WEIGHTED PLURIPOTENTIAL THEORY ON COMPLEX KÄHLER MANIFOLDS

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ABSTRACT. We introduce a weighted version of the pluripotential theory on compact Kähler manifolds developed by Guedj and Zeriahi. We give the appropriate definition of a weighted pluricomplex Green function, its basic properties and consider its behaviour under holomorphic maps. We also establish a generalization of Siciak’s H-principle.

Introduction

Recently there has been significant progress in weighted pluripotential theory on $\mathbb{C}^N$ which was originally developed in [Si1, Si2] and generalized to parabolic manifolds in [Ze]. Specifically, we refer to [BL, BI1, BI2, BZ, MS]. Concurrently, pluripotential theory on a compact Kähler manifold $X$ based on quasip plurisubharmonic functions has been explored in [GZ1, GZ2, Ko] and [HKH] (see also applications in [Be1, Be2, BB]). The goal of our article is to develop a framework which would allow for a unified treatment of both generalizations of the classical theory and would also allow one to create an analog of the psh-homogeneous pluripotential theory. We will start by showing that a weighted pluripotential theory on $\mathbb{C}^N$ extends naturally to a pluripotential theory on $\mathbb{CP}^N$ with a suitably modified weight. In turn this extends to a homogeneous pluripotential theory in the universal line bundle over $\mathbb{CP}^N$, whose charts are biholomorphic to $\mathbb{C}^{N+1}$. We will generalize these results to projective algebraic manifolds.

We define a weighted pluricomplex Green function on a compact complex manifold $X$ with a Kähler form $\omega$. The definition is formulated in terms of a mild function (see Definition 1). However, many results of our theory hold without requiring that $Q$ be mild. For a mild function $Q$ and a Borel set $K \subset X$ the weighted pluricomplex Green function is

$$V_{K, \omega, Q} = \sup\{\phi \in PSH(X, \omega) : \phi \leq Q \text{ on } K\}.$$ 

Basic properties of $V_{K, \omega, Q}$ are stated and proved in Section 4 followed by the extension of the weighted pluripotential theory in $\mathbb{C}^N$ to a suitable weighted pluripotential theory on $\mathbb{CP}^N$. We obtain more specific results, in particular a generalized Siciak’s H-principle and some approximation results, in the case when $X$ admits a positive line bundle (which by Kodaira’s imbedding theorem is equivalent to $X$ being projective algebraic).

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Preliminary version.
The initial motivation for our work was the similarity between Theorem 2.12 in [Bra] and Theorem 1 in [SH], both of which are generalized versions of Theorem 5.3.1 in [Kl]. We succeeded in proving the following result (Theorem 5, Section 2) which gives the above mentioned theorems as special cases.

**Theorem:** Let \((X,\omega)\) be a compact complex Kähler manifold and \(f : X \to X\) a holomorphic surjection. Assume that there exist \(\alpha\) and \(\beta\), \(1 < \alpha \leq \beta\), such that \(\alpha f_* (PSH(X,\omega)) \subset PSH(X,\omega)\) and \(f^* (PSH(X,\omega)) \subset \beta \cdot PSH(X,\omega)\). Then for every Borel set \(K \subset X\) and every mild function \(Q\) on \(X\),

\[
\alpha V_{f^{-1}(K),\omega,f \cdot Q/\alpha}(x) \leq V_{K,\omega, Q} \circ f(x) \leq \beta V_{f^{-1}(K),\omega,f \cdot Q/\beta}.
\]

1. **Weighted pluricomplex Green functions**

Throughout the paper we assume that \(X\) is a connected compact complex Kähler manifold. Therefore we have on \(X\) (cf. [GF], VI.3) the fundamental form \(\omega\) of a hermitian metric \(\Gamma\) on \(X\) with \(\omega = i \sum_j \gamma_j dz_j \wedge d\overline{z}_j\), satisfying \(d\omega = 0\). It follows that in each coordinate neighborhood in \(X\) we can define a \(C^\infty\) real-valued function \(\phi\) such that \(i \partial \overline{\partial} \phi = (1/2) d\omega\) on \(X\). The functions \(\phi\) are called local potentials of the Kähler metric \(\Gamma\). Existence of smooth local potentials is in fact equivalent to \(\Gamma\) being Kähler: if the fundamental form \(\omega\) of \(\Gamma\) satisfies \(i \partial \overline{\partial} \phi = d\omega\), then \(d\omega = 0\). For example, the Fubini-Study metric on \(\mathbb{C}P^n\) is Kähler, since it has local potentials given by \(\phi_j = \log(1 + \sum_{k \neq j} |z_{j,k}|^2)\) in the coordinate neighborhoods \(U_j = \{Z_j \neq 0\}\) with \(j = 0, 1, \ldots, N\). Here \([Z_0 : \ldots : Z_N]\) are homogeneous coordinates in \(\mathbb{C}P^N\) and \(z_{j,k} := Z_k/Z_j\) in \(U_j\). The set \(U_0\) is identified with \(\mathbb{C}^N\) and \(z_0, k = z_k, k = 1, \ldots, N\), are affine coordinates. We have \(\phi_0 = \log(1 + ||z||^2)\) for \(z \in \mathbb{C}^N\). Let \(\omega\) be a closed real \((1,1)\) current on \(X\) with continuous local potentials. From [GZII], the class of \(\omega\)-plurisubharmonic functions is defined as

\[
PSH(X,\omega) = \{v \in L^1(X,\mathbb{R} \cup \{-\infty\}) : dv \geq -\omega \text{ and } v \text{ is upper semicontinuous}\}.
\]

The \(\omega\)-pluricomplex Green function of a Borel set \(K \subset X\) is defined as

\[
V_{K,\omega}(x) = \sup \{v(x) : v \in PSH(X,\omega) : v|_K \leq 0\}
\]

Consider the class of \(PSH(X,\omega)\), where \(\omega\) is a Kähler form on \(X\) with local potentials \(\phi_j : U_j \to \mathbb{R}\), where \(\{U_j\}_{j=0}^m\) is an open cover of \(X\) by coordinate neighborhoods.

