Abstract

In this article we present a comprehensive account of the operator formulation of the Green-Schwarz superstring in the semi-light-cone (SLC) gauge, where the world-sheet conformal invariance is preserved. Starting from the basic action, we systematically study the symmetry structure of the theory in the SLC gauge both in the Lagrangian and the phase space formulations. After quantizing the theory in the latter formulation we construct the quantum Virasoro and the super-Poincaré generators and clarify the closure properties of these symmetry algebras. Then by making full use of this knowledge we will be able to construct the BRST-invariant vertex operators which describe the emission and the absorption of the massless quanta and show that they form the appropriate representation of the quantum symmetry algebras. Furthermore, we will construct an exact quantum similarity transformation which connects the SLC gauge and the familiar light-cone (LC) gauge. As an application BRST-invariant DDF operators in the SLC gauge are obtained starting from the corresponding physical oscillators in the LC gauge.
§1. Introduction

With the advent of the idea of the D-branes\(^1\) and the subsequent discovery of the AdS/CFT correspondence,\(^2\)–\(^4\) the choice of the worldsheet formalism of the superstring theory underwent a notable change. The Lorentz-covariant Ramond-Neveu-Schwarz (RNS) formalism,\(^5,\)\(^6\) which had been dominating over the alternative Green-Schwarz (GS) formalism,\(^7,\)\(^8\) was to be used less frequently. The reason is that in the RNS formalism the spacetime spinors are described by the composite spin fields which are not easy to handle and hence this formalism is not so suitable for the description of the Ramond-Ramond (RR) bispinor fields characteristically produced by the D-branes. It was thus inevitable that the first formulation of the superstring in the \(AdS_5 \times S^5\) background with the RR flux\(^9\) was made in the GS formalism, where the target space spinor fields are among the basic variables. Despite its non-covariance at the quantum level, the GS formalism regained its raisons d’être.

It was not long before another scheme containing fundamental spinor variables, called pure spinor (PS) formalism, was invented.\(^10\) The great advantage of this formalism is that, although rigorously speaking the super-Poincaré covariance is broken by the underlying quantization procedure, the rules for the computations of the amplitudes can be made completely covariant. Moreover, the rules for the multiloop amplitudes look quite similar to those of the bosonic or topological string.\(^11\) Consequently they are formally much simpler than those of the RNS formalism and with judicious regularization procedure a number of powerful results have been obtained.\(^12,\)\(^13\) Also since the target space spinor variables are built in, as in the GS formalism, it is suitable for the description of a superstring in curved spacetimes with RR flux relevant to AdS/CFT. In fact the action in the \(AdS_5 \times S^5\) background was written down already in the original work\(^10\) that introduced this formalism.

Although it has many advantages as sketched above, the PS formalism is not entirely without shortcomings. One feature is that the structure of the worldsheet conformal symmetry is not explicitly seen. The BRST operator does not contain the energy-momentum tensor and this presents a difficulty in constructing the string field theory based on this formalism. Correspondingly, the construction of the “\(b\)-ghost” is quite complex. Another problem is that due to the presence of the quadratic pure spinor constraints, the hermiticity property of the pure spinor variables is peculiar.\(^14\) This leads to the difficulty in constructing the D-brane boundary states together with their conjugates. In the context of the study of the AdS/CFT correspondence, PS formalism so far has not been particularly useful in actually solving the quantum dynamics in the relevant curved background. For instance, the spectrum of the superstring in the plane-wave background, which was obtained exactly in the
light-cone gauge GS formalism \cite{15} has not been reproduced in this formalism. PS formalism needs to be further developed for such purposes.

Let us now go back to the GS formalism and assess some of its features. In the past, the study and the use of the GS formalism have been made overwhelmingly in the light-cone (LC) gauge, where half of the light-cone components of the fermions are set to zero and the bosonic light-cone variable $X^+$ is identified with the worldsheet time. These conditions make the Lorentz invariance non-manifest and moreover break the conformal invariance. With the lack of these symmetries the computations of the amplitudes become less organized and cumbersome. This is certainly a big disadvantage of this formalism. On the other hand, GS formalism in the LC gauge deals directly with the physical degrees of freedom and is powerful in studying the physical property of the system, such as the spectrum. In this regard, we have already mentioned the celebrated exact solution of the spectrum for a string in the plane-wave background using the LC gauge \cite{15}, which played a crucial role in the understanding of the AdS/CFT correspondence in this background \cite{16}.

It should now be mentioned that the LC gauge is not the only gauge in which the GS string can be quantized. Although there is no way to make the Lorentz symmetry manifest in the quantum GS formalism, there exists a more symmetric gauge in which the conformal invariance can be retained. This is the so-called semi-light-cone (SLC) gauge \cite{17,18,19} where only the fermionic gauge conditions are imposed to fix the local $\kappa$-symmetry. As for the worldsheet reparametrization symmetry, the usual conformal gauge condition is adopted so that the Virasoro symmetry still remains and can be treated by the BRST formalism.

One of the main issues of the SLC gauge formalism when it was introduced was whether the theory suffers from conformal and related anomalies. Although early studies \cite{17,18,19,20,21,22} claimed that there is no anomaly, a subsequent work \cite{23} revealed the existence of the 10-dimensional Lorentz anomaly. More complete study was made in Ref. 24), 25), which confirmed the result of Ref. 23) as well as pointed out that the conformal and the related anomalies can be cancelled by adding appropriate counter terms to the action and the transformation rules.

All such studies were made in the path-integral formalism. The first operator formulation in the SLC gauge was attempted in Ref. 26). In this work, to avoid dealing with the second class constraints, the Batalin-Fradkin formalism \cite{27,28} was adopted, which makes use of graded brackets in the extended phase space with new fields. The BRST and the super-Poincaré operators were constructed with quantum modifications but their structures were quite complicated. Much more recently, a simpler BRST formalism for the GS superstring in the SLC gauge was introduced in Ref. 29) for the purpose of showing the equivalence of the PS and GS formalisms and it was subsequently utilized in some related works \cite{30,31,32}. 

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Although the work of Ref 29) contained a number of important ideas, it was not intended for a systematic development of the operator formalism for the GS superstring in the SLC gauge.

A brief sketch of the development of the SLC gauge formulation given above reveals that despite its long history surprisingly little has been known about its fundamental structures: Among other things, quantum symmetry structure has not been fully clarified and no vertex operators have yet been constructed. The primary purpose of the present work is to fill this gap and lay the foundation of the GS superstring in this important and unique gauge.

In particular, we will systematically develop the operator formalism, which is best suited for studying the quantum symmetry structure of the theory, starting from the basic action of the GS superstring. We will construct the quantum Virasoro and super-Poincaré generators and clarify the structure of their algebras in full detail for the first time. This knowledge in turn is indispensable for the construction of the BRST-invariant vertex operators. We will demonstrate this by constructing the vertex operators for the massless states of the open superstring in completely explicit manner. The way all the quantum symmetry algebras are realized in the space of these vertex operators is quite intricate and non-trivial.

Another new result achieved in this work is the construction of an exact quantum similarity transformation that connects the SLC gauge and the LC gauge. Among many expected applications of this mapping, in this article we will use it to construct all the so-called DDF operators,\textsuperscript{33} which generate the BRST-invariant states in the SLC gauge, from the simple physical oscillators in the LC gauge.

As we wished to clarify the fundamental structures of the theory fully in a self-contained manner, this article has become rather long. Therefore we will now give the outline of our work in some detail so that the reader can grasp the scope of the manuscript.

We begin, in section 2, by reviewing the action and its symmetries of the type II Green-Schwarz superstring in the flat 9 + 1 dimensional spacetime.

Then in section 3 we describe the gauge-fixing procedure for the local symmetries. To keep the worldsheet conformal invariance, the reparametrization symmetry is fixed by imposing the conformal gauge condition. As this condition is not invariant under the original local $\kappa$-symmetry transformation, one must redefine the $\kappa$-transformation by adding a judicious compensating reparametrization transformation. Subsequently, we will fix this modified $\kappa$-symmetry by imposing the semi-light-cone (SLC) gauge condition. This procedure in turn breaks the global super-Poincaré invariance and we must add appropriate compensating $\kappa$-transformations to modify the super-Poincaré transformations in order to stay in the SLC gauge. Finally we check that after all this process the conformal symmetry is still preserved.

In section 4, we develop the phase space formulation and quantize the theory. We will
compare it to the canonical quantization approach and emphasize several advantages of the phase space formulation. One practical point is that in this formulation the compensating transformations, which are often complicated, are automatically taken care of by the use of the Dirac brackets. Another feature is that the phase space formulation can be useful for quantizing a non-linear system for which the complete classical solutions are not available.

Having completed all the necessary groundwork, we will begin the detailed study of the quantum operator formulation of the GS superstring in SLC gauge. In section 5, we will first clarify the structure of the quantum symmetry algebras. Besides being important in its own right, this will be indispensable for the construction of the vertex operators. The characteristic feature of the SLC gauge is that in contrast to the full light-cone gauge the conformal symmetry is retained. The corresponding Virasoro operators are constructed first at the classical level and then at the quantum level. At the quantum level, one needs to add a quantum correction\(^29\) in order to cancel the conformal anomaly. The nilpotent BRST operator is obtained in the usual way with this modification. We then discover that this correction, the origin of which was rather mysterious previously,\(^29\) naturally shows itself up as one computes the quantum supersymmetry algebra. The algebra closes only up to a BRST-exact term, where the BRST operator automatically contains the correct quantum modification. The rest of the super-Poincaré algebra turned out to possess similar features. With a judicious quantum correction added to a part of the Lorentz generators, the algebra precisely closes up to BRST-exact terms.

We then proceed, in section 6, to the construction of the vertex operators for the massless excitations. (For simplicity we will consider the type I super-Maxwell sector.) By definition they must be BRST-invariant and form a correct representation of the quantum super-Poincaré algebra established in the previous section, up to BRST-exact expressions. The computations were quite involved, requiring various non-trivial \(\gamma\)-matrix and spinorial identities, but we could obtain the desired vertex operators which satisfy all the requirements consistently.

In order to deepen our understanding of the GS superstring in the SLC gauge further, we will study in section 7 its connection to the much-studied formulation in the usual (full) light-cone gauge. We will be able to do this in the most direct way, namely by constructing an explicit quantum similarity transformation which connects the operators in the two formulations. The basic method used is the one in Ref. 34), which was developed to relate Green-Schwarz and an extended version of the pure spinor superstring. Although the presence of the extra term in the BRST operator demanded additional new ideas and observations, we have obtained the desired similarity transformation exactly. As an application, we have been able to construct the BRST-invariant DDF operators in the SLC gauge from
the physical operators in the light-cone gauge by performing this similarity transformation.

Finally, section 8 will be devoted to discussions, where we examine some problems which are not solved in this work and indicate future directions. Several appendices are provided to describe our conventions and supply additional technical details.

§2. Classical action and its symmetries

We begin by reviewing the classical action for the Green-Schwarz superstring and its symmetries before gauge-fixing,\(^7,8\) mainly to set up our notations and to make this article self-contained.

2.1. Action

The action invariant under the super-Poincaré transformations is most easily constructed using the supercoset method.\(^3\) It consists of the kinetic part and the Wess-Zumino (WZ) part and can be expressed compactly in terms of the worldsheet differential forms in the following way:

\[
S = S_K + S_{WZ},
\]

\[
S_K = -\frac{T}{2} \int \Pi^\mu \wedge *\Pi_\mu, \tag{2.2}
\]

\[
S_{WZ} = -T \int \left( \Pi^\mu \wedge \widetilde{W}_\mu + \frac{1}{2} W^\mu \wedge \widetilde{W}_\mu \right). \tag{2.3}
\]

Here, \(T = 1/2\pi\alpha'\) is the string tension\(^3\), and the 1-forms \(\Pi^\mu, W^\mu, \widetilde{W}^\mu\) are defined as

\[
\Pi^\mu = \Pi_i^\mu d\xi^i = dX^\mu - W^\mu, \tag{2.4}
\]

\[
W^\mu = W^{1\mu} + W^{2\mu}, \quad \widetilde{W}^\mu = W^{1\mu} - W^{2\mu}, \tag{2.5}
\]

\[
dX^\mu = \partial_i X^\mu d\xi^i, \quad W^{A\mu} = W_i^{A\mu} d\xi^i = i\theta^{A\alpha} \gamma_{i\alpha\beta}^\mu \partial_i \theta^{A\beta} d\xi^i, \quad A = 1, 2. \tag{2.6}
\]

\(X^\mu\) are the string coordinates, \(\theta^{A\alpha}\) are the two sets of 16-component real chiral spinors\(^4\) and \(\xi^i = (t, \sigma)\) denote the worldsheet coordinates. The convention for the spinors and the \(\gamma\)-matrices\(^5\) are elaborated in appendix A. We take the flat target space metric to be \(\eta_{\mu\nu} = (-1, +1, \ldots, +1)\) and the signature of the worldsheet metric as \((-+, +). The wedge product \(\wedge\) and the Hodge dual \("*\)" with respect to the worldsheet metric \(g_{ij}\) are given for

\(^3\) As for the fundamental length scale, we will use the string length \(\ell_s\), related to \(T\) by \(\ell_s = 1/\sqrt{2\pi T}\).

\(^4\) In this article we specifically deal with the type IIB case. Type IIA case can be easily described by adjusting certain signs.

\(^5\) For convenience, we use different notations for the \(\gamma\)-matrices with lower and upper indices, namely, \(\tilde{\gamma}_{i\alpha\beta}^\mu\) and \(\gamma^{\mu\alpha\beta}\). See appendix A for more details.
the basic coordinate 1-form $d\xi^i$ as

$$d\xi^i \wedge d\xi^j = -\epsilon^{ij} d^2\xi, \quad \epsilon_{01} \equiv 1 = -\epsilon^{01}, \quad (2.7)$$

$$*d\xi^i = -\sqrt{-g} g^{ijk} \epsilon_{kj} d\xi^j. \quad (2.8)$$

This gives $d\xi^i \wedge *d\xi^j = \sqrt{-g} g^{ij} d^2\xi$. With these formulas, the Lagrangian density can be written in terms of components as

$$L_K = -\frac{T}{2} \sqrt{-g} g^{ij} \Pi^i \Pi^j, \quad L_{WZ} = T \epsilon^{ij} \left( \Pi^\mu_i \widetilde{W}_{\mu j} + \frac{1}{2} W_{i}^\mu \widetilde{W}_{\mu j} \right). \quad (2.9)$$

It is sometimes useful to note that the WZ term can be written in a slightly different form:

$$L_{WZ} = T \epsilon^{ij} \left( \Pi^\mu_i (W^1_{\mu j} - W^2_{\mu j}) - W^1_{i} W^2_{\mu j} \right). \quad (2.10)$$

This is due to the identity $\frac{1}{2} \epsilon^{ij} W_{i}^\mu \widetilde{W}^2_{\mu j} = -\epsilon^{ij} W^1_{i} W^2_{\mu j}$.

2.2. Symmetries of the action

The action presented above enjoys four types of symmetries, namely, the worldsheet reparametrization invariance, the target space Lorentz invariance, the $N = 2$ supersymmetry and the $\kappa$ symmetry. Since the first two symmetries are obvious, we will review the latter two.

The supercoset construction guarantees that the action is invariant under the global supersymmetry transformations

$$\delta_\chi \theta^A = \chi^A, \quad \delta_\chi X^\mu = \sum_A i \chi^A \gamma^\mu \theta^A, \quad (2.11)$$

where the supersymmetry(SUSY) parameters $\chi^{A\alpha}$ are constant real chiral spinors$^6$. In fact since $\Pi_i^\mu$ are SUSY invariant the kinetic term is manifestly invariant even at the Lagrangian level. However, the Lagrangian for the WZ part is not invariant (because $W_{i}^{\alpha \mu}$ are not invariant) and transforms into a total derivative. As this is important in deriving the Noether current, let us quickly review how this comes about.

For this purpose, it is convenient to use the form of $L_{WZ}$ given in (2.10). It is easy to see that although $W_{i}^{\alpha \mu}$ is not an invariant, it transforms into a total derivative as

$$\delta_\chi W_{i}^{\alpha \mu} = \partial_\mu V_{i}^{\alpha \mu}, \quad (2.12)$$

where $V_{i}^{\alpha \mu} = i \chi^A \gamma^{-\mu} \theta^A$. Applying this to (2.10) and rearranging, we readily obtain

$$\delta_\chi L_{WZ} = T \epsilon^{ij} \partial_j \left[ \partial_i X^\mu (V^1_\mu - V^2_\mu) \right] - T \epsilon^{ij} (W^{1 \mu}_i \partial_j V^1_\mu - W^{2 \mu}_i \partial_j V^2_\mu). \quad (2.13)$$

$^6$ We reserve the commonly used letter $\epsilon$ for the $SO(8)$ anti-chiral components of $\chi$, to appear later.
The first term is manifestly a total derivative. To show that the second term is also a total derivative, we need to use the well-known Fierz identity. Define

\[ A \equiv \epsilon^{ij} W_i^\mu \partial_j V_\mu = \epsilon^{ij} (\bar{\theta} \gamma^i \partial \theta) (\partial_j \bar{\theta} \gamma^j \chi), \] (2.14)

\[ B \equiv \epsilon^{ij} (\partial_i \theta \bar{\gamma}^i \partial_j \theta) (\bar{\gamma}^j \chi). \] (2.15)

Contracting the expression \( \epsilon^{ij} \theta^a \partial_i \theta^b \partial_j \bar{\theta} \gamma^d \) with the Fierz identity

\[ 0 = \bar{\gamma}^\mu \bar{\gamma}_\alpha \bar{\gamma}^\nu \bar{\gamma}_\beta \bar{\gamma}^\rho \bar{\gamma}_\delta, \] (2.16)

we get \( 0 = 2A + B \). Using this relation, the total derivative \( \epsilon^{ij} \partial_j (W_i^\mu V_\mu) \), which equals \(-B + A\), becomes \( 3A \). Hence, we get a non-trivial identity

\[ \epsilon^{ij} W_i^\mu \partial_j V_\mu = \frac{1}{3} \epsilon^{ij} \partial_j (W_i^\mu V_\mu). \] (2.17)

Applying this to (2.13), we readily obtain

\[ \delta \chi \mathcal{L}_{WZ} = \chi^{1a} \partial_1 A^i_\alpha + \chi^{2a} \partial_1 A^i_\alpha, \] (2.18)

\[ A^i_\alpha = -i T \epsilon^{ij} \left( \Pi_j^\mu + W_j^2 + \frac{2}{3} W_j^1 \right) (\bar{\gamma}^j \theta^1), \] (2.19)

\[ A^i_\alpha = i T \epsilon^{ij} \left( \Pi_j^\mu + W_j^1 + \frac{2}{3} W_j^2 \right) (\bar{\gamma}^j \theta^2). \] (2.20)

From this result, it is easy to get the conserved SUSY Noether currents as

\[ j^1_\alpha = 4iT (\bar{\gamma}^j \theta^1), \] (2.21)

\[ j^2_\alpha = 4iT (\bar{\gamma}^j \theta^2). \] (2.22)

Another important symmetry is the \( \kappa \)-symmetry. The action is invariant under the off-shell symmetry transformations given by

\[ \delta_\kappa X^\mu = i \theta A^\mu \kappa^A, \quad \delta_\kappa \theta^A = (\gamma_i)^{\alpha \beta} \kappa^A_i, \] (2.23)

\[ \delta_\kappa (\sqrt{-g} g^{ij}) = 8i \sqrt{-g} \left( P^k_i \partial_k \theta^1 + P^k_i \partial_k \theta^2 \right). \] (2.24)

Here \( \gamma_i \equiv \Pi_i^\mu \gamma_\mu, \kappa^A_i \) are local fermionic parameters, which are anti-chiral spinors in spacetime and vectors on the worldsheet, and the projection operators \( P^i_\pm \) are given by

\[ P^i_\pm = \frac{1}{2} \left( g^{ij} \pm \frac{\epsilon^{ij}}{\sqrt{-g}} \right). \] (2.25)

For invariance, the \( \kappa \) parameters must satisfy the conditions

\[ P^i_+ \kappa^1_j = 0, \quad P^i_- \kappa^2_j = 0, \] (2.26)

where \( \kappa^A_i \equiv g_{ik} \kappa^A_k. \)

\footnote{Up to an overall normalization and the interchange of \( \theta^1 \) and \( \theta^2 \), this agrees with the expression given in Ref. 8.}
§3. Gauge-fixing and compensating transformation

In this section, we perform the gauge-fixing of the local symmetries. We will do this in two steps: First, we pick the conformal gauge to fix the reparametrization symmetry. Since the original \( \kappa \)-transformation acts on the worldsheet metric as in (2.24), we must modify the \( \kappa \)-transformation rule by adding an appropriate compensating reparametrization in order to stay in the conformal gauge. We will then fix the \( \kappa \)-symmetry by adopting the so-called semi-light-cone (SLC) gauge. As this gauge is not invariant under the supersymmetry transformation nor the Lorentz transformation, we must include suitable compensating \( \kappa \)-transformations in these transformations in order to keep the SLC gauge. Below we describe these procedures in some detail.

3.1. Conformal gauge fixing

As we wish to keep the worldsheet conformal invariance, we will choose the conformal gauge, where the Weyl-invariant combination \( \sqrt{-g} g^{ij} \) takes the flat form \( \eta^{ij} \). Since the original \( \kappa \)-transformation (2.24) changes this value, to remain in the conformal gauge we must make a judicious compensating reparametrization transformation \( \delta f \xi^i = f^i(\xi) \) so that

\[
(\delta_\kappa + \delta_f)\sqrt{-g} g^{ij} = 0
\]

holds at the conformal gauge point. Under the reparametrization transformation, \( X^\mu, \theta^{A\alpha} \) are scalars while \( g_{ij} \) and \( g^{ij} \) are covariant and contravariant tensors. So they transform like

\[
\delta f X^\mu = f^i \partial_i X^\mu, \quad \delta \theta^{A\alpha} = f^i \partial_i \theta^{A\alpha},
\]

\[
\delta f g_{ij} = \nabla_i f_j + \nabla_j f_i, \quad \delta f g^{ij} = - (\nabla^i f^j + \nabla^j f^i),
\]

where \( f_k \equiv g_{kl} f^l \). As for \( \sqrt{-g} \), it transforms as \( \delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{kl} \delta g_{kl} = \sqrt{-g} (\nabla^k f_k) \). Combining these results, one easily finds that at the conformal gauge point, \( \sqrt{-g} g^{ij} \) transforms like

\[
\delta_f (\sqrt{-g} g^{ij}) \Big|_{\text{conf}} = - (\partial^i f^j + \partial^j f^i - \eta^{ij} \partial_k f^k).
\]

Now let us denote the \( \kappa \)-transform of \( \sqrt{-g} g^{ij} \) at the conformal gauge point by \( h^{ij} \equiv \delta_\kappa (\sqrt{-g} g^{ij}) \Big|_{\text{conf}} \). This quantity is symmetric and traceless. Then (3.1) reduces to

\[
\partial^i f^j + \partial^j f^i - \eta^{ij} \partial_k f^k = h^{ij}.
\]

Applying \( \partial_i \) on both sides, we immediately obtain \( \Box f^j = \partial_i h^{ij} \) (where \( \Box \equiv \partial_i \partial^i \)) and hence we can solve for \( f^j \) as

\[
f^j = \Box^{-1} \partial_i h^{ij}.
\]
In the present case, \( h^{ij} \) is given by (see (2.24))

\[
    h^{ij} = 8i \left( P^{ki}_{+} \partial_{k} \theta^{1} \kappa^{1ij} + P^{ki}_{-} \partial_{k} \theta^{2} \kappa^{2ij} \right),
\]

where \( P^{ki}_{\pm} \equiv \frac{1}{2}(\eta^{ki} \pm \epsilon^{ki}) \). The conditions (2.26) on the \( \kappa \) parameters \( \kappa^{Ai} \) become

\[
    -\kappa^{1,0} + \kappa^{1,1} = 0 , \quad \kappa^{2,0} + \kappa^{2,1} = 0 .
\]

Putting (3.7) into (3.6), we get the appropriate compensating reparametrization transformation. With the function \( f^{i} \) obtained in (3.6), the modified \( \kappa \)-transformations for \( \theta^{A\alpha} \) and \( X^{\mu} \) are given by

\[
    \delta_{\kappa} \theta^{A} = \delta^{0}_{\kappa} \theta^{A} + f^{i} \partial_{i} \theta^{A} ,
\]

\[
    \delta_{\kappa} X^{\mu} = \sum i \theta^{A} \bar{\gamma}^{\mu} \delta^{0}_{\kappa} \theta^{A} + f^{i} \partial_{i} X^{\mu} ,
\]

where \( \delta^{0}_{\kappa} \) denotes the original transformation (2.23). These transformations leave the conformal-gauge-fixed action invariant.

3.2. Semi-light-cone gauge fixing

3.2.1. SLC gauge condition and the Lagrangian

Next, let us fix the \( \kappa \)-symmetry by imposing the SLC gauge conditions given by

\[
    \bar{\gamma}^{+}_{\alpha \beta} \theta^{A \beta} = 0 \quad \Leftrightarrow \quad \theta^{A \bar{\alpha}} = 0 ,
\]

where \( \theta^{A \bar{\alpha}} \) denotes the \( SO(8) \) anti-chiral components. Here and hereafter, we will often make use of the \( SO(8) \) decompositions of the spinors and the \( \gamma \)-matrices. Our conventions and the properties of the \( \gamma \)-matrices are summarized in appendix A. In this gauge various terms in the action simplify considerably. First, the only non-vanishing components of \( W^{A \mu}_{i} \) are \( W^{A -}_{i} = i\sqrt{2} \theta^{A} \delta_{ab} \partial_{i} \theta^{Ab} \) because \( \bar{\gamma}^{\mu}_{ab} \) is non-vanishing only for \( \mu = - \). This immediately leads to the formulas

\[
    \Pi^{+}_{i} = \partial_{i} X^{+} , \quad \Pi^{I}_{i} = \partial_{i} X^{I} , \quad \Pi^{-}_{i} = \partial_{i} X^{-} - W^{-}_{i} , \quad W^{A \mu}_{i} W^{B \mu}_{ij} = 0 .
\]

The kinetic and the WZ parts of the Lagrangian become

\[
    \mathcal{L}_{K} = -\frac{T}{2} \left[ 2 \partial_{i} X^{+} \left( \bar{\partial}^{i} X^{-} - \sum_{A} i \theta^{A} \bar{\gamma}^{-} \partial^{A} \theta^{A} \right) + \partial_{i} X^{I} \bar{\partial}^{i} X^{I} \right] ,
\]

\[
    \mathcal{L}_{WZ} = i T e^{ij} \partial_{i} X^{+} \sum_{A} \eta_{A} \theta^{A} \bar{\gamma}^{-} \partial_{j} \theta^{A} ,
\]

\[
    \eta_{1} = -\eta_{2} = 1 .
\]

\footnote{In the historic paper\(^8\), where this was first discussed, only the case of a very special \( \kappa \) transformation was considered. Namely the authors imposed the extra conditions \( \partial_{-} \kappa^{1j} = \partial_{+} \kappa^{2j} = 0 \). In this case, the expression for \( f^{j} \) simplifies to \( f^{j} = 4i \sum_{A} \theta^{A} \kappa^{Aj} \). In general, however, one should not (and need not) impose such dynamical restrictions on the \( \kappa \)-parameters.}
Although there are still cubic terms in $L$, the equations of motion for all the fields reduce to “free field forms”, namely $\partial_i \partial^i X^\mu = 0$, $\partial_+ \theta^1 = 0$ and $\partial_- \theta^2 = 0$, because the interaction terms vanish on-shell.

3.2.2. Supersymmetry in the SLC gauge

Let us now begin the discussion of the modification of the symmetry transformations needed in the SLC gauge. First consider the SUSY transformations. As we need to distinguish between the $SO(8)$ chiral and anti-chiral parts, we will split the SUSY parameters as $\chi^{A\alpha} = (\eta^{Aa}, \epsilon^{A\dot{a}})$ and write the SUSY transformations in the form

$$\delta_\eta \theta^{Aa} = \eta^{Aa}, \quad \delta_\epsilon \theta^{A\dot{a}} = \epsilon^{A\dot{a}}. \quad (3.16)$$

Since only the anti-chiral $\epsilon^{A\dot{a}}$ transformations violate the gauge conditions, the compensating $\kappa$-transformations will involve only half of the $\kappa$-parameters. Indeed we will find that it is enough to keep $\kappa^{A\dot{a}}$ and set $\kappa^{Aa}$ to zero.

It is useful to note that in such a case we can ignore the reparametrization part of the modified $\kappa$ transformations obtained in (3.9, 3.10) and simply use the original $\kappa$ transformations. The reason is as follows: The expression $h^{ij}$ given in (3.7) is made up of the expressions of the form

$$\partial_k \theta^{A\alpha} \kappa_{a}^{\alpha i} = \partial_k \theta^{Aa} \kappa_{a}^{\alpha i} + \partial_k \theta^{A\dot{a}} \kappa_{\dot{a}}^{\alpha i}. \quad (3.17)$$

This vanishes in the SLC gauge for $\kappa_{a}^{\alpha i} = 0$ and hence the reparametrization parameter $f^i$ vanishes.

Thus, the parameter for the compensating $\kappa$-transformation should be determined by the requirement

$$\delta_\epsilon \theta^{A\dot{a}} \equiv (\delta^0_\epsilon + \delta_\kappa) \theta^{A\dot{a}} = \epsilon^{A\dot{a}} + (\Pi_i^\mu \gamma_\mu \kappa^{A\dot{a}})^\dot{a} = 0, \quad (3.18)$$

where $\delta^0_\epsilon$ denotes the original transformation (3.16). Writing this out explicitly with $\kappa_{a}^{\alpha i}$ set to zero, we obtain the equation

$$\epsilon^{A\dot{a}} - \sqrt{2} \Pi_i^+ \delta^{\dot{a}\dot{b}} \kappa_{b}^{A\dot{a}} = 0. \quad (3.19)$$

Recalling that only half of $\kappa_{b}^{A\dot{a}}$ are independent, as in (3.8), we can write this as

$$\epsilon^{1\dot{a}} = 2\sqrt{2} \partial_+ X^+ \delta^{1\dot{a}} \kappa_{b}^{1,0}, \quad (3.20)$$

$$\epsilon^{2\dot{a}} = 2\sqrt{2} \partial_- X^+ \delta^{2\dot{a}} \kappa_{b}^{2,0}, \quad (3.21)$$
where we introduced the worldsheet light-cone coordinates and the derivatives as \( \sigma^\pm = \xi^0 \pm \xi^1 \) and \( \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \). Therefore, as long as \( \partial_\pm X^+ \) do not vanish, we can solve for the \( \kappa \) parameters:

\[
\kappa_{a}^{1,0} = \kappa_{a}^{1,1} = \frac{\delta_{ab} \epsilon_{1b}}{2\sqrt{2}\partial_+ X^+},
\]

\[
\kappa_{a}^{2,0} = -\kappa_{a}^{2,1} = \frac{\delta_{ab} \epsilon_{2b}}{2\sqrt{2}\partial_- X^+}.
\]

Throughout we assume that the zero-mode part of \( T\partial_0 X^+ \), i.e. the momentum \( p^+ \), is non-vanishing and define the operators such as \( 1/\partial_+ X^+ \) by expanding around \( p^+ \).

We can now write down the \( \epsilon \)-SUSY transformations for the fields \( \theta^A a \) and \( X^\mu \), modified by the \( \kappa \)-transformations. The \( \epsilon \)-transformation for \( \theta^A a \) consists solely of the \( \kappa \)-transformation and is given by

\[
\delta_\epsilon \theta^A a = (\gamma^I)^{ab} \delta_{bc} \partial_+ X^I \epsilon_1^c + \sqrt{-g} P^0_0 \partial_+ X^I \epsilon_1^c.
\]

Substituting (3.22) and (3.23), this becomes

\[
\delta_\epsilon \theta^1 a = (\gamma^I)^{ab} \delta_{bc} \partial_+ X^I \epsilon_1^c,
\]

\[
\delta_\epsilon \theta^2 a = (\gamma^I)^{ab} \delta_{bc} \partial_+ X^I \epsilon_1^c.
\]

As for the \( \epsilon \)-transformation of \( X^\mu \), it is given by

\[
\delta_\epsilon X^\mu = i\epsilon^A a (\tilde{\gamma}^\mu)_{ab} \theta^{ab} + i\theta^A a (\tilde{\gamma}^\mu)_{ab} \delta_\epsilon \theta^{ab}.
\]

From the property of \( \tilde{\gamma}^\mu \), the first term is non-vanishing only for \( \mu = I \), while the second term exists only for \( \mu = - \). Using the results (3.24) and (3.25), we get

\[
\delta_\epsilon X^I = i\epsilon^A a \tilde{\gamma}^I \theta^A,
\]

\[
\delta_\epsilon X^- = i(\theta^I \tilde{\gamma}^\mu \epsilon^I) \frac{\partial_+ X^I}{\partial_+ X^+} + i(\theta^2 \tilde{\gamma}^\mu \epsilon^2) \frac{\partial_- X^I}{\partial_- X^+}.
\]

We will end the discussion of the SUSY in the SLC gauge by giving the form of the supercharge densities in this gauge. Before gauge-fixing, they are given as the time-components of the currents (2.21) and (2.22) as

\[
j_{a}^{10} = 4i T(\bar{\gamma}^\mu \theta^1)_{a} \left( \sqrt{-g} P^0_0 \Pi_{ij} + \frac{2}{3} W^1_{\mu j} \right),
\]

\[
j_{a}^{20} = 4i T(\bar{\gamma}^\mu \theta^2)_{a} \left( \sqrt{-g} P^0_0 \Pi_{ij} + \frac{2}{3} W^2_{\mu j} \right).
\]

The expressions in the conformal gauge are obtained by substituting \( \sqrt{-g} = 1 \), \( P^0_0 \Pi_{ij} = -\frac{1}{2}(\Pi_{i0} - \Pi_{0j}) \) and \( P^0_0 \Pi_{ij} = -\frac{1}{2}(\Pi_{i0} + \Pi_{0j}) \). As we further impose the SLC gauge condition
\(\tilde{\gamma}^+\theta^A = 0\), the following simplifications occur. First, for any vector \(A_\mu\) we have \((\tilde{\gamma}^{\mu}\theta)A_\mu = (\tilde{\gamma}^{-}\theta)A^+ + (\tilde{\gamma}^{I}\theta)A^I\). Next, \(\Pi_i^\mu\) are simplified as \(\Pi_i^+ = \partial_+ X^+\) and \(\Pi_i^I = \partial_+ X^I\). Finally, one can check the useful identity \((\tilde{\gamma}^{\mu}\theta)W_{\mu\bar{\alpha}}^A = 0\). Because of this, \(W_{\mu\bar{\alpha}}^A\) all disappear and we obtain

\[
j_{\alpha}^{01} = -4i T \left( (\tilde{\gamma}^{-}\theta)^{1}_\alpha \partial_+ X^+ + (\tilde{\gamma}^{I}\theta)^{1}_\alpha \partial_+ X^I \right), \tag{3.30}\]
\[
j_{\alpha}^{02} = -4i T \left( (\tilde{\gamma}^{-}\theta)^{2}_\alpha \partial_+ X^+ + (\tilde{\gamma}^{I}\theta)^{2}_\alpha \partial_+ X^I \right). \tag{3.31}\]

For later convenience, let us make the \(SO(8)\) decomposition. Using \((\tilde{\gamma}^-)_{ab} = \sqrt{2} \delta_{ab}\), \((\tilde{\gamma}^-)_{\bar{a}b} = 0\), we get

\[
j_a^{01} = -4\sqrt{2} i \theta_{a}^{1} T \partial_+ X^+, \tag{3.32}\]
\[
j_a^{02} = -4\sqrt{2} i \theta_{a}^{2} T \partial_+ X^+, \tag{3.33}\]
\[
j_a^{01} = -4i (\tilde{\gamma}^I)_{ab} \theta^{1b} T \partial_+ X^I, \tag{3.34}\]
\[
j_a^{02} = -4i (\tilde{\gamma}^I)_{ab} \theta^{2b} T \partial_+ X^I, \tag{3.35}\]

where we defined\(^9\) \(\theta^A_{a} \equiv \delta_{ab} \theta^{Ab}\). We will later quantize them in the phase space formulation and study the quantum algebra of the supercharges.

