Spacetime Topology Change: The Physics of Calabi-Yau Moduli Space*

Paul S. Aspinwall,† Brian R. Greene‡ and David R. Morrison§

We review recent work which has significantly sharpened our geometric understanding and interpretation of the moduli space of certain $N=2$ superconformal field theories. This has resolved some important issues in mirror symmetry and has also established that string theory admits physically smooth processes which can result in a change in topology of the spatial universe.

* Lecture delivered by B.R.G. at Strings ’93, Berkeley.
† School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540.
‡ School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540. On leave from: F.R. Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853.
§ School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540. On leave from: Department of Mathematics, Duke University, Box 90320, Durham, NC 27708.
1. Introduction

The essential lesson of general relativity is that the geometrical structure of spacetime is governed by dynamical variables. That is, the metric changes in time according to the Einstein equations. In the usual formulations of general relativity, the spacetime metric is defined on a space of fixed topological type – the “size” and “shape” of the space can smoothly change, but the underlying topology does not. A natural question to ask is whether this formulation is too restrictive; might the topology of space itself be a dynamical variable and hence possibly change in time? This issue has long been speculated upon. Heuristically, one suspects that topology might be able to change by means of the violent curvature fluctuations which would be expected in any quantum theory of gravity. Just as the fluctuations of the magnetic field in a box of size $L$ are on the order of $(\hbar c)^{1/2}/L^2$, those of the curvature of the gravitational field are on the order of $(\hbar G c)^{1/2}/L^3$. Thus, on extremely small scales, say $L \sim L_{\text{Planck}}$, huge curvature fluctuations are unsuppressed. One can imagine that such curvature fluctuations could “tear” the fabric of space resulting in a change of topology. The expected discontinuities in physical observables accompanying the discontinuous operation of a change in topology would be hidden, one hopes, behind the smoothing effects of quantum uncertainty. Of course, without a true theory of quantum gravity, one cannot make quantitative sense of such hypothesized processes.

With the advent of string theory, we are led to ask whether any new quantitative light is shed on the issue of topology change. Two works over the last year [1, 2] have carried out studies, from somewhat different points of view, which definitively establish that there are physically smooth processes in string theory which result in a change in the topology of spacetime. Furthermore, as phenomena in string theory, these processes are not at all exotic. Rather, they correspond to the most basic kind of operation arising in conformal field theory: deformation by a truly marginal operator. From a spacetime point of view, this corresponds to a slow variation in the VEV of a scalar field which has an exactly flat potential. It is crucial to emphasize that these physically smooth topology changing processes occur even at the level of classical string theory. It is not, as had been

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1. Recently, another example illustrating topology change in string theory has been proposed [3].

2. To avoid confusion, we remark that the present study focuses on static vacuum solutions to string theory. One expects that configurations involving the generic slow variation of such scalar fields are solutions as well.
suspected from point particle intuition, that quantum effects give rise to topology change, but, rather, it is the extended structure of the string which bears responsibility for this effect.

We can immediately summarize here the essential content of [1] and [2]. From the viewpoint of classical general relativity or the classical nonlinear sigma model, we know that there are constraints on the metric tensor which appears in the action. Namely, since the metric is used to measure lengths, areas, volumes, etc., it must satisfy a set of positivity conditions. For instance, if we have a nonlinear sigma model on a Kähler target space $M$ with metric (in complex coordinates) $g_{\mu\nu}$, we can write the Kähler form of the metric as $J = ig_{\mu\nu}dX^\mu \wedge dX^\nu$ (a real closed 2-form). The latter must satisfy

$$\int_{M_r} J^r > 0 \quad (1.1)$$

where $M_r$ is an $r$ (complex) dimensional submanifold of $M$ and $J^r$ represents the $r$-fold wedge product of $J$ with itself. The set of real closed 2-forms which satisfy (1.1) is a subset of $H^2(X, \mathbb{R})$ known as the Kähler cone and is schematically depicted in figure 1a. Such Kähler forms manifestly span a cone because if $J$ satisfies (1.1) then so does $sJ$ for any positive real $s$. The burden of [1] and [2] is that, in string theory, (1.1) can be relaxed and still result in perfectly well behaved physics. In fact, the Kähler form of a target Calabi-Yau space is one of the moduli fields of the associated conformal field theory. Investigation of the conformal field theory moduli space reveals that the corresponding geometrical description necessarily involves configurations in which the (supposed) Kähler form lies outside of the Kähler cone of the particular Calabi-Yau being studied. In fact, any and all choices of an element of $H^2(X, \mathbb{R})$ give rise to well-defined conformal field theories. In [1, 2] it was shown that some of these configurations can be interpreted as nonlinear sigma models on Calabi-Yau manifolds of topological type distinct from the original. With respect to this Calabi-Yau of new topology, the Kähler modulus satisfies (1.1) and hence may be thought of as residing in a new Kähler cone which shares a common wall with the original. Furthermore, there is no physical obstruction to continuously deforming the underlying conformal field theory so that its geometrical description passes from one Kähler cone to another and hence results in a change in topology of the target space – i.e. of space itself.

