The two dimensional local density of states of a Topological Insulator with an edge dislocation

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Abstract

We investigate the effect of a crystal edge dislocation on the metallic surface of a Topological Insulator. The edge dislocation gives rise to torsion which the electrons experience as a spin connection. As a result the electrons propagate along confined two dimensional regions and circular contours. Due to the edge dislocations the parity symmetry is violated resulting in a current measured by the in-plane component of the spin on the surface. The tunneling density of states for Burger vectors in the $y$ direction is maximal along the $x$ direction. The evidence of the enhanced tunneling density of states can be verified with the help of the scanning tunneling technique.
I Introduction

Seldom a new state of matter is predicted. Even less often is it unambiguously observed in the laboratory in the spectacular way was done with HgTe quantum wells [1]. Indeed, this novel two-dimensional (2D) topological insulator (TI) shows a quantized charge conductance, similar to that observed in Quantum Hall effect, without the need to apply a magnetic field. This result triggered further theoretical and experimental work that resulted in the discovery of metallic surface states in the insulating alloy Bi$_x$Sb$_{1-x}$, the first three-dimensional topological insulator. In this case, the conducting surface states are formed by topological effects that render the electrons traveling on such surfaces insensitive to scattering by impurities. The theoretical foundations of this phenomena are based on the topology of the Brillouin zone [2–6, 12, 13, 15, 17, 22, 31]. The theoretical predictions have been confirmed experimentally for the three dimensional (3D) Topological Insulators (TI) Bi$_2$Se$_3$, Bi$_2$Te$_3$ and Bi$_{1-x}$Sb$_x$. The electronic band structure of these crystals is time reversal invariant obeying Kramer’s theorem $T^2 = -1$ and have a single Dirac cone which lies in a gap [1, 2, 18, 23]. At the boundary of the 3D TI, one obtains a 2D surface with an odd number of chiral edge excitations coined helical liquid [37] and realized experimentally in the two-dimensional CdTe/HgTe/CdTe quantum wells [1, 36]. The dissipationless surface states are believed to be quantum-protected by the bulk insulator. A variety of transport experiments suggest that the conductivity of a 3D TI contains a significant metallic contribution [39]. Scanning tunneling microscopy (STM) and transmission electron spectroscopy (TEM) of the chiral metallic boundary show that crystal defects modify the density of states. The momentum-resolved Landau spectroscopy of the Dirac surface state in Bi$_2$Se$_3$ [38] reveals, in addition to the Landau spectroscopy, triangular-shaped structure caused by the presence of vacancies of Bi at the Se sites. These experiments suggest that the two dimensional surface of the TI is sensitive to defects and geometry and therefore, the quantization rules are expected to be modified, and probably are revealed through the new Berry indices $\gamma \neq \frac{1}{2}$ [42]. (In a recent Shubnikov-de Haas oscillation experiment performed on the TI Bi$_2$Se$_3$ [42] the authors show magnetic oscillations which correspond to the Landau level quantization $E_n = v_F \sqrt{2(n + \gamma)\hbar eB}$ with the Berry phase index which is different from $\gamma = \frac{1}{2}$.) The appearance of the new indices $\gamma \neq \frac{1}{2}$ might be due to topology of the defects or/and interactions. Recently [43] the effect of the $\delta^{(2)}(\vec{r})$ impurity potential on the metallic surface of the TI has been shown to modify the local density of states. The authors
in ref. [19] have proposed that crystal dislocations generate protected zero modes which give rise to perfect metallic conduction. This idea has been used to explain the thermoelectric transport in the 3D TI [25].

From the theory of quantum crystal [47] three primary types of topological defects are known: edge dislocations, screw dislocations and mixed dislocations. The strength of the dislocations is measured in units of the Burger vectors which corresponds to the shortest lattice translation in the crystal. Experimentally, the dislocations are seen as dark lines in the lighter central regions of the transmission electron micrograph.

The dislocations induce crystal stresses which are characterized by the Burger vector, shear modulus and Poisson’s ratio [33, 47]. As a result, the lattice vibrations are complicated and the electron-phonon interaction plays a crucial role in inducing new dynamical effects. We will ignore the dynamical effects and concentrate on the static effects caused by the elastic strain field on the electrons. In a crystal we have the neutrality condition, namely the sum of the Burger vectors must be zero. We consider first a single dislocation (one assumes that a second dislocation with an opposite Burger vector is located far away from the first dislocation) and in the second stage we generalize the results to an even number of dislocations.

Thirty two years ago the propagation of electrons in the presence of a dislocated crystal was investigated [26]. Those calculations, performed with the Schrödinger equation, revealed the possibility for interesting electronic transport. For a screw dislocation, the Schrödinger equation is equivalent to the Aharonov-Bohm problem in two dimensions [27] which hints that the physics of persistent currents might play a crucial role. In graphene, different topological defects such as crystal disclinations have been modeled as a vortex field [28, 29].

We start our discussion with the Peierls model [20] which is an approximation for the edge dislocations without torsion. This model gives results which are in agreement with those obtained by [19] for the 3D TI. For this model we find that at the boundary of the dislocation a zero mode exist and confines the electrons to the dislocation line.

Unfortunately, the Peierls model represents a over-simplification of the reality. The model is lacking torsion which is an important property for dislocations. A complete description of the edge dislocation must include the torsion tensor. This tensor generates spin connections [22, 44, 46] which are controlled by the Burger vector. Using the complete description of the edge dislocation we find that the electronic excitations are confined to a two dimensional
region and to a set of circular contours. Such structure can be studied using the formulation introduced in ref. [50–52]. The circular contours have a radius \( R_g(n) \) \( n = \pm 1, \pm 2, \pm 3, \ldots \) which are determined by the strength of the Burger vector. Comparing the results obtained from the model with torsion to the one obtained from the Peierls model, we find that the torsion destroys the zero mode state, but due to the Parity violation the system is only weakly affected by backscattering.

The contents of this paper is as follows: In chapter II we present the chiral model for the boundary surface of the TI. Chapter III is devoted to the edge dislocation. We devote a short discussion to the Peierls model and present an extended derivation for the effect of dislocations on the TI. Chapter IV is devoted to the solution of the wave function in the presence of an edge dislocation. Chapter V presents the explicit wave functions for the two dimensional region and circular contours. Chapter VI presents the the computation of the tunneling density of states induced by the edge dislocation. In chapter VII we show that the violation of the parity symmetry by the edge dislocation generates a current which has an in-plane spin component. We suggest that this current which confirms the presence of the edge dislocation might be measured using Magnetic Force Microscopy techniques. Chapter VIII is devoted to conclusions.

II-The chiral metal - the boundary surface of the three dimensional Topological Insulator

The low energy Hamiltonian for the bulk 3D TI in the \( Bi_2Se_3 \) family was shown to behave on the boundary surface (the \( x,y \)-plane) as a two dimensional chiral metal \([11]\) which is similar the Rashba \([10]\) model.

\[
H = \int d^2r \Psi^\dagger(\vec{r})[h^{T.I}(x,y) - \mu] \Psi(\vec{r}) \equiv \hbar v_F \int d^2r \Psi^\dagger(\vec{r})[i\sigma_1 \partial_y - i\sigma_2 \partial_x - \mu] \Psi(\vec{r})
\]

(1)

\( h^{T.I}(x,y) = \hbar v_F[i\sigma_1 \partial_y - i\sigma_2 \partial_x] \) is the chiral Dirac Hamiltonian in the first quantized language. \( v_F \approx 5 \cdot 10^5 \frac{m}{sec} \) is the Fermi velocity, \( \sigma \) is the Pauli matrix describing the electron spin and \( \mu \) is the chemical potential measured relative to the Dirac \( \Gamma \) point. The Hamiltonian for the two dimensional surface \( L \times L \) describes well the excitations smaller than the bulk gap of the 3D TI at 0.3 eV. Moving away from the \( \Gamma \) point, the Fermi velocity becomes momentum dependent; therefore, we will introduce a momentum cut off \( \Lambda \) to restrict the
validity of the Dirac model. The chiral Dirac model in the Bloch representation takes the form: \( h = \hbar v_F (\vec{K} \times \vec{\sigma}) \cdot \hat{z} \equiv \hbar v_F (-\sigma^1 k_y + \sigma^2 k_x) \) where periodic boundary conditions imply \( k_x = \frac{2\pi}{L} m_x, m_x = 0, \pm 1, \pm 2, \ldots \) and \( k_y = \frac{2\pi}{L} m_y, m_y = 0, \pm 1, \pm 2, \ldots \). The eigen-spinors for this Hamiltonian are: \( |u(\vec{K})> = |u_1(\vec{K})>, |u_2(\vec{K})> > |\vec{K}> \otimes |1, i\epsilon \chi(k_x,k_y)>^T \) where \( \chi(k_x,k_y) = \tan^{-1}(k_x/k_y) \) is the spinor phase and \( \epsilon = \hbar v_F \sqrt{k_x^2 + k_y^2} \) is the eigenvalue for particles. For holes we have the eigenvalue \( \epsilon = -\hbar v_F \sqrt{k_x^2 + k_y^2} \) and eigenvectors \( |v(\vec{K})> = |v_1(\vec{K})>, |v_2(\vec{K})> > |\vec{K}> \otimes [-1, i\epsilon \chi(k_x,k_y)>^T \). The chirality operator is defined in terms of the chiral phase \( \chi(k_x,k_y): \)

\[
(\vec{\sigma} \times \frac{\vec{K}}{|\vec{K}|}) \cdot \hat{z} \equiv \sin [\chi(k_x,k_y)] \sigma^1 - \cos [\chi(k_x,k_y)] \sigma^2
\]

The chirality operator takes the eigenvalue – (counter-clockwise) for particles \( |\sin [\chi(k_x,k_y)] \sigma^1 - \cos [\chi(k_x,k_y)] \sigma^2| |\vec{K}> > \otimes |1, i\epsilon \chi(k_x,k_y)>^T = -|\vec{K}> \otimes |1, i\epsilon \chi(k_x,k_y)>^T \) and + (clockwise) for holes \( |\sin [\chi(k_x,k_y)] \sigma^1 - \cos [\chi(k_x,k_y)] \sigma^2| |\vec{K}> > \otimes [-1, i\epsilon \chi(k_x,k_y)>^T = |\vec{K}> \otimes [-1, i\epsilon \chi(k_x,k_y)>^T \). The Hamiltonian \( h^{T.I}(-\vec{K}) = \hbar v_F (-\sigma^1 k_y + \sigma^2 k_x) \) is time reversal invariant \( h^{T.I}(-\vec{K}) = Th^{T.I}(-\vec{K})T^{-1} \) \((T = -i\sigma_y)K_0 \) is the time reversal operator and \( K_0 \) is the conjugation operator). At \( \vec{K} = 0 \) we have \( h^{T.I}(\vec{K} = 0) = Th^{T.I}(\vec{K} = 0)T^{-1} \therefore \) the eigenstate is a Kramer degenerate. The eigenstate \( |u(\vec{K})> \) obeys \( T^2|u(\vec{K})> = -|u(\vec{K})> \). The states \( |u(\vec{K})> \) and \( T|u(\vec{K})> \) are orthogonal to each other. Since \( T|u(\vec{K})>\propto |u(-\vec{K})> \) the property \( T^2 = -1 \) guarantees that backscattering is prohibited.

III-The edge dislocation on the metallic Surface of the three dimensional Topological Insulator

A-The Peierls model for the edge dislocation

We start our presentation with the Peierls phenomenological model \[20\] which might be useful for a large number of dislocations. An edge dislocation in two dimensions can be formulated in the following way: We cut the the two dimensional crystal into two halves \((y < 0 \text{ and } y > 0)\). After cutting the crystal along the line \( y = 0 \), two surfaces \([x, y < 0]\) and \([x, y > 0]\) have been created. We glue back the two halves by translating by minus one half of the lattice spacing along \( x \) for \( y < 0 \) and one half along \( x \) for \( y > 0 \). This procedure creates a dislocation along the line \( y = 0 \) with a Burger vector along \( y \), \( B^{(2)} \). We introduce
a distribution of dislocations $\rho(x, y)$ then the quantity $\rho(x, y)dxdy$ represents the Burger vector passing through the area $dxdy$. In order to understand the effect of the dislocations on the chiral fermions, we rewrite the two dimensional model on a lattice. On a lattice the model is characterized by the hopping matrix elements $t = t_0$ (for opposite spins) and $t' = t'_0$ (for parallel spins). In the absence of the Peierls deformation the lattice model takes the form:

$$H^L = \sum_{[x=na,y=ma]} \Psi^\dagger(\vec{r})[\sigma^x(-it)\hat{\partial}_y + \sigma^y(it)\hat{\partial}_x]\Psi(\vec{r}) + \Psi^\dagger(\vec{r})[\sigma^z(-2t' + M)]\Psi(\vec{r})$$

$$+ (-2t')\Psi^\dagger(\vec{r})\nabla_x \Psi(\vec{r}) + (-2t')\Psi^\dagger(\vec{r})\nabla_y \Psi(\vec{r})$$

(3)

where $\hat{\partial}_y, \hat{\partial}_x$ represents the discrete lattice derivatives and the action of the discrete operators $\nabla_x, \nabla_y$ on the field operator is: $\nabla_x \Psi(\vec{r}) \equiv \Psi(\vec{r} + a_x) + \Psi(\vec{r} - a_x) - 2\Psi(\vec{r})$ and $\nabla_y \Psi(\vec{r}) \equiv \Psi(\vec{r} + a_y) + \Psi(\vec{r} - a_y) - 2\Psi(\vec{r})$. When $2t'_0 = M$ we obtain the chiral model given in eq.(1).

$$H^L \approx \sum_{[x=na,y=ma]} \Psi^\dagger(\vec{r})[\sigma^x(-it)\hat{\partial}_y + \sigma^y(it)\hat{\partial}_x]\Psi(\vec{r})$$

(4)

Following [20] we use the explicit form of the deformation field introduced by the dislocations $u(y) = \frac{B^{(2)}}{\pi} \tan^{-1}\left(\frac{y}{W}\right)$. According to the elasticity theory, the Peierls edge dislocation [47] is determined by the half width $W$ function and Burger vector $B^{(2)}$: $\rho(y)dy = B^{(2)}$, $\rho(y) = -\frac{du}{dy}$, $u(y) \propto M(y) = \frac{B^{(2)}}{\pi} \tan^{-1}\left(\frac{y}{W}\right)$. Replacing the coordinates $[x, y]$ by the deformed one, $[X(\vec{r}), Y(\vec{r})]$ we introduce the deformed matrix elements $t' \neq t'_0 : t' = t'_0 + (\frac{\partial u}{\partial u(y)})u(y)$.