3.2.3. Lorentz symmetry in the SLC gauge

The next subject is the form of the Lorentz transformations in the SLC gauge. Before gauge-fixing, the infinitesimal Lorentz transformations for \(X^\mu\) and \(\theta^{\alpha A}\) are given by

\[
\delta X^\mu = \frac{1}{2} \xi_{\rho\sigma} (\eta^{\mu\rho} X^\sigma - \eta^{\mu\sigma} X^\rho), \tag{3.36}\]
\[
\delta \theta^{\alpha A} = \frac{1}{4} \xi_{\rho\sigma} (\gamma^{\rho\sigma})^A_{\alpha} \theta^{\beta \bar{B}}, \tag{3.37}\]

where \(\xi_{\rho\sigma}\) are infinitesimal antisymmetric parameters and \(\gamma^{\rho\sigma} \equiv \frac{1}{2} (\gamma^\rho \tilde{\gamma}^\sigma - \gamma^\sigma \tilde{\gamma}^\rho)\). The SLC gauge conditions are broken in general by the Lorentz transformations, which act on \(\theta^{\alpha A}\) as \(\delta \theta^{\alpha A} = (1/4) \xi_{\rho\sigma} (\gamma^{\rho\sigma})^A_{\alpha} \theta^{\beta \bar{B}} \neq 0\). From the \(SO(8)\) structure of the \(\gamma\)-matrices discussed in appendix A, it is not difficult to see that \((\gamma^{\rho\sigma})^A_{\alpha}\) is non-vanishing only for \((\rho\sigma) = (I-\bar{I})\) (and of course \((-I\bar{I})\)). Thus for such transformations, we must add compensating \(\kappa\) transformations in order to stay in the SLC gauge. It turns out that, just as in the case of the supersymmetry, one can find such transformations using \(\kappa^{\dot{A}i}_{b}\) only, with \(\kappa^{\dot{A}i}_{b}\) set to zero. Therefore, as explained around (3.17), the modification of the \(\kappa\)-transformation due to reparametrization can be ignored. Hence the condition to fix the compensating \(\kappa\) parameters takes the form

\[
0 = \delta_{\xi_{\dot{I}_-}} \theta^{A\dot{A}} = (\delta_{\xi_{\dot{I}_-}}^0 + \delta_{\xi_{\dot{I}_-}}^\kappa) \theta^{A\dot{A}} = \xi_{\dot{I}-} \frac{1}{2} (\tilde{\gamma}^{I^-})_{\dot{a}} \theta^{\alpha B} + (\Pi_{\dot{I}}^\mu \gamma^\mu \kappa^{\dot{A}i})_{\dot{a}}
\]
\[
= \xi_{\dot{I}-} \frac{1}{2} (\tilde{\gamma}^{I^-})_{\dot{a}} \theta^{\alpha B} - \Pi_{\dot{I}}^i \delta^{\dot{a}b} \kappa^{\dot{A}i}_{b}. \tag{3.38}\]

\(^9\) Once decomposed into \(SO(8)\) components, the spinor indices can be trivially raised and lowered.
where $\delta^0_{\xi_{l-}}$ denotes the original Lorentz transformation with the parameter $\xi_{l-}$. Recalling $\Pi_i^+ \kappa^1_b = 2 \partial_+ X^+ \kappa^1_{b,0}$ and $\Pi_i^+ \kappa^2_b = 2 \partial_- X^+ \kappa^2_{b,0}$, we easily find the solution:

$$\kappa^1_{a,0}(\xi_{l-}) = \frac{\delta_{ab} \xi_{l-} (\gamma^I)^b \gamma^1 c}{4 \partial_+ X^+},$$  \hspace{1cm} (3.39)

$$\kappa^2_{a,0}(\xi_{l-}) = \frac{\delta_{ab} \xi_{l-} (\gamma^I)^b \gamma^2 c}{4 \partial_- X^+}. \hspace{1cm} (3.40)$$

Accordingly we must change the transformation for $\theta^A a$ and $X^\mu$ by adding the $\kappa$-transformation with these field-dependent parameters.

Consider first $\delta_{\xi_{l-}} \theta^A a$. The direct Lorentz transformation $\xi_{l-} \frac{1}{2} (\gamma^I)^a b \theta^{ab}$ vanishes since $\theta^{ab} = 0$. The transformation for $\theta^1 a$ induced by the $\kappa$ transformation is

$$\delta^\kappa_{\xi_{l-}} \theta^1 a = (\Pi_i^+ \gamma^I \kappa^1_l (\xi_{l-}))^a = 2 \partial_+ X^J (\gamma^J)^{ab} \kappa^1_{b,0}(\xi_{l-})$$

$$= \frac{1}{2} \frac{\partial_+ X^J}{\partial_+ X^+} (\gamma^J)^{ab} \delta_{bc} (\gamma^I)^c d \theta^1 d \xi_{l-} . \hspace{1cm} (3.41)$$

This can be slightly simplified by noting $(\bar{\gamma}^+)^{bc} = -\sqrt{2} \delta_{bc}$ and $\gamma^+ \bar{\gamma} \theta = 2 \theta \ (\text{since } \bar{\gamma} \theta = 0)$. After the simplification, together with the similar result for $\theta^2 a$, we obtain

$$\delta^\kappa_{\xi_{l-}} \theta^1 a = \frac{1}{\sqrt{2}} \frac{\partial_+ X^J}{\partial_+ X^+} (\gamma^J)^{a} \theta^1 a \xi_{l-} ,$$  \hspace{1cm} (3.42)

$$\delta^\kappa_{\xi_{l-}} \theta^2 a = \frac{1}{\sqrt{2}} \frac{\partial_- X^J}{\partial_- X^+} (\gamma^J)^{a} \theta^2 a \xi_{l-} . \hspace{1cm} (3.43)$$

Next consider the transformation of $X^\mu$. Since the $\kappa$-transformation for $X^\mu$ in the SLC gauge is $\delta^\kappa_{\xi_{l-}} X^\mu = i \sum_A \theta^{A a} (\gamma^I)_{ab} \delta^0 \partial^a \theta^{ab}$, only the $X^-$ component is affected by this transformation since $(\bar{\gamma}^-)^{ab} = \sqrt{2} \delta_{ab}$ is the only non-vanishing component. For the case with the parameter $\xi_{l-}$, this induced piece $\delta^\kappa_{\xi_{l-}} X^-$ is in fact the only contribution to $\delta^\kappa_{\xi_{l-}} X^-$, because the original Lorentz transformation $\delta^0_{\xi_{l-}} X^-$ vanishes. The explicit expression for $\delta^\kappa_{\xi_{l-}} X^-$ is simplified by using the following relation

$$\theta^a (\gamma^-)^{ab} (\gamma^J)^{bc} (\gamma^-)^{cd} \theta^d = \frac{1}{\sqrt{2}} \theta \gamma^- \gamma^J \gamma^I \gamma^- \theta = \frac{1}{\sqrt{2}} \theta \gamma^J \gamma^I \gamma^- \gamma^- \theta$$

$$= \sqrt{2} \theta \gamma^J \gamma^I \gamma^- \theta = \sqrt{2} \theta \gamma^J \gamma^I \gamma^- \theta , \hspace{1cm} (3.44)$$

where we used the Clifford algebra and the identity $\theta \bar{\gamma} \theta = 0$. In this way we obtain

$$\delta^\kappa_{\xi_{l-}} X^- = \frac{1}{\sqrt{2}} \left( \frac{\partial_+ X^J}{\partial_+ X^+} \theta^1 \gamma^- \gamma^- \theta + \frac{\partial_- X^J}{\partial_- X^+} \theta^2 \gamma^J \gamma^- \theta^2 \right) \xi_{l-} . \hspace{1cm} (3.45)$$
3.2.4. Conformal symmetry in the SLC gauge

Finally, let us briefly touch upon the classical conformal symmetry, which is still preserved in the SLC gauge. This can easily be seen if we express the Lagrangian given in (3.13) and (3.14) in terms of the worldsheet light-cone coordinates as

\[
\mathcal{L} = 2T \left[ \partial_+ X^+ \left( \partial_- X^- - \sum_A i \theta^A \bar{\gamma}^- \partial_- \theta^A \right) \\
+ \partial_- X^+ \left( \partial_+ X^- - \sum_A i \theta^A \bar{\gamma}^- \partial_+ \theta^A \right) + \partial_+ X^I \partial_- X^I \right] \\
+ 2iT \left[ \partial_+ X^+ \sum_A \eta_A \theta^A \bar{\gamma}^- \partial_- \theta^A + \partial_- X^+ \sum_A \eta_A \theta^A \bar{\gamma}^- \partial_+ \theta^A \right],
\]

(3.46)

where the sign \( \eta_A \) is as defined in (3.15). Because of the derivative structure, under the conformal transformation \( \sigma_\pm \rightarrow f_\pm(\sigma_\pm) \), the action is invariant provided that \( X^\mu \) and \( \theta^A \) are considered as conformal scalars. The Virasoro algebra formed by the conformal transformations will be discussed in detail at the classical and the quantum level in section 5.1.

§4. Phase space formulation and quantization

Having discussed the classical symmetries and how some of the transformation laws are modified in the SLC gauge, we now wish to quantize the system and study the quantum realization of these symmetries. In this section we will develop the phase space formulation, which will be most suited for that purpose. This is mainly because the complicated compensating transformations will be automatically taken into account by the use of the Dirac bracket. This will be spelled out in section 4.2.

4.1. Poisson-Dirac brackets and quantization

We begin by setting up the Poisson-Dirac bracket between the basic fields and their conjugates. We will denote the momenta conjugate to \((X^+, X^-, X^I)\) as \((P^-, P^+, P^I)\). From the Lagrangian given in (3.13) and (3.14), they are readily obtained as

\[
P^+ = T \partial_0 X^+,
\]

(4.1)

\[
P^- = T \left[ \partial_0 X^- - 2 \sqrt{2} i (\theta^1_\alpha \partial_+ \theta^1_\alpha + \theta^2_\alpha \partial_- \theta^2_\alpha) \right],
\]

(4.2)

\[
P^I = T \partial_0 X^I.
\]

(4.3)

As for the fermionic fields, the momenta \(p^A_a\) conjugate to \(\theta^A_a\) take the form

\[
p^A_a = i \sqrt{2T} (\partial_0 X^+ - \eta_A \partial_1 X^+) \theta^A_a = i \pi^{+A} \theta^A_a,
\]

(4.4)
π ≡ \sqrt{2}(P^+ - \eta_A T\partial_1 X^+).  

As is well-known, these equations actually give the constraints

\[ d^A_a \equiv p^A_a - i\pi^+ A^A \theta^A_a = 0. \]

We define the Poisson brackets as

\[ \{ X^I(\sigma, t), P^J(\sigma', t) \}_P = \delta^{IJ}\delta(\sigma - \sigma'), \]

\[ \{ X^{\pm}(\sigma, t), P^{\mp}(\sigma', t) \}_P = \delta(\sigma - \sigma'), \]

\[ \{ \theta^A_a(\sigma, t), p^B_b(\sigma', t) \}_P = -\delta^{AB}\delta_{ab}\delta(\sigma - \sigma'), \]

\[ \text{rest} = 0. \]

Under this bracket, the fermionic constraints \( d^A_a \) form the second class algebra

\[ \{ d^A_a(\sigma, t), d^B_b(\sigma', t) \}_P = 2i\delta^{AB}\delta_{ab}\pi^+ \theta^A_a(\sigma, t)\delta(\sigma - \sigma'). \]

Defining the Dirac bracket in the standard way, \( \theta^A \)'s become self-conjugate and satisfy

\[ \{ \theta^A_a(\sigma, t), \theta^B_b(\sigma', t) \}_D = \frac{i\delta^{AB}\delta_{ab}}{2\pi^+ \theta^A_a(\sigma, t)}\delta(\sigma - \sigma'). \]

By going to the Dirac bracket, the relations (4.7) and (4.8) continue to hold, but the brackets between \( (X^-, P^-) \) and \( \theta^A \), which vanished under Poisson bracket, become non-trivial\(^{10}\). One finds

\[ \{ X^-(\sigma, t), \theta^A_a(\sigma', t) \}_D = -\frac{1}{\sqrt{2}\pi^+ A(\sigma, t)}\theta^A_a(\sigma, t)\delta(\sigma - \sigma'), \]

\[ \{ P^-(\sigma, t), \theta^A_a(\sigma', t) \}_D = -\frac{1}{\sqrt{2}\pi^+ A(\sigma', t)}\theta^A_a(\sigma', t)\delta(\sigma - \sigma'). \]

However, if we define the combination

\[ \Theta^A_a \equiv \sqrt{2}\pi^+ A\theta^A_a, \]

it is not difficult to check that they satisfy

\[ \{ \Theta^A_a(\sigma, t), \Theta^B_b(\sigma', t) \}_D = i\delta^{AB}\delta_{ab}\delta(\sigma - \sigma'), \]

and commute with all the other fields. So the fields to be used are \( (X^\pm, X^I, P^\pm, P^I, \Theta^A_a) \), which satisfy the canonical form of the Dirac bracket relations.

\(^{10}\) The bracket \( \{ X^-, P^- \}_D \) still vanishes due to \( \theta^A_a\theta^A_a = 0. \)
For later convenience, we introduce the following dimensionless fields \( \{ A, B, \Pi, \tilde{\Pi}, S \} \), with appropriate sub- and super-scripts:

\[
A = \sqrt{2\pi T} X, \quad B = \sqrt{2\pi T} P, \\
\Pi = \frac{1}{\sqrt{2}} (B + \partial_1 A), \quad \tilde{\Pi} = \frac{1}{\sqrt{2}} (B - \partial_1 A), \\
S = i\sqrt{2\pi \Theta}.
\] (4.17)

The non-vanishing Dirac brackets among them are

\[
\{ A^{+}(\sigma, t), B^{-}(\sigma', t) \}_D = \{ A^{-}(\sigma, t), B^{+}(\sigma', t) \}_D = 2\pi \delta(\sigma - \sigma'), \\
\{ A^{I}(\sigma, t), B^{J}(\sigma', t) \}_D = 2\pi \delta^{IJ} \delta(\sigma - \sigma'), \\
\{ S_{a}^{A}(\sigma, t), S_{b}^{B}(\sigma', t) \}_D = \frac{2\pi i}{\ell} \delta^{AB} \delta_{ab} \delta(\sigma - \sigma').
\] (4.19)

The quantization can be performed in the standard way, namely by replacing the Dirac brackets by the (anti-)commutators and \( \delta(\sigma - \sigma') \to i\delta(\sigma - \sigma') \). We will do this at time \( t = 0 \) and hence drop \( t \) for all the fields from now on. In particular, the non-vanishing (anti-)com mutators among the basic dimensionless fields are then given by

\[
\begin{align*}
[A^{+}(\sigma), B^{-}(\sigma')] &= [A^{-}(\sigma), B^{+}(\sigma')] = 2\pi i \delta(\sigma - \sigma'), \\
[A^{I}(\sigma), B^{J}(\sigma')] &= 2\pi i \delta^{IJ} \delta(\sigma - \sigma'), \\
\{ S_{a}^{A}(\sigma), S_{b}^{B}(\sigma') \} &= 2\pi \delta^{AB} \delta_{ab} \delta(\sigma - \sigma').
\end{align*}
\] (4.22)

We take the Fourier mode expansions of \( A^{\ast}, B^{\ast} (\ast = (\pm, I)) \) and \( S_{a}^{A} \) to be

\[
A^{\ast}(\sigma) = \sum_{n} A_{n}^{\ast} e^{-in\sigma}, \quad B^{\ast}(\sigma) = \sum_{n} B_{n}^{\ast} e^{-in\sigma}, \quad S_{a}^{A}(\sigma) = \sum_{n} S_{a,n}^{A} e^{-in\sigma}.
\] (4.25)

Then, these modes satisfy the simple (anti-)commutation relations:

\[
\begin{align*}
[A_{m}^{\pm}, B_{n}^{\mp}] &= i\delta_{m+n,0}, \quad [A_{m}^{I}, B_{n}^{J}] = i\delta^{IJ} \delta_{m+n,0}, \\
\{ S_{a,m}^{A}, S_{b,n}^{B} \} &= \delta^{AB} \delta_{ab} \delta_{m+n,0}, \quad \text{rest} = 0.
\end{align*}
\] (4.26)

It should be noted that the commutator of the modes of the original fields \( X_{\mu}(\sigma) = \sum_{n} X_{n}^{\mu} e^{-in\sigma} \) and \( P_{\mu}(\sigma) = \sum_{n} P_{n}^{\mu} e^{-in\sigma} \) has an extra factor of \( 1/2\pi \) as

\[
[X_{m}^{\mu}, P_{n}^{\nu}] = \frac{i}{2\pi} \eta^{\mu\nu} \delta_{m+n,0}.
\] (4.28)

For this reason, we will often use the following notations for the zero modes, which satisfy the canonical commutation relations:

\[
x^{\mu} \equiv X_{0}^{\mu}, \quad p^{\nu} \equiv 2\pi P_{0}^{\nu}, \quad [x^{\mu}, p^{\nu}] = i\eta^{\mu\nu}.
\] (4.29)
As for the $H$ and $\tilde{H}$ fields, we have

$$[\Pi^+(\sigma),\Pi^-(\sigma')] = -[\tilde{\Pi}^+(\sigma),\tilde{\Pi}^-(\sigma')] = 2\pi i \delta'(\sigma - \sigma'),$$

$$[\Pi^I(\sigma),\Pi^J(\sigma')] = -[\tilde{\Pi}^I(\sigma),\tilde{\Pi}^J(\sigma')] = 2\pi i \delta^{IJ} \delta'(\sigma - \sigma'),$$

$$[\Pi^I(\sigma),A^J(\sigma')] = [\tilde{\Pi}^I(\sigma),A^J(\sigma')] = -\frac{i}{\sqrt{2}} \delta^{IJ} \delta'(\sigma - \sigma'). \quad (4.30)$$

We will write the mode expansion of $\Pi^\mu$ and $\tilde{\Pi}^\mu$ in the following way:

$$\Pi^\mu(\sigma) \equiv \sum_n \alpha_n^\mu e^{-i n \sigma}, \quad (4.31)$$

$$\tilde{\Pi}^\mu(\sigma) \equiv \sum_n \tilde{\alpha}_n^\mu e^{-i n \sigma}. \quad (4.32)$$

In terms of the modes of $A^\mu$ and $B^\mu$, the oscillators $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ are given by

$$\alpha_n^\pm \equiv \frac{1}{\sqrt{2}} (B_n^\pm - i n A_n^\pm), \quad \alpha_n^I \equiv \frac{1}{\sqrt{2}} (B_n^I - i n A_n^I), \quad (4.33)$$

$$\tilde{\alpha}_n^\pm \equiv \frac{1}{\sqrt{2}} (B_n^\pm + i n A_n^\pm), \quad \tilde{\alpha}_n^I \equiv \frac{1}{\sqrt{2}} (B_n^I + i n A_n^I), \quad (4.34)$$

and they satisfy the following commutation relations:

$$[\alpha_n^m,\alpha_n^{\mp}] = m \delta_{m+n,0}, \quad [\alpha_n^m,\alpha_n^J] = m \delta^{IJ} \delta_{m+n,0}, \quad (4.35)$$

$$[\tilde{\alpha}_n^m,\tilde{\alpha}_n^{\pm}] = -m \delta_{m+n,0}, \quad [\tilde{\alpha}_n^m,\tilde{\alpha}_n^J] = -m \delta^{IJ} \delta_{m+n,0}. \quad (4.36)$$

Note that $\alpha$ and $\tilde{\alpha}$ commutators are opposite in sign.

Now in constructing the quantum symmetry generators in the subsequent section, we will have to specify the normal-ordering of operators, in particular of non-zero modes. In the present situation, it is easy to see that the Hamiltonian is quadratic in the modes and the appropriate normal-ordering is uniquely dictated by the criterion that the modes with positive (negative) energy should be regarded as creation (annihilation) operators. Specifically, up to a constant, our Hamiltonian has the structure

$$H = \ell_2^2 p^2 + \sum_{n \geq 1} \left( \alpha_n^+ \alpha_n^- + \alpha_n^- \alpha_n^+ + \alpha_n^I \alpha_n^I + \tilde{\alpha}_n^- \tilde{\alpha}_n^+ + \tilde{\alpha}_n^+ \tilde{\alpha}_n^- + \tilde{\alpha}_n^I \tilde{\alpha}_n^I \right)$$

$$+ n S_{a,n}^2 + S_{a,n}^1 + n S_{a,n}^1 S_{a,n}^1 \right). \quad (4.37)$$

From the (anti)-commutation relations of the modes, we immediately find that, for $n > 0$, $\alpha_n^\pm, \alpha_n^I, \tilde{\alpha}_n^\pm, \tilde{\alpha}_n^I, S_{a,n}^1, S_{a,n}^1, S_{a,n}^1$ are eigen-operators of $H$ with positive energy and hence should be regarded as creation operators. Note that for $\alpha_n^\mu$ and $S_{a,n}^1$ the moding is opposite to the
usual convention. For this reason, hereafter we will introduce the following notations (we will leave $\alpha^{-n}$ intact):

\[
\bar{\alpha}_{-n}^\mu \equiv \tilde{\alpha}_{n}^\mu, \quad S_{a,-n} \equiv S^2_{a,-n}, \quad \bar{S}_{a,-n} \equiv S^1_{a,n}.
\]  

(4.38)

In this notation, the modes with negative mode number are uniformly regarded as creation operators. Then, the properly normal-ordered Hamiltonian is given by

\[
H = l_p^2 p^2 + \sum_{n \geq 1} \left( \alpha_{-n}^+ \alpha_{n}^- + \alpha_{n}^+ \alpha_{-n}^- + \bar{\alpha}_{-n}^+ \bar{\alpha}_{n}^- + \bar{\alpha}_{n}^+ \bar{\alpha}_{-n}^- \right) + n S_{a,-n} S_{a,n} + n \bar{S}_{a,-n} \bar{S}_{a,n}.
\]  

(4.39)

4.2. Compensating transformation in the phase space formulation

In section 3, we gave a detailed discussion of the gauge-fixing and the associated compensating transformations needed for the super-Poincaré transformations in the Lagrangian formulation. One of the great advantages of the phase space formulation is that in this formulation it is not necessary to work out the compensating transformations, which are often complicated. Conceptually, the reason is very simple: The Dirac bracket is so designed to remove any flow out of the chosen gauge slice. Since the role of the compensating transformation is precisely to keep the system on the gauge slice, the Dirac bracket automatically fulfills this role.

It is instructive to see this explicitly. Let $\phi(x)$ be a field and $\pi(x)$ be its conjugate, with the Poisson bracket relation $\{\phi(x), \pi(y)\}_P = \delta(x - y)$. Suppose that the system is simultaneously invariant under a gauge transformation generated by a first class constraint $\Phi_1(\phi, \pi)(x)$ and under a gauge-invariant global symmetry transformation generated by $U(\phi, \pi)$. They satisfy $\{\Phi_1, U\}_P = 0$. Now let us fix a gauge by imposing a gauge condition $\Phi_2(\phi, \pi)(x) = 0$. This means that we have $\{\Phi_1(x), \Phi_2(y)\}_P = \epsilon_{ij} C(x) \delta(x - y) \neq 0$, where $i, j = 1, 2$ and $\epsilon_{ij}$ is the antisymmetric $\epsilon$-symbol with $\epsilon_{12} = 1$. We are interested in the case where $U$ breaks the gauge condition, that is, $\delta_U \Phi_2(x) \equiv \{\Phi_2(x), \epsilon U\}_P \neq 0$, where $\epsilon$ is an infinitesimal parameter. In such a case, to stay in the original gauge, we must modify the generator $U$ by adding a compensating gauge generator $\Delta U$, with a judicious gauge parameter function $\alpha(x)$, of the form

\[
\Delta U = \int dy \alpha(y) \Phi_1(y).
\]  

(4.40)

The function $\alpha(x)$ should be determined by requiring that the total transformation leaves $\Phi_2$ invariant. In other words,

\[
(\delta_{\epsilon} + \delta_{\epsilon}^{\text{gauge}}) \Phi_2(x) = \epsilon \{\Phi_2(x), U + \Delta U\}_P = \epsilon (\{\Phi_2(x), U\}_P - \alpha(x) C(x)) = 0.
\]  

(4.41)
This gives
\[ \alpha(x) = C^{-1}(x) \{ \Phi_2(x), U \}_P . \]  
(4.42)

Thus the combined total transformation for an arbitrary field \( F(\phi, \pi)(x) \) becomes
\[ \delta^{\text{total}}_\epsilon F(x) = \epsilon \{ F(x), U \}_P + \epsilon \int dy \{ F(x), \Phi_1(y) \}_P C^{-1}(y) \{ \Phi_2(y), U \}_P . \]  
(4.43)

Since \( \{ \Phi_1, U \}_P = 0 \), we can write this as
\[ \delta^{\text{total}}_\epsilon F(x) = \epsilon \left[ \{ F(x), U \}_P - \int dy \{ F(x), \Phi_i(y) \}_P C^{-1}(y)_{ij} \{ \Phi_j(y), U \}_P \right] , \]  
(4.44)

where \( (C^{-1})_{ij} = -\epsilon_{ij}C^{-1} \). This however is nothing but the Dirac bracket \( \epsilon \{ F(x), U \}_D \). So we confirm that the compensating transformation is automatically included by the use of the Dirac bracket.

4.3. Relation to the canonical quantization scheme

Starting from the next section, we will study the structure of the quantum symmetry algebras and construct the vertex operators. For these purposes, it will be convenient to make use of the powerful technique of operator product expansion (OPE). Since this technique is usually based on the canonically quantized fields defined on the complex plane, it is appropriate here to clarify the relation between the fields in the phase space formulation and those in the canonical formulation.

In the canonical quantization scheme, one begins by obtaining the general complete solutions of the equations of motion at the classical level. From this dynamical information, one tries to express the time-independent modes in terms of the fields using the completeness relation. Once this is achieved, the quantum commutation relations among these modes can be obtained from the basic quantization rule for the fields at equal time. This then allows one to compute the commutation relations between quantum fields at unequal times. So, the characteristic feature of this scheme is that the dynamics must be solved completely at the classical level before the quantization.

In contradistinction, the phase space quantization scheme does not require the solution of the equations of motion. The quantization of fields and their conjugates is done at a particular fixed time, say \( t = 0 \), by simply replacing the Poisson-Dirac brackets by the quantum brackets. To construct the quantum fields at arbitrary time \( t \), one needs to carry out the time evolution by the quantum Hamiltonian \( H \) as \( \phi(t) = e^{iHt} \phi(0) e^{-iHt} \) for every field \( \phi(t) \) (with spatial coordinates suppressed). The field \( \phi(t) \) constructed this way then satisfies the equation of motion \( \partial_t \phi(t) = -i[\phi(t), H] \). So, the characteristic feature of this scheme is that the dynamics must be worked out at the quantum level after the quantization.
The summary of the foregoing discussion is expressed, for our two dimensional case, by the equation

\[ \phi_{\text{can}}(\sigma, t) = e^{iHt} \phi_{\text{phase}}(\sigma, 0)e^{-iHt}. \]  

This formula is completely general and is particularly useful for quantizing a non-linear system for which the complete solutions of the classical equations of motions are difficult to obtain. In such a case, the canonical quantization cannot be performed but one can still quantize in the phase space formulation and then try to evaluate the right hand side of (4.45) by some appropriate means. As a matter of fact, for the computation of \(t\)-independent quantities, such as various symmetry charges, even that computation is not necessary and one can obtain quantum information by using the phase space fields quantized at a common fixed time\(^{11}\).

Now for the present system, although the Lagrangian (3.13, 3.14) contains cubic interactions, the Hamiltonian expressed in terms the phase space variables is quadratic, as given in (4.39), and one can compute the right hand side of (4.45) explicitly. The mode expansions of the basic phase space fields are given by

\[ X^\mu(\sigma, 0) = \sum_n X_n^\mu e^{-in\sigma} = x^\mu + i\ell_s \sum_{n \neq 0} \left( \frac{1}{n} \alpha_n^\mu e^{-in\sigma} + \frac{1}{n} \bar{\alpha}_n^\mu e^{in\sigma} \right), \]  

\[ P^\mu(\sigma, 0) = \sum_n P_n^\mu e^{-in\sigma} = \frac{p^\mu}{2\pi} + \frac{1}{4\pi \ell_s} \sum_{n \neq 0} \left( \alpha_n^\mu e^{-in\sigma} + \bar{\alpha}_n^\mu e^{in\sigma} \right), \]  

\[ S_a(\sigma, 0) = \sum_n S_{a,n} e^{-in\sigma}, \quad \bar{S}_a(\sigma, 0) = \sum_n \bar{S}_{a,n} e^{in\sigma}. \]  

By time-developing via the Hamiltonian (4.39), we obtain the canonical fields at time \(t\) as

\[ X^\mu(\sigma, t) = e^{iHt} X^\mu(\sigma, 0)e^{-itH} = x^\mu + 2\ell_s^2 p^\mu t + i\ell_s \sum_{n \neq 0} \left( \frac{1}{n} \alpha_n^\mu e^{-in(t+\sigma)} + \frac{1}{n} \bar{\alpha}_n^\mu e^{-in(t-\sigma)} \right), \]  

\[ P^\mu(\sigma, t) = e^{iHt} P^\mu(\sigma, 0)e^{-itH} = \frac{p^\mu}{2\pi} + \frac{1}{4\pi \ell_s} \sum_{n \neq 0} \left( \alpha_n^\mu e^{-in(t+\sigma)} + \bar{\alpha}_n^\mu e^{-in(t-\sigma)} \right), \]  

\[ S_a(\sigma, t) = e^{iHt} S_a(\sigma, 0)e^{-itH} = \sum_n S_{a,n} e^{-in(t+\sigma)}, \]  

\[ \bar{S}_a(\sigma, t) = e^{iHt} \bar{S}_a(\sigma, 0)e^{-itH} = \sum_n \bar{S}_{a,n} e^{-in(t-\sigma)}. \]

\(^{11}\) The advantage of the phase space quantization scheme was emphasized in Ref. 32) and utilized for the quantization of superstring in the plane-wave background in the SLC gauge.
By introducing the Euclidean time $\tau \equiv it$ and the variables $z \equiv \exp(\tau + i\sigma), \bar{z} \equiv \exp(\tau - i\sigma)$, $X^\mu(\sigma, t)$ can be expressed as

$$X^\mu(z, \bar{z}) = x^\mu - i\ell_2^2 p^\mu \left(\ln z + \ln \bar{z}\right) + i\ell_2 \sum_{n \neq 0} \left(\frac{1}{n} \alpha_n z^{-n} + \frac{1}{n} \bar{\alpha}_n \bar{z}^{-n}\right).$$

(4.53)

Similarly, other canonical fields can be expressed as functions of $z$ and $\bar{z}$. Later, when we regard them as conformal fields on the complex plane, we will have to take into account the effect of the conformal transformation from the cylinder to the complex plane, which shifts the power of $z$ (and $\bar{z}$) according to the conformal weights.

§5. Structure of the quantum symmetry algebras

This section will be devoted to the study of the structure of the quantum symmetry algebras of the system. Besides being important in its own right, this is an indispensable prerequisite for the construction of the vertex operators, to be performed in the subsequent section.

5.1. Virasoro algebra and BRST symmetry

We will first focus on the Virasoro algebra and the associated BRST symmetry. The reason is that, as we shall see, the super-Poincaré algebra closes only up to BRST transformations.

