ON KATO’S SMOOTHING EFFECT FOR A FRACTIONAL VERSION OF
THE ZAKHAROV-KUZNETSOV EQUATION

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Abstract. In this work we study some regularity properties associated to the initial value
problem (IVP)
\begin{equation}
\begin{cases}
\partial_t u - \partial_{x_1} (-\Delta)^{\alpha/2} u + u \partial_{x_1} u = 0, & 0 < \alpha \leq 2, \\
u(x,0) = u_0(x), & x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \ n \geq 2, \quad t \in \mathbb{R},
\end{cases}
\end{equation}
where \((-\Delta)^{\alpha/2}\) denotes the \(n\)-dimensional fractional Laplacian.

We show that solutions to the IVP (0.1) with initial data in a suitable Sobolev space
exhibit a local smoothing effect in the spatial variable of \(\alpha \) derivatives, almost everywhere in
time. One of the main difficulties that emerge when trying to obtain this regularizing effect
underlies that the operator in consideration is non-local, and the property we are trying to
describe is local, so new ideas are required. Nevertheless, to avoid these problems, we use
a perturbation argument replacing \((-\Delta)^{\alpha/2}\) by \((I - \Delta)^{\alpha/2}\), that through the use of pseudo-
differential calculus allows us to show that solutions become locally smoother by \(\frac{\alpha}{2}\) of a
derivative in all spatial directions.

As a by-product, we use this particular smoothing effect to show that the extra regularity
of the initial data on some distinguished subsets of the Euclidean space is propagated by
the flow solution with infinity speed.

1. Introduction

In this work we study the Fractional Zakharov-Kuznetsov (FZK) equation
\begin{equation}
\begin{cases}
\partial_t u - \partial_{x_1} (-\Delta)^{\frac{\alpha}{2}} u + u \partial_{x_1} u = 0, & 0 < \alpha < 2, \\
u(x,0) = u_0(x), & x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \ t \in \mathbb{R},
\end{cases}
\end{equation}
where \(u = u(x,t)\) is a real valued function and \((-\Delta)^{\frac{\alpha}{2}}\) stands for the fractional Laplacian
whose description in the Fourier space is given by
\[\mathcal{F}\left((-\Delta)^{\frac{\alpha}{2}} f\right)(\xi) := |2\pi \xi|^\alpha \mathcal{F}(f)(\xi) \quad f \in \mathcal{S}(\mathbb{R}^n).\]

The FZK equation formally satisfies the following conservation laws, at least for smooth solutions
\[\mathcal{I}(t) = \int_{\mathbb{R}^n} u(x,t) \, dx = \mathcal{I}(0),\]
\[\mathcal{M}(t) = \int_{\mathbb{R}^n} u^2(x,t) \, dx = \mathcal{M}(0),\]
and the Hamiltonian
\[\mathcal{H}(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{\alpha}{2}} u(x,t)\right)^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^n} u^3(x,t) \, dx = \mathcal{H}(0).\]
In the case \(\alpha = 1\) the equation (1.1) can be formally rewritten as
\begin{equation}
\partial_t u - \partial_1 \Delta u + u \partial_{x_1} u = 0,
\end{equation}

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where $\mathcal{R}_j$ denotes the Riesz transform in the $x_1-$variable, that is,

$$\mathcal{R}_1(f)(x) := \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi \frac{x_1}{n+1}} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_1 - y_1}{|x-y|^{n+1}} f(y) \, dy,$$

whenever $f$ belongs to a suitable class of functions.

The model (1.2) seems to be firstly deduced by Shrira [52] to describe the bi-dimensional long-wave perturbations in a boundary-layer type shear flow. More precisely, (1.2) represents the equation of the longitudinal velocity of fluid under certain conditions (see Shrira [52] for a more detailed description). We shall also refer the works [3], [9],[46] and [47] where several variants (either be extensions or reductions) of the model (1.2) has been studied.

We shall also point out that the equation in (1.2) represents a higher dimensional extension of the famous Benjamin-Ono equation

$$\partial_t u - \mathcal{H} \partial_x^2 u + u \partial_x u = 0,$$

where $\mathcal{H}$ denotes the Hilbert transform.

The equation (1.2) has called the attention in recent years and several results concerning local well-posedness have been established. In this sense, we can mention the work of Hickman, Linares, Riaño, Rogers, Wright [16], who establish local well-posedness of the IVP associated to (1.2) in the Sobolev space $H^s(\mathbb{R}^n)$, where $s > \frac{n}{2}$ in the bi-dimensional case and $s > \frac{n}{2} + \frac{1}{2}$ whenever $n \geq 3$. Also, Schippa [50] improves the work in [16], by showing that (1.1) is locally well-posed in $H^s(\mathbb{R}^n)$ for $s > \frac{n+3}{2} - \alpha$ and $1 \leq \alpha < 2$. See also Riaño [45] where some results of local well-posedness in weighted Sobolev spaces are established, as well as, some unique continuation principles for equation (1.2).

The case $\alpha = 2$ in (1.1) corresponds to the Zakharov-Kuznetsov (ZK) equation

(1.3)

$$\partial_t u + \partial_{x_1} \Delta u + u \partial_{x_1} u = 0.$$

Originally (1.3) was derived by Zakharov, Kuznetsov [58] in the three-dimensional case as a model to describe the propagation of ionic-acoustic waves in magnetized plasma. More recently, Lannes, Linares, and Saut [21] justify that the ZK equation can be formally deduced as a long wave small-amplitude limit of the Euler-Poisson system in the “cold plasma ” approximation. From the physical point of view the ZK model is not only interesting in the 3-dimensional case but also in the 2-dimensional sense. e.g., it describes under certain conditions the amplitude of long waves on the free surface of a thin film in a specific fluid with particular parameters of viscosity (see Melkonian, Maslowe [36] for more details)

The FZK equation has been little studied except in the distinguish cases we pointed out above, that is, $\alpha = 1$ and $\alpha = 2$. Nevertheless, results of local and global well-posedness of the FZK equation in the range $\alpha \in (0, 2) - \{1, 2\}$ are scarce. To our knowledge, we can only mention the work of Schippa [50] where the well-posedness problem in $H^s(\mathbb{R}^n)$ is addressed. In [50] is proved that the IVP (1.1) is locally well-posed in $H^s(\mathbb{R}^n)$ for $s > \frac{n+3}{2} - \alpha$ whenever $n > 2$ and $\alpha \in [1, 2)$. See also [50] for additional results of local well-posedness on $H^s(\mathbb{T}^n)$.

In contrast, the ZK equation has been the object of intense study in the recent years, this has lead to a enormous improvements concerning the local and global well posedness. In this sense we could mention the work of Faminskii [10] who shows local well posedness in $H^m(\mathbb{R}^2), m \in \mathbb{N}$. Also in the 2d case Linares and Pastor [31] prove local well-posedness in $H^s(\mathbb{R}^2)$ for $s > \frac{3}{4}$. Later, Molinet and Pilod [42] and Grünrock and Herr [14] extend the local well-posedness to $H^s(\mathbb{R}^2), s > \frac{1}{4}$, by using quite similar arguments based on the Fourier restriction method.

Regarding the 3d case, Molinet, Pilod [42] and also Ribaudo and Vento [49] prove local well posedness in $H^s(\mathbb{R}^3), s > 1$. In this direction, the most recent work of Kinoshita [27]
and Herr, Kinoshita [15] establish well-posedness in the best possible Sobolev range where the Picard iteration scheme can be applied, that is, \( H^s(\mathbb{R}^2) \), \( s > -\frac{1}{4} \) and \( H^s(\mathbb{R}^n) \), \( s > \frac{n-2}{2} \) when \( n > 2 \).

In this work, we do not pursue to improve results concerning local or global well-posedness, but on the contrary we need to establish local well-posedness on certain Sobolev space that allow us to describe particular properties of the solutions. The energy method proved by Bona and Smith [5] yields local well-posedness in \( H^s(\mathbb{R}^n) \) for \( s > \frac{n+2}{4} \). More precisely, the following result holds:

**Theorem 1.4.** Let \( s > s_0 \) where \( s_0 := \frac{n+2}{2} \). Then, for any \( u_0 \in H^s(\mathbb{R}^n) \), there exist \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution \( u \) to the IVP (1.1) such that

\[
(1.5) \quad u \in C\left([0, T]: H^{s_0}(\mathbb{R}^n)\right) \quad \text{and} \quad \nabla u \in L^1\left((0, T): L^\infty(\mathbb{R}^n)\right).
\]

Moreover, the flow map \( u_0 \mapsto u \) defines a continuous application from \( H^{s_0}(\mathbb{R}^n) \) into \( H^{s_0}(\mathbb{R}^n) \).

The energy method in this case does not consider the effects of the dispersion and it is mainly based on *a priori estimates* for smooth solutions, that combined with Kato-Ponce commutator estimate (see Theorem 5.3) give

\[
\|u\|_{L^\infty_t H^s_x} \lesssim \|u_0\|_{H^s}^\alpha \|\nabla u\|_{L^1_t L^\infty_x}^{1-\alpha}.
\]

The condition on the gradient in (1.5) as the reader can see later is fundamental in the solution of the problems we address in this work.

A quite remarkable property that dispersive equations satisfies is *Kato’s smoothing effect*, this is a property found by Kato [23] in the context of the *Korteweg-de Vries-(KdV)* equation

\[
(1.6) \quad \partial_t u + c^3_x u + u \partial_x u = 0.
\]

In [23] Kato proves that solutions of the KdV equation (1.6) becomes more regular locally by one derivative with respect to the initial data, that is, if \( u \) is a solution of (1.6) on a suitable Sobolev space then: for any \( r > 0 \),

\[
(1.7) \quad \int_0^T \int_{-r}^r (\partial_x u(x, t))^2 \, dx \, dt < c(T, r)\|u_0\|_{L^2_x}^2.
\]

Independently, Kruzhkov, Faminskiĭ [29] obtained a quite similar result to (1.7).

As was shown later, and almost simultaneously by Constantine, Saut [8], Sjölin [53] and Vega [57], the local smoothing is an intrinsic property of linear dispersive equations (see Chapter 4 and the references therein).

A question that arise naturally is to determine whether the solutions \( u \) of IVP (1.1) have a local smoothing effect similar to that one satisfied by the solutions of the KdV equation (1.7). Certainly, this is not an easy question to answer in the full range \( \alpha \in (0, 2) \). In the case \( \alpha = 2 \), the operator that provides the dispersion in the linear part of the equation (1.1) is a local operator and it is possible to obtain by performing energy estimates that solutions of the ZK equation in a suitable Sobolev space satisfy an inequality similar in spirit to (1.7) with \( \nabla \) instead of \( \partial_x \), but on another class of subsets of the euclidean space.

However, when we turn our attention to the case \( \alpha \in (0, 2) \), the situation is not so easy to address, since the operator \((-\Delta)^{\alpha/2}\) is fully non-local. Nevertheless, as is indicated in the work of Constantine, Saut [8] we expect a local gain of \( \alpha/2 \) of a derivative either the operator be local or not.

One of the main goals of this work is to prove *à la Kato* that solutions of the IVP (1.1) gain locally \( \alpha/2 \) of a spatial derivative. Certainly this problem has been addressed previously in the one-dimensional case *e.g.* Ponce [48] and Ginibre, Velo [12, 13] for solutions of the
Benjamin-Ono equation. We shall also mention the work of Kenig, Ponce and Vega [26] where is proved that solutions of the IVP
\begin{align}
\label{1.8}
\begin{cases}
i\partial_t u + P(D)u = 0, \\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},
\end{cases}
\end{align}
where
\[ P(D)f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi}P(\xi)\mathcal{F}(f)(\xi)\,d\xi, \]
with \( P \) satisfying certain conditions, enjoy of local smoothing effect. Also, in [26] is showed that solutions of the IVP (1.8) satisfy a global smoothing effect (see sections 3 and 4 in[26]). Their proofs are mainly based on estimates of oscillatory integrals, as well as, the use of the Fourier restriction method.

The results in [12] and its extension in [13] are quite versatile and allow us to obtain the smoothing effect in the desired range \( \alpha \in (0,2) \). The main idea behind these arguments relies on obtaining a pointwise decomposition for the commutator
\begin{align}
\label{1.9}
\left[ (-\Delta)^{\frac{\alpha}{2}} \partial_x; \varphi \right],
\end{align}
where \( \varphi \) is a real valued smooth function with certain decay at infinity.

Heuristically, the idea is to decouple (1.9) in lower-order pieces plus some non-localized error term easy to handle. However, in higher dimensions there is not known pointwise decomposition formula for the commutator
\begin{align}
\label{1.10}
\left[ (-\Delta)^{\frac{\alpha}{2}} \partial_x; \varphi \right],
\end{align}
similar to that one in [12], so that new ideas are required to obtain the desired smoothing.

After obtaining a decomposition of commutator (1.10) in pieces of lower order, we replace the operator \((-\Delta)^{\frac{\alpha}{2}}\) by \((I-\Delta)^{\frac{\alpha}{2}}\), to our proposes the main difference between both operators relies on the fact that \((I-\Delta)^{\frac{\alpha}{2}}\) is a pseudo-differential operator, instead of \((-\Delta)^{\frac{\alpha}{2}}\) that is not for \( \alpha \) in the indicated range.

Few years ago, Bourgain and Li [6] established the pointwise formula
\begin{align}
\label{1.11}
(I-\Delta)^{\frac{\alpha}{2}} = (-\Delta)^{\frac{\alpha}{2}} + \mathcal{K}_\alpha, \quad 0 < \alpha \leq 2,
\end{align}
where \( \mathcal{K}_\alpha \) an integral operator that maps \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for all \( p \in [1, \infty] \).

The expression above allow us to obtain after replacing in (1.10)
\begin{align}
\label{1.12}
\left[ (-\Delta)^{\frac{\alpha}{2}} \partial_x; \varphi \right] = \left[ (I-\Delta)^{\frac{\alpha}{2}} \partial_x; \varphi \right] + [\mathcal{K}_\alpha \partial_x; \varphi],
\end{align}
At this point, the situation is easier to handle since for the first expression on the r.h.s we use Pseudo-differential calculus to decouple pointwise the commutator expression in terms of lower order in the spirit of Ginibre and Velo decomposition [12]. Even when the situation is more manageable in comparison to (1.10), the decomposition produces a considerable amount of error terms that are not easy to handle due to the interactions between terms of higher regularity vs lower regularity.

If somehow we had to summarize in a few words the arguments used to estimate the first term in (1.12) we could refer to a maxim Roman emperor’s Julius Caesar “divide et vinces”.

The situation is quite different for the term involving \( \mathcal{K}_\alpha \), this one is by far the hardest to deal with. It contains a sum that requires to know information about the behavior of Bessel potentials at origin and infinity (see Appendix A) and several cases depending on the dimension have to be examined. In this process the Gamma function and its properties are fundamentals to guarantee control in certain Sobolev norm.

After decomposing the first term on the r.h.s of (1.12) we clearly obtain that solutions of the IVP (1.1) gain locally in space the expected \( \frac{\alpha}{2} \) of a derivative in the full range \( \alpha \in (0,2) \) without restrictions, this constitutes our first main result whose statement we show below.
Lemma 1.1. Let $u \in C((0, T) : H^s(\mathbb{R}^n))$, $s > \frac{n}{2} + 1$, be a solution of (1.1) with $0 < \alpha < 2$ and $n \geq 2$.

If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^\infty(\mathbb{R}^n)$ function satisfying:

(i) There exist a non-decreasing smooth function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ such that

$$\varphi(x) = \phi(v \cdot x + \delta) \quad x \in \mathbb{R}^n,$$

for some $\delta \in \mathbb{R}$. The vector $v$ is taken in such a way that it satisfies one and only one of the following conditions:

Case 1: $v_1 > 0$ and $v_2, v_3, \ldots, v_n = 0$.

Case 2: $v_1 > 0$, $(v_2, v_3, \ldots, v_n) \neq 0$, verify the inequality

$$0 < \sqrt{v_2^2 + v_3^2 + \cdots + v_n^2} < \min \left\{ \frac{2v_1}{\sqrt[n]{\alpha(n-1)}}, \frac{v_1(1+\alpha)}{\alpha\sqrt[n]{n-1}} \right\},$$

with

$$0 < \epsilon < \frac{v_1}{|v_1|\sqrt{n-1}} - \frac{\alpha\sqrt{n-1}|\varphi|}{4v_1} c^{-2},$$

where

$$C := \inf_{f \in L^2(\mathbb{R}^n), f \neq 0} \frac{\|J^{-1} \partial_x f\|_{L^2}^2}{\|f\|_{L^2}^2}, \quad j = 2, 3, \ldots, n,$$

and

$$|\varphi| := \sqrt{v_2^2 + v_3^2 + \cdots + v_n^2}.$$

(ii) The function $\phi$ satisfies:

$$\phi' \equiv 1 \quad \text{on} \quad [0, 1].$$

(iii) There exist a positive constant $c$ such that

$$\sup_{0 \leq j \leq 4} \sup_{x \in \mathbb{R}} \left| (\partial_x^j \phi)(x) \right| \leq c.$$

(iv) For all $x \in \mathbb{R}$,

$$\phi'(x) \geq 0.$$

The function $\phi^{1/2}$ satisfies

$$\sup_{x \in \mathbb{R}} \left| \partial_x^j \left( \sqrt{\phi(x)} \right) \right| \leq c \quad \text{for} \quad j = 1, 2.$$

Then

$$\int_0^T \int_{\mathbb{R}^n} \left( f^{s+\frac{2}{r}} u(x, t) \right)^2 \partial_{x_1} \phi(x) \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \left( \partial_{x_1} f^{s+\frac{2}{r}} u(x, t) \right)^2 \partial_{x_1} \phi(x) \, dx \, dt$$

$$\lesssim_{n, \alpha} \left( 1 + T + \|\nabla u\|_{L^\infty_t} + T \|u\|_{L^\infty_t H^\frac{\alpha}{r} \dot{H}^{\frac{n}{2}}_x} \right)^{1/2} \|u\|_{L^\infty_t H^\frac{\alpha}{r} \dot{H}^{\frac{n}{2}}_x},$$

whenever $r > \frac{n}{4}$.

The reader should keep in mind that the results Lemma (1.1) show a strong dependence on the variable $x_1$, this can be observed in the condition on $v_1$, it never can be null unlike the other coordinates that can be null but not all of them simultaneously as it is pointed out in the case 2 in Lemma 1.1.

After translating properly $\varphi$, it is possible to describe some regions where the smoothing effect is valid. We show that depending on the dimension and the possible sign of the coordinates of vector $v$ in (1.13) the geometry of the regions might change. See figures 1-3.
A question that comes out naturally from Lemma 1.1 is to determine whether a homogeneous version of smoothing also holds. Indeed, it holds and its proof is a consequence of combining Lemma 1.1 and formula (1.11).

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Regions where the smoothing effect occurs. The region on
the left corresponds to \( \nu_1 > 0, \nu_2 < 0. \)
The region on the right corresponds to the case \( \nu_1 > 0, \nu_2 > 0. \)

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Particular region where the smoothing effect is valid:
\( Q_{\{\nu, \tau_1, \nu\}} \) where \( 0 < \tau_1 < \tau_2 \) and \( \nu_1 > 0, \nu_2 = 0. \)

**Corollary 1.1.** Under the hypothesis of Lemma 1.1 the solution of the IVP (1.1) satisfies

\[
\left( -\Delta \right)^{\nu/2} u(x, t) \right)^2 \hat{c}_{x_1} \varphi \, dx \, dt \\
\lesssim \|u\|_{L^\infty_t \dot{H}^\nu_x} \left( 1 + T + \|\nabla u\|_{L^1_t L^\infty_x} + T \|u\|_{L^\infty_t \dot{H}^\nu_x} \right) \frac{1}{2} \|u\|_{L^\infty_t \dot{H}^\nu_x}. \tag{1.15}
\]

In the study of the asymptotic behavior of the solutions of the Zakharov-Kuznetsov equation in the energy space is required to know the behavior of the function and its derivatives on certain subsets of the plane e.g. channels, squares. In this sense, estimates such as (1.15) are quite useful in the description of such behavior (see Mendez, Muñoz, Poblete and Pozo [40] for more details).
Corollary 1.2. Let \( u \in C([0, T] : H^s(\mathbb{R}^n)), s > \frac{n}{2} + 1 \) with \( n \geq 2 \), and \( u \) be a solution of (1.1).

Let \( \vec{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{Z}^n \). For \( \vec{\kappa} \neq 0 \) we define

\[ \mathcal{P}_{\vec{\kappa}} := \{ x \in \mathbb{R}^n \mid \kappa_j < x_j < \kappa_j + 1, j = 1, 2, \ldots, n \}. \]

Then

\[
\int_0^T \int_{\mathcal{P}_{\vec{\kappa}}} \left( f^{s+\frac{1}{2}} u(x, t) \right)^2 \, dx \, dt + \int_0^T \int_{\mathcal{P}_{\vec{\kappa}}} \left( \partial_x f^{s+\frac{1}{2}} u(x, t) \right)^2 \, dx \, dt \\
\lesssim_{n, \alpha} \left( 1 + T + \| \nabla u \|_{L^1_x L^{s+\frac{1}{2}}_t} + T \| u \|_{L^{s+\frac{1}{2}} H^s_t} \right)^{1/2} \| u \|_{L^{s+\frac{1}{2}} H^s_t},
\]

whenever \( r > \frac{n}{2} \).

Also in the homogeneous case is possible to describe the smoothing effect in the same region as above.

Corollary 1.3. Let \( u \in C([0, T] : H^s(\mathbb{R}^n)), s > \frac{n}{2} + 1 \), with \( n \geq 2 \), and \( u \) be a solution of (1.1).

For \( \vec{\kappa} \) and \( \mathcal{P}_{\vec{\kappa}} \) as in Corollary 1.2, the solution \( u \) associated to (1.1) satisfies:

\[
\int_0^T \int_{\mathcal{P}_{\vec{\kappa}}} \left( (-\Delta)^{s+\frac{1}{2}} u \right)^2 (x, t) \, dx \, dt \\
\lesssim_{n, \alpha} \left( 1 + T + \| \nabla u \|_{L^1_x L^{s+\frac{1}{2}}_t} + T \| u \|_{L^{s+\frac{1}{2}} H^s_t} \right)^{1/2} \| u \|_{L^{s+\frac{1}{2}} H^s_t},
\]

whenever \( r > \frac{n}{2} \).

Kato’s smoothing effect has found diverse applications in the field of dispersive equations. Our intention in this part of the work is to present to the reader an application of Kato’s smoothing effect for the solutions of the IVP (1.1).
The question addressed is the following: If the initial data \( u_0 \) in the IVP (1.1) is provided with extra regularity in the half space \( \mathcal{H}_{(\epsilon,\nu)} \) where

\[
\mathcal{H}_{(\epsilon,\nu)} := \{ x \in \mathbb{R}^n \mid \nu \cdot x > \epsilon \},
\]

\( \epsilon > 0 \) and \( \nu \) is a non-null vector in \( \mathbb{R}^n \). Does the solution \( u \) preserve the same regularity for almost all time \( t > 0 \)?

