Geometry and integrability in $\mathcal{N} = 8$ supersymmetric mechanics

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We construct the $\mathcal{N} = 8$ supersymmetric mechanics with potential term whose configuration space is the special Kähler manifold of rigid type and show that it can be viewed as the Kähler counterpart of $\mathcal{N} = 4$ mechanics related to “curved WDVV equations”. Then, we consider the special case of the supersymmetric mechanics with the non-zero potential term defined on the family of $U(1)$-invariant one-(complex)dimensional special Kähler metrics. The bosonic parts of these systems include superintegrable deformations of perturbed two-dimensional oscillator and Coulomb systems.

I. INTRODUCTION

The construction of $\mathcal{N}$-extended supersymmetric mechanics has remained one of the main directions of supersymmetric community since the introduction of the concept of supersymmetry. Nevertheless, until now there has been no regular way to find the $\mathcal{N} > 2$ supersymmetric extensions of the given mechanical systems. The traditional way to increase the number of supersymmetries (without exceeding the number of fermionic degrees of freedom) is to provide the configuration space with the complex structure(s)(with appropriate specification of the potential term), i.e. to restrict the configuration space to Kähler, hyper-Kähler or quaternionic manifold. Say, on the generic configuration space with the complex structure(s)(with appropriate specification of the potential term), i.e. to increase the number of supersymmetries (without exceeding the number of fermionic degrees of freedom) is to provide superintegrable deformations of perturbed two-dimensional oscillator and Coulomb systems.

\begin{equation}
H_{(2)} = \frac{1}{2} g^{ij}(x) (p_{i} p_{j} + \partial_{i} W(x) \partial_{j} W(x)) \quad \rightarrow \quad H_{(4)} = \frac{1}{2} g^{ab}(z, \bar{z}) \left( \bar{\pi}_{a} \pi_{b} + \bar{\partial}_{a} U(\bar{z}) \partial_{b} U(z) \right),
\end{equation}

with $(p_{i}, x^{i})$, $(\pi_{a}, z^{a})$ being canonically conjugated pairs and $g^{ij}(x)$ and $g^{ab}$ being the inverse Riemann and Kähler metrics, respectively.

Another way to increase the number of supersymmetries (above $\mathcal{N} = 2$ supersymmetry) is the doubling of fermionic degrees of freedom, supplied with introducing of additional geometric objects. Say, to construct the $\mathcal{N} = 4$ supersymmetric extension of free-particle system on generic configuration space we have to double the number of fermionic degrees of freedom from 2$\mathcal{N}$ to 4$\mathcal{N}$ and introduce the third-rank symmetric tensor $F_{ijk}(x) dx^{i} dx^{j} dx^{k}$ which satisfies the curved WDVV equations [1]

\begin{equation}
F_{kmj;l} = F_{kmi;j}, \quad F_{jkp} g^{pq} F_{imq} - F_{ikp} g^{pq} F_{jmq} = R_{ijkl},
\end{equation}

where $R_{ijkl}$ are the components of Riemann tensor of $(M_{0}, g_{ij} dx^{i} dx^{j})$, and the subscript $;\;$ denotes covariant derivative with Levi-Civita connection.

Similarly, to construct the $\mathcal{N} = 8$ supersymmetric extension of free-particle system on Kähler manifold we have to increase the number of the (real) fermionic variables from 4$\mathcal{N}$ to 8$\mathcal{N}$ and introduce the third-rank holomorphic symmetric tensor $f_{abc}(z) dz^{a} dz^{b} dz^{c}$ which satisfies the equations [2]

\begin{equation}
f_{abc;d} = f_{abcd;c}, \quad R_{abcd} = -f_{ace}, g^{de} f_{cebd},
\end{equation}

where $f_{abcd} = f_{abc,d} - \Gamma^{e}_{ad} f_{ebc} - \Gamma^{e}_{bd} f_{ecb} - \Gamma^{e}_{cd} f_{eab}$, and $R_{abcd}$, $\Gamma^{a}_{bc}$ are the non-zero components of the Riemann tensor and Levi-Civita connection which are defined by the relations

\begin{equation}
\Gamma^{a}_{bc} = g^{ad} g_{bd;c}, \quad R_{abcd} = g_{ab} \left( \Gamma^{a}_{cd} \right)^{d}.
\end{equation}

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These manifolds are known as the special Kähler manifolds of the rigid type [3] and they have been extensively studied since their introduction within the context of Seiberg-Witten duality [4]. The similarity between these systems has been not noticed before.

In this paper we show that this similarity holds for the supersymmetric mechanics with the potential term as well. Namely, after reviewing the main properties of $\mathcal{N} = 4$ supersymmetric mechanics connected with the solution of modified WDVV equations [1, 5, 6](Section 2), we construct on the special Kähler manifold of the rigid type, the $\mathcal{N} = 8$ supersymmetric mechanics with potential term (Section 3). We find that for the doubling of the supersymmetries the prepotentials $W(x), U(z)$ in the bosonic Hamiltonians (1.1) should satisfy the following equations

$$W_{ij} + F_{ijk}g^{km}W_m = 0, \quad U_{ab} - f_{abc}g^{cd}U_{cd} = 0. \quad (1.5)$$

Finally, we present in the Section 3 the general solution of the one-(complex)dimensional $U(1)$-symmetric special Kähler manifold and find the admissible set of potentials for $\mathcal{N} = 8$ supersymmetric mechanics. The bosonic parts of these supersymmetric mechanics include the superintegrable perturbations of deformed two-dimensional oscillator and Coulomb system suggested in [7, 8].

