COUNTING PATHS IN PERFECT TREES

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Abstract. We present some exact expressions for the number of paths of a given length in a perfect $m$-ary tree. We first count the paths in perfect rooted $m$-ary trees and then use the results to determine the number of paths in perfect unrooted $m$-ary trees, extending a known result for binary trees.

1. Introduction

A tree $T = (V, E)$ is a connected acyclic graph with a finite vertex set $V$ and finite edge set $E \subseteq \binom{V}{2}$. The distance $d_T(u, v)$ between two vertices $u, v \in V$ is the number of edges in the (unique) path in $T$ that joins $u$ and $v$. In this paper, we focus on counting the pairs of vertices that are some given distance apart, or equivalently the paths of a given length, in a perfect tree.

Given a tree $T$, let $P(T, t)$ denote the number of pairs of vertices at distance exactly $t \geq 1$ from each other. That is,

$$P(T, t) = \left| \{ u, v \in \binom{V}{2} : d_T(u, v) = t \} \right| \quad \text{and} \quad \sum_{t \geq 1} P(T, t) = \frac{|V|(|V| - 1)}{2}.$$ 

Note immediately that $P(T, 1) = |E|$. Furthermore, from the observations that each vertex $v$ of degree $\deg(v)$ is the central vertex of $\binom{\deg(v)}{2}$ distinct paths of length 2 and that each edge $\{u, v\}$ is the central edge of $(\deg(u) - 1)(\deg(v) - 1)$ distinct paths of length 3, we obtain

$$P(T, 2) = \sum_{v \in V} \binom{\deg(v)}{2} \quad \text{and} \quad P(T, 3) = \sum_{\{u, v\} \in E} (\deg(u) - 1)(\deg(v) - 1).$$

Similar expressions for $P(T, t)$ when $t \geq 4$ become increasingly complicated.

Faudree et al. [3] constructed examples showing that two non-isomorphic trees $T_1, T_2$ can have identical path length distributions (that is, $P(T_1, t) = P(T_2, t)$ for all $t$). Tight upper and lower bounds for $P(T, t)$ were given by Dankelmann [1] in terms of $|V|$ and either the radius or diameter of $T$.

A binary tree $T$, in which every vertex has degree 1 or degree 3, is perfect (or balanced) if $T$ has the maximum number of vertices among all binary trees of the same diameter. De Jong et al. [2] used a recursive approach to show that the perfect binary tree $T$ of diameter $D$ with $n$ degree-1 vertices has

$$P(T, t) = \begin{cases} 2^{\frac{t+1}{2}} \left( n - 3 \cdot 2^{\frac{t-3}{2}} \right), & t \text{ odd,} \\ 3 \cdot 2^{\frac{t-3}{2}} - 1 \left( n - 2^{\frac{t-1}{2}} \right), & t \text{ even} \end{cases}$$

paths of length $t$ for $3 \leq t \leq D$. 

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Date: November 27, 2017.

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We adopt a different approach to extend this to perfect \( m \)-ary trees, where each vertex has degree 1 or \( m + 1 \). In particular, we prove the following theorem.

**Theorem 1.1.** Let \( T \) be the perfect unrooted \( m \)-ary tree of diameter \( D \). Then, for \( 1 \leq t \leq D \),
\[
P(T, t) = \begin{cases} m^{\frac{t-1}{2}} (V(D) - V(t-1)), & t \text{ odd,} \\ \frac{1}{2} (m+1)m^{\frac{t}{2}-1} (V(D) - V(t-1)), & t \text{ even,} \end{cases}
\]
where \( V(d) \) is the number of vertices in the perfect unrooted \( m \)-ary tree of diameter \( d \).

We first derive an analogous theorem for perfect rooted \( m \)-ary trees, where the root has degree \( m \) and all other vertices have degree 1 or \( m + 1 \). Theorem 1.2 is obtained in Section 2 by counting the paths of length \( t \) in a perfect rooted \( m \)-ary tree according to minimum depth, considering odd \( t \) and even \( t \) separately.

