Liouville first passage percolation: geodesic dimension is strictly larger than 1 at high temperatures

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Abstract

Let \( \{ \eta(v) : v \in V_N \} \) be a discrete Gaussian free field in a two-dimensional box \( V_N \) of side length \( N \) with Dirichlet boundary conditions. We study the Liouville first passage percolation, i.e., the shortest path metric where each vertex is given a weight of \( e^{\gamma \eta(v)} \) for some \( \gamma > 0 \). We show that for sufficiently small but fixed \( \gamma > 0 \), with probability tending to 1 as \( N \to \infty \) the dimensions of all geodesics between vertices of macroscopic distances are simultaneously strictly larger than 1.

1 Introduction

For \( B \subset \mathbb{Z}^2 \), the discrete Gaussian free field (DGFF) \( \{ \eta^B(v) : v \in B \} \) with Dirichlet boundary conditions is a mean-zero Gaussian process with

\[
\mathbb{E} \eta^B(x) \eta^B(y) = \mathbb{E}_x \sum_{t=0}^{\tau-1} 1\{S_t=y\} \quad \text{for all } x, y \in B,
\]

where \( \{S_t : t = 0, 1, 2, \ldots \} \) is a simple random walk starting from \( x \), and \( \tau \) is the hitting time to the boundary \( \partial B = \{ z \in B : \exists w \in B^c \text{ such that } z \text{ is a neighbor of } w \} \).

Let \( V_N = [0, N)^2 \cap \mathbb{Z}^2 \). We denote \( V_5_N = [-2N, 3N)^2 \cap \mathbb{Z}^2 \), and believe there is no ambiguity since \( [0, 5N)^2 \cap \mathbb{Z}^2 \) will not be involved. A path \( P \) in \( V_N \) is a sequence of vertices \( v_0, v_1, \ldots, v_d \) in \( V_N \), where \( v_i \) is a neighbor of \( v_{i+1} \) for all \( i \). The weight of \( P \) is defined to be \( w(P) := \sum_{z \in P} \exp\{\gamma \eta^V_5(z)\} \), where \( \gamma > 0 \) plays the role of the inverse-temperature. For \( x, y \in V_N \), the Liouville first passage percolation (FPP) distance between \( x \) and \( y \) is defined to be \( \min_P w(P) \), where the minimization is taken over all paths in \( V_N \) joining \( x \) and \( y \). The (unique with probability 1) minimizer is defined to be the geodesic between \( x \) and \( y \), and denoted by \( \text{Geo}_{N,x,y} \). For a set \( A \), we denote by \( |A| \) its cardinality. Our main result is on the (conjecturally unique) dimensions of such geodesics.

**Theorem 1.1.** There exists \( \gamma_0 > 0 \) such that the following holds. For all \( 0 < \gamma < \gamma_0 \), there exists \( \alpha = \alpha(\gamma) > 0 \) such that for all \( \kappa \in (0, 1) \),

\[
\lim_{N \to \infty} \mathbb{P} \left( |\text{Geo}_{N,x,y}| > N^{1+\alpha}, \text{ for all } x, y \in V_N \text{ with } ||x - y|| \geq \kappa N \right) = 1.
\]
Theorem 1.1 states that with probability tending to 1 as $N \to \infty$, the dimensions of all geodesics are simultaneously strictly larger than 1. We note that the question on the dimension of the (conjecturally well-defined) scaling limit of the geodesic was asked by Benjamini [3], where the author suspected that the dimension is strictly larger than 1.

Theorem 1.1 follows from a combination of [11, Theorem 1.1] and the following theorem. Suppose $P$ is a path from $x$ to $y$, then we denote $\|P\| = \|x - y\|$. Denote by $P_{\kappa,\alpha}$ the set of all paths $P$ in $V_N$ with $\|P\| \geq \kappa N$ and $|P| \leq N^{1+\alpha}$.

**Theorem 1.2.** For any $\delta \in (0, 1)$, there exists $\alpha = \alpha(\delta) > 0$ such that for all $\kappa \in (0, 1)$, 
\[
\lim_{N \to \infty} \mathbb{P}(|\{z \in P : \eta^{V_N}(z) \geq -15\sqrt{\delta \log N}\}| \geq \frac{1}{8} \kappa N, \text{ for all } P \in P_{\kappa,\alpha}) = 1.
\]

By Theorem 1.2 with probability tending to 1 as $N \to \infty$, we have that 
\[
w(P) \geq \frac{1}{8} \kappa N e^{-15\sqrt{\delta \log N}} \geq N^{1-6\gamma\sqrt{\delta}}, \text{ for all } P \in P_{\kappa,\alpha}.
\]

By [11, Theorem 1.1] and a chaining argument (see, e.g., [9, Proposition 6.5]), we have that (for $\gamma < \gamma_0$) with probability tending to 1 as $N \to \infty$, 
\[
\max_{x,y \in V_N} w(\text{Geo}_{N,x,y}) \leq N^{1-\gamma^2/(2 \cdot 10^3)}.
\]

Pick $\delta < 10^{-10} \gamma^2$, and we conclude that $\text{Geo}_{N,x,y} \notin P_{\kappa,\alpha}$, establishing Theorem 1.1. Thus, the main task of the present paper is to prove Theorem 1.2.

**Remark 1.3.** In fact, for any (not necessarily small) $\gamma > 0$, if one can show that the weight exponent for the Liouville FPP is strictly less than 1, then combined with Theorem 1.2 it yields that the geodesic has dimension strictly larger than 1.

**Remark 1.4.** If one considers the DGFF in $V_N$, Theorem 1.1 and Theorem 1.2 also hold, with an additional assumption that $x$, $y$ are away from boundary, i.e. $|x - z|$, $|y - z| \geq aN$ for all $z \in \partial V_N$, where $0 < a < 1$. The proof is essentially same. We chose to consider the DGFF in $V_5N$ to avoid unnecessary cumbersome notation.

**Remark 1.5.** The proof of Theorem 1.2 works for some other log-correlated Gaussian fields such as branching random walk (see [10]). In particular, our method should be adaptable for proving an analogue of Theorem 1.2 for a discrete approximation of the continuous Gaussian free field. A natural question there (similar to questions asked in [3]), is to show that the (conjectural) limit of the geodesics (as the discrete approximation gets finer and finer) is of dimension strictly larger than 1. We chose not to investigate that in this paper for the reason that the convergence of the geodesics is a major open problem and it is far from being clear at this point. As such, we decide that it makes more sense to leave this problem for future consideration.

### 1.1 Related works

Recently, there has been a few works on the Liouville first passage percolation metric [9, 11], where in [11] (as mentioned above) an upper bound on the weight exponent was derived and in [9] a sub-sequential scaling limit was proved for the normalized metric. Our work addresses another
important aspect of this random metric, which in particular manifests the \textit{fractality} of the metric by showing that the geodesic has dimension strictly larger than 1. In contrast, the analogous question for classical first passage percolation with i.i.d. weights has a trivial answer that the dimension of the geodesic is 1. This, in turn, emphasizes the fractal nature of Liouville FPP which is drastically different from classical FPP.

In \cite{12} the chemical distance (i.e., graph distance in the induced open cluster) for the percolation of level sets of two-dimensional DGFF was studied, and in particular \cite[Theorem 1.1]{12} implies that there exists a path of dimension 1 joining the two boundaries of an annulus which has Liouville FPP weight $O(N^{1+\alpha(1)})$. This can be seen as a complement of our Theorem 1.2. In \cite{13}, a non-universality result was proved on the weight exponent for the geodesic among log-correlated Gaussian fields, which suggests the subtlety when attempting to estimate the weight exponent. We would like to point out that the proof of our main result Theorem 1.2 is expected to be adaptable to general log-correlated Gaussian fields with $\star$-scale invariant kernels as in \cite{14}. However, in light of \cite{13} there seems to be no reason to expect that the geodesic dimension is universal among log-correlated fields.

Another related recent work \cite{15} studied a type of discrete distance associated with Liouville quantum gravity (LQG) and proved some bounds on the exponent for the weight of the geodesic. While the Liouville FPP metric is expected to be related to random metric on LQG, we refrain ourselves from an extensive discussion on the LQG metric, but simply refer an interested reader to \cite{20,19} (as well as references therein) for a body of recent works on the construction of a metric in the continuum as well as its connection to the Brownian map. However, so far we see no mathematical connection between our work to \cite{15,20,19}. Finally, we remark that in a recent work \cite{18} the authors studied a random pseudo-metric on a graph defined via the zero-set of the Gaussian free field on its metric graph.

1.2 A word on proof strategy

The general proof strategy in this paper is multi-scale analysis, which has seen powerful applications in percolation theory (see, e.g., \cite{5,8,6,7,21,2}). In particular, our proof of Theorem 1.2 is inspired by the methods employed in \cite{7,21} for showing that the dimension of the shortest crossing for fractal percolation process is strictly larger than 1, and our proof is especially inspired by \cite{21}. A particular instance of fractal percolation process is defined by removing independently (with probability $p$) all dyadic boxes with corners of dyadic forms in all scales and considering remaining vertices as open. The basic strategy of \cite{21} in lower bounding the crossing dimension is to argue that in every scale if the dimension of a path is close to 1 then with good probability some box (in that scale) along the path is removed. The framework of our proof is similar in spirit as two-dimensional GFF has a similar hierarchical structure (see Section 2.2), but with two important differences as follows.

- The analogue of a vertex being closed in our context is that its Gaussian value is at least $-15\sqrt{\alpha} \log N$ and this can only be verified by putting together information from almost all scales regarding to this vertex (as opposed to the closeness in the fractal percolation process).

- In our context it is not sufficient to demonstrate the existence of one closed point. Instead, we have to show that there exist a large number (close to dimension 1) of closed points in any short path.

In order to address the aforementioned difficulties, we associate each path with a tree where a node in the tree corresponds to a subpath in a scale corresponding to the depth of the node (as
demonstrated in Section 3.1. The key intuition is that, the number of leaves of such tree is small (since the path is short) and thus most nodes in the tree should have a degree that is given by the ratio between two neighboring scales. If so (we call the node tame in this case), the corresponding subpath is close to a straight line in that scale and we can then derive a bound on the fraction of vertices with not too small values (for Gaussian variables in the corresponding scale). This is proved in Proposition 4.1. For nodes with larger degrees in the tree, we will control its influence by using the fact that the fraction of such nodes is small. To this end, we consider a uniform flow on the tree to facilitate our analysis. This is incorporated in Proposition 3.4. Altogether, these two facts will imply that there are a large number of vertices in the path whose values are not too small.

We still need to address a main challenge which requires a simultaneous control on all possible paths. More precisely, we need to show that simultaneously for all possible paths, a typical tame node will contributes to a good fraction of children with values not too small (so, a strengthened version of Proposition 4.1). This is done by a multi-scale analysis on corresponding tree structures, where we apply a union bound in every scale in an inductive manner. Our strategy is in flavor similar to the chaining argument (see [23] for an excellent account on this topic) which is a powerful method in bounding the maximum of a random process. The success is guaranteed by the uniform flow on the tree we introduced: if a node has a large degree, on one hand the choices of the children grows exponentially, but on the other hand we will perform an average over many children and thus have a large deviation in probability that decays faster than the growth of enumeration. This step is carried out in Section 4.2.

1.3 Notation convention

For $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{R}^2$, denote by
$$|z - w| = |z_1 - w_1| \vee |z_2 - w_2|, \quad \|z - w\| = \sqrt{|z_1 - w_1|^2 + |z_2 - w_2|^2}$$
the $L^\infty$-distance and $L^2$-distance of $z$ and $w$, respectively. Denote $d(z, B) = \inf_{w \in B} \|z - w\|$. Similarly, we have $d(B_1, B_2), d_\infty(x, B)$ and $d_\infty(B_1, B_2)$.

Throughout this paper, $\kappa, \delta, \varepsilon \in (0, 1)$, $\varepsilon > 0$. Let $C_1, C_2 \cdots$ be universal constants, $k \in \mathbb{Z}_+$, $K = 2^k$, $K^m \leq N < K^{m+1}$. Let $K$ be large but fixed, i.e. there exists some $K_0$ (may depend on $\delta, \varepsilon$) such that $K \geq K_0$. We consider the limiting behavior when $m \to \infty$, i.e. $N \to \infty$. Thus, we can assume without loss of generality that the inequalities such as $C_2 \leq C_1 k, \varepsilon k^2 e^{-k} < 1, (K e^{-k})^3 m < \delta$ hold.

