STABLE AND $L^2$-COHOMOLOGY OF ARITHMETIC GROUPS

BY A. BOREL

Introduction. In [1], [2] we gave a range of dimensions in which the real cohomology of an arithmetic or $S$-arithmetic subgroup $\Gamma$ of a connected semi-simple group $G$ over $\mathbb{Q}$ is naturally isomorphic to the space of harmonic forms on the quotient $X = G(\mathbb{R})/K$ of the group $G(\mathbb{R})$ of real points of $G$ by a maximal compact subgroup $K$ which are invariant under $\Gamma$ and the identity component $G(\mathbb{R})^0$ of $G(\mathbb{R})$, and indicated some applications to the stable cohomology of classical arithmetic groups and to algebraic $K$-theory. In this note we first state an extension to nontrivial coefficients, since this has become of interest in topology and $K$-theory [7]. A chief tool in [2] was the proof that $H^*(\Gamma; \mathbb{C})$ could be computed using differential forms on $\Gamma \backslash X$ which have “logarithmic growth” at infinity. Theorem 2 extends this to more general growth conditions. This can be used to show that certain $L^2$-harmonic forms are not cohomologous to zero [9]. In §3, 4, 5 we consider the $L^2$-cohomology space $H^2(\Gamma \backslash X)$ and relate it to the spectral decomposition of the space $L^2(\Gamma \backslash G)$ of square integrable functions on $\Gamma \backslash G$. Theorem 4 gives a sufficient condition under which it is finite dimensional, hence isomorphic to the space of square integrable harmonic forms, and §5 a series of examples in which it is not. For convenience, we assume $G$ simple over $\mathbb{Q}$ and $\Gamma$ torsion-free.

1. Let $P_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of $G$, $S$ a maximal $\mathbb{Q}$-split torus of $P_0$, $N$ the unipotent radical of $P$ and $n$ the Lie algebra of $N$. Let $X(S)$ be the group of rational characters of $S$ and $\rho \in X(S)$ be such that $a^{2\rho} = \det \text{Ad} a|_n$ for $a \in S$. For $\mu \in X(S)$ let $c(G, \mu)$ be the maximum of $q$ such that $\rho - \mu - \eta > 0$, where $\eta$ runs through the weights of $S$ in $\Lambda^q n$. Let $c(G) = c(G, 0)$. If $(r, E)$ is a finite-dimensional complex representation of $G(\mathbb{C})$, we let $c(G, r)$ be the minimum of $c(G, \mu)$, where $\mu$ runs through the weights of $r$ with respect to $S$. It is easily seen that $c(G) \geq \Sigma_i c(G_i)$, where $G_i$ runs through the simple factors of $G(\mathbb{C})$, and $c(G_i)$ is defined similarly, and that $c(G_i)$ is equal to $[(l - 1)/2], l - 1, l - 2, l - 1, 7, 13, 25, 5, 1$ if $G_i$ is of type $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$.

THEOREM 1. The natural homomorphism $H^*(\Gamma; f ; E) \rightarrow H^q(\Gamma; E)$ is injective for $q \leq c(G, r)$, surjective if in addition $q < \text{rk}_R G$. If $E^G = (0)$, then $H^q(\Gamma; E) = 0$ for $q \leq c(G, r)$, $(\text{rk}_R G - 1)$. If $G$ is simply connected, these
assertions remain true if \( \Gamma \) is replaced by an \( S \)-arithmetic subgroup or by \( G(\mathbb{Q}) \).

Here \( g \) and \( \mathfrak{f} \) stand for the Lie algebras of \( G(\mathbb{R}) \) and \( K \). See [5] for relative Lie algebra cohomology. The proofs of these statements are similar in principle to those given or sketched in [1], [2] when \( r \) is the trivial representation, and moreover, take into account some results proved in [5]. If we have an inductive system of groups and representations without trivial constituents \( (G_n, \Gamma_n, r_n, E_n) \) such as \( (G, \Gamma, r, E) \) and if \( c(G_n, r_n) \to \infty \), then Theorem 1 implies that \( H^q(\lim \Gamma_n, \lim E_n) = 0 \) for \( q > 0 \).

2. On Siegel sets, we consider coefficients of differential forms with respect to special frames, as in [2]. For \( \lambda \in X(S) \) we say that \( \eta \in \Omega_{\lambda+}(\Gamma\backslash X) \) if the coefficients of \( \eta \) and of \( d\eta \) satisfy a growth condition,

\[
|f(x)| < a(x)^\lambda |P(\log a^{\alpha_1}, \ldots, \log a^{\alpha_l})|,
\]

where \( \alpha_1, \ldots, \alpha_l \) are the simple \( \mathbb{Q} \)-roots and \( P \) is a polynomial in \( l \) variables (\( l = \dim S \)). The proof of the following theorem is analogous to that of 7.4 in [2].

**Theorem 2.** If \( \lambda \) is dominant, then the injection \( \Omega_{\lambda+}(\Gamma\backslash X) \to \Omega(\Gamma\backslash X) \) induces an isomorphism in cohomology. The elements of \( \Omega_{\lambda+}^q \) are square integrable if \( q < c(G, \lambda) \). The space of square integrable harmonic \( q \)-forms contained in \( \Omega_{\lambda+}(\Gamma\backslash X) \) maps injectively into the cohomology of \( \Gamma \) for \( q < c(G, \lambda) + 1 \).

