q-deformation with $(\varphi, \Gamma)$ structure of the de Rham cohomology of the Legendre family of elliptic curves

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Abstract

In the late ’60s, B. Dwork studied a Frobenius structure compatible with the classical hypergeometric differential equation with parameters $(\frac{1}{2}, \frac{1}{2}; 1)$ by analyzing behavior of solutions of the differential equation under Frobenius transformation. Recently, P. Scholze conjectured the existence of $q$-de Rham cohomology groups for any $\mathbb{Z}$-scheme. In this paper, we give a Frobenius structure compatible with the $q$-hypergeometric differential equation with parameters $(q^i, q^j, q)$ by showing a $q$-analogue of some results of Dwork. This construction gives a $q$-deformation with $(\varphi, \Gamma)$-structure over $\mathbb{Z}_p[[q-1]][[\lambda]]$ of the de Rham cohomology of the $p$-adic Legendre family of elliptic curves which has Frobenius structure and connection.

1 Introduction

Let $p$ be an odd prime number. Let $h(\lambda) = \sum_{i=0}^{\nu-1} \left( \frac{\lambda}{p} \right)^i \lambda^i$ be the Hasse polynomial. Let $B = \mathbb{Z}_p\left( \lambda, \frac{1}{\lambda(1-\lambda)} \right)$ be the $p$-adic completion of the ring $\mathbb{Z}_p\left[ \lambda, \frac{1}{\lambda(1-\lambda)} \right]$. We consider the $p$-adic Legendre family of elliptic curves

$$E = \text{Proj} \left( B[X, Y, Z]/(Y^2 Z - X(X - Z)(X - \lambda Z)) \right) \to \text{Spec}(B).$$

Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Let $\overline{\mathbb{Z}}_p$ be the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. Let $\mathfrak{m}$ be the maximal ideal of $\overline{\mathbb{Z}}_p$. For every value $\mu \in \overline{\mathbb{Z}}_p, \mu(1 - \mu) \not\equiv 0 \mod p$ of $\lambda$, the fiber above $\mu$ is an elliptic curve with good ordinary reduction, denoted by $E_\mu$. The relative curve $E$ over $B$ with the divisor at infinity deleted is written as $\text{Spec} \left( B[x, y]/(y^2 - x(x - 1)(x - \lambda)) \right)$, where $(x, y) = \left( \frac{X}{Z}, \frac{Y}{Z} \right)$. Then the de Rham cohomology $H^1_{\text{dR}} := H^1_{\text{dR}}(E/B)$ is a free $B$-module of rank $2$ and $\text{Fil}^1 H^1_{\text{dR}} = \Gamma(E, \Omega_{E/B})$ is a free $B$-module of rank $1$ with basis $\omega = \mathfrak{d}$. In [Dw69] §6, Dwork defined a Frobenius structure $\varphi_{H^1_{\text{dR}}}$ on $H^1_{\text{dR}}$. Moreover, he found that there exists a unique direct summand $U$ of the $B$-module $H^1_{\text{dR}}$ stable under $\varphi_{H^1_{\text{dR}}}$. (See [vdP86] and [FrP04, pp. 232-233].) This $U$ is called the unit root part of $H^1_{\text{dR}}$. Let $E_{\mathfrak{m}}$ denote the reduction mod $p$ of $E$. Then $H^1_{\text{dR}}$ is canonically isomorphic to the crystalline cohomology $H^1_{\text{crys}}(E_{\mathfrak{m}}/B)$, and $\varphi_{H^1_{\text{dR}}}$ coincides with the Frobenius structure induced by the absolute Frobenius of $E_{\mathfrak{m}}$.

More precisely, we obtain $U$ as follows. Let $\varphi: B \to B$ be the unique lifting of the absolute Frobenius satisfying $\varphi(a) = a (a \in \mathbb{Z}_p)$ and $\varphi(\lambda) = \lambda^p$ (cf. §2). Then the Frobenius structure $\varphi_{H^1_{\text{dR}}}$ is realized as a $\varphi$-semilinear endomorphism of $H^1_{\text{dR}}$, which is again denoted by $\varphi_{H^1_{\text{dR}}}$ in the following. Let $\nabla: H^1_{\text{dR}} \to H^1_{\text{dR}} \otimes_B \Omega_B$ be the Gauss-Manin connection. We define the $B$-linear endomorphism $D$ of $H^1_{\text{dR}}$ by $\nabla = D \otimes d\lambda$. Then $\omega$ and $D(\omega)$ form a basis of $H^1_{\text{dR}}$ because the Kodaira-Spencer map $\Gamma(E, \Omega_{E/B}) \subset H^1_{\text{dR}} \nabla \to H^1_{\text{dR}} \otimes_B \Omega_B \to H^1_{\text{dR}}/\text{Fil}^1 \otimes_B \Omega_B$ is an isomorphism. (This follows from the following fact: For any field $F$ of characteristic
$\neq 2$, and any $a \in F \setminus \{0,1\}$, the elliptic curve $y^2 = x(x-1)(x-a-\varepsilon)$ over $F[\varepsilon]/(\varepsilon^2)$ is not constant, i.e., not isomorphic to the base change of the elliptic curve $y^2 = x(x-1)(x-a)$ over $F$ under $F \rightarrow F[\varepsilon]/(\varepsilon^2)$.

By [vdP86 Proposition 7.11.(ii)], we can write $\nabla$ on $H^1_{\text{dR}}$ explicitly as

$$\nabla \left( (\lambda(1-\lambda) \omega - \lambda(1-\lambda) D(\omega)) \right) = \left( \lambda(1-\lambda) \omega - \lambda(1-\lambda) D(\omega) \right) \frac{1}{\lambda(1-\lambda)} \left( \begin{array}{c} 1 - 2\lambda \\ -\lambda(1-\lambda) \\ 0 \end{array} \right) \otimes d\lambda. \quad (1)$$

Let $C$ be a ring extension of $B$ (e.g. $\mathbb{Q}_p[[\lambda]]$) which carries an extension of $\frac{d}{d\lambda}$. Then the above formula implies that, for $f_1, f_2 \in C$, we have $D(f_1(1-\lambda) \omega - f_2(1-\lambda) D(\omega)) = 0$ if and only if $f_1 = \frac{df}{d\lambda}$ and $f_2$ satisfies the classical hypergeometric equation with parameters $(\frac{1}{2}, \frac{1}{2}; 1)$ ([vdP86 Proposition 7.11.(iii))]

$$\lambda(1-\lambda) \frac{d^2}{d\lambda^2} f_2 + (1-2\lambda) \frac{d}{d\lambda} f_2 - \frac{1}{4} f_2 = 0. \quad (2)$$

This differential equation has the well-known solution $f(\lambda) := F(\frac{1}{2}, \frac{1}{2}; 1; \lambda) = \sum_{n=0}^{\infty} \left( \prod_{i=0}^{n-1} \left( \frac{i+\frac{1}{2}}{\lambda} \right) \right) \lambda^n \in \mathbb{Q}_p[\lambda]$, which converges on the open unit disk.

Let $\mathbb{Z}_p \left< \frac{1}{\n\lambda} \right>$ be the $p$-adic completion of $\mathbb{Z}_p \left< \lambda, \frac{1}{\n\lambda} \right>$. In [Dw69 §1-§4], Dwork showed $\frac{\varepsilon f}{\varepsilon} \in \mathbb{Z}_p \left< \frac{1}{\n\lambda} \right>$, from which he derived $\frac{df}{d\lambda} \in \mathbb{Z}_p \left< \frac{1}{\n\lambda} \right>$. The latter implies that $\lambda \n 2 \omega = D(\omega)$ form a basis of $H^1_{\text{dR}}$. The unit root part $U$ is given by $U = B \n 2$, and he further showed a formula ([Dw69 (6.29))]

$$\varphi_{H^1_{\text{dR}}} (\n 2) = \varepsilon \frac{\varphi(f)}{\varphi(f)} \n 2, \quad \varepsilon = (-1)^{\frac{n}{2}}. \quad (3)$$

By the last claim in the previous paragraph, we see $D(f \n 2) = 0$, which implies $D(\n 2) = -\frac{df}{d\lambda} \n 2$, and therefore $U$ is stable under $\nabla$.

By using the explicit formula of $\nabla$ on $H^1_{\text{dR}}$ above, we see $\nabla(\n 2 \n 2) = 0$ and therefore $(\n \lambda^2 H^1_{\text{dR} \mid \varepsilon = 0}) = \mathbb{Z}_p (\n 2 \n 2)$. Since the Frobenius endomorphism of $\n \lambda^2 H^1_{\text{dR}} (E_p) = H^2_{\text{dR}} (E_p)$ for $\mu = [a] \in \mathbb{Z}_p (a \in \mathbb{F}_p \{0,1\})$ is the multiplication by $p$, this implies $\varphi(\n 2 \n 2) = \varepsilon p \n 2 \n 2$. As $\varphi_{H^1_{\text{dR}}} (\varepsilon) H^1_{\text{dR}} \subset pH^1_{\text{dR}}$, we have

$$\varphi_{H^1_{\text{dR}}} (\n 1) = \varepsilon p \varphi(f) \n 1 + pb \n 2 \quad (4)$$

for some $b \in B$.

Let $B'$ be $\mathbb{Z}_p \left< \lambda, \frac{1}{\n\lambda} \right>$ equipped with the Frobenius $\varphi$ defined in the same way as $B$. Let $U_{B'}$ be the $B'$-submodule $B' \n 2$ of $U$. Then, by the explicit description of $\varphi_{H^1_{\text{dR}}} \mid_U$ and $\nabla \mid_U$ recalled above, we see that they induce a $\varphi$-semilinear endomorphism $\varphi_{U_{B'}}$ of $U_{B'}$, which satisfies $B' \cdot \varphi_{U_{B'}} (U_{B'}) = U_{B'}$, and a connection $\nabla: U_{B'} \rightarrow U_{B'} \otimes_{\Omega_{B'}} \Omega_{B'}$, where $\Omega_{B'} = B' \varepsilon d\lambda$. We also see that $(H^1_{\text{dR}} \mid \varepsilon, \nabla)$ has a “$B'$-structure” given by relative log de Rham cohomology as follows. Let $T$ be $\text{Spec}(B')$ equipped with the log structure defined by the divisor $\lambda(1-\lambda) = 0$. Then, by replacing $B$ with $B'$ in the definition of $E$, we obtain a log smooth extension $E'/T$ of $E/\text{Spec}(B)$. Its relative log de Rham cohomology $H_{B'} := H^1_{\text{dR}}(E'/T)$ is a free $B'$-module of rank 2 equipped with the logarithmic Gauss-Manin connection $\nabla: H_{B'} \rightarrow H_{B'} \otimes_{\Omega_{B'}} \Omega_{B'}$, where $\Omega_{B'} = B' \frac{d\lambda}{\lambda(1-\lambda)}$. The pull-back by $\text{Spec}(B) \rightarrow T$ induces a $B'$-linear isomorphism $H_{B'} \otimes_{B'} B \cong H^1_{\text{dR}}$ compatible with $\nabla$, and $\text{Fil}^\bullet$. Moreover, we have $\text{Fil}^1 H_{B'} = \Gamma(E', \Omega_{E'/T}) = B' \omega$, and the Kodaira-Spencer map $\text{Fil}^1 H_{B'} \rightarrow H_{B'}/\text{Fil}^1 \otimes_{B'} \Omega_{B'}$ is an isomorphism. Since $\nabla(\omega) = (\lambda(1-\lambda) D(\omega) \otimes \frac{d\lambda}{\lambda(1-\lambda)})$, this means
that both \((\omega, \lambda(1 - \lambda)D(\omega))\) and \((\overline{\omega}, \overline{\lambda})\) are bases of \(H_{B'}\). This implies that \(U_{B'}\) is a direct factor of \(H_{B'}\). Since \(\varphi\) of \(B'\) does not preserve the divisor \(\lambda(1 - \lambda) = 0\), the comparison isomorphism with the log crystalline cohomology of \(E^e_{D'}/T\) does not give a \(q\)-semilinear endomorphism of \(H_{B'}\).

Let \(B'' = \mathbb{Z}_p \langle \lambda, \frac{1}{1 - \lambda(1 - \lambda)H(\lambda)} \rangle\) be the \(p\)-adic completion of \(\mathbb{Z}_p \langle \lambda, \frac{1}{1 - \lambda(1 - \lambda)H(\lambda)} \rangle\), and define the Frobenius \(\varphi\) of \(B''\) in the same way as that of \(B\). Then, since \(\varphi\) of \(B''\) preserves the divisor \(\lambda = 0\), the comparison isomorphism with the log crystalline cohomology induces a \(q\)-semilinear endomorphism \(\varphi_{H_{B''}}\) of \(H_{B''} := H_{B''} \otimes B''\), which is compatible with \(\varphi_{H_{B''}}\). As \(\varphi_{H_{B''}}(\text{Fil}^1H_{B''}) \subset pH_{B''}\), we obtain \(b \in B''\).

In \([3]\) we introduce a category \(\text{MIC}_{[0, a]}(A, \varphi, \text{Fil}^*)\) for a non-negative integer \(a\) whose object is a free \(A\)-module of finite type \(M\) with a decreasing filtration, a Frobenius endomorphism and a connection satisfying certain conditions. By the above construction, we obtain an object of \(\text{MIC}_{[0, a]}(A, \varphi, \text{Fil}^*)\) in each of the cases \(A = B'\), \(a = 0\), \(M = U_{B'}\) and \(A = \mathbb{Z}_p[\lambda]\), \(a = 1\), \(M = H_{B''} \otimes B''\). In this paper, we are interested in \(q\)-analogues of these objects, namely \(q\)-deformations involving a formal variable \(q\) such that the specialization to \(q = 1\) recovers the original objects. P. Scholze made a conjecture that there exists a canonical \(q\)-deformation of de Rham cohomology, which is sometimes called \(q\)-de Rham cohomology or Aomoto-Jackson cohomology. (See [Sch17].) Especially in [Sch17] (8), he asked whether there is a relation between the \(q\)-differential equation given by the conjectured \(q\)-de Rham cohomology of the Legendre family and the \(q\)-hypergeometric equation. Our result, which is explained below, may be regarded as positive evidence to his question.

Let \(S' = \mathbb{Z}_p[[q - 1]] \langle \lambda, \frac{1}{1 - \lambda(1 - \lambda)H(\lambda)} \rangle\) be the \((p, q - 1)\)-adic completion of \(\mathbb{Z}_p[[q - 1]] \langle \lambda, \frac{1}{1 - \lambda(1 - \lambda)H(\lambda)} \rangle\). Let \(R'\) be \(\mathbb{Z}_p[[q - 1]][\lambda]\) or \(S'\). Put \(A := R'/(q - 1)\), then we can identify \(A\) with \(\mathbb{Z}_p[[\lambda]]\) (resp. \(B'\)) when \(R' = \mathbb{Z}_p[[q - 1]]\) (resp. \(S'\)).

In \([2]\) we give a Frobenius structure and a \(\Gamma\)-action \(\rho\), and then recall the definition of \(q\)-connections on \(R'\)-modules and the relation between \(\rho\)-semilinear \(\Gamma\)-actions and \(q\)-connections. In \([3]\) we introduce the category \(\text{MF}^{[p], q-1}_{[0, a]}(R', \varphi, \Gamma)\) for a non-negative integer \(a\) whose object is a free \(R'\)-module of finite type \(M\) with a decreasing filtration, a \(\varphi\)-semilinear endomorphism and a \(\rho\)-semilinear action of \(\Gamma\) satisfying certain conditions. Then the canonical surjection \(R' \to A\) induces a functor

\[
\text{MF}^{[p], q-1}_{[0, a]}(R', \varphi, \Gamma) \xrightarrow{\text{mod } q - 1} \text{MIC}_{[0, a]}(A, \varphi, \text{Fil}^*).
\]

By using the equivalence of categories in [Tsu17] §7, we further construct a canonical right inverse of this functor when \(a = 0, 1\) as

\[
\text{MIC}_{[0, a]}(A, \varphi, \text{Fil}^*) \xrightarrow{- \otimes A/(q - 1)^{a+1}} \text{MF}^{[p], q-1}_{[0, a]}(R'/(q - 1)^{a+1}, \varphi, \Gamma) \xleftarrow{\sim} \text{MG}^{[p], q-1}_{[0, a]}(R', \varphi, \Gamma)
\]

This applies to the objects \(U_{B'}\) (for \(A = B'\) and \(a = 0\)) and \(H_{B''} \otimes B''\mathbb{Z}_p[\lambda]\) (for \(A = \mathbb{Z}_p[\lambda]\) and \(a = 1\)) mentioned above.

