Motives and the Pfaffian–Grassmannian equivalence

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Abstract
We consider the Pfaffian–Grassmannian equivalence from the motivic point of view. The main result is that under certain numerical conditions, both sides of the equivalence are related on the level of Chow motives. The consequences include a verification of Orlov’s conjecture for Borisov’s Calabi–Yau threefolds, and verifications of Kimura’s finite-dimensionality conjecture, Voevodsky’s smash conjecture and the Hodge conjecture for certain linear sections of Grassmannians. We also obtain new examples of Fano varieties with infinite-dimensional Griffiths group.

1. Introduction
Given $V$ a complex vector space of dimension $n$, let

$$\text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$$

denote the Grassmannian of 2-dimensional linear subspaces of $V$ in its Plücker embedding. The projective dual to $\text{Gr}(2, V)$ is the Pfaffian of degenerate skew forms

$$\text{Pf} := \left\{ \omega \in \mathbb{P}(\wedge^2 V^\vee) \mid \text{rank} \omega < r_{\text{max}} \right\} \subset \mathbb{P}(\wedge^2 V^\vee),$$

where $r_{\text{max}} = n$ if $n$ is even, and $r_{\text{max}} = n - 1$ if $n$ is odd.

Given a linear subspace $U \subset \wedge^2 V$ of codimension $k$, one can define varieties by intersecting on the Grassmannian side and on the Pfaffian side:

$$X = X_U := \text{Gr}(2, V) \cap \mathbb{P}(U) \subset \mathbb{P}(\wedge^2 V),$$

$$Y = Y_U := \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\wedge^2 V^\vee).$$

For $U$ generic and $k$ small enough (the precise condition is that $k \leq 6$ when $n$ is even, and $k \leq 10$ when $n$ is odd, so that $Y$ avoids the singular locus of Pf), the intersections $X$ and $Y$ are smooth and dimensionally transverse, of dimension

$$\dim X = 2(n - 2) - k,$$

$$\dim Y = \begin{cases} 
  k - 2 & \text{if } n \text{ even,} \\
  k - 4 & \text{if } n \text{ odd.}
\end{cases}$$

These varieties have been intensively studied, and particularly so in the case $n = k = 7$ [11, 12, 30, 38, 49]. In this case, $X$ and $Y$ are Calabi–Yau threefolds that are L-equivalent [11, 38] and derived equivalent [2, 12, 30], while for general $U$ they are not birational. The L-equivalence of these Calabi–Yau threefolds gave rise to the first example that the affine line is a zero-divisor in the Grothendieck ring of varieties [11].

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For \((n, k)\) arbitrary, the varieties \(X\) and \(Y\) can still be related on the level of the Grothendieck ring of varieties, and hence on the level of cohomology (cf. Subsections 3.2 and 3.3), but the relation on the level of derived categories becomes conjectural (this is because the Pfaffian is singular, and one needs to find a categorical resolution of singularities, cf. Subsection 3.4). The main result of the present paper relates \(X\) and \(Y\) on the level of Chow motives:

**Theorem** (=Theorem 3.17). Let \(X\) and \(Y\) be as above smooth dimensionally transverse intersections. Assume \(k < 6\) is odd, or \((n, k) = (7, 7)\). Assume also that the transcendental cohomology \(H^*_r(X, \mathbb{Q})\) is non-zero. Then there is an isomorphism of Chow motives

\[
t(X) \xrightarrow{\cong} t(Y)(-m) \quad \text{in } \mathcal{M}_{\text{rat}},
\]

where \(m = \frac{1}{2}(\dim X - \dim Y)\).

Here \(t(X)\) is a certain motive with the property that \(h(X) = t(X) \oplus \bigoplus 1(*)\). The above is a simplified version; the actual statement of Theorem 3.17 applies to certain other values of \((n, k)\).

Theorem 3.17 means that the Grassmannian–Pfaffian equivalence is now better understood on the level of Chow motives than on the level of derived categories (which is as it should be: morally speaking, motives are easier to handle than derived categories).

In proving Theorem 3.17, we rely on the ‘spread’ argument crafted by Voisin [62–64]. This argument (used here in the form of the Franchetta-type result Proposition 2.8) consists in working with families of varieties and correspondences, and exploiting the fact that the total space of the family has a very simple structure. Actually, the starting point of the present paper was the realization that the Grassmannian–Pfaffian equivalence (and more generally, much of the set-up of HPD) is ideally suited to Voisin’s ‘spread’ argument.

The particular case \((n, k) = (7, 7)\) of Theorem 3.17 has the following consequence:

**Corollary** (=Corollary 4.1). Let \(X\) and \(Y\) be the Calabi–Yau threefolds arising from the Pfaffian–Grassmannian equivalence [11]. Then there is an isomorphism of Chow motives

\[
h(X) \cong h(Y) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]

This is in agreement with Orlov’s conjecture [43], stating that derived equivalent varieties should have isomorphic Chow motive.

For other values of \((n, k)\), Theorem 3.17 can be applied to the study of Chow groups \(A^i(X) := CH^i(X)_{\mathbb{Q}}\) (that is, the groups of codimension \(i\) algebraic cycles on \(Y\) with \(\mathbb{Q}\)-coefficients, modulo rational equivalence). Here is a sampling of some applications:

**Corollary** (=Corollaries 4.6, 4.8, 4.13 and 4.15). Let \(X\) be a smooth dimensionally transverse intersection

\[
X := \text{Gr}(2, n) \cap H_1 \cap \cdots \cap H_k \subset \mathbb{P}(2)^{-1},
\]

where the \(H_j\) are hyperplanes.

(i) Assume \(k \leq 2\). Then

\[
A^j_{\text{hom}}(X) = 0 \quad \forall \ j.
\]

(ii) Assume \(n\) is even and \(k = 3\), or \(n\) is odd and \(k = 5\). Then

\[
A^j_{\text{AJ}}(X) = 0 \quad \forall \ j,
\]

and in particular \(X\) has Kimura finite-dimensional motive. (Here \(A^j_{\text{AJ}}(X) \subset A^j(X)\) denotes the subgroup of Abel–Jacobi trivial cycles.)
Assume $n$ is even and $k = 4$, or $n$ is odd and $k = 6$. Then

$$A^j_{AJ}(X) = 0 \quad \forall \ j \neq \frac{1}{2} \dim X + 1,$$

and in particular Voevodsky’s smash conjecture is true for $X$.

(iv) Assume $n$ is even and $k = 5$, or $n$ is odd and $k = 6$. Then

$$A^j_{hom}(X) = 0 \quad \forall \ j > \frac{1}{2} (\dim X + 3),$$

and in particular the Hodge conjecture is true for $X$. (Here $A^j_{hom}(X) \subset A^j(X)$ denotes the subgroup of homologically trivial cycles.)

There is also an application to the Griffiths group (that is, homologically trivial algebraic cycles modulo algebraic equivalence):

**Corollary (=Corollary 4.21).** Let $X$ be a general complete intersection

$$X := Gr(2, 10) \cap H_1 \cap \cdots \cap H_5 \subset \mathbb{P}^{14},$$

where the $H_j$ are hyperplanes. Then $X$ is a Fano 11-fold and the Griffiths group $Grif^6(X) \mathbb{Q}$ is infinite dimensional.

It would be interesting to try and extend the results of this paper to generalized Pfaffian varieties (cf. [30, Conjecture 6] and [56, Section 5.3] and [48] for the conjectural HPD statement, and [13] for a relation on the level of Hodge numbers).

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: We denote by $A_j(Y) := CH_j(Y)_{\mathbb{Q}}$ the Chow group of $j$-dimensional cycles on $Y$ with $\mathbb{Q}$-coefficients; for $Y$ smooth of dimension $n$ the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A^j_{hom}(Y)$ and $A^j_{AJ}(X)$ will be used to indicate the subgroup of homologically trivial (respectively, Abel–Jacobi trivial) cycles.

The contravariant category of Chow motives (that is, pure motives with respect to rational equivalence as in [42, 50]) will be denoted $\mathcal{M}_{rat}$.

## 2. Preliminaries

### 2.1. Cayley’s trick and motives

**Theorem 2.1 [25].** Let $E \to U$ be a vector bundle of rank $r \geq 2$ over a smooth projective variety $U$, and let $S := s^{-1}(0) \subset U$ be the zero locus of a regular section $s \in H^0(U, E)$ such that $S$ is smooth of dimension $\dim U - \text{rank } E$. Let $X := w^{-1}(0) \subset \mathbb{P}(E)$ be the zero locus of the regular section $w \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ that corresponds to $s$ under the natural isomorphism $H^0(U, E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$, and assume $X$ is smooth. There is an isomorphism of Chow motives

$$h(X) \cong h(S)(1 - r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \quad \text{in } \mathcal{M}_{rat}.$$
Proof. This is [25, Corollary 3.2], which more precisely gives an isomorphism of integral Chow motives. Let us give some details about the isomorphism as constructed in [25]. Let

\[ \Gamma := X \times_U S \subset X \times S \]

(this is equal to \( \mathbb{P}(N_t) = \mathcal{H}_s \times_X Z \) in the notation of [25]). Let

\[ \Pi_i \in A^i(X \times U) \quad (i = 0, \ldots, r - 2) \]

be correspondences inducing the maps \( (\pi_i)_* \) of [25], that is, \( (\Pi_i)_* = (\pi_i)_* := (q_i + 1)_* \iota_* : A_j(X) \to A_j(U) \) are defined in [25] in terms of the projective bundle formula for \( q : E \to U \). As indicated in [25, Corollary 3.2] (cf. also [25, text preceding Corollary 3.2]), there is an isomorphism

\[ \left( \Gamma, \Pi_0, \Pi_1, \ldots, \Pi_{r-2} \right) : h(X) \xrightarrow{\cong} h(S)(1 - r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \quad \text{in } \mathcal{M}_{\text{rat}}. \]

\[ \Box \]

Remark 2.2. In the set-up of Theorem 2.1, a cohomological relation between \( X \) and \( S \) was established in [29, Prop. 4.3] (cf. also [22, section 3.7], as well as [8, Proposition 46] for a generalization). A relation on the level of derived categories was established in [43, Theorem 2.10] (cf. also [27, Theorem 2.4] and [8, Proposition 47]).

2.2. The Franchetta property

Definition 2.3. Let \( X \to B \) be a smooth projective morphism, where \( X, B \) are smooth quasi-projective varieties. We say that \( X \to B \) has the Franchetta property in codimension \( j \) if the following holds: for every \( \Gamma \in A^j(X) \) such that the restriction \( \Gamma|_{X_b} \) is homologically trivial for the very general \( b \in B \), the restriction \( \Gamma|_b \) is zero in \( A^j(X_b) \) for all \( b \in B \).

We say that \( X \to B \) has the Franchetta property if \( X \to B \) has the Franchetta property in codimension \( j \) for all \( j \).

This property is studied in [7, 18, 19, 44].

