The decay $b \to s\gamma$ in the 3-3-1 model

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Abstract

The 3-3-1 model, based on the gauge group $SU(3)_c \times SU(3)_L \times U(1)_X$, makes a natural prediction of three generations based on anomaly cancellation. Since this is accomplished by incorporating the third family of quarks differently from the other two, it leads to potentially large flavor changing neutral currents. A sensitive place to look for such effects is the flavor changing $b \to s\gamma$ decay, which has recently been measured at CLEO. We compute this decay rate in the 3-3-1 model and compare it with that of the two-Higgs-doublet model, a subset of the full 3-3-1 model. We find that the additional 3-3-1 physics weakens the bound on the charged Higgs mass from $M_{H^+} > 290$ GeV to $M_{H^+} \gtrsim 120$ GeV.
I. INTRODUCTION

Since most indirect effects of new physics first enter at the loop level, they are often dominated by tree level Standard Model (SM) contributions. It is for this reason that only now, with the advent of precision electroweak measurements, we are beginning to probe the structure of new physics. However an exception to this approach is the process $b \to s\gamma$. Since this Flavor Changing Neutral Current (FCNC) process first occurs at loop level in the SM, we have an interesting case where the effects of new physics may be comparable to the SM contribution.

On the experimental side, CLEO has recently announced a measurement of the inclusive decay rate

$$BR(b \to s\gamma) = (2.32 \pm 0.51 \pm 0.29 \pm 0.32) \times 10^{-4},$$ (1.1)

where the final two errors are both systematic in nature. This translates into

$$1 \times 10^{-4} < BR(b \to s\gamma) < 4 \times 10^{-4},$$ (1.2)

at the 95% confidence level. While there are still substantial theoretical uncertainties in the SM prediction for $b \to s\gamma$ (mostly related to unknown next leading order QCD corrections [2]), this experimental value agrees well with a standard model prediction of $(2.75 \pm 0.80) \times 10^{-4}$ with a top quark mass $m_t = 175$ GeV [2–5].

The sensitivity of this FCNC process to new physics thus allows us to put limits on many theories of physics beyond the SM [5]. In this paper we examine the implications of the $b \to s\gamma$ penguin on the 3-3-1 model [6]. This model, based on the gauge group $SU(3)_c \times SU(3)_L \times U(1)_X$, gives a potential answer to the question of flavor by predicting three families as a consequence of anomaly cancellation. This is accomplished by making one of the three quark families transform differently from the other two under the $SU(3)_L \times U(1)_X$ extended electroweak gauge group. Hence this model suffers generically from large FCNCs [6–8], leading to a possible large enhancement of the $b \to s\gamma$ decay rate.

While the 3-3-1 model predicts no new leptons, it predicts one new quark per family, denoted here by $D$ and $S$ with charge $-4/3$ and $T$ with charge $5/3$ [6]. These quarks interact with the ordinary quarks via the charged dilepton gauge boson doublet, $(Y^{++}, Y^+)$. In addition, there is a neutral $Z'$ gauge boson with flavor changing couplings to the usual quarks. Thus there are new processes contributing to $b \to s\gamma$ incorporating $Y$–new quark
loops as well as flavor changing $Z'$ loops. Although many Higgs multiplets are necessary in order to break the 3-3-1 gauge symmetry, we show that it is only necessary to look at an effective two-Higgs-doublet model.

An unusual feature of the 3-3-1 model are the strong bounds placed on the masses of the new particles. Of present interest is the bound $300 < M_Y \lesssim 1100$ GeV \cite{9,10} which indicates that the 3-3-1 contributions to $b \to s\gamma$ may not be suppressed by simply increasing the 3-3-1 scale, $M_Y$.

As is well known, QCD corrections lead to an important enhancement of the $b \to s\gamma$ decay rate in the SM. Thus, when examining the additional contribution from the 3-3-1 model, we divide the calculation into two parts. In section 2, we calculate the new electroweak penguin diagrams associated with the 3-3-1 model, and in section 3, we examine the effects of QCD running from the 3-3-1 scale to $b$-quark scale. In section 4 we combine the results of the previous sections and examine the implications of the current CLEO bound on the 3-3-1 model. Finally we present our conclusions in section 5.

II. ELECTROWEAK PENGUINS IN THE SM AND 3-3-1 MODEL

Although the 3-3-1 model contains tree level FCNC interactions mediated by $Z'$ exchange, the decay $b \to s\gamma$ remains a loop process in this model. Thus we expect the additional 3-3-1 contributions to $b \to s\gamma$ to be at most comparable to the SM penguin diagram. These new 3-3-1 contributions arise from dilepton gauge boson $Y$, $Z'$ and charged Higgs loops. At one-electroweak-loop order, all these contributions add linearly to the $b \to s\gamma$ amplitude. Hence we consider them one at a time.

In the gauge sector all lowest order contributions to $b \to s\gamma$ are given by the penguin diagrams of Fig. \[ where $V = W$, $Y$ or $Z'$. Since all charged gauge bosons contribute similarly, we first calculate the effective $b \to s\gamma$ vertex for an arbitrary left handed gauge boson loop and subsequently specialize to $W$ or $Y$ loops. The $Z'$ contribution is calculated separately since it involves flavor changing vertices.