5.1.1. Classical Virasoro algebra

Let us begin with the classical Virasoro algebra. Because the Lagrangian in the SLC gauge is classically conformally invariant, the $++$ and the $--$ components of the energy-momentum tensor, to be denoted by $T_\pm$, become the Virasoro constraints. Through the standard procedure one obtains

$$T_\pm = T \left(2\partial_\pm X^+ \partial_\pm X^- + (\partial_\pm X_I)^2\right) - i2\sqrt{2} T \partial_\pm X^+ \theta_a^1 \partial_\pm \theta_a^1 + \theta_a^2 \partial_\pm \theta_a^2. \tag{5.1}$$

$$T_+ = \frac{1}{2}\left(\mathcal{H} + \mathcal{P}\right) = \frac{1}{2\pi} \left(\Pi^+ \Pi^- + \frac{1}{2} \Pi_I^2 + \frac{i}{2} S^2 \partial_1 S^2\right), \tag{5.2}$$

$$T_- = \frac{1}{2}\left(\mathcal{H} - \mathcal{P}\right) = \frac{1}{2\pi} \left(\Pi^+ \Pi^- + \frac{1}{2} \Pi_I^2 - \frac{i}{2} S^1 \partial_1 S^1\right) \tag{5.3}.$$

$\mathcal{P}$ and $\mathcal{H}$ are, respectively, (the dimensionless version of) the momentum density and the Hamiltonian density. To compute the Dirac bracket relations among $\mathcal{P}$ and $\mathcal{H}$ (and hence...
The last two lines show that
\[ O(\sigma')\delta'(\sigma - \sigma') = O(\sigma)\delta'\sigma' + \partial_t O(\sigma)\delta(\sigma - \sigma'), \] (5.4)
\[ O(\sigma')\delta''(\sigma - \sigma') = O(\sigma)\delta''(\sigma - \sigma') + 2\partial_1 O(\sigma)\delta'(\sigma - \sigma') + \partial_1^2 O(\sigma)\delta(\sigma - \sigma'), \] (5.5)

which must be understood in the sense of distributions. After a straightforward calculation, we obtain the expected results:

\[ \{\mathcal{H}(\sigma, t), \mathcal{H}(\sigma', t)\}_D = \{\mathcal{P}(\sigma, t), \mathcal{P}(\sigma', t)\}_D = 0, \]
\[ \{\mathcal{P}(\sigma, t), \mathcal{H}(\sigma', t)\}_D = \{\mathcal{H}(\sigma, t), \mathcal{P}(\sigma', t)\}_D = 0, \]
\[ \{T_\pm(\sigma, t), T_\pm(\sigma', t)\}_D = \pm 2T_\pm(\sigma, t)\delta'(\sigma - \sigma') \pm \partial_1 T_\pm(\sigma, t)\delta(\sigma - \sigma'), \] (5.8)
\[ \{T_\pm(\sigma, t), T_\mp(\sigma', t)\}_D = 0. \] (5.9)

The last two lines show that \( T_\pm \) form mutually commuting Virasoro algebras\(^{12} \). Integrating the first two equations with respect to \( \sigma' \) and identifying \( \int d\sigma'\mathcal{H}(\sigma', t) \) to be the Hamiltonian \( H \), which generates the time-development of a field \( A(\sigma, t) \) as \( \partial_0 A = \{A, H\}_D \), one readily finds
\[ \partial_0 H = \{H, H\}_P = \partial_1 \mathcal{P}, \quad \partial_0 \mathcal{P} = \{\mathcal{P}, H\}_P = \partial_1 \mathcal{H}. \] (5.10)

Combining them, we get \( \partial_\mp T_\pm = 0 \), showing that \( T_\pm = T_\pm(\sigma_\pm) \). Therefore, we can define the Virasoro mode operators \( L_\pm^\pm \) by
\[ T_\pm = \frac{1}{2\pi} \sum_n L_\pm^\pm e^{-in(t\sigma)}. \] (5.11)

Putting this into (5.8), one verifies that \( L_\pm^\pm \) satisfy the usual form of the classical Virasoro algebra, namely
\[ \{L_\pm^\pm, L_\pm^\pm\}_D = \frac{1}{i}(m - n)L_\pm^\pm, \quad \{L_\pm^m, L_\pm^n\}_D = 0. \] (5.12)

What is important here is that since \( L_\pm^\pm \) are independent of \( t \) and \( \sigma \) they can be obtained from \( T_\pm \) at one time slice, which we take to be \( t = 0 \):
\[ L_\pm^\pm = \int_0^{2\pi} d\sigma e^{\pm in\sigma} T_\pm(\sigma, t = 0). \] (5.13)

\(^{12} \)\( \pm \) signs on the RHS are just right for the Virasoro mode operators \( L_\pm^\pm \) to satisfy the same algebra.
But at \( t = 0 \), we know the exact Dirac brackets for the fields composing \( T_\pm \) and hence we can quantize them in the standard way. Consequently, any quantum properties of the system which are dictated by \( L_n^\pm \) can be calculable\(^{13}\).

5.1.2. Quantum Virasoro algebra

We are now ready to study the quantum Virasoro algebra. Since \( T_+ \) and \( T_- \), being made up of mutually independent fields, commute, we will concentrate on \( T_+ \). To compute the algebra, the most convenient way is to form the generating function

\[
T(z) \equiv \sum_n L_n^+ z^{-n-2} \tag{5.14}
\]

on the complex \( z \)-plane, express it in terms of suitable chiral fields and make use of the familiar operator product expansion (OPE) method. The chiral fields which make up \( T(z) \) are given by

\[
\Pi^\mu(z) \equiv i\ell_s^{-1} \partial X^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1}, \quad S_a(z) \equiv \sum_n S_{a,n} z^{-n-1/2}, \tag{5.15}
\]

where the basic chiral field \( X^\mu(z) \) is defined, as usual, by

\[
X^\mu(z) \equiv x^\mu - i\ell_s^2 p^\mu \ln z + i\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu z^{-n}. \tag{5.16}
\]

This is obtained from (4.53) by dropping the \( \bar{z} \)-dependent parts. With the normal-ordering prescription discussed previously, these fields satisfy the OPE’s

\[
X^\mu(z)X^\nu(w) \sim -\eta^{\mu\nu}\ell_s^2 \ln(z - w), \tag{5.17}
\]

\[
X^\mu(z)\Pi'^\nu(w) \sim \frac{i\ell_s\eta^{\mu\nu}}{z - w}, \quad \Pi'^\mu(z)\Pi'^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z - w)^2}, \tag{5.18}
\]

\[
S_a(z)S_b(w) \sim \frac{\delta_{ab}}{z - w}. \tag{5.19}
\]

In terms of these fields \( T(z) \) is expressed as

\[
T(z) = \Pi^+(z)\Pi^-(z) + \frac{1}{2}(\Pi^I(z))^2 - \frac{1}{2}S_a\partial S_a(z), \tag{5.20}
\]

where the products are understood to be normal-ordered. It is straightforward to compute the \( T(z)T(w) \) with the result

\[
T(z)T(w) = \frac{7}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w}. \tag{5.21}
\]

\(^{13}\) In the present system \( L_n^\pm \) are quadratic in canonical fields and hence this statement may sound trivial. However, this feature is valid even when \( L_n^\pm \) contain non-linear terms.
This shows that $L_n^+$'s satisfy the Virasoro algebra with the central charge $14$, of which $10$ comes from the bosons and $4$ from the self-conjugate fermions. In order to construct a consistent string theory with conformal symmetry, we need to supply $12$ more units of central charge. This was achieved in Ref. 29) by adding a quantum correction of the form $\frac{1}{2} \partial^2 \ln \Pi^+$. The modified $T(z)$ is therefore given by\footnote{The Virasoro operator $\bar{T}$ for the right-going sector is obtained by putting a bar on all the fields.}

$$T(z) = \Pi^+(z)\Pi^-(z) + \frac{1}{2}(\Pi^I(z))^2 - \frac{1}{2}S_a \partial S_a(z) + \frac{1}{2} \partial^2 \ln \Pi^+, \quad (5.22)$$

where explicitly

$$\partial^2 \ln \Pi^+ = \frac{\partial^2 \Pi^+}{\Pi^+} - \left( \frac{\partial \Pi^+}{\Pi^+} \right)^2. \quad (5.23)$$

One can readily verify that $T(z)$ continues to satisfy the usual form of the Virasoro algebra, this time with $c = 26$. At this stage, the addition of the peculiar term $\frac{1}{2} \partial^2 \ln \Pi^+$ appears to be rather ad hoc and its quantum origin is unclear. One might even wonder whether it is unique. A natural and clarifying answer to these questions will emerge in the next subsection, as we examine the quantum closure of the supersymmetry algebra.

It is easy to see that $\Pi^+$ and $\Pi^I$ are primary fields of dimension $1$ and $S_a$ is a primary of dimension $1/2$. As for $\Pi^-$, it is no longer a primary field because of the presence of the $\frac{1}{2} \partial^2 \ln \Pi^+$ term. However, one can modify $\Pi^-$ to construct a primary of dimension $1$, to be called $\tilde{\Pi}^-$, in the following way:

$$\tilde{\Pi}^- \equiv \Pi^- + \partial^2 \left( \frac{1}{2\Pi^+} \right) = \Pi^- - \frac{1}{2\Pi^+} \left( (\partial \ln \Pi^+)^2 + \partial^2 \ln \Pi^+ \right)$$

$$= \Pi^-(z) - \frac{1}{2} \frac{\partial^2 \Pi^+(z)}{(\Pi^+)^2} + \frac{(\partial \Pi^+)^2(z)}{(\Pi^+)^3}. \quad (5.24)$$

This fact will be relevant when we construct the quantum Lorentz generator involving $\Pi^-$.  

5.1.3. BRST operator

With the Virasoro operator (5.22) with $c = 26$ at hand, the nilpotent BRST operator is constructed in the usual way. For the left sector on which we focus, it is given by

$$Q \equiv \int [dz] \left( cT + bc\partial c \right)(z), \quad (5.25)$$

where $c(z)$ and $b(z)$ are the standard reparametrization ghost and anti-ghost operators satisfying the OPE

$$c(z)b(w) = b(z)c(w) \sim \frac{1}{z-w}. \quad (5.26)$$
If we define the total Virasoro operator including the ghost sector by \( T^{\text{tot}} = \{Q, b\} \), \( b \) and \( c \) carry conformal weight 2 and \(-1\) with respect to \( T^{\text{tot}} \). In the next subsection, we will encounter another BRST operator \( \hat{Q} \), which is related to \( Q \) by a similarity transformation. Further, in section 6, we will show how \( Q \) can be reduced, again by a similarity transformation, to the form relevant in the full light-cone gauge formulation.

5.2. **Super-Poincaré algebra**

Next we study how the super-Poincaré algebra is realized, in particular quantum mechanically, in the SLC gauge. Although the SLC gauge condition breaks the part of the super-Poincaré symmetry, the symmetry is still preserved in the physical Hilbert space. Accordingly, we will see that the algebra closes up to BRST exact expressions.

5.2.1. **Algebra of supercharges**

We begin with the algebra of supercharges. The supercharge densities in the Lagrangian formulation were given in (3.32) \( \sim \) (3.35). Using various relations already given, it is straightforward to express them in terms of the phase space variables. The result is

\[
\begin{align*}
j^0_a(\sigma) &= -\sqrt{\frac{\pi^+ A(\sigma)}{\pi}} S^A_a(\sigma), \\
\dot{j}^0_a(\sigma) &= -\frac{1}{\sqrt{\pi^+ A(\sigma)}} \bar{\gamma}^I_{ab} S^{A_b}(\sigma) (P^I - \eta_A \partial_1 X^I)(\sigma), \quad \eta_1 = -\eta_2 = 1.
\end{align*}
\]

where \( \pi^+ A(\sigma) \) was defined in (4.5). From these expressions we see that the charge densities in the left (\( A = 2 \)) and the right (\( A = 1 \)) sectors are independent, and hence we will deal only with the left sector and suppress the index \( A = 2 \).

To facilitate the computation, we again use the OPE method. In terms of the chiral fields, the supercharges \( Q_\alpha \equiv \int d\sigma j^0_\alpha(\sigma) \) can be written as

\[
Q_\alpha = \int [dz] j_\alpha(z), \quad [dz] \equiv \frac{dz}{2\pi i},
\]

\[
\begin{align*}
j_\alpha(z) &= -\rho \sqrt{\Pi^+(z)} S_\alpha(z), \\
\dot{j}_\alpha &= -\rho \frac{\Pi^I(z)}{\sqrt{2\Pi^+(z)}} \bar{\gamma}^I_{ab} S^b(z), \\
\rho &= (32\pi T)^{1/4} = 2^{3/4} \epsilon_s^{-1/2}.
\end{align*}
\]

The expression such as \( \sqrt{\Pi^+(z)} \) is defined, as usual, by the expansion around the zero mode in the manner

\[
\sqrt{\Pi^+(z)} = \left( \frac{\alpha_0^+}{z} + \sum_{n \neq 0} \frac{\alpha_n^+ z^{-n-1}}{2} \right)^{1/2} = \sqrt{\alpha_0^+ z^{-1/2}} + \frac{z^{1/2}}{2\sqrt{\alpha_0}} \sum_{n \neq 0} \alpha_n^+ z^{-n-1} + \cdots.
\]
Therefore the currents \( j_\alpha(z) \) above do not have cuts and can be integrated over \( z \) to give \( Q_\alpha \) properly. Also it is easy to check that \( j_\alpha(z) \) are primary fields of dimension 1 so that \( Q_\alpha \) are conformally invariant.

The computations of \( \{ Q_a, Q_b \} \) and \( \{ Q_a, Q_b \} \) are trivially done by using the OPE \( S^a(z)S^b(w) \sim \delta^{ab}/(z - w) \) and we get the standard answer

\[
\{ Q_a, Q_b \} = 2\sqrt{2}\delta_{ab}p^+ , \quad (5.33)
\]

\[
\{ Q_a, Q_b \} = 2\gamma^I_{ab}p^I , \quad (5.34)
\]

where we used \( \int [dw]\Pi^\mu(w) = \ell_s\ell^\mu \).

The calculation of the anti-commutator \( \{ Q_\dot{a}, Q_\dot{b} \} \) is more involved and it receives an important quantum correction. It is convenient to split the contributions into three parts as

\[
\{ Q_\dot{a}, Q_\dot{b} \} = A_{SS} + A_{\Pi\Pi} + A_{SS\Pi\Pi} . \quad (5.35)
\]

Here \( A_{SS} \), \( A_{\Pi\Pi} \) and \( A_{SS\Pi\Pi} \) are due, respectively, to the contraction of \( S^aS^b \), the contraction of \( \Pi^I\Pi^J \) and the double contractions. Using the OPE technique, they are given by

\[
A_{SS} = \frac{1}{2}\rho^2\delta_{\dot{a}\dot{b}}\int [dw] \frac{(\Pi^I)^2}{\Pi^+} (w) , \quad (5.36)
\]

\[
A_{\Pi\Pi} = -\frac{1}{2}\rho^2\delta_{\dot{a}\dot{b}}\int [dw] \frac{S^a\partial S^a}{\Pi^+} (w) , \quad (5.37)
\]

\[
A_{SS\Pi\Pi} = -\frac{1}{2}\rho^2\delta_{\dot{a}\dot{b}}\int [dw] \frac{(\partial\Pi^+)^2}{(\Pi^+)^3} (w) . \quad (5.38)
\]

We can rewrite \( A_{SS\Pi\Pi} \) by using the identity

\[
\frac{(\partial\Pi^+)^2}{(\Pi^+)^3} = \partial^2 \ln \Pi^+ - \partial \left( \frac{\partial\Pi^+}{(\Pi^+)^2} \right) , \quad (5.39)
\]

where the second term on the right hand side vanishes upon integration. Therefore we can write the total contribution as

\[
\{ Q_\dot{a}, Q_\dot{b} \} = \rho^2\delta_{\dot{a}\dot{b}}\int [dw] \frac{1}{\Pi^+} \left( \frac{1}{2}(\Pi^I)^2 - \frac{1}{2}S^a\partial S^a - \frac{1}{2}\partial^2 \ln \Pi^+ \right) \]

\[
= -\rho^2\delta_{\dot{a}\dot{b}}\int [dw] \Pi^- + \rho^2\delta_{\dot{a}\dot{b}}\int [dw] \mathcal{T} , \quad (5.40)
\]

where we have defined the quantity

\[
\mathcal{T} \equiv \Pi^- + \frac{1}{\Pi^+} \left( \frac{1}{2}(\Pi^I)^2 - \frac{1}{2}S^a\partial S^a - \frac{1}{2}\partial^2 \ln \Pi^+ \right) . \quad (5.41)
\]

Therefore, we have found that \( \{ Q_\dot{a}, Q_\dot{b} \} \) closes into the usual momentum term plus an extra contribution proportional to \( \int [dw] \mathcal{T} \).
Let us now clarify the nature of this extra term. First note that $T$ is almost identical to $T/\Pi^+$, where $T$ is the Virasoro operator given in (5.22), except for the sign of the last logarithmic term. This difference is very important. With the minus sign as in (5.41) above, one can verify, with a slightly tedious calculation, that $T(z)T(w) \sim 0$, i.e. its OPE with itself is non-singular. Because of this property, we can define a nilpotent BRST-like operator $\hat{Q}$ by

$$\hat{Q} \equiv \int [dz] T(z)c(z),$$

and write $\int [dw] T$ in a “BRST-exact” form

$$\int [dw] T = \int [dw] \{\hat{Q}, b(w)\},$$

where $c(z)$ and $b(z)$ are the ghost-antighost pair satisfying the OPE (5.26). It is interesting to introduce the “anti-ghost for $\hat{Q}$” as $B \equiv b\Pi^+$ and define the corresponding energy-momentum tensor, including the ghost sector, by

$$\hat{T}^{\text{tot}}(z) \equiv \{\hat{Q}, B(z)\} = \Pi^+\Pi^- + \frac{1}{2}(\Pi^I)^2 - \frac{1}{2}S^a\partial S^a - \frac{1}{2}\partial^2 \ln \Pi^+ - b\partial c.$$ (5.44)

With respect to $\hat{T}^{\text{tot}}$, the $(b,c)$ ghosts carry dimensions $(1,0)$.

We now show that $\hat{Q}$ is related to the usual BRST operator $Q$ given in (5.25) by a quantum similarity transformation. Consider an operator

$$R \equiv \int [dz] cb \ln \Pi^+.$$ (5.45)

Then it is easy to see that

$$e^R ce^{-R} = \Pi^+ c, \quad e^R be^{-R} = \frac{b}{\Pi^+}.$$ (5.46)

Similarly, it is straightforward to show that

$$e^R T e^{-R} = T + R, T + \frac{1}{2}[R, [R, T]]$$

$$= T + \partial \left( \frac{cb}{\Pi^+} \right) + \frac{1}{2\Pi^+} \left( \partial^2 \ln \Pi^+ - (\partial \ln \Pi^+)^2 \right).$$ (5.47)

To compute $e^R \hat{Q} e^{-R}$ using these results, one must be careful since the products of $\Pi^+ c$ and the operators $\Pi^-$ and $\partial(cb/\Pi^+)$ contained in (5.47) are singular. To multiply them properly, one must point-split the arguments of the products and then take the coincident limit. Since

\footnote{This was first mentioned in Ref. 29). Here we display the slightly non-trivial details.}
originally there is no singularity in the product of $T$ and $c$, one expects that the singularities would cancel. Indeed, we find

$$
\Pi^-(z)(\Pi^+ c)(w) = \frac{c(w)}{(z - w)^2} + : \Pi^- \Pi^+ : c,
$$

(5.48)

$$
\partial \left( \frac{cb}{\Pi^+(z)} \right)(\Pi^+ c)(w) = -\frac{c(w)}{(z - w)^2} + \frac{1}{2} \partial^2 \left( \frac{c}{\Pi^+(w)} \right) \Pi^+(w) + : bc\partial c : (w),
$$

(5.49)

and the singularities do cancel each other. The rest of the calculations are straightforward and we obtain

$$
e^R \hat{Q} e^{-R} = Q = \int [dz] (cT + bc\partial c),
$$

(5.50)

where $T$ is precisely the expression of the Virasoro operator given in (5.22). Note that through the similarity transformation the conventional non-linear term $bc\partial c$ is generated and moreover the sign in front of $\partial^2 \ln \Pi^+$ is reversed.

Getting back to the closure $\{Q_\dot{a}, Q_\dot{b}\}$, we can now express the extra term as

$$
\int [dw] \left\{ \hat{Q}, b(w) \right\} = e^{-R} \int [dw] \left\{ Q, \frac{b}{\Pi^+(w)} \right\} e^R.
$$

(5.51)

Since $Q_\dot{a}$ and the zero mode part $\int [dw] \Pi^-$ commute with the operator $R$, we can make a similarity transformation of the form $e^R(\ast)e^{-R}$ without affecting them. In this way, we finally obtain

$$
\{Q_\dot{a}, Q_\dot{b}\} = -2\sqrt{2} \delta_{\dot{a}\dot{b}} p^- + \left\{ Q, \frac{2\sqrt{2}}{\ell_s} \delta_{\dot{a}\dot{b}} \int [dz] \frac{b}{\Pi^+(z)} \right\},
$$

(5.52)

showing that the algebra closes up to a BRST-exact contribution.

We would like to emphasize that from the direct computation of the algebra $\{Q_\dot{a}, Q_\dot{b}\}$, the structure of the Virasoro operator $T(z)$ including the correct extra $+\frac{1}{2} \partial^2 \ln \Pi^+$ contribution naturally emerged. The recognition that the proper implementation of the global algebra can dictate the correct structure of the local constraint algebra can be a very useful and powerful observation in other similar circumstances.

5.2.2. Lorentz algebra

We now construct the Lorentz part of the super-Poincaré generators and examine the structure of the closure.

First, consider the classical charge densities. By using the Noether method\textsuperscript{16} with the transformation rules given in (3.36) and (3.37), we readily obtain the charge densities in

\textsuperscript{16} For fermionic variables, we use left derivative throughout.
terms of the phase space variables in the following form:

\[ j^{0,\mu
u} = j^{0,\mu\nu}_B + j^{0,\mu\nu}_F, \quad (5.53) \]

\[ j^{0,\mu
u}_B = X^\mu P^\nu - X^\nu P^\mu, \quad (5.54) \]

\[ j^{0,\mu
u}_F = -\frac{1}{2} (\gamma^{\mu\nu})^a_\beta \theta^{\dot{A}\dot{B}} p^A_\alpha. \quad (5.55) \]

In the SLC gauge, where \( \theta^{A\dot{a}} = 0 \), the fermionic part \( j^{0,\mu\nu}_F \) simplifies to \( j^{0,\mu\nu}_F = -\frac{1}{2} ((\gamma^{\mu\nu})^a_\beta \theta^{\dot{A}\dot{B}} p^A_\alpha + (\gamma^{\mu\nu})^{\dot{a}}_\beta \theta^{\dot{A}\dot{B}} p^A_\alpha) \). The only non-vanishing components of \( (\gamma^{\mu\nu})^a_\beta \) and \( (\gamma^{\mu\nu})^{\dot{a}}_\beta \) are \( (\gamma^{IJ})^a_\beta \), \( (\gamma^{\pm})^a_\beta \) and \( (\gamma^{I-})^{\dot{a}}_\beta \). Now we must substitute the form of the momenta \( p^A_\alpha \) and \( p^A_\dot{a} \) in terms of \( \theta^{A\dot{a}} \).

As for \( p^A_\dot{a} \), we already have the appropriate expression (4.4). On the other hand, for \( p^A_\alpha \) we need to use the general definition of \( p^A_\alpha \) before \( \kappa \) symmetry gauge-fixing. It is given by

\[ p^A_\alpha = i(P^\mu - \eta_\Lambda T(\Pi^I_1 - W^{AI}_1))((\bar{\gamma}^I_\alpha)_{\dot{a}b} \theta^{\dot{A}\dot{B}}). \quad (5.56) \]

For \( \alpha = \dot{a} \) case, in the SLC gauge \( (\bar{\gamma}^I_\mu \theta^A_\dot{a})_\dot{a} \) is non-vanishing only for \( \mu = I \) and for these transverse components we have \( \Pi^I_1 = \partial_1 X^I \) and \( W^{AI}_1 = 0 \). Therefore \( p^A_\dot{a} \) simplifies to

\[ p^A_\dot{a} = i(P^I - \eta_\Lambda T \partial_1 X^I)((\bar{\gamma}^I_\dot{a})_{ab} \theta^{ab}). \quad (5.57) \]

When these formulas for \( p^A_\alpha \) are substituted, \( j^{0,\mu\nu}_F \) become

\[ j^{0,II}_F = \frac{i}{\sqrt{2}} (\theta^A_\alpha (\bar{\gamma}^{IJ})^a_\beta \theta^{\dot{A}\dot{B}} (P^+ - \eta_\Lambda T \partial_1 X^+) \), \quad (5.58) \]

\[ j^{0,+-}_F = 0, \quad (5.59) \]

\[ j^{0,J-}_F = -\frac{i}{\sqrt{2}} ((\bar{\gamma}^J_\alpha \theta^A_\dot{a}) (\bar{\gamma}^{J-}_\dot{a}) \theta^{ab} (P^- - \eta_\Lambda T \partial_1 X^J) \). \quad (5.60) \]

Finally expressing \( \theta^{A\dot{a}} \) in terms of the canonical variable \( \Theta^{A\dot{a}} \) using the definition (4.15) and further converting them into \( S^A_\alpha \) via (4.18), we obtain

\[ J^{IJ}_F = -\frac{i}{8\pi} \left( (\bar{\gamma}^{IJ}_a \theta^{ab} S^a + S^{ab} (\gamma^{IJ})^a_b S^{2b}) \right), \quad (5.61) \]

\[ J^{+ -}_F = 0, \quad (5.62) \]

\[ J^{- -}_F = \frac{i}{8\pi} \left( (\bar{\gamma}^I S^1)_a \delta^{ab} (\bar{\gamma}^J S^1)_b \bar{\Pi}^J \bar{\Pi}^I + (\bar{\gamma}^I S^2)_a \delta^{ab} (\bar{\gamma}^J S^2)_b \bar{\Pi}^J \bar{\Pi}^I \right). \quad (5.63) \]

The Lorentz generators obtained by integrating these densities will be denoted by \( M^{\mu\nu} \equiv \int d\sigma j^{0,\mu\nu}(\sigma) \).

We now wish to examine the quantum closure of the algebra of the operators \( M^{\mu\nu} \). To compute the commutators of \( M^{\mu\nu} \), we again wish to make use of the powerful OPE method.
However, contrary to the case of the supersymmetry algebra, one encounters several technical problems in doing so. Since these problems are of general nature, we will illustrate them in the context of free bosonic string in order to make the points clear.

A generic problem is that $M^{\mu\nu}_{B}$ does not split cleanly into the left and the right going parts, due to the presence of the common zero mode part. Explicitly, we have the structure

$$M^{\mu\nu}_{B} = M^{\mu\nu}_{0,B} + \tilde{M}^{\mu\nu}_{L} + \tilde{M}^{\mu\nu}_{R},$$

where

$$M^{\mu\nu}_{0,B} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu},$$

$$\tilde{M}^{\mu\nu}_{L} = \frac{1}{i} \sum_{n \geq 1} \frac{1}{n} (\alpha_{-n}^{\mu} \alpha^{\nu}_{n} - \alpha^{\nu}_{-n} \alpha^{\mu}_{n}),$$

$$\tilde{M}^{\mu\nu}_{R} = \frac{1}{i} \sum_{n \geq 1} \frac{1}{n} (\bar{\alpha}_{-n}^{\mu} \bar{\alpha}^{\nu}_{n} - \bar{\alpha}^{\nu}_{-n} \bar{\alpha}^{\mu}_{n}).$$

Since each type of the generators $M^{\mu\nu}_{0,B}$, $\tilde{M}^{\mu\nu}_{L}$ and $\tilde{M}^{\mu\nu}_{R}$ separately satisfies the same form of the Lorentz algebra, if we wish to focus just on the "left-going part", we should not consider $\frac{1}{2} M^{\mu\nu}_{0,B} + \tilde{M}^{\mu\nu}_{L}$ but rather the combination $M^{\mu\nu}_{0,B} + \tilde{M}^{\mu\nu}_{L}$. This unfortunately is not easy to describe in terms of the chiral fields on the complex $z$-plane. The candidate expression would be\(^{17}\)

$$\sqrt{2} \ell^{-1}_{s} \int [dz] (X^{\mu} \Pi^{\nu} - X^{\nu} \Pi^{\mu})(z),$$

where $\Pi^{\mu}(z)$ and $X^{\mu}(z)$ are defined in (5·15) and (5·16). However, this expression is not suitable for the following reasons: First, $X^{\mu}(z)$ is not a genuine conformal primary field. It contains $\ln z$ and the integral over $z$ is not well-defined. Second, even if we cure it with the regularization $\ln z \rightarrow \ln(z + \epsilon)$, with a small non-vanishing constant $\epsilon$, what we obtain is $\frac{1}{2} M^{\mu\nu}_{0,B} + \tilde{M}^{\mu\nu}_{L}$, which does not satisfy the proper Lorentz algebra.

Despite these difficulties, we can still make use of the OPE method by modifying the zero mode part of $X^{\mu}$ and carefully follow the proper definitions of the commutators. The new coordinate field $\tilde{X}^{\mu}$ is defined by

$$\tilde{X}^{\mu}(z) \equiv \sqrt{2} x^{\mu} + \tilde{X}^{\mu}(z), \quad \tilde{X}^{\mu}(z) = i \sum_{n \neq 0} \frac{1}{n} \alpha^{\mu}_{n} z^{-n},$$

where we put the over-check mark to denote the non-zero-mode part. Since the dependence on the scale $\ell_{s}$ is rather trivial, we have set $\ell_{s} = 1$ for simplicity. We will continue to do so for

\(^{17}\) This expression appears in some of the literatures, but it does not make sense, as we explain below.
the rest of this subsection. The field $\Pi^\mu(z)$ is unchanged and its relation to the coordinate field now reads

\begin{equation}
\tag{5.70}
i\partial \circ X^\mu = i\partial \tilde{X}^\mu = \Pi^\mu - \frac{p^\mu}{z} \equiv \tilde{\Pi}^\mu.
\end{equation}

With these definitions, one can check that the correct generator $L^{\mu\nu} = M^{\mu\nu}_{0,B} + \tilde{M}^{\mu\nu}_L$ is reproduced, including the zero mode part, in the form

\begin{equation}
\tag{5.71}
L^{\mu\nu} = \frac{1}{2} \int [dz] (\circ \Pi^\mu(z) - (\mu \leftrightarrow \nu)).
\end{equation}

Let us now sketch how one can compute the Lorentz algebra $[L^{\mu\nu}, L^{\rho\sigma}]$ by the OPE method. The basic OPE’s among the fundamental variables are

\begin{align}
\circ X^\mu(z) \circ X^\nu(w) &\sim -\eta^{\mu\nu} \ln \left(1 - \frac{w}{z}\right), \quad (5.72) \\
\circ X^\mu(z) \Pi^\nu(w) &\sim \frac{i\eta^{\mu\nu}}{z - w}, \quad (5.73) \\
\Pi^\mu(z) \circ X^\nu(w) &= \tilde{\Pi}^\mu(z) \circ X^\nu(w) + \frac{2}{z} [p^\mu, x^\nu] \sim -i\eta^{\mu\nu} \left(\frac{1}{z - w} + \frac{1}{z}\right), \quad (5.74) \\
\Pi^\mu(z) \Pi^\nu(w) &\sim \frac{\eta^{\mu\nu}}{(z - w)^2}. \quad (5.75)
\end{align}

As these rules are slightly different from the usual ones for $X^\mu(z)$, we will have to be careful at certain points. To recognize this, it is instructive to recall how the OPE method works for computing the commutators in the usual situation. Suppose we wish to compute the commutator $[A,B]$, where the operators $A, B$ are given by contour integrals around $z = 0$ over holomorphic operators $A(z), B(z)$ defined on $z$-plane as $A = \int [dz] A(z)$ and $B = \int [dz] B(z)$. One normally deals with the situation where the following two conditions are satisfied: (i) The product $A(z)B(w)$ defined for $|z| > |w|$ with renormal-ordering has only the pole singularities at the coincident points of the form $A(z)B(w) = \sum_k C(w)/(z - w)^k$. (ii) In the domain $|w| > |z|$, the product $B(w)A(z)$ is given also by $\sum_k C(w)/(z - w)^k$, i.e formally by the same expression as $A(z)B(w)$. When these two conditions are satisfied, the commutator $[A,B]$ is defined and computed as

\begin{equation}
\tag{5.76}
[A, B] = \int_{|z| > |w|} A(z)B(w) - \int_{|w| > |z|} B(w)A(z)
\end{equation}

\begin{equation}
\tag{5.77}
= \left(\int_{|z| > |w|} - \int_{|w| > |z|}\right) [dz][dw] \sum_k \frac{C_k(w)}{(z - w)^k}
\end{equation}

\begin{equation}
\tag{5.77}
= \int [dw] \int [dz] \sum_k \frac{C_k(w)}{(z - w)^k} = \int [dw] C_1(w),
\end{equation}

\textsuperscript{18} For illustration, we consider the case of bosonic operators.
where \( \int [dz]_w \) means the integral circling \( w \). This discussion shows that we have to be careful when the conditions (i) and/or (ii) are not satisfied. Looking at the basic OPE’s given in (5.72) ~ (5.75), we see that in the present case there are indeed cases where these conditions are not quite satisfied. In such cases, what we have to do is to simply go back to the basic definition, \( i.e. \) the first line (5.76) above, and treat those parts which violate (i) and/or (ii) separately. Once this is done then the rest that satisfies (i) and (ii) can be computed easily by the usual OPE method as above. In appendix B.1 we shall illustrate this method explicitly for the computation of \([L^{\mu \nu}, L^{\rho \sigma}]\), where \( L^{\mu \nu} \) is given by (5.71).

It should now be clear that by using the above trick we can study the closure of the Lorentz algebra by effectively separating the analysis into that of the holomorphic sector and the anti-holomorphic sector. For our system, the holomorphic Lorentz currents are given by

\[
J^{IJ}(z) = \frac{1}{2} (\overset{\circ}{X}^I \Pi^J(z) - \overset{\circ}{X}^J \Pi^I(z)) - \frac{i}{4} S^a (\gamma^{IJ})_{ab} S^b(z),
\]

\[
J^{\mu+}(z) = \frac{1}{2} (\overset{\circ}{X}^{\mu} \Pi^+(z) - \overset{\circ}{X}^+ \Pi^{\mu}(z)),
\]

\[
J^{I-}(z) = \frac{1}{2} (\overset{\circ}{X}^I \Pi^-(z) - \overset{\circ}{X}^- \Pi^I(z)) + \frac{i}{4} (\bar{\gamma}^I S)_a (\bar{\gamma}^K S)_{ab} \Pi^K(z),
\]

and the anti-holomorphic currents are quite similar. Note that the fermionic contributions in \( J^{IJ} \) and the last term of \( J^{I-} \), which was generated by the compensating \( \kappa \)-transformation, are naturally holomorphic as they do not involve \( X^\mu \).

