Mathematical Remarks on the Feynman Path Integral for Nonrelativistic Quantum Electrodynamics

Wataru Ichinose*

Department of Mathematical Science, Shinshu University, Matsumoto
390-8621, Japan.
E-mail: ichinose@math.shinshu-u.ac.jp

Abstract: The Feynman path integral for nonrelativistic quantum electrodynamics is studied mathematically of a standard model in physics, where the electromagnetic potential is assumed to be periodic with respect to a large box and quantized thorough its Fourier coefficients. In physics, the Feynman path integral for non-relativistic quantum electrodynamics is defined very formally. For example, as is often seen, even independent variables are not so clear. First, the Feynman path integral is defined rigorously under the constraints familiar in physics. Secondly, the Feynman path integral is also defined rigorously without the constraints, which is stated in Feynman and Hibbs’ book without any comments. So, our definition may be completely new. Thirdly, the vacuum and the state of photons of momentums and polarization states are expressed by means of concrete functions of variables consisting of the Fourier coefficients of the electromagnetic potential. Our results above have many applications as is seen in Feynman and Hibbs’ book, though the

*Research partially supported by Grant-in-Aid for Scientific Research No.16540145 and No.19540175, Ministry of Education, Culture, Sports, Science and Technology, Japanese Government.
applications are not rigorous so far. It is also proved rigorously by means of the
distribution theory that the Coulomb potentials between charged particles natu-
really appear in the Feynman path integral above. As is well known, this shows that
photons give the Coulomb forth.

1 Introduction

A number of mathematical results on the Feynman path integral for quan-
tum mechanics have been obtained. On the other hand, the author doesn’t
know any mathematical results on the Feynman path integral for quantum
electrodynamics (cf. [17]), written as QED from now on.

A functional integral representation for a nonrelativistic QED model in
[20] with imaginary time was obtained by Hiroshima [11] on the Fock spaces
in terms of the probabilistic method. It has been well known that the only
translation invariant measure on a separable infinite dimensional Banach space
is the identically zero measure (cf. Theorem 4 in §5 of Chapter 4 in [21]).
The measure defining the Feynman path integral is expected to be translation
invariant (cf. (7-29) in [8]). So, there exist no measures defining the Feynman
path integral.

In the present paper the Feynman path integral for nonrelativistic QED
is studied rigorously of a standard model in physics (cf. [4, 5, 7, 8, 20, 22]),
where the electromagnetic potential is assumed to be periodic with respect to
a large box in $R^3$ and quantized thorough its Fourier coefficients. In physics,
the Feynman path integral for nonrelativistic QED is defined very formally.
For example, as is often seen (cf. [8]), even independent variables are not so
clear. Our aim in the present paper is to give the mathematical definition
of the Feynman path integral for nonrelativistic QED of a standard model in
physics. We note that in the present paper, regrettably, the Fourier coefficients
with large wave numbers need to be arbitrarily cut off and we don’t take the
limit of a box to $R^3$. We also note that our nonrelativistic QED model is
completely different from nonrelativistic QED models on the Fock spaces (cf.
[10] [11] [24]).
First, the mathematical definition of the Feynman path integral for nonrelativistic QED is given under the constraints. These constraints are well known (cf. (9-17) in [8], (A-7) in [22], (13.10) in [24] and (7.38) in [25]).

Secondly, without the constraints we give the mathematical definition of the Feynman path integral for nonrelativistic QED, which has been given by (9-98) in [8] without any comments. Our method of giving the Feynman path integral for nonrelativistic QED without the constraints is like one used in [15] for giving the phase space Feynman path integral. The author emphasize that any definitions of (9-98) in [8] have not been given. So our result may be completely new. We note that our Feynman path integral without the constraints is proved to be equal with the Feynman path integral under the constraints before taking the limit of the discretization parameter.

Thirdly, the vacuum and the states of photons of momentums and polarization states are expressed by means of concrete functions of variables consisting of the Fourier coefficients of the electromagnetic potential. In [8] only the vacuum and the state of a photon with a momentum and a polarization state are expressed by means of the concrete functions, which our functions are equal to. Generally, in physics the vacuum and the states of photons with momentums and polarization states are not considered concretely but considered abstractly (cf. [22, 25]). To write down the state of photons concretely, we introduce creation operators and annihilation operators, which can be written concretely as partial differential operators of the first order.

The results stated above have many applications as is seen in the chapter 9 of [8], though the applications are not rigorous so far.

Fourthly, we show by means of the distribution theory that the Coulomb potentials between charged particles appear when the periods of the Fourier series tend to infinity and the cut-off of the Fourier coefficients is gotten out. This result, which shows that photons give the Coulomb forth, is well known in physics (cf. [5, 8]). In the present paper we give the rigorous proof.

The proof of giving a mathematical definition of the Feynman path integral for nonrelativistic QED under or without the constraints is obtained by means of a somewhat delicate study on oscillatory integral operators, the abstract Accoli-Arzelà theorem on the weighted Sobolev spaces and the uniqueness to
the initial problem for the Schrödinger type equations as in [13, 14, 15, 16].

The proof of expressing the vacuum and the states of photons with momentums and polarization states by means of concrete functions is as follows. We take $e^{ik \cdot x}$ $(k, x \in \mathbb{R}^3)$ as the Fourier functions. Then, annihilation operators are defined for the real parts and the imaginary parts of the Fourier coefficients, respectively. Combining the annihilation operators for the real parts and the imaginary parts, we can define the annihilation operators of photons. The creation operators are defined as the adjoint operators of the annihilation operators.

The proof of the appearance of the Coulomb potentials between charged particles is given by proving the convergence theorem for the Riemann sum of a unbounded function as the discretization parameter tends to zero, which will be stated in Proposition 4.3 in the present paper.

Our plan in the present paper is as follows. §2 is devoted to preliminaries. In §3 the main results on the Feynman path integral for nonrelativistic QED are stated. In §4 the appearance of the Coulomb potentials between charged particles is proved rigorously. In §5 the vacuum and the states of photons with momentums and polarization state are given concretely. §6 - §9 are devoted to proofs of the main results stated in §3.

2 Preliminaries

We consider $n$ charged nonrelativistic particles $x^{(j)} \in \mathbb{R}^3$ ($j = 1, 2, \ldots, n$) with mass $m_j > 0$ and charge $e_j \in \mathbb{R}$. Let $T > 0$ be an arbitrary constant, $t \in [0, T], x \in \mathbb{R}^3, \phi(t, x) \in \mathbb{R}$ a scalar potential and $A(t, x) \in \mathbb{R}^3$ a vector potential. We set

$$\vec{x} := (x^{(1)}, \ldots, x^{(n)}) \in \mathbb{R}^{3n},$$
$$\dot{\vec{x}} := (\dot{x}^{(1)}, \ldots, \dot{x}^{(n)}) \in \mathbb{R}^{3n}.$$  

Then the Lagrangian function for particles and the electromagnetic field with the charge density

$$\rho(t, x) = \sum_{j=1}^{n} e_j \delta \left( x - x^{(j)}(t) \right) \quad (2.1)$$
and the current density

\[ j(t, x) = \sum_{j=1}^{n} e_j \dot{x}^{(j)}(t) \delta \left( x - x^{(j)}(t) \right) \in \mathbb{R}^3, \quad \dot{x}^{(j)}(t) = \frac{dx^{(j)}}{dt}(t) \tag{2.2} \]

is given by

\[
\mathcal{L} \left( t, \vec{x}, \vec{v}, \vec{A}, \frac{\partial A}{\partial x}, \frac{\partial \phi}{\partial x} \right) \\
= \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 - \int \rho(t, x) \phi(t, x) dx + \frac{1}{c} \int j(t, x) \cdot A(t, x) dx \\
+ \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 - |B(t, x)|^2 \right) dx + C \\
= \sum_{j=1}^{n} \left( \frac{m_j}{2} |\dot{x}^{(j)}|^2 - e_j \phi(t, x^{(j)}) + \frac{1}{c} e_j \dot{x}^{(j)} \cdot A(t, x^{(j)}) \right) \\
+ \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( |E(t, x)|^2 - |B(t, x)|^2 \right) dx + C \tag{2.3}
\]

(cf. \[8, 24\]), where

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \frac{\partial \phi}{\partial x}, \quad B = \nabla \times A, \tag{2.4}
\]

\[
\frac{\partial \phi}{\partial x} = (\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}) \quad \text{and} \quad C \quad \text{is an indefinite constant. It seems that an indefinite constant in (2.3) has not been used by anyone before (cf. \[8, 22, 24\]).}
\]

As in \[5, 7, 20, 22\] we consider a sufficient large box

\[
V = \left[ -\frac{L_1}{2}, \frac{L_1}{2} \right] \times \left[ -\frac{L_2}{2}, \frac{L_2}{2} \right] \times \left[ -\frac{L_3}{2}, \frac{L_3}{2} \right] \subset \mathbb{R}^3.
\]

As variables we consider all periodic potentials \( \phi(t, x) \) and \( A(t, x) \) in \( x \in \mathbb{R}^3 \) with periods \( L_1, L_2 \) and \( L_3 \) satisfying

\[
\nabla \cdot A(t, x) = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^3 \quad \text{(the Coulomb gauge)} \tag{2.5}
\]

and also

\[
\int_V \phi(t, x) dx = 0, \quad \int_V A(t, x) dx = 0. \tag{2.6}
\]
Let $|V| = L_1L_2L_3$. We set
\[
    k := \left( \frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3 \right) \quad (s_1, s_2, s_3 = 0, \pm 1, \pm 2, \ldots) \tag{2.7}
\]
and take $e_j^e(k) \in R^3$ ($j = 1, 2$) such that $(e_1^e(k), e_2^e(k), k/|k|)$ for all $k \neq 0$ forms a set of mutually orthogonal unit vectors and
\[
e_j^e(-k) = -e_j^e(k) \quad (j = 1, 2). \tag{2.8}
\]
We note that we can easily determine measurable functions $e_j^e(k) \in R^3$ ($k \in R^3, j = 1, 2$) by the Gram and Schmidt method such that $(e_1^e(k), e_2^e(k), k/|k|)$ for all $k \neq 0$ forms a set of mutually orthogonal unit vectors and satisfies (2.8) (cf. p. 448 in [1]). Noting (2.5) and (2.6), we can expand $\phi(t, x)$ and $A(t, x)$ formally into the Fourier series
\[
    A(x, \{a_{lk}(t)\}) = \sqrt{\frac{4\pi}{|V|}} c \sum_{k \neq 0} \left\{ a_{1k}(t)e^{ik \cdot x}e_1^e(k) + a_{2k}(t)e^{ik \cdot x}e_2^e(k) \right\}, \tag{2.9}
\]
\[
    \phi(x, \{\phi_k(t)\}) = \frac{1}{|V|} \sum_{k \neq 0} \phi_k(t)e^{ik \cdot x}. \tag{2.10}
\]

**Remark 2.1.** In physics (cf. [8, 22]) the condition (2.6) is not assumed clearly.

We write
\[
    a_{lk} := a_{lk}^{(1)} - ia_{lk}^{(2)} \quad l = 1, 2, \tag{2.11}
\]
\[
    \phi_k := \phi_k^{(1)} - i\phi_k^{(2)}, \tag{2.12}
\]
where $a_{lk}^{(i)} \in R$ and $\phi_k^{(i)} \in R$, and also the complex conjugate of $a_{lk}$ as $a_{lk}^*$. Since $A$ and $\phi$ are real valued, the relations
\[
    a_{1-k}^{(1)} = -a_{lk}^{(1)}, \quad a_{1-k}^{(2)} = a_{lk}^{(2)}, \quad \phi_{-k}^{(1)} = \phi_k^{(1)}, \quad \phi_{-k}^{(2)} = -\phi_k^{(2)} \tag{2.13}
\]
hold from (2.8). So, from (2.9) and (2.10) we have
\[
    A(x, \{a_{lk}\}) = \sqrt{\frac{4\pi}{|V|}} c \sum_{k \neq 0} \sum_{l=1}^2 \frac{1}{\sqrt{2}} (a_{lk}^{(1)} \cos k \cdot x + a_{lk}^{(2)} \sin k \cdot x) e_1^e(k), \tag{2.14}
\]
\[
    \phi(x, \{\phi_k\}) = \frac{1}{|V|} \sum_{k \neq 0} (\phi_k^{(1)} \cos k \cdot x + \phi_k^{(2)} \sin k \cdot x). \tag{2.15}
\]
We also write
\[
\rho_k^{(1)}(\vec{x}) := \sum_{j=1}^{n} e_j \cos k \cdot x^{(j)},
\]
\[
\rho_k^{(2)}(\vec{x}) := \sum_{j=1}^{n} e_j \sin k \cdot x^{(j)}.
\]

Determining an indefinite constant \( C \) in the Lagrangian function (2.3) formally by
\[
\frac{2\pi}{|V|} \sum_{j=1}^{n} e_j^2 \sum_{k \neq 0} \frac{1}{|k|^2} + \frac{1}{2} \sum_{k \neq 0} \frac{hc|k|}{2},
\]
we can write \( \mathcal{L} \) from (2.3) by means of (2.4), (2.9), (2.10) and (2.15) as
\[
\mathcal{L}(\vec{x}, \dot{\vec{x}}, \{a_{ik}\}, \{\dot{a}_{ik}\}, \{\phi_k\}) = \sum_{j=1}^{n} \frac{mj}{2} |\dot{x}^{(j)}|^2
\]
\[
+ \frac{1}{8\pi|V|} \sum_{k \neq 0} \left\{ \sum_{i=1}^{2} \left( |k|^2 \left( \phi^{(i)}_k \right)^2 - 8\pi \rho_k^{(i)}(\vec{x}) \phi^{(i)}_k \right) \right\}
\]
\[
+ 16\pi^2 \sum_{j=1}^{n} e_j^2 |k|^2 \right\} + \frac{1}{c} \sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A(x^{(j)}, \{a_{ik}\})
\]
\[
+ \frac{1}{2} \sum_{k \neq 0, i, l} \left( \frac{\dot{a}_{ik}^{(i)}}{2|V|} - \frac{(c|k|)^2}{2|V|} \frac{\dot{a}_{ik}^{(i)}}{2|V|} + \frac{hc|k|}{2} \right).
\]

**Remark 2.2.** If we don’t assume (2.6), we must add \((-1/|V|)(\sum_{j=1}^{n} e_j)\phi_0^{(i)}\)
and \(\sum_{i=1,2} \left( \dot{a}_{i10}^{(i)} \right)^2 / (4|V|)\) to (2.19).

The reason why we determined an indefinite constant in (2.3) by (2.18) will
be explained in Remark 5.1. Taking account of the constraints
\[
|k|^2 \phi^{(i)}_k = 4\pi \rho_k^{(i)}(\vec{x}) \quad (i = 1, 2, \ k \neq 0),
\]
roughly \( \nabla \cdot E = 4\pi \rho \) from (2.1) and (2.4) as in (9-17) in [8] and (7.38) in [25],
then from (2.16) and (2.17) we have

\[
\sum_{i=1}^{2} \left( |k|^2 \left( \phi_k^{(i)} \right)^2 - 8\pi \rho_k^{(1)}(x) \phi_k^{(i)} \right) + 16\pi^2 \sum_{j=1}^{n} e_j^2 \\
= -\frac{16\pi^2}{|k|^2} \left\{ \left( \rho_k^{(1)} \right)^2 + \left( \rho_k^{(2)} \right)^2 - \sum_{j=1}^{n} e_j^2 \right\} \\
= -\frac{16\pi^2}{|k|^2} \sum_{j,l=1,j\neq l}^{n} e_j e_l e^{ik \cdot (x^{(j)} - x^{(l)})} \\
= -\frac{16\pi^2}{|k|^2} \sum_{j,l=1,j\neq l}^{n} e_j e_l \cos k \cdot (x^{(j)} - x^{(l)}).
\]

(2.21)

So we get

\[
\mathcal{L}_c(\vec{x}, \vec{x}, \{a_{ik}\}, \{\overline{a}_{ik}\}) = \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 \\
- \frac{2\pi}{|V|} \sum_{k \neq 0} \sum_{j,l=1,j\neq l}^{n} e_j e_l \cos k \cdot (x^{(j)} - x^{(l)}) \frac{1}{|k|^2} \\
+ \frac{1}{c} \sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A(x^{(j)}, \{a_{ik}\}) \\
+ \frac{1}{2} \sum_{k \neq 0, i,l} \left( \frac{\dot{a}_{ik}^{(i)}}{2|V|} - \frac{(c|k|)^2}{2|V|} + \frac{\dot{a}_{ik}^{(i)}}{2} \frac{h\epsilon |k|}{2} \right).
\]

(2.22)

For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( z = (z_1, \ldots, z_m) \in \mathbb{R}^m \) we write \( |\alpha| = \sum_{j=1}^{m} \alpha_j, z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}, \partial_z = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_m)^{\alpha_m} \) and \( < z > = \sqrt{1 + |z|^2} \). Let \( L^2 = L^2(\mathbb{R}^m) \) be the space of all square integrable functions in \( \mathbb{R}^m \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We introduce the weighted Sobolev spaces \( B^a(\mathbb{R}^m) := \{ f \in L^2; \| f \|_{B^a} := \| f \| + \sum_{|\alpha| = a} (\| z^\alpha f \| + \| (h\partial_z)^a f \|) < \infty \} \) \((a = 1, 2, \ldots). \) Let \( B^{-a}(\mathbb{R}^m) \) denote its dual space. We set \( B^a := L^2. \) Let \( S = S(\mathbb{R}^m) \) be the Schwartz space of all rapidly decreasing functions in \( \mathbb{R}^m \).

Let \( \chi \in C^\infty(\mathbb{R}^m) \) with compact support such that \( \chi(0) = 1. \) For a function \( g(z, z') \in \mathbb{R}^m \times \mathbb{R}^m \) we define the oscillatory integral \( \text{Os} - \int g(\cdot, z')dz' \) by
\[
\lim_{\epsilon \to 0} \int \chi(\epsilon z') g(\cdot, z') dz' \text{ independently of the choice of } \chi \text{ pointwise, in the topology of } B^a(R^m) \text{ or in the topology in } S(R^m) \text{ (cf. [18]).}
\]

3 Main results

We arbitrarily cut off the terms of large wave numbers \(k\) in (2.22). That is, let \(M_j (j = 1, 2, 3)\) be arbitrary positive integers such that \(M_2 \leq M_3\). We consider

\[
\Lambda_j := \left\{ k = \left( \frac{2\pi}{L_1} s_1, \frac{2\pi}{L_2} s_2, \frac{2\pi}{L_3} s_3 \right); s_1^2 + s_2^2 + s_3^2 \neq 0, \quad |s_1|, |s_2|, |s_3| \leq M_j \right\}.
\]

Then we take \(\Lambda'_j (j = 1, 2, 3)\) such that

\[
\Lambda_j =: \Lambda'_j \cup -\Lambda'_j, \quad \Lambda'_j \cap -\Lambda'_j = \text{empty set, } \Lambda'_2 \subseteq \Lambda'_3 \quad (3.2)
\]

and fix \(\Lambda'_j\) hereafter. Let \(N_j\) denote the number of elements of the set \(\Lambda'_j\). It follows from (2.13) that \(a_{\Lambda'_j} := \{a_{ik}\}_{k \in \Lambda'_j, i, l} \in R^{4N_j}\) are independent variables (cf. p.154 in [24]).

