UNSTABLE ENTROPIES AND VARIATIONAL PRINCIPLE FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We study entropies caused by the unstable part of partially hyperbolic systems. We define unstable metric entropy and unstable topological entropy, and establish a variational principle for partially hyperbolic diffeomorphisms, which states that the unstable topological entropy is the supremum of the unstable metric entropy taken over all invariant measures. The unstable metric entropy for an invariant measure is defined as a conditional entropy along unstable manifolds, and it turns out to be the same as that given by Ledrappier-Young, though we do not use increasing partitions. The unstable topological entropy is defined equivalently via separated sets, spanning sets and open covers along a piece of unstable leaf, and it coincides with the unstable volume growth along unstable foliation. We also obtain some properties for the unstable metric entropy such as affineness, upper semi-continuity and a version of Shannon-McMillan-Breiman theorem.

0. Introduction

The difference between partially hyperbolic systems and hyperbolic systems is the presence of the center direction in the former case. The original motivation of the paper is to study some ergodic properties of partially hyperbolic systems that arise from the hyperbolic part. Since entropies are the important invariants measuring the complexity of the systems, they are good objects to start with.

It is generally agreed that entropies are caused by the expansive part of dynamical systems. There are some existing notions for such measurements, including the entropies given by Ledrappier and Young ([8]) from the measure theoretic point of view and the unstable volume growth given by Hua, Saghin, and Xia ([4]) from the topological point of view. Another motivation of the paper is trying to put them in a framework that is similar to the classical entropy theory. In this paper we redefine the notion of unstable metric entropy $h^u_\mu(f)$, and define the unstable topological entropy $h^u_{\text{top}}(f)$, and prove a variational principle for them. Also, for the unstable metric entropy, we provide a version of Shannon-McMillan-Breiman theorem.

The unstable metric entropy for an invariant measure $\mu$ is defined by using $H_\mu(\vee_{i=0}^{n-1} f^{-i}\alpha|\eta)$, where $\alpha$ is a finite measurable partition of the underlying manifold $M$, and $\eta$ is a measurable partition consisting of local unstable leaves that can be obtained by refining a finite partition into pieces of unstable leaves (see details in Definition 1.1). The definition is about the same as the classical metric entropy, except for the conditional partition $\eta$, which is used to eliminate the impact from center directions. The entropy defined in [8] can be regarded as that given by $H_\mu(\xi|f\xi)$, where $\xi$ is an increasing partition, that is, $\xi \geq f\xi$, subordinate to unstable manifolds. We show in Theorem A that it is identical to the unstable metric entropy we defined. We also prove that the unstable metric entropy map,
Unstable entropies and variational principle

as a function from the set of all invariant measures to nonnegative real numbers, is affine and upper semi-continuous (Proposition 2.14 and Proposition 2.15).

Generally speaking, a good notion of entropy should satisfy a type of Shannon-McMillan-Breiman theorem. In this paper we provide a version of Shannon-McMillan-Breiman theorem for unstable metric entropy in Theorem B.

We define unstable topological entropy by using the growth rates of the cardinality of \((n, \varepsilon)\) separated sets or spanning sets of a local unstable leaf at every point \(x\) then taking the supremum over \(x \in M\) (see Definition 1.4). It measures the asymptotic rate of orbit divergence along unstable manifolds. As the classical topological entropy (see Sections 7.1 and 7.2 in [17]), the cardinality of \((n, \varepsilon)\) separated sets or spanning sets can be replaced by a subcover of an open cover of the form \(\vee_{i=0}^{n-1} f^{-i} U\), where \(U\) is an open cover of \(M\) (Definition 4.2). We show in Theorem C that the unstable topological entropy we defined coincides with the volume growth given in [4].

As same as the classical case, we can obtain a variational principle for unstable entropies (Theorem D). That is, the unstable topological entropy is the supremum of unstable metric entropy taken over all invariant probability measures, as well as all ergodic measures.

Ledrappier and Young ([8]) introduced a hierarchy of metric entropies \(h_i = h_i(f)\), each of which corresponds to a different Lyaponov exponent, and is regarded as the entropy caused by different hierarchy of unstable manifolds. If there are \(u\) different positive Lyapunov exponents in the unstable direction of a partially hyperbolic system, then \(h_u(f)\) gives the unstable metric entropy we define in this paper. The entropy has a simple form \(H_\mu(\xi|f_\xi)\), where \(\xi\) is an increasing partition subordinate to unstable manifolds. Because the partition is increasing, it is convenient to use sometimes. However, practically it takes some work to construct such partitions, such as in [12] and [19]. In our definition, instead of \(\xi\), we relax the increasing condition and use partitions \(\eta\) that can be obtained by refining any finite partitions of small diameters to unstable leaves, and is much easier to construct. Moreover, the size of the elements of \(\eta\) can be uniformly bounded from above and below, while the size of elements of \(\xi\) could be arbitrarily large or small on unstable leaves. Further, since our definition for unstable metric entropy has a similar form as the classical one, some properties can be obtained by following the same strategies, such as upper semi-continuity and variational principle (see the proof of Proposition 2.15 and Theorem D).

The unstable topological entropy we introduce can be regarded as a kind of conditional topological entropy. Conditional topological entropy is first introduced in [9], where the author uses open covers for condition. Our definition is close to the topological conditional entropy defined in [5], though they have a factor map to provide a natural partition for condition and we do not. However, the quantity that gives the same information is the unstable volume growth introduced by Hua-Saghin-Xia ([4]), reminiscent from the works by Yomdin and Newhouse ([20, 11]). With the notion they obtained a formula for upper bound of the metric entropy of an invariant measure using unstable volume growth and the sum of positive Lyapunov exponents in the center direction (see (1.1)). With unstable entropies we defined, we can give different versions of the formula in both measure theoretic and topological categories. For the former one we use unstable metric entropy and the sum of positive center Lyapunov exponents for upper bounds of metric entropy (Corollary A.1). For the latter one we conclude in Corollary C.2
that the topological entropy of a partially hyperbolic diffeomorphism is bounded by the unstable topological entropy and the growth rates of \(|Df|_{E^u}\) and its outer products (defined in (1.4) and (1.5)). Further we can define a \(\text{transversal entropy}\) (Definition 1.5) and then obtain that the upper bound can be given by the sum of the unstable topological entropy and transversal entropy.

When the paper was being written we found a paper by Jiagang Yang [19] that contains the upper semi-continuity of the unstable metric entropy with respect to both the invariant measures \(\mu\) and the dynamical systems \(f\), by constructing an increasing partition \(\xi\). It is more general than the result in Proposition 2.15. However, we still give our proof since it is much simpler and straightforward when our definition of the unstable metric entropy is used.

Though the unstable entropies we introduce here are for the unstable foliations of partially hyperbolic diffeomorphisms, it is obvious that they can be applied to more general settings. If a diffeomorphism has a hierarchy of unstable foliations, the entropies can be defined on each level as long as the map is uniformly expanding restricted to the leaves of the foliation. Also, if partial hyperbolicity holds only on a closed invariant subset \(\Lambda\), such as an Axiom A system crossing any slow motion system, then we can study unstable entropies for the system \(f|_{\Lambda}\), the diffeomorphism restricted to \(\Lambda\).

The paper is organized as following. In Section 1 we give definitions of unstable metric and topological entropy and state the main results. We prove Theorem A and provide some properties of unstable metric entropy in Section 2. Section 3 is for a proof of Shannon-McMillan-Breiman theorem (Theorem B) for unstable metric entropy. Properties of unstable topological entropy and proof of Theorem C are provided in Section 4. The last section, Section 5, is for the proof of Theorem D, the variational principle.

1. Definitions and statements of main results

Let \(M\) be an \(n\)-dimensional smooth, connected and compact Riemannian manifold without boundary and \(f : M \to M\) a \(C^1\)-diffeomorphism. \(f\) is said to be \(\text{partially hyperbolic}\) (cf. for example [15]) if there exists a nontrivial \(Tf\)-invariant splitting \(TM = E^s \oplus E^c \oplus E^u\) of the tangent bundle into stable, center, and unstable distributions, such that all unit vectors \(v^\sigma \in E^\sigma_x (\sigma = c, s, u)\) with \(x \in M\) satisfy

\[
\|T_{x}f v^s\| < \|T_{x}f v^c\| < \|T_{x}f v^u\|,
\]

and

\[
\|T_{x}f|_{E^c}\| < 1 \quad \text{and} \quad \|T_{x}f^{-1}|_{E^c}\| < 1,
\]

for some suitable Riemannian metric on \(M\). The stable distribution \(E^s\) and unstable distribution \(E^u\) are integrable to the stable and unstable foliations \(W^s\) and \(W^u\) respectively such that \(TW^s = E^s\) and \(TW^u = E^u\) (cf. [3]).

In this paper we always assume that \(f\) is a \(C^1\)-partially hyperbolic diffeomorphism of \(M\), and \(\mu\) is an \(f\)-invariant probability measure.

For a partition \(\alpha\) of \(M\), let \(\alpha(x)\) denote the element of \(\alpha\) containing \(x\). If \(\alpha\) and \(\beta\) are two partitions such that \(\alpha(x) \subseteq \beta(x)\) for all \(x \in M\), we then write \(\alpha \geq \beta\) or \(\beta \leq \alpha\). A partition \(\xi\) is \(\text{increasing}\) if \(f^{-1}\xi \supseteq \xi\). For a measurable partition \(\beta\), we denote \(\beta_m = \bigvee_{i=m}^{n} f^{-i}\beta\). In particular, \(\beta_0^{-1} = \bigvee_{i=1}^{n} f^{-i}\beta\).

Take \(\varepsilon_0 > 0\) small. Let \(\mathcal{P} = \mathcal{P}_{\varepsilon_0}\) denote the set of finite measurable partitions of \(M\) whose elements have diameters smaller than or equal to \(\varepsilon_0\), that is, \(\text{diam} \alpha = \sup \{\text{diam} A : A \in \alpha\} \leq \varepsilon_0\). For each \(\beta \in \mathcal{P}\) we can define a finer partition \(\eta\)
such that $\eta(x) = \beta(x) \cap W^u_{\text{loc}}(x)$ for each $x \in M$, where $W^u_{\text{loc}}(x)$ denotes the local unstable manifold at $x$ whose size is greater than the diameter $\varepsilon_0$ of $\beta$. Clearly $\eta$ is a measurable partition satisfying $\eta \geq \beta$. Let $\mathcal{P}^u = \mathcal{P}^u_{\eta_0}$ denote the set of partitions $\eta$ obtained this way.

A partition $\xi$ of $M$ is said to be subordinate to unstable manifolds of $\mu$ if for $\mu$-almost every $x$, $\xi(x) \subset W^u(x)$ and contains an open neighborhood of $x$ in $W^u(x)$. It is clear that if $\mu \in \mathcal{P}$ such that $\mu(\partial \alpha) = 0$ where $\partial \alpha := \bigcup_{A \in \alpha} \partial A$, then the corresponding $\eta$ given by $\eta(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$ is a partition subordinate to unstable manifolds of $f$.

Given a measure $\mu$ and measurable partitions $\alpha$ and $\eta$, let

$$H_\mu(\alpha|\eta) := -\int_M \log \mu^\alpha_\eta(\alpha(x)) d\mu(x)$$

denote the conditional entropy of $\alpha$ given $\eta$ with respect to $\mu$, where $\{\mu^n_x : x \in M\}$ is a family of conditional measures of $\mu$ relative to $\eta$. The precise meaning is given in Definition 2.1 in the next section (see also [14]).

**Definition 1.1.** The conditional entropy of $f$ with respect to a measurable partition $\alpha$ given $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f, \alpha|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha^{n-1}|\eta).$$

The conditional entropy of $f$ given $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f|\eta) = \sup_{\alpha \in \mathcal{P}} h_\mu(f, \alpha|\eta).$$

and the unstable metric entropy of $f$ is defined as

$$h^u_\mu(f) = \sup_{\eta \in \mathcal{P}^u} h_\mu(f|\eta).$$

**Remark 1.2.** In the definition of $h_\mu(f, \alpha|\eta)$ we take $\limsup$ instead of $\lim$, since $\eta$ is not invariant under $f$ and hence the sequence $\{H_\mu(\alpha^{n-1}|\eta)\}$ is not necessarily subadditive. Therefore, existence of such a limit is not obvious.

We show in Lemma 2.8 that $h_\mu(f|\eta)$ is independent of $\eta$, as long as it is in $\mathcal{P}^u$. Hence, we actually have $h^u_\mu(f) = h_\mu(f|\eta)$ for any $\eta \in \mathcal{P}^u$.

Suppose that $\mu$ is ergodic. Recall a hierarchy of metric entropies $h_\mu(f, \xi_i) := H_\mu(\xi_i|f\xi_i)$ introduced by Ledrappier and Young in [8], where $i = 1, \ldots, \tilde{u}$, and $\tilde{u}$ is the number of distinct positive Lyapunov exponents. For each $i$, $\xi_i$ is an increasing partition subordinate to the $i$th level of the unstable leaves $W^{1(i)}$, and is a generator. (See Subsection 2.2 for precise meaning.) It is proved there that $h_\mu(f, \xi_i) = h_\mu(f)$, the metric entropy of $\mu$.

If there are $u$ distinct Lyapunov exponents on unstable subbundle, then the $u$th unstable foliation are the unstable foliation of the partially hyperbolic system $f$. We show that the unstable metric entropy we define is identical to $h_\mu(f, \xi_u)$ given by Ledrappier-Young.

Denote by $\mathcal{Q}^u$ the set of increasing partitions $\xi_u$ that are subordinate to $W^u$, and are generators, that is, partitions $\xi_u$ satisfying condition (i)-(iii) in Lemma 2.9 in Subsection 2.2.

**Theorem A.** Suppose $\mu$ is an ergodic measure. Then for any $\alpha \in \mathcal{P}$, $\eta \in \mathcal{P}^u$ and $\xi \in \mathcal{Q}^u$,

$$h_\mu(f, \alpha|\eta) = h_\mu(f, \xi).$$
Hence,
\[ h_u^u(f) = h_\mu(f|\eta) = h_\mu(f, \xi). \]

It is easy to see the following relation by the definition of unstable metric entropy and a formula given by Ledrappier and Young.

