We propose a framework, named the postselected inflation framework, to obtain converging outer approximations of the sets of probability distributions that are compatible with classical multi-network scenarios. Here, a network is a bilayer directed acyclic graph with a layer of sources of classical randomness, a layer of agents, and edges specifying the connectivity between the agents and the sources. A multi-network scenario is a list of such networks, together with a specification of subsets of agents using the same strategy. We furthermore show that the postselected inflation framework is mathematically equivalent to the standard inflation framework: in that respect, our results allow to gain further insights into the convergence proof of the inflation hierarchy of Navascués and Wolfe, and extend it to the case of multi-network scenarios.
1 Introduction

Causal compatibility. The problem of causal compatibility is the problem of deciding whether a given probability distribution can arise from a given causal structure. The study of causal compatibility in quantum mechanics can be traced back to Bell’s theorem [Bel64]. In modern language, this result can be understood as a proof that there exist outcome distributions compatible with a certain quantum causal model (Alice and Bob sharing an entangled quantum state and having classical inputs) yet incompatible with the corresponding classical causal model (where the entangled quantum state is replaced with shared classical randomness). This result, apart from its fundamental implications for possible theories of nature, turns out to be crucial for quantum cryptography [Eke91, PAB+20].

Beyond the setting of Bell’s theorem, there are a number of reasons to be interested in causal compatibility in greater generality [TPKLR21]. Here, we are concerned with causal structures called networks that feature a number of independent, unobserved parameters, which we refer to as “sources”, that may influence a number of observed parameters, which we refer to as the outcomes of “agents”: this naming convention reflects the quantum information mindset where the observed parameters would be the outcomes of measurements that human agents would carry out in a lab. Such networks come in different flavors depending on the type of physical systems that implement the unobserved parameters: these are typically taken to be classical (e.g., bit strings sent out to the agents), quantum (featuring in particular quantum systems whose states may be entangled with respect to the different agent labs they are sent out to) or merely non-signaling (i.e., physical systems that are less constrained than quantum systems, whose internal description is not specified, but that nonetheless verify certain non-signaling conditions). Some important results from the perspective of quantum information theory are the demonstration of non-locality without inputs [RBB+19], of full network non-locality (where all sources have to be non-classical) [PKGT22], as well as the necessary existence of $N$-partite “entanglement” in any non-signaling theory of nature [CRWR21].

In this work, we wish to investigate the structure of achievable outcome distributions in classical multi-network scenarios. These can be understood as networks of classical observers sharing maximally entropic classical sources of randomness in a specific arrangement, allowing for some observers to be using the same strategy (i.e., responding identically to the same set of inputs). Accounting for such same-strategy constraints can be seen, from an operational standpoint, as a “markovianity” constraint on the agents. For instance, if the agents are in fact memory-less black boxes that can be plugged in various positions of the network, then such same-strategy constraints would arise. In fact, a distribution that is causally incompatible with a given same-strategy multi-network scenario but is causally compatible with the corresponding “any-strategy” multi-network scenario is a distribution in which these memory effects are non-trivial. Alternatively, the same-strategy constraints can be desirable in applications: this may be the case of a network in which each of many agents choose among two strategies depending on the correlations they wish to achieve with the rest of the network. While we focus on the theory behind classical networks, such same-strategy constraints were already investigated in the non-signaling case [BG21].

Methods. The problem of causal compatibility is typically hard to solve analytically. If one is interested in inner approximations of the set of achievable distributions of a given causal structure, then one can resort to an analytical or numerical sampling of the underlying search space — namely, the space of all agent strategies and all source behaviors within a given theory. A notably efficient tool in this direction is the neural network oracle for classical causal compatibility introduced in [KCC+20].

In order to obtain guarantees that a fair amount of the search space has been sampled, it is crucial to have access to tractable outer approximations of the set of outcome distributions. A useful review for that purpose is that of [TPKLR21]. One basic analytical tool is Finner’s inequality [RWB+19], that provides simple bounds on achievable correlations in non-signaling networks. In the case of classical networks, one can analytically certify the infeasibility of certain distributions based on so-called rigidity arguments [RB20]. A general possibility to further study classical, quantum or non-signaling networks is to use entropy-based outer approximations of the feasible correlations [CMG15, CB16, WC17]. Additionally, for quantum or classical networks, one may also use the semidefinite-programming-based relaxation of [PKRR+19], that is based on building a positive semidefinite correlation matrix aug-
mented with scalar operators that enable the incorporation of conditional independence relations, if the network at hand features such conditional independences.

The only method known to date of generating outer approximations that converge to the actual set of feasible correlations in classical networks is the technique of inflation. Inflation is a general technique that can come in three flavors for classical, quantum and non-signaling networks. The general idea is simple: given a certain network and an outcome distribution for the agents of this network, one could in principle have access to several copies of sources and agents, wire them in various ways, and then obtain a sensible probability distribution that should verify certain compatibility conditions with respect to the original outcome distribution. In the classical case, the inflation technique — which we will call “fanout inflation” — was originally introduced in [WSF19], and later proven to converge asymptotically in [NW20]. The case of quantum and non-signaling inflation is different to the classical case: while classical information can be freely cloned, this does not hold for quantum and non-signaling information. Hence, in classical inflation, one makes use of “fanout” inflation graphs with explicit cloning of classical information, while in the quantum and non-signaling cases, the inflation graphs are in that sense “non-fanout”. A description of quantum inflation can be found in [WPKG+21]. Some recent developments regarding the potential convergence of quantum inflation can be found in [LGG21], but there remains a “rank constraint” loophole to be addressed. The case of non-signaling inflation was discussed initially in [WSF19], and then further developed in e.g. [GBC+20, CRWR21].

From a practical perspective, inflation is typically handled as a linear program in the classical and non-signaling case, while it takes the form of a semidefinite program in the quantum case.

Objectives. The proof of convergence of the classical inflation [NW20] is a rather surprising result that deserves some attention. The primary objective of the present manuscript is to gain additional insights as to how the proof works. This desire eventually yielded the postselected inflation formulation, which can be understood as an equivalent formulation of fanout inflation: the equivalence holds for the outer approximations that these two schemes can generate, but also in terms of the linear programs that one would solve in either formulation. Interestingly, in the postselected inflation formulation, the convergence of the outer approximations is rather straightforward. On the other hand, the fact that the postselected inflation scheme yields outer approximations of the relevant set of outcome distributions is non-trivial. The situation is the opposite with the fanout inflation formulation, which clearly yields outer approximation, but whose convergence is rather hidden. In that sense, we strongly encourage the interested reader to gain familiarity with both formulations, as they complement each other with respect to the intuition that one gains from knowing about them. A possible structure for this manuscript would have been to start introducing fanout inflation, and then work our way towards the postselected inflation formalism. Instead, we choose to temporarily pretend, for the sake of pedagogy, that fanout inflation does not exist, and motivate and prove the soundness and convergence of the postselected inflation formalism from the bottom up. This brings the opportunity to prove the convergence of certain inflation hierarchies in the contexts of classical multi-network correlations involving same-strategy constraints. At last, we will show explicitly some working examples of the correspondence between fanout inflation and postselected inflation.

Outline. The Correlated Sleeper example is introduced first in section 2 as a motivation to the kind of problems that the inflation framework can solve. We then introduce the relevant tensor network notation in section 3. The multi-network scenarios and causal compatibility problem that we will consider are introduced in section 4. The next section 5 is there to motivate the postselected inflation framework, and to intuitively show that the resulting scheme is convergent. We collect certain basic formal results regarding the postselected inflation scheme in section 6 — namely, that the scheme generates outer approximations that are increasingly tight and that eventually converge. We apply the postselected inflation formalism to the Correlated Sleeper in section 7 as a concrete example. The correspondence between postselected and fanout inflation is made explicit in section 8.
2 The Correlated Sleeper

Let us introduce the Correlated Sleeper task: in the rest of the article, we will use it as the main example to which we will apply our framework. This task involves an agent $A$ that will be subject to two rounds of interrogation. In each of these rounds, $A$ has to output a number, say, either 1 or 2. If the outputs in the first and second round are equal, $A$ wins a prize, and $A$’s objective is to maximize her average probability of winning.

To succeed in this task, $A$ has access to two inputs, referred to as her left and right inputs. These inputs each provide $A$ with a real number between 0 and 1. One of the two inputs will contain the same number (drawn uniformly at random between 0 and 1) across the two rounds — this is the “faithful” input. The other input will contain two independently drawn numbers during the two rounds. However, $A$ does not know which of her two inputs is the faithful one: this is decided according to the toss of a fair coin to which $A$ does not have access. The situation is summarized in figure 1. On top of her inputs, $A$ may also use some local randomness (for instance, she may flip a coin to decide which input to trust).

![Figure 1](image.png)

Figure 1: In the Correlated Sleeper protocol, effectively, the agent $A$ has a uniform prior over the above two networks. By assumption, the agent $A$ has one unique strategy independent of her location in the networks. Each node denotes an independent source of randomness that generates a uniform number in [0,1]; the edges specify to which source each instance of the agent $A$ has access; and the bottom-left and bottom-right inputs of the $A$ node are distinguishable by $A$.

Now, there are some restrictions at play. The first restriction is that the agent $A$ will have no memories of the first round during the second round, and there is no information available to $A$ that allows her to distinguish between the two rounds. This is in analogy with the setting of the Sleeping Beauty paradox [Elg00], in which the Sleeping Beauty is woken up multiple times without knowing how many times she has been woken up earlier on. This justifies the fact that $A$ has to use the same strategy during the two rounds. This strategy is most generally captured by the set of conditional probabilities $p_A(a|\alpha,\beta)$ of $A$ giving the outcome $a$ upon seeing that her left (resp. right) input number is $\alpha$ (resp. $\beta$). Alternatively, and perhaps more realistically, the agent $A$ can be thought of as a computer that has been programmed to use the strategy $p_A$ and is rebooted between the two rounds. The second restriction is a marginal constraint on the strategy that $A$ may use: on average over the first round, her outcome distribution must be uniform over the two values 1 and 2. This is expressed as: for all $a \in \{1,2\}$,\footnote{We will typically leave the integration domains implicit in this work. Here, the integral runs over $[0,1]^2$.}

$$\int d\alpha d\beta \ p_A(a|\alpha,\beta) = \frac{1}{2}. \quad (1)$$

The question is then: what is the maximal probability $p^*$, optimized over the strategy $p_A$, of $A$ winning the game? For instance, a viable strategy could be the following: choose

$$p_A(a|\alpha,\beta) = \begin{cases} \delta_{a1}, & \text{if } \alpha < 0.5, \\ \delta_{a2}, & \text{if } \alpha \geq 0.5, \end{cases} \quad (2)$$

corresponding to $A$ only looking at her left input $\alpha$. The single-round output of $A$ is indeed uniform over $\{1,2\}$. Then, if we are in the case where the left input is the faithful one, it will hold that (denoting with $a_i$ the output of $A$ in the round $i \in \{1,2\}$) $a_1 = a_2$ with certainty. In the case where the right input is the faithful one, then $A$ is effectively choosing $a_1$ and $a_2$ independently and uniformly over...
{1, 2}, so that \( a_1 = a_2 \) occurs with probability \( 1/2 \). On average, the probability of success is thus of \( 3/4 \), which means that \( p^* \) is at least \( 3/4 \).

Let us denote the outcome distribution of \( A \) when the left (resp. right) input was the faithful one as \( p_1 \) (resp. \( p_2 \)) (a probability distribution over the set \( \{1, 2\} \times \{1, 2\} \)). We can thus write \( p^* \) as the following optimization problem:

\[
p^* := \sup_{p_A} \frac{1}{2} \sum_{a \in \{1, 2\}} (p_1(a, a) + p_2(a, a)),
\]

s.t. \( \forall \alpha, \beta \in [0, 1], \forall a, a_1, a_2 \in \{1, 2\} : \)

\[
p_A(a|\alpha, \beta) \geq 0, \quad \sum_{a' \in \{1, 2\}} p_A(a'|\alpha, \beta) = 1, \quad \tag{3c}
\]

\[
p_1(a_1, a_2) = \int d\alpha d\beta_1 d\beta_2 p_A(a_1|\alpha, \beta_1)p_A(a_2|\alpha, \beta_2), \quad \tag{3d}
\]

\[
p_2(a_1, a_2) = \int d\alpha_1 d\alpha_2 d\beta p_A(a_1|\alpha_1, \beta)p_A(a_2|\alpha_2, \beta), \quad \tag{3e}
\]

\[
\frac{1}{2} = \int d\alpha d\beta p_A(a|\alpha, \beta). \quad \tag{3f}
\]

As we will show in proposition 16, it turns out that \( p^* = 3/4 \), so that the strategy of (2) is in fact optimal. It is quite likely that this result can be obtained with a more straightforward proof and in some greater generality (e.g., allowing the agent \( A \) to use a different strategy during each round), but this nonetheless gives us the opportunity to see a working example of our framework at a minimal computational cost.
3 Tensor notation

In this section, we introduce the tensor notation that we will be using to present our results — it is merely a specialized tensor network notation that is reviewed more generally in e.g. [BC17].

3.1 Probability tensors

We will use probability tensors to represent conditional probability distributions. The input legs are drawn at the bottom of the boxes, while the outputs are above, allowing to think of the diagrams as a time-ordered transmission of information from bottom to top. These probability tensors can be thought of as functions from several sets, one set per leg, to the interval \([0, 1]\) such that upon summation over the outputs of the top legs, one obtains 1 to achieve the desired normalization.

Examples. A classical agent such as \(A\) in the Correlated Sleeper task (see section 2) with two inputs and one output uses a conditional probability distribution that we draw as \(\begin{array}{c}
\text{A} \\
\alpha \\
\beta
\end{array}\) which, upon evaluation, gives

\[
\begin{array}{c}
\text{A} \\
\alpha \\
\beta
\end{array} = p_A(a|\alpha, \beta).
\]

The normalization can be written as

\[
\forall \alpha, \beta: \sum_{a^\prime} \begin{array}{c}
\text{A} \\
\alpha \\
\beta
\end{array} = 1.
\]

Analogously, an outcome distribution over two outcomes, e.g. \(p_1\) in (3), corresponds to a tensor \(\begin{array}{c}
p_1 \\
a_1 \\
a_2
\end{array}\) which evaluates to

\[
\begin{array}{c}
p_1 \\
a_1 \\
a_2
\end{array} = p_1(a_1, a_2).
\]

We will make extensive use of the tensors \(\begin{array}{c}
U_n \\
i
\end{array}\), whose output leg has domain \(\{1, \ldots, n\}\) and which evaluates to, for all \(i \in \{1, \ldots, n\}\),

\[
\begin{array}{c}
U_n \\
i
\end{array} := \frac{1}{n}.
\]

We will also make use of, in a certain sense, the limit case \(n \to \infty\), which we define as the probability tensor \(\begin{array}{c}
U_\infty \\
\alpha
\end{array}\) that represents the uniform probability density over the unit interval \([0, 1]\). That is, the output leg has continuous domain \([0, 1]\), and \(\begin{array}{c}
U_\infty \\
\alpha
\end{array} := 1\) for all \(\alpha \in [0, 1]\).

Tensor domains. In principle, one should always specify the domain of a tensor leg index. Here, we will leave this implicit, as it should be relatively clear from the context and is anyway typically irrelevant for the general constructions that we describe.

3.2 Composition rules

There are several ways to combine the above tensors together, which we clarify in this section.
**Scalar multiplication.** Drawing two tensors next to each other simply implies the scalar multiplication of the probabilities. For instance,
\[
\begin{array}{c}
x \\
\hline
X
\end{array}
\begin{array}{c}
y \\
\hline
Y
\end{array} = \left(\begin{array}{c}
x \\
\hline
X
\end{array}\right) \cdot \left(\begin{array}{c}
y \\
\hline
Y
\end{array}\right).
\]
(8)

One can think of such disconnected tensors as representing parallel, independent processes.

**Contractions.** One can contract the input leg of a tensor with the output leg of another tensor, provided that they share the same domain. This contraction, indicated graphically by the corresponding connection, implies a summation or integral over the corresponding argument. In the context of an integral, we will always tacitly assume that we are dealing with a Riemann integral, allowing us to approximate e.g. \( \frac{1}{U_\infty} \) sources with limits as \( n \to \infty \) of \( \frac{1}{U_n} \) sources — this will be used in particular in appendix A. For instance, the marginal constraint of (1) can be written as
\[
\frac{1}{2} = \underbrace{\begin{array}{c}
a \\
\hline
A
\end{array}}_{U_\infty} = \int \, d\alpha \, d\beta \, \underbrace{\begin{array}{c}
\alpha \\
\hline
A
\end{array}}_{U_\infty} \underbrace{\begin{array}{c}
\beta \\
\hline
\alpha
\end{array}}_{U_\infty},
\]
(9)

We can actually be even more compact by writing equality between tensors with open legs, which corresponds to component-wise equality. The output legs of either side of an equality have to be matched from left to right. For instance, the constraint (3f) can be written as
\[
\frac{1}{2} = \underbrace{\begin{array}{c}
a \\
\hline
A
\end{array}}_{U_\infty} = \int \, d\alpha \, d\beta \, p_A(a|\alpha, \beta).
\]
(10)

We will occasionally draw tensor contractions using a dashed leg such as \( \cdots \) to better distinguish overlapping legs.

### 3.3 Special tensors

**Deterministic tensors.** A deterministic tensor is defined as a probability tensor which, upon evaluation of all input and output legs, yields either 0 or 1. They are represented by double-edged boxes, e.g. \( \underbrace{\begin{array}{c}
a \\
\hline
A
\end{array}}_{U_\infty} \). Such deterministic tensors are in one-to-one correspondence with functions from the joint values of the input legs to the joint values of the output legs, e.g., if \( A \) uses a deterministic strategy in the Correlated Sleeper task, then there must exist a function \( f : [0, 1] \times [0, 1] \to \{1, 2\} \) such that, for all \( a \in \{1, 2\}, \) for all \( \alpha, \beta \in [0, 1], \)
\[
\underbrace{\begin{array}{c}
a \\
\hline
A
\end{array}}_{\alpha \beta} = \delta(a - f(\alpha, \beta)).
\]
(11)

We will prefer using the deterministic probability tensors over the functions such as \( f \) in this work.

**Marginal node.** Another useful node is the marginal node, \( \underbrace{\begin{array}{c}
a \\
\hline
A
\end{array}}_{U_\infty} \), which takes in arbitrary inputs, has no outputs, and always evaluates to 1. This implies, through the contraction rule, that placing this node on an output leg amounts to marginalizing over this leg:
\[
\sum_a \underbrace{\begin{array}{c}
ab \\
\hline
b
\end{array}}_c = \sum_a \underbrace{\begin{array}{c}
ab \\
\hline\hline
a
\end{array}}_c = \sum_a \underbrace{\begin{array}{c}
ab \\
\hline\hline\hline
a
\end{array}}_c.
\]
(12)
Fanout nodes. Since we deal with the transmission of classical information, there is a special tensor which we will use quite often, namely the fanout node. It has one input and arbitrarily many outputs, all within the same domain, and gives probability one if and only if each output is equal to the input. For instance,

\[ x_1, x_2, x_3 \xrightarrow{\Delta} x_0 := \delta(x_1 - x_0)\delta(x_2 - x_0)\delta(x_3 - x_0). \tag{13} \]

Here, \( \delta \) denotes either a Dirac delta functional in the physicist’s notation or a Kronecker delta tensor, depending on whether the tensor leg domain is in the integers or in the reals, which are the main two options here.

