Multiphoton Emission

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We describe the emission, detection and structure of multiphoton states of light. By including frequency filtering, we describe at a fundamental level the physical detection of the quantum state. The case of spontaneous emission of Fock states is treated fully and analytically. We stress this picture by contrasting it to the numerical simulation of two-photon bundles emitted from a two-level system in a cavity. We show that dynamical factors exist that allow for a more robust multiphoton emission than spontaneous emission. We also describe how thermal light relates to multiphoton states.

Multiphoton (or multiquanta) physics is rapidly emerging in various areas of quantum technology [1–8]. In this Letter, we provide a detailed description of basic and fundamental aspects of multiphoton emission and detection. Namely, we give exact closed-form expressions for the full photon-counting probabilities of the frequency-filtered Spontaneous Emission (SE) of Fock states, from which we derive their complete temporal structure and thus all statistical observables of possible interest. We discuss particular cases for illustration and then contrast SE to Continuous Wave (CW) emission and thermal equilibrium.

To give the most general description, we compute the probability \( p(n, T; t) \) of detecting \( n \) photons in the time window between \( t \) and \( t + T \). For a single-mode source with bosonic annihilation operator \( a \) and emission rate \( \gamma_\alpha \), this is given by Mandel’s formula [9, 10]

\[
p(n, T; t) = \frac{1}{n!} \langle \Omega^{n} \exp(-\Omega) \rangle \text{ with } : : \text{ denoting normal ordering and } \Omega \text{ the time-integrated operator } \Omega(t, T) = \sum_{k=0}^{\infty} t^{k} a^{\dagger} a^{k}, \xi \text{ quantifying detection efficiency (1 for a perfect detector). This is more general than the popular Glauber correlation functions } g^{(n)} [11], \text{ that can be derived from it.}
\]

This is apparent in the expanded form for \( p(n, T; t) = \frac{1}{n!} \sum_{k=0}^{n} \langle \sum_{k=0}^{n-k} (-\xi)^{n+k} \Omega^{k} \rangle \int_{t}^{t+T} \cdots \int_{t}^{t} \langle a(t_{1}) \cdots a(t_{k+k}) \rangle \text{ with } \langle a(t_{1}) \cdots a(t_{k+k}) \rangle = a(t_{n+k}) \cdots a(t_{1}) \rangle dt_{1} dt_{2} \cdots dt_{n+k}. \text{ We first compute it for the simplest and most fundamental type of multiphoton emission, namely, that of a group, or “bundle” [12–18], of } N \text{ photons, emitted spontaneously by a source, e.g., a passive cavity of natural frequency } \omega_{n} \text{ in the state } \rho(0) = |N \rangle \langle N | \text{ at } t = 0, \text{ that freely radiates its photons at the rate } \gamma_{\alpha} \text{ without any type of stimulation or other dynamical factor. Under these conditions, the state evolves according to the Lindblad master equation (we set } \hbar = 1) \rho = -i \omega_{n} [a^{\dagger} a, \rho] + \frac{\gamma_{\alpha}}{\Omega} (2a \rho a^{\dagger} - a^{\dagger} a \rho - \rho a^{\dagger} a) \text{ for which one can find, through a tedious but systematic procedure [23], the multi-time correlators } \langle a^{\dagger}(t_{1}) \cdots a^{\dagger}(t_{m}) a(t_{m}) \cdots (t_{1}) \rangle = \frac{1}{(N-m)!} \langle \sum_{k=1}^{m} \int_{0}^{T} e^{-(\gamma_{\alpha}/2 - i\omega_{n}) t_k} \rangle \langle \sum_{k=1}^{m} e^{-(\gamma_{\alpha}/2 + i\omega_{n}) t_k} \rangle. \text{ One can then compute the quantum-average for any power of } \Omega \text{ by splitting its time-dynamics from the operator } a\alpha a \text{ in a scalar function } \mathcal{T} \text{ as } \Omega(t, T) = \mathcal{T}(T) a\alpha(t) a(T). \text{ For SE, with } \Omega(0, T) = \gamma_{\alpha} \xi \int_{0}^{T} e^{-\gamma_{\alpha} t} dt a\alpha(0) a(0), \text{ it gives:}
\]

\[
\mathcal{T}(T) = \xi (1 - e^{-\gamma_{\alpha} T}). \tag{1}
\]

With time and quantum-averages now separated, it is straightforward to compute \( \langle \Omega^{k} \rangle = \mathcal{T}^{k}(a\alpha^{k} a^{k})(0) = \xi^{k} (1 - e^{-\gamma_{\alpha} T})^{k} \frac{N!}{(N-k)!} \text{ for all } 1 \leq k \leq N. \text{ Substituting in Mandel’s formula (now keeping track in the notation of the initial state } N \text{ instead of the initial time } t), \text{ } p(n, T; N) = \sum_{k=n}^{\infty} (-1)^{n+k} \langle \Omega^{k} \rangle \langle a\alpha^{k} a^{k} \rangle/(n!(k-n)!), \text{ this simplifies to the physically transparent form:}
\]

\[
p(n, T; N) = \binom{N}{n} \mathcal{T}(T)^{n} (1 - \mathcal{T}(T))^{N-n} \tag{2},
\]

that is, a binomial distribution arising from independent Bernoulli trials for the detection of a single photon in the time \( T \). Variations of this result have been known for a long time [19, 20] and, with hindsight, Eqs. (1) and (2) could have been postulated based on physical arguments. Equation (1) is indeed the probability to detect a single photon in SE in the time window \([0, T]\) and thus corresponds to the quantum efficiency.

The value of the above derivation is, besides a rigorous and direct justification of the result, its extension to the case of filtered emission. At a fundamental level, frequency filtering describes detection [21, 22], since any physical detector has a finite temporal resolution and, consequently, a frequency bandwidth \( \Gamma \), which restricts the quantum attributes of the measured system. Ultimately, any quantum system has to be observed and, at this point, instead of being a mere techni-
cal final step, detection typically brings all the counterintuitive and disruptive nature of the theory. Including these fundamental constraints is therefore essential for a physical theory of multiphoton emission. To do that, the previous derivation can be repeated but now involving filtered-field operators $\varsigma(t) = \int_{-\infty}^{t} F(t-t_1) a(t_1) dt_1$ whereby a filter $F$ is applied to the source’s field. For SE, the time dynamics can similarly be separated from the quantum operator $\varsigma(t) = \Xi(t) a(0)$ that now combines both the dynamics of filtering and SE in $\Xi \equiv \int_{-\infty}^{t} F(t-t_1) e^{-\left(\frac{2}{\hbar} + i\omega t_1\right)} \theta(t_1) dt_1$ ($\theta$ the Heaviside function). In this way, the filtered time-integrated intensity operator from $[0,T]$ reads $\Omega_{T}(T) = \gamma a \xi \int_{0}^{T} \varsigma^{*}(t) dt = \mathcal{T}_{T}(T)(a^{'a})(0)$. This provides the single-photon-detection probability (quantum efficiency) for a filtered-field $\mathcal{T}_{T}(T) = \gamma a \xi \int_{0}^{T} \Xi_{T}(t)^{*} \Xi_{T}(t) dt$. For interference (Lorentzian) filters, $\mathcal{F}(t) = \frac{\Gamma}{\pi} e^{-(t/t_{0}^{2} + i\omega_{0} t)} \theta(t)$, and for SE, $\Xi(t) = (\Gamma/\Gamma-\Gamma)(e^{-\gamma a t/2 + i\omega_a t}) - e^{-\Gamma t/2} \theta(t)$, in which case, defining $\Gamma_{\pm} \equiv \Gamma \pm \gamma_a$, the quantum efficiency to use in Eq. (2) for a physical detector, or to describe the filtering of light, is:

$$\mathcal{T}_{T}(T) = \frac{\Gamma}{\Gamma_{+}} - \frac{\Gamma^{2} e^{-\gamma_{a} T} + \Gamma \gamma_{a} e^{-\gamma_{TT}}}{\Gamma_{+}^{2}} + 4 \Gamma^{2} \gamma_{a} e^{-\Gamma_{+} T/2}. \tag{3}$$

It generalizes Eq. (1), which it recovers in the limit $\Gamma \rightarrow \infty$ since, due to Heisenberg uncertainty, one must detect at all the frequencies and loose this information, to know precisely when the photons are being detected. On the other hand, the $T \rightarrow \infty$ limit gives $P(k,N) \equiv \lim_{T \rightarrow \infty} p(k,N; T) = \frac{N}{k} \prod_{i=1}^{N-k} \mathbb{I}_{T}(t_i)$ as the probability to detect $k$ out of N emitted photons. This is less than 1 even for a perfect detector ($\xi = 1$), due to its finite bandwidth, showing again the fundamental interplay of time and frequency in the detection. We can now describe completely the SE of N photons. This comes in the form of the joint probability distribution function (pdf) $\phi^{(N)}(t_k)$ for the kth photon to be detected at time $t_k$. We derive this quantity from its link to the photon counting probability by defining $p(t_1, \ldots, t_N)$ the probability to detect the kth photon up to time $t_k$ as $p(t_k) = \int_{0}^{t_k} \cdot \int_{0}^{t_k} \phi^{(N)}(t_k) \prod_{i=1}^{N} \nu_{t_i} dt_i$. Since this is also related to the combinatorics of single-photon detection probabilities, namely, $p(t_k) = N^T(t_1) \prod_{k=1}^{N-1} (\mathcal{T}(t_{k+1}) - \mathcal{T}(t_k)) \tag{23}$, one can finally obtain $\phi(t_1, \ldots, t_N) = \prod_{k=1}^{N} \frac{\partial^{N}}{\partial t_{i_{1}} \ldots \partial t_{i_{k}}} p(t_1, \ldots, t_N)$ from the fundamental theorem of calculus. We give directly the filtered emission result, from which unfiltered emission arises as the limiting case $\Gamma \rightarrow \infty$ (it is exploited in [23]):

$$\phi^{(N)}_{T}(t_1, \ldots, t_N) = N \Gamma_{\gamma_{a}}^{N} \times \left( \frac{\Gamma}{\Gamma_{-}} \right)^{2N} \prod_{i=1}^{N} (e^{-\Gamma t_i/2} - e^{-\gamma_{a} t_i/2}) \prod_{i=1}^{N} \mathbb{I}_{T}(t_i) \tag{4}$$

where $\mathbb{I}_{T}(t)$ is the indicator function which is 1 if $t \in T$ and 0 otherwise. This joint probability distribution provides the exact and complete temporal structure of multiphoton SE, and is one of the main results of this work (the $N = 2$ case is shown in Fig. 1(a)). From it, one can compute all the statistical quantities regarding filtered multiphoton SE. As an illustration, we derive the marginal distributions for the emission of each photon in isolation $\phi^{(N)}_{T,k}(t_k) \equiv \int \cdot \int_{t \neq t_k} \phi(t_1, \ldots, t_N) \prod_{j \neq k} \nu_{t_j} dt_j$ and, from there, emission time averages $\langle t_k^{N} \rangle \equiv \int_{0}^{\infty} t_k \phi^{(N)}_{T,k}(t_k) dt_k$ (others like variances, etc., are illustrated in the Supplementary Material [23]). Although less fundamental and general, one-photon marginals are more familiar, accessible and practical. We find $\phi^{(N)}_{T,k}(t_k) \equiv \left( \frac{\Gamma}{\Gamma_{-}} \right)^{2N} \gamma_{a}^{N} k \prod_{i=1}^{N-k} (\gamma_{a} - g(t_0) - g(t_k))$ where we defined $g(t) \equiv e^{-\gamma_{a} t} - (e^{-\gamma_{a} t} - 1)$ so that $g'(t) = -(-e^{-\gamma_{a} t} - \gamma_{a} t) \frac{\pi}{2}$ and the same that enters in Eq. (4)—makes the integration of the marginal tractable in the form of nested integrals $\int_{t_{k-1}}^{t_{k}} \int_{t_{k-2}}^{t_{k-1}} \cdots \int_{t_{1}}^{t_{N-1}} \prod_{i=1}^{N} g(t_i) dt_1 \cdots dt_{k-1} dt_{k+1} \cdots dt_{N}$. The distributions $\phi^{(N)}_{T,k}(t_k)$ are for the kth photon from a fully-detected N-photon bundle. To take into account that filtering removes some photons, one needs to turn to the conditional probabilities $\varphi^{(N)}(t_k)$ of detecting the kth photon regardless of how many photons have been detected. This is obtained as the probability to detect the photon in the $k$th position in a $N$-photon bundle of which $n$ photons have been detected, the probability of which is Eq. (2), so that, from the law of total probability [23], $\varphi^{(N)}(t_n) = \sum_{k=n}^{N} \frac{P(k,N)}{P(k,N)} \phi^{(N)}_{T,k}(t_k)$. Here, $\varphi^{(N)}$ is normalized to the fraction of $n$-photon bundles detected in the SE of N photons, $\mathcal{N}(n,N) = \sum_{k=n}^{N} P(k,N) \equiv \frac{N^{N-n}}{\Gamma_{n}} \prod_{i=1}^{N} \mathbb{I}_{T}(t_i) \tag{23}$ where $2F1$ is the Hypergeometric function. Qualitatively, the detected photons are delayed and spread, with earlier photons being more affected. A representative case is shown for $\varphi$ in Fig. 1(b) for $N = 5$ and $T = 5$ and filter widths $\Gamma_{\gamma_{a}} = 1.5$ and 25, the latter case of which starts to be close to unfiltered emission. Comparing to a full-detection case $\phi$, shown in the upper part, one sees that incomplete bundles further spread out photons and pile them up towards later times. Note that the $5$th photon distribution is the same in both cases since these are detected only when the bundle remains unbroken. From all the possible statistical observables that one can derive from these results (as illustrated in the Supplementary Material [23]), it should be enough here to contemplate the mean time of detection for the $k$th photon of a $N$-photon bundle, which, although it could seem a simple quantity, actually requires an 11-indices
FIG. 1. Filtered N-photon spontaneous emission. (a) For \( N = 2 \) photons, one can visualize the full pdf \( \phi_2^{(2)}(t_1, t_2) = 2z^2_2(\Gamma/\Gamma_-)^3g(t_1)g(t_2)1_{|t_1, \infty|}(t_2) \) in addition to its marginals \( \phi_2^{(1)}(t_k) \) which are its projections on the single-photon subspaces. (b) Marginals for \( N = 5 \) for various filter widths \( \Gamma \), conditioned \( \phi \), or up (or \( \varphi \), down) to full-bundle detection. The case \( \Gamma/\gamma_a = 1 \) has superimposed a numerical simulation. On the floor, the mean times (5) showing dilation of the bundles under filtering (solid for \( \varphi \), dashed for \( \phi \)).