**Definition 1.** Let \(Q : X \to \mathbb{R} \cup \{+\infty\}\) be a function such that \(\exp(-Q + \phi_j)\) is continuous in \(U_j, j = 1, \ldots, m\) and \(\{Q \neq +\infty\}\) is not a pluripolar subset of \(X\). We will call \(Q\) satisfying these assumptions a mild function.

Mild functions are necessarily lower semicontinuous.

**Definition 2.** For a mild function \(Q\) on \(X\) and for a Borel set \(K \subset X\) let us define the weighted \(\omega\)-pluricomplex Green function as

\[
V_{K,\omega,Q} = \sup \{\phi \in PSH(X,\omega) : \phi \leq Q \text{ on } K\}.
\]

The following properties are direct consequences of our definition of \(V_{K,\omega,Q}\).

**Proposition 1.** Let \(K, K_1, K_2\) be Borel subsets of \(X\) and \(Q, Q_1, Q_2\) be mild functions.

i) If \(Q_1 \leq Q_2\) on \(K\) then \(V_{K,\omega,Q_1} \leq V_{K,\omega,Q_2}\).
Recall that a set $\{\tilde{z} \mid \tilde{z} \in \mathbb{C} \}$ boundedness condition. The expression

\begin{align*}
\text{Recall that in the weighted theory on } \mathbb{C}^N \text{ one begins with an admissible weight function on a closed set } K \subset \mathbb{C}^N. \text{ An admissible weight } w \text{ is a nonnegative upper semicontinuous function } w \text{ on } \mathbb{C}^N \text{ with } \{z \in K : w(z) > 0\} \text{ non-pluriharmonic and satisfying the boundedness condition } \lim_{|z| \to \infty} |z|^2 w(z) = 0 \text{ if } K \text{ is an unbounded set (cf. [BL, BII, ST]). The weighted pluricomplex Green function of } K \text{ is defined as } \]

\[
V_{K,Q} = \sup\{u \in \mathcal{L} : u \leq Q \text{ on } K\},
\]

where $Q = -\log w$.

In the homogeneous coordinates $[Z_0 : \ldots : Z_N]$ on $\mathbb{CP}^N$ (with the usual identification $\mathbb{C}^N \simeq \{Z_0 \neq 0\}$ and affine coordinates $z_j = Z_j/Z_0$, $j = 1, \ldots, N$) let \(\tilde{w}(Z_0 : \ldots : Z_N) = w(z_1, \ldots, z_N)/|Z_0| \in \{Z_0 \neq 0\}$, where $w$ is nonnegative and upper semicontinuous with $\{w > 0\}$ non-pluriharmonic, but not necessarily satisfying the boundedness condition. The expression $W(Z) = ||Z||\tilde{w}(Z)$ defines a homogeneous function of order 0 in $\mathbb{C}^{N+1} \setminus \{Z_0 = 0\}$. We have $W(Z) = \varphi(z) + \log w(z)$ for $Z_0 \neq 0$, where $\varphi(z) = (1/2)\log(1 + |z|^2)$. We take

\[
\sqrt{|Z_1|^2 + \ldots + |Z_N|^2\tilde{w}(0 : Z_1 : \ldots : Z_N)} = \lim_{y_j \to 0,y_j \to Z_j} \sup_{y \neq y_0} ||Y||\tilde{w}(Y), Y = (Y_0, \ldots, Y_N)
\]
to obtain an upper semicontinuous function (still denoted by $W$) globally on $\mathbb{CP}^N$, with all values greater or equal to 0. The boundedness condition is equivalent to the property that this global function is identically zero on the hyperplane $\{Z_0 = 0\}$.

This is because $\lim_{|z| \to \infty} |z| w(z) = \lim_{|z| \to \infty} \sqrt{1 + |z|^2} w(z)$. We will assume a weaker condition, namely that $W$ is bounded in $\mathbb{CP}^N$.

The following example demonstrates that the boundedness condition is too restrictive when constructing a weighted pluripotential theory on complex manifolds.

**Example 1.** Let $\omega_{FS}$ be the Fubini-Study Kähler form on $X = \mathbb{CP}^N$ with local potentials $\phi_j$ as above and let $K$ be a subset of $\mathbb{CN} \subset \mathbb{CP}^N$. For $Z \in \mathbb{CP}^N$ define $Q_j(Z) = \phi_j(Z), \ j = 0, \ldots, N$, so that $Q_0(z) = (1/2) \log(\sqrt{1 + \|z\|^2})$ for $z \in \mathbb{CN}$.

