CONSTANT VORTICITY GEOPHYSICAL WAVES WITH
CENTRIPETAL FORCES AND AT ARBITRARY LATITUDE

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Abstract. We consider three-dimensional geophysical flows at arbitrary latitude and with constant vorticity beneath a wave train and above a flat bed in the β-plane approximation with centripetal forces. We consider the f-plane approximation as well as the β-plane approximation. For the f-plane approximation, we prove that there is no bounded solution. For the β-plane approximation, we show that the flow is necessarily irrotational and the free surface is necessarily flat if it exhibits a constant vorticity. Our results reveal some essential differences from those results in the literature, due to the presence of centripetal forces. Moreover, for the case exhibiting the surface tension, we prove that there are no flows exhibiting constant vorticity.

1. Introduction

In this paper, we focus on geophysical ocean waves in which both Coriolis and centripetal effects of the Earth’s rotation play a significant role. In recent years, the mathematical analysis of geophysical flows [22, 37] has attracted much attention for their wide applications (see the references [1, 7, 9, 10, 13, 14, 15, 25, 28, 29, 40] for the flows in the equatorial region and [2, 3] for the flows at arbitrary latitude). However, in most existed results, centripetal forces are typically neglected because they are relatively much smaller than Coriolis forces. Recently Henry in [26] showed in a remarkable way that the relatively small-scale centripetal force plays a central role in facilitating the admission of a wide range of constant underlying currents in studying the exact solution for the equatorially nonlinear waves in the β-plane approximation and with centripetal forces. Later, an explicit three-dimensional nonlinear solution for geophysical waves propagating at arbitrary latitude in the β-plane approximation with centripetal forces was presented in [3].

Compared with large studies on equatorial water waves, the study on the non-equatorial waves seems much fewer. Besides the work [3] mentioned above, an extension of the exact solution [27] for equatorial waves in the f-plane approximation to the cases at arbitrary latitude and in the presence of a constant underlying background current was presented in [29]. A β-plane approximation at arbitrary latitude in the presence of an underlying current and a Gerstner-like solution to this problem was very recently provided in [2].

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Vorticity is adequate in describing the motion of both equatorial and non-equatorial flows. The nonzero vorticity serves as a tool for describing interactions of waves with non-uniform currents. From the history perspective, the mathematical theory of rotational water waves was originally started by Gerstner in the beginning of the 19th century [24], in which an explicit family of periodic travelling waves with non-zero vorticity was constructed using Lagrangian coordinates. In recent works [6, 17, 19, 21, 29, 34, 38], the assumption of nonzero constant vorticity has been assumed, which is the simplest rotational setting and corresponds to a uniform current. Although such an assumption is for physical viewpoints (see the discussion in [31]), the main consideration lies on more convenient in mathematical analysis, for example, constant vorticity flows have the advantage that their velocity field consists of harmonic functions (see the modern discussions in [19, 20]). The importance of the vorticity in the realistic modeling of ocean flows is highlighted in the very recent papers [1, 16, 32, 33]. See [11, 12, 18, 30, 32, 39, 40] and the monograph [5] for more results on rotational water waves.

Among the results on vorticity in the literature, a feature is to determine the dimensionality of the flow. From the mathematical perspective, the study of the two-dimensionality for the rotational flow was started by the work [6], in which Constantin showed that a free surface water flow of constant nonzero vorticity beneath a wave train and above a flat bed must be two-dimensional and the vorticity must have only one nonzero component which points in the horizontal direction orthogonal to the direction of wave propagation. After [6], more results along this line has been obtained in different settings. In the presence of Coriolis forces, Martin in [33] proved the two-dimensionality of the equatorial flows in the \( f \)-plane approximation, and it was found that there is a striking difference between the geophysical flows and the classical gravity flows, that is, the two-dimensionality holds even if the vorticity vector vanishes due to the presence of Coriolis forces. Martin also proved in [36] and [35] that for the equatorial and non-equatorial flows in the \( \beta \)-plane approximation, the only flow exhibiting a constant vorticity vector is the stationary flow with vanishing velocity field and flat surface. Very recently, the authors [4] obtained several results much different from [36], and we show that the equatorial flow is necessarily irrotational, the free surface is necessarily flat, and possess non-vanishing horizontal velocity field if it exhibits a constant vorticity, owing to the presence of centripetal forces.

