Renormalization of QCD Coupling Constant in Terms of Physical Quantities

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Abstract

A renormalization scheme is suggested where QCD input parameters - quark mass and coupling constant - are expressed in terms of gauge invariant and infrared stable quantities. For the renormalization of coupling constant the quark anomalous electromagnetic moment is used; the latter is calculated in a two loop approximation. Examination of the renormalized $S$ matrix indicates confinement phenomenon already in the framework of perturbation theory.

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Usually, in the charge and the mass renormalization procedure the so-called MOM scheme is used, where the input parameters of the theory, appearing in action, are expressed in terms of on mass-shell Green’s functions (Itzykson and Zuber, 1980; Ramond, 1989). Another widely accepted scheme, MS, deals only with the divergent parts of the Green’s functions, not appealing to any condition on momenta. Evidently, due to the renormalization invariance, any scheme is admissible.

In gauge theories like quantum electrodynamics (QED) and quantum chromodynamics (QCD) the input parameters are considered to be gauge invariant and infrared stable (i.e. not containing infrared divergences, generated by a massless gauge field). Let us for brevity call any quantity with these properties a physical quantity. Obviously, the observables, being measurable, should be the physical quantities but the converse may be not true (as an example, consider any function of field strength $F_{\mu\nu}$ in QED). Since Green’s functions are not infrared stable and depend on a gauge (in general, their renormalization factors are nonlocal quantities (Bassetto et al., 1987)), the interpretation of the results obtained from the schemes mentioned above may be obscured. Renormalized quantities may share undesirable properties, originated from the dynamics of the nonphysical degrees of freedom and the extraction of the physical information may be complicated.

Therefore, from our point of view, in gauge theories the most illuminating would be a scheme operating only with the physical degrees of freedom, thus allowing to avoid complications mentioned above. The prescription is as follows: calculate in regularized theory as many physical quantities as the input parameters are and express the latter in terms of physical quantities. So, in QED as well as in QCD, containing only two input parameters - coupling constant and the fermion mass (for illustrative purposes we consider here only one flavor) - we need the expressions for two physical quantities:

$$\Sigma_\alpha = \Sigma_\alpha(g_0, m_0; n); \Sigma_\beta = \Sigma_\beta(g_0, m_0; n),$$  \hspace{1cm} (1)

where $g_0$ and $m_0$ are input parameters, the dependence of $\Sigma$ on momenta, being irrelevant here, is omitted, $n$ is a space-time dimension and throughout this paper we use dimensional regularization. In QED and QCD $\Sigma_\alpha$ and $\Sigma_\beta$ have no limit at $n = 4$ - regularization cannot be removed in (1). Now resolve $g_0$ and $m_0$ in terms of $\Sigma_\alpha$, $\Sigma_\beta$ and $n$ and substitute into the expression for any other physical quantity $\Sigma_\gamma$. In renormalizable theories, after all the calculations are performed, $\Sigma_\gamma$ becomes finite in terms of $\Sigma_\alpha$, $\Sigma_\beta$ - regularization can be removed:

$$\lim_{n \to 4} \Sigma_\gamma(g_0, m_0; n) = \lim_{n \to 4} \Sigma_\gamma(g_0(\Sigma_\alpha, \Sigma_\beta; n), m_0(\Sigma_\alpha, \Sigma_\beta; n); n) =$$

$$\lim_{n \to 4} \Sigma^*_\gamma(\Sigma_\alpha, \Sigma_\beta; n) \equiv \sigma_\gamma < \infty$$  \hspace{1cm} (2)

This scheme, describing the renormalization procedure as the expression of physical quantities in terms of physical quantities allows to avoid any significant difficulties and seems most transparent from the physical point of view.

1A similar procedure (in a different context) was discussed already in the early days of QED (Dyson, 1949).
The aim of this paper is to develop such a scheme for QCD.