For a left handed gauge boson $V$, the diagrams of Fig. \[ give rise to an effective vertex

$$
\Gamma_\mu = e \frac{\alpha}{4\pi s^2} \frac{1}{2M_V^3} \left[ (q^2\gamma_\mu - q_\mu q)F_1(q^2) + (i\sigma_{\mu\nu}q^\nu p_1 + p_2 i\sigma_{\mu\nu}q^\nu)F_2(q^2) + M_V^2\gamma_\mu F^{\nu\lambda}(q^2) \right] \gamma_L b,
$$

(2.1)

where $q_\mu$ is the momentum carried by the photon. While gauge invariance of the photon
demands a vanishing $F_{nAb}$, we will see that this arises in a subtle manner in the 3-3-1 model. We work in Feynman gauge and ignore the light quark masses. Then, for a single fermion of mass $m$ in the loop, we find the on-shell form factors

\[
F_1(0) = Q \left[ \frac{1}{9} + \frac{x(x^2 + 11x - 18)}{12(x-1)^3} + \frac{-9x^2 + 16x - 4}{6(x-1)^4} \ln x \right] + Q_V \left[ \frac{8}{9} + \frac{x(7x^2 - x - 12)}{12(x-1)^3} + \frac{x^2(x^2 - 10x + 12)}{6(x-1)^4} \ln x \right]
\]

\[
F_2(0) = Q \left[ \frac{2}{3} - f_1(x) \right] + Q_V \left[ -\frac{5}{6} - f_2(x) \right], \tag{2.2}
\]

where $x = m^2/M^2$. For later convenience, we have defined the functions

\[
f_1(x) = \frac{x(x^2 - 5x - 2)}{4(x-1)^3} + \frac{3}{2} \frac{x^2}{(x-1)^4} \ln x
\]

\[
f_2(x) = \frac{x(-2x^2 - 5x + 1)}{4(x-1)^3} + \frac{3}{2} \frac{x^3}{(x-1)^4} \ln x. \tag{2.3}
\]

In the above form factors, $Q$ is the charge of the fermion in the loop and $Q_V$ is the charge of the gauge boson. In the SM, only the $W$ loop is present, in which case $Q = 2/3$ and $Q_V = -1$. After summing over all three families, the constant terms in Eqn. (2.2) vanish by the GIM mechanism while the non-constant terms agree with the expressions given by Inami and Lim [11].

Prior to imposing the GIM mechanism, the gauge non invariant term $F_{nAb}$ is present. In Feynman gauge, to the same order as above, we find

\[
F_{nAb}(q^2) = 2Q_V \left[ \Delta_M + \frac{q^2}{6M^2} + \cdots \right], \tag{2.4}
\]

where $\Delta_M = \frac{1}{4} - \gamma - \ln \frac{M^2}{4\pi\mu^2}$ in dimensional regularization. We note that $F_{nAb}$ is independent of the mass $m$ of the quark propagating in the loop. Thus, assuming generation universality as in the case of the SM, it drops out of the final expression because of the GIM mechanism.

When we take QCD corrections into account we need to calculate the induced $b \to sg$ penguins as well. The above form factors are also valid for the gluon penguin provided we set $Q = 1$ and $Q_V = 0$ and replace $e$ by the strong coupling $g_3 T^a$. For on-shell interactions, $F_1$ is unimportant and we are left with the dipole terms given by $F_2$. By convention, we now denote the coefficient of such photon and gluon dipole terms by $C_7$ and $C_8$ respectively [4]. In the case of a $W$ loop, we take the GIM mechanism into account and assume a heavy
top quark. In this case the photon and gluon dipole coefficients become:

\[ C_7^{SM}(M_W) = -\frac{1}{2} V_{ts}^* V_{tb} \left( \frac{2}{3} f_1(x_t) - f_2(x_t) \right) \]

\[ C_8^{SM}(M_W) = -\frac{1}{2} V_{ts}^* V_{tb} f_1(x_t) , \] (2.5)

where \( x_t = m_t^2/M_W^2 \). This is the only contribution in the SM.

In the 3-3-1 model, we need to include the contributions from the dilepton–heavy quark loops. In this case, there is a generalized GIM mechanism which replaces the standard case in the SM. Denoting the unitary mixing matrix in the down quark sector by \( V_L \) with elements \( v_{ij} \), we sum over all three families (exotic quarks \( D, S, T \)) to find:

\[ C_Y^7(M_Y) = -\frac{1}{2} \sum_{i=1}^{3} v_{is} v_{ib} \left[ -\frac{4}{3} f_1(x_i) + f_2(x_i) \right] \]

\[ -\frac{1}{2} v_{3s}^* v_{3b} \left[ -\frac{9}{2} + 3 f_1(x_3) - 3 f_2(x_3) \right] \]

\[ C_Y^8(M_Y) = -\frac{1}{2} \sum_{i=1}^{3} v_{is} v_{ib} f_1(x_i) , \] (2.6)

where \( x_i = m_{Q_i}^2/M_Y^2 \) and \( Q_i = (D, S, T) \). Unlike the SM contribution, where we ignore masses of the first two families, in this case phenomenology dictates that all exotic quarks are heavy. The gluon penguin has the same form as in the SM since QCD is insensitive to the electric charges of the quarks and gauge bosons in the loop. On the other hand, the second line in Eqn. (2.6) is present because the third family couples differently. This is already well known for the tree level \( Z' \) couplings. However we see here that generation non-universality also appears in dilepton loop diagrams.

We note that this imperfect GIM cancellation has a couple of consequences. First of all, unlike the SM, the dilepton induced \( b \to s\gamma \) penguin in the 3-3-1 model may be non-vanishing even when all exotic quarks are degenerate in mass. This is simply another manifestation of potentially large FCNCs in the 3-3-1 model. Secondly, we may worry about the divergent gauge non-invariant term \( F^{nAb} \) of Eqn. (2.4). Because the GIM sum involves the gauge bosons \( Y^+ \) and \( Y^- \), we find a left over divergent term of the form \(-3v_{3s}^* v_{3b} [\Delta_M + q^2/6M^2] \). This, however, is cancelled by the \( Z' \) FCNC vertex induced by \( \gamma-Z' \) mixing as shown in Fig. 2.

\[ ^1 \text{In this section we include the CKM factors explicitly. However subsequently they will be absorbed into the definition of the effective Hamiltonian.} \]
Since this diagram does not contribute to the dipole form factor, its only purpose for the present discussion is to eliminate the unwanted $F^{nAb}$, and can otherwise be ignored.