\[
\sum_{\{k_1\}} \equiv \sum_{k_1 + k_2 + \cdots + k_N} \sum_{k_{N-k} = 1}^\infty \sum_{k_{10}+k_{11}} 2N! \left( \frac{\gamma_a \Gamma^+}{\Gamma_-} \right)^N \prod_{j=1}^{\sum k_j} \frac{1}{k_j!} \times \frac{\gamma_a^{k_1 + k_2 + \cdots + k_N} + \frac{\Gamma^+}{\Gamma_-} (k_3 + k_4) \sum \frac{1}{k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 + k_{10} + k_{11}}} {\Gamma_a \Gamma^+ \Gamma^- \left( \frac{\Gamma^+}{\Gamma_-} \right)^2} \]

where \( \Sigma_1 \equiv k_1 + k_4 + k_7, \Sigma_2 \equiv k_2 + k_5 + k_8, \Sigma_3 \equiv k_3 + k_6 + k_9, \Sigma_4 \equiv k_1 + k_5 + k_7 + k_8 + k_9 \) and \( \Sigma_5 \equiv k_3 + k_8 + k_9 \). This generalizes the unfiltered-photons result \( \langle \tau \rangle_N = (H_N - H_{N-k})/\gamma_a \) (with \( H_N \) the Nth harmonic number and assuming \( H_0 = 0 \)) and explains the observed photon-bundle time length (the time between the first and last detected photons) \( \langle \tau \rangle_N \equiv \langle \tau_k^{(N)} \rangle \) that goes to \( H_{N-1}/\gamma_a \) in this limit, as was observed in Refs. [12, 18], as well as the 4th photon lifetime measured as \( \langle \tau_k^{(4)} \rangle - \langle \tau_k^{(3)} \rangle \) as was observed in Refs. [24, 25].

In possession of the full probability distribution, we can also derive statistical quantities previously out of reach even in the limit \( \Gamma \rightarrow \infty \), such as the standard deviation \( \sigma_k^{(N)} \) for the emission time of the kth photon from a N-photon bundle, \( \gamma_a \sigma_k^{(N)} = \sqrt{\gamma_a} \) [23] with \( H_{N,2} \equiv \sum_{k=1}^N k^2 \) (generalized harmonic numbers).

In the opposite limit, the same harmonic structure is found but this time governed by the filter alone with no direct influence (to leading order) from the radiative decay, with, e.g., \( \langle \tau_k^{(N)} \rangle \sim (H_N - H_{N-k})/\Gamma \) as \( \Gamma \rightarrow 0 \), so:

\[
\langle \tau \rangle_N = \frac{H_{N-1}}{\Gamma} \quad \text{at small } \Gamma \quad \text{and } \frac{H_{N-1}}{\gamma_a} \quad \text{at large } \Gamma.
\]

To unify these two simple and symmetric limits, one has to use Eq. (5). This time-dilation of a filtered bundle is shown on the floor of Fig. 1(b). We now turn to a richer, more accessible and more insightful statistical quantity that will further allow us to generalize our discussion to other types of multiphoton emission, namely, the waiting-time distribution (wtd), i.e., the probability distribution of the time difference between successive photons. For the SE of a two-photon bundle, this is obtained from Eq. (4) as

\[
w_2(\tau) = \int_0^\infty \rho_2^{(2)}(t, \tau + t) dt,
\]

which gives

\[
w_2(\tau) = \frac{\gamma_a(G + \gamma_a)}{\gamma_a + \gamma_a + \gamma_a(3\Gamma + \gamma_a)} \left(1 + \frac{\gamma_a}{\Gamma} - \frac{3\gamma_a}{\Gamma} \right) \left(1 + \frac{\gamma_a}{\Gamma} - \frac{3\gamma_a}{\Gamma} \right) \left(1 + \frac{\gamma_a}{\Gamma} - \frac{3\gamma_a}{\Gamma} \right).
\]

This is a tri-exponential decay that balances the radiative decay with its filtering. This quantity provides another way to the average time length of a two-photon bundle as a function of filtering \( \langle \tau \rangle_2 = \int_0^\infty \tau w_2(\tau) d\tau = \left[ \frac{1}{\gamma_a} + \frac{1}{\gamma_a + \gamma_a} - \frac{9}{3(\gamma_a + \gamma_a)} \right] \left[ \frac{9}{3(\gamma_a + \gamma_a)} \right] \) (this is, alternatively, obtained from Eq. (5) as \( \langle \tau_2^{(2)} \rangle - \langle \tau_1^{(2)} \rangle \), which recovers the limits (6) with numerator \( H_2 = 1 \)).

This completes our description of filtered SE of Fock states, which is the starting point and reference for all other types of multiphoton emission, for instance, steady-state N-photon emission. We focus for now on the simplest case \( N = 2 \). We simulate numerically with a frequency-resolved Monte Carlo method [20] a filtered “bundler” (emitter of bundles) [12], i.e., a coherently driven Jaynes-Cummings Hamiltonian in the dispersive limit, that can also be understood as a Purcell-enhanced Mollow-triplet. We compare both its wtd and its photon-purity \( \pi_2 \) (the percentage of unbroken bundles or photons emitted in pairs) with a toy-model of CWSE (continuous wave spontaneous emission) that simulates the actual emission of the bundler with randomly triggered SE of two-photon Fock states. For the most part, we find that the multiphoton emission from the bundler behaves accordingly to the SE of collapsed Fock states, as has been suggested all along [27]. Their wtd, shown in Fig. 2(a) for different purities (50% and 100%), share the same qualitative main features. The multiphoton peak, in particular, i.e., the short-time excess of probabilities due to photons piling up from the multiphotons, behaves similarly under filtering. This is shown in Fig. 2(b) where numerically simulated CWSE and two-photon bundling are superimposed on the theoretical \( \langle \tau \rangle_2 \) above. There are, nevertheless, noteworthy quantitative departures, most prominently, the greater robustness of the bundles to filtering, as seen in Fig. 2(c) where the purity corresponds to that of SE with an effective decay rate of \( \approx \gamma_a/2 \). Therefore, for the same filter width, bundles originating from the driven system (bundler) are significantly more likely to be fully detected than if they were emitted by SE. This is surprising given the otherwise excellent agreement with a picture of collapsed Fock states that are subsequently spontaneously emitted. One can track this departure to the dynamics of the emitter itself (the two-level system) that, upon collapsing, resets the SE process in a mechanism akin to the quantum Zeno effect [28] that effectively slows down the emission, i.e.,...
reduces $\gamma_a$. This, in turn, makes the multiphoton detection more robust, since filtered SE is indeed better for larger $\Gamma/\gamma_a$. This dynamical enhancement depends on various parameters such as the statistics of the bundles themselves, which warrants a study on its own. This suggests, however, that the prospects of improving multiphoton emission by frequency filtering are even better than has been anticipated [29].

Finally, one cannot address multiphoton emission without considering the most famous and prevailing case, which has been foundational to quantum optics [30] and whose role remains central for applications [31], let alone this being the most common type of light, namely, the thermal state. This is obtained by supplementing the Lindblad master equation of SE with a pumping term $P_a^2(2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger)$ where $P_a < \gamma_a$ drives the cavity to a steady state $\rho = (1 - \theta) \sum n=0^\infty \theta^n |n\rangle \langle n|$ with effective temperature $\theta = P_a/\gamma_a$. We can now precise its perceived relationship to multiphoton emission by comparing thermal light to pure multiphoton emission. The Laplace transform of the wtd is obtained from the same transform $g^{(2)}(s)$ of the $g^{(2)}(\tau)$ function [32] as $\tilde{w}_{\text{th}}(s) = (1 + 1/[(\gamma_a, n_s g^{(2)}(s))]^{-1}$ or, by inverse transform, $w_{\text{th}}(\tau) = \frac{2P_a^2\gamma_a Q_a^2}{\gamma_a^2} \exp\left\{ - \frac{\gamma_a^2 + P_a^2\gamma_a}{2(\gamma_a - P_a^2)} \right\} Q_a \left[ \frac{Q_a - Q_\pi}{(\gamma_a - P_a^2)} \right] - 2P_a^2\gamma_a \sinh\left\{ - \frac{Q_a}{(\gamma_a - P_a^2)} \right\}$, where we have defined $Q_a^2 = P_a^4 - 4P_a^3\gamma_a + 10P_a^2\gamma_a^2 - 4P_a^2\gamma_a^3 + \gamma_a^4$. This has a similar shape (cf. Fig. 2(a), green dotted line) to the wtd of the multiphoton emissions previously described, but behaves distinctively under filtering. For instance, let us compare the average of the thermal multiphoton peak $\langle \gamma_{\text{th}} \rangle_{\text{th}}$ with its pure multiphoton counterpart in Eq. (6): in the limit of large-bandwidth filtering, the average is that of a thermal state, $\langle \gamma_\infty \rangle_{\text{th}} = (\gamma_a - P_a)/(P_a^2 + \gamma_a^3 + Q_a)$, shown in the inset of Fig. 2(b). It is, in units of $\gamma_a$, a quantity between 0 (when $P_a \to \gamma_a$ with the multiphoton peak becoming a Dirac $\delta$ function) and 1 (when $P_a \to 0$ with the multiphoton peak reducing to two-photon bunching and thus to the radiative lifetime). It is thus continuous, in contrast to pure multiphoton emission that is quantized (through the Harmonic numbers). For finite $\Gamma$, the average is not that of a thermal state anymore, since, surprisingly, although filtering has a tendency to thermalize quantum states, and does this exactly in the limit $\Gamma \to 0$ for all physical states, filtering a thermal state does not produce another thermal state, despite the fact that all the Glauber correlators of the filtered thermal field $g^{(n)}(0) = n!$ remain the same. Their time dynamics, however, exhibit a different evolution [23]. Since in the limit $\Gamma \to 0$, a thermal state is again recovered as a universal limit, one can obtain the asymptotic behaviour: $\gamma_a(\gamma_\infty)_{\text{th}} = (\gamma_\infty - \gamma_a)/(1 - \theta)/\Gamma$. Filtering pure multiphoton emission, on the other hand, produces bundles of various sizes, which yields a multiphoton peak in the wtd $\langle \tilde{\gamma} \rangle_N = \sum_{k=2}^N \left( \frac{k(k-1)}{2} \right) P(k, N) / \sum_{k=2}^N P(k, N)$ obtained by weighting the contribution of each sub-bundle population. This is qualitatively different from thermal emission, that retains thermal contributions from multiphotons of all sizes. Therefore, even if the wide-filter asymptotes $(\gamma_\infty)_{\text{th}}$ and $(\tilde{\gamma})_N$ match, their narrow-filter trends depart as shown in Fig. 2(b). This disconnection of the two limits, as compared to Eq. (6), shows that there are dynamical features present in thermal equilibrium that break from the paradigm of SE. In the limit $\Gamma \to 0$, another unexpected result is found: the thermal field is monochromatic though chaotic, as was already understood by Glauber [11]. The surprise is that its temperature $\theta_{\text{th}}(\Gamma) = P_a\gamma_a/(P_a^2 + (\gamma_a + \Gamma)(\gamma_a - P_a))$, as compared to that of the unfiltered field $\theta_{\text{th}} = P_a/\gamma_a$, can be higher (if $\Gamma < P_a$), although one is only imposing a passive element. This apparent paradox is due to both fields having different time scales, with the filter slowing down times by delaying photons. As a result, the filtered photons can have a flatter distribution of photon-numbers, with higher probabilities to find more excited-states, but these also have a longer lifetime, so are less likely to be emitted, thus conserving energy while increasing temperature. This could also be seen as a CW version of an equally surprising phenomenon when subtracting a single photon from a thermal field, that results in increasing its average population [33]. These various findings in multiphoton emission provide insightful relationships and departures between a thermal state and pure multiphoton sources. Their closest encounter is at vanishing temperature where a thermal state then behaves as a two-photon source but with most of its bundles broken... A fairly subtle and elusive connection!