As a result we find: $-2t' + M = (-2t'_0 + M) - 2\frac{\partial u}{\partial u(y)}u(y) = 0 - \hbar v_F\kappa u(y)$ where $\hbar v_F \equiv 2t_0$, $\kappa = \frac{1}{t_0}(\frac{\partial u}{\partial u(y)})$. As a result we obtain the Peierls Hamiltonian which is similar to the domain wall model introduced by [32].

$$\epsilon\psi(x, y) = h^{\text{Peierls}}(x, y)\psi(x, y)$$

$$h^{\text{Peierls}}(x, y) = \hbar v_F[-i\sigma^x\partial_y + i\sigma^y\partial_x - \sigma^z\kappa M(y)]$$

$$\sigma^y\psi_\lambda(x, y) = \lambda\psi_\lambda(x, y); \ \lambda = \pm 1;$$

(5)

The eigenfunction for the Peierls model obeys periodic boundary conditions in the $x$ direction $\psi(x + L, y) = \psi(x, y)$. Therefore the eigen-spinor is given by: $\psi(x, y) = e^{ipx}\Phi(y), \ \ p = \ldots$
The eigenvalue $\epsilon$ is replaced by $E$ where $E \equiv \epsilon - \hbar v_F p$. We observe that if $\psi(x, y)$ is an eigenfunction with the eigenvalue $E$, the eigenfunction $\sigma^y \psi(x, y)$ corresponds to the eigenvalue $-E$. $E = 0$ corresponds to the eigenvalue $\epsilon = \hbar v_F p$ with the zero mode wave function $\psi_0(x, y; p)$ given by:

$$\psi_0(x, y; p) = e^{ipx}(\theta[y]e^{-\frac{\pi(2)}{4\pi} \int_0^y dy' \tan^{-1}\left(\frac{y'}{2}\right)} \begin{pmatrix} 1 \\ i \end{pmatrix} + \theta[-y]e^{\frac{\pi(2)}{4\pi} \int_0^y dy' \tan^{-1}\left(\frac{y'}{2}\right)} \begin{pmatrix} 1 \\ -i \end{pmatrix})\Phi(0)$$

(6)

$\theta[y]$ represents the step function which is one for $y > 0$ and zero otherwise. As a result, we obtain for $B^{(2)} > 0$ only a single zero mode with $\lambda = 1$ for $y < 0$ and $\lambda = -1$ for $y > 0$. The eigenstate given in eq.(6), $|U_0(p)\rangle$ and $|U_0(-p)\rangle$ are related by the time reversal symmetry $T|U_0(p)\rangle > \propto |U_0(-p)\rangle$. From the equations $T^2 = -1$ and $T|U_0(p)\rangle > \propto |U_0(-p)\rangle$ we conclude that the zero mode state is stable against backscattering along the $x$ direction. (The states $|U_0(p)\rangle$ and $|U_0(-p)\rangle$ are defined according to their wave functions $<x, y|U_0(p)\rangle \equiv \psi_0(x, y; p)$, $<x, y|U_0(-p)\rangle \equiv \psi_0(x, y; -p)$.)

**B-The edge dislocation with with a non zero torsion $T^{(2)}_{\mu, \nu}$**

For the remaining part, we will limit our discussion to the microscopic formulation of dislocations. When an edge dislocation is introduced into the crystal, the lattice coordinates $\vec{r} = (x, y)$ are modified, $\vec{r} \rightarrow \vec{R} = \vec{r} + \vec{u} \equiv [X(\vec{r}), Y(\vec{r})]$ where $\vec{u}(\vec{r})$ is the local lattice deformation with the core at the dislocation centered at $\vec{r} = (0, 0)$. The elastic field contour integral around the core $\vec{r} = (0, 0)$ is determined by the Burger vector $B^{(a)}$, $a = 1, 2$:

$$\oint dx^\mu \partial_\mu u^{(a)}(\vec{r}) = -B^{(a)}.$$  

For an edge dislocation in the $x$ direction the Burger vector $B^{(2)}$ is in the $y$ direction. The value of the burger vector $B^{(2)}$ is given by the shortest translation lattice vector in the $y$ direction. (For the TI Bi$_2$Se$_3$ the length of the vector $B^{(2)}$ is 5 times the inter atomic distance ). For the two dimensional surface we use the notation $x^\mu, \mu = x, y$ for the fixed Cartesian coordinate and $X^a, a = 1, 2$ to describe the reference frame, the media with dislocations. In the media with dislocations we introduced a set of vectors $e_a$ which are orthogonal to each other ($e_a, e_b) \equiv <e^b|e_a> = \delta_a^b$. The unit vector $e_a$ can be represented in terms of the Cartesian fixed frame space with the coordinate basis $\partial_\mu, \mu = x, y$. The basis in the media frame can be expanded in terms of the fixed Cartesian frame $\partial_\mu$; we have : $e_a = e^a_\mu \partial_\mu$ (for the particular case where vectors $e_a$ are
given by \( e_a = \partial_a \) the transformation between the two basis is \( e_a^\mu = \delta_{a}^{\mu} \). Any vector \( \vec{X} \) can be represented in terms of the unit vectors \( e_a \) (in the dislocation space) and \( \partial_\mu \) in the Cartesian fixed coordinates space, \( \vec{X} = X^a e_a = X^\mu \partial_\mu \). The dual vector \( e^a \) is a one form and can be expanded in terms of the one forms \( dx^\mu \). We have \( e^a = e^a_\mu dx^\mu \), where \( e^a_\mu \) represents the matrix transformation \( e^a \equiv (\partial_\mu X^a) dx^\mu \). The scalar product of the components \( e^a_\mu e^a_\nu = g_\mu,\nu \), \( e_a^\nu e_b^\mu = \delta_{a,b} \) defines the metric tensors, \( g_\mu,\nu \) (in the Cartesian frame) and \( \delta_{a,b} \) in the dislocation frame. We will compute the matrix elements fields \( e^a_\mu \) for our problem:

\[
e^a_\mu = \partial_\mu X^a(\vec{r}); \quad a = 1, 2; \quad \mu = x, y
\]  

(7)

Following [33] we can express the Burger vector in terms of the partial derivatives with respect the coordinates \( a = 1, 2 \) in the dislocation frame and \( \mu = x, y \) for the fixed Cartesian frame:

\[
\partial_x e_y^2 - \partial_y e_x^2 = B^{(2)} \delta^2(\vec{r})
\]

(8)

Using Stokes theorem, we replace the line integral \( \oint dx^\mu c^2_\mu(\vec{r}) \) by the surface integral \( \int \int dx^\mu dx^\nu [\partial_x e_y^2 - \partial_y e_x^2] \). For a system with zero curvature and non zero torsion \( T^{(2)}_{\mu,\nu} \) we find that the surface torsion tensor integral \( \int \int dx^\mu dx^\nu T^{(2)}_{\mu,\nu} \) is equal to \( \int \int dx^\mu dx^\nu [\partial_x e_y^2 - \partial_y e_x^2] \), and therefore both integrals are equal to the Burger vector.

\[
\oint dx^\mu c^2_\mu(\vec{r}) = \int \int dx^\mu dx^\nu [\partial_x e_y^2 - \partial_y e_x^2] = B^{(2)};
\]

\[
\int \int dx^\mu dx^\nu T^{(2)}_{\mu,\nu} = \int \int dx^\mu dx^\nu [\partial_x e_y^2 - \partial_y e_x^2] = B^{(2)};
\]

\[
\partial_x e_y^2 - \partial_y e_x^2 = B^{(2)} \delta^2(\vec{r})
\]

(9)

where \( dx^\mu dx^\nu \) represents the surface element. The tangent components \( e^a_\mu \) can be expressed in terms of the Burger vector density \( B^{(2)} \delta^2(\vec{r}) \) [34]:

\[
e^2_x = \left( \frac{B^{(2)}}{2\pi} \right) \frac{y}{x^2 + y^2}; \quad e^2_y = 1 - \left( \frac{B^{(2)}}{2\pi} \right) \frac{x}{x^2 + y^2}; \quad e^1_x = 1; \quad e^1_y = 0
\]

(10)

Using the tangent components, we obtain the metric tensor \( g_{\mu,\nu} \).

\[
e^a_\mu e^a_\nu \equiv e^1_\mu e^1_\nu + e^2_\mu e^2_\nu = g_{\mu,\nu}(\vec{r}); \quad e^a_\mu e^b_\nu \equiv e^a_\mu e^b_\nu + e^a_\nu e^b_\mu = \delta_{a,b}
\]

(11)
The inverse of the metric tensor $g_{\mu,\nu}(\vec{r})$ is the tensor $g^{\nu,\mu}(\vec{r})$ defined through the equation $g_{\mu,\nu}(\vec{r})g^{\nu,\mu}(\vec{r}) = \delta_{\mu}^{\nu}$. Using the tangent vectors, we find to first order in the Burger vector the metric tensor $g_{\mu,\nu}$ and the Jacobian transformation $\sqrt{G}$:

$$g_{x,x} = 1; \quad g_{x,y} = \frac{B^{(2)} y}{2\pi x^2 + y^2}; \quad g_{y,y} = 1 - \frac{B^{(2)} y}{2\pi x^2 + y^2}; \quad g_{y,x} = 0; \quad G = \det[g_{\mu,\nu}] = 1 - \frac{B^{(2)} y}{2\pi x^2 + y^2} \tag{12}$$

The inverse tensor is given by: $g_{\mu,\nu}^{(2)} \approx 1, \quad g_{x,y}^{(2)} = \frac{B^{(2)} y}{2\pi x^2 + y^2}, \quad g_{y,y}^{(2)} = 1 + \frac{B^{(2)} y}{2\pi x^2 + y^2}$. Using the inverse tensor $g^{\mu,\nu}$ we obtain the inverse matrix $\epsilon_{a}^{\mu}$ which is given by:

$$\epsilon_{a}^{\mu} = e_{a,\nu}g^{\nu,\mu} = (\delta_{a,b}e_{\nu}^{b})g^{\nu,\mu} = e_{a}^{\nu}g^{\nu,\mu} \tag{13}$$

In figure 1 we show the coordinate transformation $\vec{r} = (x, y) \rightarrow [X(\vec{r}) = x, Y(\vec{r}) = y + \frac{B^{(2)} \pi}{2\pi} \tan^{-1}(\frac{y}{x})]$ with the core of the dislocation centered at $\vec{r} = (0, 0)$.

Next we consider the effect of the dislocation on the TI Hamiltonian. Using the components $\epsilon_{a}^{\mu}$ we compute the the transformed Pauli matrices. The Hamiltonian in the absence of the edge dislocation is given by $h_{T.I.}^{T.I.} = i\gamma^{a}\partial_{a} \equiv \sum_{a=1,2} i\gamma^{a}\partial_{a}$ where the Pauli matrices are given by $\gamma^{1} = -\sigma^{2}, \gamma^{2} = \sigma^{1}$ and $\gamma^{3} = \sigma^{3}$. (We will use the convention that when an index appears twice we perform a summation over this index.) In the presence of the edge dislocation, the term $\gamma^{a}\partial_{a}$ must be expressed in terms of the Cartesian fixed coordinates $\mu = x, y, z$. As a result, the spinor $\Psi(\vec{r})$ transforms accordingly to the $SU(2)$ transformation . If $\tilde{\Psi}(\vec{R})$ is the spinor for the deformed lattice, it can be related with the help of an $SU(2)$ transformation to the spinor $\tilde{\Psi}(\vec{r})$ in the undeformed lattice: $\tilde{\Psi}(X,Y) = e^{-i\frac{\delta\varphi(x,y)}{2}\sigma^{3}}\Psi(x,y)$, where $\delta\varphi(x,y)$ is the rotation angle between the two set of coordinates: $\delta\varphi(x,y) = \tan^{-1}(\frac{Y}{X}) - \tan^{-1}(\frac{y}{x})$. Using the relation between the coordinates (see figure 1) $X = x$, and $Y = y + \frac{B^{(2)} \pi}{2\pi} \tan^{-1}(\frac{y}{x})$ with the singularity at $x = y = 0$ gives us that the derivative of the phase which is a delta function, $\partial_{x}\delta\varphi(x, y) = -\partial_{y}\delta\varphi(x, y) \propto \delta^{2}(x, y)$. Combining the transformation of the derivative with the $SO(2)$ rotation in the plane, we obtain the form of the chiral Dirac equation in the Cartesian space (see Appendix A) given in terms of the spin connection $\omega^{\mu,2}_{\mu} \tilde{\Psi}$:

$$i\gamma_{a}\partial_{a}\tilde{\Psi}(\vec{R}) = i\delta_{a,b}\gamma^{b}\partial_{a}\tilde{\Psi}(\vec{R}) = i\gamma^{a}\epsilon_{a}^{\mu}[\partial_{\mu} + \frac{1}{4}[\gamma^{b}, \gamma^{c}]\omega^{bc}_{\mu}](\vec{r}) \tag{14}$$

The Hamiltonian $h_{T.I.}^{T.I.} \rightarrow h_{edge}$ is transformed to the dislocation edge Hamiltonian with the
explicit form given by:

\[
\begin{align*}
    h^{\text{edge}} &= i \sigma^1 \partial_2 - i \sigma^2 \partial_1 = i \sigma^1 e_2^\mu [\partial_\mu + \frac{1}{8} [\sigma^1, \sigma^1] \omega^{1,2}_\mu] - i \sigma^2 e_1^\mu [\partial_\mu + \frac{1}{8} [\sigma^1, \sigma^1] \omega^{1,2}_\mu] \\
    &= i (\sigma^1 e_2^\mu - \sigma^2 e_1^\mu)(\partial_\mu + \frac{1}{8} [\sigma^1, \sigma^2] \omega^{1,2}_\mu)
\end{align*}
\]

To first order in the Burger vector we find: \( \omega^{12}_x = -\omega^{21}_x = 0 \) and \( -\omega^{21}_y = \omega^{12}_y = -\frac{B^{(2)}}{2} \delta^2(\vec{r}) \), see eqs. (69 – 71) in Appendix A.

