A CHAIN-LEVEL HKR-TYPE MAP AND A CHERN CHARACTER FORMULA

KUERAK CHUNG, BUMSIG KIM, AND TAEJUNG KIM

Abstract. We construct a Hochschild-Kostant-Rosenberg-type quasi-isomorphism for the negative cyclic homology of the category of global matrix factorizations on a smooth separated scheme of finite type over a field. The map is explicit enough to yield a negative cyclic Chern character formula for global matrix factorizations. We also extend these results to the equivariant case of a finite group.

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1. INTRODUCTION

1.1. Overview. Fix a field $k$ of characteristic zero and let $X$ be a smooth separated scheme of finite type over $k$. Let $w$ be a global function on $X$ and let $G$ be either the group $\mathbb{Z}$ or $\mathbb{Z}/2$. The structure sheaf $\mathcal{O}_X$ of $k$-algebras is defined to be concentrated in degree 0 as a $G$-graded sheaf. We require that the degree of $w$ is 2. Hence if $w$ is nonzero, $G$ is $\mathbb{Z}/2$. If $w = 0$, we allow either $G$. Assume that the critical values of $w$ are possibly zero.

We consider a differential $G$-graded (in short dg) category $D_{dg}(X, w)$ of $G$-graded matrix factorizations for $(X, w)$. By a matrix factorization $(P, \delta_P)$
for \((X, w)\) we mean a locally free coherent \(O_X\)-module \(P\) with a \(\mathbb{G}\)-grading and \(\delta_P\) is a curved differential of \(P\). By definition, \(\delta_P\) is a degree 1 \(O_X\)-endomorphism of \(P\) satisfying \(\delta_P^2 = w \cdot \text{id}_P\). The Hom space from \(P\) to \(Q\) in \(D_{dg}(X, w)\) is taken to be a dg \(k\)-module \(\text{Hom}_{O_X}(I(P), I(Q))\) with the chosen quasi-curved injective replacements \(I(P), I(Q)\) of \(P, Q\) respectively; see \([1]\). Note that when \(\mathbb{G} = \mathbb{Z}\), then \(w = 0\) and the homotopy category of \(D_{dg}(X, w)\) is triangulated and equivalent to the derived category of bounded complexes of coherent sheaves on \(X\). When \(w \neq 0\) in every component of \(X\), the homotopy category is also equivalent to the singularity category of the hypersurface \(w^{-1}(0)\); see \([17]\) Theorem 2] and \([15]\) Proposition 2.13).

The periodic cyclic homology and the Hochschild homology of \(D_{dg}(X, w)\) play the roles of the de Rham cohomology and the Hodge cohomology of the “space” \((X, w)\), respectively. The negative cyclic homology \(HN_{-1}(D_{dg}(X, w))\) encapsulates both homology theories, which is canonically isomorphic to

\[ H^{-s}(X, (\Omega^s_{X/k}[[u]], -dw + ud)); \]

see \([7]\) Theorem 1.4. Here \(u\) is a formal variable with degree 2 and by definition the exterior derivative \(d\) has degree \(-1\). Each \((P, \delta_P)\) defines a tautological class in \(HN_0(D_{dg}(X, w))\), the so-called categorical Chern character \(Ch_{HN}(P)\). Under the isomorphism, \(Ch_{HN}(P)\) can be written as an element \(ch_{HN}(P)\) of \(H^0(X, (\Omega^0_{X/k}[[u]], -dw + ud))\). In this paper we find a formula for \(ch_{HN}(P)\) by Chern-Weil theory and a Čech resolution, which is a globalization of Brown – Walker’s formula \([3]\).

1.2. A Chern character formula via Chern-Weil theory. We fix a finite open affine covering \(\mathcal{U} := \{U_i\}_{i \in I}\) of \(X\) and a total ordering of the index set \(I\). Let \((P, \delta_P)\) be a matrix factorization for \((X, w)\) as before. Since each homogeneous part of \(P|_{U_i}\) corresponds to a finitely generated projective module over a noetherian ring \(\Gamma(U_i, O_X)\), \(P|_{U_i}\) always admits a connection

\[ \nabla_i : P|_{U_i} \to P|_{U_i} \otimes O_{U_i} \Omega ^1_{U_i/k}. \]

It is a \(k\)-linear degree \(-1\) map satisfying \(\nabla_i(pa) = \nabla_i(p)a + (-1)^{|p|}p \otimes da\) for \(p \in P|_{U_i}\), \(a \in O_{U_i}\). We define the associated total curvature

\[ R := \prod_{i \in I} (u \nabla _i^2 + [\nabla _i, \delta _P]) + \prod_{i < j, i, j \in I} (\nabla _i - \nabla _j) \]

as an element in the Čech complex \(\check{C}(\mathcal{U}, \text{End}(P) \otimes O_X \Omega ^{\bullet}_{X/k}[[u]])\); see \((6.2)\) and Remark \((5.6)\). Note that the total degree of each term in the curvature \(R\) is zero.

**Theorem 1.1.** (Theorem \((6.1)\)) The following Chern character formula holds:

\[ ch_{HN}(P) = \text{tr} \exp(-R) \]
in the ordered Čech cohomology of complexes of sheaves
\[ H^0(\mathcal{U}, (\Omega^\bullet_{X/k}[[u]], -dw + ud)). \]

Here the expression
\[ \exp(-R) = \text{id}_P - R + R^2/2 - \cdots + (-1)^{\dim X} R^{\dim X}/(\dim X)! \]
uses the product formula (4.5) in the Čech cochain complex and \( \text{tr} \) denotes the supertrace (5.10).

When \( |I| = 1 \) so that \( X \) is affine, formula (1.3) is the main result of Brown and Walker [3]. When \( u = 0 \), \( \text{ch}_{HN}(P) \) becomes the image \( \text{ch}_{HH}(P) \) of the Hochschild homology valued Chern character \( \text{Ch}_{HH}(P) \). Formula (1.3) specialized to \( u = 0 \) matches with those of Căldăraru [4] for \( G = \mathbb{Z} \); and Platt [18] and Kim and Polishchuk [14] for \( G = \mathbb{Z}/2 \), respectively (up to a sign convention which occurs in the choices of isomorphisms). We will also get a localized Chern character formula; see §6.2.

1.3. A chain-level HKR-type map. We will prove Theorem 1.1 by an explicit quasi-isomorphism, which is a globalization of the generalized Hochschild-Kostant-Rosenberg (in short HKR) map in the affine case [3, Theorem 5.19]. The ordinary HKR map [16, §1.3.15] uses the canonical de Rham differential \( d \) of \( \mathcal{O}_X \). While \( P \) does not admit a global connection in general, there is a connection on \( P|_{U_i} \). This leads us to consider the Čech model \( D_{\mathcal{C}}(X, w) \) of \( D_{dg}(X, w) \); see §4.2.2. Let \( \check{C}(P) \) denote the sheafified version of the ordered Čech \( k \)-complex \( \check{C}(\mathcal{U}, P) \) for \( P \); see (4.2). The objects of \( D_{\check{C}}(X, w) \) are those of \( D_{dg}(X, w) \). The Hom space \( \text{Hom}(P, Q) \) is defined to be the dg \( k \)-space
\[ \text{Hom}^\bullet_{\check{C}(\mathcal{O}_X)}(\check{C}(P), \check{C}(Q)) \cong \Gamma(X, \check{\text{Hom}}^\bullet_{\mathcal{O}_X}(P, Q) \otimes_{\mathcal{O}_X} \check{\text{C}}(\mathcal{O}_X)) \]
(1.4)

Its underlying graded \( k \)-space is the linear space of graded right \( \check{\text{C}}(\mathcal{O}_X) \)-linear sheaf homomorphisms from \( \check{C}(P) \) to \( \check{C}(Q) \). Its differential is given as follows. Let \( \delta_{\check{C}(P)} = \delta_P + d_{\check{C}} \), the sum of the differential \( \delta_P \) of \( P \) and the Čech differential \( d_{\check{C}} \). Then the differential of \( \text{Hom}(P, Q) \) is defined by sending \( f \in \text{Hom}(P, Q) \) to
\[ \delta_{\check{C}(Q)} \circ f - (-1)^{|f|} f \circ \delta_{\check{C}(P)} \in \text{Hom}(P, Q), \]
where \( |f| \) denotes the degree of \( f \).

For each matrix factorization \( P \) for \( (X, w) \) and \( i \in I \), choose, once and for all, a connection \( \nabla_{P,i} \) on \( P|_{U_i} \). Hence we have a zero-th global Čech element
\[ \nabla_P := \prod_{i \in I} \nabla_{P|_{U_i}} \]
of the \( k \)-sheaf \( \check{\text{Hom}}_k(\check{C}(P), \check{C}(P) \otimes \Omega^1_X) \).

When \( P = \mathcal{O}_X \), we choose the de Rham differential as a connection on \( \mathcal{O}_{U_i} \). Let \( C(D_{\mathcal{C}}(X, w)) \) denote the Hochschild complex of the dg category
$D\mathcal{C}(X, w)$. Using the supertrace $\text{tr}_\nabla$ and the product (4.5) we define a $k[[u]]$-linear map

\begin{equation}
(1.5) \quad \text{tr}_\nabla : C(D\mathcal{C}(X, w))[[u]] \to \mathcal{C}(\Omega^*_{\mathcal{O}_X/\mathcal{O}_X})[[u]]
\end{equation}

by sending

\begin{equation}
(1.6) \quad \alpha_0[\alpha_1|...|\alpha_n] \to \sum_{(j_0, ..., j_n)} (-1)^J \text{tr}\left( \frac{\alpha_0 R_{j_1}^{j_0} [\nabla, \alpha_1] R_{j_2}^{j_1} [\nabla, \alpha_2] ... R_{j_n}^{j_{n-1}} [\nabla, \alpha_n] R_{j_n}^{j_0}}{(n + J)!} \right)
\end{equation}

for $\alpha_j \in \mathcal{C}(\Omega, \mathcal{H}om_{\mathcal{O}_X}(P_{j+1}, P_j))$ with $P_{n+1} = P_0$, where $j_i \in \mathbb{Z}_{\geq 0}$ and $J = \sum_{i=0}^n j_i$; and

$$[\nabla, \alpha_j] := \nabla_{P_j} \cdot \alpha_j - (-1)^{|\alpha_j|} \alpha_j \cdot \nabla_{P_{j+1}};$$

$$R_j := u \nabla_{P_j}^2 + [\nabla_{P_j}, \delta_{P_j}] + \prod_{i_0 < i_1} (\nabla_{P_{j,i_0}} - \nabla_{P_{j,i_1}});$$

see (5.16) for unwinded expressions. Note that when $|I| = 1$ and $P_i$ is $\mathcal{O}_X$ for $i = 0, ..., n$, formula (1.5) is the usual HKR map.

**Theorem 1.2.** (Theorem 5.4) Let $b$ denote the sum of Hochschild differentials from compositions and differentials; and let $B$ be Connes’ boundary map. Then the $k[[u]]$-linear map

\begin{equation}
(1.7) \quad \text{tr}_\nabla : C(D\mathcal{C}(X, w))[[u]], b + uB) \to (\mathcal{C}(\Omega, \Omega^*_{\mathcal{O}_X/\mathcal{O}_X})[[u]], d_{\text{Cech}} - dw + ud)
\end{equation}

is a quasi-isomorphism compatible with the HKR-type isomorphism.

Again when $|I| = 1$ (i.e., the affine case), this is a theorem of Brown and Walker [3], which generalizes a work of Segal [21].

1.4. **The equivariant case by a finite group.** As an application of Theorem 1.2 we can get an equivariant Chern character formula: Let $X$ come with a finite group $G$ action and assume that $w$ is $G$-fixed. Furthermore, assume that $X$ is quasi-projective. We consider the dg category $D\mathcal{C}_G(X, w)$ of $G$-equivariant matrix factorizations for $(X, w)$. We choose a finite open affine covering $\mathcal{U} = \{U_i\}_{i \in I}$ such that each $U_i$ is $G$-invariant. For a finite open affine covering of the $g$-fixed locus $X^g$, let us use $\{X^g \cap U_i\}_{i \in I}$. Let $\nabla_{P|U_i}$ be a $G$-equivariant connection on $P|_{U_i}$. Define a connection $\nabla_{P|X^g \cap U_i}$ on $P|_{X^g \cap U_i}$ to be the restriction of $\nabla_{P|U_i}$ to $X^g \cap U_i$ followed by the natural map

$$P|_{X^g \cap U_i} \otimes_{\mathcal{O}_{X^g \cap U_i}} \Omega^1_{U_i/k}|_{X^g \cap U_i} \to P|_{X^g \cap U_i} \otimes_{\mathcal{O}_{X^g \cap U_i}} \Omega^1_{X^g \cap U_i/k}.$$ 

**Theorem 1.3.** Let $w_g := w|_{X^g}$.

1. (Theorem 6.9) The mixed Hochschild complex $(C(D\mathcal{C}_G(X, w)), b, B)$ is naturally quasi-isomorphic to the coinvariant mixed complex

$$\left( \bigoplus_{g \in G} \Gamma(X^g, \mathcal{C}(\Omega^*_{X^g/\mathcal{O}_X})), d_{\text{Cech}} - dw_g, d \right)_G.$$
(2) (Formula (6.13)) For $P \in D^G_C(X, w)$, its $G$-equivariant Chern character $\text{ch}^G_{HH}(P)$ becomes

$$\text{ch}^G_{HH}(P) = \frac{1}{|G|} \bigoplus_{g \in G} \text{tr} \left( \varphi_g \exp \left( -[\prod_i \nabla_{P|X^g,i}, \delta_{P|X^g} + d_{\text{Cech}}] \right) \right),$$

where $\varphi_g \in \text{End}_{O_{X^g}}(P|X^g)$ is defined by the multiplication by $g$.