II. $\mathcal{N} = 4$ MECHANICS ON RIEMANN MANIFOLDS

In order to construct the $\mathcal{N} = 4$ supersymmetric mechanics on $N$-dimensional Riemannian manifold $(M_0, g_{ij}(x)dx^idx^j)$ we extend the cotangent bundle $(T^*M_0, dp_i \wedge dx^i)$ by $4N$ fermionic variables $\psi^{\alpha}, \bar{\psi}_{\dot{\beta}} = (\psi^{\dot{\beta}})^\dagger$, with $su(2)$ indices $\alpha, \beta = 1, 2$ which are raised and lowered as follows: $A_\alpha = c_{\alpha\beta}A^\beta, A^\alpha = c^{\alpha\beta}A_\beta$ with $\epsilon_{12} = e^{21} = 1$, and then define the following transition maps from one chart to the other

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}p_j, \quad \tilde{\psi}^{\alpha} = \frac{\partial \tilde{x}^i(x)}{\partial x^j}\psi^{\alpha}. \quad (2.1)$$

Then we equip this supermanifold with the supersymplectic structure which is manifestly invariant with respect to (2.1)

$$\Omega = dp_i \wedge dx^i + id\left(\psi^{\alpha}g_{ij}D\bar{\psi}_\dot{\beta} - \bar{\psi}_{\dot{\beta}}g_{ij}D\psi^{\alpha}\right) = dp_i \wedge dx^i + iR_{ijkl}\psi^{\alpha}\bar{\psi}_{\dot{\beta}}dx^k \wedge dx^l + 2ig_{ij}D\psi^{\alpha} \wedge D\bar{\psi}_{\dot{\beta}}, \quad (2.2)$$

where $D\psi^{\alpha} \equiv d\psi^{\alpha} + \Gamma^{i}_{jk}\psi^{\alpha}dx^k$, $\alpha = 1, 2$ and $\Gamma^{i}_{jk}, R_{ijkl}$ are the components of the connection and curvatures of the metric $g_{ij}(x)dx^idx^j$.

The Poisson brackets corresponding to this (super)symplectic structure, are defined by the following nonzero relations

$$\{p_j, x^i\} = \delta^i_j, \quad \{p_i, \psi^{\alpha}\} = -\Gamma^{i}_{jk}\psi^{\alpha}, \quad \{p_i, p_j\} = 2iR_{ijkm}\psi^{\alpha}\bar{\psi}_m, \quad \{\psi^{\alpha}, \bar{\psi}_{\dot{\beta}}\} = -\frac{i}{2}\delta^{\alpha}_{\dot{\beta}}g^{ij}. \quad (2.3)$$

Our goal is to construct the supercharges $Q^\alpha, \bar{Q}_{\dot{\beta}}$, and the Hamiltonian $H$ which obey the $\mathcal{N} = 4, d = 1$ Poincaré superalgebra

$$\{Q^\alpha, \bar{Q}_{\dot{\beta}}\} = -\frac{1}{2}\delta^\alpha_{\dot{\beta}}H, \quad \{Q^\alpha, Q^\beta\} = \{\bar{Q}_{\alpha}, \bar{Q}_{\beta}\} = 0. \quad (2.4)$$

To this end, following [1], we firstly equip the Riemann manifold $(M_0, g_{ij}(x)dx^idx^j)$ with the third-rank symmetric tensor $F_{ijk}(x)dx^idx^jdx^k$ which obeys the equations (1.2).

The first equation in (1.2) defines the well-known Codazzi tensor, while the second equation could be viewed as a generalization of Witten-Dijkgraaf-Verlinde-Verlinde equation [9] to Riemann manifolds, and was referred as curved WDVV equation in [1, 5, 6].

To construct the supersymmetric mechanics with nontrivial potential we should define on $(M_0, g_{ij}dx^idx^j, F_{ijk}(x)dx^idx^jdx^k)$ the closed one-form obeying the following compatibility condition

$$W^{(1)} = W_1(x)dx^i : dW^{(1)} = 0, \quad W_{ij} + F_{ijk}g^{km}W_m = 0. \quad (2.5)$$

Clearly, it can be locally presented as an exact one-form $W^{(1)} = dW(x)$, with the locally defined function $W(x)$ called "prepotential".
With these objects at hands we can construct the $\mathcal{N} = 4$ supersymmetric mechanics defined by the following supercharges and Hamiltonian [5]
\[
Q^{\alpha} = p_i \psi^{i\alpha} + iW_i \psi^{i\alpha} + iF_{ijk} \psi^{j\beta} \bar{\psi}^{k\alpha}, \quad \bar{Q}_{\alpha} = p_i \bar{\psi}^{i\alpha} - iW_i \bar{\psi}^{i\alpha} + iF_{ijk} \bar{\psi}^{j\beta} \psi^{k\alpha},
\]
\[
\mathcal{H} = g^{ij}p_ip_j + g^{ij}W_iW_j + 4W_i \psi^{i\alpha} \bar{\psi}^{j\beta} - 4(F_{km;j} + R_{kimj})\psi^{i\alpha} \bar{\psi}^{m\beta} \bar{\psi}^{j\gamma} \psi^{k\alpha},
\] (2.6)

It follows from (1.2) that there exists the special coordinate frame where the metrics (and, respectively, Christoffel symbols and Riemann tensor) takes the form [6]
\[
g_{ij} = \frac{\partial^2 A}{\partial x^i \partial x^j} : \quad \Gamma_{ij}^{\alpha} = \frac{1}{2} \frac{\partial^3 A(x)}{\partial x^i \partial x^j \partial x^\alpha}, \quad R_{ijkm} = \Gamma_{impq}g^{pq}_\gamma \Gamma_{ijkm} - \Gamma_{ikpq}g^{pq}_\gamma \Gamma_{jkm}. \quad (2.8)
\]

