On worldsheet curvature coupling in pure spinor sigma-model

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Abstract

We discuss the relation between unintegrated and integrated vertex operators in string worldsheet theory, in the context of BV formalism. In particular, we clarify the origin of the Fradkin-Tseytlin term. We first consider the case of bosonic string, and then concentrate on the case of pure spinor superstring in $AdS_5 \times S^5$. In particular, we compute the action of $b_0 - \bar{b}_0$ on the beta-deformation vertex. As a by-product, we formulate some new conjectures on general finite-dimensional vertices.

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In a general curved background, the $b$-ghost of the pure spinor superstring \[1, 2\] is not holomorphic:

\[ \bar{\partial}b = Q(\ldots) \]  \hspace{1cm} (1)

On one hand, this is a problem, complicating the computation of scattering amplitudes. On the other hand, this is a tip of an interesting mathematical structure. It was suggested in \[3, 4\] that in such cases the definition of the string measure should be modified, so that the resulting measure should descend on the factorspace of metrics\footnote{or, more generally, of Lagrangian submanifolds of BV phase space} over diffeomorphisms. The method of \[4, 3\] is to first construct a pseudodifferential form equivariant with respect to diffeomorphisms, and then obtain a base form using some connection.

This procedure can be also used to study the insertion of unintegrated vertex operators. Once we inserted unintegrated vertex operators, we should then integrate over the moduli space of Riemann surfaces with marked points. Let us first integrate, for each fixed complex structure on $\Sigma$, over the positions of the marked points, postponing the integration over complex structure...
for later. We interpret the result as the insertion of the \textbf{in}tegrated vertex operator. It is usually assumed that to any \textbf{un}integrated vertex operator $V$ corresponds some integrated vertex operator $U$. The naive formula is:

$$U = b_{-1} \bar{b}_{-1} V$$

(2)

However, this naive formula does not always work correctly. First of all, in the pure spinor formalism, $b$ is a rational function of the pure spinor fields. This, generally speaking, leads to $U$ being a rational function of the pure spinors, with non-constant denominators. It is not clear if such rational expressions should be allowed in the worldsheet action. We will leave this question open. Instead, we discuss another issue: Eq. (2) does not tell us the whole truth about the curvature coupling (the Fradkin-Tseytlin term in the worldsheet action). In this paper we will explain how to derive the Fradkin-Tseytlin term in the action starting from the insertion of the \textbf{un}integrated vertex operator $V$. We will construct, following the prescription of \cite{4, 3}, the integration measure for integrating over the point of insertion of $V$. We will show that the procedure of \cite{4, 3} simplifies. This is mostly due to the existence of a relatively straightforward construction of a connection on the space of Lagrangian submanifolds, as a principal bundle with the structure group diffeomorphisms. The curvature of this connection is essentially equal to the Riemann curvature of the worldsheet metric. The curvature term in the base form generates, effectively, the dilaton coupling (the Fradkin-Tseytlin term) on the string worldsheet. Under certain conditions, this reasoning leads (Section 2) to the formula for the deformation of the dilaton superfield:

$$(b_0 - \bar{b}_0)V = Q\Phi$$

(3)

In general there are two contributions to $\Phi$: one from Eq. (2) and another from Eq. (3).

**Eqs. (2) and (3) in the case of bosonic string** In the case of bosonic string (Section 4.2), the curvature coupling, generally speaking, comes from both Eq. (2) and Eq. (3). The contribution from Eq. (2) is due to the fact that already the \textbf{un}integrated vertex operator contains the curvature coupling: $c\bar{c}\sqrt{g} R\Phi$.

**Eqs. (2) and (3) in the case of pure spinor superstring** In Section 6 we discuss Eqs. (2) and (3) in the context of the pure spinor superstring.
on $AdS_5 \times S^5$. In this case, the only source of the curvature coupling is Eq. [3] — the second line of Eq. (36).

The $b$-ghost is a rational function of the pure spinor (not a polynomial). Therefore, the OPEs $b_{-1} \tilde{b}_{-1} V$ and $(b_0 - \tilde{b}_0) V$ are also non-polynomial. We explicitly evaluate $(b_0 - \tilde{b}_0) V$ in the particular case when $V$ is the beta-deformation vertex, using the $b_0$ and $\tilde{b}_0$ from [5] — see Section 6. At this time, we do not know any specific application of the formulas of Section 6. However, these computations inspired us to make some conjectures about the unintegrated vertex operators — see Sections 5.5 and 5.6.

One interesting feature of the beta-deformation is the existence of non-physical vertex operators [6, 7]. They normally cannot be put on a curved worldsheet, because of the anomaly. However, once we allow denominators of the form $\frac{1}{\text{STr}(\lambda_L \lambda_R)}$, it seems that there is no obstacle, and the nonphysical vertices can be included. This at least means, that the first few orders in the expansion in powers of $\varepsilon$ in Eq. (54) actually make sense in string perturbation theory.

## 2 General theory of vertex insertions

In this Section we will apply the prescription of [4, 3] for the vertex operators insertion.

### 2.1 Use of BV formalism and notations

In BV formalism, instead of integrating over the worldsheet complex structures, we integrate over general families of Lagrangian submanifolds $L$ in BV phase space. The space of all Lagrangian submanifolds is denoted $\text{LAG}$. In this paper, we will only consider a $6g - 6$-dimensional subspace of $\text{LAG}$, $\text{LAG}$ which corresponds to variations of the complex structure.

We use the **notations** of [4]. The odd Poisson bracket will be denoted $\{\cdot, \cdot\}_\text{BV}$, or just $\{\cdot, \cdot\}$. For a vector field $\xi$ on the BV phase space, generated $\{\cdot, \cdot\}$ by a BV Hamiltonian, we denote that Hamiltonian $\xi$:

$$\xi = \{\xi, \cdot\}_\text{BV}$$  \hspace{1cm} (4)
2.2 Use of worldsheet metric

Classically, the string worldsheet action depends on the worldsheet metric only through its complex structure. Quantum mechanically, the computation of the path integral usually involves the choice of the worldsheet metric (and not just complex structure), and then showing that in critical dimension the result of the computation is actually Weyl-invariant (i.e., only depends on the complex structure).

In this paper, we will need a worldsheet metric also for another purpose: to define a connection \(^2\) on the space of Lagrangian submanifolds as a principal bundle:

\[
\text{LAG} \rightarrow \frac{\text{LAG}}{\text{Diff}}
\]

which we need to convert an equivariant form into a base form. Suppose that we choose a metric for every complex structure. Then, we will explain in Section 2.5 this defines a choice of horizontal directions, i.e., a connection \(^2\) on \(\text{LAG} \rightarrow \frac{\text{LAG}}{\text{Diff}}\) — see Eqs. (32) and (33).

Given a complex structure, we will use the constant curvature metric of unit volume, which always exists and is unique by the uniformization theorem \([8]\). (But other global choices of a metric would also be OK.)

2.3 String measure

Equivariant Master Equation String worldsheet theory, in the approach of \([4, 3]\), comes with a PD\(^3\) \(\Omega^{\text{base}}\) on LAG, which is base with respect to \(H = \text{Diff}\). It is obtained from the equivariant half-density \(\rho^c\), which satisfies the equivariant Master Equation:

\[
\Delta_{\text{can}} \rho^c(\xi) = \xi \rho^c(\xi)
\]

where \(\xi \in \mathfrak{h} = \text{Lie}(H)\) is the equivariant parameter, and \(\xi\) the corresponding BV Hamiltonian.

Expansion in powers of \(\xi\) Let us write \(\rho^c(\xi)\) as a product:

\[
\rho^c(\xi) = e^{\alpha(\xi)} \rho_{1/2}
\]
where $\rho_{1/2}$ is a half-density satisfying the usual (not equivariant) Master Equation:

$$\rho_{1/2} = \exp(S_{\text{BV}}) \quad (S_{\text{BV}} \text{ is string worldsheet Master Action})$$

$$\Delta_{\text{can}} \rho_{1/2} = 0$$

and $a(\xi)$ is a function on the BV phase space, $a(0) = 0$. For any function $f$ and half-density $\rho_{1/2}$, let us denote:

$$\Delta_{\rho_{1/2}} f = \rho_{1/2}^{-1} \Delta_{\text{can}}(f \rho_{1/2}) - (-)^f f \rho_{1/2}^{-1} \Delta_{\text{can}} \rho_{1/2}$$

Eqs. (6) and (9) imply:

$$\Delta_{\rho_{1/2}} a(\xi) + \frac{1}{2} \{a(\xi), a(\xi)\}_{\text{BV}} = \xi$$

2.3.1 $a(\xi)$ for bosonic string and for pure spinor string

For bosonic string $a(\xi)$ is background-independent, linear in $\xi$, and given by a simple formula:

$$a(\xi) = a^{(1)}(\xi) = \int_{\Sigma} \xi^\alpha c^*_{\alpha}$$

For pure spinor string $a(\xi)$ is a complicated background-dependent expression. For background $AdS_5 \times S^5$, the $a^{(1)}(\xi)$ was constructed in [9], where it was called $\Phi_{\xi}$. Schematically:

$$a^{(1)}(\xi) = \int_{\Sigma} (\xi \cdot \partial Z^M) A_{M\alpha} \lambda^\alpha + (\xi \cdot \partial Z^M) B^N_M Z^*_N +$$

$$+ (\partial Z^M) C^\alpha_M L_{\xi} w_\alpha + D^{\alpha\beta} w_\alpha L_{\xi} w_\beta$$

where:

- $Z$ are coordinates on super-$AdS_5 \times S^5$
- $\lambda$ are pure spinors (both $\lambda_L$ and $\lambda_R$)
- $A_{M\alpha}, B^N_M, C^\alpha_M$ and $D^{\alpha\beta}$ are some functions of $Z$,
  and rational functions of pure spinors
2.3.2 Some assumptions

BV formalism is ill-defined in field-theoretic context, because $\Delta^{(0)}$ is ill-defined. We will assume that on local functionals $\Delta^{(0)} = 0$. In other words, when $f_{loc}$ is a local functional of the string worldsheet fields:

$$\Delta_{p_{1/2}} f_{loc} = \{ S_{BV}, f_{loc} \}$$

(15)

We believe that it is possible justify this assumption in worldsheet perturbation theory, but at this time our considerations are not rigorous.

2.4 Equivariant unintegrated vertex

Stabilizer of a point  Insertion of unintegrated vertex operator $V$ at a point on $p \in \Sigma$ leads to breaking of the diffeomorphisms down to the subgroup $\text{St}(p) \subset \text{Diff}$ which preserves $p$. Let $\text{st}(p)$ denote the Lie algebra of $\text{St}(p)$:

$$\text{St}(p) = \{ g \in \text{Diff} \mid g(p) = p \}$$

(16)

$$\text{st}(p) = \text{Lie}(\text{St}(p))$$

(17)

We will now explain how to construct an $\text{St}(p)$-equivariant form on LAG, and then in Sections 2.5 and 2.6 how to construct a base form.

Equivariantization of vertex  Given an unintegrated vertex $V$, suppose that we can construct for any $\xi_0 \in \text{st}(p)$ an equivariant vertex $V^c(\xi_0)$, satisfying

$$V^c(0) = V$$

(18)

$$\Delta_{\rho^c(\xi_0)} V^c(\xi_0) = 0$$

(19)

and:

$$\{ \xi_0, V^c(\eta_0) \}_{\text{BV}} = \frac{d}{dt} \bigg|_{t=0} V^c(e^{t[\xi_0, \cdot]} \eta_0)$$

(20)

Under the conditions of Eqs. (19) and (20) the product $V^c(\xi_0)\rho^c(\xi_0)$ defines an $\text{st}(p)$-equivariant half-density satisfying the $\text{st}(p)$-equivariant Master Equation:

$$(\Delta_{\text{can}} - \xi_0) \left( V^c(\xi_0)\rho^c(\xi_0) \right) = 0$$

(21)

footnote: the subindex $c$ stands for Cartan model of equivariant cohomology
Any solution $V^c(\xi_0)$ of Eq. (21) leads to st($p$)-equivariant pseudo-differential form:

$$\Omega^c(L, dL, \xi_0) = \int_{gL_0} \exp (\sigma \langle dL \rangle) \ V^c(\xi_0) \rho^c(\xi_0)$$  (22)

Here $\sigma \langle dL \rangle$ is any BV Hamiltonian generating the infinitesimal deformation $dL$ of $L$.