Definition 2.4. Given a family \( \mathcal{X} \to B \) as above, with \( X := X_b \) a fiber, we write

\[ GDA^j_B(X) := \text{Im} \left( A^j(\mathcal{X}) \to A^j(X) \right) \]

for the subgroup of generically defined cycles. In a context where it is clear to which family we are referring, the index \( B \) will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that \( GDA^*_B(X) \) injects into cohomology, under the cycle class map.

2.3. The CK property

Definition 2.5. Let \( M \) be a smooth quasi-projective variety. We say that \( M \) has the CK property if for any smooth quasi-projective variety \( Z \), the natural map

\[ A^*(M) \otimes A^*(Z) \to A^*(M \times Z) \]

is surjective.

(NB: ‘CK’ stands for ‘Chow–Künneth’.)
Lemma 2.6. Let \( \bar{M} \) be a smooth projective variety.

(i) \( \bar{M} \) has the CK property if and only if \( \bar{M} \) has trivial Chow groups.

(ii) If \( \bar{M} \) has the CK property, any Zariski open \( M \subset \bar{M} \) has the CK property.

Proof. (i) This is well known. Assume \( \bar{M} \) has the CK property. Then the diagonal \( \Delta_{\bar{M}} \) is completely decomposable, that is, one has

\[
\Delta_{\bar{M}} = V_1 \times W_1 + \cdots + V_r \times W_r \quad \text{in} \quad A_{\dim \bar{M}}(\bar{M} \times \bar{M}),
\]

(1)

where \( V_j, W_j \subset \bar{M} \) are subvarieties and \( \dim V_j + \dim W_j = \dim \bar{M} \). Looking at the action of this decomposition on Chow groups, one finds that \( A_{\text{hom}}^*(X) = 0 \).

Conversely, assume \( \bar{M} \) has trivial Chow groups. The Bloch–Srinivas argument \([10]\) then gives a decomposition (1), which means that the motive of \( \bar{M} \) is a sum of trivial motives:

\[
h(\bar{M}) = \oplus I(*) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

Given any smooth projective variety \( \bar{Z} \), it follows that

\[
h(\bar{M} \times \bar{Z}) = \oplus h(\bar{Z})(*) \quad \text{in} \quad \mathcal{M}_{\text{rat}},
\]

and so

\[
A^*(\bar{M}) \otimes A^*(\bar{Z}) \xrightarrow{\sim} A^*(\bar{M} \times \bar{Z}).
\]

Given a smooth quasi-projective variety \( Z \), let \( \bar{Z} \supset Z \) be a smooth compactification. The commutative diagram

\[
A^*(\bar{M}) \otimes A^*(\bar{Z}) \rightarrow A^*(\bar{M} \times \bar{Z})
\]

\[
\downarrow \quad \downarrow
\]

\[
A^*(\bar{M}) \otimes A^*(Z) \rightarrow A^*(\bar{M} \times Z)
\]

(where vertical arrows are surjections) implies that the lower horizontal arrow is surjective, that is, \( \bar{M} \) has the CK property.

(ii) This is immediate in view of the commutative diagram

\[
A^*(\bar{M}) \otimes A^*(Z) \rightarrow A^*(\bar{M} \times Z)
\]

\[
\downarrow \quad \downarrow
\]

\[
A^*(M) \otimes A^*(Z) \rightarrow A^*(M \times Z).
\]

\( \square \)

Example 2.7. Here is the main example we have in mind: Let \( V \) be a vector space of dimension \( n \), and consider the Pfaffian of degenerate skew forms

\[
Pf := \left\{ \omega \in \mathbb{P}(\wedge^2 V^\vee) \left| \right. \text{rank} \omega < r_{\text{max}} \right\} \subset \mathbb{P}(\wedge^2 V^\vee)
\]

(where \( r_{\text{max}} = n \) if \( n \) is even, and \( r_{\text{max}} = n - 1 \) if \( n \) is odd). We claim that the non-singular locus

\[
Pf^\circ := Pf \setminus \text{Sing}(Pf)
\]

has the CK property.

To see this, we consider the variety

\[
\widehat{Pf} := \left\{ (\omega, K) \in Pf \times \text{Gr}(s, n) \left| K \subset \ker \omega \right\} \subset Pf \times \text{Gr}(s, n),
\]

where \( s = n \) if \( n \) is even, and \( s = n - 1 \) if \( n \) is odd.
where \( s = 2 \) if \( n \) is even, and \( s = 3 \) if \( n \) is odd. The projection \( \widetilde{\text{Pf}} \to \text{Gr}(2, n) \) is a projective bundle (and so \( \text{Pf} \) is smooth), and the projection \( \text{Pf} \to \text{Pf} \) is an isomorphism over the nonsingular locus (and so \( \text{Pf} \to \text{Pf} \) is a resolution of singularities). Because \( \text{Pf} \) (being a projective bundle over a Grassmannian) has trivial Chow groups, the claim follows from Lemma 2.6.

2.4. A Franchetta-type result

**Proposition 2.8.** Let \( M = \tilde{M} \setminus N \), where \( \tilde{M} \) is a projective variety and \( N \subset M \) is closed. Assume \( M \) is smooth and has the CK property. Let \( L_1, \ldots, L_r \to \tilde{M} \) be very ample line bundles, and let \( X \to B \) be the universal family of smooth complete intersections of type

\[
X = \tilde{M} \cap H_1 \cap \cdots \cap H_r, \quad H_j \in [L_j].
\]

Assume all fibers \( X = X_b \) are disjoint from \( N \) and have \( H_{tr}^{\dim X}(X, \mathbb{Q}) \neq 0 \). There is an inclusion

\[
\ker\left( GDA_{B}^{\dim X}(X \times X) \to H^{2 \dim X}(X \times X, \mathbb{Q}) \right) \subseteq \left( (p_1)^* GDA_{B}^*(X), (p_2)^* GDA_{B}^*(X) \right).
\]

**Proof.** In case \( M = \tilde{M} \) is projective, this is essentially Voisin’s ‘spread’ result [64, Proposition 1.6] (cf. also [35, Proposition 5.1] for a reformulation of Voisin’s result). We give a proof which is somewhat different from [64]. Let \( \bar{B} := \mathbb{P} H^0(\bar{M}, L_1 \oplus \cdots \oplus L_r) \) (so \( B \subset \bar{B} \) is a Zariski open), and let us consider the projection

\[
\pi: X \times B \to Y \times Y.
\]

Using the very ampleness assumption, one finds that \( \pi \) is a \( \mathbb{P}^s \)-bundle over \((\tilde{M} \times \tilde{M}) \setminus \Delta_{\tilde{M}}\), and a \( \mathbb{P}^t \)-bundle over \( \Delta_{\tilde{M}} \). That is, the morphism \( \pi \) is what is termed a stratified projective bundle in [18]. As such, [18, Proposition 5.2] implies the equality

\[
GDA_{B}^*(X \times X) = \text{Im}\left( A^*(M \times M) \to A^*(X \times X) \right) + \Delta_* GDA_{B}^*(X), \tag{2}
\]

where \( \Delta: X \to X \times X \) is the inclusion along the diagonal. As \( M \) has the CK property, \( A^*(M \times M) \) is generated by \( A^*(M) \otimes A^*(M) \). Base-point freeness of the \( L_j \) implies

\[
GDA_{B}^*(X) = \text{Im}(A^*(M) \to A^*(X)).
\]

The equality (2) thus reduces to

\[
GDA_{B}^*(X \times X) = \left( (p_1)^* GDA_{B}^*(X), (p_2)^* GDA_{B}^*(X), \Delta_X \right)
\]

(where \( p_1, p_2 \) denote the projection from \( S \times S \) to first resp. second factor). The assumption that \( X \) has non-zero transcendental cohomology implies that the class of \( \Delta_X \) is not decomposable in cohomology. It follows that

\[
\text{Im}\left( GDA_{B}^{\dim X}(X \times X) \to H^{2 \dim X}(X \times X, \mathbb{Q}) \right) = \text{Im}\left( \text{Dec}^{\dim X}(X \times X) \to H^{2 \dim X}(X \times X, \mathbb{Q}) \right) \oplus \mathbb{Q}[\Delta_X],
\]

where we use the shorthand

\[
\text{Dec}^j(X \times X) := \left( (p_1)^* GDA_{B}^*(X), (p_2)^* GDA_{B}^*(X) \right) \cap A^j(X \times X)
\]

for the decomposable cycles. We now see that if \( \Gamma \in GDA_{B}^{\dim X}(X \times X) \) is homologically trivial, then \( \Gamma \) does not involve the diagonal and so \( \Gamma \in \text{Dec}^{\dim X}(X \times X) \). This proves the proposition.
Remark 2.9. Proposition 2.8 has the following consequence: if the family \( \mathcal{X} \to B \) has the Franchetta property, then \( \mathcal{X} \times_B \mathcal{X} \to B \) has the Franchetta property in codimension \( \dim X \).

Note that the condition \( H_{i}^{\dim X}(X, \mathbb{Q}) \neq 0 \) is necessary to ensure that the class of the diagonal in cohomology is not decomposable. This condition is ‘not very restrictive’, cf. [64, Remark 0.8] where a similar condition is discussed.

2.5. A Hilbert schemes argument

Proposition 2.10 [62, 64]. Let \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \) be families over \( B \), and assume the morphisms to \( B \) are smooth projective and the total spaces are smooth quasi-projective. Let

\[
\Gamma \in A^{i}(\mathcal{X} \times_{B} \mathcal{Z})
\]

be a relative correspondence, with the property that for any \( b \in B \) there exist correspondences \( \Lambda_{b} \in A^{i}(Y_{b} \times Z_{b}) \) and \( \Psi_{b} \in A^{i}(X_{b} \times Y_{b}) \) such that

\[
\Gamma|_{b} = \Lambda_{b} \circ \Psi_{b} \quad \text{in} \quad H^{2i}(X_{b} \times Z_{b}, \mathbb{Q}) .
\]

Then there exist relative correspondences

\[
\Lambda \in A^{i}(\mathcal{Y} \times_{B} \mathcal{Z}) , \quad \Psi \in A^{i}(\mathcal{X} \times_{B} \mathcal{Y})
\]

with the property that for any \( b \in B \)

\[
\Gamma|_{b} = \Lambda|_{b} \circ \Psi|_{b} \quad \text{in} \quad H^{2i}(X_{b} \times Z_{b}, \mathbb{Q}) .
\]

Proof. The statement is different, but this is really the same Hilbert schemes argument as [62, Proposition 3.7], [63, Proposition 4.25] (cf. also [33, Proposition 2.10] for a similar statement). The point is that the data of all the \((b, \Lambda_{b}, \Psi_{b})\) that are solutions to the splitting problem

\[
\Gamma|_{b} = \Lambda_{b} \circ \Psi_{b} \quad \text{in} \quad H^{2i}(X_{b} \times Z_{b}, \mathbb{Q})
\]

can be encoded by a countable number of algebraic varieties \( p_{j} : M_{j} \to B \), with universal objects

\[
U_{j} = (\Lambda_{j}, \Psi_{j}) , \quad \Lambda_{j} \subset \mathcal{Y} \times_{M_{j}} \mathcal{Z} , \Psi_{j} \subset \mathcal{X} \times_{M_{j}} \mathcal{Y}
\]

with the property that for \( m \in M_{j} \) and \( b = p_{j}(m) \in B \), we have

\[
(U_{j})|_{m} = (\Lambda_{b}, \Psi_{b}) \quad \text{in} \quad H^{*}(Y_{b} \times Z_{b}) \oplus H^{*}(X_{b} \times Y_{b}) .
\]

