Dynamics of clusters of galaxies with extended $f(\chi) = \chi^{3/2}$ gravity

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ABSTRACT

In this article we present the results of the fourth order perturbation analysis of the
gravitational metric theory of gravity $f(\chi) = \chi^{3/2}$, developed by Bernal et al. (2011b)
and Mendoza et al. (2013). The theory accounts in detail for the mass from the observations
of 12 Chandra X-ray clusters of galaxies, without the need of dark matter. The dynamical
observations can be obtained in terms of the metric coefficients of the metric theory of gravity up
to the fourth order of approximation, in perturbations of $v/c$. In this sense, we calculate the first
relativistic correction of the theory, which is relevant at the outer regions of clusters of galaxies,
in order to reproduce the observations.

Subject headings: gravitation – relativity – galaxy clusters

1. Introduction

The observations of type Ia supernovae, the
anisotropies observed in the microwave back-
ground, the acoustic oscillations in the bary-
onic matter, the power–law spectrum of galaxies
and gravitational lenses among others, represent
strong evidences for the standard cosmological model, the so-called $\Lambda$CDM concordance model.
From recent observations of the European space
mission Planck, the contribution of the baryonic matter to the present content of the matter–energy
density of the universe was inferred to be only
5%, while the dark sector constitutes $\sim 95\%$, of
which 27% is Cold Dark Matter (CDM) and 68%
is dark energy or a positive cosmological constant
$\Lambda$ (Planck Collaboration et al. 2013).

The dark matter component was postulated in
order to explain the observed rotation curves of
spiral galaxies, as well as the mass to light ratios
in giant galaxies and clusters of galaxies, the observed gravitational lenses and the structure formation in the early universe, among other astrophysical and cosmological phenomena. On the other hand, the dark energy or a cosmological constant has been postulated to explain the accelerated expansion of the universe (Perlmutter et al. 1999).

The ΛCDM model adjusts quite well most of these observations. However, direct or indirect search of dark matter candidates has yielded null results. In addition, the lack of any further evidence for dark energy opens up the possibility that there are no dark entities in the universe but instead, the theory associated to these astrophysical and cosmological phenomena needs to be modified.

Current models of dark matter and dark energy are based on the assumption that Newtonian gravity and Einstein’s general relativity are valid at all scales. However, their validity has only been verified with high precision for systems which scales are no larger than the Solar System one. In that sense, is conceivable that both, the accelerated expansion of the universe and the stronger gravitational force required in different systems, represent a change in our understanding of the gravitational interactions.

From the geometrical point of view, modified theories of gravity are viable alternatives to solve the astrophysical and cosmological problems that dark matter and dark energy are trying to solve (see e.g. Schimming & Schmidt 2004, Nojiri & Odintsov 2011, Capozziello & Faraoni 2011). In this sense, any theory of modified gravity which attempts to supplant the dark components of the Universe, must account for two crucial observations: the dynamics observed for massive particles and the observations of the deflection of light for massless particles. As extensively described by Will (1993), when working with the weak field limit of a relativistic theory of gravity in a static spherically symmetric spacetime, the dynamics of massive particles determine the functional form of the time component of the metric, while the deflection of light determines the form of the radial one (see also Will 2006, and references therein).

The first successful modification in the non-relativistic regime to deal with these issues was the Modified Newtonian Dynamics (MOND) (see Famaey & McGaugh 2012 for a review). Due to its phenomenological nature and its success in the non-relativistic limit, it is understood that any fundamental theory of modified gravity should adapt to it on galactic scales in the low accelerations regime. However, from the study of groups and clusters of galaxies it has been shown that, even in the deep MOND regime, a dominant dark matter component is required in these systems (60 to 80% of the dynamical or virial mass). Angus et al. (2008) showed that the central region of galaxy clusters could be explained with a halo of neutrinos with mass of 2 eV (which is about the value of the experimental upper limit). But on the scale of groups of galaxies, the central contribution cannot be explained by a contribution of neutrinos with that mass. Moreover, MOND/AQUAL, the Lagrangian formulation of MOND, is not able to reproduce the observed gravitational lensing for different systems (see e.g. Takahashi & Chiba 2007, Natarajan & Zhao 2008), mainly because it is a non–relativistic description, and as such it cannot explain gravitational lensing and cosmological phenomena, which require a relativistic theory of gravity.

Through the years, there have been some attempts to find the relativistic extension of MOND. The first successful attempt was proposed by Bekenstein (2004) who formulated a Tensor-Vector-Scalar (TeVeS) theory. This approach presents some cumbersome mathematical complications and it cannot reproduce crucial astrophysical phenomena (see e.g. Ferreras et al. 2009). Another recent construction by Demir & Karahan (2014) propose a relativistic version of MOND through modifications in the energy-momentum tensor. In an empirical way, they recover the MONDian limit modifying the dynamics sector.

As will be discussed in section 3 the extended gravity theory proposed by Bernal et al. (2011b) is equivalent to the MONDian description in some systems, for example in spherical symmetric ones, but with remarkable advantages. Mendola et al. (2013) have performed a second order perturbation analysis of the gravitational metric theory of gravity $f(\chi) = \chi^{3/2}$ proposed first by Bernal et al. (2011b). They have shown that the theory accounts in detail for both observational facts: it is possible to recover the phenomenology of flat rotation curves and the baryonic Tully-Fisher relation.
2. Perturbations in spherical symmetry

In this section we define the relevant properties of the perturbation theory for applications to the metric theory developed by Bernal et al. (2011b) and Mendoza et al. (2013). Many of the results developed in this section were obtained by Mendoza et al. (2013). This section is kept here for completeness of the article.

Perturbations applied to metric theories of gravity, in particular general relativity, are extensively detailed in the monograph by Will (1993). In particular, for $f(R)$ metric theories, Capozziello & Stabile (2009) have developed a perturbation analysis applied to lenses and clusters of galaxies (Capozziello et al. 2009).

In this article, Einstein’s summation convention over repeated indices is used. Greek indices take values 0, 1, 2, 3 and Latin ones 1, 2, 3. In spherical coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$, where $c$ is the speed of light, $t$ is the time coordinate and $r$ the radial one, with $\theta$ and $\phi$ the polar and azimuthal angles, respectively. The angular displacement $d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2$. We use a $(+, -, -, -)$ signature for the metric of the spacetime.

Let us consider a fixed point-mass $M$ at the center of coordinates generating a gravitational field. In this case, the spacetime is static and its spherically symmetric metric $g_{\mu\nu}$ is generated by the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2 + g_{11} dr^2 - r^2 d\Omega^2,$$

(1)

where $g_{\mu\nu}$ is the metric tensor and, due to the symmetry of the problem, the unknown functions $g_{00}$ and $g_{11}$ are functions of the radial coordinate $r$ only.

The geodesic equations are given by

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,$$

(2)

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols. In the weak field limit when the speed of light $c \to \infty$, $ds = c \, dt$, and since the velocity $v \ll c$, then each component $v^i \ll dx^i/dt$ with $v^i := (dr/dt, r \, d\theta/dt, r \, \sin \theta \, d\phi/dt)$. In this case, the radial component of the geodesic equations (2), for the interval (1), is given by

$$\frac{1}{c^2} \frac{d^2 r}{dt^2} = \frac{1}{2} g^{11} g_{00, r},$$

(3)
where the subscript \((\ ),r := d/dr\) denotes the derivative with respect to the radial coordinate.