In the second quantized form the chiral Dirac Hamiltonian in the presence of an edge dislocations is given by:

\[
\begin{align*}
    H^{\text{edge}} &\approx \int d^2 r \sqrt{G} \Psi^\dagger(\vec{r}) [h^{\text{edge}} - \mu] \Psi(\vec{r}) \\
    &\equiv \hbar v_F \int d^2 r \sqrt{G} \Psi^\dagger(\vec{r}) [i \sigma^1 (\partial_y - \frac{i}{2} \sigma^3 B^{(2)} \delta^2(\vec{r})) - i \sigma^2 \partial_x - \mu] \Psi(\vec{r})
\end{align*}
\]

\( h^{\text{edge}} \) is the Hamiltonian in the first quantized language, \( \mu \) is the chemical potential and \( \Psi(\vec{r}) = [\Psi_\uparrow(\vec{r}), \Psi_\downarrow(\vec{r})]^T \) is the two component spinor field.

**IV- The solution for the metallic Surface in the presence of an edge dislocation with the torsion \( T^{(2)}_{\mu,\nu} \)**

**A-The model in the momentum space**

We work in the momentum representation where the edge Hamiltonian \( h^{\text{edge}} \) takes the form:

\[
\begin{align*}
    h^{\text{edge}} &= \hbar v_F [\vec{K} \times \vec{\sigma}] \cdot \hat{z} + \frac{-i}{2} B^{(2)} \sigma^2 \int \frac{d^2 q}{(2\pi)^2} e^{i q_x (i \partial_{k_x}) + q_y (i \partial_{k_y})} \\
\end{align*}
\]

The Hamiltonian is **time reversal invariant** but it is **not invariant under the planar parity symmetry**. The eigenstate \( |u(\vec{K})\rangle > \) obeys the secular equation: \( [\epsilon - \epsilon(\vec{K}) + i \frac{B^{(2)}}{2} \sigma^2] |W(\epsilon)\rangle >= 0 \) where \( |W(\epsilon)\rangle >= \int \frac{d^2 q}{(2\pi)^2} |u(\vec{K} - \vec{q})\rangle >. \) The the presence of the term \( \frac{-i}{2} B^{(2)} \sigma^2 \) generates unstable solutions. A stable solution can be obtained if the Pauli matrix \( \sigma^2 \) annihilates the eigenstate. Using the chirality operator we observe that states
The eigenfunctions of the edge Hamiltonian which satisfy the chirality operator must have the form: 
\[
|u(k_x, y) > = e^{i k_x x} |u_\uparrow(y) >, |u_\downarrow(y) > \]
In the next sections we will construct the explicit eigenstates for this Hamiltonian.

**B-Identification of the physical contours for the edge Hamiltonian** 

In order to identify the solutions, we will use the complex representation. The coordinates in the complex representation are given by, 
\[
z = \frac{1}{2}(x + iy), \quad \bar{z} = \frac{1}{2}(x - iy), \quad \partial_z = \partial_x - i \partial_y, \quad \partial_{\bar{z}} = \partial_x + i \partial_y \]
In this representation the two dimensional delta function \(\delta^2(\vec{r})\) is given by \(\delta^2(\vec{r}) \equiv \frac{1}{\pi} \partial_z (\frac{1}{z}) \stackrel{[48, 49]}{=} \frac{1}{\pi} \partial_x (\frac{1}{z}) \). We will use the edge Hamiltonian \(h_{edge}\) and will compute the eigenfunctions 
\[
u_\epsilon(z, \bar{z}) = [U_{\epsilon \uparrow}(z, \bar{z}), U_{\epsilon \downarrow}(z, \bar{z})]^T \]
and 
\[
u_\epsilon(z, \bar{z}) = [V_{\epsilon \uparrow}(z, \bar{z}), V_{\epsilon \downarrow}(z, \bar{z})]^T \]
The eigenvalue equation is given by:

\[
\epsilon U_{\epsilon \uparrow}(z, \bar{z}) = -[\partial_z + (\frac{B^{(2)}}{2\pi})\partial_x (\frac{1}{z})]U_{\epsilon \downarrow}(z, \bar{z})
\]
\[
\epsilon U_{\epsilon \downarrow}(z, \bar{z}) = [\partial_{\bar{z}} + (\frac{B^{(2)}}{2\pi})\partial_x (\frac{1}{z})]U_{\epsilon \uparrow}(z, \bar{z})
\]

(19)

The eigenfunctions \(u_\epsilon(z, \bar{z})\) and \(v_\epsilon(z, \bar{z})\) can be written with the help of a singular matrix \(M(z, \bar{z})\) :

\[
u_\epsilon(z, \bar{z}) = M(z, \bar{z}) F_\epsilon(z, \bar{z})
\]

\[
F_\epsilon(z, \bar{z}) = \begin{bmatrix}
0 \\
\frac{B^{(2)}}{2\pi} (\frac{1}{z})
\end{bmatrix}
\begin{bmatrix}
F_{\epsilon \uparrow}(z, \bar{z}) \\
F_{\epsilon \downarrow}(z, \bar{z})
\end{bmatrix}
\]

\((F_\epsilon(z, \bar{z})\) and \(F_{-\epsilon}(z, \bar{z})\) are the transformed eigenfunctions for \(\epsilon > 0\) and \(\epsilon < 0\) respectively.

In terms of the transformed spinors the eigenvalue equation \(h_{edge}(z, \bar{z}) u_\epsilon(z, \bar{z}) = \epsilon u_\epsilon(z, \bar{z})\) and \(F_{\epsilon \downarrow}(z, \bar{z})\) becomes:

\[
\epsilon \begin{bmatrix}
F_{\epsilon \uparrow}(z, \bar{z}) \\
F_{\epsilon \downarrow}(z, \bar{z})
\end{bmatrix} = \left[\begin{array}{cc}
I(z, \bar{z}) & 0 \\
0 & (I(z, \bar{z})^*)
\end{array}\right] \begin{bmatrix}
-\partial_z \\
0
\end{bmatrix} \left[\begin{array}{c}
F_{\epsilon \uparrow}(z, \bar{z}) \\
F_{\epsilon \downarrow}(z, \bar{z})
\end{array}\right]
\]

where \(I(z, \bar{z}) = e^{-\frac{B^{(2)}}{2\pi} (\frac{1}{z})} \equiv e^{2\frac{B^{(2)}}{\pi} (\frac{iy}{x^2+y^2})}, (I(z, \bar{z}))^* = e^{2\frac{B^{(2)}}{\pi} (\frac{-iy}{x^2+y^2})}, |I(z, \bar{z})| = 1\). We search for zero modes \(\epsilon = 0\) and find:

\[
\partial_z F_{\epsilon \downarrow}(z, \bar{z}) = 0 \quad \partial_{\bar{z}} F_{\epsilon \uparrow}(z, \bar{z}) = 0
\]

(20)
The solutions are given by the holomorphic representation \( F_{\epsilon=0\uparrow}(z,\bar{z}) = f_{\uparrow}(z) \) and the anti-holomorphic function \( F_{\epsilon=0\downarrow}(z,\bar{z}) = f_{\downarrow}(\bar{z}) \). The zero mode eigenfunctions are given by:

\[
    u_{\epsilon=0,\uparrow}(z) = e^{-\frac{B(2)}{2\pi}(\frac{1}{z})}f_{\uparrow}(z), \quad u_{\epsilon=0,\downarrow}(\bar{z}) = e^{-\frac{B(2)}{2\pi}(\frac{1}{\bar{z}})}f_{\downarrow}(\bar{z})
\]

Due to the presence of the essential singularity at \( z = 0 \) it is not possible to find analytic functions \( f_{\uparrow}(z) \) and \( f_{\downarrow}(\bar{z}) \) which vanish fast enough around \( z = 0 \) such that \( \int d^2z u_{\epsilon=0,\lambda}(z) u_{\epsilon=0,\lambda}(z) < \infty \). Therefore, we conclude that zero mode solution does not exists. The only way to remedy the problem is to allow for states with finite energy.

In the next step we look for finite energy states. We perform a coordinate transformation:

\[
    z \rightarrow W[z,\bar{z}] ; \quad \bar{z} \rightarrow \overline{W}[z,\bar{z}]
\]

We demand that the transformation is conformal and preserve the orientation. This restricts the transformations to holomorphic and anti-holomorphic functions. This means that we have the conditions \( \partial_z W[z,\bar{z}] = 0 \) and \( \partial_{\bar{z}} \overline{W}[z,\bar{z}] = 0 \). As a result we obtain \( W[z,\bar{z}] = W[z] \) and \( \overline{W}[z,\bar{z}] = \overline{W}[\bar{z}] \), which obey the eigenvalue equations:

\[
    \epsilon F_{\epsilon\uparrow}(W, \overline{W}) = -\partial_W F_{\epsilon\downarrow}(W, \overline{W})
\]

\[
    \epsilon F_{\epsilon\downarrow}(W, \overline{W}) = \partial_{\overline{W}} F_{\epsilon\uparrow}(W, \overline{W})
\]

This implies the conditions \( \frac{dW[z]}{dz} = (I(z,\bar{z}))^* \) and \( \frac{d\overline{W}[\bar{z}]}{dz} = I(z,\bar{z}) \). Since \( I(z,\bar{z}) \) is neither holomorphic or anti-holomorphic and satisfy \( |I(z,\bar{z})| = 1 \), the only solutions for \( W[z] \) and \( \overline{W}[\bar{z}] \) must obey \( I(z,\bar{z}) = 1 \):

\[
    I(z,\bar{z}) \equiv e^{2\frac{B(2)}{\pi}(\frac{iy}{x^2+y^2})} = e^{i2\pi n} ; \quad n = 0, \pm 1, \pm 2, \ldots
\]

For \( I(z,\bar{z}) \neq 1 \) one obtains solutions which are unstable. The stable solutions will be given by a one parameter \( s \) curve \((s \text{ is the length of the curve}) \). The curve allows us to define the tangent \( \vec{t}(s) \) and the normal vectors \( \vec{N}(s) \). This allows us to introduce a two-dimensional region in the vicinity of the contour of \( \vec{t}(s) \rightarrow \vec{R}(s,u) = \vec{t}(s) + u\vec{N}(s) \).

**V- The wave function**
A-The wave function for $n = 0$

The condition $I(z, \xi) = e^{2\frac{B(z)}{\pi} \left(\frac{1}{x^2 + y^2}\right)} = 1$ for $n = 0$ is satisfied for $y = 0$ and large value of $y$ which obey $2\frac{B(z)}{\pi} \left(\frac{1}{x^2 + y^2}\right) << 1$. The values of $y$ which satisfy these conditions are restricted to $I(z, \xi) = e^{2\frac{B(z)}{\pi} \left(\frac{1}{x^2 + y^2}\right)} \approx 1$. This condition is satisfied for values of $y$ in the range:

$$2\frac{B(z)}{\pi} \left(\frac{y}{x^2 + y^2}\right) \leq \eta < \frac{\pi}{4} < 1$$

(25)

We introduce the radius $R_g = \frac{B^2}{2\pi}$ and find that the condition $I(z, \xi) \approx 1$ give rise to the equation for $y$. The solution is given by $x^2 + (y \pm \frac{2\pi}{\eta} R_g)^2 = (\frac{2\pi}{\eta} R_g)^2$. Therefore, for $|y| > |d| \geq (\frac{2\pi}{\eta}) 2R_g > 2R_g$ we have $I \approx 1$ which corresponds to a free particle eigenvalue equations.

$$\epsilon F_{\uparrow}(x, y) = e^{\frac{B(z)}{\pi} \left(\frac{1}{x^2 + y^2}\right)} [-\partial_x + i\partial_y] F_{\downarrow}(x, y)$$

$$\approx [-\partial_x + i\partial_y] F_{\downarrow}(x, y);$$

$$\epsilon F_{\downarrow}(x, y) = e^{\frac{B(z)}{\pi} \left(\frac{1}{x^2 + y^2}\right)} \partial_x + i\partial_y] F_{\uparrow}(x, y)$$

$$\approx [\partial_x + i\partial_y] F_{\uparrow}(x, y)$$

(26)

For $|y| > d$ the eigenfunctions are given by: $U_{\epsilon, \uparrow}(x, y) = e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + y^2}\right)} F_{\epsilon, \uparrow}(x, y)$, $U_{\epsilon, \downarrow}(x, y) = e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + y^2}\right)} F_{\epsilon, \downarrow}(x, y)$ where $F_{\epsilon, \uparrow}(x, y)$ and $F_{\epsilon, \downarrow}(x, y)$ are the eigenfunctions of equation (26). The envelope functions $e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + y^2}\right)}$, $e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + y^2}\right)}$ which multiply the wave functions $F_{\epsilon, \uparrow}(x, y)$, $F_{\epsilon, \downarrow}(x, y)$ impose vanishing boundary conditions for the eigenfunctions $U_{\epsilon, \uparrow}(x, y)$ and $U_{\epsilon, \downarrow}(x, y)$ at $y \to \pm \infty$. Therefore, we demand that the eigenfunctions $U_{\epsilon, \uparrow}(x, y)$, $U_{\epsilon, \downarrow}(x, y)$ should vanish at the boundaries $y = \pm \frac{L}{2}$. Since the multiplicative envelope functions for opposite spins is complex conjugate to each other we have to make the choice that one of the spin components vanishes at one side and the other component at the opposite side. Two possible choices can be made:

$$U_{\epsilon, \uparrow}(x, y = \frac{L}{2}) \equiv e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + \left(\frac{L}{2}\right)^2}\right)} F_{\epsilon, \uparrow}(x, \frac{L}{2}) = U_{\epsilon, \downarrow}(x, y = -\frac{L}{2}) \equiv e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + \left(\frac{L}{2}\right)^2}\right)} F_{\epsilon, \downarrow}(x, -\frac{L}{2}) = 0$$

or

$$U_{\epsilon, \uparrow}(x, y = -\frac{L}{2}) \equiv e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + \left(-\frac{L}{2}\right)^2}\right)} F_{\epsilon, \uparrow}(x, -\frac{L}{2}) = U_{\epsilon, \downarrow}(x, y = \frac{L}{2}) \equiv e^{-\frac{B(z)}{2\pi} \left(\frac{1}{x^2 + \left(-\frac{L}{2}\right)^2}\right)} F_{\epsilon, \downarrow}(x, \frac{L}{2}) = 0$$

Making the first choice, (both choices give the same eigenvalues and eigenfunction) we compute the eigenfunctions $F_{\epsilon, \uparrow}(x, y)$ and $F_{\epsilon, \downarrow}(x, y)$ for $|y| > d$ using the boundary conditions
Due to the fact that the solutions are restricted to $|y| > d$ no conditions need to be imposed at $x = y = 0$. In the present case we consider a situation with a single dislocation. This is justified for a dilute concentration of dislocations typically separated by a distance $l \approx 10^{-6}m$. (In principle we need at least two dislocations in order to satisfy the condition that the sum of the Burger vectors is zero.) The eigenvalues are given by $\epsilon = \pm \hbar v_F\sqrt{p^2 + q^2}$. The value of $p$ is determined by the periodic boundary condition in the $x$ direction $p(m) = \frac{2\pi}{L} m \equiv \frac{2\pi}{Na} m$, $m = 0, 1, ..., (N - 2), (N - 1)$ and $a$ is the lattice constant $a \approx \frac{2\pi}{\Lambda}$. The value of $q$ will be obtained from the vanishing boundary conditions at $y = \pm \frac{L}{2}$. The eigenfunctions $F_{\epsilon,\sigma}(x, y)$ will be obtained using the linear combination of the spinors introduced in chapter II. In the Cartesian representation we can build four spinors $\Gamma_{p,q}(x, y)$, $\Gamma_{p,-q}(x, y)$, $\Gamma_{-p,q}(x, y)$, $\Gamma_{-p,-q}(x, y)$ which are eigenstates of the chirality operator and are given by:

$$
\Gamma_{p,q}(x, y) = e^{ipx} e^{iqy} \begin{pmatrix} 1 \\ i e^{i\chi(p,q)} \end{pmatrix} \\
\Gamma_{p,-q}(x, y) = e^{ipx} e^{-iqy} \begin{pmatrix} 1 \\ i e^{-i\chi(p,q)} \end{pmatrix} \\
\Gamma_{-p,q}(x, y) = e^{ipx} e^{iqy} \begin{pmatrix} 1 \\ -i e^{i\chi(p,q)} \end{pmatrix} \\
\Gamma_{-p,-q}(x, y) = e^{-ipx} e^{-iqy} \begin{pmatrix} 1 \\ -i e^{-i\chi(p,q)} \end{pmatrix}
$$

(28)

where $\tan[\chi(p, q)] = \frac{q}{p}$.