We consider

\[
\tilde{L}_c(\tilde{x}, \dot{\tilde{x}}; \{a_{ik}\}, \{\dot{a}_{ik}\}) := \sum_{j=1}^{n} \frac{m_j}{2} |\tilde{x}^{(j)}|^2
\]

\[
- \frac{2\pi}{|V|} \sum_{k \in \Lambda_1, j, l=1, j \neq l} e_j e_l \cos k \cdot (\tilde{x}^{(j)} - \tilde{x}^{(l)}) \frac{1}{|k|^2}
\]

\[
+ \frac{1}{c} \sum_{j=1}^{n} e_j \tilde{x}^{(j)} \cdot \tilde{A}(\tilde{x}^{(j)}, a_{\Lambda'_2})
\]

\[
+ \frac{1}{2} \sum_{k \in \Lambda_3, i, l} \left( \frac{(\dot{a}_{ik})^2}{2|V|} - \frac{(c|k|)^2 (a_{ik})^2}{2|V|} + \frac{hc|k|^2}{2} \right)
\]

(3.3)
in place of \( \mathcal{L}_c \) given by (2.22), where \( A \) given by (2.14) is replaced with
\[
\tilde{A}(x,a_{\Lambda_2}) = \frac{\sqrt{4\pi}}{|V|} c g(x) \sum_{k \in \Lambda_2} \sum_{l=1}^{2} \frac{1}{\sqrt{2}} \left( \psi(a^{(1)}_l) \cos k \cdot x + \psi(a^{(2)}_l) \sin k \cdot x \right) e^t(k).
\]

We assume \( \psi(-\theta) = -\psi(\theta) \) \((\theta \in \mathbb{R})\).

For the sake of simplicity we write \( \Lambda' := \Lambda'_3 \) and \( N := N_3 \). We consider a subdivision
\[
\Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T, \quad |\Delta| := \max_{1 \leq l \leq \nu} (\tau_l - \tau_{l-1})
\]
of \([0,T]\). Let \( \vec{x} \in \mathbb{R}^{3n} \) and \( a_{\Lambda'} \in \mathbb{R}^{4N} \) be fixed. We take arbitrarily
\[
\vec{x}^{(0)}, \ldots, \vec{x}^{(\nu-1)} \in \mathbb{R}^{3n}
\]
and
\[ a^{(0)}_{\Lambda'}, \ldots, a^{(\nu-1)}_{\Lambda'} \in \mathbb{R}^{4N}. \]
Then, we write the broken line path on \([0,T]\) connecting \( \vec{x}^{(l)}(\theta) \) at \( \theta = \tau_l \) \((l = 0, 1, \ldots, \nu, \vec{x}^{(\nu)} = \vec{x})\) in order as \( \vec{q}_\Delta(\theta) \in \mathbb{R}^{3n} \). Of course, \( d\vec{q}_\Delta(\theta)/d\theta =: \dot{\vec{q}}_\Delta(\theta) \) in the distribution sense is in \( L^2([0,T]) \). In the same way we define the broken line path \( a_{\Lambda_{\Delta}}(\theta) \in \mathbb{R}^{4N} \) on \([0,T]\) for \( a^{(0)}_{\Lambda'}, \ldots, a^{(\nu-1)}_{\Lambda'} \) and \( a_{\Lambda'} \). We define \( a_{\Lambda_{\Delta}}(\theta) \in \mathbb{R}^{8N} \) by means of (2.13). We write the classical action
\[
S_c(T,0; \vec{q}_\Delta, a_{\Lambda_{\Delta}}) = \int_0^T \mathcal{L}_c(\vec{q}_\Delta(\theta), \dot{\vec{q}}_\Delta(\theta), a_{\Lambda_{\Delta}}(\theta), \dot{a}_{\Lambda_{\Delta}}(\theta)) d\theta.
\]

Let \( \rho^* > 0 \) be the constant, which will be defined for \( \Lambda'_1, \Lambda'_2 \) and \( \Lambda'_3 \) in Proposition 7.2 of the present paper. See also Remark 7.1. Then we have

**Theorem 3.1.** We assume for \( g(x) \) and \( \psi(\theta) \) in (3.4) that for any \( l = 1, 2, \ldots \) and any multi-index \( \alpha \) there exist constants \( \delta_l > 0 \) and \( \delta_\alpha > 0 \) satisfying
\[
|\partial^\alpha \psi(\theta)| \leq C_l < \theta >^{-(1+\delta_l)}, \quad \theta \in \mathbb{R}
\]
(3.6)
and
\[
|\partial^2_x g(x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)}, \quad x \in \mathbb{R}^3. \tag{3.7}
\]

Let \(|\Delta| \leq \rho^*\) and \(f(x, a_{\lambda'}) \in B^a(R^{3n+4N})\) \((a = 0, 1, 2, \ldots)\). Then, there exists the function
\[
\left\{ \prod_{l=1}^\nu \left( \prod_{j=1}^n \frac{m_j}{2\pi i h(t_j - t_{j-1})} \right)^{3} \left( \frac{1}{2\pi |h(t_j - t_{j-1})|} \right)^{4N} \right\} \times Os - \int \cdots \int (\exp ih^{-1}S_c(T, 0; \bar{q}, a_{\lambda\Delta})) f(\bar{q}_\Delta(0), a_{\lambda\Delta}(0)) d\bar{x}^{(0)} \cdots d\bar{x}^{(n-1)} da_{\lambda\Delta}^{(0)} \cdots da_{\lambda\Delta}^{(n-1)} \tag{3.8}
\]

in \(B^a(R^{3n+4N})\), which we write as \((C_{\Delta}(T, 0)f)(\bar{x}, a_{\lambda\Delta})\) or \(\iint (\exp ih^{-1}S_c(T, 0; \bar{q}, a_{\lambda\Delta})) f(\bar{q}_\Delta(0), a_{\lambda\Delta}(0)) d\bar{q}_\Delta d\lambda_{\Delta}.\) In addition, as \(|\Delta|\) tends to 0, the function \((C_{\Delta}(T, 0)f)(\bar{x}, a_{\lambda\Delta})\) converges to the so-called Feynman path integral \(\iint (\exp ih^{-1}S_c(T, 0; \bar{q}, a_{\lambda\Delta})) f(\bar{q}(0), a_{\lambda\Delta}(0)) d\bar{q} d\lambda_{\Delta}\) in \(B^a(R^{3n+4N}).\) We also see that this limit, which is \(B^a\)-valued continuous and \(B^{a-2}\)-valued continuously differentiable in \(T \in (0, \infty),\) satisfies the Schrödinger type equation
\[
\i h \frac{\partial}{\partial t} u(t) = H(t)u(t) \quad \tag{3.9}
\]

with \(u(0) = f,\) where
\[
H(t) = \sum_{j=1}^n \frac{1}{2m_j} \left| \frac{h}{i} \frac{\partial}{\partial x^{(j)}} - \frac{e_j}{c} A(x^{(j)}, a_{\lambda\Delta}) \right|^2 + \frac{2\pi}{|V|} \sum_{k \in \Lambda_1} \sum_{f=1,j \neq f}^n e_j e_f \cos k \cdot (x^{(j)} - x^{(f)}) |k|^2 + \sum_{k \in \Lambda', l} \left\{ \frac{|V|}{2} \left( \frac{h}{i} \frac{\partial}{\partial a_{ik}^{(l)}} \right)^2 + \frac{(c|k|)^2}{2|V|} \left( a_{ik}^{(l)} \right)^2 - \frac{hc|k|}{2} \right\}. \tag{3.10}
\]

Remark 3.1. We determine an indefinite constant \(C\) in (2.3) by
\[
\frac{2\pi}{|V|} \sum_{j=1}^n e_j^2 \sum_{k \in \Lambda_1} \frac{1}{|k|^2} + \frac{1}{2} \sum_{k \in \Lambda_3} \frac{hc|k|}{2}
\]
and cut off the terms of large wave numbers \( k \) of (2.19) by introducing \( \Lambda_j \) \((j = 1, 2, 3)\). Then we get (3.3) again, taking the account of the constraints (2.20).

**Remark 3.2.** Let \( 0 \leq t_0 \leq t \leq T \). For \( f \in B^a(R^{3n+4N}) \) \((a = 0, 1, 2, \ldots)\) we define \( C_\Delta(t, t_0) f \) as in (3.8). See (9.3) in the present paper for the precise definition. As will be seen in the proof of Theorem 3.1, under the assumptions of Theorem 3.1 there exist \((C_\Delta(t, t_0)f) (\vec{x}, a_\Lambda)\) in \( B^a \) and \( \lim_{|\Delta| \to 0} C_\Delta(t, t_0) f \) in \( B^a \) uniformly in \( 0 \leq t_0 \leq t \leq T \), which satisfies the Schrödinger type equation (3.9) with \( u(t_0) = f \).

In place of \( \mathcal{L} \) expressed by (2.19) we consider

\[
\tilde{\mathcal{L}}(\vec{x}, \vec{x}'; \{a_{l_k}\}, \{\dot{a}_{l_k}\}, \{\phi_k\}) := \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 \\
+ \frac{1}{8\pi|V|} \sum_{k \in \Lambda_1} \left\{ \sum_{i=1}^{2} \left( |k|^2 \left( \phi_k^{(i)} \right)^2 - 8\pi \rho_k^{(i)}(\vec{x}) \phi_k^{(i)} \right) \\
+ 16\pi^2 \frac{\sum_{j=1}^{n} e_j}{|k|^2} \right\} + \frac{1}{c} \sum_{j=1}^{n} e_j \cdot \vec{x}^{(j)} \cdot \vec{A}(\vec{x}(\tau), a_\Lambda) \\
+ \frac{1}{2} \sum_{k \in \Lambda_3} \left( \frac{\left( a_{l_k}^{(0)} \right)^2}{2|V|} - \frac{(c|k|)^2 \left( a_{l_k}^{(1)} \right)^2}{2|V|} + \frac{hc|k|}{2} \right)
\]

(3.11)

by means of (3.4) as in \( \tilde{\mathcal{L}} \).

Let \( \overrightarrow{q}_\Delta(\theta) \in R^{3n} \), \( a_\Lambda(\theta) \in R^{4n} \) and \( a_\Lambda(\theta) \in R^{8n} \) be the broken line paths defined before. Let \( \overrightarrow{\xi}_k := \{\xi_k^{(i)}\}_{i=1,2} \in R^2 \) for \( k \in \Lambda'_1 \). Take \( \overrightarrow{\xi}_k^{(0)}, \overrightarrow{\xi}_k^{(1)}, \ldots \) and \( \overrightarrow{\xi}_k^{(\nu-1)} \) in \( R^2 \) arbitrarily. Set \( \rho_k(\vec{x}) := (\rho_k^{(1)}(\vec{x}), \rho_k^{(2)}(\vec{x})) \) by means of (2.16) and (2.17). Then, we define the path

\[
\phi_{k\Delta}(\theta) := \overrightarrow{\xi}_k^{(l)} + \frac{4\pi \rho_k(\overrightarrow{q}_\Delta(\theta))}{|k|^2} \in R^2, \; \tau_{l-1} < \theta \leq \tau_l
\]

(3.12)

\((l = 1, 2, \ldots, \nu)\), where \( \phi_{k\Delta}(0) := \lim_{\theta \to 0+0} \phi_{k\Delta}(\theta) \). We set \( \phi_{\Lambda'_1\Delta}(\theta) := \{\phi_{k\Delta}(\theta)\}_{k \in \Lambda'_1} \in R^{2N_1} \). We define \( \phi_{\Lambda_1\Delta}(\theta) \in R^{4N_1} \) by means of (2.13). Let \( S(T, 0; \overrightarrow{q}_\Delta, a_\Lambda, \phi_{\Lambda_1\Delta}) \) be the classical action for \( \tilde{\mathcal{L}}(\vec{x}, \vec{x}'; \{a_{l_k}\}, \{\dot{a}_{l_k}\}, \{\phi_k\}) \) given by (3.11).
Theorem 3.2. Let \(|\Delta| \leq \rho^a\) and \(f(\vec{x}, a_{\Lambda'}) \in B^a(R^{3n+4N}) (a = 0, 1, 2, \ldots)\). Then, under the assumption of Theorem 3.1 there exists the function

\[
\left[ \prod_{l=1}^{\nu} \left\{ \prod_{j=1}^{n} \sqrt{\frac{m_j}{2\pi i h (\tau_l - \tau_{l-1})}} \right\} \sqrt{\frac{1}{2\pi i h |V| (\tau_l - \tau_{l-1})}} \right] \times \prod_{k \in \Lambda'_i} \left\{ \frac{|k|^2 (\tau_l - \tau_{l-1})}{4i\pi^2 h |V|} \right\} \right] \exp - \int \cdots \int \left( \exp i\hbar^{-1} S(T, 0; \vec{q}_{\Lambda'}, a_{\Lambda}) \right) f (\vec{q}_{\Lambda}(0), a_{\Lambda}(0)) d\vec{x}^0 \cdots d\vec{x}^{\nu-1} \cdot da_{\Lambda'}^{(0)} \cdots da_{\Lambda'}^{(\nu-1)} \prod_{k \in \Lambda'_i} d\xi_k^{(0)} d\xi_k^{(1)} \cdots d\xi_k^{(\nu-1)}
\]

(3.13)
in \(B^a(R^{3n+4N})\), which is equal to

\[
\iint (\exp i\hbar^{-1} S_c(T, 0; \vec{q}_{\Lambda}, a_{\Lambda})) \times f (\vec{q}_{\Lambda}(0), a_{\Lambda}(0)) \mathcal{D} \vec{q}_{\Lambda} \mathcal{D} a_{\Lambda}
\]
defined by (3.8) in Theorem 3.1. So it follows from Theorem 3.1 that as \(|\Delta| \to 0\), then (3.13) converges to the Feynman path integral \((G(T, 0) f) (\vec{x}, a_{\Lambda'})\) or

\[
\iint (\exp i\hbar^{-1} S(T, 0; \vec{q}, a_{\Lambda}, \phi_{\Lambda})) f (\vec{q}(0), a_{\Lambda}(0)) \mathcal{D} \vec{q} \mathcal{D} a_{\Lambda} \mathcal{D} \phi_{\Lambda}
\]

(3.14)
in \(B^a(R^{3n+4N})\), which satisfies the Schrödinger type equation (3.9) with \(u(0) = f\). This expression (3.14) is given in §9-8 in Feynman-Hibbs [8] without any comments on its definition.

Remark 3.3. As was noted in the introduction, the constraints (2.20) are not needed in Theorem 3.2 above. The path \(\phi_{k\Delta}(\theta)\) defined by (3.12) is determined so that \(\partial \mathcal{L}(\vec{q}_{\Delta}(\theta), \vec{\phi}_{\Delta}(\theta), a_{\Lambda}(\theta), \phi_{\Lambda}(\theta))/\partial \phi_{k}^{(i)} (i = 1, 2)\) is piecewise constant.

Remark 3.4. We take \(f \in \mathcal{S}(R^{3n+4N})\) and set \(M_0 = [(3n+4N)/2] + 1\), where \([\cdot]\) denotes Gauss’ symbol. Let \(\zeta = (x, X)\), and \(\alpha\) and \(\beta\) multi-indices. Then, the Sobolev inequality shows

\[
\sup_{\zeta \in R^{3n+4N}} \left| \zeta^\alpha \partial_\zeta^\beta f(\zeta) \right| \leq \left\| \zeta^\alpha \partial_\zeta^\beta f \right\| + \sum_{|\kappa| = M_0} \left\| \partial_\zeta^\kappa \left( \zeta^\alpha \partial_\zeta^\beta f \right) \right\|
\]

13
It follows from Lemma 2.4 with \(a = b = 1\) in \([12]\) or as in the proof of \((7.14)\) in the present paper that the rhs of the above is bounded by \(C_{\alpha,\beta} \|f\|_{B^{\alpha+\beta+M_0}}\) with a constant \(C_{\alpha,\beta}\). Hence, for \(|\Delta| \leq \rho^*\) there exist \((3.8)\), \((3.13)\), the limit of \((3.8)\) as \(|\Delta| \to 0\) and the limit of \((3.13)\) as \(|\Delta| \to 0\) in the topology of \(S\), so pointwise.

**Remark 3.5.** Let \(0 \leq t_0 \leq t \leq T\). For \(f \in B^a(R^{3n+4N})\) \((a = 0, 1, 2, \ldots)\) we can define \(G_\Delta(t,t_0) f\) as in \((3.13)\) in the same way that \(C_\Delta(t,t_0) f\) is defined in Remark 3.2. See also \((9.20)\) in the present paper. As will be seen in the proof of Theorem 3.2, under the assumptions of Theorem 3.1 there exists \(G_\Delta(t,t_0) f\) in \(B^a\) and \(G_\Delta(t,t_0) f\) is equal to \(C_\Delta(t,t_0) f\).

We consider an external electromagnetic field \(E_{ex}(t,x) = (E_{ex1}, E_{ex2}, E_{ex3}) \in R^3\) and \(B_{ex}(t,x) = (B_{ex1}, B_{ex2}, B_{ex3}) \in R^3\) such that \(\partial^\alpha_x E_{exj}(t,x), \partial^\alpha_x B_{exj}(t,x)\) and \(\partial_t B_{exj}(t,x)\) \((j = 1, 2, 3)\) are continuous in \([0,T] \times R^n\) for all \(\alpha\). Let \(\phi_{ex}(t,x) \in R\) and \(A_{ex}(t,x) \in R^3\) be the electromagnetic potentials to \(E_{ex}\) and \(B_{ex}\). Then we get Theorem 3.3 below. Though Theorem 3.3 gives the generalization of Theorems 3.1 and 3.2, the results are stated separately from Theorems 3.1 and 3.2 to avoid confusion.