We can think of $V^c(\xi_0) \rho^c(\xi_0)$ as correction of the first order in $\epsilon$ to $\rho^c(\xi_0)$ under the deformation:

$$\rho \exp (a(\xi_0)) \rightarrow \rho \exp (a(\xi_0) + \epsilon V^c(\xi_0))$$  (23)

Eqs. (19) and (20) imply:

$$\left( \Delta_{\rho_{1/2}} + \{a(\xi_0), \cdot \}_{BV} \right) V^c(\xi_0) = 0$$  (24)

$$\{\xi_0, V^c(\eta_0)\}_{BV} = \left. \frac{d}{dt} \right|_{t=0} V^c(e^{t[\xi_0, \cdot]} \eta_0)$$  (25)

The exact deformations, of the form:

$$V^c_{\text{exact}}(\xi_0) = \left( \Delta_{\rho_{1/2}} + \{a(\xi_0), \cdot \}_{BV} \right) v^c(\xi_0)$$  (26)

with $v^c$ satisfying the equivariance condition $\{\xi_0, v^c(\eta_0)\}_{BV} = \left. \frac{d}{dt} \right|_{t=0} v^c(e^{t[\xi_0, \cdot]} \eta_0)$ are considered trivial.

Consider the expansion of $V^c(\xi_0)$ in powers of $\xi_0$:

$$V^c(\xi_0) = V^{(0)} + V^{(1)} \langle \xi_0 \rangle + V^{(2)} \langle \xi_0 \otimes \xi_0 \rangle + \ldots$$  (27)

(We use angular brackets $\langle \ldots \rangle$ to highlight linearity, i.e. $f \langle x \rangle$ instead of $f(x)$ when $f$ is a linear functions of $x$.) In particular, Eq. (24) implies at the linear order in $\xi$:

$$\Delta_{\rho_{1/2}} V^{(1)} \langle \xi_0 \rangle + \{a^{(1)} \langle \xi_0 \rangle, V^{(0)} \}_{BV} = 0$$  (28)

Equivariant vertex operators form a representation of the $Dg$ algebra discussed in [10], the differential $d$ of [10] being represented by $\Delta_{\rho_{1/2}}$.

For our purpose, we will use a slightly different form of Eq. (28). Let us return to Eq. (21). At the linear order in $\xi_0$ it becomes:

$$\Delta_{\rho_{1/2}} \left( a^{(1)} \langle \xi_0 \rangle V^{(0)} + V^{(1)} \langle \xi_0 \rangle \right) = \xi_0 V^{(0)}$$  (29)
An exact $V$ corresponds to (see Eq. (26)):

\[
V^{(0)}_{\text{exact}} = \Delta_{\rho_{1/2}} v^{(0)}
\]
\[
V^{(1)}_{\text{exact}} \langle \xi_0 \rangle = \Delta_{\rho_{1/2}} (a^{(1)} \langle \xi_0 \rangle v^{(0)} + v^{(1)} \langle \xi_0 \rangle) - \xi^{(0)} v^{(0)}
\]

Eq. (29) is an equivalent form of Eq. (28). We will explain in Section 4 that in case of bosonic string it is more convenient to use Eq. (28). But in case of pure spinor string we use Eq. (29).

2.5 A connection on $\Lambda \to \Lambda/\text{St}(p)$

In order to integrate, we need to pass from equivariant $\Omega^c$ to base $\Omega^{\text{base}}$. This requires a choice of a connection in the principal $\text{St}(p)$-bundle $\text{LAG} \to \text{LAG}/\text{St}(p)$. We will now define the connection by specifying the distribution $\mathcal{H}_0 \subset \Omega^1|S$ of horizontal vectors. We say that the vector belongs to $\mathcal{H}_0$, if it is a linear combination of vectors of the following two classes:

- The **first class** consists of the variations of the metric satisfying:

\[
h^{\alpha\beta} \delta h_{\alpha\beta} = 0
\]
\[
\nabla^{\alpha} \delta h_{\alpha\beta} = 0
\]

Such $\delta h_{\alpha\beta}$ can be identified as holomorphic or antiholomorphic quadratic differentials.

- The **second class** by definition consists of infinitesimal isometric ("rigid") translations of the disk $D_{\epsilon}$ of the small radius $\epsilon$. These are delta-function-like variations of the metric with the support on $\partial D_{\epsilon}$. They are always trivial in $\text{LAG}/\text{Diff}$, but nontrivial in $\text{LAG}/\text{St}(p)$ when genus is greater than one.

(This definition only works for the metric of constant negative curvature, because for generic metric $D_{\epsilon}$ does not have any infinitesimal isometries. In such cases, we can choose some lift to a vector field $v$ which is approximately isometry, in the sense that $L_v g_{\alpha\beta} = O(|z|^2)$. Formulas do not change.)
2.6 Base form and its integration

Given a connection, we can construct a base form out of the equivariant form of Eq. (22); it is given by the following expression [4, 3]:

$$\Omega^{\text{base}}(L, dL) = \int_L \exp(\sigma\langle dL|_{\text{hor}} \rangle) V^c(F) \rho^c(F)$$

(34)

where $V^c$ must satisfy Eqs. (19) and (20), and $F$ is the curvature of our connection. Here, as in Eq. (22), $\sigma\langle dL|_{\text{hor}} \rangle$ is any BV Hamiltonian generating the infinitesimal deformation, but we have to “project” the variation $dL$ to the horizontal subspace (using our connection).

Let us consider the fiber bundle:

$$\begin{array}{ccc}
\text{MET} & \pi & \text{MET} \\
St(p) & \rightarrow & \text{Diff}
\end{array}$$

(35)

We want to integrate $\Omega$ over the cycle of the form $\pi^{-1}c_{6g-6}$ where $c_{6g-6}$ is the fundamental cycle of the moduli space of Riemann surfaces. Let us first integrate over the fiber (which is $\Sigma$). Our connection, described in Section 2.5, lifts the tangent vectors to the fiber as horizontal vectors of the second class, i.e. as infinitesimal rigid translations of $D_\epsilon$. The curvature of our connection, evaluated on a pair of vectors tangent to the fiber, takes values in infinitesimal rigid rotations of $D_\epsilon$ and equals to the curvature of $\Sigma$. Therefore $\Omega^{\text{base}}$ is:

$$\Omega^{\text{base}} = \int e^S \left[ V^{(0)} \sigma \langle dL_{\text{hor}} \rangle \wedge \sigma \langle dL_{\text{hor}} \rangle + V^{(0)} a^{(1)} \langle R \rangle + V^{(1)} \langle R \rangle \right]$$

(36)

We will now explain this equation, first line first, and then the second.

2.6.1 First line of Eq. (36)

With our definition of the connection in Section 2.5, the horizontal projection $dL|_{\text{hor}}$ is an infinitesimal diffeomorphism: an infinitesimal translation of the disk $D_\epsilon$ by $\left[ \begin{array}{c} dz \\
\frac{d\bar{z}}{d\bar{z}} \end{array} \right]$. Therefore, the corresponding BV Hamiltonian $\sigma\langle dL|_{\text{hor}} \rangle$ is actually $\Delta$-exact. Indeed, Eq. (11) implies that:

$$\sigma\langle dL|_{\text{hor}} \rangle = \Delta_{\rho_{1/2}} a^{(1)} \langle u(dz, d\bar{z}) \rangle$$

(37)
Here $u(dz, d\bar{z})$ is the vector field on $\Sigma$ which is:
- at the center of $D_\epsilon$ equals to $\begin{bmatrix} dz \\ d\bar{z} \end{bmatrix}$
- inside $D_\epsilon$ is an infinitesimal rigid translation
- outside of $D_\epsilon$ is zero

Since $a^{(1)}$ is a local functional on the string worldsheet, Eqs. (37) and (15) imply:
\[ \sigma \langle dL \rangle_{\text{hor}} = \{ S_{BV}, a^{(1)} \langle u(dz, d\bar{z}) \rangle \} \]  \hspace{1cm} (38)

**Lemma-definition 1:** For any vector field $v$, the restriction of $\{ S_{BV}, a^{(1)} \langle v \rangle \}$ on $L$ is $\int b^{\alpha \beta} \nabla_\alpha v_\beta$:
\[ \{ S_{BV}, a^{(1)} \langle v \rangle \} \big|_L = \int b^{\alpha \beta} \nabla_\alpha v_\beta \]  \hspace{1cm} (39)

We take Eq. (39) as the definition of $b^{\alpha \beta}$ (which is otherwise defined only up to a $Q$-closed expression).

**Proof** Let us consider the expansion of $S_{BV}$ and the expansion of $\mathcal{V} = \{ S_{BV}, a^{(1)} \}$:
\[ S_{BV} = S_0 + Q^A \phi_A^* + \ldots \]  \hspace{1cm} (40)
\[ \{ S_{BV}, a^{(1)} \} = \mathcal{V}_0 + \mathcal{V}_1^A \phi_A^* + \ldots \]  \hspace{1cm} (41)

From $\{ S_{BV}, \{ S_{BV}, a^{(1)} \} \} = 0$ we derive:
\[ \mathcal{L}_Q \mathcal{V}_0 = \mathcal{L}_{\mathcal{V}_1} S_0 \]  \hspace{1cm} (42)

Eq. (39) follows from the variation of $S_0$ under infinitesimal diffeomorphism being equal to $\int T^{\alpha \beta} \nabla_\alpha v_\beta$, and from the vanishing of the off-shell cohomology in ghost number $-1$ (we are working off-shell!).

Returning to Eq. (38), Since $u$ is an isometry inside $D_\epsilon$ and zero outside $D_\epsilon$, we have:
\[ \{ S_{BV}, a^{(1)} \langle u \rangle \} \big|_L = \int_\Sigma \sqrt{g} b^{\alpha \beta} \nabla_\alpha u_\beta = \oint_{\partial D_\epsilon} dz^\alpha b_{\alpha \beta} u^\beta \]  \hspace{1cm} (43)

Therefore the first line in Eq. (36) contributes:
\[ b_{-1} \bar{b}_{-1} V^{(0)} \]  \hspace{1cm} (44)
2.6.2 Second line of Eq. (36)

Expressions like $a^{(1)}\langle R \rangle$ and $V^{(1)}\langle R \rangle$ should be understood in the following way. We think of the curvature $R$ as a two-form on the worldsheet with values in rotations of the tangent space:

$$R \in \Gamma \left( \Omega^2 \Sigma \otimes \text{so}(T\Sigma) \right)$$

In particular, if $\xi \in T_p \Sigma$ and $\eta \in T_p \Sigma$ are two tangent vectors, then $R(\xi, \eta)$ at the point $p$ is an infinitesimal rotations of $T_p \Sigma$. This infinitesimal rotation can be represented by a vector field $v$ with zero at the point $p$. Let us “truncate” $v$ by putting it to zero outside $D_\epsilon$, i.e. multiply $v$ by the function $\chi_{D_\epsilon}$ which is 1 inside $D_\epsilon$ and 0 outside. By definition:

$$a^{(1)}\langle R(\xi, \eta) \rangle \overset{\text{def}}{=} a^{(1)}\langle \chi_{D_\epsilon} v \rangle$$

$$V^{(1)}\langle R(\xi, \eta) \rangle \overset{\text{def}}{=} V^{(1)}\langle \chi_{D_\epsilon} v \rangle$$

(This is an abbreviation, rather than a definition.) In this context, Eq. (29) becomes:

$$\Delta_{\rho_1/2} \left( V^{(0)} a^{(1)}\langle R \rangle + V^{(1)}\langle R \rangle \right) = \{ S_{BV}, a^{(1)}\langle R \rangle \} V^{(0)}$$

In the case of pure spinor string $\{ a^{(1)}\langle R \rangle, V^{(0)} \} = 0$, because $v$ in Eq. (46) is a vector field vanishing at the point of insertion of $V^{(0)}$, and $V^{(0)}$ does not contain derivatives. Therefore, the left hand side of Eq. (47) is $\{ S_{BV}, a^{(1)}\langle R \rangle V^{(0)} + V^{(1)}\langle R \rangle \}$. When restricted to the Lagrangian submanifold, up to equations of motion\footnote{In spite of the fact that $\chi_{D_\epsilon} v$ of Eq. (46) is zero at the point of insertion of $V^{(0)}$, we cannot claim that $a^{(1)}\langle R \rangle |_L V^{(0)} |_L$ is zero. This is because of the singularities in the OPE of the integrand of $a^{(1)}_L$ and $V^{(0)}$.}

$$Q \left( a^{(1)}\langle R \rangle |_L V^{(0)} |_L + V^{(1)}\langle R \rangle |_L \right) = \{ S_{BV}, a^{(1)}\langle R \rangle \} |_L V^{(0)} |_L$$

