C-parameter Distribution at N$^3$LL’ including Power Corrections

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We compute the $e^+e^-$ C-parameter distribution using the Soft-Collinear Effective Theory with a resummation to N$^3$LL’ accuracy of the most singular partonic terms. This includes the known fixed-order QCD results up to $\mathcal{O}(\alpha_s^3)$, a numerical determination of the two loop non-logarithmic term of the soft function, and all logarithmic terms in the jet and soft functions up to three loops. Our result holds for $C$ in the peak, tail, and far tail regions. Additionally, we treat hadronization effects using a field theoretic nonperturbative soft function, with moments $\Omega_n$. In order to eliminate an $\mathcal{O}(\Lambda_{QCD})$ renormalon ambiguity in the soft function, we switch from the MS to a short distance “$R_{gap}$” scheme to define the leading power correction parameter $\Omega_1$. We show how to simultaneously account for running effects in $\Omega_1$ due to renormalon subtractions and hadron mass effects, enabling power correction universality between C-parameter and thrust to be tested in our setup. We discuss in detail the impact of resummation and renormalon subtractions on the convergence. In the relevant fit region for $\alpha_s(m_Z)$ and $\Omega_1$, the perturbative uncertainty in our cross section is $\approx 3\%$ at $Q = m_Z$.

I. INTRODUCTION

The study of event shape distributions in $e^+e^-$ colliders has served as an excellent avenue to understand the structure of jets in QCD. Currently they also provide an important testing ground for new achievements in theoretical formalism, that can then also be extended to applications at hadron colliders. Moreover, event shapes provide accurate determinations of the strong coupling constant, see for example [1] and [2]. Experimentally they have been measured with high accuracy, and there exist perturbative computations at $\mathcal{O}(\alpha_s^3)$ [3–8]. The use of the Soft-Collinear Effective Theory (SCET) [9–12] has simplified the analysis of factorization theorems for event shapes [13–16], enabling a resummation of large perturbative logs at next-to-next-to-leading log (N$^3$LL’) order and high-precision analyses for thrust and the heavy jet mass distributions [17–20]. The corresponding factorization theorems have recently been extended to oriented event shapes in [21] and to account for the effects of virtual and real secondary massive quark radiation in [22,23].

Our main motivations for studying the C-parameter distribution are:

a) Extend the theoretical precision of the logarithmic resummation for C-parameter from NLL + $\mathcal{O}(\alpha_s^2)$ to N$^3$LL + $\mathcal{O}(\alpha_s^3)$.

b) Implement the leading power correction $\Omega_1$ using only field theory and with sufficient theoretical precision to provide a serious test of universality between C-parameter and thrust.

c) Determine $\alpha_s(m_Z)$ using N$^3$LL’ + $\mathcal{O}(\alpha_s^3)$ + $\Omega_1$ theoretical precision for $C$, to make this independent extraction competitive with the thrust analysis carried out at this level in Refs. [19,20].

In this article we present the theoretical calculation and analysis that yields a N$^3$LL’ + $\mathcal{O}(\alpha_s^3)$ + $\Omega_1$ cross section for C-parameter, and we analyze its convergence and perturbative uncertainties. A numerical analysis that obtains $\alpha_s(m_Z)$ from a fit to a global C-parameter dataset, and investigates the power correction universality will be presented in a companion paper [24]. Preliminary versions of these results were presented in [25,26].

A nice property of C-parameter is that its definition does not involve any minimization procedure, unlike thrust. This makes its determination in data or Monte Carlo simulations computationally inexpensive. Unfortunately, this does not translate into a simplification of perturbative theoretical computations, which are similar to those for thrust.

The resummation of singular logarithms in C-parameter was first studied by Catani and Webber in Ref. [27] using the coherent branching formalism [28], where next-to-leading log (NLL) accuracy was achieved. Making use of SCET in this article we achieve a resummation at N$^3$LL order. The relation between thrust and C-parameter in SCET discussed here has been used in the Monte Carlo event generator GENEVA [29], where a N$^3$LL’ (next-to-next-to-leading log) C-parameter result was presented. Nonperturbative effects for the C-parameter distribution have been studied by a number of authors: Gardi and Magnea [30], in the context of the dressed gluon approximation; Korchemsky and Tafat [31], in the context of a shape function; and by Dokshitzer and Webber [32] in the context of the dispersive model.

Catani and Webber [27] showed that up to NLL the cross sections for thrust and the reduced C-parameter

$$\bar{C} = \frac{C}{6},$$ (1)

are identical. Gardi and Magnea [30] showed that this
relation breaks down beyond NLL due to soft radiation at large angles. Using SCET we confirm and extend these observations by demonstrating that the hard and jet functions, along with all anomalous dimensions, are identical for thrust and $\bar{C}$ to all orders in perturbation theory. At any order in perturbation theory the perturbative non-universality of the singular terms appears only through fixed-order terms in the soft function, which differ starting at $O(\alpha_s)$.

There is also a universality between the leading power corrections for thrust and $C$-parameter which has been widely discussed [32–35]. This universality has been proven nonperturbatively in Ref. [35] using the field theory definition of the leading power correction with massless particle kinematics. In our notation this relation is

$$\Omega_1^C = \frac{3\pi}{2} \Omega_1^T.$$  \hspace{1cm} (2)

Here $\Omega_1^T$ is the first moment of the nonperturbative soft functions for the event shape $e$, and in the tail of the distribution acts to shift the event shape variable

$$\hat{\sigma}(e) \rightarrow \hat{\sigma}(e - \Omega_1^T/Q),$$  \hspace{1cm} (3)

at leading power. The exact equality in Eq. (2) can be spoiled by hadron mass effects [35], which have been formulated using a field theoretic definition of the $\Omega_1^T$ parameters in Ref. [37]. Even though nonzero hadron masses can yield quite large effects for some event shapes, the universality breaking corrections between thrust and $C$-parameter are at the 2.5% level, and hence for our purposes are small relative to other uncertainties related to determining $\Omega_1$. Since relations like Eq. (2) do not hold for higher moments $\Omega_{\alpha > 1}$ of the nonperturbative soft functions, these are generically different for thrust and $C$-parameter.

Following Ref. [19], a rough estimate of the impact of power corrections can be obtained from the experimental data with very little theoretical input. We write $(1/\sigma) d\sigma/dC \simeq h(C) - \Omega_1^C/Q + \cdots$ for the tail region, and assume the perturbative function $h(C)$ is proportional to $\alpha_s$. Then one can easily derive that if a value $\alpha_s$ is extracted from data by setting $\Omega_1^C = 0$, then the change in the extracted value $\delta\alpha_s$ when $\Omega_1^C$ is present will be

$$\frac{\delta\alpha_s}{\alpha_s} \simeq \frac{\Omega_1^C}{Q} \frac{h'(C)}{h(C)},$$  \hspace{1cm} (4)

where the slope factor $h'(C)/h(C)$ should be constant at the level of these approximations. By looking at the experimental results at the Z-pole shown in Fig. 1 we see that this is true at the level expected from these approximations, finding $h'(C)/h(C) \simeq -3.3 \pm 0.8$. This same analysis for thrust $T = 1 - \tau$ involves a different function $h$ and yields $[h'(\tau)/h(\tau)] \tau \simeq -14 \pm 4$ [19]. It is interesting to note that even this very simple analysis gives a value $[h'(\tau)/h(\tau)] \tau [h'(C)/h(C)]_C \simeq 4.2$ that is very close to the universality prediction of $3\pi/2 = 4.7$. In the context of Eq. (2) this already implies that in a $C$-parameter analysis we can anticipate the impact of the power correction in the extraction of the strong coupling to be quite similar to that in the thrust analysis [19], where $\delta\alpha_s/\alpha_s \simeq -9\%$.

In Ref. [38] it was shown that within perturbation theory the $C$-parameter distribution reaches an infinite value at a point in the physical spectrum $0 < C < 1$, despite being an infrared and collinear safe observable. This happens for the configuration that distinguishes planar and non-planar partonic events and first occurs at $O(\alpha_s^2)$ where one has enough partons to create a non-planar event at the value $C_{\text{shoulder}} = 3/4$. However, this singularity is integrable and related to the fact that at $O(\alpha_s)$ the cross section does not vanish at the three-parton endpoint $C_{\text{shoulder}}$. In Ref. [38] this deficiency was cured by performing soft gluon resummation at $C_{\text{shoulder}}$ to achieve a smooth distribution at LL order. Since $C_{\text{shoulder}}$ is far away from the dijet limit (in fact, it is a pure three-jet configuration) we will not include this resummation. In our analysis the shoulder effect is included in the non-singular contributions in fixed-order, and when the partonic distribution is convoluted with a nonperturbative shape function the shoulder effect is smoothed out, providing a continuous cross section across the entire $C$ range.

This paper is organized as follows: In Sec. II we review the definition of $C$-parameter and highlight some of its properties and relations to other event shapes, and in Sec. III we study simplifications that occur for the calculation of $C$-parameter when we consider soft and collinear radiation in the dijet limit. The latter are necessary to derive the $C$-parameter factorization theorem and to address the question of which perturbative objects are common with thrust. In Sec. IV we establish the factorization theorem for $C$ and show how to perform large log

![FIG. 1. Plot of $d/dC \ln[(1/\sigma)d\sigma/dC] \simeq h'(C)/h(C)$, computed from experimental data at $Q = m_Z$. The derivative is calculated using the central difference with neighboring points.](image-url)
resummation using the renormalization group evolution (RGE). Some details on the form of the kinematically power suppressed terms known as nonsingular contributions to the C-parameter distribution are discussed in Sec. XI and in Sec. XII we show results for our numerical determination of the non-logarithmic coefficients of the two-loop soft function. Section XII contains a presentation of the form of the nonperturbative power corrections and their definition in a renormalon-free scheme. Our method to implement scale variation using profile functions is discussed in Sec. VII. In Sec. VIII we present a comparison of our SCET prediction, expanded beyond those in the main body of the paper. In App. B we present a discussion of how the Rgap scheme is handled in the conclusions and an outlook.

The work contains seven appendices: Appendix A provides all needed formulae for the singular cross section beyond those in the main body of the paper. In App. B we present a comparison of our SCET prediction, expanded at fixed-order, with the numerical results in full QCD at $O(\alpha_s^2)$ and $O(\alpha_s^3)$ from EVENT2 and EERAD3, respectively. In App. C we give a general formula for the one-loop soft function, valid for any event shape which is not recoil-sensitive. In App. D we give analytic expressions for the $B_i$ and $G_{ij}$ coefficients of the fixed-order singular logs up to $O(\alpha_s^3)$ according to the exponential formula of Sec. IX. The R-evolution of the renormalon-free gap parameter is described in App. E. Appendix F is devoted to a discussion of how the Rgap scheme is handled in the shoulder region above $C_{\text{shoulder}} = 3/4$. Finally, in App. G we show results for the perturbative gap subtraction series based on the C-parameter soft function.

II. DEFINITION AND PROPERTIES OF C-PARAMETER

C-parameter is defined in terms of the linearized momentum tensor:

$$\Theta^{\alpha\beta} = \frac{1}{\sum_i |\vec{p}_i|} \sum_i \frac{p_i^\alpha p_i^\beta}{|\vec{p}_i|},$$

(5)

where $\alpha = 1, 2, 3$ are spacial indices and $i$ sums over all final state particles. Since $\Theta$ is a symmetric positive semi-definite matrix, its eigenvalues are real and nonnegative. Let us denote them by $\lambda_i$, $i = 1, 2, 3$. As $\Theta$ has unit trace, $\sum_i \lambda_i = 1$, which implies that the eigenvalues are bounded $0 \leq \lambda_i \leq 1$. Without loss of generality we can assume $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. The characteristic polynomial for the eigenvalues of $\Theta$ is:

$$x^3 - x^2 + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) x - \lambda_1 \lambda_2 \lambda_3 = 0.$$ 

(6)

C-parameter is defined to be proportional to the coefficient of the term linear in $x$:

$$C = 3(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)$$

$$= 3(\lambda_1 + \lambda_2)(1 - \lambda_1 - \lambda_2^2),$$

(7)

where we have used the unit trace property to write $\lambda_3$ in terms of $\lambda_1$ and $\lambda_2$ in order to get the second line. Similarly one defines $D$-parameter as $D = 27 \lambda_1 \lambda_2 \lambda_3$, proportional to the $x$-independent term in the characteristic equation. Trivially one also finds that

$$\text{Tr}[\Theta^2] = \sum_i \lambda_i^2 = 1 - \frac{2}{3} C,$$ 

(8)

where again we have used $\lambda_3 = 1 - \lambda_1 - \lambda_2$. We can easily compute $\text{Tr}[\Theta^2]$ using Eq. (5)

$$\text{Tr}[\Theta^2] = \frac{1}{(\sum_i |\vec{p}_i|)^2} \sum_{ij} (\vec{p}_i \cdot \vec{p}_j)^2$$

$$= \frac{1}{(\sum_i |\vec{p}_i|)^2} \sum_{ij} |\vec{p}_i||\vec{p}_j| \cos^2 \theta_{ij}$$

$$= 1 - \frac{1}{(\sum_i |\vec{p}_i|)^2} \sum_{ij} |\vec{p}_i||\vec{p}_j| \sin^2 \theta_{ij}.$$ 

(9)

From the last relation one gets the familiar expression:

$$C = \frac{3 \sum_{ij} |\vec{p}_i||\vec{p}_j| \sin^2 \theta_{ij}}{(\sum_i |\vec{p}_i|)^2}.$$ 

(10)

From Eq. (10) and the properties of $\lambda_i$ it follows that $C \geq 0$, and from the second line of Eq. (7) one finds that $C \leq 1$, and the maximum value is achieved for the symmetric configuration $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$. Hence $0 \leq C \leq 1$. Planar events have $\lambda_3 = 0$. To see this simply consider that the planar event defines the $x-y$ plane and then any vector in the $z$ direction is eigenstate of $\Theta$ with zero eigenvalue. Hence planar events have $D = 0$ and $C = 3 \lambda_1 (1 - \lambda_1)$, which gives a maximum value for $\lambda_1 = \lambda_2 = 1/2$, and one has $0 \leq C_{\text{planar}} \leq 3/4$. Thus $C > 3/4$ needs at least four particles in the final state. C-parameter is related to the first non-trivial Fox-Wolfram parameter $H_1$. The Fox-Wolfram event shapes are defined as follows:

$$H_i = \sum_{ij} \frac{|\vec{p}_i||\vec{p}_j|}{Q^2} P_i(\cos \theta_{ij}).$$ 

(11)

One has $H_0 = 1$, $H_1 = 0$ and

$$H_2 = 1 - \frac{3}{2} \sum_{ij} |\vec{p}_i||\vec{p}_j| \sin^2 \theta_{ij},$$ 

(12)

which is similar to Eq. (10). It turns out that for massless partonic particles they are related in a simple way: $H_2 = 1 - C_{\text{part}}$. As a closing remark we note that for massless particles C-parameter can be easily expressed in terms of scalar products with four vectors:

$$C_{\text{part}} = 3 - \frac{3}{2} \sum_{ij} \frac{(p_i \cdot p_j)^2}{E_i E_j}.$$ 

(13)
III. C-PARAMETER KINEMATICS IN THE DIJET LIMIT

We now show that in a dijet configuration with only soft\(^2\) n-collinear, and \(\bar{n}\)-collinear particles, the value of C-parameter can, up to corrections of higher power in the SCET counting parameter \(\lambda\), be written as the sum of contributions from these three kinds of particles:

\[
C_{\text{dijet}} = C_n + C_{\bar{n}} + C_s + O(\lambda^4).
\]  

To that end we define

\[
C_{\text{dijet}} = C_{n,n} + C_{n,\bar{n}} + 2 C_{n,s} + 2 C_{\bar{n},s} + C_{s,s} - \frac{3}{2} \left( \sum_{i<j} |p_i| |p_j| \sin^2 \theta_{ij} \right).
\]  

The various factors of 2 take into account that for \(a \neq b\) one has to add the symmetric term \(a \leftrightarrow b\).

The SCET\(_1\) power counting rules implies the following scaling for momenta: \(p_i^\mu \sim Q \lambda^2\), \(p_i \sim Q (\lambda^2, 1, \lambda)\), \(p_n \sim Q (1, \lambda^2, \lambda)\), where we use the light-cone components \(p_i^\mu = (p_i^+ p^- p^\perp)\). Each one of the terms in Eq. (15), as well as \(C_{\text{dijet}}\) itself, can be expanded in powers of \(\lambda\):

\[
a_{a,b} = \sum_{i=0} C_{a,b}^{(i)}, \quad C_{a,b}^{(i)} \sim O(\lambda^{i+2}).
\]  

The power counting implies that \(C_{\text{dijet}}\) starts at \(O(\lambda^2)\) and \(C_{s,s}\) is a power correction since \(C_{s,s}^{(0)} = 0\), while \(C_{s,s}^{(2)} \neq 0\). All \(n\) and \(\bar{n}\)-collinear particles together will be denoted as the collinear particles with \(c = n \cup \bar{n}\). The collinear particles have masses much smaller than \(Q \lambda\) and can be taken as massless at leading power. For soft particles we have perturbative components that can be treated as massless when \(Q \lambda^2 \gg \Lambda_{\text{QCD}}\), and non-perturbative components that always should be treated as massive. Also at leading order one can use \(\sum_i |\vec{p}_i| = \sum_{i\in E} |\vec{p}_i| + O(\lambda^2) = Q + O(\lambda^2)\) and \(|\vec{p}_i| = E_{i\in E} + O(\lambda^2)\).

Defining \(C_s = 2 C_{n,s} + 2 C_{\bar{n},s}\) we find:

\[
C_s = \frac{3}{Q^2} \sum_{i\in E,j\in \bar{E}} |\vec{p}_i| |\vec{p}_j| \sin^2 \theta_{ij} - \frac{3}{Q^2} \sum_{i\in E} |\vec{p}_i| \sum_{j\in \bar{E}} |\vec{p}_j| \left[ \sin^2 \theta_{ij} + O(\lambda^2) \right] = \frac{3}{Q^2} \sum_{j\in \bar{E}} (p_j^+ p^- |\vec{p}_j|) + O(\lambda^4) = \frac{3}{Q} \sum_{j\in \bar{E}} p_j^+ \cosh \eta_j + O(\lambda^4),
\]  

where the last displayed term will be denoted as \(C_s^{(0)}\).

Here \(\theta_{ij}\) is the angle between the three-momenta of a particle and the thrust axis, and hence is directly related to the pseudo-rapidity \(\eta_j\). Also \(p_j^+ = |\vec{p}_j^+|\) is the magnitude of the three-momentum projection normal to the thrust axis. To get to the second line we have used that \(\sin \theta_{ij} = \sin \theta_j + O(\lambda^2)\), and to get the last line we have used \(\sin \theta_j = |p_j^-|/|\vec{p}_j^+| = 1/\cosh \eta_j\). In order to compute the partonic soft function it is useful to consider \(C_s^{(0)}\) for the case of massless particles:

\[
C_s^{(0)} \bigg|_{m=0} = \frac{6}{Q} \sum_{j\in \bar{E}} p_j^+ p_{\bar{j}}^-.
\]

Let us next consider \(C_{n,n}\) and \(C_{n,\bar{n}}\). Using energy conservation and momentum conservation in the thrust direction one can show that up to \(O(\lambda^2)\), \(E_n = E_{\bar{n}} = Q/2\). All \(n\)-collinear particles are in the plus-hemisphere, and all the \(\bar{n}\)-collinear particles are in the minus-hemisphere. Here the plus- and minus-hemispheres are defined by the thrust axis. For later convenience we define \(\bar{p}_n^\mu = \sum_{i\in n} p_i^\mu\) and \(E_{\bar{n}} = p_{\bar{n}}^0\) with \(a \in \{n, \bar{n}\}\) denoting the set of collinear particles in each hemisphere. We also define \(s_a = p_a^2\).