In this limit, a particle bound to a circular orbit about the mass \(M\) experiences a centrifugal radial acceleration given by equation (3), such that:

\[
a_c = \frac{v^2}{r} = \frac{c^2}{2} g_{00,r},
\]

(4)

for a circular or tangential velocity \(v\). At this point, it is important to note that the last equation is a general kinematic relation, and does not introduce any particular assumption about the specific gravitational theory. In other words, it is completely independent of the field equations associated to the structure of spacetime produced by the energy-momentum tensor.

In the weak field limit of the theory, the metric takes the form (see e.g. [Landau & Lifshitz 1975]):

\[
g_{00} = 1 + 2\phi/c^2, \quad g_{11} = -1 + 2\psi/c^2, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad (5)
\]

for a Newtonian gravitational potential \(\phi\) and an extra gravitational potential \(\psi\). At the weakest order of the theory, the motion of material particles is described by the potential \(\phi\), taking into account the fact that \(\psi = 0\) (e.g. Landau & Lifshitz 1975). The motion of relativistic massless photon particles is described by taking into consideration not only the second order corrections to the potential \(\phi\), but also the same order perturbations of the potential \(\psi\) (cf. Will 1993).

For circular motion about a mass \(M\) in the weak field limit of the theory, the equations of motion are obtained when the left-hand side of equation (3) is of order \(c^2/v^2\) and when the right-hand side is of order \(\phi/c^2\). Both are orders \(O(c^{-2})\) of the theory, or simply \(O(2)\). When lower or higher order corrections of the theory are introduced we use the notation \(O(n)\) for \(n = 0, 1, 2, \ldots\) meaning \(O(c^0)\), \(O(c^{-1})\), \(O(c^{-2})\), \ldots, respectively.

The extended regions of clusters of galaxies need a huge amount of dark matter to explain the velocity dispersions observed for the stars and gas in these systems. At these regions, the velocity dispersions are typically of order \(10^{-4} - 10^{-3}\) times the speed of light. Hence, the “Newtonian” physics given by an \(O(2)\) approximation should be extended to post-Newtonian \(O(4)\) corrections or, equivalently in our model, “post-MONDian” dynamics.

In order to test a gravitational theory through different astrophysical observations (e.g. the motion of material particles, the bending of light, massless particles, etc.), the metric tensor \(g_{\mu\nu}\) is expanded about the flat Minkowski metric \(\eta_{\mu\nu}\) for corrections \(h_{\mu\nu} \ll \eta_{\mu\nu}\) in the following way:

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (6)
\]

The metric \(g_{\mu\nu}\) is approximated up to second perturbation order \(O(2)\) for the time and radial components and up to zeroth order for the angular components, in accordance with the spherical symmetry of the problem. At this lowest perturbation order, Mendoza et al. (2013) found the time \(g_{00}^{(2)}\) and radial \(g_{11}^{(2)}\) metric components, for the \(f(\chi) = \chi^{3/2}\) metric theory of gravity. Our notation is such that the superscript \((n)\) denotes the order \(O(n)\) at which a particular quantity is approximated. These metric values are necessary to compare with the astrophysical observations of the motion of material particles and that of massless photon particles through the bending of light (Will 1993). In fact, through the observations of the rotation curves of galaxies and the Tully-Fisher relation, and the details of the gravitational lensing in individual, groups and clusters of galaxies, Mendoza et al (2013) fixed the unknown potentials \(\phi\) and \(\psi\) of the theory.

In this paper, we develop perturbations of the relativistic extended model \(f(\chi) = \chi^{3/2}\) up to the fourth order in the time component \(g_{00}^{(4)}\), corresponding to the next order of approximation to describe the motion of massive particles (Will 1993). In this case, the metric components can be written as

\[
g_{00} = 1 + g_{00}^{(2)} + g_{00}^{(4)} + O(6),
\]

\[
g_{11} = -1 + g_{11}^{(2)} + O(4),
\]

\[
g_{22} = g_{22}^{(0)} = -r^2, \quad g_{33} = g_{33}^{(0)} = -r^2 \sin^2 \theta. \quad (7)
\]

In other words, the metric is written up to the fourth order in the time component and up to the second order in the radial one. The contravariant
metric components of the previous set of equations are given by

\[ g^{00} = 1 - \frac{\chi}{2} - \frac{\chi^4}{4} + \mathcal{O}(6), \]
\[ g^{11} = 1 - \frac{2}{(1 + \chi)^4} + \mathcal{O}(4), \]
\[ g^{22} = g^{22(0)} = -\frac{1}{r^2}, \]
\[ g^{33} = g^{22}/\sin^2 \theta. \]

### 3. Extended \( f(\chi) = \chi^{3/2} \) gravity

#### 3.1. Field equations

The \( f(\chi) \) metric theory, proposed by Bernal et al. (2011b), is constructed through the inclusion of MOND’s acceleration scale \( a_0 \) (Milgrom 1983) as a fundamental physical constant, which has been shown to be of astrophysical and cosmological relevance (see e.g. Bernal et al. 2011a; Carranza et al. 2013; Mendoza et al. 2011; Mendoza 2012; Hernandez et al. 2010, 2012; Hernandez & Jiménez 2012; Mendoza et al. 2013 Mendoza & Olmo 2014, Mendoza 2015).

The correct dimensional Hilbert’s action in the metric approach, for a point-mass source generating the gravitational field, is given by (Bernal et al. 2011a)

\[ S_I = -\frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g} \, d^4 x, \] (9)

for any arbitrary dimensionless function \( f(\chi) \) of the dimensionless Ricci scalar

\[ \chi := L_M^2 R, \] (10)

where \( R \) is the standard Ricci scalar and

\[ L_M := \zeta (r_g l_M)^{1/2}, \] (11)

is a length scale where

\[ r_g := \frac{GM}{c^2}, \quad l_M := \left( \frac{GM}{a_0} \right)^{1/2}, \] (12)

with \( r_g \) the gravitational radius of the system, \( l_M \) the mass-length scale of the system (Mendoza et al. 2011), \( a_0 := 1.2 \times 10^{-10} \) m/s\(^2\), the Milgrom’s acceleration constant (see e.g. Famaey & McGaugh 2012, and references therein) and \( \zeta \) is a coupling constant of order one calibrated through astrophysical observations.

Equation (9) is a particular case of a full gravity-field action formulation in which the details of the mass distribution appear inside the gravitational action (see e.g. Carranza et al. 2013; Mendoza 2015 for further investigation). In the present work, we assume the solution for the gravitational potential of a point-mass source and generalize the result to a mass distribution, for applications to spherically symmetric systems, in particular clusters of galaxies.

Now, the matter action takes its ordinary form

\[ S_m = -\frac{1}{2c} \int L_m \sqrt{-g} \, d^4 x, \] (13)

with \( L_m \) the Lagrangian matter density of the system. The null variation of the complete action, i.e. \( \delta (S_I + S_m) = 0 \), with respect to the metric tensor \( g_{\mu \nu} \), yields the following field equations:

\[ f'(\chi) \chi_{\mu \nu} - \frac{1}{2} f(\chi) g_{\mu \nu} - L_M^2 (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Delta) f'(\chi) = \frac{8\pi G L_M^2 T_{\mu \nu}}{c^4}, \]

(14)

where the prime denotes the derivative with respect to the argument, the Laplace-Beltrami operator is \( \Delta := \nabla^\mu \nabla_\mu \) and the energy-momentum tensor \( T_{\mu \nu} \) is defined through the standard relation \( \delta S_m = -(1/2c) T_{\alpha \beta} \delta g^{\alpha \beta} \). Also, in equation (14), the dimensionless Ricci tensor is defined as

\[ \chi_{\mu \nu} := L_M^2 R_{\mu \nu}, \] (15)

where \( R_{\mu \nu} \) is the standard Ricci tensor. The trace of equations (14) is given by

\[ f'(\chi) \chi - 2 f(\chi) + 3 L_M^2 \Delta f'(\chi) = \frac{8\pi G L_M^2 T}{c^4}, \] (16)

where \( T := T^\alpha_\alpha \).

