THE DUAL SPACE OF PRECOMPACT GROUPS

M. FERRER, S. HERNÁNDEZ, AND V. USPENSKIJ

ABSTRACT. For any topological group $G$ the dual object $\widehat{G}$ is defined as the set of equivalence classes of irreducible unitary representations of $G$ equipped with the Fell topology. If $G$ is compact, $\widehat{G}$ is discrete. In an earlier paper we proved that $\widehat{G}$ is discrete for every metrizable precompact group, i.e. a dense subgroup of a compact metrizable group. We generalize this result to the case when $G$ is an almost metrizable precompact group.

1. Introduction

For a topological group $G$ let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$. The set $\widehat{G}$ can be equipped with a natural topology, the so-called Fell topology (see Section 2 for a definition).

A topological group $G$ is precompact if it is isomorphic (as a topological group) to a subgroup of a compact group $H$ (we may assume that $G$ is dense in $H$). If $H$ is compact, then $\widehat{H}$ is discrete. If $G$ is a dense subgroup of $H$, the natural mapping $\widehat{H} \to \widehat{G}$ is a bijection but in general need not be a homeomorphism. Moreover, for every countable non-metrizable precompact group $G$ the space $\widehat{G}$ is not discrete [12, Theorem 5.1], and every non-metrizable compact group $H$ has a dense subgroup $G$ such that $\widehat{G}$ is not discrete [12, Theorem 5.2]. (The Abelian case was considered in...
On the other hand, if \( G \) is a precompact metrizable group, then \( \hat{G} \) is discrete [12, Theorem 4.1]. (The Abelian case was considered in [2, 4]). The aim of the present paper is to generalize this result to the almost metrizable case: \( \hat{G} \) is discrete for every almost metrizable precompact topological group \( G \). A topological group \( G \) is \textit{almost metrizable} if it has a compact subgroup \( K \) such that the quotient space \( G/K \) is metrizable. According to Pasynkov’s theorem [1, 4.3.20], a topological group is almost metrizable if and only if it is feathered in the sense of Arhangel’skii.

We reduce the almost metrizable case to the metrizable case considered in [12, Theorem 4.1].

2. Preliminaries: Fell topologies

All topological spaces and groups that we consider are assumed to be Hausdorff. For a (complex) Hilbert space \( \mathcal{H} \) the unitary group \( U(\mathcal{H}) \) of all linear isometries of \( \mathcal{H} \) is equipped with the strong operator topology (this is the topology of pointwise convergence). With this topology, \( U(\mathcal{H}) \) is a topological group.

A \textit{unitary representation} \( \rho \) of the topological group \( G \) is a continuous homomorphism \( G \to U(\mathcal{H}) \), where \( \mathcal{H} \) is a complex Hilbert space. A closed linear subspace \( E \subseteq \mathcal{H} \) is an \textit{invariant} subspace for \( \mathcal{S} \subseteq U(\mathcal{H}) \) if \( ME \subseteq E \) for all \( M \in \mathcal{S} \). If there is a closed subspace \( E \) with \( \{0\} \subsetneq E \subseteq \mathcal{H} \) which is invariant for \( \mathcal{S} \), then \( \mathcal{S} \) is called \textit{reducible}; otherwise \( \mathcal{S} \) is \textit{irreducible}. An \textit{irreducible representation} of \( G \) is a unitary representation \( \rho \) such that \( \rho(G) \) is irreducible.

If \( \mathcal{H} = \mathbb{C}^n \), we identify \( U(\mathcal{H}) \) with the \textit{unitary group of order} \( n \), that is, the compact Lie group of all complex \( n \times n \) matrices \( M \) for which \( M^{-1} = M^* \). We denote this group by \( \mathbb{U}(n) \).
Two unitary representations $\rho : G \to U(\mathcal{H}_1)$ and $\psi : G \to U(\mathcal{H}_2)$ are equivalent if there exists a Hilbert space isomorphism $M : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$. The dual object of a topological group $G$ is the set $\hat{G}$ of equivalence classes of irreducible unitary representations of $G$.