The natural 1-to-1 correspondence between $PSH(X, \omega_{FS})$ and the class $L(\mathbb{CN})$ of plurisubharmonic functions with logarithmic growth at infinity, presented explicitly in Example 1.2 in [GZ1], gives the following:

$$V_{K,Q_0}(x) = \sup\{u(x) : u \in \mathcal{L}(\mathbb{CN}), u(z) \leq \log \sqrt{1 + \|z\|^2} \ \forall z \in K\}$$

$$= \sup\{u(x) : u \in \mathcal{L}(\mathbb{CN}), u(z) - (1/2) \log(1 + |z|^2) \leq 0 \ \forall z \in K\}$$

$$= \sup\{v(x) + (1/2) \log(1 + |x|^2), v \in PSH(\mathbb{CP}^N, \omega_{FS}) : v|_{K \leq 0}\}$$

$$= V_{K,\omega_{FS}}(x) + (1/2) \log(1 + |x|^2)$$

for every $x \in \mathbb{CN}$. Assume now that $K$ is not $PSH(\mathbb{CP}^N, \omega_{FS})$-polar. Then $V_{K,\omega_{FS}} \in PSH(\mathbb{CP}^N, \omega_{FS})$ and $V_{K,Q_0} \in \mathcal{L}(\mathbb{CN})$. For a point $Z$ on the hyperplane at infinity $\{Z_0 = 0\}$ we get

$$V_{K,\omega_{FS}}(Z) = \limsup_{x \to Z, x \in \mathbb{CN}} (V_{K,Q_0}(x) - (1/2) \log(1 + |x|^2)).$$

Note that the function $w(z) = \exp(-Q_0(z))$ in our example does not satisfy the boundedness condition in $\mathbb{CN}$. Indeed, the function $\|Z\| w(z) = \exp(-Q_j(Z) + \phi_j(Z))$ for $Z \in U_j, \ j = 0, \ldots, N$ is a constant function 1 on $\mathbb{CP}^N$ (which of course is continuous, but never 0). We draw the reader’s attention to the paper [B12], in which a relation between weighted theory in $\mathbb{CN}$ and standard pluripotential theory in $\mathbb{CN}+1$ is outlined. Examples considered in the Section 5 of that paper deal with a weight function $v$ which is given as the Hartogs radius of a domain with balanced fibers in $\mathbb{CN}+1$ (for the definition and basic properties, see [Sh]). Such a function is upper semicontinuous, but as shown in [B12], does not have to satisfy the boundedness condition on $\mathbb{CN}$. Furthermore, the results of [S22] as well as [MS] were obtained without assuming the boundedness condition. It thus seems reasonable to weaken this condition when working on complex manifolds. In [Gu] a notion of a ‘convex’ hull with respect to a closed real $(1,1)$-current $T$ is considered where the functions $f$ defining the hull satisfy the condition that $\exp(f + \phi)$ are continuous, with $\phi$ continuous local potentials for $T$. We adopted an analogous condition as a part of our definition of a mild function.

The method demonstrated in Example 1 can also be used to prove the following:

**Proposition 4.** Let $K \subset \mathbb{CN} \cong \{Z_0 \neq 0\}$. For a mild function $Q$ on $\mathbb{CP}^N$ with respect to $\omega = \omega_{FS}$ define

$$q(z_1, \ldots, z_N) = q(Z_1/Z_0, \ldots, Z_N/Z_0) = Q(Z) - \log(\|Z\|/|Z_0|), \ Z_0 \neq 0.$$
Conversely, for a lower semicontinuous \( q \) on \( \mathbb{C}^N \), consider

\[
Q(Z) = q(Z_{1}/Z_{0}, ..., Z_{N}/Z_{0}) + \log \| Z \| + \log \| Z_{0} \|,
\]

together with its lower semicontinuous regularization as \( Z_{0} \to 0 \). Then \( V_{K,q}(x) = V_{K,\omega,Q}(x) = (1/2) \log(1 + \| x \|^2), \quad x \in \mathbb{C}^N \).

Consider now a holomorphic line bundle \( L \) over a compact Kähler manifold \( X \). Recall that a (singular) metric on \( L \) can be given (cf. [De], [DPS]) by a collection of real-valued functions \( h = \{ h_{i} \} \) on \( X \), defined in a trivializing cover \( \{ U_{j} \} \), such that \( h_{j} = h_{i} + \log | g_{ij} | \), where \( g_{ij} \) are transition functions for \( L \). The metric is called positive if all \( h_{j} \) are plurisubharmonic. (The notion of positivity is used here in the weak sense.) In particular, a smooth metric \( \{ \phi_{j} \} \) such that \( \omega = dd^{c} \phi_{j} \) is a Kähler form will be positive.

If \( L \) is a positive line bundle and \( \omega = [c_{1}(L)] \), there is a 1-to-1 correspondence between the family of all positive metrics on \( L \) and the class \( \text{PSH}(X, \omega) \). In the case of \( X = \mathbb{P}^{N} \) with the Fubini-Study form \( \omega \), this correspondence is equivalent to the \( H \)-principle due to Siciak ([SZ]).

**Proposition 5.** (cf. [G], property (iv) pg 456): Let \( h \) be a logarithmically homogeneous plurisubharmonic nonnegative function on \( \mathbb{C}^{N+1} \). Then \( h \) defines a positive metric on \( \mathbb{C}^{N} \). Conversely, each positive metric on \( \mathbb{C}^{N} \) defines a logarithmically homogeneous \( \text{psh} \) function on \( \mathbb{C}^{N+1} \).