For \(C_{n,n}\) one finds

\[
C_{n,n} = \frac{3}{2Q^2} \sum_{i,j \in n} |\vec{p}_i| |\vec{p}_j| (1 - \cos \theta_{ij})(1 + \cos \theta_{ij}) = \frac{3}{Q^2} \sum_{i,j \in n} p_i \cdot p_j = \frac{3}{Q^2} \left( \sum_{i \in n} p_i \right)^2 + O(\lambda^4) = \frac{3}{Q^2} s_n + O(\lambda^4) = \frac{3}{Q} \sum_{i \in n} p_i^+ + O(\lambda^4) = \frac{3}{Q} p_n^+ + O(\lambda^4),
\]

and we can identify \(C_{n,n}^{(0)} = 3 \bar{p}_n^+ / Q\). To get to the second line we have used that for collinear particles in the same direction \(\cos \theta_{ij} = 1 + O(\lambda^2)\). In the third line we use the property that the total perpendicular momenta of each hemisphere is exactly zero, and that \(0 = \sum_{i \in \bar{E}} p_i^+ = \sum_{i \in n} p_i^+ + O(\lambda^2)\) and \(s_n = Q \bar{p}_n^+\). In a completely analogous way we get

\[
C_{n,\bar{n}}^{(0)} = \frac{3}{Q} \sum_{i \in \bar{E}} p_i^- = \frac{3}{Q} \bar{p}_{\bar{n}}^-.
\]

The last configuration to consider is \(C_{\bar{n},\bar{n}}\):

\[
2 C_{\bar{n},\bar{n}} = \frac{3}{Q^2} \sum_{i,j \in \bar{n}, \bar{E}} |\vec{p}_i| |\vec{p}_j| (1 - \cos \theta_{ij})(1 + \cos \theta_{ij}) = \frac{6}{Q^2} \sum_{i,j \in \bar{E}} (2E_i E_j - p_i \cdot p_j) (1 + O(\lambda^2)) = \frac{3}{Q^2} (\bar{p}_n^+ \bar{p}_{\bar{n}}^+ - \bar{p}_n^- \bar{p}_{\bar{n}}^- - 2 \bar{p}_n^+ \bar{p}_{\bar{n}}^+ + O(\lambda^4)) = \frac{3}{Q} \left( \bar{p}_n^+ \bar{p}_{\bar{n}}^- + O(\lambda^4) \right) = C_{n,n}^{(0)} + C_{\bar{n},\bar{n}}^{(0)} + O(\lambda^4).
\]

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\(^2\) In this paper, for simplicity we denote as soft particles what are sometimes called ultrasoft particles in SCET\(_1\).
In the second equality we have used that for collinear particles in opposite directions \( \cos \theta_{ij} = -1 + \mathcal{O}(\lambda^2) \), in the third equality we have written \( 2E = p^+ + p^- \), and in the fourth equality we have discarded the scalar product of perpendicular momenta since it is \( \mathcal{O}(\lambda^2) \) and also used that at leading order \( \not{E}_n^- = \not{E}_n^+ = Q + \mathcal{O}(\lambda^2) \) and \( \not{E}_n^+ \sim \lambda^2, \not{E}_n^- \sim \lambda^2 \). For the final equality we use the results obtained in Eqs. (19) and (20). Because the final result in Eq. (21) just doubles those from Eqs. (19) and (20) we can define

\[
C^{(0)}_{n,\alpha} = 2 C^{(0)}_{\bar{n},\bar{\alpha}}, \quad C^{(0)}_{\bar{n},\alpha} = 2 C^{(0)}_{n,\bar{\alpha}}.
\]  

(22)

Using Eqs. (19), (20), and (21) we then have

\[
C^{(0)}_{\alpha} = \frac{6}{Q} \not{E}_n^+, \quad C^{(0)}_{\bar{\alpha}} = \frac{6}{Q} \not{E}_n^-.
\]  

(23)

Equation (17) together with Eq. (23) finalize the proof of Eq. (14). As a final comment we note that one can express \( p^\pm = p^\pm \exp(\mp \eta) \) and since for \( n \)-collinear particles \( 2 \cosh \eta = \exp(\eta )[1 + \mathcal{O}(\lambda^2)] \) whereas for \( \bar{n} \)-collinear particles \( 2 \cosh \eta = \exp(-\eta)[1 + \mathcal{O}(\lambda^2)] \) one can also write

\[
C^{(0)}_{\alpha} = \frac{3}{Q} \sum_{i \in n} \frac{p^i_\perp}{\cosh \eta_i}, \quad C^{(0)}_{\bar{\alpha}} = \frac{3}{Q} \sum_{i \in \bar{n}} \frac{p^i_\perp}{\cosh \eta_i},
\]  

(24)

such that the same master formula applies for soft and collinear particles in the dijet limit and we can write

\[
C^{(0)}_{\text{dijet}} = \frac{3}{Q} \sum_i \frac{p^i_\perp}{\cosh \eta_i}.
\]  

(25)

### IV. FACTORIZATION AND RESUMMATION

The result in Eq. (14) leads to a factorization in terms of hard, jet, and soft functions. The dominant nonperturbative corrections at the order we are working come from the soft function and can be factorized with the following formula in the \( \overline{\text{MS}} \) scheme for the power corrections [31] [32] [33]:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dC} = \int dp \frac{1}{\sigma_0} \frac{d\hat{\sigma}}{dC} \left( C - \frac{p}{\bar{C}} \right) F_{C}(p),
\]  

(26)

\[
\frac{d\hat{\sigma}}{dC} = \frac{d\hat{\sigma}_s}{dC} + \frac{d\hat{\sigma}_\text{ns}}{dC}.
\]

Here \( F_{C} \) is a shape function describing hadronic effects, and whose first moment \( \Omega_2^C \) is the leading nonperturbative power correction in the tail of the distribution. \( \Omega_2^C \) and \( \Omega_2^\bar{C} \) are related to each other, as will be discussed further along with other aspects of power corrections in Sec. VII. The terms \( d\hat{\sigma}_s/dC, d\hat{\sigma}_\text{ns}/dC, \) and \( d\hat{\sigma}_\text{ns}/dC \) are the total partonic cross section and the singular and nonsingular contributions, respectively. The latter will be discussed in Sec. V.

After having shown Eq. (14) we can use the general results of [10] for the factorization theorem for the singular terms of the partonic cross section that splits into a sum of soft and collinear components. One finds:

\[
\frac{1}{\sigma_0} \frac{d\hat{\sigma}}{dC} = \frac{1}{6} \frac{d\hat{\sigma}_s}{\sigma_0} \frac{d\hat{\sigma}_s}{dC} + \frac{1}{6} \frac{d\hat{\sigma}_\text{ns}}{\sigma_0} \frac{d\hat{\sigma}_\text{ns}}{dC} + \frac{1}{6} \frac{d\hat{\sigma}_\text{ns}}{\sigma_0} \frac{d\hat{\sigma}_\text{ns}}{dC}.
\]  

\[
= \frac{Q}{6} H(Q, \mu) \int ds J_\tau(s, \mu) \bar{S}_C \left( Q\bar{C} - \frac{s}{Q\mu} \right),
\]  

(27)

where in order to make the connection to thrust more explicit we have switched to the variable \( \bar{C} = C/6 \). Here \( J_\tau \) is the thrust jet function which is obtained by the convolution of the two hemisphere jet functions, and where our definition for \( J_\tau \) coincides with that of Ref. [19]. It describes the collinear radiation in the direction of the two jets. Expressions up to \( \mathcal{O}(\alpha_s^2) \) and the logarithmic terms determined by its anomalous dimension at three loops are summarized in App. [A].

The hard factor \( H \) contains short-distance QCD effects and is obtained from the Wilson coefficient of the SCET to QCD matching for the vector and axial vector currents. The hard function is the same for all event shapes, and its expression up to \( \mathcal{O}(\alpha_s^2) \) is summarized in App. [A] together with the full anomalous dimension for \( H \) at three loops.

The soft function \( S_C \) describes wide-angle soft radiation between the two jets. It is defined as

\[
S_C(\ell, \mu) = \langle 0 | \bar{Y}_n Y_\ell \delta(\ell - Q\bar{C}) Y_n^\dagger Y_\ell^\dagger | 0 \rangle,
\]  

(28)

\[
S_\bar{C}(\ell, \mu) = \langle 0 | \bar{Y}_\ell Y_n \delta(\ell - \frac{Q\bar{C}}{6}) Y_n^\dagger Y_\ell^\dagger | 0 \rangle = 6 S_C(6\ell, \mu),
\]

where \( Y_n \) and \( Y_\ell \) are Wilson lines in the fundamental representation from 0 to \( \infty \), and \( \bar{Y}_n \) and \( \bar{Y}_\ell \) are Wilson lines in the anti-fundamental representation from 0 to \( \infty \). Here \( \bar{C} \) is an operator whose eigenvalues on physical states correspond to the value of \( C \)-parameter for that state: \( \bar{C} \mid X \rangle = C(\lambda) \mid X \rangle \). Since the hard and jet functions are the same as for thrust, the anomalous dimension of the \( C \)-parameter soft function has to coincide with the anomalous dimension of the thrust soft function to all orders in \( \alpha_s \) by consistency of the RGE. This allows us to determine all logarithmic terms of \( S_C \) up to \( \mathcal{O}(\alpha_s^2) \). Hence one only needs to determine the non-logarithmic terms of \( S_C \). We compute it analytically at one loop and use EVENT2 to numerically determine the two-loop constant, \( s_3^C \). The three-loop constant \( s_3^C \) is currently not known and we estimate it with a Padé, assigning a very conservative error. We vary this constant in our theoretical uncertainty analysis, but it only has a noticeable impact in the peak region, outside of where one fits for \( \alpha_s \).

In Eq. (27) the hard, jet and soft functions are evaluated at a common scale \( \mu \). There is no choice that simultaneously minimizes the logarithms of these three matrix
elements. One can use the renormalization group equations to evolve to \( \mu \) from the scales \( \mu_H \sim Q, \mu_J \sim Q^{1/2} \) and \( \mu_S \sim Q^{3/2} \) at which logs are minimized in each piece. In this way, large logs of ratios of scales are summed up in the renormalization group factors:

\[
\frac{1}{\sigma_0} \frac{d\hat{\sigma}}{dC} = \frac{Q}{6} H(Q, \mu_H) U_H(Q, \mu_H, \mu) \int ds ds' \frac{d}{dk} J_r(s, \mu_J) U^J_r(s - s', \mu_J, \mu) U^J_\mu(k, \mu, \mu_S) \exp \left( 3\pi \frac{\mu_j(s, \mu)}{\mu} \right) \tilde{S}_C \left( \frac{Q C - 3 \pi \Delta(R, \mu_S)}{6} - \frac{s}{Q} - \frac{k}{\mu} \right).
\]

The terms \( \delta \) and \( \Delta \) are related to the definition of the leading power correction in a renormalon-free scheme, as explained in Sec. VII below.

**V. NONSINGULAR TERMS**

We include the kinematically power suppressed terms in the C-parameter distribution using the nonsingular partonic distribution, \( d\hat{\sigma}_{\text{ns}}/dC \). We calculate the nonsingular distribution using

\[
\frac{d\hat{\sigma}_{\text{ns}}}{dC}(Q, \mu_{\text{ns}}) = \frac{d\hat{\sigma}_{\text{FO}}}{dC}(Q, \mu_{\text{ns}}) - \frac{d\hat{\sigma}_s}{dC}(Q, \mu_{\text{ns}}). \tag{30}
\]

This nonsingular distribution is independent of the scale \( \mu_{\text{ns}} \) order-by-order in perturbation theory as an expansion in \( \alpha_s(\mu_{\text{ns}}) \). We can identify the nontrivial ingredients in the nonsingular distribution by choosing \( \mu_{\text{ns}} = Q \) to give

\[
\frac{1}{\sigma_0} \frac{d\hat{\sigma}_{\text{ns}}}{dC} = \alpha_s(Q) \frac{Q}{2\pi} f_1(C) + \left( \frac{\alpha_s(Q)}{2\pi} \right)^2 f_2(C) + \left( \frac{\alpha_s(Q)}{2\pi} \right)^3 f_3(C) + \ldots
\]

We can calculate each \( f_i(C) \) using an order by order subtraction of the fixed-order singular distribution from the full fixed-order distribution as displayed in Eq. [30].

At one loop, we can write down the exact form of the full distribution as a two-dimensional integral

\[
\frac{1}{\sigma_0} \left| \frac{d\hat{\sigma}}{dC} \right|^{1-\text{loop}} = \alpha_s \frac{2\pi}{C} \int x_1 \int c_1 c_2 \theta(1 - x_1 - x_2) \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \times \delta \left( C - \frac{6(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}{x_1 x_2(2 - x_1 - x_2)} \right),
\]

which has support for \( 0 < C < 3/4 \), and jumps to zero for \( C > 3/4 \). After resolving the delta function, it becomes a one-dimensional integral that can be easily evaluated numerically. After subtracting off the one-loop singular piece discussed in Sec. IV we obtain the result for \( f_1 \) shown in Fig. 2. For \( C > 3/4 \) the nonsingular distribution at this order is simply given by the negative of the singular and for practical purposes one can find a parameterization for \( f_1 \) for \( C < 3/4 \), so we use

\[
f_1(C < 0.75) = -2.25168 + 0.506802 C + 0.184585 C^2 + 0.121051 C^3 + (0.890476 - 0.544484 C - 0.252937 C^2 - 0.0327797 C^3) \ln(C),
\]

\[
f_1(C > 0.75) = \frac{4}{3C} \left[ 3 + 4 \ln \left( \frac{C}{\pi} \right) \right]. \tag{33}
\]

For an average over \( C \) this result for \( f_1(C) \) is accurate to \( 10^{-7} \), and at worst for a particular \( C \) is accurate at \( 10^{-5} \). An exact closed form in terms of elliptic functions for the integral in Eq. (32) has been found in Ref. 30.

The full \( \mathcal{O}(\alpha_s^2) \) and \( \mathcal{O}(\alpha_s^3) \) fixed-order distributions can be obtained numerically from the Fortran programs.
EVENT2 $^{45}~^{46}$ and EERAD3 $^{5}~^{6}$, respectively. At $O(\alpha_s^3)$ we use log-binning EVENT2 results for $C < 0.2$ and linear-binning (with bin size of 0.02) results for $0.2 < C < 0.75$. We then have additional log-binning from $0.75 < C < 0.775$ (using bins in $\ln(C - 0.75)$) before returning to linear binning for $C > 0.775$. We used runs with a total of $3 \times 10^{11}$ events and an infrared cutoff $y_0 = 10^{-8}$. In the regions of linear binning, the statistical uncertainties are quite low and we can use a numerical interpolation for $f_2(C)$. For $C < 0.15$, we use the ansatz $f_2(C) = \sum_{i=0}^{3} a_i \ln^i C + a_4 C \ln C$ and fit the coefficients from EVENT2 output, including the constraint that the total fixed-order cross section gives the known $O(\alpha_s^2)$ coefficient for the total cross section. The resulting values for the $a_i$ are given as functions of $s_2^C$, the non-logarithmic coefficient in the partonic soft function. Details on the determination of this fit function and the determination of $s_2^C$ can be found below in Sec. VI. We find

$$s_2^C = -43.2 \pm 1.0,$$

(34)

whose central value is used in Figs. $^{3}~^{4}~^{5}$ and $^{11}$ and whose uncertainty is included in our uncertainty analysis. For $0.75 < C < 0.8$, we employ another ansatz $f_2(C) = \sum_{i=0}^{3} \sum_{k=0}^{i} b_{ik} (C - 0.75)^i \ln^k [8 (C - 0.75)/3]$. We use the values calculated in Ref. $^{38}$ $b_{01} = 59.8728$, $b_{02} = 43.3122$ and $b_{12} = -115.499$ and fit the rest of the coefficients to EVENT2 output. The final result for the two-loop nonsingular cross section coefficient then has the form

$$f_2(C) + \delta_{2}^{\text{low}} \sigma_{2}^{\text{low}(C)} + \delta_{2}^{\text{high}} \sigma_{2}^{\text{high}(C)}.$$

Here $f_2(C)$ gives the best fit in all regions, and $\delta_{2}^{\text{low}}$ and $\delta_{2}^{\text{high}}$ give the 1-\sigma error functions for the lower fit ($C < 0.75$) and upper fit ($C > 0.75$), respectively. The two variables $\delta_{2}^{\text{low}}$ and $\delta_{2}^{\text{high}}$ are varied during our theory scans in order to account for the error in the nonsingular function. In Fig. $^{3}$ we show the EVENT2 data as dots and the best-fit nonsingular function as a solid blue line. The uncertainties are almost invisible on the scale of this plot.

In order to determine the $O(\alpha_s^3)$ nonsingular cross section $f_3(C)$ we follow a similar procedure. The EERAD3 numerical output is based on an infrared cutoff $y_0 = 10^{-5}$ and calculated with $6 \times 10^5$ events for the three leading color structures and $10^7$ events for the three subleading color structures. The results are linearly binned with bin size of 0.02 for $C < 0.835$ and bin size of 0.01 for $C > 0.835$. As the three-loop numerical results have larger uncertainties than the two-loop results, we employ a fit for all $C < 0.835$ and use interpolation only above that value. The fit is split into two parts for $C$ below and above 0.75. For the lower fit we use the ansatz

$$f_3(C) = \sum_{i=1}^{5} a_i \ln^i (C - 0.75).$$

The results for the $a_i$ depend on the $O(\alpha_s^3)$ partonic soft function coefficient $s_3^C$, and a combination of the three-loop coefficients in the partonic soft and jet function, $s_3^C + 2 j_3$. Due to the amount of numerical uncertainty in the EERAD3 results, it is not feasible to fit for this combination, so each of these parameters is left as a variable that is separately varied in our theory scans. Above $C = 0.75$ we carry out a second fit, using the fit form $f_3(C) = \sum_{i=0}^{3} b_i \ln^i (C - 0.75)$. We use the fit $b_4 = 122.718$ predicted by exponentiation in Ref. $^{38}$. The rest of the $b_i$'s depend on $s_3^C$. The final result for the three-loop nonsingular cross section coefficient can once again be written in the form

$$f_3(C) + \delta_{3}^{\text{low}} \sigma_{3}^{\text{low}(C)} + \delta_{3}^{\text{high}} \sigma_{3}^{\text{high}(C)},$$

(36)

where $f_3(C)$ is the best-fit function and the $\delta_{3}^{\text{low}}$'s give the 1-\sigma error function for the low ($C < 0.75$) and high ($C > 0.75$) fits. Exactly like for the $O(\alpha_s^2)$ case, the $\delta_{3}^{\text{low}}$'s are varied in the final error analysis. In Fig. $^{4}$ we plot the EERAD3 data as dots, the best-fit function $f_3$ as a solid line, and the nonsingular results with $\delta_{3}^{\text{low}} = \delta_{3}^{\text{high}} = \pm 1$ as dashed lines. In this plot, we take $s_3^C$ to its best-fit value and $j_3 = s_3^C = 0$.

In the final error analysis we vary the nonsingular parameters encoding the numerical extraction uncertainty $\epsilon_i$'s, as well as the profile parameter $\mu_{\text{inx}}$. The uncertainties in our nonsingular fitting are obtained by taking $\epsilon_1^{\text{low}}$, $\epsilon_1^{\text{high}}$, $\epsilon_2^{\text{low}}$, and $\epsilon_2^{\text{high}}$ to be $-1$, 0, and 1 independently. The effects of $\epsilon_1^{\text{low}}$, $\epsilon_2^{\text{high}}$, and $\epsilon_3^{\text{high}}$ are essentially negligible in the tail region. Due to the high noise in the EERAD3 results, the variation of $\epsilon_3^{\text{high}}$ is not negligible in the tail region, but because it comes in at $O(\alpha_s^3)$, it is still small. We vary the nonsingular renormalization scale $\mu_{\text{inx}}$ in a way described in Sec. VIII.