2 Preliminaries on two-dimensional DGFF

In the following, $\{S_t : t = 0, 1, 2, \cdots\}$ is a simple random walk on $\mathbb{Z}^2$, and $\tau_D$ is the time it hits $D$. For $x = (x_1, x_2) \in \mathbb{Z}^2$ and $\ell \in 2\mathbb{Z}_+$, let
$$B_\ell(x) := ([x_1 - \ell/2, x_1 + \ell/2] \times [x_2 - \ell/2, x_2 + \ell/2]) \cap \mathbb{Z}^2$$
be the box centered at $x$ and of side length $\ell$. 

4
2.1 Log-correlation

Recall the DGFF in $B$ has covariance as the Green function of $\{S_t\}$ in $B$. An explicit formula of the Green function is given in [16 Theorem 4.6.2].

$$\mathbb{E} \eta^B(x) \eta^B(y) = \mathbb{E}_x \sum_{t=0}^{\tau-1} 1_{\{S_t = y\}} = \sum_{z \in \partial B} \mathbb{P}_x(S_\tau = z) a(z - y) - a(x - y),$$

where

$$a(x) = \left( \frac{2}{\pi} \log |x|_\infty + \frac{2\gamma + \log 8}{\pi} \right) + O(|x|_\infty^2) \quad \text{with} \quad a(0) = 0,$$

and $\gamma$ is the Euler constant [16 Theorem 4.4.4].

Suppose $B = B_\ell(x_0)$ for $\ell \in 2\mathbb{Z}_+$. Denote $B^{(\frac{1}{10})} := \{z \in B : d_\infty(z, \partial B) > \frac{1}{10} \ell\}$ (here $\frac{1}{10}$ is a somewhat arbitrary choice). We call $B^{(\frac{1}{10})}$ the deep inside part of $B$. Then, one can check that the DGFF is log-correlated, i.e. there are universal constant $C_1, C_2 > 0$ such that

$$\left| \mathbb{E} \eta^B(x) \eta^B(y) - C_1 \log \frac{\ell}{|x-y|_\infty \vee 1} \right| \leq C_2, \quad \text{for all} \quad x, y \in B^{(\frac{1}{10})},$$

by noting $\mathbb{P}_x(S_\tau = z) \leq C_3 \ell^{-1}$ for all $x \in B^{(\frac{1}{10})}$ (see, e.g., [16 Proposition 8.1.4]). Moreover, one can set $C_2, C_3$ such that

$$\mathbb{E} \eta^B(x_0) \eta^B(y) \leq C_2, \quad \text{for all} \quad y \in B \setminus B^{(\frac{1}{10})},$$

$$|a(x) - \left( \frac{2}{\pi} \log |x|_\infty + \frac{2\gamma + \log 8}{\pi} \right)| \leq C_3 |x|_\infty^2.$$  

Then, a refined estimate of the variance at the center $x_0$ can be given as below.

$$\left| \mathbb{E} \eta^B(x_0)^2 - \left( \frac{2}{\pi} \log \ell - \frac{2}{\pi} \log 2 + \frac{2\gamma + \log 8}{\pi} \right) \right| \leq \frac{C_3}{\ell^2}. \quad (5)$$

**Remark 2.1.** In fact, it is well-known that $C_1 = \frac{2}{\pi} \log 2$.

2.2 Hierarchy structure

For $B \subset V_{3N}$, we denote by $H^B$ the Gaussian field satisfying (a) $H^B$ coincides with $\eta^{V_{5N}}$ on $B^c \cup \partial B$; (b) $H^B$ is harmonic in $B \setminus \partial B$. That is, $H^B$ is the harmonic extension of $\eta^{V_{5N}} \mid_{B^c \cup \partial B}$. The well-known Markov field property of DGFF states that $\eta^{V_{5N}} - H^B = \eta^B$ (i.e. it is a version of $\eta^B$), and it is independent with $H^B$. This property also holds when $V_{3N}$ is replaced by any $D$ with $B \subset D$. We are now prepared to introduce the hierarchy structure of $\eta^{V_{5N}} \mid_{V_N}$.

Let $K = 2^k$ be large enough but fixed, and suppose $K^m \leq N < K^{m+1}$. Denote $H_{2^r}(x) := H^{2^r \eta}(x)$, $r \geq 1$ for brevity, and $H_1(x) := \eta^{V_{5N}}(x)$. Let $X_0(x) := \eta^{V_{5N}}(x) - H_2(x)$, and $X_r(x) := H_{2^r}(x) - H_{2^{r+1}}(x)$, for all $1 \leq r \leq mk - 1$. Define $\eta_j(x) = \sum_{r=j}^{j+1} X_r(x)$, $j = 0, \ldots, m - 1$. Then

$$\eta^{V_{5N}}(x) = \eta(x) + H_{K^m}(x), \quad \text{where} \quad \eta(x) := \sum_{j=0}^{m-1} \eta_j(x). \quad (6)$$

Basically, we will investigate $\eta(x) = \sum_{j=0}^{m-1} \eta_j(x)$. The tail term $H_{K^m}(x)$ will be dealt with at last.
By the Markov field property,
\[
\begin{align*}
X_r(x), \quad r = 0, 1, \ldots, mk-1 & \text{ are mutually independent.} \\
X_r(x) \text{ and } X_r(y) & \text{ are independent if } |x-y|_\infty \geq 2^{r+1}, \\
\eta_j(x) \text{ and } \eta_j(y) & \text{ are independent if } |x-y|_\infty \geq K^{j+1}.
\end{align*}
\] (7)

Furthermore, note \(\sum_{i=0}^{r-1} X_i(x) = \eta^{B_2}(x)\). By (2) and (7), we have
\[
E X_r(x)^2 \leq C_1 + 2C_2, \quad |E \eta_j(x)^2 - C_1 k| \leq 2C_2.
\] (8)

Next, we show that the covariances of all \(\eta_j\) are bounded from below.

**Lemma 2.2.** There exists a universal constant \(C_4\) such that \(E \eta_j(x) \eta_j(y) \geq -C_4\) for all \(x, y \in V_N\).

**Proof.** We only give the proof for \(j \geq 1\), since that for \(j = 0\) is similar by setting \(B_1 = \{x\}\) below. Denote \(B_1 = B_{K^i}(x)\) and \(B_2 = B_{K^{i+1}}(x)\) for brevity. Recall \(\eta_j(x) = \eta^{B_2}(x) - \eta^{B_1}(x)\). By the Markov field property,
\[
E \eta_j(x) \eta^{V_N}(z) = \begin{cases} 0, & \text{for all } z \in V_N \setminus B_2, \\ E \eta^{B_2}(x) \eta^{B_2}(z), & \text{for all } z \in B_2 \setminus B_1, \\ E \eta^{B_2}(x) \eta^{B_2}(z) - E \eta^{B_1}(x) \eta^{B_1}(z), & \text{for all } z \in B_1. \end{cases}
\] (9)

In all these cases, \(E \eta_j(x) \eta^{V_N}(z) \geq 0\).

Note \(\eta_j(y) = H_{K^j}(y) - H_{K^{j+1}}(y)\). By the positive correlated property and (9),
\[
E \eta_j(x) \eta_j(y) \geq - \sum_{z \in \partial B_2} \mathbb{P}_y(S_\tau = z)E \eta_j(x) \eta^{V_N}(z) = - \sum_{z \in \partial} \mathbb{P}_y(S_\tau = z)E \eta_j(x) \eta^{V_N}(z),
\]

where \(\tilde{B}_2 = B_{K^{j+1}}(y), \quad \tau = \tau_{\partial \tilde{B}_2}\), and \(\partial = B_2 \cap \partial \tilde{B}_2\). Note \(\partial\) lies in the union of a horizon line and a vertical line. We will show that for any horizon or vertical line \(L\),
\[
\sum_{z \in L} \mathbb{E} \eta_j(x) \eta^{V_N}(z) \leq (3C_1 + 2C_2 + 1)K^{j+1}.
\] (10)

Assuming (10), we have \(-\mathbb{E} \eta_j(x) \eta_j(y) \geq -2(3C_1 + 2C_2 + 1)C_3\), completing the proof.

It remains to prove (10). Without loss of generality, suppose \(x = (0, 0)\), and \(L = \{(z_1, z_2) \in B_2 : z_2 = b\}\). Furthermore, we assume \(b \in (-\frac{1}{2}K^{j+1}, \frac{1}{2}K^{j+1}) \cap \mathbb{Z}\), noting (9). We will respectively investigate the contributions of the following three parts.

Part 1. By (3) and (9),
\[
\sum_{z \in L \setminus B_2^{(\frac{3}{4})}} \mathbb{E} \eta_j(x) \eta^{V_N}(z) \leq \sum_{z \in L \setminus B_2^{(\frac{3}{4})}} \mathbb{E} \eta^{B_2}(x) \eta^{B_2}(z) \leq C_2 |L \setminus B_2^{(\frac{3}{4})}| \leq C_2 K^{j+1}.
\]

Part 2. Denote \(L_0 = \{(z_1, z_2) \in L : z_1 \in [-\frac{1}{2}K^j, \frac{1}{2}K^j] \cap \mathbb{Z}\}\). By (2) and (9),
\[
\sum_{z \in (L \
 \setminus L_0) \setminus B_2^{(\frac{3}{4})}} \mathbb{E} \eta_j(x) \eta^{V_N}(z) \leq 2 \sum_{r=K^{j/2}+1}^{K^{j+1}/2} (C_1 \log_2 \frac{K^{j+1}}{r} + C_2) \leq (3C_1 + C_2)K^{j+1}.
\]
Part 3. Suppose \( z \in L_0 \cap B_{2^{-\frac{1}{10}}} \). If \(|b| \geq K^j / 2\), one has \(|x - z|_\infty \geq K^j / 2\). Otherwise, we have \( z \in B_1 \). By \([10]\),
\[
\mathbb{E}_{\eta_j}(x) \eta_{V^N}^{\tau_2}(z) = \mathbb{E}_z \sum_{t=\tau_1}^{\tau_2-1} 1_{\{S_t = x\}} \leq \max_{w \in \partial B_1} \mathbb{E}_w \sum_{t=0}^{\tau_2-1} 1_{\{S_t = x\}} = \max_{w \in \partial B_1} \mathbb{E}_w \eta_{B^2}(x) \eta_{B^2}(w),
\]
where \( \tau_1 \) and \( \tau_2 \) are respectively the times \( \{S_t\} \) hits \( \partial B_1 \) and \( \partial B_2 \). Note for any \( w \in \partial B_1 \), \(|w - x|_\infty = K^j / 2\). By \([2]\), in any situation, we have \( \mathbb{E}_{\eta_j}(x) \eta_{V^N}^{\tau_2}(z) \leq C_1 \log_2 \frac{K^{j+1}}{K^j} + C_2 \). Therefore,
\[
\sum_{z \in L_0 \cap B_{2^{-\frac{1}{10}}}} \mathbb{E}_{\eta_j}(x) \eta_{V^N}^{\tau_2}(z) \leq (K^j + 1) \times (C_1 \log_2(2K) + C_2) \leq K^{j+1}.
\]
Sum the upper bounds in these three parts, and we conclude \([10]\). \( \square \)

**Lemma 2.3.** There is a universal constant \( C_5 \) such that for \( j \geq 1 \) and \(|u - v|_\infty \leq K^j \), \( \mathbb{E}(H_{K^j}(u) - H_{K^j}(v))^2 \) and \( \mathbb{E}(\eta_j(u) - \eta_j(v))^2 \) are both less than \( C_5 \frac{|u - v|_\infty}{K^j} \).

