FLEXIBLE WEINSTEIN MANIFOLDS

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To Alan Weinstein with admiration.

Abstract. This survey on flexible Weinstein manifolds is, essentially, an extract from the book [4].

1. Introduction

The notion of a Weinstein manifold was introduced in [11], formalizing the symplectic handlebody construction from Alan Weinstein’s paper [23] and the Stein handlebody construction from [9]. Since then, the notion of a Weinstein manifold has become one of the central notions in symplectic and contact topology. The existence question for Weinstein structures on manifolds of dimension > 4 was settled in [9]. The past five years have brought two major breakthroughs on the uniqueness question: By work of McLean [21] and others we now know that, on any manifold of dimension > 4 which admits a Weinstein structure, there exist infinitely many Weinstein structures that are pairwise non-homotopic (but formally homotopic). On the other hand, Murphy’s h-principle for loose Legendrian knots [22] has led to the notion of flexible Weinstein structures, which are unique up to homotopy in their formal class. In this survey, which is essentially an extract from the book [4], we discuss this uniqueness result and some of its applications.

1.1. Weinstein manifolds and cobordisms.

Definition. A Weinstein structure on an open manifold $V$ is a triple $(\omega, X, \phi)$, where

- $\omega$ is a symplectic form on $V$,
- $\phi : V \to \mathbb{R}$ is an exhausting generalized Morse function,
- $X$ is a complete vector field which is Liouville for $\omega$ and gradient-like for $\phi$.

The quadruple $(V, \omega, X, \phi)$ is then called a Weinstein manifold.

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Let us explain all the terms in this definition. A \textit{symplectic form} is a nondegenerate closed 2-form $\omega$. A \textit{Liouville field} for $\omega$ is a vector field $X$ satisfying $L_X \omega = \omega$; by Cartan’s formula, this is equivalent to saying that the associated \textit{Liouville form} $\lambda := i_X \omega$ satisfies $d\lambda = \omega$. A function $\phi : V \to \mathbb{R}$ is called \textit{exhausting} if it is proper (i.e., preimages of compact sets are compact) and bounded from below. It is called \textit{Morse} if all its critical points are nondegenerate, and \textit{generalized Morse} if its critical points are either nondegenerate or \textit{embryonic}, where the latter condition means that in some local coordinates $x_1, \ldots, x_m$ near the critical point $p$ the function looks like the function $\phi_0$ in the \textit{birth–death family}

$$\phi_t(x) = \phi_t(p) \pm tx_1 + x_1^3 - \sum_{i=2}^{k} x_i^2 + \sum_{j=k+1}^{m} x_j^2.$$  

A vector field $X$ is called \textit{complete} if its flow exists for all times. It is called \textit{gradient-like} for a function $\phi$ if

$$d\phi(X) \geq \delta(|X|^2 + |d\phi|^2),$$

where $\delta : V \to \mathbb{R}_+$ is a positive function and the norms are taken with respect to any Riemannian metric on $V$. Note that away from critical points this just means $d\phi(X) > 0$. Critical points $p$ of $\phi$ agree with zeroes of $X$, and $p$ is nondegenerate (resp. embryonic) as a critical point of $\phi$ if it is nondegenerate (resp. embryonic) as a zero of $X$. Here a zero $p$ of a vector field $X$ is called embryonic if $X$ agrees near $p$, up to higher order terms, with the gradient of a function having $p$ as an embryonic critical point.

It is not hard to see that any Weinstein structure $(\omega, X, \phi)$ can be perturbed to make the function $\phi$ Morse. However, in 1-parameter families of Weinstein structures embryonic zeroes are generically unavoidable. Since we wish to study such families, we allow for embryonic zeroes in the definition of a Weinstein structure.

We will also consider Weinstein structures on a \textit{cobordism}, i.e., a compact manifold $W$ with boundary $\partial W = \partial_+ W \cup \partial_- W$. The definition of a \textit{Weinstein cobordism} $(W, \omega, X, \phi)$ differs from that of a Weinstein manifold only in replacing the condition that $\phi$ is exhausting by the requirement that $\partial_\pm W$ are regular level sets of $\phi$ with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$, and completeness of $X$ by the condition that $X$ points inward along $\partial_- W$ and outward along $\partial_+ W$.

A Weinstein cobordism with $\partial_- W = \emptyset$ is called a \textit{Weinstein domain}. Thus any Weinstein manifold $(V, \omega, X, \phi)$ can be exhausted by Weinstein domains $W_k = \{ \phi \leq c_k \}$, where $c_k \nearrow \infty$ is a sequence of regular values of the function $\phi$. 
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The Liouville form $\lambda = i_X \omega$ induces contact forms $\alpha_c := \lambda|_{\Sigma_c}$ and contact structures $\xi_c := \ker(\alpha_c)$ on all regular level sets $\Sigma_c := \phi^{-1}(c)$ of $\phi$. In particular, the boundary components of a Weinstein cobordism carry contact forms which make $\partial_+ W$ a symplectically convex and $\partial_- W$ a symplectically concave boundary (i.e., the orientation induced by the contact form agrees with the boundary orientation on $\partial_+ W$ and is opposite to it on $\partial_- W$). Contact manifolds which appear as boundaries of Weinstein domains are called Weinstein fillable.

A Weinstein manifold $(V, \omega, X, \phi)$ is said to be of finite type if $\phi$ has only finitely many critical points. By attaching a cylindrical end $(\mathbb{R}^+ \times \partial W, d(e^r \lambda|_{\partial W}), \frac{\partial}{\partial r}, f(r))$ (i.e., the positive half of the symplectization of the contact structure on the boundary) to the boundary, any Weinstein domain $(W, \omega, X, \phi)$ can be completed to a finite type Weinstein manifold, called its completion. Conversely, any finite type Weinstein manifold can be obtained by attaching a cylindrical end to a Weinstein domain.

Here are some basic examples of Weinstein manifolds.

(1) $\mathbb{C}^n$ with complex coordinates $x_j + iy_j$ carries the canonical Weinstein structure

\[
\left( \sum_j dx_j \wedge dy_j, \frac{1}{2} \sum_j (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}), \sum_j (x_j^2 + y_j^2) \right).
\]

(2) The cotangent bundle $T^*Q$ of a closed manifold $Q$ carries a canonical Weinstein structure which in canonical local coordinates $(q_j, p_j)$ is given by

\[
\left( \sum_j dp_j \wedge dq_j, \sum_j p_j \frac{\partial}{\partial p_j}, \sum_j p_j^2 \right).
\]

(As it stands, this is not yet a Weinstein structure because $\sum_j p_j^2$ is not a generalized Morse function, but it can be easily perturbed to make the function Morse.)

(3) The product of two Weinstein manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ has a canonical Weinstein structure $(V_1 \times V_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$. The product $V \times \mathbb{C}$ with its canonical Weinstein structure is called the stabilization of the Weinstein manifold $(V, \omega, X, \phi)$.

In a Weinstein manifold $(V, \omega, X, \phi)$, there is an intriguing interplay between Morse theoretic properties of $\phi$ and symplectic geometry: the stable manifold $W^-_p$ (with respect to the vector field $X$) of a critical point $p$ is isotropic in the symplectic sense (i.e., $\omega|_{W^-_p} = 0$), and its intersection with every regular level set $\phi^{-1}(c)$ is isotropic in the contact sense (i.e., it is tangent to $\xi_c$). In particular, the Morse indices of critical points of $\phi$ are $\leq \frac{1}{2} \dim V$. 
1.2. Stein – Weinstein – Morse. Weinstein structures are related to several other interesting structures as shown in the following diagram:

\[
\begin{array}{c}
\text{Stein} \xrightarrow{\mathcal{M}} \text{Weinstein} \xrightarrow{\mathcal{M}} \text{Morse} \\
\downarrow \\
\text{Liouville}
\end{array}
\]

Here \text{Weinstein} denotes the space of Weinstein structures and \text{Morse} the space of generalized Morse functions on a fixed manifold \(V\) or a cobordism \(W\). As before, we require the function \(\phi\) to be exhausting in the manifold case, and to have \(\partial \pm W\) as regular level sets with \(\phi|_{\partial \pm W} = \min \phi\) and \(\phi|_{\partial \pm W} = \max \phi\) in the cobordism case. The map \(\mathcal{M} : \text{Weinstein} \to \text{Morse}\) is the obvious one \((\omega, X, \phi) \mapsto \phi\).

The space \text{Liouville} of Liouville structures consists of pairs \((\omega, X)\) of a symplectic form \(\omega\) and a vector field \(X\) (the Liouville field) satisfying \(L_X \omega = \omega\). Moreover, in the cobordism case we require that the Liouville field \(X\) points inward along \(\partial_- W\) and outward along \(\partial_+ W\), and in the manifold case we require that \(X\) is complete and there exists an exhaustion \(V_1 \subset V_2 \subset \cdots\) of \(V = \bigcup_k V_k\) by compact sets with smooth boundary \(\partial V_k\) along which \(X\) points outward. The map \(\mathcal{M} : \text{Stein} \to \text{Liouville}\) sends \((\omega, X, \phi)\) to \((\omega, X)\).

The space \text{Stein} of Stein structures consists of pairs \((J, \phi)\) of an integrable complex structure \(J\) and a generalized Morse function \(\phi\) (exhausting resp. constant on the boundary components) such that \(-dd^C \phi(v, Jv) > 0\) for all \(0 \neq v \in TV\), where \(d^C \phi := d\circ J\). If \((J, \phi)\) is a Stein structure, then \(\omega_\phi := -dd^C \phi\) is a symplectic form compatible with \(J\). Moreover, the Liouville field \(X_\phi\) defined by

\[
i_{X_\phi} \omega_\phi = -d^C \phi
\]

is the gradient of \(\phi\) with respect to the Riemannian metric \(g_\phi := \omega_\phi(\cdot, J\cdot)\). In the manifold case, completeness of \(X_\phi\) can be arranged by replacing \(\phi\) by \(f \circ \phi\) for a diffeomorphism \(f : \mathbb{R} \to \mathbb{R}\) with \(f'' \geq 0\) and \(\lim_{x \to \infty} f'(x) = \infty\); we will suppress the function \(f\) from the notation. So we have a canonical map

\[
\mathcal{W} : \text{Stein} \to \text{Weinstein}, \quad (J, \phi) \mapsto (\omega_\phi, X_\phi, \phi).
\]

It is interesting to compare the homotopy types of these spaces. For simplicity, let us consider the case of a compact domain \(W\) and equip all spaces with the \(C^\infty\) topology. The results which we discuss below remain true in the manifold case, but one needs to define the topology more carefully; see Section 4.3 below. Since all the
spaces have the homotopy types of CW complexes, any weak homotopy equivalence between them is a homotopy equivalence.

The spaces Liouville and Weinstein are very different: there exist many examples of Liouville domains that admit no Weinstein structure, and of contact manifolds that bound a Liouville domain but no Weinstein domain. The first such example was constructed by McDuff [20]: the manifold \([0, 1] \times \Sigma\), where \(\Sigma\) is the unit cotangent bundle of a closed oriented surface of genus \(> 1\), carries a Liouville structure, but its boundary is disconnected and hence cannot bound a Weinstein domain. Many more such examples are discussed in [16].

By contrast, the spaces of Stein and Weinstein structures turn out to be closely related. One of the main results of the book [4] is

**Theorem 1.1.** The map \(\mathcal{W} : \text{Stein} \to \text{Weinstein}\) induces an isomorphism on \(\pi_0\) and a surjection on \(\pi_1\).

It lends evidence to the conjecture that \(\mathcal{W} : \text{Stein} \to \text{Weinstein}\) is a homotopy equivalence.

The relation between the spaces Morse and Weinstein is the subject of this article. Note first that, since for a Weinstein domain \((W, \omega, X, \phi)\) of real dimension \(2n\) all critical points of \(\phi\) have index \(\leq n\), one should only consider the subset \(\text{Morse}_n \subset \text{Morse}\) of functions all of whose critical points have index \(\leq n\). Moreover, one should restrict to the subset \(\text{Weinstein}_{\eta}^{\text{flex}} \subset \text{Weinstein}\) of Weinstein structures \((\omega, X, \phi)\) with \(\omega\) in a fixed given homotopy class \(\eta\) of nondegenerate 2-forms which are flexible in the sense of Section 2 below. The following sections are devoted to the proof of

**Theorem 1.2 (4).** Let \(\eta\) be a nonempty homotopy class of nondegenerate 2-forms on a domain or manifold of dimension \(2n > 4\). Then:

(a) Any Morse function \(\phi \in \text{Morse}_n\) can be lifted to a flexible Weinstein structure \((\omega, X, \phi)\) with \(\omega \in \eta\).

(b) Given two flexible Weinstein structures \((\omega_0, X_0, \phi_0), (\omega_1, X_1, \phi_1)\) \(\in \text{Weinstein}_{\eta}^{\text{flex}}\), any path \(\phi_t \in \text{Morse}_n, t \in [0, 1]\), connecting \(\phi_0\) and \(\phi_1\) can be lifted to a path of flexible Weinstein structures \((\omega_t, X_t, \phi_t)\) connecting \((\omega_0, X_0, \phi_0)\) and \((\omega_1, X_1, \phi_1)\).

In other words, the map \(\mathcal{M} : \text{Weinstein}_{\eta}^{\text{flex}} \to \text{Morse}_n\) has the following properties:

- \(\mathcal{M}\) is surjective;
- the fibers of \(\mathcal{M}\) are path connected;
- \(\mathcal{M}\) has the path lifting property.

This motivates the following
Conjecture. On a domain or manifold of dimension \(2n > 4\), the map \(\mathcal{M} : \text{Weinstein}^{\text{flex}}_n \to \text{Morse}_n\) is a Serre fibration with contractible fibers.

2. Flexible Weinstein structures

Roughly speaking, a Weinstein structure is “flexible” if all its attaching spheres obey an \(h\)-principle. More precisely, note that each Weinstein manifold or cobordism can be cut along regular level sets of the function into Weinstein cobordisms that are elementary in the sense that there are no trajectories of the vector field connecting different critical points. An elementary \(2n\)-dimensional Weinstein cobordism \((W, \omega, X, \phi)\), \(n > 2\), is called flexible if the attaching spheres of all index \(n\) handles form in \(\partial - W\) a loose Legendrian link in the sense of Section 2.3 below. A Weinstein cobordism or manifold structure \((\omega, X, \phi)\) is called flexible if it can be decomposed into elementary flexible cobordisms.

A \(2n\)-dimensional Weinstein structure \((\omega, X, \phi)\), \(n \geq 2\), is called subcritical if all critical points of the function \(\phi\) have index < \(n\). In particular, any subcritical Weinstein structure in dimension \(2n > 4\) is flexible.

The notion of flexibility can be extended to dimension 4 as follows. We call a \(4\)-dimensional Weinstein cobordism flexible if it is either subcritical, or the contact structure on \(\partial - W\) is overtwisted (or both); see Section 2.2 below. In particular, a \(4\)-dimensional Weinstein manifold is then flexible if and only if it is subcritical.

Remark 2.1. The property of a Weinstein structure being subcritical is not preserved under Weinstein homotopies because one can always create index \(n\) critical points (see Proposition [4.7 below]. We do not know whether flexibility is preserved under Weinstein homotopies. In fact, it is not even clear to us whether every decomposition of a flexible Weinstein cobordism \(W\) into elementary cobordisms consists of flexible elementary cobordisms. Indeed, if \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are two partitions of \(W\) into elementary cobordisms and \(\mathcal{P}_2\) is finer than \(\mathcal{P}_1\), then flexibility of \(\mathcal{P}_1\) implies flexibility of \(\mathcal{P}_2\) (in particular the partition for which each elementary cobordism contains only one critical value is then flexible), but we do not know whether flexibility of \(\mathcal{P}_2\) implies flexibility of \(\mathcal{P}_1\).

The remainder of this section is devoted to the definition of loose Legendrian links and a discussion of the relevant \(h\)-principles.

2.1. Gromov’s \(h\)-principle for subcritical isotropic embeddings. Consider a contact manifold \((M, \xi = \ker \alpha)\) of dimension \(2n - 1\) and a manifold \(\Lambda\) of dimension \(k - 1 \leq n - 1\). A monomorphism \(F : T\Lambda \to TM\) is a fiberwise injective bundle homomorphism covering a smooth map \(f : \Lambda \to M\). It is called isotropic if it
sends each $T_x\Lambda$ to a symplectically isotropic subspace of $\xi_{f(x)}$ (with respect to the symplectic form $d\alpha|_\xi$). A formal isotropic embedding of $\Lambda$ into $(M, \xi)$ is a pair $(f, F^s)$, where $f : \Lambda \hookrightarrow M$ is a smooth embedding and $F^s : T\Lambda \to TM$, $s \in [0, 1]$, is a homotopy of monomorphisms covering $f$ that starts at $F^0 = df$ and ends at an isotropic monomorphism $F^1 : T\Lambda \to \xi$. In the case $k = n$ we also call this a formal Legendrian embedding.