In conclusion, we have provided a comprehensive and analytical description of the fundamental case of SE of Fock states, illustrated with statistical observables of interest. The treatment through frequency filtering allows to describe at a fundamental level the detection process. While we have focused on time observables, the same analysis could be done in the frequency domain by Fourier transform of Eq. (4), providing probabilities of detections at given frequencies. Compared to other types of multiphoton emission, we have shown that although a bundler behaves in all respect as SE of collapsed Fock states, dynamical features exist that protect bundles, making them significantly more resilient to filtering than if they were generated by SE. This calls for further studies to understand, characterize and exploit such dynamical advantages. We have also highlighted connections and departures with thermal light, which features multiphoton emission to all orders of a different character than pure multiphoton emission with broken bundles. Such characterizations could also be made for superchaotic light, superbunching, leapfrog processes in resonance fluorescence, etc., which should enlighten on the nature, character and relationship of these sources with pure multiphoton emission, thus allowing to establish a zoology of its various types.
FIG. 2. Dynamical multiphoton emission. (a) wtd for the toy-model (CWSE of Fock states, purple), the bundler (blue) and a thermal state (dotted green), all featuring a multiphoton peak sitting on the background of successive multiphotons. The toy model and bundler would be identical if they had the same purity. (b) Multiphoton peak average as a function of filtering, for the case of pure multiphoton emission \( \langle \tau \rangle_N \) for \( N \) from 2 to 5, exhibiting quantization at large \( \Gamma \) and a unique asymptote at small \( \Gamma \) when all bundles are broken into two-photon ones. The toy-model (purple diamonds) and the bundler (blue circles) behave according to the SE picture into two-photon ones. The toy-model (purple diamonds) and the bundler (circles) which higher coefficients for the asymptotes. Dashed gray lines: \( N\)-photon bundle time length \( \langle \tau \rangle_N \) also featuring same coefficients for the asymptotes. (c) Two-photon purity [percentage of fully detected bundles] from Fock state SE (line), the toy-model (diamonds) and the bundler (circles) which higher purity fits well with a SE rate of \( \approx \gamma_a/2 \) (dotted lines).

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[1] Deléglise, S. et al. Reconstruction of non-classical cavity field states with snapshots of their decoherence.

[2] Hofheinz, M. et al. Generation of Fock states in a superconducting quantum circuit. Nature 454, 310 (2008).

[3] Sayrin, C. et al. Real-time quantum feedback prepares and stabilizes photon number states. Nature 477, 73 (2011).

[4] Chu, Y. et al. Multiphonon interactions in a bulk acoustic-wave resonator. Nature 563, 666 (2018).

[5] Sánchez Muñoz, C. & Schlawin, F. Photon correlation spectroscopy as a witness for quantum coherence. Phys. Rev. Lett. 124, 203601 (2020).

[6] Vírias Boström, E., D’Andrea, A., Cini, M. & Verdolizzi, C. Time-resolved multiphoton effects in the fluorescence spectra of two-level systems at rest and in motion. Phys. Rev. A 102, 103719 (2020).

[7] Zhong, H.-S. et al. Quantum computational advantage using photons. Science 370, 1460 (2020).

[8] Uria, M., Solano, P. & Hermann-Aviglio, C. Deterministic generation of large fock states. Phys. Rev. Lett. 125, 093603 (2020).

[9] Mandel, L. Fluctuations of photon beams: The distribution of the photoelectrons. Proc. Roy. Soc. 74, 233 (1959).

[10] Kelley, P. L. & Kleiner, W. H. Theory of electromagnetic field measurement and photoelectron counting. Phys. Rev. 136, A316 (1964).

[11] Glauber, R. J. Photon correlations. Phys. Rev. Lett. 10, 84 (1963).

[12] Sánchez Muñoz, C. et al. Emitters of N-photon bundles. Nature Photon. 8, 550 (2014).

[13] Dong, X.-L. & Li, P.-B. Multiphonon interactions between nitrogen-vacancy centers and nanomechanical resonators. Phys. Rev. A 100, 043825 (2019).

[14] Bin, Q., Lü, X.-Y., Laussy, F. P., Nori, F. & Wu, Y. n-photon bundle emission via the stokes process. Phys. Rev. Lett. 124, 053601 (2020).

[15] Ma, S., Li, X., Ren, Y., Xie, J. & Li, F. Antibunched n-photon bundles emitted by a josephson photonic device. Phys. Rev. Res. 3, 043020 (2021).

[16] Q. Bin, Y. W. & Lü, X.-Y. Parity-symmetry-protected multiphoton bundle emission. Phys. Rev. Lett. 127, 073602 (2021).

[17] Ma, S., Li, X., Ren, Y., Xie, J. & Li, F. Antibunched n-photon bundles emitted by a josephson photonic device. Phys. Rev. Res. 3, 043020 (2021).

[18] Deng, Y., Shi, T. & Yi, S. Motional n-photon bundle states of a trapped atom with clock transitions. Phot. Res. 9, 1289 (2021).

[19] Sticca, D. M. O. & W. E. Lamb, J. Quantum theory of an optical maser. III. Theory of photoelectron counting
statistics. Phys. Rev. 179, 368 (1969). http://doi.org/10.1103/PhysRev.179.368.

[20] Arnoldus, H. F. & Nienhuis, G. Photon correlations between the lines in the spectrum of resonance fluorescence. J. Phys. B.: At. Mol. Phys. 17, 963 (1984). http://doi.org/10.1088/022-3700/17/6/011.

[21] Eberly, J. & Wo´dkiewicz, K. The time-dependent physical spectrum of light. J. Opt. Soc. Am. 67, 1252 (1977). http://doi.org/10.1364/JOSA.67.001252. TDS1.3.

[22] del Valle, E., González-Tudela, A., Laussy, F. P., Tejedor, C. & Hartmann, M. J. Theory of frequency-filtered and time-resolved n-photon correlations. Phys. Rev. Lett. 109, 183601 (2012). http://doi.org/10.1103/PhysRevLett.109.183601.

[23] Díaz Camacho, G. et al. Supplementary material. (online) (2021).

[24] Wang, H. et al. Measurement of the decay of Fock states in a superconducting quantum circuit. Phys. Rev. Lett. 101, 240401 (2008). http://doi.org/10.1103/PhysRevLett.101.240401.

[25] Brune, M. et al. Process tomography of field damping and measurement of Fock state lifetimes by quantum nondemolition photon counting in a cavity. Phys. Rev. Lett. 101, 240402 (2008). http://doi.org/10.1103/PhysRevLett.101.240402.

[26] Lópe´z Carreño, J. C., del Valle, E. & Laussy, F. P. Frequency-resolved Monte Carlo. Sci. Rep. 8, 6975 (2018). http://doi.org/10.1038/s41598-018-24975-y.

[27] Strekalov, D. V. Cavity quantum electrodynamics: A bundle of photons, please. Nature Photon. 8, 500 (2014). http://doi.org/10.1038/nphoton.2014.144.

[28] Misra, B. & Sudarshan, E. C. G. The Zeno’s paradox in quantum theory. J. Math. Phys. 18, 756 (1977). http://doi.org/10.1063/1.523304.

[29] Sánchez Muñoz, C., Laussy, F. P., del Valle, E., Tejedor, C. & González-Tudela, A. Filtering multiphoton emission from state-of-the-art cavity QED. Optica 5, 14 (2018). http://doi.org/10.1364/OPTICA.5.000014.

[30] Hanbury Brown, R. & Twiss, R. Q. Correlation between photons in two coherent beams of light. Nature 177, 27 (1956). http://doi.org/10.1038/177027a0.

[31] Valencia, A., Scarcelli, G., D’Angelo, M. & Shih, Y. Two-photon imaging with thermal light. Phys. Rev. Lett. 94, 063601 (2005). http://doi.org/10.1103/PhysRevLett.94.063601.

[32] Kim, M., Knight, P. & Wodkiewicz, K. Correlations between successively emitted photons in resonance fluorescence. Opt. Commun. 62, 385 (1987). http://doi.org/10.1016/0030-4018(87)90005-8.

[33] Parigi, V., Zavatta, A., Kim, M. & Bellini, M. Probing quantum commutation rules by addition and subtraction of single photons to/from a light field. Science 317, 1890 (2007). http://doi.org/10.1126/science.1146204.
Multiphoton Emission
Supplementary Material

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We provide details on the theoretical derivations which, although not necessary to the skilled
Mathematician to derive the results in the text, may help in reproducing or extending them. In
addition to exact but awkward combinatoric sums, we also provide explicit results—both for con-
venience and illustration—for all photons up to \( N = 5 \) or, for particularly voluminous expressions,
\( N = 3 \). We finally provide the spectrum and \( g^{(2)}(\tau) \) function of a filtered thermal field, so as to
show that this is not a thermal field, featuring departures in its dynamics although it has the same
density matrix.

JOINT PROBABILITY DISTRIBUTION OF SE MULTIPHOTON EMISSION

We derive \( \phi^{(N)}(\theta_1, \ldots, \theta_N) \) the joint probability distribution function (pdf) that the \( k \)th photon
of a Fock state \( |N \rangle \) at \( t = 0 \) in a freely-radiating cavity, is emitted at time \( \theta_k \). To fix ideas with
a concrete case, we also discuss in parallel the case \( N = 2 \) for which we compute \( \phi^{(2)}(\theta_1, \theta_2) \)
that the first photon is emitted at time \( \theta_1 \) and the second photon is emitted at time \( \theta_2 \). By
construction, since the \( k \)th photon is emitted after the \( (k−1) \)th and before the \( (k+1) \)th:

\[
0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_N
\]  
(S1)

with \( \theta_N \) unbounded. The distribution is defined on the corresponding domain or, alternatively,
the probability being zero that a different order be observed, one can define the distribution on \( \mathbb{R}^n \)
but enforce the time-ordering (S1) by multiplying the distribution with the support function

\[
S(t_1, \ldots, t_N) \equiv 1_{[0, \theta_2]}(\theta_1)1_{[\theta_1, \theta_3]}(\theta_2) \cdots 1_{[\theta_{N−2}, \theta_N]}(\theta_{N−1})1_{[\theta_{N−1}, \infty)}(\theta_N)
\]  
(S2)

where \( 1_T(t) \) is the indicator function which is 1 if \( t \in T \) and is 0 otherwise. For \( N = 2 \), the
support is \( S(t_1, t_2) = 1_{[t_1, \infty)}(t_2) \) and restricts the distribution to the upper-triangular part of
the \( (t_1, t_2) \) space, enforcing that the probability that the 1st photon comes after the 2nd is zero.
The integral of the probability distribution gives the probability that a given number of photons
have been detected up to some time, which one can relate to the photon-counting formula \( P \) derived from Mandel’s formula. Namely

\[
P(t_1, \ldots, t_N) = \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_N} \phi^{(N)}(\theta_1, \ldots, \theta_N) \, d\theta_1 \cdots d\theta_N
\]  
(S3)

is the probability to have detected \( N \) photons, at the respective times \( t_i \). We can invert Eq. (S3)
to obtain the pdf as the derivatives of its primitive from the fundamental theorem of calculus:

\[
\phi^{(N)}(\theta_1, \ldots, \theta_N) = \frac{\partial^N}{\partial t_1 \cdots \partial t_N} P(t_1, \ldots, t_N).
\]  
(S4)

The photon-counting probability \( P(\{t_i\}) \equiv P(t_1, \ldots, t_N) \) is found in the main text from the
Mandel photon-counting version \( P(n, T; N) \) of detecting \( n \) of them up to \( T \) for both filtered
and unfiltered Spontaneous Emission (SE). It is given by a binomial distribution whose success parameter is the quantum efficiency $\mathcal{T}(T)$, or probability to detect a single photon in SE between times 0 and $T$. From $P(n, T; N)$, and since the structure of SE is that of Bernouilli trials in a sequence of independent success (emission)/failure (no-emission) experiments, one can get access to the probability of any multiphoton detection configuration, e.g., $S(t_1, t_2) \equiv \mathcal{T}(t_2) - \mathcal{T}(t_1)$ is the probability to detect a single photon between times $t_1$ and $t_2$, while the probability to detect one photon of a two-photon bundle, between time 0 and $t$ with the other not having yet been detected is $2\mathcal{T}(t)(1 - \mathcal{T}(t))$, and $\mathcal{T}(t)^2$ is the probability to detect the two photons of a two-photon bundle anywhere between times 0 and $t$. In this way one can complete the link between $P$ and $\phi$. We also give an alternative, geometric derivation for the case $N = 2$, with $P(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} \phi(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2$ being the probability to have detected two photons, the first one between 0 and $t_1$ and the second between 0 and $t_2$ or, more precisely and by construction, between $t_1$ and $t_2$. One can visualize this event in the 2D temporal plane as the reddish-areas shown in Fig. (S1), where the dark-red triangle is the probability to detect the two photons in the time window $[0, t_1]$, i.e., $\mathcal{T}(t_1)^2$, and the upper, lighter-red rectangle is the probability to detect the 1st photon up to time $t_1$ and the 2nd between time $t_1$ and $t_2$, i.e., $\binom{2}{1} \mathcal{T}(t_1) S(t_1, t_2)$. The lower triangle is zero, by definition (1st photon comes first!) So the area of interest integrates to a total probability of:

$$P(t_1, t_2) \equiv \mathcal{T}(t_1)^2 + 2\mathcal{T}(t_1) S(t_1, t_2). \tag{S5}$$

Applying Eq. (S4) to this result, i.e., $\phi^{(2)}(t_1, t_2) = \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \int_0^{t_2} \int_0^{t_1} \phi(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2$, derives the two-photon probability distribution from its counting probabilities:

$$\phi^{(2)}(t_1, t_2) = \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \left( \mathcal{T}(t_1)^2 + 2\mathcal{T}(t_1) S(t_1, t_2) \right). \tag{S6}$$

Interestingly, the darker-red triangle, i.e., $\mathcal{T}(t_1)^2$, does not contribute directly, which means that the two photons being detected at exactly the same time is negligible as compared to other possibilities. The general case is not so easy to visualize geometrically but has otherwise the exact same formulation in a hyperspace of photon time emissions which links $\phi$ to $P$ for $N$ photons as:

$$\phi^{(N)}(t_1, \ldots, t_N) = \frac{\partial^N}{\partial \theta_1 \ldots \partial \theta_N} \left[ \binom{N}{1} \mathcal{T}(t) \prod_{k=1}^{N-1} S(t_k, t_{k+1}) \right]. \tag{S7}$$

Now it is just a matter to compute this expression for a given choice of quantum efficiency $\mathcal{T}$, that are given by Eqs. (1) and (3) of the main text for unfiltered and for filtered SE, respectively.