We replace \(\tilde{A}(x^{(j)}, a\Lambda')\) in \((3.3)\), \((3.10)\) and \((3.11)\) with \(\tilde{A}(x^{(j)}, a\Lambda') + A_{ex}(t, x^{(j)})\). Moreover we add \(- \sum_{j=1}^n e_j \phi_{ex}(t, x^{(j)})\) to \((3.3)\) and \((3.11)\), and \(\sum_{j=1}^n e_j \phi_{ex}(t, x^{(j)})\) to \((3.10)\), respectively. Then we have

**Theorem 3.3.** Besides the assumptions of Theorem 3.1 we suppose as in Ichinose \([14, 15, 16]\) that for any \(\alpha \neq 0\) there exist constants \(C_{\alpha}\) and \(\delta_{\alpha} > 0\) satisfying

\[
|\partial^\alpha_x E_{exj}(t,x)| \leq C_{\alpha}, \ |\partial^\alpha_x B_{exj}(t,x)| \leq C_{\alpha} < x > ^{-\alpha - \delta_{\alpha}} \quad (3.15)
\]

and

\[
|\partial^\alpha_x A_{exj}(t,x)| \leq C_{\alpha}, \ |\partial^\alpha_x \phi_{ex}(t,x)| \leq C_{\alpha} < x > \quad (3.16)
\]

for \(j = 1, 2\) and \(3\) in \([0,T] \times R^n\). Then, the same assertions as in Theorems 3.1 and 3.2 hold.

**Remark 3.6.** It follows from Lemma 6.1 in \([14]\) that under the assumptions \((3.15)\) there exist \(A_{ex}\) and \(\phi_{ex}\) satisfying \((3.16)\).
4 The appearance of the Coulomb potentials

We will show rigorously that the Coulomb potentials appear as the limit of the second term on the rhs of (3.3) and the limit of the second term on the rhs of (3.10). This result is well known in physics (cf. [5, 8]). We will give the rigorous proof. Our proof is somewhat delicate.

**Theorem 4.1.** Let \( L_j \ (j = 1, 2, 3) \) tend to \( \infty \) under the condition

\[
\lim_{L_1, L_2, L_3 \to \infty} \frac{L_i}{L_j L_k} = 0, \ (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).
\]

(4.1)

Then we have

\[
\lim_{L_1, L_2, L_3 \to \infty} \lim_{M_1 \to \infty} \frac{2\pi}{|V|} \sum_{k \in \Lambda_1} \sum_{j, l = 1, j \neq l}^{n} e_j e_l \cos k \cdot (x^{(j)} - x^{(l)}) \left| \frac{k}{|k|^2} \right|
\]

\[
= \frac{1}{2} \sum_{j, l = 1, j \neq l}^{n} \left( \frac{e_j e_l}{|x^{(j)} - x^{(l)}|} \right) \text{ in } S'(\mathbb{R}^{3n}).
\]

(4.2)

Let \( \chi_0(k) \) be the function in \( \mathbb{R}^3 \) defined by

\[
\chi_0(k) := \begin{cases} 
1, & |k| \leq 1, \\
0, & |k| > 1.
\end{cases}
\]

(4.3)

We first prove

**Lemma 4.2.** Let \( \epsilon > 0 \). Then we have

\[
\lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \sum_{j, l = 1, j \neq l}^{n} e_j e_l \int \frac{\cos k \cdot (x^{(j)} - x^{(l)})}{|k|^2} \chi_0(\epsilon k) dk
\]

\[
= \frac{1}{2} \sum_{j, l = 1, j \neq l}^{n} \left( \frac{e_j e_l}{|x^{(j)} - x^{(l)}|} \right) \text{ in } S'(\mathbb{R}^{3n}).
\]

(4.4)

**Proof.** Let \( x \) and \( k \) be in \( \mathbb{R}^3 \). Then, it is well known that

\[
\frac{1}{(2\pi)^2} \int \frac{e^{i k \cdot x}}{|k|^2} dk = \frac{1}{2|x|} \text{ in } S'(\mathbb{R}^3)
\]

(4.5)
(cf. §5.9 in [19]).

For the sake of simplicity we consider the case of \( n = 2 \). Let \( x = x^{(1)} \) and \( y = x^{(2)} \). We will prove

\[
\lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int e^{i k \cdot (x - y)} |k|^2 \chi_0(\epsilon k) dk = \frac{1}{2|x - y|} \quad \text{in} \; \mathcal{S}'(R^6). \tag{4.6}
\]

We write the operation as \( \langle T, f \rangle \) for \( T \in \mathcal{S}' \) and \( f \in \mathcal{S} \). Let \( \varphi(x, y) \in \mathcal{S}(R^6) \).

Then from (4.6) we have

\[
\lim_{\epsilon \to 0} \left\langle \frac{1}{(2\pi)^2} \int \frac{e^{i k \cdot (x - y)}}{|k|^2} \chi_0(\epsilon k) dk, \varphi(x, y) \right\rangle = \lim_{\epsilon \to 0} \left\langle \frac{1}{(2\pi)^2} \int \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k) dk, \varphi(x, y) \right\rangle
+ i \left\langle \frac{1}{(2\pi)^2} \int \frac{\sin k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k) dk, \varphi(x, y) \right\rangle
\]

\[
= \lim_{\epsilon \to 0} \left\langle \frac{1}{(2\pi)^2} \int \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k) dk, \varphi(x, y) \right\rangle
\]

\[
= \left\langle \frac{1}{2|x - y|}, \varphi(x, y) \right\rangle.
\]

Consequently we obtain (4.4).

The equation (4.6) is equivalent to

\[
\lim_{\epsilon \to 0} \left\langle \frac{1}{2\pi^2} \int e^{i k \cdot (x - y)} \chi_0(\epsilon k) dk, \varphi(x, y) \right\rangle = \int \int \frac{1}{|x - y|} \varphi(x, y) dx dy \quad \text{for all} \; \varphi(x, y) \in \mathcal{S}(R^6). \tag{4.7}
\]

We set \( x' = (x - y)/\sqrt{2} \) and \( y' = (x + y)/\sqrt{2} \). Let \( \psi_1(x') \) and \( \psi_2(y') \) be in \( \mathcal{S}(R^3) \). We take \( \varphi(x, y) = \tilde{\varphi}(x', y') := \psi_1(x')\psi_2(y') \) in the lhs of (4.7). Then the lhs of (4.7) is equal to

\[
\lim_{\epsilon \to 0} \frac{1}{2\pi^2} \int \int e^{i k \cdot (x - y)} \chi_0(\epsilon k) \psi_1(x')\psi_2(y') dk dx' dy' = \lim_{\epsilon \to 0} \frac{1}{2\pi^2} \int \psi_1(x') dx' \int \frac{e^{i k \cdot \sqrt{2} x'}}{|k|^2} \chi_0(\epsilon k) dk \int \psi_2(y') dy',
\]

16
which is also equal to

\[
\int \frac{1}{\sqrt{2|x'|}} \psi_1(x')dx' \int \psi_2(y')dy' = \int \int \frac{\varphi(x', y')}{\sqrt{2|x'|}} dx' dy' = \int \int \frac{\varphi(x, y)}{|x - y|} dxdy
\]

from (4.5). So, (4.7) holds for \( \varphi(x, y) = \psi_1(x')\psi_2(y') \). Since the set of all linear combinations of \( \psi_1(x')\psi_2(y') \) for all \( \psi_1 \) and \( \psi_2 \) in \( S(R^3) \) is dense in \( S(R^6) \), so (4.7) holds for all \( \varphi(x, y) \in S(R^6) \). Hence we get (4.6).

\[\text{Proposition 4.3.}\] Let \( c \geq 0 \) be a constant. Let \( \Phi(k) \) be continuous in \( R^3 \setminus \{0\} \cup \{|k| = c\} \). We suppose \( |\Phi(k)| \leq \phi(|k|) \) (\( k \in R^3 \)), where \( \phi(r) \) is non-increasing in \((0, \infty)\), and \( r^2 \phi(r) \) is in \( L^1([0, \infty)) \) and bounded in \((0, \infty)\). Then, \( \sum_{k \neq 0} \Phi(k) \) is convergent absolutely, where the sum of \( k \) is taken over \( (2\pi s_1/L_1, 2\pi s_2/L_2, 2\pi s_3/L_3) \) \((s_1, s_2, s_3 = 0, \pm 1, \pm 2, \ldots)\). We also get

\[
\lim_{L_1, L_2, L_3 \to \infty} \frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \Phi(k) = \int \Phi(k)dk \tag{4.8}
\]

under the condition (4.1).

**Proof.** We write \( L = (L_1, L_2, L_3) \). Let’s define the step function \( \Phi_L(k) \) by

\[
\Phi_L(k) = \Phi \left( \frac{2\pi s_1}{L_1}, \frac{2\pi s_2}{L_2}, \frac{2\pi s_3}{L_3} \right), \quad k \in \left( \frac{2\pi(s_1 - 1)}{L_1}, \frac{2\pi s_1}{L_1} \right]
\]

\[
\times \left( \frac{2\pi(s_2 - 1)}{L_2}, \frac{2\pi s_2}{L_2} \right], \quad \Phi_L(k) = \Phi \left( \frac{2\pi s_1}{L_1}, -\frac{2\pi s_2}{L_2}, \frac{2\pi s_3}{L_3} \right), \quad k \in \left( \frac{2\pi(s_1 - 1)}{L_1}, \frac{2\pi s_1}{L_1} \right]
\]

\[
\times \left[ -\frac{2\pi s_2}{L_2}, -\frac{2\pi(s_2 - 1)}{L_2} \right] \right) \times \left( \frac{2\pi(s_3 - 1)}{L_3}, \frac{2\pi s_3}{L_3} \right],
\]

for \( s_1, s_2, s_3 = 1, 2, \ldots \). Then, for \( k \in (2\pi(s_1 - 1)/L_1, 2\pi s_1/L_1) \times (2\pi(s_2 - 1)/L_2, 2\pi s_2/L_2) \times (2\pi(s_3 - 1)/L_3, 2\pi s_3/L_3) \) we have

\[
|\Phi_L(k)| = \left| \Phi \left( \frac{2\pi s_1}{L_1}, \frac{2\pi s_2}{L_2}, \frac{2\pi s_3}{L_3} \right) \right|
\]

\[
\leq \phi \left( \left| \left( \frac{2\pi s_1}{L_1}, \frac{2\pi s_2}{L_2}, \frac{2\pi s_3}{L_3} \right) \right| \right) \leq \phi(|k|)
\]
since $\phi(r)$ is non-increasing. In the same way, for $k \in (2\pi(s_1-1)/L_1, 2\pi s_1/L_1] \times [-2\pi s_2/L_2, -2\pi(s_2-1)/L_2) \times (2\pi(s_3-1)/L_3, 2\pi s_3/L_3]$ we get

$$|\Phi_L(k)| \leq \phi(|k|).$$  \hspace{1cm} (4.9)

In the same way as in the above we can define the step function $\Phi_L(k)$ for all $k \in R^3 \setminus \{0\}$ such that (4.9) and (4.10) below hold. It holds that

$$\frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \Phi(k) = \int_{R^3} \Phi_L(k)dk + \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_1 s_2 s_3 = 0} \Phi(k).$$  \hspace{1cm} (4.10)

For a short while we suppose $L_1 \leq L_2 \leq L_3$. Since $\phi(r)$ is non-increasing, it holds that for $s_1 \geq 2$ we have

$$\phi \left( \left( \frac{2\pi s_1}{L_1}, \frac{2\pi s_2}{L_2}, 0 \right) \right) \leq \phi \left( \left( \frac{2\pi(s_1-1)}{L_1}, \frac{2\pi}{L_2}, \frac{2\pi}{L_3} \right) \right) \leq \phi(|k|),$$

$$k \in \left( \frac{2\pi s_2}{L_1}, \frac{2\pi s_1}{L_1} \right] \times \left( 0, \frac{2\pi}{L_2} \right) \times \left( 0, \frac{2\pi}{L_3} \right].$$

and also for $s_1 \geq 2$ and $s_2 \geq 1$

$$\phi \left( \left( \frac{2\pi s_1}{L_1}, \frac{2\pi s_2}{L_2}, 0 \right) \right) \leq \phi \left( \left( \frac{2\pi(s_1-1)}{L_1}, \frac{2\pi s_2}{L_2}, \frac{2\pi}{L_3} \right) \right) \leq \phi(|k|),$$

$$k \in \left( \frac{2\pi(s_1-2)}{L_1}, \frac{2\pi(s_1-1)}{L_1} \right] \times \left( \frac{2\pi(s_2-1)}{L_2}, \frac{2\pi s_2}{L_2} \right) \times \left( 0, \frac{2\pi}{L_3} \right].$$

For $s_2 \geq 2$ we also have

$$\phi \left( \left( \frac{2\pi}{L_1}, \frac{2\pi s_2}{L_2}, 0 \right) \right) \leq \phi \left( \left( \frac{2\pi}{L_1}, \frac{2\pi(s_2-1)}{L_2}, \frac{2\pi}{L_3} \right) \right) \leq \phi(|k|),$$

$$k \in \left( 0, \frac{2\pi}{L_1} \right] \times \left( \frac{2\pi(s_2-2)}{L_2}, \frac{2\pi(s_2-1)}{L_2} \right] \times \left( 0, \frac{2\pi}{L_3} \right].$$

Consequently we get

$$\frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_3 = 0} |\Phi(k)| \leq \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_3 = 0} \phi(|k|) \leq \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_3 = 0, s_1 \neq 0, s_2 \neq 0, \pm 1} \phi(|k|)$$

$$+ \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_3 = 0} \phi(|k|) + 3 \int_{0 \leq k_3 \leq (2\pi)/L_3} \phi(|k|)dk.$$  \hspace{1cm} (4.11)
We take a constant \( m \geq 1 \) such that \( L_3 \geq mL_1 \geq L_2 \). We add the refinement
\[
\{(2\pi)/(mL_1), (2\pi s_2)/L_2, (2\pi s_3)/L_3) ; s_2, s_3 = 0, \pm 1, \pm 2, \ldots \}\]
to \( \{(2\pi s_1)/L_1, (2\pi s_2)/L_2, (2\pi s_3)/L_3) ; s_1, s_2, s_3 = 0, \pm 1, \pm 2, \ldots \}\). Then, for \( s_2 \geq 2 \) noting
\[
\phi \left( \left( \frac{0, 2\pi s_2}{L_2}, 0 \right) \right) \leq \phi \left( \left( \frac{2\pi}{mL_1}, \frac{2\pi(s_2 - 1)}{L_2}, \frac{2\pi}{L_3} \right) \right),
\]
as in the proof of \((4.11)\) we have
\[
\frac{(2\pi)^3}{m|V|} \sum_{k \neq 0, s_3 = s_1 = 0} \phi(|k|) \leq \int_{0 \leq k_1 \leq (2\pi)/(mL_1), 0 \leq k_3 \leq (2\pi)/L_3} \phi(|k|)dk
\]
\[
\leq \frac{1}{m} \int_{0 \leq k_3 \leq (2\pi)/L_3} \phi(|k|)dk.
\]
Consequently from \((4.11)\) we get
\[
\frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_3 = 0} |\Phi(k)| \leq \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_1, s_2 = 0, \pm 1} \phi(|k|)
\]
\[
+ 4 \int_{0 \leq k_3 \leq (2\pi)/L_3} \phi(|k|)dk. \quad (4.12)
\]

Consider the case of \( L_1 \leq L_2 \) generally. We add the refinement \( \{(2\pi s_1)/L_1, (2\pi s_2)/L_2, (2\pi)/(mL_3) ; s_1, s_2 = 0, \pm 1, \pm 2, \ldots \}\) to \( \{(2\pi s_1)/L_1, (2\pi s_2)/L_2, (2\pi s_3)/L_3) ; s_1, s_2, s_3 = 0, \pm 1, \pm 2, \ldots \}\), where \( m \geq 1 \) is a constant such that \( L_2 \leq mL_3 \). Then, as in the proof of \((4.12)\) we can also prove \((4.12)\) for this case. Hence, for general \( L_1, L_2 \) and \( L_3 \) we obtain
\[
\frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_j = 0} |\Phi(k)| \leq \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_1, s_2, s_3 = 0, \pm 1} \phi(|k|)
\]
\[
+ 4 \int_{0 \leq k_j \leq (2\pi)/L_j} \phi(|k|)dk \quad (j = 1, 2, 3). \quad (4.13)
\]

From \((4.9)\), \((4.10)\) and \((4.13)\) we can prove that \( \sum_{k \neq 0} |\Phi(k)| \) is convergent. It follows from the assumptions that
\[
\phi(|k|) \leq \text{Const.} \frac{1}{|k|^2}, \quad k \neq 0.
\]
So we see that \((\frac{(2\pi)^3}{|V|}) \sum_{k \neq 0, s_1, s_2, s_3 = 0, \pm 1} \phi(|k|)\) tends to zero as \(L_1, L_2\) and \(L_3\) tend to the infinity under the condition (4.1). We note that we first used the condition (4.1) here. Consequently, from (4.13) and the assumptions we have
\[
\lim_{L_1, L_2, L_3 \to \infty} \frac{(2\pi)^3}{|V|} \sum_{k \neq 0, s_j = 0} \Phi(k) = 0, \quad j = 1, 2, 3
\]
under (4.1). Hence, noting (4.9), from (4.10) we obtain (4.8) by means of the Lebesgue dominated convergence theorem.

**Remark 4.1.** The condition (4.1) is necessary for \(\Phi(k)\) in Proposition 4.3 to satisfy (4.8) in general. In fact, for example, suppose \(\lim_{L_1, L_2, L_3 \to \infty} \frac{L_1}{L_2 L_3} > 0\). Let \(0 \leq \chi_1(k) \in C^\infty(R^3)\) with compact support such that \(\chi_1(k) = 1\) \((|k| \leq 1)\). We consider \(\Phi(k) = |k|^{-2} \chi_1(k) \geq 0\). Then, from (4.9) and (4.10) we have
\[
\lim_{L_1, L_2, L_3 \to \infty} \frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \Phi(k) \geq \int \Phi(k)dk + \lim_{L_1, L_2, L_3 \to \infty} \frac{(2\pi)^3}{|V|} \Phi \left( \frac{2\pi}{L_1}, 0, 0 \right)
\]
\[
> \int \Phi(k)dk.
\]
So, (4.8) doesn’t hold.