Specializing to the case where $X$ is affine space, this recovers the results [19, Theorem 2.5.4 & Theorem 3.3.3] of Polishchuk and Vaintrob. When $w = 0$ and $G = \mathbb{Z}$, Theorem 1.3 (1) is the main theorem of Baranovsky [2]. In fact we will establish Theorem 1.3 combining Theorem 1.2 with his works in [2]. Somewhere else we will treat the case of smooth separated DM stacks of finite type over $k$.

1.5. Outline of the paper. In §2 we collect definitions and fundamental facts of curved dg $k$-categories; and their modules, mixed Hochschild complexes, and negative cyclic complexes. In §3 we recall the definitions of categorical Chern characters and find their alternative expressions; see Proposition 3.3. In §4 we discuss the Čech model of the dg category of global matrix factorizations and introduce the sheafifications of mixed Hochschild complexes. We also collect various invariance results which we will use later. In §5 we construct connections on a Čech resolution of global matrix factorizations. Using those connections we define a cochain map $\text{tr}_\nabla$ and prove Theorem 1.2. In §6 we prove Theorems 1.1 and 1.3.

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1.7. Conventions. Unless otherwise stated, “graded” means $\mathbb{G}$-graded. For a (graded) sheaf $\mathcal{F}$ on a topological space and an open covering $\mathcal{U} = \{U_i\}_{i \in I}$, let $U_{i_0...i_p} = U_{i_0} \cap ... \cap U_{i_p}$ and let $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < ... < i_p} \mathcal{F}(U_{i_0...i_p})$. For a $k$-category $\mathcal{A}$, by $x \in \mathcal{A}$ we mean an object of $\mathcal{A}$. For $x, y \in \mathcal{A}$, by $\mathcal{A}(x, y)$, $\text{Hom}(x, y)$ or $\text{Hom}_A(x, y)$ we mean the Hom space from $x$ to $y$ in $\mathcal{A}$. For an element $a$ of a (bi-)graded $k$-space, the (total) degree of $a$ is denoted by $|a|$. The commutator $[\, , \, ]$ is the graded commutator. Often we write $1_P$ for $\text{id}_P$. The symbol $\simeq$ indicates a quasi-isomorphism.

2. Hochschild-type invariants

In this section we collect definitions of curved dg $k$-categories, their mixed Hochschild complexes, and their negative cyclic complexes. The main references of this section are [3, 20].

2.1. Curved dg $k$-categories.
2.1.1. Curved dg algebras. Let $k$ be a field of characteristic zero. A $k$-category is a $k$-algebra with several objects. A $k$-algebra is a $k$-module with a unital associative $k$-linear multiplication. Let $\mathcal{A}$ be a graded algebra over $k$ and let $h$ be a degree 2 element of $\mathcal{A}$. Let $d_\mathcal{A}$ be a degree 1, $k$-linear map such that (1) the Leibniz rule holds (i.e., $d_\mathcal{A}(a_1a_2) = d_\mathcal{A}(a_1)a_2 + (-1)^{|a_1|}a_1d_\mathcal{A}(a_2)$ for $a_1, a_2 \in \mathcal{A}$), (2) $d_\mathcal{A}^2 = [h, ?]$ (i.e., $d_\mathcal{A}^2(a) = ha - ah$ for $a \in \mathcal{A}$), and (3) $d_\mathcal{A}(h) = 0$. Then we call $(\mathcal{A}, d_\mathcal{A}, h)$ a curved dg (in short cdg) $k$-algebra. Furthermore, if $h = 0$, then we call $(\mathcal{A}, d_\mathcal{A})$ a dg $k$-algebra.

2.1.2. Curved dg categories. There is the notion of a cdg $k$-category $\mathcal{A}$ as a cdg $k$-algebra with several objects. It is a graded $k$-category with a differential $d_{x,y}$ of $\mathcal{A}(x, y)$ and a degree 2 element $h_x \in \mathcal{A}(x, x)$ for every $x, y \in \mathcal{A}$ satisfying the Leibniz rule, $d_{x,y}^2(f) = h_y \circ f - f \circ h_x$ for $f \in \mathcal{A}(x, y)$, and $d_{x,x}(h_x) = 0$. The element $h_x$ is called the curvature of $x$. If $h_x = 0$ for all $x \in \mathcal{A}$, we call $\mathcal{A}$ a dg $k$-category.

2.1.3. Precomplexes. There is a cdg category $\text{Com}_{cdg}(k)$, the category of precomplexes of $k$-modules; an object is a graded $k$-module $C$ with a $k$-linear degree 1 map $\delta_C$ and the Hom space from $C$ to $D$ has a differential $[\delta_C, ?]$. The curvature of $C$ is by definition $\delta_C$. The full subcategory $\text{Com}_{cdg}(k)$ of all objects with vanishing curvatures is called the dg category of $k$-complexes.

2.1.4. Cdg functors. A pair $(F, \alpha)$ is called a quasi-cdg functor from a cdg category $\mathcal{A}$ to another cdg category $\mathcal{B}$ if $F$ is a $k$-linear degree 0 homogeneous functor and $\alpha$ is an assignment of a degree 1 element $\alpha_x \in \mathcal{B}(Fx, Fx)$ for each $x \in \mathcal{A}$ such that

$$F(d_{x,y}(f)) = d_{F_x,F_y}(F(f)) + \alpha_y \circ F(f) - (-1)^{|f|}F(f) \circ \alpha_x$$

for all $x, y \in \mathcal{A}$ and $f \in \mathcal{A}(x, y)$. Furthermore if $F(h_x) = h_{Fx} + d_{Fx,Fx}(\alpha_x) + \alpha_x^2$, $(F, \alpha)$ is called a cdg functor. We often say simply $F$ is a (quasi-)cdg functor for a (quasi-)cdg functor $(F, \alpha)$. If $\alpha_x = 0$ for every $x$, we call $F$ strict. In particular if $\mathcal{A, B}$ are dg categories and $F$ is a strict cdg functor, $F$ is just called a dg functor. Informally a dg functor is a set of several “cochain” maps. If $(G, \beta) : \mathcal{B} \to \mathcal{C}$ is a quasi-cdg functor, then the composition of $(G, \beta)$ and $(F, \alpha)$ is defined as $(G \circ F, G \circ \alpha + \beta \circ F)$. Note that the composition is a quasi-cdg functor and the composition of cdg functors becomes a cdg functor.

2.1.5. The opposite category. The opposite category $\mathcal{A}^{op}$ of $\mathcal{A}$ is a cdg category whose objects are the objects of $\mathcal{A}$ and whose morphisms are $\mathcal{A}^{op}(x, y) := \mathcal{A}(y, x)$ with differential $d_{y,x}$. The composition $g \circ f$ in $\mathcal{A}^{op}$ is $(-1)^{|f||g|}f \circ g$ in $\mathcal{A}$ and the curvature of $x$ in $\mathcal{A}^{op}$ is $-h_x$.

2.1.6. Left and right (quasi-)cdg modules. A left (resp., right) quasi-cdg module over a cdg category $\mathcal{A}$ is a strict quasi-cdg functor $F : \mathcal{A}$ (resp., $\mathcal{A}^{op}$) $\to \text{Com}_{cdg}(k)$. A left (resp., right) cdg module over $\mathcal{A}$ is a strict cdg functor $F : \mathcal{A}$ (resp., $\mathcal{A}^{op}$) $\to \text{Com}_{cdg}(k)$. A left (resp., right) cdg module over $\mathcal{A}$ is a strict cdg functor $F : \mathcal{A}$ (resp., $\mathcal{A}^{op}$) $\to \text{Com}_{cdg}(k)$. A left (resp., right) cdg module over $\mathcal{A}$ is a strict cdg functor $F : \mathcal{A}$ (resp., $\mathcal{A}^{op}$) $\to \text{Com}_{cdg}(k)$.
\( \mathcal{A} \) (resp., \( \mathcal{A}^{op} \)) \to \text{Com}_{cdg}(k). \) If \( \mathcal{A} \) is a dg category and \( F : \mathcal{A} \) (resp., \( \mathcal{A}^{op} \)) \to \text{Com}_{cdg}(k) \) is a dg functor, we call \( F \) a left (resp., right) dg module over \( \mathcal{A} \).

The category \( q\text{Mod}_{cdg}(\mathcal{A}) \) of right quasi-dg modules over \( \mathcal{A} \) has a natural cdg structure: The Hom space \( \text{Hom}(F, G) \) of two quasi-dg modules \( F, G : \mathcal{A}^{op} \to \text{Com}_{cdg}(k) \) is the \( \mathbb{G} \)-graded \( k \)-space of \( k \)-linear homogenous natural transformations \( n : F \Rightarrow G \). The differential \( \delta \) of Hom space is given by \( (\delta n)(x) := [d_F(x), G(x), n(x)] \) and the curvature \( h_F \in \text{Hom}(F, F) \) of \( F \) is given by \( h_F(x) = h_F(x) - F(h_x) \). The full subcategory \( \text{Mod}_{cdg}(\mathcal{A}) \) of \( q\text{Mod}_{cdg}(\mathcal{A}) \) consisting of right cdg modules over \( \mathcal{A} \) has a natural dg structure. If \( \mathcal{A} \) is a dg category, we write \( \text{Mod}_{dg}(\mathcal{A}) \) for \( \text{Mod}_{cdg}(\mathcal{A}) \).

### 2.1.7. Matrix factorizations.

By unwinding the definition, a right quasi-dg module \( M \) over \( (A, d_A, -h) \) is a right graded \( A \)-module \( M \) with a degree 1, \( k \)-linear map \( \delta : M \to M \) such that \( \delta(ma) = \delta(m)a + (-1)^{|m|}d_A(a) \) for every \( m \in M, a \in A \). Its curvature is defined to be \( \delta^2 + \rho_{-h} \in \text{End}_A(M) \) where \( \rho_{-h} \) is the right multiplication by \( -h \). (1) For example, \((A, d_A)\) can be regarded as a right quasi-dg module over \((A, d_A, -h)\) with curvature \( \lambda_{-h} \). (2) Exactly when the curvature \( \delta^2 + \rho_{-h} \) of \( M \) is zero, \( M \) is called a right cdg-module over \((A, d_A, -h)\). Furthermore we call the right cdg module \((M, \delta)\) over \((A, 0, -h)\) a matrix factorization for \((A, h)\) or for \((\text{Spec} A, h)\) when \( A \) is a commutative \( k \)-algebra concentrated in degree 0, and \( M \) is a finitely generated projective \( A \)-module. The full subcategory of \( \text{Mod}_{cdg}(\mathcal{A}) \) consisting of matrix factorizations will be denoted by \( D_{dg}(A, h) \) (and also by \( D_{cdg}(\text{Spec} A, h) \)).

### 2.1.8. Quasi-Yoneda.

There is a quasi-Yoneda embedding \( \mathcal{A} \to q\text{Mod}_{cdg}(\mathcal{A}), \ x \mapsto \mathcal{A}(\ , x) \) generalizing the usual Yoneda embedding in the case of dg categories. We define the full cdg-subcategory \( q\text{Perf}(\mathcal{A}) \) of \( q\text{Mod}_{cdg}(\mathcal{A}) \) as the smallest cdg-subcategory containing \( \mathcal{A} \) and closed under finite operations of finite direct sum, shift, twist, and passage to a direct summand; see [3, 20] for details. We call an object of \( q\text{Perf}(\mathcal{A}) \) a perfect right quasi-\( \mathcal{A} \)-module. If its curvature vanishes, then call it a perfect right \( \mathcal{A} \)-module. We define \( \text{Perf}(\mathcal{A}) \) as the full dg-subcategory of \( \text{Mod}_{cdg}(\mathcal{A}) \) consisting of all perfect right \( \mathcal{A} \)-modules.

### 2.2. Mixed Hochschild complexes.

#### 2.2.1. The category of mixed complexes.

Consider a \( \mathbb{G} \)-graded dg algebra \( k[B] \) defined as a 2-dimensional graded \( k \)-algebra generated by \( B \) with relation \( B^2 = 0 \), degree \( |B| = -1 \), and trivial differential. A dg \( k[B] \)-module is called a mixed complex. For example, for a cdg category \( \mathcal{A} \) we will have a mixed complex \( \text{MC}(\mathcal{A}) \) as defined in [22] below. A morphism \( \phi : M \to M' \) between mixed complexes is a dg \( k[B] \)-module homomorphism, i.e., a \( k \)-linear map preserving degree and both differentials. This defines the category \( \text{Com}(k[B]) \) of mixed complexes. We call \( \phi \) a quasi-isomorphism if it is so as a cochain map \((M, b) \to (M', b') \).
Consider Connes’ boundary map.

2.2.3. Hochschild homology. The homology with differential \( b \) by \( 1.2 \). For any \( \text{cdg} k \)-category \( \mathcal{A} \), we may consider the Hochschild complex \( C(\mathcal{A}) \) of \( \mathcal{A} \). Let \( \mathcal{A}(x,y)[1] \) denote the degree shifted \( \mathcal{A}(x,y) \) by 1. The canonical degree \(-1\) map \( \mathcal{A}(x,y) \to \mathcal{A}(x,y)[1] \) is denoted by \( s \) so that \( |sa| = |a| - 1 \). Let

\[
C(\mathcal{A}) = \bigoplus_{x \in \mathcal{A}} \mathcal{A}(x,x) \oplus \bigoplus_{n \geq 1} \bigoplus_{(x_0, \ldots, x_n) \in \mathcal{A}^{n+1}} \mathcal{A}(x_1, x_0) \otimes_k \mathcal{A}(x_2, x_1)[1] \otimes_k \ldots \otimes_k \mathcal{A}(x_0, x_n)[1],
\]

with differential \( b := b_2 + b_1 + b_0 \) defined as follows. Denote

\[
(1) \quad a_0[a_1| \ldots |a_n] := a_0 \otimes a_1 \otimes \ldots \otimes a_n \in \mathcal{A}(x_1, x_0) \otimes \mathcal{A}(x_2, x_1)[1] \otimes \ldots \otimes \mathcal{A}(x_0, x_n)[1].
\]

Then define

\[
b_2(a_0[a_1| \ldots |a_n]) := (-1)^{|a_0|}a_0a_1[a_2| \ldots |a_n] + (-1)^{|a_0|+|a_1|-1}a_0[a_1a_2[a_3| \ldots |a_n]
\]
\[+ \cdots + (-1)^{\sum_{i=0}^{n-1}|a_i|-(n-1)}a_0[a_1| \ldots |a_{n-1}a_n]
\]
\[+ (-1)^{|a_0|-1}(\sum_{i=0}^{n} |a_i|-(n-1))a_0a_1[a_2| \ldots |a_{n-1}a_n];
\]

\[
b_1(a_0[a_1| \ldots |a_n]) := d(a_0)[a_1| \ldots |a_n] + (-1)^{|a_0|-1}a_0[d(a_1)a_2| \ldots |a_n]
\]
\[+ \cdots + (-1)^{\sum_{i=0}^{n-1}|a_i|-n}a_0[a_1| \ldots |a_{n-1}d(a_n)];
\]

\[
b_0(a_0[a_1| \ldots |a_n]) := (-1)^{|a_0|}a_0[h|a_1| \ldots |a_n] + \cdots + (-1)^{\sum_{i=0}^{n}|a_i|-n}a_0[a_1| \ldots |a_n|h].
\]

The homology

\[
HH_\ast(\mathcal{A}) = H^{-\ast}(C(\mathcal{A}), b)
\]

is called the Hochschild homology of \( \mathcal{A} \).