One can ask whether there is a relationship between the canonical lifts (\(q\)-deformations) of \(U_{B'}\) and \(H_{B''} \otimes B''\mathbb{Z}_p[\lambda]\) constructed as above and the \(q\)-hypergeometric differential equation [GR99] with parameters \((q^2, q^2, q^2, q^2, q^2)\) defined by

\[
q\lambda(1 - q\lambda)d^2f + (1 - (1 + [2| q] - 2 [\frac{1}{q}])\lambda)d_qf - [\frac{1}{q}]^2f = 0,
\]

which is a \(q\)-analogue of the differential equation \([2]\). We give a positive answer to this question as follows. By “\(q\)-deforming” the relations of \(\nabla\) and \(\varphi\) on \(U_{B'}\) (resp. \(H_{B''} \otimes B''\mathbb{Z}_p[\lambda]\)) to the hypergeometric equation \([2]\) recalled above, we construct a Frobenius endomorphism and a \(q\)-connection on a free \(S''\)-module of rank 1 (resp. a free \(\mathbb{Z}_p[[q - 1]][\lambda]\)-module of rank 2) associated with the \(q\)-hypergeometric differential equation \([5]\), and show that it gives the desired canonical \(q\)-deformation of \(U_{B'}\) (resp. \(H_{B''} \otimes B''\mathbb{Z}_p[\lambda]\)).
In we state the main theorems. In we give explicit solutions of the \( q \)-hypergeometric equation \( 5 \), one of which involves a \( q \)-analogue of the logarithmic functions \( \log(\lambda) \) and \( \log(1 - \lambda) \), and compute a \( q \)-analogue of Wronskian of the explicit solutions. In we construct a \( q \)-deformation of \( \nabla \) on \( H_{B^a} \) which is related to the \( q \)-differential equation \( 5 \) similarly to the relation between \( \nabla \) on \( H_{B^a} \) and the differential equation \( 2 \) recalled above. In we show a \( (p, q - 1) \)-adic formal congruence, which is a \( q \)-analogue of Dwork’s results in \( \text{Dw69} \). In and \( 9 \) by applying the formal congruence to the explicit solutions constructed in we give Frobenius structures to the \( q \)-deformations of the connections on \( U_{B^a} \) and \( M_{B^a} \otimes B^a \mathbb{Z}[\lambda]|_\mathbb{Z} \) (constructed in \( 8 \)), and show that they give the desired canonical \( q \)-deformations. In we further show that the \( q \)-deformation of \( \varphi \) and \( \nabla \) on \( U_{B^a} \) admits an “arithmetic \( \Gamma \)-structure”.

**Remark 1.1.** The \( B^a \)-module \( H_{B^a} \) with \( \nabla, \varphi_{B^a}, \) and the filtration is an object of MIC\([0,1]\)(\( B^a, \varphi, \text{Fil}^* \)), and the functor \( 10 \) in \( 8 \) for \( a = 1 \) and \( R' = S'' := \mathbb{Z}_p[[q - 1]] \left( \lambda, \frac{1}{1 - \lambda} \right) \) gives a canonical \( q \)-deformation of \( H_{B^a} \) in MF\([\lambda]\)^{\text{\( q \)-1}}(\( S'', \varphi, \Gamma \)). Therefore one may ask whether its \( q \)-connection is related to the \( q \)-differential equation \( 3 \) similarly to the relation between \( \nabla \) on \( H^{1\text{\( dR \)}}_{B^a} \) and the differential equation \( 2 \). We can also apply the same construction to the log smooth extension of the Legendre family over the base \( \mathbb{Z}_p \left( \lambda, \frac{1}{1 - \lambda} \right) \) (without removing the supersingular locus), and ask the same question. It is natural to expect that this canonical \( q \)-deformation coincides with the conjectured (log) \( q \)-de Rham cohomology of the family. (We can compare the two in the category MF\([\lambda]\)^{\text{\( q \)-1}}(\( \mathbb{Z}_p[q - 1]/(q - 1)^2 \left( \lambda, \frac{1}{1 - \lambda} \right), \varphi, \Gamma \)), where our canonical \( q \)-deformation is reduced to the scalar extension by \( \mathbb{Z}_p \left( \lambda, \frac{1}{1 - \lambda} \right) \rightarrow \mathbb{Z}_p[q - 1]/(q - 1)^2 \left( \lambda, \frac{1}{1 - \lambda} \right) \). Thus our question is connected with the question by Scholze mentioned above.

**Notation.** We fix some notation used throughout this paper. Let \( p \) be an odd prime number. Let \( v_p \) be the \( p \)-adic valuation of \( \mathbb{Q}_p \), normalized by \( v_p(p) = 1 \). Let \( q \) be a formal variable. Let \( R = \mathbb{Z}_p[[q - 1]] \). Let \( Q \) be the quotient field of \( R \). Let \( B = \mathbb{Z}_p \left( \lambda, \frac{1}{1 - \lambda} \right) \). Let \( B' = \mathbb{Z}_p \left( \lambda, \frac{1}{1 - \lambda} \right) \). We equip \( B \) and \( B' \) with the \( p \)-adic topology. Let \( S' = R \left( \lambda, \frac{1}{1 - \lambda} \right) \). We equip \( R[[\lambda]] \) and \( S' \) with the \( (p, q - 1) \)-adic topology. For \( a \in \mathbb{Q} \cap \mathbb{Z}_p \), let \([a]_q\) be the \( q \)-number (the \( q \)-analogue of the rational number \( a \)) defined by

\[
[a]_q := \frac{q^a - 1}{q - 1} = \sum_{i=1}^{\infty} \binom{a}{i} (q - 1)^{i-1}.
\]

If \( a \) is a positive integer, \([a]_q\) is equal to \( 1 + q + q^2 + \cdots + q^{a-1} \).

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## 2 \( q \)-connection

We define \( R(\lambda) \) to be the completion of the polynomial ring \( R[\lambda] \) with respect to the \( (p, q - 1) \)-adic topology, namely,

\[
R(\lambda) := \varprojlim_n R[\lambda]/(p, q - 1)^n R[\lambda].
\]

For \( g(\lambda) \in R[\lambda] \setminus (p, q - 1)R[\lambda] \), we define \( R \left( \lambda, \frac{1}{g(\lambda)} \right) \) to be the completion of the ring \( R \left[ \lambda, \frac{1}{g(\lambda)} \right] \) with respect to the \( (p, q - 1) \)-adic topology. In this section, we construct a \( q \)-analogue of the differential operator \( \frac{d}{d\lambda} \) on \( R' = R[[\lambda]], R(\lambda) \) and \( R \left( \lambda, \frac{1}{g(\lambda)} \right) \) (\( g(\lambda) \in R[\lambda] \setminus (p, q - 1)R[\lambda] \)).
Definition 2.1. We define the Frobenius endomorphism $\varphi$ of $R[[\lambda]], R\langle\lambda\rangle$ and $R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$ $(g(\lambda) \in R[\lambda] \setminus (p, q-1)R[\lambda])$ as follows: First, we define the endomorphism $\varphi$ of $R[\lambda]$ satisfying $\varphi(x) \equiv x^p \mod p$ by $\varphi(a) = a (a \in Z_p)$, $\varphi(q) = q^p$, and $\varphi(\lambda) = \lambda^p$. Since it maps the ideals $(\lambda)$ and $(p, q-1)$ of $R[\lambda]$ into themselves, we can define $\varphi$ of $R[[\lambda]]$ and $R\langle\lambda\rangle$ by taking its $\lambda$- and $(p, q-1)$-adic completions, respectively. The endomorphism $\varphi$ of $R[\lambda]$ induces a homomorphism $\varphi: R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle \rightarrow R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$, and its $(p, q-1)$-adic completion gives the Frobenius endomorphism $\varphi$ on $R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$ by Lemma 2.2 below.

Lemma 2.2. For $g(\lambda), h(\lambda) \in R[\lambda]$ satisfying $g(\lambda) \equiv h(\lambda)^n \not\equiv 0 \mod (p, q-1)R[\lambda]$ for some integer $n > 0$, we have

$$R\left\langle\lambda, \frac{1}{g(\lambda)}\right\rangle = R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle.$$  

Proof. The congruence $g(\lambda) \equiv h(\lambda)^n \mod (p, q-1)$ implies that the image of $g(\lambda)$ in the quotient $R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle/\langle p, q-1\rangle R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle$ is a unit. Since $R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle$ is $(p, q-1)$-adically complete, $g(\lambda)$ is a unit of $R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle$. By the same argument, we see that $h(\lambda)$ is a unit of $R\left\langle\lambda, \frac{1}{g(\lambda)}\right\rangle$. Thus the natural homomorphisms $R[\lambda] \rightarrow R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle, R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle$ extend to homomorphisms

$$f: R\left\langle\lambda, \frac{1}{g(\lambda)}\right\rangle \rightarrow R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle, \quad g: R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle \rightarrow R\left\langle\lambda, \frac{1}{g(\lambda)}\right\rangle,$$

which satisfy $g \circ f = \text{id}$ and $f \circ g = \text{id}$. □

Definition 2.3. Let $\Gamma$ be a group isomorphic to $Z$, and let $\gamma$ be a generator of $\Gamma$. We define the action of $\Gamma$ on $R[\lambda], R\langle\lambda\rangle$ and $R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$ $(g(\lambda) \in R[\lambda] \setminus (p, q-1)R[\lambda])$ as follows. First, we define the action of $\Gamma$ on $R[\lambda]$ by $\gamma(\lambda) = q\lambda$ and $\gamma(a) = a (a \in R)$. Since this action of $\Gamma$ preserves the ideals $(\lambda)$ and $(p, q-1)$ of $R[\lambda]$, it induces an action of $\Gamma$ on $R[[\lambda]]$ and $R\langle\lambda\rangle$. The action of $\gamma$ on $R[\lambda]$ extends to an isomorphism $\gamma: R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle \rightarrow R\left\langle\lambda, \frac{1}{h(\lambda)}\right\rangle$, whose $(p, q-1)$-adic completion gives an action of $\gamma$ on $R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$ by Lemma 2.2.

Let $R'$ be one of the $R$-algebras $R[[\lambda]], R\langle\lambda\rangle$ and $R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$. Since $R'$ is noetherian, $(q-1)\lambda R'$ is a closed ideal of $R'$ with respect to the $\lambda$-adic topology for $R' = R[[\lambda]],$ and the $(p, q-1)$-adic topology for $R' = R\langle\lambda\rangle, R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$. Hence $(\gamma - 1)(\lambda^n) = (q-1)[n]q\lambda^n \in (q-1)\lambda R[\lambda]$ and $(\gamma - 1)(g(\lambda)^{-n}) = -g(\lambda)^{-n}\gamma(g(\lambda))^{-n}(\gamma - 1)(g(\lambda)^{-n}) \in (q-1)\lambda R\left\langle\lambda, \frac{1}{q\lambda}\right\rangle$ imply the inclusion $(\gamma - 1)(R') \subset (q-1)\lambda R'$.

Definition 2.4. We define the $q$-differential operator $d_q: R' \rightarrow R'$ by $d_q = \frac{\gamma - 1}{(q-1)\lambda}$.

This is a $q$-analogue of the differential operator $\frac{d}{d\lambda}$. Clearly we have $\gamma = 1 + (q-1)\lambda d_q$.

Proposition 2.5. We have a $q$-analogue of Leibniz rule:

$$d_q(xy) = d_q(x)\gamma(y) + xd_q(y), \quad x, y \in R'.$$

Proof. The formula follows from $(\gamma - 1)(xy) = (\gamma - 1)(x)\gamma(y) + x(\gamma - 1)(y)$. □

Definition 2.6. We define the $q$-differential module $q\Omega_{R'/R}$ to be the free $R'$-module $R'd\log\lambda$ and the $q$-derivation $\delta_q: R' \rightarrow q\Omega_{R'/R}$ by $d_q(x) = d_q(x) \cdot d\lambda$, where $d\lambda = \lambda d\log\lambda$. 

5
Let \( A = R'/\langle q - 1 \rangle R' \). Then \( q\Omega_{R'/R} \) mod \( q - 1 \) is naturally identified with the differential module \( \Omega_{A/\mathbb{Z}_p, \log} \) with log poles along the divisor \( \lambda = 0 \). Since \( \delta_q \) is \( R \)-linear, we can define \( \delta : A \to \Omega_{A/\mathbb{Z}_p, \log} \) as \( \delta_q \) mod \( q - 1 \). We have \( \delta(x) = \frac{dx}{\Delta} - d\lambda \), i.e., \( \delta \) is the universal continuous \( R \)-linear derivation.

**Definition 2.7.** Let \( M \) be an \( R' \)-module. An \( R \)-linear map \( \nabla_q : M \to M \otimes_{R'} q\Omega_{R'/R} \) is called a \( q \)-connection on \( M \) if it satisfies

\[
\nabla_q(\mathbf{a}m) = m \otimes \delta_q(a) + \gamma(a)\nabla_q(m), \quad a \in R', m \in M.
\]

Let \(( M, \nabla_q)\) be an \( R' \)-module with a \( q \)-connection. Let \( A = R'/\langle q - 1 \rangle R', N = M/\langle q - 1 \rangle M \) and let \( \nabla : N \to N \otimes_A \Omega_{A/\mathbb{Z}_p, \log} \) be \( \nabla_q \) mod \( q = 1 \). Then \( \nabla \) satisfies

\[
\nabla(\mathbf{a}m) = m \otimes \delta(a) + a\nabla(m)
\]

for all \( a \in A, m \in N \), i.e., \( \nabla \) is a connection on \( N \).

**Definition 2.8.** For an \( R' \)-module \( M \) with a \( q \)-connection \( \nabla_q \), we define the \( R \)-linear endomorphisms \( D^\log_q \) and \( \gamma_M \) of \( M \) by \( \nabla_q = D^\log_q \otimes d\log\lambda \) and \( \gamma_M = 1 + (q - 1)D^\log_q \). We define \( D_q \) to be \( \lambda^{-1}D^\log_q \) if \( \lambda \) is invertible in \( R' \).

We see that \( \gamma_M \) is \( \gamma \)-semilinear as follows: For all \( r \in R', m \in M \), we have

\[
\gamma_M(rm) = rm + (q - 1)D^\log_q(rm) = rm + (q - 1)(\lambda d_q(r)m + \gamma(r)D^\log_q(m)) = rm + (q - 1)(r)m + (q - 1)\gamma(r)D^\log_q(m) = \gamma(r)\gamma_M(m).
\]

The endomorphism \( \gamma_M \) is bijective and defines a \( \rho \)-semilinear action of \( \Gamma \) on \( M \) continuous with respect to the \( (p, q - 1) \)-adic topology.

**Remark 2.9.** For any \( \ell(\lambda) \in R' \setminus \{0\} \), we can define a \( q \)-connection \( \nabla_q : M \to M \otimes_{R'} \frac{1}{\ell(\lambda)} q\Omega_{R'/R} \) in the same way as in Definition 2.8. However we cannot construct the \( \Gamma \)-action associated to \( \nabla_q \) in general unless \( \ell(\lambda) \) is invertible in \( R' \).

We give some properties necessary for the \( q \)-analogue calculation in the following sections.

**Proposition 2.10.** (i) \([a + b]_q = [a]_q + q^a [b]_q\); (ii) \([ap]_q = [p]_q \varphi ([a]_q)\); (iii) \([p^n]_q \in (p, q - 1)^n R\); (iv) \(\gamma \circ \varphi = \varphi \circ \gamma \) on \( R' \); (v) \(d_q \varphi = [p]_q \lambda^{p - 1} \varphi d_q \) on \( R' \).

**Proof.** We can verify the equalities (i) and (ii) by simple computations. The claim (iii) follows from \([p^n]_q \equiv p^n \mod (q - 1)R\). For the claim (iv), since both sides are continuous endomorphisms, it suffices to show the commutativity for \( a \in \mathbb{Z}_p, q, \) and \( \lambda \), which is verified as follows:

\[
\gamma \circ \varphi(a) = a = \varphi \circ \gamma(a), \quad \gamma \circ \varphi(q) = q^p = \varphi \circ \gamma(q). \quad \gamma \circ \varphi(\lambda) = q^p \lambda^p = \varphi \circ \gamma(q).
\]

We obtain the equality (v) by substituting \( \gamma = 1 + (q - 1)\lambda d_q \) into the equality (iv).

## 3 Background

First, we define a category \( \text{MIC}_{[0, \infty)}(A, \varphi, \text{Fil}^\bullet) \). Let \( a \) be a non-negative integer. Let \( A \) be a \( p \)-torsion free commutative ring with an endomorphism \( \varphi \), and let \( \text{Fil}^\bullet A \) be the trivial decreasing filtration of \( A \) defined by

\[
\text{Fil}^r A = \begin{cases} A & r \in \mathbb{Z} \cap (-\infty, 0] \\ 0 & r \in \mathbb{Z} \cap (0, \infty). \end{cases}
\]
Let $\Omega_A$ be a free $A$-module of rank 1, let $\delta: A \to \Omega_A$ be a derivation, and let $\varphi^1: \Omega_A \to \Omega_A$ be a $\varphi$-semilinear homomorphism satisfying $\varphi^1 \circ \delta = \delta \circ \varphi$.

**Definition 3.1.** We define the category $\text{MIC}_{[0,a]}(A, \varphi, \text{Fil}^*)$ as follows. An object is a quartet $(M, \text{Fil}^*M, \varphi_M, \nabla)$ consisting of the following.