The aim of this paper is to show that, assuming that the non-equatorial flows admit a constant vorticity vector, the centripetal force can lead to a better outcome, especially compared with the existed results without the centripetal term. Both \( f \)-plane approximation and \( \beta \)-plane approximation are studied. We will extend the results in [4] to the flows at arbitrary latitude. In particular, for the \( f \)-plane approximation, we prove that there is no bounded solution, while for the \( \beta \)-plane approximation, we show that the flow is necessarily irrotational and the free surface is necessarily flat if it exhibits a constant vorticity. Moreover, for the case exhibiting the surface tension, we prove that there are no flows exhibiting constant vorticity.
2. Preliminary

We recall the following governing equations derived by Constantin and Johnson in [2] for geophysical fluid dynamics in the cylindrical coordinates

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu + \frac{v}{R} + wu + \frac{v}{R} + wu + 2\Omega(w \cos \phi - v \sin \phi) &= -\frac{1}{\rho} P_x, \\
\frac{\partial \rho}{\partial t} + uu + \frac{v}{R} + wu + \frac{v}{R} + wu + 2\Omega u \sin \phi + (R + z)\Omega^2 \sin \phi \cos \phi &= -\frac{1}{\rho} \frac{\partial P}{\partial z}, \\
\frac{\partial w}{\partial t} + uw + \frac{w}{R} + wv + wv - \frac{v}{R} + wv - 2\Omega u \cos \phi - (R + z)\Omega^2 \cos^2 \phi &= -\frac{1}{\rho} P_z - g,
\end{align*}
\]

together with the equation of incompressibility

\[
u_x + \frac{1}{R + z} \frac{\partial P}{\partial z}[(R + z)w] = 0.
\]

Here the origin in the cylindrical coordinates is located at the centre of the Earth, \(x\)-axis with the positive \(x\)-direction going from west to east, \(\phi\) is the angle of latitude and \(z = r - R\) is the variation in the locally vertical direction of the radial variable from the Earth’s surface, \((u, v, w)\) is the fluid velocity field, \(P\) is the pressure, \(\rho\) is the water’s density, \(t\) is the time, \(g\) is the standard gravitational acceleration at the Earth’s surface and \(\Omega = 7.29 \times 10^{-5}\) rad/s is the rotational speed of the Earth and \(R = 6378\) km is the radius of the Earth.

The Coriolis parameters, defined by:

\[
f = 2\Omega \sin \phi, \quad \hat{f} = 2\Omega \cos \phi,
\]
depend on the variable latitude \(\phi\). At the Equator \(f = 0, \hat{f} = 2\Omega\). For water waves propagating zonally in a relatively narrow ocean strip less than a few degrees of latitude wide, it is adequate to use the \(f\)- or \(\beta\)-plane approximations. Within the \(f\)-plane approximation the Coriolis parameters are treated as constants, and in terms of the Cartesian coordinate system \((x, y, z)\), we obtain the governing equations

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu + vu + uu + \frac{v}{R} + wu + \frac{v}{R} + wu + 2\Omega w - f w - f v &= -\frac{1}{\rho} P_x, \\
\frac{\partial v}{\partial t} + vu + vv + vv + (f + \beta y)u + \frac{v}{R} + wv + \frac{v}{R} + wv + 2\Omega w &= -\frac{1}{\rho} P_y, \\
\frac{\partial w}{\partial t} + uw + vu + vv + wv + wv - \frac{v}{R} + wv - f u - f^2 R &= -\frac{1}{\rho} P_z - g.
\end{align*}
\]

Within the \(\beta\)-plane approximation, we consider that, at the fixed latitude \(\phi\), \(\hat{f}\) is constant and \(f\) has a linear variation with the latitude. Defining \(y = R\alpha\) and retaining only terms of linear order in the expansion of \(\sin(\phi + \alpha)\), this linear variation has the form \(\hat{f} + \beta y\), with

\[
\beta = \frac{\hat{f}}{R} = \frac{2\Omega \cos \phi}{R}.
\]