**Proof.** For \( u \neq v \), denote \( B_{K^j}(u) \), \( B_{K^j}(v) \), \( B_{K^j+2|u - v|_\infty}(u) \) and \( B_{K^j+2|u - v|_\infty}(v) \) as \( B_u \), \( B_v \), \( \tilde{B}_u \) and \( \tilde{B}_v \) for brevity. Without loss of generality, we suppose \( u_1 < v_1 \) and \( u_2 < v_2 \) (here we denote \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \)). Let \( B \) be the box of side length \( K^j + |u - v|_\infty \) and lower left corner \((u_1 - K^j/2, u_2 - K^j/2)\). Then \( B_u \cup B_v \subset B \subset \tilde{B}_u \cap \tilde{B}_v \). Note
\[
H_{K^j}(u) - H_{K^j}(v) = (H^{B_u}(u) - H^B(u)) - (H^B(v) - H^{B_v}(v)) + (H^B(u) - H^B(v)).
\]
For the first two terms, note
\[
\mathbb{E}(H^{B_u}(u) - H^B(u))^2 = \mathbb{E}(\eta^{B}(u))^2 - \mathbb{E}(\eta^{B_u}(u))^2 \leq \mathbb{E}(\eta^{\tilde{B}_u}(u))^2 - \mathbb{E}(\eta^{B_u}(u))^2.
\]
Consequently, by \([3]\),
\[
\mathbb{E}(H^{B_u}(u) - H^B(u))^2 \leq \frac{2}{\pi} \log \frac{K^j + 2|u - v|_\infty}{K^j} + 2C_3 \frac{1}{K^j} \leq 2 \frac{|u - v|_\infty}{K^j}.
\]
One can have the same estimate for the second term, by the same reasoning.

For the third term, let
\[
\varphi(D) = \sum_{z \in \partial D} (\mathbb{P}_u(S_\tau = z) - \mathbb{P}_v(S_\tau = z)) (a(z - u) - a(z - v)),
\]
where \( \tau = \tau_{\partial D} \) for any box \( D \) containing \( u, v \). Following the arguments in \([4] \) Lemma 3.10], we have
\[
\mathbb{E}(H^{B}(u) - H^B(v))^2 \leq |\varphi(V^N)| + |\varphi(D)|.
\]
Let \( D = B \) or \( D = V_{5N} \). One can estimate \(|\mathbb{P}_u(S_\tau = z) - \mathbb{P}_v(S_\tau = z)|\) by Harnack’s inequality \([10]\) as in \([4]\), and estimate \(|a(z - u) - a(z - v)|\) directly by \([4]\). Then, one can conclude that
\[
|\mathbb{P}_u(S_\tau = z) - \mathbb{P}_v(S_\tau = z)| \leq 5 \times 10^3 \frac{C_3}{|\partial D|}, \quad |a(z - u) - a(z - v)| \leq 2 \frac{|u - v|_\infty}{K^j},
\]
for any \( z \in \partial D \). It follows that \(|\varphi(D)| \leq 10^4 C_3 |u - v|_\infty / K^j \) for \( D = B \) and \( D = V_{5N} \). Consequently, the third term has variance less than \( 2 \times 10^4 C_3 |u - v|_\infty / K^j \).

Altogether, we get that \( \mathbb{E}(H_{K^j}(u) - H_{K^j}(v))^2 \leq C|u - v|_\infty / K^j \), where \( C = 2(1 + 10^4 C_3) \). Note \( \eta_j = H_{K^j} - H_{K^{j+1}} \). Thus, the result holds, with \( C_5 := 4C \). \( \square \)
At the end of this subsection, we state two lemmas. We will use them to control the maximum Gaussian value in a box in next sections.

Lemma 2.4. ([1], Theorem 4.1) Let $B \subset \mathbb{Z}^2$ be a box of side length $\ell$ and $\{G_w : w \in B\}$ be a mean zero Gaussian field satisfying

$$\mathbb{E}(G_z - G_w)^2 \leq |z - w|_\infty / \ell \quad \text{for all } z, w \in B.$$ 

Then $\mathbb{E} \max_{w \in B} G_w \leq C_6$, where $C_6$ is a universal constant.

Lemma 2.5. ([7], Theorem 7.1, Equation (7.4)) Let $\{G_z : z \in B\}$ be a Gaussian field on a finite index set $B$. Set $\sigma^2 = \max_{z \in B} \text{Var}(G_z)$. Then

$$\mathbb{P}(\max_{z \in B} G_z - \mathbb{E} \max_{z \in B} G_z \geq x) \leq 2e^{-\frac{x^2}{2\sigma^2}}.$$ 

3 Hierarchical structure of a path

In this section, we explore a certain hierarchical structure of a path, on which a certain multi-scale analysis will be carried out later in Section 4. In Subsection 3.1, we will consider scales in the form of $K^j$ for a fixed large integer $K$ and $j \geq 0$. We will give the definition of a path in scale $K^j$, and introduce a procedure to extract subpaths in scale $K^j$ from a path in scale $K^{j+1}$. With this procedure, we can obtain some properties of the subpaths (see Proposition 3.1), which are crucial for our goal. Then, we will associate a tree to each path, where the nodes of the tree correspond to subpaths and parent-child relation in the tree corresponds to containment relation of two subpaths under the procedure. In Subsection 3.2, we will prove that the tree associated to a path with dimension close to 1 satisfies a certain regularity condition (see Proposition 3.4).

3.1 Construction of the tree associated to a path

In this subsection, we will devise a procedure to extract subpaths $P^{(i)}$'s from a path $P$ (where $\cup_i P^{(i)}$ is not necessarily $P$). The main result is Proposition 3.1. Then, we will associate a tree $T_Q$ with a path $Q$, where a node $u$ in the tree is identified with a subpath $Q^u$ of $Q$ and the children $v$'s of $u$ are identified with subpaths of $P := Q^u$ in Proposition 3.1.

We start with a few definitions. For an integer $r \geq 1$, denote $[r] = \{1, \cdots, r\}$. Let $BD_r$ be the collection of closed boxes of form $B = [ar - \frac{1}{2}, (a + 1)r - \frac{1}{2}] \times [br - \frac{1}{2}, (b + 1)r - \frac{1}{2}]$ with $a, b \in \mathbb{Z}$. For technical reason, we will always use closures of boxes, which partition $\mathbb{R}^2$. Thus we shift the integer boxes by $\frac{1}{2}$ in each axis, in order to make the integers lie in the interior of a box $B$. Consequently, $(B \cap \mathbb{Z}^2)$'s partition $\mathbb{Z}^2$, where $B$ is taken over $BD_r$. For $\ell > 0$, let

$$B(x, \ell) = \{z \in \mathbb{R}^2 : ||x - z|| \leq \ell\}$$

be the $\ell_2$-ball centered at $x$ and of radius $\ell$. Denote the starting point and end point of a path $P$ respectively by $x_P$ and $y_P$. Define $||P|| := ||x_P - y_P||$, and call it the ($\ell_2$)-distance of $P$. Note that $||P||$ is different from $|P|$, the cardinality of $P$. We say two paths $P, Q$ are disjoint if $P \setminus \{x_P, y_P\}$ and $Q \setminus \{x_Q, y_Q\}$ are disjoint.

To make subpaths have the same distance as stated in Proposition 3.1, we will view a discrete path $P = (z_0, z_1, z_2, \cdots)$ on $\mathbb{Z}^2$ as a continuous path in $\mathbb{R}^2$, where we incorporate all points on the
edge connecting \( v_i \) and \( v_{i+1} \) for each \( i \). To allow us further flexibility, we in what follows allow a path to start or end at a point that is internal in an edge. For a path in \( \mathbb{R}^2 \), \( |P| := |P \cap \mathbb{Z}^2| \).

Recall \( K^m \leq N < K^{m+1} \). Let

\[
\mathcal{SL}_j := \begin{cases} \{ P : 1 \leq \|P\|/K^j \leq 1 + 1/K, \ P \subset B(x_P, \|P\|) \}, & j \in [m - 1], \\ \{ P : \|P\| \geq \kappa N \}, & j = m, \\ \mathbb{Z}^2, & j = 0. \end{cases} \tag{12}
\]

The definition of \( \mathcal{SL}_m \) serves for \( \mathcal{P}_{\kappa,\alpha} \), so it is a bit different from that of \( \mathcal{SL}_j \) for \( j \in [m - 1] \). Actually, \( \mathcal{SL}_j \), \( j < m \) consists of subpaths of \( P \in \mathcal{SL}_m \) constructed in Proposition 3.1 and Corollary 3.2 below. The assumption \( P \subset B(x_P, \|P\|) \) is Property (d) therein, which is useful in the proofs later. In the definition of \( \mathcal{SL}_0 \) and what follows, we identify a point \( z \) as \( \{ z \} \), and regard it as a path in the smallest scale. We say \( P \) is a path in scale \( K^j \) if \( P \in \mathcal{SL}_j \), so that \( \|P\| \) is comparable to \( K^j \) for \( j \in [m - 1] \).

For \( 0 \leq j \leq m - 2 \) and any \( P \in \mathcal{SL}_{j+1} \), let

\[
E(P) := \{ z \in \mathbb{R}^2 : \|x_P - z\| + \|y_P - z\| \leq (1 + 2/K^2)\|P\| \},
\]

which is an ellipse of width \( O(K^{j+1}) \) and height \( O(K^j) \). Define

\[
\tilde{E}(P) = \{ z \in \mathbb{R}^2 : d(z, E(P)) \leq 4K^j \}. \tag{13}
\]

We say \( P \) is tame if \( P \subseteq \tilde{E}(P) \), and untame otherwise. We would like to mention that we do not say a path in \( \mathcal{SL}_0 \) or \( \mathcal{SL}_m \) is tame or untame. The tame property implies that \( P \) is roughly a straight line of length \( O(K) \) if we rescale \( \mathbb{R}^2 \) by regarding \( K^j \) as the unit. The untame property implies that one travels a long way along \( P \). Hence, we can extract more or longer subpaths, as Property (b) in Proposition 3.1 states. This implies that a path with dimension close to \( 1 \) can not contain many untame subpaths, which will be proved in Proposition 3.4. We call Property (c) in Proposition 3.1 the 12-times-rule (here the number 12 is a somewhat arbitrary choice). With it, the subpaths are disperse in some sense, so that we can take advantage of \( (7) \) and employ a large deviation estimate for \( h_j|_{P(i)} \)’s later.

**Proposition 3.1.** Suppose \( 1 \leq j \leq m - 2 \) and \( P \in \mathcal{SL}_{j+1} \). Then, there exists \( \ell \in [K^j, (1 + \frac{1}{K})K^j] \), a positive integer \( d \), and disjoint subpaths \( P^{(i)} \in \mathcal{SL}_j \) of \( P \) for \( i \in [d] \) such that

(a) \( d \geq K \).

(b) \( \ell \geq \frac{1}{2}\|P\| \). Furthermore, \( \ell \geq (1 + \frac{1}{K^2})\frac{1}{2}\|P\| \) if \( P \) is untame.

(c) Each box in \( \mathcal{BD}_{K^j} \) is visited by at most 12 subpaths.

(d) \( \|P^{(i)}\| = \ell \) and \( P^{(i)} \subset B(x_{P^{(i)}}, \ell) \), for each \( i \in [d] \).

**Proof.** In the following, \( x, y \) stand for the starting point and end point of \( P \), and \( x_i, y_i \) stand for those of \( P^{(i)} \). For \( z \in P \), we let \( P_z \) be the subpath of \( P \) starting at \( z \), and \( P_{z,\ell} \) be that starting at \( z \) until \( P \) reaches \( \partial B(z, \ell) \) for the first time. We will set \( \ell \), pick \( x_i \)’s, and define \( y_i \) as the point where \( P_{x_i} \) first reaches \( \partial B(x_i, \ell) \), i.e. \( P^{(i)} = P_{x_i,\ell} \). Then, Property (d) will be satisfied naturally, and only Properties (a), (b) and (c) are to be checked. We next describe the procedure and verify our construction respectively in the tame and untame cases.
Suppose $P$ is tame. Set $d := K$ and $\ell := \frac{1}{K}\|P\|$. Let $L(x, y)$ be the straight line connecting $x$ and $y$. Denote by $z_i$ the point in $L(x, y)$ with $\|z_i - x\| = i\ell$, and by $L_i$ the line perpendicular to $L(x, y)$ at $z_i$. Then we take $x_i$ to be the point where $P$ hits $L_{i-1}$ for the first time, for $i \in [d]$, with $x_1 = x$. It is not hard to check that $P^{(i)}$’s are disjoint, and Properties (a), (b), (c) all hold.