When \( \mathcal{A} \) is a curved dg algebra \( (A, d, h) \), we write \( C(A, d, h) \) for \( C(\mathcal{A}) \).

2.2.3. Connes’ boundary map. On the graded \( k \)-module \( C(\mathcal{A}) \), there is another boundary map, the Connes boundary map, \( B = (1 - t^{-1})sN \), where

\[
t(a_0[a_1| \ldots |a_n]) := (-1)^{|a_0|-1}(\sum_{i=0}^{n}|a_i|-1)a_1[a_2| \ldots |a_n]a_0;
\]

\[
s(a_0[a_1| \ldots |a_n]) := I[a_0[a_1| \ldots |a_n];
\]

\[
N(a_0[a_1| \ldots |a_n]) := \sum_{i=0}^{n} t^i(a_0[a_1| \ldots |a_n]).
\]

Consider

\[
(2) \quad \operatorname{MC}(\mathcal{A}) := (C(\mathcal{A}), b, B).
\]
Since $bb + Bb = 0$, $MC(A)$ is a mixed complex, called the mixed Hochschild complex of $A$.

It is known that $HH^*_A(A)$ vanishes if the cdg category $A$ contains an object with a nonzero curvature; see [5, 20]. This motivates the following.

2.2.4. Hochschild complexes of the second kind. For a cdg category $A$ one can take the underlying graded $k$-module to be

$$C^{II}(A) = \bigoplus_{x \in A} A(x, x) \oplus \prod_{n \geq 1} \left( \bigoplus_{(x_0, ..., x_n) \in A^{\otimes n+1}} A(x_1, x_0) \otimes_k A(x_2, x_1)[1] \otimes_k ... \otimes_k A(x_0, x_n)[1] \right),$$

which has the corresponding differentials $b_i$, $i = 0, 1, 2$. This complex is called the Hochschild complex of the second kind.

2.2.5. Normalized Hochschild complexes. Consider a subcomplex $D$ of $C(A)$ generated by elements $a_0[a_1|...|a_n]$ for which $a_i = t \cdot \id_x$ for $x \in A$, $t \in k$ and some $i \geq 1$. Define the normalized Hochschild complex

$$\overline{C}(A) = (C(A), b)/D =: \bigoplus_{n \geq 0} \overline{C}_n(A),$$

where $n$ denotes the tensor degree as in (2.1). The Connes operator descends to an operator on $\overline{C}(A)$, which will be also denoted by $\overline{B}$ by the abuse of notation. We let $\overline{MC}(A) := (\overline{C}(A), b, \overline{B})$.

Proposition 2.2. ([16 §5.3] and [9 §3].) For a dg category $A$ the natural map

$$MC(A) \xrightarrow{\sim} \overline{MC}(A)$$

is a quasi-isomorphism.

Likewise, there is the normalized Hochschild complex $\overline{C}^{II}(A)$ of the second kind. There are corresponding Connes' operators $\overline{B}$ and mixed complexes $\overline{MC}^{II}(A) := (\overline{C}^{II}(A), b, \overline{B}), \overline{MC}^{II}(A) := (\overline{C}^{II}(A), b, \overline{B})$.

Proposition 2.3. ([3 Proposition 3.15]) For a cdg category $A$ the quotient map

$$\text{quot}^{II} : \overline{MC}^{II}(A) \xrightarrow{\sim} \overline{MC}(A)$$

is a quasi-isomorphism.

2.2.6. Negative cyclic complexes. Let $u$ be a formal variable with degree 2. Then we can make a total complex

$$(C(A)[[u]] := C(A) \otimes_k k[[u]], b + uB)$$

It is called the negative cyclic complex of $A$ and its homology $HN_u(A) = H^{-*}(C(A)[[u]], b + uB)$
is called the *negative cyclic homology* of \( \mathcal{A} \). Similarly, one may define its variants

\[
\overline{HN}_* (\mathcal{A}), HN^\text{II}_* (\mathcal{A}), \overline{HN}^\text{II}_* (\mathcal{A}).
\]

### 2.2.7. Functoriality.

Let \( \text{cdg-cat}_k \) denote the category of small cdg \( k \)-categories. The morphisms between them are cdg functors. We define a functor

\[
\text{cdg-cat}_k \xrightarrow{\text{MC}^\text{II}} \text{Com}_k; \mathcal{A}, (F, \alpha) \mapsto (\overline{C}^\text{II}(\mathcal{A}), b, B), (F, \alpha)_*
\]

by letting

\[
(F, \alpha)_*(a_0[a_1|...|a_n]) := \sum_{n=0}^{\infty} \sum_{(j_0, ..., j_n) \in \mathbb{Z}_{\geq 0}^{n+1}} (-1)^{j_0 + ... + j_n} F(a_0)[a_0|...|a_j] F(a_1)[a_j|...|a_{j_1}] F(a_2)[a_{j_1}|...|a_{j_2}] \cdots F(a_n)[a_{j_n}|...|a_n].
\]

Here \( \alpha_i := \alpha_{\text{domain of } F(a_i)} \). This is indeed a functor; see [3, 20].

**Remark 2.4.** For a cdg functor \((F, \alpha)\), it induces a cochain map \((C^\text{II}(\mathcal{A}), b) \rightarrow (\overline{C}^\text{II}(\mathcal{A}), b')\) but not necessarily a morphism of mixed complexes \(\text{MC}^\text{II}(\mathcal{A}) \rightarrow \text{MC}^\text{II}(\mathcal{A}')\); see [3, Remark 3.21].

Let \( \text{cdg-cat}_{k}^\text{st} \) denote the category of small cdg \( k \)-categories whose morphisms are *strict* cdg functors. Let \( \text{dg-cat}_k \) be the full subcategory of \( \text{cdg-cat}_{k}^\text{st} \) whose objects are dg categories. In the same way we have natural functors

\[
\text{cdg-cat}_{k}^\text{st} \xrightarrow{\text{MC}} \text{Com}_k; \mathcal{A}, F \mapsto (\overline{C}(\mathcal{A}), b, B), F_*
\]

\[
\text{dg-cat}_k \xrightarrow{\text{MC}} \text{Com}_k; \mathcal{A}, F \mapsto (C(\mathcal{A}), b, B), F_*
\]

where \( F_* := (F, 0)_* \).

### 2.3. Some invariances.

#### 2.3.1. Morita invariances.

Let \( F : \mathcal{A} \rightarrow \mathcal{B} \) be a dg functor between dg categories \( \mathcal{A}, \mathcal{B} \). We say that it is *cohomologically full and faithful* if for every \( x, y \in \mathcal{A} \), the cochain map \( F : \mathcal{A}(x, y) \rightarrow \mathcal{B}(Fx, Fy) \) is a quasi-isomorphism. Furthermore, if \( \mathcal{B} \) is split-generated by the image of \( F \), then \( F \) induces a Morita equivalence between \( \mathcal{A} \) and \( \mathcal{B} \). Thus the morphism

\[
F_* : \text{MC}(\mathcal{A}) \xrightarrow{\sim} \text{MC}(\mathcal{B})
\]

is a quasi-isomorphism; see [23, Lemma 4.12] and references therein. The Yoneda embedding \( \mathcal{A} \rightarrow \text{Perf}\mathcal{A} \) is such an example. Furthermore if \( F \) is essentially surjective, \( F \) is called a *quasi-equivalence*. 

2.3.2. Invariance under Pseudo-equivalences. Let \((F, \alpha) : \mathcal{A} \to \mathcal{B}\) be a cdg functor between cdg categories \(\mathcal{A}, \mathcal{B}\). We say that \((F, \alpha)\) is a pseudo-equivalence if \(\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)\) is an isomorphism for all \(x, y \in \mathcal{A}\) and every object of \(\mathcal{B}\) can be constructed from the image of \(F\) by a finite sequence of operations of finite direct sum, passage to a direct summand, twist, and shift.

**Theorem 2.5.** (Polishchuk-Positselski [20]) For a pseudo-equivalence \((F, \alpha)\)
\[
(F, \alpha)_* : \overline{MC}^{II}(\mathcal{A}) \xrightarrow{\sim} \overline{MC}^{II}(\mathcal{B})
\]
is a quasi-isomorphism.

The embeddings \(\mathcal{A} \to q\text{Perf}\mathcal{A}\) and \(\text{Perf}\mathcal{A} \to q\text{Perf}\mathcal{A}\) are pseudo-equivalences; see [20].

2.3.3. **Comparison.** Let \(X\) be an affine \(k\)-scheme and let \(w : X \to \mathbb{A}^1_k\) be a regular function. Then we consider the dg category \(D_{dg}(X, w)\) of matrix factorizations for \((X, w)\); see §2.1.7.

**Proposition 2.6.** ([20] §4.8 Corollary A) If \(X\) is a smooth affine scheme finite type over \(k\) and \(w\) has no other critical values but zero, then the natural map
\[
C(D_{dg}(X, w)) \xrightarrow{\sim} C^{II}(D_{dg}(X, w))
\]
is a quasi-isomorphism.

**Proof.** Without loss of generality we may assume that \(X\) is connected. When \(w \neq 0\), this follows from [20 Corollary A], since \(k\) is a perfect field. When \(w = 0\), this follows from considering the tensor grading of \(C^{II}(\Gamma(X, \mathcal{O}_X))\) and the HKR isomorphism of [10]. \(\square\)

3. **Categorical Chern characters**

In this section we recall the definition of categorical Chern characters and discuss alternative expressions of them which we will use in §6.3 for Chern character formulae.

3.1. **Chern characters** \(\text{Ch}_{HH}(P)\) and \(\text{Ch}_{HN}(P)\). For every object \(P\) of a dg category \(\mathcal{A}\), note that the identity map \(1_P\) is a degree 0 cocycle of the Hochschild complex of \(\mathcal{A}\). Hence each \(P\) determines the class \([1_P]\) in \(HH_0(\mathcal{A})\). We call \([1_P]\) the Hochschild homology valued Chern character of \(P\), denoted by \(\text{Ch}_{HH}(P)\). Likewise, since \(1_P\) is a cocycle of the normalized negative cyclic complex of \(\mathcal{A}\), we simply define the negative cyclic homology valued Chern character \(\text{Ch}_{HN}(P)\) of \(P\) to be the class \([1_P]\) in \(HN_0(\mathcal{A}) \cong HH_0(\mathcal{A})\).

For a cdg category \(\mathcal{A}\) and \(P \in q\text{Perf}\mathcal{A}\) we define Chern characters \(\text{Ch}_{HH}^{II}(P)\), \(\text{Ch}_{HN}^{II}(P)\) of the second kind in the same manner using \(1_P\) in the normalized complexes of the second kind.
3.2. Alternative expression via direct summands. Let $P$ and $N$ be objects of a dg category $\mathcal{A}$. Suppose that $P$ is a direct summand of $N$, i.e., there are $g : P \to N$ and $f : N \to P$ degree 0 closed homomorphisms such that $f \circ g = 1_P$. Note that $1_P$ and the idempotent $\pi := g \circ f$ are homologous (3.1)

$$1_P \sim \pi$$

in the Hochschild complex of $\mathcal{A}$; see [24]. If $N$ is simpler than $P$, then $\pi$ is often easier to handle than $1_P$.

3.2.1. More generally, every element of $C(\operatorname{End}_\mathcal{A} P)$ is homologous to an element of $C(\operatorname{End}_\mathcal{A} N)$ in the Hochschild complex of $\mathcal{A}$: Consider the inclusion functor $\operatorname{inc}$ from $\{N\}$ to $\{P, N\}$ between the full dg subcategories of $\mathcal{A}$ consisting only one indicated object, two indicated objects respectively. As a semifunctor (i.e., a dg functor not necessarily preserving identities between dg categories), $\operatorname{inc}$ has a left inverse $F$.

$$\{N\} \xleftarrow{F} \{P, N\} \xrightarrow{\operatorname{inc}} \{N\}.$$ The functor $F : \{P, N\} \to \{N\}$ is determined by sending $\alpha \mapsto g \circ \alpha \circ f$ for $\alpha \in \operatorname{End} P$, $\alpha' \mapsto g \circ \alpha'$ for $\alpha' \in \operatorname{Hom}(N, P)$, $\alpha'' \mapsto \alpha'' \circ f$ for $\alpha'' \in \operatorname{Hom}(P, N)$, and $\beta \mapsto \beta$ for $\beta \in \operatorname{End}(N)$.

**Lemma 3.1.** In the Hochschild complex $(C\{P, N\}, b)$, every element $\alpha_0[\alpha_1|\ldots|\alpha_n]$ is homologous to $F(\alpha_0)[F(\alpha_1)|\ldots|F(\alpha_n)]$.

