On feedback in Gaussian multi-hop networks

Bobbie Chern, Farzan Farnia, Ayfer Özgür
Stanford University
{bgchern, farnia, aozgur}@stanford.edu

Abstract

The study of feedback has been mostly limited to single-hop communication settings. In this paper, we consider Gaussian networks where sources and destinations can communicate with the help of intermediate relays over multiple hops. We assume that links in the network can be bidirected providing opportunities for feedback. We ask the following question: can the information transfer in both directions of a link be critical to maximizing the end-to-end communication rates in the network? Equivalently, could one of the directions in each bidirected link (and more generally at least one of the links forming a cycle) be shut down and the capacity of the network still be approximately maintained? We show that in any arbitrary Gaussian network with bidirected edges and cycles and unicast traffic, we can always identify a directed acyclic subnetwork that approximately maintains the capacity of the original network. For Gaussian networks with multiple-access and broadcast traffic, an acyclic subnetwork is sufficient to achieve every rate point in the capacity region of the original network, however, there may not be a single acyclic subnetwork that maintains the whole capacity region. For networks with multicast and multiple unicast traffic, on the other hand, bidirected information flow across certain links can be critically needed to maximize the end-to-end capacity region. These results can be regarded as generalizations of the conclusions regarding the usefulness of feedback in various single-hop Gaussian settings and can provide opportunities for simplifying operation in Gaussian multi-hop networks.

I. INTRODUCTION

Feedback has been studied extensively for single-hop communication channels. While feedback cannot increase the capacity of the discrete memoryless point-to-point channel [1], it is well understood that it can increase the capacity of the Gaussian multiple access (MAC), broadcast and relay channels, but only through a power gain [2], [3], [4]. More recently, it has been shown in [5] that feedback can provide degrees of freedom gain in the Gaussian interference channel, which translates to an unbounded gain in capacity when SNR increases. In the recent years, there has been significant interest in larger networks where communication between nodes is established in multiple hops [6], [7], [8]. However, the study of the usefulness of feedback has been mostly limited to the above single-hop settings of a few nodes.

In this paper, we aim to understand the role of feedback in general Gaussian networks. We consider a Gaussian network where sources communicate to destinations in multiple-hops with the help of intermediate relay nodes. In wireless, if a given node can send information to another node, typically it can also receive information from that node, thus communication links between pairs of nodes are often bidirectional. Therefore, inherently there are a lot of opportunities for “feeding back” information in wireless networks, though the nature of these feedback links is significantly different from the idealized feedback models considered in the single-hop settings. First, transmissions, and therefore also feedback, may not be isolated but subject to broadcast and superposition. Second, while in single hop networks the links originating from destinations and/or arriving at source nodes can be clearly identified as feedback, in multihop networks there can be “feedback” between any pair of nodes. Bidirected links and cycles in the network can be used to feedback information, however it is not a priori possible to designate links as communication links and feedback. Therefore, in these new multi-hop settings it is not totally clear how to think about feedback and how to study its usefulness.

In this paper, we adopt the following approach. We consider a general Gaussian relay network with arbitrary topology and channel gains, possibly with bidirected links and cycles, where some links can be

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subject to broadcast and superposition and some can be isolated in a completely arbitrary fashion. We ask the following question: can the information transfer in both directions of any link in the network be critical to maximizing the end-to-end communication rate? Equivalently, could one of the directions in each bidirected link (and more generally at least one of the links forming a cycle) be shut down and the capacity of the network still be approximately maintained?

We show that when there is only a single source-destination pair in the network (unicast traffic), we can always identify a directed acyclic subnetwork that approximately preserves the capacity of the original network. More precisely, if any of the links that do not belong to this subgraph could be disabled, the capacity of the resultant network would still remain within a bounded gap to the capacity of the original network. See Figure 1. The main technical step is to show that in every Gaussian relay network, there exists a directed acyclic subnetwork for which the information theoretic cutset upper bound evaluated under i.i.d. input distributions is exactly the same as that for the original network.

Conceptually, identifying the directed acyclic “skeleton” subnetwork that approximately carries the capacity of the network can be used to classify links as information carriers (critical to information transfer) and feedback links (of limited contribution to capacity). It also allows one to associate a direction with the information flow in an undirected wireless network. From a practical perspective, this result provides possibilities for simplifying network operation (in terms of delay and complexity) by identifying feedback links that can be potentially shut down without significantly impacting capacity. The simplification is immediate in networks with isolated links, such as graphical networks, which form a special case of the model we consider in this paper. For wireless networks, shutting down individual links may be nontrivial since these links may represent interference or overheard transmissions over other links. In Section VII we provide examples which illustrate possibilities for simplifying network operation even in the wireless case. This simplification aspect of our result is similar in spirit to [9] and [10], where [9] seeks a high-capacity small core in a wireless relay network that carries a good fraction of the overall capacity and [10] investigates the impact of removing a single edge on the capacity region of a graphical (wired) network.

After discussing the unicast case, we extend our result to more general traffic models. We show that for multiple-access (multiple sources communicating to the same destination node) and broadcast (a single source node communicating independent information to multiple destinations) traffic, each rate point in the capacity region of the original network can be approximately achieved by using an acyclic directed subnetwork. However, a single acyclic directed subnetwork that allows to approximately achieve all the rate points in the original capacity region may not exist. For multicast (a single source node communicates the same information to multiple destinations) and multiple unicast (multiple source-destination pairs communicating independent information with each other) traffic, we provide examples which illustrate that bidirected communication over certain links can be critical to achieving capacity. These results provide a generalization of the conclusions for the single-hop case, where it is known that the capacity gain from feedback is bounded (or absent) for point-to-point, multiple-access and broadcast Gaussian channels and can be unbounded in the case of interference channels.

