Navigating catastrophes: Local but not global optimisation allows for macro-economic navigation of crises

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Abstract. Two aspects of modern economic theory have dominated the recent discussion on the state of the global economy: Crashes in financial markets and whether or not traditional notions of economic equilibrium have any validity. We have all seen the consequences of market crashes: plummeting share prices, businesses collapsing and considerable uncertainty throughout the global economy. This seems contrary to what might be expected of a system in equilibrium where growth dominates the relatively minor fluctuations in prices. Recent work from within economics as well as by physicists, psychologists and computational scientists has significantly improved our understanding of the more complex aspects of these systems. With this interdisciplinary approach in mind, a behavioural economics model of local optimisation is introduced and three general properties are proven. The first is that under very specific conditions local optimisation leads to a conventional macro-economic notion of a global equilibrium. The second is that if both global optimisation and economic growth are required then under very mild assumptions market catastrophes are an unavoidable consequence. Third, if only local optimisation and economic growth are required then there is sufficient parametric freedom for macro-economic policy makers to steer an economy around catastrophes without overtly disrupting local optimisation.

1. Optimisation in Localised and Global Economies

In this study the Maximum Entropy (MaxEnt) technique is applied to a simple economy in which production output is set through a competitive financial market mechanism. A justification for the use of MaxEnt in economics is given in [1] but an alternative is introduced here. MaxEnt provides a way of finding the least structured distribution over output choices for a given set of economic constraints. As entropy over production choices decreases, this lowers the level of output for products with a lower profit and the minimum entropy is achieved when only the most profitable products are produced. Across an entire economy this may not be desirable as variety in economic output provides a level of economic security. MaxEnt provides a mechanism to achieve this: Given a minimum Gross Domestic Product (GDP) target to reach (which may increase over time), produce the most diverse range and largest quantity of different products across the economy. If economic targets allow for it, producers should tend to produce anything and everything in equal proportion. However, in practice economic constraints require choosing to manufacture some products and not others, and shifting economic targets and competition may mean some products are not economically viable. This study considers the most basic elements of a model that instantiates these ideas.
1.1. Global Optimisation of a Simple Economy

Following [1] there are two groups of economic agent, \( i \) and \(-i\), in a joint strategy economy \( x_j, y_k, j, k \in \{1, 2\} \). Each group contributes to economic output (GDP) based on their choices:

\[
\begin{bmatrix}
u^i_{x,y}^1 \\
u^i_{x,y}^2 \\
u^{-i}_{x,y}^1 \\
u^{-i}_{x,y}^2 
\end{bmatrix} = 
\begin{bmatrix}
u^i_{x_1,y_1} \\
u^i_{x_2,y_1} \\
u^{-i}_{x_1,y_2} \\
u^{-i}_{x_2,y_2} 
\end{bmatrix},
\]

The expected GDP is found via the probability distributions \( q_i \) and \( q_{-i} \) over choices:

\[
E(u^i) = GDP_i = \sum_{x,y} q_i(x)q_{-i}(y)u^i_{x,y}, \quad E(u^{-i}) = GDP_{-i} = \sum_{x,y} q_i(x)q_{-i}(y)u^{-i}_{x,y}
\]

The constrained optimisation is to maximise the entropy subject to a minimum GDP:

\[
\max_{f(q_i,q_{-i})} S(f(q_i,q_{-i})) = \max_{f(q_i,q_{-i})} \left(-\sum_{x,y} f(q_i,q_{-i}) \log(f(q_i,q_{-i}))\right)
\]

subject to the constraints:

\[
f(q_i,q_{-i}) \geq 0 \forall x,y
\]

\[
\sum_{x,y} f(q_i,q_{-i}) = 1
\]

\[
\sum_{x,y} f(q_i,q_{-i})u^i_{x,y} + \sum_{x,y} f(q_i,q_{-i})u^{-i}_{x,y} = \sum_{x,y} f(q_i,q_{-i})(u^i_{x,y} + u^{-i}_{x,y}) \geq GDP
\]

