OPENNESS OF UNIFORM K-STABILITY IN THE KÄHLER CONE

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ABSTRACT. We prove continuity results for new stability thresholds related to uniform K-stability and deduce that uniform K-stability is an open condition in the Kähler cone of any compact Kähler manifold, thus establishing an algebro-geometric counterpart to a classical result of LeBrun-Simanca for constant scalar curvature (cscK) metrics. This settles a folklore conjecture in the field, and in particular implies openness of uniform K-stability for smooth polarized varieties. Moreover, it strengthens evidence supporting the uniform version of the Yau-Tian-Donaldson conjecture for arbitrary polarizations, including the case of irrational polarizations and non-projective Kähler manifolds. As a key tool we introduce a new norm on test configurations and establish estimates for non-archimedean energy functionals in terms of this norm. This leads to new characterizations of uniform K-stability by restricting to test configurations that satisfy certain uniform bounds. As a byproduct we obtain continuity results for a stability threshold related to non-archimedean entropy and deduce openness of uniform J-stability, as well as openness of J-stability in the projective case.

1. INTRODUCTION

Let \((X,\omega)\) be a compact Kähler manifold, and \(\mathcal{C}\) the cone of Kähler cohomology classes in \(H^{1,1}(X,\mathbb{R})\). Uniform K-stability is an algebro-geometric condition which was introduced as a refinement of the classical K-stability notion of Tian \([53]\) and Donaldson \([30]\), motivated by examples in \([1]\). It has played a central part in recent advances related to the uniform Yau-Tian-Donaldson conjecture, which predicts that uniform K-stability is equivalent to existence of constant scalar curvature (cscK) metrics in \(\omega \in H^{1,1}(X,\mathbb{R})\). This conjecture, in the spirit of the Hilbert-Mumford criterion, has been confirmed in the case of Fano manifolds and for arbitrary polarizations in the case of toric, spherical, and cohomogeneity one manifolds (see e.g \([14, 54, 22, 16, 25, 8, 41, 34, 23, 24, 39]\) for background and recent progress), but is considered wide open in general.

A fundamental question about uniform K-stability is whether or not it is preserved under various perturbations. In one direction, related to moduli problems, it is known that uniform K-stability is a Zariski open condition in \(\mathbb{Q}\)-Gorenstein families of \(\mathbb{Q}\)-Fano varieties, see \([6, 31, 44, 7]\) and references therein. In the direction concerned with the Yau-Tian-Donaldson conjecture we instead ask whether uniform K-stability is preserved under small perturbations of the underlying polarization: If \((X, L)\) is uniformly K-stable, is there a Euclidean open neighbourhood \(U_L \subset N^1(X)_{\mathbb{R}}\) such that \((X, L')\) is uniformly K-stable for every ample line bundle \(L' \in U_L\)? This parallels a classical result of LeBrun-Simanca \([37]\) which establishes analogous openness results for perturbations of cscK metrics. In particular, if the (uniform) YTD conjecture holds, then (uniform) K-stability must be an open condition in the appropriate sense. More generally, establishing openness results for K-stability (filtration stability, valuative stability etc) is closely related to understanding
which stability notions are the correct ones with respect to the Yau-Tian-Donaldson correspondence, shedding light on discussions such as the one sparked by the examples in [1]. For uniform K-stability, Fujita [33] recently proved a perturbation result on polarized varieties, in case the polarization is anti-canonical. In general it is a fundamental open problem how any of the above mentioned K-stability notions perturb for arbitrary polarizations, that are “far away” from the anti-canonical class.

To discuss openness of uniform K-stability it is natural to consider the extension of this notion to arbitrary Kähler classes (rather than just Q-line bundles). Indeed, while uniform K-stability was initially defined for smooth polarized varieties, it can be made sense of more generally for arbitrary Kähler classes on arbitrary Kähler manifolds, following [28, 48], building on an idea in [3]. We will always write \( \alpha := [\omega] \in H^{1,1}(X, \mathbb{R}) \) for the underlying Kähler class. Extending the usual projective setup we then have a notion of test configuration for \( (X, \alpha) \), and say that \( (X, \alpha) \) is uniformly K-stable if there is a \( \delta > 0 \) such that \( DF(X, \mathcal{A}) \geq \delta(1 - J)^{NA}(X, \mathcal{A}) \) for all test configurations \( (X, \mathcal{A}) \in TC_\alpha \), where the latter denotes the space of all non-trivial relatively Kähler test configurations for \( (X, \alpha) \). The natural connection to the usual setting is that if \( (X, L) \) is a smooth polarized variety and \( (X, L) \) is a test configuration for \( (X, L) \), then the property of being uniformly K-stable clearly depends only on \( c_1(L) \), and this is then emphasized by the observation that the compactification \( (X, c_1(L)) \) over \( \mathbb{P}^1 \), adding a trivial fiber at infinity, is a test configuration for \( (X, c_1(L)) \) in the generalized sense. For the problem at hand it is therefore possible to consider uniform K-stability as a condition on the underlying Kähler classes.

A main difficulty in obtaining general openness results for uniform K-stability in the Kähler cone is that if \( \alpha, \alpha' \in C \) are two nearby Kähler classes, then there is in general no sufficient understanding of the relationship between the spaces \( TC_\alpha \) and \( TC_{\alpha'} \) of test configurations for \( (X, \alpha) \) and \( (X, \alpha') \) respectively. In addition, care has to be taken as to how both the norm and the Donaldson-Futaki invariant vary as we pass from \( TC_\alpha \) to \( TC_{\alpha'} \), which is difficult to achieve with the required uniform control.

In this paper we show how to circumvent the above difficulties, establishing the following algebraic analogue of LeBrun-Simanca [37]:

**Theorem 1.1.** The set

\[
UKs := \{ \alpha \in C : (X, \alpha) \text{ is uniformly K-stable} \}
\]

is an open subcone of \( C \).

Note that this result is new even in the case of smooth polarized varieties \( (X, L) \). It then specializes to the following statement for ample line bundles:

**Theorem 1.2.** Let \( (X, L) \) be a uniformly K-stable smooth polarized variety. Then there is a Euclidean open neighbourhood \( U_L \subset N^1(X, \mathbb{R}) \) such that \( (X, L') \) is uniformly K-stable for any ample line bundle \( L' \in U_L \).

As previously mentioned, the above further supports the expectation that uniform K-stability is a good refinement of the original K-stability notion, in the sense of being the appropriate stability notion such that a Yau-Tian-Donaldson type correspondence holds for general polarizations. Such a correspondence can however only hold as stated if \( \text{Aut}(X) \) contains no \( \mathbb{C}^* \)-subgroup (or in the polarized case, if the reduced automorphism group \( \text{Aut}(X, L)/\mathbb{C}^* \) is finite). This reflects ideas in the field saying that one should then instead consider a notion similar to that of K-polystability, taking into account the non-trivial product test configurations (see e.g. [34] for a discussion on such notions), and one...
would then expect to prove openness in the Futaki vanishing locus $C_F \subseteq C$, rather than in the full Kähler cone. The uniform Yau-Tian-Donaldson conjecture has moreover been recently studied on singular polarized varieties in [40]. Obtaining perturbation results in these settings are natural followup questions which are left for future work.

In what follows we discuss elements of the proof, and further results obtained along the way. The reader interested in the case of smooth polarized varieties may always replace statements about test configurations $(X, \alpha)$ for $(X, \alpha)$ by the corresponding statement involving test configurations $(X, \mathcal{L})$ for $(X, \mathcal{L})$, in the usual sense. We further underline that while several elements of the proof of Theorem 1.1 are analytic in nature, our arguments make use only of the geometry of the space of Kähler potentials (and energy functionals on this space), combined with a study of new stability thresholds that we introduce. In particular, it does not use the inverse function theorem, and it is interesting to ask which parts of this paper may be adapted to purely algebro-geometric arguments.

**Elements of proof and further results.** Fix $\alpha \in C$ and let $\alpha' \in C$ be a nearby Kähler class. In order to understand how uniform K-stability varies across Kähler classes one needs to understand how to associate to each $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha$ a test configuration $(\mathcal{X}, \mathcal{A}') \in \mathcal{T}C_{\alpha'}$ in such a way that we can find a relationship between their respective Donaldson-Futaki invariants that takes into account in a suitable way the norm of the test configurations. More precisely, the main difficulty in proving openness results for uniform K-stability lies in controlling the change in Donaldson-Futaki invariant divided by a suitable norm, such that the control is uniform over all test configurations that need to be considered. In particular, even if one assumes that $(\mathcal{X}, \mathcal{A})$ is close to realizing the optimal threshold

$$\Delta(\alpha) := \sup\{\delta \in \mathbb{R} : \text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta(I - J)^{NA}(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha\},$$

it is not clear how to associate a test configuration $(\mathcal{X}, \mathcal{A}')$ which is close to optimal for $(X, \alpha')$. In order to associate a new test configuration to the given one $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha$, we assume that it is smooth and dominating, i.e. $\mathcal{X}$ is smooth and dominates $X \times \mathbb{P}^1$ via a bimeromorphic morphism $\mu$, such that $\mathcal{A} = \mu^*\mathcal{A} + [D]$ for some $\mathbb{R}$-divisor $D$ on $X'$ (see Definition 2.2 and Proposition 2.4). If $\alpha' \geq \alpha$ (i.e. $\alpha' - \alpha$ is nef), we may then associate to $(\mathcal{X}, \mathcal{A})$ the test configuration $(\mathcal{X}, \mathcal{A}' := \mu^*\mathcal{A}' + [D])$, which in this setup remains relatively Kähler. A key observation is that keeping the divisor $D$ unchanged corresponds to saying that $(\mathcal{X}, \mathcal{A}')$ has the same associated geodesic ray for each $\alpha' \in C$ with $\alpha' > \alpha$, even though some care needs to be taken in making sense of this statement (see Section 2.4 Remark 2.10 and Lemma 3.1).

Along these lines, we may now deal with perturbation of the first term in the decomposition

$$M^{NA}(\mathcal{X}, \mathcal{A}) := J^{NA}_{-\mathcal{A}_{\mathcal{X} negative}}(\mathcal{X}, \mathcal{A}) + H^{NA}_{\mathcal{A}_{\mathcal{X} positive}}(\mathcal{X}, \mathcal{A}),$$

of the non-archimedean K-energy, which is closely related to the Donaldson-Futaki invariant (in fact we can assume that they are equal, up to restricting to smooth and dominating test configuration whose central fiber is reduced, see Proposition 2.8). Indeed, noting that we may without loss of generality work over the space $\mathcal{T}C' \subseteq \mathcal{T}C_\alpha$ of smooth and dominating relatively Kähler test configurations for $(X, \alpha)$, we have the following result:

**Theorem 1.3.** Suppose that $\alpha \in C$. Fix $\delta > 0$. Then there exists an open set $U_\delta \subset C$ such that the open subcone $\mathbb{R}_+U_\delta \subset C$ contains $\alpha$, and moreover, for every $\alpha' \in U_\delta$, and
every test configuration $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_{\alpha}^*$, there exists a test configuration $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_{\alpha}$ satisfying the inequalities

$$|\mathcal{J}_{c^1(\mathcal{X})}^N(\mathcal{X}, \mathcal{A}) - \mathcal{J}_{c^1(\mathcal{X})}^N(\mathcal{X}, \mathcal{A}')| \leq \delta J_{\alpha}^N(\mathcal{X}, \mathcal{A})$$

and

$$|\mathcal{J}_{\alpha}^N(\mathcal{X}, \mathcal{A}) - \mathcal{J}_{\alpha}^N(\mathcal{X}, \mathcal{A}')| \leq \delta J_{\alpha}^N(\mathcal{X}, \mathcal{A}).$$

The proof relies on obtaining estimates for the corresponding functionals on the space of all Kähler metrics using techniques for the $d_1$-distance of Darvas [17] [18], and then passing to the above non-archimedean functionals using results on energy functional asymptotics along (sub)geodesic rays. Note that since the underlying Kähler class is not fixed, we need to consider a suitable space of all Kähler potentials, which is a straightforward adaptation of the usual pluripotential theory of energy functionals on the space of Kähler metrics.

This part of the argument is outlined in the first few paragraphs of Section 3, to which we refer for further details.

We now focus on the main difficulty in proving openness results for uniform K-stability, which is to control the change in non-archimedean entropy $H_{\alpha}^N$ in a similar way, such that the control is uniform over all test configurations that need to be considered. A central idea of the paper is that the above problems can be dealt with by allowing extra flexibility in the choice of norm. The point of view taken is to study the variation of the central idea of the paper is that the above problems can be dealt with by allowing extra flexibility in the choice of norm. The point of view taken is to study the variation of the non-archimedean (I−J)-functional by $\mathcal{J}_{\alpha}^N$, emphasizing that it can be seen as a special case of the non-archimedean analogue of Donaldson’s J-functional, as pointed out in [25].

We may also note that the first term in (1) is precisely the minimum norm of Dervan [25].

The above new norm satisfies $||'(\mathcal{X}, \mathcal{A})||_{\alpha} \geq 0$ with equality precisely for the trivial test configuration, justifying the terminology, and it clearly dominates the minimum norm, i.e. $\mathcal{J}_{\alpha}^N(\mathcal{X}, \mathcal{A}) < ||'(\mathcal{X}, \mathcal{A})||_{\alpha}$. On the other hand, it is not a priori clear whether the resulting uniform K-stability notion using the new norm is equivalent to the usual notions in the literature.

Using this norm $||.||_{\alpha}$ we now discuss how to control the change of non-archimedean functionals along the following family of test configurations $(\mathcal{X}, \mathcal{A}_t := \mu p^*_t10 + [D])$ for $(\mathcal{X}, t\alpha)$, which are relatively Kähler for $t \geq 1$. In order to underline certain fundamental properties of $||.||_{\alpha}$, we introduce some terminology: The variation of a test configuration with respect to the norm $||.||$ is the following quantity

$$v_{\alpha,||}(\mathcal{X}, \mathcal{A}) := \sup_{t \in [1,2]} \left\{ \frac{d}{dt} H_{\alpha}^N(\mathcal{X}, \mathcal{A}_t)||'(\mathcal{X}, \mathcal{A}_t)||_{\alpha}^{-1} \right\},$$

differentiating a rational function whose denominator never vanishes. Similarly, we also introduce the height of a test configuration with respect to the norm $||.||$ as the quantity

$$h_{\alpha,||}(\mathcal{X}, \mathcal{A}) := \sup_{t \in [1,2]} \left\{ \frac{d}{dt} ||'(\mathcal{X}, \mathcal{A}_t)||_{\alpha}||'(\mathcal{X}, \mathcal{A}_t)||_{\alpha}^{-1} \right\},$$

differentiating a polynomial.
We also make use of the normalized entropy of the test configuration \((X, A) \in \mathcal{T}C_\alpha\), given by
\[
\text{Ent}_{\alpha, J_{\alpha}^{NA}}(X, A) := H_{\alpha}^{NA}(X, A) J_{\alpha}^{NA}(X, A)^{-1}
\]
As a main ingredient of the proof of Theorem 1.1 we then show that, with respect to the new norm, it is possible to restrict attention to test configurations that satisfy certain uniform bounds on the above height, variation and normalized entropy. We also show that using the new norm or any of the usual norms is equivalent:

**Theorem 1.4.** Let \(K \subset C\) be a compact subset of the Kähler cone. Then there is a constant \(B := B(K, n) > 0\), depending only on \(K\) and the dimension of \(X\), such that the following are equivalent for any \(\alpha \in K\):

1. \((X, \alpha)\) is uniformly K-stable.
2. There is a \(\delta_\alpha > 0\) such that
   \[
   \text{DF}(X, A) \geq \delta_\alpha ||(X, A)||_\alpha
   \]
   for all relatively Kähler test configurations \((X, A) \in \mathcal{T}C_\alpha\).
3. There is a \(\delta_\alpha > 0\) such that
   \[
   \text{DF}(X, A) \geq \delta_\alpha ||(X, A)||_\alpha
   \]
   for all smooth and dominating relatively Kähler test configurations \((X, A)\) for \((X, \alpha)\) satisfying the following uniform variation, height and normalized entropy bounds \(v_\alpha(X, A) \leq B\), \(h_\alpha(X, A) \leq B\) and \(\text{Ent}_{\alpha, J_{\alpha}^{NA}}(X, A) \leq B\).

This gives a new characterization of what it means to be uniformly K-stable, which may potentially be of independent interest.

**Remark 1.5.** The point is that the test configurations in (3) are precisely those for which the variation of non-archimedean entropy can be uniformly controlled (see Theorem 1.7 below).

The idea of the proof of Theorem 1.4 is to show that any test configuration which is “nearly optimal” in the sense that \(\text{DF}(X, A) \leq (\Delta(\alpha) + c)|| (X, A)||_\alpha\) for some absolute constant \(c > 0\), must satisfy uniform bounds on height, variation and normalized entropy on compact subset of the Kähler cone. In order to prove this we consider auxiliary twisted stability thresholds depending on a parameter \(\beta \in H^{1,1}(X, \mathbb{R})\) (keeping \(\alpha \in C\) fixed), given by
\[
\hat{\Delta}_{\beta}(\alpha) := \sup\{\delta \in \mathbb{R} : M_{\beta}^{NA}(X, A) \geq \delta ||(X, A)||_\alpha, \forall (X', A) \in \mathcal{T}C_\alpha\}
\]
and
\[
\hat{\Delta}_{\beta}^{B}(\alpha) := \sup\{\delta \in \mathbb{R} : M_{\beta}^{NA}(X, A) \geq \delta ||(X, A)||_\alpha, \forall (X, A) \in \mathcal{T}C_\alpha(B)\},
\]
where \(\mathcal{T}C_\alpha^{(B)} \subset \mathcal{T}C_\alpha\) denotes the subset of test configurations with height, variation and normalized entropy bounded above by \(B > 0\), and \(M_{\beta}^{NA}\) is a twisted version of the non-archimedean K-energy (see Section 2). We then show that
\[
\hat{\Delta}_{\beta}^{B}(\alpha) = \hat{\Delta}_{\beta}(\alpha)
\]
for all \(B := B(K, \beta, n) > 0\) large enough (in particular this holds in our main case of interest \(\beta = -c_1(X)\) corresponding to untwisted stability, and the choice of \(B\) is uniform over all \(\alpha \in K\)). We finally show that their sign equals that of \(\Delta(\alpha)\), even if the threshold functions are a priori different in general. We refer to Section 3 Steps 3-5 for the details of the proof.
To put the above approach into context, let us more generally consider uniform K-stability with respect to a subspace $S_\alpha \subset TC_\alpha$ of test configurations and with respect to a given norm functional $||| \cdot |||_{\alpha}$, and say that $(X, \alpha)$ is $(S_\alpha, ||| \cdot |||_{\alpha})$-uniformly K-stable if there exists $\delta > 0$ such that $DF(X, A) \geq \delta |||(X, A)|||_{\alpha}$ for all test configurations $(X, A) \in S_\alpha$. It is clear that not all choices of pairs $(S_\alpha, ||| \cdot |||_{\alpha})$ yield equivalent uniform stability notions. As we will see in this paper there are sometimes advantages in allowing for different (not necessarily equivalent) norms, and the idea of restricting to smaller sets of test configurations goes back at least to the well-known special test configurations (see [41]). This notion, or the recent notion of uniform integral K-stability (which was shown to be equivalent to valuative stability in [27]) are both examples of notions of interest that fall under this general approach, for the appropriate choice of the set $S_\alpha$. In the spirit of the above discussion, the statement of Theorem 1.4 can be understood as saying that with $X, \alpha$ is positive if and only if $(X, \alpha)$ is uniformly K-stable. For the purpose of discussion, we may say that two stability thresholds are equivalent, and write

$$\Delta^{(S_\alpha, ||| \cdot |||_{\alpha})} := \sup\{ \delta \in \mathbb{R} : DF(X, A) \geq \delta |||(X, A)|||_{\alpha}, \forall (X, A) \in S_\alpha\}$$

is positive if and only if $(X, \alpha)$ is uniformly K-stable. For the purpose of discussion, we may say that two stability thresholds are equivalent, and write

$$\Delta^{(S_1, ||| \cdot |||_1)} \sim \Delta^{(S_2, ||| \cdot |||_2)},$$

precisely if they always have the same sign. One of the main goals of this paper is to prove that there is a suitable choice of $S_\alpha$ and $||| \cdot |||_{\alpha}$ such that the resulting stability threshold function is equivalent to $\Delta$, and moreover continuous on the Kähler cone:

**Theorem 1.6.** There is a pair $(S_\alpha, ||| \cdot |||_{\alpha})$ with $S_\alpha \subset TC_\alpha$ and $||| \cdot |||_{\alpha} : TC_\alpha \to \mathbb{R}$, such that the following holds:

1. $\Delta^{(S_\alpha, ||| \cdot |||_{\alpha})} \sim \Delta$
2. $C \ni \alpha \mapsto \Delta^{(S_\alpha, ||| \cdot |||_{\alpha})}$ is continuous.

As a first step towards this result, the idea behind the the choice of norm [41] and the uniformly B-bounded test configurations in Theorem 1.4, is that this setup leads to the following main technical result, which can be interpreted as a generalized version of [33] Theorem 1.2 and Proposition 1.3 to the setting of arbitrary polarizations:

**Theorem 1.7.** Suppose that $\alpha \in C$. Fix $\delta > 0$. Then there exists an open set $U_\delta \subset C$ and a constant $B > 0$, such that the open subcone $R_\delta U_\delta \subset C$ contains $\alpha$, and moreover, for every $\alpha' \in U_\delta$, and every test configuration $(X, A')$ for $(X, \alpha')$ of uniformly B-bounded height, variation and normalized entropy, there exists a test configuration $(X, A) \in TC_\alpha$ satisfying the inequalities

$$DF_\alpha(X, A) - DF_\alpha'(X, A') \leq \delta |||(X, A)|||_{\alpha}$$

and

$$H_{\alpha}'^{NA}(X, A) - H_{\alpha}'^{NA}(X, A') \leq \delta |||(X, A)|||_{\alpha}.$$

**Remark 1.8.** Analogous results hold also for $J_\alpha^{NA}$ and $J_\alpha^{NA}$ as a direct consequence of Theorem 1.3, since $J_\alpha^{NA} \leq nJ_\alpha^{NA} \leq n||| \cdot |||_{\alpha}$.