**Theorem 1.2.** Let \( T \) be the perfect rooted \( m \)-ary tree of depth \( r \), and let \( t \) satisfy \( 1 \leq t \leq 2r \). If \( t \) is odd, then
\[
P(T, t) = \begin{cases} m^{\frac{t-1}{2}} (V_R(r) - V_R(\frac{t-1}{2})) - \frac{1}{2} m^{t-1}, & t \leq r, \\ m^{\frac{t}{2}} (V_R(r) - V_R(\frac{t}{2})) - (r - \frac{t}{2} + 1) m^{t-1}, & t > r, \end{cases}
\]
and if \( t \) is even, then
\[
P(T, t) = \begin{cases} \frac{1}{2} (m+1)m^{\frac{t-1}{2}} (V_R(r) - V_R(\frac{t}{2} - 1)) - \frac{1}{2} m^{t-1}, & t \leq r, \\ \frac{1}{2} (m+1)m^{\frac{t}{2}-1} (V_R(r) - V_R(\frac{t}{2} - 1)) - (r - \frac{t}{2} + 1) m^{t-1}, & t > r, \end{cases}
\]
where \( V_R(d) \) is the number of vertices in the perfect rooted \( m \)-ary tree of depth \( d \).

In Section 3 we use the results from Section 2 to prove Theorem 1.1.

2. **Perfect rooted \( m \)-ary trees**

In a rooted \( m \)-ary tree \( T = (V, E) \), there is a distinguished vertex \( \rho \) of degree \( m \) called the root, while every other vertex has degree 1 or \( m + 1 \). The depth \( r \) of \( T \) is the maximum value of \( d_T(\rho, v) \) over all vertices \( v \in V \). We call \( T \) perfect if and only if every degree-1 vertex is distance \( r \) from the root \( \rho \).

Let \( T \) be the perfect rooted \( m \)-ary tree of depth \( r \). For \( 0 \leq s \leq r \), there are exactly \( m^s \) vertices \( v \in V \) for which \( d_T(\rho, v) = s \). Let \( p = v_0 \cdots v_t \) be a path of length \( t \) in \( T \). Then there is a unique vertex \( v_s \), \( 0 \leq s \leq t/2 \), such that \( d_T(\rho, v_s) \leq d_T(\rho, v_i) \) for all \( 0 \leq i \leq t \). We call \( p \) a type-\([s, t-s] \) path rooted at \( v_s \).

**Lemma 2.1.** Let \( T \) be the perfect rooted \( m \)-ary tree of depth \( r \). If \( r < t - s \), then the number of type-\([s, t-s] \) paths in \( T \) rooted at \( \rho \) is 0. If \( r \geq t - s \), then the number of type-\([s, t-s] \) paths in \( T \) rooted at \( \rho \) is
\[
\begin{align*}
\begin{cases} m^t, & s = 0, \\
(m-1)m^{t-1}, & 0 < s < \frac{t}{2}, \\
\frac{1}{2} (m-1)m^{t-1}, & s = \frac{t}{2}. 
\end{cases}
\end{align*}
\]

**Proof.** The case \( r < t - s \) is obvious, as is the case \( r \geq t - s \) with \( s = 0 \). Assume that \( r \geq t - s \) and that \( 0 < s < \frac{t}{2} \). Then any type-\([s, t-s] \) path can be decomposed into a type-\([0, s] \) path rooted at \( \rho \) and a type-\([0, t-s] \) path rooted at \( \rho \), where these two paths are disjoint. There are \( m^s \) choices for
the type-\([0, s]\) path. Once this choice has been made, there are \((m - 1)m^{t-s-1}\) choices for the type-\([0, t-s]\) path, so the total number of type-\([s, t-s]\) paths rooted at \(\rho\) is \(2\binom{m}{2}m^{t-2} = (m - 1)m^{t-1}\). If \(s = \frac{t}{2}\), then this argument counts each type-\([s, s]\) path twice, hence the third equality.