In order to have an idea about the size of the nonsingular distributions with respect to the singular terms, we quote numbers for the average value of the one-, two-, and
three-loop distributions between $C = 0.2$ and $C = 0.6$: 5, 21, 76 (singular at 1-, 2- and 3-loops); $-0.4$, $-1$, $-4$ (nonsingular at 1-, 2- and 3-loops). Hence, in the region to be used for fitting $\alpha_s(n_f)$, the singular distribution is twelve (at 1-loop) to twenty (at 2-, 3-loops) times larger than the nonsingular one, and has the opposite sign. Plots comparing the singular and nonsingular cross sections for all $C$ values are given below in Fig. [3]

VI. DETERMINATION OF TWO-LOOP SOFT FUNCTION PARAMETERS

In this section we will expand on the procedure used to extract the $O(\alpha_s^2)$ non-logarithmic coefficient in the soft function using EVENT2. For a general event shape, we can separate the partonic cross section into a singular part where the cross section involves $\delta(e)$ or $\ln^k(e)/e$, and a nonsingular part with integrable functions, that diverge at most as $\ln^k(e)$. Of course, when these are added together and integrated over the whole spectrum of the event shape distribution, we get the correct fixed-order normalization:

$$\hat{\sigma}_{\text{had}} = \int_{e_{\text{max}}}^{e_{\text{min}}} de \left( \frac{d\hat{\sigma}_s}{de} + \frac{d\hat{\sigma}_{\text{ns}}}{de} \right). \tag{37}$$

Here $e_{\text{max}}$ is the maximum value for the given event shape (for C-parameter, $e_{\text{max}} = 1$). Using SCET, we can calculate the singular cross section at $O(\alpha_s^2)$, having the form

$$\frac{1}{\sigma_0} \frac{d\hat{\sigma}_s^{(2)}}{de} = A_8 \delta(e) + \sum_{n=0}^{3} D_n \left[ \ln^n e \right], \tag{38}$$

To define $d\hat{\sigma}_s^{(2)}/de$ we factor out $\alpha_s^2/16\pi$ and set $\mu = Q$. The only unknown term at $O(\alpha_s^2)$ for the C-parameter distribution is the two-loop constant $s_2^C$ in the soft function, which contributes to $A_8$. The explicit result for the terms in Eq. [38] can be obtained from $H(Q, \mu)P(Q, QC/6, \mu)$ which are given in App. [A]. This allows us to write the singular integral in Eq. [37] as a function of $s_2^C$ and known constants.

We extract the two-loop nonsingular portion of the cross section from EVENT2 data. Looking now at the specific case of C-parameter, we use both log-binning (in the small C region, which is then described with high accuracy) and linear binning (for the rest). By default we use logarithmic binning for $C < C_{\text{fit}} = 0.2$, but this boundary is changed between 0.15 and 0.25 in order to estimate systematic uncertainties of our method. In the logarithmically binned region we use a fit function to extrapolate for the full behavior of the nonsingular cross section. In order to determine the coefficients of the fit function we use data between $C_{\text{cut}}$ and $C_{\text{cut}}$. By default we take the value $C_{\text{cut}} = 10^{-4}$, but we also explore different values between $5 \times 10^{-5}$ and $7.625 \times 10^{-4}$ to estimate systematic uncertainties. We employ the following functional form, motivated by the expected nonsingular logarithms

$$\frac{1}{\sigma_0} \frac{d\hat{\sigma}_{\text{ns}}^{(2)}}{dC} = \sum_{i=0}^{3} a_i \ln^i C + a_4 C \ln^2 C, \tag{39}$$

taking the value $n = 1$ as default and exploring values $0 \leq n \leq 3$ as an additional source of systematic uncertainty.

For the region with linear binning, we can simply calculate the relevant integrals by summing over the bins. One can also sum bins that contain the shoulder region as its singular behavior is integrable. These various pieces all combine into a final formula that can be used to extract the two-loop constant piece of the soft function:

$$\hat{\sigma}_{\text{had}}^{(2)} = \int_0^1 dC \frac{d\hat{\sigma}_s^{(2)}}{dC} + \int_0^{C_{\text{fit}}} dC \frac{d\hat{\sigma}_{\text{fit}}^{(2)}}{dC} + \int_{C_{\text{fit}}}^1 dC \frac{d\hat{\sigma}_{\text{int}}^{(2)}}{dC}. \tag{40}$$

Using Eq. [40] one can extract $s_2^C$, which can be decomposed into its various color components as

$$s_2^C = C_F^2 s_2^{[C_F^2]} + C_F C_A s_2^{[C_F C_A]} + C_F n_f T s_2^{[n_f]}. \tag{41}$$

The results of this extraction are

$$s_2^{[C_F^2]} = -0.46 \pm 0.75, \quad s_2^{[C_F C_A]} = -29.08 \pm 0.13, \quad s_2^{[n_f]} = 21.87 \pm 0.03. \tag{42}$$

The quoted uncertainties include a statistical component coming from the fitting procedure, and a systematical component coming from the parameter variations explained above, added in quadrature. Note that the value for $s_2^{[C_F^2]}$ is consistent with zero, as expected from exponentiation [47]. For our analysis we will always take $s_2^{[C_F^2]} = 0$. We have cross checked that, when a similar extraction is repeated for the case of thrust, the extracted values are consistent with those calculated analytically in [18, 49]. This indicates a high level of accuracy in the fitting procedure. We have also confirmed that following the alternate fit procedure of [47] gives compatible results, as shown in App. [B].

VII. POWER CORRECTIONS AND RENORMALON-FREE SCHEME

The expressions for the theoretical prediction of the C-parameter distribution in the dijet region shown in
Eqs. [26] and [27] incorporate that the full soft function can be written as a convolution of the partonic soft function $S_C$ and the nonperturbative shape function $F_C$:

$$S_C(k, \mu) = \int dk' \hat{S}_C(k - k', \mu) F_C(k', \Delta_C).$$  \hfill (43)

Here, the partonic soft function $\hat{S}_C$ is defined in fixed order in $\overline{\text{MS}}$. The shape function $F_C$ allows a smooth transition between the peak and tail regions, where different kinematical expansions are valid, and $\Delta_C$ is a parameter of the shape function that represents an offset from zero momentum, and which will be discussed further below. By definition, the shape function satisfies the relations $F_C(k, \Delta_C) = F_C(k - 2\Delta_C)$ and $F_C(k < 0) = 0$. In the tail region, where $QC/6 \gg \Lambda_{\text{QCD}}$, this soft function can be expanded to give

$$S_C(k, \mu) = \hat{S}_C(k) - \frac{d\hat{S}_C(k)}{dk} \Omega^C_1 + O\left(\frac{\alpha_s \Lambda_{\text{QCD}}}{QC}, \frac{\Lambda_{\text{QCD}}^2}{Q^2 C^2}\right),$$  \hfill (44)

where $\Omega^C_1$ is the leading nonperturbative power correction in $\overline{\text{MS}}$ which effectively introduces a shift of the distribution in the tail region $[19]$. The $\Omega^C_1$ power correction

$$\Omega^C_1(\mu) = \langle 0 | Y^\dagger_n Y_n | (QC) Y_n Y_n | 0 \rangle,$$  \hfill (45)

is related to $\Omega^C_1$, the first moment of the thrust shape function, as given in Eq. [2]. In addition to the normalization difference that involves a factor of $3\pi/2$, their relation is further affected by hadron mass effects which cause an additional deviation at the 2.5% level (computed in Sec. [VII A]). The dominant contributions of the $O(\alpha_s \Lambda_{\text{QCD}}/QC)$ corrections indicated in Eq. [44] are log-enhanced and will be captured once we include the $\mu$-anomalous dimension for $\Omega^C_1$ that is induced by hadron mass effects [37]. There are additional $O(\alpha_s \Lambda_{\text{QCD}}/QC)$ corrections, which we neglect, that do not induce a shift. We consider hadron mass effects in detail in Sec. [VII A].

From Eq. [43] and the OPE of Eq. [44] we can immediately read off the relations

$$\int dk' k' F_C(k', \Delta_C) = 2 \Delta_C + \int dk' k' F_C(k') = \Omega^C_1,$$  \hfill (46)

which state that the first moment of the shape function provides the leading power correction and that the shape function is normalized. In the peak, it is no longer sufficient to keep only the first moment, as there is no OPE when $QC/6 \sim \Lambda_{\text{QCD}}$ and we must keep the full dependence on the model function in Eq. [43].

The partonic soft function in $\overline{\text{MS}}$ has an $O(\Lambda_{\text{QCD}})$ renormalon, an ambiguity which is related to a linear sensitivity in its perturbative series. This renormalon ambiguity is in turn inherited to the numerical values for $\Omega^C_1$ obtained in fits to the experimental data. It is possible to avoid this renormalon issue by switching to a different scheme for $\Omega^C_1$, which involves subtractions in the partonic soft function that remove this type of infrared sensitivity. Following the results of Ref. [42], we write $\Delta_C$ as

$$\Delta_C = \frac{3\pi}{2} \left[ \hat{\Delta}(R, \mu) + (\delta(R, \mu) \right].$$  \hfill (47)

The term $\delta(R, \mu)$ is a perturbative series in $\alpha_s(\mu)$ which has the same renormalon behavior as $\Omega^C_1$. In the factorization formula it is grouped into the partonic soft function $S_C$ through the exponential factor involving $\delta(R, \mu)$ shown in Eq. [29]. Upon simultaneous perturbative expansion of the exponential together with $S_C$, the $O(\Lambda_{\text{QCD}})$ renormalon is subtracted. The term $\hat{\Delta}(R, \mu)$ then becomes a nonperturbative parameter which is free of the $O(\Lambda_{\text{QCD}})$ renormalon. Its dependence on the subtraction scale $R$ and on $\mu$ is dictated by $\delta(R, \mu)$ since $\Delta_C$ is $R$- and $\mu$-independent. The subtraction scale $R$ encodes the momentum scale associated with the removal of the linearly infrared-sensitive fluctuations. The factor $3\pi/2$ is a normalization coefficient that relates the $O(\Lambda_{\text{QCD}})$ renormalon ambiguity of the $C$ soft function $S_C$ to the one for the thrust soft function. Taking into account this normalization we can use for $\delta(R, \mu)$ the scheme for the thrust soft function already defined in Ref. [19].

$$\delta(R, \mu) = \frac{R}{2} e^{\gamma_E} \frac{d}{d\ln(\tau_{R, \mu})} \ln \left[ S^\text{part}_T(x, \mu) \right]_{x=(iR e^{\gamma_E})^{-1}}.$$  \hfill (48)

where $S^\text{part}_T(x, \mu)$ is the position-space thrust partonic soft function. From this, we find that the perturbative series for the subtraction is

$$\delta(R, \mu) = Re^{\gamma_E} \sum_{i=1}^{\infty} \alpha^i_s(\mu) \delta^i(R, \mu).$$  \hfill (49)

Here the $\delta^i_{\geq 2}$ depend on both the adjoint Casimir $C_A = 3$ and the number of light flavors in combinations that are unrelated to the QCD beta function. Using five light flavors the first three coefficients have been calculated in Ref. [17] as

$$\frac{\pi}{2} \delta^1(R, \mu) = -1.33333 L_R,$$

$$\frac{\pi}{2} \delta^2(R, \mu) = -0.245482 - 0.732981 L_R - 0.813459 L_R^2,$$

$$\frac{\pi}{2} \delta^3(R, \mu) = -0.886828 - 0.977769 L_R$$

$$-1.22085 L_R^2 - 0.661715 L_R^3,$$  \hfill (50)

where $L_R = \ln(\mu/R)$. Using these $\delta$’s, we can make a scheme change on the first moment to what we call the
Rgap scheme:

\[ \Omega_i^G(R, \mu) = \Omega_i^G(\mu) - 3\pi \delta(R, \mu). \]  

(51)

In contrast to the \( \overline{\text{MS}} \) scheme \( \Omega_i^G(\mu) \) the Rgap scheme \( \Omega_i^G(R, \mu) \) is free of the \( \Lambda_{QCD} \) renormalon. From Eq. (46) it is then easy to see that the first moment of the shape function becomes

\[ \int dk \kappa F_C(k - 3\pi \Delta(R, \mu)) = \Omega_i^G(R, \mu). \]  

(52)

The factorization in Eq. (49) can now be written as

\[ S_C(k, \mu) = \int dk' e^{-3\pi \delta(R, \mu)} \frac{\pi}{n!} \hat{S}_C(k - k', \mu) \times F_C(k' - 3\pi \Delta(R, \mu)). \]  

(53)

The logs in Eq. (50) can become large when \( \mu \) and \( R \) are far apart. This imposes a constraint that \( R \sim \mu \), which will require the subtraction scale to depend on \( C \) in a way similar to \( \mu \). On the other hand, we also must consider the power counting \( \Omega^C \sim \Lambda_{QCD} \), which leads us to desire using \( R \approx 1 \) GeV. In order to satisfy both of these constraints in the tail region, where \( \mu \sim QC/6 \gg 1 \) GeV, we (i) employ \( R \sim \mu \) for the subtractions in \( \delta(R, \mu) \) that are part of the Rgap par-tonic soft function and (ii) use the R-evolution to relate the gap parameter \( \Delta(R, \mu) \) to the reference gap parameter \( \Delta(R, \mu_\Delta) \) with \( R_\Delta \sim \mu_\Delta \sim O(1 \) GeV\) where the \( \Lambda_{QCD} \) counting applies \cite{47, 50, 51}. The formulae for the \( R \)-RGE and \( \mu \)-RGE are

\[ R \frac{d}{dR} \Delta(R, \mu) = -R \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \Gamma_{\text{cusp}}(\alpha_s(\mu)/4\pi)^{n+1}, \]  

(54)

where for five flavors the \( \Gamma_{\text{cusp}} \) is given in App. A and the \( \gamma \) and the \( \gamma \) coefficients are given by

\[ \gamma_0 = 0, \gamma_1 = -43.954260, \gamma_2 = -606.52329. \]  

(55)

The solution to Eq. (54) is given at \( N^k \) LL by

\[ \tilde{\Delta}(R, \mu) = \tilde{\Delta}(R_\Delta, \mu_\Delta) + R e^{\gamma_e} \omega' [\Gamma_{\text{cusp}}, \mu, R] \]  

\[ + R_\Delta e^{\gamma_e} \omega' [\Gamma_{\text{cusp}}, R_\Delta, \mu_\Delta] \]  

\[ + \Lambda_{QCD}^{(k)} \sum_{j=0}^{k} (-1)^j S_j e^{i\pi b_1} \]  

\[ \times \left[ \Gamma(-b_1, t_1) - \Gamma(-b_1, t_0) \right] \]  

\[ \equiv \tilde{\Delta}(R_\Delta, \mu_\Delta) + \Delta^{\text{diff}}(R_\Delta, R, \mu_\Delta, \mu). \]  

(56)

The solution to Eq. (56) is given at \( N^k \) LL by

\[ \tilde{\Delta}(R, \mu) = \tilde{\Delta}(R_\Delta, \mu_\Delta) + R e^{\gamma_e} \omega' [\Gamma_{\text{cusp}}, \mu, R] \]  

\[ + R_\Delta e^{\gamma_e} \omega' [\Gamma_{\text{cusp}}, R_\Delta, \mu_\Delta] \]  

\[ + \Lambda_{QCD}^{(k)} \sum_{j=0}^{k} (-1)^j S_j e^{i\pi b_1} \]  

\[ \times \left[ \Gamma(-b_1, t_1) - \Gamma(-b_1, t_0) \right] \]  

\[ \equiv \tilde{\Delta}(R_\Delta, \mu_\Delta) + \Delta^{\text{diff}}(R_\Delta, R, \mu_\Delta, \mu). \]  

(56)

For the convenience of the reader the definition for \( \omega \) is provided in Eq. (A10) and the values for \( b_1 \) and the \( S_j \) are given in Eq. (51). In order to satisfy the power counting criterion for \( R \), we specify the parameter \( \Delta(R_\Delta, \mu_\Delta) \) at the low reference scales \( R_\Delta = \mu_\Delta = 2 \) GeV. We then use Eq. (56) to evolve this parameter up to a scale \( R(C) \), which is given in Sec. VIII of the paper, and satisfies the condition \( R(C) \sim \mu_\Delta) \) in order to avoid large logs. This R-evolution equation yields a similar equation for the running of \( \Omega_i^G(R, \mu_S) \), which is easily found from Eqs. (54) and (55).

We also apply the Rgap scheme in the nonsingular part of the cross section by using the convolution

\[ \int dk' e^{-3\pi \delta(R, \mu)} \frac{\pi}{n!} \hat{S}_C(k' - k, \mu) \times F_C(k' - 3\pi \Delta(R, \mu)). \]  

(57)

By employing the Rgap scheme for both the singular and nonsingular pieces, the sum correctly recombines in a smooth manner to the fixed-order result in the far-tail region.

Note that by using Eq. (47) we have defined the renormalon-free moment parameter \( \Omega_i^G(R, \mu) \) in a scheme directly related to the one used for the thrust analyses in Refs. \cite{19, 20}. This is convenient as it allows for a direct comparison to the \( \Omega_i \) fit results we obtained in both these analyses. However, many other renormalon-free schemes can be devised, and all these schemes are perturbatively related to each other through their relation to the \( \overline{\text{MS}} \) scheme \( \Delta_C \). As an alternative, we could have defined a renormalon-free scheme for \( \Omega_i^G \) by determining the subtraction \( \delta \) directly from the \( \hat{C} \) soft function \( \hat{S}_C \) using the analogue to Eq. (48). For future reference we give the results for the resulting subtraction function \( \delta_C \) in App. C.

In close analogy to Ref. \cite{19} we parametrize the shape function \( F_C \) in terms of the basis functions introduced in \cite{48, 49, 50}. In this expansion the shape function has the
form

\[ F_C(k, \lambda, \{c_i\}) = \frac{1}{\lambda} \left[ \sum_{n=0}^{N} c_n f_n \left( \frac{k}{\lambda} \right) \right]^2, \]

(58)

where the \( f_n \) are given by

\[ f_n(z) = 8 \frac{2z^3(2n + 1)}{3} e^{-2z} P_n[g(z)], \]

\[ g(z) = \frac{2}{3} [3 - e^{-4z}(3 + 12z + 24z^2 + 32z^3)] - 1, \]

(59)

and \( P_n \) denote the Legendre polynomials. The additional parameter \( \lambda \) is irrelevant when \( N \to \infty \). For finite \( N \) it is strongly correlated with the first moment \( \Omega^C_1 \) (and with \( c_1 \)). The normalization of the shape function requires that \( \sum_{n=0}^{N} c_n^2 = 1 \). When plotting and fitting in the tail region, where the first moment of the shape function \( \Omega^C_1 \) is the only important parameter, it suffices to take \( c_0 = 1 \) and all \( c_{i>0} = 0 \). In this case the parameter \( \lambda \) directly specifies our \( \Omega^C_1 \) according to

\[ \Omega^C_1(R_\Delta, \mu_\Delta) = \lambda + 3\pi \bar{\Delta}(R_\Delta, \mu_\Delta). \]

(60)

In the tail region where one fits for \( \alpha_s(m_\pm) \) there is not separate dependence on the nonperturbative parameters \( \lambda \) and \( \bar{\Delta}(R_\Delta, \mu_\Delta) \), they only appear together through the parameter \( \Omega^C_1(R_\Delta, \mu_\Delta) \). In the peak region, one should keep more \( c_i \)’s in order to correctly parametrize the nonperturbative behavior.

In Fig. 6 we plot the absolute value of the four components of the partonic fixed-order \( C \) distribution at \( \mathcal{O}(\alpha_s^3) \) in the \( \text{Rgap} \) scheme at \( Q = m_Z \). Resummation has been turned off. The cross section components include the singular terms (solid blue), nonsingular terms (dashed blue), and separately the contributions from terms that involve the subtraction coefficients \( \delta_i \) for both singular subtractions (solid red) and nonsingular subtractions (dashed red). The sum of these four components gives the total cross section (solid black line). One can observe that the nonsingular terms are significantly smaller than the singular ones in the tail region below the shoulder, i.e. for \( C < 0.7 \). Hence the tail region is completely dominated by the part of the cross section described by the SCET factorization theorem, where resummation matters most. Above the shoulder the singular and nonsingular \( C \) results have comparable sizes. An analogous plot for the thrust cross section is shown in Fig. 6. We see that the portion of the \( C \)-parameter distribution where the logarithmic resummation in the singular terms is important, is substantially larger compared to the thrust distribution.