We state the main results of our paper in Section III, prove them in Sections IV and V and discuss their implications in more detail in Section VII.
II. Model

We consider a bidirected Gaussian relay network $G$ consisting of a set of nodes $V$ and communication links $E$. We let $|V|$ denote the total number of nodes. All nodes in the network are able to send and receive, thus, for each pair of nodes $u, v \in V$ we can potentially have links $(u, v) \in E$ and $(v, u) \in E$ with arbitrary channel gains. We assume the links with non-zero channel gains are represented with directed edges as in Figure[1] giving rise to a directed graph with potentially bidirected edges and cycles. We assume nodes can have multiple transmit and receive antennas. Let $X_v \in \mathbb{C}^{M_v}$ denote the signal transmitted by node $v \in V$ with $M_v$ transmit antennas. Similarly, let $Y_v \in \mathbb{C}^{N_v}$ denote the signal received by node $v \in V$ with $N_v$ receive antennas. We have

$$Y_v = \sum_{u \in V} H_{vu} X_u + Z_v,$$

where $H_{vu}$ denotes the channel matrix from node $u$ to node $v$. This multiple-input multiple-output channel model can also be used to incorporate networks where different channels operate on different frequencies as well as networks with isolated links.\(^1\) The noise $Z_v$ are independent and circularly symmetric Gaussian random vectors $\mathcal{N}(0, I)$. All nodes are subject to an average power constraint $P$. Note that the equal power constraint assumption is without loss of generality as the channel coefficients are arbitrary.

We consider the following traffic scenarios over this network:

- **Unicast**: Source node $s \in V$ wants to communicate to destination node $d \in V$. The capacity of the network $G$, denoted by $C(G)$, is the largest rate at which $s$ can reliably communicate to $d$.
- **Multiple-Access**: Source nodes $s_1, s_2, \ldots, s_n \in V$ want to communicate independent messages to a destination node $d \in V$. The capacity region $C(G)$ is the closure of jointly achievable rate pairs $R_1, \ldots, R_n$ where $R_i$ is the reliable communication rate from $s_i$ to $d$.
- **Broadcast**: Source node $s \in V$ wants to communicate independent messages to destination nodes $d_1, \ldots, d_n \in V$. The capacity region $C(G)$ is the closure of jointly achievable rate pairs $R_1, \ldots, R_n$ where $R_i$ is the reliable communication rate from $s$ to $d_i$.
- **Multicast**: Source node $s \in V$ wants to communicate the same message to destination nodes $d_1, \ldots, d_n \in V$. The capacity $C(G)$ is the largest rate $R$ at which the message can be simultaneously communicated to all destinations.
- **Multiple-Unicast**: Source node $s_i \in V$ wants to communicate an independent message to its destination node $d_i \in V$ for $i = 1, \ldots, n$. The capacity region $C(G)$ is the closure of jointly achievable rate pairs $R_1, \ldots, R_n$ where $R_i$ is the reliable communication rate from $s_i$ to $d_i$.

Note that we slightly abuse notation here by using $C(G)$ to refer to a single number in the case of unicast and multicast traffic and a region in the case of multiple-access, broadcast and multiple-unicast traffic. The same is true for $C_{i,i,d}(G)$ we define in the next section. The usage should be clear from the context.

III. Main Results

For an arbitrary bidirected Gaussian relay network $G$ with a set of nodes $V$ and communication links $E$, we define a directed acyclic subnetwork $\tilde{G}$ to be one which consists of the same set of nodes $V$ and a subset of the communication links $\tilde{E} \subseteq E$. For the Gaussian relay network, this corresponds to setting the channel coefficients corresponding to the edges in $E \setminus \tilde{E}$ to zero. A directed acyclic subnetwork satisfies the property that for any pair of nodes $u, v \in V$, if $(u, v) \in \tilde{E}$ then $(v, u) \notin \tilde{E}$. In other words, if there is a link in one direction between any two nodes, there cannot be a link in the opposite direction. Moreover, it contains no cycles. That is, for every set of nodes $v_1, \ldots, v_N \in V$, at least one of the edges $(v_1, v_2), \ldots, (v_k, v_{k+1}), \ldots, (v_N, v_1) \notin \tilde{E}$ for any value of $N$.

The main conclusions of this paper are summarized in the following three theorems.

\(^1\)Indeed, the conclusions of the paper also hold for wired networks (in this case with no gap) and a mixture of wireless and wired networks.
**Theorem 3.1:** Let \( C(G) \) be the capacity of a Gaussian network \( G \) with unicast traffic. We can identify a directed acyclic subnetwork \( \tilde{G} \) of \( G \) whose capacity \( C(\tilde{G}) \) in bits/s/Hz is bounded by

\[
C(G) - g_1 \leq C(\tilde{G}) \leq C(G) + g_1
\]

where \( g_1 \) is a constant independent of the channel gains and SNRs and can be upper bounded by \( 3.3M \) where \( M = \sum_{v \in V} M_v + N_v \) is the total number of antennas in the network.

The fact that the gap between the capacity of \( \tilde{G} \) and that of the original network \( G \) can be bounded independent of the channel gains and SNRs implies that the gain due to using the additional links in \( G \) remains bounded as SNR grows. The core of our argument for proving this theorem is summarized in the following proposition, which indeed only involves the information-theoretic cutset upper bound on the capacity of the network evaluated under i.i.d. Gaussian input distribution. Then

\[
\text{Proposition 3.1: Consider a Gaussian network } G \text{ with unicast traffic. Let }
\]

\[
C_{i.i.d.}(G) = \min_S f(G; S),
\]

where \( S \subseteq V : s \in S, d \notin S \) is a source-destination cut of the network and \( f(G; S) \) for all \( S \subseteq V \) is defined as

\[
f(G; S) = I(X_S; Y_{S^c}|X_{S^c}),
\]

where \( X_v, v \in V \) are i.i.d. \( \mathcal{CN}(0, (P/M_v)I) \). In other words, \( C_{i.i.d.}(G) \) is the information-theoretic cutset upper bound on the capacity of the network evaluated under an i.i.d. Gaussian input distribution. Then in every bidirected network \( G \) with \( C_{i.i.d.}(G) \), we can identify a directed acyclic subnetwork \( \tilde{G} \) with \( C_{i.i.d.}(\tilde{G}) = C_{i.i.d.}(G) \).

The proof of Theorem 3.1 follows by combining this proposition with the existing results in the literature which show that the capacity \( C(G) \) of any Gaussian relay network with unicast traffic is within a bounded gap to \( C_{i.i.d.}(G) \) [7], [11], [17]. We recall the following result from [17]:

\[
\text{Theorem 3.2 (Theorem 4, [17]): In any Gaussian relay network } G \text{ with unicast traffic, we can achieve all rates}
\]

\[
R \leq C_{i.i.d.}(G) - g_2
\]

where \( g_2 \leq 1.3M \). Equivalently, \( C(G) \geq C_{i.i.d.}(G) - g_2 \).