We must stress that this equation is only valid under assumption $\{ a^{(1)}\langle R \rangle, V^{(0)} \} = 0$. Generally speaking, instead of Eq. (48):

$$Q \left( a^{(1)}\langle R \rangle |_L V^{(0)} |_L + V^{(1)}\langle R \rangle |_L \right) = \{ S_{BV}, a^{(1)}\langle R \rangle \} |_L V^{(0)} |_L - \{ a^{(1)}\langle R \rangle, V^{(0)} \} |_L$$

5
The computation of \( \{ S_{BV}, a^{(1)}(R) \}_L \) uses Eq. (43):

\[
\{ S_{BV}, a^{(1)}(R) \}_L V^{(0)}|_L = (b_0 - \bar{b}_0)V^{(0)}
\]  

(50)

Therefore:

\[
a^{(1)}(R)|_L V^{(0)}|_L + V^{(1)}(R)|_L = \sqrt{g}R\Phi
\]  

(51)

where \( \Phi \) satisfies:

\[
Q\Phi = (b_0 - \bar{b}_0)V^{(0)}
\]  

(52)

To summarize, the total integrated vertex insertion corresponding to the unintegrated vertex \( V^{(0)} \) is given by the expression:

\[
\int \Sigma d^2z \left( b_{-1}\bar{b}_{-1}V^{(0)} + \sqrt{g}R\Phi \right)
\]  

where \( \Phi \) satisfies: \( Q\Phi = (b_0 - \bar{b}_0)V^{(0)} \)

3 Brief review of the conventional description of the curvature coupling

Here we will briefly review the “standard” derivation of the curvature coupling.

Consider the deformation of the worldsheet action by adding the integrated vertex operator:

\[
S \mapsto S + \epsilon \int U
\]  

(54)

where \( \epsilon \) is a small “deformation parameter”. Suppose that the deformed action is classically BRST invariant. At the one loop level, we get:

\[
\partial^\mu j_\mu^{\text{BRST}} = \alpha'(X + \sqrt{g}RY)
\]  

(55)

where \( X \) is a BRST-closed operator of conformal dimension \((1,1)\) and ghost number one, and \( Y \) is a BRST-closed expression of conformal dimension zero and ghost number one\(^6\). In generic curved target-spaces, there is no BRST

\(^6\) Notice that there is no \( \sqrt{g}RV \) term in Eq. (54), because there are no BRST-closed scalar operators \( V \) of ghost number zero, other than 1 (the 1 corresponding to the change in string coupling).
cohomology at ghost number 1 and conformal dimension zero. Therefore, exists $\Phi$ such that:

$$Y = -Q_{\text{BRST}}\Phi$$  \hspace{1cm} (56)$$

Also, there is no cohomology in conformal dimension $(1,1)$ and ghost number 1, therefore exists $U'$ such that $X = -QU'$. These $U'$ and $\Phi$ can be absorbed into $U$:

$$U \mapsto U + \alpha' U' + \alpha' \sqrt{g} R \Phi$$  \hspace{1cm} (57)$$

and the term $\Phi$ is the deformation of the dilaton.

## 4 Bosonic string vs pure spinor string

### 4.1 Main differences

In **pure spinor** string theory on $AdS_5 \times S^5$:

- simplification: $V^{(0)}$ does not contain derivatives

- complication: restriction of $a(\xi)$ on “standard” family of Lagrangian submanifolds is nonzero

In this case we need compute $(a^{(1)}V^{(0)} + V^{(1)})|_L$ (this is what deforms the equivariant density), and we get it from Eq. (48)

In **bosonic** string theory:

- complication: $V^{(0)}$ contains at least derivatives of matter fields, and sometimes derivatives of ghosts

- simplification: $a(\xi)$ is given by a simple formula: $a(\xi) = \xi^\alpha c^*_\alpha$, and in particular its restriction to the standard Lagrangian submanifold is zero

In this situation we compute $V^{(1)}$ from Eq. (24):

$$\{S_{\text{BV}}, V^{(1)}\} = \{-\xi^\alpha c^*_\alpha, V^{(0)}\} = -\xi^\alpha \frac{\partial}{\partial c^\alpha} V^{(0)}$$  \hspace{1cm} (58)$$

The exact vertex has:

$V^{(0)}_{\text{exact}} = \{S_{\text{BV}}, v^{(0)}\}$  \hspace{1cm} (59)$$

$V^{(1)}_{\text{exact}}(\xi) = \{S_{\text{BV}}, v^{(1)}(\xi)\} + \xi^\alpha \frac{\partial}{\partial c^\alpha} v^{(0)}$  \hspace{1cm} (60)$$

\ldots  \hspace{1cm} (61)$$
4.2 Bosonic string vertices as functions on BV phase space

Consider bosonic string on a general curved worldsheet. We work in BV formalism, our vertex operators are functions on the BV phase space of bosonic string worldsheet.

Let us start by considering the vertex corresponding to a “gravitational wave”, i.e. an infinitesimal deformation of the target space metric $G_{\mu\nu}$. We assume that $G_{\mu\nu}$ satisfies transversality and linearized Einstein equations:

$$\partial^\mu G_{\mu\nu} = 0$$
$$\Box G_{\mu\nu} = 0$$

(almost all gravitational waves can be obtained like this, except for some zero modes). Let $h_{\alpha\beta}$ be the worldsheet metric, and $I^\alpha_\beta$ the corresponding complex structure. We claim that the following vertex operator:

$$V^{(0)} = (I^c \cdot \partial X^\mu)(c \cdot \partial X^\mu)G_{\mu\nu}(x)$$

satisfies:

$$\{S_{BV}, V^{(0)}\} = 0$$

Let us prove this. The odd Poisson brackets with BV Master Action are:

$$\{S_{BV}, X\} = \mathcal{L}_c X$$
$$\{S_{BV}, c\} = \frac{1}{2}[c, c]$$
$$\{S_{BV}, I\} = \mathcal{L}_c I$$

(Here $\mathcal{L}_c X$ is the same as $c \cdot \partial X$ — the Lie derivative of $X$.)

$$\{S_{BV}, (I^c \cdot \partial X)(c \cdot \partial X)\} =$$

$$= (([\mathcal{L}_c, \mathcal{L}_{I^c}] - \mathcal{L}_{I^{[c, c]}})X) \mathcal{L}_c X - (\mathcal{L}_{I^c} \mathcal{L}_c X) \mathcal{L}_c X + \frac{1}{2}(\mathcal{L}_{I^{[c, c]}}) \mathcal{L}_c X =$$

$$= (\mathcal{L}_c \mathcal{L}_{I^c} X) \mathcal{L}_c X - \frac{1}{2}(\mathcal{L}_{I^{[c, c]}}) \mathcal{L}_c X$$

Eq. (69) follows from:

$$\mathcal{L}_c \mathcal{L}_{I^c} X - \frac{1}{2} \mathcal{L}_{I^{[c, c]}} X = \frac{1}{2} i^2_c d * dX = \frac{1}{2} \{S_{BV}, i^2_c X^*\}$$
In Eq. (70) we identify $X^*$ as a 2-form on the worldsheet, and contract it two times with $c$. This operation can be characterized by saying that for every local (i.e. given by a single integral over the worldsheet $\Sigma$) functional $F[X]$: \[ \{ t_\eta X^*, F[X] \} = t_\eta \frac{\delta F}{\delta X} \] (71)

To prove Eq. (70), let us choose the coordinates $(z, \bar{z})$ where the complex structure is: $I \frac{\partial}{\partial z} = i \frac{\partial}{\partial z}$. We denote $C = c^z$ and $\bar{C} = c^{\bar{z}}$, i.e. $c \cdot \partial = C \partial + \bar{C} \bar{\partial}$ (with a slight abuse of notations, we let $\partial$ denote also $\partial_z$). With these notations:

\[ (C \partial + \bar{C} \bar{\partial})(iC \partial - i\bar{C} \bar{\partial})X - I(C \partial + \bar{C} \bar{\partial})^2 X = 2i\bar{C}C \partial \bar{\partial}X \] (72)

In order to actually insert $V^{(0)}$ we have to regularize it. (Even when Eqs. (62) and (63) are satisfied, we have the product of two $\partial X$ at the same point, which does not make sense without regularization.)

**Regularization** We regularize $V^{(0)}$ by replacing every $X^\mu$ (including those acted on by $\partial$) with the averaged value:

\[ X(0, 0) \mapsto N_\epsilon \int d^2z \sqrt{g} \exp \left( -\frac{1}{\epsilon} \text{dist}^2((z, \bar{z}), (0, 0)) \right) X(z, \bar{z}) \] (73)

where dist is the distance measured by the worldsheet metric, $\epsilon \to 0$ the regularization parameter, and $N_\epsilon$ is the normalization factor:

\[ N_\epsilon = \left[ \int d^2z \sqrt{g} \exp \left( -\frac{1}{\epsilon} \text{dist}^2((z, \bar{z}), (0, 0)) \right) \right]^{-1} \] (74)

When $c$ gets contracted with $\partial x$, we take the average of $c^\alpha \partial_\alpha x$.

**Renormalization** After specifying the regularization prescription, we have to subtract infinities. Actually, with Eqs. (62) and (63) the subtraction is not even needed, because the regularized $V^{(0)}$ remains finite when $\epsilon \to 0$.

But suppose that (having in mind extensions to string field theory) we want to define our vertex in a way which requires smooth extension off-shell, i.e.
relaxing of Eqs (62) and (63). Then, for our expression to remain finite off-shell, we have to do a regularization. We define the subtraction as follows:

$$ O_{\text{ren}} = \exp \left( - \int d^2 z \int d^2 w \frac{\alpha'}{2} \ln \text{dist}^2(z, \bar{z}; w, \bar{w}) \frac{\delta}{\delta X^\mu(z, \bar{z})} \frac{\delta}{\delta X_\mu(w, \bar{w})} \right) O $$

— this removes the short distance singularity in $\langle X(z, \bar{z})X(w, \bar{w}) \rangle$. Although this subtraction is diffeomorphism invariant, it is not Weyl invariant, and therefore it does not commute with $\{S_{BV}, \cdot \}$. The actual effect of the subtraction is:

$$ \lim_{x \to y} \left( c^\alpha(x)c^\beta(y) \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \log \text{dist}^2(x, y) \right) = \frac{\alpha'}{3} (c, Ic) R(x) $$

This implies, that the unintegrated vertex annihilated by $\{S_{BV}, \cdot \}$ is:

$$ (\mathcal{L}_c x^\mu \mathcal{L}_I c x^\nu G_{\mu \nu}(x))_{\text{ren}} + \frac{\alpha'}{3} (c, Ic) \Phi_{\text{ren}} R $$

where $\Phi = G_{\mu \nu}^\mu$ (78)

Therefore the curvature coupling arises from Eq. (2), as $b_1 \bar{b}_1 ((c, Ic) R \Phi) = R \Phi$. (And this source of curvature coupling is not present in the pure spinor case.)

If we do not impose the condition (62), then Eq. (64) requires modification. Additional terms should be added, such as $\text{e.g.} \text{ div } c (\mathcal{L}_c x^\mu) A_\mu(x)$. With these extra terms, Eq. (3) also contributes to the curvature coupling.