\section{Hadron Mass Effects}

Following the analysis in Ref. [37], we include the effects of hadron masses by including the dependence of \( \Omega^C_1 \) on the distributions of transverse velocities

\[ r \equiv \frac{p_\perp}{\sqrt{p_\perp^2 + m_H^2}}, \]

(61)

where \( m_H \) is the nonzero hadron mass and \( p_\perp \) is the transverse velocity with respect to the thrust axis. For the massless case one has \( r = 1 \). However, when the hadron masses are nonzero, \( r \) can take any value in the range 0 to 1. The additional effects of the finite hadron masses cause non-trivial modifications in the form of the first moment of the shape function:

\[ \bar{\Omega}^C_1(\mu) = c_e \int_0^1 dr g_e(r) \bar{\Omega}_1(r, \mu), \]

(62)

where \( c_e \) denotes the specific event shape that we are studying, \( c_e \) is an event-shape dependent constant, \( g_e(r) \) is an event-shape dependent function \( \bar{\Omega}_1(r, \mu) \) is universal for all recoil-insensitive event shapes. Note that once hadron masses are included, there is no limit of the hadronic parameters that reduces to the case where hadron masses are not accounted for.

For the cases of thrust and \( C \)-parameter, we have

\[ c_C = 3\pi, \quad g_C(r) = \frac{2r^2}{\pi} K(1 - r^2), \]

(63)

\[ c_r = 2, \quad g_r(r) = 1 - E(1 - r^2) + r^2 K(1 - r^2), \]

\[ K(x) = \frac{4x^{2/3}}{\Gamma(1/3)^2} \int_0^1 \frac{1}{(1 - y)^{1/3}(1 - xy)^{1/3}} (y - x)^{1/3} dy, \]

(64)

\[ E(x) = \frac{1 - x^{1/3}}{x^{1/3}} \int_0^1 \frac{1}{(1 - y)^{1/3}(1 - xy)^{1/3}} (y - x)^{1/3} dy. \]

(65)

As discussed in Ref. [37], event shapes with a common \( g_e \) function belong to the same universality class. This means that their leading power corrections are simply related by the \( c_e \) factors. In this sense, \( g_e \) is universality-class dependent rather than event-shape dependent.
where $E(x)$ and $K(x)$ are complete elliptic integrals, whose definition can be found in Ref. [36]. Notice that $g_C(r)$ and $g_r(r)$ are within a few percent of each other over the entire $r$ range, so we expect the relation

$$2 \Omega_1^C(\mu) = 3 \pi \Omega_1^r(\mu) \left[ 1 + O(2.5\%) \right], \quad (64)$$

where the $O(2.5\%)$ captures the breaking of universality due to the effects of hadron masses. We have determined the size of the breaking by

$$\int_0^1 dr \left[ g_C(r) - g_r(r) \right] / \int_0^1 dr \left[ g_C(r) + g_r(r) \right] / 2 = 0.025. \quad (65)$$

All $r \in [0,1]$ contribute roughly an equal amount to this deviation, which is therefore well captured by this integral.

As indicated in Eq. (62) the moment $\Omega_1(r,\mu)$ in the MS scheme is renormalization-scale dependent and at LL satisfies the RGE of the form [37]

$$\dot{\Omega}_1(r,\mu) = \Omega_1(r,\mu_0) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_1(r)}. \quad (66)$$

In App. [E] we show how to extend this running to the Rgap scheme in order to remove the $O(\Lambda_{\text{QCD}})$ renormalon. The result in the Rgap scheme is

$$g_C(r) \Delta_1^C(R,\mu,\tau) = g_C(r) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_1(r)} \Omega_1^C(R,\mu,\tau,\mu) + R e^{\gamma_E} \left( \frac{\alpha_s(\mu)}{\alpha_s(R)} \right)^{\gamma_1(r)} \omega \left[ \Gamma^{\text{cusp}}, \mu, R \right] + R \Delta e^{\gamma_E} \left( \frac{\alpha_s(\mu)}{\alpha_s(R)} \right)^{\gamma_1(r)} \omega \left[ \Gamma^{\text{cusp}}, R, \mu \right] + \Lambda^{(k)}_{\text{QCD}} \left( \frac{\beta_0 \alpha_s(\mu)}{2\pi} \right) \sum_{j=0}^k S_j^\gamma(r)(-1)^j e^{i\pi [b_j - \gamma_1(r)]} \times \left[ \Gamma(-b_j + \gamma_1(r) - j, t_1) - \Gamma(-b_j + \gamma_1(r) - j, t_0) \right] \equiv g_C(r) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\gamma_1(r)} \Omega_1(R,\mu,\tau,\mu) + \Delta^{\text{diff}}(R,\mu,\mu,\tau,\mu) \quad (67)$$

Here, the formula is resummed to $N^k$LL and $\Lambda^{(k)}_{\text{QCD}}$ is the familiar $N^k$LO perturbative expression for $\Lambda_{\text{QCD}}$. We always use $R_\Delta = \mu_\tau = 2 \text{ GeV}$ to define the initial hadronic parameter. The values for $b_1$, $\gamma_1(r)$, $t_1$, $t_0$, and the $S_j$ can all be found in App. [E] and the resummed $\omega$ is given in Eq. (A19).

In order to implement this running, we pick an ansatz for the form of the moment at the low scales, $R_\Delta$ and $\mu_\Delta$, given by

$$\Omega_1(R_\Delta,\mu_\Delta,\tau) = \left[ a(R_\Delta,\mu_\Delta) f_a(r) + b(R_\Delta,\mu_\Delta) f_b(r) \right]^2, \quad f_a(r) = 3.510 e^{-\frac{r^2}{2\tau^2}}. \quad (68)$$

The form of $\Omega_1(R_\Delta,\mu_\Delta,\tau)$ was chosen to be always positive and to smoothly go to zero at the endpoint $r = 1$. In the Rgap scheme $\Omega_1$ can be interpreted in a Wilsonian manner as a physical hadronic average momentum parameter, and hence it is natural to impose positivity. As $\mu \rightarrow 1$ we are asking about the vacuum-fluctuation-induced distribution of hadrons with large $p_\perp$ which is anticipated to fall off rapidly. We also check other ansätze that satisfied these conditions, but choosing different positive definite functions has a minimal effect on the distribution. The functions $f_a$ and $f_b$ were chosen to satisfy $\int_0^1 dy g_C(r) f_a(r)^2 = \int_0^1 dy g_C(r) f_b(r)^2 = 1$ and $\int_0^1 dy g_C(r) f_a(r) f_b(r) = 0$. This allows us to write

$$\Omega_1^C(R_\Delta,\mu_\Delta) = c_C \int_0^1 dy g_C(r) \Omega_1(R_\Delta,\mu_\Delta,\tau) = 3 \pi \left[ a(R_\Delta,\mu_\Delta)^2 + b(R_\Delta,\mu_\Delta)^2 \right], \quad (69)$$

and to define an orthogonal variable

$$\theta(R_\Delta,\mu_\Delta) = \arctan \left( b(R_\Delta,\mu_\Delta) / a(R_\Delta,\mu_\Delta) \right). \quad (70)$$

The parameters $a$ and $b$ can therefore be swapped for $\Omega_1^C(R_\Delta,\mu_\Delta)$ and $\theta(R_\Delta,\mu_\Delta)$. This $\theta$ is defined as part of the model for the universal function $\Omega_1(R,\mu,\tau)$ and so should also exhibit universality between event shapes. In Sec. [X] below we will demonstrate that $\theta$ has a small effect on the cross section for the C-parameter, and hence that $\Omega_1^C(R_\Delta,\mu_\Delta)$ is the most important hadronic parameter.

VIII. PROFILE FUNCTIONS

The ingredients required for cross section predictions at various resummed perturbative orders are given in Table [I]. This includes the order for the cusp and non-cusp anomalous dimensions for $H$, $J_r$, and $S_C$, their perturbative matching order, the beta function $\beta[\alpha_s]$ for the running coupling, and the order for the nonsingular corrections discussed in Sec. [X]. It also includes the anomalous dimensions $\gamma_\Delta$ and subtractions $\delta$ discussed in this section. In our analysis we only use primed orders with the factorization theorem for the distribution. For the unprimed orders only the formula for the cumulant cross section properly resums the logarithms, see [52], but for the reasons discussed in [19] we need to use the distribution cross section for our analysis. The primed order distribution factorization theorem properly resums the desired series of logarithms for $C$, and was also used in Refs. [19] [20] to make predictions for thrust.

The factorization formula in Eq. (29) contains three characteristic renormalization scales, the hard scale $\mu_H$, the jet scale $\mu_J$, and the soft scale $\mu_S$. In order to avoid
large logarithms, these scales must satisfy certain constraints in the different $C$ regions

1) nonperturbative: $C \lesssim 3\pi \frac{\Lambda_{\text{QCD}}}{Q}$

\[ \mu_H \sim Q, \mu_J \sim \sqrt{\Lambda_{\text{QCD}}Q}, \mu_S \sim R \sim \Lambda_{\text{QCD}}, \]

2) resummation: $3\pi \frac{\Lambda_{\text{QCD}}}{Q} \ll C < 0.75$

\[ \mu_H \sim Q, \mu_J \sim Q \sqrt{\frac{C}{6}}, \mu_S \sim R \sim \frac{QC}{6} \gg \Lambda_{\text{QCD}}, \]

3) fixed-order: $C > 0.75$

\[ \mu_H = \mu_J = \mu_S = R \sim Q \gg \Lambda_{\text{QCD}}. \]

In order to meet these constraints and have a continuous factorization formula, we make each scale a smooth function of $C$ using profile functions.

When one looks at the physical $C$-parameter cross-section, it is easy to identify the peak, tail, and far-tail as distinct physical regions of the distribution. How much of the physical peak belongs to the nonperturbative versus resummation region is in general a process dependent statement, as is the location of the transition between the resummation and fixed-order regions. For example, in $b \to s \gamma$ the entire peak is in the nonperturbative region [43], whereas for $pp \to H +1$ gluon initiated jet with $p_T \sim 400$ GeV the entire peak is in the resummation region [53]. For thrust with $Q = m_Z$ [19], and similarly here for $C$-parameter with $Q = m_Z$, the transition between the nonperturbative and resummation regions occurs near the maximum of the physical peak. Note that, despite the naming, in the nonperturbative region, where the full form of the shape function is needed, resummation is always important. The tail for the thrust and $C$-parameter distributions is located in the resummation region, and the far-tail, which is dominated by events with three or more jets, exists in the fixed-order region.

For the renormalization scale in the hard function, we use

\[ \mu_H = \epsilon_H Q, \]

where $\epsilon_H$ is a parameter that we vary from 0.5 to 2.0 in order to account for theory uncertainties.

The profile function for the soft scale is more complicated, and we adopt the following form:

\[ \mu_S = \begin{cases} 
\mu_0 & 0 \leq C < t_0 \\
\zeta(\mu_0, 0, 0, \frac{r_s \mu_H}{6}, t_0, t_1, C) & t_0 \leq C < t_1 \\
\frac{r_s \mu_H}{6} & t_1 \leq C < t_2 \\
\zeta(0, \frac{r_s \mu_H}{6}, \mu_H, 0, t_2, t_s, C) & t_2 \leq C < t_s \\
\mu_H & t_s \leq C < 1 
\end{cases} \]

where $\zeta$ is a piecewise function made out of two linear segments with $r_s \mu_H/6$ as the soft scale.

Here the 1st, 3rd, and 5th lines satisfy the three constraints in Eq. (71). In particular, $\mu_0$ controls the intercept of the soft scale at $C = 0$. The term $t_0$ controls the boundary of the purely nonperturbative region and the start of the transition to the resummation region, and $t_1$ represents the end of this transition. As the border between the nonperturbative and perturbative regions is $Q$ dependent, we actually use $n_0 \equiv t_0(Q/1\text{GeV})$ and $n_1 \equiv t_1(Q/1\text{GeV})$ as the profile parameters. In the resummation region $t_1 < C < t_2$ the parameter $r_s$ determines the linear slope with which $\mu_S$ rises. The parameter $t_2$ controls the border and transition between the resummation and fixed-order regions. Finally, the $t_s$ parameter sets the value of $C$ where the renormalization scales all join. We require both $\mu_S$ and its first derivative to be continuous, and to this end we have defined the function $\zeta(a_1, b_1, a_2, b_2, t_1, t_2, l) = a_1 + b_1 l$ for $t < t_1$ and $l_2(t) = a_2 + b_2 t$ for $t > t_2$, which smoothly connects two straight lines of the form.

| cusp non-cusp matching $\beta[a_s]$ nonsingular $\gamma_{D,R,r}^\delta$ |
|-------------------------|-----|-----|-----|-----|
| LL | 1 | - | tree | 1 | - | - |
| NLL | 2 | 1 | tree | 2 | - | 1 | - |
| N^2LL | 3 | 2 | 1 | 3 | 1 | 2 | 1 |
| N^3LL | 3 | 2 | 4 | 3 | 3 | 2 | 2 |

**TABLE I.** Loop corrections required for specified orders.

![FIG. 7. Solid lines are the central results for the profile functions for the renormalization scales $\mu_H$, $\mu_J(C)$, $\mu_S(C)$ at $Q = m_Z$. The bands and up-down arrow indicate the results of varying the profile parameters. The result for $R(C)$ is identical to $\mu_S(C)$ at the resolution of this figure, differing only at small $C$. Above $C = t_s \approx 0.8$ all the scales merge, $\mu_H = \mu_J = \mu_S = R$.](image-url)
Moreover, in Ref. \[19\] there was only one quadratic form. The soft scale profile in Eq. (73) was also used in Ref. \[54\].

These two second-order polynomials join at the middle point \( t_m = (t_1 + t_2)/2 \):

\[
\zeta(t) = \begin{cases} 
\hat{a}_1 + b_1(t-t_1) + c_1(t-t_1)^2 & t_1 \leq t \leq t_m \\
\hat{a}_2 + b_2(t-t_2) + c_2(t-t_2)^2 & t_m \leq t \leq t_2 
\end{cases}
\]

where \( \hat{a}_1 = a_1 + b_1 t_1 \), \( \hat{a}_2 = a_2 + b_2 t_2 \), \( c_1 = \frac{4 (\hat{a}_2 - \hat{a}_1) - (3 b_1 + b_2) (t_2 - t_1)}{2 (t_2 - t_1)^2} \), and \( c_2 = \frac{4 (\hat{a}_1 - \hat{a}_2) + (3 b_2 + b_1) (t_2 - t_1)}{2 (t_2 - t_1)^2} \). \hspace{1cm} (74)

The soft scale profile in Eq. (73) was also used in Ref. \[54\] for jet-mass distributions in \( pp \to Z + 1\)-jet.

In Ref. \[19\] slightly different profiles were used. For instance there was no region of constant soft scale. This can be reproduced from our new profiles by choosing \( t_0 = 0 \). Moreover, in Ref. \[19\] there was only one quadratic form after the linear term, and the slope was completely determined by other parameters. These new profiles have several advantages. The most obvious is a variable slope, which allows us to balance the introduction of logs and the smoothness of the profiles. Additionally, in the new set up, the parameters for different regions are more independent. For example, the \( n_0 \) parameter will only affect the nonperturbative region in the new profiles while in the old profiles, changing \( n_0 \) would have an impact on the resummation region. This independence makes analyzing the different regions more transparent.

For the jet scale we introduce a “trumpeting” factor that modifies the natural relation to the hard and soft scales in the following way:

\[
\mu_J(C) = \begin{cases} 
1 + \epsilon_J (C - t_s)^2 & \sqrt{\mu H \mu S(C)} \leq t_s \\
\mu H & \mu H > t_s \end{cases} \hspace{1cm} (75)
\]

The parameter \( \epsilon_J \) is varied in our theory scans.

The subtraction scale \( R(C) \) can be chosen to be the same as \( \mu_S(C) \) in the resummation region to avoid large logarithms in the subtractions for the soft function. In the nonperturbative region we do not want the \( \mathcal{O}(\alpha_s) \) subtraction piece to vanish, see Eq. (50), so we choose

\begin{table}
\begin{tabular}{|l|l|l|}
\hline
parameter & default value & range of values \\
\hline
\( \mu_0 \) & 1.1 GeV & - \\
\hline
\( R_0 \) & 0.7 GeV & - \\
\hline
\( n_0 \) & 12 & 10 to 16 \\
\hline
\( n_1 \) & 25 & 22 to 28 \\
\hline
\( t_2 \) & 0.67 & 0.64 to 0.7 \\
\hline
\( t_s \) & 0.83 & 0.8 to 0.86 \\
\hline
\( r_s \) & 2 & 1.78 to 2.26 \\
\hline
\( c_J \) & 0 & -0.5 to 0.5 \\
\hline
\( c_H \) & 1 & 0.5 to 2.0 \\
\hline
\( n_s \) & 0 & -1, 0, 1 \\
\hline
\hline
\end{tabular}
\end{table}

\begin{table}
\begin{tabular}{|l|l|l|}
\hline
\( \Gamma_3^{\text{cusp}} \) & 1553.06 & -1553.06 to +4659.18 \\
\hline
\( s_2^\zeta \) & -43.2 & -44.2 to -42.2 \\
\hline
\( j_3 \) & 0 & -3000 to +3000 \\
\hline
\( s_3^\zeta \) & 0 & -500 to +500 \\
\hline
\( \epsilon_2^{\text{low}} \) & 0 & -1, 0, 1 \\
\hline
\( \epsilon_2^{\text{high}} \) & 0 & -1, 0, 1 \\
\hline
\( \epsilon_3^{\text{low}} \) & 0 & -1, 0, 1 \\
\hline
\( \epsilon_3^{\text{high}} \) & 0 & -1, 0, 1 \\
\hline
\end{tabular}
\end{table}

\begin{table}
\begin{tabular}{|l|l|l|}
\hline
parameter & default value & range of values \\
\hline
\( \mu_0 \) & 1.1 GeV & - \\
\hline
\( R_0 \) & 0.7 GeV & - \\
\hline
\( n_0 \) & 2 & 1.5 to 2.5 \\
\hline
\( n_1 \) & 10 & 8.5 to 11.5 \\
\hline
\( t_2 \) & 0.25 & 0.225 to 0.275 \\
\hline
\( t_s \) & 0.4 & 0.375 to 0.425 \\
\hline
\( r_s \) & 2 & 1.77 to 2.26 \\
\hline
\( c_J \) & 0 & -1.5 to 1.5 \\
\hline
\( c_H \) & 1 & 0.5 to 2.0 \\
\hline
\( n_s \) & 0 & -1, 0, 1 \\
\hline
\hline
\( j_3 \) & 0 & -3000 to +3000 \\
\hline
\( s_3^\tau \) & 0 & -500 to +500 \\
\hline
\( \epsilon_2 \) & 0 & -1, 0, 1 \\
\hline
\( \epsilon_3 \) & 0 & -1, 0, 1 \\
\hline
\end{tabular}
\end{table}
In order to account for resummation effects in the non-singular partonic cross section, we can specify the theory parameters for thrust, which are varied coherently, we vary $\mu_n$. We use three possibilities,

$$
\begin{align*}
\mu_n(C) = & \begin{cases} 
\frac{1}{2} [ \mu_H(C) + \mu_J(C) ] & n_s = 1 \\
\mu_H & n_s = 0 \\
\frac{1}{2} [ 3 \mu_H(C) - \mu_J(C) ] & n_s = -1
\end{cases} \quad (77)
\end{align*}
$$

Using these variations, as opposed to those in Ref. [19], gives more symmetric uncertainty bands for the nonsingular distribution.

The only free parameter in this equation, $R_0$, simply sets the value of $R$ at $C = 0$. The requirement of continuity at $t_1$ in both $R(C)$ and its first derivative are again ensured by the $\zeta$ function.

