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Aₚ Weights in Directionally (γ, m) Limited Space and Applications

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Abstract: Let (X, d) be a directionally (γ, m)-limited space with every γ ∈ (0, ∞). In this setting, we aim to study an analogue of the classical theory of Aₚ(µ) weights. As an application, we establish some weighted estimates for the Hardy–Littlewood maximal operator. Then, we introduce the relationship between directionally (γ, m)-limited space and geometric doubling. Finally, we obtain the weighted norm inequalities of the Calderón–Zygmund operator and commutator in non-homogeneous space.

Keywords: directionally (γ, m)-limited; Aₚ weight, non-homogeneous space; Hardy–Littlewood maximal operator; Calderón–Zygmund-type operator

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1. Introduction

Let (X, d) be a metric space. A set B = B(cₓ, rₓ) := {x ∈ X : d(x, cₓ) < rₓ} with cₓ ∈ X and rₓ ∈ (0, ∞) is called a ball in X, and, moreover, cₓ and rₓ are its center and radius, respectively.

Many classical theories in ℝⁿ of harmonic analysis have included the assumption of the doubling measure, such as mu(B(x, r)) ≤ Cµ(B(x, r/2)) as the space of homogeneous type, introduced by Coifman and Weiss [1, 2]. Another set is that in which the metric spaces are equipped with non-doubling measures. Specially, let µ be a non-negative Radon measure on ℝⁿ. µ satisfies the polynomial growth condition as there exist some positive constants C and m ∈ (0, n] such that, for all x ∈ ℝⁿ and r ∈ (0, ∞), we have

\[ µ(B(x, r)) ≤ C r^m. \]  \hfill (1)

The result of the non-doubling condition plays an important role in solving several long-standing problems related to analytic capacity, (e.g., Vitushkin’s conjecture or Painlevé’s problem; see [3]).

The proposition of the Aₚ weight under the above settings can be found in [4], and the authors assumed that any cube Q with sides parallel to the coordinate axes satisfies the property: ∂Q = 0. We have to prove our main results with the assumption that for any ball B, ∂B = 0.

In a metric space X, a set E is said to be directionally (γ, m)-limited if for each a ∈ A there are at most M distinct points b₁, . . . , bₚ in B(a, γ) ∩ A such that for i ≠ j and x ∈ X

\[ d(a, x) + d(x, b_i) = d(a, b_j) = d(a, x) \]  \hfill (2)
imply
\[ d(x, b_j) \geq \frac{1}{4} d(a, b_j). \] (3)

Then we have the following covering theorem; see [5].

**Lemma 1 (The Besicovitch–Federer covering theorem).** If \( A \) is a directionally \((\gamma, m)\)-limited subset of \( X \) and if \( F \) is a family of closed balls centered at \( A \) with radii bounded by \( \gamma > 0 \), then there are \( 2M + 1 \) disjointed subfamilies \( G_1, \ldots, G_{2M+1} \) of \( F \) such that
\[
A \subset \bigcup_{i=1}^{2M+1} \bigcup_{B \in G_i} B.
\]

With [6] and Theorem 3, we have the following lemma.

**Lemma 2.** The balls in \( G_i \) in the Besicovitch–Federer covering theorem are at most countable.

The main objective of this paper was to establish bounded estimates for Hardy–Littlewood maximal operators that do not depend on the measure condition and bounded estimates for Calderón-Zygmund commutators that combine the measure conditions.

The authors in ([7]) gave the weighted boundedness of Calderón-Zygmund in the non-homogeneous metric measure space. The authors in ([8]) gave the weighted boundedness of Calderón-Zygmund in Morrey space in the non-homogeneous metric measure space. Other results of Calderón-Zygmund operators and Hardy–Littlewood maximal operators can be found in [4,9–12]. The boundedness of commutators has also attracted a lot of attention, for example, in [13,14].

This paper is organized as follows. In Section 2, we discuss the classical theory of \( A_p \) weights in a directionally \((\gamma, m)\)-limited space.

In Section 3, we establish the boundedness of the Hardy–Littlewood maximal operator on \( L^p(w) \) space. Specifically, we need not use the boundedness of the Hardy–Littlewood maximal operator on weighted Lebesgue space to have the above weighted estimate.

The (Definition 3) of geometric doubling is well known in the analysis of metric spaces. It can be found in Coifman and Weiss [1], and is also known as metric doubling (see, for example, [5]). In Section 4, we prove that a directionally \((\gamma, m)\)-limited space with \( M \) being a positive integer is a sufficient and necessary condition for \( X \) to be described as geometrically doubling; see Theorem 3. We also introduce an interesting illustration in Example 1.

In Section 5, we assume the occurrence of upper doubling; see Definition 5. Then we prove the strong weighted norm inequalities of the Calderón–Zygmund operator and commutator.

### 2. The \( A_p \) Weight in Directionally \((\gamma, m)\)-Limited Space

The classical \( A_p \) weight in Euclidean space was first introduced by Muckenhoupt in [15]. The \( A^\rho_p(\mu) \) weights of the Muckenhoupt type with non-doubling measures were discussed by Orobitg and Pérez [4] and Komori [16]. The setting of the following \( A^\rho_p(\mu) \) weight was introduced by Hu et al. [7]. We consider \( \mu \) as a Borel measure.