We now describe the actual analysis of the closure of the quantum Lorentz algebra. Since only the boost generator \( M^{I-} = \int [dz] J^{I-}(z) \) is affected non-trivially by the SLC gauge condition, we will focus on this generator. Before we begin the computation of the commutators, however, we must examine if the generators are physical, \( i.e. \) if they commute with the BRST charge \( Q \). By using our OPE technique, it is easily found that, while \( M^{IJ} \) and \( M^{\mu+} \) are physical, \( M^{I-} \) is not. (This phenomenon was noted previously in Ref. 29,31.) This problem is cured by adding a simple quantum correction. The modified generator \( \mathcal{M}^{I-} \) is given by

\[
\mathcal{M}^{I-} = M^{I-} + \Delta M^{I-},
\]

\[
M^{I-} = \int [dz] \left\{ \frac{1}{2} (\overset{\circ}{X}^I \Pi^-(z) - \overset{\circ}{X}^- \Pi^I(z)) + \frac{i}{4} (\bar{\gamma}^I S)_{a} (\bar{\gamma}^K S)_{ab} \Pi^K(z) \right\},
\]

\[
\Delta M^{I-} = - \int [dz] \frac{i \partial \Pi^I(z)}{2 \Pi^+(z)},
\]

which can be checked to satisfy the desired physical condition \([Q, \mathcal{M}^{I-}] = 0\). The necessity of the added term \( \Delta M^{I-} \) can be understood in the following way. Up to a zero mode part,
\[ \int [dz] \frac{1}{2} (\dot{X}^I \Pi^- - \dot{X}^- \Pi^I) \] equals \[ \int [dz] \dot{X}^I \Pi^- \] and to promote \( \Pi^- \) to a primary field we should make a replacement \( \Pi^- \to \Pi^\dagger \), where \( \Pi^\dagger \) is given in (5.24). This adds to \( M^I^- \) the extra term \( \int [dz] \dot{X}^I \partial^2 (1/2 \Pi^+) \), which upon integration by parts twice becomes precisely \( \Delta M^I^- \) above.

Let us now compute the commutator \([M^I-, M^J^-]\). From the experience of the computation in the full light-cone gauge,\(^{37}\) it is expected to yield a non-trivial result. Using the OPE technique, we obtain

\[
[M^I-, M^J^-] = \int [dw] \left[ \frac{1}{2} \frac{\Pi^+ \Pi^-(w) (\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)^2} \right. \\
- \frac{1}{4} \frac{\Pi^I \Pi^K (w) (\gamma^I S)_a (\gamma^K S)_a (w)}{\Pi^+(w)^2} + \frac{1}{4} \frac{\Pi^J \Pi^K (w) (\gamma^J S)_a (\gamma^K S)_a (w)}{\Pi^+(w)^2} \\
+ \frac{1}{2} \frac{\Pi^I \partial \Pi^J (w) - \Pi^I \partial \Pi^J (w)}{\Pi^+(w)^2} + \frac{1}{4} \left\{ \partial^2 \left( \frac{1}{\Pi^+(w)} \right) \right\} \left( \frac{(\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)} \right) \\
- \frac{1}{16} \left\{ \partial \left( \frac{(\gamma^I S)_a (\gamma^K S)_a (w)}{\Pi^+(w)} \right) \right\} \left( \frac{(\gamma^J S)_b (\gamma^K S)_b (w)}{\Pi^+(w)} \right) \\
- \frac{1}{4} \left( \frac{\Pi^K \Pi^L (w)}{\Pi^+(w)^2} \right) \{ S_a (w) (\gamma^I \gamma^J \gamma^L a \ b) S_b (w) \} \\
+ \frac{1}{8} \left\{ \partial \left( \frac{S_a (w)}{\Pi^+(w)} \right) \right\} \left( \gamma^I \gamma^J \gamma^L a \ b \right) \left\{ \partial \left( \frac{S_b (w)}{\Pi^+(w)} \right) \right\} \\
- \frac{1}{8} \left\{ \partial \left( \frac{\Pi^K (w)}{\Pi^+(w)} \right) \right\} \left( \frac{\Pi^L (w)}{\Pi^+(w)} \right) \text{Tr} \left( \gamma^I \gamma^J \gamma^L \right) \right]. \quad (5.83)
\]

In the course of this calculation, we have performed appropriate integration by parts and discarded the resultant total-derivative terms. This type of manipulation will often be made in the subsequent calculations as well. By using various \( \gamma \)-matrix formulas given in appendix C.1, this can be simplified to

\[
[M^I-, M^J^-] = \int [dw] \left[ \frac{1}{2} \frac{\Pi^+ \Pi^-(w) (\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)^2} \right. \\
+ \frac{1}{4} \frac{\Pi^K \Pi^K (w) (\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)^2} \frac{1}{4} \frac{S_a \partial S_a (\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)^2} \\
+ \left\{ \partial^2 \left( \frac{1}{\Pi^+(w)} \right) \right\} \left( \frac{(\gamma^I S)_a (\gamma^J S)_a (w)}{\Pi^+(w)} \right) \\
+ \frac{3}{4} \left( \frac{(\gamma^I \partial^2 S)_a (\gamma^J S)_a (w)}{\Pi^+(w)^2} \right) - \frac{3}{2} \left( \frac{\partial \Pi^+(w) (\gamma^I \partial S)_a (\gamma^J S)_a (w)}{\Pi^+(w)} \right) \right]. \quad (5.84)
\]

Some detail of this computation is given in appendix B.2.
So the commutator does not vanish. However, we can now show that it is actually a
BRST-exact term. To this end, consider the fermionic operator
\[
\Psi \equiv \frac{1}{2} \int [dw] \left( \frac{b(w)(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} \right).
\] (5.85)
The anti-commutator of \( \Psi \) and the BRST charge \( Q \) can be calculated straightforwardly and yields
\[
\{Q, \Psi\} = \int [dw] \left[ \frac{1}{2} \frac{\Pi^+\Pi^-(w)(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} - \frac{1}{4} \frac{S_a \partial S_a(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} \right.
\]
\[
+ \frac{1}{4} \frac{\Pi^K \Pi^K(w)(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} - \frac{1}{4} \frac{S_a \partial S_a(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} \]
\[
- \frac{1}{4} \frac{1}{(\Pi^+(w))^2} (\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w) \left( \partial \Pi^+(w) \right) - \frac{1}{4} \frac{1}{(\Pi^+(w))^3} (\tilde{\gamma}^I S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w) \]
\[
+ \frac{3}{8} \frac{(\tilde{\gamma}^I \partial^2 S)_a(\tilde{\gamma}^J S)_{\dot{a}}(w)}{(\Pi^+(w))^2} + \frac{3}{8} \frac{(\tilde{\gamma}^I S)_a(\tilde{\gamma}^J \partial^2 S)_{\dot{a}}(w)}{(\Pi^+(w))^2} \right].
\] (5.86)
(Although ghost-dependent terms also appear in the OPE, such terms become total-derivatives
in \( \partial_w \) and drop out\(^19\). ) Now by performing partial integration twice for the last term of
(5.86), one finds that this expression becomes identical to the commutator (5.84) we found
previously. Thus, we have proved the desired relation
\[
[\mathcal{M}^I^-, \mathcal{M}^J^-] = 0 + \{Q, \Psi\}.
\] (5.87)
This constitutes a highly non-trivial check of the closure of the Lorentz algebra in the BRST-
cohomology of the superstring in the SLC gauge. We have also calculated the other commu-
tators involving \( \mathcal{M}^I^- \) in a similar way and found that they are consistent with the Lorentz
algebra without BRST-exact terms.

5.2.3. Spinorial property of supercharges

As the final step of the study of the realization of the super-Poincaré algebra, let us
examine if the supercharges transform properly under the Lorentz transformations. Since
only the boost generators \( \mathcal{M}^I^- \) are non-trivially affected by the SLC gauge condition, we
again concentrate on the commutators \([\mathcal{M}^I^-, Q_a]\) and \([\mathcal{M}^I^-, Q_{\dot{a}}]\). From the expressions
of the supercurrents (5.30), the supercharges are given by (we continue to set \( \ell_s = 1 \) for
simplicity)
\[
Q_a = -2^{3/4} \int [dz] \sqrt{\Pi^+(z)} S_a(z), \quad Q_{\dot{a}} = -2^{1/4} \int [dz] \frac{(\tilde{\gamma}^I S)_{\dot{a}} \Pi_I(z)}{\sqrt{\Pi^+(z)}},
\] (5.88)
\(^{19}\) This is consistent with the fact that the Lorentz current and their commutator do not include ghosts.
Using these formulas, we finally obtain

\[ [\mathcal{M}^l, Q_a] = i2^{-1/4} \int [dw] \left\{ S_a \Pi^I(w) \frac{\sqrt{\Pi^+}}{\sqrt{\Pi^-}} - \left( \frac{\Pi^I(w)}{\sqrt{\Pi^-}} \right) (\gamma^K)_{ab} S_b(w) \right\} \]

\[ = i2^{-1/4} \int [dw] \left\{ \gamma_{aa}^I \frac{(\bar{\gamma}^KS)_{\dot{a}} \Pi^K(w)}{\sqrt{\Pi^+}} \right\} = - \frac{i}{\sqrt{2}} \gamma_{aa}^I Q_a. \]  

(5.89)

Since \( \gamma_{aa}^I = -\sqrt{2} \gamma_a^I \) in our convention, this result gives the correct commutator

\[ [\mathcal{M}^l, Q_a] = i2 \frac{\gamma_{aa}^I}{\sqrt{2}} Q_a. \]  

(5.90)

Next, we proceed to compute the commutator of \( \mathcal{M}^l \) and \( Q_a \). Since both generators contain non-trivial compensation terms due to the SLC gauge-fixing, this commutator is expected to yield the correct algebra up to BRS exact terms, just as in the case of \([\mathcal{M}^l, \mathcal{M}^l] \).

With the expressions of \( \mathcal{M}^l \) and \( Q_a \) given in (5.80) and (5.88), we can straightforwardly compute the commutator and obtain

\[ [\mathcal{M}^l, Q_a] = (-i)2^{1/4} \int [dw] \left\{ \Pi^- \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}} + \frac{1}{2} \frac{\Pi^I \Pi^K \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}}}{(\Pi^+)^{3/2}} \right. 

\[ + \frac{1}{16} \left( \partial \Pi^+ \right)^2 \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\Pi^+} - \frac{7}{8} \left( \partial \Pi^+ \right)^2 \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{(\Pi^+)^{5/2}} + \frac{1}{2} \partial^2 \left( \frac{1}{\Pi^+} \right) \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}} 

\[ + \frac{1}{4} \partial \left( \frac{(\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b(w)}{\Pi^+} \right) \right\} . \]  

(5.91)

In getting the first line, the identity (C.7) has been utilized. Now we can rewrite the last term, cubic in \( S^a \), by judiciously performing integration by parts and get

\[ \partial \left( \frac{(\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b}{\Pi^+} \right) \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}} = \frac{1}{3} \left\{ \partial \left[ (\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b \right] (\bar{\gamma}^I S)_{\dot{a}} - 2(\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b \partial [(\bar{\gamma}^K S)_{\dot{a}}] \right\} \right\} . \]  

(5.92)

Further, by using the \( \gamma \)-matrix identities (C.3) and (C.4), the numerator of this expression can be simplified to

\[ \partial \left[ (\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b \right] (\bar{\gamma}^I S)_{\dot{a}} - 2(\bar{\gamma}^I S)_b(\bar{\gamma}^K S)_b \partial [(\bar{\gamma}^K S)_{\dot{a}}] = -6S_b \partial S_b (\bar{\gamma}^I S)_{\dot{a}}. \]  

(5.93)

Using these formulas, we finally obtain

\[ [\mathcal{M}^l, Q_a] = (-i)2^{1/4} \int [dw] \left\{ \Pi^- \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}} + \frac{1}{2} \frac{\Pi^I \Pi^K \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{\sqrt{\Pi^+}}}{(\Pi^+)^{3/2}} \right. 

\[ + \left( \frac{37}{16} \right) \left( \partial \Pi^+ \right)^2 \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{(\Pi^+)^{7/2}} - \left( \frac{11}{8} \right) \frac{\partial^2 \Pi^+ \frac{(\bar{\gamma}^I S)_{\dot{a}}(w)}{(\Pi^+)^{5/2}}}{(\Pi^+)^{3/2}} \right\} . \]  

(5.94)
Now, analogously to the case of the boost commutator, we define the following bosonic operator $\Phi$:

$$\Phi = (-i 2^{1/4}) \int [dw] \left\{ \frac{b(w)(\bar{\gamma}^{I} S)_{\dot{a}}(w)}{(\Pi^{+}(w))^{3/2}} \right\}. \quad (5.95)$$

Then, the commutator with the BRST operator yields

$$[Q, \Phi] = (-i 2^{1/4}) \int [dw] \left\{ \frac{\Pi^{-} (\bar{\gamma}^{I} S)_{\dot{a}}(w)}{\sqrt{\Pi^{+}}} + \frac{1}{2} \frac{\Pi^{K} \Pi^{K} (\bar{\gamma}^{I} S)_{\dot{a}}(w)}{(\Pi^{+})^{3/2}} - \frac{1}{2} \frac{S_{\dot{a}} S_{\dot{b}} (\bar{\gamma}^{I} S)_{\dot{a}}(w)}{(\Pi^{+})^{3/2}} \right\}.

(5.96)$$

By rewriting the last term of (5.96) by performing partial integration twice, we find that this expression coincides with the commutator (5.94). Thus, we have obtained the desired result

$$[\mathcal{M}^{I-}, Q_{\dot{a}}] = 0 + [Q, \Phi], \quad (5.97)$$

which shows that $Q_{\dot{a}}$ transforms correctly as a spinor in the physical Hilbert space. In checking these non-trivial relations, one notes that the quantum correction terms in $T(z)$ and in $\mathcal{M}^{I-}$ play crucial roles.

This completes the demonstration that the super-Poincaré algebra is properly represented in the quantum Green-Schwarz superstring in the SLC gauge in the sense of BRST cohomology.

§6. Vertex operators for massless states

Having clarified the structure of the quantum Virasoro and super-Poincaré algebras in the SLC gauge, we are now ready to construct the vertex operators of the theory. Although we have been dealing with the type II closed superstring, to avoid unnecessary technical complications we will study the type I open string vertices at the massless level for illustration. Extension to the closed string case is straightforward.

As is well-known, the vertex operators at the massless level of the open superstring should describe the “photon” and the “photino” excitations of the super- Maxwell theory in 10 dimensions.\textsuperscript{20} The basic principle for the construction is that these vertex operators must be BRST invariant and form the appropriate representation of the super-Poincaré algebra, up to BRST-exact terms. As we shall see, these requirements will indeed fix the form of the vertex operators albeit in a fairly intricate manner.

\textsuperscript{20} This of course can be regarded as the linear part of the super-Yang-Mills theory, with the Chan-Paton factor suppressed.
6.1. General form of the BRST-invariant vertex operators

In the following, we will be dealing with the integrated vertex operator of the form \( V = \int [dz] \mathcal{V}(z) \). (The unintegrated vertex operator \( \mathcal{U}(z) \) is related to \( \mathcal{V}(z) \) by \( Q \mathcal{V}(z) = \partial \mathcal{U}(z) \), where \( Q \) is the BRST operator\(^{21}\).) For \( V \) to be BRST invariant, \( \mathcal{V}(z) \) must be a primary operator of dimension \( \Delta = 1 \) on the mass-shell, where the momentum \( k^\mu \) satisfies \( k^\mu k_\mu = 0 \) and the wave functions satisfy appropriate equations of motion. In addition, for a technical reason to be explained shortly, we need to consider the vertex operators in the Lorentz frame in which \( k^+ = 0 \). \( \mathcal{V}(z) \) must also be consistent with the symmetries which remain manifest in the SLC gauge. One is of course the SO(8) symmetry. Another useful symmetry is the boost symmetry generated by \( M^+M^- \). From the transformation property under \( M^+M^- \), one can read off the boost charges for various quantities as

\[
X^+, \zeta^+ : +1, \quad X^-, k^-, \zeta^- : -1, \quad X^I, k^I, \zeta^I : 0, \quad (6.1)
\]

\[
u^a : -1/2, \quad u^a : +1/2, \quad S_a : 0. \quad (6.2)
\]

Consider first the candidates for the boson emission vertex operator, which must be linear in the polarization vector \( \zeta^\mu \). If we temporarily disregard the common structure \( \exp(ik^\mu X^\mu) \), there are two types of operators which satisfy all the requirements above. One type consists of\(^{22}\)

\[
\zeta^- \Pi^+, \zeta^+ \Pi^-, \zeta^I \Pi^I, \zeta^I R^I, \zeta^+ k^- k^I \Pi^I, \quad (6.3)
\]

where \( R^I \) is a fermion bilinear defined by

\[
R^I = k_J S_{\gamma^IJ} S. \quad (6.4)
\]

Another type of operators are of the form \((\zeta^+ / \Pi^+) \times (\Delta = 2 \text{ transverse primaries})\), which are\(^{23}\)

\[
\frac{\zeta^+}{\Pi^+} \times (\Pi^I R^I, R^I R^I, (k^I \Pi^I)(k^J \Pi^J)) \quad (6.5)
\]

Now we have to combine these operators with the exponential factor \( \exp(ik^\mu X^\mu) \). If \( k^+ \neq 0 \), a problem arises. As we already mentioned at the end of section 5.1.2, the field \( \Pi^-(z) = i\partial X^-(z) \) and hence \( X^-(z) \) as well do not have good conformal properties and consequently the factor \( \exp(ik^+ X^-) \) is not a primary operator. One might try to use the

\(^{21}\) This is the general definition of the unintegrated vertex operator applicable to any theories. In the present case it reduces to the familiar relation \( \mathcal{U} = \alpha \mathcal{V} \).

\(^{22}\) \( \zeta^I k^J k^I \Pi^I \) is omitted since \( \zeta^I k^I = -\zeta^+ k^- \).

\(^{23}\) Note that \( \Pi^I \Pi^I \) and \( k^I \partial \Pi^I \) are not included in the list since they produce 4th and 3rd order poles in the OPE with \( T(z) \) and are not primary operators.
modified field $\hat{X}^- \equiv X^- - \partial (1/2 \partial X^+)$, which is the integral of $(1/i)\hat{\Pi}^-$ and has a good conformal property, but unfortunately $\hat{X}^-(z)$ has a severe singularity with itself and it is difficult to define the operator $\exp(ik^+ \hat{X}^-)$. Therefore in this work we will only deal with the configuration (or the frame) where $k^+ = 0$ and $\exp(ik^- X^+ + ik^I X^I)$.

How one might be able to improve on this situation will be discussed in the discussion section.

Combined with this special type of exponential factor, it is easy to check that, while the terms $\zeta^- \Pi^I(z)$ and $\zeta^I \Pi^I(z)$ must appear together as $(\zeta^- \Pi^I + \zeta^I \Pi^I) e^{ik^- X^+ + ik^I X^I}$ in order to produce a genuine primary operator of dimension one, each of the remaining structures continues to be an independent primary operator of dimension one. Therefore, the general form of the BRST-invariant vertex operator for the photon is

$$V_B(\zeta) = \int [dz] e^{i k_\mu X^\mu(z)} \left\{ A \zeta^- \Pi^+ + B (\zeta^I \Pi^I(z) + \zeta^- \hat{\Pi}^-(z)) + C \zeta^I R^I(z) + \zeta^+ \left( D \frac{\Pi^I R^I(z)}{\Pi^+} + E \frac{R^I R^I(z)}{\Pi^+} + Y k^- k_1 \Pi^I(z) + Z \frac{(k_1 \Pi^I)(k_J \Pi^J)(z)}{\Pi^+} \right) \right\},$$

(6.6)

where $\zeta^- \Pi^I$ satisfies the transverse condition $\zeta^\mu k_\mu = 0$. The relative magnitude of the coefficients $A, B, C, D, E, Y, Z$, which are arbitrary at this stage, should be determination solely from the requirement of the primarity, had we been able to deal with the general $k^+ \neq 0$ case.

For lack of such means, we will show that these coefficients can alternatively be uniquely determined from the requirement of super-Poincaré covariance.

Next, consider the candidates for the fermion vertex, describing the photino, linear in the fermionic wave function $u^a$. Again there are two types. The one containing $u^a$ is unique and is given by $u^a \sqrt{\Pi^+} S_a$. The other type proportional to $u^\dot{a}$ consists of two operators

$$u^\dot{a} \left( \bar{\gamma}^I S \right) \int \left\{ G \frac{\Pi^I(z)}{\sqrt{\Pi^+}} + \frac{R^I(z)}{\sqrt{\Pi^+}} \right\}.$$

(6.7)

Thus the general form of the fermion vertex (which is a bosonic operator as a whole) is

$$V_F(u) = \int [dz] e^{i k_\mu X^\mu(z)} \left\{ u^a \left( G \sqrt{\Pi^+} S_a(z) \right) + u^\dot{a} \left( K \bar{\gamma}^I S \right) \frac{\Pi^I(z)}{\sqrt{\Pi^+}} + \frac{L \bar{\gamma}^I S R^I(z)}{\sqrt{\Pi^+}} \right\}.$$

(6.8)

$G, K, L$ are coefficients to be determined and $u^\alpha$ must satisfy the on-shell condition $k_\mu \gamma^\mu u = 0$.

These vertex operators must transform properly under the supersymmetry transformations. This requires

$$[\eta^\alpha Q_a, V_B(\zeta)] = V_F(\tilde{u}), \quad [\eta^\alpha Q_a, V_F(u)] = -V_B(\tilde{\zeta}),$$

(6.9)
\[ [\epsilon \dot{Q}_a, V_B(\zeta)] = V_F(\tilde{u}), \quad [\epsilon \dot{Q}_a, V_F(u)] = -V_B(\tilde{\zeta}), \] (6.10)

where \( \tilde{\zeta}, \tilde{\zeta} \) and \( \tilde{u}, \tilde{u} \) are, respectively, the polarization vectors and the spinor wave functions obtained by the supersymmetry transformations of the corresponding super-Maxwell fields. Their explicit form will be given in the next subsection. Note the minus signs in front of the \( V_B \)'s on the right hand side. We will see below that it is dictated by the consistent realization of the super-Poincaré symmetries on the vertex operators.

6.2. SUSY transformation of the wave functions

To obtain the transformation rules for the wave functions \( \zeta^\mu \) and \( u^\alpha \) under the \( \epsilon \)- and \( \eta \)-supersymmetries, we must examine the super-Maxwell theory in 10 dimensions. The action is given by

\[ S = \int d^{10}x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \Gamma^\mu \partial_\mu \psi \right), \] (6.11)

where \( \psi^\alpha(x) \) is a sixteen-component Majorana-Weyl spinor. It is invariant under the SUSY transformation with a spinor parameter \( \epsilon^\alpha \):

\[ \delta A^\mu = i\bar{\epsilon} \Gamma^\mu \psi = i\epsilon^\alpha \tilde{\gamma}^\alpha \psi^\beta, \] (6.12)

\[ \delta \psi^\alpha = \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon = \frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})^\alpha_\beta \epsilon^\beta. \] (6.13)

The equation of motion for \( A^\mu \), in the Lorentz gauge \( \partial_\mu A^\mu = 0 \), is

\[ \Box A^\mu = 0, \] (6.14)

while that of the fermion \( \psi^\alpha \) is

\[ \Gamma^\mu D_\mu \psi = 0. \] (6.15)

The SUSY transformation above is compatible with these equations of motion and the Lorentz gauge condition.

To separate the \( \epsilon \)- and \( \eta \)-SUSY transformations, we make the \( SO(8) \) decomposition of spinors and \( \gamma \)-matrices described in appendix A.2. In particular, the spinor field and the SUSY parameter are decomposed as \( \psi^\alpha = (\psi^a, \psi^{\dot{a}}) \) and \( \epsilon^\alpha = (\eta^a, \dot{\epsilon}^{\dot{a}}) \). Then, the SUSY transformation (6.12) for \( A^\mu \) is decomposed as

\[ \delta A^+ = i\epsilon \tilde{\gamma}^+ \psi = i\epsilon \tilde{\gamma}^+ \psi^b = -i\sqrt{2} \epsilon^b \psi_a, \] (6.16)

\[ \delta A^- = i\epsilon \tilde{\gamma}^- \psi = i\eta^a \tilde{\gamma}_{ab} \psi^b = i\sqrt{2} \eta^a \psi_a, \] (6.17)

\[ \delta A_I = i\epsilon \tilde{\gamma}^I \psi = i\epsilon \tilde{\gamma}^I \psi^b + i\eta^a \tilde{\gamma}^I_{ab} \psi^b. \] (6.18)
Similarly, the SUSY transformation (6.13) for $\psi^\alpha$ becomes
\[
\delta \psi^\alpha = \frac{1}{2} F_{\mu\nu} \left( (\gamma^{\mu\nu})^a b\eta^b + (\gamma^{\mu\nu})^a \delta^b \right)
= \frac{1}{2} F_{IJ} \left( \gamma^I \right)^a a_{IJ} b\eta^b + F_{-\eta^a} + \sqrt{2} F_{+I} \left( \gamma^I \right)^a \delta^b,
\]
(6.19)
\[
\delta \bar{\psi} = \frac{1}{2} F_{\mu\nu} \left( (\gamma^{\mu\nu})^a b\eta^b + (\gamma^{\mu\nu})^a \delta^b \right)
= \frac{1}{2} F_{IJ} \left( \gamma^I \right)^{\dot{a}} \delta^b + F_{-\eta^{\dot{a}}} + \sqrt{2} F_{+I} \left( \gamma^I \right)^{\dot{a}} b\eta^b.
\]
(6.20)

Now the polarization vector $\zeta^\mu(k)$ and the fermionic wave function $u^\alpha(k)$ are defined through the Fourier transform of $A^\mu$ and $\psi^\alpha$ as
\[
A^\mu(x) = \int [dk] \zeta^\mu(k)e^{ikx},
\]
(6.21)
\[
\psi^\alpha(x) = \int [dk] u^\alpha(k)e^{ikx}.
\]
(6.22)

We emphasize that since $\psi^\alpha$ is Grassmann odd $u^\alpha$ should also be regarded as Grassmann odd quantity for consistency. From the transformation rules for $A^\mu$ and $\psi^\alpha$, it is now straightforward to read off the SUSY transformation properties of the components of $\zeta^\mu$ and $u^\alpha$. For the SUSY transformation with the parameter $\eta^a$ ($\eta$–SUSY), we get
\[
\delta_\eta \zeta^+ = 0, \quad \delta_\eta \zeta^- = i\sqrt{2} \eta^a u_a, \quad \delta_\eta \zeta^I = i\eta^b \gamma^I_{ab} u^b,
\]
(6.23)
\[
\delta_\eta u^a = i k_{IJ} \zeta_J \left( \gamma^I \right)^a b\eta^b + i \left( k^- \zeta^+ - k^+ \zeta^- \right) \eta^a,
\]
(6.24)
\[
\delta_\eta \bar{u} = i\sqrt{2} \left( k^I \zeta^+ - k^+ \zeta^I \right) \left( \gamma^I \right)^{\dot{a}} b\eta^b.
\]

Similarly, the SUSY transformation with the parameter $e^{\dot{a}}$ ($\epsilon$–SUSY) is given by
\[
\delta_\epsilon \zeta^+ = -i\sqrt{2} \epsilon^{\dot{a}} u_{\dot{a}}, \quad \delta_\epsilon \zeta^- = 0, \quad \delta_\epsilon \zeta^I = i\epsilon^{\dot{a}} \gamma^I_{ab} u^b,
\]
(6.25)
\[
\delta_\epsilon u^a = i\sqrt{2} \left( k^- \zeta^I - k^I \zeta^- \right) \left( \gamma^I \right)^a \delta^b,
\]
(6.26)
\[
\delta_\epsilon \bar{u} = i k_{IJ} \zeta_J \left( \gamma^I \right)^{\dot{a}} \delta^b - i \left( k^- \zeta^+ - k^+ \zeta^- \right) \delta^a.
\]
(6.27)

Then $\tilde{\zeta}, \tilde{u}$ etc. introduced in the previous subsection are identified as $\tilde{\zeta} \equiv \delta_\eta \zeta, \tilde{u} \equiv \delta_\eta u$ and $\tilde{\zeta} \equiv \delta_\epsilon \zeta, \tilde{u} \equiv \delta_\epsilon u$.

The on-shell conditions for the polarization and fermionic wave function can be also obtained by the Fourier transformation of the equations of motion, (6.14) and (6.15), and the Lorentz gauge condition. For the polarization, we have
\[
k_{\mu} k^\mu = 2k^+ k^- + k^I k^I = 0,
\]
(6.28)
\[
k_{\mu} \zeta^\mu = k^+ \zeta^- + k^- \zeta^+ + k^I \zeta^I = 0.
\]
(6.29)
As for the fermionic wave function, using the SO(8) decomposition, we get

\[
\sqrt{2}k^+ u_a + k^I \gamma^I \eta_a = 0, \tag{6.30}
\]

\[
-\sqrt{2}k^- u_a + k^I \gamma^I \eta_a = 0. \tag{6.31}
\]

In the frame where \( k^+ = 0 \), these equations become

\[
k^I k^I = 0, \quad k^- \zeta^+ + k^J \zeta^J = 0, \tag{6.32}
\]

\[
k^I \gamma^I \eta^I = 0, \quad \sqrt{2}k^- u_a = k^I \gamma^I \eta^I. \tag{6.33}
\]

6.3. Transformation of the vertex operators under \( \eta\)-SUSY

In order to determine the coefficients in the vertex operators, we first compute the commutator \([\eta^a Q_a, V_B(\zeta)]\) by the OPE method, where the generator \( Q_a \) of the \( \eta\)-SUSY is is given in (5.88). Using the OPE formulas (5.18) and (5.19) with \( \ell_a = 1 \) and partial integration, we get the following result:

\[
[\eta^a Q_a, V_B(\zeta)] = \int [dw] e^{ik \cdot X(w)} \left\{ \left( \frac{2}{7} C \right) k_I \zeta^I \sqrt{\Pi^+} (\eta \gamma^{IJ} S)(w) + \left( 2^{\frac{1}{4}} B \right) \zeta^+ k^- \sqrt{\Pi^+} (\eta^a S_a(w)) \right\}.
\]

(6.34)

Here and hereafter, \( \exp(ik \cdot X) \) means \( \exp(ik \cdot X^+ + ik^J X^J) \) i.e. with \( k^+ = 0 \).

This should be compared with the fermionic vertex \( W_F(\bar{u}) \). Inserting \( \bar{u}^a \) given in (6.24) (with \( k^+ = 0 \)) into the general form of \( W_F(u) \) given in (6.8), we get

\[
W_F(\bar{u}^a, \bar{u}^{a'}) = \int [dz] e^{ik \cdot X(z)} \left\{ (-iG)k_I \zeta^J \sqrt{\Pi^+} (\eta \gamma^{IJ} S)(z) + (iG)k^- \zeta^+ \sqrt{\Pi^+} (\eta^a S_a(z)) \right\}
\]

\[
+ \left( i\sqrt{2} \right) k^I \gamma^I \eta^I \left( \gamma^{IJ} S_a(z) \right) \right\}
\]

\[
= \int [dz] e^{ik \cdot X(z)} \left\{ (-iG)k_I \zeta^J \sqrt{\Pi^+} (\eta \gamma^{IJ} S)(z) + (iG)k^- \zeta^+ \sqrt{\Pi^+} (\eta^a S_a(z)) \right\}
\]

\[
+ \left( i\sqrt{2} \right) k^I \gamma^I \eta^I \left( \gamma^{IJ} S_a(z) \right) \right\}
\]

\[
\tag{6.35}
\]

For the second equality we have used the \( \gamma\)-matrix identities (C.4) and the relation \( k_I R^J(z) = k_I k_J (S \gamma^{IJ} S)(z) = 0 \).
With these results, the SUSY relation \([\eta^a Q_a, V_B(\zeta)] = V_F(\tilde{u})\) yields the following relations among the coefficients:

\[
\begin{align*}
2\tilde{\tau}C &= -iG, \quad 2\tilde{\tau}B = iG, \quad 2\tilde{\tau}B = i\sqrt{2}K, \\
2\tilde{\tau}D &= i\sqrt{2}K, \quad 2\tilde{\tau}E = i\sqrt{2}L.
\end{align*}
\] (6.36)

Next, we consider the second part of the SUSY relation (6.9), namely \([\eta^a Q_a, V_F(u)] = -V_B(\tilde{\zeta}).\) The commutator can be calculated straightforwardly as

\[
[\eta^a Q_a, V_F(u)] = \int [dw] e^{ik \cdot X(w)} \left\{ \left( 2\tilde{\tau} G \right) \Pi^+(w) (\eta^a u_a) + \left( 2\tilde{\tau} K \right) \Pi^I(w) (\eta \gamma^I u) \right.
\]
\[
\left. + \left( 2\tilde{\tau} L \right) (R^I(w) (\eta \gamma^I u) - 2k_J (u \gamma^I J S) (\eta \gamma^I J S)) \right\}. \tag{6.37}
\]

Reshuffling the spinor indices by the identity (C.3) and using the on-shell condition (6.33) for the fermion, the last term can be rearranged into

\[
k_J (u \gamma^I J S) (\eta \gamma^I J S) = - (\eta \gamma^I u) k_J (S \gamma^I J S) = -R^I (\eta \gamma^I u). \tag{6.38}
\]

Then, the commutator simplifies to

\[
[\eta^a Q_a, V_F(u)] = \int [dw] e^{ik \cdot X(w)} \left\{ \left( 2\tilde{\tau} G \right) \Pi^+(w) (\eta^a u_a) + \left( 2\tilde{\tau} K \right) \Pi^I(w) (\eta \gamma^I u) \right.
\]
\[
\left. + \left( 2\tilde{\tau} L \right) \left( \eta \gamma^I u \right) R^I(w) \right\}. \tag{6.39}
\]

On the other hand, with the form of the polarization \(\tilde{\zeta}^\mu\) given in (6.23) inserted into \(V_B(\zeta)\) in (6.6), \(V_B(\tilde{\zeta})\) is obtained as

\[
V_B(\tilde{\zeta}) = \int [dw] e^{ik \cdot X(w)} \left\{ i\sqrt{2} (\eta^a u_a) A \Pi^+(w) + i (\eta \gamma^I u) \left( B \Pi^I(w) + C R^I(w) \right) \right\}. \tag{6.40}
\]

Comparing these results, we obtain further relations among the coefficients as

\[
-i\sqrt{2} A = 2\tilde{\tau} G, \quad -iB = 2\tilde{\tau} K, \quad -iC = 2\tilde{\tau} 3 L. \tag{6.41}
\]

Now from the set of relations (6.36) and (6.41) we can determine many of the coefficients. Let us fix the overall normalization by setting \(B = 1\), which is the conventional choice in the case of the full light-cone gauge vertex. Then the unique solution satisfying the above relations is

\[
A = 1, \quad B = 1, \quad C = \frac{1}{4}, \quad D = \frac{1}{4}, \quad E = \frac{1}{96},
\]
\[
G = -i2^{-\frac{3}{4}}, \quad K = -i2^{-\frac{3}{4}}, \quad L = i \frac{2^{-\frac{3}{4}}}{12}. \tag{6.42}
\]
It should be remarked that the coefficients $A = B = 1, C = -\frac{1}{4}$ for $V_B(\zeta)$ and $G, K, L$ in (6.42) for $V_F(u)$ are consistent with the result in the full light-cone gauge.\(^\text{36,37}\)

On the other hand, the coefficients $Y$ and $Z$ in $V_B(\zeta)$ have not been determined by the foregoing analysis. We will see that they will be fixed by the analysis of the property under $\epsilon$-SUSY.