**Proof.** The semifunctor $F$ induces a cochain map $F_* : (C\{P, N\}, b) \to (C\{N\}, b)$, which is a left inverse of the quasi-isomorphism $\operatorname{inc}_* : (C\{N\}, b) \to (C\{P, N\}, b)$ induced from $\operatorname{inc}$. Therefore for every $m \in C\{P, N\}$, $m$ is homologous to $\operatorname{inc}_*(F_*(m))$, since $F_*(m) = F_*(\operatorname{inc}_*(F_*(m)))$. □

**Remark 3.2.** We can generalize Lemma 3.1. Consider two full dg subcategories $\mathcal{C}$ and $\mathcal{D}$ of Mod$_{dg}\mathcal{A}$ such that $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and any object of $\mathcal{D}$ is a direct summand of an object of $\mathcal{C}$. Then we can construct a semifunctor from $\mathcal{D} \to \mathcal{C}$ which is a left inverse of the inclusion $\mathcal{C} \to \mathcal{D}$ and hence a left inverse of the quasi-isomorphism $(C\{\mathcal{C}\}, b) \to (C\{\mathcal{D}\}, b)$.

3.2.2. There is a negative cyclic homology version of (3.1): Consider a cycle

$$\gamma_P := 1_P + \sum_{i=1}^{\infty} (-1)^i \frac{(2i)!}{2(2i)} \cdot 1_P[1_P|\ldots|1_P]u^i$$

in $(C\{P\}[[u]], b + uB)$. It projects to $1_P$ in $(\overline{C}\{P\}[[u]], b + uB)$. Let

$$\eta_\pi := \pi + \sum_{i=1}^{\infty} (-1)^i \frac{(2i)!}{2(2i)} (2\pi - 1_\mathcal{N})[\pi|\ldots|\pi]u^i$$

$$= \pi - (2\pi - 1_\mathcal{N})[\pi]\pi u + 6(2\pi - 1_\mathcal{N})[\pi]\pi[\pi]\pi u^2 + \cdots .$$

It is straightforward to check that $\eta_\pi$ is also a cycle in $(\overline{C}\{N\}[[u]], b + uB)$. 

Proposition 3.3. (Proposition A.1) The two cycles $1_P$ and $\eta_\pi$ are homologous in $(\mathbb{C}[P,N][u], b + uB)$.

Proposition 3.3 might be known to the experts. Due to lack of a suitable reference, we will give a proof in Appendix A. Another proof will be given in [6].

Remark 3.4. A combination of Proposition 3.3 and [3, Theorem 5.7] immediately provides an answer to a question in [3, Remark 5.23].

4. Global matrix factorizations

Let $X$ be a smooth separated scheme of finite type over $k$. In this section we introduce the notion of global matrix factorizations as certain cdg modules over the sheaf $(\mathcal{O}_X, 0, -w)$ of cdg algebras. And we define the Čech model for a dg enhancement of the derived category of global matrix factorizations. We introduce the sheafifications of mixed Hochschild complexes that are defined in §2. All of these will be used in §5.

4.1. Matrix factorizations and injective models. We may sheafify the notion of cdg algebras and (quasi-)cdg modules. For example, we have

$$\mathcal{A}_w := (\mathcal{O}_X, 0, -w)$$

a sheaf of cdg $k$-algebras: $U \mapsto (\mathcal{O}_X(U), 0, -w|_U)$ for open subsets $U$ of $X$. A global matrix factorization, or simply matrix factorization for $(X, w)$ amounts to a cdg $\mathcal{A}_w$-module $(P, \delta_P)$ such that $P$ is a locally free and coherent as an $\mathcal{O}_X$-module. There is a notion of the derived category of matrix factorizations for $(X, w)$; see [17]. It is equivalent to the homotopy category of $D_{dg}(X, w)$ if we take $D_{dg}(X, w)$ to be the dg category of matrix factorizations whose objects are matrix factorizations and whose Hom from $P$ to $Q$ is defined to be $\text{Hom}_{\mathcal{O}_X}(I_P, I_Q)$ using quasi-coherent curved injective replacements $I_P, I_Q$ of $P, Q$, respectively.

4.2. Čech models. We will consider another model using Čech resolutions of matrix factorizations instead of injective resolutions. We fix a finite open affine covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of $X$. Fix a total ordering of $I$. For an $\mathcal{O}_X$-sheaf $\mathcal{F}$, let $\check{\mathcal{C}}(\mathcal{F})$ the sheafified version of the (ordered) Čech complex of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$:

$$\check{\mathcal{C}}^p(\mathcal{F}) := \prod_{i_0 < \ldots < i_p} f_*(\mathcal{F}|_{U_{i_0 \ldots i_p}}) = \mathcal{F} \otimes_{\mathcal{O}_X} \prod_{i_0 < \ldots < i_p} f_*\mathcal{O}_{U_{i_0 \ldots i_p}}$$

where $f$ denotes the immersions $U_{i_0 \ldots i_p} := U_{i_0} \cap \ldots \cap U_{i_p} \rightarrow X$. Here the second equality follows from the projection formula. The Čech differential $1F \otimes d_{\check{\mathcal{C}}}$ for

$$\check{\mathcal{C}}(\mathcal{F}) = \mathcal{F} \otimes \check{\mathcal{C}}(\mathcal{O}_X)$$

will be written simply $d_{\check{\mathcal{C}}}$ by abuse of notation. We follow the Koszul sign rule so that $d_{\check{\mathcal{C}}}(x \otimes a) = (-1)^{|x|} x \otimes d_{\check{\mathcal{C}}}(a)$ for $x \otimes a \in \mathcal{F} \otimes \check{\mathcal{C}}(\mathcal{O}_X)$. 

4.2.1. **Alexander-Čech-Whitney products.** The Alexander-Čech-Whitney product (see [26]) is an $\mathcal{O}_X$-homomorphism
\[
\cdot : \check{\mathcal{C}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \check{\mathcal{C}}(\mathcal{O}_X) \rightarrow \check{\mathcal{C}}(\mathcal{O}_X)
\]
defined by setting
\[
(a \cdot b)_{i_0, \ldots, i_{p+q}} := a_{i_0, \ldots, i_p} | u_{i_0, \ldots, i_{p+q}} b_{i_p, \ldots, i_{p+q}} | u_{i_0, \ldots, i_{p+q}}
\]
for $a \in \check{\mathcal{C}}^p(\mathcal{O}_X), b \in \check{\mathcal{C}}^q(\mathcal{O}_X)$.

If $\mathcal{F}$ is a sheaf of graded $\mathcal{O}_X$-modules, we have the right $\check{\mathcal{C}}(\mathcal{O}_X)$-module structure on the Čech resolution $\check{\mathcal{C}}(\mathcal{F}) = \mathcal{F} \otimes \check{\mathcal{C}}(\mathcal{O}_X)$. If $\mathcal{F}$ is a sheaf of graded $\mathcal{O}_X$-algebras, we define the product structure on $\check{\mathcal{C}}(\mathcal{F}) = \mathcal{F} \otimes \check{\mathcal{C}}(\mathcal{O}_X)$ by the formula
\[
(x \otimes a) \cdot (y \otimes b) := (-1)^{|y|} x y \otimes a \cdot b
\]
for $x, y \in \mathcal{F}$, $a \in \check{\mathcal{C}}^p(\mathcal{O}_X), b \in \check{\mathcal{C}}^q(\mathcal{O}_X)$, and the degree $|y|$ of $y$.

Note that $d_{\text{Čech}}(\alpha \cdot \beta) = (d_{\text{Čech}}\alpha) \cdot \beta + (\alpha \cdot (d_{\text{Čech}}\beta))$ for $\alpha, \beta \in \check{\mathcal{C}}(\mathcal{F})$. Therefore
\[
\mathcal{A}_{\check{\mathcal{C}}, w} := (\check{\mathcal{C}}(\mathcal{O}_X), d_{\text{Čech}}, -w)
\]
is a sheaf of cdg $k$-algebras on $X$ as $[-w, -1] = d_{\text{Čech}}^2$.

4.2.2. **The category $\mathcal{D}_{\check{\mathcal{C}}}(X, w)$.** For each $P \in \mathcal{D}_{dg}(X, w)$, consider $\check{\mathcal{C}}(P)$. It comes with the total curved differential $\delta_P \otimes 1 + 1 \otimes d_{\text{Čech}}$ by the identification $\check{\mathcal{C}}(P) = P \otimes_{\mathcal{O}_X} \check{\mathcal{C}}(\mathcal{O}_X)$. Abusing notation we simply write
\[
\delta_P + d_{\text{Čech}} \quad \text{for} \quad \delta_P \otimes 1 + 1 \otimes d_{\text{Čech}},
\]
but we need to remember the Koszul sign rule when we apply $d_{\text{Čech}}$. Hence $\check{\mathcal{C}}(P)$ can be regarded as a certain cdg-module over $\mathcal{A}_{\check{\mathcal{C}}, w}$. In the dg category $\text{Mod}_{dg}^\mathbb{C}(\mathcal{A}_{\check{\mathcal{C}}, w})$ of right cdg-modules over $\mathcal{A}_{\check{\mathcal{C}}, w}$, let us take all the objects of form $(\check{\mathcal{C}}(P), \delta_P + d_{\text{Čech}})$, for some $P \in \mathcal{D}_{dg}(X, w)$. This yields the full subcategory
\[
\mathcal{D}_{\check{\mathcal{C}}}(X, w),
\]
which is easily seen to be quasi-equivalent to $\mathcal{D}_{dg}(X, w)$. We call $\mathcal{D}_{\check{\mathcal{C}}}(X, w)$ the Čech model for $\mathcal{D}_{dg}(X, w)$ with respect to $\mathfrak{U}$. We may regard the objects of $\mathcal{D}_{dg}(X, w)$ as the objects of $\mathcal{D}_{\check{\mathcal{C}}}(X, w)$.

4.2.3. **The category $q\mathcal{D}_{\check{\mathcal{C}}}(X, w)$.** We will also consider a full subcategory $q\mathcal{D}_{\check{\mathcal{C}}}(X, w)$ of $\mathcal{D}_{\check{\mathcal{C}}}(X, w)$ consisting of $(\check{\mathcal{C}}(P), \delta_P + d_{\text{Čech}})$ such that $(P, \delta_P)$ is a locally free coherent right quasi-cdg module over $\mathcal{A}_w$. Here a locally free coherent right quasi-cdg module $(P, \delta_P)$ means that $P$ is a $\mathbb{G}$-graded locally free coherent $\mathcal{O}_X$-module and $\delta_P$ is an $\mathcal{O}_X$-linear, degree 1 map. It is not required that $\delta_P^2 = -\rho_- w$. The curvature element of $(P, \delta_P)$ is defined to be $\delta_P^2 + \rho_- w$.

4.3. **Sheafification.**
4.3.1. Let \( \mathcal{P} \) be a presheaf of cdg categories on \( X \):

\[
U \mapsto \mathcal{P}(U) \in \text{cdg}_k
\]

for each open subset \( U \) of \( X \). For example, \( \mathcal{A}_w \) and \( \mathcal{A}_{\check{\mathcal{C}}_w} \) are sheaves of cdg \( k \)-algebras on \( X \), that is \( U \mapsto (\mathcal{O}(U), 0, -w|_U) \) and \( U \mapsto (\check{\mathcal{C}}(\mathcal{O}_X)(U), d_{\check{\mathcal{C}}\text{ech}}, -w|_U) \), respectively. Also, we have a presheaf

\[
U \mapsto D_{\check{\mathcal{C}}}(U, w|_U)
\]

(resp., \( qD_{\check{\mathcal{C}}}(U, w|_U) \)) of dg (resp., cdg) categories, which will be denoted by

\[
\check{\mathcal{C}}_{\mathcal{P}}(X, w|_U)
\]

(resp., \( q\check{\mathcal{C}}_{\mathcal{P}}(X, w|_U) \)). Here for each \( U \) we use the induced covering \( \{U_i \cap U\}_{i \in I} \) of \( U \) to construct \( \check{\mathcal{C}}(Q) \) for \( Q \in (q)D_{dg}(U, w|_U) \). Note that for \( P \in (q)D_{dg}(X, w) \), \( \check{\mathcal{C}}(P|_U) \cong \check{\mathcal{C}}(P|_U) \).

4.3.2. We may sheafify the mixed Hochschild complex of \( \mathcal{P}(U) \): the sheaf

\[
C_{\mathcal{P}}(U)
\]

associated to the presheaf

\[
U \mapsto C(\mathcal{P}(U)).
\]

Let \( \underline{MC}(\mathcal{P}) \) denote this complex of sheaves, i.e., \( C(\mathcal{P}) \) with the induced differential and Connes’ operator, denoted by \( b, B \) by abuse of notation. Similarly we have the complex \( \underline{MC}^{II}(\mathcal{P}) \) of sheaves associated to \( U \mapsto \check{\mathcal{C}}^{II}(\mathcal{P}(U)) \). For example, a sheafification of complexes \( U \mapsto C(D_{\check{\mathcal{C}}}(U, w|_U)) \)

is denoted by \( \check{\mathcal{C}}_{\mathcal{P}}(X, w|_U) \).

4.3.3. Let \( G \) denote the Godement resolution functor so that for a sheaf \( \mathcal{F} \) of abelian groups, the canonical flasque resolution of \( \mathcal{F} \) is denoted by \( G(\mathcal{F}) \).

Let \( G_{\leq d} \) denote the Godement resolution functor canonically truncated at the amplitude \( d \) with \( d := \dim X \). Since \( H^i(X, \mathcal{F}) = 0 \) for all \( i > d \) by a vanishing theorem of Grothendieck ([8, III. Theorem 2.7]), \( G_{\leq d}(\mathcal{F}) \) is a \( \Gamma \)-acyclic resolution of \( \mathcal{F} \). For a sheaf \( (\underline{\mathcal{C}}, b, B) \) of mixed (unbounded) complexes on \( X \) we define a mixed complex

\[
R\Gamma(\underline{\mathcal{C}}, b, B) := (\Gamma(X, G_{\leq d}\underline{\mathcal{C}}), G_{\leq d}b, G_{\leq d}B)
\]

following [7, Definition 3.22]. The functor \( R\Gamma \) preserves quasi-isomorphisms.