Any genuine isotropic embedding can be viewed as a formal isotropic embedding $(f, F^s \equiv df)$. We will not distinguish between an isotropic embedding and its canonical lift to the space of formal isotropic embeddings. A homotopy of formal isotropic embeddings $(f_t, F^s_t)$, $t \in [0, 1]$, will be called a formal isotropic isotopy. Note that the maps $f_t$ underlying a formal isotropic isotopy form a smooth isotopy.

In the subcritical case $k < n$, Gromov proved the following $h$-principle.

**Theorem 2.2** ($h$-principle for subcritical isotropic embeddings [17, 12]).

Let $(M, \xi)$ be a contact manifold of dimension $2n - 1$ and $\Lambda$ a manifold of dimension $k - 1 < n - 1$. Then the inclusion of the space of isotropic embeddings $\Lambda \hookrightarrow (M, \xi)$ into the space of formal isotropic embeddings is a weak homotopy equivalence. In particular:

(a) Given any formal isotropic embedding $(f, F^s)$ of $\Lambda$ into $(M, \xi)$, there exists an isotropic embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is $C^0$-close to $f$ and formally isotropically isotopic to $(f, F^s)$.

(b) Let $(f_t, F_t^s)$, $t \in [0, 1]$, be a formal isotropic isotopy connecting two isotropic embeddings $f_0, f_1 : \Lambda \hookrightarrow M$. Then there exists an isotropic isotopy $\tilde{f}_t$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is $C^0$-close to $f_t$ and is homotopic to the formal isotopy $(f_t, F^s_t)$ through formal isotropic isotopies with fixed endpoints.

Let us discuss what happens with this theorem in the critical case $k = n$. Part (a) remains true in all higher dimensions $k = n > 2$:

**Theorem 2.3** (Existence theorem for Legendrian embeddings for $n > 2$ [9, 11]).

Let $(M, \xi)$ be a contact manifold of dimension $2n - 1 \geq 5$ and $\Lambda$ a manifold of dimension $n - 1$. Then given any formal Legendrian embedding $(f, F^s)$ of $\Lambda$ into $(M, \xi)$, there exists a Legendrian embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is $C^0$-close to $f$ and formally Legendrian isotopic to $(f, F^s)$.

Part (b) of Theorem 2.2 does not carry over to the critical case $k = n$: For any $n \geq 2$, there are many examples of pairs of Legendrian knots in $(\mathbb{R}^{2n-1}, \xi_{st})$ which are formally Legendrian isotopic but not Legendrian isotopic; see e.g. [3, 7].

1 The hypothesis in [4] that $\Lambda$ is simply connected can be easily removed.
2.2. Legendrian knots in overtwisted contact manifolds. Finally, let us consider Theorem 2.2 in the case $k = n = 2$, i.e., for Legendrian knots (or links) in contact 3-manifolds. Recall that in dimension 3 there is a dichotomy between tight and overtwisted contact structures, which was introduced in [8]. A contact structure on a 3-dimensional manifold $M$ is called overtwisted if there exists an embedded disc $D \subset M$ which is tangent to $\xi$ along its boundary $\partial D$. Equivalently, one can require the existence of an embedded disc with Legendrian boundary $\partial D$ which is transverse to $\xi$ along $\partial D$. A disc with such properties is called an overtwisted disc.

Part (a) of Theorem 2.2 becomes false for $k = n = 2$ due to Bennequin’s inequality. Let us explain this for $\mathbb{R}^3$ with its standard (tight) contact structure $\xi_{st} = \ker \alpha_{st}$, $\alpha_{st} = dz - pdq$. To any formal Legendrian embedding $(f, F^s)$ of $S^1$ into $(\mathbb{R}^3, \xi_{st})$ we can associate two integers as follows. Identifying $\xi_{st} \cong \mathbb{R}^2$ via the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the $(q,p)$-plane, the fiberwise injective bundle homomorphism $F^1 : TS^1 \cong S^1 \times \mathbb{R} \to \xi_{st} \cong \mathbb{R}^2$ gives rise to a map $S^1 \to \mathbb{R}^2 \setminus 0$, $t \mapsto F^1(t, 1)$. The winding number of this map around $0 \in \mathbb{R}^2$ is called the rotation number $r(f, F^1)$. On the other hand, $(F^1, iF^1, \partial_z)$ defines a trivialization of the bundle $f^*T\mathbb{R}^3$, where $i$ is the standard complex structure on $\mathbb{R}^3 \cong \mathbb{C}$. Using the homotopy $F^s$, we homotope this to a trivialization $(e_1, e_2, e_3)$ of $f^*T\mathbb{R}^3$ with $e_1 = \dot{f}$ (unique up to homotopy). The Thurston–Bennequin invariant $tb(f, F^s)$ is the linking number of $f$ with a push-off in direction $e_2$. It is not hard to see that the pair of invariants $(r, tb)$ yields a bijection between homotopy classes of formal Legendrian embeddings covering a fixed smooth embedding $f$ and $\mathbb{Z}^2$. In particular, the pair $(r, tb)$ can take arbitrary values on formal Legendrian embeddings, while for genuine Legendrian embeddings $f : S^1 \hookrightarrow (\mathbb{R}^3, \xi_{st})$ the values of $(r, tb)$ are constrained by Bennequin’s inequality \cite{14}

\[ tb(f) + |r(f)| \leq -\chi(\Sigma), \]

where $\Sigma$ is a Seifert surface for $f$.

Bennequin’s inequality, and thus the failure of part (a), carry over to all tight contact 3-manifolds. On the other hand, Bennequin’s inequality fails, and except for the $C^0$-closeness Theorem 2.2 remains true, on overtwisted contact 3-manifolds:

**Theorem 2.4** \cite{6 \cite{10}}. Let $(M, \xi)$ be a closed connected overtwisted contact 3-manifold, and $D \subset M$ an overtwisted disc.

(a) Any formal Legendrian knot $(f, F^s)$ in $M$ is formally Legendrian isotopic to a Legendrian knot $\tilde{f} : S^1 \hookrightarrow M \setminus D$.

(b) Let $(f_t, F_t^s)$, $s, t \in [0, 1]$, be a formal Legendrian isotopy in $M$ connecting two Legendrian knots $f_0, f_1 : S^1 \hookrightarrow M \setminus D$. Then there exists a Legendrian isotopy $\tilde{f}_t : S^1 \hookrightarrow M \setminus D$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is homotopic to $(f_t, F_t^s)$ through formal Legendrian isotopies with fixed endpoints.
Although Theorem 2.2 (b) generally fails for knots in tight contact 3-manifolds, there are some remnants for special classes of Legendrian knots:

- any two formally Legendrian isotopic unknots in $(\mathbb{R}^3, \xi_{st})$ are Legendrian isotopic [10];
- any two formally Legendrian isotopic knots become Legendrian isotopic after sufficiently many stabilizations (whose number depends on the knots) [15].

In [22], E. Murphy discovered that the situation becomes much cleaner for $n > 2$: on any contact manifold of dimension $\geq 5$ there exists a class of Legendrian knots, called loose, which satisfy both parts of Theorem 2.2. Let us now describe this class.

2.3. Murphy’s $h$-principle for loose Legendrian knots. In order to define loose Legendrian knots we need to describe a local model. Throughout this section we assume $n > 2$.

Consider first a Legendrian arc $\lambda_0$ in the standard contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ with front projection as shown in Figure 2.1 for some $a > 0$. Suppose that the slopes at the self-intersection point are $\pm 1$ and the slope is everywhere in the interval $[-1, 1]$, so the Legendrian arc $\lambda_0$ is contained in the box

$$Q_a := \{|q_1|, |p_1|, |z| \leq a\}$$

and $\partial \lambda_0 \subset \partial Q_a$. Consider now the standard contact space $(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i)$, which we view as the product of the contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ and the Liouville space $(\mathbb{R}^{2n-4}, - \sum_{i=2}^{n-1} p_i dq_i)$. We set $q' := (q_2, \ldots, q_{n-1})$ and $p' := (p_2, \ldots, p_{n-1})$. For $b, c > 0$ we define

$$P_{bc} := \{|q'| \leq b, |p'| \leq c\} \subset \mathbb{R}^{2n-4},$$

$$R_{abc} := Q_a \times P_{bc} = \{|q_1|, |p_1|, |z| \leq a, |q'| \leq b, |p'| \leq c\}.$$
Let the Legendrian solid cylinder \( \Lambda_0 \subset (\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i) \) be the product of \( \lambda_0 \subset \mathbb{R}^3 \) with the Lagrangian disc \( \{ p' = 0, |q'| \leq b \} \subset \mathbb{R}^{2n-4} \). Note that \( \Lambda_0 \subset R_{abc} \) and \( \partial \Lambda_0 \subset \partial R_{abc} \). The front of \( \Lambda_0 \) is obtained by translating the front of \( \lambda_0 \) in the \( q' \)-directions; see Figure 2.2. The pair \( (R_{abc}, \Lambda_0) \) is called a standard loose Legendrian chart if \[ a < bc. \]

Given any contact manifold \( (M^{2n-1}, \xi) \), a Legendrian submanifold \( \Lambda \subset M \) with connected components \( \Lambda_1, \ldots, \Lambda_k \) is called loose if there exist Darboux charts \( U_1, \ldots, U_k \subset M \) such that \( \Lambda_i \cap U_j = \emptyset \) for \( i \neq j \) and each pair \( (U_i, \Lambda_i \cap U_i) \), \( i = 1, \ldots, k \), is isomorphic to a standard loose Legendrian chart \( (R_{abc}, \Lambda_0) \). A Legendrian embedding \( f : \Lambda \hookrightarrow M \) is called loose if its image is a loose Legendrian submanifold.

**Remark 2.5.** (1) By the contact isotopy extension theorem, looseness is preserved under Legendrian isotopies within a fixed contact manifold. Since the model \( \Lambda_0 \) above can be slightly extended to a Legendrian disc in standard \( \mathbb{R}^{2n-1} \), and any two Legendrian discs are isotopic, it follows that any Legendrian disc is loose.

(2) By rescaling \( q' \) and \( p' \) with inverse factors one can always achieve \( c = 1 \) in the definition of a standard loose Legendrian chart. However, the inequality \( a < bc \) is absolutely crucial in the definition. Indeed, it follows from Gromov’s isocontact embedding theorem that around any point in any Legendrian submanifold \( \Lambda \) one can find a Darboux neighborhood \( U \) such that the pair \( (U, \Lambda \cap U) \) is isomorphic to \( (R_{1b1}, \Lambda_0) \) for some sufficiently small \( b > 0 \).

(3) Figure 2.3 taken from [22] shows that the definition of looseness does not depend on the precise choice of the standard loose Legendrian chart \( (R_{abc}, \Lambda_0) \): Given a standard loose Legendrian chart with \( c = 1 \), the condition \( a < b \) allows us to shrink its front in the \( q' \)-directions, keeping it fixed near the boundary and with all partial derivatives in \([-1, 1]\) (so the deformation remains in the Darboux chart \( R_{a'b1} \), to another standard loose Legendrian chart \( (R_{a'b1}, \Lambda'_0) \) with \( b' \geq (b - a)/2 \) and arbitrarily small \( a' > 0 \). Moreover, we can arbitrarily prescribe the shape of the cross section.
Figure 2.3. Shrinking a standard loose Legendrian chart (picture is courtesy of E. Murphy).

$\lambda'$ of $\Lambda'_0$ in this process. So if a Legendrian submanifold is loose for some model $(R_{abc}, \Lambda_0)$, then it is also loose for any other model. In particular, fixing $b, c$ we can make $a$ arbitrarily small, and we can create arbitrarily many disjoint standard loose Legendrian charts.

Now we can state the main result from [22].

**Theorem 2.6 (Murphy’s h-principle for loose embeddings [22]).**

Let $(M, \xi)$ be a contact manifold of dimension $2n - 1 \geq 5$ and $\Lambda$ a manifold of dimension $n - 1$. Then:

(a) Given any formal Legendrian embedding $(f, F^*)$ of $\Lambda$ into $(M, \xi)$, there exists a loose Legendrian embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is $C^0$-close to $f$ and formally Legendrian isotopic to $(f, F^*)$.

(b) Let $(f_t, F_t^*)$, $t \in [0, 1]$, be a formal Legendrian isotopy connecting two loose Legendrian embeddings $f_0 : f_1 : \Lambda \hookrightarrow M$. Then there exists a Legendrian isotopy $\tilde{f}_t$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is $C^0$-close to $f_t$ and is homotopic to the formal isotopy $(f_t, F_t^*)$ through formal Legendrian isotopies with fixed endpoints.

Part (a) of this theorem is a consequence of Theorem 2.3 and the stabilization construction which we describe next.

**2.4. Stabilization of Legendrian submanifolds.** Consider a Legendrian submanifold $\Lambda_0$ in a contact manifold $(M, \xi)$ of dimension $2n - 1$. Near a point of $\Lambda_0$, pick Darboux coordinates $(q_1, p_1, \ldots, q_{n-1}, p_{n-1}, z)$ in which $\xi = \ker(dz - \sum_j p_j dq_j)$ and the front projection of $\Lambda_0$ is a standard cusp $z^2 = q_1^3$. Deform the two branches of the front to make them parallel over some open ball $B^{n-1} \subset \mathbb{R}^{n-1}$. After rescaling, we
may thus assume that the front of $\Lambda_0$ has two parallel branches $\{z = 0\}$ and $\{z = 1\}$ over $B^{n-1}$, see Figure 2.4.

Pick a non-negative function $\phi: B^{n-1} \to \mathbb{R}$ with compact support and 1 as a regular value, so $N := \{\phi \geq 1\} \subset B^{n-1}$ is a compact manifold with boundary. Replacing for each $t \in [0, 1]$ the lower branch $\{z = 0\}$ by the graph $\{z = t\phi(q)\}$ of the function $t\phi$ yields the fronts of a path of Legendrian immersions $\Lambda_t \subset M$ connecting $\Lambda_0$ to a new Legendrian submanifold $\Lambda_1$. Note that $\Lambda_t$ has a self-intersection for each critical point of $t\phi$ on level 1.

We count the self-intersections with signs as follows. Consider the immersion $\Gamma := \bigcup_{t \in [0, 1]} \Lambda_t \times \{t\} \subset M \times [0, 1]$. After a generic perturbation, we may assume that $\Gamma$ has finitely many transverse self-intersections and define its self-intersection index

$$I_{\Gamma} := \sum_p I_{\Gamma}(p) \in \begin{cases} \mathbb{Z} & \text{if } n \text{ is even}, \\ \mathbb{Z}_2 & \text{if } n \text{ is odd} \end{cases}$$

as the sum over the indices of all self-intersection points $p$. Here the index $I_{\Gamma}(p) = \pm 1$ is defined by comparing the orientations of the two intersecting branches of $\Gamma$ to the orientation of $M \times [0, 1]$. For $n$ even this does not depend on the order of the branches and thus gives a well-defined integer, while for $n$ odd it is only well-defined mod 2.

By a theorem of Whitney [24], for $n > 2$, the regular homotopy $\Lambda_t$ can be deformed through regular homotopies fixed at $t = 0, 1$ to an isotopy iff $I_{\Gamma} = 0$.

**Proposition 2.7** ([22]). For $n > 2$, the Legendrian regular homotopy $\Lambda_t$ obtained from the stabilization construction over a nonempty domain $N \subset B^{n-1}$ has the following properties.

(a) $\Lambda_1$ is loose.

(b) If $\chi(N) = 0$, then $\Lambda_1$ is formally Legendrian isotopic to $\Lambda_0$. 
(c) The regular homotopy \((\Lambda_t)_{t \in [0,1]}\) has self-intersection index \((-1)^{(n-1)(n-2)/2}\chi(N)\).

Proof. (a) Recall that in the stabilization construction we choose a Darboux chart in which the front of \(\Lambda_0\) consists of the two branches \(\{ z = \pm q_3^{3/2} \}\) of a standard cusp, and then deform the lower branch to the graph of a function \(\phi\) which is bigger than \(q_3^{3/2}\) over a domain \(N \subset \mathbb{R}^{n-1}\); see Figure 2.5. Performing this construction sufficiently close to the cusp edge, we can keep the values and the differential of the function \(\phi\) arbitrarily small. Then the deformation is localized within the chosen Darboux neighborhood, and comparing Figures 2.5 and 2.2 we see that \(\Lambda_1\) is loose.