Thus we get the following \(\beta\)-plane approximation equations for geophysical fluid dynamics with centripetal terms:

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu + vu + vu + wu + \hat{f} w - (f + \beta y)v &= -\frac{1}{\rho} P_x, \\
\frac{\partial v}{\partial t} + vu + vv + vv + (f + \beta y)u + \frac{v}{R} + wv + \frac{v}{R} + wv + 2\Omega w &= -\frac{1}{\rho} P_y, \\
\frac{\partial w}{\partial t} + uw + vu + vv + wv - \hat{f} u - \hat{f}^2 R &= -\frac{1}{\rho} P_z - g.
\end{align*}
\]

In both cases, we have the condition of incompressibility

\[
u_x + v_y + w_z = 0.
\]

We will consider regular wave trains of water waves propagating steadily in the direction of the horizontal \(x\)-axis, \(L\)-periodic in the variable \(x\), and presents no
variation in the $y$-direction. The fluid domain is bounded below by the impermeable flat bed $z = -d$, and above by the free surface $z = \eta(x - ct)$, where $\eta$ gives the wave profile with the zero mean $\int_0^L \eta(s) ds = 0$ and $c > 0$ is the wave speed. We assume that the wave crest is located at $x = 0$, and thus obviously we know $\eta(0) > 0$.

Complementing the equations of motion are the boundary conditions, of which

\begin{equation}
P = P_{atm} \quad \text{on} \quad z = \eta(x - ct),
\end{equation}

with $P_{atm}$ being the constant atmospheric pressure, decouples the motion of the water from that of the air. In addition to (2.4), we have the kinematic boundary conditions

\begin{equation}
w = (u - c)\eta_x \quad \text{on} \quad z = \eta(x - ct),
\end{equation}

and

\begin{equation}
w = 0 \quad \text{on} \quad z = -d.
\end{equation}

In the presence of surface tension, (2.4) is replaced by

\begin{equation}
P = P_{atm} - \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \quad \text{on} \quad z = \eta(x - ct),
\end{equation}

where the constant $\sigma > 0$ is the surface tension coefficient, and we assume that $\eta \in C^2(\mathbb{R}^2)$ in (2.7).

The vorticity vector $\Upsilon$ is defined as the curl of the velocity field $\mathbf{u} = (u, v, w)$:

\begin{equation}
\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3) = (w_y - v_z, u_z - w_x, v_x - u_y).
\end{equation}

In this paper, we assume that the vorticity vector is constant and satisfies

\begin{equation}
\Upsilon_2 + \hat{f} \neq 0, \quad \Upsilon_3 + f \neq 0,
\end{equation}

which are reasonable since the magnitude of the equatorial undercurrent’s relative vorticity is much larger than that of the planetary vorticity (see the discussions in [8]).

3. $f$-plane approximation

In this Section, we consider the $f$-plane approximation, which corresponds to the governing equations (2.1) with the conditions (2.3)-(2.6). The main result of this Section reads as follows.

**Theorem 3.1.** Assume that the vorticity vector $\Upsilon$ is constant and satisfies (2.9). Then there is no bounded solution to the equations (2.1) with (2.3)-(2.6).

**Proof.** It is easy to verify that the constant vorticity vector $\Upsilon$ satisfies the equation

\begin{equation}
(\Upsilon \cdot \nabla) \mathbf{u} + \hat{f}(u_y, v_y, w_y) + f(u_z, v_z, w_z) = 0,
\end{equation}

which is equivalent to the following three equalities

\begin{equation}
\Upsilon_1 u_x + (\Upsilon_2 + \hat{f}) u_y + (\Upsilon_3 + f) u_z = 0,
\end{equation}

\begin{equation}
\Upsilon_1 v_x + (\Upsilon_2 + \hat{f}) v_y + (\Upsilon_3 + f) v_z = 0,
\end{equation}

\begin{equation}
\Upsilon_1 w_x + (\Upsilon_2 + \hat{f}) w_y + (\Upsilon_3 + f) w_z = 0.
\end{equation}