We will now precise which case is which with a subscript. In the first case (unfiltered SE or $\Gamma \to \infty$), we find:

$$\phi^{(N)}(t_1, \ldots, t_N) = N! \gamma_a^N e^{-\Gamma(t_1 + \cdots + t_N)} \prod_{i=1}^{N} S(t_{i-1}, t_{i+1}), \tag{S8}$$
while in the filtered case, we find:

$$\phi^{(N)}_\Gamma(t_1, \ldots, t_N) = N!\gamma_a^N \left( \frac{\Gamma}{\Gamma_+} \right)^{2N} \prod_{i=1}^{N} \left( e^{-\Gamma_i t_i/2} - e^{-\gamma_a t_i/2} \right)^2 \mathbb{I}_{[t_{i-1}, t_i+1]}(t_i)$$

(S9)

where, in both cases, we have defined $t_0 \equiv 0$ and $t_N+1 \equiv \infty$. Equation (S9) is Eq. (4) of the main text. As should be, Eq. (S8) is a particular case of (S9), in the limit $\Gamma \to \infty$ (whence the notation). Equation (S8) makes particularly obvious the intrinsic symmetry of the emission process: the photons are emitted independently the ones from the others as far as the radiation mechanism is concerned, the joint pdf being a simple product of their single-photon probabilities. It is the domain restrictions $\prod_{i=1}^{N} \mathbb{I}_{[t_{i-1}, t_i+1]}(t_i)$ that correlate the photons in a nontrivial way, opening the radiation window for the $k$th photon when the $(k-1)$th has been emitted and closing it for all successives photons. The same structure is observed for Eq. (S9) which is a product of one-dimensional distributions $\varrho_T(t)$ with

$$\varrho_T(t) \equiv \sqrt{N!} \gamma_a \left( \frac{\Gamma}{\Gamma_+} \right)^2 (e^{-\Gamma t/2} - e^{-\gamma_a t/2})^2$$

(S10)

(that reduces to $\varrho_{\infty}(t) = \sqrt{N!} \gamma_a e^{-\gamma a t}$ for unfiltered emission). Equation (S10) can thus be seen as the (unnormalized) probability density of photon-detection for a filtered SE photon. There is a clear balance in this expression between the limits where the filter and the radiation dominate the dynamics, respectively, which leads to the two types of behaviour captured, e.g., in Eq. (6) of the main text. One could possibly, from these results, postulate the general form of SE under any type of detection once its single-photon response is known.

In addition to the impact of filtering on the temporal structure of the emission, there is an even stronger effect, namely, that it actually removes photons. Because of this, the probability to detect all the $N$ photons from a filtered $N$-photon bundle emission is not one. It is, instead, given by the integral of Eq. (S9)

$$N \equiv \int_0^\infty \cdots \int_0^\infty \phi^{(N)}_\Gamma(t_1, \ldots, t_N) dt_1 \cdots dt_N = P(N, N)$$

(S11)

which is, as should be expected, the fraction of fully detected $N$-photon bundles, providing the normalization constant $N = (\Gamma/\Gamma_+)^N$ where we remind that $\Gamma_+ \equiv \Gamma \pm \gamma_a$. Depending on whether one is interested in probabilities or in percentages, the distribution has to be normalized, or not, as we will illustrate below on some particular cases. The general case is given in the main text with $\mathcal{T}$ given by Eq. (3) in the limit of infinite times, $P(k, N) \equiv \lim_{T \to \infty} p(k, T; N) = \binom{N}{k} \left( \frac{\Gamma}{\Gamma_+} \right)^k (1 - \frac{\Gamma}{\Gamma_+})^{N-k}$, i.e.,

$$P(k, N) \equiv \binom{N}{k} \frac{\gamma_a^{N-k} \Gamma^k}{\Gamma_+^{N-k}}.$$  

(S12)

This gives the probability to detect $k$ photons out of $N$ from a filtered $N$-photon bundle, and will be needed to normalize bundles broken by filtering. We will turn, below, to the marginal distributions $\phi^{(N)}_{\Gamma,k}(t_k)$ for the $k$th photon from a $N$-photon bundle to be emitted at time $t_k$. Unfortunately, it turns out that all the indices $(k, N, \Gamma)$ are required to avoid confusions in the notations. This is, by definition:

$$\phi^{(N)}_{\Gamma,k}(t_k) \equiv \int_0^\infty \cdots \int_0^\infty \phi(\theta_1, \ldots, \theta_N) d\theta_1 \cdots d\theta_{k-1} d\theta_{k+1} \cdots d\theta_N,$$

(S13)

where the integral is taken over all but the $k$th variable. The calculation for the general filtered case, starting with Eq. (S9), proceeds as follows: integrating first over $t_N$ (the last time in the expression) reduces to the 1D integral:

$$\int_{t_{N-1}}^{\infty} (e^{-\Gamma t_N/2} - e^{-\gamma_a t_N/2})^2 dt_N = \frac{e^{\gamma_a t_N} - e^{-\Gamma t_N/2}}{\gamma_a} + \frac{e^{-\Gamma t_N/2} - 4 e^{-\Gamma t_N/2}}{\Gamma_+}$$

(S14)
where the previous photon only enters as the lower bound of integration. Now integrating over
the new last time in the expression, \( t_{N-1} \), we are met with a new 1D integral whose previous
photon (third to last) also enters as the lower bound, while the current photon, although not
the last, is left free to wander up to infinity (there is enough room “beyond infinity” for the last
photon to be emitted after that):

\[
\int_{t_{N-2}}^{\infty} \left( e^{-\Gamma t_{N-1}/2} - e^{-\gamma_a t_{N-1}/2} \right) \left( e^{-\gamma_a t_{N-1}/2} + e^{-\Gamma t_{N-1}} - 4 e^{-\Gamma t_{N-1}/2} \right) \, dt_{N-1}.
\] (S15)

As merely a sum of exponentials, this is definitely integrable for any particular case. To find the
general case, we observe that Eq. (S9) can be rewritten as:

\[
\phi_T(t_1, \ldots, t_N) = (-1)^N N! \gamma_a^N \left( \frac{\Gamma}{\gamma_a} \right)^{2N} \prod_{i=1}^{N} g'(t_i) \prod_{t_{i-1}, t_{i+1}}(t_i)
\] (S16)

where we defined

\[
g(t) \equiv \frac{e^{-\gamma_a t}}{\gamma_a} + \frac{e^{-\Gamma t}}{\Gamma} - 4 \frac{e^{-\Gamma t/2}}{\Gamma_+}
\] (S17)

from Eq. (S14), so that indeed

\[
g'(t) = -(e^{-\Gamma t/2} - e^{-\gamma_a t/2})^2.
\] (S18)

The marginal distribution for the \( k \)th photon is thus obtained as (where the overline means “skip
this” or “don’t perform”):

\[
\phi^{(N)}_{T, \infty}(t_k) = (-1)^N N! \gamma_a^N \left( \frac{\Gamma}{\gamma_a} \right)^{2N} \times
\int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} \int_{t\infty}^{t_{k-2}} \int_{t_{k-1}}^{t_{k-2}} \cdots \int_{t_{2}}^{t_{1}} \int_{t_{1}}^{t_k} \cdots \int_{t_{N-2}}^{t_{N-1}} \int_{t_{N-1}}^{\infty} g'(t_1) g'(t_2) \cdots g'(t_{k-1}) g'(t_k) g'(t_{k+1}) \cdots
go \int_{t_{k-1}}^{\infty} g'(t_N) \, dt_1 \cdots \, dt_{k-1} \, dt_{k+1} \cdots \, dt_N
\] (S19)

which reduces the multi-dimensional integral to two series of recursive 1D integrals of the type
of Eq. (S15), one post and the other prior to the \( k \)th photon emission. They both provide a
functional dependence in addition to the \( k \)th photon emission itself, \( g'(t_k) \). We consider first
the post-emission series. Integrating backward (last photon first), we meet with the recurring
pattern:

\[
\int_{t_{j-1}}^{\infty} g'(t_j) g^{N-j}(t_j) \, dt_j = \frac{g^{N-j+1}(t_{j-1})}{N-j+1}
\] (S20)

(since the \( t \to \infty \) limit vanishes), whose recursive application from the last \( j = N \) to the first
post-emitted photon \( j = k + 1 \), yields:

\[
\prod_{j=N}^{k+1} \int_{t_{j-1}}^{\infty} g'(t_j) \, dt_j = \frac{(-1)^{N-k}}{(N-k)!} g^{N-k}(t_k)
\] (S21)

where the decreasing product is to be performed in this order and understood as implying nested
integrals as shown in Eq. (S19). This \( t_k \) dependence, from the post-emission, factors out along
with \( g'(t_k) \) the dynamical emission of the \( k \)th photon itself, and we are left to integrate the
pre-emission, for which we use the recurrence relation:

\[
\int_{t_{j-1}}^{t_k} g'(t_j)(g(t_k) - g(t_j))^n \, dt_j = \frac{1}{n+1}(g(t_k) - g(t_{j-1}))^{n+1}
\] (S22)
so that:

\[ \prod_{i=k-1}^{t_k} \int_{t_{i-1}}^{t_k} g'(t_i) \, dt_i = \frac{1}{(k-1)!} (g(t_k) - g(t_0))^{k-1} \]  \hspace{1cm} (S23)

where the decreasing product is also to be understood as nesting integrals. Bringing back the various pieces—the normalization constant, Eq. (S21), \( g'(t_k) \) and Eq. (S23)—together, we find:

\[ \phi_{\Gamma,k}^{(N)}(t_k) = (-1)^N N! \gamma_a^N \left( \frac{\Gamma}{N} \right)^{2N} \frac{(-1)^{N-k}}{(N-k)!} g^{N-k}(t_k) g'(t_k) \frac{1}{k-1} (g(t_k) - g(t_0))^{k-1} \]  \hspace{1cm} (S24)

which simplifies to the filtered one-photon marginal as:

\[ \phi_{\Gamma,k}^{(N)}(t_k) = -\left( \frac{\Gamma}{N} \right)^{2N} \frac{N^N}{k^k} \gamma_a^N g(t_k)^{N-k} (g(t_0))^{k-1} g'(t_k). \]  \hspace{1cm} (S25)

One could either reproduce the same calculation with Eq. (S8) instead of Eq. (S9), or take the limit \( \Gamma \to \infty \), to obtain the marginal of the unfiltered SE. In both cases, one finds the unfiltered probability distributions of detecting the \( k \)th photon as:

\[ \phi_{\infty,k}^{(N)}(t_k) = k \gamma_a \binom{N}{k} e^{-N \gamma_a t_k} (e^{\gamma_a t_k} - 1)^{k-1}. \]  \hspace{1cm} (S26)

While Eq. (S26) is normalized, for the same reason as before, the marginals for the filtered emission, Eq. (S25) are not normalized to one but to:

\[ \int_0^\infty \phi_{\Gamma,k}^{(N)}(t) \, dt = P(N,N) \]  \hspace{1cm} (S27)

which is the same result as Eq. (S11). The reason is that these marginals provide the probability distributions when considering all the \( N \) photons from a given photon bundle, i.e., \( \phi_{\Gamma,k}^{(N)}(t) \) is the probability to detect the \( k \)th photon from a \( N \)-photon bundle which has been detected in its entirety (or conditioned to its full-detection). To take into account that filtering occasionally removes some of the photons (any number, from none to all), we introduce still another probability distribution function, namely \( \varphi_{\Gamma,k}^{(N)} \), which is the probability density to detect the \( k \)th photon from a \( N \)-photon bundle of which any number from \( k \) to \( N \) photons have been detected. For all \( N, \phi_{\Gamma,k}^{(N)} = \varphi_{\Gamma,k}^{(N)} \) since a \( N \)th photon is detected if and only if all photons have been detected. The general relationship is otherwise obtained from the law of total probability

\[ P(A_n) = \sum_k P(A_n|B_k)P(B_k) \]  \hspace{1cm} (S28)

where we define

- \( A_n \) as “the detected photon is in the \( n \)th position of those detected”,
- \( B_k \) as “there is a total of \( k \) photons detected out of the initial \( N \)”

with the detection assumed to be in the interval \([t,t+\Delta t]\), in which case, since by definitions \( P(A_n) = \varphi_{\Gamma,n}^{(N)}(t)\Delta t \), \( P(A_n|B_k) = (\varphi_{\Gamma,k}(t)/P(k,k))\Delta t \) (with \( P(k,k) \) the normalization of \( \phi_n^{(k)} \) from Eq. (S27)) and \( P(B_k) = P(k,N) \), then Eq. (S28) becomes \( \varphi_{\Gamma,n}^{(N)} = \sum_{k=1}^{N} \frac{\phi_n^{(k)}}{P(k,k)} P(k,N) \) for all \( t \) and simplifying \( \Delta t \). Since \( \phi_n^{(k)} \) is zero if \( n > k \), the sum can also be taken from \( k = n \) to \( N \). In this case, since \( \cup_{k=n}^{N} B_k \) does not form a partition of the space of possible detection events anymore (the union from \( k = 1 \) to \( N \) does), therefore \( \varphi_{\Gamma,n}^{(N)} \) is not normalized to one but to:

\[ \mathcal{N}(n,N) \equiv \sum_{k=n}^{N} P(k,N) = \frac{\Gamma^{n-n-N}_n}{\Gamma^N_n} \binom{N}{n} \cdot \binom{2F_1(1, n-N, n+1, -\Gamma/\gamma_a)}{n} \]  \hspace{1cm} (S29)
which gives the fraction of detections with at least \( n \) photons in the SE of \( N \) photons, generalizing Eq. (S11) that was the particular case \( \mathcal{N}(N,N) \). The result is also clear on physical grounds as it states that any bundle which has been detected with exactly \( n \) photons contributes to all fractions with at least \( n \) photons. One can compute \( P(k,N)/P(k,k) \) from Eq. (S12), so that the \( k \)th photon normalized probability marginal from potentially broken bundles is finally obtained as

\[
\phi_{\Gamma,n}(t_n) = \frac{1}{\mathcal{N}(n,N)} \sum_{k=n}^{N} \binom{N}{k} \left( \frac{\gamma_a}{\Gamma_+} \right)^{N-k} \phi_{\Gamma,n}^{(k)}(t_n).
\]  