To explain the starting point of the proof, we fix as before the Kähler class $\alpha \in C$ and consider the open neighbourhood defined by $(1 - \epsilon)\alpha < \alpha' < (1 + \epsilon)\alpha$. Assuming that $(X, A')$ is a test configuration with height, variation and normalized entropy bounded above by a given constant $B > 0$, we show in Section 3 Step 3, that there is a test
configuration \((\mathcal{X}, \mathcal{A})\) for \((X, \alpha)\), and a continuous function \(f_B : \mathbb{R}_+ \to \mathbb{R}_+\) with \(f_B(0) = 0\) such that

\[
H^\mathcal{NA}_\alpha(\mathcal{X}, \mathcal{A}) \leq H^\mathcal{NA}_{\alpha'}(\mathcal{X}, \mathcal{A}') + f_B(\epsilon).
\]

Combined with Theorem 1.3 an analogous estimate can be obtained also for the Donaldson Futaki invariant. Even though these inequalities hold a priori only for the \(B\)-bounded test configurations considered here, we then show (see Proposition 3.2 and Theorem 3.30 for precise formulations) that this is enough to conclude the openness result in Theorem 1.1, by shrinking \(\epsilon > 0\) as necessary.

In fact, building on these considerations we can moreover prove the following continuity result for the associated stability thresholds:

**Theorem 1.9.** Let \(F^\mathcal{NA}\) be any of the non-archimedean functionals \(\mathcal{M}^\mathcal{NA}_\beta\), \(\mathcal{J}^\mathcal{NA}_\beta\), \(\mathcal{J}^\mathcal{NA}\) and \(H^\mathcal{NA}\). Then the stability thresholds

\[
\alpha \mapsto \hat{\Delta}(\alpha) := \sup\{\delta \in \mathbb{R} : DF(\mathcal{X}, \mathcal{A}) \geq \delta \| (\mathcal{X}, \mathcal{A}) \|_\alpha \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha\}
\]

and

\[
\alpha \mapsto \hat{\Delta}^F(\alpha) := \sup\{\delta \in \mathbb{R} : F^\mathcal{NA}(\mathcal{X}, \mathcal{A}) \geq \delta \| (\mathcal{X}, \mathcal{A}) \|_\alpha \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha\}
\]

are continuous as functions on the Kähler cone.

**Remark 1.10.** Clearly Theorem 1.9 implies the main Theorem 1.1. For \(F^\mathcal{NA} = H^\mathcal{NA}\) this can be seen as an algebraic analogue of the recent continuity result of [56] for the analytic delta invariant. If \(F = \mathcal{J}^\mathcal{NA}_\beta\) then this proves continuity of the algebraic threshold corresponding to uniform \(J\)-stability, see Theorem 1.12 below.

As a further noteworthy, but more straightforward, consequence of the definition of the norm \(\| . \|_\alpha\) (and the proof of Theorem 1.9) we have the following uniform bounds on the quotient \(F^\mathcal{NA}/\| . \|_\alpha\) in terms of the stability thresholds associated to any one of the above functionals:

**Theorem 1.11.** Let \(K \subset \mathcal{C}\) be a compact set. Let \(F^\mathcal{NA}\) be any of the non-archimedean functionals \(\mathcal{M}^\mathcal{NA}_\beta\), \(\mathcal{J}^\mathcal{NA}_\beta\), \(\mathcal{J}^\mathcal{NA}\) and \(H^\mathcal{NA}\). Then there exists \(\lambda := \lambda(K) > 0\) such that the following double inequality

\[
\hat{\Delta}^F(\alpha) \leq \frac{F^\mathcal{NA}(\mathcal{X}, \mathcal{A})}{\| (\mathcal{X}, \mathcal{A}) \|_\alpha} \leq \hat{\Delta}^F(\alpha) + \lambda
\]

holds for all \((\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha\) and all \(\alpha \in K\). In other words

\[
\left| \hat{\Delta}^F(\alpha) - \frac{F^\mathcal{NA}(\mathcal{X}, \mathcal{A})}{\| (\mathcal{X}, \mathcal{A}) \|_\alpha} \right| \leq \lambda
\]

uniformly over all \((\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha\) and all \(\alpha \in K\).

The lower bounds here follow by definition, whereas the upper bounds show that the value of the quotient \(F^\mathcal{NA}/\| . \|_\alpha\) must stay close to its minimal value \(\hat{\Delta}^F\), which to our knowledge contrasts what is known for any of the usual norms in the literature.

Finally, it is interesting to note that along the lines of Theorem 1.3 we obtain a proof of openness of uniform \(J\)-stability, which is the stability notion related to the \(J\)-equation introduced by Donaldson [29] (see also [11]), and which has been studied by numerous authors, including [51, 38, 16, 10].
Theorem 1.12. For every $\beta \in H^{1,1}(X, \mathbb{R})$, the set
\[ \text{UJS}_\beta := \{ \alpha \in C : (X, \beta, \alpha) \text{ is uniformly J -- stable} \} \]
is an open subcone of $C$.

This complements the explicit formula obtained in \[50\], and extends the case of surfaces, when openness follows from the numerical existence criterion of \[11\]. By a recent result of Datar and Pingali \[21\] uniform J-stability is moreover equivalent to J-stability on projective manifolds, so in that case also J-stability is an open condition in the Kähler cone.

Organization of the paper. The paper is structured as follows: In Section 2 we establish the necessary preliminaries on test configuration. In Section 2.4 we review classical and non-archimedean functionals and their interplay, in particular asymptotics along subgeodesic rays in the space of Kähler potentials. In Section 3 we begin by explaining the starting point of the proof, and defining the candidate open set (together with a brief outline of the main objectives of the proof). The proof of Theorem 1.1 is then presented in six steps: Step 1 is about establishing analytic estimates for the I and J functionals, and transferring these estimates to their non-archimedean counterparts. In Step 2 we do the same with the entropy and non-archimedean entropy. As an important step of the proof we also introduce a new norm $||.|\|_\alpha$ on test configurations along with a notion of \emph{height}, \emph{variation} and \emph{normalized entropy} of test configurations. In Step 3 we introduce a special subset $\mathcal{T}_{\beta, o}$ of nearly optimal test configuration and prove that they have bounded normalized entropy in this setting. In Steps 4-5 we prove that any nearly optimal test configuration moreover satisfies uniform bounds on height and variation. This in particular prove Theorem 1.4. In Step 6 we combine Steps 3-5 to finish the proof of openness of uniform K-stability, by proving Theorem 1.7 and deducing Theorem 1.1. In Section 4 we finally discuss some applications and point out some immediate corollaries of our techniques, and strengthen the openness results to continuity of the corresponding stability thresholds, thus proving Theorem 1.9.

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2. Uniform K-stability, norms, and stability thresholds

We here introduce our notation and recall a number of results from the literature that we will use. The main references for this section are \[48, 28, 50, 8, 9\], even though the results go back much longer in time, to \[53, 30\], and the work of many others.

2.1. Test configurations. The definition of uniform K-stability involves the notion of test configuration. In the classical situation one considers the case of $(X, L)$ a polarized manifold, i.e. $X$ is a compact Kähler manifold and $L \to X$ is an ample line bundle on $X$. In the sense of \[30\] and followup work, a test configuration for $(X, L)$ is then a normal polarized manifold $(\mathcal{X}, \mathcal{L})$ with 1) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting to $L$, and 2) a flat $\mathbb{C}^*$-equivariant projective morphism $\pi : \mathcal{X} \to \mathbb{C}$ where $\mathbb{C}$ is given the standard $\mathbb{C}^*$-action, such that the fiber $\pi^{-1}(t)$ is isomorphic to $(X, L^r)$ for all $t \in \mathbb{C}^*$ and some $r \in \mathbb{N}$. If $(\mathcal{X}, \mathcal{L})$ is a test configuration for $(X, L)$, it is convenient to refer to the total space $\mathcal{X}$.
simply as a test configuration for $X$. We say that $(\mathcal{X}, \mathcal{L})$ is relatively ample (respectively relatively nef) if $\mathcal{L}$ is. Any test configuration $\mathcal{X}$ for $X$ can moreover be compactified in a canonical way by gluing together $\mathcal{X}$ and $X \times \mathbb{P}^1 \setminus \{0\}$ along the open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{C} \setminus \{0\})$, using the $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{C} \setminus \{0\})$. The compactification of a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is defined similarly, where $\mathcal{L}$ is a $\mathbb{C}^*$-linearized ($\mathbb{Q}$)-line bundle on $\mathcal{X}$. The polarization $\mathcal{L}$ is relatively (semi)ample iff $\mathcal{L}$ is.

In order to discuss openness it is natural to use the generalized notion introduced in [HS, 25], based on an idea in [3], which contains the above classical notion of test configuration as a special case. To recall this more general notion, suppose that $(X, \omega)$ is any compact Kähler manifold, not necessarily projective, and write $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ for the associated cohomology class.

We first introduce the notion of test configuration $\mathcal{X}$ for $X$, working directly over $\mathbb{P}^1$. In what follows, we refer to [17] for further background and discussion.

**Definition 2.1.** A test configuration $\mathcal{X}$ for $X$ consists of

- a normal compact Kähler complex space $\mathcal{X}$ with a flat (i.e. surjective) morphism $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$
- a $\mathbb{C}^*$-action $\lambda$ on $\mathcal{X}$ lifting the canonical action on $\mathbb{P}^1$
- a $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\})$.  \(2\)

The isomorphism (2) gives an open embedding of $X \times (\mathbb{P}^1 \setminus \{0\})$ into $\mathcal{X}$, hence induces a canonical $\mathbb{C}^*$-equivariant bimeromorphic map $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$. We say that $\mathcal{X}$ dominates $X \times \mathbb{P}^1$ if the above bimeromorphic map $\mu$ is a morphism. Taking $\mathcal{X}'$ to be the normalisation of the graph of $\mathcal{X} \rightarrow X \times \mathbb{P}^1$ we obtain a $\mathbb{C}^*$-equivariant bimeromorphic morphism $\rho : \mathcal{X}' \rightarrow \mathcal{X}$ with $\mathcal{X}'$ normal and dominating $X \times \mathbb{P}^1$. By the above considerations we will often, up to replacing $\mathcal{X}$ by $\mathcal{X}'$, be able to assume that the given test configuration for $X$ dominates $X \times \mathbb{P}^1$. Moreover, any test configuration $\mathcal{X}$ for $X$ can be dominated by a smooth test configuration $\mathcal{X}'$ for $X$. Indeed, by Hironaka (see [36, Theorem 45] for the precise statement concerning normal complex spaces) there is a $\mathbb{C}^*$-equivariant proper bimeromorphic map $\rho : \mathcal{X}' \rightarrow \mathcal{X}$, with $\mathcal{X}'$ smooth, such that $\mathcal{X}_0$ has simple normal crossings and $\rho$ is an isomorphism outside of the central fiber $\mathcal{X}_0$.

**Definition 2.2.** A test configuration for $(X, \alpha)$ is a pair $((\mathcal{X}, \mathcal{A}_\alpha))$ where $\mathcal{X}$ is a test configuration for $X$ and $\mathcal{A}_\alpha \in H^{1,1}_{BC}(\mathcal{X}, \mathbb{R})^{\mathbb{C}^*}$ is a $\mathbb{C}^*$-invariant $(1, 1)$–Bott-Chern cohomology class whose image under the $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\})$ is $p_1^* \alpha$, where $p_1 : X \times \mathbb{P}^1 \rightarrow X$ is the first projection.

The trivial test configuration for $(X, \alpha)$ is given by $(\mathcal{X} := X \times \mathbb{P}^1, p_1^* \alpha, \lambda_{\text{triv}}, p_2)$, where $p_1 : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $p_2 : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are the projections on the 1st and 2nd factor respectively, and $\lambda_{\text{triv}} : \mathbb{C}^* \times X \rightarrow X, (\tau, (x, z)) \mapsto (x, \tau z)$ is the $\mathbb{C}^*$-action that acts trivially on the first factor.

**Definition 2.3.** A test configuration $((\mathcal{X}, \mathcal{A}_\alpha))$ for $(X, \alpha)$ is said to be smooth if the total space $\mathcal{X}$ is smooth. It is said to be dominating $X \times \mathbb{P}^1$ if the canonical isomorphism $\mu : \mathcal{X} \setminus \mathcal{X}_0 \rightarrow X \times \mathbb{P}^1 \setminus \{0\}$ extends to a morphism $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$. It is said to be semistable if it is smooth and the central fiber $\mathcal{X}_0$ is a reduced divisor with simple normal
crossings. Finally, a test configuration \((\mathcal{X}, \mathcal{A})\) is said to be relatively Kähler if \(\mathcal{A}\) is a relatively Kähler class, i.e. if there is a Kähler class \(\eta \in H^{1,1}(\mathbb{P}^1)\) such that \(\mathcal{A} + \pi^* \eta\) is Kähler on \(\mathcal{X}\).

**Notation 1.** Write \(\mathcal{T}C_\alpha\) for the space of non-trivial relatively Kähler test configurations for \((X, \alpha)\). We denote by \(\mathcal{T}C_\alpha^* \subset \mathcal{T}C_\alpha\) the subset of (non-trivial) relatively Kähler smooth and dominating test configurations.

We refer the reader to \([48, 28, 47]\) for background and details. All test configurations considered in the proof of our main results are of this smooth and dominating type, and this assumption can be made without loss of generality (see Proposition 2.8 below).

For comparison with the general literature we note that if \((\mathcal{X}, \mathcal{L})\) is a test configuration for a smooth polarized variety \((X, L)\), then \((\mathcal{X}, c_1(\mathcal{L}))\) is a test configuration for \((X, c_1(L))\), where \((\mathcal{X}, \mathcal{L})\) is the compactification over \(\mathbb{P}^1\). We moreover recall from the algebraic setting that if \((\mathcal{X}, \mathcal{L})\) dominates \((X, L) \times \mathbb{C}\), then \(\mathcal{L} = \mu^* p_1^* L + D\) for a unique \(\mathbb{Q}\)-Cartier divisor \(D\) supported on \(\mathcal{X}_0\), see \([8]\), and this divisor plays a key role with respect to the filtration stability notion, see \([8\text{, Lemma 5.17}]\). We recall that the analogous result holds on any compact Kähler manifold:

**Proposition 2.4.** (\([8\text{, Proposition 3.10}]\)) Let \((\mathcal{X}, \mathcal{A})\) be a smooth cohomological test configuration for \((X, \alpha)\) dominating \(X \times \mathbb{P}^1\), with \(\mu: \mathcal{X} \rightarrow X \times \mathbb{P}^1\) the corresponding canonical \(\mathbb{C}^*\)-equivariant bimeromorphic morphism. Then there exists a unique \(\mathbb{R}\)-divisor \(D\) supported on the central fiber \(\mathcal{X}_0\) such that

\[
\mathcal{A} = \mu^* p_1^* \alpha + [D]
\]

in \(H^{1,1}(\mathcal{X}, \mathbb{R})\).

### 2.2. Non-Archimedean functionals and norms on \(\mathcal{T}C_\alpha\). We further recall our notation for the Donaldson-Futaki invariant and other non-archimedean functionals. These invariants are cohomological quantities depending on \(\alpha \in \mathcal{C}\), and a second given cohomology class \(\beta \in H^{1,1}(X, \mathbb{R})\). They are defined as intersection numbers on the (smooth and dominating) total space \(\mathcal{X}\), which we for convenience compile in the following list:

\[
\text{DF}(\mathcal{X}, \mathcal{A}) := \frac{\bar{S}_\alpha}{(n+1)V_\alpha}(\mathcal{A}^{n+1})_\mathcal{X} + \frac{1}{V_\alpha} (K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}_\alpha^0)_\mathcal{X},
\]

\[
\mathcal{J}_{\beta}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \frac{\mu_{\beta, \alpha}}{(n+1)V_\alpha}(\mathcal{A}^{n+1})_\mathcal{X} + \frac{1}{V_\alpha} (\mu^* p_1^* \beta \cdot \mathcal{A}_\alpha^n)_\mathcal{X},
\]

\[
\mathcal{H}_{\alpha}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \frac{1}{V_\alpha} (K_{\mathcal{X}/\mathbb{P}^1}^{\log} \cdot \mathcal{A}_\alpha^n) - \frac{1}{V_\alpha} (\mu^* K_{X \times \mathbb{P}^1/X}^{\log} \cdot \mathcal{A}_\alpha^n),
\]

\[
\mathcal{M}_{\beta}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \mathcal{J}_{\beta}^{\text{NA}}(\mathcal{X}, \mathcal{A}) + \mathcal{H}_{\alpha}^{\text{NA}}(\mathcal{X}, \mathcal{A}),
\]

\[
\mathcal{I}_{\alpha}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \frac{1}{V_\alpha}(\mathcal{A}_\alpha \cdot \mu^* p_1^* \alpha^n)_\mathcal{X} - \frac{1}{(n+1)V_\alpha}(\mathcal{A}^{n+1})_\mathcal{X},
\]

where

\[
K_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1}, \quad K_{\mathcal{X}/\mathbb{P}^1}^{\log} = K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1} + \mathcal{X}_{0,\text{red}} - \mathcal{X}_0,
\]
and the cohomological constants $V_\alpha$, $\mu_{\beta,\alpha}$ and $\bar{S}_\alpha$ are defined by

$$V_\alpha := (\alpha^n)_X, \quad \mu_{\beta,\alpha} := -n\frac{\beta \cdot \alpha^{n-1}}{(\alpha^n)_X}, \quad \bar{S}_\alpha := \mu_{-c_1(X),\alpha} = n\frac{(c_1(X) \cdot \alpha^{n-1})_X}{(\alpha^n)_X}.$$  

Note that $\bar{S}_\alpha$ is the average scalar curvature of $(X, \omega)$, and $V_\alpha$ is the Kähler volume.

**Remark 2.5.** By [55] and [33] the above $DF$-invariant generalizes the classical one. Indeed, if $(\mathcal{X}, \mathcal{L})$ is a test configuration for $(X, L)$, then $DF(\mathcal{X}, c_1(\mathcal{L})) = DF(\mathcal{X}, \mathcal{L})$.

Further recall that if we take $\beta = -c_1(X)$ above, then

$$M_{-c_1(X)}^{NA}(\mathcal{X}, \mathcal{A}_\alpha) = DF(\mathcal{X}, \mathcal{A}_\alpha) + (\mathcal{X}_0, \text{red}) \cdot \mathcal{A}_\alpha,$$

so the Donaldson-Futaki invariant $DF$ is closely related to the non-archimedean K-energy functional $M_{-c_1(X)}^{NA}$, and we have equality whenever the central fiber $X_0$ is reduced. We also consider norms on test configurations:

**Definition 2.6.** A functional $||.|.| : \mathcal{T}_\alpha \to \mathbb{R}$ is called a norm if $||(|(\mathcal{X}, \mathcal{A})|| \geq 0$ for all relatively Kähler test configurations for $(X, \alpha)$, with equality precisely if $(\mathcal{X}, \mathcal{A})$ is the trivial test configuration.

Examples of norm functionals commonly seen in the literature include $J_{\alpha}^{NA}$ introduced above, as well as

$$J_{\alpha}^{NA} = \frac{\mu_{\alpha,\alpha}}{(n+1)V_\alpha}(\mathcal{A}_\alpha^{n+1})_X + \frac{1}{V_\alpha}(\mu^* p^*_\alpha \cdot \mathcal{A}_\alpha^n)_X$$

which coincides with the non-archimedean $(I - J)$-functional (thus also the minimum norm of Dervan [25]), as follows from a simple computation, see [50]. These norms are in fact equivalent, as a consequence of the following double inequality, which is a non-archimedean analogue of a well-known inequality involving the energy functionals $J_\alpha$ and $J_\alpha$ (see Section 2.21 for the notation):

$$\frac{1}{n} J_{\alpha}^{NA} \leq J_{\alpha}^{NA} \leq n J_{\alpha}^{NA}.$$  

We are finally ready to define uniform K-stability:

**Definition 2.7.** ([28, 47]) We say that $(X, \alpha)$ is uniformly K-stable if and only if there is a $\delta > 0$ such that

$$DF(\mathcal{X}, \mathcal{A}_\alpha) \geq \delta J_{\alpha}^{NA}(\mathcal{X}, \mathcal{A}_\alpha)$$

for all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A}_\alpha)$ for $(X, \alpha)$.

The following result says that it suffices to restrict to smooth and dominating test configurations with reduced central fiber, and appeared in various contexts in [47 Proposition 3.2.20, Proposition 4.3.6], [18 Proposition 3.12], [28 Proposition 2.23], [41 Section 3], [8 Proposition 7.15]:

**Proposition 2.8.** The pair $(X, \alpha)$ is uniformly K-stable if and only if (3) holds for all relatively Kähler semistable and dominating test configurations $(\mathcal{X}, \mathcal{A}_\alpha)$ for $(X, \alpha)$.

**Remark 2.9.** Equivalently, we may ask that there is a $\delta > 0$ such that

$$DF(\mathcal{X}, \mathcal{A}_\alpha) \geq \delta J_{\alpha}^{NA}(\mathcal{X}, \mathcal{A}_\alpha)$$

for all relatively Kähler semistable and dominating test configurations $(\mathcal{X}, \mathcal{A}_\alpha)$ for $(X, \alpha)$, since the norms $J_{\alpha}^{NA}$ and $J_{\alpha}^{NA}$ are equivalent.
Finally, for comparison with the case of smooth polarized varieties we point out that the uniform K-stability notion we use then coincides with the classical one, i.e. $(X, L)$ is uniformly K-stable in the classical sense precisely if $(X, c_1(L))$ is uniformly K-stable in the sense of Definition 2.7. We refer to [47, Proposition 3.2.24] for the proof of this result.