Let \( \{\lambda_i^c\} \) denote distinct Lyapunov exponents of \( \mu \) in the center direction, and \( m_i \) denote the multiplicity of \( \lambda_i^c \).

**Corollary A.1.** \( h_\mu^u(f) \leq h_\mu(f) \).

Moreover, if \( f \) is \( C^{1+\alpha} \), then \( h_\mu(f) \leq h_\mu^u(f) + \sum_{\lambda_i^c>0} \lambda_i^c m_i \). In particular, if there is no positive Lyapunov exponent in the center direction at \( \mu \)-a.e. \( x \in M \), then \( h_\mu^u(f) = h_\mu(f) \).

In [4] the authors proved that for any ergodic measure \( \mu \),
\[
(1.1) \quad h_\mu(f) \leq \chi^u(f) + \sum_{\lambda_i^c>0} \lambda_i^c m_i,
\]
where \( \chi^u(f) \) denotes the volume growth of the unstable foliation (see (1.2) and (1.3) below for precise meaning). The part of the inequality for the upper bound of \( h_\mu(f) \) in the corollary can be regarded as a version of (1.1) in measure theoretic category.

The first equation of Theorem A gives that \( h_\mu(f, \alpha|\eta) \) is independent of \( \alpha \). The proof of the theorem also gives the following:

**Corollary A.2.** \( h_\mu^u(f) = h_\mu(f, \alpha|\eta) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \) for any \( \alpha \in \mathcal{P} \) and \( \eta \in \mathcal{P}^u \).

The next result is a version of Shannon-McMillan-Breiman theorem for the unstable metric entropy. In the proof of Theorem A we actually showed that the sequence of the integrals of functions \( \left\{ \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta) \right\} \) converges to \( h_\mu(f, \alpha|\eta) \). This theorem states that the functions converge almost everywhere.

**Theorem B.** Suppose \( \mu \) is an ergodic measure of \( f \). Let \( \eta \in \mathcal{P}^u \) be given. Then for any partition \( \alpha \) with \( H_\mu(\alpha|\eta) < \infty \), we have
\[
\lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) = h_\mu(f, \alpha|\eta) \quad \mu\text{-a.e.} x \in M.
\]

We can use \( \xi \in \mathcal{Q}^u \) for the given partition as well.

**Corollary B.1.** Let \( \mu \) be \( f \)-ergodic and \( \xi \in \mathcal{Q}^u \). Then for any partition \( \alpha \) with \( H_\mu(\alpha|\xi) < \infty \), we have
\[
\lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = h_\mu(f, \alpha|\xi) \quad \mu\text{-a.e.} x \in M,
\]
where \( h_\mu(f, \alpha|\xi) \) is defined as in Definition 1.1 with \( \eta \) replaced by \( \xi \).

**Remark 1.3.** We mention here that in Lemma 3.7 we also obtain
\[
\lim_{n \to \infty} \frac{1}{n} I_\mu(\xi_0^{n-1}|\xi)(x) = h_\mu(f, \xi)
\]
when \( \mu \) is ergodic.
Now we start to define the unstable topological entropy.

We denote by $d_u$ the metric induced by the Riemannian structure on the unstable manifold and let $d_u^n(x, y) = \max_{0 \leq j \leq n-1} d_u(f^j(x), f^j(y))$. Let $W^u(x, \delta)$ be the open ball inside $W^u(x)$ centered at $x$ of radius $\delta$ with respect to the metric $d_u$. Let $N^u(f, \epsilon, n, x, \delta)$ be the maximal number of points in $W^u(x, \delta)$ with pairwise $d_u^n$-distances at least $\epsilon$. We call such set an $(n, \epsilon)$ $u$-separated set of $W^u(x, \delta)$.

**Definition 1.4.** The unstable topological entropy of $f$ on $M$ is defined by

$$h^u_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h^u_{\text{top}}(f, W^u(x, \delta)),$$

where

$$h^u_{\text{top}}(f, W^u(x, \delta)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N^u(f, \epsilon, n, x, \delta).$$

We can also define unstable topological entropy by using $(n, \epsilon)$ $u$-spanning sets or open covers to get equivalent definitions.

Unstable topological entropy defined here can be regarded as the asymptotic rate of orbit divergence along unstable manifolds. Since $f$ is expanding restricted to unstable manifolds, the rate of orbit divergence can also be reflected by the asymptotic rate of the volume growth of unstable manifolds under iterations of $f$.

Volume growth was first used by Yomdin and Newhouse for the entropy of diffeomorphisms (cf. [20], [11]). The unstable volume growth for partially hyperbolic systems is used in [4], which is defined as following:

$$\chi_u(f) = \sup_{x \in M} \chi_u(x, \delta)$$

where

$$\chi_u(x, \delta) = \lim_{n \to \infty} \frac{1}{n} \log (\text{Vol}(f^n(W^u(x, \delta))).$$

Note that the unstable volume growth is independent of $\delta$ and the Riemannian metric (cf. Lemma 1.1 in [16]). We show that the unstable topological entropy actually coincides with the unstable volume growth.

**Theorem C.** $h^u_{\text{top}}(f) = \chi_u(f)$.

By the definition and the equality, we have the following facts.

**Corollary C.1.** $h^u_{\text{top}}(f) \leq h_{\text{top}}(f)$.

The equation holds if there is no positive Lyapunov exponent in the center direction at $\nu$-a.e. with respect to any ergodic measure $\nu$.

We can also give a version of the inequality formula (1.1) in terms of unstable topological entropy and growth rates of $\|\bigwedge^i Df^n|_{E^c}\|$ in the center direction. For $1 \leq i \leq \dim E^c$, let

$$\sigma^{(i)} = \lim_{n \to \infty} \frac{1}{n} \log \|\bigwedge^i Df^n|_{E^c}\|,$$

where $\bigwedge^i Df^n|_{E^c}$ denotes the $i$th outer product of the differential $Df^n|_{E^c}$. The limit exists because of subadditivity of $\log \|\bigwedge^i Df^n|_{E^c}\|$. Then we denote

$$\sigma = \max\{\sigma^{(i)} : i = 1, \cdots, \dim E^c\}.$$

Note that $\sigma^{(i)}$ is greater than the sum of the largest $i$ Lyapunov exponents in the center direction at any point $x$ whenever they exist.
Huyi Hu, Yongxia Hua and Weisheng Wu

Corollary C.2. $h_{\text{top}}(f) \leq h^u_{\text{top}}(f) + \sigma$.
The equation holds if $\sigma^{(1)} \leq 0$.

Similar to the quantity $h_{i+1} - h_i$, the difference between consecutive hierarchy entropies, used by Ledrappier and Young in [8], we can define transversal topological entropy as following.

Let $N(f, \epsilon, n, x, \delta)$ be the maximal number of points in $B(x, \delta)$ with pairwise $d_n$-distances at least $\epsilon$, where $B(x, \delta)$ denotes the open ball about $x$ of radius $\delta$, and $d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y))$.

Definition 1.5. The transversal topological entropy of $f$ on $M$ is defined by

$$h_{\text{top}}^t(f) = \lim_{\delta \to 0} \sup_{x \in M} h_{\text{top}}^t(f, B(x, \delta)),$$

where

$$h_{\text{top}}^t(f, B(x, \delta)) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \left[ \log N(f, \epsilon, n, x, \delta) - \log N^u(f, \epsilon, n, x, \delta) \right].$$

With the notion we can give another version of formula in [4] in topological category.

Corollary C.3. $h_{\text{top}}(f) \leq h^u_{\text{top}}(f) + h_{\text{top}}^t(f)$.

We mention here that in [18] the authors proved that if $f$ is a partially hyperbolic diffeomorphism with a uniformly compact center foliation, then $h_{\text{top}}(f) \leq p^c(f) + h_{\text{top}}(f, W^c)$, where $p^c(f)$ is the growth rate of periodic center leaves, i.e., the leaves $W^c(x)$ with $f^nW^c(x) = W^c(f^n x)$, and $h_{\text{top}}(f, W^c)$ is given by the growth rate of $(n, \epsilon)$ separated sets on center leaves.

For partially hyperbolic diffeomorphisms, we can also build a variational principle for unstable metric entropy and unstable topological entropy.

Let $M_f(M)$ and $M^e_f(M)$ denote the set of all $f$-invariant and ergodic probability measures on $M$ respectively.

Theorem D. Let $f : M \to M$ be a $C^1$-partially hyperbolic diffeomorphism. Then

$$h^u_{\text{top}}(f) = \sup \{ h^u_\mu(f) : \mu \in M_f(M) \}.$$ 

Moreover,

$$h^u_{\text{top}}(f) = \sup \{ h^u_\nu(f) : \nu \in M^e_f(M) \}.$$ 

2. Unstable metric entropy

2.1. Conditional entropy. In this subsection we provide more detailed information about conditional entropy and some properties as a supplement of Definition 1.1.

Recall that for a measurable partition $\eta$ of a measure space $X$ and a probability measure $\nu$ on $X$, the canonical system of conditional measures for $\nu$ and $\eta$ is a family of probability measures $\{ \nu^\eta_x : x \in X \}$ with $\nu^\eta_x(\eta(x)) = 1$, such that for every measurable set $B \subset X$, $x \mapsto \nu^\eta_x(B)$ is measurable and

$$\nu(B) = \int_X \nu^\eta_x(B) d\nu(x).$$

(See e.g. [14] for reference.)

The following notions are standard.
Definition 2.1. The information function of $\alpha \in \mathcal{P}$ are defined as

$$I_\mu(\alpha)(x) := -\log \mu(\alpha(x)),$$

and the entropy of partition $\alpha$ as

$$H_\mu(\alpha) := \int_M I_\mu(\alpha)(x) d\mu(x) = -\int_M \log \mu(\alpha(x)) d\mu(x).$$

The conditional information function of $\alpha \in \mathcal{P}$ with respect to a measurable partition $\eta$ of $M$ is defined as

$$I_\mu(\alpha|\eta)(x) := -\log \mu^\eta_\sigma(\alpha(x)).$$

Then the conditional entropy of $\alpha$ with respect to $\eta$ is defined as

$$H_\mu(\alpha|\eta) := \int_M I_\mu(\alpha|\eta)(x) d\mu(x) = -\int_M \log \mu^\eta_\sigma(\alpha(x)) d\mu(x).$$

The properties in the following lemma is well known (see e.g. [14]).

Lemma 2.2. Let $\alpha$, $\beta$ and $\gamma$ be measurable partitions with $H_\mu(\alpha|\gamma), H_\mu(\beta|\gamma) < \infty$.

(i) If $\alpha \leq \beta$, then $I_\mu(\alpha|\gamma)(x) \leq I_\mu(\beta|\gamma)(x)$ and $H_\mu(\alpha|\gamma) \leq H_\mu(\beta|\gamma)$.

(ii) $I_\mu(\alpha \vee \beta|\gamma)(x) = I_\mu(\alpha|\gamma)(x) + I_\mu(\beta|\gamma \vee \gamma)(x)$ and $H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\alpha|\gamma) + H_\mu(\beta|\gamma \vee \gamma)$.

(iii) $H_\mu(\alpha \vee \beta|\gamma) \leq H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha)$.

(iv) If $\beta \leq \gamma$, then $H_\mu(\alpha|\beta) \geq H_\mu(\alpha|\gamma)$.

Remark 2.3. We mention here that $\beta \leq \gamma$ does not imply $I_\mu(\alpha|\beta)(x) \geq I_\mu(\alpha|\gamma)(x)$ for $\mu$-a.e. $x$, though we have (v) in the above lemma.

Recall that for a probability space $(X, \mathcal{B}, \nu)$ and a sequence of increasing sub-$\sigma$-algebras $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}$, a sequence of functions $\{\phi_n\}$ is a martingale with respect to $\{\mathcal{B}_n\}$ if

(i) $\phi_n$ is $\mathcal{B}_n$ measurable for all $n > 0$; and

(ii) $E_\nu(\phi_{n+1}|\mathcal{B}_n) = \phi_n$ $\nu$-a.e. $x$, where $E_\nu$ denotes the expectation.

If “$=$” in Condition (ii) is replaced by “$\leq$” or “$\geq$”, then the sequence $\{\phi_n\}$ is called a supermartingale or submartingale respectively.

A supermartingale $\{\phi_n\}$ is $L^1$ bounded if $\sup_n E_\nu(|\phi_n|) < \infty$.

Note that if $\{\phi_n\}$ is a supermartingale, then $\{-\phi_n\}$ is a submartingale. $\{\phi_n\}$ is $L^1$ bounded if and only if $\{-\phi_n\}$ is $L^1$ bounded. So Doob’s martingale convergence theorem can be stated in the following way.

Theorem (Doob’s martingale convergence theorem). Every $L^1$ bounded supermartingale or submartingale $\{\phi_n\}$ converges almost everywhere.

Since a martingale is also a supermartingale, the theorem gives convergence of $L^1$ bounded martingales.

Lemma 2.4. Let $\alpha \in \mathcal{P}$ and $\{\zeta_n\}$ be a sequence of increasing measurable partitions with $\zeta_n \nearrow \zeta$. Then for $\phi_n(x) = I_\mu(\alpha|\zeta_n)(x)$, $\phi^* := \sup_n \phi_n \in L^1(\mu)$.

The proof of the lemma can be found in the proof of Lemma 14.27 in [2] or Lemma 2.1 and Corollary 2.2 on p.261 in [13].

For partitions $\{\zeta_n\}$ and $\zeta$ given in the last lemma, let $\{\mathcal{B}(\zeta_n)\}$ and $\mathcal{B}(\zeta)$ be the sub-$\sigma$-algebras generated by $\{\zeta_n\}$ and $\zeta$ respectively, that is, $\{\mathcal{B}(\zeta_n)\}$ is the smallest sub-$\sigma$-algebra containing elements of $\zeta_n$. Let $\phi_n(x) = I_\mu(\alpha|\zeta_n)(x) =$
Proof. (i) Replacing \( \alpha \) and \( \beta \) by \( \beta_0^n \) and \( f^{-i} \beta \) in Lemma 2.2(ii) respectively, we have
\[
\begin{align*}
I_\mu(\beta_0^n \mid \gamma)(x) &= I_\mu(\alpha f^{-i} \mid \beta_0^n \mid \gamma)(x) + I_\mu(\beta f^i \mid \beta_0^n \mid \gamma)(x) \\
&= I_\mu(\alpha f^{-i} \mid \gamma)(x) + I_\mu(\beta f^i \mid \beta_0^n \mid \gamma)(x).
\end{align*}
\]
Summing the equality over \( i \) from 1 to \( n - 1 \) we get the first equality in part (i). The second one follows by integrating the first equality.