Surprisingly, that extra regularity is propagated by the flow solution with infinity speed and this property has been shown to be true in several nonlinear dispersive models, in fact, this property is known nowadays as principle of propagation of regularity.

The description of such phenomena depends strongly on Kato’s smoothing effect. Indeed, the method to establish this particular property is mainly based on weighted energy estimates where it is also possible to show that the localized regularity entails the gain of extra derivatives on the channel \( \mathcal{Q}_{(\epsilon,\tau,\nu)} \) traveling in some specific direction, where

\[
\mathcal{Q}_{(\epsilon,\tau,\nu)} := \{ x \in \mathbb{R}^n \mid \epsilon < \nu \cdot x < \tau \},
\]

being \( \tau > \epsilon \). The figure 4 below intends to describe this particular phenomena in dimension 2.

![Diagram](https://via.placeholder.com/150)

**Figure 4.** Propagation of regularity sense in the 2 dimensional case with \( v_1 > 0, v_2 > 0 \). The dashed arrows denotes the propagation sense.

This question was originally address by Isaza, Linares and Ponce [18, 19, 20] for solutions of the KdV equation and later it was studied for solutions of the Benjamin-Ono equation and the Kadomset-Petviashvili equation resp. In the case of one dimensional models where the dispersion is weak this was established in [37, 38] for solutions of the dispersive generalized Benjamin-Ono and the fractional Korteweg-de Vries equation resp. In the quasi-linear type equations case Linares, Smith and Ponce show under certain conditions that this principle also holds. In the case of higher dispersion, Segata and Smith show that for the fifth order
KdV equation. The extension to the case where the regularity of the initial data was fully addressed by Kenig, Linares, Ponce and Vega [25] for solutions of the KdV.

The results in [25] were later extended in [39] to the $n$-dimensional case and subsequently these techniques were applied by Freire, Mendez and Riaño [11] for solutions of the dispersive generalized Benjamin-Ono-Zakharov-Kuznetsov equation, that is,

$$\partial_t u - \partial_x \left( -\partial_x^2 \right)^{\frac{\alpha+1}{2}} u + \partial_x^2 \partial_x u + u \partial_x u = 0 \quad 0 \leq \alpha \leq 1.$$  

The case $\alpha = 0$ in (1.17) was first addressed by Nascimento in [43] in the spirit of [19].

For the most recent compendium of propagation of regularity results we refer to Linares and Ponce [34] and the references therein.

Our second main result is devoted to show that solution of the FZK also satisfies the propagation of regularity principle and it is summarized in the following theorem:

**Theorem 1.18.** Let $u_0 \in H^s(\mathbb{R}^n)$ with $s > s_n$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$, $n \geq 2$ with $\nu$ satisfying (1.13)-(1.14). If the initial data $u_0$ satisfies

$$\int_{\mathcal{H}(\nu \check{\ell})} \left( \int_{\mathbb{R}^n} |\partial_x^s u_0(x)|^2 \, dx \right) \, dt < \infty,$$

then the corresponding solution $u = u(x, t)$ of the IVP (1.1) with $1 \leq \alpha < 2$: satisfies that for any $\omega > 0$, $\epsilon > 0$ and $\tau \geq 5\epsilon$,

$$\sup_{0 \leq t \leq T} \int_{\mathcal{H}(\nu \check{\ell} - \omega \check{\ell}, \nu \check{\ell} - \omega \check{\ell} \check{\ell})} \left( \int_{\mathbb{R}^n} |\partial_x^s u(x, t)|^2 \, dx \right) \, dt \leq c^*,$$

for any $r \in (0, \tilde{s}]$ with $c^* = c^* \left( \nu; \tau; \omega; \|u_0\|_{H^{s_n}}; \|\int dt u_0\|_{L^2(\mathcal{H}(\nu \check{\ell}))} \right)$.

In addition, for any $\omega > 0$, $\epsilon > 0$ and $\tau \geq 5\epsilon$

$$\int_0^T \int_{\mathcal{Q}(\nu \check{\ell} - \omega \check{\ell}, \nu \check{\ell} - \omega \check{\ell} \check{\ell})} \left( \int_{\mathbb{R}^n} |\partial_x^{s+1} u(x, t)|^2 \, dx \right) \, dt \leq c^*,$$

with $c^* = c^* \left( \nu; \tau; \omega; \|u_0\|_{H^{s_n}}; \|\int dt u_0\|_{L^2(\mathcal{H}(\nu \check{\ell}))} \right)$.

If in addition to (1.19) there exists $\beta > 0$, such that

$$\int_{\mathbb{R}^n} |\partial_x^{s+\frac{\beta}{2}} u_0| \, dx \in L^2 \left( \mathcal{H}(\nu \check{\ell}) \right),$$

then for any $\omega > 0$, $\epsilon > 0$ and $\tau \geq 5\epsilon$

$$\sup_{0 \leq t \leq T} \int_{\mathcal{H}(\nu \check{\ell} - \omega \check{\ell}, \nu \check{\ell} - \omega \check{\ell} \check{\ell})} \left( \int_{\mathbb{R}^n} |\partial_x^{s+\frac{\beta}{2}} u(x, t)|^2 \, dx \right) \, dt \leq c,$$

with $c = c(T; \epsilon; \omega; c; \|u_0\|_{H^{s_n}}; \|\int dt u_0\|_{L^2((x_0, \infty))}) > 0$.

The proof of of Theorem 1.18 is based on weighted energy estimates combined with an inductive argument, that due to the weak effects of dispersion it has to be carried out in two steps.

**Remark 1.1.** The result in Theorem 1.18 is also true in the case where the dispersion is even weaker e.g. $0 < \alpha < 1$, the proof in this case follows by combining the ideas of the proof of Theorem 1.18 and the bi-inductive argument applied in [38, 41] for solutions of the iKdV.

As a corollary we obtain that in the case that the extra regularity of the initial data is provided on an integer scale the result also holds true.
Corollary 1.4. Let \( u_0 \in H^s(\mathbb{R}^n) \) with \( s > s_n \). If for some \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n, n \geq 2 \) with \( v \) satisfying (1.13)-(1.14).

If there exist \( m \in \mathbb{N}, m > 1 + \left[ \frac{n}{2} \right] \) such that, initial data \( u_0 \) satisfy

\[
(1.24) \quad \partial_t^m u_0 \in L^2(\mathcal{H}_1) < \infty, \quad \text{such that} \quad |\alpha| = m, 
\]

then the corresponding solution \( u = u(x, t) \) of the IVP (1.1) satisfies: for any \( \nu > 0, \epsilon > 0 \) and \( \tau \geq 5\epsilon \),

\[
(1.25) \quad \sup_{0 \leq t \leq T} \int_{\mathcal{H}_1} (j^\nu u(x, t))^2 \, dx \leq c^*, 
\]

for any \( r \in (0, m] \) with \( c^* = c^*(\epsilon; \nu; u_0; \|u_0\|_{H^m}; \|\partial_t^m u_0\|_{L^2(\mathcal{H}_1)}) \).

In addition, for any \( \omega > 0, \epsilon > 0 \) and \( \tau \geq 5\epsilon \)

\[
(1.26) \quad \int_0^T \int_{\mathcal{Q}_1} (j^{m+\frac{\nu}{2}} u)^2 \, dx \, dt \leq c^*, 
\]

with \( c^* = c^*(\epsilon; \nu; u_0; \|u_0\|_{H^m}; \|\partial_t^m u_0\|_{L^2(\mathcal{H}_1)}) \).

If in addition to (1.19) there exists \( \beta > 0, \) such that

\[
(1.27) \quad j^{\frac{\nu}{2}} \partial_t^m u_0 \in L^2(\mathcal{H}_1),
\]

then for any \( \omega > 0, \epsilon > 0 \) and \( \tau \geq 5\epsilon \),

\[
(1.28) \quad \sup_{0 \leq t \leq T} \int_{\mathcal{H}_1} (j^\nu u)^2 \, dx \leq c,
\]

with \( r \in \left(0, m + \frac{1}{2}\right] \) and \( c = c(T; \epsilon; \omega; \alpha; \|u_0\|_{H^m}; \|\partial_t^m u_0\|_{L^2(\mathcal{H}_1)}) > 0 \).

Also, the solutions associated to the IVP (1.1) satisfies the following transversal propagation of regularity, which is summarized in the following theorem.

Corollary 1.5. Let \( u_0 \in H^s(\mathbb{R}^n) \). If for some \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \) with

\[
\int_{\mathcal{H}_1} (j^\nu u_0(x))^2 \, dx < \infty, 
\]

then the corresponding solution \( u = u(x, t) \) of the IVP provided by Theorem 1.4 satisfies that for any \( \omega > 0, \epsilon > 0 \) and \( \tau \geq 5\epsilon \)

\[
(1.29) \quad \sup_{0 \leq t \leq T} \int_{\mathcal{H}_1} (j^\nu u)^2 \, dx \leq c^*, 
\]

for any \( r \in (0, s] \) with \( c^* = c^*(\epsilon; \nu; u_0; \|u_0\|_{H^s}; \|\partial_t^s u_0\|_{L^2(\mathcal{H}_1)}) \).

In addition, for any \( \omega > 0, \epsilon > 0 \) and \( \tau \geq 5\epsilon \)

\[
(1.30) \quad \int_0^T \int_{\mathcal{Q}_1} (j^{s+1} u)^2 \, dx \, dt \leq c 
\]

with \( c = c(T; \epsilon; \nu; \omega; \|u_0\|_{H^s}; \|\partial_t^s u_0\|_{L^2(\mathcal{H}_1)}) > 0 \).
1.1. **Organization of the paper.** In section 1.2 we introduce the notation to be used in this work. In the section 2 we provide a detailed manner the arguments of the proof of our first main result. Finally in section 3 we provide an application of Lemma 1.1 by proving that solutions of the IVP (1.1) satisfies the propagation of regularity principle.

1.2. **Notation.** For two quantities $A$ and $B$, we denote $A \lesssim B$ if $A \leq cB$ for some constant $c > 0$. Similarly, $A \gtrsim B$ if $A \geq cB$ for some $c > 0$. Also for two positive quantities, $A$, $B$ we say that are *comparable* if $A \lesssim B$ and $B \lesssim A$, when such conditions are satisfied we indicate it by writing $A \approx B$. The dependence of the constant $c$ on other parameters or constants are usually clear from the context and we will often suppress this dependence whenever it is possible.

For any pair of quantities $X$ and $Y$, we denote $X \ll Y$ if $X \leq cY$ for some sufficiently small positive constant $c$. The smallness of such constant is usually clear from the contex.

The notation $X \gg Y$ is similarly defined.

For $f$ in a suitable class is defined the *Fourier transform of $f$* as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) \, dx.$$ 

For $x \in \mathbb{R}^n$ we denote

$$\langle x \rangle := \left(1 + |x|^2\right)^{\frac{1}{2}}.$$ 

For $s \in \mathbb{R}$ is defined the *Bessel potential of order $-s$* as $f^s := (1 - \Delta)^{-\frac{s}{2}}$, following this notation the operator $J^s$ admits representation via Fourier transform as

$$\mathcal{F}(J^s f)(\xi) = \left(2\pi \xi\right)^s \mathcal{F}(f)(\xi).$$

We denote by $S(\mathbb{R}^n)$ the Schwartz functions space and the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$. Additionally, for $s \in \mathbb{R}$ we consider the *Sobolev spaces* $H^s(\mathbb{R}^n)$ that are defined as

$$H^s(\mathbb{R}^n) := J^{-s}L^2(\mathbb{R}^n).$$

For $p \in [1, \infty]$ we consider the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. Also, we shall often use mixed-norm spaces notation. For example, for $f : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$, we will denote

$$\|f\|_{L^p_tL^q_x} := \left(\int_0^T \|f(\cdot, t)\|_{L^q_x}^p \, dt\right)^{\frac{1}{p}},$$

with the obvious modifications in the cases $p = \infty$ or $q = \infty$. Additionally, the *mixed Sobolev spaces*

$$\|f\|_{L^p_tH^s_x} := \left(\int_0^T \|f(\cdot, t)\|_{H^s_x}^p \, dt\right)^{\frac{1}{p}}.$$ 

We recall for operators $A$ and $B$ we define the *commutator* between the operator $A$ and $B$ as $[A, B] = AB - BA$.

Let $\epsilon \in \mathbb{R}$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_{n}) \in \mathbb{R}^n$ we define the *half space*

$$\mathcal{H}_{\nu, \epsilon} := \{x \in \mathbb{R}^n | \nu \cdot x > \epsilon\},$$

where $\cdot$ denotes the canonical inner product in $\mathbb{R}^n$.

Let $\tau > \epsilon$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_{n}) \in \mathbb{R}^n$ we define the *channel* as the set $\mathcal{Q}_{\nu, \epsilon, \tau}$ satisfying

$$\mathcal{Q}_{\nu, \epsilon, \tau} := \{x \in \mathbb{R}^n | \epsilon < \nu \cdot x < \tau\}.$$
2. Proof of Lemma 1.1

In this section we describe the details of the proof of the main lemma in this paper.

Proof. Let \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) be a smooth function such that

\[
\sup_{\gamma \in (\mathbb{N}_0)^n, |\gamma| \leq 4} \sup_{x \in \mathbb{R}^n} |\partial_x^\gamma \varphi(x)| \leq c,
\]

for some positive constant \( c \).

By standard arguments we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (J^s u)^2 \varphi \, dx - \int_{\mathbb{R}^n} J^s \partial_x \frac{(-\Delta)^{\alpha}/2}{2} u J^s u \varphi(x) \, dx \geq \Theta_1(t)
\]

\[
+ \int_{\mathbb{R}^n} J^s (u \partial_x u) J^s u \varphi \, dx = 0.
\]

First, we handle the term providing the dispersive part of the equation, by noticing that after apply integration by parts we obtain

\[
\Theta_1(t) = \frac{1}{2} \int_{\mathbb{R}^n} J^s u \left[ \left( -\Delta \right)^{\alpha}/2 \partial_x \varphi \right] J^s u \, dx.
\]

The next step is crucial in our argument mainly because we will replace the operator \((-\Delta)^{\frac{\alpha}{2}}\)

by

\[
(-\Delta)^{\frac{\alpha}{2}} = (I - \Delta)^{\frac{\alpha}{2}} + \mathcal{K}_{\alpha},
\]

where \( \mathcal{K}_{\alpha} \) is an operator that satisfies the following properties:

(i) There exists a kernel \( k_{\alpha} \) such that

\[
(\mathcal{K}_{\alpha} f)(x) := \int_{\mathbb{R}^n} k_{\alpha}(x, x - y) f(y) \, dy, \quad f \in \mathcal{S}(\mathbb{R}^n).
\]

(ii) For \( 1 \leq p \leq \infty \),

\[
\mathcal{K}_{\alpha} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)
\]

with

\[
\|\mathcal{K}_{\alpha} f\|_{L^p} \lesssim \|f^{\alpha - 2}\|_{L^p}.
\]

For more details on the decomposition (2.2) see Bourgain, Li [6].

Thus, after replacing (2.2) yield

\[
\Theta_1(t) = \frac{1}{2} \int_{\mathbb{R}^n} J^s u \left[ J^a \partial_x \varphi \right] J^s u \, dx + \frac{1}{2} \int_{\mathbb{R}^n} J^s u \left[ \mathcal{K}_{\alpha} \partial_x \varphi \right] J^s u \, dx
\]

\[
= \Theta_{1,1}(t) + \Theta_{1,2}(t).
\]

To handle the term \( \Theta_{1,1} \) we define

\[
\Psi_{\alpha} := \left[ J^a \partial_x \varphi \right].
\]
Clearly $\Psi_{c_n} \in \text{OPS}^a$. In virtue of pseudo-differential calculus (see appendix 4), its principal symbol has the following decomposition

\begin{equation}
(2.7) \quad c_a(x, \xi) = \sum_{|\beta|=1} \frac{1}{2\pi i} \partial^{\beta}_\xi (2\pi i \xi_1^2 \partial^{\alpha}_x) \partial^{\beta}_x \varphi + \sum_{|\beta|=2} \frac{1}{(2\pi i)^2} \partial^{\beta}_\xi (2\pi i \xi_1^2 \partial^{\alpha}_x) \partial^{\beta}_x \varphi \\
+ r_{\alpha-2}(x, \xi) = p_a(x, \xi) + p_{a-1}(x, \xi) + r_{\alpha-2}(x, \xi),
\end{equation}

where $r_{\alpha-2} \in \text{S}^{a-2} \subset \text{S}^0$.

After rearranging the expressions for $p_a$ and $p_{a-1}$ yield

\begin{equation}
(2.8) \quad p_a(x, \xi) = \langle 2\pi \xi^\alpha \partial_x \varphi - \alpha \sum_{|\beta|=1} \langle 2\pi \xi^\alpha \partial_x \varphi \rangle \beta \partial_x \varphi \rangle \xi_1 \varphi - \frac{\alpha}{2\pi} \sum_{|\beta|=1} \langle 2\pi \xi^\alpha \partial_x \varphi \rangle \beta \partial_x \varphi \rangle \xi_1 \varphi \\
- \frac{\alpha}{2\pi} \sum_{|\beta|=1} \langle 2\pi \xi^\alpha \partial_x \varphi \rangle \beta \partial_x \varphi \rangle \xi_1 \varphi \\
+ \frac{\alpha(\alpha-2)}{2\pi} \sum_{|\beta_2|=1 |\beta_1|=1} \langle 2\pi \xi^\alpha \partial_x \varphi \rangle \beta_1 \partial_x \varphi \rangle \xi_1 \varphi.
\end{equation}

Therefore,

\begin{equation}
(2.10) \quad \Psi_{c_n} = p_a(x, D) + p_{a-1}(x, D) + r_{\alpha-2}(x, D),
\end{equation}

where

\begin{equation}
(2.11) \quad p_a(x, D) = \partial_x \varphi J^\alpha - \alpha \partial_x \varphi J^{\alpha-2} \partial_x^2 - \alpha \sum_{|\beta|=1} \partial_x^\beta \varphi J^{\alpha-2} \partial_x^2 \partial_x \varphi.
\end{equation}

\begin{equation}
(2.12) \quad p_{a-1}(x, D) = -\alpha \sum_{|\beta|=1} \partial_x^\beta \varphi \partial_x \varphi J^{\alpha-2} - \frac{\alpha}{2\pi} \sum_{|\beta|=1} \partial_x^\beta \varphi \partial_x \varphi J^{\alpha-2} \\
- \frac{\alpha}{2\pi} \sum_{|\beta|=1} \partial_x^\beta \varphi \partial_x \varphi J^{\alpha-2} + \frac{\alpha(\alpha-2)}{2\pi} \sum_{|\beta_2|=1 |\beta_1|=1} \partial_x^\beta \varphi \partial_x \varphi \partial_x^\beta \varphi J^{\alpha-4},
\end{equation}

and $r_{\alpha-2}(x, D) \in \text{OPS}^{a-2} \subset \text{OPS}^0$. 
Thus, after replacing the operators $p_a(x, D)$, $p_{a-1}(x, D)$ and $p_{a-2}(x, D)$ in $\Theta_{1,1}$ we get

\[
\Theta_{1,1}(t) = \frac{1}{2} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx - \frac{\alpha}{2} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx
\]

\[
= \frac{\alpha}{2} \sum_{|\beta| \leq 1} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx - \frac{\alpha}{2} \sum_{|\beta| \leq 1} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx
\]

\[
= \frac{\alpha}{4\pi} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx - \frac{\alpha}{4\pi} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx
\]

\[
+ \frac{\alpha}{2} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx
\]

\[
= \Theta_{1,1,1}(t) + \Theta_{1,1,2}(t) + \Theta_{1,1,3}(t) + \Theta_{1,1,4}(t) + \Theta_{1,1,5}(t) + \Theta_{1,1,6}(t)
\]

\[
+ \Theta_{1,1,7}(t) + \Theta_{1,1,8}(t).
\]

In the first place, we rewrite $\Theta_{1,1,1}$ as

\[
\Theta_{1,1,1}(t) = \frac{1}{2} \int_{\mathbb{R}^n} (J^s u J^{\alpha+\beta \varphi})^2 \partial_x \varphi dx + \frac{1}{2} \int_{\mathbb{R}^n} J^s u J^{\alpha+\beta \varphi} \partial_x \varphi dx
\]

\[
= \Theta_{1,1,1,1}(t) + \Theta_{1,1,1,2}(t).
\]

The first expression in the r.h.s above represents after integrating in time the smoothing effect.

Nevertheless, the price to pay for such expression becomes reflected in estimating $\Theta_{1,1,1,2}$, which is not an easy task to tackle down.

For $s \in \mathbb{R}$ we set

\[
a_s(x, D) := [J^s; \varphi], \quad \varphi \in C^\infty(\mathbb{R}^n)
\]

with

\[
\sup_x |\partial_x^\gamma \varphi| \leq c_\gamma \quad \text{for all} \quad \gamma \in (\mathbb{N}_0)^n.
\]

By the pseudo-differential calculus, its principal symbol admits the decomposition

\[
a_s(x, D) = -s \sum_{|\beta| = 1} \partial_x^\beta \varphi \partial_x^\beta J^{\alpha+\beta \varphi} + r_{s-2}(x, D),
\]

where $r_{s-2}(x, D) \in \text{OPS}^{s-2}$.
Back to our case
\[ \Theta_{1,1,2}(t) \]
\[ = -\frac{\alpha}{4} \sum_{|\beta|=1} \left( \int_{\mathbb{R}^n} f^s u \left[ \frac{\partial f^s}{\partial x} \right] \partial_x^\beta \partial_t \left[ \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right] u dx \right) - \frac{\alpha}{4} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u \partial_x^\beta \partial_t \left[ \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right] u dx 
+ \frac{1}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u \left( \partial_x^\beta \right) u dx 
- \frac{\alpha}{4} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u \left( \partial_x^\beta \right) u dx 
+ \frac{\alpha}{8} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u \partial_x^\beta \partial_t \left( \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right) u dx. \]

At this point we have by Theorem 4.3 that
\[ (2.16) \]
\[ \int_0^T |\Theta_{1,1,2}(t)| \, dt \lesssim \|u\|_{L^\infty_x H^s}. \]

On the other hand we get after rearranging
\[ \Theta_{1,2}(t) \]
\[ = \frac{\alpha}{2} \sum_{|\beta|=1} \left( \int_{\mathbb{R}^n} \partial_x \left[ \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right] u \partial_x^\beta \partial_t \left[ \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right] u dx \right) + \frac{\alpha}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} \partial_x^\beta \partial_t \left( \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right) u dx 
+ \frac{\alpha}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} \partial_x \left( \frac{f^{s+\frac{\alpha}{2}}}{f^s} \right) u dx 
= \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2} (t). \]

We shall point out that an argument as the one used in (2.14)-(2.16) allows us to estimate \( \Theta_{1,1,2}, \Theta_{1,1,2}, \Theta_{1,1,2} \) and \( \Theta_{1,1,2} \) with the bound
\[ \int_0^T \max \{ |\Theta_{1,1,2}(t)|, |\Theta_{1,1,2,3}(t)|, |\Theta_{1,1,2,4}(t)| \} \, dt \lesssim T \|u\|_{L^\infty_x H^s}. \]

Notice that \( \Theta_{1,1,2} \) has the correct sing in front and the correct regularity desired. More precisely, it provides the smoothing effect after integrating in time.