These results were established in [2] by means of mirror symmetry. Hence, in the next section we shall briefly review the phenomenon of mirror manifolds. In section III we will give a discussion of moduli spaces of both conformal theories and Calabi-Yau manifolds in
order to fill in a bit more detail required for the discussion of topology change. We will see that this discussion raises an interesting puzzle whose resolution, discussed in section IV, directly leads to the necessity of physically smooth topology changing processes. In section V we shall verify the abstract discussion of the preceding sections in an explicit example which provides a highly sensitive confirming test of the picture we present. Finally, in section VI we shall give our conclusions.

2. Mirror Manifolds

Nonlinear sigma models on Calabi-Yau target spaces, at their infrared fixed point, provide the geometric interpretation for a class of conformal field theories. As is well known, these conformal field theories have $N = (2, 2)$ world sheet supersymmetry. One can turn this identification around and inquire as to whether every $N = (2, 2)$ superconformal field theory with central charge, say, equal to 9 is interpretable as a nonlinear sigma model with some Calabi-Yau as target. The answer to this question is not known; however, it is known that one conformal field theory can sometimes be interpretable in terms of nonlinear sigma models on two very different Calabi-Yau target spaces [4]. The possible existence of this phenomenon was raised in [5] and in [6] based on the fact that there is an unnatural asymmetry between the identification of an abstract conformal field theory with a Calabi-Yau nonlinear sigma model which is resolved when a second Calabi-Yau interpretation exists. Namely, the truly marginal operators in the abstract conformal field theory can be labeled with the $U(1)_L \times U(1)_R$ quantum numbers of the lowest components of the supermultiplet to which they belong. These eigenvalues divide the space of truly marginal operators into those with charges $(1, 1)$ and $(-1, 1)$ (and their complex conjugates). Now, a Calabi-Yau sigma model also has two types of truly marginal operators: the complex
structure deformations and the Kähler deformations. Mathematically, these two types are vastly different objects; nonetheless, since they correspond to the truly marginal operators in the associated conformal field theory, the abstract formulation only distinguishes them by the sign of a $U(1)$ charge. It is surprising that an important mathematical distinction finds such a trivial conformal field theory manifestation. It was suggested in [5] and [6] that a natural resolution of this asymmetry would be the existence of a second Calabi-Yau manifold giving rise to the same conformal field theory but with the identification of geometrical deformations and conformal field theory marginal operators reversed (relative to the $U(1)$ charges) with respect to the first Calabi-Yau. One consequence of the existence of such a second Calabi-Yau interpretation is that the Hodge numbers of the first, say $M$, and those of the second, say $\tilde{M}$, are related via

$$h^{p,q}_M = h^{3-p,q}_{\tilde{M}}. \quad (2.1)$$

Although an interesting speculation, there was no evidence for the existence of this phenomenon until the simultaneous works of [7] and [4]. The authors of [7] performed a computer survey of a large number of Calabi-Yau manifolds realized as hypersurfaces in weighted projective four space. They found that almost every Calabi-Yau in the resulting list had a counterpart with Hodge numbers related as above. This falls short of establishing that these pairs of Calabi-Yau manifolds correspond to the same conformal field theory but it is at least consistent with this possibility. In [4], on the other hand, a constructive proof was given for the existence of certain pairs of Calabi-Yau manifolds $M$ and $\tilde{M}$ whose Hodge numbers are related by (2.1) and which give rise to isomorphic conformal field theories. Such pairs of Calabi-Yau spaces were named “mirror manifolds” in [4] because the relation between the Hodge numbers corresponds to a reflection in a diagonal plane of the corresponding Hodge diamonds. The construction of [4] applies to any conformal field theory built up from the $N=2$ minimal models and these include Fermat hypersurfaces in weighted projective space. At the present time, this construction supplies the only known examples of mirror manifolds.

Before proceeding, there are two points which, although not directly relevant to our present study, are worth emphasizing here. First, mirror manifolds are not the first nor

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3 There have been other conjectured constructions of mirror manifolds in both the physics [8] and mathematics [9,10] literatures, but as yet no one has been able to establish that these constructions yield pairs of Calabi-Yau manifolds corresponding to the same conformal field theory.
the only examples of distinct geometrical spaces which give rise to the same conformal field theory. For example, string theory on a circle of radius $R$ and on a distinct circle of radius $\alpha'/R$ give rise to isomorphic physics [11]. The same is true for certain pairs of toroidal orbifolds [12]. The most general term describing the phenomenon in quantum geometry of distinct spaces giving rise to identical physical models is string equivalent spaces. That is, two distinct background spaces $X$ and $Y$ on which string propagation is physically isomorphic are called string equivalent. From this definition, it is clear that $M$ and $\tilde{M}$ are a mirror pair if they are string equivalent and if they are Calabi-Yau manifolds whose Hodge diamonds are mirror reflected and hence related by (2.1). In keeping with this definition, for instance, circles of radii $R$ and $\alpha'/R$ are string equivalent but are not a mirror pair. Second, it has sometimes been asserted that the phenomenon of mirror manifolds amounts to nothing more than the fact that there is a trivial automorphism of $N = (2, 2)$ conformal field theory obtained by changing the sign of one of the $U(1)$ charges. There is a misleading imprecision here. It is true that there is a trivial automorphism of these conformal theories arising from such a change in sign. However, this trivial conformal field theory operation has an equally trivial geometrical interpretation. Namely, to specify a supersymmetric nonlinear sigma model we need to supply not only a Calabi-Yau manifold but also a vector bundle on it (to which the world sheet fermions couple), meeting certain conditions. The simplest solution to these conditions and the solution implicitly chosen in most studies is that of the tangent bundle to the Calabi-Yau manifold. There is, however another equally valid and physically equivalent choice: the cotangent bundle to the same Calabi-Yau space. These two equivalent choices differ, from the conformal field theory viewpoint, by a change in sign of one of the $U(1)$ charges in the theory. Thus, as promised, a trivial conformal field theory operation has a trivial geometric interpretation. This is not mirror symmetry. Rather, mirror symmetry is a phenomenon in which the space changes, not simply the bundle. It is true that this isomorphism is proved [4] by making use of the fact that the two relevant conformal theories differ by the trivial automorphism associated with the $U(1)$ charge. However, the existence of such a trivial automorphism does not (at our present level of understanding) by any means establish the existence of mirror manifolds — in fact, as just mentioned, there is a far more trivial geometric interpretation which immediately presents itself.