The Hamiltonian $h^{\text{edge}}$ is not invariant under the symmetry operation $P_x : x \to -x, \sigma^1 \to \sigma^1, \sigma^2 \to -\sigma^2; [h^{\text{edge}}, P_x] \neq 0$ therefore; we need to construct two independent eigenfunctions $F_{p>0,q}^{(n=0,R)}(x, y)$ for $p > 0$ and $F_{p<0,q}^{(n=0,L)}(x, y)$ $p < 0$.

$$
F_{p>0,q}^{(n=0,R)}(x, y) = A(q) \Gamma_{p,q}(x, y) + B(q) \Gamma_{p,-q}(x, y) \\
F_{p<0,q}^{(n=0,L)}(x, y) = C(q) \Gamma_{-p,q}(x, y) + D(q) \Gamma_{-p,-q}(x, y)
$$

(29)
Employing the boundary conditions given in equation (27) we obtain the amplitudes \( \frac{D(q)}{A(q)} \), \( \frac{B(q)}{A(q)} \) and the conditions for the momenta \( q \). For the pair \( \Gamma_{p,q}(x,y), \Gamma_{p,-q}(x,y) \) \( p > 0 \) we obtain:

\[
F_{\epsilon(p>0,q+)}^{(n=0,R)}(x,y) = e^{ipx} e^{\frac{i}{2} \chi(p,q+)} [e^{i(q+y- \frac{1}{2} \chi(p,q+)}) + (-1)^{k+1} e^{-i(q+y- \frac{1}{2} \chi(p,q+))}]; |y| > d \\
F_{\epsilon(p>0,q+)}^{(n=0,R)}(x,y) = i e^{ipx} e^{\frac{i}{2} \chi(p,q+)} [e^{i(q+y+ \frac{1}{2} \chi(p,q+)}) + (-1)^{k+1} e^{-i(q+y+ \frac{1}{2} \chi(p,q+))}]; |y| > d \\
q \equiv q_+ = \frac{\pi}{L} k + \frac{1}{L} \tan^{-1}(\frac{q_+}{p}); k = 1, 2, 3...; \tan[\chi(p,q_+)] = \left(\frac{q_+}{p}\right) \\
\epsilon(p,q_+) = \pm \hbar v_F \sqrt{\left(\frac{2\pi}{L m}\right)^2 + q_+^2} \\
(30)
\]

Similarly, for the second pair \( \Gamma_{-p,q}(x,y), \Gamma_{-p,-q}(x,y) \), \( p > 0 \) we obtain:

\[
F_{\epsilon(-p>0,q-)}^{(n=0,L)}(x,y) = e^{-ipx} e^{\frac{i}{2} \chi(p,q-)} [e^{i(q-y+ \frac{1}{2} \chi(p,q-))} + (-1)^{k+1} e^{-i(q-y+ \frac{1}{2} \chi(p,q-))}]; |y| > d \\
F_{\epsilon(-p>0,q-)}^{(n=0,L)}(x,y) = -i e^{-ipx} e^{\frac{i}{2} \chi(p,q-)} [e^{i(q-y- \frac{1}{2} \chi(p,q-))} + (-1)^{k+1} e^{-i(q-y- \frac{1}{2} \chi(p,q-))}]; |y| > d \\
q \equiv q_- = \frac{\pi}{L} k - \frac{1}{L} \tan^{-1}(\frac{q_-}{p}); k = 1, 2, 3...; \tan[\chi(p,q_-)] = \left(\frac{q_-}{p}\right) \\
\epsilon(-p,q_-) = \pm \hbar v_F \sqrt{\left(\frac{2\pi}{L m}\right)^2 + q_-^2} \\
(31)
\]

For the state with zero momentum \( p = 0 \) we find:

\[
F_{\epsilon(p=0,q)}^{(n=0,0)}(x,y) = 2 e^{\frac{i \pi}{4}} \cos[qy + \frac{\pi}{4}]; |y| > d \\
F_{\epsilon(p=0,q)}^{(n=0,0)}(x,y) = i 2 e^{\frac{i \pi}{4}} \cos[qy - \frac{\pi}{4}]; |y| > d \\
q = \frac{\pi}{2L} + \frac{\pi}{L} k; k = 0, 1, 2, 3... \\
\epsilon(p=0,q) = \pm \hbar v_F |q| \\
(32)
\]

The eigenfunctions for the dislocation problem will be given for \( |y| > d \) by: \( u^{(n=0,R)}_\epsilon(x,y) \equiv \)
\[ [U^{(n=0,R)}_{\epsilon_+^\uparrow}(x,y), U^{(n=0,L)}_{\epsilon_+^\downarrow}(x,y)]^T, \ u^{(n=0,R)}_{\epsilon_+^\downarrow}(x,y) \equiv [U^{(n=0,L)}_{\epsilon_+^\uparrow}(x,y), U^{(n=0,L)}_{\epsilon_+^\downarrow}(x,y)]^T. \]

\[
U^{(n=0,R)}_{\epsilon_+^\uparrow}(x,y) \approx \frac{2\text{const.}(B^{(2)})}{G^{\uparrow}(x,y)L}e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,R)}_{\epsilon_+^\uparrow}(x,y)
\]

\[
U^{(n=0,R)}_{\epsilon_+^\downarrow}(x,y) \approx \frac{2\text{const.}(B^{(2)})}{G^{\downarrow}(x,y)L}e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,R)}_{\epsilon_+^\downarrow}(x,y)
\]

\[
U^{(n=0,L)}_{\epsilon_+^\uparrow}(x,y) \approx \frac{2\text{const.}(B^{(2)})}{G^{\uparrow}(x,y)L}e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,L)}_{\epsilon_+^\uparrow}(x,y)
\]

\[
U^{(n=0,L)}_{\epsilon_+^\downarrow}(x,y) \approx \frac{2\text{const.}(B^{(2)})}{G^{\downarrow}(x,y)L}e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,L)}_{\epsilon_+^\downarrow}(x,y)
\]

\[
U^{(n=0,0)}_{\epsilon_+^\uparrow}(x,y) \approx e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,0)}_{\epsilon_+^\uparrow}(x,y)
\]

\[
U^{(n=0,0)}_{\epsilon_+^\downarrow}(x,y) \approx e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})}F^{(n=0,0)}_{\epsilon_+^\downarrow}(x,y)
\]

(33)

where \( G(x,y) = 1 - \frac{B^{(2)}}{2\pi} \frac{y}{\sqrt{x^2+y^2}} \) is the Jacobian introduced by the edge dislocation. The eigenstates are normalized and obey: \( \int dx \int dy \sqrt{G(x,y)}(U^{(n=0,R)}_{\sigma^\uparrow}(x,y))^*U^{(n=0,R)}_{\sigma^\downarrow}(x,y) \approx \delta_{\sigma,\sigma^\prime}, \int dx \int dy \sqrt{G(x,y)}(U^{(n=0,L)}_{\sigma^\downarrow}(x,y))^*U^{(n=0,L)}_{\sigma^\downarrow}(x,y) \approx \delta_{\sigma,\sigma^\prime} \). The normalization factor \( \frac{2\text{const.}(B^{(2)})}{L} \approx \frac{2}{L} \), has a weak dependence on the Burger vector \( B^{(2)} \). This dependence is a consequence of the Jacobian \( \sqrt{G} \) which affects the normalization constant (see appendix B).

(The multiplicative factor \( e^{-\frac{\beta^{(2)}}{2\pi}(\frac{1}{x+y})} \) gives rise to a weak non-orthogonality between the states. This non-orthogonality of the linear independent eigenfunctions can be corrected with the help of the Grahm-Shmidt method.)

For the present case, backscattering is allowed but it is much weaker in comparison to regular metals. This is seen as follows: Time reversal is not violated; due to the parity violation, the eigenstates \( u^{(n=0,R)}_{\epsilon_+^\uparrow}(x,y) \) are not related by a time reversal symmetry \( (Tu^{(n=0,R)}_{\epsilon_+^\downarrow})(x,y) \neq u^{(n=0,L)}_{\epsilon_+^\downarrow}(x,y) \). As a result, the backscattering potential \( V_{p,-p} \) is controlled by a finite matrix element between states with different eigenvalues \( \epsilon(-p,q_-) \neq \epsilon(p,q_+) \) (contrary to regular metals where the impurity potential \( V_{p,-p} \) connects states with the same energy). In the present case \( |\epsilon(-p,q_-) - \epsilon(p,q_+)| = h\nu_F[\sqrt{(2\pi m)^2 + q^2} - \sqrt{(2\pi m)^2 + q^2}] \neq 0 \) the eigenvalues are not equal, therefore the finite matrix element controlled by the backscattering potential \( V_{p,-p} \) gives rise only to a second order backscattering effect!

**B- The circular contours-the wave function for \( n \neq 0 \)**

The equation \( I(z,\bar{z}) = e^{\frac{\beta^{(2)}}{y}(\frac{1}{x+y})} = e^{i2\pi n} \) gives the set of ring contours for \( n = \ldots \)
\[ (x(s))^2 + (y(s) \pm R_g(n))^2 = (R_g(n))^2 \]

The centers of the contours are given by \([\bar{x}, \bar{y}] = [0, R_g(n)]\) for \(n \neq 0\). When \(n > 0\) the center of the contours has positive coordinates (upper contour) and for \(n < 0\) the center has negative coordinates (lower contour). Each contour is characterized by a circle with a radius \(R_g(n) \equiv \frac{R_g}{|n|}\) centered at \([\bar{x}, \bar{y}] = [0, R_g(n)]\) (see figure 2). The contour is parametrized in terms of the arc length \(0 \leq s < 2\pi \frac{R_g}{|n|}\) which is equivalent to \(0 \leq \varphi < 2\pi\). Each contour is parametrized by \(\bar{r}(s) \equiv [x(s), R_g(n) + y(s)]\) where \(x(s) = R_g(n) \cos\left[\frac{s}{R_g(n)}\right] \equiv R_g(n) \cos[\varphi]\) and \(y(s) = R_g(n) \sin\left[\frac{s}{R_g(n)}\right] \equiv R_g(n) \sin[\varphi]\). We will extend this curve to a two dimensional strip with the coordinate \(u\) in the normal direction: For the curve curve \(\bar{r}(s) = [x(s), y(s)]\) we will use the tangent \(\vec{t}(s)\) and the normal vector \(\vec{N}(s)\). Therefore, the two dimensional region in the vicinity of the one parameter curve \(\bar{r}(s)\) is replaced by \(\bar{r}(s) \rightarrow \bar{R}(s, u) = \bar{r}(s) + u\vec{N}(s)\).

\[
\begin{align*}
  x(s, u) &= R_g(n) \cos\left[\frac{s}{R_g(n)}\right] + u \cos\left[\frac{s}{R_g(n)}\right] \\
  y(s, u) &= R_g(n) \sin\left[\frac{s}{R_g(n)}\right] + u \sin\left[\frac{s}{R_g(n)}\right]
\end{align*}
\]

We will restrict the width \(|u|\) such that \(e^{i2\pi n} e^{i\eta} \approx 1\) where \(\eta\) obeys \(\eta < \frac{\pi}{4} < 1\), \(|u| \leq \frac{R_g(n)}{2\pi} - R_g(n) \approx R_g(n) n \frac{\pi}{2\pi n} < \frac{R_g(n)}{8n}\). In these new coordinates, the Dirac equation is approximated for \(|u| \leq R_g(n) n \frac{\pi}{2\pi n} = \frac{D(n)}{2}\) by:

\[
\begin{align*}
  \epsilon F_{\downarrow}(s, u) &= -I(s, u) e^{-i \frac{\pi}{R_g(n)} [\partial_u - \frac{i}{u \frac{1}{R_g(n)}} \partial_s] F_{\downarrow}(s, u)} \approx -e^{-i \frac{\pi}{R_g(n)} [\partial_u - i \partial_s] F_{\downarrow}(s, u)} \\
  \epsilon F_{\uparrow}(s, u) &= (I(s, u))^* e^{i \frac{\pi}{R_g(n)} [\partial_u + \frac{i}{u \frac{1}{R_g(n)}} \partial_s] F_{\uparrow}(s, u)} \approx e^{i \frac{\pi}{R_g(n)} [\partial_u + i \partial_s] F_{\uparrow}(s, u)}
\end{align*}
\]

The solution for the contour \(n \neq 0, 0 \leq s < 2\pi R_g(n); |u| \leq \frac{D(n)}{2}\)