By assumption, the union of the \( M_{j} \) dominates \( B \). Since there is a countable number of \( M_{j} \), one of the \( M_{j} \) (say \( M_{0} \)) must dominate \( B \). Taking hyperplane sections, we may assume \( M_{0} \to B \) is generically finite (say of degree \( d \)). Projecting \( \Lambda_{0} \) to \( \mathcal{Y} \times_{B} \mathcal{Z} \) (respectively, projecting \( \Psi_{0} \) to \( \mathcal{X} \times_{B} \mathcal{Y} \)) and dividing by \( d \), we have obtained \( \Lambda \) (respectively, \( \Psi \)) as requested. □

This is a variant of Proposition 2.10:

Proposition 2.11. Let \( \mathcal{X}, \mathcal{Y} \) be families over \( B \), and assume the morphisms to \( B \) are smooth projective and the total spaces are smooth quasi-projective. Let

\[
\Gamma_{\mathcal{X}} \in A^{i}(\mathcal{X} \times_{B} \mathcal{X}) , \quad \Gamma_{\mathcal{Y}} \in A^{i}(\mathcal{Y} \times_{B} \mathcal{Y})
\]

be relative correspondences, with the property that for any \( b \in B \) there exist correspondences \( \Lambda_{b} \in A^{i}(Y_{b} \times X_{b}) \) and \( \Psi_{b} \in A^{i}(X_{b} \times Y_{b}) \) such that

\[
\Gamma_{\mathcal{X}}|_{b} = \Lambda_{b} \circ \Psi_{b} \quad \text{in} \quad H^{2i}(X_{b} \times X_{b}, \mathbb{Q}) ,
\]

\[
\Gamma_{\mathcal{Y}}|_{b} = \Psi_{b} \circ \Lambda_{b} \quad \text{in} \quad H^{2i}(Y_{b} \times Y_{b}, \mathbb{Q}) .
\]
Then there exist relative correspondences
\[ \Lambda \in A^*(Y \times_B X), \quad \Psi \in A^*(X \times_B Y) \]
with the property that for any \( b \in B \)
\[ \Gamma_X|_b = \Lambda|_b \circ \Psi|_b \text{ in } H^{2i}(X_b \times X_b, \mathbb{Q}), \]
\[ \Gamma_Y|_b = \Psi|_b \circ \Lambda|_b \text{ in } H^{2i}(Y_b \times Y_b, \mathbb{Q}). \]

Proof. As in the proof of Proposition 2.10, the data of all \((b, \Lambda_b, \Psi_b)\) are encoded by a countable number of algebraic varieties carrying universal objects. The same argument then applies. \(\square\)

2.6. Transcendental motive and variable motive

**Proposition 2.12.** Let \( X \) be a smooth projective variety. There exists a splitting
\[ h(X) = t(X) \oplus h_{\text{alg}}(X) \text{ in } \mathcal{M}_{\text{rat}}, \]
such that
\[ H^*(t(X), \mathbb{Q}) = H^*_{\text{tr}}(X, \mathbb{Q}) \]
(where \( H^*_{\text{tr}}(X, \mathbb{Q}) \) is defined as the orthogonal complement of the algebraic part of cohomology \( H^*_{\text{alg}}(X, \mathbb{Q}) \)), and
\[ A^*(t(X)) = A^*_{\text{hom}}(X). \]

Proof. This is a standard construction, cf., for instance, [57] where projectors on \( H^*_{\text{alg}}(X, \mathbb{Q}) \) are constructed. \(\square\)

The inconvenience of the decomposition of Proposition 2.12 is that \( t(X) \) is not canonically defined (the construction depends on choices). Also, if \( X \) varies in a family, it is not clear if the decomposition is generically defined. For complete intersections \( X \subset M \), there is another decomposition which does not present these inconveniences. This decomposition is a motivic incarnation of the so-called ‘variable cohomology’, that is, the cohomology of \( X \) not coming from the ambient space \( M \) (for more on this notion, and the relation with primitive cohomology, cf. [63, Section 4.3.4] or [46]).

**Proposition 2.13.** Let \( M \) be a smooth projective variety with trivial Chow groups, and let \( X \subset M \) be a smooth complete intersection (defined by very ample line bundles on \( M \)). There is a canonical splitting
\[ h(X) = h_{\text{var}}(X) \oplus h_{\text{fix}}(X) \text{ in } \mathcal{M}_{\text{rat}}, \]
such that
\[ H^*(h_{\text{var}}(X), \mathbb{Q}) = H_{\text{dim}}^X(X, \mathbb{Q}) := \text{Coker} \left( H_{\text{dim}}^X(M, \mathbb{Q}) \to H_{\text{dim}}^X(X, \mathbb{Q}) \right). \]

Moreover, there exists \( t(X) \) as in Proposition 2.12 such that there is an inclusion as submotive
\[ t(X) \hookrightarrow h_{\text{var}}(X) \text{ in } \mathcal{M}_{\text{rat}}. \]

Proof. This is a standard construction, cf., for instance, [46] (where the hypotheses on \( M \) are less stringent).

For the ‘moreover’ part, one can start from \( h_{\text{var}}(X) \) and take out trivial motives generating the algebraic part of \( H^*(h_{\text{var}}(X), \mathbb{Q}) \); the difference \( t(X) := h_{\text{var}}(X) - \oplus 1(*) \) is as requested. \(\square\)
3. The Pfaffian–Grassmannian equivalence

3.1. Set-up

**Notation 3.1.** Given $V$ a vector space of dimension $n$, let
\[
\text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)
\]
denote the Grassmannian of 2-dimensional linear subspaces in $V$ in its Plücker embedding. The projective dual to $\text{Gr}(2, V)$ is the Pfaffian of degenerate skew forms
\[
\text{Pf} := \left\{ \omega \in \mathbb{P}(\bigwedge^2 V^\vee) \middle| \text{rank} \, \omega < n \right\} \subset \mathbb{P}(\bigwedge^2 V^\vee).
\]
Given a linear subspace $U \subset \bigwedge^2 V$ of codimension $k$, we define varieties by intersecting
\[
X = X_U := \text{Gr}(2, V) \cap \mathbb{P}(U) \subset \mathbb{P}(\bigwedge^2 V),
\]
\[
Y = Y_U := \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\bigwedge^2 V^\vee).
\]
For $U$ generic and $k$ small enough (the precise condition is that $k \leq 6$ when $n$ is even, and $k \leq 10$ when $n$ is odd, so that $Y$ avoids the singular locus of Pf), the intersections $X$ and $Y$ are smooth and dimensionally transverse, of dimension
\[
\dim X = 2(n - 2) - k,
\]
\[
\dim Y = \begin{cases} 
  k - 2 & \text{if } n \text{ even,} \\
  k - 4 & \text{if } n \text{ odd.}
\end{cases}
\]

3.2. Relation in the Grothendieck ring

**Lemma 3.2.** Assume $n \geq 4$. There is a relation in $K_0(\text{Var}_C)$
\[
[\text{Gr}(2, n)] = \begin{cases} 
  [\mathbb{P}^{n-2}] \cdot \sum_{k=0}^{(n-2)/2} L^{2k} & \text{if } n \text{ even,} \\
  [\mathbb{P}^{n-1}] \cdot \sum_{k=0}^{(n-3)/2} L^{2k} & \text{if } n \text{ odd.}
\end{cases}
\]
Moreover, let $H(2, n) \subset \text{Gr}(2, n)$ denote a smooth Plücker hyperplane section. There is a relation in $K_0(\text{Var}_C)$
\[
[H(2, n)] = \begin{cases} 
  [\mathbb{P}^{n-3}] \cdot \sum_{k=0}^{(n-2)/2} L^{2k} & \text{if } n \text{ even,} \\
  [\mathbb{P}^{n-2}] \cdot \sum_{k=0}^{(n-3)/2} L^{2k} & \text{if } n \text{ odd.}
\end{cases}
\]

**Proof.** The formula for $[\text{Gr}(2, n)]$ is [38, Section 2]. As for the second formula, it is well known that there is no variable cohomology:
\[
H^{2(n-2)-1}(H(2, n), \mathbb{Q}) = 0
\]
[16, Proposition 2.3]. Since the cohomology in degree $< 2(n - 2) - 1$ is isomorphic to that of $\text{Gr}(2, n)$ (weak Lefschetz), this gives the formula for $[H(2, n)]$. \(\square\)

**Theorem 3.3.** Given $n, k \in \mathbb{N}$ such that $n$ is even and $k \leq 6$, or $n$ is odd and $k \leq 10$, let
\[
X \subset \text{Gr}(2, V), \quad Y \subset \text{Pf}
\]
be smooth dimensionally transverse intersections as in Notation 3.1. There is a relation in the Grothendieck ring of varieties

\[ [X] \cdot L^{k-1} + [P^{k-2}] \cdot [\text{Gr}(2, n)] = [Y] \cdot L^s + [P^{k-1}] \cdot [H(2, n)] \quad \text{in } K_0(\text{Var}_C), \]

where

\[ s = \begin{cases} n - 1 & \text{if } n \text{ odd,} \\ n - 2 & \text{if } n \text{ even.} \end{cases} \]

**Proof** (This proof is inspired by [38], where the case \((n, k) = (7, 7)\) is done). Let us consider

\[ Q := \{ (T, C_\omega) \in \text{Gr}(2, V) \times \mathbb{P}(U^\perp) \mid \omega|_T = 0 \} \subset \text{Gr}(2, V) \times \mathbb{P}(U^\perp), \]

the so-called Cayley hypersurface. There is a diagram

\[ Q_X \hookrightarrow Q \leftrightarrow Q_Y \]

\[ \xrightarrow{p} \quad \xleftarrow{q} \quad \xrightarrow{\iota} \]

\[ X \hookrightarrow \text{Gr}(2, V) \quad \quad \mathbb{P}(U^\perp) \leftrightarrow Y \]

Here, the morphisms \(p\) and \(q\) are induced by the natural projections, and the closed subvarieties \(Q_X, Q_Y \subset Q\) are defined as \(p^{-1}(X)\), respectively, \(q^{-1}(Y)\).

The restriction of \(p\) to \(Q \setminus Q_X\) is trivial with fiber \(Q_u \cong \mathbb{P}^{k-2}\), while the restriction of \(p\) to \(Q_X\) is Zariski locally trivial with fiber \(Q_{X,x} \cong \mathbb{P}^{k-1}\). This allows us to relate \(Q\) and \(X\) in the Grothendieck ring:

\[ [Q] = [Q \setminus Q_X] + [Q_X] \]

\[ = [\text{Gr}(2, n) \setminus X] \cdot [\mathbb{P}^{k-2}] + [X] \cdot [\mathbb{P}^{k-1}] \]

\[ = \left( [\text{Gr}(2, n)] - [X] \right) \cdot \sum_{j=0}^{k-2} L^j + [X] \cdot \sum_{j=0}^{k-1} L^j \]

\[ = [\text{Gr}(2, n)] \cdot [\mathbb{P}^{k-2}] + [X] \cdot L^{k-1} \quad \text{in } K_0(\text{Var}_C). \]

This gives us the left-hand side of the relation of Theorem 3.3.