If $G$ is a precompact group, the Peter-Weyl Theorem (see [15]) implies that all irreducible unitary representation of $G$ are finite-dimensional and determine an embedding of $G$ into the product of unitary groups $U(n)$.

If $\rho : G \to U(\mathcal{H})$ is a unitary representation, a complex-valued function $f$ on $G$ is called a function of positive type (or positive-definite function) associated with $\rho$ if there exists a vector $v \in \mathcal{H}$ such that $f(g) = (\rho(g)v,v)$ (here $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$). We denote by $P'_\rho$ the set of all functions of positive type associated with $\rho$. Let $P_\rho$ be the convex cone generated by $P'_\rho$, that is, the set of sums of elements of $P'_\rho$.

Let $G$ be a topological group, $\mathcal{R}$ a set of equivalence classes of unitary representations of $G$. The Fell topology on $\mathcal{R}$ is defined as follows: a typical neighborhood of $[\rho] \in \mathcal{R}$ has the form

$$W(f_1, \cdots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \cdots, g_n \in P_\sigma \forall x \in C \, |f_i(x) - g_i(x)| < \epsilon \},$$

where $f_1, \cdots, f_n \in P_\rho$ (or $\in P'_\rho$), $C$ is a compact subspace of $G$, and $\epsilon > 0$. In particular, the Fell topology is defined on the dual object $\hat{G}$. If $G$ is locally compact, the Fell topology on $\hat{G}$ can be derived from the Jacobson topology on the primitive ideal space of $C^*(G)$, the $C^*$-algebra of $G$ [7, section 18], [3, Remark F.4.5].

Every onto homomorphism $f : G \to H$ of topological groups gives rise to a continuous injective dual map $\hat{f} : \hat{H} \to \hat{G}$. A mapping $h : X \to Y$ between topological
spaces is \textit{compact-covering} if for every compact set \( L \subseteq Y \) there exists a compact set \( K \subseteq X \) such that \( h(K) = L \).

\textbf{Lemma 2.1.} If \( f : G \to H \) is a compact-covering onto homomorphism of topological groups, the dual map \( \hat{f} : \widehat{H} \to \widehat{G} \) is a homeomorphic embedding.

\textit{Proof.} This easily follows from the definition of Fell topology. \hfill \Box

Let \( \pi \) be a unitary representation of a topological group \( G \) on a Hilbert space \( \mathcal{H} \). Let \( F \subseteq G \) and \( \epsilon > 0 \). A unit vector \( v \in \mathcal{H} \) is called \((F, \epsilon)\)-\textit{invariant} if \( \|\pi(g)v - v\| < \epsilon \) for every \( g \in F \).

A topological group \( G \) has \textit{property (T)} if and only if there exists a pair \((Q, \epsilon)\) (called a \textit{Kazhdan pair}), where \( Q \) is a compact subset of \( G \) and \( \epsilon > 0 \), such that for every unitary representation \( \rho \) having a unit \((Q, \epsilon)\)-invariant vector there exists a non-zero invariant vector. Equivalently, \( G \) has property (T) if and only if the trivial representation \( 1_G \) is isolated in \( \mathcal{R} \cup \{1_G\} \) for every set \( \mathcal{R} \) of equivalence classes of unitary representations of \( G \) without non-zero invariant vectors [3, Proposition 1.2.3].

Compact groups have property (T) [3, Proposition 1.1.5], but countable Abelian precompact groups do not have property (T) [12, Theorem 6.1].

We refer to Fell’s papers [9, 10], the classical text by Dixmier [7] and the recent monographs by de la Harpe and Valette [13], and Bekka, de la Harpe and Valette [3] for basic definitions and results concerning Fell topologies and property (T).