It has been also shown in Lemma 6.6. of [7] that the restriction to i.i.d. Gaussian input distributions is within \( g_3 = 2M \) bits/s/Hz of the actual information-theoretic cutset upper bound \( \tilde{C}(G) \), i.e. for any Gaussian network \( G \)

\[
C(G) \leq \tilde{C}(G) \leq C_{i.i.d.}(G) + g_3.
\]

This shows that within a total gap of \( g_2 + g_3 \), the capacity of the network is approximately given by \( C_{i.i.d.}(G) \). More precisely,

\[
C_{i.i.d.}(G) - g_2 \leq C(G) \leq C_{i.i.d.}(G) + g_3.
\]

The proof of Theorem 3.1 follows immediately by combining (5) with Proposition 3.1 where \( g_1 = g_2 + g_3 \).

\(^2\) The result in [17] is stronger than what is stated here as it shows that \( C(G) \geq \tilde{C}(G) - g_2 \) where \( \tilde{C} \) is the actual information-theoretic cutset upper bound on the capacity of the network. We use the weaker form \( C(G) \geq C_{i.i.d.}(G) - g_2 \) here as we need a lower and an upper bound on \( C(G) \) in terms \( C_{i.i.d.}(G) \) in order to connect Theorem 3.1 and Proposition 3.1. Also, the result in [17] is for Gaussian channels with real input and outputs. The gap stated here is for complex channels which is twice the gap for the real case.
The following theorems state analogous results for the multiple access and broadcast case. The proofs of these theorems follow a similar structure to the unicast case.

**Theorem 3.3:** Let \( C(G) \) be the capacity region of a Gaussian network \( G \) with multiple access traffic. If \( (R_1, R_2, \ldots, R_n) \in C(G) \), then there exists an acyclic subnetwork \( \tilde{G} \) of \( G \) such that
\[
(R_1 - g_1, R_2 - g_1, \ldots, R_n - g_1) \in C(\tilde{G}),
\]
where \( g_1 \) is a constant independent of the channel gains and SNRs. \( g_1 \) can be upper bounded by \( 3.3M \).

**Theorem 3.4:** Let \( C(G) \) be the capacity region of a Gaussian network \( G \) with broadcast traffic. If \( (R_1, R_2, \ldots, R_n) \in C(G) \), then there exists an acyclic subnetwork \( \tilde{G} \) of \( G \) such that
\[
(R_1 - g_4, R_2 - g_4, \ldots, R_n - g_4) \in C(\tilde{G}),
\]
where \( g_4 \) is a constant independent of the channel gains and SNRs. \( g_4 = O(M \log M) \) where \( M \) again is the total number of antennas in the network.

Analogous to the unicast case, the proofs of these theorems are based on the following propositions which only involve the information-theoretic cutset upper bound on the capacity region of the network when evaluated under i.i.d. input distributions.

**Proposition 3.2:** Consider a Gaussian network \( G \) with multiple access traffic. Let \( C_{i.i.d.}(G) \) be the set of rate tuples \( (R_1, R_2, \ldots, R_n) \) such that
\[
\sum_{s \in S} R_i \leq f(G; S),
\]
\( \forall S \subseteq V : d \not\in S \) and \( f(G; S) \) is defined in (2). Then for each \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \), we can identify a directed acyclic subnetwork \( G \) of \( G \) such that \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(\tilde{G}) \) (where \( C_{i.i.d.}(\tilde{G}) \) is defined analogously according to (6) for \( \tilde{G} \)).

**Proposition 3.3:** Consider a Gaussian network \( G \) with broadcast traffic. Let \( C_{i.i.d.}(G) \) be the set of rate tuples \( (R_1, R_2, \ldots, R_n) \) such that
\[
\sum_{d \notin S} R_i \leq f(G; S),
\]
\( \forall S \subseteq V : s \in S \) and \( f(G; S) \) is given in (2). Then for each \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \), we can identify a directed acyclic subnetwork \( G \) of \( G \) such that \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(\tilde{G}) \) (where \( C_{i.i.d.}(\tilde{G}) \) is defined analogously according to (7) for \( \tilde{G} \)).

**Remark 3.1:** Note that Propositions 3.2 and 3.3 do not imply that \( C_{i.i.d.}(G) \subseteq C_{i.i.d.}(\tilde{G}) \), since the subgraphs \( \tilde{G} \) which we identify here may not be the same for different rate points \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \). In other words, in both cases there may not be a single subnetwork \( \tilde{G} \) which achieves all the rate points \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \). In Section V, we provide examples which illustrate this point.

The proofs of Theorems 3.3 and 3.4 similarly follow by combining Propositions 3.2 and 3.3 respectively with the existing results in the literature which show that the capacity region of a Gaussian network \( G \) with multiple-access [17] or broadcast [19] traffic is within a bounded gap to \( C_{i.i.d.}(G) \). We restate Theorem 4 of [17] now in its more general form which holds for multiple-access traffic and also recall the main result of [19] for broadcast traffic.

**Theorem 3.5 (Theorem 4, [17]):** Consider any Gaussian network with multiple-access traffic and let \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \). Then \( (R_1 - g_2, R_2 - g_2, \ldots, R_n - g_2) \in C(G) \) where \( g_2 \leq 1.3M \).

**Theorem 3.6 (Theorem 1, [19]):** Consider any Gaussian network with broadcast traffic and let \( (R_1, R_2, \ldots, R_n) \in C_{i.i.d.}(G) \). Then \( (R_1 - g_5, R_2 - g_5, \ldots, R_n - g_5) \in C(G) \) where \( g_5 = O(M \log M) \).
For broadcast and multiple-access traffic, the fact that the restriction to i.i.d. Gaussian input distributions is within $g_3 = 2M$ bits/s/Hz of the actual information-theoretic cut-set upper bound $\hat{C}(G)$ in Lemma 6.6 of [12] implies that

$$C(G) \leq \hat{C}(G) \leq C_{i.i.d.}(G) + g_3,$$

which implies that for any $(R_1, R_2, \ldots, R_n) \in C(G)$, $(R_1 - g_3, R_2 - g_3, \ldots, R_n - g_3) \in C_{i.i.d.}(G)$. Together with the results in the last two theorems, this implies that within a gap independent of the channel gains and SNRs, the capacity region of a Gaussian network with multiple access and broadcast traffic is approximately given by $C_{i.i.d.}(G)$.