**Definition 1.** Let \( p \in [1, \infty), \rho \in (1, \infty) \) and \( w \) be a non-negative \( \mu \)-measurable function. Then, \( w \) is called an \( A^\rho_p(\mu) \) weight if there exists a positive constant \( C \) such that, for all balls \( B \subset X \) (where \( 1/p + 1/p' = 1 \)),
\[
\left\{ \frac{1}{\mu(qB)} \int_B w(x) d\mu(x) \right\} \left\{ \frac{1}{\mu(qB)} \int_B [w(x)]^{1-p'} d\mu(x) \right\}^{p-1} \leq C.
\] (4)
Furthermore, for a function \( w \), if a positive constant \( C \) exists such that, for all balls \( B \subset X \),

\[
\frac{1}{\mu(B)} \int_B w(x) \, d\mu(x) \leq C \inf_{y \in B} w(y). \tag{5}
\]

\( w \) is called an \( A^p_1(\mu) \) weight. Similarly to the classical setting, let \( A^p_\infty(\mu) := \bigcup_{p=1}^\infty A^p_p(\mu) \).

Then, we denote \( A^p_1 \) as the \( A_p \) weight.

**Lemma 3.** For any \( \gamma \in (0, \infty) \), let \( X \) be directionally \( (\gamma, m) \)-limited. Let \( \mu \) be a nonnegative measure in \( X \) and \( \mu(\partial B) = 0 \) for every \( B = B(x, r) \) in \( X \). Suppose that \( f \in L^1(\mu) \) and a positive number \( \lambda \) with \( \lambda > \|f\|_{L^1(\mu(X))} \). Then, there exists a decomposition of function \( f \) as \( f = g + b \), and a sequence of balls \( \{B_j\} \), s. t.

- \( |g(x)| \leq C\lambda \) for a.e. \( x \);
- \( b = \sum b_j \), where each \( b_j \) is supported in \( B_j \), \( \int f = 0 \), and \( \int |b_j| \, d\mu \leq 2\lambda \mu(B_j) \);
- \( \{B_j\} \) is almost disjoint with a constant \( 2M + 1 \);
- \( \sum \mu(B_j) \leq \frac{C}{\lambda} \sum_{B_j} \int_{B_j} |f| \, d\mu \).

**Proof.** Firstly, let \( Mf \) be the centered Hardy–Littlewood maximal function

\[
Mf(x) := \sup_{r > 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| \, d\mu(y).
\]

For each \( x \in E_\lambda = \{ x : Mf(x) > \lambda \} \), we consider a ball \( B(x, r_x) \) such that \( \int_{B(x, r_x)} |f| \, d\mu > \lambda \mu(B(x, r_x)) \). Then, we proceed as in the proof of the covering lemma. We define

\[
H(r) := \frac{1}{B(x, r)} \int_{B(x, r)} |f| \, d\mu(y). \tag{6}
\]

Then, \( H(r_x) > \lambda \) and \( \lim_{r \to \infty} H(r) = \|f\|_{L^1(\mu(X))} < \lambda \). Therefore, \( H(r) \) is a continuous function. Since we assume that \( \mu(\partial B) = 0 \), \( H(r) \) is continuous with respect to \( r \).

On \((r_x, \infty)\), we obtain a ball \( B_x \) centered at \( x \) such that

\[
\lambda = \frac{1}{B_x} \int_{B_x} |f| \, d\mu. \tag{7}
\]

With the Besicovitch–Federer covering theorem, there is an almost disjoint sequence \( \{B_j\} \) of balls such that \( E_\lambda = \bigcup_j B_j \) and (7) holds for each \( B_j \).

Consider the following functions (\( \chi_{B_j} \) is a function, i.e., 1 in the ball and 0 outside the ball.)

\[
\varphi_j = \frac{\chi_{B_j}}{\sum \chi_{B_j}}, \quad \frac{1}{2M + 1} \leq \varphi_j \leq 1 \text{ on } B_j,
\]

and \( \sum \varphi_j = 1 \) on \( \bigcap_j B_j \), and define

\[
b_j = (f \varphi_j - (f \varphi_j)_{B_j}) \chi_{B_j}.
\]

Clearly,

\[
\int_{B_j} |b_j| \, d\mu \leq \int_{B_j} |f| \, d\mu + \int_{B_j} |f \varphi_j| \, d\mu \leq 2 \int_{B_j} |f| \, d\mu = 2\lambda \mu(B_j)
\]

and

\[
\int_{B_j} |b_j| \, d\mu = 0.
\]

Finally, we take \( g = f - \sum b_j \) and \( b = \sum b_j \).
Now, if \( x \notin \cap B_j \), then \( g(x) = f(x) \) and the differentiation theorem gives \( |g(x)| \leq \lambda \) for \( \mu \)-a.e. \( x \notin \cap B_j \). When \( x \in \cap B_j \),
\[
g(x) = \sum_{j} (f \varphi_j)_{B_j \chi_{B_j}} \leq \lambda(2M + 1)
\]
because \( \{B_j\} \) is almost disjoint and \( (f \varphi_j)_{B_j} = \lambda \).

