UNIFORM BOUNDEDNESS DECIDING SETS, AND A QUESTION OF M. VALDIVIA

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Abstract. We prove that if a set $B$ in a Banach space $X$ can be written as an increasing, countable union $B = \cup_n B_n$ of sets $B_n$ such that no $B_n$ is uniform boundedness deciding, then also $B$ is not uniform boundedness deciding. From this we can give a positive answer to a question of M. Valdivia.

1. Introduction

Recall the classical Nikodým-Dieudonné-Grothendieck theorem (the NDG-theorem): If a family $(\mu_\alpha)$ of bounded additive measures on a $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$ satisfies that $\sup_\alpha |\mu_\alpha(E)| < \infty$ for all $E \in \Sigma$, then it is uniformly bounded, i.e., $\sup_\alpha |\mu_\alpha|(|\Omega|) < \infty$.

M. Valdivia proved in [8] the following: Whenever a sigma-algebra $\Sigma$ is represented as an increasing countable union, $\Sigma = \cup_n B_n$, then there is an index $N$ so that the NDG-theorem holds with $B_N$ instead of $\Sigma$. It is common to say that a family of sets has the Nikodým property if the NDG-theorem holds on it. Many non-$\sigma$-algebras are known to enjoy the Nikodým property while for example the algebra of finite and co-finite subsets of $\mathbb{N}$ fails the Nikodým property ([3, Ex. 5 page 18]). Let us for a moment say that a family of subsets of a set has the strong Nikodým property if the above Valdivia-version of the NDG-theorem holds on it.

Recently, in [9], Valdivia proved that the non-$\sigma$-algebra of Jordan measurable sets in $\mathbb{R}^k$ has the strong Nikodým property and asked whether every algebra with the Nikodým property automatically has the strong Nikodým property. We will see that this is indeed so, and that this is not at all a measure theory result, but a consequence of a category-like theorem valid in all Banach spaces (Theorem 1.1).

We now reformulate Valdivia’s question in Banach space language: Let $\mathcal{A}$ be an algebra of subsets of a set $\Omega$ (a set-algebra), and let $B(\mathcal{A})$ be the Banach space formed by the uniform closure of the linear span of the characteristic functions on $\mathcal{A}$. Put $\mathcal{X}(\mathcal{A}) = \{\chi_E : E \in \mathcal{A}\}$. The dual of $B(\mathcal{A})$ (see e.g., [3, Theorem 1.13 page 6]) is the space $ba(\mathcal{A})$ of finitely additive measures on $\mathcal{A}$ with bounded variation, and the dual norm to the sup-norm in $B(\mathcal{A})$ is just the variation, $|\cdot|(|\Omega|)$. Naturally, if $f \in B(\mathcal{A})$ and $\mu \in ba(\mathcal{A})$, then the action of $\mu$ on $f$ is $\mu(f) = \int_{\Omega} f \, d\mu$.

A reformulation of the NDG-theorem is, in this terms: Let $\Sigma$ be a set $\sigma$-algebra. If a family $(\mu_\alpha) \subset B(\Sigma)^*$ is pointwise bounded on $\mathcal{X}(\mathcal{A}) \subset B(\Sigma)$, then it is uniformly bounded. Note that $\mathcal{X}(\mathcal{A})$ is a nowhere-dense subset of

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$B(\Sigma)$, so we observe a situation where the Uniform boundedness principle works without the “test-set” $\mathcal{X}(A)$ being of second Baire-category.

