Construction of non-CSS quantum codes using measurements on cluster states

Swayangprabha Shaw*, Harsh Gupta†, Shahid Mehraj Shah‡ and Ankur Raina§

*Department of EECS, Indian Institute of Science Education and Research Bhopal, India
†Department of Physics, Indian Institute of Science Education and Research Srinagar, India
‡Department of ECE, National Institute of Technology Srinagar, India
§Department of Science Education and Research Bhopal, India

Email: *swayangprabha17@iiserb.ac.in, †harsh22@iiserb.ac.in, ‡shahidshah@nitsri.net, §ankur@iiserb.ac.in

Abstract—The Measurement-based quantum computation provides an alternate model for quantum computation compared to the well-known gate-based model. It uses qubits prepared in a specific entangled state followed by single-qubit measurements. The stabilizers of cluster states are well defined because of their graph structure. We exploit this graph structure extensively to design non-CSS codes using measurement in a specific basis on the cluster state. The procedure is general and can be used specifically as an encoding technique to design any non-CSS codes with one logical qubit. We show there exists a specific entangled state followed by single-qubit measurements. A 2D grid cluster state is shown in Fig. 1. This cluster state can be parameterized by mathematical structure given by $G = (V, E)$, where $G$ denotes the graph state associated with the cluster, $V$ indicates the set of vertices and $E$ represents the set of edges. If one of the qubits in this cluster state is used as a message qubit then the resultant quantum state becomes a quantum error correcting code with one logical qubit.

I. INTRODUCTION

Quantum computing relies on the error-free operations of qubits. Inevitably, qubits suffer from noise and need to be preserved using Quantum error correcting codes (QECC). In the realm of quantum error correction, one of the elegant families of quantum codes is the Calderbank-Shor-Steane (CSS) codes named after their inventors [1] whose structure motivates the construction of quantum codes using classical codes. One of the important properties of CSS codes is that they consist of purely $X$ and $Z$ stabilizers [2] which is particularly useful for the design of elegant decoders. An interesting quantum code with an even better coding rate is the five-qubit code, however that is a non-CSS code [5]. This non-CSS code with least number of physical qubits is capable of correcting any single-qubit error. Motivation for the construction of such non-CSS codes stems from Measurement-based quantum computing (MBQC) proposed by Raussendorf et al. [6]. This scheme is shown to be viable for universal quantum computing. It makes use of a resource state called cluster state consisting of entangled qubits, initially prepared in the $\left|+\right\rangle$ state followed by single qubit measurement. The model of computation is therefore also called cluster state computation [7]. The success of this approach is based on the creative utilisation of qubit measurements. A 2D grid cluster state is shown in Fig. 1. This cluster state can be parameterized by mathematical structure given by $G = (V, E)$, where $G$ denotes the graph state associated with the cluster, $V$ indicates the set of vertices and $E$ represents the set of edges. If one of the qubits in this cluster state is used as a message qubit then the resultant quantum state becomes a quantum error correcting code with one logical qubit.

In this paper, we propose an algorithm showing explicit construction of encoding one logical qubit into $n$ physical qubits. Then we show how one can construct a $(n+1)$ qubit cluster associated with a $[n,1]$ code. We propose a general algorithm to construct a $(n+1)$ qubit cluster state from the parity check matrix of a given $[n,1]$ stabilizer code. Qubit measurement on $(n+1)$ qubit cluster gives the desired $[n,1]$ code, independent of the distance of the code. In this paper, we are not interested in increasing the distance of the code.

The paper is organized as follows. We present the algorithm for the evolution of stabilizers for cluster states under measurement in Section II and show the procedure for construction of non-CSS codes in Section III with circuit simulations of the codes. We have shown our approach to construct the cluster corresponding to a given non-CSS code in Section IV. We conclude the paper in Section V.