With a similar proof, we can also have the following lemma.

**Lemma 4.** For any \( \gamma \in (0, \infty) \), \( X \) is directionally \( (\gamma, m) \)-limited. Let \( \mu \) be a nonnegative measure in \( X \) with \( \mu(\partial B) = 0 \) for every \( B = B(x, r) \) in \( X \). Suppose that \( f \in L^1(\mu) \) and a positive number \( \lambda \) with \( \lambda > \|f\|_{L^1}/(\mu(X)) \). There exists a sequence of balls \( \{B_j\} \), so that
\[
\bullet \quad |f(x)| \leq \lambda \text{ for } \mu\text{-a.e. } x \notin \cap B_j;
\bullet \quad \{B_j\} \text{ is almost disjoint with constant } 2M + 1;
\bullet \quad \frac{1}{\mu(B_j)} \int_{B_j} |f| \, d\mu = \lambda.
\]
Furthermore, we prove some of the results that hold in our situation.

**Lemma 5.** For a weight \( w \), the following conditions are equivalent:

\( a \) \( w \in A_\infty(\mu) \).

\( b \) For any ball \( B \),
\[
\frac{1}{\mu(B)} \int_B wd\mu \approx \exp \left( \frac{1}{\mu(B)} \int_B \log wd\mu \right).
\]

\( c \) There are constants \( 0 < \alpha, \beta < 1 \) such that for every ball \( B \)
\[
\mu(\{x \in B : w(x) \leq \beta w_B\}) \leq \alpha \mu(B). \tag{8}
\]

\( d \) There are positive constants \( C \) and \( \beta \) such that for any ball \( B \) and for any \( \lambda > w_B \)
\[
w(\{x \in B : w(x) \leq \lambda\}) \leq C\lambda \mu(\{x \in B : w(x) \leq \lambda\}).
\]

\( e \) \( w \) satisfies a reverse Hölder inequality: there are positive constants \( c \) and \( \sigma \) such that for any ball \( B \):
\[
\left( \frac{c}{\mu(B)} \int_B w^{1+\sigma} \, d\mu \right)^{\frac{1}{1+\sigma}} \leq \frac{c}{\mu(B)} \int_B wd\mu.
\]

\( f \) There are positive constants \( c \) and \( \rho \) such that for every ball \( B \) and any measurable set \( E \) contained in \( B \):
\[
\frac{w(E)}{w(B)} \leq c \left( \frac{\mu(E)}{\mu(B)} \right)^{\rho}. \tag{9}
\]

\( g \) There are constants \( 0 < \alpha, \beta < 1 \) such that for every ball \( B \) and any measurable set \( E \) contained in \( B \):
\[
\frac{\mu(E)}{\mu(B)} < \alpha \text{ implies } \frac{w(E)}{w(B)} < \beta. \tag{10}
\]

**Proof.** \( a \Rightarrow b \). Jensen’s inequality means that
\[
\frac{1}{\mu(B)} \int_B wd\mu \geq \exp \left( \frac{C}{\mu(B)} \int_B \log wd\mu \right).
\]
Because the $A_p$ weights are increasing on $p$, $w \in A_\infty(\mu)$ means that there exists $p_0 > 1$ such that $w \in A_p$ for $p > p_0$. Therefore, there exists a constant $C$ such that for $p \geq p_0$

$$\left[ \frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right] \left\{ \frac{1}{\mu(B)} \int_B |w(x)|^{1-p'} d\mu(x) \right\}^{p-1} \leq C.$$ 

Letting $p \to \infty$, we obtain (b).

(b) $\Rightarrow$ (c). Dividing $w$ by an appropriate constant (to be precise, $\exp \left( \frac{C}{\mu(B)} \int_B \log wd\mu \right)$), we assume that $\int_B \log wd\mu = 0$ and, consequently, $w_B \leq C$. Therefore,

$$\mu(\{x \in B : w(x) \leq \beta w_B\}) \leq \mu(\{x \in B : w(x) \leq \beta C\}) \leq \mu(\{x \in B : \log(1 + \frac{1}{w(x)}) \leq \log(1 + \frac{1}{\beta C})\}) \leq \frac{1}{\log(1 + \frac{1}{\beta C})} \int_B \log(1 + \frac{1}{w(x)}) d\mu \leq \frac{1}{\log(1 + \frac{1}{\beta C})} \int_B \log(1 + w(x)) d\mu.$$ 

With $\log(1 + t) \leq t, t \geq 0$, we can deduce

$$\mu(\{x \in B : w(x) \leq \beta w_B\}) \leq \frac{1}{\log(1 + \frac{1}{\beta C})} \int_B w(x) d\mu \leq \frac{C \mu(B)}{\log(1 + \frac{1}{\beta C})} \leq \frac{1}{2} \mu(B)$$

with a small enough $\beta$.

