satisfying the following conditions:
1) $S\beta = \alpha$.

2) $m_H(S \otimes id)\Delta = \beta \epsilon S : H \to H$, where $m_H : H \otimes H \to H$ is the multiplication map of $H$.

3) There exists a linear map $\gamma : H \otimes_R H \to H$ satisfying $\pi \circ \gamma = id_H \otimes_R \beta : H \otimes_R H \to H$ where $\pi : H \otimes H \to H$ is the natural projection.

Note that while the operator $m_H(S \otimes id)\Delta : H \to H$ is well defined, i.e., independent of the choice of any section for the projection map $H \otimes_R H \to H$, the operator $m_H(id \otimes S)\Delta$ is not well defined and one has to fix a linear section $\gamma$ for $\pi$ first. Also, unlike Hopf algebras, the antialgebra property of $S$ does not follow from axioms $HA1) - HA3)$, and has to be assumed. Since the operator $H \otimes_R H \to H$, $x \otimes_R y \mapsto y \otimes_R x$ is not even well defined, the anti-coalgebra property of $S$ does not even make sense.

Definition 2.1. A bialgebroid $(H, R, \alpha, \beta, \Delta, \epsilon)$ is called a Para-Hopf algebroid if there is an antialgebra map $T : H \to H$, called a para-antipode, satisfying the following conditions:

PH1) $T\beta = \alpha$.

PH2) $m_H(T \otimes id)\Delta = \beta \epsilon T : H \to H$, where $m_H : H \otimes H \to H$ is the multiplication map of $H$.

PH3) $T^2 = id_H$, and for all $h \in H$

$$T(h^{(1)}) \otimes_R T(h^{(2)}) = 1 \otimes_R T(h). \tag{1}$$

Remark 1. It follows from Theorem 3.1 that in terms of the cyclic operator $\tau$ in Theorem 3.1, axiom PH3 can be expressed as:

$$\tau_1^2(h) = h, \quad \text{and} \quad \tau_2^2(1_H \otimes_R h) = 1_H \otimes_R h,$$

for all $h \in H$.

We label the result of the following lemma (cu3) because some authors assume it as an additional axiom for the counit in the definition of a bialgebroid. We need (cu3) in several proofs in this paper, especially in the proof of Theorem 3.1.

Lemma 2.1. Let $(H, R)$ be a Para-Hopf algebroid. Then for all $h, g$ in $H$ we have

$$(\text{cu3}) \quad \epsilon(hg) = \epsilon(h\alpha(\epsilon(g))) = \epsilon(h\beta(\epsilon(g))).$$
Proof. By using (PH1), (PH2) and $T^2 = 1_{d_H}$, we have:
\[
\epsilon(hg) = \epsilon(T^2(hg)) = T(T(hg)^{(1)})T(hg)^{(2)} = T(T(h)^{(1)})T(T(g)^{(1)})T(g)^{(2)}T(h)^{(2)} = T(T(h)^{(1)})\beta(\epsilon(g))T(h)^{(2)} = T(T(h\beta(\epsilon(g)))^{(1)})T(h\beta(\epsilon(g)))^{(2)} = \epsilon(h\beta(\epsilon(g))).
\]
The other equality can be proven the same way. \(\square\)

Throughout this paper we make use of the following two actions. First, the right action of $H^\otimes n$ on $H^\otimes n$ defined by
\[
(h_1 \otimes_R \ldots \otimes_R h_n) \cdot (g_1 \otimes_R \ldots \otimes_R g_n) = (h_1g_1 \otimes_R h_2g_2 \otimes_R \ldots \otimes_R h_ng_n).
\]

It is evident that this action is well defined because the $R$-bimodule structure of $H$ is defined by using $\alpha$, $\beta$ and left multiplication. The next action is the left action of $H$ on $H^\otimes n$ defined by
\[
h \triangleright (g_1 \otimes_R \ldots \otimes_R g_n) = h^{(1)}g_1 \otimes_R \ldots \otimes_R h^{(n)}g_n.
\]

This is a well defined action because of property cp3) of the coproduct of a bialgebroid. One can show that $H^\otimes n$ is an $H - H^\otimes n$ bimodule. We use this bimodule structure in some proofs in this paper.

The following lemma will prove useful in verifying that certain examples satisfy condition (1) of definition 2.2.

**Lemma 2.2.** Condition (1) above is multiplicative. That is, if it is satisfied by $h$ and $g$ then it is satisfied by $hg$.