A. Notation

$X, Y, Z$ Measurement in $\sigma_x, \sigma_y, \sigma_z$ basis.
$CZ$ diag$(1, 1, 1, -1)$. 
$CZ_{ij}$ CZ Operation between $i^{th}$ and $j^{th}$ qubit.
$N(j)$ Neighbourhood of site $j$.
$S_j$ $j^{th}$ Stabilizer.
$S_i^{(i)}$ Set consists of $<S_j>_{j=1}^{n}$, $n$ is the number of qubits.
$S_i^{(i)}$, $S_f^{(i)}$ The initial and final $j^{th}$ stabilizer respectively.
$s$ Parameter mapped to measurement outcome.
$s_i$ $s$ associated with $i^{th}$ qubit.
$\mathcal{I}$ Set containing label of qubits need to measure.
$\mathcal{M}$ Set of measurable observables.
$M_a$ Elements in $\mathcal{M}$, where $a \in \mathcal{I}$.
$[S_j, M_a]$ Commutation operation between $S_j$ and $M_a$.
$\chi_1, \chi_2$ $[S_j, X_a]$ and $[S_j, Z_a]$ respectively.
$A, H$ Adjacency and Parity check matrix respectively.

Fig. 1: Cluster as graph $G = (V, E)$ where $V$ is the set of vertices and $E$ is the set of edges.
II. EVOLUTION OF STABILIZERS

Consider the cluster state shown in Fig. 1, as a result of this graphical structure, the cluster state can be written as [6]:

$$|G\rangle = \prod_{(i,j) \in E} CZ_{ij} |+\rangle^\otimes n, \tag{1}$$

where $CZ_{ij}$ is the entangling gate applied between every pair of nodes connected by an edge $(i,j) \in E$. Here the nodes correspond to qubits prepared in $|+\rangle$ state. In this paper, we use nodes and qubits interchangeably. An important feature of cluster states of $n$ qubits is that they have stabilizers of the form [6]:

$$S_j = X_j \prod_{k \in \mathcal{N}(j)} Z_k, \ j = 1, 2, \cdots, n. \tag{2}$$

The unique nature of these stabilizers being not purely $X$ or $Z$ type has motivated us to construct non-CSS codes using the cluster states.

Before going to non-CSS code construction, we first observe how the stabilizers evolve in response to a set of measurements on the cluster. After performing $X$ or $Z$ measurement on qubits of different types of clusters, we notice the final set of stabilizers contains only those stabilizers that commute with the measurement observables. Stabilizers having operator with same qubit index as measurement observables results in a phase factor $(-1)^s$ on the stabilizer where $s$ is the parameter mapped to measurement outcomes. The mapping of $s$ is such that $s = 0$ when measurement outcome is $+1$ and $s = 1$ when measurement output is $-1$. This phase factor is helpful in adapting to the post-measurement operations on unmeasured qubits. These observations lead us to the Algorithm 1 that gives the evolution of stabilizers upon measurement of qubits in the cluster.

In this algorithm, the initial stabilizer generator group $S^{(i)}$ contains $< S_j >_{j=1}^n$, whose expression is given in Eq. 2. The number of measurements is equal to the cardinality of $\mathcal{I}$ ($|\mathcal{I}|$) defined in the notation I-A. Let consider $S^{(k)} \subset S^{(i)}$ such that the elements in $S^{(k)}$ anti-commutes with any of the measurement observable in the set $\mathcal{M}$ i.e., $\{S_1, M_a\} = 0$, where $S_1 \in S^{(k)}$ and $M_a \in \{X, Y, Z\}$. If $|S^{(k)}|$ is 2 or more, $S_p, S_q \in S^{(k)}$ are replaced with $S_p S_q$. The replacement stabilizer is appended to $S^{(i)}$. Then, to determine whether the measurement observable commutes with the updated $S^{(i)}$ or not, a new variable $\chi_1$ and $\chi_2$ (defined in notation I-A) is introduced. For a commuting stabilizer ($\chi_1 = 0$ or $\chi_2 = 0$) with qubit index same as that of the measurement observable, the elements of $S^{(i)}$ modifies as follows:

- $M_a = X_a$, and $a = j$, the stabilizer modifies as $(-1)^s S_j$,
- $M_a = Z_a$ and $a = k$ where $k \in \mathcal{N}(j)$, $S_j$ then transforms as $(-1)^s S_j$.

The elements of $S^{(i)}$ is appended then to a new generator set $S^{(m)}$, which will be called as final set of stabilizers after measurement.

This algorithm is helpful in visualising the impact of $X$ and $Z$ measurement on cluster. For example, in case of three-qubit cluster state which has the initial stabilizer group as $< X_1 Z_2, X_2 Z_1 Z_3, X_3 Z_2 >$, the stabilizer generator set $S^{(m)}$ is measured in $X$ basis resulting in a Bell pair between qubit 1 and 3. (b) The first qubit of a three qubit linear cluster state is measured in $Z$ basis, resulting in the removal of the first qubit with a phase factor.

