Critical Theories of the Dissipative Hofstadter Model

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Abstract

It has recently been shown that the dissipative Hofstadter model (dissipative quantum mechanics of an electron subject to uniform magnetic field and periodic potential in two dimensions) exhibits critical behavior on a network of lines in the dissipation/magnetic field plane. Apart from their obvious condensed matter interest, the corresponding critical theories represent non-trivial solutions of open string field theory, and a detailed account of their properties would be interesting from several points of view. A subject of particular interest is the dependence of physical quantities on the magnetic field since it, much like $\theta_{\text{QCD}}$, serves only to give relative phases to different sectors of the partition sum. In this paper we report the results of an initial investigation of the free energy, $N$-point functions and boundary state of this type of critical theory. Although our primary goal is the study of the magnetic field dependence of these quantities, we will present some new results which bear on the zero magnetic field case as well.

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1. Introduction

The Hofstadter problem concerns the quantum mechanics of an electron moving in two dimensions subject to a magnetic field and a periodic potential. The energy bands of this model show a remarkable fractal structure as a function of the number of flux quanta per lattice unit cell. In a previous paper, the Caldeira-Leggett model of dissipative quantum mechanics (DQM) was used to study how this discontinuous behavior is smoothed out by the unavoidable elements of randomness in real physical systems. A complicated phase diagram was discovered which showed precisely how this “smoothing out” works: Above a certain critical dissipation, the particle is localized, but, as the dissipation is reduced, there is an increasingly dense system of phase-transition lines which have a fractal structure in the zero-dissipation limit. The topic of this paper is the study of the properties of the one-dimensional critical theories corresponding to the phase transitions themselves. In addition to their relevance to the Hofstadter problem, these theories represent new solutions of open string theory in a non-trivial background of tachyons and gauge fields. The string theory connection strongly suggests that the critical theory should have an enhanced $SL(2,R)$ invariance in addition to the usual scale invariance and we will verify that this is so.

We have found a simple regulator which reduces the calculation of most quantities of interest, to any order of perturbation in the potential, to the purely algebraic exercise of extracting the residues of poles in a rational function associated with each perturbation theory diagram. Using this regulator, for some special points in the phase diagram we explicitly demonstrate the absence of logarithmic divergences, and show that the only renormalization needed is a rescaling of the potential strength and the subtraction of an infinite constant from the free energy. We give evidence for the criticality of the circular arcs in the phase diagram which join these special points to one another, although we cannot give a proof to all orders in this case. In addition we have done some explicit calculations of the magnetic field dependence of free energies and $N$-point functions to various orders in the potential strength. We find that many of the connected higher $N$-point functions are zero, up to contact terms. Although this suggests that the critical theories are “almost” free and therefore ought to be soluble, we have not been able to exploit this hint to obtain exact solutions and must for the moment content ourselves with the rather clumsy perturbative approach presented here.

In the interests of making this paper relatively self-contained, we devote the first two sections to a brief review of background material that has, by and large, appeared
elsewhere. In Section 2 we review dissipative quantum mechanics and its relationship with open string theory. In Section 3 we specialize to the Hofstadter model. We show that it is equivalent to a Coulomb gas, demonstrate that it has phase transitions and show that the critical theories have $SL(2, R)$ invariance. We then move on to the considerations which are new to this paper. In Section 4 we give a fairly complete discussion of the unique critical theory at zero magnetic field, and show that it is a free theory with certain nontrivial contact interactions. In Section 5 we calculate the free energies and $N$-point functions of critical theories at non-zero magnetic field in the zero-charge sector of the Coulomb gas. An interesting feature of our results is that at the previously-mentioned special points in the phase diagram, most of the $N$-point functions reduce to contact terms. There are, however, some few that do not and we give their $SL(2, R)$-invariant form. Section 6 is devoted to the non-zero charge sectors of the Coulomb gas. They are crucial to the construction of the open string boundary state, and we find that no new renormalizations are needed, beyond those needed to deal with the zero-charge sector. Our conclusions are in Section 7. In one appendix we present details of the proof that all the higher $N$-point functions reduce to contact terms in the absence of a magnetic field. In another, we show that the $N$-point functions satisfy the very non-trivial string theory reparametrization invariance Ward identities.

2. Background: Dissipative Hofstadter Model and Open String Theory

We begin with a brief outline of dissipative quantum mechanics and its connection with open string theory. For details the reader is referred to [5] and [6]. A macroscopic object is typically subject to dissipative forces caused by its interaction with its environment. Classically, these forces can be described by including the phenomenological term $-\eta \dot{X}^i$ in the equation of motion for the particle, where $\eta$ is the coefficient of friction and $X^i$ the particle’s coordinate. In order to describe dissipation quantum mechanically, one can model the environment by a bath of an infinite number of harmonic oscillators coupled linearly to the $\vec{X}$. The coupling constants, $C_\alpha$, and the distribution of the frequencies, $\omega_\alpha$, of the oscillators can be chosen so that when the oscillator coordinates are eliminated via their equations of motion, the resulting equations of motion for the $X^i$ contain the required friction term. The functional condition on the parameters is

$$J(\omega) = \pi \sum_\alpha \frac{C_\alpha^2}{2\omega_\alpha} \delta(\omega-\omega_\alpha) = \eta \omega,$$  \hspace{1cm} (2.1)
which represents Ohmic dissipation in the system. This is the Caldeira-Leggett model \[5\].

Since the dependence on the oscillator coordinates in the lagrangian is quadratic, it is also possible to integrate them out from the quantum mechanical path integral. This results in a quantum effective action for the \(X^i\) variables, which includes a non-local piece containing the effect of dissipation. For a particle coupled to scalar (\(V\)) and vector (\(\vec{A}\)) potentials the full action reads

\[
S[\vec{X}] = \int dt \left\{ \frac{1}{2} M \ddot{\vec{X}}^2 + V(\vec{X}) + i A_i(\vec{X}) \dot{X}^i + \frac{\eta}{4\pi} \int_{-\infty}^{\infty} dt' \frac{\left(\dot{X}(t) - \dot{X}(t')\right)^2}{(t-t')^2} \right\}.
\]

(2.2)

Remarkably, the only dependence on the oscillator parameters that remains in the action is through the \(\eta\)-term. Because this term is non-local, the path integral is effectively that of a one-dimensional statistical system with long-range interactions. Such systems, unlike one-dimensional local systems, have phase transitions (the classic example being the Ising chain with \(1/r^2\) interactions \[\text{1}\]). In the DQM context, the phase transitions are between different regimes of long-time behavior of Green’s functions (typically between localized and delocalized behavior).

At these critical points, the 1-D field theories describing dissipative quantum mechanics correspond to solutions of open string theory \[\text{2}\]. In the presence of open string background fields, interactions between a string and the background take place at the boundary of the string and their effects can be represented by a boundary state \(|B\rangle\). In \[\text{3}\] it is shown that this boundary state is given by

\[
|B\rangle = \exp \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m} \right\} \int [D\vec{X}(s)]' \exp(-S_R - S_{KE} - S_I - S_{LS}) |0\rangle,
\]

(2.3)

where

\[
S_R = \int_0^T ds \frac{1}{2} M \ddot{\vec{X}}^2(s);
\]

(2.4)

\[
S_{KE} = \frac{1}{8\pi^2\alpha'} \int_0^T ds \int_{-\infty}^{\infty} ds' \frac{(\vec{X}(s) - \vec{X}(s'))^2}{(s-s')^2};
\]

(2.5)

\[
S_I = i \int_0^T ds A_\mu(\vec{X}) \dot{X}^\mu + \int_0^T ds \mathcal{T}(\vec{X});
\]

(2.6)

\[\text{1}\] The Dyson chain with \(1/r^n\) interactions for \(n \leq 2\) is known to have phase transitions. See references \[\text{1,2}\] and references therein.
and
\[ S_{LS} = \sqrt{\frac{2}{\alpha'}} \int ds \; \alpha(s) \cdot \vec{X}(s) \quad \text{with} \quad \alpha^\mu(s) = \sum_{m=1}^{\infty} i(\tilde{\alpha}_{-m}^\mu e^{-ims} + \alpha_{-m}^\mu e^{ims}). \quad (2.7) \]

In these expressions, \( T \) is the parameter length of the boundary (when appropriate, we must regard \( \vec{X}(s) \) as periodic in \( s \) with period \( T \)), \( \alpha' \) is the string constant and \( A_\mu(\vec{X}) \) and \( T(\vec{X}) \) are the gauge fields and tachyon fields, respectively. The creation operators of the left- and right-moving modes of the closed string, \( \tilde{\alpha}_{-m} \) and \( \alpha_{-m} \), act on the closed string vacuum \( |0\rangle \) to create some state in the closed string Hilbert space. The notation \( [D\vec{X}(s)]' \) means that the zero-mode, \( \vec{X}_0 \) is not integrated out. The commuting objects \( \tilde{\alpha}_{-m}, \alpha_{-m} \) and \( \vec{X}_0 \) together make up a set of coordinates which specify where the boundary lies in the target space and the boundary state is just a functional of these coordinates.

As an example of the utility of this construct, we note that the projection of \( |B\rangle \) onto the graviton state is essentially the energy-momentum tensor of the open string object under study. This gives us a string-theoretic way to define such important notions as gravitational and inertial mass.

This path integral is the generating functional for a renormalizable “one-dimensional” field theory described by the underlying action \( S_{KE} + S_I \): \( (S_{LS} \) is the linear source term in the generating function and the kinetic term \( S_R \) functions as a regulator for divergences). Leaving aside the linear source term, it is clear that the DQM action and the action defining the string theory boundary state are the same if we relate the coefficient of friction to the string tension by \( \eta = 1/(2\pi\alpha') \). The full string theory prescription for \( |B\rangle \) requires that we take the cut-off, \( M \), to zero. In order for this limit to be meaningful, the field theory must lie at a renormalization group fixed point, which is to say that the gauge and tachyon fields must satisfy some “vanishing beta function” equations of motion for open string background fields. This means that the associated DQM must lie at a phase transition.

The upshot is that, modulo technical details, any solution of the open string equations of motion is equivalent to a particular critical DQM: The background gauge and tachyon fields of the string theory become the vector and scalar potentials to which the DQM electron is subject.

In string theory we require worldsheet reparametrization invariance, and this includes reparametrizations of the boundary. The condition that the boundary state be reparametrization invariant can be written
\[ (L_n - \tilde{L}_{-n})|B\rangle = 0. \quad (2.8) \]
where the $L$ operators are the closed string Virasoro generators (they act in a known way on the coordinates $\alpha_{-m}$, $\bar{\alpha}_{-m}$ and $\bar{X}_0$ on which the boundary path integral depends). This symmetry generates a set of Ward identities which turns out to be very useful both in string theory and in DQM (details can be found in [14]). In fact, the one-dimensional field theory has what amounts to broken reparametrization invariance, because of the non-local dissipation term. There is nonetheless a remaining manifest $SL(2, R)$ symmetry ($SU(1, 1)$ on the circle) which tightly constrains the allowed form of the correlation functions. These extra symmetries would not have been expected at the one-dimensional critical points without the string theory connection and we will look for them in our explicit calculations.

3. General Properties of the Dissipative Hofstadter Model

3.1. Equivalence to a Coulomb Gas

To specialize to the dissipative Hofstadter model, we consider a particle moving in a periodic potential in two dimensions (with $\vec{X} = (X, Y)$) and subject to a uniform magnetic field $B$. The Euclidean action for this problem is the sum of a quadratic piece and a more complicated potential term:

$$ S = S_q + S_V $$  \hspace{1cm} (3.1) 

where

$$ \frac{1}{\hbar} S_q = \frac{1}{\hbar} \int_{-T/2}^{T/2} dt \left\{ \frac{M}{2} \dot{X}^2 + \frac{i e B}{2c} (\dot{X}Y - \dot{Y}X) + \frac{\eta}{4\pi} \int_{-\infty}^{\infty} dt' \frac{(\bar{X}(t) - \bar{X}(t'))^2}{(t-t')^2} \right\} $$  \hspace{1cm} (3.2) 

and

$$ S_V = \int_{-T/2}^{T/2} dt \ V(X, Y) \ . $$  \hspace{1cm} (3.3) 

For the periodic potential we take

$$ V(X, Y) = -V_0 \cos\left(\frac{2\pi X(t)}{a}\right) - V_0 \cos\left(\frac{2\pi Y(t)}{a}\right) \ . $$  \hspace{1cm} (3.4) 

Nothing dramatically new happens if we take the strength and period of the potential to be different in the X- and Y- directions. It is convenient to define the dimensionless parameters

$$ 2\pi \alpha = \frac{\eta a^2}{\hbar}, \quad 2\pi \beta = \frac{eB}{\hbar c a^2} \ . $$  \hspace{1cm} (3.5)
to rescale $X$ and $Y$ by $a/2\pi$, and to rescale $V_0$ by $\hbar$. Until we come to consider the infrared regulation of the theory in Section 3.4, we take $T$ to be infinite, which means that we are at zero temperature, and that the particle lives on a line. Then the action $S_q$ can be written as

$$\frac{1}{\hbar} S_q = \frac{1}{2} \int \frac{d\omega}{2\pi} \left\{ \frac{\alpha}{2\pi} |\omega| + \frac{M a^2}{\hbar^2} \omega^2 \right\} \delta_{\mu\nu} + \frac{\beta}{2\pi} \epsilon_{\mu\nu\omega} \tilde{X}_\mu(\omega) \tilde{X}_\nu(\omega).$$

(3.6)