Bundle nodes. It will be useful to think of special types of legs which represent tuples of legs. For instance, suppose an agent \( A \) receives a number of inputs \( x_1, \ldots, x_n \) from sources \( X_1, \ldots, X_n \):

\[
\begin{array}{c}
\vdots \\
X_1 \cdots X_n \\
\vdots
\end{array}
\xrightarrow{\Delta} A
\]

(14)

It will be convenient to have a prescription for the notation in this sort of situation. We can achieve this by denoting

\[
\begin{array}{c}
x_1 \ldots x_n \\
X_1 \cdots X_n
\end{array}
\xrightarrow{\Lambda} \bar{x},
\]

(15)

where the leg style \( | \) indicates a tuple of values normally carried by \( | \) legs, and where \( \bar{x} = (x_1, \ldots, x_n) \). The special bundle node \( \Lambda \) is responsible of bundling all its input legs into the outgoing tuple of values. Formally speaking, we can define this \( \Lambda \) tensor as

\[
\bar{y} = (y_1, \ldots, y_n)
\]

\[
\begin{array}{c}
x_1 \ldots x_n \\
X_1 \cdots X_n
\end{array}
\xrightarrow{\Lambda} \bar{x} := \delta(y_1 - x_1) \ldots \delta(y_n - x_n).
\]

(16)

The diagram of equation (14) now becomes

\[
\begin{array}{c}
\vdots \\
X_1 \cdots X_n \\
\vdots
\end{array}
\xrightarrow{\Lambda} A
\quad = \quad
\begin{array}{c}
\vdots \\
X_1 \cdots X_n \\
\vdots
\end{array}
\xrightarrow{\Lambda} A
\]

(17)

Selector nodes. The above construction allows to capture conditional tensor contractions once we introduce the selector node. The selector node, \( \diamond \), takes a tuple of legs as its bottom input, and receive another input, a discrete one, call it \( i \), on the side. The output is then the \( i \)-th component of the input tuple. Formally, this reads

\[
\begin{array}{c}
\vdots \\
X_1 \cdots X_n \\
\vdots
\end{array}
\xrightarrow{\diamond} i := \delta(y - x_i).
\]

(18)
One can make this even more complete by allowing the selection to pick an ordered subset of \( k \leq n \) of the input legs, so that one can write:

\[
\vec{y} = (y_1, ..., y_k) \\
\vec{i} = (i_1, ..., i_k) := \delta(y_1 - x_{i_1}) \cdots \delta(y_k - x_{i_k}).
\]  

(19)

3.4 Postselection

We will make use of postselection over tensors with \{\text{False, True}\}-valued output. We will always postselect on the output \text{True}. For instance, the tensor \( f \) induces the postselection:

\[
\text{True} \quad f \\
\text{X} \\
\text{X}
\]

\[
(20)
\]

assuming that the postselection has a chance of succeeding, that is, assuming that

\[
\text{True} \quad f
\]

\[
\neq 0.
\]  

(21)

3.5 Correlated Sleeper in tensor notation

We may now rewrite the optimization problem of (3) in tensor notation. Notice that the non-negativity and normalization constraint of equation (3c) are now omitted because they are implied by \( A \) being denoted as a probability tensor. This yields the compact form:

\[
p^* = \sup \left( \frac{1}{2} \sum_{a \in \{1, 2\}} \left( \frac{a}{p_1} \frac{a}{p_2} + \frac{a}{p_2} \frac{a}{p_1} \right) \right)
\]  

(22a)

s.t. \[
\left\{ \begin{array}{l}
p_1 \quad A \\
U_{\infty} \quad U_{\infty} \quad U_{\infty}
\end{array} \right.,
\left\{ \begin{array}{l}
p_2 \quad A \\
U_{\infty} \quad U_{\infty} \quad U_{\infty}
\end{array} \right.,
\left\{ \begin{array}{l}
u_2 \quad A \\
U_{\infty} \quad U_{\infty}
\end{array} \right..
\]

(22b)

Let us in fact take the opportunity to further simplify this problem with the following proposition, which allows us to restrict the optimization to deterministic strategies for \( A \). The proof is given in appendix A; it primarily relies on the Riemann integrability assumption over \( A \)'s strategy.
Proposition 1. It holds that one can restrict the optimization variable of (22), namely, the probability tensor $A$, to range over the deterministic probability tensors only:

\[
p^* = \sup_{A} \frac{1}{2} \sum_{a \in \{1, 2\}} \left( p_{1a}^* + p_{2a}^* \right)
\]

\[
\text{s.t. \hspace{1cm} } \\
\begin{align*}
A & = \begin{cases}
U_{\infty} & \text{if } a = 1 \\
U_{\infty} & \text{if } a = 2
\end{cases} \\
A & = \begin{cases}
U_{\infty} & \text{if } a = 1 \\
U_{\infty} & \text{if } a = 2
\end{cases} \\
A & = \begin{cases}
U_{\infty} & \text{if } a = 1 \\
U_{\infty} & \text{if } a = 2
\end{cases}
\end{align*}
\]

(23a, 23b)
4 Causal compatibility

We now turn to the problem of causal compatibility. It is worth mentioning that in the work of [NW20], one can find an excellent introduction to the notion of causal unpacking, which describes the tools that one can use to translate the problem of causal compatibility with a causal structure into a related problem of causal compatibility with another simpler causal structure. In the case of a classical causal structure, where all the nodes (agent, sources etc.) have an associated probability tensor, one can actually unpack this causal structure (featuring e.g. direct causal influence between observed agents, several layers of unobserved sources interconnected in arbitrary ways, measurement settings à la CHSH, etc.) into a bi-layer causal structure with no inputs. However, we will be considering multi-network scenarios with some agents using the same strategies, and it is now unclear whether bilayer structures are most general in this extended case. Let us nonetheless restrict our attention to these cases, since this is an interesting generalization of the work of [NW20].

4.1 Network scenarios

Single-network scenarios. The most general causal structure that we shall consider will be called a network. The network consists of several ingredients. There are three integer parameters: $K$ labels the number of strategies that may be used by the agents, $P$ labels the number of agents in the network (we can assume in the case of a single-network scenario that $K \leq P$), and $S$ labels the number of sources (sometimes called latent nodes in the literature) that exist in the network. There are now two maps to specify: one is the strategy assignment map, $\nu : \{1, \ldots, P\} \rightarrow \{1, \ldots, K\}$, which says that the agent $p \in \{1, \ldots, P\}$ must use the strategy $\nu(p) \in \{1, \ldots, K\}$. The other is the connectivity map, $\vec{\mu} : \{1, \ldots, P\} \rightarrow \vec{\mathcal{P}}(\{1, \ldots, S\}) := \{(1), (2, 3), (3, 2), (1, 4, 5), (1, 2, S - 2, S), \ldots\}$, where $\vec{\mathcal{P}}(\{1, \ldots, S\})$ is the set of all sequences coming from the subsets of $\{1, \ldots, S\}$. This map $\vec{\mu}$ specifies that the agent $p$ receives the sources $\vec{\mu}(p)$, in this order, as inputs to their strategy. We implicitly assume, for consistency, that whenever two agents $p \neq p'$ are using the same strategy $\nu(p) = \nu(p')$, it must be that $\vec{\mu}(p)$ and $\vec{\mu}(p')$ are sequences of equal length, since the strategy $\nu(p)$ has a well-defined number of inputs (occasionally denoted “(#in) $\nu(p)$”).

For instance, consider the bilocal network (also known as the “three-on-a-line” network), represented graphically in figure 2, where $P = 3$ agents share $S = 2$ sources, so that the agent $p = 2$ has access to the two sources $s = 1, 2$, while the agent $p = 1$ has only access to the source $s = 1$ and the agent $p = 3$ has access to the $s = 2$ source. If the three agents are allowed to use $K = 3$ arbitrary strategies, then this network will be specified as

\begin{align}
N_{\text{biloc}} &= (K = 3, P = 3, S = 2, \nu, \vec{\mu}), \\
\nu(1) &= 1, \quad \nu(2) = 2, \quad \nu(3) = 3, \quad (3 \text{ different strategies}) \\
\vec{\mu}(1) &= (1), \quad \vec{\mu}(2) = (1, 2), \quad \vec{\mu}(3) = (2). \quad (\text{connectivity of the bilocal network})
\end{align}

![Figure 2: The graph corresponding to the $N_{\text{biloc}}$ single-network scenario. The labeling of the sources and of the different agent strategies is chosen to match the parametrization of (25) — more conventionally, one would prefer to think of $A_1$ as Alice, $A_2$ as Bob and $A_3$ as Charlie.](image-url)
Multi-network scenarios. We will also be interested in cases where several networks are involved, with the set of available strategies being globally shared across these networks: such scenarios will be called multi-network scenarios. While a multi-network scenario can always be embedded into a single-network scenario whose associated graph features several disconnected components, the inflation framework is most conveniently applied to the multi-network scenario formulation.

To fix the notation, consider a number $C$ of networks. Each network $c \in \{1, \ldots, C\}$ will have its own number of agents $P_c$, number of sources $S_c$, strategy assignment map $\nu_c : \{1, \ldots, P_c\} \to \{1, \ldots, K\}$, and connectivity map $\vec{\mu}_c : \{1, \ldots, P_c\} \to \mathcal{P}\{1, \ldots, S_c\}$, but the number of strategy $K$ does not depend on $c$. There is a consistency condition that is implicitly assumed: for all $c, c' \in \{1, \ldots, C\}$, for any pair of agents $p \in \{1, \ldots, P_c\}$ and $p' \in \{1, \ldots, P_{c'}\}$ that use the same strategy, i.e., $\nu_c(p) = \nu_{c'}(p')$, it must be that these two agents receive the same number of inputs, i.e., $\vec{\mu}_c(p)$ and $\vec{\mu}_{c'}(p')$ must be sequences of the same length, since the agent strategy $\nu_c(p)$ has a well-defined number of inputs $\#\text{in}_{\nu_c(p)}$. Such a multi-network scenario will be denoted as $(\mathcal{N}_c)_{c=1}^C$ (the sequence notation is to emphasize the fact that we pick a specific ordering of the networks), or more explicitly as

$$
(\mathcal{N}_c = (K, P_c, S_c, \nu_c, \vec{\mu}_c))_{c=1}^C.
$$

The Correlated Sleeper’s multi-network scenario. In the case of the Correlated Sleeper, there are three relevant networks, whose graphs are represented in figure 3: the first network is the one where the two instances of $A$ are connected through the left input, which corresponds to the network

$$
\mathcal{N}_1^{(s)} = (K = 1, P_1 = 2, S_1 = 3, \nu_1, \vec{\mu}_1),
$$

(one strategy, two agents, three sources) \hspace{1cm} (27a)

$$
\nu_1(1) = 1, \quad \nu_1(2) = 1,
$$

(agents use same strategy) \hspace{1cm} (27b)

$$
\vec{\mu}_1(1) = (1, 2), \quad \vec{\mu}_1(2) = (1, 3).
$$

(agents’ first inputs connected to first source) \hspace{1cm} (27c)

The second network is the one where the two $A$’s are connected through the right input, i.e.,

$$
\mathcal{N}_2^{(s)} = (K = 1, P_2 = 2, S_2 = 3, \nu_2, \vec{\mu}_2),
$$

(28a)

$$
\nu_2(1) = 1, \quad \nu_2(2) = 1,
$$

(28b)

$$
\vec{\mu}_2(1) = (1, 3), \quad \vec{\mu}_2(2) = (2, 3).
$$

(agents’ second inputs connected to third source) \hspace{1cm} (28c)

The last network is the one that allows us to express the marginal constraint, where we only look at one isolated agent $A$:

$$
\mathcal{N}_3^{(s)} = (K = 1, P_3 = 1, S_3 = 2, \nu_3, \vec{\mu}_3),
$$

(29a)

$$
\nu_3(1) = 1,
$$

(29b)

$$
\vec{\mu}_3(1) = (1, 2).
$$

(29c)

![Figure 3: The Correlated Sleeper’s multi-network scenario.](image)

4.2 Causal compatibility

A note on deterministic strategies. Our framework deals best with deterministic agent strategies as basic primitives. This may sound restrictive, but fundamentally speaking, it is not: any non-deterministic strategy can be achieved with a deterministic strategy upon giving each agent access to
an additional local randomness source, which can be captured by an appropriate update of the network scenario. From a computational perspective, this explicit addition of additional sources can turn out to be costly — we will return to this aspect in section 6.2. However, this addition of local sources is not always necessary: depending on the sources shared between the agents, local randomness can sometimes be extracted without adding additional local sources. This is for instance demonstrated in proposition 1 for the Correlated Sleeper. In the following bilocal network example, although we do not give an explicit construction, it is also the case that local randomness can be extracted from the shared sources.

**Causal compatibility: example.** We first present causal compatibility with the bilocal network before generalizing to arbitrary networks. A probability tensor \( \mathcal{P} \), which we will typically call an outcome distribution, is compatible with the bilocal network \( \mathcal{N}_{\text{biloc}} \) (equation \( (25) \)), which we denote as

\[
\mathcal{P} \in \mathcal{L}(\mathcal{N}_{\text{biloc}}),
\]

if there exist deterministic probability tensors

\[
\begin{pmatrix} A \end{pmatrix}, \begin{pmatrix} B \end{pmatrix}, \begin{pmatrix} C \end{pmatrix},
\]

such that for all \( a, b, c, \)

\[
\begin{pmatrix} A \end{pmatrix} \begin{pmatrix} B \end{pmatrix} \begin{pmatrix} C \end{pmatrix} = \mathcal{P}.
\]

**Causal compatibility: single-network scenarios.** More generally, a probability tensor \( \mathcal{P} \) is compatible with a single-network scenario \( \mathcal{N} = (K, P, S, \nu, \bar{\mu}) \), denoted

\[
\mathcal{P} \in \mathcal{L}(\mathcal{N}),
\]

if there exist deterministic probability tensors

\[
\left\{ \begin{pmatrix} A_k \end{pmatrix} \right\}_{k=1}^K
\]

such that for all \( a_1, a_2, \ldots, a_P, \)

\[
\begin{pmatrix} A_{\nu(1)} \end{pmatrix} \begin{pmatrix} A_{\nu(2)} \end{pmatrix} \cdots \begin{pmatrix} A_{\nu(P)} \end{pmatrix} = \mathcal{P}.
\]

The output domains of the strategies are some finite subsets of the integers (this is anyway the only possibility from a computational perspective) that are also left implicit here. Recall the special selector tensor \( \diamond \), allowing us to parametrize the sources that each agent has access to, and the bundle tensor \( \triangle \), collecting the source outputs in a single vector, both introduced more precisely in section 3.3.
Causal compatibility: multi-network scenarios. We now introduce causal compatibility for multi-network scenarios: this will be the problem formulation that we shall use in the rest of this work. In fact, the rest of this work will be concerned with computationally tractable supersets of the following $L\left(\left\{(N_c)_c\right\}_{c=1}^C\right)$.

**Definition 2.** Consider a multi-network scenario

$$\left(\left\{(N_c = (K, P_c, S_c, \nu_c, \mu_c)\right\}_{c=1}^C\right. \right.$$

(this notation is explained in section 4.1). We say that a sequence of outcome distributions $\left(\frac{\cdots}{P_c}\right)_{c=1}^C$, where the outcome distribution $\frac{\cdots}{P_c}$ must have $P_c$ output legs, is causally compatible with the multi-network scenario $\left(\left\{(N_c)_c\right\}_{c=1}^C\right)$, denoted

$$\left(\frac{\cdots}{P_c}\right)_{c=1}^C \in L\left(\left\{(N_c)_c\right\}_{c=1}^C\right),$$

if there exist deterministic probability tensors

$$\left\{A_{k}\right\}_{k=1}^K,$$

such that, for all $c = 1, \ldots, C$,

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A_{\nu_c(1)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A_{\nu_c(2)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A_{\nu_c(P_c)}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mu_c(1)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mu_c(2)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mu_c(P_c)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U_{\infty}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U_{\infty}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(S_c - 3)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U_{\infty}
\end{array}
\end{array}
\end{array} = \frac{\cdots}{P_c}. \quad (36b)
\end{array}$$

In equation (36b), we used the compact notation where equality of two tensors with open legs simply corresponds to component-wise equality. Furthermore, we use “…” in the same sense that $\{1, \ldots, 4\} = \{1, 2, 3, 4\}$, but with the sources, the number of terms omitted is not very explicit, so we write “(S_c – 3)” to indicate that we omitted $S_c – 3$ sources and drew the 3 remaining one explicitly. Note that the tensors $\diamond$ and $\triangle$ were introduced in section 3.3.

Causal compatibility: Correlated Sleeper. In this notation, thanks to proposition 1 and the multi-network scenario of equations (27)-(29), we can rewrite the feasible region of the optimization problem (23) as a causal compatibility problem:

$$p^* = \sup_{\frac{1}{P_1}, \frac{1}{P_2}} 1 \sum_{a=1,2} \left(\frac{a}{P_1} + \frac{a}{P_2}\right) \quad (37a)$$

s.t. $\left(\frac{1}{P_1}, \frac{1}{P_2}, \frac{1}{U_2}\right) \in L\left(\left\{(N_c)_{c=1}^3\right\}\right). \quad (37b)$

We emphasize the fact that the marginal constraint induced from (1) on $\frac{1}{P_1}$ and $\frac{1}{P_2}$ is indeed contained in the above equation (37b).

\(^2\)Strictly speaking, we should also specify the domain of the outputs of each agent (i.e., how many different outcomes they may output) to have a well-defined set $L\left(\left\{(N_c)_{c=1}^C\right\}\right)$, but we leave this dependence implicit.
4.3 Further generalizations

**Complex source behavior.** We are currently allowing all the sources to be “maximally entropic”, so that any other source distribution can be obtained by the agents upon applying the relevant post-processing. One thing that our framework can not deal with (currently, at least — it is unclear whether this can be nicely incorporated in) is the possibility to constrain the sources to a specific type of distribution. This could either be a network-wide constraint, e.g. restrict all sources to be uniform over a fixed number of values, or context-dependent constraints, allowing to capture e.g. the performance of a strategy faced with different source distributions.

**Partial constraints, optimization.** Our framework can relatively straightforwardly deal with partial constraints over the network probabilities, as well as optimizing polynomials of the network probabilities. These ideas and techniques are explored more systematically in [NW20], and can be adapted to the present framework easily. For instance, in the problem (36), one may not know the full statistics $p_c$ for all $c$, but perhaps only an average value for $c = 1$, the probability of one event (one tuple of outcomes) only for $c = 2$, a lower bound of a certain polynomial over the probabilities of the events for $c = 3$, etc. We shall not attempt to parameterize these sorts of problems in order to remain somewhat concise, but we will deal with the explicit example of the Correlated Sleeper in section 7.
5 Postselected inflation: motivation

In this section, we give an intuitive motivation for the postselected inflation outer approximations in the context of a single-network scenario, namely, the bilocal network $\mathcal{N}_{\text{biloc}}$. In the next section 6, we will state general proofs of soundness of this approach, before explicitly applying these techniques in section 7. We will return to the correspondence with the usual fanout inflation formalism in section 8.

