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with Deformed Antibracket

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Path Integral Formulation with Deformed Antibracket

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Abstract

We propose how to incorporate the Leites-Shchepochkina-Konstein-Tyutin deformed antibracket into the quantum field-antifield formalism.

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1 Introduction

The concept of deformations in the field-antifield formalism [1, 2, 3] based on a nilpotent higher-order $\Delta_*$ operator was developed in a series of papers [4, 5, 6, 7, 8, 9, 10, 11, 12]. Such deformations typically modify the Jacobi identity with BRST-exact terms. In contrast, we shall in this paper only discuss local deformations of the antibracket with a Grassmann-even deformation parameter such that the Jacobi identity holds strongly, and without assuming an underlying $\Delta_*$ operator a priori. Recently [13, 14, 15], a non-trivially deformed antibracket $(f, g)_* := (f, g) + (-1)^f \left( \frac{\kappa c(\kappa)}{1 + \frac{\kappa c(\kappa)}{N}} \Delta f \right) \cdot (1 - \frac{N}{2}) g + ((1 - \frac{N}{2}) f) \cdot \frac{\kappa c(\kappa)}{1 + \frac{\kappa c(\kappa)}{N}} \Delta g , \quad (1.1)$ for functions $f, g$ of finitely many variables $z^A$ was constructed inside various algebras $\mathcal{A}$ (e.g., polynomial algebra, algebra of smooth functions with compact support, etc.). Here $\kappa$ is a deformation parameter; $c(\kappa) = \sum_{k=0}^{\infty} c_k \kappa^k$ is an arbitrary formal power series in $\kappa$; and $N := z^A \partial / \partial z^A$ is the Euler/conformal vector field. Moreover, it was shown [14] that this deformed antibracket (1.1) is unique modulo trivial deformations and reparametrizations of the deformation parameter $\kappa$. Thus, it is expected to play a central role.

In this paper, we propose how to incorporate the non-trivially* deformed antibracket (1.1) into the quantum field-antifield formalism [1, 2, 3]. Concretely, we suggest a $\kappa$-deformed odd Laplacian; quantum master action $W = S + O(\bar{\hbar})$; quantum master equation; and partition function $\tilde{Z}$, such that, the classical master equation is given in terms of the above $\kappa$-deformed antibracket $$(S, S)_* = 0 ; \quad (1.2)$$ the classical BRST symmetry is $s = (S, \cdot)_*$; and the partition function $\tilde{Z}$ is formally independent of the gauge-fixing $X$.

How would a $\kappa$-deformation be realized in practice? Firstly, we stress that field theory implies infinitely many $z^A$-variables, so that both the Euler vector field $N$ and the odd Laplacian $\Delta$ would need regularization. Nevertheless, it is reasonable to assume that the naive finite-dimensional $N$-deformation (1.1) still serves as a model of what to come in field theory. Secondly, we note that the traditional field-antifield approach [1, 2, 3] (where one starts from a classical action, which is independent of ghosts and antifields, and one introduces ghosts and antifields as generators of gauge- and BRST-symmetry, respectively) is not expected to produce a $\kappa$-deformation, as the antibracket traditionally remains on Darboux form. Rather, a relevant physical system should have an antisymplectic phase space built in from the beginning, like, e.g., closed string field theory [16], or generalized Poisson sigma models [17, 18, 19]. It is believed that the $\kappa$-deformation here could be caused by a choice of regularization scheme that manifestly preserves the Jacobi identity.

The new construction is motivated by two key ideas, which may be symbolized with the introduction of a Bosonic and Fermionic variable, $t$ and $\theta$, respectively, with collective notation $\tau := \{t; \theta\}$. Mathematically, they are, in fact, intimately tied to Lie cohomology theory. We will only here sketch the Lie cohomology argument, and defer a more detailed explanation to an accompanying paper [20]. Recall that the ambiguity/uniqueness of deformations of a Lie-bracket is measured by the second Lie cohomology group, while the first Lie cohomology group classifies outer (=non-Hamiltonian) Lie algebra derivations. Konstein and Tyutin have calculated [14] the first and second Lie cohomology group for

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* A trivial deformation $(f, g)_* = T^{-1}(T f, T g)$ of the antibracket with $T = 1 + O(\kappa)$ amounts to a trivial deformation $\Delta_* = T^{-1} \Delta T$ and $f \ast g = T^{-1}(T f \cdot T g)$ of the underlying BV algebra $(\mathcal{A}; \Delta_*; \ast)$. 

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the constant, non-degenerated antibracket $(\cdot,\cdot)$. The first Lie cohomology group is two-dimensional, and, in detail, it is generated by the odd Laplacian $\Delta$ and the affine operator $N-2$. The second Lie cohomology group is two-dimensional as well, and, in accordance with the Künneth formula, it is generated by all possible non-zero\footnote{The last $\Delta\cup\Delta=0$ of the $2\times2=4$ possibilities vanishes identically, because the cup product $\cup$ is (graded) commutative.} cup product combinations of the first cohomology. These are $\Delta \cup (N-2) = (N-2) \cup \Delta$ and $(N-2) \cup (N-2)$, which lead to two deformed antibrackets, with an even and an odd deformation parameter, respectively, where we here will only consider the former.

The first key idea is to suspend the algebra $\mathcal{A}$ by introducing a suspension parameter $t$ to turn the affine operator $N-2$ into a genuine vector field $N_r = N + t\partial/\partial t$, which satisfies the Leibniz rule. The non-triviality of the $N_r$ vector field in the $\{z^A; t\}$ space means that it is not a Hamiltonian vector field. The second key idea is to complement the $\{z^A; t\}$ space with an antisymplectic partner $\theta$, in such a way, that $\theta$ becomes (minus) the Hamiltonian generator for the vector field $N_c = -(\theta, \cdot)_{\tau}$, and hence, so that the vector field $N_r$ becomes trivial, and, in turn, it makes the corresponding $(t; \theta)$-extended deformed antibracket $(\cdot,\cdot)_{\tau}$ trivial.