where total \( GDP = GDP_i + GDP_{-i} \). The solution to this optimisation is to form the Lagrangian \( \mathcal{L}(f(q_i,q_{-i})) \), find the stationary point and solve for \( f(q_i,q_{-i}) \):

\[
\mathcal{L}(f(q_i,q_{-i})) = S(f(q_i,q_{-i})) + \beta \left( \sum_{x,y} f(q_i,q_{-i})(u^i_{x,y} + u^{-i}_{x,y}) \right) + \beta_0 \sum_{x,y} f(q_i,q_{-i})
\]

\[
f(q_i,q_{-i}) = Z^{-1} \exp \left( \beta \sum_{x,y} (u^i_{x,y} + u^{-i}_{x,y}) \right)
\]

where \( Z^{-1} \) is the normalisation factor and \( \beta \) enforces the minimum GDP constraint. In a free market with no collusion between participants \( q_i \) and \( q_{-i} \) are independent: \( f(q_i,q_{-i}) = q_iq_{-i} \) and the distributions are separable: \( q_iq_{-i} \propto \exp \left( \beta \sum_{x,y} u^i_{x,y} \right) \exp \left( \beta \sum_{x,y} u^{-i}_{x,y} \right) \). To find the joint probability when it is in a global equilibrium state simply substitute in all the values for \( u^i_{x,y} \) and \( u^{-i}_{x,y} \), which are known from the problem definition. However, to then ask “What is the probability that \( q_i \) chooses \( x = x_j \)” is an ill-posed question, setting \( x = x_j \) and using \( q_i(x = x_j) \propto \exp \left( \beta \sum_{x,y} u^i_{x,y} \right) \) has no unique solution as \( y \) is still undetermined, i.e. global optimisation has insufficient information for local decisions. This is fixed by assuming that \( i \) is able to estimate \( q_{-i} \), learned through previous interactions or some other means. The problem of finding a clearly defined local distribution is now well-posed: \( q_i(x = x_j|q_{-i}) \propto \exp \left( \beta \sum_y q_{-i}u^i_{x,j,y} \right) \).

1.2. Local Optimisation Within a Simple Economy

Now we start from the other extreme, given a local optimisation problem what is the form of \( q_i \) and \( q_{-i} \)? Each set of economic agents has to maximise an entropy function subject to the constraint that their contribution to the GDP = \( GDP_i + GDP_{-i} \) reaches a minimum level:

\[
\max_{q_i} S(q_i) = \max_{q_i} \left(-\sum_x q_i \log(q_i)\right)
\]
subject to the constraints:

\[ q_i \geq 0 \forall x \]  
\[ \sum_x q_i = 1 \]  
\[ \sum_{x,y} q_i q_{-i} u_{x,y}^i \geq \text{GDP}_i \]

Following an analogous process to the global case we have:

\[ \mathcal{L}(q_i) = S(q_i) + \beta_i \left( \sum_{x,y} q_i, q_{-i} u_{x,y}^i \right) + \beta_0 \sum_x q_i \]

\[ q_i = Z_i^{-1} \exp \left( \beta_i \sum_{x,y} q_{-i} u_{x,y}^i \right), \quad q_{-i} = Z_{-i}^{-1} \exp \left( \beta_{-i} \sum_{x,y} q_i u_{x,y}^i \right) \]

Note that the need to post-hoc stipulate an estimate for \( q_{-i} \) (when we optimise for \( i \)) disappears as \( q_{-i} \) does not vanish when we differentiate with respect to \( q_i \) as it does when we differentiated with respect to \( f(q_i, q_{-i}) \). We also note that Equation 12 is identical with the Quantal Response Equilibrium (QRE) of MacKelvey and Palfrey [2] although the derivation used here was only introduced recently [1] and that Equation 6 is a macroscopic (MaxEnt) and previously unreported equivalent of the derivation of the QRE.