Now we will prove Theorem 4.1. For the sake of simplicity let \(n = 2\). Let \(\chi_0(k)\) be the function defined by (4.3). We write \(x = x^{(1)}\) and \(y = x^{(2)}\). We take \(\varphi(x, y) \in S(R^6)\). Then, we have
\[
\left\langle \frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k), \varphi(x, y) \right\rangle
\]
\[
= \frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \int \int \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k) \varphi(x, y)dx dy
\]
\[
= \frac{(2\pi)^3}{|V|} \sum_{k \neq 0} \int \int \frac{\cos k \cdot (x - y)}{|k|^2 < k >^2} \chi_0(\epsilon k) < D_x >^2 \varphi(x, y)dx dy, \quad (4.14)
\]
where \(< D_x >^2 = \left(1 - \sum_{j=1}^n \partial_x^2 \right)\). Let \(\Phi(k) = |k|^{-2} < k >^{-2} \int \int \cos k \cdot (x - y) \times < D_x >^2 \varphi(x, y)dx dy\) and
\[
\phi(|k|) := \frac{1}{|k|^2 < k >^2} \int \int |< D_x >^2 \varphi(x, y)| dx dy.
\]
Then from (4.14) Proposition 4.3 shows
\[
\lim_{L_1, L_2, L_3 \to \infty} \lim_{\epsilon \to 0} \left( 2\pi \frac{3}{|V|} \sum_{k \neq 0} \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k), \varphi(x, y) \right)
\]
\[= \lim_{L_1, L_2, L_3 \to \infty} \frac{2\pi}{|V|} \sum_{k \neq 0} \frac{\cos k \cdot (x - y)}{|k|^2} \int \int \frac{\cos k \cdot (x - y)}{|k|^2 < k >^2} < D_x >^2 \varphi(x, y) dxdy
\]
\[= \int \int \frac{1}{|k|^2 < k >^2} dk \int \int (\cos k \cdot (x - y)) < D_x >^2 \varphi(x, y) dxdy. \] (4.15)

In the same way from (4.14) we also have
\[
\lim_{\epsilon \to 0} \lim_{L_1, L_2, L_3 \to \infty} \left( 2\pi \frac{3}{|V|} \sum_{k \neq 0} \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k), \varphi(x, y) \right)
\]
\[= \int \int \frac{1}{|k|^2 < k >^2} dk \int \int (\cos k \cdot (x - y)) < D_x >^2 \varphi(x, y) dxdy. \] (4.16)

On the other hand, Lemma 4.2 and Proposition 4.3 indicate
\[
\lim_{\epsilon \to 0} \lim_{L_1, L_2, L_3 \to \infty} \left( 2\pi \frac{3}{|V|} \sum_{k \neq 0} \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k), \varphi(x, y) \right)
\]
\[= \lim_{\epsilon \to 0} \int \int \frac{\cos k \cdot (x - y)}{|k|^2} \chi_0(\epsilon k) \varphi(x, y) dxdydk
\]
\[= 2\pi^2 \int \int \frac{\varphi(x, y)}{|x - y|} dxdy. \] (4.17)

Hence we obtain (4.2) together with (4.15) and (4.16).

**Remark 4.2.** Let \( \chi(k) \in \mathcal{S}(R^3) \) such that \( \chi(0) = 1 \) and \( \chi(-k) = \chi(k) \). We take the limit of \( L_j \) \((j = 1, 2, 3)\) under the condition (4.1). Then it holds that

\[
\lim_{\epsilon \to 0} \lim_{L_1, L_2, L_3 \to \infty} \frac{2\pi}{|V|} \sum_{k \neq 0} \sum_{j,l=1,j \neq l}^{n} \chi(\epsilon k) e_j e_l \cos k \cdot (x^{(j)} - x^{(l)})
\]
\[= \frac{1}{2} \sum_{j,l=1,j \neq l}^{n} \frac{e_j e_l}{|x^{(j)} - x^{(l)}|}. \] (4.18)

pointwise for \( x \in R^{3n} \) such that \( x^{(j)} - x^{(l)} \neq 0 \) \((j, l = 1, 2, \ldots, n, j \neq l)\). The proof is easy. Consider the case of \( n = 2 \) and \( e_1 = e_2 = 1 \). Let’s write \( x = x^{(1)} \)
and $y = x^{(2)}$. We take $\chi_1(k) \in C^\infty(R^3)$ such that $\chi_1(k) = 1 (|k| \leq 1)$ and $\chi_1(k) = 0 (|k| \geq 2)$. Then, Proposition 4.3 says for $x \neq y$ that the lhs of (4.18) is equal to

$$\frac{1}{(2\pi)^2} \lim_{\epsilon \to 0} \int \chi(\epsilon k) \frac{\cos k \cdot (x - y)}{|k|^2} dk$$

$$= \frac{1}{(2\pi)^2} \lim_{\epsilon \to 0} \left\{ \int \chi_1(k) \chi(\epsilon k) \frac{\cos k \cdot (x - y)}{|k|^2} dk \right\}$$

$$- \frac{1}{|x - y|^2} \int \left( \cos k \cdot (x - y) \right) \Delta_k \left\{ (1 - \chi_1(k)) \chi(\epsilon k) |k|^{-2} \right\} dk$$

$$= \frac{1}{(2\pi)^2} \left\{ \int \chi_1(k) \frac{\cos k \cdot (x - y)}{|k|^2} dk \right\}$$

$$- \frac{1}{|x - y|^2} \int \left( \cos k \cdot (x - y) \right) \Delta_k \left\{ (1 - \chi_1(k)) |k|^{-2} \right\} dk$$

(4.19)

pointwise, where $\Delta_k$ denotes the Laplacian operator with respect to $k \in R^3$ and we used $|\epsilon \chi'(\epsilon k)| = \epsilon^{1/3} |k|^{-2/3} |\epsilon k|^{2/3} |\chi'(\epsilon k)| \leq \text{Const.} \epsilon^{1/3} |k|^{-2/3}$. Since we have $|\Delta_k \left\{ (1 - \chi_1(k)) \chi(\epsilon k) |k|^{-2} \right\}| \leq C < k^{-3-1/3}$ with a constant $C$ independent of $\epsilon$, so we can prove that the equation (4.19) is also true in the distribution sense $S'(R^6)$. On the other hand, we see as in the proof of Lemma 4.2 that the lhs of (4.19) is equal to $1/(2|x - y|)$ in $S'(R^6)$. Consequently we can prove that (4.19) is equal to $1/(2|x - y|)$. Hence (4.18) holds pointwise.

5 The expression of the vacuum and the states of photons

In this section we express the vacuum and the states of photons of momentums and polarization states by means of concrete functions in terms of variables $a_{N'}$ consisting of the Fourier coefficients of the electromagnetic potential. In Problem 9-8 of [8] only the vacuum and the state of a photon of momentum $\hbar k$ and polarization state $l$ are expressed concretely. In this section we generalize this result in [8] for the general states of photons. In physics the vacuum and the state of photons are not considered concretely but considered abstractly (cf. [22, 23]). We also note that the state of photons of given momentums and
polarization states can't be considered in the study for QED models on the Fock spaces (cf. [10, 11, 24]).

To write down the vacuum and the state of photons concretely, we will introduce the creation operators and the annihilation operators concretely. Let's define

\[
\hat{a}^{(i)\dagger}_{lk} := i \sqrt{\frac{|V|}{2hc|k|}} \left( \frac{h}{i} \frac{\partial}{\partial \alpha^{(i)\dagger}_{lk}} - i \frac{c|k|}{|V|} \alpha^{(i)\dagger}_{lk} \right)
= \sqrt{\frac{|V|}{2hc|k|}} \left( \frac{h}{i} \frac{\partial}{\partial \alpha^{(i)\dagger}_{lk}} + c|k| \alpha^{(i)\dagger}_{lk} \right)
\]

(5.1)

for \( k \in \Lambda \) and \( i, l = 1, 2 \). From (2.13) we have

\[
\hat{a}^{(1)}_{-k} = -\hat{a}^{(1)}_{k}, \quad \hat{a}^{(2)}_{-k} = \hat{a}^{(2)}_{k}.
\]

Let \( \hat{a}^{(i)\dagger}_{lk} \) denote the adjoint operator of \( \hat{a}^{(i)}_{lk} \). Then we know that the commutator relations

\[
[\hat{a}^{(i)}_{lk}, \hat{a}^{(i')\dagger}_{l'k'}] = \delta_{ll'}\delta_{kk'}, \quad [\hat{a}^{(i)}_{lk}, \hat{a}^{(i')}_{l'k'}] = 0
\]

hold for \( k \) and \( k' \) in \( \Lambda' \) (cf. §34 in [34]). We define the operator \( \hat{a}_{lk} \) for \( k \in \Lambda \) and \( l = 1, 2 \) by

\[
\hat{a}_{lk} := \hat{a}^{(1)}_{lk} - i\hat{a}^{(2)}_{lk}
\]

(5.2)

(cf. (2.11)). We call \( \hat{a}_{lk} \) the annihilation operator and \( \hat{a}_{lk}^{\dagger} \) the creation operator.

We can easily see from the commutator relations for \( \hat{a}^{(i)}_{lk} \) that the operators \( \hat{a}_{lk} \) and \( \hat{a}_{lk}^{\dagger} \) also satisfy the commutator relations

\[
[\hat{a}_{lk}, \hat{a}_{l'k'}^{\dagger}] = \delta_{ll'}\delta_{kk'}, \quad [\hat{a}_{lk}, \hat{a}_{l'k'}] = 0
\]

(5.3)

for \( k \) and \( k' \) in \( \Lambda \) (cf. (2.26) in [22]). It follows from the commutator relations (5.3) that we have

\[
\hat{a}_{lk}(\hat{a}_{lk}^{\dagger})^{n'} - (\hat{a}_{lk}^{\dagger})^{n'}\hat{a}_{lk} = n'(\hat{a}_{lk}^{\dagger})^{n'-1}
\]

(5.4)

(cf. §34 in [34]). Then we get the following expression as in physics (cf. p.198 in [10], (2.60) and (2.64) in [22]).
Proposition 5.1. We can write the last term of $H(t)$ defined by (3.10) as

$$H_{rad} := \sum_{k \in \Lambda \setminus 1} \sum_{i=1}^{\mathcal{N}} \left\{ \frac{|V|}{2} \left( \frac{h}{i} \frac{\partial}{\partial \alpha_{i_{jk}}^{(i)}} \right)^{2} + \frac{(c|k|)^2}{2|V|} \left( \alpha_{i_{jk}}^{(i)} \right)^2 - \frac{hc|k|}{2} \right\}$$

$$= \sum_{k \in \Lambda \setminus 1} hc|k| \hat{a}_{i_{jk}}^{\dagger} \hat{a}_{i_{jk}}.$$

(5.5)

The vector potential $A(x, a_{\Lambda_{2}})$ defined by (2.9) or (2.14), where the sum of $k$ is taken over $\Lambda_{2}$, is given by the expression

$$A(x, a_{\Lambda_{2}}) = \sqrt{\frac{4\pi h}{|V|}} \sum_{k \in \Lambda_{2}} \sum_{i=1}^{\mathcal{N}} \frac{1}{\sqrt{2|c|k|}} (\hat{a}_{i_{jk}} e^{|i-k|} + \hat{a}_{i_{jk}}^{\dagger} e^{-|i-k|}) \bar{e}_{l}^{(i)}(k).$$

(5.6)

Proof. Since from (5.1) and (5.2) we have

$$hc|k| (\hat{a}_{i_{jk}}^{\dagger} \hat{a}_{i_{jk}} + \hat{a}_{i_{jk}}^{\dagger} \hat{a}_{i_{jk}}^{\dagger})$$

$$= \frac{hc|k|}{2} \left\{ \left( \hat{a}_{i_{jk}}^{(1)} + i\hat{a}_{i_{jk}}^{(2)} \right)^{\dagger} \left( \hat{a}_{i_{jk}}^{(1)} - i\hat{a}_{i_{jk}}^{(2)} \right) + \left( -\hat{a}_{i_{jk}}^{(1)} + i\hat{a}_{i_{jk}}^{(2)} \right)^{\dagger} \left( -\hat{a}_{i_{jk}}^{(1)} - i\hat{a}_{i_{jk}}^{(2)} \right) \right\}$$

$$= \frac{hc|k|}{2} \left( \hat{a}_{i_{jk}}^{(1)} \hat{a}_{i_{jk}}^{(1)} + \hat{a}_{i_{jk}}^{(2)} \hat{a}_{i_{jk}}^{(2)} \right)$$

$$= \sum_{i=1}^{\mathcal{N}} \left\{ \frac{|V|}{2} \left( \frac{h}{i} \frac{\partial}{\partial \alpha_{i_{jk}}^{(i)}} \right)^{2} + \frac{(c|k|)^2}{2|V|} \left( \alpha_{i_{jk}}^{(i)} \right)^2 - \frac{hc|k|}{2} \right\}$$

for $k \in \Lambda$, so we get (5.5).

From (5.1) and (5.2) we have

$$\hat{a}_{i_{jk}} e^{|i-k|} + \hat{a}_{i_{jk}}^{\dagger} e^{-|i-k|}$$

$$= \frac{1}{\sqrt{2}} \left\{ \left( \hat{a}_{i_{jk}}^{(1)} + \hat{a}_{i_{jk}}^{(1)} \right) \cos k \cdot x - i \left( \hat{a}_{i_{jk}}^{(2)} + \hat{a}_{i_{jk}}^{(2)} \right) \cos k \cdot x$$

$$+ i \left( \hat{a}_{i_{jk}}^{(1)} - \hat{a}_{i_{jk}}^{(1)} \right) \sin k \cdot x + \left( \hat{a}_{i_{jk}}^{(2)} + \hat{a}_{i_{jk}}^{(2)} \right) \sin k \cdot x \right\}$$

$$= \sqrt{|V|} \frac{c|k|}{hc|k|} \hat{a}_{i_{jk}}^{(1)} \cos k \cdot x + \frac{c|k|}{|V|} \hat{a}_{i_{jk}}^{(2)} \sin k \cdot x$$

$$- ih(\cos k \cdot x) \frac{\partial}{\partial a_{i_{jk}}^{(2)}} + ih(\sin k \cdot x) \frac{\partial}{\partial a_{i_{jk}}^{(1)}}.$$
So, it is shown from (2.8) and (2.13) that
\[
\sum_{k \in \Lambda_{2}} \frac{1}{\sqrt{2c|k|}} \left( \hat{a}_{ik} e^{ik \cdot x} + \hat{a}_{ik}^{\dagger} e^{-ik \cdot x} \right) \overline{e}_{i}(k)
\]
\[
= \sum_{k \in \Lambda_{2}} \frac{1}{\sqrt{2\hbar|V|}} \left( a_{ik}^{(1)} \cos k \cdot x + a_{ik}^{(2)} \sin k \cdot x \right) \overline{e}_{i}(k).
\]

Hence, we see that the rhs of (5.6) is equal to
\[
\sqrt{4\pi} \sum_{k \in \Lambda_{2}} \sum_{l=1}^{2} \frac{1}{\sqrt{2\hbar|V|}} \left( a_{ik}^{(1)} \cos k \cdot x + a_{ik}^{(2)} \sin k \cdot x \right) \overline{e}_{i}(k),
\]
which is equal to the lhs of (5.6) from (2.14). \(\square\)

We know
\[
\int_{-\infty}^{\infty} e^{-a\theta^{2}} d\theta = \sqrt{\frac{\pi}{a}}
\]
for a constant \(a > 0\). So, we can easily see from (5.2) and (5.3) that
\[
\Psi_{0}(a_{\lambda'}) := \prod_{k \in \Lambda', l} \sqrt{\frac{c|k|}{\pi\hbar|V|}} \exp \left\{ -\frac{c|k|}{2\hbar|V|} \left( a_{ik}^{(1)^{2}} + a_{ik}^{(2)^{2}} \right) \right\}
\]
is the normal ground state of \(H_{rad}\), called vacuum, whose energy is 0, i.e.
\[
H_{rad} \Psi_{0} = 0
\]
and that we have
\[
\hat{a}_{ik}^{\dagger} \Psi_{0} = \sqrt{\frac{2c|k|}{\hbar|V|}} a_{ik}^{\ast} \Psi_{0}, \quad \hat{a}_{ik} \Psi_{0} = 0 \quad (k \in \Lambda)
\]
(cf. §8-1, (9-43) and Problem 9-8 in [8]). We know that the eigenvalue 0 of (5.8) is simple (cf. Theorem 3.4 in Chapter 3 of [2]).

The function \(\Psi_{n'\lambda}(a_{\lambda'}) := (\hat{a}_{ik}^{\dagger})^{n'} \Psi_{0}(a_{\lambda'}) \) \((k \in \Lambda, n' = 0, 1, 2, \ldots)\), which can be written concretely from (5.1), (5.2) and (5.7), expresses the state of \(n'\) photons of momentum \(\hbar k\) and polarization state \(l\) (cf. §9-2 in [8] and §2-2 in [22]) and satisfies
\[
\left( \sum_{k \in \Lambda, l} \hat{a}_{ik}^{\dagger} \hat{a}_{ik} \right) \Psi_{n'\lambda\lambda'} = n' \Psi_{n'\lambda\lambda'},
\]
25
\[
\left( \sum_{k \in \Lambda} \bar{h} k \hat{a}_{ik}^\dagger \hat{a}_{ik} \right) \Psi_{n'k'} = n'(hk') \Psi_{n'k'}
\]
and
\[
H_{\text{rad}} \Psi_{n'k'} = n'(hc|k'|) \Psi_{n'k'}
\]
from (5.4), (5.5) and (5.9). The operators \( \sum_{k \in \Lambda, l} \hat{a}_{ik}^\dagger \hat{a}_{ik} \) and \( \sum_{k \in \Lambda} \bar{h} k \hat{a}_{ik}^\dagger \hat{a}_{ik} \) are called the total number operator and the momentum operator, respectively (cf. (2.68) and (2.80) in [22]). In the same way, \( \prod_{k \in \Lambda, l} (\hat{a}_{ik}^\dagger)^{n'(l,k)} \Psi_0(a_{\Lambda'}) \) \( (n'(1,k) = 0,1,\ldots) \) denotes the state of \( n'(1,k) \) photons of momentum \( \bar{h} k \) and polarization state \( l \). Then, setting \( \Psi(a_{\Lambda'}) = \prod_{k \in \Lambda, l} (\hat{a}_{ik}^\dagger)^{n'(l,k)} \Psi_0(a_{\Lambda'}) \), we get
\[
\left( \sum_{k \in \Lambda, l} \hat{a}_{ik}^\dagger \hat{a}_{ik} \right) \Psi = \left( \sum_{k \in \Lambda, l} n'(1,k) \right) \Psi,
\]
(5.10)
\[
\left( \sum_{k \in \Lambda} \bar{h} k \hat{a}_{ik}^\dagger \hat{a}_{ik} \right) \Psi = \left( \sum_{k \in \Lambda, l} n'(1,k) \bar{h} k \right) \Psi
\]
(5.11)
and
\[
H_{\text{rad}} \Psi = \left( \sum_{k \in \Lambda, l} n'(1,k) hc|k| \right) \Psi.
\]
(5.12)
The family
\[
\left\{ \prod_{k \in \Lambda', l} \left( \hat{a}_{ik}^\dagger \right)^{n'(l,k,i)} \Psi_0 \right\}_{n'(l,k,i)=0}^{\infty}
\]
makes a complete orthogonal system in \( L^2(R^{4N}) \) (cf. Theorem 3.1 in Chapter 3 of [2] and §34 in [4]). We have
\[
\hat{a}_{ik}^{(1)} = \frac{\hat{a}_{ik} - \hat{a}_{1-k}}{\sqrt{2}}, \quad \hat{a}_{ik}^{(2)} = \frac{i(\hat{a}_{ik} + \hat{a}_{1-k})}{\sqrt{2}}
\]
from (2.13) and (5.2). So we see together with (5.4) and the second equation in (5.9) that the family
\[
\left\{ \prod_{k \in \Lambda, l} \frac{1}{\sqrt{n'(1,k)!}} (\hat{a}_{ik}^\dagger)^{n'(l,k)} \Psi_0 \right\}_{n'(l,k)=0}^{\infty}
\]
(5.13)
also makes a complete orthonormal system in $L^2(R^{4N})$ (cf. §34 in [4] and (2.46) in [22]). For example, we have
\[
\left( \hat{a}_{lk}^\dagger \Psi_0, \left( \hat{a}_{lk}^\dagger \right)^2 \Psi_0 \right) = \left( \Psi_0, \hat{a}_{lk} \left( \hat{a}_{lk}^\dagger \right)^2 \Psi_0 \right) = \left( \Psi_0, \left( \hat{a}_{lk}^\dagger \right)^2 \hat{a}_{lk} \Psi_0 \right) + 2 \left( \Psi_0, \hat{a}_{lk}^\dagger \Psi_0 \right) = 2 \left( \hat{a}_{lk} \Psi_0, \Psi_0 \right) = 0.
\]

Remark 5.1. We considered the Lagrangian function (3.3) and the Hamiltonian operator (3.10), determining an indefinite constant in (2.3) by (2.18) or in Remark 3.1. On the other hand, in many literatures (cf. [8], [22] and [24]) an indefinite constant is determined to be 0. Consequently, the term \( \frac{1}{2} \sum_{j=1}^{n} e_j^2 / |x^{(j)} - x^{(j)}| \) appears in (4.2) from (2.21) and the ground state energy of \( H_{\text{rad}} \) is \( \sum_{k \in \Lambda^*} |k|/2 \), which tends to infinity as \( M_3 \) tends to infinity. Arguments are had about these infinities in §9-3 and §9-5 of [8]. In the present paper we could see that the term \( \frac{1}{2} \sum_{j=1}^{n} e_j^2 / |x^{(j)} - x^{(j)}| \) disappears in (4.2) and that the ground state energy of \( H_{\text{rad}} \) is 0.