**Condition (\( \ast \)):** \( X \) is a smooth separated scheme finite type over \( k \) and \( w \) has no other critical values but zero.

**Proposition 4.1.** ([7 Proposition 5.1 & Proposition 3.24])

1. Assume (\( \ast \)) and let \( C' \) denote either \( C, \check{\mathcal{C}}, C^{II} \) or \( \check{\mathcal{C}}^{II} \). The natural morphism

\[
(C'(D_{\check{\mathcal{C}}}(X, w)), b, B) \to R\Gamma(\underline{C'}, b, B)
\]

is a quasi-isomorphism between the mixed complexes.

2. For a sheaf \( \mathcal{F} \) of \( k \)-vector spaces the natural morphism

\[
G_{\leq d}(\mathcal{F} \otimes k[[u]]) \to (G_{\leq d}\mathcal{F}) \otimes k[[u]]
\]

is a quasi-isomorphism between the complexes of \( \Gamma \)-acyclic sheaves.
Proof. (1) The proof of Proposition 5.1 of Efimov [7] works for the various $C'$. (2) This is the contents of [7, Proposition 3.24 & Lemma 3.25].

Combining Proposition 2.6 and Proposition 4.1 (1) we may remove the affine condition on $X$ in Proposition 2.6.

**Corollary 4.2.** Under the condition (•) the natural map

\[ C(D_C(X,w)) \xrightarrow{\sim} C^{II}(D_C(X,w)) \]

is a quasi-isomorphism.

From now on we will assume the condition (•) unless otherwise stated.

4.4. **Applications of invariances.** Note that for each open affine subset $U$ of $X$, we have the following.

1. The HKR map

\[
I_{HKR} : \mathcal{MC}^{II}(\mathcal{A}_w(U)) \to (\Omega^\bullet_{U/k}, d, -dw); \\
a_0[a_1|...|a_n] \mapsto \frac{1}{n!}a_0da_1...da_n
\]

is a quasi-isomorphism of mixed complexes; see [5, 21].

2. The natural morphism

\[
\mathcal{MC}^{II}(\mathcal{A}_w(U)) \to \mathcal{MC}^{II}(\mathcal{A}_{C,w}(U))
\]

is a quasi-isomorphism by the spectral sequence argument with the filtration of the complex $(C^{II}(\check{\mathcal{O}}(X)(U)), b)$ by the Čech grading.

3. The quasi-Yoneda embedding

\[
\mathcal{A}_{C,w}(U) \to qD_C(U, w|U)
\]

is a pseudo-equivalence and hence induces a quasi-isomorphism in the mixed normalized Hochschild complexes of the second kind by (2.6).

4. The natural embedding

\[
D_C(U, w|U) \to qD_C(U, w|U)
\]

is a pseudo-equivalence and hence induces a quasi-isomorphism in the mixed normalized Hochschild complexes of the second kind again by (2.6).

Thus we have quasi-isomorphisms between sheaves of mixed complexes

\[
\mathcal{MC}^{II}(D_C(X,w)) \simeq \mathcal{MC}^{II}(qD_C(X,w)) \simeq \mathcal{MC}^{II}(\mathcal{A}_{C,w}) \simeq (\Omega^\bullet_{X/k}, -dw, d) \simeq (\check{\mathcal{O}}(\Omega^\bullet_{X/k}), -dw, d).
\]
5. Connections and the map $\text{tr}_V$

5.1. $V$-Connections. Let $(A, d_A, -h)$ be a curved dg $k$-algebra. Let $V$ be a subalgebra of the $k$-algebra $A$ such that

$$d_A|_V = 0, \quad h \in V,$$

and $V$ is contained in the even degree part of $A$. Furthermore assume that

$$va = av \quad \text{for every } v \in V, \ a \in A.$$

Note that $d_A^2 = [-h,?] = 0$. Consider a $k$-linear map

$$d : A \to \Omega^1_{V/k} \otimes_V A$$

satisfying the Leibniz rule

$$(5.1) \quad d(a_0 a_1) = (da_0)a_1 + (-1)^{|a_0|}a_0 da_1$$

for $a_0, a_1 \in A$. Here if $da_i = dv_i \otimes a'_i$ with $v_i \in V, a'_i \in A$, we define

$$(da_0)a_1 := dv_0 \otimes a'_0 a_1 \quad \text{and} \quad a_0 da_1 := (-1)^{|a_0|}dv_1 \otimes a_0 a'_1.$$

On $\Omega^1_{V/k} \otimes_V A$ we define a differential (again denoted by) $d_A$ by extension of scalars:

$$d_A(dv \otimes a) = -dv \otimes d_A(a).$$

Assume that for every $a \in A$

$$d_A d(a) + dd_A(a) = 0. \quad (5.2)$$

Example 1. When $V = A$, then $d$ is the usual $k$-derivation of $V$.

Example 2. From the de Rham differential $\Omega_X \to \Omega^1_{X/k}$ we have

$$d : \mathcal{C}(\mathcal{O}_X) \to \Omega^1_{X/k} \otimes \mathcal{C}(\mathcal{O}_X) \quad (5.3)$$

by the projection formula. Let $U$ be an affine open subset of $X$ and let

$$V = \Gamma(U, \mathcal{O}_X), \ A = \Gamma(U, \mathcal{C}(\mathcal{O}_X)), \ d_A = d_{\text{Cech}}, \ h = w.$$

Then taking $\Gamma(U, -)$ at (5.3) (and slightly abusing notation) we obtain a canonical $k$-derivation

$$d : \Gamma(U, \mathcal{C}(\mathcal{O}_X)) \to \Omega^1_{U/k} \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{C}(\mathcal{O}_X)) \quad (5.4)$$

satisfying (5.1) and (5.2).

Definition 5.1. Let $M$ be a graded $A$-module. We call a $k$-linear degree $-1$ map

$$\nabla : M \to \Omega^1_{V/k} \otimes_V M$$

a $V$-connection with respect to $d$ above if

$$\nabla(ma) = \nabla(m)a + (-1)^{|m|}m \otimes da$$

for every $a \in A$. Here $m \otimes da \in M \otimes_A (A \otimes_V \Omega^1_{V/k}) \cong M \otimes_V \Omega^1_{V/k}$. 
Example 3. Note that if $\nabla'$ is another $V$-connection on $M$, then $\nabla - \nabla'$ is an $A$-linear map $M \to \Omega^1_{V/k} \otimes_V M$ as usual. If $M = A\otimes n$, then $M$ has the obvious connection $d_F$ induced from $d$. So a connection $\nabla$ on $A\otimes n$ can be written as $d_F + C$ for some $n \times n$ matrix $C$ whose entries are elements of $\Omega^1_{V} \otimes_V A$.

5.2. The map $\text{tr}_V$. Consider a perfect right quasi-cdg module $(P, \delta_P)$ over $(\mathbb{V}, 0, -h)$. Note that $\delta_P$ is $V$-linear. Let $\mathcal{P} := P \otimes_V A$ and note that $\text{End}_A(\mathcal{P}) = \text{End}_V(P) \otimes_V A$. Then we have a cdg algebra

$$(\text{End}_A(\mathcal{P}), [\delta_P, -], \delta^2_P + \rho_{-h}),$$

where $\delta_P := \delta_P \otimes 1 + 1 \otimes d_A$. Suppose that $\delta^2_P = \rho_{\nu}$ for some $\nu \in V$. Here $\rho_{\nu}$ denotes the right multiplication by $\nu$.

Assume that a $V$-connection $\nabla$ on $\mathcal{P}$ is given and let

$$(5.5) \quad R := u\nabla^2 + [\nabla, \delta_P] \in \Omega^2_V \otimes_V \text{End}_A(\mathcal{P})[[u]].$$

Here $\nabla^2$ denotes $(\text{id} \otimes \nabla) \circ \nabla$ followed by the wedge operation:

$$\mathcal{P} \xrightarrow{\nabla} \Omega^1_{V/k} \otimes_V \mathcal{P} \xrightarrow{\text{id} \otimes \nabla} \Omega^1_{V/k} \otimes_V (\Omega^1_{V/k} \otimes_V \mathcal{P}) \xrightarrow{\cdot \otimes 1} \Omega^2_{V/k} \otimes_V \mathcal{P}.$$  

The composition $\nabla^2$ is $A$-linear. The $A$-linearity of $[\nabla, \delta_P]$ requires (5.2).

We regard $\Omega^1_{V/k} \otimes_V \text{End}_A(\mathcal{P})[[u]]$ as a $k[[u]]$-algebra defined by

$$(\gamma_1 \otimes \alpha_1) \cdot (\gamma_2 \otimes \alpha_2) = (-1)^{\alpha_1|\gamma_2}(\gamma_1 \wedge \gamma_2) \otimes (\alpha_1 \circ \alpha_2)$$

for $\gamma_i \in \Omega^1_{V/k}$, $\alpha_i \in \text{End}_A(\mathcal{P})$.

Define a $k[[u]]$-linear map $\text{tr}_V$ as the composition

$$(5.6) \quad C(\text{End}_A(\mathcal{P})[[u]]) \to \Omega^1_{V/k} \otimes_V \text{End}_A(\mathcal{P})[[u]] \xrightarrow{\text{tr}} \Omega^1_{V/k}[[u]] \otimes_V A,$$

where the first map is defined by sending $a_0[\alpha_1|\ldots|\alpha_n]$ to

$$(5.7) \quad \sum_{(j_0, \ldots, j_n)} (-1)^{\gamma_0 \gamma_1} \gamma_0 R_{j_0}[\nabla, \alpha_1] R_{j_1}[\nabla, \alpha_2] \ldots R_{j_{n-1}}[\nabla, \alpha_n] R_{j_n} \left( \frac{n + J}{n!} \right),$$

with $j_i \in \mathbb{Z}_{\geq 0}$ and $J = \sum_{i=0}^{n} j_i$; and the second map $\text{tr}$ is the $\Omega^1_{V/k}[[u]]$-linear extension the supertrace

$$(5.8) \quad \text{End}_A(\mathcal{P}) \xrightarrow{\text{canonically}} (\text{Hom}_A(\mathcal{P}, A) \otimes_A \mathcal{P}) \xrightarrow{\text{ev}} A.$$

Here the canonical isomorphism $\cong$ follows from the fact that $\mathcal{P}$ is a finitely generated projective $A$-module. Note that in the sum (5.7), the terms for $j_0 + \ldots + j_n > \dim X$ vanish.

Similarly we may define the version of $\text{tr}_V$ for the negative cyclic complex of the second kind as

$$(5.9) \quad \text{tr}^H_V : C^H(\text{End}_A(\mathcal{P}))[[u]] \to A \otimes_V \Omega^1_{V/k}[[u]]$$

by the same formula (5.7).
**Theorem 5.2.** (Theorem B.1) The map $\text{tr}_Y$ satisfies 
\[(ud + dA - dh) \circ \text{tr}_Y = \text{tr}_Y \circ (uB + b_2 + b_1 + b_0).\]
The same cochain map equality also holds for $\text{tr}^I_U$.

The proof of Theorem 5.2 is rather technical and will be given in Appendix B closely following the proof of [3, Theorem 5.19].

5.3. **Proof of Theorem 1.2.** As in §4.2 we fix a finite open affine covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ and fix a total ordering of $I$.

From Example 2 for each affine open subset $U$ of $X$ we consider the case:

\[V = \Gamma(U, \mathcal{O}_X), A = \Gamma(U, \tilde{\mathcal{C}}(\mathcal{O}_X)), d_A = d_{\text{Cech}}, h = w\]

with the $V$-connection $d$ on $A$ from the de Rham differential of $V$.

For each $P \in qD_C(X, w)$, we choose, once and for all, a $k$-linear sheaf homomorphism

\[\nabla_P : \tilde{\mathcal{C}}(P) \to \tilde{\mathcal{C}}(P) \otimes \mathcal{O}_X \Omega^1_{X/k}\]

such that $\nabla_P(U)$ is an $\mathcal{O}_X(U)$-connection with respect to $d$. Such a $\nabla_P$ we call an $\mathcal{O}_X$-connection on $\tilde{\mathcal{C}}(P)$ and will construct in §5.4 as $\nabla_E$ in (5.17) for $E = P$. The supertrace map (5.8) becomes

\[(5.10) \quad \text{tr} : \Gamma(U, \mathcal{O}_X^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(P) \otimes \tilde{\mathcal{C}}(\mathcal{O}_X)(U)) \to \Gamma(U, \mathcal{O}_X^* \otimes \tilde{\mathcal{C}}(\mathcal{O}_X)(U)).\]

Since any finite sum of objects of $qD_C(U, w|U)$ is allowed, the definition of $\text{tr}^I_U$ in (5.9) can be modified to give the map:

\[\text{tr}^I_U : \mathcal{C}^I_U(qD_C(U, w|U))[u]] \to \mathcal{O}^*_{X/k} \otimes \mathcal{O}_X \tilde{\mathcal{C}}(\mathcal{O}_X)(U)[u].\]

By Theorem 5.2 it is a cochain map. Hence its sheafified version

\[\text{tr}^I_U : \mathcal{C}^I_U(qD_C(X, w))[u]] \to \mathcal{O}^*_{X/k} \otimes \mathcal{O}_X \tilde{\mathcal{C}}(\mathcal{O}_X)(u][u]\]

is also a cochain map between the sheaves of complexes over $k[u]$.