5.1 Convexification of the causal compatibility problem

Deciding the causal compatibility of a distribution $p$ with a single-network scenario is generally hard. There are two reasons behind this: one is that the problem in its standard formulation as in equation (33) is not convex, in the sense that a convex combination of the solution tensors of equation (33a) cannot be used as a new solution of the problem (33). In fact, allowing for convex combinations of these tensors is equivalent to sending the output of a source $\Lambda$ to all the agents. Let us make the corresponding causal compatibility problem explicit in the case of the bilocal network (this is to be compared with equations (30)-(31)): we let $\mathcal{I}_{\text{GR}}(\mathcal{N}_{\text{biloc}})$ (for “global randomness”) be the set of all distributions $p$ for which there exist

\[
\begin{array}{cccc}
A & B & C & \Lambda \\
\end{array}
\]

such that (the style difference between the dashed and solid edges is there to guide the eye but implies the same operation of tensor contraction)

\[
\begin{array}{cccc}
\Lambda & B & C & U_{\infty} \\
\end{array}
\]

This modified causal compatibility is too permissive: it holds that $\mathcal{L}(\mathcal{N}_{\text{biloc}}) \subseteq \mathcal{I}_{\text{GR}}(\mathcal{N}_{\text{biloc}})$, which is a general feature of allowing global randomness. One needs to think of something else to obtain a causal compatibility problem that is both convex, i.e., which allows for a global randomness source $\Lambda$, and that yields a good outer approximation of the set $\mathcal{L}(\mathcal{N}_{\text{biloc}})$. The trick is the following: define the set $\mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}})$ of all distributions $p$ for which there exist

\[
\begin{array}{cccc}
A & B & C & \Lambda \\
\end{array}
\]

such that

\[
\begin{array}{cccc}
\Lambda & B & C & \Lambda \\
\end{array}
\]

This modified causal compatibility is too permissive: it holds that $\mathcal{L}(\mathcal{N}_{\text{biloc}}) \subseteq \mathcal{I}_{\text{GR}}(\mathcal{N}_{\text{biloc}})$, which is a general feature of allowing global randomness. One needs to think of something else to obtain a causal compatibility problem that is both convex, i.e., which allows for a global randomness source $\Lambda$, and that yields a good outer approximation of the set $\mathcal{L}(\mathcal{N}_{\text{biloc}})$. The trick is the following: define the set $\mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}})$ of all distributions $p$ for which there exist

\[
\begin{array}{cccc}
A & B & C & \Lambda \\
\end{array}
\]

such that

\[
\begin{array}{cccc}
\Lambda & B & C & \Lambda \\
\end{array}
\]
Indeed, it holds that $\mathcal{L}(\mathcal{N}_{\text{biloc}}) = \mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}})$. The fact that $\mathcal{L}(\mathcal{N}_{\text{biloc}}) \subseteq \mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}})$ is trivial — the agents may simply discard the input from $\Lambda$. For the other direction, if the strategies $A$, $B$, $C$ are making a non-trivial use of the global randomness from $\Lambda$, then, they will do so on both ends of the diagram, and will thus necessarily become correlated, whereas the constraint of (39b) imposes the two halves of the diagram to be uncorrelated. Formally speaking, this follows from our “main lemma”, whose proof is given in appendix B, along with the definition of the relevant norms.

**Lemma 3 (Main lemma).** For any $k \in \mathbb{N}$, for any probability tensors

$$\Lambda, \, p, \in \mathbb{R}^k, \left\{ \lambda \in \mathbb{R}^k \right\}_\lambda,$$

it holds that

$$\int d\lambda \frac{\lambda}{\Lambda} \left\| \frac{\lambda}{\Lambda} p - \frac{q}{\lambda} \right\|_2^2 \leq 3 \left\| \frac{p}{\Lambda} - \frac{q}{\Lambda} \right\|_1. \tag{41}$$

In our case, we can apply lemma 3 to equation (39b) to read out that we must have

$$\int d\lambda \frac{\lambda}{\Lambda} \left\| \frac{\lambda}{\Lambda} p - \frac{q}{\lambda} \right\|_2^2 = 0. \tag{42}$$

This implies in particular that there exists a value $\lambda_0$ such that

$$\Lambda_{\lambda_0} \frac{A}{B} \frac{C}{\Lambda}, \frac{A}{B} \frac{C}{\Lambda} = \frac{p}{\Lambda}, \tag{43}$$

which proves that $\mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}}) \subseteq \mathcal{L}(\mathcal{N}_{\text{biloc}})$.

Now, it is important to notice how we have effectively turned the causal compatibility problem into a convex one. For each output of the source $\Lambda$, the agents of (39) are using some strategies in the original network scenario. Thus, solving for the problem (39) is equivalent to optimizing the distribution $\Lambda$ whose output domain is the set of possible tuples of strategies, and the only constraint in place is that of (39b), which is linear in $\Lambda$ for fixed $\frac{p}{\Lambda}$.\(^3\) However, this convexification of the original problem is trivial so far: one has to enumerate all possible tuples of strategies, and there are infinitely many of them due to the fact that the sources output an unbounded number of values to which the agents may react differently. To tackle this, we move on to finding a way to restrict the output of the sources to take very few values, e.g., 2 or 3 each, while still having a chance of certifying that $\frac{p}{\Lambda} \notin \mathcal{L}(\mathcal{N}_{\text{biloc}})$.

\(^3\)Note that in (39b), one may assume without loss of generality that $\Lambda$ outputs at most $\sim k^2$ distinct values, given $k$ outcomes for $\frac{p}{\Lambda}$ — this is the content of Carathéodory’s theorem, see e.g. theorem 4.3.2 in [Pan93].
5.2 Restricting the output cardinality of the sources

Still focusing on the example of the bilocal network, let us now attempt to add to the problem of (39) the constraint that the sources may only take \( n \in \mathbb{N}, n \geq 2 \) different values (potentially, say, only 2 or 3 values — in fact, we could also use a different cardinality for each individual source, but we do not make this option explicit here for simplicity). All sources will thus be distributed as the uniform distribution \( \frac{1}{U_n} \) over \( n \) values. We let \( \mathcal{I}^{(n)}_{\text{restr}}(\mathcal{N}_{\text{biloc}}) \) be the set of all \( p \) for which there exist

\[
\begin{align*}
|A|, |B|, |C|, |\Lambda|.
\end{align*}
\]

such that

\[
\begin{align*}
|A| |B| |C| |A| |B| |C| U_n U_n U_n U_n = |P| |P|.
\end{align*}
\]

This problem can be formulated without loss of generality as a linear program over \( \frac{1}{A} \), whose outputs can be taken to range over the tuple of deterministic strategies that the agents should use, and there are now finitely many such strategies. It is apparent that the above set \( \mathcal{I}^{(n)}_{\text{restr}}(\mathcal{N}_{\text{biloc}}) \) taken in the limit of arbitrarily large \( n \) will coincide with \( \mathcal{L}_{\text{alt}}(\mathcal{N}_{\text{biloc}}) \) as in equation (39), which by the arguments of section 5.1 equals to \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \), so the convergence of the above scheme is under control. However, we have restricted the possibilities for the agents by going from (39) to (44), so it is clear that \( \mathcal{I}^{(n)}_{\text{restr}}(\mathcal{N}_{\text{biloc}}) \subseteq \mathcal{L}(\mathcal{N}_{\text{biloc}}) \): we need to give the agents more possibilities to obtain outer approximations of \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \).

5.3 Adding possibilities through postselection

So far, the agents cannot really use the source \( \frac{1}{A} \) because of the independence condition of (44b) and the argument surrounding lemma 3. To bypass lemma 3, let us add additional correlations between the two halves of the left-hand side diagram of (44b) besides the source \( \frac{1}{A} \). More precisely, let us add correlations between the four i.i.d. sources \( \frac{1}{U_n} \). We could in principle denote this by a new source with four output legs, but it is more adequate to in fact add correlations in the form of postselection on the outputs of the sources \( \frac{1}{U_n} \). Let us for now leave the postselection arbitrary: we denote it with \( \bigtriangleup f \), meaning that we postselect on the output True of a tensor \( f \) which has outputs True and False — see also section 3.4 for more details on postselection. We say that a distribution \( \frac{1}{P} \) is compatible with a postselected inflation, denoted for now

\[
\frac{1}{P} \in \mathcal{I}^{(n)}_{\text{restr}}(\mathcal{N}_{\text{biloc}}),
\]

if there exist

\[
\begin{align*}
|A|, |B|, |C|, |\Lambda|.
\end{align*}
\]
such that
\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (B) at (1,0) {$B$};
\node (C) at (2,0) {$C$};
\node (Un) at (0,-1) {$U_n$};
\node (Un) at (1,-1) {$U_n$};
\node (Un) at (2,-1) {$U_n$};
\node (Un) at (3,-1) {$U_n$};
\node (A) at (4,0) {$A$};
\node (B) at (5,0) {$B$};
\node (C) at (6,0) {$C$};
\draw [->] (A) -- (Un);
\draw [->] (B) -- (Un);
\draw [->] (C) -- (Un);
\draw [->] (A) -- (Un);
\draw [->] (B) -- (Un);
\draw [->] (C) -- (Un);
\end{tikzpicture}
\end{array}
\]
\[= \frac{p}{\hat{p}}. \quad (46b)\]

Lemma 3 no longer applies in this case, since the left-hand side of (46b) no longer has the form of a mixture of i.i.d. distributions. The convergence of the above problem is still under control: if the postselection \( \hat{f} \) becomes closer and closer to being trivial while \( n \) becomes larger and larger, the set \( I_f^{(n)}(\mathcal{N}_{\text{biloc}}) \) will converge to the set \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \) as characterized in equation (39). However, compared with the original causal compatibility problem (39), we took two contradicting steps, and the sets \( I_f^{(n)}(\mathcal{N}_{\text{biloc}}) \) and \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \) seem incomparable: on the one hand, we let the agents share some additional correlations through the postselection \( \hat{f} \), but we also restricted the cardinality of the sources \( U_n \). Ideally, we would like a postselection \( f \) such that, overall, the “extra possibilities” given by the postselection win over the restriction of the source cardinalities, so that we obtain an outer approximation \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \subseteq I_f^{(n)}(\mathcal{N}_{\text{biloc}}) \).

5.4 Fixing the postselection

It turns out that the two main criteria that the postselection \( \hat{f} \) should fulfill are the following.

The main one is that only distinct values for the inputs of \( \hat{f} \) should pass the postselection: this will guarantee that \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \subseteq I_f^{(n)}(\mathcal{N}_{\text{biloc}}) \). The second important feature is that \( \hat{f} \) should be as unimportant as possible; that is, the impact of introducing the postselection in the network should be as low as possible. This will guarantee that \( I_f^{(n)}(\mathcal{N}_{\text{biloc}}) \) is a good outer approximation of \( \mathcal{L}(\mathcal{N}_{\text{biloc}}) \). For these reasons, we make the choice (the label “2” refers to the numbers of inputs of the tensor)

\[
\hat{f} = \frac{\frac{\delta}{\delta}}{\frac{\delta}{\delta}}, \quad (47)
\]

where the right hand-side denotes the postselection over the outcome \textbf{True} of the tensor \( \frac{\delta}{\delta} \) defined, for all \( b \in \{\text{False, True}\} \), for all \( u_1, u_2 \in \{1, \ldots, n\} \), through

\[
\begin{array}{c}
\begin{tikzpicture}
\node (b) at (0,0) {$b$};
\node (u1) at (-0.5,-0.5) {$u_1$};
\node (u2) at (0.5,-0.5) {$u_2$};
\node (p) at (0,0.5) {$p$};
\node (q) at (0,-0.5) {$q$};
\draw [->] (b) -- (p);
\draw [->] (b) -- (q);
\end{tikzpicture}
\end{array}
\]
\[= \delta_b \text{False} \delta_{u_1, u_2} + \delta_b \text{True}(1 - \delta_{u_1, u_2}). \quad (48)\]
The effect of this postselection on the sources is the following:

\[ \forall i, j \in \{1, \ldots, n\} : \quad \frac{1}{n(n-1)} (1 - \delta_{ij}). \quad (49) \]

The set \( \mathcal{I}_f^{(n)}(N_{biloc}) \) with this choice of postselection is denoted as \( \mathcal{I}_n^{(n)}(N_{biloc}) \). It is almost obvious that we will have, roughly speaking, that \( \lim_{n \to \infty} \mathcal{I}_n^{(n)}(N_{biloc}) = \mathcal{L}(N_{biloc}) \): as \( n \to \infty \), the postselection strategy \( \frac{\not{\chi}}{2} \) has almost no effect (the inputs are anyway not equal to one another with high probability), so that \( \lim_{n \to \infty} \mathcal{I}_n^{(n)}(N_{biloc}) \) is essentially equal to \( \lim_{n \to \infty} \mathcal{I}_{\text{rest}}^{(n)}(N_{biloc}) \), which is itself equal to the set \( \mathcal{L}(N_{biloc}) \) as characterized in equation (39). This claim will be made general and formal in theorem 8.

Perhaps more surprising is the fact that, although \( \mathcal{I}_f^{(n)}(N_{biloc}) \) and \( \mathcal{L}(N_{biloc}) \) seemed incomparable, we have \( \mathcal{L}(N_{biloc}) \subseteq \mathcal{I}_f^{(n)}(N_{biloc}) \), i.e., this specific postselection guarantees that we obtain an outer approximation as desired. Let us prove this fact in the following lemma, which is, at least conceptually, a corollary of the more general theorem 6 that we will give in the next section 6 — however, an explicit proof in this simple context captures the general proof idea.

**Lemma 4.** It holds that

\[ \mathcal{L}(N_{biloc}) \subseteq \mathcal{I}_f^{(n=2)}(N_{biloc}). \quad (50) \]

**Proof.** Let \( p \in \mathcal{L}(N_{biloc}) \) so that we have probability tensors \( A_0, B_0, C_0 \) such that

\[ A_0 \quad B_0 \quad C_0 \quad = \quad p. \quad (51) \]

Let us choose the tensors of (46a) to be as follows. First off, the source \( \Lambda \) will actually be sending a tuple of four values, sampled from four independent \( U_\infty \) sources:

\[ \Lambda \quad (x_1, y_1, x_2, y_2) \quad = \quad \frac{x_1}{U_\infty} \quad \frac{y_1}{U_\infty} \quad \frac{x_2}{U_\infty} \quad \frac{y_2}{U_\infty}. \quad (52) \]

The strategy \( \overline{A} \) will consist in using the input \( i \) coming from the \( U_2 \) source of (46b) to choose to use the value \( x_i \) coming from the source \( \Lambda \) as the input to the original strategy \( \overline{A_0} \). The other two strategies are analogous: we let

\[ \overline{A} \quad := \quad \overline{A_0}. \quad (53a) \]
We can now show that (46b) is indeed verified: for all \(a, b, c, \tilde{a}, \tilde{b}, \tilde{c}\),

\[
\begin{align*}
\frac{1}{4} & \sum_{i_1, i_2, j_1, j_2 \in \{1, 2\}} \int dx_1 \, dx_2 \, dy_1 \, dy_2 \\
& = \frac{1}{\mathcal{P}} \cdot \frac{1}{\mathcal{P}}.
\end{align*}
\]

so that indeed \(p \in \mathcal{T}_{\neq}^{(n=2)}(\mathcal{N}_{\text{biloc}})\).

Note that this proof idea would work regardless of the precise behavior of the \(U_2\) and \(\neq\) tensors, as long as the postselection enables \(i_1 \neq i_2\) and \(j_1 \neq j_2\).
6 Postselected inflation: formal aspects

In this section, we give a general description of the sort of outer approximations that we consider, and we prove a number of results that characterize these. The previous section 5 gave some intuition for the idea of this construction.

6.1 Definition

Consider a multi-network scenario \((N_c)_{c=1}^C\) as in equation (26). The set \(L\left((N_c)_{c=1}^C\right)\) was defined in definition 2. Let us define the postselected inflation set \(I\left((N_c)_{c=1}^C, n, m\right)\) — parametrized by two integers \(n\) (number of output values for the discretized sources) and \(m\) (order of the tensor product constraints) — which is meant to be the outer approximation of the set \(L\left((N_c)_{c=1}^C\right)\). In the following definition, for all integers \(k\), the tensor \(\frac{\delta}{\delta^k}\) denotes the postselection of the \(\text{True}\) outcome of the probability tensor \(\frac{\delta}{\delta^k}\) (see section 3.4 for the definition of postselection). This probability tensor \(\frac{\delta}{\delta^k}\) has \(k\) input legs, each with domain \(\{1, \ldots, n\}\) in this context, and is defined, for all \(b \in \{\text{False, True}\}, i_1, \ldots, i_k \in \{1, \ldots, n\}\), as

\[
\frac{\delta}{\delta^k} = \begin{cases} 
\delta_{b, \text{True}}, & \text{if all the values } i_1, i_2, \ldots, i_k \text{ are pairwise distinct,} \\
\delta_{b, \text{False}} & \text{else.}
\end{cases}
\]

Now, the following diagrammatic constraint (57b) is quite large, so let us describe how to obtain it: for each network \(c \in \{1, \ldots, C\}\),

(i) Take the corresponding tensor contraction, as in equation (36b).

(ii) Replace each source \(U_\infty\) by a uniform source \(U_n\) over \(n\) values.

(iii) Add an additional input to all the strategy tensors, e.g. \(A_{1} \rightarrow A_1\).

(iv) Duplicate \(m\) times (take the \(m\)-fold tensor product of) both the network tensor and the target distribution \(p\) in equation (36b).

(v) Connect all the agent’s additional inputs to a global randomness source \(\Lambda\). Importantly, the tensor \(\Lambda\) is the same for all \(c = 1, \ldots, C\).

(vi) At last, postselect on the \(S_c \cdot m\) sources \(U_n\) having different values. This is the step which requires \(n \geq S_c \cdot m\).

This yields the following definition (recall that the style difference between the dashed and solid edges is there to guide the eye but implies the same operation of tensor contraction).
**Definition 5** (Postselected inflation). Consider a multi-network scenario \( \mathcal{N}_c = (\mathcal{K}, P_c, S_c, \nu_c, \tilde{\mu}_c)_{c=1}^C \). Let \( m \) be an integer parameter, and let \( n \) be any integer such that

\[
n \geq \max_{c \in \{1, \ldots, C\}} S_c \cdot m.
\]  
(56)

A list of distributions \( \left( \sum_{P_c} \right)^C_{c=1} \) belongs to the set \( \mathcal{I}\left( (\mathcal{N}_c)_{c=1}^C, n, m \right) \) if and only if there exist probability tensors

\[
\left\{ \begin{array}{c}
A_{k} \\
\Lambda
\end{array} \right\}_{k=1}^{K},
\]  
(57a)

such that, for all \( c = 1, \ldots, C \), it holds that

\[
\sum_{P_c} \cdots \sum_{P_c} \cdots \sum_{P_c} \cdots = \sum_{P_c} \left( \sum_{P_c} \right)^{(m-3)}_{P_c} \cdots \sum_{P_c}.
\]  
(57b)

6.2 Remarks about the implementation

The idea behind the above definition 5 is that testing for \( \left( \sum_{P_c} \right)^C_{c=1} \in \mathcal{I}\left( (\mathcal{N}_c)_{c=1}^C, n, m \right) \) can be easily formulated as a linear program over \( \Lambda \) only. Indeed, it is clear from (57b) that one can assume without loss of generality that the output domain of \( \Lambda \) is the set of all tuples of \( K \) deterministic strategies such that each agent \( p \) then chooses the \( \nu(p) \)-th element of this tuple to use as their strategy. This output domain is quite large. For each \( k \in \{1, \ldots, K\} \), we denote with \((\#\text{out})_k\) the number of outcomes of the strategy \( k \), and with \((\#\text{in})_k\) the number of input legs of this strategy. This strategy will then receive from the sources \( U_{n} \) a tuple of \((\#\text{in})_k\) values each in the set \( \{1, \ldots, n\} \), and there are \( n^{(\#\text{in})_k} \) such tuples. Since for each such tuple, the deterministic strategy may give \((\#\text{out})_k\) different outcomes, there are thus \( \exp \left( \ln[(\#\text{out})_k]n^{(\#\text{in})_k} \right) \) such deterministic strategies. Overall, the

\[\text{This number } (\#\text{in})_k \text{ would be the length of the list } \tilde{\mu}_c(p) \text{ for any } c \text{ and } p \text{ such that } \nu_c(p) = k.\]
distribution $A$ should thus have an output (indexing the possible strategies) with cardinality
\[
\exp \left( \sum_{k=1}^{K} \ln[(\#\text{out})_k]n^{(\#\text{in})_k} \right),
\]
which is quite a large number as $n$ grows.