(ii) Replacing \( \alpha \) and \( \beta \) by \( \alpha_1^{n-1} \) and \( \alpha \) in Lemma 2.2(ii) respectively, we have
\[
\begin{align*}
I_\mu(\alpha_1^{n-1} \mid \gamma)(x) &= I_\mu(\alpha_1^{n-1} \mid \gamma)(x) + I_\mu(\alpha \mid \alpha_1^{n-1} \mid \gamma)(x) \\
&= I_\mu(\alpha_1^{n-2} \mid f \gamma)(f(x)) + I_\mu(\alpha \mid \alpha_1^{n-1} \mid \gamma)(x).
\end{align*}
\]
By induction, we have
\[
I_\mu(\alpha_0^n \mid \gamma)(x) = I_\mu(\alpha f^{n-1} \mid \gamma)(f^{n-1}(x)) + \sum_{i=0}^{n-2} I_\mu(\alpha \mid \alpha_1^{n-1} \mid \gamma)(f^i(x)).
\]
Integrating both sides of the formula, we have the second equality of the part. \( \square \)

Lemma 2.7.

(i) For any \( \eta_1, \eta_2 \in \mathcal{P}_u \), \( H_\mu(\eta_2 \mid \eta_1), H_\mu(\eta_1 \mid \eta_2) < \infty \). Hence
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_2 \mid \eta_1) = 0 = \lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_1 \mid \eta_2).
\]

(ii) For any \( \alpha, \beta \in \mathcal{P} \) and \( \eta \in \mathcal{P}_u \),
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^n \mid \beta_0^n \mid \gamma \mid \eta) = 0 = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^n \mid \beta_0^n \mid \gamma \mid \eta).
\]

Proof. (i) Recall that \( \mathcal{P} = \mathcal{P}_{\varepsilon_0} \) is the set of finite measurable partitions of diameter less than or equal to \( \varepsilon_0 \). For any \( \eta_1, \eta_2 \in \mathcal{P}_u \), there exist \( \alpha_1, \alpha_2 \in \mathcal{P} \) such that \( \eta_i(x) = \alpha_i(x) \cap W_{\alpha_i}(x), i = 1, 2 \), for all \( x \in M \). Let \( N_i \) be the cardinality of \( \alpha_i \). Then for any \( x \in M \), \( \eta_1(x) \) intersects at most \( N_2 \) elements of \( \alpha_2 \), and therefore intersects at most \( N_2 \) elements of \( \eta_2 \). Similarly, \( \eta_2(x) \) intersects at most \( N_1 \) elements of \( \eta_1 \). So we have \( H_\mu(\eta_2 \mid \eta_1) \leq \log N_2 \) and \( H_\mu(\eta_1 \mid \eta_2) \leq \log N_1 \).
Proof. (i) By Lemma 2.6(ii) with \( \gamma = \beta_0^n \vee \eta \), and using the fact \( f^i \beta_0^{n-1} = \beta_0^{n-i-1} \geq \beta_1^{n-i-1} \), we have

\[
H_\mu(\alpha_0^{n-1}|\beta_0^{n-1} \vee \eta) \leq H_\mu(\alpha_0^{n-1} \beta_0^{n-1} \vee f^{n-1} \eta) + \sum_{i=0}^{n-2} H_\mu(\alpha_0^{n-1-i} \beta_1^{n-i-1} \vee f^i \eta)
\]

\[
\leq H_\mu(\alpha_1 \beta \vee f^{n-1} \eta) + \sum_{i=1}^{n-1} H_\mu(\alpha_1^i \beta_1^i \vee f^{n-i-1} \eta)
\]

Since for any \( x \), \((\alpha_1^n \vee \beta_1^n \vee \eta) \in W_{\text{loc}}^u(x) \) and \( \text{diam}(\alpha_1^n \vee \beta_1^n \vee \eta) \to 0 \) as \( n \to \infty \), we have \( \lim_{n \to \infty} H_\mu(\alpha_1^n \vee \beta_1^n \vee \eta) = 0 \) by Lemma 2.5(ii). It means that the terms in the summation in the last inequality tend to 0. Hence, we get that

\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\beta_0^{n-1} \vee \eta) = 0.
\]

Lemma 2.8. (i) For any \( \alpha \in \mathcal{P} \) and \( \eta_1, \eta_2 \in \mathcal{P}^u \), \( h_\mu(f, \alpha|\eta_1) = h_\mu(f, \alpha|\eta_2) \).

(ii) For any \( \alpha, \beta \in \mathcal{P} \) and \( \eta \in \mathcal{P}^u \),

\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1}|\eta),
\]

\[
\liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = \liminf_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1}|\eta).
\]

Hence, \( h_\mu(f, \alpha|\eta) = h_\mu(f, \beta|\eta) \).

Proof. (i) By Lemma 2.2(iv) we have

\[
H_\mu(\alpha_0^{n-1}|\eta_1) \leq H_\mu(\alpha_0^{n-1}|\eta_2) + H_\mu(\eta_2|\eta_1),
\]

\[
H_\mu(\alpha_0^{n-1}|\eta_2) \leq H_\mu(\alpha_0^{n-1}|\eta_1) + H_\mu(\eta_1|\eta_2).
\]

Hence by using Lemma 2.7(i) we get

\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_1) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_2).
\]

Then the result of part (i) of the lemma follows.

(ii) Similarly by Lemma 2.2(i) and (ii) we have

\[
H_\mu(\alpha_0^{n-1}|\eta) \leq H_\mu(\beta_0^{n-1}|\eta) + H_\mu(\alpha_0^{n-1} \beta_0^{n-1} \vee \eta),
\]

\[
H_\mu(\beta_0^{n-1}|\eta) \leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta_0^{n-1} \alpha_0^{n-1} \vee \eta).
\]

By dividing the inequalities by \( n \), and taking \( \limsup \) and \( \liminf \), and then by Lemma 2.7(ii) we get equalities of the lemma.

By this lemma, \( h_\mu(f, \alpha|\eta) \) is independent of \( \alpha \) and \( \eta \) as long as \( \alpha \in \mathcal{P} \) and \( \eta \in \mathcal{P}^u \). Hence we can define the unstable metric entropy \( h^u_\mu(f, \alpha|\eta) \) for any \( \alpha \in \mathcal{P} \) and \( \eta \in \mathcal{P}^u \).

2.2. Increasing partitions \( \xi_u \). For an ergodic measure \( \mu \) with positive Lyapunov exponents \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \), let \( E^{(1)} \subset E^{(2)} \subset \cdots \subset E^{(n)} \) denote the subbundles in the tangent bundle consisting of vectors whose Lyapunov exponents are greater than or equal to \( \lambda_1, \lambda_2, \cdots, \lambda_n \) respectively. It is well known that if \( f \) is \( C^{1+\alpha} \), then for almost every \( x \) there exist unstable manifolds \( W^{(1)}(x) \subset W^{(2)}(x) \subset \cdots \subset W^{(n)}(x) \) such that if \( y \in W^{(i)}(x) \), then \( \lim_{n \to \infty} \frac{1}{n} \log d(f^{-n}y, f^{-n}x) \leq -\lambda_i \).
for any $1 \leq i \leq u$. The entropies $h_\mu(f, \xi_i)$ are determined by a hierarchy of partitions given in the next lemma.

We mention that a partition $\beta$ of $M$ is a generator if $\sqrt[n]{f^{-n}\beta} = \varepsilon$ where $\varepsilon$ is a partition of $M$ into points up to a set of zero measure.

**Lemma 2.9** (Lemma 9.1.1 in [8]). Assume that $f$ is $C^{1+\alpha}$. Then there exist measurable partitions $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_u$ on $M$ such that for each $1 \leq i \leq u$,

1. $\xi_i$ is subordinate to $W^{(i)}$,
2. $\xi_i$ is increasing,
3. $\xi_i$ is a generator.

To construct such a partition, the authors in [8] first take a point $z$ and then take

$$S_i(z, r) = \bigcup_{y \in W(z, r)} W^{(i)}(y, r)$$

where $W(z, r)$ is an open ball of radius $r$ centered at $z$ inside a local manifold $W$ passing through $z$ transversally to the unstable foliation $W^{(i)}$, and $W^{(i)}(y, r)$ are local unstable manifolds. Moreover, $z$ and $S_i(z, r)$ are taken in such a way that $\mu(S_i(z, r)) > 0$ for any $r > 0$. Then define a partition $\xi_{i,z}$ such that $\xi_{i,z}(y) = W^{(i)}(\tilde{y}, r)$ if $y \in S_i(z, r)$, where $\tilde{y} \in W(z, r)$ and $y \in W^{(i)}(\tilde{y}, r)$, and $\xi_{i,z}(y) = M \setminus S_i(z, r)$ otherwise. Next take $\xi_i = \xi_{i,z} := \bigvee_{j \geq 0} f^j \xi_{i,z}$. It has been proven (see e.g. [8]) that if $\mu$ is ergodic, then for almost every small real number $r > 0$ in the sense of Lebesgue measure, $\xi_i$ is subordinate to unstable manifolds $W^{(i)}$ and therefore is a partition satisfying Lemma 2.9.

The above construction of such partitions is also carried out in [6] and [7]. Though by the construction it is unclear whether the diameter of $\xi_i(x)$ is bounded above or below in the metric $d^i$, the Riemannian metric restricted to $W^{(i)}$, it has been proven in [8] that

$$H_\mu(f^{-1}\xi_i|\xi_i) = -\int_M \log \mu^\xi_x \left( (f^{-1}\xi_i)(x) \right) \, d\mu(x)$$

is finite. It is also proved that $h_\mu(f, \xi_i) := H_\mu(\xi_i|f\xi_i) = H_\mu(f^{-1}\xi_i|\xi_i)$ is independent of the choice of $\xi_i$ as long as $\xi_i$ satisfies the above conditions (cf. Subsection (3.1) in [7]).

If there are $u$ distinct Lyapunov exponents on unstable subbundle, then the $u$th unstable foliation are the unstable foliation of the partially hyperbolic system $f$. Recall that $Q^u$ denote the set of partitions $\xi_u$ satisfying (i)-(iii) above with $i = u$. The above construction of $\xi_u$ still applies even if $f$ is only assumed to be $C^1$, since the unstable foliation of $f$ always exists under $C^1$ regularity.

Denote $S = S_u(z, r)$, the set given in (2.1). Recall that $\xi_{i,z}$ is a partition defined above such that $\xi_i = \xi_{i,z} := \bigvee_{j \geq 0} f^j \xi_{i,z}$. For $i = u$ we denote $\xi = \xi_{u,z}$. Recall by the notation we use,

$$\hat{\xi}_{-k} = \bigvee_{j=0}^k f^j \hat{\xi}_{u,z}.$$ 

We further denote $\xi_{-k} = \hat{\xi}_{-k}$. Hence $\xi = \xi_{-\infty}$.

**Lemma 2.10.** Suppose $\mu$ is an ergodic measure and $\alpha \in P$. For any $\varepsilon > 0$, there exists $K > 0$ such that for any $k \geq K$,

$$\limsup_{n \to \infty} H_\mu(\alpha \cap \hat{\xi}_{-k}^n) \leq \varepsilon.$$
Proof. Denote $S_{-k} = \cup_{i=0}^{k} f^i S$, where $S = S_a(z, r)$ is given by (2.1).

Let $\varepsilon > 0$. Since $\mu$ is ergodic, $\mu S_{-k} \to 1$ as $k \to \infty$. So there exists $K > 0$ such that for any $k \geq K$, $\mu(M \setminus S_{-k}) \leq \varepsilon/\log N$, where $N_\alpha$ is the cardinality of the partition $\alpha$.