6 Preliminaries for the proofs of main results

From §6 to §9 we often write \( \overrightarrow{x} \) and \( \overrightarrow{y} \) in \( R^{3n} \) as \( x \) and \( y \), respectively for the sake of simplicity when no confusion arises.

Let \( 0 \leq s \leq t \leq T \). For \( x \) and \( y \) in \( R^{3n} \) we define
\[
\overrightarrow{q}_{x,y}^{t,s}(\theta) := x - \frac{t - \theta}{t - s} (x - y), \quad s \leq \theta \leq t. \tag{6.1}
\]

For \( X \) and \( Y \) in \( R^{4N} \) we also define
\[
a_{x,y}^{t,s}(\theta) := X - \frac{t - \theta}{t - s} (X - Y), \quad s \leq \theta \leq t. \tag{6.2}
\]

Then \( a_{x,y}^{t,s}(\theta) \in R^{8N} \) is defined by means of (2.13). We set
\[
V_1(x) := \frac{2\pi}{|V|} \sum_{k \in \Lambda_1} \sum_{j=1, j \neq l}^{n} e_j e_l \cos k \cdot (x^{(j)} - x^{(l)}) \tag{6.3}
\]

27
and

\[ V_2(a_{\Lambda'}) := \sum_{k \in \Lambda', i, l} \left( \frac{\left( |c| |k| \right)^2}{2|V|} \left( d_{ik}^{(i)} \right)^2 - \frac{hc|k|}{2} \right). \]  

(6.4)

For the sake of simplicity we suppose \( \Lambda' = \Lambda'_{3} (= \Lambda') \) from §6 to §9. We write \( x = (x, X) \in R^{3n+4N} \) and

\[ q_{x,y}^{t,s}(\theta) = (\theta, q_{x,y}^{t,s}(\theta), a_{\Lambda'X,Y}^{t,s}(\theta)) \in R^{1+3n+4N}, \ s \leq \theta \leq t. \]  

(6.5)

Then from (6.3) and (6.5) we have

\[ S_{c}(t, s; q_{x,y}^{t,s}, a_{\Lambda'X,Y}^{t,s}) = \frac{1}{2(t-s)} \sum_{j=1}^{n} m_{j}|x^{(j)} - y^{(j)}|^{2} \]

\[ + \int_{q_{x,y}^{t,s}} \left( -V_{1}(x)dt + \frac{1}{c} \sum_{j=1}^{n} e_{j} \tilde{A}(x^{(j)}, a_{\Lambda'}) \cdot dx^{(j)} - V_{2}(a_{\Lambda'})dt \right) + \frac{|X - Y|^{2}}{2|V|(t-s)} \]

\[ = \frac{1}{2(t-s)} \sum_{j=1}^{n} m_{j}|x^{(j)} - y^{(j)}|^{2} - \int_{s}^{t} V_{1}(x - \frac{t-\theta}{t-s}(x - y))d\theta \]

\[ + \frac{1}{c} \sum_{j=1}^{n} e_{j}(x^{(j)} - y^{(j)}) \cdot \int_{0}^{1} \tilde{A}(x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y))d\theta \]

\[ + \frac{|X - Y|^{2}}{2|V|(t-s)} - \int_{s}^{t} V_{2}(x - \frac{t-\theta}{t-s}(X - Y))d\theta \]

(6.6)

Let \( M \geq 0 \) and \( p(x, w, X, W) \) a \( C^{\infty} \) function in \( R^{6n} \times R^{8N} \) such that

\[ |\partial_{x}^{\alpha} \partial_{x}^{\beta} \partial_{W}^{\gamma} \partial_{X}^{\delta} p(x, w, X, W)| \leq C_{\alpha, \beta, \alpha', \beta'} (< x; w > < X; W >)^{M} \]  

(6.7)

for all multi-indices \( \alpha, \beta, \alpha' \) and \( \beta' \) with constants \( C_{\alpha, \beta, \alpha', \beta'} \), where \(< x; w > := \ 

28
\( \sqrt{1 + |x|^2 + |w|^2} \). For \( f(x, X) \in \mathcal{S}(R^{3n+4N}) \) we define the operator \( P(t, s) \) by

\[
\left\{ \begin{array}{l}
\left( \prod_{j=1}^{n} \sqrt{\frac{m_j}{2\pi i h}} \right)^3 \left( \frac{1}{2\pi i |V|} \right)^{4N} \int_{0}^{t} \left( \exp^{-1} S_c(t, s; \tilde{q}_{t, s}, \tilde{a}_{t, s}^{t, s}) \right) \times p(x, w, X, W) f(x, X, W) dwdW f(x, X), \\
\left( \prod_{j=1}^{n} \sqrt{\frac{m_j}{2\pi i h}} \right)^3 \left( \frac{1}{2\pi i |V|} \right)^{4N} \int_{0}^{t} \left( \exp^{-1} \left( \sum_{j=1}^{n} \frac{m_j|w_j|^2}{2} \right) \right) \times p(x, w, X, W) f(x, X, W) dwdW f(x, X), \\
\end{array} \right.
\]

(6.8)

When \( p(x, w, X, W) = 1 \), \( P(t, s) \) is called the fundamental operator and denoted by \( C(t, s) \).

**Lemma 6.1.** Let \( M_1 \) and \( M_2 \) be non-negative constants. Suppose that \( g(x) (x \in R^3) \) and \( \psi(\theta) (\theta \in R) \) in (3.4) satisfy

\[ |\partial_x^\alpha \partial_y^{\alpha'} g(x)| \leq C_\alpha < x >^{M_1}, \ x \in R^3 \]

for all \( \alpha \) and

\[ |\frac{d^k}{d\theta^k} \psi(\theta)| \leq C_k < \theta >^{M_2}, \ \theta \in R \]

for all \( k = 0, 1, \ldots \). Let \( f \in \mathcal{S}(R^{3n+4N}) \). Then, \( \partial_x^\alpha \partial_y^{\alpha'} (P(t, s)f)(x, X) \) are continuous in \( 0 \leq s \leq t \leq T \) and \( (x, X) \in R^{3n+4N} \) for all \( \alpha \) and \( \alpha' \).

**Proof.** Let \( s < t \) and make the change of variables: \( y \rightarrow w = (x - y)/\sqrt{t - s} \) and \( Y \rightarrow W = (X - Y)/\sqrt{t - s} \) in (6.8). Then from (6.6) we have

\[
P(t, s)f = \left( \prod_{j=1}^{n} \sqrt{\frac{m_j}{2\pi i h}} \right)^3 \left( \frac{1}{2\pi i |V|} \right)^{4N} \int_{0}^{t} \left( \exp^{-1} \phi(t, s; x, w, X, W) \right) \times p(x, w, X, W) f(x - \sqrt{\rho}w, X - \sqrt{\rho}W) dwdW, \quad \rho = t - s,
\]

(6.9)
where

\[
\phi(t, s; x, w, X, W) := \sum_{j=1}^{n} \frac{m_j}{2} |w^{(j)}|^2 + \frac{1}{2|V|} |W|^2 + \psi(t, s; x, \sqrt{\rho}w, X, \sqrt{\rho}W)
\]

\[
:= \sum_{j=1}^{n} \frac{m_j}{2} |w^{(j)}|^2 + \frac{1}{2|V|} |W|^2 - \rho \int_{0}^{1} V_1(x - \theta \sqrt{\rho}w) d\theta + \frac{1}{c} \sum_{j=1}^{n} e_j \sqrt{\rho}w^{(j)}
\]

\[
\cdot \int_{0}^{1} \tilde{A}(x^{(j)} - \theta \sqrt{\rho}w^{(j)}, X - \theta \sqrt{\rho}W) d\theta - \rho \int_{0}^{1} V_2(X - \theta \sqrt{\rho}W) d\theta. \quad (6.10)
\]

We note from (6.8) that (6.9) is also true for \( t = s \).

Let \( L^{(j)} := <w^{(j)}> - 2(1 - i\hbar m_j^{-1} \sum_{k=1}^{3} w_k^{(j)} \partial / \partial w_k^{(j)}) \) \((j = 1, 2, \ldots, n)\) and \( L^{(j)}\) its transposed operator. We also let \( L_1 := <W> - 2(1 - i\hbar |V| \sum_{k=1}^{4N} W_k \partial W_k) \).

Then, integrating by parts with respect to \( w \) and \( W \) in (6.9) by means of \( L^{(j)} \) and \( L_1 \), we see that the integrand is bounded by

\[
\text{Const.} \cdot <x; X> <w> - (3n+1) <W> - (4N+1)
\]

for some real constant \( l \). See the proof of Lemma 2.1 in [14] for further details.

Consequently, we see that \((P(t, s) f)(x, X)\) is continuous in \( 0 \leq s \leq t \leq T \) and \((x, X) \in R^{3n+4N}\). In the same way we can complete the proof. \( \square \)

For \( 0 \leq \sigma_1, \sigma_2 \leq 1 \) we set \( \sigma := (\sigma_1, \sigma_2) \) and

\[
\tau^{(j)}(\sigma) := t - \sigma_1(t - s) \in R,
\]

\[
\zeta^{(j)}(\sigma) := z^{(j)} + \sigma_1(x^{(j)} - z^{(j)}) + \sigma_1 \sigma_2(y^{(j)} - x^{(j)}) \in R^3, \quad j = 1, 2, \ldots, n,
\]

\[
\zeta(\sigma) := z + \sigma_1(x - z) + \sigma_1 \sigma_2(y - x) \in R^3,
\]

\[
\zeta(\sigma) := Z + \sigma_1(X - Z) + \sigma_1 \sigma_2(Y - X) \in R^{4N}. \quad (6.11)
\]

We also set

\[
B_{ml}(x^{(j)}, a_N^*) = \frac{\partial \tilde{A}_l}{\partial x_m}(x^{(j)}, a_N^*) - \frac{\partial \tilde{A}_m}{\partial x_l}(x^{(j)}, a_N^*) \quad (6.12)
\]

for \( l, m = 1, 2, 3 \) and \( j = 1, 2, \ldots, n \). Then from (6.6) we have
Lemma 6.2. We can write

\[ S_c(t, s; \overrightarrow{t}^{s, t}_z, \overrightarrow{a}^{t, s}_{\Lambda Z}) - S_c(t, s; \overrightarrow{q}^{s, t}_x, a^{t, s}_{\Lambda Z}) \]

\[ = \frac{1}{t - s} \sum_{j=1}^{n} m_j (x^{(j)} - y^{(j)}) \cdot \left( z^{(j)} - \frac{x^{(j)} + y^{(j)}}{2} \right) \]

\[ + (t - s)(x - y) \cdot \int_0^1 \int_0^1 \sigma_1 \frac{\partial V_1}{\partial x} (\zeta(\sigma)) \, d\sigma_1 d\sigma_2 \]

\[ + \frac{1}{c} \sum_{j=1}^{n} e_j (x^{(j)} - y^{(j)}) \cdot \int_0^1 \tilde{A}(x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y)) \, d\theta \]

\[ + \frac{1}{c} \sum_{j=1}^{n} \sum_{L,m=1}^{3} e_j (x^{(j)} - y^{(j)}) (x^{(j)} - z^{(j)}_l) \int_0^1 \int_0^1 \sigma_1 B_{ml}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) \, d\sigma_1 d\sigma_2 \]

\[ + (X - Y) \cdot \frac{1}{c} \sum_{j=1}^{n} \sum_{m=1}^{3} e_j (x^{(j)} - z^{(j)}_m) \int_0^1 \int_0^1 \sigma_1 \frac{\partial \tilde{A}_m}{\partial a_N}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) \, d\sigma_1 d\sigma_2 \]

\[ + \frac{1}{(t - s)|V|} (X - Y) \cdot \left( Z - \frac{X + Y}{2} \right) \]

\[ + (t - s)(X - Y) \cdot \int_0^1 \int_0^1 \sigma_1 \frac{\partial V_2}{\partial a_N}(\tilde{\zeta}(\sigma)) \, d\sigma_1 d\sigma_2. \quad (6.13) \]

Proof. We use (6.6). From (6.5) and (6.11) we see

\[ \int_{q^{t,x}_y} (-V_1(x)) \, dt - \int_{q^{t,x}_y} (-V_1(x)) \, dt = \sum_{j=1}^{n} \sum_{l=1}^{3} \int_0^1 \partial V_1/\partial x^{(j)}_l \, dt \wedge dx^{(j)}_l \]

\[ = \sum_{j=1}^{n} \sum_{l=1}^{3} \int_0^1 \int_0^1 \partial V_1(\zeta(\sigma))/\partial x^{(j)}_l \, d\sigma_1 d\sigma_2 \]

\[ = \sum_{j=1}^{n} \sum_{l=1}^{3} \int_0^1 \int_0^1 \sigma_1 \partial V_1(\zeta(\sigma))/\partial x^{(j)}_l \, d\sigma_1 d\sigma_2 \]

\[ = (t - s)(x - y) \cdot \int_0^1 \int_0^1 \sigma_1 \frac{\partial V_1}{\partial x}(\zeta(\sigma)) \, d\sigma_1 d\sigma_2, \quad (6.14) \]

where \( \Delta = \Delta(t, s, x, y, z) \) is the 2-dimensional plane with oriented boundary consisting of \((\theta, \overrightarrow{t}^{s, t}_z(\theta)), (-\theta, \overrightarrow{q}^{s, t}_x(\theta)) \) and \((\theta, \overrightarrow{q}^{s, t}_y(\theta)) \) \((s \leq \theta \leq t)\), and \( \sigma \) in
gives the positive orientation of $\Delta$. So the second term on the rhs of (6.13) appears. In the same way the last term appears. It is easy to show that the first and the 7th terms appear.

As in the proof of (6.14) we have

$$\int_{q^*_z, y^*_z} A_j(x, a_N) \cdot dx - \int_{q^*_z, y^*_z} A_j(x, a_N) \cdot dx$$

$$= \int_{q^*_z, y^*_z} A_j(x, a_N) \cdot dx + \int_{\Delta} d \left( A_j(x, a_N) \cdot dx \right)$$

$$= (x^{(j)} - y^{(j)}) \cdot \int_0^1 A_j \left( x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y) \right) d\theta$$

$$+ \sum_{1 \leq m < l \leq 3} \int_{\Delta} B_{ml} dx_m \wedge dx_l - \sum_{k \in \Lambda'} \sum_{i=1}^3 \int_{\Delta} \left( \frac{\partial A_m}{\partial a_k} \right) dx_m \wedge da_k$$

$$= (x^{(j)} - y^{(j)}) \cdot \int_0^1 A_j \left( x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y) \right) d\theta$$

$$+ \sum_{1 \leq m < l \leq 3} \left\{ (x^{(j)}_m - y^{(j)}_m)(x^{(j)}_l - z^{(j)}_l) - (x^{(j)}_l - y^{(j)}_l)(x^{(j)}_m - z^{(j)}_m) \right\}$$

$$\times \int_0^1 \int_0^1 \sigma_1 B_{ml} \left( \zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma) \right) d\sigma_1 d\sigma_2$$

$$- \sum_{m=1}^3 \left\{ (x^{(j)}_m - y^{(j)}_m)(X - Z) - (X - Y)(x^{(j)}_m - z^{(j)}_m) \right\}$$

$$\cdot \int_0^1 \int_0^1 \sigma_1 \frac{\partial A_m}{\partial a_N} \left( \zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma) \right) d\sigma_1 d\sigma_2. \quad (6.15)$$

So we can complete the proof of (6.13) from (6.6).