Bernal et al. (2011b) and Mendoza et al. (2013) have shown that the function \( f(\chi) \) must satisfy the following limits:

\[ f(\chi) = \begin{cases} \chi, & \text{when } \chi \gg 1, \\ \chi^{3/2}, & \text{when } \chi \ll 1. \end{cases} \] (17)
in order to recover Einstein’s general relativity in the limit \( \chi \gg 1 \) and a relativistic version of MOND in the regime \( \chi \ll 1 \). In the latter case, the first two terms on the left-hand side of the trace (16) are much smaller than the third one (Bernal et al. 2011b), and so

\[
f'(\chi) \chi - 2f(\chi) \ll 3L_M^2 \Delta f'(\chi),
\]

at all orders of approximation. All these facts mean that the trace (16) is given by:

\[
3L_M^2 \Delta f'(\chi) = \frac{8\pi G L^2}{c^4 M} T.
\]

For the field produced by a point mass \( M \), the right-hand side of equation (19) is null far from the source and so, the last relation in vacuum can be rewritten as:

\[
\Delta f'(\chi) = 0.
\]

Now, for simplicity, we assume a power-law form for the function \( f(\chi) \):

\[
f(\chi) = \chi^b,
\]

for a real power \( b \). In this case, relation (20) is equivalent to

\[
\Delta f'(R) = 0,
\]

at all orders of approximation for a power-law function of the Ricci scalar

\[
f(R) = R^b.
\]

Substitution of the power-law function (21) in the null variations of the gravitational field’s action (9) in vacuum leads to

\[
\delta S_t = -\frac{c^3}{16\pi G} L_M^{2(b-1)} \int R^b \sqrt{-g} \, d^4x = 0,
\]

and so

\[
\delta \int R^b \sqrt{-g} \, d^4x = 0.
\]

From the last relation, the same field equations as the null variation of the action for a standard power-law metric \( f(R) \) theory (23) in vacuum are obtained, but with the important restriction (22) needed to yield the correct relativistic extension of MOND. Mendoza et al. (2013) showed that this condition is crucial to describe the details observed for gravitational lensing for individuals, groups and clusters of galaxies, and differs from the results obtained by Capozziello et al. (2007), for a standard \( f(R) \) power-law description in vacuum.

As discussed in Appendix A of Mendoza et al. (2013), such discrepancy occurs from the sign convention used in the definition of the Riemann tensor, giving two different choices of signature that effectively bifurcate on the solution space, a property which does not appear in Einstein’s general relativity. This is due to higher order derivatives with respect to the metric tensor that appear on metric theories of gravity (cf. equations (14) and (16)). Following the results by Mendoza et al. (2013), we use the same definition and branch of solutions which recover the correct weak field limit of the theory to explain the rotation curves of spiral galaxies, based on the Tully-Fisher relation, and the gravitational lensing observed in groups and clusters of galaxies, at the outer regions of these systems.

The standard perturbation analysis for \( f(R) \) metric theories restricted by the constraint equation (22) is developed for a power-law description of gravity in the weak field limit and in the first MOND-like relativistic correction (cf. equation (17)), by Mendoza et al. (2013). The standard field equations (14) can be written as (see e.g. Capozziello & Faraoni 2011)

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = 0,
\]

where the fourth-order terms are grouped into the term

\[
\mathcal{H}_{\mu\nu} := -\left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta\right) f'(R).
\]

The trace of equation (20) is given by

\[
f'(R)R - 2f(R) + \mathcal{H} = 0,
\]

with

\[
\mathcal{H} := \mathcal{H}_{\mu\nu}g^{\mu\nu} = 3\Delta f'(R).
\]
For the case of the static spherically symmetric spacetime \[ M(\mathbf{r}) \], it follows that (Mendoza et al. 2013)

\[
\mathcal{H}_{\mu\nu} = -f'' \left\{ R_{\mu\nu} - \Gamma^1_{\mu\nu} R_{,r} - g_{\mu\nu} \left[ \left( g^{11} \right)_{,r} + g^{11} \left( \ln \sqrt{-g} \right)_{,r} R_{,r} + g^{11} R_{,rr} \right] \right\} - f''' \left\{ R_{\mu\rho} R_{,\nu} - g_{\mu\nu} g^{11} R_{,r}^2 \right\},
\]

(30)

and

\[
\mathcal{H} = 3f'' \left[ \left( g^{11} \right)_{,r} + g^{11} \left( \ln \sqrt{-g} \right)_{,r} R_{,r} + g^{11} R_{,rr} \right] + 3f''' g^{11} R_{,r}^2.
\]

(31)

The general field equations are of fourth order in the derivatives of the metric tensor. In dealing with the algebraic manipulations for the perturbations to our \( f(R) \) theory of gravity, T. Bernal, S. Mendoza and L.A. Torres have developed a code in the Computer Algebra System (CAS) Maxima, the MEXICAS (Metric EXtended-gravity Incorporated through a Computer Algebraic System) code (licensed with a GNU Public License Version 3). The code is described in Appendix B of Mendoza et al. (2013) and can be downloaded from: [http://www.mendoza.org/sergio/mexicas](http://www.mendoza.org/sergio/mexicas).

In a more general description of the gravitational field, the mathematical form of the field’s action \[ S \] includes the Schwarzschild mass (through \( L_M \)) into the integral. This means a modification to the standard Lagrangian description of the gravitational field since the matter content is generally assumed to appear only in the matter action \[ \mathcal{L}_M \]. However, according to Sobouti (2007); Rosas-Guevara (2006); Mendoza et al. (2013), modifications at the very fundamental level of Hilbert’s action could be expected when discussing extensions of gravity.

### 3.2. Weakest field limit and post-MONDian correction

We present in this subsection fourth order perturbation results for the metric coefficients and the Ricci scalar.

Ricci’s scalar can be written as follows:

\[
R = R^{(2)} + R^{(4)} + O(6),
\]

(32)

which has a non-null second and higher perturbation orders. This scalar is a function of the metric components and their derivatives with respect to the coordinates up to the second order. The fact that \( R^{(0)} = 0 \), is consistent with the flatness of spacetime assumption at the lowest zeroth perturbation order. The second order component of Ricci’s scalar \( R^{(2)} \) from the metric components \[ \mathbf{g} \] is given by

\[
R^{(2)} = - \frac{2}{r} \left[ g^{(2)}_{11,rr} + g^{(2)}_{11,rr} \right] - g^{(2)}_{00,rrr} - \frac{2}{r} g^{(2)}_{00,rrr},
\]

(33)

and the fourth order component is:

\[
R^{(4)} = - \frac{2}{r} \left[ g^{(2)}_{11,rr} + g^{(2)}_{11,rr} + g^{(2)}_{11,rr} \right] + g^{(2)}_{00,rrr} \left[ g^{(2)}_{11,rr} - g^{(2)}_{11,rr} \right] - \frac{2}{r} g^{(4)}_{00,rr} - g^{(4)}_{00,rr},
\]

(34)