Write

$$H_\mu(\alpha | \alpha^u_1 \vee \hat{\xi}^n_{-k}) = \int_{S_{-k}} I_\mu(\alpha | \alpha^u_1 \vee \hat{\xi}^n_{-k})d\mu(x) + \int_{M \setminus S_{-k}} I_\mu(\alpha | \alpha^u_1 \vee \hat{\xi}^n_{-k})d\mu(x).$$

For $x \in S_{-k}$, $(\alpha^u_1 \vee \hat{\xi}^n_{-k})(x) \subset W^u_{\log}(x)$. Hence for almost every $x \in S_{-k}$, there exists $N = N(x) > 0$ such that for any $n \geq N$, $(\alpha^u_1 \vee \hat{\xi}^n_{-k})(x) \subset \alpha(x)$ and therefore

$$\log \mu_x(\alpha^u_1 \vee \hat{\xi}^n_{-k}) = 0.$$ Lemma 2.4 with $\zeta_n = \alpha^u_1 \vee \hat{\xi}^n_{-k}$ implies that Lebesgue dominated convergence theorem can be applied to integration over $S_{-k}$. So it follows

$$\limsup_{n \to \infty} \int_{S_{-k}} I_\mu(\alpha | \alpha^u_1 \vee \hat{\xi}^n_{-k})d\mu(x) = 0.$$

For $x \in M \setminus S_{-k}$, $(\alpha^u_1 \vee \hat{\xi}^n_{-k})(x) \subset S_{-k}$. We know that on $(\alpha^u_1 \vee \hat{\xi}^n_{-k})(x),$

$$\int_{\alpha^u_1 \vee \hat{\xi}^n_{-k}(x)} - \log \mu_x(\alpha^u_1 \vee \hat{\xi}^n_{-k}(\alpha(x)))d\mu_x \leq \log N_\alpha.$$ It gives that

$$\int_{M \setminus S_{-k}} - \log \mu_x(\alpha^u_1 \vee \hat{\xi}^n_{-k}(\alpha(x)))d\mu(x) \leq \mu(M \setminus S_{-k}) \cdot \log N_\alpha \leq \varepsilon.$$

So the result of the lemma follows. \qed

Lemma 2.11. Let $\mu$ be an ergodic measure. Suppose $\eta \in P^u$ that is subordinate to the unstable manifold, and $\hat{\xi}_{-k}$ is a partition described as above, where $k \in \mathbb{N} \cup \{\infty\}$. Then for almost every $x$, there is $N = N(x) > 0$ such that for any $i > N$,

$$(\hat{\xi}_{-k-i} \vee f^i(\eta))(f^i(x)) = (\hat{\xi}_{-k-i})(f^i(x)).$$

Hence, for any partition $\beta$ with $H_\mu(\beta | \hat{\xi}_{-k}) < \infty$,

$$I_\mu(\beta | \hat{\xi}_{-k} \vee f^i(\eta))(f^i(x)) = I_\mu(\beta | \hat{\xi}_{-k-i})(f^i(x))$$

and therefore

$$\lim_{i \to \infty} H_\mu(\beta | \hat{\xi}_{-k-i} \vee f^i(\eta)) = H_\mu(\beta | \beta).$$

In particular, if we take $k = \infty$, then the last two equalities become

$$I_\mu(\beta | \xi \vee f^i(\eta))(f^i(x)) = I_\mu(\beta | \xi)(f^i(x))$$

and

$$\lim_{i \to \infty} H_\mu(\beta | \xi \vee f^i(\eta)) = H_\mu(\beta | \xi).$$

Proof. Since $\eta$ is subordinate to $W^u$, for $\mu$-a.e. $x$, there is $r = r(x) > 0$ such that $B^u(x, r) \subset \eta(x)$. Since $\mu$ is ergodic, for $\mu$-a.e. $x$, there are infinite many $n > 0$ such that $f^n(x) \in S$. Take $n_0 = n_0(x)$ large enough, such that $f^n(x) \in S$ and $f^{-n_0}(\hat{\xi}(f^{n_0}(x))) \subset B^u(x, r) \subset \eta(x)$. It follows that $-i(f^{-i-n_0}(\hat{\xi}(f^i(x)))) \subset \eta(x)$ for any $i \geq n_0$. Since $\hat{\xi}_{-k-i} = \vee_{j=0}^{k+i} f^j \hat{\xi} \geq f^{-n_0} \hat{\xi} \setminus f^{-i}(\hat{\xi}_{-k-i})(f^i(x)) \subset \eta(x)$. That is, $\hat{\xi}_{-k-i}(f^i(x)) \subset (f^i(\eta))(f^i(x))$. It implies that $(\hat{\xi}_{-k-i} \vee f^i(\eta))(f^i(x)) = (\hat{\xi}_{-k-i})(f^i(x))$ for all $i > N$.

By definition we can get directly $I_\mu(\beta | \hat{\xi}_{-k-i} \vee f^i(\eta))(f^i(x)) = I_\mu(\beta | \hat{\xi}_{-k-i})(f^i(x)).$
Let \( \phi_i = (I_\mu(\beta|\hat{\xi}_{-k+i} \vee f^i \eta) - I_\mu(\beta|\hat{\xi}_{-k-i})) \circ f^i \). The above fact gives \( \lim_{i \to \infty} \phi_i(x) = 0 \) for almost every \( x \). By Fatou’s lemma,

\[
\liminf_{i \to \infty} \int \phi_i d\mu \geq \int \liminf_{i \to \infty} \phi_i d\mu = 0.
\]

It means

\[
\liminf_{i \to \infty} H_\mu(\beta|\hat{\xi}_{-k-i} \vee f^i \eta) \geq \lim_{i \to \infty} H_\mu(\beta|\hat{\xi}_{-k-i}) = H_\mu(\beta|\xi),
\]

where in the last step we use Lemma 2.3. Proof of Theorem A and its corollary. Note that \( h_\mu(f, \xi) \) can be written as \( \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi_0^{n-1}|f \xi) \), and \( h_\mu(f, \alpha|\eta) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|f \eta) \). To prove Theorem A we need to know that the difference \( H_\mu(\xi_0^{n-1}|f \xi) - H_\mu(\alpha_0^{n-1}|f \eta) \) increases at most subexponentially. It is natural to compare both \( H_\mu(\xi_0^{n-1}|f \xi) \) and \( H_\mu(\alpha_0^{n-1}|f \eta) \) with \( H_\mu(\xi_0^{n-1}|\eta) \). Since the size of elements of \( \xi \) can be arbitrarily small, it is unknown whether \( H_\mu(\xi|\eta) \) is finite and whether \( H_\mu(\xi_0^{n-1}|\alpha_0^{n-1}) \) increases at most subexponentially. So we cannot obtain the result as easy as the same way we use in the proof in Lemma 2.8.

**Proposition 2.12.** Suppose \( \mu \) is an ergodic measure. Then for any \( \alpha \in \mathcal{P}, \eta \in \mathcal{P}^u \) subordinate to unstable manifolds, and \( \xi \in \mathcal{Q}^u \),

\[
h_\mu(f, \alpha|\eta) \leq h_\mu(f, \xi)
\]

**Proof.** Applying Lemma 2.6(i) with \( \gamma = \eta, \beta = \hat{\xi}_{-k} \), and using the fact \( \beta_0^{n-1} = \hat{\xi}_{-k}^{n-1} \) and \( f^i|\beta_0^{i-1} = f^i|\hat{\xi}_{-k-i+1} \) we have that for any \( \eta \in \mathcal{P}^u, n > 0, \)

\[
\frac{1}{n} H_\mu(\hat{\xi}_{-k}^{n-1}|\eta) = \frac{1}{n} H_\mu(\hat{\xi}_{-k}|\eta) + \frac{1}{n} \sum_{i=1}^{n-1} H_\mu(\hat{\xi}_{-k+i}|f^i(\hat{\xi}_{-k-i+1} \vee f^i \eta)).
\]

Applying Lemma 2.11 with \( \beta = \hat{\xi}_{-k} \), we get that the terms in the summation converge to \( H_\mu(\hat{\xi}_{-k}|f \xi) \) as \( i \to \infty \). It is easy to see by the construction of \( \hat{\xi}_{-k} \), each element of \( \eta \) intersects at most \( 2^k \) elements of \( \hat{\xi}_{-k} \). Hence \( H_\mu(\hat{\xi}_{-k}|\eta) \leq 2^k \) and \( \frac{1}{n} H_\mu(\hat{\xi}_{-k}|\eta) \to 0 \). We get

\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\hat{\xi}_{-k}^{n-1}|\eta) = H_\mu(\hat{\xi}_{-k}|f \xi) \leq H_\mu(\xi|f \xi).
\]
On the other hand, taking $\gamma = \hat{\xi}_n^{-1}$ in Lemma 2.6(ii) we have
\[
H_\mu(\alpha_0^{-1} | \hat{\xi}_n^{-1}) = H_\mu(\alpha | \hat{\xi}_{n-k+1}) + \sum_{i=0}^{n-2} H_\mu(\alpha | \alpha_1^{n-1-i} \lor \hat{\xi}_{n-k+i})
\]
\[
= H_\mu(\alpha | \hat{\xi}_{n-k+1}) + \sum_{i=1}^{n-1} H_\mu(\alpha | \alpha_1^i \lor \hat{\xi}_{n-k+i-1})
\]
\[
\leq H_\mu(\alpha) + \sum_{i=1}^{n-1} H_\mu(\alpha | \alpha_1^i \lor \hat{\xi}_{n-k}),
\]
where we use the fact that $f \hat{\xi}_{n-k}^{-1} = \xi_{n-k}^{-1}$ and therefore $f^{-1} \hat{\xi}_{n-k}^{-1} = \hat{\xi}_{n-k+1}$. For any $\epsilon > 0$ we take $k > 0$ as in Lemma 2.10. By the lemma we know that $\limsup_{n \to \infty} H_\mu(\alpha | \alpha_1 \lor \hat{\xi}_{n-k}^{-1}) \leq \limsup_{n \to \infty} H_\mu(\alpha | \hat{\xi}_{n-k}^{-1}) \leq \epsilon$. Since $H_\mu(\alpha) < \infty$, we get
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{-1} | \hat{\xi}_n^{-1}) \leq \epsilon.
\]
By Lemma 2.2
\[
H_\mu(\alpha_0^{-1} | \eta) \leq H_\mu(\hat{\xi}_n^{-1} | \eta) + H_\mu(\alpha_0^{-1} | \hat{\xi}_{n-k}^{-1}).
\]
By (2.3), (2.4), and then by (2.2), we get
\[
h_\mu(f, \alpha | \eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{-1} | \eta)
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{n} H_\mu(\hat{\xi}_n^{-1} | \eta) + \epsilon = H_\mu(\hat{\xi}_n | f \xi) + \epsilon = h_\mu(f, \xi) + \epsilon.
\]
Since $\epsilon$ is arbitrary, we obtain the result of the proposition. $\square$

**Proposition 2.13.** Suppose $\mu$ is an ergodic measure. Then for any $\eta \in \mathcal{P}^a$ subordinate to unstable manifolds, and $\xi \in \mathcal{Q}^u$,
\[
h_\mu(f, \xi) \leq \sup_{\alpha \in \mathcal{P}} h_\mu(f, \alpha | \eta).
\]

**Proof.** Recall that $\xi \in \mathcal{Q}^u$ is constructed after the statement of Lemma 2.9. We take finite number of points $z^{(1)}, \ldots, z^{(K)} \in M$ and real numbers $r^{(1)}, \ldots, r^{(K)} \leq \hat{\epsilon}$ such that $\{S_u(z^{(j)}, r^{(j)}) \mid 1 \leq j \leq K\}$ form a cover of $M$, where $S_u(z^{(j)}, r^{(j)})$ are the sets with the form given by (2.1) for each $1 \leq j \leq K$. Construct $\xi^{(j)}$ the same way we described, and denote $\xi = \xi^{(1)} \lor \cdots \lor \xi^{(K)}$. Clearly $\xi$ is also a partition satisfying Lemma 2.9, and every element of $\xi$ has diameter smaller than $\epsilon_0$ if $\epsilon$ is small enough. It is in fact proved in Lemma 3.1.2 in [7] that for any two such partitions $\xi'$ and $\xi''$, $h_\mu(f, \xi' \lor \xi'') = h_\mu(f, \xi')$. By induction we can show that $h_\mu(f, \xi) = h_\mu(f, \xi^{(j)})$ for any $1 \leq j \leq K$. So we only need to prove the result for $\xi$. For the sake of notational simplicity, we will drop the tildes and write $\xi$ instead.

Since $f^{-1} \xi$ is a measurable partition of manifold $M$, there exists a sequence of partitions $\alpha_n \in \mathcal{P}$ such that $\alpha_n \not\sim f^{-1} \xi$ as $n \to \infty$. Hence, $\lim_{n \to \infty} H_\mu(\alpha_n | \xi) = H_\mu(f^{-1} \xi | \xi)$. So we have
\[
\sup_{\alpha \in \mathcal{P}, \alpha \not\sim f^{-1} \xi} H_\mu(\alpha | \xi) = H_\mu(f^{-1} \xi | \xi).
\]
On the other hand, if $\alpha \in \mathcal{P}$ with $\alpha < f^{-1}\xi$, then for any $i \geq 1$, $f^i\alpha_0^{-1} < f(f^{-1}\xi)_0^{-i} = \xi$. By Lemma 2.6(i) with $\beta = \alpha$, $\gamma = \xi$,

$$H_\mu(\alpha_0^{n-1}|\eta) = H_\mu(\alpha|\eta) + \sum_{i=1}^{n-1} H_\mu(\alpha|f^i\alpha_0^{-1} \lor f^i\eta) \geq H_\mu(\alpha|\eta) + \sum_{i=1}^{n-1} H_\mu(\alpha|\xi \lor f^i\eta)$$

By Lemma 2.11 with $\beta = \alpha$ we have $\lim_{i \to \infty} H_\mu(\alpha|\xi \lor f^i\eta) = H_\mu(\alpha|\xi)$. Hence

$$\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq \liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq H_\mu(\alpha|\xi).$$

So we get

$$\sup_{\alpha \in \mathcal{P}} h_\mu(f, \alpha|\eta) = \sup_{\alpha \in \mathcal{P}, \alpha < f^{-1}\xi} h_\mu(f, \alpha|\eta) = \sup_{\alpha \in \mathcal{P}, \alpha < f^{-1}\xi} \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq \sup_{\alpha \in \mathcal{P}, \alpha < f^{-1}\xi} \liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq H_\mu(\alpha|\xi) \geq H_\mu(f^{-1}\xi|\xi) = h_\mu(f, \xi).$$

This is what we need. □

**Proof of Theorem A.** Proposition 2.12 and Proposition 2.13 gives that for any $\alpha \in \mathcal{P}$, $\eta \in \mathcal{P}^u$ subordinate to unstable manifolds, and $\xi \in \mathcal{Q}^u$,

$$h_\mu(f, \alpha|\eta) \leq h_\mu(f, \xi) \leq \sup_{\beta \in \mathcal{P}} h_\mu(f, \beta|\eta).$$

By Lemma 2.8, $\sup_{\beta \in \mathcal{P}} h_\mu(f, \beta|\eta) = h_\mu(f, \alpha|\eta)$. So the result follows.

By Lemma 2.8, $h_\mu(f, \alpha|\eta)$ is independent of choice of $\eta$ as long as $\eta \in \mathcal{P}^u$. So the result is true for any $\eta \in \mathcal{P}^u$, not necessary subordinate to unstable manifolds. □

**Proof of Corollary A.1.** Since $H_\mu(\alpha_0^{n-1}|\eta) \leq H_\mu(\alpha_0^{n-1})$ for any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$, by definition we get $h_\mu^n(f) \leq h_\mu(f)$.

If $f$ is $C^r$ with $r > 1$, then Ledrappier-Young’s formula can be applied, that is,

$$h_\mu(f) = \sum_{i=1}^{\dim u} \lambda_i \gamma_i,$$

where $\lambda_1 > \cdots > \lambda_u > 0$ are the positive Lyapunov exponents, $0 \leq \gamma_1 \leq \dim E_i$, and $E_i$ are the subspaces whose nonzero vectors have Lyapunov exponents $\lambda_i$. If there are $u$ distinct Lyapunov exponents on the unstable subspace, then $u \leq u$. Ledrappier-Young’s formula also gives $H_\mu(\xi|f\xi) = \sum_{i=1}^{\dim u} \lambda_i \gamma_i$, where $\xi \in \mathcal{Q}^u$. Since by Theorem A, $h_\mu^n(f) = H_\mu(\xi|f\xi)$, we get the inequalities.

