Spatial correlations of primordial density fluctuations in the standard cosmological model

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We revisit the origin of structures problem of standard Friedmann-Robertson-Walker cosmology to point out an unjustified approximation in the prevalent analysis. We follow common procedures in statistical mechanics to revise the issue without the disputed approximation. Our conclusions contradict the current wisdom and reveal an unexpected scenario for the origin of primordial cosmological structures. We show that standard physics operating in the cosmic plasma during the radiation dominated expansion of the universe produce at the time of decoupling scale invariant density anisotropies over cosmologically large comoving volumes. Scale invariance is shown to be a direct consequence of the causality constraints imposed by the short FRW comoving horizon at decoupling, which strongly suppress the power spectrum of density fluctuations with cosmologically large comoving wavelength. The global amplitude of these cosmological density anisotropies is fixed by the power spectrum in comoving modes whose wavelength is shorter than the causal horizon at the time and can be comparable to the amplitude of the primordial cosmological inhomogeneities imprinted in the cosmic microwave background radiation.

I. INTRODUCTION.

The cosmic microwave background radiation (CMBR) is a fossil evidence of the early history of the expanding and cooling Friedmann-Robertson-Walker (FRW) universe. Indeed, this radiation is a high-resolution picture of the cosmological density field at the instant of recombination, when the temperature of the universe was $T_{\text{eq}} \sim 1$ eV and this radiation last scattered and decoupled from the cosmic plasma. At that instant the size of the universe was $10^{-3}$ of its current size. The red shifted radiation observed at present is extremely homogeneous and isotropic over the sky, with a perfect blackbody spectrum of temperature
$T = 2.725 \text{ K} \sim 10^{-3} \text{ eV}$ up to tiny fluctuations $\Delta T/T \lesssim 10^{-5}$. Encoded in the statistical correlations of these tiny temperature anisotropies are the statistical features of a primordial seed of very small cosmological density anisotropies in the, otherwise, homogeneous and isotropic universe at the instant of decoupling. Current cosmological models work within the paradigm that large scale mass structures in the distribution of matter in the present universe are the result of gravitational evolution operating since the time of matter radiation equality on that primordial seed (for a recent claim of observational evidence of this correlation see [5]).

The most characteristic statistical feature of the primordial seed of density anisotropies at decoupling is its scale invariance over cosmologically large comoving scales [6]: the variance $(\Delta M_V)^2$ of density anisotropies over volumes $V$ of cosmological comoving size $L$ is proportional to the area $S \sim L^2$ of the surface that bounds the considered region,

$$(\Delta M_V)^2 \simeq \kappa^2 \rho_{\text{eq}}^2 H_{\text{eq}}^{-4} S.$$  \hfill (1)

The dimensionless numerical factor $\kappa \sim 10^{-5}$ measures the global amplitude of the primordial density anisotropies in terms of the cosmological density at decoupling $\rho_{\text{eq}} = \frac{3H_{\text{eq}}^2}{8\pi G}$ and the comoving causal horizon $H_{\text{eq}}^{-1}$ at that time. Comoving scales of cosmological size are much larger than this horizon,

$$H_{\text{eq}}^{-1} \ll L \lesssim H_0^{-1}. \hfill (2)$$

The upper bound $H_0^{-1} \approx 10^4 H_{\text{eq}}^{-1}$ is the comoving size of the present causal horizon. For the sake of simplicity we have chosen the scale factor of the expansion of the universe at decoupling $a(t_{\text{eq}}) = 1$ as reference.

Scale invariant macroscopic density anisotropies produce a gravitational potential independent of the scale $L$,

$$(\Delta \phi|_L)^2 = G^2 \frac{(\Delta M_V)^2}{L^2} \simeq \kappa^2.$$ \hfill (3)

This feature is observable, through the Wolfe-Sachs effect, as the renowned Harrison-Zeldovich plateau in the spectrum of temperature anisotropies in the CMBR at angular scales larger than the angle subtended on the sphere of last scattering by the comoving horizon at decoupling.
\[ \frac{\Delta T(\theta)}{T} \simeq \frac{1}{3} \Delta \phi \simeq \kappa, \quad \text{at } \theta \gg 1^\circ. \quad (4) \]

The amplitude of temperature fluctuations over large angular scales is roughly constant.

A. The prevalent analysis.

The pattern of scale invariant primordial density anisotropies at decoupling over comoving volumes of size much larger than the causal horizon at that time has puzzled cosmologists since long ago. Theoretical analysis reproduced in textbooks as well as research papers [8, 9, 10] seem to conclusively prove that the pattern cannot be reproduced in the (otherwise very successful) standard FRW cosmology without violating causal correlations over scales beyond the horizon. The argument goes as follows (see, for example, Eq. 9.1, 9.2 and 9.3 in [11]).

The variance of macroscopic density anisotropies at decoupling over volumes \( V \) of comoving size \( L \), which is related to their power spectrum \( P(k) \) by the equality (see section II.B)

\[
(\Delta M_V)^2 = \rho_{eq}^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} P(k) \left| \int_V d^3 \vec{x} \ e^{-i\vec{k} \cdot \vec{x}} \right|^2,
\]

is approximated by the expression:

\[
(\Delta M_V)^2 \simeq \rho_{eq}^2 P(k \sim 1/L) V,
\]

after noticing that

\[
\left| \int_V d^3 \vec{x} \ e^{-i\vec{k} \cdot \vec{x}} \right|^2 \simeq V^2 \quad \text{for } k \lesssim 1/L,
\]

\[
\left| \int_V d^3 \vec{x} \ e^{-i\vec{k} \cdot \vec{x}} \right|^2 \ll V^2 \quad \text{for } k \gg 1/L.
\]

Approximation relies on the suppression of the geometric factor for modes \( k \gg 1/L \) to estimate that the contribution of these modes to macroscopic inhomogeneities over comoving volumes of size \( L \) is negligible and conclude that the dominant contribution comes necessarily from density fluctuations in modes \( k \lesssim 1/L \) with comparable comoving wavelength.
According to expression (6) the pattern (1,2) of scale invariant primordial density anisotropies over cosmologically large comoving volumes corresponds univocally to a power spectrum at decoupling

$$p_*(k) \simeq (\kappa^2 H_{eq}^{-4}) k, \quad \text{for} \quad k \ll H_{eq}, \quad (8)$$

which decreases linearly over the range of modes with cosmologically short comoving momenta. To check this inference plug (8) into (6) to reproduce the scale invariant pattern (12): 

$$\left(\Delta M_V\right)^2 \simeq \rho_{eq}^2 p_*(k \sim 1/L) V \simeq \kappa^2 \rho_{eq}^2 H_{eq}^{-4} \frac{1}{L} V \sim \kappa^2 \rho_{eq}^2 H_{eq}^{-4} S.$$ 

Inference (8) is bewildering within the theoretical framework of standard FRW cosmology. The power spectrum of density anisotropies that standard physics operating during the radiation dominated expansion of the universe can produce at decoupling is known to be suppressed much strongly than linearly (8) over the range \(k \ll H_{eq}\). The suppression is known to be at least quartic [10],

$$p(k) \leq p(k \sim H_{eq}) \left(\frac{k^4}{H_{eq}^4}\right), \quad \text{for} \quad k \ll H_{eq}. \quad (9)$$

This feature is forced by the constrain of global mass (energy) conservation and by the causal structure of the FRW universe, in which sites separated by cosmologically large comoving distances have not ever been in causal contact before decoupling. Hence, (9) applies to the power spectrum of density anisotropies generated by any mechanism operating in the cosmic baryon-photon plasma before decoupling as long as it does not violate causality.