2 Basic Setting: Constant Non-Degenerate Antibracket

Let $\mathcal{A} := C[[z]]$ be the algebra of formal power series $f = f(z)$ in $2n$ variables $z^A$ of Grassmann parity $\varepsilon(z^A) := \varepsilon_A$, equipped with a constant, non-degenerate antibracket $E^{AB} = (z^A, z^B)$ with Grassmann parity $\varepsilon(E^{AB}) = \varepsilon_A + 1 + \varepsilon_B$ corresponding to the odd Laplacian

$$\Delta := \frac{(-1)^{\varepsilon_A}}{2} \frac{\partial^2 f}{\partial z^A} E^{AB} \frac{\partial^2 f}{\partial z^B}, \quad \Delta^2 = 0, \quad \varepsilon(\Delta) = 1. \quad (2.1)$$

The antibracket

$$(f, g) := (-1)^{\varepsilon_f}[\overleftarrow{\Delta}, f], g] 1 = -(-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)}(g, f), \quad f, g \in \mathcal{A}, \quad (2.2)$$

satisfies skewsymmetry $(2.2)$, the Jacobi identity

$$\sum_{\text{cyl.} f,g,h} (-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)}(f, (g, h)) = 0, \quad f, g, h \in \mathcal{A}, \quad (2.3)$$

and the Leibniz rule/Poisson property

$$(f, g, h) = f(g, h) + (-1)^{\varepsilon_f \varepsilon_g} g(f, h), \quad f, g, h \in \mathcal{A}. \quad (2.4)$$

3 Non-Trivially Deformed Algebra $\mathcal{A}$

We will from now on use the simplifying convention that the power series from eq. $(1.1)$ is $c(\kappa) = -2$. To reintroduce the whole $c(\kappa)$ series, just replace $\kappa \rightarrow -\frac{\kappa c(\kappa)}{2}$. The deformed odd Laplacian $\Delta_\kappa$ and antibracket $(\cdot,\cdot)_\kappa$, cf. eq. $(1.1)$, read

$$\Delta_\kappa := \frac{1}{1-K} \Delta = \frac{1}{1-\kappa N} \Delta, \quad \Delta_\kappa^2 = 0, \quad (3.1)$$

$$(f, g)_\kappa := (f, g) + (-1)^{\varepsilon_f}(\Delta_\kappa f) \cdot (K g) + (K f) \cdot (\Delta_\kappa g)$$

$$= (-1)^{\varepsilon_f}(\Delta_\kappa f) - (1-K)((-1)^{\varepsilon_f}(\Delta_\kappa f) + f(\Delta_\kappa g))$$

$$= -(-1)^{(\varepsilon_f + 1)(\varepsilon_g + 1)}(g, f)_\kappa, \quad f, g \in \mathcal{A}, \quad (3.3)$$

$$K := \kappa(N-2), \quad N := z^A \frac{\partial^2 f}{\partial z^A}, \quad [\Delta, N] = 2\Delta, \quad \varepsilon(\kappa) = 0. \quad (3.5)$$
Within the algebra $\mathcal{A}$, the deformed odd Laplacian $\Delta_\ast$ is characterized by nilpotency, and the property

$$\Delta_\ast(f,g)_\ast = (\Delta_\ast f,g)_\ast - (-1)^{\ell f}(f,\Delta_\ast g)_\ast , \quad f,g \in \mathcal{A} ,$$

i.e., that $\Delta_\ast$ differentiates the deformed antibracket $(\cdot,\cdot)_\ast$. The standard Witten formula (2.2), cf. Ref. [21], is deformed into (3.3), which, in turn, can be used to prove the Jacobi identity (3.7) for the deformed antibracket $(\cdot,\cdot)_\ast$,

$$\sum_{\text{cyc}.f,g,h} (-1)^{(\epsilon_f+1)(\epsilon_h+1)}(f,(g,h))_\ast = 0 , \quad f,g,h \in \mathcal{A} .$$

Note that the deformed antibracket $(\cdot,\cdot)_\ast$ does not satisfy the Leibniz rule/Poisson property, cf. eq. (2.4), and hence the deformed antibracket $(\cdot,\cdot)_\ast$ is, technically speaking, not an odd Poisson bracket. Therefore, the deformation and the corresponding cohomology must be treated within the framework of (infinite-dimensional, graded) Lie algebras instead of (finitely generated, graded) Poisson algebras.

## 4 $k$-Suspended Deformed Operators

Define for later convenience a $k$-suspended deformed odd Laplacian $\Delta_\ast^{(k)}$ and a $(k,\ell)$-suspended deformed antibracket $(\cdot,\cdot)_\ast^{(k,\ell)}$,

$$\Delta_\ast^{(k)} := \Delta \frac{1}{1-K(k)} , \quad (\Delta_\ast^{(k)})^2 = 0 , \quad K^{(k)} \Delta = \Delta K^{(k-2)} ,$$

$$\begin{align*}
(f,g)^{(k,\ell)}_\ast &:= (f,g) + (-1)^{\epsilon_f}(\Delta_\ast^{(k)} f) \cdot (K^{(\ell)} g) + (K^{(k)} f) \cdot (\Delta_\ast^{(\ell)} g) \\
&= (-1)^{\epsilon_f} \Delta (fg) - (1-K^{(k+\ell+2)})\{(-1)^{\epsilon_f} (\Delta_\ast^{(k)} f) g + f(\Delta_\ast^{(\ell)} g)\}
\end{align*}$$

$$K^{(k)} := \kappa N^{(k)} , \quad N^{(k)} := N+k , \quad N_s^{(k)} := N(k) \frac{1}{1-K(k)} , \quad K_s^{(k)} := \kappa N_s^{(k)} ,$$

where $k,\ell$ are integers. In particular, the $k$-suspended definitions (4.1)-(4.5) generalize the definitions (3.1)-(3.5) of Section 3 in the following way,

$$\Delta_\ast^{(-2)} \equiv \Delta_\ast , \quad (f,g)^{(-2,-2)}_\ast \equiv (f,g)_\ast , \quad K^{(-2)} \equiv K , \quad N^{(0)} \equiv N .$$

Equation (4.3) is a $(k,\ell)$-suspended deformed Witten formula [21]. Note also the elementary, but useful, formula

$$K^{(k+\ell)}(fg) = (K^{(k)} f)g + f(K^{(\ell)} g) , \quad f \in \mathcal{A} .$$