Let \(P_m(r, t)\) denote the number of paths of length \(t\) in the perfect rooted \(m\)-ary tree of depth \(r\). The preceding lemma can be used to derive exact expressions for \(P_m(r, t)\). We consider paths of odd length and paths of even length separately, and make repeated use of the identity

\[\sum_{i=a}^{b} m^i = \frac{m^{b+1} - m^a}{m - 1}.\]

**Proposition 2.2.** The number of paths of length \(t = 2k - 1\) in the perfect rooted \(m\)-ary tree of depth \(r\), where \(1 \leq k \leq r\), is

\[P_m(r, t) = \begin{cases} m^{2k-2} \left( \frac{m^{r-s-1}}{m-1} - (r - k + 2) \right), & r < 2k - 1, \\ m^{2k-2} \left( \frac{m^{r-s-1}}{m-1} - k \right), & r \geq 2k - 1. \end{cases}\]

**Proof.** Let \(T\) be the perfect rooted \(m\)-ary tree of depth \(r\). If \(r < k\), then the longest path in \(T\) has length \(2r < t\), and so \(P_m(r, t) = 0\).

If \(k \leq r < 2k - 1\) and \(2k - r - 1 \leq s \leq k - 1\), then by Lemma 2.1 there are \((m - 1)m^{t-1}\) type-\([s, t-s]\) paths rooted at each vertex \(v\) for which \(d_T(\rho, v) \leq r - t + s\). Therefore,

\[P_m(r, t) = (m - 1)m^{2k-2} \sum_{s=2k-r-1}^{k-1} \left( \sum_{d=0}^{r-2k+s-1} m^d \right) \]

\[= (m - 1)m^{2k-2} \sum_{s=2k-r-1}^{k-1} \frac{m^{r-2k+s-1} - 1}{m - 1} \]

\[= m^{2k-2} \sum_{i=1}^{r-k+1} (m^i - 1) \]

\[= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - (r - k + 2) \right). \]

If \(r \geq 2k - 1\), then there are \(m^t\) type-\([0, t]\) paths rooted at each vertex \(v\) for which \(d_T(\rho, v) \leq r - t\). Furthermore, for \(1 \leq s \leq k - 1\), there are \((m - 1)m^{t-1}\) type-\([s, t-s]\) paths rooted at each vertex.
for which $d_T(p, v) \leq r - t + s$. Therefore,

$$P_m(r, t) = m^{2k-1} \sum_{d=0}^{r-2k+1} m^d + (m-1)m^{2k-2} \sum_{s=1}^{r-2k+s+1} \left( \sum_{d=0}^{s-2k+s+1} m^d \right)$$

$$= m^{2k-2} \left( m \left( \frac{m^{r-2k+2} - 1}{m - 1} \right) + (m-1) \sum_{s=1}^{k-1} \frac{m^{r-2k+s+2} - 1}{m - 1} \right)$$

$$= m^{2k-2} \left( \frac{m^{r-2k+3} - m}{m - 1} + \sum_{i=r-2k+3}^{r-k+1} (m^i - 1) \right)$$

$$= m^{2k-2} \left( \frac{m^{r-2k+3} - m}{m - 1} + \frac{m^{r-k+2} - m^{r-2k+3}}{m - 1} - (k-1) \right)$$

$$= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - k \right).$$

\[\square\]

**Proposition 2.3.** The number of paths of length $t = 2k$ in the perfect rooted $m$-ary tree of depth $r$, where $1 \leq k \leq r$, is

$$P_m(r, t) = \begin{cases} 
  m^{2k-1} \left( \frac{1}{2}(m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - (r - k + 1) \right), & r < 2k; \\
  m^{2k-1} \left( \frac{1}{2}(m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - k \right), & r \geq 2k. 
\end{cases}$$

**Proof.** Let $T$ be the perfect rooted $m$-ary tree of depth $r$. If $r < k$, then the longest path in $T$ has length $2r < t$, and so $P(d, t) = 0$.