\textbf{3. Almost metrizable groups}

If \( A \) is a subset of a topological space \( X \), the \textit{character} \( \chi(A, X) \) of \( A \) in \( X \) is the least cardinality of a base of neighborhoods of \( A \) in \( X \). (If this definition leads to a finite value of \( \chi(A, X) \), we replace it by \( \omega \), the first infinite cardinal, and similarly for
other cardinal invariants.) If $A$ is a closed subset of a compact space $X$, the character $\chi(A, X)$ equals the pseudocharacter $\psi(A, X)$ – the least cardinality of a family $\gamma$ of open subsets of $X$ such that $\cap \gamma = A$. In particular, if $A$ is a closed $G_\delta$-subset of a compact space $X$, then $\chi(A, X) = \omega$.

If $K$ is a compact subgroup of a topological group, then $G/K$ is metrizable if and only if $\chi(K, G) = \omega$ [1, Lemma 4.3.19]. Let $G$ be an almost metrizable topological group, $\mathcal{K}$ the collection of all compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then for every neighborhood $O$ of the neutral element there is $K \in \mathcal{K}$ such that $K \subset O$ [1, Proposition 4.3.11]. We now show that if $G$ is additionally $\omega$-narrow, then $K$ can be chosen normal (in the algebraic sense). Recall that a topological group $G$ is $\omega$-narrow [1] if for every neighborhood $U$ of the neutral element there exists a countable set $A \subset G$ such that $AU = G$.

**Lemma 3.1.** Let $G$ be an $\omega$-narrow almost metrizable group, $\mathcal{N}$ the collection of all normal (= invariant under inner automorphisms) compact subgroups $K$ of $G$ such that the quotient group $G/K$ is metrizable (equivalently, $\chi(K, G) = \omega$). Then for every neighborhood $O$ of the neutral element there exists $K \in \mathcal{N}$ such that $K \subset O$.

**Proof.** Let $L \subset O$ be a compact subgroup of $G$ such that the quotient space $G/L = \{xL : x \in G\}$ is metrizable. It suffices to prove that $K = \cap \{gLg^{-1} : g \in G\}$, the largest normal subgroup of $G$ contained in $L$, belongs to $\mathcal{N}$.

There exists a compatible metric on $G/L$ which is invariant under the action of $G$ by left translations. To construct such a metric, consider a countable base $U_1, U_2, \ldots$ of neighborhoods of $L$ in $G$. We may assume that for each $n$ we have $U_n = U_n^{-1} = U_nL$ and $U_{n+1}^2 \subset U_n$. Let $\gamma_n = \{gU_n : g \in G\}$. The open cover $\gamma_n$ of $G$ is invariant under left $G$-translations and under right $L$-translations, and $\gamma_{n+1}$ is a barycentric refinement
of $\gamma_n$. The pseudometric on $G$ that can be constructed in a canonical way from the sequence $(\gamma_n)$ of open covers (see [8, Theorem 8.1.10]) gives rise to a compatible $G$-invariant metric on $G/L$. A similar construction was used in [1, Lemma 4.3.19].

If an $\omega$-narrow group transitively acts on a metric space $X$ by isometries, then $X$ is separable [1, 10.3.2]. Thus $X = G/L$ is separable. Let $Y$ be a dense countable subset of $X$. Then $K = \{g \in G : gx = x \text{ for every } x \in X\} = \{g \in G : gx = x \text{ for every } x \in Y\}$ is a $G_\delta$-subset of $L$, hence $\chi(K, L) = \omega$. It follows that $\chi(K, G) \leq \chi(K, L)\chi(L, G) = \omega$ ([8, Exercise 3.1.E]).

4. Main theorem

**Theorem 4.1.** If $G$ is a precompact almost metrizable group, then $\widehat{G}$ is discrete.

**Proof.** Let $\rho$ be an irreducible unitary representation of $G$. We must prove that $[\rho]$ is isolated in $\widehat{G}$. It suffices to find a discrete open subset $D \subset \widehat{G}$ such that $[\rho] \in D$.