**Proof.** Let $h$ and $g$ satisfy (1). We have
\[
T((hg)^{(1)})^{(1)}(hg)^{(2)} \otimes_R T((hg)^{(1)})^{(2)} = \ldots
\]

This shows that $hg$ satisfies (1) as well. \(\square\)

The following proposition shows that for Hopf algebras our para-antipodes are simply twisted antipodes in the sense of Connes and Moscovici [1]. We would like to emphasize that no analogue of this result exists in the world of bialgebroids.

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Proposition 2.1. Let $H$ be a Hopf algebra over $k$ with antipode $S$. Then $T : H \to H$ is an antialgebra map and satisfies (1) if and only if $T = \delta \ast S$, where $\ast$ denotes the convolution multiplication and $\delta : H \to k$ is an algebra map.

Proof. Let $\delta$ be an algebra map. It is obvious that $T = \delta \ast S : H \to H$ is an antialgebra map. We have

$$T(h^{(1)})(1) \otimes T(h^{(1)})(2) = \delta(h^{(1)})S(h^{(2)})(1)h^{(3)} \otimes S(h^{(2)})(2)$$

$$= \delta(h^{(1)})S(h^{(3)})h^{(4)} \otimes S(h^{(2)}) = 1_H \otimes \delta(h^{(1)})S(h^{(2)}) = 1_H \otimes T(h).$$

On the other hand, let $T$ be an antialgebra map that satisfies condition (1). We define $\delta = \epsilon \circ T : H \to k$. It is evident that $\delta$ is an algebra map. We verify that $\delta \ast S = T$. Indeed, by our assumption we have

$$T(h^{(1)})(1)h^{(2)} \otimes T(h^{(1)})(2) = 1 \otimes T(h).$$

Applying $m \circ (S \otimes id_H)$, where $m$ denotes the multiplication map of $H$, to both sides of the above equation, we obtain

$$S(h^{(2)})S(T(h^{(1)})(1)T(h^{(1)})(2) = T(h),$$

or, equivalently, $\delta(h^{(1)})S(h^{(2)}) = T(h)$. \hfill $\square$

We give a few examples of para-Hopf algebroids.

Example 2.1. Let $H$ be a Hopf algebra over a field $k$, $\delta : H \to k$ an algebra map, and $\tilde{S}_\delta = \delta \ast S$ the $\delta$-twisted antipode defined by $\tilde{S}_\delta(h) = \delta(h^{(1)})S(h^{(2)})$. Assume that $\tilde{S}_\delta^2 = id$. Let $R$ be any algebra over $k$. We define a para-Hopf algebroid as follows. Let $H = R \otimes H \otimes R^{op}$, where $R^{op}$ denotes the opposite algebra of $R$. One can check that with the following structure $(H, R)$ is an extended Hopf algebra:

$$\alpha(a) = a \otimes 1 \otimes 1$$

$$\beta(a) = 1 \otimes 1 \otimes a$$

$$\Delta(a \otimes h \otimes b) = a \otimes h^{(1)} \otimes 1 \otimes_R 1 \otimes h^{(2)} \otimes b$$

$$\epsilon(a \otimes h \otimes b) = \epsilon(h)ab$$

$$T(a \otimes h \otimes b) = (b \otimes \tilde{S}_\delta(h) \otimes a).$$

Example 2.2. Let $G$ be a groupoid over a finite base. Equivalently, $G$ is a category with a finite set of objects, such that each morphism is invertible. The groupoid algebra of $G$, denoted by $H = kG$, is freely generated over $k$:
by morphisms $g \in \mathcal{G}$ with unit $1 = \sum_{X \in \text{Obj}(\mathcal{G})} \text{id}_X$. The product of two morphisms is equal to their composition if the latter is defined and 0 otherwise. We show that $k\mathcal{G}$ is an extended Hopf algebra over the base algebra $R = k\mathcal{S}$, where $\mathcal{S}$ is the subgroupoid of $\mathcal{G}$ whose objects are those of $\mathcal{G}$ and $\text{Mor}(X,Y) = \text{id}_X$ whenever $X = Y$ and $\emptyset$ otherwise. The relevant maps are defined as follows: $\alpha = \beta : R \to H$ is the natural embedding, and

$$\Delta(g) = g \otimes_R g, \quad \epsilon(g) = \text{id}_{\text{target}(g)}, \quad T(g) = g^{-1},$$

for any $g \in \mathcal{G}$. To show that it is an para-Hopf algebroid, we see that all conditions are obvious except possibly the condition (1). To check this, we compute $\tau^3_2(1_H \otimes_R g)$. We have $\tau_2(1_H \otimes_R g) = g \otimes_R 1_H$, and $\tau_2(g \otimes_R 1_H) = g^{-1} \otimes_R g^{-1}$. Hence $\tau^3_2(1_H \otimes_R g) = gg^{-1} \otimes_R g = 1_H \otimes_R g$.