There are a number of interesting and important implications of mirror symmetry which we will not have time to discuss here. However, one particular result will be useful in our later discussion, so we briefly record it now.
Since a mirror pair $M$ and $\widetilde{M}$ correspond to the same conformal field theory, every correlation function in the latter has two geometric interpretations: one on $M$ and one on $\widetilde{M}$. Typically, these geometric realizations will be quite different; however, since they mathematically represent one and the same correlation function they must be identically equal. This fact gives rise to some highly nontrivial identities between particular geometrical formulas on $M$ and others on $\widetilde{M}$. One such identity, originally shown in [4] and later employed to remarkable ends in [13], arises from the study of three point functions amongst fields $\{O_i\}$ associated with, say, the (chiral, chiral) primary fields in the conformal theory. On $\widetilde{M}$ such fields are associated with harmonic $(0, 1)$ forms taking values in the tangent bundle, $B^\alpha_{(i)}$, and it has been shown that [14]

$$\langle O_i O_j O_k \rangle = \int_{\widetilde{M}} \Omega \wedge B^\alpha_{(i)} \wedge B^\beta_{(j)} \wedge B^\gamma_{(k)} \Omega_{\alpha\beta\gamma}. \quad (2.2)$$

On $M$, the $O_i$ are associated with harmonic $(0, 1)$ forms taking values in the cotangent bundle which are isomorphic to harmonic $(1, 1)$ forms $A_{(i)}$ and it has been shown that [13,16,13,17]

$$\langle O_i O_j O_k \rangle = \int_{M} A_{(i)} \wedge A_{(j)} \wedge A_{(k)} + \sum_{m, \{u\}} e^{\int_{\mathbb{P}^1} u^*_m K} \left( \int_{\mathbb{P}^1} u^*_A \int_{\mathbb{P}^1} u^*_A \int_{\mathbb{P}^1} u^*_A \right), \quad (2.3)$$

where $\{u\}$ is the set of holomorphic maps to rational curves on $M$, $u : \mathbb{P}^1 \to C$ (with $C$ such a holomorphic curve), $\pi_m$ is an $m$-fold cover $\mathbb{P}^1 \to \mathbb{P}^1$ and $u_m = u \circ \pi_m$.

As each of these mathematical expressions on the right hand side is equal to the same correlation function in a single conformal field theory, they must be equal to each other. Hence we have

$$\int_{\widetilde{M}} \Omega \wedge B^\alpha_{(i)} \wedge B^\beta_{(j)} \wedge B^\gamma_{(k)} \Omega_{\alpha\beta\gamma} = \int_{M} A_{(i)} \wedge A_{(j)} \wedge A_{(k)} + \sum_{m, \{u\}} e^{\int_{\mathbb{P}^1} u^*_m K} \left( \int_{\mathbb{P}^1} u^*_A \int_{\mathbb{P}^1} u^*_A \int_{\mathbb{P}^1} u^*_A \right). \quad (2.4)$$

In fact, although for ease of discussion we have focused on a single conformal field theory, if we deform that theory to any point in its moduli space, there exist corresponding choices for the Kähler class on $M$ and the complex structure of $\widetilde{M}$ such that this equality continues to hold. Notice that the leading term in (2.2) is the topological intersection form on $M$ and that this term is the only one which contributes to the correlation function in (2.2) if the integral $\int_C K$ goes to infinity, for every rational curve $C$ on $M$. This occurs if the Kähler form $K$ approaches a “large radius limit” — a concept which will be made precise in the sequel.
3. Moduli Spaces

Quite generally, as mentioned, the conformal field theories we study here come in continuously connected families related via deformations by truly marginal operators. For the \(N=2\) theories, more specifically, these truly marginal operators come in two varieties which are distinguished by their \(U(1) \times U(1)\) charges with the latter being a subalgebra of the \(N=2\) superconformal algebra. In particular, the two types of marginal operators have charges \((1, 1)\) and \((-1, 1)\) respectively. (Actually, the marginal operators are chargeless — they lie in supermultiplets whose lowest component has the given charges.) Invoking standard usage, we refer to the space of all conformal theories related by such truly marginal deformations as the \textit{conformal theory moduli space}.

When an \(N=2\) conformal theory arises from a nonlinear sigma model with a Calabi-Yau target space, the marginal operators just referred to have geometrical counterparts. The two types of marginal operators correspond to the two types of deformations of the Calabi-Yau space which preserve the Calabi-Yau condition (of Ricci flatness). These are deformations of the complex structure and deformations of the Kähler structure. Concretely, one can think of the latter as Ricci flat deformations of the metric of the form \(\delta g_{\mu\nu}\) and the while the former are of the form \(\delta g_{\mu\nu}\).