A direct evaluation, according to approximation (6), of the variance of cosmological density anisotropies at decoupling that correspond to the power spectrum (8) leads to

$$\left(\Delta M_V\right)^2 \simeq \left(\Delta M_{V_{eq}}\right)^2 \frac{H_{eq}^{-1}}{L}. \quad (10)$$

According to this estimation, the variance of density anisotropies that standard physics produce at decoupling over cosmologically large comoving volumes \(V\) of size \(L \gg H_{eq}^{-1}\) is predicted to be suppressed by a factor \(H_{eq}^{-1}/L\) with respect to the variance of the anisotropies \(\left(\Delta M_{V_{eq}}\right)^2\) that the same physics generate at that time over comoving volumes \(V_{eq}\) of the size of the causal horizon. The manifest disagreement between the theoretical prediction (10) and the observed scale invariant pattern (1) is known in the literature as the origin of structures problem of standard FRW cosmology.
In order to solve this problem cosmologists have worked out during the last twenty years a quite unexpected cosmological paradigm: a period of exponential inflation when the temperature of the universe was close to the Planck scale would have supposedly preceded the standard radiation dominated expansion of the universe \[12\]. It is claimed that a very early period of cosmological inflation would naturally explain the exceptional conditions of the universe at the beginning of the standard FRW expansion: namely, its huge size and its flatness, as well as its homogeneity and isotropy over cosmologically large comoving scales. In addition, the density anisotropies present in the very first instants of the history of the universe would be stretched by the inflation to comoving cosmological scales. At decoupling these anisotropies would have a characteristic linear power spectrum

$$P_{\text{inflat}}(k) \sim A_k, \quad \text{for} \quad k \ll H_{eq}$$

over the range of cosmologically short comoving momenta. According to inference \[8\] these anisotropies would be scale invariant, \((\Delta M_V)^2_{\text{inflat}} \simeq \rho_{eq}^2 P_{\text{inflat}}(k \sim 1/L) V \simeq \rho_{eq}^2 A S\). The global amplitude of the anisotropies left by inflation is fixed by the parameter \(A\). In most models of inflation, \(A\) is an adjustable parameter that depends on the scale at which inflation takes place. This parameter is tuned to the value \(A \simeq \kappa^2 H_{eq}^{-4}\) by additional constrains on the inflationary setup.

The widely accepted conclusion of the analysis that we have briefly reviewed is that primordial scale invariant cosmological anisotropies \[12\] are a pristine signature of inflation, which cannot have been altered by anisotropies \[10\] generated by standard physics during the radiation dominated expansion of the universe. Inflation is currently regarded by most cosmologists as the natural solution to the origin of structures problem of FRW cosmology, even though multiple technical problems (mostly related with the need to causally connect two temperature scales \(T_{eq} \sim 1\ eV\) and \(T_{\text{Planck}} \sim 10^{18}\ \text{GeV}\) separated by 28 orders of magnitude) have prevented the inflationary paradigm to get successfully implemented in models of particle cosmology.

B. An unjustified approximation.

The motivation of this paper is to contend that the above analysis of the origin of structures problem of standard FRW cosmology is flawed by an unjustified approximation. We
point out that the estimation (10) of the variance of cosmological density anisotropies produced at decoupling by standard physics (9) is deceptive and leads to erroneous conclusions. The estimation (10) is based on approximation (6), which relies on the suppression of the geometric factor (7) in modes $k \gg 1/L$ in order to neglect the contribution of these modes to the integral expression (5) for the variance of density anisotropies over comoving volumes of size $L$. According to estimation (10), the suppression of the geometric factor in modes $k \gg 1/L$ implies that the largest contribution to the integral expression (5) must come from modes $k \lesssim 1/L$. We notice that this argument is invalid when the considered comoving volume of integration is cosmologically large $L \gg H_{eq}^{-1}$, because the power spectrum (9) of density anisotropies left at decoupling by standard physics is greatly enhanced in modes $k \approx H_{eq} \gg 1/L$ and suppressed in modes $k \approx 1/L$.

$$P(k \sim 1/L) \lesssim P(k \sim H_{eq}) \left( \frac{H_{eq}^{-1}}{L} \right)^4, \quad \text{for} \quad L \gg H_{eq}^{-1}. \quad (12)$$

In order to get a reliable estimation of the contribution of modes $k \gg H_{eq}$ to the variance (5) of density anisotropies produced at decoupling by standard physics (9) over cosmologically large comoving volumes of size $L \gg H_{eq}^{-1}$, we must compare the suppression of the geometric factor (7) in these modes against their enhanced power spectrum (12).

Indeed we have proven in [13], Section III, that approximation (6) does not give a correct estimation of the variance (5) of cosmological density anisotropies whenever the power spectrum $P(k) \sim k^n$ decreases faster than linearly, $n \geq 1$, over the range $k \ll H_{eq}$ of comoving modes with cosmologically large comoving wavelength. For these power spectra the largest contribution to the variance of cosmological density inhomogeneities comes from modes $k \gtrsim H_{eq}$ with comoving wavelength within the horizon, while the contribution from modes $k \lesssim 1/L$ with cosmologically large comoving wavelength is negligible. Approximation (6) carelessly neglects the contribution of modes with comoving wavelength within the horizon to the variance of density anisotropies over cosmologically large comoving volumes. This forgotten contribution accounts for random density fluctuations at the boundary surface of the cosmological volume of integration. Once this contribution is taken into account the variance (5) of cosmological density anisotropies associated to the power spectra (9) is shown to be scale invariant,
\[(\Delta M_V)^2 \simeq (\Delta M_{\text{eq}})^2 \left( \frac{L}{H_{\text{eq}}^{-1}} \right)^2, \quad \text{for} \quad L \gg H_{\text{eq}}^{-1}. \] (13)

This theoretical prediction mends the flawed estimation (10). In next Section II, subsections II.B and II.C, we prove and discuss this prediction in clear physical terms.

In summary, we claim that standard physics operating during the radiation dominated expansion of the universe generate at decoupling scale invariant density anisotropies over cosmologically large comoving volumes. The cosmological anisotropies are produced by random density fluctuations with comoving wavelength shorter than the comoving causal horizon at the time. According to prediction (13), the pattern (12) of scale invariant primordial cosmological anisotropies is not bewildering in the framework of standard FRW cosmology. On the contrary, this pattern is the clear signature of the strong suppression of the power spectrum of standard physics (9) in modes with cosmologically large comoving wavelength. Scale invariance is dictated by the causality constraints of standard FRW cosmology. This conclusion obviously contradicts the current wisdom in cosmology, expressed in the flawed Eq. (10).

Equation (13) also proves that the inference (8) of a linear power spectrum of primordial cosmological density anisotropies at the time of decoupling is absolutely unjustified. Scale invariance (12) of primordial cosmological density anisotropies at decoupling only implies that their power spectrum decreases faster than linearly over the range of cosmologically short comoving momenta. The power spectrum (9) fulfills this constrain.