Equations (4.3) and (4.7) can be used to prove the Jacobi identity

$$\sum_{\text{cyc}.(f,k),(g,\ell),(h,m)} (-1)^{(\epsilon_f+1)(\epsilon_h+1)}(f,(g,h))_\ast^{(k,\ell+m+2)} = 0 , \quad f,g,h \in \mathcal{A} ,$$

and the differentiation rule

$$\Delta_\ast^{(k+\ell+2)}(f,g)_\ast^{(k,\ell)} = (\Delta_\ast^{(k)} f,g)_\ast^{(k+m,\ell)} - (-1)^{\epsilon_f}(f,\Delta_\ast^{(\ell)} g)_\ast^{(k,m+\ell)} , \quad f,g \in \mathcal{A} .$$
5 \( \tau \)-Extended Algebra \( \mathcal{A}_\tau \)

Let us now introduce a \( \tau \)-extended algebra \( \mathcal{A}_\tau := C[[z; t; \theta]][[\frac{1}{t}]] \) of formal (lower truncated) Laurent series

\[
F = \sum_{k=-M_F}^{\infty} F_{(k)}(z; \theta)t^k , \quad F_{(k)}(z; \theta) = F_{(k|0)}(z) + \theta F_{(k|1)}(z) ,
\]

where the lower limit \( k = -M_F \) may depend on the series \( F \), and \( \tau := \{ t; \theta \} \) is a collective notation for the two new variables \( t \) and \( \theta \) of Grassmann parity \( \varepsilon(t) = 0 \) and \( \varepsilon(\theta) = 1 \), respectively. One introduces a suspension map \([ \cdot ] : \mathcal{A} \to \mathcal{A}_\tau \) as

\[
[f] := \frac{f}{t^2} , \quad f \in \mathcal{A} .
\]

The residue map \( \pi : \mathcal{A}_\tau \to \mathcal{A} \) reads \( \pi(F) := \int_0^{\frac{1}{2\pi}} f d\theta \theta F = F_{(-2|0)} \) with Berezin integral convention \( f d\theta \theta = 1 \). One has \( \pi \circ [ \cdot ] = \text{id}_\mathcal{A} \), or equivalently, \( \pi \circ [f] = f \) for \( f \in \mathcal{A} \).

6 \( \tau \)-Extended Antisymplectic Structure

Define generalized Darboux\(^\dagger\) coordinates \( \{ z_0^A; t_0; t_0^* \} \) as

\[
z_0^A := \frac{z^A}{t} , \quad t_0 := \ln(t) , \quad t_0^* := \theta ,
\]

with inverse transformation

\[
z^A = e^{t_0} z_0^A , \quad t = e^{t_0} , \quad \theta = t_0^* .
\]

The Berezin volume densities for the generalized Darboux and original coordinates are chosen as

\[
\rho_0 := 1 , \quad \rho_\tau := \frac{\rho_0}{t} , \quad J := \text{sdet} \frac{\partial \{ z^A; t; \theta \}}{\partial \{ z_0^A; t_0; t_0^* \}} = t .
\]

The algebra \( \mathcal{A}_\tau \) is equipped with the second-order odd Laplacian\(^5\)

\[
\Delta_\tau := (\tau)^2 \frac{\partial^2}{\partial z_0^A} E^{AB} \frac{\partial^2}{\partial z_0^B} + \frac{\partial^2}{\partial t_0^*} \frac{\partial^2}{\partial t_0} = t^2 \Delta + N_\tau \frac{\partial^2}{\partial \theta} , \quad \Delta_\tau^2 = 0 ,
\]

\[
N_\tau := N + t \frac{\partial^2}{\partial t} = - (\theta, \cdot)_\tau , \quad [N_\tau, \Delta_\tau] = 0 ,
\]

\[
(F, G)_\tau := (\tau)^2 \int (\Delta_\tau, F, G) d1 ,
\]

such that the suspension map \([ \cdot ]\) intertwines between an operation and its \( \tau \)-extended counterpart,

\[
\Delta_\tau [f] = \Delta f , \quad N_\tau [f] = [(N-2)f] , \quad f \in \mathcal{A} ,
\]

\[
([f], [g])_\tau = [(f, g)] , \quad (f, [g])_\tau = (f, g) , \quad f, g \in \mathcal{A} .
\]

The non-vanishing antibrackets \( (\cdot, \cdot)_\tau \) of the fundamental variables \( \{ z^A; t; \theta \} \) read

\[
(z^A, z^B)_\tau = t^2 E^{AB} , \quad (z^A, \theta)_\tau = z^A , \quad (t, \theta)_\tau = t , \quad (t_0^*, t_0^*)_\tau = 1 .
\]

\(^\dagger\)Generalized Darboux coordinates are coordinates in which the (odd) Poisson bi-vector is constant, cf. eq. (6.10).

\(^5\)Theoretically, the parameter \( t \) serves as a unit of suspension. In practice, it may be more convenient to expand in terms of its square \( t_2 := t^2 \), so that \([ f ] := f / t_2 / N_\tau := N + 2 t_2 \partial / \partial t_2 ; \Delta_\tau := t_2 \Delta + N_\tau \partial / \partial \theta ; \) etc.
7 Trivially Deformed $\tau$-Extended Odd Poisson Algebra $\mathcal{A}_\tau$

Define a trivially deformed odd Laplacian

$$\Delta_{\tau^*} := \Delta \frac{1}{1 - K_\tau} = T^{-1} \Delta_T, \quad \Delta_{\tau^*}^2 = 0, \quad (7.1)$$

$$K_\tau := \kappa N_\tau, \quad [K_\tau, \Delta_T] = 0, \quad (7.2)$$

cf. Appendix A, where $T$ is the trivialization map in the $\tau$-extended algebra $\mathcal{A}_\tau$,

$$T := 1 + \kappa \theta \Delta_{\tau^*}, \quad T^{-1} := 1 - \kappa \theta \Delta_T, \quad T^{-1} T = 1 = T T^{-1}, \quad (7.3)$$

cf. Appendix B, so that in the suspended sector,

$$\Delta_{\tau^*}[f] = \Delta_T f, \quad K_\tau[f] = [K_T f], \quad f \in \mathcal{A}. \quad (7.4)$$