Taking Lemma 2.1 into account, we see that $\text{tr}^I_U$ fits into the following commuting diagram of cochain maps between the sheaves of complexes over $k[u]$:

\[(5.11)\]

The diagram visualizes the relationships between the sheaves and the cochain maps involved in the proof of Theorem 1.2.
**Proposition 5.3.** The map \( \overrightarrow{tr}^{II} \) and the map 
\[
\mathbb{R}\Gamma(\overrightarrow{tr}^{II} \circ inc) : \mathbb{R}\Gamma(C^{II}(D_C(X, w)))[[u]] \to \mathbb{R}\Gamma(C^{\bullet}(\Omega^\bullet_{X/k}))[[u]]
\]
are quasi-isomorphisms.

**Proof.** It follows from Proposition 4.1 (2) and the fact that in diagram (5.11) all the arrows but \( \overrightarrow{tr}^{II} \) are shown to be quasi-isomorphisms in § 4.4. \( \square \)

We define a \( k[[u]] \)-linear cochain map
\[
tr_v : C(D_C(X, w))[[u]] \to \check{C}(\Omega^\bullet_{X/k})[[u]]
\]
by composing
\[
C(D_C(X, w))[[u]] \to \overline{C}^{II}(D_C(X, w))[[u]] \xrightarrow{\text{natural}} \Gamma(X, \overline{C}^{II}(D_C(X, w)))[[u]]
\]

\[
\mathbb{R}\Gamma(\overrightarrow{\Pi}^{II}_{\text{ince}}) : \mathbb{R}\Gamma(\check{C}(\Omega^\bullet_{X/k}))[[u]].
\]

**Theorem 5.4.** (Theorem 1.2) The \( k[[u]] \)-linear map
\[
tr_v : (C(D_C(X, w))[[u]], b + uB) \to (\check{C}(\Omega^\bullet_{X/k})[[u]], d_{\check{C}ech} - dw + ud)
\]
is a quasi-isomorphism compatible with the HKR-type isomorphism.

**Proof.** The proof follows from the following commuting diagram of cochain maps
\[
\begin{array}{cccc}
\overset{\sim}{\text{Prop. 4.1}} & (C(D_C(X, w))[[u]], b + uB) & \overset{\sim}{\text{Prop. 5.3}} & \mathbb{R}\Gamma(\check{C}(\Omega^\bullet_{X/k}))[[u]].
\end{array}
\]

where the arrows with \( \sim \) are meant to be quasi-isomorphisms. \( \square \)

For \( \alpha_j \in \check{C}(\Omega^\bullet_{X/k}) \) with \( P_{n+1} = P_0 \), we note that \( tr_v(\alpha_0[\alpha_1|...|\alpha_n]) \) is as in formula (1.6)
\[
\sum_{(j_0, ..., j_n)} (-1)^J tr(\alpha_0 R^{j_0}_{[\nabla, \alpha_1]} R^{j_1}_{[\nabla, \alpha_2]} ... R^{j_{n-1}}_{[\nabla, \alpha_n]} R^{j_n}_{0}(n + J)!)
\]
where $j_i \in \mathbb{Z}_{\geq 0}$ and $J = \sum_{i=0}^{m} j_i$. From Remark 5.5 we see that

\begin{equation}
\[ \nabla, \alpha_j \] = \prod_{i_0 < \ldots < i_p} \left( \nabla_{P_j, i_0 | U_{i_0 \ldots i_p}} \circ (\alpha_j)_{i_0 \ldots i_p} - (-1)^{|\alpha_j|} (\alpha_j)_{i_0 \ldots i_p} \circ \nabla_{P_j+1, i_0 | U_{i_0 \ldots i_p}} \right);
\end{equation}

\[ R_j = \prod_{i} \left( u
abla^2_{P_j, i} + [\nabla_{P_j, i}, \delta_{P_j} | U_i] \right) + \prod_{i_0 < i_1} \left( \nabla_{P_j, i_0} - \nabla_{P_j, i_1} \right). \]

### 5.4. Construction of $\mathcal{O}_X$-connections on $\check{C}(E)$

Let $E$ be a vector bundle on $X$. Choose a connection $\nabla_i$ of $E|U_i$ for $i \in I$:

\[ \nabla_i : E|U_i \to \Omega^1_{U_i/k} \otimes \mathcal{O}_{U_i} E|U_i, \]

which exists, since $U_i$ is affine. We consider

\begin{equation}
\nabla_E := \prod_{i \in I} \nabla_i \in C^0(\mathcal{U}, \text{Hom}_{\check{C}}(\check{C}(E), \Omega^1_{X/k} \otimes \check{C}(E)))
\end{equation}

a Čech element of a $k$-sheaf $\text{Hom}_{\check{C}}(\check{C}(E), \Omega^1_{X/k} \otimes \check{C}(E))$ on $X$. Note that $\nabla_E$ satisfies

\[ \nabla_E(sa) = \nabla_E(s)a - (-1)^{|s|} s \otimes da \]

for $s \in \check{C}(E)$, $a \in \check{C}(\mathcal{O}_X)$. Thus for each affine $U$ open subset of $X$, $\nabla_E(U)$ is an $\mathcal{O}_X(U)$-connection of the $\mathcal{O}_X(U)$-module $\check{C}(E)(U)$ with respect to $d$.

We have constructed an $\mathcal{O}_X$-connection on $\check{C}(E)$.

**Remark 5.5.** Note that

\[ \nabla_E^2 \in \text{Hom}_{\check{C}(\mathcal{O}_X)}(\check{C}(E), \Omega^2_{X/k} \otimes \check{C}(E)) = \text{Hom}_{\mathcal{O}_X}(E, \Omega^2_{X/k} \otimes \check{C}(E)); \]

\[ [\nabla_E, d_{\text{Čech}}] \in \text{Hom}_{\check{C}(\mathcal{O}_X)}(\check{C}(E), \Omega^1_{X/k} \otimes \check{C}(E)) = \text{Hom}_{\mathcal{O}_X}(E, \Omega^1_{X/k} \otimes \check{C}(E)). \]

In fact they are elements in $\text{Hom}_{\mathcal{O}_X}(E, \Omega^2_{X/k} \otimes \mathcal{O}_X \check{C}(E))$ and $\text{Hom}_{\mathcal{O}_X}(E, \Omega^1_{X/k} \otimes \mathcal{O}_X \check{C}(E))$, respectively. Now it is easy to see that they are expressed as

\begin{equation}
\quad \prod_i \nabla_i^2 \in \check{C}^0(\mathcal{U}, \Omega^2_{X/k} \otimes \mathcal{O}_X \mathcal{E}nd\mathcal{O}_X(E));
\end{equation}

\begin{equation}
\quad \prod_{i < j} (\nabla_i - \nabla_j) \in \check{C}^1(\mathcal{U}, \Omega^1_{X/k} \otimes \mathcal{O}_X \mathcal{E}nd\mathcal{O}_X(E)).
\end{equation}

Let $\alpha \in \check{C}^p(\mathcal{U}, \mathcal{E}nd\mathcal{O}_X(E)) \subseteq \text{End}_{\check{C}(\mathcal{O}_X)}(\check{C}(E))$. Then $[\nabla_E, \alpha]$ is an element in $\text{Hom}_{\check{C}(\mathcal{O}_X)}(\check{C}(E), \Omega^1_{X/k} \otimes \mathcal{O}_X \check{C}(E))$, which is expressed as

\[ \prod_{i_0 < \ldots < i_p} [\nabla_{i_0 | U_{i_0 \ldots i_p}}, \alpha_{i_0 \ldots i_p}] \in \check{C}^p(\mathcal{U}, \Omega^1_{X/k} \otimes \mathcal{O}_X \mathcal{E}nd\mathcal{O}_X(E)). \]

**Remark 5.6.** Note that (5.18) is the curvature $\nabla_E^2$ of the connection $\nabla_E$ on $\check{C}(E)$ and (5.19) is the Čech representative of the Atiyah class of the vector bundle $E$ (see [11] for example).
6. Applications

In this section we will apply Theorem 1.2 to prove a Chern character formula (1.3). We also generalize the formula to the localized case as well as the equivariant case of a finite group action.

6.1. A Chern character formula. Denote by \( \text{ch}_{HN}(P) \) and \( \text{ch}_{HH}(P) \) the classes in

\[
\mathbb{H}^{\bullet}(X, (\Omega^\bullet_{X/k}[[u]], ud - dw)) \quad \text{and} \quad \mathbb{H}^{\bullet}(X, (\Omega^\bullet_{X/k}, -dw))
\]

corresponding to \( \text{Ch}_{HN}(P) \) and \( \text{Ch}_{HH}(P) \), respectively via the isomorphism induced from (5.14).

6.1.1. Proof of Theorem 1.1. Recall that under the natural cochain map there is an isomorphism

\[
\mathcal{H}_{N}(D_{\hat{C}}(X, w)) \cong \mathcal{H}_{N}^{II}(D_{\hat{C}}(X, w)).
\]

Now considering the class \([1_P]\) of \( \mathcal{H}_{N}^{II}(D_{\hat{C}}(X, w)) \) and denoting by \( \text{tr}_{\nabla,X}^{II} \) the composition of the last two maps in (5.13), we get

\[
\text{ch}_{HN}(P) = \text{tr}_{\nabla,X}^{II}(1_P)
\]

in the cohomology \( \hat{H}^0(\Omega, (\Omega^\bullet_X[[u]], ud - dw)) \). For an \( \mathcal{O}_X \)-connection \( \nabla_P \) on \( \hat{C}(P) \) let

\[
R := u\nabla_P^2 + [\nabla_P, \delta_P + d_{\text{cach}}],
\]

which we call the total curvature of \( \nabla_P \). Then by formula (5.1) we have

\[
\text{tr}_{\nabla,X}^{II}(1_P) = \text{tr} \exp(-R).
\]

Therefore we have established the following.

**Theorem 6.1.** (Theorem 1.1) The Chern character \( \text{ch}_{HN}(P) \) is representable by

\[
\text{ch}_{HN}(P) = \text{tr} \exp(-R).
\]

Remark 5.5 shows expression (1.2).

6.1.2. When \( k = \mathbb{C} \). Assume furthermore that the critical locus of \( w \) is proper over \( \mathbb{C} \). Let \( X^{an} \) denote the complex manifold associated to \( X \). Due to GAGA, instead of \( \hat{C}(P) \), we may use the Dolbeaut resolution

\[
(\mathcal{O}^{\bullet}_{X^{an}}, \partial) \otimes \mathcal{O}_{X^{an}} P^{an}
\]

to construct the Dolbeaut dg enhancement \( D_{\text{Dol}}(X, w) \) of the derived category \( D(X, w) \). Here \( P^{an} \) denotes the holomorphic version of the matrix factorization \( P \). The objects of \( D_{\text{Dol}}(X, w) \) are the objects of \( D_{dy}(X, w) \). However \( \text{Hom}(P, Q) \) in \( D_{\text{Dol}}(X, w) \) is, by definition, a complex

\[
\Gamma(X, \mathcal{O}^{\bullet}_{X^{an}} \otimes \mathcal{O}_{X^{an}} \text{Hom}_{\mathcal{O}_{X^{an}}}(P^{an}, Q^{an})).
\]
It is straightforward to check that the Dolbeault version of Theorem \ref{thm:HKR} holds once we choose a differentiable \((1,0)\)-connection \(\nabla\) for each \(P \in D_{\text{Dol}}(X, w)\). Hence we also have a Chern character formula
\[(6.4) \quad \text{ch}_{HN}(P) = \text{tr} \exp(-u \nabla^2 - [\nabla, \delta P + \cbar])\]
in the cohomology
\[H^0(\Gamma(X^{an}, \mathcal{O}_{X^{an}}^{\bullet}[[u]]), u\cbar + \cbar - \delta w \wedge).\]

6.2. Cohomology with supports. A matrix factorization \(P\) for \((X, w)\) is called acyclic if \(1_P\) is null-homotopic in \(\text{End}_{D_{dg}(X, w)}(P)\). Let \(Z\) be a closed subset of \(X\). We say that a matrix factorization \(P\) for \((X, w)\) is supported on \(Z\) if \(P|_{X - Z}\) is an acyclic matrix factorization for \((X - Z, w|_{X - Z})\). We may consider a dg category \(D_{dg}(X, w)_Z\) of matrix factorizations for \((X, w)\) supported on \(Z\). It is a full subcategory of \(D_{dg}(X, w)\).

Let \(\mathcal{U}_1 = \{U_i\}_{i \in I_1}\) be a finite collection of open affine subsets \(U_i\) of \(X\) such that
\[\bigcup_{i \in I_1} U_i \supseteq Z\]
and let \(\mathcal{U}_2 = \{U_i\}_{i \in I_2}\) be a finite open affine covering of \(X - Z\). Let \(I := I_1 \sqcup I_2\) and \(\mathfrak{U} := \{U_i\}_{i \in I}\), a finite open affine covering of \(X\). As before \(\check{C}(P)\) and \(\check{C}(P)|_{X - Z}\) denote the ordered \(\check{C}\)ech complexes of \(P\) and \(P|_{X - Z}\) with respect to \(\mathfrak{U}\) and \(\mathcal{U}_2\), respectively. Let \(\check{C}(P)_Z\) denote the kernel of the natural projection cochain map \(\check{C}(P) \to j_* \check{C}(P)|_{X - Z}\) where \(j : X - Z \to X\) is the open inclusion.