C. A revised analysis.

The global amplitude of the scale invariant cosmological density anisotropies (13) produced at decoupling by standard physics operating during the radiation dominated expansion of the universe is fixed by their power spectrum \(P(k)\) in modes \(k \gtrsim H_{\text{eq}}\) with comoving wavelength shorter than the causal horizon at that time. Hence, their amplitude is proportional to the amplitude \((\Delta M_{\text{eq}})^2 \simeq \rho_{\text{eq}}^2 P(\sim H_{\text{eq}}) V_{\text{eq}}\) of the inhomogeneities that the same physics produce over comoving volumes \(V_{\text{eq}}\) of the size of this causal horizon. A comparison of (13) with (1) implies that it is necessary

\[(\Delta M_{\text{eq}}) \simeq \kappa \rho_{\text{eq}} V_{\text{eq}} \simeq 10^{-5} \rho_{\text{eq}} V_{\text{eq}}, \] (14)
for standard physics operating during the radiation dominated expansion of the universe to produce at decoupling scale invariant cosmological density anisotropies with global amplitude comparable to that of the primordial scale invariant density inhomogeneities imprinted in the CMBR. This conclusion reveals an unexplored, but feasible, scenario for the origin of primordial cosmological structures within the framework of FRW cosmology [14]. This scenario had been previously disregarded in the literature, in favour of the inflationary paradigm, due to the flawed estimation [10].

During the radiation dominated expansion of the universe the cosmic plasma is a dense and opaque soup of coupled baryons and photons in thermal equilibrium. Density (pressure) waves can propagate in this fluid with sound velocity \( c_s \approx \sqrt{\frac{1}{3}} \) (in natural units). These waves homogenize the cosmic density field, eroding all random density fluctuations with comoving wavelength much shorter than the sound causal horizon at the time. Obviously, causality constrains prevents this mechanism to operate over comoving distances larger than the sound causal horizon. Hence, the power spectrum of the density anisotropies left by standard physics in the cosmic plasma at decoupling is peaked at \( k \approx H_{eq} \) and suppressed \( P(k) \lesssim P(k \sim H_{eq}) \left( \frac{H_{eq}}{k} \right)^3 \) over modes \( k \gg H_{eq} \) by the action of these non-linear wave phenomena. We have discussed above [9] that the power spectrum of the density anisotropies left by standard physics is also suppressed over modes \( k \ll H_{eq} \) because of causal constrains. Such power spectrum corresponds to a pattern of macroscopic density anisotropies \( (\Delta M_V)^2 \approx (\Delta M_{V_{eq}})^2 \left( \frac{V_{eq}}{V} \right)^2 \) over volumes of comoving size \( L \lesssim H_{eq}^{-1} \), which turns scale invariant [13] over volumes of comoving size \( L \gg H_{eq}^{-1} \). We will show in next Section II, subsection II.E, that the amplitude [14] is roughly given by

\[
(\Delta M_{V_{eq}})^2 \approx 2 \langle [\rho(\vec{x})]^2 - \rho_{eq}^2 \rangle V_{eq}^2 = 2 \rho_{eq}^2 \left( \int \frac{d^3k}{(2\pi)^3} P(k) \right) V_{eq}^2,
\]

where \( \rho(\vec{x}) \) is the cosmological density field at decoupling. The dimensionless factor can be estimated by \( \int \frac{d^3k}{(2\pi)^3} P(k) \approx P(k \sim H_{eq}) H_{eq}^3 \). Thus, Eq. (14) is satisfied if

\[
P(k \sim H_{eq}) H_{eq}^3 \approx \kappa^2 \approx 10^{-10}.
\]

The dimensionless factor \( \int \frac{d^3k}{(2\pi)^3} P(k) \) that fixes [15] the global amplitude of the cosmological density anisotropies at decoupling [13] is a measure also of the entropy of the fluctuations. If the radiation dominated expansion of the universe is adiabatic and the en-
tropy of the density fluctuations is conserved, then the global amplitude of the cosmological
density anisotropies at decoupling is set by the dynamics of the universe at a much earlier
stage. We discuss condition (13) in further detail below in Section II, subsection II.E.

The implications of this revised analysis are far reaching. The standard FRW cosmolog-
cal framework that we are discussing here assumes, implicitly, that the exceptional initial
conditions at the beginning of the radiation dominated expansion of the universe (namely, its
huge size and flatness, as well as its homogeneity and isotropy over cosmologically large co-
moving scales) have been set by some unspecified mechanism (e.g. inflation) at some earlier
stage. According to the prevalent analysis of the origin of primordial cosmological struc-
tures, see subsection I.A, the mechanism that fixes the initial homogeneous and isotropic
conditions of the universe needs to be tuned to seed at once density fluctuations with cosmo-
logically large comoving wavelength and the right amplitude, see Eq. (11). The difficulties
to get the mechanism appropriately tuned have an obvious origin: cosmological homogene-
ity and isotropy means, by definition, that all density fluctuations with cosmologically large
comoving wavelength should be negligibly small. We realize now that, according to Eq.

- The primordial cosmological density anisotropies (12) have not to be necessarily
  seeded by the same mechanism that fixes the initial homogeneous and isotropic
  conditions at the beginning of the standard FRW expansion of the universe. The
  anisotropies may be seeded later on, during the radiation dominated expansion of
  the universe. Thus, we should consider as feasible models of inflation (or any other
  mechanism) that can fix the initial size, flatness, homogeneity and isotropy of the
  FRW universe, and leaves negligibly small density fluctuations with cosmologically
  large wavelength. Such models should be easier to implement in models of particle
  cosmology. The price to pay is obvious: the physics of such a mechanism would not
  be readily testable on the spectrum of temperature anisotropies in the CMBR.

- Scale invariant primordial density anisotropies (12) over cosmologically large comov-
ing volumes are not necessarily a pristine signature of relic inhomogeneities left over
by the mechanism that fixed the initial conditions at the beginning of the radiation
dominated expansion of the universe (e.g. inflation). Even in the case that relic scale
invariant cosmological inhomogeneities were left by that mechanism at the beginning
of the radiation dominated expansion, standard physics operating later on produce scale invariant cosmological anisotropies that are overlaid upon them by the time of decoupling.

II. SCALE INVARIANT DENSITY ANISOTROPIES.

The main aim of this paper is to prove that the characteristic scale invariance of the primordial cosmological density anisotropies at decoupling \([1,2]\) is an unequivocal signature of the standard FRW cosmology. In the standard cosmology the causal comoving horizon grows monotonically with cosmic time. Comoving scales of cosmological size are much larger than the comoving causal horizon at decoupling. The short comoving causal horizon prevents the generation of density fluctuations with cosmologically large comoving wavelength during the radiation dominated expansion of the universe \([9]\). As a consequence the dominant contribution to density anisotropies at decoupling over cosmologically large comoving volumes comes from random density fluctuations with comoving wavelength shorter than the horizon. This contribution corresponds to random density fluctuations at the boundary surface of the volume of integration and, hence, the variance of these anisotropies grows with the area of the boundary surface. Below we compare the geometric dependence of the variance of the primordial cosmological density anisotropies on the boundary area \([1]\) with the dependence of the variance of thermal anisotropies in systems in equilibrium on some power \(\beta \in [1, 2]\) of the volume of integration, \((\Delta M_V)^2 \sim V^\beta\).

It has been formally proved \([15]\) that in any homogeneous and isotropic statistical system the variance \((\Delta M_V)^2\) of the spatial anisotropies of any extensive magnitude \(M_V\) over macroscopically large, but finite, volumes \(V\) must grow with their size \(L\) as \((\Delta M_V)^2 \sim L^\alpha\), with \(\alpha \geq 2\). The derivation of equation (26) below is a brief proof of this result. Scale invariant statistical systems like \([1]\), for which \(\alpha = 2\), have been dubbed for this reason superhomogeneous \([16]\). We can say that, in this sense, scale invariant systems are the most ordered homogeneous and isotropic statistical systems and thermal systems in equilibrium, for which \(\alpha \geq 3\), are more disordered than scale invariant systems.