It is clear that we need two physical quantities. In (Tarrach, 1981) it was demonstrated that in the covariant gauge the solution of the equation

\[ G^{-1}(p, m_0, g_0, \xi; n)\Psi_{in}(p) = 0 \]  

(3)

can be expressed in the framework of perturbation theory as

\[ m = m_0 + g_0^2\delta m_1(m_0; n) + g_0^4\delta m_2(m_0; n) + ..., \]  

(4)

where \( \delta m_1 \) and \( \delta m_2 \) do not depend on a gauge parameter \( \xi \) and are infrared stable. In (3), \( G \) is a quark propagator and \( \Psi_{in}(p) \) is the solution of the Dirac equation \((\gamma_\mu p_\mu - m)\Psi_{in}(p) = 0\). The investigation in an axial gauge (Japaridze et al., 1991) confirms the gauge invariance and the infrared stability of \( m \). In (Japaridze et al., 1991) it was shown that up to order \( g_0^6 \) the quark propagator has a simple pole at \( p^2 = m^2 \). Therefore, the quark pole mass can be considered as one of the \( \Sigma \)'s in relations (1) and to complete the scheme we have to point out the another physical quantity in QCD.

We propose the quark anomalous electromagnetic moment, defined, as usual, from the amplitude of the quark elastic scattering on an external electromagnetic field:

\[ \langle p|j_\mu|p + k\rangle A^\mu(k) = \bar{\Psi}_{in}(p)\left(\gamma_\mu F_1(k^2) + \frac{i}{2}[\gamma_\mu, \gamma_\nu]k_\nu F_2(k^2)\right)\Psi_{in}(p + k)A^\mu(k) \]  

(5)

The anomalous electromagnetic moment \( \chi \) is defined as \( F_2(0) \).

We calculate \( \chi \) up to order \( g_0^6 \) regularizing all the divergences (ultraviolet and infrared) in terms of space-time dimension \( n \):

\[ \chi = g_0^2\chi_1(m_0; n) + g_0^4\chi_2(m_0; n), \]  

(6)

where \( \chi_i \) is the \( i \)-loop contribution. The gauge dependent terms cancel in \( \chi_i \); \( \chi_1 \) is infrared stable and the infrared divergence appears in \( \chi_2 \). To obtain the expression of order \( g_0^6 \), containing only \( g_0 \)-associated divergences, we must use the relation (see (4))

\[ m_0 = m - g_0^2\delta m_1(m_0; n) + O(g_0^4) = m - g_0^2\delta m_1(m; n) + O(g_0^4) \]  

in (6), i.e. reexpand the loop expressions:

\[ \chi = g_0^2\chi_1(m - g_0^2\delta m_1(m; n); n) + g_0^4\chi_2(m; n) = \]

\[ = g_0^2\chi_1(m; n) + g_0^4\left(\chi_2(m; n) - \delta m_1\frac{\partial \chi_1(m; n)}{\partial m}\right) \]  

(7)

This reexpansion generates the infrared divergent term \( \partial \chi_1/\partial m \) which cancels the infrared divergent term in \( \chi_2 \).

So, \( \chi \) is gauge invariant and infrared stable. Omitting the intermediary calculations, we quote the final result:

\[ \chi = \frac{C_F g^2}{8\pi} \left(1 - \frac{g^2}{8\pi^2} \left[\frac{11C_A}{3} - 2n_f \left(\frac{1}{n - 4} + \ln \frac{m^2}{4\pi\nu^2}\right)\right] + \Phi + O(g_0^2; n - 4)\right), \]  

(8)
where $C_A \equiv N$, $C_F \equiv (N^2 - 1)/2N$ are the invariants of $SU(N)$ group, $g^2 \equiv g_0^2 \nu^{n-4}$ is the dimensionless coupling constant, $\nu$ is the mass scale, appearing in the framework of dimensional regularization (Itzykson and Zuber, 1980; Ramond, 1989), $n_f$ is the number of flavors, $m$ is the pole mass of the scattered quark, the external momenta obey $p^2 = (p + k)^2 = m^2$ and for the finite part $\Phi$ see the Appendix.