For the right-hand side of the relation, we express \([Q]\) in terms of \([Y]\) by exploiting the right-hand side of diagram (3). This is more involved (for one thing, the morphisms on the right-hand side are not known to be Zariski locally trivial fibrations), but can be done with some patience. The first thing to note is that the morphism \(q\) has only two types of fibers: □

**Lemma 3.4.** The morphism \(Q_Y \to \mathbb{P}(U^\perp) \setminus Y\) (obtained by restricting \(q\)) is piecewise trivial (in the sense of [51, Section 4.2]) with constant fiber \(F_1\).

Likewise, the morphism \(Q_Y \to Y\) (obtained by restricting \(q\)) is piecewise trivial with constant fiber \(F_2\).

**Proof.** The argument of [38, Lemma 3.3] for the case \((n, k) = (7, 7)\) extends. Let us give a sketch for the morphism \(Q_Y \to Y\). In view of [51, Theorem 4.2.3], it suffices to check that there exists \(F_2\) such that for all \(y \in Y\) the fiber \(q^{-1}(y)\) is a \(\mathbb{C}(x)\)-scheme isomorphic to \(F_2 \times_\mathbb{C} \mathbb{C}(x)\).
To check this, one observes that a skew form of rank $2r$ (where $2r = n - 2$ if $n$ is even, and $2r = n - 3$ if $n$ is odd) with coefficients in a field $K \supset \mathbb{C}$ is congruent to the skew form
\[
\begin{bmatrix}
0_r & I_r \\
-I_r & 0_r \\
0_{n-2r}
\end{bmatrix}
\]
(where $0_r$ and $I_r$ denote the $r \times r$ zero-matrix resp. identity matrix, and all empty entries are zero), with a base change having coefficients in $K$. \hfill \qed

The next step consists in finding expressions for the classes of the fibers $F_1$ and $F_2$ in the Grothendieck ring:

**Lemma 3.5.** One has
\[
[F_1] = [H(2, n)],
\]
\[
[F_2] = [H(2, n)] + L^s \quad \text{in } K_0(\text{Var}_\mathbb{C}),
\]
where
\[
s = \begin{cases} 
  n - 1 & \text{if } n \text{ odd,} \\
  n - 2 & \text{if } n \text{ even.}
\end{cases}
\]

**Proof.** For $F_1$, we note that the open $\mathbb{P}(U^\perp) \setminus Y$ is exactly the locus where $q$ has smooth fibers, and (since the Cayley hypersurface $Q \subset \text{Gr}(2, n) \times \mathbb{P}(U^\perp)$ is a (1,1)-divisor), a fiber of $q$ is a Plücker hyperplane section of $\text{Gr}(2, n)$. (Alternatively, one can do a cut-and-paste argument similar to [38, Lemma 3.5], which is the case $(n, k) = (7, 7)$.)

For $F_2$, we can apply the stratification argument of [38, Lemma 3.4] (which is the case $(n, k) = (7, 7)$); a little computation gives the formula. Alternatively, one can reason as in [36, Proposition 2.6]: a singular Plücker hyperplane section $F_2$ of $\text{Gr}(2, n)$ is isomorphic to a Schubert divisor, and so $F_2$ has a cell decomposition given by all the cells of $\text{Gr}(2, n)$ minus the big cell. It follows that
\[
[F_2] = [\text{Gr}(2, n)] - L^{2n-4} \quad \text{in } K_0(\text{Var}_\mathbb{C}),
\]
and the formula follows from Lemma 3.2.

Armed with Lemmata 3.4 and 3.5, we can relate $Q$ and $Y$ in the Grothendieck group:
\[
[Q] = [Q \setminus Q_Y] + [Q_Y] = \left[\mathbb{P}^{k-1} \setminus Y\right] \cdot [F_1] + [Y] \cdot [F_2] = \left[\mathbb{P}^{k-1} \setminus [Y]\right] \cdot [H(2, n)] + [Y] \cdot\left([H(2, n)] + L^s\right)
\]
\[
= \left[\mathbb{P}^{k-1}\right] \cdot [H(2, n)] + [Y] \cdot L^s \quad \text{in } K_0(\text{Var}_\mathbb{C}). (5)
\]
(Here we have used the general fact that if $M \to N$ is a piecewise trivial morphism with constant fiber $F$, one has equality
\[
[M] = [N] \cdot [F] \quad \text{in } K_0(\text{Var}_\mathbb{C}),
\]
which is [51, Lemma 4.2.2].)

Theorem 3.3 now follows from writing
\[
[Q] = [Q] \quad \text{in } K_0(\text{Var}_\mathbb{C}),
\]
and applying equalities (4) and (5). \hfill \qed
**COROLLARY 3.6.** Let $X, Y$ be as in Theorem 3.3, and assume $n = k$ is odd. Then $X$ and $Y$ are Calabi–Yau varieties of dimension $n - 4$ and

$$(b = 0 \text{ in } K_0(Var_\mathbb{C}))$$

(that is, $X$ and $Y$ are L-equivalent).

**Proof.** This follows from Theorem 3.3, combined with the fact that

$$[\mathbb{P}^{n-1}] \cdot [H(2, n)] = [\mathbb{P}^{n-2}] \cdot [\text{Gr}(2, n)] \text{ in } K_0(Var_\mathbb{C}).$$

(This equality follows from Lemma 3.2.)

**REMARK 3.7.** In case $n = k = 7$, $X$ and $Y$ are the famous Calabi–Yau threefolds of Borisov [11]. In this case, the relation of Corollary 3.6 was proven by Martin [38], improving on work of Borisov. (For the fact that in general $[X] \neq [Y]$ in the Grothendieck ring, so that $L$ is a zero-divisor, this follows from the fact that $X$ and $Y$ are not stably birational, cf. proof of [11, Theorem 2.12].)

For $n = k = 9$, $X$ and $Y$ are Calabi–Yau fivefolds that are derived equivalent (cf. Theorem 3.11(iii)). For this case, the L-equivalence of Corollary 3.3 appears to be new.

### 3.3. Relation in cohomology

The relation in the Grothendieck ring (Theorem 3.3) has consequences for the cohomology of the Pfaffian–Grassmannian equivalence. We first deduce a numerical statement:

**LEMMA 3.8.** Assume $n$ is even and $k \in \{2, 4\}$, or $n$ is odd and $k \in \{2, 4, 6\}$. Let

$$X \subset \text{Gr}(2, V), \ Y \subset \text{Pf}$$

be smooth dimensionally transverse intersections as in Notation 3.1. Then

$$\dim H^\dim X_{\text{var}}(X, \mathbb{Q}) = \dim H^\dim Y(Y, \mathbb{Q}) - 1.$$  

**Proof.** Let us write $k = 2k'$. Since there is a functor from $K_0(Var_\mathbb{C})$ to the category of Hodge structures, the relation in $K_0(Var_\mathbb{C})$ of Theorem 3.3 gives an equality of Betti numbers

$$b_{\dim X}(X) + \sum_{j=n-1-k'} b_{2j}(\text{Gr}(2, n)) = b_{\dim Y}(Y) + \sum_{j=n-2-k'} b_{2j}(H(2, n)),$$

(where for any variety $M$ we write $b_j(M) := \dim H^j(M, \mathbb{Q})$).

Using the equalities $b_j(\text{Gr}(2, n)) = b_j(H(2, n))$ for $j < 2n - 4$ and $b_j(H(2, n)) = b_{j+2}(\text{Gr}(2, n))$ for $j \geq 2n - 4$ (weak Lefschetz), this simplifies to

$$b_{\dim X}(X) + b_{2n-4}(\text{Gr}(2, n)) = b_{\dim Y}(Y) + b_{2n+2k'-6}(H(2, n)) + b_{2n-2k'-4}(H(2, n)).$$

Expressing everything in terms of Betti numbers of $\text{Gr}(2, n)$ (weak Lefschetz), this becomes

$$b_{\dim X}(X) + b_{2n-4}(\text{Gr}(2, n)) = b_{\dim Y}(Y) + 2b_{2n-4-2k'}(\text{Gr}(2, n)).$$

Using the equality

$$b_{2n-4}(\text{Gr}(2, n)) - b_{2n-4-2k'}(\text{Gr}(2, n)) = \begin{cases} \lceil \frac{k'}{2} \rceil & \text{if } n \text{ is even,} \\ \lfloor \frac{k'}{2} \rfloor & \text{if } n \text{ is odd} \end{cases},$$

(which can be deduced from Lemma 3.2), one finds that

$$b_{\dim X}(X) + 1 = b_{\dim Y}(Y) + b_{2n-4-2k'}(\text{Gr}(2, n)),$$
provided either \( n \) is even and \( k \in \{2, 4\} \), or \( n \) is odd and \( k \in \{2, 4, 6\} \). Since 
\[
\dim H^\dim X_{\text{var}}(X, \mathbb{Q}) = b_{\dim X}(X) - b_{2n-4-2k}^\bullet(\text{Gr}(2, n)),
\]
this proves the lemma. 
\( \square \)

Restricting to the transcendental cohomology, one obtains a stronger statement: there is a correspondence-induced isomorphism of transcendental cohomology:

**Proposition 3.9.** Assume either \( k \leq 6 \), or \((n, k) = (7, 7)\). Let \( X \subset \text{Gr}(2, V), Y \subset \text{Pf} \) be smooth dimensionally transverse intersections as in Notation 3.1. There is an isomorphism in cohomology
\[
H^\dim X_{\text{tr}}(X, \mathbb{Q}) \cong H^\dim Y_{\text{tr}}(Y, \mathbb{Q}).
\]
(Here \( H^\dim X_{\text{tr}}(X, \mathbb{Q}) \) is defined as the orthogonal complement of the algebraic part of cohomology. In particular, \( H^\dim X_{\text{tr}}(X, \mathbb{Q}) = H^\dim X_{\text{tr}}(X, \mathbb{Q}) \) if \( X \) is of odd dimension.)

More precisely, there is an isomorphism of homological motives
\[
t(X) \cong t(Y)(-m) \text{ in } \mathcal{M}_{\text{hom}},
\]
where \( m := \frac{1}{2}(\dim X - \dim Y) \).