From (3.3), we know that $w$ is constant in the direction of the vector $(\Upsilon_1, \Upsilon_2 + \hat{f}, \Upsilon_3 + f)$, which is not parallel to the flat bed $z = -d$ due to the condition
\( \Upsilon_3 + f \neq 0 \). Using the kinematic boundary condition (2.6), we obtain that \( w = 0 \) throughout the fluid domain. Thus, we obtain from (2.8) that
\[
 u_z = \Upsilon_2 \quad \text{and} \quad v_z = -\Upsilon_1.
\]
From the above relations, we can infer that there exist two functions \( \hat{u} = \hat{u}(x, y, t) \), \( \hat{v} = \hat{v}(x, y, t) \) such that
\[
 u(x, y, z, t) = \hat{u}(x, y, t) + \Upsilon_2 z,
\]
\[
 v(x, y, z, t) = \hat{v}(x, y, t) - \Upsilon_1 z,
\]
for all \( x, y, z, t \) with \(-d \leq z \leq \eta(x - ct)\). Due to (2.3), the functions \( \hat{u} \) and \( \hat{v} \) satisfy the equation
\[
 \hat{u}_x + \hat{v}_y = 0,
\]
which admits us to choose a function \( \psi = \psi(x, y, t) \) satisfying
\[
 (3.6) \quad \hat{u} = \psi_y \quad \text{and} \quad \hat{v} = -\psi_x.
\]
Consequently, from the equations (3.1)-(3.2), we deduce that
\[
 (3.7) \quad \begin{cases}
 \Upsilon_1 \psi_{xy} + (\Upsilon_2 + \hat{f})\psi_{yy} + (\Upsilon_3 + f)\Upsilon_2 = 0, \\
 \Upsilon_1 \psi_{xx} + (\Upsilon_2 + \hat{f})\psi_{xy} + (\Upsilon_3 + f)\Upsilon_1 = 0.
\end{cases}
\]
We also obtain from the definition of \( \Upsilon_3 \) that
\[
 (3.8) \quad \psi_{xx} + \psi_{yy} = -\Upsilon_3.
\]
Using the relations (3.7) and (3.8), we have
\[
 \psi_{xx} = \frac{\hat{\Upsilon}_2(f\Upsilon_2 + \hat{f}\Upsilon_3) - \Upsilon_1^2\Upsilon_3 - f\Upsilon_1^2}{\Upsilon_1^2 + \Upsilon_2^2} := A,
\]
\[
 \psi_{xy} = \frac{-\Upsilon_1(\hat{\Upsilon}_2\hat{\Upsilon}_3 + f\hat{\Upsilon}_2)}{\Upsilon_1^2 + \Upsilon_2^2} := B,
\]
\[
 \psi_{yy} = \frac{f\Upsilon_1^2 - \hat{\Upsilon}_2\hat{\Upsilon}_3\Upsilon_2}{\Upsilon_1^2 + \Upsilon_2^2} := C.
\]
where
\[
 \hat{\Upsilon}_2 = \Upsilon_2 + \hat{f}, \quad \hat{\Upsilon}_3 = \Upsilon_3 + f.
\]
Therefore, there exist functions \( d(t), e(t), g(t) \) such that
\[
 \psi(x, y, t) = \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2 + d(t)x + e(t)y + g(t).
\]
By (3.9), we find that
\[
 \hat{u}(x, y, t) = Bx + Cy + e(t),
\]
\[
 \hat{v}(x, y, t) = -Ax - By - d(t).
\]
Since the functions \( \hat{u} \) and \( \hat{v} \) are bounded, we can infer that
\[
 A = B = C = 0.
\]
Now, we claim that \( \Upsilon_1 = 0 \). On the contrary, we assume that \( \Upsilon_1 \neq 0 \). Since \( B = 0 \), we conclude that
\[
 \Upsilon_2\hat{\Upsilon}_3 + f\hat{\Upsilon}_2 = 0,
\]
and thus
\[
 \Upsilon_2(\Upsilon_2\hat{\Upsilon}_3 + f\hat{\Upsilon}_2) = 0.
\]
Using the fact $C = 0$, the above equation becomes

$$f(\Upsilon_1^2 + \Upsilon_2^2) = 0,$$

which is impossible. Therefore $\Upsilon_1 = 0$.