Because the ordinary kinetic term, $\frac{1}{2} M \dot{\vec{X}}^2$, is a dimension-two operator, it is irrelevant and acts only as a regulator as far as the large-time behavior is concerned. Since we are studying critical behavior, it will be legitimate to set $M = 0$ and use some other, more convenient, regulator where needed. The Fourier-transformed propagator defined by (3.6) (with $M = 0$) is

$$G^{\mu\nu}(\omega) = \frac{2\pi}{\alpha^2 + \beta^2} \frac{1}{|\omega|} \delta^{\mu\nu} - \frac{2\pi}{\alpha^2 + \beta^2} \frac{1}{\omega} \epsilon^{\mu\nu}. $$

(3.7)

In the time domain, this becomes

$$G^{\mu\nu}(t_i - t_j) = -\frac{\alpha}{\alpha^2 + \beta^2} \ln(t_i - t_j)^2 \delta^{\mu\nu} - \frac{i}{2} \frac{2\pi}{\alpha^2 + \beta^2} \text{sign}(t_i - t_j) \epsilon^{\mu\nu}. $$

(3.8)

For future reference, we note that in one dimension and for vanishing magnetic field this propagator reduces to

$$G(t_1 - t_2) = -\frac{1}{\alpha} \ln(t_1 - t_2)^2. $$

(3.9)

Except for the cosine potential, the action (3.1) is quadratic, so we will treat the potential as a perturbation. We proceed by expanding

$$\exp\left(V_0 \int \cos X(t) dt\right) = \sum_{n=0}^{\infty} \int d\tau_1 \ldots d\tau_n \left(\frac{V_0}{2}\right)^n \frac{1}{n!} \sum_{q_j=\pm 1} \prod_{j=1}^{n} e^{i q_j \cdot \vec{X}(\tau_j)}. $$

(3.10)

Then the partition function is given by

$$Z = \exp\left(-\frac{1}{\hbar} S\right) = \int D\vec{X}(t) \sum_{n=0}^{\infty} \int d\tau_1 \ldots d\tau_n \left(\frac{V_0}{2}\right)^n \frac{1}{n!} \sum_{q_j=\pm 1, \pm 0} \prod_{j=1}^{n} e^{i q_j \cdot \vec{X}(\tau_j)} e^{-S_q/\hbar} $$

(3.11)

$$= \sum_{n=0}^{\infty} \int d\tau_1 \ldots d\tau_n \left(\frac{V_0}{2}\right)^n \frac{1}{n!} \sum_{q_j} \left\langle \prod_{j=1}^{n} e^{i q_j \cdot \vec{X}(\tau_j)} \right\rangle_0 ,$$

where the functional integral is over periodic paths, and the correlation functions of the operators $e^{i q \cdot \vec{X}(t)}$ are to be computed with the propagator (3.8). A subtlety to be borne in
mind is that for the dissipative quantum mechanics system, we must integrate over the zero mode in equation (3.11), which imposes the charge conservation requirement \[ \sum \vec{q}_i = 0. \]

For the string theory boundary state path-integral, we omit the integration over the zero mode, and the \( q_i \)'s are unconstrained.

If we restrict to one dimension and drop the magnetic field, the \( O(V_0^n) \) term in (3.11) is a sum over \( q_j = \pm 1 \) of

\[
Z_n = \frac{1}{n!} \left( \frac{V_0}{2} \right)^n \int \prod_{i=1}^{n} dt_i \left\langle e^{i q_1 X(t_1)} e^{i q_2 X(t_2)} \ldots e^{i q_n X(t_n)} \right\rangle_0
\]

Using (3.9), this evaluates to

\[
Z_n = \frac{1}{n!} \left( \frac{V_0}{2} e^{-\frac{1}{2} \langle X^2(0) \rangle} \right)^n \int \prod_{i=1}^{n} dt_i \exp \left( \frac{1}{\alpha} \sum_{i < j} q_i q_j \ln(t_i - t_j)^2 \right). \tag{3.13}
\]

The expression in the exponent is the free energy for \( n \) particles with charges \( q_j = \pm 1 \), interacting logarithmically and restricted to a line. For the dissipative quantum mechanics system, the condition that \( \sum q_j = 0 \) just requires the gas to be neutral. For the boundary state path integral, the gas can have any charge. The full partition function describes a one-dimensional Coulomb gas of particles with fugacity proportional to \( V_0 \).

Similarly, in the case of most interest to us (two dimensions and non-zero field), the \( O(V_0^n) \) term for \( Z \) is a sum over \( \vec{q}_j = (\pm 1, 0) \) and \( (0, \pm 1) \) of

\[
Z_n = \frac{1}{n!} \left( \frac{V_0}{2} e^{-\frac{1}{2} \langle X^2(0) \rangle} \right)^n \int \prod_{i=1}^{n} dt_i \exp \left( \frac{1}{\alpha} \sum_{i < j} \left\{ \frac{\alpha}{\alpha^2 + \beta^2} \vec{q}_i \cdot \vec{q}_j \ln(t_i - t_j)^2 + \frac{i}{2} \frac{2\pi \beta}{\alpha^2 + \beta^2} \epsilon_{\mu \nu} q_i^\mu q_j^\nu \operatorname{sign}(t_i - t_j) \right\} \right). \tag{3.14}
\]

This has an interpretation as a generalized “Coulomb” gas. Now there are two species of particles, one corresponding to the X-component of \( \vec{q} \) and one to the Y-component of \( \vec{q} \). Charges of the same species still interact logarithmically (through the \( \delta_{\mu \nu} \) piece of (3.8)). Charges of differing species only interact through a sign function (the \( \epsilon_{\mu \nu} \) piece of (3.8)) and the wave-function picks up a phase factor when they are interchanged. One of our main points is that this simple generalization of the Coulomb gas has a very rich phase structure.

This Coulomb gas sum has an additional interpretation. When there is no dissipation, it was shown in ref. [11] that the partition function (3.14) describes an electron in a Landau
orbit centered on a dual lattice site, \( m(a/\beta)\hat{x} + n(a/\beta)\hat{y} \). Whenever the potential acts (via an insertion of a \((V_0/2)e^{i\vec{q}_j\cdot\vec{X}(t_j)}\)), the center of the Landau orbit hops by \( \vec{q}_j \) units in the reciprocal lattice and the action picks up the Aharonov-Bohm phase due to this hop. Once we turn on the dissipation, according to equation (3.14), there are two major changes. The first is that the dual lattice becomes

\[
    m\frac{a}{\alpha^2 + \beta^2}\hat{x} + n\frac{a}{\alpha^2 + \beta^2}\hat{y} ,
\]

so that now, when the center of the particle’s orbit hops through a square in the reciprocal lattice, it picks up a phase of \( \pm2\pi\beta/(\alpha^2 + \beta^2) \). Additionally, there is now a logarithmic interaction between the particle’s hops at time \( t_i \) and time \( t_j \).

### 3.2. Phase Transitions in the Dissipative Hofstadter Model

In standard quantum mechanics, a particle in a periodic potential is delocalized by coherent quantum tunneling effects. In the presence of strong enough dissipation one expects coherence between tunneling events to be lost and the particle to become localized. The signature of this localization-delocalization phase transition can be looked for in the asymptotic behavior of the two-point function, \( \langle X(t_1)X(t_2) \rangle \): Let the long-time behavior of the two-point function be \( (t_1 - t_2)^\gamma + \text{const.} \). Localization corresponds to \( \gamma < 0 \), delocalization to \( \gamma > 0 \). When \( \gamma = 0 \), so that \( \langle X(t_1)X(t_2) \rangle \sim \ln|t_1 - t_2| \), the system is at a critical point.

Fisher and Zwerger \[12\] have given a renormalization group argument which shows that DQM with a periodic potential and zero magnetic field is critical at \( \alpha = 1 \). Their argument is valid for small values of the potential strength, \( V_0 \), since they treat the potential perturbatively, but sum over all loops. Following this calculation closely, we can extend it to include a constant magnetic field. We will present here the calculation to first order in \( V_0 \) in order to identify a candidate for a critical circle on the \( \alpha-\beta \) plane. We can show that the results given here remain true at order \( V_0^2 \), and we expect that they will hold for all orders in \( V_0 \).

In the action defined in (3.1) and the following equations, we set the mass term to zero and regulate with a high frequency cut-off, \( \Lambda \), instead. The action is then given by

\[
    \frac{1}{\hbar}S = \frac{1}{2} \int \frac{d\omega}{2\pi} \tilde{X}_\rho(\omega)S^\rho(\omega)\tilde{X}_\lambda(\omega) + \Lambda V_0(\Lambda) \int d\tau [\cos X(\tau) + \cos Y(\tau)],
\]

(3.16)
with

\[ S^{\mu\nu}(\omega) = \frac{\alpha}{2\pi} |\omega| \delta^{\mu\nu} + \frac{\beta}{2\pi} \omega \epsilon^{\mu\nu} \]  

(3.17)

and \( \Lambda V_0(\Lambda) \) replacing the bare coupling \( V_0 \). At this point, we want to perform the functional integral over all the fast modes, \( \tilde{X}(\omega) \), with \( \Lambda > \omega > \mu \), for some \( \mu \ll \Lambda \). To do so, we divide the field into fast and slow modes,

\[ \tilde{X}(\tau) = \tilde{X}_s(\tau) + \tilde{X}_f(\tau) , \]  

where

\[ \tilde{X}(\omega) \approx \tilde{X}_s(\omega) \quad \text{for} \quad |\omega| \leq \mu \quad \text{and} \quad \tilde{X}(\omega) \approx \tilde{X}_f(\omega) \quad \text{for} \quad \mu \leq |\omega| \leq \Lambda . \]  

(3.18)

We would like to calculate \( \tilde{S} \), where \( \tilde{S} \) is defined by the relation

\[ \int D\tilde{X}(\tau) e^{-\frac{1}{\hbar} \tilde{S}} = \int D\tilde{X}_s(\tau) e^{-\frac{1}{\hbar} \tilde{S}} . \]  

(3.19)

The coefficients of \( |\omega||\tilde{X}(\omega)|^2 \), \( \omega \epsilon^{\sigma\nu} \tilde{X}_s^\sigma(\omega) \tilde{X}_s^\nu(\omega) \), and \( \cos X_s(\tau) + \cos Y_s(\tau) \) in \( \tilde{S} \) determine the flows of \( \alpha \), \( \beta \) and \( V_0 \), respectively. To calculate \( \tilde{S} \), we treat the potential term perturbatively, just as we did to obtain equation (3.11). It is not too hard to show that, to first order in \( V_0 \),

\[ \tilde{S} = \frac{1}{2} \int_{-\mu}^{\mu} \frac{d\omega}{2\pi} \tilde{X}_s^\dagger(\omega) S^{\rho\lambda}(\omega) \tilde{X}_s^\lambda(\omega) + \langle S_V \rangle_f . \]  

(3.20)

where \( S_V = \Lambda V_0(\Lambda) \int d\tau [\cos X(\tau) + \cos Y(\tau)] \), and the gaussian average over the fast modes is calculated with the following two-point function:

\[ G^{\rho\lambda}(\tau) = \left \langle X_\rho^\mu(\tau) X_\lambda^\mu(0) \right \rangle = \int_{-\Lambda}^{\Lambda} d\omega \left ( S^{-1} \right )^{\rho\lambda}(\omega)e^{i\omega \tau} W(\omega/\mu) . \]  

(3.21)

\( W(x) \) must be \( \approx 0 \) for \( x << 1 \) and \( \approx 1 \) for \( x >> 1 \). To the order in \( V_0 \) we are calculating here, it is sufficient to take \( W(x) \) to be a step function. (If we wish to continue the calculation to higher orders in \( V \), then \( W(x) \) must be smooth enough to avoid the generation of spurious long-range behavior in \( G(\tau) \).) With our choice for \( W(x) \), we can calculate the diagonal part of \( G^{\rho\lambda}(0) \) with the result

\[ G^{\rho\lambda}(0) = \frac{2\alpha}{\alpha^2 + \beta^2} \ln(\Lambda/\mu) \quad \text{for} \quad \rho = \lambda . \]  

(3.22)

\( \langle S_V \rangle_f \) can be written solely as a function of \( G^{XX}(0) = G(0) \) as follows:

\[ \langle S_V \rangle_f = \Lambda V_0(\Lambda)e^{-\frac{1}{\hbar} G(0)} \int d\tau [\cos X_s(\tau) + \cos Y_s(\tau)] . \]  

(3.23)
Finally, we rescale $\tau$ by $\Lambda/\mu$ to restore $\tilde{S}$ to its original form and find that the coefficient of the potential term becomes

$$V_0(\mu) = V_0(\Lambda)(\Lambda/\mu)e^{-\frac{1}{2}G(0)}$$

$$= V_0(\Lambda)(\mu/\Lambda)^{\left(\frac{\alpha}{\alpha^2+\beta^2} - 1\right)},$$

(3.24)

while the coefficients of the friction term and magnetic field term do not change. We conclude that, as we take $\mu/\Lambda$ to 0, the potential term flows to zero if $\alpha/(\alpha^2 + \beta^2) > 1$, remains fixed when $\alpha/(\alpha^2 + \beta^2) = 1$ and grows when $\alpha/(\alpha^2 + \beta^2) < 1$. Also, to this order in $V_0$, we have just shown that $\alpha$ and $\beta$ do not flow. Therefore, we have demonstrated the existence of a critical circle in the $\alpha$-$\beta$ plane, to first order in $V_0$. We note that we obtain a critical circle for any initial value of $V_0$, as long as $V_0$ is small enough to justify the perturbative expansion. Inside this circle, $V_0$ is irrelevant, so the particle should be delocalized, and, outside the circle, $V_0$ is relevant.