Finally, the diagrams with a $Z'$ in the loop also contribute to $b \to s\gamma$ due to the flavor changing $Z'$ couplings. In this neutral current case, since the quark in the loop is light, we ignore its mass and find

\[
C_7^{Z'}(M_{Z'}) = -\frac{1}{3} C_8^{Z'}(M_{Z'})
\]
\[
C_8^{Z'}(M_{Z'}) = -\frac{1}{2} v_{3s}^* v_{3b} - \frac{16}{9} \frac{s^2}{1 - 4s^2},
\]  

(2.7)

where $s^2 = 1 - c^2 = \sin^2 \theta_W$ arises from the $Z'$ coupling to quarks \[12\]. In principle, this introduces yet another scale into the problem, namely $M_{Z'}$. Since the $Z'$ is considerably heavier than the dileptons, we would in principle need to worry about all three heavy scales, $M_{Z'}$, $M_Y$ and $M_W$ (in addition to heavy quark thresholds) when including QCD corrections. However, as a simplification we ignore the difference between the two 3-3-1 scales since the QCD running effects are less pronounced at higher energies. In this case, we find it convenient to rewrite the $Z'$ contribution to change the mass scale in the effective vertex, Eqn. (2.1), from $M_{Z'}$ to $M_Y$. Using the relation $M_{Z'}^2 = M_Y^2/\rho_{3-3-1} \sin^2 \theta_{3-3-1}$ where $\cos^2 \theta_{3-3-1} = 3 \tan^2 \theta_W$ \[10\], we arrive at

\[
C_7^{Z'}(M_Y) = -\frac{1}{3} C_8^{Z'}(M_Y)
\]
\[
C_8^{Z'}(M_Y) = -\frac{1}{2} v_{3s}^* v_{3b} - \frac{16}{9} \rho_{3-3-1} \frac{s^2}{c^2}.
\]  

(2.8)

As in the SM, the generalized rho parameter, $\rho_{3-3-1}$, depends on the specifics of the extended Higgs sector. $\rho_{3-3-1} = 3/4$ in the minimal 3-3-1 model where $SU(3)_L \times U(1)_X$ is broken by a single $SU(3)_L$ triplet Higgs VEV \[10\].

In addition to the gauge boson loop contributions, additional scalars may also induce a $b \to s\gamma$ dipole transition. The reduction of the minimal Higgs sector of the 3-3-1 model has been considered in \[14\]. Three $SU(3)_L$ triplet and one sextet Higgs fields are required to break the symmetries and generate all fermion masses. While the general Higgs structure is quite complicated (and includes flavor changing neutral Higgs interactions), we make the assumption that only interactions proportional to the (large) top Yukawa coupling are important. Along this line, we note that the minimal 3-3-1 Higgs sector reduces to a three-Higgs-doublet SM with additional fields carrying lepton number. In the quark sector, this reduces further into a two-Higgs-doublet model with the added feature that the couplings
of the two Higgs doublets to the third family are interchanged compared to the first two families.

Assuming an approximately diagonal family structure, since only loops involving the top quark are important, we only consider the third family Higgs boson couplings. In this case, the scalar contribution is equivalent to that of an ordinary two-Higgs-doublet model (at this level of approximation), with the well known result

\[
C_{2\text{HD}}^7(M_W) = -\frac{1}{6} V_{ts}^* V_{tb} \left[ \left( \frac{2}{3} f_1(y_t) - f_2(y_t) \right) \cot^2 \beta - \left( \frac{2}{3} f_3(y_t) - f_4(y_t) \right) \right]
\]

\[
C_{2\text{HD}}^8(M_W) = -\frac{1}{6} V_{ts}^* V_{tb} \left[ f_1(y_t) \cot^2 \beta - f_3(y_t) \right],
\]  

(2.9)

where

\[
f_3(y) = \frac{3y(-y + 3)}{2(y-1)^2} - \frac{3y}{(y-1)^3} \ln y
\]

\[
f_4(y) = \frac{3y(y + 1)}{2(y-1)^2} - \frac{3y^2}{(y-1)^3} \ln y.
\]  

(2.10)

In this case, \( y_t = m_t^2/M_{H_t}^2 \) and \( \tan \beta = v_2/v_1 \) where \( v_2 \) gives rise to \( m_t \).

The complete electroweak contribution to \( b \to s\gamma \) in the 3-3-1 model is simply a sum of the individual contributions given in Eqns. (2.5), (2.6), (2.8) and (2.9)

\[
C_i(M_W) = C_i^{\text{SM}}(M_W) + C_i^{2\text{HD}}(M_W) + \frac{M_W^2}{M_Y^2} [C_i^Y(M_Y) + C_i^{2\gamma'}(M_Y)].
\]  

(2.11)

This shows explicitly that the 3-3-1 contributions are suppressed by the higher dilepton mass scale. Nevertheless, as shown in the next section, the 3-3-1 coefficient \( C_i^Y(M_Y) \) is quite large because of the family non-universality. This and the upper bound on \( M_Y \) ensure that the new 3-3-1 effects are generally of the same order as that of the SM and cannot be ignored.