(b) Consider again the stabilization construction on the two parallel branches \(\{ z = 0 \}\) and \(\{ z = 1 \}\) of \(\Lambda_0\) over the domain \(N = \{ \phi \geq 1 \}\). Since \(\chi(N) = 0\), there exists a nowhere vanishing vector field \(v\) on \(N\) which agrees with \(\nabla \phi\) near \(\partial N\). Linearly interpolating the \(p\)-coordinate of \(\Lambda_1\) from \(\nabla \phi(q)\) to \(v(q)\) (keeping \((q,z)\) fixed), then pushing the \(z\)-coordinate down to 0 (keeping \((q,p)\) fixed), and finally linearly interpolating \(v(q)\) to 0 (keeping \((q,z)\) fixed) defines a smooth isotopy \(f_t: \Lambda_0 \hookrightarrow M\) from \(f_0 = \mathbb{1}: \Lambda_0 \to \Lambda_0\) to a parametrization \(f_1: \Lambda_0 \to \Lambda_1\). On the other hand, the graphs of the functions \(t\phi\) define a Legendrian regular homotopy from \(f_0\) to \(f_1\), so their differentials give a path of Legendrian monomorphisms \(F_t\) from \(F_0 = df_0\) to \(F_1 = df_1\). Now note that over the region \(N\) all the \(df_i\) and \(F_i\) project as the identity onto the \(q\)-plane, so linearly connecting \(df_t\) and \(F_t\) yields a path of monomorphisms \(F_t^s, s \in [0,1]\), and hence the desired formal Legendrian isotopy \((f_t, F_t^s)\) from \(f_0\) to \(f_1\).
To prove (c), make the function $\phi$ Morse on $N$ and apply the Poincaré–Hopf index theorem. □

Since for $n > 2$ there exist domains $N \subset \mathbb{R}^{n-1}$ of arbitrary Euler characteristic $\chi(N) \in \mathbb{Z}$, we can apply Proposition 2.7 in two ways: Choosing $\chi(N) = 0$, we can $C^0$-approximate every Legendrian submanifold $\Lambda_0$ by a loose one which is formally Legendrian homotopic to $\Lambda_0$. Combined with Theorem 2.3, this proves Theorem 2.6(a). Choosing $\chi(N) \neq 0$, we can connect each Legendrian submanifold $\Lambda_0$ to a (loose) Legendrian submanifold $\Lambda_1$ by a Legendrian regular homotopy $\Lambda_t$ with any prescribed self-intersection index. This will be a crucial ingredient in the proof of existence of Weinstein structures.

Remark 2.8. For $n = 2$ we can still perform the stabilization construction. However, since every domain $N \subset \mathbb{R}$ is a union of intervals, the self-intersection index $\chi(N)$ in Proposition 2.7 is now always positive and hence cannot be arbitrarily prescribed. Figure 2.6 shows two front projections of the stabilization over an interval (related by Legendrian Reidemeister moves, see e.g. [14]). Thus our stabilization introduces a downward and an upward zigzag, which corresponds to a “positive and a negative stabilization” in the usual terminology of 3-dimensional contact topology. It leaves the rotation number unchanged and decreases the Thurston–Bennequin invariant by 2, in accordance with Bennequin’s inequality. In particular, stabilization in dimension 3 never preserves the formal Legendrian isotopy class.

2.5. **Totally real discs attached to a contact boundary.** The following Theorems 2.9 and 2.10 are combinations of the $h$-principles discussed in Sections 2.1 and 2.3 with Gromov’s $h$-principle for totally real embeddings [17].

Let $(V,J)$ be an almost complex manifold and $W \subset V$ a domain with smooth boundary $\partial W$. Let $L$ be a compact manifold with boundary. Let $f : L \hookrightarrow V \setminus \text{Int} W$ be an embedding with $f(\partial L) = f(L) \cap \partial W$ which is transverse to $\partial W$ along $\partial L$. We say in this case that $f$ transversely attaches $L$ to $W$ along $\partial L$. If, in addition, $Jdf(TL|_{\partial L}) \subset T(\partial W)$, then we say that $f$ attaches $L$ to $W$ $J$-orthogonally. Note that this implies that $df(\partial L)$ is tangent to the maximal $J$-invariant distribution.
\[ \xi = T(\partial W) \cap JT(\partial W). \] In particular, if the distribution \( \xi \) is a contact structure, then \( f(\partial L) \) is an isotropic submanifold for the contact structure \( \xi \).

**Theorem 2.9.** Suppose that \((V, J)\) is an almost complex manifold of real dimension \(2n\), and \(W \subset V\) is a domain such that the distribution \( \xi = T(\partial W) \cap JT(\partial W) \) is contact. Suppose that an embedding \( f : D^k \hookrightarrow V, k \leq n \), transversely attaches \( D^k \) to \( W \) along \( \partial D^k \). If \( k = n = 2 \) we assume, in addition, that the induced contact structure on \( \partial W \) is overtwisted. Then there exists an isotopy \( f_t : D^k \hookrightarrow V, t \in [0, 1], \) through embeddings transversely attaching \( D^k \) to \( W \), such that \( f_0 = f \), and \( f_1 \) is totally real and \( J \)-orthogonally attached to \( W \). Moreover, in the case \( k = n = 2 \) we can arrange that the Legendrian embedding \( f_1|_{\partial D^k} : \partial D^k \hookrightarrow \partial W \) is loose, while for \( k = n = 2 \) we can arrange that the complement \( \partial W \setminus f_t(\partial D^2) \) is overtwisted for all \( t \in [0, 1] \).

We will also need the following 1-parametric version of Theorem 2.9 for totally real discs attached along loose knots.

**Theorem 2.10.** Let \( J_t, t \in [0, 1], \) be a family of almost complex structures on a \(2n\)-dimensional manifold \( V \). Let \( W \subset V \) be a domain with smooth boundary such that the distribution \( \xi_t = T(\partial W) \cap J_tT(\partial W) \) is contact for each \( t \in [0, 1] \). Let \( f_t : D^k \hookrightarrow V \setminus \text{Int} W, t \in [0, 1], k \leq n, \) be an isotopy of embeddings transversely attaching \( D^k \) to \( W \) along \( \partial D^k \). Suppose that for \( i = 0, 1 \) the embedding \( f_i \) is \( J_i \)-totally real and \( J_i \)-orthogonally attached to \( W \). Suppose that either \( k < n \), or \( k = n > 2 \) and the Legendrian embeddings \( f_i|_{\partial D}, i = 0, 1 \) are loose. Then there exists a 2-parameter family of embeddings \( f^s_t : D^k \hookrightarrow V \setminus \text{Int} W \) with the following properties:

- \( f^s_t \) is transversely attached to \( W \) along \( \partial D^k \) and \( C^0 \)-close to \( f_t \) for all \( t, s \in [0, 1] \);
- \( f^0_t = f_t \) for all \( t \in [0, 1] \) and \( f^s_0 = f_0, f^1_t = f_1 \) for all \( s \in [0, 1] \);
- \( f^1_t \) is \( J_i \)-totally real and \( J_i \)-orthogonally attached to \( W \) along \( \partial D^k \) for all \( t \in [0, 1] \).

### 3. Morse preliminaries

In this section we gather some notions and results from Morse theory that are needed for our main results. We omit most of the proofs and refer the reader to the corresponding chapter of our book [4]. Throughout this section, \( V \) denotes a smooth manifold and \( W \) a cobordism, both of dimension \( m \).

#### 3.1. Gradient-like vector fields. A smooth function \( \phi : V \to \mathbb{R} \) is called Lyapunov for a vector field for \( X \), and \( X \) is called gradient-like for \( \phi \), if

\[ X \cdot \phi \geq \delta (|X|^2 + |d\phi|^2) \]
for a positive function \( \delta : V \to \mathbb{R}_+ \), where \(|X|\) is the norm with respect to some Riemannian metric on \( V \) and \(|d\phi|\) is the dual norm. By the Cauchy–Schwarz inequality, condition (1) implies

\[
\delta |X| \leq |d\phi| \leq \frac{1}{\delta} |X|.
\]

In particular, zeroes of \( X \) coincide with critical points of \( \phi \).

**Lemma 3.1.** (a) If \( X_0, X_1 \) are gradient-like vector fields for \( \phi \), then so is \( f_0 X_0 + f_1 X_1 \) for any nonnegative functions \( f_0, f_1 \) with \( f_0 + f_1 > 0 \).

(b) If \( \phi_0, \phi_1 \) are Lyapunov functions for \( X \), then so is \( \lambda_0 \phi_0 + \lambda_1 \phi_1 \) for any nonnegative constants \( \lambda_0, \lambda_1 \) with \( \lambda_0 + \lambda_1 > 0 \).

In particular, the following spaces are convex cones and hence contractible:

- the space of Lyapunov functions for a given vector field \( X \);
- the space of gradient-like vector fields for a given function \( \phi \).

**Proof.** Consider two vector fields \( X_0, X_1 \) satisfying \( X_i \cdot \phi \geq \delta_i (|X_i|^2 + |d\phi|^2) \) and nonnegative functions \( f_0, f_1 \) with \( f_0 + f_1 > 0 \). Then the vector field \( X = f_0 X_0 + f_1 X_1 \) satisfies (1) with \( \delta := \min \left\{ \frac{\delta_0}{2f_0}, \frac{\delta_1}{2f_1}, f_0 \delta_0 + f_1 \delta_1 \right\} : \)

\[
X \cdot \phi \geq f_0 \delta_0 |X_0|^2 + f_1 \delta_1 |X_1|^2 + (f_0 \delta_0 + f_1 \delta_1) |d\phi|^2 \\
\geq 2\delta (|f_0 X_0|^2 + |f_1 X_1|^2) + \delta |d\phi|^2 \\
\geq \delta (|X|^2 + |d\phi|^2).
\]

Positive combinations of functions are treated analogously. \( \square \)

### 3.2. Morse and Smale cobordisms

A (generalized) **Morse cobordism** is a pair \((W, \phi)\), where \( W \) is a cobordism and \( \phi : W \to \mathbb{R} \) is a (generalized) Morse function which has \( \partial_{-} W \) as its regular level sets such that \( \phi|_{\partial_{-} W} < \phi|_{\partial_{+} W} \). A (generalized) **Smale cobordism** is a triple \((W, \phi, X)\), where \((W, \phi)\) is a (generalized) Morse cobordism and \( X \) is a gradient-like vector field for \( \phi \). Note that \( X \) points inward along \( \partial_{-} W \) and outward along \( \partial_{+} W \). A generalized Smale cobordism \((W, \phi, X)\) is called **elementary** if there are no \( X \)-trajectories between different critical points of \( \phi \).

If \((W, \phi, X)\) is an elementary generalized Smale cobordism, then the stable manifold of each nondegenerate critical point \( p \) is a disc \( D_{-}^p \) which intersects \( \partial_{-} W \) along a sphere \( S_{-}^p = \partial D_{-}^p \). We call \( D_{-}^p \) and \( S_{-}^p \) the stable disc (resp. sphere) of \( p \). Similarly, the unstable manifolds and their intersections with \( \partial_{+} W \) are called unstable discs and spheres. For an embryonic critical point \( p \), the closure of the (un)stable manifold is the (un)stable half-disc \( \widehat{D}_{-}^p \) intersecting \( \partial_{\pm} W \) along the hemisphere \( \widehat{S}_{\pm}^p \).
An admissible partition of a generalized Smale cobordism \((W, \phi, X)\) is a finite sequence \(m = c_0 < c_1 < \cdots < c_N = M\) of regular values of \(\phi\), where we denote \(\phi|_{\partial^- W} = m\) and \(\phi|_{\partial^+ W} = M\), such that each subcobordism \(W_k = \{c_{k-1} \leq \phi \leq c_k\}\), \(k = 1, \ldots, N\), is elementary. The following lemma is straightforward.

**Lemma 3.2.** Any generalized Smale cobordism admits an admissible partition into elementary cobordisms. Similarly, for any exhausting generalized Morse function \(\phi\) and gradient-like vector field \(X\) on a noncompact manifold \(V\), one can find regular values \(c_0 < \min \phi < c_1 < \cdots \to \infty\) such that the cobordisms \(W_k = \{c_{k-1} \leq \phi \leq c_k\}\), \(k = 1, \ldots, N\), are elementary. If \(\phi\) has finitely many critical points, then all but finitely many of these cobordisms have no critical points.

### 3.3. Morse and Smale homotopies.

A smooth family \((W, \phi_t), t \in [0, 1]\), of generalized Morse cobordism structures is called a Morse homotopy if there is a finite set \(A \subset (0, 1)\) with the following properties:

- for each \(t \in A\) the function \(\phi_t\) has a unique birth-death type critical point \(e_t\) such that \(\phi_t(e_t) \neq \phi_t(q)\) for all other critical points \(q\) of \(\phi_t\);
- for each \(t \notin A\) the function \(\phi_t\) is Morse.

A Smale homotopy is a smooth family \((W, X_t, \phi_t), t \in [0, 1]\), of generalized Smale cobordism structures such that \((W, \phi_t)\) a Morse homotopy. A Smale homotopy \(S_t = (W, X_t, \phi_t), t \in [0, 1]\) is called an elementary Smale homotopy of Type I, IIb, IIId, respectively, if the following holds:

- Type I. \(S_t\) is an elementary Smale cobordism for all \(t \in [0, 1]\).
- Type IIb (birth). There is \(t_0 \in (0, 1)\) such that for \(t < t_0\) the function \(\phi_t\) has no critical points, \(\phi_{t_0}\) has a birth type critical point, and for \(t > t_0\) the function \(\phi_t\) has has two critical points \(p_t\) and \(q_t\) of index \(i\) and \(i-1\), respectively, connected by a unique \(X_t\)-trajectory.
- Type IIId (death). There is \(t_0 \in (0, 1)\) such that for \(t > t_0\) the function \(\phi_t\) has no critical points, \(\phi_{t_0}\) has a death type critical point, and for \(t < t_0\) the function \(\phi_t\) has two critical points \(p_t\) and \(q_t\) of index \(i\) and \(i-1\), respectively, connected by a unique \(X_t\)-trajectory.

We will also refer to an elementary Smale homotopy of Type IIb (resp. IIId) as a creation (resp. cancellation) family.

An admissible partition of a Smale homotopy \(S_t = (W, X_t, \phi_t), t \in [0, 1]\), is a sequence \(0 = t_0 < t_1 < \cdots < t_p = 1\) of parameter values, and for each \(k = 1, \ldots, p\) a finite sequence of functions

\[
m(t) = c^k_0(t) < c^k_1(t) < \cdots < c^k_{N_k}(t) = M(t), \quad t \in [t_{k-1}, t_k],
\]
where \( m(t) := \phi_t(\partial_-W) \) and \( M(t) := \phi_t(\partial_+W) \), such that \( c^j_k(t) \), \( j = 0,\ldots,N_k \) are regular values of \( \phi_t \) and each Smale homotopy
\[
\mathcal{S}_j^k := \left( W_{j}^k(t) := \{ c^j_{k-1}(t) \leq \phi_t \leq c^j_k(t) \}, X_t|_{W_{j}^k(t)}, \phi_t|_{W_{j}^k(t)} \right)_{t \in [t_{k-1},t_k]}
\]
is elementary.

**Lemma 3.3.** Any Smale homotopy admits an admissible partition.

*Proof.* Let \( A \subset (0,1) \) be the finite subset in the definition of a Smale homotopy. Using Lemma 3.2, we now first construct an admissible partition on \( \mathcal{O}p A \) and then extend it over \([0,1] \setminus \mathcal{O}p A\). \( \square \)

### 3.4. Equivalence of elementary Smale homotopies

We define the profile (or Cerf diagram) of a Smale homotopy \( \mathcal{S}_t = (W,X_t,\phi_t) \), \( t \in [0,1] \), as the subset \( C(\{\phi_t\}) \subset \mathbb{R} \times \mathbb{R} \) such that \( C(\{\phi_t\}) \cap (t \times \mathbb{R}) \) is the set of critical values of the function \( \phi_t \). We will use the notion of profile only for elementary homotopies.

The following two easy lemmas are proved in [1]. The first one shows that if two elementary Smale homotopies have the same profile, then their functions are related by diffeomorphisms.

**Lemma 3.4.** Let \( \mathcal{S}_t = (W,X_t,\phi_t) \) and \( \widetilde{\mathcal{S}}_t = (W,\widetilde{X}_t,\widetilde{\phi}_t) \), \( t \in [0,1] \), be two elementary Smale homotopies with the same profile such that \( \mathcal{S}_0 = \widetilde{\mathcal{S}}_0 \). Then there exists a diffeotopy \( h_t : W \to W \) with \( h_0 = \mathbb{I} \) such that \( \phi_t = \phi_t \circ h_t \) for all \( t \in [0,1] \). Moreover, if \( \phi_t = \widetilde{\phi}_t \) near \( \partial_+W \) and/or \( \partial_-W \) we can arrange \( h_t = \mathbb{I} \) near \( \partial_+W \) and/or \( \partial_-W \).