If one expands with respect to the $t$ variable, one gets

$$\Delta_{\tau^*} F = \sum_k \left( t^2 \Delta^{(k)} f^{(k)} + N^{(k)} f^{(k+1)} \right) t^k, \quad F \in \mathcal{A}, \quad (7.5)$$

$$K_\tau F = \sum_k (K^{(k)} f^{(k)}) t^k, \quad F \in \mathcal{A}. \quad (7.6)$$

Define a trivially deformed antibracket

$$(F, G)_{\tau} := T^{-1}(TF, TG)_\tau = (F, G)_\tau + (-1)^{\varepsilon_F} (\Delta_{\tau^*} F) \cdot K_\tau G + (K_\tau F) \cdot \Delta_{\tau^*} G \quad (7.7)$$

$$= (-1)^{\varepsilon_F} \Delta_T (FG) - (1 - K_\tau) \{( -1)^{\varepsilon_F} (\Delta_{\tau^*} F) G + F \Delta_{\tau^*} G \} \quad (7.8)$$

$$= -(-1)^{(\varepsilon_F + 1)(\varepsilon_G + 1)} (G, F)_{\tau}, \quad F, G \in \mathcal{A}. \quad (7.9)$$

cf. Appendix C, so that in the suspended sector,

$$([f], [g])_{\tau^*} = [(f, g)_{\tau}], \quad f, g \in \mathcal{A}. \quad (7.10)$$

If one expands with respect to the $t$ variable, one gets

$$(F, G)_{\tau} = \sum_{k, \ell} \left( t^2 (F^{(k)}, G^{(\ell)})_{\tau^*}^{(k,\ell)} + (-1)^{\varepsilon_F} \left( \frac{1}{1 - K^{(k)}} \right) N^{(k)} F^{(\ell)} \right) t^k t^\ell, \quad F, G \in \mathcal{A}. \quad (7.11)$$

The trivially deformed antibracket $(\cdot, \cdot)_{\tau^*}$ satisfies the Jacobi identity,

$$\sum_{cycl. \ F, G, H} (-1)^{(\varepsilon_F + 1)(\varepsilon_G + 1)} ([F, (G, H)]_{\tau^*})_{\tau^*} = 0, \quad F, G, H \in \mathcal{A}. \quad (7.12)$$

Equation (7.10) therefore gives an alternative derivation of the Jacobi identity (3.7). Define a trivial associative and commutative star product as

$$F \ast G := T^{-1}(TF \cdot TG) = FG - (-1)^{\varepsilon_F \kappa \theta} (F, G)_{\tau}, \quad F, G \in \mathcal{A}, \quad \varepsilon(*) = 0, \quad (7.13)$$

cf. Appendix D, so that in the suspended sector,

$$[f] \ast [g] = [[fg]] - (-1)^{\varepsilon f \kappa \theta} [(f, g)_{\tau}], \quad f, g \in \mathcal{A}. \quad (7.14)$$
The trivially deformed Witten formula [21] reads
\[(F,G)_{\tau s} = (-1)^{\tilde{\epsilon} F} \Delta_{\tau s}(F \ast G) - (-1)^{\tilde{\epsilon} F} (\Delta_{\tau s} F) \ast G - F \ast \Delta_{\tau s} G , \quad F,G \in A_{\tau}.\] (7.15)

The Leibniz rule/Poisson property reads
\[(F \ast G,H)_{\tau s} = F \ast (G,H)_{\tau s} + (-1)^{\tilde{\epsilon} F} \tilde{c} G \ast (F,H)_{\tau s} , \quad F,G,H \in \mathcal{A}_{\tau}.\] (7.16)

The Getzler identity [22] for the BV algebra \((\mathcal{A}_\tau; \Delta_{\tau s}; \ast)\) reads
\[0 = \Delta_{\tau s}(F \ast G \ast H) - \Delta_{\tau s}(F \ast G \ast H) + (-1)^{\tilde{\epsilon} F} F \ast \Delta_{\tau s}(G \ast H) - (-1)^{\tilde{\epsilon} F} \tilde{c} G \ast \Delta_{\tau s}(F \ast H) \ast G + (\Delta_{\tau s} F) \ast G \ast H + (-1)^{\tilde{\epsilon} F} F \ast (\Delta_{\tau s} G) \ast H + (-1)^{\tilde{\epsilon} F + \tilde{\epsilon} \tilde{c} F} G \ast \Delta_{\tau s} H , \quad F,G,H \in \mathcal{A}_{\tau}.\] (7.17)

which encodes the vanishing of higher antibrackets [7, 8, 23]. The \textit{star exponential} is defined as
\[e^B := 1 + B + \frac{1}{2} B \ast B + \frac{1}{3!} B \ast B \ast B + \frac{1}{4!} B \ast B \ast B \ast B + \ldots = T^{-1} e^{(TB)}
= e^B \left(1 - \frac{1}{2} \kappa \theta(B, B)_{\tau s}\right) = e^{B - \frac{1}{2} \kappa \theta(B, B)_{\tau s}}, \quad B \in \mathcal{A}_{\tau}, \quad \varepsilon(B) = 0,\] (7.18)
cf. Appendix E. The star exponential satisfies
\[e^{-B} \ast e^B = 1, \quad e^{-B} \ast (\Delta_{\tau s} e^B) = (\Delta_{\tau s} B) + \frac{1}{2} (B, B)_{\tau s}, \quad \delta e^B = e^B \ast \delta B,\] (7.19)
\[e^{B + B'} = e^B \ast e^{B'}, \quad B, B' \in \mathcal{A}_{\tau}, \quad \varepsilon(B) = 0 = \varepsilon(B').\] (7.20)

If we want to stress the deformation parameter \(\kappa\), we write a subindex \(\kappa\), i.e.,
\[T \equiv T_{\kappa}, \quad \Delta_{\tau s} \equiv \Delta_{\tau s}(\kappa), \quad (\cdot, \cdot)_{\tau s} \equiv (\cdot, \cdot)_{\tau s}(\kappa), \quad F \ast G \equiv F \ast (\kappa) G, \quad e^B \equiv e^B_{\kappa} .\] (7.21)