1. $M$ is a free $A$-module of finite type. (Let $N$ be the rank of $M$.)
2. A decreasing filtration $\text{Fil}^r M$ ($r \in \mathbb{N} \cap [0,a]$) of $M$ satisfying the following conditions.
   - (ii-a) There exists a basis $e_\nu$ ($\nu \in \mathbb{N} \cap [1,N]$) of $M$ and $r_\nu \in \mathbb{N} \cap [0,a]$ for each $\nu \in \mathbb{N} \cap [1,N]$ such that $\text{Fil}^r M = \bigoplus_{\nu \in \mathbb{N} \cap [1,N]} \text{Fil}^{r_\nu} \cdot e_\nu$ for $r \in \mathbb{N} \cap [0,a]$.
3. $\varphi_M: M \to M$ is a $\varphi$-semilinear endomorphism satisfying the following condition.
   - (iii-a) $\varphi_M(\text{Fil}^r M) \subseteq p^r M$ for $r \in \mathbb{N} \cap [0,a]$.
   - (iii-b) $M = \sum_{r \in \mathbb{N} \cap [0,a]} A \cdot p^{-r} \varphi_M(\text{Fil}^r M)$.
4. $\nabla: M \to M \otimes_A \Omega_A$ is a connection on $M$ satisfying the following condition.
   - (iv-a) $\nabla(\text{Fil}^r M) \subseteq \text{Fil}^{r-1} M \otimes_A \Omega_A$ for $r \in \mathbb{N} \cap [1,a]$.
   - (iv-b) $\nabla \circ \varphi_M = (\varphi_M \otimes \varphi^1) \circ \nabla$.

A morphism is an $A$-linear homomorphism preserving the filtration, and compatible with $\varphi_M$ and $\nabla$.

The Frobenius structure $\varphi^1_{H^1}$ recalled in [11] satisfies the following.

1. Assume that $a = 0$, $A = B'$, and $\Omega_A = Ad \lambda$. Then we have $(U_{B'}, \nabla, \varphi_{U_{B'}}) \in \text{MIC}_{[0,0]}(A, \varphi, \text{Fil}^*)$. (In the case $a = 0$, we can forget filtrations because the condition (ii-a) implies $\text{Fil}^0 U = U$ and $\text{Fil}^1 U = 0$.)
2. Assume that $a = 1$, $A = \mathbb{Z}_p[[\lambda]]$, and $\Omega_A = Ad \log \lambda$. Put $H_0 = H_{B'} \otimes_{B'} \mathbb{Z}_p[[\lambda]]$, which has the connection $\nabla$, the Frobenius structure $\varphi_{H_0}$, and the filtration induced by those of $H_{B'}$. Then we have $(H_0, \nabla, \varphi_{H_0}, \text{Fil}^*H_0) \in \text{MIC}_{[0,1]}(A, \varphi, \text{Fil}^*)$.
3. Assume that $a = 1$, $A = B''$, and $\Omega_A = Ad \log \lambda$. Then we have $(H_{B''}, \nabla, \varphi_{H_{B''}}, \text{Fil}^*) \in \text{MIC}_{[0,1]}(A, \varphi, \text{Fil}^*)$.

Next, we define a category $\text{MF}^{[p]a}_{[0,a]}(R', \varphi, \Gamma)$. Let $a$ be the same as above. As before Definition 2.4, let $R'$ be one of the rings $R[[\lambda]]$, $R(\lambda)$, and $R\left<\lambda, \frac{1}{p^m}\right>$ for $(g(\lambda) \in R[[\lambda]]\setminus(p,q-1)R(\lambda))$ equipped with the $(p,q-1)$-adic topology, and put $A = R'/(q-1)$. Then $R'$ is a commutative ring with an endomorphism $\varphi$ and an action $\rho$ of $\Gamma$ (See §2). Let $\text{Fil}^* R'$ be the decreasing filtration of $R'$ defined by

$$\text{Fil}^r R' = \begin{cases} R' & r \in \mathbb{Z} \cap (-\infty, 0) \\ (q-1)^r R' & r \in \mathbb{Z} \cap [0, \infty) \end{cases}.$$ 

We give $\Gamma$ a discrete topology. Then, as $\varphi(q-1) = (q-1)[p]_{q}$ and $\gamma \circ \varphi = \varphi \circ \gamma$, the quartet $(R', [p]_{q}, \text{Fil}^* R', \varphi, \rho)$ satisfies all conditions in §6, 7 of [Tsu17].

Thus we can define the category $\text{MF}^{[p]a}_{[0,a]}(R', \varphi, \Gamma)$.

**Definition 3.2 ([Tsu17], §7)**. We define the category $\text{MF}^{[p]a}_{[0,a]}(R', \varphi, \Gamma)$ as follows. An object is a quartet $(M, \text{Fil}^* M, \varphi_M, \rho_M)$ consisting of the following.
(i) $M$ is a free $R'$-module of finite type. (Let $N$ be the rank of $M$.)

(ii) A decreasing filtration $\text{Fil}^r M$ ($r \in \mathbb{N} \cap [0, a]$) of $M$ satisfies the following conditions.

(ii-a) There exists a basis $e_r$ ($\nu \in \mathbb{N} \cap [1, N]$) of $M$ and $r_r \in \mathbb{N} \cap [0, a]$ for each $\nu \in \mathbb{N} \cap [1, N]$ such that $\text{Fil}^r M = \oplus_{\nu \in \mathbb{N} \cap [1, N]} \text{Fil}^{r-r_r} R' e_r$ for $r \in \mathbb{N} \cap [0, a]$.

(iii) $\varphi_M : M \to M$ is a $\varphi$-semilinear endomorphism satisfying the following conditions.

(iii-a) $\varphi_M (\text{Fil}^r M) \subset [p]_q^r M$ for $r \in \mathbb{N} \cap [0, a]$.

(iii-b) $M = \sum_{r \in \mathbb{N}_0} r' \cdot [p]^{-r'}_q \varphi_M (\text{Fil}^r M)$.

(iv) $\rho_M : \Gamma \to \text{Aut}(M)$ is a $\rho$-semilinear action and satisfies the following conditions.

(iv-a) $\rho_M (g) (\text{Fil}^r M) = \text{Fil}^r M$ for $r \in \mathbb{N} \cap [0, a]$ and $g \in \Gamma$.

(iv-b) $\rho_M (g) \varphi_M = \varphi_M \rho_M (g)$ for $g \in \Gamma$.

(v) $\Gamma \times M \to M; (g, m) \mapsto \rho_M (g)m$ is continuous.

A morphism is an $R'$-linear homomorphism preserving the filtrations, compatible with $\varphi_M$’s , and moreover $\Gamma$-equivariant.

**Remark 3.3.** Let $M$ be a free $R'$-module equipped with a $q$-connection $\nabla_q : M \to M \otimes_R q \Omega_{R'/R}$, and a $\varphi$-semilinear endomorphism $\varphi_M$. Let $\rho_M$ be the $\rho$-semilinear continuous action of $\Gamma$ on $M$ associated to $\nabla_q$. Then $\varphi_M$ is $\Gamma$-equivariant, i.e., satisfy the condition (iv-b) in Definition 3.2 if and only if $(\varphi_M \otimes \varphi^1) \circ \nabla_q = \nabla_q \circ \varphi_M$. Here $\varphi^1$ is the $\varphi$-semilinear endomorphism of $q \Omega_{R'/R}$ defined by $\varphi^1 (d \log \lambda) = [p]_q d \log \lambda$. Note that we have $\varphi^1 \circ \delta_q = \delta_q \circ \varphi$ by Proposition 2.10(v).

**Definition 3.4.** We define $\text{MF}^{[p]_q, q^{-1}}_{[0, a]} (R', \varphi, \Gamma)$ to be the full subcategory of $\text{MF}^{[p]_q, \text{cont}}_{[0, a]} (R', \varphi, \Gamma)$ consisting of $M$ such that the $\Gamma$-action $\rho_M$ on $M$ satisfies $(\rho_M (\gamma) - 1)(M) \subset (q - 1)M$.

The Frobenius $\varphi$ and the $\Gamma$-action on $R'$ induce those on $R'(q - 1)^{a+1}$, and we can define the decreasing filtration of $R'$ by $\text{Fil}^\bullet (R'(q - 1)^{a+1}) = (\text{Fil}^\bullet R') \cdot (R'(q - 1)^{a+1})$. Therefore we can define the categories $\text{MF}^{[p]_q, \text{cont}}_{[0, a]} (R'(q - 1)^{a+1}, \varphi, \Gamma)$ and $\text{MF}^{[p]_q, q^{-1}}_{[0, a]} (R'(q - 1)^{a+1}, \varphi, \Gamma)$ in the same way.

We equip $A$ with the derivation $\delta : A \to \Omega_{A, \log} = Ad \log \lambda$, the reduction mod $q - 1$ of $\delta_q : R' \to q \Omega_{R'/R}$. Then the scalar extension by $R' \to A$ induces a functor

$$
\text{MF}^{[p]_q, q^{-1}}_{[0, a]} (R', \varphi, \Gamma) \xrightarrow{\text{mod } q - 1} \text{MIC}_{[0, a]} (A, \varphi, \text{Fil}^\bullet).
$$

(For $(M, \text{Fil}^\bullet M, \varphi_M, \rho_M) \in \text{MF}^{[p]_q, q^{-1}}_{[0, a]} (R', \varphi, \Gamma)$, we define the connection of $M/(q - 1)M$ by $\frac{\rho_M (\gamma) - 1}{q-1} \text{mod } q - 1 \otimes d \log \lambda$.)

For $a = 0, 1$, we have a functor induced by the base change from $A$ to $R'(q - 1)^{a+1}$:

$$
\text{MIC}_{[0, a]} (A, \varphi, \text{Fil}^\bullet) \to \text{MF}^{[p]_q, q^{-1}}_{[0, a]} (R'(q - 1)^{a+1}, \varphi, \Gamma).
$$

(7) For $(M, \text{Fil}^\bullet M, \varphi_M, \nabla) \in \text{MIC}_{[0, a]} (A, \varphi, \text{Fil}^\bullet)$, we define the $\rho$-semilinear action of $\Gamma$ on $M \otimes_A R'(q - 1)^{a+1}$ by $1 + (q - 1)D q^{-1} a$, if $a = 1$, and by $1$, i.e., the trivial action if $a = 0$. For the Frobenius structure, note that we have $[p]_q q = p \cdot (\text{unit})$ in $R'(q - 1)^{a+1}$ because $a + 1 \leq p - 1$. The reduction mod $(q - 1)^{a+1}$ gives an equivalence of categories [Tsu17]

$$
\text{MF}^{[p]_q, \text{cont}}_{[0, a]} (R', \varphi, \Gamma) \to \text{MF}^{[p]_q, \text{cont}}_{[0, a]} (R'(q - 1)^{a+1}, \varphi, \Gamma),
$$

(8)
which induces an equivalence between full subcategories
\[ MF_{[0,a]}^{[p,q-1]}(R',\varphi,\Gamma) \rightarrow MF_{[0,a]}^{[p,q-1]}(R'(q-1)^{a+1},\varphi,\Gamma). \] (9)

By combining (7) and (9), we obtain a right inverse of the functor (6)
\[ \text{MIC}_{[0,a]}(A,\varphi,\text{Fil}^\bullet) \rightarrow MF_{[0,a]}^{[p,q-1]}(R',\varphi,\Gamma). \] (10)

If \( a = 0 \), then the functor (7) is an equivalence of categories. Hence the functors (6) and (10) are equivalences of categories quasi-inverse of each other. By applying this functor, we obtain the canonical \( q \)-deformations of the objects \((U'_B,\nabla,\varphi_{U'_B})\) and \((H_0,\nabla,\varphi_{H_0},\text{Fil}^\bullet H_0)\) to \( MF_{[0,a]}^{[p,q-1]}(R',\varphi,\Gamma) \) for \( R' = S' \), \( a = 0 \) and \( R' = R[[\lambda]] \), \( a = 1 \), respectively.

We want to know whether the two canonical deformations are related to the \( q \)-hypergeometric differential equation (9).

**Remark 3.5.** To show the equivalence of categories (5), we have to check that \( R' \) with the \( \Gamma \)-action, \( \varphi \), and the filtration satisfies the conditions in [Tsu17, §6,§7], specifically [Tsu17, Conditions 39 and 54], while they are trivial.

4 Main theorems

As before Definition 2.4 let \( R' \) be one of the rings \( R[[\lambda]] \), \( R(\lambda) \), and \( R\left< \lambda, \frac{1}{g(\lambda)} \right> (g(\lambda) \in R[\lambda]\backslash\{(p,q-1)R[\lambda]\}) \) equipped with the \((p,q-1)\)-adic topology. Let \( \nabla_q: M' \rightarrow M' \otimes_R q\Omega_{R'/R} \) be the \( q \)-connection associated with the \( q \)-hypergeometric differential equation on a free \( R' \)-module \( M' \) of rank 2 introduced later in (6). The \( R' \)-module \( M' \) is also equipped with a filtration, and the reduction mod \( q = 1 \) of \((M',\nabla_q,\text{Fil}^\bullet)\) is canonically isomorphic to \((H_{B'} \otimes_{B'} A, \nabla, \text{Fil}^\bullet)\) (see (21)):
\[ (M',\nabla_q,\text{Fil}^\bullet) \otimes_{R'} A \cong (H_{B'} \otimes_{B'} A, \nabla, \text{Fil}^\bullet). \] (11)

Let \( S' = R\left< \lambda, \frac{1}{g(\lambda)} \right> \) be the \((p,q-1)\)-adic completion of \( R\left< \lambda, \frac{1}{g(\lambda)} \right> \), where \( h(\lambda) \) is the Hasse polynomial (see 3). In this paper, we give an appropriate Frobenius structure which is compatible with the \( q \)-connection on a rank 1 and \( \nabla_q \)-stable submodule of \( M' \) (resp. \( M' \) itself) in the case \( R' = S' \) (resp. \( R[[\lambda]] \)).

**Theorem 4.1.** Assume that \( R' = S' \). There exists a pair \((U',\varphi_{U'})\) which satisfies the following conditions.

(i) \( U' \) is a direct factor of the \( S' \)-module \( M' \) of rank 1 satisfying \( \nabla_q(U') \subset U' \otimes_R q\Omega_{R'/R} \). Let \( \rho_{U'} \) be the \( \rho \)-semilinear continuous action of \( \Gamma \) on \( U' \) associated to \( \nabla_q U' \).

(ii) \( \varphi_{U'} \) is a \( \varphi \)-semilinear endomorphism of \( U' \) and satisfies \( S' \cdot \varphi_{U'}(U') = U' \).

(iii) The triple \((U',\varphi_{U'},\rho_{U'})\) is an object of \( MF_{[0,0]}^{[p,q-1]}(S',\varphi,\Gamma) \), i.e., \( \rho_{U'}(\gamma) \circ \varphi_{U'} = \varphi_{U'} \circ \rho_{U'}(\gamma) \).

(iv) The isomorphism (11) induces a \( B' \)-linear isomorphism \( U' \otimes_{S'} B' \rightarrow U_{B'} \), and it defines an isomorphism between \((U_{B'},\nabla,\varphi_{U_{B'}})\) and the image of \((U',\varphi_{U'},\rho_{U'})\) under the equivalence of categories (see after (10))
\[ MF_{[0,0]}^{[p,q-1]}(S',\varphi,\Gamma) \rightarrow \text{MIC}_{[0,0]}(B',\varphi,\text{Fil}^\bullet). \]

**Theorem 4.2.** Assume that \( R' = R[[\lambda]] \), and let \( \rho_{M'} \) be the continuous \( \rho \)-semilinear action of \( \Gamma \) on \( M' \) associated to \( \nabla_q \). (Note that we have \( \nabla_q \Omega_{R'/R} = q\Omega_{R'/R} \) because \( 1 - \lambda \in R[[\lambda]]^\times \).) Then there exists a \( \varphi \)-semilinear endomorphism \( \varphi_{M'} \) on \( M' \) which satisfies the following conditions.
In this section, we give explicit solutions of the $q$-hypergeometric differential equation and its solutions

(i) There exists a basis $(f_1, f_2)$ of $M'$ such that $\varphi(e_1') = [p]_q f_1, \varphi(e_2') = f_2$.

(ii) The quartet $(M', \text{Fil}^* M', \varphi_{M'}, \rho_{M'})$ is an object of $\text{Fil}^*[p]_q [-1] (\mathcal{R}_{[\lambda]}, \varphi, \Gamma)$, i.e., $\rho_{M'}(\gamma)(\text{Fil}^1 M') = \text{Fil}^1 M'$ and $\rho_{M'}(\gamma) \circ \varphi_{M'} = \varphi_{M'} \circ \rho_{M'}(\gamma)$.