In order to account for resummation effects in the non-singular partonic cross section, we vary $\mu_n$. We use three possibilities,

$$
R(C) = \begin{cases} 
R_0 & 0 \leq C < t_0 \\
\zeta(R_0, 0, 0, \frac{\mu_0 t_0}{6}, t_0, t_1, C) & t_0 \leq C < t_1 \cdot (76) \\
\mu_S(C) & t_1 \leq C \leq 1
\end{cases}
$$

The only free parameter in this equation, $R_0$, simply sets the value of $R$ at $C = 0$. The requirement of continuity at $t_1$ in both $R(C)$ and its first derivative are again ensured by the $\zeta$ function.

In order to account for resummation effects in the non-singular partonic cross section, which we cannot treat coherently, we vary $\mu_n$. We use three possibilities,

$$
\begin{align*}
\mu_n(C) = & \begin{cases} 
\frac{1}{2} [ \mu_H(C) + \mu_J(C) ] & n_s = 1 \\
\mu_H & n_s = 0 \\
\frac{1}{2} [ 3 \mu_H(C) - \mu_J(C) ] & n_s = -1
\end{cases} \quad (77)
\end{align*}
$$

Using these variations, as opposed to those in Ref. [19], gives more symmetric uncertainty bands for the nonsingular distribution.

The plot in Fig. 7 shows the scales for the default parameters for the case $Q = m_Z$ (thick lines). Also shown (in gray) are plots of $QC/6$ and $Q\sqrt{C/6}$. In the resummation region, these correspond fairly well with the profile functions, indicating that in this region our analysis will avoid large logarithms. Note that the soft and jet scales in the plot would exactly match the gray dashed lines in the region $0.25 < C < 0.67$ if we took $r_s = 1$ as our default. For reasons discussed in Sec. [XB] we use $r_s = 2$ as our default value. We also set as default values $\mu_0 = 1.1$ GeV, $R_0 = 0.7$ GeV, $\epsilon_H = 1$, $\epsilon_J = 0$, and $n_s = 0$. Default central values for other profile parameters for $C$ are listed in Table [11].

Perturbative uncertainties are obtained by varying the profile parameters. We hold $\mu_0$ and $R_0$ fixed, which are the parameters relevant in the region impacted by the entire nonperturbative shape function. They influence the meaning of the nonperturbative soft function parameters in $F_C$. The difference of the two parameters is important for renormalon subtractions, and hence should not be varied ($\mu_0 - R_0 = 0.4$ GeV) to avoid changing the meaning of $F_C$. Varying $\mu_0$ and $R_0$ keeping the difference fixed has a very small impact compared to variations from $F_C$ parameters, as well as other profile parameters, and hence is also kept constant. We are then left with eight profile parameters to vary during the theory scan, whose central values and variation ranges used in our analysis are: $r_s = 2 \times 1.13^{\pm 1}$, $n_0 = 12 \pm 2$, $n_1 = 25 \pm 3$, $t_2 = 0.67 \pm 0.03$, $t_s = 0.83 \pm 0.03$, $\epsilon_J = 0 \pm 0.5$, $\epsilon_H = 2^{\pm 1}$ and $n_s = 0 \pm 1$. The resulting ranges are also listed in Table [11] and the effect of these variations on the scales is plotted in Fig. 7. Since we have so many events in our EVENT2 runs, the effect of $e_2^{\text{rms}}$ is completely negligible in the theory uncertainty scan. Likewise, the effect of $e_2^{\text{high}}$ is also tiny above the shoulder region.

Due to the advantages of the new profile functions, we have implemented them for the thrust predictions from Refs. [19] as well. For thrust we redefine $r_s \rightarrow 6 r_s$, which eliminates all four appearances of the factor of $1/6$ in Eqs. (73) and (76). After making this substitution, we can specify the theory parameters for thrust, which are summarized in Table [11]. These choices create profiles and profile variations that are very similar to those used in [19]. The only noticeable difference is the flat $\mu_S$ in the nonperturbative region (which is relevant for a fit to the full shape function, but is irrelevant for the $\alpha_s$ tail fit).

### IX. RESULTS

In this section we present our final results for the C-parameter cross section, comparing to the thrust cross section when appropriate. We will use different levels of theoretical accuracy for these analyses. When indicating the perturbative precision, and whether or not the power correction $\Omega_1$ is included and at what level of precision, we follow the notation:

| Resummation Order | Calculable $G_{ij}$’s and $B_i$’s |
|-------------------|----------------------------------|
| LL                | $G_{i,i+1}$                      |
| NLL’              | $G_{i,i}$ and $B_1$              |
| N^2LL’            | $G_{i,i-1}$ and $B_2$            |
| N^3LL’            | $G_{i,i-2}$ and $B_3$            |

Table IV. By doing resummation to the given order, we can access results for the entire hierarchy of $G_{ij}$’s listed.

In first three subsections we discuss the determination of higher order perturbative coefficients in the cross section, the impact of the slope parameter $r_s$, and the order-by-order convergence and uncertainties. Since the effect of the additional running induced by the presence of hadron masses is a relatively small effect on the cross section, we leave their discussion to the final fourth subsection.
FIG. 8. Theory scan for C-parameter [panels (a) to (f)] and Thrust [panels (g) to (l)] cross section uncertainties at various orders, for few center of mass energies $Q$ and slopes $r_s$. The theoretical predictions include log resummation and are purely perturbative. The respective upper and lower rows use $r_s = 1$ and $r_s = 2$, respectively. The left, center and right columns correspond to $Q = 40 \text{ GeV}$, $91.2 \text{ GeV}$, and $200 \text{ GeV}$, respectively. Here we use $\alpha_s(m_Z) = 0.1141$.

A. $G_{ij}$ expansion

Our $N^3\text{LL}'$ resummed predictions can be used to compute various coefficients of the most singular terms in the cross section. In this determination only perturbative results are used. In order to exhibit the terms that
are determined by the logarithmic resummation, one can take \( \mu = Q \) and write the cumulative function in the following way:

\[
\Sigma_0(C) = \frac{1}{\sigma_0} \int_0^C dC' \frac{d\sigma}{dC'} = \left( 1 + \sum_{m=1}^{\infty} B_m^{[0]} \left( \frac{\alpha_s(Q)}{2\pi} \right)^m \right) \times \exp \left( \sum_{i=1}^{\infty} \sum_{j=1}^{i+1} G_{ij} \left( \frac{\alpha_s(Q)}{2\pi} \right) \ln^i \left( \frac{6}{C} \right) \right),
\]

(78)

where \( \sigma_0 \) is the tree-level total cross section. A different normalization with respect to the total cross section including all QCD corrections is also used in the literature and reads

\[
\Sigma(C) = \frac{1}{\sigma_{\text{had}}} \int_0^C dC' \frac{d\sigma}{dC'} = \left( 1 + \sum_{m=1}^{\infty} B_m \left( \frac{\alpha_s(Q)}{2\pi} \right)^m \right) \times \exp \left( \sum_{i=1}^{\infty} \sum_{j=1}^{i+1} G_{ij} \left( \frac{\alpha_s(Q)}{2\pi} \right) \ln^i \left( \frac{6}{C} \right) \right).
\]

(79)

Notice that the different normalizations do not affect the \( G_{ij} \)'s and only change the non-logarithmic pieces. Our N\( ^3 \)LL’ result allows us to calculate the \( B_m^{[0]} \)'s and \( B_m \)'s to third order and entire hierarchies of the \( G_{ij} \) coefficients as illustrated in Table IV.

The results for these coefficients through \( G_{34} \) are collected in App. [D]. Note that, due to the equivalence of the thrust and C-parameter distributions at NLL (when using \( \hat{C} \)), we know that the \( G_{1,i+1} \) and \( G_{i,1} \) series are equal for these two event shapes [24]. From our higher order resummation analysis we find that the \( G_{i,i-1} \) and \( G_{i,i-2} \) coefficients differ for the C-parameter and thrust event shapes because the fixed-order \( s_1 \hat{C}, \tau \) and \( s_2 \hat{C}, \tau \) constants differ, and enter into the resummed results at N\( ^2 \)LL and N\( ^3 \)LL’ respectively. The values of all the \( B_i \) coefficients are also different. (If we had instead used the unprimed counting for logarithms, N\( ^3 \)LL, there is less precision obtained at a given order, and each index of a \( B_i \) in Table IV would be lowered by one.)

The \( G_{ij} \) and \( B_m \) serve to illustrate the type of terms that are included by having resummation and fixed order terms at a given order. They are not used explicitly for the resummed analyses in the following sections, which instead exploit the full resummed factorization theorem in Eq. (29).

FIG. 9. C-parameter (left panel) and thrust (right panel) cross section predictions at N\( ^3 \)LL’ + \( \mathcal{O}(\alpha_s^2) \) + \( \Omega_1(R, \mu) \) for \( r_s = 1 \) (blue) and \( r_s = 2 \) (red). For these plots we use our most complete setup, with power corrections in the renormalon-free Rgap scheme. Here we use \( \alpha_s(m_Z) = 0.1141 \) and \( \Omega_1(R, \mu) = 0.33 \text{ GeV} \).

B. The slope \( r_s \) for C-parameter and Thrust

In the profiles of Sec. [VIII] the parameter \( r_s \) was defined as the dimensionless slope of \( \mu_s \) in the resummation region. It would seem natural to pick \( r_s = 1 \) to eliminate the powers of \( \ln(6 \mu_s/(Q C)) \) and \( \ln(\mu_s/(Q \tau)) \) that appear in the cross section formula for \( C \) and \( \tau \). However, having a slightly steeper rise may also yield benefits by smoothing out the profile. Using \( r_s = 2 \) that is larger than 1, such as \( r_s = 2 \), will only shift small \( \ln(r_s) \) factors between different orders of the resummed cross section. Just like other profile parameters the dependence on \( r_s \) will decrease as we go to higher orders in perturbation theory, but the central value choice may improve the accuracy of lower order predictions.

In order to determine whether \( r_s = 1 \) or \( r_s = 2 \) is a better choice for the slope parameter, we examine the convergence of the cross section between different orders of resummation. For this analysis we will compare the three perturbative orders N\( ^3 \)LL’ + \( \mathcal{O}(\alpha_s^2) \), N\( ^2 \)LL’ + \( \mathcal{O}(\alpha_s^2) \), NLL’ + \( \mathcal{O}(\alpha_s) \). We also fix \( \alpha_s(m_Z) = 0.1141 \), which is the value favored by the QCD only thrust fits [19]. (Use of larger values of \( \alpha_s(m_Z) \) leads to the same conclusions.
FIG. 10. Total hadronic cross section obtained from integrating the resummed cross section. The top two panels show the prediction for \( r_s = 1 \) and \( r_s = 2 \) for C-parameter, respectively. Likewise, the bottom two panels show the thrust results. Green squares correspond to the prediction with log-resummation and the power correction in the \( \overline{\text{MS}} \) scheme, whereas red triangles have log-resummation and the power correction in the Rgap scheme. The blue points correspond to the fixed-order prediction, and the blue line shows the highest order FO prediction.

FIG. 11. Components of the C-parameter cross section with resummation at \( \mathcal{N}^3\text{LL}'+\mathcal{O}(\alpha_s^3) + \Omega_1(R,\mu) \) with \( \Omega_1(R_\Delta,\mu_\Delta) = 0.33 \text{ GeV} \) and \( \alpha_s(m_Z) = 0.1141 \).

That we draw below.) In Fig. 8 we show the perturbative C-parameter cross section (upper two rows) and thrust cross section (lower two rows) with a scan over theory parameters (without including \( \Omega_1 \) or the shape function) for both \( r_s = 1 \) (first respective row) and \( r_s = 2 \) (second respective row). Additionally, we plot with different values of \( Q \), using \( Q = 40 \text{ GeV} \) in the first column, \( Q = 91.2 \text{ GeV} \) in the second column and \( Q = 200 \text{ GeV} \) in the third column. The bands here correspond to a theory parameter scan with 500 random points taken from Tabs. II and III. We conclude from these plots that \( r_s = 2 \) has better convergence between different orders than \( r_s = 1 \). For all of the values of \( Q \), we can see that in the slope 1 case, the \( \mathcal{N}^3\text{LL}' \) band lies near the outside of the edge of the NLL' band, while in the slope 2 plots, the scan for \( \mathcal{N}^3\text{LL}' \) is entirely contained within the scan for NLL'. A similar picture can be seen for the transition from \( \mathcal{N}^3\text{LL}' \) to \( \mathcal{N}^5\text{LL}' \). This leads us to the conclusion that the resummed cross section prefers \( r_s = 2 \) profiles which we choose as our central value for the remainder of the analysis. (In the thrust analysis of Ref. 19, the profiles did not include an independent slope parameter, but in the resummation region their profiles are closer to taking \( r_s = 2 \) than \( r_s = 1 \).)

At the highest order, \( \mathcal{N}^3\text{LL}'+\mathcal{O}(\alpha_s^3) \), the choice of
FIG. 12. Theory scan for cross-section uncertainties in C-parameter. The panels are (a) fixed order, (b) resummation with no nonperturbative function, (c) resummation with a nonperturbative function using the MS scheme for $\bar{\Omega}_C^1$ without renormalon subtraction, (d) resummation with a nonperturbative function using the Rgap scheme for $\Omega_C^1$ with renormalon subtraction.

$r_s = 1$ or $r_s = 2$ has very little impact on the resulting cross section, both for C-parameter and thrust. Indeed, the difference between these two choices is smaller than the remaining (small) perturbative uncertainty at this order. This is illustrated in Fig. 9 which shows the complete $N^{3LL'} + \mathcal{O}(\alpha_s^3) + \Omega_1(R,\mu)$ distributions for $r_s = 1$ (blue) and $r_s = 2$ (red) for C-parameter in the left panel, and for thrust in the right panel. Here we use $\Omega_1(R,\mu) = 0.33$ GeV. Thus the choice of $r_s$ is essentially irrelevant for our highest order predictions, but has a bit of impact on conclusions drawn about the convergence of the lower to highest orders. Another thing that is clear from this figure is that the perturbative uncertainties for the C-parameter cross section at $Q = m_Z$, which are on average $\pm 2.5\%$ in the region $0.25 < C < 0.65$, are a bit larger than those for thrust where we have on average $\pm 1.8\%$ in the region $0.1 < \tau < 0.3$. This $\pm 1.8\%$, obtained with the profile and variations discussed here, agrees well with the $\pm 1.7\%$ quoted in Ref. [19].

One can also look at the effect that the choice of $r_s$ has on the total integral over the C-parameter and thrust distributions at $N^{3LL'} + \mathcal{O}(\alpha_s^3) + \Omega_1(R,\mu)$, which should reproduce the total hadronic cross section. For C-parameter the outcome is shown in the first row of Fig. 10, where green squares and red triangles represent resummed cross sections with power corrections in the MS and Rgap schemes, respectively. In blue we display the fixed-order prediction. Comparing the predictions for $r_s = 1$ (left panel) and $r_s = 2$ (right panel), we observe that the former achieves a better description of the
fixed-order prediction at $N^2\text{LL}'$, in agreement with observations made in Ref. [29]. For the case of thrust (second row of Fig. 10) similar conclusions as for C-parameter can be drawn by observing the behavior of the total cross section. Again, at the highest order the result is independent of the choice of the slope within uncertainties for both $C$ and thrust. Since our desired fit to determine $\alpha_s(m_Z)$ and $\Omega_C^{R}$ requires the best predictions for the shape of the normalized cross section, we do not use the better convergence for the normalization as a criteria for using $r_s = 1$. Our results for cross section shapes are self-normalized using the central profile result.

In Fig. 11 we present a plot analogous to Fig. 5 but including resummation at $N^3\text{LL}' + \mathcal{O}(\alpha^3_s) + \Omega_1(R, \mu)$ with $r_s = 2$ (a similar plot for thrust can be found in Ref. [19]). The suppression of the dashed blue nonsingular curve relative to the solid upper blue singular curve is essentially the same as observed earlier in Fig. 4. The subtraction components are a small part of the cross section in the resummation region, but have an impact at the level of precision obtained in our computation. Above the shoulder region the singular and nonsingular terms appear with opposite signs and largely cancel. This is clear from the figure where the individual singular and nonsingular lines are larger than the total cross section in this region. The same cancellation occurs for the singular subtraction and nonsingular subtraction terms. The black curve labeled total in Fig. 11 shows the central value for our full prediction. Note that the small dip in this black curve, visible at $C \simeq 0.75$, is what survives for the log-singular terms in the shoulder after the convolution with $F_C$.

C. Convergence and Uncertainties: Impact of Resummation and Renormalon Subtractions

Results for the C-parameter cross sections at $Q = m_Z$ are shown at various levels of theoretical sophistication in Fig. 12. The simplest setup is the purely perturbative fixed-order $\mathcal{O}(\alpha^k_s)$ QCD prediction (i.e. no resummation, no power corrections), shown in panel (a), which does not make use of the new perturbative results in this paper. Not unexpectedly, the most salient feature at fixed order is the lack of overlap between the $O(\alpha_s)$ (green), $O(\alpha^2_s)$ (blue), and $O(\alpha^3_s)$ (red) results. This problem is cured once the perturbative resummation is included with $N^k\text{LL}' + \mathcal{O}(\alpha^k_s)$ predictions shown in panel (b): the NLL’ (green), $N^2\text{LL}'$ (blue) and $N^3\text{LL}'$ (red) bands now nicely overlap. To achieve this convergence and overlap with our setup it is important to normalize the cross section bands with the integrated norm at a given order using the default profiles. (The convergence for the normalization in Fig. 11 was slower. Further discussion of this can be found in Ref. 12.)

The panel (b) results neglect power corrections. Including them in the $\overline{\text{MS}}$ scheme, $N^k\text{LL}' + \mathcal{O}(\alpha^k_s) + \bar{\Omega}_1$, as shown in panel (c), does not affect the convergence of the series, but rather simply shifts the bands towards larger $C$ values. This was mentioned above in Eq. (3).

In panel (d) we show our results, which use the Rgap scheme for the power correction, at $N^k\text{LL}' + \mathcal{O}(\alpha^k_s) + \bar{\Omega}_1(R, \mu)$. In this scheme a perturbative series is subtracted from the partonic soft function to remove its $\mathcal{O}(\Lambda_{\text{QCD}})$ renormalon. This subtraction entails a corresponding scheme change for the parameter $\bar{\Omega}_1$ which becomes a subtraction-scale dependent quantity. In general the use of renormalon-free schemes stabilizes the perturbative behavior of cross sections. The main feature visible in panel (d) is the noticeable reduction of the perturbative uncertainty band at the two highest orders, with the bands still essentially contained inside lower order ones.

We can see the improvement in convergence numerically by comparing the average percent uncertainty between different orders at $Q = m_Z$. If we first look at the results without the renormalon subtraction, at $N^k\text{LL}' + \mathcal{O}(\alpha^k_s) + \Omega_1$, we see that in the region of interest for $\alpha_s(m_Z)$ fits (0.25 $< C < 0.65$), the NLL’ distribution has an average percent error of $\pm 11.7\%$, the $N^2\text{LL}'$ distribution has an average percent error of $\pm 7.0\%$ and the highest order $N^3\text{LL}'$ distribution has an average percent error of only $\pm 4.3\%$. Once we implement the Rgap scheme to remove the renormalon, giving $N^k\text{LL}' + \mathcal{O}(\alpha^k_s) + \bar{\Omega}_1(R, \mu)$, we see that the NLL’ distribution has an average percent error of $\pm 11.8\%$, the $N^2\text{LL}'$ distribution has an average percent error of $\pm 4.9\%$ and the most precise $N^3\text{LL}'$ distribution has an average percent error of only $\pm 2.5\%$. Although the renormalon subtractions for C-parameter induce a trend towards the lower edge of the perturbative band of the predictions at one lower order, the improved convergence of the perturbative series makes the use of these more accurate predictions desirable.