\section{Deformed Quantum Master Equations}

We will here for simplicity use the strong first-level\footnote{The strong first-level gauge-fixing action \(\tilde{X}\) also depends on first-level Lagrange multipliers \(\{\lambda^\alpha_n\} = \{\lambda^\alpha; \lambda_n\}\), and is capable of incorporating all Abelian gauge-fixing constraints \((G_n, G^\beta)_{\tau s} = 0\). For non-Abelian gauge-fixing constraints, it is necessary to add weak terms in the quantum master equation [27], or still better, to go to the second-level formalism, which introduces antifields \(\lambda^\alpha_n\) for the first-level Lagrange multipliers; second-level Lagrange multipliers \(\lambda^\alpha_{(2)}\); odd Laplacian \(\Delta_{(1)}\) for the first-level Lagrange multipliers; second-level Lagrange multipliers \(\lambda^\alpha_{(2)}\); odd Laplacian \(\Delta_{(1)} = \Delta_{\tau s} + (-1)^s \partial/\partial \lambda^\alpha \partial/\partial \lambda^\alpha\); and action \(\tilde{W} = \lambda^\alpha_{(2)} \lambda^\alpha_{(2)} + \tilde{W}\).} \(W\)-\(X\)-formalism, which consists of a gauge-generating and a gauge-fixing action, \(W\) and \(X\) [24, 25, 26, 27, 28, 8, 29, 30]. In the \(\tau\)-extended case, we adorn the two actions with tildes. The two quantum master equations are
\[\Delta_{\tau s}(\kappa) e^\frac{\partial}{\partial \kappa} \tilde{W} = 0, \quad \Delta_{\tau s}(-\kappa) e^\frac{\partial}{\partial (-\kappa)} \tilde{X} = 0, \quad \tilde{W}, \tilde{X} \in \mathcal{A}_{\tau}, \quad \varepsilon(\tilde{W}) = 0 = \varepsilon(\tilde{X}),\] (8.1)
or equivalently,
\[\frac{1}{2} (\tilde{W}, \tilde{W})_{\tau s(\kappa)} = i \hbar \Delta_{\tau s(\kappa)} \tilde{W}, \quad \frac{1}{2} (\tilde{X}, \tilde{X})_{\tau s(-\kappa)} = i \hbar \Delta_{\tau s(-\kappa)} \tilde{X}.\] (8.2)

From now on, it is implicitly assumed that the star deformations in the \(\tilde{W}\) - \(\tilde{X}\)-sector refer to the deformation parameter \(\kappa\) and \(-\kappa\), respectively, to avoid clutter. Consider first the \(\tilde{W}\) action. Let us mention that \(\tilde{W}\) satisfies the \(\kappa\)-deformed quantum master equation if and only if \(T \tilde{W}\) satisfies
the undeformed quantum master equation. If one expands the quantum master equation for $\widetilde{W} = \sum_{k=-\infty}^{\infty} \widetilde{W}_k t^k$ with respect to the $t$ variable, one gets
\[ \frac{1}{2} \sum_{\ell=-\infty}^{\infty} (\widetilde{W}_\ell, \widetilde{W}_{k-\ell})_{(\ell,k-\ell)} + \sum_{\ell=-\infty}^{\infty} N^{(\ell)} \widetilde{W}_\ell \cdot \frac{1}{1-K(k+2\ell)} \widetilde{W}_{k+2\ell} = i\hbar \Delta^S_{(k)} \widetilde{W}_k + i\hbar N_{(k+2)} \widetilde{W}_{k+2} \cdot \quad (8.3) \]

We next identify the component $\widetilde{W}_{(-2|0)} = S$ with the proper\footnote{An action is called \textit{proper} (with respect to a set of antisymplectic variables) if its corresponding Hessian has rank equal to half the number of variables at the stationary surface, see \textit{e.g.}, Ref. [31].} classical action $S$ from eq. (1.2). To have the classical master equation (1.2) within the $t$-hierarchy (8.3), the Laurent series $\widetilde{W}$ must truncate from below as
\[ \widetilde{W} = \sum_{k=-2}^{\infty} \widetilde{W}_{(k|0)} t^k + \theta \sum_{k=1}^{\infty} \widetilde{W}_{(k|1)} t^k \cdot \quad (8.4) \]

The minimal Ansatz for the gauge-generating and gauge-fixing actions, $\widetilde{W}$ and $\widetilde{X}$ reads\footnote{Note that while the leading term $[S]$ in the $W$ action is proper in the original antisymplectic phase space \{\textit{z}^{\textit{A}}\}, it is in general \textit{not} proper in the $t$-extended antisymplectic phase space \{\textit{z}^{\textit{A}}; t; \theta\}. Thus if one would like to treat the $t$ variable perturbatively, it is necessary to include $t$-dependent classical (=\hbar-independent) terms in the $\widetilde{W}$ action, which necessarily must violate the minimal Ansatz (8.5). We analyze here the minimal Ansatz (8.5) for simplicity, as the Ansatz is consistent with the quantum master equation (8.2), but with the caveat that $t$ may acquire a non-perturbative status.}
\[ \widetilde{W} = \frac{1}{t^2} W(z; h t^2; \kappa) = [S] + h M_1 + \mathcal{O}(h^2 t^2) \quad \text{and} \quad \frac{\partial \widetilde{W}}{\partial \theta} = 0 \cdot \quad (8.5) \]
\[ \widetilde{X} = X(\frac{z}{t}; \lambda; h) + i\hbar \theta \lambda \theta = X(z_0; \lambda; h) + i\hbar t_0 \lambda_0 \quad \text{and} \quad N_r \widetilde{X} = 0 \cdot \quad (8.6) \]

where $\lambda_0 \equiv \lambda_0$ is a Fermionic first-level Lagrange multiplier to gauge-fix the $\theta$ variable, and where
\[ W = W(z; h t^2; \kappa) = S + \sum_{k=1}^{\infty} (t^2 h^2)^k M_k \quad \text{and} \quad S = S(z; \kappa) \quad \text{for} \quad k \geq 1 \cdot \quad (8.7) \]