(iii) The isomorphism (11) induces an isomorphism

$$
(M', \varphi_{M'}, \nabla_q, \text{Fil}^* M') \otimes \mathcal{R}_{[\lambda]}[\lambda] / \mathbb{Z}_p[\lambda] \cong (H_0, \nabla, \varphi_{H_0}, \text{Fil}^* H_0).
$$

Moreover, it can be lifted to an isomorphism between $(M', \text{Fil}^* M', \varphi_{M'}, \rho_{M'})$ and the canonical $q$-deformation of $(H_0, \nabla, \varphi_{H_0}, \text{Fil}^* H_0)$ constructed after (10).

5 $q$-hypergeometric differential equation and its solutions

In this section, we give explicit solutions of the $q$-hypergeometric differential equation [GR90] defined by

$$
L[f] = q\lambda(1-q\lambda)d^2_q f + (1 - (1 + [2]_q - 2 [\frac{1}{2}]_q)\lambda)d_q f - [\frac{1}{2}]_q^2 f = 0,
$$

which is a $q$-analogue of the classical hypergeometric differential equation [vDP86]

$$
\lambda(1-\lambda)\frac{d^2}{d\lambda^2} f + (1 - 2\lambda)\frac{d}{d\lambda} f - \frac{1}{4} f = 0.
$$

For convenience, we put $\alpha = 1 + [2]_q - 2 [\frac{1}{2}]_q$.

To describe the solutions, we introduce $q$-logarithmic function $\log_q(-)$. (Note that the “$q$” in “log$_q$” does not mean a base.) Since $d_q$ does not have compatibility with the translation $\lambda \mapsto \lambda + a$ for $a \in \mathbb{Z}_p$, we have to define $\log_q \lambda$ and $\log_q(1-\lambda)$ respectively. Since $\varphi$ and $\gamma$ are injective as endomorphisms of $\mathcal{R}_{[\lambda]}$, we can extend them to endomorphisms of $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$. First, put $\log_q(1-\lambda) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{[n]_q}$. Then, $\log_q(1-\lambda)$ is an element of $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$ and $d_q \log_q(1-\lambda) = \frac{1}{1-q}$. Next, let $\log_q \lambda$ be a formal variable and extend $\varphi$ and $\gamma$ to endomorphisms of $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$ by

$$
\varphi(\log_q \lambda) = [p]_q \log_q \lambda \quad \text{and} \quad \gamma(\log_q \lambda) = q - 1 + \log_q \lambda.
$$

Then, by

$$
\gamma(\varphi(\log_q \lambda)) = (q-1)[p]_q + [p]_q \log_q \lambda = \varphi(\gamma(\log_q \lambda)),
$$

the commutativity $\gamma \circ \varphi = \varphi \circ \gamma$ is satisfied on $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$. Moreover, since $(q-1)\lambda \in \mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$, we can extend $d_q$ of $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$ to $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$ by $d_q = \frac{q-1}{(q-1)^2}$. We have $d_q \log_q \lambda = \frac{1}{1-q}$. In the theorem below, we give explicit solutions of (12) in $\mathcal{R}_{[\lambda]}(\mathcal{R}_{[\lambda]})$. Put $a_n = \prod_{i=0}^{n-1} \left( \frac{[i+\frac{1}{2}]_q}{[i+\frac{1}{2}]_q} \right)^2$ for a non-negative integer $n$.

**Theorem 5.1.** We have $L[F] = L[H] = 0$, where

$$
F = \sum_{n=0}^{\infty} a_n \lambda^n, \quad H = F \log_q \lambda - F \log_q(1-\lambda) - \sum_{n=0}^{\infty} a_n \lambda^n \sum_{i=1}^{n} \left( \frac{2}{[i]_q} + q - 1 \right).
$$

First, we give some lemma.

**Lemma 5.2.** (i) $d_q \gamma = q \gamma d_q$; (ii) $d_q \gamma + \gamma d_q = (1+q)(d_q + (q-1)\lambda d_q)$; (iii) $\gamma^2 = 1 + (q^2-1)\lambda d_q + q(q-1)^2\lambda^2 d_q^2$.


Proof. We can verify the equalities by simple computations. □

Proof of Theorem 5.1 First, we prove that \( L[F] = 0 \). Put \( y = \sum_{n=0}^{\infty} c_n \lambda^n \in Q[[\lambda]] \) \( (c_n \in Q) \). Then, the coefficient of \( \lambda^n \) in \(-\left[ \frac{1}{2} \right]_q y \), and in \( q \lambda(1 - q \lambda) d_q^2 y \), are \(-\left[ \frac{1}{2} \right]_q c_n, [n + 1]_q c_{n+1} - \alpha [n]_q c_n, \) and \( q [n + 1]_q [n]_q c_{n+1} - q^2 [n]_q [n-1]_q c_n \) respectively. By adding all of them, we see that the coefficient of \( \lambda^n \) in \( L[y] \) is \([n + 1]_q c_{n+1} - [n + \frac{1}{2}]_q c_n \). Since \( \{a_n\}_{n \geq 0} \) has the property \([n + 1]_q a_{n+1} = [n + \frac{1}{2}]_q a_n \) for \( n \geq 0 \), we obtain \( L[F] = 0 \).

For a non-negative integer \( r \), we put \( F_{\geq r} = \sum_{n=r}^{\infty} a_n \lambda^n \). Then by the same calculation as \( L[F] \), we have \( L[F_{\geq r}] = [r]_q a_r \lambda^{r-1} \). To prove \( L[H] = 0 \), we calculate \( L[F \log_q \lambda - F \log_q(1 - \lambda)] \) by using the following two claims.

Claim.

\[ L[F \log_q \lambda] = -2 \left[ \frac{1}{2} \right]_q F + 2(1 - q^{\frac{1}{2}} \lambda) d_q F. \] (13)

First, we calculate \( d_q(F \log_q \lambda) \) and \( d_q^2(F \log_q \lambda) \).

\[
d_q(F \log_q \lambda) = d_q F \log_q \lambda + \frac{1}{\lambda} F + (q - 1) d_q F;
\]

\[
d_q^2(F \log_q \lambda) = d_q^2 F \log_q \lambda + (\gamma d_q + d_q \gamma) F \left[ \frac{1}{\lambda} - \gamma^2(F) \frac{1}{q \lambda^2} \right]
= d_q^2 F \log_q \lambda - \frac{1}{q \lambda^2} F + \frac{q + 1}{q \lambda} d_q F + 2(q - 1) d_q^2 F.
\]

Thus,

\[
L[F \log_q \lambda] = q \lambda(1 - q \lambda) d_q^2(F \log_q \lambda) + (1 - \alpha \lambda)d_q(F \log_q \lambda) - \left[ \frac{1}{2} \right]_q F \log_q \lambda
= q \lambda(1 - q \lambda) \left( -\frac{1}{q \lambda^2} F + \frac{q + 1}{q \lambda} d_q F + 2(q - 1) d_q^2 F \right)
+ (1 - \alpha \lambda) \left( \frac{1}{\lambda} F + (q - 1) d_q F \right) \quad \text{(by } L[F] = 0)\n= (q - \alpha + 2(q - 1) \left[ \frac{1}{2} \right]_q) F + (2(-q^2 - q + \alpha(q - 1))) d_q F
= -2 \left[ \frac{1}{2} \right]_q F + 2(1 - q^{\frac{1}{2}} \lambda) d_q F.
\]

Claim.

\[ L[F \log_q(1 - \lambda)] = -\frac{2q \left[ \frac{1}{2} \right]_q \lambda - 1}{q \lambda - 1} F - 2q \lambda \frac{q^{\frac{1}{2}} \lambda - 1}{q \lambda - 1} d_q F. \] (14)

In the same way as above, we obtain

\[
d_q(F \log_q(1 - \lambda)) = d_q F \log_q(1 - \lambda) + \frac{1}{\lambda - 1} F + \frac{(q - 1)\lambda}{\lambda - 1} d_q F;
\]

\[
d_q^2(F \log_q(1 - \lambda)) = d_q^2 F \log_q(1 - \lambda) - \frac{1}{(\lambda - 1)(q \lambda - 1)} F + \frac{q + 1}{q \lambda - 1} d_q F + \frac{(q - 1)\lambda}{(\lambda - 1)(q \lambda - 1)} d_q^2 F.
\]
Thus, 

\[
L[F \log_q (1 - \lambda)] = q \lambda (1 - q \lambda) \left( -\frac{1}{\lambda - 1} \frac{F}{(q \lambda - 1)} + \frac{q + 1}{q \lambda - 1} d_q F + \frac{(q - 1) \lambda (2q \lambda - q - 1)}{(\lambda - 1)(q \lambda - 1)} d_q^2 F \right) \\
+ (1 - \alpha \lambda) \left( \frac{1}{\lambda - 1} F + \frac{(q - 1) \lambda}{\lambda - 1} d_q F \right) \quad \text{(by } L[F] = 0) \\
= \frac{(1 + (q - \alpha) \lambda)(q \lambda - 1) + \left[ \frac{1}{q} \right]^2 (q - 1) \lambda (2q \lambda - q - 1)}{(\lambda - 1)(q \lambda - 1)} F \\
+ \frac{(1 - \alpha \lambda)(q - 1) \lambda - q(q + 1) \lambda (\lambda - 1)(q \lambda - 1) - (1 - \alpha \lambda)(q - 1) \lambda (2q \lambda - q - 1)}{(\lambda - 1)(q \lambda - 1)} d_q F.
\]

Now the proofs of the two claims are completed. By claims (13) and (14),

\[
L[F \log_q \lambda - F \log_q (1 - \lambda)] \\
= 2(1 - q \lambda) d_q F - 2 \left( \left[ \frac{1}{q} \right] \lambda - 1 \text{ ) } \frac{2q \lambda - q - 1}{q \lambda - 1} d_q F \\
= 2q \left[ \frac{1}{q} \right] \lambda - 1 d_q F + 2 \left[ \frac{1}{q} \right] \frac{q \lambda - 1}{q \lambda - 1} F.
\]

We describe (15) as an $Q$-linear combination of $L[F_{\geq n+1}]$ for $n$ (and $d_q F$). We start by writing $\frac{1}{1 - q \lambda} F$ and $\frac{q \lambda}{1 - q \lambda} d_q F$ as $Q$-linear combinations of $L[F_{\geq n+1}]$.

\[
\frac{1}{1 - q \lambda} F = \left( \sum_{n=0}^{\infty} a_n \lambda^n \right) \left( \sum_{m=0}^{\infty} q^m \lambda^m \right) = \sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} a_m q^{n-m} = \sum_{n=0}^{\infty} L[F_{\geq n+1}] \left( \sum_{m=0}^{n} a_m q^{n-m} \right).
\]

\[
\frac{q \lambda}{1 - q \lambda} d_q F = \left( \sum_{n=0}^{\infty} \left[ \frac{n}{q} \right] a_n \lambda^{n-1} \right) \left( \sum_{m=0}^{\infty} q^m \lambda^m \right) = \sum_{n=0}^{\infty} L[F_{\geq n+1}] \left( \sum_{m=0}^{n} \left[ \frac{m}{q} \right] a_m q^{n+1-m} \right).
\]
Thus,

\[
2q^{\frac{1}{2}}\lambda - 1 \overline{d_q} F + \frac{2}{q} \left[ \frac{1}{2} \right] \frac{\lambda - 1}{d_q} F
\]

\[
= 2d_q F + 2 \sum_{n=0}^{\infty} L[F_{\geq n+1}] \sum_{m=0}^{n} \left( \frac{2}{q} - q^{\frac{1}{2}} \right) [m] q a_m q^{n+1-m} - 2 \left[ \frac{1}{2} \right] q - 1 \right) \frac{1}{1-q\lambda} F
\]

\[
= 2d_q F + \sum_{n=0}^{\infty} L[F_{\geq n+1}] \sum_{m=0}^{n} \left( 2(q - q^{\frac{1}{2}}) [m] q - 2 \left[ \frac{1}{2} \right] - 1) a_m q^{n-m} \right)
\]

\[
= 2d_q F + (q-1) \sum_{n=0}^{\infty} L[F_{\geq n+1}] \quad \text{(by Lemma 5.3 below)}
\]

\[
= 2 \sum_{r=0}^{\infty} \frac{1}{[n+1]_q} L[F_{\geq n+1}] + (q-1) \sum_{n=0}^{\infty} (2(q - q^{\frac{1}{2}}) [m] q - 2 \left[ \frac{1}{2} \right] - 1) a_m q^{n-m}
\]

\[
= L \left[ \sum_{r=0}^{\infty} a_n \lambda^n \sum_{n=1}^{\infty} \left( \frac{2}{[n]_q} - 1 \right) \right].
\]

Hence we obtain

\[
L \left[ F \log_q \lambda - F \log_q (1 - \lambda) \right] - L \left[ \sum_{n=1}^{\infty} a_n \lambda^n \sum_{i=1}^{n} \left( \frac{2}{[n]_q} - 1 \right) \right] = 0.
\]

\[\square\]

**Lemma 5.3.** For all \(n \in \mathbb{N},\)

\[
\sum_{m=0}^{n} \left( 2(q - q^{\frac{1}{2}}) [m] q - 2 \left[ \frac{1}{2} \right] - 1) a_m q^{n-m} = (q - 1) [n+1]_q^2 a_{n+1}. \tag{16}\]

**Proof.** We prove this by induction on \(n.\) If \(n = 0,\) the left-hand side of (16) is \(-\left[ \frac{1}{2} \right]_q = (q-1) \left[ \frac{1}{2} \right]_q^2.\)

So it is equal to the right-hand side of (16). We assume (16) for \(n.\) To prove (16) for \(n + 1,\) it suffices to show

\[
(2(q - q^{\frac{1}{2}}) [n] q - 2 \left[ \frac{1}{2} \right] q + 1) a_n = (q - 1) \left( [n+1]_q^2 a_{n+1} - [n]_q^2 a_n \right) \tag{17}.
\]

We can verify the equation (17) by simple computations. \[\square\]

**Remark 5.4.** We have another description of \(H:\)

\[
H = F \log_q \lambda + \sum_{n=1}^{\infty} a_n \lambda^n \left( \sum_{i=1}^{n} \frac{2}{[i]_q - 1} \right).
\]

One can show that the right-hand side is annihilated by \(L\) in the same way as the proof of \(L[H] = 0\) in Theorem 5.1.

In the rest of this section, we calculate a \(q\)-analogue of Wronskian \(W(F, H) = F d_q H - H d_q F.\) in the preparation for the computation in §8.

**Lemma 5.5.** We have \(F d_q H - H d_q F = \frac{1}{\lambda(1-\lambda)}.\)
Proof. We have
\[
q(1 - \lambda) dqF = (1 - [2]_q \lambda) dqF + \gamma(1 - \lambda) dq^2 F
= (1 - [2]_q \lambda) dqF + q(1 - \lambda) dq^2 F
= (1 - [2]_q \lambda) dqF - (1 - \alpha \lambda) dqF + [\frac{1}{2}]_q dq^2 F
= \left[\frac{1}{2}\right]_q^2 \gamma(F).
\]
Similarly, \(dq(1 - \lambda) dqH = \left[\frac{1}{2}\right]_q^2 \gamma(H)\). Thus,
\[
dq(1 - \lambda)(Fd_q H - Hd_q F) = dqF \cdot (1 - \lambda) dqH + \gamma(F) \cdot dq(1 - \lambda) dqH
- dqH \cdot (1 - \lambda) dqF - \gamma(H) \cdot dq(1 - \lambda) dqF
= \gamma(F) \cdot dq(1 - \lambda) dqH - \gamma(H) \cdot dq(1 - \lambda) dqF
= 0.
\]
Since \(\lambda(1 - \lambda)(Fd_q H - Hd_q F) \in Q[[\lambda]]\) and \(Q[[\lambda]]^{dq=0} = Q\), we see that \(\lambda(1 - \lambda)(Fd_q H - Hd_q F)\) is constant. For all \(g \in Q[[\lambda]]\), we have \(\lambda g = 0\), \(\lambda dq(g) = 0\) when \(\lambda = 0\). Therefore, we have
\[
\lambda(1 - \lambda)(Fd_q H - Hd_q F) |_{\lambda=0} = \lambda(1 - \lambda)(Fd_q F \log_q \lambda - Fd_q F \log_q \lambda) |_{\lambda=0}
= \lambda(1 - \lambda) \left( Fd_q F \log_q \lambda + \gamma(F) \frac{1}{\lambda} - Fd_q F \log_q \lambda \right) |_{\lambda=0}
= (1 - \lambda) F \gamma(F) |_{\lambda=0}
= 1.
\]

6 \(q\)-connection and \(q\)-hypergeometric differential equation

Let \(R'\) be one of the rings \(R[[\lambda]], R(\lambda),\) and \(R \left( \lambda, \frac{1}{\gamma(\lambda)} \right) \) \((g(\lambda) \in R[\lambda] \setminus (p, q - 1) R[\lambda])\). Let \(M''\) be the free \(R'\) module \(R'e_1 \oplus R'e_2\) of rank 2. In this section, we determine a \(q\)-connection \(\nabla_q : M'' \to M'' \otimes_R q \Omega_{R'/R}\)
(\text{Remark 2.9}) which satisfies

\[
\nabla_q \left( e_1 \quad e_2 \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = 0 \iff dq(f_2) = f_1 \text{ and } L[f_2] = 0,
\]