2.3. More on stability thresholds, norms and uniform K-stability. In this paper it will be useful to take a more flexible approach to uniform K-stability, allowing in particular for changing the norm and restrict to special subsets of test configurations, but also to allow for uniform stability notions with regards to different non-archimedean functionals. To this end, let $F^{NA} : TC_\alpha \to \mathbb{R}$ be any of the non-archimedean functionals in the previous subsection, let $S_\alpha \subset TC_\alpha$ be any subset of test configurations, and let $||.|| : TC_\alpha \to \mathbb{R}_+$ be any norm functional (Definition 2.9). To each triple $(F^{NA}, S_\alpha, ||.||)$ above we then have an associated numerical invariant (stability threshold) given by

$$\Delta^{(F^{NA}, S_\alpha, ||.||)}(\alpha) := \sup\{ \delta \in \mathbb{R} : F^{NA}(\mathcal{X}, \mathcal{A}) \geq \delta ||(\mathcal{X}, \mathcal{A})||, \forall (\mathcal{X}, \mathcal{A}) \in S_\alpha \}.$$ 

It is a common technique in the literature to restrict to test configurations of a certain kind (e.g. special, valuative, slope test configurations), and these approaches all fit into this general framework. To avoid carrying the indices around we will often employ the following shorthand notation for the cases of interest to us here (with $\beta \in H^{1,1}(X, \mathbb{R})$):

$$\Delta(\alpha) := \sup\{ \delta \in \mathbb{R} : DF(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in TC_\alpha \}$$

$$\Delta^{pp}(\alpha) := \sup\{ \delta \in \mathbb{R} : J^{NA}_{c_1}(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in TC_\alpha \}$$

$$\Delta^H(\alpha) := \sup\{ \delta \in \mathbb{R} : H^{NA}_\alpha(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in TC_\alpha \}$$

$$\Delta_{\beta}(\alpha) := \sup\{ \delta \in \mathbb{R} : M^{NA}_\beta(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in TC_\alpha \}$$

$$\Delta^{pp}_{\beta}(\alpha) := \sup\{ \delta \in \mathbb{R} : J^{NA}_{\beta}(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in TC_\alpha \}$$

With this setup the uniform Yau-Tian-Donaldson conjecture is naturally formulated as saying that existence of cscK metrics in $\alpha \in C$ is characterized by the condition $\Delta(\alpha) > 0$. We moreover say that $(X, \alpha)$ is uniformly twisted K-stable with respect to $[\theta] \in H^{1,1}(X, \mathbb{R})$ if there exists $\delta > 0$ such that

$$M^{NA}_{c_1(\mathcal{X}) + |\theta|}(\mathcal{X}, \mathcal{A}) \geq \delta J^{NA}_\alpha(\mathcal{X}, \mathcal{A})$$

for all $(\mathcal{X}, \mathcal{A}) \in TC_\alpha$. This is the uniform stability notion which is naturally associated with existence of so called twisted cscK metrics, and it corresponds precisely to $\Delta_{c_1(\mathcal{X}) + |\theta|}(\alpha) > 0$ (see e.g. [25] for background in this direction). The condition $\Delta^{pp}_{\beta}(\alpha) > 0$ is related to uniform J-stability and the Lejmi-Székelyhidi conjecture on the J-equation (proven recently for projective manifolds in [21], see also [38]). Moreover, $\Delta^H(\alpha)$ can be considered an algebraic analogue of the quantity referred to as the ‘analytic delta invariant’ in [56], and is a quantity of interest in its own right.

Note moreover that the non-archimedean entropy $H^{NA}_\alpha$ is coercive with a lower bound in terms of Tian’s alpha invariant, which induces the inequality $\Delta^H(\alpha) \geq \frac{n+1}{n} \alpha_X(\alpha) > 0$, where the latter denotes Tian’s alpha invariant (see [53]).
We also point out the following relationship between the above stability thresholds, which is easily established from their definition:

$$\Delta(\alpha) \geq \Delta_{pp}(\alpha) + \Delta_H(\alpha) > \Delta_{pp}(\alpha).$$

Since there always exists a lower bound for $\Delta_{pp}(\alpha)$ (see [50, Proposition 12]) it therefore follows that also $\Delta(\alpha)$ is bounded below. This shows that the supremum in the definition is not taken over the empty set, and $\Delta(\alpha)$ is a well-defined real-valued function on the Kähler cone $\mathcal{C}$.

2.4. Relation to energy functionals on the space of all Kähler potentials. As preparation for the proof of openness of uniform K-stability we briefly discuss non-archimedean energy functionals and their interplay with classical energy functionals, via asymptotic slope formulas along certain (sub)geodesic rays associated to test configurations. For this we use the compatibility notions introduced in [48], which will be central to the proof.

First, given a Kähler form $\tilde{\omega}$, denote by $H_{[\tilde{\omega}]}$ the space of Kähler metrics cohomologous to $\tilde{\omega}$. By the $\partial \bar{\partial}$–lemma of Hodge theory, up to a constant, the metrics of $H_{[\tilde{\omega}]}$ can be identified with elements of $P_{\tilde{\omega}}$, the space of $\omega$–Kähler potentials:

$$P_{\tilde{\omega}} = \{ \varphi \in C^\infty(X) \text{ s.t. } \tilde{\omega} + i\partial \bar{\partial} \varphi > 0 \}.$$
where \( \alpha \to \gamma_\alpha \), \( \alpha \in \mathcal{C} \) is a smooth map of closed forms. Note that \( E_{\omega_\alpha}(0_\alpha) = 0 \) and \( E_{\gamma_\alpha}(0_\alpha) = 0 \). We further caution the reader that we here use a different notation than in \[50\], where \( E_\alpha^{\gamma_\alpha} \) denoted a related but different functional. As in \[3\] Section 2, we moreover introduce the J-functional of \[29\] \[11\] associated to \( \gamma_\alpha \) and \( \omega_\alpha \):

\[
\mathcal{J}_{\omega_\alpha}(\varphi_\alpha) := E_{\omega_\alpha}(\varphi_\alpha) - n \int_X \gamma_\alpha \Lambda \bar{\omega}_\alpha^{n-1} E_{\omega_\alpha}(\varphi_\alpha) = E_{\omega_\alpha}(\varphi_\alpha) - \frac{1}{n} \int_X \omega_\alpha^{n} E_{\omega_\alpha}(\varphi_\alpha), \quad \varphi_\alpha \in \mathcal{P}.
\]

and the closely related Aubin I and J-functionals that are defined as follows:

\[
I_{\omega_\alpha}(\varphi_\alpha) := \frac{1}{n} \int_X \omega_\alpha^{n} \int_X \varphi_\alpha [\omega_\alpha^{n} - (\omega + i\partial \bar{\partial} \varphi_\alpha)^n], \quad \varphi_\alpha \in \mathcal{P},
\]

\[
J_{\omega_\alpha}(\varphi_\alpha) = \frac{1}{n} \int_X \omega_\alpha^{n} \left[ \int_X \varphi_\alpha \omega_\alpha^{n} - E_{\omega_\alpha}(\varphi_\alpha) \right], \quad \varphi_\alpha \in \mathcal{P}.
\]

When \( \gamma_\alpha = \omega_\alpha \), these last two functionals are related by the following well known inequalities (with absolute constant \( C > 1 \)):

\[
\frac{1}{C} \mathcal{J}_{\omega_\alpha}(\varphi_\alpha) \leq \int_X \omega_\alpha^{n} \cdot J_{\omega_\alpha}(\varphi_\alpha) \leq C \mathcal{J}_{\omega_\alpha}(\varphi_\alpha).
\]

In relation with the \( d^{\omega_\alpha}_{1}\)-geometry of \( \mathcal{P}_{\omega_\alpha} \) (or its completion \( \mathcal{E}_{\omega_\alpha}^{1} \)) we have the following double inequality from \[18\] Remark 6.3 (See \[19\] Proposition 3.44) (with absolute constant \( C > 1 \)):

\[
\frac{1}{C} d^{\omega_\alpha}_{1}(0_\alpha, \varphi_\alpha) - C \leq J_{\omega_\alpha}(\varphi_\alpha) \leq C d^{\omega_\alpha}_{1}(0_\alpha, \varphi_\alpha) + C, \quad \varphi_\alpha \in \mathcal{P}_{\omega_\alpha} \cap E_{\omega_\alpha}^{-1}(0).
\]

Next, we extend the K-energy functional \[12\] to the Kähler cone, using our reference metrics \( \alpha \to \omega_\alpha \):

\[
M_\alpha(\varphi_\alpha) = H_{\omega_\alpha}(\varphi_\alpha) + \frac{1}{n} \int_X \omega_\alpha^{n} \left[ \bar{S}_{\omega_\alpha} E_{\omega_\alpha}(\varphi_\alpha) + E_{\omega_\alpha}^{1-Ric(\omega_\alpha)}(\varphi_\alpha) \right], \quad \varphi_\alpha \in \mathcal{P},
\]

where \( \bar{S}_{\omega_\alpha} \) is the average scalar curvature of \( \omega_\alpha \), \( E_{\omega_\alpha} \) and \( E_{\omega_\alpha}^{1-Ric(\omega_\alpha)} \) are the Monge-Ampère energy and its contracted version (see \[17\] and \[18\] below). Moreover, \( H \) is the relative entropy functional of probability measures given by

\[
H_{\omega_\alpha}(\varphi_\alpha) := \frac{1}{n} \int_X \omega_\alpha^{n} \int_X \log \left[ \left( \frac{\omega_\alpha + i\partial \bar{\partial} \varphi_\alpha}{\omega_\alpha^{n}} \right)^n \right] (\omega_\alpha + i\partial \bar{\partial} \varphi_\alpha)^n.
\]

Notice that our choice of \( M_\alpha \) guarantees that \( M_\alpha(0_\alpha) = 0 \), where \( 0_\alpha \) is the “zero potential” of the class \( \mathcal{P}_{\omega_\alpha} \). Finally, since \( \varphi_\alpha \to M_\alpha(\varphi_\alpha) \) is constant invariant, it descends to a map on \( \mathcal{M} \).

### 2.5. Analytic data associated to test configurations and asymptotic slope formulas.

A standard technique in Kähler geometry consists in relating non-archimedean energy functionals to the asymptotic slope of the corresponding energy functionals on the space of Kähler potentials. In the following paragraph we briefly recall a variant of this formalism, borrowed from \[18\], which will be central to the proof of our main result.

In what follows, let \((X, \mathcal{A})\) be a smooth test configuration for \((X, \alpha)\) dominating \( X \times \mathbb{P}^1 \), with \( \mu : X \to X \times \mathbb{P}^1 \) the corresponding canonical \( \mathbb{C}^*\)-equivariant bimeromorphic morphism. We then have \( \mathcal{A} = \mu^* p_1^* \alpha + [D] \) for a unique \( \mathbb{R}\)-divisor \( D \) supported on \( X_0 \), with \( p_1 : X \times \mathbb{P}^1 \to X \) denoting the first projection, cf. Proposition \[24\]. Fix moreover...
a choice of $S^1$-invariant function $\psi_D$ for $D$, so that $\delta_D = \theta_D + i\partial\bar{\partial}\psi_D$, with $\theta_D$ a smooth $S^1$-invariant closed $(1,1)$-form on $\mathcal{X}$. Locally, we thus have

$$\psi_D = \sum_j a_j \log |f_j| \mod C^\infty,$$

where (writing $D := \sum_j a_j D_j$ for the decomposition of $D$ into irreducible components) the $f_j$ are local defining equations for the $D_j$ respectively. In particular, the choice of $\psi_D$ is uniquely determined modulo a smooth function.

We then have the following notion of smooth (not necessarily subgeodesic) rays that are $C^\infty$-compatible with the given test configuration $(\mathcal{X}, \mathcal{A})$ for $(X, \alpha)$.

**Definition 2.11.** ([HS Definition 4.3]) Let $(\varphi_t)_{t \geq 0}$ be a smooth ray in $C^\infty(X)$, and denote by $\Phi$ the corresponding smooth $S^1$-invariant function on $X \times \Delta^*$. We say that $(\varphi_t)$ and $(\mathcal{X}, \mathcal{A})$ are compatible if $\Phi \circ \mu + \psi_D$ extends smoothly across $\mathcal{X}_0$.

**Remark 2.12.** The condition is indeed independent of the choice of $\psi_D$, as the latter is well-defined modulo a smooth function. In the case of a polarised manifold $(X, L)$ with an (algebraic) test configuration $(\mathcal{X}, \mathcal{L})$ this condition amounts to demanding that the metric on $\mathcal{L}$ associated to the ray $(\varphi_t)_{t \geq 0}$ extends smoothly across the central fiber.

We will use the following existence result from [HS] (for the reader’s convenience we recall the proof):

**Lemma 2.13.** ([HS Lemma 4.4]) If $\mathcal{A}$ is relatively Kähler, then $(\mathcal{X}, \mathcal{A})$ is compatible with some smooth subgeodesic ray $(\varphi_t)$.

**Proof.** Since $\mathcal{A}$ is relatively Kähler, it admits a smooth $S^1$-invariant representative $\Omega$ with $\Omega + \pi^* \eta > 0$ for some $S^1$-invariant Kähler form $\eta$ on $\mathbb{P}^1$. By the $\partial\bar{\partial}$-lemma on $\mathcal{X}$, we have $\Omega = \mu^* p_1^* \omega + \theta_D + i\partial\bar{\partial} u$ for some $S^1$-invariant $u \in C^\infty(X)$, which may be assumed to be 0 after replacing $\psi_D$ with $\psi_D - u$. As a result, we get $\Omega = \mu^* p_1^* \omega + \delta_D - i\partial\bar{\partial}\psi_D$. We may also choose a smooth $S^1$-invariant function $f$ on a neighborhood $U$ of $\Delta$ such that $\eta_U = i\partial\bar{\partial} f$, and a constant $A \gg 1$ such that $D \leq A\mathcal{X}_0$. Using the Lelong-Poincaré formula $\delta_{\mathcal{X}_0} = i\partial\bar{\partial} \log |f|$ we get

$$0 < \Omega + \pi^* \eta = \mu^* p_1^* \omega + \delta_D - A\mathcal{X}_0 + i\partial\bar{\partial} (f \circ \pi + A \log |\tau| - \psi_D)$$

on $\pi^{-1}(U)$. Since $D - A\mathcal{X}_0 \leq 0$, it follows that $f \circ \pi + A \log |\tau| - \psi_D$ is $\mu^* p_1^* \omega$-psh, and hence descends to an $S^1$-invariant $p_1^* \omega$-psh function $\tilde{\Phi}$ on $X \times U$ (because the fibers of $\mu$ are compact and connected, by Zariski’s main theorem). The ray associated with the $S^1$-invariant function $\Phi := \tilde{\Phi} - A \log |\tau|$ has the desired properties. \hfill $\square$

We finally recall the well-known fact that non-archimedean energy functionals can be realized as the asymptotic slope of a certain ray of potentials associated to the given test configuration. To state these results it is convenient to recall the notion of Deligne functional, since all the functionals of interest here (except the Mabuchi energy) belongs to this class. Now let $\theta_0, \ldots, \theta_n$ be closed $(1,1)$-forms on $X$. Motivated by corresponding properties for the Deligne pairing (cf. e.g. [2, 43, 32] for background) we would like to consider a functional $\langle \cdot, \ldots, \cdot \rangle(\theta_0, \ldots, \theta_n)$ on the space $\text{PSH}(X, \theta_0) \cap L^\infty_{\text{loc}} \times \cdots \times \text{PSH}(X, \theta_n) \cap L^\infty_{\text{loc}} (n + 1 \times)$ that is symmetric, i.e. for any permutation $\sigma$ of the set $\{0, 1, \ldots, n\}$, we have

$$\langle \varphi_{\sigma(0)}, \ldots, \varphi_{\sigma(n)} \rangle(\theta_{\sigma(0)}, \ldots, \theta_{\sigma(n)}) = \langle \varphi_0, \ldots, \varphi_n \rangle(\theta_0, \ldots, \theta_n).$$
and satisfies the ‘change of function’ property
\[ \langle \varphi_0', \varphi_1', \ldots, \varphi_n' \rangle - \langle \varphi_0, \varphi_1, \ldots, \varphi_n \rangle = \int_X (\varphi_0' - \varphi_0) (\omega_1 + i\partial\bar{\partial}\varphi_1) \wedge \cdots \wedge (\omega_n + i\partial\bar{\partial}\varphi_n). \]

analogous to that for Deligne pairings. Demanding that the above properties hold necessarily leads to the following definition:

**Definition 2.14.** ([48, Definition 2.1]) Let \( \theta_0, \ldots, \theta_n \) be closed \((1,1)\)-forms on \( X \). Define a multivariate energy functional \( \langle \cdot, \ldots, \cdot \rangle_{(\theta_0, \ldots, \theta_n)} \) on the space \( \text{PSH}(X, \theta_0) \cap L^\infty_{\text{loc}} \times \cdots \times \text{PSH}(X, \theta_n) \cap L^\infty_{\text{loc}} \) \((n + 1)\) times by
\[
\langle \varphi_0, \ldots, \varphi_n \rangle_{(\theta_0, \ldots, \theta_n)} := \int_X \varphi_0 (\theta_1 + i\partial\bar{\partial}\varphi_1) \wedge \cdots \wedge (\theta_n + i\partial\bar{\partial}\varphi_n) + \int_X \varphi_1 \theta_0 \wedge (\theta_2 + i\partial\bar{\partial}\varphi_2) \wedge \cdots \wedge (\theta + i\partial\bar{\partial}\varphi_n) + \cdots + \int_X \varphi_n \theta_0 \wedge \cdots \wedge \theta_{n-1}.
\]

The multivariate energy functional \( \langle \cdot, \ldots, \cdot \rangle_{(\theta_0, \ldots, \theta_n)} \) can also be defined on \( \mathcal{C}^\infty(X) \times \cdots \times \mathcal{C}^\infty(X) \) by the same formula.

The following result builds on many of the techniques in [9] for polarized manifolds. Results in this spirit were also used independently in [28].

**Theorem 2.15.** ([48, Theorem B]) Let \( X \) be a compact Kähler manifold of dimension \( n \) and let \( \theta_i, 0 \leq i \leq n \), be closed \((1,1)\)-forms on \( X \). Set \( \alpha_i := [\theta_i] \in H^{1,1}(X, \mathbb{R}) \). Consider smooth cohomological test configurations \((\mathcal{X}_i, \mathcal{A}_i)\) for \((X, \alpha_i)\) dominating \( X \times \mathbb{P}^1 \). For each collection of smooth rays \((\varphi_i^t)_{t \geq 0}\) compatible with \((\mathcal{X}_i, \mathcal{A}_i)\) respectively, the asymptotic slope of the multivariate energy functional \( \langle \cdot, \ldots, \cdot \rangle := \langle \cdot, \ldots, \cdot \rangle_{(\theta_0, \ldots, \theta_n)} \) is well-defined and satisfies
\[
\frac{\langle \varphi_i^0, \ldots, \varphi_i^n \rangle}{t} \rightarrow (A_0 \cdot \ldots \cdot A_n)
\]
as \( t \to +\infty \).

The above result in particular applies to the Deligne functionals \( J_{\omega}, \mathcal{J}_{\omega}', L_{\omega}, E_{\omega}, E_{\omega}' \) from the previous sections. Moreover, it is proven in [48, Section 5.1] that there is a Deligne functional (closely related, but not identical to the non-archimedean Mabuchi K-energy) whose asymptotic slope along smooth compatible subgeodesic rays is precisely the Donaldson-Futaki invariant. We refer to [48] for more background and details of this result.

### 3. Proof of openness of uniform K-stability

We now proceed to the proof of the main result, Theorem 1.1. It is conducted in six steps, and will occupy us for the rest of the paper.

We first give a brief outline of the setup: Fix \( \alpha \in \mathcal{C} \) a uniformly K-stable Kähler class, and pick \( \delta > 0 \) such that \( \text{DF}(\mathcal{X}, \mathcal{A}, \alpha) \geq \delta \mathcal{F}_{\alpha}^{NA}(\mathcal{X}, \mathcal{A}) \) for all relatively Kähler test configurations \((\mathcal{X}, \mathcal{A}, \alpha)\) for \((X, \alpha)\). We aim to show that there exists an open neighbourhood \( \mathcal{U} \) of \( \alpha \) in \( \mathcal{C} \), and a positive real number \( \delta' > 0 \), such that
\[
\text{DF}(\mathcal{X}, \mathcal{A}, \alpha) \geq \delta' \mathcal{F}_{\alpha}^{NA}(\mathcal{X}, \mathcal{A})
\]
for all \( \alpha' \in \mathcal{U} \) and all relatively Kähler test configurations \((\mathcal{X}, \mathcal{A}, \alpha')\) for \((X, \alpha')\). We may without loss of generality restrict attention to relatively Kähler smooth and dominating
test configurations \((\mathcal{X}, \mathcal{A}_{\alpha'})\) for \((X, \alpha')\), with reduced central fiber \(\mathcal{X}_0\) (see Definition \ref{def:centralfiber} and Proposition \ref{prop:centralfiber}).