The periodicity in \(s\) allows us to represent the eigenfunctions in the form: \(F_{\downarrow}(s, u) = \sum_{j=-\infty}^{\infty} \sum_{q} e^{i j (\pi g(qn))} e^{iu} F_{\downarrow}(j, q)\) and \(F_{\uparrow}(s, u) = \sum_{j=-\infty}^{\infty} \sum_{q} e^{i (j+1) (\pi g(qn))} e^{iu} F_{\uparrow}(j, q)\). We
\[ \epsilon F_\uparrow(\epsilon; j, q) = (iq + \frac{j}{R_g(n)})F_\uparrow(\epsilon; j, q) \]
\[ \epsilon F_\downarrow(\epsilon; j, q) = (iq + \frac{j+1}{R_g(n)})F_\downarrow(\epsilon; j, q) \]

(37)

The determinant of the two equations determines the relation between the eigenvalue \( \epsilon \), the transverse momentum \( Q(\epsilon) \) and the eigenfunctions \( F_\downarrow(j, q), F_\uparrow(j, q) \). The eigenvalues are degenerate and obey: \( \epsilon(j = l; k) = \epsilon(j = -(l+1); k) \), where \( l \geq 0 \).

\[ q \equiv -\frac{i}{2R_g(n)} \pm Q(\epsilon); \quad Q(\epsilon) = \sqrt{\epsilon^2 - \left(\frac{l+\frac{1}{2}}{R_g(n)}\right)^2} \]
\[ F_\epsilon(l, q) \equiv [F_\epsilon(\uparrow)(l, q), F_\epsilon(\downarrow)(l, q)]^T \propto [1, e^{-i\kappa(Q,l)}]^T; \quad \kappa(Q,l) = \tan^{-1}\left(\frac{QR_g(n)}{l+\frac{1}{2}}\right) \]

(38)

The value of the transversal momentum \( Q(\epsilon) \) will be determined from the boundary conditions at \( \pm \frac{D(n)}{2} \). We will introduce a polar angle \( \theta \) measured with respect the Cartesian axes: The angle \( 0 < \varphi(n = 1) \leq 2\pi \) for the upper contour \( n = 1 \) centered at \([\overline{x} = 0, \overline{y} = R_g]\) is described by the polar coordinate \( 0 < \theta \leq \pi \) measured from the center of the Cartesian coordinate \([0, 0]\). The lower contour centered at \([\overline{x} = 0, \overline{y} = -R_g]\) characterized by the angle \( 0 < \varphi(n = -1) \leq 2\pi \) is described by the polar angle \( \theta \) restricted to \( \pi < \theta \leq 2\pi \). We establish the correspondence between \( \varphi(n = \pm 1) \) and \( \theta \):

\[ \varphi(n = 1) = 2\theta + \frac{3\pi}{2} \text{ for the upper contour } n = 1, \ 0 < \theta \leq \pi \]
\[ \varphi(n = -1) = 2\theta + \frac{3\pi}{2} + \pi \text{ for the lower contour } n = -1, \ 0 < \theta \leq \pi \]

(39)

Following the discussion from the previous chapter we will introduce the following boundary conditions:

\[ F^{(n=1)}_\epsilon(s, u = \frac{D}{2}) = 0; \quad F^{(n=1)}_\epsilon(s, y = -\frac{D}{2}) = 0 \]
\[ F^{(n=-1)}_\epsilon(s, u = -\frac{D}{2}) = 0; \quad F^{(n=-1)}_\epsilon(s, y = \frac{D}{2}) = 0 \]
\[ D(n = \pm 1) \equiv D \]

(40)
For the two contours $n = \pm 1$ we introduce eight spinors $\Gamma^{(n=\pm1)}_{l,Q}(\varphi(n = \pm1), u), \Gamma^{(n=\pm1)}_{l,-Q}(\varphi(n = \pm1), u), \Gamma^{(n=\pm1)}_{-l,Q}(\varphi(n = \pm1), u), \Gamma^{(n=\pm1)}_{-l,-Q}(\varphi(n = \pm1), u)$. Using this spinor we will compute the eigenfunctions: $F^{(n=1,R)}_{l,Q}(\varphi(n = 1), u), F^{(n=-1,L)}_{l,Q}(\varphi(n = -1), u)$ characterized by the transverse momentum $Q_-$ and the pair $F^{(n=-1,R)}_{l,Q}(\varphi(n = -1), u), F^{(n=1,L)}_{l,Q}(\varphi(n = 1), u)$ characterized by the momentum $Q_+$.

$$
\Gamma^{(n=\pm1)}_{l,Q}(\varphi(n = \pm1), u) = e^{il(\varphi(n=\pm1))}e^{iQ_u} \begin{pmatrix} 1 \\ e^{i(\varphi(n=\pm1))}e^{-ik(l,Q)} \end{pmatrix} \\
\Gamma^{(n=\pm1)}_{l,-Q}(\varphi(n = \pm1), u) = e^{il(\varphi(n=\pm1))}e^{-iQ_u} \begin{pmatrix} 1 \\ e^{i(\varphi(n=\pm1))}e^{ik(l,Q)} \end{pmatrix} \\
\Gamma^{(n=\pm1)}_{-l,Q}(\varphi(n = \pm1), u) = e^{-il(\varphi(n=\pm1))}e^{iQ_u} \begin{pmatrix} 1 \\ -e^{-i(\varphi(n=\pm1))}e^{ik(l,Q)} \end{pmatrix} \\
\Gamma^{(n=\pm1)}_{-l,-Q}(\varphi(n = \pm1), u) = e^{-il(\varphi(n=\pm1))}e^{-iQ_u} \begin{pmatrix} 1 \\ -e^{-i(\varphi(n=\pm1))}e^{-ik(l,Q)} \end{pmatrix}
$$

(41)

Using the vanishing boundary condition given in equation (40), we find for the upper contour $n = 1$ and positive angular momentum $l$ the wave functions:

$$
F^{(n=1,R)}_{\epsilon(l,Q_-),1}(\varphi(n = 1), u) = e^{il(\varphi(n=1))}e^{-\frac{1}{2}k(l,Q_-)}[e^{i(Q_-u+\frac{1}{2}k(l,Q_-))} + (-1)^{k+1}e^{-i(Q_-u+\frac{1}{2}k(l,Q_-))}]
$$

$$
F^{(n=1,R)}_{\epsilon(l,Q_-),-1}(\varphi(n = 1), u) = e^{il(\varphi(n=1))}e^{\frac{1}{2}k(l,Q_-)}e^{i\varphi(n=1)}[e^{i(Q_-u-\frac{1}{2}k(l,Q_-))} + (-1)^{k+1}e^{-i(Q_-u-\frac{1}{2}k(l,Q_-))}]
$$

(42)

For the lower contour $n = -1$ and negative angular momentum we find:

$$
F^{(n=-1,L)}_{\epsilon(-l,Q_-),1}(\varphi(n = -1), u) = e^{-il(\varphi(n=-1))}e^{\frac{1}{2}k(l,Q_-)}[e^{i(Q_-u-\frac{1}{2}k(l,Q_-))} + (-1)^{k+1}e^{-i(Q_-u-\frac{1}{2}k(l,Q_-))}]
$$

$$
F^{(n=-1,L)}_{\epsilon(-l,Q_-),-1}(\varphi(n = -1), u) = e^{-il(\varphi(n=-1))}e^{-\frac{1}{2}k(l,Q_-)}(-1)e^{-i\varphi(-1)}[e^{i(Q_-u+\frac{1}{2}k(l,Q_-))} + (-1)^{k+1}e^{-i(Q_-u+\frac{1}{2}k(l,Q_-))}]
$$

(43)

For both set of eigenfunctions we obtain the quantization conditions for the momentum $Q_-$ and eigenvalues $\epsilon(l, Q_-)$.

$$
Q_- = \frac{\pi}{D} k - \frac{1}{D} \tan^{-1}\left(\frac{Q_-R_g(1)}{l+\frac{1}{2}}\right), k = 1, 2, 3...; \tan[\kappa(l, Q_-)] = \left(\frac{Q_-R_g(1)}{l+\frac{1}{2}}\right)
$$

$$
\epsilon(l, Q_-) = \pm \hbar v_F \sqrt{\left(\frac{l+\frac{1}{2}}{R_g(1)^2}\right)^2 + Q_-^2}
$$

(44)
The presence of the singular transformation $e^{-\frac{\theta}{2\pi}(\frac{1}{x+y})}$ and $e^{-\frac{\theta}{2\pi}(\frac{1}{x-y})}$ demands that the eigenfunction should vanish at $x = y = 0$, which corresponds to the point $u = 0$, $\varphi(n = 1) = -\frac{\pi}{2}$ for the upper contour and $u = 0$, $\varphi(n = -1) = \frac{\pi}{2}$ for the lower contour. Using the mapping given eq.(39) we can transform from the variables $\varphi(n = \pm 1)$ to the polar coordinate $\theta$ (the wave function must vanishes at $\theta = 0$ and $\theta = \pi$). In order to obtain a finite wave function, we combine the spinors $F_{l,Q-}^{(n=1,R)}(\varphi(n = 1), u)$, $F_{l,Q-}^{(n=-1,L)}(\varphi(n = -1), u)$ with the singular transformation functions $e^{-\frac{\theta}{2\pi}(\frac{1}{x+y})}$, $e^{-\frac{\theta}{2\pi}(\frac{1}{x-y})}$:

$$U_{e(l,Q-)}^{\uparrow}(\theta, u) = e^{-\frac{\theta}{2\pi}[(R_y+u)\cos[\varphi(n=1)]+i(R_y+(R_y+u)\sin[\varphi(n=1)]]} F_{l,Q-}^{(n=1,R)}(\varphi(n = 1), u)$$

$$+ r(l, \uparrow)e^{-\frac{\theta}{2\pi}[(R_y+u)\cos[\varphi(n=1)]+i(-R_y+(R_y+u)\sin[\varphi(n=1)])]} F_{-l,Q-}^{(n=1,L)}(\varphi(n = -1), u)$$

and

$$U_{e(l,Q-)}^{\downarrow}(\theta, u) = e^{-\frac{\theta}{2\pi}[(R_y+u)\cos[\varphi(n=1)]-i(R_y+(R_y+u)\sin[\varphi(n=1)]]} F_{l,Q-}^{(n=1,R)}(\varphi(n = 1), u)$$

$$+ r(l, \downarrow)e^{-\frac{\theta}{2\pi}[(R_y+u)\cos[\varphi(n=1)]-i(-R_y+(R_y+u)\sin[\varphi(n=1)])]} F_{-l,Q-}^{(n=1,L)}(\varphi(n = -1), u)$$

The amplitudes $r(l, \uparrow)$, $r(l, \downarrow)$ are determined by demanding the vanishing of the wave function $U_{e(l,Q-)}^{\uparrow}(\theta, u)$ and $U_{e(l,Q-)}^{\downarrow}(\theta, u)$ at $x = y = 0$. As a result, we obtain the explicit form of the wave functions $U_{e(l,Q-)}^{\uparrow}(\theta, u)$, $U_{e(l,Q-)}^{\downarrow}(\theta, u)$ (see equation (76) in Appendix-C) which depend of the parameters $\eta(u)$ and $\zeta(\theta, u) : \eta(u) = \frac{R_y(1)}{R_y+u}$, $\frac{|u|}{R_y(1)} < 1$; $\zeta(\theta, u) = \frac{2\pi(R_y+u)!((\sin[2\theta])^2+(\eta(u)-\cos[2\theta])^2)}{2\pi(R_y+u)!((\sin[2\theta])^2+(\eta(u)-\cos[2\theta])^2)}$.

For the second pair $F_{l,Q+}^{(n=-1,R)}(\varphi(n = -1), u)$, $F_{-l,Q+}^{(n=1,L)}(\varphi(n = 1), u)$ we obtain the quantization conditions: $Q_+ = \frac{n}{4}\tan^{-1}(\frac{Q_+R_y(1)}{R_y(n)})$, $k=1,2,3... \tan[k(l, Q_+)] = (\frac{Q_+R_y(1)}{R_y(n)})$ with the eigenvalues: $\epsilon(l, Q_+) = \pm\hbar v_F \sqrt{(\frac{n}{L})^2 + Q_+^2}$. Following the same procedure as we used for the pairs $F_{l,Q+}^{(n=-1,R)}(\varphi(n = -1), u)$, $F_{-l,Q-}^{(n=1,L)}(\varphi(n = -1), u)$ we obtain the wave functions $U_{e(l,Q+)}^{\uparrow}(\theta, u)$, $U_{e(l,Q+)}^{\downarrow}(\theta, u)$ which are given in equation (77) Appendix-C.