The second lemma provides elementary Smale homotopies with prescribed profile.

**Lemma 3.5.** Let \((W,X,\phi)\) be an elementary Smale cobordism with \( \phi|_{\partial_+W} = a_- \) and critical points \( p_1,\ldots,p_n \) of values \( \phi(p_i) = c_i \). For \( i = 1,\ldots,n \), let \( c_i : [0,1] \to (a_-,a_+) \) be smooth functions with \( c_i(0) = c_i \). Then there exists a smooth family \( \phi_t, t \in [0,1], \) of Lyapunov functions for \( X \) with \( \phi_0 = \phi \) and \( \phi_t = \phi \) on \( \mathcal{O}p \partial W \) such that \( \phi_t(p_i) = c_i(t) \).

### 3.5. Holonomy of Smale cobordisms

Let \((W,X,\phi)\) be a Smale cobordism such that the function \( \phi \) has no critical points. The *holonomy* of \( X \) is the diffeomorphism
\[
\Gamma_X : \partial_+W \to \partial_-W,
\]
which maps \( x \in \partial_+W \) to the intersection of its trajectory under the flow of \(-X\) with \( \partial_-W \).

Consider now a Morse cobordism \((W,\phi)\) without critical points. Denote by \( \mathcal{X}(W,\phi) \) the space of all gradient-like vector fields for \( \phi \). Note that the holonomies of all \( X \in \mathcal{X}(W,\phi) \) are isotopic. We denote by \( \mathcal{D}(\partial_+W,\partial_-W) \) the corresponding path
component in the space of diffeomorphisms from \(\partial_+ W\) to \(\partial_- W\). All spaces are equipped with the \(C^\infty\)-topology.

Recall that a continuous map \(p : E \to B\) is a Serre fibration if it has the homotopy lifting property for all closed discs \(D^k\), i.e., given a homotopy \(h_t : D^k \to B\), \(t \in [0, 1]\), and a lift \(\widetilde{h}_0 : D^k \to E\) with \(p \circ \widetilde{h}_0 = h_0\), there exists a homotopy \(\widetilde{h}_t : D^k \to E\) with \(p \circ \widetilde{h}_t = h_t\). We omit the proof of the following easy lemma.

**Lemma 3.6.** Let \((W, \phi)\) be a Morse cobordism without critical points. Then the map \(\mathcal{X}(W, \phi) \to \mathcal{D}(\partial_+ W, \partial_- W)\) that assigns to \(X\) its holonomy \(\Gamma_X\) is a Serre fibration. In particular:

(i) Given \(X \in \mathcal{X}(W, \phi)\) and an isotopy \(h_t \in \mathcal{D}(\partial_+ W, \partial_- W)\), \(t \in [0, 1]\), with \(h_0 = \Gamma_X\), there exists a path \(X_t \in \mathcal{X}(W, \phi)\) with \(X_0 = X\) such that \(\Gamma_{X_t} = h_t\) for all \(t \in [0, 1]\).

(ii) Given a path \(X_t \in \mathcal{X}(W, \phi)\), \(t \in [0, 1]\), and a path \(h_t \in \mathcal{D}(\partial_+ W, \partial_- W)\) which is homotopic to \(\Gamma_{X_t}\) with fixed endpoints, there exists a path \(\widetilde{X}_t \in \mathcal{X}(W, \phi)\) with \(\widetilde{X}_0 = X_0\) and \(\widetilde{X}_1 = X_1\) such that \(\Gamma_{\widetilde{X}_t} = h_t\) for all \(t \in [0, 1]\).

As a consequence, we obtain

**Lemma 3.7.** Let \(X_t, Y_t\) be two paths in \(\mathcal{X}(W, \phi)\) starting at the same point \(X_0 = Y_0\). Suppose that for a subset \(A \subset \partial_+ W\), one has \(\Gamma_{X_t}(A) = \Gamma_{Y_t}(A)\) for all \(t \in [0, 1]\). Then there exists a path \(\widetilde{X}_t \in \mathcal{X}(W, \phi)\) such that

(i) \(\widetilde{X}_t = X_{2t}\) for \(t \in [0, \frac{1}{2}]\);

(ii) \(\widetilde{X}_1 = Y_1\);

(iii) \(\Gamma_{\widetilde{X}_t}(A) = \Gamma_{Y_t}(A)\) for \(t \in [\frac{1}{2}, 1]\).

**Proof.** Consider the path \(\gamma : [0, 1] \to \mathcal{D}(\partial_+ W, \partial_- W)\) given by the formula

\[
\gamma(t) := \Gamma_{X_t} \circ \Gamma_{X_t}^{-1} \circ \Gamma_{Y_t}.
\]

We have \(\gamma(0) = \Gamma_{X_1}\) and \(\gamma(1) = \Gamma_{Y_1}\). The path \(\gamma\) is homotopic with fixed endpoints to the concatenation of the paths \(\Gamma_{X_{1-t}}\) and \(\Gamma_{Y_1}\). Hence by Lemma 3.6 we conclude that there exists a path \(X_t' \in \mathcal{X}(W, \phi)\) such that \(X_0' = X_1, X_1' = Y_1,\) and \(\Gamma_{X_t'} = \gamma(t)\) for all \(t \in [0, 1]\). Since

\[
\Gamma_{X_t'}(A) = \Gamma_{X_t} \left( \Gamma_{X_t}^{-1} (\Gamma_{Y_1}(A)) \right) = \Gamma_{X_1}(A) = \Gamma_{Y_1}(A),
\]

the concatenation \(\widetilde{X}_t\) of the paths \(X_t\) and \(X_t'\) has the required properties. \(\square\)
In this section we collect some facts about Weinstein structures needed for the proofs of our main results. Most of the proofs are omitted and we refer the reader to our book [4] for a more systematic treatment of the subject.

4.1. Holonomy of Weinstein cobordisms. In this subsection we consider Weinstein cobordisms $\mathcal{W} = (W, \omega, X, \phi)$ without critical points (of the function $\phi$). Its holonomy along trajectories of $X$ defines a contactomorphism

$$\Gamma_{\mathcal{W}} : (\partial_+ W, \xi_+) \to (\partial_- W, \xi_-)$$

for the contact structures $\xi_{\pm}$ on $\partial_{\pm} W$ induced by the Liouville form $\lambda = i_X \omega$.

We say that two Weinstein structures $\mathcal{W} = (\omega, X, \phi)$ and $\widetilde{\mathcal{W}}$ agree up to scaling on a subset $A \subset W$ if $\widetilde{\mathcal{W}}|_A = (C\omega, X, \phi)$ for a constant $C > 0$. Note that in this case $\widetilde{\mathcal{W}}|_A$ has Liouville form $C\lambda$.

Let us fix a Weinstein cobordism $\mathcal{W} = (W, \omega, X, \phi)$ without critical points. We denote by $\mathcal{W}(\mathcal{W})$ the space of all Weinstein structures $\mathcal{W} = (W, \omega, X, \phi)$ with the same function $\phi$ such that

- $\mathcal{W}$ coincides with $\mathcal{W}$ on $O_p \partial_- W$ and up to scaling on $O_p \partial_+ W$;
- $\mathcal{W}$ and $\mathcal{W}$ induce the same contact structures on level sets of $\phi$.

Equivalently, $\mathcal{W}(\mathcal{W})$ can be viewed as the space of Liouville forms $\lambda = f\bar{\lambda} + g d\phi$ with $f \equiv 1$ near $\partial_- W$, $f \equiv C$ near $\partial_+ W$, and $g \equiv 0$ near $\partial W$, where $\bar{\lambda}$ denotes the Liouville form of $\mathcal{W}$.

Denote by $\mathcal{D}(\mathcal{W})$ the space of contactomorphisms $(\partial_+ W, \xi_+) \to (\partial_- W, \xi_-)$, where $\xi_{\pm}$ is the contact structure induced on $\partial_{\pm} W$ by $\mathcal{W}$. Note that $\Gamma_{\mathcal{W}} \in \mathcal{D}(\mathcal{W})$ for any $\mathcal{W} \in \mathcal{W}(\mathcal{W})$. The following lemma is the analogue of Lemma 3.6 in the context of Weinstein cobordisms.

**Lemma 4.1.** Let $\mathcal{W}$ be a Weinstein cobordism without critical points. Then the map $\mathcal{W}(\mathcal{W}) \to \mathcal{D}(\mathcal{W})$ that assigns to $\mathcal{W}$ its holonomy $\Gamma_{\mathcal{W}}$ is a Serre fibration. In particular:

(i) Given $\mathcal{W} \in \mathcal{W}(\mathcal{W})$ and an isotopy $h_t \in \mathcal{D}(\mathcal{W})$, $t \in [0, 1]$, with $h_0 = \Gamma_{\mathcal{W}}$, there exists a path $\mathcal{W}_t \in \mathcal{W}(\mathcal{W})$ with $\mathcal{W}_0 = \mathcal{W}$ such that $\Gamma_{\mathcal{W}_t} = h_t$ for all $t \in [0, 1]$.

(ii) Given a path $\mathcal{W}_t \in \mathcal{W}(\mathcal{W})$, $t \in [0, 1]$, and a path $h_t \in \mathcal{D}(\mathcal{W})$ which is homotopic to $\Gamma_{\mathcal{W}_t}$ with fixed endpoints, there exists a path $\mathcal{W}_t \in \mathcal{W}(\mathcal{W})$ with $\mathcal{W}_0 = \mathcal{W}_0$ and $\mathcal{W}_1 = \mathcal{W}_1$ such that $\Gamma_{\mathcal{W}_t} = h_t$ for all $t \in [0, 1]$. 
4.2. Weinstein structures near stable discs. The following two lemmas concern the construction of Weinstein structures near stable discs of Smale cobordisms.

Lemma 4.2. Let $\mathcal{S} = (W, X, \phi)$ be an elementary Smale cobordism and $\omega$ a nondegenerate 2-form on $W$. Let $D_1, \ldots, D_k$ be the stable discs of critical points of $\phi$, and set $\Delta := \bigcup_{j=1}^k D_j$. Suppose that the discs $D_1, \ldots, D_k$ are $\omega$-isotropic and the pair $(\omega, X)$ is Liouville on $\mathcal{O}_p(\partial W)$. Then for any neighborhood $U$ of $\partial W \cup \Delta$ there exists a homotopy $(\omega_t, X_t)$, $t \in [0, 1]$, with the following properties:

(i) $X_t$ is a gradient-like vector field for $\phi$ and $\omega_t$ is a nondegenerate 2-form on $W$ for all $t \in [0, 1]$;
(ii) $(\omega_0, X_0) = (\omega, X)$, and $(\omega_t, X_t) = (\omega, X)$ outside $U$ and on $\Delta \cup \mathcal{O}_p(\partial W)$ for all $t \in [0, 1]$;
(iii) $(\omega_1, X_1)$ is a Liouville structure on $\mathcal{O}_p(\partial W \cup \Delta)$.

Lemma 4.2 has the following version for homotopies.

Lemma 4.3. Let $\mathcal{S}_t = (W, X_t, \phi_t)$, $t \in [0, 1]$, be an elementary Smale homotopy and $\omega_t$, $t \in [0, 1]$, a family of nondegenerate 2-forms on $W$. Let $\Delta_t$ be the union of the stable (half-)discs of critical points of $X_t$ and set

$$\Delta := \bigcup_{t \in [0, 1]} \{t\} \times \Delta_t \subset [0, 1] \times W.$$  

Suppose that $\Delta_t$ is $\omega_t$-isotropic for all $t \in [0, 1]$, the pair $(\omega_t, X_t)$ is Liouville on $\mathcal{O}_p(\partial W)$ for all $t \in [0, 1]$, and $(\omega_0, X_0)$ and $(\omega_1, X_1)$ are Liouville on all of $W$. Then for any open neighborhood $V = \bigcup_{t \in [0, 1]} \{t\} \times V_t$ of $\Delta$ there exists an open neighborhood $U = \bigcup_{t \in [0, 1]} \{t\} \times U_t \subset V$ of $\Delta$ and a 2-parameter family $(\omega^s_t, X^s_t)$, $s, t \in [0, 1]$, with the following properties:

(i) $X^s_t$ is a gradient-like vector field for $\phi_t$ and $\omega^s_t$ is a nondegenerate 2-form on $W$ for all $s, t \in [0, 1]$;
(ii) $(\omega^0_t, X^0_t) = (\omega_t, X_t)$ for all $t \in [0, 1]$, $(\omega^s_0, X^s_0) = (\omega_0, X_0)$ and $(\omega^s_t, X^s_t) = (\omega_t, X_t)$ for all $s \in [0, 1]$, and $(\omega^s_t, X^s_t)$ is a Liouville structure on $U_t$ for all $t \in [0, 1]$.

4.3. Weinstein homotopies. A smooth family $(W, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, of Weinstein cobordism structures is called Weinstein homotopy if the family $(W, X_t, \phi_t)$ is a Smale homotopy in the sense of Section 3.3. Recall that this means in particular that the functions $\phi_t$ have $\partial_t W$ as regular level sets, and they are Morse except for finitely many $t \in (0, 1)$ at which a birth-death type critical point occurs.
The definition of a Weinstein homotopy on a manifold $V$ requires more care. Consider first a smooth family $\phi_t : V \to \mathbb{R}$, $t \in [0, 1]$, of exhausting generalized Morse functions such that there exists a finite set $A \subset (0, 1)$ satisfying the conditions stated at the beginning of Section 3.3. We call $\phi_t$ a simple Morse homotopy if there exists a sequence of smooth functions $c_1 < c_2 < \cdots$ on the interval $[0, 1]$ such that for each $t \in [0, 1]$, $c_i(t)$ is a regular value of the function $\phi_t$ and $\bigcup_k \{ \phi_t \leq c_k(t) \} = V$. A Morse homotopy is a composition of finitely many simple Morse homotopies. Then a Weinstein homotopy on the manifold $V$ is a family of Weinstein manifold structures $(V, \omega_t, X_t, \phi_t)$ such that the associated functions $\phi_t$ form a Morse homotopy.

The main motivation for this definition of a Weinstein homotopy is the following result from [11] (see also [4]).

**Proposition 4.4.** If two Weinstein manifolds $\mathcal{W}_0 = (V, \omega_0, X_0, \phi_0)$ and $\mathcal{W}_1 = (V, \omega_1, X_1, \phi_1)$ are Weinstein homotopic, then they are symplectomorphic. More precisely, there exists a diffeotopy $h_t : V \to V$ with $h_0 = 1$ such that $h_t^* \lambda_1 - \lambda_0$ is exact, where $\lambda_i = i_{X_i} \omega_i$ are the Liouville forms. If $\mathcal{W}_0$ and $\mathcal{W}_1$ are the completions of homotopic Weinstein domains, then we can achieve $h_t^* \lambda_1 - \lambda_0 = 0$ outside a compact set.

**Remark 4.5.** Without the hypothesis on the functions $c_k(t)$ in the definition of a Weinstein homotopy, Proposition 4.4 would fail. Indeed, it is not hard to see that without this hypothesis all Weinstein structures on $\mathbb{R}^{2n}$ would be “homotopic”. But according to McLean [21], there are infinitely many Weinstein structures on $\mathbb{R}^{2n}$ which are pairwise non-symplectomorphic.

**Remark 4.6.** It is not entirely obvious but true (see [4]) that any two exhausting Morse functions on the same manifold can be connected by a Morse homotopy.

The notion of Weinstein (or Stein) homotopy can be formulated in more topological terms. Let us denote by $\text{Weinstein}$ the space of Weinstein structures on a fixed manifold $V$. For any $\mathcal{W}_0 \in \text{Weinstein}$, $\varepsilon > 0$, $A \subset V$ compact, $k \in \mathbb{N}$, and any unbounded sequence $c_1 < c_2 < \cdots$, we define the set

$$U(\mathcal{W}_0, \varepsilon, A, k, c) := \{ \mathcal{W} = (\omega, X, \phi) \in \text{Weinstein} | \| \mathcal{W} - \mathcal{W}_0 \|_{C^k(A)} < \varepsilon, c_i \text{ regular values of } \phi \}.$$ 

It is easy to see that these sets are the basis of a topology on $\text{Weinstein}$, and a smooth family of Weinstein structures satisfying the conditions at the beginning of Section 3.3 defines a continuous path $[0, 1] \to \text{Weinstein}$ with respect to this topology if and only if (possibly after target reparametrization of the functions) it is a Weinstein homotopy according to the definition above. A topology on the space $\text{Morse}$ of exhausting generalized Morse functions can be defined similarly.
4.4. Creation and cancellation of critical points of Weinstein structures.