At the lowest perturbation order \( O(2b-2) \), the trace of the field equations for a theory given by equation \[ (23) \], can be written as (Mendoza et al. 2013):

\[
\mathcal{H}^{(2b-2)} = 3\Delta f^{(2b-2)}(R) = 0.
\]

(35)

Note that this is the only independent equation at this perturbation order. Substitution of expressions \[ (3), (23) \] and \[ (32) \] in the previous equation leads to a differential equation for Ricci’s scalar at order \( O(2) \), which has a solution given by (Mendoza et al. 2013)

\[
R^{(2)}(r) = \left[ (b - 1) \left( \frac{A}{r} + B \right) \right]^{1/(b-1)},
\]

(36)

where \( A \) and \( B \) are constants of integration. Far away from the central mass spacetime is flat, and so, Ricci’s scalar must vanish at large distances from the origin, i.e. the constant \( B = 0 \). The case \( b = 3/2 \) yields a MONDian-like behavior for
the gravitational field in the limit \( r \gg l_M \gg r_s \) (Bernal et al. 2011b; Mendoza et al. 2013) and so, substituting \( b = 3/2 \) in relation (36) and \( B = 0 \) yields:

\[
R^{(2)}(r) = \frac{\hat{R}}{r^2},
\]

where \( \hat{R} := A^2/4 \).

At the next perturbation order \( \mathcal{O}(2b) \), the metric components \( g_{00}^{(2)} \), \( g_{11}^{(2)} \), \( g_{00}^{(4)} \) and Ricci’s scalar \( R^{(4)} \) can be obtained. The field equations (25) at this order are given by (Mendoza et al. 2013)

\[
bR^{(2)b-1} R^{(2)}_{\mu
u} - \frac{1}{2} R^{(2)b} g_{\mu
u}^{(0)} + \mathcal{H}_{\mu
u}^{(2b)} = 0, \tag{38}
\]

where

\[
\mathcal{H}_{\mu
u}^{(2b)} = - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta) f'^{(2b)}(R). \tag{39}
\]

From equation (22) it follows that \( \Delta f'^{(2b)} = 0 \) and the last equation simplifies greatly. Using relations (7) and (8) in the 00 component of equation (38) leads to (Mendoza et al. 2013)

\[
bR^{(2)b-1} R_{00}^{(2)} - \frac{1}{2} R^{(2)b} + \frac{1}{2} b(b-1) g_{00}^{(2)} R^{(2)b-2} R^{(2)}_{,r} = 0, \tag{40}
\]

where the 00 component of the Ricci tensor at \( \mathcal{O}(2) \) is

\[
R_{00}^{(2)} = - \frac{r g_{00,rr}^{(2)} + 2 g_{00,rr}^{(2)}}{2r}. \tag{41}
\]

Substituting this last expression, \( b = 3/2 \) and result (37) into equation (40), the following differential equation for \( g_{00}^{(2)} \) is obtained (Mendoza et al. 2013):

\[
r^2 g_{00,rr}^{(2)} + 3r g_{00,r}^{(2)} + \frac{2\hat{R}}{3} = 0, \tag{42}
\]

which has a solution given by

\[
g_{00}^{(2)}(r) = - \frac{\hat{R}}{3} \ln \left( \frac{r}{r_s} \right) + \frac{k_1}{r^2}, \tag{43}
\]

where \( k_1 \) and \( r_s \) are constants of integration. By substitution of this result into equation (38) and using equation (37), the following differential equation for \( g_{11}^{(2)} \) is obtained (Mendoza et al. 2013):

\[
r g_{11,rr}^{(2)} + g_{11,r}^{(2)} + \frac{k_1}{r^2} + \frac{\hat{R}}{3} = 0, \tag{44}
\]

with analytic solution

\[
g_{11}^{(2)}(r) = \frac{k_1}{r^2} + \frac{k_2}{r} - \frac{\hat{R}}{3}, \tag{45}
\]

where \( k_2 \) is a constant of integration.

Now, to derive the first relativistic correction of the metric theory (21) with \( b = 3/2 \), i.e. to obtain \( g_{00}^{(4)} \) and \( R^{(4)} \) having in mind further applications to the outer regions of clusters of galaxies, we use another independent field equation, namely the 22 component of equation (38). In fact, substitution of relations (7) and (8) in the 22 component of equation (38) yields

\[
b(b-1) r R^{(2)b-2} \left[ R^{(4)}_{,r} + g_{11,r}^{(2)} R^{(2)}_{,r} + (b-2) R^{(2)b-2} R^{(2)}_{,rr} \right] - b R^{(2)b-1} R^{(2)}_{22} - \frac{r^2}{2} R^{(2)b} = 0, \tag{46}
\]

where the 22 component of the Ricci tensor at \( \mathcal{O}(2) \) is given by

\[
R^{(2)}_{22} = g_{11}^{(2)} + \frac{r}{2} \left[ g_{00,rr}^{(2)} + g_{11,rr}^{(2)} \right]. \tag{47}
\]

Using the fact that \( b = 3/2 \), and substituting the last equation together with relations (37) and (45) into equation (40), we obtain the following differential equation for the Ricci scalar at \( \mathcal{O}(4) \):

\[
r^4 R^{(4)}_{,r} + r^3 R^{(4)} + \hat{R}^2 r - 3k_2 \hat{R} = 0, \tag{48}
\]

which has the following exact solution:

\[
R^{(4)}(r) = \frac{\hat{R}^2}{r^2} - \frac{3k_3 \hat{R}}{2r^3} + \frac{4k_3}{r}, \tag{49}
\]

where \( k_3 \) is a constant of integration. Now, the expression for the Ricci scalar \( R^{(4)} \) from the metric components (7) is given by expression (33). Substituting such equation in (49), together with relations (37), (43) and (45), we obtain the following differential equation for \( g_{00}^{(4)} \):
\[ -9g^{(4)}_{00,rr} + \frac{18}{r} \left( g^{(4)}_{00,r} + 2k_3 \right) + \frac{\hat{R}^2}{r^2} \left[ \ln \left( \frac{r}{r_s} \right) - \frac{23}{2} \right] \\
+ \frac{3k_2}{r^3} \left( 5\hat{R} + \frac{6k_2}{r} \right) - \frac{3k_1}{r^4} \left[ 2 \ln \left( \frac{r}{r_s} \right) + 1 \right] \\
+ \frac{45k_1k_2}{r^5} + \frac{54k^2_2}{r^6} = 0, \]

with the following exact solution:

\[ g^{(4)}_{00} = \frac{\hat{R}^2}{18} \ln^2 \left( \frac{r}{r_s} \right) - \frac{25\hat{R}^2}{18} \ln \left( \frac{r}{r_s} \right) - 2k_3 r - \frac{2k_4}{r}, \]

\[ - k_1 \frac{\hat{R}}{3r^2} \left[ \ln \left( \frac{r}{r_s} \right) + 2 \right] - \frac{5k_2}{3r} \hat{R} \left[ \ln \left( \frac{r}{r_s} \right) + 1 \right] \\
+ \frac{k_2^2}{r^2} + \frac{5k_1k_2}{6r^3} + \frac{k_1^2}{2r^4}, \]

where \( k_4 \) and \( k_5 \) are constants of integration.

To fix the free parameters in relations \( k_3 \text{ and } k_4 \), Mendoza et al. (2013) compared the metric coefficients with observations of rotation curves of spiral galaxies and the Tully-Fisher relation and with gravitational lensing results of individual, groups and clusters of galaxies. Their results are summarized in Table 1.