We expect this behavior to continue at higher orders in $V_0$ because the only relevant terms that are generated in $\tilde{S}$ are of the form $\cos X_s(\tau) + \cos Y_s(\tau)$. In particular, the non-local kinetic term and the magnetic term are not generated, so the friction per unit cell and the magnetic flux cannot flow. Thus, we expect the circle $\alpha^2 + \beta^2$ to be critical to all orders in $V_0$.

3.3. Symmetries of the Critical Theory

On the critical circle, where $\frac{\alpha}{\alpha^2+\beta^2} = 1$, the unregulated theory has several nice properties. First, the dissipative quantum mechanics system displays $SL(2, R)$ invariance, not just scale invariance. As mentioned in Section 2, we expected this larger symmetry group at the phase transition because of the connection with open string theory. This invariance means that, under the transformation

$$t \rightarrow \tilde{t} = \frac{at + b}{ct + d}, \quad ad - bc = 1, \quad a, b, c, d, \in R,$$

(3.25)

the form of the partition function remains unchanged while $\dot{X}(t)$ transforms as a dimension one operator and $e^{ikX(t)}$ transforms as a dimension $k^2$ operator.

To show this, we first consider the $O(V_0^{2n})$ term of the partition function for a neutral gas. For simplicity, we study the zero-field case. Note, however, that the $SL(2, R)$ invariance remains when the magnetic field is non-zero, because the magnetic-field-dependent
contributions to $Z_n$ and to the correlation functions depend only on the ordering of the $t_i$, which remains unchanged under the $SL(2, R)$ transformation.

In the zero-field case ($\beta = 0$) one has a critical point at $\alpha = 1$. At this point, we can rewrite equation (3.13) for $Z_{2n}$ as

$$Z_{2n} = \frac{1}{(2n)!} \left( \frac{V_0}{2} e^{-\frac{1}{2} \langle X^2(0) \rangle} \right)^{2n} \int \frac{dt_i}{2\pi} \frac{ds_i}{2\pi} \frac{\prod_{i, j < i} (t_i - t_j)^2 (s_i - s_j)^2}{\prod_{i, j} (t_i - s_j)^2}. \quad (3.26)$$

This is the $O(V_0^{2n})$ term of the partition function with $+1$ charges at the $t_i$’s and $-1$ charges at the $s_j$’s. This expression clearly remains unchanged under the translation $t_i \to t_i + a$ and $s_j \to s_j + a$. It also remains unchanged when all the variables are rescaled by the same factor. The third transformation needed to generate $SL(2, R)$ can be taken to be $t \to -1/t$, and it can easily be seen that $Z_{2n}$ is invariant under this inversion as well. Thus, the partition function is invariant under $SL(2, R)$ transformations.

Operator $N$-point functions are only slightly more complicated. Using the properties of gaussian propagators, one can show that the connected correlation functions $\langle \hat{X}(r_1) \ldots \hat{X}(r_m) \rangle$ are given by the connected part of the partition function with the insertion of

$$(-1)^{m/2} \prod_{j=1}^{m} \left\{ \sum_i \left( \langle \hat{X}(r_j)X(t_i) \rangle - \langle \hat{X}(r_j)X(s_i) \rangle \right) \right\}$$

$$= (-1)^{m/2} \prod_{j=1}^{m} \left\{ \sum_i \left( \frac{2}{r_j - t_i} - \frac{2}{r_j - s_i} \right) \right\} \quad (3.27)$$

$$= (-4)^{m/2} \prod_{j=1}^{m} \left\{ \sum_i \frac{t_i - s_i}{(r_j - t_i)(r_j - s_i)} \right\}.$$

Because $Z_{2n}$ is invariant under $SL(2, R)$ transformations, we only need to see how the expression in curly brackets transforms under $r_j \to \tilde{r}_j = (ar_j + b)/(cr_j + d)$ with $ad - bc = 1$ (along with a simultaneous transformation of $t_i$ and $s_i$). We find

$$\frac{t_i - s_i}{(r_j - t_i)(r_j - s_i)} \to \frac{\tilde{t}_i - \tilde{s}_i}{(\tilde{r}_j - \tilde{t}_i)(\tilde{r}_j - \tilde{s}_i)} = (cr_j + d)^2 \frac{t_i - s_i}{(r_j - t_i)(r_j - s_i)}. \quad (3.28)$$

Taking the derivative of $\tilde{r}$ with respect to $r$, we have

$$\frac{d\tilde{r}}{dr} = \frac{1}{(cr + d)^2}, \quad (3.29)$$

from which it follows that

$$\langle \hat{X}(r_1) \ldots \hat{X}(r_m) \rangle = \prod_{i=1}^{m} \left( \frac{d\tilde{r}_i}{dr_i} \right) \langle \hat{X}(\tilde{r}_1) \ldots \hat{X}(\tilde{r}_m) \rangle. \quad (3.30)$$

Thus, insertions of $\hat{X}(r)$ transform as dimension-1 operators. For correlation functions like $\langle e^{ik_1 X(t_1)} \ldots e^{ik_2 X(t_m)} \rangle$ with $\sum k_j = 0$, the calculation is similar.
3.4. The Regulated Theory

So far, we have only considered the unregulated theory. However, it has both infrared and ultraviolet divergences which we must regulate. The ultraviolet divergence was originally regulated by the $\dot{M}\dot{X}^2$ term, which acts as a high frequency cutoff by multiplying the Fourier-space propagator, $\tilde{G}(\omega)$, by $1/(|M\omega| + \eta)$. We find it more convenient to use an $e^{-\delta|\omega|}$ regulator, where $\delta$ is a dimensionful ultraviolet cutoff. To take care of the infrared divergence, we put the particle on a circle of circumference $T$, which, in Euclidean space, is equivalent to looking at finite temperature. In this case, the propagator is

$$G^{\mu\nu}(t_1 - t_2) = \frac{\alpha}{\alpha^2 + \beta^2} \sum_{m \neq 0} \frac{1}{|m|} e^{im(t_1-t_2)2\pi/T} e^{-\epsilon|m|}$$

when $\mu = \nu$ and

$$G^{\mu\nu}(t_1 - t_2) = -\frac{\alpha}{\alpha^2 + \beta^2} \ln \left(1 + e^{-2\epsilon} - 2e^{-\epsilon} \cos \frac{2\pi}{T}(t_1 - t_2)\right)$$

when $\mu \neq \nu$. In these expressions, $\epsilon$ is a dimensionless cutoff which is the ratio of the ultraviolet cutoff $\delta$ to the infrared scale $T$. The expressions for $G^{\mu\nu}$ take on much simpler forms if we change variables to $z_j = e^{2\pi i t_j/T}$. With this definition, the exponentiated propagator is

$$e^{-q_1^\mu \dot{q}_2^\nu} G^{\mu\nu}(t_1 - t_2) = \left[\frac{-e^\epsilon z_1 z_2}{(z_1 - e^\epsilon z_2)(z_1 - e^{-\epsilon} z_2)}\right]^{\zeta_\alpha} \left[\frac{z_1 - e^\epsilon z_2}{z_2 - z_1 e^\epsilon z_2}\right]^{\zeta_\beta},$$

where the exponents are

$$\zeta_\alpha = -q_1^\mu \cdot \dot{q}_2^\nu \frac{\alpha}{\alpha^2 + \beta^2}, \quad \zeta_\beta = -q_1^\mu q_2^\nu e^{\mu\nu} \frac{\beta}{\alpha^2 + \beta^2},$$

and we have used the identity

$$\arctan x = \frac{1}{2i} \ln \left(\frac{1 + ix}{1 - ix}\right),$$

to rewrite the off-diagonal part of $G$. Using the same variables we also have

$$\langle \dot{X}(t_1) X(t_2) \rangle_0 = \frac{\alpha}{\alpha^2 + \beta^2} \left( -\frac{2\pi i}{T} \right) \frac{z_1^2 - z_2^2}{(z_1 - e^{-\epsilon} z_2)(z_1 - e^\epsilon z_2)}.$$
and we can obtain similar expressions for \( \langle \dot{Y}(t_1)Y(t_2) \rangle_0, \langle \dot{X}(t_1)Y(t_2) \rangle_0 \), etc.

Now we exploit simplifications which occur on the critical circle, i.e. when \( \frac{\alpha}{\alpha^2 + \beta^2} = 1 \).

First, we note that as \( \epsilon \to 0 \) the exponentiated propagator becomes

\[
e^{(X(t_1)X(t_2))} = -\frac{z_1z_2}{(z_1 - z_2)^2} .
\]

This is very similar to the expression for \( e^{(X(t_1)X(t_2))} \) when \( t \) lies on a line. Therefore, we can give a similar argument to the one illustrating \( SL(2, R) \) invariance on a line to show that on the circle, the theory has \( SU(1, 1) \) invariance. This means that under

\[
z = e^{2\pi it/T} \to \tilde{z} = \frac{az + b}{bz + \bar{a}} \quad \text{with} \quad |a|^2 - |b|^2 = 1 ,
\]

the partition function remains invariant, \( \dot{X}(t) \) transforms as a dimension one operator and so on. As a result, even though at finite temperature the system does not exhibit scale invariance, it still possesses a larger symmetry group, \( SU(1, 1) \).

In order to consider what happens when \( \epsilon \neq 0 \), we begin by specializing to the case when \( \vec{q}_i \cdot \vec{q}_j \frac{\alpha}{\alpha^2 + \beta^2} \) and \( q_i^n q_j^\mu \epsilon_{\mu \nu} \frac{\beta}{\alpha^2 + \beta^2} \) are integers for all \( \vec{q}_i \). This condition occurs on the critical circle \( \frac{\alpha}{\alpha^2 + \beta^2} = 1 \) at the “special points” where \( \beta/\alpha \in Z \). At these points, the partition function and correlation functions have a very simple form even before we take the regulator to zero. This is because \( e^{q_i^n q_j^\mu \langle X^\mu(t_i)X^\nu(t_j) \rangle} \) and \( \langle \dot{X}(t_i)X^\nu(t_j) \rangle \) are just rational functions in the \( z_i \)'s with coefficients of the form \( e^{n\epsilon} \). For example, at the zero-field critical point, where \( \alpha = 1 \) and \( \beta = 0 \), the regulated version of the integral in (3.26) which gives the \( O(V_0^{2n}) \) term of the partition function is

\[
Q_{2n} = \oint \prod_{j=1}^n \left( \frac{dz_j dw_j}{2\pi i 2\pi i} \right) I_{2n} ,
\]

where the definition of \( I_{2n} \) is

\[
I_{2n} = (-1)^n \frac{\prod_{i<j}(z_i - e^\epsilon z_j)(z_i - e^{-\epsilon} z_j)(w_i - e^\epsilon w_j)(w_i - e^{-\epsilon} w_j)}{\prod_{i,j}(z_i - e^\epsilon w_j)(z_i - e^{-\epsilon} w_j)} .
\]
denominator that is completely factored into linear terms. This property of the integrand has several important consequences.

The first conclusion is that it is straightforward, although tedious, to analytically evaluate any term in the perturbation expansions for the free energy and correlation functions. More generally, there is a well-defined set of rules for integrating any term in the perturbation series. They tell us how to go from a graph (or integrand) with \( n \) vertices to several graphs with \( n - 1 \) vertices by evaluating the residues of a rational function. With these rules, we can program a computer to calculate correlation functions analytically. We have done a few examples with the help of Mathematica, and some of the results are given in Section 4.

We can also use these rules to prove that the free energy and correlation functions have certain properties. The most important such property is that there are no logarithmic divergences, which can be seen as follows. After we have performed all the integrals, we must obtain a rational function in \( e^\epsilon \) for the free energy (or a rational function of the variables \( e^{2\pi i r_j} \) for correlation functions of arbitrary numbers of \( \bar{X}(r_j) \) and \( e^{iX(r_j)} \) fields). This implies that, in the limit as \( \epsilon \to 0 \), we obtain only rational functions in \( \epsilon \). Thus it is obvious that, to all orders in the periodic potential (at the special points), no logarithmic divergences in \( \epsilon \) are possible. Pole divergences can of course still occur.

This result is actually quite general. The structure of the partition function integrand will be the same as above for any doubly periodic potential of the form

\[
V = V_X \cos \frac{2\pi X}{a} + V_Y \cos \frac{2\pi Y}{a} + \text{higher harmonics} \tag{3.41}
\]

at the special points \( \frac{\alpha}{\alpha^2 + \beta^2} = 1 \) and \( \frac{\beta}{\alpha^2 + \beta^2} \in \mathbb{Z} \). It appears that imposing the discrete symmetry \( X \to X + \frac{n\alpha}{2\pi}, \ Y \to Y + \frac{m\alpha}{2\pi} \) guarantees that there are no logarithmic divergences at these points in the phase diagram. For the general problem of open string tachyon scattering, others [13] have found logarithmic divergences, but here we are looking only at exceptional values of momenta for which their integrals are ill-defined. One of the things we have accomplished here is to provide a regularization scheme for which calculations at these usually singular values of momenta are possible.

A careful analysis of the graphs shows that for the case at hand, the graphs for the free energy with total charge 0 (i.e. \( \sum |q_j^X| + |\sum q_j^Y| = 0 \)) diverge at most as \( 1/\epsilon \) while all other connected graphs are finite. Naive power-counting would predict that the charge-one graphs should diverge logarithmically (which is equivalent to saying that the \( \beta \)-function
for $e^{iX(t)}$ does not vanish). Quite the contrary, we have proved here that the $\beta$-function for $e^{iX(t)}$ vanishes at these special points. In other work, we have also shown that these theories exactly satisfy the infinite set of Ward identities predicted by the connection with string theory outlined in Section 2. An example is included in Appendix B, but the details of this statement must await another paper.

This general analysis cannot say very much about the points in the phase diagram where $\frac{\beta}{\alpha^2 + \beta^2} \notin Z$, but in Section 5 we will offer some evidence in support of the claim, suggested by the results of our renormalization group calculation in Section 3.2, that there is a circle of critical theories in the phase diagram to all orders in the periodic potential.