Thus from the point of view of renormalization procedure QED and QCD do not differ - in both of theories the input parameters can be expressed in terms of physical quantities. Surely, any scheme can be used but our goal was to demonstrate that it is possible to renormalize QCD coupling constant in terms of physical degrees of freedom. The scheme is described by the relations:

$$m_0 = m_0(m, \chi; n), \quad g_0 = g_0(m, \chi; n) \quad (9)$$

To use (9) we need the numerical values of $\chi$ and $m$. The gauge invariance and the infrared stability of these quantities does not mean necessarily that they are directly measurable, but these properties guarantee that the numerical values of $\chi$ and $m$ can be extracted from the experimental data. Of course, the numerical values of $\chi$ and $m$ depend not only on $m_0$ and $g_0$ but also on all other input parameters of, say, the Standard Model, but for the considered problem it is enough to analyze only QCD corrections.

The scheme (9) may be not suitable from the point of view of numerical convergence - the rate of convergence depends on the numerical values of $\chi$ and $m$. To improve the convergence, one has to introduce the so-called effective parameters (Itzykson and Zuber, 1980; Ramond, 1989) $g_R$ and $m_R$, i.e. to move to another scheme. It has to be pointed out that some statements formulated in terms of effective parameters (say, the increasing of the of the QCD effective coupling constant in the infrared region, sometimes interpreted as a physical effect of increasing the force between quarks at large distances) are scheme dependent and are not valid in another scheme. In other words, since the effective parameters are chosen arbitrarily, they can not affect the physical results.

To see this, let us illustrate how the renormalization group equation and the renormalization scheme arise in a quantum field theory. It is transparent in the framework of dimensional regularization, where we have two parameters $m_0$ and $g^2 \equiv g_0^2 \nu^{n-4}$. The mass scale $\nu$ defines the dimension of $g_0$, providing the dimensionless action. Parameters $g$ and $\nu$ are not independent:

$$g_0^2 = g^2(\nu_1)\nu_1^{4-n} = g^2(\nu_2)\nu_2^{4-n}, \quad (10)$$

i.e.

$$\frac{dg^2(\nu)}{d\nu} + \frac{4 - n}{\nu} g^2(\nu) = 0 \quad (11)$$

The renormalization group equation (11) can be presented in a familiar form by introducing the effective parameter $g_R(\nu)$ by means of the relation

$$g^2 = g^2(g_R^2(\nu), \nu) \quad (12)$$

leading to

$$\nu \frac{dg_R^2(\nu)}{d\nu} = \beta(g_R^2, \nu) \quad (13)$$
where

$$\beta = \lim_{n \to 4} \frac{1}{\partial g^2 / \partial g_R^2} \left[ (n - 4)g^2 - \frac{\partial g^2}{\partial \nu} \right]$$

(14)

The renormalization scheme is specified by the relation (12); then from (14) we obtain the appropriate $\beta$-function. For example, the choice

$$g^2 = c_1(n)g_R^2(\nu) + c_2(n)g_R^4(\nu) + ...$$

(15)

results in

$$\beta = \lim_{n \to 4} \frac{(n - 4)g_R^2}{c_1(n)} \left( 1 - \frac{c_2(n)}{c_1(n)} g_R^2 + ... \right)$$

(16)

The schemes are labelled by particular choices of $c_i$ (e.g MS is obtained if we choose $c_i = a_i/(n - 4)$). The introduction of $m_R(\nu)$ is based on the same argumentation.

So, the behavior of the effective charge defined through any particular scheme (e.g. leading to asymptotic freedom), being scheme dependent, may not lead to any valuable results - the behavior and numerical values of effective parameters depend on our choice and are not defined from the theory alone.

Thus, the advantage of scheme (9), besides gauge invariant and infrared stability is that it does not operates with the effective parameters, allowing us to avoid conclusions which are insignificant from the physical point of view.

As it becomes evident, the behavior of $g_R$ and $m_R$ does not lead to a scheme-independent statements, e.g. the absence of quarks and gluons in the asymptotic states. On the other hand, from the renormalizability of QCD it follows that in the expansion of any physical quantity $\sigma_\gamma$ (say, Wilson Loop)

$$\sigma_\gamma = \sum_{j=0}^{\infty} \chi^j \sigma_{\gamma, j}$$

(17)

the coefficients $\sigma_{\gamma, j}$ are gauge invariant, infrared stable and contain no ultraviolet divergences at $n = 4$, i.e. $\sigma_{\gamma, j}$ are finite.