For this reason, it is crucial to be able to formulate versions of definition 5 with $n$ being a small integer. Depending on the number $S_c$ of sources in each network $c$, this may be problematic with respect to the postselection constraint of equation (56). It is thus useful to keep in mind that although definition 5 is fully general for our purposes, one should in practice investigate the network’s structure to see if it is possible to have less postselection than in (57b). Additionally, as already alluded to in section 5.4, having less postselection is also helpful for the purpose of letting the outer approximation generated by the postselected inflation procedure be as tight as possible. In this work, two examples of enforcing less postselection than in definition 5 are given: the case of a single-network scenario ($C = 1$) where all agents use independent strategies ($K = P$) — see section 5.4 as well as section 8.2 — and the case of the Correlated Sleeper — see section 7.2. Furthermore, in the latter, we show how letting the parameter $m$ depend on $c$ can be convenient with respect to these requirements. It is also possible to think of taking different value of $n$ for each source $\frac{1}{U_n}$ appearing in the left-hand side of (57b).

6.3 Results

We now prove a number of results regarding the postselected inflation scheme. As will be discussed in section 8, these results extend those of [NW20] to the case of our multi-network scenarios with subsets of agents using the same strategy. The proofs are gathered in appendix B. The first result establishes that the postselected inflation scheme indeed allows to certify causal incompatibility. The proof is the generalization of that of lemma 4 and lemma 13.

**Theorem 6** (Certification). Let $(\mathcal{N}_c)_{c=1}^C$, $n$, $m$ be as in definition 5. It holds that
\[
\mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right) \subseteq \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right).
\]

The next theorem establishes that increasing the parameters $n$ and $m$ that appear in the postselected inflation problem will make the outer approximation $\mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right)$ smaller and smaller, while remaining an outer approximation of $\mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right)$ thanks to theorem 6.

**Theorem 7** (Hierarchy). For all $(\mathcal{N}_c = (K, P_c, S_c, \nu_c, \bar\mu_c))_{c=1}^C$ and $n$, $m$ ($n \geq \max_c S_c \cdot m$) as well as $n'$, $m'$ ($n' \geq \max_c S_c \cdot m'$) such that
\[
n \geq n' \text{ and } m \geq m',
\]
it holds that
\[
\mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right) \subseteq \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n', m' \right).
\]

To discuss convergence, we will make use of the $p$-norms on $\mathbb{R}^k$ (defined explicitly in definition 20 in appendix B). We will take the 1-norm as the operationally most meaningful one, since it is related to an operational measure of distinguishability of two distributions. The following theorem makes precise the fact that $\mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right)$ is a tight inner approximation of $\mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right)$. The fact that increasing the parameter $m$ to its maximal value, $\lfloor n / \max_c S_c \rfloor$, does not improve the below convergence rate suggests that the convergence rate is actually better than the one we give here.

**Theorem 8** (Convergence). Let $(\mathcal{N}_c = (K, P_c, S_c, \nu_c, \bar\mu_c))_{c=1}^C$, $n$ and $m$ be as in definition 5. We assume that $m \geq 2$. Then, for any list of distributions
\[
\left( \frac{\mathcal{N}_c^{\nu_c}}{P_c} \right)_{c=1}^C \in \mathbb{R}^{d_c} \in \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right)
\]
where we made explicit the total number of outcomes \( d_c \) of each distribution,
\(^5\) it holds that

\[
\inf \left\{ \left( \frac{1}{\sqrt{2}} \right)^C \right\}_{c=1}^C \frac{1}{C} \sum_{c=1}^C \left\| \tilde{p}_c - \tilde{q}_c \right\|_1 \leq \frac{\sqrt{12(C-1)} \sum_{c=1}^C d_c S_c \left( S_c - \frac{1}{2} \right)}{\sqrt{n}} + O\left( \frac{1}{n^{1/2}} \right). \tag{63}
\]

A corollary of theorem 6 and theorem 8 is the following:

**Corollary 9.** Consider some \( \left( N_c = (K, P_c, S_c, \nu_c, \bar{\mu}_c) \right)_{c=1}^C \) and \( m \geq 2 \), and let \( n_0 := \max_c S_c \cdot m \). In the topology induced by the metric

\[
d \left[ \left( \frac{1}{\sqrt{2}} \right)^C \left( \begin{array}{c} \cdots \cdots \\ \tilde{p}_c \\ \cdots \cdots \\ \tilde{q}_c \\ \cdots \cdots \\ \end{array} \right)_{c=1}^C, \left( \begin{array}{c} \cdots \cdots \\ \tilde{p}_c \\ \cdots \cdots \\ \tilde{q}_c \\ \cdots \cdots \\ \end{array} \right)_{c=1}^C \right] := \frac{1}{C} \sum_{c=1}^C \left\| \tilde{p}_c - \tilde{q}_c \right\|_1, \tag{64}
\]

and denoting \( \overline{X} \) the closure of a set \( X \) in this topology, it holds that

\[
\mathcal{L} \left( \left( N_c \right)_{c=1}^C \right) \subseteq \bigcap_{n=n_0}^\infty \mathcal{I} \left( \left( N_c \right)_{c=1}^C, n, m \right) \subseteq \mathcal{L} \left( \left( N_c \right)_{c=1}^C \right). \tag{65}
\]

---

\(^5\)To tie the notation together: we have that \( d_c = \prod_{p=1}^P (\#\text{out})_{\nu_c(p)} \).
7 Application: Correlated Sleeper

In this section, we apply the postselected inflation formalism introduced in sections 5 and 6 to the Correlated Sleeper protocol, which was introduced in section 2. This allows to demonstrate the use of the formalism in a simple example.

7.1 Feasible region

**Parametrization.** It will be useful to parametrize the achievable distributions in the Correlated Sleeper protocol as follows. The symmetry of the protocol and the marginal constraint (1) first give the following lemma. See appendix C.1 for the proof.

**Lemma 10.** Let \( p_1 \) and \( p_2 \) be as in (23b). Define for all \( c \in \{1, 2\} \)

\[
\lambda_c := \frac{1}{p_c}.
\]  

(66)

It then holds that

\[
\begin{align*}
\frac{1}{p_c} &= \lambda_c, \\
\frac{2}{p_c} &= \lambda_c, \\
\frac{2}{p_c} &= 1 - \lambda_c.
\end{align*}
\]

(67)

with

\[
\lambda_c \in \left[0, \frac{1}{2}\right].
\]  

(68)

It is thus sufficient for us to specify the value of the pair \((\lambda_1, \lambda_2)\) to characterize the output behavior of a given strategy used by the agent \(A\) in the Correlated Sleeper protocol. In this parametrization, the objective function of the optimization problem (23) is just \(\lambda_1 + \lambda_2\). For instance, the strategy of (2), where \(A\) only looks at her first input, leads to \((\lambda_1 = 1/2, \lambda_2 = 1/4)\) corresponding to a score of 3/4, while a completely mixed behavior (where \(A\) ignores her inputs and outputs a random bit) leads to \((\lambda_1 = 1/4, \lambda_2 = 1/4)\) corresponding to a score of 1/2. Generally speaking, if \((\lambda_1, \lambda_2)\) is feasible with a certain strategy, then using the same strategy while exchanging the role of the two inputs that \(A\) receives shows that also \((\lambda_2, \lambda_1)\) is feasible. Furthermore, we prove the following lemma in appendix C.1.

**Lemma 11.** Let \((\lambda_1, \lambda_2)\) be as in lemma 10. It holds for all \(c \in \{1, 2\}\) that

\[
\lambda_c \geq \frac{1}{4}.
\]  

(69)

This bound can be equivalently stated as \(\frac{2}{p_c} \leq 1/4\), which means that the best chance that \(A\) has to obtain distinct outcomes in the two round is to adopt a completely mixed behavior — any deviation from this will inevitably lead to increased correlations.

**Multi-network scenario.** Recall that in the formulation (37), we defined the feasible region of the Correlated Sleeper protocol with the causal compatibility problem

\[
\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{U_2}\right) \in \mathcal{L} \left(\mathcal{N}_1^{(s)}, \mathcal{N}_2^{(s)}, \mathcal{N}_3^{(s)}\right),
\]

(70)

where the relevant networks were defined in equations (27)-(29). Here, we want to add an extra network that will look redundant at first, but that will eventually yield non-trivial feasibility constraints once considered from the perspective of the postselected inflation test. This network, whose graph is shown
in figure 4, is one with three disconnected components where, in the first, the two $A$'s are connected with their left input, in the second the two $A$'s are connected with their right input, and the third is an isolated $A$:

$$\mathcal{N}_4^{(s)} = (K = 1, P_4 = 5, S_4 = 8, \nu_4, \vec{\mu}_4),$$  \quad (71a)$$
$$\forall p \in \{1, \ldots, 5\}, \ \nu_4(p) = 1, \quad \text{(only one strategy)} \quad (71b)$$
$$\vec{\mu}_4(1) = (1, 2), \ \vec{\mu}_4(2) = (1, 3), \quad \text{(first input connected)} \quad (71c)$$
$$\vec{\mu}_4(3) = (4, 6), \ \vec{\mu}_4(4) = (5, 6), \quad \text{(second input connected)} \quad (71d)$$
$$\vec{\mu}_4(5) = (7, 8). \quad \text{(isolated agent)} \quad (71e)$$

In this network, we want the distribution \( \frac{1}{p_1} \frac{1}{p_2} \frac{1}{U_2} \) with this output ordering, to be feasible.

We are now ready to define the feasible region \( L^{(s)} \) in the form that we will feed in to the postselected inflation:

$$L^{(s)} := \left\{ \left( \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{U_2} \right) \mid \left( \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{U_2} \right) \in L \left( \mathcal{N}_4^{(s)} \right)^4 \right\}. \quad (72)$$

Furthermore, the subset of \([1/4, 1/2] \times 2\) that corresponds to the parametrization of \( L^{(s)} \) according to lemmas 10 and 11 will be denoted \( L^{(s)} \).

7.2 Outer approximation of the feasible region

The outer approximation of the feasible region will be constructed thanks to a postselected inflation feasibility problem, but with two twists with respect to definition 5:

- We will not use the same parameter \( m \) across the four networks: this is to allow \( n \) to be not too large. Essentially, we will take the reasonable value of \( n = 4 \) and, for each network, we will let \( m \) be as large as possible given the postselection and \( n = 4 \). We indicate these values of \( m \) in the equations (73).
- We will not postselect on all the sources taking different values as in definition 5, but rather introduce a more minimal postselection scheme which allows, according to the arguments of section 5.4, to have a tighter outer approximation. In that sense, the mention of “inflation” in the equations (73) is to be understood as a small generalization of definition 5.

Recall that the style difference between the dashed and solid edges is there to guide the eye but implies the same operation of tensor contraction.

**Definition 12.** We let our outer approximation of the feasible region \( \mathcal{L}^{(s)} \) be the set \( \mathcal{I}^{(s)} \) defined as

$$\mathcal{I}^{(s)} := \left\{ \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \mid \exists \frac{1}{A}, \frac{1}{A} \text{ s.t.} \right\}. \quad (73a)$$
The corresponding subset of $[0, 1/2]^2$ in the parametrization of lemma 10 is denoted $I^{(s)}$.

Since we used a slightly different postselection compared to definition 5, let us briefly state that we still have an outer approximation of the feasible region:

**Lemma 13.** It holds that

$$L^{(s)} \subseteq I^{(s)},$$

or, in the parametrization of lemma 10, that $L^{(s)} \subseteq I^{(s)}$.

**Proof.** The proof is analogous to those of lemma 4 and theorem 6. If we let $A_0$ be the strategy that establishes that $(p_1, p_2) \in \mathcal{L}^{(s)}$ (as in equation (23b)), then the following tensors (defined
for all \( x_1, \ldots, x_4, y_1, \ldots, y_4 \in [0, 1], a \in \{1, 2\}, i, j \in \{1, \ldots, 4\} \)

\[
\begin{bmatrix}
A
\end{bmatrix} := \begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 & y_4
\end{bmatrix},
\tag{75a}
\]

\[
\begin{bmatrix}
a & a
\end{bmatrix} := \begin{bmatrix}
A_0
\end{bmatrix},
\tag{75b}
\]

will verify equations (73b)-(73e), thus proving that \( \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \in I^{(s)} \).

\[\square\]

7.3 Details about the implementation

**Semi-explicit linear program.** Let us give some additional details about the formulation of the question \((\lambda_1, \lambda_2) \in I^{(s)}\) as a linear program. We can assume without loss of generality that the source \( A \) appearing in equations (73) has as outputs the possible deterministic strategies that \( A \) will then use. We can conveniently represent these deterministic strategies as \( 4 \times 4 \) matrices with entries 1 or 2, corresponding to \( A \) outputting the value of the matrix at position \( i, j \) upon receiving the inputs \( i, j \) from the sources \( \begin{bmatrix} U_4 \end{bmatrix} \). Let us denote such matrices as \( M \) and the set thereof as \( \mathcal{M} \) (containing a priori \( 2^{16} = 65536 \) elements). We can thus rewrite the integral over the outputs of \( \Lambda \) as a finite sum where we make the first couple terms explicit:

\[
\int d\lambda \begin{bmatrix} \lambda \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix} = \sum_{M \in \mathcal{M}} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix} = \begin{bmatrix}
(1 & 1 & 1)
(1 & 1 & 1)
(1 & 1 & 1)
\end{bmatrix} + \begin{bmatrix}
(1 & 1 & 1)
(1 & 1 & 2)
(1 & 1 & 2)
\end{bmatrix} + \ldots
\]

(76)

Testing for \((\lambda_1, \lambda_2) \in I^{(s)}\) thus amounts to optimizing the coefficients \( \begin{bmatrix} M \end{bmatrix} \) so that they fulfill the conditions (73b)-(73e). For instance, the 1, 1, 1, 1 component of equation (73b) looks like:

\[
\sum_{M \in \mathcal{M}} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{4!} \sum_{i_1, i_2, j_1, j_2 \in \{1, \ldots, 4\}} \delta(M_{i_1, j_1} = 1) \delta(M_{i_2, j_2} = 1) \delta(M_{i_2, j_3} = 1) \delta(M_{i_2, j_4} = 1)
\]

\[
= 1 \cdot \begin{bmatrix}
1 & 1 & 1 & 1
1 & 1 & 1 & 1
\end{bmatrix} + \frac{3}{4} \cdot \begin{bmatrix}
1 & 1 & 1 & 1
1 & 1 & 1 & 2
\end{bmatrix} + \ldots
\]

(77)

**Symmetry reduction.** We can in fact reduce the variable set by noting that any two \( M_1, M_2 \in \mathcal{M} \) where \( M_2 \) can be obtained by permuting the rows and columns of \( M_1 \) will yield the same behavior within any of equations (73b)-(73e). We can thus restrict without loss of generality the source \( \begin{bmatrix} \Lambda \end{bmatrix} \) to only have outputs in the subset \( \mathcal{M}_{\text{red}} \subset \mathcal{M} \) of representatives of the orbits of \( \mathcal{M} \) under the group action induced by swapping rows and columns. This subset \( \mathcal{M}_{\text{red}} \) is found numerically to have cardinality 317. We can furthermore remove certain redundant components of the equations (73b)-(73e): indeed, we can see from the structure of the left-hand side tensor contractions that certain permutations of the outputs of the agents gives the very same constraints. For instance, in (73b), we have that the component \((a_1, a_2, a_3, a_4)\) will give the same constraint as that of \((a_2, a_1, a_3, a_4)\) as well as \((a_1, a_2, a_4, a_3)\) and \((a_3, a_4, a_1, a_2)\). This only gives as useful constraints the \((1, 1, 1, 1), (1, 1, 1, 2), (1, 2, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2)\) and \((2, 2, 2, 2)\) components of (73b). However, one of these equations is trivially verified.

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30
thanks to the normalization constraint on $\Lambda$, so that we can for instance remove the (2, 2, 2, 2) component. We summarize the linear program taking into account these symmetry reductions in appendix C.2.

Figure 5: Numerical results: we show a sample of points belonging to $L(s)$, as well as a sample of points of $[1/4, 1/2] \times 2 \setminus I(s)$, which are points that are guaranteed to lie outside of $L(s)$ and thus correspond to infeasible correlations.

7.4 Numerical results: inner and outer approximations

We explore the inside of $L(s)$ by looking at deterministic strategies that process their continuous left (resp. right) input into a uniformly distributed discrete random variable over the set \{1, ..., $n$\} (resp. \{1, ..., $m$\}), and refer to the corresponding set of strategies as “the $n \times m$ strategies”. There are $2^{nm}$ such strategies without the marginal constraint (1). To implement the marginal constraint, we first make sure that $nm$ is an even integer, and we then fill the strategy with $nm/2$ 1’s and $nm/2$ 2’s, for a total of $\binom{nm/2}{2}$ deterministic $n \times m$ strategies that verify the marginal constraint (1). The resulting points, for reasonable values of $n$ and $m$, are shown in figure 5: they populate the bottom left corner of the parameter space, corresponding to a score $\lambda_1 + \lambda_2$ less than $3/4$. The density of these points is relatively low — it could well be at this stage that higher values of $n$ and $m$ would allow for better scores.

To certify that this is not the case, and gain further insights into the geometrical structure of the feasible region, we scan the $[1/4, 1/2] \times 2$ square by discretizing it into a uniform mesh, and keep in memory the points of the mesh that are found numerically to be incompatible with an inflation of the form of definition 12, i.e., those points for which the corresponding linear program is found numerically...
to be infeasible: such points are guaranteed to lie outside of $L^{(s)}$. A sample of these points are shown in figure 5. The infeasible region $[1/4, 1/2] \times I^{(s)}$ seems to have a smooth shape: let us assume that this is the case. We can then track the boundary $\partial I^{(s)}$ of the set $I^{(s)}$ efficiently: for each fixed $\lambda_1 \in [1/4, 1/2]$, we run a dichotomic search on $\lambda_2$ to find the threshold value between feasibility and infeasibility of the linear program corresponding to definition 12. The resulting line (which is strictly speaking a dense mesh of data points) is shown in figure 5 as well. We can already see from the data of the infeasible region, assuming that all our meshes were sufficiently fine-grained, that it looks like the best achievable score is $\lambda_1 + \lambda_2 = 3/4$ — we draw the corresponding parameter range to guide the eye. Additionally, in appendix C.3, we provide an extended plot that shows the feasibility with respect to the inflation of definition 12 of more general distributions in the extended square $[0, 1/2] \times [2]$; indeed, the inflation of definition 12 is not expected to, and does not, capture exactly the bound $\lambda_c \geq 1/4$ of lemma 11.