If $k \leq r < 2k$, then by Lemma 2.1 there are $\frac{1}{2}(m - 1)m^{t-1}$ type-$[k, k]$ paths rooted at each vertex $v$ for which $d_T(p, v) \leq r - k$. Furthermore, for $2k - r \leq s \leq k - 1$, there are $(m - 1)m^{t-1}$ type-$[s, t - s]$ paths rooted at each vertex $v$ for which $d_T(p, v) \leq r - t + s$. Therefore,

$$P_m(r, t) = \frac{1}{2} (m - 1)m^{2k-1} \sum_{d=0}^{r-k} m^d + (m-1)m^{2k-1} \sum_{s=2k-r}^{r-k-1} \left( \sum_{d=0}^{s-2k+s+1} m^d \right)$$

$$= m^{2k-1} \left( \frac{1}{2}(m - 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) + (m-1) \sum_{s=2k-r}^{k-1} \frac{m^{r-2k+s+1} - 1}{m - 1} \right)$$

$$= m^{2k-1} \left( \frac{1}{2}(m - 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) + \sum_{i=1}^{r-k} (m^i - 1) \right)$$

$$= m^{2k-1} \left( \frac{1}{2} \left( m^{r-k+1} - 1 \right) + \frac{m^{r-k+1} - m}{m - 1} - (r - k) \right)$$

$$= m^{2k-1} \left( \frac{1}{2}(m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - (r - k + 1) \right).$$

If $r \geq 2k$, then by Lemma 2.1 there are $\frac{1}{2}(m - 1)m^{t-1}$ type-$[k, k]$ paths rooted at each vertex $v$ for which $d_T(p, v) \leq r - k$ and $m^t$ type-$[0, t]$ paths rooted at each vertex $v$ for which $d_T(p, v) \leq r - 2k$. Furthermore, for $1 \leq s \leq k - 1$, there are $(m - 1)m^{t-1}$ type-$[s, t - s]$ paths rooted at each vertex
Proposition 3.1. The number of paths of length \( r \) in a perfect unrooted \( m \)-ary tree is \( P_m(r, t) = \frac{1}{2} (m - 1) m^{2k-1} \sum_{d=0}^{r-k} m^d + (m - 1) m^{2k-1} \sum_{d=0}^{r-2k+s} m^d + m^{2k} \sum_{d=0}^{r-2k} m^d \). Therefore,

\[
P_m(r, t) = m^{2k-1} \left( \frac{1}{2} (m - 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) + (m - 1) \sum_{s=1}^{k-1} m^{r-2k+s+1} \frac{1}{m - 1} + m \left( \frac{m^{r-2k+1} - 1}{m - 1} \right) \right)
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m - 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) + \sum_{s=1}^{k-1} m^{r-2k+s} \frac{1}{m - 1} + m \left( \frac{m^{r-2k+1} - 1}{m - 1} \right) \right)
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m - 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) + \sum_{i=r-2k+2}^{r-k} (m^i - 1) + \frac{m^{r-2k+2} - m}{m - 1} \right)
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - k \right).
\]

Theorem 1.2 now follows by combining Propositions 2.2 and 2.3 with the observation that

\[
V_R(d) = \frac{m^{d+1} - 1}{m - 1}.
\]

3. Perfect unrooted \( m \)-ary trees

In an unrooted \( m \)-ary tree \( T = (V, E) \), every vertex has degree 1 or \( m + 1 \). The diameter \( D \) of \( T \) is the maximum value of \( d_T(u, v) \) over all pairs of vertices \( u, v \in V \). We call \( T \) perfect if and only if every degree-1 vertex is distance \( D \) from some other vertex.