**Proof.** Theorem 3.3 gives a relation in the Grothendieck ring
\[
[X] \cdot L^{k-1} + \sum_j L^{n_j} = [Y] \cdot L^s + \sum_j L^{m_j} \text{ in } K_0(\text{Var}_C),
\]
where
\[
s = \begin{cases} 
  n-1 & \text{if } n \text{ odd,} \\
  n-2 & \text{if } n \text{ even.}
\end{cases}
\]
Since there is a functor \( K_0(\text{Var}_C) \to K_0(\mathcal{M}_{\text{num}}) \) sending \([M]\) to \( h(M)\) (for any smooth projective variety \( M \)) and \( L^j \) to \( \mathbb{1}(-j)\), this gives a relation
\[
h(X) + \bigoplus \mathbb{1}(*) = h(Y)(-m) + \bigoplus \mathbb{1}(*) \text{ in } K_0(\mathcal{M}_{\text{num}}),
\]
(where we have used the equality \( m = s - k + 1 \)). Because the category \( \mathcal{M}_{\text{num}} \) is semi-simple [24], the above induces an isomorphism of motives
\[
h(X) + \bigoplus \mathbb{1}(*) \cong h(Y)(-m) + \bigoplus \mathbb{1}(*) \text{ in } \mathcal{M}_{\text{num}}.
\]
Using that (by construction) \( h(X) = t(X) \oplus \oplus \mathbb{1}(*) \), this gives an isomorphism
\[
t(X) \oplus \bigoplus_j \mathbb{1}(r_j) \cong t(Y)(-m) \oplus \bigoplus_k \mathbb{1}(s_k) \text{ in } \mathcal{M}_{\text{num}}.
\]

At this point, we note that \( X \) and \( Y \) verify the standard conjectures (for \( X \), this is just because all complete intersections in Grassmannians verify the standard conjectures; for \( Y \), this is clear when \( n \) is even [because then \( Y \subset \mathbb{P}^{k-1} \) is a hypersurface], and when \( n \) is odd it is true because \( \dim Y \leq 2 \) or \( Y \) is a 3-dimensional Calabi–Yau variety, and the standard conjectures are known for threefolds not of general type [55]). In particular, homological and numerical equivalence coincide for all self-powers and products of \( X \) and \( Y \). It follows that the
above relation actually takes place in the subcategory $\mathcal{M}_{\text{hom}}^\circ$ generated by homological motives of varieties satisfying the standard conjectures:

$$t(X) \oplus \bigoplus_j \mathbb{I}(r_j) \cong t(Y)(-m) \oplus \bigoplus_k 1(s_k) \quad \text{in } \mathcal{M}_{\text{hom}}^\circ.$$ 

Taking algebraic cohomology $H^*_{\text{alg}}()$ on both sides (which excludes the transcendental motives), we find that

$$\bigoplus_j 1(r_j) \cong \bigoplus_k 1(s_k) \quad \text{in } \mathcal{M}_{\text{hom}}^\circ.$$

Since $\mathcal{M}_{\text{hom}}^\circ \subset \mathcal{M}_{\text{num}}$ is semi-simple, it follows that there is also an isomorphism

$$t(X) \cong t(Y)(-m) \quad \text{in } \mathcal{M}_{\text{hom}}^\circ.$$

Since $\mathcal{M}_{\text{hom}}^\circ$ is a subcategory of $\mathcal{M}_{\text{hom}}$, this proves the result.

**Remark 3.10.** Let $X, Y$ be as in Proposition 3.9, and assume $n$ is even and $k = 3$ (that is, $X$ has dimension $2n - 7$ and $Y$ is a curve). In this case, Donagi has proven in his thesis that the intermediate Jacobian of $X$ is naturally isomorphic to the Jacobian of $Y$ [16, Theorem 2.5].

### 3.4. Relation of derived categories

The following is included merely for illustrative purposes; this will not be used in this paper:

**Theorem 3.11** [30, 48, 52]. Let

$$X \subset \text{Gr}(2, V), \ Y \subset \text{Pf}$$

be smooth dimensionally transverse intersections as in Notation 3.1.

(i) Assume $n = 6$ or $n = 7$. There exist semi-orthogonal decompositions

$$D^b(X) = \langle E_1, \ldots, E_r, \text{Kuz}(X) \rangle,$$

$$D^b(Y) = \langle E_1, \ldots, E_s, \text{Kuz}(Y) \rangle$$

where the $E_j$ are exceptional objects, and there is an equivalence of categories

$$\text{Kuz}(X) \cong \text{Kuz}(Y).$$

(ii) Assume $n$ is odd and $k \leq \min(n, 10)$, or $n$ is even and $k \leq \min(n/2, 6)$. Then there is an embedding

$$D^b(Y) \hookrightarrow D^b(X)$$

admitting a right adjoint.

(iii) Assume $(n, k) = (9, 9)$. Then there is an equivalence of derived categories

$$D^b(Y) \cong D^b(X).$$

**Proof.** Point (i) is one of the first instances of the famous theory of homological projective duality [30].

Point (ii) is [52, Theorem 2.9]; the argument is motivated by but distinct from HPD, which explains why the statement is weaker than that of (i): it is still an open question whether the orthogonal to $D^b(Y)$ inside $D^b(X)$ is generated by exceptional objects (cf. [52, Remark 3.8]).

Point (iii) is a special case of [48, Theorem 1.1]. □
Remark 3.12. Conjecturally, the HPD program applies in full generality, and so Theorem 3.11(i) should be true for all \((n,k)\). This is [30, Conjecture 5] (cf. also [56, Section 5.3] and the introduction of [52]).

3.5. The Noether–Lefschetz condition

Definition 3.13. Given \(V\) a vector space of dimension \(n\), let

\[ X \to B, \quad Y \to B \]

denote the universal families of smooth dimensionally transverse intersections of type

\[ X = X_U := \text{Gr}(2,V) \cap \mathbb{P}(U) \subset \mathbb{P}(\wedge^2 V) , \]

\[ Y = Y_U := \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\wedge^2 V^\vee) , \]

where \(U \subset \wedge^2 V\) is a codimension \(k\) linear subspace. (In particular, \(k \leq 6\) if \(n\) is even, and \(k \leq 10\) if \(n\) is odd.)

We write \(B^\circ \subset B\) for the Zariski open over which both \(X_b\) and \(Y_b\) are smooth dimensionally transverse.

Definition 3.14. We say that the family \(X \to B\) satisfies the condition (NL) if the following holds: for the very general fiber \(X_b\), the inclusion

\[ H^\dim X_b(\text{Gr}(2,V), \mathbb{Q}) \to H^\dim X_b(X_b, \mathbb{Q}) \supset H^\dim X_b(X_b, \mathbb{Q}) \]

is an equality, that is, all Hodge classes in \(H^\dim X_b(X_b, \mathbb{Q})\) come from \(\text{Gr}(2,V)\).

(NB: ‘NL’ stands for Noether–Lefschetz.)

Likewise, we say that \(Y \to B\) satisfies (NL) if the following holds: for the very general fiber \(Y_b\), the inclusion

\[ H^\dim Y_b(\mathbb{P}(\wedge^2 V^\vee), \mathbb{Q}) \to H^\dim Y_b(Y_b, \mathbb{Q}) \supset H^\dim Y_b(Y_b, \mathbb{Q}) \]

is an equality.

Given two integers \((n,k)\), we say the pair \((n,k)\) satisfies condition (NL) if the family \(X \to B\) and the family \(Y \to B\) both satisfy (NL).

The condition (NL) is trivially fulfilled in case \(k\) is odd. Also, at least for small \(k\) it suffices to test condition (NL) on one side of the Pfaffian–Grassmanian equivalence:

Lemma 3.15. Assume \(n\) is even and \(k \leq 4\), or \(n\) is odd and \(k \leq 6\). The family \(X \to B\) satisfies condition (NL) if and only if \(Y \to B\) satisfies (NL).

Proof. We may restrict to the common base \(B^\circ\). For \(b \in B^\circ\) very general, we look at the diagram

\[ H^\dim X_b(X_b, \mathbb{Q}) \to H^\dim Y_b(Y_b, \mathbb{Q}) \]

\[ \downarrow \quad \downarrow \]

\[ H^\dim X_b(X_b, \mathbb{Q}) \quad H^\dim Y_b(Y_b, \mathbb{Q}) , \]

where the horizontal arrow is the isomorphism of Proposition 3.9, and the vertical arrows are the inclusions. Under the hypothesis on \(k\), we know that the two spaces at the bottom of this diagram have the same dimension (Lemma 3.8). It follows that if one of the vertical arrows is an isomorphism, the other vertical arrow is an isomorphism as well. \(\square\)
Here are some examples:

**Lemma 3.16.** The following pairs satisfy condition (NL).

(0) \((n, k)\) with \(k\) odd.
(i) \((2m, 4)\) where \(m \geq 4\) and \((2m + 1, 6)\) where \(m \geq 3\).
(ii) \((6, 6), (7, 8)\).

**Proof.** (i) The assumptions imply that \(Y_b\) is a surface with \(H^{2,0}(Y_b) \neq 0\) (indeed, the canonical bundle of \(Y_b\) is either trivial or ample). For the very general \(b\), there is then an isomorphism

\[
A^1(\text{Pf}^o) \cong A^1(Y_b),
\]

this follows from [39].

(ii) The assumptions imply that \(X_b\) is a surface with \(H^{2,0}(X_b) \neq 0\). The argument of (i) then applies to the very general \(X_b\). \(\square\)

3.6. **Relation of Chow motives**

We now proceed to prove the main result of this paper. In view of Lemma 3.16, the theorem stated in the introduction is a special case of the following:

**Theorem 3.17.** Given \(V\) an \(n\)-dimensional vector space, let

\[
X = X_U := \text{Gr}(2, V) \cap \mathbb{P}(U) \subset \mathbb{P}(\wedge^2 V),
\]

\[
Y = Y_U := \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\wedge^2 V^\vee).
\]

be smooth dimensionally transverse intersections, where \(U \subset \wedge^2 V\) is a codimension \(k\) linear subspace. Assume \(k \leq 6\) or \((n, k) = (7, 7)\). Assume also that \((n, k)\) satisfies condition (NL), and that \(H^*_c(X, \mathbb{Q}) \neq 0\). Then there exist \(t(X), t(Y)\) as in Proposition 2.12 such that there is an isomorphism of Chow motives

\[
t(X) \overset{\cong}{\to} t(Y)(-m) \quad \text{in } M_{\text{rat}},
\]

where \(m = \frac{1}{2}(\dim X - \dim Y)\).

In particular, there are isomorphisms

\[
A^i_{\text{hom}}(X) \cong A^{i-m}_{\text{hom}}(Y) \quad \forall j.
\]

**Proof.** Due to Proposition 3.9, we know that there is an isomorphism of homological motives

\[
\Psi : t(X) \overset{\cong}{\to} t(Y)(-m) \quad \text{in } M_{\text{hom}}.
\]

Let us now consider things family-wise, that is, we use the universal families \(X \to B^o, Y \to B^o\) as in Definition 3.13. For each \(b \in B^o\), there is an isomorphism

\[
\Psi_b : t(X_b) \overset{\cong}{\to} t(Y_b)(-m) \quad \text{in } M_{\text{hom}}, \quad (6)
\]

with an inverse which we will call \(\Lambda_b\).