In $t$-components, the minimal Ansatz (8.5) for $\widetilde{W}$ reads
\[ \widetilde{W}_{(-2)} = S \quad \text{and} \quad \widetilde{W}_{(2k-2)} = \hbar^k M_k \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \widetilde{W}_{(-2k)} = 0 \quad \text{for} \quad k \geq 2 \quad \text{and} \quad \widetilde{W}_{(2k+1)} = 0 \cdot \quad (8.8) \]

The quantum hierarchy (8.3) for $\widetilde{W}$ becomes
\[ (S, S)_k = 0 \quad \text{and} \quad (M_1, S)_k^{(0,-2)} = i\Delta_S S \cdot \quad (8.9) \]
\[ (M_k, S)_k^{(2k-2,-2)} = i\Delta_S^{(2k-2)} M_{k-1} - \frac{1}{2} \sum_{\ell=1}^{k-1} (M_{\ell}, M_{k-\ell})_k^{(2\ell-2k-2\ell-2)} \quad \text{for} \quad k \geq 2 \cdot \quad (8.10) \]

The hierarchy (8.9)-(8.10) successively determines $S$ and $M_k$ for $k \geq 1$. The untilded gauge-fixing action $X$ satisfies an ordinary quantum master equation
\[ \Delta e^{i\pi X} = 0 \quad \Leftrightarrow \quad \frac{1}{2} (X, X) = i\hbar \Delta X \cdot \quad (8.11) \]

which is undeformed in the deformation parameter $-\kappa$.\footnote{An action is called \textit{proper} (with respect to a set of antisymplectic variables) if its corresponding Hessian has rank equal to half the number of variables at the stationary surface, see \textit{e.g.}, Ref. [31].}
9 Deformed Path Integral

The first-level path integral measure is

$$d\mu = \rho_\tau \, dt \, d\theta \, d\lambda_\theta [dz] [d\lambda] = \rho_0 \, dt_0 \, d\lambda_0 [dz_0] [d\lambda] \; ,$$

(9.1)
cf. eq. (6.3). The transposed operator $A^T$ of an operator $A$ is defined via [8]

$$\int d\mu \, (A^T F) \cdot G = (-1)^F c_F \int d\mu \, F \cdot (AG) \; ,$$

(9.2)
where $F, G$ are two arbitrary functions. The transposed odd Laplacians and transposed Euler vector fields are

$$\Delta^T = \Delta \, , \quad N^T = -N \, , \quad \Delta^T = \Delta_\tau \, , \quad N^T = -N_\tau \, , \quad \Delta^T_{\tau^s(\kappa)} = \Delta_{\tau^s(-\kappa)} \; .$$

(9.3)
The first-level path integral $\tilde{Z}$ in the $\tau$-extended antisymplectic phase space is defined as

$$\tilde{Z} = \int d\mu \, e^{\frac{i}{\hbar} \tilde{W}} \cdot e^{\frac{i}{\hbar} \tilde{X}} = \int d\mu \, e^{\frac{i}{\hbar} \tilde{X}} \; ,$$

(9.4)
where the total first-level action $\tilde{A}$ is

$$\tilde{A} = \tilde{W} - \frac{i \kappa \theta}{2\hbar} (\tilde{W}, \tilde{X})_{\tau^s(\kappa)} + \tilde{X} + \frac{i \kappa \theta}{2\hbar} (\tilde{X}, \tilde{X})_{\tau^s(-\kappa)}$$

$$= \tilde{W} + \kappa \theta \Delta_{\tau^s(\kappa)} \tilde{W} + \tilde{X} - \kappa \theta \Delta_{\tau^s(-\kappa)} \tilde{X} = T_{\tau^s(\kappa)} \tilde{W} + T_{\tau^s(-\kappa)} \tilde{X} \; .$$

(9.5)
Note that the total action $\tilde{A}$ does not contain inverse powers of $\hbar$ due to the quantum master equations (8.2) for $\tilde{W}$ and $\tilde{X}$.

10 Independence of Gauge-Fixing $\tilde{X}$

The quantum BRST operator for $\tilde{X}$ is defined as

$$(\sigma_{\tilde{X}} F) := \frac{\hbar}{i} e^{\frac{i}{\hbar} \tilde{X}} \cdot \Delta_{\tau^s(\kappa)} (e^{\frac{i}{\hbar} \tilde{X}} \cdot F) - \frac{\hbar}{i} e^{\frac{i}{\hbar} \tilde{X}} \cdot (\Delta_{\tau^s(\kappa)} e^{\frac{i}{\hbar} \tilde{X}}) \cdot F$$

$$= \hbar \frac{i}{2} (\Delta_{\tau^s(\kappa)} F + (\tilde{F} \cdot (\Delta_{\tau^s(\kappa)} e^{\frac{i}{\hbar} \tilde{X}}) \cdot F \in A_{\tau^s}, \quad \sigma_{\tilde{X}}^2 = 0 \; .$$

(10.1)
Since the $\sigma_{\tilde{X}}$ operator is nilpotent, one may argue on general grounds that an arbitrary infinitesimal variation $\delta \tilde{X}$ of the action $\tilde{X}$ should be BRST exact,

$$(\sigma_{\tilde{X}} \delta \tilde{X}) = 0 \; , \quad \delta \tilde{X} = (\sigma_{\tilde{X}} \delta \Psi) \; ,$$

(10.2)
for some infinitesimal Fermion $\delta \Psi$, or equivalently,

$$\frac{i}{\hbar} e^{\frac{i}{\hbar} \tilde{X}} \cdot \delta \tilde{X} = \delta e^{\frac{i}{\hbar} \tilde{X}} = \Delta_{\tau^s(\kappa)} (e^{\frac{i}{\hbar} \tilde{X}} \cdot \delta \Psi) - (\Delta_{\tau^s(\kappa)} e^{\frac{i}{\hbar} \tilde{X}}) \cdot \delta \Psi \; .$$

(10.3)
By using properties (9.3) of transposed operators, and the quantum master equations (8.1), one may deduce that the $\tilde{Z}$ partition function (9.4) is independent of the gauge-fixing $\tilde{X}$.