A key ingredient in Smale’s proof of the $h$-cobordism theorem is the creation and cancellation of pairs of critical points of a Morse function. The following two propositions describe analogues of these operations for Weinstein structures.

**Proposition 4.7** (Creation of critical points). Let $(W,\omega,X,\phi)$ be a Weinstein cobordism without critical points. Then given any point $p \in \text{Int } W$ and any integer $k \in \{1,\ldots,n\}$, there exists a Weinstein homotopy $(\omega,X_t,\phi_t)$ with the following properties:

(i) $(X_0,\phi_0) = (X,\phi)$ and $(X_t,\phi_t) = (X,\phi)$ outside a neighborhood of $p$; 

(ii) $\phi_t$ is a creation family such that $\phi_1$ has a pair of critical points of index $k$ and $k-1$.

**Proposition 4.8** (Cancellation of critical points). Let $(W,\omega,X,\phi)$ be a Weinstein cobordism with exactly two critical points $p,q$ of index $k$ and $k-1$, respectively, which are connected by a unique gradient trajectory along which the stable and unstable manifolds intersect transversely. Let $\Delta$ be the skeleton of $(W,X)$, i.e., the closure of the stable manifold of the critical point $p$. Then there exists a Weinstein homotopy $(\omega,X_t,\phi_t)$ with the following properties:

(i) $(X_0,\phi_0) = (X,\phi)$, and $(X_t,\phi_t) = (X,\phi)$ near $\partial W$ and outside a neighborhood of $\Delta$; 

(ii) $\phi_t$ has no critical points outside $\Delta$; 

(iii) $\phi_t$ is a cancellation family such that $\phi_1$ has no critical points.

5. Existence and deformations of flexible Weinstein structures

In this section we prove Theorem 1.2 from the Introduction and some other results about flexible Weinstein manifolds and cobordisms. For simplicity, we assume that individual functions are Morse and 1-parameter families are Morse homotopies in the sense of Section 3.3. The more general case of arbitrary (1-parameter families of) generalized Morse functions is treated similarly.

5.1. Existence of Weinstein structures. The following two theorems imply Theorem 1.2(a) from the Introduction.

**Theorem 5.1** (Weinstein existence theorem). Let $(W,\phi)$ be a $2n$-dimensional Morse cobordism such that $\phi$ has no critical points of index $> n$. Let $\eta$ be a nondegenerate (not necessarily closed) 2-form on $W$ and $Y$ a vector field near $\partial W$ such that $(\eta,Y,\phi)$ defines a Weinstein structure on $\Omega p \partial W$. Suppose that either $n > 2$, or $n = 2$ and the contact structure induced by the Liouville form $\lambda = \iota_Y \eta$ on $\partial W$
is overtwisted. Then there exists a Weinstein structure \((\omega, X, \phi)\) on \(W\) with the following properties:

(i) \((\omega, X) = (\eta, Y)\) on \(\partial_+ W\);  
(ii) the nondegenerate 2-forms \(\omega\) and \(\eta\) on \(W\) are homotopic rel \(\partial_+ W\).

Moreover, we can arrange that \((\omega, X, \phi)\) is flexible.

Theorem 5.1 immediately implies the following version for manifolds.

**Theorem 5.2.** Let \((V, \phi)\) be a \(2n\)-dimensional manifold with an exhausting Morse function \(\phi\) that has no critical points of index > \(n\). Let \(\eta\) be a nondegenerate (not necessarily closed) 2-form on \(V\). Suppose that \(n > 2\). Then there exists a Weinstein structure \((\omega, X, \phi)\) on \(V\) such that the nondegenerate 2-forms \(\omega\) and \(\eta\) on \(V\) are homotopic. Moreover, we can arrange that \((\omega, X, \phi)\) is flexible. \(\square\)

**Proof of Theorem 5.1.** By decomposing the Morse cobordism \(\mathcal{M} = (W, \phi)\) into elementary ones, \(W = W_1 \cup \cdots \cup W_N\), and inductively extending the Weinstein structure over \(W_1, \ldots, W_N\), it suffices to consider the case of an elementary cobordism. To simplify the notation, we will assume that \(\phi\) has a unique critical point \(p\); the general case is similar. Let us extend \(Y\) to a gradient-like vector field for \(\phi\) on \(W\) and denote by \(\Delta\) the stable disc of \(p\).

**Step 1.** We first show that, after a homotopy of \((\eta, Y)\) fixed on \(\partial_+ W\), we may assume that \(\Delta\) is \(\eta\)-isotropic.

The Liouville form \(\lambda = i_Y \eta\) on \(\partial_+ W\) defines a contact structure \(\xi := \ker(\lambda|_{\partial_+ W})\) on \(\partial_+ W\). We choose an auxiliary \(\eta\)-compatible almost complex structure \(J\) on \(W\) which preserves \(\xi\) and maps \(Y\) along \(\partial_+ W\) to the Reeb vector field \(R\) of \(\lambda|_{\partial_+ W}\). We apply Theorem 2.9 to find a diffeotopy \(f_t : W \to W\) such that the disc \(\Delta' = f_1(\Delta)\) is \(J\)-totally real and \(J\)-orthogonal attached to \(\partial_+ W\). This is the only point in the proof where the overtwistedness assumption for \(n = 2\) is needed. Moreover, according to Theorem 2.9 in the case \(\text{dim } \Delta = n\) we can arrange that the Legendrian sphere \(\partial \Delta'\) in \((\partial_+ W, \xi)\) is loose (meaning that \(\partial_+ W \setminus \partial \Delta'\) is overtwisted in the case \(n = 2\)).

Next we modify the homotopy \(f_t^* J\) to keep it fixed near \(\partial_+ W\). \(J\)-orthogonality implies that \(\partial \Delta'\) is tangent to the maximal \(J\)-invariant distribution \(\xi \subset T(\partial_+ W)\) and thus \(\lambda|_{\partial \Delta'} = 0\). Since the spaces \(T \Delta'\) and \(\text{span}\{T \partial \Delta', Y\}\) are both totally real and \(J\)-orthogonal to \(T(\partial_+ W)\), we can further adjust the disc \(\Delta'\) (keeping \(\partial \Delta'\) fixed) to make it tangent to \(Y\) in a neighborhood of \(\partial \Delta'\). It follows that we can modify \(f_t\) such that it preserves the function \(\phi\) and the vector field \(Y\) on a neighborhood \(U\) of \(\partial_+ W\) (extend \(f_t\) from \(\partial_+ W\) to \(U\) using the flow of \(Y\)).

Hence, there exists a diffeotopy \(g_t : W \to W\), \(t \in [0, 1]\), which equals \(f_t\) on \(W \setminus U\), the identity on \(\partial_+ W\), and preserves \(\phi\) (but not \(Y!\)) on \(U\); see Figure 5.1. Then the
diffeotopy \( k_t := f_t^{-1} \circ g_t \) equals the identity on \( W \setminus U \), \( f_t^{-1} \) on \( O_p \partial W \), and preserves \( \phi \) on all of \( W \). Thus the vector fields \( Y_t := k_t^*Y \) are gradient-like for \( \phi = k_t^*\phi \) and coincide with \( Y \) on \((W \setminus U) \cup O_p \partial W \). The nondegenerate 2-forms \( \eta_t := g_t^*\eta \) are compatible with \( J_t := g_t^*J \) and coincide with \( \eta \) on \( O_p \partial W \). Moreover, since \( \Delta' \) is \( J \)-totally real, the stable disc \( \Delta_1 := k_1^{-1}(\Delta) = g_1^{-1}(\Delta') \) of \( p \) with respect to \( Y_1 \) is \( J_1 \)-totally real and \( J_1 \)-orthogonally attached to \( \partial W \).

After renaming \((\eta_1,Y_1,\Delta_1)\) back to \((\eta,Y,\Delta)\), we may hence assume that \( \Delta \) is \( J \)-totally real and \( J \)-orthogonally attached to \( \partial W \) for some \( \eta \)-compatible almost complex structure \( J \) on \( W \) which preserves \( \xi \) and maps \( Y \) to the Reeb vector field \( R \) along \( \partial W \). In particular, \( \partial \Delta \) is \( \lambda \)-isotropic and \( \Delta \cap O_p \partial W \) is \( \eta \)-isotropic. Since the space of nondegenerate 2-forms compatible with \( J \) is contractible, after a further homotopy of \( \eta \) fixed on \( O_p \partial W \) and outside a neighborhood of \( \Delta \), we may assume that \( \Delta \) is \( \eta \)-isotropic.

**Step 2.** By Lemma 4.2 there exists a homotopy \((\eta_t,Y_t), t \in [0,1]\), of gradient-like vector fields for \( \phi \) and nondegenerate 2-forms on \( W \), fixed on \( \Delta \cup O_p \partial W \) and outside a neighborhood of \( \Delta \), such that \((\eta_0,Y_0) = (\eta,Y)\) and \((\eta_1,Y_1)\) is Liouville on \( O_p (\partial W \cup \Delta) \). After renaming \((\eta_1,Y_1)\) back to \((\eta,Y)\), we may hence assume that \((\eta,Y)\) is Liouville on a neighborhood \( U \) of \( \partial W \cup \Delta \).

**Step 3.** Pushing down along trajectories of \( Y \), we construct an isotopy of embeddings \( h_t : W \to W, t \in [0,1] \), with \( h_0 = \text{Id} \) and \( h_t = \text{Id} \) on \( O_p (\partial W \cup \Delta) \), which preserves trajectories of \( Y \) and such that \( h_1(W) \subset U \). Then \((\eta_t,Y_t) := (h_t^*\eta,h_t^*Y)\) defines a homotopy of nondegenerate 2-forms and vector fields on \( W \), fixed on \( O_p (\partial W \cup \Delta) \), from \((\eta_0,Y_0) = (\eta,Y)\) to the Liouville structure \((\eta_1,Y_1) =: (\omega,X)\). Since the \( Y_t \) are proportional to \( Y \), they are gradient-like for \( \phi \) for all \( t \in [0,1] \).

The Weinstein structure \((\omega,X,\phi)\) will be flexible if we choose the stable sphere \( \partial \Delta \) in Step 1 to be loose, so Theorem 5.1 is proved. \( \square \)
5.2. Homotopies of flexible Weinstein structures. The following Theorems 5.3 and 5.4 for cobordisms, and Theorems 5.5 and 5.6 for manifolds, are our main results concerning deformations of flexible Weinstein structures. They imply Theorem 1.2(b) from the Introduction.

**Theorem 5.3** (First Weinstein deformation theorem). Let \( \mathcal{W} = (W, \omega, X, \phi) \) be a flexible Weinstein cobordism of dimension \( 2n \). Let \( \phi_t, t \in [0, 1], \) be a Morse homotopy without critical points of index \( > n \) with \( \phi_0 = \phi \) and \( \phi_1 = \phi \) near \( \partial W \). In the case \( 2n = 4 \) assume that either \( \partial_- W \) is overtwisted, or \( \phi_t \) has no critical points of index \( > 1 \). Then there exists a homotopy \( \mathcal{W}_t = (W, \omega_t, X_t, \phi_t), t \in [0, 1], \) of flexible Weinstein structures, starting at \( \mathcal{W}_0 = \mathcal{W} \), which is fixed near \( \partial_- W \) and fixed up to scaling near \( \partial_+ W \).

**Theorem 5.4** (Second Weinstein deformation theorem). Let \( \mathcal{W}_0 = (\omega_0, X_0, \phi_0) \) and \( \mathcal{W}_1 = (\omega_1, X_1, \phi_1) \) be two flexible Weinstein structures on a cobordism \( W \) of dimension \( 2n \). Let \( \phi_t, t \in [0, 1], \) be a Morse homotopy without critical points of index \( > n \) connecting \( \phi_0 \) and \( \phi_1 \). In the case \( 2n = 4 \) assume that either \( \partial_- W \) is overtwisted, or \( \phi_t \) has no critical points of index \( > 1 \). Let \( \eta_t, t \in [0, 1], \) be a homotopy of nondegenerate (not necessarily closed) 2-forms connecting \( \omega_0 \) and \( \omega_1 \) such that \( (\eta_t, Y_t, \phi_t) \) is Weinstein near \( \partial_- W \) for a homotopy of vector fields \( Y_t \) on \( \partial p \partial_- W \) connecting \( X_0 \) and \( X_1 \).

Then \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \) can be connected by a homotopy \( \mathcal{W}_t = (\omega_t, X_t, \phi_t), t \in [0, 1], \) of flexible Weinstein structures, agreeing with \( (\eta_t, Y_t, \phi_t) \) on \( \partial p \partial_- W \), such that the paths of nondegenerate 2-forms \( t \mapsto \eta_t \) and \( t \mapsto \omega_t, t \in [0, 1], \) are homotopic rel \( \partial p \partial_- W \) with fixed endpoints.

Theorems 5.3 and 5.4 will be proved in Sections 5.3 and 5.4. They have the following analogues for deformations of flexible Weinstein manifolds, which are derived from the cobordism versions by induction over sublevel sets.

**Theorem 5.5.** Let \( \mathcal{W} = (V, \omega, X, \phi) \) be a flexible Weinstein manifold of dimension \( 2n \). Let \( \phi_t, t \in [0, 1], \) be a Morse homotopy without critical points of index \( > n \) with \( \phi_0 = \phi \). In the case \( 2n = 4 \) assume that \( \phi_t \) has no critical points of index \( > 1 \). Then there exists a homotopy \( \mathcal{W}_t = (V, \omega_t, X_t, \phi_t), t \in [0, 1], \) of flexible Weinstein structures such that \( \mathcal{W}_0 = \mathcal{W} \).

If the Morse homotopy \( \phi_t \) are fixed outside a compact set, then the Weinstein homotopy \( \mathcal{W}_t \) can be chosen fixed outside a compact set.

**Theorem 5.6.** Let \( \mathcal{W}_0 = (\omega_0, X_0, \phi_0) \) and \( \mathcal{W}_1 = (\omega_1, X_1, \phi_1) \) be two flexible Weinstein structures on the same manifold \( V \) of dimension \( 2n \). Let \( \phi_t, t \in [0, 1], \) be a Morse homotopy without critical points of index \( > n \) connecting \( \phi_0 \) and \( \phi_1 \). In the case \( 2n = 4 \), assume that \( \phi_t \) has no critical points of index \( > 1 \). Let \( \eta_t \) be a homotopy
of nondegenerate 2-forms on $V$ connecting $\omega_0$ and $\omega_1$. Then there exists a homotopy $\mathcal{W}_t = (\omega_t, X_t, \phi_t)$ of flexible Weinstein structures connecting $\mathcal{W}_0$ and $\mathcal{W}_1$ such that the paths $\omega_t$ and $\eta_t$ of nondegenerate 2-forms are homotopic with fixed endpoints. $\Box$

5.3. Proof of the first Weinstein deformation theorem. The proof of Theorem 5.3 is based on the following three lemmas.

Lemma 5.7. Let $\mathcal{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism and $Y$ a gradient-like vector field for $\phi$ such that the Smale cobordism $(W, Y, \phi)$ is elementary. Then there exists a family $X_t$, $t \in [0, 1]$, of gradient-like vector fields for $\phi$ and a family $\omega_t$, $t \in [0, \frac{1}{2}]$, of symplectic forms on $W$ such that

- $\mathcal{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0, \frac{1}{2}]$, is a Weinstein homotopy with $\mathcal{W}_0 = \mathcal{W}$, fixed on $O_p \partial_- W$ and fixed up to scaling on $O_p \partial_+ W$;
- $X_1 = Y$ and the Smale cobordisms $(W, X_t, \phi_t)$, $t \in [\frac{1}{2}, 1]$, are elementary.

Proof. Step 1. Let $c_1 < \cdots < c_N$ be the critical values of the function $\phi$. Set $c_0 := \phi|_{\partial_- W}$ and $c_{N+1} := \phi|_{\partial_+ W}$. Choose $\varepsilon \in \left(0, \min_{j=0,\ldots,N} \frac{c_{j+1} - c_j}{2}\right)$ and define

$$W_j := \{c_j - \varepsilon \leq \phi \leq c_j + \varepsilon\}, \quad j = 2, \ldots, N - 1,$$

$$W_1 := \{\phi \leq c_1 + \varepsilon\}, \quad W_N := \{\phi \geq c_N - \varepsilon\},$$

$$V_j := \{c_j + \varepsilon \leq \phi \leq c_{j+1} - \varepsilon\}, \quad j = 1, \ldots, N - 1,$$

$$\Sigma_j^+ := \{\phi = c_j + \varepsilon\}, \quad j = 1, \ldots, N;$$

see Figure 5.2.