The symmetry of a perfect unrooted \( m \)-ary tree \( T \) of diameter \( D \) depends on whether \( D \) is odd or even. We introduce some notation to be used in this respect. If \( D = 2r - 1 \) is odd, then \( T \) can be constructed by connecting the roots \( p_1, p_2 \) of two perfect rooted \( m \)-ary trees \( T_1, T_2 \) of depth \( r - 1 \) with an edge \( e = \{p_1, p_2\} \). If \( D = 2r \) is even, then \( T \) can be constructed by connecting the roots \( p_1, p_2 \) of the perfect rooted \( m \)-ary tree \( T_1 \) of depth \( r \) and the perfect rooted \( m \)-ary tree \( T_2 \) of depth \( r - 1 \) with an edge \( e = \{p_1, p_2\} \). In either case, a path in \( T \) is either contained in \( T_1 \), contained in \( T_2 \), or contains \( e \).

Let \( U_m(D; t) \) denote the number of paths of length \( t \) in the perfect unrooted \( m \)-ary tree of diameter \( D \). We consider four cases, depending on the parities of \( t \) and \( D \). The proofs of the propositions below make repeated use of Lemma 2.4 and Propositions 2.2 and 2.3.

Proposition 3.1. The number of paths of length \( t = 2k - 1 \) in the perfect unrooted \( m \)-ary tree of diameter \( D = 2r - 1 \), where \( 1 \leq k \leq r \) is

\[
U_m(D, t) = \frac{m^{2k-2}}{m - 1} \left( 2m^{r-k+1} - (m + 1) \right).
\]

Proof. Let \( T \) be the perfect unrooted \( m \)-ary tree of diameter \( D = 2r - 1 \). We use the decomposition of \( T \) into \( T_1, T_2 \), and \( e \). The number of paths in \( T \) of length \( t \) that contain \( e \) is

\[
\sum_{s=\max\{0, t-r\}}^{\min\{t-1, r-1\}} m^{t-1} = \begin{cases} 0, & r < k, \\ (2r - 2k + 1) m^{2k-2}, & k \leq r < 2k - 1 \\ (2k - 1) m^{2k-2}, & r \geq 2k - 1. \end{cases}
\]
If $r < k$, then $U_m(D, t) = 0$. If $r = k$, then the depth of $T_1$ (and of $T_2$) is $r - 1 < k$, so $P_m(T_1, t) = P_m(T_2, t) = 0$. Also, $2r - 2k + 1 = 1$, and hence $U_m(D, t) = m^{2k-2}$. If $k < r < 2k - 1$, then $k \leq r - 1 < 2k - 2$, and so

$$
U_m(D, t) = 2P_m(r - 1, 2k - 1) + (2r - 2k + 1)m^{2k-2}
= 2m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - (r - k + 1) \right) + (2r - 2k + 1)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} \left( 2m^{r-k+1} - (m + 1) \right).
$$

If $r = 2k - 1$, then $r - 1 < 2k - 1$, and so

$$
U_m(D, t) = 2P_m(r - 1, 2k - 1) + (2k - 1)m^{2k-2}
= 2m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - (r - k + 1) \right) + (2k - 1)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} \left( 2m^{r-k+1} - (m + 1) \right).
$$

If $r > 2k - 1$, then $r - 1 \geq 2k - 1$, and so

$$
U_m(D, t) = 2P_m(r - 1, 2k - 1) + (2k - 1)m^{2k-2}
= 2m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - k \right) + (2k - 1)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} \left( 2m^{r-k+1} - (m + 1) \right).
$$

\[ \square \]

**Proposition 3.2.** The number of paths of length $t = 2k$ in the perfect unrooted $m$-ary tree of diameter $D = 2r - 1$, where $1 \leq k < r$, is

$$
U_m(D, t) = \frac{m^{2k-1}}{m - 1} \left( (m + 1)m^{r-k} - (m + 1) \right).
$$