4.3 Ghost number one

Cohomology at ghost number one is (cp Eq. (70)):

$$ W^\mu = \mathcal{L}_{Ic} X^\mu - \frac{1}{2} t_c^2 X^{\mu *} $$

$$ W^{\mu \nu} = X^{[\mu} \mathcal{L}_{Ic} X^{\nu]} - \frac{1}{2} X^{[\mu} t_c^2 X^{\nu]*} $$

They are both already equivariant, because $\{a(\xi), W\} = 0$, since $W$ does not contain derivatives of $c$. Notice that:

$$ dW^\mu = \{S_{BV}, U^\mu\} $$

where $U^\mu = *dX^\mu - t_c X^{\mu *}$ (82)
The proof of Eq. (81) uses:
\[ d\mathcal{L}_{Ic}X^\mu - \mathcal{L}_c * dX^\mu = \{S_{\text{BV}}, \iota_c X^\mu\} \]  
(83)

As a consistency check, it should be true, at least in restriction to a reasonable Lagrangian submanifold, that:
\[ \iota_\xi U^\mu = \left( \int_D \{ S_{\text{BV}}, a^{(1)}(\xi) \} \right) W^\mu \]  
(84)

where
\[ \{ S_{\text{BV}}, a^{(1)}(\xi) \} = (\mathcal{L}_\xi X^\mu)X^*_\mu + [\xi, c]^* + (\mathcal{L}_\xi g_{\alpha\beta})b^{\alpha\beta} \]  
(85)

This is true on the standard Lagrangian submanifold, i.e. \( c^* = 0, X^* = 0 \). We did not explicitly check this for other Lagrangian submanifolds.

### 4.4 Dilaton zero mode

**Ghost dilaton** Let us lift the expression \( \partial c - \bar{\partial}\bar{c} \) of [11] to the BV phase space as \( v = \text{div}(Ic) \). The Cartan differential of \( v \) is (see Eqs. (30) and (31)):
\[ V^{(0)} = \{ S_{\text{BV}}, v \} = \mathcal{L}_c(\text{div}(Ic)) - \frac{1}{2}\text{div}(I[c, c]) \]  
(86)
\[ V^{(1)}(\xi_0) = \{ a^{(1)}(\xi_0), v \} = \text{div}(I\xi_0) \]  
(87)
\[ V^{(\geq 2)}(\ldots) = 0 \]

The restriction of \( V^{(0)} \) on the standard family is, on-shell, \( c\partial^2 c - \bar{c}\partial^2\bar{c} \).

The base form corresponding to \( V^{(1)} \) by the procedure of Section 2.6 is \( \sqrt{g}R \). Therefore, we should interpret \( V^{(1)} \) as the unintegrated vertex operator corresponding to the dilaton zero mode. However, \( V^{(1)} \) by itself is not \( \{S_{\text{BV}}, -\} \)-closed:
\[ \{ S_{\text{BV}}, V^{(1)} \} = \text{tr}\left( I [\mathcal{L}_c I, \mathcal{L}_{\xi_0} I] \right) \neq 0 \]  
(88)

**(commutator as matrices in \( T_p\Sigma \))**

(89)

**What is going on?** The construction of the base form consists of the substitution of the curvature 2-form in place of \( \xi_0 \). The way we construct connection in Section 2.5 it actually takes values in a smaller subalgebra
st(p, Ip) ⊂ st(p), which consists of those vector fields which preserve the complex structure in the tangent space to the point p of insertion, i.e. Ip ∈ gl(TpΣ). We observe that:

\[ \xi_0 ∈ st(p, Ip) ⊂ st(p) \Rightarrow \{ S_{BV}, V^{(1)}(\xi_0) \} = 0 \]  

(90)

(We must stress that, since I is one of the BV fields, st(p, Ip) varies from point to point in the BV phase space.)

**Equivalence of V(0) and V(1)** Eqs. (86) and (87) imply that the integrated vertex obtained from \( V^{(1)} \) should be same as the one obtained from \( V^{(0)} \). We can check this explicitly:

\[
\left( \oint dz^\alpha b_{\alpha\beta} \xi^\beta \right) \left( \oint dz^\alpha b_{\alpha\beta} \eta^\beta \right) V^{(0)} = \left( \mathcal{L}_\xi \text{div}(I\eta) - \frac{1}{2} \text{div}(I[\xi, \eta]) \right) - (\xi \leftrightarrow \eta) = \text{div}(I[\xi, \eta]) = R(\xi, \eta)
\]

(91)

(92)

(93)

We used the fact that, by the prescription of Section 2.5, \( \xi \) and \( \eta \) are lifted as isometries of a small neighborhood of the insertion point; in particular, the Lie derivative \( \mathcal{L}_\xi \) commutes with the operations \( I \) and div.

### 4.5 Semirelative cohomology

In our paper we identify the space of states as the cohomology of the equivariant complex, as defined in Section 2.4.

The usual definition is via the semirelative complex \( \Pi \). In the case of bosonic string, the cohomology is the same. Indeed, imposing the semirelative condition \((b_0 - \bar{b}_0)V = 0\) leads to two effects:

**Effect 1** There are ghost number 2 cocycles, which should be thrown away because they are not annihilated by \( b_0 - \bar{b}_0 \).

Those are non-physical beta-deformations. \( \] 

**Effect 2** The ghost-dilaton is \( Q(\partial C - \bar{\partial} \bar{C}) \) — would be BRST exact in the naive BRST complex, but \( Q(\partial C - \bar{\partial} \bar{C}) \) is not annihilated by \( b_0 - \bar{b}_0 \). Therefore, the ghost-dilaton is actually nontrivial.

The equivariant complex gives the same result. For \( V \) a nonphysical beta-deformation (Effect 1), \( \{ a(\xi), V \} \) is not just nonzero, but actually not even
{S_{BV}, \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot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It should satisfy:
\[ \rho(\lambda_L + \lambda_R)v = 0 \]  
(97)
where \( \rho(\lambda_L + \lambda_R) \) is the action of the element \( \lambda_L^\alpha \epsilon_\alpha + \lambda_R^\alpha \bar{\epsilon}_\alpha \in \mathfrak{g} \) in \( \mathcal{H}' \). In this sense, the pure spinor BRST operator acts on \( \mathcal{H}' \):
\[ Q = \rho(\lambda_3 + \lambda_1) \]  
(98)
In this Section we will study the case when \( \mathcal{H} \) is finite-dimensional. Then \( \mathcal{H}' = \mathcal{H} \). We will consider those \( \mathcal{H} \) which can be constructed products of adjoint representations of \( \mathfrak{g} \), the simplest example being the beta-deformation \( (\mathfrak{g} \wedge \mathfrak{g})_0 \). Such spaces are naturally related to the cochain complex of \( \mathfrak{g} \), which we will now discuss.

5.2 Lie algebra cohomology complex

Let us consider the Lie algebra cohomology complex of \( \mathfrak{g} = \mathfrak{psu}(2, 2|4) \) with coefficients in a trivial representation. As a linear space, it is the direct sum \( \bigoplus_{i=0}^{\infty} \Lambda^n \mathfrak{g}' \), where \( \mathfrak{g}' \) is the dual space of \( \mathfrak{g} \). We use the fact that \( \mathfrak{g} \) has a supertrace, and identify \( \mathfrak{g}' \) with \( \mathfrak{g} \). The supertrace induces the pairing
\[ \Lambda^n \mathfrak{g} \otimes \Lambda^n \mathfrak{g} \longrightarrow \mathbb{C} \]  
(99)
For example:
\[ \langle x \wedge y, z \wedge w \rangle = \]
\[ = \text{STr}(yz)\text{STr}(xw) - (-1)^{\bar{y}\bar{z}}\text{STr}([x,z])\text{STr}([y,w]) \]  
(100)
(101)
The Lie superalgebra cohomology differential \( d_{\text{Lie}} \) acts as follows:
\[ d_{\text{Lie}} : \Lambda^n \mathfrak{g} \rightarrow \Lambda^{n+1} \mathfrak{g} \]  
(102)
\[ \langle d_{\text{Lie}} x, y \wedge w \rangle \overset{\text{def}}{=} d_{\text{Lie}} \langle x, [y, w] \rangle = \text{STr}(x[y, w]) \]  
(103)

5.3 Vertex operators corresponding to global symmetries

The following element:
\[ \lambda_3 - \lambda_1 \in C^1 \mathfrak{g} = \mathfrak{g} \]  
(104)
is a nontrivial cocycle of \( Q \). It corresponds to the unintegrated vertex operator:
\[ V_a^{(0)} = \text{STr}(t_a g^{-1} (\lambda_3 - \lambda_1)g) \]  
(105)
5.4 Interplay between Lie algebra cohomology and pure spinor cohomology

The $Q$-cocycle $\lambda_3 - \lambda_1$ is not a $Q$-coboundary. However the Lie algebra differential applied to it is a coboundary, if we allow denominator $\frac{1}{\text{STr}(\lambda_3 \lambda_1)}$:

$$d_{\text{Lie}}(\lambda_3 - \lambda_1) = Q \left( k^{\alpha \dot{\alpha} t^3_\alpha} \wedge (1 - 2P_{13}) t^1_{\dot{\alpha}} \right)$$

(106)

The internal commutator of $k^{\alpha \dot{\alpha} t^3_\alpha} \wedge (1 - 2P_{13}) t^1_{\dot{\alpha}}$ is nonzero, but is $Q$-exact:

$$k^{\alpha \dot{\alpha} (t^3_\alpha, (1 - 2P_{13}) t^1_{\dot{\alpha}}) = \frac{3}{2} \{\lambda_3, \lambda_1\} = \frac{3}{4} Q(\lambda_3 + \lambda_1)$$

(107)

5.5 Beta-deformation and its generalizations

5.5.1 Definition

The definition of the unintegrated vertex for beta-deformation given in [15, 6] is:

$$V = B^{ab} W_a W_b$$

(108)

where

$$W_a = \text{STr} \left( t_a g^{-1} (\lambda_3 - \lambda_1) g \right)$$

(109)

where $B^{ab}$ is a constant antisymmetric tensor, defined up to the equivalence relation:

$$B^{ab} \sim B^{ab} + f^{abc} A_c$$

(110)

The beta-deformation transforms in the following the following representation of $\text{psu}(2,2|4)$:

$$\frac{(g \wedge g)_0}{g}$$

(111)

where the factor over $g$ accounts for the equivalence relation defined by the Eq. [110].

This vertex operator defined in Eq. (108) is not strictly speaking covariant, for the following reason. When we change $B^{ab}$ to $B^{ab} + f^{abc} A_c$, it changes by a BRST exact expression:

$$V \rightarrow V + QW$$

(112)

where

$$W = \text{STr} \left( A g^{-1} (\lambda_3 + \lambda_1) g \right)$$

(113)
It is possible to define the vertex which is strictly covariant:

\[ V' = V - \langle B, g^{-1} ([\Sigma, \lambda_3 + \lambda_1] \wedge [\Sigma, \lambda_3 + \lambda_1]) g \rangle \]  

(114)

where

\[ \Sigma = \text{diag}(1, 1, 1, -1, -1, -1, -1, -1) \]  

(115)

The difference between \( V \) and \( V' \) is a BRST-exact expression:

\[ \langle B, g^{-1} ([\Sigma, \lambda_3 + \lambda_1] \wedge [\Sigma, \lambda_3 + \lambda_1]) g \rangle = QX \]  

(116)

where

\[ X = - \langle B, g^{-1} (\Sigma \wedge [\Sigma, \lambda_3 + \lambda_1]) g \rangle \]  

(117)

The definition of \( X \) requires some work, because \( \Sigma \) is not an element of \( g = \text{psu}(2, 2|4) \), because \( \text{STr} \Sigma \neq 0 \). Therefore, in order to define \( X \), we need to lift \( B \) from \( g \wedge g \) to \( \text{su}(2, 2|4) \wedge \text{su}(2, 2|4) \). There is no way to do it while preserving the \( \text{psu}(2, 2|4) \)-invariance. Therefore, \( X \) does not transform as Eq. (111). Still, Eq. (116) holds, thus \( V' \) is BRST-equivalent to \( V \).

### 5.5.2 Alternative definition

When \( B \) satisfies the “physicality” condition \( B^{ab} f_{abc} = 0 \), we can use the alternative vertex:

\[ \tilde{V} = \text{STr}(\lambda_3 \lambda_1) B^{ab} \left( t_a \wedge t_b, g^{-1} (k^{\alpha \dot{\alpha}} t_{\alpha} \wedge P_{13} t_{\dot{\alpha}}) g \right) \]  

(118)

This alternative beta-deformation vertex is “homogeneous”, in the sense that it has a definite ghost number \((1, 1)\). It is linear in \( \lambda_3 \) and in \( \lambda_1 \), because the pre-factor \( \text{STr}(\lambda_3 \lambda_1) \) cancels the denominator in \( P_{13} \).

**Conjecture**  The vertex operator \( \tilde{V} \) defined by Eq. (118) is not BRST-exact. If this is the case, then \( \tilde{V} \) is proportional to the beta-deformation vertex of Eq. (108). We leave the proof of this conjecture, and the computation of the proportionality coefficient, for future work.