With respect to \(\mathfrak{U}\), we have the \(\check{C}\)ech model \(D_{\check{C}}(X, w)\) for \(D_{dg}(X, w)\); see §4.2.2. We define \(D_{\check{C}}(X, w)_Z\) as the full subcategory of \(D_{\check{C}}(X, w)\) consisting of all objects supported on \(Z\). We will define another version \(D_{\check{C}}(X, w)_Z^{\text{rel}}\) of \(D_{dg}(X, w)_Z\), which is a relative version. The objects remain the same. However the Hom space \(\text{Hom}(P, Q)\) for \(P, Q \in D_{\check{C}}(X, w)_Z^{\text{rel}}\) is defined to be
\[(6.5) \quad \text{Hom}_{\check{C}(\mathcal{O}_X)_Z}^\bullet(\check{C}(P)_Z, \check{C}(Q)_Z) \cong \Gamma(X, \check{\text{Hom}}_{\mathcal{O}_X}^\bullet(P, Q) \otimes_{\mathcal{O}_X} \check{\mathcal{O}}(\mathcal{O}_X)_Z).\]
This is a relative version of \((1.4)\). Note that the natural inclusion
\[\text{Hom}_{\check{C}(\mathcal{O}_X)_Z}^\bullet(\check{C}(P)_Z, \check{C}(Q)_Z) \to \text{Hom}^\bullet_{\check{C}(\mathcal{O}_X)}(\check{C}(P), \check{C}(Q))\]
is a quasi-isomorphism. The relative version \(D_{\check{C}}(X, w)_Z^{\text{rel}}\) is not necessarily unital, which is required in the definition of dg categories at §2.1. It is however cohomologically unital, i.e., it has units in its cohomological category; for the definition see [22 §1.1].

For a not-necessarily unital dg category \(\mathcal{C}\), with no changes the definition of Hochschild complex \((\mathcal{C}^{an}(\mathcal{C}), b)\) works. There is another version, the so-called non-unital Hochschild complex
\[(C^{an}(\mathcal{C}), b)\]
which includes \((C'(\mathcal{C}), b)\) as a subcomplex and the inclusion is a quasi-isomorphism. The complex \((C'(\mathcal{C}), b)\) however has a suitable Connes differential \(B^e\) making a mixed complex \((C'(\mathcal{C}), b^e, B^e)\); see \[23\] §3.5 where \(C'(\mathcal{C})\) is denoted by \(CC_{\ast}^m(\mathcal{C})\). On the other hand for a unital dg category \(\mathcal{A}\), there is a quasi-isomorphism

\[
p(u) : (C'(\mathcal{A})[[u]], b^e + uB^e) \xrightarrow{\sim} (C(\mathcal{A})[[u]], b + uB)
\]
defined in \[25\].

Therefore we may consider a natural commutative diagram in the category of cochain complexes

\[
\begin{array}{cccc}
C'^e(D_{\mathcal{C}}(X, w)Z_j)[[u]] & \xrightarrow{inc} & C'^e(D_{\mathcal{C}}(X, w)Z)[[u]] & \\
\downarrow & & \downarrow & \\
C'(D_{\mathcal{C}}(X, w)Z)[[u]] & \xrightarrow{\tilde{\gamma}} & C'(D_{\mathcal{C}}(U, w|_U))[[u]] & \\
\downarrow & & \downarrow & \\
\Gamma(X, C(\Omega^\ast_{X/k})Z)[[u]] & \xrightarrow{\tilde{\gamma}, X} & \Gamma(U, C(\Omega^\ast_{U/k})[[u]]) & \\
\end{array}
\]

Here \(\tilde{\gamma}_{\ast, U}\) (resp. \(\tilde{\gamma}_{\ast, X}\)) denotes \(\tilde{\gamma}\) for \(X\) with respect to \(U\) (resp. for \(U\) with respect to \(\Omega_2\)) and the dotted arrow \(\derivative\tilde{\gamma}\) denotes \(\tilde{\gamma}_{\ast, X}\) restricted to \(C'^e(D_{\mathcal{C}}(X, w)Z)[[u]]\), which lands on the subcomplex \(\Gamma(X, C(\Omega^\ast_{X/k})Z)[[u]]\).

**Theorem 6.2.** The cochain map

\[
\derivative\tilde{\gamma} : (C'^e(D_{\mathcal{C}}(X, w)Z)[[u]], b^e + uB^e) \rightarrow (\Gamma(X, \Omega^\ast_{X/k} \otimes_{\mathcal{O}_X} \bar{C}(\mathcal{O}_X)Z)[[u]], d_{\text{Cech}} - dw + ud)
\]

is a quasi-isomorphism compatible with the HKR-type isomorphism \(\tilde{\gamma}_{\ast, X}\).

Furthermore, in the relative Čech cohomology

\[
H^0(\Gamma(X, \Omega^\ast_{X/k} \otimes_{\mathcal{O}_X} \bar{C}(\mathcal{O}_X)Z)[[u]], d_{\text{Cech}} - dw + ud),
\]

the localized Chern character \(\text{ch}_{HN}^\ast(P)\) for \(P \in D_{\mathcal{C}}(X, w)Z\) with local connections \([\text{1.1}\)] is representable by

\[
\text{tr} \left( \prod_{i \in I_1} 1_{P|_{U_i}} \exp \left( -u(\nabla_i^2)_{i \in I} - (\nabla_i, \delta_P)_{i \in I} - (\nabla_i - \nabla_j)_{i < j, i, j \in I} \right) \right).
\]

**Proof.** First we invoke two facts that \(inc\) in \((6.6)\) is a quasi-isomorphism; see \[23\] Corollary 4.13 and the third line of \((6.6)\) is a distinguished triangle; see \[13\] §5.7). Note that the last line of \((6.6)\) is obviously a short exact sequence. By these facts together with the quasi-isomorphisms \(p(u)\) it follows that \(\derivative\tilde{\gamma}\)
is also a quasi-isomorphism. Since the unit $\prod_{i \in I} 1_{P|_{U_i}}$ for $P$ in $D_C(X,w)_Z$ is homologous to $1_P^\circ := \prod_{i \in I} 1_{P|_{U_i}}$ and, by the definition of $p(u)$,

$$p(u) \circ inc(1_P^\circ) = 1_P^\circ,$$

we conclude (6.8).

6.3. **Global quotients by a finite group.** Let a finite group $G$ act on a smooth quasi-projective variety $X$ from the left. Suppose that $w$ is $G$-fixed. We consider the dg category $D^G_C(X,w)$ of $G$-equivariant matrix factorizations for $(X,w)$. The Hom dg-space from $P$ to $Q$ is the $G$-fixed part of $\text{Hom}_{D_C(X,w)}(P,Q)$.

Denote by $t$ the action map and by $s$ the projection map

$$t : G \times X \to X, \ (g,x) \mapsto gx; \quad s : G \times X \to X, \ (g,x) \mapsto x.$$ 

For $E \in D_C(X,w)$ denote $(g^{-1})^*E$ by $gE$ and let

$$\tilde{E} := s_* t^* E = \bigoplus_{g \in G} gE = E \otimes \bigoplus_{g} gO_X.$$ 

We will consider $\tilde{E}$ as an object in $D^G_C(X,w)$ endowed with the natural $G$-equivariant structure.

Let $\tilde{D}^G_C(X,w)$ be the full dg-subcategory of $D^G_C(X,w)$ consisting of all the objects of form $\tilde{E}$ for some $E \in D_C(X,w)$. In fact this gives rise to a functor $D_C(X,w) \to \tilde{D}^G_C(X,w)$. We will see that every object $P$ of $D^G_C(X,w)$ is a direct summand of $\tilde{P}$ as follows.

For $P \in D^G_C(X,w)$, its $G$-equivariant structure is a compatible isomorphism $\varphi : t^* P \to s^* P$. Let $\varphi_g$ be the $g$-component of $\varphi$,

$$\varphi_g : gP \xrightarrow{\sim} P.$$ 

Then we get a map $i : P \to s_* t^* P$ given by the composition

$$i : P \xrightarrow{\text{natural}} s_* s^* P \xrightarrow{s_* \varphi^{-1}} s_* t^* P.$$ 

Let $\omega$ be a map $s_* t^* P \to P$ given by the composition

$$\omega : s_* t^* P \xrightarrow{s_* \varphi} s_* s^* P \xrightarrow{\text{pin}} P,$$

making $\omega \circ i = 1_P$. We impose a $G$-equivariant structure on $s_* t^* P$ by letting

$$s_* t^* P = P^\tilde{g} = P \otimes \bigoplus_{g} gO_X$$

where $P^\tilde{g}$ denotes the object in $D_C(X,w)$ corresponding to $P$. Then it is straightforward to check that $i, \omega$ are closed maps in $D^G_C(X,w)$. By Morita invariance (2.5) we have the following lemma.
Lemma 6.3. The inclusion
\[ \text{MC}(D_C^G(X, w)) \xleftarrow{\text{inc}} \text{MC}(\widetilde{D}_C^G(X, w)) \]
is a quasi-isomorphism.

For \( E, F \in D_C(X, w) \) every map \( \phi : \widetilde{E} \to \widetilde{F} \) in \( \text{MC}(D_C^G(X, w)) \) is expressed as a square matrix \((\phi_{g,h})_{g,h\in G}\) whose entries \( \phi_{g,h} : gF \to hE \) is a morphism in \( D_C(X, w) \). Since \( \phi \) is \( G \)-equivariant, \( h(\phi_{h^{-1},g,\text{id}}) = \phi_{g,h} \) if we let \( h(\phi_{h^{-1},g,\text{id}}) := (h^{-1})^*(\phi_{g,h}) \). Hence \( \phi \) is uniquely determined by the components
\[ \phi_g := \phi_{g,\text{id}} : gF \to E, \ g \in G. \]

We formally write \( \phi = \sum_{g\in G} \phi_g \otimes g \).

Example 4. For \( \alpha \in \text{End}^G_C(X) \), let \( \widetilde{\alpha} := \iota \circ \alpha \circ \varpi \), which is an endomorphism of \( \widetilde{P}^1 = s \ast t^* P \) in \( D_C^G(X, w) \). Since \( \widetilde{\alpha} \) is a \( G \)-equivariant map, it is determined by each component \( gP \to P \), which is
\[ \frac{1}{|G|} \alpha \circ \varphi_g. \]

Proposition 6.4. The morphism \( \Psi \) between the mixed complexes is a quasi-isomorphism.

Proof. When \( w = 0 \), the map \( \Psi \) was defined and shown to be a quasi-isomorphism in [2]. For general \( w \) we may assume, by the Mayer-Vietoris sequence argument, that \( X \) is affine. In this case, for the Hochschild homology of the second kind, it is a quasi-isomorphism by a spectral sequence argument as in [5, Theorem 6.3]. Now again by [2,1], we conclude that it
is a quasi-isomorphism in the Hochschild complex of the first kind. Taking Lemma 2.1 into account, we complete the proof. □

Fix a finite open affine covering \( U = \{ U_i \}_{i \in I} \) of \( X \) such that each \( U_i \) is \( G \)-invariant. This induces a finite open affine covering \( \{ X^g \cap U_i \}_{i \in I} \) of \( X^g \). For each \( g \) and each \( Q \in D_C(X^g, w) \), we choose, once and for all, a connection \( \nabla_{Q,i} \) of \( Q|_{X^g \cap U_i} \) such that the chosen connection \( \nabla_{h^* Q,i} \) of \( (h^* Q)|_{X^{h^{-1} g h} \cap U_i} \) is isomorphic to \( h^*(\nabla_{Q,i}) \). This is possible by Zorn’s lemma. Now we have a quasi-isomorphism

\[
(6.10) \quad \bigoplus_{g \in G} C(D_C(X^g, w_g))[[u]] \xrightarrow{\text{tr}_g} \bigoplus_{g \in G} \Gamma(X^g, \mathcal{O}_{X^g} \otimes \Omega^*_{X^g/k})[[u]]
\]

defined by applying \( \text{tr}_g \) component-wise for each \( g \). The cochain map \( \text{tr}_g \) in (6.10) is \( G \)-equivariant. Hence it induces a cochain map in \( G \)-coinvariants, also denoted by \( \text{tr}_g \) by abuse of notation.

**Theorem 6.5.** There is a natural isomorphism

\[
H^*_H(D_C^G(X, w)) \cong (\bigoplus_{g \in G} H^*(-(\Gamma(X^g, \mathcal{O}_{X^g} \otimes \Omega^*_{X^g/k})))[[u]], d_{\text{Coch}} - dw_g + ud))_G.
\]

Under the isomorphism, the \( G \)-equivariant Chern character becomes

\[
(6.12) \quad \text{ch}^G_{H^*_H}(P) = \text{tr}_g \Psi(\eta_{\pi}),
\]

where \( \pi := \iota \circ \varpi \) and \( \eta_{\pi} \) is from (3.2). In particular,

\[
(6.13) \quad \text{ch}^G_{H^*_H}(P) = \frac{1}{|G|} \bigoplus_{g \in G} \text{tr} \left( \varphi_{g|X^g} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \right) \right) \Delta \right)
\]

for \( G \)-equivariant connections \( \nabla_{P|U_i} \) of \( P|U_i \).

**Proof.** The isomorphism (6.11) follows by isomorphisms

\[
H^*_H(D_C^G(X, w)) \xrightarrow{\text{tr}_g \circ \Psi} \text{RHS of } (6.11).
\]

The specialization of (6.12) at \( u = 0 \) becomes (6.13) by (6.9). Thus it is enough to show (6.12). However this is clear from Proposition 3.3 which says that \( \eta_{\pi} \) is homologous to \( 1_P \) in \( \mathcal{C}(D_C^G(X, w))[[u]], b + u B) \). □

**Remark 6.6.** Since \( \alpha \) is homologous to \( \hat{\alpha} = \sum_{g \in G} \frac{1}{|G|} (\alpha \circ \varphi_g) \otimes g \), we also get a boundary bulk map formula for Hochschild homology:

\[
(6.14) \quad \alpha \mapsto \frac{1}{|G|} \bigoplus_{g \in G} \text{tr} \left( (\alpha \circ \varphi_g)|_{X^g} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \exp\left( -\prod_{i \in I} \nabla_{P|X^g,i} \right) \right) \Delta \right) \).
\]

**Remark 6.7.** Let \( X \) be an open subscheme of \( \mathbb{A}^n_k \) containing the origin and let \( w \) have only one critical point at the origin. Under the natural isomorphism \( (\bigoplus_{g \in G} \Omega^*_{X^g/k})_G \rightarrow (\bigoplus_{g \in G} \Omega^*_{X^g/k})_G \) and the completion, (6.14) coincides with the corresponding formula (3.16) in [19, Theorem 3.3.3] up to a normalization factor \( |G| \) and the sign convention in HKR-type isomorphisms.
Remark 6.8. When \( k = \mathbb{C} \), using a \( G \)-equivariant differentiable \((1, 0)\)-connection on \( P \) we get a \( G \)-equivariant Chern character formula of \( ch_H^G(P) \) in the Dolbeault cohomology. It is the \( G \)-equivariant version of (6.4).