Using these solutions of the metric coefficients at order \( O(2) \), the metric coefficient \( g^{(4)}_{00} \) and Ricci’s scalar \( R^{(4)} \) reduce to

\[ g^{(4)}_{00} = 2 \left( \frac{\rho_s}{l_M} \right)^2 \ln \left( \frac{r}{r_s} \right) \left[ \ln \left( \frac{r}{r_s} \right) - 25 \right] - 2k_3 r \\
- \frac{2k_4}{r}, \]

\[ R^{(4)} = \left( \frac{6\rho_s}{l_M} \right)^2 \frac{1}{r^2} + \frac{4k_3}{r}, \]

for \( \frac{\rho_s}{l_M} = \left( \frac{GMa_0}{c^2} \right)^{1/2} \).

To fix the constants of integration \( k_3 \) and \( k_4 \) \((k_5 \text{ vanishes upon derivation of } g^{(4)}_{00} \text{ with respect to } r – \text{cf. equation (53)})\), we adjusted the observational data of 12 clusters of galaxies, as described in the next subsection.

### 3.3. Generalization to extended systems

To compare the correction \( g^{(4)}_{00} \) with the observations of clusters of galaxies, let us take the radial component \( \hat{R} \) of the geodesic equations \( \hat{R} \) in the weak field limit of the theory. In this limit, the rotation curve for test particles bound to a circular orbit about a mass \( M \) with circular velocity is \( v(r) \) given by equation (4). Such equation, up to the fourth order of approximation, is given by

\[ \frac{1}{c^2} \frac{d^2r}{dt^2} = -\frac{1}{2} \left[ \frac{\hat{g}_{00}}{r} + \frac{\hat{g}_{11}}{r^2} + \frac{\hat{g}_{00,r}}{r^2} \right]. \]

Substitution of the \( O(2) \) perturbation values of the metric coefficients from Table 1 and equation (51) in (53), results in the following expression for the acceleration of a test mass particle in the gravitational field generated by the point-mass \( M \):

\[ a_c(r) = \frac{\left( \frac{GMa_0}{r} \right)^{1/2}}{r} \]

\[ + \frac{1}{c^2} \left[ \frac{23GMa_0}{r} - \frac{2GMa_0}{r} \ln \left( \frac{r}{r_s} \right) + c^4k_3 - \frac{c^4k_4}{r^2} \right]. \]

where \( a_c := |a_c| \). The first term on the right-hand side of last equation corresponds to the “deep-MONDian” acceleration (see e.g. Famaey & McGaugh 2012). The remaining \( O(2) \) terms are the first relativistic correction to the gravitational acceleration.

In order to apply these results to clusters of galaxies, it is necessary to generalize the gravitational acceleration \( \hat{R} \) to a spherical mass distribution \( M(r) \). The first term of such equation can be easily generalized according to Mendoza et al. (2011). In this case, the deep-MONDian acceleration can be written as \( f(x) = a/a_0 = x \), for \( x := l_M/r \). As discussed by Mendoza et al. (2011) this function, and in general any analytical function which depends only on the parameter \( x \), guarantees Newton’s theorems. In other words, the gravitational acceleration exerted by the outer shells at position \( r \) cancels out and depends only on the mass \( M(r) \) interior to \( r \):

\[ M(r) = 4\pi \int_0^r \rho(r) r^2 dr, \]
where \( \rho \) is the mass density of the system. Thus, the first term of the gravitational acceleration \( a_c(r) \) due to a mass distribution can be written as

\[
a_c(r) = \frac{GM(r)a_0^{1/2}}{r}. \tag{56}
\]

For the second order \( \mathcal{O}(2) \) terms on the right-hand side of acceleration \( \frac{1}{r^2} \), let us take the gravitational potential generated by a point-mass particle \( M \) for the model \( f(\chi) = \chi^{3/2} \). Integrating the acceleration \( \{54\} \) with respect to the radius \( r \), according to the spherical symmetry assumption, we obtain

\[
|\phi(r)| = \int a_c(r) dr,
\]

\[
= (GMa_0)^{1/2} \ln \left( \frac{r}{rs} \right) + \frac{1}{c^2} \left( GMa_0 \ln \left( \frac{r}{rs} \right) \left[ 23 - \ln \left( \frac{r}{rs} \right) \right] + \frac{c^4 k_3 r}{4} \right) \cdot \phi^{(2)}(r) = \frac{GMa_0}{c^2} \left\{ \ln \left( \frac{r}{rs} \right) \left[ 23 - \ln \left( \frac{r}{rs} \right) \right] + Ar - \frac{B}{r} \right\}, \tag{57}
\]

This point-mass gravitational potential can be generalized considering that the extended system is composed of many infinitesimal mass elements \( dm \), each one contributing with a point-like gravitational potential \( \phi^{(2)}(r) \), such that:

\[
M(r) = \int_V dm = \int_V \rho(r') dV', \tag{58}
\]

where the volume element is \( dV' = r'^2 \sin \theta' d\theta' d\varphi' dr' \), integrated over the volume \( V \). Notice that the mass of the system \( M \) appears elevated to the one half power in the first term of equation \( \{57\} \), and that the potential depends linearly with the mass in the second and third terms. Since the constants \( k_3 \) and \( k_4 \) are proportional to \( 1/c^4 \) in order to have second order terms in the acceleration, let us assume that they are proportional to \( GMa_0/c^4 \). Thus, the \( \mathcal{O}(2) \) point-gravitational potential can be written as

\[
\phi^{(2)}(r) = GMa_0 \left\{ \ln \left( \frac{r}{rs} \right) \left[ 23 - \ln \left( \frac{r}{rs} \right) \right] + Ar - \frac{B}{r} \right\}, \tag{59}
\]

where \( A \) and \( B \) are constants.

From equation \( \{59\} \), the generalized gravita-

---

**Table 1**

| Metric coefficient | \( g_{(2)}^{(2)} \) | \( g_{(2)}^{(2)} \) |
|--------------------|-------------------|-------------------|
| Observations       | \(-\frac{2\pi \chi}{r^2} \ln \left( \frac{r}{r_s} \right)\) | \(-\frac{2\pi \chi}{r^2} \ln \left( \frac{r}{r_s} \right)\) |
| (Tully-Fisher)     | (Lensing)         | (Lensing)         |
| Theory             | \(-\frac{2\pi \chi}{r^2} \ln \left( \frac{r}{r_s} \right) + \frac{k_1}{r} + \frac{k_2}{r^2} - \frac{4}{r}\) | \(-\frac{2\pi \chi}{r^2} \ln \left( \frac{r}{r_s} \right) + \frac{k_1}{r} + \frac{k_2}{r^2} - \frac{4}{r}\) |
| \( f(\chi) = \chi^{3/2} \) | \( \hat{R} = 6r_s/l_M, \ k_1 = 0 \) | \( \hat{R} = 6r_s/l_M, \ k_2 = 0 \) |