Suppose $P$ is untame. Take

$$d := \begin{cases} K, & \text{if } \|P\| \leq \frac{1+1/K}{1+1/K^2}K^{j+1}, \\ K+1, & \text{otherwise}, \end{cases} \quad \ell := \frac{1}{d}(1 + \frac{1}{K^2})\|P\|.$$ 

Since $P \in SL_{j+1}$, we always have $\ell \in [K^j, K^j(1 + \frac{1}{K})]$.

We set $x_1 = x$. The key is to describe the choice of $x_{i+1}$ for $i \geq 1$ recursively. Suppose we have defined $x_i$ and $y_i$. To obtain Property (c) and that subpaths are disjoint. By Property (d), in Case 1 and 2, $x_i$ and $z_i$ hits $L_i$. Of course, $x_i$, $y_i$ and $z_i$ can not be defined if $P_{x_{i+1}}$ is contained in the interior of $B(x_{i+1}, \ell)$. Let

$$d := \min \{i : x_{i+1} \text{ or } y_{i+1} \text{ is not well-defined} \}.$$ 

In Case 1 and 2, $y_{i+1}$ can not be defined if $P_{x_{i+1}}$ is contained in the interior of $B(x_{i+1}, \ell)$. Let $d := \min \{i : x_{i+1} \text{ or } y_{i+1} \text{ is not well-defined} \}$.

That is, the procedure stops at $i = d$ naturally, when it can not continue. By our construction, the subpaths are disjoint, and any box $B \in BD_{K^j}$ can be visited by at most 12 subpaths, verifying Property (c). Thus, it now remains to check Properties (a) and (b).

Next, we will prove $d \geq \hat{d}$ in three steps, which would then immediately imply Property (a) and (b) by the definitions of $\hat{d}$ and $\ell$. Then, the proof is completed.

Step 1. We show that the sequence of subpaths spreads out in a regular way such that

$$\|z - x\| \leq i\ell, \quad \text{for all } z \in P^{(i)} \text{ and } i \in [d].$$ 

We check it by induction on $i$. Note (16) holds for $i = 1$ by the definition of $P^{(1)}$. Suppose it holds for all $i' \in [i]$, then we need to show it also holds for $i + 1 (\leq d)$. By Property (d) and
triangle inequality, we only need to check \( \|x_{i+1} - x\| \leq i\ell \). If \( x_{i+1} = y_i \), it holds by the induction assumption and \( y_i \in P^{(i)} \). Otherwise, Case 2 in (15) is true, and \( x_{i+1} = z_i \), which lies in some box \( B \in BD^i \). By the definition of \( BD^i \), one can find \( i' \leq i - 11 \) and a point \( w \in B \cap P^{(i')} \) such that \( \|x_{i+1} - w\| \leq \sqrt{2}K^j \leq 2\ell \). This, together with the induction assumption, yields that \( \|x_{i+1} - x\| \leq \|x_{i+1} - w\| + \|w - x\| \leq 2\ell + i\ell \leq i\ell \).

**Step 2.** We show that
\[
\|yd - y\| \leq 5\ell. \tag{17}
\]
If \( x_{d+1} \) is well-defined, \( \|x_{d+1} - y_d\| \leq 4\ell \) by (14) and (15). In this case, \( \|x_{d+1} - y\| < \ell \) since \( y_{d+1} \) can not be well-defined. By triangle inequality, \( \|yd - y\| < 4\ell + \ell = 5\ell \). Otherwise, \( x_{d+1} \) is not well-defined. Consequently, Case 3 in (15) is true, i.e. \( y \in B^d \). Then, (17) follows from (14).

**Step 3.** We show \( d \geq \tilde{d} \), using the untame property and results in the former two steps.
If some box in \( BD_{Kj} \) is visited by 12 subpaths, we claim that
\[
\|x - y\| \leq (d - 2)\ell. \tag{18}
\]
Assuming (18), we then conclude \( d \geq 2 + \frac{\|x - y\|}{\ell} \geq K + 1 \geq \tilde{d} \), where we use \( \frac{\|x - y\|}{\ell} \geq \frac{K}{1 + \frac{\kappa N}{K^{m-1}}} > K - 1 \) by the definitions of \( \tilde{d} \) and \( \ell \). We now verify (18). Let \( i \) satisfy \( B^i \neq \emptyset \) and \( B^{i'} = \emptyset \) for all \( i' > i \). We pick a box \( B \in BD^i \), and let \( P^{(i)} \) be the first subpath visiting \( B \). Then, \( i_0 \leq i - 11 \). Pick \( z \in B \cap P^{(i)} \), we have \( \|x - z\| \leq i_0\ell \leq (i - 11)\ell \) by (16). If \( i = d \), combing this with (17) and \( \|y_i - z\| \leq 4\ell \) by (13), one can conclude (18). Next, we assume \( i < d \). In this case, \( B \) is chosen as the box containing \( z_i \). Then, \( \|z - x_{i+1}\| \leq \max\{4\ell, \sqrt{2}K^j\} = 4\ell \), where we use (14) and the fact \( x_{i+1} = y_i \) or \( z_i \). By the choice of \( i \), \( \|x_{i+1} - y_d\| \leq (d - i)\ell \). Collecting the above inequalities, we have \( \|x - y\| \leq \|x - z\| + \|z - x_{i+1}\| + \|x_{i+1} - y_d\| + \|y_d - y\| \leq (i - 11)\ell + 4\ell + (d - i)\ell + 5\ell = (d - 2)\ell \), where inequality (17) is used.

Otherwise, none of the boxes in \( BD_{Kj} \) is visited by 12 subpaths. Note we never remove any part of \( P \) in the procedure. By the definition of \( d \), we conclude \( \|y_d - y_d\| < \ell \), and thus \( P_{y_d} \subset B(y, 2\ell) \). It follows that any point in \( P \setminus B(y, 2\ell) \) must lie in some subpath. Note \( B(y, 2\ell) \subset E(P) \), since \( 2\ell \leq 4K^j \). Since \( P \) is untame, there exist \( w \notin E(P) \) and \( i \leq d - 1 \) such that \( w \in P^{(i+1)} \). By Property (d) and \( \ell < 4K^j \), we have \( x_{i+1} \notin E(P) \), i.e. \( \|x - z\| + \|z - y\| \geq (1 + \frac{\kappa N}{K^{m-1}})\|P_{y_d}\| > d\ell \), where \( z = x_{i+1} = y_i \). Since \( x_{i'} = y_{i'} \) for all \( i' \), we have \( i\ell \geq \|x_{i'} - y_{i'}\| = \|x - z\| \) and \( (d - i)\ell \geq \|x_{i+1} - y_d\| = \|z - y_d\| \) by Property (d) and triangle inequality. Collecting the inequalities above, we conclude \( (d + 1)\ell = i\ell + (d - i)\ell \geq \|x - z\| + \|z - y_d\| + \|y_d - y\| \geq \|x - z\| + \|z - y\| > d\ell \). Therefore, \( d + 1 > \tilde{d} \), i.e. \( d \geq \tilde{d} \). \( \square \)

For \( j = m - 1 \) and \( P \in SL_{j+1} = SL_m \), one can follow the procedure to extract subpaths from a tame path in the proof of Proposition 3.1 with \( \ell := K^{m-1} \) in the situation here. Consequently, we can extract
\[
d = d_0 := \left\lfloor \frac{\kappa N}{K^{m-1}} \right\rfloor
\]
subpaths from \( P \), with \( P_{y_d} \) being removed totally. Then, we obtain (i) in the following Corollary 3.2. For \( j = 0 \) and \( P \in SL_{j+1} = SL_1 \), we regard each \( \{z\} \) as a subpath of \( P \), where \( z \in (Z^2 \cap P) \setminus \{x_P, y_P\} \). We remove the ends to match the definition that paths are disjoint. Note \( K \leq \|P\| \leq K + 1 \) for \( P \in SL_1 \). It follows that \( |P| \geq K - 2 \). Consequently, \( P \) has at least \( K - 4 \geq \frac{1}{2}\|P\| \) subpaths.
Corollary 3.2. (i) Suppose \( j = m - 1 \) and \( P \in SL_m \). Then, we can extract \( d = d_0 \) disjoint subpaths (of \( P \)) in \( SL_j \), such that Property (c) and (d) in Proposition 3.1 hold. (ii) Suppose \( j = 0 \) and \( P \in SL_1 \). Then, we can find \( d \geq \frac{1}{2}\|P\| \) different subpaths in the internal of \( P \).

Next, we will associate any path \( P \in SL_j \) with a tree \( T_P \) of depth \( j \), for any \( j \in [m] \). We will first introduce the construction of \( T_P \), then state the conclusion in Corollary 3.3 at the end of this subsection. In what follows, we denote the number \( d \) in Proposition 3.1 and Corollary 3.2 by \( d_P \) for clearer dependence on the path \( P \).

**Constructing the tree** \( T_P \). We construct the trees by induction of \( j \). For \( j = 1 \), we identify \( P \) as the root \( \rho \), and each subpath constructed in Corollary 3.2 as a child of \( \rho \). In this situation, each child is a leaf and is identified as a point in \( \mathbb{Z}^2 \cap P \setminus \{x_P, y_P\} \). Suppose \( P \in SL_j \), and we have associated any path \( Q \in SL_{j-1} \) with a tree \( T_Q \) of depth \( j - 1 \). By Proposition 3.1 and Corollary 3.2, we extract \( d = d_P \) disjoint subpaths \( P^{(1)}, \ldots, P^{(d)} \in SL_{j-1} \) from \( P \). Now, we let the root \( \rho \) give births to \( d \) children \( u_1, \ldots, u_d \). Then, we identify \( P \) as \( \rho \), and \( P^{(i)} \) as \( u_i \) for all \( i \in [d] \). By attaching the root of \( T_{P^{(i)}} \) to the node \( u_i \), we obtain the tree \( T_P \).

Now we have associated a path \( P \in SL_j \) with a tree \( T_P \) with depth \( j \). Denote the root by \( \rho \), and the set of leaves by \( \mathcal{L} \). Any node \( u \) is identified with a subpath of \( P \), which is denoted by \( P^u \). Let \( L(u) \) be the level of a node \( u \), with \( L(\rho) = 0 \) and \( L(v) = j \) for leaves, then \( P^u \in SL_{j - L(u)} \). Denote \( \|u\| := \|P^u\| \) if \( L(u) \leq j - 1 \). Denote \( d_u = d_{P^u} \), which is the number of children of \( u \). We would like to mention that different leaves correspond to different points, since subpaths are disjoint in Proposition 3.1 as well as Corollary 3.2 and we remove ends of paths in \( SL_1 \) in constructing leaves.

Corollary 3.3. Suppose \( P \in SL_j \), \( j \in [m] \). Then, the tree \( T_P \) associated with \( P \) constructed above satisfies the following properties.

(a) \( d_\rho = d_0 \) for \( j = m \).
(b) Suppose \( L(u) \leq j - 2 \) and \( u \neq \rho \) as \( j = m \). Then, \( d_u \geq K \). For any child \( v \) of \( u \), \( \|v\| \geq \frac{1}{d_u}\|u\| \). Furthermore, \( \|v\| \geq (1 - \frac{1}{K})\|u\| \) if \( u \) is untame, i.e. \( P^u \) is untame.
(c) Suppose \( L(u) = j - 1 \), then \( d_u \geq \frac{1}{2}\|u\| \).
(d) \( |\mathcal{L}| \leq |P| \).

3.2 The total flow through untame nodes

Recall any \( P \in SL_j \) is associated with a tree \( T_P \), for all \( j \in [m] \). Let \( \theta_P \) be the unit uniform flow on \( T_P \) from \( \rho \) to \( \mathcal{L} \), with \( \theta_P(\rho) = 1 \) and \( \theta_P(v) = \frac{1}{d_u} \theta_P(u) \) if \( v \) is a child of \( u \). In this subsection, we show the following proposition.