For instance, in the case shown in the main text for a five-photon bundle, reproduced in more details in Fig. S2, one has for, say, the second detected photon from all the possible detections:

\[
\phi_{\Gamma,2}^{(5)}(t_2)\mathcal{N}(2,5) = \phi_{\Gamma,2}^{(5)}(t_2) + 5\frac{\gamma_a}{\Gamma_+} \phi_{\Gamma,2}^{(4)}(t_2) + 10 \left( \frac{\gamma_a}{\Gamma_+} \right)^2 \phi_{\Gamma,2}^{(3)}(t_2) + 10 \left( \frac{\gamma_a}{\Gamma_+} \right)^3 \phi_{\Gamma,2}^{(2)}(t_2)
\]  

with normalization \( \mathcal{N}(2,5) = \Gamma^2(\Gamma^3 + 5\Gamma^2\gamma_a + 10\Gamma\gamma_a^2 + 10\gamma_a^3)/\Gamma_+^5 \) the sum of the last four \((5-2+1)\) terms in the binomial expansion of \( \Gamma_+^5 = (\gamma_a + \Gamma)^5 \), cf. Eq. (S29). Such a result has also been checked numerically with a Monte Carlo simulation that included \( N_{\text{traj}} = 400\,000 \) photons, where “traj” stands for “trajectory”. The corresponding data is also plotted in Fig. S2, which also illustrates how the noise is distributed among the various cases. According to Eq. (S29), for \( \Gamma = \gamma_a \), the expected numbers of fully-detected bundles is \((\Gamma/\Gamma_+)^5 N_{\text{traj}} = N_{\text{traj}}/32 = 12\,500 \) (numerically 11\,542 events have been recorded) while one expects (resp. finds) 62\,500 bundles with one-missing photon (vs 12\,238), 125\,000 with two-missing (vs 126\,663) and 62\,500 with four-missing photons, i.e., only one out of the five photons, has been detected (numerically, 66\,093). The difference from the number of attempts goes with the completely missed bundles: 12\,500 expected (13\,909 numerically). It is clear how, due to the smaller number of events, the noise is greater for \( \phi_{\Gamma,k} \) as compared to \( \varphi_{\Gamma,k} \), except for the case \( k = 5 \) where, not only the amount, but the noise itself is the same in both cases since \( \varphi_{\Gamma,5}^{(5)} = \phi_{\Gamma,5}^{(5)} \) and both distributions are identically reconstructed from the available signal. The amount of noise otherwise remains the same for lower \( k \) for \( \phi \) as all marginals are reconstructed from the same number of events, \( P(N,N) \), but decreases for \( \varphi \) since each event
with $\kappa$ photons contributes to all $\varphi^{(N)}_{\Gamma,k}$ with $k \geq \kappa$. Consequently, $\varphi^{(5)}_{\Gamma,1}$ has the most signal for its reconstruction, and indeed displays an excellent quantitative agreement with very little noise.

One can similarly derive multiphoton marginals. In the main text, we focus in particular on the waiting time distribution (wtd) since it is both popular and rich in information related to multiphoton emission, particularly for two-photon emission which is the focus of the main text. Without filtering, the SE of two-photons is trivial, with a wtd $w_2(\tau) = \gamma_a e^{-\gamma_a \tau}$ that does not feature any multiphoton effect, since the first-emitted photon, that gets a boost in its decay time, sets the starting point for the next one, which is just spontaneous emission. With filtering, however, the result becomes sensibly richer:

$$w_2(\tau) = \frac{1}{P(2,2)} \int_0^\infty \phi_t^{(2)}(t_1, \tau + t_1) \, dt_1$$

$$= \frac{1}{P(2,2)} \int_0^\infty g(t_1) g(\tau + t_1) \, dt_1$$

$$= \frac{2\gamma_a^2}{P(2,2)} \frac{\Gamma}{\Gamma - \kappa} \int_0^\infty [(e^{-\Gamma t_1/2} - e^{-\gamma_a t_1/2})(e^{-\Gamma (\tau + t_1)/2} - e^{-\gamma_a (\tau + t_1)/2})]^2 \, dt_1$$

since $\mathbb{1}_{t_1 \leq t_N} = 1$. With the normalization $P(2,2) = (\Gamma/\Gamma_+)^2$ to account for photon losses from filtering, Eq. (S32c) gives:

$$w_2(\tau) = \frac{\Gamma \gamma_a (\Gamma + \gamma)}{(\Gamma - \gamma_a)^2 (3\Gamma + \gamma) (\Gamma + 3\gamma_a)} \int_0^\infty \left( (\Gamma (3\Gamma + \gamma_a)e^{-\gamma_a \tau} + \gamma_a (\Gamma + 3\gamma_a)e^{-\Gamma \tau} - 8\Gamma \gamma_a e^{-\frac{\Gamma}{\Gamma + \gamma_a} \tau} \right).$$

This is the distribution of time differences of a two-photon bundle that is spontaneously emitted and filtered through a filter of width $\Gamma$. That reduces to $\gamma_a e^{-\gamma_a \tau}$ for $\Gamma \rightarrow \infty$ but for finite $\Gamma$, this acquires the tri-exponential decay form that bridges over the two limits of strong and no filtering. Note that one-photon events, or broken bundles, do not alter the result as the wtd is a two-photon observable. The situation is different for higher $N$ where photon losses should be taken into account, or the observable conditioned to full-photon detection. For instance, another two-photon marginal of interest is the one that retains the first and last photon from a $N$-photon bundle, tracing over the intermediate ones. This is useful to characterize the statistics of observables such as the bundle time-length. We find (some steps of the calculation will be detailed below on other but similar cases):

$$\phi_T(t_1, t_N) = 2N! \left( \frac{\Gamma}{\Gamma_a} \right)^{2N} \sum_{k_1 = \ldots = k_6}^{N-2} \sum_{k_7 = \ldots = k_{10}}^{N} \prod_{i=1}^{10} \frac{1}{k_i!} \times$$

$$e^{-\frac{\Gamma}{\Gamma_a} (\gamma_a (2k_1 + k_7 + k_8) + \Gamma (2k_2 + k_9 + k_{10}) + \Gamma_+ k_3) \left( (\gamma_a (2k_4 + k_7 + k_8) + \Gamma (2k_5 + k_9 + k_{10}) + \Gamma_+ k_3) \right)^{k_3 + k_6}}$$

from the marginals, one can then compute statistical averages, for instance the average times of detection, keeping in mind the normalization Eq. (S11). We work out some cases in detail and otherwise list the results. We start with $\langle t_1^{(N)} \rangle$, the average time of detection of the first photon from a $N$-photon bundle. This is obtained, by definition, as $\langle t_1^{(N)} \rangle \equiv \frac{1}{N} \int_0^\infty t_1 \varphi_1^{(N)}(t_1) \, dt_1$ and, from Eq. (S25), we have to compute

$$\langle t_1^{(N)} \rangle = \left( \frac{\gamma_a \Gamma \Gamma_+}{\Gamma_a^2} \right)^N \frac{N}{\Gamma_a} \int_0^\infty t_1 g(t) (t_1)^{N-1} g'(t_1) \, dt_1$$

which can be done by parts, since the integrand is $uv'$ where

$$u = t \quad u' = 1$$

$$v = (1/N) g(t)^N \quad v' = g(t)^{N-1} g'(t).$$
so that

\( \langle t_{1}^{(N)} \rangle \propto uv \bigg|_{0}^{\infty} - \int_{0}^{\infty} u' v \)  

(S37a)

\( = - \frac{1}{N} \int_{0}^{\infty} g(t)^N \, dt . \)  

(S37b)

Given that \( g(t) \) is a sum of three easily integrable terms, cf. Eq. (S17), we can use the multinomial theorem to find:

\[ \int_{0}^{\infty} g(t)^N \, dt = \int_{0}^{\infty} \left( \frac{e^{-\gamma a t}}{\Gamma} + \frac{e^{-\Gamma t}}{\Gamma} - 4 \frac{e^{-\Gamma_t t / 2}}{\Gamma} \right)^N \, dt \]  

(S38a)

\[ = \sum_{k_1+k_2+k_3=N} \left( \frac{\Gamma}{k_1, k_2, k_3} \right) \int_{0}^{\infty} \frac{e^{-k_1 \gamma a t}}{\Gamma^{k_1}} e^{-k_2 \Gamma} \, dt \]  

(S38b)

\[ = \sum_{k_1+k_2+k_3=N} \frac{(-4)^{k_3} \left( \frac{N}{k_1, k_2, k_3} \right)}{\Gamma^{k_1} \Gamma^{k_2} \Gamma^{k_3} (k_1 \gamma a + k_2 \Gamma + k_3 \Gamma + \frac{1}{2})} . \]  

(S38c)

The sum \( k_1 + k_2 + k_3 = N \) is taken over all of the \( (N + 1)(N + 2)/2 \) possible combinations of positive (including zero) integers that satisfy this condition, e.g., for \( N = 2 \), \( (k_1, k_2, k_3) \in \{(0, 0, 2), (0, 2, 0), (2, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1)\} \). With all pieces put together back again, the general expression for average time of detection of the first photon from a \( N \)-photon bundle, reads:

\[ \langle t_{1}^{(N)} \rangle = \left( \frac{\gamma a \Gamma^{N+1}}{\Gamma^2} \right)^N \sum_{k_1+k_2+k_3=N} \frac{(-4)^{k_3} \left( \frac{N}{k_1, k_2, k_3} \right)}{\Gamma^{k_1} \Gamma^{k_2} \Gamma^{k_3} (k_1 \gamma a + k_2 \Gamma + k_3 \Gamma + \frac{1}{2})} . \]  

(S39)

Explicit cases for the first photon are given in Eqs. (S45a), (S45b) and (S45d) below. We now tackle the case \( k = N \), i.e., the last photon:

\[ \langle t_{N}^{(N)} \rangle \propto \int_{0}^{\infty} t_N (g(t_0) - g(t_N))^{N-1} g'(t_N) \, dt_N \]  

(S40a)

\[ = uv \bigg|_{0}^{\infty} - \int_{0}^{\infty} u' v \]  

(S40b)

where

\[ u = t \quad u' = 1 \]  

(S41a)

\[ v = (-1/N)(g(t_0) - g(t))^{N} \quad v' = (g(t_0) - g(t))^{N-1} g'(t) . \]  

(S41b)

So we now basically have to compute:

\[ \frac{(-1)^N N}{N} \int_{0}^{\infty} g(t)^N \, dt \]  

(S42)

since the \( g(t_0) \) offset of the curve cancels in the calculation of its area. Following the same procedure as detailed above, we arrive to:

\[ \langle t_{N}^{(N)} \rangle = N \left( \frac{\gamma a \Gamma^{N+1}}{\Gamma^2} \right)^N \sum_{k_1+k_2+k_3+k_4+k_5+k_6} \frac{(-1)^{k_3+k_4+k_5+k_6}}{\gamma a^{k_1+k_4} \Gamma^{k_2+k_5} \Gamma^{k_3+k_6}} \left( \frac{\Gamma}{\Gamma} \right)^{k_1+k_4} \left( a_{\gamma a + k_2 \Gamma + k_5 \Gamma + \frac{1}{2}} + \Gamma(k_5 + k_7 + \frac{1}{2}) + \Gamma k_9 \right)^2 . \]  