### III. THE EFFECTIVE HAMILTONIAN AND QCD CORRECTIONS

In the SM, the QCD corrections to \( b \to s\gamma \) soften the GIM mechanism to yield a logarithmic GIM cancellation. This effect leads to an enhancement of the \( b \to s\gamma \) rate of about a factor of three. For the 3-3-1 model, however, the generalized GIM mechanism present in dilepton exchange diagrams, Eqn. (2.6), is imperfect, even in the absence of QCD corrections. Thus in this case, additional QCD corrections are not expected to dominate the 3-3-1 contribution to \( b \to s\gamma \). Nevertheless, we include them here for completeness.
We deal with the QCD corrections using the standard technique of integrating out the heavy degrees of freedom at each scale, using the renormalization group approach with effective hamiltonians. We work with three scales, $m_b$, $M_W$ and $M_Y$, where the latter two can also be thought of as the electroweak and the 3-3-1 scales respectively. Starting at $M_Y$, we first integrate out the 3-3-1 degrees of freedom, and then at $M_W$ integrate out the $W$ and top simultaneously\(^2\). Since the running between $M_W$ and $m_b$ has been extensively studied, we use the well known results of the leading order calculation \([4,16,17]\) and generalize them to take into account additional operators present in the 3-3-1 case.

A. The effective Hamiltonian for $M_W \leq \mu \leq M_Y$

Below $M_Y$, we integrate out both dilepton and $Z'$ loops, yielding an effective Hamiltonian

$$\mathcal{H}_{\text{eff}}^{3-3-1} = -2\sqrt{2}G_F \frac{M_W^2}{M_Y^2} v_{33} v_{30} \sum_i C_i^{3-3-1}(\mu) O_i^{3-3-1}(\mu),$$

where the set of operators $O_i^{3-3-1}$ consist of both four-Fermi and penguin operators relevant to the flavor changing $\Delta B = -\Delta S = -1$ interaction. Since the 3-3-1 model has additional FCNC interactions, we must use the extended operator basis

$$O_1^t = (\bar{s}L \gamma^\mu \gamma^\nu t_L)(\bar{t}_L \gamma^\rho t^\alpha_L)$$
$$O_2^t = (\bar{s}L \gamma^\mu \gamma^\nu t_L)(\bar{t}_L \gamma^\rho b_L)$$
$$O_1^b = (\bar{s}L \gamma^\mu \gamma^\nu b_L)(\bar{b}_L \gamma^\rho b_L)$$
$$O_3 = (\bar{s}L \gamma^\mu \gamma^\nu b_L) \sum_q (\bar{q}_L \gamma^\rho q_L)$$
$$O_4 = (\bar{s}L \gamma^\mu \gamma^\nu b_L) \sum_q (\bar{q}_L \gamma^\rho \gamma^\mu q_L)$$
$$O_5 = (\bar{s}L \gamma^\mu \gamma^\nu b_L) \sum_q (\bar{q}_R \gamma^\rho q_R)$$
$$O_6 = (\bar{s}L \gamma^\mu \gamma^\nu b_L) \sum_q (\bar{q}_R \gamma^\rho \gamma^\mu q_R)$$
$$O_8^Q = (\bar{s}L \gamma^\mu \gamma^\nu b_L) \sum_q e_q (\bar{q}_R \gamma^\rho q_R)$$

\(^2\)In principle, there are additional corrections arising from QCD running between $m_t$ and $M_W$ \([14,15]\). However such corrections are presently dominated by the uncertainty arising from QCD scale dependence in the leading order calculation and may be ignored.
\[ O_6^Q = (\bar{s}_L \gamma_\mu b_R^\beta) \sum_q e_q (\bar{u}_R \gamma^\mu q_R^\alpha) \]
\[ O_7 = \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma_{\mu\nu} b_R) F^{\mu\nu} \]
\[ O_8 = \frac{g_3}{16\pi^2} m_b (\bar{s}_L \sigma_{\mu\nu} T^a b_R) G^{a\mu\nu}. \] (3.2)

Color non-singlet channels are indicated explicitly via the color indices \( \alpha \) and \( \beta \). Since we are effectively above \( m_t \), the sums are over all six quarks.

The Wilson coefficients, \( C_{3-3-1}^{3-3-1}(M_Y) \), are given by the matching conditions at the 3-3-1 scale. Integrating out the dileptons and exotic quarks gives rise to the penguin operators as shown in the previous section. Integrating out the \( Z' \) gives rise to effective four-Fermi in addition to the penguin operators. To leading order, the non-zero Wilson coefficients are

\[
\begin{align*}
C_{1t}^{3-3-1}(M_Y) &= 2 \rho_{3-3-1} \\
C_{1b}^{3-3-1}(M_Y) &= \frac{2}{3} \rho_{3-3-1} \\
C_3^{3-3-1}(M_Y) &= \frac{1}{3} \rho_{3-3-1} \frac{2s^2 - 1}{c^2} \\
C_{5Q}^{3-3-1}(M_Y) &= 2\rho_{3-3-1} \frac{s^2}{c^2} \\
C_7^{3-3-1}(M_Y) &= \frac{9}{4} - \frac{8}{27} \rho_{3-3-1} \frac{s^2}{c^2} - \left[ \frac{2}{3} f_1(x_S) - \frac{1}{2} f_2(x_S) \right] + \left[ -\frac{5}{6} f_1(x_T) + f_2(x_T) \right] \\
C_8^{3-3-1}(M_Y) &= \frac{8}{9} \rho_{3-3-1} \frac{s^2}{c^2} + \frac{1}{2} \left[ f_1(x_S) - f_1(x_T) \right] ,
\end{align*}
\] (3.3)

where \( x_S = m_S^2/M_Y^2 \) and \( x_T = m_T^2/M_Y^2 \). We have assumed negligible mixing to the first family so that \( v_{s_2}^* v_{2b} \approx -v_{3s}^* v_{3b} \). This is also the reason why the exotic \( D \) quark does not appear. Although we take \( \rho_{3-3-1} = 3/4 \) for our numerical results, it is shown explicitly above to indicate that those terms are due to \( Z' \) exchange.