Scale invariant anisotropies are known in mathematics, for example, in the distribution over the complex plane of zeroes of statistical analytic functions \([17]\). In physics, scale invariant anisotropies happen generically in homogeneous and isotropic statistical systems with
very long relaxation times to thermal equilibrium. The reader familiar with the arguments that justify Eq. \(9\) will appreciate the following illustrative example.

A. A simple example of scale invariant anisotropies.

Let consider an homogeneous and isotropic distribution of a large number of individuals over a very large (infinite) domain \(\Omega \subset \mathbb{R}^3\) and assume that this initial distribution of individuals does not contain random density fluctuations with wavelength longer than the shortest accessible resolution scale \(d\).

We now let each individual to walk randomly and assume the same statistical features for the individual random walks everywhere in \(\Omega\). The typical distance \(D(t) \equiv \langle |\vec{r}(t) - \vec{r}_0|^2 \rangle^{1/2}\) between the actual position of each individual at time \(t\) with respect to its original position grows monotonically with the time interval \(D(t) = \sqrt{\gamma} t^{1/2}\) as a function determined by the specific properties of the stochastic process. Let us call the distance \(D(t)\) a particle horizon: at any instant each individual is constrained to be located within a finite region of typical size \(D(t)\) around its original position. Let us also assume that we have waited enough, such that \(D(t) \gg d\).

Let us denote by \(N_V\) the number of individuals contained within a macroscopic volume \(V \subset \Omega\) at time \(t\). Macroscopic volumes are defined as volumes of typical size \(L \gtrsim d\). Random fluctuations in the extensive statistical magnitude \(N_V\) result from the individual random walks. We wish to characterize their variance \((\Delta N_V)^2\) as a function of the size \(L\) of the considered macroscopic volume.

If the macroscopic volume has a typical size \(L\) in the range \(d \lesssim L \lesssim D(t)\) (of the order of the particle horizon \(D(t)\) or shorter) the fluctuations are obviously thermal \((\Delta N_V)^2 \propto V\), as we have nothing but an example of brownian motion of independent particles in a macro-canonical ensemble. Indeed the particle horizon \(D(t)\) is not evident locally, over volumes of smaller size. The temperature of these thermal anisotropies is dictated by the statistical features of the random walks. If the laws of the individual random walks are invariant under spatial translations and rotations in \(\Omega\) the system remains statistically homogeneous and isotropic at any time. The temperature of the local anisotropies may change with time, but at any instant it is the same over the whole space \(\Omega\) (even though the existence of a finite particle horizon implies that the statistical system cannot have relaxed to thermal
equilibrium over scales beyond it).

The existence of a finite particle horizon is manifest in the fluctuations in the number of individuals $N_V$ in volumes $V$ of size $L \gg D(t)$. The density anisotropies over volumes much larger than the particle horizon are scale invariant $(\Delta N_V)^2 \sim S$, similar to the primordial cosmological density anisotropies at decoupling (11) that are the focus of our interest in this paper. In order to justify this claim let us first define the cover of width $D(t)$ of the boundary surface of a macroscopic finite volume $V$, and denote it as $\partial D(t)V$, as the union of all balls of radius $D(t)$ around any point $x \in \partial V$: the cover $\partial D(t)V$ of a volume $V$ is the set of all points whose distance to its boundary surface $\partial V$ is smaller than $D(t)$. Individuals that are originally located out of this cover cannot contribute at the instant $t$ to fluctuations in the number of individuals within the volume $V$, even though they also are walking randomly. Only individuals that were originally located within $\partial D(t)V$ can produce, when they cross the border $\partial V$, fluctuations in the total number of individuals in $V$. That is, only the degrees of freedom in $\partial D(t)V$ are apparent in these fluctuations. Naturally, then, the anisotropies that the random walks of these degrees of freedom induce in the number of particles $N_V$ are scale invariant $(\Delta N_V)^2 \sim D(t) S \sim D(t) L^2$, rather than thermal $(\Delta N_V)^2 \sim V \sim L^3$.

The power spectrum of the number density anisotropies generated by the random walks of individual particles has been described in the literature [9, 10],

$$P(k; t) = \begin{cases} 
C, & \text{for } k \gtrsim D(t)^{-1} \\
C [k D(t)]^4, & \text{for } k \ll D(t)^{-1},
\end{cases}$$

(17)

where $C$ is a constant determined by the specific features of the random walks. This power spectrum is roughly constant over the range of modes with wavelength shorter than the particle horizon $D(t)$, but it decreases as $\sim k^4$ for modes with longer wavelength. The suppression of this power spectrum in modes with wavelength longer than the horizon $D(t)$ is similar to the suppression of the power spectrum (9) of cosmological density anisotropies left at decoupling by standard physics operating during the radiation dominated expansion of the universe. Hence, on scales $L \gg D(t)$ condition (12) holds for power spectrum (17), with $H_{eq}^{-1}$ replaced by $D(t)$. In fact, this condition defines the presence of a causal horizon in homogeneous and isotropic statistical systems and implies that the system has not reached thermal equilibrium over scales beyond it.

It is clear from this example that the scale invariance of spatial random anisotropies
over volumes of size $L \gg D(t)$ is a direct consequence of the fact that individual degrees of freedom are constrained within a finite particle horizon $D(t)$. The horizon prevents the development of random density fluctuations with longer wavelength. This constrain is expressed through the suppression of the power spectrum in modes $k \ll D(t)^{-1}$. As a result of the strong suppression of the power spectrum in these modes, the largest contribution to the integral expression for the variance of density anisotropies over volumes of size $L \gg D(t)$ comes from modes $k \sim D(t)^{-1}$. Their contribution is scale invariant because it corresponds to density fluctuations at the boundary surface of the considered volume of integration.

We wish to remark that the contribution from modes $k \sim D(t)^{-1}$ to macroscopic density anisotropies over volumes of size $L \gg D(t)$ is not a mathematical pathology associated to a boundary surface that needs to be resolved with infinite precision, as sometimes claimed in the literature in cosmology. Scale invariant anisotropies are produced by boundary fluctuations with typical wavelength $d \ll \lambda \sim D(t)$, much shorter than the size $L$ of the considered volume of integration but still much longer than the resolution scale $d$ of the boundary surface.

**B. The formal arguments.**

We now wish to formalize these arguments using simple tools and consider an statistical density field on $\mathbb{R}^3$,

$$\rho(\vec{x}; d) = \rho_{eq} + \rho_{eq} \int \frac{d^3k}{(2\pi)^3} \delta_k(d) e^{-i\vec{k} \cdot \vec{x}},$$

(18)

to describe, in comoving coordinates, the cosmological density field at the instant of decoupling resolved at a certain finite length scale $d$. The scale factor of the universe at decoupling $a(t_{eq}) = 1$ is chosen as reference. The density field is assumed to be homogeneous and isotropic,

$$\langle \rho(\vec{x}; d) \rangle = \rho_{eq}$$

(19)

(or equivalently, $\langle \delta_k(d) \rangle = 0$), up to tiny statistical fluctuations. We assume that the random modes $\delta_k^+(d) = \delta_{-k}^-(d)$ are independent statistical complex variables with gaussian distribution,
\begin{equation}
\langle \delta_{\vec{k}_1}^*(d) \delta_{\vec{k}_2}(d) \rangle = (2\pi)^3 \, \mathcal{P}(\vec{k}_1; d) \, \delta^3(\vec{k}_1 - \vec{k}_2).
\end{equation}

The function \( \mathcal{P}(\vec{k}; d) \) is known as the power spectrum of the statistical fluctuations and is always positive \( \mathcal{P}(\vec{k}; d) \geq 0 \). In isotropic statistical systems the power spectrum is a function \( \mathcal{P}(\vec{k}; d) = \mathcal{P}(k; d) \) only of the modulus of the wave vector.