From the last equation it becomes clear that the choice $F_{ijk} = \pm \Gamma_{ijk}$ solves curved WDVV equations (1.2). Then, solving the equations (2.5) we get the two sets of solutions
\[
\left( F_{ijk} = -\Gamma_{ijk}, \ W = \sum_i c_i x^i \right), \quad \left( F_{ijk} = \Gamma_{ijk}, \ W = \sum_i c_i x^i \right), \quad \text{with} \quad c_i = \text{const.} \quad (2.9)
\]

The first solution is the one obtained in [10]. The second solution could be transformed to the first one by the Legendre transformation
\[
x^i \rightarrow u_i = \partial_i A(x), \quad A(x) \rightarrow \tilde{A}(u) = (u_i x^i - A(x))|_{u_i = \partial_i A(x)}. \quad (2.10)
\]

In this coordinate frame the system (2.6),(2.7) coincides with $N$-dimensional $\mathcal{N} = 4$ supersymmetric mechanics constructed by using the $N$ scalar supermultiplets [10] (the respective system with single supermultiplet was investigated in [11]).

However, in many cases it is more convenient to solve the equations (1.2),(2.5) in other frames. Below we will exemplify this by presenting their solutions on $so(N)$-invariant special Riemann manifolds.

**$SO(N)$-invariant Riemann manifolds**

Let us consider curved WDVV and potential equations (1.2), (2.5) on isotropic ($so(N)$-invariant) spaces with the metric represented in conformal flat form
\[
g_{ij} dx^i dx^j = \sum_{i=1}^{N} g(r) dx^i dx^j, \quad \text{where} \quad r^2 = \sum_{i=1}^{N} x^i x^i. \quad (2.11)
\]

Let us show that these manifolds always admit nontrivial solutions.

Indeed, let $F^{(0)}_{ijk}(x)$ and $W^{(0)}_{ij}(x)$ be the solutions of the WDVV and potential equations on the Euclidian space which obey some additional condition
\[
F^{(0)}_{ijk} F^{(0)}_{pjm} = F^{(0)}_{ijkp} F^{(0)}_{pjm}, \quad \partial_i \partial_j W^{(0)} = F^{(0)}_{ijk} \partial_k W^{(0)} = 0, \quad (2.12)
\]

with
\[
\sum_{i=1}^{N} x^i F^{(0)}_{ijk} = \delta_{jk}, \quad \sum_{i=1}^{N} x^i \partial_i W^{(0)} = \alpha_0, \quad (2.13)
\]

where $F^{(0)}_{ijk} = \frac{\partial^3 F^{(0)}}{\partial x^i \partial x^j \partial x^k}$ and $\alpha_0$ is some constant. The variety of pairs $(F^{(0)}, W^{(0)})$ obeying these equations was presented in [12].

These flat solutions can be lifted to the solutions of curved WDVV and potential equations on isotropic spaces as follows (we use here the notation which slightly differs from that in [1, 5])
\[
F^\kappa_{ijk} = g(r) \left( F^{(0)}_{ijk} + \Gamma(r) \frac{\delta_i x^k + \delta_j x^i + \delta_k x^j}{r^2} - A(r) \frac{x^i x^j x^k}{r^4} \right), \quad (2.14)
\]

where
\[
\Gamma(r) = \frac{r d\log g}{2 dr}, \quad A(r) = 2\Gamma - \frac{r \Gamma'}{\Gamma + 1}. \quad (2.15)
\]
Corresponding solution of curved potential equation and the respective Hamiltonian are given by the expressions

\[ W = W(0) - \alpha_0 \int \frac{\Gamma}{1 + \Gamma} \frac{d\rho}{r}, \quad \Rightarrow \quad H = g^{-1}(r) \sum_{i=1}^{N} (p_i^2 + W(0)iW(0)i) - \frac{2\alpha_0^2}{2g(r)} \left( 1 - \frac{1}{(1 + \Gamma)^2} \right). \] (2.16)

Note that the "curved" counterpart of the initial Hamiltonian yields an additional potential term with coupling constant \( \alpha_0^2 \). In the particular case of sphere and two-sheet hyperboloid (pseudosphere), when \( g = (1 + \epsilon r^2)^{-\frac{1}{2}} \) (with \( \epsilon = 1 \) corresponding to the sphere and \( \epsilon = -1 \) to the pseudosphere), it coincides with the potential of the superintegrable (pseudo)spherical generalization of harmonic oscillator known as Higgs oscillator [13].

Thus, with the specific choice of initial prepotential \( W(0)(x) \) we can construct \( N = 4 \) supersymmetric superintegrable deformations of Higgs oscillator. For example, the choice \( W(0) = \sum_{i=1}^{N} \alpha_i \log x_i \), \( F(0) = \frac{1}{2} \sum_{i=1}^{N} (x_i)^2 \log x_i \), yields superintegrable (pseudo)spherical deformation of \( N \)-dimensional oscillator with extra centrifugal terms (which is also known as Rosochatius system) with additional restriction to the oscillator frequency [5]

\[ H_{Ros} = (1 + \epsilon r^2)^2 \left( \sum_{i=1}^{N} p_i^2 + \sum_{i}^{N} \frac{\alpha_i^2}{x_i^2} + \frac{4(\sum_i \alpha_i)^2}{(1 - \epsilon r^2)^2} \right). \] (2.17)

Taking the solutions of (2.12) corresponding to the three-particle rational Calogero model [14], we will get the following (pseudo)spherical Hamiltonian

\[ H_{3\text{Calogero}} = (1 + \epsilon r^2)^2 \left( \sum_{i=1}^{3} p_i^2 + \sum_{i > j}^{3} \frac{2g^2}{(x_i - x_j)^2} + \epsilon \frac{36g^2}{(1 - \epsilon r^2)^2} \right). \] (2.18)

It is a particular case of superintegrable (pseudo)spherical Calogero-Higgs oscillator [15].