Since $C = 0$, we can infer that $\Upsilon_2 \Upsilon_3 \Upsilon_2 = 0$, owing to (2.3), we can conclude that $\Upsilon_2 = 0$. Moreover, we can obtain from $A = 0$ that $f\Upsilon_2 \Upsilon_3 = 0$. Because $\Upsilon_2 \neq 0$ and $\hat{f} \neq 0$, we derive that $\Upsilon_3 = 0$.

From (3.3) and (3.5), we obtain that

$$\hat{\Upsilon} \neq 0,$$

which means that $u, v$ are only dependent of $t$. Moreover, from (2.1), we obtain

$$\begin{cases}
    P_x = -\rho[u'(t) - f v(t)], \\
    P_y = -\rho[v'(t) + fu(t) + \frac{f^2}{8} R], \\
    P_z = \rho[f u(t) + \frac{f^2}{4} R - g].
\end{cases}$$

Therefore, the pressure can be given as

$$P(x, y, z, t) = -\rho[u'(t) - f v(t)]x - \rho \left[ (v'(t) + fu(t) + \frac{f^2}{4} R) y + \frac{f^2}{8} y^2 \right]$$

$$+ \rho \left[ f u(t) + \frac{f^2}{4} R - g \right] z + \tilde{p}(t).$$

Now the kinematic boundary condition (2.4) becomes

$$P_{atm} = -\rho[u'(t) - f v(t)]x - \rho \left[ (v'(t) + fu(t) + \frac{f^2}{4} R) y + \frac{f^2}{8} y^2 \right]$$

$$+ \rho \left[ f u(t) + \frac{f^2}{4} R - g \right] \eta(x - ct) + \tilde{p}(t),$$

for all $x, y, t$. We infer from the above equation that the coefficient of $y$ must vanish, which is impossible. Therefore, we conclude that there is no solution to the equations (2.1) with (2.3) - (2.6).

\[\square\]

4. $\beta$-PLANE APPROXIMATION

In this section, we consider the $\beta$-plane approximation, which corresponds to the equations (2.2) - (2.3) with the conditions (2.4) - (2.6). Using (2.2) and (2.3), the vorticity equation becomes

$$\Upsilon_t + (\mathbf{u} \cdot \nabla) \Upsilon - \hat{f}(u_y, v_y, w_y) - (f + \beta y)(u_z, v_z, w_z) + \beta(0, 0, v) = (\Upsilon \cdot \nabla) \mathbf{u}.$$  

For the constant vorticity, we can obtain

$$(\Upsilon \cdot \nabla) \mathbf{u} + \hat{f}(u_y, v_y, w_y) + (f + \beta y)(u_z, v_z, w_z) - \beta(0, 0, v) = 0,$$

which is equivalent to the following equalities

$$\begin{align*}
(4.1) \quad \Upsilon_1 u_x + (\Upsilon_2 + \hat{f}) u_y + (\Upsilon_3 + f + \beta y) u_z &= 0, \\
(4.2) \quad \Upsilon_1 v_x + (\Upsilon_2 + \hat{f}) v_y + (\Upsilon_3 + f + \beta y) v_z &= 0, \\
(4.3) \quad \Upsilon_1 w_x + (\Upsilon_2 + \hat{f}) w_y + (\Upsilon_3 + f + \beta y) w_z - \beta v &= 0.
\end{align*}$$
Theorem 4.1. There is no water flow exhibiting non-zero constant vorticity vector and with a flat surface. Indeed, any flow with a flat surface and constant vorticity vector must have the vanishing vorticity vector, that is $\Upsilon = (0, 0, 0)$.