If there is no positive Lyapunov exponent in the center direction, then $u = u$, hence we can take $\xi_u = \xi_u$ to get $h_\mu^n(f) = h_\mu(f, \xi_u) = h_\mu(f, \xi_u) = h_\mu(f)$. □

**Proof of Corollary A.2.** The equality $h_\mu^n(f) = h_\mu^n(f, \alpha|\eta)$ for any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$ is implied in Lemma 2.8, as well as in the two equalities given in Theorem A. So we only need to prove that the limit $\lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta)$ exists.

First, by Theorem A, all “≥” in (2.5) becomes “=”. Also, by Lemma 2.8, all the supremum in (2.5) can be dropped. So (2.5) becomes

$$h_\mu(f, \alpha|\eta) = \limsup_{n \to \infty} \frac{1}{n} H(\alpha_0^{n-1}|\eta) = \liminf_{n \to \infty} \frac{1}{n} H(\alpha_0^{n-1}|\eta) = h_\mu(f, \xi).$$

We get existence of $\lim_{n \to \infty} \frac{1}{n} H(\alpha_0^{n-1}|\eta)$.
2.4. Further Properties. In this subsection we show that the unstable metric entropy is affine and upper semi-continuous with respect to measures.

Recall that $\mathcal{M}_f(M)$ and $\mathcal{M}_f^s(M)$ denote the set of all $f$-invariant and ergodic probability measures on $M$ respectively. Let $\mathcal{M}(M)$ denote the set of all probability measures on $M$.

Note that any partition $\gamma$ generates a sub-$\sigma$-algebra $\mathcal{B}(\gamma)$, that is, $\mathcal{B}(\gamma)$ is the smallest sub-$\sigma$-algebra that contains the elements in the partition $\gamma$. Clearly, if $\{\gamma_n\}$ is a sequence of increasing measurable partitions, then $\{\mathcal{B}(\gamma_n)\}$ is a sequence of increasing sub-$\sigma$-algebras.

Proposition 2.14. For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^n$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is concave.

Furthermore, the map $\mu \mapsto h_\mu(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is affine.

Proof. For $\mu = a\mu_1 + (1-a)\mu_2$ where $\mu_1, \mu_2 \in \mathcal{M}(M)$ and $0 < a < 1$, and for any $\alpha, \beta \in \mathcal{P}$, it is well known that (cf. Lemma 3.3 in [5])

$$0 \leq H_{\mu}(\alpha|\beta) - aH_{\mu_1}(\alpha|\beta) - (1-a)H_{\mu_2}(\alpha|\beta) \leq \phi(a) + \phi(1-a)$$

where $\phi(x) = -x \log x$. For $\eta \in \mathcal{P}^n$, we can find a sequence of partitions $\beta_n \in \mathcal{P}$ such that $\beta_1 < \beta_2 < \cdots$. Using Lemma 2.5 with $\zeta_n = \beta_n$ and $\zeta = \eta$, we have

$$H_\mu(\alpha|\eta) = \lim_{n \to \infty} H_\mu(\alpha|\beta_n) \geq \lim_{n \to \infty} (aH_{\mu_1}(\alpha|\beta_n) + (1-a)H_{\mu_2}(\alpha|\beta_n))$$

$$= aH_{\mu_1}(\alpha|\eta) + (1-a)H_{\mu_2}(\alpha|\eta).$$

The first part of the proposition follows. Similarly, we have

$$H_\mu(\alpha|\eta) \leq aH_{\mu_1}(\alpha|\eta) + (1-a)H_{\mu_2}(\alpha|\eta) + \phi(a) + \phi(1-a).$$

Hence

$$aH_{\mu_1}(\alpha_0^{n-1}|\eta) + (1-a)H_{\mu_2}(\alpha_0^{n-1}|\eta) \leq H_\mu(\alpha_0^{n-1}|\eta)$$

$$\leq aH_{\mu_1}(\alpha_0^{n-1}|\eta) + (1-a)H_{\mu_2}(\alpha_0^{n-1}|\eta) + \phi(a) + \phi(1-a).$$

Dividing by $n$ and taking limit, we have $h_\mu(f, \alpha|\eta) = ah_{\mu_1}(f, \alpha|\eta) + (1-a)h_{\mu_2}(f, \alpha|\eta)$. Then the second part of the proposition follows by Corollary A.2. □

Recall that for each partition $\alpha \in \mathcal{P}$, the partition $\zeta$ given by $\zeta(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$ for any $x \in M$ is an element in $\mathcal{P}^n$. Denote such $\zeta$ by $\alpha^\eta$. Conversely, for each partition $\eta \in \mathcal{P}^n$, there is a partition $\beta \in \mathcal{P}$ such that $\eta(x) = \beta(x) \cap W^u_{\text{loc}}(x)$ for any $x \in M$. Denote such $\beta$ by $\eta^\beta$.

Proposition 2.15. (a) Let $\nu \in \mathcal{M}(M)$. For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^n$ with $\mu(\partial \alpha) = 0$ and $\mu(\partial \eta^\beta) = 0$, the map $\mu \mapsto H_\nu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semi-continuous at $\mu$, i.e.

$$\limsup_{\nu \to \mu} H_\nu(\alpha|\eta) \leq H_\mu(\alpha|\eta).$$

(b) The unstable entropy map $\mu \mapsto h_\mu^u(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semi-continuous at $\mu$, i.e.

$$\limsup_{\nu \to \mu} h_\nu^u(f) \leq h_\mu^u(f).$$

Proof. (a) Since $\mu(\partial \eta^\beta) = 0$, we can take a sequence of partitions $\{\beta_n\} \subset \mathcal{P}$ such that $\beta_1 < \beta_2 < \cdots$ and $\mathcal{B}(\beta_n) \not\supset \mathcal{B}(\eta)$, and moreover, $\mu(\partial \beta^\eta_n) = 0$ for $n = 1, 2, \cdots$. □
Since $\mu(\partial \alpha) = 0 = \mu(\partial \beta_n)$, and for any invariant measure $\nu$,
\[ H_\nu(\alpha|\beta_n) = -\sum_{A_i \in \alpha, B_j \in \beta_n} \nu(A_i \cap B_j) \log \frac{\nu(A_i \cap B_j)}{\nu(B_j)}, \]
we have $\lim_{\nu \to \mu} H_\nu(\alpha|\beta_n) = H_\mu(\alpha|\beta_n)$ for any $n \in \mathbb{N}$. By martingale convergence theorem, $H_\nu(\alpha|\eta) = \lim_{n \to \infty} H_\nu(\alpha|\beta_n)$. So for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $H_\mu(\alpha|\beta_N) \leq H_\mu(\alpha|\eta) + \epsilon$. One has
\[ \limsup_{\nu \to \mu} H_\nu(\alpha|\eta) \leq \limsup_{\nu \to \mu} H_\nu(\alpha|\beta_N) = H_\mu(\alpha|\beta_N) \leq H_\mu(\alpha|\eta) + \epsilon. \]
Since $\epsilon > 0$ is arbitrary, we get the inequality.

(b) To get the upper semi-continuity for unstable entropy map $\mu \mapsto h_\mu^n(f)$, we take $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$ with $\mu(\partial \alpha) = 0$ and $\mu(\partial \eta^u) = 0$.

By Lemma 2.2, for any $f$-invariant measure $\nu$, we have
\[ H_\nu(\alpha^{m+n-1} \mid \eta) = H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(f^{-n} \alpha^{n-1} \mid f^n \eta) \]
\[ = H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\alpha^{m-1} \mid f^n \eta) \]
\[ \leq H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\alpha^{m-1} \mid \eta) + H_\nu(\eta^u \mid f^n \eta). \]

Note that for any $\zeta \in \mathcal{P}^u$, $x \in M$, $\eta^u(x) \cap \zeta(x) = \eta(x) \cap \zeta(x)$. The definition of conditional entropy gives
\[ H_\nu(\eta^u \mid f^n \eta) = H_\nu(\eta^u \mid \eta^u \mid f^n \eta) \leq H_\nu(\eta^u). \]
So by (2.6),
\[ H_\nu(\alpha^{m+n-1} \mid \eta) \leq H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\alpha^{m-1} \mid \eta) + H_\nu(\eta^u). \]
That is, $\{H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\eta^u)\}$ is a subadditive sequence. Hence we have
\[ \lim_{n \to \infty} \frac{1}{n} (H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\eta^u)) = \inf_{n \in \mathbb{N}} \frac{H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\eta^u)}{n}. \]

Let $\epsilon > 0$ be arbitrary. By Corollary A.2, we can take $N \in \mathbb{N}$ large enough such that
\[ \frac{H_\mu(\alpha^{N-1} \mid \eta) + H_\mu(\eta^u)}{N} \leq h_\mu(f, \alpha|\eta) + \epsilon = h_\mu^n(f) + \epsilon. \]
Since $\mu(\partial \alpha) = 0$, $\mu(\partial \alpha^{n-1}) = 0$ for any $n \geq 1$ by invariance of measure $\mu$. So we can use Corollary A.2 and (2.7), and then apply the conclusion in part (a) with $\alpha$ replaced by $\alpha^{n-1}$ to get
\[ \limsup_{\nu \to \mu} h_\nu^n(f) = \limsup_{\nu \to \mu} h_\nu^n(f, \alpha|\eta) = \limsup_{\nu \to \mu} \inf_{n \in \mathbb{N}} \frac{H_\nu(\alpha^{n-1} \mid \eta) + H_\nu(\eta^u)}{n} \leq \limsup_{\nu \to \mu} \frac{H_\nu(\alpha^{N-1} \mid \eta) + H_\nu(\eta^u)}{N} \leq h_\mu^n(f) + \epsilon. \]
Since $\epsilon > 0$ is arbitrary, we get the result. \qed

3. Shannon-McMillan-Breiman Theorem

In this section, we prove Theorem B, a version of Shannon-McMillan-Breiman theorem for unstable metric entropy. Throughout the section we assume that $\mu$ is an ergodic measure of $f$ since we have such an assumption in the theorem.
3.1. Proof of Theorem B: Lower Limits. Let \( d^u \) denote the metric induced by the Riemannian structure on unstable manifolds. Let \( B^u(y, r) \) denote the open ball centered at \( y \) with radius \( r > 0 \) in the unstable manifold \( W^u(y) \) with respect to \( d^u \).

Then let \( d^u_k(x, y) = \max_{i \in \mathbb{Z}} |f^i(x) - f^i(y)| \), and \( B^u_k(x, r) \) be the open ball centered at \( x \) of radius \( r \) with respect to the metric \( d^u_k \), i.e. an \((n, \epsilon)\) Bowen ball in \( W^u(x) \) about \( x \).

Recall for any \( \xi \in \mathcal{Q}^u \), the entropy of \( \xi \) is given by \( h_\mu(f, \xi) = H_\mu(f^{-1}\xi|\xi) \).

**Lemma 3.1.** For any \( \xi \in \mathcal{Q}^u, \epsilon > 0 \),

\[
h_\mu(f, \xi) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)), \quad \mu - a.e. x.
\]

**Proof.** Denote

\[
\underline{h}_n(f, x, \epsilon, \xi) = \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)),
\]

\[
\overline{h}_n(f, x, \epsilon, \xi) = \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)).
\]

(3.1)

It is proved in (9.2) and (9.3) in [8] that

\[
\lim_{\epsilon \to 0} \underline{h}_n(f, x, \epsilon, \xi) = \lim_{\epsilon \to 0} \overline{h}_n(f, x, \epsilon, \xi),
\]

and hence the common value can be denoted as \( h_\mu(f, x, \xi) \). When \( \mu \) is ergodic, \( x \mapsto h_\mu(f, x, \xi) \) is constant almost everywhere. This constant coincides with \( h_\mu(f, \xi) \).

Now we show that the upper and lower limits in (3.1) are limit.

Since \( f \) is uniformly expanding restricted to the unstable manifolds, for any \( 0 < \delta < \epsilon \), there exists \( k > 0 \) such that \( B^u_k(x, \epsilon) \subset B^u(x, \delta) \) and therefore \( B^u_n(x, \delta) \subset B^u_n(x, \epsilon) \) for any \( n > 0 \) and \( x \in M \). It implies

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \delta)) = \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)),
\]

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \delta)) = \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)).
\]

It means that both \( \underline{h}_n(f, x, \epsilon, \xi) \) and \( \overline{h}_n(f, x, \epsilon, \xi) \) given in (3.1) are independent of \( \epsilon \). So (3.2) becomes \( \underline{h}_n(f, x, \epsilon, \xi) = \underline{h}_n(f, x, \epsilon, \xi) \). We get that for \( \mu \)-a.e. \( x \),

\[
H_\mu(f^{-1}\xi|\xi) = h_\mu(f, x, \xi) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_\xi(B^u_n(x, \epsilon)). \quad \square
\]

**Corollary 3.2.** For any \( \eta \in \mathcal{P}^u \) subordinate to unstable manifolds and any \( \epsilon > 0 \),

\[
h_\mu(f, \eta) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_\eta(B^u_n(x, \epsilon)) \quad \mu - a.e. x.
\]

**Proof.** Take \( \xi \in \mathcal{Q}^u, \epsilon > 0 \). Let \( x \in M \) be a generic point. Since \( \eta \) is subordinate to unstable manifolds, there exists \( N > 0 \) such that for any \( n > N \), \( B^u_n(x, \epsilon) \subset \eta(x) \).