**Note** — The table reproduced from Mendoza et al. (2013) shows the results obtained for the metric components \( g_{(2)}^{(2)} \) and \( g_{(2)}^{(2)} \) for a static spherically symmetric spacetime. The metric coefficients are empirically obtained from astronomical observations in the scales of galaxies (Tully-Fisher relation) and lensing in the outer regions of individual, groups and clusters of galaxies, and compared to the ones predicted by the metric \( f(\chi) = \chi^{3/2} \) theory of gravity. Any proposed metric of a theory of modified gravity must converge to the observational values presented in the table. As shown by the authors, the theory \( f(\chi) = \chi^{3/2} \) is in perfect agreement with the observational metric components. Since the metric components up to the order \( \mathcal{O}(2) \) determine the “MONDian gravitational potential” of the system, the length \( r_s \) is undetermined. However, as explained in section 4, its value is necessary to describe the dynamics up to perturbation order \( \mathcal{O}(4) \) of the theory and so it will be necessary to fix its value with observational data.
tional potential \( \Phi(2)(r) \) in spherical symmetry is the convolution

\[
\int f(r - r') \rho(r') r'^2 \sin \theta' \, d\varphi' \, d\theta' \, dr',
\]

of the function

\[
f(r - r') = \frac{G a_0}{c^2} \left\{ \ln \left( \frac{|r - r'|}{r_s} \right) \left[ 23 - \ln \left( \frac{|r - r'|}{r_s} \right) \right] \right.
\]

\[
+ A |r - r'| - \frac{B}{|r - r'|} \right\}, \tag{61}
\]

with the differential \( dm \), given by equation (63) (see e.g. Vladimirov 2002), for \( f \) and \( \rho \) locally integrable functions for \( r > 0 \). Due to the spherical symmetry of the problem, the integration can be done in one direction, for example the \( z \) axis, where the polar angle \( \theta = 0 \) and \( |r - r'| = \sqrt{r^2 + r'^2 - 2rr' \cos \theta'} \). Thus, the \( O(2) \) generalized gravitational potential for a mass distribution can be written as

\[
\Phi(2)(r) = \frac{G a_0}{c^2} \int_{0}^{R} \int_{0}^{\pi} \int_{0}^{2\pi} \left\{ \ln \left( \frac{|r - r'|}{r_s} \right) \left[ 23 - \ln \left( \frac{|r - r'|}{r_s} \right) \right] \right.
\]

\[
+ A |r - r'| - \frac{B}{|r - r'|} \right\} \rho(r') r'^2 \sin \theta' \, d\varphi' \, d\theta' \, dr', \tag{62}
\]

integrated over the whole volume. If the density distribution is known, the extended potential \( \Phi(2) \) can be numerically integrated to obtain the gravitational acceleration, from \( 0 < r < r' \) and \( r' < r < R \), where \( R \) is the radius of the spherical configuration. It is a well-known result that the matter outside the spherical shell of radius \( r \) does not contribute to the potential for the fourth term on the right-hand side of equation (62) (see e.g. Vladimirov 2002). For the other terms, the integration is done for the interior and exterior shells of mass with respect to the radius \( r \), giving as result the following expression:

\[
\Phi_c(2)(r) = \frac{2\pi G a_0}{c^2 r} \int_{0}^{R} \left\{ \left( r - r' \right)^2 \ln \left( \frac{|r - r'|}{r_s} \right) \left[ \frac{1}{2} \ln \left( \frac{|r - r'|}{r_s} \right) - 24 \right] \right.
\]

\[
- \left( r + r' \right)^2 \ln \left( \frac{r + r'}{r_s} \right) \left[ \frac{1}{2} \ln \left( \frac{r + r'}{r_s} \right) - 24 \right]
\]

\[
+ \frac{A}{3} \left( \left( r + r' \right)^3 - |r - r'|^3 \right) \rho(r') r' \, dr',
\]

\[
- \frac{12G a_0}{c^2} \int_{0}^{R} 4\pi \rho(r') r'^2 \, dr', \tag{63}
\]

where the last term is constant. Performing the derivation of the potential \( \Phi(2) \) with respect to \( r \) and simplifying some terms, the generalized gravitational acceleration can be written as

\[
a_{\varphi}(r) = \frac{\left[ GM(r) a_0 \right]^{1/2}}{r} + \frac{d\Phi(2)(r)}{dr},
\]

\[
= \frac{\left[ GM(r) a_0 \right]^{1/2}}{r} + \frac{GM(r) a_0 B}{c^2 r^2} + \frac{d\Phi_c(2)(r)}{dr}, \tag{64}
\]

which can be obtained for a given \( \rho(r) \). Notice that the parameters \( A \) and \( r_s \) appear only on the last term of equation (63).

4. Cluster mass profiles

4.1. Fit with observations of clusters of galaxies

To apply the results of the last subsection to the spherically symmetric X-ray clusters of galaxies reported by Vikhlinin et al. (2006), notice that there are two observables: the ionized gas profile \( \rho_g(r) \) and the temperature profile \( T(r) \). Under the hypothesis of hydrostatic equilibrium, the hydrodynamic equation can be derived from the collisionless isotropic Boltzmann equation for spherically symmetric systems in the weak field limit of approximation (see e.g. Binney & Tremaine 2008):

\[
\frac{d \left[ \sigma_r^2 \rho_g(r) \right]}{dr} + \frac{\rho_g(r)}{r} \left[ 2\sigma_r^2 - \left( \sigma_\theta^2 + \sigma_\varphi^2 \right) \right] = -\rho_g(r) \frac{d\Phi(r)}{dr}, \tag{65}
\]

where \( \Phi \) is the gravitational potential and \( \sigma_r, \sigma_\theta \) and \( \sigma_\varphi \) are the mass-weighted velocity dispersions in the radial and tangential directions respectively.
For an isotropic system with rotational symmetry there is no preferred transverse direction, and so \( \sigma_{\theta} = \sigma_{\phi} \). For an isotropic distribution of the velocities, we also have \( \sigma_r = \sigma_\theta \).

The radial velocity dispersion can be related to the pressure profile \( P(r) \), the gas mass density \( \rho_g(r) \) and the temperature profile \( T(r) \) by means of the ideal gas law to obtain:

\[
\sigma_r^2 = \frac{P(r)}{\rho_g(r)} = \frac{k_B T(r)}{\mu m_p}, \tag{66}
\]

where \( k_B \) is the Boltzmann constant, \( \mu \approx 0.609 \) is the mean molecular mass per particle and \( m_p \) is the mass of the proton. Direct substitution of equation (65) into (66) yields the gravitational equilibrium relation:

\[
|a(r)| = \left| \frac{d\Phi(r)}{dr} \right| = \frac{k_B T(r)}{\mu m_p r} \left[ \frac{d \ln \rho_g(r)}{d \ln r} + \frac{d \ln T(r)}{d \ln r} \right]. \tag{67}
\]

The right-hand side of the previous equation is determined by observational data, while the left-hand side should be consistent through a given gravitational acceleration and distribution of matter. In standard Newtonian gravity, the total mass of the cluster is given by the mass of the gas and the mass of the galaxies inside it, with the addition of an unknown dark matter component to avoid a discrepancy of one order of magnitude on both sides of the last equation.

For the particular Newtonian gravity case, the “dynamical” mass \( M_{\text{dyn}} \) of the system is determined by the Newtonian acceleration \( a_N \):

\[
M_{\text{dyn}} := \frac{r^2 a_N}{G} = \frac{k_B T(r)}{\mu m_p G} \left[ \frac{d \ln \rho_g(r)}{d \ln r} + \frac{d \ln T(r)}{d \ln r} \right]. \tag{68}
\]

In the \( f(\chi) = \chi^{3/2} \) model, the acceleration will be given by equation (64), thus we can define the “theoretical” mass \( M_{\text{th}} \) as:

\[
M_{\text{th}} := \frac{r^2 a_c(r)}{G} = \left[ \frac{M(r) a_0}{G} \right]^{1/2} r + \frac{M(r) a_0 B}{c^2} + \frac{r^2 d \Phi_c^{(2)}(r)}{G} \tag{69}
\]

where the mass \( M(r) \) is given only by the baryonic mass in the system, i.e.:

\[
M(r) = M_{\text{gas}}(r) + M_{\text{stars}}(r). \tag{70}
\]

In the previous equation, \( M_{\text{gas}} \) is the mass of the gas and \( M_{\text{stars}} \) is the stellar mass in the cluster.