**Algorithm 1** Stabilizer Evolution, $S^{(i)} \rightarrow S^{(m)}$

1. Initialize $S^{(i)} = < S_j >_{j=1}^n$ where $S_j = X_j \prod_{k \in \mathcal{N}(j)} Z_k$
2. $\mathcal{M} = \{M_a, a \in \mathcal{I}\}$
3. $S^{(k)} \subset S^{(i)}$ s.t $\{S_1, M_a\} = 0 \ \forall \ S_1 \in S^{(k)}$
4. if $|S^{(k)}| \geq 2$ then
5. for $S_q \in S^{(k)}$ do
6. for $S_p \in S^{(k)} \setminus \{S_q\}$ do
7. $S_j \leftarrow S_p S_q$ where $S_j \in S^{(i)}$
8. end for
9. end for
10. end if
11. for $M_a \in \mathcal{M}$ do
12. $\chi_1 := [S_j, M_a = X_a]$
13. $\chi_2 := [S_j, M_a = Z_a]$
14. if $\chi_1 = 0$ then
15. if $a = j$ then
16. $(-1)^s S_j \leftarrow S_j$
17. $S_j^f = S_j^i$
18. else
19. $S_j^i \leftarrow S_j^i$
20. $S_j^f = S_j^i$
21. end if
22. else if $\chi_2 = 0$ then
23. if $Z_a \in \prod_{k \in \mathcal{N}(j)} Z_k$ then
24. $(-1)^s S_j \leftarrow S_j$
25. $S_j^f = S_j^i$
26. else
27. $S_j^i \leftarrow S_j^i$
28. $S_j^f = S_j^i$
29. end if
30. else
31. $S_j$ vanishes
32. end if
33. end for
34. $S^{(m)} = < S_j^f >_{j=1}^n$
evolves as $< (-1)^{s_1}X_2Z_3$, $X_1X_3 >$ after $X$ measurement on qubit labelled 2. The stabilizer set are similar to the stabilizer generator of the Bell state. Therefore, $X$ measurement results in fusion of the remaining physical qubit in cluster state into a logical qubit. In Fig. 2 (a) the grey shaded cluster state refers to this logical qubit.

Now if we consider the $Z$ measurement on the first qubit of the cluster state as shown in Fig. 2 (b), after measurement the stabilizer group transforms as $< (-1)^{s_1}X_2Z_3$, $X_3Z_2 >$. However for $s_1 = 0$ the stabilizer group is equivalent to a two qubit cluster state and for $s_1 = 1$ the stabilizer group still remains same as a two qubit cluster state with a phase factor. Therefore we can conclude that $Z$ measurement is equivalent to cutting out a qubit from the cluster state.

We can generalise this example to $n$ qubit linear cluster state also. For a cluster state consists of $\{1, 2, ..., i-1, i, i+1, ..., n \}$ qubits if we measure the $i^{th}$ qubit in $Z$ basis then the $i^{th}$ qubit will be removed from the cluster and if we measure the $i^{th}$ qubit in $X$ basis the neighboring states $\{(i-1)^{th}, (i+1)^{th}\}$ will fuse together. Thus the algorithm has offered a taste of the impact of various measurements on cluster state.

III. CONSTRUCTION OF NON-CSS CODES USING MEASUREMENTS

Using Algorithm 1, we now aim to build non-CSS codes by measuring qubits of the cluster state. The steps for constructing non-CSS codes through measurement is as follows:

- A set of stabilizers $S' = < S_1, S_2, \cdots, S_{n-1} >$ associated with an $[[n, 1]]$ non-CSS code is given.
- We construct a cluster state of $(n+1)$ qubits by strategically placing the message qubit in the cluster which we shall call as parent cluster.
- Let the stabilizer generator group for $(n+1)$ qubit cluster to be $S$ which will be of form Eq. 2.
- We measure the message qubit in the $X$ basis.
- Depending upon the measurement outcome, perform local unitary corrections if required. This evolves the stabilizer set from $S$ to $S'$.