**Example 2.3.** It is known that the algebraic quantum torus $A_\theta$ is not a Hopf algebra although it is a deformation of the Hopf algebra of Laurent polynomials in two variables. We show that $A_\theta$ is a para-Hopf algebroid. Recall that $A_\theta$ is the unital $\mathbb{C}$-algebra generated by two invertible elements $U$ and $V$ subject to the relation $UV = qVU$, where $q = e^{2\pi i \theta}$ and $\theta \in \mathbb{R}$. Let $R = \mathbb{C}[U, U^{-1}]$ be the algebra of Laurent polynomials embedded in $A_\theta$. Let $\alpha = \beta : R \to A_\theta$ be the natural embedding. Define the coproduct $\Delta : A_\theta \to A_\theta \otimes_R A_\theta$ by

$$\Delta(U^n V^m) = U^n V^m \otimes_R V^m$$

and the counit $\epsilon : A_\theta \to R$ by $\epsilon(U^n V^m) = U^n$. Define the extended antipode $T : A_\theta \to A_\theta$ by

$$T(U^n V^m) = q^{nm} U^n V^{-m}.$$  

We check that $A_\theta$ is a para-Hopf algebroid over $R$. Among the axioms we just check the validity of the condition (1). By Lemma 2.2, it is enough to check this condition just for the generators $U$ and $V$. We have

$$\tau^3_2(1 \otimes_R U) = \tau^3_2(U \otimes_R 1_H) = \tau_2(U \otimes_R 1_H) = (U \otimes_R 1_H) = 1_H \otimes_R U,$$

and

$$\tau^3_2(1 \otimes_R V) = \tau^3_2(V \otimes_R 1_H) = \tau(V^{-1} \otimes_R V^{-1}) = 1_H \otimes_R V.$$  

Next we show that the Connes-Moscovici algebra $\mathcal{H}_{FM}$, introduced in [4], is a para-Hopf algebroid. In fact it is already shown in [4] that $\mathcal{H}_{FM}$ is a bialgebroid and an antialgebra map $\tilde{S} : \mathcal{H}_{FM} \to \mathcal{H}_{FM}$ is defined such that $\tilde{S}^2 = \text{id}$, $\tilde{S} \beta = \alpha$, and $m_H(\tilde{S} \otimes \text{id}) \Delta = \beta \epsilon \tilde{S} : \mathcal{H}_{FM} \to \mathcal{H}_{FM}$. All we have to do then is to check that condition (1) is satisfied for $T = \tilde{S}$. First let us briefly recall the definition of $\mathcal{H}_{FM}$ from [4].

Let $M$ be an smooth $n$-dimensional manifold that admits a finite open cover by coordinate charts and let $FM$ denote the frame bundle of $M$. A
local diffeomorphism of $FM$ is called a prolonged diffeomorphism if it is the natural prolongation of a local diffeomorphism of $M$. Let $\mathcal{G}$ denote the set of germs of prolonged local diffeomorphisms of $FM$. Then $\mathcal{G}$ is a smooth étale groupoid with $FM$ as its set of objects. Let $\mathcal{A} = \mathcal{A}_{FM} := C^\infty(\mathcal{G})$ denote the smooth convolution algebra of $\mathcal{G}$. Its elements are linear combinations of elements of the form $fU^*_\phi$, with $f \in C^\infty(Dom\phi)$. Here $\phi$ denotes the prolongation of a local diffeomorphism $\phi$ of $M$ and the asterisk denotes the inverse. The product is defined by