**Proof.** Using the decomposition of $T$ into $T_1$, $T_2$, and $e$, the number of paths of length $t = 2k$ in $T$ that contain $e$ is

$$
\sum_{s = \max\{0, t-r\}}^{\min\{t-1, r-1\}} m^{t-1} = \begin{cases} 
0, & r \leq k, \\
(2r - 2k)m^{2k-1}, & k < r \leq 2k \\
2km^{2k-1}, & r > 2k,
\end{cases}
$$

and again the result is immediate for $r \leq k$. If $k < r \leq 2k$, then $k \leq r - 1 < 2k$, and so

$$
U_m(D, t) = 2P_m(r - 1, 2k) + (2r - 2k)m^{2k-1}
= 2m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k} - 1}{m - 1} \right) - (r - k) \right) + (2r - 2k)m^{2k-1}
= \frac{m^{2k-1}}{m - 1} \left( (m + 1)m^{r-k} - (m + 1) \right).
$$
If \( r > 2k \), then \( r - 1 \geq 2k - 1 \), and so
\[
U_m(D, t) = 2P_m(r-1, 2k) + 2km^{2k-1}
= 2m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^r - k - 1}{m - 1} \right) - k \right) + 2km^{2k-1}
= \frac{m^{2k-1}}{m - 1} ((m + 1)m^{r-k} - (m + 1)).
\]

\[
\square
\]

**Proposition 3.3.** The number of paths of length \( t = 2k - 1 \) in the perfect unrooted \( m \)-ary tree of diameter \( D = 2r \), where \( 1 \leq k \leq r \), is
\[
U_m(D, t) = \frac{m^{2k-2}}{m - 1} ((m + 1)m^{r-k+1} - (m + 1)).
\]

**Proof.** Using the decomposition of \( T \) into \( T_1 \) (depth \( r \)), \( T_2 \) (depth \( r - 1 \)), and \( e \), the number of paths of length \( t \) paths in \( T \) that contain \( e \) is
\[
\sum_{s = \max(0, t - r)}^{\min(t - 1, r)} m^{t-1} = \begin{cases} 
0, & r < k, \\
(2r - 2k + 2)m^{2k-2}, & k \leq r < 2k - 1 \\
(2k - 1)m^{2k-2}, & r \geq 2k - 1.
\end{cases}
\]

If \( r < k \), then \( U_m(D, t) = 0 \). If \( r = k \), then \( r - 1 < k \), and so
\[
U_m(D, t) = P_m(r, 2k - 1) + P_m(r-1, 2k - 1) + (2r - 2k + 2)m^{2k-2}
= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - (r - k + 2) \right) + (2r - 2k + 2)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} ((m + 1)m^{r-k+1} - (m + 1)).
\]

If \( k < r < 2k - 1 \), then \( k \leq r - 1 < 2k - 1 \), and so
\[
U_m(D, t) = P_m(r, 2k - 1) + P_m(r-1, 2k - 1) + (2r - 2k + 2)m^{2k-2}
= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - (r - k + 2) \right) + m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - (r - k + 1) \right)
+ (2r - 2k + 2)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} ((m + 1)m^{r-k+1} - (m + 1)).
\]

If \( r = 2k - 1 \), then \( r - 1 < 2k - 1 \), and so
\[
U_m(D, t) = P_m(r, 2k - 1) + P_m(r-1, 2k - 1) + (2k - 1)m^{2k-2}
= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - k \right) + m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - (r - k + 1) \right) + (2k - 1)m^{2k-2}
= \frac{m^{2k-2}}{m - 1} ((m + 1)m^{r-k+1} - (m + 1)).
\]
If $r > 2k - 1$, then $r - 1 \geq 2k - 1$, and so

\[
U_m(D, t) = P_m(r, 2k - 1) + P_m(r - 1, 2k - 1) + (2k - 1)m^{2k-2}
\]

\[
= m^{2k-2} \left( \frac{m^{r-k+2} - 1}{m - 1} - k \right) + m^{2k-2} \left( \frac{m^{r-k+1} - 1}{m - 1} - k \right) + (2k - 1)m^{2k-2}
\]

\[
= \frac{m^{2k-2}}{m - 1} \left( (m + 1)m^{r-k+1} - (m + 1) \right).
\]