Finally, by the \( L^2 \)-continuity (see Theorem 4.3)
\[ \int_0^T |\Theta_{1,1,2,5}(t)| \, dt \lesssim T \|u\|_{L^\infty_x H^s} \|\partial_x^3 \phi\|_{L^\infty}. \]
For the term $\Theta_{1,1,3}$ we have

$$
\Theta_{1,1,3}(t) = \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s u J^{s+a-2} \partial_x u \partial_x^\beta \varphi \, dx + \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} J^s u J^{s+a-2} \partial_x \partial_x^\beta \varphi \, dx
$$

$$
= \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s u \left[ J^{\frac{s+a}{2}} \partial_x \partial_x^\beta \right] \partial_x J^{s+a-2} u \, dx
$$

$$
+ \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x J^s \partial_x J^{s+\frac{s+a}{2}} u \partial_x \partial_x^\beta \varphi \, dx
$$

$$
+ \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x J^s (J^{s+\frac{s+a}{2}} u \partial_x \partial_x^\beta \varphi) \, dx
$$

$$
= \Theta_{1,1,3,1}(t) + \Theta_{1,1,3,2}(t) + \Theta_{1,1,3,3}(t) + \Theta_{1,1,3,4}(t).
$$

Thus, after applying the decomposition (2.15) is clear that

$$
\Theta_{1,1,3,1}(t) = \frac{\alpha (a - 2)}{4} \sum_{|\beta| = 1 \atop \beta \neq e_1} \sum_{|\gamma| = 1 \atop \gamma \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s \partial_x^\gamma \varphi \partial_x J^{s+\frac{s+a}{2}} u \partial_x^\beta \partial_x^\gamma \varphi \, dx
$$

$$
+ \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s u \partial_x J^{s+\frac{s+a}{2}} \varphi \, dx
$$

$$
= \frac{\alpha (a - 2)}{4} \sum_{|\beta| = 1 \atop \beta \neq e_1} \sum_{|\gamma| = 1 \atop \gamma \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s \partial_x^\gamma \varphi \partial_x J^{s+\frac{s+a}{2}} u \partial_x^\beta \partial_x^\gamma \varphi \, dx
$$

$$
+ \frac{\alpha}{2} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s u \partial_x J^{s+\frac{s+a}{2}} \varphi \, dx
$$

$$
= \Theta_{1,1,3,1,1}(t) + \Theta_{1,1,3,1,2}(t),
$$

where $r_{\frac{s+a}{2}} (x, D) \in \text{OPS}_{\frac{s+a}{2}} \subset \text{OPS}^0$.

Since

$$
\Theta_{1,1,3,1,1}(t) = \frac{\alpha (a - 2)}{4} \sum_{|\beta| = 1 \atop \beta \neq e_1} \sum_{|\gamma| = 1 \atop \gamma \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s \partial_x^\gamma \varphi \partial_x J^{s+\frac{s+a}{2}} u \partial_x^\beta \partial_x^\gamma \varphi \, dx
$$

$$
+ \frac{\alpha (a - 2)}{4} \sum_{|\beta| = 1 \atop \beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^\beta J^s \partial_x J^{s+\frac{s+a}{2}} u \partial_x^\beta \partial_x^\gamma \varphi \, dx
$$

$$
= \Xi_1(t) + \Xi_2(t).
$$
The next terms are quite important since these ones determine the kind of sets where the smoothing can take place. More precisely, it force us to impose conditions on the weighted function \( \varphi \) to decouple certain terms.

To handle \( \Xi_1 \) notice that there exists a skew symmetric operator \( \Psi_{-1} \in \text{OPS}^{-1} \), such that

\[
\Xi_1(t) = \frac{\alpha(\alpha-2)}{8} \int_{\mathbb{R}^n} \partial_x^{2\beta} \partial_x^\gamma \varphi \partial_x f \Psi_{-1} f \, dx,
\]

where

\[
f := \sum_{|\beta|=1} \partial_x^{\beta} f^{s+\frac{\alpha-2}{2}} u.
\]

If we assume that \( \varphi \) has the following representation

\[
(2.17) \quad \varphi(x) = \varphi (v \cdot x + \delta) \quad x \in \mathbb{R}^n, \delta \in \mathbb{R},
\]

where \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \) is a non-null vector and \( \varphi : \mathbb{R} \to \mathbb{R} \) satisfies \((2.1)\), this assumption allow us to say

\[
\Xi_1(t) = \frac{\alpha(2-\alpha)}{16} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^{\beta} f^{s+\frac{\alpha-2}{2}} u [\Psi_{-1}; \varphi] \partial_x^\gamma f^{s+\frac{\alpha-2}{2}} u \, dx
\]

\[
= \frac{\alpha(2-\alpha)}{16} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} f^{s+\frac{\alpha-2}{2}} u \partial_x^{\beta} [\Psi_{-1}; \varphi] \partial_x^\gamma f^{s+\frac{\alpha-2}{2}} u \, dx.
\]

Hence, combining pseudo-differential calculus with Theorem \((4.3)\) imply that

\[
\int_0^T |\Xi_1(t)| \, dt \lesssim_n T \left( \frac{\alpha(\alpha-2)}{16} + 1 \right) \|u\|_{L^r H^s_t}^2,
\]

and

\[
\int_0^T |\Theta_{1,1,3,1,2}(t)| \, dt \lesssim_n (n-1) \|u\|_{L^r H^s_t}^2.
\]

To provide some control over \( \Theta_{1,1,3,1,1} \) is not an easy task at all since several interactions between the variables have to be taken into consideration.

Thus, after using \((2.15)\) yield

\[
\Theta_{1,1,3,3}(t) = \frac{\alpha(2-\alpha)}{8} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^{2\beta} \partial_x^\gamma \varphi \partial_x f^{s+\frac{\alpha-2}{2}} u \partial_x^\gamma f^{s+\frac{\alpha-2}{2}} u \, dx
\]

\[
- \frac{\alpha}{8\pi} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} \partial_x f^{s+\frac{\alpha-2}{2}} u \frac{f^{s+\frac{\alpha-2}{2}}}{f^{s+\frac{\alpha-2}{2}}} (x, D) \, f^{s+\frac{\alpha-2}{2}} u \, dx
\]

\[
= \frac{\alpha(2-\alpha)}{8} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} \partial_x^{2\beta} \partial_x^\gamma \varphi \partial_x f^{s+\frac{\alpha-2}{2}} u \partial_x^\gamma f^{s+\frac{\alpha-2}{2}} u \, dx
\]

\[
+ \frac{\alpha}{8\pi} \sum_{|\beta|=1} \sum_{|\gamma|=1} \sum_{\beta \neq e_1} \int_{\mathbb{R}^n} f^{s+\frac{\alpha-2}{2}} u \partial_x^\gamma r_{\frac{n-\alpha}{2}} (x, D) \, f^{s+\frac{\alpha-2}{2}} u \, dx
\]

where \( \partial_x^\gamma r_{\frac{n-\alpha}{2}} (x, D) \in \text{OPS}^{\frac{\alpha-2}{2}} \subset \text{OPS}^0 \).
Since $0 < \alpha < 2$, we obtain by the $L^2$–continuity (Theorem 4.3)
\[
\int_0^T |\Theta_{1,1,3}(t)| \, dt \lesssim T \|u\|_{L^2_t H^2_x}^2 \left\{ \alpha(2 - \alpha) \sum_{|\beta| = 1} \sum_{|\gamma| = 1} \|\partial_x^{2\beta} \partial_x^{\gamma} \varphi\|_{L^2_t} + \alpha(n - 1) \right\}
\]
and
\[
\int_0^T |\Theta_{1,1,3,4}(t)| \, dt \lesssim T \|u\|_{L^2_t H^2_x}^2 \sum_{|\beta| = 1} \left\|\partial_x^{2\beta} \partial_x^{\gamma} \varphi\right\|_{L^2_t}.
\]

The term $\Theta_{1,1,3,2}$ will be crucial in our analysis, since from it we shall extract the desired smoothing effect after integrating in time. Nevertheless, we postpone this tedious task for the next section where we unify all the estimates.

First, we rewrite the term as follows
\[
\Theta_{1,1,4}(t) = -\frac{\alpha}{2} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} f_t^s + \frac{\alpha - 2}{2} u \left[ \frac{2 - \alpha}{\pi} \partial_x^{2\beta} \partial_x^{\gamma} \varphi \right] \partial_x^{\gamma} J^{\alpha - 2 + s} u \, dx
\]
\[+ \frac{\alpha}{4} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} \left( f_t^s + \frac{\alpha - 2}{2} u \right)^2 \partial_x^{2\beta} \partial_x^{\gamma} \varphi \, dx
\]
\[= \Theta_{1,1,4,1}(t) + \Theta_{1,1,4,2}(t).
\]

By Theorem 4.3 it follows
\[
\int_0^T |\Theta_{1,1,4,2}(t)| \, dt \lesssim T \|u\|_{L^2_t H^2_x}^2 \sum_{|\beta| = 1} \left\|\partial_x^{2\beta} \partial_x^{\gamma} \varphi\right\|_{L^2_t},
\]
and for the remainder term we use decomposition (2.14)-(2.15) to obtain
\[
\Theta_{1,1,4,1}(t) = \frac{\pi \alpha(2 - \alpha)}{2} \sum_{|\beta| = 1} \sum_{|\gamma| = 1} \int_{\mathbb{R}^n} f_t^s + \frac{\alpha - 2}{2} u \partial_x^{2\beta} \partial_x^{\gamma} \varphi \partial_x^{\gamma} J^{\alpha - 2 + s} \, dx
\]
\[+ \frac{\alpha}{2} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} f_t^s + \frac{\alpha - 2}{2} u \varphi (x, D) \partial_x^{2\beta} J^{\alpha - 2 + s} u \, dx,
\]
where $\varphi_{(\frac{n+2}{2})}(x, D) \in \text{OPS}^{-\frac{n+2}{2}} \subset \text{OPS}^0$.

In virtue of Theorem 4.3
\[
\int_0^T |\Theta_{1,1,4,1}(t)| \, dt \lesssim T \|u\|_{L^2_t H^2_x}^2 \left\{ \frac{\alpha(2 - \alpha)}{2} \sum_{|\beta| = 1} \sum_{|\gamma| = 1} \left\|\partial_x^{2\beta} \partial_x^{\gamma} \varphi\right\|_{L^2_t} + \frac{(n - 1) \alpha}{4 \pi} \right\}.
\]

On the other hand
\[
\Theta_{1,1,5}(t) = -\frac{\alpha}{4 \pi} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} f_t^s + \frac{\alpha - 2}{2} u \left[ f_t^s + \frac{2\alpha}{\pi} \partial_x^{2\beta} \partial_x^{\gamma} \varphi \right] \partial_x^{\gamma} J^{\alpha - 2 + s} u \, dx
\]
\[+ \frac{\alpha}{8 \pi} \sum_{|\beta| = 1} \int_{\mathbb{R}^n} \left( f_t^s + \frac{\alpha - 2}{2} u \right)^2 \partial_x^{2\beta} \partial_x^{\gamma} \varphi \, dx
\]
\[= \Theta_{1,1,5,1}(t) + \Theta_{1,1,5,2}(t).
\]
For the first term above we have by Theorem 4.3

\[
\int_0^T |\Theta_{1,1,5,2}(t)| \, dt \lesssim T\|u\|_{L^\infty T H_0^s}^2 \left( \frac{\alpha}{8\pi} \sum_{|\beta|=1} \|\partial_x^\beta \partial_x^{2}\varphi\|_{L^\infty_T} \right).
\]

Instead, for \(\Theta_{1,1,5,2}\) we require an extra work that can be handled by using (2.14)-(2.15).

More precisely,

\[
\Theta_{1,1,5,1}(t) = \frac{\alpha(2-\alpha)}{4} \sum_{|\beta|=1} \sum_{|\gamma|=1} \int_{\mathbb{R}^n} f^{s+\frac{a-2}{2}} u c_\gamma \partial_x \beta c_\gamma \partial_x^{2}\phi \partial_x \partial_x^{2}\partial_x^\gamma f^{s+\frac{a-2}{2}} \, dx
\]

(2.18)

\[
- \frac{\alpha}{4\pi} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^{s+\frac{a-2}{2}} ur - (\frac{a+2}{2}) (x, D) \partial_x^\beta f^{a-2+s} u \, dx,
\]

where \(r - (\frac{a+2}{2})(x, D) \in \text{OPS}^{-\frac{a+2}{2}} \subset \text{OPS}^0\).

At this point we obtain by Theorem 4.3

\[
\int_0^T |\Theta_{1,1,5,1}(t)| \, dt \lesssim T\|u\|_{L^\infty T H_0^s}^2 \left( \frac{\alpha(2-\alpha)}{4} \sum_{|\beta|=1} \sum_{|\gamma|=1} \|\partial_x^\beta \partial_x^{2}\varphi\|_{L^\infty_T} + \frac{n\alpha}{4\pi} \right).
\]

Notice that this term is quite similar to the previous one and the way to bound it follows the same. Indeed,

\[
\Theta_{1,1,6}(t) = - \frac{\alpha}{4\pi} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^{s+\frac{a-2}{2}} u \left( \frac{2-a}{2} \partial_x \beta \partial_x^{2}\phi \right) \partial_x \partial_x^{2}\partial_x^\gamma f^{a-2+s} u \, dx
\]

\[
+ \frac{\alpha}{8\pi} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^{s+\frac{a-2}{2}} u \partial_x \beta c_\gamma \partial_x^{2}\phi \, dx
\]

\[
= \Theta_{1,1,6,1}(t) + \Theta_{1,1,6,2}(t).
\]

Applying an argument similar to the one in (2.18)-(2.19) \textit{mutatis mutandis} yield

\[
\int_0^T |\Theta_{1,1,6,2}(t)| \, dt \lesssim T\|u\|_{L^\infty T H_0^s}^2 \left( \frac{\alpha}{8\pi} \sum_{|\beta|=1} \|\partial_x^\beta \partial_x^{2}\varphi\|_{L^\infty_T} \right)
\]

and

\[
\int_0^T |\Theta_{1,1,6,1}(t)| \, dt \lesssim T\|u\|_{L^\infty T H_0^s}^2 \left( \frac{\alpha(2-\alpha)}{4} \sum_{|\beta|=1} \sum_{|\gamma|=1} \|\partial_x^\beta \partial_x^{2}\varphi\|_{L^\infty_T} + \frac{n\alpha}{4\pi} \right).
\]
After rewriting

\[ \Theta_{1,1,7}(t) = \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} J^s u \partial_{x_1} \partial_{x_2}^2 \partial_{x}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1}^2 \partial_{x_2}^2 \partial_{x}^2 \phi \, dx \]

\[ = - \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} \partial_{x_1}^2 \partial_{x}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x_2}^2 \partial_{x}^2 \phi \, dx \]

\[ = - \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} \partial_{x_1}^2 \partial_{x_2}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x}^2 \partial_{x}^2 \phi \, dx \]

\[ = - \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} \partial_{x_1}^2 \partial_{x_2}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x}^2 \partial_{x}^2 \phi \, dx \]

\[ = \Theta_{1,1,7,1}(t) + \Theta_{1,1,7,2}(t) + \Theta_{1,1,7,3}(t). \]

In virtue of (2.17)

\[ \Theta_{1,1,7,1}(t) = - \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} J^s \frac{\alpha - 4}{2} \partial_{x_1}^2 \partial_{x}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x}^2 \partial_{x}^2 \phi \, dx \]

\[ = - \frac{\alpha(\alpha - 2)}{4\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \int_{\mathbb{R}^n} J^s \frac{\alpha - 4}{2} \partial_{x_1}^2 \partial_{x}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x}^2 \partial_{x}^2 \phi \, dx \]

\[ = - \frac{\alpha(\alpha - 2)}{4\pi} \int_{\mathbb{R}^n} \left( \sum_{|\beta_2|=1} \partial_{x_1}^2 \partial_{x}^2 \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x_1} \partial_{x}^2 \partial_{x}^2 \phi \right) \, dx \]

\[ = - \frac{\alpha(\alpha - 2)\nu_1}{8\pi} \int_{\mathbb{R}^n} \left( \sum_{|\beta|=1} \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x}^2 \phi \right)^2 \, dx, \]

which by continuity implies

\[ \int_0^T |\Theta_{1,1,7,1}(t)| \, dt \lesssim_{a,n} \int_0^T \sum_{|\beta|=1} \int_{\mathbb{R}^n} v^{2\beta} \left( \partial_{x}^2 f^{\alpha - 4 + s} u \partial_{x}^2 \phi \right)^2 \, dx \, dt \]

\[ \lesssim_{a,n,v} T \|u\|^2_{L^\infty_T H^\beta_x} \|\phi''\|_{L^\infty_T}, \]

and

\[ \int_0^T |\Theta_{1,1,7,3}(t)| \, dt \lesssim_T \frac{\alpha(2 - \alpha)}{4\pi} \|u\|^2_{L^\infty_T H^\beta_x} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \left( \partial_{x_1}^2 \partial_{x}^2 \partial_{x}^2 \phi \right)_{L^\infty_T}. \]
After adapting the decomposition (2.14)-(2.15) to our case we can rewrite the term above as
\[
\Theta_{1,1,7,2}(t) = \frac{\alpha(\alpha-2)}{4} \sum_{|\gamma|=1} \sum_{|\beta|=1} \sum_{|\beta_{1}|=1} \int_{\mathbb{R}^n} \partial_{x}^{\gamma} \partial_{x}^{\beta} \partial_{x}^{\beta_{1}} u \int f^{s+\frac{\alpha-4}{2}} \partial_{x}^{\beta_{1}} u dx
\]
\[
- \frac{\alpha(\alpha-2)}{4\pi} \sum_{|\gamma|=1} \sum_{|\beta|=1} \sum_{|\beta_{1}|=1} \int_{\mathbb{R}^n} f^{s+\frac{\alpha-4}{2}} \partial_{x}^{\beta_{1}} u r_{-\frac{2}{4}}(x,D) \partial_{x_{1}} f^{s+\alpha-4} \partial_{x}^{\beta_{1}} u dx.
\]
Hence, by continuity
\[
\int_{0}^{T} |\Theta_{1,1,7,2}(t)| dt \leq \frac{\alpha(\alpha-2)}{4} \sum_{|\gamma|=1} \sum_{|\beta|=1} \sum_{|\beta_{1}|=1} \left( \| \partial_{x}^{\gamma} \partial_{x}^{\beta} \partial_{x}^{\beta_{1}} u \|_{L^{2}} + \frac{\alpha(\alpha-2)}{4\pi} H^{3} \right).
\]
Since \( r_{\alpha-2}(x,D) \in \text{OPS}^{a-2} \subset \text{OPS}^{0} \), the \( L^{2} \)-continuity of the order zero pseudo-differential operators implies
\[
\int_{0}^{T} |\Theta_{1,1,8}(t)| dt \leq \frac{\alpha(\alpha-2)}{4} \sum_{|\gamma|=1} \sum_{|\beta|=1} \sum_{|\beta_{1}|=1} \left( \| \partial_{x}^{\gamma} \partial_{x}^{\beta} \partial_{x}^{\beta_{1}} u \|_{L^{2}} + \frac{\alpha(\alpha-2)}{4\pi} H^{3} \right).
\]

**Claim 1:**

There exist a constant \( \lambda = \lambda(n,v,\alpha) > 0 \), such that
\[
\lambda \left( \int_{\mathbb{R}^n} \partial_{x_{1}} \varphi d\mathbf{x} + \int_{\mathbb{R}^n} \partial_{x_{1}} f^{s+\frac{\alpha-2}{2}} \partial_{x_{1}} \varphi d\mathbf{x} \right) \leq \Theta_{1,1,1,1}(t) + \Theta_{1,1,2,2}(t) + \Theta_{1,1,3,2}(t).
\]

We shall remind that from (2.17)
\[
\varphi(x) = \varphi(v \cdot x + \delta) \quad x \in \mathbb{R}^n.
\]
Therefore,
\[
\Theta_{1,1,3,2}(t) \leq \frac{\alpha}{2} \sum_{|\beta|=1 \atop \beta \neq v_{1}} \int_{\mathbb{R}^n} \partial_{x}^{\beta} f^{s+\frac{\alpha-2}{2}} u \partial_{x_{1}} f^{s+\frac{\alpha-2}{2}} u \partial_{x_{1}} \varphi d\mathbf{x}
\]
\[
\leq \frac{\alpha}{2} \sum_{|\beta|=1 \atop \beta \neq v_{1}} \left( \| \eta \Psi_{\beta} f^{s+\frac{\alpha}{2}} u \|_{L^{2}} + \| \eta \partial_{x_{1}} f^{s+\frac{\alpha}{2}} u \|_{L^{2}} \right) \left( \| \eta \partial_{x_{1}} f^{s+\frac{\alpha}{2}} u \|_{L^{2}} \right)
\]
\[
= \frac{\alpha}{2} \sum_{|\beta|=1 \atop \beta \neq v_{1}} \| \Psi_{\beta} (f^{s+\frac{\alpha}{2}} u \eta) \|_{L^{2}} + \| \eta \partial_{x_{1}} f^{s+\frac{\alpha}{2}} u \|_{L^{2}} \left( \| \eta \partial_{x_{1}} f^{s+\frac{\alpha}{2}} u \|_{L^{2}} \right)
\]
where \( \eta := (\varphi')^{\frac{1}{2}} \) and \( \Psi_{\beta} := \partial_{x}^{\beta} f^{-1} \).