To define such an open neighbourhood of \(\alpha\) we fix as before (and for the rest of the section) a smooth choice of background metrics on the Kähler cone, by letting \(\mathcal{M}\) be the space of all Kähler metrics on \(X\), considering a map \([\cdot] : \mathcal{M} \to H^2(X, \mathbb{R})\) whose image is the Kähler cone \(\mathcal{C}\), and fixing \(\omega(\cdot) : \mathcal{C} \to \mathcal{M}\) a smooth right inverse of the projection \(\mathcal{M} \to \mathcal{C}\). Now fix \(\epsilon > 0\) and introduce the following open subset of \(\mathcal{C}\) defined by

\[\mathcal{U}_\epsilon := \{\alpha' \in \mathcal{C} : (1 - \epsilon)\omega_\alpha < \omega_{\alpha'} < (1 + \epsilon)\omega_\alpha\}.
\]

Clearly

\[\mathcal{U}_\epsilon \subset \{\alpha' \in \mathcal{C} : (1 - \epsilon)\alpha < \alpha' < (1 + \epsilon)\alpha\},
\]

and \(\mathcal{U}_\epsilon\) contains the fixed class \(\alpha\).

Now consider \(\alpha' \in \mathcal{U}_\epsilon\), \(\alpha' \neq \alpha\), and let \((\mathcal{X}, \mathcal{A}_{\alpha'})\) be any relatively Kähler semistable and dominating test configuration for \((X, \alpha')\). Given our choice of \(\mathcal{U}_\epsilon\), we notice that \(\Omega_{\alpha', \epsilon} := (1 + \epsilon)\omega_\alpha - \omega_{\alpha'}\) (the “\(\epsilon\)-deviation” form) is Kähler and trivially

\[\omega_{\alpha'} + \Omega_{\alpha', \epsilon} = (1 + \epsilon)\omega_\alpha \quad \text{and} \quad 0 < \Omega_{\alpha', \epsilon} < 2\epsilon\omega_\alpha.
\]

As a consequence, \((\mathcal{X}, \mathcal{A}_{\alpha'} + \mu^* p_1^*(\Omega_{\alpha', \epsilon}))\) is a smooth relatively Kähler test configuration for \((X, (1 + \epsilon)\omega_\alpha)\). Since \((X, \omega_\alpha)\) is \(\delta\)-uniformly K-stable then (up to a small error controlled by \(\epsilon\)) so is \((X, (1 + \epsilon)\omega_\alpha)\), i.e. we can write that

\[\text{DF}(\mathcal{X}, \mathcal{A}_{\alpha'} + \mu^* p_1^*(\Omega_{\alpha', \epsilon})) \geq \delta(1 - f(\epsilon))\mathcal{J}_{(1 + \epsilon)\alpha}^{\text{NA}}(\mathcal{X}, \mathcal{A}_{\alpha'} + \mu^* p_1^*(\Omega_{\alpha', \epsilon}))\]

(13)

where \(f : \mathbb{R}^+ \to \mathbb{R}^+\) is some continuous function with \(f(0) = 0\). Using this inequality, in order to conclude (12), it clearly suffices to prove the estimates

\[|\mathcal{J}_{\alpha'}^{\text{NA}}(\mathcal{X}, \mathcal{A}_{\alpha'}) - \mathcal{J}_{(1 + \epsilon)\alpha}^{\text{NA}}(\mathcal{X}, \mathcal{A}_{\alpha'} + \mu^* p_1^*(\Omega_{\alpha', \epsilon}))| \leq f(\epsilon)\mathcal{J}_{\alpha'}^{\text{NA}}(\mathcal{X}, \mathcal{A}_{\alpha'})\]

(14)

and

\[\text{DF}(\mathcal{X}, \mathcal{A}_{\alpha'}) - \text{DF}(\mathcal{X}, \mathcal{A}_{\alpha'} + \mu^* p_1^*(\Omega_{\alpha', \epsilon})) \geq -f(\epsilon)\mathcal{J}_{\alpha'}^{\text{NA}}(\mathcal{X}, \mathcal{A}_{\alpha'})\]

(15)

where \(f : \mathbb{R}^+ \to \mathbb{R}^+\) is again some continuous function with \(f(0) = 0\).

The first of the above inequalities (14) is established for all \(\alpha' \in \mathcal{U}_\epsilon\) in Step 1 of the below proof. In order to deal with the estimate (15) the argument is more involved (see Steps 2-6 below), and we in fact have to replace the right hand side with a new choice of norm \(\|\cdot\|_{\alpha}\) on test configurations that we introduce. This norm has certain desirable properties, and we consider the associated stability threshold \(\hat{\Delta}(\alpha) := \sup\{\delta > 0 : \text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta\|\mathcal{X}, \mathcal{A}\|_{\alpha}\}\) with respect to this norm. We then show that \(\hat{\Delta}(\alpha) > 0\) if and only if \((X, \alpha)\) is uniformly K-stable (with respect to the usual norm \(\mathcal{J}_\alpha^{\text{NA}}\)), see Corollary \ref{cor:uniformKstability}. For technical reasons we are also led to restricting further the open set we consider, and we finally show that if \(\alpha'\) lies in the smaller open set

\[\mathcal{U}_{\epsilon'} := \{\beta \in \mathcal{C} : \omega_\alpha < \omega_\beta < \sqrt{1 + \epsilon}\omega_\alpha\} \subset \mathcal{U}_\epsilon,
\]

(16)

then we obtain an inequality relating \(\hat{\Delta}(\alpha)\) and \(\hat{\Delta}(\alpha')\) for all such \(\alpha' \in \mathcal{U}_{\epsilon'}\) (see Proposition \ref{prop:uniformKstability}), that is enough to conclude. Indeed, this will show that if \(\Delta(\alpha) > 0\), then there exists \(\epsilon := \epsilon(\alpha) > 0\) such that \(\Delta(\alpha') > 0\) for all \(\alpha' \in \mathcal{U}_{\epsilon'}\). While \(\alpha \notin \mathcal{U}_{\epsilon'}\), the cone \(\mathbb{R}_+ \mathcal{U}_{\epsilon'}\) is then an open neighbourhood of \(\alpha\) contained in the uniformly K-stable locus, thus proving the desired openness of uniform K-stability.

A central idea of the proof is to obtain estimates for the relevant energy functionals on the space of Kähler metrics, and use Kempf-Ness type formulas (in particular \ref{prop:KempfNess}).
Theorems B]) to transfer these estimates onto their non-Archimedean counterpart. In
Step 1 below we use this to establish (14).

**Step 1: Perturbation of \(I^{NA}, J^{NA}\) and related functionals.** In this subsection we
give a proof of \([14]\) and prove some preliminary results that help in establishing \([15]\).
In particular we establish estimates controlling the change of the norms here considered,
under perturbation of the underlying Kähler class.

Let \(\theta_1\) and \(\theta_2\) be two Kähler forms on \(X\) and let \(\gamma\) be a closed \((1,1)\)-form. As before we
consider the energy functionals \(E_{\theta_1}, E_{\theta_2}\) and \(E_{\gamma}, E_{\gamma}\) defined by
\[
E_{\theta}(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X \varphi \theta^j \wedge (\theta + i\bar{\partial}\varphi)^{n-j}, \ \varphi \in P_{\theta}.
\]
(17)
\[
E_{\gamma}(\varphi) = \sum_{j=0}^{n-1} \int_X \varphi \gamma \wedge \theta^j \wedge (\theta + i\bar{\partial}\varphi)^{n-j-1}, \ \varphi \in P_{\theta}.
\]
(18)

Note that compared to \([17]\) (and the general literature) we did not use any “volume
normalization” for these functionals. As in \([11]\) and \([5, \text{Section 2}]\) we consider the \(J\)-
functional associated to \(\gamma\) and \(\theta_1\):
\[
\mathcal{J}_{\theta_1}^\gamma(\varphi) := E_{\theta_1}^\gamma(\varphi) - \frac{1}{n} \int_X \gamma \wedge \theta^{n-1}_1 E_{\theta_1}(\varphi) = E_{\theta_1}^\gamma\left( \varphi - \frac{1}{\int_X \theta^n_1} E_{\theta_1}(\varphi) \right), \ \varphi \in P_{\theta_1}.
\]
and the closely related Aubin J-functional that is defined as follows:
\[
J_{\theta_1}(\varphi) = \frac{1}{\int_X \theta^n_1} \left[ \int_X \varphi \theta^n_1 - E_{\theta_1}(\varphi) \right] \geq 0, \ \varphi \in P_{\theta_1}.
\]

When \(\gamma = \theta_1\), these last two functionals are related by the following well known
inequalities:
\[
\frac{1}{n} \mathcal{J}_{\theta_1}^\theta (\varphi) \leq \int_X \theta^n_1 \cdot J_{\theta_1}(\varphi) \leq n \mathcal{J}_{\theta_1}^\theta (\varphi).
\]

In relation with the \(d^n_\theta\)-geometry of \(P_{\theta_1}\) (or more generally \(E_{\theta}^\gamma\)) we have the following
double inequality from \([18]\) Remark 6.3 (See \([19]\) Proposition 3.44) (with absolute constant
\(C > 1\)):
\[
\frac{1}{C} d^n_{\theta_1}(0, \varphi) - C \leq J_{\theta_1}(\varphi) \leq C d^n_{\theta_1}(0, \varphi) + C, \ \varphi \in P_{\theta_1} \cap E_{\theta_1}^{-1}(0).
\]
(19)

The following estimates also follow from the results of \([18]\):

**Lemma 3.1.** There exists \(C := C(n) > 1\) such that
\[
\frac{1}{C} d^n_\theta(\varphi, \psi) \leq \frac{1}{\int_X \theta^n_1} \sum_{j=0}^{n} \int_X |\varphi - \psi| (\theta_1 + i\bar{\partial}\varphi)^j \wedge (\theta_1 + i\bar{\partial}\psi)^{n-j} \leq C d^n_\theta(\varphi, \psi), \ \varphi, \psi \in P_{\theta_1}.
\]

Proof. The first estimate follows from \([18]\) Theorem 3 (See also \([19]\) Theorem 3.32).
\([19]\) Lemma 3.33] implies that \(d^n_\theta(\varphi, \psi/2 + \varphi/2) \leq d^n_\theta(\varphi, \psi).\) On the other hand, using
\([18]\) Theorem 3] and the multilinearity of non-pluripolar measures we can write:
\[
C \sum_{j=0}^{n} \int_X |\varphi - \psi| (\theta_1 + i\bar{\partial}\varphi)^j \wedge (\theta_1 + i\bar{\partial}\psi)^{n-j} \leq 2^{n+2} d^n_\theta(\varphi, \psi/2 + \varphi/2)
\]
for some \(C \in (0, 1)\). The second estimate of the lemma now follows after putting together
the last two. \(\square\)
In order to study the asymptotics of our functions we prove the estimates in the following proposition:

**Proposition 3.2.** Let \( \theta_1 \) and \( \theta_2 \) be Kähler forms and \( \gamma \) is a smooth closed \((1,1)\)-form. Suppose that \( \theta_1 \leq \theta_2 \) and \( \varphi \in \text{PSH}(X, \theta_1) \cap L^\infty \subset \text{PSH}(X, \theta_2) \cap L^\infty \). Then there exists a constant \( C := C(\|\gamma\|_{\theta_1}, \theta_1, \theta_2, X) > 0 \) such that the following estimates hold:

\[
\left| \frac{1}{\int_X \theta_2^n} \mathcal{E}_{\theta_2}(\varphi) - \frac{1}{\int_X \theta_1^n} \mathcal{E}_{\theta_1}(\varphi) \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C, \tag{20}
\]

\[
\left| \frac{1}{\int_X \theta_2^n} \int_X \varphi \theta_2^n - \frac{1}{\int_X \theta_1^n} \int_X \varphi \theta_1^n \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C, \tag{21}
\]

\[
\left| \mathcal{J}_{\theta_2}(\varphi) - \mathcal{J}_{\theta_1}(\varphi) \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C. \tag{22}
\]

\[
\left| \mathcal{J}_{\theta_2}(\varphi) - \mathcal{J}_{\theta_1}(\varphi) \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C. \tag{23}
\]

**Proof.** Since all the estimates are constant invariant, we can assume that \( \mathcal{E}_{\theta_1}(\varphi) = 0 \) throughout the proof. With this choice, using the multilinearity of the non-pluripolar products we can start writing:

\[
\left| \frac{1}{\int_X \theta_2^n} \mathcal{E}_{\theta_2}(\varphi) \right| = \left| \frac{1}{(n+1) \int_X \theta_2^n} \sum_{j=0}^n \int_X \varphi \theta_2^j \wedge (\theta_2 + i\partial \bar{\partial} \varphi)^{n-j} \right|
\]

\[
\leq \frac{C}{\int_X \theta_2^n} \sum_{0 \leq j + k \leq n} \int_X |\varphi|(\theta_2 - \theta_1)^k \wedge \theta_1^j \wedge (\theta_1 + i\partial \bar{\partial} \varphi)^{n-j-k}
\]

\[
\leq \frac{C\|\theta_2 - \theta_1\|_{\theta_1}}{\int_X \theta_2^n} \sum_{j=0}^n \int_X |\varphi|\theta_1^j \wedge (\theta_1 + i\partial \bar{\partial} \varphi)^{n-j} \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_1(0, \varphi),
\]

where in the last estimate we have used Lemma 3.1. Using (19), this estimate establishes (20). Next, since \( 0 \leq n(\theta_2 - \theta_1) \leq \|\theta_2 - \theta_1\|_{\theta_1} \theta_1 \) we can start writing that:

\[
\left| \frac{1}{\int_X \theta_2^n} \int_X \varphi \theta_2^n - \frac{1}{\int_X \theta_1^n} \int_X \varphi \theta_1^n \right| \leq C\|\theta_2 - \theta_1\|_{\theta_1} \left| \frac{1}{\int_X \theta_1^n} \int_X \varphi \theta_1^n \right| \leq C\|\theta_2 - \theta_1\|_{\theta_1} \frac{1}{\int_X \theta_1^n} \int_X |\varphi|\theta_1^n \leq C\|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_1(0, \varphi).
\]

Using (19) again, this estimate establishes (21). (22) is a consequence of adding up (21) and (20).

To prove (23), we start with the following sequence of identities and estimates:

\[
\left| \mathcal{J}_{\theta_2}(\varphi) - \mathcal{J}_{\theta_1}(\varphi) \right| \leq \left| \mathcal{E}_{\theta_2}(\varphi) - \mathcal{E}_{\theta_1}(\varphi) \right| + n \left| \frac{1}{\int_X \theta_2^n} \int_X \gamma \wedge \theta_2^{n-1} \mathcal{E}_{\theta_2}(\varphi) \right|
\]

By multilinearity in (18), via Lemma 3.1 and (19), we conclude that

\[
\left| \mathcal{E}_{\theta_2}(\varphi) - \mathcal{E}_{\theta_1}(\varphi) \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C.
\]

Similarly, (20) implies that

\[
n \left| \frac{1}{\int_X \theta_2^n} \int_X \gamma \wedge \theta_2^{n-1} \mathcal{E}_{\theta_2}(\varphi) \right| \leq \|\theta_2 - \theta_1\|_{\theta_1} \mathcal{C}_J(\varphi) + C.
\]

From putting together the last tree estimates, (23) follows. \( \square \)
Our last estimate says that the growth of $\mathcal{J}_{\theta_1}^\gamma$ is always controlled by $J_{\theta_1}$, regardless of the choice of $\gamma$:

**Lemma 3.3.** Let $\gamma$ be a closed $(1,1)$-form. Then there exists $C := C(\|\gamma\|_{\theta_1}, X) > 0$ such that

$$|\mathcal{J}_{\theta_1}^\gamma(\varphi)| \leq CJ_{\theta_1}(\varphi) + C, \quad \varphi \in \text{PSH}(X, \theta_1) \cap L^\infty.$$  

**Proof.** We can again assume that $E_{\theta_1}(\varphi) = 0$. Since $-C\theta_1 \leq \gamma \leq C\theta_1$, we have the following sequence of estimates:

$$|\mathcal{J}_{\theta_1}^\gamma(\varphi)| = |E_{\theta_1}^\gamma(\varphi)| \leq \sum_{j=1}^{n} \int_X |\varphi|\theta_1^n \wedge (\theta_1 + i\partial\bar{\partial}\varphi)^{n-j} \leq Cd_1(0, \varphi),$$

where in the last inequality we have used Lemma 3.1. Finally, the estimate of the proposition now follows from (19). □

**Perturbation of $J_{NA}$, $J^{NA}$ and related functionals.** Let $\mathcal{T}_C$ be the space of relatively Kähler semistable and dominating test configurations for $(X, \alpha)$, defined in Section 2. Using the estimates from the last proposition applied to $\theta_1 = \omega_\alpha$ and $\theta_2 = \omega_\alpha$, an analysis of the asymptotic behaviour of the corresponding Deligne functionals along geodesic rays yields the following:

**Corollary 3.4.** Let $\gamma$ be any smooth $(1,1)$-form $\gamma$ on $X$ and $\alpha$, $(X, \mathcal{A}_\alpha)$ as described at the beginning of the section. Then there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$, such that the following estimates hold:

1. $$\left| (\mathcal{A}_\alpha + \mu^*p_1[\Omega_{\alpha', \epsilon}])^{n+1} - \mathcal{A}_{\alpha'}^{n+1} \right| \leq f(\epsilon)J^{NA}_\alpha(X, \mathcal{A}_\alpha),$$

2. $$\left| \mu^*p_1(\alpha' + [\Omega_{\alpha', \epsilon}]) \cdot (\mathcal{A}_\alpha + \mu^*p_1[\Omega_{\alpha', \epsilon}]) - \mu^*p_1^n \cdot \mathcal{A}_\alpha \right| \leq f(\epsilon)J^{NA}_\alpha(X, \mathcal{A}_\alpha),$$

3. $$\left| \left[ \mu^*p_1[\gamma] \cdot (\mathcal{A}_\alpha + \mu^*p_1[\Omega_{\alpha', \epsilon}]) - \frac{\left[ \mu^*p_1[\gamma] \cdot \mathcal{A}_\alpha^{n+1} - \frac{1}{\alpha'} \right]}{\alpha'} \right] \leq f(\epsilon)J^{NA}_\alpha(X, \mathcal{A}_\alpha).$$

**Proof.** By [17], Proposition 3.2.18] we can write $\mathcal{A}_\alpha' = \mu^*p_1[\alpha'] + [D_{\alpha'}]$, for some unique $\mathbb{R}$-divisor $[D_{\alpha'}]$ supported on $X_0$. Consequently we get that

$$\mathcal{A}_\alpha + \mu^*p_1[\Omega_{\alpha', \epsilon}] = \mu^*p_1[\alpha'] + \mu^*p_1[\Omega_{\alpha', \epsilon}] + [D_{\alpha'}] = \mu^*p_1(1 + \epsilon)\alpha + [D_{\alpha'}],$$

and hence the unique divisor associated to $\mathcal{A}_\alpha'$ and $\mathcal{A}_\alpha + \mu^*p_1[\Omega_{\alpha', \epsilon}]$ is the same. It follows from this and [17, Definition 4.1.3] that if $t \to \varphi_t$ is a subgeodesic ray compatible with $(X, \mathcal{A}_\alpha)$ then it is a subgeodesic ray compatible with $(X, \mathcal{A}_\alpha' + \mu^*p_1[\Omega_{\alpha', \epsilon}])$ as well (since $\Omega_{\alpha', \epsilon}$ is a Kähler form), but not necessarily vice versa. In other words, we have the following:

**Lemma 3.5.** Suppose that $(\varphi_t)_{t \geq 0}$ is a subgeodesic ray in $\mathcal{P}_{\omega_{\alpha'}}$ which is compatible with $(X, \mathcal{A}_\alpha') \in \mathcal{T}_C$. Then the same potentials $(\varphi_t)_{t \geq 0}$ define a subgeodesic ray in $\mathcal{P}_{\omega_{(1+\epsilon)\alpha'}}$, compatible with $(X, \mathcal{A}_\alpha' + \mu^*p_1[\Omega_{\alpha', \epsilon}]) \in \mathcal{T}_C(1+\epsilon)\alpha'$. 

This observation allows us to compare the algebro-geometric invariants associated to \((X,\mathcal{A}_\alpha)\) and \((X,\mathcal{A}_\alpha + \mu^*p^!_1[0_{\alpha'\cdot\alpha}])\), using the asymptotic behavior of the corresponding Deligne functionals. Given the definition of \(\mathcal{U}_\alpha\) in terms of a given smooth choice of background metrics \(\alpha \to \omega_\alpha\), the result is then an immediate consequence of Proposition 3.2 and Lemma 3.3 and [48, Theorem B].

**Remark 3.6.** Since \(J^\alpha_{NA}\) and \(\mathcal{J}^\alpha_{NA}\) are equivalent norms, the functionals in the above estimates can also be uniformly controlled by \(\mathcal{J}^\alpha_{NA}\).

The above result applies to any non-archimedean energy functional that can be written as a linear combination of the above intersection numbers. This include in particular \(J^\beta_{NA}\), which represents the energy part in the Chen-Tian decomposition of the twisted non-Archimedean Mabuchi functional. For future use we now record the following application of Corollary 3.4 to the stability threshold

\[
\Delta^\text{pp}_\beta(\alpha) := \sup \{ \delta \in \mathbb{R} : \mathcal{J}^\alpha_{NA}(X,\mathcal{A}) \geq \delta \mathcal{J}^\alpha_{NA}(X,\mathcal{A}), \forall (X,\mathcal{A}) \in TC_\alpha \},
\]

The following simple estimate is an immediate consequence:

**Corollary 3.7.** Suppose that \(\beta \in H^{1,1}(X,\mathbb{R})\) and \(\alpha \in \mathcal{C}\) be given. Then there exists a continuous function \(f_\beta : \mathbb{R}_+ \to \mathbb{R}_+, \forall f(0) = 0\), such that every \(\alpha' \in \mathcal{C}\) with \((1 - \epsilon)\omega_\alpha < \omega_{\alpha'} < (1 + \epsilon)\omega_\alpha\) satisfies the inequality

\[
|\Delta^\text{pp}_\beta(\alpha') - \Delta^\text{pp}_\beta(\alpha)| \leq f_\beta(\epsilon).
\]

**Proof.** \(\mathcal{J}^\alpha_{NA}\) is a linear combination of the intersection numbers in Corollary 3.4. \(\square\)

In other words, the stability threshold relevant to J-stability is continuous, and in particular J-stability is an open condition (these results may be compared to those of [50] where an explicit formula for \(\Delta^\text{pp}_\beta\) was obtained). Moreover, the stability threshold \(\Delta^\text{pp}_\beta\) is uniformly bounded below on compacts:

**Corollary 3.8.** For each compact set \(K \subset \mathcal{C}\) there is a constant \(C_{\beta,K} > 0\) such that \(\Delta^\text{pp}_\beta(\alpha') > -C_{\beta,K}\) for all \(\alpha' \in K\).