D. Impact of Hadron Mass Effects

In this section we discuss the impact of adding hadron mass effects, which gives the orders denoted $N^k\text{LL} + \mathcal{O}(\alpha^k_s) + \Omega_1(R, \mu, r)$. Hadron masses induce an additional anomalous dimension for $\Omega_1$, and an associated series of logarithms of the form $\ln(QC/\Lambda_{\text{QCD}})$ that need to be resummed. They also impact the definition of $\Omega_1$ and its RGE equations in the $\overline{\text{MS}}$ and Rgap schemes. Since the overall effect of hadron masses on C-parameter and thrust are rather small, they do not change the perturbative convergence discussed in the previous section. Therefore we study these effects here making use of only the highest order perturbative results at $N^3\text{LL}' + \mathcal{O}(\alpha^3_s)$.

In Fig. 13 we show the effect of hadron mass running on the cross section. We compare differential cross sections with and without hadron masses, at the same center of mass energies. When the running effects from hadron masses are turned off, the value of $\Omega_1(R, \mu_s)$ preferred by the experimental data will attempt to average away these effects by absorbing them into the value of the ini-
FIG. 13. Effects of hadron masses on the differential cross section for C-parameter (left) and thrust (right). Curves correspond to the percent difference between the cross-section with and without hadron mass effects, for three center of mass energies: $Q = 91.2$ GeV (solid red) 200 GeV (dashed blue) and 40 GeV (dotted green). The cross section with hadron mass effects uses $\theta (R_\Delta, \mu_\Delta) = 0$ and $\Omega_1 (R_\Delta, \mu_\Delta) = 0.32$ (0.30) GeV for C-parameter (thrust), while the cross section without hadron mass effects has $\Omega_1 (R_\Delta, \mu_\Delta) = 0.33$ GeV. We set $\alpha_s (m_Z) = 0.1141$.

FIG. 14. Effects of $\theta (R_\Delta, \mu_\Delta)$ on the cross section for C-parameter (left) and thrust (right). The lines correspond to the largest variation achievable by varying $\theta (R_\Delta, \mu_\Delta)$ in both directions, (which happen for $\theta = 0.23 \pi$ and $\theta = -0.27 \pi$), with respect to the cross section with $\theta (R_\Delta, \mu_\Delta) = 0$. The solid (red), dashed (blue) and dotted (green) lines correspond to $Q = 91.2$ GeV, 200 GeV and 40 GeV, respectively. We use $\Omega_1 (R_\Delta, \mu_\Delta) = 0.32$ (0.30) GeV for C-parameter (thrust), and $\alpha_s (m_Z) = 0.1141$.

tial parameter. (The running of $\Omega_1$ with and without hadron masses is shown below in Fig. 19). Therefore, the specific values used in Fig. 13 are obtained by fixing $\alpha_s (m_Z)$ and then fitting for $\Omega_1 (R_\Delta, \mu_\Delta)$ to minimize the difference between the cross section with and without hadron masses in the tail region. As the values for $\alpha_s (m_Z)$ and $\Omega_1$ in the case of no hadron mass effects come from a fit to data (from [19]), choosing $\Omega_1 (R_\Delta, \mu_\Delta)$ with the outlined procedure is similar to a full fit to data and will give results that allow comparison between the two cases. With hadron masses on we fix $\theta (R_\Delta, \mu_\Delta) = 0$, so the effects observed in Fig. 13 are related to the additional log resummation for $\Omega_1$. The effect is largest at $Q = 40$ GeV where it varies between a $-1.25\%$ and $+1.25\%$ shift for C-parameter and between a $-1.5\%$ and $+0.5\%$ shift for thrust. For $Q = m_Z$ it amounts to a $1.0\%$ shift for C-parameter and a shift of $0\%$ to $1.3\%$ for thrust.

With hadron masses the additional hadronic parameter $\theta (R_\Delta, \mu_\Delta)$ encodes the fact that the extra resummation takes place in $r$-space, and therefore induces some dependence on the shape of the $\Omega_1 (R, \mu, r)$ parameter. In contrast the dominant hadronic parameter $\Omega_1 (R_\Delta, \mu_\Delta)$ is the normalization of this $r$-space hadronic function. In Fig. 14 we show the effect that varying $-\pi/2 < \theta (R_\Delta, \mu_\Delta) < \pi/2$ has on the
cross section for three center of mass energies, fixing 
\( \Omega \left(R_\Delta, \mu_\Delta \right) = 320 \text{ MeV} \) for C-parameter and 300 MeV for thrust. The sensitivity of \( \Delta \sigma \) here is proportional 
to \( \Omega \left(R_\Delta, \mu_\Delta \right) \). In these plots we pick the value of \( \theta \) that 
gives the largest deviation from the \( \theta = 0 \) cross section 
(thes values are listed in the figure caption). This max-
imum deviation is only 1.5\% for C-parameter and occurs 
at \( Q = 40 \text{ GeV} \) at larger \( C \). For thrust, the largest devi-
ation also occurs for \( Q = 40 \text{ GeV} \) and for higher values of 
\( \tau \), and is \( \lesssim 2.0\% \) for \( \tau \leq 0.3 \). For \( Q = m_Z \) the effect 
is roughly \( \pm 1.0\% \) for C-parameter, and around \( 1.0\% \) for 
thrust when \( \tau < 0.25 \), growing to \( 2\% \) by \( \tau = 0.33 \).

We conclude that the effect of hadron masses on log 
resummation should be included if one wishes to avoid 
an additional \( \sim 1.5\% \) uncertainty on the cross section. 
Furthermore one should consider fitting \( \theta \left(R_\Delta, \mu_\Delta \right) \) as an 
additional parameter if one wants to avoid another \( \sim 1\% 
uncertainty in the cross section that it induces. (Recall 
that Fig. 14 shows the worst case scenario.)

X. CONCLUSIONS

We have provided a factorization formula for the 
C-parameter distribution in \( e^+ e^- \) annihilation and ana-
alyzed its utility in making precision cross-section predic-
tions, extending earlier work on thrust. We have de-
determined or computed the ingredients needed to achieve 
a summation of large logarithms at N\( ^3 \)LL order for the 
singular terms in the dijet limit where \( C \) is small, pushing 
beyond the classical resummed computations in Ref. [27] 
by two additional orders. To achieve this goal, we demon-
strated that the anomalous dimensions, as well as the 
hard and jet functions, are identical in the C-parameter 
and thrust distributions to all orders in perturbation 
theory. We then computed the only distinct piece, the 
soft function, to one-loop order analytically, two-loop or-
der numerically, and with logarithmic accuracy at three-
loops using its anomalous dimension. (We also presented 
a master formula for the one-loop soft function, valid for 
any recoil-free dijet event shape in App. C.) Our factor-
ization formula also incorporates the previously known 
\( \mathcal{O}(\alpha_s^2) \) and \( \mathcal{O}(\alpha_s^3) \) perturbative QCD corrections in fixed 
order [3][8].

Using our factorization theorem and computation for 
the soft function, we are able to analytically predict the 
fixed-order log-singular terms of C-parameter up to 
\( \mathcal{O}(\alpha_s^3) \). These results are used to consistently incorporate 
fixed-order results at the same order by subtracting them 
from the numerical \( \mathcal{O}(\alpha_s^3) \) results, to obtain the nonsin-
gular terms plotted in Figs. 2, 3, and compared to the 
singular cross section in Fig. 3.

The factorization formula we use incorporates a sys-
tematic description of nonperturbative power corrections 
using a shape function \( F_C \), whose moments are given by 
quantum field theory matrix elements. This nonpertur-
bative shape function describes the dynamics of soft particle 
radiation at large angles. Although the shape function 
of thrust and C-parameter are a priori unrelated, one 
can show that, up to small deviations caused by hadron masses, 
their respective first moments are proportional 
to each other involving a simple calculable coefficient, as 
shown in Eq. [2]. Tests of this universality are very im-
portant, since the first moment of the shape function con-
stitutes the most important nonperturbative power correc-
tion in the tail of the distribution, where one can write 
down an OPE for the event-shape distributions. The 
results in this paper allow for universality to be tested 
with data accounting for a high degree of perturbative 
precision, and incorporating hadron mass effects.

In order to reduce the sensitivity to an \( \mathcal{O}(\lambda_{\text{QCD}}) \) renor-
malon present in the soft function, we switch to a short-
distance scheme for the leading power correction, which 
translates into a more stable perturbative behavior. Us-
ing universality relations, it is possible (and desirable) 
to use thrust gap subtractions in the C-parameter dis-
tribution, with the appropriate prefactor. Furthermore, 
we introduce hadron mass running effects into this for-
malism, and show how one can consistently account for 
Rgap and hadron mass running simultaneously.

We have used profile functions, which are \( C \)-dependent 
renormalization scales, to both properly implement the 
required conditions for the scales in different regions in 
Eq. (71), including the resummation of large logarithms 
in the nonperturbative and resummation regions, and to 
smoothly carry out the transition between regions. There 
are two significant differences for the C-parameter rela-
tive to thrust: the nonperturbative region is enlarged by 
a factor \( 3\pi/2 \), and the slope of \( \mu_{S} \) in the resummation 
region is reduced by a factor of 6. In Sec. IX B we made 
a numerical investigation of possible choices for the slope 
of the soft function in the resummation region, and con-
ccluded both for thrust and C-parameter that having a 
slope that is twice the canonical value, \( r_s = 2 \) rather than 
\( r_s = 1 \), is advantageous to have a better order-by-order 
convergence. At the highest order we achieve, which is 
N\( ^3 \)LL', we also observe that the choice of \( r_s = 1 \) versus 
\( r_s = 2 \) becomes irrelevant.

Our most accurate theoretical description has a very 
nice perturbative convergence, and achieves a perturba-
tive uncertainty of on average 2.5\% in the tail of the 
distribution for \( Q = m_Z \), see Fig. 12(d). We have also 
shown that the effect of hadron mass corrections is at 
the < 2\% level (depending on \( Q \), and differing somewhat 
between C-parameter and thrust), see Figs. 13 and 14.
The theory developments made here are necessary for a 
high-precision determination of the strong coupling con-
tant \( \alpha_s(m_Z) \) by fits to experimental data. Moreover, 
by simultaneously fitting to the leading power correction 
\( \Omega \), our formalism allows us to check for universality be-
tween thrust and C-parameter, exploiting the high level 
of perturbative precision, and also accounting for hadron 
mass effects. These results will be presented in a separate 
paper [24].
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Appendix A: Formulae

In this appendix we collect all the remaining formulæ used in our analysis for the case of massless quarks. Since we want to compare to experimental data, which is normalized to the total number of events, we need to calculate \((1/\sigma)d\sigma/dC\), our normalized cross section. To do this, we can either self-normalize our results by integrating over \(C\), or use the fixed-order result for the total hadronic cross section, which, at three loops for massless quarks at \(\mu = Q\) is (see Ref. [55] for further discussion),

\[
R_{\text{had}} = 1 + 0.3183099 \alpha_s(Q) + 0.1427849 \alpha_s^3(Q) - 0.411757 \alpha_s^4(Q). \tag{A1}
\]

Throughout our analysis, we use \(m_Z = 91.187\) GeV and all numerical results quoted below are for SU(3) color with \(n_f = 5\) active light flavors, for simplicity.

Singular Cross Section Formula

For the singular part of the differential cross section given in Eq. (29) we simplify the numerical evaluation using our freedom to take \(\mu = \mu_J\). This means \(U_J^\delta(s - s', \mu_J, \mu_J) = \delta(s - s')\) and we can write

\[
\frac{1}{\sigma_0} \int d\vec{k} \frac{d\sigma}{dC}(C - \frac{k}{Q}) F_C(k - 3\pi \Delta(R, \mu_S)) = \frac{Q}{6} H(Q, \mu_H) U_H(Q, \mu_H, \mu_J) \times \int d\vec{k} P(Q, QC/6 - k/6, \mu_J) \times e^{-3\pi \delta(R, \mu_s) \frac{d}{dC} F_C(k - 3\pi \Delta(R, \mu_S))}. \tag{A2}
\]

Here \(\sigma_0\) is the tree-level (Born) cross section for \(e^+e^- \rightarrow q\bar{q}\). Here we have combined the perturbative corrections from the partonic soft function, jet function, and soft evolution factor into a single function,

\[
P(Q, k, \mu_J) = \int ds \int d\vec{k}' J_{\delta}(s, \mu_J) U_3^\delta(k', \mu_J, \mu_S) \times \hat{S}_C(k - k' - s/Q, \mu_S). \tag{A3}
\]

The large logarithms of \(C/6\) are summed up in the evolution factors \(U_H\) and \(U_S\). We can carry out the integrals in \(P\) exactly and the results are enumerated below. The shape function \(F_C(k - 3\pi \Delta)\) is discussed in Sec. VII, and we have used integration by parts in Eq. (A2) to have the derivative in the exponential act on \(F_C\), which is simpler than acting the derivative on the perturbative soft function. For our numerical calculation, we expand \(H, J, \) and \(S_C\) order-by-order as a series in \(\alpha_s(\mu_H), \alpha_s(\mu_J), \) and \(\alpha_s(\mu_S)\) respectively, with no large logs. Additionally, we expand \(\exp(-3\pi \delta(R, \mu_S)d/dC)\) (see Eq. (49)) as a series in \(\alpha_s(\mu_S)\), which must be done consistently to cancel the renormalon present in \(S_C\).

The hard function to \(O(\alpha_s^3)\) with \(n_f = 5\) is [19][20][22],

\[
H(Q, \mu_H) = 1 + \alpha_s(\mu_H) \left(0.745808 - 1.27324L_Q - 0.848826L_Q^2\right)
+ \alpha_s^2(\mu_H) \left(2.27587 - 0.0251035L_Q - 1.06592L_Q^2ight)
+ 0.735517L_Q^3 + 0.360253L_Q^4
+ \alpha_s(\mu_H) \left(0.00050393h_3 + 2.78092L_Q - 2.85654L_Q^2ight)
- 0.147051L_Q^3 + 0.865045L_Q^4 - 0.165638L_Q^5
- 0.101931L_Q^6, \tag{A4}
\]

where \(L_Q = \ln \frac{Q^2}{\Lambda^2} + h_3 = 8998.080\) from [01].

The resummation of large logs between \(\mu_H\) and \(\mu_J\) is given by \(U_H(Q, \mu_H, \mu_J)\), the solution of the RGE for the hard function, which can be written as [10],

\[
U_H(Q, \mu_H, \mu_J) = e^{2K(\Gamma_H, \gamma_H, \mu_H, \mu_H)} \left(\frac{\mu_H^2}{Q^2}\right)^{\omega(\Gamma_H, \mu_H, \mu_H)}, \tag{A5}
\]

where \(\omega\) and \(K\) are given in Eqs. (A19) and (A20) below.

In momentum space, we can use the results from Ref. [33] to calculate the convolution of the plus-functions in \(P\) to give the form,

\[
P(Q, k, \mu_J) = \frac{1}{\xi} E_S(\xi, \mu_J, \mu_S) \times \sum_{m,n,k,l} V_{k,m}^{\tau n} \int J_{\xi}(\mu_J, \xi, \mu_J, \mu_S) \times \hat{S}_n^{\tau} \left[\alpha_s(\mu_S), \frac{\xi}{\mu_S}\right] \times V_\tau^L \left[-2\omega(\Gamma_S, \mu_J, \mu_S)\right] \times L_{\xi}^{-2\omega(\Gamma_S, \mu_J, \mu_S)} \left(\frac{k}{\xi}\right). \tag{A6}
\]

Here \(\xi\) is a dummy variable that does not affect the value of the result. \(\xi E_S(\xi, \mu_J, \mu_S)\) is given by [63][64],

\[
E_S(\xi, \mu_J, \mu_S) = \exp\left[2K(\Gamma_S, \gamma_S, \mu_J, \mu_S)\right] \tag{A7}
\]

\footnote{When convolved with \(F_C\) we use our freedom in choosing \(\xi\) to simplify the final numerical integration, picking \(\xi = QC/6 - 3\pi \Delta(R, \mu_S)\).}
and the coefficients 

\[ S^\tilde{\alpha}_S(\mu_S) = \exp\left[ \frac{2\gamma_E}{\xi} \omega(\Gamma, \mu_S) \right]\frac{\Gamma}{1 - 2\omega(\Gamma, \mu_S)}, \]

and encodes part of the running between \( \mu_S \) and \( \mu_J \). The rest of the running is included in the \( V \) coefficients and the plus-functions, \( \mathcal{L}_I \).

The \( J_m \) and \( S_n \) in Eq. (A6) are the coefficients of the momentum-space soft and jet functions, given by

\[
J_r(p^- k, \mu_J) = \frac{1}{p^- \xi} \sum_{m=1}^{\infty} J_m(\alpha_s(\mu_J), \frac{p^- \xi}{\mu_J^2}) \mathcal{L}_m\left(\frac{k}{\xi}\right),
\]

\[
S_n^\tilde{\alpha}(k, \mu_S) = \frac{1}{\xi} \sum_{n=1}^{\infty} S_n^\tilde{\alpha}(\alpha_s(\mu_S), \frac{\xi}{\mu_S^2}) \mathcal{L}_n\left(\frac{k}{\xi}\right). \tag{A8}
\]

Here the C-parameter soft function coefficients are

\[
S_n^\tilde{\alpha}(-1, x) = S_n^\tilde{\alpha}(0, x) + \sum_{n=0}^{\infty} S_n(\alpha_s) \frac{\ln^{n+1} x}{n!},
\]

\[
S_n^\tilde{\alpha}(0, x) = \sum_{k=0}^{\infty} \frac{(n + k)!}{n! k!} S_n(\alpha_s) \ln^k x. \tag{A9}
\]

which for \( n_f = 5 \) can be written as

\[
S_n^\tilde{\alpha}(0, x) = 1 + 1.0472 \alpha_s + (1.75598 + 0.012666 s^2_2) \alpha_s^2 + (2.59883 + 0.013269 s^2_2 + 0.00100786 s_3^2) \alpha_s^3,
\]

\[
S_0^\tilde{\alpha}(0, x) = 1.22136 \alpha_s^2 + (2.63481 - 0.030977 s^2_2) \alpha_s^3,
\]

\[
S_1^\tilde{\alpha}(0, x) = -1.69765 \alpha_s - 7.45178 \alpha_s^2 - (19.1773 + 0.021501 s^2_2) \alpha_s^3,
\]

\[
S_2^\tilde{\alpha}(0, x) = 1.03573 \alpha_s^2 + 2.3245 \alpha_s^3,
\]

\[
S_3^\tilde{\alpha}(0, x) = 1.44101 \alpha_s^2 + 10.299 \alpha_s^3,
\]

\[
S_4^\tilde{\alpha}(0, x) = -1.46525 \alpha_s^3,
\]

\[ S^\tilde{\alpha}_2(\alpha_s) = -0.611585 \alpha_s^3. \tag{A10} \]

Note that \( s^2_2 \) and \( s^3_3 \) are the \( \mathcal{O}(\alpha_s^2) \) coefficients of the non-logarithmic terms in the series expansion of the logarithm of the position space C-parameter soft function. The coefficients for the jet function are

\[
J_{-1}(\alpha_s, x) = J_{-1}(\alpha_s) + \sum_{n=0}^{\infty} J_n(\alpha_s) \frac{\ln^{n+1} x}{n!},
\]

\[
J_n(\alpha_s, x) = \sum_{k=0}^{n} \frac{(n + k)!}{n! k!} J_n(\alpha_s) \ln^k x, \tag{A11}
\]

and are known up to \( \mathcal{O}(\alpha_s^3) \) except for the constant \( j_3 \) term \[17] [69] [69]. With \( n_f = 5 \) we have

\[
J_{-1}(\alpha_s) = 1 - 0.603949 \alpha_s - 2.26795 \alpha_s^2 + (2.21087 + 0.00100786 j_3) \alpha_s^3,
\]

\[
J_0(\alpha_s) = -0.63662 \alpha_s + 3.04010 \alpha_s^2 + 4.54566 \alpha_s^3,
\]

\[
J_1(\alpha_s) = 0.848826 \alpha_s - 0.447165 \alpha_s^2 - 11.905 \alpha_s^3,
\]

\[
J_2(\alpha_s) = -1.0695 \alpha_s^2 + 5.36297 \alpha_s^3,
\]

\[
J_3(\alpha_s) = 0.360253 \alpha_s^2 + 1.69497 \alpha_s^3,
\]

\[
J_4(\alpha_s) = -0.469837 \alpha_s^3,
\]

\[
J_5(\alpha_s) = 0.0764481 \alpha_s^3. \tag{A12}
\]