The proof of Theorems 3.3 and 3.4 for the multiple access and broadcast traffic scenarios follow immediately by combining Theorems 3.5 and 3.6 and Eq. (8) with Propositions 3.2 and 3.3. Let $(R_1, R_2, \ldots, R_n)$ be in the capacity region of the original Gaussian network $G$ with multiple access traffic. (8) implies that $(R_1 - g_3, R_2 - g_3, \ldots, R_n - g_3) \in C_{i.i.d.}(G)$. In turn, Proposition 3.2 implies that there exists a directed acyclic subnetwork for which $(R_1 - g_3, R_2 - g_3, \ldots, R_n - g_3) \in C_{i.i.d.}(G)$ and Theorem 3.5 implies that $(R_1 - g_1, R_2 - g_1, \ldots, R_n - g_1) \in \hat{C}(G)$ with $g_1 = g_2 + g_3$ which gives the result in Theorem 3.3. A similar argument holds for Theorem 3.4.

Note that the core of our argument in Propositions 3.1, 3.2, and 3.3 holds with no gap. The gaps in Theorems 3.1, 3.3 and 3.4 are due to the current approximation gap of the capacity of Gaussian relay networks with respect to the i.i.d. cutset upper bound. Better approximations for the capacity of Gaussian relay networks in terms of $C_{i.i.d.}$ can immediately improve the gap in our main results. For example, in [13], [14] and [15] it is shown that the approximations in [7], [17] can be significantly tightened for certain network configurations.

Proposition 3.1 is proved in Section IV and Propositions 3.2 and 3.3 are proved in Section V.

IV. UNICAST NETWORKS

In this section, we concentrate on proving Proposition 3.1. We divide our proof into two parts. In the first part of the proof, we will show that for any pair of links $(u, v)$ and $(v, u)$, we can remove one of the links without changing $C_{i.i.d.}$. Given this new network, we can iterate this procedure for each bidirected link until we are left with a directed network that contains no bidirected edges. In the second part of the proof we show that given a directed network with cycles, we can remove at least one of the links in the cycle without changing $C_{i.i.d.}$. Iterating this procedure for each cycle, we can obtain a directed subnetwork of the same $C_{i.i.d.}$ that contains no cycles.

Our proof is based on two important properties of the cut function in (2):

1) For a fixed cut $S \subset V$, the cut values of a network $G$ and subnetwork $G'$ are the same if all outgoing links from $S$ are in both $G$ and $G'$:

$$f(G; S) = f(G'; S), \text{ if } \forall (u, v) \in G : (u \in S, v \notin S), (u, v) \in G'.$$

(Note that because $G'$ is a subgraph of $G$, the channel coefficients corresponding to $(u, v)$ are the same in both $G$ and $G'$ if $(u, v) \in G'$.)

2) $f(G; S)$ is a submodular function on $2^V$:

$$f(G; S_1) + f(G; S_2) \geq f(G; S_1 \cup S_2) + f(G; S_1 \cap S_2), \forall S_1, S_2 \subseteq V.$$

The first property follows from the fact that when all outgoing links are in both $G$ and $G'$, the MIMO matrix between $X_S$ and $Y_S$ are the same, and thus $I(X_S; Y_S | X_S)$ which corresponds to the capacity of this MIMO matrix is the same for both networks. A proof of the second property is given in Theorem 1 of [12].
A. Reduction of bidirected network to directed network

Given a bidirected network $G$ and any pair of links $(u, v)$ and $(v, u)$, we create the subnetworks $G'$, $G''$, and $G'''$, where the link $(v, u)$, $(u, v)$, and both $(u, v)$ and $(v, u)$ are removed from $G$, respectively. See Figure 2.

Define $S_v$, $S_u$, $S_{uv}$, and $S_{uv}$ to be the following:

$$S_v = \arg \min_{\{S \mid s, v \in S, t, u \notin S\}} f(G; S)$$

$$S_u = \arg \min_{\{S \mid s, u \in S, t, v \notin S\}} f(G; S)$$

$$S_{uv} = \arg \min_{\{S \mid s, u, v \in S, t \notin S\}} f(G; S)$$

$$S_{uv} = \arg \min_{\{S \mid s \in S, t, u, v \notin S\}} f(G; S).$$

$S_v$ is the cut with the minimum cut value among all cuts for which $v$ remains on the source side and $u$ remains on the destination side; $S_u$ is the cut with the minimum cut value among all cuts for which $u$ remains on the source side and $v$ remains on the destination side; $S_{uv}$ is the cut with the minimum cut value among all cuts for which both $u$ and $v$ are on the source side; and $S_{uv}$ is the cut with the minimum cut value among all cuts for which $u$ and $v$ remain on the destination side. See Figure 3. A cut that achieves the minimum cut value need not be unique; we choose an arbitrary one in such cases. Note that

$$C_{i.i.d.}(G) = \min \left( f(G; S_v), f(G; S_u), f(G; S_{uv}), f(G; S_{uv}) \right).$$

(9)

We also define $S'_v$, $S'_u$, $S'_{uv}$, $S'_{uv}$, $S''_v$, $S''_u$, $S''_{uv}$, $S''_{uv}$ in a similar fashion for graphs $G'$ and $G''$, respectively.

Proposition 3.2 claims that either $C_{i.i.d.}(G) = C_{i.i.d.}(G')$ or $C_{i.i.d.}(G) = C_{i.i.d.}(G'')$. We prove this by showing that each of the following assumptions lead to a contradiction:
(a) \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) < C_{i.i.d.}(G'') \);
(b) \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) > C_{i.i.d.}(G'') \) (or \( C_{i.i.d.}(G) > C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) < C_{i.i.d.}(G'') \));
(c) \( C_{i.i.d.}(G) > C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) > C_{i.i.d.}(G'') \).

Case (a): Assume \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) < C_{i.i.d.}(G'') \).

If \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \), then \( C_{i.i.d.}(G) = f(G; S_v) \), and

\[
f(G; S_v) < \min\left(f(G; S_u), f(G; S_{uv}), f(G; S_{uv}^*)\right).
\]

This can be seen as follows. Note that the minimums in the definitions of \( S_u, S_{uv} \) and \( S_{uv}^* \) are taken over a set of cuts that cannot cross the link \((v, u)\) and \( G \) and \( G' \) only differ by the existence of the link \((v, u)\).

By Property (1), any cut that does not cross the edge \((v, u)\) has the same value in \( G \) and \( G' \). Therefore, \( f(G; S_u) = f(G'; S_u) \), \( f(G; S_{uv}) = f(G'; S_{uv}) \) and \( f(G; S_{uv}^*) = f(G'; S_{uv}^*) \). Now, if the minimum in (9) were to be achieved by any term other than \( f(G; S_v) \), this would imply that \( C_{i.i.d.}(G') \leq C_{i.i.d.}(G) \), which would contradict the assumption that \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \). Therefore, we have (10).