(18)

which is a \(q\)-analogue of \([vdP86], \text{Proposition 7.11 (iii)}\) (see the claim before (1)). Here \(f_1\) and \(f_2\) are elements of any extension \(C\) of \(\text{Frac}R'\) which is \((q - 1)\)-torsion free and carries an extension of \(\Gamma\)-action satisfying \((\gamma - 1)C \subset (q - 1)C\). Let \(P \in \frac{1}{\lambda - 1} M_2(R')\) and define a \(q\)-connection \(\nabla_q : M'' \to M'' \otimes_R q \Omega_{R'/R}\)
by \(\nabla_q (e_1 \quad e_2) = \left( e_1 \quad e_2 \right) P \otimes dq\). Then,
\[
\nabla_q \left( e_1 \quad e_2 \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = 0 \iff (e_1 \quad e_2) P \gamma \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) + (e_1 \quad e_2) dq \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = 0
\]

\[
\iff (1 + (q - 1) \lambda P) \left( \begin{array}{c} \gamma(f_1) \\ \gamma(f_2) \end{array} \right) = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right).
\]

(19)

We define \(P' \in M_2(\text{Frac}R')\) by \(1 + (q - 1) \lambda P = (1 + (q - 1) \lambda P')^{-1}\). Then the equation (19) is equivalent to \(dq \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = P' \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)\). Hence (18) holds when
\[
P' = \frac{1}{q \lambda(1 - q \lambda)} \left( \begin{array}{cc} -1 + \alpha \lambda & [\frac{1}{2}]_q^2 \\ q \lambda(1 - q \lambda) & 0 \end{array} \right).
\]

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Therefore,
\[
1 + (q - 1)\lambda P = (1 + (q - 1)\lambda P')^{-1}
\]
\[
= q\lambda(1 - q\lambda) \left(\frac{q\lambda(1 - q\lambda) - (q - 1)\lambda(1 - \alpha\lambda)}{(q - 1)\lambda q\lambda(1 - q\lambda)} \left(\frac{q\lambda(1 - q\lambda)}{\lambda(1 - \lambda)}\right)^2\right)^{-1}
\]
\[
= \frac{1}{\lambda(1 - \lambda)} \left(\frac{q\lambda(1 - q\lambda)}{\lambda(1 - \lambda)} - (q - 1)\lambda \left(\frac{1}{\lambda}\right)^2 (q - 1)\lambda\right)
\]
and we obtain
\[
P = \frac{1}{\lambda(1 - \lambda)} \left(\frac{1 - [2]_q \lambda}{-q\lambda(1 - q\lambda)} - \left(\frac{1}{\lambda}\right)^2 (q - 1)\lambda\right).
\]

Let \(\overline{\tau}_1\) and \(\overline{\tau}_2\) be the elements \(\lambda(1 - \lambda)\omega\) and \(-\lambda(1 - \lambda)D(\omega)\) of \(H^1_{\text{dR}}\), respectively (see \(\mathcal{H}\)). Then, in the case \(R' = R \left< \lambda, \frac{1}{\lambda(1 - \lambda)}\right>\), the comparison with the formula \(\mathcal{I}\) shows that we have the following isomorphism of \(B'\)-modules with connection and filtration
\[
(M'', \nabla_q) \otimes_{R'} B \xrightarrow{\cong} (H^1_{\text{dR}}, \nabla); e_1 \otimes 1, e_2 \otimes 1 \mapsto \overline{\tau}_1, \overline{\tau}_2.
\]

(20)

Put \(e'_1 = \frac{1}{\lambda(1 - \lambda)}\right)e_1 \in M'' \left< \frac{1}{\lambda(1 - \lambda)}\right>,\) and let \(M'\) be the free \(R'\)-submodule \(R'e'_1 \oplus R'e_2\) of \(M'' \left< \frac{1}{\lambda(1 - \lambda)}\right>.\)

Then the \(q\)-connection \(\nabla_q\) on \(M''\) uniquely extends to a \(q\)-connection on \(M'' \left< \frac{1}{\lambda(1 - \lambda)}\right>,\) and by a straightforward computation, we see that its restriction to \(M'\) gives the \(q\)-connection \(\nabla_q': M' \to M' \otimes_{R'} \frac{1}{\lambda(1 - \lambda)}q\Omega_{R'/R}\) on \(M'\) defined by
\[
\nabla_q(e'_1, e_2) = (e'_1, e_2) \left(\frac{0}{\lambda(1 - \lambda)} - \left(\frac{1}{\lambda}\right)^2 (q - 1)\lambda\right) \otimes d\lambda.
\]

We define the filtration on \(M'\) by
\[
\text{Fil}^r M' = \begin{cases} 
M' & r = 0 \\
R'e'_1 \oplus (q - 1)R'e_2 & r = 1.
\end{cases}
\]

Then, in the case \(R' = R \left< \lambda, \frac{1}{\lambda(1 - \lambda)}\right>\), we have an isomorphism of \(B'\)-modules with connection and filtration
\[
(M', \nabla_q, \text{Fil}^\bullet) \otimes_{R'} B' \xrightarrow{\cong} (H^1_{B'}, \nabla, \text{Fil}^\bullet); e'_1, e_2 \mapsto \overline{\tau}'_1, \overline{\tau}'_2.
\]

(21)

7 \(q\)-analogue of \(p\)-adic formal congruence

In this section, we prove some formal congruence between power series in \(R[[\lambda]]\) and show that certain constructions give elements of a ring smaller than \(R[[\lambda]]\) by constructing \(q\)-analogues of some results of [Dw09] \(\S 1-3\). Put
\[
C_\theta(n) = \prod_{\nu=0}^{n-1} \left[\theta + \nu\right]_q.
\]

Let \(\theta \in \mathbb{Z}_p\) be neither zero nor negative rational integer. We define \(\theta' \in \mathbb{Q} \cap \mathbb{Z}_p\) to be the unique number such that \(p\theta' - \theta\) is an ordinary integer in \([0, p - 1]\). For all \(x \in \mathbb{Q}\), we put
\[
\rho(x) = \begin{cases}
0 & x \leq 0 \\
1 & x > 0.
\end{cases}
\]
Lemma 7.1. [Dw69, §1, Lemma 1 (1,1)] Let $\mu, s$ be positive integers. Let $a \in \mathbb{N} \cap [0, p-1]$. Then,

$$
\frac{C_\theta(a + \mu p + mp^{s+1})}{\varphi(C_\theta(\mu + mp^s))} = \frac{C_\theta(mp^{s+1})}{\varphi(C_\theta(mp^s))} \left( \frac{[mp^s]_q}{[\theta + \mu]_q} \right)^{p(a + \theta - mp^s)} \mod 1 + [p^{s+1}]_q R.
$$

Proof. By the definition of $C_\theta$,

$$
\frac{C_\theta(a + \mu p + mp^{s+1})}{C_\theta(mp^{s+1})} = \prod_{\nu=0}^{a+mp-1} \left[ \theta + mp^{s+1} + \nu \right]_q
$$

and

$$
\frac{C_\theta(mp^{s+1})}{C_\theta(mp^s)C_\theta(\mu + mp^s)} = \prod_{\nu=0}^{a+mp-1} \left[ \theta + mp^{s+1} + \nu \right]_q \frac{\theta + \nu}{[\theta + \nu]_q} = \prod_{\nu=0}^{a+mp-1} \left( 1 + q^{\theta + \nu} \frac{[mp^s]_q}{[\theta + \nu]_q} \right).
$$

We have

$$
[mp^{s+1}]_q = [p^{s+1}]_q \left( 1 + q^{\nu_1 + 1} + q^{\nu_2 + 1} + \cdots + q^{\nu_m + 1} \right) \in [p^{s+1}]_q R,
$$

so the proof of Lemma 1.1 is almost the same as the Dwork’s proof in [Dw69, §1 Lemma 1 p.31].

Lemma 7.2. [Dw69, §1, Lemma 1 (1,2)] Let $\mu, s$ be positive integers. Then,

$$
\frac{C_\theta(mp^{s+1})}{\varphi(C_\theta(mp^s))} \equiv \frac{C_1(mp^{s+1})}{\varphi(C_1(mp^s))} \mod 1 + [p^{s+1}]_q R.
$$

Proof. By putting $a = 0, \mu = p^s$, the proof of Lemma 7.2 is reduced to the case $m = 1$. For $\nu \in \mathbb{N} \cap [0, p^{s+1} - 1]$, the condition

$$
\theta + \nu \equiv 0 \mod p
$$

is equivalent to the condition that there exists $\nu' \in \mathbb{N} \cap [0, p^s - 1]$ such that $\nu = (p\theta' - \theta) + p\nu'$. This condition implies that $\theta + \nu = p(\theta + \nu')$. Thus, we have

$$
\frac{C_\theta(p^{s+1})}{\varphi(C_\theta(p^s))} = \prod_{\nu', \theta + \nu'} \left[ p(\theta' + \nu') \right]_q \prod_{\nu, \theta + \nu} \left[ \theta + \nu \right]_q = \prod_{\nu', \theta + \nu'} \left[ p^{\theta + \nu'} \right]_q \prod_{\nu, \theta + \nu} \left[ \theta + \nu \right]_q.
$$

Especially, by putting $\theta = \theta' = 1$, we have (replacing $\nu$ with $\nu_1$)

$$
\frac{C_1(p^{s+1})}{\varphi(C_1(p^s))} = [p]_q^{p\nu_1} \prod_{\nu_1 \in \mathbb{N} \cap [0, p^{s+1} - 1]} [1 + \nu_1]_q.
$$

The sets $\{ \theta + \nu \mid p \nmid \theta + \nu, \nu \in \mathbb{N} \cap [0, p^{s+1} - 1] \}$ and $\{ 1 + \nu_1 \mid p \nmid 1 + \nu_1, \nu_1 \in \mathbb{N} \cap [0, p^{s+1} - 1] \}$ are both representatives of $(\mathbb{Z}/p^{s+1}\mathbb{Z})^\times$, so they have one-to-one correspondence. Namely for all $\nu$, there is a unique $\nu_1$ such that

$$
\theta + \nu \equiv 1 + \nu_1 \mod p^{s+1}.
$$

Since $(1 + \nu_1, p) = 1$, we have $\theta + \nu = (1 + \nu_1)(1 + p^{s+1}a_{\nu})$ for some $a_{\nu} \in \mathbb{Z}_q$. Thus, we have

$$
\frac{C_\theta(p^{s+1})}{\varphi(C_\theta(p^s))} = [p]_q^{p\nu_1} \prod_{1 + \nu_1} \left[ 1 + \nu_1 \right]_q\left(1 + q^{1 + \nu_1} \frac{[p^{s+1}a_{\nu}(1 + \nu_1)]_q}{[1 + \nu_1]_q} \right).
$$

This is congruent to $\frac{C_1(p^{s+1})}{\varphi(C_1(p^s))} \mod 1 + [p^{s+1}]_q R$, which follows from $\frac{[p^{s+1}a_{\nu}(1 + \nu_1)]_q}{[1 + \nu_1]_q} \in [p^s + 1]_q R$. 

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Corollary 7.3. [Dw69 §1, Corollary 2] Assume that \( \theta = \frac{1}{2} \). (Then we have \( \theta' = \frac{1}{2} \) by the definition.)

(i) Assume \( a > \frac{v_{p+1}}{2} \). Then
\[
\frac{A_{\frac{1}{2}}(a + \mu p)}{\varphi(\mu)} \equiv 0 \mod \left[ p^{1+v_p\left(\mu + \frac{1}{2}\right)} \right]_q R.
\]

(ii) We have
\[
\frac{A_{\frac{1}{2}}(n + mp^{+1})}{\varphi(\mu)} \equiv \frac{A_{\frac{1}{2}}(n)}{\varphi(\mu)} \mod \left[ p^{s+1} \right]_q R.
\]

Proof. (i) Put
\[
r(N, m) = \left| \left\{ \frac{1}{2} + i \mid i \in \mathbb{N} \cap [0, N - 1], v_p\left(\frac{1}{2} + i\right) \geq m \right\} \right| - \left| \left\{ i \mid i \in \mathbb{N} \cap [0, N - 1], v_p(i) \geq m \right\} \right|.
\]
If we write \( N = b + p^m c \) \((b, c \in \mathbb{N}, b \in [0, p^m - 1])\), we can rewrite \( r(N, m) \) by
\[
r(N, m) = \begin{cases} 1 & b \geq \frac{1}{2}(p^m + 1) \\ 0 & b < \frac{1}{2}(p^m + 1). \end{cases}
\]
Put
\[
\Phi_p(q) = \frac{q^{p^m} - 1}{q^{p^m - 1} - 1}.
\]
Then modulo units, we have
\[
A_{\frac{1}{2}}(a + \mu) = \left( \prod_{m=1}^{\infty} \Phi_p(q)^{r(\mu, m)} \right) \varphi \left( \frac{\mu}{p} \right)
\]
and
\[
\varphi \left( \frac{\mu}{p} \right) = \varphi \left( \prod_{m=1}^{\infty} \Phi_p(q)^{r(\mu, m)} \right) = \prod_{m=1}^{\infty} \Phi_p(q)^{r(\mu, m)} = \prod_{m=1}^{\infty} \Phi_p(q)^{r(\mu, m - 1)}.
\]
Therefore, by setting \( r(N, 0) = 0 \) for \( N \in \mathbb{N} \) we obtain
\[
\frac{A_{\frac{1}{2}}(a + \mu p)}{\varphi(\mu)} = \left( \prod_{m=1}^{\infty} \Phi_p(q)^{r(\mu, m - 1)} \right).
\]
So we have to determine \( r(a + \mu p, m) - r(\mu, m - 1) \) for each \( m \).

Thus,
\[
\nu = \left[ \frac{a + \nu p}{p} \right] = \left[ \frac{p^m - 1}{2p} \right] \leq \frac{1}{2}(p^m - 1).
\]

So \( m - 1 \geq p^m - 1 \) implies \( r(a + \mu p, m) - r(\mu, m - 1) \geq 0 \) for all \( m. \) Assume \( m - 1 \leq \nu \left(\mu + \frac{1}{2}\right) \), then we obtain
\[
\nu \left(\mu + \frac{1}{2}\right) \geq m - 1,
\]
from which we obtain
\[
\mu + \frac{1}{2} \geq \frac{p + 1}{2},
\]
and
\[
\nu \geq \frac{1}{2}(p^m - 1).
\]

Put \( A_\theta(n) = C_\theta(n)/C_1(n) \) for a non-negative number \( n. \)
The inequality \( [23] \) means \( r(a + \mu m, m) = 1 \) and \( [24] \) means \( r(\mu, m - 1) = 0 \), so we obtain \( r(a + \mu p, m) - r(\mu, m - 1) = 1 \) under the condition \( m \leq 1 + v_p(\mu + \frac{a}{m}) \). Therefore,

\[
\frac{A_\frac{a}{m}(n)}{\varphi\left(A_\frac{a}{m}\left(\frac{n}{m}\right)\right)} \in \prod_{m=1}^{\infty} \Phi_p(q) = \left[p^{1+v_p(\mu+\frac{a}{m})}\right]_q.
\]

(ii) Put \( n = a + \mu p \ (a, \mu \in \mathbb{N}, a \in \{0, p - 1\}) \). Then,

\[
\frac{A_\frac{a}{m}(n + mp^{s+1})}{\varphi\left(A_\frac{a}{m}\left(\frac{n}{m} + mp\right)\right)} = \frac{\varphi(C_\frac{a}{m}(\mu + mp^{s+1}))}{\varphi\left(A_\frac{a}{m}(\mu + mp)\right)}
\]

\[
= \frac{\varphi(C_\frac{a}{m}(\mu + mp))}{\varphi(A_\frac{a}{m}(\mu))}
\]

\[
= \left[\varphi\left(1 + q^{\frac{a}{m} + \mu}\left[\frac{mp^s}{q^{1+\mu}}\right]_q\right)\right]^{o(a - \frac{a}{m})} \mod [p^{s+1}]_q R \quad \text{(by Lemma 7.1)}
\]

\[
= \frac{A_\frac{a}{m}(a + \mu p)}{\varphi(A_\frac{a}{m}(\mu))} \left[\varphi\left(1 + q^{\frac{a}{m} + \mu}\left[\frac{mp^s}{q^{1+\mu}}\right]_q\right)\right]^{o(a - \frac{a}{m})} \mod [p^{s+1}]_q R
\]

by Lemma 7.1 again. (Use for \( \theta = \theta' = 1 \).) Thus, if \( a - \frac{a}{m} \leq 0 \), \( \text{ii} \) is clear. Assume \( a - \frac{a}{m} > 0 \), then

\[
\frac{A_\frac{a}{m}(a + \mu p)}{\varphi(A_\frac{a}{m}(\mu))} \left[\varphi\left(1 + q^{\frac{a}{m} + \mu}\left[\frac{mp^s}{q^{1+\mu}}\right]_q\right)\right]^{o(a - \frac{a}{m})} \mod [p^{s+1}]_q R
\]