Additionally,
\[
\left\| \Psi_{\beta} \left( f^{s+\frac{\alpha}{2}} u \eta \right) \right\|_{L^{2}} \leq C \left\| f^{s+\frac{\alpha}{2}} u \eta \right\|_{L^{2}},
\]
\[
(2.22)
\]
where
\[ C := \inf_{f \in L^2(\mathbb{R}^n), f \neq 0} \frac{\| J^{-1} \hat{\varphi}_j f \|_{L^2}}{\| f \|_{L^2}}, \quad j = 2, 3, \ldots, n. \]

Also,
\[ \left\| \begin{bmatrix} \Psi_{\beta} \eta \end{bmatrix} f^s + \frac{\alpha}{2} u \right\|_{L^2} \leq c(\alpha, \beta) \left\| f^s u(t) \right\|_{L^2}. \]

Thus, for \( \epsilon > 0 \),
\[(2.23)\]
\[ \frac{\alpha}{2} \sum_{|\beta| = 1} v^\beta \left( \left\| \begin{bmatrix} \Psi_{\beta} \left( f^s + \frac{\alpha}{2} u \right) \right\|_{L^2} + \left\| \begin{bmatrix} \Psi_{\beta} \eta \end{bmatrix} f^s + \frac{\alpha}{2} u \right\|_{L^2} \right) \left\| \eta \partial_{x_1} f^s + \frac{\alpha}{2} u \right\|_{L^2} \leq \frac{\alpha \sqrt{n - 1} |\mathcal{V}| C}{2} \left\| f^s + \frac{\alpha}{2} u \right\|_{L^2} \]
\[ + \frac{\alpha \epsilon \sqrt{n - 1} |\mathcal{V}|}{2} \left\| \eta \partial_{x_1} f^s + \frac{\alpha}{2} u \right\|_{L^2} \]

where \( |\mathcal{V}| := \sqrt{v_2^2 + v_3^2 + \cdots + v_n^2} \).

Hence,
\[(2.24)\]
\[ \frac{v_1}{2} \left\| f^s + \frac{\alpha}{2} u \right\|_{L^2}^2 + \frac{av_1}{2} \left\| \eta \partial_{x_1} f^s + \frac{\alpha}{2} u \right\|_{L^2}^2 - \frac{\alpha \epsilon \sqrt{n - 1} |\mathcal{V}|}{2} \left\| f^s + \frac{\alpha}{2} u \right\|_{L^2} \left\| \eta \partial_{x_1} f^s + \frac{\alpha}{2} u \right\|_{L^2} \leq \frac{\alpha}{8} \sum_{|\beta| = 1} v^\beta \frac{c(\alpha, \beta)}{2} \left\| f^s u(t) \right\|_{L^2}^2 + \Theta_{1,1,1}(t) + \Theta_{1,1,2,2}(t) + \Theta_{1,1,3,2}(t). \]

At this point we shall make emphasis in two possible situations that we proceed to discuss below.

(i) If \( v_1 > 0 \) and \( v_2 = v_3 = \cdots = v_n = 0 \), then (2.20) holds with
\[ \lambda(\alpha, v, n) = \frac{av_1}{2} > 0. \]

We call the reader attention on the dependence on the dispersion, as well as, on the direction \( x_1 \).

(ii) If \( v_1 > 0 \) and
\[ 0 < \sqrt{v_2^2 + v_3^2 + \cdots + v_n^2} < \min \left\{ \frac{2v_1}{C/\alpha(n - 1)}, \frac{v_1(1 + \alpha)}{\alpha \epsilon \sqrt{n - 1} |\mathcal{V}|} \right\}, \]

with \( \epsilon \) satisfying
\[ 0 < \epsilon < \frac{v_1}{|\mathcal{V}| \sqrt{n - 1}} - \frac{\alpha \epsilon \sqrt{n - 1} |\mathcal{V}|}{4v_1} c^2, \]
implies that
\[
\lambda = \lambda(\alpha, \nu, n, \epsilon) = \frac{n^2(1 - \alpha)^2}{4} + |\nu_1| \frac{\alpha(1 - \alpha)e \sqrt{n - 1}}{2} + |\nu|^2 \frac{\alpha^2(n - 1)(\epsilon^2 + C^2)}{4},
\]
in (2.20).

In some sense we recover the results obtained by Linares & Ponce in [33] for the Zakharov-Kuznetsov equation in the 2d and 3d cases. More precisely, in [33] is shown that inequality (2.20) holds true whenever
\[
\sqrt{3} v_1 > \sqrt{v_2^2 + v_3^2 + \cdots + v_n^2} > 0, \quad v_1 > 0, v_2, v_3, \ldots, v_n \geq 0,
\]
for dimension \( n = 2 \) and \( n = 3 \).

Nevertheless, a quick inspection shows that for \( \alpha = 2 \) the value \( \sqrt{3} \) is not obtained directly from our calculations. We shall point that this particular number is quite related to a specific cone where the radiation part of the solutions falls into (see [33] and also the recent work in [44], that provides certain numerical simulations for solutions of (1.1) where this situation is described).

Now we show how to deal with the term that presents the major difficulties.

(2.25) 
\[
\Theta_{1,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} |K_{\alpha}(\xi, \phi)|^2 u dx.
\]

As is pointed out by Bourgain & Li [6], the operator \( K_{\alpha} \) can be rewritten as
\[
K_{\alpha}f(x) = \langle 2\pi \xi \rangle^\alpha \langle 2\pi \xi \rangle^a \hat{f}(\xi)
\]
(2.26)

where
\[
\psi(\xi) := \sum_{j=1}^{\infty} \left( \frac{a}{2} \right)^j \langle 2\pi \xi \rangle^{2-2j}.
\]

For \( \beta > 0 \), the binomial coefficient has the following asymptotic equivalence
\[
\binom{\beta}{k} = \frac{(-1)^{k}}{\Gamma(-\beta)k^{k+\beta}} (1 + o(1)) \quad \text{as} \quad k \to \infty.
\]
(2.28)

More precisely,
\[
\left| \binom{\beta}{k} \right| \approx \frac{1}{k^{\beta+1}} \quad \text{for} \quad k \gg 1.
\]
(2.29)

From (2.28)-(2.29) is clear that
\[
|\psi(\xi)| < \infty, \quad \forall \xi \in \mathbb{R}^n.
\]

The decomposition (2.26) allow us to write
\[
K_{\alpha}f(x) = \left( T_{q} |x|^{2-a} \right) f(x),
\]
(2.30)

where \( (T_{q} f)(\xi) := \psi(\xi) \hat{f}(\xi), f \in \mathcal{S}(\mathbb{R}^n) \).
From (2.30) is clear that
\[
\Theta_{1,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^x u [K_n \partial_{x_1}; \varphi] f^s u \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^n} f^x u \mathcal{T}_\varphi \left[ f^{s-2}; \varphi \right] \partial_{x_1} f^s u \, dx + \frac{1}{2} \int_{\mathbb{R}^n} f^x u \left[ \mathcal{T}_\varphi; \varphi \right] f^{s-2} \partial_{x_1} f^s u \, dx
\]
\[\Theta_{1,2}(t) + \Theta_{1,2,2}(t).\]

After combining Plancherel's Theorem, Theorem 4.3 and (2.14)-(2.15)
\[
\int_0^T |\Theta_{1,2,1}(t)| \, dt \lesssim T \|u\|_{L^p \mathcal{F} H^s_3} \left\| f^{s-2}; \varphi \right\| \partial_{x_1} f^s u \|_{L^p \mathcal{F} L^2_3}
\]
\lesssim_{a, T} \|u\|^2_{L^p \mathcal{F} H^s}.

At this point only reminds to estimate \(\Theta_{1,2,2}\). In this sense, we consider for \(b > 0\), the function
\[
B_b(y) := \frac{1}{\text{vol} B(2b)^\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\frac{y^2}{2b^2}} e^{-\frac{\rho^2}{2}} \, d\rho, \quad y \in \mathbb{R}^n, \quad b > 0.
\]

It is well known that for any \(b > 0\) the following properties are true:
(i) \(B_b \in L^1(\mathbb{R}^n)\) with \(\|B_b\|_{L^1} = 1\).
(ii) \(B_b\) has Fourier transform and it is given by the formula
\[
\hat{B}_b(\xi) = \frac{1}{(2\pi)^n} \, b^n.
\]
(iii) \(B_b\) is smooth in \(\mathbb{R}^n - \{0\}\).
(iv) \(B_b\) is strictly positive.

For the proof of these and other properties see Stein [54], chapter V.

The consideration of this function allow us to write
\[
\mathcal{T}_\varphi \left[ f^{s-2}; \varphi \right] \partial_{x_1} f^s u
\]
\[= \sum_{j=2}^{\infty} \left( \frac{\alpha}{2j} \right) \left( B_{2j-2} * (\varphi f^{s-2} \partial_{x_1} f^s u) - \varphi B_{2j-2} * f^{s-2} \partial_{x_1} f^s u \right) (x).
\]
Thus, we restrict our attention to estimate the commutator in the expression above. More precisely, for \(x \in \mathbb{R}^n\),
\[
\left( B_{2j-2} * (\varphi f^{s-2} \partial_{x_1} f^s u) - \varphi B_{2j-2} * f^{s-2} \partial_{x_1} f^s u \right) (x)
\]
\[=- \int_{\mathbb{R}^n} B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( \partial_{x_1} f^{s-2} u \right) (x - y) \, dy.
\]

Now, to our propose we require make use of the following smooth partition of unity: let \(\rho \in C_0^\infty(\mathbb{R}^n)\) such that \(\rho(y) = 1\) on \(\{|y| \leq \frac{1}{2}\}\) and \(\rho(y) = 0\) on \(\{|y| \geq 1\}\).

Setting \(\chi(y) = \rho(y/2) - \rho(y)\) then, for all \(y \in \mathbb{R}^n\)
\[
1 = \rho \left( \frac{y}{2} \right) + \sum_{\rho > 0} \chi_\rho(y),
\]
where \(\chi_\rho(y) := \chi \left( \frac{y}{2\rho + 1} \right), y \in \mathbb{R}^n\).
Thus,

\[
\begin{align*}
(B_{2j-2} * (\varphi f^{s-2} \partial_x f^s u) - \varphi B_{2j-2} * f^{s-2} \partial_x f^s u)(x) &= -\int_{\mathbb{R}^2} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( \partial_y f^{s+\alpha-2} u \right)(x - y) dy \\
&\quad - \sum_{j' \neq j} \int_{\mathbb{R}^2} \chi_{j'}(y) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( \partial_y f^{s+\alpha-2} u \right)(x - y) dy \\
&= I + II.
\end{align*}
\] (2.32)

At this stage we shall take into consideration the different behaviors of $B_{2j-2}$ depending on $j$.

**Case $j = 2$**: 

**Sub Case: $n = 2$**: 

By Lemma 4.1, the function $B_2$ can be represented as

\[
B_2(y) = \frac{e^{-|y|}}{2\sqrt{2\pi}} \int_0^{\infty} e^{-|y|^2/2} ds.
\]

In addition, Lemma 4.1 provides the following estimate: for $\beta$ multi-index with $|\beta| = 1$,

\[
|\partial_y^\beta B_2(y)| \leq c_\beta e^{-|y|} \left( 1 + |y|^{-1} \right), \quad y \in \mathbb{R}^2 - \{0\}.
\] (2.33)

Hence

\[
I = \lim_{\varepsilon \to 0} \int_{B(0,2) \setminus B(0,\varepsilon)} \rho \left( \frac{y}{2} \right) B_2(y) (\varphi(x - y) - \varphi(x)) \left( \partial_y f^{s+\alpha-2} u \right)(x - y) dy \\
= -\frac{1}{2} \int_{B(0,2)} (\partial_y \rho) \left( \frac{y}{2} \right) B_2(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+\alpha-2} u \right)(x - y) dy \\
- \lim_{\varepsilon \to 0} \int_{B(0,2) \setminus B(0,\varepsilon)} \rho \left( \frac{y}{2} \right) (\partial_y f^{s+\alpha-2} u)(x - y) dy \\
+ \int_{B(0,2)} \rho \left( \frac{y}{2} \right) B_2(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+\alpha-2} u \right)(x - y) dy \\
+ \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} \theta \rho \left( \frac{y}{2} \right) B_2(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+\alpha-2} u \right)(x - y) dS_y \\
= I_1 + I_2 + I_3 + I_4,
\]

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ denotes the inward pointing unit normal along $\partial B(0; \varepsilon)$.

Thus, by the mean value Theorem

\[
|\varphi(x - y) - \varphi(y)| \leq \int_0^1 |\nabla \varphi(x + (\theta - 1)y) \cdot y| \, d\theta \leq \|\nabla \varphi\|_{L^\infty} |y|, \quad x, y \in \mathbb{R}^n.
\] (2.34)

Hence combining (2.33) and (2.34) we get

\[
I_1 = -\frac{1}{2} \int_{B(0,2)} (\partial_y \rho) \left( \frac{y}{2} \right) B_2(y) (\varphi(x - y) - \varphi(x)) \left| f^{s+\alpha-2} u(x - y) \right| dy \\
\lesssim \|\nabla \varphi\|_{L^2} \int_{B(0,2)} |\partial_y \rho \left( \frac{y}{2} \right) B_2(y) \left| f^{s+\alpha-2} u(x - y) \right| dy,
\]
which by Young’s inequality allow us to obtain
\[ \int_0^T \| I_1(t) \|_{L^2} \, dt \lesssim \| \nabla \varphi \|_{L^p} \| u \|_{L^p H^s_x}. \]
For the second term, that is, \( I_2 \) we have in virtue of (2.33) that
\[ |I_2(t)| \lesssim \| \nabla \varphi \|_{L^p} \lim_{\epsilon \downarrow 0} \int_{B(0,2) \setminus B(0,x)} \rho \left( \frac{y}{\epsilon} \right) e^{-|y|/(|y|+1)} \left( f^{s+\alpha-2} u \right) (x-y) \, dy \]
\[ \lesssim \| \nabla \varphi \|_{L^p} \int_{B(0,2)} \rho \left( \frac{y}{\epsilon} \right) e^{-|y|/(|y|+1)} \left( f^{s+\alpha-2} u \right) (x-y) \, dy. \]
Then by Young’s inequality
\[ \int_0^T \| I_2(t) \|_{L^2} \, dt \lesssim T \| \nabla \varphi \|_{L^p} \| u \|_{L^p H^s_x}. \]
The term \( I_3 \) is quite straightforward to handle since a direct application of young inequality yield
\[ \int_0^T \| I_3(t) \|_{L^2} \, dt \lesssim T \| \tilde{c}_{x_1} \varphi \|_{L^p} \| u \|_{L^p H^s_x}. \]
On the other hand Lemma 4.2 implies
\[ |I_4| = \lim_{\epsilon \downarrow 0} \int_{\partial B(0,\epsilon)} \partial_1 \rho \left( \frac{y}{\epsilon} \right) B_2(y) (\varphi(x-y) - \varphi(x)) \left( f^{s+\alpha-2} u \right) (x-y) \, dS_y \]
\[ \lesssim \| \nabla \varphi \|_{L^p} \lim_{\epsilon \downarrow 0} e^2 e^{-\epsilon} \left( 1 + \log^+ \left( \frac{1}{\epsilon} \right) \right) \int_{\partial B(0,\epsilon)} \frac{|f^{s+\alpha-2} u (x-y)|}{|y|} \, dS_y \]
and since
\[ \lim_{\epsilon \downarrow 0} \int_{\partial B(x,\epsilon)} |f^{s+\alpha-2} u (y)| \, dS_y = \left| f^{s+\alpha-2} u (x) \right|, \]
whenever \( x \in \mathbb{R}^n \) be a Lebesgue point

Therefore, \( I_4 = 0 \).
Next,
\[ II = \sum_{p \geq 0} \frac{1}{2^p} \int_{\mathbb{R}^n} (\partial_y \varphi, \chi_p(y) B_2(y) (\varphi(x-y) - \varphi(x)) \left( f^{s+\alpha-2} u \right) (x-y) \, dy \]
\[ + \sum_{p \geq 0} \int_{\mathbb{R}^n} \chi_p(y) (\partial_y \varphi, B_2(y) (\varphi(x-y) - \varphi(x)) \left( f^{s+\alpha-2} u \right) (x-y) \, dy \]
\[ - \sum_{p \geq 0} \int_{\mathbb{R}^n} \chi_p(y) B_2(y) (\varphi(x-y) \partial_y \varphi) \left( f^{s+\alpha-2} u \right) (x-y) \, dy \]
\[ = II_1 + II_2 + II_3. \]
The terms \( II_1 \) and \( II_2 \) satisfy
\[ |II_1| \lesssim \| \varphi \|_{L^p} \sum_{p \geq 0} \frac{1}{2^p} \left( B_2 * \left| f^{s+\alpha-2} u \right| \right)(x), \]
\[ \text{and} \quad |II_2| \lesssim \| \varphi \|_{L^p} \sum_{p \geq 0} \frac{1}{2^p} \left( B_2 \ast \left| f^{s+\alpha-2} u \right| \right)(x), \]
\[ |II_3| \lesssim \| \varphi \|_{L^p} \sum_{p \geq 0} \frac{1}{2^p} \left( B_2 + \left| f^{s+\alpha-2} u \right| \right)(x), \]

\footnote{If \( \omega_n \) denotes the volume of the unitary \( n \)-dimensional sphere we set \( f_{B(x,\epsilon)} f(y) dS_y := \frac{1}{\omega_n \epsilon^n} \int_{B(0,\epsilon)} f(y) dS_y. \)
Therefore, Young’s inequality ensure that
\[ |I_2| \lesssim \| \varphi \|_{L^\infty} \sum_{p \geq 0} \left( \chi_p e^{-|\cdot|} \ast \left| f^{s+\alpha-2} u \right| \right)(x), \]
where we have used (2.33).

On the other hand we have after rewriting the following bound for \( I_3 \)
\[ |I_3| \leq \sum_{p \geq 0} \left( \left( \chi_p B_{\frac{1}{2}} \right) \ast \left| \partial_x \varphi f^{s+\alpha-2} u \right| \right)(x). \]

Therefore, Young’s inequality ensure that
\[
\int_0^T \max \left( \| I_1(t) \|_{L^\infty}, \| I_2(t) \|_{L^\infty}, \| I_3(t) \|_{L^\infty} \right) dt \lesssim _T \| u \|_{L^\infty H^s} \| \nabla \varphi \|_{L^\infty}. 
\]

**Sub case: \( n > 2 \)**

This case is quite similar to the sub-case \( j > 1 + \frac{n}{2} \) below, so for the sake of brevity we omit here and we refer to reader to the pointed out case where are indicated all the details.

**Case \( j > 2 \)**

The following sub-cases are examined in the case that such integer \( j \) satisfies the indicated condition. The reader shall notice that in some cases for lower dimensions e.g. \( n = 2, 3, 4 \) some cases are empty.

We start by the easier case.

**Sub case: \( j > 1 + \frac{n}{2} \)**

\[
I = \lim_{\epsilon \downarrow 0} \int_{B(0,2) \setminus B(0,\epsilon)} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) \left( \varphi(x - y) - \varphi(x) \right) \left( \partial_x f^{s+\alpha-2} u \right)(x - y) \ dy \\
= - \lim_{\epsilon \downarrow 0} \int_{B(0,2) \setminus B(0,\epsilon)} \partial_{x_1} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) \left( \varphi(x - y) - \varphi(x) \right) \left( f^{s+\alpha-2} u \right)(x - y) \ dy \\
\quad + \nu_1 \lim_{\epsilon \downarrow 0} \int_{\partial B(0,\epsilon)} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) \left( \varphi(x - y) - \varphi(x) \right) \left( f^{s+\alpha-2} u \right)(x - y) \ dS_y \\
= -\frac{1}{2} \int_{B(0,2)} \left( \partial_{x_1} \rho \right) \left( \frac{y}{2} \right) B_{2j-2}(y) \left( \varphi(x - y) - \varphi(x) \right) \left( f^{s+\alpha-2} u \right)(x - y) \ dy \\
\quad - \lim_{\epsilon \downarrow 0} \int_{B(0,2) \setminus B(0,\epsilon)} \rho \left( \frac{y}{2} \right) \left( \partial_{y_1} B_{2j-2}(y) \right) \left( \varphi(x - y) - \varphi(x) \right) \left( f^{s+\alpha-2} u \right)(x - y) \ dy \\
\quad \left( \partial_{y_1} \rho \right) \left( \frac{y}{2} \right) B_{2j-2}(y) \left( \varphi(x - y) - \varphi(x) \right) \left( f^{s+\alpha-2} u \right)(x - y) \ dS_y \\
= I_1 + I_2 + I_3 + I_4.
\]

The terms \( I_1 \) and \( I_3 \) can be easily bounded by using Young’s inequality. Indeed,
\[
\int_0^T \| I_1(t) \|_{L^\infty} dt \lesssim _T \| \varphi \|_{L^\infty} \| u \|_{L^\infty H^s},
\]
and
\[
\int_0^T \| I_3(t) \|_{L^\infty} dt \lesssim _T \| \partial_x \varphi \|_{L^\infty} \| u \|_{L^\infty H^s}.
\]
Since
\[ |\partial_{y_i} B_{2j-2}(y)| \lesssim \frac{\Gamma (j - \frac{4}{\alpha})}{\Gamma (j - 1)}, \quad \text{for } 0 < |y| \leq 2, \]
then
\[ \int_0^T \| I_2(t) \|_{L^2} \, dt \lesssim T \frac{\Gamma (j - \frac{4}{\alpha})}{\Gamma (j - 1)} \| \varphi \|_{L^\infty} \| u \|_{L^\infty H^s}. \]
By (2.34) we provide the following upper bound
\[ |I_4| \lesssim \| \partial_\alpha \varphi \|_{L^\infty} \lim_{\epsilon \to 0} \int_{\partial\Omega(\epsilon, x)} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) \| f^{s+\alpha-2} u \| (x - y) \, dS_y \]
(2.35)
\[ \approx \frac{\Gamma (j - \frac{4}{\alpha})}{\Gamma (j - 1)} \| \partial_\alpha \varphi \|_{L^\infty} \lim_{\epsilon \to 0} \int_{\partial\Omega(\epsilon, x)} \rho \left( \frac{y}{2} \right) |y| \| f^{s+\alpha-2} u \| (x - y) \, dS_y, \]
where we have used that
\[ B_{2j-2}(y) \approx \frac{\pi^{\frac{n}{2}} \Gamma (j - \frac{4}{\alpha})}{\Gamma (j - 1)}, \quad \text{for } y \to 0, \]
see Lemma 4.3.
Notice that
\[ \lim_{\epsilon \to 0} \int_{\partial\Omega(\epsilon, x)} \rho \left( \frac{y}{2} \right) |y| \| f^{s+\alpha-2} u \| (x - y) \, dS_y \]
\[ \lesssim \lim_{\epsilon \to 0} \epsilon^{\alpha} \int_{\partial\Omega(\epsilon, x)} \| f^{s+\alpha-2} u \| (x - y) \, dS_y, \]
where
\[ \lim_{\epsilon \to 0} \int_{\partial\Omega(\epsilon, x)} \| f^{s+\alpha-2} u(y) \| \, dS_y = \| f^{s+\alpha-2} u(x) \|, \]
whenever \( x \) be a Lebesgue point.
Therefore, the estimates (2.35)-(2.38) imply \( I_3 = 0. \)
An argument quite similar to the one used above also applies to prove that \( I_4 = 0, \) and to avoid repeating the same arguments we will omit the details.
On the other hand
\[ II = \sum_{p \geq 0} \frac{1}{2p+1} \int_{\mathbb{R}^n} (\partial_{y_i} \chi_p(y)) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+\alpha-2} u \right) (x - y) \, dy \]
\[ + \sum_{p \geq 0} \int_{\mathbb{R}^n} \chi_p(y) (\partial_{y_i} B_{2j-2})(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+\alpha-2} u \right) (x - y) \, dy \]
\[ - \sum_{p \geq 0} \int_{\mathbb{R}^n} \chi_p(y) B_{2j-2}(y) (\partial_{y_i} \chi_p)(x - y) \left( f^{s+\alpha-2} u \right) (x - y) \, dy \]
\[ = II_1 + II_2 + II_3. \]
The expression \( II_1 \) clear that it satisfies the inequality
\[ |II_1| \lesssim \| \varphi \|_{L^\infty} \sum_{p \geq 0} \frac{1}{2p} \left( B_{2j-2} \ast |f^{s+\alpha-2} u| \right) (x). \]
(2.39)
Since \( B_{2j-2} \) is smooth away from zero, the following identity holds: for \( y \in \mathbb{R}^n \setminus \{0\}, \)
\[ (\partial_{y_i} B_{2j-2})(y) = -\frac{y_{j-2(\alpha-2)}}{2(\alpha-2)} B_{2j-4}(y), \quad \text{whenever } j > 2. \]
If we set
\[ \widetilde{B}_j(y) := y_1 B_{j-2}(y), \quad \text{for } j > 2, \]
then
\[ |I_{I_2}| \lesssim \frac{\|\varphi\|_{L^\infty}}{j - 2} \sum_{\beta \geq 0} \left( \chi_\beta \widetilde{B}_j * \left| f^{s + \alpha - 2} u \right| \right)(x), \tag{2.40} \]
and
\[ |I_{I_3}| \leq \sum_{\beta \geq 0} \left( \chi_\beta \widetilde{B}_{2j-2} * \left| \partial_{x_1} \varphi f^{s + \alpha - 2} u \right| \right)(x), \quad \forall x \in \mathbb{R}^n. \tag{2.41} \]