Let \(\pi_X^\text{var}\) be the projector defining the motives \(h_{\text{var}}(X)\) as in Proposition 2.13. On the Pfaffian side, let \(h_{\text{var}}(Y)\) be the motive defined by the projector

\[
\pi_Y^\text{var} := \Delta_Y - \sum_{j \neq \dim Y} \pi_Y^{j} - \frac{1}{d} \dim Y/2 \times \dim Y/2 \in A^{\dim Y}(Y \times Y),
\]
where the $\pi^j_Y$ for $j \neq \dim Y$ are the (completely decomposed) projectors given by Proposition 2.12, and $d$ is the degree of $Y$, and it is understood the term $h^{\dim Y/2}$ is zero for $\dim Y$ odd.

The nice thing about these projectors is that they are generically defined with respect to $B^\circ$, that is, there exist relative cycles
\[
\pi^\var X_b \in A^{\dim X_b}(X \times B^\circ X), \quad \pi^\var Y_b \in A^{\dim Y_b}(Y \times B^\circ Y)
\]
such that
\[
\pi^\var X|_b = \pi^\var X_b, \quad \pi^\var Y|_b = \pi^\var Y_b \quad \forall b \in B^\circ.
\]
Due to the condition (NL), we have equalities $t(X) = h_{\var}(X)$ and $t(Y) = h_{\var}(Y)$ in $\mathcal{M}_{\hom}$ for all fibers over some $B^{\circ_0} \subset B^\circ$ (where $B^{\circ_0}$ is a countable intersection of Zariski opens). Combined with the isomorphism (6), this means that for all $b \in B^{\circ_0}$ there are isomorphisms
\[
\Psi_b: \ h_{\var}(X_b) \xrightarrow{\cong} h_{\var}(Y_b)(-m) \quad \text{in } \mathcal{M}_{\hom},
\]
with inverse $\Lambda_b$.

Using Proposition 2.11 (with $\Gamma_X = \pi^\var X$ and $\Gamma_Y = \pi^\var Y$), we may assume the isomorphism (7) is generically defined, that is, there exist relative cycles $\Psi \in A^*(X \times B^\circ Y), \Lambda \in A^*(Y \times B^\circ X)$ such that
\[
(\Lambda \circ \pi^\var Y \circ \Psi)|_b = \pi^\var X|_b \quad \text{in } H^{2\dim X_b}(X_b \times X_b, \mathbb{Q}),
\]
\[
(\Psi \circ \pi^\var X \circ \Lambda)|_b = \pi^\var Y|_b \quad \text{in } H^{2\dim Y_b}(Y_b \times Y_b, \mathbb{Q}) \quad \forall b \in B^{\circ_0}.
\]
Applying Proposition 2.8 (with $M = \tilde{M} = \text{Gr}(2, V)$) to the generically defined cycle
\[
(\pi^\var X - \Lambda \circ \pi^\var Y \circ \Psi)|_b \in \text{GDA}_{\hom}^{\dim X_b}(X_b \times X_b),
\]
we find that there exists a decomposed cycle $\gamma_b \in A^*(X_b) \otimes A^*(X_b)$ such that
\[
\pi^\var X_b = (\Lambda \circ \pi^\var Y \circ \Psi)|_b + \gamma_b \quad \text{in } A^{\dim X_b}(X_b \times X_b).
\]
This means there is a split injection of motives
\[
(\Psi_b, \Xi_b): \ h_{\var}(X_b) \hookrightarrow h_{\var}(Y_b)(-m) \oplus \bigoplus 1(*) \quad \text{in } \mathcal{M}_{\rat}
\]
(with the map from $h_{\var}(Y_b)(-m)$ to $h_{\var}(X_b)$ given by $\Lambda_b$).

On the Pfaffian side, we can apply Proposition 2.8 (with $M = \text{Pf}^\circ := \text{Pf} \setminus \text{Sing}(\text{Pf})$, which is possible due to Example 2.7; note that the condition $H^1_{\var}(Y_b, \mathbb{Q}) \neq 0$ of Proposition 2.8 is satisfied due to Proposition 2.9) to the generically defined cycle
\[
(\pi^\var Y - \Psi \circ \pi^\var X \circ \Lambda)|_b \in \text{GDA}_{\hom}^{\dim Y_b}(Y_b \times Y_b).
\]
The result is that there exists a decomposed cycle $\gamma'_b \in A^*(Y_b) \otimes A^*(Y_b)$ such that
\[
\pi^\var Y_b = (\Psi \circ \pi^\var X \circ \Lambda)|_b + \gamma'_b \quad \text{in } A^{\dim Y_b}(Y_b \times Y_b),
\]
and hence there is also a split injection of motives
\[
(\Lambda_b, \Xi'_b): \ h_{\var}(Y_b)(-m) \hookrightarrow h_{\var}(X_b) \oplus \bigoplus 1(*) \quad \text{in } \mathcal{M}_{\rat}
\]
(with the map from $h_{\var}(X_b)$ to $h_{\var}(Y_b)(-m)$ given by $\Psi_b$).

Taking Chow groups on both sides of (8) (and exploiting that $t(X) = h_{\var}(X)$ has the property that $A^*(t(X)) = A^*_{\hom}(X)$), we find an equality of actions
\[
\left(\pi^\var X_b \circ \Lambda_b \circ \pi^\var Y_b \circ \Psi_b \circ \pi^\var X_b - \pi^\var X_b\right)_*: \ A^*(X_b) \to A^*(X_b) \quad \forall b \in B^{\circ_0}. \quad (10)
\]
Likewise, taking Chow groups on both sides of (9), we find equality of actions

\[
\left( \pi_{Y_b}^{\text{var}} \circ \Psi_b \circ \pi_{X_b}^{\text{var}} \circ \Lambda_b \circ \pi_{Y_b}^{\text{var}} - \pi_{Y_b}^{\text{var}} \right)_* = 0: \ A^*(Y_b) \rightarrow A^*(Y_b) \quad \forall \ b \in B^{\circ\circ}.
\]

Applying [53, Proposition 3.2] (which is inspired by the Bloch–Srinivas decomposition of the diagonal argument [10]), this means that the expressions in parentheses in (10) and (11) are both nilpotent. Taking the largest of the two nilpotence indices, it follows that

\[
\Psi_b: h_{\text{var}}(X_b) \rightarrow h_{\text{var}}(Y_b)(-m) \quad \text{in } M_{\text{rat}}
\]

is an isomorphism (with inverse \( \Lambda'_b \), which is a sum of expressions of the form \( \Lambda_b \circ \pi_{Y_b}^{\text{var}} \circ \Psi_b \circ \pi_{X_b}^{\text{var}} \circ \cdots \circ \Psi_b \)), for any \( b \in B^{\circ}. \)

This means that for any \( b \in B^{\circ} \), there exist submotives \( t(X) \subset h_{\text{var}}(X), t(Y) \subset h_{\text{var}}(Y) \) as in Proposition 2.12, and such that there is an isomorphism

\[
t(X_b) \oplus \bigoplus \mathbb{I}(\ast) \xrightarrow{\cong} t(Y_b)(-m) \oplus \bigoplus \mathbb{I}(\ast) \quad \text{in } M_{\text{rat}}.
\]

Taking Chow groups (and exploiting that \( A^*(t(X)) = A^*_{\text{hom}}(X) \)), we deduce that there is equality of actions

\[
\begin{align*}
\left( \pi_{X_b}^{\text{tr}} \circ \Lambda'_b \circ \pi_{Y_b}^{\text{tr}} \circ \Psi_b \circ \pi_{X_b}^{\text{tr}} - \pi_{X_b}^{\text{tr}} \right)_* &= 0: \ A^*(X_b) \rightarrow A^*(X_b), \\
\left( \pi_{Y_b}^{\text{tr}} \circ \Psi_b \circ \pi_{X_b}^{\text{tr}} \circ \Lambda'_b \circ \pi_{Y_b}^{\text{tr}} - \pi_{Y_b}^{\text{tr}} \right)_* &= 0: \ A^*(Y_b) \rightarrow A^*(Y_b).
\end{align*}
\]

Applying once more [53, Proposition 3.2], this means that the expressions in parentheses are both nilpotent. Taking the largest of the nilpotence indices, we find there is an isomorphism

\[
\Psi_b: t(X_b) \rightarrow t(Y_b)(-m) \quad \text{in } M_{\text{rat}} \quad \forall \ b \in B^{\circ}.
\]

The theorem is proven.

\[\square\]

**Remark 3.18.** Concerning the smoothness assumption in Theorem 3.17, we add the following observation: if \( X \) and \( Y \) both have the expected dimension, then \( X \) is smooth if and only if \( Y \) is smooth. For \((n,k) = (7,7)\), this was observed in [12], the general case follows from [26, Lemma 3.10 and Remark 3.3].

4. Applications

This section presents some applications of the motivic relation of Theorem 3.17.

4.1. Calabi–Yau threefolds

A first consequence concerns the famous Calabi–Yau threefolds of Borisov [11, 12]:

**Corollary 4.1.** Let

\[
X := \text{Gr}(2, 7) \cap \mathbb{P}(U) \subset \mathbb{P}^{20}, \\
Y := \text{Pf} \cap \mathbb{P}(U^\perp) \subset (\mathbb{P}^{20})^\vee
\]

be smooth dimensionally transverse intersections, where \( \mathbb{P}(U) \subset \mathbb{P}^{20} \) is a codimension 7 linear subspace. Then \( X \) and \( Y \) are Calabi–Yau threefolds, and

\[
h(X) \cong h(Y) \quad \text{in } M_{\text{rat}}.
\]
Proof. Let $h^3(X) := h_{var}(X)$ (which is equal to $t(X)$ because $X$ is odd-dimensional), and similarly for $Y$. According to Theorem 3.17 with $(n, k) = (7, 7)$, there is an isomorphism $h^3(X) \cong h^3(Y)$. Both $X$ and $Y$ have Picard number 1, and so the result follows. □

4.2. Cubic threefolds and Fano threefolds of genus 8

Corollary 4.2. Let $Y$ be a general cubic threefold. There exists a prime Fano threefold $X$ of genus 8, and an isomorphism of Chow motives

$$h(X) \cong h(Y)$$

in $\mathcal{M}_{rat}$.

Moreover, the general prime Fano threefold of genus 8 arises in this way.

Proof. The general cubic threefold is Pfaffian [5], and so this is the case $(n, k) = (6, 5)$ of Theorem 3.17. The ‘moreover’ statement follows from work of Mukai [41]. □

Remark 4.3. In the set-up of Corollary 4.2, the varieties $X$ and $Y$ are actually birational [47], and the isomorphism of motives can be readily obtained by exploiting the specific form of the birationality [34]. However, the proof given here does not rely on the birationality.

4.3. Pfaffian cubic fourfolds

Corollary 4.4. Let $Y \subset \mathbb{P}^5$ be a general Pfaffian cubic fourfold. There exists a genus 8 $K3$ surface $X$ and an isomorphism of motives

$$h(Y) \cong h(X)(-1) \oplus \mathbb{L} \oplus \mathbb{L}(-4)$$

in $\mathcal{M}_{rat}$.

Moreover, the general genus 8 $K3$ surface arises in this way.