Suppose \( \eta(x) \subset \xi(x) \), then

\[
\mu_\xi(B^u_n(x, \epsilon)) = \mu^\eta(B^u_n(x, \epsilon)|\mu_\xi(\eta(x)) = \mu_\xi(B^u_n(x, \epsilon))\mu_\xi(\eta(x)).
\]

Since for \( \mu \)-a.e. \( x \), \( \mu_\xi(\eta(x)) \) is finite, \( \lim_{n \to \infty} -\frac{1}{n} \log \mu_\xi(\eta(x)) = 0 \). So by Theorem A and Lemma 3.1 we get

\[
h_\mu(f, \eta) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_\eta(B^u_n(x, \epsilon)).
\]

Suppose \( \eta(x) \not\subset \xi(x) \). Then for \( \mu \) almost every \( x \), we can take \( k = k(x) > 0 \) such that \( f^{-k}(\eta(x)) \subset \xi(f^{-k}(x)) \). In fact, let \( \Omega_r := \{y \in M : B^u(y, r) \subset \xi(y) \} \). Since \( \xi \)
is subordinate to \(W^u, \mu(\cup_{r>0} \Omega_r) = 1\). So there exists \(r > 0\) such that \(\mu(\Omega_r) > 0\). For \(\mu\text{-a.e. } x\), there exist infinitely many \(k_i = k_i(x)\) such that \(f^{-k_i}x \in \Omega_r\). Let \(k_i > 0\) be large enough such that \(f^{-k_i}(\eta(x)) \subset B^u(f^{-k_i}(x), r) \subset \xi(f^{-k_i}(x))\). Let \(k = k_i\).

Now we have
\[
h_\mu(f|f^{-k}\eta) = \lim_{n \to \infty} \frac{1}{n} \log \mu_{f^{-k}x}^u(B^u_n(f^{-k}(x), \epsilon)) = \lim_{n \to \infty} \frac{1}{n} \log \mu_{f^{-k}x}^u(B^u_{n+k}(f^{-k}(x), \epsilon)),
\]
Since \(f^{-k}\eta \in \mathcal{P}^u\), we have \(h_\mu(f|f^{-k}\eta) = h_\mu(f|\eta)\) by Lemma 2.8. Also, since \(\mu\) is an invariant measure and \(f^k(B^u_{n+k}(f^{-k}(x), \epsilon)) = B^u_n(x, \epsilon)\), it follows that
\[
mu_{f^{-k}x}^u(B^u_{n+k}(f^{-k}(x), \epsilon)) = \mu_\mu^n(B^u_n(x, \epsilon)).
\]
So the result of the corollary follows. \(\square\)

**Lemma 3.3.** Let \(\alpha \in \mathcal{P}, \eta \in \mathcal{P}^u\). Then for any \(\xi \in \mathcal{Q}^u\),
\[
h_\mu(f, \alpha|\eta) \leq \lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) \quad \mu \text{-a.e. } x.
\]

Proof. Let \(\varepsilon > 0\). Take \(k > 0\) such that \(\text{diam } \alpha_0^k \vee \xi \leq \varepsilon\). Hence, for any \(n > 0\),
\[
(\alpha_0^{k+n-1} \vee \xi)(x) = \vee_{i=0}^{n-1} (f^{-i}\alpha_0^k \vee \xi)(x) \subset B^u_n(x, \varepsilon).
\]
By Theorem A and Lemma 3.1,
\[
h_\mu(f, \alpha|\eta) = h_\mu(f, \xi) = \lim_{n \to \infty} \frac{1}{n} \log \mu^u_\xi(B^u_n(x, \epsilon)) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu^u_\xi(\alpha_0^{k+n-1}(x))
\]
for \(\mu\text{-a.e. } x\). \(\square\)

Next, we need to pass the given partition from \(\xi\) to \(\eta\) to obtain the estimates we want.

**Lemma 3.4.** Let \(\alpha \in \mathcal{P}, \eta \in \mathcal{P}^u\) and \(\xi \in \mathcal{Q}^u\). Then for \(\mu\text{-a.e. } x\),
\[
\liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = \liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x),
\]
\[
\limsup_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = \limsup_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x).
\]

Proof. By Lemma 2.2, for \(\mu\text{-a.e. } x\),
\[
I_\mu(\alpha_0^{n-1}|\xi) + I_\mu(\eta|\alpha_0^{n-1} \vee \xi) = I_\mu(\alpha_0^{n-1} \vee \eta|\xi) = I_\mu(\alpha_0^{n-1} \vee \eta|\xi \vee \eta) + I_\mu(\eta|\xi),
\]
(3.3)
\[
I_\mu(\alpha_0^{n-1}|\eta) + I_\mu(\xi|\alpha_0^{n-1} \vee \eta) = I_\mu(\alpha_0^{n-1} \vee \xi|\eta) = I_\mu(\alpha_0^{n-1} \vee \xi|\eta \vee \xi) + I_\mu(\xi|\eta).
\]
Since \(\text{diam } (\alpha_0^{n-1} \vee \xi)(x) \to 0\) and \(\text{diam } (\alpha_0^{n-1} \vee \eta)(x) \to 0\) as \(n \to \infty\) for any \(x\),
\[
\lim_{n \to \infty} \frac{1}{n} I_\mu(\eta|\alpha_0^{n-1} \vee \xi)(x) = 0 = \lim_{n \to \infty} \frac{1}{n} I_\mu(\xi|\alpha_0^{n-1} \vee \eta)(x) \quad \mu \text{-a.e. } x.
\]
Also, since \(I_\mu(\xi|\eta)\) and \(I_\mu(\eta|\xi)\) are finite \(\mu\) almost everywhere,
\[
\lim_{n \to \infty} \frac{1}{n} I_\mu(\xi|\eta)(x) = 0 = \lim_{n \to \infty} \frac{1}{n} I_\mu(\eta|\xi)(x) \quad \mu \text{-a.e. } x.
\]
By (3.3), for \(\mu\) almost every \(x\),
\[
I_\mu(\alpha_0^{n-1}|\xi) + I_\mu(\eta|\alpha_0^{n-1} \vee \xi) + I_\mu(\xi|\eta) = I_\mu(\alpha_0^{n-1}|\eta) + I_\mu(\xi|\alpha_0^{n-1} \vee \eta) + I_\mu(\eta|\xi).
\]
Dividing by \(n\), and taking \(\liminf\) and \(\limsup\), we can get the equalities. \(\square\)
Unstable entropies and variational principle

Proof of Theorem B: the Lower Limits. By Lemma 3.3 and 3.4 we get directly
\[ \liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_{0}^{n-1} | \eta)(x). \]

3.2. A generalized ergodic theorem. The next results can be viewed as generalizations of Birkhoff ergodic theorem. The results and methods of proof can be seen in references for some particular sequence of functions (e.g. [13], proof of Theorem 2.3 on p.261). We state it in a more general setting.

Proposition 3.5. Let \( T : X \to X \) be a transformation preserving an ergodic measure \( \mu \). Suppose \( \{ \phi_n \} \) is a sequence of functions on \( X \) satisfying the following:

(i) \( \lim_{n \to \infty} \phi_n(x) = \phi_0(x) \) \( \mu \)-a.e. \( x \), for some function \( \phi_0 \in L^1(\mu) \);

(ii) \( \phi^* := \sup_n |\phi_n| \in L^1(\mu) \).

Then for \( \mu \)-a.e. \( x \in X \),
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_{n-i}(T^i(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_0(T^i(x)) = \int \phi_0 d\mu, \]
and
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_i(T^i(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_0(T^i(x)) = \int \phi_0 d\mu. \]

Proof. We may assume \( \phi_0 = 0 \), otherwise we can replace \( \phi_n \) by \( \phi_n - \phi_0 \) and take \( \phi^* = \sup_n |\phi_n - \phi_0| \).

Let \( \varepsilon > 0 \) be given. Denote \( \phi_N^* = \sup_{n \geq N} \phi_n \), then \( \phi_N^* \leq \phi^* \) and \( \phi_N^* \to 0 \) \( \mu \)-a.e. as \( N \to \infty \). By Lebesgue’s dominated convergence theorem, we have
\[ \lim_{N \to \infty} \int \phi_N^* d\mu = \int \lim_{N \to \infty} \phi_N^* d\mu = 0. \]
So we can take \( N_0 > 0 \) such that for any \( N \geq N_0 \), \( \int \phi_N^* d\mu < \varepsilon \).

By Birkhoff ergodic theorem there exists \( N_1 = N_1(x, N) > N \) such that for any \( n > N_1 \),
\[ \frac{1}{n} \sum_{i=0}^{n-1} \phi_N^*(T^i x) \leq \frac{1}{n} \sum_{i=0}^{n-1} \phi_N^*(T^i x) - \int \phi_N^* d\mu + \int \phi_N^* d\mu < 2\varepsilon. \]
Hence
\[ \frac{1}{n} \sum_{i=0}^{n-N-1} \phi_{n-i}(T^i x) \leq \frac{1}{n} \sum_{i=0}^{n-N-1} \phi_N^*(T^i x) \leq \frac{1}{n} \sum_{i=0}^{n-1} \phi_N^*(T^i x) < 2\varepsilon. \]

On the other hand, Birkhoff ergodic theorem implies \( \frac{1}{n} \phi^*(T^n y) \to 0 \) \( \mu \)-a.e. \( y \). Hence, there exists \( N_2 = N_2(x, N) > N \) such that for any \( n > N_2 \),
\[ \frac{1}{n} \sum_{i=0}^{n-1} \phi_{n-i}(T^i x) \leq \frac{1}{n} \sum_{i=1}^{N} |\phi_1(T^{n-i} x)| \leq \frac{1}{n} \sum_{i=1}^{N} \phi^*(T^{n-i} x) < \varepsilon. \]
So we get that for any \( n > \max\{N_0, N_1, N_2\}, \frac{1}{n} \sum_{i=0}^{n-1} \phi_{n-i}(T^i x) < 3\varepsilon. \) Hence we obtain the first equality.

The second formula can be proved in a similar way. \( \Box \)
Remark 3.6. If $\phi_n = \phi$ for all $n$ in the above proposition, then it is Birkhoff ergodic theorem.

3.3. Proof of Theorem B: Upper Limits. First we show the result with $\alpha$ replaced by an increasing partition $\xi$, which is easy to get. Then we use Lemma 3.8 below to pass the result from $H_\mu(\xi_0^{-1} | \eta)$ to $H_\mu(\alpha_0^{-1} | \eta)$.

**Lemma 3.7.** For any $\eta \in \mathcal{P}^n$ and $\xi \in \mathcal{Q}^n$,

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(f^{-n}\xi | \eta)(x) = \lim_{n \to \infty} \frac{1}{n} I_\mu(f^{-n}\xi)(x) = h(\mu, \xi) \quad \mu-a.e. \quad x.$$

**Proof.** Applying Lemma 2.6(i) with $\beta = \gamma = \xi$, and then Birkhoff ergodic theorem, we have that for almost every $x$,

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(f^{-n+1}\xi | \xi)(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} I_\mu(\xi | f\xi)(f^i(x)) = H_\mu(\xi | f\xi) = h(\mu, \xi),$$

where we used the fact $\xi^{i-1} = f^{-i+1}\xi$ and $f\xi \vee f^i\xi = f\xi$ for all $i \geq 1$.

With $\beta = \xi, \gamma = \eta$, we use Lemma 2.6(i) again to get

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(f^{-n+1}\xi | \eta)(x) = \lim_{n \to \infty} \frac{1}{n} \left[ I_\mu(\xi | \eta)(x) + \sum_{i=1}^{n-1} I_\mu(\xi | f\xi \vee f^i\eta)(f^i(x)) \right].$$

By Lemma 2.11, for $\mu-a.e. \ x$, there exist $N > 0$ such that for any $i > N$, $I_\mu(\xi | f\xi \vee f^i\eta)(f^i(x)) = I_\mu(\xi | f\xi)(f^i(x))$. Therefore the limit is also equal to $h(f, \xi)$. \qed

**Lemma 3.8.** Let $\alpha \in \mathcal{P}$, $\eta \in \mathcal{P}^n$ and $\xi \in \mathcal{Q}^n$. Then for $\mu$-a.e. $x$,

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{-1} | \xi_0^{-1} \vee \eta)(x) = 0.$$

**Proof.** By Lemma 2.6(ii) with $\gamma = \xi_0^{-1} \vee \eta$,

$$I_\mu(\alpha_0^{-1} | \xi_0^{-1} \vee \eta)(x)$$

\[= I_\mu(\alpha | \xi \vee f^{-1}\eta)(f^{-1}(x)) + \sum_{i=0}^{n-2} I_\mu(\alpha_1 | \alpha_1^{-1} \vee \xi_1^{-1} \vee f^i\eta)(f^i(x)), \tag{3.4}\]

where we use the fact $f^i\xi_0^{-1} = \xi_0^{-1} - i = \xi_1^{-1} - i$ since $\xi$ is increasing.

Take $\phi_1 = I_\mu(\alpha | \xi)(x)$, and $\phi_n(x) = I_\mu(\alpha_1 | \alpha_1^{-1} \vee \xi_1^{-1})(x)$. Since diam$(\alpha_1^{-1} \vee \xi_1^{-1})(x) \to 0$ as $n \to \infty$, $\phi_n \to 0$ as $n \to \infty$ almost everywhere.

Also, by Lemma 2.4, $\phi^* = \sup_{\xi} \phi_n \in L^1(\mu)$. Hence we can apply Lemma 3.5 to get that $\mu$-a.e. $x$,

$$\lim_{n \to \infty} \frac{1}{n} \left[ I_\mu(\alpha_0 | \xi \vee f^{-1}\eta)(f^{-n}(x)) + \sum_{i=0}^{n-2} I_\mu(\alpha_1 | \alpha_1^{-1} \vee \xi_1^{-1} \vee f^i\eta)(f^i(x)) \right] = 0.$$

By Lemma 2.11, for almost every $x$, there is $N > 0$ such that for all $i > N$, $(f\xi \vee f^i\eta)(f^i(x)) = (f\xi)(f^i(x))$. Therefore, the equation is still true if the partition $f\xi$ is replaced by finer ones. So we have that

$$I_\mu(\alpha_0 | \alpha_1^{-1} \vee \xi_1^{-1})(f^i(x)) = I_\mu(\alpha_1 | \alpha_1^{-1} \vee \xi_1^{-1} \vee f^i\eta)(f^i(x))$$

for all large $i$. Therefore, we can get

$$\lim_{n \to \infty} \frac{1}{n} \left[ I_\mu(\alpha_0 | \xi \vee f^{-n}\eta)(f^{-n}(x)) + \sum_{i=0}^{n-2} I_\mu(\alpha_1 | \alpha_1^{-1} \vee \xi_1^{-1} \vee f^i\eta)(f^i(x)) \right] = 0.$$
By (3.4) we get the result of the lemma.