Finally, gathering (2.39), (2.40) and (2.41) and taking into consideration (2.31)
\[ \|I_1\|_{L^2_x} \lesssim \|\varphi\|_{L^\infty}_J \|B_{2j-2} * \left| f^{s + \alpha - 2} u \right|\|_{L^2_x} + \frac{\|\varphi\|_{L^\infty}}{2(j - 2)} \|\widetilde{B}_j * \left| f^{s + \alpha - 2} u \right|\|_{L^2_x}, \]
which by Young’s inequality allow us to obtain the bound
\[ \int_0^T \|I_1(t)\|_{L^2_x} \, dt \lesssim \int_0^T \left\{ \|\varphi\|_{L^\infty} \|f^2 u(t)\|_{L^2_x} + \frac{\|\varphi\|_{L^\infty}}{2(j - 2)} \|\widetilde{B}_j * \|f^2 u(t)\|_{L^2_x} \right\} \, dt \]
\[ + T \|\partial_{x_1} \varphi\|_{L^\infty} \|u\|_{L^\infty \times H^1_t} \lesssim_{n,T} \|u\|_{L^\infty \times H^1_t} \|\nabla \varphi\|_{W^{1,\infty}_t}, \]
where we have used that
\[ \|\widetilde{B}_j\|_{L^1_x} \approx_n \frac{(j - 1)(j - 2)}{j^2}, \quad \text{whenever } j > 2. \]

**Sub case: \( j = 1 + \frac{q}{2} \):**

This case is quite similar to the sub case \( n = 2 \), so that for the sake of brevity we omit the details.

**Sub case: \( j < 1 + \frac{q}{2} \):**

We start by specifying the implicit constant in inequality (c) in Lemma 4.2. More precisely, for any multi-index \( \beta \in (\mathbb{N}_0)^n \) with \( |eta| = 1 \), the following identity holds for \( y \in \mathbb{R}^n \setminus \{0\} \),
\[ \left( \partial_{y_\beta} \tilde{B}_{2j-2} \right)(y) = -\frac{y_\beta}{|y|} \left( B_{2j-2}(y) + \gamma_j e^{-|y|} \int_0^\infty e^{-s|y|} s^{\frac{n-(2j-2)-1}{2}} ds \right) \]
where
\[ \gamma_j := \frac{1}{(2\pi)^{\frac{n-1}{2}}} 2^{j-1} \Gamma(j - 1) \Gamma\left( \frac{n-2j+3}{2} \right). \]

Therefore, a rough upper bound for (2.42) is
\[ \left| \left( \partial_{y_\beta} \tilde{B}_{2j-2} \right)(y) \right| \leq B_{2j-2}(y) + e^{-|y|} \gamma_j \left( \frac{3}{2} \right)^{\frac{n+1}{2}} \left( \left( \frac{2}{3} \right)^j + 2^j \frac{\Gamma(n - 2j + 3)}{|y|^n 2^{j+3}} \right), \quad y \in \mathbb{R}^n \setminus \{0\}. \tag{2.43} \]
Now, we shall pay attention to the terms in the r.h.s above. By Legendre’s\(^2\) duplication formula for Gamma function we obtain
\[
\sup_{j \geq 1} \left\{ \gamma_j \left( \frac{2}{3} \right)^j, \gamma_j 2^j \Gamma(n - 2j + 3) \right\} \lesssim 1.
\]
Next, we turn our attention to \(I\),
\[
I = \lim_{\varepsilon \to 0} \int_{B(0,2) \setminus B(0,\varepsilon)} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( \partial_{y_1} f^{s+a-2} u \right) (x - y) \, dy
\]
\[
= -\frac{1}{2} \int_{B(0,2)} \left( \partial_{y_1} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) \right) (\varphi(x - y) - \varphi(x)) \left( f^{s+a-2} u \right) (x - y) \, dy
\]
\[
- \lim_{\varepsilon \to 0} \int_{B(0,2) \setminus B(0,\varepsilon)} \rho \left( \frac{y}{2} \right) (\partial_{y_1} B_{2j-2})(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+a-2} u \right) (x - y) \, dy
\]
\[
+ \int_{B(0,2)} \rho \left( \frac{y}{2} \right) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+a-2} u \right) (x - y) \, dy
\]
\[
+ \lim_{\varepsilon \to 0} \int_{\partial B(0,2) \setminus B(0,\varepsilon)} \partial_1 \rho \left( \frac{y}{2} \right) B_{2j-2}(y) (\varphi(x - y) - \varphi(x)) \left( f^{s+a-2} u \right) (x - y) \, dS_y
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]
Nevertheless, we realize that after incorporating the bound obtained in (2.43) combined with the arguments already described in the case \(n = 2\) implies that \(I_3 = 0\),
\[
\int_0^T \max \left( \|I_1(t)\|_{L^2_x}, \|I_2(t)\|_{L^2_x}, \|I_4(t)\|_{L^2_x} \right) \, dt \lesssim_T \|u\|_{L_T^\infty \mathcal{H}_2} \|\varphi\|_{W^{1,\infty}_x}.
\]
and
\[
\int_0^T \max \left( \|II_1(t)\|_{L^2_x}, \|II_2(t)\|_{L^2_x}, \|II_3(t)\|_{L^2_x} \right) \, dt \lesssim_T \|u\|_{L_T^\infty \mathcal{H}_2} \|\varphi\|_{W^{1,\infty}_x}.
\]
In summary, we have obtained
\[
\int_0^T |\Theta(t)| \, dt \lesssim_{\alpha, \eta, T, \nu} \|u\|_{L_T^\infty \mathcal{H}_2} \|\varphi\|_{W^{1,\infty}_x} \left( \sum_{j=2}^\infty \frac{1}{j^{\alpha+1}} \right)
\]
\[
\lesssim_{\alpha} \|u\|_{L_T^\infty \mathcal{H}_2} \|\varphi\|_{W^{1,\infty}_x},
\]
for any \(\alpha \in (0, 2)\).

\textbf{Remark 2.1.} Notice that the last inequality above is quite instructive to understand the effects of the dispersion on the solutions, since the argument above suggest that for \(\alpha \geq 0\), the pursuit smoothing effect does not hold.

Finally, by inequality (5.3)
\[
\Theta_2(t) = \int_{\mathbb{R}^n} f^s (u \partial_t u) f^s u \varphi \, dx
\]
\[
= \int_{\mathbb{R}^n} f^s u \varphi |f^s u| \partial_t u \partial_{x_1} u \, dx - \frac{1}{2} \int_{\mathbb{R}^n} (f^s u)^2 (\partial_{x_1} u \varphi + u \partial_{x_1} \varphi) \, dx
\]
\[
\lesssim \|\varphi\|_{L^\infty_x} \|f^s u(t)\|_{L^2_x}^2 \|\nabla u(t)\|_{L^2_x} + |f^s u(t)|_{L^2_x}^2 \left( \|\partial_{x_1} u(t)\|_{L^\infty_x} + \|\partial_{x_1} \varphi\|_{L^\infty_x} \|u(t)\|_{L^\infty_x} \right).
\]

\(^2\)For \(z \in \mathbb{C}\) the Legendre’s duplication Gamma formula is given by
\[
\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma \left( z + \frac{1}{2} \right).
\]
Finally, gathering the estimates above we obtain after integrating in the time variable the expression (2.11)

$$
\int_0^T \int_{\mathbb{R}^n} \left( \left( \kappa^s + \frac{8}{\omega} u(x,t) \right)^2 + \left( \partial_x \kappa^s + \frac{4\kappa_0}{\omega} u(x,t) \right)^2 \right) \partial_x \varphi \, dx \, dt
\lesssim n, \alpha, \nu, T \left( 1 + T + \|\nabla u\|_{L^1_\omega L^\infty_T} + T \|u\|_{L^\infty_\omega H^2_T} \right) \|u\|_{L^\infty_\omega H^2_T},
$$

whenever $r > \frac{\mu}{T}$. □

3. Proof of Theorem 1.18

In this section we focus our attention in to provide some immediate applications of the smoothing effect deduced in the previous section. in this sense, we prove that solution of the IVP satisfies the principle of propagation of regularity in dispersive equations. More precisely, we prove Theorem 1.18 and its method of proof follows the ideas from [18], [25],[37], and [39].

For the proof we consider the following standard notation, that it has shown to be versatile. In detail, for $\epsilon > 0$ and $\tau \equiv 5\epsilon$ we consider the following families of functions

$$
\chi_{\epsilon,\tau}, \widetilde{\phi}_{\epsilon,\tau}, \phi_{\epsilon,\tau}, \psi_{\epsilon} \in C^\infty(\mathbb{R}),
$$
satisfying the conditions indicated below:

(i) $\chi_{\epsilon,\tau}(x) = \begin{cases} 1 & x \geq \tau \\ 0 & x \leq \epsilon \end{cases}$,

(ii) $\text{supp}(\chi_{\epsilon,\tau}) \subset [\epsilon, \tau]$,

(iii) $\chi_{\epsilon,\tau}'(x) \geq 0$,

(iv) $\chi_{\epsilon,\tau}'(x) \geq \frac{1}{10(\tau - \epsilon)}[2\epsilon, \tau - 2\epsilon](x)$,

(v) $\text{supp}(\widetilde{\phi}_{\epsilon,\tau})$, $\text{supp}(\phi_{\epsilon,\tau}) \subset [\frac{\epsilon}{3}, \tau]$,

(vi) $\phi_{\epsilon,\tau}(x) = \phi_{\epsilon,\tau}(x) = 1$, if $x \in [\frac{\epsilon}{3}, \epsilon]$,

(vii) $\text{supp}(\psi_{\epsilon}) \subset (-\infty, \frac{\epsilon}{3})$.

(viii) For all $x \in \mathbb{R}$ the following quadratic partition of the unity holds

$$
\chi_{\epsilon,\tau}^2(x) + \widetilde{\phi}_{\epsilon,\tau}^2(x) + \psi_{\epsilon}(x) = 1,
$$

(ix) also for all $x \in \mathbb{R}$

$$
\chi_{\epsilon,\tau}(x) + \phi_{\epsilon,\tau}(x) + \psi_{\epsilon}(x) = 1, \quad x \in \mathbb{R}.
$$

For a more detailed construction of these families of weighted functions see [18].

Proof. In order to not saturate the notation the weighted functions $\chi_{\epsilon,\tau}, \phi_{\epsilon,\tau}, \widetilde{\phi}_{\epsilon,\tau}$ and $\psi_{\epsilon}$ will be considered as functions of the variable $\nu \cdot x + \omega t$ e.g. in the case of $\chi_{\epsilon,\tau}$ we understand it as

$$
\chi_{\epsilon,\tau}(x,t) := \chi_{\epsilon,\tau}(\nu \cdot x + \omega t),
$$

and the dependence on $x, t$ will be suppressed.
Performing energy estimates allow us to obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^n} \left( f^s \right)^2 \chi_{\epsilon, \tau}^2 \, dx - \frac{\omega}{2} \int_{\mathbb{R}^n} \left( f^s \right)^2 \chi_{\epsilon, \tau} \chi_{\epsilon^\prime, \tau} \, dx - \int_{\mathbb{R}^n} f^s u \partial_x (-\Delta) \left( f^s \right)^2 \chi_{\epsilon, \tau} \, dx \\
+ \int_{\mathbb{R}^n} f^s f^s \left( u \partial_x u \right) \chi_{\epsilon, \tau}^2 \, dx = 0.
\]
(3.2)

The proof follows by using an inductive argument we describe below: the idea is to start by
\[
\begin{array}{c}
 f^s \\
\downarrow \\
 f^{s+\frac{\alpha}{2}} \\
\downarrow \\
 f^{s+1} \\
\downarrow \\
 f^{s+1+\frac{\alpha}{2}}
\end{array}
\]

**Figure 5.** Description of the argument to reach more regularity at steps of order $\alpha/2$ in the inductive process. The abbreviations PR stands for *Propagated Regularity* and the down arrows indicate local regularity gained. The downward arrows show the local gain of regularity at the corresponding step showing that regularity of the solutions close to Sobolev index where there exist local well-posedness, is propagated with infinity speed over the moving half spaces. In this process, the smoothing effect is fundamental to estimate several terms when we perform the energy estimate (3.2) e.g. the to show that $\Theta_1 \in L^1$ we requires the extra regularity provided by the smoothing. Since the smoothness provided is too weak we require two steps (compare to ZK equation see [33]) to reach one extra derivative. The figure 3 describes the two steps inductive process.

**Case:** $s \in (s_n, s_n + \frac{\alpha}{2})$

**Step 1**

Notice that
\[
\int_0^T |\Theta_1(t)| \, dt \lesssim \int_0^T \int_{\mathbb{R}^n} \left( f^s \right)^2 \chi_{\epsilon, \tau} \chi_{\epsilon^\prime, \tau} \, dx \, dt \\
\lesssim \int_0^T \int_{\mathbb{R}^n} 1_{\mathcal{H}_{(\epsilon - \omega, \tau)} \cap \mathcal{H}_{(\tau - \omega, \tau)}^c} \left( f^s \right)^2 \, dx \, dt.
\]
(3.3)

Although, by Lemma 1.1 the solutions enjoys of extra regularity on the channel
\[
\mathcal{H}_{(\epsilon - \omega, \tau)} \cap \mathcal{H}_{(\tau - \omega, \tau)}^c, \quad \text{for } \epsilon > 0, \tau > 5\epsilon,
\]
More precisely, we can choose $\varphi$ in Lemma 1.1 properly to obtain
\[
\int_0^T \int_{\mathcal{H}_{(\epsilon - \omega, \tau)} \cap \mathcal{H}_{(\tau - \omega, \tau)}^c} \left( f^{s+\frac{\alpha}{2}} \right)^2 \, dx \, dt \lesssim c.
\]
(3.4)

In virtue of Lemma 5.5 *mutatis mutandis* in the weighted functions we get
\[
\int_0^T \int_{\mathcal{H}_{(\epsilon - \omega, \tau)} \cap \mathcal{H}_{(\tau - \omega, \tau)}^c} \left( f^s \right)^2 \, dx \, dt \lesssim c,
\]
(3.5)
for \( r \in (0, s_\alpha + \frac{\epsilon}{2}) \), for \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \).

Then \( \Theta_1 \in L^2_T \). The arguments in (2.2)-(2.5) allow us to rewrite \( \Theta_2 \) as follows:

\[
\Theta_2(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^u \left[ f^u \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 \right] f^u \, dx + \frac{1}{2} \int_{\mathbb{R}^n} f^u \left[ \mathcal{K}_a \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 \right] f^u \, dx
\]
\[
= \Theta_{2,1}(t) + \Theta_{2,2}(t).
\]

Next, we consider the operator

\[
c_a(x, D) := \left[ f^u \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 \right].
\]

Thus, by using pseudo-differential calculus there exist operators \( p_{\alpha-k}(x, D), j \in \{1, 2, \ldots, m\} \) for some \( m \in \mathbb{N} \) such that

\[
c_a(x, D) = p_{\alpha}(x, D) + p_{\alpha-1}(x, D) + \cdots + p_{\alpha-m}(x, D) + r_{\alpha-m-1}(x, D),
\]

where \( p_{\alpha-j} \in \text{OPS}^{\alpha-j} \) and \( r_{\alpha-m-1} \in \text{OPS}^{\alpha-1-m} \).

The representation above presents two main difficulties. The first one consist into describe accurately the terms \( p_{\alpha-j}(x, D) \) for each \( j \). The second problem deals into determine \( m \) adequately.

We will show later that it is only required to estimate \( p_{\alpha}(x, D) \) and \( p_{\alpha-1}(x, D) \). According to (2.6)-(2.12)

\[
p_{\alpha}(x, D) = \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^\alpha - \alpha \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^2 - \alpha \sum_{|\beta|=1} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^\beta \hat{c}_{x_1}^\beta,
\]

and

\[
p_{\alpha-1}(x, D) = -\alpha \sum_{|\beta|=1} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^\beta \hat{c}_{x_1}^\beta f^{\alpha-2} - \frac{\alpha}{2\pi} \sum_{|\beta|=1} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^\beta \hat{c}_{x_1}^\beta f^{\alpha-2}
\]
\[
- \frac{\alpha}{2\pi} \sum_{|\beta|=1} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^\beta \hat{c}_{x_1}^\beta f^{\alpha-2} + \frac{\alpha(\alpha - 2)}{2\pi} \sum_{|\beta_2|=1} \sum_{|\beta_1|=1} \sum_{|\beta_1|=1} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 f^{\alpha-2} c_{x_1}^\beta \hat{c}_{x_1}^\beta f^{\alpha-2}.\]

The remainder terms are obtained by using pseudo-differential calculus and these ones can be rewritten as

\[
p_{\alpha-j}(x, D) = \sum_{|\beta|=j} \hat{c}_{x_1} \chi_{\epsilon,\tau}^2 \Psi_{\beta,j} f^{\alpha-j}, \quad j \geq 2,
\]

where \( \Psi_{\beta,j} \in \text{OPS}^0 \) for \( j \in \{2, \ldots, m\} \).

Setting \( m \) as being

\[
m = [2s + \alpha - 1 - s_\alpha].
\]

Thus,

\[
\int_0^T |\Theta_{2,1,m+1}(t)| \, dt \leq T \|u_0\|_{L^2_T} \|f^u r_{\alpha-m-1}(x, D) f^u u\|_{L^\infty_T L^2_T}
\]
\[
\lesssim_{\epsilon,\tau,\alpha,n} T \|u_0\|_{L^2_T} \|u\|_{L^\infty_T H^{s_m+}}.
\]
When replacing (3.9) into $\Theta_{2,1}$ we obtain

\begin{equation}
\Theta_{2,1}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^s u [\xi^{s+\alpha} u \delta x_1 (\chi_{r,\tau}^2)] dx - \frac{\alpha}{2} \int_{\mathbb{R}^n} f^s u [\xi^{s+\alpha-2} \delta x_1 (\chi_{r,\tau}^2)] dx
\end{equation}

\begin{equation}
- \frac{\alpha}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u [\xi^{s+\alpha-2} \delta x_1 \partial_{x_i}^\beta \delta x_1 (\chi_{r,\tau}^2)] dx + \frac{1}{2} \sum_{j=2}^m \int_{\mathbb{R}^n} f^s u p_{\alpha-j} (x, D) f^s u dx
\end{equation}

\begin{equation}
+ \frac{1}{2} \int_{\mathbb{R}^n} f^s u r_{\alpha-m-1} (x, D) f^s u dx
\end{equation}

By using an argument similar to the one described in (2.13)-(2.15) there exists $r_{\frac{1}{2}-2} (x, D) \in \text{OPS}^{\frac{1}{2}-2}$, such that

\begin{equation}
\Theta_{2,1,1}(t) = v_1 \int_{\mathbb{R}^n} \left( f^{s+\tau} u \right)^2 \chi_{r,\tau} \chi_{r,\tau} dx + \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\tau} u \left[ f^{s+\tau} u \partial_{x_1} (\chi_{r,\tau}^2) \right] f^s u dx
\end{equation}

\begin{equation}
= \Theta_{2,1,1,1}(t) + \Theta_{2,1,1,2}(t).
\end{equation}

The term containing the commutator expression is quite more complicated to handle since at first sight some upper bound would require more regularity. However, we will show that this is not the case, since there are several cancellations that allow close the argument without any additional assumption.

First, we rewrite $\Theta_{2,1,1,2}$ as follows

\begin{equation}
\Theta_{2,1,1,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^s u \left[ f^{s+\tau} u \partial_{x_1} (\chi_{r,\tau}^2) \right] f^s u dx
\end{equation}

\begin{equation}
- \frac{1}{2} \int_{\mathbb{R}^n} f^s u \left[ f^{s+\tau} u \partial_{x_1} (\chi_{r,\tau}^2) \right] f^{s+\tau} u dx
\end{equation}

\begin{equation}
= \Lambda_1(t) + \Lambda_2(t).
\end{equation}

We focus our attention on $\Lambda_1$. In the same spirit of the decomposition used in (3.10), is clear that for some $m_1 \in \mathbb{N}$ there exist operators $q_{\alpha-1} (x, D), q_{\alpha-2} (x, D), \ldots, q_{\alpha-m_1} (x, D), r_{\alpha-m_1} (x, D)$ such that

\begin{equation}
\left[ f^{s+\tau} u \partial_{x_1} (\chi_{r,\tau}^2) \right] = q_{\alpha-1} (x, D) + q_{\alpha-2} (x, D) + \cdots + q_{\alpha-m_1} (x, D) + r_{\alpha-m_1} (x, D),
\end{equation}

where

\begin{equation}
q_{\alpha-j} (x, D) = \sum_{|\beta| = j} c_{\beta,j} \delta x_1 (\chi_{r,\tau}^2) \Psi_{\beta,j} f^s u,
\end{equation}

and $\Psi_{\beta,j} \in \text{OPS}^{0}$ for all $\beta$. 

and $\Psi_{\beta,j} \in \text{OPS}^{0}$ for all $\beta$. 