$$f_1U^*_\phi \cdot f_2U^*_\phi = f_1(f_2 \circ \tilde{\phi}_1)U^*_\phi.$$  

Let $\mathcal{R} = \mathcal{R}_{FM} := C^\infty(FM)$ denote the algebra of smooth functions on the frame bundle $FM$. $\mathcal{R}$ acts on $\mathcal{A}$ by left and right multiplication operators: $\alpha(b)(fU^*_\phi) = b \cdot fU^*_\phi$ and $\beta(b)(fU^*_\phi) = fU^*_\phi \cdot b = b \circ \phi \cdot fU^*_\phi$, for all $b \in \mathcal{R}$. It is easily seen that $\alpha : \mathcal{R} \to \text{End}(\mathcal{A})$ is an algebra map, $\beta : \mathcal{R} \to \text{End}(\mathcal{A})$ is an antialgebra map, and the images of $\alpha$ and $\beta$ commute. We also have the action of vector fields on $FM$ on the algebra $\mathcal{A}$ by the formula $Z(fU^*_\phi) = Z(f)U^*_\phi$, where $Z$ is a vector field. Note that while vector fields act by derivations on functions on the frame bundle, their action on $\mathcal{A}$ does not satisfy the derivation property. In fact the failure of derivation property is responsible for the non-cocommutativity of the coproduct of $\mathcal{H}_{FM}$.

Let $\mathcal{H}_{FM} \subset \text{End}(\mathcal{A})$ be the subalgebra of the algebra of linear operators on $\mathcal{A}$ generated by the images of $\alpha$, $\beta$ and actions of vector fields as above. Its elements are called transverse differential operators on the étale groupoid $\mathcal{G}$.

It is shown in \cite{4} that $\mathcal{H}_{FM}$ is a free $\mathcal{R} \otimes \mathcal{R}$-module. In fact fixing a torsion free connection on $FM$, one obtains a Poincaré-Birkhoff-Witt-type basis for $\mathcal{H}_{FM}$ over $\mathcal{R} \otimes \mathcal{R}$ as follows. Let $\{Y^i\}$ denote the fundamental vertical vector fields corresponding to the standard basis of $gl(n, \mathbb{R})$ and $X_1, \ldots, X_n$ denote the standard horizontal vector fields corresponding to the standard basis of $\mathbb{R}^n$. These $n^2 + n$ vector fields form a basis for the tangent space of $FM$ at all points. It is shown in \cite{4} that the operators $Z_I \cdot \delta_\kappa$, where

$$Z_I = X_{i_1} \cdots X_{i_p} Y^{j_1}_{k_1} \cdots Y^{j_q}_{k_q}, \quad \delta_\kappa = \delta^{j_1}_{j_1 k_1 \ell_1} \cdots \delta^{j_q}_{j_q k_q \ell_q},$$

$$\delta^{j}_{j k \ell_1 \cdots \ell_p} = [X_\ell, \cdots [X_{\ell_1}, \delta^{j}_{j k}]],$$

form a basis for $\mathcal{H}_{FM}$ over $\mathcal{R} \otimes \mathcal{R}$. (See Proposition 3 and Lemma 2 in \cite{4} for precise range of multi-indices $I$ and $\kappa$ as well as the definition of the operators $\delta^{i}_{i k}$.)

A coproduct $\Delta$ and an antialgebra map $\tilde{S}$ with $\tilde{S}^2 = id$ are already defined in \cite{4} and all the identities of a bialgebroid as well as axioms $PH1, PH2$ in Definition 2.2 are verified. We check that condition (11) is satisfied as well.
Lemma 2.3. With $T = \tilde{S}$ condition (1) is satisfied for the Connes-Moscovici algebra $\mathcal{H}_{FM}$. 

Proof. Thanks to Lemma 2.2, we just need to check the condition (1) for the generators of $H = \mathcal{H}_{FM}$. Let $R = \mathcal{R}$. We just check the validity of this condition for generators $X_k$, the rest being straightforward to check. We know that 

$$\Delta(X_k) = X_k \otimes_R 1_H + 1_H \otimes_R X_k + \delta^i_j k \otimes_R Y^j_i.$$ 

So the right hand side of the condition (1) converts to 

$$\tilde{S}(X_k)^{(1)} \otimes_R \tilde{S}(X_k)^{(2)} + X_k \otimes_R 1_H + \tilde{S}(\delta^i_j k)^{(1)} Y^j_i \otimes_R \tilde{S}(\delta^i_j k).$$