### III. \( N = 8 \) Mechanics on Special Kähler Manifolds

In this Section we generalize the system presented in [2], and construct, on the special Kähler manifolds of the rigid type, the \((N|4N)\)-dimensional mechanics with potential term, which possesses the \( N = 8 \) supersymmetry

\[ \{ Q_{ia}, Q_{j\beta} \} = \{ \bar{Q}_{ia}, \bar{Q}_{j\beta} \} = 0, \quad \{ Q_{ia}, \bar{Q}_{j\beta} \} = -i \epsilon_{ij} \epsilon_{a\beta} \mathcal{H}. \] (3.1)

For this purpose we define the \((2N|4N)\)-dimensional phase superspace equipped by the supersymplectic structure

\[ \Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - R_{abcd} \bar{\eta}^c_{ia} \eta^{d\bar{c}}_{\bar{a}a} dz^d \wedge d\bar{z}^\bar{d} + g_{ab} D_{i\alpha} \chi^{a\alpha} \wedge D_{\bar{b}\bar{\alpha}} \bar{\eta}^{b\bar{b}}_{\bar{a}a} \chi^{\bar{b}\bar{b}}, \quad D_{i\alpha} a^a = d\pi_{i\alpha} a^a + \Gamma_{abc} \eta^{b\bar{b}}_{\bar{a}a} d\bar{z}^\bar{b}, \] (3.2)

with fermionic variables \( \eta_{ia} \) and \( \bar{\eta}_{ia} \) related as follows (\( \eta_{ia} \)) and \( \bar{\eta}_{ia} \). Here \( a, i = 1, 2 \) are su(2) indices which are raised and lowered as follows: \( A_a = \epsilon_{a\beta} \bar{A}_\beta, A^a = \epsilon^{a\beta} A_\beta, A_i = \epsilon_{ij} A_j, A^i = \epsilon^{ij} A_j \)), with \( \epsilon_{12} = \epsilon^{21} = 1 \).

This supersymplectic structure is manifestly invariant under the coordinate transformation

\[ z^a = \tilde{z}^a (z), \quad \bar{z}^a = \frac{\partial z^\bar{b}}{\partial \bar{z}^\bar{b}} \tilde{z}^\bar{b}, \quad \eta_{ia} = \frac{\partial z^a}{\partial \bar{z}^\bar{b}} \tilde{\eta}_{\bar{b}a}, \] (3.3)

i.e. \( \eta_{ia} \) transforms as \( dz^a \).

The Poisson brackets corresponding to (3.2) are defined by the relations

\[ \{ \pi_a, z^b \} = \delta^b_a, \quad \{ \pi_a, \bar{\eta}_{i\alpha} \} = -\Gamma_{abc} \eta^{c\bar{c}}_{\bar{a}a} \pi_{i\alpha}, \quad \{ \pi_a, \bar{\pi}_b \} = i R_{abcd} \eta^{c\bar{c}}_{\bar{a}a} \bar{\eta}^{d\bar{b}}_{\bar{a}a}, \quad \{ \eta_{ia}, \bar{\eta}_{j\beta} \} = -i \epsilon_{ij} \epsilon_{a\beta} \delta^\alpha_\beta. \] (3.4)

For the construction of supersymmetric mechanics with nonzero potential we have to equip the Kähler manifold by the closed holomorphic one-form

\[ U^{(1)} = U_a (z) dz^a, \quad U_a = \frac{\partial U(z)}{\partial z^a}, \] (3.5)

where \( U(z) \) is a locally defined holomorphic function which is called “prepotential.”
With these ingredients at hand we can construct the $\mathcal{N} = 8$ supersymmetric mechanics with potential term. Having in mind the structure of supercharges of the $\mathcal{N} = 4$ supersymmetric mechanics on generic Kähler manifold [17], and of the the $\mathcal{N} = 8$ supersymmetric mechanics (without potential term) on special Kähler manifolds [2], we choose the following anzats for supercharges:

$$Q_{\alpha} = \pi_{a}\eta_{\alpha} + \overline{U}_{a} T_{\alpha} \tilde{\pi}_{\beta}_{a} + \frac{1}{3} \overline{f}_{a\beta}\epsilon_{\tilde{a}\beta}\eta_{\alpha}, \quad \overline{Q}_{\alpha} = \bar{\pi}_{\alpha}\eta_{\alpha} - U_{a} T_{\alpha} \gamma_{\beta}_{a} + \frac{1}{3} f_{a\beta}\eta_{\alpha}\eta_{\beta}\eta_{\alpha},$$

where the matrix $T^{\beta}_{\alpha}$ collects the parameters which control the explicit breaking of the $su(2)$ symmetry realized on the Greek indices, and, without loss of generality, is parameterized by the two angle-like parameters $\alpha_{0}, \beta_{0}$

$$T^{\beta}_{\alpha} = \begin{pmatrix} \cos \alpha_{0} & e^{i\beta_{0}} \sin \alpha_{0} \\ e^{-i\beta_{0}} \sin \alpha_{0} & -\cos \alpha_{0} \end{pmatrix}. \quad (3.6)$$

One should stress that it is impossible to introduce the interaction which preserved both $su(2)$ symmetries (realized on the Greek and Latin indices from the middle of alphabet $(i, j, k)$). However the simultaneous breaking of both these symmetries results in the appearance of the central charges in the super Poincaré algebra [18].

The components of (anti)holomorphic symmetric tensors $f_{abc}(z)$ and $\overline{f}_{a\beta}(\overline{z})$ have to obey the constraints (1.3), and $U_{a}$ and $\overline{U}_{a}$ were defined in (3.5).