4. The Free Energy and Correlation Functions for Zero Magnetic Field

We will now use the methods of the previous section to explicitly calculate the free energy and correlation functions at the critical point with $\beta = 0$ and $\alpha = 1$. The mobility, or two-point function, at this critical point has been calculated by Fisher and Zwerger [12] for a weak potential and by Guinea et al. [14] for the tight-binding model. We provide a new calculation here for the weak potential case which is in agreement with the previous ones, and extend it to all $N$-point functions. Our method has the advantage that it is free of approximation and works at some other points on the critical circle with non-zero magnetic field. In addition, it can be used to calculate correlation functions in the charged sector, which are of interest in determining the renormalization-group flow and can be used to calculate the boundary state in open string theory.

Even though we can perform any integral in the perturbation series for the free energy and correlation functions, the one draw-back with our choice of regulator is that calculations with it become quite time-consuming at higher orders in $V_0$. To the order in $V_0$ that we have calculated, the main features of the $\beta = 0$ neutral sector which emerge are that: the free energy is proportional to $1/\epsilon + O(\epsilon)$ (the significance of this will be explained Section 4.1); $\langle \dot{X}(t)\dot{X}(0) \rangle$ is proportional to its value at $V_0 = 0$; all higher $m$-point functions of $\dot{X}$ are zero except for possible contact interactions; and in the limit $\epsilon \to 0$ all the correlation functions are $SU(1,1)$ invariant. For both the neutral and charged graphs, we find that the only needed renormalizations are the subtraction of an infinite constant from the free energy and a rescaling of the periodic potential strength. For our choice of potential, this rescaling is equivalent to holding $V_0 T\epsilon$ finite. We will now describe some of the calculations which lead to these conclusions.

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4.1. The Free Energy

If we follow the steps described in Section 3 to introduce convenient ultraviolet and infrared cutoffs, the partition function (3.26) can be rewritten as

\[ Z = \sum_n \frac{1}{(2n)!} \left[ (\frac{V_0 T}{2})^2 (e^\epsilon + e^{-\epsilon} - 2) \right]^n \left( \frac{2n}{n} \right) Q_{2n}, \]

where \( Q_{2n} \) is the integral defined in (3.39). The factor \( e^\epsilon + e^{-\epsilon} - 2 \) comes from self-contractions of \( X(t) \) with itself within a single potential insertion \( e^{\pm iX} \), and the factor of \( T \) multiplying \( V_0 \) comes from the change of variables from \( t_i \) to \( z_i \). Performing the integrals and expanding in powers of \( \epsilon \), we have found that

\[ Q_2 = \frac{1}{e^\epsilon - e^{-\epsilon}} = \frac{1}{2\epsilon} - \frac{1}{12} \epsilon + \frac{7}{720} \epsilon^3 + O(\epsilon^5), \]

\[ Q_4 = \frac{1}{2\epsilon^2} - \frac{7}{32\epsilon} - \frac{1}{6} + \frac{25}{192} \epsilon + O(\epsilon^2), \]

and

\[ Q_6 = \frac{3}{4\epsilon^3} - \frac{63}{64\epsilon^2} + \frac{7161}{65536\epsilon} + \frac{3}{4} + O(\epsilon). \]

Putting (4.1) through (4.4) together, we can obtain the first few terms in an expansion of the free energy, \( F = -\frac{1}{T} \ln Z \), in powers of the potential strength:

\[ F = -\frac{1}{T} \left( \frac{V_0 T\epsilon}{2} \right)^2 \left( \frac{1}{2\epsilon} + O(\epsilon) \right) - \frac{1}{T} \left( \frac{V_0 T\epsilon}{2} \right)^4 \left( -\frac{7}{128\epsilon} + O(\epsilon) \right) - \frac{1}{T} \left( \frac{V_0 T\epsilon}{2} \right)^6 \left( \frac{10579}{786432\epsilon} + O(\epsilon) \right) \cdots. \]

Since \( \epsilon \) is the dimensionless ratio of the ultraviolet cutoff \( \delta \) and the infrared cutoff \( T \), it is useful to define a dimensionless potential strength, \( V_r = V_0 \delta = V_0 T\epsilon \), which makes no reference to the infrared scale \( T \). One of our main points will be that the large-\( T \) limit defines a critical theory, independent of the short-distance scale \( \delta \), once we have rescaled the potential strength in this way. The free energy is then a power series in \( \epsilon \) with coefficients which are themselves power series in \( V_r \):

\[ F = -\frac{1}{T} \sum_{n=-1}^{\infty} f_n(V_r) \epsilon^n. \]

In the case at hand, the \( f_0 \) term happens to vanish, but we do not expect this to be a general feature of the free energy.
It is instructive to re-express the free energy expansion in terms of the dimensional scales $\delta$ and $T$:

$$F = -f_{-1}(V_r) \frac{1}{\delta} - f_0(V_r) \frac{1}{T} - f_1(V_r) \frac{\delta}{T^2} + \cdots. \quad (4.7)$$

The only universal term is the $f_0$ term: all the others depend on $\delta$ and hence on the details of the definition of the theory at short distances. Furthermore, the $f_{-1}$ term can be completely removed by an appropriate $V_r$-dependent constant shift of the original periodic potential (such a shift obviously has no observable effect on the particle dynamics). The meaningful critical physics of the free energy is therefore entirely contained in $f_0(V_r)$: In the string theory context, it provides the overall normalization of the boundary state (see [9] for examples). There is also a more conventional thermodynamic interpretation: In setting up our functional integral in Section 2, instead of integrating out the oscillators, we could have shown that (2.1) makes the oscillators equivalent to a free massless scalar field living on a one-dimensional world with a boundary (basically a spatial slice of the string worldsheet) and with some non-trivial, but perfectly local, boundary action for the massless scalar field induced by the action of the original quantum particle. The path integral over paths periodic in Euclidean time with period $T$ then generates the partition function of this system at temperature $\tau = 1/T$. Equation (4.7) can then be interpreted as the expansion about zero temperature of the thermodynamic free energy and $f_0$ can be identified as the zero-temperature limit of the entropy associated with the boundary. (Affleck and Ludwig [15] have recently calculated the boundary entropy of Ising and Kondo systems, and their paper gives a clear explanation of how to separate boundary from bulk contributions to the entropy.) In some sense, this entropy counts the number of dynamically active degrees of freedom living on the boundary. It should (and does) vanish for dynamically trivial boundary conditions like Dirichlet or Neumann. What it should do for the case at hand, where there is a complicated boundary action, is not obvious. The calculations we have summarized in (4.5) show, perhaps surprisingly, that $f_0(V_r)$ is identically zero. We will later see that, if the magnetic field is turned on as well as the potential, $f_0$ becomes a non-zero function of the magnetic field and of $V_r$. In view of this discussion, in the rest of this paper we shall identify the free energy with the $f_0$ term in the appropriate analog of the expansion (4.7). It would be more accurate to speak of the zero-temperature entropy, but this abuse of language should cause no confusion.
4.2. The \(N\)-Point Functions

Next, we turn our attention to the \(m\)-point functions. When \(m\) is odd, the \(m\)-point functions are zero by symmetry. For the connected \(2m\)-point functions, we insert the regulated version of (3.27) into the partition function, and absorb the self contractions of the potential insertions into \(V_r\). The result is that we must compute the connected part of

\[
\langle \dot{X}(r_1) \ldots \dot{X}(r_{2m}) \rangle = \langle \dot{X}(r_1) \ldots \dot{X}(r_{2m}) \rangle_0 + \sum_{n=1}^{\infty} \frac{(V_r/2)^{2n}}{(2n)!} \left( \frac{2\pi}{iT} \right)^{2m} A_{2n} .
\] (4.8)

Here, \(\langle \dot{X}(r_1) \ldots \dot{X}(r_{2m}) \rangle_0\) is the \(2m\)-point function in the absence of the periodic potential and \(A_{2n}\) is defined to be

\[
A_{2n} = \oint \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \frac{dw_j}{2\pi i} I_{2n} R_{n,m} ,
\] (4.9)

where \(I_{2n}\) is the function defined in (3.40) and

\[
R_{n,m} = (-1)^m \prod_{j=1}^{2m} \prod_{i=1}^{n} \left( \frac{\xi_j^2 - z_i^2}{(\xi_j - e^{-\epsilon} z_i)(\xi_j - e^{\epsilon} z_i)} - \frac{\xi_j^2 - w_i^2}{(\xi_j - e^{-\epsilon} w_i)(\xi_j - e^{\epsilon} w_i)} \right) ,
\] (4.10)

with \(\xi_j = e^{2\pi i r_j/T}\).

For the two-point function, we have done explicit calculations out to fourth order in the potential, finding

\[
A_2 = -\frac{1}{2 \sin^2 \left[ \frac{2\pi}{T} \left( \frac{t_1 - t_2}{2} \right) \right]} + O(\epsilon^2) ;
\] (4.11)

and

\[
A_4 = \left( -\frac{2}{\epsilon} + 3 \right) \frac{1}{2 \sin^2 \left[ \frac{2\pi}{T} \left( \frac{t_1 - t_2}{2} \right) \right]} .
\] (4.12)

These two expressions are both proportional to the two-point function in the absence of the periodic potential, which is given by

\[
\langle \dot{X}(t_1) \dot{X}(t_2) \rangle_0 = - \left( \frac{2\pi}{T} \right)^2 \frac{1}{2 \sin^2 \left[ \frac{2\pi}{T} \left( \frac{t_1 - t_2}{2} \right) \right]} .
\] (4.13)

To this order in \(V_0\), \(A_{2n}\) contains one non-zero disconnected graph, coming from \(Z_2 \cdot A_2\). When we substitute the expression for \(A_0\), \(A_2\) and \(A_4\) back into equation (4.8) for the \(2m\)-point function, and then subtract off the disconnected graph, we find that as \(\epsilon \to 0\),

\[
\langle \dot{X}(t_1) \dot{X}(t_2) \rangle_{\text{conn}} = \mu(V_r) \langle \dot{X}(t_1) \dot{X}(t_2) \rangle_0 ,
\] (4.14)

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where $\mu$ is given by
\begin{equation}
\mu(V_r) = 1 - \left(\frac{V_r}{2}\right)^2 + \frac{3}{4} \left(\frac{V_r}{2}\right)^4 + \cdots.
\end{equation}

Therefore, the renormalized two-point function is equal to the “free” two-point function, rescaled by $\mu(V_r)$. For small enough $V_0$, we see that $\mu < 1$ but, lacking an all-orders calculation, we cannot show that $\mu$ is always less than 1. (In ref. [4], we sketch an alternate calculation in which we fermionize the theory. In that case, we can calculate the mobility to all orders in $V_r$ and we find that $0 \leq \mu \leq 1$.) We note here that, after rescaling $V_0$, the answer is finite (except when $t_1 = t_2$). This means that we do not need to renormalize the kinetic part (2.5) of the action, despite the fact that the propagator correction diagrams appeared to have a divergence $\propto 1/\epsilon$. These results are consistent with what was found in the tight-binding approximation some time ago [14].

We have also used Mathematica to evaluate the integrals for the four-point function to order $V_0^4$ and have found that the connected four-point function, $\langle \dot{X}(r_1) \cdots \dot{X}(r_4) \rangle$, is zero as $\epsilon \to 0$, except when $r_i = r_j$. In addition, we have calculated the connected $2m$-point functions for any $m > 1$ at order $V_0^2$ and found them to vanish as $\epsilon \to 0$, as long as none of the points are coincident. Both calculations are quite involved and the details of the latter are given in Appendix A.

In summary, the critical theory looks basically like the free theory, except that the two-point function is rescaled and the $2m$-point functions have contact terms. The critical theory is obtained as the large-$T$ (zero-temperature!) limit of the theory with a fixed short-distance scale. The only needed renormalizations are a rescaling of the potential strength (the critical theory depends only on $V_r = V_0 \delta$) and the subtraction of a temperature-independent constant from $F$. The two-point function differs from the free two-point function by a finite mobility factor $\mu(V_r)$ which decreases from unity as $V_r$ increases from zero. We expect that the mobility should be less than one for any $V_r$ because the periodic potential should inhibit the particle’s motion. Lastly, the critical theory is $SU(1,1)$-invariant: The interacting two-point function is proportional to the free two-point function, which is $SU(1,1)$-invariant, and the higher-point functions, barring contact terms, are zero. (Actually, using the fermionic regulator of [4], we can show that even the contact terms are $SU(1,1)$-invariant.)
5. Magnetic Field Effects

We now turn to our main problem, that of calculating the critical properties of a particle moving in two dimensions and simultaneously subject to a magnetic field and a periodic potential. As before, we will calculate both the free energy and a variety of $N$-point functions in a perturbation expansion in the potential strength (but the dependence on the magnetic field will be exactly accounted for). We will also restrict ourselves to the neutral sector of the Coulomb gas arising from the expansion in powers of the periodic potential. In what follows we assume that we are on the critical circle \( \frac{\alpha^2 + \beta^2}{\alpha^2} = 1 \) and identify where we are on that circle by the value of \( \gamma = \beta/\alpha \). So far we have proved to all orders in \( V_0 \) that the points where \( \gamma \in \mathbb{Z} \) are exactly critical, and have shown in Section 3.2 that to the first few orders in \( V_0 \) the theory is also critical for \( \gamma \notin \mathbb{Z} \). We now support this with some explicit calculations, valid for any \( \gamma \), which again show no logarithmic divergences.