Now by replacing $\tilde{S}(X_k)$ and $\tilde{S}(\delta^i_j k)$ in the above expression by their equals $-X_k + \delta^i_j k \otimes_R Y^j_i$ and $-\delta^i_j k$, respectively, and using the fact that $\Delta$ is multiplicative, we find that the above expression is equal to 

$$X_k \otimes_R 1_H - 1_H \otimes_R X_k - \delta^i_j k \otimes_R Y^j_i + \delta^i_j k \otimes_R 1_H + \delta^i_j k \otimes_R Y^j_i$$

$$+ Y^j_i \otimes_R \delta^i_j k + 1_H \otimes_R \delta^i_j k Y^j_i + X_k \otimes_R 1_H - \delta^i_j k Y^j_i \otimes_R 1_H - Y^j_i \otimes_R \delta^i_j k.$$ 

After cancelling the identical terms with opposite signs we obtain $-1_H \otimes_R X_k + 1_H \otimes_R \delta^i_j k Y^j_i$, which is $1_H \otimes_R \tilde{S}(X_k)$. \hfill \Box

Remark 2. The fact that the Connes-Moscovici algebra $\mathcal{H}_{FM}$ is a para-Hopf algebra can also be directly derived by combining Theorem 3.1 with the fact, proved in [4], that $\mathcal{H}_{FM}$ affords a cocyclic module.

If $M = \mathbb{R}^n$ is the flat Euclidean space, then $\mathcal{H}_{FM} = \mathcal{R} \otimes \mathcal{H}_n \otimes \mathcal{R}$, where $\mathcal{H}_n$ is the Connes-Moscovici Hopf algebra in dimension $n$. The para-Hopf algebroid structure on $\mathcal{H}_{FM}$ is induced from the Hopf algebra structure on $\mathcal{H}_n$ as in Example 2.1

Example 2.4. Let $\mathcal{H}$ be a Hopf algebra and $\mathcal{P}$ be a left $\mathcal{H}$-module algebra. We generalize Example 2.1 by turning the double crossed product algebra $H := \mathcal{P} \rtimes \mathcal{H} \rtimes \mathcal{P}^{op}$, introduced in [5], into a Para-Hopf algebroid. To this end let us first recall from [5] its algebra structure. Equipped with the following multiplication and $1 \times 1 \times 1$ as its unit, $\mathcal{P} \otimes \mathcal{H} \otimes \mathcal{P}$ is a unital associative algebra:

$$(P_1 \times h_1 \times Q_1) \cdot (P_2 \times h_2 \times Q_2) = P_1 h_1^{(1)} (P_2) \times h_1^{(2)} h_2 \times h_1^{(3)} (Q_2) Q_1.$$ 

The source and target maps are defined by $\alpha : \mathcal{P} \to H$ defined by $\alpha(P) = P \times 1 \times 1$ and $\beta : \mathcal{P} \to H$ defined by $\beta(Q) = 1 \times 1 \times Q$. The comultiplication $\Delta : H \to H \otimes H$ is defined by 

$$\Delta(P \times h \times Q) = P \times h^{(1)} \times 1 \otimes \mathcal{P} 1 \times h^{(2)} \times Q.$$ 

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The above data together with the counit $\epsilon : \mathcal{P} \to H$ defined by $\epsilon(P \times h \times Q) = P \epsilon(h)Q$ turn $H$ into a bialgebroid. Now if $\mathcal{H}$ admits a character such that $\widetilde{S}_\delta^2 = id_\mathcal{H}$ then we define a para-antipode for $H$ and show that $H$ is a para-Hopf algebroid over $\mathcal{P}$. Let $T : H \to H$ be defined by

$$T(P \times h \times Q) = S(h^{(3)})(Q) \times S(h^{(2)}) \times \widetilde{S}_\delta(h^{(1)})(P)$$

Evidently $T^2 = id$, and $T$ is an antialgebra map. The other identities are straightforward to check and we leave them to the reader except the crucial identity $T$. Let $h = P \times h \times Q \in H$. We have

$$T(h^{(1)}) \otimes_P T(h^{(1)}) = (1 \times S(h^{(3)}) \times 1) \cdot (1 \times h^{(4)} \times Q) \otimes_P$$

$$1 \times S(h^{(2)}) \times \widetilde{S}_\delta(h^{(1)})(P)$$

$$= 1 \times 1 \times S(h^{(3)})(Q) \otimes_P 1 \times S(h^{(2)}) \times \widetilde{S}_\delta(h^{(1)})(P)$$

$$= 1 \times 1 \times 1 \otimes_P S(h^{(3)})(Q) \times S(h^{(2)}) \times \widetilde{S}_\delta(h^{(1)})(P)$$