$$\delta \tilde{Z} = \int d\mu \, e^{\frac{i}{\hbar} \tilde{W}} \cdot \delta e^{\frac{i}{\hbar} \tilde{X}} = \int d\mu \, e^{\frac{i}{\hbar} \tilde{W}} \cdot \Delta_{\tau^s(\kappa)} (e^{\frac{i}{\hbar} \tilde{X}}) \cdot \delta \Psi = \int d\mu \, (\Delta_{\tau^s(\kappa)} e^{\frac{i}{\hbar} \tilde{X}}) \cdot (e^{\frac{i}{\hbar} \tilde{X}} \cdot \delta \Psi) = 0 \; .$$

(10.4)
11 Integrating Out The \( \tau \)-Extended Sector

One can always integrate out the new variable \( \theta \equiv t^*_0 \). The boundary condition (8.6) creates a delta-function
\[
\int d\lambda_0 \ e^{i \Phi t^*_0} = \int d\lambda_0 \ e^{\lambda_0 \theta} = \delta(\theta) ,
\]
and therefore one implements the condition \( \theta = 0 \). The other new variable \( t_0 \equiv \ln(t) \) is a Schwinger proper time variable in a world-line formalism [32]. Let us for simplicity use Darboux coordinates \( \{ z^A; t_0; t^*_0 \} = \{ \phi^0_0; \phi^a_0; t_0; t^*_0 \} \), and integrate out the first-level Lagrange multipliers \( \{ \lambda^0 \} \equiv \{ \lambda^a; \lambda_0 \} \), such that the resulting zero-level total action \( A \) is a lower truncated Laurent series in the \( t \equiv e^{t^*_0} \) variable
\[
A = \tilde{A} \left( \phi_0^0; \phi^0_0 = \frac{\partial \tilde{\psi}}{\partial \phi^a_0}; t_0; t^*_0 = \frac{\partial \tilde{\psi}}{\partial t^*_0}; \lambda = 0; \lambda_0 = 0; \hbar; \kappa \right) = \sum_{k=-M}^{\infty} A(k) e^{kt^*_0} , \quad A(k) = A(k)(\phi_0^0; \hbar; \kappa) .
\]

For a theory that is perturbative in the original \( z \)-variables, (minus) the lower limit is \( M \leq 2 \). If we furthermore integrate out the Schwinger proper time variable \( t_0 \), then the \( \tilde{Z} \) partition function (9.4) becomes
\[
\tilde{Z} = \int_{\infty}^{0} dt^*_0 \ [d\phi_0^0] \ e^{\tilde{A}}
= \begin{cases}
\frac{1}{M} \sum_{m=0}^{M} \sum_{k_1, \ldots, k_m \geq 1 - M} \int [d\phi_0^0] \left( \frac{i}{\kappa}A_{(-M)} \right) \sum_{k=1}^{\kappa} \Gamma \left( \frac{\Sigma k}{M}, -\frac{i}{\kappa}A_{(-M)} \right) \prod_{k=1}^{\kappa} A(\kappa) & \text{for } M > 0 , \\
\sum_{m=0}^{\infty} \frac{1}{M} \sum_{k_1, \ldots, k_m \geq 1} \int [d\phi_0^0] e^{\tilde{A}(0)} \sum_{k=1}^{\kappa} \Gamma \left( \frac{\Sigma k}{M}, -\frac{i}{\kappa}A(k) \right) & \text{for } M = 0 , \\
\sum_{m=0}^{\infty} \frac{1}{M} \sum_{k_1, \ldots, k_m \geq -M} \int [d\phi_0^0] \sum_{k=1}^{\kappa} \Gamma \left( \frac{\Sigma k}{M}, -\frac{i}{\kappa}A(k) \right) & \text{for } M < 0 ,
\end{cases}
\]

where \( \Sigma k := \sum_{k=1}^{\kappa} k_i \); \( \Gamma(\varepsilon; \kappa) := \int_{\varepsilon}^{\infty} \frac{du}{u} \exp(-u) \) is the incomplete Gamma function; and in the case \( M > 0 \), it has been assumed that \( \Im(\tilde{A}_{(-M)}) > 0 \). The case \( M < 0 \) can be viewed as the case \( M = 0 \) with \( A(0) = 0 \). The formula (11.3) is an expansion in Planck’s constant \( \hbar \) if all the subleading terms \( A_{(k > -M)} = O(\hbar) \) are quantum corrections. We stress that the world-line path integral \( \tilde{Z} \) does not reproduce the standard field-antifield path integral [1] in the undeformed limit \( \kappa \to 0 \), as only the former contains a Schwinger proper time integration.\(^{11}\)

\(^{11}\)However, we mention an alternative procedure in the special situation where \( \Delta, S = 0 \), which includes both (i) the undeformed case \( \kappa = 0 \) with action \( W = \frac{W^F}{\tau^F} \), and (ii) the truncated case \( W = \frac{W^F}{\tau^F} = [S] \) with \( \Delta, S = 0 \). In these two cases, shift the \( \tilde{W} \) action with a one-loop contribution \( \tilde{W} = \frac{W^F}{\tau^F} \rightarrow \tilde{W} = \frac{W^F}{\tau^F} + i \hbar \ln(1 - \tau^F) = \frac{W^F}{\tau^F} + i \sum_{k=1}^{\infty} \frac{\tau^F}{k} \). One may check that the shifted \( \tilde{W} \) action also satisfies the quantum master equation (8.2). Now choose the \( t \) integration contour as a small circle around \( t = 1 \). The one-loop correction \( \oint f \frac{dt}{\tau^F} e^{\tilde{W}} \) creates a simple pole at \( t = 1 \), and thereby one implements the condition \( t = 1 \). Therefore the \( \tilde{Z} \) path integral (9.4) reduces (up to a constant multiplicative factor) to the standard \( W \)-X-form \( \tilde{Z} = \int [d\tau] [d\lambda] e^{(W^F + X)} = Z \). In the undeformed case \( \kappa = 0 \), the \( W \) action (8.7) at \( t = 1 \) becomes the standard loop expansion, which satisfies the standard quantum master equation \( \Delta e^{\tilde{W}} = 0 \).
A Proof of eq. (7.1)

\[ T^{-1} \Delta_\tau T = \Delta_\tau T = \Delta_\tau (1 + \kappa \theta \Delta_{\tau s}) = \Delta_\tau + \kappa \theta [\Delta_\tau, \theta] \Delta_{\tau s} = \Delta_\tau + K_\tau \Delta_\tau \frac{1}{1 - K_\tau} \]

\[ = \Delta_\tau \frac{1}{1 - K_\tau} = \Delta_{\tau s}. \quad (A.1) \]