The plus-distributions, denoted by \( \mathcal{L}(x) \), are given by

\[
\mathcal{L}_n^a(x) = \left[ \frac{\theta(x) \ln^n x}{x^{1-a}} \right]_+ = \frac{d^n}{d a^n} \mathcal{L}^a(x) \quad [n \geq 0], \tag{A13}
\]

\[
\mathcal{L}^a(x) = \left[ \frac{\theta(x) x^{1-a}}{x^{1-a}} \right]_+ = \lim_{\epsilon \to 0} \frac{d}{dx} \left[ \theta(x - \epsilon) x^{1-a} - 1 \right]. \tag{A14}
\]

In Eq. (A6) we also take advantage of the shorthand for the \( V \) coefficients presented in [13]:

\[
V_k^n(a) = \begin{cases} 
\delta_{kn} \left( a \frac{d^n}{da^n} V(a, b) \right)_{b=0}, & k = -1, \\
\delta_{kn} \left( a \frac{d^{n-k}}{da^{n-k}} V(a, b) \right)_{b=0}, & 0 \leq k \leq n, \\
\frac{a}{n+1}, & k = n+1,
\end{cases}
\]

and the coefficients

\[
V_k^{mn} = \begin{cases} 
\frac{d^m}{da^m} \frac{d^n}{db^n} V(a, b)_{a=b=0}, & k = -1, \\
\sum_{p=0}^{m} \sum_{q=0}^{n} \delta_{p+q,k} \left( a \frac{d^{m-p}}{da^{m-p}} \frac{d^{n-q}}{db^{n-q}} V(a, b)_{a=b=0} \right), & 0 \leq k \leq m + n, \\
\frac{1}{m+1} + \frac{1}{n+1}, & k = m + n + 1,
\end{cases}
\]

\[ S^\tilde{\alpha}_2(\alpha_s) = -0.611585 \alpha_s^3. \tag{A10} \]
for
\[ V(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} - \frac{1}{a} - \frac{1}{b}. \] (A17)

We also need the special cases
\[ V_{-1}^{-1}(a) = 1, \quad V_{0}^{-1}(a) = a, \quad V_{k \geq 1}^{-1}(a) = 0, \quad V_{k}^{-n} = V_{k}^{n-1} = \delta_{nk}. \] (A18)

**Evolution factors and Anomalous Dimensions**

The running between scales is encoded in just a few functions. In Eqs. (A5), (A6), and (A7) we use,
\[ \omega(\Gamma, \mu, \mu_0) = 2 \int_{a_{\mu_0}(\mu_0)}^{a_{\mu}(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma(\alpha) \]
\[ = -\frac{\Gamma_0}{\beta_0} \left\{ \ln \kappa + \frac{\alpha_s(\mu)}{4\pi} \left( \frac{\Gamma_1 - \beta_1}{\beta_0} \right) (\kappa - 1) + \frac{1}{2} \left( \frac{\Gamma_0}{\beta_0} \right)^2 \left( \frac{\beta_2}{\beta_0} - \frac{\beta_2}{\beta_0} \right) (\kappa^2 - 1) \right\}, \] (A19)
and
\[ K(\Gamma, \gamma, \mu, \mu_0) = 2 \int_{a_{\mu_0}(\mu_0)}^{a_{\mu}(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma(\alpha) \int_{a_{\mu_0}(\mu_0)}^{a_{\mu}(\mu)} \frac{d\alpha'}{\beta(\alpha')} \]
\[ = \frac{\Gamma_0}{2\beta_0^2} \left\{ \frac{4\pi}{\alpha_s(\mu_0)} \ln \kappa + \frac{\Gamma_0}{\beta_0} (\kappa - 1) \right\}, \] (A20)
where here \( \kappa = \alpha_s(\mu)/\alpha_s(\mu_0) \) requires the known 4-loop running couplings, and we have defined \( B_2 \equiv \beta_2^2/\beta_0^2 - 2/\beta_0 \) and \( B_3 \equiv -\beta_3^2/\beta_0^2 + 2\beta_2^2/\beta_0^2 - \beta_3/\beta_0 \).

These results are expressed in terms of the coefficients of series expansions of the QCD \( \beta \) function \( \beta(\alpha_s) \), \( \Gamma(\alpha_s) \) (which is given by a constant of proportionality times the QCD cusp anomalous dimension), and of a non-cusp anomalous dimension \( \gamma(\alpha_s) \). These expansion coefficients are defined by the equations,
\[ \beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}, \] (A21)
\[ \Gamma(\alpha_s) = \sum_{n=0}^{\infty} \Gamma_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}, \quad \gamma(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}. \]

For \( n_f = 5 \), the relevant coefficients are \( 69, 74 \)
\[ \beta_0 = 23/3, \quad \beta_1 = 116/3, \quad \beta_2 = 180.907, \] (A22)
\[ \beta_3 = 4826.16, \quad \Gamma_0^{\text{cusp}} = 16/3, \quad \Gamma_1^{\text{cusp}} = 36.8436, \quad \Gamma_2^{\text{cusp}} = 239.208. \]

As was mentioned in the text, we use a Padé approximation for the unknown four-loop cusp anomalous dimension, assigning a large uncertainty to this estimate:
\[ \Gamma_3^{\text{cusp}} = (1 \pm 2) \frac{\Gamma_2^{\text{cusp}}}{\Gamma_1^{\text{cusp}}} . \] (A23)

For the hard, jet, and soft functions, the anomalous dimensions are the same as in the thrust case and are given by \( 28, 57, 59, 64, 69, 75, 78 \)
\[ \Gamma_n^{H} = -\Gamma_n^{\text{cusp}}, \quad \Gamma_n^{J} = 2\Gamma_n^{\text{cusp}}, \quad \Gamma_n^{S} = -\Gamma_n^{\text{cusp}}, \]
\[ \gamma_0^{H} = -8, \quad \gamma_1^{H} = 1.14194, \quad \gamma_2^{H} = -249.388, \]
\[ \gamma_0^{J} = 8, \quad \gamma_1^{J} = -77.3527, \quad \gamma_2^{J} = -409.631, \]
\[ \gamma_0^{S} = -\gamma_0^{H} - \gamma_0^{J}. \] (A24)

For the 4-loop running of the strong coupling constant, we use a form that agrees very well numerically with the
Appendix B: Comparison to Parton Level Monte Carlo

A useful way to validate our results is to compare the SCET prediction with fixed scales $\mu_H = \mu_J = \mu_S = Q$ to the fixed-order prediction, for small values of $C$. This also constitutes an important test on the accuracy of parton level MC’s such as EVENT2 and EERAD3.

At $O(\alpha_s^3)$ we compare our runs to the EVENT2 parton level MC output, again splitting the results in the various color structures. Once again we use logarithmically binned distributions for this exercise. The results are shown in Fig. 16. The comparison looks very good for the numerically most relevant color structures (which also have the biggest uncertainties), and quite good for other structures. Slight deviations are observed in some cases for $C < 10^{-3}$, presumably indicating systematic uncertainties due the numerical infrared cutoff. The dominant color structures do not have this problem and have larger uncertainties, so one can still use the distribution even for $C \sim 10^{-4}$ as long as all color structures are summed up.

In Fig. 17 we compare our numerical determination of $s_2^C$, that was described above in Sec. 16 to the alternate method used in Ref. 17, and show that both procedures yield very similar results. The result from Sec. 16 is shown as an orange line (whose width is its uncertainty). The method of Ref. 17 gives the points. In 17 $s_2^C$ is computed from a relation very similar to Eq. (40), written in the following form for C-parameter:

$$s_2^C = \frac{1}{\sigma_0} \left\{ \sigma_{\text{had}}^{(2)} - \Sigma_2^{(2)}(1) \right\}_{|s_2^C=0}$$

Eq. (B1) can be broken down into various color factors. The limit in Eq. (B1) has to be taken numerically from the output of EVENT2. This is best achieved if events are distributed in logarithmic bins, such that the $C \to 0$ region is enhanced, as can be seen in Fig. 17. The limit can be identified as the value at which the log-binned distribution reaches a plateau, which in the case of C-parameter happens for $10^{-4} \lesssim C \lesssim 10^{-3}$.

Fig. 17 shows that our determination of $s_2^C$ as described...
FIG. 16. Comparison of the fixed-order analytic SCET prediction for the $\mathcal{O}(\alpha_s^2)$ piece with the parton level MC EERAD3. The decomposition in the three color structures $N_C^2$, $1$ and $N_C^{-2}$ are shown in panels (a), (b) and (c), of the first row, respectively, and $N_C n_f$, $n_f/N_C$ and $n_f^2$ are shown in panels (d), (e) and (f), respectively, on the second row. The factor $\alpha_s^2/(2\pi)^3$ has been divided out. We use a log binning in the horizontal axis to emphasize the dijet region.

FIG. 17. Comparison of the determination of the soft function non-logarithmic constants at $\mathcal{O}(\alpha_s^2)$ as explained in Sec. VI (shown as horizontal lines), with a determination employing the method used in Ref. [17] (plateau at $10^{-4} \lesssim C \lesssim 10^{-3}$). The function $s_2^f(C)$ is defined in Eq. (B1). The $C_F C_A$ and $C_F n_f T_f$ color structures are shown in left and right panels, respectively. The logarithmic horizontal axis emphasizes the $C \to 0$ extrapolation. Error bars are included and the blue and red points correspond to data from a different binning.

in Sec. VI] represented by an orange line, agrees with the plateau for the two nontrivial color structures.

Appendix C: Computation of 1-loop Soft Function

In this section we present a general computation of the 1-loop soft function for any event shape $e$ which can be expressed in the dijet limit as

$$e = \frac{1}{Q} \sum_i p_i^4 f_e(y_i),$$

where the sum is over all particles in the final state, $p_i$ is the magnitude of the transverse momentum and $y_i$ is the rapidity of the particle, both measured with respect to the thrust axis. For thrust one has $f_T(y) = \exp(-|y|)$, for angularities one has $f^{\tau}(y) = \exp[-(1-a)|y|]$ and for $C$-parameter one has $f_C(y) = 3/\cosh y$ and $f_C(y) = 1/(2\cosh y)$. One needs to compute the four diagrams in Fig. 18.

7 For perturbative computations partons are taken as massless and hence rapidity $y$ and pseudo-rapidity $\eta$ coincide.
After integrating the angular variables, it is convenient to determine the soft function. We take the space-time number of dimensions to be $d = 4 - 2\epsilon$. A direct computation in momentum space gives

$$S_{\epsilon}^{\text{1-loop}}(\ell) = 4 \, g^2 C_F \int \frac{d^3 2p}{(2\pi)^3 2|p|} \frac{\delta(\ell - p T f_\epsilon(y))}{p^+ p^-}. \tag{C2}$$

After integrating the angular variables, it is convenient to make a change of variables from $p^\pm$ to $(p_T, y)$:

$$\frac{d^3 2p}{(2\pi)^3 2|p|} = \frac{2}{(4\pi)^2 - \epsilon} \frac{p_T^{1-2\epsilon}}{\Gamma(1-\epsilon)} \, dp_T \, dy. \tag{C3}$$

Using Eq. (C3) in (C2) and imposing the on-shell condition $p^+ p^- = p_T^2$, where $p^2 = p_T^2 + 2\epsilon$, we obtain

$$S_{\epsilon}^{\text{1-loop}}(\ell) = 2 \frac{\alpha_s(\mu) C_F e^{-\gamma_E}}{\mu \pi \Gamma(1-\epsilon)} \int dp_T \, dy \left( \frac{p_T}{\mu} \right)^{1-2\epsilon} \delta[\ell - p_T f_\epsilon(y)]$$

$$= 2 \frac{\alpha_s(\mu) C_F e^{-\gamma_E}}{\mu \pi \Gamma(1-\epsilon)} \left( \frac{\ell}{\mu} \right)^{-1-2\epsilon} I_\epsilon(\ell), \tag{C4}$$

where we have defined

$$I_\epsilon(\ell) = \int_{-\infty}^{\infty} dy \, [f_\epsilon(y)]^{2\epsilon}. \tag{C5}$$

Similarly in Fourier space one gets:

$$\tilde{S}_{\epsilon}^{\text{1-loop}}(x) = \frac{2 \alpha_s C_F}{\pi} \frac{\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)} (ix \mu)^{2\epsilon} e^{-\gamma_E} I_\epsilon(\ell). \tag{C6}$$

For thrust and angularities one trivially obtains

$$I_\epsilon(\ell) = \frac{1}{\epsilon}, \quad I_\epsilon^\ast(\ell) = \frac{1}{1 - a \epsilon}. \tag{C7}$$

For C-parameter it is convenient to perform a change of variables

$$y = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \quad \cosh y = \frac{1}{\sqrt{1 - x^2}}. \tag{C8}$$

obtaining for the integral:

$$I_{\tilde{C}}(\epsilon) = 2^{1-2\epsilon} \int_0^1 dx (1 - x^2)^{-1+\epsilon} = \frac{\Gamma(\epsilon)^2}{2 \Gamma(2\epsilon)}. \tag{C9}$$

Expanding in $\epsilon \to 0$ and upon renormalization in $\overline{MS}$ we find for the position space soft function:

$$\tilde{S}_{\epsilon}^{1\text{-loop}} = -\frac{\alpha_s(\mu)}{4\pi} C_F \left[ \frac{\pi^2}{3} + 8 \ln^2(i x \mu e^{-\gamma_E}) \right], \tag{C10}$$

$$\tilde{S}_{\epsilon}^{2\text{-loop}} = -\frac{1}{1 - a} \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \pi^2 + 8 \ln^2(i x \mu e^{-\gamma_E}) \right].$$

Writing the logarithm of the soft function in Fourier space evaluated at the point $x = -i \exp(-\gamma_E)/\mu$ in a generic form as

$$\ln \tilde{S}_\epsilon = 2 \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^n s_n^\epsilon. \tag{C11}$$

we obtain:

$$s_1^\epsilon = -\frac{\pi^2}{6} C_F, \quad s_2^\epsilon = -\frac{1}{1 - a} \frac{\pi^2}{2} C_F. \tag{C12}$$

Fourier transforming the result one obtains the renormalized momentum space soft function:

$$S_{\epsilon}^{1\text{-loop}} = \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \pi^2 \delta(\ell) - \frac{16}{\mu} \left[ \ln(\ell/\mu) \right]_+ \right],$$

$$S_{\epsilon}^{2\text{-loop}} = \frac{1}{1 - a} \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \pi^2 \delta(\ell) - \frac{16}{\mu} \left[ \ln(\ell/\mu) \right]_+ \right]. \tag{C13}$$

### Appendix D: $B_i, G_{ij}$ coefficients

The resummed cross section at N$^3$LL$'$ can be used to compute various fixed-order coefficients, as in Eqs. (78) and (79). The results for coefficients up to $O(\alpha_s^3)$ in perturbation theory are summarized here.

The $B_i$ coefficients read:

$$B_{i,0}^{[i]} = C_F \left( \frac{2\pi^2}{3} - 1 \right), \tag{D1}$$

$$B_{2,0}^{[o]} = C_F \left( -6 \zeta_3 + 1 - \frac{17\pi^2}{24} + \frac{11\pi^4}{36} \right)$$

$$+ C_A C_F \left( \frac{s_t^{[C_F C_A]}}{4} + \frac{283\zeta_3}{18} - \frac{73\pi^4}{360} + \frac{85\pi^2}{24} + \frac{493}{324} \right)$$

$$+ C_F n_f T_F \left( \frac{s_t^{[G_{ij}]}}{4} - \frac{22\zeta_3}{9} - \frac{7\pi^2}{6} + \frac{7}{81} \right).$$

(continued)
\[ B_3^{[0]} = C_A^2 C_F \left( \frac{620179\zeta_3}{1944} - \frac{41\pi^2\zeta_3}{2} - \frac{284\zeta_3^2}{9} - \frac{217\zeta_5}{18} \\ - \frac{51082685}{209952} + \frac{1294933\pi^2}{-34992} - \frac{3641\pi^4}{7776} + \frac{4471\pi^6}{102060} \right) + C_A C_F^2 \left( \frac{5\pi^2 s_2^{[C_F C_A]}}{24} - \frac{s_2^{[C_F C_A]}}{3} + \frac{227\zeta_5}{9} + \frac{2\zeta_3^2}{3} \\ + \frac{248\pi^2\zeta_3}{9} - \frac{89\zeta_3}{27} - \frac{14887\pi^6}{68040} + \frac{23093\pi^4}{19440} + \frac{172585\pi^2}{3888} \\ - \frac{185039}{1296} \right) + C_A C_F n_f T_F \left( -\frac{352\zeta_3}{3} + \frac{13\pi^2\zeta_3}{3} - \frac{3\zeta_5}{3} \right) + \frac{1700171}{13122} - \frac{103903\pi^2}{4374} + \frac{227\pi^4}{4860} \right) + C_F^3 \left( -167\zeta_3 \right) + \frac{38\pi^2\zeta_3}{3} - \frac{4\zeta_3^2}{3} + 22\zeta_5 - \frac{4679}{96} + \frac{139\pi^2}{18} - \frac{109\pi^4}{40} + \frac{4275\pi^6}{136080} \right) + C_F^2 n_f T_F \left( \frac{5\pi^2 s_2^{[n_f]} - s_2^{[n_f]}}{24} - \frac{s_2^{[n_f]}}{4} + \frac{368\zeta_5}{9} \right) \]

\[ + \frac{326\zeta_5}{9} - \frac{326\zeta_5}{9} + \frac{2\zeta_3^2}{3} \\ + \frac{314\pi^2\zeta_3}{9} + \frac{67\zeta_3}{108} - \frac{14887\pi^6}{68040} + \frac{14503\pi^4}{9720} + \frac{56039\pi^2}{1944} \\ - \frac{87719}{1296} \right) + C_A C_F n_f T_F \left( -\frac{1952\zeta_3}{9} + \frac{13\pi^2\zeta_3}{3} - \frac{22\zeta_5}{3} \right) + \frac{3585851}{13122} - \frac{109249\pi^2}{4374} + \frac{227\pi^4}{4860} \right) + C_F^3 \left( -158\zeta_3 \right) + \frac{38\pi^2\zeta_3}{3} - \frac{4\zeta_3^2}{3} + 22\zeta_5 - \frac{5093}{96} + \frac{151\pi^2}{144} - \frac{191\pi^4}{60} \right) + \frac{4275\pi^6}{136080} \right) + C_F^2 n_f T_F \left( \frac{5\pi^2 s_2^{[n_f]} - s_2^{[n_f]}}{24} - \frac{s_2^{[n_f]}}{4} + \frac{368\zeta_5}{9} \right) \]