### 5.6 Conjectures about higher finite-dimensional vertices

#### 5.6.1 Recurrent construction of vertices

Eq. (118) calls for generalization for higher finite-dimensional vertices [13]. Let us consider the bicomplex:

\[ d_{\text{tot}} = Q + d_{\text{Lie}} \]  

(119)
Eq. (106) shows that:

\[ Qv_2 = -d_{\text{Lie}}v_1 \]

where \( v_1 = \lambda_3 - \lambda_1 \)  \hspace{1cm} (121)
\[ v_2 = t^3_\alpha \wedge (1 - 2P_{13})t^1_\alpha \]  \hspace{1cm} (122)

Notice that the ghost number of \( v_n \) is \( 2 - n \).

**Conjecture:**

1. Exist \( v_3, v_4, \ldots \) such that:

\[ d_{\text{tot}} \sum_{j=1}^{\infty} v_j = 0 \]  \hspace{1cm} (123)

2. For \( j \geq 2 \): \((\text{STr}(\lambda_3 \lambda_1))^j v_{2j}\) is a polynomial in \( \lambda_3 \) and \( \lambda_1 \), and is a covariant ghost number 2 vertex for the deformation corresponding to \( \int d^4x \text{tr} Z^{2+j} \)

3. For \( j \geq 2 \): \((\text{STr}(\lambda_3 \lambda_1))^{j+1} v_{2j+1}\) is a polynomial in \( \lambda_3 \) and \( \lambda_1 \), and is a covariant ghost number 3 vertex, also corresponding to \( \int d^4x \text{tr} Z^{2+j} \) as explained in [12].

We leave the verification of these conjectures for future work.

**5.6.2 Infinitesimal deformations of worldsheet BV Master Action**

We will now describe another recurrent construction. As explained in [9], the pure spinor superstring in \( AdS_5 \times S^5 \) is quasiisomorphic to the theory with the following Master Action:

\[ S_{\text{BV}} = \int \text{STr} (J_1 \wedge (1 - 2P_{31})J_3) \]  \hspace{1cm} (124)

This is the integral over the worldsheet of the 2-form \( \mathcal{B} = \text{STr} (J_1 \wedge (1 - 2P_{31})J_3) \) which satisfies the property:

\[ \mathcal{L}_Q \mathcal{B} = dA \]  \hspace{1cm} (125)

where \( A = \text{STr}(\lambda_3 J_1 - \lambda_1 J_3) = \text{Str} ((\lambda_3 - \lambda_1)J) \)  \hspace{1cm} (126)
It is natural to conjecture that a vertex operator will correspond to an infinitesimal deformation of the action defined by Eq. (124):

$$\Delta S_{BV} = \int \langle \beta, J \wedge J \rangle$$  \hspace{1cm} (127)

Here $\beta$ is a rational function of $\lambda$ with values in Hom$(\mathcal{H}, \mathfrak{g} \wedge \mathfrak{g})$, where $\mathcal{H}$ is the space of deformations. The BRST invariance of the deformed action implies:

$$Q\beta = d_{\text{Lie}} \alpha$$  \hspace{1cm} (128)

Suppose that $\text{STr}(\lambda_3 \lambda_1) \beta$ is a polynomial in $\lambda$. Then Eq. (128) implies that $\text{STr}(\lambda_3 \lambda_1) \beta$ defines a $Q$-closed equivariant vertex for $\mathcal{H} \otimes (\mathfrak{g} \wedge \mathfrak{g})_0$. We conjecture that this vertex is nontrivial (i.e. not BRST exact), although it may be BRST exact on a proper subspace $L \subset \mathcal{H} \otimes (\mathfrak{g} \wedge \mathfrak{g})_0$. That means that, given a covariant vertex transforming in the representation $\mathcal{H}$, we can build a new covariant vertex on the space of the larger spin representation $\tilde{\mathcal{H}} = \frac{\mathcal{H} \otimes (\mathfrak{g} \wedge \mathfrak{g})_0}{L}$. This gives a recurrent procedure for producing covariant vertices. We leave verification of these conjectures for future work.

6 OPE of $b$-ghost with beta-deformation vertex

We will use the explicit formulas for the $b$-ghost collected in Section A.3.

6.1 General considerations

At the leading order in $\alpha'$, we should have:

$$b_{zz} W_a = \frac{1}{z}(j_{az} + Ql_{az})$$  \hspace{1cm} (129)

$$b_{\bar{z}z} W_a = -\frac{1}{\bar{z}}(j_{a\bar{z}} + Ql_{a\bar{z}})$$  \hspace{1cm} (130)

where $l_{az}$, $l_{a\bar{z}}$ are some operators, and $j_{az} dz + j_{a\bar{z}} d\bar{z}$ is the global charge density; our definition of the charge density is such that:

$$\left(\frac{1}{2\pi i} \oint j_{az} dz + j_{a\bar{z}} d\bar{z}\right) W_b = f_{abc} W_c$$  \hspace{1cm} (131)
Notice:

\[ j_{za} W_b = \frac{1}{2z} f_{ab} c W_c + \ldots \]  
\[ j_{\bar{z}a} W_b = -\frac{1}{2\bar{z}} f_{ab} c W_c + \ldots \]  

(132) (133)

(where \ldots can include log \( z \) but not \( z^{-1} \)) Therefore:

\[(b_0 - \bar{b}_0)V = \oint (dz z b_{zz} - d\bar{z} \bar{z} b_{\bar{z}\bar{z}}) V = \]

\[= B^{ab} f_{ab} c W_c + Q \left[ B^{ab} \left( \oint l_a \right) W_b \right] \]  
(134)

One is tempted to say that Eq. (134) implies that \( V \) is annihilated by \( b_0 - \bar{b}_0 \), in cohomology, once \( B \) satisfies the physicality condition \( B^{ab} f_{ab} c = 0 \). However, notice that the expression \( f_{ab} c W_c \) is anyway \( Q \)-exact (and even \( g \)-covariantly \( Q \)-exact) since we allow denominator \( \frac{1}{\text{STr}(\lambda_3 \lambda_1)} \), see Section B.

6.2 Explicit computation

The operator \((b_0 - \bar{b}_0)V\) is a sum of two terms: the term with the ghost number \((1, 0)\) and the term with the ghost number \((0, 1)\). The term with the ghost number \((0, 1)\) is:

\[
\frac{2}{\text{STr}(\lambda_3 \lambda_1)} \left( \{[\lambda_1, t^2_{-m}], \lambda_3 \} \wedge [t^2_m, \lambda_1] - \{[\lambda_3, t^2_{-m}], \lambda_1 \} \wedge [t^2_m, \lambda_1] \right) \]

\[ - \kappa^{\beta\tilde{\beta}} \{\tilde{t}^3_{\beta}, \lambda_1 \} \wedge t^1_{\tilde{\beta}} \]  
(135)

and the term with the ghost number \((1, 0)\) is equal, with the minus sign, to the same expression with \( \lambda_3 \leftrightarrow \lambda_1 \) and exchanged dotted and undotted
indices. Transform:

\[ - \frac{2}{\text{STr}(\lambda_3 \lambda_1)} \{[\lambda_3, \overline{t^2_{-m}}], \lambda_1 \} \wedge [t^2_m, \lambda_1] \]

\[ = - \frac{2 \kappa \dot{\beta}}{\text{STr}(\lambda_3 \lambda_1)} \{[\lambda_3, \{\lambda_1, t^1_{\beta}\}_{\text{STL}}], \lambda_1 \} \wedge t^3_{\beta} \]

\[ = \frac{2 \kappa \dot{\beta}}{\text{STr}(\lambda_1 \lambda_3)} \{[\lambda_3, \{\lambda_1, t^1_{\beta}\}_{\text{STL}}], \lambda_1 \} \wedge t^3_{\beta} \]

\[ = - \frac{2 \kappa \dot{\beta}}{\text{STr}(\lambda_1 \lambda_3)} \{[\{\lambda_1, t^1_{\beta}\}_{\text{STL}}, \lambda_3], \lambda_1 \} \wedge t^3_{\beta} \]

\[ = \kappa \dot{\beta} \{t^1_{\beta}, \lambda_1 \} \wedge t^3_{\beta} \]  \hspace{1cm} (136)

where we used the explicit form of the pure spinor projector \(P_{31}\) that can be found in [9]. Thus we arrive at:

\[ \frac{2}{\text{STr}(\lambda_3 \lambda_1)} \{[\lambda_1, \overline{t^2_{-m}}], \lambda_3 \} \wedge [t^2_m, \lambda_1] - Q_R \left( \kappa \dot{\beta} t^3_{\beta} \wedge t^1_{\beta} \right) \]  \hspace{1cm} (137)

Adding the “mirror” term with the ghost number \((1, 0)\), we arrive at:

\[ (b_0 - \bar{b}_0) \left< B^{ab} g(t_a \wedge t_b) g^{-1}, (\lambda_3 - \lambda_1) \wedge (\lambda_3 - \lambda_1) \right> = Q \Phi \]  \hspace{1cm} (138)

where:

\[ \Phi = \left< B^{ab} g(t_a \wedge t_b) g^{-1}, \begin{array}{c} 2[\overline{t^2_{-m}}, \lambda_1] \wedge [t^2_m, \lambda_1] + 2[t^2_{-m}, \lambda_3] \wedge [t^2_m, \lambda_3] - \kappa \dot{\beta} t^3_{\beta} \wedge t^1_{\beta} \end{array} \right> \]  \hspace{1cm} (139)

Up to \(Q\)-exact terms, we can also take:

\[ \Phi = \left< B^{ab} g(t_a \wedge t_b) g^{-1}, \begin{array}{c} -4[\overline{t^2_{-m}}, \lambda_3] \wedge [t^2_m, \lambda_1] + 2[\{\lambda_3, \lambda_1\}, t^2_{-m}] \wedge t^2_m - \kappa \dot{\beta} t^3_{\beta} \wedge t^1_{\beta} \end{array} \right> \]  \hspace{1cm} (140)
6.3 Discussion

In this Section we will compare our proposed Eq. (53):

\[
\int U = \int \Sigma d^2 z \left( b_{-1} \bar{b}_{-1} V^{(0)} + \sqrt{g} R \Phi \right)
\]

where \( \Phi \) is given by Eq. (139)

with the standard approach to the beta-deformation [6]. The most obvious observation is that the “dilaton superfield” \( \Phi \) of Eq. (139) contains pure spinors (while the “standard” dilaton superfield, obviously, does not). Therefore, they are certainly not the same. We will now explain that there are two reasons for the difference.

**First reason:** \( b_{-1} \bar{b}_{-1} V^{(0)} \) is different from the standard integrated vertex on flat worldsheet. The standard integrated vertex on flat worldsheet is [15, 6]:

\[
B_{ab} j^a \wedge j^b
\]

In our approach here, it is the \( b_{-1} \bar{b}_{-1} V^{(0)} \) of Eq. (141). This is *not* equal to \( B_{ab} j^a \wedge j^b \), but differs from it by a \( Q \)-exact expression, which we have not explicitly computed:

\[
B_{ab} j^a \wedge j^b = dz \wedge d\bar{z} b_{-1} \bar{b}_{-1} V^{(0)} + QX
\]

Notice that the BRST operator is only nilpotent on-shell:

\[
Q^2 = \frac{\partial S}{\partial w_1} \frac{\partial}{\partial w_3} + (1 \leftrightarrow 3)
\]

Therefore, the \( QX \) on the RHS of Eq. (144) deforms the BRST operator:

\[
Q \mapsto Q + \left( \frac{\partial X}{\partial w_1} \frac{\partial}{\partial w_3} + (1 \leftrightarrow 3) \right)
\]

This leads to the change in the BRST anomaly, and, by the mechanism of Eqs. (56), (57), to the change of the Fradkin-Tseytlin term.