Proof. We can obtain from (2.8) and the equation (2.3) that
\[ \Delta w = w_{xx} + w_{yy} + w_{zz} = u_{xx} + v_{yy} + w_{zz} = (u_x + v_y + w_z)z = 0. \]
Analogously, $\Delta u = \Delta v = 0$.
Therefore, the velocity components $u, v, w$ are harmonic function within the fluid domain. Moreover, it is obvious that all partial derivatives of $u, v, w$ are harmonic functions. Then it follows from (4.1) that
\[ \Delta(yu_z) = 0, \]
which can be written as
\[ y\Delta(u_z) + 2u_{zy} = 0, \]
from which we obtain that $u_{zy} = 0$.
Similarly, using the equations (4.2) and (4.3), we can infer that $v_{zy} = 0$ and $w_{zy} = 0$.
Then, by the definitions of $\Upsilon_1$ and $\Upsilon_2$, we have
\[ w_{yy} = v_{zy} = 0 \quad \text{and} \quad w_{xy} = u_{zy} = 0. \]
Using the above relations, we conclude that $w_y = f(t)$ for some function $f$, combined with the kinematic boundary condition (2.6), we have
\[ w_y = 0 \quad \text{on} \quad z = -d. \]
Therefore, we conclude that $w_y \equiv 0$,
which implies that $v_z = -\Upsilon_1$.
Differentiating with respect to $y$ in (4.3), we obtain
\[ \Upsilon_1 w_{xy} + (\Upsilon_2 + \hat{f})w_{yy} + (\Upsilon_3 + f + \beta y)w_{zy} + \beta w_z - \beta v_y = 0. \]
Since $w_y \equiv 0$, we can infer that
\[ w_z = v_y, \]
from which we have
\[ w_{zz} = v_{yz} = (-\Upsilon_1)_y = 0, \]
and
\[ v_{yy} = w_{zy} = 0. \]
Moreover, due to $v_{zz} = (-\Upsilon_1)_z \equiv 0$ and $\Delta v = \Delta w = 0$, we conclude that $v_{xx} \equiv 0$ and $w_{xx} \equiv 0$.
Differentiating with respect to $z$ in the equation of mass conservation (2.3) we get
\[ u_{xx} + v_{yz} + w_{zz} = 0. \]
Due to (4.3), we get
\[ u_{xz} = 0. \]
Analogously, differentiating with respect to \( y \) in (2.3), we can obtain that
\[
 u_{xy} = 0.
\]
Now, let us differentiate with respect to \( z \) in the equation (4.1), we have
\[
 \Upsilon_1 u_{xz} + (\Upsilon_2 + \hat{f}) u_{yz} + (\Upsilon_3 + f + \beta y) u_{zz} = 0,
\]
which becomes
\[
 (\Upsilon_3 + f + \beta y) u_{zz} = 0,
\]
since \( u_{xz} = u_{yz} = 0 \). Thus \( u_{zz}(x,y,z) = 0 \) for all \((x,y,z)\) with \( y \neq -\frac{\Upsilon_3 + f}{\beta} \). From the continuity of \( u_{zz} \), we have
\[
 u_{zz} = 0 \quad \text{within the fluid domain.}
\]
Recalling that \( \Upsilon_2 = u_z - w_x \), it is easy to obtain that
\[
 w_{xz} = 0.
\]
By the fact that \( w_{xx} = w_{xy} = 0 \), we conclude that \( w_x = a(t) \) for some function \( a \).
Since \( w_x = 0 \) on the flat bed \( z = -d \), we obtain
\[
 w_x = 0 \quad \text{within the fluid domain,}
\]
which implies that \( w_{xx} = 0 \) and \( \Upsilon_2 = u_z \). Moreover, by the fact \( w_{xy} = w_{zz} = 0 \), we conclude that
\[
 w_z \text{ is constant within the fluid domain.}
\]
Differentiating the equation (4.2) with respect to \( x \), we have
\[
 \Upsilon_1 v_{xx} + (\Upsilon_2 + \hat{f}) v_{yx} + (\Upsilon_3 + f + \beta y) v_{zx} = 0.
\]
Since \( v_{xx} = (-\Upsilon_1)_x = 0 \), \( v_{yx} = 0 \), we get
\[
 (\Upsilon_2 + \hat{f}) v_{yx} = 0.
\]
Using the assumption \( \Upsilon_2 + \hat{f} \neq 0 \), we deduce that
\[
 v_{xy} = 0 \quad \text{within the fluid,}
\]
which, by the expression of \( \Upsilon_3 = v_x - u_y \), implies that
\[
 u_{yy} = 0 \quad \text{within the fluid.}
\]
Using the previous relations \( u_{xy} = u_{yy} = v_{xy} = v_{yy} = 0 \), we can obtain from the equations (4.1) and (4.2) that
\[
 u_z = v_z = 0 \quad \text{throughout the flow,}
\]
which yields that
\[
 \Upsilon_1 = \Upsilon_2 = 0.
\]
Thus, equations (4.1) and (4.2) become
\[
 \hat{f} u_y = 0, \quad \text{and} \quad \hat{f} v_y = 0,
\]
which allow us to conclude that
\[
 u_y = v_y = 0.
\]
Notice that (4.3) holds, so we have
\[
 w_z = 0 \quad \text{within the fluid.}
\]
Due to \( \Upsilon_1 = \Upsilon_2 = 0 \) and \( w_x = w_y = 0 \), the equation (4.3) can be simplified to
\[
 (\Upsilon_3 + f + \beta y) w_z = \beta v.
\]
By (4.8), we obtain
\[ v = 0 \] within the fluid.