Proof. This is Theorem 3.17 with $(n, k) = (6, 6)$. The ‘moreover’ statement is work of Mukai [40]. □

Remark 4.5. Pfaffian cubic fourfolds and genus 8 $K3$ surfaces are also related on the level of derived categories [30, Theorem 10.4]. The relation of motives of Corollary 4.4 is well known, and is also valid for other cubic fourfolds with an associated $K3$ surface [14, Theorem 0.3]. Contrary to [14], however, the argument proving Corollary 4.4 is direct and geometric, and does not rely on derived category results.

4.4. Varieties with trivial Chow groups

The following is not a corollary of the main result. However, we include it because it fits in well here:

Proposition 4.6. Let $X$ be a smooth dimensionally transverse intersection

$$X := \text{Gr}(2, n) \cap H_1 \cap \cdots \cap H_k \subset \mathbb{P}_{\mathbb{Q}}^{(\frac{n}{2})-1},$$

where $H_j$ are hyperplanes. Assume $k \leq 2$. Then

$$A^i_{hom}(X) = 0 \quad \forall \ i.$$

Proof. It is well-known that $H^*_{tr}(X, \mathbb{Q}) = 0$ in this case [16, Propositions 2.3 and 2.4], and so Theorem 3.17 does not apply. However, one can understand the Chow groups of $X$ by using a straightforward geometric argument, inspired by [16].

Let $P \subset \mathbb{P}(V_n)$ be a fixed hyperplane, and consider (as in [16, Section 2.3]) the rational map

$$\text{Gr}(2, V_n) \dashrightarrow P.$$
sending a line in $\mathbb{P}(V_n)$ to its intersection with $P$. This map is resolved by blowing up a subvariety $\sigma_{11}(P) \cong \text{Gr}(2, n - 1)$, resulting in a morphism

$$\Gamma: \tilde{\text{Gr}} \to P$$

(where $\tilde{\text{Gr}} \to \text{Gr}(2, V_n)$ denotes the blow-up with center $\sigma_{11}(P)$).

Let $\tilde{X} \to X$ be the blow-up of $X$ with center $\sigma_{11}(P) \cap X$, and let us consider the morphism

$$\Gamma_X: \tilde{X} \to P,$$

obtained by restricting $\Gamma$.

In case $k = 1$ and $P$ is generic with respect to $X$, the morphism $\Gamma_X$ is a $\mathbb{P}^{n-3}$-fibration over $P$. It follows that $\tilde{X}$, and hence $X$, has trivial Chow groups.

In case $k = 2$, and $P$ chosen generically with respect to $X$, the morphism $\Gamma_X$ is generically a $\mathbb{P}^{n-4}$-fibration over $P$, and there are finitely many points in $P$ where the fiber is $\mathbb{P}^{n-3}$. Applying Theorem 2.1, this implies that $\tilde{X}$, and hence $X$, has trivial Chow groups.

\begin{remark}
In case $n = 6$ or 7, and $X \subset \text{Gr}(2, n)$ as in Proposition 4.6, it follows from HPD that the derived category of $X$ has a full exceptional collection [30, Corollaries 10.2 and 11.2]. Since it is known that varieties admitting a full exceptional collection have trivial Chow groups [37], this gives another proof of Proposition 4.6 for $n = 6$ or $n = 7$.
\end{remark}

4.5. Fano varieties with finite-dimensional motive

\begin{corollary}
Let $X$ be a smooth dimensionally transverse intersection

$$X := \text{Gr}(2, n) \cap H_1 \cap \cdots \cap H_k \subset \mathbb{P}^{(\frac{n}{2})-1},$$

where $H_j$ are hyperplanes. Assume either $k = 3$ and $n$ is even, or $k = 5$ and $n$ is odd, or $(n, k) = (8, 5)$. Then

$$A^*_{\mathbb{A}, J}(X) = 0,$$

and in particular $X$ has finite-dimensional motive (in the sense of [28]).
\end{corollary}

\begin{proof}
We first treat the cases $k = 3$ and $n$ is even, or $k = 5$ and $n$ is odd. As a first step, let us assume $X$ is such that the dual $Y \subset (\mathbb{P}^{(\frac{n}{2})-1})^\vee$ is smooth and dimensionally transverse, that is, $Y$ has dimension 1. Theorem 3.17 implies the isomorphism

$$h(X) \cong t(Y)(-m) \oplus \bigoplus \mathbb{I}(*) \quad \text{in } M_{\text{rat}}.$$

Since curves have finite-dimensional motive and injective Abel–Jacobi maps, this implies both statements for $X$.

Next, let $\mathcal{X} \to B$ denote the universal family of all smooth complete intersections of the type under consideration, and let $B^0 \subset B$ denote the Zariski open subset parametrizing smooth $X$ for which the dual $Y$ is a smooth curve. By definition of finite-dimensionality, the above means exactly that for all $b \in B^0$ one has vanishing

$$\text{Sym}^{b_{tr}} t(X_b) = 0 \quad \text{in } A^{b_{tr} - \dim X_b}(X_b^{\text{tr}} \times X_b^{\text{tr}}),$$

where the symmetric product of a motive is as in [28], and $b_{tr} := \dim H^{\dim X_b}(X_b, \mathbb{Q})$. But the projector defining $t(X_b)$ is generically defined, and so the spread lemma [63, Lemma 3.2] then implies that the vanishing (13) holds for all $b \in B$, that is, all $X_b$ have finite-dimensional motive.

To prove that $A^*_{\mathbb{A}, J}(X_b) = 0$ for all $X_b$, we observe that the above implies that for all $b \in B^0$, one has

$$\text{Niveau}(A^*(X_b)) \leq 1 \quad \text{(14)}$$

in the sense of [32]. This means that for each $b \in B^\circ$, there is a decomposition of the diagonal
\[ \Delta_{X_b} = \gamma^0_b + \cdots + \gamma^r_b \quad \text{in} \quad A^{\dim X_b}(X_b \times X_b), \]
where $\gamma^j_b$ is supported on $V^j_b \times W^j_b \subset X_b \times X_b$ and $\dim V^j_b + \dim W^j_b = \dim X_b + 1$. Using the Hilbert schemes argument of [62, Proposition 3.7] (cf. also [35, Proposition A.1] for the precise form used here), the $\gamma^j_b, V^j_b, W^j_b$ exist relatively, that is, one can find subvarieties $V^j, W^j \subset X$ with $\codim V^j + \codim W^j = \dim X_b - 1$, and cycles $\gamma^j$ supported on $V^j \times_{B^\circ} W^j$ such that
\[ \Delta_{X|b} = \gamma^0|_b + \cdots + \gamma^r|_b \quad \text{in} \quad A^{\dim X_b}(X_b \times X_b) \quad \forall \, b \in B^\circ. \]

Let $\tilde{\gamma}^j \in A^{\dim X_b}(X \times_B X)$ be cycles that restrict to $\gamma^j \in A^{\dim X_b}(X \times_B X)$, and let $\tilde{V}^j \times_B V^j$ be the support of $\tilde{\gamma}^j$. Given $b_1 \in B \setminus B^\circ$, it may happen that $\dim \tilde{V}^j|_{b_1} + \dim V^j|_{b_1}$ is larger than $\dim X_{b_1} + 1$. However, using the moving lemma, one can find representatives for $\gamma^j$ such that the supports verify the condition
\[ \dim \tilde{V}^j|_{b_1} + \dim V^j|_{b_1} = \dim X_{b_1} + 1, \]
that is, (14) holds for $X_{b_1}$.

We have ascertained that (14) holds for all $b \in B$. Letting the decomposition of the diagonal act on Chow groups, this shows that
\[ A^*_{A^j}(X_b) = 0 \quad \forall \, b \in B. \]

Finally, the argument for $(n, k) = (8, 5)$ is similar: in this case, $X$ is a Fano sevenfold and $Y$ is a Fano threefold. Since Fano threefolds have finite-dimensional motive, Theorem 3.17 implies the same for $X$. □

**Remark 4.9.** Corollary 4.8 improves on results of Tabuada, who established (using HPD results and non-commutative motives) Schur-finiteness for linear sections of $\Gr(2, 7)$ and $\Gr(2, 7)$ [54, Theorems 1.3 and 1.4].

### 4.6. Fano varieties satisfying Voevodsky’s conjecture

**Definition 4.10 [60].** Let $X$ be a smooth projective variety. A cycle $a \in A^i(X)$ is called smash-nilpotent if there exists $m \in \mathbb{N}$ such that
\[ a^m := a \times \cdots \times a = 0 \quad \text{in} \quad A^{mi}(X \times \cdots \times X). \]

Two cycles $a, a'$ are called smash-equivalent if their difference $a - a'$ is smash-nilpotent. We will write $A^i_{\circ}(X) \subseteq A^i(X)$ for the subgroup of smash-nilpotent cycles.

**Conjecture 4.11 [60].** Let $X$ be a smooth projective variety. Then
\[ A^i_{\num}(X) \subseteq A^i_{\circ}(X) \quad \text{for all} \, i. \]

**Remark 4.12.** It is known [4, Théorème 3.33] that Conjecture 4.11 for all smooth projective varieties implies (and is strictly stronger than) Kimura’s conjecture ‘all smooth projective varieties have finite-dimensional motive’ [28].

**Corollary 4.13.** Let $X$ be a smooth dimensionally transverse intersection
\[ X := \Gr(2, n) \cap H_1 \cap \cdots \cap H_k \subset \mathbb{P}^{(2) - 1}, \]
where $H_j$ are hyperplanes and $k \in \{3, 4\}$ if $n$ is even and $k \in \{5, 6\}$ if $n$ is odd. Then Voevodsky’s conjecture is true for $X$. Moreover,

$$A^j_{A^j}(X) = 0 \quad \forall j \neq \frac{1}{2} \dim X + 1.$$ 

**Proof.** In a first step, we assume $X$ is such that the dual $Y \subset Pf$ on the Pfaffian side is smooth and dimensionally transverse. Theorem 3.17 applies and gives a split injection

$$h(X) \hookrightarrow h(Y)(-m) \oplus \bigoplus 1(*) \text{ in } \mathcal{M}_{\text{rat}}.$$ 

Since $\dim Y \leq 2$, this implies

$$\text{Niveau}(A^*(X)) \leq 2$$

(in the sense of [32]), that is, there is a decomposition of the diagonal

$$\Delta_X = \gamma_0 + \cdots + \gamma_r \text{ in } A^{\dim X}(X \times X),$$

where $\gamma_j$ is supported on $V_j \times W_j \subset X \times X$ and $\dim V_j + \dim W_j = \dim X + 2$. Looking at the action on Chow groups, this decomposition implies that homological and algebraic equivalence coincide on $X$. Since it is known that $A^*_{\text{alg}}(X) \subset A^*\otimes(X)$ [60, 61], this implies that Conjecture 4.11 holds for $X$.

Next, let us extend to arbitrary $X$. Let $X \to B$ denote the universal family, and $B^\circ \subset B$ the Zariski open where $X$ and its dual $Y$ are simultaneously smooth and dimensionally transverse. The first step above shows that $\text{Niveau}(A^*(X_b)) \leq 2$ for all $b \in B^\circ$. As in the proof of Corollary 4.8, this property extends from $B^\circ$ to $B$ and we find that

$$\text{Niveau}(A^*(X_b)) \leq 2 \quad \forall b \in B.$$ 

In particular, Voevodsky’s conjecture holds for all $X_b$ in the family.