5.1. Magnetic Field Contribution to Free Energy

Let us first deal with the free energy, expanding it in powers of the potential strength: \( F(B,V_r) = -\sum_0^\infty F_n(B)V_r^n \). The term of zeroth-order in \( V_r \) has been computed elsewhere \[16,17\] and has the interesting form

\[
F_0(B) = \frac{1}{2T} \ln(1 + (2\pi\alpha'B)^2) .
\] (5.1)

As explained in \[17\] and elsewhere, if this expression is treated as an action functional, it generates stringy corrections to Maxwell’s equations. It is at least an existence proof that the boundary entropy discussed in the previous section can have non-trivial dependence on the parameters of the critical theory. We will not discuss it further, as we are really interested in the joint effects of potential and magnetic field which are responsible for the unusual spectrum of the Hofstadter problem with no dissipation. Such effects do not occur until fourth-order in the expansion in powers of the periodic potential. This is the first order at which we can have the insertion of both $X$ and $Y$ “charges”, so that the system can feel the effect of the phases due to the magnetic field. (By the zero-charge condition, each $e^{iX}$ must be accompanied by an $e^{-iX}$ and each $e^{iY}$ by an $e^{-iY}$.) In what follows, we will evaluate the free energy to order $V_0^4$ (it is possible, but tedious to go to higher orders).

For calculations involving non-zero magnetic field, the contour integral technique introduced in Section 3 encounters difficulties at fourth order in the potential strength. This
is precisely because the interesting interplay between $B$ and $V$ begins at this order, and the origin of the problem can be seen explicitly in the part of (3.33) which is the exponential of the off-diagonal propagator. Diagrams with both $X$ and $Y$ internal charges will involve integrals over expressions containing factors of this kind, and whenever $\gamma$ is not an integer, branch cuts will be present which make evaluation difficult. To avoid this complication, we choose to use a new method of calculation starting directly with equation (3.14) for the partition function. In the unregulated diagrams, the exponentiated off-diagonal propagators which connect $X$ and $Y$ charges simply contribute a phase factor when an $X$ charge moves past a $Y$ charge. Except for the above-mentioned $V_0$-independent piece, the magnetic field only contributes to the free energy through these phases. Thus we will take into account the interaction between the the $X$ and $Y$ charges simply by keeping track of the phase factors. We will regulate the diagonal propagator in the usual way. Unfortunately, the necessity to maintain the ordering of the charges makes it impossible to use contour integration and so we resort to series expansion of the diagonal propagator instead. The advantage of this technique of course is that it works for non-integer values of $\gamma$.

One might worry that we are not regulating the sign-function which generates the phases, but this is only of concern in the calculation of correlation functions. If we simply wish to calculate the free energy, the off-diagonal propagator contributes no divergences, and we expect that the phase prescription will work correctly. In calculating correlation functions, however, we take derivatives of the off-diagonal propagator and regulation becomes necessary because the derivative of a sign-function is a $\delta$-function.

As we indicated above, the first $B$-dependence in the free energy comes at order $V_0^4$, and we now proceed to calculate this term. The contribution to the partition function of the diagram with $n_X$ of the $X$ charges and $n_Y$ of the $Y$ charges is (with $n = n_X + n_Y$)

$$Z(n_X, n_Y) = \left( \frac{V_0}{2} e^{-\frac{1}{2} G(0)} \right)^n \sum_{\bar{q}_i = \pm} \int d\tau_1 \ldots d\tau_n \exp \sum_{i<j} (i\phi_{ij} - \bar{q}_i \cdot \bar{q}_j G(\tau_i - \tau_j))$$

(5.2)

where the phase factor $\phi_{ij}$ is defined by

$$\phi_{ij} = \pi \gamma \epsilon^{\mu\nu} q_i^\mu q_j^\nu \text{sign}(\tau_i - \tau_j),$$

(5.3)
the integration range is $0 \leq \tau_i \leq \tau_{i+1} \leq T$ and $G(t)$ is the diagonal element of the propagator (3.8). The fourth-order piece of the free energy is given by

$$F_4 = -\frac{4}{T} \left( \frac{V_0}{2} e^{-\frac{1}{2}G(0)} \right)^4 \left( e^{2\pi i \gamma} + e^{-2\pi i \gamma} - 2 \right)$$

$$\int dt_1 dt_2 ds_1 ds_2 \exp(G(t_1 - t_2) + G(s_1 - s_2))$$

(5.4)

where: $0 \leq t_1 \leq s_1 \leq t_2 \leq s_2 \leq T$; the overall factor of 4 takes account of the fact that the first charge can be a plus or a minus and an $X$ or a $Y$; the $-2$ in the $\gamma$-dependent factor subtracts the contribution due to the disconnected graphs, which are also present in the absence of a magnetic field.

To get finite results we must, of course, regulate the propagator. Because we are on the critical circle, we can use the strategy described in Section 3 following (3.31). This amounts to the replacement

$$G(z) = -\ln \left( 1 - ze^{-\epsilon} \right) \left( 1 - \bar{z}e^{-\epsilon} \right)$$

(5.5)

with $z = \exp(2\pi i t/T)$. For our later calculation it will be useful to have the following power series expansion of the exponentiated propagator

$$e^{G(z)} = \frac{1}{1 - e^{-2\epsilon}} \sum_{m=-\infty}^{\infty} z^m e^{-|m|\epsilon}.$$

(5.6)

For coincident points this reduces to

$$e^{-G(0)} = (1 - e^{-\epsilon})^2.$$

(5.7)

At this point we note the translational invariance and periodicity of $G$ and use the formula

$$\int_0^T \frac{d\tau_1 \ldots d\tau_n}{n!} f(\tau_i - \tau_j) = \frac{T}{n} \int d\tau_2 \ldots d\tau_n f(\tau_i - \tau_j) \bigg|_{\tau_1=0}$$

(5.8)

where on the right hand side $0 \leq \tau_i \leq \tau_{i+1} \leq T$ and $f(\tau)$ is periodic in all its arguments with period $T$. This allows us to write

$$F_4 = 4 \sin^2 \pi \gamma \left( \frac{V_0}{2} e^{-\frac{1}{2}G(0)} \right)^4 \int ds_1 dt_2 ds_2 \exp(G(t_2) + G(s_1 - s_2))$$

(5.9)
where again $0 \leq s_1 \leq t_2 \leq s_2 \leq T$. Transforming to angular variables on a circle and using (5.6) we obtain

$$F_4 = 4 \sin^2 \pi \gamma \left( \frac{V_0}{2} e^{-\frac{1}{2}G(0)} \right)^4 \left( \frac{T}{2\pi} \right)^3 \left( \frac{1}{1-e^{-2\epsilon}} \right)^2 \sum_{m,n=-\infty}^{\infty} e^{-|m|\epsilon} e^{-|n|\epsilon} I_{mn}$$

(5.10)

where the integral is given by

$$I_{mn} = \int_0^{2\pi} d\phi_2 \int_0^{\phi_2} d\theta_2 \int_0^{\theta_2} d\phi_1 e^{im\theta_2} e^{in(\phi_2-\phi_1)} .$$

(5.11)

Doing the integral we find

$$I_{mn} = \frac{4\pi^3}{3} \delta_{m,0} \delta_{n,0} - \frac{4\pi}{n^2} \delta_{m,0} (1 - \delta_{m,0}) - \frac{4\pi}{n^2} \delta_{m,0} (1 - \delta_{n,0})$$

$$+ \frac{2\pi}{n^2} (1 - \delta_{n,0}) (\delta_{m,n} + \delta_{m,-n})$$

(5.12)

and therefore that

$$\sum_{m,n} e^{-|m|\epsilon} e^{-|n|\epsilon} I_{mn} = \frac{4\pi^3}{3} - 8\pi \sum_{m=1}^{\infty} \frac{1}{m^2} (2e^{-m\epsilon} - e^{-2m\epsilon}) .$$

(5.13)

The sum can be evaluated for small $\epsilon$ and we find that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} (2e^{-m\epsilon} - e^{-2m\epsilon}) = \frac{\pi^2}{6} - 2\epsilon \ln 2 + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{12} + O(\epsilon^4) .$$

(5.14)

Finally, using the renormalization of the potential strength introduced in the previous section, the free energy becomes

$$F_4 = \frac{V_4}{32\pi^2 T} \sin^2 \pi \gamma \left( \frac{4 \ln 2}{\epsilon} - 1 + O(\epsilon) \right) .$$

(5.15)

The $\epsilon^{-1}$ term must of course be subtracted away, but a finite magnetic-field-dependent piece is left over. This finite part vanishes for any integer $\gamma = k$. This is physically reasonable since, at these points, the magnetic phase associated with transporting the center of the electron’s orbit around a unit cell of the reciprocal lattice (as defined in equation (3.15)) is a multiple of $2\pi$. Therefore we expect this situation to look equivalent to the zero field case. Consideration of the phase prescription for higher orders in the potential shows that all such terms will also be zero for these special points on the critical circle. Of particular importance is the observation that there is no logarithmic divergence even when $\gamma$ is not an integer. If this remains true for higher orders, as we believe it will, then it confirms the result of Section 3.2 that we have a critical circle, $\alpha = \alpha^2 + \beta^2$. 

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5.2. *N*-Point Functions and Magnetic Field

We now turn to the calculation of the \(m\)-point functions on the critical circle. Again, the \((2m + 1)\)-point functions are all trivially zero by symmetry arguments. We present explicit calculations of the \(2m\)-point functions to \(O(V_0^2)\) using the contour integral techniques outlined in Section 3. Our results are valid for any value of \(\gamma\), not just integer ones.

As we saw above, the extension of these results to higher orders in the strength of the periodic potential is more troublesome because, unless \(\gamma\) is an integer, there are branch cuts in the integrands whenever there is an internal line connecting internal \(X\) and \(Y\) charges. However, our earlier calculations show that when \(\gamma\) is an integer, nothing remarkable happens at higher orders, and certainly that there will be no logarithmic divergences. For non-integer \(\gamma\), the presence of the branch cuts makes it harder to be certain, but the absence of logarithmic divergences in the free energy is strong evidence that even here the \(O(V_0^2)\) behavior will carry on essentially unchanged at higher orders.

The \(O(V_0^2)\) contribution to the \(2m\)-point functions has the form

\[
\left\langle \dot{X}^{\mu_1}(r_1) \ldots \dot{X}^{\mu_{2m}}(r_{2m}) \right\rangle_2 = \left( \frac{V_0}{2} e^{-\frac{1}{2}G(0)} \right)^2 \int dt_1 dt_2 \exp(G^{XX}(t_1 - t_2)) \left[ R^X(2m) + R^Y(2m) \right],
\]

where

\[
R^\nu(2m) = (-1)^m \prod_{i=1}^{2m} \left[ \frac{d}{dr_i} G^{\mu_i,\nu}(r_i - t_1) - \frac{d}{dr_i} G^{\mu_i,\nu}(r_i - t_2) \right].
\]

\(R^X(2m)\) comes from two \(\cos X\) potential insertions, and \(R^Y(2m)\) comes from two \(\cos Y\) potential insertions. Now let \(G\) be equal to \(G^{XX}\) and \(N\) equal to \(G^{XY}\), where the propagators have been regulated as in Section 3. Then, with the definition \(z_j = \exp(2\pi i t_j / T)\), we have

\[
\partial_1 G(t_1 - t_2) = \frac{2\pi i}{T} \left[ \frac{z_2}{z_2 - e^{-\epsilon} z_1} - \frac{z_1}{z_1 - e^{-\epsilon} z_2} \right],
\]

and

\[
\partial_1 N(t_1 - t_2) = \frac{\gamma}{T} \left[ \frac{z_1}{z_1 - e^{\epsilon} z_2} + \frac{z_2}{z_2 - e^{\epsilon} z_1} \right].
\]

We will first restrict our attention to the calculation of the \(\langle \dot{X} \dot{X} \rangle\) and \(\langle \dot{X} \dot{Y} \rangle\) two-point functions. Because our system is invariant under rotations in the \(X-Y\) plane, \(\langle \dot{Y} \dot{Y} \rangle\) has the form
the same form as $\langle \dot{X} \dot{X} \rangle$. The integrals for the second order contributions to the two-point functions can be written as

$$
\langle \dot{X}(t_1)\dot{X}(t_2) \rangle_2 = -\left( \frac{V_0}{2} e^{-iG(0)} \right)^2 \int_0^T du dv \; e^{G(u-v)} I_\alpha(u,v)
$$

and

$$
\langle \dot{X}(t_1)\dot{Y}(t_2) \rangle_2 = -\left( \frac{V_0}{2} e^{-iG(0)} \right)^2 \int_0^T du dv \; e^{G(u-v)} I_\beta(u,v)
$$

where $I_\alpha$ is defined by

$$
I_\alpha(u,v) = 2\partial_1 G(t_1 - u)\partial_2 (G(t_2 - u) - G(t_2 - v))
+ 2\partial_1 N(t_1 - u)\partial_2 (N(t_2 - u) - N(t_2 - v)),
$$

and $I_\beta$ is given by

$$
I_\beta(u,v) = 2\partial_1 G(t_1 - u)\partial_2 (N(t_2 - v) - N(t_2 - u))
+ 2\partial_1 N(t_1 - u)\partial_2 (G(t_2 - u) - G(t_2 - v)).
$$

The calculation by contour integration is straight-forward and gives for the diagonal contribution

$$
\langle \dot{X}(t_1)\dot{X}(t_2) \rangle_2 = \left( \frac{V_r}{2} \right)^2 \left( \frac{2\pi}{T} \right)^2 (1 - \gamma^2) \left[ 2 \sin^2 \left( \frac{\pi}{T}(t_1 - t_2) \right) \right]^{-1}
$$

$$
= - (1 - \gamma^2) \left( \frac{V_r}{2} \right)^2 \langle \dot{X}(t_1)\dot{X}(t_2) \rangle_0
$$

where $V_r$ is the renormalized potential, given by $V_r = V_0 T_\epsilon$. This two-point function is the same as the free propagator with a rescaled coefficient depending on the renormalized potential strength and the magnetic field.