(S43)
The general case for the $k$th photon from a $N$-photon bundle is obtained similarly, and features a cancellation of the multinomial terms which results in a fairly compact notation despite three summations over the combinatorics:

$$
\langle t_k^{(N)} \rangle = 2N! \left( \frac{\gamma_a \Gamma + \Gamma}{\Gamma_a} \right)^N \sum_{k_1+k_2+k_3 = \ldots = N-k} \sum_{k_4 + \ldots + k_9} \sum_{j=1}^{11} \prod_{j=1}^{k_j} \frac{1}{k_j!} \times
\left( -1 \right)^{k_1+k_4+k_7+k_6+k_9} \left( \begin{array}{c} \Gamma \k_1+k_5+k_9 \k_3+k_6+k_9 \end{array} \right) \left( \gamma_a (k_1 + k_7 + \frac{k_4}{2}) + \Gamma (k_2 + k_8 + \frac{k_6}{2}) + \frac{\Gamma}{2} (k_3 + k_9) \right)^2.
$$

(S44)

Summations are, as before (and from the multinomial formula), taken over all positive integers that satisfy the condition and with $\cdots$ indicating successive terms, i.e., $k_4 + \cdots + k_9 = \sum_{j=4}^{9} k_j$. Equations (S39) and (S43) are, of course, recovered as particular cases of Eq. (S44), which is the general result, Eq. (5), retained in the main text. From combinatorics (stars and bars), there are $(N+k-1)^\alpha$ ways to add $k$ positive or zero integers to sum up to $N$. We give explicitly the six expressions for $\langle t_k^{(N)} \rangle$ for all $k$ up to $N = 3$ (these expressions are also plotted in Fig. S3):

$$
\langle t_1^{(1)} \rangle = \frac{\gamma_a^2 + 4 \gamma_a \Gamma + \Gamma^2}{\Gamma \gamma_a (\Gamma + \gamma_a)},
$$

(S45a)

$$
\langle t_2^{(2)} \rangle = \frac{3 \gamma_a^4 + 31 \gamma_a^2 \Gamma + 64 \gamma_a^2 \Gamma^2 + 34 \gamma_a \Gamma^3 + 3 \Gamma^4}{2 \Gamma \gamma_a (\Gamma + \gamma_a) (\Gamma + 3 \gamma_a) (3 \Gamma + \gamma_a)},
$$

(S45b)

$$
\langle t_2^{(2)} \rangle = \frac{3 (3 \gamma_a^4 + 19 \gamma_a^2 \Gamma + 40 \gamma_a^2 \Gamma^2 + 19 \gamma_a \Gamma^3 + 3 \Gamma^4)}{2 \Gamma \gamma_a (\gamma_a + \Gamma) (3 \gamma_a + \Gamma) (\gamma_a + 3 \Gamma)},
$$

(S45c)

$$
\langle t_3^{(3)} \rangle = \frac{10 \gamma_a^6 + 177 \gamma_a^4 \Gamma + 800 \gamma_a^4 \Gamma^2 + 1298 \gamma_a^2 \Gamma^3 + 800 \gamma_a^2 \Gamma^4 + 177 \gamma_a \Gamma^5 + 10 \Gamma^6}{3 \Gamma \gamma_a (\Gamma + \gamma_a) (\Gamma + 2 \gamma_a) (2 \Gamma + \gamma_a) (5 \Gamma + \gamma_a) (\Gamma + 5 \gamma_a)},
$$

(S45d)

$$
\langle t_2^{(2)} \rangle = \left( \frac{150 \gamma_a^8 + 2345 \gamma_a^7 \Gamma + 14493 \gamma_a^6 \Gamma^2 + 41371 \gamma_a^5 \Gamma^3}{6 \Gamma \gamma_a (\gamma_a + \Gamma) (2 \gamma_a + \Gamma) (3 \gamma_a + \Gamma) (5 \Gamma + \gamma_a) (\Gamma + 5 \gamma_a)} \right) + \frac{5876 \gamma_a^4 \Gamma^4 + 41371 \gamma_a^3 \Gamma^5 + 14493 \gamma_a^2 \Gamma^6 + 2345 \gamma_a \Gamma^7 + 150 \Gamma^8}{6 \Gamma \gamma_a (\gamma_a + \Gamma) (2 \gamma_a + \Gamma) (3 \gamma_a + \Gamma) (5 \Gamma + \gamma_a) (\Gamma + 5 \gamma_a)}. \tag{S45e}
$$

This shows how the complexity of these simple and fundamental observables grows quickly for multiphotons. We could not find a further-simplified expression for Eq. (S44) but do not exclude that it exists. There are, indeed, alternative ways to write the particular results, e.g., Eq. (S45a) can also be written as $\langle t_1^{(1)} \rangle = \frac{1}{\gamma_a} + \frac{1}{\Gamma_a} + \frac{1}{\gamma_a + \Gamma}$ with an hoc description that the time of emission of a filtered single photon results from adding to its radiative time, the filtering time $1/\Gamma$ and the time from their combined (averaged) emission rate $1/(\gamma_a + 1/\Gamma)$. Such a forced reading of the equation however becomes obscure for multiphotons, e.g., $\langle t_1^{(1)} \rangle = \frac{1}{\gamma_a} + \frac{1}{\Gamma_a} + \frac{s}{2 \gamma_a + 4 \Gamma} + \frac{s}{3 \gamma_a + 6 \Gamma} + \frac{s}{4 \gamma_a + 8 \Gamma} + \frac{s}{5 \gamma_a + 10 \Gamma}$. Similar decompositions exist for other averages but we could not find them in closed-form. We will discuss below how the unfiltered results, however, simplify considerably.
We now compute, also for illustration that this can be done, other statistical observables of possible interest. Considering, for instance, the variance, one needs to find the squared arrival time for the \( k \)th photon from a \( N \)-photon bundle, which is:

\[
\langle (t_k^{(N)})^2 \rangle = 4N! \left( \frac{\gamma_a \Gamma + \Gamma}{\Gamma-1} \right)^N \sum_{k_1+k_2+\cdots+k_2} \sum_{k_1+k_2+\cdots+k_2} \prod_{i=1}^{10} \frac{1}{k_i!} \times \\
\gamma_a^{k_1+k_2+\cdots+k_2} \Gamma^{k_1+k_2+\cdots+k_2} \left( \frac{\Gamma}{\Gamma} \right)^{k_1+k_2+\cdots+k_2} \left( \gamma_a (k_1 + k_2 + k_4) + \Gamma (k_2 + k_6 + k_8) + \frac{\Gamma}{2} (k_3 + k_9) \right)^3
\]

from which one obtains the standard deviation for the time of emission as:

\[ \sigma_k^{(N)} = \sqrt{\langle (t_k^{(N)})^2 \rangle - \langle t_k^{(N)} \rangle^2}. \]  

(S46)

Particular cases for the averaged squared times of detections are:

\[
\langle (t_1^{(1)})^2 \rangle = \frac{2 (\gamma_a^2 + 5 \gamma_a \Gamma^2 + 12 \gamma_a ^2 \Gamma^2 + 5 \gamma_a ^3 \Gamma^2 + \Gamma^4)}{\gamma_a \Gamma (\gamma_a + \Gamma)^2},
\]

(S48a)

\[
\langle (t_1^{(2)})^2 \rangle = \frac{9 \gamma_a^8 + 132 \gamma_a \Gamma^2 + 86 \gamma_a \Gamma (3 \gamma_a \Gamma + \Gamma^2)^2 (\gamma_a + 3 \Gamma)^2}{2 \gamma_a \Gamma (\gamma_a + \Gamma)^2 (3 \gamma_a \Gamma + \Gamma^2)^2 (\gamma_a + 3 \Gamma)^2},
\]

(S48b)

\[
\langle (t_2^{(2)})^2 \rangle = \frac{63 \gamma_a^8 + 708 \gamma_a \Gamma^2 + 332 \gamma_a \Gamma (3 \gamma_a \Gamma + \Gamma^2)^2 (\gamma_a + 3 \Gamma)^2}{2 \gamma_a \Gamma (\gamma_a + \Gamma)^2 (3 \gamma_a \Gamma + \Gamma^2)^2 (\gamma_a + 3 \Gamma)^2},
\]

(S48c)

e tc. Also in this case, one can find a decomposition in terms of inverse squared rates for \( \langle (t_k^{(N)})^2 \rangle \), e.g., \( \langle (t_1^{(1)})^2 \rangle/2 \frac{1}{\gamma_a^2} + \frac{1}{\gamma_a \Gamma} - \frac{1}{\gamma_a \Gamma} + \frac{4}{\gamma_a + \Gamma} \) (cf. Eq. (S48a)) or \( 4 \langle (t_1^{(2)})^2 \rangle \frac{2}{\gamma_a^2} + \frac{2}{\gamma_a \Gamma} + \frac{8}{\gamma_a + \Gamma} + \frac{9}{\gamma_a + \Gamma} + \frac{9}{\gamma_a + \Gamma} + \frac{18}{\gamma_a \Gamma + \Gamma^2} \) (cf. Eq. (S48b)), which again suggests possible simplifications of the general expression (S46). The squared times in themselves are not of great intrinsic interest, but the standard deviations are. We similarly provide them explicitly for the detection times of all photons up to three-photon bundle. The expressions are so bulky, however, that a notational device needs being developed. Standard deviations are of the type, for the \( k \)th photon of a \( N \)-photon bundle:

\[
\sigma_k^{(N)} = \sqrt{\sum_{i=0}^{\mu_k^{(N)}} \frac{\alpha_{i,k}^{(N)} \gamma_a \Gamma^{\mu_k^{(N)}-i}}{\gamma_a \Gamma^{(\mu_k^{(N)}/2)-1}} \sum_{i=0}^{\beta_{i,k}^{(N)} \gamma_a \Gamma^{(\mu_k^{(N)}/2)-1}}} \}
\]

(S49)

where, for each \( k \), some constants \( \alpha_{i,k}^{(N)} \) for the numerator and \( \beta_{i,k}^{(N)} \) for the denominator are defined (with \( \mu_k^{(N)} + 1 \) terms for the numerator and, to keep the dimensionality correct, \( \mu_k^{(N)} / 2 \) terms for the denominator). For instance:

\[
\sigma_1^{(1)} = \sqrt{\gamma_a^4 + 2 \gamma_a \Gamma + 6 \gamma_a \Gamma^2 + 2 \gamma_a \Gamma^3 + \Gamma^4},
\]

(S50a)

\[
\sigma_1^{(2)} = \sqrt{9 \gamma_a^8 + 78 \gamma_a \Gamma^2 + 427 \gamma_a \Gamma^4 + 1118 \gamma_a \Gamma^6 + 1584 \gamma_a \Gamma^8 + 1118 \gamma_a \Gamma^8 + 427 \gamma_a \Gamma^6 + 78 \gamma_a \Gamma^4 + 9 \Gamma^8},
\]

(S50b)
By construction, $\alpha_{i,k}^{(N)} = \alpha_{N-i,k}^{(N)}$ for all $0 \leq i \leq \mu_k^{(N)}$ and $\beta_{i,k}^{(N)} = \beta_{N-i,k}^{(N)}$ for all $0 \leq i \leq (\mu_k^{(N)}/2) - 1$. It is therefore more concise to only display the minimal set of required coefficients, for instance as follows:

$$\sigma_k^{(N)} \to \frac{(\alpha_{1,k}^{(N)}, \alpha_{2,k}^{(N)}, \ldots, \alpha_{(\mu_k^{(N)})/2,k}^{(N)})}{(\beta_{1,k}^{(N)}, \beta_{2,k}^{(N)}, \ldots, \beta_{(\mu_k^{(N)})/2-1,k}^{(N)})}.$$  (S51)

If a term can be factored out, then we write it in front of the list. Since this structure holds for other statistical quantities, e.g., Eqs. (S45) or Eqs. (S48), we could have adopted this convention for these quantities as well, and would higher photon-number ever prove to be needed, such tabulations (similar to Clebsch–Gordan tables in the problem of angular momentum) would certainly be generalized. For the case of the standard deviation, the coefficients grow so fast that this more concise notation is, this time, mandatory. For all photons up to 3-photon bundles, the standard deviations are found to be:

$$\sigma_1^{(1)} \to \frac{(1, 2, 6)}{1},$$  (S52a)
$$\sigma_2^{(2)} \to \frac{(9, 78, 427, 1118, 1584)}{2(3, 13)},$$  (S52b)
$$\sigma_2^{(2)} \to \frac{(45, 390, 1235, 2662, 3864)}{2(3, 13)},$$  (S52c)
$$\sigma_1^{(3)} \to \frac{(100, 1740, 16609, 895638, 2911672, 585918, 738854)}{3(10, 87, 227)},$$  (S52d)
$$\sigma_2^{(3)} \to \frac{(11700, 281580, 2927353, 18667626, 82053985, 256611346, 576542235, 936554328, 1101372067)}{6(30, 361, 1581, 3212)},$$  (S52e)
$$\sigma_3^{(3)} \to \frac{(44400, 1061340, 11033869, 65375898, 250957345, 683779714, 1399513767, 2177984712, 2534107982)}{6(30, 361, 1581, 3212)},$$  (S52f)

where Eqs. (S52a) and (S52b) encode respectively the expressions (S50) written in full (this should help clarify the notation). In these lists, for each $k$ and $N$, $\mu_k^{(N)}$ is given by twice the numbers of terms in the numerator minus one, i.e., five terms for $(1, 2, 6)$ (three terms in the numerator of Eq. (S52a)) so $2 \times 3 - 1$ terms in the numerator of Eq. (S50a)); nine for $(9, 78, 427, 1118, 1584)$, etc. The resulting standard deviations are shown as bands surrounding the means in Fig. S3, scaled by $1/10$ to avoid cluttering the plot. One must also keep in mind the log-scale in reading the magnitude of the standard deviation. Closer inspection shows, as can be seen on the figure, that the relative deviation, $\sigma_k^{(N)}/(t_k^{(N)})$, is minimum at $\Gamma = \gamma_0$.