Thus we have $\Sigma_j^+ = \partial_- V_j = \partial_+ W_j$ for $j = 1, \ldots, N - 1$ and $\Sigma_j^- = \partial_+ V_j = \partial_- W_j$ for $j = 2, \ldots, N$. We denote by $\xi_j^\pm$ the contact structure induced by the Liouville form $i_X \omega$ on $\Sigma_j^\pm$, $j = 1, \ldots, N$.

For $k \geq j$ we denote by $S_j^{k-}$ the intersection of the union of the $Y$-stable manifolds of the critical points on level $c_k$ with the hypersurface $\Sigma_j^-$. Similarly, for $i \leq j$ we denote by $S_j^{i+}$ the intersection of the union of the $Y$-unstable manifolds of the critical points on level $c_i$ with the hypersurface $\Sigma_j^+$; see Figure 5.2. Set

$$S_j^- := \bigcup_{k \geq j} S_j^{k-}, \quad S_j^+ := \bigcup_{i \leq j} S_j^{i+}.$$

The assumption that the Smale cobordism $(Y, \phi)$ is elementary implies that $S_j^\pm$ is a union of spheres in $\Sigma_j^\pm$.

Consider on $\bigcup_{j=1}^N W_j$ the gradient-like vector fields $Y_t := (1 - t)Y + tX$, $t \in [0, 1]$, for $\phi$. Let us pick $\varepsilon$ so small that for all $t \in [0, 1]$ the $Y_t$-unstable spheres in $\Sigma_j^+$
of the critical points on level $c_j$ do not intersect the $Y$-stable spheres in $\Sigma_j^+$ of any critical points on higher levels. By Lemma 3.1 we can extend the $Y_t$ to gradient-like vector fields for $\phi$ on $W$ such that $Y_0 = Y$ and $Y_t = Y$ outside $\mathcal{O}_p \bigcup_{j=1}^{N} W_j$ for all $t \in [0, 1]$. By Lemma 3.6 this can be done in such a way that the intersection of the $Y_t$-stable manifold of the critical point locus on level $c_i$ with the hypersurface $\Sigma_j^+$ remains unchanged. This implies that the cobordisms $(W, Y_t, \phi)$ are elementary for all $t \in [0, 1]$. After renaming $Y_1$ back to $Y$ and shrinking the $W_j$, we may hence assume that $Y = X$ on $\mathcal{O}_p \bigcup_{j=1}^{N} W_j$. Moreover, after modifying $Y$ near $\partial W$ we may assume that $Y = X$ on $\mathcal{O}_p \partial W$.

We will construct the required homotopies $X_t$, $t \in [0, 1]$, and $\omega_t$, $t \in [0, \frac{1}{2}]$, separately on each $V_j$, $j = 1, \ldots, N - 1$, in such a way that $X_t$ is fixed near $\partial V_j$ for all $t \in [0, 1]$ and $\omega_t$ is fixed up to scaling near $\partial V_j$ for $t \in [0, \frac{1}{2}]$. This will allow us then to extend the homotopies $X_t$ and $\omega_t$ to $\bigcup_{j=1}^{N} W_j$ as constant (resp. constant up to scaling).

**Figure 5.2.** The partition of $W$ into subcobordisms.
Step 2. Consider $V_j$ for $1 \leq j \leq N - 1$. To simplify the notation, we denote the restriction of objects to $V_j$ by the same symbol as the original objects, omitting the index $j$. Let us denote by $\mathcal{X}(V_j, \phi)$ the space of all gradient-like vector fields for $\phi$ on $V_j$ that agree with $X$ near $\partial V_j$. We connect $X$ and $Y$ by the path $Y_t := (1-t)X + tY$ in $\mathcal{X}(V_j, \phi)$.

Denote by $\Gamma_{V_j} : \Sigma_{j+1}^- \rightarrow \Sigma_j^+$ the holonomy of the vector field $Y_t$ on $V_j$ and consider the isotopy $g_t := \Gamma_{V_j}|_{\Sigma_{j+1}^-} : \Sigma_{j+1}^- \hookrightarrow \Sigma_j^+$. Suppose for the moment that $\Sigma_{j+1}^- \subset \Sigma_j^-$ is isotropic and loose (this hypothesis will be satisfied below when we perform induction on descending values of $j$).

Since $\Gamma_{V_0} = \Gamma_X$ is a contactomorphism, this implies that the embedding $g_0$ is loose isotropic. Hence, by Theorem 2.2 for the subcritical case, Theorem 2.4 for the Legendrian overtwisted case in dimension 4, and Theorem 2.6 in the Legendrian loose case in dimension $2n > 4$, the isotopy $g_t$ can be $C^0$-approximated by an isotropic isotopy. More precisely, there exists a $C^0$-small diffeotopy $\delta_t : \Sigma_j^+ \rightarrow \Sigma_j^+$ with $\delta_0 = Id$ such that $\delta_t \circ g_t$, $t \in [0, 1]$, is loose isotropic with respect to the contact structure $\xi_j^+$.

The path $\Gamma_{V_t}$, $t \in [0, 1]$, in $\text{Diff}(\Sigma_{j+1}^-)$ is homotopic with fixed endpoints to the concatenation of the paths $\delta_t \circ \Gamma_{V_j}$ (from $\Gamma_{V_0}$ to $\delta_1 \circ \Gamma_{V_j}$) and $\delta_{t-1} \circ \delta_1 \circ \Gamma_{V_j}$ (from $\delta_1 \circ \Gamma_{V_j}$ to $\Gamma_{V_1}$.) Hence by Lemma 3.6 we find paths $Y_t'$ and $Y''_t$, $t \in [0, 1]$, in $\mathcal{X}(V_j, \phi)$ such that

- $Y_0' = X$, $Y_1' = Y_0''$ and $Y_1'' = Y$;
- $\Gamma_{V_t}' = \delta_t \circ \Gamma_{V_j}$ and $\Gamma_{V_t}'' = \delta_t^{-1} \circ \delta_1 \circ \Gamma_{V_j}$, $t \in [0, 1]$.

Note that $\Gamma_{V_t}'|_{\Sigma_{j+1}^-}$ is loose isotropic. Moreover, by choosing $\delta_t$ sufficiently $C^0$-small, we can ensure that $\Gamma_{V_t}''(S_{j+1}^-) \cap S_j^+ = \emptyset$ in $\Sigma_j^+$ and $\Gamma_Y(S_{j+1}^-)$ is loose in $\Sigma_j^+ \setminus S_j^+$. We extend the vector fields $Y_t'$ and $Y''_t$ to $W$ by setting $Y_t := (1-t)X + tY$ and $Y''_t := Y$ on $W \setminus V_j$. The preceding discussion shows that the cobordisms $(W, Y_t'', \phi)$ are elementary for all $t \in [0, 1]$. Hence it is sufficient to prove the lemma with the original vector field $Y$ replaced by $Y''_t = Y_0''$. To simplify the notation, we rename $Y'_t$ to $Y$ and the homotopy $Y_t'$ to $Y_t$. The new homotopy now has the property that the isotopy $\Gamma_{V_t}|_{S_{j+1}^-} : S_{j+1}^- \hookrightarrow \Sigma_j^+$ is loose isotropic and $\Gamma_Y(S_{j+1}^-)$ is loose in $\Sigma_j^+ \setminus S_j^+$. So the image of $\Gamma_Y(S_{j+1}^-)$ under the holonomy of the elementary Weinstein cobordism $(W_j, X = Y, \phi)$ is loose isotropic in $\Sigma_j^-$. Since the union $S_j^-$ of the stable spheres of $(W_j, Y)$ are loose by the flexibility hypothesis on $\mathfrak{W}$, this implies that $S_j^- \subset \Sigma_j^-$ is loose isotropic.

Now we perform this construction inductively in descending order over $V_j$ for $j = N - 1, N - 2, \ldots, 1$, always renaming the new vector fields back to $Y$. The resulting vector field $Y$ is then connected to $X$ by a homotopy $Y_t$ such that the manifolds
\[ S_{j+1}^- \subset \Sigma_{j+1} \] and the isotopies \( \Gamma_Y|_{S_{j+1}^-} : S_{j+1}^- \hookrightarrow \Sigma_j^+ \), \( t \in [0,1] \), are loose isotropic for all \( j = 1, \ldots, N - 1 \).

**Step 3.** Let \( Y \) and \( Y_t \) be as constructed in Step 2. Now we construct the desired homotopies \( X_t \) and \( \omega_t \) separately on each \( V_j, \ j = 1, \ldots, N - 1 \), keeping them fixed near \( \partial V_j \). We keep the notation from Step 2. By the contact isotopy extension theorem, we can extend the isotropic isotopy \( \Gamma_Y|_{S_{j+1}^-} : S_{j+1}^- \hookrightarrow \Sigma_j^+ \) to a contact isotopy \( G_t : (\Sigma_{j+1}^-, \xi_{j+1}^-) \to (\Sigma_j^+, \xi_j^+) \) starting at \( G_0 = \Gamma_{Y_0} = \Gamma_X \). By Lemma 4.1, we find a Weinstein homotopy \( W_t = (V_j, \tilde{\omega}_t, \tilde{X}_t, \phi) \) beginning at \( \tilde{W}_0 = \tilde{W} \) with holonomy \( \Gamma_{\tilde{W}_t} = G_t \) for all \( t \in [0,1] \). Now Lemma 3.7 provides a path \( X_t \in \mathcal{X}(V_j, \phi) \) such that

\[
\begin{align*}
(i) & \ X_t = \tilde{X}_t \text{ for } t \in [0, \frac{1}{2}]; \\
(ii) & \ X_1 = Y_1 = Y; \\
(iii) & \ \Gamma_{X_t}(S_{j+1}^-) = \Gamma_Y(S_{j+1}^-) \text{ for } t \in [\frac{1}{2}, 1].
\end{align*}
\]

Over the interval \([0, \frac{1}{2}]\) the Smale homotopy \( \mathcal{G}_t = (V_j, X_t, \phi) \) can be lifted to the Weinstein homotopy \( W_t = (V_j, \omega_t, X_t, \phi) \), where \( \omega_t := \tilde{\omega}_{2t} \).

Condition (iii) implies that \( \Gamma_{X_t}(S_{j+1}^-) \cap S_j^+ = \emptyset \) for all \( t \in [\frac{1}{2}, 1] \), so the resulting Smale homotopy on \( W \) is elementary over the interval \([\frac{1}{2}, 1]\). \( \square \)

The following lemma is the analogue of Lemma 5.7 in the case that the Smale cobordism \( (W, Y, \phi) \) is not elementary, but has exactly two critical points connected by a unique trajectory.

**Lemma 5.8.** Let \( \mathcal{W} = (W, \omega, X, \phi) \) be a flexible Weinstein cobordism and \( Y \) a gradient-like vector field for \( \phi \). Suppose that the function \( \phi \) has exactly two critical points connected by a unique \( Y \)-trajectory along which the stable and unstable manifolds intersect transversely. Then there exists a family \( X_t, \ t \in [0, 1] \), of gradient-like vector fields for \( \phi \) and a family \( \omega_t, \ t \in [0, \frac{1}{2}] \), of symplectic forms on \( W \) such that

- \( \mathcal{W}_t = (W, \omega_t, X_t, \phi), \ t \in [0, \frac{1}{2}] \), is a homotopy with \( \mathcal{W}_0 = \mathcal{W} \), fixed on \( \mathcal{O}_P \partial_- W \) and fixed up to scaling on \( \mathcal{O}_P \partial_+ W \);
- \( X_1 = Y \) and for \( t \in [\frac{1}{2}, 1] \) the critical points of the function \( \phi \) are connected by a unique \( X_t \)-trajectory.

**Proof.** Let us denote the critical points of the function \( \phi \) by \( p_1 \) and \( p_2 \) and the corresponding critical values by \( c_1 < c_2 \). As in the proof of Lemma 5.7, for sufficiently small \( \varepsilon > 0 \), we split the cobordism \( W \) into three parts:

\[
W_1 := \{ \phi \leq c_1 + \varepsilon \}, \quad V := \{ c_1 + \varepsilon \leq \phi \leq c_2 - \varepsilon \}, \quad W_2 := \{ \phi \geq c_2 - \varepsilon \}.
\]
Arguing as in Step 1 of the proof of Lemma 5.7, we reduce to the case that $Y = X$ on $\mathcal{O} p (W_1 \cup W_2)$.

On $V$ consider the gradient-like vector fields $Y_t := (1 - t)X + tY$ for $\phi$. Let $\Sigma := \{ \phi = c_1 + \varepsilon \} = \partial_- V$. Denote by $S_t \subset \Sigma$ the $Y_t$-stable sphere of $p_2$ and by $S^+ \subset \Sigma$ the $Y$-unstable sphere of $p_1$. Note that $S^+$ is coisotropic, $S_0$ is isotropic, and $S_1$ intersects $S^+$ transversely in a unique point $q$. We deform $S_1$ to $S'_1$ by a $C^0$-small deformation, keeping the unique transverse intersection point $q$ with $S^+$, such that $S'_1$ is isotropic near $q$. Connect $S_0$ to $S'_1$ by an isotopy $S'_t$ which is $C^0$-close to $S_t$. Due to the flexibility hypothesis on $\mathcal{W}$, the isotropic sphere $S'_0 = S_0$ is loose. Hence by Theorems 2.2, 2.4, and 2.6 we can $C^0$-approximate $S'_t$ by an isotropic isotopy $\tilde{S}_t$ such that $\tilde{S}_0 = S'_0 = S_0$, and $\tilde{S}_1$ coincides with $S'_1$ near $q$. In particular, $\tilde{S}_1$ has $q$ as the unique transverse intersection point with $S^+$. Arguing as in Steps 2 and 3 of the proof of Lemma 5.7, we now construct a Weinstein homotopy $\mathcal{W}_t = (V, \omega_t, X_t, \phi)$, $t \in [0, \frac{1}{2}]$, fixed near $\partial_- V$ and fixed up to scaling near $\partial_+ V$, and Smale cobordisms $(V, X_t, \phi)$, $t \in [\frac{1}{2}, 1]$, fixed near $\partial V$, such that

- $\mathcal{W}_0 = \mathcal{W}|_V$ and $X_1 = Y|_V$;
- the $X_t$-stable sphere of $p_2$ in $\Sigma$ equals $\tilde{S}_{2t}$ for $t \in [0, \frac{1}{2}]$, and $\tilde{S}_1$ for $t \in [\frac{1}{2}, 1]$.

In particular, for $t \in [\frac{1}{2}, 1]$ the $X_t$-stable sphere of $p_2$ in $\Sigma$ intersects $S^+$ transversely in the unique point $q$, so the two critical points $p_1, p_2$ are connected by a unique $X_t$-trajectory for $t \in [\frac{1}{2}, 1]$. \hfill \Box

The following lemma will serve as induction step in proving Theorem 5.3.

**Lemma 5.9.** Let $\mathcal{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism of dimension $2n$. Let $\mathcal{G}_t = (W, Y_t, \phi_t)$, $t \in [0, 1]$, be an elementary Smale homotopy without critical points of index $> n$ such that $\phi_0 = \phi$ on $W$ and $\phi_t = \phi$ near $\partial W$ (but not necessarily $Y_0 = X!$). If $2n = 4$ and $\mathcal{G}_t$ is of Type IIb assume that either $\partial_- W$ is overtwisted, or $\phi$ has no critical points of index $> 1$. Then there exists a homotopy $\mathcal{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, of flexible Weinstein structures, starting at $\mathcal{W}_0 = \mathcal{W}$, which is fixed near $\partial_- W$ and fixed up to scaling near $\partial_+ W$.

**Proof.** Type I. Consider first the case when the homotopy $\mathcal{G}_t$ is elementary of Type I. We point out that $(W, X, \phi)$ need not be elementary. To remedy this, we apply Lemma 5.7 to construct families $X_t$ and $\omega_t$ such that

- $\mathcal{W}_t = (W, \omega_t, X_t, \phi)$, $t \in [0, \frac{1}{2}]$, is a Weinstein homotopy with $\mathcal{W}_0 = \mathcal{W}$, fixed on $\mathcal{O} p \partial_- W$ and fixed up to scaling on $\mathcal{O} p \partial_+ W$;
- $X_1 = Y_0$ and the Smale cobordisms $(W, X_t, \phi)$, $t \in [\frac{1}{2}, 1]$, are elementary.
Thus it is sufficient to prove the lemma for the Weinstein cobordism \((\omega^1, X^1, \phi)\) instead of \(W\), and the concatenation of the Smale homotopies \((X_t, \phi_t)_{t \in [\frac{1}{2}, 1]}\) and \((Y_t, \phi_t)_{t \in [0, 1]}\) instead of \((Y_t, \phi_t)\). To simplify the notation, we rename the new Weinstein cobordism and Smale homotopy back to \(W\) instead of \((\omega, X, \phi)\) and \((Y_t, \phi_t)\). So in the new notation we now have \(X = Y_0\).