**Proposition 3.4.** Let \( \mathcal{P}_{\kappa, \delta, K} := \{P : \|P\| \geq \kappa N, |P| \leq N^{1 + \frac{\delta}{K^2}} \delta m \} \). Then, for any \( P \in \mathcal{P}_{\kappa, \delta, K} \),

\[
\sum_{u : 1 \leq L(u) \leq m - 1} \theta_P(u)1_{\{u \text{ is untame}\}} \leq 2\delta m.
\]
Proof. Note $\mathcal{P}_{\kappa, \delta, K} \subset \mathcal{S}L_m$. For any leaf $v$, denote nodes on the ray in $T_P$ from $\rho$ to $v$ by $v_0(= \rho), v_1, \cdots, v_{m-1}, v_m(= v)$ in order. Let $C_v = \{|1 \leq r \leq m - 1 : v_r$ is untame$\}$. Then, by (b) and (c) in Corollary 3.3

$$
\prod_{r=1}^{m-2} \frac{\|v_{r+1}\|}{\|v_r\|} \geq (1 + \frac{1}{K^2})^{C_v-1} \prod_{r=1}^{m-2} \frac{1}{d_{v_r}}, \quad \frac{1}{\|v_{m-1}\|} \geq \frac{1}{2d_{v_{m-1}}}.
$$

Note $\|v_1\| \geq K^{m-1}$, since $P^{v_1} \in \mathcal{S}L_{m-1}$. It follows that

$$
\frac{1}{K^{m-1}} \geq \frac{1}{\|v_1\|} = (\prod_{r=1}^{m-2} \frac{\|v_{r+1}\|}{\|v_r\|}) \frac{1}{\|v_{m-1}\|} \geq \frac{1}{2}(1 + \frac{1}{K^2})^{C_v-1} \prod_{r=1}^{m-1} \frac{1}{d_{v_r}}.
$$

By definition, $\theta_P(v) = \frac{1}{d_0} \prod_{r=1}^{m-1} \frac{1}{d_{v_r}}$, and $d_0 \geq \frac{1}{4}\kappa K$ for $N \geq K^m$. Consequently,

$$
\theta_P(v) \leq \frac{2}{K^{m-1}}d_0 \frac{K^2}{(K^2 + 1)^{C_v-1}} \leq \frac{8}{\kappa K^m}(1 - \frac{1}{K^2 + 1})^{C_v} \leq \frac{8}{\kappa K^m}e^{-\frac{1}{K^2+1}C_v}. \tag{19}
$$

It follows that

$$
\sum_{u : 1 \leq L(u) \leq m-1} \theta_P(u)1_{\{u \text{ is untame}\}} = \sum_{1 \leq L(u) \leq m-1} \left( \sum_{v \in \mathcal{L}} \theta_P(v)1_{\{u \text{ is an ancestor of } v\}} \right)1_{\{u \text{ is untame}\}} = \sum_{v \in \mathcal{L}} \theta_P(v)C_v = \sum_{v \in \mathcal{L}} \theta_P(v)C_v1_{\{C_v \geq \delta m\}} + \sum_{v \in \mathcal{L}} \theta_P(v)C_v1_{\{C_v < \delta m\}} \leq \frac{8}{\kappa K^m}e^{-\frac{1}{K^2+1}\delta m}|\mathcal{L}|m + \delta m,
$$

where in the last inequality, we use $C_v \leq m$ as well as (19) for the case of $C_v \geq \delta m$, and $\sum_{v \in \mathcal{L}} \theta_P(v) = 1$ for the case of $C_v < \delta m$.

Finally, we show that $\frac{8}{\kappa K^m}e^{-\frac{1}{K^2+1}\delta m}|\mathcal{L}| \leq \delta$, which yields the result. By (d) in Corollary 3.3 and the assumption $P \in \mathcal{P}_{\kappa, \delta, K}$, we have $|\mathcal{L}| \leq |P| \leq N^{1+\frac{\delta}{\kappa}} \leq K((m+1)(1+\frac{\delta}{\kappa}))$. Therefore,

$$
\frac{8}{\kappa K^m}e^{-\frac{1}{K^2+1}\delta m}|\mathcal{L}| \leq \frac{8}{\kappa} K^{1+\frac{\delta}{\kappa}} e^{-\frac{1}{K^2+1}\delta m} \leq \delta
$$

since $K^{1+\frac{\delta}{\kappa}}e^{-\frac{1}{K^2+1}} < 1$ and $m \to \infty$. \qed

4 Multi-scale analysis on hierarchical structure of the path

In this section, we will prove Theorem 1.2. To this end, we say a path $P$ in $\mathcal{S}L_j$ is open if $\eta_j(z) \geq \varepsilon k$ for some $z \in P \cap \mathbb{Z}^2$, where $\varepsilon > 0$ is to be chosen and $\eta_j$ is defined before (6). We say a node $u$ is open if $P^u$ is. We will show that with high probability, each tame node has a small fraction of open children, which is Proposition 4.1. Combining this with Proposition 3.4, we will then show in Proposition 4.3 that open nodes in the tree are rare. In subsection 4.3, we will show that many leaves have a small number of open ancestors. For such a leaf $z$, $\eta_j(z) \leq \varepsilon k$ for most $j$. Then, we will show $\eta_{V_N}(z) \leq 15\sqrt{\delta} \log N$, by choosing $\varepsilon$ according to $\delta$. Consequently, Theorem 1.2 will follow by symmetry.
4.1 The fraction of open subpaths of a tame path

Let us begin with some definitions. Suppose $\varepsilon > 0$, which will be chosen later. We say a path in $\mathcal{S}L_j$ is open if $\eta_j(z) \geq \varepsilon k$ for some $z \in \mathbb{Z}^2$ on it. Suppose $P \in \mathcal{S}L_j$. Let $P^{(i)}$, $i \in [d_P]$ be the subpaths constructed in Proposition 3.1 (or Corollary 3.2). Define

$$\Delta_P = \frac{1}{d_P} |\{i \in [d_P] : P^{(i)} \text{ is open}\}|.$$

We would like to mention that $\Delta_P$ for $P \in \mathcal{S}L_j$ relies on the field $\eta_{j-1}$, since $P^{(i)} \in \mathcal{S}L_{j-1}$. In what follows, we will often deal with tame paths $P$'s with fixed ends, so that $\Delta_P$'s rely on the Gaussian field in a local region. Define

$$P_j(B_1, B_2) := \{P \in \mathcal{S}L_j : x_P \in B_1, y_P \in B_2\}, \quad T_j(B_1, B_2) := \{P \in P_j(B_1, B_2) : P \text{ is tame}\}.$$

Denote

$$\text{END}_j := \{(B_1, B_2) \in \mathcal{B}D_{Kj-2} \times \mathcal{B}D_{Kj-2} : P_j(B_1, B_2) \neq \emptyset\}.$$

In fact, we regard each box in $\mathcal{B}D_{Kj-2}$ as an end-box of paths in $\mathcal{S}L_j$, and we classify $\mathcal{S}L_j$ via end-boxes. The main goal in this subsection is to show $\Delta_P$'s for paths with a fixed pair of end-boxes are uniformly small (see Proposition 4.1). Later, we will use (7) and employ a large deviation estimate to prove Proposition 4.3 via counting possible pairs of end-boxes.

**Proposition 4.1.** There exist universal constants $C_7$ and $C_8$ such that for any $j \in [m-1]$ and $(B_1, B_2) \in \text{END}_j$,

$$\mathbb{P}(\Delta_P \geq \delta \text{ for some } P \in T_j(B_1, B_2)) \leq K^2 e^{-C_8 \varepsilon^2 k^2 \delta^2}, \text{ for all } \delta \geq \frac{C_8}{\sqrt{k\varepsilon}}.$$

To this goal, we rescale $\mathbb{R}^2$ by regarding $K^{j-1}$ as the unit. Then, under the new scale, paths in $T_j(B_1, B_2)$ all lie in the same ellipse with width $O(K)$ and height $O(1)$, roughly speaking. This ellipse is what the set $D$ below stands for, where (20) below describes the width and the height. We will first show in Lemma 4.2 that a stronger result holds in the new scale of $\mathbb{R}^2$, and then prove Proposition 4.1.

Suppose $D \subset \mathbb{Z}^2$. Let $D_{1,a} := |\{b \in \mathbb{Z} : (a, b) \in D\}|$, $D_1 = \max_{a \in \mathbb{Z}} D_{1,a}$, $D_{2,b} = |\{a \in \mathbb{Z} : (a, b) \in D\}|$, and $D_2 = \max_{b \in \mathbb{Z}} D_{2,b}$.

**Lemma 4.2.** Let $c_1, c_2, c_3 > 0$. Suppose $D \subset \mathbb{Z}^2$ satisfies

$$|D| \leq c_1 K, \quad D_1 \wedge D_2 \leq c_1. \quad (20)$$

Suppose $\varphi_r$, $0 \leq r \leq k-1$ are mean zero Gaussian fields in $D$ satisfying the following (i)-(iii).

(i) For any $z \in D$, $\varphi_0(z), \ldots, \varphi_{k-1}(z)$ are independent. For any $r$, $\varphi_r(z)$ is independent with $\varphi_{r'}(w)$ if $|z-w|_\infty \geq 2^{r+1}$.

(ii) $\mathbb{E}\varphi_r(z)^2 \leq c_2$ for all $r, z$.

(iii) For any $z, w \in D$, $\mathbb{E}\varphi(z) \varphi(w) \geq -c_3$, where $\varepsilon := \sum_{r=0}^{k-1} \varphi_r$. 

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Then, for \( k \geq c_3 \vee 6 \),
\[
\Pr \left( \left\{ z \in D : \varphi(z) \geq \varepsilon k \right\} \geq \delta K \right) \leq 5ke^{-c_2k^2\delta^2}, \quad \text{for all } \frac{8c_1\sqrt{2c_2}}{\sqrt{k\varepsilon}} \leq \delta \leq 1,
\]
where \( c = \min\{\frac{1}{128c_3}, \frac{1}{128c_3^{\frac{1}{2}}}, \frac{1}{2c_1c_2}\} \).

**Proof.** To investigate the event \( \left\{ z \in D : \varphi(z) \geq \varepsilon k \right\} \geq \delta K \), we will compare the value of \( \sum_{z \in D} \varphi(z) \) with \( \varepsilon k \delta K \). To this goal, we add a common variable to the field \( \varphi \) in order to obtain a positively correlated Gaussian field and employ the FKG inequality. Let \( Z \sim N(0, c_3) \) be independent of the Gaussian field \( \varphi \). Let
\[
E_\delta = \left\{ \left\{ z \in D : \varphi(z) + Z \geq \frac{7}{8} \varepsilon k \right\} \geq \delta K \right\}.
\]

Then,
\[
\Pr \left( \left\{ z \in D : \varphi(z) \geq \varepsilon k \right\} \geq \delta K \right) \leq P(-Z \geq \frac{1}{8} \varepsilon k) + \Pr(E_\delta) \leq 2e^{-\frac{1}{128c_3}c_2k^2\delta^2} + \Pr(E_\delta),
\]
where in the second inequality, we use Lemma 2.5 and (20).

By independence,
\[
E_\delta = \left\{ \left\{ z \in D : \varphi(z) \geq \varepsilon k \right\} \geq \delta K \right\},
\]
and \( \delta \leq 1 \). Next, we estimate \( \Pr(E_\delta) \). Define
\[
E = \left\{ \sum_{z \in D} \left| \varphi(z) + Z \right| 1_{\{\varphi(z) + Z < 0\}} \leq 2c_1\sqrt{2c_2kK} \right\}.
\]

By independence, \( \mathbb{E}(\varphi(z) + Z)^2 = \mathbb{E}(\varphi(z))^2 + \mathbb{E}Z^2 \leq c_2k + c_3 \leq 2c_2k \), where we use (i), (ii) as well as the assumption \( k \geq c_3/c_2 \). Consequently, \( \mathbb{E}(\varphi(z) + Z) \leq \sqrt{2c_2k} \). It follows that
\[
\Pr(E^c) \leq \Pr \left( \sum_{z \in D} \left| \varphi(z) + Z \right| > 2c_1\sqrt{2c_2kK} \right) \leq \frac{1}{2c_1\sqrt{2c_2kK}} |D| \sqrt{2c_2k} \leq \frac{1}{2},
\]
by (21). That is, \( \Pr(E) \geq \frac{1}{2} \). Since both \( E_\delta \) and \( E \) are increasing events of the positive correlated Gaussian field \( \{\varphi(z) + Z, z \in D\} \), by FKG inequality \( \Pr(E_\delta) \Pr(E) \leq \Pr(E_\delta E) \). Thus,
\[
\Pr(E_\delta) \leq \frac{\Pr(E_\delta E)}{\Pr(E)} \leq 2\Pr(E_\delta E) \leq 2\Pr \left( \sum_{z \in D} \left( \varphi(z) + Z \right) \geq \frac{7}{8} \varepsilon k \delta K - 2c_1\sqrt{2c_2kK} \right).
\]

Note \( \frac{7}{8} \varepsilon k \delta K - 2c_1\sqrt{2c_2kK} \geq \frac{5}{8} \varepsilon k \delta K \) since \( \delta \geq 8c_1\sqrt{2c_2}/(\sqrt{k\varepsilon}) \) by assumption. Then, it follows
\[
\Pr(E_\delta) \leq 2\Pr \left( |D| Z \geq \frac{1}{8} \varepsilon k \delta K k \right) + 2\Pr \left( \sum_{z \in D} \varphi(z) \geq \frac{1}{2} \varepsilon k \delta K k \right)
\leq 4e^{-\frac{1}{c_2c_3}c_2k^2\delta^2} + 2\Pr \left( \sum_{z \in D} \varphi(z) \geq \frac{1}{2} \varepsilon k \delta K k \right), \tag{22}
\]
where in the second inequality, we use Lemma 2.5 and (20).