Let’s define $\Phi^{(j)}_m(t, s; x^{(j)}, y^{(j)}, z^{(j)}, X, Y, Z) \in R \ (m = 1, 2, 3, j = 1, 2, \ldots, n)$
and $\Phi_1(t, s; x, y, z, X, Y, Z) \in R^{4N}$ by

$$\Phi_m^{(j)} = \left( z_m^{(j)} - \frac{x_m^{(j)} + y_m^{(j)}}{2} \right) + \frac{e_j(t - s)}{m_j c} \sum_{l=1}^{3} (x_l^{(j)} - z_l^{(j)}) \int_{0}^{1} \int_{0}^{1} \sigma_1 B_{ml}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2$$

$$- \frac{e_j(t - s)}{m_j c} (X - Z) \cdot \int_{0}^{1} \int_{0}^{1} \sigma_1 \partial \tilde{A}_m(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2$$

$$+ \frac{e_j(t - s)}{m_j c} \int_{0}^{1} \tilde{A}_m(x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y)) d\theta$$

$$+ \frac{(t - s)^2}{m_j} \int_{0}^{1} \int_{0}^{1} \sigma_1 \partial V_1(\zeta(\sigma)) / \partial x_m^{(j)} d\sigma_1 d\sigma_2$$

\[(6.16)\]

and

$$\Phi_1 = \left( Z - \frac{X + Y}{2} \right) + \frac{(t - s)|V|}{c} \sum_{j=1}^{3} \sum_{m=1}^{3} e_j (x_m^{(j)} - z_m^{(j)})$$

$$\times \int_{0}^{1} \int_{0}^{1} \sigma_1 \partial \tilde{A}_m(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2$$

$$+ \frac{(t - s)^2|V|}{c} \int_{0}^{1} \int_{0}^{1} \sigma_1 \partial V_2(\zeta(\sigma)) d\sigma_1 d\sigma_2,$$

\[(6.17)\]

respectively. Let $\Phi^{(j)} := \left( \Phi_1^{(j)}, \Phi_2^{(j)}, \Phi_3^{(j)} \right) \in R^3$. Then it follows from $\textit{6.13}$, $\textit{6.16}$ and $\textit{6.17}$ that

$$S_c(t, s; \overline{a}_{z,y}^{t,s}, \overline{a}_{\Delta Z,Y}^{t,s}) - S_c(t, s; \overline{a}_{z,x}^{t,s}, \overline{a}_{\Delta Z,X}^{t,s})$$

$$= \frac{1}{t - s} \sum_{j=1}^{n} m_j (x_j^{(j)} - y_j^{(j)}) \cdot \Phi^{(j)}(t, s; x^{(j)}, y^{(j)}, z^{(j)}, X, Y, Z)$$

$$+ \frac{1}{(t - s)|V|} (X - Y) \cdot \Phi_1(t, s; x, y, z, X, Y, Z).$$

\[(6.18)\]

7 The stability of the fundamental operator

**Lemma 7.1.** Let $f \in C^1(R^d)$ and $|\partial_x^\alpha f| \leq C_\alpha < x >^{-(1 + \delta_\alpha)}$ for all $|\alpha| = 1$, where $\delta_\alpha > 0$ are constants. Then we have: (1) $f$ is a bounded function in
We have

\[ |x - z| \left| \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \int_0^1 \int_0^1 \sigma_1 f(z + \sigma_1(x - z) + \sigma_1 \sigma_2(y - x)) d\sigma_1 d\sigma_2 \right| \leq C_{\alpha,\beta,\gamma}, \quad |\alpha + \beta + \gamma| = 1, \quad x, y, z \in R^d. \]

The proof is easy. See the proof of Lemma 3.5 in [13] for the proof of Lemma 7.1.

We note (3.4) and (6.11). Then, it follows from Lemma 7.1 that under the assumptions of Theorem 3.1 we have:

\[ (1) \text{ We have } \]

\[ \left| \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \int_0^1 \int_0^1 \sigma_1 f(z + \sigma_1(x - z) + \sigma_1 \sigma_2(y - x)) d\sigma_1 d\sigma_2 \right| \leq C_{\alpha,\beta,\gamma}, \quad \alpha + \beta + \gamma \geq 0 \]

for \( x^{(j)}, y^{(j)}, z^{(j)} \in R^3 \) and \( X, Y, Z \in R^4N \). In the same way we have the same estimates as the above for \( (x_t^{(j)} - z_t^{(j)}) \int_0^1 \int_0^1 \sigma_1 B_{m l}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2 \) and \( (x_m^{(j)} - z_m^{(j)}) \int_0^1 \int_0^1 \sigma_1 B_{m l}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2 \). To obtain these estimates we assumed (3.6) and (3.7). Consequently, letting \( \Theta \) be a component of \( \Phi^{(j)} \) and \( \Phi_1 \), and \( |\alpha + \beta + \gamma + \alpha' + \beta' + \gamma'| \geq 1 \), then from (6.16) and (6.17) we obtain

\[ \left| \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \int_0^1 \int_0^1 \sigma_1 \frac{\partial A_m}{\partial a_N}(\zeta^{(j)}(\sigma), \tilde{\zeta}(\sigma)) d\sigma_1 d\sigma_2 \right| \leq C_{\alpha,\beta,\gamma}, \quad \alpha + \beta + \gamma \geq 0 \]

(7.1)

and

(7.2)

together with (6.3) and (6.4) for \( 0 \leq s \leq t \leq T, x, y, z \in R^3n \) and \( X, Y, Z \in R^4N \).

**Proposition 7.2.** Under the assumptions of Theorem 3.1 we have:

1. There exists a constant \( \rho^* > 0 \) such that the mapping \( R^{3n + 4N} \ni (z, Z) \rightarrow (\xi, \Xi) = (\Phi, \Phi_1) \) is homeomorphic and \( \det \partial (\xi, \Xi)/\partial (z, Z) \geq 1/2 \) for each fixed \( 0 \leq t - s \leq \rho^* \), \( x, y, X \) and \( Y \). We write its inverse mapping as \( R^{3n + 4N} \ni (\xi, \Xi) \rightarrow (z, Z) = (z(t, s; x, \xi, y, X, \Xi, Y), Z(t, s; x, \xi, y, X, \Xi, Y)) \in R^{3n + 4N} \).

2. Let \( \eta(t, s; x, \xi, y, X, \Xi, Y) \) be a component of \( z \) and \( Z \). Then, letting \( |\alpha + \beta + \gamma + \alpha' + \beta' + \gamma'| \geq 1 \), we have

\[ \left| \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \partial_{\Xi}^\alpha \partial_{Y}^\beta \eta(t, s; x, \xi, y, X, \Xi, Y) \right| \leq C_{\alpha,\beta,\gamma}, \quad \alpha + \beta + \gamma \geq 0 \]

(7.3)

for \( 0 \leq t - s \leq \rho^* \), \( x, \xi, y \in R^3n \) and \( X, \Xi, Y \in R^4N \).
Proof. (1) From (6.16) and (6.17) we write
\[
\partial(\Phi, \Phi_1)/\partial(z, Z) = I + (t - s)d(t, s; x, y, z, X, Y, Z),
\]
where \(I\) is the identity matrix of degree \(3n + 4N\). We can see as in the proof of (7.2) that each component of \(d\) satisfies (7.2) for all \(\alpha, \beta, \gamma, \alpha', \beta'\) and \(\gamma'\). Hence, applying Theorem 1.22 in [23] to the mapping : \((z, Z) \rightarrow (\Phi, \Phi_1)\), we can prove (1).

(2) We see
\[
(\xi, \Xi) = (\Phi(t, s; x, y, z, X, Y, Z), \Phi_1(t, s; x, y, z, X, Y, Z))
\]
with \(z = z(t, s; x, \xi, y, X, \Xi, Y)\) and \(Z = Z(t, s; x, \xi, y, X, \Xi, Y)\). So, (7.3) follows from (7.2) and \(\det \partial(\xi, \Xi)/\partial(z, Z) \geq 1/2\). 

Remark 7.1. Let’s consider the general case of \(\Lambda_2 \subseteq \Lambda_3\). Then from (3.4) and (6.12) we take \(\hat{A}(x, a_{\Lambda_2})\) and \(B_{ml}(x, a_{\Lambda_2})\) in (6.16) and (6.17). Let \(\Lambda'_1\) and \(\Lambda'_2\) be fixed. When \(\Lambda'_3 = \Lambda'_2\), we could determine \(\rho^* > 0\) from (7.4) such that we get \(\det \partial(\Phi, \Phi_1)/\partial(z, Z) \geq 1/2\) for \(0 \leq t - s \leq \rho^*, x, y, z \in R^{3n}\) and \(X, Y, Z \in R^{4N_3}\). Let \(\Lambda'_3 \supseteq \Lambda'_2\). Then, the direct calculations show
\[
\det \partial(\Phi, \Phi_1)/\partial(z, Z) \geq 1/2
\]
for \(0 \leq t - s \leq \rho^*, x, y, z \in R^{3n}\) and \(X, Y, Z \in R^{4N_3}\) from (6.16) and (6.17) since \(|V|\partial^2V_2(a_{\Lambda'})/\partial(a_{kl}(t)) = (2\rho)|k|^2\) are positive. Consequently, we can see that when \(\Lambda'_1\) and \(\Lambda'_2\) are fixed, the constant \(\rho^* > 0\) is taken independently of \(\Lambda'_3\). See the Pauli-Fierz hamiltonian in [20] for the condition \(\Lambda'_3 \supseteq \Lambda'_2\).

Theorem 7.3. Let \(\rho^* > 0\) be the constant determined in Proposition 7.2. Then under the assumptions of Theorem 3.1 we can find constants \(K_a \geq 0\) \((a = 0, 1, 2, \ldots)\) such that
\[
\|C(t, s)f\|_{B^a} \leq e^{K_a(t-s)}\|f\|_{B^a}, \quad 0 \leq t - s \leq \rho^*
\]
for all \(f(x, a_{\Lambda'}) \in B^a(R^{3n+4N})\).
Proof. The definition (6.8) says
\[ C(s, s) = \text{Identity}. \] (7.6)
So (7.5) holds for \( t = s \).

Let \( 0 < t - s \leq \rho^* \). We take \( \chi \in C^\infty(R^{3n+4N}) \) with compact support such that \( \chi(0) = 1 \). Let \( \epsilon > 0 \) and \( f \in \mathcal{S}(R^{3n+4N}) \). Then from (6.8) and (6.18) we can write
\[
C(t, s)^* \chi(\epsilon \cdot)^2 C(t, s) f = \left\{ \prod_{j=1}^{n} \left( \frac{m_j}{2\pi h(t-s)} \right)^3 \right\} \left( \frac{1}{2\pi h |V|(t-s)} \right)^{4N} \iint f(y, Y) dydY \\
\times \iint \chi(\epsilon z, \epsilon Z)^2 \exp \left\{ i \hbar^{-1} S_c(t, s; \bar{q}^{t, s}_{z\bar{h}}, a^{t,s}_{A_{Z,Y}}) - i \hbar^{-1} S_c(t, s; \bar{q}^{t, s}_{z,\bar{x}}, a^{t,s}_{\Lambda Z,\lambda}) \right\} dzdZ \\
= \left\{ \prod_{j=1}^{n} \left( \frac{m_j}{2\pi h(t-s)} \right)^3 \right\} \left( \frac{1}{2\pi h |V|(t-s)} \right)^{4N} \iint f(y, Y) dydY \iint \chi(\epsilon z, \epsilon Z)^2 \\
\times \exp \left( i \sum_{j=1}^{n} (x^{(j)} - y^{(j)}) \cdot \frac{m_j \Phi^{(j)}}{h(t-s)} + i (X - Y) \cdot \frac{\Phi_1}{h |V|(t-s)} \right) dzdZ. \] (7.7)

We can make the change of variables : \((z, Z) \rightarrow (\xi, \Xi) = (\Phi, \Phi_1)\) in (7.7) from Proposition 7.2. Then
\[
C(t, s)^* \chi(\epsilon \cdot)^2 C(t, s) f = \left\{ \prod_{j=1}^{n} \left( \frac{m_j}{2\pi h(t-s)} \right)^3 \right\} \left( \frac{1}{2\pi h |V|(t-s)} \right)^{4N} \\
\times \iint f(y, Y) dydY \iint \chi(\epsilon z, \epsilon Z)^2 \left\{ \exp \left( i \sum_{j=1}^{n} (x^{(j)} - y^{(j)}) \cdot \frac{m_j \xi^{(j)}}{h(t-s)} \right) \\
+ i (X - Y) \cdot \frac{\Xi}{h |V|(t-s)} \right\} \frac{\partial(z, Z)}{\partial(\xi, \Xi)} d\xi d\Xi. \]
The equation (7.4) and (2) of Proposition 7.2 show
\[
\frac{\partial(z, Z)}{\partial(\xi, \Xi)} = 1 + (t-s) h(t, s; x, \xi, y, X, \Xi, Y), \] (7.8)
where \( h(t, s; x, \xi, y, X, \Xi, Y) \) satisfies (7.3) for all \( \alpha, \beta, \gamma, \alpha', \beta' \) and \( \gamma' \). Conse-
quently from (2) of Proposition 7.2 we have
\[
\lim_{\varepsilon \to 0} C(t, s)^* \chi(\varepsilon \cdot)^2 C(t, s) f = \left( \frac{1}{2\pi} \right)^{3n+4N} \lim_{\varepsilon \to 0} \iint f(y, Y) dydY \iint \chi(\varepsilon z, \varepsilon Z)^2
\]
\[
\times \{ \exp\left( i(x - y) \cdot \gamma + i(X - Y) \cdot \Gamma \right) \} \det \frac{\partial (z, Z)}{\partial (\xi, \Xi)} d\gamma d\Gamma
\]
\[
= f(x, X) + (t - s) \left( \frac{1}{2\pi} \right)^{3n+4N} \text{Os} - \iint \iint \{ \exp\left( i(x - y) \cdot \gamma + i(X - Y) \cdot \Gamma \right) \}
\]
\[
\times h(t, s; x, \xi, y, X, \Xi, Y) f(y, Y) dydY d\gamma d\Gamma,
\]
(7.9)
where \( \xi^{(j)} = \bar{h} (t - s) \gamma^{(j)}/m_j \) \((j = 1, \ldots, n)\) and \( \Xi = \bar{h} |V|(t - s)\Gamma \). We note that the second term on the rhs of (7.9) is a pseudo-differential operator. So, applying the Calderón-Vaillancourt theorem \([3]\), we obtain
\[
\lim_{\varepsilon \to 0} \| \chi(\varepsilon \cdot) C(t, s) f \|^2 = \lim_{\varepsilon \to 0} (C(t, s)^* \chi(\varepsilon \cdot)^2 C(t, s) f, f)
\]
\[
= (\lim_{\varepsilon \to 0} C(t, s)^* \chi(\varepsilon \cdot)^2 C(t, s) f, f) \leq (1 + 2K_0(t - s)) \| f \|^2
\]
\[
\leq e^{2K_0(t - s)} \| f \|^2
\]
with a constant \( K_0 \geq 0 \). Hence we get (7.5) with \( a = 0 \) by Fatou’s lemma.

Let \( p(x, w, X, W) \) be a \( C^\infty \) function satisfying (6.7) with an integer \( M \geq 0 \). Then we obtain
\[
\| P(t, s) f \| \leq \text{Const.} \| f \|_{BM}
\]
(7.10)
as in the proof of (7.5) with \( a = 0 \). See the proof of Proposition 4.3 in \([14]\) for further details.

Let’s remember the expression (6.9) of \( C(t, s) f \). Set \( \zeta := (x, X) \) and let \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{3n+4N}) \) be an arbitrary multi-index. Then we can see that \( \partial_{\zeta}^\kappa (\tilde{C}(t, s) f) - \tilde{C}(t, s) (\partial_{\zeta}^\kappa f) \) and \( \zeta^\kappa (\tilde{C}(t, s) f) - \tilde{C}(t, s) (\zeta^\kappa f) \) are written in the form
\[
(t - s) \sum_{|\gamma| \leq |\kappa|} \tilde{P}_\gamma(t, s; x) (\partial_{\zeta}^\kappa f) := (t - s) \sum_{|\gamma| \leq |\kappa|} \left( \prod_{j=1}^{n} \sqrt{\frac{m_j}{2\pi i h}} \right)^3 \sqrt{\frac{1}{2\pi i h |V|}}^{4N}
\]
\[
\times \text{Os} - \iint (\exp ih^{-1} \phi(t, s; x, w, X, W)) p_\gamma(t, s; x, \sqrt{p} w, X, \sqrt{p} W)
\]
\[
\times (\partial_{\zeta}^\kappa f)(x - \sqrt{p} w, X - \sqrt{p} W) dwdW
\]
(7.11)
respectively, where \( p_\gamma(t, s; x, w, X, W) \) satisfies (6.7) with \( M = |\kappa| - |\gamma| \) for all \( \alpha, \beta, \alpha' \) and \( \beta' \). We can prove these results by induction with respect to \( |\kappa| \), using \( \bar{h} \partial_w (j) e^{im \bar{h}^{-1}|w(j)|^2/2} = im_j w(j) e^{im \bar{h}^{-1}|w(j)|^2/2} \), \( h \partial_W e^{i\bar{h}^{-1}|W|^2/(2|V|)} = (iW/|V|) e^{i\bar{h}^{-1}|W|^2/(2|V|)} \) and the integration by parts in (6.9). See the proof of Lemma 3.2 in [16] for further details.

Let \( |\kappa| = a \) \((a = 0, 1, 2, \ldots)\). Then we have

\[
\| \partial_\kappa^\alpha (C(t, s)f) \| \leq \| C(t, s)(\partial_\kappa^\alpha f) \| + (t - s) \sum_{|\gamma| \leq a} \| \partial_\gamma^\beta (C(t, s)f) \|.
\]

Applying (7.5) with \( a = 0 \) and (7.10) to the rhs above, we get

\[
\| \partial_\kappa^\alpha (C(t, s)f) \| \leq e^{K_0(t-s)} \| \partial_\kappa^\alpha f \| + \text{Const.} (t - s) \sum_{|\gamma| \leq a} \| \partial_\gamma^\beta f \|_{B^a-|\gamma|}.
\]

We know from Lemma 2.3 with \( s = 1 \) and \( a = b \) in [12] that there exist a constant \( \mu_a \geq 0 \) and \( \lambda_a(\zeta, \eta) \) satisfying

\[
|\partial_\eta^\alpha \partial_\zeta^\beta \lambda_a(\zeta, \eta)| \leq C_{\alpha, \beta} < \zeta; \eta >^{-a} \tag{7.12}
\]

for all \( \alpha \) and \( \beta \), and

\[
\Lambda_a(\zeta, D_\zeta) = (\mu_a+ < \zeta >^a + < D_\zeta >^a)^{-1} \tag{7.13}
\]
on \( S \), where \( \Lambda_a(\zeta, D_\zeta) \) is the pseudo-differential operator with symbol \( \lambda_a(\zeta, \eta) \).