**Proof of Theorem B: Upper Limits.** By Lemma 2.2

\[ I_\mu(\alpha_0^{n-1}|\eta)(x) \leq I_\mu(\alpha_0^{n-1} \vee \xi_0^{n-1}|\eta)(x) = I_\mu(\xi_0^{n-1}|\eta)(x) + I_\mu(\alpha_0^{n-1}|\xi_0^{n-1} \vee \eta)(x). \]

Hence, by Lemma 3.8, Lemma 3.7 and Theorem A,

\[ \limsup_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) \leq \limsup_{n \to \infty} \frac{1}{n} I_\mu(\xi_0^{n-1}|\eta)(x) = h_\mu(f, \xi) = h_\mu(f, \alpha|\eta). \]

We get the same bound for the upper limit.

**Proof of Corollary B.1.** Theorem B implies that \( \lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) \) exists. So Lemma 3.4 gives

\[ \lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) = \lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) \quad \mu\text{-a.e. } x. \]

By Theorem B we have

\[ h_\mu(f, \alpha|\eta) = \int \limsup_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta) d\mu. \]

Hence by Fatou’s lemma, (3.5) and (3.6),

\[ h_\mu(f, \alpha|\xi) \geq \lim \inf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi) \geq \int \lim \inf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi) d\mu \]

\[ = \int \lim \inf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta) d\mu = h_\mu(f, \alpha|\eta). \]

On the other hand, since

\[ H_\mu(\alpha_0^{n-1}|\xi) \leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\eta|\xi), \]

and \( H_\mu(\eta|\xi) < \infty \), we have

\[ h_\mu(f, \alpha|\xi) = \lim \sup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi) \leq \lim \sup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = h_\mu(f, \alpha|\eta). \]

Together with (3.7), we obtain \( h_\mu(f, \alpha|\xi) = h_\mu(f, \alpha|\eta) \). Now the conclusion of the corollary follows from Theorem B and (3.5).

4. **Unstable topological entropy**

4.1. **Definition using spanning sets.** Recall that unstable topological entropy is defined in Definition 1.4 using \((n, \epsilon)\) u-separated sets. We can also define unstable topological entropy by using \((n, \epsilon)\) u-spanning sets as follows. A set \( E \subset W^u(x) \) is called an \((n, \epsilon)\) u-spanning set of \( W^u(x, \delta) \) if \( W^u(x, \delta) \subset \bigcup_{y \in E} B_n^u(y, \epsilon) \), where \( B_n^u(y, \epsilon) = \{ z \in W^u(x) : d_n^u(y, z) \leq \epsilon \} \) is the \((n, \epsilon)\) u-Bowen ball around \( y \). Let \( S^u(f, \epsilon, n, x, \delta) \) be the cardinality of a minimal \((n, \epsilon)\) u-spanning set of \( W^u(x, \delta) \). It is standard to verify that

\[ N^u(f, 2\epsilon, n, x, \delta) \leq S^u(f, \epsilon, n, x, \delta) \leq N^u(f, \epsilon, n, x, \delta). \]

So in Definition 1.4 we can also use

\[ h^u_{\text{top}}(f, W^u(x, \delta)) = \lim \sup_{\epsilon \to 0} \frac{1}{n} \log S^u(f, \epsilon, n, x, \delta). \]

The following lemma tells that in the definition we do not have to let \( \delta \to 0 \).

**Lemma 4.1.** \( h^u_{\text{top}}(f) = \sup_{x \in M} h^u_{\text{top}}(f, W^u(x, \delta)) \) for any \( \delta > 0 \).
Proof. It is easy to see that \( h_{\text{top}}^u(f) \leq \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \) for any \( \delta > 0 \) since \( \delta \mapsto \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \) is increasing.

Let us prove the other direction for some fixed \( \delta > 0 \). For any \( \rho > 0 \), let \( y \in M \) be such that
\[
(4.1) \quad \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \leq h_{\text{top}}^u(f, W^u(y, \delta)) + \frac{\rho}{3}.
\]
We can choose \( \epsilon_0 > 0 \) such that
\[
(4.2) \quad h_{\text{top}}^u(f, W^u(y, \delta)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^n(f, \epsilon, n, y, \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log S^n(f, \epsilon_0, n, y, \delta) + \frac{\rho}{3}.
\]
Choose \( \delta_1 > 0 \) small enough such that \( \delta_1 < \delta \) and
\[
(4.3) \quad h_{\text{top}}^u(f) \geq \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta_1)) - \frac{\rho}{3}.
\]
Then there exist \( y_j \in W^u(y, \delta), 1 \leq j \leq N \) where \( N \) only depends on \( \delta, \delta_1 \), and the Riemannian structure on \( W^u(y, \delta) \), such that
\[
W^u(y, \delta) \subset \bigcup_{j=1}^N W^u(y_j, \delta_1).
\]
It follows that
\[
\limsup_{n \to \infty} \frac{1}{n} \log S^n(f, \epsilon_0, n, y, \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{j=1}^N S^n(f, \epsilon_0, n, y_j, \delta_1) \right)
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log \left( n S^n(f, \epsilon_0, n, y_j, \delta_1) \right) = \limsup_{n \to \infty} \frac{1}{n} \log S^n(f, \epsilon_0, n, y_j, \delta_1) = h_{\text{top}}^u(f, W^u(y_j, \delta_1))
\]
for some \( 1 \leq i \leq N \). Combining (4.1), (4.2), (4.4) and (4.3),
\[
\sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \leq h_{\text{top}}^u(f, W^u(y, \delta)) + \frac{\rho}{3}
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log S^n(f, \epsilon_0, n, y, \delta) + \frac{2\rho}{3} \leq h_{\text{top}}^u(f, W^u(y, \delta_1)) + \frac{2\rho}{3}
\]
\[
\leq \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta_1)) + \frac{2\rho}{3} \leq h_{\text{top}}^u(f) + \rho.
\]
Since \( \rho > 0 \) is arbitrary, one has \( \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \leq h_{\text{top}}^u(f) \). \( \square \)

4.2. Definition using open covers. We proceed to define the unstable topological entropy by using open covers. Let \( C_M \) denote the set of open covers of \( M \). For any \( K \subset M \), set \( N(K) := \min \{ \text{the cardinality of } V : V \subset K \} \) and \( H(K) := \log N(K) \).

Definition 4.2. We define
\[
\bar{h}_{\text{top}}^u(f) = \lim_{\delta \to 0} \sup_{x \in M} \bar{h}_{\text{top}}^u(f, W^u(x, \delta)),
\]
where
\[ \tilde{h}_\text{top}^u(f, \overline{W^u(x, \delta)}) = \sup_{U \in C^o_M} \limsup_{n \to \infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}\overline{W^u(x, \delta)}). \]

**Remark 4.3.** It is easy to see that \( H(\mathcal{U}\overline{W^u(x, \delta)}) = H(f^{-1}\mathcal{U}(\overline{W^u(x, \delta)})) \). But we don’t know whether the sequence \( H(\mathcal{U}_0^{n-1}\overline{W^u(x, \delta)}) \) is subadditive or not, and so we use \( \limsup \) in the definition above. That is the main difference from the case for classical topological entropy.

Now we verify that the two definitions in Definition 1.4 and 4.2 for unstable topological entropy coincide.

**Lemma 4.4.** Let \( \delta > 0 \) be small enough. Then there exists a constant \( C > 1 \) such that for any \( \epsilon > 0 \) small enough, any \( \mathcal{U}_\epsilon \in C^o_M \) with Lebesgue number \( 2\epsilon \), and any \( \mathcal{V}_\epsilon \in C^o_M \) with \( \text{diam}(\mathcal{V}_\epsilon) \leq \frac{\epsilon}{C} \),
\[ N((\mathcal{U}_\epsilon)^{n-1}\overline{W^u(x, \delta)}) \leq S^u(f, \epsilon, n, x, \delta) \leq N^u(f, \epsilon, n, x, \delta) \leq N((\mathcal{V}_\epsilon)^{n-1}\overline{W^u(x, \delta)}). \]

**Proof.** Observe that for \( \delta > 0 \) small enough, there exists \( C > 1 \) such that for any \( x \in M \),
\[ d(y, z) \leq d^n(y, z) \leq Cd(y, z) \quad \text{for any } y, z \in \overline{W^u(x, \delta)}. \]

Then the lemma follows by a similar argument as in the proof of Theorem 7.7 in [17].

**Corollary 4.5.** \( \tilde{h}_\text{top}^u(f, \overline{W^u(x, \delta)}) = h^u_\text{top}(f, \overline{W^u(x, \delta)}) \). As a consequence,
\[ \tilde{h}_\text{top}^u(f) = h^u_\text{top}(f). \]

### 4.3. Proof of Theorem C: relation to unstable volume growth.
In this subsection, we prove Theorem C, which states that the unstable topological entropy actually coincides with the unstable volume growth defined in [4]. The notation \( \chi_u(f) \) for unstable volume growth is used in [4].

**Proof of Theorem C.** Choose a small \( \delta > 0 \). By the definition of \( \chi_u(f) \), for any given \( \rho > 0 \), there exists a point \( x \) such that
\[ \chi_u(x, \delta) \geq \chi_u(f) - \rho. \]

For \( \varepsilon > 0 \), let \( E \) be a minimal \((n, \varepsilon)\) u-spanning set of \( \overline{W^u(x, \delta)} \), then \( f^n(E) \) is an \( \varepsilon \)-spanning set of \( f^n(\overline{W^u(x, \delta)}) \). Thus \( f^n(\overline{W^u(x, \delta)}) \subset \bigcup_{y \in f^n(E)} W^u(y, \varepsilon) \). The volume of any \( \varepsilon \) u-ball is bounded from above by \( c_1 \varepsilon^{k} \), where \( c_1 > 0 \) is a constant depending on the Riemannian metric and \( k \) is the dimension of the unstable manifolds. Then the total volume covered by the \( \varepsilon \) u-balls around the points in \( f^n(E) \) is less than \( c_1 \varepsilon^k S^u(f, \epsilon, n, x, \delta) \). Therefore
\[ \text{Vol}(f^n(\overline{W^u(x, \delta)})) \leq c_1 \varepsilon^k S^u(f, \epsilon, n, x, \delta). \]

We get
\[ h^u_\text{top}(f, \overline{W^u(x, \delta)}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^u(f, \epsilon, n, x, \delta) \]
\[ \geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(\text{Vol}(f^n(\overline{W^u(x, \delta)}))/(c_1 \varepsilon^k)) \]
\[ = \chi_u(x, \delta) \geq \chi_u(f) - \rho. \]
Since $\rho > 0$ is arbitrary,
\[ h_{\text{top}}^u(f) = \sup_{x \in M} h_{\text{top}}^u(f, W^u(x, \delta)) \geq \chi_u(f). \]

On the other hand, for any given $\rho > 0$, by the definition of $h_{\text{top}}^u(f)$, there exist a point $x$ and $0 < \varepsilon_0 < \delta$ such that
\[ \limsup_{n \to \infty} \frac{1}{n} N(f, \varepsilon_0, n, x, \delta) > h_{\text{top}}^u(f) - \rho. \]

Let $F \subset W^u(x, \delta)$ be an $(n, \varepsilon_0)$ separated set, then $f^n(F)$ is $\varepsilon_0$ separated. The volume of any $\varepsilon_0/2$ u-ball can be bounded from below by $c_2 \varepsilon_0^k$, where $c_2 > 0$ is a constant depending on the Riemannian metric. Since the $\varepsilon_0/2$ u-balls around the points in $f^n(F)$ are disjoint subsets of $f^n(W^u(x, \delta + \varepsilon_0))$, we get
\[ \text{Vol}(f^n(W^u(x, \delta + \varepsilon_0))) \geq c_2 \varepsilon_0^k N(f, \varepsilon_0, n, x, \delta). \]

Therefore,
\[ \chi_u(f) \geq \chi_u(x, \delta + \varepsilon_0) = \limsup_{n \to \infty} \frac{1}{n} \log \text{Vol}(f^n(W^u(x, \delta + \varepsilon_0))) \geq \limsup_{n \to \infty} \frac{1}{n} \log (c_2 \varepsilon_0^k N(f, \varepsilon_0, n, x, \delta)) > h_{\text{top}}^u(f) - \rho. \]

Since $\rho > 0$ is arbitrary,
\[ \chi_u(f) \geq h_{\text{top}}^u(f). \]

This completes the proof. \qed

**Proof of Corollary C.1.** The inequality $h_{\text{top}}^u(f) \leq h_{\text{top}}(f)$ follows from the definition directly. If there is no positive Lyapunov exponents in the center direction, then by (1.1) and Theorem C, we also have $h_{\text{top}}(f) \leq \chi^u(f) = h_{\text{top}}^u(f)$. \qed

**Proof of Corollary C.2.** Clearly for any invariant measure $\mu$, $\sum \lambda^\mu_i > 0$ the variational principle for entropy.

Then we use the variational principle for entropy. \qed

**Proof of Corollary C.3.** For any $x \in M$, $\delta > 0$, denote
\[ h_{\text{top}}(f, B(x, \delta)) = \lim_{\varepsilon \to 0} \sup_{n \to \infty} \frac{1}{n} \log N(f, \varepsilon, n, x, \delta), \]
where $N(f, \varepsilon, n, x, \delta)$ denote the maximal number of points in $B(x, \delta)$ with pairwise $d_n$-distances at least $\varepsilon$. By using a finite cover argument we know that for any $\delta > 0$, there is an $x \in M$ such that
\[ h_{\text{top}}(f, B(x, \delta)) = \lim_{\varepsilon \to 0} \sup_{n \to \infty} \frac{1}{n} \log N(f, \varepsilon, n, M), \]
where $N(f, \varepsilon, n, M)$ denote the maximal number of points in $M$ with pairwise $d_n$-distances at least $\varepsilon$. Hence we can get that
\[ h_{\text{top}}(f) = \lim_{\delta \to 0} \sup_{x \in M} h_{\text{top}}(f, B(x, \delta)). \]
By the definition of unstable and transversal topological entropies, Definition 1.4 and Definition 1.5, we have
\[ h_{\text{top}}(f, B(x, \delta)) \leq h_{\text{top}}^u(f, B(x, \delta)) + h_{\text{top}}^t(f, B(x, \delta)). \]
Then taking supremum over \( x \in M \) and letting \( \delta \) go to 0, we get the inequality. □

5. The variational principle

In this section, we prove Theorem D, the variational principle for unstable entropies \( h_{\text{top}}^u(f) \) and \( h_{\text{top}}^u(f) \).