Finally, the vanishing of $A^j_{A^j}(X), j \neq \frac{1}{2} \dim X + 1$ is a straightforward consequence of the decomposition (15). \hfill \Box

**Remark 4.14.** In [9, Theorem 1.7], the special case of Corollary 4.13 where $n = 6$ or $n = 7$ was proven (in [9], the restriction to $n = 6, 7$ is necessary because the argument relies on HPD for Grassmannians via non-commutative motives). The argument proving Corollary 4.13 is more elementary, in that we do not rely on derived category arguments at all.

### 4.7. Fano varieties satisfying the Hodge conjecture

**Corollary 4.15.** Let $X$ be a smooth dimensionally transverse intersection

$$X := \text{Gr}(2, n) \cap H_1 \cap \cdots \cap H_k \subset \mathbb{P}^{\binom{n}{2}}-1,$$

where $H_j$ are hyperplanes and either $n$ is even and $k \in \{3, 4, 5\}$ or $n$ is odd and $k \in \{5, 6\}$. Then

$$A^j_{h,\text{om}}(X) = 0 \quad \forall j > \frac{1}{2} \dim X + 3,$$

and in particular the Hodge conjecture is true for $X$.

**Proof.** First, let us assume $X$ is such that the dual $Y \subset Pf$ on the Pfaffian side is smooth and dimensionally transverse. Theorem 3.17 applies and gives a split injection

$$h(X) \hookrightarrow h(Y)(-m) \oplus \bigoplus 1(*) \text{ in } \mathcal{M}_{\text{rat}}.$$ 

Since $\dim Y \leq 3$, this implies

$$\text{Niveau}(A^*(X)) \leq 3.$$
(in the sense of [32]), that is, there is a decomposition of the diagonal
\[ \Delta_X = \gamma_0 + \cdots + \gamma_r \] in \( A^{\dim X}(X \times X) \),
where \( \gamma_j \) is supported on \( V_j \times W_j \subset X \times X \) and \( \dim V_j + \dim W_j = \dim X + 3 \).

Next, let us extend to arbitrary \( X \). Let \( X \to B \) denote the universal family, and \( B^o \subset B \) the Zariski open where \( X \) and its dual \( Y \) are simultaneously smooth and dimensionally transverse. The above shows that \( \Niveau(A^i(X_b)) \leq 3 \) for all \( b \in B^o \). As in the proof of Corollary 4.8, this property extends from \( B^o \) to \( B \) and we find that
\[ \Niveau(A^i(X_b)) \leq 3 \quad \forall \ b \in B. \]

It is well known (cf., for instance, [32]) that this implies the vanishing of \( A^j_{\operatorname{hom}}(X) \) in the indicated range, as well as the truth of the Hodge conjecture for \( X_b \). \( \square \)

In the special case \((n, k) = (6, 3)\), one can say more:

**Corollary 4.16.** Let \( X \) be a general intersection
\[ X := \operatorname{Gr}(2, 6) \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^{14}, \]
where \( H_j \) are hyperplanes. Then \( X \) is a Fano fivefold, and the generalized Hodge conjecture is true for \( X^m \) for all \( m \in \mathbb{N} \).

**Proof.** \( X \) being general, the dual \( Y \subset \operatorname{Pf} \) on the Pfaffian side is an elliptic curve. The isomorphism of motives of Theorem 3.17 implies there is an isomorphism of Hodge structures
\[ H^j(X^m, \mathbb{Q}) \cong H^{j-4m}(Y^m, \mathbb{Q})(-2m) \oplus \bigoplus H^*(Y^{m-1}, \mathbb{Q})(*) \oplus \cdots \oplus \bigoplus \mathbb{Q}(*) . \]
Since this isomorphism is also compatible with the coniveau filtration [58, Proposition 1.2], one is reduced to proving the generalized Hodge conjecture for powers of an elliptic curve \( Y \). This is known due to work of Abdulali [1, Section 8.1] (cf. also [59, Corollary 3.13]). \( \square \)

### 4.8. Fano eightfolds of K3 type

**Corollary 4.17.** Let \( X \) be a general complete intersection\n\[ X := \operatorname{Gr}(2, 8) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \subset \mathbb{P}^{27}, \]
where \( H_j \) are hyperplanes. Then there exists a quartic K3 surface \( S \) and an isomorphism of motives
\[ h(X) \cong h(S)(-3) \oplus \bigoplus 1(*) \quad \text{in } \mathcal{M}_{\text{rat}}. \]

**Proof.** This is the case \((n, k) = (8, 4)\) of Theorem 3.17. \( \square \)

**Corollary 4.18.** For any \( 3 \leq \rho \leq 22 \), there exist Fano eightfolds \( X \) as in Corollary 4.17 with
\[ \dim \operatorname{Im}(A^4(X) \to H^8(X, \mathbb{Q})) = \rho . \]
For \( \rho \geq 21 \), \( X \) has finite-dimensional motive.

**Proof.** Inside the moduli space \( \mathcal{F}_3 \) of genus 3 K3 surfaces, let \( \mathcal{F}_3^\rho \subset \mathcal{F}_3 \) denote the locus of K3 surfaces which are Pfaffian quartics and for which one of the duals \( X \subset \operatorname{Gr}(2, 8) \) is smooth and dimensionally transverse. As the general quartic K3 surface is Pfaffian [5], \( \mathcal{F}_3^\rho \) is a dense open subset. Let \( \mathcal{F}_3^{\rho, 0} \subset \mathcal{F}_3 \) denote the locus parametrizing K3 surfaces with Picard number \( \geq \rho \).
The Noether–Lefschetz theory for K3 surfaces [21, Chapter 17] implies that the locus \( F_{\geq \rho} \) has dimension \( 20 - \rho \) and is analytically dense in \( F_{\geq \rho - 1} \), and so in particular for each \( 1 \leq \rho \leq 20 \), the locus \( F_{\geq \rho} \) meets the open \( F_{3} \). Given a K3 surface in \( F_{\geq \rho} \cap F_{3} \), let \( X \subset \text{Gr}(2,8) \) be a dual eightfold. The isomorphism of Corollary 4.17 induces an isomorphism

\[
H^8_{\text{var}}(X, \mathbb{Q}) \cap \text{Im} \left( A^4(X) \to H^8(X, \mathbb{Q}) \right) \cong H^2_{\text{var}}(Y, \mathbb{Q}) \cap \text{Im} \left( A^1(Y) \to H^2(Y, \mathbb{Q}) \right),
\]

and so

\[
\dim \text{Im} \left( A^4(X) \to H^8(X, \mathbb{Q}) \right) = \rho + 2.
\]

The last statement follows from Corollary 4.17 plus the fact that K3 surfaces with Picard number \( \geq 19 \) have finite-dimensional motive [45]. □

Remark 4.19. In particular, Corollary 4.18 with \( \rho = 22 \) gives examples of \( \rho \)-maximal varieties, that is, \( 2m \)-dimensional varieties \( X \) with the property that

\[
\dim \text{Im} \left( A^m(X) \to H^{2m}(X, \mathbb{Q}) \right) = \dim H^{m,m}(X, \mathbb{C}).
\]

This notion is discussed in [6, Section 8].

4.9. Fano varieties of Calabi–Yau type with infinite-dimensional Griffiths group

Definition 4.20. Let \( X \) be a smooth projective variety. The Griffiths groups of \( X \) are defined as

\[
\text{Grif}^j(X) := \frac{z^j_{\text{hom}}(X)}{z^j_{\text{alg}}(X)},
\]

where \( z^j_{\text{hom}}(X) \) and \( z^j_{\text{alg}}(X) \) denote the groups of codimension \( j \) algebraic cycles on \( X \) that are homologically trivial, respectively, algebraically trivial.

Corollary 4.21. Let \( X \) be a general complete intersection

\[
X := \text{Gr}(2,10) \cap H_1 \cap \cdots \cap H_5 \subset \mathbb{P}^{44},
\]

where the \( H_j \) are Plücker hyperplanes. Then \( X \) is a Fano 11-fold and the Griffiths group \( \text{Grif}^6(X, \mathbb{Q}) \) is infinite dimensional.

Proof. For sufficiently general hyperplanes \( H_j \), both \( X \) and its dual

\[
Y := \text{Pf} \cap \mathbb{P}(U^\perp) \subset (\mathbb{P}^{44})^\perp
\]

are smooth and dimensionally transverse. Theorem 3.17 with \( (n,k) = (10,5) \) then gives an isomorphism of motives

\[
h(X) \cong t(Y)(-4) \oplus \mathbb{I}(*) \quad \text{in } M_{\text{rat}}.
\]

Taking Griffiths groups, this implies

\[
\text{Grif}^j(X, \mathbb{Q}) = 0 \quad \forall j \neq 6, \quad \text{Grif}^6(X, \mathbb{Q}) \cong \text{Grif}^2(Y, \mathbb{Q}).
\]

Here \( Y \) is a quintic threefold, and the general quintic threefold arises in this way [5, Proposition 8.9]. The corollary thus follows from Clemens’ celebrated result that \( \text{Grif}^2(Y, \mathbb{Q}) \) is infinite dimensional for a general quintic threefold \( Y \) [15]. □

Corollary 4.22. Let \( Z \) be an intersection

\[
Z := \left( \text{Gr}(2,10) \times \mathbb{P}^4 \right) \cap H_{(1,1)} \subset \mathbb{P}^{44} \times \mathbb{P}^4,
\]
where \( H_{1,1} \) is a general bidegree \((1,1)\) hypersurface. Then \( Z \) is a Fano 19-fold and the Griffiths group \( \text{Grif}^{10}(Z)_{\mathbb{Q}} \) is infinite-dimensional.

**Proof.** Using the Cayley trick in the form of Theorem 2.1, we find there is an isomorphism of motives

\[
h(Z) \oplus \bigoplus (-4) \cong h(X) \oplus \bigoplus (-4) \quad \text{in } \mathcal{M}_{\text{rat}},
\]

where \( X \) is an intersection of \( \text{Gr}(2,10) \) with five general hyperplanes. Taking Griffiths groups, this implies in particular

\[
\text{Grif}^j(Z)_{\mathbb{Q}} = 0 \quad \forall \ j \neq 10, \quad \text{Grif}^{10}(Z)_{\mathbb{Q}} \cong \text{Grif}^6(X)_{\mathbb{Q}}.
\]

The corollary now follows from Corollary 4.21. \( \square \)

**Remark 4.23.** Following up on Clemens’ famous result about the quintic threefold, infinite-dimensionality of the Griffiths group has been proven in [3] for the cubic sevenfold, and in [17] for certain other Fano varieties of Calabi–Yau type. The results in [3, 17] are more difficult and remarkable than those of Corollary 4.21 and 4.22: indeed, the varieties of [3, 17] correspond to a non-commutative Calabi–Yau threefold (that is, the interesting part of the derived category is a CY3 category without geometric incarnation), while the varieties of Corollary 4.21 and 4.22 are (motivically and categorically) related to an honest Calabi–Yau threefold.

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