VI - Computation of the STM density of states

The STM tunneling current $I$ is a function of the bias voltage $V$ which gives spatial and spectroscopic information about the electronic surface states. At zero temperature, the derivative of the current with respect the bias voltage $V$ is given in term of the single particles eigenvalues: $\epsilon(m, q_-) = \pm\hbar v_F \sqrt{(\frac{2m}{L})^2 + q_-^2}$, $\epsilon(m, q_+) = \pm\hbar v_F \sqrt{(\frac{2m}{L})^2 + q_+^2}$, $m = 0,1,2,3...$ for contour $n = 0$. For the upper and lower circular contours $n = \pm 1$, we have
\[ \varepsilon(l, Q_-) = \pm \hbar v_F \sqrt{\left( \frac{l + \frac{1}{2}}{R_g(1)} \right)^2 + Q_-^2}, \varepsilon(l, Q_+) = \pm \hbar v_F \sqrt{\left( \frac{l + \frac{1}{2}}{R_g(1)} \right)^2 + Q_+^2}, \quad l = 0, 1, 2, 3 \ldots \]

The \textit{STM} density of states is computed for a voltage \( V \) between the \textit{STM} tip and the sample. The tunneling current is a function of the bias voltage \( V \) and the chemical potential \( \mu > 0 \) \[30]:

\[
\frac{dI}{dV} \propto D(E = eV; s, u) \equiv \sum_n D^{(n)}(E = eV; s, u) =
\]

\[
\sum_{n=\pm1} \sum_{l} \sum_{Q_+} \sum_{\sigma} |U^{(n=\pm1;l,Q_+)}(\theta, u)|^2 \delta[eV + \mu - \eta \hbar v_F \sqrt{\left( \frac{l + \frac{1}{2}}{R_g(1)} \right)^2 + Q_+^2}]
\]

\[45\]

\( \eta = + \) corresponds to electrons with energy \( 0 < \epsilon \leq \mu \) and \( \eta = - \) corresponds to electrons below the Dirac point \( \epsilon < 0 \). For the rest of this paper we will take the chemical potentials to be \( \mu = 120mV \) (this is typical value for the TI). We will neglect the states with \( \eta = - \) which correspond to particles below the Dirac cone (see Appendix - B). The density of states at the tunneling energy \( eV \) is weighted by the probability density of the \textit{STM} tip at position \([x, y]\) for \( n=0 \). The contours for \( n = \pm 1 \) will be parametrized in terms of the polar angle \( \theta \) and transverse coordinate \( u \). The proportionality factor \( J \) for the tunneling probability (not shown in the equation) \( \frac{dI}{dV} = JD(V; x, y) \) is a function of the distance between the tip and the sample. The notation \( D^{(n)}(V; x, y) \) represents the tunneling density for the different contours.

\[A-\] The tunneling density of states for the Peierls model \( D^{\text{Peierls}}(V; x, y) \)

We consider first the Peierls model introduced in section \( III - A \). This model has a zero mode for which we have computed the wave function in equation \( 6 \). We find that the tunneling density of states density is confined to the quantum strip \( y = 0 \) with a varying width \( \tilde{W} \); \( |y| < \tilde{W} \) determined by the dislocations distribution \[47\]

\[
D^{\text{Peierls}}(V; x, y) \propto \left( \frac{L}{\hbar v_F} \right)^2 (B^{(2)} L^2 e^{-\frac{2\kappa b^{(2)}}{\pi} \int_0^{|y|} dy' \tan^{-1}(\frac{x'}{L})})
\]

\[46\]

In figure 3 we show the tunneling density of states for the parameters \( \frac{2\kappa b^{(2)}}{\pi} = 0.1 \) in units of the half width \( \tilde{W} \).
B-The tunneling density of states $D^{(n=0)}(V; x, y)$ for $n = 0$

Summing up the single particle states weighted with occupation probability $|U_0^{(n=0;m,qv)}(x, y)|^2$, we obtain a space dependent density of states for the two dimensional boundary surface $\frac{d}{2} \leq x \leq \frac{d}{2}$ and the coordinate $y$ is restricted to the regions $\frac{d}{2} < y \leq \frac{d}{2}$ and $\frac{d}{2} < y \leq \frac{d}{2}$. We will perform the computation at the thermodynamic limit, namely we replace the discrete momentum $\frac{\pi}{L}k$ by $Y = \frac{k}{N}$ and $\frac{2\pi}{L}m$ by $X = \frac{m}{N}$ where $N = \frac{L}{a}$. We find for the dimensionless momentum $\hat{q} \equiv qa$ the equations : $\hat{q}_\pm(Y) = \pi Y \pm \frac{1}{\pi} \tan^{-1}[\hat{q}_\pm(Y)]$ where $2\pi X = pa = \hat{p}$. As a result we obtain the following density of states $\frac{\partial \hat{q}_\pm}{\partial Y}$

\[
\begin{align*}
\left[\frac{\partial \hat{q}_+}{\partial Y}\right]^{-1} &= \frac{1}{\pi} \frac{\hat{q}_+^2 + \hat{p}^2 - \frac{1}{N} \hat{p}}{\hat{q}_+^2 + \hat{p}^2} \\
\left[\frac{\partial \hat{q}_-}{\partial Y}\right]^{-1} &= \frac{1}{\pi} \frac{\hat{q}_-^2 + \hat{p}^2 + \frac{1}{N} \hat{p}}{\hat{q}_-^2 + \hat{p}^2}
\end{align*}
\]

(47)

Using this results, we compute the tunneling density of states in terms of the energy $\mu + eV$ measured with respect the chemical potential $\mu$ and the transverse energy $\epsilon_\perp \equiv hv_F \epsilon_\perp$.

\[
\begin{align*}
D^{(n=0)}(V; x, y) &= \left(\frac{L}{hv_F}\right)^2 B(2) \frac{1}{L} \int_{0}^{E_{\max}} d\epsilon_\perp \frac{(\mu + eV)}{\sqrt{(\mu + eV)^2 - \epsilon_\perp^2}} \\
&\times \left\{ \frac{1}{2} \left( 1 + \frac{hv_F}{L(\mu + V)} \right) \sqrt{1 - \left( \frac{\epsilon_\perp}{\mu + V} \right)^2} + \frac{1}{2} \left( 1 - \frac{hv_F}{L(\mu + V)} \right) \sqrt{1 - \left( \frac{\epsilon_\perp}{\mu + V} \right)^2} \right\} \\
&\times \left\{ \frac{hv_F}{L} (H[\mu + V - \frac{hv_F}{2L}] - H[\mu + eV - E_{\max}]) \cdot ((\cos[\frac{(\mu + eV)}{hv_F} y - \frac{\epsilon_\perp}{4}])^2 + (\cos[\frac{(\mu + eV)}{hv_F} y - \frac{\epsilon_\perp}{4}])^2) \right\} \\
&= \left(\frac{L}{hv_F}\right)^2 B(2) \frac{1}{L} \int_{0}^{E_{\max}} d\epsilon_\perp \frac{(\mu + eV)}{\sqrt{(\mu + eV)^2 - \epsilon_\perp^2}} \\
&\times \left\{ \frac{hv_F}{L} (H[\mu + V - \frac{hv_F}{2L}] - H[\mu + eV - E_{\max}]) \cdot ((\cos[\frac{(\mu + eV)}{hv_F} y - \frac{\epsilon_\perp}{4}])^2 + (\cos[\frac{(\mu + eV)}{hv_F} y - \frac{\epsilon_\perp}{4}])^2) \right\} \\
&\times \left(\frac{L}{hv_F}\right)^2 B(2) \frac{1}{L} \int_{0}^{E_{\max}} d\epsilon_\perp \frac{(\mu + eV)}{\sqrt{(\mu + eV)^2 - \epsilon_\perp^2}} \\
&\times \left\{ \frac{hv_F}{L} (H[\mu + V - \frac{hv_F}{2L}] - H[\mu + eV - E_{\max}]) \right\}
\end{align*}
\]

for $|y| > d$

(48)

$H[\mu + eV - \frac{hv_F}{2L}]$ is the step function which is one for $\mu + eV - \frac{hv_F}{2L} \geq 0$ and zero otherwise.

$a = \frac{2\pi}{L}$ is the short distance cut-off and $E_{\max} = hv_F \Lambda < 0.3eV$ is the maximal energy which restricts the validity of the Dirac model. We observe in the second line that the asymmetry in the density of states $1 \pm \frac{1}{\pi} \frac{hv_F}{L(\mu + V)} \sqrt{1 - \left( \frac{\epsilon_\perp}{\mu + V} \right)^2}$ cancels.
Equation (48) shows that the tunneling density of states is linear in the energy $\mu + eV$ (in the present case we have looked only for energies above the Dirac cone). For the chemical potential $\mu = 120mV$, the zero energy corresponds to the Voltage $V = -120mV$. The tunneling density of states has a constant part at energies $\frac{\hbar v_F}{2e} \approx 0.2mV$ for $-120mV < V < -119.8mV$. For $V > -119.8mV$ the density of states is proportional to $\mu + eV$.

In figure 4 we have plotted the tunneling density of states as a function of the coordinates $x$ and $y$. The shape of the plot is governed by the the multiplicative factor $e^{-\frac{\mu(2)}{\pi}(\frac{x}{x^2+y^2})}$ which governs the solutions in eq.(35). We observe that the density of state is maximal in the region $|y| < 10B^{(2)}$.

Figure 5 shows the dependence on the voltage $V$ and coordinate $y$. We observe the linear increase in the tunneling density of states which is maximal in the region $|y| < 10B^{(2)}$.

**C-The tunneling density of states $D^{(n=0)}(V, x, y; \vec{r}_1, \ldots \vec{r}_{2M})$ for 2M dislocations.**

For many dislocations which satisfy $\sum_{w=1}^{2M} B^{(2,w)} = 0$ (sum of the Burger vectors is zero) with the core centered at $[x_w, y_w]$ , $w = 1,2,\ldots,2M$ the coordinate $\vec{r} = (x, y) \rightarrow [X(\vec{r}), Y(\vec{r})]$ is replaced by $[X(\vec{r}) = x, Y(\vec{r}) = y + \sum_w B^{(2,w)} \tan^{-1}(\frac{y-y_w}{x-x_w})]$. Following the method used previously, we find the edge Hamiltonian with many dislocations takes the form:

$$h^{\text{edge}}(w = 1,2\ldots2M) \approx i\sigma^1[\partial_y - \frac{i}{2} \sum_{w=1}^{2M} \sigma^3 B^{(2,w)} \delta^2(\vec{r} - \vec{r}_w)] - i\sigma^2 \partial_x$$  (49)

As a result, the wave functions are given by:

$$U^{(n=0,w=1,2\ldots2M)}_{\uparrow}(x,y) \propto \prod_{w=1,2\ldots2M} e^{-\frac{\mu(2)}{2\pi}(\frac{1}{x-x_w}+\frac{1}{y-y_w})} F^{(n=0)}_{\uparrow}(x,y)$$

$$U^{(n=0,w=1,2\ldots2M)}_{\downarrow}(x,y) \propto \prod_{w=1,2\ldots2M} e^{-\frac{\mu(2)}{2\pi}(\frac{1}{x-x_w}+\frac{1}{y-y_w})} F^{(n=0)}_{\downarrow}(x,y)$$  (50)

Using these wave functions, we find that the tunneling density of states is given by:

$$D^{(n=0)}(V, x, y; \vec{r}_1, \ldots \vec{r}_{2M}) \propto \prod_{w=1,2\ldots2M} e^{-\frac{\mu(2)}{\pi}(\frac{x-x_w}{x-x_w+y-y_w})}$$  (51)

In figure 6 we show the tunneling density of states for an even number of dislocations in the $y$ directions which have the core on the $y = 0$ axes ($\vec{r}_w = [x_w, y_w = 0], w = 1,2,3,\ldots2M$). We observe that the tunneling density of states is confined to the axes $y = 0$ and resembles the structure obtained from the Peierls model given in figure 3.
D-The tunneling density of states $D^{(n=\pm 1)}(V; \theta, u)$ for the $n = \pm 1$ contours.

Following the same procedure as used for the $n = 0$ case and the eigenfunction given in Appendix-C, we find for the tunneling density of states:

$$D^{(n=\pm 1)}(V; \theta, u) \equiv D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{even}} + D^{(n=\pm 1)}(\mu, V; \theta, u, \mu)_{\text{odd}}$$  \hspace{1cm} (52)

For the even $k$'s, we solve for the momentum $Q_+$ and $Q_-$ and find:

$$D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{even}} = \frac{(B^{(2)})^2}{2\pi R_g(1) D(1) \sqrt{G(\theta, u)}} \sum_{Q_r=Q_+,Q_-} \sum_{l=0}^{\infty} \delta(eV + \mu - \hbar v_F \sqrt{(\frac{l + \frac{1}{2}}{R_g(1)})^2 + Q_r^2})$$ \hspace{1cm} (53)


due \begin{equation} \begin{aligned} ([e^{-2\zeta(\theta,u)\sin[2\theta]} + e^{2\zeta(\theta,u)\sin[2\theta]}](\cos[Q_r u - \frac{1}{2}\kappa(l, Q_r)])^2 + (\sin[Q_r u + \frac{1}{2}\kappa(l, Q_r)])^2) + \\
2(-1)^l \cos[Q_r u + \frac{1}{2}\kappa(l, Q_r)] \cos[Q_r u - \frac{1}{2}\kappa(l, Q_r)] \cdot \\
(\cos[l(\theta + \frac{3\pi}{2}) - \zeta(\theta,u)(-\eta(u) + \cos[2\theta])] - \cos[(l + 1)(\theta + \frac{3\pi}{2}) + \zeta(\theta,u)(-\eta(u) + \cos[2\theta])])
\end{aligned} \end{equation}

Similarly for the odd $k$'s we find:

$$D^{(n=\pm 1)}(\mu, V; \theta, u)_{\text{odd}} = \frac{(B^{(2)})^2}{2\pi R_g(1) D(1) \sqrt{G(\theta, u)}} \sum_{Q_r=Q_+,Q_-} \sum_{l=0}^{\infty} \delta(eV + \mu - \hbar v_F \sqrt{(\frac{l + \frac{1}{2}}{R_g(1)})^2 + Q_r^2})$$ \hspace{1cm} (54)


due \begin{equation} \begin{aligned} ([e^{-2\zeta(\theta,u)\sin[2\theta]} + e^{2\zeta(\theta,u)\sin[2\theta]}](\cos[Q_r u - \frac{1}{2}\kappa(l, Q_r)])^2 + (\sin[Q_r u + \frac{1}{2}\kappa(l, Q_r)])^2) + \\
2(-1)^l \cos[Q_r u + \frac{1}{2}\kappa(l, Q_r)] \cos[Q_r u - \frac{1}{2}\kappa(l, Q_r)] \cdot \\
(\cos[l(\theta + \frac{3\pi}{2}) - \zeta(\theta,u)(-\eta(u) + \cos[2\theta])] - \cos[(l + 1)(\theta + \frac{3\pi}{2}) + \zeta(\theta,u)(-\eta(u) + \cos[2\theta])])
\end{aligned} \end{equation}

For the present case the energy scale of the excitations is governed by the radius $R_g(1)$ and width $D$. The spectrum is discrete and we can’t replace it by a continuum density of states as we did for the case $n = 0$.

In figure 7 we show the tunneling density of states at a fixed polar angle $\theta = \frac{\pi}{2}$ as a function of the voltage $V$. We observe that the density of states is dominated by high energy eigenvalues. This solutions are localized in energy. The range of the spectrum is above $\mu + eV > 200mV$ which is well separated from the low energy spectrum controlled by the $n = 0$ contour (which ranges from $-120mV$ to $70mV$).

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Figure 8 shows the tunneling density of states as a function of the polar angle $\theta$ for a fixed energy. The periodicity in $\theta$ is controlled by the discrete energy eigenvalues.

In figure 9 we show the tunneling density of states at a fixed voltage $V$ as a function of the polar angle $0 < \theta < \pi$ and width $|u| < 0.1$.