Finally, we estimate \( \Pr \left( \sum_{z \in D} \varphi(z) \geq \frac{1}{2} \varepsilon k \delta K k \right) \). Denote
\[
\Gamma := \frac{1}{2} \varepsilon k \delta K k, \quad \Gamma_r := 2^{-\frac{1}{2}(k-r)-3} \varepsilon k \delta K k.
\]
Note \( \Gamma_r \leq (1 - \frac{1}{\sqrt{2}})^2 \frac{1}{2} (k-r) \Gamma \). So, \( \sum_{r=0}^{k-1} \Gamma_r \leq \Gamma \). It follows that
\[
P \left( \sum_{z \in D} \varphi(z) \geq \Gamma \right) \leq \sum_{r=0}^{k-1} P \left( \sum_{z \in D} \varphi_r(z) \geq \Gamma_r \right).
\] (23)

Next, we fix \( r \) and deal with \( P(\sum_{z \in D} \varphi_r(z) \geq \Gamma_r) \) by calculating \( E(\sum_{z \in D} \varphi_r(z))^2 \). By (i) and (20), for any \( z \), there are at most \( c_1 2^{r+2} \) points \( w \in D \) such that \( \varphi_r(w) \) is not independent with \( \varphi_r(z) \). Hence, \( E(\varphi_r(z) \sum_{w \in D} \varphi_r(w)) \leq c_1 2^{r+2} \), by (ii). This, together with (20), yields that
\[
\sigma_r^2 := E \left( \sum_{z \in D} \varphi_r(z) \right)^2 = \sum_{z \in D} E \left( \varphi_r(z) \sum_{w \in D} \varphi_r(w) \right) \leq c_1 K c_2 2^{r+2} = c_1 2 c_2 2^{k+r+2}.
\]
By Lemma \[2.5\] for all \( r = 0, \ldots, k-1 \),
\[
P \left( \sum_{z \in D} \varphi_r(z) \geq \Gamma_r \right) \leq 2 e \frac{r^2}{2 \sigma_r^2} = 2 e \left( \frac{2-2^{-(k-r)}-6 \delta^2 k^2}{c_1 2 c_2 2^{k+r+2}} \right) = 2 e \frac{1}{2 \pi c_1^2 \sigma_r^2} \delta^2.
\]
This, together with (21), (22), (23) and the assumption \( k \geq 6 \), yields the result. \( \square \)

**Proof of Proposition 4.1**. Equivalently, we will show the result for \( j+1 \), where \( 0 \leq j \leq m-2 \).

Suppose \( (B_1, B_2) \in \text{END}_{j+1} \). Denote
\[
A = \bigcup_{P \in T_{j+1}(B_1, B_2)} P, \quad \hat{D} = \mathbb{Z}^2 \cap A, \quad \hat{D} = \{l_B : B \in \mathcal{BD}_{K_j}^A\},
\]
where \( \mathcal{BD}_{K_j}^A \) consists of boxes in \( \mathcal{BD}_{K_j} \) intersecting with \( A \), and \( l_B \) is the lower left corner of \( B \cap \mathbb{Z}^d \). By the definition of the same path, there is a universal constant \( \hat{C}_1 \) such that for all \( (B_1, B_2) \in \text{END}_{j+1} \),
\[
|\hat{D}| \leq \hat{C}_1 K^{2j+1}, \quad \hat{D}_1 \cap \hat{D}_2 \leq \hat{C}_1 K^j, \quad |\hat{D}| \leq \hat{C}_1 K, \quad \hat{D}_1 \cap \hat{D}_2 \leq \hat{C}_1.
\] (24)
We set \( c_1 = \hat{C}_1, c_2 = C_1 + 2 C_2, \) and \( c_3 = C_4 \) in Lemma \[4.2\]. Consequently, \( c \) therein is defined correspondingly, which is a universal constant here. Then \( k \geq (c_3/c_2) \sqrt{6} \) is satisfied, since \( K \) is large. Let \( c_7 = \min\{1, \frac{1}{18 \sqrt{8}}, \frac{1}{8 \sqrt{2}}, \frac{1}{8 \sqrt{2}}\} \), \( c_8 = 288 c_1 \sqrt{2} c_2 \), which are universal constants. In the following setting, (7), (8), Lemma \[2.2\] and \[24\] respectively correspond to (i)-(iii) and (20) in Lemma \[4.2\]. That is, the assumptions in Lemma \[4.2\] all hold. Therefore, we will directly apply Lemma \[4.2\].

For \( j = 0 \), let \( D = \hat{D} \) and \( \varphi_r := \varphi_r \). In this situation, \( \varphi = \eta_0 \). By Corollary \[3.2\], \( d_P \geq K/2 \) for all \( P \in T_1(B_1, B_2) \). It follows that
\[
P(\Delta_P \geq \delta \text{ for some } P \in T_1(B_1, B_2)) \leq \mathbb{P}(\{z \in D : \varphi(z) \geq \varepsilon k\} \geq \frac{1}{2} \delta K).
\]

By the definition of \( c_8 \), \( \delta \geq \frac{c_8}{\sqrt{\varepsilon k}} \) implies that \( \delta/2 \geq \frac{8 c_1 \sqrt{2} \varepsilon k^2}{\sqrt{\varepsilon k} / 2} \). We apply Lemma \[4.2\] to \( \delta/2 \) (rather than \( \delta \)), and conclude
\[
P(\Delta_P \geq \delta \text{ for some } P \in T_1(B_1, B_2)) \leq 5 k e^{-c_2 k^2 (\delta/2)^2} \leq K^2 e^{-C_7 \varepsilon^2 k^3 \delta^2},
\]
where in the last inequality we use the definition of \( C_7 \).
For \( j \geq 1 \), let \( D = K^{-j} \tilde{D} \) and \( \varphi_r(z) := X_{jk+r}(K^jz) \), for all \( z \in D \). In this situation, \( \varphi(z) = \eta_j(K^jz) \). We apply Lemma 4.2 to \( \varepsilon/2 \) (rather than \( \varepsilon \)). It follows that for \( \delta \geq \frac{8c_1\sqrt{c_2}}{\sqrt{k\varepsilon}/2} \),
\[
\mathbb{P}(\tilde{E}_\delta) \leq 5ke^{-c(\varepsilon/2)^2k^2\delta^2}, \quad \text{where } \tilde{E}_\delta := \left\{ |\{ B \in BD_{K,j}^A : \eta_j(l_B) \geq \frac{1}{2} \varepsilon k \} | \geq \delta K \right\}.
\]

Next, we estimate the fluctuation of \( \eta_j \) within a box \( B \in BD_{K,j} \). Denote \( M_B := \max_{z \in B \cap Z^2}(\eta_j(z) - \eta_j(l_B)) \). Then, by Lemma 2.3 and 2.4, \( \mathbb{E}M_B \leq \sqrt{C_5C_6} \), and \( \max_{z \in B \cap Z^2} \mathbb{E}(\eta_j(z) - \eta_j(l_B))^2 \leq C_5 \).

By Lemma 2.5,
\[
\mathbb{P}(M_B > \frac{1}{2} \varepsilon k) \leq \mathbb{P}(M_B - \mathbb{E}M_B > \frac{1}{3} \varepsilon k) \leq 2e^{-\frac{1}{18c_5} \varepsilon^2 k^2}.
\]

Recall \( |\tilde{D}| \leq \tilde{C}_1 K \) by (24), i.e. there are at most \( \tilde{C}_1 K \) boxes \( B \in BD_{K,j} \) intersecting with \( A \). By the union bound,
\[
\mathbb{P}(E^M) \leq \tilde{C}_1 K \times 2e^{-\frac{1}{18c_5} \varepsilon^2 k^2}, \quad \text{where } E^M := \left\{ M_B > \frac{1}{2} \varepsilon k \text{ for some } B \in BD_{K,j} \right\}.
\]

Note
\[
\tilde{E}_\delta \subseteq E^M \cup \tilde{E}_\delta, \quad \text{where } \tilde{E}_\delta := \{|\{ B \in BD_{K,j}^A : \eta_j(z) \geq \varepsilon k \text{ for some } z \in B \}| \geq \delta K \}.
\]

It follows that
\[
\mathbb{P}(\tilde{E}_\delta) \leq \mathbb{P}(E^M) + \mathbb{P}(\tilde{E}_\delta) \leq 2\tilde{C}_1 Ke^{-\frac{1}{18c_5} \varepsilon^2 k^2} + 5ke^{-\frac{1}{4}c\varepsilon^2 k^2\delta^2}, \quad \text{for all } \delta \geq \frac{8c_1\sqrt{c_2}}{\sqrt{k\varepsilon}/2}.
\]

Finally, by Properties (a) and (c) in Proposition 5.1 the event \( \Delta_P \geq \delta \) for some \( P \in T_{j+1}(B_1, B_2) \) implies that \( \tilde{E}_{\delta/12} \) happens. For \( \delta \geq C_8/\sqrt{k\varepsilon} \), it holds that \( \delta/12 \geq \frac{8c_1\sqrt{c_2}}{\sqrt{k\varepsilon}/2} \). Consequently,
\[
\mathbb{P}(\Delta_P \geq \delta \text{ for some } P \in T_{j+1}(B_1, B_2)) \leq 2\tilde{C}_1 Ke^{-\frac{1}{18c_5} \varepsilon^2 k^2\delta^2} + 5ke^{-\frac{1}{4}c\varepsilon^2 k^2(\delta/12)^2} \leq K e^{-C_7\varepsilon^2 k^2\delta^2},
\]
completing the proof. \( \square \)

4.2 The fraction of points with high Gaussian values

Recall any \( P \in \mathcal{S}\mathcal{L}_j \) is associated with a tree \( T_P \) of depth \( j \), and \( \theta_P \) is the unit uniform flow on \( T_P \). A node \( u \in T_P \) is identified with a subpath \( P^u \) of \( P \), and \( u \) is said to be tame/open if \( P^u \) is. Denote \( \Delta_u = \Delta_{P^u} \), which depends on \( \eta_{j-1} \) if \( P^u \in \mathcal{S}\mathcal{L}_j \). Recall \( \mathcal{P}_{\kappa, \delta, K} \) is defined in Proposition 3.4 and \( \mathcal{P}_{\kappa, \delta, K} \subseteq \mathcal{S}\mathcal{L}_m \). In this subsection, we aim to show the following proposition.