**Step 2:** Perturbation of \(H^\text{NA}\) along test configurations of bounded height and variation. The next step is to establish a simple growth inequality for the stability threshold associated to non-Archimedean entropy. As before, we write

\[
H_{\omega_\alpha}(\varphi) := \frac{1}{V_\alpha} \int_X \log \left[ \frac{(\omega_\alpha + i\partial\bar{\partial}\varphi)^n}{\omega_\alpha^n} \right] (\omega_\alpha + i\partial\bar{\partial}\varphi)^n
\]

for the relative entropy functional. Recall also that the corresponding non-Archimedean entropy functional of a test configuration \((X,\mathcal{A})\) for \((X,\alpha)\) is given precisely by \(H^\text{NA}_{\alpha}(X,\mathcal{A})\) (see Section 2 for the definition). The following is then an immediate consequence of Theorem 2.15.

**Lemma 3.9.** Let \(\alpha,\alpha' \in \mathcal{C}\) be two Kähler classes such that \(\omega_\alpha < \omega_{\alpha'} < A\omega_\alpha\) for some \(A > 1\). Then for any smooth and dominating relatively Kähler test configuration \((X,\mathcal{A}_\alpha := \mu^*p^!_1[\alpha] + [D])\) for \((X,\alpha')\), we have the inequality

\[
H^\text{NA}_{\mathcal{A}_{\alpha'}}(X,\mu^*p^!_1A\alpha + [D]) < \frac{V_{\alpha'}}{V_\alpha} H^\text{NA}_{\mathcal{A}_{\alpha}}(X,\mu^*p^!_1A\alpha + [D]).
\]
Proof. Write $\omega := \omega_0$ and $\omega' := \omega_0'$ for short, and let $V_{\omega'} := \int_X \omega'^n$ and $V_{\omega} := \int_X \omega^n$ be the associated Kähler volumes of $(X, \omega)$ and $(X, \omega')$. For any $\varphi \in \text{PSH}(X, \omega')$ we have $\varphi \in \text{PSH}(X, A\omega) \cap \text{PSH}(X, A\omega')$. Moreover, because $\omega < \omega' < A\omega$, we have $\omega^n < \omega'^n < A^n \omega^n$ (for instance by simultaneous diagonalization of positive definite Hermitian matrices, see e.g. [35, p. 226]) and likewise $(A\omega' + i\partial\bar{\partial}\varphi)^n > (A\omega + i\partial\bar{\partial}\varphi)^n$. Hence there is a constant $C > 0$ depending on $A$, such that

$$\frac{V_{\omega'}}{V_{\omega}} H_{A\omega'}(\varphi) = \frac{V_{\omega'}}{V_{\omega}} \int_X \log \left( \frac{(A\omega' + i\partial\bar{\partial}\varphi)^n}{A^n \omega^n} \right) \frac{(A\omega + i\partial\bar{\partial}\varphi)^n}{V_{A\omega}} >$$

$$> \int_X \log \left( \frac{(A\omega + i\partial\bar{\partial}\varphi)^n}{A^n \omega^n} \right) \frac{(A\omega + i\partial\bar{\partial}\varphi)^n}{V_{A\omega}} - C = H_{A\omega}(\varphi) - C.$$ 

If $(\mathcal{X}, \mu^* p_1^* A\alpha' + [D])$ is a relatively Kähler test configuration for $(X, \alpha')$ and $(\varphi_s)$ is an associated compatible subgeodesic ray, then the test configurations $(\mathcal{X}, \mu^* p_1^* A\alpha + [D])$ for $(X, A\alpha)$ and $(\mathcal{X}, \mu^* p_1^* A\alpha + [D])$ for $(X, A\alpha)$ are both relatively Kähler and compatible with the same subgeodesic ray. By considering the asymptotic slope, passing to the limit we see that [Writing $H = M - \mathcal{J}_\omega \text{Ric}(\omega)$ as a difference of the K-energy functional and the energy part (which is a Deligne functional, see the terminology in Section 2.4), the asymptotic slope can be computed as the limit $\lim_{s \to +\infty} H_{A\omega'}(\varphi_s)/s$]

$$H_{A\alpha}^{NA}(\mathcal{X}, \mu^* p_1^* A\alpha + [D]) = \lim_{s \to +\infty} s^{-1} H_{A\omega}(\varphi_s) <$$

$$< \frac{V_{\omega'}}{V_{\omega}} \lim_{s \to +\infty} s^{-1} H_{A\omega'}(\varphi_s) = \frac{V_{\omega'}}{V_{\omega}} H_{A\alpha'}^{NA}(\mathcal{X}, \mu^* p_1^* A\alpha' + [D]).$$

\[\square\]

Norms, variation and height of test configurations. With our approach to proving continuity of the uniform K-stability factor, a technical issue is to control the behaviour of energy functionals as we rescale the potential $\varphi \mapsto t^{-1} \varphi$ (or equivalently $\alpha \mapsto t\alpha$ and $\omega \mapsto t\omega$ for $t > 1$). To overcome this obstacle, we introduce a new norm functional on the space of test configurations, given by the following iteration number

$$||(\mathcal{X}, \mathcal{A})||_\alpha := \mathcal{J}_\alpha^{NA}(\mathcal{X}, \mathcal{A}) + H_{A\alpha}^{NA}(\mathcal{X}, \mathcal{A})$$

computed on the total space $\mathcal{X}$, see for instance [38, 28]. In passing we check that the introduced quantity satisfies the following basic property, motivating why we refer to it as a norm functional:

**Proposition 3.10.** Let $\alpha \in \mathcal{C}$ and let $(\mathcal{X}, \mathcal{A})$ be a relatively Kähler test configuration for $(X, \alpha)$. Then

$$||(\mathcal{X}, \mathcal{A})||_\alpha \geq 0$$

with equality if and only if $(\mathcal{X}, \mathcal{A})$ is the trivial test configuration, i.e. $\mathcal{X} \simeq X \times \mathbb{P}^1$.

**Proof.** By definition of $H^{NA}$ we have

$$||(\mathcal{X}, \mathcal{A})||_\alpha = M_{c_1(X)}^{NA}(\mathcal{X}, \mathcal{A}) + \mathcal{J}_\alpha^{NA}(\mathcal{X}, \mathcal{A}) - \mathcal{J}_\alpha^{J}(\mathcal{X}, \mathcal{A}).$$

If $(\mathcal{X}, \mathcal{A})$ is trivial it is easy to check that each of the above terms must vanish; in particular we have $(\mu^* c_1(X) \cdot \mu^* \alpha^n)_X = 0$ for dimension reasons.

Conversely, if $(\mathcal{X}, \mathcal{A})$ is non-trivial, it follows from the observation that $\mathcal{J}_\alpha^{NA}$ is precisely the classical $(I - J)$-functional in the literature, that $\mathcal{J}_\alpha^{NA}(\mathcal{X}, \mathcal{A}) > 0$ holds. Finally, it follows from coercivity of $H_\omega$ (see [53]), that

$$H_{A\alpha}^{NA}(\mathcal{X}, \mathcal{A}) > 0,$$
using also that

\[ H_{\alpha}^{NA}(X, A) = \lim_{t \to +\infty} t^{-1} H_{\omega}(\varphi_t), \]

see [48, Theorems B and C]. In other words, we have \||\!(X, A)\|_{\alpha} > 0\|\ for non-trivial test configurations, concluding the proof. \(\square\)

In the following paragraphs we temporarily consider an alternative definition of uniform K-stability, where we have made two modifications: One is that we consider uniform K-stability with respect to the new norm \||\!||_{\alpha} rather than \(\mathcal{K}^a_{\alpha}^{NA}\). The other modification is restricting the definition of uniform K-stability to test configurations satisfying an additional few properties described below (as a justification for this approach we later show that, in the situation of interest in this paper, the resulting stability notions are in fact all equivalent).

To define the special subset of all test configurations that we will consider, we wish to exclude for the moment test configurations that vary “too fast” under rescaling \(\varphi \mapsto t^{-1}\varphi\) of the potential (equivalently this corresponds to rescaling \(\alpha \mapsto t\alpha\)). To make this precise, let \((X, A := \mu^*p_1^*\alpha + [D]) \in \mathcal{T}C_\alpha^*\) be a given smooth and dominating non-trivial relatively Kähler test configurations for \((X, \alpha)\) and introduce the following auxiliary terminology: Consider the family of test configurations \((X, \mu^*p_1^*t\alpha + [D]) = (X, A + \mu^*p_1^*\Delta_{\alpha, t})\) for \((X, t\alpha)\), and the associated rational function

\[ R_\alpha(t) := \frac{H_{\alpha}^{NA}(X, \mu^*p_1^*t\alpha + [D])}{||\!(X, \mu^*p_1^*\alpha + [D])\|_{\alpha}}, \]

where \(t \in [1, +\infty)\). We then let the variation \(v_\alpha : \mathcal{T}C_\alpha \to \mathbb{R}\) of the non-trivial smooth and dominating test configuration \((X, A)\) for \((X, \alpha)\) be the numerical quantity

\[ v_\alpha(X, A) := \sup_{t \in [1, 2]} \left\{ \left| \frac{d}{dt} R_\alpha(t) \right| \right\}. \]

Note that the variation is well-defined with values in \(\mathbb{R}\), except when \((X, A)\) is the trivial test configuration, in which case \(|\!(X, A)\|_{\alpha} = 0\) and we say that the variation is not defined. Taking the supremum over any interval \([1, c]\) where \(c > 1\) is fixed would be equivalent for the purpose of our proof.

Similarly we may introduce the height of a test configuration by considering

\[ Q_\alpha(t) := \frac{||\!(X, \mu^*p_1^*t\alpha + [D])\|_{\alpha}}{||\!(X, \mu^*p_1^*\alpha + [D])\|_{\alpha}}, \]

where \(t \in [1, +\infty)\), and letting

\[ h_\alpha(X, A) := \sup_{t \in [1, 2]} \left\{ \left| \frac{d}{dt} Q_\alpha(t) \right| \right\}. \]

Restricting attention to test configurations with a given height and variation bound significantly simplifies matters. Indeed, the first property motivating the use of such test configurations and such a choice of norm is the following:

**Lemma 3.11.** Let \(\alpha \in \mathcal{C}\) and \(A \in (1, 2)\) be given. Then every relatively Kähler smooth and dominating test configuration \((X, A := \mu^*p_1^*\alpha + [D])\) for \((X, \alpha)\) satisfies the following inequality

\[ H_{\alpha}^{NA}(X, \mu^*p_1^*A\alpha + [D]) - H_{\alpha}^{NA}(X, A) \leq g(A)||\!(X, A)\|_{\alpha}, \]

where
where
\[ g(A) := \max \left\{ A - 1 + (A - 1)^2 h_\alpha(\mathcal{X}, A); (A - 1)h_\alpha(\mathcal{X}, A) \right\} \left[ v_\alpha(\mathcal{X}, A) + \frac{H^N_A(\mathcal{X}, A)}{\| (\mathcal{X}, A) \|} \right]. \]

**Proof.** Let us use the shorthand notation
\[ (\mathcal{X}, A) := (\mathcal{X}, \mu^* p^*_t \alpha + [D]), \quad t \geq 1, \]
so that in particular \((\mathcal{X}, A) = (\mathcal{X}, A_1)\). By definition of the variation of a test configuration, we have
\[ \frac{H^N_A(\mathcal{X}, A_1)}{\| (\mathcal{X}, A_1) \|} - \frac{H^N_A(\mathcal{X}, A)}{\| (\mathcal{X}, A) \|} = R_\alpha(A) - R_\alpha(1) \leq (A - 1)v_\alpha(\mathcal{X}, A_1). \]
Similarly, it follows by definition of the height function that
\[ \| (\mathcal{X}, A) \| - (1 + (A - 1)h_\alpha(\mathcal{X}, A_1))\| (\mathcal{X}, A_1) \| = \]
Putting this together, we have the following string of inequalities
\[ \frac{H^N_A(\mathcal{X}, A_1)}{\| (\mathcal{X}, A_1) \|} - \frac{H^N_A(\mathcal{X}, A)}{\| (\mathcal{X}, A) \|} = \]
\[ = \| (\mathcal{X}, A) \| - (A - 1)v_\alpha(\mathcal{X}, A_1) + (A - 1)h_\alpha(\mathcal{X}, A_1) + (A - 1)v_\alpha(\mathcal{X}, A_1) \]
In other words, taking
\[ g(A) := \max \{ A - 1 + (A - 1)^2 h_\alpha(\mathcal{X}, A_1); (A - 1)h_\alpha(\mathcal{X}, A_1) \} \left[ v_\alpha(\mathcal{X}, A_1) + \frac{H^N_A(\mathcal{X}, A_1)}{\| (\mathcal{X}, A_1) \|} \right] \]
we have
\[ H^N_A(\mathcal{X}, A_1) - H^N_A(\mathcal{X}, A_1) \leq g(A)\| (\mathcal{X}, A_1) \|, \]
which is what we wanted to prove. \(\Box\)

With the latter observation we obtain the following improved version of Lemma 3.9, taking into account the height and variation of the test configuration in question:

**Corollary 3.12.** Let \(\alpha, \alpha' \in \mathcal{C}\) be two Kähler classes such that \(\omega_\alpha < \omega'_\alpha < A\omega_\alpha\) for some \(A > 1\). Then for any smooth and dominating relatively Kähler test configuration \((\mathcal{X}, \mathcal{A} := \mu^* p^*_t \alpha + [D])\) for \((X, \alpha')\) we have the inequality
\[ \frac{H^N_A(\mathcal{X}, \mathcal{A})}{\| (\mathcal{X}, \mathcal{A}) \|} \leq \frac{\omega_\alpha}{\| (\mathcal{X}, \mathcal{A}) \|} + \frac{g(A)}{\| (\mathcal{X}, \mathcal{A}) \|}. \]

**Proof.** This follows immediately by combining Lemma 3.9 and Lemma 3.11. \(\Box\)
In what follows it will be central to control $g(A)$ uniformly over the space $\mathcal{T}C_\alpha$ of test configurations. In this direction we will later show that for the purpose of proving Theorem 1.1 we may, for $B > 0$ large enough, without loss of generality restrict to test configurations that satisfy the following conditions

1. $v_\alpha(\mathcal{X}, \mathcal{A}) \leq B$
2. $h_\alpha(\mathcal{X}, \mathcal{A}) \leq B$

uniformly for all $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha$ and all $\alpha \in K$ where $K \subset \mathcal{C}$ is some compact set. We refer to test configurations satisfying conditions 1) and 2) as having bounded variation and bounded height respectively. The role of these conditions will be to guarantee uniform control on the change of $H^\alpha_{NA}$, thus also $M^\alpha_{NA}$, in terms of the norm $||.||_\alpha$. These aspects are treated in detail in Section 3, Steps 5-6 below.

We refer to a test configuration satisfying conditions 1) and 2) as having bounded variation and bounded height respectively. We moreover introduce a third quantity that we refer to as the normalized entropy of the test configuration $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha$, given by

$$\text{Ent}_{\alpha, J^\alpha_{NA}}(\mathcal{X}, \mathcal{A}) := \frac{H^\alpha_{NA}(\mathcal{X}, \mathcal{A})}{J^\alpha_{NA}(\mathcal{X}, \mathcal{A})}.$$

In this terminology we obtain an an a priori different (but closely related) uniform stability notion, with associated stability threshold

$$\hat{\Delta}^B(\alpha) := \sup \left\{ \delta \in \mathbb{R} : DF_\alpha(\mathcal{X}, \mathcal{A}) \geq \delta ||(\mathcal{X}, \mathcal{A})||_\alpha, \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha^{(B)} \right\},$$

defined by taking the supremum over the set $\mathcal{T}C_\alpha^{(B)} \subset \mathcal{T}C_\alpha$ of relatively Kähler smooth and dominating test configurations with height, variation and normalized entropy all bounded above by $B$ (and the hat indicates the change of norm to considering $||.||_\alpha$ rather than $J^\alpha_{NA}$ here). This quantity may then be compared to

$$\hat{\Delta}(\alpha) := \sup \{ \delta \in \mathbb{R} : DF_\alpha(\mathcal{X}, \mathcal{A}) \geq \delta ||(\mathcal{X}, \mathcal{A})||_\alpha, \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C_\alpha^* \}.$$

In Section 3, Step 5, we will show that for sufficiently large $B > 0$ these thresholds coincide on compact subsets, in a suitable sense.

**Remark 3.13.** We may moreover consider the set

$$\hat{\mathcal{U}}K_{SB} := \{ \alpha \in \mathcal{C}_X : \hat{\Delta}^B(\alpha) > 0 \},$$

which is the uniform K-stability locus associated to the above stability notion. Unlike the usual uniform K-stability loci $\hat{\mathcal{U}}K's := \{ \alpha \in \mathcal{C} : \Delta(\alpha) > 0 \}$ and $\hat{\mathcal{U}}K's := \{ \alpha \in \mathcal{C} : \hat{\Delta}(\alpha) > 0 \}$ one may note that there is a priori no reason why $\hat{\mathcal{U}}K_{SB}$ should be a cone. However, we have a trivial inclusion $\hat{\mathcal{U}}K's \subseteq \hat{\mathcal{U}}K_{SB}$ coming from the fact that we restrict attention to a smaller set of test configurations. Likewise $\hat{\mathcal{U}}K_{SB_1} \subseteq \hat{\mathcal{U}}K_{SB_2}$ if $B_1 \geq B_2$.

**Step 3: Sufficiency of bounded normalized entropy.** As explained above we aim to show that uniformly bounded height, variation and normalized entropy can be assumed without loss of generality, in the sense that it suffices to test uniform K-stability on such test configurations. The goal of the following paragraphs is to deal with this question for normalized entropy.

For purely technical reasons it will be useful to introduce a certain subset of test configurations on which it is clearly sufficient to test for uniform K-stability, namely the “nearly optimal” ones:
**Definition 3.14.** Let $\beta \in H^{1,1}(X, \mathbb{R})$ and $\alpha \in \mathcal{C}$. Then let

$$\mathcal{Tc}^{opt}_{\beta,\alpha} := \{(\mathcal{X}, \mathcal{A}) \in \mathcal{Tc}_\alpha : M^\mathcal{NA}_\beta(\mathcal{X}, \mathcal{A}) \leq (\Delta_\beta(\alpha) + 1)\mathcal{J}^\mathcal{NA}_\alpha(\mathcal{X}, \mathcal{A})\}.$$ 

Here we might as well ask $M^\mathcal{NA}_\beta(\mathcal{X}, \mathcal{A}) \leq (\Delta_\beta(\alpha) + c)\mathcal{J}^\mathcal{NA}_\alpha(\mathcal{X}, \mathcal{A})$ for any given $c > 0$, and the choice $c = 1$ is made for convenience. Whichever $c > 0$ we choose not all test configurations are in this set, but for the purpose of testing (twisted) uniform K-stability against the usual norms $\mathcal{J}^\mathcal{NA}_\alpha$ or $\mathcal{J}^\mathcal{NA}_\alpha$ it clearly suffices to restrict attention to $\mathcal{Tc}^{opt}_{\beta,\alpha}$.

This definition is motivated by the following lemma, which yields a uniform control on compact subsets of the Kähler cone:

**Proposition 3.15.** Let $X$ be a compact Kähler manifold and let $K \subset \mathcal{C}$ be a compact subset of the Kähler cone. Fix $\beta \in H^{1,1}(X, \mathbb{R})$. Then there exists $C_\beta := C(K, \beta) > 0$ such that the following double inequality holds

$$\Delta_{\beta}^{pp}(\alpha) + \Delta_{\beta}^{H}(\alpha) \leq \Delta_{\beta}(\alpha) \leq \Delta_{\beta}^{pp}(\alpha) + \Delta_{\beta}^{H}(\alpha) + C_\beta$$

for every $\alpha \in K$.

**Proof.** The first inequality follows immediately by considering

$$\frac{M^\mathcal{NA}_\beta}{\mathcal{J}^\mathcal{NA}_\alpha} = \frac{\mathcal{J}^\mathcal{NA}_\beta}{\mathcal{J}^\mathcal{NA}_\alpha} + \frac{\mathcal{H}^\mathcal{NA}_\beta}{\mathcal{J}^\mathcal{NA}_\alpha}$$

and using that the infimum of the sum is larger than or equal to the sum of the infima.