The observation that the NDG-theorem is a strong Uniform boundedness principle is of course well-known and classical (it is made a point of in [4]). Now say that a subset $X$ of a Banach space $X$ is uniform boundedness deciding (a ubd-set for short) if it has the following property: Whenever a family $(x^*_\alpha) \subset X^*$ is pointwise bounded on $X$, i.e., $\sup_\alpha |x^*_\alpha(x)| < \infty$ for all $x \in X$, then it is uniformly bounded, i.e., $\sup_\alpha \|x^*_\alpha\| < \infty$. Informally, $X$ is ubd if the Uniform boundedness principle holds on it. By the NDG-theorem, $\mathcal{X}(\Sigma)$ is always a ubd-set in $B(\Sigma)$, and an algebra $A$ has the Nikodým property exactly when $\mathcal{X}(A)$ is a ubd-set in $B(A)$. The concept of a ubd-set is not widely used, but one can find it e.g. in [1].

Of course the definition of a ubd-set can be just as well given in normed spaces. But then there may very well be no ubd-sets. It is not difficult to verify that a normed space is itself a ubd-set if and only if it is barrelled. Anyway, the reformulation of the NDG-theorem is a Banach space theorem and Valdivia’s question is a special case of a general Banach space question: Assume a ubd-set $\mathcal{X}$ in a Banach space $X$ is written as a countable, increasing union, $\mathcal{X} = \bigcup_n B_n$. Does there then exist an index $N$, so that $B_N$ is a ubd-set? Our main result is that this is indeed true, from which it follows that the answer to Valdivia’s question is “yes”.

What we will prove is the following statement, equivalent to the statement above:

**Theorem 1.1.** If a set $B$ in a Banach space $X$ can be written as an increasing, countable union, $B = \bigcup_n B_n$, of non-ubd-sets $B_n$, then also $B$ is non-ubd.

We will mostly put together more or less known and quite elementary facts from [7], that all in all add up to a proof of Theorem 1.1

Throughout $X$ will be a Banach space, $B_X$ and $S_X$ its unit ball and unit sphere, respectively, and the absolute convex hull of a set $A$ in $X$ will be denoted $\text{absco}(A)$. In the real case we have $\text{absco}(A) = \text{co}(\pm A)$, while in the complex case $\text{absco}(A) = \text{co}\{rA, |r| = 1\}$, where, as usual, $\text{co}(A)$ refers to the convex hull of a set $A$. For the linear span of a set $A \subset X$ we write $\text{span}(A)$. We mark the closure of a set by an “overline”; thus, e.g., $\overline{\text{absco}(A)}$ is the closed absolute convex hull of the set $A$.

2. **Characterizations of ubd-sets and a proof of the main theorem**

Most candidates of being ubd-sets are subset of the unit sphere. And if they are not, there is really no loss of generality in assuming that the set $B$ is a subset of the unit sphere $S_X$ because a family $(x^*_\alpha) \subset X^*$ is pointwise bounded on the set $B$ if and only if it is pointwise bounded on

$$B' = \{ \frac{x}{\|x\|} : x \in B \} \subset S_X.$$ 

Thus, we have
Lemma 2.1. If a set $B$ in a Banach space $X$ is a ubd-set in $X$, then also the set

$$B' = \left\{ \frac{x}{\|x\|} : x \in B \right\} \subset S_X$$

is a ubd-set in $X$. If $B$ is not ubd, then also $B'$ is not ubd.

Having Lemma 2.1 in mind, from now on we restrict ourselves to study ubd-sets consisting of norm-one elements.

Remark 2.1. What follows is a theory where unfortunately rather few concrete examples are known. We know from the NDG-theorem that the characteristic functions form a ubd-set in $B(\Sigma)$. In particular, we deduce from this that the extreme points of the unit balls in $\ell_\infty$ and $L_\infty[0,1]$ are ubd-sets. It is also known that the extreme points (even the exposed points) of the unit ball of a reflexive space forms a ubd-set (this follows from [2] and the characterizations below). If we go to spaces like $c_0$ or $\ell_1$ no proper ubd-set of $S_X$ is known to the author. The NDG-theorem makes it natural to guess that the inner functions in $H^\infty(D)$ is also a ubd-set, but whether it really is so is not known. This problem has however been investigated, see [1].