Now, if also \( C_{i.i.d.}(G) < C_{i.i.d.}(G'') \), by the same argument above we should have \( C_{i.i.d.}(G) = f(G; S_u) \), and

\[
f(G; S_u) < \min\left(f(G; S_v), f(G; S_{uv}), f(G; S_{uv}^*)\right).
\]

But (10) and (11) are contradictory.

Case (b): Assume \( C_{i.i.d.}(G) < C_{i.i.d.}(G') \) and \( C_{i.i.d.}(G) > C_{i.i.d.}(G'') \). Then, by the same argument in case (a), we have \( C_{i.i.d.}(G) = f(G; S_v) \), and

\[
f(G; S_v) < \min\left(f(G; S_u), f(G; S_{uv}), f(G; S_{uv}^*)\right).
\]

Similarly, the assumption \( C_{i.i.d.}(G) > C_{i.i.d.}(G'') \) implies that \( C_{i.i.d.}(G'') = f(G''; S''_u) \), and

\[
f(G''; S''_u) < \min\left(f(G''; S'_v), f(G''; S''_{uv}), f(G''; S''_{uv}^*)\right).
\]

This follows by the same argument for (10): Since \( G \) and \( G'' \) only differ by the existence of \((u, v)\), the value of the cut \( S_u \) should be different in \( G \) and \( G'' \) while the values of the remaining three cuts are the same in both \( G \) and \( G'' \).

Note that (12) implies that

\[
f(G; S_v) < f(G; S_{uv}^*) \leq f(G; S_v \cap S''_u),
\]

where the last inequality follows from the fact that since \( u \notin S_v \) and \( v \notin S''_u \), \( u, v \notin S_v \cap S''_u \) and the definition of \( S_{uv}^* \) which implies that among all such cuts of \( G \), \( S_{uv}^* \) is the one with mincut value. Now, by Property (1), \( f(G, S_v) = f(G'', S'_v) \) and \( f(G; S_v \cap S''_u) = f(G''; S_v \cap S''_u) \) since \( G \) and \( G'' \) only differ by the existence of the link \((u, v)\) and both \( S_v \) and \( S_v \cap S''_u \) correspond to cuts that cannot cross this link. Therefore, we have \( f(G''; S_v) < f(G''; S_v \cap S''_u) \). On the other hand, (13) implies that

\[
f(G''; S''_u) < f(G''; S''_{uv}) \leq f(G''; S_v \cup S''_u),
\]

since \( v \in S_v \) and \( u \in S''_u \), \( u, v \in S_v \cup S''_u \). Combining the last two inequalities we obtain

\[
f(G''; S_v) + f(G''; S''_u) < f(G''; S_v \cap S''_u) + f(G''; S_v \cup S''_u).
\]

However, submodularity (Property (2)) for \( f \) implies that

\[
f(G''; S_v) + f(G''; S''_u) \geq f(G''; S_v \cap S''_u) + f(G''; S_v \cup S''_u),
\]

which leads to a contradiction.
Case (c): Finally, we assume $C_{i.i.d}(G) > C_{i.i.d}(G')$ and $C_{i.i.d}(G) > C_{i.i.d}(G'')$.
By similar arguments as in case (b), the first assumption implies that $C_{i.i.d}(G') = f(G'; S'_u)$, and the second one implies that $C_{i.i.d}(G'') = f(G''; S''_u)$. Moreover,

$$f(G'; S'_u) < f(G'; S'_v \cup S''_u),$$

(14)

and

$$f(G''; S''_u) < f(G'; S'_v \cap S''_u).$$

(15)

The last two inequalities follow from our assumption, $C_{i.i.d}(G) > C_{i.i.d}(G')$ and $C_{i.i.d}(G) > C_{i.i.d}(G'')$, which implies that the minimum cut values for $G'$ and $G''$ are strictly less than any cut value of $G$. Combining (14) and (15), we have

$$f(G'; S'_u) + f(G''; S''_u) < f(G'; S'_v \cap S''_u) + f(G'; S'_v \cup S''_u),$$

Observing that by Property (1)

$$f(G'; S'_u) = f(G''; S'_u)$$

(16)

$$f(G''; S''_u) = f(G''; S''_u)$$

(17)

$$f(G'; S'_v \cap S''_u) = f(G''; S'_v \cap S''_u)$$

(18)

$$f(G'; S'_v \cup S''_u) = f(G''; S'_v \cup S''_u),$$

(19)

we get

$$f(G''; S''_u) + f(G''; S''_u) < f(G''; S''_v \cap S''_u) + f(G''; S''_v \cup S''_u).$$

This contradicts with the submodularity of $f$ in $G''$. Since cases (a), (b) and (c) are eliminated, we conclude that either $C_{i.i.d}(G) = C_{i.i.d}(G')$ or $C_{i.i.d}(G) = C_{i.i.d}(G'')$.

B. Removing cycles in a directed network

Consider a directed network $G$, where the nodes $\{v_1, v_2, \ldots, v_N\}$ form a length $N$ cycle, and let $v_{N+1} = v_1$. Define $G_k, k = 1, 2, \ldots, N$ to be a subnetwork of $G$ with the link $(v_k, v_{k+1})$ removed. In our proof, we denote subnetworks with both links $(v_{k-1}, v_k)$ and $(v_k, v_{k+1})$ removed as $G_{k-1,k}$. See Figure 4 for an example. Let $S^*$ and $S_k$ denote cuts that achieve the minimum cut values of networks $G$ and $G_k$, respectively:

$$S^* = \arg \min_{S \in \mathcal{S}} f(G; S),$$

$$S_k = \arg \min_{S \in \mathcal{S}} f(G_k; S).$$

We prove that $C_{i.i.d}(G) = C_{i.i.d}(G_k)$ for at least one value of $k, k = 1, 2, \ldots, N$ by showing that each of the following assumptions lead to a contradiction:

(a) $C_{i.i.d}(G) > C_{i.i.d}(G_k)$ for $k = 1, 2, \ldots, N$;

(b) $C_{i.i.d}(G) \neq C_{i.i.d}(G_k)$ for $k = 1, 2, \ldots, N$ and $C_{i.i.d}(G) < C_{i.i.d}(G_k)$ for at least one value of $k$.

Case (a): Assume $C_{i.i.d}(G) > C_{i.i.d}(G_k)$ for $k = 1, 2, \ldots, N$.