Then, taking their Poisson brackets we find that these supercharges span the $\mathcal{N} = 8$ Poincaré superalgebra (3.1) if $U_{a}$ and $\overline{U}_{a}$ obey the equations

$$U_{a;b} - f_{abc}g^{dc}\overline{U}_{d} = 0. \quad (3.8)$$

with $U_{a;b} = U_{a;b} - \Gamma^{c}_{ab}U_{c}$. In such a case, the Hamiltonian reads

$$\mathcal{H} = g^{ab}(\pi_{a}\overline{\pi}_{b} + U_{a}\overline{U}_{b}) - \frac{1}{2} U_{a}g^{ae}\overline{f}_{\tilde{a}e}\overline{\eta}_{\tilde{a}e} T^{\alpha}_{a} \overline{\eta}_{\beta e} - \frac{1}{2} \overline{U}_{a}g^{ae}f_{\tilde{a}e}\eta_{\tilde{a}e} T^{\alpha}_{a} \eta_{\beta e} - \frac{1}{12} f_{a\beta\delta}g^{\alpha\beta\delta} \eta_{\alpha e} T^{\beta}_{a} \eta_{\delta e} - \frac{1}{12} \overline{f}_{a\beta\delta}\overline{g}^{\alpha\beta\delta} \overline{\eta}_{\alpha e} \overline{T^{\beta}_{a} \overline{\eta}_{\delta e}} - \frac{1}{4} f_{a\beta\delta}g^{\epsilon\beta\delta} \eta_{\epsilon e} \eta_{\beta e} T^{\alpha}_{a} \eta_{\delta e} + \frac{1}{4} \overline{f}_{a\beta\delta}\overline{g}^{\epsilon\beta\delta} \overline{\eta}_{\epsilon e} \overline{\eta}_{\beta e} \overline{T^{\alpha}_{a} \overline{\eta}_{\delta e}}.$$ \hspace{1cm} (3.9)

The equations (1.3) can be expressed in the distinguished coordinate frame via single holomorphic function (“Seiberg-Witten potential”) $F(z)$ (see, e.g. [3])

$$g_{ab} = \Re \partial_{a}\partial_{b}F(z), \quad \Gamma_{abc} = \partial_{a}\partial_{b}\partial_{c}F(z), \quad \Rightarrow \quad f_{abc} = \Gamma_{abc}. \quad (3.10)$$

In this coordinate frame the equation (3.8) looks as follows

$$\partial_{a}\partial_{b}U - (\partial_{d}U + \partial_{d}\overline{U}) g^{de}\partial_{a}\partial_{b}\partial_{c}F = 0. \quad (3.11)$$

From this equation we immediately get the following solution

$$U(z) = \sum_{a=1}^{N} (m^{a}\partial_{a}F(z) + in_{a}z^{a}), \quad (3.12)$$

with $m^{a}, n_{a}$ being real constants.

The bosonic part of constructed $\mathcal{N} = 8$ supersymmetric mechanics respects “T-duality” transformation, which is the complex counterpart of Legendre transformation (2.10)

$$z^{a} \rightarrow u_{a} = \partial_{a}F, \quad F(z) \rightarrow \tilde{F}(u) = (z^{a}u_{a} - F(z)) |_{u_{a} = \partial_{u}F}.$$ \hspace{1cm} (3.13)

For the potential it reads

$$U(z) = \sum_{a=1}^{N} m^{a}\partial_{a}F(z) + in_{a}z^{a}, \quad \rightarrow \quad U(u) = \sum_{a=1}^{N} m^{a}u_{a} + in_{a}\partial^{a}\tilde{F}(u) \quad (3.14)$$

The extension of duality transformation to the whole phase superspace is as follows

$$(z^{a}, \pi_{a}, \eta_{ia}) \rightarrow (u_{a}, p^{a}, \epsilon_{ia}), \quad \text{where} \quad u_{a} = \partial_{a}F(z), \quad p^{a} = \partial^{2}F \partial_{a}z^{b}, \quad \epsilon_{ia} = \partial_{z^{a}}\partial_{\overline{z}}^{b}\epsilon_{ia}. \quad (3.15)$$
Looking back at the presented model of $\mathcal{N} = 8$ supersymmetric mechanics we can observe many similarities with $\mathcal{N} = 4$ supersymmetric mechanics described in the previous Section, which prompts us to consider it as a complex counterpart of the latter one. In particular, the notion of "special Kähler manifold of the rigid type" (1.3) can be viewed as the complex analog of curved WDVV equations (1.2), and as well as the restriction on the prepotential $U(z)$ can be viewed as a complex counterpart of those to the real one (1.5). In both cases there exist special coordinate frames where the metrics and the respective third-rank tensors are expressed via single function, cf. (2.8) and (3.10). Further visible similarities can be noticed comparing (3.12) and (2.9).

However, the requirement of “special Kähleriarity” (1.3) is more restrictive than (1.2). Say, special Kähler manifold of the rigid type necessarily has a negative curvature, while “curved WDVV equations” does not yield such restriction; “curved WDVV equations” admit the nontrivial solutions on the generic $so(N)$-invariant Riemann manifolds (including $N$-dimensional spheres and hyperboloids). In contrast to this, complex projective spaces (and their non-compact counterparts) cannot be equipped by the structure of special Kähler manifold. Moreover, it seems that special Kähler metrics could possess the $U(N)$ isometry only in the simplest case $N = 1$ to be considered in the next Section.