According to Lemma 3.5, there exists a family \(\tilde{\phi}_t\), \(t \in [0, 1]\), of Lyapunov functions for \(X\) with the same profile as the family \(\phi_t\), and such that \(\tilde{\phi}_0 = \phi\) and \(\tilde{\phi}_t = \phi_t\) on \(\partial p \partial W\). Then Lemma 3.4 provides a diffeotopy \(h_t : W \rightarrow W\), \(t \in [0, 1]\), such that \(h_0 = \text{Id}\), \(h_t|_{\partial p \partial W} = \text{Id}\), and \(\phi_t = \tilde{\phi}_t \circ h_t\) for all \(t \in [0, 1]\). Thus the Weinstein homotopy \((W, \omega_t = h_t^* \omega, X_t = h_t^* X, \phi_t = h_t^* \tilde{\phi}_t), t \in [0, 1]\), has the desired properties. It is flexible because the \(X_t\)-stable spheres in \(\partial_- W\) are loose for \(t = 0\) and moved by an isotropic isotopy, so they remain loose for all \(t \in [0, 1]\).

**Type IIId.** Suppose now that the homotopy \(\mathcal{S}_t\) is of Type IIId. Let \(t_0 \in [0, 1]\) be the parameter value for which the function \(\phi_t\) has a death-type critical point. In this case the function \(\phi\) has exactly two critical points \(p\) and \(q\) connected by a unique \(Y_0\)-trajectory. Arguing as in the Type I case, using Lemma 5.8 instead of Lemma 5.7, we can again reduce to the case that \(X = Y_0\).

Then Proposition 4.8 provides an elementary Weinstein homotopy \((W, \omega, \tilde{X}_t, \tilde{\phi}_t)\) of Type IIId starting from \(W\) and killing the critical points \(p\) and \(q\) at time \(t_0\). One can also arrange that \((\tilde{X}_t, \tilde{\phi}_t)\) coincides with \((X, \phi)\) on \(\partial p \partial W\), and (by composing \(\tilde{\phi}_t\) with suitable functions \(\mathbb{R} \rightarrow \mathbb{R}\)) that the homotopies \(\tilde{\phi}_t\) and \(\phi_t\) have equal profiles. Then Lemma 3.4 provides a diffeotopy \(h_t : W \rightarrow W\), \(t \in [0, 1]\), such that \(h_0 = \text{Id}\), \(h_t|_{\partial p \partial W} = \text{Id}\), and \(\phi_t = \tilde{\phi}_t \circ h_t\) for all \(t \in [0, 1]\). Thus the Weinstein homotopy \((W, \omega_t = h_t^* \omega, X_t = h_t^* \tilde{X}_t, \phi_t = h_t^* \tilde{\phi}_t), t \in [0, 1]\), has the desired properties. It is flexible because the intersections of the \(X_t\)-stable manifolds with regular level sets remain loose for \(t \in [0, t_0]\) and there are no critical points for \(t > t_0\).

**Type IIb.** The argument in the case of Type IIb is similar, except that we use Proposition 4.7 instead of Proposition 4.8 and we do not need a preliminary homotopy. However, the flexibility of \(W_t\) for \(t \geq t_0\) requires an additional argument.

Consider first the case \(2n > 4\). Suppose \(\phi_1\) has critical points \(p\) and \(q\) of index \(n\) and \(n - 1\), respectively (if they have smaller indices flexibility is automatic). Then the closure \(\Delta\) of the \(X_1\)-stable manifold of the point \(p\) intersects \(\partial_- W\) along a Legendrian disc \(\partial_\Delta \Delta\). The boundary \(S^-_q\) of this disc is the intersection with \(\partial_- W\) of the \(X_1\)-stable manifold \(D^-_q\) of \(q\). According to Remark 2.5(1) all Legendrian discs are loose, or more precisely, \(\partial_- \Delta \setminus S^-_q\) is loose in \(\partial_- W \setminus S^-_q\). Let \(c\) be a regular value of \(\phi_1\) which separates \(\phi_1(q)\) and \(\phi_1(p)\) and consider the level set \(\Sigma := \{\phi_1 = c\}\). Flowing along \(X_1\)-trajectories defines a contactomorphism \(\partial_- W \setminus S^-_q \rightarrow \Sigma \setminus D^-_q\) mapping \(\partial_- \Delta \setminus S^-_q\)
onto $\Delta \cap \Sigma \setminus \{r\}$, where $r$ is the unique intersection point of $\Delta$ and the $X_t$-unstable manifold $D_q^+$ in the level set $\Sigma$. It follows that $\Delta \cap \Sigma \setminus \{r\}$ is loose in $\Sigma \setminus \{r\}$, and hence $\Delta \cap \Sigma$ is loose in $\Sigma$. This proves flexibility of $\mathcal{W}_1$, and thus of $\mathcal{W}_t$ for $t \geq t_0$.

Finally, consider the case $2n = 4$. If the critical points have indices 1 and 0, flexibility is automatic. If they have indices 2 and 1 and $\partial_- W$ is overtwisted, we can arrange that $\partial_- \Delta \subset \partial_- W$ (in the notation above) has an overtwisted disc in its complement, hence so does the intersection of $\Delta$ with the regular level set $\{\phi = c\}$.

\[ \square \]

**Proof of Theorem 5.3.** Let us pick gradient-like vector fields $Y_t$ for $\phi_t$ with $Y_0 = X$ and $Y_t = X$ near $\partial W$ to get a Smale homotopy $\mathcal{G}_t = (W, Y_t, \phi_t)$, $t \in [0, 1]$. By Lemma 3.3 we find an admissible partition for the Smale homotopy $\mathcal{G}_t$. Thus we get a sequence $0 = t_0 < t_1 < \cdots < t_p = 1$ of parameter values and smooth families of partitions

$$W = \bigcup_{j=1}^{N_k} W_j^k(t), \quad W_j^k(t) := \{c_j^k(t) \leq \phi_t \leq c_j^k(t)\}, \quad t \in [t_{k-1}, t_k],$$

such that each Smale homotopy

$$\mathcal{G}^k_j := \left( (W_j^k(t), Y_t|_{W_j^k(t)}, \phi_t|_{W_j^k(t)}) \right)_{t \in [t_{k-1}, t_k]}$$

is elementary. We will construct the Weinstein homotopy $(\omega_t, X_t, \phi_t)$ on the cobordisms $\bigcup_{j \in [t_{k-1}, t_k]} W_j^k(t)$ inductively over $k = 1, \ldots, p$, and for fixed $k$ over $j = 1, \ldots, N_k$.

Suppose the required Weinstein homotopy is already constructed on $W$ for $t \leq t_{k-1}$. To simplify the notation we rename $\phi_{t_{k-1}}$ to $\phi$, the vector fields $X_{t_{k-1}}$ and $Y_{t_{k-1}}$ to $X$ and $Y$, and the symplectic form $\omega_{t_{k-1}}$ to $\omega$. We also write $N$ instead of $N_k$, $W_j$ and $W_j(t)$ instead of $W_j^k(t_{k-1})$ and $W_j^k(t)$, and replace the interval $[t_{k-1}, t_k]$ by $[0, 1]$.

There exists a diffeotopy $f_t : W \to W$, fixed on $\partial p \partial W$, with $f_0 = 1$ and such that $f_t(W_j) = W_j(t)$ for all $t \in [0, 1]$. Moreover, we can choose $f_t$ and a diffeotopy $g_t : \mathbb{R} \to \mathbb{R}$ with $g_0 = 1$ such that the function $g_t := g_t \circ \phi_t \circ f_t$ coincides with $\phi$ on $\partial p \partial W_j$ for all $t \in [0, 1], j = 1, \ldots, N$. Set $Y_t := f_t^* Y_t$. So we have a flexible Weinstein cobordism $\mathcal{W} = (W = \bigcup_{j=1}^{N} W_j, \omega, X, \phi = \widehat{\phi}_0)$ and a Smale homotopy $(\widehat{Y}_t, \widehat{\phi}_t)$, $t \in [0, 1]$, whose restriction to each $W_j$ is elementary. (But the restriction of $\mathcal{W}$ to $W$ need not be elementary.)

Now we apply Lemma 5.9 inductively for $j = 1, \ldots, N$ to construct Weinstein homotopies $\widehat{\mathcal{W}}^j_t = (W_j, \widehat{\omega}_t, \widehat{X}_t, \widehat{\phi}_t)$, fixed near $\partial_- W_j$ and fixed up to scaling near
\[ \partial_s W_j, \text{ with } \hat{\mathcal{M}}_0^j = \mathcal{M}|_{W_j}. \] Thus the \( \mathcal{M}_t^j \) fit together to form a Weinstein homotopy \( \hat{\mathcal{M}}_t = (\hat{\omega}_t, \hat{X}_t, \hat{\phi}_t) \) on \( W \). The desired Weinstein homotopy on \( W \) is now given by

\[
\mathcal{M}_t := \left( f_t \hat{\omega}_t, f_t \hat{X}_t, g_t^{-1} \circ \hat{\phi}_t \circ f_t^{-1} \right). \qedhere
\]

5.4. **Proof of the second Weinstein deformation theorem.** Let us extend the vector fields \( Y_t \) from \( O_p \partial_- W \) to a path of gradient-like vector fields for \( \phi_t \) on \( W \) connecting \( X_0 \) and \( X_1 \). We will deduce Theorem 5.4 from Theorem 5.3 and the following special case, which is just a 1-parametric version of the Weinstein Existence Theorem 5.1.

**Lemma 5.10.** Theorem 5.4 holds under the additional hypothesis that \( \phi_t = \phi \) is independent of \( t \in [0, 1] \) and the Smale homotopy \((W, Y_t, \phi)\) is elementary.

**Proof.** The proof is just a 1-parametric version of the proof of Theorem 5.1 using Theorem 2.10 and Lemma 4.3 instead of Theorem 2.9 and Lemma 4.2. \( \Box \)

**Lemma 5.11.** Theorem 5.4 holds under the additional hypothesis that \( \phi_t = \phi \) is independent of \( t \in [0, 1] \).

**Proof.** Let us pick regular values

\[ \phi|_{\partial_- W} = c_0 < c_1 < \cdots < c_N = \phi|_{\partial_+ W} \]

such that each \( (c_{k-1}, c_k) \) contains at most one critical value. Then the restriction of the homotopy \((Y_t, \phi), t \in [0, 1]\), to each cobordism \( W^k := \{ c_{k-1} \leq \phi \leq c_k \} \) is elementary.

We apply Lemma 5.10 to the restriction of the homotopy \((\eta_t, Y_t, \phi)\) to \( W^1 \). Hence \( \mathcal{M}_0|_{W^1} \) and \( \mathcal{M}_1|_{W^1} \) are connected by a homotopy \( \mathcal{M}_t^1 = (\omega_t^1, X_t^1, \phi), t \in [0, 1] \), of flexible Weinstein structures on \( W^1 \), agreeing with \((\eta_t, Y_t, \phi_t)\) on \( O_p \partial_- W \), such that the paths \( t \mapsto \omega_t^1 \) and \( t \mapsto \eta_t, t \in [0, 1] \), of nondegenerate 2-forms on \( W^1 \) are connected by a homotopy \( \eta_t^k, s, t \in [0, 1] \) rel \( O_p \partial_- W \) with fixed endpoints. We use the homotopy \( \omega_t^1 \) to extend \( \omega_t^1 \) to nondegenerate 2-forms \( \eta_t^1 \) on \( W \) such that \( \eta_0^1 = \omega_0 \), \( \eta_t^1 = \omega_1 \), \( \eta_t^1 = \eta_t \) outside a neighborhood of \( W^1 \), and the paths \( t \mapsto \eta_t^1 \) and \( t \mapsto \eta_t \), \( t \in [0, 1] \), of nondegenerate 2-forms on \( W \) are homotopic rel \( O_p \partial_- W \) with fixed endpoints. By Lemma 3.1 we can extend \( X_t^1 \) to gradient-like vector fields \( Y_t^1 \) for \( \phi \) on \( W \) such that \( Y_0^1 = X_0 \) and \( Y_1^1 = X_1 \). Now we can apply Lemma 5.10 to the restriction of the homotopy \((\eta_t^1, Y_t^1, \phi)\) to the elementary cobordism \( W^2 \) and continue inductively to construct homotopies \((\eta_t^k, Y_t^k, \phi)\) on \( W \) which are Weinstein on \( W^k \), so \((\eta_t^N, Y_t^N, \phi)\) is the desired Weinstein homotopy. Note that \((\eta_t^N, Y_t^N, \phi)\) is flexible because its restriction to each \( W^k \) is flexible. \( \Box \)
Proof of Theorem 5.4. Let us reparametrize the given homotopy \((\eta_t, Y_t, \phi_t), t \in [0, 1]\), to make it constant for \(t \in \left[\frac{1}{2}, 1\right]\). After pulling back \((\eta_t, Y_t, \phi_t)\) by a diffeotopy and target reparametrizing \(\phi_t\), we may further assume that \(\phi_t\) is independent of \(t\) on \(O_p \partial W\).

By Theorem 5.3, \(M_0\) can be extended to a homotopy \(M_t = (\omega_t, X_t, \phi_t), t \in [0, \frac{1}{2}]\), of flexible Weinstein structures on \(W\), fixed on \(O_p \partial W\). We can modify \(M_t\) to make it agree with \((\eta_t, Y_t, \phi_t)\) on \(O_p \partial W\). Note that \(M_{\frac{1}{2}}\) and \(M_1\) share the same function \(\phi_{\frac{1}{2}} = \phi_1\). We connect \(\omega_{\frac{1}{2}}\) and \(\omega_1\) by a path \(\eta_t, t \in \left[\frac{1}{2}, 1\right]\) of nondegenerate 2-forms by following the path \(\omega_t\) backward and then \(\eta_t\) forward. Since \(\omega_t = \eta_t\) on \(O_p \partial W\) for \(t \in \left[0, \frac{1}{2}\right]\), we can apply Lemma 5.11 to the homotopy \((\eta_t, X_t, \phi_1), t \in \left[\frac{1}{2}, 1\right]\) of gradient-like vector fields for \(\phi_1\) which agree with \(X_{\frac{1}{2}} = X_1\) on \(O_p \partial W\). By Lemma 3.1, we can connect \(X_{\frac{1}{2}}\) and \(X_1\) by a homotopy \(Y_t, t \in \left[\frac{1}{2}, 1\right]\), of gradient-like vector fields for \(\phi_1\) which agree with \(X_{\frac{1}{2}} = X_1\) on \(O_p \partial W\). Thus the concatenated Weinstein homotopy \(M_t, t \in [0, 1]\), has the desired properties. \(\square\)

6. Applications

6.1. The Weinstein h-cobordism theorem. Most of your applications are based on the following result, which is a direct consequence of the “two-index theorem” of Hatcher and Wagoner; see [4] for its formal derivation from the results in [18, 19].

Theorem 6.1. Any two Morse functions without critical points of index \(> n\) on a cobordism or a manifold of dimension \(2n > 4\) can be connected by a Morse homotopy without critical points of index \(> n\) (where, as usual, functions on a cobordism \(W\) are required to have \(\partial_W \) as regular level sets and functions on a manifold are required to be exhausting).

Corollary 6.2. In the case \(2n > 4\), we can remove the hypothesis on the existence of a Morse homotopy \(\phi_t\) from Theorems 5.3, 5.4, 5.5 and 5.5 and still conclude the existence of the stated Weinstein homotopies. \(\square\)

In particular, we have the following Weinstein version of the h-cobordism theorem.

Corollary 6.3 (Weinstein h-cobordism theorem). Any flexible Weinstein structure on a product cobordism \(W = Y \times [0, 1]\) of dimension \(2n > 4\) is homotopic to a
Weinstein structure \((W, \omega, X, \phi)\), where \(\phi : W \to [0, 1]\) is a function without critical points.

6.2. Symplectomorphisms of flexible Weinstein manifolds. Theorem 5.6 has the following consequence for symplectomorphisms of flexible Weinstein manifolds.

**Theorem 6.4.** Let \(\mathcal{W} = (V, \omega, X, \phi)\) be a flexible Weinstein manifold of dimension \(2n > 4\), and \(f : V \to V\) a diffeomorphism such that \(f^* \omega\) is homotopic to \(\omega\) through nondegenerate 2-forms. Then there exists a diffeotopy \(f_t : V \to V, t \in [0, 1]\), such that \(f_0 = f\), and \(f_1\) is an exact symplectomorphism of \((V, \omega)\).