For the off-diagonal part we get zero for separated points and the contact term looks like the derivative of a delta function, or two derivatives of a step function, as we would expect if this were also just a rescaled version of the free off-diagonal propagator. Explicitly, we find

$$
\langle \dot{X}(t_1)\dot{Y}(t_2) \rangle_2 = -i\pi\gamma V_r^2 \delta'(t_1 - t_2).$


Integration of the regulated version of this result, or a direct calculation using the series expansion (5.6) for \( G \) (and a similar one for \( N \)) gives the result that

\[
\langle X(t_1)Y(t_2) \rangle_2 = -2\pi i\gamma \left( \frac{V_r}{2} \right)^2 \left( \frac{2}{T}(t_1 - t_2) - \text{sign}(t_1 - t_2) \right)
\]

\[
= -2 \left( \frac{V_r}{2} \right)^2 \langle X(t_1)Y(t_2) \rangle_0 .
\]

The constants of integration which arise upon integrating (5.25) twice to obtain (5.26) are fixed by the requirement that all functions be periodic with period \( T \). Thus the second order contribution is just \(-2(V_r/2)^2\) times the free propagator. The conclusion here is that when we turn on the potential in the presence of the magnetic field, the zeroth-order two-point function \( \langle \dot{X}^\mu \dot{X}^\nu \rangle \) is just multiplied by a constant matrix whose elements are functions of \( \gamma \) and \( V_r \). For the special points in the phase diagram where \( \gamma \) is integer we know for sure that this remains true at higher orders in the potential, but we have not proved this when \( \gamma \) is not an integer.

In reference [4], we have shown that the correlation functions on the critical circle at integer \( \gamma \) are all related by a duality symmetry. Given the form of the two-point function for \( \gamma = 1 \), this symmetry completely determines the form of the two-point functions at all the other special critical points. It is interesting to compare the duality prediction with the explicit functions we have just calculated. To do this, it is convenient to specify location in the \( \alpha - \beta \) plane by the complex number \( z = \alpha + i\beta \) and to rewrite the free two-point function (3.8) as follows:

\[
G^{\mu\nu}(t; z) = -\text{Re}(1/z) \ln(t^2) \delta^{\mu\nu} + i\pi \text{ Im}(1/z) \text{ sign}(t) \epsilon^{\mu\nu}.
\]

(For simplicity we have removed the infrared cutoff and rewritten this as a two-point function on the open line.) The content of the duality relation derived in [4] is that we can express the interacting two-point function at the special points on the critical circle solely in terms of the function \( G \) at various values of \( z \), and of the zero-magnetic-field mobility function, \( \mu(V_r) \), displayed in (4.15). Define \( z_\gamma = 1/(1-i\gamma) \). Then when \( \alpha/(\alpha^2 + \beta^2) = 1 \) and \( \beta/\alpha = \gamma \), we have \( z \) equal to \( z_\gamma \). According to equation 6.2 in [4], when \( \gamma \) is an integer, the two-point function with potential strength \( V_0 \) is given by

\[
\langle \vec{X}(t) \vec{X}(0) \rangle (\gamma, V_0) = G(t; z_\gamma) + (\mu(V_r) - 1) G(t; z_\gamma^2) .
\]
The first term on the right-hand side is just the expression for the two-point function in the absence of the potential. The second term contains all the effects due to the potential. We note that 
\[
\frac{1}{z_1 z_2} = (1 - \gamma^2) - \gamma^2 z_1 z_2 \gamma^2 \]
and that 
\[
\mu(V_r) - 1 = - (V_r/2)^2 + O(V_r^4). \]
Putting this together with (5.27), we see that (5.28) predicts the order \(V_0^2\) part of the two-point function to be

\[
\langle X^\mu(t) X^\nu(0) \rangle (\gamma, V_0) = \left(\frac{V_r}{2}\right)^2 (1 - \gamma^2) \ln(t^2) \delta^{\mu\nu} + i\pi \left(\frac{V_r}{2}\right)^2 2 \gamma \text{sign}(t) e^{\mu\nu}. \]

This can be expressed in terms of the free two-point function (5.27) at \(\gamma\) as follows:

\[
\langle \dot{X}(t) \dot{X}(0) \rangle_2 (\gamma, V_0) = -(1 - \gamma^2) \left(\frac{V_r}{2}\right)^2 \langle \dot{X}(t) \dot{X}(0) \rangle_0, \]

and

\[
\langle \dot{X}(t) \dot{Y}(0) \rangle_2 (\gamma, V_0) = -2 \left(\frac{V_r}{2}\right)^2 \langle \dot{X}(t) \dot{Y}(0) \rangle_0. \]

On comparing this expression with equations (5.24) and (5.26), we see that it agrees with our direct computation. We conclude that not only do the two-point functions exhibit \(SL(2, R)\) covariance as a function of the time variable, but they also satisfy a duality symmetry that relates two-point functions at different values of friction and flux.

For the \(2m\)-point functions with \(m \geq 2\), we can again perform the contour integrals and sum over all the diagrams, exactly as we do in Appendix A for the \(2m\)-point functions with zero magnetic field. This time, we find that not all the correlation functions vanish when the points are non-coincident. To this order in the potential, any correlation function with exactly two \(\dot{X}\)’s or two \(\dot{Y}\)’s is finite as the cut-off goes to zero and all the other correlation functions are zero except for contact terms. When there are exactly two \(\dot{X}\)’s, the result is

\[
\langle \dot{X}(t_1) \dot{X}(t_2) \dot{Y}(s_1) \ldots \dot{Y}(s_{2m}) \rangle_2 = C_m \prod_{j=1}^{2m} \left(\frac{z_1 + w_j}{z_1 - w_j} - \frac{z_2 + w_j}{z_2 - w_j}\right) \frac{z_1 z_2}{(z_1 - z_2)^2}, \]

for \(m \geq 2\), and similarly when there are exactly two \(\dot{Y}\)’s. In this equation, we have defined

\[
C_m = \left(\frac{2\pi}{T}\right)^{(2m+2)} \left(\frac{V_r}{2}\right)^2 \gamma^2, \]

for \(m \geq 2\), and similarly when there are exactly two \(\dot{Y}\)’s. In this equation, we have defined
The four-point function can be written more simply as

\[
\langle \dot{X}(t_1) \dot{X}(t_2) \dot{Y}(s_1) \dot{Y}(s_2) \rangle_2 = C_1 \left( \frac{16 z_1 z_2 w_1 w_2}{(z_1 - w_1)(w_1 - z_2)(z_2 - w_2)(w_2 - z_2)} \right).
\]

We note that these correlation functions have the same form as the integrand of the \(O(V_0^2)\) contribution of the \(2m\)-point function with no magnetic field. Using arguments similar to those in Section 3, we can show that such functions are \(SU(1,1)\) covariant (or \(SL(2,R)\) covariant in the limit as \(T \to \infty\)). Thus, we have found non-trivial critical theories which exhibit not only scale invariance, but also the higher symmetry group, \(SL(2,R)\), as predicted by their connection with string theory. At higher orders in \(V_0\), when \(\gamma\) is an integer, we expect that additional correlation functions will have a finite, \(SL(2,R)\)-covariant limit as the cut-off is taken to zero.

**Section 6. Charged Sector Diagrams**

So far in this paper we have restricted our attention to the neutral sector of the Coulomb gas arising from the perturbative expansion in the periodic potential. In the DQM path integral, this restriction is enforced by the integral over the zero-mode. In string theory, the zero-mode integral usually serves to enforce momentum conservation in S-matrix elements. However, in calculating the open string boundary state, the zero-mode integration is left undone, and the charged sector contributes. Charged diagrams are also of interest in their own right from a Coulomb gas point of view and are useful in calculating the renormalization-group flow (e.g. as in Section 3.2 or in [13]). We shall see that in the critical theory, the connected diagrams of the charged sectors are all completely finite functions of the rescaled potential \(V_r\). For simplicity, we restrict ourselves in this section to zero magnetic field.

**Subsection 6.1. Charged Sector Free Energy**

As shown previously for the critical theory at zero magnetic field, all diagrams can be calculated by contour integration tricks. The \(n\)th order contribution to the partition function is given by

\[
Z_n = \frac{1}{n!} \left( \frac{V_0}{2} e^{-\frac{1}{2} G(0)} \right)^n \sum_{\{q_j = \pm 1\}} e^{iQ_n X_0} \int_{-T/2}^{T/2} dt_1 \cdots \int_{-T/2}^{T/2} dt_n \exp \left( - \sum_{j<k} q_j q_k G(t_j - t_k) \right)
\]
where $Q_n = \sum q_j$, $G(t)$ is defined by eqn. (5.3) and $X_0$ is the zero-mode. We have done the calculation to fourth order in $V_0$ and found the following results for $Z$ (we express everything in terms of the usual rescaled potential strength $V_r = V_0 T \epsilon$):

\[
Z_0 = 1 ,
\]

\[
Z_1 = 2 \left( \frac{V_r}{2} \right) \cos X_0 \left[ 1 - \frac{\epsilon}{2} + O(\epsilon^2) \right] ,
\]

\[
Z_2 = 2 \left( \frac{V_r}{2} \right)^2 \left[ \frac{1}{4\epsilon} + (1 - 2\epsilon) \cos 2X_0 - \frac{\epsilon}{48} + O(\epsilon^2) \right] ,
\]

\[
Z_3 = 2 \left( \frac{V_r}{2} \right)^3 \left[ \frac{1}{2\epsilon} \cos X_0 + \cos 3X_0 - \frac{5}{8} \cos X_0 + O(\epsilon) \right] ,
\]

\[
Z_4 = 2 \left( \frac{V_r}{2} \right)^4 \left[ \left( \frac{1}{4\epsilon} \right)^2 + \frac{1}{2\epsilon} \left( \cos 2X_0 - \frac{7}{128} \right) - \frac{1}{96} + \cos 4X_0 - \frac{5}{3} \cos 2X_0 + O(\epsilon) \right] .
\]

The free energy is related to the connected part of these diagrams in the usual way:

\[
F = -\frac{1}{T} \ln Z = -\frac{2}{T} \sum_{n=1}^{\infty} \left( \frac{V_r}{2} \right)^n F_n .
\]

Rearranging the series for $Z$, we find that the first four $F_n$ are

\[
F_1 = \cos X_0 ,
\]

\[
F_2 = \frac{1}{4\epsilon} + \frac{1}{2} \cos 2X_0 - \frac{1}{2} ,
\]

\[
F_3 = \frac{1}{3} \cos 3X_0 - \frac{3}{8} \cos X_0 ,
\]

\[
F_4 = -\frac{7}{256\epsilon} + \frac{1}{4} \cos 4X_0 - \frac{7}{24} \cos 2X_0 + \frac{1}{8} .
\]

The contribution of the charge-$Q$ diagrams is given by the coefficient of $e^{iQX_0}$ in these expressions. All the divergences come from the neutral sector and can be removed by the subtraction of a $V_r$-dependent $1/\epsilon$ term. Using general arguments about the form of the diagrams, we can show that this behavior continues to all orders. The key thing to note here is that the logarithmic divergences expected in such diagrams according to naive power-counting arguments are completely absent. Using our method of calculation, this is of course guaranteed from the beginning.
subsection 6.2 6.2. Charged Sector Two-Point Function

Calculation of the $N$-point functions can be done in a very similar way. The two-point function is particularly important for the open string boundary state (one extracts the energy-momentum tensor from it) and also for checking consistency with the Ward identity, so we show the calculation here as an example. The explicit formula for the $n$-th order contribution to the two-point function is

$$
\langle \dot{X}(\tau_1) \dot{X}(\tau_2) \rangle_n = -\frac{1}{Z} \frac{1}{n!} \left( \frac{V_0}{2} e^{-\frac{1}{2}G(0)} \right)^n \sum_{q_j = \pm 1} \sum_{i,j=1}^n e^{iQ_nX_0} \sum_{i,j=1}^n q_i q_j I_{ij}(n, \vec{q})
$$

where we define $I_{ij}(n, \vec{q})$ by

$$
I_{ij}(n, \vec{q}) = \int dt_1 \ldots dt_n \partial_1 G(\tau_1 - t_i) \partial_2 G(\tau_2 - t_j) \prod_{k=2}^n \prod_{l=1}^{k-1} \exp(q_k q_l G(t_k - t_l)).
$$

Explicit results for the first three terms in an expansion in powers of the potential are

$$
\langle \dot{X}(\tau_1) \dot{X}(\tau_2) \rangle_0 = -\left( \frac{2\pi}{T} \right)^2 2 \sin^2 \left( \frac{\pi}{T} \tau_{12} \right)^{-1},
$$

$$
\langle \dot{X}(\tau_1) \dot{X}(\tau_2) \rangle_1 = V_r \cos X_0 \left( \frac{2\pi}{T} \right)^2 \left( 1 - \Delta_2 \left( \frac{2\pi}{T} \tau_{12} \right) \right),
$$

$$
\langle \dot{X}(\tau_1) \dot{X}(\tau_2) \rangle_2 = \left( \frac{V_r}{2} \right)^2 \left( \frac{2\pi}{T} \right)^2 \left[ 8 \cos 2X_0 \cos^2 \left( \frac{\pi}{T} \tau_{12} \right) - 4 \cos^2 X_0 + 4 \sin^2 X_0 \Delta_2 \left( \frac{2\pi}{T} \tau_{12} \right) + \left( 2 \sin^2 \left( \frac{\pi}{T} \tau_{12} \right) \right)^{-1} \right],
$$

where $\Delta$ is a regulated delta-function defined by

$$
\Delta_n(\tau) = \lim_{\epsilon \to 0} \left( \frac{\sinh(n\epsilon)}{\cosh(n\epsilon) - \cos \tau} \right)
$$

and $\tau_{12} = \tau_1 - \tau_2$. Note that while all divergences are eliminated by rescaling the potential, the answer contains contact terms.