The proof of the second inequality relies on linearity of the function $\beta \mapsto \mathcal{J}^\mathcal{NA}_\beta(\mathcal{X}, \mathcal{A})$ for any given test configuration $(\mathcal{X}, \mathcal{A})$ for $(X, \alpha)$. It also requires sufficient continuity properties to bound uniformly $\Delta_{\beta}^{pp}(\alpha)$ and $\Delta_{\beta}^{pp}(\alpha)$ from below (using Corollary 3.8, see also the main result of [50]) and $\Delta_{\beta}^{H}(\alpha)$ from above. More precisely, fix $\beta \in H^{1,1}(X, \mathbb{R})$ and consider the line $\beta_t := (1-t)\beta + t\alpha, t \in \mathbb{R}$, such that $\beta_0 = \beta$, $\beta_1 = \alpha$ and $\beta_2 = 2\alpha - \beta$. Now set

$$L(t) := \frac{\mathcal{J}^\mathcal{NA}_{\beta_t}(\mathcal{X}, \mathcal{A})}{\mathcal{J}^\mathcal{NA}_{\alpha}(\mathcal{X}, \mathcal{A})}, \ t \in \mathbb{R}.$$ 

Then $L(t)$ is linear, and

$$L(0) = \frac{\mathcal{J}^\mathcal{NA}_{\beta}(\mathcal{X}, \mathcal{A})}{\mathcal{J}^\mathcal{NA}_{\alpha}(\mathcal{X}, \mathcal{A})}, \ L(1) = 1, \ L(2) = 2 + \frac{\mathcal{J}^\mathcal{NA}_{\beta}(\mathcal{X}, \mathcal{A})}{\mathcal{J}^\mathcal{NA}_{\alpha}(\mathcal{X}, \mathcal{A})}.$$ 

Moreover

$$\Delta_{2\alpha - \beta}^{pp}(\alpha) \leq \frac{\mathcal{J}^\mathcal{NA}_{\beta}(\mathcal{X}, \mathcal{A})}{\mathcal{J}^\mathcal{NA}_{\alpha}(\mathcal{X}, \mathcal{A})} = L(2).$$

It was shown in [50] Lemma 15 that the function $t \mapsto \Delta_{\beta_t}^{pp}(\alpha)$ is only piecewise linear (it is linear for $t \geq 1$ and for $t \leq 1$ respectively, but not differentiable in $t = 1$, see [50] for details). We may however let $P(t)$ be the unique linear function that satisfies $P(t) := \Delta_{\beta_t}^{pp}(\alpha)$ for all $t \geq 1$. Then $P(2) \leq L(2)$ and $P(1) = L(1)$, which together implies that $P(t) \geq L(t)$ for all $t \leq 1$ by linearity. Hence

$$\Delta_{\beta}^{pp}(\alpha) \leq \frac{\mathcal{J}^\mathcal{NA}_{\beta}(\mathcal{X}, \mathcal{A})}{\mathcal{J}^\mathcal{NA}_{\alpha}(\mathcal{X}, \mathcal{A})} \leq P(t)$$

where the bounds are independent of the test configuration $(\mathcal{X}, \mathcal{A})$ for $(X, \alpha)$. 

In order to obtain the desired bound we finally note that by Corollary 3.8 there exists a $C_\beta := C(K, \beta) > 0$ such that $\Delta_{\beta}(\alpha) \geq -C_\beta$ for all $\alpha \in K$, so also $P(2) \geq -C_\beta$ uniformly, up to changing the constant. Since $P(1) = 1$ always, this means that we have

$$\Delta_{\beta}(\alpha) \leq J_{\alpha}(\mathcal{X}, A) - J_{\alpha}(\mathcal{X}, A) \leq \Delta_{\beta}(\alpha) + C_\beta$$

for all $\alpha \in K$. As a consequence, if $H_{\alpha}(\mathcal{X}, A) \leq H_{\alpha}(\mathcal{X}, A) \leq \Delta_{\beta}(\alpha) + 1$.

By Corollary 3.8 there is moreover a $D > 0$ such that

$$\Delta_{\beta}(\alpha) \leq \Delta_{\beta}(\alpha) \leq \Delta_{\beta}(\alpha) + 1.$$
Proof. Fix $\alpha \in K$ and let $(\mathcal{X}, \mathcal{A})$ be any smooth and dominating test configuration for $(X, \alpha)$. In the notation of Proposition $2.4$ we then have $\mathcal{A} = \mu^* p_1^* \alpha + [D]$ for some $\mathbb{R}$-divisor $D$ on $\mathcal{X}$ supported on the central fiber $\mathcal{X}_0$. Let

$$K^+ := \{ \alpha' \in K : \alpha' > \alpha \},$$

so that $(\mathcal{X}, \mu^* p_1^* \alpha' + [D])$ is a relatively Kähler semistable and dominating test configuration for $(X, \alpha')$ whenever $\alpha' \in K^+$. We claim moreover that the function

$$K^+ \ni \alpha' \mapsto R_{(\mathcal{X}, \mathcal{A})}(\alpha') := \frac{H^{NA}_\alpha(\mathcal{X}, \mu^* p_1^* \alpha' + [D])}{J^{NA}_\alpha(\mathcal{X}, \mu^* p_1^* \alpha' + [D])}$$

is continuous (by continuity of the intersection numbers involved, noting in particular that the denominator never vanishes in $K^+$). We can therefore pick a constant $C := C(K) > 0$ such that $R_{(\mathcal{X}, \mathcal{A})}(\alpha') \leq C$ for all $\alpha' \in K^+$, and it is deduced directly from the definition that $\Delta^H(\alpha) \leq C$ for all $\alpha' \in K^+$.

It remains to treat the case when $\alpha \in K \setminus K^+$. For this we note that there is a constant $D := D(K) > 0$ such that $\lambda \alpha' \in K$ for some $\lambda := \lambda(\alpha) \leq D$. In conclusion

$$C \geq \Delta^H(\lambda \alpha') = \lambda^{-1} \Delta^H(\alpha'),$$

and hence

$$\Delta^H(\alpha') \leq \lambda C \leq D \cdot C$$

for all $\alpha' \in K$. This gives a uniform upper bound on $K$. \hfill $\square$

This shows that restricting attention to test configurations of satisfying uniform normalized entropy bounds on compact sets is enough:

**Proposition 3.18.** Let $K \subset \mathcal{C}$ be any compact subset of the Kähler cone. Then there is a constant $B := B(K, \beta) > 0$ such that

$$\text{Ent}_{\alpha, \mathcal{J}^{NA}_\beta}(\mathcal{X}, \mathcal{A}) \leq B$$

for all $\alpha \in K$ and all test configurations $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^\text{opt}_{\beta, \alpha}$.

In the following Steps 4 and 5 we proceed to prove similar results for the height and variation functions.

**Step 4: Uniform bounds for $H^{NA}$, $\mathcal{J}^{NA}_\beta$ and $DF$ in terms of $\|\|_\alpha$.** As a next step we establish some uniform bounds on non-archimedean energy functionals that will play a central role in our proof, and serve as a preparation to reduce the proof of our main result to the case of test configurations of uniformly bounded height and variation.

**Linearity properties of the norm $\|\|_\alpha$.** We now provide a control of the variation of the Donaldson-Futaki invariant in terms of our new choice of norm $\|\|_\alpha$ on test configurations. More precisely, the next goal is to show that for each compact $K \subset \mathcal{C}$ we have a constant $C := C(K) > 0$ such that the threshold

$$\hat{\Delta}(\alpha) := \sup \{ \delta \in \mathbb{R} : DF_\alpha(\mathcal{X}, \mathcal{A}) \geq \delta \|\|_\alpha(\mathcal{X}, \mathcal{A}), \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha \}$$

satisfies the double inequality

$$\hat{\Delta}(\alpha) \|\|_\alpha(\mathcal{X}, \mathcal{A}) \leq DF(\mathcal{X}, \mathcal{A}) \leq \left[ \hat{\Delta}(\alpha) + C \right] \|\|_\alpha(\mathcal{X}, \mathcal{A})$$

for all $\alpha \in K$ and all $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}C^*_\alpha$. Another main goal of the section is to prove that the norms $\|\|_\alpha$ and $\mathcal{J}^{NA}_\alpha$ are comparable on the subspace $\mathcal{T}C^\text{opt}_\alpha$, and more generally on the space of test configurations with $\text{Ent}_{\alpha, \mathcal{J}^{NA}_\alpha}(\mathcal{X}, \mathcal{A}) \leq B$. 
As in the previous section, recall that the norm can be realized as the following asymptotic slope

$$||(X, A)||_\alpha = \lim_{t \to +\infty} t^{-1} \mathcal{J}_\alpha^\omega (\varphi_t) + H_\omega (\varphi_t)$$

where $(\varphi_t)_{t \geq 0}$ is any geodesic ray compatible with $(X, A)$ in the sense of Section 2.4.

One of the key observations about the norm functional $||.||_\alpha : \mathcal{T}C_\alpha \to \mathbb{R}$ is that it makes the stability threshold

$$\hat{\Delta}_\beta (\alpha) := \sup \{ \delta \in \mathbb{R} : M^\alpha_{\beta} (X, A) \geq \delta ||(X, A)||_\alpha, \forall (X, A) \in \mathcal{T}C_\alpha \}$$

linear along the line $\beta_t := (1 - t)\beta + ta$ for $t \leq 1$ (and analogously to what was proven for the J-functional in [50] it will be linear also for $t \geq 1$, but it is not differentiable in $t = 1$ in general). We then have the following result, based on the observation that a functional $(1 - t)F + t||.||$ has linear coercivity threshold for all $t \leq 1$, depending on the stability threshold associated with $F$.

**Lemma 3.19.** Let $\alpha \in C$ and $\beta \in H^{1,1}(X, \mathbb{R})$. Then

$$\hat{\Delta}_{(1-t)\beta+ta} (\alpha) = (1-t)\hat{\Delta}_\beta (\alpha) + t$$

holds for all $t \in (-\infty, 1]$.

**Proof.** As previously noted $M^\alpha_{\beta} = ||.||_\alpha$, and since $(1 - t) + t = 1$ we have

$$M^\alpha_{(1-t)\beta+ta} = \mathcal{J}^\alpha_{(1-t)\beta+ta} + H^\alpha_{(1-t)\beta+ta} = (1 - t) (\mathcal{J}^\alpha_\beta + H^\alpha_\beta) + t (\mathcal{J}^\alpha_\alpha + H^\alpha_\alpha) =$$

$$= (1 - t)M^\alpha_\beta + t||.||_\alpha.$$  

If $t = 1$ the formula is clearly true. Now assume that $1 - t > 0$. It then follows from the above that

$$M^\alpha_{(1-t)\beta+ta} \geq \delta ||.||_\alpha$$

for some $\delta \in \mathbb{R}$, precisely if

$$M^\alpha_\beta \geq \frac{\delta - t}{1 - t} ||.||_\alpha.$$  

We therefore conclude that

$$\hat{\Delta}_{(1-t)\beta+ta} (\alpha) = (1-t)\hat{\Delta}_\beta (\alpha) + t\hat{\Delta}_\alpha (\alpha) = (1-t)\hat{\Delta}_\beta (\alpha) + t,$$

for all $t \leq 1$, which is what we wanted to prove. \qed

**Uniform bounds on $H^\alpha_{\beta}$, $\mathcal{J}^\alpha_\beta$ and $DF$ in terms of $||.||_\alpha$.** We now use the above observation to prove upper and lower bounds for the non-archimedean energy functionals of interest, in terms of the norm $||.||_\alpha$:

**Lemma 3.20.** Let $K \subset C$ be a compact subset of the Kähler cone and fix $\beta \in H^{1,1}(X, \mathbb{R})$. Then there exist $C_\beta := C(K, \beta) > 0$ and $C := C(K) > 0$ such that the following inequalities hold for every $\alpha \in K$ and every test configuration $(X, A) \in \mathcal{T}C_\alpha$:

$$\hat{\Delta}_\beta (\alpha) ||(X, A)||_\alpha \leq M^\alpha_{\beta} (X, A) \leq (\hat{\Delta}_\beta (\alpha) + C_\beta) ||(X, A)||_\alpha,$$  

$$\hat{\Delta}^H (\alpha) ||(X, A)||_\alpha \leq H^\alpha_{\beta} (X, A) \leq (\hat{\Delta}^H (\alpha) + C) ||(X, A)||_\alpha,$$  

$$\hat{\Delta}^{pp}_{\beta} (\alpha) ||(X, A)||_\alpha \leq \mathcal{J}^\alpha_{\beta} (X, A) \leq (\hat{\Delta}^{pp}_{\beta} (\alpha) + C_\beta) ||(X, A)||_\alpha.$$  

(24)
Proof. The lower bounds hold by definition of the thresholds $\hat{\Delta}_\beta(\alpha)$, $\hat{\Delta}_H(\alpha)$ and $\hat{\Delta}_{pp}^\beta(\alpha)$. For the upper bounds, the proof relies on linearity of the function $\beta \mapsto M_{\beta}^{\alpha}(\mathcal{X}, \mathcal{A})$ for any given test configuration $(\mathcal{X}, \mathcal{A})$ for $(\mathcal{X}, \alpha)$, as well as sufficient continuity properties to bound uniformly $\hat{\Delta}_{pp}^\beta(\alpha)$ and $\hat{\Delta}_{pp}^{\beta}(\alpha)$ from below: Indeed, fix $\beta \in H^{1,1}(X, \mathbb{R})$ and consider the line $\beta_t := (1-t)\beta + t\alpha$, for $t \in \mathbb{R}$, such that $\beta_0 = \beta$, $\beta_1 = \alpha$ and $\beta_2 = 2\alpha - \beta$. Now fix any test configuration $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}\mathcal{C}_\alpha$ and write

$$P(\mathcal{X}, \mathcal{A})(t) := \frac{M_{\beta_t}^{\alpha}(\mathcal{X}, \mathcal{A})}{\|(\mathcal{X}, \mathcal{A})\|_\alpha}, \quad t \in \mathbb{R}.\leqno{(27)}$$

Then $P(t) := P(\mathcal{X}, \mathcal{A})(t)$ is linear and satisfies

$$P(0) = \frac{M_{\beta}^{\alpha}(\mathcal{X}, \mathcal{A})}{\|(\mathcal{X}, \mathcal{A})\|_\alpha}, \quad P(1) = 1, \quad P(2) = 2 + \frac{M_{\beta}^{\alpha}(\mathcal{X}, \mathcal{A})}{\|(\mathcal{X}, \mathcal{A})\|_\alpha},$$

where in the last equality we have used that $\|(\mathcal{X}, \mathcal{A})\|_\alpha = M_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})$.

Moreover, let $L(t)$ be the linear function that satisfies $L(0) = \hat{\Delta}_\beta(\alpha)$ and $L(1) = \hat{\Delta}_\alpha(\alpha)(= 1)$, and let $R(t)$ be the linear function that satisfies $R(2) = \hat{\Delta}_{2\alpha-\beta}(\alpha)$ and $R(1) = \hat{\Delta}_\alpha(\alpha)$. Then

$$\min\{L(t), R(t)\} \leq P(\mathcal{X}, \mathcal{A})(t) \leq \max\{L(t), R(t)\} \leq \min\{L(t), R(t)\} \leq \max\{L(t), R(t)\}\leqno{(27)}$$

and this holds for any test configuration $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}\mathcal{C}_\alpha$ (note also that in fact the left hand side above equals $\hat{\Delta}_{\beta_t}(\alpha)$ for all $t \in \mathbb{R}$). It therefore suffices to show that the right hand side in the inequality (27) is bounded above by some $C_\beta' := C'(K, \beta) > 0$ for all $\alpha \in K$ (and then any $C_\beta$ such that $C_\beta \geq C_\beta' - \hat{\Delta}_\beta(\alpha)$ for all $\alpha \in K$ will do the job). To see this we note that (up to changing the constants if necessary) $R(2) \geq -C_\beta$ implies that $R(0) \leq C_\beta$ by linearity, and similarly $L(0) \geq -C_\beta$ implies that $L(2) \leq C_\beta$. Since

$$R(2) \geq \hat{\Delta}_{pp}^{\beta}(\alpha) + \hat{\Delta}_H(\alpha) = 2 + \hat{\Delta}_{pp}^{\beta}(\alpha) + \hat{\Delta}_H(\alpha),$$

it is enough to show that $\hat{\Delta}_{pp}^{\beta}(\alpha) + \hat{\Delta}_H(\alpha) \geq -C_\beta$ for all $\alpha \in K$. But we always have $\hat{\Delta}_H(\alpha) > 0$ and

$$|\hat{\Delta}_{pp}^{\beta}(\alpha)| \leq |\hat{\Delta}_{pp}^{\beta}(\alpha)|.$$\leqno{(28)}$$

Hence it suffices to bound from below $\Delta_{pp}^{\beta}(\alpha)$ using the norm $J_{\alpha}^{\alpha}$, as was established in Corollary 3.8. We thus obtain the required bounds on compact sets, finishing the proof of (24).

The upper bound (25) on $H_{\alpha}^{\alpha}$ follows trivially, since

$$\frac{\|(\mathcal{X}, \mathcal{A})\|_\alpha}{H_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})} = 1 + \frac{J_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})}{H_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})} > 1,$$

and hence

$$\frac{H_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})}{\|(\mathcal{X}, \mathcal{A})\|_\alpha} < 1.$$\leqno{(29)}$$

Finally, to prove the upper inequality in (26) we first note that

$$\frac{J_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})}{\|(\mathcal{X}, \mathcal{A})\|_\alpha} \leq \frac{J_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})}{J_{\alpha}^{\alpha}(\mathcal{X}, \mathcal{A})}.$$
As a consequence it follows from Corollary 3.8 and Proposition 3.15 that
\[ \hat{\Delta}^{\text{pp}}_{\beta}(\alpha) \leq \frac{J^\text{NA}_{\beta}(\mathcal{X}, \mathcal{A})}{||\mathcal{X}, \mathcal{A}||_{\alpha}} \leq \Delta^{\text{pp}}_{\beta}(\alpha) + C_{\beta} \leq \hat{\Delta}^{\text{pp}}_{\beta}(\alpha) + C_{\beta}, \]
up to changing the constant \( C \) as necessary (using that \( \Delta^{\text{pp}}_{\beta}(\alpha) \) is clearly uniformly bounded from above on the compact set \( K \), by upper semi continuity, see Corollary 3.7).

A delicate point is that the threshold \( \hat{\Delta}^{\text{pp}}_{\beta}(\alpha) \) may a priori be zero even if \( \beta = \alpha \). This is in stark contrast to the situation when stability thresholds are considered with respect to the minimum norm, in which case \( \Delta^{\text{pp}}_{\beta}(\alpha) = 1 \) always. On the other hand, restricting to test configurations of bounded normalized entropy we may write
\[ \mathcal{T}^{(B)}_{\alpha}: \{ (\mathcal{X}, \mathcal{A}) \in \mathcal{T}_{\alpha} : \text{Ent}_{\alpha, J^\text{NA}}(\mathcal{X}, \mathcal{A}) \leq B \}, \]
and for the arguments of the following paragraph we temporarily consider the associated threshold
\[ \hat{\Delta}^{\text{pp}, B}_{\beta}(\alpha) := \sup \{ \delta \in \mathbb{R} : J^\text{NA}_{\beta}(\mathcal{X}, \mathcal{A}) \geq \delta ||(\mathcal{X}, \mathcal{A})||_{\alpha} \forall (\mathcal{X}, \mathcal{A}) \in \mathcal{T}^{(B)}_{\alpha} \} \]
In the special case \( \beta = \alpha \) we then have the following lemma:

**Lemma 3.21.** Let \( \alpha \in \mathcal{C} \), \( B > 0 \) and \( t \geq 1 \). Then the stability threshold \( \hat{\Delta}^{\text{pp}, B}_{\alpha} \) satisfies the following properties
\[ \hat{\Delta}^{\text{pp}, B}_{\alpha}(t\alpha) \geq \hat{\Delta}^{\text{pp}, Bt}_{\alpha}(\alpha) \]
and
\[ \hat{\Delta}^{\text{pp}, B}_{\alpha}(\alpha) \geq (1 + B)^{-1}. \]
In particular \( \hat{\Delta}^{\text{pp}, B}_{\alpha}(\alpha) \) is strictly positive.

**Proof.** Consider any smooth and dominating test configuration \( (\mathcal{X}, \mathcal{A}) := (\mu, \mu^*p^\alpha + \hat{D}) \in \mathcal{T}_{\alpha} \). For any \( t \geq 1 \) we may moreover consider the rescaled test configuration \( (\mathcal{X}, t\mathcal{A}) = (\mathcal{X}, \mu^*p^\alpha t\alpha + t[D]) \in \mathcal{T}_{t\alpha} \), which is readily seen to satisfy the following properties
\[ J^\text{NA}_{t\alpha}(\mathcal{X}, t\mathcal{A}) = tJ^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}), \]
and
\[ H^\text{NA}_{t\alpha}(\mathcal{X}, t\mathcal{A}) = H^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}), \]
using that the volume rescales as \( V_{t\alpha} = t^nV_\alpha \). Since the non-archimedean entropy \( H^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}) \) is always positive, it follows that
\[ \frac{J^\text{NA}_{t\alpha}(\mathcal{X}, t\mathcal{A})}{||(\mathcal{X}, t\mathcal{A})||_{t\alpha}} \geq \frac{J^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A})}{||(\mathcal{X}, \mathcal{A})||_{\alpha}}. \]  \( \square \)

Now if \( (\mathcal{X}, t\mathcal{A}) \in \mathcal{T}^{(B)}_{\alpha} \) then by the above rescaling properties we have \( (\mathcal{X}, \mathcal{A}) \in \mathcal{T}^{(Bt)}_{\alpha} \), and taking the infimum on both sides of the inequality (28) it follows that
\[ \hat{\Delta}^{\text{pp}, B}_{t\alpha}(t\alpha) \geq \hat{\Delta}^{\text{pp}, Bt}_{\alpha}(\alpha). \]
Finally, the last part follows since if \( (\mathcal{X}, \mathcal{A}) \in \mathcal{T}^{(B)}_{\alpha} \), then by definition
\[ H^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}) \leq BJ^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}) \]
From this we deduce the bound
\[ J^\text{NA}_{\alpha}(\mathcal{X}, \mathcal{A}) \geq (1 + B)^{-1}||(\mathcal{X}, \mathcal{A})||_{\alpha}. \]
Corollary 3.22. Fix $B > 0$. Let $K \subset C$ be a compact set and for each $t > 1$, let $V_{a,t}$ be the open set defined by $V_{a,t} := \{ \alpha' \in C : \alpha \leq \alpha' < t\alpha \}$. Assume moreover that $t\alpha \in K$. Then there exists $C := C(K,B) > 0$ such that the following inequality holds

$$|||(\mathcal{X}, \mu^* p_t^* \alpha') + [D]|||_{a'} \leq C |||(\mathcal{X}, \mu^* p_t^* t\alpha + [D])|||_{ta}$$

for all $\alpha' \in V_{a,t} \cap K$ and every test configuration $(\mathcal{X}, \mu^* p_t^* \alpha' + [D]) \in \mathcal{T}\mathcal{C}\alpha'$. If in addition $\alpha' = \alpha$, then we have a double inequality

$$C^{-1} |||(\mathcal{X}, \mu^* p_t^* t\alpha + [D])|||_{ta} \leq |||(\mathcal{X}, \mu^* p_t^* \alpha' + [D])|||_{a'} \leq C |||(\mathcal{X}, \mu^* p_t^* t\alpha + [D])|||_{ta}$$

for every test configuration $(\mathcal{X}, \mu^* p_t^* t\alpha + [D]) \in \mathcal{T}\mathcal{C}\alpha$. 