Therefore, combined with (4.7) we have
\[ \Upsilon_3 = v_x - u_y = 0. \]

Now the proof is finished. \( \square \)

**Theorem 4.2.** Assume that the vorticity vector \( \Upsilon \) is constant and satisfies \( \Upsilon_2 + \hat{f} \neq 0 \). Then the only bounded solution to the equations (2.2)-(2.3) with the conditions (2.4)-(2.6) is the one with flat surface, velocity field and the pressure given as
\[ (u, v, w) = \left( -\frac{\hat{f}^2}{4\beta}, 0, 0 \right), \]
\[ P(x, y, z, t) = \rho \left[ -\frac{\hat{f}^3}{4\beta} + \frac{\hat{f}^2}{4} R - g \right] (z - \eta_0) + P_{atm}, \]
where \( \eta_0 \) is a constant.

**Proof.** From the proof of Theorem 4.1, we can deduce that \( w = 0 \) within the fluid domain, since \( w_z = 0 \) and \( w = 0 \) on the flat bed \( z = -d \). In addition, we have shown that \( v = 0 \). Thus we only need to find the horizontally velocity \( u \) for the velocity field.

From the equation (2.3) and the fact \( v_y = w_z = 0 \), we obtain that \( u_x = 0 \). Going back to (4.6) and (4.7), we conclude that
\[ u(x, y, z, t) = b(t) \]
for some function \( b \).

Note that \( (u, v, w) = (b(t), 0, 0) \), the Euler equations (2.2) become
\[ \begin{align*}
P_x &= -\rho b'(t), \\
P_y &= -\rho \left[ (f + \beta y)b(t) + \frac{\hat{f}^2}{4} y + \frac{\hat{f}^2}{4} R \right], \\
P_z &= \rho \left[ \hat{f} b(t) + \frac{\hat{f}^2}{4} R - g \right].
\end{align*} \]

Therefore, the pressure can be given as
\[ P(x, y, z, t) = -\rho b'(t) x - \rho \left[ \frac{\beta b(t)}{2} y^2 + \frac{\hat{f}^2}{8} y^2 + \frac{\hat{f}^2}{4} R y \right] + \rho \left[ \hat{f} b(t) + \frac{\hat{f}^2}{4} R - g \right] z + p(t). \]

Now the kinematic boundary condition (2.4) becomes
\[ P_{atm} + \rho \left[ \frac{\beta b(t)}{2} y^2 + \frac{\hat{f}^2}{8} y^2 + \frac{\hat{f}^2}{4} R y \right] = -\rho b'(t) x + \rho \left[ \hat{f} b(t) + \frac{\hat{f}^2}{4} R - g \right] \eta(x - ct) + p(t), \]
for all \( x, y, t \). We infer from the above equation that the coefficient of \( y \) must vanish, which means that \( b(t) = -\frac{\hat{f}^2}{4\beta} \). Now the equality (4.11) simplifies to
\[ P_{atm} = \rho \left[ -\frac{\hat{f}^3}{4\beta} + \frac{\hat{f}^2}{4} R - g \right] \eta(x - ct) + p(t) \text{ for all } x, t, \]
which is only possible if both functions \( p \) and \( \eta \) are constants \( p_0, \eta_0 \). Thus the pressure function can be given as the form (4.9). \( \square \)
Remark 4.3. The above result is also true for the case that the fluid domain bounded below by the flat bed $z = -d$ and above by the free surface $z = \eta(x, y, t)$ (not the wave trains). In fact, the velocity field, the pressure and the free surface given by

$$
\begin{align*}
(u, v, w) &= \left( -\frac{f^2}{2\hat{\nu}}, 0, 0 \right), \\
P(x, y, z, t) &= \rho \left[ -\frac{f^2}{4\hat{\nu}} + \frac{f^2}{4} R - g \right] (z - \bar{\eta}_0) + P_{atm}, \\
\eta(x, y, t) &= \bar{\eta}_0,
\end{align*}
$$