$$= 1_H \otimes_P T(h).$$

3 Cyclic cohomology of para-Hopf algebroids

In [4], Connes and Moscovici used the natural action of $\mathcal{H}_{FM}$ on the algebra $\mathcal{A}_{FM}$ and an invariant faithful trace $Tr$ on $\mathcal{A}_{FM}$ to define a cocyclic module for $\mathcal{H}_{FM}$. More precisely, they showed that the maps

$$\gamma_{Tr} : \mathcal{H}_{FM}^\otimes(n+1) \longrightarrow Hom_\mathbb{C}(\mathcal{A}_{FM}, \mathbb{C}),$$

are linear isomorphisms for each $n \geq 0$ and their images form a cocyclic submodule of the cocyclic module of the algebra $\mathcal{A}_{FM}$. This cocyclic submodule was then transferred, via $\gamma_{Tr}$, to a cocyclic module based on $\mathcal{H}_{FM}$.

In this section our aim is to show that the formulas discovered by Connes and Moscovici define a cocyclic module for any para-Hopf algebroid and provide several computations.

Let $(H, R)$ be a bialgebroid. It is easily checked that the following module $H^\circ_\natural$ is a cosimplicial module. We put

$$H^0_\natural = R, \text{ and } H^n_\natural = H \otimes_R H \otimes_R \cdots \otimes_R H \quad (n\text{-fold tensor product}).$$
The cofaces $\delta_i$ and codegeneracies $\sigma_i$ are defined by:

$$
\delta_0(a) = \alpha(a), \quad \delta_1(a) = \beta(a) \quad \text{for all } a \in R = H^0
$$

$$
\delta_i(h_1 \otimes_R \cdots \otimes_R h_n) = 1_H \otimes_R h_1 \otimes_R \cdots \otimes_R h_n
$$

$$
\delta_i(h_1 \otimes_R \cdots \otimes_R h_n) = h_1 \otimes_R \cdots \otimes_R \Delta(h_i) \otimes_R \cdots \otimes_R h_n \quad \text{for } 1 \leq i \leq n
$$

$$
\delta_{n+1}(h_1 \otimes_R \cdots \otimes_R h_n) = h_1 \otimes_R \cdots \otimes_R h_m \otimes_R 1_H
$$

$$
\sigma_i(h_1 \otimes_R \cdots \otimes_R h_n) = h_1 \otimes_R \cdots \otimes_R \epsilon(h_{i+1}) \otimes_R \cdots \otimes_R h_n \quad \text{for } 0 \leq i \leq n.
$$

Let $T : H \to H$ be an antialgebra map such that axioms $PH1$ and $PH2$ of Definition 2.2 are satisfied. That is $T \beta = \alpha$ and $m_H(T \otimes \text{id}) \Delta = \beta \epsilon T : H \to H$. Then one can check that the cyclic operator $\tau$ of Connes and Moscovici defined by

$$
\tau_n(h_1 \otimes_R \cdots \otimes_R h_n) = \Delta^{n-1} T(h_1) \cdot (h_2 \otimes_R \cdots \otimes_R h_n \otimes 1_H),
$$

is well defined. Note that the right action of $H \otimes^n$ on $H \otimes_R^n$ by right multiplication is used. The question of when $H^\natural$, endowed with the above cyclic operator, is a cocyclic module is completely answered by the following theorem.

**Theorem 3.1.** Let $(H, R)$ be a bialgebroid and $T : H \to H$ be as above. Then $H^\natural$ is a cocyclic module if and only if $(H, R, T)$ is a para-Hopf algebroid, that is $T^2 = \text{id}_H$, and for all $h \in H$,

$$
T(h^{(1)})^{(1)} h^{(2)} \otimes_R T(h^{(1)})^{(2)} = 1 \otimes_R T(h).
$$

**Proof.** We should verify the following identities:

- $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$
- $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i < j$
- $\sigma_j \delta_i = \begin{cases} 
\delta_i \sigma_{j-1} & i < j \\
\text{id} & i = j \text{ or } i = j + 1 \\
\delta_{i-1} \sigma_j & i > j + 1
\end{cases}$
- $\tau_{n+1} \delta_i = \delta_{i-1} \tau_n$ for $1 \leq i \leq n$
- $\tau_{n-1} \sigma_i = \sigma_i \tau_n$ for $1 \leq i \leq n$
- $\tau_{n+1} = \text{id}_n$

The cosimplicial relations are not hard to prove. We just verify the cyclic relations $\tau^{n+1}_{n+1} = \text{id}_n$ and $\tau \delta_1 = \delta_0 \tau$, and leave the others to the reader.