Hence,

\[
\Lambda_1(t) = \frac{1}{2} \sum_{j=1}^{n_1} f^s u q_{\alpha-j}(x, D) f^s u \, dx + \frac{1}{2} \int_{\mathbb{R}^n} f^s u r_{\alpha-m_1}(x, D) f^s u \, dx
\]

\[
= \sum_{j=1}^{n_1} \Lambda_{1,j}(t) + \frac{1}{2} \int_{\mathbb{R}^n} f^s u r_{\alpha-m_1}(x, D) f^s u \, dx
\]

A straightforward calculus shows that

\[
(3.16) \quad q_{\alpha-1}(x, D) = -\frac{\alpha}{2} \sum_{|\beta|=1} \partial_x^\beta \partial_x \left( \chi_{x,\tau}^2 \right) f^{\alpha-2} \partial_x^\beta
\]

that after replacing it into (3.13) yield

\[
\Lambda_{1,1}(t) = -\frac{\alpha}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^s u f^{s+\alpha-2} \partial_x^\beta \partial_x \left( \chi_{x,\tau}^2 \right) \, dx
\]

\[
= -\frac{\alpha}{2} \sum_{|\beta|=1} \int_{\mathbb{R}^n} f^{s+\alpha-2} \partial_x^\beta \partial_x \left( \chi_{x,\tau}^2 \right) \, dx
\]

\[
= \frac{\alpha}{4} \sum_{|\beta|=1} \int_{\mathbb{R}^n} \left( f^{s+\alpha-2} \right)^2 \partial_x^\beta \partial_x \left( \chi_{x,\tau}^2 \right) \, dx
\]

\[
= \Lambda_{1,1}(t) + \Lambda_{1,2}(t).
\]

From (3.5) we obtain after fixing properly \( \epsilon \) and \( \tau \) that

\[
\int_0^T |\Lambda_{1,1}(t)| \, dt < \infty.
\]

In the case of \( \Lambda_{1,2} \) we avoid a new commutator decomposition as follows

\[
\Lambda_{1,2}(t) = -\frac{\alpha}{2} \sum_{|\beta|=1} \left\{ \int_{\mathbb{R}^n} f^{s+\alpha-2} (u \chi_{e,\tau}) \left[ f^{s+\alpha-2} \partial_x^\beta \chi_{x,\tau} \right] f^{s+\alpha-2} \partial_x^\beta (u \chi_{e,\tau} + u \psi_{e,\tau}) \, dx
\]

\[
+ \int_{\mathbb{R}^n} f^{s+\alpha-2} (u \psi_{e,\tau}) \left[ f^{s+\alpha-2} \partial_x^\beta \chi_{x,\tau} \right] f^{s+\alpha-2} \partial_x^\beta (u \chi_{e,\tau} + u \psi_{e,\tau}) \, dx
\]

\[
+ \int_{\mathbb{R}^n} f^{s+\alpha-2} (u \psi_{e,\tau}) \left[ f^{s+\alpha-2} \partial_x^\beta \chi_{x,\tau} \right] f^{s+\alpha-2} \partial_x^\beta (u \chi_{e,\tau} + u \psi_{e,\tau}) \, dx
\]

In virtue of Theorem 4.3 and Lemma 5.4 we obtain

\[
|\Lambda_{1,2}(t)| \lesssim \|f^s (u \chi_{e,\tau})\|_{\mathcal{L}_2}^2 + \|f^s (u \psi_{e,\tau})\|_{\mathcal{L}_2}^2 + \|u_0\|_{\mathcal{L}_2}^2,
\]

although

\[
(3.17) \quad f^s (u \chi_{e,\tau}) = \chi_{e,\tau}^s u + [f^s ; \chi_{e,\tau}^s] (u \chi_{e,\tau} + u \psi_{e,\tau}),
\]
and the first term in the r.h.s is the quantity to estimate after applying Gronwall’s inequality. The remainder terms are of order $s - 1$ and these are estimated by using (3.5), Lemma 5.3 and Lemma 5.4 after integrating in time.

For $\Lambda_{1,j}$ with $j > 1$ is easily handled since the regularity required for such terms is less than $s$ and therefore after integrating in time, the inequality (3.5) is the key part. More precisely, if we omit the constants in front and we replace (3.16) yield

$$
\Lambda_{1,j}(t) = \sum_{j=2}^{m_1} \sum_{|\rho| = j} \int_{\mathbb{R}^n} f^j(u\chi_{e,\tau} + u\phi_{i,\tau} + u\psi_e) \phi^j \partial_{\tau}^r \partial_1 (\chi_{e,\tau}^2) \Psi_{\beta_j} f^{a-j+\epsilon} (u\chi_{e,\tau} + u\phi_{i,\tau} + u\psi_e) \, dx.
$$

Thus, by combining Lemma 5.1 and Theorem 4.3 produce

$$
|\Lambda_{1,j}(t) | \lesssim \| f^j(u\chi_{e,\tau}) \|_{L^2_t}^2 + \| f^j(u\phi_{i,\tau}) \|_{L^2_t}^2 + \| u \|_{L^2_t}^2,
$$

for $j = 2, 3, \ldots, m_1$.

At this point we apply (3.17) as we did above to bound $\| f^j(u\chi_{e,\tau}) \|_{L^2_t}$, and for $\| f^j(u\phi_{i,\tau}) \|_{L^2_t}$ it is only required to combine Lemma 5.3 together with (3.5).

In addition,

$$
\frac{1}{2} \int_{\mathbb{R}^n} f^j u \partial_{\alpha} r_{\beta-1}(x, D) f^j u \, dx = \frac{1}{2} \int_{\mathbb{R}^n} u \partial_{\alpha} r_{\beta-1}(x, D) f^j u \, dx
$$

which implies after setting

$$
m_1 = [2s + \alpha - 1 - s_n]
$$

then

$$
\int_0^T \frac{1}{2} \int_{\mathbb{R}^n} u \partial_{\alpha} r_{\beta-1}(x, D) f^j u \, dx \, dt \lesssim T \| u_0 \|_{L^2_t}^2 \| u \|_{L^\infty_t H^s_{\alpha, \beta}} < \infty,
$$

where we have used Theorem 4.3 in the last inequality above.

A quite similar argument applies to $\Lambda_2$ in (3.13), although for the seek of brevity we omit the details.

An idea quite similar to the one used to bound the term $\Lambda_1$ also applies for $\Theta_{2,1,j+2}$ for $j = 2, 3, \ldots, m$ in (3.11). Indeed,

$$
|\Theta_{2,1,j+2}(t)| \lesssim \| f^j(u\chi_{e,\tau}) \|_{L^2_t}^2 + \| f^j(u\phi_{i,\tau}) \|_{L^2_t}^2 + \| u \|_{L^2_t}^2, \quad \text{for} \quad j = 2, 3, \ldots, m.
$$

This term is quite important since it contains part of the smoothing effect we desire to obtain. In the first place, we rewrite the term as

$$
\Theta_{2,1,2}(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \left( f^j \partial_{\alpha} \chi_{e,\tau}^2 \right) \partial_{\tau} \partial_1 (\chi_{e,\tau}^2) \, dx
$$

$$
+ \frac{\alpha}{2} \int_{\mathbb{R}^n} f^j \partial_{\alpha} \chi_{e,\tau}^2 \partial_1 \partial_{\tau} \partial_1 (\chi_{e,\tau}^2) \, dx
$$

$$
= \Lambda_3(t) + \Lambda_4(t).
$$

Notice that $\Lambda_{3,1}$ is the term to be estimated after integrating in time (it contains the desired smoothing).

On the other hand, $\Lambda_4$ does not contains terms that will provide some useful information in our analysis. In fact, to provide upper bounds for $\Lambda_4$ we require apply a decomposition of the commutator expression quite similar to that in (3.13) and following the arguments used to bound $\Lambda_1$. The expression of $\Theta_{2,1,3}$ contains several interactions that make up the smoothing effect.

Nevertheless, we require to decouple such interactions to close the argument.
In this sense, we claim that there exist $\lambda > 0$ such that
\[
\lambda \left( \int_{\mathbb{R}^n} \left( f^{s+\frac{3}{2}} u \right)^2 \bar{\partial}_{x_1} \left( \chi_{e,\tau}^2 \right) dx + \int_{\mathbb{R}^n} \left( f^{s+\frac{3}{2}} \bar{u} \big| \bar{\partial}_{x_1} \left( \chi_{e,\tau}^2 \right) dx \right) \right) \\
\leq \Theta_{2,1,1}(t) + \Lambda_{3,1}(t) + \Theta_{2,1,3}(t),
\]
whenever the condition holds true: $v_1 > 0$ and
\[
0 < v_1 < \frac{2v_1}{C \sqrt{\alpha(n-1)}}, \quad \frac{v_1(1+a)}{\alpha \sqrt{n-1}},
\]
with $\epsilon$ satisfying
\[
0 < \epsilon < \frac{v_1}{|\mathbb{R}|^{1/2} - 1} - \frac{\alpha \sqrt{n-1}|\mathbb{R}|}{4v_1} c^2,
\]
where $|\mathbb{R}| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ and
\[
C := \inf_{f \in L^2(\mathbb{R}^n), f \neq 0} \frac{\| f^{-1/2} \|_2}{\| f \|_2}, \quad j = 2, 3, \ldots, n.
\]

The reader can check that the proof is analogous to the one furnished in Claim 1 in the proof of Lemma 1.1 (see (2.20) for more details). Since
\[
\Theta_{2,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^s(u \chi_{e,\tau} + u \phi_{e,\tau} + u \psi_e) \left[ \chi_{e,\tau}^2 \bar{\partial}_{x_1} \chi_{e,\tau}^2 \right] f^s(u \chi_{e,\tau} + u \phi_{e,\tau} + u \psi_e) dx.
\]
An argument similar to the used to handle (2.25) combined with Lemma 5.1 and corollary 5.1 implies that
\[
|\Theta_{2,2}(t)| \lesssim \left\{ \| f^s(u \chi_{e,\tau}) \|_{L^2_{x_1}}^2 + \| f^s(u \phi_{e,\tau}) \|_{L^2_{x_1}}^2 + \| u_0 \|_{L^2_{x_1}}^2 \right\}.
\]
Notice that
\[
f^s(u \chi_{e,\tau}) = \chi_{e,\tau} f^s u + [f^s; \chi_{e,\tau}] (u \chi_{e,\tau} + u \phi_{e,\tau} + u \psi_e),
\]
the first term in the r.h.s above is the quantity to be estimated after taking the $L^2$-norm and the remainder terms are of order $s-1$. Although, to control $\| f^s(u \phi_{e,\tau}) \|_{L^2_{x_1}}$, we only require to use Lemma 5.3 combined with (3.5), we skip the details, in such a case we obtain
\[
\int_0^T \| f^s(u(\cdot, t) \phi_{e,\tau}(\cdot, t)) \|_{L^2_{x_1}}^2 dt < c,
\]
for some positive constant $c$.

We decompose the nonlinear term as follows
\[
\Theta_3(t) = -\int_{\mathbb{R}^n} \chi_{e,\tau} f^s u [f^s; \chi_{e,\tau}] u \bar{\partial}_{x_1} u dx + \int_{\mathbb{R}^n} \chi_{e,\tau} f^s u [f^s; u \chi_{e,\tau}] \bar{\partial}_{x_1} u dx \\
- \frac{1}{2} \int_{\mathbb{R}^n} \left( f^s u \right)^2 \chi_{e,\tau}^2 dx - v_1 \int_{\mathbb{R}^n} u \left( f^s u \right)^2 \chi_{e,\tau}^2 \bar{\partial}_{x_1} dx
\]
\[
= \Theta_{3,1}(t) + \Theta_{3,2}(t) + \Theta_{3,3}(t) + \Theta_{3,4}(t).
\]
The term $\Theta_{3,1}$ requires to describe the commutator $[f^s; \chi_{e,\tau}]$. Although, to obtain such expression have been a reiterative argument in this paper and for the sake of brevity we will show the crucial steps. Thus, if we set
\[
b_{3-1}(x, D) := [f^s; \chi_{e,\tau}],
\]
then $b_{s-1} \in \text{OPS}^{s-1}$, and its principal symbol admits the following decomposition

$$b_{s-1}(x, \xi) = \sum_{1 \leq |\alpha| \leq l} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \left\{ \hat{c}_\alpha^{\pm} (\langle \xi \rangle^s) \hat{c}_\alpha^{\pm} (\chi_{e, \tau}(x, \xi)) \right\} + \kappa_{s-1}(x, \xi)$$

$$= \sum_{j=1}^{l} \sum_{|\alpha| = j} \frac{(2\pi i)^{|\alpha|}}{\alpha!} \left\{ \hat{c}_\alpha^{\pm} (\langle \xi \rangle^s) \hat{c}_\alpha^{\pm} (\chi_{e, \tau}(x, \xi)) \right\} + \kappa_{s-1}(x, \xi)$$

where we choose $l > |s - 2 - \frac{n}{2}|$.

Additionally, for multi-index $\alpha, \beta$ with $\beta \leq \alpha$ we define

$$\eta_{\alpha, \beta}(x, \xi) := \frac{(2\pi i \xi)^\beta}{(1 + |\xi|^2)^{\frac{n}{2}}}, \quad x, \xi \in \mathbb{R}^n.$$ 

Thus, $\eta_{\alpha, \beta} \in S^{\beta-|\alpha|} \subset S^0$, and

$$(3.22) \quad \Psi_{\eta_{\alpha, \beta}} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \eta_{\alpha, \beta}(x, \xi) \hat{g} (\xi) \, d\xi, \quad g \in \mathcal{S}(\mathbb{R}^n),$$

with this at hand we rearrange the terms in the decomposition of the symbol $q_{s-1}$ to obtain

$$\Psi_{\eta_{s-1}} f(x) = \sum_{j=1}^{l} \sum_{|\alpha| = j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, v, s} \hat{c}_\alpha^{\pm} \chi_{e, \tau}(x, \xi) \Psi_{\eta_{\alpha, \beta}, f}^{s-|\alpha|} \hat{c}_\alpha^{\pm} \left( (u_\xi e, \tau)^2 \right) \Psi_{\chi_{e, \tau}} f^{s-|\alpha|} \hat{c}_\alpha^{\pm} \left( (u_\xi e, \tau)^2 \right)$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $\omega_{\alpha, \beta, v, s}$ denotes a constant depending on the parameters indicated.

Now, we turn back to $\Theta_{3,1}$ from where we get after combining Theorem 4.3, Lemma 5.1 and Theorem 5.4

$$|\Theta_{3,1}(t)| \lesssim \|\chi_{e, \tau} f^s u\|_{L^2_t} \sum_{j=1}^{l} \sum_{|\alpha| = j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, v, s} \left\| \hat{c}_\alpha^{\pm} \chi_{e, \tau} \Psi_{\eta_{\alpha, \beta}, f}^{s-|\alpha|} \hat{c}_\alpha^{\pm} \left( (u_\xi e, \tau)^2 \right) \right\|_{L^2_t}$$

$$+ \|\chi_{e, \tau} f^s u\|_{L^2_t} \sum_{j=1}^{l} \sum_{|\alpha| = j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, v, s} \left\| \hat{c}_\alpha^{\pm} \chi_{e, \tau} \Psi_{\eta_{\alpha, \beta}, f}^{s-|\alpha|} \hat{c}_\alpha^{\pm} \left( (u_\xi e, \tau)^2 \right) \right\|_{L^2_t}$$

$$+ \|\chi_{e, \tau} f^s u\|_{L^2_t} \sum_{j=1}^{l} \sum_{|\alpha| = j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, v, s} \left\| \hat{c}_\alpha^{\pm} \chi_{e, \tau} \Psi_{\eta_{\alpha, \beta}, f}^{s-|\alpha|} \hat{c}_\alpha^{\pm} \left( (u_\xi e, \tau)^2 \right) \right\|_{L^2_t}$$

$$\lesssim \|\chi_{e, \tau} f^s u\|_{L^2_t} \left\{ \|f^s (u_\xi e, \tau)^2\|_{L^2_t} + \|f^s (u_\xi e, \tau)^2\|_{L^2_t} + \|u_\xi e, \tau^2\|_{L^2_t} \right\}$$

$$\lesssim \|u\|_{L^\infty} \|\chi_{e, \tau} f^s u\|_{L^2_t} \left\{ \|f^s (u_\xi e, \tau)^2\|_{L^2_t} + \|f^s (u_\xi e, \tau)^2\|_{L^2_t} + \|u_\xi e, \tau^2\|_{L^2_t} \right\},$$

from this last inequality we have that $\|\chi_{e, \tau} f^s u\|_{L^2_t}$ is the quantity to be estimated and if additionally we make use of (3.21) it follows that

$$(3.23) \quad f^s (u_\xi e, \tau) = \chi_{e, \tau} f^s u + q_{s-1}(x, D)(u_\xi e, \tau + u_\xi e, \tau + u_\xi e),$$

and notice that from the decomposition above the first term in the r.h.s above is the quantity to be estimated by Gronwall’s inequality, and the remainder terms are of lower order and localized.
To handle the expression \( \| F^s (u \phi_{\epsilon, \tau}) \|_{L^2_{\tau}} \) we only require to consider a suitable cut-off function and combine Lemma 5.3 together with (3.5) to show that

\[
(3.24) \quad \int_0^T \| F^s (u(\cdot, t) \phi_{\epsilon, \tau}(\cdot, t)) \|_{L^2_{\tau}}^2 dt < \infty.
\]

a quite similar analysis is used to show that

\[
(3.25) \quad \int_0^T \| F^s (u(\cdot, t) \phi_{\epsilon, \tau}(\cdot, t)) \|_{L^2_{\tau}}^2 dt < \infty.
\]

Since

\[
\Theta_{3,2}(t) = \int_{\mathbb{R}^n} \chi_{\epsilon, \tau} F^s u [F^s; u \chi_{\epsilon, \tau}] \partial_x u dx
\]

\[
= \int_{\mathbb{R}^n} \chi_{\epsilon, \tau} F^s u [F^s; u \chi_{\epsilon, \tau}] \partial_x (u \chi_{\epsilon, \tau} + u \phi_{\epsilon, \tau} + u \psi_{\epsilon}) dx
\]

\[
= \Theta_{3,2,1}(t) + \Theta_{3,2,2}(t) + \Theta_{3,2,3}(t).
\]

The arguments to bound \( \Theta_{3,2,1} \) and \( \Theta_{3,2,2} \) are quite similar and are obtained by using Theorem 5.3

\[
| \Theta_{3,2,1}(t) | \lesssim \| \chi_{\epsilon, \tau} F^s u \|_{L^2_{\tau}} \| J^s (u \chi_{\epsilon, \tau}) \|_{L^2_{\tau}} \| \nabla u \|_{L^\infty_{\tau}}
\]

and

\[
| \Theta_{3,2,2}(t) | \lesssim \| \nabla u \|_{L^\infty_{\tau}} \left( \| \chi_{\epsilon, \tau} F^s u \|_{L^2_{\tau}}^2 + \| J^s (u \phi_{\epsilon, \tau}) \|_{L^2_{\tau}}^2 + \| J^s (u \chi_{\epsilon, \tau}) \|_{L^2_{\tau}}^2 \right).
\]

The arguments used control the terms \( \| J^s (u \chi_{\epsilon, \tau}) \|_{L^2_{\tau}}^2 \) and \( \| J^s (u \phi_{\epsilon, \tau}) \|_{L^2_{\tau}}^2 \) were already described in (3.23) resp. (3.25).

For \( \Theta_{3,2,3} \), Lemma 5.1 yields

\[
| \Theta_{3,2,3}(t) | \lesssim \| \chi_{\epsilon, \tau} F^s u \|_{L^2_{\tau}} \| u \|_{L^2_{\tau}} \| u_0 \|_{L^2_{\tau}}.
\]

Notice that \( \Theta_{3,3} \) is the term to estimate by Gronwall’s inequality and \( \Theta_{3,4} \) can be bounded above by using Sobolev embedding and (3.5).

Finally, we gather the information corresponding to this step after applying Gronwall’s inequality combined with integration in the time variable imply that for any \( \epsilon > 0 \) and \( \tau \geq 5\epsilon, \)

\[
\sup_{0 < t < T} \int_{\mathbb{R}^n} (F^s u(x, t))^2 \chi_{\epsilon, \tau}^2 (v \cdot x + \omega t) dx
\]

\[
+ \int_0^T \int_{\mathbb{R}^n} (F^{s+\frac{1}{2}} u(x, t))^2 \chi_{\epsilon, \tau}^2 (s + \omega t) dx dt \leq c.
\]

This estimate finish the step 1.

**Step 2:**

In this case we consider \( s \in (s_n, s_n + \frac{\alpha}{2}) \). Thus,

\[
s_n + 1 - \frac{\alpha}{2} < s + 1 - \frac{\alpha}{2} < s_n + 1.
\]
As we did in step 1 we perform energy estimates, the main difference is that now are at level $s + \frac{2-a}{2}$. More precisely,

\begin{align}
\frac{d}{dt} \int_{\mathbb{R}^n} \left( J^{s+\frac{2-a}{2}} u \right)^2 \chi_{\epsilon,\tau}' \, dx - \frac{\omega}{2} \int_{\mathbb{R}^n} \left( J^{s+\frac{2-a}{2}} u \right)^2 \chi_{\epsilon,\tau} \chi_{\epsilon,\tau}' \, dx \\
- \int_{\mathbb{R}^n} J^{s+\frac{2-a}{2}} u \partial_{x_1} (-\Delta)^{\frac{s}{2}} J^{s+\frac{2-a}{2}} u \chi_{\epsilon,\tau} \, dx + \int_{\mathbb{R}^n} J^{s+\frac{2-a}{2}} (u \partial_{x_1}, u) \chi_{\epsilon,\tau} \, dx = 0.
\end{align}

In this step we will focus only on the more difficult terms to handle, these correspond to $\Theta_1$ and $\Theta_2$; the term $\Theta_3$ can be estimated by using similar arguments to the ones described in the previous step. Nevertheless, for the reader’s convenience we will indicate the steps that are needed to provided the desired bounds. An argument similar to the one used in (3.3)-(3.5) applied to the second term in (3.26) produce: for $\varepsilon > 0$ and $\tau \geq 5\varepsilon$,

\begin{align}
\int_0^T \int_{\mathcal{H}_{(t-\varepsilon,\tau)} \cap \mathcal{H}_{(t,\tau)}} (J' u(x,t))^2 \, dx \, dt \lesssim c_r,
\end{align}

for some positive constant $c_r$, and $r \in (0, s + \frac{\alpha}{2}]$.