**Proposition 4.3.** \( \lim_{m \to \infty} \mathbb{P}\left( \sum_{u \in T_P \setminus \{\rho\}} \theta_P(u)1_{\{u \text{ is open}\}} \leq 4\delta m, \text{ for all } P \in \mathcal{P}_{\kappa, \delta, K} \right) = 1. \)

We express \( \sum_{u \in T_P \setminus \{\rho\}} \theta_P(u)1_{\{u \text{ is open}\}} \) in terms of
\[
Y_{P,r} := \sum_{u \in T_{P,r}} \theta_P(u)1_{\{u \text{ is tame}\}},
\]
(25)
where \( \mathcal{T}_{P,r} := \{ u \in \mathcal{T}_P : L(u) = r \} \). Note that \( Y_{P,r} \)'s satisfy the following recursion formula:

\[
Y_{P,r+1} = \frac{1}{d} \sum_{i=1}^{d} \sum_{u \in \mathcal{T}_{P,(i),r}} \theta \mathcal{P}(u) \Delta_u 1_{\{u \text{ is tame}\}} = \frac{1}{d} \sum_{i=1}^{d} Y_{P(i),r},
\]

where \( d = d_P \) and \( P(i), i \in [d] \) are subpaths of the path \( P \) constructed in Proposition 3.1 or Corollary 3.2. Denote

\[
\xi_{r,j,B_1,B_2} := \max \{ Y_{P,r}, P \in \mathcal{P}_j(B_1,B_2) \}.
\]

Recall Property (c) of Proposition 3.1. We let \( \text{END}_{j,d} \) consist of sequences \( (B_{i,1},B_{i,2}), i \in [d] \) in \( \text{END}_j \) such that the 12-times-rule holds, namely,

\[
\text{END}_{j,d} := \{ \{(B_{i,1},B_{i,2})\}_{i \in [d]} \subset \text{END}_j : |\{ i : B_{i,1} \subset B \}| \leq 12, \text{ for all } B \in \mathcal{BD}_{K_j} \}.
\]

**Lemma 4.4.** Denote \( \beta = 1/2^9 \),

\[
c_r = C_7 \varepsilon^2 (\beta K)^r, \quad \delta_0 = \frac{C_k \sqrt{2/C_7}}{\varepsilon \sqrt{k}}, \quad \delta_r = \delta_0 + \frac{4(1+2/2s/3)}{v_c \beta K}.
\]

Then, the following (i) and (ii) hold for \( j \in [m-1] \) and \( 0 \leq r \leq j-1 \).

(i) Suppose \( (B_1,B_2) \in \text{END}_j \). Then, \( \xi := \xi_{r,j,B_1,B_2} \) satisfies

\[
\mathbb{P}(\xi > \delta) \leq 2e^{-c_r k^2 (\delta - \delta_0)^2}, \quad \text{for all } \delta \geq \delta_r.
\]

(ii) Let \( d \geq 1 \). Suppose \( \{(B_{i,1},B_{i,2})\}_{i \in [d]} \in \text{END}_{j,d} \), and denote \( \xi_i = \xi_{r,j,B_{i,1},B_{i,2}}, i \in [d] \). Then,

\[
\mathbb{P}\left( \frac{1}{d}(\xi_1 + \cdots + \xi_d) > \delta \right) \leq \left( K^{-16} e^{-\beta c_r k^2 (\delta - \delta_{r+1})^2} \right)^d, \quad \text{for all } \delta \geq \delta_{r+1}.
\]

**Proof.** We prove the results by the induction of \( r \). In Step 1, we will check (i) for \( r = 0 \) and all \( j \in [m-1] \). In Step 2, we will show that (i) implies (ii). Then, in Step 3, we will show (i) for \( r+1 \) and all \( j \in [r+2,m-1] \cap \mathbb{Z} \), provided that (ii) holds for all \( j \in [r+1,m-1] \cap \mathbb{Z} \). Assuming these, we obtain the lemma, by the induction.

**Step 1.** Let \( r = 0 \). Note \( Y_{P,0} = \Delta_P 1_{\{P \text{ is tame}\}} \), hence \( \xi = \max \{ \Delta_P, P \in \mathcal{T}_j(B_1,B_2) \} \). We need to check (i) for all \( j \in [m-1] \). This follows from Proposition 4.1 directly, since \( \delta \geq \delta_0 \) implies \( K^2 e^{-C_7 \varepsilon^2 k^2 \delta^2} \leq e^{-2k-C_7 \varepsilon^2 k^2 \delta_0^2-C_7 \varepsilon^2 k^2 (\delta-\delta_0)^2} \leq 2e^{-c_r k^2 (\delta-\delta_0)^2} \).

**Step 2.** Assuming (i) holds, we will show (ii). By the definition of \( \mathcal{SL}_j \) for \( j \leq m-1 \), \( P \subset B(x_P,2K^j) \). If \( d_{\infty}(B_{i,1},B_{i,1}) \geq 5K^j \), we have \( d_{\infty}(\cup_{P \in \mathcal{P}_j(B_{i,1},B_{i,2})} P, \cup_{P \in \mathcal{P}(B_{i,1},B_{i,2})} P) \geq K^j \). Consequently, \( \xi_i \) and \( \xi_{i'} \) are mutually independent by \( 7 \), since they rely on the field \( \eta_{j-1-r} \).

Next, we classify \( \xi_i \)'s into 432 groups, such that \( \xi_i \)'s in each group are mutually independent. Concretely, we classify \( \mathcal{BD}_{K_j} \) into 36 families \( \mathcal{F}_s, s \in [36] \), where \( \mathcal{F}_1 \) consists of boxes respectively containing \( (6aK^j,6bK^j) \), \( a,b \in \mathbb{Z} \) and other \( \mathcal{F}_s \)'s are their shifts. Accordingly, let \( \mathcal{G}_s := \{ B_{i,1} : B_{i,1} \subset B \text{ for some } B \in \mathcal{F}_s \} \). By the 12-times-rule, we can classify each \( \mathcal{G}_s \) into 12 groups \( \mathcal{G}_{s,t}, t \in [12] \), such that for any \( s,t \), a box in \( \mathcal{F}_s \) contains at most one \( B_{i,1} \) in \( \mathcal{G}_{s,t} \). Finally, we classify \( B_{i,1} \)'s (equivalently, \( \xi_i \)'s) into 432 (\( \leq 2^9 = 1/\beta \)) groups \( \mathcal{G}_{s,t} \), such that all \( \xi_i \)'s in each \( \mathcal{G}_{s,t} \) are mutually independent.
Denote \( W_{s,t} := \prod_{B_{i,1} \in G_{s,t}} e^{\alpha \beta (\xi_i - \delta)} \), where \( \alpha > 0 \). Then

\[
\mathbb{E}e^{\alpha \beta \sum_{i} (\xi_i - \delta)} = \prod_{s=1}^{36} \prod_{t=1}^{12} W_{s,t} \leq \prod_{s=1}^{36} \prod_{t=1}^{12} \left( \mathbb{E}W_{s,t}^{1/\beta} \right)^{\beta} \leq \prod_{s=1}^{36} \prod_{t=1}^{12} \prod_{B_{i,1} \in G_{s,t}} \left( \mathbb{E}e^{\alpha (\xi_i - \delta)} \right)^{\beta} = \prod_{i=1}^{d} \left( \mathbb{E}e^{\alpha (\xi_i - \delta)} \right)^{\beta}.
\]

It follows that for any \( \alpha > 0 \),

\[
P \left( \frac{1}{d} (\xi_1 + \cdots + \xi_d) > \delta \right) \leq \mathbb{E}e^{\alpha \beta \sum_{i} (\xi_i - \delta)} \leq \prod_{i=1}^{d} \left( \mathbb{E}e^{\alpha (\xi_i - \delta)} \right)^{\beta}.
\]

Denote \( \xi = \xi_i \) for any \( i \). Next, we estimate \( \mathbb{E}e^{\alpha \xi} \). By (i) and the fact \( 0 \leq \xi \leq 1 \), it holds that

\[e^{\alpha \xi} = e^{\alpha \delta r} + \int_{\delta r}^{1} P(\xi > z)ae^{az} dz \leq e^{\alpha \delta r} + 2ae^{\alpha \delta r} \int_{-\infty}^{\infty} e^{-c_r k^2 (z - \delta r)^2} e^{a(z - \delta r)} dz \leq e^{\alpha \delta r} + \frac{a}{\sqrt{c_r}} e^{\alpha \delta r + a^2/4c_r k^2} \leq (1 + \frac{a}{\sqrt{c_r}}) e^{\frac{a^2}{4c_r k^2} + \alpha \delta r}.
\]

Consequently, \( \mathbb{E}e^{\alpha (\xi - \delta)} \leq (1 + \frac{a}{\sqrt{c_r}}) e^{\frac{a^2}{4c_r k^2} - a((\delta - \delta r)} \). For any \( \delta \geq \delta r \), we set \( a = 2c_r k^2 (\delta - \delta r) \) to optimize the exponent. Then,

\[\mathbb{E}e^{\alpha (\xi - \delta)} \leq (1 + \frac{2c_r k^2 (\delta - \delta r)}{\sqrt{c_r}}) e^{-c_r k^2 (\delta - \delta r)^2} \leq 3\sqrt{c_r k^2} e^{-c_r k^2 (\delta - \delta r)^2}.
\]

Note

\[\delta_{r+1} - \delta_r = a_r + b_r, \quad \text{where} \quad a_r = \frac{4}{\sqrt{c_r \beta K}}, \quad b_r = \frac{1 + \sqrt{2r/3}}{\sqrt{c_r \beta K}},
\]

\[e^{-\beta c_r k^2 a_r^2} = e^{-16k} \leq K^{-16},
\]

and

\[3\sqrt{c_r k^2} e^{-\beta c_r k^2 b_r^2} = 3\sqrt{C_T e} k^2 (\beta K)^2 e^{-1(1+2r/3)k} \leq 3\sqrt{C_T e} k^2 e^{-\frac{1}{3}k} \leq 1.
\]

Combining the four formulas above, we conclude that for \( \delta \geq \delta_{r+1} \),

\[\left( \mathbb{E}e^{\alpha (\xi - \delta)} \right)^{\beta} \leq 3\sqrt{c_r k^2} e^{-\beta c_r k^2 ((\delta - \delta_{r+1})^2 + a_r^2 + b_r^2)} \leq K^{-16} e^{-\beta c_r k^2 (\delta - \delta_{r+1})^2}.
\]

The above inequality holds for \( \xi = \xi_i \), for all \( i \). This, together with (27), implies (ii).

**Step 3.** Assume (ii) holds for all \( r + 1 \leq j \leq m - 1 \). Then, we will show (i) for \( r + 1 \) and all \( m - 1 \leq j \leq r + 2 \). Suppose \((B_1, B_2) \in \text{END}_j \). First, we fix the end-boxes of the subpaths. Concretely, for \( d \geq K \) and any sequence \( S := \{(B_{i,1}, B_{i,2})\}_{i \in [d]} \) in \( \text{END}_{j-1,d} \), we denote

\[P_{j,d} := \{ P \in P_j(B_1, B_2) : d_P = d \}, \quad \zeta_d := \max \{ Y_{P,r} : P \in P_{j,d} \},
\]

\[P_{j,S} := \{ P \in P_{j,d} : P^{(i)} \in P_{j-1}(B_{i,1}, B_{i,2}), \text{ for all } i \in [d] \}, \quad \zeta_S := \max \{ Y_{P,r} : P \in P_{j,S} \}.
\]
By (26), \( \zeta \leq \frac{1}{d} \sum_{i=1}^{d} \xi_i \), where \( \xi_i := \xi_{r,j-1,B_i,1,0} \), \( i \in [d] \). For any \( r + 2 \leq j \leq m - 1 \), we have \( r + 1 \leq j - 1 \leq m - 1 \). For any \( \delta \geq \delta_{r+1} \), we apply (ii) to \( j - 1 \), and have

\[
P(\zeta_s > \delta) \leq \left( K^{-16} \frac{e^{-\beta \varepsilon c k^2(\delta - \delta_{r+1})}}{2} \right)^d, \quad \text{for all } S \in \text{END}_{j-1,d}.
\]

By the definition of \( \text{END}_{j-1} \) and \( S\mathcal{L}_j \) for \( j \leq m - 1 \), we have \( B_{i,1}, B_{i,2} \in \mathcal{BD}_{K^3} \) and there are at most \( K^7 \) boxes in \( \mathcal{BD}_{K^3} \) intersecting with some path \( P \in \mathcal{P}_j(B_1, B_2) \). Therefore, we can find at most \( K^{14d} \) sequences \( S_i \in \text{END}_{j-1,d} \) such that \( \mathcal{P}_{j,d} \subset \bigcup_i \mathcal{P}_{j,S_i} \). By union bound,

\[
P(\zeta_d > \delta) \leq K^{14d} \max_{i} P(\zeta_{S_i} > \delta) \leq \left( K^{-16} \frac{e^{-\beta \varepsilon c k^2(\delta - \delta_{r+1})}}{2} \right)^d.
\] \hfill (28)