(c) $\Rightarrow$ (d). With Lemma 4 and $\lambda > w_B$, we find a family of quasi-disjoint balls satisfying

$$\frac{1}{\mu(B_j)} \int_{B_j} |f| d\mu = \lambda$$

for each $j$. Then, we can deduce that

$$w\{x \in B : w(x) > \lambda\} \leq \sum_{k=1}^{2M+1} \sum_i w(B_i) \leq \lambda \sum_{k=1}^{2M+1} \sum_i \mu(B_i) \leq \frac{\lambda}{1-\alpha} \sum_{k=1}^{2M+1} \sum_i \mu(\{x \in B_i : w(x) > \beta w_B\}) \leq \frac{\lambda(2M+1)}{1-\alpha} \mu(\{x \in B : w(x) > \beta w_B\})$$

with $w_{B_j} = \lambda > w_B$.

(d) $\Rightarrow$ (e). We will use the equation

$$\int_X w(x)^p d\mu = p \int_0^\infty \lambda^p \mu(\{x \in B : w(x) > \lambda\}) \frac{d\lambda}{\lambda} \quad (11)$$

For an arbitrary positive $\sigma$, we have

$$\frac{1}{\mu(B)} \int_B w^{1+\sigma} d\mu = \frac{\sigma}{\mu(B)} \int_0^\infty \lambda^\sigma w(\{x \in B : w(x) > \lambda\}) \frac{d\lambda}{\lambda}$$
Therefore, if we take \( \eta \) such that
\[
\int_B 1_{B} w(x) \, d\mu(x) = 0,
\]
then we can use the Hölder inequality to obtain
\[
\int_B 1_{B} w(x) \, d\mu(x) = 0.
\]

If we choose a \( \sigma \) value small enough that \( \frac{C\sigma}{\mu(B)} < 1 \), we can obtain (e).

(e) \( \Rightarrow \) (f). With the Hölder inequality, \( E \subset B \) and letting \( p = 1 + \sigma \), we can deduce
\[
\frac{w(E)}{w(B)} = \int_B 1_{E} w \, d\mu \leq \left( \frac{1}{\mu(B)} \int_B w \, d\mu \right)^{1/r} \left( \frac{\mu(E)}{\mu(B)} \right)^{1/r},
\]
and this implies (f) with \( p = 1/r' \).

(f) \( \Rightarrow \) (g). This is directional.

(g) \( \Rightarrow \) (c). At first, (g) means that there are positive constants \( \alpha', \beta' < 1 \) such that \( E \) is a measurable set of a ball \( B \),
\[
\frac{w(E)}{w(B)} < \alpha' \text{ implies } \frac{\mu(E)}{\mu(B)} < \beta'.
\]

Then, let \( E = \{ x \in B : w(x) > bw \} \), where \( b \in (0, 1) \) is going to be chosen now, and let \( E' = E \setminus E = \{ x \in B : w(x) \leq bw \} \). We can deduce that \( w(E') \leq bww(B) \leq bw(B) \). Therefore, if we take \( b = \beta' \), we have \( \mu(E') < \alpha' \mu(B) \). This yields (c).

(e) \( \Rightarrow \) (a). When we write \( d\mu = w^{-1} \, d\mu \), due to (e), there are positive constants \( c, \sigma \) such that
\[
\left( \frac{c}{w(B)} \int_B (w^{-1})^{1+\sigma} \, d\mu \right)^{1/\sigma} \leq \frac{c}{w(B)} \int_B w^{-1} \, d\mu.
\]

Hence,
\[
\frac{w(B)}{\mu(B)} \left( \frac{1}{\mu(B)} \int_B w^{-\sigma} \right)^{1/\sigma} \leq C.
\]

Then, letting \( \sigma = \frac{1}{p}(p = 1 + \frac{1}{p} > 1) \), we may observe that \( w \in A_p \). The proof of this lemma is complete. \( \square \)

**Corollary 1.** Let \( p > 1 \) and \( w \in A_p \). Then

- There is an \( \sigma > 0 \) such that \( w \in A_{p-\sigma} \), and hence
\[
A_p(\mu) = \bigcup_{q < p} A_q(\mu).
\]

- There is an \( \eta > 0 \) such that \( w^{1+\eta} \in A_p(\mu) \).
Definition 2. BMO($\mu$) space is the set of functions with the following property:

$$\sup_B \frac{1}{\mu(B)} \int_B |f - f_B|d\mu < \infty.$$ 

Then we have the following corollary:

Corollary 2. Let $p > 1$ and $w \in A_p$. Then,
- If $w \in A_\infty(\mu)$, then $\log(w) \in \text{BMO}$. 
- Fix $p > 1$ and let $b \in \text{BMO}(\mu)$. Then, there exists a $\delta > 0$ depending only on the BMO($\mu$) constant of $b$ such that $e^{tb} \in A_p(\mu)$ for $|x| < \delta$.