In the next section a self-consistent economic solution is one whereby both locally optimising economic agents as well as globally optimising economic agents can find numerical values for their probability distributions. A parametric path through an optimised economy is a slow variation in \( \beta \) for Equation 6 or a joint variation in \( \beta_i \) and \( \beta_{-i} \) in Equation 12 and its \( q_{-i} \) counterpart. The QRE (Equation 12) contains the Nash Equilibria (NE) as a special case where both \( \lim_{\beta_i \rightarrow \infty} q_i \) and \( \lim_{\beta_{-i} \rightarrow \infty} q_{-i} \) [2]. A disordered economic state is when \( \beta = 0 \) (global case) or \( \beta_i = 0 \) and \( \beta_{-i} = 0 \) (local case), at which point there is only ever one equilibrium point [2] and it is a uniform distribution over choices. An Economic Catastrophe [3] is where a disproportionate shift in the optimised solution occurs for a relatively small change in a macroscopic economic parameter [1] such as \( \beta, \beta_i \) or \( \beta_{-i} \), illustrations of this are shown below.

2. Main Results

Theorem 1. Local Maximum Entropy optimisation contains self-consistent global Maximum Entropy optimisation as a special case.

Proof: From the global optimisation case, Equation 6, we have:

\[ f(q_i, q_{-i}) = q_i q_{-i} = Z^{-1} \exp \left( \beta \sum_{x,y} (u_{x,y}^i + u_{x,y}^{-i}) \right) \]

\[ = Z^{-1} \exp \left( \beta \sum_{x,y} u_{x,y}^i \right) \exp \left( \beta \sum_{x,y} u_{x,y}^{-i} \right) \text{ not self-consistent} \]

\[ \rightarrow q_i q_{-i} = Z^{-1} \exp \left( \beta \sum_{x,y} q_{-i} u_{x,y}^i \right) \exp \left( \beta \sum_{x,y} q_i u_{x,y}^{-i} \right) \text{ self-consistent} \]

From the local optimisation case, Equation 12, we have:

\[ q_i q_{-i} = Z_i^{-1} Z_{-i}^{-1} \exp \left( \beta_i \sum_{x,y} q_{-i} u_{x,y}^i \right) \exp \left( \beta_{-i} \sum_{x,y} q_i u_{x,y}^{-i} \right) \]

For the special case where \( \beta_i = \beta_{-i} \) the two exponentials in Equation 14 are equivalent to those in Equation 13. As both of these representations of \( q_i q_{-i} \) are normalised then \( Z = Z_i Z_{-i} \) and therefore self-consistent global optimisation is a special case of local optimisation.
Theorem 2. If an equilibrium point is not a Pareto Efficient equilibrium point then there is no parametric path in global optimisation that can reach a Pareto Efficient point without going through an Economic Catastrophe.

Proof: If there are 3 unique NE then there is always a bifurcation point on the 45° path $\beta_i = \beta_{-i} = \beta$ at which multiple equilibria appear. At least one of these points is a Pareto Efficient point and at least one other is not, if the economy is not at the Pareto Efficient point for a fixed $\beta$, no variation in $\beta$ can arrive at a Pareto Efficient point without a discontinuous step change in $Q_i$, see Figure 1.

Theorem 3. Theorem 2 is not always true for local optimisation.

Proof: By inspection the left plot of Figure 1 shows a path is possible from the lower surface to the upper surface that does not require a discontinuous step in $Q_i$.

**Figure 1.** Two Equilibrium surfaces for two different economies each with 3 NE. The vertical axis is a rescaled probability: $Q_i = 1 - 2q_i$. The 45° path always passes through a bifurcation if there are 3 NE. The left figure has a path from the top surface to the bottom surface for suitable variation $\beta_i$ and $\beta_{-i}$, this is not possible for the right figure.

3. Discussion

Bifurcations occur in financial markets [4] and macro-economics [3]. Recent evidence from econophysics has shown signatures of phase-transitions in both the macroscopic [5] and microscopic [6] interactions in financial markets. This work shows that the way in which such systems are optimised has significant impact on the flexibility and manageability of an economy. Importantly, early warning signatures are detectable before a catastrophic crash [7] and so the possibility of ‘steering’ an economy around such phase transitions as described in this work is possible.

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