The \( Z' \) induced terms arise where one vertex gives the \( b \to s \) flavor changing interaction and the other vertex is flavor conserving. However, since the left handed \( Z' \) interaction is non-universal, it singles out one of the families for special treatment. Up to small mixing (which we ignore), it must be the third family \([7,8]\), which is why the third family operators \( O_t^l \) and \( O_b^l \) are singled out. The other new four-Fermi operators, \( O_2, O_5^Q \) and \( O_6^Q \), must then be included to account for operator mixing as well as the right handed \( Z' \) vertex. It is straightforward to extend the results presented in \([17]\) to determine the anomalous dimension matrix corresponding to the mixing of the operators in Eqn. (3.2). The results are shown in the Appendix.
In order to examine the significance of the additional 3-3-1 contributions to \( b \to s\gamma \), we show values for the initial Wilson coefficient \( C_{7}^{3-3-1}(M_Y) \) in Fig. 3. From the figure, it is obvious that the generalized GIM mechanism is quite different from the ordinary case. Instead of vanishing as in the usual case, the Wilson coefficient takes on its largest values when the exotic quarks are relatively light and degenerate in mass. Since the functions \( f_1 \) and \( f_2 \) are bounded by \( 0 \leq f_1(x) \leq 1/4 \) and \( -1/2 \leq f_2(x) \leq 0 \), we find the limits

\[
1.06 \leq C_{7}^{3-3-1}(M_Y) \leq 2.18 \\
0.076 \leq C_{8}^{3-3-1}(M_Y) \leq 0.326 ,
\]

although for realistic values of \( m_S \) and \( m_T \), we expect a more limited range, \( 1.3 \lesssim C_{7}^{3-3-1}(M_Y) \lesssim 2.0 \) as indicated by the figure. Note that this may be contrasted with the SM value \( C_{7}^{\text{SM}}(M_W) \approx -0.20 \) for \( m_t = 175 \text{ GeV} \). After accounting for the additional \( M^2_W/M^2_Y \) factor in Eqn. (3.1) that arises from the difference in mass scales, we see that the new 3-3-1 effects are comparable to that of the SM.

**B. The matching conditions at \( M_W \) and \( H_{\text{eff}} \) for \( m_b \leq \mu \leq M_W \)**

When we reach the electroweak scale, \( M_W \), we further integrate out the top and \( W \). In this case, we match the effective hamiltonian \( H_{\text{eff}}^{3-3-1}(M_W) \) onto a second one, \( H_{\text{eff}} \), without the top degrees of freedom. At this stage, since we include the SM contributions to \( b \to s\gamma \), we choose the conventional form

\[
H_{\text{eff}} = -2\sqrt{2}G_F V_{ts} V_{tb} \sum_i C_i(\mu)O(\mu) .
\]

However, since we have included a larger set of operators in \( H_{\text{eff}}^{3-3-1} \), they must be retained when running to \( m_b \). Thus a complete set of operators below \( M_W \) consist of those of Eqn. (3.2), with the exception that \( O_1^t \) and \( O_2^t \) are replaced by the conventional operators

\[
O_1 = (\bar{s}_L \gamma_\mu c_L^\beta)(\bar{c}_L^\beta \gamma^\mu b_L^a) \\
O_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L) ,
\]

and we only take five active quarks in \( O_3 \) through \( O_6^Q \).

At \( M_W \), the Wilson coefficients get contributions both from integrating out the top and from matching onto \( H_{\text{eff}}^{3-3-1} \). We find
\[ C_1(M_W) = C_1^{2\text{HDSM}}(M_W) \]
\[ C_2(M_W) = C_2^{2\text{HDSM}}(M_W) \]
\[ C_i(M_W) = C_i^{2\text{HDSM}}(M_W) + \chi \frac{M_W^2}{M_Y^2} C_i^{3-3-1}(M_W) \quad (\text{all other operators}) \], \quad (3.7)

where \( \chi = v_{ts}^* v_{tb} / V_{ts} V_{tb} \) is the ratio of 3-3-1 and SM mixing angles. The SM coefficients include the contributions from the two-Higgs-doublet model and are given by
\[
\begin{align*}
C_2^{2\text{HDSM}}(M_W) & = 1 \\
C_7^{2\text{HDSM}}(M_W) & = -\frac{1}{2} \left[ \frac{2}{3} f_1(x_t) - f_2(x_t) \right] \\
& - \frac{1}{6} \left[ \left( \frac{2}{3} f_1(y_t) - f_2(y_t) \right) \cot^2 \beta - \left( \frac{2}{3} f_3(y_t) - f_4(y_t) \right) \right] \\
C_8^{2\text{HDSM}}(M_W) & = -\frac{1}{2} f_1(x_t) \\
& - \frac{1}{6} \left[ f_1(y_t) \cot^2 \beta - f_3(y_t) \right]. \quad (3.8)
\end{align*}
\]

IV. THE \( b \to s\gamma \) RATE AND LIMITS ON 3-3-1 PHYSICS

Once the 3-3-1 and SM matching conditions, Eqns. (3.3) and (3.8), are given, it is straightforward to solve the renormalization group equations to arrive at \( C_7(\mu) \). The \( b \to s\gamma \) decay rate is then calculated in the ratio
\[
\frac{\Gamma(b \to s\gamma)}{\Gamma(b \to c\ell \nu_\ell)} = \frac{|V_{ts} V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi I(z)} |C_7(\mu)|^2, \quad (4.1)
\]
where \( z = m_c / m_b \) and \( I(z) = 1 - 8z^2 + 8z^6 - z^8 - 24z^4 \ln z \) is the phase space factor for the charged current decay; \( I(z) = 0.485 \pm 0.028 \) for \( z = 0.316 \pm 0.013 \) [18]. Following [2], we do not include the \( O(\alpha_3) \) corrections to \( b \to c\ell \nu_\ell \) at this order. In the absence of a complete next to leading order calculation, when necessary we vary the renormalization scale \( \mu \) by a factor of two around \( m_b \) to estimate the effects of the QCD scale ambiguity [2]. The resulting large \( \mu \) dependence of the leading order calculation is the dominant theoretical uncertainty in the predicted \( b \to s\gamma \) decay rate.