The proof of this corollary is similar to the classical one; see [17]

3. Boundedness of Hardy–Littlewood Maximal-Type Operators

For any given $\eta \in (1, \infty)$, we aim to define the Hardy–Littlewood maximal operator $M_\eta$. For all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$,

$$M_\eta(f)(x) := \sup_{x \in B} \frac{1}{\mu(\eta B)} \int_B |f(y)|d\mu(y). \quad (12)$$

We also define the center Hardy–Littlewood maximal operator $M^c_\eta$ by setting, for any $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$,

$$M^c_\eta(f)(x) := \sup_{B(x,r)} \frac{1}{\mu(\eta B)} \int_B |f(y)|d\mu(y). \quad (13)$$

Theorem 1. Let $\rho \in [1, \infty)$, $p \in [1, \infty)$, $w \in A^\rho_p(\mu)$, $\mu$ be a non-negative measure in $\mathcal{X}$ and $\mu(\partial B) = 0$ for any $B = B(x, r)$ in $\mathcal{X}$. Then, $M^c_\eta$ is bounded from $L^p(w)$ into $\text{WL}^p(w)$ with $\eta \in [q, \infty)$ ($\text{WL}^p(w)$ is a weak Lebesgue space).

Proof of Theorem 1. For any ball $B \subset \mathcal{X}$, $\eta \in [q, \infty)$ and any $R \in (0, \infty)$, we define the operator $M_{\eta,R}$ by means of the following settings. For any $x \in \mathcal{X}$,

$$M_{\eta,R}(f)(x) := \sup_{S := B(x, r_x), r_x < R} \frac{1}{\mu(\eta S)} \int_S |f(y)|d\mu(y).$$

For any $t \in (0, \infty)$, we define

$$E_R(t) := \{x \in B : M_{\eta,R}(f)(x) > t\}.$$

With (13), forevery $x \in E_R(t)$, there exists a ball $B(x, r_x)$ and

$$\frac{1}{\mu(\eta B(z_x, r_x))} \int_{B(z_x, r_x)} |f(y)|d\mu(y) > t. \quad (14)$$

To prove the theorem, we first consider the case of $p \in (1, \infty)$ and the following two cases of $\{B_i\}$.

Case (1): There exists a ball $\tilde{B} \in \{B_i\}$, such that $\tilde{B} \supset B$. With (14), the Hölder inequality, (4) and the assumption of $\eta \in [q, \infty)$, we can deduce that

$$\int_{E_R(t)} w(y)d\mu(y) \leq \int_B w(y)d\mu(y)$$
Let $X = L^p \in (\mathfrak{M}_{\rho}, \mu)$. Combining Theorem 1, Corollary 1, and Lemma 6, we have the following theorem.

Then, we can immediately obtain the following corollary.

**Case (II):** For all $B_i \in \{B_i\}_i$, $B_i \nsubseteq B$. Through the Besicovitch–Federer covering theorem, there exists a disjointed subfamily $\{B_i := B(z_{x_i}, r_{x_i})\}_i$ such that

$$\left[ \bigcup_{x \in E_\mathcal{X}(i)} B(z_{x_i}, r_{x_i}) \right] \subseteq \left[ \bigcup_{i} B_i \right].$$

From this, together with (15), (14), the Hölder inequality, and (4), we have

$$\int_{E_\mathcal{X}(i)} w(y) d\mu(y) \leq \sum_i w(B_i) \leq \frac{1}{\mu(\eta B_i)} \int_{B_i} |f(y)| d\mu(y) \left( \int_{B_i} |w(y)|^{-\rho'} d\mu(y) \right)^{\rho'/\rho} w(B_i) \leq \frac{1}{\mu(\eta B_i)^p} \int_{B_i} |f(y)|^p w(y) d\mu(y) \leq \frac{1}{\mu(\eta B_i)^p} \int_{B_i} |f(y)|^p w(y) d\mu(y).$$

Letting $R \to \infty$ and $r_B \to \infty$, we obtain the proof of $p \in (1, \infty)$. In fact, the process for $p = 1$ is similar. We only need to replace the Hölder inequality and (4) with (5). We have thus completed the proof of Theorem 1. \hfill $\Box$

To obtain a better result, we provide the following lemma; see [17].

**Lemma 6 ([17]).** Let $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ be two measure spaces, $1 \leq p_0 < p_1 \leq \infty$, and let $T$ be a sublinear operator from $L^{p_0, \infty} \rho(\mathcal{X}, \mu) + L^{p_1, \infty} \rho(\mathcal{X}, \mu)$ to the measurable functions on $Y$. If $T$ is bounded from $L^{p_0}(\mathcal{X}, \mu)$ into $L^{p_0}(\mathcal{X}, \mu)$ and bounded from $L^{p_1}(\mathcal{X}, \mu)$ into $W^{p_1}(\mathcal{X}, \mu)$, then $T$ is bounded on $L^p(\mathcal{X}, \mu)$ for $p_0 < p < p_1$.

Combining Theorem 1, Corollary 1, and Lemma 6, we have the following theorem.

**Theorem 2.** Let $\rho \in [1, \infty)$, $p \in [1, \infty)$, $w \in A^p_p(\mu)$, and $\mu$ be a non-negative measure in $\mathcal{X}$ and $\mu(\partial B) = 0$ for any $B = B(x, r)$ in $\mathcal{X}$. Then, $M^1_1$ is bounded on $L^p(w)$ with $p \in (1, \infty)$ and from $L^p(w)$ into $W^{p_1}(w)$ with $p = 1$.

Then, we can immediately obtain the following corollary.