Let us call a set $A \subset S_X$ with the property that there exists some number $\delta > 0$ such that $\text{absco}(A) \supset \delta \cdot B_X$ norming for the dual. The reason for this term is that the function $\psi(x^*) = \sup_{x \in A} |x^*(x)|$ defines an equivalent norm on $X^*$ exactly when there exists a $\delta > 0$ such that $\text{absco}(A) \supset \delta \cdot B_X$. The next simple result links the ubd-property to increasing countable unions, it was maybe first observed in [6]. We include the proof here for the reader’s convenience.

Lemma 2.2. A set $B$ in a Banach space $X$ is ubd if and only if whenever $B$ is written as an increasing countable union $B = \bigcup B_n$, then there exists an index $N$ such that $B_N$ is norming for the dual.

Proof. Assume $B$ has the property that whenever $B$ is written as an increasing countable union $B = \bigcup B_n$, then there exists an index $N$ such that $B_N$ is norming for the dual. Let $(x_\alpha)$ be pointwise bounded on $B$ and put $B_n = \{ x \in B : \sup_n |x_\alpha^*(x)| \leq n \}$. Then $B$ is the increasing countable union of the $B_n$’s. Now find $N$ and $\delta > 0$ such that $\text{absco}(B_N) \supset \delta \cdot B_X$. Then, for any $\alpha$,

$$\delta \|x_\alpha^*\| = \sup_{x \in \delta B_X} |x_\alpha^*(x)| \leq \sup_{x \in \text{absco}(B_N)} |x_\alpha^*(x)| \leq N,$$

and $(x_\alpha^*)$ is uniformly bounded by $N/\delta$.

For the converse assume $B = \bigcup B_n$ where no $B_n$ is norming for the dual. Then, for each $n$, there is $x_n^* \in S_{X^*}$ with $\sup_{x \in B_n} |x_n^*(x)| < 1/n$. Thus $(nx_n^*)$ is an unbounded sequence in $X^*$ which is easily seen to be pointwise bounded on $B$. \hfill \Box

We need the contrapositive statement of Lemma 2.2.

Corollary 2.3. A set $B$ in a Banach space $X$ is non-ubd if and only if it can be written as an increasing countable union, $B = \bigcup B_n$, of sets which are non-norming for the dual.
The next Lemma will, together with Corollary 2.3, be the operating tool in the proof of Theorem 1.1. It can be seen by combining 7 (Theorem 1.1) (a), (c) and (d) with the technique of proof of Theorem 1 in [5]. We give a direct proof here.

**Lemma 2.4.** A set $B$ in $S_X$ is non-ubd if and only if there exists a set $A \subset X$ such that $A$ is non-norming for the dual, but still $\text{span}(A) = \text{span}(B)$.

**Proof.** First assume $\text{span}(A) = \text{span}(B)$ for some non-norming set $A$. Then also $\overline{\text{absco}}(A)$ is non-norming and $\text{span}(A) = \bigcup_n n \cdot \overline{\text{absco}}(A)$. By Corollary 2.3, $\text{span}(B)$ is non-ubd. Thus $B$ is non-ubd.

Next assume $B$ is non-ubd. We are going to construct $A$. By Corollary 2.3, we can find an increasing family $B_n$ of sets which are non-norming for the dual, but $B = \bigcup_n B_n$. Let $A_1 = B_1$, $A_n = B_n \setminus B_{n-1}$, $n \geq 2$ and put

$$A = \bigcup_n \frac{1}{n} A_n.$$ 

Let $x = \sum_{j=1}^k a_j x_j \in \text{span}(B)$. For each $j$ there is $n(j)$ such that $x_j \in A_{n(j)}$. Thus, for each $j$, $x_j \in n(j) \cdot A$, and so $x \in \text{span}(A)$. It remains to argue that $A$ is not norming for the dual. Let $\varepsilon > 0$ and take $N$ so big that $1/N < \varepsilon$. Use that $B_N$ is non-norming for the dual to pick $x^* \in S_X^*$ with $\sup_{x \in B_N} |x^*(x)| < \varepsilon$. Since $(B_n)$ is an increasing sequence, we get $\sup_{x \in A_k} |x^*(x)| < \varepsilon$ whenever $k \leq N$. For $k > N$ we have

$$\sup_{x \in A_k} |x^*(x)| = \frac{1}{k} \sup_{x \in A_k} |x^*(x)| \leq \frac{1}{k} \frac{1}{N} < \varepsilon.$$ 