This is congruent to 0 modulo \( [p^{s+1}]_q R \), which follows from

\[
\frac{A_\frac{a}{m}(a + \mu p)}{\varphi(A_\frac{a}{m}(\mu))} \in \left[p^{1+v_p(\mu+\frac{a}{m})}\right]_q R
\]

and

\[
[p^{1+v_p(\mu+\frac{a}{m})}]_q \varphi\left(\frac{[p^s]_q}{[p^{s+1}]_q}\right) = [p]_q \varphi\left([p^s]_q\right) = [p^{s+1}]_q.
\]

\]

\[
\square
\]

**Theorem 7.4.** \([\text{Dw69}, \text{Theorem 2}]\) Let \( A = B^{(-1)}, B = B^{(0)}, B^{(1)}, B^{(2)}, \ldots \), be a sequence of functions on \( \mathbb{N} \) with values in \( Q := \text{Frac}(R) \). Put

\[
F(\lambda) = \sum_{n=0}^{\infty} A(n) \lambda^n, \quad G(\lambda) = \sum_{n=0}^{\infty} B(n) \lambda^n.
\]

Assume for all \( n, m, s \in \mathbb{N}, i \geq -1, \)
(a) \[ \frac{B^{(i)}(n+mp^s)}{\varphi(B^{(i)}([\frac{n}{p}]+mp^s))} \equiv \frac{B^{(i)}(n)}{\varphi(B^{(i)}([\frac{n}{p}]))} \mod [p^{s+1}]_q R. \]

(b) \[ \frac{B^{(i)}(n)}{\varphi(B^{(i)}([\frac{n}{p}]))} \in R. \]

(c) \[ B^{(i)}(n) \in R. \]

(d) \[ B^{(i)}(0) \in R^\times. \]

Then,

\[ F(\lambda) \varphi \left( \sum_{j=mp^s}^{(m+1)p^s-1} B(j) \lambda^j \right) \equiv \varphi(G(\lambda)) \sum_{j=mp^s+1}^{(m+1)p^s-1} A(j) \lambda^j \mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m)) R[[\lambda]]. \]

**Proof.** Let \( n = a + pN \) (\( a \in \mathbb{N} \cap [0, p - 1] \)). So the coefficient of \( \lambda^n \) on the left side of (7.4) is

\[ \sum_{j=mp^s}^{(m+1)p^s-1} A(n - pj) \varphi(B(j)), \]

and the coefficient of \( \lambda^n \) on the right side of (7.4) is

\[ \sum_{j=mp^s}^{(m+1)p^s-1} \varphi(B(N - j)) A(a + pj). \]

Let

\[ U_a(j, N) = A(a + p(N - j)) \varphi(B(j)) - \varphi(B(N - j)) A(a + pj), \]

\[ H_a(m, s, N) = \sum_{j=mp^s}^{(m+1)p^s-1} U_a(j, N). \]

Then what we have to show is

\[ H_a(m, s, N) \equiv 0 \mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m)) R. \] (25)

Since \( a \in [0, p - 1] \), we have \( U_a(j, N) = 0 \) for \( j > N \). So

\[ H_a(m, s, N) = 0 \quad \text{for} \quad N < mp^s. \] (26)

In preparation for the proof of this theorem, we note some facts. The proof of these facts is almost the same as [Dw69].

\[ \sum_{m=0}^{T} H_a(m, s, N) = 0 \quad \text{for} \quad (T + 1)p^s > N. \] (27)

\[ H_a(m, s, N) = \sum_{\mu=0}^{p-1} H_a(\mu + mp^s, s - 1, N) \quad \text{for} \quad s \geq 1. \] (28)

\[ B^{(i)}(i + mp^s) \equiv 0 \mod \varphi^s(B^{(s+t)}(m)) R \quad \text{for} \quad i \in \mathbb{N} \cap [0, p^s - 1], s \geq 0, t \geq -1. \] (29)
We prove \([24]\) using induction on \(s\). Put the induction hypothesis

\[(\alpha)_s : \hat{H}_a(m,u,N) \equiv 0 \mod \ [p^{u+1}]_q \varphi^{u+1}(B^{(u)}(m))R \quad \text{for} \quad 0 \leq u < s,
\]

and the supplementary hypothesis

\[(\beta)_{t,s} : \hat{H}_a(m,s,N+mp^s) \equiv \sum_{j=0}^{p^{t-1}} \frac{\varphi^{t+1}(B^{(1)}(j+mp^{s-1}))H_a(j,t,N)}{\varphi^{t+1}(B^{(1)}(j))} \mod \ [p^{t+1}]_q \varphi^{t+1}(B^{(s)}(m))R \quad \text{for} \quad t \in \mathbb{N} \cap [0,s].
\]

Then \((\alpha)_s\) for all \(s \geq 1\) is reduced to the following four claims:

(i) \((\alpha)_1\); (ii) \((\beta)_{0,s}\); (iii) \((\beta)_{t,s}\) and \((\alpha)_s\) imply \((\beta)_{t+1,s}\); (iv) \((\beta)_{s,s}\) implies \((\alpha)_{s+1}\).

(i) By \([20]\), we may assume \(N \geq m\). By hypothesis \([\mathbb{1}]\),

\[
\frac{A(a+p(N-m))}{\varphi(B(N-m))} = \frac{A(a)}{\varphi(B(0))} \mod \ [p]_q R.
\]

Especially if \(N = 2m\), \(\frac{A(a+pmn)}{\varphi(B(m))} \equiv \frac{A(a)}{\varphi(B(0))} \mod \ [p]_q R\). Thus,

\[
\frac{U_a(m,N)}{\varphi(B(m))\varphi(B(N-m))} = \frac{A(a+p(N-m))}{\varphi(B(N-m))} \equiv \frac{A(a+pmn)}{\varphi(B(m))} \equiv 0 \mod \ [p]_q R.
\]

Then by \([\mathbb{1}]\), \(U_a(m,N) \equiv 0 \mod \ [p]_q \varphi(B(m))R\).

(ii) We have \(H_a(m,s,N+mp^s) = \sum_{j=0}^{p^{s-1}} U_a(j+mp^s,N+mp^s)\) and by definition of \(U_a\),

\[
U_a(j+mp^s,N+mp^s) = A(a+p(N-j))\varphi(B(j+mp^s)) - \varphi(B(N-j))A(a+mp+mp^s).
\]

By hypothesis \([\mathbb{2}]\),

\[
A(a+mp+mp^s) = \frac{A(a+mpj)\varphi(B(j+mp^s))}{\varphi(B(j))} + X_j\varphi(B(j+mp^s))
\]

for some \(X_j \in [p^{s+1}]_q R\). Then the right-hand side of \((30)\) is

\[
A(a+p(N-j))\varphi(B(j+mp^s)) - \varphi(B(N-j))\left(\frac{A(a+mpj)\varphi(B(j+mp^s))}{\varphi(B(j))} - X_j\varphi(B(j+mp^s))\right)
\]

\[
= \varphi(B(j+mp^s))\left(\frac{U_a(j,N)}{\varphi(B(j))} - X_j\varphi(B(N-j))\right)
\]

\[
= \varphi(B(j+mp^s))\left(\frac{H_a(j,0,N)}{\varphi(B(j))} - X_j\varphi(B(N-j))\right)
\]

\[
= \frac{\varphi(B(j+mp^s))H_a(j,0,N)}{\varphi(B(j))} - X_j\varphi(B(j+mp^s))\varphi(B(N-j)).
\]

We have \(\varphi(B(j+mp^s)) \equiv 0 \mod \varphi^{s+1}(B^{(s)}(m))R\) by \([23]\). So, by combining \(X_j \in [p^{s+1}]_q R\), we obtain \(X_j\varphi(B(j+mp^s)) \equiv 0 \mod \ [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R\), namely

\[
U_a(j+mp^s,N+mp^s) \equiv \frac{\varphi(B(j+mp^s))H_a(j,0,N)}{\varphi(B(j))} \mod \ [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R.
\]
Therefore, we obtain

\[ H_a(m, s, N + mp^s) \equiv \sum_{j=0}^{p^s-1} \frac{\varphi(B(j + mp^s))H_a(j, 0, N)}{\varphi(B(j))} \mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R, \]

which is \((\beta)_{0,s}\).

(iii) Put \(j = \mu + pi\), then the right-hand side of \((\beta)_{t,s}\) is

\[ \sum_{\mu=0}^{p-1} \sum_{i=0}^{p^s t - 1 - 1} \frac{\varphi^{t+1}(B^{(t)}(\mu + pi + mp^{s-t}))H_a(\mu + pi, t, N)}{\varphi^{t+1}(B^{(t)}(\mu + pi))}, \tag{31} \]

By hypothesis (a),

\[ B^{(t)}(\mu + pi + mp^{s-t}) = B^{(t)}(\mu + pi)\varphi(B^{(t+1)}(i + mp^{s-t-1})) + X_{i,\mu} \varphi(B^{(t+1)}(i + mp^{s-t-1})) \]

for some \(X_{i,\mu} \in [p^{s-t}]_q R\). Thus, the general term in the double sum of (31) is

\[ \frac{\varphi^{t+2}(B^{(t+1)}(i + mp^{s-t-1}))H_a(\mu + pi, t, N)}{\varphi^{t+2}(B^{(t+1)}(i))} + Y_{i,\mu}, \]

where

\[ Y_{i,\mu} = \varphi(X_{i,\mu}) \frac{\varphi^{t+2}(B^{(t+1)}(i + mp^{s-t-1}))H_a(\mu + pi, t, N)}{\varphi^{t+2}(B^{(t+1)}(\mu + pi))}. \]

By (a), \(H_a(\mu + pi, t, N) \equiv 0 \mod [p^{s+1}]_q \varphi^{t+1}(B^{(t)}(\mu + pi))\), so combining \(X_{i,\mu} \in [p^{s-t}]_q R\), we obtain

\[ Y_{i,\mu} \equiv 0 \mod [p^{s+1}]_q \varphi^{t+2}(B^{(t+1)}(i + mp^{s-t-1}))R \]

\[ \equiv 0 \mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R \quad \text{by (29)}, \]

Therefore, the right-hand side of \((\beta)_{t,s}\) \(\mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R\) is

\[ \sum_{\mu=0}^{p-1} \sum_{i=0}^{p^s t - 1 - 1} \frac{\varphi^{t+2}(B^{(t+1)}(i + mp^{s-t-1}))H_a(\mu + pi, t, N)}{\varphi^{t+2}(B^{(t+1)}(i))} = \sum_{i=0}^{p^s t - 1 - 1} \frac{\varphi^{t+2}(B^{(t+1)}(i + mp^{s-t-1}))H_a(i, t + 1, N)}{\varphi^{t+2}(B^{(t+1)}(i))} \quad \text{by (29)}, \]

which is \((\beta)_{t+1,s}\).

(iv) Let us think about the hypothesis

\[ (\gamma)_N : H_a(0, s, N) \equiv 0 \mod [p^{s+1}]_q R \]

for \(n \in \mathbb{Z}\). We know \((\gamma)_N\) is true for \(N \leq 0\). Suppose that \(\{N \in \mathbb{N} \mid (\gamma)_N \text{ fails }\} \neq \emptyset\) and put \(N' = \min\{N \in \mathbb{N} \mid (\gamma)_N \text{ fails }\}\). Then by \((\beta)_{s,s}\) and hypothesis (a), we have

\[ \varphi^{s+1}(B^{(s)}(0))H_a(m, s, N') \equiv \varphi^{s+1}(B^{(s)}(m))H_a(0, s, N' - mp^s) \mod [p^{s+1}]_q \varphi^{s+1}(B^{(s)}(m))R \]

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Theorem 7.5. [Dw69, §3, Theorem 3] Let $B: \mathbb{N} \to R$ be a map satisfying conditions (a), (b), (c) of Theorem 7.4 and a condition

(d') $B(0) = 1$.

(We set $B(i) = B$ for all $i \geq -1$.) Put

$$F_λ = \sum_{j=0}^{\infty} B(j)λ^j, \quad F_s(λ) = \sum_{j=0}^{p^s-1} B(j)λ^j.$$ 

Let $T = R\left<\lambda, \frac{1}{F_λ}\right>$. Then,

$$\frac{F_{s+1}(λ)}{φ(F_s(λ))}$$

converges to an element of $T^\times$.

Proof. By Theorem 7.4 for $i \geq 0, s \geq 0$, $F_λφ(F_λ) \equiv φ(F_λ)F_{s+1}(λ) \mod [p^{s+1}]_q R[[λ]]$. Since $F, F_s \in R[[λ]]^\times$ by (d'),

$$\frac{F_{s+1}(λ)}{φ(F_s(λ))} \equiv \frac{F(λ)}{φ(F(λ))} \mod [p^{s+1}]_q R[[λ]]. \quad (32)$$

Especially for $s = 0$, $F_1(λ) \equiv \frac{F(λ)}{φ(F(λ))} \mod [p]_q R[[λ]]$. Hence,

$$\frac{F_{s+1}(λ)}{φ(F_s(λ))} \equiv F_1(λ) \mod [p]_q R[[λ]].$$

Therefore, we have

$$F_{s+1}(λ) \equiv φ(F_s(λ))F_1(λ) \mod [p]_q R[[λ]], \quad (33)$$

since the congruence holds mod $[p]_q R[[λ]]$ and both sides are polynomials. Clearly we have $F_1 \in T^\times$. Since $[p]_{q} \in (p, q - 1)T$ (Proposition 2.10), we obtain $F_s \in T^\times$ ($s ∈ \mathbb{N}$) from (33) by induction on $s$. For $s ∈ \mathbb{N}$, we put

$$f_s(λ) = \frac{F_{s+1}(λ)}{φ(F_s(λ))}.$$ 

Then $f_s \in T^\times$. By (32), we have

$$f_{s+1} \equiv f_s \mod (p, q - 1)^{s+1} R[[λ]]. \quad (34)$$

Let $R_{s+1} = R/(p, q - 1)^{s+1}$. Since $F_1(λ) ∈ R_{s+1}[[λ]]^\times$, the natural homomorphism $R_{s+1}[λ]F_1(λ) → R_{s+1}[[λ]]$ is injective. Then by $R_{s+1}[λ]F_1(λ) = R_{s+1}[\lambda, \frac{1}{F_1(λ)}] = T/(p, q - 1)^{s+1}T$, the natural homomorphism

$$T/(p, q - 1)^{s+1}T → R_{s+1}[[λ]] = R[[λ]]/(p, q - 1)^{s+1}R[[λ]]$$

is injective. Therefore by (34), we have $f_{s+1} \equiv f_s \mod (p, q - 1)^{s+1}T$. So by the completeness of $T$, \{f_s\}_{s∈\mathbb{N}} converges to a unit of $T$. □
Corollary 7.6. Let $F, F_1$ be same as Theorem 7.5. Then, $\frac{d_q F}{F}$ and $\frac{d_q F}{\gamma(F)}$ are elements of $T = R \left( \lambda, \frac{1}{\gamma(\lambda)} \right)$.

Proof. Put $f = \frac{F}{\varphi(F)}$. By Theorem 7.5, $f, f^{-1} \in T$. So we have

$$\frac{d_q F}{F} = \frac{1}{F} d_q (f \varphi(F))$$

$$= \frac{1}{F} \left( d_q (f) \varphi(F) + \gamma(f) d_q (\varphi(F)) \right)$$

$$= \frac{1}{F} \left( d_q (f) \varphi(F) + [p_q] \lambda^p - 1 \gamma(f) \varphi(d_q(F)) \right) \quad \text{(by Proposition 2.11)}$$

$$= \frac{d_q f}{f} + [p_q] \lambda^p - 1 \gamma(f) \varphi \left( \frac{d_q F}{F} \right).$$

By repeating this computation,

$$\frac{d_q F}{F} = \sum_{j=0}^{s-1} [p_j] \lambda^p - 1 \left( \prod_{i=0}^{j-1} \varphi \left( \frac{\gamma(f)}{f} \right) \right) \varphi^j \left( \frac{d_q f}{f} \right) + [p^s] \lambda^p - 1 \left( \prod_{j=0}^{s-1} \varphi \left( \frac{\gamma(f)}{f} \right) \right) \varphi^s \left( \frac{d_q F}{F} \right).$$

The last term of the right side converges to 0 in $R[[\lambda]] \ (s \to \infty)$. Put

$$\eta_s = \sum_{j=0}^{s-1} [p_j] \lambda^p - 1 \left( \prod_{i=0}^{j-1} \varphi \left( \frac{\gamma(f)}{f} \right) \right) \varphi^j \left( \frac{d_q f}{f} \right).$$

Then by the completeness of $T$, we obtain $\frac{d_q F}{F} = \lim_{s \to \infty} \eta_s \in T$.

To prove the latter, put $g = f^{-1} = \frac{\varphi(F)}{f}$. We have $\frac{d_q F}{\gamma(F)} = -F d_q \left( \frac{1}{f} \right)$, and $F d_q \left( \frac{1}{f} \right) = F d_q \left( \frac{g}{\varphi(F)} \right) = \frac{d_q g}{g} + [p_q] \lambda^p - 1 \gamma(g) \varphi \left( F d_q \frac{1}{f} \right).$ So discussing similarly, we obtain $\frac{d_q F}{\gamma(F)} \in T$. \qed

8 Proof of the main theorems I

In this section, we prove Theorem 4.1 and construct $\varphi_{M'}$ of $M'$ satisfying the conditions (i) and (ii) in Corollary 2.2.