The finite resolution scale of the density field is introduced in the description through a convolution kernel (window function) with typical width \( d > 0 \),

\[ \rho(\vec{x}; d) = \int \rho(\vec{y}) W(\vec{x} - \vec{y}; d) \, d^3 \vec{y} \] (see section VIII in [13]). The specific features of the window function \( W(\vec{x} - \vec{y}; d) \) are not of particular interest here, other than it introduces a natural ultraviolet cutoff in the power spectrum \( \mathcal{P}(k; d) = \mathcal{P}(k) |\tilde{W}(k; d)|^2 \) over scales shorter than the resolution length:

\[ |\tilde{W}(k; d)|^2 \ll 1, \text{ for } k \gg d^{-1} \text{ and } |\tilde{W}(k; d)|^2 \sim 1, \text{ for } k \lesssim d^{-1}. \]

In other words, the convolution kernel in the definition of \( \rho(\vec{x}; d) \) simply integrates out all modes with wavelength shorter than the resolution scale.

Macroscopic volumes are defined as volumes whose typical size is much larger than the resolution scale \( d \). According to the central limit theorem the statistical extensive magnitudes

\[ M_V(d) = \int_V d^3 \vec{x} \, \rho(\vec{x}; d) \]

have gaussian distribution. Hence, they are completely specified by their first two momenta: the average value \( \langle M_V \rangle \) and the variance of its random fluctuations \( (\Delta M_V)^2 \), which is related to their power spectrum by expression (5).

Expression (5) emphasizes that the variance of random density anisotropies over macroscopic volumes of any size \( L \) receives positive contributions from random fluctuations in all modes \( \delta_{\vec{k}}(d) \). We want to evaluate separately the contribution from modes \( k \gg L^{-1} \) and compare it to the contribution from modes \( k \lesssim L^{-1} \),

\begin{equation}
(\Delta M_V)^2 \simeq \rho_{eq}^2 \int_0^{k \lesssim L^{-1}} \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) \left| \int_V d^3 \vec{x} \, e^{-i\vec{k} \cdot \vec{x}} \right|^2 + \rho_{eq}^2 \int_{k \gg L^{-1}}^{k \lesssim d^{-1}} \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) \left| \int_V d^3 \vec{x} \, e^{-i\vec{k} \cdot \vec{x}} \right|^2.
\end{equation}

The first contribution, from modes \( k \lesssim 1/L \), coincides with approximation (6):

\begin{equation}
\rho_{eq}^2 \int_0^{k \lesssim L^{-1}} \frac{d^3 \vec{k}}{(2\pi)^3} \mathcal{P}(k) \left| \int_V d^3 \vec{x} \, e^{-i\vec{k} \cdot \vec{x}} \right|^2 \simeq \rho_{eq}^2 \frac{1}{V} \mathcal{P}(k \sim 1/L) \, V^2 \sim \rho_{eq}^2 \mathcal{P}(k \sim 1/L) \, V.
\end{equation}
The second contribution, from modes $k \gg L^{-1}$, is neglected in approximation (6). This contribution is the focus of our interest in this paper because it is scale invariant, 

$$\rho_{eq}^2 \int_{k \gg L^{-1}}^{k < \sim d^{-1}} \frac{d^3k}{(2\pi)^3} P(k) \left| \int_V d^3\vec{x} \ e^{-i\vec{k} \cdot \vec{x}} \right|^2 \simeq \rho_{eq}^2 \left[ \int_{k \gg L^{-1}}^{k < \sim d^{-1}} \frac{d^3k}{k^4} P(k) \right] S. \quad (24)$$

As above, $S$ stands for the area of the surface that bounds the volume of integration $V$. Equation (24) is very simple to derive in the case when the volume of integration $V$ is a sphere of radius $L$,

$$\left| \int_V d^3\vec{x} \ e^{-i\vec{k} \cdot \vec{x}} \right|^2 = 4\pi \frac{S}{k^4} \left( \frac{1}{kL} \sin(kL) - \cos(kL) \right)^2, \quad (25)$$

and the power spectrum $P(k)$ in the range $k \gg L^{-1}$ is smooth over scales $dk \simeq L^{-1}$.

The result can be immediately extended to any other connected volume, assuming that the surface that bounds it is smooth and its area does not depend on the resolution scale.

In summary, the variance of spatial random anisotropies of extensive magnitudes over macroscopic volumes $V$ of any size

$$(\Delta M_V)^2 \simeq \rho_{eq}^2 P(k \sim 1/L) \ V + \rho_{eq}^2 P(k \sim H_{eq}) \ \frac{V}{\pi} \left[ \int_{k \gg L^{-1}}^{k < \sim d^{-1}} \frac{dk}{k^2} \frac{P(k)}{k^4} \right] S, \quad (26)$$

receives always a bulk contribution from fluctuations with wavelength comparable to the size of the volume of integration and an additional scale invariant contribution from fluctuations at its boundary surface with much shorter wavelength. We notice that the global amplitudes of each of these contributions depend, respectively, on the power spectrum in modes $k \lesssim 1/L$ and $k \gg 1/L$. Obviously, if the first contribution is negligible compared to the second, then the anisotropies are scale invariant.

Let us now consider, in particular, the density anisotropies produced at decoupling by standard physics operating during the radiation dominated expansion of the universe. Their power spectrum (9) is strongly suppressed in modes $k \ll H_{eq}$ with comoving wavelength much larger than the causal horizon. Over cosmologically large comoving volumes, of size $L \gg H_{eq}^{-1}$, condition (12) holds. An explicit evaluation of the two contributions in (26) gives

$$\rho_{eq}^2 P(k \sim 1/L) \ V \lesssim \rho_{eq}^2 P(k \sim H_{eq}) \left( \frac{H_{eq}^{-1}}{L} \right)^4 V \sim \rho_{eq}^2 P(k \sim H_{eq}) \ \frac{H_{eq}^4}{L} L^{-1}, \quad (27)$$

for the bulk contribution and
\[
\frac{\rho_{eq}^2}{\pi} \left[ \int_{k \gtrapprox L^{-1}}^{k \lesssim d^{-1}} dk \frac{\mathcal{P}(k)}{k^2} \right] S \simeq \frac{\rho_{eq}^2}{\pi} \left[ \int_{k \gtrapprox L^{-1}}^{k \lesssim H_{eq}} dk \frac{\mathcal{P}(k)}{k^2} \right] S \simeq \rho_{eq}^2 \mathcal{P}(k \sim H_{eq}) H_{eq}^{-1} L^2, \tag{28}
\]

for the scale invariant contribution. We notice that (27) corresponds to the estimation (10) of the variance of primordial cosmological density anisotropies at decoupling in standard FRW cosmology, which led to the formulation of the origin of structures problem. This estimation neglects the second contribution (28). Nevertheless, a direct comparison clearly shows that over volumes of comoving size \( L \gg H_{eq}^{-1} \) the second, scale invariant contribution (28), from modes \( k \gtrsim H_{eq} \), prevails:

\[
(\Delta M_V)^2 \simeq \rho_{eq}^2 \mathcal{P}(k \sim H_{eq}) H_{eq}^{-1} L^2, \tag{29}
\]

while the first, bulk contribution (27), from modes \( k \lesssim 1/L \), is negligible. The flawed estimation (10) keeps the negligible contribution (27) and neglects the dominant one (28).