So, the following question arises: are the quark and gluon degrees of freedom observable, i.e. do quarks and gluons appear in asymptotic states?

The answer may be obtained from the examination of the scattering matrix. We suggest the following criterion: if at least one $S$-matrix element built up in terms of fields is finite, the appropriate quanta appear in asymptotic states as a particles.

In QED it is well known that the electron elastic scattering amplitude is infrared divergent and taking into account the emission of photons leads to the cancellation of these infrared singularities - only the inclusive cross sections are finite (Itzykson and Zuber, 1980; Ramond, 1989). That is why we can say that from the Dirac-Maxwell equations it follows the existence of electrons and photons as an observable particles. Of course, we \textit{a priori} know that they exist and the $S$-matrix analysis is in accordance with the experimental data.

We consider the $S$-matrix element of scattering of quark on an external electromagnetic field. Note first that the existence of a physical quantity $\chi \equiv F_2(0)$ (see (5)) does not mean at
all that the amplitude of the elastic scattering is finite - in a full analogy with QED, the infrared divergences remain in the elastic amplitude after the charge and the mass renormalization is performed. Let us consider the gluon emission, assuming that the inclusive cross section might be finite. According to the LSZ reduction technique (Itzykson and Zuber, 1980; Ramond, 1989), for a gluon with the momentum $q$ and the polarization $\epsilon_\mu(q)$ in an asymptotic state we have

$$\frac{\epsilon_\mu(q)}{Z_3^{1/2}q^2}D_{\mu\rho}(q) = \frac{\epsilon_\mu(q)}{Z_3^{1/2}q^2}Z_3^{1/2} = Z_3^{1/2}\epsilon_\mu(q),$$

(18)

where $D_{\mu\rho}$ is the gluon propagator and $Z_3^{1/2}$ is the gluon wave function renormalization factor. In order $g^2$ the contribution of the gauge field in the residue $Z_3$ is (Itzykson and Zuber, 1980; Ramond, 1989)

$$Z_3^A(q^2) = iC_A\frac{g^2\nu^{A-n}}{2^{n+1}\pi^{n/2}}\frac{3n-2}{n-1}\frac{\Gamma^2(n/2-1)\Gamma(2-n/2)}{\Gamma(n-2)}(q^2)^{(n-4)/2},$$

(19)

where $\Gamma$ is the Euler’s function (Bateman and Erdelyi, 1973). The factor $Z_3$ contributes to the QCD coupling constant renormalization. To obtain the expression for the amplitude we have to perform all the calculations before the regularization is removed: renormalize $m_0$ and $g_0$ using (9), use (as we did for the amplitude (5)) the conditions $p^2 = (p + k)^2 = m^2$, $q^2 = 0$ and then set $n = 4$. The condition $q^2 = 0$ means that $Z_3^A(0)$ can be equated to zero. This is analogous to the procedure of vanishing of integrals corresponding to a tadpole-type diagrams in a massless theory- because of the analyticity of theory in $n$ we can always find the region where the result is well-defined and then continue analytically in the desired region of $n$ (Itzykson and Zuber, 1980; Ramond, 1989). So

$$Z_3(q^2) = Z_3^A(q^2) + Z_3^{\text{fermion}}(q^2) \to Z_3^{\text{fermion}}(0)$$

(20)

at $q^2 \to 0$. This means that only part of the $g_0$ renormalization constant, namely $C_A/2 - 2n_f/3$ is restored (compare to $(11C_A - 2n_f)/3$ in expression (8)). In other words, $Z_3(0)$ does not renormalize $g_0$.

Therefore, if we consider the gluon emission in order to cancel the infrared divergences of the elastic scattering amplitude, the ultraviolet divergence appears and the cross section in order $g^4$ contains an unavoidable infinity. The $S$-matrix is the same in any scheme but the result is transparent while using the scheme (9), manipulating only with the physical degrees of freedom and not using the effective parameters.