To show the efficacy of our scheme, we consider two codes, namely $[[4, 1]]$ and $[[5, 1]]$ that uses four and five physical qubits respectively to encode the information of one logical qubit. Motivated by these examples, we generalize the technique and present it in Algorithm 2.

A. Building the $[[4, 1]]$ code

The stabilizer set for $[[4, 1]]$ code is given as [10]:

$$S' = < Y_1Z_2Z_4Y_5, Y_1Z_2Y_4Z_5, Z_1Y_2Y_4Z_5 >.$$  (3)

For building the $[[4, 1]]$ non-CSS stabilizer code using measurement, we reiterate the procedure given in [10] by considering a five-qubit cluster state shown in Fig. 4 (a). The quantum state associated with the five-qubit cluster state can be written as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle_3 + \frac{1}{\sqrt{2}}|1\rangle_3.$$  (4)

where

![Fig. 3: (a) Encoding an arbitrary state $|\psi\rangle$ into five-qubit cluster. Qubit labelled 3 is measured in the $X$ basis. (b) Measurement in $X$ basis removes the third qubit and encodes $|\psi\rangle$ into the cluster. (c) Circuit diagram of cluster created in Qiskit where $X$ measurement is done on the third qubit labelled as $q_2$ because numbering of qubit will start from 0. (d) Results of stabilizer set before and after the measurement using Algorithm 1 and generation of parity check matrix. The generated stabilizer-set has redundant stabilizers. The stabilizers of the $[[4, 1]]$ can be obtained from this set.](image)
\[ |0_L\rangle = \frac{1}{\sqrt{2}} (|\Phi^-\rangle_{15} |\Phi^-\rangle_{42} - |\Psi^-\rangle_{15} |\Psi^-\rangle_{42}), \quad (5) \]
\[ |1_L\rangle = \frac{1}{\sqrt{2}} (|\Phi^+\rangle_{15} |\Phi^+\rangle_{42} + |\Psi^+\rangle_{15} |\Psi^+\rangle_{42}), \quad (6) \]
\[ |\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm \frac{1}{\sqrt{2}} |11\rangle), \quad (7) \]

Using Eq. 2, the stabilizers associated with the cluster state of five qubits can be written as
\[ S = <X_1Z_2Z_3Z_4Z_5, X_1Z_2Z_3X_4Z_5, \]
\[ Z_1Z_2Z_3X_4, Z_1Z_2Z_3X_5 >. \quad (8) \]

To encode an arbitrary message, we replace the \(|+\rangle\) state of the qubit at location 3 with \(\alpha |0 \rangle + \beta |1 \rangle\) and measure it in the X basis. Due to X measurement the cluster state changes to \(|\phi'\rangle = X^{s_1}(\alpha |+\rangle + \beta |-_L\rangle\). After X measurement, the stabilizer generator for the modified cluster state is
\[ S' = <Y_1Z_2Z_4Y_5, Y_1Z_2Z_4Z_5, Z_1Y_2Z_4Z_5 >, \quad (9) \]

which is equal to the stabilizer generator of the non-CSS code that we initially wanted to construct in Eq. 3. We also note that the logical X and logical Z operators associated with the \([4,1]\) code are \(X = Z_1Z_2X_4, Z = Z_1Z_2Z_3\) as verified by this construction. The parity check matrix i.e \(H = [H_x|H_z]\)
\[ [9] \]
associated with Eq. 3 is given as :
\[ H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}. \quad (10) \]

By considering the whole parity check matrix in Eq. 10, one can show that the minimum number of linearly dependant columns are 2. Therefore the distance for this code is 2 and it can be written as \([4, 1, 2]\) code. By taking \(H_x\) and \(H_z\) matrix individually, we can calculate the minimum distance of this code for correcting X error \(d_x\) and Z error \(d_z\) respectively \([9]\). It is easy to show that two columns of \(H_z\) and four columns of \(H_x\) are linearly dependent. Therefore in this code, \(d_z = 2\) and \(d_x = 4\). Also, error-correction capability is verified in simulations corroborating the fact that this code corrects Z, Y errors but detects X error \([10]\). The simulation in Qiskit is done using stabilizer formation explained in \([11]\). We use Algorithm 1 to generate the stabilizer group for the \([4,1]\) code. Note that, numbering of qubits in Qiskit is started from zero.