**Proof.** By logarithmic homogeneity we have,

\[
v(Z_{0}/Z_{1}, ..., 1, ..., Z_{N}/Z_{k}) = v(Z) - \log | Z_{k} | \quad \text{in} \quad \{ Z_{k} \neq 0 \}
\]

Hence \( v_{k} = v_{i} + \log | Z_{k}/Z_{i} | \) in \( U_{i} \cap U_{k} \) and all \( v_{i} \) are plurisubharmonic. To prove the converse, take \( h_{0} = h \mid v_{0} \). The function \( v(Z) = h_{0}(Z) + \log | Z_{0} | \) in \( U_{0} \), and \( v(0, Z_{1}, ..., Z_{N}) = \limsup_{\lambda \to 0} v(0, Z_{1}, ..., Z_{N}) \) is plurisubharmonic. Since it also satisfies \( v(\lambda Z) = v(Z) + \log | \lambda | \) for \( \lambda \in \mathbb{C} \) our proof is complete.

By Example 1.2 in [GZ], the class \( \mathcal{L}(\mathbb{C}^{N}) \) corresponds in a 1-to-1 manner with the class of \( \text{PSH}(\mathbb{C}^{N}, \omega) \) functions, which in turn correspond in a 1-to-1 manner with positive metrics on the (positive) hyperplane bundle over \( \mathbb{P}^{N} \). Thus Proposition 5 establishes a 1-to-1 correspondence between logarithmically homogeneous functions \( \tilde{v} \) on \( \mathbb{C}^{N+1} \) and functions \( v \) in the class \( \mathcal{L}(\mathbb{C}^{N}) \) so that \( \tilde{v}(1, z) = v(z) \) for \( z \in \mathbb{C}^{N} \), that is, the \( H \)-principle.

If \( L \) is a positive line bundle over \( X \), then its dual \( L' \) is negative (cf. [G], Prop. VI.6.1 and VI.6.2). Hence there exists a system of trivializations \( \theta_{i} : L' \mid U_{i} \to U_{i} \times \mathbb{C} \) with transition functions \( G_{ik} = g_{ik}^{-1} = g_{ki} \) and a smooth metric \( \{ h_{i} \} \) on \( L \) such that the smooth function \( \chi_{k} : L' \to \mathbb{R} \), defined as \( \chi_{k} \circ \theta_{k}^{-1}(x, t) = H_{i}(x) \cdot | t |^{2} \), is strictly plurisubharmonic outside the zero section of \( L' \), where \( H_{i}(x) = \exp 2h_{i}(x), \quad x \in U_{i} \). As a simple example of a negative line bundle we can take the universal line bundle over \( \mathbb{P}^{N} \), \( O(-1) := \{(Z, \xi) \in \mathbb{P}^{N} \times \mathbb{C}^{N} : \xi \in \mathbb{C} \cdot Z, | Z | \in \mathbb{C}^{N+1} \} \). That is, the fiber of \( O(-1) \) over a point \( [Z] \in \mathbb{P}^{N} \) is the complex line in \( \mathbb{C}^{N+1} \) generated by \( (Z_{0}, ..., Z_{N}) \). The function \( \chi \circ \theta_{1}^{-1}(Z, t) = | t |^{2} | Z_{i} |^{-2} | Z |^{2} \) for \( Z_{i} \neq 0 \), associated with the Fubini-Study metric on the dual line bundle \( O(1) \) over \( \mathbb{P}^{N} \), is plurisubharmonic.

Next we establish a generalization of Siciak’s \( H \)-principle.
Theorem 1. (cf. [GF], Prop. VI.6.1): Let $L$ be a positive line bundle over a compact Kähler manifold $X$ and let $d > 0$. Let $\mathcal{H}^+_d$ denote the family of all functions $\chi \in PSH(L')$ which are nonnegative, not identically 0 and absolutely homogeneous of order $d$ in each fiber. Then there is a one-to-one correspondence between $\mathcal{H}^+_d$ and the class of positive metrics on $L$.

Proof. Consider a system of trivializations $\theta_i : L'|_{U_i} \rightarrow U_i \times \mathbb{C}$ with transition functions $G_{ik} = g_{ik} = 1/g_{ik}$. Let $\chi \in \mathcal{H}^+_d$. For $x \in U_i$, $t \neq 0$ define
$$H_i(x) := \chi \circ \theta_i^{-1}(x, t)/|t|^d.$$ 
Note that this expression does not depend on $t$. We have $\chi \circ \theta_i^{-1}(x, t) = \chi \circ \theta_k^{-1}(x, G_{ik}(x)t)$, hence by absolute homogeneity of order $d$, $H_k(x) = |G_{ik}(x)|^d H_i(x)$ in $U_i \cap U_k$. Taking $h_i = (1/d) \log H_i$ in $U_i$ we get a collection of plurisubharmonic functions satisfying $h_k = \log g_{ik} + h_i$, i.e., a positive metric on $L$. Conversely, let $\{h_i\}$ be a metric on $L$. The function $\chi$ on $L'$ defined as $\chi \circ \theta_i^{-1}(x, t) = \exp(dh_i(x)) \cdot |t|^d$ is plurisubharmonic if and only if $h_i$ are, so for a positive metric the associated function $\chi$ is in $\mathcal{H}^+_d$. \hfill \square

Unless otherwise indicated, we will work with $\mathcal{H}^+ := \mathcal{H}^+_1$. Note that if we take $L'$ in Theorem 1 to be the universal line bundle $\mathcal{U}$ over $\mathbb{C} \mathbb{P}^N$, then the trivialization $\theta_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$ is given as $\theta_i(t(Z)) = ([Z_0 : ... : Z_N], tZ_i)$. Hence for a function $\chi \in \mathcal{H}^+$ we have $\chi \circ \theta_i^{-1}([Z_0 : ... : Z_N], t) = h_i([Z_0 : ... : Z_N]) + \log |Z_i| + \log |t|$ for $Z_i \neq 0$, where $h_i$ define a metric on $\mathbb{C} \mathbb{P}^N$. By Proposition 5 over the chart $Z_0 \neq 0$ we get $\chi(tZ) = v(Z_1/Z_0, ..., Z_N/Z_0) + \log |t|$ for $t \neq 0$ with $v$ plurisubharmonic. That is, $\chi$ defines a logarithmically homogeneous psh function on $\mathbb{C} \mathbb{P}^{N+1}$.