VII-The charge current—the in plane spin on the surface

A-The current in the absence of the edge dislocation

From the Hamiltonian given in equation 1 we compute the equation of motion for the velocity operator:

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, h] = v_F \sigma^y, \quad \frac{dy}{dt} = \frac{1}{i\hbar}[y, h] = -v_F \sigma^x.$$  

We multiply the velocity operator by the charge $(-e)$ and identify the charge current operators:

$$\hat{J}_x = (-e)v_F \sigma^2, \quad \hat{J}_y = (-e)(-v_F)\sigma^1.$$  

This also represents the ”real” spin on the surface. Therefore, the charge current is a measure of the in-plane spin on the surface.

Integrating over the $y$ coordinate we obtain the current $I_{TI,x}$ in the $x$ direction. Using the eigenstates $\Gamma_{p,q}(x,y)$ and $\Gamma_{-p,q}(x,y)$ of the $h_{TI}$ Hamiltonian

$$\Gamma_{p,q}(x,y) = e^{ipx} e^{i\chi} \begin{pmatrix} 1 \\ ie^{i\chi(p,q)} \end{pmatrix}$$

$$\Gamma_{-p,q}(x,y) = e^{-ipx} e^{i\chi} \begin{pmatrix} 1 \\ -ie^{-i\chi(p,q)} \end{pmatrix}$$

we find $\langle \Gamma_{p,q}(x,y) | (\sigma^2) | \Gamma_{p,q}(x,y) \rangle = -(\Gamma_{-p,q}(x,y) | (\sigma^2) | \Gamma_{-p,q}(x,y) \rangle$ therefore, we conclude that the current $I_{TI,x} = 0$ is zero.

B-The current in the presence of the edge dislocation

We will compute the current in the presence of the edge dislocation. The current operator $\hat{J}_{edge,x}(x,y)$ will be given in terms of the transformed currents. We find that the current density operator $\hat{J}_{edge,x}(x,y)$ is given by:

$$\hat{J}_{edge,x}(x,y) = (-e)v_F [\sigma_2 e_1^x - \sigma_1 e_2^x] = (-e)v_F \sigma^2 - (-e)v_F \frac{B^{(2)}}{2\pi} \left( \frac{y\sigma_1 + x\sigma_2}{x^2 + y^2} \right) \approx (-e)v_F \sigma^2 \quad (55)$$

We use the zero order current operator $\hat{J}_{edge,x}(x,y) \approx (-e)v_F \sigma^2$ to construct the second quantization form for the current density. The operator is defined with respect the to
shifted ground state \( |\mu > = |\vec{0} > \) with the energy \( E = \epsilon - \mu \) measured with respect the chemical potential and spinor field \( \Psi_{n=0}(x, y) \).

\[
\langle x, y | J^\text{edge}_x(x, y) = \langle \mu | \Psi_{n=0}^\dagger(x, y) J^\text{edge}_x(x, y) \Psi_{n=0}(x, y) |\mu >
\]

Using the spinor eigenfunction given in equation (35) and the second quantized form given in Appendix -B we find:

\[
\Psi_{n=0}(x, y; t) \approx \sum_{E>0} [\alpha_{E,R} \left( \begin{array}{c} U_\uparrow^{(n=0,R)}(x, y) \\ U_\downarrow^{(n=0,R)}(x, y) \end{array} \right) e^{-i \frac{E}{\hbar}t} + \beta_{E,R}^\dagger \left( \begin{array}{c} U_\uparrow^{(n=0,R)}(x, y) \\ U_\downarrow^{(n=0,R)}(x, y) \end{array} \right) e^{i \frac{E}{\hbar}t} \\
+ \alpha_{E,L} \left( \begin{array}{c} U_\uparrow^{(n=0,L)}(x, y) \\ U_\downarrow^{(n=0,L)}(x, y) \end{array} \right) e^{-i \frac{E}{\hbar}t} + \beta_{E,L}^\dagger \left( \begin{array}{c} U_\uparrow^{(n=0,L)}(x, y) \\ U_\downarrow^{(n=0,L)}(x, y) \end{array} \right) e^{i \frac{E}{\hbar}t} ]
\]

The current is a sum of two terms computed with the eigen spinor obtained in equation (33): \( [U_\uparrow^{(n=0,R)}(x, y), U_\downarrow^{(n=0,R)}(x, y)]^T \sigma^2 [U_\uparrow^{(n=0,R)}(x, y), U_\downarrow^{(n=0,R)}(x, y)] \) and \( [U_\uparrow^{(n=0,L)}(x, y), U_\downarrow^{(n=0,L)}(x, y)]^T \sigma^2 [U_\uparrow^{(n=0,L)}(x, y), U_\downarrow^{(n=0,L)}(x, y)] \) which have opposite signs.

Due to the parity violation caused by the dislocation, the density of states is asymmetric \( 1 \pm \frac{1}{\pi} \frac{hv_F}{L(\mu+V)} \sqrt{1 - \left( \frac{\epsilon}{\mu+V} \right)^2} \) resulting in a finite current. We integrate over the transversal direction \( y \) and obtain the edge current \( I_{x=0,\text{edge}} \).

\[
I_{n=0,\text{edge}} = (-e)v_F \int \frac{dx}{L} \int_{-L/2}^{L/2} dy \langle \mu | J^\text{edge}(x, y) |\mu > =
\]

\[
\frac{(-e)v_F}{4\pi} \left( \frac{L}{hv_F} \right)^2 \left( \frac{1}{L} \right) \int_{-L/2}^{L/2} \frac{dx}{L} \int_{-L/2}^{L/2} \frac{dy e^{-i \frac{\epsilon}{\mu+V} \left( x^2 + y^2 + a^2 \right)}}{\sqrt{G(x, y)}} \int d\epsilon| \int d\epsilon H[\mu - \sqrt{(\epsilon||)^2 + (\epsilon_\perp)^2}] \left( \frac{hv_F}{L} \right) \cdot \epsilon|| \left( (\epsilon||)^2 + (\epsilon_\perp)^2 \right)
\]

\[
= \frac{1}{4\pi} \left( -ev_F \right) \left( \frac{\mu}{hv_F} \right) \left( \frac{B_\perp}{L} \right) \left( \frac{B_\parallel}{L} \right) \left[ (H[\mu + eV - hv_F / L]) - H[\mu + eV - E_{\text{max}}] \right] \approx 6.22
\]

\( H[\mu - \sqrt{(\epsilon||)^2 + (\epsilon_\perp)^2}] \) is the step function which is one for \( \sqrt{(\epsilon||)^2 + (\epsilon_\perp)^2} \leq \mu \). The single particle energies are \( \epsilon_\perp = hv_F q_\perp \) and \( \epsilon|| = hv_F p \). For \( L \approx 10^{-6} m \), chemical potential \( \mu = 120 mV \) and \( \frac{L}{B_\perp} \approx 100 \) we find that the current \( I_{x=0,\text{edge}} \) is in the range of mA.

To conclude, we have shown that the presence of an edge dislocation gives rise to a non-zero current which is a manifestation of the in-plane component of the spin on the two dimensional surface. Therefore a nonzero value \( I_{x=0,\text{edge}} \neq 0 \) will be an indication of the presence of the edge dislocation. This effect might be measured using a coated tip with magnetic material used by the technique of Magnetic Force Microscopy.
VIII-Conclusions

Using the full description of the edge dislocation in terms of the torsion tensor, we have shown that the singularity at the core center eliminates the zero mode. As a result, only weak backscattering effect is allowed. Using this formulation, we have shown that for the case $n = 0$ the tunneling density of states is confined along $y = 0$. At high energies, the tunneling density of states is confined to circular contours governed by the Burger vector. For a large number of dislocations, we obtain a result similar to the one obtained from the Peierls domain wall model. The in-plane spin orientation is a manifestation of the parity violation induced by the edge dislocation. We propose that scanning tunneling and Magnetic Force Microscopy are advanced experimental techniques which can verify our predictions.
Appendix -A

We consider that a two dimensional manifold with a mapping from the curved space $X^a$, $a = 1, 2$, to the local flat space $x^\mu$, $\mu = x, y$ exists. We introduce the tangent vector $e^a_\mu(\vec{x}) = \frac{\partial X^a(\vec{x})}{\partial x^\mu}$, $\mu = x, y$ which satisfies the orthonormality relation $e^a_\mu(\vec{x}) e^b_\nu(\vec{x}) = \delta_{a,b}$ (here we use the convention that we sum over indices which appear twice). The metric tensor for the curved space is given in terms of the flat metric $\delta_{a,b}$ and the scalar product of the tangent vectors: $e^a_\mu(\vec{x}) e^a_\nu(\vec{x}) = g_{\mu,\nu}(\vec{x})$. The linear connection is determined by the Christoffel tensor $\Gamma^\lambda_{\mu,\nu}$:

$$\nabla_\mu \partial_\nu = -\Gamma^\lambda_{\mu,\nu} \partial_\lambda \tag{59}$$

The Christoffel tensor is constructed from the metric tensor $g_{\mu,\nu}(\vec{x})$.

$$\Gamma^\lambda_{\mu,\nu} = -\frac{1}{2} \sum_{\tau=x,y} g^{\lambda,\tau}(\vec{x}) \left[ \partial_\nu g_{\nu,\tau}(\vec{x}) + \partial_\mu g_{\nu,\tau}(\vec{x}) - \partial_\tau g_{\mu,\nu}(\vec{x}) \right] \tag{60}$$

Next, we introduce the vector field $\vec{V} = V^a \partial_a = V^\mu \partial_\mu$ where $a = 1, 2$ are the components in the curved space and $\mu = x, y$ represents the coordinate in the fixed cartesian frame. The covariant derivative of the vector field $V^a$ is determined by the spin connection $\omega^\mu_{a,b}$ which needs to be computed:

$$D_\mu V^a(\vec{x}) = \partial_\mu V^a(\vec{x}) + \omega^\mu_{a,b} V^b \tag{61}$$

For a two component spinor, we can identify the spin connection in the following way: The spinor in the the curved space (generated by the dislocation) is represented by $\tilde{\Psi}(\vec{X})$ and in the Cartesian space it is given by is given by $\Psi(\vec{x}) \equiv e^{-\frac{i}{2} \omega_{1,2} \sigma_3}$ [52]. The two component spinor represents a chiral fermion which transform under spatial rotation as spin half fermion:

$$\tilde{\Psi}(\vec{X}) = e^{\frac{-i}{2} \omega_{1,2} \sigma_3} \Psi(\vec{x})$$

$$e^{\frac{-i}{2} \omega_{1,2} \sigma_3} \equiv e^{\frac{i}{2} \omega_{a,b} \Sigma^{a,b}} \equiv e^{\sum_{a=1,2} \sum_{b=1,2} \frac{i}{2} \omega_{a,b} \Sigma_{a,b}}$$

$$\omega_{a,b} \equiv -\omega_{b,a}$$

$$\Sigma_{a,b} \equiv \frac{1}{4} [\sigma^a, \sigma^b]$$

$$\tag{62}$$
We have used the anti symmetric property of the rotation matrix \( \omega_{a,b} \equiv -\omega_{b,a} \), and the representation of the generator \( \Sigma^{a,b} \) in terms of the Pauli matrices.

Therefore for a two component spinor we obtain the connection:

\[
D_\mu \Psi(\vec{x}) = (\partial_\mu + \frac{1}{2} \omega^{a,b}_\mu \Sigma_{a,b}) \Psi(\vec{x}) \equiv (\partial_\mu + \frac{1}{8} \omega^{a,b}_\mu [\sigma_a, \sigma_b]) \Psi(\vec{x})
\]  

(63)

Next we will compute the spin connection \( \omega^{a,b}_\mu \) using the Christoffel tensor. In the physical coordinate basis \( x^\mu \) the covariant derivative \( D_\mu V^\nu(\vec{x}) \) is determined by the Christoffel tensor:

\[
D_\mu V^\nu(\vec{x}) = \partial_\mu V^\nu(\vec{x}) + \Gamma^\lambda_{\mu,\nu} V^\lambda
\]  

(64)

The relation between the spin connection and the linear connection can be obtained from the fact that the two covariant derivative of the vector \( \tilde{V} \) are equivalent.

\[
D_\mu V^a = e^a_\mu D_\mu V^\nu
\]  

(65)

Since we have the relation \( V^a = e^a_\nu V^\nu \) it follows from the last equation

\[
D_\mu[e^a_\nu] = D_\mu \partial_\nu e^a = (D_\mu \partial_\nu) e^a + \partial_\nu (D_\mu e^a) = 0
\]  

(66)

Using the definition of the Christoffel index and the differential geometry relation \( \nabla_\partial \partial_\nu \frac{\omega^{a,b}}{1/2} \mu = -\Gamma^\lambda_{\mu,\nu} \partial_{\lambda} \) \[45\], we obtain the relation between the spin connection and the linear connection:

\[
D_\mu[e^a_\nu] = \partial_\mu e^a_\nu(\vec{x}) - \Gamma^\lambda_{\mu,\nu} e^a_\lambda(\vec{x}) + \omega_{\mu,\nu} e^b_\nu(\vec{x}) \equiv 0
\]  

(67)

Solving this equation, we obtain the spin connection given in terms of the Burger vector. We multiply from left equation by the tangent vector \( e^a_\nu \), replace \( \Gamma^\lambda_{\mu,\nu} \) by equation (54) use the metric tensor relations \( e^a_\mu(\vec{x}) e^b_\nu(\vec{x}) = \delta_{a,b} \), \( e^a_\mu(\vec{x}) e^a_\nu(\vec{x}) = g_{\mu,\nu}(\vec{x}) \). As a result, we find \[45\]:

\[
\omega^{a,b} = \frac{1}{2} e^{\nu,a}(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu) - \frac{1}{2} e^{\nu,b}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)
\]

\[
-\frac{1}{2} e^{\rho,a} e^\sigma,b(\partial_\rho e_{\sigma,c} - \partial_\sigma e_{\rho,c}) e^c_\mu 
\]  

(68)

We notice the asymmetry between \( e^{\nu,a} \) and \( e_{a,\nu} \):

\[
e^{\nu,a} \equiv g^{\nu,\lambda} e^a_\lambda \quad \text{and} \quad e_{a,\nu} \equiv \delta_{a,b} e^b_\nu
\]
For our case we have a two component the spin connection \( \omega_x^{12} \) and \( \omega_y^{12} \)

\[
\omega_x^{12} = \frac{1}{2} e^{\nu,1}_x (\partial_x e^2_\nu - \partial_\nu e^2_x) - \frac{1}{2} e^{\nu,2}_x (\partial_x e^1_\nu - \partial_\nu e^1_x) - \frac{1}{2} e^{\rho,a}_x e^{\sigma,b}_x (\partial_\rho e_{\sigma,c} - \partial_\sigma e_{\rho,c}) e^c_x,
\]

\[
\omega_y^{12} = \frac{1}{2} e^{\nu,1}_y (\partial_y e^2_\nu - \partial_\nu e^2_y) - \frac{1}{2} e^{\nu,2}_y (\partial_y e^1_\nu - \partial_\nu e^1_y) - \frac{1}{2} e^{\rho,a}_y e^{\sigma,b}_y (\partial_\rho e_{\sigma,c} - \partial_\sigma e_{\rho,c}) e^c_y
\]