**Proof.** By Theorem 5.6 and Corollary 6.2, there exists a Weinstein homotopy \(\mathcal{W}_t\) connecting \(\mathcal{W}_0 = \mathcal{W}\) and \(\mathcal{W}_1 = f^* \mathcal{W}\). Thus Proposition 4.4 provides a diffeotopy \(h_t : V \to V\) such that \(h_0 = 1\) and \(h_t^* \lambda - \lambda\) is exact, where \(\lambda\) is the Liouville form of \(\mathcal{W}\). Now \(f_t = f \circ h_t\) is the desired diffeotopy. □

**Remark 6.5.** Even if \(\mathcal{W}\) is of finite type and \(f = 1\) outside a compact set, the diffeotopy \(f_t\) provided by Theorem 6.4 will in general not equal the identity outside a compact set.

6.3. Symplectic pseudo-isotopies. Let us recall the basic notions of pseudo-isotopy theory from [2, 15]. For a manifold \(W\) (possibly with boundary) and a closed subset \(A \subset W\), we denote by \(\text{Diff}(W, A)\) the space of diffeomorphisms of \(W\) fixed on \(\text{Op}(A)\), equipped with the \(C^\infty\)-topology. For a cobordism \(W\), the restriction map to \(\partial W\) defines a fibration

\[
\text{Diff}(W, \partial W) \to \text{Diff}(W, \partial_- W) \to \text{Diff}_{\mathcal{P}}(\partial_+ W),
\]

where \(\text{Diff}_{\mathcal{P}}(\partial_+ W)\) denotes the image of the restriction map \(\text{Diff}(W, \partial_- W) \to \text{Diff}(\partial_+ W)\).

For the product cobordism \(I \times M, I = [0, 1], \partial M = \emptyset\),

\[
\mathcal{P}(M) := \text{Diff}(I \times M, 0 \times M)
\]

is the group of pseudo-isotopies of \(M\). Denote by \(\text{Diff}_{\mathcal{P}}(M)\) the group of diffeomorphisms of \(M\) that are pseudo-isotopic to the identity, i.e., that appear as the restriction to \(1 \times M\) of an element in \(\mathcal{P}(M)\). Restriction to \(1 \times M\) defines the fibration

\[
\text{Diff}(I \times M, \partial I \times M) \to \mathcal{P}(M) \to \text{Diff}_{\mathcal{P}}(M),
\]

and thus a homotopy exact sequence

\[
\cdots \to \pi_0 \text{Diff}(I \times M, \partial I \times M) \to \pi_0 \mathcal{P}(M) \to \pi_0 \text{Diff}_{\mathcal{P}}(M) \to 0.
\]

We will use the following alternative description of \(\mathcal{P}(M)\); see [2]. Denote by \(\mathcal{E}(M)\) the space of all smooth functions \(f : I \times M \to I\) without critical points and satisfying
$f(r, x) = r$ on $\mathcal{O}p(\partial I \times M)$. We have a homotopy equivalence

$$\mathcal{P}(M) \to \mathcal{E}(M), \quad F \mapsto p \circ F,$$

where $p : I \times M \to I$ is the projection. A homotopy inverse is given by fixing a metric and sending $f \in \mathcal{E}(M)$ to the unique diffeomorphism $F$ mapping levels of $f$ to levels of $p$ and gradient trajectories of $f$ to straight lines $I \times \{x\}$. Note that the last map in the homotopy exact sequence

$$\cdots \to \pi_0 \text{Diff}(I \times M, \partial I \times M) \to \pi_0 \mathcal{E}(M) \to \pi_0 \text{Diff}_P(M)$$

associates to $f \in \mathcal{E}(M)$ the flow from $0 \times M$ to $1 \times M$ along trajectories of a gradient-like vector field (whose isotopy class does not depend on the gradient-like vector field).

For the symplectic version of the pseudo-isotopy spaces, it will be convenient to replace $I \times M$ by $\mathbb{R} \times M$ as follows: We replace $\mathcal{E}(M)$ by the space of functions $f : \mathbb{R} \times M \to \mathbb{R}$ without critical points and satisfying $f(r, x) = r$ outside a compact set; $\text{Diff}(I \times M, \partial I \times M)$ by the space $\text{Diff}_c(\mathbb{R} \times M)$ of diffeomorphisms that equal the identity outside a compact set; and $\mathcal{P}(M)$ by the space of diffeomorphisms of $\mathbb{R} \times M$ that equal the identity near $\{-\infty\} \times M$ and have the form $(r, x) \mapsto (r + f(x), g(x))$ near $\{+\infty\} \times M$. The last map in the exact sequence

$$\cdots \to \pi_0 \text{Diff}_c(\mathbb{R} \times M) \to \pi_0 \mathcal{E}(M) \to \pi_0 \text{Diff}_P(M)$$

then associates to $f \in \mathcal{E}(M)$ the flow from $\{-\infty\} \times M$ to $\{+\infty\} \times M$ along trajectories of a gradient-like vector field which equals $\partial_r$ outside a compact set.

We endow the spaces $\mathcal{P}(M)$, $\mathcal{E}(M)$ and $\text{Diff}_c(\mathbb{R} \times M)$ with the topology of uniform $C^\infty$-convergence on $\mathbb{R} \times M$ (and not the topology of uniform $C^\infty$-convergence on compact sets), with respect to the product of the Euclidean metric on $\mathbb{R}$ and any Riemannian metric on $M$. In other words, a sequence $F_n \in \mathcal{P}(M)$ converges to $F \in \mathcal{P}(M)$ if and only if $\|F_n - F\|_{C^k(\mathbb{R} \times M)} \to 0$ for every $k = 0, 1, \ldots$. For example, consider any non-identity element $F \in \mathcal{P}(M)$ and the translations $\tau_c(r, x) = (r + c, x)$, $c \in \mathbb{R}$, on $\mathbb{R} \times M$. Then the sequence $F_n := \tau_n \circ F \circ \tau_{-n}$ does not converge as $n \to \infty$ to the identity in $\mathcal{P}(M)$, although it does converge uniformly on compact sets. With this topology, the obvious inclusion maps from the spaces on $I \times M$ to the corresponding spaces on $\mathbb{R} \times M$ are weak homotopy equivalences.

**Remark 6.6.** It was proven by Cerf in [2] that $\pi_0 \mathcal{P}(M)$ is trivial if $\dim M \geq 5$ and $M$ is simply connected. In the non-simply connected case and for $\dim M \geq 6$, Hatcher and Wagoner ([18], see also [19]) have expressed $\pi_0 \mathcal{P}(M)$ in terms of algebraic K-theory of the group ring of $\pi_1(M)$. In particular, there are many fundamental groups for which $\pi_1 \mathcal{P}(M)$ is not trivial.
Let us now fix a contact manifold \((M^{2n-1}, \xi)\) and denote by \((\mathcal{SM}, \lambda_{\text{st}})\) its symplectization with its canonical Liouville structure \((\omega_{\text{st}} = d\lambda_{\text{st}}, X_{\text{st}})\). Any choice of a contact form \(\alpha\) for \(\xi\) yields an identification of \(\mathcal{SM}\) with \(\mathbb{R} \times M\) and the Liouville structure \(\lambda_{\text{st}} = e^\alpha, \omega_{\text{st}} = d\lambda_{\text{st}}, X_{\text{st}} = \partial_r\). However, the following constructions do not require the choice of a contact form. We will refer to the two ends of \(\mathcal{SM}\) as \(\{\pm \infty\} \times M\).

We define the group of symplectic pseudo-isotopies of \((M, \xi)\) as

\[
P(M, \xi) := \{ F \in \text{Diff}(\mathcal{SM}) \mid F^*\omega_{\text{st}} = \omega_{\text{st}}, \ F = \mathbb{I} \text{ near } \{-\infty\} \times M, \ F^*\lambda_{\text{st}} = \lambda_{\text{st}} \text{ near } \{+\infty\} \times M \}.
\]

Moreover, we introduce the space

\[
\mathcal{E}(M, \xi) := \{ (\lambda, \phi) \mid \text{Weinstein structure on } \mathcal{SM} \text{ without critical points} \mid d\lambda = \omega_{\text{st}}, \ (\lambda, \phi) = (\lambda_{\text{st}}, \phi_{\text{st}}) \text{ outside a compact set} \}
\]

and its image \(\mathcal{E}(M, \xi)\) under the projection \((\lambda, \phi) \mapsto \lambda\). We endow the spaces \(\mathcal{P}(M, \xi), \mathcal{E}(M, \xi),\) and \(\bar{\mathcal{E}}(M, \xi)\) with the topology of uniform \(C^\infty\)-convergence on \(SM = \mathbb{R} \times M\) as explained above.

**Lemma 6.7.** The map

\[\mathcal{E}(M, \xi) \to \bar{\mathcal{E}}(M, \xi), \quad (\lambda, \phi) \mapsto \lambda,\]

is a homotopy equivalence and the map

\[\mathcal{P}(M, \xi) \to \bar{\mathcal{E}}(M, \xi), \quad F \mapsto F^*\lambda_{\text{st}},\]

is a homeomorphism.

**Proof.** The first map defines a fibration whose fiber over \(\lambda\) is the contractible space of Lyapunov functions for \(X\) which are standard at infinity. The inverse of the second map associates to \(\lambda\) the unique \(F \in \text{Diff}(SM)\) satisfying \(F_*X = X_{\text{st}}\) on \(SM\) and \(F = \mathbb{I}\) near \(\{-\infty\} \times M\) (which implies \(F^*\lambda_{\text{st}} = \lambda_{\text{st}}\) on \(SM\)).

Since \(F \in \mathcal{P}(M, \xi)\) satisfies \(F^*\lambda_{\text{st}} = \lambda_{\text{st}}\) near \(\{+\infty\} \times M\), it descends there to a contactomorphism \(F_+ : M \to M\). By construction, \(F_+\) belongs to the group \(\text{Diff}_P(M)\) of diffeomorphisms that are pseudo-isotopic to the identity, so it defines an element in

\[\text{Diff}_P(M, \xi) := \{ F_+ \in \text{Diff}_P(M) \mid F_+^*\xi = \xi \}.
\]

Moreover, \(F_+ = \mathbb{I}\) if and only if \(F\) belongs to the space

\[\text{Diff}_c(SM, \omega_{\text{st}}) := \{ F \in \text{Diff}_c(SM) \mid F^*\omega_{\text{st}} = \omega_{\text{st}} \}
\]

of compactly supported symplectomorphisms of \((SM, \omega_{\text{st}})\). Thus we have a fibration

\[\text{Diff}_c(SM, \omega_{\text{st}}) \to \mathcal{P}(M, \xi) \to \text{Diff}_P(M, \xi).\]
The corresponding homotopy exact sequence fits into a commuting diagram
\[
\pi_0 \text{Diff}_c(SM, \omega_{st}) \longrightarrow \pi_0 \mathcal{P}(M, \xi) \longrightarrow \pi_0 \text{Diff}_c(M, \xi) \longrightarrow 0
\]
where the vertical maps are induced by the obvious inclusions.

The following is the main result of this section.

**Theorem 6.8.** For any closed contact manifold \((M, \xi)\) of dimension \(2n - 1 \geq 5\), the map \(\pi_0 \mathcal{P}(M, \xi) \rightarrow \pi_0 \mathcal{P}(M)\) is surjective.

**Proof.** By the discussion above, it suffices to show that the map \(\pi_0 \mathcal{E}(M, \xi) \rightarrow \pi_0 \mathcal{E}(M)\) induced by the projection \((\lambda, \phi) \mapsto \phi\) is surjective. So let \(\psi \in \mathcal{E}(M)\), i.e., \(\psi : \mathbb{R} \times M \rightarrow \mathbb{R}\) is a function without critical points which agrees with \(\phi_{st}(r, x) = r\) outside a compact set \(W = [a, b] \times M\). By Theorem 6.1, there exists a Morse homotopy \(\phi_t : \mathbb{R} \times M \rightarrow \mathbb{R}\) without critical points of index > \(n\) connecting \(\phi_0 = \phi_{st}\) with \(\phi_1 = \psi\) such that \(\phi_t = \phi_{st}\) outside \(W\) for all \(t \in [0, 1]\). We apply Theorem 5.3 to the Weinstein cobordism \(\mathfrak{W} = (W, \omega_{st}, X_{st}, \phi_{st})\) and the homotopy \(\phi_t : W \rightarrow \mathbb{R}\).

Hence there exists a Weinstein homotopy \(\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)\), fixed on \(\mathcal{O}_p \partial_- W\) and fixed up to scaling on \(\mathcal{O}_p \partial_+ W\), such that \(\mathfrak{W}_0 = \mathfrak{W}\). Note that \(\lambda_t = c_t \lambda_{st}\) on \(\mathcal{O}_p \partial_+ W\) for constants \(c_t\) with \(c_0 = 1\). So we can extend \(\mathfrak{W}_t\) over the rest of \(\mathbb{R} \times M\) by the function \(\phi_{st}\) and Liouville forms \(f_t(r) \lambda_{st}\) such that \(\mathfrak{W}_t = \mathfrak{W}\) on \(\{r \leq a\}\) and on \(\{r \geq c\}\) for some sufficiently large \(c > b\). By Moser’s stability theorem, we find a diffeotopy \(h_t : SM \rightarrow SM\) with \(h_0 = \text{Id}, h_t = \text{Id}\) outside \([a, c] \times M\), and \(h_t^* \mathfrak{W}_t = \mathfrak{W}\). Thus \(h_t^* \mathfrak{W}_t = (\lambda, \phi)\) with the function \(\phi := \psi \circ h_t\) and a Liouville form \(\lambda\) which agrees with \(\lambda_{st}\) outside \([a, c] \times M\) and satisfies \(d\lambda = \omega_{st}\). Hence \((\lambda, \phi) \in \mathcal{E}(M, \xi)\) and \(\phi\) is homotopic (via \(\psi \circ h_t\)) to \(\psi\) in \(\mathcal{E}(M)\), i.e., \([\phi] = [\psi] \in \pi_0 \mathcal{E}(M)\). \(\square\)

By Theorem 6.8 the second vertical map in the diagram (3) is surjective and we obtain

**Corollary 6.9.** Let \((M, \xi)\) be a closed contact manifold of dimension \(2n - 1 \geq 5\). Then every diffeomorphism of \(M\) that is pseudo-isotopic to the identity is smoothly isotopic to a contactomorphism of \((M, \xi)\).

**Remark 6.10.** Considering in the diagram (3) elements in \(\pi_0 \mathcal{P}(M)\) that map to \(\text{Id} \in \pi_0 \text{Diff}_c(M, \xi)\), we obtain the following (non-exclusive) dichotomy for a contact manifold \((M, \xi)\) of dimension \(\geq 5\) for which the map \(\pi_0 \text{Diff}_c(\mathbb{R} \times M) \rightarrow \pi_0 \mathcal{P}(M)\) is nontrivial: Either there exists a contactomorphism of \((M, \xi)\) that is smoothly but
not contactly isotopic to the identity; or there exists a compactly supported symplectomorphism of \((SM, \omega_m)\) which represents a nontrivial smooth pseudo-isotopy class in \(P(M)\). Unfortunately, we cannot decide which of the two cases occurs.

6.4. Equidimensional symplectic embeddings of flexible Weinstein manifolds. Finally, let us mention a recent result concerning equidimensional symplectic embeddings of flexible Weinstein manifolds. Its proof goes beyond the methods discussed in this paper.

**Theorem 6.11** ([13]). Let \((W, \omega, X, \phi)\) be a flexible Weinstein domain with Liouville form \(\lambda\). Let \(\Lambda\) be any other Liouville form on \(W\) such that the symplectic forms \(\omega\) and \(\Omega := d\Lambda\) are homotopic as non-degenerate (not necessarily closed) 2-forms. Then there exists an isotopy \(h_t : W \to W\) such that \(h_0 = \text{Id}\) and \(h_t^* \Lambda = \varepsilon \lambda + dH\) for some small \(\varepsilon > 0\) and some smooth function \(H : W \to \mathbb{R}\). In particular, \(h_1\) defines a symplectic embedding \((W, \varepsilon \omega) \hookrightarrow (W, \Omega)\).

**Corollary 6.12** ([13]). Let \((W, \omega, X, \phi)\) be a flexible Weinstein domain and \((X, \Omega)\) any symplectic manifold of the same dimension. Then any smooth embedding \(f_0 : W \hookrightarrow X\) such that the form \(f_0^* \Omega\) is exact and the differential \(df : TW \to TX\) is homotopic to a symplectic homomorphism is isotopic to a symplectic embedding \(f_1 : (W, \varepsilon \omega) \hookrightarrow (X, \Omega)\) for some small \(\varepsilon > 0\). Moreover, if \(\Omega = d\Lambda\), then the embedding \(f_1\) can be chosen in such a way that the 1-form \(f_1^* \Lambda - i_X \omega\) is exact. If, moreover, the Liouville vector field dual to \(\Lambda\) is complete, then the embedding \(f_1\) exists for arbitrarily large constant \(\varepsilon\).

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