For a positive holomorphic line bundle $L$ over a compact Kähler manifold $X$ there is a precise relation between the weighted pluricomplex Green function with respect to $\omega = [c_1(L)]$ of a Borel set $K$ in $X$ and a $\mathcal{H}^+$-envelope of some associated set $\tilde{K}$ in the dual bundle $L'$. It generalizes the formulas obtained by Bloom in ([B2]).

For the weight $Q$ on $X$ consider the collection $q_i = Q - \phi_i$, where $\omega = dd^c \phi_i$ in $U_i$ and $U_i$ form a trivializing cover for $L$. For $K \subset X$ define $\tilde{K} \subset L'$ by taking
$$\tilde{K} \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x, t) : x \in U_i \cap K, |t| = \exp(-q_i(x))\}.$$ 
This set is well defined, since $\theta_k^{-1}(x, t) = \theta_i^{-1}(x, G_{ik}(x)t)$. Hence if $x \in U_i \cap U_k \cap K$, then $|G_{ik}(x)|t| = \exp(-q_i(x))$ if and only if $|t| = \exp(-q_k(x))$. Consider
$$H_{\tilde{K}} = \sup\{u \in PSH(L') : \exp u \in \mathcal{H}^+, u |_{\tilde{K}} \leq 0\}.$$ 
The following theorem gives the relationship between functions $H_{\tilde{K}}$ and $V_{K, \omega, Q}.$

Theorem 2. (cf. [B2], Thm 2.1): For all $i$,
$$H_{\tilde{K}} \circ \theta_i^{-1}(x, t) = V_{K, \omega, Q}(x) + \log |t| + \phi_i(x)$$

Proof. By Theorem 1
$$H_{\tilde{K}} = \sup\{u : u \circ \theta_i^{-1}(x, t) = h_i(x) + \log |t|, u |_{U_i \cap K} \leq 0\}$$ 
$$= \sup\{u : u \circ \theta_i^{-1}(x, t) = h_i(x) + \log |t|, h_i(x) \leq q_i, \forall i\}$$
where \( h_i \) define a positive metric on \( L \). Hence, for such \( h_i \),
\[
H_K \circ \theta_i^{-1}(x, t) = \sup \{ h_i(x) : h_i(x) |_{K \cap U_i} \leq q_i \} + \log |t|
\]
\[
= \sup \{ \phi_i(x) : \phi_i(x) \in \text{PSH}(X, \omega), \phi_i \leq Q, \phi_i(x) \leq |t| \}
\]
\[
= \mathcal{V}_{K, \omega, Q}(x) + \log |t| + \phi_i(x), \quad \forall i.
\]

Theorem 2 allows us to study the behavior of the weighted pluricomplex Green functions as we vary the weight. Namely, we have the following:

**Proposition 6.** (cf. [B2], Cor 2.2): Let \( K \subset X \) be a Borel set. Suppose \( \{Q_n\}, Q \) are mild functions with \( Q_n \not\subset Q \). Then \( \lim_{n \to \infty} \mathcal{V}_{K, \omega, Q_n} = \mathcal{V}_{K, \omega, Q} \).

**Proof.** Consider the sets \( K_n, M_n \subset L' \), where

\[
M_n \cap \pi^{-1}(U_i) = \{ \theta_i^{-1}(x, t) : x \in U_i \cap K, |t| \leq \exp(-q_i(\nu(x))) \},
\]
\[
K_n \cap \pi^{-1}(U_i) = \{ \theta_i^{-1}(x, t) : x \in U_i \cap K, |t| = \exp(-q_i(\nu(x))) \}
\]

where \( q_i(\nu) = Q_n - \phi_i, n \geq 0 \). The sequence \( M_n \) is decreasing, with \( \bigcap_{n=1}^{\infty} M_n = \tilde{M}_0 \).

By maximum principle (applied in each fiber), \( H_{M_n} = H_{K_n}, n \geq 0 \) (here we use the assumption of all \( Q_n \) being mild). For a function \( u \in \mathcal{H}^+ \) such that \( u \leq 0 \) on \( M_0 \) and an arbitrary \( \varepsilon > 0 \), there exists an \( n_0 \) such that for all \( n \geq n_0 \) we have \( M_n \subset \{ u < \varepsilon \} \). The function \( u - \varepsilon \) is in \( \mathcal{H}^+ \) and for \( n \geq n_0 \) it satisfies \( u - \varepsilon \leq H_{M_n} \leq \lim_{n \to \infty} H_{M_n} \leq H_{M_0} \), hence \( \lim_{n \to \infty} H_{M_n} = H_{K_0} \). By Theorem 2

\[
\lim_{n \to \infty} \mathcal{V}_{K, \omega, Q_n} = \mathcal{V}_{K, \omega, Q_0}.
\]

**Proposition 7.** (cf. [B2], Cor 2.4) Let \( Q_n, n \geq 0 \) be mild functions on \( X \) such that \( Q_n \not\subset Q_0 \). Then \( \mathcal{V}_{K, \omega, Q_n} \to \mathcal{V}_{K, \omega, Q_0} \).