When we modify the unintegrated vertex:

\[
V^{(0)} \mapsto \tilde{V}^{(0)} = V^{(0)} + QW^{(0)}
\]

\[\text{since we have not explicitly computed } b_{-1} \bar{b}_{-1} V^{(0)}\]
The change in $\Phi$, i.e. $\tilde{\Phi} - \Phi$, should satisfy:

$$Q(\tilde{\Phi} - \Phi) = (b_0 - \bar{b}_0)QW^{(0)}$$  \hspace{1cm} (148)

Under the assumption that $(L_0 - \bar{L}_0)W^{(0)} = 0$ this can be solved by taking:

$$\tilde{\Phi} = \Phi - (b_0 - \bar{b}_0)W^{(0)}$$  \hspace{1cm} (149)

Suppose that we were able to find such $W^{(0)}$ that $\tilde{V}^{(0)}$ is polynomial in pure spinors. Then, the curvature coupling also changes, according to Eq. (149),

**Second reason: we have not required the vanishing of $B^{ab}f_{abc}$.** In fact, $\Phi$ of Eq. (140) can be presented as:

$$\Phi = B^{ab} \left( X_{[ab]} + \left< g(t_a \wedge t_b)g^{-1}, \frac{2\{\{\lambda_3, \lambda_1\}, t^2_{-m}] \wedge t^2_{m}}{\text{Str}(\lambda_3\lambda_1)} \right> \right)$$  \hspace{1cm} (150)

where $X_{[ab]}$ is defined in Eq. (193). Since $QX_{[ab]}$ is proportional to $f_{abc}$, the term $B^{ab}X_{[ab]}$ can be dropped when $B$ has zero internal commutator, i.e. $B^{ab}f_{abc} = 0$. In that case, we have just:

$$\Phi = B^{ab} \left< g(t_a \wedge t_b)g^{-1}, \frac{2\{\{\lambda_3, \lambda_1\}, t^2_{-m}] \wedge t^2_{m}}{\text{Str}(\lambda_3\lambda_1)} \right>$$  \hspace{1cm} (151)

We see that imposing the condition $B^{ab}f_{abc} = 0$ “considerably simplifies” the expression for the dilaton superfield. But still the resulting expression is a rational function of $\lambda$’s.

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**A   Technical details**

**A.1   MATHEMATICA code**

MATHEMATICA code for computations in $AdS_5 \times S^5$ sigma-model is available on GitHub.
A.2 Conventions and notations for $AdS_5 \times S^5$ string

We begin introducing some notation that will be useful throughout the calculation. Our notation is largely based on references [16, 6].

Constant Grassmann parameters The target space is a supermanifold, a coset of the Lie supergroup $PSU(2, 2|4)$. As usual [17], treating the supermanifold, we introduce a “pool” of constant Grassmann parameters $\epsilon, \epsilon', \epsilon'', \ldots$. We can construct the “$\epsilon, \epsilon', \epsilon''\ldots$-points” of the supermanifold $PSU(2, 2|4)$ as formal expressions of the form, for example $\exp (\epsilon \mu^\alpha t^3_\alpha + \epsilon' \mu'^\dot{\alpha} t^1_\dot{\alpha})$ where $\mu^\alpha$ and $\mu'^\dot{\alpha}$ are some spinors with real number components. In addition to these constant Grassmann parameters, there are string worldsheet fields $\theta^\alpha_L$ and $\theta^\dot{\alpha}_R$; therefore we also have: $\exp (\theta^\alpha_L t^3_\alpha)$ — another element of the supergroup.

Superconformal generators and Casimir conventions An element in the superconformal algebra $g = psu(2, 2|4)$ will be represented according to its $Z_4$ grading,

$$t = t^0_{[mn]} \oplus t^1_\alpha \oplus t^2_m \oplus t^3_\alpha$$

where

$$t^0_{[mn]} \in g_0, \quad t^1_\alpha \in g_1, \quad t^2_m \in g_2 \quad \text{and} \quad t^3_\alpha \in g_3$$  (152)

Latin letters are vector indices and greek letters are spinor indices. The bosonic generators are boosts and rotations, given by $t^0_{[mn]}$, and translations denoted $t^2_m$. The fermionic generators are the right supersymmetries, $t^1_\alpha$, and the left supersymmetries, $t^3_\alpha$, with both spinors in the $d = 10$ Majorana-Weyl representation. The vector space $g_2$ is the sum of the tangent vector spaces of $AdS_5$ and $S^5; m \in \{0, \ldots, 9\}$.

For a finite-dimensional representation, the invariant bilinear form is given by the supertrace:

$$\str (t^2_m t^2_n) = \kappa_{mn}, \quad \str (t^3_\alpha t^1_\dot{\alpha}) = \kappa_{\alpha\dot{\alpha}} \quad \text{and} \quad \str (t^1_\alpha t^3_\dot{\alpha}) = \kappa_{\dot{\alpha}\alpha}$$  (153)

where $\kappa_{\alpha\dot{\alpha}}$ and $\kappa_{mn}$ are Casimir tensors.
A.3 The $b$-ghost

The $b$-ghost satisfies:

\begin{align}
Q_L b_{zz} &= T_{zz}, \quad (154) \\
Q_R b_{zz} &= 0 \quad (155)
\end{align}

where $T_{zz}$ is the holomorphic stress-energy tensor. The $\bar{b}_{\bar{z}\bar{z}}$ is defined by the same formula with $Q_L$ exchanged with $Q_R$ and $T_{zz}$ replaced with $T_{\bar{z}\bar{z}}$. The solutions of these equations are given by \[18, 5\]:

\begin{align}
b_{zz} &= -\frac{\text{str} (\lambda_1 [J_{z\bar{z}} \Sigma, J_{1\bar{z}}]) }{\text{str} (\lambda_3 \lambda_1) } + \frac{1}{2} \text{str} (P_{13} \omega_{1\bar{z}} J_{3\bar{z}}) \quad (156)
\end{align}

and

\begin{align}
\bar{b}_{\bar{z}\bar{z}} &= +\frac{\text{str} (\lambda_3 [J_{z\bar{z}} \Sigma, J_{3\bar{z}}]) }{\text{str} (\lambda_3 \lambda_1) } + \frac{1}{2} \text{str} (P_{31} \omega_{3\bar{z}} J_{1\bar{z}}) \quad (157)
\end{align}

where $P_{13}$ and $P_{31}$ are some projectors. These projectors are needed because the pure spinor momenta $\omega_{1\bar{z}}$ and $\omega_{3\bar{z}}$ are defined up to gauge transformations of the form:

\begin{align}
\delta_u \omega_{1\bar{z}} = [u_{\bar{z}}, \lambda_1], \quad \text{and} \quad \delta_u \omega_{3\bar{z}} = [u_{\bar{z}}, \lambda_3],
\end{align}

for both $u_{\bar{z}}$ and $u_{\bar{z}}$ in $g_2$. Therefore, the projectors are constructed to satisfy

\begin{align}
P_{13} \delta_u \omega_{1\bar{z}} = 0 \quad \text{and} \quad P_{31} \delta_u \omega_{3\bar{z}} = 0. \quad (159)
\end{align}

Explicit formulas for $P_{13}$ and $P_{31}$ as rational functions of the pure spinor variables can be found in \[9\].

It is an open question to prove that the expressions $\text{str} (P_{13} \omega_{1\bar{z}} J_{3\bar{z}})$ and $\text{str} (P_{31} \omega_{3\bar{z}} J_{1\bar{z}})$ are well-defined in the quantum theory.

Lemma 2.6.1 implies that $b$ given by Eqs. (156) and (157) coincides with $\Delta \Psi|_L$ up to a $Q$-closed expression. We have not verified this explicitly.

Parametrization of $\text{AdS}_5 \times S^5$ We will work with the conventions of \[16\].

The coordinates in $\text{AdS}_5 \times S^5$ are given by \((x, \theta, \hat{\theta})\) such that

\begin{align}
x = x^m (z, \bar{z}) t^m, \quad \theta = \theta^a (z, \bar{z}) t^a, \quad \hat{\theta} = \hat{\theta}^\dot{a} (z, \bar{z}) t^\dot{a}. \quad (160)
\end{align}
Each of these coordinates lifts to an element in $PSU(2,2|4)$ given by

$$g(x, \theta, \hat{\theta}) = \exp \left( \frac{1}{R} \theta + \frac{1}{R} \hat{\theta} \right) \exp \left( \frac{1}{R} x \right)$$  \hspace{1cm} (161)$$

where $R$ is the AdS radius.

**The pure spinor action**  The $AdS_5 \times S^5$ pure spinor string action is

$$S = \frac{R^2}{\pi} \int d^2 z \text{str} \left( \frac{1}{2} J_{zz} J_{zz} + \frac{3}{4} J_{zz} J_{zz} + \frac{1}{4} J_{zz} J_{zz} + \omega_{1z} D_z \lambda_3 + \omega_{3z} D_z \lambda_1 + N_{0z} N_{0z} \right)$$  \hspace{1cm} (162)$$

with the covariant derivatives defined as

$$D_z \lambda_3 = \partial_z \lambda_3 + [J_{0z}, \lambda_3], \quad D_z \lambda_1 = \partial_z \lambda_1 + [J_{0z}, \lambda_1]$$  \hspace{1cm} (163a)$$

and the Lorentz currents for the ghosts given by

$$N_{0z} = -\{\omega_{1z}, \lambda_1\}, \quad N_{0z} = -\{\omega_{3z}, \lambda_3\}.$$  \hspace{1cm} (163b)$$

The pure spinor action is built out of the right-invariant currents:

$$J = -dg g^{-1} = -\partial_z g g^{-1} dz - \partial_{\bar{z}} g g^{-1} d\bar{z},$$  \hspace{1cm} (164)$$

where $g$ is given by Eq. (161). These currents decompose according to the conformal weight and the $\mathbb{Z}_4$ grading. We write $J = J_0 + J_1 + J_2 + J_3$ to highlight the grading structure, and we observe that under local Lorentz symmetry $J_0$ transforms as a connection while $J_1, J_2$ and $J_3$ transform in the adjoint representation.

**OPE between $b$-ghost and global vertex**  With these definitions, the OPE between the $b$-ghost and unintegrated global symmetry becomes to 1-loop order:

$$\langle \epsilon (b_0 - \bar{b}_0) V[\hat{\epsilon}](0) e^{-S_i} \rangle = \langle \left( \frac{d}{2\pi i} z \epsilon b_{zz}(z) - \frac{d}{2\pi i} \bar{z} \epsilon b_{\bar{z}\bar{z}} \right) V[\hat{\epsilon}](0) \rangle - \langle \left( \frac{d}{2\pi i} z \epsilon b_{zz}(z) - \frac{d}{2\pi i} \bar{z} \epsilon b_{\bar{z}\bar{z}} \right) V[\hat{\epsilon}](0) S_i \rangle.$$  \hspace{1cm} (165)$$
We will calculate all Feynman diagrams considering the pure spinor action and the $b$-ghost as a power series in the $AdS$ radius. For the parametrization [161], the expansion of the action can be found in reference [16]. In the above equation $S_i$ represents all contributions of order $1/R$ or greater.

**A.4 Computation.**

The free field propagators can be read from [16]:

$$
\langle x^m(z, \bar{z})x^n(0) \rangle = -\kappa_{mn} \log |z|^2 \quad (166)
$$

$$
\langle \theta^\alpha_L(z, \bar{z})\theta^\beta_R(0) \rangle = -\kappa^{\alpha\beta} \log |z|^2 \quad (167)
$$

$$
\langle \theta^\alpha_R(z, \bar{z})\theta^\beta_L(0) \rangle = -\kappa^{\dot{\alpha}\beta} \log |z|^2 . \quad (168)
$$

The propagator $\lambda w$ can be characterized by saying that for any $A^\alpha(\lambda)$ such that $A^\alpha \Gamma^m_{\alpha\beta} \lambda^\beta = 0$ (i.e. tangent to the pure spinor cone):

$$
\langle A_{\dot{\alpha}}(\lambda(z, \bar{z})) \ w^\dot{\alpha}(z, \bar{z}) \ \lambda^\beta \rangle = -\kappa^{\dot{\alpha}\beta} z^{-1} \quad (169)
$$

**A.4.1 Current Vertex**

Let us focus, for the moment, on contractions that take only one $V$ in $V \wedge V$; that is, we are going to compute the OPE of $(b_0 - \bar{b}_0)$ with $\epsilon(\lambda_3 - \lambda_1)$. The contributions we are interested are represented in the diagrams below:

![Figure 1: Disconnected contractions for the OPE between $b_{zz}$ and $V[\tilde{\epsilon}]$.](image)

$$
\frac{1}{\operatorname{str}(\lambda_3 \lambda_1)} \frac{\kappa^{mn}\kappa^{\dot{\alpha}\alpha}}{R^4(z - w)^2} \operatorname{str} \left( \epsilon \lambda_1 \left[ t_2^m \Sigma, t_1^n \right] \right) \left[ t_2^m, \{ t_3^\alpha, \tilde{\epsilon} (\lambda_3 - \lambda_1) \} \right] . \quad (170)
$$
\[ \text{str}(\epsilon \beta_{13} \omega_{1} t_{3}) \to -\text{str}(\epsilon \beta_{13} \omega_{1} \partial \theta) \]

\[ g^{-1} \tilde{\epsilon}(\lambda_{3} - \lambda_{1}) g \to -[\tilde{\theta}, \tilde{\epsilon} \lambda_{3}] \]

Figure 2: Disconnected contractions for the OPE between \( b_{zz} \) and \( V[\tilde{\epsilon}] \).

\[ \text{str}(\tilde{\epsilon} \lambda_{1}[\partial x \Sigma, \partial \theta]) \]

\[ \begin{array}{c}
\hat{\partial} \\
\hat{\partial x} \\
\hat{\theta}
\end{array} \]

\[ [\hat{\theta}, \epsilon(\lambda_{3} - \lambda_{1})] \]

Figure 3: Vertex contribution \( b_{zz} \) and \( V[\tilde{\epsilon}] \).