7.5 Solving the optimization

We now present the linear program relaxation\(^6\) to the optimization (23) that we will use:

\[
p^{(\text{inf})} := \max_{\Lambda, A} \frac{1}{2} \sum_{a \in \{1, 2\}} \left( \begin{array}{c}
A \\
1/2
\end{array} \right)
\]

\[
\text{s.t.} \quad \left( \begin{array}{c}
A \\
1/2
\end{array} \right) = \left( \begin{array}{c}
U_4 \\
U_4
\end{array} \right)
\]

(78a)

This is indeed a linear program according to the same logic as section 7.3: one can assume without loss of generality that the source $\Lambda$ tells the agent $A$ what to do, so that one can remove the optimization over $\{A, 1/2\}$. The objective and constraint are then linear functions of $\Lambda$. Let us state that the linear program (78) is indeed an upper bound to the original value $p^*$ of (23):

**Lemma 14.** It holds that

\[
p^* \leq p^{(\text{inf})}.
\]

(79)

**Proof.** The proof of lemma 13 can be used to show that any feasible value in the optimization problem defining $p^*$ induces a feasible value for the optimization problem defining $p^{(\text{inf})}$. Indeed, consider an objective value $\tilde{p}$ (less than or equal to $p^*$) that is feasible in equation (23), and let the probability

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\(6\)Since (78) is a feasible linear program, we know the maximum is achievable, so we replace the supremum with a maximum.
tensor that achieves it be $A_0$. Then, the probability tensors $\Lambda$ and $A$ constructed from $A_0$ as in equation (75) are feasible in the linear program (78) and yield the same objective value $\tilde{p}$. □

Now, let us in fact focus on the dual problem to (78):

$$d(\text{inf}) := \min_{z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{R}} \frac{1}{4} \sum_{a, b \in \{1, 2\}} z_{ab}$$  \hspace{1cm} (80a)

s.t. $\forall A$:

$$\sum_{a,b \in \{1,2\}} z_{ab} \geq \frac{1}{2} \sum_{c \in \{1,2\}} \frac{a}{2} U_4 U_4 U_4 U_4 + \frac{b}{2} U_4 U_4 U_4 U_4$$  \hspace{1cm} (80b)

This dual problem is written in standard notation in appendix C.2. The relation between the primal and the dual can be understood intuitively as follows. Consider an agent $A$ who made some choice of strategy $A$ and shared randomness $\Lambda$ that are feasible in the primal (78) (i.e., that verify (78b)). This agent $A$ was told that she will receive an amount of money equal, in some units, to the objective of (78a). Now, suppose the organizer of the protocol, who does not know the choice of strategy of the agent $A$, will actually run an alternative protocol where the agent $A$ is put in the network of (78b), and want to give $A$ some amount of money $z_{ab}$ (in the same units) when the two $A$’s outputs the outcomes $a$ and $b$ in this network. The organizer wants to ensure that no matter what $A$ is actually doing, the amounts $\{z_{ab}\}_{ab}$ are chosen fairly so that $A$ will receive at least the amount of money that she would have gotten in the original protocol — this is what (80b) captures. The organizer wants however to minimize the average cost of this alternative protocol, if performed on an honest $A$ that respects (78b) — this is what the minimization of (80a) captures. In particular, weak duality holds:

**Lemma 15.** It holds that

$$p(\text{inf}) \leq d(\text{inf}).$$  \hspace{1cm} (81)

**Proof.** Consider some $\Lambda$, $A$ that are feasible in the primal problem of $p(\text{inf})$ (i.e., they verify (78b) but do not necessarily achieve the maximum value of $p(\text{inf})$), and some $\{z_{ab}\}_{ab}$ that are feasible in the dual problem (i.e., they verify (80b) but do not necessarily achieve the minimum value of $d(\text{inf})$). Then, using the non-negativity and normalization of $\Lambda$ (recall that these constraints are always implicitly assumed when we draw such probability tensors), we can compute:
Maximizing (resp. minimizing) the first (resp. last) term of this inequality yields that indeed \( p^{(\inf)} \leq d^{(\inf)} \).

It is now easy to solve the optimization:

**Proposition 16.** It holds that
\[
p^* = \frac{3}{4}.
\]

**Proof.** We saw in section 2 that \( p^* \geq 3/4 \). Combining lemmas 14 and 15 together gives \( p^* \leq d^{(\inf)} \).

The choice of
\[
z_{11} = z_{22} = 1, \quad z_{12} = z_{21} = \frac{1}{2}
\]
is feasible in the dual problem (80): indeed, for all \( \begin{bmatrix} A \\ \Lambda \end{bmatrix} \), which one may enumerate since there are \( 2^{16} \) of them,\(^7\) the condition of (80b) is verified — this can be checked numerically using rational arithmetic. The choice of (84) corresponds to an objective value in (80a) of 3/4, which proves at first that \( d^{(\inf)} \leq 3/4 \), and then also \( p^* = p^{(\inf)} = d^{(\inf)} = 3/4 \).

---

\(^7\)This follows from \( \begin{bmatrix} A \\ \Lambda \end{bmatrix} \) having 2 outcomes and 4 · 4 different inputs. Alternatively, one may simply enumerate the inequivalent strategies belonging to \( \mathcal{M}_{\text{red}} \) — see section 7.3.
8 The fanout inflation correspondence

In this section, we show through three examples of the fact that to each postselected inflation corresponds a fanout inflation (in the usual sense of e.g. [WSF19, NW20]). This implies that the framework that we propose can be seen as a reformulation of the fanout inflation framework. Each formulation comes with its own insights: arguably, the principles underlying the fanout inflation frameworks are easy to explain from physical principles, clearly yield outer approximations, and the same principles allow to formulate quantum or non-signaling inflation schemes [WPKG+21, GBC+20]. The postselected inflation formulation has the benefit that establishing convergence is relatively intuitive as explained in section 5. It also allows to prove straightforwardly the formal convergence of inflation in multi-network scenarios with subsets of agents using the same strategy. Now, for each multi-network scenario, given the specific proof of convergence formulated in the context of postselected inflation, a fanout inflation proof is of course readily available once one extract the correct hierarchy of fanout inflations. In that sense, the postselected inflation framework can be seen as a convenient parametrization of fanout inflation in which the general rule to produce convergent hierarchies is clear.

8.1 Correlated Sleeper

Recall that in definition 12, we introduced the set $I(s)$. We claim that this set has an alternative definition in terms of a fanout inflation sketched in figure 6.

![Figure 6: The fanout inflation graph of $I^{(s)}_{alt}$. We highlight with hyperedges the subsets of nodes whose marginal distribution is expressible in terms of the target distributions — see equations (85c)-(85f).](image)

In the following definition, $q$ is a probability distribution over $\{1, 2\}^{16}$, namely, over the possible outcome tuples of 16 copies of the agent $A$. The copies of the agent $A$ are indexed in a matrix by two indices $i, j \in \{1, 2, 3, 4\}$. We denote by $q(\{A_{ij} = a_{ij}\}_{i,j=1}^{4})$ the probability of an atomic event, while e.g. $q(\{A_{11} = a_{11}, A_{22} = a_{22}\})$ represents the marginal probability that the agent $A_{ii}$ outputs $a_{ii}$ for $i \in \{1, 2\}$. We also let $S_4$ denote the group of permutations of 4 elements.
Lemma 17. Define

\[ \mathcal{I}^{(s)}_{\text{alt}} := \left\{ \left( \frac{p_1}{p_2} \right) \bigg| \exists \text{ a probability distribution } q \text{ over the outcomes of the agents } A_{ij} \right\} \]

of the graph of figure 6 such that \( \forall \sigma, \pi \in S_4, \forall \{a_{ij} \in \{1, 2\}\}_{i,j=1}^4 : \)

\[ q(\{A_{ij} = a_{ij}\}_{i,j=1}^4) = q(\{A_{ij} = a_{\sigma(i)\pi(j)}\}_{i,j=1}^4), \quad (85a) \]

\[ q(\{A_{11} = a_{11}, A_{12} = a_{12}, A_{23} = a_{23}, A_{24} = a_{24}\}) = \frac{a_{11}}{p_1} \frac{a_{11}}{p_1} \frac{a_{23}}{p_1} \frac{a_{24}}{p_1}, \quad (85b) \]

\[ q(\{A_{11} = a_{11}, A_{21} = a_{21}, A_{32} = a_{32}, A_{42} = a_{42}\}) = \frac{a_{11}}{p_2} \frac{a_{21}}{p_2} \frac{a_{32}}{p_2} \frac{a_{42}}{p_2}, \quad (85c) \]

\[ q(\{A_{ii} = a_{ii}\}_{i=1}^4) = \frac{a_{11}}{U_2} \frac{a_{21}}{U_2} \frac{a_{33}}{U_2} \frac{a_{44}}{U_2}, \quad (85d) \]

\[ q(\{A_{11} = a_{11}, A_{12} = a_{12}, A_{23} = a_{23}, A_{33} = a_{33}, A_{44} = a_{44}\}) = \frac{a_{11}}{p_1} \frac{a_{11}}{p_1} \frac{a_{23}}{p_2} \frac{a_{33}}{p_2} \frac{a_{44}}{U_2}. \quad (85e) \]

Then, it holds that

\[ \mathcal{I}^{(s)}_{\text{alt}} = \mathcal{I}^{(s)}. \quad (86) \]

Proof. \( \mathcal{I}^{(s)}_{\text{alt}} \subseteq \mathcal{I}^{(s)} \): for any \( \left( \frac{p_1}{p_2} \right) \in \mathcal{I}^{(s)}_{\text{alt}} \), consider the \( q \) distribution associated to it. Let us write \( q \) as a mixture of deterministic probability distributions \( q^{(\lambda)} \) with convex weights \( A^{(\lambda)} \):

\[ q := \sum_{\lambda} \frac{A^{(\lambda)}}{\Lambda} q^{(\lambda)}. \quad (87) \]

Define the deterministic\(^8\) probability tensor, for all \( a \in \{1, 2\}, i, j \in \{1, 2, 3, 4\} \), for all \( \lambda \),

\[ A^{(\lambda)} := q^{(\lambda)}(A_{ij} = a). \quad (88) \]

This combination of \( A^{(\lambda)} \) and \( A^{(\lambda)} \) solves the postselected inflation for \( \left( \frac{p_1}{p_2} \right) \): for instance, we can verify equation (73b) explicitly: for all \( a_{11}, a_{12}, a_{23}, a_{24} \in \{1, 2\} \),

\[ \text{Figure 6} \]  

---

\(^8\)Indeed, the marginal of a deterministic distribution is still deterministic.
\[
\sum_\lambda \frac{1}{12 \cdot 4!} \sum_{i_1, i_2 \; \text{all } \neq j_1, \ldots, j_4 \; \text{all } \neq} \text{a}_{i_1 j_1} \text{a}_{i_1 j_2} = \text{a}_{12}, \text{a}_{i_2 j_3} = \text{a}_{23}, \text{a}_{i_2 j_4} = \text{a}_{24})
\]

\[
eqs (87), (88) = \frac{1}{12 \cdot 4!} \sum_{i_1, i_2 \; \text{all } \neq j_1, \ldots, j_4 \; \text{all } \neq} \text{q}(\{\text{a}_{i_1 j_1} = \text{a}_{11}, \text{a}_{i_1 j_2} = \text{a}_{12}, \text{a}_{i_2 j_3} = \text{a}_{23}, \text{a}_{i_2 j_4} = \text{a}_{24}\})
\]

\[
\sigma(1) := i_1, \ldots, \pi(4) := j_4 = \frac{1}{4! \cdot 4!} \sum_{\sigma, \pi \in S_4} \text{q}(\{\text{a}_{\sigma(1) \pi(1)} = \text{a}_{11}, \text{a}_{\sigma(1) \pi(2)} = \text{a}_{12}, \text{a}_{\sigma(2) \pi(3)} = \text{a}_{23}, \text{a}_{\sigma(2) \pi(4)} = \text{a}_{24}\})
\]

(89b)

The other cases are proven analogously, so that indeed \( (\begin{array}{c} p_1 \\ p_2 \end{array} \) \( \in \mathcal{I}^{(s)} \).

\[\mathcal{I}^{(s)} \supseteq \mathcal{I}^{(s)}: \text{ for any } (\begin{array}{c} p_1 \\ p_2 \end{array} \) \( \in \mathcal{I}^{(s)} \), consider the probability tensors \( \begin{array}{c} A \\ 1 \end{array} \) \( \) and \( \begin{array}{c} A \\ 1 \end{array} \) \( \) (the latter can be assumed to have finitely many outputs \( \lambda \) without loss of generality — see section 7.3) associated to it. Define the distribution \( \text{q} \) through, for all \{\text{a}_{ij} \in \{1, 2\}\}_{i,j=1}^4

\[
\text{q}(\{\text{a}_{ij} = \text{a}_{ij}\}_{i,j=1}^4) := \sum_\lambda \frac{1}{4! \cdot 4!} \sum_{\sigma, \pi \in S_4} \prod_{i,j=1}^{4} \text{a}_{\sigma(i) \pi(j)}
\]

(90)

This \( \text{q} \) clearly verifies the symmetry condition (85b). It furthermore verifies the desired marginal constraints: for instance, for (85c), we have

\[
\text{q}(\{\text{a}_{11} = \text{a}_{11}, \text{a}_{12} = \text{a}_{12}, \text{a}_{23} = \text{a}_{23}, \text{a}_{24} = \text{a}_{24}\})
\]

(91a)

\[
\sum_\lambda \frac{1}{4! \cdot 4!} \sum_{\sigma, \pi \in S_4} \text{a}_{11} \text{a}_{12} \text{a}_{23} \text{a}_{24}
\]

(91b)

The other marginal constraints are verified analogously, and indeed we see that (\begin{array}{c} p_1 \\ p_2 \end{array} \) \( \in \mathcal{I}^{(s)} \).
8.2 Triangle network: three strategies

Let us give two additional examples. First off, consider the triangle network, sketched in figure 7:

\[ N^{(tr)} = (K = 3, P = 3, S = 3, \nu, \mu), \]  

\[ \forall p \in \{1, 2, 3\}, \nu(p) = p, \]  

\[ \mu(1) = (2, 3), \mu(2) = (3, 1), \mu(3) = (1, 2). \]  

We can formulate a postselected inflation with a more minimal postselection than that of definition 5 following the argument of lemma 4 (this type of postselected inflation is generic in the context of a single-network scenario with all agents using distinct strategies): let

\[ \mathcal{I}^{(tr)}(n, m = 2) := \left\{ \exists A, B, C, \mu \text{ s.t.} \right\}. \]  

The above characterization coincides with the fanout inflation of the triangle network as in [NW20]. Leaving the number of outcomes implicit, let, for any \( n \geq 2 \),

\[ \mathcal{I}^{alt}_{(tr)}(n, m = 2) := \left\{ \exists a probability distribution \: q \: \text{over the outcomes of the agents} \: \{A_{ij}\}_{i,j=1}, \{B_{kl}\}_{k,l=1}, \{C_{pq}\}_{p,q=1} \text{ such that} \forall \{a_{ij}\}_{i,j=1}, \{b_{kl}\}_{k,l=1}, \{c_{pq}\}_{p,q=1} \right\}. \]  

The corresponding fanout inflation graph is shown in figure 8 for the case of \( n = 2 \).
Figure 8: The fanout inflation graph for of $\mathcal{I}_{\text{alt}}^{(tr)}(n,m = 2)$ for $n = 2$. The red hyperedge represents the known marginal of equation (94c).

Let us briefly sketch the equality proof (which can easily be generalized to arbitrary $m$).

**Lemma 18.** It holds that $\mathcal{I}_{\text{alt}}^{(tr)}(n,m = 2) = \mathcal{I}^{(tr)}(n,m = 2)$.

**Proof.** Given $\|p\| \in \mathcal{I}_{\text{alt}}^{(tr)}(n,m = 2)$ and the $q$ distribution associated to it, write $q$ as a mixture of deterministic distributions $q^{(\lambda)}$ with convex weights $\frac{\lambda}{\Lambda}$:

$$q =: \sum_{\lambda} \frac{\lambda}{\Lambda} q^{(\lambda)}.$$  \hspace{1cm} (95)

Then, define the deterministic tensors, for all $i,j,k,l,p,q \in \{1, \ldots, n\}$ and $a,b,c,\lambda$:

$$\begin{align*}
\begin{array}{c|c}
\hline
A & a \\
\hline
i,j & \lambda \\
\hline
\end{array} & := q^{(\lambda)}(A_{ij} = a), \\
\begin{array}{c|c}
\hline
B & b \\
\hline
k,l & \lambda \\
\hline
\end{array} & := q^{(\lambda)}(B_{kl} = b), \\
\begin{array}{c|c}
\hline
C & c \\
\hline
p,q & \lambda \\
\hline
\end{array} & := q^{(\lambda)}(C_{pq} = c). \hspace{1cm} (96)
\end{align*}$$

The probability tensors $\begin{array}{c|c}
\hline
A & \Lambda \\
\hline
\end{array}$, $\begin{array}{c|c}
\hline
B & \Lambda \\
\hline
\end{array}$, $\begin{array}{c|c}
\hline
C & \Lambda \\
\hline
\end{array}$, $\Lambda$ verify equation (93b) thanks to equations (94b) and (94c) — the manipulations are analogous to those of the proof of lemma 17 — so that $\|p\| \in \mathcal{I}^{(tr)}(n,m = 2)$.

Conversely, given $\|p\| \in \mathcal{I}^{(tr)}(n,m = 2)$ and the tensors $\begin{array}{c|c}
\hline
A & \Lambda \\
\hline
\end{array}$, $\begin{array}{c|c}
\hline
B & \Lambda \\
\hline
\end{array}$, $\begin{array}{c|c}
\hline
C & \Lambda \\
\hline
\end{array}$, $\Lambda$ associated to it, define the probability distribution $q$ such that, for all $\{a_{ij}\}_{i,j=1}^{n}$, $\{b_{kl}\}_{k,l=1}^{n}$, $\{c_{pq}\}_{p,q=1}^{n}$,
\[ q(\{A_{ij} = a_{ij}, B_{kl} = b_{kl}, C_{pq} = c_{pq}\}) := \lambda^{\sum_{\lambda} \prod_{i,j=1}^{n} A_{ij} \prod_{k,l=1}^{n} B_{kl} \prod_{p,q=1}^{n} C_{pq}}. \]  

This distribution \( q \) verifies the symmetry condition (94b) by design, and verifies (94c) thanks to (93b) after some manipulations similar to those of lemma 17. This implies \( p \in I_{\text{alt}}^{(n,m = 2)} \).

Notice that as \( n \) grows, the outer approximation \( I_{\text{alt}}^{(n,m = 2)} \) will converge to the set \( L(\mathcal{N}_{\text{tr}}) \); this is a special case of the proof of [NW20]. It can also be seen intuitively from equation (93b) together with the arguments of section 5, and formally from theorem 8 after adapting the proof to the more minimal postselection of equation (93b).

### 8.3 Triangle network: one strategy

Now, what about the case of the triangle network, but with only one strategy for the agents? This defines the network

\[
\mathcal{N}_{\text{tr}} = (K = 1, P = 3, S = 3, \nu, \bar{\mu}),
\]

\[
\forall p \in \{1, 2, 3\}, \quad \nu(p) = 1, \quad (\text{one strategy})
\]

\[
\bar{\mu}(1) = (2, 3), \quad \bar{\mu}(2) = (3, 1), \quad \bar{\mu}(3) = (1, 2),
\]

which we sketch in figure 9.

![Figure 9: The network \( \mathcal{N}_{\text{tr}} \), corresponding to the triangle network with only one strategy for the three agents.](image)

A set of outer approximations that converges to the set \( L(\mathcal{N}_{\text{tr}}) \) would be those defined in definition 5 — in this case, it does not seem to be possible to reduce the amount of postselection. Let us choose \( m = 2 \); we then need \( n \geq 6 \) to have a feasible postselection on the \( m \cdot S = 6 \) sources present in the postselected inflation. The general construction of definition 5 directly yields the following outer approximation:

\[
I_{\text{alt}}^{(n,m = 2)} := \left\{ \left| \frac{\Lambda}{p} \right|: \exists \left| \frac{A}{\prod_{\lambda}} \right|, \quad \text{s.t.} \right\}. \]

![Figure 9: The network \( \mathcal{N}_{\text{tr}} \), corresponding to the triangle network with only one strategy for the three agents.](image)

A set of outer approximations that converges to the set \( L(\mathcal{N}_{\text{tr}}) \) would be those defined in definition 5 — in this case, it does not seem to be possible to reduce the amount of postselection. Let us choose \( m = 2 \); we then need \( n \geq 6 \) to have a feasible postselection on the \( m \cdot S = 6 \) sources present in the postselected inflation. The general construction of definition 5 directly yields the following outer approximation:

\[
I_{\text{alt}}^{(n,m = 2)} := \left\{ \left| \frac{\Lambda}{p} \right|: \exists \left| \frac{A}{\prod_{\lambda}} \right|, \quad \text{s.t.} \right\}. \]

![Figure 9: The network \( \mathcal{N}_{\text{tr}} \), corresponding to the triangle network with only one strategy for the three agents.](image)
Notice that the agent $A$ will never see the two inputs from the $U_6$ sources being equal in the network of equation (99): computationally speaking, one can thus safely assume that the agent output a fixed, default outcome in this case, or even avoid storing that information entirely. This is apparent in the fanout inflation formulation.