IV. TWO-DIMENSIONAL SYSTEMS

In this Section we construct the one-(complex)dimensional special Kähler manifolds which is invariant under $U(1)$-transformation $z \to e^{\lambda} z$, and then find the potentials admitting $\mathcal{N} = 8$ supersymmetric extension.

Choosing the metric $g$ to be the function of $z \bar{z}$ only, i.e. putting $g = g(z \bar{z}) dz d\bar{z}$ one may explicitly solve the second equation in (1.3) as

$$g(z \bar{z}) dz d\bar{z} = (c_1(z \bar{z})^{n_1} + c_2(z \bar{z})^{n_2}) dz d\bar{z}, \quad f(z)[dz]^3 = -c_1 c_2 (n_1 - n_2) z^{n_1 + n_2 - 1} [dz]^3,$$

Corresponding Kähler potential reads

$$K(z, \bar{z}) = c_1 (z \bar{z})^{n_1 + 1} + c_2 (z \bar{z})^{n_2 + 1} \left(\frac{1}{n_1 + 1}\right)^2.$$

Then, performing transformation $\frac{\sqrt{c_1}}{n_1 + 1} z^{n_1 + 1} \to z$, we can simplify these structures as follow

$$ds^2 = \left(1 - \kappa^2 (z \bar{z})^m\right) dz d\bar{z}, \quad f(z)[dz]^3 = \kappa m z^{m-1} [dz]^3,$$ with $|z| \in [0, \kappa^{-1/m})$.

The Christoffel symbol and the Riemann curvature are

$$\Gamma^1_{11} = -\kappa^2 m z^{m-1} \bar{z} m / \left(1 - \kappa^2 (z \bar{z})^m\right), \quad R_{1111} = -\kappa^2 m^2 (z \bar{z})^{m-1} / \left(1 - \kappa^2 (z \bar{z})^m\right).$$

For this special case the potential equation (3.8) takes the form

$$U'' + \frac{\kappa^2 m z^{m-1} \bar{z} m}{1 - \kappa^2 (z \bar{z})^m} U' - \frac{\kappa m z^{m-1}}{1 - \kappa^2 (z \bar{z})^m} U' = 0.$$ (4.20)

Then we obtain

$$\frac{dU'(z)}{dz} = \frac{d}{dz} \left(\frac{1 - \kappa^2 (z \bar{z})^m}{\kappa m z^{m-1}} U'(z) + \kappa \bar{z}^m U'(z)\right) = 0,$$ (4.21)

From this equation we immediately get the solution

$$U'(z) = \kappa a z^m + \bar{a}$$ (4.22)

with $a$ being an arbitrary complex constant.

Thus, the one-(complex)dimensional $\mathcal{N} = 8$ supersymmetric mechanics is defined by the following bosonic Hamiltonian

$$H_{\kappa, m, a} = \frac{\pi \bar{\pi} + |\kappa a z^m + \bar{a}|^2}{1 - \kappa^2 (z \bar{z})^m},$$ with $\{\pi, z\}_0 = \{\bar{\pi}, \bar{z}\}_0 = 1, \{\pi, \bar{\pi}\}_0 = \{z, \bar{z}\}_0 = 0$. (4.23)
The presence of nonzero potential breaks the kinematical $U(1)$- symmetry $z \to e^{i\lambda} z, \pi \to e^{i\lambda} \pi$. But in the free particle case $a = 0$ the hamiltonian becomes manifestly invariant under this transformation and thus defines the integrable system

$$H_{\kappa, m, 0} = \frac{\pi \hat{\pi}}{1 - \kappa^2 (z \bar{z})^m} \quad J = i (z \pi - \bar{z} \bar{\pi}) : \quad \{H_0, J\} = 0$$

(4.24)

where $J$ is the generator of $U(1)$-symmetry.

However, for the specific values of $m$ the system could have the hidden symmetries. The simplest example corresponds to the $m = -2$ case.

- $m = -2$

In this case the Hamiltonian (4.23) admits the separation of variables in the polar coordinates

$$z = r e^{i\varphi}, \quad \pi = \frac{e^{-i\varphi}}{2} \left( p_r - i \frac{p_\varphi}{r} \right) \quad ; \quad H_{\kappa, -2, a} = \frac{\dot{r}^2 + |a|^2 (1 + \frac{\kappa^2}{r^2})}{4(1 - \frac{\kappa^2}{r^2})} + \frac{\kappa^2 + \kappa |a|^2 \cos(\varphi + \arg a)}{4(r^2 - \kappa^2)}$$

(4.25)

which allows immediately find the quadratic constant of motion

$$H_{\kappa, -2, a} = \frac{\pi \hat{\pi} + |a|^2}{1 - \frac{\kappa^2}{|z|^2}}, \quad I = \frac{\dot{r}^2 + 2\kappa |a|^2 \cos(\varphi + \arg a)}{(z \pi - \bar{z} \bar{\pi})^2} = \frac{4\kappa a^2 z^2 + a^2 \bar{z}^2}{z \bar{z}}$$

(4.26)

To find additional values of the parameter $m$ leading to the (super)integrable systems, one has to do the following. Fixing the energy surface of the Hamiltonian (4.23) one may re-write it as

$$\pi \bar{\pi} + \kappa^2 (|a|^2 + E_{\kappa, m, a}) |z|^2 m + \kappa a^2 z^m + \kappa \bar{a}^2 \bar{z}^m = E_{\kappa, m, a} - |a|^2$$

(4.27)

From this expression we immediately deduce that for $m = 1$ it coincides with the energy surface of the two-dimensional oscillator interacting with linear electric field which could be absorbed by the trivial sift of complex coordinate $z$, while for the $m = -1/2$ it can be easily transform to the $m = 1$ case by the Bohlin-Levi-Civita transformation $z = \bar{z}$ which relates the energy surfaces of two-dimensional oscillator and Coulomb problem. Hence, for the particular values of $m = 1, -1/2$ the Hamiltonian (4.23) possesses two functionally independent constants of motion and hence becomes superintegrable. Let us consider these cases in the full details.