Given our assumption, we first show that for each subnetwork $G_k$, there exists a cut $S'_k$ that achieves the minimum cut value, i.e.,

$$C_{i.i.d}(G_k) = f(G_k; S'_k)$$

(20)

with the property $v_1, v_2, \ldots, v_k \in S'_k$ and $v_{k+1} \notin S'_k$. This will lead to a contradiction when we take $k = N$.

If $C_{i.i.d}(G) > C_{i.i.d}(G_k)$, then $v_k \in S_k$ and $v_{k+1} \notin S'_k$. This can be seen as follows. Any cut that does not cross the link $(v_k, v_{k+1})$ has the same cut value for both $G$ and $G_k$ by Property (1). So the minimum
Fig. 4. An example of a directed network with a length 3 cycle and subnetworks with some links removed.

cut value attained by \( G_k \) must be for a cut which crosses the link \((v_k, v_{k+1})\) and yields a cut value strictly less than any cut which does not cross that link. Thus, for \( k = 1 \) we can choose \( S'_1 = S_1 \).

We will discover the sets \( S'_k \) for larger \( k \) by induction. We will show that if the claim in (20) holds for \( k - 1 \), it should also hold for \( k \).

First note that since \( S'_{k-1} \) and \( S_k \) achieve the minimum cut values for \( G_{k-1} \) and \( G_k \), they must be less than or equal to any other cut in \( G_{k-1} \) and \( G_k \) respectively. In particular,

\[
f(G_{k-1}; S'_{k-1}) \leq f(G_{k-1}; S'_{k-1} \cap S_k),
\]

(21)

\[
f(G_k; S_k) \leq f(G_k; S'_k \cup S_k).
\]

(22)

Now, since \( v_k \in S'_{k-1} \) and \( v_k \in S'_{k-1} \cap S_k \), \((v_k, v_{k+1})\) cannot be an outgoing link in either of the cuts \( S'_{k-1} \) and \( S'_{k-1} \cap S_k \), and all other links in \( G_{k-1} \) are also in \( G_{k-1,k} \), so by Property (1) of \( f \), we have

\[
f(G_{k-1,k}; S'_{k-1}) = f(G_{k-1,k}; S'_{k-1} \cap S_k),
\]

(23)

\[
f(G_{k-1,k}; S'_{k-1} \cap S_k) = f(G_{k-1}; S'_{k-1} \cap S_k).
\]

(24)

Also, \( v_k \in S_k \) and \( v_k \in S'_{k-1} \cup S_k \), so \((v_k, v_{k+1})\) cannot be an outgoing link in either of those cuts, and all other links in \( G_k \) are also in \( G_{k-1,k} \). So again by Property (1) of \( f \), we have

\[
f(G_{k-1,k}; S_k) = f(G_k; S_k),
\]

(25)

\[
f(G_{k-1,k}; S'_{k-1} \cup S_k) = f(G_k; S'_k \cup S_k).
\]

(26)

By the submodular property of \( f \) on \( G_{k-1,k} \) we have

\[
f(G_{k-1,k}; S'_{k-1}) + f(G_{k-1,k}; S_k) \geq f(G_{k-1,k}; S'_{k-1} \cap S_k) + f(G_{k-1,k}; S'_{k-1} \cup S_k),
\]

and equations (23)-(26) yield

\[
f(G_{k-1}; S'_{k-1}) + f(G_k; S_k) \geq f(G_{k-1}; S'_{k-1} \cap S_k) + f(G_k; S'_k \cup S_k).
\]

Combining this result and equations (21) and (22) yields

\[
f(G_k; S_k) = f(G_k; S'_k \cup S_k).
\]

Thus the cut \( S'_k = S'_{k-1} \cup S_k \) achieves the minimum cut value for network \( G_k \) and has the property \( v_1, \ldots, v_k \in S'_k \). Now suppose \( v_{k+1} \in S'_k \). Then the cut \( S'_k \) cannot cross the link \((v_k, v_{k+1})\), and thus
\[ f(G_k; S'_k) = f(G; S'_k) \]. But since \( S'_k \) achieves the minimum cut value for network \( G_k \), we have the following:

\[ C_{\text{i.i.d}}(G_k) = f(G_k; S'_k) = f(G; S'_k) \geq C_{\text{i.i.d}}(G), \]

which would contradict our assumption \( C_{\text{i.i.d}}(G) > C_{\text{i.i.d}}(G_k) \). The last inequality follows because \( C_{\text{i.i.d}}(G) \) must be less than or equal to any cut value of \( G \). Thus, \( v_{k+1} \notin S'_k \).

Letting \( k = N \), we have \( v_1, \ldots, v_N \in S'_N \), but \( v_{k+1} = v_1 \notin S'_N \), which is a contradiction.

**Case (b):** Assume \( C_{\text{i.i.d}}(G) \neq C_{\text{i.i.d}}(G_k) \) for \( k = 1, 2, \ldots, N \) and \( C_{\text{i.i.d}}(G) < C_{\text{i.i.d}}(G_k) \) for at least one value of \( k \).

Without loss of generality, let \( C_{\text{i.i.d}}(G) < C_{\text{i.i.d}}(G_1) \). By the same arguments as in the previous case, \( v_1 \in S^* \), \( v_2 \notin S^* \).

We now show that for \( k > 1 \), if \( v_k \in S^* \), then \( v_{k+1} \in S^* \). This will lead to a contradiction when we take \( k = N \).

Assume \( v_k \in S^* \) and consider \( C_{\text{i.i.d}}(G_k) \):

\[ C_{\text{i.i.d}}(G_k) = f(G; S^*) \]

\[ \overset{(a)}{=} f(G_k; S^*) \]

\[ \overset{(b)}{>} C_{\text{i.i.d}}(G_k). \]

(a) follows by Property (1) and the fact that \( v_k \in S^* \), and so \( S^* \) cannot cross the link \((v_k, v_{k+1})\). (b) follows from the fact that \( C_{\text{i.i.d}}(G_k) \) must be less than or equal to any cut value of \( G_k \), i.e. \( f(G_k; S^*) \geq C_{\text{i.i.d}}(G_k) \) and our assumption that \( C_{\text{i.i.d}}(G) \neq C_{\text{i.i.d}}(G_k) \) for \( k = 1, 2, \ldots, N \), thus making the inequality strict. Now since \( C_{\text{i.i.d}}(G_k) < C_{\text{i.i.d}}(G) \), for the minimum cut \( S_k \), we must have \( v_k \in S_k \), \( v_{k+1} \in S_k \). Next, consider \( S^* \cap S_k \). We have

\[ f(G_k; S^* \cap S_k) \overset{(a)}{=} f(G; S^* \cap S_k) \]

\[ \overset{(b)}{>} f(G_k; S_k), \quad (27) \]

\[ \overset{(c)}{=} f(G_k; S^*) \]