Proof. Suppose that $(\mathcal{X}, \mathcal{A}') \in \mathcal{T}\mathcal{C}\alpha'$. By Lemma 3.21 we have $\tilde{\Delta}^{\alpha, B}_a(\mathcal{A}') \geq (1 + B)^{-1}$, so in particular $\mathcal{F}_{t\alpha}(\mathcal{X}, \mathcal{A}) \geq (1 + B)^{-1} |||(\mathcal{X}, \mathcal{A})|||_{a'}$. By Corollary 3.4 there is moreover a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$, such that

$$|||(\mathcal{X}, \mathcal{A}')|||_{a'} \leq (1 + B)\mathcal{J}_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A}') \leq (1 + B) n J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A}')$$

$$(1 + B)n(1 + f(\epsilon))J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mu^* p_t^* t\alpha + [D]) \leq (1 + B)n^2(1 + f(\epsilon))J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mu^* p_t^* t\alpha + [D]) \leq 2(1 + B)n^2(1 + f(\epsilon))|||(\mathcal{X}, \mu^* p_t^* t\alpha + [D])|||_{ta}.$$

Since $2(1 + B)n^2(1 + f(\epsilon)) \leq C$ can be uniformly bounded from above on the compact $K$, this concludes that $|||(\mathcal{X}, \mathcal{A}')|||_{a'} \leq C |||(\mathcal{X}, \mu^* p_t^* t\alpha + [D])|||_{ta}$ for some uniform constant $C := C(K,B) > 0$.

In the special case when $\alpha = \alpha'$ the same argument can be applied reversing the roles of $t\alpha$ and $\alpha'$. Indeed, by Lemma 3.21 we have

$$\tilde{\Delta}^{\alpha, B}_a(t\alpha) \geq \tilde{\Delta}^{\alpha, B}_a(\alpha) \geq (1 + Bt)^{-1} \geq C > 0$$

for all $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}\mathcal{C}\alpha'$ and for some uniform constant $C := C(K,B) > 0$. Repeating the above argument then yields

$$|||(\mathcal{X}, \mathcal{A})|||_{a} \leq (1 + B)\mathcal{J}_{t\alpha}^\mathcal{N}(\mathcal{X}, t\mathcal{A}) \leq 2(1 + B)n^2(1 + f(\epsilon))|||(\mathcal{X}, \mu^* p_t^* \alpha + [D])|||_{a},$$

which as before gives a uniform upper bound on the compact set $K$, thus finishing the proof of the double inequality. 

Another key input to our argument is the following, showing in particular that the norm functionals $J_{t\alpha}^\mathcal{N}$, $\mathcal{J}_{t\alpha}^\mathcal{N}$ and $|||\cdot|||_a$ are all equivalent on compact sets, provided that uniform K-stability is restricted to $\mathcal{T}\mathcal{C}\alpha$. 

Corollary 3.23. Fix $B > 0$ and let $K \subset C$ be a compact set. Then there exists $C := C(K,B) > 0$ such that the double inequalities

$$-C J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A}) \leq DF(\mathcal{X}, \mathcal{A}) \leq C J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A})$$

and

$$C^{-1} J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A}) \leq |||(\mathcal{X}, \mathcal{A})|||_{a} \leq C J_{t\alpha}^\mathcal{N}(\mathcal{X}, \mathcal{A}).$$

hold for every $\alpha \in K$ and every $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}\mathcal{C}\alpha$. 

so in particular $\tilde{\Delta}^{pp,B}_a(\alpha)$ has a strictly positive lower bound $(1 + B)^{-1} > 0$ on $K$. □
Proof. This is an immediate consequence of Lemma 3.20 and Lemma 3.21 with $\beta = -c_1(X)$ and $\beta = \alpha$ respectively. □

Step 5: Equivalence of bounded and unbounded stability. Fix $\beta \in H^{1,1}(X, \mathbb{R})$ for the rest of the subsection. As a direct application of previous sections we now show that uniform bounds on height and variation (as well as normalized entropy, by Step 3) are always achieved by nearly optimal test configurations. As a consequence, assuming these conditions (for suitably large bounds) can be made without loss of generality.

Note that the required uniform bounds on normalized entropy were established in Proposition 3.24.

Nearly optimal test configurations have uniform bounds on normalized entropy over compact subsets of $C$. The uniform bounds on normalized entropy were established in Corollary 3.16.

Proposition 3.24. Let $K \subset C$ be any compact subset of the Kähler cone. Then there is a constant $B := B(K, \beta) > 0$ such that

$$\text{Ent}_{\alpha, J^{NA}}(\mathcal{X}, \mathcal{A}) \leq B$$

for all $\alpha \in K$ and all test configurations $(\mathcal{X}, \mathcal{A}) \in T_{C_{\beta, \alpha}}^{\text{opt}}$.

We proceed to prove similar results for the height and variation functions.

Nearly optimal test configurations have uniform height bounds on compact subsets of $C$. Let $K \subset C$ be a compact set and consider $\alpha \in K$. Let $(\mathcal{X}, \mathcal{A} := \mu^* p^*_1 \alpha + D) \in T_{\beta, \alpha}^{\text{opt}}$ be a relatively Kähler semistable and dominating test configuration for $(X, \alpha)$. As in previous sections, consider the family of test configurations given by

$$(\mathcal{X}, \mathcal{A}_t) := (\mathcal{X}, \mu^* p^*_1 t \alpha + [D]) \in T_{C_{t \alpha}}^{\text{opt}}, \quad t \geq 1.$$

Note that $(\mathcal{X}, \mathcal{A}_t)$ is not necessarily in $T_{C_{t \alpha}}^{\text{opt}}$ for all $t > 1$. On the other hand, by Proposition 3.24 we have the inclusion

$$T_{C_{\beta, \alpha}}^{\text{opt}} \subseteq T_{C_{t \alpha}}^{(B)},$$

so we may apply Corollary 3.22 with $\alpha' = \alpha$. Hence there is a constant $C := C(K) > 0$ such that the polynomial

$$P(t) := \frac{||(\mathcal{X}, \mathcal{A}_t)||_{t \alpha}}{||(\mathcal{X}, \mathcal{A}_1)||_{\alpha}}$$

satisfies

$$C^{-1} < P(t) \leq C$$

for all $t \in [1, 2]$ such that $t \alpha \in K$. Up to replacing $K$ with the larger compact set $K_{[1,2]} := \{t \alpha : \alpha \in C, t \in [1, 2]\}$, and replacing $C(K)$ by $C(K_{[1,2]})$, we obtain the above estimate for all $\alpha \in K$.

Using that derivation is a linear, thus bounded, operator on the finite dimensional vector space of polynomials of degree at most $d$, it is a standard exercise in functional analysis to establish the following fact about polynomials, adapted to the situation at hand:
Lemma 3.25. Suppose that $P \in \mathbb{R}[t]$ is a polynomial of degree at most $d$. Then there exists a constant $C = C(d)$ such that
\[
\max_{t \in [1,2]} \left| \frac{d}{dt} P(t) \right| \leq C(d) \max_{t \in [1,2]} |P(t)|.
\]

Recalling that
\[
h_\alpha(X, A) := \max_{t \in [1,2]} \left| \frac{d}{dt} P(t) \right|,
\]
we have thus proved the following:

Proposition 3.26. Let $K \subset C$ be any compact subset of the Kähler cone. Then there is a constant $B := B(K, \beta) > 0$ such that
\[
h_\alpha(X, A) \leq B
\]
for all $\alpha \in K$ and all test configurations $(X, A) \in TC_{\beta,\alpha}^{opt}$.

In other words, there is a suitable uniform bound $B > 0$ on the compact set $K$ such that restricting to test configurations with bounded height $h_\alpha(X, A) \leq B$ yields an equivalent notion of uniform K-stability for all $\alpha$ in $K$.

Nearly optimal configurations have uniform variation bounds on compact subsets of $C$. Fix as before a compact subset $K \subset C$ and let $\alpha \in K$. Let $(X, A) \in TC_{\beta,\alpha}^{opt}$ be any test configuration for $(X, \alpha)$. Consider the rational function
\[
R_\alpha(t) := \frac{H_{t\alpha}^{NA}(X, \mu^*P^*t\alpha + [D])}{||(X, \mu^*P^*t\alpha + [D])||_{t\alpha}}, \quad t \in [1, +\infty).
\]

Introduce also the following polynomials
\[
P_\alpha(t) := \frac{H_{t\alpha}^{NA}(X, A)}{||(X, A)||_{t\alpha}}, \quad Q_\alpha(t) := \frac{||(X, A)||_{t\alpha}}{||(X, A)||_{t\alpha}}, \quad t \in [1, +\infty),
\]
of degrees $\deg P \leq n + 1$ and $\deg Q \leq n + 1$, which are strictly positive for all $t$. Whenever $t\alpha \in K$ it then follows from Lemma 3.20 that
\[
|R_\alpha(t)| = \left| \frac{P_\alpha(t)}{Q_\alpha(t)} \right| \leq C
\]
As in the previous subsection there is a constant $C := C(K) > 0$ such that the following double inequality holds
\[
C^{-1} < Q_\alpha(t) < C
\]
for all $\alpha \in K$ and all $t \in [1, 2]$. As a consequence $P_\alpha$ is bounded from above, so $0 < P_\alpha(t) \leq D$ for some $D > 0$. In order to study the variation $v_\alpha(X, A) := \max_{t \in [1,2]} |\frac{d}{dt} R_\alpha(t)|$, we then bound
\[
\left| \frac{d}{dt} R(t) \right| = \frac{\frac{d}{dt} P(t) Q(t) - P(t) \frac{d}{dt} Q(t)}{Q^2(t)} \leq C^2 \left( \left| \frac{d}{dt} P(t) Q(t) \right| + \left| P(t) \frac{d}{dt} Q(t) \right| \right) \leq C^3 \left| \frac{d}{dt} P(t) \right| + C^2 D \left| \frac{d}{dt} Q(t) \right|.
\]

Invoking again the standard fact on polynomials, see Lemma 3.25, yields bounds also on $|\frac{d}{dt} P(t)|$ and $|\frac{d}{dt} Q(t)|$. We have therefore proven the following uniform variation bound on compact subsets of the Kähler cone:
Proposition 3.27. Let $K \subset C$ be any compact subset of the Kähler cone. Then there is a constant $B := B(K, n) > 0$ such that

$$v_\alpha(X, A) \leq B$$

for all $\alpha \in K$ and all $(X, A) \in \mathcal{T}C^\text{opt}_\alpha$.

Equivalence of stability under change in variation, height, normalized entropy and choice of norm. Recall that if $B$ is a positive real number, then by definition $\mathcal{T}C^{(B)}_{\alpha} \subset \mathcal{T}C_\alpha$ consists precisely of the test configurations for $(X, \alpha)$ that satisfy the height, variation and normalized entropy bounds $h_\alpha(X, A) \leq B$, $v_\alpha(X, A) \leq B$ and $\text{Ent}_{\alpha,\mathcal{J}^{\mathcal{N}A}}(X, A) \leq B$. Combining the above Proposition 3.24, Proposition 3.26 and Proposition 3.27 we obtain the following comparison result for the spaces of bounded and unbounded relatively Kähler test configurations respectively:

Proposition 3.28. For any $\beta \in H^{1,1}(X, \mathbb{R})$ and any compact subset $K \subset C$ there exists $B := B(K, \beta, n) > 0$ such that

$$\mathcal{T}C_{\beta,\alpha}^{\text{opt}} \subseteq \mathcal{T}C^{(B)}_{\alpha}$$

for every $\alpha \in K$.

As a direct consequence of Proposition 3.28 and the equivalence of norms (Corollary 3.23) we may now summarize our findings as follows:

Theorem 3.29. Let $\beta \in H^{1,1}(X, \mathbb{R})$ and $K \subset C$ a compact subset of the Kähler cone. Then there is a constant $B := B(K, \beta, n) > 0$, depending only on $K$ and the dimension of $X$, such that the following are equivalent for any $\alpha \in K$:

1. There is a $\delta_\alpha > 0$ such that

   $$M_{\beta}^{\mathcal{N}A}(X, A) \geq \delta_\alpha \| (X, A) \|_\alpha$$

   for all test configurations $(X, A) \in \mathcal{T}C^{(B)}_{\alpha}$ of bounded variation $v_\alpha(X, A) \leq B$, bounded height $h_\alpha(X, A) \leq B$, and bounded normalized entropy $\text{Ent}_{\alpha,\mathcal{J}^{\mathcal{N}A}}(X, A) \leq B$.

2. There is a $\delta_\alpha > 0$ such that

   $$M_{\beta}^{\mathcal{N}A}(X, A) \geq \delta_\alpha \| (X, A) \|_\alpha$$

   for all test configurations $(X, A) \in \mathcal{T}C_\alpha$.

3. There is a $\delta_\alpha > 0$ such that

   $$M_{\beta}^{\mathcal{N}A}(X, A) \geq \delta_\alpha \mathcal{J}^{\mathcal{N}A}_\alpha(X, A)$$

   for all test configurations $(X, A) \in \mathcal{T}C_\alpha$.

4. There is a $\delta_\alpha > 0$ such that

   $$M_{\beta}^{\mathcal{N}A}(X, A) \geq \delta_\alpha \mathcal{J}^{\mathcal{N}A}_\alpha(X, A)$$

   for all test configurations $(X, A) \in \mathcal{T}C_\alpha$.

If $\beta = -c_1(X)$, then $(X, \alpha)$ is uniformly K-stable if and only if there is a $\delta_\alpha > 0$ such that

$$DF(X, A) \geq \delta_\alpha \| (X, A) \|_\alpha$$

for all test configurations $(X, A) \in \mathcal{T}C^{(B)}_\alpha$. 

Theorem 3.30. Let $K \subset C$ be a compact subset and let $\beta \in H^{1,1}(X, \mathbb{R})$ be given. Let moreover $\sigma : \mathbb{R} \to \{-1, 0, 1\}$ be the sign function defined by $\sigma(0) = 0$ and $\sigma(x) = x/|x|$ if $x \neq 0$. Then

$$\sigma(\Delta(\alpha)) = \sigma(\Delta(\alpha))$$
for all $\alpha \in K$. Moreover, there exists $B = B(K, \beta, n) > 0$ such that
\[
\hat{\Delta}_\beta(\alpha) = \hat{\Delta}^B_\beta(\alpha)
\]
for all $\alpha \in K$.

**Remark 3.31.** For the proof of Theorem 1.1 we will apply this result to $\beta = -c_1(X)$.

**Proof.** By points 3) and 4) of Theorem 3.29 the threshold $\hat{\Delta}_\beta(\alpha) > 0$ if and only if $\Delta_\beta(\alpha) > 0$. The fact that the thresholds are (strictly) negative if and only if the other one is, follows from the positivity of the norms. The thresholds must therefore also vanish at the same time, concluding the proof of the first part.

For the second part, it can be proven exactly in the same way as in the proof of Proposition 3.19 that $(-\infty, 1] \ni t \mapsto \hat{\Delta}^B_{\beta_t}(\alpha)$ is linear (this is true more generally if we restrict to any subset $S_\alpha \subset \mathcal{T}C_\alpha$). It is moreover an immediate consequence of the definition that $\hat{\Delta}^B_{\beta_t}(\alpha) \geq \hat{\Delta}_{\beta_t}(\alpha)$ always holds, where $\beta_t := (1 - t)\beta + t\alpha$. Since
\[
\hat{\Delta}^B_\alpha(\alpha) = \hat{\Delta}_\alpha(\alpha) = 1
\]
it hence suffices to show that
\[
\sigma(\hat{\Delta}^B_\alpha(\alpha)) = \sigma(\hat{\Delta}_\alpha(\alpha))
\]
for all $t \leq 1$. But it follows from Theorem 3.29 that there exists $B = B(K, \beta, n)$ such that $\hat{\Delta}^B_{\beta_t}(\alpha) > 0$ precisely if $\hat{\Delta}_{\beta_t}(\alpha)$. Since they are both linear this yields (29), finishing the proof. \(\square\)

**Step 6: Completing the proof of openness of uniform K-stability.** Putting together the work of the above sections we now complete the proof of openness of uniform K-stability (Theorem 1.1). Fix as before a fixed choice of background metrics $\alpha \mapsto \omega_\alpha$ and consider the open set
\[
\check{U}_\epsilon := \{\eta \in C : \omega_\alpha < \omega_\eta < \sqrt{1 + \epsilon}\omega_\alpha\} \subset U_\epsilon.
\]
We now fix $\alpha \in C$ and a compact subset $K \subset C$ containing $\alpha$. The goal is to estimate the difference between $\hat{\Delta}(\alpha)$ and $\hat{\Delta}^B(\alpha)$, where $\alpha' \in C$ is a nearby Kähler class to $\alpha$ (here the hats refer to the corresponding stability thresholds with respect to $||.||_a$ rather than $\mathcal{J}_a^{\mathrm{NA}}$).

To recall the set up, consider any smooth and dominating relatively Kähler test configuration $(\mathcal{X}, \mathcal{A}', \mu, \pi)$ for $(X, \alpha')$, where $\mathcal{A}' = \mu^*p^*_1\alpha' + [D]$ for some $\mathbb{R}$-divisor $D$ supported on the central fiber $X_0 := \pi^{-1}(0)$, where $\pi : \mathcal{X} \to \mathbb{P}^1$ is the projection, $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ and $p_1 : X \times \mathbb{P}^1 \to X$ is the projection on the first factor, following the notation introduced at the beginning of the section. As before we assume that $\omega_{\alpha'} > \omega_\alpha$ and $A > 1$ is chosen so that $A\omega_\alpha > \omega_{\alpha'}$, which in particular implies that $\alpha' > \alpha$ and $A\alpha > \alpha'$. In particular, if we let $\eta$ be any of the Kähler classes $A\alpha, A\alpha'$ or $\alpha'$, then $(\mathcal{X}, \mu^*p^*_1\eta + [D])$ is a relatively Kähler test configuration for $(X, \eta)$.

By Theorem 3.30 we may moreover assume that the test configuration $(\mathcal{X}, \mathcal{A}')$ for $(X, \alpha)$ has height, variation and normalized entropy bounded above by some constant $B := B(K, \beta, n) > 0$ as explained in the previous section. Note that $\mathcal{T}C^{(B)}_\alpha$ is naturally included in the set $\mathcal{T}C_{A\alpha}$ of all relatively Kähler test configurations for $(X, A\alpha)$, since $A\alpha > \alpha'$ and we have explicitly $\mathcal{T}C^{(B)}_\alpha \to \mathcal{T}C_{A\alpha}$ given by $(\mathcal{X}, \mu^*p^*_1\alpha' + D) \mapsto (\mathcal{X}, \mu^*p^*_1\alpha + D)$. Concerning the desired estimates for the $M^{\mathrm{NA}}_\beta$ functional, it now suffices to obtain suitable
estimates for the energy and the entropy parts separately. For the energy part \( \mathcal{J}^{\eta}_{\beta} \) we use the estimates of Section 2.2, Step 2 (keeping in mind along the way that the Ricci curvature form is invariant under rescaling, i.e. \( \text{Ric}(A\omega) = \text{Ric}(\omega) \)). Indeed, let \( \epsilon > 0 \) and consider the open neighbourhood \( \mathcal{U} \) of \( \alpha \) defined as before by
\[
\mathcal{U} := \{ \eta \in C : \{ 1 - \epsilon \} \omega_\alpha < \omega_\eta < \{ 1 + \epsilon \} \omega_\alpha \}.
\]
Now assume that \( \alpha, \alpha', A\alpha, A\alpha' \) are all in \( \mathcal{U} \) (as is always possible while still satisfying \( \omega_\alpha < \omega_{\alpha'} < \sqrt{1 + \epsilon} \omega_\alpha \), take e.g. \( \alpha' \in \{ \beta \in C : \omega_\beta < \omega_\beta < \sqrt{1 + \epsilon} \omega_\alpha \} \) and \( A = \sqrt{1 + \epsilon} \).

By Corollary 3.4 there exists a continuous function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( f(0) = 0 \) such that
\[
\mathcal{J}^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha' + [D]) - \mathcal{J}^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) < f(\epsilon) (\mathcal{J}^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha + [D]))^2 \leq f(\epsilon) ||(\mathcal{X}, \mu^* p^*_A \alpha + [D])||_{\alpha A}
\]
for every relatively Kähler semistable and dominating test configuration \( (\mathcal{X}, \mu^* p^*_A \alpha + [D]) \) for \( (X, \alpha') \). In particular, we here used that \( H^{\eta}_{\alpha A} \) is always positive, and
\[
\mathcal{J}^{\eta}_{\alpha A} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) - \mathcal{J}^{\eta}_{\alpha A} (\mathcal{X}, \mu^* p^*_A \alpha' + [D]) < f(\epsilon) \mathcal{J}^{\eta}_{\alpha A} (\mathcal{X}, \mu^* p^*_A \alpha' + [D])
\]
holds for any \( \alpha' \in \mathcal{U} \).