- $m = 1$

In this particular case the Hamiltonian (4.23) takes a form

$$H_{\kappa, 1, a} = \pi \bar{\pi} + \kappa a \bar{a}$$

(4.28)

It possesses the hidden symmetry given by the deformed $U(1)$-generator $J$ presented in (4.24)

$$J_{\kappa, 1} = i \left[ \left( z + \frac{a^2}{\kappa (|a|^2 + H_{\kappa, 1, a})} \right) \bar{\pi} - \left( \bar{z} + \frac{a^2}{\kappa (|a|^2 + H_{\kappa, 1, a})} \right) \pi \right]$$

(4.29)

and by the complex constant of motion

$$F_\kappa = \pi^2 + \kappa^2 (|a|^2 + H_{\kappa, 1, a}) \left( \bar{z} + \frac{a^2}{\kappa (|a|^2 + H_{\kappa, 1, a})} \right)^2$$

(4.30)

which can be interpreted as a deformation of the so-called Fradkin tensor written in complex coordinates $z = (x_1 + i x_2)/\sqrt{2}$ and conjugated momentum.

They form the nonlinear algebra

$$\{J_{\kappa, 1}, F_\kappa\} = 2i F, \quad \{J_{\kappa, 1}, \bar{F}_\kappa\} = -2i \bar{F}_\kappa, \quad \{F_\kappa, \bar{F}_\kappa\} = 4i \kappa^2 (|a|^2 + H_{\kappa, 1, a}) J_{\kappa, 1.}$$(4.31)

For emphasizing the relation of this system with oscillator, let us re-write the Hamiltonian (4.28) it as follows

$$H_{\kappa, 1, a} = H_{osc}^\kappa + \frac{|\omega|^2}{2 \kappa^2}, \quad H_{osc}^\kappa = \frac{\pi \bar{\pi} + |\omega|^2 \bar{z} \bar{\pi} E \bar{z} + E \bar{z}}{1 - \kappa^2 (z \bar{z})}, \quad \omega := \sqrt{2\kappa} |a|, \quad E := \kappa a^2.$$ (4.32)
The function in numerator can be interpreted as a two-dimensional isotropic oscillator with the frequency $|\omega|$ interacting with electric field $\mathbf{E} = (E_1, E_2)$ with $E = (E_1 + iE_2)/2$. The parameters $\kappa, a$ can be expressed via $\omega, E$ as follows

$$\kappa = \frac{1}{2} \frac{|\omega|^2}{|E|}, \quad a = \sqrt{2}e^{-i\frac{\pi E}{|\omega|}}.$$

(4.33)

- $m = -1/2$

In this case the Hamiltonian (4.23) acquires the form

$$H_{\kappa,-1/2,a} = \frac{\pi \bar{\pi} + |\kappa a \frac{1}{\sqrt{z}} + \bar{a}|^2}{1 - (\pi^2)}.$$

(4.34)

It possesses the hidden symmetry given by the deformed $U(1)$-generator $J_{\kappa,-1/2}$

$$J_{\kappa,-1/2} = 2i \left[ z - \frac{\kappa a^2 \sqrt{z}}{H_{\kappa,-1/2,a} - |a|^2} \right] \pi - \left( \frac{\kappa a^2 \sqrt{z}}{H_{\kappa,-1/2,a} - |a|^2} \sqrt{z} \right) \bar{\pi}.$$

(4.35)

and by the complex constant of motion being the deformation of two-dimensional Runge-Lenz vector $A = (A_1, A_2)$ with $A_{\kappa} = (A_1 + iA_2)/2$:

$$A_{\kappa} = z \pi^2 - (H_{\kappa,-1/2,a} - |a|^2)^2 \left( \frac{\kappa a^2}{H_{\kappa,-1/2,a} - |a|^2} \right).$$

(4.36)

They form the non-linear algebra

$$\{J_{\kappa,-1/2}, A_{\kappa}\} = 2iA_{\kappa}, \quad \{J_{\kappa,-1/2}, \bar{A}_{\kappa}\} = -2i\bar{A}_{\kappa}, \quad \{A_{\kappa}, \bar{A}_{\kappa}\} = -i (H_{\kappa,-1/2,a} - |a|^2) J_{\kappa,-1/2}.$$

(4.37)

The Hamiltonian (4.34) can be interpreted as a deformation of the two-dimensional Coulomb problem perturbed by the potential $\delta V = k (\frac{a^2}{\sqrt{z}} + \frac{\bar{a}^2}{\sqrt{z}})$.

The bosonic Hamiltonian $H$ (4.23) possesses the following duality transformation:

$$H_{\kappa,m,a} = \frac{\pi \bar{\pi} + |\kappa az + \bar{a}|^2}{(1 - \kappa^2 |z|^2)^m} = \frac{\pi \bar{\pi} + |\kappa a \frac{1}{\sqrt{z}} + \bar{a}|^2}{1 - (\pi^2)} = -H_{\kappa,\bar{m},\bar{a}},$$

(4.38)

where the variables are related as

$$z = \kappa \omega \bar{z}^{\bar{m}+1}, \quad \pi = \frac{\pi}{\kappa \omega \bar{m}},$$

(4.39)

and the following constraints on the parameters are imposed

$$(m+1)(\bar{m}+1) = 1, \quad \kappa \bar{m}^{m+1} = |\bar{m}+1|^m, \quad \bar{a} = \bar{a}.$$

(4.40)