### 6.4. Transformation of the vertex operators under $\epsilon$-SUSY

From the $\eta$–SUSY relations, we have been able to fix the form of the massless vertex operators except for a few terms. We will now check if these vertex operators also satisfy the $\epsilon$–SUSY relations and fix the remaining coefficients.

This analysis is rather non-trivial since the $\epsilon$–SUSY transformation contains the compensating $\kappa$ transformation induced by the SLC gauge fixing condition. As a result, there will appear certain BRST-exact terms in the consistency relations. Explicitly, we will find that the following relations are realized:

\[ [\epsilon^a Q_a, V_F(u)] = -V_B(\tilde\zeta) + \{Q, \Psi_F(\epsilon, u)\}, \quad (6.43) \]
\[ [\epsilon^a Q_a, V_B(\zeta)] = V_F(\tilde\mu) + \{Q, \Psi_B(\epsilon, \zeta)\}. \quad (6.44) \]

Here $Q$ is the BRST charge defined in (5.25) and $\Psi_F$ and $\Psi_B$ are appropriate fermionic operators. Note that since $V_F(u)$ has already been completely fixed by the analysis of the previous subsection, the minus sign in front of $V_B(\tilde\zeta)$ is dictated by the calculation of the commutator $[\epsilon^a Q_a, V_F(u)]$.

First, let us derive the relation (6.43). The supercharge $Q_a$ for $\epsilon$–SUSY is given by (5.88) and the fermion vertex operator $V_F(u)$, which has been determined completely from the previous analysis, is of the form

\[ V_F(u) = \int [dz] \, e^{ik \cdot X(z)} \left\{ u^a \left( -i 2^{-1/4} \sqrt{\Pi^+} S_a(z) \right) + u^a \left( -i 2^{-3/4} \frac{\bar{\gamma}^I S^a_{\dot{\gamma}} \Pi_I(z)}{\sqrt{\Pi^+}} + i \left( \frac{2^{-3/4}}{12} \right) \frac{\bar{\gamma}^I S^a_{\dot{\gamma}} R_I(z)}{\sqrt{\Pi^+}} \right) \right\}. \quad (6.45) \]

Using the OPE technique, the relevant commutator $[\epsilon^a Q_a, V_F(u)]$ is computed as\(^{24}\)

\[ [\epsilon Q, V_F(u)] = \int [dw] \, e^{ik \cdot X} \left[ -i \Pi_I (\epsilon \bar{\gamma}^I u) - i \partial \left( \frac{1}{\sqrt{\Pi^+}} \right) k_I (\epsilon \bar{\gamma}^I u) \right. \]
\[ \left. - \frac{i}{\sqrt{2}} \frac{\Pi_I \Pi^I}{\Pi^+} (\epsilon u) - i \sqrt{2} \partial \left( \frac{1}{\sqrt{\Pi^+}} \right) k_I \Pi^I \left( \frac{1}{\sqrt{\Pi^+}} \right)^2 \frac{1}{\Pi^+} (\epsilon u) \right] \]

\(^{24}\)For simplicity, we use the abbreviation such as $\epsilon \bar{\gamma}^I u = \epsilon^a \bar{\gamma}^I_{\dot{a}b} u^b$, $\epsilon u = \epsilon^a u_{\dot{a}}$, $u S = u^a S_a$, $u \bar{\gamma}^I S = u^a \bar{\gamma}^I_{\dot{a}b} S_b$, etc. and omit the functional dependence on $w$ in the integrand.
\[
+ \left( \frac{i}{12\sqrt{2}} \right) \frac{\Pi_I R^I}{\Pi^+} (\epsilon u) + \left( \frac{i}{12\sqrt{2}} \right) \frac{\Pi_I R_J}{\Pi^+} (\epsilon^{ij} u)
+ A_1 + A_2 + A_3 + A_4 + A_5 \right],
\]  

(6.46)

where

\[
A_1 = i k_I (\epsilon^{ij} S) (u S),
\]

(6.47)

\[
A_2 = \frac{i}{\sqrt{2}} k_I (\epsilon^{ij} S) \frac{\Pi_J (u \gamma^J S)}{\Pi^+},
\]

(6.48)

\[
A_3 = - \left( \frac{i\sqrt{2}}{12} \right) \frac{\Pi_I (u \gamma^J S) k_L (\epsilon^{ij} \gamma^{JL} S)}{\Pi^+},
\]

(6.49)

\[
A_4 = \frac{i}{\sqrt{2}} \partial \left( \frac{(\epsilon^{ij} S)}{\sqrt{\Pi^+}} \right) \frac{(u \gamma^J S)}{\sqrt{\Pi^+}},
\]

(6.50)

\[
A_5 = - \left( \frac{i}{12\sqrt{2}} \right) \frac{k_I (\epsilon^{ij} S) R_J (u \gamma^J S)}{\Pi^+}.
\]

(6.51)

(We have singled out the last 5 terms for later convenience.) In this computation, we have made appropriate use of the equation of motion for the fermion wave function (6.33).

On the other hand, by inserting the form of the polarization vector \( \tilde{\zeta}^\mu \) given in (6.25), the boson vertex operator \( V_B(\tilde{\zeta}) \) with the coefficients (6.42) becomes

\[
V_B(\tilde{\zeta}) = \int [dw] e^{ik \cdot X} \left[ -i\sqrt{2} (\epsilon u) \left( \Pi^- + \frac{1}{4} \frac{\Pi_I R^I}{\Pi^+} - \frac{1}{96} \frac{R_I R^I}{\Pi^+} \right)
+ Y k^- k_I \Pi^I + Z \left( k_I R^I \right) \left( k_J R^J \right) \right] + i (\epsilon^{ij} S) \left( \Pi_I - \frac{1}{4} R_I \right).\]

(6.52)

We now wish to compare the results (6.46) and (6.52). This turned out to require heavy use of various \( \gamma \)-matrix identities summarized in appendix C.1 to rewrite the terms denoted by \( A_1 \sim A_5 \) in (6.46). Below we sketch this procedure.

First, focus on the expression \( A_1 \), which is quadratic in the fermion operator \( S \). By using the Fierz identity (C.5), it can be rewritten as

\[
i k_I (\epsilon^{ij} S) (u S) = - \frac{i}{16} (S \gamma^{KL} S) k_I (\epsilon^{ij} \gamma^{KL} u)
= - \frac{i}{16} (S \gamma^{KL} S) k_I (\epsilon [\gamma^i, \gamma^{KL}] u) - \frac{i}{16} (S \gamma^{KL} S) k_I (\epsilon^{ij} \gamma^{KL} u).
\]

(6.53)

Recalling the definition of the operator \( R_I \) given in (6.4), applying the Fierz identity (C.6), and making use of the on-shell condition (6.33) for the fermion, this expression can be
\[ A_1 = \frac{i}{4} R_I \left( \epsilon \gamma^I u \right) + \frac{i}{2\sqrt{2}} k^- \left( \epsilon \gamma^I S \right) \left( u \gamma^I S \right). \] (6.54)

In a similar manner, the numerator of the expression \( A_2 \) can be transformed, with the help of (C.4), (C.5), (C.7) and (6.33), into

\[ k_I \left( \epsilon \gamma^I S \right) \left( u \gamma^I S \right) = -\frac{1}{16} \left( S \gamma^{KL} \right) k_I \left( \epsilon \gamma^I \gamma^{KL} \gamma^J u \right) \]
\[ = -\frac{1}{16} \left( S \gamma^{KL} \right) k_I \left[ \epsilon \left( \bar{\gamma}^I, \gamma^{KL} \right) \gamma^J + \bar{\gamma}^{KL} \left\{ \bar{\gamma}^I, \gamma^J \right\} - \bar{\gamma}^{KL} \bar{\gamma}^I \gamma^J \right] u \]
\[ = \frac{1}{4} R_J (\epsilon u) + \frac{1}{4} R_I \left( \epsilon \gamma^I \gamma^J u \right) + \frac{1}{2} k_J \left( \epsilon \gamma^I S \right) \left( u \gamma^I S \right). \] (6.55)

Then \( A_2 \) becomes

\[ A_2 = \frac{i}{4\sqrt{2}} \frac{\Pi^J R_J}{\Pi^+} (\epsilon u) + \frac{i}{4\sqrt{2}} \frac{\Pi_J R_I}{\Pi^+} \left( \epsilon \gamma^I \gamma^J u \right) + \frac{i}{2\sqrt{2}} \frac{k_J \Pi^J}{\Pi^+} \left( \epsilon \gamma^I S \right) \left( u \gamma^I S \right). \] (6.56)

Now we move on to the expression \( A_3 \). By using (C.5) it can be rewritten as

\[ k_L \left( u \gamma^I S \right) \left( \epsilon \gamma^I \gamma^{LM} \right) = -\frac{1}{16} k_L \left( S \gamma^{MN} \right) u \left( \bar{\gamma}^J \gamma^{MN} \gamma^{LM} \gamma^I \right) \epsilon \]
\[ = -\frac{1}{16} k_L \left( S \gamma^{MN} \right) u \left( \bar{\gamma}^J \left[ \gamma^{MN}, \gamma^{LM} \right] \gamma^I \right) \epsilon \]
\[ = -\frac{1}{16} k_L \left( S \gamma^{MN} \right) u \left( \bar{\gamma}^J \gamma^{MN} \gamma^I \right) \epsilon. \] (6.57)

where, actually, the last term vanishes due to the identity \( \bar{\gamma}^J \gamma^{LM} = -7 \bar{\gamma}^L \) and (6.33). Using the \( \gamma \)-commutator (C.9), the first term becomes

\[ -\frac{1}{16} k_L \left( S \gamma^{MN} \right) u \left( \bar{\gamma}^J \left[ \gamma^{MN}, \gamma^I \right] \right) \epsilon \]
\[ = -\frac{1}{4} R_M \left( u \gamma^J \gamma^{ML} \gamma^I \epsilon \right) - \frac{1}{4} k_L \left( S \gamma^{JM} \right) \left( u \gamma^J \gamma^{ML} \gamma^I \epsilon \right) = R_J \left( u \gamma^J \gamma^I \epsilon \right). \] (6.58)

For the last equality, the fermionic on-shell condition (6.33) was used. In this way, \( A_3 \) can be re-expressed as

\[ A_3 = - \left( \frac{i\sqrt{2}}{12} \right) \frac{\Pi_I}{\Pi^+} R_J \left( u \gamma^J \gamma^I \epsilon \right) = \left( \frac{i\sqrt{2}}{12} \right) \frac{\Pi_I R_I}{\Pi^+} (\epsilon u) + \left( \frac{i\sqrt{2}}{12} \right) \frac{\Pi_I R_J}{\Pi^+} \left( \epsilon \gamma^I \gamma^J u \right). \] (6.59)

Next consider the expression \( A_4 \). For this term, we perform an integration by parts appropriately and rewrite terms in a manner similar to the manipulation leading from (5.92) to (5.93). We then get

\[ A_4 = \frac{i}{\sqrt{2}} \frac{S \partial S}{\Pi^+} (\epsilon u) - \frac{i}{2\sqrt{2}} \frac{k^- \Pi^+ + k_J \Pi^J}{\Pi^+} \left( \epsilon \gamma^I S \right) \left( u \gamma^I S \right). \] (6.60)
Finally consider $A_5$. The numerator of this expression can be rearranged by the Fierz identity (C.5) and the (anti-)commutators of $\gamma$-matrices (C.2) and (C.7) in the following way:

$$k_I (\epsilon \bar{\gamma}^I S) R_J (u \bar{\gamma}^J S) = -\frac{1}{16} k_I R_J (S \gamma_{KL} S) (\epsilon \bar{\gamma}^I \gamma^K \gamma^I u)$$

$$= -\frac{1}{16} k_I R_J (S \gamma_{KL} S) \epsilon \left( [\bar{\gamma}^I , \gamma^K] \gamma^J + \bar{\gamma}^K \{ \gamma^I , \gamma^J \} - \bar{\gamma}^K \bar{\gamma}^J \gamma^I \right) u = \frac{1}{4} R_I R^I (\epsilon u) ,$$

(6.61)

Here we have used the identity $k_I R^I = 0$ and the on-shell condition (6.33) for the last equality. This leads to the simple result

$$A_5 = -i\sqrt{2} (\epsilon u) \left( \frac{1}{96} R_I R^I \right).$$

(6.62)

Substituting the results (6.54), (6.56), (6.59), (6.60) and (6.62) for $A_1 \sim A_5$ into the commutator (6.46) and using the on-shell condition (6.33) for the fermion, we obtain

$$[\epsilon Q, V_F(u)] = \int [dw] e^{ik \cdot X} \left( -i (\epsilon \bar{\gamma}^I u) \left( \Pi_I - \frac{1}{4} R_I \right) + i \sqrt{2} (\epsilon u) \left( \frac{1}{4} R_I R^I - \frac{1}{96} R_I R^I \right) \right)$$

$$- i\sqrt{2} (\epsilon u) \left\{ \frac{1}{2} \left( \frac{\Pi_I \Pi^I}{\Pi^+} - S \partial S \right) + \partial \left( \frac{1}{\sqrt{\Pi^+}} \frac{k^- \Pi^+ + k^- \Pi^I}{\sqrt{\Pi^+}} + \frac{2}{\sqrt{\Pi^+}} \partial^2 \left( \frac{1}{\sqrt{\Pi^+}} \right) \right) \right\}. \quad (6.63)$$

We see that the terms in the first line constitute a part of $-V_B(\tilde{\zeta})$, where $V_B(\tilde{\zeta})$ is given in (6.52). This means that the relation (6.43) holds if the remaining terms in the second line supply the rest of $-V_B(\tilde{\zeta})$, plus an expression of the form $\{ Q, \Psi_F(\epsilon, u) \}$ for some fermionic operator $\Psi_F(\epsilon, u)$. In the next subsection we shall see that this is indeed the case for a particular choice of the coefficients $Y$ and $Z$.

6.5. BRST-exact terms in $\epsilon$-SUSY relations

Inspection of some of the terms in the second line of (6.63) suggests that the candidate fermionic operator $\Psi_F(\epsilon, u)$ would be

$$\Psi_F(\epsilon, u) = \int [dw] \left( -i\sqrt{2} \frac{b(w)}{\Pi^+(w)} e^{ik \cdot X(w)} \right) (\epsilon u) ,$$

(6.64)

where $b(w)$ is the reparametrization anti-ghost. The anticommutator $\{ Q, \Psi_F(\epsilon, u) \}$ can then be computed straightforwardly and gives

$$\{ Q, \Psi_F(\epsilon, u) \} = \int [dw] e^{ik \cdot X} (\epsilon u) \left[ -i \sqrt{2} \frac{\Pi_I \Pi^I(w)}{\Pi^+} + i \sqrt{2} \frac{S \partial S}{\Pi^+} \right.

\left. - i\sqrt{2} \Pi^- + i \sqrt{2} \frac{(\partial \Pi^+)^2}{(\Pi^+)^3} - i\sqrt{2} \frac{(k^- \partial \Pi^+ + k^- \Pi^I)}{\Pi^+} \right]. \quad (6.65)$$
Note that the first two terms are precisely those that appear in (6.63), while the third term can be recognized as a part of $V_B(\tilde{\zeta})$. This indicates that indeed the operator $\Psi_F(\epsilon, u)$ above should be relevant. (We should remark that the terms which depend on the ghosts together form a total derivative inside the integral and drop out. )

To make the statement more precise, let us compute the difference of the left and the right hand sides of the equation (6.43), using the results (6.63) and (6.65) above. After performing a partial integration\(^{25}\) we obtain

$$[\epsilon Q, V_F(u)] + V_B(\tilde{\zeta}) - \{Q, \Psi_F(\epsilon, u)\}$$

$$= \left(-i\sqrt{2}\right) \int [dw] \ e^{i k \cdot X} (\epsilon u) \left[ \partial \left( \frac{1}{\sqrt{\Pi^+}} \right) \left( \frac{k^- \Pi^+ + k_I \Pi^I}{\sqrt{\Pi^+}} \right) + \frac{2}{\sqrt{\Pi^+}} \partial^2 \left( \frac{1}{\sqrt{\Pi^+}} \right) + \frac{3}{2} \left( \partial \Pi^+ \right)^2 \right.$$

$$+ \frac{1}{2} \left( \partial^2 \Pi^+ \right) - \frac{1}{2} \left( \partial \Pi^+ \right)^2 - \frac{k^- \partial \Pi^+ + k_I \partial \Pi^I}{\Pi^+} - \partial^2 \Pi^+ + \frac{1}{96} \left( k_I \Pi^I \right) \left( k_J \Pi^J \right) \right) \left. \right]$$

$$= \left(-i\sqrt{2}\right) \int [dw] \ e^{i k \cdot X} (\epsilon u) \left[ (Y + 1)k^- k_I \Pi^I + (Z + 1) \frac{\left( k_I \Pi^I \right) \left( k_J \Pi^J \right)}{\Pi^+} \right]. \quad (6.66)$$

This expression vanishes as desired if and only if we make the choice

$$Y = -1 \quad \text{and} \quad Z = -1. \quad (6.67)$$

This fixes the boson vertex operator completely with the result

$$V_B(\zeta) = \int [dz] \ e^{i k \cdot X(z)} \left[ \zeta^- \Pi^+(z) + \zeta^I \left( \Pi_I(z) - \frac{1}{4} R_I(z) \right) \right.$$

$$+ \zeta^+ \left( \Pi^-(z) + \frac{1}{4} \frac{\Pi^I R_I(z)}{\Pi^+} - \frac{1}{96} \frac{R^I R_I(z)}{\Pi^+} - k^- k_I \Pi^I(z) \right. - \left. \frac{\left( k_I \Pi^I \right) \left( k_J \Pi^J \right)(z)}{\Pi^+} \right]. \quad (6.68)$$

Our last remaining task is to prove that the relation (6.44) for the $\epsilon$-SUSY transformation holds for our vertices with some choice of a fermionic operator $\Psi_B(\epsilon, \zeta)$. After some non-trivial calculations similar to the above (some details are given in appendix C.3), we find that indeed the desired $\epsilon$-SUSY relation (6.44) holds, with the fermionic operator $\Psi_B(\epsilon, \zeta)$ is given by

$$\Psi_B(\epsilon, \zeta) = -2^{1/4} \zeta^+ \int [dw] \left( \frac{k_I \left( \epsilon \bar{z} \frac{1}{S} b(w) \right)}{(\Pi^+)^{3/2}} \right) e^{i k \cdot X(w)}. \quad (6.69)$$

\(^{25}\) For instance we use the formula $\partial \left( e^{i k \cdot X} \right) = (k^- \Pi^+ + k_I \Pi^I) e^{i k \cdot X}$. 48
Note that the fermionic operators $\Psi_B(\epsilon, \zeta)$ and $\Psi_F(\epsilon, u)$ are proportional to $\zeta^+$ and $\tilde{\zeta}^+$ respectively. This shows that these anomalous terms are indeed unphysical, since $\zeta^+$ and hence $\tilde{\zeta}^+$ vanish in the physical light-cone gauge in the super-Maxwell theory.

As a further check of our vertex operators, we have explicitly computed the following double commutators, which show that the SUSY algebras (5.33) and (5.34) are properly realized on our vertex operators:

\[
[\eta Q, \Psi_B(\epsilon, \zeta)] = 0 = [\eta Q, \Psi_F(\epsilon, u)].
\]  

For the verification of (6.71), the following identities have been used:

\[
[\eta Q, \Psi_B(\epsilon, \zeta)] = \Psi_F(\epsilon, \tilde{\epsilon}), \quad [\eta Q, \Psi_F(\epsilon, u)] = 0 = \Psi_B(\epsilon, \tilde{\zeta}).
\]  

We have also checked that the remaining double commutators are consistent with the closure relation (5.52):

\[
[\epsilon_1 Q, [\epsilon_2 Q, V_B(F)]] = [\epsilon_2 Q, [\epsilon_1 Q, V_B(F)]] = 2\sqrt{2}(\epsilon_1 \epsilon_2) \left[\int [dw] \frac{b(w)}{\Pi^+}, V_B(F)\right],
\]  

For this calculation, we have used $[Q, V_B(F)] = 0$ and the following identities:

\[
[\epsilon_1 Q, \Psi_B(\epsilon_2, \zeta)] + \Psi_F(\epsilon_1, \delta_{\epsilon_2} u) - (1 \leftrightarrow 2) = -2\sqrt{2}(\epsilon_1 \epsilon_2) \left[\int [dw] \frac{b(w)}{\Pi^+}, V_B(\zeta)\right],
\]

\[
[\epsilon_1 Q, \Psi_F(\epsilon_2, u)] = \Psi_B(\epsilon_1, \delta_{\epsilon_2} \zeta).
\]

Note that these double commutator relations confirm once more that the choice of the negative signs on the right-hand side of the SUSY relations, (6.9) and (6.10), is the correct one.

In summary, we have constructed the BRST-invariant vertex operators, (6.68) for $V_B(\zeta)$ and (6.45) for $V_F(u)$, of the Green-Schwarz superstring at the lowest (massless) level, which describe the photons and photinos in the ten-dimensional super-Maxwell theory. Also, we have shown that these vertex operators indeed form the representation of the quantum super-Poincaré algebra on the physical Hilbert space.

Finally, let us make an important remark on how one can use these vertex operators for the computation of the scattering amplitudes. Although the formulation in the SLC gauge has the advantage that the conformal symmetry is preserved and the BRST invariance can be
utilized, some of the features remain to be the same as in the case of the LC gauge. Notably, the vertex operators can be constructed so far only for special external momenta satisfying $k^+ = 0$. Also the inverse powers of $\partial X^+$ appear in some parts of the vertex operators, which are well-defined only when the zero mode part of $\partial X^+$ has non-vanishing eigenvalue. Therefore to compute the amplitudes, we should follow the scheme used in the LC gauge calculation.$^{36,37}$ Namely, the $N$-point amplitude $A_N$ should be computed, schematically, as

$$A_N \sim \langle k_1 | V_2(k_2) V_3(k_3) \cdots V_{N-1}(k_{N-1}) | k_N \rangle$$

for special momentum configuration where $k_i^+ = k_N^+ \neq 0$ for the initial and the final momenta and $k_i^+ = 0$ for $i = 2, 3, \ldots, N - 1$. (Note that $k_i^\dagger$ should be allowed to be complex due to the on-shell condition.) In such a configuration $(\partial X^+)^{-1}$s in $V_i$ are well-defined and the amplitude can be computed properly. Finally, for low values of $N$, the amplitude so obtained for special configuration can be uniquely Lorentz-covariantized to yield the result valid for general configuration. It is an important future problem to improve on this situation by removing the restriction $k^+ = 0$ for the vertex operators.

§7. Similarity transformation to the LC gauge and construction of the DDF operators

In this section, we will discuss the relation between the formulation in the SLC-conformal gauge we have developed to the conventional (i.e. full) LC gauge formulation. Explicitly, we will be able to find the quantum similarity transformation which connects the operators in the two formulations. As an application, the spectrum-generating DDF(Del Giudice-Di Vecchia-Fubini) operators$^{33}$ in the SLC gauge will be constructed using this method.

7.1. Similarity transformation to the LC gauge

The most basic connection between the two formulations is of course that the physical spectrum is the same. In other words, the cohomology of the BRST operator $Q$ of the SLC gauge given in (5.25) coincides with the states generated by the physical oscillators, $\alpha^L_n$ and $S^a_{-n}$, of the LC gauge. One can prove this by adapting the standard argument, given for example in the Polchinski’s book$^{38}$ for the bosonic string, to the case of the superstring. An apparent complication, however, is that our Virasoro operator $T$ contains a complicated operator proportional to $\partial^2 \ln \Pi^+$, which is not quadratic in the fields as in the bosonic string case. Essentially similar situation was already encountered for the GS formulation of the superstring in the plane wave background in the SLC gauge studied in Ref. 32). There it was shown that as long as the non-linear term (such as $\partial^2 \ln \Pi^+$) contains at least one non-zero mode of $\Pi^\pm$, the cohomology of $Q$ is independent of such a term and coincides with
the light-cone Hilbert space. As this argument directly applies to the present case, we will not repeat the proof. For more details see Ref. 32).

The purpose of the present subsection is to demonstrate that actually a much more concrete relation between the SLC gauge and LC gauge formulations can be established. Namely, we will be able to construct an explicit quantum similarity transformation which maps the corresponding operators in the two gauges. A powerful technique for constructing such a similarity transformation was developed in Ref. 34) first for the bosonic string case and then extended for the superstring case to prove the equivalence between a variant of the pure spinor formalism and the LC gauge GS formalism. This method can be applied to the present case as well, although the construction will require a new rather non-trivial twist due to the presence of the term $\partial^2 \ln \Pi^+$.  

7.1.1. Description of the method

The main idea of the method developed in Ref. 34) is to systematically remove the set of unphysical modes $(b_{n}, c_{n}, \alpha_{n}^{+}, \alpha_{n}^{-})_{n \neq 0}$ from the BRST operator $Q$ in the form of a similarity transformation. This guarantees that the cohomology is unchanged. As these modes are absent in the LC formalism, they must form a “quartet” with respect to an appropriate nilpotent operator, to be called $\delta$, and decouple from the cohomology of $Q$. Such a $\delta$ can easily be found within $Q$. When $\sum_{n \neq 0} c_{-n} L_{n}$ in $Q$ contains the operator

$$\delta \equiv \sum_{n \neq 0} c_{-n} \alpha_{n}^{-},$$

and it satisfies the relations

$$\{\delta, \delta\} = 0,$$  \hspace{1cm} (7.2)

$$[\delta, \alpha_{n}^{+}] = p^{+} n c_{-n}, \quad \{\delta, c_{-n}\} = 0,$$  \hspace{1cm} (7.3)

$$[\delta, b_{-n}] = p^{+} \alpha_{n}^{-}, \quad [\delta, \alpha_{n}^{-}] = 0.$$  \hspace{1cm} (7.4)

This shows that indeed $(b_{n}, c_{n}, \alpha_{n}^{+}, \alpha_{n}^{-})_{n \neq 0}$ form a quartet (i.e. two doublets) with respect to $\delta$. To remove these modes systematically, it is convenient to assign non-vanishing degrees to these unphysical modes in such a way that (i) $\delta$ will carry degree $-1$ and (ii) the remaining part of $Q$ will carry non-negative degrees. Separating the zero-mode and the non-zero mode parts (indicated by the overcheck) of the unphysical fields as

$$\Pi^+(z) = \frac{p^+}{z} + \tilde{\Pi}(z), \quad \Pi^-(z) = \frac{p^-}{z} + \tilde{\Pi}^-(z),$$  \hspace{1cm} (7.5)

$$c(z) = c_0 z + \tilde{c}(z), \quad b(z) = \frac{b_0}{z^2} + \tilde{b}(z),$$  \hspace{1cm} (7.6)
it is achieved by the assignment
\[(\hat{\Pi}^+) = 2, \quad (\hat{\Pi}^-) = -2 \quad (7.7)
\]
\[
(\hat{c}) = 1, \quad (\hat{b}) = -1, \quad \text{deg(rest)} = 0. \quad (7.8)
\]

Unlike the case of the bosonic string, due to the presence of the term \(\varpi \equiv \frac{1}{2} \partial^2 \ln \Pi^+\) in the Virasoro operator, our \(Q\) contains \(\hat{\Pi}^+\) to arbitrary high powers and hence we will have to deal with infinitely high degrees. Explicitly, we have
\[
\varpi = \frac{1}{2} \partial^2 \ln \left( \frac{p^+}{z} + \hat{\Pi}^+ \right) = \frac{1}{2z^2} + \frac{1}{2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \partial^2 \left( \frac{z \hat{\Pi}^+}{p^+} \right)^n, \quad (7.9)
\]
and it contributes to \(Q\) as
\[
\int [dz] \varpi = e_0 + \sum_{n=1}^{\infty} e_{2n+1}, \quad (7.10)
\]
where \(e_m\), carrying degree \(m\), is given by
\[
e_0 = \int [dz] c_0 z \varpi = \frac{1}{2} c_0, \quad (7.11)
\]
\[
e_{2n+1} = \frac{(-1)^{n-1}}{2n} \int [dz] \partial^2 \hat{c} \left( \frac{z \hat{\Pi}^+}{p^+} \right)^n, \quad (n \geq 1). \quad (7.12)
\]
For later purposes, let us express \(e_{2n+1}\) in terms of the modes \(\alpha^+_n\) of \(\hat{\Pi}^+\). Expanding \(\hat{c}(z)\) and \(\hat{\Pi}^+(z)\) into modes and performing the integral we obtain
\[
e_{2n+1} \equiv \frac{(-1)^{n-1}}{2n(p^+)^n} \sum' m(m+1)c_{-m}[(\alpha^+)^n]_m, \quad (7.13)
\]
where we introduced a convenient notation
\[
[(\alpha^+)^n]_m \equiv \sum_{\sum^+_k = m} \alpha^+_1 \alpha^+_2 \cdots \alpha^+_m. \quad (7.14)
\]

Let us now decompose the BRST operator \(Q\) according to the degree. Indicating the degree by the subscripts (except for \(\delta\), which carries degree \(-1\)), the result is
\[
Q = \delta + Q_0 + d_1 + d_2 + d_3 + e_{\geq 3}, \quad (7.15)
\]

\[\text{26 This grading is essentially a refined version of the one used in Ref. 39).}\]
where

\[ \delta = p^+ \int [dz] \frac{1}{z} \bar{c} \bar{H}^- = p^+ \sum_{n \neq 0} c_{-n} \alpha_{-n}, \]  

(7.16)

\[ Q_0 = c_0 \left( \frac{1}{2} + \int [dz] z (T^{(0)} - b \partial \bar{c}) \right), \]  

(7.17)

\[ d_1 = \int [dz] (\bar{c} T^{(0)} + b \partial \bar{c}) , \]  

(7.18)

\[ d_2 = b_0 \int [dz] \frac{1}{z^2} \bar{c} \partial \bar{c} , \]  

(7.19)

\[ d_3 = p^- \int [dz] \frac{1}{z} \bar{c} \bar{H}^+ , \]  

(7.20)

\[ e_{\geq 3} \equiv \sum_{n \geq 1} e_{2n+1} . \]  

(7.21)

In the above, \( T^{(0)} \) is the degree 0 part of \( T \) and is given by

\[ T^{(0)} = z^{-2} \left( p^+ p^- + \frac{1}{2} p' p' \right) + \bar{H}^+ \bar{H}^- + \frac{1}{2} \bar{H}' \bar{H}' - \frac{1}{2} S_a \partial S_4 . \]  

(7.22)

Since the physical fields are contained in \( Q_0 \), the first and the main task is to remove the unphysical components with positive degrees, namely \( d_m \) and \( e_m \), by a suitable similarity transformation. Once this is done, the rest of the unphysical fields in \( Q_0 \) can be removed easily by another similarity transformation, to be discussed later.

Therefore, we seek the similarity transformation of the form

\[ Q = e^{-3 \mathfrak{R}} (\delta + Q_0) e^{3 \mathfrak{R}} = \delta + Q_0 + [\delta + Q_0, \mathfrak{R}] + \frac{1}{2} [[\delta + Q_0, \mathfrak{R}], \mathfrak{R}] + \cdots . \]  

(7.23)

Hereafter, we will write the graded commutator \( [A, B] \) simply as \( AB \) or \( (AB) \). In this notation the graded Jacobi identity reads \( A(BC) = (AB)C + (-1)^{AB} B(AC) \), where \((-1)^{AB}\) is \(-1\) if \( A \) and \( B \) are both fermionic and \( 1 \) otherwise. Then, decomposing the operator \( \mathfrak{R} \) according to the degree as

\[ \mathfrak{R} = R_2 + R_3 + \cdots , \]  

(7.24)

the equation above becomes

\[ Q = \delta + Q_0 + d_1 + d_2 + d_3 + e_{\geq 3} \]

\[ = \delta + Q_0 + \delta R_2 + \delta R_3 + Q_0 R_2 + \delta R_4 + \frac{1}{2} (\delta R_2) R_2 + \cdots . \]  

(7.25)

From this we get an equation at each degree, like \( d_1 = \delta R_2, \ d_2 = \delta R_3 + Q_0 R_2 \), etc., and the question is whether we can determine \( R_n \)'s which consistently solve these infinite number of equations.
There are two important ingredients for solving these equations. One is the nilpotency of $Q$. By decomposing $Q^2 = 0$ in degrees, we obtain a relation at every degree, starting from degree $-2$. Up to degree 7, these equations read

\begin{align}
(E_{-2}) \quad &\delta\delta = 0, \quad (7.26) \\
(E_{-1}) \quad &\delta Q_0 = 0, \quad (7.27) \\
(E_0) \quad &\frac{1}{2}Q_0 Q_0 + \delta d_1 = 0, \quad (7.28) \\
(E_1) \quad &Q_0 d_1 + \delta d_2 = 0, \quad (7.29) \\
(E_2) \quad &\delta(d_3 + e_3) + Q_0 d_2 + \frac{1}{2} d_1 d_1 = 0, \quad (7.30) \\
(E_3) \quad &Q_0(d_3 + e_3) + d_1 d_2 = 0, \quad (7.31) \\
(E_4) \quad &\delta e_5 + d_1(d_3 + e_3) + \frac{1}{2} d_2 d_2 = 0, \quad (7.32) \\
(E_5) \quad &Q_0 e_5 + d_2(d_3 + e_3) = 0, \quad (7.33) \\
(E_6) \quad &\delta e_7 + d_1 e_5 + \frac{1}{2}(d_3 + e_3)(d_3 + e_3) = 0, \quad (7.34) \\
(E_7) \quad &Q_0 e_7 + d_2 e_5 = 0. \quad (7.35)
\end{align}

The other ingredient is the well-known fact about the (co)homology of the nilpotent operator $\delta$. Consider the (homotopy) operator $\hat{K}$ of degree 1 given by

$$
\hat{K} \equiv \frac{1}{p^*} \sum_{n \neq 0} \frac{1}{n} \alpha_+^n b_n. \quad (7.36)
$$

Further define

$$
\hat{N} \equiv \delta \hat{K} = \sum_{n \neq 0} : (c_n b_n + \frac{1}{n} \alpha_+^n \alpha_n^-) :. \quad (7.37)
$$

This is an extension of the ghost number operator and assigns the “$\hat{N}$-number” $(1, -1, 1, -1)$ to the quartet $(c_n, b_n, \alpha_+^n, \alpha_n^-)$. Now let $\mathcal{O}$ be a $\delta$-closed operator carrying $\hat{N}$-number $n$, i.e. $\delta \mathcal{O} = 0$ and $\hat{N} \mathcal{O} = n \mathcal{O}$. If $n \neq 0$, we can solve for $\mathcal{O}$ as an $\delta$-exact form as

$$
\mathcal{O} = \frac{1}{n} \hat{N} \mathcal{O} = \frac{1}{n} (\delta \hat{K}) \mathcal{O} = \delta \left( \frac{1}{n} \hat{K} \mathcal{O} \right). \quad (7.38)
$$

This formula will be utilized repeatedly.