Figure 10: The fanout inflation graph of $\bar{I}_{\text{alt}}(n = 6, m = 2)$. The red hyperedge represents the known marginal of equation (100c).

The corresponding fanout inflation, whose graph is shown in figure 10, is the following:

$$\bar{I}_{\text{alt}}(n = 6, m = 2) := \left\{ \begin{array}{c} p \\ \exists \text{ a probability distribution } q \text{ over the outcomes of the agents } A_{ij} \\
\forall \sigma \in S_6 : q(\{A_{ij} = a_{ij}\}_{i \neq j \in \{1, \ldots, 6\}}) = q(\{A_{ij} = a_{\sigma(i)\sigma(j)}\}_{i \neq j \in \{1, \ldots, 6\}}), \\
q(\{A_{12} = a_{12}, A_{23} = a_{23}, A_{31} = a_{31}, A_{45} = a_{45}, A_{56} = a_{56}, A_{64} = a_{64}\}) = \left\{ \begin{array}{c} a_{12} a_{21} a_{23} a_{31} a_{45} a_{56} a_{64} \\
p \end{array} \right\}. \right\} \tag{100c}$$

The equality between the two characterizations works as in the previous two examples:

**Lemma 19.** It holds that $\bar{I}_{\text{alt}}(n = 6, m = 2) = \bar{I}(n = 6, m = 2)$.

**Proof.** Given $\left\{ \begin{array}{c} p \\
p \end{array} \right\} \in \bar{I}_{\text{alt}}(n = 6, m = 2)$ and the associated $q$, decompose $q$ as a mixture of deterministic
behaviors

\[ q := \sum_{\lambda} A^{(\lambda)} q^{(\lambda)} \quad (101) \]

and define the deterministic tensor \( A \) for all \( i \neq j \in \{1, \ldots, 6\} \) and \( a, \lambda \) through 9

\[ \begin{array}{c}
\begin{array}{c}
\overline{A} \\
i_j \lambda
\end{array}
\end{array} := q^{(\lambda)}(A_{ij} = a). \quad (102) \]

This will show that \( \overline{p} \in \overline{I}^{(tr)}(n = 6, m = 2) \).

Conversely, given \( \overline{p} \in \overline{I}^{(tr)}(n = 6, m = 2) \) and the associated \( \overline{A}, \Lambda \), define the probability distribution \( q \) for all \( \{a_{ij}\}_{i \neq j \in \{1, \ldots, 6\}} \) through

\[ q(\{A_{ij} = a_{ij}\}_{i \neq j \in \{1, \ldots, 6\}}) := \sum_{\lambda} A \prod_{\sigma \in S_6 \setminus i \neq j \in \{1, \ldots, 6\}} \frac{1}{6!} \prod_{i \neq j \in \{1, \ldots, 6\}} A_{\sigma(i) \sigma(j)}^{a_{\sigma(i) \sigma(j)}}. \quad (103) \]

This will show that \( \overline{p} \in \overline{I}^{(tr)}_{alt}(n = 6, m = 2) \).

The above set \( \overline{I}^{(tr)}(n = 6, m = 2) \), whose generalization to arbitrary \( n \) is clear from definition 5, would indeed converge to the set \( \mathcal{L}(\overline{N}^{(tr)}) \) thanks to theorem 8. However, at the finite order of \( n = 6 \) that we are considering here, we are missing one constraint: this is the constraint that the postselected inflation of equation (99) should additionally verify

\[ \begin{array}{c}
\begin{array}{c}
\overline{A} \\
n \neq 6
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
p \quad p \quad p \quad p \quad p \quad p
\end{array}
\end{array}. \quad (104) \]

The corresponding fanout inflation of equation (100) should verify

\[ \forall a_{12}, a_{34}, a_{56} : q(\{A_{12} = a_{12}, A_{34} = a_{34}, A_{56} = a_{56}\}) = \begin{array}{c}
\begin{array}{c}
p \\
a_{12} \quad a_{14} \quad a_{56}
\end{array}
\end{array}. \quad (105) \]

The behavior whenever \( i = j \) is irrelevant.
9 Outlook

In this work, we introduced the postselected inflation framework that can be seen as a reformulation of the fanout inflation framework as exemplified in section 8. Despite the mathematical equivalence, the postselected inflation framework allows to conveniently devise converging outer approximations of the set of distributions causally compatible with a given classical multi-network scenario, in particular in the case where several agents are using the same strategy. The general idea behind the convergence of these outer approximations was presented in section 5 and formally proven in section 6.

Further developments? Certain basic problems of causal compatibility remain open to this day. An interesting example is the outcome distribution of \cite{Gis17} that is causally compatible with the quantum triangle network. There, although it is believed that the distribution is not causally compatible with the classical triangle network, the inflation technique is not able to certify this causal incompatibility with modern computing power. Successfully proving this incompatibility may involve the formulation of efficient outer approximation schemes to supplement the inflation framework.

Quantum analogues? The quantum analogues of fanout inflation in the context of networks featuring quantum sources are very natural to formulate, and were extensively studied \cite{WPKG+21, LGG21}. Whether the postselected inflation formulation may open the door to alternative outer approximation schemes in the quantum case is open.
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Software

The numerical simulations were run in Python v3.8.10 (Python Software Foundation, python.org). The linear programming library used is MOSEK’s Python Optimizer API v9.2.45 (MOSEK ApS, mosek.com). Please contact vgitton@ethz.ch to gain access to the code and the detailed data. The tensor networks, the network graphs and the plots were generated thanks to the TikZ and PGF packages v3.1.9a (The TikZ and PGF packages, pgf-tikz.github.io).

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A Deterministic strategies are sufficient for the Correlated Sleeper

Here we state explicitly the definition of the norms we shall use.

Definition 20 (p-norms). Let $p \in \mathbb{R}$, $p \geq 1$, $k \in \mathbb{N}$, and $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$. We define

$$\|x\|_p = \left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}}.$$  \hspace{1cm} (106)

This definition will be extended in the obvious way to linear combinations of probability tensors that share the same finite output domain and that have no inputs. For instance, we can write

$$\|p_1 - p_2\|_1 = \sum_{a_1, a_2} \left|\frac{a_1}{p_1} - \frac{a_2}{p_2}\right|.$$  \hspace{1cm} (107)

A.1 Deterministic approximation

We first prove a result regarding the set of feasible distributions in the context of the multi-network scenario described by the two networks $N_1^{(s)}$ and $N_2^{(s)}$ of equations (27) and (28); namely, that the set of feasible distributions allowing for non-deterministic strategies is the closure of the set $\mathcal{L}(N_1^{(s)}, N_2^{(s)})$, which only allows for deterministic strategies (see definition 2). The proof is based on the idea that a deterministic strategy taking the sum mod 2 of the two inputs (discretized into bits) can generate local randomness. This lemma does not yet take into account the uniform-marginal constraint: this will be covered in lemma 23.

Lemma 21. Let $\left(\begin{array}{c}p_1 \\ p_2 \end{array}\right)$ be such that there exists a (Riemann integrable, as usual) probability tensor $A_0$ with

$$\begin{array}{c}
p_1 \\ U_\infty \end{array} = \begin{array}{ccc}
A_0 & A_0 \\ U_\infty & U_\infty \end{array}, \quad \begin{array}{c}
p_2 \\ U_\infty \end{array} = \begin{array}{ccc}
A_0 & A_0 \\ U_\infty & U_\infty \end{array}.$$

Then, for all $\epsilon > 0$, there exists a deterministic probability tensor $A$ such that

$$\left\|\begin{array}{c}p_1 \\ U_\infty \end{array} - \begin{array}{c}A \\ U_\infty \end{array}\right\|_1 \leq \epsilon, \quad \left\|\begin{array}{c}p_2 \\ U_\infty \end{array} - \begin{array}{c}A \\ U_\infty \end{array}\right\|_1 \leq \epsilon.$$  \hspace{1cm} (109)

Proof. We can always rewrite $\begin{array}{c}A_0 \\ U_\infty \end{array}$ as a deterministic strategy $\begin{array}{c}A_1 \\ U_\infty \end{array}$ with an extra input connected to a local source of randomness:

$$\begin{array}{c}
A_0 \\ U_\infty \end{array} = \begin{array}{c}
A_1 \\ U_\infty \end{array}.$$  \hspace{1cm} (110)
We will make use of our assumption of Riemann integrability to approximate this newly introduced source. In particular, there exists \( n \in \mathbb{N} \) such that
\[
\left\| p_1 - A_1 A_1 \right\|_1 \leq \epsilon, \tag{111a}
\]
\[
\left\| p_2 - A_1 A_1 \right\|_1 \leq \epsilon. \tag{111b}
\]

The deterministic tensor \( A_1 \) that achieves (109) will be one that is such that
\[
A_1 A_1 = A_1 A_1, \tag{112a}
\]
\[
A_1 A_1 = A_1 A_1. \tag{112b}
\]

How can \( A_1 \) simulate a local source of randomness using only a deterministic function of the two inputs, \( \alpha \) and \( \beta \)? Surely, using, say, the left input \( \alpha \) only as a tentative source of local randomness will not do the trick when trying to reproduce \( \frac{p_1}{\epsilon} \). However, suppose that we define an extractor function labeled \( E_n \) defined through: for all \( i \in \{1, \ldots, n\} \), for all \( \alpha, \alpha' \in [0, 1] \),
\[
E_n^{\alpha \alpha'} := \delta(i - (1 + n\alpha))\delta(\alpha' - (n\alpha - |n\alpha|)). \tag{113}
\]

\(^{10}\)Technically, going from (110) to (111), the tensor \( A_1 \) needs to apply a rescaling of the input \( i \in \{1, \ldots, n\} \) coming from the \( U_n \) to map it to \( i/n \in [0, 1] \), but we leave this \( n \)-dependence implicit.
For instance, in the case \( n = 3 \),
\[
\begin{bmatrix}
  i \\
  \frac{3}{2} + \frac{1}{6}
\end{bmatrix}
\begin{bmatrix}
  \alpha' \\
  2
\end{bmatrix}
= \delta (i - 3) \delta \left( \alpha' - \frac{1}{3} \right).
\]
(114)

We furthermore define a sum modulo \( n \) function as, for all \( i_1, i_2, i_3 \in \{1, \ldots, n\} \):
\[
\oplus_n \begin{bmatrix}
  i_3 \\
  i_1 \\
  i_2
\end{bmatrix}
:= \delta \left( i_3 - (i_1 + i_2 \mod n) \right).
\]
(115)

With these new tensors at hand, let us define our desired tensor \( \mathbf{A} \), corresponding to \( \mathbf{A} \) extracting two discrete values from her continuous inputs and taking their sum mod \( n \), before forwarding the resulting three values into the strategy \( \mathbf{A}_1 \) of (111):
\[
\begin{align*}
\mathbf{A} := \mathbf{A}_1 \oplus_n \mathbf{E}_n \mathbf{E}_n \mathbf{U}_\alpha \mathbf{U}_\beta
\end{align*}
\]
(116)

It now remains to verify (112). To do so, we will make use of three useful tensor identities.

Claim: it holds that
\[
\begin{align*}
\mathbf{E}_n U_\alpha U_\beta = U_n U_\infty, \\
\mathbf{U}_n = U_n, \\
\mathbf{E}_n \mathbf{E}_n U_\infty = U_\infty.
\end{align*}
\]
(117)

To prove (117a), it suffices to realize that the map \( \mathbf{E}_n \) is invertible, and so it must map the uniform distribution over \([0, 1]\) to the uniform distribution over \([0, 1] \times [0, 1] \). To prove (117b), it suffices to see that for all \( i_2, i_3 \in \{1, \ldots, n\} \),
\[
\frac{1}{n} \sum_{i_1=1}^{n} \delta (i_3 - (i_1 + i_2 \mod n)) = \frac{1}{n} = \frac{i_3}{i_2}.
\]
(118)
Equation (117c) is true since, using the invertibility of $E_n$ and then (117a), it holds that for all $\alpha_1, \alpha_2 \in [0, 1]$,

$$\begin{align*}
E_n^{\alpha_1} U_\infty = \delta(\alpha_1 - \alpha_2) E_n^{\alpha_1} = \delta(\alpha_1 - \alpha_2) = U_\infty^\alpha_1. \quad (119)
\end{align*}$$

Let us now establish equation (112a) using the choice (116) and the identities (117):

This establishes (112a), and the case of (112b) is completely analogous. □

A.2 Deterministic approximation, exact marginal

We will make use of the following lemma relating the trace distance between the marginals of arbitrary distributions:

**Lemma 22.** For all $\frac{p}{A}$, $\frac{q}{B}$, it holds that

$$\left\| \frac{p}{A} - \frac{q}{B} \right\|_1 \leq \left\| \frac{p}{A} - \frac{q}{B} \right\|_1. \quad (121)$$

**Proof.** Using the triangle inequality and the definition of the $\|\cdot\|_1$ norm (see definition 20), we get:

$$\begin{align*}
\left\| \frac{p}{A} - \frac{q}{B} \right\|_1 &= \sum_a \left| \frac{p}{A} - \frac{q}{B} \right| \\
&= \sum_a \sum_b \left| \frac{a}{p} - \frac{a}{q} \right| \\
&= \sum_a \sum_b \left( \frac{a}{p} - \frac{a}{q} \right) \quad (122b)
\end{align*}$$

49
\[
\begin{align*}
\leq & \sum_{a,b} \left| \frac{a}{p} - \frac{b}{q} \right| \\
= & \left\| \frac{p}{q} - \frac{q}{p} \right\|_1,
\end{align*}
\]  

which concludes the proof.

We now prove a slightly stronger result, which takes into account the marginal constraint of the Correlated Sleeper task. The idea is to slightly deform the strategy obtained in lemma 21 to maintain closeness with the target output distributions while \textit{exactly} achieving the desired marginal constraint in the network \(N_3^{(\epsilon)}\) of equation (29).

\textbf{Lemma 23.} Let \(\left( \frac{p_1}{A_0}, \frac{p_2}{A_0} \right)\) be such that there exists a (Riemann integrable, as usual) probability tensor \(A_0\) with

\[
\begin{align*}
\frac{p_1}{A_0} = U_\infty U_\infty U_\infty, & \quad \frac{p_2}{A_0} = U_\infty U_\infty U_\infty, \quad U_2 = U_\infty U_\infty.
\end{align*}
\]

(123)

Then, for all \(\epsilon > 0\), there exists a deterministic probability tensor \(A\) such that

\[
\begin{align*}
\frac{U_2}{A} = & \quad U_\infty U_\infty
\end{align*}
\]

(124)

and

\[
\begin{align*}
\frac{1}{2} \left\| \frac{p_1}{A} - \frac{A}{1} \right\|_1 + \frac{1}{2} \left\| \frac{p_2}{A} - \frac{A}{1} \right\|_1 \leq \epsilon.
\end{align*}
\]

(125)

\textit{Proof.} Thanks to lemma 21, we know that for all \(\epsilon > 0\), there exists \(A_1\) such that (the factor of 1/3 is chosen for later convenience):

\[
\begin{align*}
\left\| \frac{p_1}{A_1} - \frac{A_1}{1} \right\|_1 \leq \frac{\epsilon}{3}, & \quad \left\| \frac{p_2}{A_1} - \frac{A_1}{1} \right\|_1 \leq \frac{\epsilon}{3}.
\end{align*}
\]

(126)

This \(A_1\) does not have the desired marginal of (124), but is close to it: using lemma 22, the last equation of (123) and the first inequality of equation (126), we get
If we parametrize the marginal distribution of $A_1$ with some $\nu \in \mathbb{R}$ such that

$$A_1 = \frac{1}{2} + \nu, \quad (128a)$$

and

$$A_1 = \frac{1}{2} - \nu, \quad (128b)$$

then (127) implies that

$$|\nu| \leq \epsilon/6. \quad (129)$$

Suppose that $\nu \geq 0$ (otherwise, swap the role of the outcomes 1 and 2 in the following argument). Let $\mathcal{Y}$ be a subset of area $\nu$ of the unit square $[0, 1]^2$ such that, for all $(\alpha, \beta) \in \mathcal{Y}$, we have

$$\delta_{a1}, \quad (130)$$

that is, $\mathcal{Y}$ corresponds to an input range where $A_1$ always outputs 1. Such a $\mathcal{Y}$ always exists: $(128a)$ states that there exists a subset of area $1/2 + \nu$ of the unit square which has this property, so we may simply choose a subset thereof with the right area.

Now, consider the modified strategy $A$ defined as follows:

$$A_{a\alpha\beta} := \begin{cases} A_{a\alpha\beta}, & \text{if } (\alpha, \beta) \in [0, 1] \times Y, \\ \delta_{a2}, & \text{if } (\alpha, \beta) \in \mathcal{Y}, \end{cases} \quad (131)$$

that is, we reverse the output in the region $\mathcal{Y}$ but otherwise leave the strategy unchanged. This new strategy clearly verifies the marginal constraint (124): for all $a \in \{1, 2\}$,

$$A_{a\alpha\beta} = \int_{[0,1]^2} d\alpha d\beta A_{a\alpha\beta} \quad (132a)$$
\[
\begin{align*}
\phi & = \int_{[0,1]^2 \setminus Y} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} + \int_{Y} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} \\
& = \int_{[0,1]^2 \setminus Y} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} + \nu \delta_{a2} \\
& = \int_{[0,1]^2} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} - \int_{Y} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} + \nu \delta_{a2} \\
& = \nu \delta_{a1} - \nu \delta_{a2} \\
& = \frac{\nu}{2}.
\end{align*}
\]

Furthermore, we can show that this small modification of the strategy has a small impact on the output correlations.

Claim: it holds that

\[
\begin{align*}
\| p_{1} - A \|_{1} & \leq \varepsilon, \quad (133a) \\
\| p_{2} - A \|_{1} & \leq \varepsilon. \quad (133b)
\end{align*}
\]

Using in particular the triangle inequality, we verify explicitly (133a):

\[
\begin{align*}
\leq \| p_{1} - A \|_{1} & \leq \| p_{1} - A \|_{1} + \| A - A \|_{1} + \| A - A \|_{1} \\
& \leq \frac{\varepsilon}{3} + \sum_{a,b \in \{1,2\}} \left[ \int_{[0,1]^2} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} - \left[ A \right]_{\alpha \beta} - \left[ A \right]_{\alpha \beta} \right] \\
& = \frac{\varepsilon}{3} + \sum_{a,b} \left[ \int_{[0,1]^2} \mathrm{d} \alpha \, \mathrm{d} \beta \left[ A \right]_{\alpha \beta} - \left[ A \right]_{\alpha \beta} - \left[ A \right]_{\alpha \beta} \right] \quad (134d)
\end{align*}
\]

(see eq. (128))
\[
\leq \frac{\epsilon}{3} + \sum_b \int_a d \beta_2 \begin{bmatrix} A_1 \quad \beta \quad b \\ \alpha \quad \beta_2 \end{bmatrix} - \begin{bmatrix} A \quad \beta \quad b \\ \alpha \quad \beta_2 \end{bmatrix} + \sum_a \int_a d \beta_1 \begin{bmatrix} A_1 \quad \alpha \quad \beta_1 \\ \alpha \quad \beta \end{bmatrix} - \begin{bmatrix} A \quad \alpha \quad \beta_1 \\ \alpha \quad \beta \end{bmatrix} \tag{134e}
\]

\[
= \frac{\epsilon}{3} + 2 \sum_a \int_{[0,1] \times Y} d \alpha d \beta \begin{bmatrix} A_1 \quad \alpha \quad \beta \\ \alpha \quad \beta \end{bmatrix} - \begin{bmatrix} A \quad \alpha \quad \beta \\ \alpha \quad \beta \end{bmatrix} + 2 \sum_a \int_Y d \alpha d \beta \begin{bmatrix} A_1 \quad \alpha \quad \beta \\ \alpha \quad \beta \end{bmatrix} - \begin{bmatrix} A \quad \alpha \quad \beta \\ \alpha \quad \beta \end{bmatrix} \tag{134f}
\]

\[
= \frac{\epsilon}{3} + 2 \cdot 0 + 2 \sum_{a \in \{1,2\}} \nu \delta_{a1} - \delta_{a2} \tag{134g}
\]

\[
= \epsilon + 4 \nu \tag{134h}
\]

\[
\leq \frac{\epsilon}{3} + \frac{4 \epsilon}{6} \quad \text{(see eq. (129))} \tag{134i}
\]

\[
= \epsilon \tag{134j}
\]

The case of (133b) can be obtained by applying the same argument to the strategies \(A_1\) and \(A\) under the exchange of the role of the two inputs \(\alpha\) and \(\beta\).