An important aspect of this result is that the zero-mode-dependent pieces are not $SU(1,1)$-invariant. This has to do with the fact, explained in [10], that reparametrization invariance of the boundary state path integral is not manifest: it is in some sense “softly
broken” by the non-local kinetic term in the boundary state action and the fixing of the zero mode in the boundary state path integral measure. There is nonetheless a complete set of “broken reparametrization invariance” Ward identities in which all of these effects are included [10]. As it turns out, the full Ward identity for the zero-mode-dependent two-point function involves the zero-mode-dependent part of the the free energy as well. The calculations reported in this section can therefore be used to make a rather nontrivial check of the Ward identities (not to speak of our whole renormalization scheme). We have checked that the above expressions satisfy the identity. The presence of the contact terms (terms involving $\Delta$) in the two-point function turns out to be a crucial element in the consistency of the Ward identity (details are relegated to an appendix). We conclude that our renormalization scheme produces a genuine solution to open string theory.

Given these results we can also explicitly construct the tachyon and massless particle contributions to the boundary state, defined in equation (2.3), up to second order in the tachyon strength, $V_0$. From the massless contribution, space-time equations of motion can be derived for the graviton and dilaton in the presence of this background field, and the stress energy tensor in Einstein’s equations due to the tachyon source can be constructed explicitly. However, efforts to discover a space-time effective action from which the equations of motion could be derived were unsuccessful, probably because the tachyon fluctuates on a short scale, rendering meaningless a low-energy description of its effects.

$$\text{(5.18) (5.18) equation 7.1 section 77. Conclusions}$$

The state of affairs described in this paper is promising, but far from fully satisfactory. Our goal is to find as complete a characterization of the “Hofstadter” critical theories as one has for the Ising model or the WZW models. The purely perturbative approach described in this paper is obviously not going to take us that far, but it does give a pretty clear idea of what we would eventually like to prove. Since one of the marginal terms (the dissipation term) in the action is non-local, standard field theory intuition does not necessarily apply and we have had to invent special methods and re-examine the whole question of critical behavior from the ground up. Our primary claim, supported by a variety of perturbative calculations, is that there is a two-parameter critical surface: The starting action has three marginal parameters (dissipation constant, magnetic field strength and strength of the periodic potential) and we have offered evidence that there is a critical theory for any value of the renormalized potential strength so long as the dissipation constant and
magnetic field satisfy one functional condition. (Basically the periodic potential has to be a dimension-one operator in the one-loop approximation! This is reminiscent of the situation in the Liouville theory.) We have also calculated some of the critical $N$-point functions. Because we are working in the rather unfamiliar context of critical theories with nonlocal interactions, we have found it worthwhile to use these $N$-point functions to explicitly check the reparametrization invariance Ward identities (this ensures that the theory is not just critical, but also a solution of string theory, a much more stringent requirement). A further, surprising, finding is that many of the higher $N$-point functions (all of them in the case of zero magnetic field) reduce to contact terms. This means that these critical theories are almost, but not quite, trivial. This is a broad hint that an exact solution for this system should be possible, but we have yet to see how to exploit it. We hope to return to this point in a future publication.

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Appendix (5.18) N-Point Functions

In this appendix, for zero magnetic field, we evaluate the $O(V_0^2)$ contribution to the higher $2m$-point functions of $\dot{X}$ and demonstrate that, except for contact terms, they are zero as $\epsilon \to 0$. As described in Section 4, the integral we want to evaluate is $A_2$ (4.3):

$$A_2 = \oint \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} I_2 R_{1,m},$$

with

$$I_2 = -\frac{1}{(z_1 - e^{-\epsilon} z_2)(z_1 - e^{\epsilon} z_2)}$$

and

$$R_{1,m} = (-1)^m \prod_{j=1}^{2m} \left( \frac{w_j^2 - z_1^2}{(w_j - e^{-\epsilon} z_1)(w_j - e^{\epsilon} z_1)} - \frac{w_j^2 - z_2^2}{(w_j - e^{-\epsilon} z_2)(w_j - e^{\epsilon} z_2)} \right)$$

$$= (-1)^m \sum_{M=0}^{2m} \sum_{\sigma_M} \prod_{j \in \sigma_M} \frac{w_j^2 - z_1^2}{(w_j - e^{-\epsilon} z_1)(w_j - e^{\epsilon} z_1)} \prod_{i \notin \sigma_M} \frac{w_i^2 - z_2^2}{(w_i - e^{-\epsilon} z_2)(w_i - e^{\epsilon} z_2)} (-1)^M,$$
where $\sigma_M$ is summed over all subsets of the first $2m$ integers which contain $M$ elements.

We will proceed by performing the integral

$$
\oint \frac{dz_1 \, dz_2}{2\pi i \, 2\pi i} \prod_{j=1}^{M} \frac{w_j^2 - z_1^2}{(w_j - e^{-\epsilon}z_1)(w_j - e^{-\epsilon}z_2)} \prod_{i=M+1}^{2m} \frac{w_i^2 - z_2^2}{(w_i - e^{-\epsilon}z_2)} I_2
$$

and then symmetrize later. If we first perform the $z_1$ integral, we must evaluate the residues when $z_1 = e^{-\epsilon}z_2$ and, if $M > 0$, also when $z_1 = e^{-\epsilon}w_k$, for $1 \leq k \leq M$. We will call the residue from the $z_1 = e^{-\epsilon}z_2$ pole $r_z$, and the residues from the $z_1 = e^{-\epsilon}w_k$ poles $r_k$. They are given by

$$
r_z = \frac{1}{z_2(e^\epsilon - e^{-\epsilon})} \prod_{i=1}^{M} \frac{(z_2^2 e^{-\epsilon} - w_i^2 e^\epsilon)(z_2^2 - w_i^2)(z_2 - e^{-\epsilon}w_i)}{(z_2 - w_i)(z_2 e^{-\epsilon} - w_i e^\epsilon)},
$$

and

$$
r_k = \frac{w_k}{(w_k - z_2)(e^\epsilon z_2 - e^{-\epsilon}w_k)} \prod_{i=1, i \neq k}^{M} \frac{w_k^2 e^{-\epsilon} - w_i^2 e^\epsilon}{(w_k e^{-\epsilon} - w_i e^\epsilon)(w_k - w_i)} \times \prod_{j=M+1}^{2m} \frac{z_2^2 - w_j^2}{(z_2 - e^{-\epsilon}w_j)(z_2 - e^\epsilon w_j)},
$$

for $1 \leq k \leq M$.

$r_k$ and $r_z$ both appear to have poles on the $z_2$ contour when $z_2 = w_k$ or $w_i$, respectively. However, we started with a completely well-defined, convergent integral for any value of $z_2$ and $w_i$, so we know that the result, $r_2 + \sum_k r_k$, must be finite. Consequently, all the poles on the $z_2$ contour must cancel each other when we add up all the residues. (We have checked this explicitly for $m \leq 2$.) Therefore, we can integrate each residue separately and just ignore the poles on the contour. Then, for $\oint r_z \, dz_2$, we must evaluate residues when $z_2 = 0$ and $z_2 = e^{-\epsilon}w_l$ for $M + 1 \leq l \leq 2m$. The residue when $z_2 = 0$ is

$$
r_{zz} = \frac{1}{e^\epsilon - e^{-\epsilon}}.
$$

The residue when $z_2 = e^{-\epsilon}w_l$ is

$$
r_{zl} = \frac{1}{e^\epsilon - e^{-\epsilon}} \prod_{i=1}^{M} \frac{w_i^2 e^{-2\epsilon} - w_i^2 e^{2\epsilon}}{(w_i e^{-\epsilon}/2 - w_i e^\epsilon/2)(w_i e^{-3\epsilon/2} - w_i e^{3\epsilon/2})} \times \prod_{j=M+1}^{2m} \frac{w_l^2 e^{-\epsilon} - w_j^2 e^\epsilon}{(w_l - w_j)(w_l e^{-\epsilon} - w_j e^\epsilon)}.
$$
The integral $\oint r_k \frac{dz}{2\pi i}$ has poles inside the contour when $z_2 = e^{-2\epsilon}w_k$ and $z_2 = e^{-\epsilon}w_l$ for $M + 1 \leq l \leq 2m$. The residue when $z_2 = e^{-2\epsilon}w_k$ is

$$
r_{kk} = \frac{1}{e^{\epsilon} - e^{-\epsilon}} \prod_{i=1}^{M} \frac{w_k^{2}e^{\epsilon} - w_i^{2}e^{\epsilon}}{(w_ke^{\epsilon} - w_ie^{\epsilon})(w_k - w_i)} \times \prod_{j=M+1}^{2m} \frac{w_k^{2}e^{-2\epsilon} - w_j^{2}e^{2\epsilon}}{(w_ke^{-2\epsilon} - w_je^{2\epsilon})(w_k - w_i)} ;
$$

and, lastly, the residue when $z_2 = e^{-\epsilon}w_l$ for $M + 1 \leq l \leq 2m$ is

$$
r_{kl} = \frac{w_kw_l}{(w_ke^{\epsilon/2} - w_le^{-\epsilon/2})(w_le^{\epsilon/2} - w_ke^{-\epsilon/2})} \prod_{i=1}^{M} \frac{w_k^{2}e^{\epsilon} - w_i^{2}e^{\epsilon}}{(w_ke^{\epsilon} - w_ie^{\epsilon})(w_k - w_i)} \times \prod_{j=M+1}^{2m} \frac{w_k^{2}e^{-2\epsilon} - w_j^{2}e^{2\epsilon}}{(w_l - w_j)(w_le^{-\epsilon} - w_je^{\epsilon})} .
$$

The total 2m-point function is then given by a sum over all the residues, symmetrized in the $w_k$’s:

$$
A_2 = (-1)^m \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^M \left[ r_{zz} + \sum_{l \notin \sigma_M} r_{zl} + \sum_{k \in \sigma_M} r_{kk} + \sum_{k \in \sigma_M} \sum_{l \notin \sigma_M} r_{kl} \right] .
$$

To evaluate $A_2$ as $\epsilon \to 0$, we Taylor expand $r_{zz}$, $r_{zl}$, $r_{kk}$ and $r_{kl}$. After some algebra, we find

$$
r_{zz} = \frac{1}{e^{\epsilon} - e^{-\epsilon}} = \frac{1}{2\epsilon} + O(\epsilon) ;
$$

$$
r_{zl} = \frac{1}{2\epsilon} f(w_l) + 2 \sum_{p \in \sigma_M} h(w_l, w_p) + \sum_{p \notin \sigma_M} h(w_l, w_p) + O(\epsilon) ;
$$

$$
r_{kk} = \frac{1}{2\epsilon} f(w_k) + \sum_{p \in \sigma_M \text{ and } p \neq k} h(w_k, w_p) + 2 \sum_{p \notin \sigma_M} h(w_k, w_p) + O(\epsilon) ;
$$

and

$$
r_{kl} = -H_{\sigma_M}(k, l) ,
$$

where $H_{\sigma_M}(k, l)$ is given by equation A.15.
where we have defined

\[ f(w) = \prod_{i=1}^{2m} \frac{w_i + w_i}{w_i - w_i} ; \quad (A.16) \]

\[ h(w_k, w_p) = \frac{w_k w_p}{(w_k - w_p)^2} \prod_{i=1}^{2m} \frac{w_k + w_i}{w_k - w_i} ; \quad (A.17) \]

and

\[ H_{\sigma_M}(k, l) = \frac{w_k w_l}{(w_k - w_l)^2} \prod_{i \in \sigma_M} \frac{w_k + w_i}{w_k - w_i} \prod_{j \notin \sigma_M} \frac{w_i + w_j}{w_l - w_j} . \quad (A.18) \]

We will also define the functions

\[ F = \sum_{j=1}^{2m} f(w_j) ; \quad (A.19) \]

\[ g(w_k) = \sum_{p=1, p \neq k}^{2m} h(w_k, w_p) ; \quad (A.20) \]

and

\[ G = \sum_{k=1}^{2m} g(w_k) . \quad (A.21) \]

Then we can write \( A_2 \) as the sum of the following three terms. The first comes from \( r_{zz} \) and is independent of all \( w_j \)'s and \( M \):

\[ A_{zz} = (-1)^{M} \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^{M} r_{zz} . \quad (A.22) \]

The second term comes from \( r_{zl} \) and \( r_{kk} \) and depends only on \( F \).

\[ A_F = (-1)^{M} \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^{M} \frac{1}{2\epsilon} F . \quad (A.23) \]

The last term depends on \( h, g, \) and \( H \). It is

\[ A_H = (-1)^{M} \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^{M} \left[ \sum_{i \notin \sigma_M} \sum_{p \in \sigma_M} h(w_i, w_p) + \sum_{k \in \sigma_M} \sum_{p \notin \sigma_M} h(w_k, w_p) \\
+ \sum_{k=1}^{2m} g(w_k) - \sum_{k \in \sigma_M} \sum_{l \notin \sigma_M} H(k, l) \right] . \quad (A.24) \]
First, we will evaluate $A_F$ by demonstrating that $F = 0$. $F$ is given by

$$F = \sum_{k=1}^{2m} \prod_{i=1, i \neq k}^{2m} \frac{w_k + w_i}{w_k - w_i}. \quad (A.25)$$