To illustrate the method, let us briefly review the construction of $R_n$ up to degree 3, which is formally the same as for the bosonic string described in Ref. 34. The first two nilpotency equations $(E_{-2})$ and $(E_{-1})$ simply say that $\delta$ is nilpotent and $\delta$ and $Q_0$ anticommute. The
first non-trivial equation is \((E_0)\). Since \(Q_0\) itself is nilpotent by inspection, it says \(\delta d_1 = 0\). Since the \(\tilde{N}\)-number of \(d_1\) is 1, we can use the formula (7.38) to get \(d_1 = \delta R_2\), where \(R_2 \equiv \tilde{K}d_1\). This solves the degree 1 part of the basic equation (7.25). Next, look at the relation \((E_1)\). Since \(\delta d_2 = 0\) holds by inspection, \((E_1)\) actually splits into two separate equations \(\delta d_2 = 0\) and \(Q_0 d_1 = 0\). Now since \(\tilde{N}d_2 = 2d_2\), one can again use (7.38) to write \(d_2 = \delta R_3\), where \(R_3 \equiv \frac{1}{2}\tilde{K}d_2\). For this to be consistent with the degree 2 part of (7.25), we must show that \(Q_0 R_2 = 0\). Using the form of \(R_2\) and the Jacobi identity, we can write this as \(Q_0 R_2 = Q_0 (\tilde{K}d_1) = (Q_0 \tilde{K}) d_1 - \tilde{K}(Q_0d_1)\). While we already know that \(Q_0d_1 = 0\), it is easy to check explicitly that the anti-commutator \(Q_0 \tilde{K}\) vanishes.

This completes the construction of \(R_2\) and \(R_3\). As they play a crucial role in the construction of the DDF operator in section 7.2, we shall give its explicit form:

\[
R \equiv R_2 + R_3 = \frac{1}{p^+} \sum_{k \neq 0} \frac{1}{k^{\alpha^+_k}} L^{\text{tot}}_k .
\]

(7.39)

Here \(L^{\text{tot}}_k\) is the degree 0 part of the Virasoro generator and consists of the bosonic, the fermionic and the ghost part given by

\[
L^{\text{tot}}_k = L^b_k + L^f_k + L^g_k ,
\]

(7.40)

\[
\bar{L}^b_k = p^I \alpha^I_k + \frac{1}{2} \sum_{n \neq 0} \alpha^\mu_{k-n} \alpha_{n}^\mu ,
\]

(7.41)

\[
\bar{L}^f_k = \frac{1}{2} \sum_{n \neq 0} n S_{k-n}^a S^a_n ,
\]

(7.42)

\[
\bar{L}^g_k = \sum_{n \neq 0} n c_{k-n} b_{k+n}.
\]

(7.43)

In the case of the bosonic string (i.e. with \(L^f_k\) absent) it was shown in Ref. 34) that \(R\) is actually the exact exponent \(\Re\). In other words \(R_n\) for \(n \geq 4\) all vanish. In the present case, however, they do not vanish due to the presence of the terms \(e_{\geq 3}\) in \(Q\) with arbitrarily high degrees. We shall now describe the construction of \(R_n\) for \(n \geq 4\).

7.1.2. Construction of the similarity transformation at higher degrees

In order to construct \(R_{n \geq 4}\) by using the method above, we must first examine the structure of the nilpotency relations at higher degrees. Although the relations \((E_n)\) displayed previously up to degree 7 look rather complicated, many of the terms actually vanish and in particular the relations above degree 4 turned out to be quite simple.

Let us list the anticommutators that vanish. First, the following ones among \(Q_0\) and \(d_i\) can be shown to vanish, either directly or by the reasoning similar to those made in Ref. 34):

\[
Q_0 d_3 = d_1 d_2 = d_1 d_3 = d_2 d_2 = d_2 d_3 = 0 .
\]

(7.44)
Next, by inspection we can immediately check the relations
\[ d_2e_{2n+1} = d_3e_{2n+1} = e_{2n+1}e_{2m+1} = 0. \]  
(7.45)

Another useful relation is
\[ Q_0e_{2n+1} = 0. \]  
(7.46)

A proof, which is slightly non-trivial, will be provided in appendix D.1.

If we substitute these vanishing relations, we see that the nilpotency relations at higher degrees simplify drastically. \((E_3)\) becomes trivial and above \((E_4)\) it reduces simply to the following important relations
\[ (E_{2n+2}) \quad d_1e_{2n+1} = -\delta e_{2n+3}, \quad n \geq 1. \]  
(7.47)

We now discuss how one can determine \(R_n\) for \(n \geq 4\). Although we have explicitly constructed \(R_n\) up to \(R_{10}\), we will not show all the details of the computations as they are rather involved and not so illuminating. What we will do is to describe in some detail the construction of \(R_4\) and a number of important general formulas, which will also be needed in the *all order analysis* to be presented in the next subsection. Then we will quote the results up to \(R_{10}\).

To determine \(R_4\), we must look at the degree 3 part of the basic equation (7.25). It reads
\[ d_3 + e_3 = \delta R_4 + Q_0 R_3 + \frac{1}{2}(\delta R_2)R_2. \]  
(7.48)

First focus on the double commutator \(\frac{1}{2}(\delta R_2)R_2\) on the right hand side. By using the already established relations, we can rewrite it as
\[ \frac{1}{2}(\delta R_2)R_2 = \frac{1}{2}d_1R_2 = \frac{1}{2}d_1(\hat{K}d_1) = \frac{1}{2}(d_1\hat{K})d_1 - \frac{1}{2}\hat{K}(d_1d_1) = -\frac{1}{4}\hat{K}(d_1d_1). \]  
(7.49)

Now applying the relation \((E_2)\) to the expression \(\frac{1}{4}(d_1d_1)\), this becomes
\[ \frac{1}{2}(\delta R_2)R_2 = \frac{1}{2}\hat{K}(\delta(d_3 + e_3)) + \frac{1}{2}\hat{K}(Q_0d_2). \]  
(7.50)

The second term on the right hand side equals \(-\frac{1}{2}Q_0(\hat{K}d_2) = -Q_0R_3\) and cancels \(Q_0R_3\) in (7.48). On the other hand, the first term can be rewritten as
\[ \frac{1}{2}\hat{K}(\delta(d_3 + e_3)) = \frac{1}{2}\hat{N}(d_3 + e_3) - \frac{1}{2}\delta(\hat{K}(d_3 + e_3)) = d_3 + e_3 - \frac{1}{2}\delta(\hat{K}e_3), \]  
(7.51)

where we used \(\hat{K}d_3 = 0\), which can be easily checked directly. Since the first two terms on the right hand side match the left hand side of (7.48), the degree 3 equation (7.48) reduces to \(\delta(R_4 - \frac{1}{2}\hat{K}e_3) = 0\) and this gives \(R_4 = \frac{1}{2}\hat{K}e_3\).
This type of expression turns out to be quite basic and it is very useful to define the following quantity:

\[ r_{2n} = \frac{1}{n} \hat{K}e_{2n-1} = \frac{(-1)^n}{2n(n-1)(p^+)^n}[(\alpha^+)^n]_0, \quad n \geq 2. \quad (7.52) \]

In this notation our result for \( R_4 \) is simply \( R_4 = r_4 \). The quantity \( r_{2n} \) has many nice properties:

(i) Since \( \hat{K}^2 = 0 \), we have \( \hat{K}r_{2n} = 0 \).

(ii) Since \( Q_0\hat{K} = 0 \) and \( Q_0e_{2n-1} = 0 \), we have \( Q_0r_{2n} = 0 \).

(iii) By explicit computation, it is easy to prove \( \delta r_{2n+2} = -d_1r_{2n} \).

(iv) Applying \( \hat{K} \) to the relation \( (E_{2n}) \) shown in (7.47), i.e. \( d_1e_{2n-1} = -\delta e_{2n+1} \), and using the Jacobi identities, we easily obtain \( R_2e_{2n-1} = (n+1)(\delta e_{2n+1} - e_{2n+1}) \). Applying \( \hat{K} \) again to this equation we establish an important equation

\[ r_{2n}R_2 = (n+1)r_{2n+2}. \quad (7.53) \]

In the case of the bosonic string, the reason why \( \mathfrak{R} = R_2 + R_3 \) was the exact answer was because various higher multiple commutators of \( \delta \) and \( R_i \) \( (i = 2, 3) \) vanished. For the present case, it is no longer true in general and in particular the following relation plays an important role in the higher degree analysis. It reads

\[ \frac{1}{n!} \delta(\text{ad}_{R_2})^n = e_{2n-1} - \delta r_{2n}, \quad (n \geq 2), \quad (7.54) \]

where the multiple commutator is denoted by \( \delta(\text{ad}_{R_2})^n \equiv ((((\delta R_2)R_2\cdots)R_2) \). The proof of this formula is given in appendix D.1.

Although rather tedious, making use of these formulas, one can analyse the equation (7.25) at each higher degree, similarly to the case of degree 3, and determine the operators \( R_n \). The results up to degree 10 turned out to be

\[ R_4 = r_4, \quad R_5 = 0, \quad R_6 = \frac{1}{2}r_6, \quad R_7 = 0, \]
\[ R_8 = 0, \quad R_9 = 0, \quad R_{10} = -\frac{1}{6}r_{10}. \quad (7.55) \]

One immediately notes that the operators at odd degrees vanish. Actually we can prove that this is a general property. In contrast, it is apparently not possible to guess the pattern for \( R_{2n} \). Nevertheless, in what follows we will be able to construct the exact similarity transformation to all degrees by a different approach and confirm that it reproduces the result up to degree 10 given above.
7.1.3. Exact form of the similarity transformation

We now describe a new idea which lets us construct the similarity transformation exactly to all degrees. Consider an ansatz of the form

\[ Q = e^{-R_2-R_3} e^{-\tilde{R} (\delta + Q_0)} e^{R_2+R_3}, \]  

(7.56)

where \( \tilde{R} \) is taken to be of the form

\[ \tilde{R} = \sum_{n \geq 2} \xi_{2n} r_{2n}, \]  

(7.57)

with \( \xi_{2n} \) being appropriate coefficients to be determined. Then, since \( Q_0 \) and \( \delta r_{2n} \) commute with \( r_{2n} \), the similarity transformation by \( \tilde{R} \) simply gives

\[ e^{-\tilde{R} (\delta + Q_0)} e^{\tilde{R}} = \delta + Q_0 + \sum_{n \geq 2} \xi_{2n} \delta r_{2n}. \]  

(7.58)

For the \( \delta + Q_0 \) part, the subsequent similarity transformation by \( R_2 + R_3 \) is almost the same as for the bosonic string case. The only difference is that, while \( \delta (\text{ad}_{R_2})^n \) vanished for \( n \geq 3 \) for the bosonic string, in the present case it is non-vanishing for all \( n \) and is given by (7.54) for \( n \geq 2 \). Therefore we have

\[ e^{-R_2-R_3} (\delta + Q_0) e^{R_2+R_3} = \delta + Q_0 + d_1 + d_2 + d_3 + \sum_{n \geq 2} e_{2n-1} - \sum_{n \geq 2} \delta r_{2n} = Q - \sum_{n \geq 2} \delta r_{2n}. \]  

(7.59)

where \( Q \) is the full BRST operator we want.

The remaining contribution is \( e^{-R_2-R_3} \sum_{n \geq 2} \xi_{2n} \delta r_{2n} e^{R_2+R_3} = \sum_{n \geq 2} e^{-R_2} \delta r_{2n} e^{R_2} \), where we used the fact that \( R_3 \) commutes with both \( R_2 \) and \( \delta r_{2n} \). If this contribution cancels the term \( -\sum_{n \geq 2} \delta r_{2n} \) in (7.59) we get the desired result. Now as we have already encountered in (D.10) in appendix D.1, there is a formula

\[ (\delta r_{2n}) R_2 = n \delta r_{2n+2}. \]  

(7.60)

Applying this repeatedly \( m \) times with the weight factor \( 1/m! \), we get the multiple commutator needed in the computation of \( e^{-R_2} \delta r_{2n} e^{R_2} \) as

\[ \frac{1}{m!} (\delta r_{2n}) (\text{ad}_{R_2})^m = \frac{1}{m!} n(n+1) \cdots (n+m-1) \delta r_{2(n+m)} \]

\[ = \frac{(n+m-1)!}{m!(n-1)!} \delta r_{2(n+m)} = \binom{n+m-1}{m} \delta r_{2(n+m)}. \]  

(7.61)
Altogether the total similarity transformation gives

\[ e^{-R} e^{-\tilde{R}} (\delta + Q_0) e^{\tilde{R}} e^R = Q + \sum_{n \geq 2} \xi_{2n} \sum_{m \geq 0} \binom{n + m - 1}{m} \delta r_{2(n+m)} - \sum_{k \geq 2} \delta r_{2k} \]

\[ = Q + \sum_{k \geq 2} \left( \sum_{n=2}^{k} \xi_{2n} \binom{k-1}{k-n} - 1 \right) \delta r_{2k} . \quad (7.62) \]

So we get the desired result, namely \( Q \), if \( \xi_{2n} \) can be chosen to satisfy the relation

\[ \sum_{n=2}^{k} \xi_{2n} \binom{k-1}{k-n} = 1 . \quad (7.63) \]

Inspection of this equation for low values of \( k \) suggests that the general answer is \( \xi_{2n} = (-1)^n \). In fact such a formula is easily proved by evaluating the integral \( \int_0^1 dx (1-x)^l = 1/(l+1) \) by expanding in powers of \( x \).

Thus, we have obtained a remarkably simple answer for the exact similarity transformation:

\[ Q = e^{-R_2-R_3} e^{-\tilde{R}} (\delta + Q_0) e^{\tilde{R}} e^{R_2+R_3} , \quad (7.64) \]

\[ \tilde{R} = \sum_{n \geq 2} (-1)^n r_{2n} . \quad (7.65) \]

Actually, \( \tilde{R} \) can be written in a closed form. Going back to the definition of \( r_{2n} \), it can be expressed in terms of a contour integral in the \( z \)-plane as

\[ \tilde{R} = \sum_{n \geq 2} (-1)^n r_{2n} = \int [dz] \sum_{n \geq 2} \frac{1}{2n(n-1)} \left( \frac{z \tilde{H}^+}{p^+} \right)^n \]

\[ = \frac{1}{2} \int [dz] \left[ f(z) + (1 - f(z)) \ln(1 - f(z)) \right] , \quad (7.66) \]

where \( f(z) \) is defined as \( f(z) \equiv z \tilde{H}^+(z)/p^+ \).

As a check of the formula above, we wish to compare it with the (partial) answer obtained by the degree-wise analysis. To do this, we must rewrite the product \( e^{\tilde{R}} e^{R_2+R_3} \) into the form \( e^{\mathfrak{R}} \), using the Baker-Campbel-Hausdorff (BCH) formula. (Actually, as \( R_3 \) commutes with \( R_2 \) and \( \tilde{R} \), we only need to deal with the product \( e^{\tilde{R}} e^{R_2} \).) Since this computation is rather non-trivial and technical, we shall only quote the result here and relegate its derivation to appendix D.2. The exact result for \( \mathfrak{R} \) can be written rather compactly as

\[ \mathfrak{R} = R_2 + R_3 + \tilde{R} , \quad (7.67) \]

\[ \tilde{R} = r_4 + \sum_{n \geq 1} (-1)^{n-1} (2n + 1) B_n r_{2(2n+1)} , \quad (7.68) \]
where $B_n$ are the Bernoulli numbers. Substituting $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, the first few terms of $\hat{R}$ read

$$\hat{R} = r_4 + \frac{1}{2}r_6 - \frac{1}{6}r_{10} + \frac{1}{6}r_{14} + \cdots.$$  \hspace{1cm} (7.69)

Remarkably, this precisely reproduces the rather sporadic-looking result of the degree-wise analysis given in (7.55). Also, the formula (7.68) is consistent with the general feature $R_{2n+1} = 0$ deduced previously.

Our final task is to remove the unphysical oscillators still remaining in $Q_0$, given in (7.17), by another similarity transformation and connect to the light-cone gauge formulation. Here we have to be a bit careful. First, in the ghost sector we have to change from the $sl(2)$-invariant vacuum $|0\rangle_{inv}$, tacitly used in the plane coordinate treatment adopted so far, to the so-called “physical” or “down” vacuum given by $|\downarrow\rangle = c_1|0\rangle_{inv}$. This adds the well-known intercept $-1$ to the zero mode of the Virasoro operator. After that $Q_0$ can be decomposed as $Q_0 = Q_{lc} + c_0\tilde{N}$, where

$$Q_{lc} = c_0 \left( \frac{1}{2} - 1 + p^+ p^- + \int [dz] z T_{lc}(z) \right),$$  \hspace{1cm} (7.70)

$$T_{lc}(z) = -\frac{1}{2} \partial X^I \partial X^I(z) - \frac{1}{2} S^a \partial S^a(z),$$  \hspace{1cm} (7.71)

$$\tilde{N} = \sum_{n \geq 1} \left( \alpha_+^n \alpha_-^n + \alpha_-^n \alpha_+^n + nc_{-n} b_n + nb_{-n} c_n \right).$$  \hspace{1cm} (7.72)

Here $T_{lc}(z)$ is the energy-momentum tensor for the physical matter fields in the plane coordinate, which carries the central charge $C_{lc} = 12$. Now since the LC gauge formulation is constructed in the cylinder coordinate $\zeta = \tau + i\sigma$, we must make the familiar conformal transformation $z \rightarrow \zeta = \ln z$. Then we get $T_{lc}(z) = (1/z^2)(T_{lc}(\zeta) + (C_{lc}/24)) = (1/z^2)(T_{lc}(\zeta) + (1/2))$, yielding an additional intercept of $1/2$. Altogether the intercepts add up to zero and we obtain the form of $Q_{lc}$ in the cylinder frame as

$$Q_{lc} = c_0 \left( \frac{1}{2} p^\mu p_\mu + \sum_{n \geq 1} \alpha_+^l \alpha_-^l + \sum_{n \geq 1} n S^a_{-n} S^a_n \right).$$  \hspace{1cm} (7.73)

We now construct the similarity transformation which removes the $c_0\tilde{N}$ part consisting of unphysical oscillators. To this end introduce the following operator $\tilde{K}$ which is similar to but different from $\hat{K}$:

$$\tilde{K} \equiv \frac{1}{p^+} \sum_{n \neq 0} \alpha_+^n b_n.$$  \hspace{1cm} (7.74)

Note that the normal-ordering for the ghost part in $\tilde{N}$ is now appropriate for the down vacuum.
Using this operator, it is easy to check that \( \tilde{N} \) can be expressed as the anticommutator

\[
\tilde{N} = \{ \tilde{K}, \delta \}. \tag{7.75}
\]

Then the following similarity transformation does the required job:

\[
e^{-c_0 \tilde{K}} (\delta + Q_0) e^{c_0 \tilde{K}} = \delta + Q_0 - [c_0 \tilde{K}, \delta] = \delta + Q_0 - c_0 \tilde{N} = \delta + Q_{cl}. \tag{7.76}
\]

Finally, by a rather standard argument one can show that the cohomology of the operator \( \delta + Q_{cl} \) indeed coincides with the physical space of the light-cone theory. We will not reproduce the argument here and refer the reader to the description in Ref. 34).

7.2. Construction of the DDF operators

Although there should be many applications of the similarity transformation that connects the SLC and the LC gauges, let us discuss below one direct application, namely the construction of the so-called DDF operators in the SLC gauge using this machinery.

7.2.1. Basic idea

In the case of the bosonic string, such a method was developed in Ref. 34) and the bosonic DDF operator \( A^I_n \), which commutes with the Virasoro generators in the conformal gauge, was generated from the transverse physical oscillator \( \alpha^I_n \) in the LC gauge by the similarity transformation. More precisely, it was demonstrated that the following relation holds:

\[
A^I_n = e^{inx/p} \tilde{A}^I_n, \tag{7.77}
\]

\[
\tilde{A}^I_n = \int [d\tau] e^{in\tau} \partial \tau X^I(\tau) e^{i(n/p^+)} \tilde{X}^I(\tau) = e^{-R} \alpha^I_n e^R. \tag{7.78}
\]

Here, the operator \( R \) is the same as \( R_2 + R_3 \) given in (7.39) (without fermions), \( \tau \) is a variable in the interval \([0, 2\pi]\), the integration measure is \([d\tau] \equiv d\tau/2\pi\) and \( X^\mu(\tau) \) is given by

\[
X^\mu(\tau) = x^\mu + p^\mu \tau + \tilde{X}^\mu(\tau), \quad \tilde{X}^\mu(\tau) = i \sum_{n \neq 0} 1/n \alpha^\mu_n e^{-in\tau}. \tag{7.79}
\]

What we wish to do is to construct the fermionic DDF operator corresponding to the LC oscillator \( S^n_a \), to be denoted by \( S^n_a \), by using our similarity transformation constructed in the previous subsection. Due to the presence of the \( \partial^2 \ln \Pi^+ \) term in the Virasoro operator, the similarity transformation is more involved than in the bosonic string case. However, since the additional transformation, which consists solely of \( \Pi^+ \), does not affect \( S^n_a \), we should be able to compute the main part\(^28\) of the DDF operator \( \tilde{S}^a_n \) by

\[
\tilde{S}^a_n = e^{-R} S^n_a e^R, \tag{7.80}
\]

\(^28\) To get the genuine DDF operator, we need to multiply by a zero-mode factor as in (7.77).
where $R$ is given by $R_2 + R_3$ of (7.39). The only difference from the bosonic string case is the presence of the fermionic part $\tilde{L}_k^f$ given in (7.42). Therefore, the computation of the bosonic DDF operator via $e^{-R_2 \alpha^I_n e^R}$ quoted in (7.78) is exactly the same as in Ref. 34).

In fact, it is useful to recall the salient feature of the calculation in Ref. 34) before we launch on the computation of $e^{-R_2 S_a^\tau e^R}$. The important point was that as one successively evaluates the terms in the expansion $e^{-R_2 \alpha^I_n e^R} = \alpha^I_n - [R, \alpha^I_n] + \frac{1}{2} [R, [R, \alpha^I_n]] + \cdots$, the oscillator $\alpha^I_n$ gets multiplied by powers of $\hat{X}^+$ precisely in such a way to get dressed exponentially. The simplicity of this process, in turn, was due to the fact that $\hat{X}^+$ and $\partial_\tau \hat{X}^I$ are primaries of dimension 0 and 1, respectively, with respect to $\tilde{L}_k^{tot}$. Consequently, the commutators of $R$ and these fields take the form

$$[R, \hat{X}^+(\tau)] = \frac{1}{2p^+} \partial_\tau (\hat{X}^+)^2, \quad (7.81)$$

$$[R, \partial_\tau \hat{X}^I(\tau)] = \frac{1}{p^+} \partial_\tau \left( \hat{X}^+ \partial_\tau \hat{X}^I \right), \quad (7.82)$$

which are both total derivatives. This allows one to perform integration by parts repeatedly to reach the simple result (7.78).

The situation for the case of the operator $S_a^\tau$ is qualitatively different. If we form the field $S_a^\tau(\tau) = \sum_n S_a^\tau e^{-in\tau}$, it is also a primary field with respect to $\tilde{L}_k^{tot}$ but is of dimension $1/2$, i.e.

$$\left[ \tilde{L}_k^{tot}, S_a^\tau(\tau) \right] = e^{ik\tau} \left( \frac{1}{i} \partial_\tau + \frac{1}{2} k \right) S_a^\tau(\tau). \quad (7.83)$$

Because of this the commutator with $R$ is given by

$$[R, S_a^\tau(\tau)] = \frac{1}{p^+} \left( \hat{X}^+ \partial_\tau S_a + \frac{1}{2} \partial_\tau \hat{X}^+ S_a \right), \quad (7.84)$$

which is not a total derivative. This makes the computation of $e^{-R_2 S_a^\tau e^R}$ much more involved.

Nevertheless, by using the formula (7.84) as well as (7.81) and integrating by parts appropriately, we can perform the computation at low orders. Up to the second order in $R$ the result is

$$\hat{S}_n^a = \int [d\tau] e^{in\tau} \left( S_a^\tau - [R, S_a^\tau] + \frac{1}{2} [R, [R, S_a^\tau]] + \cdots \right)$$

$$= \int [d\tau] e^{in\tau} S_a^\tau(\tau) \left( 1 + \frac{in}{p^+} \hat{X}^+ + \frac{1}{2p^+} \partial_\tau \hat{X}^+ - \frac{1}{2} \left( \frac{n}{p^+} \right)^2 (\hat{X}^+)^2 \right.$$

$$\left. - \frac{1}{8p^2} (\partial_\tau \hat{X}^+)^2 + \frac{in}{4p^2} \partial_\tau (\hat{X}^+)^2 + \cdots \right). \quad (7.85)$$
Although it looks a little complicated at first sight, one can actually guess what this series should develop into. An important property of the DDF operator is that it should commute with the Virasoro generators. Thus, it should be given by an integral of a conformal primary of dimension 1. In the case of $\partial_\tau X_I$, it is already of dimension 1 and hence we only needed to dress it by dimension 0 operators, as in (7.78). In the present case, as $S^a$ is of dimension $1/2$, the dressing factor must carry dimension $1/2$, not zero. The natural guess is to first promote $S^a$ into a dimension 1 primary by multiplying by $\sqrt{\partial_\tau X^+}$ and then dress it further with the dimension zero operator. To define $\sqrt{\partial_\tau X^+}S^a$ properly, we must expand it around the zero mode. Therefore more precise form of the guess would be

$$\tilde{S}^a_n = e^{-R} S^a_n e^R = \int [d\tau] e^{i\tau} e^{i(n/p)X^+} \left(1 + \frac{\partial_\tau X^+}{p^+}\right)^{1/2} S^a(\tau).$$  

Indeed it is easy to see that, to the second order in $R$, the expansion of the square root in powers of $\partial_\tau X^+$ exactly reproduces the expression (7.85).

7.2.2. A theorem on finite operatorial conformal transformation

We now wish to prove the formula (7.86) to all orders in $R$. Clearly the brute force computation of $e^{-R} S^a_n e^R$ to higher orders would not be useful for this purpose. It turned out that understanding of the meaning of our similarity transformation is crucial for achieving the goal. It leads to a theorem on finite operatorial conformal transformation.

To state the theorem precisely, we first need to describe the set-up. Let the mode operators $\phi_n$ and $\chi_n$ enjoy the following commutation relations with a set of operators $L_k$:

$$[L_k, \phi_n] = -(n + (1 - h)k)\phi_{n+k},$$  

$$[L_k, \chi_n] = -(n + k)\chi_{n+k}. \tag{7.88}$$

These relations are of the same form as the commutation relations of the modes of Virasoro generators $L_k$ and the modes of primary fields $\phi(\tau)$ and $\chi(\tau)$ of dimensions $h$ and 0 respectively. However, we will not require the algebra of $L_k$ themselves. For our present problem, we need to consider the situation where the field $\chi(\tau)$ is formed without the zero mode. Namely we define

$$\phi(\tau) \equiv \sum_n \phi_n e^{-i\tau}, \quad \chi(\tau) \equiv \sum_n \chi_n e^{-i\tau}, \tag{7.89}$$

where the prime on the summation symbol signifies that the zero mode is omitted. Then, multiplying (7.87) and (7.88) by $e^{-i\tau}$ and summing over the respective range as above, we get

$$[L_k, \phi(\tau)] = e^{ik\tau} \left(\frac{1}{i} \partial_\tau + hk\right) \phi(\tau), \tag{7.90}$$

63
\[ [L_k, \chi(\tau)] = e^{ik\frac{1}{2} \partial_\tau \chi(\tau)} . \]  

(7.91)

Now consider an infinitesimal conformal transformation of \( \phi(\tau) \) by \( \tau' = \tau - \epsilon(\tau) \), or equivalently \( \tau = \tau' + \epsilon(\tau') \), where the parameter \( \epsilon(\tau) \) has no zero mode part. Then, from the basic equation \( \phi'(\tau')(d\tau')^h = \phi(\tau)(d\tau)^h \) one easily derives the change of the functional form as

\[
\delta \phi(\tau) = \phi'(\tau) - \phi(\tau) = \epsilon(\tau) \partial_\tau \phi(\tau) + h \partial_\tau \epsilon(\tau) \phi(\tau). \tag{7.92}
\]

In terms of modes, this reads

\[
\delta \phi_n = -i \sum_k 'n + (1 - h)k) \epsilon_{-k} \phi_{n+k}. \tag{7.93}
\]

Using (7.87) this can be written as the commutator

\[
\delta \phi_n = \left[ i \sum_k ' -k L_k, \phi_n \right]. \tag{7.94}
\]

Therefore, the operator which generates an infinitesimal conformal transformation associated to \( \tau \rightarrow \tau' = \tau - \epsilon(\tau) \) is given by

\[
T_\epsilon = i \sum_k ' -k L_k. \tag{7.95}
\]

In this formula, \( \epsilon(\tau) \) is a function, not a field. In what follows, we replace it by a primary field \( \chi(\tau) \) of dimension 0 and consider the operator

\[
T_\chi = i \sum_k ' -k L_k. \tag{7.96}
\]

We can now state the theorem.

**Theorem:** Let \( T_\chi \) be given by (7.96), where \( \chi \) is finite, not infinitesimal. Then we can generate a finite conformal transformation associated with \( \tau' = \tau - \chi(\tau) \) in the form of the similarity transformation

\[
e^{T_\chi \phi(\tau)} e^{-T_\chi} = \phi'(\tau), \tag{7.97}
\]

where the functional form of \( \phi' \) is given by the formula for the finite conformal transformation

\[
\phi'(\tau')(d\tau')^h = \phi(\tau)(d\tau)^h. \tag{7.98}
\]

It is extremely important that \( \chi(\tau) \) is a primary field of dimension 0, not just a parameter. The reason is that otherwise \([T_\chi, \phi], [T_\chi, [T_\chi, \phi]], \) etc will not be proper conformal descendants and the theorem does not hold. Despite the simplicity of the form of the theorem, the proof is rather involved and we will describe it in appendix D.3.
7.2.3. Exact form of the DDF operator

Now we apply this theorem with the identification \( L_k = \tilde{L}_k^{\text{tot}}, \phi(\tau) = S^a(\tau), \chi(\tau) = -\tilde{X}^+(\tau)/p^+ \) and \( h = 1/2 \). Then, \( T_\chi \) is identified with \(-R\), where \( R \) is given in (7.39) and the associated conformal transformation is given by \( \tau' = \tau + \tilde{X}^+(\tau)/p^+ \). Then from the theorem we obtain

\[
e^{-R}S^a(\tau)e^R = S^a(\tau).
\] (7.99)

What remains to be shown is that \( S^a(\tau) \) is exactly the same function as \( \hat{S}^a(\tau) \equiv \sum_n \hat{S}^a_ne^{-in\tau} \), where \( \hat{S}^a_n \) was given in (7.86). In the following manipulation, we set \( p^+ = 1 \) for notational simplicity. From (7.86) we can compute \( \hat{S}^a(\rho) \) at an arbitrary argument \( \rho \) as

\[
\hat{S}^a(\rho) = \int [d\tau] \sum_n e^{in(\tau + \tilde{X}^+(\tau) - \rho)} (1 + \partial_\tau \tilde{X}^+)^{1/2} S^a(\tau)
\]

\[
= \int [d\tau] 2\pi \delta(\tau + \tilde{X}^+(\tau) - \rho) (1 + \partial_\tau \tilde{X}^+)^{1/2} S^a(\tau).
\] (7.100)

Let us set \( \rho = \bar{\tau} + \tilde{X}^+(\bar{\tau}) \), where \( \bar{\tau} \) is an arbitrary new variable. Then, the \( \delta \)-function can be rewritten as

\[
\delta(\tau + \tilde{X}^+(\tau) - (\bar{\tau} + \tilde{X}^+(\bar{\tau}))) = \frac{\delta(\tau - \bar{\tau})}{d\tau} = \delta(\tau - \bar{\tau}) (1 + \partial_\tau \tilde{X}^+)^{-1}.
\] (7.101)

Put this into (7.100), integrate it over \( \tau \) and then rename \( \bar{\tau} \) as \( \tau \). This gives

\[
\hat{S}^a(\tau + \tilde{X}^+(\tau)) = (1 + \partial_\tau \tilde{X}^+)^{-1/2} S^a(\tau).
\] (7.102)

Note that this is nothing but the conformal transformation of \( S^a(\tau) \) and hence the left-hand side equals \( S^a_m(\tau + \tilde{X}^+(\tau)) \). In other words, as functions, \( \hat{S}^a(\tau) = S^a(\tau) \). Reinstating \( p^+ \), we have proved (7.86) as an identity.

Just as in the case of the bosonic string (7.77), the actual DDF operator \( S^a_n \) which commutes with the Virasoro generators is obtained by adjoining the zero-mode piece, namely,

\[
S^a_n = e^{inx/p^+} \hat{S}^a_n.
\] (7.103)

Including this factor, the formula for the DDF operator can be written more compactly as

\[
e^{-R}e^{inx/p^+}S^a_ne^R = S^a_n = \frac{1}{\sqrt{p^+}} \int [d\tau] e^{i(n/p^+)X^+} \sqrt{\partial_\tau X^+} S^a(\tau),
\] (7.104)

using the full field \( X^+(\tau) \) including the zero mode pieces. This is our final answer for the DDF operator. It is clear that \( S^a_n \)'s satisfy the same anti-commutation relations as \( S^a_m \)'s, namely \( \{S^a_m, S^b_n\} = \delta^{ab} \delta_{m+n,0} \). Also, the operator \( S^a_n \) commutes with the Virasoro algebra.
§8. Discussions

In this article, starting from the basic action we have developed the operator formulation of the Green-Schwarz superstring in the SLC gauge in a fairly systematic manner. We have clarified the structure of the quantum symmetry algebras of the theory, constructed the vertex operators for the massless excitations and obtained the exact quantum map between the operators of the SLC gauge formulation and those of the familiar LC gauge formulation.