One could carry on to higher photon-numbers, higher orders and/or for other statistical quantities. It should be clear how to do this following the examples that we have provided. Independently of whether all such quantities will prove useful in the future, it is remarkable how intricate even the simplest of them are for so fundamental observables. Equation (S50a), for instance, is the standard deviation for the detection time of a single photon. Single-photon emission is by far one of the most investigated type of quantum light, with considerable in-depth analyses performed over decades by countless groups worldwide. One would assume that the standard deviation of its detection time would be well-known and confirmed experimentally by now (from pulsed-emission, for instance). It is, again, remarkable how such basic and fundamental quantities have, however, resisted an exact description to date and proved to be particularly complex. We will turn to limiting cases below, which are naturally much simpler.

By the symmetry discussed in the Main Text, visible in the expressions (and spelled out to tabulate the coefficients for the standard deviations (S52)), both $\Gamma \to \infty$ (no filtering) or $\Gamma \to 0$ (extreme filtering) display symmetric behaviors, for which simple closed-form expressions can be given. The general case might similarly admit such simplified versions, e.g., providing explicitly...
FIG. S3. Mean time of detection \( \langle t_k^{(N)} \rangle \), Eqs. (S45), and its standard deviation \((\times 1/10)\,\) Eqs. (S52), for every photon of a single-photon (blue), two-photon (pink) and three-photon bundle (red) as a function of filtering, conditioned to (a) full-photon detection, i.e., average is on \( \phi_{k}^{(N)} \) or to (b) broken-bundles due to filtering, i.e., average is on \( \phi_{k_k}^{(N)} \), without standard deviation for clarity. The unconditioned mean times from (a) are also shown in dotted lines in (b) for comparison. In (c), the case \( N = 5 \) is shown with normalization to the asymptotes, to evidence the strong departures brought by filtering in the intermediate region.

For higher \( N \), this allows us to compute, for instance, the covariance or quantities derived, the computation of averages from two-photon marginals. We have computed a formula for each \( \alpha_{i_k}^{(N)} \) or \( \beta_{i_k}^{(N)} \), but we have not been able to find them in the general case, although from the results above, one can provide any particular case, for instance \( \langle \sigma_1^{(1)} \rangle^2 = \gamma_0^{-1} \). In particular, the corresponding coefficients, in one representation or another, are not (yet) in the OEIS database of remarkable integer sequences. Regardless of possible future simplifications and/or enlightening interpretations, we wish to stress, however, that our results already provide an exact and comprehensive description of spontaneous emission of multiphotons.

The same exact treatment for these fundamental quantities, which one would wish to have such an exact treatment for these fundamental quantities, which may ultimately be involved in the operating details of, e.g., a photonic computer.

We finally provide, also both for illustration and for the intrinsic interest of the statistical quantities derived, the computation of averages from two-photon marginals. We have computed in Eq. (S32) the waiting time distribution for a biphoton and in Eq. (S34) the joint first-and-last photons distributions. Corresponding two-photon marginals include, e.g.,

\[
\langle t_1^{(N)} t_N^{(N)} \rangle = 32N! \left( -\frac{\gamma_0 \Gamma^+}{\Gamma^-} \right)^N \sum_{k_1 + \cdots + k_N = N} \sum_{k_1 + \cdots + k_N = 2} \prod_{i=1}^{10} \frac{1}{k_i!} \left( \frac{\Gamma^-}{\gamma_0^2} \right)^{k_1+k_2} \times \frac{\gamma_0 (6k_1 + 2k_4 + 4k_7 + 3k_8 + k_9) + \Gamma (6k_2 + 2k_5 + k_8 + 3k_9 + 4k_{10}) + \Gamma^+ (3k_3 + k_6)}{\gamma_0 (2k_4 + k_7 + k_8) + \Gamma (2k_2 + k_5 + k_8 + 3k_9 + 4k_{10}) + \Gamma^+ (3k_3 + k_6)} \times \frac{(\gamma_0 (2k_1 + 2k_4 + 4k_7 + 3k_8 + k_9) + \Gamma (2k_2 + 2k_5 + k_8 + 3k_9 + 4k_{10}) + \Gamma^+ (3k_3 + k_6))^3}{(\gamma_0 (2k_1 + 2k_4 + 4k_7 + 3k_8 + k_9) + \Gamma (2k_2 + 2k_5 + k_8 + 3k_9 + 4k_{10}) + \Gamma^+ (3k_3 + k_6))^3}.
\]

This is a genuine multiphoton observable, that is valid for \( N \geq 2 \), with, e.g.,

\[
\langle t_1^{(2)} t_2^{(2)} \rangle = \frac{(\Gamma^2 + 4\gamma_0^2)^2}{\Gamma^2 \gamma_0^2}.
\]

The case \( N = 1 \) evaluates to zero due to the first sum running its positive coefficients over a total of \(-1\), which is impossible. It must thus be obtained directly from the single-photon marginal Eq. (S46). For higher \( N \), this allows us to compute, for instance, the covariance or...
related statistical indicators, such as, to measure correlations, the normalized covariance, known as the Pearson correlation coefficient:

\[
\rho_{t_1, t_N} \equiv \frac{\langle t_1^{(N)} t_N^{(N)} \rangle - \langle t_1^{(N)} \rangle \langle t_N^{(N)} \rangle}{\sqrt{\langle (t_1^{(N)})^2 \rangle - \langle t_1^{(N)} \rangle^2} \sqrt{\langle (t_N^{(N)})^2 \rangle - \langle t_N^{(N)} \rangle^2}}. \tag{S56}
\]

For the case \(N = 2\), the bulky expression shows an essentially constant coefficient with exact limits \(\rho_{t_1, t_2} = 1/\sqrt{3} \approx 0.57\) at both \(\Gamma \to 0\) and \(\Gamma \to \infty\) with a slight increase to \(25/\sqrt{2929} \approx 0.46\) for \(\Gamma = \gamma_a\) where the correlations are maximum. It can also be checked that these are largely due to the geometrical constrain in the distribution (S9) imposed by the indicator function, as previously commented. The Pearson correlator indeed basically cancels if their ordering in time is randomly shuffled. On the other hand, this is also impacted from the fact that the distribution of events is not centered around their means. In this case, a stronger indicator of correlation is given by the reflective correlation coefficient, defined as:

\[
\tilde{\rho}_{t_1, t_N} \equiv \frac{\langle t_1^{(N)} t_N^{(N)} \rangle}{\sqrt{\langle (t_1^{(N)})^2 \rangle \langle (t_N^{(N)})^2 \rangle}}. \tag{S57}
\]

and, for \(N = 2\), with both stronger correlations and stronger variations of those correlations, from \(2/\sqrt{7} \approx 0.76\) at both \(\Gamma \to 0\) and \(\Gamma \to \infty\) with a now sensibly greater increase to \(16/\sqrt{319} \approx 0.9\) for \(\Gamma = \gamma_a\) where the correlations are also maximum. We do not discuss, but note, a systematic increased degree of correlations between photons when filtering them with a bandwidth that matches their radiative rate. This is shown in Fig S4 for both the theory and a Monte Carlo simulation. This precises one’s physical intuition of strong correlations but with no direct causality between the events, i.e., if the first photon is emitted late in time, then the second also has to, by construction, but without otherwise actively influencing this from the radiating mechanism itself. Correlations are smaller but remain sizable for the reflective coefficient if shuffling the orders of arrival times. This means that if one photon has been emitted late in time, chances are that the other will also have been emitted late in time, again, with no causality. The problem of photon correlations would be more interesting for quantum correlated light, in which case these results will provide a benchmark against which to evidence genuine radiation-correlations.

Also of interest is the statistics of the bundle time-length, i.e., over which period of time \(\tau_N\) does a \(N\)-photon bundle extend. For the averages, there is no need of extra results, since, defining:

\[
\tau_N \equiv t_N^{(N)} - t_1^{(N)} \tag{S58}
\]

the time difference between the last and first photons from the bundles, then clearly \(\langle \tau_N \rangle = \langle t_N^{(N)} \rangle - \langle t_1^{(N)} \rangle\) follows from Eqs. (S44). For a serious statistical treatment of the data, however, one would also need the variance or standard deviation of Eq. (S58), which is \(\sigma_{\tau_N}^2 \equiv \langle \tau_N^2 \rangle - \langle \tau_N \rangle^2\), i.e.,

\[
\sigma_{\tau_N}^2 = (\sigma_1^{(N)})^2 + (\sigma_N^{(N)})^2 + 2\langle t_1^{(N)} \rangle \langle t_N^{(N)} \rangle - \langle t_1^{(N)} \rangle \langle t_N^{(N)} \rangle \tag{S59}
\]

which involves the covariance and thus requires Eq. (S54). We do not provide explicitly the corresponding expressions, which can be obtained from the previous results, but plot them up to \(N = 5\) in Fig. S5. From the various results that can be discussed there, we will content to highlight that correlations between the first and last photons (defining the bundle’s duration or time length) weaken with increasing \(N\), as could be expected. Also interestingly, if normalized to their common asymptotes, unlike Fig. S3(c) where there were strong departures for the various \(k\), we find that the bundle lengths depart very little from each others, namely \(\tau_N/H_{N-1}\) differs from \(\tau_2\) by less than 1% of \(\tau_2\), for \(N\) up to 5 (one needs \(N = 6\) so that \(\tau_6/H_5 - \tau_2) / \tau_2\) reaches an extremum of \(\approx -1.12\%). In all cases, the greatest departures always occur at \(\Gamma = \gamma_a\).

We have seen how the general results involve complex combinatorics interplay of the filter width and free radiation, which have been expressed in exact closed-forms. Considerable simplifications are obtained for the limiting cases. The unfiltered results, for instance (which by symmetry
FIG. S4. Two-photon emission correlation (a) as measured by the reflective Pearson correlator Eq. (S57) (theory, red, vs Monte Carlo, joined dots) and (b) as visualized through a Monte Carlo scatter-plot. At small $\Gamma$, the smaller available signal results in departures from the mean but these are due to exploding uncertainties (shaded area).

FIG. S5. Average time length $\langle \tau_N \rangle$ of a $N$-photon bundle $\pm \sigma^{(N)}_{\tau}/10$ for $N = 2$ (red) till 5 (blue). Dots are Monte Carlo simulations, which depart from the exact theoretical result (lines) when $\Gamma$ gets too small due to increasing numerical error caused by the small signal. In inset, a zoom shows the exact standard deviation (shaded) as compared to that obtained by assuming $\langle t_N^1(t) \rangle / \langle t_N^N(t) \rangle = (t_1^{(N)}/t_N^{(N)})$, showing that the larger the $N$, the smaller the photon correlations.

are also those of vanishing filtering widths $\Gamma \to 0$) are also of intrinsic and special interest. Their general case has not been addressed either in the literature, to the best of our knowledge, despite its tremendous fundamental interest. They, too, can be obtained either directly from the unfiltered pdf, Eq. (S26), or by taking the limit of the filtered result:

$$\langle t_{\infty,k}^{(N)} \rangle \equiv \int_0^\infty t \phi_k^{(N)}(t) \, dt = \lim_{\Gamma \to \infty} \langle t_k^{(N)} \rangle.$$  \hspace{1cm} (S60)

In both cases, we find:

$$\langle t_{\infty,k}^{(N)} \rangle = \frac{k}{\gamma_a} \binom{N}{k} \sum_{l=0}^{k-1} (-1)^{k-1-l} \frac{(k-1)_{l}}{(N-l)^2}.$$  \hspace{1cm} (S61)

This result is a treasure trove of combinatoric, and the table of coefficients that it gives rise to, Table I, has both a rich structure and a simpler one than is conveyed by Eq. (S61) which we can unravel from a physical understanding of the nature of SE. In particular, the first column is simply $\langle t_{\infty,1}^{(N)} \rangle = 1/(N\gamma_a)$ as is expected on physical grounds: the first photon to be emitted can be any of the $N$ available, each given an opportunity independent from the others, thus speeding up its emission rate by a factor $N$. Similarly, the diagonal $\langle t_{\infty,N}^{(N)} \rangle = H_N/\gamma_a$ is the time duration or time-length of a bundle, and was already noted from the bundler’s emission as the result of
TABLE I. Values of $\langle t_{\infty,k}^{N} \rangle_{\gamma_{a}}$, cf. Eq. (S61), up to $N = 10$. One can recognize several famous numerical series in these combinatorics, prominently, the reciprocals on the 1st column and the harmonic numbers on the diagonal. We show in the text how the intermediate values follow a simple pattern, cf. Eq. (S63).

| N   | $\langle t_{\infty,1}^{N} \rangle$ | $\langle t_{\infty,2}^{N} \rangle$ |
|-----|---------------------------------|---------------------------------|
| 1   | 1                               |                                 |
| 2   | $\frac{1}{2}$                   |                                 |
| 3   | $\frac{1}{3} + \frac{1}{6}$     |                                 |
| 4   | $\frac{1}{4} + \frac{1}{6} + \frac{1}{12}$ | $\frac{1}{12}$         |

These values can be built in Table I by summing the Harmonic progression backward, summing from the smallest number, e.g., for the $N$th row: $\frac{1}{N}, \frac{1}{N} + \frac{1}{N-1}, \frac{1}{N} + \frac{1}{N-1} + \frac{1}{N-2}, \ldots, H_{N}$. This means that we can also write (S61) as $\langle t_{\infty,k}^{N} \rangle = \frac{1}{k} \sum_{i=N-k+1}^{N} \frac{1}{i}$ which simplifies to the physically transparent form:

$$\langle t_{\infty,k}^{N} \rangle = \frac{H_{N} - H_{N-k}}{\gamma_{a}}.$$  \hspace{1cm} (S63)

In particular:

$$\langle t_{\infty,1}^{N} \rangle = \frac{1}{N \gamma_{a}},$$ \hspace{1cm} (S64a)

$$\langle t_{\infty,2}^{N} \rangle = \frac{2N - 1}{N(N - 1) \gamma_{a}},$$ \hspace{1cm} (S64b)

$$\ldots$$

$$\langle t_{\infty,\infty}^{N} \rangle = \frac{H_{N}}{\gamma_{a}}.$$ \hspace{1cm} (S64c)

Once such a relationship between Eq. (S61) and Eq. (S63) is established, it is easy to prove its mathematical validity for all cases by recurrence. Equation (S63) is remarkably simple despite its rich structure. In this triangular tabulation, its numerators are given by the OEIS sequence A213998 (numerator of the triangle of fractions read by row) while its denominators are given by the OEIS sequence A093919 (least-common multiple of integers listed in reversed order). Again, while such simplifications or links to combinatorics can be established in the limit $\Gamma \to \infty$, we could find no such description for the general case of finite $\Gamma$, cf. Eqs. (S45), although it appears likely that similar reductions exist given the occurrences of remarkable sequences (one can spot, for instance, Catalan numbers and number-theoretic partitions of integers, such as the...
totient functions and greatest-common divisors). Those are mainly considerations of elegance and aesthetic, however, which do not remove to Eq. (S44) its character of generality, providing the exact result for all \( k \) and \( N \).