B Proof of eq. (7.3)

\[ T^{-1} T := (1 - \kappa \theta \Delta_\tau)(1 + \kappa \theta \Delta_{\tau s}) = 1 - \kappa \theta \Delta_\tau + \kappa \theta \Delta_{\tau s} - \kappa^2 \theta \Delta_\tau \theta \Delta_{\tau s} = 1. \quad (B.1) \]

\[ TT^{-1} := (1 + \kappa \theta \Delta_{\tau s})(1 - \kappa \theta \Delta_\tau) = 1 - \kappa \theta \Delta_\tau + \kappa \theta \Delta_{\tau s} - \kappa^2 \theta \Delta_\tau \frac{1}{1 - K_\tau} \theta \Delta_{\tau s} = 1. \quad (B.2) \]

C Proof of eq. (7.7)

\[ (B, B)_{\tau s} := T^{-1}(TB, TB)_{\tau} = (1 - \kappa \theta \Delta_\tau)(TB, TB)_{\tau} = I - II \]

\[ = (B, B)_{\tau} + 2(\Delta_{\tau s} B) \cdot (K_\tau B), \quad B \in A_\tau, \quad \varepsilon(B) = 0, \quad (C.1) \]

where

\[ I := (TB, TB)_{\tau} = (B + \kappa \theta \Delta_{\tau s} B, B + \kappa \theta \Delta_{\tau s} B)_{\tau} \]

\[ = (B, B)_{\tau} - 2 \kappa (\Delta_{\tau s} B) \cdot (\theta(B), B)_{\tau} + 2 \kappa \theta (\Delta_{\tau s} B, B)_{\tau} + 2 \kappa^2 \theta (\Delta_{\tau s} B, B)_{\tau} \cdot \Delta_{\tau s} B, \quad (C.2) \]

\[ II := \kappa \theta \Delta_\tau (TB, TB)_{\tau} = 2 \kappa \theta (\Delta_{\tau s} TB, TB)_{\tau} = 2 \kappa \theta (\Delta_{\tau s} B + \kappa [\Delta_\tau, \theta] \Delta_{\tau s} B, TB)_{\tau} \]

\[ = 2 \kappa \theta (\Delta_{\tau s} B + K_\tau \Delta_{\tau} \frac{1}{1 - K_\tau} B, TB)_{\tau} = 2 \kappa \theta (\Delta_{\tau s} B, B + \kappa \theta \Delta_{\tau s} B)_{\tau} \]

\[ = 2 \kappa \theta (\Delta_{\tau s} B, B)_{\tau} + 2 \kappa^2 \theta (\Delta_{\tau s} B, \theta)_{\tau} \cdot \Delta_{\tau s} B. \quad (C.3) \]

Now use polarization of eq. (C.1) to prove eq. (7.7), cf. e.g., Ref. [23].

D Proof of eq. (7.13)

\[ B * B = T^{-1}(TB)^2 = T^{-1}(B + \kappa \theta (\Delta_{\tau s} B))^2 = (1 - \kappa \theta \Delta_\tau) \left( B^2 + 2 \kappa \theta B \Delta_{\tau s} B \right) \]

\[ = I - II - III = B^2 - \kappa \theta (B, B)_{\tau s} - 2 \kappa \theta (K_\tau B) \cdot \Delta_{\tau s} B \]

\[ = B^2 - \kappa \theta (B, B)_{\tau s}, \quad B \in A_\tau, \quad \varepsilon(B) = 0, \quad (D.1) \]

where

\[ I := B^2 + 2 \kappa \theta B \Delta_{\tau s} B = B^2 + 2 \kappa \theta B \Delta_{\tau} \frac{1}{1 - K_\tau} B, \quad (D.2) \]

\[ II := \kappa \theta \Delta_\tau (B^2) = 2 \kappa \theta B \Delta_{\tau s} B + \kappa \theta (B, B)_{\tau s}, \quad (D.3) \]

\[ III := 2 \kappa^2 \theta \Delta_{\tau s} B \Delta_{\tau s} B = 2 \kappa^2 \theta (\Delta_{\tau s} B, B) \Delta_{\tau s} B = 2 \kappa \theta (K_\tau B) \cdot \Delta_{\tau s} B + 2 \kappa \theta B K_\tau \Delta_{\tau} \frac{1}{1 - K_\tau} B. \quad (D.4) \]

Now use polarization of eq. (D.1) to prove eq. (7.13).
E Proof of eq. (7.18)

\[ e^B_\tau = T^{-1} e^{(TB)} = T^{-1} e^{B + \kappa \theta (\Delta_\tau, B)} = (1 - \kappa \theta \Delta_\tau) e^{B(1 + \kappa \theta \Delta_\tau B)} \]

\[ = I - II - III = e^B \left( 1 - \frac{1}{2} \kappa \theta (B, B)_\tau - \kappa \theta (K_\tau B) \cdot \Delta_\tau B \right) \]

\[ = e^B \left( 1 - \frac{1}{2} \kappa \theta (B, B)_\tau \right), \quad B \in \mathcal{A}_\tau, \quad \varepsilon(B) = 0 , \quad (E.1) \]

where

\[ I := e^B(1 + \kappa \theta \Delta_\tau B) = e^B \left( 1 + \kappa \theta \Delta_\tau \frac{1}{1-K_\tau} B \right) , \quad (E.2) \]

\[ II := \kappa \theta (\Delta_\tau e^B) = \kappa \theta e^B \left( \Delta_\tau B + \frac{1}{2} (B, B)_\tau \right) , \quad (E.3) \]

\[ III := \kappa^2 \theta \Delta_\tau \theta e^B \Delta_\tau B = \kappa^2 \theta [\Delta_\tau, \theta] e^B \Delta_\tau B = \kappa \theta K_\tau e^B \Delta_\tau B \]

\[ = \kappa \theta e^B (K_\tau B) \cdot \Delta_\tau B + \kappa \theta e^B K_\tau \Delta_\tau \frac{1}{1-K_\tau} B . \quad (E.4) \]

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