We need to take into account that

$$s + \frac{\alpha}{2} \geq s + \frac{2-a}{2},$$

so that, the regularity in the previous step is enough to control the terms with localized regularity.

Thus, in virtue of (3.28) we obtain for $\varepsilon > 0$ and $\tau \geq 5\varepsilon$;

\begin{align}
\int_0^T |\Theta_1(t)| \, dt \lesssim \int_{\mathcal{H}_{(\varepsilon,\tau)} \cap \mathcal{H}_{(t,\tau)}} (f^{s+\frac{2-a}{2}} u(x,t))^2 \, dx \, dt \lesssim c_{s,a}.
\end{align}

A procedure similar to the one used for $\Theta_2$ in the step 1 implies that

\begin{align}
\Theta_2(t) = \frac{1}{2} \int_{\mathbb{R}^n} J^{s+\frac{2-a}{2}} u \left[ J^{s+\frac{2-a}{2}} u \right]^2 \chi_{\epsilon,\tau}' \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^n} J^{s+\frac{2-a}{2}} u \left[ J^{s+\frac{2-a}{2}} u \right]^2 \chi_{\epsilon,\tau}' \, dx
\end{align}

\begin{align}
= \Theta_{2,1}(t) + \Theta_{2,2}(t).
\end{align}

We shall remind from (3.7) that there exist pseudo-differential operators $p_{a-j}(x,D)$, where $j \in \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$ satisfying

\begin{align}
c_{a}(x,D) = p_{a}(x,D) + p_{a-1}(x,D) + \cdots + p_{a-m}(x,D) + r_{a-m-1}(x,D),
\end{align}

where $p_{a-j} \in \text{OPS}^{a-j}$ and $r_{a-m-1} \in \text{OPS}^{a-1-m}$.

We choose $m$ as being

\begin{align}
m = \lfloor 2s + 1 - s_n \rfloor.
\end{align}
Thus,
\[
\Theta_{2,1}(t) = \frac{1}{2} \sum_{j=0}^{m} \int_{\mathbb{R}^n} f^{s+\frac{2s}{a}} u p_{a-j}(x, D) f^{s+\frac{2s}{a}} u \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\frac{2s}{a}} u r_{a-1-m}(x, D) f^{s+\frac{2s}{a}} u \, dx
\]

(3.33)
\[
= \frac{1}{2} \sum_{j=0}^{m} \int_{\mathbb{R}^n} f^{s+\frac{2s}{a}} u p_{a-j}(x, D) f^{s+\frac{2s}{a}} u \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^n} u f^{s+\frac{2s}{a}} r_{a-1-m}(x, D) f^{s+\frac{2s}{a}} u \, dx
\]
\[
= \sum_{j=0}^{m} \Theta_{2,1,j}(t) + \Theta_{2,1,m+1}(t).
\]

We shall remind from (3.10) that for each \( j \in \{1, 2, \ldots, m\} \),
\[
p_{a-j}(x, D) = \sum_{|\beta|=j} c_{\beta,j} \Phi_{\beta,j}(\chi^2_h) \Psi_{\beta,j} f^{a-j},
\]
with \( \Psi_{\beta,j} \in \text{OPS}^0 \).

The decomposition above allow us to estimate \( \Theta_{2,1,j} \) for all \( j > 1 \). Indeed,
\[
\int_{0}^{T} |\Theta_{2,1,j}(t)| \, dt \lesssim \sum_{|\beta|=j} |c_{\beta,j}| \int_{0}^{T} \int_{\mathbb{R}^n} \left| f^{s+\frac{2s}{a}} u f^{s+\frac{2s}{a}} \chi_{\epsilon,\tau} \Psi_{\beta,j} f^{s+\frac{2s}{a}} u \right| \, dx \, dt
\]
\[
\lesssim \sum_{|\beta|=j} |c_{\beta,j}| \int_{0}^{T} \int_{\mathbb{R}^n} \left| f^{s+\frac{2s}{a}} u f^{s+\frac{2s}{a}} \chi_{\epsilon,\tau} \Psi_{\beta,j} f^{s+\frac{2s}{a}} u \right| \, dx \, dt
\]
\[
\lesssim \int_{0}^{T} \left\| \theta_1 f^{s+\frac{2s}{a}} u \right\|_{L^2_x}^2 \, dt + \sum_{|\beta|=j} |c_{\beta,j}| \int_{0}^{T} \left\| \theta_1 \Psi_{\beta,j} f^{s+\frac{2s}{a}} u \right\|_{L^2_x}^2 \, dt,
\]
where \( \theta_1 \in C^\infty(\mathbb{R}^n) \), such that for all \( \beta \) multi-index with \( |\beta|=j \), the following relationship holds
\[
\theta_1 \equiv 1 \quad \text{on \ supp}_x \left( \chi^2_h(\cdot, \cdot, t) \right)
\]
and
\[
\text{supp}_x \theta_1 \subset H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}} \cap H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}} \quad \forall \, t > 0,
\]
whenever \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \).

Then,
\[
\int_{0}^{T} \left\| \theta_1 f^{s+\frac{2s}{a}} u \right\|_{L^2_x}^2 \, dt \lesssim \int_{0}^{T} \int_{H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}} \cap H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}}} \left( f^{s+\frac{2s}{a}} u \right)^2 \, dx \, dt < c,
\]
after choosing \((\epsilon, \tau) = (\epsilon', \tau')\) properly in (3.29).

We consider \( \theta_2 \in C^\infty(\mathbb{R}^n) \) satisfying
\[
\theta_2 \equiv 1 \quad \text{on \ supp} \theta_1
\]
and
\[
\text{supp} \theta_2 \subset H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}} \cap H_{r+\frac{5s}{16}}^{t+\frac{5s}{16}} \quad \forall \, t > 0,
\]
whenever \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \).
Thus, combining Lemma 5.4 and (3.28) implies that

$$\sum_{|\beta| = j} |c_{\beta,j}| \int_0^T \left\| \theta_1 \Psi_{\beta,j} f^{s + 1 - j + \frac{\alpha}{2}} u \right\|_{L_x^2}^2 \, dt < c,$$

for all \( j \geq 2 \).

From (3.32) we obtain

$$\int_0^T |\theta_0(t)| \, dt \lesssim T \|u_0\|_{L_x^2} \|u\|_{L^\infty_T H^{\frac{\alpha}{2}}_x}.$$  

It remains to estimate \( \Theta_{2,1,1} \) that did not fall into the scope of the previous analysis. Although, when we replace (3.34) in (3.33) we find

$$\Theta_{2,1,1}(t) = \frac{1}{2} \sum_{|\beta| = 1} c_{\beta,1} \int_{\mathbb{R}^n} f^{s + \frac{\alpha}{2}} u \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \, dx$$

(3.37)

$$= \frac{1}{2} \sum_{|\beta| = 1} c_{\beta,1} \int_{\mathbb{R}^n} f^{s + \frac{\alpha}{2}} u \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \, dx$$

$$+ \frac{1}{2} \sum_{|\beta| = 1} c_{\beta,1} \int_{\mathbb{R}^n} f^{s + \frac{\alpha}{2}} u \left[ f^{s + \frac{\alpha}{2}} ; \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \right] \, dx$$

$$= \Theta_{2,1,1,1}(t) + \Theta_{2,1,1,2}(t).$$

Since \( s + \frac{\alpha}{2} \leq s + \frac{\alpha}{2} \) for \( \alpha \in [1, 2) \) we get up to constants

$$\int_0^T |\Theta_{2,1,1,1}(t)| \, dt \lesssim \int_0^T \int_{\mathbb{R}^n} \left( f^{s + \frac{\alpha}{2}} u \right)^2 \chi_{\frac{\alpha}{2}, \tau} \, dx \, dt$$

(3.38)

$$+ \sum_{|\beta| = 1} \int_0^T \int_{\mathbb{R}^n} \left( \Psi_{\beta,1} f^{s + \frac{\alpha}{2}} u \right)^2 \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \, dx \, dt < \infty,$$

where we have used (3.28) in the first term in the r.h.s above and for the second one we combine Lemma 5.4 together with (3.28).

Instead for \( \Theta_{2,1,1,2} \) we decompose the commutator expression as

$$\Theta_{2,1,1,2}(t) = \frac{1}{2} \sum_{|\beta| = 1} c_{\beta,1} \int_{\mathbb{R}^n} f^{s} u \left[ f^{s} ; \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \right] \, dx$$

(3.39)

$$- \frac{1}{2} \sum_{|\beta| = 1} c_{\beta,1} \int_{\mathbb{R}^n} f^{s} u \left[ f^{s} ; \bar{\phi}_\beta^2 (x; \epsilon) \chi_{\frac{\alpha}{2}, \tau} \right] \, dx.$$

After use a decomposition as the one in (3.14) (replacing \( \frac{\alpha}{2} \) in (3.14)) combined with Theorem 4.3 and inequality (3.28) produce

$$\int_0^T |\Theta_{2,1,1,2}(t)| \, dt < \infty.$$  

(3.40)
Next, we focus our attention into $\Theta_{2,1,0}$. Indeed, from (3.9) we have the explicit expression for $\rho_n(x,D)$, so that after replacing it in (3.33) yield

$$\Theta_{2,1,0}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2n}} u f^{s-\frac{\alpha}{2n}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) dx - \frac{\alpha}{2} \int_{\mathbb{R}^n} f^{s+\frac{2\alpha}{n}} u f^{s+\frac{\alpha}{n}} \hat{\varphi}_x^{2} u \hat{\varphi}_x \left( \frac{2}{\alpha} \right) dx$$

$$- \frac{\alpha}{2} \sum_{|\beta|=1, \beta \neq e_1} \int_{\mathbb{R}^n} f^{s+\frac{2\alpha}{n}} u f^{s+\frac{\alpha}{n}} \hat{\varphi}_x^{\beta} u \hat{\varphi}_x^{\beta} \left( \frac{2}{\alpha} \right) dx$$

$$= \Theta_{2,1,0,1}(t) + \Theta_{2,1,0,2}(t) + \Theta_{2,1,0,3}(t).$$

In the first place

$$\Theta_{2,1,0,1}(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left( f^{s+1} u \right)^2 \hat{\varphi}_x \left( \frac{2}{\alpha} \right) dx + \frac{1}{2} \int_{\mathbb{R}^n} f^{s+1} u \left[ f^{\frac{s+1}{2}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+\frac{2\alpha}{n}} u dx$$

$$= \Theta_{2,1,0,1,1}(t) + \Theta_{2,1,0,1,2}(t).$$

The term $\Theta_{2,1,0,1,1}(t)$ represents after integrating in time the pursuit smoothing effect. For $\Theta_{2,1,0,1,2}$ we rewrite it as

$$\Theta_{2,1,0,1,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^s u \left[ f^{\frac{s+1}{2}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+1} u dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^n} f^s u \left[ f \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+1} u dx$$

$$= \Lambda_1(t) + \Lambda_2(t).$$

To handle $\Lambda_1$ we use the expression (3.14) (after replacing $\frac{a}{n}$ by $\frac{a}{n} + 1$), the main difference relies in the fact that we shall we shall fix the positive integer $m_1$ as being

$$m_1 = \left[ 2s + \frac{3\alpha}{2} - 2s_n \right].$$

Notice that from such decomposition we obtain terms of lower order which are controlled by using (3.26).

$$\Theta_{2,1,0,2}(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \left( \hat{\varphi}_x f^s u \right)^2 \hat{\varphi}_x \left( \frac{2}{\alpha} \right) dx$$

$$+ \frac{\alpha}{2} \int_{\mathbb{R}^n} \hat{\varphi}_x f^s u \left[ f^{\frac{s+1}{2}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+\frac{2\alpha}{n}} u dx$$

$$= \Theta_{2,1,0,2,1}(t) + \Theta_{2,1,0,2,2}(t).$$

The term $\Theta_{2,1,0,2,1}$ represents the smoothing effect after integrating in the temporal variable. Instead, the remainder does not contain terms that yield some smoothing, thus we require to estimate it. In this sense, we rewrite it as

$$\Theta_{2,1,0,2,2}(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \hat{\varphi}_x f^s u \left[ f^{\frac{s+1}{2}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+\frac{2\alpha}{n}} u dx$$

$$- \frac{\alpha}{2} \int_{\mathbb{R}^n} f^s u \left[ f^{\frac{2s+\alpha}{n}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+\frac{2\alpha}{n}} u dx$$

$$- \frac{\alpha}{2} \int_{\mathbb{R}^n} f^s u \left[ f^{\frac{s+1}{2}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] \hat{\varphi}_x f^{s+\frac{2\alpha}{n}} u dx$$

$$= \Lambda_3(t) + \Lambda_4(t).$$

We indicate how to estimate $\Lambda_3$. We start by writing

$$\Lambda_3 = \frac{\alpha}{2} \int_{\mathbb{R}^n} f^s (u \chi_{\epsilon,T} + u \phi_{\epsilon,T} + u \psi_{\epsilon}) \left[ f^{\frac{2s+\alpha}{n}} \hat{\varphi}_x \left( \frac{2}{\alpha} \right) \right] f^{s+\frac{2\alpha}{n}} (u \chi_{\epsilon,T} + u \phi_{\epsilon,T} + u \psi_{\epsilon}) dx$$
Hence by Theorem 4.3 and Lemma 5.1 we obtain
\[ |\Omega_{2,1,0,2,2}(t)| \lesssim \|F^s(u\chi_{\epsilon,\tau})\|_{L^2}^2 + \|F^s(u\phi_{\epsilon,\tau})\|_{L^2}^2 + \|u_0\|_{L^2}^2. \]

Unlike the previous step to estimate \( \|F^s(u\chi_{\epsilon,\tau})\|_{L^2} \) we only require to apply Lemma 5.5 and (3.26) we get
\[ \sup_{t \in (0,T)} \|F^s(u\chi_{\epsilon,\tau})\|_{L^2}^2 < \infty. \]

Instead, to estimate \( \|F^s(u\phi_{\epsilon,\tau})\|_{L^2} \) we combine (3.26) and the arguments described in (3.24)-(3.25) to obtain
\[ \|F^s(u\phi_{\epsilon,\tau})\|_{L^2} < \infty. \]

In summary,
\[ \int_0^T |\Lambda_3(t)| \, dt < \infty. \]

The arguments of the proof are similar to the ones described in the proof of (2.20). To handle the nonlinear part we use the decomposition
\[ \Theta_3(t) = -\int_{\mathbb{R}^n} \chi_{\epsilon,\tau} f^{s+2} \partial_{x_1} \left( \lambda_{\epsilon,\tau}^2 \right) u \partial_{x_1} u \, dx + \int_{\mathbb{R}^n} \chi_{\epsilon,\tau} f^{s+1} \partial_{x_1} \left( \lambda_{\epsilon,\tau}^2 \right) u \partial_{x_1} u \, dx \]
\[ + \int_{\mathbb{R}^n} \chi_{\epsilon,\tau} f^{s+2} \partial_{x_1} \left( \lambda_{\epsilon,\tau}^2 \right) u \partial_{x_1} u \, dx - \frac{1}{2} \int_{\mathbb{R}^n} \left( f^{s+2} \partial_{x_1} u \right)^2 \chi_{\epsilon,\tau}^2 \partial_{x_1} \chi_{\epsilon,\tau} \, dx - v_1 \int_{\mathbb{R}^n} u \left( f^{s+2} \partial_{x_1} u \right)^2 \chi_{\epsilon,\tau} \partial_{x_1} \lambda_{\epsilon,\tau} \, dx. \]

The reader can notice that this decomposition is just adapted to the regularity of the step. Although, the process required to estimate the terms above is similar as the ones used in the \( \Theta_3 \) in the previous step, so for the sake of brevity we will omit the details.

In summary, we find after applying Gronwall’s inequality and integrating in time the following: for any \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \),
\[ \sup_{0 < t < T} \int_{\mathbb{R}^n} \left( f^{s+2} \partial_{x_1} u(x, t) \right)^2 \chi_{\epsilon,\tau}^2 (v \cdot x + \omega t) \, dx \]
\[ + \int_0^T \int_{\mathbb{R}^n} (f^{s+1} u(x, t))^2 (\chi_{\epsilon,\tau} \lambda_{\epsilon,\tau}') (v \cdot x + \omega t) \, dx \, dt \leq c. \]
We shall remind that there exist pseudo-differential operators $\alpha$ that for any $\epsilon > 0$ and $\tau \geq 5\epsilon$ the following estimate holds true:

$$
\sup_{0 < t < T} \int_{\mathbb{R}^n} \left( f^s u(x, t) \right)^2 \chi_{\epsilon, \tau}^2 (v \cdot x + \omega t) \, dx
$$

$$
+ \int_0^T \int_{\mathbb{R}^n} \left( f^{s+\frac{\alpha}{2}} u(x, t) \right)^2 \left( \chi_{\epsilon, \tau} \chi_{\epsilon, \tau}' \right) (v \cdot x + \omega t) \, dx \, dt \leq c.
$$

(3.42)

As usual our starting point is the energy estimate (3.27).

As we did in the previous case we only describe the part that will provide the smoothing effect since the remainder terms can be estimated in a standard way with arguments described previously.

First, we rewrite $\Theta_2$ from (3.27) as follows

$$
\Theta_2(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2}} u \left[ f^a \partial_{x_1} \chi_{\epsilon, \tau}^2 \right] f^{s+\frac{\alpha}{2}} u \, dx
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2}} u \left[ K_a \partial_{x_1} \chi_{\epsilon, \tau}^2 \right] f^{s+\frac{\alpha}{2}} u \, dx
$$

(3.43)

$$
= \Theta_{2,1}(t) + \Theta_{2,2}(t).
$$

We shall remind that there exist pseudo-differential operators $p_{a-j}(x, D)$ for each $j \in \{1, 2, \ldots, m\}$ and some $m \in \mathbb{N}$ satisfying

$$
c_a(x, D) = p_a(x, D) + p_{a-1}(x, D) + \cdots + p_{a-m}(x, D) + r_{a-m-1}(x, D),
$$

where $p_{a-j} \in \text{OPS}^{a-j}$ and $r_{a-m-1} \in \text{OPS}^{a-1-m}$.

We choose $m$ as being

$$
m = \lfloor 2s + 1 - s_a \rfloor.
$$

Thus,

$$
\Theta_{2,1}(t) = \frac{1}{2} \sum_{j=0}^m \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2}} u p_{a-j}(x, D) f^{s+\frac{\alpha}{2}} u \, dx
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2}} u r_{a-1-m}(x, D) f^{s+\frac{\alpha}{2}} u \, dx
$$

$$
= \frac{1}{2} \sum_{j=0}^m \int_{\mathbb{R}^n} f^{s+\frac{\alpha}{2}} u p_{a-j}(x, D) f^{s+\frac{\alpha}{2}} u \, dx
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^n} u f^{s+\frac{\alpha}{2}} r_{a-1-m}(x, D) f^{s+\frac{\alpha}{2}} u \, dx
$$

$$
= \sum_{j=0}^m \Theta_{2,1,j}(t) + \Theta_{2,1,m+1}(t).
$$

where

$$
p_{a-j}(x, D) = \sum_{|\beta|=j} c_{\beta,j} \chi_a^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau} \chi_{\epsilon, \tau}) \Psi_{\beta,j} f^{a-j},
$$

for each $j \in \{1, 2, \ldots, m\}$, with $\Psi_{\beta,j} \in \text{OPS}^0$.

The key part to control the terms in the expression above relies into combine assumption (3.46) and apply Lemma 5.5 properly to obtain that $\epsilon > 0$ and $\tau \geq 5\epsilon$, the following estimates holds true:

$$
\sup_{0 < t < T} \int_{\mathbb{R}^n} \left( f^{s+\frac{\alpha}{2}} u(x, t) \right)^2 \chi_{\epsilon, \tau}^2 (v \cdot x + \omega t) \, dx \leq c
$$

(3.44)
for any \( r_1 \in (0, s] \), and

\[
(3.45) \quad \int_0^T \int_{\mathbb{R}^n} (f^2 u(x, t))^2 (\chi_{\epsilon, T} \chi'_{\epsilon, T}) (v \cdot x + \omega t) \, dx \, dt \leq c.
\]

for all \( r_2 \in (0, s + \frac{\alpha}{2}] \).

The term \( \Theta_{2,1,j} \) can be estimated combining (3.44)-(3.45) together the arguments already described in (3.35)-(3.36).

Next, the distinguished term in our decomposition is \( \Theta_{2,1,0} \) and it is given by

\[
\Theta_{2,1,0}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^{s+1} u \frac{\partial u}{\partial t} \hat{c}_{x_1} \left( \lambda^2 \right) \, dx + \frac{1}{2} \int_{\mathbb{R}^n} f^{s+1} u \frac{\partial u}{\partial t} \hat{c}_{x_1} \left( \lambda^2 \right) \, dx
\]

\[
= \Theta_{2,1,0,1}(t) + \Theta_{2,1,0,2}(t) + \Theta_{2,1,0,3}(t).
\]

In the first place

\[
\Theta_{2,1,0,1}(t) = \frac{1}{2} \int_{\mathbb{R}^n} (f^{s+1} u) \hat{c}_{x_1} \left( \lambda^2 \right) \, dx + \frac{1}{2} \int_{\mathbb{R}^n} f^{s+1} u \left[ \frac{\partial}{\partial t} ; \hat{c}_{x_1} \left( \lambda^2 \right) \right] f^{s+1} u \, dx
\]

The term \( \Theta_{2,1,0,1} \) represents after integrating in time the pursuit smoothing effect.