The same limits for the unfiltered emission of average squares yields the general result

\[
\langle (t_{\infty,k})^2 \rangle = \frac{2k}{\gamma_a^2} \left( \frac{N}{k} \right) \sum_{l=0}^{k-1} (-1)^{k-l-1} \left( \frac{k-1}{N-1} \right)^l
\]

from which the corresponding variances can be found. Some particular cases can be obtained in a considerably simplified form, for instance:

\[
\langle (t_{\infty,1})^2 \rangle = \frac{2}{(N\gamma_a)^2}, \quad (S66a)
\]

\[
\langle (t_{\infty,2})^2 \rangle = \frac{2 + 3N(N - 1)}{(\gamma_a N(N - 1))^2}, \quad (S66b)
\]

\[
\ldots
\]

\[
\langle (t_{\infty,N})^2 \rangle = \frac{1}{\gamma_a^2} (H_N^2 + H_{N,2}), \quad (S66c)
\]

where \( H_{N,2} \equiv \sum_{k=1}^{N} \frac{1}{k^2} \) is the 2nd generalized harmonic number. As we shall now see, there is a contained closed-form expression for this expression as well, namely:

\[
\langle (t_{\infty,k})^2 \rangle = \frac{1}{\gamma_a^2} (H_{N,2} - H_{N-k,2} - (H_N - H_{N-k})^2). \quad (S67)
\]

This result can be inferred by inspection of the standard deviation for the detection times of photons from unfiltered \( N \)-photon bundles, obtained from Eqs. (S64) and (S66), for which one finds:

\[
\sigma_{\infty,1}^{(N)} = \frac{1}{\gamma_a N}, \quad (S68a)
\]

\[
\sigma_{\infty,2}^{(N)} = \frac{1}{\gamma_a} \sqrt{\frac{1}{N^2} + \frac{1}{(N - 1)^2}}, \quad (S68b)
\]

\[
\ldots
\]

\[
\sigma_{\infty,N}^{(N)} = \frac{\sqrt{H_{N,2}}}{\gamma_a}. \quad (S68c)
\]

This suggests, which can be confirmed a posteriori, that the contained closed-form expression reads:

\[
\sigma_{\infty,k}^{(N)} = \frac{\sqrt{H_{N,2} - H_{N-k,2}}}{\gamma_a}. \quad (S69)
\]

From Eq. (S69) and the definition of the standard deviation (S47), one can therefore deduce Eq. (S66) above. As previously, once the result is found and checked for particular values, it can be firmly established by recurrence.

Similarly, the statistics for the time-length of a fully-detected photon bundle involves the computation of \( \langle (t_{\infty,N} - t_{\infty,1})^2 \rangle \) which itself requires:

\[
\langle (t_{\infty,1} - t_{\infty,N})^2 \rangle = \frac{(-1)^N}{\gamma_a^2} (N - 1) \sum_{l=0}^{N-2} \binom{N-2}{l} \frac{N + 2l + 2}{(-1)^{N-2+l} N^2 (l+1)^2} \quad (S70)
\]

which can be further simplified as:

\[
\langle (t_{\infty,1} t_{\infty,N}) \rangle = \frac{1}{\gamma_a^2} \left( \frac{H_N}{N} + \frac{1}{N^2} \right). \quad (S71)
\]
From the above, one can now obtain the standard deviation for the unfiltered bundle time length as the $\Gamma \to \infty$ limit of Eq. (S59):

$$
\sigma_{\infty,\tau_N}^2 = (\sigma_{\infty,1}^N)^2 + (\sigma_{\infty,2}^N)^2 + 2\left(\langle t_{\infty,1}^{(N)} \rangle \langle t_{\infty,2}^{(N)} \rangle - \langle t_{\infty,1}^{(N)} \rangle^2 \right)
$$

(S72)

which, from Eqs. (S63), (S69), and (S71), simplify to the simple general result:

$$
\sigma_{\infty,\tau_N} = \frac{\sqrt{H_{N-1,2}}}{\gamma_a}
$$

(S73)

i.e., $\sigma_{\infty,\tau_2} = \frac{1}{\gamma_a}$, $\sigma_{\infty,\tau_3} = \frac{\sqrt{5}}{2\gamma_a}$, $\sigma_{\infty,\tau_4} = \frac{3}{4\gamma_a}$, $\sigma_{\infty,\tau_5} = \frac{\sqrt{15}}{2\gamma_a}$, etc. That summarizes nicely the statistics of $N$-photon bundles time lengths:

$$
\langle \tau_N \rangle_{\gamma_a} = H_{N-1} \pm \sqrt{H_{N-1,2}}. 
$$

(S74)

We believe that the above results provide a fairly comprehensive overview of the statistics of bundle emission. What is not provided explicitly can be either obtained from the general expressions or obtained along similar lines of computation if not available here (e.g., temporal correlations in broken bundles, skewness of emission times, kurtosis, etc.). The unfiltered results happen to exhibit beautiful connections to the generalized harmonic numbers. We can imagine how still higher, $k$-orders statistical quantities would likewise involve $H_{N,k}$. We leave such generalizations for separate investigations.

“THERMALIZING” A THERMAL STATE

We write “thermalizing” to describe the effect of filtering because a filter that is narrow enough provides a thermal state out of any quantum field, even single-photon emission [1, 2]. The only exception is if the field has unphysical attributes, e.g., a $c$-number laser field with no spectral width [2]. It is therefore a bit surprising that filtering a thermal field does not produce another thermal field, except, of course, in the limit of a vanishing filter bandwidth since this applies to all fields. This is even more surprising given that all the Glauber correlators at zero delay remain those of a thermal field, namely:

$$
g^{(n)}(0) = n! \quad \text{for all } n \geq 2.
$$

(S75)

The field remains thus closely related to a thermal field in any case, but with some important departures in the finite time correlations, i.e., the $g^{(n)}(\tau)$, including $g^{(1)}(\tau)$ whose Fourier transform provides the normalized power spectrum:

$$
S_{\text{th},\Gamma}(\omega) = \frac{\pi}{2} S_{\text{th}}(\omega) S_{\Gamma}(\omega)(\Gamma + \gamma_a - P_a)
$$

(S76)

where $S_{\text{th}}(\omega)$ and $S_{\Gamma}(\omega)$ are the normalized spectral shapes of the thermal state and of the filters, respectively:

$$
S_{\text{th}}(\omega) = \frac{1}{\pi} \frac{(\gamma_a - P_a)/2}{((\gamma_a - P_a)/2)^2 + \omega^2},
$$

(S77a)

$$
S_{\Gamma}(\omega) = \frac{1}{\pi} \frac{\Gamma/2}{(\Gamma/2)^2 + \omega^2}.
$$

(S77b)

This shows how a thermal field, created under incoherent driving (at rate $P_a$) and spontaneous decay (at rate $\gamma_a$), that is filtered with a Lorentzian filter of bandwidth $\Gamma$, does not itself have a Lorentzian spectral line, and thus cannot be the result of a thermal equilibrium in a cavity. Instead, filtering produces a spectrum with less-fat tails than the thermal field itself, as shown in Fig. S6(a) where the narrower field, thanks to filtering, is, however, non-Lorentzian. To the best of our knowledge, this spectral shape has not reported in the literature, despite its importance prior to the laser and the modern theory of optical coherence, since this was the privileged way to produce a monochromatic field: by filtering thermal light.
FIG. S6. A thermal field of effective temperature $\theta = 0.25$ with $P_a = \theta \gamma a$ (blue) as seen through its power spectrum (a) and second-order correlation function (b), as compared to its filtered field with a filter of width $\Gamma = \gamma a/2$ (yellow) and in the limit of vanishing filtering $\Gamma \to 0$ (dark red). In the latter case, the spectrum becomes a Dirac $\delta$ function while the correlation becomes constant. The dotted line shows the best-fit of the filtered thermal field by an actual thermal field, showing that the spectrum is not Lorentzian and that the decay of the correlation function is of a wholly different character. The limiting case $\Gamma \to 0$ can recover, however, an exact thermal field, by brushing off these departures to infinity.

We also give the second-order correlation of the filtered thermal field to show how these departures take place also for higher-order correlators:

$$g^{(2)}_{\text{th}, \Gamma}(\tau) = 1 + \frac{1}{(\gamma_a - P_a + \Gamma)^2} \left( \Gamma^2 e^{-(\gamma_a - P_a)\tau} + (\gamma_a - P_a)^2 e^{-\Gamma \tau} - 2\Gamma (\gamma_a - P_a) e^{-(\Gamma + \gamma_a - P_a)\tau/2} \right).$$

(S78)

This shows that although $g^{(2)}_{\text{th}, \Gamma}(0) = 2$, the subsequent $\tau$ dynamics differs from that of a thermal field, that is the one recovered when filtering at all frequencies:

$$g^{(2)}_{\text{th}}(\tau) = \lim_{\Gamma \to \infty} g^{(2)}_{\text{th}, \Gamma}(\tau) = 1 + \exp \left( - (\gamma_a - P_a)\tau \right).$$

(S79)

The qualitative difference for this quantity with the two types of fields is shown in Fig. S6(b). The other limit is also interesting:

$$g^{(2)}_{\text{th}, 0}(\tau) = \lim_{\Gamma \to 0} g^{(2)}_{\text{th}, \Gamma}(\tau) = 2.$$  

(S80)

We postulate that $g^{(n)}_{\text{th}, 0}(\tau) = n!$ for all $n$. The correlations are constant because in this limit, the filter vanishes faster than the system’s inverse coherence time, as seen in Eq. (S79). The power spectrum also acquires an extreme form in this limit, the only one that allows it to qualify as an exact thermal state:

$$S_{\text{th}, 0}(\omega) \equiv \lim_{\Gamma \to 0} S_{\text{th}, \Gamma}(\omega) = \delta(\omega).$$

(S81)

The drawback is, of course, that the signal itself is zero. For finite $\Gamma$, one indeed obtains the emission rate of the filtered thermal field as $\gamma_a P_a \Gamma/[(\gamma_a - P_a + \Gamma)(\gamma_a - P_a)]$ that is always smaller than the unfiltered ($\Gamma \to \infty$) rate $I_{\text{th}}$, as should be by conservation of energy, and decays linearly with filtering with a small enough $\Gamma$, $I_{\text{th}, \Gamma} \approx I_{\text{th}} \Gamma$. Ignoring the $\tau$ dynamics, that becomes constant in this limit, since the density matrix itself is that of a thermal state, and the dynamics is “frozen”, one can speak of the effective temperature for the filtered thermal field, with the interesting result that it can actually be higher than that of the original field. Indeed, the filtered-field temperature is $\theta_{\text{th}, \Gamma} = P_a \gamma a/(P_a^2 + (\gamma_a + \Gamma)(\gamma_a - P_a))$ as compared to that of the unfiltered field $\theta_{\text{th}} = P_a / \gamma a$. One can see how, when $\Gamma < P_a$, the filtered field thus has a higher temperature than the field that it is filtering. This apparent paradox is explained in the main text.
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[1] del Valle, E., González-Tudela, A., Laussy, F. P., Tejedor, C. & Hartmann, M. J. Theory of frequency-filtered and time-resolved $n$-photon correlations. *Phys. Rev. Lett.* **109**, 183601 (2012). [http://doi.org/10.1103/PhysRevLett.109.183601](http://doi.org/10.1103/PhysRevLett.109.183601).

[2] González-Tudela, A., Laussy, F. P., Tejedor, C., Hartmann, M. J. & del Valle, E. Two-photon spectra of quantum emitters. *New J. Phys.* **15**, 033036 (2013). [http://doi.org/10.1088/1367-2630/15/3/033036](http://doi.org/10.1088/1367-2630/15/3/033036).