In order to reproduce the observations, the theoretical mass obtained from our modification to the gravitational acceleration must be equal to the dynamical one coming from the observations, without the inclusion of dark matter, i.e. \( M_{\text{th}} = M_{\text{dyn}} \). This provides an observational procedure to fit the three free parameters \( r_s \), \( A \) and \( B \) of our model.

### 4.2. Chandra clusters sample

Chandra X-ray clusters of galaxies sample can be found in the article of Vikhlinin et al. (2006) and include the full analysis of 12 clusters: gas density, temperature and total mass profiles.

The observed keV temperatures in clusters of galaxies are interpreted in such a way that the gas is fully ionized and the hot plasma is mainly emitted by free-free radiation processes. There is also line emission by the ionized heavy elements. The radiation process generated by these mechanisms is proportional to the emission measure profile \( n_p n_e(r) \). Vikhlinin et al. (2006) introduced a modification to the standard \( \beta \)-model of Cavaliere & Fusco-Femiano (1978), in order to reproduce the observed features from the surface brightness profiles, the gas density at the centers of relaxed clusters and the observed X-ray brightness profiles at large radii. A second \( \beta \)-model component (with small core radius) is added, to increase accuracy near the cluster centers. Vikhlinin et al. (2006) introduced modifications to the standard \( \beta \)-model of Cavaliere & Fusco-Femiano (1978), in order to reproduce the observed features from the surface brightness profiles, the gas density at the centers of relaxed clusters and the observed X-ray brightness profiles at large radii. A second \( \beta \)-model component (with small core radius) is added, to increase accuracy near the cluster centers. With these modifications, the complete expression for the emission measure profile has 9 free parameters. The 12 clusters can be adequately fitted by this model. The best fit values to the emission measure for the 12 clusters of galaxies can be found in Table 2 of Vikhlinin et al. (2006).

To obtain the baryonic density of the gas, the primordial abundance of He and the relative metallicity \( Z = 0.2Z_\odot \) are taken into account and so,

\[
\rho_g(r) = 1.624 m_p \sqrt{n_p n_e(r)}. \tag{71}
\]
For the stellar component of the clusters, we used the empirical relation between the stellar and the total mass (baryonic plus dark matter) in the Newtonian approximation (Lin et al. 2012):

\[
\frac{M_{\text{stars}}}{10^{12}M_\odot} = (1.8 \pm 0.1) \left( \frac{M_{500}}{10^{14}M_\odot} \right)^{0.71 \pm 0.04}, \quad (72)
\]

where \(M_{500}\) is the mass of the cluster in the standard dark matter description.

For the temperature profile \(T(r)\), Vikhlinin et al. (2006) used a different approach from the polytropic law, i.e. \(T(r) \propto r^{-\gamma}\), to model non-constant cluster temperature profiles at large radii. All the projected temperature profiles show a broad peak near the centers and decreases at larger radii, with a temperature decline toward the cluster center, probably because of the presence of radiative cooling (Vikhlinin et al. 2000). To model the temperature profile in three dimensions, they constructed an analytic function such that outside the central cooling region, the temperature profile can be represented as a broken power law with a transition region. The 8 best-fit parameters for the temperature profiles of the X-ray clusters of galaxies can be found in Table 3 of Vikhlinin et al. (2006).

The total dynamical masses obtained with the use of equation (68) from the derived gas density and temperature profiles for 12 Chandra X-ray clusters of galaxies, were kindly provided by A. Vikhlinin.

4.3. Parameters estimation method

We conceptualized the three parameters calibration, \(A, B\), and \(r_s\), as an optimization problem and propose to resolve it using Genetic Algorithms (GAs), which are evolutionary based stochastic search algorithms that, in some sense, mimic natural evolution. In this heuristic search technique, points in the search space are considered as individuals (solution candidates), which as a whole form a population. The particular fitness of an individual is a number, indicating their quality for the problem at hand. As in nature, GAs include a set of fundamental genetic operations that work on the genotype (solution candidate codification): mutation, recombination and selection operators (Mitchell 1998).

These algorithms operate with a population of individuals \(P_t = x_1^t, \ldots, x_N^t\), for the \(t\)-th iteration, where the fitness of each \(x_i\) individual is evaluated according to a set of objective functions \(f_j(x_i)\). These objective functions allow to order, from best to worst, individuals of the population in a continuum of degrees of adaptation. Then, individuals with higher fitness, recombine their genotypes to form the gene pool of the next generation, in which random mutations are also introduced to produce new variability.

A fundamental advantage of GAs versus traditional methods is that GAs solve discrete, non-convex, discontinuous, and non-smooth problems successfully, and thus they have been widely used in Ecology, Natural Resources Management, among other fields, but not so much in Astrophysics (López-Corona et al. 2013). Nevertheless, they have been recently used by Nesseris (2011) for parameters searches in \(\Lambda\)CDM models with SNIa data.

It is important to note that, as it is well known from Taylor series, any (normal) function may be well approximated by a polynomial up to certain correct order of approximation. Of course, even that this is correct from a mathematical point of view, it is possible to consider that a polynomial approximation is not universal for any physical phenomenon. In this line of thought, one may fit any data using a model with many free parameters, and even in this approximation we may have a great performance in a statistical sense, but it could be incorrect from the physical perspective.

In this sense, an important question to ask is, how much better a complex (more parameters) model must perform in a fitting process, justifying the incorporation of extra parameters? In a more straightforward sense, how do we trade off fit with simplicity? Such question has been the motivation in the recent years for new model selection criteria development in statistics, all of which define simplicity in terms of the number of parameters, or the dimension of a model (see e.g. Forster & Sober 1994, for a non-technical introduction). These criteria include Akaike’s Information Criterion (AIC) (Akaike 1974, 1983), the Bayesian Information Criterion (BIC) (Schwarz 1978) and the Minimum Description Length (MDL) (Rissanen 1989). They fit the parameters of a model a little different between them, but all of them address the same
problem as a significance test: Which of the estimated “curves" from competing models best represent reality? (Forster & Sober 1994).

Akaike (1974, 1985) has shown that choosing the model with the lowest expected information loss (i.e., the model that minimizes the expected Kullback-Leibler discrepancy) is asymptotically equivalent to choosing a model $M_j$, from a set of models $j = 1, 2, ..., k$, that has the lowest AIC value, defined by:

\[
AIC = -2 \ln (\mathcal{L}_j) + 2V_j, 
\]

(73)

where $\mathcal{L}_j$ is the maximum likelihood for the candidate model and is determined by adjusting the $V_j$ free parameters in such a way that they maximize the probability that the candidate model has generated the observed data. This equation shows that AIC rewards descriptive accuracy via the maximum likelihood, and penalizes lack of parsimony according to the number of free parameters (note that models with smaller AIC values are to be preferred). In that sense, Akaike (1974, 1985) extended this paradigm by considering a framework in which the model dimension is also unknown, and must therefore be determined from the data. Thus, Akaike proposed a framework where both model estimation and selection could be simultaneously accomplished. For those reasons, AIC is generally regarded as the first, most widely known and used model selection tool.