The power spectrum \( \mathcal{P}(k \sim H_{eq}) \) in modes with comoving wavelength comparable to the comoving causal horizon is not suppressed with respect to the power in modes with shorter wavelength. Therefore, over comoving volumes \( V_{eq} \) of the size of the horizon approximation (6) is valid, \( (\Delta M_{V_{eq}})^2 \simeq \rho_{eq}^2 \mathcal{P}(k \sim H_{eq}) V_{eq} \simeq \rho_{eq}^2 \mathcal{P}(k \sim H_{eq}) H_{eq}^{-3} \). Equation (13) is obtained straightforward from (29).

Finally, we wish to remark that in the framework we have laid in this section we did not even need to worry about the technical difficulties that appear when trying to manage the coupled dynamics of density fluctuations with comoving wavelength longer than the causal horizon and space-time metric perturbations \( [11] \), because the power spectrum in these modes is strongly suppressed and their contribution to cosmological anisotropies is sub-leading.

### C. Scale invariant gravitational potential.

We have claimed in the Introduction to this paper that scale invariant primordial density anisotropies over cosmologically large comoving volumes \( [12] \) produce a statistically constant gravitational potential \( [3] \) over cosmologically large comoving distances. In this subsection we prove that this result is a consequence of the strong suppression \( [12] \) of the
power spectrum of primordial density anisotropies in modes with cosmologically large co-
moving wavelength.

The statistical gravitational potential at the time of decoupling

\[ \phi(\vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \phi_\vec{k} \ e^{-i \vec{k} \cdot \vec{x}}, \]  \hspace{1cm} (30)

is related to the statistical density field at the time (18) by the equation

\[ \vec{\nabla}^2 \phi(\vec{x}) = 4\pi G (\rho(\vec{x}) - \rho_{eq}). \]  \hspace{1cm} (31)

In momentum space the same equation can be written

\[ \phi_\vec{k} = -4\pi G \rho_{eq} \frac{1}{|\vec{k}|^2} \delta_\vec{k}. \]  \hspace{1cm} (32)

The gravitational field (30) is real because

\[ \phi^*_{-\vec{k}} = \phi_\vec{k} \]  \hspace{1cm} (33)

and it is zero everywhere on average \( \langle \phi(\vec{x}) \rangle = 0 \), because

\[ \langle \phi_\vec{k} \rangle = 0. \]  \hspace{1cm} (34)

Thus, the average potential difference between any two points is also zero,

\[ \langle \phi(\vec{x}) - \phi(\vec{y}) \rangle = 0. \]  \hspace{1cm} (35)

The two points correlation function of the statistical gravitational potential is related to the power spectrum of density fluctuations by the expression,

\[ \langle \phi(\vec{x}) \phi(\vec{y}) \rangle = \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \langle \phi^*_{\vec{k}_1} \phi_{\vec{k}_2} \rangle e^{+i \vec{k}_1 \cdot \vec{x} - i \vec{k}_2 \cdot \vec{y}} = (4\pi G \rho_{eq})^2 \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \frac{\delta_{\vec{k}_1} \delta_{\vec{k}_2}}{|\vec{k}_1|^2 |\vec{k}_2|^2} e^{+i \vec{k}_1 \cdot \vec{x} - i \vec{k}_2 \cdot \vec{y}} = (4\pi G \rho_{eq})^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{P(|\vec{k}|)}{|\vec{k}|^4} e^{+i \vec{k} \cdot (\vec{x} - \vec{y})} \]  \hspace{1cm} (36)

and the variance of random fluctuations in the potential difference between any two points is given by,
\[
\langle [\phi(\vec{x}) - \phi(\vec{y})]^2 \rangle = 2\langle \phi(\vec{x})^2 \rangle - \langle \phi(\vec{x}) \phi(\vec{y}) \rangle - \langle \phi(\vec{y}) \phi(\vec{x}) \rangle = (4G\rho_{eq})^2 \int \frac{dk}{k^2} \frac{P(k)}{k^2} \left(1 - \frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|}\right).
\] (37)

We are interested in the asymptotic behaviour of this variance when \(|\vec{x} - \vec{y}| \gg H_{eq}^{-1}\), the comoving distance between the two considered points is cosmologically large. To evaluate the integral expression (37) we separate the contribution from modes \(k \ll H_{eq}\), whose comoving wavelength is much larger than the horizon, from the contribution of modes \(k \gtrsim H_{eq}\), whose comoving wavelength is within the horizon,

\[
\langle [\phi(\vec{x}) - \phi(\vec{y})]^2 \rangle \simeq (4G\rho_{eq})^2 \left[\int_0^{k \ll H_{eq}} \frac{dk}{k^2} \frac{P(k)}{k^2} \left(1 - \frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|}\right) + \int_{k \gtrsim H_{eq}} \frac{dk}{k^2} \frac{P(k)}{k^2}\right].
\] (38)

The latter contribution is roughly constant \((4G\rho_{eq})^2 \frac{P(k \sim H_{eq})}{H_{eq}}\) because \(\frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|} \ll 1\), for \(k \gtrsim H_{eq}\) and \(|\vec{x} - \vec{y}| \gg H_{eq}^{-1}\). We want now to explore the contribution from comoving modes with cosmologically large wavelength when the power spectrum of density fluctuations is largely suppressed in this range (12),

\[
\int_0^{k \ll H_{eq}} \frac{dk}{k^2} \frac{P(k)}{k^2} \left(1 - \frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|}\right) \ll \frac{P(k \sim H_{eq})}{H_{eq}} \int_0^{k \ll H_{eq}} \frac{dk}{k^2} \left(1 - \frac{\sin[k|\vec{x} - \vec{y}|]}{k|\vec{x} - \vec{y}|}\right) \ll \frac{P(k \sim H_{eq})}{H_{eq}}.
\]

We find again that the contribution from random density fluctuations with cosmologically large comoving wavelength is suppressed relative to the contribution from density fluctuations with comoving wavelength within the horizon. In consequence, \(\langle \phi(\vec{x})\phi(\vec{y}) \rangle \ll \langle [\phi(\vec{x})]^2 \rangle\) when \(|\vec{x} - \vec{y}| \gg H_{eq}^{-1}\) and

\[
\langle [\phi(\vec{x}) - \phi(\vec{y})]^2 \rangle \simeq \langle [\phi(\vec{x})]^2 \rangle + \langle [\phi(\vec{y})]^2 \rangle \simeq (4G\rho_{eq})^2 \frac{P(k \sim H_{eq})}{H_{eq}}
\] (39)

the random gravitational potential is statistically constant over cosmologically large comoving distances.
D. Scale invariant anisotropies vs. thermal anisotropies.

The result (13) that we have proved in subsection II.B above is at odds with the intuition inferred from the common experience with thermodynamic systems in equilibrium, in which the variance of random density anisotropies of extensive magnitudes grows with the size of the considered region of integration as some power of its volume $\left( \Delta M_V \right)_n^2 \sim V^\beta$, with $1 \leq \beta < 2$. In this section we want to compare the bulk dependence of the variance of thermal anisotropies with the surface dependence of the variance of scale invariant anisotropies.