To be self-consistence, the transformations (4.39) should be supplied by the changing of the admitted values of the coordinates

$$\text{From } |z| \in [0, \kappa^{-1/m}] \text{ to } |\bar{z}| \in [\kappa^{-1/\bar{m}}, \infty).$$

(4.41)

Explicitly, the supercharges of the $\mathcal{N} = 8$ supersymmetric extensions of the presented bosonic systems read

$$Q_{i\alpha} = \pi \eta_{i\alpha} + (\kappa a \bar{z} + a) T^\beta_{\alpha} \eta_{i\beta} + \frac{i}{3} \kappa m z^{m-1} \bar{\eta}_{i\beta} \bar{\eta}_{j\beta} \eta_{i\alpha},$$

while the Hamiltonian has the form

$$H_{\kappa,m,a} = \frac{\pi \bar{\pi} + |ka z + \bar{a}|^2}{1 - \kappa^2 (z \bar{z})^m} + \frac{m \kappa z^{m-1} (a + \kappa a \bar{z})}{2(1 - \kappa^2 (z \bar{z})^m)} \eta^i T^\beta_{\alpha} \eta_{i\beta} - \frac{m \kappa z^{m-1} (\bar{a} + \kappa a \bar{z})}{2(1 - \kappa^2 (z \bar{z})^m)} \bar{\eta}^i T^\beta_{\alpha} \eta_{i\beta} - \frac{\kappa m}{12 (1 - \kappa^2 (z \bar{z})^m)} \left( z^{-m} \bar{\eta}_{i\alpha} \bar{\eta}_{j\beta} \eta_{i\beta} \eta_{j\alpha} + \bar{z}^{-m} \eta_{i\alpha} \eta_{j\beta} \eta_{i\beta} \eta_{j\alpha} \right) - \frac{\kappa^2 m^2 (z \bar{z})^{m-1}}{4 (1 - \kappa^2 (z \bar{z})^m)} \left( \bar{\eta}_{i\alpha} \eta_{j\beta} \eta_{i\beta} \eta_{j\alpha} + \eta_{i\alpha} \eta_{j\beta} \bar{\eta}_{i\beta} \bar{\eta}_{j\alpha} \right).$$

(4.42)
The $U(1)$-transformation $z \to e^{i\lambda}z$ extended to the supersymplectic structure (3.2) looks as follows

$$z \to e^{i\lambda}z, \pi \to e^{-i\lambda}\pi, \eta_\alpha \to e^{i\lambda}\eta_\alpha.$$  \hspace{1cm} (4.43)

It is defined by the generator

$$\mathcal{J} = i(z\pi - \bar{z}\bar{\pi}) - \frac{\partial^2 h(z\bar{z})}{\partial z\partial\bar{z}}\eta^{\alpha}\eta_\alpha, \hspace{1cm} h(z, \bar{z}) = z\bar{z} - \frac{\kappa^2(z\bar{z})^{m+1}}{m+1}$$  \hspace{1cm} (4.44)

where $h(z, \bar{z})$ is the Killing potential for $U(1)$-isometry.

It is seen that the supercharges (4.42) and the Hamiltonian (4.42) are not invariant under this transformation for the generic $m$ even for $a = 0$. It is not surprising, since the third-order tensor $f(a)[dz]^3$ in (4.18) is invariant under $U(1)$-transformation $z \to e^{i\nu}$ only for the $m = -2$, while the one-form (4.22) is not $U(1)$-invariant at all. Since $\eta_\alpha$ transforms as $dz$, we conclude that the supercharges and supersymmetric Hamiltonian fail to be $U(1)$-invariant in the generic case. Hence, only in the case $a = 0, m = -2$, corresponding to the bosonic Hamiltonian (4.26) we can construct the $\mathcal{N} = 8$ supersymmetric extension with the supercharges and Hamiltonian which are invariant under transformation (4.43).

On the other hand, the Hamiltonian $\mathcal{H}_{\kappa, m, 0}$, in contrast to supercharges $Q_{(\kappa, m, 0)\alpha}$, is invariant under transformation

$$z \to e^{i\lambda}z, \pi \to e^{-i\lambda}\pi, \eta_\alpha \to e^{i\frac{2-m}{4}\lambda}\eta_\alpha.$$  \hspace{1cm} (4.45)

Hence, it commutes with the generator

$$\tilde{\mathcal{J}} = i(z\pi - \bar{z}\bar{\pi}) - \frac{\partial^2 h(z\bar{z})}{\partial z\partial\bar{z}}\eta^{\alpha}\eta_\alpha + \frac{m+2}{4}g(z\bar{z})\eta_\alpha\bar{\eta}^{\alpha} : \{\tilde{\mathcal{J}}, \mathcal{H}_{m, 0}\} = 0,$$  \hspace{1cm} (4.46)

where $h(z\bar{z})$ is Killing potential (4.44) and $g(z\bar{z}) = 1 - \kappa^2(z\bar{z})^m$ is the component of special Kähler metrics (4.16).

V. CONCLUDING REMARKS

In this paper we have constructed the $\mathcal{N} = 8$ supersymmetric mechanics with potential term, whose configuration space is a special Kähler manifold of the rigid type. We observed that it can be viewed as a complex counterpart of the recently suggested $\mathcal{N} = 4$ supersymmetric mechanics [1, 5]. Then we constructed the $U(1)$-invariant one-dimensional special Kähler manifold and corresponding $\mathcal{N} = 8$ supersymmetric mechanics, including $\mathcal{N} = 8$ supersymmetric extensions of superintegrable perturbations of deformed two-dimensional oscillator and Coulomb systems considered in [8] as particular cases. It is an open question whether a $\mathcal{N} = 8$ supersymmetric counterparts of the hidden symmetries of these superintegrable systems exist.

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