So, using Lemma 2.4 in [12] and the Calderón-Vaillancourt theorem, we have

\[
\| \partial_\zeta^\alpha f \|_{B^a-|\gamma|} \leq \text{Const.} \| \left( \mu_a-|\gamma| + < \zeta >^a-|\gamma| + < D_\zeta >^a-|\gamma| \right) \partial_\zeta^\alpha f \|
\]

\[
= \text{Const.} \| \left( \mu_a-|\gamma| + < \zeta >^a-|\gamma| + < D_\zeta >^a-|\gamma| \right) \partial_\zeta^\alpha \Lambda_a \left( \mu_a+ < \zeta >^a + < D_\zeta >^a \right) \| \leq \text{Const.} \| f \|_{B^a}.
\]

Hence we get

\[
\| \partial_\kappa^\alpha (C(t, s)f) \| \leq e^{K_0(t-s)} \| \partial_\kappa^\alpha f \| + \text{Const.} (t - s) \| f \|_{B^a}. \tag{7.15}
\]

In the same way we get

\[
\| \zeta^\kappa (C(t, s)f) \| \leq e^{K_0(t-s)} \| \zeta^\kappa f \| + \text{Const.} (t - s) \| f \|_{B^a}. \tag{7.16}
\]
Thus we obtain
\[
\|C(t, s)f\|_{B^a} \leq e^{K_a(t-s)}\|f\|_{B^a} + \text{Const.}(t-s)\|f\|_{B^a}
\]
\[
\leq e^{K_a(t-s)}\|f\|_{B^a} + \text{Const.}(t-s)\|f\|_{B^a}.
\]
This completes the proof of Theorem 7.3. \qed

**Proposition 7.4.** Let \(0 \leq t-s \leq \rho^*\) and \(p(x, w, X, W)\) satisfy (6.7) with an integer \(M \geq 0\). Then \(P(t, s)\) is a continuous operator from \(B^a\) (\(a = 0, 1, 2, \ldots\)) into \(B^{a+M}\).

*Proof.* Let \(\zeta = (x, X)\) and \(f \in S(R^{3n+4N})\). We also use (6.9) as in the proof of Theorem 7.3. Then we have
\[
\partial_\xi^\kappa P(t, s)f = \sum_{\gamma \leq \kappa} P_\gamma(t, s)\partial_\xi^\gamma f,
\]
where \(\gamma \leq \kappa\) denotes \(\gamma_j \leq \kappa_j\) for all \(j\) and \(p_\gamma(t, s; x, w, X, W)\) satisfy (6.7) with \(M + |\kappa| - |\gamma|\) as \(M\). Using \(\zeta = (x, X) = (x - \sqrt{p}w, X - \sqrt{p}W) + \sqrt{p}(w, W)\), we also have
\[
\zeta^\kappa P(t, s)f = \sum_{\gamma \leq \kappa} Q_\gamma(t, s)\zeta^\gamma f,
\]
where \(q_\gamma(t, s; x, w, X, W)\) satisfy (6.7) with \(M + |\kappa| - |\gamma|\) as \(M\). Hence from (7.10) and (7.14) we see
\[
\|P(t, s)f\|_{B^a} = \|P(t, s)f\| + \sum_{|\kappa| = a} (\|\zeta^\kappa P(t, s)f\| + \|\partial_\xi^\kappa P(t, s)f\|)
\]
\[
\leq \text{Const.}\|f\|_{B^{a+M}}.
\]
(7.17)

So we could complete the proof. \qed

## 8 The consistency of the fundamental operator

Let \(C(t, s)\) and \(H(t)\) be the fundamental operator defined in §6 and the operator defined by (3.10) with \(a_{\Lambda'} = a_{\Lambda_2'} = X\), respectively.
Theorem 8.1. Under the assumptions of Theorem 3.1 there exist integers $M \geq 0, M' \geq 0, C^\infty$ functions $r(t, s; x, w, X, W)$ and $r'(t, s; x, w, X, W)$ in $0 \leq s \leq t \leq T$, $(x, w) \in \mathbb{R}^{6n}$ and $(X, W) \in \mathbb{R}^{8N}$ satisfying (6.7) for all $\alpha, \beta, \alpha'$ and $\beta'$, respectively such that

$$\left(i\hbar \frac{\partial}{\partial t} - H(t)\right) C(t, s) f = \sqrt{t-s} R(t, s) f$$

(8.1) and

$$i\hbar \frac{\partial}{\partial s} C(t, s) f + C(t, s) H(s) f = \sqrt{t-s} R'(t, s) f,$$

(8.2)

where $R(t, s)$ and $R'(t, s)$ are the operators defined by (6.8).

Proof. In this proof we write $x$ and $y$ as $\vec{x}$ and $\vec{y}$, respectively. Let $x$ denote variables in $\mathbb{R}^3$. It follows from (3.10), (6.6) and (6.8) that the direct calculations show

$$\left(i\hbar \frac{\partial}{\partial t} - H(t)\right) C(t, s) f = -\left(\prod_{j=1}^n \int \frac{m_j}{2\pi i\hbar(t-s)} \right)^4 N \times \int \left( \exp i\hbar^{-1} S_c(t, s; \vec{q}^{t,s}_{\vec{x},\vec{y}}, a^{t,s}_{\Lambda X,Y}) \{ r_1(t, s; \vec{x}, \vec{y}, X, Y) + \frac{i\hbar}{2} r_2(t, s; \vec{x}, \vec{y}, X, Y) \right) f(\vec{y}, Y) d\vec{y} dY$$

(8.3)

by means of (6.3) and (6.4), where

$$r_1(t, s; \vec{x}, \vec{y}, X, Y) = \partial_t S_c(t, s; \vec{q}^{t,s}_{\vec{x},\vec{y}}, a^{t,s}_{\Lambda X,Y}) + \sum_{j=1}^n \frac{1}{2m_j} |\partial x^{(j)} S_c - \frac{e_j}{c} \vec{A}(x^{(j)}, X)|^2 + V_1(\vec{x}) + \frac{|V|}{2} |\partial X S_c|^2 + V_2(X)$$

(8.4) and

$$r_2 = \frac{3n + 4N}{t-s} - \sum_{j=1}^n \frac{1}{m_j} \Delta x^{(j)} S_c$$

$$+ \frac{1}{c} \sum_{j=1}^n \frac{e_j}{m_j} (\nabla_x \cdot \vec{A})(x^{(j)}, X) - |V| \Delta X S_c, \quad x \in \mathbb{R}^3$$

(8.5)

(cf. the proof of Proposition 2.3 in [13]).
Set $\rho = t - s$. From (6.6) we can write

$$
\partial_{x^{(j)}} S_c - \frac{e_j}{c} \tilde{A}(x^{(j)}, X) = \frac{m_j(x^{(j)} - y^{(j)})}{\rho} \\
+ \frac{e_j}{c} \int_0^1 \left\{ \tilde{A}(x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y)) - \tilde{A}(x^{(j)}, X) \right\} d\theta \\
+ \frac{e_j}{c} \sum_{l=1}^3 (x^{(j)}_l - y^{(j)}_l) \int_0^1 (1 - \theta) \frac{\partial \tilde{A}_l}{\partial x} (x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y)) d\theta \\
- \rho \int_0^1 (1 - \theta) \frac{\partial V_1}{\partial x^{(j)}} (\overrightarrow{x} - \overrightarrow{\theta(\overrightarrow{x} - \overrightarrow{y})}) d\theta \\
= \frac{m_j(x^{(j)} - y^{(j)})}{\rho} - \frac{e_j}{2c} \sum_{m=1}^3 (x^{(j)}_m - y^{(j)}_m) \frac{\partial \tilde{A}}{\partial x_m}(x^{(j)}, X) \\
- \frac{e_j}{2c} \sum_{m=1}^{4N} (X_m - Y_m) \frac{\partial \tilde{A}}{\partial X_m}(x^{(j)}, X) + \frac{e_j}{2c} \sum_{l=1}^3 (x^{(j)}_l - y^{(j)}_l) \frac{\partial \tilde{A}_l}{\partial x}(x^{(j)}, X) \\
+ \rho q_1(t, s; \overrightarrow{x}, \overrightarrow{\overrightarrow{x} - \overrightarrow{y}}, X, \frac{X - Y}{\sqrt{\rho}}) \tag{8.6}
$$

and

$$
\partial_{x} S_c = \frac{X - Y}{|V|\rho} - \rho \int_0^1 (1 - \theta) \frac{\partial V_2}{\partial X} (X - \theta(X - Y)) d\theta \\
+ \frac{1}{c} \sum_{j=1}^n \sum_{l=1}^3 e_j(x^{(j)}_l - y^{(j)}_l) \int_0^1 (1 - \theta) \frac{\partial \tilde{A}_l}{\partial X} (x^{(j)} - \theta(x^{(j)} - y^{(j)}), X - \theta(X - Y)) d\theta \\
= \frac{X - Y}{|V|\rho} + \frac{1}{2c} \sum_{j=1}^n \sum_{l=1}^3 e_j(x^{(j)}_l - y^{(j)}_l) \frac{\partial \tilde{A}_l}{\partial X}(x^{(j)}, X) + \rho q_2(t, s; \overrightarrow{x}, \overrightarrow{\overrightarrow{x} - \overrightarrow{y}}, X, \frac{X - Y}{\sqrt{\rho}}). \tag{8.7}
$$

It holds that

$$
- \sum_{k,m=1}^3 \left( x^{(j)}_k - y^{(j)}_k \right) \left( x^{(j)}_m - y^{(j)}_m \right) \frac{\partial A_k}{\partial x_m}(x^{(j)}, X) \\
+ \sum_{k,l=1}^3 \left( x^{(j)}_k - y^{(j)}_k \right) \left( x^{(j)}_l - y^{(j)}_l \right) \frac{\partial A_l}{\partial x_k}(x^{(j)}, X) = 0. \tag{8.8}
$$
The equations (8.6) - (8.8) show
\[
\sum_{j=1}^{n} \frac{1}{2m_j} \left| \partial_{x(j)} S_c - \frac{e_j}{c} \tilde{A}(x(j), X) \right|^2 + \frac{|V|}{2} |\partial_X S_c|^2 \\
= \frac{1}{2\rho^2} \sum_{j=1}^{n} m_j \left| x(j) - y(j) \right|^2 + \frac{|X - Y|^2}{2|V|^2} + \sqrt{\rho q_4(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})}.
\]

(8.9)

From (8.6) we also have
\[
\partial_t S_c(t, s; \overrightarrow{q_t}, a_t, \Lambda_{X,Y}) = -\frac{1}{2\rho^2} \sum_{j=1}^{n} m_j \left| x(j) - y(j) \right|^2 - V_1(\overrightarrow{x}) - \frac{|X - Y|^2}{2|V|^2} \\
- V_2(X) + \sqrt{\rho q_4(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})}.
\]

(8.10)

Hence together with (8.4) we obtain
\[
r_1(t, s; \overrightarrow{x}, \overrightarrow{y}, X, Y) = \sqrt{\rho q_5(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})}.
\]

(8.11)

From (6.6) or (8.6) - (8.7) the same arguments as for \( r_1 \) show
\[
\sum_{j=1}^{n} \frac{1}{m_j} \Delta_{x(j)} S_c + |V| \Delta_X S_c = \frac{3n + 4N}{\rho} + \frac{2}{c} \sum_{j=1}^{n} \frac{e_j}{m_j} \int_{0}^{1} (1 - \theta) \\
\times (\nabla_x \cdot A) (x(j) - \theta(x(j) - y(j)), X - \theta(X - Y)) d\theta \\
+ \sqrt{\rho q_6(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})} = \frac{3n + 4N}{\rho} \\
+ \frac{1}{c} \sum_{j=1}^{n} \frac{e_j}{m_j} \left( \nabla_x \cdot \tilde{A} \right) (x(j), X) + \sqrt{\rho q_7(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})}.
\]

(8.12)

Hence together with (8.5) we get
\[
r_2(t, s; \overrightarrow{x}, \overrightarrow{y}, X, Y) = -\sqrt{\rho q_7(t, s; \overrightarrow{x}, \overrightarrow{\frac{x - y}{\sqrt{\rho}}}, X, \overrightarrow{X - Y})}.
\]

(8.13)

Thus we could complete the proof of (8.1) from (8.3), (8.11) and (8.13).
We consider (8.2). By direct calculations we see that the lhs of (8.2) is equal to

\[- \left( \prod_{j=1}^{n} \sqrt[3]{\frac{m_j}{2\pi i h(t-s)}} \right) \sqrt{\frac{1}{2\pi i h|V|(t-s)}} \]

\[\times \int \int \left( \exp \frac{i}{\hbar} S_c(t,s; \vec{q}^{t,s}_{x,y},a_{\Lambda X,Y}^{t,s}) \right) \left\{ r'_1(t,s; \vec{x}, \vec{y}, X, Y) + \frac{i}{2} r'_2(t,s; \vec{x}, \vec{y}, X, Y) \right\} f(\vec{y}, Y)d\vec{y}dY, \]  

(8.14)

where

\[r'_1(t,s; \vec{x}, \vec{y}, X, Y) = \partial_s S_c(t,s; \vec{q}^{t,s}_{x,y},a_{\Lambda X,Y}^{t,s}) - \sum_{j=1}^{n} \frac{1}{2m_j} |\partial_{y(j)} S_c + \frac{e_j}{c} \tilde{A}(y^{(j)}, Y)|^2 \]

\[- V_1(\vec{y}) - \frac{|V|}{2} |\partial_Y S_c|^2 - V_2(Y) \]

(8.15)

and

\[r'_2 = - \frac{3n + 4N}{t-s} + \sum_{j=1}^{n} \frac{1}{m_j} \Delta_{y(j)} S_c \]

\[+ \frac{1}{c} \sum_{j=1}^{n} \frac{e_j}{m_j} (\nabla_x \cdot \tilde{A})(y^{(j)}, Y) + |V| \Delta_Y S_c. \]  

(8.16)

Consequently we can prove (8.2) as in the proof of (8.1).

\[\square\]

9 The proofs of the main results

We first prove Theorem 3.1. Let \(\rho^* > 0\) be the constant determined in Proposition 7.2 and \(\chi \in C^\infty(R^{3n+4N})\) with compact support such that \(\chi(0) = 1\). We consider bounded operators \(K_j\) and \(K'_j\) \((j = 1, 2, \ldots, \nu)\) on \(B^a(R^{3n+4N})\).
Then, it holds for \( f \in B^a(R^{3n+4N}) \) that

\[
K_\nu \chi(\epsilon)K_{\nu-1} \chi(\epsilon) \cdots \chi(\epsilon)K_{1} \chi(\epsilon)f - K_{\nu} K_{\nu-1} K_{\nu-2} \cdots K_{1} f
\]

\[
= \sum_{j=1}^{\nu} K_\nu \chi(\epsilon) \cdots \chi(\epsilon)K_{j+1} \chi(\epsilon) \left(K_j - K'_j\right) K'_{j-1} \cdots K'_{1} f
\]

\[
+ \sum_{j=0}^{\nu-1} K_\nu \chi(\epsilon) \cdots \chi(\epsilon)K_{j+1} \chi(\epsilon) \left(\chi(\epsilon) - 1\right) K'_j \cdots K'_{1} f.
\] (9.1)

Noting (6.1) and (6.2), from (3.5) we have

\[
\|\chi(\epsilon)f\|_{B^a} \leq \text{Const.} \|f\|_{B^a}
\]

and

\[
\lim_{\varepsilon \to 0} \|\chi(\epsilon) - 1\|_{B^a} = 0.
\]

Consequently, using Theorem 7.3 and (9.1), we can see that there exists (3.8) in \( B^a \), which is written as

\[
C(T, \tau_{\nu-1})C(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C(\tau_2, \tau_1)C(\tau_1, 0)f (= C_{\Delta}(T, 0)f).
\] (9.2)

We also see from Remark 3.4 that there exists (3.8) in \( S \).

Let \( 0 \leq s \leq t \leq T \). For a subdivision \( \Delta \) of \([0,T] \) we can find \( j \) and \( l \) such that \( j \leq l, \tau_{j-1} < s \leq \tau_j \) and \( \tau_{j-1} < t \leq \tau_j \), where we take \( j = 1 \) for \( s = 0 \). Then we define

\[
C_{\Delta}(t, s)f = \lim_{\varepsilon \to 0} C(t, \tau_{\nu-1})\chi(\epsilon)fC(\tau_{\nu-1}, \tau_{\nu-2})\chi(\epsilon) \cdots \chi(\epsilon)C(\tau_{j+1}, \tau_j)\chi(\epsilon)C(\tau_j, s)\chi(\epsilon)f
\] (9.3)
for \( f \in B^a \) as was stated in Remark 3.2. Then we have

\[
C(\Delta)(t, s) f = C(t, \tau_{l-1})C(\tau_{l-1}, \tau_{l-2}) \cdots C(\tau_{j+1}, \tau_j)C(\tau_j, s) f
\]
as in the proof of (9.2). Consequently, from (7.5) we have

\[
\|C(\Delta)(t, s) f\|_{B^a} \leq e^{K_a(t-s)}\|f\|_{B^a} \quad (a = 0, 1, 2, \ldots)
\]
(9.4)
for \( |\Delta| \leq \rho^* \) under the assumptions of Theorem 3.1.

**Proposition 9.1.** Let \( |\Delta| \leq \rho^* \). Then, under the assumptions of Theorem 3.1 we can find an integer \( M \geq 2 \) such that

\[
\|C(\Delta)(t, s) f - C(\Delta)(t', s') f\|_{B^a} \leq C_a(|t - t'| + |s - s'|)\|f\|_{B^{a+M}}
\]
(9.5)
for \( 0 \leq s \leq t \leq T \), \( 0 \leq s' \leq t' \leq T \) and \( a = 0, 1, 2, \ldots \).

**Proof.** Let \( R(t, s) \) and \( R'(t, s) \) be the operators defined by (8.1) and (8.2), respectively. We determine \( M \) in Proposition 9.1 by \( \max(M, M', 2) \) for \( M \) and \( M' \) in Theorem 8.1. We can easily see

\[
i\hbar(C(t, s) f - C(t', s') f) = \int_{t'}^t (H(\theta)C(\theta, s) f + \sqrt{\theta - s}R(\theta, s) f)d\theta
\]
(9.6)
from (8.1) for \( s \leq t' \leq t \). Let \( \tau_j < t \leq \tau_{j+1} \) and \( \tau_k < t' \leq \tau_{k+1} \). So \( j \geq k \) holds. Using the equation just after (9.3) and (9.6), we get

\[
i\hbar(C(\Delta)(t, s) f - C(\Delta)(t', s') f)
= \int_{t'}^t H(\theta)C(\theta, s) f d\theta + \int_{t}^{\tau_j} \sqrt{\theta - \tau_j}R(\theta, \tau_j) d\theta C(\tau_j, s) f
+ \sum_{l=1}^{j-k-1} \int_{\tau_{j-l}}^{\tau_{j-l+1}} \sqrt{\theta - \tau_{j-l}}R(\theta, \tau_{j-l}) d\theta C(\tau_{j-l}, s) f
+ \int_{\tau_k}^{t' \kappa} \sqrt{\theta - \tau_k}R(\theta, \tau_k) d\theta C(\tau_k, s) f
\]
(9.7)

See the proof of Theorem 4.2 in [16] for further details.