Consider a thermodynamic system in equilibrium far from a phase transition. Thermal fluctuations in the local densities are described by Poisson two point correlation functions $F(\vec{x} - \vec{y}) \equiv \frac{1}{\rho_0} [\langle \rho(\vec{x}) \rho(\vec{y}) \rangle - \langle \rho(\vec{x}) \rangle \langle \rho(\vec{y}) \rangle] = \zeta^2 d^3 \delta^3(\vec{x} - \vec{y})$, or the more realistic

$$F(r) = \zeta^2 \frac{1}{\pi^{3/2}} e^{-(r/d)^2}, \quad (40)$$

where $d \simeq 1/T$ is the finite resolution scale and $\zeta$ is a dimensionless factor that measures the global amplitude of the fluctuations. The corresponding power spectrum,

$$P(k) = \int d^3 \vec{r} F(\vec{r}) e^{+ik \cdot \vec{r}} \simeq \zeta^2 d^3 \quad (41)$$

is roughly constant over the whole range $0 < k \simeq d^{-1}$ of accessible scales. Hence, over spheres of radius $L$ the variance of spatial thermal anisotropies is

$$\left( \Delta M_V \right)_n^2 = 8\zeta^2 \rho_0^2 d^3 L^3 \int_0^{L/d} d(kL) \left( \frac{1}{(kL)^{3/2}} \sin(kL) - \frac{1}{kL} \cos(kL) \right)^2 \simeq \zeta^2 \rho_0^2 d^3 V. \quad (42)$$

It can be readily seen that, independently of the size of the region $L$, the largest contribution to the last integral comes from modes $kL \sim 1$ (the integrand vanishes for $kL \ll 1$ and for $kL \gg 1$) and the variance is linearly proportional to the volume of integration, in good agreement with the estimation (14).

The linear dependence of the variance (12) on the volume of the considered region of integration results from the short range of the Poisson two points correlation function (11). Poisson correlation is characteristic of thermodynamics systems in equilibrium far from a phase transition. For thermodynamic systems in equilibrium but close to a phase transition the power spectrum of thermal fluctuations is peaked at modes with very long wavelength
\( \mathcal{P}(k) \sim \zeta^2 \, d^{3-\gamma} \, k^{-\gamma} \), with \( \gamma \in (0, 3) \), and the associated two-points correlation function is long-range. When this is the case

\[
(\Delta M_V)^2 = 8\zeta^2 \rho_0^2 \, L^{3+\gamma} \, d^{3-\gamma} \int_0^{L/d} d(kL) \frac{1}{(kL)^\gamma} \left( \frac{1}{(kL)^2} \sin(kL) - \frac{1}{kL} \cos(kL) \right)^2 \simeq \zeta^2 \rho_0^2 \, d^{3-\gamma} \, V^{1+\gamma/3}. \tag{43}
\]

The largest contribution to the variance of spatial density anisotropies still comes from modes \( kL \sim 1 \) whose wavelength is comparable to the size of the volume of integration, but now the variance grows with a higher power of the volume of integration. Estimation (3) still gets it right.

We have brought these examples of thermal anisotropies into consideration because we want to notice that their power spectra are not suppressed over any range of short momenta, \( \lim_{k \to 0} \mathcal{P}(k) \neq 0 \). In the statistical systems described by these power spectra there is no causal horizon. These systems are theoretically allowed to excite with finite probability random density fluctuations with any wavelength longer than the resolution scale \( d \). In consequence, in these statistical systems the bulk contribution (27) to the variance (26) of spatial random anisotropies in macroscopic volumes \( V \) of any size \( L \) is always dominant over the scale invariant contribution (28).

The power spectrum of an statistical system that is constrained by a finite causal horizon \( D \) looks like

\[
\mathcal{P}(k) = p(k / D) \, \mathcal{P}_0(k \sim D^{-1}), \tag{44}
\]

where the dimensionless function \( p(k / D) \simeq o(k / D) \) decreases to zero faster than linearly for \( k \ll D^{-1} \) and \( p(k / D) \sim 1 \) for \( k \approx D^{-1} \). The pattern of anisotropies associated to these power spectra are typically of the form (12) or (13) over volumes of size \( L \lesssim D \) shorter than the horizon, but it turns scale invariant (13) over volumes of size \( L \gg D \) as a result of the suppression of the power spectrum (14) in modes \( k \ll D^{-1} \).

In the model discussed in subsection II.A, for example, the power spectrum (17) is suppressed in modes \( k \ll D^{-1} \) by a factor \( p(k / D) \sim (k / D)^4 \). A similar suppression factor appears in the power spectrum of cosmological density anisotropies produced at decoupling by standard physics operating during the radiation dominated expansion. A suppression
factor $p(x) = \frac{x^n}{1+x^n}$, $n \geq 1$, appears in the power spectrum that describes the statistical distribution of topological defects in a sample cooled down homogeneously, but very fast, below its critical temperature (Kibble mechanism) \[18\]. The super fast cooling rate of the sample, rather than the cosmological expansion, introduces an "horizon" in this example, if we define the horizon $D$ of an homogeneous and isotropic statistical system as the scale $D^{-1}$ beyond which its power spectrum is suppressed. Spatial random anisotropies in the distribution of defects over regions of the sample of size much larger than the "cooling" horizon are theoretically known to be scale invariant.

The suppression of the power spectrum \[44\] over scales $k \ll D^{-1}$ introduces the global constrain

$$\left[ \int_{\Omega} d^3\vec{r} F(r) = \lim_{k \to 0} P(k) \right] \ll P(k \sim D^{-1}).$$

This condition $\int_{\Omega} d^3\vec{r} F(r) \simeq 0$, which we have associated directly to the scale invariance of random anisotropies over very large volumes, is known in the literature as a demand of incompressibility of the statistical density fluctuations. Incompressibility is understood in our context as a constrain $(\Delta M_\Omega)^2 = 0$ of total conservation of the extensive magnitude $M$ over the whole volume $\Omega$, see section IV in \[13\]. This condition can be evidently satisfied when the sub-volumes $V$ of any size are assumed to be embedded in a larger isolated micro-canonical ensemble. In the framework of a quantum field theory this condition can be easily obtained if the state of the system is assumed to be an eigenstate of $M_\Omega$. In quantum Hall effect in condensed matter physics, for example, the incompressibility of the electron fluid is known to be associated to the low-energy border exitations of the system \[19\].

E. Amplitude of primordial cosmological density anisotropies.

In this subsection we examine the amplitude of the scale invariant cosmological density anisotropies left at decoupling by standard physics \[13\] and compare it with the amplitude of the scale invariant anisotropies \[1\] imprinted in the CMBR.

We describe the radiation dominated density field at (physical) cosmological time $t \leq t_{eq}$ as a statistically homogeneous and isotropic random field,
\[
\rho(\vec{x}; t) = \rho_t + \rho_t \int \frac{d^3k}{(2\pi)^3} \delta_{\vec{k}} \ e^{-i\vec{k} \cdot \vec{x}},
\]
with

\[
\langle \delta_{\vec{k}} \rangle = 0, \quad (\langle \rho(\vec{x}, t) \rangle = \rho_t),
\]
and

\[
\langle \delta_{\vec{k_1}}^{*} \delta_{\vec{k_2}} \rangle = (2\pi)^3 \ P(|\vec{k_1}|; t) \ \delta^3(\vec{k_1} - \vec{k_2}).
\]

The Fourier modes are assumed to be statistically independent and gaussian. We keep using comoving spatial coordinates defined with the scale factor of the universe at decoupling \(a(t_{eq}) = 1\) as reference.