**Corollary 3.** Let $\rho \in [1, \infty)$, $p \in [1, \infty)$, $w \in A^p_p(\mu)$, and $\mu$ be a non-negative measure in $\mathcal{X}$ and $\mu(\partial B) = 0$ for any $B = B(x, r)$ in $\mathcal{X}$. Then, $M^1_1$ and $M$ are bounded on $L^p(w)$ with $p \in (1, \infty)$ and from $L^p(w)$ into $W^{p_1}(w)$ with $p = 1$.\hfill $\Box$
4. Relation between Directionally \((\gamma, m)\)-Limited Space and Geometric Doubling

In fact, homogeneous-type spaces are geometrically doubling, which was proven by Coifman and Weiss in [1].

**Definition 3.** If there exists an \(N_0 \in \mathbb{N} := \{1, 2, \ldots\}\) such that, for any ball \(B(x, r) \subset \mathcal{X}\) with \(x \in \mathcal{X}\) and \(r \in (0, \infty)\), there exists a finite ball covering \(\{B(x_i, r/2)\}\), of \(B(x, r)\) such that the cardinality of this covering is at most \(N_0\). A metric space \((\mathcal{X}, d)\) is thus said to be geometrically doubling.

**Remark 1.** For a metric space \((\mathcal{X}, d)\), Hytönen in [6] showed that geometric doubling is equivalent to the following condition: for any \(e \in (0, 1)\) and any ball \(B(x, r) \subset \mathcal{X}\) with \(x \in \mathcal{X}\) and \(r \in (0, \infty)\), there exists a finite ball covering \(\{B(x_i, er)\}\) of \(B(x, r)\) such that the cardinality of this covering is at most \(e^{-n_0}\); here and hereafter, \(N_0\) is as in Definition 3 and \(n_0 := \log_2 N_0\).

**Theorem 3.** For any \(\gamma > 0\), \(\mathcal{X}\) is a directionally \((\gamma, m)\)-limited space with \(M\) being a positive integer. This is a sufficient and necessary condition that \(\mathcal{X}\) is a geometrically doubling space.

**Example 1.** Let \(\mathcal{X} = \{(x_1, x_2, \ldots, x_n, \ldots) \in \mathbb{R} : x_n \in \{0, 2^{-n}\}\}\). \(\mathcal{X}\) satisfies the geometric doubling condition, but it is not directionally \((\gamma, m)\)-limited.

With the boundness of Hardy–Littlewood Maximal Operators in [7], we have the following corollary.

**Corollary 4.** Let \(\mu\) be a nonnegative measure in \(\mathcal{X}\) and \(\mu(\partial B) = 0\) for any ball \(B = B(x, r)\) in \(\mathcal{X}\). Let \(p \in [1, \infty), \eta \in [5, \infty)\) and \(w \in A_p(\mu)\). Then, \(M_\eta\) is bounded on \(L^p(w)\) with \(p \in (1, \infty)\) and from \(L^p(w)\) into \(W(L^p(w))\) with \(p = 1\).

**Proof.** First, we prove that \(\mathcal{X}\) is a geometrically doubling condition space. We consider a ball \(B(x, r) := B((x_1, x_2, \ldots, x_n, \ldots), r)\). Let \(k\) be an integer such that \(r \in (2^{-(k+1)}, 2^{-k})\). We only need to show that \(B := B((x_1, x_2, \ldots, x_n, \ldots), 2^{-k})\) could be covered by a finite ball covering \(\{B(x_i, 2^{-(k-2)})\}\) such that the cardinality of this covering is at most \(N_0\), which implies that \(\{B(x_i, r/2)\}\) is a ball covering of \(B((x_1, x_2, \ldots, x_n, \ldots), r)\).

Without the loss of generality, we could assume that \(k > 2\). Then we can deduce that for any point \(y := (y_1, y_2, \ldots, y_n, \ldots) \in B\) and \(j < k\), \(x_j = y_j\) because if \(x_j \neq y_j\), then

\[
d(x_j, y_j) > d(x_j, (x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots)) = 2^{-j} \geq 2^{-k}.
\]

Then we consider a set \(E := \{z = (z_1, z_2, \ldots, z_n, \ldots) \in \mathcal{X} : \text{for } j < k \text{ or } j > k + 4, z_j = x_j\}\) and the cardinality of \(E\) is at most 8. Let \(z_0 = (z^*_1, z^*_2, \ldots, z^*_n, \ldots) \in E\) and for \(j < k\) or \(j > k + 3\), \(z^*_j = y_j\). Moreover, we have

\[
d(y, z_0) \leq \frac{1}{2^{k+4}} + \frac{1}{2^{k+5}} + \frac{1}{2^{k+6}} \cdots = \frac{1}{2^{k+3}}
\]

which implies \(y \in B(z_0, \frac{1}{2^{k+3}})\). Finally, we can deduce that

\[
B \subset \bigcup_{x \in E} B(x, 2^{-(k+2)}).
\]