(28)

where (a) follows by Property (1) and the fact that \( v_k \in S^* \cap S_k \), and so \( S^* \cap S_k \) cannot cross the link \((v_k, v_{k+1})\). (b) follows because \( C_{\text{i.i.d}}(G_k) < C_{\text{i.i.d}}(G) \), so the minimum cut value of \( C_{\text{i.i.d}}(G_k) = f(G_k; S_k) \), must be strictly less than any cut value of \( G \). Now suppose \( v_{k+1} \in S^* \). Then

\[ f(G_k; S^* \cup S_k) \overset{(a)}{=} f(G; S^* \cup S_k) \]

\[ \overset{(b)}{=} f(G_k; S^*) \]

\[ \overset{(c)}{=} f(G; S^*). \quad (29) \]

(30)

(a) and (c) follow because \( v_{k+1} \in S^* \cup S_k \) and \( v_{k+1} \in S^* \), so neither of those cuts can cross the link \((v_k, v_{k+1})\). (b) follows because \( S^* \) achieves the minimum cut value of graph \( G \).

Combining \((27)-(31)\), we have

\[ f(G_k; S^* \cup S_k) + f(G_k; S^* \cap S_k) > f(G_k; S_k) + f(G_k; S^*). \]

This contradicts the submodularity of \( f \) in \( G_k \). Thus \( v_{k+1} \in S^* \).

Letting \( k = N \), we have \( v_1 \in S^*, v_2, v_3, \ldots, v_N \in S^* \). However, because \( v_1, \ldots, v_N \) form a cycle, the node \( v_1 \) can be thought of \( v_{N+1} \) and \( v_N \in S^* \) by the above iteration implies that \( v_1 \in S^* \). This contradicts with the fact that \( v_1 \in S^* \) and shows that the initial assumptions for case (b) necessarily lead to a contradiction.

Since we have eliminated cases (a) and (b) above, we conclude that \( C_{\text{i.i.d}}(G) = C_{\text{i.i.d}}(G_k) \) for at least one value of \( k = 1, 2, \ldots, N \).
V. MULTIPLE ACCESS AND BROADCAST NETWORKS

A. Multiple Access Networks

In this section, we use Proposition 3.1 to prove Proposition 3.2. Consider a Gaussian network $G$ with multiple access traffic between the sources $s_1, s_2, \ldots s_n$ and the destination $d$. Assume that $(R_1, R_2, \ldots R_n) \in C_{i.i.d.}(G)$. We will show that there exists an acyclic subnetwork $\tilde{G}$ of $G$ such that $(R_1, R_2, \ldots R_n) \in C_{i.i.d.}(\tilde{G})$.

Starting from $G$, we first construct an extended directed graph $G' = (V', E')$ as follows:

1) Let $V' = V \cup \{s'\}$ where $s'$ is an added auxiliary vertex.
2) Let $E' = E \cup \{(s', s_i) \mid 1 \leq i \leq n\}$.

We assume that each edge $(s', s_i)$ represents an isolated edge of capacity $R_i$, for $i = 1, 2, \ldots, n$. This can be done within the Gaussian network model we defined in Section II by assuming, for example, that $s'$ is equipped with $n$ transmit antennas where each transmit antenna is connected only to the corresponding $s_i$ with a Gaussian channel of capacity $R_i$ (the channel coefficient of this channel is chosen accordingly). Consider this Gaussian network $G'$ with unicast traffic from $s'$ to $d$. We next lower bound $C_{i.i.d.}(G')$ for this unicast network. Note that for any cut $S \subseteq V' : s' \in S$,

$$f(G'; S) = f(G; S \setminus \{s'\}) + \sum_{s_i \notin S} R_i \geq \sum_{s_i \in S} R_i + \sum_{s_i \notin S} R_i = \sum_{i=1}^{n} R_i,$$

(32)

where the first equality follows from the fact that $f(G'; S) = I(X_S; Y_S | X_{S'})$ under i.i.d. input distributions decomposes into $f(G'; S \setminus \{s'\}) + \sum_{s_i \notin S} R_i$ since $(s', s_i)$ are isolated from other channels in $G'$ and also from each other. In turn, $f(G'; S \setminus \{s'\}) = f(G; S \setminus \{s'\})$ since due to the way we constructed $G'$ the outgoing edges from $S \setminus \{s'\}$ are the same in both $G$ and $G'$. The second line follows from our assumption that $(R_1, R_2, \ldots R_n) \in C_{i.i.d.}(G)$ and the definition of $C_{i.i.d.}(G)$ for multiple access networks in (6).

Therefore, we can conclude that for the constructed unicast network $G'$, we have

$$C_{i.i.d.}(G') \geq \sum_{i=1}^{n} R_i.$$

Now, due to Proposition 3.1 we know that can find an acyclic subnetwork $\tilde{G}$ of $G'$ for which $C_{i.i.d.}(\tilde{G}) = C_{i.i.d.}(G')$. Let $\tilde{G}$ be the graph obtained by removing $s'$ and $\{(s', s_i) \mid 1 \leq i \leq n\}$ from $G'$. Note that $\tilde{G}$ is an acyclic subnetwork of our original multiple access network $G$.

To complete the proof, we show that $(R_1, R_2, \ldots R_n) \in C_{i.i.d.}(\tilde{G})$. Consider an arbitrary $S \subseteq V$. Let $S' = S \cup \{s'\}$. Since $C_{i.i.d.}(G')$ is not less than $\sum_{i=1}^{n} R_i$, we have

$$f(\tilde{G}; S) = f(\tilde{G}; S) + \sum_{s_i \notin S} R_i - \sum_{s_i \notin S} R_i = f(G'; S') - \sum_{s_i \notin S} R_i \geq \sum_{i=1}^{n} R_i - \sum_{s_i \notin S} R_i = \sum_{s_i \in S} R_i.$$  

(33)
where the second equality again follows from the fact that \( f(\tilde{G}'; S') \) decomposes into \( f(\tilde{G}'; S) + \sum_{s_i \notin S} R_i \) and \( f(\tilde{G}; S) = f(\tilde{G}'; S) \). Thus, according to the definition of \( C_{\text{i.i.d.}}(\tilde{G}) \) for a multiple access network in (6) we have shown that \((R_1, R_2, \ldots, R_n) \in C_{\text{i.i.d.}}(\tilde{G})\), and this completes the proof for Proposition 3.2.