Using the current value for the charged current decay mode, \( \text{BR}(b \to c\ell \nu_\ell) = (0.104 \pm 0.004) \) [19], Eqn. (4.1) may be rewritten as
\[
\text{BR}(b \to s\gamma) = (2.84 \pm 0.23) \times 10^{-3} |C_7(\mu)|^2, \quad (4.2)
\]
so that the current CLEO limits, Eqn. (1.2), correspond to
0.18 \leq |C_7(\mu)| \leq 0.38 \quad \text{(CLEO)}. \quad (4.3)

From now on we focus on $C_7(\mu)$ instead of the branching ratio since the SM $W-t, H^+-t$ and new 3-3-1 contributions may simply be added together in the amplitude (this will always be true, even with higher order QCD corrections, as long as we work only at the one-electroweak-loop order).

The 3-3-1 model introduces many new parameters into the $b \to s\gamma$ calculation. Explicitly, we write

\[ C_7(\mu) = C_7(\mu; m_t, M_{H^+}, \tan \beta; m_S, m_T, M_Y, \chi) = C_{7}^{2\text{HDSM}}(\mu; m_t, M_{H^+}, \tan \beta) + \Delta C_7(\mu; m_S, m_T, M_Y, \chi), \quad (4.4) \]

where in the second line we have separated out the two-Higgs-doublet SM and new 3-3-1 contributions. The SM prediction for $C_7(m_b)$ is shown by the solid line in Fig. 4 along with the CLEO bounds. We have taken $\alpha_3(M_Z) = 0.120$ and $m_b = 5$ GeV. As can be seen, the present experimental data is consistent with the unadorned SM. When the charged Higgs loop is included, it always contributes with the same sign and hence can only increase the predicted branching ratio. As an example, we plot $C_7(m_b)$ for several values of the charged Higgs mass in Fig. 4 (dotted lines) when $\tan \beta = 3$. The two-Higgs-doublet results are fairly insensitive to $\tan \beta$ provided $\tan \beta \gtrsim 1$. For small $\tan \beta$, on the other hand, the charged Higgs contribution is enhanced and often falls outside the CLEO bound.

The effect of the 3-3-1 contributions, $\Delta C_7(m_b)$, may be estimated from the renormalization group analysis as

\[ \Delta C_7(m_b) \approx \chi \frac{M_Y^2}{M_W^2} \left[ 0.063 + 0.59 C_7^{3-3-1}(M_Y) + 0.11 C_8^{3-3-1}(M_Y) \right], \quad (4.5) \]

using 400 GeV as the 3-3-1 scale. Since both Wilson coefficients are bounded according to Eqn. (3.4), we find the size of the new 3-3-1 effects to be

\[ \Delta C_7(m_b) \approx \chi \frac{M_Y^2}{M_W^2} (0.7 - 1.4). \quad (4.6) \]

This range is given by the dotted lines in Fig. 5, assuming $\chi = 1$. The actual predictions for degenerate exotic quarks are given by the solid lines in the figure.

While the mixing parameter, $\chi = v_3^* v_3 v_3^* / V_{ts} V_{tb}$, is in principle undetermined, we expect both the numerator and denominator to be comparable, giving a value $|\chi| \sim O(1)$. Unrealistic cases of vanishing mixing in the down quark sector ($V_L = 1$) and up quark sector
(\(U_L = 1\)) give \(\chi = 0\) and 1 respectively. In order to put an experimental limit on \(\chi\), we may use the neutral meson mixing data to restrict the product of 3-3-1 mixing angles to be \(|v^*_{3s}v_{3b}| \leq 0.25\) at 90\% C.L. \(|\chi| \leq 8.3\), although values near the upper bound may be somewhat unnatural from a theoretical point of view.

As in the two-Higgs-doublet model, the additional contribution from the 3-3-1 model, \(\Delta C_7(\mu)\), always enters with the same sign. However, unlike the charged Higgs contribution which always increases the \(b \to s\gamma\) rate, the effect of the new 3-3-1 contribution depends on the quark mixing parameter \(\chi\). From Eqn. (4.6) we see that \(\chi > 0\) \((< 0)\) leads to a suppression (enhancement) of the overall \(b \to s\gamma\) decay rate. In general, when \(\chi\) is complex, the results lie somewhere in the middle. This feature seems to be shared with other models incorporating two Higgs doublets. Namely, while the charged Higgs loop alone always increases the \(b \to s\gamma\) decay rate, the additional new particles, whether superpartners in the SUSY case or dilepton gauge bosons in the 3-3-1 model, contribute with arbitrary sign and may compensate for the increase arising from the Higgs loop. This is also the case for radiative corrections to the \(Z \to b\bar{b}\) vertex.