**B. Building the \([5,1]\) code**

The stabilizer associated with the \([5,1]\) non-CSS code is:
\[ S' = <X_1Z_2Z_3X_4, X_2Z_3Z_4X_5, \]
\[ X_1X_4Z_3Z_5, Z_1X_2X_4Z_5 >. \quad (11) \]

Using the same logical flow for the construction of \([4,1]\) code, we have constructed a \([5,1]\) code with stabilizer generator \(S'\).

---

**Fig. 4:** (a) Encoding an arbitrary state \(|\psi\rangle\) into six-qubit cluster. Qubit labelled 6 is measured in the X basis. (b) Measurement in X basis removes the sixth qubit and encodes \(|\psi\rangle\) into the cluster. (c) Circuit diagram of cluster created in Qiskit where X measurement is done on the sixth qubit labelled as \(q_5\) and numbering of qubit will start from 0. (d) Results of Stabilizer set before and after the measurement using Algorithm 1 and generation of parity check matrix. The generated stabilizer-set has redundant stabilizers. The stabilizers of the \([5,1]\) can be obtained from this set.
We consider a six-qubit cluster state as our parent cluster shown in Fig. 4. From the six-qubit cluster, we obtain the $[[5,1]]$ non-CSS code by measuring the message qubit in the $X$ basis. The quantum state associated with this six qubit cluster state $|\phi\rangle$ can be written as:

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left[ |0\rangle_6 |-_L \rangle + |1\rangle_6 |+_L \rangle \right], \tag{12}$$

where,

$$|0\rangle_L = \frac{1}{4} \left[ |00000\rangle + |01010\rangle + |01010\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11110\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle \right], \tag{13}$$

$$|1\rangle_L = \frac{1}{4} \left[ |11111\rangle + |01101\rangle + |10110\rangle + |01101\rangle - |00100\rangle - |11001\rangle - |01111\rangle - |00110\rangle - |10001\rangle - |11010\rangle + |10111\rangle \right]. \tag{14}$$

The stabilizer generator for the six-qubit cluster is:

$$S = < X_1 Z_2 Z_3 Z_6, Z_1 X_2 Z_3 Z_6, Z_2 X_3 Z_4 Z_6, Z_3 X_4 Z_5 Z_6, Z_4 X_5 Z_6, Z_5 Z_2 Z_3 Z_4 Z_6 >.$$ \tag{15}

Now instead of the $|+_\rangle$ state, we use any arbitrary state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ as the message to be encoded in the cluster and the encoded state in Eq. 12 modifies as $|\phi\rangle = \alpha |0\rangle_6 |-_L \rangle + \beta |1\rangle_6 |+_L \rangle$. Due to $X$ measurement the cluster state modifies as $|\phi\rangle = -X^{st} (\alpha |-_L \rangle + \beta |+_L \rangle)$. After $X$ measurement the generator of stabilizer for the modified cluster state is

$$S' = < X_1 Z_2 Z_3 X_4, X_2 Z_3 Z_4 X_5, X_1 X_2 Z_3 Z_5, Z_1 Z_2 X_3 Z_4 Z_6 >$$

which is the desired set for the non-CSS code. The Parity check matrix associated with Eq. 11 is given as:

$$H = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}. \tag{16}$$

Also, the logical $X$ and logical $Z$ operators associated with the $[[5,1]]$ code are

$$X = X_1 X_2 X_3 X_4 X_5, Z = Z_1 Z_2 Z_3 Z_4 Z_6. \tag{17}$$

Similarly to $[[4,1]]$ code, one can find out $d_x = 5$ , $d_z = 5$ and $d = 3$. Minimum distance is $3$ because of there are minimum three linearly dependent columns of parity check matrix. Therefore $[[5,1]]$ code can also be written as $[[5, 1, 3]]$. Since $d = 3$, it can correct up to $(d-1)/2 \approx 1$ Pauli error i.e 1 Pauli error. This result is verified by us with the help of QISKit as shown in Fig. 4.

From the two examples, we can observe that constructing an $[[n,1]]$ QECC by measuring a qubit in the $(n+1)$ qubit cluster is a viable technique for non-CSS codes. We generalize this approach for any arbitrary $[[n,1]]$ non-CSS QECC by invoking the connections of the adjacency matrix of a cluster to the stabilizer group of a QECC.