Secondly, we prove that \(\mathcal{X}\) is not directionally \((\gamma, m)\)-limited. Let \(a = (0, 0, 0, \ldots)\) and \(b_j = (0, 0, \ldots, 0, 2^{-j}, 0, \ldots)\). For any \(j\), there is no \(x \in \mathcal{X}\) such that \(d(a, x) + d(x, b_j) = d(a, b_j)\) and \(d(a, b_j) = d(a, x)\). Because there are infinite \(j\) such that \(2^{-j} < \gamma\), we have proven that \(\mathcal{X}\) is not directionally \((\gamma, m)\)-limited. □
**Theorem 3.** With Example 1, we only need to prove that if \( \mathcal{X} \) is directionally \((\gamma, m)\)-limited, then \( \mathcal{X} \) satisfies the geometric doubling condition. We assume that \( \mathcal{X} \) does not satisfy the geometric doubling condition and, with Remark 1 (iii), we can observe that for every \( \varepsilon \in (0, 1) \), there exists a ball \( B(x, r) \subset \mathcal{X} \) with \( x \in \mathcal{X} \) and \( r \in (0, \infty) \) containing infinite disjoint balls \( \{B(x_i, \varepsilon r)\}_i \). Let \( a = x_1 \) and \( b_i = x_{i-1} \). For any \( i, j \), if there is \( x \in \mathcal{X} \),

\[
d(a, x) + d(x, b_i) = d(a, b_i) \quad \text{and} \quad d(a, b_j) = d(a, x).
\]

(16)

Then we have

\[
d(x, a) = d(a, b_j) \leq d(a, c_B) + d(c_B, b_j) < 2r,
\]

which implies

\[
\frac{1}{4}d(x, a) \leq r.
\]

(17)

Together with Remark 1 (iii), we have

\[
d(x, b_j) > \varepsilon r
\]

(18)

and

\[
d(x, a) = d(a, b_j) > 2\varepsilon r.
\]

(19)

To prove \( d(x, b_j) \geq \frac{r}{2} \), let \( \varepsilon = 0.9 \) and assume that \( d(x, b_j) < \frac{r}{2} \). Then, we can observe that \( d(x, b_i) > 0.9r \) and \( d(x, a) > 0.9r \). With (2), (18) and

\[
d(a, b_i) \leq d(a, c_B) + d(c_B, b_i) < 2r,
\]

we have

\[
d(x, a) < 2r - \varepsilon r = 1.1r,
\]

(20)

which is a contradiction with (18). Therefore, we have \( d(x, b_j) \geq \frac{r}{2} \), together with (17), which implies that (3).

In summary, for \( \gamma > r \), \( \mathcal{X} \) is not directionally \((\gamma, m)\)-limited; we have thus completed the proof. \( \square \)

To obtain a better result, we can consider a better definition.

**Definition 4.** For any \( a, b, c \in \mathcal{X} \), we have \( x \in \mathcal{X} \) such that

\[
d(a, x) + d(x, b) = d(a, b) \quad \text{and} \quad d(a, c) = d(a, x).
\]

(21)

5. **Boundedness of Calderón-Zygmund Operators and Commutators in Non-Homogeneous Space**

However, in [6] Hytönen pointed out that the measure \( \mu \) satisfying (1) is different from the doubling measure. Hytönen [6] introduced a new class of metric measure spaces. The new spaces satisfy the following upper doubling condition and geometric doubling condition.

These spaces are called non-homogeneous metric measure spaces. These spaces include both homogeneous-type spaces and non-doubling measure spaces.

**Definition 5.** If \( \mu \) is a Borel measure on \( \mathcal{X} \) and there exist a dominating function \( \lambda : \mathcal{X} \times (0, \infty) \to (0, \infty) \) and a positive constant \( C_{(\lambda)} \), depending on \( \lambda \), such that, for each \( x \in \mathcal{X} \), \( r \to \lambda(x, r) \) is non-decreasing and, for all \( x \in \mathcal{X} \) and \( r \in (0, \infty) \),

\[
\mu(B(x, r)) \leq \lambda(x, r) \leq C_{(\lambda)}\lambda(x, r/2).
\]

(22)

Thus a metric measure space \((\mathcal{X}, d, \mu)\) is said to be upper doubling.
Definition 6. Let a function $K \in L^1_{\text{loc}}(\{X \times X\} \setminus \{(x, x) : x \in X\})$. If there exists a positive constant $C$, such that,

(i) for all $x, y \in X$ with $x \neq y$,

$$|K(x, y)| \leq C \frac{1}{\lambda(x, d(x, y))};$$

(23)

(ii) there exist positive constants $\delta \in (0, 1]$ and $c(K)$, depending on $K$, such that, for all $x, \tilde{x}, y \in X$ with $d(x, \tilde{x}) \geq c(K)d(x, \tilde{x}),$

$$|K(x, y) - K(\tilde{x}, y) + |K(y, x) - K(y, \tilde{x})| \leq C \frac{[d(x, \tilde{x})]^{\delta}}{[d(x, y)]^{\delta} \lambda(x, d(x, y))},$$

(24)

then, $K$ is called a Calderón–Zygmund kernel.