As our analysis will involve a close study of these moduli spaces, let us now describe each in a bit more detail.

3.1. Kähler Moduli Space

Given a Kähler metric \(g_{\mu\nu}\) we can construct the Kähler form \(J = ig_{\mu\nu}dX^\mu \wedge dX^\nu\). As discussed earlier, the set of allowed \(J\)'s forms a cone known as the Kähler cone of \(M\). One additional important fact is that string theory instructs us to work not just with \(J\) but also with \(B = B_{\mu\nu}\) the antisymmetric tensor field. The latter, which is a closed two-form, combines with \(J\) in the form \(B + iJ\) to yield the highest component of a complex chiral multiplet we shall call \(K\). \(K\) can therefore be thought of as a \textit{complexified} Kähler form. The precise way in which \(B\) enters the conformal field theory is such that if \(B\) is replaced by \(B + Q\) with \(Q \in H^2(M, \mathbb{Z})\), then the resulting physical model does not change. Thus, a convenient way to parametrize the space of allowed and physically distinct \(K\)'s is to introduce

\[ w_l = e^{2\pi i(B_l + iJ_l)} \]  

(3.1)
where we have expressed

\[ B + iJ = \sum_l (B_l + iJ_l)e^l \]  

(3.2)

with the \( e^l \) forming an integral basis for \( H^2(M, \mathbb{Z}) \). The \( \omega_l \) have the invariance of the antisymmetric tensor field under integral shifts built in; the constraint that \( J \) lie in the Kähler cone bounds the norm of the \( \omega_l \). Thus, the Kähler cone and space of allowed and distinct \( \omega_l \) are schematically shown in figures 1a and 1b. Notice that any choice of complexified Kähler form in the interior of figure 1b is physically admissible. Choices of \( K \) which correspond to points on the walls in figure 1b (or 1a) correspond to metrics on \( M \) which fail to meet (1.1) and hence are degenerate in some manner.

3.2. Complex Structure Moduli Space

All of the Calabi-Yau spaces we shall concern ourselves with here are given by the vanishing locus of homogeneous polynomial constraints in some projective space (or possibly a weighted projective space and products thereof). For ease of discussion, and in preparation for an explicit example we will examine shortly, let’s assume we are dealing with a Calabi-Yau manifold given by the vanishing locus of a homogeneous polynomial \( P \) of degree \( d \) in weighted projective four space \( \mathbb{P}^4_{k_1, \ldots, k_5} \). The Calabi-Yau condition translates into the requirement that \( d = \sum_i k_i \). Let’s call the homogeneous weighted projective space coordinates \( (z_1, \ldots, z_5) \) and write down the most general form for \( P \):

\[ P = \sum a_{i_1i_2\ldots i_5} z_1^{i_1} \ldots z_5^{i_5} \]  

(3.3)

where \( \sum_j k_j i_j = d \). Different choices for the constants \( a_{i_1i_2\ldots i_5} \) correspond to different choices for the complex structure of the underlying Calabi-Yau manifold. There are two important points worthy of emphasis in this regard. First, not all choices of the \( a_{i_1i_2\ldots i_5} \) give rise to distinct complex structures. For instance, distinct choices of the \( a_{i_1i_2\ldots i_5} \) which can be related by a rescaling of the \( z_j \) of the form \( z_j \rightarrow \lambda_j z_j \) with \( \lambda_j \in \mathbb{C}^* \) manifestly correspond to the same complex structure (as they differ only by a trivial coordinate transformation). The most general situation would require that we consider \( a_{i_1i_2\ldots i_5} \)’s related by general linear transformations on the \( z_j \)’s. Second, not all choices of \( a_{i_1i_2\ldots i_5} \) give rise to smooth Calabi-Yau manifolds. Specifically, if the \( a_{i_1i_2\ldots i_5} \) are such that \( P \) and \( \frac{\partial P}{\partial z_j} \) have a common zero (for all \( j \)), then the space given by the vanishing locus of \( P \) is not smooth. The set of all choices of the coefficients \( a_{i_1i_2\ldots i_5} \) which correspond to such singular spaces comprise the discriminant locus of the family of Calabi-Yau spaces associated with \( P \). The precise
equation of the discriminant locus is generally quite complicated; however, the only fact we need is that it forms a complex codimension one subspace of the complex structure moduli space. From the viewpoint of conformal field theory, the nonlinear sigma model associated to points on the discriminant locus appears to be ill defined. For example, the chiral ring becomes infinite dimensional. It is an interesting and important question to thoroughly understand whether there might be some way of making sense of such theories. For the present purposes, though, all we need to know is that at worst the space of badly behaved physical models is complex codimension one in the complex structure moduli space. We illustrate the form of the complex structure moduli space in figure 2.