This then implies the desired property (125), which concludes the proof. \(\square\)

A.3 Conclusion

The last lemma that we need is a basic topological result that we will use in the following proof.

**Lemma 24.** Let \(X\) be a subset of a metric space \(M\) with metric \(d : M \times M \rightarrow \mathbb{R}\). Let \(\phi : M \rightarrow \mathbb{R}\) be a continuous function. Then, it holds that the closure of the image of \(X\) equals the closure of the image of the closure of \(X\):

\[
\overline{\phi(X)} = \overline{\phi(\overline{X})}. \tag{135}
\]

**Proof.** Since \(\phi(X) \subseteq \phi(\overline{X})\), we also have \(\overline{\phi(X)} \subseteq \overline{\phi(\overline{X})}\). To show the other direction, let \(u \in \overline{\phi(X)}\). In particular, there exists a sequence \((x_k \in \overline{X})_{k \in \mathbb{N}}\) such that

\[
u = \lim_{k \to \infty} \phi(x_k). \tag{136}
\]

For each \(k\), since \(x_k \in \overline{X}\), it holds that for all \(\delta_k > 0\) there exists \(y_k \in X\) such that

\[
d(x_k, y_k) \leq \delta_k. \tag{137}
\]

Choose \(\delta_k\) such that \(|\phi(x_k) - \phi(y_k)| \leq 1/k\) (this is possible since \(\phi\) is continuous).

**Claim:** It holds that

\[
u = \lim_{k \to \infty} \phi(y_k). \tag{138}
\]

We compute:

\[
|u - \phi(y_k)| = |u - \phi(x_k) + \phi(x_k) - \phi(y_k)| \tag{139}
\]

\[
\leq |u - \phi(x_k)| + |\phi(x_k) - \phi(y_k)| \tag{140}
\]

\[
\leq |u - \phi(x_k)| + \frac{1}{k} \xrightarrow{k \to \infty} 0, \tag{141}
\]

where we used equation (136) in the last step.

This proves that \(u \in \overline{\phi(X)}\). \(\square\)
Proposition 1. It holds that one can restrict the optimization variable of (22), namely, the probability tensor \( A \), to range over the deterministic probability tensors only:

\[
p^* = \sup_{\mathcal{A}} \frac{1}{2} \sum_{a \in \{1,2\}} \left( \frac{a}{p_1} \right) + \left( \frac{a}{p_2} \right)
\]

(23a)

\[
\text{s.t. } p_1 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \quad p_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \quad U_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}.
\]

(23b)

Proof. Consider the metric space \( M \) with metric \( d : M \times M \to \mathbb{R} \) such that

\[
M := \left\{ \left( \frac{p_1}{p_2} \right) \mid (p_1, p_2) \in \mathbb{R}^{2 \times 2} \text{ are probability tensors} \right\},
\]

\[
d \left( \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right), \left( \begin{array}{c} q_1 \\ q_2 \end{array} \right) \right) := \frac{1}{2} \left\| \frac{p_1}{p_2} - \frac{q_1}{q_2} \right\|_1 + \frac{1}{2} \left\| \frac{p_2}{p_1} - \frac{q_2}{q_1} \right\|_1.
\]

(142a)

(142b)

Consider the two subsets \( \mathcal{R} \subset M \) (standing for “allowing the use of local Randomness”) and \( \mathcal{L} \subset M \) (a special case of definition 2) defined as

\[
\mathcal{R} := \left\{ \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) \mid \exists \mathcal{A} \text{ s.t. } \begin{cases} p_1 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \\
                    p_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \\
                    U_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases} \end{cases} \right\},
\]

(143a)

\[
\mathcal{L} := \left\{ \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) \mid \exists \mathcal{A} \text{ s.t. } \begin{cases} p_1 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \\
                    p_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases}, \\
                    U_2 = \begin{cases} \mathcal{A} \mid \mathcal{U}_\infty \end{cases} \end{cases} \right\}.
\]

(143b)

Claim: \( \mathcal{R} = \mathcal{L} \).

Lemma 23 proved that \( \mathcal{R} \subseteq \mathcal{L} \). Indeed, in this notation, lemma 23 reads: “for all \( r \in \mathcal{R} \), for all \( \epsilon > 0 \), there exists \( l \in \mathcal{L} \) such that \( d(r, l) < \epsilon \).” This implies in turn that \( \mathcal{R} \subseteq \mathcal{L} \). Further more, it is clear that \( \mathcal{L} \subseteq \mathcal{R} \) (deterministic probability tensors are a special case of probability tensors), which implies \( \mathcal{L} \subseteq \mathcal{R} \). Altogether we see that \( \mathcal{R} = \mathcal{L} \).
Let us now introduce the map \( \phi : M \to \mathbb{R} \) defined as

\[
\phi \left( \left( \frac{p_1}{p_2}, \frac{q_1}{q_2} \right) \right) := \frac{1}{2} \sum_{a \in \{1, 2\}} \left( \frac{a}{p_1} + \frac{a}{q_2} \right).
\]  

(144)

Claim: \( \phi \) is a continuous map.

It suffices to verify that

\[
\left| \phi \left( \left( \frac{p_1}{p_2}, \frac{q_1}{q_2} \right) \right) - \phi \left( \left( \frac{q_1}{q_2}, \frac{p_1}{p_2} \right) \right) \right| \leq \frac{1}{2} \sum_{a \in \{1, 2\}} \left| \frac{a}{p_1} - \frac{a}{q_1} \right| + \left| \frac{a}{p_2} - \frac{a}{q_2} \right| \right.
\]  

\[
\leq d \left[ \left( \frac{p_1}{p_2}, \frac{q_1}{q_2} \right), \left( \frac{q_1}{q_2}, \frac{p_1}{p_2} \right) \right] \]  

(145)

(146)

which implies the continuity of \( \phi \).

Now, using lemma 24, we know that \( \overline{\phi(\mathcal{R})} = \overline{\phi(\mathcal{L})} \) and \( \overline{\phi(\mathcal{L})} = \overline{\phi(\mathcal{L})} \). Since \( \mathcal{R} = \mathcal{L} \), we in fact have \( \overline{\phi(\mathcal{R})} = \overline{\phi(\mathcal{L})} \). The result now follows easily: the original definition of \( p^* \) in (22) can be rewritten in this notation as:

\[
p^* = \sup_{u \in \phi(\mathcal{R})} u = \sup_{u \in \phi(\mathcal{L})} u = \sup_{u \in \phi(\mathcal{L})} u \]  

(147)

where the last supremum is the same optimization problem as equation (23).
B Postselected inflation: proofs

B.1 Main lemma

We now restate and prove lemma 3 (recall the definitions of the $p$-norms in definition 20).

**Lemma 3** (Main lemma). For any $k \in \mathbb{N}$, for any probability tensors

\[ \Lambda, \ p, q \in \mathbb{R}^k, \left\{ \begin{array}{l} \frac{q}{\lambda} \in \mathbb{R}^k \end{array} \right\}, \]

it holds that

\[ \int d\lambda \left\| \frac{\lambda}{\Lambda} p - \frac{\lambda}{\Lambda} q \right\|_2^2 \leq 3 \left\| \frac{p}{\Lambda} - \frac{q}{\Lambda} \right\|_1. \]

**Proof.** Let us label the outcomes of the probability tensors with $i = 1, \ldots, k$.

\[
\int d\lambda \left\| \frac{\lambda}{\Lambda} p - \frac{\lambda}{\Lambda} q \right\|_2^2 = \int d\lambda \left\| \frac{\lambda}{\Lambda} \sum_{i=1}^{k} \left[ \frac{i}{p} - \frac{i}{q} \right]^2 \right\|_2.
\]

\[
= \int d\lambda \left\| \frac{\lambda}{\Lambda} \sum_{i=1}^{k} \left[ \frac{i}{p} \left( \frac{i}{p} - \frac{i}{q} \right) - \left( \frac{i}{p} - \frac{i}{q} \right) \right] \right\|_2.
\]

\[
\leq 2 \sum_{i=1}^{k} \left\| \frac{i}{p} - \frac{i}{q} \right\|_1 + \sum_{i=1}^{k} \left\| \frac{i}{p} \right\|_1 - \left\| \frac{i}{q} \right\|_1.
\]

\[
\leq 2 \left\| \frac{p}{\Lambda} - \frac{q}{\Lambda} \right\|_1 + \left\| \frac{p}{\Lambda} - \frac{q}{\Lambda} \right\|_1.
\]

\[
\leq 3 \left\| \frac{p}{\Lambda} - \frac{q}{\Lambda} \right\|_1.
\]

where we used the definition of the 1- and 2-norms (definition 20), the triangle inequality, and lemma 22.

\[ \square \]

B.2 Certification

Let us now prove theorem 6.

**Theorem 6** (Certification). Let $(\mathcal{N}_c)_{c=1}^C$, $n, m$ be as in definition 5. It holds that

\[ \mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right) \subseteq \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right). \]
Proof. Suppose that \( \left( \sum_{c=1}^{C} p_c \right)_c \in \mathcal{L} \left( (\mathcal{N}_c)_c = 1 \right) \). Then, there exist deterministic probability tensors
\[
\left\{ \begin{array}{c}
A_{k}^{(0)} \\
k = 1
\end{array} \right\}^{K}
\]
which verify the condition of equation (36b). We added the superscript “0” to make clear that these are the original agent strategies, solving the causal compatibility problem, not to be confused with the agent strategies of (57a) that are to solve the postselected inflation problem.

It remains to write down the probability tensors that solve the postselected inflation problem (57a) to establish the inclusion relation (59). The random variable \( \Lambda \) is actually distributed a tuple \( \vec{\lambda} \) of \( n \) independent values sampled from \( n \) sources \( \mathcal{U}_\infty \), which we can represent as, for any sequence \( \vec{\lambda} \) of \( n \) real numbers in \([0,1]\),

\[
\Lambda = \begin{cases} 
\vec{\lambda} & \text{at} \mathcal{U}_\infty \\
\mathcal{U}_\infty & \text{at} \mathcal{U}_\infty \\
\{n-3\} & \text{at} \mathcal{U}_\infty \\
\end{cases}
\]

Then, the agents should use their tuple of values \( \vec{i} \), which is received from the sources \( \mathcal{U}_n \) that they have access to, to select which of the \( n \) source distributed by \( \Lambda \) they should use as inputs to the original agent strategies. This can be represented as, for each \( k \in \{1,\ldots,K\} \),

\[
\begin{array}{c}
\Lambda \\
\vec{i} \\
\vec{\lambda}
\end{array}
\begin{array}{c}
A_k \\
\end{array}
:= \begin{array}{c}
\Lambda \\
\vec{\lambda} \\
\vec{i}
\end{array}
\]

Recall the definition of the selector node \( \diamond \) in section 3.3. It remains to argue why this construction implies the desired inclusion of equation (59), that is, why this construction solves the postselected inflation problem of equation (57b). Let us fix \( c \in \{1,\ldots,C\} \). Consider the left-hand side diagram of (57b) but for a fixed value assignment for each of the outputs of the \( S_c \cdot m \) sources \( \mathcal{U}_n \), such that this value assignment is compatible with the postselection, i.e., such that all the values therein are pairwise distinct. Let us refer to this assignment as a “\( \mathcal{U}_n \)-conditioning” for brevity. Furthermore, let us refer to the left-hand side diagram as being formed from \( m \) “groups”, where a group consists of \( S_c \) sources \( \mathcal{U}_n \) and \( P_c \) agents.

Claim: with equations (150) and (151), the left-hand side diagram of (57b) under any \( \mathcal{U}_n \)-conditioning factorizes in the same way as the right-hand side diagram of (57b) factorizes.

There is no value of the sources \( \mathcal{U}_n \) of one group in common with any of the value of the sources \( \mathcal{U}_n \) of any other group under any \( \mathcal{U}_n \)-conditioning thanks to the postselection. Given (151),

---

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this means that the agents of one group are guaranteed to only look at sources \( U_\infty \) distributed through \( \Lambda \) which are independent from those that the agents of any other group look at. Since the only potential correlations between two groups, after \( U_n \)-conditioning, would come from the shared source \( \Lambda \), the claim follows.

Claim: with equations (150) and (151), the marginal of the left-hand side diagram of (57b) under any \( U_n \)-conditioning where all groups except one are ignored equals to \( \sum_{pc} \).

In this marginal, each agent will be sampling several sources \( U_\infty \) sent out by \( \Lambda \). Which of the \( n \) such sources \( U_\infty \) the agent samples depends on the values sent out by the subset of the sources \( U_n \) the agent has access to. Thus, all the agents that are receiving the value of a given source \( U_n \) will be sampling from the same source \( U_\infty \) sent out by \( \Lambda \), and so the corresponding inputs of the original strategies (equation (149)) will be connected to the same source \( U_\infty \), exactly as they should in the original causal compatibility diagram of (36b). Furthermore, because the values of the different sources \( U_n \) are all pairwise distinct under \( U_n \)-conditioning (thanks to the postselection), it follows that the agents will never be using the same source \( U_\infty \) sent out by \( \Lambda \), except if they are connected to the same source \( U_n \). Thus, the marginal under consideration will be equal to the tensor contraction of the left-hand side of equation (36b), which is, by assumption on the original agent strategies of (149), equal to the right-hand side of equation (36b), which is just \( \sum_{pc} \).

Using the two claims together, we see that under \( U_n \)-conditioning, the left-hand side of (57b) equals the right-hand side. Averaging over the possible \( U_n \)-conditioning, this property remains true, so that we obtain that (57b) holds true, and \( c \) was arbitrary in \( \{1, \ldots, C\} \).

\section*{B.3 Hierarchy}

Let us now turn to theorem 7.

\begin{theorem}[Hierarchy] For all \( (N_c = (K, P_c, S_c, \nu_c, \bar{\nu}_c))_{c=1}^C \) and \( n, m (n \geq \max_c S_c \cdot m) \) as well as \( n', m' (n' \geq \max_c S_c \cdot m') \) such that
\[ n \geq n' \quad \text{and} \quad m \geq m', \tag{60} \]

it holds that
\[ \mathcal{I} \left( \left( N_c \right)_{c=1}^C, n, m \right) \subseteq \mathcal{I} \left( \left( N_c \right)_{c=1}^C, n', m' \right). \tag{61} \]
\end{theorem}

\textbf{Proof.} We first prove that the relation (61) works in the case \( n' = n \) and \( m' = m - 1 \), and then in the case \( n' < n \) and \( m' = m \). The general relation (61) then follows by repeated iteration of the arguments.

Claim: the relation (61) holds in the case \( n' = n \) and \( m' = m - 1 \).
For any outcome distributions \( \sum_{p_c} \mathbb{C} \in \mathcal{I} \left( (\mathcal{N}_c)^C_{c=1}, n, m \right) \), consider equation (57b) with \( m \) groups, as displayed. The marginal where the last group is ignored is simply the condition of equation (57b) with \( m - 1 \) groups. This is clear for the right-hand side. For the left-hand side, this follows from the fact that one can safely marginalize sources that are inputs to the postselection, that is, for any \( k \in \mathbb{N} \) such that \( k + 1 \leq n \),

\[
\begin{array}{l}
\mathcal{U}_n \setminus \{k+1\} = \mathcal{U}_n \setminus \{k\} \quad \text{(152)}
\end{array}
\]

Thus, the tensors of (57a) that solve the problem of (57) with \( m \) groups also solve the problem of (57) with \( m - 1 \) groups, and hence \( \left( \sum_{p_c} \mathbb{C} \in \mathcal{I} \left( (\mathcal{N}_c)^C_{c=1}, n, m - 1 \right) \right) \).

**Claim:** the relation (61) holds in the case where \( n' < n \) and \( m' = m \).

Consider some \( \sum_{p_c} \mathbb{C} \in \mathcal{I} \left( (\mathcal{N}_c)^C_{c=1}, n, m \right) \), and let \( \left\{ A_k \right\}_{k=1}^K \) and \( \Lambda \) be some choice of tensors that solve equation (57). We will construct new tensors \( \left\{ A_{k}' \right\}_{k=1}^K \) and \( \Lambda' \) that solve the problem of (57) with parameters \( n' < n \) and \( m \), thus showing that also \( \sum_{p_c} \mathbb{C} \in \mathcal{I} \left( (\mathcal{N}_c)^C_{c=1}, n', m \right) \). The construction is the following: let the new \( \Lambda' \) tensor be a tuple of the original \( \Lambda \) tensor together with a uniformly sampled permutation \( \pi \) of \( n \) indices (such permutations are in one-to-one correspondence with the set \( \{1, \ldots, n!\} \)):

\[
\hat{X}_{= (\lambda, \pi)}^{\Lambda'} = \lambda \Theta^{\Lambda} \Gamma^{\pi \left( U_{(n)} \right)}.
\]  

(153a)

The new agent strategies \( \left\{ A_{k}' \right\}_{k=1}^K \) are then obtained by letting the agents first apply the permutation \( \pi \) on all the values they receive from the sources \( \mathcal{U}_{n'} \), and then using the original strategies:

\[
\hat{X}_{= (\lambda, \pi)}^{A_k} = \lambda \Theta^{A_k} \Gamma^{\pi(i)}.
\]  

(153b)

where we introduced the notation \( \pi(i) \) to denote the application of the permutation \( \pi \) to all the components of \( \hat{X} \), e.g. if \( \pi = (1 \leftrightarrow 2)(3 \leftrightarrow 4) \) and \( \hat{X} = (1, 6, 2, 3) \), then \( \pi(i) = (2, 6, 1, 4) \). It remains to proves that the choice of (153) does solve the problem of (57) for \( n' < n \). Using the terminology of the proof of theorem 6, the diagram of the left-hand side of (57b) with \( n \) replaced by \( n' < n \), and under \( \mathcal{U}_{n'} \)-conditioning, i.e., under a choice of value assignment for all the sources \( \mathcal{U}_{n'} \) where the value assignment is compatible with the postselection, does in fact already verifies the condition (57b). Thus, the average over all \( \mathcal{U}_{n'} \)-conditioning will also verify this condition (57b),
and the claim follows. Consider for instance the value assignment where the sources \( \frac{1}{U_{n'}} \) take the values \( 1, 2, \ldots, S_c \cdot m \) (recall that by assumption, \( S_c \cdot m \leq n' \)). After all agents apply the random permutation \( \pi \), the sources effectively take the values \( \pi(1), \pi(2), \ldots, \pi(S_c \cdot m) \), which is a uniformly distributed tuple of values all pairwise distinct and in the range \( \{1, \ldots, n\} \), exactly as those obtained from the \( S_c \cdot m \) sources \( \frac{1}{U_n} \) postselected with the tensor \( \prod_{i \neq \cdot} \).

This concludes the proof.

\( \square \)

B.4 Convergence

The following lemma gives a useful relation between norms.

**Lemma 25.** For all \( k \in \mathbb{N} \), \( x \in \mathbb{R}^k \), it holds that

\[
\|x\|_1 \leq \sqrt{k} \|x\|_2 .
\] (154)

**Proof.** Let \( y = (1, \ldots, 1) \in \mathbb{R}^k \), and \( \tilde{x} = ([x_1], \ldots, [x_k]) \in \mathbb{R}^k \). We see that

\[
\|x\|_1 = \sum_{i=1}^{k} y_i \tilde{x}_i \leq \|y\|_2 \|\tilde{x}\|_2 = \sqrt{k} \|x\|_2 ,
\] (155)

where we used the Cauchy-Schwartz inequality for the canonical inner product of \( \mathbb{R}^k \).