6.3.1. The equivariant mixed complex. Let \( \Psi^I \) be the version of \( \Psi \) for the mixed normalized Hochschild complex of the second kind. We have a diagram of quasi-isomorphisms of mixed complexes

\[
\begin{array}{ccc}
\Psi^I(D_C^G(X, w)) & \xleftarrow{\text{inc}} & \Psi^I(D_C^G(X, w)) \\
(\oplus_{g \in G}\Psi^I(D_C^G(X^g, w_g)))_G & \xleftarrow{\text{inc}} & (\oplus_{g \in G}\Psi^I(gD_C^G(X^g, w_g)))_G \\
(\oplus_{g \in G}\Psi^I(\Gamma(X^g, \mathcal{O}_{X^g})))_G & \xrightarrow{\text{inc}} & (\oplus_{g \in G}(\Gamma(X^g, \mathcal{O}_{X^g/k})))_G.
\end{array}
\]

The third inclusion map \( \text{inc} \) is a quasi-isomorphism, since it is the case for the affine case.

Theorem 6.9. The natural morphism \( \overline{MC}^I(D_C^G(X, w)) \rightarrow \overline{MC}^I(D_C^G(X, w)) \) is a quasi-isomorphism and \( \overline{MC}^I(D_C^G(X, w)) \) is quasi-isomorphic to

\[
\left( (\oplus_{g \in G}(\Gamma(X^g, \mathcal{O}_{X^g/k}))), d, d \right)_G.
\]

Proof. Since the second statement follows from diagram (6.15), it remains to prove the first statement. When \( X \) is affine, its proof follows from \( 2.2.3 \) \( 2.7 \), and the first two quasi-isomorphisms in diagram (6.15). For the general \( X \), note first that there exists a finite collection of \( G \)-invariant open affine schemes that cover \( X \), since \( X \) is quasi-projective. We may now apply the Mayer-Vietoris exact triangle argument as in \( 2.7 \) to reduce the proof to the affine case.

Appendix A. Proof of Proposition 3.3

Proposition A.1. (Proposition 3.3) The two cycles \( 1_P \) and \( \eta \) are homologous in \( (\overline{C}(P, N))[[u]], b + uB \).

Proof. Let \( \mathcal{M} \) be a \( k \)-category consisting of two formal objects, say \( P \) and \( N \), and arrows generated by morphisms \( 1_P := \text{id}_P, g : P \rightarrow N, f : N \rightarrow P, 1_N := \text{id}_N, \pi : N \rightarrow N \), together with the obvious relations by \( 1_P, 1_N \) and retract relations

\[
g \circ f = \pi, \quad f \circ g = 1_P.
\]

Let \( \overline{\mathcal{M}}(x, y) \) denote \( \mathcal{M}(x, y) \) if \( x \neq y \) or \( \mathcal{M}(x, y)/k \cdot 1_x \) if \( x = y \). The proof of Proposition A.1 amounts to finding \( \{\xi_i\}_{i=1}^x \) in

\[
\xi_i \in \overline{C}_{2i-1}(\mathcal{M}) := \bigoplus_{x_0, \ldots, x_{2i-1} \in \mathcal{M}} \mathcal{M}(x_1, x_0) \otimes \overline{\mathcal{M}}(x_2, x_1)[1] \otimes \cdots \otimes \overline{\mathcal{M}}(x_0, x_{2i-1})[1]
\]
such that $\xi_1 := g[f]$ and the $\xi_i$ satisfy the following relation

\[(A.1) \quad b(\xi_{i+1}) = \eta_k - B(\xi_i) \text{ where } \eta_\pi := \sum_{i=0}^{\infty} \eta_i u^i.\]

Let

\[
\overline{D'} := \bigoplus_{n=0}^{\infty} \{a_0[a_1|...|a_n] \mid a_0 = t_{k_1} \pi + t_{k_2} 1_N \text{ and } a_i = t_{k_i} \pi \text{ for } t_{k_i} \in k, \text{ all } i \geq 1 \} \subseteq \overline{\mathcal{C}(M)}.
\]

One may check that it is an acyclic subcomplex in $\overline{\mathcal{C}(M)}$. Letting $\overline{\mathcal{C}^2(M)} := \overline{\mathcal{C}(M)/D'}$, define

\[
\overline{D''} := \bigoplus_{n=0}^{\infty} \{a_0[a_1|...|a_n] \mid a_i = t_i \pi \text{ and } a_{i+1} = t_{i+1} \pi \text{ for some } i + 1 \text{ mod } n + 1 \text{ and } t_i \in k \} \subseteq \overline{\mathcal{C}^2(M)}.
\]

By some computation, it can be also seen that it is an acyclic subcomplex in $\overline{\mathcal{C}^2(M)}$. In the quotient space

\[
\overline{\mathcal{C}^3(M)} := \overline{\mathcal{C}(M)/\overline{D''}},
\]

relation \((A.1)\) becomes

\[(A.2) \quad b(\xi_{i+1}) = -B(\xi_i) \text{ for } i \geq 1.\]

In $\overline{\mathcal{C}^3(M)}$, letting $\xi_1 = g[f]$ and for $i \geq 2$

\[
\xi_i := (i-1)!(-1)^{i-1} \left( g[f| g|...|f] - (1_N[\pi|g|...|f] + 1_N[\pi|\pi| g|...|f] + \cdots + 1_N[\pi|...|\pi| g|f] \right) \in \overline{\mathcal{C}^3_{2i-1}(M)},
\]

one may check that they satisfy the recursive relation \((A.2)\). \(\square\)

**Appendix B. Proof of Theorem 5.2**

**Theorem B.1.** \(\text{(Theorem 5.2)}\) The map $\text{tr}_V$ satisfies

\[
(ud + d_A - dh) \circ \text{tr}_V = \text{tr}_V \circ (uB + b_2 + b_1 + b_0).
\]

The same cochain map equality also holds for $\text{tr}_V$.\(^U\)

**Proof.** It is enough to show the equality for $\text{tr}_V$. The proof of [3 Theorem 5.19] for when $A = V$, $V$ is commutative, and $\delta_P = 0$ also works for the general case. We provide some details following the proof. In what follows, let

\[(B.1) \quad \alpha' := [\nabla, \alpha] \text{ for } \alpha \in \text{End}_A(P),\]

Lemma B.3.\[ \Omega \]

Recalling \( \delta_P = \delta_P \otimes 1 + 1 \otimes d_A \) and the V-linearity of \( \delta_P \), we see that
\[
(ud + d_A) \circ \text{tr}_V^J(a_0[a_1] \cdots |a_n])
\]
\[
= \text{tr}(u \nabla + \delta_P, \sum_{(j_0, \ldots, j_n)} (-1)^j a_0 R^{j_0} a'_1 \cdots a'_n R^{j_n} (J + n) !)
\]
\[
= \text{tr}( \sum_{(j_0, \ldots, j_n)} (-1)^j \frac{[\nabla, a_0] R^{j_0} a'_1 \cdots a'_n R^{j_n}}{(J + n)!} )
\]
\[
+ \text{tr}( \sum_{i=1}^n (-1)^{-i+1} \sum_{(j_0, \ldots, j_n)} (-1)^i \frac{a_0 R^{j_i} a'_1 \cdots a'_n R^{j_n}}{(J + n)!} )
\]
\[
+ \text{tr}( \sum_{i=0}^n (-1)^{i+1} \sum_{(j_0, \ldots, j_n)} (-1)^i \frac{[\nabla, R^{j_i} |a'|_{i+1} R^{j_{i+1}} \cdots a'_n R^{j_n}}{(J + n)!} )
\]
In the last equality, by \( \text{(B.7)} \) the first two terms become \( \text{tr}_V^J \circ (b_2 + b_1 + uB) \) and by \( \text{(B.4)} \) the last term becomes
\[
\text{(B.2)} \quad \text{tr}( \sum_{(j_0, \ldots, j_n)} (-1)^j a_0 R^{j_0} a'_1 \cdots a'_n R^{j_n} (J + n) !)
\]
After letting \( K := J - 1 \) and reindexing by \( K = k_0 + \ldots + k_n \), \( \text{(B.2)} \) becomes
\[
d\gamma \wedge \text{tr}( \sum_{k_0, \ldots, k_n} (-1)^K a_0 R^{k_0} a'_1 \cdots R^{k_{i+1}} a'_i+1 \cdots a'_n R^{k_n} (K + n) !)
\]
Writing \( \nu \) as the sum of \( h \) and the curvature element \( (\nu - h) \) of \( \mathcal{P} \), we conclude the proof. \( \square \)

Lemma B.2. \( \text{tr} : \text{End}_A(\mathcal{P}) \otimes \Omega^*_{V/k}[[u]] \rightarrow A \otimes \Omega^*_{V/k}[[u]] \)

\[
(ud + d_A) \text{tr}(\gamma) = \text{tr}[u \nabla + \delta_P, \gamma]
\]

Proof. As a V-module, \( P \) is a direct summand of a finite sum \( N \) of shifted V’s. Hence we have \( A \)-linear maps \( g : \mathcal{P} \otimes_V A \rightarrow N \otimes_V A \) and \( f : N \otimes_V A \rightarrow P \otimes_V A \) such that \( f \circ g = 1_{P \otimes_V A} \). Then we have
\[
ud \text{tr}_P(\gamma) = ud \text{tr}_N(g \circ (u \nabla + \delta_P)) = \text{tr}_N(ud, g \circ (u \nabla + \delta_P))
\]
\[
= \text{tr}_N(g \circ [u \nabla, \gamma] \circ f) = \text{tr}_N(g \circ (u \nabla, \gamma) \circ f) = \text{tr}_P[u \nabla, \gamma],
\]
where \( \text{tr}_P, \text{tr}_N \) indicate the supertrace for \( \text{End}_V(P), \text{End}_V(N) \), respectively. Similarly, \( d_A \text{tr}_P(\gamma) = d_A \text{tr}_P(\delta_P, \gamma) = \text{tr}_P[\delta_P, \gamma] \). \( \square \)

Lemma B.3. With the notation from \( \text{(B.1)} \), in the algebra \( \text{End}_A(\mathcal{P}) \otimes_V \Omega^*_{V/k}[[u]] \) we have
\[
\text{(B.3)} \quad [u \nabla + \delta_P, \alpha'] = [R, \alpha] - ([\delta_P, \alpha])';
\]
\[
\text{(B.4)} \quad [u \nabla + \delta_P, R'] = -j(d\nu) R'^{-1}.
\]
Proof. The proofs are straightforward. □

Lemma B.4. In $A \otimes_{V} \Omega_{V/k}^{*}\llbracket[u]\rrbracket$ we have

\begin{equation}
\text{tr}\left(\sum_{i=1}^{n}(-1)^{i-1+\sum_{k=0}^{i-1}|\alpha_{k}|} \sum_{(j_{0},...,j_{n})}(-1)^{J} \frac{\alpha_{0}R_{i0\alpha_{1}}^{j_{0}}\cdots R_{i1\alpha_{n}'}^{j_{1}}[R_{i,j}\alpha_{2}R_{i2\alpha_{n}'}^{j_{2}}\cdots R_{i,j'}^{j_{n}}]}{(J+n)!}\right)
\end{equation}

\begin{equation}
= \text{tr}_{V}^{H} \circ b_{2}(\alpha_{0}[\alpha_{1}|...|\alpha_{n}]);
\end{equation}

\begin{equation}
\text{tr}\left(\sum_{(j_{0},...,j_{n})}(-1)^{J} \frac{\alpha_{i_{0}}'alpha_{i_{1}}'\cdots alpha_{i_{n}}'R_{j_{n}}^{j_{n}}}{(J+n)!}\right) = \text{tr}_{V}^{H} \circ (uB)(\alpha_{0}[\alpha_{1}|...|\alpha_{n}]); \text{ and}
\end{equation}

\begin{equation}
\text{tr}\left(\sum_{(j_{0},...,j_{n})}(-1)^{J} \frac{\alpha_{i_{0}}'alpha_{i_{1}}'\cdots alpha_{i_{n}}'R_{j_{n}}^{j_{n}}}{(J+n)!}\right)
\end{equation}

\begin{equation}
+ \text{tr}\left(\sum_{i=1}^{n}(-1)^{i-1+\sum_{k=0}^{i-1}|\alpha_{k}|} \sum_{(j_{0},...,j_{n})}(-1)^{J} \frac{\alpha_{0}R_{i0\alpha_{1}}^{j_{0}}\cdots R_{i1\alpha_{n}'}^{j_{1}}[\nabla,\alpha_{i_{2}}']R_{i2\alpha_{n}'}^{j_{2}}\cdots R_{i,j'}^{j_{n}}}{(J+n)!}\right)
\end{equation}

\begin{equation}
= \text{tr}_{V}^{H} \circ (b_{2} + b_{1} + uB),
\end{equation}

where $\nabla := u\nabla + \delta p$.

Proof. Equations (B.5) and (B.6) are straightforward up to some combinatorics which are checked in the Appendix in [3]. Equation (B.7) follows from (B.3), (B.5), and (B.6). □

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Department of Mathematics Education, Korea National University of Education, 250 Taeseongtabyeon-ro, Gangnae-myeon, Heungdeok-gu, Cheongju-si, Chungbuk 28173, Republic of Korea

Email address: krchung@kias.re.kr

Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

Email address: bumsig@kias.re.kr

Department of Mathematics Education, Korea National University of Education, 250 Taeseongtabyeon-ro, Gangnae-myeon, Heungdeok-gu, Cheongju-si, Chungbuk 28173, Republic of Korea

Email address: tjkim@kias.re.kr