![Figure 2. The moduli space of complex structures.](image)

### 3.3. Implications of Mirror Manifolds

Locally the moduli space of Calabi-Yau deformations is a product space of the complex and Kähler deformations (in fact, up to subtleties which will not be relevant here, we can think of the moduli space as a global product). Thus, we expect

\[ \mathcal{M}_{\text{CFT}} \equiv \mathcal{M}_{\text{complex structure}} \times \mathcal{M}_{\text{Kähler structure}} \]

with \( \mathcal{M}(\ldots) \) denoting the moduli space of \((\ldots)\). Pictorially, we can paraphrase this by saying that the conformal field theory moduli space is expected to be the product of figure 1b and figure 2.
This, in fact, was the picture which had emerged from much work over the last few years and was generally accepted. The advent of mirror symmetry, however, raised a serious puzzle related to this description (as first observed in [18]). Let \( M \) and \( \tilde{M} \) be a mirror pair of Calabi-Yau spaces. As we discussed before, such a pair correspond to isomorphic conformal theories with the explicit isomorphism being a change in sign of, say, the right moving \( U(1) \) charge. From our description of the moduli space, it then follows that the moduli space of Kähler structures on \( M \) should be isomorphic to the moduli space of complex structures on \( \tilde{M} \) and vice versa. That is, both \( M \) and \( \tilde{M} \) correspond to the same family of conformal theories and hence yield the same moduli space on the left hand side of (3.4). Therefore, the right hand side of (3.4) must also be the same for both \( M \) and \( \tilde{M} \).

The explicit isomorphism of mirror symmetry shows this to be true with the two factors on the right hand side of (3.4) being interchanged for \( M \) relative to \( \tilde{M} \).

The isomorphism of the Kähler moduli space of one Calabi-Yau and the complex structure of its mirror is a statement which appears to be in direct conflict with the form of figure 1b and that of figure 2. Namely, the former is a bounded domain while the latter is a quasi-projective variety. More concretely, the subspace of theories which appear possibly to be badly behaved are the boundary points in figure 1b (where the metric on the associated Calabi-Yau fails to meet (1.1)) and the points on the discriminant locus in figure 2. The former are real codimension 1 while the latter are real codimension 2. Therefore, how can these two spaces be isomorphic as implied by mirror symmetry?

4. Topology Change

As the puzzle raised in the last section was phrased in terms of those points in the moduli space which have the potential to correspond to badly behaved theories, it proves worthwhile to study the nature of such points in more detail. We will first do this from the point of view of the Kähler moduli space of \( M \).

Consider a path in the Kähler moduli space which begins deep in the interior and moves towards and finally reaches a boundary wall as illustrated in figure 3. More specifically, we follow a path in which the area of a \( \mathbb{P}^1 \) (a rational curve) on \( M \) is continuously shrunk down to zero, attaining the latter value on the wall itself. The question we ask ourselves is: does this choice for the Kähler form on \( M \) yield an ill defined conformal theory and furthermore, what would happen if we try to extend our path beyond the wall where it appears that the area of the rational curve would become negative? (We note the
linguistically awkward phrase “area of a curve” arises since we are dealing with complex
curves which therefore are real dimension two.)

As a prelude to answering this physical question, we note that precisely this operation
is well known and thoroughly studied from the viewpoint of mathematics. Namely, in
algebraic geometry there is an operation called a \textit{flop} in which the area of a rational curve
is shrunk down to zero (\textit{blown down}) and then expanded back to positive volume (\textit{blown
up}) in a “transverse” direction. Typically (although not always) this operation results
in a change of the topology of the space in which the curve is embedded. Thus, when
we say that the blown up curve has positive volume we mean positive with respect to
the Kähler metric on the new ambient space. That is, the flop operation involves first
following a path like that in figure 3 which blows the curve down, and then continuing
through the wall (as in figure 4) by blowing the curve up to positive volume on a new
Calabi-Yau space. The latter space, \( M' \) also has a Kähler cone whose complexification in
the exponentiated \( w_l \) coordinates is another bounded domain. Thus, the operation of the
flop corresponds to a path in moduli space beginning in the Kähler cone of \( M \), passing
through one of its walls and landing in the \textit{adjoining} Kähler cone of \( M' \). Although \( M \) and
\( M' \) can be topologically distinct, their Hodge numbers are the same; they differ in more
subtle topological invariants such as the intersection form governing the classical homology
ring. Mathematically, they are said to be topologically distinct but in the same birational
equivalence class.

The mathematical formulation of what it means to pass to a wall in the Kähler moduli
space has led us to a more detailed framework for studying the corresponding description
in conformal field theory. We see that from the mathematical point of view, distinct
Kähler moduli spaces naturally adjoin along common walls. We can rephrase our initial motivating question of two paragraphs ago as: does the operation of flopping a rational curve (and thereby changing the topology of the Calabi-Yau under study) have a physical manifestation? That is, does a path such as that in figure 4 correspond to a family of well behaved conformal theories?

This is a hard question to answer directly because our main tool for analyzing nonlinear sigma models is perturbation theory. The expansion parameters of such perturbative studies are of the form $\sqrt{\alpha}/R$ where $R$ refers to the set of Kähler moduli on the target manifold. Now, when we approach or reach a wall in the Kähler moduli space, at least one such moduli field $R$ is going to zero (namely the one which sets the size of the blown down rational curve). Hence, sigma model perturbation theory breaks down and we are hard pressed to answer directly whether the associated conformal theory makes nonperturbative sense.