**IV. CONSTRUCTION OF THE CLUSTER TO ENCODE NON-CSS CODES**

A stabilizer code with parity check matrix $H = [H_x | H_z]$ satisfies

$$H_x H_x^T + H_z H_z^T = 0, \tag{18}$$

where $0$ denotes the null matrix. Also the parity check matrix for a stabilizer code associated with a cluster or graph state should have the structure given below [12]:

$$H = [I_{n-k \times n-k}: A_{n-k \times n-k}^{cc}: A_{n-k \times k}^{cm}], \tag{19}$$

where $A_{n-k \times n-k}$ is an adjacency matrix associated with the check nodes (qubits without message encoded on them) of an $n$ qubit cluster state, $A_{n-k \times k}$ is the adjacency matrix indicating the connection of the check nodes to the message nodes.

We propose the construction of an $(n + 1)$ qubit parent cluster which upon measurement of the message qubit leads us to the desired non-CSS code. To construct an $(n + 1)$-qubit cluster, we augment the matrix $H$ by adding a row and column to both $H_x$ and $H_z$ of $H$, and call it $H'$. We solve for the unknowns in this augmented matrix using Eq. 18 and the fact that $H'$ should correspond to the structure of a cluster given in Eq. 19.

```
c_p = 0 \text{ where } p = 0,1,2,3,4,5,6.
q = 0 \text{ where } q = 1,5,7.
t = 1 \text{ where } t = 0,2,3,4,6.
```

```
\begin{bmatrix}
1 & 0 & 0 & 1 & c_0 & 1 & 1 & 1 & 1 & c_4 \\
1 & 0 & 1 & 0 & c_1 & 1 & 1 & 1 & 1 & c_5 \\
0 & 1 & 1 & 0 & c_2 & 1 & 1 & 1 & 1 & c_6 \\
r_0 & r_1 & r_2 & r_3 & c_3 & r_4 & r_5 & r_6 & r_7 & c_7 \\
\end{bmatrix}
```

(a)

```
\begin{bmatrix}
1 & 0 & 0 & 1 & c'_0 & 0 & 1 & 1 & 0 & c'_5 \\
0 & 1 & 0 & 0 & c'_1 & 0 & 0 & 1 & 1 & c'_6 \\
1 & 0 & 1 & 0 & c'_2 & 0 & 0 & 1 & 1 & c'_7 \\
0 & 1 & 0 & 1 & c'_3 & 1 & 0 & 0 & 0 & c'_8 \\
r'_0 & r'_1 & r'_2 & r'_3 & c'_4 & r'_4 & r'_5 & r'_6 & r'_7 & c'_9 \\
\end{bmatrix}
```

(b)

Fig. 5: (a) and (b) are the parity check matrix for 5 qubit and 6 qubit parent cluster for $[[4,1]]$ and $[[5,1]]$ QECC respectively.
Algorithm 2 Cluster formation from non-CSS stabilizer code

i.e., forming \((n+1)\) qubit cluster from \([n,1]\) stabilizer code

1. \(S' = \langle S_1, S_2, \cdots, S_{n-1} \rangle = \langle S_j \rangle_{j=1}^{n-1}\)
2. Construct the \(H = [H_x | H_z]\) associated with \(S'\).
3. Add a new row and column by inserting binary valued variables in \(H_x\) and \(H_z\) matrix. Call it \(H'\).
4. Formulate equations of constraints for unknown variables using Eq. 18.
5. Solve for the variables doing mod 2 row addition and column exchange operation on \(H'\) such that it satisfies Eq. 19.
6. The matrix \(H'\) associated with parent cluster state is recovered. Construct the cluster from \(H'\) using the adjacency matrices \(A^{cc}\) and \(A^{cm}\).
7. Measure the message qubit in \(X\) basis and depending upon the measurement outcome, perform local unitary corrections if required to get \(S'\) containing \((n-1)\) stabilizer generators of the desired \([n,1]\) non-CSS code.

To this end, we perform row operations on \(H'\) giving us the required \(n+1\)-qubit cluster and its \(A^{cc}\) and \(A^{cm}\). In this context of parity check matrix, one should consider the operations as explained in Table. I.

| OPERATIONS                  | EQUIVALENCE                  |
|-----------------------------|------------------------------|
| Modulo 2 row operation      | Multiplying two stabilizers of the set. |
| Column exchange operations  | Relabelling of qubits.       |

TABLE I

For example the parity check matrix associated with the \([4,1]\) code and \([5,1]\) code are given in Eq. 10 and Eq. 16.