(where $\bar{\eta}_0$ is a constant) is the only solution satisfying the equations (2.2)-(2.3) with the boundary conditions

$$
P = P_{atm} \quad \text{on} \quad z = \eta(x, y, t),
$$

$$
w = \eta_t + uu_x + vv_y + wu_z + \hat{f}w - (f + \beta y)v = -\frac{1}{\rho}P_x, \\
v = \eta_t + uu_x + vv_y + wv_z + (f + \beta y)u = -\frac{1}{\rho}P_y, \\
w = \eta_t + uu_x + vv_y + ww_z - \hat{f}u = -\frac{1}{\rho}P_z - g,
$$

together with the conditions (2.5)-(2.7) and (2.13) - (2.16).

Finally in this section, we will prove a result for capillary-gravity waves, which correspond to the equations (2.2) - (2.3) and the boundary conditions (2.5) - (2.7).

Theorem 4.5. Assume that the vorticity vector is constant and also $\Upsilon_2 + \hat{\beta} \neq 0$. Then there is no bounded solution to the equations (2.2)-(2.3) with (2.5)-(2.7).

Proof. Note that (4.10) in Theorem 4.2 still holds here. Thus, the pressure should be given as

$$
P(x, y, z, t) = -\rho b'(t)x - \rho \left[ f b(t)y + \frac{\beta b(t)}{2} y^2 + \frac{\hat{f}}{8} y^2 + \frac{\hat{f}}{4} R y \right]
$$

Using the condition (2.7), we obtain that

$$
P_{atm} - p(t) = -\rho b'(t)x - \rho \left[ f b(t)y + \frac{\beta b(t)}{2} y^2 + \frac{\hat{f}}{8} y^2 + \frac{\hat{f}}{4} R y \right]
$$

$$
+ \rho \left[ \hat{f} b(t) + \frac{f^2}{4} R - g \right] \eta(x - ct) + \sigma \frac{\eta x}{(1 + \eta^2)^{3/2}}.
$$
Therefore, we conclude that
\[ \frac{\beta b(t)}{2} + \frac{\hat{f}^2}{8} = 0, \quad f b(t) + \frac{\hat{f} f}{4} R = 0. \]
Then
\[ b(t) = -\hat{f}^2 / (4 \beta) \]
and
\[ P_{\text{atm}} - p(t) = \rho \left[ -\frac{\hat{f}^3}{4 \beta} + \frac{\hat{f}^2}{4} R - g \right] \eta(x - ct) + \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \]
for all \( x, t \). Notice that the function
\[ x \to \frac{\eta_x(x)}{\sqrt{1 + \eta_x^2(x)}} \]
is periodic and \( \int_0^L \eta(s) ds = 0 \), then we obtain upon integration from 0 to \( L \) in (4.12) that
\[ \rho \left[ -\frac{\hat{f}^3}{4 \beta} + \frac{\hat{f}^2}{4} R + g \right] \eta(x) = \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \quad \text{for all } x. \]
Due to
\[ -\frac{\hat{f}^3}{4 \beta} + \frac{\hat{f}^2}{4} R + g > 0 \]
and \( \eta(0) > 0 \) implies (due to the continuity of \( \eta \)) that \( \eta(x) > 0 \) in a neighborhood \( B_r(0) \) of \( x = 0 \), we deduce from (4.13) that \( \eta_{xx} > 0 \) in \( B_r(0) \), which yields that the function \( \eta \) is convex in \( B_r(0) \), this contradicts the maximality of \( \eta \) at the crest. Therefore, there is no bounded solutions to (2.2)-(2.3) with (2.5)-(2.7). \( \square \)

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