We make formal the intuition that the postselection has almost no effect for very large \( n \) in the following lemma:

**Lemma 26.** For any \( n, S \in \mathbb{N} \) with \( n \geq S \), for any probability tensor \( \frac{1}{M} \) with \( S \) input legs, it holds that

\[
\frac{1}{\| U_n \|_1} \frac{1}{\| U_n \|_1} \leq \frac{S(S-1)}{n} + O \left( \frac{1}{n^2} \right) .
\] (156)

**Proof.** We label the outcomes of the tensor \( \frac{1}{M} \) with \( a \), and the combined inputs form the \( \frac{1}{U_n} \) sources with \( \tilde{i} \). We further use the notation “ \( \tilde{i} \) all \( \neq \) ” in case the tuple of inputs \( \tilde{i} \) is compatible with the postselection, i.e., all the components of \( \tilde{i} \) are pairwise distinct, and the notation “ \( \tilde{i} \) not all \( \neq \) ” otherwise, i.e., if at least two components of \( \tilde{i} \) are equal. Note that with \( S \) sources outputting \( n \) distinct values, it holds that

\[
|\{ \tilde{i} \text{ all } \neq \}| = n(n-1) \cdots (n-(S-1)) = \frac{n!}{(n-S)!} .
\] (157)

Expanding the norm and the source contractions, we obtain

\[
\frac{1}{\| U_n \|_1} \frac{1}{\| U_n \|_1} = \sum_{a} \sum_{i \text{ all } \neq} \frac{a}{M_i} \frac{(n-S)!}{n!} - \sum_{i} \frac{a}{M_i} \frac{1}{n^S} .
\] (158a)
\[
\begin{align*}
&= \sum_a \left| \sum_{i \text{ all } \neq a} \frac{a}{i} \left( \frac{(n-S)!}{n!} - \frac{1}{n^S} \right) - \sum_{i \text{ not all } \neq a} \frac{a}{i} \frac{1}{n^S} \right| \\
&\leq \sum_{i \text{ all } \neq a} \sum_a \frac{a}{i} \left( \frac{(n-S)!}{n!} - \frac{1}{n^S} \right) + \sum_{i \text{ not all } \neq a} \sum_a \frac{a}{i} \frac{1}{n^S} \\
&= \frac{n!}{(n-S)!} \left( \frac{(n-S)!}{n!} - \frac{1}{n^S} \right) + \left( n^S - \frac{n!}{(n-S)!} \right) \frac{1}{n^S} \\
&= 2 \left( 1 - \frac{n!}{n^S(n-S)!} \right) \\
&= 2 \left( 1 - \prod_{k=1}^{S-1} \left( 1 - \frac{k}{n} \right) \right) \\
&= 2 \frac{S-1}{n} \sum_{k=1}^{S-1} k + \mathcal{O} \left( \frac{1}{n^2} \right) \\
&= \frac{S(S-1)}{n} + \mathcal{O} \left( \frac{1}{n^2} \right),
\end{align*}
\]
as expected. \(\square\)

We now prove theorem 8.

**Theorem 8** (Convergence). Let \(\mathcal{N}_c = (K, P_c, S_c, \nu_c, \bar{\mu}_c)\)\(^c\) for \(c = 1\) to \(n\), and let \(m\) be as in definition 5. We assume that \(m \geq 2\). Then, for any list of distributions

\[
\left( \begin{array}{c}
\mathcal{P}_c \\
\mathcal{Q}_c
\end{array} \right)_{c=1}^C \in \mathcal{L} \left( \mathcal{N}_c \right)_{c=1}^C, n, m
\]

where we made explicit the total number of outcomes \(d_c\) of each distribution,\(^{11}\) it holds that

\[
\inf \left( \begin{array}{c}
\mathcal{P}_c \\
\mathcal{Q}_c
\end{array} \right)_{c=1}^C \left\| \begin{array}{c}
\mathcal{P}_1 \cdots \mathcal{P}_c \\
\mathcal{Q}_1 \cdots \mathcal{Q}_C
\end{array} \right\|_1 \leq \sqrt{12C^{-1} \sum_{c=1}^C d_c \left( S_c - \frac{1}{2} \right)} + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).
\]

\(1\)To tie the notation together: we have that \(d_c = \prod_{p=1}^{P_c} (\# \text{out})_{\nu_c(p)}\).

\(\sqrt{12C^{-1} \sum_{c=1}^C d_c \left( S_c - \frac{1}{2} \right)} + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).\)
We now introduce some notation. We know that \( \left( \frac{p_c}{c} \right)_{c=1}^C \) is in \( I \left( (N_c)_{c=1}^C, n, m \right) \) with \( m \geq 2 \), so thanks to theorem 7, we have in particular \( \left( \frac{p_c}{c} \right)_{c=1}^C \in I \left( (N_c)_{c=1}^C, n, 2 \right) \): consider the tensors \( \{ A_k \}_{k=1}^K \) and \( \Lambda \) of equation (57a) that establish that \( \left( \frac{p_c}{c} \right)_{c=1}^C \in I \left( (N_c)_{c=1}^C, n, 2 \right) \). Define, for all \( \lambda \) in the output domain of the source \( \Lambda \) and for all \( c \), the distributions \( q_c \):

\[
\begin{align*}
q_c \lambda := & A_{\nu(1)} \cdots A_{\nu(P_c)} \bar{\mu}_c(1) \cdots \bar{\mu}_c(P_c) \\
& \cdots \\
& \cdots \\
& U_n (S_c - 2) U_n
\end{align*}
\]

By construction, for any \( \lambda \), it holds that

\[
\left( \frac{q_c}{c} \right)_{c=1}^C \in \mathcal{L} \left( (N_c)_{c=1}^C \right).
\]

Thus, we can upper bound the infimum of (159c) by the convex combination

\[
(159c) \leq \int d\lambda \frac{1}{\Lambda} \sum_{c=1}^C d_c \left\| \frac{p_c}{c} - \frac{q_c}{\lambda} \right\|_2^2.
\]

Now, using lemma 3:

\[
(159c) \leq \sum_{c=1}^C d_c \left\| \frac{p_c}{c} - \frac{q_c}{c} \right\|_1.
\]

Let us now introduce, for all \( c = 1, \ldots, C \), a new probability tensor \( M_c \) that allows us to rewrite the constraint of equation (57b) as

\[
M_c
\]

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and looking back at the definition of \( q_c \) in equation (160), we also have

\[
\lambda \in \mathcal{M}_c = \frac{q_c^* - q_c}{U_n}.
\]

(165)

Using equations (164) and (165) together with lemma 26, we have that

\[
\|p_c - q_c\|_1 \leq \frac{2S_c(2S_c - 1)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{166}
\]

Inserting this bound into equation (163) yields the desired result.

The corollary 9 is now easy to obtain:

**Corollary 9.** Consider some \( (N_c = (K, P_c, S_c, \nu_c, \vec{\mu}_c))^c_{c=1} \) and \( m \geq 2 \), and let \( n_0 := \max_c S_c \cdot m \). In the topology induced by the metric

\[
d \left[ \left( \begin{array}{c} p_c^C \\ \vdots \\ p_c^C \end{array} \right)_{c=1}^C, \left( \begin{array}{c} q_c^C \\ \vdots \\ q_c^C \end{array} \right)_{c=1}^C \right] \leq \frac{1}{C} \sum_{c=1}^C \left\| p_c^C - q_c^C \right\|_1,
\]

and denoting \( \overline{X} \) the closure of a set \( X \) in this topology, it holds that

\[
\mathcal{L} \left( (N_c)_{c=1}^C \right) \subseteq \bigcap_{n=n_0}^{\infty} \mathcal{I} \left( (N_c)_{c=1}^C, n, m \right) \subseteq \overline{\mathcal{L} \left( (N_c)_{c=1}^C \right)}.
\]

(65)

**Proof.** Theorem 6 already implies that

\[
\mathcal{L} \left( (N_c)_{c=1}^C \right) \subseteq \bigcap_{n=n_0}^{\infty} \mathcal{I} \left( (N_c)_{c=1}^C, n, m \right). \tag{167}
\]

Now, let

\[
\left( \begin{array}{c} p_c^C \\ \vdots \\ p_c^C \end{array} \right)_{c=1}^C \in \bigcap_{n=n_0}^{\infty} \mathcal{I} \left( (N_c)_{c=1}^C, n, m \right). \tag{168}
\]

For all \( \epsilon > 0 \), choose \( n \) sufficiently large such that the right-hand side of equation (63) is less than or equal to \( \epsilon \). Since we have in particular that

\[
\left( \begin{array}{c} p_c^C \\ \vdots \\ p_c^C \end{array} \right)_{c=1}^C \in \mathcal{I} \left( (N_c)_{c=1}^C, n, m \right), \tag{169}
\]

theorem 8 implies that

\[
\inf_{\left( \begin{array}{c} q_c^C \\ \vdots \\ q_c^C \end{array} \right)_{c=1}^C \in \mathcal{L} \left( (N_c)_{c=1}^C \right)} d \left[ \left( \begin{array}{c} p_c^C \\ \vdots \\ p_c^C \end{array} \right)_{c=1}^C, \left( \begin{array}{c} q_c^C \\ \vdots \\ q_c^C \end{array} \right)_{c=1}^C \right] \leq \epsilon. \tag{170}
\]

Since equation (170) holds for all \( \epsilon > 0 \), and since the metric \( d \) is positive definite, we must have in fact

\[
\inf_{\left( \begin{array}{c} q_c^C \\ \vdots \\ q_c^C \end{array} \right)_{c=1}^C \in \mathcal{L} \left( (N_c)_{c=1}^C \right)} d \left[ \left( \begin{array}{c} p_c^C \\ \vdots \\ p_c^C \end{array} \right)_{c=1}^C, \left( \begin{array}{c} q_c^C \\ \vdots \\ q_c^C \end{array} \right)_{c=1}^C \right] = 0. \tag{171}
\]
This is equivalent to the statement that there exists a sequence \( (l_k \in \mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right))_{k \in \mathbb{N}} \) such that

\[
\lim_{k \to \infty} l_k = \left( \prod_{c=1}^C \prod_{n=1}^{\infty} \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C \right) \right)
\]  

in the metric \( d \), so that

\[
\bigcap_{n=n_0}^{\infty} \mathcal{I} \left( (\mathcal{N}_c)_{c=1}^C, n, m \right) \subseteq \overline{\mathcal{L} \left( (\mathcal{N}_c)_{c=1}^C \right)}
\]

holds.
Correlated Sleeper: additional material

C.1 Parametrization

Lemma 10. Let $\frac{1}{p_1}$ and $\frac{1}{p_2}$ be as in (23b). Define for all $c \in \{1, 2\}$

$$\lambda_c := \frac{1}{p_c},$$

(66)

It then holds that

$$\frac{1}{p_c} = \frac{1}{2 - \lambda_c},$$

(67a)

$$\frac{2}{p_c} = \frac{2}{2 - \lambda_c},$$

(67b)

with

$$\lambda_c \in \left[0, \frac{1}{2}\right].$$

(68)

Proof. First, the symmetry of each $\frac{1}{p_c}$, apparent from (23b), implies

$$\frac{1}{p_c} = \frac{2}{p_c}.$$  

(174)

Additionally, the marginal constraint implied by the last equality of (23b) yields

$$\frac{1}{p_c} + \frac{1}{p_c} = \frac{1}{2},$$

(175a)

$$\frac{2}{p_c} + \frac{2}{p_c} = \frac{1}{2},$$

(175b)

which together with (174) implies that

$$\frac{1}{p_c} = \frac{2}{p_c}.$$  

(176)

Thus, the only free parameter in the distribution $\frac{1}{p_c}$ is $\lambda_c := \frac{1}{p_c}$. It is clear from (175a) that

$$0 \leq \frac{1}{p_c} \leq \frac{1}{2},$$

(177)

which concludes the proof.

Lemma 11. Let $(\lambda_1, \lambda_2)$ be as in lemma 10. It holds for all $c \in \{1, 2\}$ that

$$\lambda_c \geq \frac{1}{4}.$$  

(69)

Proof. To see this, consider the 1, 2 component of the distribution:

$$\frac{1}{p_c} = \int d\alpha f_1^{(c)}(\alpha)f_2^{(c)}(\alpha),$$

(178)
where we defined
\[
\begin{align*}
  f_i^{(c)}(\alpha) := \begin{cases}
    A_i \ U_\infty, & \text{if } c = 1, \\
    A_i \ U_\infty, & \text{if } c = 2.
  \end{cases}
\end{align*}
\] (179)

The right-hand side of equation (178) is an inner product on the function space \(L^2([0, 1])\), so that we can apply the Cauchy-Schwartz inequality:

\[
\frac{1}{p_c} = \langle f_1^{(c)} \big| f_2^{(c)} \rangle_{L^2([0, 1])} \leq \| f_1^{(c)} \|_2 \| f_2^{(c)} \|_2.
\] (180)

Now, these norms can be evaluated explicitly:

\[
\| f_i^{(c)} \|_2^2 = \int d\alpha f_i^{(c)}(\alpha)f_i^{(c)}(\alpha) = \frac{1}{p_c} = \frac{1}{p_c}.
\] (181)

We hence have

\[
\frac{1}{p_c} \leq \frac{1}{p_c},
\] (182)

so that

\[
\frac{1}{p_c} = \frac{1}{2} - \frac{1}{p_c} \geq \frac{1}{2} - \frac{1}{p_c}
\] (183)

which implies

\[
\frac{1}{p_c} \geq \frac{1}{4}.
\] (184)

This inequality expresses the intuitive fact that the best \(A\) can do to obtain opposite output bits across the two rounds is to ignore her inputs and output a random bit instead.

C.2 Explicit linear programs

**Explicit feasibility problem.** Continuing with the notation of section 7, we say that \((\lambda_1, \lambda_2) \in I^{(s)}\) if and only if there exists

\[
\left\{ \begin{array}{c}
  M \\
  \frac{M}{A}
\end{array} \right\}_{M \in \mathcal{M}_{\text{red}}}
\] (185)

such that (the distributions \(\frac{M}{p_c}\) for \(c \in \{1, 2\}\) are obtained from \((\lambda_1, \lambda_2)\) as in lemma 10, the indices \(i_1, \ldots, j_4\) are always in the set \(\{1, \ldots, 4\}\))

\[
\forall M \in \mathcal{M}_{\text{red}} : \quad \frac{M}{A} \geq 0, \quad \sum_{M' \in \mathcal{M}_{\text{red}}} \frac{M'}{A} = 1,
\] (186a)

\[
\forall (a_1, a_2, a_3, a_4) \in \{(1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 2)\} : \\
\sum_{M \in \mathcal{M}_{\text{red}}} \frac{M}{A} \frac{1}{12 \cdot 4!} \sum_{j_1, j_2, j_3, j_4 \text{ all } \neq} \delta(M_{i_1, j_1} = a_1)\delta(M_{i_1, j_2} = a_2)\delta(M_{i_2, j_3} = a_3)\delta(M_{i_2, j_4} = a_4)
\] (186b)

\[
= \frac{a_1}{p_1} \frac{a_2}{p_1} \frac{a_3}{p_1} \frac{a_4}{p_1},
\] (186b)
\[
\sum_{M \in \mathcal{M}_{\text{red}}} \frac{M!}{4! \cdot 12} \sum_{i_1, \ldots, i_4 \text{ all } \neq j_1, j_2 \text{ all } \neq} \delta(M_{i_1, j_1} = a_1) \delta(M_{i_2, j_2} = a_2) \delta(M_{i_3, j_3} = a_3) \delta(M_{i_4, j_4} = a_4)
\]
\[
= \frac{a_1}{p_2} \frac{a_2}{p_2} \frac{a_3}{p_2} \frac{a_4}{p_2},
\]  
(186c)

\[
\forall (a_1, a_2, a_3, a_4) \in \{(1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2)\}:
\]
\[
\sum_{M \in \mathcal{M}_{\text{red}}} \frac{M!}{4! \cdot 4!} \sum_{i_1, \ldots, i_4 \text{ all } \neq j_1, j_2 \text{ all } \neq} \delta(M_{i_1, j_1} = a_1) \delta(M_{i_2, j_2} = a_2) \delta(M_{i_3, j_3} = a_3) \delta(M_{i_4, j_4} = a_4)
\]
\[
= \frac{1}{16},
\]  
(186d)

\[
\forall (a_1, a_2), (a_3, a_4) \in \{(1, 1), (1, 2), (2, 2)\}, a_5 \in \{1, 2\}, (a_1, a_2, a_3, a_4, a_5) \neq (2, 2, 2, 2, 2):
\]
\[
\sum_{M \in \mathcal{M}_{\text{red}}} \frac{M!}{4! \cdot 4!} \sum_{i_1, \ldots, i_4 \text{ all } \neq j_1, j_2 \text{ all } \neq} \delta(M_{i_1, j_1} = a_1) \delta(M_{i_2, j_2} = a_2) \delta(M_{i_3, j_3} = a_3) \delta(M_{i_4, j_4} = a_4)
\]
\[
= \frac{1}{2} \frac{a_1}{p_1} \frac{a_2}{p_1} \frac{a_3}{p_2} \frac{a_4}{p_2}.
\]  
(186e)

**Explicit dual optimization problem.** The problem of equation (80) can be rewritten as

\[
d^{(\text{inf})} = \min_{z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{R}} \frac{1}{4} \sum_{a, b \in \{1, 2\}} z_{ab}
\]
\[
\text{s.t. } \forall M \in \mathcal{M} \ (\text{or } \mathcal{M}_{\text{red}}):
\]
\[
\sum_{a, b} z_{ab} \frac{1}{12} \frac{1}{12} \sum_{i_1, i_2 \text{ all } \neq j_1, j_2 \text{ all } \neq} \delta(M_{i_1, j_1} = a) \delta(M_{i_2, j_2} = b)
\]
\[
\geq \frac{1}{2} \sum_{a} \left[ \frac{1}{4} \frac{1}{12} \sum_{j_1 \neq j_2} \delta(M_{i_1, j_1} = a) \delta(M_{i_2, j_2} = a) + \frac{1}{12} \frac{4}{4} \sum_{i_1 \neq i_2} \delta(M_{i_1, j_1} = a) \delta(M_{i_2, j_2} = a) \right].
\]  
(187b)
C.3 Extended plot of the feasible region

In figure 11, we show how the inflation described in definition 12 behaves on the whole region \([0, 1/2] \times 2\) which corresponds, thanks to lemma 10, to the set of pairs of distributions with \(2 \times 2\) outcomes that are symmetric under the exchange of the two outcomes, and that have uniform marginals. We saw in lemma 11 that if this pair of distributions is in \(\mathcal{L}^{(s)}\) (see e.g. equation (72)), then it must further verify \(\lambda_1, \lambda_2 \geq 1/4\). This is not the case according to our outer approximation of definition 12: as we see in figure 11, the postselected inflation only enforces that \(\lambda_1, \lambda_2\) are greater than \(\approx 0.17\).

Figure 11: Behavior of the postselected inflation of definition 12 on the whole \([0, 1/2] \times 2\) region. In fact, if either of the two \(\lambda_1, \lambda_2\) is less than 0.15, we find that the corresponding distribution does not admit a postselected inflation, so we restrict the range of the plot to better visualize the boundary \(\partial \mathcal{I}^{(s)}\). This boundary is obtained similarly to that of figure 5: we ran a dichotomic search of the threshold radius (with respect to feasibility of the corresponding linear program) in polar coordinates centered around \((\lambda_1 = 0.25, \lambda_2 = 0.25)\), which makes sense given the overall shape of \(\mathcal{I}^{(s)}\) shown with the scan of infeasible points.