Proposition 3.4. The number of paths of length $t = 2k$ in the perfect unrooted $m$-ary tree of diameter $D = 2r$, where $1 \leq k \leq r$, is

\[
U_m(D, t) = \frac{m^{2k-1}}{m - 1} \left( \frac{1}{2} (m + 1)^2 m^{r-k} - (m + 1) \right).
\]

Proof. Using the decomposition of $T$ into $T_1$ (depth $r$), $T_2$ (depth $r - 1$), and $e$, the number of paths of length $t$ in $T$ that contain $e$ is

\[
\sum_{s=\max(0,t-r)}^{\min(t-1,r)} m^{t-1} = \left\{
\begin{array}{ll}
0, & r < k,
(2r - 2k + 1)m^{2k-1} - (r - k + 1) + (2r - 2k + 1)m^{2k-1}, & k \leq r < 2k,
2km^{2k-1}, & r \geq 2k,
\end{array}
\right.
\]

and again the result is immediate for $r < k$. If $r = k$, then $r - 1 < k$, and so

\[
U_m(D, t) = P_m(r, 2k) + P_m(r - 1, 2k) + (2r - 2k + 1)m^{2k-1}
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - (r - k + 1) \right) + (2r - 2k + 1)m^{2k-1}
\]

\[
= \frac{m^{2k-1}}{m - 1} \left( \frac{1}{2} (m + 1)^2 m^{r-k} - (m + 1) \right).
\]

If $k < r < 2k$, then $k \leq r - 1 < 2k$, and so

\[
U_m(D, t) = P_m(r, 2k) + P_m(r - 1, 2k) + (2r - 2k + 1)m^{2k-1}
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - (r - k + 1) \right)
\]

\[
+ m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k} - 1}{m - 1} \right) - (r - k) \right) + (2r - 2k + 1)m^{2k-1}
\]

\[
= \frac{m^{2k-1}}{m - 1} \left( \frac{1}{2} (m + 1)^2 m^{r-k} - (m + 1) \right).
\]

If $r = 2k$, then $r - 1 < 2k$, and so

\[
U_m(D, t) = P_m(r, 2k) + P_m(r - 1, 2k) + 2km^{2k-1}
\]

\[
= m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k+1} - 1}{m - 1} \right) - k \right)
\]

\[
+ m^{2k-1} \left( \frac{1}{2} (m + 1) \left( \frac{m^{r-k} - 1}{m - 1} \right) - (r - k) \right) + 2km^{2k-1}
\]

\[
= \frac{m^{2k-1}}{m - 1} \left( \frac{1}{2} (m + 1)^2 m^{r-k} - (m + 1) \right).
\]
If $r > 2k$, then $r - 1 \geq 2k$, and so

$$U_m(D, t) = P_m(r, 2k) + P_m(r - 1, 2k) + 2km^{2k-1}$$

$$= m^{2k-1} \left( \frac{1}{2}(m+1) \left( \frac{m^{r-k+1} - 1}{m-1} \right) - k \right)$$

$$+ m^{2k-1} \left( \frac{1}{2}(m+1) \left( \frac{m^{r-k} - 1}{m-1} \right) - k \right) + 2km^{2k-1}$$

$$= \frac{m^{2k-1}}{m-1} \left( \frac{1}{2}(m+1)^2 m^{r-k} - (m+1) \right).$$

Theorem 1.1 now follows by combining Propositions 3.1 to 3.4 with the observation that

$$V(d) = \begin{cases} 
\frac{2m^{d+1}}{m-1} - 2, & d \text{ odd}, \\
\frac{(m+1)m^{d+1}}{m-1} - 2, & d \text{ even}.
\end{cases}$$

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