This situation — one in which we require a nonperturbative understanding of observables on $M$ — is tailor made for an analysis based upon mirror symmetry. Perturbation theory breaks down on $M$ because of the degenerate (or nearly degenerate) choice of its Kähler structure. Note that all of our discussion could be carried through for any convenient (smooth) choice of its complex structure. Via mirror symmetry, this implies that the relevant analysis for answering the question raised two paragraphs ago should be carried out on $\tilde{M}$ for a particular form of the complex structure (namely, that which is mirror
to the degenerate Kähler structure on \( M \) but for any convenient choice of the Kähler structure. The latter, though, determines the applicability of sigma model perturbation theory on \( \tilde{M} \). Thus, we can choose this Kähler structure to be arbitrarily “large” (that is, distant from any walls in the Kähler cone) and hence arrange things so that we can completely trust perturbative reasoning. In other words, by using mirror symmetry we have rephrased the difficult and necessarily nonperturbative question of whether conformal field theory continues to make sense for degenerating Kähler structures in terms of a purely perturbative question on the mirror manifold.

This latter perturbative question is one which is easy to answer and, in fact, we have already done so in our discussion of the complex structure moduli space. For large values of the Kähler structure (again, this simply means that we are far from the walls of the Kähler cone) the only choices of the complex structure which yield (possibly) badly behaved conformal theories are those which lie on the discriminant locus. As noted earlier, the discriminant locus is complex codimension one in the moduli space (real codimension two). Thus, the complex structure moduli space is, in particular, path connected. Any two points can be joined by a path which only passes through well behaved theories; in fact, the generic path in the complex structure moduli space has the latter property. This is the answer to our question. By mirror symmetry, this conclusion must hold for a generic path in Kähler moduli space and hence it would seem that a topology changing path such as that of figure 4 (by a suitable small jiggle at worst) is a physically well behaved process. Even though the metric degenerates, the physics of string theory continues to make sense. We are already familiar from the foundational work on orbifolds \(^{19}\) that degenerate metrics can lead to sensible string physics. Now we see that physically sensible degenerations of other types (associated to flops) can alter the topology of the universe. In fact, the operation being described — deformation by a truly marginal operator — is amongst the most basic and common physical processes in conformal field theory.

To summarize the picture of moduli space which has emerged from this discussion we refer to figure 5. The conformal field theory moduli space is geometrically interpretable in terms of the product of a complex structure moduli space and an enlarged Kähler moduli space \( M_{\text{enlarged Kähler}} \). The latter contains numerous complexified Kähler cones of birationally equivalent yet topologically distinct Calabi-Yau manifolds adjoined along common walls.\(^{4}\) There are two such geometric interpretations, via mirror symmetry, with

\(^{4}\) The union of such regions constitutes what we call the “partially enlarged” Kähler moduli space. The enlarged Kähler moduli space includes additional regions as we shall mention shortly.
the roles of complex structure and Kähler structure being interchanged. This is also indicated in figure 5.

We should stress that from an abstract point of view this is a compelling picture. Although we do not have time or space to discuss it here, the augmentation of the Kähler moduli space in the manner presented (and, more precisely, as we will generalize shortly) gives it a mathematical structure which is identical to that of the complex structure moduli space of its mirror. In the important case of Calabi-Yau’s which are toric hypersurfaces, both of these moduli spaces are realized as identical compact toric varieties. Hence, the picture presented resolves the previous troubling asymmetry between the structure of these two spaces which are predicted to be isomorphic by mirror symmetry.

Although compelling, we have not proven that the picture we are presenting is correct. We have found a natural mathematical structure in algebraic geometry which if realized by the physics of conformal field theory resolves some thorny issues in mirror symmetry. We have not established, as yet, whether conformal field theory makes use of this compelling mathematical structure. If conformal field theory does avail itself of this structure, though, there is a very precise and concrete conclusion we can draw: every point in the (partially) enlarged Kähler moduli space of $M$ must correspond under mirror symmetry to some point in the complex structure moduli space of $\tilde{M}$. This implies, of course, that any and all observables calculated in the theories associated to these corresponding points must be identically equal. Let’s concentrate on the three point functions we introduced earlier in (2.4). As we discussed, if we choose a point in the Kähler moduli space for which the instanton corrections are suppressed, the correlation function approaches the topological intersection form on the Calabi-Yau manifold. For ease of calculation, we shall study the correlation functions of (2.3) in this limit. This analysis will be similar to that presented in [20] although in this case in the (partially) enlarged Kähler moduli space, there is not a single unique “large radius” point of the sort we are looking for. Rather, every cell in the (partially) enlarged moduli space supplies us with one such point. Since these cells are the complexified Kähler cones of topologically distinct spaces, the intersection forms associated with these large radius points are different. If the moduli space picture we are presenting in figure 5 is correct, then there must be points in the complex structure moduli space of the mirror whose correlation functions exactly reproduce each and every one of these intersection forms. This is a precise and concrete statement whose veracity would provide a strong verification of the picture presented in figure 5. In the next section we carry out this verification in a particular example.
Figure 5. The conformal field theory moduli space.
5. An Example

In this section we briefly carry out the abstract program discussed in the last few sections in a specific example. We will see that the delicate predictions just discussed can be explicitly verified.