Though the discussion above is not a proof, it can be considered as an indication on a confinement phenomenon already in the framework of perturbation theory. In other words, QCD might be an example of a field theory where fields do not necessarily refer to the physical particles.

2 Long time ago Schwinger (Schwinger, 1962), starting from the idea that fields are more fundamental entities than particles, realized this conjecture in a two-dimensional electrodynamics, where the particles, corresponding to fermionic degrees of freedom, are absent in asymptotic states.
The next step would be the consideration of the scattering of colorless bound states. At the present time we have no definite result for this problem.

To conclude, in QCD it is possible to define renormalization procedure operating only in the space of physical degrees of freedom. Though the relations (9) guarantee that the physical quantities built up in terms of quark and gluon fields are finite, this is not enough for the existence of these field quanta as physical particles. The examination of the quark scattering amplitude indicates that quark and gluon field quanta do not appear in asymptotic states.

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APPENDIX A:

The finite part $\Phi$ is

$$
\Phi = C_F \left( -\frac{55\pi^2}{9} - 8\gamma_E^2 + 2\pi^2 \ln 2 - 3\zeta(3) + \frac{493}{12} \right) - \frac{2n_f \gamma_E}{3} + \frac{49n_f}{18} - 22 +
\frac{26\pi^2}{9} + 4\gamma_E^2 + CA \left( \frac{131\pi^2}{18} + \frac{1675\gamma_E^2}{108} + \frac{17\gamma_E}{9} - \pi^2 \ln 2 + \frac{3}{2} \zeta(3) - \frac{959}{72} \right) - \frac{\pi^2}{2} \Delta_i^{1/2} -
\Delta_i (6 + 4\ln \Delta_i) + \frac{5\pi^2}{2} \Delta_i^{3/2} +
\Delta_i^2 \left[ 4\gamma_E - \frac{2\pi^2}{3} - 3\ln^2 \Delta_i - 8 + \left( \frac{4}{9} \ln \Delta_i - \frac{20}{27} \right) F_{i1} + \frac{4}{9} F'_{i1} + \left( \frac{4}{9} \ln \Delta_i - \frac{38}{27} \right) F_{i2} + \frac{4}{9} F'_{i2} \right]
+ \Delta_i^3 \left[ \left( \frac{2}{3} \ln \Delta_i - \frac{11}{9} \right) F_{i3} + \frac{1}{3} F'_{i3} + (3 - 2\ln \Delta_i) F_{i4} - 2F'_{i4} \right], \quad (A1)
$$

where $m$ is the mass of the scattered quark, $\Delta_i \equiv m_i^2/m^2$, $m_i \neq m$, $\gamma_E \simeq 0.5771$ is the Euler constant and

$$
\zeta(3) = \sum_{j=1}^{\infty} \frac{1}{j^3} \quad (A2)
$$

is the Riemann zeta function.

The $F_{ij}$'s are the generalized hypergeometric functions (Bateman and Erdelyi, 1973) of $\Delta_i$ and the parameters of $F_{ij}$ are as follows:

$$
F_{i1} =_3 F_4 \left( 1, \frac{n+2}{2}, \frac{n-3}{2}, \frac{n-2}{2}, \frac{n+1}{2} \right), \quad (A3)
$$

$$
F_{i2} =_2 F_3 \left( 1, \frac{n-1}{2}, \frac{n-2}{2} \right), \quad (A4)
$$

$$
F_{i3} =_3 F_4 \left( 1, \frac{n+2}{n+1}, \frac{n-1}{2}, \frac{n-2}{2}, \frac{n}{2} \right), \quad (A5)
$$

$$
F_{i4} =_1 F_2 \left( 1, \frac{n-2}{n+2}, \frac{n}{2} \right), \quad (A6)
$$

and

$$
F'_{ij} \equiv \frac{dF_{ij}}{dn} \quad (A7)
$$

The functions $F_{ij}$ and $F'_{ij}$ are considered at $n = 4$. 

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