A linear operator $T$ is called the Calderón–Zygmund operator with kernel $K$ satisfying (23) and (24) if, for all $f \in L^\infty(\mu) := \{f \in L^\infty(\mu) : \text{supp}(f) \text{ is bounded}\},$

$$Tf(x) := \int_X K(x, y)f(y) \, d\mu(y), \quad x \notin \text{supp}(f).$$

(25)

The following boundedness of $T$ on $L^p(w)$ with $w$ and the $A^\infty_p(\mu)$ weight was first obtained in [7].

Lemma 7 ([7]). Let $p \in [1, \infty)$, $\varrho \in [1, \infty)$ and $w \in A^\infty_p(\mu)$. Assume that the Calderón-Zygmund operator $T$ defined by (25) with kernel $K$ as a Calderón–Zygmund kernel is bounded on $L^2(\mu)$. Then, $T$ is bounded from $L^p(w)$ into $W^p(\omega)$.

Similarly to the proof of Theorem 2, we can immediately obtain the following theorem.

Theorem 4. Let $\mu$ be a nonnegative measure in $X$ and $\mu$ satisfies $\mu(\partial B) = 0$ for any ball $B = B(x, r)$ in $X$. Let $p \in [1, \infty)$ and $w \in A^\infty_p(\mu)$. Let the Calderón–Zygmund operator $T$, defined by (25) be associated with kernel $K$ and $K$, as a Calderón–Zygmund kernel, is bounded on $L^2(\mu)$. Then, $T$ is bounded on $L^p(w)$ with $p \in (1, \infty)$ and from $L^p(w)$ into $W^p(\omega)$ with $p = 1$

As an application, we can attempt to estimate the commutator $[b, T]$ as

$$[b, T] = bT(f) - T(bf).$$

This is a bounded operator in $L^p(w)$ when $b \in BMO(\mu)$.

Theorem 5. Let $b \in BMO(\mu), w \in A_p(\mu), p > 1$ and $T$ be a Calderón-Zygmund operator. Then,

$$\|[b, T]f\|_{L^p(w)} \leq C\|b\|_1 \|f\|_{L^p(\omega)}$$

Proof. According to Corollary 1 (ii), there is a constant $\eta > 0$ such that $w^{1+\eta} \in A_p(\mu)$. Then, with Corollary 2 (ii), we choose $\delta > 0$ to ensure exp$(spb(1 + \eta)/\eta) \in A_p(\mu)$ if $0 \leq s(1 + \eta)/\eta < \delta$ with a uniform constant.

For any $z \in \mathcal{C}$, we define the operator

$$S_z f = e^{zh}T(e^{-zh} f).$$

It is easy to see that

$$\|S_z\|_{L^p(w)} \leq C\|f\|_{L^p(\omega)}$$

uniformly on $|z| \leq \sigma\eta/(1 + \eta)$. 
The function $s \mapsto S_z f$ is analytic, and according to the Cauchy theorem, if $s \leq \sigma \eta / (1 + \eta)$, 
\[
\frac{d}{dz} S_z f |_{z=0} = \frac{1}{2\pi i} \int_{|z|=s} \frac{S_z f}{z^2} dz.
\]

It is easy to see that 
\[
\frac{d}{dz} S_z f |_{z=0} = [b, T] f
\]
with the application of Minkowski inequality, we can deduce that
\[
\| [b, T] f \|_{L^p(w)} \leq \frac{1}{2\pi i} \int_{|z|=s} \frac{\| S_z f \|_{L^p(w)}}{z^2} dz \leq \frac{C}{s} \| f \|_{L^p(w)}.
\]

Therefore, we are left with the task of proving the claim, which is equivalent to proving that $\mathcal{R}(z)$ is the polar form of $z$:
\[
\int |Tf(x)|^p e^{\mathcal{R}(z) p \eta(x)} w(x) d\mu(x) \leq C \int |f(x)|^p e^{\mathcal{R}(z) p \eta(x)} w(x) d\mu(x).
\]

We denote $w_0 := e^{\mathcal{R}(z) p \eta(x)}$ and $w_1 := w^{1+\eta}$. Since $w_0$ and $w_1$ belong to $A_p(\mu)$, we have 
\[
\int |Tf(x)|^p w_0(x) d\mu(x) \leq C \int |f(x)|^p w_0(x) d\mu(x)
\]
and 
\[
\int |Tf(x)|^p w_1(x) d\mu(x) \leq C \int |f(x)|^p w_1(x) d\mu(x).
\]

Finally, based on the Stein–Weiss interpolation theorem [18], we can complete the proof of this theorem. 

6. Conclusions

We have provided a weighted boundedness estimate for the Hardy–Littlewood maximal operator in $(\gamma, m)$-limited space. This proof does not rely on any measure properties. It is worth mentioning that $\gamma$ is directionally $(\gamma, m)$-limited, with $M$ being a positive integer, and this is a sufficient and necessary condition to conclude that $\gamma$ is a geometrically doubling space.

Furthermore, on the basis that $\gamma$ is directionally $(\gamma, m)$-limited, we assumed that the measure satisfies the upper double, and we obtained a result regarding the boundedness of the Calderón–Zygmund operator and its commutator (a typical example is the Hilbert transform in higher-dimensional spaces).

We hope that this research will help in the related study of differential equations. In the meantime, it would be valuable if some researchers could further relax the assumptions on space and measure.

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