We focus on the Calabi-Yau manifold $M$ given by the vanishing locus of a degree 18 homogeneous polynomial in the weighted projective space $\mathbb{P}^4_{\{6,6,3,2,1\}}$ and its mirror $\tilde{M}$. For the former we can take the polynomial constraint to be

$$z_0^3 + z_1^3 + z_2 + z_3^9 + z_4^{18} + a_0 z_0 z_1 z_2 z_3 z_4 = 0 \quad (5.1)$$

where the $z_i$ are the homogeneous weighted space coordinates and $a_0$ is a large and positive constant (whose value, in fact, is inconsequential to the calculations which follow). The mirror to this family of Calabi-Yau spaces is constructed via the method of [4] by taking an orbifold of $M$ by the maximal scaling symmetry group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

A study of the Kähler structure of $M$ reveals that there are five cells in its (partially) enlarged Kähler moduli space, each corresponding to a sigma model on a smooth topologically distinct Calabi-Yau manifold. In each of these cells there is a large radius point for which instanton corrections are suppressed and hence the correlation functions of (2.3) are just the intersection numbers of the respective Calabi-Yau’s. We have calculated these for each of the five birationally equivalent yet topologically distinct Calabi-Yau spaces and we record the results in table 1. To avoid having to deal with issues associated with normalizing fields in the subsequent discussion, in table 1 we have chosen to list our results in terms of ratios of correlation functions for which such normalizations are irrelevant. (The $D_i$ and $H$ are divisors on $M$, corresponding to elements in $H^1(M, T^*)$ by Poincaré duality.)

| Resolution | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\Delta_4$ | $\Delta_5$ |
|------------|------------|------------|------------|------------|------------|
| $(D_2^2 D_4^3) / (D_2^2 D_3^2 D_4)$ | $-7$       | $0/0$      | $0/0$      | $\infty$  | $9$        |
| $(D_2 D_3 D_4) / (D_2 D_3 D_4)$    | $2$        | $4$        | $0$        | $0/0$      | $0/0$      |
| $(D_2 D_3 D_4) / (H D_2^2 D_3^2)$  | $1$        | $1$        | $1$        | $0$        | $0/0$      |
| $(D_2 D_3 D_4) / (H D_2 D_3)$      | $2$        | $1$        | $\infty$  | $0/0$      | $0$        |

Table 1: Ratios of intersection numbers
Following the discussion of the last section, our goal now is to find five limit points in the complex structure moduli space of $\tilde{M}$ such that appropriate ratios of correlation functions yield the same results as in table 1. To do so, we note that the most general complex structure on $\tilde{M}$ can be written

$$W = z_0^3 + z_1^3 + z_2^6 + z_3^9 + z_4^{18} + a_0 z_0 z_1 z_2 z_3 z_4 + a_1 z_2^3 z_4^9 + a_2 z_3^6 z_4 + a_3 z_3^3 z_4^{12} + a_4 z_2^3 z_3 z_4^3 = 0.$$ (5.2)

We will describe these limit points by parametrizing the complex structure as $M$ for real parameters $s$ and $r_i$ and we send $s$ to infinity. The limit points are therefore distinguished by the rates at which the $a_i$ approach infinity. Our task, therefore, is to find appropriate values for the $r_i$ (if they exist) such that we obtain mirrors to the five large radius Calabi-Yau spaces of the last paragraph. The technique we use to do this is to describe both the complex structure moduli space of $\tilde{M}$ and the enlarged Kähler moduli space of $M$ in terms of toric geometry. This description, at a fundamental level, makes it manifest that these two moduli spaces are isomorphic. We do not have time to present such analysis here — rather, we refer the reader to $[2]$. For the present purpose we note that a direct outcome of this analysis is a prediction for five choices of the vector $(r_0, \ldots, r_4)$ which should yield the desired mirrors. As we have discussed, a sensitive test of these predictions is to calculate the mirror of the ratios of correlation functions in table 1 (using $(2.2)$ and the method of $[21]$) for each of these complex structure limits and see if we get the same answers. We have done this and we show the results in table 2. Note that in the limit $s$ goes to infinity we get precisely the same results. (The $\varphi_i$ are elements of $H^1(\tilde{M}, T).$)

| Resolution | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\Delta_4$ | $\Delta_5$ |
|------------|------------|------------|------------|------------|------------|
| Direction  | $\left(\frac{11}{7}, 1, \frac{4}{7}, \frac{5}{7}, \frac{5}{7}\right)$ | $\left(\frac{3}{7}, \frac{1}{7}, 1, 1, \frac{5}{7}\right)$ | $\left(\frac{13}{7}, \frac{2}{7}, \frac{4}{7}, \frac{4}{7}\right)$ | $\left(\frac{11}{7}, \frac{7}{7}, 1, 1, \frac{3}{7}\right)$ |
| $\langle \varphi_1^3 \rangle$ | $-7 - 181s^{-1} + \ldots$ | $-\frac{2}{3}s^2 - \frac{129}{280} s + \ldots$ | $9 + 289s^{-1} + \ldots$ |
| $\langle \varphi_2^3 \rangle$ | $2 - 5s^{-1} + \ldots$ | $4 - 22s^{-1} + \ldots$ | $0 + 2s^{-1} + \ldots$ |
| $\langle \varphi_3^3 \rangle$ | $1 + \frac{1}{2}s^{-1} + \ldots$ | $1 + \frac{3}{2}s^{-2} + \ldots$ | $1 + 4s^{-1} + \ldots$ |
| $\langle \varphi_4^3 \rangle$ | $2 + 27s^{-1} + \ldots$ | $1 - \frac{1}{2}s^{-1} + \ldots$ | $-2s - 33 + \ldots$ |