We can put all the terms in $F$ over a common denominator to obtain

$$F = \prod_{i>j} \frac{1}{w_i - w_j} \sum_k (-1)^k \left[ \prod_{i=1, i \neq k}^{2m} (w_k + w_i) \right] \left[ \prod_{i>j, i \neq k} (w_i - w_j) \right]. \quad (A.26)$$

Now we can use the fact that $\prod_{i>j=1; i,j \neq k}^{2m} (w_i - w_j)$ is the discriminant, which equals the Van der Monde determinant. Consequently, we can write it as

$$\prod_{i>j, i,j \neq k} (w_i - w_j) = \det M_k, \quad (A.27)$$

where $M_k$ is a $(2m - 1)$ by $(2m - 1)$ matrix with

$$(M_k)_{ij} = \begin{cases} (w_i)^{(j-1)} & \text{for } i < k; \\ (w_{i+1})^{(j-1)} & \text{for } i \geq k. \end{cases} \quad (A.28)$$

Then we can write $F$ as

$$F = \prod_{i>j} \frac{1}{w_i - w_j} \left\{ \sum_k (-1)^k \left[ \prod_{i \neq k}^{2m} (w_k + w_i) \right] \det M_k \right\}. \quad (A.29)$$

The expression in curly brackets looks like the determinant of the matrix $A$ when it is calculated by expanding the last row, where $A$ is given by

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ w_1 & w_2 & \ldots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{2m-2} & w_2^{2m-2} & \ldots & w_{2m}^{2m-2} \\ \Pi_{i \neq 1} (w_1 + w_i) & \Pi_{i \neq 2} (w_2 + w_i) & \ldots & \Pi_{i \neq 2m} (w_{2m} + w_i) \end{pmatrix}. \quad (A.30)$$

We can write the $k$th entry of the last row as

$$\prod_{i \neq 1} (w_k + w_i) = w_k^{2m-2} \left[ \sum_{i=1}^{2m} w_i \right] + w_k^{2m-4} \left[ \sum_{i<j<l} w_i w_j w_l \right] + \cdots + 1 \left[ \sum_{i=1}^{2m} \prod_{j \neq i} w_j \right]. \quad (A.31)$$

We can use the fact that $\prod_{i>j=1; i,j \neq k}^{2m} (w_i - w_j)$ is the discriminant, which equals the Van der Monde determinant. Consequently, we can write it as

$$\prod_{i>j, i,j \neq k} (w_i - w_j) = \det M_k, \quad (A.27)$$
From this equation we can see that the last row of $A$ is equal to a linear combination of the first $2m - 1$ rows of $A$, so $\det A = 0$. Therefore, since $F \propto \det A$, $F$ is zero. Then we can conclude that the contribution, $A_F$, to the $2m$-point function is also zero.

Next, we will evaluate $A_{zz}$ by performing the sum

$$S = \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^M .$$  \hspace{1cm} ((A.20) (A.20) equationA.32 equationA.32 A.32)

The sum over $\sigma_M$ is a sum over all ways to choose $M$ objects from a set of $2m$ objects, so

$$S = \sum_{M=0}^{2m} \binom{2m}{M} (-1)^M = (1 - 1)^{2m} = 0 .$$  \hspace{1cm} ((A.20) (A.20) equationA.33 equationA.33 A.33)

Then

$$A_{zz} = -(-1)^m r_{zz} S = 0 .$$  \hspace{1cm} ((A.20) (A.20) equationA.34 equationA.34 A.34)

Lastly, we must evaluate $A_H$. We will begin by proving the identity

$$\sum_{k \in \pi} H_\pi(k, l) = \sum_{k \in \pi} h(w_l, w_k) ,$$  \hspace{1cm} ((A.20) (A.20) equationA.35 equationA.35 A.35)

where $\pi$ is a subset of $\{1, \ldots, 2m\}$ containing $M$ elements, and $l \notin \pi$. Using the definitions of $H_\pi$ and $h$, we can write

$$\sum_{k \in \pi} H_\pi(k, l) = \left[ w_l \prod_{j \notin \pi, j \neq l} \frac{w_l + w_j}{w_l - w_j} \prod_{i \in \pi} \frac{1}{(w_i - w_l)^2} \right] \sum_{k \in \pi} w_k \prod_{i \in \pi, i \neq k} \frac{w_k - w_i}{w_k - w_i} (w_i - w_l)^2 ;$$  \hspace{1cm} ((A.20) (A.20) equationA.36 equationA.36 A.36)

and

$$\sum_{k \in \pi} h(w_l, w_k) = \left[ w_l \prod_{j \notin \pi, j \neq l} \frac{w_l + w_j}{w_l - w_j} \prod_{i \in \pi} \frac{1}{(w_i - w_l)^2} \right] \sum_{k \in \pi} w_k \prod_{i \in \pi, i \neq k} (w_l + w_i)(w_l - w_i) .$$  \hspace{1cm} ((A.20) (A.20) equationA.37 equationA.37 A.37)

From these two equations, we can conclude that

$$\sum_{k \in \pi} H_\pi(k, l) = \sum_{k \in \pi} h(w_l, w_k) \hspace{1cm} \text{if and only if}$$

((A.20) (A.20) equationA.39 page37 A.39)
\[
\sum_{k \in \pi} w_k \prod_{i \in \pi, i \neq k} \frac{w_k + w_i}{w_k - w_i} (w_i - w_l)^2 = \sum_{k \in \pi} w_k \prod_{i \in \pi, i \neq k} (w_i^2 - w_k^2) .
\]

Subtracting the right-hand side from the left-hand side of this equation and simplifying, we find that the condition becomes

\[
0 = \left[ 2^{M-1} \prod_{i \in \pi} (w_i - w_l) \right] \sum_{k \in \pi} (w_k - w_l)^{M-2} \prod_{i \in \pi, i \neq k} \frac{1}{w_k - w_i} .
\]

Because the expression in square brackets does not equal zero for arbitrary values of \( w \), the above expression is true if and only if

\[
0 = \sum_{k \in \pi} (w_k - w_l)^{M-2} \prod_{i \in \pi, i \neq k} \frac{1}{w_k - w_i} = B .
\]

To make the argument simpler, we will take \( \pi = \{1, 2, \ldots, M\} \). Then we can write the expression on the right-hand side of the above equation as

\[
B = \left( \prod_{i, j \in \pi, i > j} \frac{1}{w_i - w_j} \right) \sum_{k \in \pi} (-1)^{M+k}(w_k - w_l)^{M-2} \prod_{i \in \pi, i > j, i \neq k} (w_i - w_j)
\]

\[
= \prod_{i, j \in \pi, i > j} \frac{1}{w_i - w_j} \sum_{k \in \pi} (-1)^{M+k}(w_k - w_l)^{M-2} \det M_k ,
\]

where \( M_k \) is an \((M-1)\) by \((M-1)\) matrix defined as in equation (A.28). The sum in the expression for \( B \) is just the row expansion of the following determinant:

\[
\det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{w_1} & \frac{1}{w_2} & \cdots & \frac{1}{w_M} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{w_1} & \frac{1}{w_2} & \cdots & \frac{1}{w_M} \\
(w_1 - w_l)^{M-2} & (w_2 - w_l)^{M-2} & \cdots & (w_M - w_l)^{M-2}
\end{pmatrix} = 0 .
\]

The last row in the matrix is a linear combination of the other rows, so \( B = 0 \). As a result, we have shown that the identity in equation (A.35) is true.

Using this identity, we can write \( A_H \) as

\[
A_H = (-1)^m \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^M \left[ G + \sum_{k \in \sigma_M} \sum_{p \notin \sigma_M} h(w_k, w_p) \right] .
\]
Because \( G \) is independent of \( M \) and \( \sigma_M \), we can perform the sum over \( G \) using equations (A.32) and (A.33) to obtain

\[
A_H = (-1)^m \sum_{M=0}^{2m} \sum_{\sigma_M} (-1)^M \sum_{k \in \sigma_M} \sum_{p \notin \sigma_M} h(w_k, w_p).
\]

When we sum over all subsets, \( \sigma_M \), the pair \((w_k, w_p)\) will take on all possible values, and each particular \((w_k, w_p)\) will occur \( \binom{2m}{M-1} \) times. Then

\[
A_H = (-1)^m \sum_{M=1}^{2m-1} \binom{2m-2}{M-1} (-1)^M \sum_{k, p=1 \atop k \neq p}^{2m} h(w_k, w_p)
= (-1)^m (1 - 1)^{2m-2} G,
\]

so \( A_H = 0 \) for \( 2m > 2 \). Therefore, the \( 2m \)-point functions for \( 2m > 2 \) are all zero as \( \epsilon \to 0 \) at order \( V_0^2 \).

The calculation for the \( O(V_0^4) \) contribution to the four-point function is similar but longer. There are two key differences. The first is that we must be careful to subtract out the disconnected diagrams. The second is that we must also make use of the identity

\[
\sum_{\sigma} \prod_i \frac{1}{(w_{\sigma(i)} - w_{\sigma(i+1)})} = 0,
\]

where the sum is over all permutations, \( \sigma \), of \( 2m \) elements, with \( \sigma(2m+1) = \sigma(1) \).

In all the \( m \)-point function calculations we have done, the only function that we have obtained that is SU(1,1) invariant is \( \prod_i w_i/(w_i - w_{i+1}) \). For example, \( F \) and \( G \) are not SU(1,1) invariant. Additionally, the \( m \)-point functions of \( \dot{X}(t_i) \) must be symmetric under interchange of the \( t_i \) (at least when \( \alpha = 1 \)). We find that when we symmetrize the SU(1,1) invariant function, we get 0. This suggests that \( \langle \dot{X}(t_1) \ldots \dot{X}(t_{2m}) \rangle = 0 \) to all orders in \( V_0 \), but does not guarantee it.

**Appendix (A.20) equation A.1 appendix BB. Ward Identity Check**

In Section 6, we noted that there is a complete set of “broken reparametrization invariance” Ward identities which must be satisfied by the free energy and correlation functions in the theory \( \Pi \). These can be derived from the condition that the string theory boundary state to which our theory corresponds must have reparametrization invariance. This condition is implemented by requiring (2.8) to be satisfied, and in this appendix, for
zero magnetic field, we check the simplest form of this identity in order to show that our
renormalization scheme is correct. It turns out that we need to know the charged sector
contributions to the free energy and to the two-point function. Both can be read off from
the results in Section 6.

A special case of (2.8) is

\[(L_1 - \tilde{L}_{-1})|B\rangle = 0,\]

and it is shown in [10] that this is equivalent to the following condition on the connected
diagram generating functional \(W\):

\[
\sum_{m = -\infty}^{\infty} (m + 1)\alpha_{-m} \frac{\partial W}{\partial \alpha_{m-1}} = il \left[ \frac{\partial^2 W}{\partial X_0 \partial \alpha_{-1}} - \frac{\partial W}{\partial \alpha_{-1}} \frac{\partial W}{\partial X_0} \right],
\]

where the \(\alpha_m\) are the closed string oscillators (see (2.7)), and \(l = \sqrt{2\alpha'}\). If we differentiate
with respect to \(\alpha_p\) and then set each \(\alpha_m = 0\) we get (using the fact that the one-point
function \(\frac{\partial W}{\partial \alpha_m} = 0\) on any solution)

\[
\left. \frac{\partial}{\partial X_0} \frac{\partial^2 W}{\partial \alpha_p \partial \alpha_{-1}} - \frac{\partial^2 W}{\partial \alpha_p \partial \alpha_{-1}} \frac{\partial W}{\partial X_0} \right|_{\alpha_m = 0} = 0.
\]

Now \(\frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_{-1}}\) is the first Fourier mode of \(\langle \dot{X}(\tau) \dot{X}(0) \rangle\), so (B.3) with \(p = 1\) is equivalent to
(we take \(T = 2\pi\))

\[
\frac{\partial}{\partial X_0} \int_0^{2\pi} \frac{d\tau}{2\pi} e^{i\tau} \langle \dot{X}(\tau) \dot{X}(0) \rangle = \left( \int_0^{2\pi} \frac{d\tau}{2\pi} e^{i\tau} \langle \dot{X}(\tau) \dot{X}(0) \rangle \right) \frac{\partial W}{\partial X_0}.
\]

This is the equation we must verify. We now substitute the calculated forms for \(W\) (equal
to \(TF\) when the sources are set to zero) and \(\langle \dot{X}(\tau) \dot{X}(0) \rangle\) from equations (6.7), (6.15) and
(6.16) to check whether (B.4) is satisfied. The \(m\)th Fourier mode of the two-point function
for \(m > 0\) is

\[
\int_0^{2\pi} \frac{d\tau}{2\pi} e^{im\tau} \langle \dot{X}(\tau) \dot{X}(0) \rangle = m - V_r \cos X_0
\]

\[
+ \frac{V^2}{4} \left[ 2\cos 2X_0 \delta_{m,1} + 4\sin^2 X_0 - m \right] + \cdots
\]

\[(A.20)(A.20)equationB.5equationB.5B.5\]
and for \( m = 1 \) the result is

\[
\int_0^{2\pi} \frac{d\tau}{2\pi} e^{i\tau} \left\langle \dot{X}(\tau) \dot{X}(0) \right\rangle = 1 - V_r \cos X_0 + \frac{1}{4} V_r^2 + \cdots .
\]

The generating functional is given by

\[
W = -V_r \cos X_0 - \frac{1}{2} V_r^2 \left( \frac{1}{4 \epsilon} - \sin^2 X_0 \right) + O(V_r^3),
\]

and it is easy to see that (B.4) is satisfied to \( O(V_r^2) \). Note that the contact terms played a crucial role in this calculation, confirming that they are an essential part of the physical content of the critical theory. This Ward identity represents a very non-trivial check that our renormalization scheme is consistent.
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