Let us use the identity:

\[ -\kappa^{m n} \kappa^{\hat{\alpha} \hat{\beta}} \text{str}\left( \tilde{\epsilon} \lambda_{1}[t_{m}^{2} \Sigma, t_{\alpha}^{1}] \right) \left[ t_{n}^{2}, \{ t_{\beta}^{3}, \tilde{\epsilon} (\lambda_{3} - \lambda_{1}) \} \right] = (171) \]

\[ = \kappa^{m n}\left[ t_{n}^{2}, \left[ [t_{m}^{2} \Sigma, \epsilon \lambda_{1}], \tilde{\epsilon} (\lambda_{3} - \lambda_{1}) \right] \right] \]

\[ = \kappa^{m n}\left[ t_{n}^{2}, \left[ [t_{m}^{2} \Sigma, \epsilon \lambda_{1}], \tilde{\epsilon} \lambda_{3} \right] \right]. (172) \]

Contribution from the diagram of Fig. 2

\[ \frac{\kappa^{\alpha \hat{\alpha}} \kappa^{\hat{\beta} \beta}}{2 R^{4}(z - w)^{2}} \text{str} \left( \epsilon \beta_{13} t_{\beta}^{1} t_{\alpha}^{3} \right) \left\{ t_{\beta}^{3}, \tilde{\epsilon} t_{\alpha}^{1} \right\} = \frac{\kappa^{\hat{\beta} \beta}}{2 R^{4}(z - w)^{2}} \left[ \epsilon t_{\beta}^{3}, \epsilon \beta_{13} t_{\beta}^{1} \right]. (173) \]
Sum of first and second diagram

\[
\epsilon b_0 V[\bar{\epsilon}] = + \frac{1}{R^4} \kappa_{mn} \left[ t_n^2, \left[ t_m^2, \epsilon \lambda_1 \right], \bar{\epsilon} \lambda_3 \right] - \frac{\kappa_{\beta\beta}^3}{2R^4} \left[ \bar{\epsilon} P_{13} t_\beta^1, \epsilon t_\beta^3 \right]. \tag{174}
\]

**Anti-holomorphic b-ghost**  
A similar computation gives for the anti-holomorphic term:

\[
\epsilon \bar{b}_0 V[\bar{\epsilon}] = + \frac{1}{R^4} \kappa_{mn} \left[ t_n^2, \left[ t_m^2, \epsilon \lambda_3 \right], \bar{\epsilon} \lambda_1 \right] - \frac{\kappa_{\beta\beta}^3}{2R^4} \left[ \bar{\epsilon} P_{13} t_\beta^1, \epsilon t_\beta^3 \right]. \tag{175}
\]

**Contribution of \( b_0 - \bar{b}_0 \)**  
We can simplify the total contribution of the diagrams of Figures 1 and 2 to

\[
\epsilon (b_0 - \bar{b}_0) V[\bar{\epsilon}] = \left( \kappa_{mn} \left[ t_n^2, \left[ t_m^2, \epsilon \lambda_1 \right], \bar{\epsilon} \lambda_3 \right] - \kappa_{mn} \left[ t_n^2, \left[ t_m^2, \epsilon \lambda_3 \right], \bar{\epsilon} \lambda_1 \right] \right)
\]

\[
= \left( \frac{5}{2} \left[ \epsilon \lambda_1, \bar{\epsilon} \lambda_3 \right] - \frac{5}{2} \left[ \epsilon \lambda_3, \bar{\epsilon} \lambda_1 \right] \right)
\]

\[
+ \left( \kappa_{mn} \left[ \left[ t_m^2, \epsilon \lambda_1 \right], t_n^2, \bar{\epsilon} \lambda_3 \right] - \kappa_{mn} \left[ \left[ t_m^2, \epsilon \lambda_3 \right], t_n^2, \bar{\epsilon} \lambda_1 \right] \right)
\]

\[
= \left( \kappa_{mn} \left[ \left[ t_m^2, \epsilon \lambda_1 \right], t_n^2, \bar{\epsilon} \lambda_3 \right] - \kappa_{mn} \left[ \left[ t_m^2, \epsilon \lambda_3 \right], t_n^2, \bar{\epsilon} \lambda_1 \right] \right)
\]

\[
= \frac{3}{2} \left[ \epsilon \lambda_1, \bar{\epsilon} \lambda_3 \right] + \frac{3}{2} \left[ \epsilon \lambda_3, \bar{\epsilon} \lambda_1 \right] = 0 \tag{176}
\]
In this derivation, we used the identities

\[
\kappa^{mn} \left[ t^2_n, [t^2_m, \Sigma, \epsilon \lambda_1] \right] = \frac{\Sigma}{2} \kappa^{mn} \left[ \{ t^2_m, t^2_n \}, \epsilon \lambda_1 \right] = \kappa^{mn} \kappa_{mn} \frac{\Sigma}{8} \left[ \Sigma, \epsilon \lambda_1 \right] = \frac{1}{4} \kappa^{mn} \kappa_{mn} \epsilon \lambda_1 = \frac{5}{2} \epsilon \lambda_1 \tag{177a}
\]

together with

\[
\kappa^{mn} \left[ \left[ t^2_m \Sigma, \epsilon \lambda_1 \right], \left[ t^2_n, \tilde{\epsilon} \lambda_3 \right] \right] = -\frac{3}{2} \left[ \epsilon \lambda_1, \tilde{\epsilon} \lambda_3 \right] \tag{177b}
\]

and

\[
- \kappa^{mn} \left[ \left[ t^2_m \Sigma, \epsilon \lambda_3 \right], \left[ t^2_n, \tilde{\epsilon} \lambda_1 \right] \right] = \frac{3}{2} \left[ \epsilon \lambda_3, \tilde{\epsilon} \lambda_1 \right] \tag{177c}
\]

**Contribution of the diagram of Figure 3** There only remains the contractions that get contributions from the interaction vertices:

\[
\frac{1}{2\pi R^4} \frac{1}{\text{str}(\lambda_3 \lambda_1)} \text{str} \left( \epsilon \lambda_1 \left[ \partial x \Sigma, \partial \bar{\theta} \right] \right) (z) \left[ \bar{\theta}, \tilde{\epsilon} (\lambda_3 - \lambda_1) \right] (w) \int d^2 u \text{ str} \left( \partial x \left[ \theta, \bar{\theta} \right] \right)
\]

We use:

\[
\int d^2 u \frac{1}{(z-u)^3} \frac{1}{(w-u)} = \frac{\pi}{2} \frac{1}{(z-w)^2} \tag{179}
\]

and obtain:

\[
- \kappa^{mn} \kappa^{\beta \gamma} \kappa^{\alpha \delta} \text{ str} \left( \epsilon \lambda_1 \left[ t^2_m \Sigma, t^2_\beta \right] \left\{ t^4_\alpha, \tilde{\epsilon} (\lambda_3 - \lambda_1) \right\} \text{ str} \left( t^2_n \left\{ t^3_\gamma, t^3_\delta \right\} \right) \right). \tag{180}
\]

We temporarily do not write the factor of $1/4R^2 \text{str}(\lambda_3 \lambda_1)$ since it only observes the calculation. This answer can be rewritten as
\[ -\kappa^{mn} \kappa^{\beta\beta} \kappa^{\alpha\alpha} \text{str} \left( \epsilon \lambda_1 \left[ t_m^2 \Sigma, t_{\beta}^1 \right] \right) \left\{ t_{1\alpha}^1, \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right\} \text{str} \left( t_n^2 \left\{ t_3^3, t_{\alpha}^3 \right\} \right) = \]

\[ -\kappa^{mn} \kappa^{\beta\beta} \kappa^{\alpha\alpha} \text{str} \left( \left[ \epsilon \lambda_1, t_m^2 \Sigma \right] t_{\beta}^1 \right) \left\{ t_{1\alpha}^1, \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right\} \text{str} \left( t_n^2 \left[ t_3^3, t_{\alpha}^3 \right] \right) = \]

\[ -\kappa^{mn} \kappa^{\beta\beta} \text{str} \left( \left[ \epsilon \lambda_1, t_m^2 \Sigma \right] t_{\beta}^1 \right) \left[ t_n^2, t_{1\beta}^1 \right], \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right] = \]

\[ -\kappa^{mn} \left[ t_n^2, \left[ \epsilon \lambda_1, t_m^2 \Sigma \right] \right], \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right] = \]

\[ +\kappa^{mn} \left[ t_n^2, \left[ t_m^2 \Sigma, \epsilon \lambda_1 \right] \right], \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right] = \frac{5}{2} \left[ \epsilon \lambda_1, \tilde{\epsilon} \left( \lambda_3 - \lambda_1 \right) \right] = \frac{5}{2} \left[ \epsilon \lambda_1, \tilde{\epsilon} \lambda_3 \right] = \frac{5}{2} \left[ \epsilon \lambda_3, \tilde{\epsilon} \lambda_1 \right] \tag{181} \]

to give the contribution – with all factors restored –

\[ \frac{5}{8R^4} \frac{g^{-1} \left[ \epsilon \lambda_3, \tilde{\epsilon} \lambda_1 \right] g}{\text{str}(\lambda_3 \lambda_1)}. \tag{182} \]

Notice that in deriving equation (181) we used identity (177a). To summarize, the contribution of Figure 3 is given by equation (182).

**Anti-holormorphic b-ghost**  One can compute the contribution of \( \bar{b}_{2\bar{z}}(\bar{z}) \) in the same way and it gives

\[ \tilde{\epsilon} \bar{b}_0 V[\tilde{\epsilon}](w) = \frac{5}{8R^4} \frac{g^{-1} \left[ \epsilon \lambda_3, \tilde{\epsilon} \lambda_1 \right] g}{\text{str}(\lambda_3 \lambda_1)} \tag{183} \]

**Final answer**  Combining the three diagrams we arrive at
\[ \epsilon (b_0 - \tilde{b}_0) V[\tilde{\epsilon}] = 0 \quad (184) \]

for the current vertex.

A.4.2 Beta-deformation Vertex.

In order to finish the calculation, we only have to compute contractions where the \( b \)-ghost hits both \( V \) in \( V \wedge V \). These mixed contractions are given by the diagrams below:

\[
V \wedge V \Rightarrow \frac{2}{R^2} [\theta, \tilde{\epsilon}(\lambda_3 - \lambda_1)] \wedge [x, \epsilon(\lambda_3 - \lambda_1)]
\]

\[
b_{zz} \Rightarrow -\frac{1}{R^2} \frac{\text{str}(\epsilon' \lambda_1 [\partial_x \Sigma, \partial \theta])}{\text{str}(\lambda_3 \lambda_1)}
\]

Figure 4:Disconnected contractions for the OPE between \( b_{zz} \) and \( V[\tilde{\epsilon}] \wedge V[\epsilon] \).

\[
-\frac{1}{2R^4} \text{str}(\epsilon' P_{13} \omega_{1z} \partial \theta)
\]

\[
-\frac{2}{R^4} [\tilde{\theta}, \epsilon(\lambda_3 - \lambda_1)] \wedge \tilde{\epsilon}(\lambda_3 - \lambda_1)
\]

Figure 5:Disconnected contractions for the OPE between \( b_{zz} \) and \( V[\tilde{\epsilon}] \wedge V[\epsilon] \).