In order to examine how the new 3-3-1 physics weakens the \(b \to s\gamma\) limits on the pure two-Higgs-doublet model, we plot the allowed region in the \(\tan \beta - M_{H^+}\) parameter space in Fig. 6. In this case we have fixed the top quark mass to be 175 GeV. The solid line corresponds to the two-Higgs-doublet model and shows that \(M_{H^+} > 290\) GeV for the case \(m_t = 175\) GeV. In arriving at this limit we have estimated the theoretical uncertainty by varying the QCD scale \(\mu\) from \(m_b/2\) to \(2m_b\). As noted previously, the limits are insensitive to \(\tan \beta\) when \(\tan \beta \gtrsim 1\). Inclusion of 3-3-1 physics with \(|\chi| \leq 1\) lowers this bound as indicated by the dotted lines in the figure. For a dilepton gauge boson mass \(M_Y = 300\) GeV, the corresponding limit on the charged Higgs mass is weakened to \(M_{H^+} > 120\) GeV for light exotic quarks. The 3-3-1 limits correspond to real positive \(\chi\), which is the region of maximum cancellation between the charged Higgs and dilepton gauge boson loops. Larger values of \(\chi\) naturally weaken the limits further. However the further we are below the two-Higgs-doublet limit (solid line), the more tuning is required between the two-Higgs-doublet and 3-3-1 parameters to achieve large cancellations. Thus from a naturalness point of view, we expect the charged Higgs mass in the 3-3-1 model to be no lighter than \(\sim 120\) GeV.
V. CONCLUSION

Because it is a loop process, the FCNC decay $b \rightarrow s\gamma$ presents an interesting test of both SM and new physics. In the SM, this process is GIM suppressed and proceeds through a heavy top quark. While the theoretical calculation suffers from large uncertainties due to unknown next to leading order QCD corrections, it is in excellent agreement with the current CLEO data \[1\]. Nevertheless, at the present level, the results are not yet sensitive to $m_t$ in the SM, as may be seen from Fig. 4. Future work, on both the theoretical and experimental side, may bring the uncertainties down to the point where $b \rightarrow s\gamma$ would become more sensitive to both $m_t$ and new physics.

In anticipation of such future improvements, we may estimate the size of the contribution of new physics to $b \rightarrow s\gamma$. Compared to the SM value, $C_7(m_b) \sim -0.3$, new physics at a scale $M_{\text{new}}$ is expected to contribute roughly to the $b \rightarrow s\gamma$ vertex as $|\delta C_7(m_b)| \sim (M_W^2/M_{\text{new}}^2)(\Delta M_Q^2/M_{\text{new}}^2)$ where $\Delta M_Q$ is a typical mass splitting between the new fermions in the loop and arises via a generalized GIM mechanism. Thus in general (assuming the absence of tree level FCNCs in the extended model) new physics is suppressed by both the heavier mass scale and a generalized GIM mechanism and is hence dominated by the larger SM contribution.

In order to evade this conclusion, we need to either have $M_{\text{new}} \approx M_W$ or somehow avoid the generalized GIM cancellation. An example of the former case is the two-Higgs-doublet model where a light charged Higgs particle may be eliminated by the current CLEO data. As an example of the latter case, we have performed a detailed calculation of $b \rightarrow s\gamma$ in the 3-3-1 model. Due to the different representation of the third quark family, the generalized GIM cancellation is imperfect and $|\Delta C_7(m_b)| \sim M_W^2/M_Y^2$ is non-vanishing even for degenerate exotic quark masses. Compared to the two-Higgs-doublet model where $M_{H^+} > 290$ GeV, this limit is weakened to $M_{H^+} > \sim 120$ GeV in the full 3-3-1 model.

Although the 3-3-1 model appears tightly constrained both by FCNC limits and by the large $U(1)_X$ coupling \[1\]\[13\]\[20\], the model survives the test of $b \rightarrow s\gamma$, at least up to the current level of precision. As experimental evidence for a heavy top continues to build, the curious feature of a different third generation in the 3-3-1 model takes on more significance. Hence we look forward with anticipation to what future experiments in the $B$ system will bring.
This work was supported in part by the National Science Foundation under Grant No. PHY-9411543, and by the Department of Energy under Grant No. DE-FG05-85ER-40219. JTL wishes to thank D. Ng for enlightening discussions in the initial stages of this work.

APPENDIX: THE ANOMALOUS DIMENSION MATRICES

At leading order, the renormalization group equation for the Wilson coefficients $\tilde{C}(\mu)$ is

\[ \mu \frac{d}{d\mu} \tilde{C}(\mu) = \left[ \gamma^T(\mu) \right] \tilde{C}(\mu), \tag{A1} \]

where

\[ \gamma(\mu) = \frac{\alpha_3(\mu)}{2\pi} \gamma^0, \tag{A2} \]

and $\alpha_3(\mu) \equiv g_3^2(\mu)/4\pi$ satisfies the $\beta$-function equation

\[ \mu \frac{d}{d\mu} \alpha_3^{-1}(\mu) = -\frac{b}{2\pi}. \tag{A3} \]

Here $b = -11N/3 + 2f/3$ is the one-loop QCD beta function coefficient for $f$ quarks ($b(6) = -7$ and $b(5) = -23/3$). Eqn. (A1) is exactly solved by

\[ \tilde{C}(\mu) = V \left[ \eta^{-\gamma^0_D/b} \right] V^{-1} \tilde{C}(M), \tag{A4} \]

where $\eta = \alpha_3(M)/\alpha_3(\mu)$. For consistency at this order, the running of $\alpha_3(\mu)$ is only calculated to one loop. In this notation, $\gamma^0_D = V^{-1} \gamma^0 V$ is the diagonal matrix of eigenvalues of $\gamma^0$.