We now discuss some issues to be investigated in the future. One issue is how to remove the restriction on the choice of the momentum frame, namely the condition \( k^+ = 0 \), for the vertex operators. When we started our investigation, our hope was that, as we need not identify \( X^+ \) with the worldsheet time in the SLC gauge, we would be able to relax such a condition in contrast to the conventional LC gauge. In the LC gauge, as one imposes the condition \( X^+ = x^+ + p^+ \tau \), the remaining longitudinal field \( X^- \) is expressed, through Virasoro constraints, as bilinears in the physical fields. This makes it extremely difficult to include the term \( ik^+ X^- \) in the exponential \( \exp(ik_\mu X^\mu) \) and hence one is forced to adopt the \( k^+ = 0 \) frame. This problem does not exist in the SLC gauge because \( X^- \) is a genuine independent field. Unfortunately, however, a similar problem arises from a different origin. Due to the existence of the quantum correction \( \frac{1}{2} \partial^2 \ln \Pi^+ \) in the Virasoro generator, the operator \( \exp(ik_\mu X^\mu) \) is no longer a primary field if we include the \( ik^+ X^- \) term. Therefore to avoid complication we have decided to impose \( k^+ = 0 \) condition in this work. In fact, as indicated in section 6, at least for the consistency of the vertex operators constructed in this work, this condition appears to be necessary.

There seem to be several possibilities to cure this problem. One is that, as we have the quantum Lorentz generator \( \mathcal{M}^{+-} \) at hand, it might be possible to perform a finite Lorentz transformation \( \exp(\xi \mathcal{M}^{+-}) V \exp(-\xi \mathcal{M}^{+-}) \) on the vertex operator \( V \) to go to the frame where \( k^+ \) is non-vanishing.

Another possibility is suggested by the recent work,\(^{40}\) which introduced a conformal field theory composed of the longitudinal fields \( X^\pm \), for the purpose of defining the light-cone gauge string field theory in non-critical dimensions in the path integral formalism. This system is essentially the same as the one that occured in our work and the Virasoro generator contains terms made up of \( \Pi^+ \)'s with two derivatives. Consequently, the conformal property of \( X^- \) becomes rather complicated as in our case. Nevertheless the authors of Ref. 40) formally succeeded in computing the amplitudes containing the vertex operators of the type \( e^{ik^+ \hat{X}^-} \) with non-vanishing \( k^+ \), where \( \hat{X}^- \) is the modified field mentioned in setion 6.1. As discussed there, \( \hat{X}^- \), while having a good conformal property, has a severe singularity with itself. Although it is not straightforward to relate the path integral formulation they adopted
and our operator formalism, this indicates that there may be a way to construct the vertex operator for non-zero $k^+$ in the operator language as well. This is left as a problem for the future.

Another problem worth studying in the future is the construction of the $Dp$-brane boundary state in the SLC gauge. Within the GS formalism, the construction of such a boundary state was performed in Ref. 41), 42) in the LC gauge. In that gauge, however, since the "time" coordinate $X^+$ must satisfy the Dirichlet boundary condition, what one obtains is the "$(p+1)$-instanton" rather than the $Dp$-brane. One must perform the double Wick rotation to get the genuine $Dp$-brane. In contrast, in the SLC-conformal gauge, it should be possible to construct the genuine $Dp$-brane with the Neumann boundary condition for $X^0$ much more naturally (at least for $p \geq 1$). For this study, the structure of the super-Poincaré and the BRST symmetry clarified in the present work will be indispensable.

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Appendix A

--- Conventions ---

A.1. Metrics and light-cone coordinates

The metric of the target 10 dimensional Minkowski space is taken to be $\eta_{\mu\nu} = (-1, +1, \ldots, +1)$, where $\mu, \nu = 0 \sim 9$. The light-cone components of a vector $A^\mu$ are defined by $A^\pm \equiv \frac{1}{\sqrt{2}}(A^0 \pm A^9)$. Accordingly the contraction of two vectors is expressed as $A^\mu B_\mu = A^+ B^- + A^- B^+ + A^I B^I$, where the subscript $I$ for the transverse components runs from 1 to 8.

The worldsheet coordinates are denoted by $\xi^i = (\xi^0, \xi^1) = (t, \sigma)$. The metric $\eta_{ij}$, the antisymmetric tensor $\epsilon_{ij}$, the light-cone coordinates $\sigma_\pm$ and the derivatives $\partial_\pm$ are taken as $\eta_{ij} = (-1, +1)$, $\epsilon_{01} = -\epsilon_{10} \equiv 1$, $\sigma_\pm = \xi^0 \pm \xi^1$ and $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. Therefore $\partial_i A \partial^j B = -2(\partial_+ A \partial_- B + \partial_- A \partial_+ B)$ and $\partial_i \partial^i = -4\partial_+ \partial_-$.

A.2. Gamma matrices and spinors

$32 \times 32$ $SO(9,1)$ Gamma matrices are denoted by $\Gamma^\mu, (\mu = 0 \sim 9)$ and they obey the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. The 10 dimensional chirality operator is taken to be $\bar{\Gamma}_{1,9} \equiv \Gamma^0 \Gamma^1 \cdots \Gamma^9$ and it satisfies $(\bar{\Gamma}_{1,9})^2 = 1$. We use the Majorana basis, where $\Gamma^\mu$ are all
real and unitary. Within the Majorana basis, we define the Weyl basis to be the one in which \( \bar{\mathbf{1}}_{1,9} = \text{diag}(1_{16}, -1_{16}) \), where \( 1_{16} \) denotes the \( 16 \times 16 \) unit matrix. In this basis, a general 32-component spinor \( \Lambda \) is written as \( \left( \begin{array}{c} \lambda^\alpha \\ \bar{\lambda}_\alpha \end{array} \right) \), where \( \lambda^\alpha \) and \( \bar{\lambda}_\alpha \) are chiral and anti-chiral respectively, with \( \alpha = 1 \sim 16 \). Correspondingly, \( \Gamma^\mu \), which flips chirality, can be expressed in terms of \( 16 \times 16 \) matrices \( \gamma^\mu \) and \( \bar{\gamma}^\mu \) as

\[
\Gamma^\mu = \left( \begin{array}{cc} 0 & (\gamma^\mu)_{\alpha\beta} \\ (\bar{\gamma}^\mu)_{\alpha\beta} & 0 \end{array} \right).
\]  

From the symmetry property of \( \Gamma^\mu \), we have \( \gamma_\mu = \bar{\gamma}_\mu \) \((\mu = 1 \sim 9)\) and \( \gamma^0 = -\bar{\gamma}^0 \), as matrices. However, we shall distinguish \( \gamma^\mu \) and \( \bar{\gamma}^\mu \) in order to keep track of the 10D chirality structure. From the Clifford algebra satisfied by \( \Gamma^\mu \), we obtain the algebra for \( \gamma^\mu \) and \( \bar{\gamma}^\mu \) as

\[
\gamma^\mu \bar{\gamma}^\nu + \bar{\gamma}^\mu \gamma^\nu = 2\eta^{\mu\nu}, \quad \bar{\gamma}^\mu \gamma^\nu + \gamma^\mu \bar{\gamma}^\nu = 2\eta^{\mu\nu}.
\]  

Antisymmetrized products of two \( \gamma^\mu \)'s are defined with appropriate index positions as

\[
(\gamma^\mu \gamma^\nu)^\alpha_\beta = \frac{1}{2} (\gamma^\mu \bar{\gamma}^\nu - \gamma^\nu \bar{\gamma}^\mu)^\alpha_\beta, \quad (\bar{\gamma}^\mu \bar{\gamma}^\nu)^\alpha_\beta = \frac{1}{2} (\bar{\gamma}^\mu \gamma^\nu - \bar{\gamma}^\nu \gamma^\mu)^\alpha_\beta.
\]  

SO(8) decomposition of the \( \gamma \)-matrices and spinors is taken as follows. The SO(8) chirality operator \( \bar{\Gamma}_8 \), which anticommutes with \( \Gamma^I \) and satisfies \( \bar{\Gamma}^2_8 = 1 \), is defined by

\[
\bar{\Gamma}_8 \equiv \Gamma^1 \Gamma^2 \cdots \Gamma^8 = \left( \begin{array}{cc} \chi_8 & 0 \\ 0 & \bar{\chi}_8 \end{array} \right) \]  

where \( \chi_8 \) and \( \bar{\chi}_8 \), which act respectively on the chiral and the anti-chiral sectors, are given

\[
(\chi_8)^\alpha_\beta \equiv (\gamma^1 \gamma^2 \cdots \gamma^7 \gamma_8)^\alpha_\beta, \quad \chi_8^2 = 1, \quad (\bar{\chi}_8)^\alpha_\beta \equiv (\bar{\gamma}^1 \bar{\gamma}^2 \cdots \bar{\gamma}^7 \bar{\gamma}_8)^\alpha_\beta, \quad \bar{\chi}_8^2 = 1.
\]  

Since \( \chi_8^2 = \bar{\chi}_8^2 = 1 \), we can choose the basis where they are diagonal, namely

\[
(\chi_8)^\alpha_\beta = \left( \begin{array}{cc} 1_8 & 0 \\ 0 & -1_8 \end{array} \right) = \left( \begin{array}{cc} \delta^a_b & 0 \\ 0 & -\delta^a_b \end{array} \right), \quad (\bar{\chi}_8)^\alpha_\beta = \left( \begin{array}{cc} 1_8 & 0 \\ 0 & -1_8 \end{array} \right) = \left( \begin{array}{cc} \delta_a^b & 0 \\ 0 & -\delta_a^b \end{array} \right),
\]  

where SO(8) chiral and anti-chiral sectors are indexed by the letters \( a, b, \ldots \) and \( \dot{a}, \dot{b}, \ldots \) respectively. These indices run from 1 to 8. In this basis each of the chiral and anti-chiral spinors \( \lambda^\alpha \) and \( \bar{\lambda}_\alpha \) is decomposed into SO(8)-chiral and anti-chiral components as

\[
\lambda^\alpha = \left( \begin{array}{c} \lambda^a \\ \lambda_{\dot{a}} \end{array} \right), \quad \bar{\lambda}_\alpha = \left( \begin{array}{c} \lambda_a \\ \lambda_{\dot{a}} \end{array} \right).
\]
In 16 component notation, these decompositions are effected by the following two types of $SO(8)$ chiral projection operators:

$$
(P^\pm_8)_{\alpha \beta} = \frac{1}{2}(1 \pm \chi_8)^\alpha_\beta, \quad (\bar{P}^\pm_8)_{\alpha \beta} = \frac{1}{2}(1 \pm \bar{\chi}_8)^\alpha_\beta.
$$

(A-10)

Correspondingly, we can make the $SO(8)$ decomposition of $\gamma^\mu$ matrices in the following way. First since $\gamma^I$ and $\bar{\gamma}^I$ “anti-commute” with the $SO(8)$ chirality operators in the manner $\gamma^I \chi_8 = -\chi_8 \gamma^I$ and $\bar{\gamma}^I \chi_8 = -\bar{\chi}_8 \bar{\gamma}^I$, they must be block off-diagonal as

$$
\gamma^I = \begin{pmatrix}
0 & (\gamma^I)_{ab} \\
(\gamma^I)_{\dot{a}\dot{b}} & 0
\end{pmatrix}, \quad \bar{\gamma}^I = \begin{pmatrix}
0 & (\bar{\gamma}^I)_{\dot{a}\dot{b}} \\
(\bar{\gamma}^I)_{ab} & 0
\end{pmatrix}.
$$

(A-11)

Next we examine the $SO(8)$ content of the light-cone components $\gamma^\pm$ and $\bar{\gamma}^\pm$, which are defined by

$$
\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^9 \pm \Gamma^0) = \begin{pmatrix} 0 & \gamma^\pm \\ \bar{\gamma}^\pm & 0 \end{pmatrix},
$$

(A-12)

$$
\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^9 \pm \gamma^0), \quad \bar{\gamma}^\pm = \frac{1}{\sqrt{2}}(\bar{\gamma}^9 \pm \bar{\gamma}^0).
$$

(A-13)

The relations $\gamma^+ \bar{\gamma}^- + \gamma^- \bar{\gamma}^+ = \bar{\gamma}^+ \gamma^- + \bar{\gamma}^- \gamma^+ = 2$, which follow from the Clifford algebra, are often useful. We now fix the explicit form of $\gamma^\pm$ and $\bar{\gamma}^\pm$ by adopting a suitable convention. From the relation $\bar{\Gamma}_{1,9} = \Gamma^0 \bar{\Gamma}_8 \Gamma^9$, we get $\gamma^9 = \gamma^0 \gamma^1 \bar{\gamma}^2 \cdots \bar{\gamma}^8$ and $\bar{\gamma}^9 = -\bar{\gamma}^0 \gamma^1 \gamma^2 \cdots \gamma^8$. They are actually equal since, as was already mentioned, $\gamma^m = \bar{\gamma}^m$ for $m = 1 \sim 9$ and $\gamma^0 = -\bar{\gamma}^0$. One consistent convention we adopt is $\gamma^0 = 1 = -\bar{\gamma}^0$. Then from the definition (A-13) the non-vanishing $SO(8)$ components of $\gamma^\pm$ and $\bar{\gamma}^\pm$ take the form

$$
(\gamma^+)_{ab} = \sqrt{2} \delta^{ab}, \quad (\gamma^-)_{\dot{a}\dot{b}} = -\sqrt{2} \delta^{\dot{a}\dot{b}},
$$

(A-14)

$$
(\bar{\gamma}^+)_{\dot{a}\dot{b}} = -\sqrt{2} \delta^{\dot{a}\dot{b}}, \quad (\bar{\gamma}^-)_{ab} = \sqrt{2} \delta_{ab}.
$$

(A-15)

### Appendix B

---

**Some details of the Lorentz algebra**

---

#### B.1. Illustration of OPE method for Lorentz algebra

Let us illustrate our OPE method for the calculation of the Lorentz algebra by giving an example from the computation of $[L^{\mu\nu}, L^{\rho\sigma}]$ using the form of $L^{\mu\nu}$ given in (5.71). It suffices to discuss the basic commutator

$$
\left[ \int [dz] \hat{X}^\mu \Pi^\nu(z), \int [dw] \hat{X}^\rho \Pi^\sigma(w) \right]
$$

(B.1)
and we shall focus on the contribution coming from the contraction of $\hat{X}^\nu(z)$ and $\hat{X}^\rho(w)$.

As this produces a logarithm that violates the condition (i) described in the main text, we go back to the definition (5-76) and denote the integrals for the two regions as $I_{|z|>|w|}$ and $I_{|w|>|z|}$. Then, $I_{|z|>|w|}$ is given by

$$I_{|z|>|w|} = \int_{|z|>|w|} [dz][dw] \left[ -\eta^{\mu\rho} \ln \left( 1 - \frac{w}{z} \right) \right] : \Pi^\nu(z) \Pi^\sigma(w) : . \quad (B.2)$$

Although a logarithm is present, the cut in the $z$-plane is from 0 to $w$ and there is no singularity along the $z$-contour. Therefore we can substitute $\Pi^\nu(z) = i\partial \hat{X}^\nu(z) + (p^\nu/z)$ and perform integration by parts to rewrite it as

$$I_{|z|>|w|} = \eta^{\mu\rho} \int_{|z|>|w|} [dz][dw] \partial_z \ln \left( 1 - \frac{w}{z} \right) i : \hat{X}^\nu(z) \Pi^\sigma(w) :$$

$$- \eta^{\mu\rho} \int_{|z|>|w|} [dz][dw] \ln \left( 1 - \frac{w}{z} \right) \frac{p^\nu}{z} \Pi^\sigma(w) . \quad (B.3)$$

The second term actually vanishes since the $z$-contour can be moved to infinity and then we have $\ln 1 = 0$. Therefore

$$I_{|z|>|w|} = i\eta^{\mu\rho} \int_{|z|>|w|} [dz][dw] \left( \frac{1}{z-w} - \frac{1}{z} \right) : \hat{X}^\nu(z) \Pi^\sigma(w) :$$

$$= i\eta^{\mu\rho} \int_{|z|>|w|} [dz][dw] \frac{1}{z-w} : \hat{X}^\nu(z) \Pi^\sigma(w) : - 2i\eta^{\mu\rho} x^\nu \pi^\sigma . \quad (B.4)$$

Note that we pick up an additional zero mode contribution. As for the $I_{|w|>|z|}$ integral, by similar reasoning we can rewrite it as

$$I_{|w|>|z|} = - \int_{|w|>|z|} [dz][dw] \eta^{\mu\rho} \ln \left( 1 - \frac{z}{w} \right) \Pi^\sigma(w) \left( i\partial \hat{X}^\nu(z) + \frac{p^\nu}{z} \right)$$

$$= i\eta^{\mu\rho} \int_{|w|>|z|} [dz][dw] \partial_z \ln \left( 1 - \frac{z}{w} \right) \Pi^\sigma(w) \hat{X}^\nu(z)$$

$$- \int_{|w|>|z|} [dz][dw] \eta^{\mu\rho} \ln \left( 1 - \frac{z}{w} \right) \Pi^\sigma(w) \frac{p^\nu}{z} . \quad (B.5)$$

Again the $z$ integral in the second term vanishes since $z$ contour is around 0 and we have the factor $\ln 1 = 0$. Therefore we obtain

$$I_{|w|>|z|} = i\eta^{\mu\rho} \int_{|w|>|z|} [dz][dw] \frac{1}{z-w} \Pi^\sigma(w) \hat{X}^\nu(z) . \quad (B.6)$$

We can now put the two contributions (B.4) and (B.6) together. The parts containing the simple pole can be combined in the usual way and we easily obtain

$$I_{|z|>|w|} - I_{|w|>|z|} = i\eta^{\mu\rho} \int [dw] : \hat{X}^\nu \Pi^\sigma(w) : - 2i\eta^{\mu\rho} x^\nu \pi^\sigma . \quad (B.7)$$
Contributions from all the other contractions can be computed in a similar manner and we find\textsuperscript{29}

\[
\left[ \int [dz] \overset{\circ}{X}^\mu \Pi^\nu (z), \int [dw] \overset{\circ}{X}^\rho \Pi^\sigma (w) \right] = i \int [dw] \left[ \eta^{\mu\sigma} \overset{\circ}{X}^\mu \Pi^\nu - \eta^{\nu\rho} \overset{\circ}{X}^\rho \Pi^\sigma + i\eta^{\mu\rho} \overset{\circ}{X}^\nu \Pi^\sigma (w) - \eta^{\nu\sigma} \overset{\circ}{X}^\rho \Pi^\mu \right] + 2i(-\eta^{\mu\rho} x^\nu p^\sigma + \eta^{\nu\sigma} x^\rho p^\mu + \eta^{\mu\rho} x^\nu p^\sigma - \eta^{\nu\sigma} x^\rho p^\mu).
\]

(B.8)

Note that the extra zero mode part in the second line is symmetric under the interchange \(\mu \leftrightarrow \nu\). Thus it vanishes upon antisymmetrization and we recover the usual closure for the algebra \([L^\mu, L^\nu]\).

Having explained the basic method of computation, we now comment on the calculation of the commutator between a term involving \(\overset{\circ}{X}^\mu\) and a term free of \(\overset{\circ}{X}^\mu\). As an illustration, let us consider the following commutator which occurs in \([\mathcal{M}^I-, \mathcal{M}^J-]\):

\[
\left[ \int [dz] \overset{\circ}{X}^I \Pi^-(z), \int [dw] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right] = \int \left[ dz \right] \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \Pi^-(z) - \int \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \overset{\circ}{X}^I \Pi^-(z)
\]

(B.9)

In this case, since the relevant basic OPE’s have no logarithmic singularities, we only need to be careful about the zero mode parts coming from (5.70) and (5.74). We then get

\[
\left[ \int [dz] \overset{\circ}{X}^I \Pi^-(z), \int [dw] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right] = \int \left[ dz \right] \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \Pi^-(z) - \int \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \overset{\circ}{X}^I \Pi^-(z)
\]

\begin{align*}
&= i \int \left[ dz \right] \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \Pi^-(z) - i \int \left[ dw \right] \left( \frac{\bar{\gamma}^J S}{} \right)_a \left( \frac{\bar{\gamma}^K S}{} \right)_a \Pi^K (w) \right) \overset{\circ}{X}^I \Pi^-(z)
\end{align*}

(B.10)

\textsuperscript{29} Double contraction contributions are easily seen to vanish.
where we have used the relation (5.70) for the last equality. Another expression which pairs with the previous one in $[\mathcal{M}^I, \mathcal{M}^J]$ is
\[
\left[ \int [dz] \breve{X}^I (z), \int [dw] \frac{(\bar{\gamma}^J S)_{\dot{a}} (\bar{\gamma}^K S)_{\dot{a}} \Pi^K (w)}{\Pi^+} \right] = i \int [dw] \left( \frac{(\bar{\gamma}^J S)_{\dot{a}} (\bar{\gamma}^J S)_{\dot{a}} \Pi^- (w)}{\Pi^+} - \frac{(\bar{\gamma}^J S)_{\dot{a}} (\bar{\gamma}^K S)_{\dot{a}} \Pi^I \Pi^K (w)}{(\Pi^+)^2} \right) - ip^- \frac{(\bar{\gamma}^J S)_{\dot{a}} (\bar{\gamma}^I S)_{\dot{a}} (\bar{\gamma}^K S)_{\dot{a}} \Pi^K (0)}{(\Pi^+)^2}.
\]
We see that the zero mode parts exactly cancel in the difference of (B.10) and (B.11) and the result is the same as what one obtains by applying the standard OPE formula (5.18). This phenomenon occurs also in other commutators of similar types.

B.2. Calculation of $[\mathcal{M}^I, \mathcal{M}^J]$ 

Here we display some details of the non-trivial computation leading from (5.83) to (5.84) in the calculation of the commutator $[\mathcal{M}^I, \mathcal{M}^J]$.

First, using the formula (C.9) for the commutator of $\gamma^I$'s, we can rewrite the 7th term of (5.83) as
\[
- \frac{1}{4} \frac{\Pi^I \Pi^L}{(\Pi^+)^2} \left\{ S_a (\gamma^I \gamma^J)^a \right\} = - \frac{1}{8} \frac{\Pi^K \Pi^L}{(\Pi^+)^2} \left\{ S_a \left[ \gamma^I, \gamma^J \right]^a \right\} = \frac{1}{4} \frac{\Pi^I \Pi^K}{(\Pi^+)^2} (\gamma^I S)_{\dot{a}} (\gamma^K S)_{\dot{a}} - \frac{1}{4} \frac{\Pi^I \Pi^L (\gamma^I S)_{\dot{a}} (\gamma^L S)_{\dot{a}}}{(\Pi^+)^2} + \frac{1}{4} \frac{\Pi^I \Pi^K (\gamma^I S)_{\dot{a}} (\gamma^J S)_{\dot{a}}}{(\Pi^+)^2}.
\]

Next, the 8th term of (5.83) can be rewritten, up to total derivative with respect to $w$, as\textsuperscript{30},
\[
\frac{1}{16} \left\{ \partial \left( \frac{S_a}{\Pi^+} \right) \right\} \left[ \gamma^I, \gamma^J \right]^a \left\{ \partial \left( \frac{S^b}{\Pi^+} \right) \right\} = - \frac{3}{4} \left\{ \partial \left( \frac{(\bar{\gamma}^I S)_{\dot{a}}}{\Pi^+} \right) \right\} \left\{ \partial \left( \frac{(\bar{\gamma}^J S)_{\dot{a}}}{\Pi^+} \right) \right\} = \frac{3}{4} \left\{ \partial^2 \left( \frac{1}{\Pi^+} \right) \right\} \left( \frac{(\bar{\gamma}^I S)_{\dot{a}}}{\Pi^+} \right) \left( \frac{(\bar{\gamma}^J S)_{\dot{a}}}{\Pi^+} \right) + \frac{3}{4} \frac{(\bar{\gamma}^I \partial^2 S)_{\dot{a}} (\bar{\gamma}^J S)_{\dot{a}}}{(\Pi^+)^2} - \frac{3}{2} \frac{\partial \Pi^+ (\bar{\gamma}^I \partial S)_{\dot{a}} (\bar{\gamma}^J S)_{\dot{a}}}{(\Pi^+)^3}.
\]
\textsuperscript{30} The symbol “$\simeq$” represents the equality up to total derivatives.
Thirdly, by using the formula \( \text{Tr} \{ \{ \gamma^I, \gamma^J \} \} = 16 (\delta^{IL} \delta^{JK} - \delta^{IJ} \delta^{KL}) \), which follows from the identity (C.10), the last term of (5.83) can be simplified, again up to total derivatives, to

\[
-\frac{1}{8} \left\{ \partial \left( \frac{\Pi^K}{\Pi^+} \right) \right\} \left( \frac{\Pi^L}{\Pi^+} \right) \text{Tr} (\gamma^I \gamma^J)
= -\frac{1}{16} \left\{ \partial \left( \frac{\Pi^K}{\Pi^+} \right) \right\} \left( \frac{\Pi^L}{\Pi^+} \right) \text{Tr} (\{ \gamma^I, \gamma^J \})
\approx -\left\{ \partial \left( \frac{\Pi^I}{\Pi^+} \right) \right\} \left( \frac{\Pi^J}{\Pi^+} \right) \approx -\frac{1}{2} \frac{\Pi^I \partial \Pi^J - \Pi^J \partial \Pi^I}{(\Pi^+)^2}.
\]  

(B.14)

Finally, we can rewrite the 6th term of (5.83), which is quartic in fermions, as (up to a total derivative)

\[
-\frac{1}{16} \left\{ \partial \left( \frac{(\gamma^I S)_a (\gamma^K S)_a}{\Pi^+} \right) \right\} \left( \frac{(\gamma^J S)_b (\gamma^K S)_b}{\Pi^+} \right)
\approx -\frac{1}{32} \frac{1}{(\Pi^+)^2} \left[ \partial_w \left\{ (\gamma^I S)_a (\gamma^K S)_a \right\} \cdot (\gamma^J S)_b (\gamma^K S)_b - (\gamma^I S)_a (\gamma^K S)_a \cdot \partial_w \left\{ (\gamma^J S)_b (\gamma^K S)_b \right\} \right]
= -\frac{1}{4} \frac{S_a \partial S_a (\gamma^I S)_a (\gamma^J S)_a}{(\Pi^+)^2}.
\]  

(B.15)

In this process we have used the basic \( \gamma \)-matrix identity (C.3) and the following formula (which itself can be verified with repeated use of (C.3) and (C.4).):

\[
(\gamma^I \partial S)_a (\gamma^K S)_a (\gamma^J S)_b (\gamma^K S)_b - (\gamma^I S)_a (\gamma^K S)_a (\gamma^J \partial S)_b (\gamma^K S)_b = 6 S_a \partial S_a (\gamma^I S)_a (\gamma^J S)_a.
\]  

(B.16)

Using these rearrangements, (5.83) can be brought to the simpler form (5.84).

Appendix C

Construction of vertex operators

In the construction of the vertex operators, one needs various identities involving \( SO(8) \) \( \gamma \)-matrices and spinors, some of which are rather non-trivial. In this appendix, we record and discuss such identities.

C.1. Useful \( SO(8) \) \( \gamma \)-matrix identities

In this appendix, we collect some useful identities involving the \( SO(8) \) \( \gamma \)-matrices, which are frequently used in the calculations described in the main text and in other parts of the appendices.
First we give the definitions of the anti-symmetrized products of the $SO(8)$ $\gamma$-matrices:

$$
\gamma^{IJ} \equiv \frac{1}{2} (\gamma^I \gamma^J - \gamma^J \gamma^I), \quad \bar{\gamma}^{IJ} \equiv \frac{1}{2} (\bar{\gamma}^I \gamma^J - \gamma^J \bar{\gamma}^I),
$$

$$
\gamma^{IJK} \equiv \frac{1}{3!} (\gamma^I \gamma^J \gamma^K \pm \text{(cycl.)}), \quad \bar{\gamma}^{IJK} \equiv \frac{1}{3!} (\bar{\gamma}^I \gamma^J \bar{\gamma}^K \pm \text{(cycl.)}),
$$

$$
\gamma^{IJKL} \equiv \frac{1}{4!} (\gamma^I \gamma^J \gamma^K \gamma^L \pm \text{(cycl.)}), \quad \bar{\gamma}^{IJKL} \equiv \frac{1}{4!} (\bar{\gamma}^I \gamma^J \bar{\gamma}^K \gamma^L \pm \text{(cycl.)}). \quad (C.1)
$$

(As remarked earlier, once we decompose the spinors and the $\gamma$-matrices into $SO(8)$ components, we can raise and lower the indices by $\delta^{ab}$, $\delta_{ab}$ and so on. Below we use this freedom to lower all the indices.) The $SO(8)$ Clifford relations read

$$
\{\gamma^I, \bar{\gamma}^J\}_{ab} \equiv (\gamma^I \bar{\gamma}^J + \gamma^J \bar{\gamma}^I)_{ab} = 2 \delta_{ab} \delta^{IJ}, \quad \{\bar{\gamma}^I, \gamma^J\}_{\dot{a} \dot{b}} \equiv (\bar{\gamma}^I \gamma^J + \bar{\gamma}^J \gamma^I)_{\dot{a} \dot{b}} = 2 \delta_{\dot{a} \dot{b}} \delta^{IJ}. \quad (C.2)
$$

Often used fundamental identities are

$$
2 \delta_{ab} \delta_{\dot{a} \dot{b}} = \gamma^I_{ab} \delta^{IJ}_{\dot{a} \dot{b}} + \gamma^I_{ab} \delta^{IJ}_{\dot{a} \dot{b}}, \quad (C.3)
$$

$$
(\gamma^I \bar{\gamma}^J\}_{ab} = \delta^{IJ} \delta_{ab} + (\gamma^I \gamma^J)_{ab} = \delta^{IJ} \delta_{ab} + \bar{\gamma}^{IJ}_{ab}. \quad (C.4)
$$

Fundamental Fierz identities read

$$
S_a S_b = \frac{1}{16} (S \gamma_{KL} S) (\gamma^{KL})_{ab}, \quad (C.5)
$$

$$
\gamma^I_{ab} \gamma^J_{\dot{a} \dot{b}} = \delta_{ab} \delta_{\dot{a} \dot{b}} + \frac{1}{4} \gamma^{KL}_{ab} \gamma_{ab}. \quad (C.6)
$$

Useful formulas for various commutators and anti-commutators are given by

$$
[\bar{\gamma}^I, \gamma^{JK}] \equiv \bar{\gamma}^I \gamma^{JK} - \gamma^{JK} \bar{\gamma}^I = 2 (\delta^{IJ} \bar{\gamma}^K - \delta^{IK} \bar{\gamma}^J), \quad (C.7)
$$

$$
\{\bar{\gamma}^I, \gamma^{JK}\} \equiv \bar{\gamma}^I \gamma^{JK} + \gamma^{JK} \bar{\gamma}^I = 2 \bar{\gamma}^{IJ}, \quad (C.8)
$$

$$
[\gamma^{IJ}, \bar{\gamma}^{KL}] \equiv 2 (\delta^{IL} \gamma^{JK} + \delta^{IK} \gamma^{JL} - \delta^{IL} \gamma^{JK} - \delta^{IK} \gamma^{JL}), \quad (C.9)
$$

$$
\{\gamma^{IJ}, \bar{\gamma}^{KL}\} \equiv 2 (\delta^{IL} \delta^{JK} - \delta^{IK} \delta^{JL}) + 2 \gamma^{IJKL}. \quad (C.10)
$$

C.2. **Identities involving $SO(8)$ spinors**

First we discuss the identity involving a product of five $S_a$’s of the form

$$
T \equiv k_A k_B k_C (S \gamma^{JA} S) (S \gamma^{JB} S) (S \gamma^{JC} S) = 0, \quad (C.11)
$$

which holds for $k_A k_A = 0$. All the capital Roman indices refer to the transverse components, running from 1 to 8. We will give a sketch of the proof, which requires some amount of computation.
In terms of components, we have \( T = k_A k_B k_C S_a (S_b S_c) (S_d S_e) \gamma_{ab} \gamma_{cd} \gamma_{ef} \). The basic idea is to expand the antisymmetric products \( S_b S_c \) and \( S_d S_e \) in terms of \( \gamma^{AB} \), which span the space of \( 8 \times 8 \) antisymmetric quantities. Explicitly, \( S_b S_c = (1/16)(\gamma^{PQ})_{ik}(S_i \gamma^{PQ} S_j) \) and similarly for \( S_d S_e \). Substituting them into \( T \), we get

\[
T = \left( \frac{1}{16} \right)^2 k_A k_B k_C (\gamma^{IAB} \gamma^{PQ} \gamma^{JBC} \gamma^{MN} \gamma^C)_{ij} (S_i \gamma^{PQ} S_j) (S_i \gamma_{MN} S_j) \delta_{ij}.
\]

The product of five \( \gamma \) matrices appearing in this expression can be reduced by repeated use of the familiar identity \( \gamma^{AB} \gamma^{CD} = \gamma^{ABCD} + \delta^{BC} \gamma^{AD} + \delta^{AD} \gamma^{BC} - \delta^{AC} \gamma^{BD} - \delta^{BD} \gamma^{AC} \). In the course of this reduction, because of the contraction with \( k_A k_B k_C \), some of the terms drop out due to the condition \( k_A k_B = 0 \). Simplification also occurs when a symmetric quantity is contracted with an antisymmetric quantity such as \( S_i \gamma^{PQ} S_j \). After some straightforward but tedious computation one proves \( T = 0 \).

Another non-trivial identity one needs involves products of three \( S_i \)’s. It reads

\[
k^B R^A (\epsilon \bar{\gamma}^I \gamma^{AB} S) + 2k^I R^C (\epsilon \bar{\gamma}^C S) - 6k^C R^I (\epsilon \bar{\gamma}^C S) = 0,
\]

where \( R \) is an expression quadratic in \( S \) given by

\[
R^A = k^D S \gamma^{AD} S.
\]

Just as before, this identity holds only when \( k_A k_B = 0 \). To prove it, we first make it into a slightly better form by applying the simple formula \( \bar{\gamma}^I \gamma^{AB} = \bar{\gamma}^A \gamma^{BI} + 2\delta^{IA} \gamma^{B} - \delta^{IB} \gamma^{A} - \delta^{AB} \gamma^I \) to the first term of (C.13). Then, what we need to prove becomes \( A_1 = 4A_3 - A_2 \), where

\[
A_1 = k^B R^A (\epsilon \bar{\gamma}^A \gamma^{BI} S), \quad A_2 = k^I R^C (\epsilon \bar{\gamma}^C S), \quad A_3 = k^C R^I (\epsilon \bar{\gamma}^C S).
\]

We now make use of the basic \( SO(8) \) \( \gamma \)-matrix identity (C.6). Contract this with \( k^B k^C S_a S_b \gamma_{ac} \gamma_{bd} \delta_{\epsilon b S_d} \). The left hand side precisely yields \( A_1 \). The right hand side gives \( A_3 + B \), where \( B = (1/4)k^B k^C (S_i \gamma^{PQ} \gamma^{BI} S)(\epsilon \bar{\gamma}^{PQ} \bar{\gamma}^{C} S) \). We can reduce the product of \( \gamma \)-matrices in this expression by using the familiar formulas. Again in the course of this reduction, some of the terms vanish due to \( k_A k_B = 0 \). After some computation, we obtain \( B = (1/2)(A_1 + 2A_3 - A_2) \). Together we find \( A_1 = A_3 + (1/2)(A_1 + 2A_3 - A_2) \), which gives the desired relation \( A_1 = 4A_3 - A_2 \).