We also note that
\[
-\mathcal{J}^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) \leq -\Delta^{pp}_{\beta} (A\alpha) \mathcal{J}^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) \leq \leq ||\Delta^{pp}_{\beta} (A\alpha)||_{\alpha A} ||(\mathcal{X}, \mu^* p^*_A \alpha + [D])||_{\alpha A}.
\]
For the entropy part \( H^{\eta}_{\alpha A} \) we make a similar estimate: By normalized entropy we may apply the same exact argument as in the proof of Corollary 3.22 to deduce that there is a constant \( C(\epsilon) := C(\beta, \epsilon) > 0 \) such that
\[
||((\mathcal{X}, \mu^* p^*_A \alpha + [D]))||_{\alpha A} \leq C(\epsilon) ||((\mathcal{X}, \mu^* p^*_A \alpha + [D]))||_{\alpha A}.
\]
Using that \( \mathcal{J}^{\eta}_{\alpha A} (\mathcal{X}, \mathcal{A}) \leq C||((\mathcal{X}, \mathcal{A}))||_{\alpha A} \) for some absolute constant \( C > 0 \), it moreover follows from Corollary 3.4, Corollary 3.12 that
\[
H^{\eta}_{\alpha A} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) < V_{\alpha'} V_{\alpha}^{-1} H^{\eta}_{\alpha A}(\mathcal{X}, \mu^* p^*_A \alpha' + [D])
\]
\[
\leq V_{\alpha'} V_{\alpha}^{-1} H^{\eta}_{\alpha A}(\mathcal{X}, \mathcal{A}) + G(\epsilon, \mathcal{A}, \mathcal{A}') ||((\mathcal{X}, \mu^* p^*_A \alpha + [D]))||_{\alpha A},
\]
where
\[
G(\epsilon, \mathcal{A}, \mathcal{A}') := C(\epsilon)(1 + f(\epsilon)) V_{\alpha'} V_{\alpha}^{-1} g(A)
\]
for some continuous function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( f(0) = 0 \), and
\[
g(A) = \max(A - 1 + (A - 1)^2 h_{\alpha'}(\mathcal{X}, \mathcal{A}'); (A - 1) h_{\alpha'}(\mathcal{X}, \mathcal{A}')) \left[ v_{\alpha'}(\mathcal{X}, \mathcal{A}') + \frac{H^{\eta}_{\alpha A}(\mathcal{X}, \mathcal{A})}{||(\mathcal{X}, \mathcal{A}')||_{\alpha A}} \right].
\]
using the notation in Corollary 3.12. Putting together the above estimates we obtain
\[
M^{\eta}_{\beta} (\mathcal{X}, \mu^* p^*_A \alpha + [D]) - V_{\alpha'} V_{\alpha}^{-1} M^{\eta}_{\beta}(\mathcal{X}, \mu^* p^*_A \alpha' + [D]) \leq
\]
\[
\leq V_{\alpha'} V_{\alpha}^{-1} \left( \mathcal{J}^{\eta}_{\beta}(\mathcal{X}, \mu^* p^*_A \alpha + [D]) - \mathcal{J}^{\eta}_{\beta}(\mathcal{X}, \mu^* p^*_A \alpha' + [D]) \right) -
\]
\[
(V_{\alpha'} V_{\alpha}^{-1} - 1) \mathcal{J}^{\eta}_{\beta}(\mathcal{X}, \mu^* p^*_A \alpha + [D]) +
\]
\[
H^{\eta}_{\alpha A}(\mathcal{X}, \mu^* p^*_A \alpha + [D]) - V_{\alpha'} V_{\alpha}^{-1} H^{\eta}_{\alpha A}(\mathcal{X}, \mu^* p^*_A \alpha' + [D]) \leq
\]
Because of the assumed height and variation bounds we have
\[ G(\epsilon, A, X, \mathcal{A}') \leq C(\epsilon)(1 + f(\epsilon))VR_{\alpha}V_{\alpha}^{-1}(B + 1) \max(A - 1 + (A - 1)^2; (A - 1)B). \]

We can always fix a large enough compact set \( K \subset C \) containing \( \mathcal{U}_c \) for all \( \epsilon \) we consider (take e.g. \( K := \overline{\mathcal{U}_c} \)), and then all the quantities \( C(\epsilon), f(\epsilon) \) and \( G(\epsilon, A, X, \mathcal{A}') \) are non-negative and bounded above on this compact. By (30) we can thus pick a \( \lambda := \lambda(K) > 0 \) such that
\[ \hat{\Delta}_\beta(A\alpha) < \lambda VR_{\alpha}V_{\alpha}^{-1}\hat{\Delta}_\beta(\alpha') + (V_{\alpha}V_{\alpha}^{-1} + \lambda_B)f(\epsilon) + (V_{\alpha}V_{\alpha}^{-1} - 1)A^{-1}|\Delta^{pp}_\beta(\alpha)|, \] where the constant \( \lambda_B > 0 \) is any uniform upper bound for \( G(\epsilon, A, X, \mathcal{A}') \) over \( K \). Now note that if \( (X, \alpha) \) is uniformly K-stable, then \( \hat{\Delta}(\alpha) \geq 0 \) (in fact \( \hat{\Delta}(\alpha) > 0 \), see Step 5). As a consequence, it can then be easily checked that
\[ A^{-1}\hat{\Delta}_\beta(\alpha) < \hat{\Delta}_\beta(A\alpha) \]
for \( A \geq 1 \). The above inequality (31) can therefore be rewritten in the following form
\[ \hat{\Delta}_\beta(\alpha) < \lambda AV_{\alpha}V_{\alpha}^{-1}\hat{\Delta}_\beta(\alpha') + (V_{\alpha}V_{\alpha}^{-1} - 1)|\Delta^{pp}_\beta(\alpha)| + f_B(\epsilon), \]
where \( f_B : \mathbb{R}_+ \to \mathbb{R} \) is a continuous function satisfying \( f_B(0) = 0 \), incorporating the constants \( \lambda \) and \( \lambda_B \) into the notation, and recalling that if \( B := B(K, \beta, n) > 0 \) is chosen large enough, then we have equality of stability thresholds \( \hat{\Delta}^B = \hat{\Delta} \) by Theorem 3.30.

Taking \( A = \sqrt{1 + \epsilon} \) and \( \omega_\alpha < \omega_{\alpha'} < \sqrt{1 + \epsilon}\omega_\alpha \) as described at the beginning of the section (and setting \( f_K := f_{B(K, \beta, n)} \)) the following relationship between stability thresholds can be deduced for any Kähler class \( \alpha' \in \hat{\mathcal{U}}_c := \mathcal{U}_c \cap K \):

**Proposition 3.32.** Suppose that \( K \) is a compact subset of the \( \mathcal{U}_c \)-cone and let \( \beta \in H^{1,1}(X, \mathbb{R}) \). Fix \( \alpha \in K \) such that \( (X, \alpha) \) is uniformly K-stable. Then there exists \( \lambda := \lambda(K) > 0 \) and a continuous function \( f_K : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( f_K(0) = 0 \), such that any \( \alpha' \in \hat{\mathcal{U}}_c \cap K \) satisfies the following inequality
\[ \hat{\Delta}_\beta(\alpha) < \lambda \sqrt{1 + \epsilon}V_{\alpha}V_{\alpha}^{-1}\hat{\Delta}_\beta(\alpha') + (V_{\alpha}V_{\alpha}^{-1} - 1)|\Delta^{pp}_\beta(\alpha)| + f_K(\epsilon). \]

In light of the above we are now finally ready to state and prove the following main result on openness of uniform K-stability:

**Theorem 3.33.** The uniform K-stability locus
\[ \text{UKs} := \{ \alpha \in C : \Delta(\alpha) > 0 \} \]
is an open subcone of \( C \).

**Proof.** Fix \( \alpha \in \text{UKs} \) and apply the above results with \( \beta = -c_1(X) \) (omitting the indices to alleviate notation). Suppose that \( \Delta(\alpha) > 0 \). By Proposition 3.32 we then have
\[ \lambda \hat{\Delta}(\alpha') > \frac{V_{\alpha}}{\sqrt{1 + \epsilon}V_{\alpha'}} \left( \hat{\Delta}(\alpha') - (V_{\alpha}V_{\alpha}^{-1} - 1)|\Delta^{pp}_\beta(\alpha)| - f_K(\epsilon) \right) \]
for all \( \alpha' \in \hat{\mathcal{U}}_c \). By continuity of \( f_K(\epsilon) \) and the volume \( V_{\alpha'} \), it follows that there exists an \( \epsilon := \epsilon(\alpha) > 0 \) such that \( \hat{\Delta}(\alpha') > 0 \) for all \( \alpha' \) in the open set \( \mathcal{U}_c \). By the sign comparison in Theorem 3.29 it moreover follows that \( \Delta(\alpha') > 0 \) for all \( \alpha' \in \hat{\mathcal{U}}_c \). Finally, this means that \( \Delta(\alpha') = r^{-1}\Delta(\alpha') > 0 \) for all \( r > 0 \), so the cone \( \text{Cone}(\mathcal{U}_c) := \mathbb{R}_+\mathcal{U}_c \) is an
open neighbourhood of $\alpha$ which is contained in the uniform K-stability locus $\text{UKs}$. This completes the proof.

The same kind of arguments clearly hold also when $\beta \neq -c_1(X)$, leading to openness also of twisted uniform K-stability, as remarked on below.

**The case of twisted uniform K-stability.** We say that $(X, \alpha)$ is uniformly twisted K-stable with respect to $[\theta] \in H^{1,1}(X, \mathbb{R})$ if there exists $\delta > 0$ such that

$$M_N^{\alpha c_1(X)+[\theta]}(\mathcal{X}, \mathcal{A}) \geq \delta J_N^{\alpha c_1(X)+[\theta]}(\mathcal{X}, \mathcal{A})$$

for all $(\mathcal{X}, \mathcal{A}) \in \mathcal{T}_C\alpha$. Following [25] this is the uniform stability notion naturally associated with existence of so called twisted cscK metrics, which in turn is known to be equivalent to coercivity of the $J_{\gamma}^{\omega}$-functional, where $\gamma := -\text{Ric}(\omega) + \theta$, see [12, 13].

Studying this uniform stability notion corresponds to understanding the stability threshold $\Delta_{c_1(X)+[\theta]}$. Since the same exact arguments apply for $\Delta_{\beta}(\alpha)$ for any $\beta \in H^{1,1}(X, \mathbb{R})$, we obtain the analogous estimates and openness results also in this slightly more general setting. In particular, we have the following:

**Theorem 3.34.** Fix $[\theta] \in H^{1,1}(X, \mathbb{R})$. Then the set

$$\text{UKs}_{c_1(X)+[\theta]} := \{ \alpha \in \mathcal{C} : \Delta_{c_1(X)+[\theta]}(\alpha) > 0 \}$$

is an open subcone of $\mathcal{C}$.

As a byproduct of the proof of Theorem 1.1 we also obtained estimates involving several other functionals of independent interest, yielding openness of uniform $J$-stability, giving estimates involving non-archimedean entropy (the analytic delta invariant), and proving continuity of several stability thresholds of interest.

4. Continuity and further results

Along the course of the proof we have proved estimates that have a number of applications of their own, some of which are briefly highlighted below.

**Proof of continuity of $\hat{\Delta}$.** We now note that the proof of Theorem 3.33 in fact shows that the stability threshold associated to the norm $||\cdot||_{\alpha}$ is continuous:

**Theorem 4.1.** For any $\beta \in H^{1,1}(X, \mathbb{R})$ the stability threshold function

$$\mathcal{C} \ni \alpha \mapsto \hat{\Delta}_\beta(\alpha)$$

is continuous.

The proof first requires the following extension of Proposition 3.32 to the case when $\hat{\Delta}(\alpha)$ is not assumed to be positive:

**Proposition 4.2.** Suppose that $K$ is a compact subset of the Kähler cone and let $\beta \in H^{1,1}(X, \mathbb{R})$. Fix $\alpha \in K$ such that $(X, \alpha)$ is uniformly K-stable. Then there exists $\lambda := \lambda(K) > 0$ and a continuous function $f_K : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_K(0) = 0$, such that any $\alpha' \in \hat{U}_\epsilon \cap K$ satisfies the inequality

$$\hat{\Delta}_\beta(\alpha) < \max\{ \sqrt{1 + \epsilon} L_\epsilon(\alpha'), L_\epsilon(\alpha') \},$$

where

$$L_\epsilon(\alpha') := \lambda V_\alpha V_\alpha^{-1} \hat{\Delta}_\beta(\alpha') + (V_\alpha V_\alpha^{-1} - 1) \frac{1}{\sqrt{1 + \epsilon}} |\Delta_{\beta}^{pp}(\alpha)| + f_K(\epsilon).$$
Proof. Using the notation of Section 3, Step 6, we then recall that
\[
\tilde{\Delta}_\beta(\sqrt{1+\epsilon \alpha}) < \lambda V'_\alpha V^{-1}_\alpha \tilde{\Delta}_\beta(B'(\alpha')) + (V'_\alpha V^{-1}_\alpha + \lambda_B) f(\epsilon) + \lambda(\alpha V'_\alpha V^{-1}_\alpha - 1) \frac{1}{\sqrt{1+\epsilon}} |\Delta_{\beta}^{pp}(\alpha)|.
\]
By Theorem 3.30, we may as before choose \(B_K := B(K, n) > 0\) large enough so that \(\tilde{\Delta}_\beta(B'(\alpha')) = \tilde{\Delta}_\beta(\alpha')\) and we set \(f_K(\epsilon) := (V'_\alpha V^{-1}_\alpha + \lambda_{B_K}) f(\epsilon)\). With this setup we thus have
\[
\tilde{\Delta}_\beta(\sqrt{1+\epsilon \alpha}) < L_\epsilon(\alpha').
\]
To complete the proof it remains to compare \(\tilde{\Delta}(\sqrt{1+\epsilon \alpha})\) and \(\tilde{\Delta}(\alpha)\). If \(t \geq 1\) and \(M_{\beta}^{NA}(\mathcal{X}, \mathcal{A}) \geq 0\) then
\[
\frac{M_{\beta}^{NA}(\mathcal{X}, t\mathcal{A})}{|\mathcal{X}, t\mathcal{A}|} = \frac{J_{\beta}^{NA}(\mathcal{X}, \mathcal{A}) + H_{\beta}^{NA}(\mathcal{X}, \mathcal{A})}{t J_{\beta}^{NA}(\mathcal{X}, \mathcal{A}) + H_{\beta}^{NA}(\mathcal{X}, \mathcal{A})} \leq \frac{M_{\beta}^{NA}(\mathcal{X}, \mathcal{A})}{|\mathcal{X}, \mathcal{A}|}.
\]
If instead \(t \geq 1\) and \(M_{\beta}^{NA}(\mathcal{X}, \mathcal{A}) < 0\), then in the same way
\[
\frac{M_{\beta}^{NA}(\mathcal{X}, \mathcal{A})}{|\mathcal{X}, \mathcal{A}|} \leq t^{-1} \frac{M_{\beta}^{NA}(\mathcal{X}, \mathcal{A})}{|\mathcal{X}, \mathcal{A}|}.
\]
Along these lines it is straightforward to check that we have the following double inequality
\[
\min \left\{ t\tilde{\Delta}_\beta(t\alpha), \tilde{\Delta}_\beta(t\alpha) \right\} \leq \tilde{\Delta}_\beta(\alpha) \leq \max \left\{ t\tilde{\Delta}_\beta(t\alpha), \tilde{\Delta}_\beta(t\alpha) \right\}
\]
for all \(\alpha \in \mathcal{C}\) and all \(t \geq 1\). The upper inequality then yields
\[
\tilde{\Delta}_\beta(\alpha) < \max\{\sqrt{1+\epsilon L_\epsilon(\alpha')}, L_\epsilon(\alpha')\},
\]
finishing the proof. \(\square\)

The main continuity result follows:

Proof of Theorem 4.1. Consider the open set \(\mathcal{V}_\epsilon = \{\beta \in \mathcal{C} : (1-\epsilon)\omega_\alpha < \omega_\beta < (1+\epsilon)\alpha\} \subset \mathcal{C}\), then for any \(\epsilon > 0\) this is an open neighbourhood of \(\alpha\). Furthermore, if \(\delta > 0\) is any given positive real number, then for any \(\alpha' \in \mathcal{V}_\epsilon\) and all \(\epsilon > 0\) small enough, we have \((1-\epsilon)^{-1}\alpha' \in \mathcal{U}_{r(\epsilon)}\), where
\[
\nu(\epsilon) := \left(\frac{1+\epsilon}{1-\epsilon}\right)^2 - 1
\]
and \(\mathcal{U}_{r}\) is defined as in (16). Since \(r(\epsilon) \to 0\) as \(\epsilon \to 0\) it follows from Proposition 4.2 that we may choose \(\epsilon > 0\) small enough so that
\[
\tilde{\Delta}_\beta(\alpha) < \max\{\sqrt{1+\epsilon L_\epsilon(\alpha')}, L_\epsilon(\alpha')\} < \tilde{\Delta}_\beta(\alpha') + \delta,
\]
for every \(\alpha' \in \mathcal{V}_\epsilon\). This proves lower semi-continuity of \(\alpha \mapsto \tilde{\Delta}_\beta(\alpha)\).

To prove upper semi-continuity we fix \(\alpha \in \mathcal{C}\) and associate to each \(\alpha' \in \mathcal{C}\) the smallest positive real number \(\lambda_{\alpha'}\) for which \(\lambda_{\alpha'} \alpha' \geq \alpha\) (i.e. \(\lambda_{\alpha'} \alpha' - \alpha\) is nef). It then suffices to note that if \((\mathcal{X}, \mathcal{A}) := \mu^* \mu_\alpha^{-1} + [D]) \in \mathcal{T}\mathcal{C}_\alpha\) is a relatively Kähler smooth and dominating test configuration for \((X, \alpha)\), then \((\mathcal{X}, \mu^* \mu_\alpha^{-1} + \lambda_{\alpha'}^{-1}[D])\) is a relatively Kähler smooth and dominating test configuration for \((X, \alpha')\), and moreover the function
\[
\mathcal{C} \ni \alpha' \mapsto \frac{M_{\beta}^{NA}(\mathcal{X}, \mu^* \mu_\alpha^{-1} + \lambda_{\alpha'}^{-1}[D])}{|\mathcal{X}, \mu^* \mu_\alpha^{-1} + \lambda_{\alpha'}^{-1}[D]|} ||| \alpha'
\]
is continuous. By definition of \(\tilde{\Delta}_\beta(\alpha')\) the upper semi-continuity follows. This finishes the proof. \(\square\)
On delta invariants and continuity of $\hat{\Delta}^H$. It is interesting to study the variation also of threshold associated with the non-archimedean entropy $H_{NA}$, i.e.

$$\hat{\Delta}^H(\alpha) := \sup\{\delta \in \mathbb{R} : H_{NA}(\mathcal{X}, A_\alpha) \geq \delta ||(\mathcal{X}, A_\alpha)||_\alpha, \forall (\mathcal{X}, A_\alpha) \in TC_\alpha\}$$

By Proposition 3.28 and the estimates for $H_{NA}$ used in Section 3, Step 5, we obtain for every $\alpha' \in \tilde{U}_\epsilon \subset K$ an inequality of the following type:

**Proposition 4.3.** Fix $\alpha \in C$ and consider the open set $\tilde{U}_\epsilon$ as before. Then there exists $\lambda := \lambda(K) > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ such that

$$\hat{\Delta}^H(\alpha) < \lambda \sqrt{1 + \epsilon} \hat{\Delta}^H(\alpha') + f(\epsilon)$$

for every $\alpha' \in \tilde{U}_\epsilon$.

As a consequence we have proved the following:

**Theorem 4.4.** The function

$$\mathcal{C} \ni \alpha \mapsto \hat{\Delta}^H(\alpha)$$

is continuous.

The above continuity result can be compared to the recent work of Zhang [56], which establishes continuity of an analytic analogue of the threshold $\Delta^H$, referred to as the analytic delta invariant. It would be an interesting question to investigate the precise relationship between $\hat{\Delta}^H$, $\Delta^H$, the analytic delta invariant and Tian’s delta invariant.

**Openness of J-stability.** It is also worth pointing out that the above techniques yield openness results for a natural uniform stability notion related the J-equation introduce by Donaldson [29] and X.X. Chen [11]. We refer to [51, 38, 16, 10] and references therein for background on this equation, which is in turn related to the deformed Hermitian-Yang-Mills equation (see e.g. the survey [15]). In the case of surfaces this follows immediately from the numerical existence criterion of [11]. In general, we here note that Corollary 3.4 immediately implies openness of the natural uniform stability notion related to $J_{\beta}^{NA}$, which yields the following result:

**Theorem 4.5.** For any fixed $\beta \in H^{1,1}(X, \mathbb{R})$ the set

$$UJ_{\beta} := \{\alpha \in \mathcal{C} : (X, \beta, \alpha) \text{ is uniformly J-stable}\}
= \{\alpha \in \mathcal{C} : \Delta^{pp}_\beta(\alpha) > 0\}$$

is an open subcone of $\mathcal{C}$.

It is moreover a direct consequence of Step 1 of the proof (see in particular Corollary 3.4) that the associated stability threshold function

$$\mathcal{C} \ni \alpha \mapsto \Delta^{pp}_\beta(\alpha)$$

is continuous.

**Remark 4.6.** By recent work of Datar-Pingali [21] it moreover follows that uniform J-stability is equivalent to J-stability for projective mani folds: Suppose that $X$ is projective and fix $\beta \in H^{1,1}(X, \mathbb{R})$. Then the set of $\alpha \in \mathcal{C}$ such that $(X, \beta, \alpha)$ is J-stable is an open subcone of $\mathcal{C}$. 


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