For the extended operator basis of Eqn. (3.2), we write the scheme independent anomalous dimension matrix in the form

\[ \gamma^0 = \begin{pmatrix} \gamma_{44} & \gamma_{4P} \\ 0 & \gamma_{PP} \end{pmatrix}. \tag{A5} \]

Then, for $u$ up and $d$ down type quarks, we find
where \( f = u + d \) and \( f = e_u u + e_d d = (2u - d)/3 \) and \( C_2 = (N^2 - 1)/2N \). The mixing of the penguin operators are given by

\[
\gamma_{PP} = \begin{pmatrix}
4C_2 & 0 \\
-\frac{1}{3}(4C_2) & 8C_2 - 2N
\end{pmatrix}.
\]  

(A7)

For \( \gamma_{4P} \), we use the scheme independent formalism of \( [17] \) (which, at this level, is equivalent to using the 't Hooft–Veltman regularization scheme). The result is

\[
\gamma_{4P} = \begin{pmatrix}
0 & 3S_2 \\
(3e_u + \frac{2}{9}e_d)C_2 & \frac{20}{9}C_2 - N \\
(3 + \frac{2}{9})e_dC_2 & 3S_2 + \frac{20}{9}C_2 - N \\
2(3 + \frac{2}{9})e_dC_2 & 3S_2 + 2(\frac{20}{9}C_2 - N) \\
(3f + \frac{2}{9}f e_d)C_2 & 6S_2 + f(\frac{20}{9}C_2 - N) \\
-4e_d C_2 & -3fS_2 - 4C_2 + N \\
(-3f + \frac{2}{9}f e_d)C_2 & -4S_2 + f(-\frac{25}{9}C_2 + \frac{N}{2}) \\
-4e_d^2 C_2 & -3fS_2 + e_d (-4C_2 + N) \\
(-3(u e_u^2 + d e_d^2) + \frac{2}{9}f e_d)C_2 & -4e_d S_2 + f(-\frac{25}{9}C_2 + \frac{N}{2})
\end{pmatrix},
\]  

(A8)

where \( S_2 = 1/2 \).

For \( M_W \leq \mu \leq M_Y \), we set \( N = 3 \) and six quarks are active \( (u = d = 3) \). In this case the explicit anomalous dimension matrix is
Below $M_W$, the top quark is integrated out, and we are left with a five quark anomalous dimension matrix

$$\gamma^0_{(5)} = \begin{pmatrix}
-1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \\
3 & -1 & 0 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 & \frac{208}{81} & \frac{25}{27} \\
0 & 0 & 2 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 & -\frac{116}{81} & \frac{151}{54} \\
0 & 0 & 0 & -\frac{11}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 0 & -\frac{232}{81} & \frac{313}{27} \\
0 & 0 & 0 & \frac{7}{3} & 1 & -\frac{2}{3} & 2 & 0 & 0 & \frac{92}{27} & \frac{97}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & \frac{16}{9} & -\frac{34}{3} \\
0 & 0 & 0 & -\frac{2}{3} & 2 & -\frac{2}{3} & -6 & 0 & 0 & -\frac{124}{27} & -\frac{137}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & -\frac{16}{27} & -\frac{13}{18} \\
0 & 0 & 0 & -\frac{9}{9} & \frac{11}{3} & -\frac{1}{9} & \frac{3}{3} & 0 & -8 & -\frac{548}{81} & -\frac{83}{54} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & \frac{14}{3} \\
\end{pmatrix} \quad \text{(A9)}$$

$$\gamma^0_{(6)} = \begin{pmatrix}
-1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \\
3 & -1 & 0 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 & \frac{208}{81} & \frac{25}{27} \\
0 & 0 & 2 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 & -\frac{116}{81} & \frac{151}{54} \\
0 & 0 & 0 & -\frac{11}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 0 & -\frac{232}{81} & \frac{545}{54} \\
0 & 0 & 0 & \frac{22}{9} & \frac{2}{3} & -\frac{5}{9} & \frac{5}{3} & 0 & 0 & \frac{68}{81} & \frac{256}{27} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & \frac{16}{9} & -\frac{59}{6} \\
0 & 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{19}{3} & 0 & 0 & -\frac{148}{81} & -\frac{763}{54} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & -\frac{16}{27} & \frac{1}{18} \\
0 & 0 & 0 & -\frac{1}{27} & \frac{1}{9} & -\frac{1}{27} & \frac{1}{9} & 0 & -8 & -\frac{1196}{243} & -\frac{11}{162} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{9} & \frac{14}{3} \\
\end{pmatrix} \quad \text{(A10)}$$
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FIGURES

FIG. 1. The gauge boson contributions to $b \to s\gamma$. In the 3-3-1 model, $V$ may be either $W$, $Y$ or $Z'$.

FIG. 2. Additional diagram needed to cancel divergences and restore gauge invariance in the $b \to s\gamma$ vertex in the 3-3-1 model.

FIG. 3. Contours of constant $C_{3-3-1}^{3-3-1}(M_Y)$ in the $x_{T}-x_{S}$ plane. Note that $C_{3-3-1}^{3-3-1}(M_Y)$ is everywhere positive and does not vanish on the diagonal $x_{T} = x_{S}$.

FIG. 4. The Wilson coefficient $C_{7}(m_b)$ in the SM (solid line) and two-Higgs-doublet model (dotted lines) plotted as a function of $m_t$. For the two-Higgs-doublet model, we have taken $\tan\beta = 3$. The current CLEO bounds are shown by the dashed lines.

FIG. 5. The 3-3-1 model contribution to $C_{7}(m_b)$ (with the mixing parameter $\chi$ removed) plotted as a function of dilepton gauge boson mass for degenerate exotic quark masses $m_Q = 250, 500, 750$ and 1000 GeV (solid lines). The dotted lines indicate the minimum and maximum possible values arising from 3-3-1 physics.

FIG. 6. The allowed region in $\tan\beta-M_{H^{\pm}}$ parameter space for the two-Higgs-doublet model (solid line) and the 3-3-1 model (dotted lines). For each pair of dotted lines, the upper one corresponds to $M_Q = 1000$ GeV and the lower one to $M_Q = 250$ GeV. For the 3-3-1 model we have restricted the ratio of mixing angles by $|\chi| \leq 1$. 
Figure 2
Figure 3
Figure 4
Figure 5