The augmented parity check matrix \(H'\) that satisfies Eq. 18 and Eq. 19 are given in Fig. 5. With the help of row additions and column exchange, we solve unknown variables and obtain the 5-qubit cluster and 6-qubit cluster corresponding to \([4,1]\) and \([5,1]\) code respectively. As a result we can we construct the \((n+1)\) qubit parent cluster and measure the message qubit in the \(X\) basis giving an \([n,1]\) non-CSS code.

We summarize this approach in Algorithm 2 which builds \((n+1)\) qubit cluster given \((n-1)\) stabilizers of the \([n,1]\) non-CSS code.

V. CONCLUSION

In this paper we used cluster states to design non-CSS codes by clever single-qubit measurements. We showed the evolution of stabilizers under single-qubit measurements and the impact of measurements on cluster state. If we are given non-CSS code stabilizers with parameters \([n,1]\), we consider \((n+1)\) qubit cluster with one message qubit, that is strategically located in this cluster. By measuring the message qubit in the \(X\) basis, followed by appropriate local operations, we project the unmeasured qubits into the given non-CSS code. To summarise, any non-CSS stabilizer code can be constructed from an \((n+1)\)-qubit cluster state.

ACKNOWLEDGMENT

A. R. thanks the start-up grant from IISER Bhopal. S.S acknowledges funding support for Chanakya -PG fellowship from the National Mission on Interdisciplinary Cyber Physical Systems, of the Department of Science and Technology, Govt. of India through the I-HUB Quantum Technology Foundation.

REFERENCES

[1] M. A. Nielsen and I. Chuang, “Quantum computation and quantum information,” 2002.
[2] D. Gottesman, Stabilizer codes and quantum error correction. California Institute of Technology, 1997.
[3] P. W. Shor, “Algorithms for quantum computation: discrete logarithms and factoring,” in Proceedings 35th annual symposium on foundations of computer science. Ieee, 1994, pp. 124–134.
[4] L.-M. Duan and G.-C. Guo, “Preserving coherence in quantum computation by pairwise quantum bits,” Physical Review Letters, vol. 79, no. 10, p. 1953, 1997.
[5] R. Laflorencie, C. Miquel, J. P. Paz, and W. H. Zurek, “Perfect quantum error correcting code,” Physical Review Letters, vol. 86, no. 1, p. 198, 1996.
[6] R. Raussendorf, D. E. Browne, and H. J. Briegel, “Measurement-based quantum computation on cluster states,” Physical review A, vol. 68, no. 2, p. 022312, 2003.
[7] D. Schlingemann and R. F. Werner, “Quantum error-correcting codes associated with graphs,” Physical Review A, vol. 65, no. 1, dec 2001. [Online]. Available: https://doi.org/10.1103%2Fphysreva.65.012308
[8] D. Schlingemann, “Stabilizer codes can be realized as graph codes,” 2001. [Online]. Available: https://arxiv.org/abs/quant-ph/0111080

[9] M. Epping, H. Kampermann, and D. Bruß, “Robust entanglement distribution via quantum network coding,” New Journal of Physics, vol. 18, no. 10, p. 103052, Oct. 2016.

[10] B. Bell, D. Herrera-Martí, M. Tame, D. Markham, W. Wadsworth, and J. Rarity, “Experimental demonstration of a graph state quantum error-correction code, nat. commun. 5, 3658 (2014),” arXiv preprint ArXiv:1404.5498.

[11] S. Aaronson and D. Gottesman, “Improved simulation of stabilizer circuits,” Physical Review A, vol. 70, no. 5, p. 052328, 2004.

[12] P. J. Nadkarni, A. Raina, and S. G. Srinivasa, “Recovery of distributed quantum information using graph states from a node failure,” in 2017 IEEE Globecom Workshops (GC Wkshps). IEEE, 2017, pp. 1–6.

[13] D. Gottesman, “Class of quantum error-correcting codes saturating the quantum hamming bound,” Physical Review A, vol. 54, no. 3, p. 1862, 1996.