Precompact groups are $\omega$-narrow, so Proposition 3.1 applies to $G$. Let $\mathcal{N}$, as above, be the collection of all normal compact subgroups $K \subset G$ such that $\chi(K, G) = \omega$. Then $\mathcal{N}$ is closed under countable intersections, and it follows from Proposition 3.1 that for every $G_\delta$-subset $A$ of $G$ containing the neutral element there exists $K \in \mathcal{N}$ such that $K \subset A$. In particular, there exists $K \in \mathcal{N}$ such that $K$ lies in the kernel of $\rho$. Let $D \subset \widehat{G}$ be the set of all classes $[\sigma] \in \widehat{G}$ such that $K$ is contained in the kernel of $\sigma$. Then $[\rho] \in D$. It suffices to verify that $D$ is open and discrete.

Step 1. We verify that $D$ is open. Let $\mathcal{R}$ be the set of equivalence classes of all finite-dimensional unitary representations (which may be reducible) of $K$ without non-zero invariant vectors. Let $\tau_n$ be the trivial $n$-dimensional representation $1_K \oplus \cdots \oplus 1_K$ ($n$ summands) of $K$, $n = 1, 2, \ldots$. In the notation of section 2, $P_{\tau_n}$ does not depend
on \( n \) and is the set of non-negative constant functions on \( K \). It follows that in the space \( \mathcal{S} = \mathcal{R} \cup \{ [\tau_n] : n = 1, 2, \ldots \} \), equipped with the Fell topology, the points \([\tau_n]\) are indistinguishable: any open set containing one of these points contains all the others. Since \( K \) has property (T), \([\tau_1] = [1_K]\) is not in the closure of \( \mathcal{R} \). Therefore \( \mathcal{R} \) is closed in \( \mathcal{S} \) and \( \mathcal{S} \setminus \mathcal{R} \) is open in \( \mathcal{S} \).

We claim that for every irreducible unitary representation \( \sigma \) of \( G \) the class of the restriction \( \sigma|_K \) belongs to \( \mathcal{S} \). In other words, the claim is that \( \sigma|_K \) is trivial if it admits a non-zero invariant vector. Let \( V \) be the (finite-dimensional) space of the representation \( \sigma \). For \( g \in G \) and \( x \in V \) we write \( gx \) instead of \( \sigma(g)x \). The space \( V' = \{ x \in V : gx = x \text{ for all } g \in K \} \) of all \( K \)-invariant vectors is \( G \)-invariant. Indeed, if \( x \in V' \), \( g \in G \) and \( h \in K \), then \( g^{-1}hx = x \) because \( g^{-1}h \in K \) and \( x \) is \( K \)-invariant. It follows that \( hgx = gx \) which proves that \( gx \in V' \). Since \( \sigma \) is irreducible, either \( V' = \{0\} \) or \( V' = V \). Accordingly, either \( \sigma|_K \) admits no non-zero invariant vectors or else is trivial.

We have just proved that the restriction map \( r : \hat{G} \to \mathcal{S} \) is well-defined. Clearly \( r \) is continuous, and therefore \( D = r^{-1}(\mathcal{S} \setminus \mathcal{R}) \) is open in \( \hat{G} \).

Step 2. We verify that \( D \) is discrete. Let \( p : G \to G/K \) be the quotient map. Then \( D \) is the image of the dual map \( \hat{p} : \widehat{G/K} \to \hat{G} \). According to [12, Theorem 4.1], the dual space of a metrizable precompact group is discrete. Thus \( \widehat{G/K} \) is discrete. Since \( p \) is a perfect map, it is compact-covering, and Lemma 2.1 implies that \( \hat{p} : \widehat{G/K} \to \hat{G} \) is a homeomorphic embedding. Therefore, \( D = \hat{p}(\widehat{G/K}) \) is discrete. \( \square \)

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Universitat Jaume I, Instituto de Matemàtiques de Castellón, Campus de Riu Sec, 12071 Castellón, Spain.
E-mail address: mferrer@mat.uji.es

Universitat Jaume I, INIT and Departamento de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain.
E-mail address: hernande@mat.uji.es

Department of mathematics, 321 Morton Hall, Ohio University, Athens, Ohio 45701, USA
E-mail address: uspenski@ohio.edu