This is evident for $n = 1$ because $\tau_1 = T$. Let $n \geq 2$, and define

$$
\Phi_n : H \otimes_R H \longrightarrow H \otimes_H^n,
$$

$$
h \otimes_R g \mapsto \tau^n_1(h \otimes_R g \otimes_R 1 \otimes_R \cdots \otimes_R 1).
$$
We have
\[ \tau^2_n(h_1 \otimes_R \cdots \otimes_R h_n) = \Phi_n(h_1 \otimes_R h_2) \cdot (h_3 \otimes \cdots \otimes h_n \otimes 1_H \otimes 1_H). \]

On the other hand we can compute \( \Phi_n(h_1 \otimes_R h_2) \) in terms of the diagonal action of \( H \) on \( H^{\otimes n} \), i.e.,
\[ \Phi_n(h_1 \otimes_R h_2) = T(h_2) \triangleright \Phi_n(h_1 \otimes_R 1_H). \]

Using condition (1), one has
\[ \Phi_n(h_1 \otimes 1_H) = 1_H \otimes_R 1_H \ldots \otimes_R 1_H \otimes_R h_1. \]

We can simplify \( \tau^2_n(h_1 \otimes_R \ldots \otimes_R h_n) \) as follows:
\[ \tau^2_n(h_1 \otimes_R \ldots \otimes_R h_n) = T(h_2)^{(1)} h_3 \otimes_R \ldots \otimes_R T(h_2)^{(n-2)} h_n \otimes_R T(h_2)^{(n-1)} \otimes_R T(h_2)^{(n)} h_1. \]

By repeating the same argument as above we obtain:
\[ \tau^n_n(h_1 \otimes_R \ldots \otimes_R h_n) = T(h_n)^{(1)} \otimes_R T(h_n)^{(2)} h_1 \otimes_R \ldots \otimes_R h_n \otimes_R T(h_n)^{(n)} h_{n-1}. \]

Applying \( \tau_n \) to both side, we obtain:
\[ \tau^n_{n+1}(h_1 \otimes_R \ldots \otimes_R h_n) = \Phi(h_n \otimes 1_H) \cdot (h_1 \otimes \ldots \otimes h_{n-1} \otimes 1_H). \]

Now since \( \Phi(h_n \otimes 1_H) = 1_H \otimes_R 1_H \ldots \otimes_R 1_H \otimes_R h_n \), we have
\[ \tau^n_{n+1} = id_n. \]

Next, we check the identity between \( \tau_n \) and \( \delta_1 \), i.e. \( \tau_{n+1} \delta_1 = \delta_0 \tau_n \).
\[ \tau_{n+1} \delta_1(h_1 \otimes_R h_2 \ldots \otimes_R h_n) = \tau_{n+1}(h_1^{(1)} \otimes_R h_2^{(2)} \otimes_R h_2 \otimes_R \ldots \otimes_R h_n) \]
\[ = T(h_1^{(1)}) h_2^{(2)} \otimes_R T(h_1^{(1)}) h_2 \otimes_R \ldots \otimes_R T(h_1^{(1)}) h_n \otimes_R T(h_1^{(2)} \otimes_R \ldots \otimes_R T(h_1^{(1)}) h_{n-1} \otimes 1_H) \]
\[ = (T(h_1^{(1)}) h_2^{(2)} \otimes_R T(h_1^{(1)})) \ldots \otimes_R T(h_1^{(1)}) h_{n-1} \otimes 1_H \]
\[ \ldots \otimes_R T(h_1^{(1)}) h_n \otimes 1_H \]
\[ = ((id_H \otimes_R \Delta^{(n-2)})(T(h_1^{(1)}) h_2^{(2)} \otimes_R T(h_1^{(2)})) \ldots \otimes_R 1_H \otimes_R h_n \otimes 1_H) \]
\[ = (id_H \otimes_R \Delta^{(n-2)})(1_H \otimes_R T(h_1)) \otimes_R h_2 \ldots \otimes_R h_{n-1} \otimes 1_H \]
\[ = 1_H \otimes_R T(h_1^{(1)}) h_2 \otimes_R \ldots \otimes_R T(h_2)^{(n-1)} h_n \otimes_R T(h_1^{(2)} h_2) \]
\[ = \delta_0 \tau_n(h_1 \otimes_R h_2 \ldots \otimes_R h_n). \]
Finally we check the relation between $\tau_n$ and $\sigma_i$. Using Lemma ?? one has:

$$\tau_{n-1}\sigma_i(h_1 \otimes_R \cdots \otimes_R h_n) = \tau_{n-1}(h_1 \otimes_R \cdots \otimes_R \beta(\epsilon(h_i))h_{i-1} \otimes_R \cdots \otimes_R h_n)$$

$$= S(h_1)^{(1)}h_2 \otimes_R \cdots \otimes_R S(h_1)^{(i-2)}\beta(\epsilon(h_i))h_{i-1} \otimes_R \cdots \otimes_R S(h_1)^{(n-2)}h_n$$

$$= S(h_1)^{(1)}h_2 \otimes_R \cdots \otimes_R \beta((S(h_1)^{(i-2)}\beta(\epsilon(h_i))))(S(h_1)^{(i-2)}\beta(\epsilon(h_i)))^{(1)}h_{i-1} \otimes_R \cdots \otimes_R h_n$$

The converse is evident because

$$T(h^{(1)})^{(1)}h^{(2)} \otimes_R T(h^{(1)})^{(2)} = \tau_2^3(1_H \otimes_R T(h))$$

Let $R$ be a $k$-algebra and $\mathcal{H}$ a $k$-Hopf algebra endowed with a twisted antipode $S^\delta$ such that $S^\delta_2 = id_\mathcal{H}$. We have a para-Hopf algebroid structure on $R \otimes T \otimes R^{op}$ as in Example 2.1. In the next proposition we recall the computation of the cyclic cohomology of $R \otimes \mathcal{H} \otimes R^{op}$ as a para-Hopf algebroid in terms of Hopf-cyclic cohomology of $\mathcal{H}$.

**Proposition 3.1.** (??) Let $\mathcal{H}$ be a Hopf algebra as above. Then

$$HC^*(R \otimes \mathcal{H} \otimes R^{op}) = HC^*_{(\delta^1)}(\mathcal{H}).$$

**Definition 3.1.** (Haar system for bialgebroids). Let $(H, R)$ be a bialgebroid. Let $\tau : H \rightarrow R$ be a right $R$-module map. We call $\tau$ a left Haar system for $H$ if for all $h \in H$,

$$\alpha(\tau(h^{(1)}))h^{(2)} = \alpha(\tau(h))1_H,$$

and $\alpha\tau = \beta\tau$. We call $\tau$ a normal left Haar system if $\tau(1_H) = 1_R$.

We give a few examples of Haar systems. Let $H$ be the extended Hopf algebra of a groupoid with finite base (Example 2.2). Then it is easy to see that $\tau : H \rightarrow R$ defined by $\tau(id_x) = id_x$ for all $x \in Obj(\mathcal{G})$ and $\tau(\gamma) = 0$ if $\gamma$ is not an identity morphism, is a normal Haar system for $H$. For a second example, one can directly check that the map $\tau : A_\theta \rightarrow C[U, U^{-1}]$ defined by

$$\tau(U^nV^m) = \delta_{m,0}U^n$$

is a normal Haar system for the noncommutative torus $A_\theta$. 

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Proposition 3.2. Let $H$ be a para-Hopf algebroid that admits a normal left Haar system. Then $HC^{2i+1}(H) = 0$ and $HC^{2i}(H) = \ker(\alpha - \beta)$ for all $i \geq 0$.

Proof. Let $\eta : H^\otimes_R n \longrightarrow H^\otimes_R (n-1)$ be the map

$$\eta(h_1 \otimes_R \cdots \otimes_R h_n) = \alpha(\tau(h_1)h_2 \otimes_R \cdots \otimes_R h_n).$$

It is easy to check that $\eta$ is a contracting homotopy for the Hochschild complex of $H$, and hence $HH^n(H) = 0$ for $n > 0$ and $HH^0(H) = \ker(\alpha - \beta)$. The rest follows from Connes’s long exact sequence relating Hochschild and cyclic cohomology.

Corollary 3.1. Let $H$ be the para-Hopf algebroid of Example 2.2. Then $HC^{2i+1}(H) = 0$ and $HC^{2i}(H) = R$ for all $i \geq 0$.

Corollary 3.2. We have $HC^{2i+1}(A_\theta) = 0$ and $HC^{2i}(A_\theta) = C[U, U^{-1}]$ for all $i \geq 0$.

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