We stress that there are no contributions from the action up to 1-loop.

**Contribution of diagram in figure 4** The diagram in figure 4 contributes as

\[
-\kappa^{\hat{a}\alpha} \kappa^{mn} \frac{2}{R^4 \text{str}(\lambda_3 \lambda_1)} \text{str} (\epsilon' \lambda_1 \left[ t_m^2 \Sigma, \epsilon_\alpha^1 \right]) \left\{ t^3_\alpha, \tilde{\epsilon}(\lambda_3 - \lambda_1) \right\} \wedge \left[ t_n^2, \epsilon(\lambda_3 - \lambda_1) \right] \quad (185)
\]

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And this result can be simplified to:

\[-\frac{2\kappa^{\dot{\alpha}\alpha} \kappa^{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \text{str} (\epsilon' \lambda_1 [t^2_m \Sigma, t^1_\alpha] \{t^3_n, \bar{\epsilon}(\lambda_3 - \lambda_1)\} \land [t^2_n, \epsilon(\lambda_3 - \lambda_1)] =

\[-\frac{2\kappa^{\dot{\alpha}\alpha} \kappa^{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \text{str} ([\epsilon' \lambda_1, t^2_m \Sigma] \{\tilde{t}^3_\alpha, \bar{\epsilon}(\lambda_3 - \lambda_1)\} \land [t^2_n, \epsilon(\lambda_3 - \lambda_1)] =

\[-\frac{2\kappa^{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \left[\epsilon' \lambda_1, t^2_m \Sigma\right], \bar{\epsilon}(\lambda_3 - \lambda_1) \right] \land \left[ t^2_n, \epsilon(\lambda_3 - \lambda_1) \right] =

\[-\frac{2\kappa^{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \left[\epsilon' \lambda_1, t^2_m \Sigma\right], \bar{\epsilon}\lambda_3 \right] \land \left[ t^2_n, \epsilon(\lambda_3 - \lambda_1) \right] =

\text{(186)}

**Contribution of diagram in figure 5**

Likewise, we obtain:

\[\frac{1}{R^2} \text{str} (\epsilon' P_{13} \omega_1 \partial \theta) \left[\hat{\theta}, \bar{\epsilon}(\lambda_3 - \lambda_1)\right] \land \epsilon(\lambda_3 - \lambda_1) =

\[-\frac{1}{R^2} \kappa^{\dot{\alpha}\alpha} \text{str} (\epsilon' P_{13} \omega_1 t^3_\alpha) \{\tilde{t}^1_\alpha, \bar{\epsilon}(\lambda_3 - \lambda_1)\} \land \epsilon(\lambda_3 - \lambda_1) =

\[\frac{1}{R^4} \kappa^{\dot{\alpha}\dot{\alpha} \dot{\beta}\beta} \text{str} (\epsilon' P_{13} t^1_\beta t^3_\alpha) \{\tilde{t}^1_\alpha, \bar{\epsilon}(\lambda_3 - \lambda_1)\} \land \epsilon t^3_\beta =

\[\frac{1}{R^4} \kappa^{\dot{\beta}\beta} \left[\epsilon' P_{13} t^1_\beta, \bar{\epsilon}\lambda_3 \right] \land \epsilon t^3_\beta =

\[\frac{1}{R^4} \kappa^{\dot{\beta}\beta} \left[\epsilon' t^1_\beta, \bar{\epsilon}\lambda_3 \right] \land \epsilon t^3_\beta =

\[\frac{1}{R^4} \kappa^{\dot{\beta}\beta} \left[\epsilon' t^1_\beta, \bar{\epsilon}\lambda_3 \right] \land \epsilon t^3_\beta =

\text{(187)}

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Holomorphic $b$-ghost  The sum of these contributions gives us:

$$
\epsilon' b_0 V[\bar{\epsilon}] \wedge V[\epsilon] = - \frac{2\kappa_{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \left[ [\epsilon' \lambda_1, t^2_m \Sigma], \bar{\epsilon} \lambda_3 \right] \wedge \left[ t^2_n, \epsilon (\lambda_3 - \lambda_1) \right]
$$

$$
+ \frac{1}{R} \kappa^{\beta \bar{\beta}} \left[ \epsilon' t^1_{\beta \bar{\beta}}, \bar{\epsilon} \lambda_3 \right] \wedge \epsilon t^1_{\bar{\beta}}
$$

(188)

Anti-holomorphic $b$-ghost  The same can be done for the anti-holomorphic $b$-ghost, and we obtain

$$
\epsilon' \bar{b}_0 V[\bar{\epsilon}] \wedge V[\epsilon] = - \frac{2\kappa_{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \left[ [\epsilon' \lambda_3, t^2_m \Sigma], \bar{\epsilon} \lambda_1 \right] \wedge \left[ t^2_n, \epsilon (\lambda_3 - \lambda_1) \right]
$$

$$
+ \frac{1}{R} \kappa^{\beta \bar{\beta}} \left[ \epsilon' t^3_{\beta \bar{\beta}}, \bar{\epsilon} \lambda_1 \right] \wedge \epsilon t^3_{\bar{\beta}}
$$

(189)

A.4.3 Final answer

The sum of all contributions from the current and the mixed contractions gives us the final answer:

$$
\epsilon' \left( b_0 - \bar{b}_0 \right) V[\bar{\epsilon}] \wedge V[\epsilon] =
$$

$$
- \frac{2\kappa_{mn}}{R^4 \text{str} (\lambda_3 \lambda_1)} \left( \left[ [\epsilon' \lambda_1, t^2_m \Sigma], \bar{\epsilon} \lambda_3 \right] \wedge \left[ t^2_n, \epsilon (\lambda_3 - \lambda_1) \right] - \left[ [\epsilon' \lambda_3, t^2_m \Sigma], \bar{\epsilon} \lambda_1 \right] \wedge \left[ t^2_n, \epsilon (\lambda_3 - \lambda_1) \right] \right)
$$

$$
+ \frac{1}{R^4} \left( \kappa^{\beta \bar{\beta}} \left[ \epsilon' t^1_{\beta \bar{\beta}}, \bar{\epsilon} \lambda_3 \right] \wedge \epsilon t^1_{\bar{\beta}} - \kappa^{\beta \bar{\beta}} \left[ \epsilon' t^3_{\beta \bar{\beta}}, \bar{\epsilon} \lambda_1 \right] \wedge \epsilon t^3_{\bar{\beta}} \right)
$$

(190)

B  BRST triviality of $f_{ab}^c W_c$

The projectors $P$ were used in [6] to prove that BRST triviality of the ghost number 1 vertices corresponding to the global symmetries. Once we allow
denominators, the BRST cohomology is zero anyway. But in highly super-
symmetric backgrounds, it is meaningful to ask to which extent resolving
\( Q\phi = \psi \) preserves the global supersymmetries. The ghost number 1 vertex
for a global symmetry \( t_a \in \text{psu}(2,2|4) \) is:

\[
W_a(\epsilon) = (g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g)_a
\]  

(191)

for a Grassmann odd constant parameter \( \epsilon \). It was proven in [6] that
\( f_{ab}^c W_c = -\epsilon Q X_{ab} = -\epsilon Q X_{[ab]} \) where

\[
X_{ab} = \text{Str} \left( gt_ag^{-1} \left( (gt_bg^{-1})_3 + 2(gt_bg^{-1})_2 + 3(gt_bg^{-1})_1 - 4P_{13}(gt_bg^{-1})_1 \right) \right)
\]

where \( f_{ab}^c \) are the structure constants of \( \text{psu}(2,2|4) \). This implies that \( f_{ab}^c W_c \) is Q-exact in a way preserving symmetries. However, \( W_c \) cannot be obtained
from \( f_{ab}^c W_c \) preserving symmetries. (Notice that \( f_{abc} f_{ab}^d = 0 \).) In this sense, \( f_{ab}^c W_c \) is BRST-exact but \( W_c \) is not.

Notice that:

\[
X_{[ab]} = \left\langle t_a \wedge t_b , g^{-1}Ag \right\rangle
\]  

(193)

where \( A = -2k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge (1 - 2P_{13})t_\tilde{\alpha}^1 =
\]

\[
= 2k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge t_\tilde{\alpha}^1 + 8\frac{k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge \{[\lambda_1, t_\alpha^1]_{STL}, \lambda_3\}}{\text{STr} \lambda_1 \lambda_3} =
\]

\[
= 2k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge t_\tilde{\alpha}^1 + 8\frac{[\lambda_1, t_m^1] \wedge [t^2_m, \lambda_3]}{\text{STr} \lambda_1 \lambda_3}
\]  

(195)

In other words, in the covariant complex (see Section 5.1, Eq. (98)):

\[
Q \left( k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge (1 - 2P_{13})t_\tilde{\alpha}^1 \right) = d_{\text{Lie}}(\lambda_3 - \lambda_1)
\]  

(196)

where \( d_{\text{Lie}} \) is defined in Section 5.2

**Relation to the “minimalistic action”**  We will now explain that Eq. (196) is equivalent to the BV Master Equation for the minimalistic action of [9]. Let us consider the scalar product, as defined in Section 5.2, with \( J_3 \wedge J_1 \):

\[
\left\langle J_3 \wedge J_1 , k^{\alpha\tilde{\alpha}}t_\alpha^3 \wedge (1 - 2P_{13})t_\tilde{\alpha}^1 \right\rangle =
\]

\[
= \text{STr} \left( J_1 t_\alpha^3 \right) k^{\alpha\tilde{\alpha}} \wedge \text{STr} \left( t_\alpha^1 (1 - 2P_{31})J_3 \right) =
\]

\[
= \text{STr} \left( J_1 \wedge (1 - 2P_{31})J_3 \right)
\]  

(199)

---

\(^9\)As usual in supergeometry, we use a sufficiently large pool of constant fermionic parameters
\[ \epsilon Q \left\langle J_3 \wedge J_1 , \ k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge (1 - 2P_{13}) t^{1}_{\alpha} \right\rangle = \] (200)

\[ = \left\langle [\epsilon \lambda_1 , J_2] \wedge J_1 + J_3 \wedge [\epsilon \lambda_2 , J_2] , \ k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge (1 - 2P_{13}) t^{1}_{\alpha} \right\rangle + \]

\[ + \left\langle - \epsilon D_0 \lambda_3 \wedge J_1 - J_3 \wedge \epsilon D_0 \lambda_3 , \ k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge (1 - 2P_{13}) t^{1}_{\alpha} \right\rangle \]

The first line of the RHS of Eq. (200) equals to (in the sense of Section 5.1, Eq. (98)):

\[ - \left\langle J_2 \wedge J_1 + J_3 \wedge J_2 , \epsilon Q \left( k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge (1 - 2P_{13}) t^{1}_{\alpha} \right) \right\rangle = \]

\[ = \left\langle J_2 \wedge J_1 + J_3 \wedge J_2 , \ d_{\text{Lie}}(\epsilon \lambda_3 - \epsilon \lambda_1) \right\rangle = \] (201)

\[ = \left\langle [J_2, J_1] + [J_3, J_2] , \ \epsilon \lambda_3 - \epsilon \lambda_1 \right\rangle = \text{STr} ([J_3, J_2] \epsilon \lambda_3 - [J_2, J_1] \epsilon \lambda_1) \]

The second line of the RHS of Eq. (200) is:

\[ \left\langle - \epsilon D_0 \lambda_3 \wedge J_1 - J_3 \wedge \epsilon D_0 \lambda_3 , \ k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge (1 - 2P_{13}) t^{1}_{\alpha} \right\rangle = \] (202)

\[ = \left\langle - \epsilon D_0 \lambda_3 \wedge J_1 - J_3 \wedge \epsilon D_0 \lambda_3 , \ k^{\alpha \dot{\alpha}_3} t^{1}_{\alpha} \wedge t^{1}_{\alpha} \right\rangle = \]

\[ = \text{STr} ((D_0 \lambda_1) J_3 - (D_0 \lambda_3) J_1) \] (203)

The sum is a total derivative:

\[ Q \text{STr} (J_1 \wedge (1 - 2P_{31}) J_3) = d \text{STr} ((\lambda_3 - \lambda_1) J) \] (204)

This shows that Eq. (124) is \( Q \)-invariant.

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