Note \( K^{-2} e^{-\beta \varepsilon c k^2(\delta - \delta_{r+1})^2} \leq K^{-2} \leq \frac{1}{2} \). By Property (a) of Proposition 3.1

\[
P(\xi > \delta) \leq \sum_{d=K}^{\infty} \sum_{v \in \mathcal{P}_{r-\delta}(\rho)} \theta_{P}(v) \mathbf{1}_{\{v \text{ is open}\}} \leq \sum_{u \in \mathcal{P}_{r-\delta}(\rho)} \sum_{v \text{ is an open child of } u} \theta_{P}(v) \Delta_u \leq \theta_{P}(\rho) \sum_{1 \leq L(u) \leq m-1} \theta_{P}(u) \Delta_u + \sum_{1 \leq L(u) \leq m-1} \theta_{P}(u) \mathbf{1}_{\{u \text{ is untame}\}},
\]

where we use \( \Delta_u \leq 1 \) for all \( u \). By Proposition 3.4 and the fact \( \theta_{P}(\rho) = 1 \leq \delta m \),

\[
\sum_{v \in \mathcal{P}_{r-\delta}(\rho)} \theta_{P}(v) \mathbf{1}_{\{v \text{ is open}\}} \leq \sum_{r=1}^{m-1} \mathbf{Y}_{P,r} + 3 \delta m, \quad \text{for all } P \in \mathcal{P}_{\kappa, \delta, K}.
\]

Therefore, to prove Proposition 4.3, we only need to check

\[
\lim_{m \to \infty} P \left( \sum_{r=1}^{m-1} \mathbf{Y}_{P,r} > \delta m \text{ for some } P \in \mathcal{P}_{\kappa, \delta, K} \right) = 0. \quad \hfill (29)
\]

Now we check (29). Recall \( d_P = d_0 \), and there are at most \( K^8 \) boxes in \( \mathcal{BD}_{m-3} \) intersecting with \( V_N \) for \( N < K^{m+1} \). Then, we can follow the arguments in Step 3 in the proof of Lemma 4.4 and conclude

\[
P(\mathbf{Y}_{P,r+1} > \delta \text{ for some } P \in \mathcal{P}_{\kappa, \delta, K}) \leq K^{16d_0} \left( K^{-16} e^{-\beta \varepsilon c k^2(\delta - \delta_{r+1})^2} \right)^{d_0} = e^{-d_0 \beta \varepsilon c k^2(\delta - \delta_{r+1})^2}
\]

for all \( \delta \geq \delta_{r+1} \), similar to (28). Consequently, since \( d_0 \geq \frac{1}{2} \kappa K \),

\[
P(\mathbf{Y}_{P,r+1} > \delta \text{ for some } P \in \mathcal{P}_{\kappa, \delta, K}) \leq e^{-\frac{\kappa^2}{2} c_{r+1}(\delta - \delta_{r+1})^2}, \quad \text{for all } \delta \geq \delta_{r+1}.
\] \hfill (30)
This holds for all \( r \geq 0 \), i.e. \( r + 1 \geq 1 \).

Recall \( c_r = C_r \varepsilon^2 (\beta K)^r \), where \( \beta = 1/2^9 \). The definition of \( \delta_r \) in Lemma 4.4 implies that 
\[
\delta_r \leq \frac{1}{r} \delta(K) \quad \text{for all } r \geq 0,
\]
with
\[
\delta(K) := \frac{1}{\sqrt{K}} \left( C_8 \sqrt{\frac{2}{C_7}} + \sum_{s=0}^{\infty} 4 + \frac{1}{\sqrt{2s+3}} \frac{1}{\sqrt{C_7 \beta (\beta K)^s}} \right).
\]

Note \( \delta(K) < \frac{1}{2} \delta \epsilon \), since \( K \) is large enough. It follows that \( \sum_{r=0}^{m-1} (\delta_r + \frac{1}{4} 2^{-r} m \delta) \leq m \frac{1}{2} \delta + \frac{1}{2} m \delta = m \delta \).

Therefore, by (30),
\[
\text{Therefore, (29) holds.}
\]

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Therefore, by (30),
\[
\text{Therefore, (29) holds.}
\]

For \( K > 4 \epsilon / \beta \), the right hand side above is less than
\[
\sum_{r=1}^{m-1} e^{-\frac{1}{32} C_r \varepsilon^2 (\frac{1}{4} 2^{-r} m \delta)^2} < \frac{1}{e^{\frac{1}{32} C_r \varepsilon^2 (m \delta)^2} - 1},
\]
which converges to 0 as \( m \to \infty \). Therefore, (29) holds. \( \square \)

### 4.3 Proof of Theorem 1.2

Recall that any \( P \in \mathcal{P}_{\kappa, \delta, K} \) is associated with a tree with depth \( m \). For a leaf \( v \), denote by \( O_v \) the number of open ancestors of \( v \) (including itself, excluding the root). We say \( v \) is heavy if \( O_v \geq 8 \delta m \).

Define
\[
E_1 := \left\{ \sum_{v \in \mathcal{L}} \theta_P(v) 1_{\{v \text{ is heavy}\}} \leq \frac{1}{2}, \quad \text{for all } P \in \mathcal{P}_{\kappa, \delta, K} \right\},
\]
\[
E_2 := \left\{ \sum_{i=1}^{r} \eta_{j_i}(z) \leq \sqrt{320 C_1 \delta \log N}, \quad \text{for all } z \in V_N, r \leq 8 \delta m \text{ and } 0 \leq j_1 < \cdots < j_r \leq m-1 \right\},
\]
\[
E_3 := \left\{ \max_{u \in V_N} H_{K^m}(u) \leq \epsilon \log N \right\}.
\]

On one hand, suppose \( E_1 \) happens. By (b) of Corollary 3.3, \( \theta_P(v) \leq 2 \frac{1}{K} \frac{1}{d_0} = 2 K^{-1} \frac{1}{d_0} \geq \frac{1}{8} \kappa N \) un-heavy leaves on \( \mathcal{L} \), for each \( P \in \mathcal{P}_{\kappa, \delta, K} \).

On the other hand, suppose \( E_2 \) and \( E_3 \) happen. We call \( z \) a good point if \( \eta^{5N}(z) \leq C \sqrt{\delta} \log N \), where \( C := \sqrt{320 C_1 + 2} \). Set \( \epsilon = \frac{1}{4} \sqrt{\delta} \). Then, for any un-heavy leaf \( v \), \( z := P^v \) is a good point since
\[
\eta^{5N}(z) = \eta(z) + H_{K^m}(z) \leq \sum_{j=0}^{m-1} \eta_j(z) 1_{\{\eta_j(z) \geq \epsilon k\}} + \sum_{j=0}^{m-1} \eta_j(z) 1_{\{\eta_j(z) < \epsilon k\}} + \epsilon \log N
\]
\[
\leq \sqrt{320 C_1 \delta \log N} + m \epsilon k + \epsilon \log N = (\sqrt{320 C_1 \delta} + 3 \epsilon) \log N \leq C \sqrt{\delta} \log N.
\]
Therefore, on $E_1 \cap E_2 \cap E_3$, we can find at least $\frac{1}{8}KN$ good points on $P$, for each $P \in \mathcal{P}_{\kappa,\delta,K}$. We will show later that $\mathbb{P}(E_1), \mathbb{P}(E_2)$ and $\mathbb{P}(E_3)$ all converge to 1 as $m \to \infty$. It then follows by symmetry that $\lim_{N \to \infty} \mathbb{P}(\tilde{E}) = 1$, where

$$\tilde{E} := \left\{ \{z \in P : \eta^{V_N}(z) \geq -C\sqrt{\delta \log N}\} \geq \frac{1}{8}KN, \text{ for all } P \in \mathcal{P}_{\kappa,\delta,K} \right\}.$$ 

By Remark 2.1, $C_1 \leq 1/2$. Thus $C \leq \sqrt{320/2} + 2 = 15$. Taking $\alpha = \frac{\delta}{K^2}$, one has the result of Theorem 1.2.

Finally, we will show that $\mathbb{P}(E_1), \mathbb{P}(E_2)$ and $\mathbb{P}(E_3)$ all converge to 1 as $m \to \infty$, completing the proof.

**Lemma 4.5.** $\lim_{m \to \infty} \mathbb{P}(E_1) = 1$.

**Proof.** Note that

$$\sum_{u \in T \setminus \{\rho\}} \theta_P(u) \mathbb{1}_{\{u \text{ is open}\}} = \sum_{u \in T \setminus \{\rho\}} \sum_{v \in \mathcal{L}} \theta_P(v) \mathbb{1}_{\{u \text{ is an open ancestor of } v\}}$$

$$= \sum_{v \in \mathcal{L}} \theta_P(v) O_v \geq 8\delta m \sum_{v \in \mathcal{L}} \theta_P(v) \mathbb{1}_{\{v \text{ is heavy}\}}.$$ 

Consequently, $E_1$ happens if $\sum_{u \in T \setminus \{\rho\}} \theta_P(u) \mathbb{1}_{\{u \text{ is open}\}} \leq 4\delta m$ for all $P \in \mathcal{P}_{\kappa,\delta,K}$. By Proposition 2.3 the result holds. \(\square\)

**Lemma 4.6.** $\lim_{m \to \infty} \mathbb{P}(E_2) = 1$.

**Proof.** Denote $G(z) = \sum_{i=1}^{r} \eta_{j_i}(z)$. Note $\eta_j(z)$’s are mutually independent. By (5), $\mathbb{E}G(z)^2 = \sum_{i=1}^{r} \mathbb{E}^2 \eta_{j_i}(z) \leq r(C_1k + 2C_2) \leq 16C_1\delta \log N$, where we use $r \leq 8\delta m$. Thus,

$$\mathbb{P}(G(z) > \sqrt{320C_1\delta \log N}) \leq 2e^{-\frac{1}{32C_1\delta \log N} \cdot 320C_1\delta \log N} = 2N^{-10}.$$ 

A union bound implies $\mathbb{P}(G(z) > \sqrt{320C_1\delta \log N}$ for some $z \in V_N)$ $\leq 2N^{-8}$. It follows that

$$1 - \mathbb{P}(E_2) \leq \sum_{r \leq 8\delta m} \frac{m!}{r!(m-r)!} \times 2N^{-8} \leq a_\delta \sqrt{m} (b_\delta K^{-8})^m,$$

where $a_\delta$ and $b_\delta$ are constants depending on $\delta$. Then, the right hand side above converges to 0 as $m \to \infty$ since $K$ is large. \(\square\)

**Lemma 4.7.** $\lim_{m \to \infty} \mathbb{P}(E_3) = 1$.

**Proof.** Denote by $B_i$’s the boxes in $\mathcal{B}D_{K^m}$ intersecting with $V_N$. For any $i$, we have $u \in V_{5N}^N$ and $B_{K^m}(u) \subset V_{5N}$ for all $u \in B_i \cap V_N$. Then, by (2),

$$\mathbb{E}H_{K^m}(u)^2 = \mathbb{E} \eta^{V_N}(u)^2 - \mathbb{E} \eta^{B_{K^m}(u)}(u)^2 \leq C_1 \log_2 \frac{3N}{K^m} + 2C_2 \leq 2C_1k,$$

for all $u \in B_i \cap V_N$.

Denote $M_i := \max_{u \in B_i \cap V_N} H_{K^m}(u)$. By Lemma 2.3 and Lemma 2.4 $\mathbb{E}M_i \leq C_6 \sqrt{C_5}$ for all $i$. Consequently, by Lemma 2.5 it holds that

$$\mathbb{P}(M_i > \epsilon \log N) \leq \mathbb{P}(M_i - \mathbb{E}M_i > \frac{1}{2} \epsilon \log N) \leq 2e^{-\frac{1}{16C_1k} \epsilon^2 \log^2 N}, \text{ for all } i.$$ 

Since $N < K^{m+1}$, there are at most $K^2$ such $B_i$’s. A union bound implies that $\mathbb{P}(E_3^c) \leq K^2 2e^{-\frac{1}{16C_1k} \epsilon^2 \log^2 N}$. Thus, $\lim_{m \to \infty} \mathbb{P}(E_3) = 1$. \(\square\)
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