Taking as objective function the AIC information index, we performed a Genetic Algorithm analysis using a modified version of the Sastry (2007) code in C++. The GA we used evaluates numerically equation (69) in order to compute the numerical results from the model with the observational cluster data by Vikhlinin et al. (2006), as explained in subsection 4.2. All parameters were searched in a broad range from $-1 \times 10^4$ to $1 \times 10^{10}$, generating populations of 1,000 possible solutions over a maximum of 500,000 generation search processes. We selected standard genetic algorithms: tournament selection with replacement (Goldberg et al. 1989; Sastry & Goldberg 2001), simulated binary crossover (SBX) (Deb & Agrawal 1995; Deb & Kumar 1995), and polynomial mutation (Deb & Agrawal 1995; Deb & Kumar 1995; Deb 2001). The parameters were estimated taking the average from the first best population decile, checking the consistency of the second order corrections with respect to the zeroth order term in the gravitational acceleration (61). Finally, since we obtained:

\[
\Delta_{AIC} := AIC_i - \min \{AIC_i\} < 2, 
\]

(74)

for the parameters estimation, then the model was accepted as a good one (Burnham & Anderson 2002).

5. Results and discussion

The results for the best fits as explained in the last subsection are summarized in Table 2. Figures 1 and 2 show the best fits compared to the total dynamical mass obtained by Vikhlinin et al. (2006), as mentioned in section 4.

From the best-fit analysis, we see that our model is capable to account for the total dynamical mass of the 12 clusters of galaxies, except at the very inner regions for some of them, a persistent behavior more accentuated for A907 and A1991. Furthermore, as can be seen in table 2 there are 2 clusters, A1413 and A2029, for which the parameter $r_s$ estimated is very far from the mean value of the other clusters. This parameter appears in the gravitational acceleration (61) only through the integral (63). Comparing its contribution to the acceleration with respect to the other two terms in equation (61), we found that the dominant second order term is the one with the parameter $B$, and the contribution of the derivative of the integral (63) is very small (because $r_s$ appears inside a logarithm and the particular combination of the functions in such equation). Moreover, the $\Delta_{AIC} < 2$ obtained for all the clusters indicates that, in general, the fitted model is very good.

From the figures, we see that our “MOND-like” relativistic correction model is better at the outer regions of these systems, exactly where dark matter is introduced in the standard Newtonian gravity scenario. In this sense, our model is better than standard MOND, which needs huge amounts of extra matter to fit the observations in these systems. Also, the second order perturbation analysis of the metric theory (21) with $b = 3/2$, was capable to account for the observations of the rotation curves of spiral galaxies and the Tully-Fisher relation, and the gravitational lensing in individual, groups
Fig. 1.— Dynamical mass vs. radius for the first 6 clusters of galaxies, with the obtained parameters summarized in Table 1. The points with uncertainty bars are the total mass obtained by the fitting of [Vikhlinin et al. (2006)]. The solid line is the best fit obtained with our model.
Table 2

| Cluster   | $r_{\text{min}}$(kpc) | $r_{\text{max}}$(kpc) | $M_b(10^{13}M_\odot)$ | $M_{\text{th}}(10^{14}M_\odot)$ | $M_{\text{dyn}}(10^{14}M_\odot)$ | $A$(kpc$^{-1}$) | $B$(10$^9$kpc) | $r_s$(10$^{-8}$kpc) |
|-----------|------------------------|------------------------|------------------------|-------------------------------|-------------------------------|----------------|----------------|------------------|
| A133      | 92.10                  | 1005.81                | 3.193                  | 3.269                         | 3.359                         | -96.563        | 2.0727         | 3.42             |
| A262      | 62.33                  | 648.36                 | 1.141                  | 0.825                         | 0.8645                        | -358.82        | 1.7825         | 2.40             |
| A383      | 51.28                  | 957.92                 | 4.406                  | 2.966                         | 3.17                          | -151.99        | 1.9714         | 2.96             |
| A478      | 62.33                  | 1347.89                | 10.501                 | 7.665                         | 8.18                          | -61.436        | 1.9271         | 3.91             |
| A907      | 62.33                  | 1108.91                | 6.530                  | 4.499                         | 4.872                         | -101.05        | 1.9024         | 3.56             |
| A1443     | 40.18                  | 1347.89                | 9.606                  | 7.915                         | 8.155                         | -51.159        | 1.9423         | 71293            |
| A1795     | 92.10                  | 1222.57                | 6.980                  | 6.071                         | 6.159                         | -61.757        | 1.9255         | 3.79             |
| A1991     | 40.18                  | 750.55                 | 1.582                  | 1.198                         | 1.324                         | -340.56        | 2.3703         | 2.49             |
| A2029     | 31.48                  | 1347.89                | 10.985                 | 7.872                         | 8.384                         | -75.339        | 2.1837         | 440.9            |
| A2390     | 92.10                  | 1415.58                | 16.621                 | 11.151                        | 11.21                         | -16.935        | 1.1517         | 4.55             |
| MKW4      | 72.16                  | 648.36                 | 0.676                  | 0.805                         | 0.8338                        | -367.55        | 2.4413         | 2.28             |
| RXJ1159+5531 | 72.16               | 680.77                 | 0.753                  | 1.105                         | 1.119                         | -171.94        | 2.3460         | 2.39             |

Mean value: $-154.59$, $2.0014$, $5980$

< SD >: $1.6462$, $0.00868$, $937.0$

Note.—Parameters estimation for the 12 clusters of galaxies studied. From left to right, the columns represent the name of the cluster, the minimal $r_{\text{min}}$ and maximal $r_{\text{max}}$ radii for the integration, the total baryonic mass $M_b$, the total theoretical mass $M_{\text{th}}$ derived from our model, the total dynamical mass $M_{\text{dyn}}$ from Vikhlinin et al. (2006), the best-fit parameters $A$, $B$ and $r_s$, respectively. Also, at the bottom of the table, we show the best-fit parameters obtained from the 12 clusters of galaxies data taken as a set of independent objective functions together, with their obtained Mean Standard Deviations < SD >.

Fig. 2.— The same as Figure 1 for the last 6 clusters of galaxies.
and clusters of galaxies (Mendoza et al. 2013). In this work, we kept fixed these parameters at perturbation order $O(2)$ to obtain the $O(4)$ of the model, with the additional result that it is possible to fit the dynamical mass of clusters of galaxies without the need of extra dark matter.

Up to now it has generally been thought that a MOND-like extended theory of gravity was not able to explain the dynamics of clusters of galaxies without the necessary introduction of some sort of unknown dark matter component. Our aim has been to show that in order to account for this dynamical description without the inclusion of dark matter, it is necessary to introduce relativistic corrections in the proposed extended theory. To do so, we have chosen the particular $f(\chi) = \chi^{3/2}$ MOND-like metric extension of Bernal et al. (2011b), which has also shown to be in good agreement with gravitational lensing of individual, groups and clusters of galaxies and with the dynamics of the universe providing an accelerated expansion without the introduction of any dark matter and/or energy entities (see e.g. the review of Mendoza 2015, and references therein).

A similar analogy occurred when studying the orbit of Mercury about a century ago. Its motions are mostly understood with Newton’s theory of gravity. However it was necessary to add relativistic corrections to the underlying gravitational theory to account for the precession of its orbit. Mercury orbits at a velocity $\sim 50\text{km/s}$, implying a Lorentz factor of $\sim 10^{-4}$ and already relativistic corrections are required. Typical velocities of clusters of galaxies are $\sim 10^3\text{km/s}$ with a Lorentz factor $\sim 10^{-3}$. This means that the dynamics of clusters of galaxies are about one order of magnitude more relativistic than the orbital velocity of Mercury and so, if the latter required relativistic corrections, then the necessity to describe the dynamics of clusters of galaxies with relativistic corrections are even more important.

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