Note that although we have proved that every rate tuple in the capacity region of a multiple access network can be achieved by using an acyclic subnetwork, we cannot conclude that there exists an acyclic subnetwork which has the same capacity region as the original network. For an example, consider the network in Figure 5 which depicts a multiple access network with isolated edges of corresponding capacities. Observe that both \((2, 0)\) and \((0, 2)\) are in the capacity region of the original network, however neither of two acyclic subnetworks can have both of these rate points in its capacity region. In other words, despite the fact that for each achievable rate point there exists an acyclic subnetwork achieving that rate point, these subnetworks may differ for different rate points, leading to cases where the capacity regions of all the acyclic subnetworks of a network are strictly smaller than the capacity of the original network.

### B. Broadcast Networks

Proposition 3.3 for broadcast traffic can be proved by using a similar approach to Proposition 3.2. Consider a Gaussian network \( G = (V, E) \) with broadcast traffic where source \( s \) communicates independent messages to destinations \( d_1, d_2, \ldots, d_n \). Let \( C_{\text{i.i.d.}}(G) \) be the associated rate region and let \((R_1, R_2, \ldots, R_n) \in C_{\text{i.i.d.}}(G)\). As before, we first create a unicast network \( G' = (V', E') \) from \( G = (V, E) \) by adding an auxiliary vertex \( d' \) to \( G \) such that \( V' = V \cup \{d'\} \), \( E' = E \cup \{(d_i, d')\} \). Let each \((d_i, d')\) be an isolated edge of capacity \( R_i \). Then for any \( s - d' \) cut of the unicast network \( G'\), \( S \subseteq V' \) and \( d' \notin S \), we have

\[
f(G'; S) = f(G; S) + \sum_{d_i \in S} R_i \geq \sum_{d_i \notin S} R_i + \sum_{d_i \in S} R_i = \sum_{i=1}^{n} R_i,
\]

where the inequality follows from our assumption that \((R_1, R_2, \ldots, R_n) \in C_{\text{i.i.d.}}(G)\). Therefore, \( C_{\text{i.i.d.}}(G') \geq \sum_{i=1}^{n} R_i \) and using Proposition 3.1 we can find an acyclic subnetwork \( \tilde{G'} \) in \( G' \) for which \( C_{\text{i.i.d.}}(\tilde{G'}) = C_{\text{i.i.d.}}(G') \geq \sum_{i=1}^{n} R_i \). Let \( \tilde{G} \) be the broadcast network obtained by removing the additional node \( d' \) and...
Fig. 7. Bidirected network with some of the links removed

the edges \((d_i, d')\) from \(\tilde{G}'\). As before, we can argue that \((R_1, R_2, \ldots R_n) \in C_{i.i.d.}(\tilde{G})\). For any \(S \subseteq V\), we have

\[
f(\tilde{G}; S) = f(\tilde{G}; S) + \sum_{d_i \in S} R_i - \sum_{d_i \in S} R_i
\]

\[
= f(\tilde{G}'; S) - \sum_{d_i \in S} R_i
\]

\[
\geq \sum_{d_i \in S} R_i.
\]

Thus, \((R_1, R_2, \ldots R_n) \in C_{i.i.d.}(\tilde{G})\), and the proof of Proposition 3.3 is complete.

Note that again as in the case of multiple access traffic, the above result does not imply the existence of a single acyclic subnetwork whose capacity region is as large as the original network. For a counter example one can consider the network in Figure 5 with the directions of the edges reversed. See Figure 6.

VI. MULTICAST AND MULTIPLE UNICAST NETWORKS

As opposed to the multiple access and broadcast networks discussed in the earlier sections, bidirected communication across certain links can be necessary to achieve certain rate points in the capacity regions of multicast and multiple unicast networks. For multicast, consider the network in Figure 6 but assume that the source wants to multicast the same information to both of the destination nodes. The multicast capacity of this network is 2, however the multicast capacity of any of its acyclic subnetworks is equal to 1. For the multiple unicast case, the classical Gaussian interference channel with feedback readily provides an example where feedback (i.e. bidirected communication) is necessary for achieving capacity. It also straightforward to construct simple examples of multiple unicast networks with isolated edges which illustrate this point.

VII. CONCLUDING DISCUSSION

In this paper, we discussed the usefulness of feeding back information through cycles in Gaussian multi-hop networks. We showed that for unicast, broadcast and multiple-access networks, every rate point in the capacity region of the original network can be approximately achieved in a cycle-free manner, i.e. by using an acyclic subnetwork of the original network. The approximation here is within a bounded gap
which is independent of the channel coefficients and the SNRs in the network which implies that feeding back information through cycles in such networks can only provide a bounded improvement in achievable rates as SNR grows.

As studied in [16] and [17], cycles significantly increase the delay and complexity of (approximately) optimal relaying strategies. By identifying a directed acyclic subnetwork that is sufficient to approximately maintain capacity, our result can be used to reduce the delay and complexity of such schemes by suggesting links that could be potentially shut down. Although shutting down individual links in wireless networks may be nontrivial since these links may represent overheard transmissions over other links, certain networks such as Gaussian networks consisting of isolated links or only MAC and broadcast components (as studied in [8] and [18]) provide some freedom in controlling individual links. Indeed, simplification can be possible even in more general networks.

Consider the example in Figure 7-(a) where the edges in the graph indicate the wireless links with non-zero channel gains. Assume that the backward links from the second layer of relays (nodes C and D) to the first (nodes A and B) operate over a separate frequency, so that while signals arriving over the same colored edges superpose at a node, signals over different colored edges arrive separately. Similarly, while signals over the same colored edges emanating from a single node represent broadcast, different signals can be transmitted over different colored edges. If the directed acyclic network in Figure 7-(b) is identified as sufficient for preserving the capacity of the network, this implies that the backward channel from the second layer to the first need not be used at all. On the other hand, if the directed subnetwork is the one in (c), there is no operational way to reduce the wireless network in (a) to (c). The forward link from node A to D cannot be avoided. However, the communication over the backward channel can still be simplified by not transmitting over the blue frequency from node C and by ignoring the received signal over the blue frequency at node B.

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