C.3. Some details of the \( \epsilon \)-SUSY transformation of \( V_B(\zeta) \)

In this appendix we provide some details of the \( \epsilon \)-SUSY transformation property of the bosonic vertex operator \( V_B(\zeta) \) given in (6.68).
C.3.1. Calculation of the commutator $[e^a Q_a, V_B(\zeta)]$

Using the OPE technique, it is tedious but straightforward to compute the commutator $[e^a Q_a, V_B(\zeta)]$, with the result

\[
[eQ, V_B(\zeta)] = \int [dw] \, e^{ik \cdot x} \left\{ (-2^{1/4} \zeta^-) \sqrt{\Pi^+} \, k_l \, (e^I S) + (2^{1/4} \zeta_I) \, k^- \sqrt{\Pi^+} \, (\epsilon^I \Gamma S) \\
+ (2^{1/4} \zeta_I) \, k^I j \frac{k_I (e^I S)}{\sqrt{\Pi^+}} + (-2^{1/4} \zeta^I) \, \Pi I k_l (e^I S) \sqrt{\Pi^+} + (2^{-3/4} \zeta^I) \, \Pi I k_l (e^I \Gamma^L S) \sqrt{\Pi^+} \\
+ (2^{-3/4} \zeta^+ \partial \frac{\Pi I (e^I S)}{(\Pi^+)^{3/2}} + (2^{-3/4} \zeta^I) \sqrt{\Pi^+} \, k^I k_L (e^L S) \\
+ (2^{-7/4} \zeta^+ \partial^2 \frac{k_I (e^I S)}{(\Pi^+)^{3/2}} + (2^{-3/4} \zeta^I) \, \partial \frac{1}{\sqrt{\Pi^+}} \, k_I k_L (e^L S) + 7 \cdot 2^{-7/4} \zeta^+ \partial^2 \frac{1}{\sqrt{\Pi^+}} \, k_L (e^L S) \\
+ (2^{1/4} \zeta^+ \frac{k^- k_I k_I (e^I S)}{\sqrt{\Pi^+}} + (2^{1/4} \zeta^I) \, k^- \partial \frac{k_I (e^I S)}{\sqrt{\Pi^+}} \\
+ (2^{1/4} \zeta_I) \, \frac{k_I k_I (e^I S)}{(\Pi^+)^{3/2}} + (2^{5/4} \zeta^I) \, \partial \frac{k_I (e^I S)}{\sqrt{\Pi^+}} \, k_I k_I \frac{\Pi^I}{\Pi^+} \\
+ (-2^{-7/4} \zeta^+ \partial \frac{e^I S}{\sqrt{\Pi^+}} \, R_I \frac{2^{1/4} \zeta^I}{24} + (2^{-7/4} \zeta^I \partial \frac{\Pi I k_I (e^I \Gamma^L S)}{(\Pi^+)^{3/2}} \\
+ (2^{-7/4} \zeta^I) \, \frac{R_I k_I (e^I S)}{\sqrt{\Pi^+}} + (2^{-7/4} \zeta^I \partial \frac{\Pi I k_I (e^I S)}{(\Pi^+)^{3/2}} \\
+ \left( \frac{2^{1/4}}{96} \zeta^+ \right) \frac{R_I R_I k_I (e^I S)}{(\Pi^+)^{3/2}} \right\}. \tag{C-16}
\]

In this calculation, we have performed an integration by parts for the first line and made use of the on-shell condition $k_I k^I = 0$ and the $\gamma$-matrix identities (C.7), (C.8) appropriately. Another identity we used is

\[
\epsilon (\bar{\epsilon}^I \Gamma^L) S = \frac{1}{2} \epsilon \left( \{ \bar{\gamma}^I, \gamma^J \} + [\bar{\gamma}^I, \gamma^J] \right) S. \tag{C-17}
\]

For example a term proportional to $k_I R_I k_L (e^I \Gamma^L S)$ was shown to vanish due to the above identity as well as the relation $k_I R^I = 0$. 

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C.3.2. Fermion vertex $V_F(\tilde{u})$

We now want to compare the above commutator with $V_F(\tilde{u})$. Using the form of $\tilde{u}^a$, given in (6.27) (with $k^+$ set to zero), we obtain

$$V_F(\tilde{u}) = \int [dw] \ e^{ik \cdot X} \left\{ \frac{1}{2} \frac{\Pi^l}{\sqrt{\Pi^+}} k^l (\epsilon^{iJ} S) + \frac{1}{2} \frac{\Pi^l}{\sqrt{\Pi^+}} k^l (\epsilon^{Ji} S) \right\} .$$

First, one easily sees the terms in the first line are consistent with the corresponding terms in the commutator (C.16). Next, the third term of $V_F(\tilde{u})$ can be rewritten, using the $\gamma$-identity (C.7), as

$$\Pi^l k^J \zeta_L (\epsilon^{iJ} \gamma^i \sigma^L S) = \Pi^l k^J \zeta_L (\epsilon^{iJ} \gamma^i \sigma^L S) - 2 k^j \Pi^j \zeta_L (\epsilon^{Lj} \gamma^j S) + 2 \zeta^i \Pi^l k^j (\epsilon^{iJ} S) .$$

This is consistent with the terms in the second line of the commutator (C.16). Now for the fifth term of $V_F(\tilde{u})$, we can make use of the same $\gamma$-identity (C.7) and the property $k^i R^j = 0$ to bring it to the form

$$R^l k^J \zeta_L (\epsilon^{iJ} \gamma^i \sigma^L S) = R^l k^J \zeta_L (\epsilon^{iJ} \gamma^i \sigma^L S) + 2 \zeta^i R^l k^j (\epsilon^{iJ} S) .$$

This will be useful below.

C.3.3. Difference as a BRST term

We now wish to show that the difference $[\epsilon^a Q_4, V_B(\zeta)] - V_F(\tilde{u})$ is equal to the BRST-exact expression $\{Q, \Psi_B(\epsilon, \zeta)\}$. Using the form of $\Psi_B(\epsilon, \zeta)$ given in (6.69) we find

$$\{Q, \Psi_B(\zeta)\} = \left\{ -2^{1/4} \zeta^+ \right\} \int [dw] \ e^{ik \cdot X} \left\{ \frac{\Pi^l k^l (\epsilon^{iJ} S)}{\sqrt{\Pi^+}} + \frac{1}{2} \frac{\Pi^l k^l (\epsilon^{iJ} S)}{(\Pi^+)^{3/2}} \right\}$$

$$- \frac{1}{2} \frac{(S \partial S) k^l (\epsilon^{iJ} S)}{(\Pi^+)^{3/2}} + \frac{3}{4} \frac{k^l (\epsilon^{iJ} \partial^2 S)}{(\Pi^+)^{3/2}} + \frac{k^l \partial \Pi^+ k^l (\epsilon^{iJ} S)}{(\Pi^+)^{3/2}}$$

$$+ \frac{k^l \partial \Pi^+ k^l (\epsilon^{iJ} S)}{(\Pi^+)^{3/2}} - \frac{1}{4} \frac{(\partial^2 \Pi^+)^2 k^l (\epsilon^{iJ} S)}{(\Pi^+)^{5/2}} - \frac{1}{2} \frac{(\partial \Pi^+)^2 k^l (\epsilon^{iJ} S)}{(\Pi^+)^{7/2}} \right\} .$$

We will organize the proof of the relation $[\epsilon^a Q_4, V_B(\zeta)] - V_F(\tilde{u}) = \{Q, \Psi_B(\epsilon, \zeta)\}$ according to the number of fermions $S_a$.

First consider the term in the commutator consisting of five fermions (the last term of (C.16)). Since no such term exists in $V_F(\tilde{u})$ nor in the BRST-exact expression (C.21), it
must vanish by itself. This is indeed the case due to the identity (C.11) proved in appendix C.2.

Next, consider the terms cubic in $S_a$. For the first such term in (C.16), we perform the following partial integration,

$$
\int [dw] \partial \left( \frac{\epsilon^{-1} S}{\sqrt{\Pi^+}} \right) \frac{R_I}{\Pi^+} e^{ik \cdot X} = \int [dw] \left[ \frac{1}{3} \left\{ 2 \left( \epsilon^{-1} \partial S \right) R_I - \left( \epsilon^{-1} S \right) \partial R_I \right\} \frac{1}{(\Pi^+)^{3/2}} e^{ik \cdot X} - \frac{1}{3} \left( k^- \Pi^+ + k_J \Pi^J \right) \left( \epsilon^{-1} S \right) \frac{R_I}{(\Pi^+)^{3/2}} e^{ik \cdot X} \right], \quad (C.22)
$$

and rearrange the numerator of the first term on the RHS\textsuperscript{31} by using the identity (5.93). Then the RHS of (C.22) becomes

$$
- \int [dw] \left[ \frac{2^{-7/4}}{3} \left\{ 2 \left( \epsilon^{-1} \partial S \right) R_I - \left( \epsilon^{-1} S \right) \partial R_I \right\} \frac{1}{(\Pi^+)^{3/2}} \right] = \int [dw] \left[ 2^{-3/4} S_b \partial S_b k_L \left( \epsilon^{-1} S \right) \frac{1}{(\Pi^+)^{3/2}} \right]. \quad (C.23)
$$

This precisely matches the three-fermion term in the BRST-exact expression (C.21).

As there are no more three-fermion terms in (C.21), the rest of the three-fermion terms must match between those in the commutator (C.16) and the ones in $V_F(\tilde{u})$. This means that the following relation must hold:

$$
\left( \frac{21/4}{24} \right) \zeta^+ \frac{\Pi_I R_J k_L \left( \epsilon^{-1} \gamma^J \gamma^L S \right)}{(\Pi^+)^{3/2}} + \left( \frac{2^{-7/4}}{3} \right) \zeta^+ \frac{k^- \left( \epsilon^{-1} S \right) R_I}{\sqrt{\Pi^+}} \quad (C.24)
$$

$$
+ \left( \frac{2^{-7/4}}{3} \right) \zeta^+ \frac{k_J \Pi^J \left( \epsilon^{-1} S \right) R_I}{(\Pi^+)^{3/2}} + \left( 2^{-7/4} \right) \zeta^+ \frac{R_J k_L \left( \epsilon^{-1} S \right) \Pi^J}{\Pi^+} \quad (C.25)
$$

$$
+ \left( -2^{-7/4} \right) \zeta^+ \frac{\Pi_J R^J k_L \left( \epsilon^{-1} S \right)}{(\Pi^+)^{3/2}} \quad (C.26)
$$

$$
= \left( \frac{2^{-3/4}}{12} \right) \zeta_L \frac{R_I}{\sqrt{\Pi^+}} k_J \left( \epsilon^{-1} \gamma^J \gamma^I S \right) + \left( \frac{2^{-3/4}}{12} \right) \zeta^+ \frac{R_I}{\sqrt{\Pi^+}} k^- \left( \epsilon^{-1} S \right). \quad (C.27)
$$

Using (C.20) and the on-shell condition $\zeta^+ k^- = -\zeta^J k_J$, we can reorganize this equation in powers of $\Pi^+$ in the following way:

$$
\frac{\zeta^J}{(\Pi^+)^{1/2}} \left\{ 4 R_J k_L \left( \epsilon^{-1} S \right) - k_J R_I \left( \epsilon^{-1} S \right) + R_I k_L \left( \epsilon^{-1} \gamma^J \gamma^L S \right) \right\}
$$

$$
+ \frac{\zeta^+}{(\Pi^+)^{3/2}} \left\{ \Pi_J R_J k_L \left( \epsilon^{-1} \gamma^J \gamma^L S \right) + 2 k_J \Pi^J R_I \left( \epsilon^{-1} S \right) - 6 \Pi_J R^J k_L \left( \epsilon^{-1} S \right) \right\} = 0. \quad (C.28)
$$

\textsuperscript{31} Note that $R^I = k_L(\hat{\gamma}^I S)_{\bar{a}}(\hat{\gamma}^L S)_{\bar{a}}$.  

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Indeed, we can show that the coefficient of $1/(\Pi+)^{1/2}$ and $1/(\Pi+)^{3/2}$ vanishes separately, thanks to the identities (C.13) and the relation $A_1 - 4A_3 + A_2 = 0$, where $A_1 \sim A_3$ are defined in appendix C.2. Therefore, all the three-fermion terms are consistent with the $\epsilon$-SUSY relation.

Finally, consider the single-fermion terms. First, we focus on the two terms which constitute the third line of the commutator (C.16). By using an integration by parts, they can be combined as

$$
\int [dw] e^{ik \cdot X} \left\{ (2^{-3/4}\zeta^+) \partial \left( \frac{\Pi_I (\epsilon \gamma^I S)}{(\Pi+)^{3/2}} \right) + (2^{-3/4}\zeta^+) k_I \Pi^I \Pi_J (\epsilon \gamma^J S) \right\}
= \int [dw] e^{ik \cdot X} \left\{ - (2^{-3/4}\zeta^+) \frac{k^- \Pi_I (\epsilon \gamma^I S) \sqrt{\Pi+}}{\sqrt{(\Pi+)^{3/2}}} \right\}.
$$

This is recognized as the single-fermion term in $V_F(\tilde{u})$. The remaining single-fermion terms in the commutator (C.16) can be written in the form

$$
\zeta^+ \int [dw] k_I (\epsilon \gamma^I S) e^{ik \cdot X} \left\{ (-2^{1/4}) \frac{\Pi^-}{\sqrt{(\Pi+)^{3/2}}} + (-2^{-3/4}) \frac{\Pi_J \Pi^J}{(\Pi+)^{3/2}} + (-2^{-3/4}) \frac{1}{\sqrt{(\Pi+)^3}} \partial^2 \left( \frac{1}{\Pi+} \right)
+ (2^{-7/4}) \frac{1}{(\Pi+)^{3/2}} \partial^2 (e^{ik \cdot X}) e^{-ik \cdot X} - (2^{-3/4}) k^- \partial \left( \frac{1}{\sqrt{(\Pi+)^{3/2}}} \right) + (2^{-3/4}) \frac{k^- \Pi_I}{\Pi+} \partial \left( \frac{1}{\Pi+} \right)
+ (7 \cdot 2^{-7/4}) \frac{1}{\Pi+} \partial^2 \left( \frac{1}{\Pi+} \right) + (2^{1/4}) \frac{k_- k_I \Pi^J}{\sqrt{(\Pi+)^{3/2}}} + (2^{-1/4}) \frac{k^2}{\sqrt{(\Pi+)^{3/2}}} \partial (e^{ik \cdot X}) e^{-ik \cdot X}
+ (2^{1/4}) \frac{(k_I \Pi^J) (k_I \Pi^L)}{(\Pi+)^{3/2}} + (2^{5/4}) \frac{1}{\Pi+} \partial \left( \frac{k_I \Pi^J e^{ik \cdot X}}{\Pi+} \right) e^{-ik \cdot X} \right\},
$$

where the on-shell condition $\zeta^+ k_j = -\zeta^+ k^-$ has been used for the fifth term. After some further manipulations, this can be brought to the following form:

$$
\zeta^+ \int [dw] k_I (\epsilon \gamma^I S) e^{ik \cdot X} \left\{ (-2^{1/4}) \frac{\Pi^-}{\sqrt{(\Pi+)^{3/2}}} + (-2^{-3/4}) \frac{\Pi_J \Pi^J}{(\Pi+)^{3/2}}
+ (-37 \cdot 2^{-15/4}) \frac{(\partial \Pi^J)^2}{(\Pi+)^{7/2}} + (11 \cdot 2^{-11/4}) \frac{\partial^2 \Pi^+}{(\Pi+)^{5/2}} + (2^{-3/4}) k^- \partial \Pi^+ + (\Pi+)^{3/2}
+ (-7 \cdot 2^{-7/4}) \frac{k_I \partial \Pi^J}{(\Pi+)^{3/2}} + (9 \cdot 2^{-7/4}) \frac{k_I \Pi^J \partial \Pi^+}{(\Pi+)^{3/2}} + (-3 \cdot 2^{-7/4}) \frac{k^2 \Pi^+ + k_I \Pi^J}{(\Pi+)^{3/2}} \right\}.
$$

Now, again after some partial integrations, we find that this expression matches precisely with the remaining single-fermion terms in the BRST-exact contribution (C.21), including all the non-trivial coefficients.
This completes the demonstration that our vertex operators satisfy the desired $\epsilon$-SUSY relation

$$[\epsilon^{\alpha}\hat{Q}_\alpha, V_B(\zeta)] = V_F(\tilde{u}) + \{Q, \Psi_B(\epsilon, \zeta)\}. \quad (C.32)$$

### Appendix D

— Similarity transformation to the light-cone gauge and construction of DDF operators —

**D.1. Some details of the degree-wise analysis**

First we give a proof of the relation (7.46), namely $Q_0e^{2n+1} = 0$. For this purpose, let us list some useful formulas involving the quantity $[(\alpha^+)^n]_m$ introduced in (7.14), which can be proved by appropriate symmetrization procedure:

- \[(A_1)\quad \alpha_i^+[(\alpha^+)^n]_m = nl[(\alpha^+)^{n-1}]_{l+m}, \quad (D.1)\]
- \[(A_2)\quad \sum'\alpha^+_{-m}[(\alpha^+)^n]_m = [(\alpha^+)^{n+1}]_0, \quad (D.2)\]
- \[(A_3)\quad \sum_{m\neq0} m\alpha^+_{-m}[(\alpha^+)^n]_m = 0, \quad (D.3)\]
- \[(A_4)\quad \sum_{m\neq0} m\alpha^+_{-m}[(\alpha^+)^n]_{m+k} = -\frac{k}{n+1}[(\alpha^+)^{n+1}]_k. \quad (D.4)\]

Now the part of $Q_0$ which acts on $e^{2n+1}$ is $c_0 \sum'_k (\alpha^+_{-k}\alpha^-_k + kc_{-k}b_k)$ and the main part of $e^{2n+1}$ is $\sum' m(m+1)c_{-m}[(\alpha^+)^n]_m$. As for the action of $\sum'_k \alpha^+_{-k}\alpha^-_k$, using $A_1$ we obtain

$$\left(\sum'_k \alpha^+_{-k}\alpha^-_k\right) \sum' m(m+1)c_{-m}[(\alpha^+)^n]_m$$
$$= \sum' m(m+1)c_{-m} \sum'_k nk\alpha^+_{-k}[(\alpha^+)^{n-1}]_{k+m}. \quad (D.5)$$

By applying $A_4$ (with $m$ and $k$ interchanged) this can be further simplified to

$$-\sum' m(m+1)c_{-m}m[(\alpha^+)^n]_m. \quad (D.6)$$

On the other hand, the action of $kc_{-k}b_k$ part gives

$$\left(\sum_k kc_{-k}b_k\right) \sum' m(m+1)c_{-m}[(\alpha^+)^n]_m$$
$$= \sum' m(m+1)c_{-m}m[(\alpha^+)^n]_m. \quad (D.7)$$

As this cancels (D.6) we get $Q_0e^{2n+1} = 0$.

Next, we give a proof of the formula (7.54) by mathematical induction. Suppose the formula holds for $n$ commutators and apply $R_2$ from right on both sides to form $n + 1$
commutators. This gives

\[
\frac{1}{n!} \delta (\text{ad} \, R_2)^{n+1} = e_{2n-1} R_2 - (\delta r_{2n}) R_2. \tag{D.8}
\]

Then by using various relations already established and the Jacobi identities, the term \(e_{2n-1} R_2\) on the right hand side can be transformed successively as

\[
e_{2n-1} R_2 = -R_2 e_{2n-1} = -(\dot{K} d_1) e_{2n-1} = -\dot{K}(d_1 e_{2n-1}) + (\dot{K} e_{2n-1}) d_1
\]

\[
= \dot{K}(\delta e_{2n+1}) + nr_2 d_1 = \dot{N} e_{2n+1} - \delta(\dot{K} e_{2n+1}) + nr_2 \delta_{2n+2}
\]

\[
= (n+1)e_{2n+1} - \delta r_{2n+2}. \tag{D.9}
\]

As for the second term on the right hand side, we can rewrite it as

\[
(\delta r_{2n}) R_2 = \delta(r_{2n} R_2) - d_1 r_{2n} = (n+1)\delta r_{2n+2} + \delta r_{2n+2} = n\delta r_{2n+2}. \tag{D.10}
\]

Adding up we get

\[
(e_{2n-1} - \delta r_{2n}) R_2 = (n+1)(e_{2n+1} - \delta r_{2n+2}). \tag{D.11}
\]

Dividing by the factor \(n+1\), it becomes the right hand side of the formula (7.54) for \(n+1\) case and hence the mathematical induction is completed.

D.2. Derivation of the formula (7.68) for \(\tilde{R}\)

Here we describe some details of the computation of \(e^{\tilde{R}} e^{R_2}\), which leads to the formula (7.68) for \(\tilde{R}\).

We need to make use of the general form of the BCH formula, which reads

\[
e^{\lambda} e^{\mu} = e^{E}, \tag{D.12}
\]

\[
E = \lambda + \int_0^1 dt \psi(e^{\lambda} e^{t\mu}) \mu, \tag{D.13}
\]

\[
\psi(z) = \frac{z \ln z}{z - 1}, \tag{D.14}
\]

where \(\lambda\) and \(\mu\) are arbitrary operators. Here and hereafter, products of \(\lambda\)'s and \(\mu\)'s are to be understood as successive adjoint actions. For example, \(\lambda \mu\) means \(\text{ad}_\lambda \mu = [\lambda, \mu]\), and \(e^{\lambda} \mu\) means \(e^{\text{ad}_\lambda \mu} = \mu + [\lambda, \mu] + \frac{1}{2}[\lambda, [\lambda, \mu]] + \cdots\), and so on. We shall apply this BCH formula with \(\lambda = \tilde{R}, \mu = R_2\). One simplifying feature for the present case is that \(\lambda\) commutes with the commutator \(\lambda \mu\), and therefore we have \(\lambda^n \mu = 0\) for \(n \geq 2\).

Let us introduce the quantity \(x \equiv e^{\lambda} e^{\mu} - 1\). Then, the \(\psi\) function in (D.13) becomes \(\psi(1 + x)\), which can be readily expanded in powers of \(x\). It reads

\[
\psi(1 + x) = \frac{(1 + x) \ln(1 + x)}{x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n. \tag{D.15}
\]
We need to act this expression on $\mu$. Because $e^{\mu} \mu = \mu$ and $\lambda \lambda \mu = 0$, we get $x \mu = e^{\lambda} \mu - \mu = (1 + \lambda) \mu - \mu = \lambda \mu$, etc. Then, it is easy to see that $x^n \mu$ can be written as

$$x^n \mu = y^{n-1}(\lambda \mu),$$

(D.16)

where $y \equiv e^{\mu} - 1$. Combining, the exponent of the BCH formula is simplified as

$$E = \lambda + \mu + \int_0^1 dt \sum_{n=1}^\infty \frac{(-1)^{n+1} y^{n-1}}{n(n+1)}(\lambda \mu)$$

$$= \lambda + \mu + \int_0^1 dt \frac{1}{y}(\psi(1+y) - 1)(\lambda \mu).$$

(D.17)

Furthermore, the integral over $t$ can be done explicitly. By changing the variable from $t$ to $t \mu$ and further to $y$ itself, we can compute it as

$$I \equiv \int_0^1 dt \frac{1}{y}(\psi(1+y) - 1)$$

$$= \frac{1}{\mu} \int_0^{e^{\mu}-1} dy \left( \frac{\ln(1+y)}{y^2} - \frac{1}{y(1+y)} \right) = \frac{1}{\mu} \left( e^{\mu} - 1 - \mu \right).$$

(D.18)

This can be further rewritten in terms of $\coth(\mu/2)$ and is expanded in powers of $\mu$ in the following way:

$$I = \frac{1}{2} + \frac{1}{\mu} - \frac{1}{2} \coth \frac{\mu}{2} = \frac{1}{2} + \sum_{m=1}^\infty (-1)^m \frac{B_m \mu^{2m-1}}{(2m)!}.$$  

(D.19)

Here $B_n$ are the Bernoulli numbers and $\mu$ really means $\text{ad}_\mu$. Now we apply this to $(\lambda \mu)$. Being careful about the sign, we have $\mu^{2m-1}(\lambda \mu) = -\lambda (\text{ad}_\mu)^{2m}$. Substituting $\lambda = (-1)^n r_{2n}, \mu = R_2$ and using the basic formula $r_{2n} R_2 = (n+1) r_{2n+2}$ repeatedly, we easily get

$$-\frac{1}{(2m)!} \lambda (\text{ad}_\mu)^{2m} = -(-1)^n \left( \frac{n+2m}{n} \right) r_{2(n+2m)}. $$

(D.20)

In this way, we finally obtain the full exponent $\mathfrak{R} (= E)$ as

$$\mathfrak{R} = R_2 + R_3 + \hat{R},$$

(D.21)

$$\hat{R} = \sum_{n \geq 2} (-1)^n r_{2n} + \frac{1}{2} \sum_{n \geq 2} (n+1) r_{2(n+1)}$$

$$+ \sum_{n \geq 2} \sum_{m \geq 1} (-1)^{m-1} B_m (-1)^n \left( \frac{n+2m}{n} \right) r_{2(n+2m)}. $$

(D.22)
We can rewrite $\hat{R}$ into a more convenient form by grouping terms into $r_{2\cdot2n}$ type and $r_{2\cdot(2n+1)}$ type. We will call them “even” and “odd” types. This separation gives

$$\hat{R} = \hat{R}_{\text{even}} + \hat{R}_{\text{odd}}, \quad (D.23)$$

$$\hat{R}_{\text{even}} = r_4 + \sum_{n \geq 2} \left( -(n - 1) + \sum_{m=1}^{n-1} (-1)^{m-1} B_m \left( \frac{2n}{2m} \right) \right) r_{2\cdot2n}, \quad (D.24)$$

$$\hat{R}_{\text{odd}} = \frac{1}{2} r_6 + \sum_{n \geq 2} \left( n - \frac{1}{2} - \sum_{m=1}^{n-1} (-1)^{m-1} B_m \left( \frac{2n+1}{2m} \right) \right) r_{2\cdot(2n+1)}. \quad (D.25)$$

By using appropriate identities for the Bernoulli numbers $B_m$, we may bring $\hat{R}_{\text{even}}$ and $\hat{R}_{\text{odd}}$ into much simpler forms. The relevant identities are

$$(i) \quad \sum_{m=1}^{n-1} (-1)^{m-1} B_m \left( \frac{2n}{2m} \right) = n - 1, \quad (D.26)$$

$$(ii) \quad \sum_{m=1}^{n} (-1)^{m-1} B_m \left( \frac{2n+1}{2m} \right) = n - \frac{1}{2}. \quad (D.27)$$

The first identity can be applied directly to $\hat{R}_{\text{even}}$ and yields a remarkable result $\hat{R}_{\text{even}} = r_4$. As for $\hat{R}_{\text{odd}}$, the use of the second identity reveals that all except $m = n$ term cancel in the sum. It turns out that the first term $r_6/2$ can be identified with $n = 1$ term of the remaining expression and hence we can write the result succinctly as

$$\hat{R}_{\text{odd}} = \sum_{n \geq 1} (-1)^{n-1} (2n + 1) B_n r_{2\cdot(2n+1)}. \quad (D.28)$$

Summarizing, $\hat{R}$ is obtained as

$$\hat{R} = r_4 + \sum_{n \geq 1} (-1)^{n-1} (2n + 1) B_n r_{2\cdot(2n+1)}, \quad (D.29)$$

which was quoted in (7-68).

D.3. Proof of the theorem on finite conformal transformation

In this appendix, we give a proof of the theorem on finite conformal transformation stated in (7-97).

To prove the theorem, it is convenient to rescale the field $\chi$ in $T_\chi$ by a parameter $\lambda$ and introduce the following quantities:

$$f(\lambda, \tau) \equiv e^{\lambda T_\chi} \phi(\tau)e^{-\lambda T_\chi}, \quad g(\lambda, \tau) \equiv e^{\lambda T_\chi} \chi(\tau)e^{-\lambda T_\chi}. \quad (D.30)$$
Then, the theorem we want to show is expressed as \( f(1, \tau) = \phi'(\tau) \). Our strategy is to make use of a set of equations for \( f(\lambda, \tau) \) and \( g(\lambda, \tau) \), obtained by differentiating them with respect to \( \lambda \). Using the commutation relations

\[
[T_x, \phi] = \chi \partial_\tau \phi + h \partial_\tau \chi \phi, \quad [T_x, \chi] = \chi \partial_\tau \chi , \tag{D.31}
\]

which follow easily from (7.90) and (7.91), we readily obtain

\[
(i) \quad \partial_\lambda g = g \partial_\tau g, \quad g(0, \tau) = \chi(\tau) , \tag{D.32}
(ii) \quad \partial_\lambda f = g \partial_\tau f + h f \partial_\tau g, \quad f(0, \tau) = \phi(\tau) . \tag{D.33}
\]

As the first step, we will prove the relation \( g(1, \tau) = \chi'(\tau) \), namely the special case of the theorem for the field \( \chi \) itself. Since the parameter field \( \chi(\tau) \) of the conformal transformation is rescaled by \( \lambda \), the corresponding transformation of the coordinate is given by \( \tau' = \tau - \lambda \chi(\tau) \). But since \( \chi(\tau) \), being a primary of dimension 0, satisfies the property \( \chi'(\tau') = \chi(\tau) \), we can rewrite this relation and express \( \tau \) in terms of \( \tau' \) as \( \tau = \tau' + \lambda \chi'(\tau') \). To emphasize that the form of the function \( \chi'(\tau) \) actually depends on \( \lambda \), we will write this as

\[
\tau = \tau' + \lambda \chi'(\lambda, \tau') . \tag{D.34}
\]

Let us now substitute this expression of \( \tau \) into the right-hand-side of the equation \( \chi'(\tau') = \chi(\tau) \). Then we get an equation where all the arguments are \( \tau' \). To make it look simpler, we rename \( \tau' \) as \( \tau \). Then, it reads

\[
\chi'(\lambda, \tau) = \chi(\tau + \lambda \chi'(\lambda, \tau)) . \tag{D.35}
\]

Although it looks like a complicated nested functional equation characterizing \( \chi'(\lambda, \tau) \), we can convert it into a simpler differential equation. By differentiating with respect to \( \tau \) and \( \lambda \), we obtain

\[
\partial_\tau \chi' = \partial_\tau \chi(\tau + \lambda \chi'(\lambda, \tau))(1 + \lambda \partial_\tau \chi') , \tag{D.36}
\partial_\lambda \chi' = \partial_\tau \chi(\tau + \lambda \chi'(\lambda, \tau))(\chi' + \lambda \partial_\lambda \chi') . \tag{D.37}
\]

Taking their ratio, we get

\[
(\partial_\lambda \chi')(1 + \lambda \partial_\tau \chi') = (\partial_\tau \chi')(\chi' + \lambda \partial_\lambda \chi') . \tag{D.38}
\]

Finally, expanding this relation we easily obtain the equation

\[
\partial_\lambda \chi' = \chi' \partial_\tau \chi' . \tag{D.39}
\]
But this is nothing but the differential equation \((i)\) for \(g(\lambda, \tau)\). Furthermore, at \(\lambda = 0\), we obviously have \(\chi'(\tau) = \chi(\tau)\), which is the same initial condition satisfied by \(g(\lambda, \tau)\). Hence, we can identify \(g(\lambda, \tau) = \chi'(\lambda, \tau)\) and in particular \(g(1, \tau) = \chi'(1, \tau) = \chi'(\tau)\).

Next we consider the equation \((ii)\). The counterpart of the equation (D.35) for \(\phi\) is

\[
\phi'(\lambda, \tau) = \phi(\tau + \lambda \chi'(\lambda, \tau))(1 + \lambda \partial_\tau \chi')^h. \tag{D.40}
\]

Just as before, we compute \(\partial_\lambda \phi'\) and \(\partial_\tau \phi'\), making use of the fact that \(\chi' = g\). We then get

\[
\begin{align*}
\partial_\lambda \phi' &= (\partial_\tau \phi)(g + \lambda \partial_\lambda g)(1 + \lambda \partial_\tau g)^h + h\phi(1 + \lambda \partial_\tau g)^{-1} \partial_\tau(g + \lambda \partial_\lambda g), \tag{D.41} \\
\partial_\tau \phi' &= (\partial_\tau \phi)(1 + \lambda \partial_\tau g)^{h+1} + h\phi(1 + \lambda \partial_\tau g)^{-1} \lambda \partial_\tau^2 g. \tag{D.42}
\end{align*}
\]

Now we use the property of \(g\) to rewrite \(g + \lambda \partial_\lambda g = g + \lambda g \partial_\tau g = g(1 + \lambda \partial_\tau g)\). Also, we use (D.40) to identify \(\phi(1 + \lambda \partial_\tau g)^{-1} = \phi'/ (1 + \lambda \partial_\tau g)\). Then, the above equations simplify to

\[
\begin{align*}
\partial_\lambda \phi' &= (\partial_\tau \phi)(1 + \lambda \partial_\tau g)^{h+1} g + \frac{h \phi'}{1 + \lambda \partial_\tau g} \partial_\tau(g(1 + \lambda \partial_\tau g)), \tag{D.43} \\
\partial_\tau \phi' &= (\partial_\tau \phi)(1 + \lambda \partial_\tau g)^{h+1} + \frac{h \phi'}{1 + \lambda \partial_\tau g} \lambda \partial_\tau^2 g. \tag{D.44}
\end{align*}
\]

From these equations, it is not difficult to verify that the following equations hold

\[
\partial_\lambda \phi' = h \partial_\tau g \phi' + g \partial_\tau \phi'. \tag{D.45}
\]

But this is identical to the equation \((ii)\) satisfied by \(f\), with the proper initial condition. Thus we can identify \(f(1, \tau) = \phi'(\tau)\), which completes the proof of the theorem.

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