If \( \lambda < 0 \), then \( H^*(\Omega_{\lambda+}) \) is canonically isomorphic to the complex cohomology with compact supports of \( \Gamma\backslash X \).

3. Let \( M \) be a Riemannian manifold. Let \( \Omega^2(M) \) be the complex of differential forms \( \eta \) on \( M \) such that \( \eta \) and \( d\eta \) are square integrable. By definition \( H^2(M) = H^*(\Omega^2(M)) \) is the space of \( L^2 \)-cohomology of \( M \). (See [6], where equivalent Hilbert space definitions are given.) Let \( H^2(M) \) be the space of \( L^2 \)-harmonic forms. It is known that if \( M \) is complete, then the natural map \( j : H^2(M) \to H^2(M) \) is injective. If \( M \) is compact, then \( j \) is an isomorphism and \( H^2(M) = H^*(M; \mathbb{C}) \).

**Theorem 3.** There are canonical isomorphisms

\[
H^2(\Gamma\backslash X) = H^*(g, K; L^2(\Gamma\backslash G)^\infty) \]

and

\[
H^2(\Gamma\backslash G) = H^*(g; L^2(\Gamma\backslash G)^\infty) \]

As usual, if \( (\pi, V) \) denotes a unitary representation of \( G(\mathbb{R}) \) then \( V^\infty \) denotes the space of \( C^\infty \)-vectors in \( V \). To establish Theorem 3, one proves first the second statement using a homotopy operator defined by the convolution by a compactly supported smooth function on \( G \), and then deduces the first one by the comparison theorem for spectral sequences, applied to suitable spectral sequences in relative Lie algebra cohomology.
4. The space \( L^2(\Gamma \backslash G) \) is the sum of the discrete spectrum \( L^2(\Gamma \backslash G)_d \) and the continuous spectrum \( L^2(\Gamma \backslash G)_c \). By results obtained jointly with H. Garland [3], [4], \( H(2)(\Gamma \backslash X) \) is finite dimensional and is the direct sum of the spaces \( H^*(g, K; H^2_i) \), where \( H \) runs through a set of irreducible constituents of \( L^2(\Gamma \backslash G)_d \). By [8], \( L^2(\Gamma \backslash G)_{ct} \) is a Hilbert direct sum of invariant subspaces, say \( V_i \) \((i \in I)\), each of which is a continuous integral of unitarily induced principal series (from parabolic Q-subgroups). By [4], \( H^*(g, K; L^2(\Gamma \backslash G)_{ct}^m) \) is the sum of the \( H^*(g, K; V_i^m) \) and can be nonzero only for finitely many terms. Those spaces can be computed as in [5, III] and can be nonzero only if the underlying parabolic subgroup is fundamental [5, IV] in \( G(\mathbb{R}) \). Together with Theorem 3, this proves

**THEOREM 4.** The map \( j: H(2)(\Gamma \backslash X) \rightarrow H(2)(\Gamma \backslash X) \) is an isomorphism if \( G \) has no proper parabolic Q-subgroup which is fundamental in \( G(\mathbb{R}) \), in particular if rank \( G = \text{rank } K \).

5. It is rather likely that if \( G \) has a proper fundamental parabolic subgroup \( P_1 \) defined over \( \mathbb{Q} \), then \( \Gamma \) has a subgroup \( \Gamma' \) of finite index such that \( H(2)(\Gamma \backslash X) \) is infinite dimensional. This has been checked in a number of cases: (i) \( G = \text{SO}(n, 1) \) for \( n \geq 3 \) odd (with \( \Gamma = \Gamma' \)); (ii) the group \( P_1 \) is minimal over \( \mathbb{R} \); (iii) \( G = \text{SL}_n(\mathbb{R}) \) and \( \Gamma \subset \text{SL}_n(\mathbb{Z}) \). In those cases, infinite-dimensional cohomology occurs exactly in the dimensions \( q \) such that \( \dim X - l_0 < 2q < \dim X + l_0 \), where \( l_0 = \text{rank } G - \text{rank } K \).

**REFERENCES**

1. A. Borel, *Cohomologie réelle stable de groupes S-arithmétiques classiques*, C.R. Acad. Sci. Paris 274 (1972), 1700–1702.
2. ———, *Stable real cohomology of arithmetic groups*, Annales E.N.S. Paris (4) 7 (1974), 235–272.
3. ———, *Cohomology of arithmetic groups*, Proc. Internat. Congr. Math. Vancouver, 1974, vol. 1, pp. 435–442.
4. A. Borel and H. Garland, *Laplacian and discrete spectrum of an arithmetic group* (in preparation).
5. A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies, No. 94, Princeton Univ. Press, Princeton, N.J., 1980.
6. J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.
7. T. Farrell and W. C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Proc. Sympos. Pure Math. vol. 32, part 1, Amer. Math. Soc., Providence, R.I., 1978, pp. 403–415.
8. R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math., vol. 544, Springer-Verlag, Berlin and New York, 1976.
9. N. Wallach, *L² automorphic forms and cohomology classes on arithmetic quotients of SU(p, q)* (to appear).

**SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540**