The results for the first few $G_{ij}$ coefficients read:

\[ G_{12} = -2 C_F, \quad G_{11} = 3 C_F, \quad (D3) \]

\[ G_{23} = C_F \left[ \frac{4 n_f T_F}{3} - \frac{11 C_A}{3} \right], \quad (D4) \]

\[ G_{22} = C_F \left[ C_A \left( \frac{\pi^2}{3} - \frac{169}{36} \right) - \frac{4}{3} \pi^2 C_F + \frac{11 n_f T_F}{9} \right], \quad (D5) \]

\[ G_{21} = C_F \left[ C_A \left( -6 \zeta_3 + \frac{57}{4} + \frac{11\pi^2}{9} \right) + C_F \left( -4 \zeta_3 + \frac{3}{4} + \pi^2 \right) + n_f T_F \left( -5 - \frac{4\pi^2}{9} \right) \right], \quad (D6) \]

\[ G_{34} = C_F \left[ -\frac{847 C_A^2}{108} + \frac{154}{27} C_A n_f T_F - \frac{28}{27} n_f^2 T_F^2 \right], \quad (D7) \]

\[ G_{33} = C_F \left[ C_A \left( \frac{11\pi^2}{9} - \frac{3197}{108} \right) - \frac{22\pi^2}{3} C_A C_F \right. \]

\[ + \frac{C_A n_f T_F}{27} - \frac{154}{27} + \frac{64 C_F^2}{3} \zeta_3 + \frac{C_A n_f T_F}{27} - \frac{154}{27} + \frac{64 C_F^2}{3} \zeta_3 \]

\[ + C_F n_f T_F \left( 2 + \frac{8\pi^2}{3} - \frac{68}{27} n_f^2 T_F^2 \right), \quad (D8) \]

\[ G_{32} = C_F \left[ C_A \left( 11 \zeta_3 - \frac{11323}{648} + \frac{497\pi^2}{54} - \frac{11\pi^4}{90} \right) \right] + C_A C_F \left[ -110 \zeta_3 + \frac{11}{8} - \frac{70\pi^2}{27} + \frac{4\pi^4}{9} \right] \]

\[ + C_A n_f T_F \left( 4 \zeta_3 + \frac{673}{162} - \frac{152\pi^2}{27} \right) + C_F^2 \left( \frac{8\pi^4}{45} - \frac{48\pi^2}{3} \right) + C_F n_f T_F \left( 32 \zeta_3 + \frac{43}{6} + \frac{8\pi^2}{27} \right) + \frac{n_f^2 T_F^2}{27} \]

\[ + \frac{70}{81} + \frac{8\pi^2}{9} \right), \quad (D9) \]

\[ G_{31} = C_A^2 C_F \left[ \frac{11 s_2^{[C_F C_A]}}{6} + 10 \zeta_5 - \frac{361\zeta_3}{27} - \frac{541\pi^4}{540} + \frac{892\pi^2}{81} \right] + \frac{77099}{486} \right] + C_A C_F^2 \left( \frac{452\zeta_3}{9} + \frac{2\pi^2\zeta_3}{3} + \frac{30\zeta_5}{27} + \frac{23}{2} \right) + \frac{161\pi^2}{72} - \frac{49\pi^4}{135} \right] + C_A C_F n_f T_F \left( -\frac{2 s_2^{[C_F C_A]}}{3} \right) \]
The result for R-evolution in the case of no hadron masses is given in Eq. \(56\). The resummed \(\omega\) appearing in this equation was given in App. \(A\). The remaining coefficients and variables that appear in this equation are

\[
S_0 = \frac{\gamma_0^R}{2\beta_0}, \quad S_1 = \frac{\gamma_1^R}{(2\beta_0)^2} - (\hat{b}_1 + \hat{b}_2) \frac{\gamma_0^R}{2\beta_0},
\]

\[
S_2 = \frac{\gamma_2^R}{(2\beta_0)^3} - (\hat{b}_1 + \hat{b}_2) \frac{\gamma_1^R}{(2\beta_0)^2},
\]

\[
\hat{b}_1 = \frac{\beta_1}{2\beta_0^2}, \quad \hat{b}_2 = \frac{\beta_1^2 - \beta_0 \beta_2}{4\beta_0^2}, \quad \hat{b}_3 = \frac{\beta_3^2 - 2\beta_0 \beta_1 \beta_2 + 2\beta_0^2 \beta_3}{8\beta_0^3},
\]

\[
t_1 = -\frac{2\pi}{\beta_0 \alpha_s(R)}, \quad t_0 = -\frac{2\pi}{\beta_0 \alpha_s(R_\Delta)}.
\]

The impact of R-evolution on the value of the hadronic parameter \(\Omega_1(R, R)\) is shown in Fig. \(19\) (with and without hadron mass effects). Here we have set \(\mu = R\) and used the default profile function for \(R(C)\). We actually plot the hybrid scheme \(\Omega_1^C(R, R, h)\) which accounts for a reasonable treatment of threshold effects at the shoulder \(C = 0.75\), and which is discussed in detail below in App. \(F\). The effect of using the hybrid scheme rather than the Rgap scheme is quite small in the region of this figure, but does cause the bending over of the curves that is visible at \(C = 0.6\). Above the shoulder the hybrid scheme uses the MS power correction, which has a flat behavior.

When we introduce hadron mass effects, it is necessary to extend these results to account for the value of \(r\), the transverse velocity. Due to the running in Eq. \(60\), the scheme change result in Eq. \(51\) now becomes

\[
g_C(r) \Omega_1^C(R, \mu, r) = g_C(r) \Omega_1(R, \mu, r) - \delta(R, \mu, r),
\]

where there is additional \(\mu\) dependence from the hadron mass induced running. Our scheme for \(\delta\) now becomes

\[
\delta(R, \mu, r) = \left[ \frac{\alpha_s(\mu)}{\alpha_s(R)} \right]^{\tilde{\gamma}_1(r)} \delta(R, \mu),
\]

and the \(r\) dependence is encoded in the known one-loop anomalous dimension \(\frac{C_\gamma}{\beta_0} \ln (1 - r^2)\). (E4)

With this \(r\)-dependent scheme change, we can once again derive the equations that govern the \(R\) evolution and \(\mu\) running for \(\Omega_1^C\), which following \(81\) become

\[
R \frac{d}{dR} \left\{ g_C(r) \frac{\Omega_1^C(R, \mu, r) (\alpha_s(\mu))^{-\tilde{\gamma}_1(r)}(E5)}{\omega(R_\Delta)} \right\} = -R \left( \frac{\alpha_s(\mu)}{4\pi} \right)^{\gamma_1(r)} \sum_{n=0}^{\infty} \frac{\Gamma_n \Gamma_n}{\Gamma_{n+1}} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^{\gamma_1(r)}
\]

\[
\mu \frac{d}{d\mu} \left\{ g_C(r) \frac{\Omega_1^C(R, \mu, r) (\alpha_s(\mu))^{-\tilde{\gamma}_1(r)}(E6)}{\omega(R_\Delta)} \right\} = 2 R \frac{\alpha_s(\mu)}{4\pi} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^{\gamma_1(r)}
\]

for

\[
\gamma_1^R(r) = \gamma_0^R + \gamma_1^R(r) = \frac{\gamma_1^R + 2\beta_0 \tilde{\gamma}_1(r) \gamma_0^R}{2\beta_0},
\]

\[
\gamma_2^R(r) = \gamma_2^R + 2\beta_0 \tilde{\gamma}_1(r) \gamma_0^R.
\]

Solving this set of equations gives Eq. \(67\), with the \(S_i^r\) given by

\[
S_0^r = S_0, \quad S_1^r = S_1 + \frac{\gamma_0^R}{2\beta_0}, \quad S_2^r = S_2 + \frac{\gamma_0^R + 2\beta_0 \gamma_0^R}{2\beta_0^2}.
\]

FIG. 19. Running of the short-distance power correction \(\Omega_1^C(R, R, h)\) with respect to the reference value \(\Omega_1^C(R_\Lambda, R_\Lambda, 1)\). The scale \(R\) is set to the default profile, and the scheme parameter \(h\) is set to the function displayed in Eq. \(60\). Red, blue and green correspond to center of mass energies of 91.2 GeV, 40 GeV and 200 GeV, respectively. The solid lines do not include hadron mass effects, whereas the dashed ones do.
Appendix F: Gap Scheme in the Fixed-Order Region

In the fixed-order (or far-tail) region there is no longer a hierarchy between the hard, jet and soft scales, and one sets them equal to reproduce the fixed-order QCD predictions exactly. This is done through our profiles. In this region the singular and non-singular terms are of similar size, and hence the factorization of the cross section in a hard factor and a convolution of jet and soft functions is no longer relevant. The issue is further complicated by the analytic structure of the shoulder region, which is located in the fixed-order region and contains the integrable singularity at $C = 0.75$ which has its own logarithmic series (see Fig. 5 and accompanying discussion there). Thus in the far-tail region $C \gtrsim 0.75$ the structure of nonperturbative corrections is likely to differ from that of the shape function $F_C$, and thus is unknown at this time. Nevertheless, we do expect a smearing by a function whose width is $\sim \Lambda_{QCD}$, and hence our smearing by $F_C$ is simply a proxy for a more detailed analysis in this region. Since fits for $\alpha_s$ can be carried out with $C < 0.75$ the treatment of this region, and the discussion below, are not relevant for predicting the shape in the fit region. The region $C \gtrsim 0.75$ does contribute when computing the total cross section from our resummed result, and this motivates us to use a cross section formula that still obtains realistic results in this region.

Due to the shoulder at $C = 0.75$, the use of the infrared subtractions $\delta$ in this region can cause an unphysical behavior of the cross section. The reason for this is that these subtractions yield derivatives acting on the partonic cross section. In the shoulder region with the singular discontinuity starting at $O(\alpha_s^2)$, these derivatives can cause an artificially enhanced singular behavior if the subtraction is not carefully defined in this region. If this is not done, the convolution with the shape function $F_C$ may be insufficient to achieve a smooth cross section near $C \approx 0.75$. In Ref. [27] it was shown that the singularities at $C = 0.75$ can be cured by including soft gluon resummation which makes the cross section smooth. However, this treatment does not resolve the question of the proper field theoretic nonperturbative function for this region, nor any accompanying infrared subtractions due to renormalons.

To deal with the region $C \gtrsim 0.75$ we take an alternative approach, which is to implement a smooth transition between the Rgap scheme in the dijet region to $\overline{\text{MS}}$ in the fixed-order region. This avoids any subtractions at $C = 0.75$. To that end we define a new scheme which depends on a continuous parameter $h$ which takes values between 0 and 1, and that smoothly switches off the gap subtractions when we get near $C = 0.75$. We start by rewriting Eq. (47) as

$$\Delta_C = \frac{3\pi}{2} \left[ \Delta^h(R, \mu, h) + h \delta(R, \mu) \right], \quad (F1)$$

This defines a hybrid short-distance scheme for $\Omega_1$ which we call $\Omega_1^h$:

$$\Omega_1^h(R, \mu, h) = \Omega_1^C - 3\pi h \delta(R, \mu), \quad (F2)$$

which becomes the $\overline{\text{MS}}$ scheme for $h = 0$ and the Rgap scheme for $h = 1$. One can easily derive RGE equations in $\mu$ and $R$, for $\Delta^h$, which we write in the following convenient form:

$$R \frac{d}{dR} \Delta^h(R, R, h) = -h R \gamma(R_\alpha, \mu) \Omega_1^h(R, \mu, h), \quad (F3)$$

$$\mu \frac{d}{d\mu} \Delta^h(R, \mu, h) = 2 R h e^{\gamma_h} \Gamma_{\text{cusp}}(\alpha_s(\mu)), \quad (F4)$$

and a relation to switch from different $h$ schemes:

$$\Delta^h(R, \mu, h_1) - \Delta^h(R, \mu, h_2) = (h_2 - h_1) \delta(R, \mu). \quad (F5)$$

The solution to these three equations is rather simple:

$$\Delta^h(R, \mu, h) = \Delta^h(R_\Delta, \mu_\Delta, h_\Delta) - (h - h_\Delta) \delta(R, \mu) + h_\Delta \Delta^{\text{diff}}(R_\Delta, \mu_\Delta, \mu), \quad (F6)$$

where $\Delta^{\text{diff}}$ has been defined in Eq. [56]. We choose to evolve first in $R$ and $\mu$ in the $h_\Delta$ scheme, where above the peak region (see Eq. [76]) there is only a single evolution since $\mu_\Delta(C) = R(C)$. Close to the shoulder region $C \approx 0.75$ we then smoothly transform from the $h_\Delta$ scheme to the $h$ scheme. This implements the transition from the Rgap scheme with $O(\Lambda_{QCD})$ renormalon subtraction to the $\overline{\text{MS}}$ scheme where this renormalon is not subtracted. The procedure entails a residual dependence on $R$ in the region $C \gtrsim 0.75$ even once $h = 0$, which comes from the fact that we are transforming from Rgap to $\overline{\text{MS}}$ at the scale $R > R_\Delta$. This residual dependence leads to a somewhat smaller effect of the $O(\Lambda_{QCD})$ renormalon even though one employs $\Omega_1^h$ in this region.

In the hybrid scheme described above the first moment of the shape function reads

$$\int dk k F_C(k) = \Omega_1^h(R_\Delta, \mu_\Delta, h_\Delta) - 3\pi \Delta^h(R_\Delta, \mu_\Delta, h_\Delta). \quad (F6)$$

For the practical implementation we choose $h_\Delta = 1$, and thus identify $\Delta^h(R_\Delta, \mu_\Delta, 1) = \Delta(R_\Delta, \mu_\Delta)$ as well as $\Omega_1^h(R_\Delta, \mu_\Delta, 1) = \Omega_1(R_\Delta, \mu_\Delta)$. In the numerical codes this amounts to inserting a factor $h$ in front of each $\delta$ and substituting each $\Delta(R, \mu)$ of Eq. [56] appearing in any of the equations shown in the main text by $\Delta^h(R, \mu, h)$ of Eq. [F5]. Note again, that all of the changes induced by the use of the hybrid scheme rather than the Rgap scheme only influence the shape of the C-parameter cross section for $C \gtrsim 0.75$.

---

8 The right hand sides of the running equations for the power correction $\Omega_1^h(R, \mu, s, h)$ are simply $3\pi$ times the right hand sides of Eqs. [F3] and [F4].
One can also easily extend the hybrid scheme to account for hadron mass effects by defining
\[
g_C(r) \tilde{\Omega}^{C,h}_1(R, \mu, h, r) = g_C(r) \Omega^{C,h}_1(\mu, r) - h \delta(R, \mu, r),
\]
and the evolution and scheme-transformation equations simply read
\[
\begin{align*}
R \frac{d}{dR} \left\{ \left[ g_C(r) \left( \alpha_s(\mu) \right) \right]^{-\hat{\gamma}_1(\mu)} \tilde{\Omega}^{C,h}_1(R, \mu, h, r) \right\}_{\mu = R} \\
= -h R \left( \alpha_s(R) \right)^{-\hat{\gamma}_1(r)} \gamma_R \left[ \alpha_s(\mu), r \right], \\
\mu \frac{d}{d\mu} \left\{ g_C(r) \left( \alpha_s(\mu) \right) \right\}^{-\hat{\gamma}_1(\mu)} \bar{\Omega}^{C,h}_1(R, \mu, h, r) \\
= 2h R e^{\gamma_E} \left( \alpha_s(R) \right)^{-\hat{\gamma}_1(r)} \Gamma_{\text{cusp}} \left[ \delta(\mu) \right], \\
g_C(r) \left[ \bar{\Omega}^{C,h}_1(R, \mu, h_1, r) - \tilde{\Omega}^{C,h}_1(R, \mu, h_2, r) \right] \\
= (h_2 - h_1) \delta(R, \mu, r).
\end{align*}
\]
The solution to these equations is again simple
\[
\begin{align*}
g_C(r) \tilde{\Omega}^{C,h}_1(R, \mu, h, r) = \\
g_C(r) \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_{\Delta})} \right]^{\hat{\gamma}_1(r)} \Omega^{C,h}_1(R_{\Delta, \mu_{\Delta}, h_{\Delta}, r)} \\
- \left( h_{\Delta} - h \right) \delta(R_{\Delta, \mu_{\Delta}, r}) + h_{\Delta} \Delta^{\text{diff}}(R_{\Delta, R, \mu_{\Delta}, r}).
\end{align*}
\]
For phenomenological analyses that consider \( C \gtrsim 0.75 \) or integrate over \( C \) to compute a normalization, we must specify \( h = h(C) \). It must be a function of \( C \) which smoothly interpolates between the values 1 and 0 as one transitions from the resummation to the fixed order region near \( C \sim 0.75 \). To achieve this we use the following simple form:
\[
h(C) = \frac{1}{2} - \frac{1}{\pi} \arctan \left[ \eta \left( C - C_0 \right) \right],
\]
\[
C_0 = 0.7, \quad \eta = 30.
\]

**Appendix G: Rgap Scheme based on the C-Parameter Soft Function**

As an alternative to the Rgap scheme subtraction function used in this work and defined in Eq. (48) one may employ the analogue relation based on the C-parameter partonic soft function:
\[
\delta_{\tilde{\mathcal{C}}}(R, \mu, k) = R e^{\gamma_E} \frac{d}{d\ln(k \mu)} \left[ \ln S_{\mathcal{C}}^{\text{part}}(x, \mu) \right]_{x = (i R e^{\gamma_E})^{-1}},
\]
In this scheme the analogue of Eq. (53) reads
\[
\begin{align*}
S_C(k, \mu) &= \int dk' e^{-6 \delta_0 \frac{d}{dk'} C(k - k', \mu) F_C(k - k' - 6 \Delta_{\tilde{\mathcal{C}}})},
\end{align*}
\]
with
\[
\Delta_{\tilde{\mathcal{C}}} = \Delta_{\tilde{\mathcal{C}}}(R, \mu) + \delta_{\tilde{\mathcal{C}}}(R, \mu).
\]
Here the subtraction function \( \delta_{\tilde{\mathcal{C}}} \) can be written as
\[
\delta_{\tilde{\mathcal{C}}}(R, \mu) = R e^{\gamma_E} \sum_{i=1}^{\infty} \alpha_s^{(i)}(\mu) \delta_{\tilde{\mathcal{C}}}^{(i)}(R, \mu),
\]
where the coefficients for five light flavors read
\[
\begin{align*}
\delta_{\tilde{\mathcal{C}}}^{(1)}(R, \mu) &= -1.69765 \ L_R, \\
\delta_{\tilde{\mathcal{C}}}^{(2)}(R, \mu) &= 0.539295 - 0.933259 \ L_R - 1.03573 \ L^2, \\
\delta_{\tilde{\mathcal{C}}}^{(3)}(R, \mu) &= 0.493255 + 0.0309077 \ s_0^C + 0.833905 \ L_R \\
&- 1.55444 \ L^2 - 0.842522 \ L^3, \quad (G5)
\end{align*}
\]
for \( L_R = \ln(\mu/R) \).

Fig. 20 shows the effect of the renormalon subtractions on the soft function \( S_C(k, \mu) \) from Eqs. (53) and (G2) (we will refer to using Eq. (53) as thrust subtractions and to using Eq. (G2) as C-parameter subtractions), which are compared with the result in the MS scheme.
scheme with any subtractions. The key thing to consider is the stability of the soft function when higher orders in perturbation theory are included, illustrated by the green, blue, and red curves at 1-, 2-, and 3-loop orders. In the left most panel of Fig. 20 we show the C-parameter soft function in the MS scheme. Here the presence of the $A_{QCD}$ renormalon is apparent from the shifting of the soft function to the right as we increase the perturbative order. The MS result also exhibits a large negative dip at small momentum, which makes predictions for the cross section at small $C$ inaccurate in this scheme. With either the C-parameter subtractions (middle panel) or the thrust subtractions (rightmost panel) one achieves significantly better convergence for the soft function, and alleviates most of the negative dip. Making an even closer comparison of the C-parameter and thrust subtraction results, it becomes evident that the thrust subtractions exhibit better convergence near the peak (comparing the difference between the blue and red lines in the two panels) and also more completely remove the negative dip at small momenta. (Similar conclusions hold for the thrust soft function, where again thrust subtractions are preferred.) This improvement for the thrust subtractions can be traced back to the fact that the sign of the non-logarithmic 2-loop term in the C-parameter subtractions in Eq. (G5) is positive, which is opposite to the sign of the renormalon. In the resummation region $R(\tau) = \mu_0(\tau)$, so $L_R \to 0$, and numerically the subtraction goes in the opposite direction to the renormalon in this scheme at 2-loops. This term has an impact even when the logarithmic terms are active in the small $C$ non-perturbative region, which we can see by taking $R = R_0$ and $\mu_2 = \mu_0$. For the thrust subtractions this gives $\pi/2(\delta^1, \delta^2, \delta^3) = \{-0.603, -0.743, -1.621\}$, whereas for the C-parameter subtractions we have $\{\delta^1_C, \delta^2_C, \delta^3_C\} = \{-0.767, -0.994, -0.861\}$. The small value for this 2-loop C-parameter subtraction coefficient is due to the positive constant term.

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