**Proof.** Since the potentials \( \phi_i \) of \( \omega \) are continuous, we have \( H_K^* \circ \theta_i^{-1}(x, t) = \mathcal{V}_{K, \omega, Q_0} + \log |t| + \phi_i \) for all \( i \). We can assume that the set \( \tilde{M}_1 \) (see Proposition 0) is not \( \omega \)-polar. By Proposition 2 \( H_K^* \) is plurisubharmonic on \( L' \). Let \( H = \lim_{n \to \infty} H_{M_n} \). The function \( H \) is in \( \mathcal{H}^+ \) and satisfies \( H \leq 0 \) on \( K_0 \setminus P \), where \( P \) is some pluripolar set. Hence \( H \leq \mathcal{V}_{K_0}^* \).

**Corollary 1.** Proposition 2 holds when the convergence \( Q_n \not\subset Q_0 \) takes place quasi-everywhere on \( X \), that is, outside some \( \omega \)-polar set.

**Corollary 2.** Proposition 2 holds when the convergence \( Q_n \not\subset Q_0 \) takes place quasi-everywhere on \( X \).

2. APPROXIMATION AND PULLBACKS BY HOLOMORPHIC MAPS

In standard pluripotential theory in \( \mathbb{C}^N \) and its weighted generalization there is a function \( \Phi_K \) such that \( \log \Phi_K = \mathcal{V}_{K, \omega, Q} \). The function \( \Phi_K \) is given as the supremum of certain functions with 'regular' growth, that is, polynomials (when \( Q \equiv 0 \)) or weighted polynomials (see Theorem 6.2 in [SI]). Theorem 2.8 in [B1], and Théorème 5.1 in [Z2]. In [Z1] it is proven that \( \mathcal{V}_{K, \omega, Q} = \sup \{ (1/n) \log \| s \|_{n, \omega} : n \geq 1, s \in \Gamma(X, L^n), \sup_K \| s \|_{n, \omega} \leq 1 \} \), where \( L \) is a positive holomorphic line bundle over a compact manifold \( X, \omega = dd^c \varphi \) in a trivializing cover \( U_j \) is a (global) Kähler form and the norm \( \| s \|_{n, \omega} \) of a section \( s \) of the tensor power \( L^n \) is
Theorem 3. Let \( K, \omega, \varphi, \omega \) be as above. Let \( Q \) be a mild function on \( X \) and let \( K \) be a compact subset of \( X \). Then

\[
V_{K, \omega, Q} = \log \Phi_{K, \omega, Q} \text{ where } \Phi_K(x) = \sup_{n \geq 1} (\Phi_n(x))^{1/n}
\]

with \( \Phi_n(x) = \sup_{s \in \Gamma(X, L^n), n \geq 1} \exp(-nQ)\|s\|_{n, \varphi} \leq 1 \).

Unlike [GZ1], in which the theorem was proved for \( Q \equiv 0 \), we will not use \( L^2 \)-estimates for the \( \bar{\partial} \)-operator. Instead, we will apply the following lemma (cf. [Ze], Lemma 5.2, [Bel], Lemma 2.1 and 3.2):

**Approximation Lemma.** Let \( X, \omega, L \) be as above and let \( v \in \text{PSH}(X, \omega) \cap C^\infty \) be such that \( dd^c v + \omega \) is strictly positive. Then for every \( \varepsilon > 0 \) and every compact \( K \subset X \) there exist \( N_1, \ldots, N_m \) and \( s_1, \ldots, s_m \) such that \( s_j \in \Gamma(X, L^{N_j}), j = 1, \ldots, m \) and

\[
v(x) - \varepsilon \leq \sup_{1 \leq j \leq m} (1/N_j) \log \|s_j(x)\|_{N_j, \varphi} \leq v(x) \text{ for all } x \in K
\]

where the norm of the section \( s_j \) is computed as above.

**Proof.** (of the Approximation Lemma): Let \( \varphi_i \) be local potentials for the Kähler form \( \omega \) and let \( h = \{h_i = v + \varphi_i\} \) be the positive metric corresponding to \( v \). The inequality in the statement of the lemma is equivalent to

\[
h_i - \varepsilon \leq \sup_{1 \leq j \leq m} (1/N_j) \log |s_j(x)| \leq |h_i(x)|, \quad x \in K \cap U_i, \quad i = 1, \ldots, l
\]