For \( \Theta_{2,1,0,1,2} \) we rewrite it as

\[
\Theta_{2,1,0,1,2}(t) = \frac{1}{2} \int_{\mathbb{R}^n} f^t u \left[ \frac{\partial}{\partial t} ; \hat{c}_{x_1} \left( \lambda^2 \right) \right] f^{s+1} u \, dx
\]

To handle \( \Lambda_1 \) we split the commutator expression as in (3.14) (after replacing \( \frac{\alpha}{2} \) by \( \frac{\alpha}{2} + 1 \), but fixing \( m_1 \in \mathbb{N} \) as being

\[
m_1 = \left[ 2s + 3 \frac{\alpha}{2} - 2 - s_n \right].
\]

Notice that from such decomposition we obtain terms of lower order which are controlled by using (3.44)-(3.45).

\[
\Theta_{2,1,0,2}(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \hat{c}_{x_1} f^t u \hat{c}_{x_1} \left( \lambda^2 \right) \, dx
\]

\[
= \Theta_{2,1,0,2,1}(t) + \Theta_{2,1,0,2,2}(t).
\]
Next, we rewrite
\[
\Theta_{2,1,0,2,2}(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \partial_\alpha \int \left[ \frac{2}{\alpha} \partial_\alpha \left( \lambda^2 \right) \right] u \, dx
\]
\[
= -\frac{\alpha}{2} \int_{\mathbb{R}^n} \int \left[ \frac{2}{\alpha} \partial_\alpha \left( \lambda^2 \right) \right] u \, dx
\]
\[
- \frac{\alpha}{2} \int_{\mathbb{R}^n} \int \left[ \frac{2}{\alpha} \partial_\alpha \left( \lambda^2 \right) \right] \partial_\alpha u \, dx
\]
\[
= \Lambda_3(t) + \Lambda_4(t).
\]

As we did previously we only indicate how to estimate \( \Lambda_3 \) since for \( \Lambda_4 \) the situation is analogous. We start by writing
\[
\Lambda_3(t) = \frac{\alpha}{2} \int_{\mathbb{R}^n} \int \left[ \frac{2}{\alpha} \partial_\alpha \left( \lambda^2 \right) \right] u \, dx
\]
\[
\text{Hence by Theorem 4.3 and Lemma 5.1 we obtain}
\]
\[
|\Theta_{2,1,0,2,2}(t)| \lesssim \|f^\alpha(u \chi_{e,\tau})\|_1^2 + \|f^\alpha(u \phi_{e,\tau})\|_1^2 + \|u_0\|_2^2.
\]

Unlike the previous step to estimate \( \|f^\alpha(u \chi_{e,\tau})\|_1 \) we only require to apply Lemma 5.5 and (3.26) we get
\[
\sup_{t \in [0,T]} \|f^\alpha(u \chi_{e,\tau})\|_1^2 < \infty.
\]

Instead, to estimate \( \|f^\alpha(u \phi_{e,\tau})\|_1 \) we combine 3.26 and the arguments described in (3.24)-(3.25) to obtain
\[
\|f^\alpha(u \phi_{e,\tau})\|_1 \lesssim \infty.
\]

In summary,
\[
\int_0^T |\Lambda_3(t)| \, dt < \infty.
\]

The same arguments apply to estimate \( \Lambda_4 \). Indeed,
\[
\int_0^T |\Lambda_4(t)| \, dt < \infty.
\]

In the case of \( \Theta_{2,1,1} \) the arguments described in (3.37)-(3.39) implies together with (3.44) and (3.45) that
\[
\int_0^T |\Theta_{2,1,1}(t)| \, dt < \infty
\]

for some positive constant \( c \).

Hence, we focus our attention on \( \Theta_{2,1,m+1} \), the error term containing \( r_m - x_{m-1}(x,D) \) satisfies in virtue of Theorem 4.3
\[
\int_0^T |\Theta_{2,1,m+1}(t)| \, dt \lesssim T \|u_0\|_2 \|u\|_L^\infty H_t^m + \cdot.
\]

Under the conditions (3.18)- (3.19) we there exist \( \lambda > 0 \), such that
\[
\lambda \left( \int_{\mathbb{R}^n} \left( f^{s+1} u \right)^2 \partial_\alpha \left( \lambda^2 \right) \, dx + \int_{\mathbb{R}^n} \left( f^{s-2} \partial_\alpha u \right)^2 \partial_\alpha \left( \lambda^2 \right) \, dx \right)
\]
\[
\leq \Theta_{2,1,1}(t) + \Lambda_3(t) + \Theta_{2,1,0,3}(t).
\]

The analysis of this term remains unchained from the one in the previous steps.
Finally, we gather the estimates above from where we find that for any $e > 0$ and $\tau \geq 5e,$

$$\sup_{0 < t < T} \int_{\mathbb{R}^n} \left( f^{3+2\frac{m}{2e}} u(x, t) \right)^2 \chi_{\mathbb{R}^n}^2(\nu \cdot x + \omega t) \, dx$$

$$+ \int_0^T \int_{\mathbb{R}^n} \left( f^{\hat{3}+1} u(x, t) \right)^2 \left( \chi_{\mathbb{R}^n}^2 \right) (\nu \cdot x + \omega t) \, dx \, dt \leq c. \tag{3.46}$$

This estimate finishes the inductive argument, and we conclude the proof. \hfill \square

The attentive reader might naturally wonder if it is possible to obtain a regularity propagation of regularity result as the one proved previously if the dispersion is weaker when compared with that of the ZK equation. More precisely, if we consider the equation

$$\partial_t u - \partial_{x_1} ( (-\Delta)^{\frac{\alpha}{2}} u + u \partial_{x_1} u = 0, \quad 0 < \alpha < 1.$$ 

This question was firstly addressed in [38] in the one dimensional case (see [41] for a more detailed exposition). It was proved in [38] that even in the case that the dispersion is too weak, the propagation of regularity phenomena occurs. Although, for higher dimensions this question has not been addressed before since it was unknown how to obtain Kato’s smoothing in these cases.

**Remark 3.1.** The arguments described in Lemma 1.1 are strong enough and allows us to describe

4. Appendix A

The following appendix intends to provide a summary of the main results of Pseudo differential operators we use in this work.

**Definition 4.1.** Let $m \in \mathbb{R}$. Let $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ denote the set of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all $\alpha$ and all $\beta$ multi-index

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim_{\alpha, \beta} (1 + |\xi|)^{m-|\beta|}, \quad \text{for all } x, \xi \in \mathbb{R}^n.$$

An element $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is called a symbol of order $m$.

**Remark 4.1.** For the sake of simplicity in the notation from here on we will suppress the dependence of the space $\mathbb{R}^n$ when we make reference to a symbol in a particular class.

**Remark 4.2.** For $m \in \mathbb{R}$, the class of symbols $S^m$ can be described as

$$S^m = \left\{ a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \mid |a|^{(j)}_{S^m} < \infty, \quad j \in \mathbb{N} \right\},$$

where

$$|a|^{(j)}_{S^m} := \sup_{x, \xi \in \mathbb{R}^n} \left\{ |\langle \xi \rangle^{|\alpha|} \partial_\xi^\alpha a(x, \xi) |_{L^2_x(\mathbb{R}^n)} \right\}, \quad |\alpha| + |\beta| \leq j \right\}.$$

**Definition 4.2.** A pseudo-differential operator is a mapping $f \mapsto \Psi f$ given by

$$(\Psi f)(x) = \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi,$$

where $a(x, \xi)$ is the symbol of $\Psi$.

**Remark 4.3.** In order to emphasize the role of the symbol $a$ we will often write $\Psi_a$. Also, we use the notation $a(x, D)$ to denote the operator $\Psi_a$.

**Definition 4.3.** If $a(x, \xi) \in S^m$, the operator $\Psi_a$ is said to belong to $\text{OPS}^m$. More precisely, if $\nu$ is any symbol class and $a(x, \xi) \in \nu$, we say that $\Psi_a \in \text{OPS}^m$. 

A quite remarkable property that pseudo-differential operators enjoy is the existence of the adjoint operator, that is described below in terms of its asymptotic decomposition.

**Theorem 4.1.** Let \( a \in S^m \). Then, there exist \( a^* \in S^m \) such that \( \Psi_a^* = \Psi_{a^*} \), and for all \( N \geq 0 \),

\[
a^*(x, \xi) = \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_x^\alpha a^\xi \Psi(x, \xi) \in S^{m-N}.
\]

**Proof.** See Stein [55] chapter VI or Taylor [56]. \( \square \)

Additionally the product \( \Psi_a \Psi_b \) of two operators with symbols \( a(x, \xi) \) and \( b(x, \xi) \) respectively is a pseudo-differential operator \( \Psi_c \) with symbol \( c(x, \xi) \). More precisely, the description of the symbol \( c \) is summarized in the following Theorem:

**Theorem 4.2.** Suppose \( a \) and \( b \) symbols belonging to \( S^m \) and \( S' \) respectively. Then, there is a symbol \( c \) in \( S^{m+r} \) so that

\[
\Psi_c = \Psi_a \circ \Psi_b.
\]

Moreover,

\[
c \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_x^\alpha a \partial_\xi^\alpha b,
\]

in the sense that

\[
c - \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_x^\alpha a \partial_\xi^\alpha b \in S^{m+r-N}, \quad \text{for all integer} \ N, N \geq 0.
\]

**Proof.** For the proof see Stein [55] chapter VI or Taylor [56]. \( \square \)

**Remark 4.4.** Note that \( c - ab \in S^{m+r-1} \). Moreover, each symbol of the form \( \partial_x^\alpha a \partial_\xi^\alpha b \) lies in the class \( S^{m+r-|\alpha|} \).

A direct consequence of the decomposition above is that it allows to describe explicitly up to an error term, operators such as commutators between pseudo-differential operators as is described below:

**Proposition 1.** For \( a \in S^m \) and \( b \in S' \) we define the commutator \([\Psi_a, \Psi_b]\) by

\[
[\Psi_a, \Psi_b] = \Psi_a \Psi_b - \Psi_b \Psi_a.
\]

Then, the operator \([\Psi_a, \Psi_b]\) is \( \text{OPS}^{m+r-2} \), has by principal symbol the Poisson bracket, i.e.,

\[
\sum_{|\alpha| = 1} \frac{1}{2\pi i} \left( \partial_x^\alpha a \partial_\xi^\alpha b - \partial_x^\alpha b \partial_\xi^\alpha a \right) \text{ mod } S^{m+r-2}.
\]

Also, certain class the pseudo-differential operators enjoy of some continuity properties as the described below.

**Proof.** The proof can be consulted in Stein [55] Chapter VI, Theorem 1. \( \square \)

An interesting and useful continuity result in the Sobolev spaces is described below.

**Theorem 4.3.** Let \( m \in \mathbb{R}, a \in S^m, \) and \( s \in \mathbb{R}. \) Then, the operator \( \Psi_a \) extends to a bounded linear operator from \( H^{s+m}(\mathbb{R}^n) \) to \( H^s(\mathbb{R}^k) \). Moreover, there exists \( j = j(n; m, s) \in \mathbb{N} \) and \( c = c(n; m; s) > 0 \) such that

\[
\|\Psi_a f\|_{H^s} \leq c |a|^{(j)} \|f\|_{H^{s+m}_c}.
\]

**Proof.** See Kumano-go [30] or Stein [55], Chapter VI. \( \square \)
An alternative formula for the Bessel kernel.

**Lemma 4.1.** Let \( 0 < \delta < n + 1 \), and \( f \) be a tempered distribution. Then \( \mathcal{I}^{-\delta} f = \mathcal{B}_{\delta} \ast f \), where

\[
\mathcal{B}_{\delta}(y) = \frac{1}{(2\pi)^{\frac{n-\delta}{2}} \Gamma \left( \frac{n-\delta}{2} \right)} e^{-|y|^{\delta-n}} \int_0^{\infty} e^{-s|y|^\delta} \left( s + \frac{s^2}{2} \right)^{\frac{n-\delta}{2}-1} ds
\]

**Proof.** See Calderon & Zygmund [7], Lemma 4.1. \( \square \)

**Lemma 4.2.** The function \( \Theta_{\delta} \) is non-negative and it satisfies the following properties:

(a) For \( 0 < \delta < n \),

\[
\mathcal{B}_{\delta}(x) \lesssim \delta^{-|x|} \left( 1 + |x|^{\delta-n} \right);
\]

(b) for \( \delta = n \)

\[
\mathcal{B}_{\delta}(x) \lesssim e^{-|x|} \left( 1 + \log^+ \left( \frac{1}{|x|} \right) \right);
\]

(c) for \( \beta \) multi-index with \( |\beta| > 0 \) and \( 0 < \delta < n + 1 \)

\[
\left| \partial^\beta \mathcal{B}_{\delta}(x) \right| \lesssim c_{\delta,\beta} e^{-|x|} \left( 1 + |x|^{-n+\delta-|\beta|} \right), \quad x \neq 0.
\]

**Proof.** We refer to Calderon & Zygmund [7], Lemma 4.2. \( \square \)

Also, the behavior of the Bessel potentials in the following cases is necessary for our arguments.

**Lemma 4.3.** Let \( \delta > 0 \). The function \( \mathcal{B}_{\delta} \) satisfies the following estimates:

(i) For \( \delta < n \) and \( |y| \to 0 \),

\[
\mathcal{B}_{\delta}(y) \approx \frac{\pi^{\frac{\delta}{2}} \Gamma \left( \frac{n-\delta}{2} \right) |2\pi y|^{\delta-n}}{2^{\delta-n} \Gamma \left( \frac{\delta}{2} \right)}. \tag{4.4}
\]

(ii) For \( \delta = n \) and \( |y| \to 0 \),

\[
\mathcal{B}_{\delta}(y) \approx \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)} \log \left( \frac{1}{|2\pi y|} \right). \tag{4.5}
\]

(iii) For \( \delta > n \) and \( |y| \to 0 \),

\[
\mathcal{B}_{\delta}(y) \approx \frac{\pi^{\frac{\delta}{2}} \Gamma \left( \frac{\delta-n}{2} \right)}{\Gamma \left( \frac{\delta}{2} \right)}. \tag{4.6}
\]

(iv) For \( \delta > 0 \) and \( |y| \to \infty \),

\[
\mathcal{B}_{\delta}(y) \approx \frac{(2\pi)^{\frac{\delta}{2}}}{2^{\frac{n+1}{2}} \pi^{-\frac{\delta}{2}} \Gamma \left( \frac{\delta}{2} \right)} |2\pi|^{\delta-n-1} e^{-|2\pi y|}. \tag{4.7}
\]

**Proof.** For the proof see Aronszajn & Smith [4]. \( \square \)
5. Appendix B

In this section we present some localization tools that are quite useful to describe the regularity phenomena we are working on.

**Lemma 5.1.** Let \( \Psi \in \text{OPS}^r \). Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a multi-index with \(|\alpha| \geq 0\). If \( f \in L^2(\mathbb{R}^n) \) and \( g \in L^p(\mathbb{R}^n) \), \( p \in [2, \infty] \) with

\[
\text{dist} (\text{supp}(f), \text{supp}(g)) \geq \delta > 0,
\]

(5.1) then,

\[
\|g \tilde{c}_x^{\alpha} \Psi f\|_{L^2} \lesssim \|g\|_{L^p} \|f\|_{L^2},
\]

where \( \tilde{c}_x^{\alpha} := \tilde{c}_x^{\alpha_1} \cdots \tilde{c}_x^{\alpha_n} \).

**Proof.** See Mendez [39]. \( \square \)

The next result can be proved by using the ideas from the proof of Lemma above. Nevertheless, for the reader convenience we describe the main details.

**Corollary 5.1.** Let \( f, g \) be functions such that

\[
\text{dist} (\text{supp}(f), \text{supp}(g)) = \delta > 0.
\]

Then, the operator

\[
(T_{\psi} f) (\xi) := \psi(\xi) \hat{f}(\xi), \ f \in \mathcal{S}(\mathbb{R}^n)
\]

where \( \psi \) is defined as in (2.27), that is,

\[
\psi(\xi) = \sum_{j=1}^{\infty} \left( \frac{\alpha / 2}{j} \right) (2\pi \xi)^{2-2j}, \ \xi \in \mathbb{R}^n.
\]

If \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^2(\mathbb{R}^n) \), then

\[
\|[T_{\psi} f] g\|_{L^2} \lesssim \|f\|_{L^p} \|g\|_{L^2},
\]

for \( p \in [2, \infty] \).

**Proof.** According to condition (5.2) we have

\[
([T_{\psi} f] g) (x) = \sum_{j=2}^{\infty} \left( \frac{\alpha / 2}{j} \right) f(x) \int_{\{|x-y|>\delta\}} g(y) B_{2j-2}(x-y) \, dy.
\]

In virtue of Lemma 4.3 is clear that for \( j > 1 \),

\[
\left| f(x) \int_{\{|x-y|>\delta\}} g(y) B_{2j-2}(x-y) \, dy \right| \lesssim n^{\frac{1}{2}} j^{(j-2)!} \int_{\{|x-y|>\delta\}} |f(x)||g(y)||2\pi(x-y)|^{\frac{2j-3}{2j}} e^{-2\pi|x-y|} \, dy.
\]

Thus, by Young’s inequality

\[
\|[T_{\psi} f] g\|_{L^2} \lesssim n \|f\|_{L^p} \|g\|_{L^2} \left( \sum_{j=2}^{\infty} \frac{\Gamma(j + \frac{n-3}{2})}{(j-2)! j^{1+n}} \right)
\]

\[
\lesssim n, \|f\|_{L^p} \|g\|_{L^2},
\]

for \( p \in [2, \infty] \). \( \square \)
The following formulas were firstly obtained by Linares, Kenig, Ponce and Vega in the one dimensional case in their study about propagation of regularity for solutions of the KdV equation. Nevertheless, these results where later extended to dimension $n, n \geq 2$ in the work of [39] for solutions of the Zakharov-Kuznetsov equation.

**Lemma 5.2** (Localization formulas). Let $f \in L^2(\mathbb{R}^n)$. Let $\nu = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ a non-null vector such that $v_j \geq 0, j = 1, 2, \ldots, n$. Let $\epsilon > 0$, we consider the function $\varphi_{\nu, \epsilon} \in C^\infty(\mathbb{R}^n)$ to satisfy: $0 \leq \varphi_{\nu, \epsilon} \leq 1$,

$$\varphi_{\nu, \epsilon}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{H}_\nu^{\epsilon} \\ 1 & \text{if } x \in \mathcal{H}_{\{\nu, \epsilon\}} \end{cases}$$

and the following increasing property: for every multi-index $\alpha$ with $|\alpha| = 1$

$$\partial_\alpha^k \varphi_{\nu, \epsilon}(x) \geq 0, \quad x \in \mathbb{R}^n.$$

(I) If $m \in \mathbb{Z}^+$ and $\varphi_{\nu, \epsilon} \partial_m f \in L^2(\mathbb{R}^n)$, then for all $\epsilon' > 2\epsilon$ and all multi-index $\alpha$ with $0 \leq |\alpha| \leq m$, the derivatives of $f$ satisfy

$$\varphi_{\nu, \epsilon} \partial_\alpha^k f \in L^2(\mathbb{R}^n).$$

(II) If $m \in \mathbb{Z}^+$ and $\varphi_{\nu, \epsilon} \partial_m^k f \in L^2(\mathbb{R}^n)$ for all multi-index $\alpha$ with $0 \leq |\alpha| \leq m$, then for all $\epsilon' > 2\epsilon$

$$\varphi_{\nu, \epsilon} \partial_m f \in L^2(\mathbb{R}^n).$$

(III) If $s > 0$, and $f'(\varphi_{\nu, \epsilon} f) \in L^2(\mathbb{R}^n)$, then for any $\epsilon' > 2\epsilon$

$$f'(\varphi_{\nu, \epsilon} f) \in L^2(\mathbb{R}^n).$$

(IV) If $s > 0$, and $\varphi_{\nu, \epsilon} s f \in L^2(\mathbb{R}^n)$, then for any $\epsilon' > 2\epsilon$

$$\varphi_{\nu, \epsilon} s f \in L^2(\mathbb{R}^n).$$

**Proof.** See [39].

A more general version that we are going to use in this work.

**Lemma 5.3.** Let $f \in L^2(\mathbb{R}^n)$. If $\theta_1, \theta_2 \in C^\infty(\mathbb{R}^n)$ are functions such that: $0 \leq \theta_1, \theta_2 \leq 1$, their respective supports satisfy

$$\text{dist}(\text{supp} (1 - \theta_1), \text{supp} (\theta_2)) \geq \delta,$$

for some positive number $\delta$, and for all multi-index $\beta$, the functions $\partial_\beta \theta_1, \partial_\beta \theta_2 \in C^\infty(\mathbb{R}^n)$. Then, the following identity holds:

(I) If $m \in \mathbb{Z}^+$ and $\theta_1 \partial_m f \in L^2(\mathbb{R}^n)$, then for all multi-index $\alpha$ with $0 \leq |\alpha| \leq m$, the derivatives of $f$ satisfy

$$\theta_2 \partial_\alpha^k f \in L^2(\mathbb{R}^n).$$

(II) If $m \in \mathbb{Z}^+$ and $\theta_1 \partial_m^k f \in L^2(\mathbb{R}^n)$ for all multi-index $\alpha$ with $0 \leq |\alpha| \leq m$, then

$$\theta_2 \partial_m f \in L^2(\mathbb{R}^n).$$

(III) If $s > 0$, and $f'(\theta_1 f) \in L^2(\mathbb{R}^n)$, then

$$\theta_2 f' f \in L^2(\mathbb{R}^n).$$

(IV) If $s > 0$, and $\theta_1 f^s f \in L^2(\mathbb{R}^n)$, then

$$f^s(\theta_2 f) \in L^2(\mathbb{R}^n).$$

**Proof.** See [39].
Lemma 5.4. Let $\Psi_a \in \text{OPS}^0$. Let $\theta_1, \theta_2 : \mathbb{R}^n \to \mathbb{R}$ be smooth functions such that
\[
\text{dist} (\text{supp} (1 - \theta_1), \text{supp} \theta_2) > \delta,
\]
for some $\delta > 0$.
Assume that $f \in H^s(\mathbb{R}^n), s < 0$. If $\theta_1 f \in L^2(\mathbb{R}^n)$, then
\[
\theta_2 \Psi_a f \in L^2(\mathbb{R}^n).
\]

Proof. For the one dimensional case see [25] and the extension to the $n-$dimensional case see [39].

Lemma 5.5. Let $f \in L^2(\mathbb{R}^n)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$ such that $\nu_1 > 0$, for
\[j = 1, 2, \ldots, n.\] Also assume that
\[
J^s f \in L^2 \left( \mathcal{H}_{(\alpha, \nu)} \right), \quad s > 0.
\]
Then, for any $\epsilon > 0$ and any $r \in (0, s]$
\[
J^r f \in L^2 \left( \mathcal{H}_{(\alpha + \epsilon, \nu)} \right).
\]

Proof. For the one dimensional case see [25] and the extension to the $n-$dimensional case see [39].

Theorem 5.3. Let $s > 0$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then,
\[
\|J^s f; g\|_{L^2} \lesssim \|J^{s-1} f\|_{L^2} \|\nabla g\|_{L^\infty} + \|J^s g\|_{L^2} \|f\|_{L^\infty},
\]
where the implicit constant does not depends on $f$ nor $g$.

Proof. For the proof see the appendix in Kato and Ponce [24].

Also, the following Leibniz rule for the operator $J^s$ is quite useful in our arguments

Theorem 5.4. Let $s > \frac{n}{2}$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$, then
\[
\|J^s (f \cdot g)\|_{L^2} \lesssim \|J^s f\|_{L^2} \|g\|_{L^\infty} + \|J^s g\|_{L^2} \|f\|_{L^\infty},
\]
where the implicit constant does not depends on $f$ nor $g$.

Proof. See the appendix in Kato and Ponce [24].

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