Let $B(n) = a_n = \prod_{i=0}^{n-1} \left( \frac{1 + i}{1 + n} \right)^2 \in R$. Then by Corollary 7.3, $\{B(n)\}_{n \in N}$ satisfies the conditions (a), (b), (c) of Theorem 7.3 and the condition (d') of Theorem 7.3. Therefore, we can apply Theorem 7.4 and Corollary 7.6 to $\{B(n)\}_{n \in N}$.

Let $F(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \in R[[\lambda]]$ and $F_1(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \in R[\lambda]$. Then, by Theorem 5.1, $F(\lambda)$ is a solution of the $q$-hypergeometric differential equation (12). Put $S' = R \left( \lambda, \frac{1}{\alpha(\lambda)} \right)$ as in (11). Then by Lemma 2.2, we have $S' = R \left( \lambda, \frac{1}{\gamma(\lambda)} \right)$. \qed

Proof of Theorem 7.7. Put $\eta = \frac{d_q F}{S'} \in S'$ (Corollary 7.6), $e_1' = \eta e_1 + e_2 \in M'$, and $U' = S' e_1' \subseteq M'$. Then by (18), $\nabla_q (F e_1') = \nabla_q (d_q F e_1 + F e_2) = 0$. Put

$$\nabla_q (e_1') = -\eta' e_1' \otimes d\lambda \quad \left( \eta' \in \frac{1}{\lambda(1-\lambda)} S' \right).$$

Then $\nabla_q (F e_1') = d_q F e_1' \otimes d\lambda - \gamma(F) \eta' e_2' \otimes d\lambda = 0$, which implies $\eta' = \frac{d_q F}{\gamma(F)} = \eta' F e_1'$. (We can determine $\eta'$ directly by the matrix $P$ of (19).) Corollary 7.6 implies $\eta' \in S'$. By Theorem 7.5, we have $\frac{F}{\varphi(F)} \in S'^e$. 23
Define $\varphi_U'$ by $\varphi_U'(e'_2) = \varepsilon \frac{F}{\varphi(F)} e'_2$, where $\varepsilon = (-1)^{\frac{p-1}{2}}$ comes from the classical Frobenius structure $\mathfrak{M}$ of $U_B'$. Then the pair $(U', \varphi_U')$ satisfies the four conditions in Theorem 4.1. 

**Proof of Theorems 4.2 (i), (ii).** Let $e'_2 = \eta e_1 + e_2 \in M'$ and $U' = R[[\lambda]]e'_2 \subset M'$ as in the proof of Theorem 4.1. Let us consider the natural projection $M' \to M'/U'$. Let $e'_1 \in M'/U'$ be the image of $e'_1$ by this projection. Then,

$$\nabla_q(e'_1) = - \frac{1}{\lambda(1 - \lambda)} e_2 \otimes d\lambda = \left( - \frac{1}{\lambda(1 - \lambda)} e'_2 + \eta e'_1 \right) \otimes d\lambda.$$  

Therefore, $\nabla_q(e'_1) \equiv \eta e'_1 \otimes d\lambda$ and

$$\nabla_q \left( \frac{1}{F} e'_1 \right) = d_q \left( \frac{1}{F} \right) e'_1 \otimes d\lambda + \gamma \left( \frac{1}{F} \right) \nabla_q \left( e'_1 \right) = - \frac{d_q(F)}{F \gamma(F)} e'_1 \otimes d\lambda + \frac{1}{\gamma(F)} \eta e'_1 \otimes d\lambda = 0.$$  

Thus, we define a $\varphi$-semilinear endomorphism $\varphi_{M'/U'}$ of $M'/U'$ by $\varphi_{M'/U'}(e'_1) = \varepsilon [p] \frac{\varphi(F)}{\varphi(H)} e'_1$ so that $\varphi_{M'/U'} \left( \frac{1}{F} e'_1 \right)$ is a solution of $\nabla_q = 0$ and that the reduction modulo $q - 1$ of $\varphi_{M'/U'}$ coincides with $\varphi$ of $(H_{U'})/(U_B' \otimes B''')$ via (11) (see (4)). Then its lifting $\varphi_{M'}(e'_1)$ should be of the form $\varepsilon \left( [p] \frac{\varphi(F)}{\varphi(H)} e'_1 + [p] \frac{\varphi(F)}{\varphi(F)} e'_2 \right)$ for some $a \in R[[\lambda]]$ by the condition (i) in Theorem 4.2.

Let $H$ be the solution of $L = 0$ given in Theorem 5.1. We have $\nabla_q(d_q H e_1 + H e_2) = 0$ by (18) and

$$d_q H e_1 + H e_2 = \lambda(1 - \lambda) d_q H e'_1 + H (e'_2 - \lambda(1 - \lambda) \eta e'_1) = \lambda(1 - \lambda) \frac{F d_q H - H d_q F}{F} e'_1 + H e'_2$$

$$= \frac{1}{F} e'_1 + H e'_2.$$  

Here the last equality follows from Lemma 4.3. Then,

$$\varphi_{M'} \left( \begin{pmatrix} 0 & \frac{1}{F} \lambda(1 - \lambda) \end{pmatrix} e'_1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{F} \lambda(1 - \lambda) \end{pmatrix} \begin{pmatrix} 0 & \frac{\varphi(F)}{\varphi(H)} \frac{\varphi'(F)}{\varphi'(H)} \end{pmatrix}$$

Since $\varphi_{M'} \left( \frac{1}{F} e'_1 + H e'_2 \right)$ has to be in the kernel of $\nabla_q$ by the condition (ii) of Theorem 4.2, we try to find $a \in R[[\lambda]]$ such that $\varphi_{M'} \left( \frac{1}{F} e'_1 + H e'_2 \right)$ is an $R$-linear combination of $F e'_2$ and $\frac{1}{F} e'_1 + H e'_2$. Suppose that there exists $c \in R$ satisfying

$$ \begin{pmatrix} c e'_1 & e'_2 \end{pmatrix} \begin{pmatrix} |p| c \frac{\varphi(F)}{\varphi(H)} & |p| c \frac{\varphi'(F)}{\varphi'(H)} \end{pmatrix} = \begin{pmatrix} c e'_1 & e'_2 \end{pmatrix} \begin{pmatrix} \frac{1}{F} \lambda(1 - \lambda) \frac{\varphi(F)}{\varphi(H)} & c \frac{\varphi(F)}{\varphi(H)} \end{pmatrix},$$

which is equivalent to

$$a = \varphi(F) H - \frac{1}{|p|} F \varphi(H) + c F \varphi(F).$$  

We prove

$$\varphi(F) H - \frac{1}{|p|} F \varphi(H) \in R[[\lambda]].$$  

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Put
\[ G_1 = \sum_{n=1}^{\infty} a_n \lambda^n \sum_{i=1}^{n} \frac{2}{[n]_q}, \quad G_2 = \sum_{n=1}^{\infty} a_n \lambda^n \sum_{i=1}^{n} (q - 1), \]
then \( H = F \log_q \lambda - F \log_q (1 - \lambda) - G_1 - G_2 \). We prove (iii) by dividing \( H \) into these four terms.

(i) \( \varphi(F)F \log_q \lambda - \frac{1}{[p]_q} F \varphi(F \log_q \lambda) = \varphi(F)F \log_q \lambda - \frac{1}{[p]_q} F \varphi(F) \varphi(\log_q \lambda) = 0. \)

(ii)
\[
-\varphi(F)F \log_q(1 - \lambda) + \frac{1}{[p]_q} F \varphi(F \log_q(1 - \lambda)) = F \varphi(F) \left( \sum_{n=1}^{\infty} \frac{\lambda^n}{[n]_q} - \frac{1}{[p]_q} \sum_{n=1}^{\infty} \lambda^{pn} \right) = F \varphi(F) \sum_{(n,p)=1}^{\infty} \frac{\lambda^n}{[n]_q}.
\]

This is an element of \( R[[\lambda]] \).

(iii)
\[
-\varphi(F)G_1 + \frac{1}{[p]_q} F \varphi(G_1) = -\varphi(F) \sum_{n=1}^{\infty} a_n \lambda^n \sum_{i'=1}^{n} \frac{2}{[i']_q} + \frac{1}{[p]_q} F \varphi \left( \sum_{n=1}^{\infty} a_n \lambda^n \sum_{i'=1}^{n} \frac{2}{[i']_q} \right) = -2\varphi(F) \sum_{i=1}^{\infty} \frac{1}{[i]_q} \sum_{n=1}^{\infty} a_n \lambda^n + 2\varphi(F) \sum_{n=1}^{\infty} \frac{1}{[n]_q} \left( \sum_{n=1}^{\infty} a_n \lambda^n \right) = -2\varphi(F) \sum_{(i,n)=1}^{\infty} \frac{1}{[i]_q} \sum_{n=1}^{\infty} a_n \lambda^n - 2\varphi(F) \sum_{(n,p)=1}^{\infty} \frac{1}{[n]_q} \sum_{n=1}^{\infty} a_n \lambda^n + 2\varphi(F) \sum_{(n,p')=1}^{\infty} \frac{1}{[n]_q} \left( \sum_{n=1}^{\infty} a_n \lambda^n \right).
\]

The first term is an element of \( R[[\lambda]] \), and the remaining two terms are equal to
\[
2 \sum_{i'=1}^{\infty} \frac{1}{[i']_q} \left( -\varphi(F) \sum_{n=1}^{\infty} a_n \lambda^n + F \varphi \left( \sum_{n=1}^{\infty} a_n \lambda^n \right) \right).
\]

By Theorem 7.3,
\[
-\varphi(F) \sum_{n=0}^{p_i'-1} a_n \lambda^n + F \varphi \left( \sum_{n=0}^{p_i'-1} a_n \lambda^n \right) \in \left[ p_i^{p_i'(i') + 1} \right]_q R[[\lambda]] = [p_i']_q R[[\lambda]],
\]
which implies \( -\varphi(F) \sum_{n=0}^{\infty} a_n \lambda^n + F \varphi \left( \sum_{n=0}^{\infty} a_n \lambda^n \right) \in [p_i']_q R[[\lambda]] \). Therefore, we have \( -\varphi(F)G_1 + \frac{1}{[p]_q} F \varphi(G_1) \in R[[\lambda]] \).

(iv) \( -\varphi(F)G_2 + \frac{1}{[p]_q} F \varphi(G_2) = -(q - 1)\varphi(F) \sum_{n=1}^{\infty} na_n \lambda^n + (q - 1)F \varphi \left( \sum_{n=1}^{\infty} na_n \lambda^n \right) \in R[[\lambda]]. \)

By adding all of them, we obtain \( -\varphi(F)H + \frac{1}{[p]_q} F \varphi(H) \in R[[\lambda]]. \)
We define \( \varphi_M' \) by choosing \( c \in R \) and using \( a \in R[[\lambda]] \) defined by (37); we set \( \varphi_M'(e'_1) = \epsilon \frac{F}{\varphi(F)} e'_1 \) as in the proof of Theorem 4.1. Then, \( \varphi_M' \) satisfies the condition (i) of Theorem 4.2. We show the condition (ii) of Theorem 4.2. We have

\[
\begin{align*}
\gamma_M(e'_1) &= \frac{\gamma(F)}{F} e'_1 - (q - 1) \frac{1}{1 - \lambda} e'_2 \quad \text{by (36)} \\
\gamma_M(e'_2) &= \frac{F}{\gamma(F)} e'_2 \quad \text{by (35)} \\
\varphi_M'(e'_1) &= \epsilon [p]_q \frac{\varphi(F)}{F} e'_1 + \epsilon [p]_q a e'_2 \\
\varphi_M'(e'_2) &= \epsilon \frac{F}{\varphi(F)} e'_2 .
\end{align*}
\]
(39)

The equation \( \varphi_M'(\gamma_M(e'_2)) = \gamma_M(\varphi_M'(e'_2)) \) follows from \( \varphi \circ \gamma = \gamma \circ \varphi \) on \( Q((\lambda)) \). By calculating \( \varphi_M'(\gamma_M(e'_1)) \) and \( \gamma_M(\varphi_M'(e'_1)) \), the equality \( \varphi_M' \circ \gamma_M(e'_1) = \gamma_M \circ \varphi_M'(e'_1) \) holds if and only if

\[
\epsilon [p]_q a \frac{\varphi(F)}{\varphi(F)} (q - 1) [p]_q \frac{1}{1 - \lambda} \frac{F}{\varphi(F)} = -(q - 1) [p]_q \frac{1}{1 - \lambda} \frac{\gamma(F)}{\gamma(F)} + [p]_q a \frac{F}{\gamma(F)}
\]
\[
\iff \lambda d_q \left( \frac{a}{F \varphi(F)} \right) = \frac{1}{1 - \lambda F \gamma(F)} - \varphi \left( \frac{1}{1 - \lambda F \gamma(F)} \right).
\]

On the other hand, by (37), we have \( \frac{a}{F \varphi(F)} = \frac{H}{F} - \frac{1}{[p]_q} \varphi \left( \frac{H}{F} \right) + c \). We calculate \( d_q \left( \frac{H}{F} \right) \) and \( d_q \left( \frac{1}{[p]_q} \varphi \left( \frac{H}{F} \right) \right) \). Note that \( d_q e = 0 \) by \( c \in R \). We have

\[
d_q \left( \frac{H}{F} \right) = d_q H - \frac{1}{\gamma(F)} F d_q H = \frac{F d_q H - H d_q F}{F \gamma(F)} = \frac{1}{\lambda(1 - \lambda) F \gamma(F)} \quad \text{(by Lemma 5.5),}
\]
\[
d_q \left( \frac{1}{[p]_q} \varphi \left( \frac{H}{F} \right) \right) = \left[ \frac{1}{[p]_q} d_q \right] \varphi \left( \frac{H}{F} \right) = \frac{1}{[p]_q} \left[ p \right]_q \lambda^{p-1} \varphi \left( \frac{H}{F} \right) = \lambda^{p-1} \varphi \left( \frac{1}{\lambda(1 - \lambda) F \gamma(F)} \right) .
\]

Thus, we obtain

\[
\lambda d_q \left( \frac{a}{F \varphi(F)} \right) = \lambda d_q \left( \frac{H}{F} - \frac{1}{[p]_q} \varphi \left( \frac{H}{F} \right) + c \right) = \frac{1}{(1 - \lambda) F \gamma(F)} - \varphi \left( \frac{1}{(1 - \lambda) F \gamma(F)} \right).
\]

Therefore, we have \( \varphi_M' \circ \gamma_M(e'_1) = \gamma_M \circ \varphi_M'(e'_1) \).

In conclusion, if we choose \( c \in R \) and define \( a \) by (37). Then \( \varphi_M' \), defined by (39), satisfies the conditions (i) and (ii) of Theorem 4.2. \( \square \)

9 Proof of the main theorems II

Assume that \( R' = R[[\lambda]] \). In this section, we prove the condition (iii) in Theorem 4.2 holds for \( \varphi_M' \) constructed in (38) for a suitable \( c \in R \). First, by taking the image of \( (H_0, \nabla, \varphi H_0, \text{Fil}^* H_0) \) under the functor (7) with \( a = 1 \), we obtain an \( R/(q - 1)^2[[\lambda]] \)-module \( H_0 \otimes R/(q - 1)^2[[\lambda]] \) with a filtration, a Frobenius endomorphism and a \( \Gamma \)-action. Second, by taking the image of \( (M', \text{Fil}^* M', \varphi M', \rho M) \) in \( M^{[p], \varphi^{-1}}_0 (R'/q - 1)^2 \varphi, \Gamma) \) under the equivalence of categories (9) with \( a = 1 \), we obtain an \( R/(q - 1)^2[[\lambda]] \)-module \( M'/q - 1)^2 M' \) with a filtration, a Frobenius endomorphism, and a \( \Gamma \)-action.

By the construction of the canonical \( q \)-deformation of \( (H_0, \nabla, \varphi H_0, \text{Fil}^* H_0) \), it suffices to show that there exists an isomorphism \( g : M'/q - 1)^2 M' \rightarrow H_0 \otimes R/(q - 1)^2[[\lambda]] \) in \( M^{[p], \varphi^{-1}}_0 (R'/q - 1)^2, \varphi, \Gamma) \) for a
suitable choice of $c \in R$ such that $g \mod q - 1$ coincides with the isomorphism (11). Let $B_1 \in M_2(\mathbb{Z}_p[[\lambda]])$, and define an $R/(q - 1)^2[[\lambda]]$-linear lifting $g$ of (11) by

$$g(e_1', e_2') = (\overline{e}_1 \otimes 1 \, \overline{e}_2 \otimes 1)(1 + (q - 1)B_1).$$

It is clear that $g$ is a filtered isomorphism. Since $\gamma_{M'} = 1 + (q - 1)D_{q}^{\text{log}}$, the compatibility of $g$ with the $\Gamma$-actions is equivalent to that of the compatibility with the connections mod $q - 1$. The latter is clear by (11).