where \( |\cdot| \) is the usual absolute value of a complex number. Let \( r \in (0, 1) \) and \( \chi_r \) be the function in the class \( \mathcal{H}^+ \) on \( L' \) associated with the metric \( r \cdot h \). For every \( r \) the set \( \Omega_r = \{\chi_r < 1\} \) is a strictly pseudoconvex neighborhood of the zero section in \( L' \) (cf. [GR], VI.6.1). Fix a point \( x_0 \in K \) and \( \zeta_0 = \theta_r^{-1}(x_0, 1) \). Then \( |t| < \chi_r(\zeta_0) \) if and only if \( (x_0, t) \in \Omega := \Omega_r \). The function \( f(t) = \sum_{n=1}^\infty (\chi_r(\zeta_0))^n t^n \), \( |t| < 1/\chi_r(\zeta_0) \), \( f(0) = 0 \) is holomorphic on the analytic set \( (\Omega \cap L'_{x_0}) \cap X \) and is identically 0 on \( X \). We consider the Remmert reduction of \( \Omega \) (see [G], Satz 1, or [P], Theorem 2.7 and preceding discussion). That is, we have a Stein space \( Y \) and a proper surjective holomorphic map \( \Phi : \Omega \rightarrow Y \) with the following properties: (i) \( \Phi \) has connected fibers; (ii) \( \Phi_* (O_{\Omega}) = O_Y \); (iii) the canonical map \( O_Y(Y) \rightarrow O_{\Omega}(\Omega) \) is an isomorphism; (iv) if \( \sigma : \Omega \rightarrow Z \) is a holomorphic map into a Stein space \( Z \) then there exists a uniquely determined holomorphic map \( \tau : Y \rightarrow Z \) such that \( \tau \circ \Phi = \sigma \). The map \( \Phi \) blows down the zero section of \( L' \). Note that the set \( A = \Phi(L'_{x_0} \cup X) = \Phi(L'_{x_0}) \) is analytic in \( Y \) (by Remmert’s Proper Mapping Theorem) and the function \( f(\Phi(t)) := f(t) \) is holomorphic on \( A \) (by property (ii) of Remmert’s reduction). Every analytic set in a compact space is the support of a closed complex subspace (cf. [GR], A.3.5), so we can apply Theorem V.4.4 in
to conclude that the function \( \tilde{f} \) is the restriction to \( A \) of a function \( \tilde{F} \) that is holomorphic on the Stein space \( Y \). By the properties (ii) and (iii) above, there exists a function \( F \) holomorphic on \( \Omega \) such that \( \tilde{F} \circ \Phi = F \). For \( t \neq 0 \) one can represent \( F \) as \( F = \tilde{G} \circ \theta_t^{-1}(x,t) = \sum_{n=1}^{\infty} F_n^{(i)}(x)t^n \), with \( F_n^{(i)} \) holomorphic in \( U_i \). We have \( F \circ \theta_t^{-1}(x,t) = F \circ \theta_t^{-1}(x,G(t,x)) \), which gives \( F_n^{(i)}(x) = (g_{ik}(x))^{n} F_n^{(k)}(x) \), i.e., \( F_n \) are cocycles corresponding to holomorphic sections of the tensor product \( L^n \) over \( \Omega_t \). Considering the domain of convergence of the representation for \( F \circ \theta_t^{-1} \), \( k = 1, \ldots, l \), we get

\[
\limsup_{n \to \infty} |F_n(x)|^{1/n} \leq \exp{rh(x)}, x \in X.
\]

By Hartogs’s lemma, there exists an \( n_\delta > 1 \) such that \((1/n) \log |F_n(x)| \leq r \cdot h(x) + \delta, \quad x \in K, n \geq n_\delta \). For the estimate from below, note that \( F_n(x_0) = \chi_r(\zeta_0) = rh(x_0) \) for all \( n \). Since \( rh = r(v + \varphi) \) is continuous, there exists an \( n_0 \geq n_\delta \) and a neighborhood \( W_{x_0} \) of \( x_0 \) such that \((1/n_0) \log |F_n(x_0)| > rh(x) - \delta, \quad x \in W_{x_0} \). Compactness of \( K \) and suitable relations between \( \epsilon, \delta \) and \( r \) give holomorphic sections satisfying the conclusion of the lemma.

Proof. (of Theorem 9): We mimic the method of proof of Theorem 2.8i in [Bl1]. Let \( u \in PSH(X,\omega) \). By Theorem 7.1 in [GZ2], there is a sequence \( u_k \in PSH(X,\omega) \cap C^\infty(X) \) such that \( u_k \downarrow u \). Let \( \epsilon > 0 \). By Dini’s theorem, there exists an integer \( k_0 \) such that \( u(x) \leq u_k(x) \leq Q(x) + \epsilon \) for all \( x \in K, k \geq k_0 \). By adding a small multiple of a local Kähler potential in some coordinate neighborhood, we can assume that \( d\omega u_k + \omega \) is strictly positive. By the Approximation Lemma, \( \exists s_j^{(k)} \in \Gamma(X, L_j^{(k)}) \), \( j = 1, \ldots, m_k \) such that

\[
\sup_{j=1, \ldots, m_k} (\log |\exp(-2N_j^{(k)} \epsilon s_j^{(k)})|)/(N_j^{(k)}) \leq (1/n) \log \Phi_n(x),
\]

where \( n = \max_j N_j^{(k)} \), \( j = 1, \ldots, m_k \). Hence \( u - 4\epsilon \leq \log \Phi \). The reverse inequality is obvious, since \((1/N) \log \|s\|_\nu \) defines a positive singular metric on \( L \).

Under the assumptions of Theorem 8 we also have the following:

**Proposition 8.** Let \( \Psi(x) = \lim_{n \to \infty} \psi_n(x) = \sup_{n \geq 1} \psi_n(x) \), with \( \psi_n(x) = \sup\{\|s\|_\nu | s \in \Gamma(X, L^n) \}, \sup^*_K(\exp(-nQ) \|s\|_\nu) \leq 1 \). If \( P \subset K, P \) is \( PSH(X,\omega) - polar \). Then

\[
V^*_K,\omega,Q = (\log \Psi_K)^*.
\]

The proof proceeds exactly like that of [Bl1], Theorem 2.8(ii), provided we have the domination principle on a compact Kähler manifold of dimension \( N \) (cf. [K]), cor. 3.7.5 and prop.5.5.1 [BT2], cor.4.5, [La], for versions on open subsets of \( \mathbb{C}^N \). In our proof we will assume that one of functions is in \( L^\infty(X) \), since this is the case we need. A more general version was recently proved independently as Proposition 2.7 in [BH].