(69)

These equations are further simplified with the help of equations (11 – 13) with \( e_y^1 = 0 \), \( e_x^1 = 1 \) and the Burger tensor \( \partial_x e_y^2 - \partial_y e_x^2 = B^{(2)} \delta^2(\vec{r}) \).

\[
\omega_x^{12} = \frac{1}{2} g^{\nu,\lambda}_x e^{1}_x (\partial_x e^2_\nu - \partial_\nu e^2_x) - \frac{1}{2} g^{\rho,r}_x e^{1}_x g^{\sigma,s}_x [\partial_\rho (\delta_{c,b} e^b_\sigma) - \partial_\sigma (\delta_{c,d} e^d_\rho)] e^c_x =
\]

\[
\frac{1}{2} B^{(2)} (r^2) [g y^x e^1_x + g^y y^x e^1_y - (g x^r g y^s - g y^r g x^s) (e^1_x e^2_x)] =
\]

\[
\frac{1}{2} B^{(2)} (r^2) [g y^x e^1_x - (g x^r g y^s - g y^r g x^s) e^1_x e^2_x] \approx
\]

\[
\frac{1}{2} B^{(2)} (r^2) [-\frac{B^{(2)} y}{2\pi y^2 + x^2}] \approx
\]

\[
\frac{1}{2} B^{(2)} (r^2) [-\frac{B^{(2)} y - x}{2\pi y^2 + x^2}]
\]

(70)

and

\[
\omega_y^{12} = \frac{1}{2} e^{\nu,1}_y (\partial_y e^2_\nu - \partial_\nu e^2_y) - \frac{1}{2} e^{\nu,2}_y (\partial_y e^1_\nu - \partial_\nu e^1_y) - \frac{1}{2} e^{\rho,a}_y e^{\sigma,b}_y [\partial_\rho (\delta_{c,b} e^b_\sigma) - \partial_\sigma (\delta_{c,d} e^d_\rho)] e^c_y =
\]

\[
\frac{1}{2} g^{\nu,\lambda}_y (\partial_y e^2_\nu - \partial_\nu e^2_y) - \frac{1}{2} g^{\rho,r}_y e^{1}_y [\partial_\rho e^1_\nu - \partial_\nu e^1_y] - \frac{1}{2} g^{\rho,r}_y e^{1}_y [\partial_\rho e^c_\sigma - \partial_\sigma e^c_\rho] e^c_y =
\]

\[
-\frac{B^{(2)} (r^2)}{2} g x^r y^s e^1_x - \frac{B^{(2)} (r^2)}{2} g y^r x^s e^1_y + \frac{B^{(2)} (r^2)}{2} \delta^2(\vec{r})
\]

(71)

To first order first the Burger vector \( B^{(2)} \) the spin connections are given by: \( \omega_x^{12} = -\omega_x^{21} \approx 0 \) and \( \omega_y^{12} = -\omega_y^{21} \approx -\frac{1}{2} B^{(2)} \delta^2(\vec{r}) \).

**Appendix-B**

The spinor field operator \( \Psi(z,\bar{z};t) \) is decomposed into a sum with different localization contours \( n \), \( \Psi(z,\bar{z};t) = \sum_{n=0,\pm 1,..} \Psi_n(z,\bar{z};t) \). The energy levels for \( n \neq 0 \) are controlled by the the inverse of the Burger vector.

We will consider the case \( n = 0 \). For \( \mu = 0 \), we have the expansion in terms of the eigenspinor \( u_e(s,u) \) for electrons with positive chirality \( v_e(s,u) \) for holes with negative chirality.
Using the eigenfunctions $u_\epsilon(s, u), v_\epsilon(s, u)$, we construct the field operator $\Psi_{n=0}(s, u; t)$ as a superposing of particles and holes.

$$\Psi_{n=0}(s, u; t) = \sum_{\epsilon>0} c_\epsilon u_\epsilon(s, u)e^{-i\tilde{\epsilon}t} + \sum_{\epsilon<0} b^\dagger_{-\epsilon} v_\epsilon(s, u)e^{-i\tilde{\epsilon}t} = \sum_{\epsilon>0} [c_\epsilon u_\epsilon(s, u)e^{-i\tilde{\epsilon}t} + b^\dagger_{-\epsilon} v_\epsilon(s, u)e^{i\tilde{\epsilon}t}]$$

(73)

where $c_\epsilon$ is the annihilation operator for particles with energy $\epsilon$ and $b^\dagger_{-\epsilon}$ is the creation operator for a hole with energy $-\epsilon$. $c_\epsilon$ and $b_{-\epsilon}$ annihilates the ground state $|0>\equiv |\tilde{0}> = 0$. The operators obey anti-commutation relations: $[c_\epsilon, c^\dagger_{\epsilon'}]_+ = [b_{-\epsilon}, b^\dagger_{-\epsilon'}]_+ = \delta(\epsilon, \epsilon')$.

The material properties of the topological insulators are such that the Fermi energy $\mu$ is positive. As a result we have electrons and holes with positive chirality, and deep holes with negative spin chiralities. The energy is measured with respect to the chemical potential $\mu$, $E \equiv \epsilon - \mu$. As a result one obtains a shifted ground state $|\mu>\equiv |\tilde{\mu}>$.

We define new operators using the holes operators $b_{\epsilon;\pm}$ and electron operator $c_{\epsilon;+}$:

$$c_{E+\mu;+} = \alpha_E$$ for $E > 0$, $c_{E+\mu;+} = \beta^\dagger_{-E}$ for $-\mu < E < 0$ and $b^\dagger_{-(E+\mu);-} = \gamma_{-E}$ for $-2\mu < E - \mu$. These operators annihilate the ground state: $\alpha_E|\tilde{0}> = 0$, $\beta_{-E}|\tilde{0}> = 0$ and $\gamma_{-E}|\tilde{0}> = 0$. 

31
\[
\Psi_{n=0}(s, u; t) =
\sum_{E=E_A}^{E=0} c_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t} + \sum_{E=0}^{E=\mu} c_{(E+\mu):}(s, u) e^{-i\frac{E}{\hbar} t} + \sum_{-\mu}^{-\mu-E_A} \tilde{b}^\dagger_{-\mu}(s, u) e^{-i\frac{E}{\hbar} t}
\]

\[
= \sum_{E=0}^{E=E_A} \alpha_E u_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t} + \sum_{E=0}^{E=\mu} \beta^\dagger_{-E} u_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t} + \sum_{E=-\mu}^{-\mu-E_A} \gamma^\dagger_{-E} v_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t}
\]

\[
= \sum_{E=0}^{E=E_A} \alpha_E u_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t} + \sum_{E=0}^{E=\mu} \beta^\dagger_{E} u_{E-\mu}(s, u) e^{i\frac{E}{\hbar} t} + \sum_{E=\mu}^{E=\mu+E_A} \gamma^\dagger_{\mu} v_{E+\mu}(s, u) e^{i\frac{E}{\hbar} t}
\]

(74)

where \( E_A \) is energy the below the Dirac point. As a result, the edge Hamiltonian is given by:

\[
H_{\text{edge}} \approx \left[ \sum_{E=E_A}^{E=0} \alpha^\dagger_E \alpha_E E + \sum_{E=\mu}^{E=0} \beta^\dagger_E \beta_E E + \sum_{E=\mu}^{E=\mu+E_A} \gamma^\dagger_E \gamma_E E \right]
\]

For most of the cases, the chemical potential is large, and we can approximate the spinor operator by a sum of states for particles and holes, ignoring the deep hole band:

\[
\Psi_{n=0}(s, u; t) \approx \sum_{E>0} \left[ \alpha_E u_{E+\mu}(s, u) e^{-i\frac{E}{\hbar} t} + \beta^\dagger_{E} u_{E-\mu}(s, u) e^{i\frac{E}{\hbar} t} \right]
\]

(75)
Appendix-C

The wave functions are given by:

\[ U_{\ell,Q_{-}}(\theta, u) = G_{\ell,Q_{-}}(\theta, u) \cdot \left[ U_{\ell,Q_{-}}^{(even,k)}(\theta, u) + U_{\ell,Q_{-}}^{(odd,k)}(\theta, u) \right]; \]
\[ U_{\ell,Q_{-}}^{(even,k)}(\theta, u) = 2ie^{\frac{i}{2}\kappa(l,Q_{-})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{-}u + \frac{1}{2}\kappa(l,Q_{-})]} \]
\[ + (-1)^le^{-\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{-}u - \frac{1}{2}\kappa(l,Q_{-})]}; \]
\[ U_{\ell,Q_{-}}^{(odd,k)}(\theta, u) = 2e^{\frac{i}{2}\kappa(l,Q_{-})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{-}u + \frac{1}{2}\kappa(l,Q_{-})]} \]
\[ + (-1)^le^{-\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{-}u - \frac{1}{2}\kappa(l,Q_{-})]}; \]
\[ U_{\ell,Q_{+}}(\theta, u) = G_{\ell,Q_{+}}(\theta, u) \cdot \left[ U_{\ell,Q_{+}}^{(even,k)}(\theta, u) + U_{\ell,Q_{+}}^{(odd,k)}(\theta, u) \right]; \]
\[ U_{\ell,Q_{+}}^{(even,k)}(\theta, u) = 2ie^{\frac{i}{2}\kappa(l,Q_{+})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{+}u + \frac{1}{2}\kappa(l,Q_{+})]} \]
\[ + e^{\zeta(\theta,u)}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{+}u - \frac{1}{2}\kappa(l,Q_{+})]}; \]
\[ U_{\ell,Q_{+}}^{(odd,k)}(\theta, u) = 2e^{\frac{i}{2}\kappa(l,Q_{+})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{+}u + \frac{1}{2}\kappa(l,Q_{+})]} \]
\[ + (-1)^le^{-\zeta(\theta,u)}e^{-i(\eta(u) - \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{+}u - \frac{1}{2}\kappa(l,Q_{+})]}; \]
\[ (76) \]

Similarly for the second pair we obtain the wave function:

\[ U_{\ell,Q_{+}}^{(even,k)}(\theta, u) = 2ie^{\frac{i}{2}\kappa(l,Q_{+})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) + \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{+}u + \frac{1}{2}\kappa(l,Q_{+})]} \]
\[ + e^{\zeta(\theta,u)}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) + \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \sin[Q_{+}u - \frac{1}{2}\kappa(l,Q_{+})]}; \]
\[ U_{\ell,Q_{+}}^{(odd,k)}(\theta, u) = 2e^{\frac{i}{2}\kappa(l,Q_{+})}e^{-i\zeta(\theta,u)}e^{-i(\eta(u) + \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{+}u + \frac{1}{2}\kappa(l,Q_{+})]} \]
\[ + (-1)^le^{-\zeta(\theta,u)}e^{-i(\eta(u) + \cos[2\theta]) + il(2\theta + \frac{3\pi}{2}) \cos[Q_{+}u - \frac{1}{2}\kappa(l,Q_{+})]}; \]
\[ (77) \]
where $G^{-\frac{1}{3}}(\theta, u)$ is the Jacobian transformation induced by the metric tensor.
FIG. 1: The edge dislocation with the core at $x = y = 0$, modifies the coordinate $x,y$ to $X, Y$ in the presence of the edge dislocation with the Burger vector $B^{(2)}$. 
FIG. 2: The contours \((x(s))^2 + (y(s) - \frac{R_g}{n})^2 = \left(\frac{R_g}{n}\right)^2\) for \(n = \pm 1, \pm 2, \pm 3\) (in decreasing size). \(R_g(n) = \frac{R_g}{n}\). \(n = 0\) corresponds to the equation \(y(s) = 0\) and \(|y| > d\) (see the text). The distance is measured in units of the Burger vector \(B^{(2)}\).
FIG. 3: The tunnelling density of states for $D^{Peierls}(\frac{x}{W}, \frac{y}{W})$ for the Peierls dislocation confined to the line $|x| < \bar{W}$ for the STM voltage $V, \mu + eV < 0$.
FIG. 4: The tunneling density of states for \( n = 0 \), \( \frac{dI}{dV} \propto D^{(n=0)}(\frac{x}{B^{(2)}}, \frac{y}{B^{(2)}}; \mu = 120 mV) \). The right corner represents the intersection of the \( x \) coordinate which runs from 30 (right corner) to \(-30\) and the \( y \) coordinate which runs from \(-30\) (right corner) to 30 in units of the Burger vector.
FIG. 5: The tunneling density of states for $n = 0$ as a function of $y$ and $V$ 
\[ \frac{dI}{dV} \propto D^{(n=0)}(\frac{x}{B(2)} = -2, \frac{y}{B(2)}; \mu = 120mV) \]. The voltage range is $-120 \leq V \leq 50$ and the $y$ coordinate is in the range $-30 \leq \frac{y}{B(2)} \leq 30$. 
FIG. 6: Many Dislocations - with the core of the dislocations at \([x_w, y = 0], w = 1, 2, \ldots, 2M\);
The maximum of the tunneling density of states is confined along \(y = 0\). The coordinates of the
tunneling density of states are restricted to: \(-40 \leq \frac{x}{B^{(2)}} \leq 40\) and \(-20 \leq \frac{y}{B^{(2)}} \leq 20\).
FIG. 7: The discrete tunneling density of states for \( n = 1 \), as a function of the voltage \( V \)

\[
D^{(n=1)}(V; \theta = \frac{\pi}{2}, \frac{u}{B(2)}, \mu = 120\,mV)
\]
FIG. 8: The tunneling density of states as a function of $\theta$ $D^{(n=1)}(\theta; \frac{u}{B^{(2)}} = 0.01, V = 280mV, \mu = 120mV)$
FIG. 9: The tunneling density of states as a function of $\theta$ and $u$ at a fixed voltage $V = 280mV$

$D^{(n=1)}(\theta, \frac{u}{B(\xi)}; V = 280mV, \mu = 120mV)$
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