Set $\varphi_{M/(q-1)^2M} \, (e_1', e_2') = (e_1' \, e_2')(A_0 + (q - 1)A_1)$ for $A_0, A_1 \in M_2(\mathbb{Z}_p[[\lambda]])$. We choose $c \in R$ such that a defined by (13) is the lift of $b$ in (11); we can show that such a $c$ exists by taking the reduction modulo $q - 1$ of the proof of Theorem 4.2 (i), (ii) in [3]. Then we have

$$\varphi_{M} \otimes \varphi \, (\overline{e}_1 \otimes 1 \, \overline{e}_2 \otimes 1) = (\overline{e}_1 \otimes 1 \, \overline{e}_2 \otimes 1) \, A_0.$$ We determine $B_1 \in M_2(\mathbb{Z}_p[[\lambda]])$ satisfying $\varphi \, g = g \circ \varphi_{M/(q-1)^2M}$. We have

\[
\begin{align*}
(\varphi_{M} \otimes \varphi) \circ g \, (e_1', e_2') &= (\overline{e}_1 \otimes 1 \, \overline{e}_2 \otimes 1) \, A_0 \varphi(1 + (q - 1)B_1), \\
g \circ \varphi_{M/(q-1)^2M} \, (e_1', e_2') &= (\overline{e}_1 \otimes 1 \, \overline{e}_2 \otimes 1)(1 + (q - 1)B_1)(A_0 + (q - 1)A_1).
\end{align*}
\]

Therefore the compatibility of $g$ with Frobenius is equivalent to

$$A_0 \varphi(1 + (q - 1)B_1) = (1 + (q - 1)B_1)(A_0 + (q - 1)A_1) \equiv 0 \mod (q - 1)^2M_2(R/(q - 1)R[[\lambda]])$$

$$\Leftrightarrow A_0 + (q^p - 1)A_0 \varphi(B_1) = A_0 + (q - 1)(A_1 + B_1A_0) \quad \text{by (q - 1)^2 = 0}$$

$$\Leftrightarrow [p]_q \, A_0 \varphi(B_1) = A_1 + B_1A_0 \quad \text{in } M_2(R/(q - 1)R[[\lambda]])$$

$$\Leftrightarrow B_1 - pA_0 \varphi(B_1)A_0^{-1} = -A_1A_0^{-1} \quad \text{in } M_2(\mathbb{Z}_p[[\lambda]]).$$

Let us consider the $\varphi$-semilinear map $\mathcal{F} : M_2(\mathbb{Z}_p[[\lambda]]) \rightarrow M_2(\mathbb{Z}_p[[\lambda]])$ defined by $\mathcal{F}(X) = pA_0 \varphi(X)A_0^{-1}$. (Since det $A_0 = p^{-1}$, $pA_0 \varphi(X)A_0^{-1}$ is an element of $M_2(\mathbb{Z}_p[[\lambda]])).$

Then what we want to show is that we can choose $c \in R$ so that $(a \mod q - 1) = b$ and there exists $B_1 \in M_2(\mathbb{Z}_p[[\lambda]])$ satisfying $(1 - \mathcal{F})(B_1) = -A_1A_0^{-1}$. (40)

Note that $A_1A_0^{-1} \in M_2(\mathbb{Z}_p[[\lambda]])$ because $(A_0 + (q - 1)A_1)A_0^{-1} = 1 + (q - 1)A_1A_0^{-1}$ and, by letting $f = F_{|_{q=1}}$, we have

\[
(A_0 + (q - 1)A_1)A_0^{-1} \equiv \begin{pmatrix} [p]_q \varphi(F) & 0 \\ [p]_q a & \varphi(F) \end{pmatrix} \begin{pmatrix} f & 0 \\ \frac{f}{\varphi(F)} - b & \varphi(F) \end{pmatrix} \mod (q - 1)^2M_2(R[[\lambda]])
\]

\[
\equiv \begin{pmatrix} [p]_q \varphi(F) & 0 \\ [p]_q a & \varphi(F) \end{pmatrix} \begin{pmatrix} f & 0 \\ \frac{f}{\varphi(F)} - b & \varphi(F) \end{pmatrix} \mod (q - 1)^2M_2(R[[\lambda]])
\]

which is an element of $M_2(R/(q - 1)^2[[\lambda]])$ by $[p]_q = p \cdot \text{unit in } R/(q - 1)^2$.

We first prove the claim modulo $\lambda$. Let $A_1(0), A_2(1) \in R$ be the values of $A_1$ and $A_2$ at $\lambda = 0$.

**Lemma 9.1.** We can choose $c \in R$ so that $(a \mod q - 1) = b$ and there exists $C \in M_2(\mathbb{Z}_p)$ satisfying $C - pA_0(0)CA_0(0)^{-1} = -A_1(0)A_0(0)^{-1}$.

**Proof.** Choose $c \in R$ such that $(a \mod q - 1) = b$. By (8) we have

$$A_0 + (q - 1)A_1 \equiv \begin{pmatrix} [p]_q \varphi(F) & 0 \\ [p]_q a & \varphi(F) \end{pmatrix} \mod (q - 1)^2R.$$
Let $b_0, b_1 \in \mathbb{Z}_p$ satisfy $a |\lambda=0=b_0 + (q-1)b_1 \mod (q-1)^2 R$. (We have $b |\lambda=0=b_0$.) Then, we have

$$A_0(0) = \begin{pmatrix} p & 0 \\ pb_0 & 1 \end{pmatrix}, \quad A_1(0) = \begin{pmatrix} \frac{p(p-1)}{2} & 0 \\ pb_1 & 0 \end{pmatrix}$$

because $F(0)=1$ and $[p]_q=p+(q-1)(\frac{p}{2})+ (q-1)^2 (\frac{p}{6}) + \cdots$. Put $C = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then,

$$C - pA_0(0)CA_0(0)^{-1} = \begin{pmatrix} (1-p)x + p^2b_0y & (1-p^2)y \\ -pb_0x + p^2b_0^2y + pb_0w & -p^2b_0y + (1-p)w \end{pmatrix},$$

$$-A_1(0)A_0(0)^{-1} = -\left( \begin{pmatrix} \frac{p(p-1)}{2} & 0 \\ pb_1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ -b_0 \end{pmatrix} = -\begin{pmatrix} \frac{p(p-1)}{2} & 0 \\ b_1 & 0 \end{pmatrix}.$$  

Hence the equality in the lemma holds if and only if

$$(1-p)x + p^2b_0y = \frac{1}{2}(p-1)$$

$$(1-p^2)y = 0$$

$$-pb_0x + p^2b_0^2y + pb_0w = -b_1$$

$$-p^2b_0y + (1-p)w = 0.$$  

The equations (11), (12), and (13) are equivalent to $x = \frac{1}{2}, y = w = 0$. This solution satisfies the equation (13) if and only if $b_1 = \frac{1}{2}pb_0$. Since $I_2(F) \in 1 + (q-1)\mathbb{R}[\lambda]$, one can replace $c$ by $c + (q-1)c'$ for some $c' \in \mathbb{R}[\lambda]$ so that this equality holds.

Choose $c \in R$ satisfying the condition in Lemma 9.1. We show that (40) holds for $\varphi_M$ defined by $c$. We have $F^n(\lambda M_2(\mathbb{Z}_p[[\lambda]])) \subset \lambda^n M_2(\mathbb{Z}_p[[\lambda]])$ for all $n \in \mathbb{N}$. Thus, $\sum_{n=0}^{\infty} F^n$ converges to an endomorphism of $\lambda M_2(\mathbb{Z}_p[[\lambda]])$. Therefore, $1 - F$ is bijective on $\lambda M_2(\mathbb{Z}_p[[\lambda]])$, because $\sum_{n=0}^{\infty} F^n$ is the inverse of $1 - F$.

We have the following commutative diagram whose two horizontal lines are exact.

$$\begin{array}{ccccccccc}
0 & \rightarrow & \lambda M_2(\mathbb{Z}_p[[\lambda]]) & \rightarrow & M_2(\mathbb{Z}_p[[\lambda]]) & \rightarrow & M_2(\mathbb{Z}_p) & \rightarrow & 0 \\
\uparrow{\cong} & & \uparrow{\pi_1} & & \uparrow{\pi_2} & & \uparrow{1-F(0)} & & \\
0 & \rightarrow & \lambda M_2(\mathbb{Z}_p[[\lambda]]) & \rightarrow & M_2(\mathbb{Z}_p[[\lambda]]) & \rightarrow & M_2(\mathbb{Z}_p) & \rightarrow & 0
\end{array}$$

By the choice of $c$, there exists $C \in M_2(\mathbb{Z}_p)$ satisfying $(1-F(0))(C) = -A_1(0)A_0(0)^{-1}$. Then by the surjectivity of $\pi_1$, there exists $\tilde{C} \in M_2(\mathbb{Z}_p[[\lambda]])$ such that $\pi_1(\tilde{C}) = C$. By the commutativity of the right square, $\pi_2((1-F)(\tilde{C})) = -A_1A_0^{-1}$, and therefore $-A_1A_0^{-1} = (1-F)(\tilde{C})$ lies in the kernel of $\pi_2$. By the exactness of the lower horizontal line, there exists $D \in \lambda M_2(\mathbb{Z}_p[[\lambda]])$ which satisfies $\pi_2(D) = -A_1A_0^{-1} - (1-F)(\tilde{C})$. Put $E = (1-F)^{-1}(D)$ in $\lambda M_2(\mathbb{Z}_p[[\lambda]])$. Then $(1-F)(\{1\}(E) + \tilde{C}) = -A_1A_0^{-1} - (1-F)(\tilde{C}) = -A_1A_0^{-1}$. This completes the proof.

10 A further topic

Let $R'$ be one of the rings $S'$ and $R[[\lambda]]$. The $\Gamma$-action on $R'$ is geometric in the sense that it defines through the coordinate $\lambda$ relevant to $q$-connection. In this section, we introduce an arithmetic action on $R'$ via the coefficient ring $R$ and show that the unit root part $U'$ of $M'$ admits an arithmetic action.

Let $U'$ be the unit root part of $M'$ given in Theorem 11.1. For $l \in \mathbb{Z}_p^\times$, we define an automorphism $\sigma_l$ of $S'$, and also of $R[[\lambda]]$, by $\sigma_l(a) = a (a \in \mathbb{Z}_p)$, $\sigma_l(q) = q^l$, and $\sigma_l(\lambda) = \lambda$. This $\sigma_l$ satisfies $\sigma_l \circ \varphi = \varphi \circ \sigma_l$. Let $F \in \mathbb{R}[[\lambda]]$ be the solution of the $q$-differential equation (12) given in Theorem 5.1.
Lemma 10.1. \( \bar{F}_{\sigma(l)} \) is an element of \( S'^{\times} \).

Proof. For each \( n \), put \( a'_{n} = a_{n} \mid_{q = 1} \in \mathbb{Z}_{p} \). Then there is a unique \( a''_{n} \in R \) such that \( a_{n} = a'_{n} + (q - 1)a''_{n} \).

For \( r \geq 0 \), we have

\[
\varphi^{r+1}(F) \equiv \sum_{n=0}^{\infty} \varphi^{r+1}(a_{n})\lambda^{p^{r+1}n} = \sum_{n=0}^{\infty} (a'_{n} + \varphi^{r+1}(q-1)(a''_{n}))\lambda^{p^{r+1}n}
\]

\[
\equiv \sum_{n=0}^{\infty} a'_{n}\lambda^{p^{r+1}n} \mod [p^{r+1}]_{q},
\]

and similarly we have \( \sigma_{l}(\varphi^{r+1}(F)) \equiv \sum_{n=0}^{\infty} a'_{n}\lambda^{p^{r+1}n} \mod [p^{r+1}]_{q} \). Therefore, \( \varphi^{r+1}(F) \equiv \sigma_{l}(\varphi^{r+1}(F)) \mod (p, q - 1)^{r+1} \). Since \( \sigma_{l}(\varphi^{r+1}(F)) \) is a unit of \( R[[\lambda]] \), we have

\[
\frac{\varphi^{r+1}(F)}{\sigma_{l}(\varphi^{r+1}(F))} \equiv 1 \mod (p, q - 1)^{r+1},
\]

(45)

Put \( f = \frac{F}{\sigma_{l}(F)} \in S'^{\times} \). Then we have \( \frac{F}{\sigma_{l}(F)} = \frac{f^{q^{r}(F)} - f^{q^{r+1}(F)}}{\sigma_{l}(f^{q^{r}(F)} - f^{q^{r+1}(F)})} \cdot \frac{\varphi^{r+1}(F)}{\sigma_{l}(\varphi^{r+1}(F))} \), and the first term of the right-hand side is contained in \( S'^{\times} \). Hence, the same argument as the proof of Theorem 12.3 shows that \( \frac{F}{\sigma_{l}(F)} \) is an element of \( S'^{\times} \).

Let \( \widehat{\Gamma} \) be the inverse limit \( \lim_{\leftarrow} \Gamma/\Gamma^{p^{n}} \), which is isomorphic to \( \mathbb{Z}_{p} \). For \( R' = S'^{\times}, R[[\lambda]] \), the triviality modulo \( q - 1 \) of \( \Gamma \) on \( R' \) implies that the action is continuous with respect to the \( p \)-adic topology of \( \Gamma \) and the \((p, q - 1)\)-adic topology of \( R' \). Therefore the action of \( \Gamma \) on \( R' \) uniquely extends to a continuous action \( \widehat{\rho} : \widehat{\Gamma} \to \text{Aut}(R') \) of \( \Gamma \) on \( R' \). We have \( \widehat{\rho}(\gamma)(a) = a \) (\( a \in R \)) and \( \widehat{\rho}(\gamma)(\lambda) = q^{m} \lambda \) for \( m \in \mathbb{Z}_{p} \).

Similarly, the action \( \rho_{U'} \) of \( \Gamma \) on \( U' \) uniquely extends to a continuous \( \widehat{\rho} \)-linear action \( \rho_{U'} : \widehat{\Gamma} \to \text{Aut}(U') \) of \( \Gamma \) on \( U' \). The formula \( \nabla_{q}(e_{2}^{l}) = -q\frac{F}{\gamma^{m}(F)}e_{2}^{l} \otimes d\lambda \) shown in the proof of Theorem 4.1 implies \( \rho_{U'}(e_{2}^{l}) = \frac{F}{\gamma^{m}(F)}e_{2}^{l} \) for \( m \in \mathbb{Z}_{p} \).

Let \( \Gamma'^{\times} \) be the group \( \mathbb{Z}_{p}^{\times} \). We define a homomorphism \( \rho' : \Gamma' \to \text{Aut}(S') \) trivial modulo \( q - 1 \) by \( \rho'(l)(s) = \sigma_{l}(s) \). By Lemma 10.1, we can define a \( \sigma_{l} \)-semilinear automorphism \( \sigma_{l,U'} \) of \( U' \) by

\[
\sigma_{l,U'}(e_{2}^{l}) = \frac{F}{\sigma_{l}(F)}e_{2}^{l}.
\]

We define the \( \rho' \)-semilinear action \( \rho'_{U'} : \Gamma' \to \text{Aut}(U') \) of \( \Gamma' \) on \( U' \) by \( \rho'_{U'}(l)(u) = \sigma_{l,U'}(u) \).

Let \( \Gamma' \cong \widehat{\Gamma} \) be the semi-direct product defined by the canonical action of \( \Gamma' = \mathbb{Z}_{p}^{\times} \) on \( \widehat{\Gamma} \cong \mathbb{Z}_{p} \). Since \( \sigma_{l}\rho(\gamma)\sigma_{l}^{-1} = \rho(\gamma^{m}) \) on \( S' \) and on \( R[[\lambda]] \) for \( l \in \mathbb{Z}_{p}^{\times} \) and \( m \in \mathbb{Z}_{p} \), we can define an action \( \rho' \ast \widehat{\rho} : \Gamma' \cong \widehat{\Gamma} \to \text{Aut}(S') \) by \( \rho' \ast \widehat{\rho} \) and \( \rho'_{U'} \).

By using \( \varphi \circ \sigma_{l} = \sigma_{l} \circ \varphi \) on \( R[[\lambda]] \), we see that the triplet \( (U', \varphi_{U'}, (\rho' \ast \widehat{\rho})_{U'}) \) is an object of the category \( \text{MF}^{[\varphi]}_{[0, q^{-1}]}(S', \varphi, \Gamma' \cong \widehat{\Gamma}) \), whose image under the equivalence of categories (Tsu17, Proposition 56)

\[
\text{MF}^{[\varphi]}_{[0, q^{-1}]}(S', \varphi, \Gamma' \cong \widehat{\Gamma}) \xrightarrow{\text{mod } q^{-1}} \text{MF}^{[\varphi]}_{[0, 0]}(B', \varphi)
\]

is \( (U_{B'}, \varphi_{U_{B'}}) \).
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