Proposition 9. If \( K \) is not \( PSH(X,\omega) - polar \) and \( Q \) is continuous, then \( V^*_K,\omega,Q \in PSH(X,\omega) \cap L^\infty(X) \). In particular, the complex Monge-Ampère operator \( (\omega_{V,K,Q})^n \) is well defined and satisfies \((\omega_{V,K,Q})^n = 0 \) in \( X \setminus K \).
Proof. The proof proceeds as that of GZ1, Theorem 4.2.2, and uses Proposition 2.

Now we may state and prove the required domination principle.

**Theorem 4.** (Domination Principle): Let $u, v \in \text{PSH}(X, \omega)$ with $v \in L^\infty(X)$ be such that

$$\int_{\{u < v\}} (\omega + dd^c u)^N = 0.$$ 

Then $u \geq v$ in $X$.

Proof. The following argument was communicated to us by Ahmed Zeriahi as a replacement for an earlier incorrect proof. It is enough to prove that $u \geq v$ on a set of full $\omega$-volume in $X$. We can assume that $v$ is negative everywhere on $X$. Then for all $s, t > 0$, \{ $u - v \leq -s - t$ \} $\subset$ \{ $u - v \leq -s - tv$ \}, which for small $t$ is still a subset of \{ $u - v < 0$ \}. Then, by Lemma 2.2 in EGZ, $0 = \int_{\{u - v < -s - tv\}} (\omega + dd^c u)^N \geq t^N \text{Cap}\{u - v \leq -s - t\}$, where Cap is the Monge-Ampere capacity defined in GZ1 (Definition 2.4). Proposition 2.5(1) in GZ1 implies that $\text{Vol}\{u - v \leq -s - t\}$ for $s, t > 0$, $t$ small, hence $\text{Vol}\{u - v < 0\} = 0$. □

Finally, we are interested in how weighted pluricomplex Green functions change under a holomorphic map $f : X \to X$, where $X$ is a compact Kähler manifold (not necessarily projective algebraic) with a closed real $(1,1)$-current $\omega$ on $X$ with continuous local potentials (not necessarily a Kähler form). Proposition 4.4.5 in GZ1 states that if $f : X \to X$ is holomorphic, and $K \subset X$ is a Borel set, then $V_{f^{-1}(K), \omega} f \circ f \leq V_{K, f^* \omega}$. The proof applies also to the weighted pluricomplex Green function and gives the following:

**Proposition 10.** Let $X, \omega, K$ be as above and let $Q$ be a mild function on $X$. Then $V_{f^{-1}(K), \omega, Q} f \circ f \leq V_{K, f^* \omega, Q} f$ in $X$.

Below, we establish a relation between the pullback of $V_{K, \omega, Q}$ by a surjective holomorphic map $f : X \to X$ and $V_{f^{-1}(K), \omega, Q}$ with an appropriate function $Q$. For a function $u : X \to \mathbb{R} \cup \{-\infty\}$ let us define $f_*u(x) = \sup\{u(y) : y \in f^{-1}(x)\}$. This is a well defined function, since $f^{-1}(x)$ is compact. Also, let $f^*u = u \circ f$. The following theorem generalizes Theorem 2.12 in Bra and Theorem 1 in St1 (it yields both as special cases):

**Theorem 5.** Assume that there exist $\alpha$ and $\beta$, $1 < \alpha \leq \beta$, such that

$$\alpha f_*(\text{PSH}(X, \omega)) \subset \text{PSH}(X, \omega)$$

and

$$f^*(\text{PSH}(X, \omega)) \subset \beta \cdot \text{PSH}(X, \omega).$$

Then for every Borel set $K \subset X$ and every mild function $Q$ on $X$,

$$\alpha V_{f^{-1}(K), \omega, f^*Q/\alpha} f \leq V_{K, \omega, Q} f \leq \beta V_{f^{-1}(K), \omega, f^*Q/\beta} f.$$
Proof. Let \( u \in PSH(X, \omega) \) be such that \( \alpha u \leq f^*Q \) on \( f^{-1}(K) \). Then \( v = \alpha f_*u \) is in \( PSH(X, \omega) \) and satisfies \( v \leq Q \) on \( X \). Moreover, \( \alpha u(x) \leq \nu(f(x)) \leq V_{K, \omega, Q}(f(x)) \), which gives the first inequality. For the second one, if \( u \in PSH(X, \omega) \) satisfies \( u \leq Q \) on \( K \), then by assumption \((1/\beta)f^*u \) is in \( PSH(X, \omega) \) and \((1/\beta)f^*u \leq (1/\beta)f^*Q \) on \( f^{-1}(K) \), which gives the conclusion.

On \( X = \mathbb{CP}^N \), the assumptions of Theorem 5 are equivalent to assumptions about growth of \( f \) made in Theorem 2.12 in [Bra] or its unweighted counterpart, Theorem 5.3.1 in [Kl]. Details may be found in Theorem 1 in [St1] and its proof. The main theorem in [St2] has conditions equivalent to the assumption \( \alpha f_*PSH(X, \omega) \subset PSH(X, \omega) \) when \( X \hookrightarrow \mathbb{CP}^N \) is a projective algebraic manifold and \( \omega \) is the pullback of the Fubini-Study form by the embedding \( \hookrightarrow \). One of the conditions is that \( f \) has an attracting divisor in \( X \), so in fact the assumption is quite strong.

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