Table 2: Asymptotic ratios of 3-point functions

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This, in conjunction with the abstract and general isomorphism we find between the complex structure moduli space of a Calabi-Yau and the enlarged Kähler moduli space of its mirror (using toric geometry), provides us with strong evidence that our understanding of Calabi-Yau conformal field theory moduli space is correct. In particular, as our earlier discussion has emphasized, this implies that the basic operation of deformation by a truly marginal operator (from a spacetime point of view, this corresponds to a slow variation in the vacuum expectation value of a scalar field with an exactly flat potential) can result in a change in the topology of the Calabi-Yau target space. This discontinuous mathematical change, however, is perfectly smooth from the point of view of physics. In fact, using mirror symmetry, such an evolution can be reinterpreted as a smooth, topology preserving, change in the “shape” (complex structure) of the mirror space.

There are two important points we need to mention. First, for ease of discussion we have focused on the case in which the only deformations are those associated with the Kähler structure of $M$ and, correspondingly, only the complex structure of $\tilde{M}$. This may have given the incorrect impression that the topology changing transitions under study can always be reinterpreted in a topology preserving manner in the mirror description. The generic situation, however, is one in which the complex structure and the Kähler structure of $M$ and $\tilde{M}$ both change. Again, from a spacetime point of view this simply corresponds to a slow variation of the expectation values of a set of scalar fields with flat potentials. Under such circumstances, topology change can occur in both the original and the mirror description. Our reasoning will ensure that such changes are physically smooth. Clearly there is no interpretation — the original or the mirror — which can avoid the topology changing character of the processes.

Second, we have used the terms “enlarged” and “partially enlarged” in our discussion of the Kähler moduli space. We now briefly indicate the distinction. The central result of the present work (and that of Witten [1]) is that the proper geometric interpretation of conformal field theory moduli space requires that we augment the previously held notion of a single complexified Kähler cone associated with a single topological type of Calabi-Yau space. In the previous sections we have focused on part of the requisite augmentation: we need to include the complexified Kähler cones of Calabi-Yau spaces related to the original by flops of rational curves (of course, it is arbitrary as to which Calabi-Yau we call the original). These Kähler cones adjoin each other along common walls. The space so created is the partially enlarged Kähler moduli space. It turns out, though, that conformal field theory moduli space requires that even more regions be added. Equivalently, the partially
enlarged Kähler moduli space is only a subregion of the moduli space which is mirror to the complex structure moduli space of the mirror Calabi-Yau manifold. The extra regions which need to be added arise directly from the toric geometric description and were first identified in the two dimensional supersymmetric gauge theory approach of [1]. These regions correspond to the moduli spaces of conformal theories on orbifolds of the original smooth Calabi-Yau, Landau-Ginzburg orbifolds, gauged Landau-Ginzburg theories and hybrids of the above. The union of all of these regions (which also join along common walls) constitutes the enlarged Kähler moduli space. For instance, in the example studied in section V we found that there were five regions in the partially enlarged Kähler moduli space. The enlarged Kähler moduli space, as it turns out, has 100 regions. One of these is a Landau-Ginzburg orbifold region, 27 of these are sigma models on Calabi-Yau orbifolds, and 67 of these are hybrid theories consisting of Landau-Ginzburg models fibered over various compact spaces. It is worthwhile emphasizing that in contrast with previously held notions, orbifold theories are not simply boundary points in the moduli space of smooth Calabi-Yau sigma models but, rather, they have their own regions in the enlarged Kähler moduli space and hence are more on equal footing with the smooth examples.

6. Conclusions

In this talk we have focused on the proper and complete geometric interpretation of points in the moduli spaces of a class of $N=2$ superconformal field theories. We have uncovered a surprisingly rich structure. Previously it was believed that any such moduli space was interpretable in terms of the complex structure and Kähler structure of an associated Calabi-Yau manifold. We now see that this is but a small fragment of the full story. The Kähler moduli space must be augmented to the enlarged Kähler moduli space of which the former is one of many cells adjoined along various common walls. The new cells correspond to nonlinear sigma models on smooth topologically distinct Calabi-Yau’s (related by flops of rational curves) as well Calabi-Yau orbifolds, Landau-Ginzburg theories and hybrid models.

There are a number of implications of this augmented picture. First, we have shown that deformations by truly marginal operators can take us in a physically smooth manner from any region to any other. In particular, this means that the topology of the target space (the universe in a theory of strings) can change with no more exotic physical impact than mere geometric expansion. Second, the enlargement of the moduli space harmoniously
clears up some troubling puzzles in mirror symmetry. More precisely, whereas it proved difficult to understand how the complexified Kähler moduli space of a Calabi-Yau could be isomorphic to the complex structure moduli space of its mirror, there is a manifest isomorphism when the enlarged Kähler moduli space is used. Third, we have seen how we are led to a shift in perspective regarding orbifolds. Rather than being boundary points in moduli space — and hence less than generic — orbifolds occupy their own regions just like the smooth Calabi-Yau manifolds. Fourth and finally, we have mentioned that the enlarged Kähler moduli space generally contains numerous regions whose most natural interpretation is not in terms of nonlinear sigma model field theories. The geometric properties of such models are presently under study.

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