As in the proof of (7.14) we see

\[
\|H(t) f\|_{B^a} \leq \text{Const.} \|f\|_{B^{a+M}}
\]
(9.8)
from (3.10) because of $M \geq 2$. We also see
\[ \|R(t, s)f\|_{B^a} \leq \text{Const.} \|f\|_{B^{a+M}} \] (9.9)
from Proposition 7.4 for $0 \leq t - s \leq \rho^*$. Consequently, (9.4) and (9.7) show
\[
\begin{align*}
&h\|C_\Delta(t, s)f - C_\Delta(t', s)f\|_{B^a} \\
&\leq \text{Const.} e^{K_{a+M}T(1 + \sqrt{\rho^*})} |t - t'| \|f\|_{B^{a+M}}
\end{align*}
\]
for $0 \leq s \leq t' \leq t \leq T$. The inequality above holds for $0 \leq s \leq t', t \leq T$. In the same way we get
\[
\begin{align*}
&h\|C_\Delta(t, s)f - C_\Delta(t, s')f\|_{B^a} \\
&\leq \text{Const.} e^{K_{a+M}T(1 + \sqrt{\rho^*})} |s - s'| \|f\|_{B^{a+M}}
\end{align*}
\]
for $0 \leq s, s' \leq t \leq T$. Hence, we can complete the proof of Proposition 9.1. 

Let $M \geq 2$ be the integer determined in Proposition 9.1. Let $\{\Delta_j\}_{j=1}^\infty$ be a family of subdivisions of $[0, T]$ such that $|\Delta_j| \leq \rho^*$ and $\lim_{j \to \infty} |\Delta_j| = 0$. Take an arbitrary $f \in B^{a+2M}$ ($a = 0, 1, 2, \ldots$). Then we see from (9.4) and (9.5) that $\{C_\Delta(t, s)f\}_{j=1}^\infty$ is uniformly bounded as a family of $B^{a+2M}$-valued continuous functions and equicontinuous as a family of $B^{a+M}$-valued functions in $0 \leq s \leq t \leq T$, respectively. It follows from the Rellich criterion (cf. Theorem XIII. 65 in [21]) that the embedding map from $B^M$ into $L^2$ is compact. So is the embedding map from $B^{a+2M}$ into $B^{a+M}$ from (7.12), (7.13) and Lemma 2.5 in [12] with $a = b = 1$. Consequently, from Ascoli-Arzelà theorem we can find a subsequence $\{\Delta_{j_k}\}_{k=1}^\infty$, which may depend on $f$, such that $C_{\Delta_{j_k}}(t, s)f$ converges in $B^{a+M}$ uniformly in $0 \leq s \leq t \leq T$ as $k \to \infty$. Since $C_{\Delta_j}(s, s)f = f$ follows from Lemma 6.1, so (9.7) - (9.9) show that $\lim_{k \to \infty} C_{\Delta_{j_k}}(t, s)f = U(t, s)f$, where $U(t, s)f$ is $B^{a+M}$-valued continuous and $B^a$-valued continuously differentiable function in $0 \leq s \leq t \leq T$ satisfying (3.9) with $u(s) = f$. Noting $M \geq 2$, we can easily see from the energy inequality that the solutions to (3.9) are unique in the class of $B^{a+M}$-valued continuous and $B^a$-valued continuously differentiable functions. Hence, we can see that $C_\Delta(t, s)f$ converges to $U(t, s)f$ in $B^{a+M}$ uniformly in $0 \leq s \leq t \leq T$ as $|\Delta| \to 0$. 

46
Take an arbitrary \( f \in B^a \). Let \( \Delta \) and \( \Delta' \) be subdivisions such that \( |\Delta| \leq \rho^* \) and \( |\Delta'| \leq \rho^* \). For any \( \epsilon > 0 \) we can take a \( g \in B^{a+2M} \) such that \( \|g - f\|_{B^a} < \epsilon \). Then from (9.4) we have

\[
\|C_\Delta(t,s)f - C_{\Delta'}(t,s)f\|_{B^a} \leq \|C_\Delta(t,s)g - C_{\Delta'}(t,s)g\|_{B^a} + \|C_\Delta(t,s)(f - g)\|_{B^a} + \|C_{\Delta'}(t,s)(f - g)\|_{B^{a+M}} + 2e^{K_aT}\epsilon.
\]

So,

\[
\lim_{|\Delta|,|\Delta'| \to 0} \max_{0 \leq s \leq t \leq T} \|C_\Delta(t,s)f - C_{\Delta'}(t,s)f\|_{B^a} \leq 2e^{K_aT}\epsilon. \tag{9.10}
\]

Hence, we can see that \( C_\Delta(t,s)f \) converges in \( B^a \) uniformly in \( 0 \leq s \leq t \leq T \) as \( |\Delta| \to 0 \). We write this limit as \( W(t,s)f \).

Let \( f \in B^a \). Take \( f_j \in B^{a+M} \) such that \( \lim_{j \to \infty} f_j = f \) in \( B^a \). From (9.7) we have

\[
i\bar{h}(W(t,s)f_j - f_j) = \int_s^t H(\theta) W(\theta, s) f_j d\theta.
\]

The inequality \( \|W(t,s)f\|_{B^a} \leq e^{K_a(t-s)}\|f\|_{B^a} \) holds from (9.4). So, from Lemma 2.5 in [12] with \( a = b = 1 \) we can see

\[
i\bar{h}(W(t,s)f - f) = \int_s^t H(\theta) W(\theta, s) f d\theta
\]

in \( B^{a-2} \) and that \( W(t,s)f \) is \( B^a \)-valued continuous and \( B^{a-2} \)-valued continuously differentiable in \( 0 \leq s \leq t \leq T \). Hence \( \lim_{|\Delta| \to 0} C_\Delta(t,s)f (= W(t,s)f) \) satisfies (3.9) with \( u(s) = f \). Thus, we could complete the proof of Theorem 3.1.

We shall consider the proof of Theorem 3.2. Let \( \overrightarrow{q}_{x,y}^{t,s}(\theta) \) and \( \overrightarrow{a}_{\Lambda X,Y}^{t,s}(\theta) \) be the paths defined by (6.1) and (6.2), respectively. For \( \xi_k \in R^2 \) (\( k \in \Lambda_1 \)) we define the path by

\[
\phi_{\xi_k}^{t,s}(\theta) := \xi_k + \frac{4\pi \rho_k(\overrightarrow{q}_{x,y}^{t,s}(\theta))}{|k|^2} \in R^2, \quad s \leq \theta \leq t \tag{9.11}
\]

as in (3.12). The path \( \phi_{\xi_k}^{t,s}(\theta) \in R^2 \) (\( k \in \Lambda_1 \)) is defined by (2.13). So from (2.16) and (2.17) we have

\[
\xi_{-k}^{(1)} = \xi_k^{(1)}, \quad \xi_{-k}^{(2)} = -\xi_k^{(2)}.
\]
For \( k \in \Lambda_1 \) we can easily see
\[
|k|^2 \left| \phi_{t,s}^{t,s}(\theta) \right|^2 - 8\pi \rho_k (\mathbf{q}_{x,y}^{t,s}(\theta)) \cdot \phi_{t,s}^{t,s}(\theta)
\]
\[
= |k|^2 \left| \phi_{t,s}^{t,s} - \frac{4\pi \rho_k}{|k|^2} \right|^2 - \frac{16\pi^2}{|k|^2} |\rho_k|^2
\]
\[
= |k|^2 \left| \xi_k \right|^2 - \frac{16\pi^2}{|k|^2} |\rho_k (\mathbf{q}_{x,y}^{t,s}(\theta))|^2.
\] (9.12)

So, the classical action for \( \tilde{L} \) defined by (3.11) is written as
\[
S(t,s; \mathbf{q}_{x,y}^{t,s}, a_{AX,Y}^{t,s}, \phi_{t,s}^{t,s}, \xi_k)_{k \in \Lambda_1} = S_c(t,s; \mathbf{q}_{x,y}^{t,s}, a_{AX,Y}^{t,s}) + \frac{(t-s)}{4\pi |V|} \sum_{k \in \Lambda'_1} |k|^2 |\xi_k|^2
\] (9.13)
from (2.21) and (3.3).

Let \( \chi_1 \in C^\infty(R^{2N_1}) \) with compact support such that \( \chi_1(0) = 1 \). Let \( \epsilon > 0 \) and \( \xi := \{ \xi_k \}_{k \in \Lambda'_1} \in R^{2N_1} \). For \( f \in S(R^{3n+4N}) \) we define \( G_\epsilon(t,s)f \) \((0 \leq s \leq t \leq T)\) by
\[
\left\{ \begin{array}{l}
\left( \prod_{j=1}^n \sqrt{\frac{m_j}{2\pi i(t-s)}} \right) \sqrt{\frac{1}{2\pi i|V|(t-s)}} \left( \prod_{k \in \Lambda'_1} \sqrt{\frac{|k|^2(t-s)}{4\pi^2 \hbar |V|}} \right) \\
\times \int \cdots \int e^{i\hbar^{-1}S} \chi_1(\epsilon \xi)f(y,Y)dydY \prod_{k \in \Lambda'_1} d\xi_k,
\end{array} \right.
\] 
\( s < t, \)
\[
f,
\] 
\( s = t, \) (9.14)
where \( S = S(t,s; \mathbf{q}_{x,y}^{t,s}, a_{AX,Y}^{t,s}, \phi_{t,s}^{t,s}, \xi_k)_{k \in \Lambda_1} \).

**Proposition 9.2.** Let \( f \in B^a(R^{3n+4N}) (a = 0, 1, 2, \ldots) \). Then, under the assumptions of Theorem 3.1 we have
\[
\lim_{\epsilon \to 0} G_\epsilon(t,s)f = C(t,s)f
\] (9.15)
in \( B^a \) for \( 0 \leq t - s \leq \rho^* \).
Proof. In the case of \( t = s \) (9.13) is clear from (7.6). Let \( 0 < t - s \leq \rho^* \) and \( \mathcal{S}(R^{3n+4N}) \). From (9.13) we have

\[
G_\epsilon(t, s)f = \left( \prod_{j=1}^{n} \frac{m_j}{2\pi i\hbar(t - s)} \right)^3 \frac{1}{2\pi i\hbar|V|(t - s)} \]

\[
\times \int \int (\exp i\hbar^{-1} \xi(t, \bar{s}; \alpha_{x,y}^{t,s}, \alpha_{\Lambda X,Y}^{t,s})) f(y, Y)dydY
\]

\[
\times \left( \prod_{k \in \Lambda_1} \frac{|k|^2(t - s)}{4\pi^2 \hbar|V|} \right) \int \cdots \int \left( \exp i(t - s) \sum_{k \in \Lambda_1} |k|^2 |\xi_k|^2 \right) \chi_1(\epsilon \xi) \prod_{k \in \Lambda_1} d\xi_k.
\]

Let \( \eta_k := (\eta_k^{(1)}, \eta_k^{(2)}) \in \mathbb{R}^2 \) and \( \eta := \{\eta_k\}_{k \in \Lambda_1} \). We know

\[
\int_{-\infty}^{\infty} e^{ia\theta^2} d\theta = \sqrt{\frac{1}{a}}
\]

for a constant \( a > 0 \). So we write

\[
G_\epsilon(t, s)f = P_\epsilon(t, s)f,
\]

where

\[
p_\epsilon(t, s) = \left( \prod_{k \in \Lambda_1} \frac{|k|^2}{i\pi} \right) \int \cdots \int \left( \exp i \sum_{k \in \Lambda_1} |k|^2 |\eta_k|^2 \right)
\]

\[
\times \chi(\epsilon \sqrt{4\pi \hbar|V|/(t - s)} \eta) \prod_{k \in \Lambda_1} d\eta_k.
\]

We see that

\[
\lim_{\epsilon \to 0} p_\epsilon(t, s) = 1
\]

pointwise. Letting \( q_\epsilon(t, s) = p_\epsilon(t, s) - 1 \), we have

\[
P_\epsilon(t, s)f - C(t, s)f = Q_\epsilon(t, s)f.
\]

We consider

\[
\|G_\epsilon(t, s)f - C(t, s)f\|^2 = \|P_\epsilon(t, s)f - C(t, s)f\|^2
\]

\[
= \left( (P_\epsilon(t, s) - C(t, s))^\dagger (P_\epsilon(t, s) - C(t, s))f, f \right)
\]

\[
= (Q_\epsilon(t, s)^\dagger Q_\epsilon(t, s) f, f).
\]
Hence, we obtain (9.15) as in the proof of Theorem 7.3 in the present paper together with Lemma 2.2 in [12]. See the proof of Lemma 4.1 in [15] for further details.

We can write (3.13) as

\[
\lim_{\epsilon \to 0} G_{\epsilon}(T, \tau_{\nu-1}) \chi(\epsilon) G_{\epsilon}(\tau_{\nu-1}, \tau_{\nu-2}) \chi(\epsilon) \cdots G_{\epsilon}(\tau_2, \tau_1) \chi(\epsilon) G_{\epsilon}(\tau_1, 0) \chi(\epsilon) f
\]

in the same way that (3.8) is written in the above of (9.2). Integrating by parts in (9.18), we see that \(\sup_{0 < \epsilon \leq 1} |p_{\epsilon}(t, s)|\) is finite. So the same proof as for (7.5) shows

\[
\sup_{0 < \epsilon \leq 1} \|G_{\epsilon}(t, s)f\|_{B^a} \leq C_a \|f\|_{B^a}, \quad a = 0, 1, 2, \ldots
\]

with constants \(C_a\) from (9.17). Hence, using (9.1), we can prove Theorem 3.2 as in the proof of the convergence of (3.8) to (9.2) together with (9.15).

Finally, we will prove Theorem 3.3. As in the proof of (6.15) we get

\[
\left( \int_0^{L_{s_{\nu}}} - \int_0^{L_{s_{\nu}}} \right) \left\{ \frac{1}{c} A_{ex}(t, x^{(j)}) \cdot d x^{(j)} - \phi_{ex}(t, x^{(j)}) dt \right\}
\]

\[
= \frac{1}{c} (x^{(j)} - y^{(j)}) \int_0^1 \int_0^1 A_{ex}(s, x^{(j)} - \theta(x^{(j)} - y^{(j)})) d \theta
\]

\[- (t - s)(x^{(j)} - y^{(j)}) \int_0^1 \sigma_1 E_{ex}(\tau(\sigma), \zeta^{(j)}(\sigma)) d \sigma_1 d \sigma_2
\]

\[- \frac{1}{c} \sum_{m=1}^3 \sigma_1 B_{ml}^{ex}(\tau(\sigma), \zeta^{(j)}(\sigma)) d \sigma_1 d \sigma_2,
\]

(9.21)

where \((B_{23}'(t, x), B_{31}'(t, x), B_{12}'(t, x)) = B_{ex}(t, x), B_{lm}' = -B_{ml}'\), and \(\tau(\sigma)\) and \(\zeta^{(j)}(\sigma)\) were defined by (6.11). See the proof of Proposition 3.3 in [13] for further details. So, we get the equation (6.18) where the sum over \(j = 1, 2, \ldots, n\) of (9.21) multiplied by \(m_j e_j / (t - s)\) is added to. Hence, under the assumptions of Theorem 3.3 we obtain the same assertion as in Theorem 3.1 in the same way that Theorem 3.1 is proved. As in the same way of the proof of Theorem 3.2 we also get the same assertion as in Theorem 3.2 under the assumptions of Theorem 3.3. Thus, we could complete the proof of the main results.
References

[1] A. Arai, *Fock Space and Quantum Field* (in Japanese), Nihon Hyoron Co., Tokyo, 2000.

[2] F. A. Berezin and M. A. Shubin, *The Schrödinger Equation*, Kluwer Academic Publishers, Dordrecht, 1983.

[3] A. P. Calderón and R. Vaillancourt, On the boundedness of pseudo-differential operators, *J. Math. Soc. Japan* 23 (1971), 374-378.

[4] P. A. M. Dirac, *The Principles of Quantum Mechanics* 4th ed, Oxford Univ. Press, London, 1958.

[5] E. Fermi, Quantum theory of radiation, *Rev. Modern Phys.* 4 (1932), 87-132.

[6] R. P. Feynman, Space-time approach to nonrelativistic quantum mechanics, *Rev. Modern Phys.* 20 (1948), 367-387.

[7] R. P. Feynman, Mathematical formulation of the quantum theory of electrodynamic interaction, *Phys. Rev.* 80 (1950), 440-457.

[8] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[9] I. M. Gel’fand and N. Y. Vilenkin, *Generalized Functions. Vol. IV, Applications of Harmonic Analysis*, Academic Press, New York-London, 1964.

[10] S. J. Gustafson and I. M. Sigal, *Mathematical Concepts of Quantum Mechanics*, Springer, Berlin, 2003.

[11] F. Hiroshima, Functional integral representation of a model in quantum electrodynamics, *Rev. Math. Phys.* 9 (1997), 489-530.

[12] W. Ichinose, A note on the existence and $h$-dependency of the solution of equations in quantum mechanics, *Osaka J. Math.* 32 (1995), 327-345.
[13] W. Ichinose, On the formulation of the Feynman path integral through broken line paths, *Commun. Math. Phys.* **189** (1997), 17-33.

[14] W. Ichinose, On convergence of the Feynman path integral formulated through broken line paths, *Rev. Math. Phys.* **11** (1999), 1001-1025.

[15] W. Ichinose, The phase space Feynman path integral with gauge invariance and its convergence, *Rev. Math. Phys.* **12** (2000), 1451-1463.

[16] W. Ichinose, Convergence of the Feynman path integral in the weighted Sobolev spaces and the representation of correlation functions, *J. Math. Soc. Japan* **55** (2003), 957-983.

[17] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman’s Operational Calculus*, Oxford Univ. Press, Oxford, 2000.

[18] H. Kumano-go, *Pseudo-Differential Operators*, MIT Press, Cambridge, 1981.

[19] E. H. Lieb and M. Loss, *Analysis*, AMS, Providence, 1997.

[20] W. Pauli and M. Fierz, Zur theorie der Emission langwellinger Lichtquanten, *Nuovo Cimento* **15** (1938), 167-188.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[22] J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Massachusetts, 1967.

[23] J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach Science Publishers, New York, 1969.

[24] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, Cambridge, 2004.

[25] M. S. Swanson, *Path Integrals and Quantum Processes*, Academic Press, San Diego, 1992.