Thus $\sup_{x \in A} |x^*(x)| < \varepsilon$, and since $\varepsilon$ was arbitrary, we are done. \hfill \Box

Note that, by taking $\overline{\text{absco}}(A)$ instead of $A$ in the proof of Lemma 2.4, we may even assume that the set $A$ is closed and absolutely convex. We need a small formality later on:

**Lemma 2.5.** If $B_1$ and $B_2$ are non-ubd-subsets of $X$ and $B_1 \subset B_2$, then the sets $A_1$ and $A_2$ guaranteed by Lemma 2.4 can be found in such a way that $A_1 \subset A_2$.

**Proof.** We know that $B_1$ and $B_2$ can be written $B_1 = \bigcup_n C_n$ and $B_2 = \bigcup_n D_n$ for non-norming sets $C_n, D_n$. Write $E_n = C_n \cup D_n$. If $A_1$ has been produced from the representation $\bigcup_n C_n$, produce $A_2$ from the representation $B_2 = \bigcup_n E_n$. \hfill \Box

The proof of our main theorem is now just to put things together:

**Proof of Theorem 1.1.** Assume $B$ is an increasing countable sets of sets, $B = \bigcup B_n$, where each $B_n$ is non-ubd. By Lemma 2.4 it is enough to find a non-ubd set $A$ such that $\text{span}(A) = \text{span}(B)$. By Lemmas 2.4 and 2.5 we can find sets $A_1 \subset A_2 \subset \ldots$, non-norming for the dual, such that $\text{span}(A_n) = \text{span}(B_n)$ for all $n$. Put $A = \bigcup A_n$. Then, by Corollary 2.3, $A$ is non-ubd. But $\text{span}(A) = \text{span}(B)$ and the proof is over. \hfill \Box

Remark 2.2. Note that we for the proof of Theorem 1.1 have used no theorems at all. Everything is just playing with non-norming sets. An alternative approach to the Banach-Steinhaus theorem could be to first prove
Lemma 2.2 and then to prove that if $B$ is a countable increasing union of sets which are non-norming for the dual, then $B$ is of first Baire-category (this is immediate as a countable union of first category sets is again first category).

3. Another Consequence of Theorem 1.1

We have already seen that Theorem 1.1 implies that Valdivia’s stronger version of the Nikodým property is only apparently stronger. Let us now look at another result that follow and that seems to be new.

First, observe that by Lemma 2.4 and the comment just after its proof, there is a natural Banach space associated with every non-ubd-set $B$. Namely, let $Y$ be the span of $A$ with $\text{absco}(A)$ as unit ball (the Banach disc). Note that the range of the embedding $j : Y \to X$ is $\text{span}(B)$. A little work now shows that an alternative version of Lemma 2.4 is

**Lemma 3.1.** A set $B \subset S_X$ is non-ubd if and only if $\text{span}(B)$ is an operator range (of a 1-1 operator) into, but not onto, $X$.

Combining Theorem 1.1 with Lemma 3.1 now gives us

**Theorem 3.2.** A countable increasing union of dense operator ranges is again a dense operator range.

**Remark 3.1.** If $B$ is a subset in a dual space $X^*$, we may of course study ubd-sets with respect to elements of $X$. Using the term norming for $X$ instead of norming for the dual produces completely analogous results as in Section 2 for weak-star ubd-sets. Concerning Lemma 3.1 we get the same result, but now with “operator range (of a 1-1 weak-star) continuous (=adjoint) operator”.

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