At first, we prove one direction of the variational principle as follows.

**Proposition 5.1.** Let \( \mu \) be any \( f \)-invariant probability measure. Then
\[ h^u_{\mu}(f) \leq h_{\text{top}}^u(f). \]

**Proof.** Let \( \mu = \int_{\mathcal{M}^u_f(M)} \nu d\tau(\nu) \) be the unique ergodic decomposition where \( \tau \) is a probability measure on the Borel subsets of \( \mathcal{M}_f(M) \) and \( \tau(\mathcal{M}^u_f(M)) = 1 \). Since \( \mu \mapsto h_{\mu}^u(f) \) is affine and upper semi-continuous by Propositions 2.14 and 2.15, then
\[
h^u_{\mu}(f) = \int_{\mathcal{M}^u_f(M)} h^u_{\mu}(f) d\tau(\nu)
\]
by a classical result in convex analysis (cf. Fact A.2.10 on p. 356 in [1]). Therefore, we only need to prove the proposition for ergodic measures.

Suppose \( \mu \) is ergodic. Let \( \rho > 0 \). Take \( \eta \in \mathcal{P}^u \) subordinate to unstable manifolds, and take \( \varepsilon > 0 \). By Corollary 3.2, we have
\[
\lim_{n \to \infty} -\frac{1}{n} \log \mu^u_y(B^u_n(y, \varepsilon)) \geq h^u_{\mu}(f\eta) \quad \mu\text{-a.e. } y.
\]
Hence for \( \mu\text{-a.e. } y \), there exists \( N(y) = N(y, \varepsilon) > 0 \) such that if \( n \geq N(y) \), then
\[
\mu^u_y(B^u_n(y, \varepsilon)) \leq e^{-n(h^u_{\mu}(f\eta) - \rho)}.
\]
Denote \( E_n = E_n(\varepsilon) = \{y \in M : N(y) = N(y, \varepsilon) \leq n\} \). Then \( \mu(\bigcup_{n=1}^{\infty} E_n) = 1 \) by the corollary. So there exists \( n > 0 \) large enough such that \( \mu(E_n) > 1 - \rho \). Hence, there exists \( x \in M \) such that \( \mu^u_x(E_n) = \mu^u_x(E_n \cap \eta(x)) > 1 - \rho \). Fix such \( n \) and \( x \). Note that if \( y \in \eta(x) \), then \( \mu^u_y = \mu^u_x \). We have
\[
\mu^u_x(B^u_n(y, \varepsilon)) \leq e^{-n(h^u_{\mu}(f\eta) - \rho)} \quad \forall y \in E_n \cap \eta(x).
\]
Let \( S^u(f, \varepsilon, n, \eta(x)) \) be the cardinality of a minimal \( (n, \varepsilon) \) u-spanning set of \( \eta(x) \). Then there exists a set \( S \subset \eta(x) \) with cardinality no more than \( S^u(f, \varepsilon/2, n, \eta(x)) \) such that \( \eta(x) \cap E_n \subset \bigcup_{z \in S} B^u_n(z, \varepsilon/2) \).
and \( B^u_n(z, \varepsilon/2) \cap E_n \neq \emptyset \). Let \( y(z) \) be an arbitrary point in \( B^u_n(z, \varepsilon/2) \cap E_n \). We have
\[
1 - \rho < \mu^u_x(\eta(x) \cap E_n) \leq \mu^u_x(\bigcup_{z \in S} B^u_n(z, \varepsilon/2)) \leq \sum_{z \in S} \mu^u_x(B^u_n(z, \varepsilon/2)) \leq \sum_{z \in S} \mu^u_x(B^u_n(y(z), \varepsilon)) \leq S^u(f, \varepsilon/2, n, \eta(x))e^{-n(h^u_{\mu}(f\eta) - \rho)}.
\]
Hence \( S^u(f, \varepsilon/2, n, \eta(x)) \geq (1 - \rho)e^{n(h^u_{\mu}(f\eta) - \rho)} \).
Now we take $\delta > 0$ such that with $W^u(x, \delta) \supset \eta(x)$. Recall that $S^n(f, \varepsilon, n, x, \delta)$ denotes the cardinality of a minimal $(n, \varepsilon)$ u-spanning set of $W^u(x, \delta)$. Clearly $S^n(f, \varepsilon, n, x, \delta) \geq S^n(f, \varepsilon, n, \eta(x))$. So we get $S^n(f, \varepsilon/2, n, x, \delta) \geq (1 - \rho)e^{n(h_{\mu}(f|\eta) - \rho)}$. It follows that
\[
h^u_{\text{top}}(f, W^u(x, \delta)) \geq h_{\mu}(f|\eta) - \rho.
\]
Then by definition,
\[
h^u_{\text{top}}(f) \geq h^u_{\text{top}}(f, W^u(x, \delta)) \geq h_{\mu}(f|\eta) - \rho = h^u_{\mu}(f) - \rho.
\]
Since $\rho$ is arbitrary, we have
\[
h^u_{\text{top}}(f) \geq h^u_{\mu}(f).
\]

Next we use the ideas in Misiurewicz’s proof of the classical variational principle ([10]) to prove Theorem D.

**Proof of Theorem D.** By Proposition 5.1, it is enough to prove that for any $\rho > 0$, there exists $\mu \in \mathcal{M}_f(M)$ such that $h^u_{\mu}(f) \geq h^u_{\text{top}}(f) - \rho$.

For some $\delta > 0$ small enough, we can find a point $x \in M$ such that
\[
h^u_{\text{top}}(f, W^u(x, \delta)) \geq h^u_{\text{top}}(f) - \rho.
\]
Take $\varepsilon > 0$ small enough. Let $S_n$ be an $(n, \varepsilon)$ u-separated set of $W^u(x, \delta)$ with cardinality $N^n(f, \varepsilon, n, x, \delta)$. Define
\[
\nu_n := \frac{1}{N^n(f, \varepsilon, n, x, \delta)} \sum_{y \in S_n} \delta_y,
\]
and
\[
\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} f^i \nu_n.
\]
Since the set $\mathcal{M}(M)$ of all probability measures of $M$ is a compact space with weak* topology, there exists a subsequence $\{n_k\}$ of natural numbers such that $\lim_{k \to \infty} \mu_{n_k} = \mu$. Obviously $\mu \in \mathcal{M}_f(M)$.

As $\delta$ is very small, we can choose a partition $\eta \in \mathcal{P}^u$ such that $W^u(x, \delta) \subset \eta(x)$. That is, $W^u(x, \delta)$ is contained in a single element of $\eta$. Then choose $\alpha \in \mathcal{P}$ such that $\mu(\partial \alpha) = 0$, and $\text{diam}(\alpha) < \frac{\varepsilon}{C}$ where $C > 1$ is as in the proof of Lemma 4.4. Hence we have $\log N^n(f, \varepsilon, n, x, \delta) = H_{\nu_n}(\alpha^{-1}_0 | \eta)$.

Fix a natural numbers $q > 1$. For any natural number $n > q$, $j = 0, 1, \ldots, q - 1$, put $a(j) = \lceil \frac{n - j}{q} \rceil$, where $[a]$ denotes the integer part of $a > 0$. Then
\[
\bigvee_{i=0}^{n-1} f^{-i} \alpha = \bigvee_{r=0}^{a(j)-1} f^{-r(q+j)} \alpha_0^{q-1} \vee \bigvee_{t \in S_j} f^{-t} \alpha,
\]
where $S_j = \{0, 1, \ldots, j - 1\} \cup \{j + qa(j), \ldots, n - 1\}$.

For a partition $\alpha \in \mathcal{P}$, denote by $\alpha^u$ the partition in $\mathcal{P}^u$ whose elements are given by $\alpha^u(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$. Note that
\[
f^r \eta \left( \bigvee_{i=0}^{r-1} f^{-i} \alpha_0^{q-1} \vee f^j \eta \right) = f^r \left( f^{r(q-1)} \alpha \vee f^j \eta \right) = f\alpha \vee \cdots \vee f^{r(q-1)} \alpha \vee f^{r(q+j)} \eta \geq f\alpha^u.
\]
Note also that the same arguments as for Lemma 2.6(i) can be applied for any probability measure \( \nu \) that is not necessary invariant. We can get that

\[
H_\nu \left( \sum_{r=0}^{\alpha(j)-1} f^{-r} q \alpha_0^{q-1} | f^j \eta \right)
\]

(5.2) \[= H_\nu (\alpha_0^{q-1} | f^j \eta) + \sum_{r=1}^{\alpha(j)-1} H_{f^r \nu} (\alpha_0^{q-1} | f^{-r} q \alpha_0^{q-1} \lor f^j \eta)\]

\[\leq H_\nu (\alpha_0^{q-1} | f^j \eta) + \sum_{r=1}^{\alpha(j)-1} H_{f^r \nu} (\alpha_0^{q-1} | f^j \eta).\]

Also,

\[
H_\nu \left( \sum_{r=0}^{\alpha(j)-1} f^{-(r+1)} q \alpha_0^{q-1} | \eta \right) = H_{f^j \nu} \left( \sum_{r=0}^{\alpha(j)-1} f^{-r} q \alpha_0^{q-1} | f^j \eta \right).
\]

Replacing \( \nu \) by \( \nu_n \) and \( f^j \nu_n \) in (5.3) and (5.2) respectively we get

\[
\log N^u(f, \varepsilon, n, x, \delta) = H_{\nu_n} (\alpha_0^{n-1} | \eta) = H_{\nu_n} \left( \sum_{r=0}^{\alpha(j)-1} f^{-r} q \alpha_0^{q-1} \lor \sum_{t \in S_j} f^{-t} \alpha | \eta \right)
\]

\[\leq \sum_{t \in S_j} H_{\nu_n} (f^{-t} \alpha | \eta) + H_{\nu_n} \left( \sum_{r=0}^{\alpha(j)-1} f^{-r} q \alpha_0^{q-1} | \eta \right)
\]

\[\leq \sum_{t \in S_j} H_{\nu_n} (f^{-t} \alpha | \eta) + H_{f^j \nu_n} \left( \sum_{r=0}^{\alpha(j)-1} f^{-r} q \alpha_0^{q-1} | f^j \eta \right)
\]

\[\leq \sum_{t \in S_j} H_{\nu_n} (f^{-t} \alpha | \eta) + H_{f^j \nu_n} (\alpha_0^{q-1} | f^j \eta) + n \sum_{r=1}^{\alpha(j)-1} H_{f^{r+j} \nu_n} (\alpha_0^{q-1} | f^j \eta).
\]

It is clear that \( \text{card} S_j \leq 2q \). Denote by \( d \) the number of elements of \( \alpha \). Summing the inequalities over \( f \) from 0 to \( q-1 \) and dividing by \( n \), by Proposition 2.14 we get

(5.4) \[\frac{d}{n} \log N^u(f, \varepsilon, n, x, \delta)
\]

\[\leq \frac{1}{n} \sum_{j=0}^{q-1} H_{\nu_n} \left( f^{-t} \alpha | \eta \right) + \frac{1}{n} \sum_{j=0}^{q-1} H_{f^j \nu_n} (\alpha_0^{q-1} | f^j \eta) + \frac{1}{n} \sum_{j=0}^{n-1} H_{f^j \nu_n} (\alpha_0^{q-1} | f^j \eta) + \frac{a(j)-1}{n} \sum_{r=1}^{\alpha(j)-1} H_{f^{r+j} \nu_n} (\alpha_0^{q-1} | f^j \eta).
\]

Let \( \{n_k\} \) be a sequence of natural numbers such that

1. \( \mu_{n_k} \to \mu \) as \( k \to \infty \); 

2. \( \lim_{k \to \infty} \frac{1}{n_k} \log N^u(f, \varepsilon, n_k, x, \delta) = \limsup \frac{1}{n} \log N^u(f, \varepsilon, n, x, \delta) \).

Since \( \mu (\partial \alpha) = 0 \), and \( \mu \) is invariant, \( \mu (\partial \alpha_0^{q-1}) = 0 \) for any \( q \in \mathbb{N} \). Hence by Proposition 2.15,

\[\limsup_{k \to \infty} H_{\mu_{n_k}} (\alpha_0^{q-1} | f^j \eta) \leq H_{\mu} (\alpha_0^{q-1} | f^j \eta).
\]
Thus replacing $n$ by $n_k$ in (5.4) and letting $k \to \infty$, we get
\[ qh_{\text{top}}^u(f, W^u(x, \delta)) \leq H_\mu(\alpha_0^{q-1}|f \alpha^u). \]

By Corollary A.2,
\[ h_{\text{top}}^u(f, W^u(x, \delta)) \leq \lim_{q \to \infty} H_\mu(\alpha_0^{q-1}|f \alpha^u) = h_\mu(f, \alpha|f \alpha^u). \]

We may choose $\alpha \in P$ such that $f \alpha^u \in P^u$. By Theorem A, $h_\mu(f, \alpha|f \alpha^u) = h_\mu(f/f \alpha^u) = h_\mu(f)$. Thus $h_\mu(f) \geq h_{\text{top}}^u(f) - \rho$. Since $\rho$ is arbitrary, we get the first equation of the theorem.

We prove the second equation in the theorem.

Let $\rho > 0$ be sufficiently small. Then there exists an invariant measure $\mu$ such that $h_\mu(f) > h_{\text{top}}^u(f) - \rho/2$. By (5.1), there exists an ergodic measure $\nu$ such that $h_\nu(f) > h_\mu(f) - \rho/2 > h_{\text{top}}^u(f) - \rho$.

Since $\rho$ is arbitrary, we have $h_{\text{top}}^u(f) = \sup \{h_\nu(f) : \nu \in \mathcal{M}_\mu(M)\}$. \qed

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