The statistical variable \(\rho(\vec{x}; t) - \rho(\vec{y}; t)\) describes the random fluctuations in the density at some comoving site \(\vec{x}\) with respect to a different comoving site \(\vec{y}\), at the given cosmological time \(t\). Statistical homogeneity (47) demands that this statistical variable has zero average value,

\[
\langle \rho(\vec{x}; t) - \rho(\vec{y}; t) \rangle = 0.
\]

We are interested in the variance of its statistical fluctuations,

\[
\begin{align*}
\langle [\rho(\vec{x}; t) - \rho(\vec{y}; t)]^2 \rangle &= 2 \left[ \langle \rho(\vec{x}; t)^2 \rangle - \langle \rho(\vec{x}; t) \rho(\vec{y}; t) \rangle \right] \\
&= 2 \left[ \langle \rho(\vec{x}; t)^2 \rangle - \rho_t^2 \right] - 2 \left[ \langle \rho(\vec{x}; t) \rho(\vec{y}; t) \rangle - \rho_t^2 \right] \\
&= 2\rho_t^2 \int \frac{d^3k}{(2\pi)^3} P(k; t) \left( 1 - e^{+i\vec{k} \cdot (\vec{x} - \vec{y})} \right) \ \\
&= \frac{\rho_t^2}{\pi^2} \int dk \ k^2 \ P(k; t) \left( 1 - \frac{\sin(\vec{k} \cdot |\vec{x} - \vec{y}|)}{|\vec{k}| \ |\vec{x} - \vec{y}|} \right).
\end{align*}
\]

During the radiation dominated expansion of the universe the cosmic fluid supports the propagation of non-linear density waves, which travel with sound velocity \(c_s = 1/\sqrt{3}\) (in natural units) and operate to erase density inhomogeneities. Therefore, we must expect maximal statistical correlation between causally connected comoving sites:
\[
\langle [\rho(\vec{x}; t) - \rho(\vec{y}; t)]^2 \rangle \ll 2 \left( \langle [\rho(\vec{x}; t)]^2 \rangle - \rho_t^2 \right), \quad \text{for} \quad |\vec{x} - \vec{y}| \lesssim H_t^{-1}. \tag{51}
\]

On the other side, causality constrains forbide the existence of statistical correlation between causally disconnected sites:

\[
\langle [\rho(\vec{x}; t) - \rho(\vec{y}; t)]^2 \rangle \simeq 2 \left( \langle [\rho(\vec{x}; t)]^2 \rangle - \rho_t^2 \right), \quad \text{for} \quad |\vec{x} - \vec{y}| \gg H_t^{-1}. \tag{52}
\]

Actually, causality does constrain the two points correlation function to satisfy the more stringent condition (51). The comoving causal horizon \( H_t^{-1} = (t/t_{eq})^{1/2} H_{eq}^{-1} \) grows monotonically during the radiation dominated expansion of the universe.

These features correspond to a power spectrum \( \mathcal{P}(k; t) \) peaked at \( k \sim H_t \),

\[ \mathcal{P}(k \sim H_t; t) \equiv \mathcal{P}_t, \tag{53} \]

and suppressed both at \( k \gg H_t \) and at \( k \ll H_t \),

\[
\mathcal{P}(k; t) \leq \begin{cases} 
\mathcal{P}_t \frac{H_t^3}{k^3}, & \text{for} \quad k \gg H_t, \\
\mathcal{P}_t \frac{k^4}{H_t^4}, & \text{for} \quad k \ll H_t, 
\end{cases} \tag{54}
\]

We plug this power spectrum in expression (5) to obtain the variance of the density anisotropies left in the cosmic plasma at cosmological time \( t \):

\[ (\Delta M_V)^2(t) \simeq \rho_t^2 \left( H_t^3 \mathcal{P}_t \right) V^2 \simeq (\Delta M_V)^2 \frac{V^2}{V_t^2}. \tag{55} \]

In particular, over comoving volumes \( V_t \) of size \( L \sim H_t^{-1} \) comparable to the causal horizon

\[ (\Delta M_V)^2(t) \simeq \left( \rho_t^2 H_t^3 \mathcal{P}_t \right) V_t^2 \simeq \rho_t^2 H_t^{-3} \mathcal{P}_t. \tag{56} \]

II) The suppression of the power in modes \( k \ll H_t \) implies that over volumes \( V \) of comoving size \( L \gg H_t^{-1} \) much larger than the causal horizon the anisotropies are scale invariant.
\[(\Delta M_V)^2(t) \simeq \rho_t^2 H_t^{-1} \mathcal{P}_t L^2 \simeq (\Delta M_V)_{\text{t}}^2 \frac{L^2}{H_t^{-2}}.\]  

(57)

The amplitude of the cosmological density anisotropies (55), (57) at decoupling \( t = t_{\text{eq}} \) is commensurate with the amplitude of the primordial anisotropies imprinted in the CMBR if, and only if, condition (14) is fulfilled. From (56) we obtain the condition \( H_{\text{eq}}^3 \mathcal{P}_{t_{\text{eq}}} \sim 10^{-10} \).

In order to give it a clear physical meaning we now define

\[ \kappa_t^2 \equiv \frac{1}{\rho_t^2} \left[ \left\langle (\rho(\vec{x}; t))^2 \right\rangle - \rho_t^2 \right] = \int \frac{d^3 k}{(2\pi)^3} \mathcal{P}(k; t) \simeq H_t^3 \mathcal{P}_t, \]

(58)

and notice that it is the normalized variance of the density fluctuations between comoving sites \( \vec{x} \) and \( \vec{y} \) that are causally disconnected at time \( t \),

\[ \kappa_t^2 \simeq \frac{1}{2} \frac{1}{\rho_t^2} \left\langle (\rho(\vec{x}; t) - \rho(\vec{y}; t))^2 \right\rangle, \quad \text{for} \quad |\vec{x} - \vec{y}| \gg H_t^{-1}. \]

(59)

We have from (56) and (58)

\[(\Delta M_V)^2(t) \simeq \kappa_t^2 \rho_t^2 V_t^2.\]

(60)

Hence, condition (14) simply requires

\[ \kappa_{t_{\text{eq}}}^2 \simeq 10^{-10}. \]

(61)

As there does not seem to exist any obvious theoretical or observational constraint that rules out (61) in the framework of standard FRW cosmology, we should conclude from this analysis that primordial cosmological structures (1) could have been seeded during the radiation dominated expansion of the universe.

In a forthcoming paper we will explore the mechanisms involved in actually fixing \( \kappa_t \) at the time \( t = t_{\text{eq}} \) of decoupling. We will show that \( \kappa_t^2 \) is a measure of the entropy of the random density fluctuations in the cosmic plasma at time \( t \). This observation is important if the entropy of the density fluctuations is preserved during the adiabatic radiation dominated expansion of the universe. In such a case, \( \kappa_t^2 \) is constant since the much earlier time \( t_* \) when the universe entered the stage of adiabatic expansion,

\[ \kappa_t^2 \simeq \text{constant}, \quad \text{for} \quad t \in (t_*, t_{\text{eq}}). \]

(62)
This would imply that the amplitude of the anisotropies (55), (56), (57) at decoupling \( t = t_{eq} \) was indeed fixed by entropy-producing mechanisms at that earlier time \( t_* \). Notice that condition (62) is satisfied if the relative amplitude of the density fluctuations between any two causally disconnected sites (59) is preserved during the adiabatic expansion.

**III. CONCLUSIONS.**

We have shown that the prevalent analysis of the origin of primordial structures in standard FRW cosmology is flawed by an unjustified approximation, which has led researchers to erroneous conclusions. Our revised analysis of the issue shows that standard physics operating in the cosmic plasma during the radiation dominated expansion of the universe can produce scale invariant cosmological density anisotropies at decoupling with amplitude comparable to that of the primordial anisotropies imprinted in the CMBR. Indeed, the characteristic scale invariance of the primordial cosmological density anisotropies at decoupling seems to be a signature of the causal structure of the standard FRW universe, which prevents the development in the radiation dominated cosmic plasma of large density fluctuations with cosmologically large comoving wavelength. This might explain the reported absence of statistical correlations at large angular scales in the temperature anisotropies measured by the WMAP [20].

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