Co-Betweenness: A Pairwise Notion of Centrality

Eric D. Kolaczyk, David B. Chua, and Marc Barthélemy

1Dept. of Mathematics and Statistics
Boston University
Boston, MA, USA
2Boston, MA

3CEA-Centre d’Etudes de Bruyères-le-Châtel, Département de Physique
Théorique et Appliquée BP12, 91680 Bruyères-Le-Châtel, France

(Dated: February 2, 2008)

Betweenness centrality is a metric that seeks to quantify a sense of the importance of a vertex in a network graph in terms of its ‘control’ on the distribution of information along geodesic paths throughout that network. This quantity however does not capture how different vertices participate together in such control. In order to allow for the uncovering of finer details in this regard, we introduce here an extension of betweenness centrality to pairs of vertices, which we term co-betweenness, that provides the basis for quantifying various analogous pairwise notions of importance and control. More specifically, we motivate and define a precise notion of co-betweenness, we present an efficient algorithm for its computation, extending the algorithm of [1] in a natural manner, and we illustrate the utilization of this co-betweenness on a handful of different communication networks. From these real-world examples, we show that the co-betweenness allows one to identify certain vertices which are not the most central vertices but which, nevertheless, act as important actors in the relaying and dispatching of information in the network.

I. INTRODUCTION

In social network analysis, the problem of determining the importance of actors in a network has been studied for a long time (see, for example, [2]). It is in this context that the concept of the centrality of a vertex in a network emerged. There are numerous measures that have been proposed to numerically quantify centrality which differ both in the nature of the underlying notion of vertex importance that they seek to capture, and in the manner in which that notion is encoded through some functional of the network graph. See [3], for example, for a recent review and categorization of centrality measures.

Paths – as the routes by which flows (e.g., of information or commodities) travel over a network – are fundamental to the functioning of many networks. Therefore, not surprisingly, a number of centrality measures quantify importance with respect to the sharing of paths in the network. One popular measure is betweenness centrality. First introduced in its modern form by [4], the betweenness centrality is essentially a measure of how many geodesic (i.e., shortest) paths run over a given vertex. In other words, in a social network for example, the betweenness centrality measures the extent to which an actor “lies between” other individuals in the network, with respect to the network path structure. As such, it is a measure of the control that actor has over the distribution of information in the network.

The betweenness centrality – as well as all other centrality measures of which we are aware – is defined specifically with respect to a single given vertex. In particular, vertex centralities produce an ordering of the vertices in terms of their individual importance, but do not provide insight into the manner in which vertices act together in the spread of information across the network. Insight of this kind can be important in presenting an appropriately more nuanced view of the roles of the different vertices, beyond their individual importance. A first natural extension of the idea of centrality in this manner is to pairs of vertices.

In this paper, we introduce such an extension, which we term the co-betweenness centrality, or simply the co-betweenness. The co-betweenness of two vertices is essentially a measure of how many geodesic paths are shared by the vertices, and as such provides us with a sense of the interplay of vertices across the network. For example, the co-betweenness alone quantifies the extent to which pairs of vertices jointly control the distribution of information in the network. Alternatively, a standardized version of co-betweenness produces a well-defined measure of correlation between flows over the two vertices. Finally, an alternative normalization quantifies the extent to which one vertex controls the distribution of information to another vertex.

This paper is organized as follows. In Section II, we briefly review necessary technical background. In Section III, we provide a precise definition for the co-betweenness and related measures, and motivate each in the context of an Internet communication network. An algorithm for the efficient computation of co-betweenness, for all pairs of vertices in a network, is sketched in Section IV and its properties are discussed. In Section V, we further illustrate our measures using two social networks whose ties are reflective of communication. Some additional discussion is provided in Section VI. Finally, a formal description of our algorithm, as well as pseudo-code, may be found in the appendix.
II. BACKGROUND

Let $G = (V, E)$ denote an undirected, connected network graph with $n_v$ vertices in $V$ and $n_e$ edges in $E$. A walk on $G$, from a vertex $v_0$ to another vertex $v_l$, is an alternating sequence of vertices and edges, say $(v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l)$, where the endpoints of $e_i$ are $(v_{i-1}, v_i)$. The length of this walk is said to be $l$. A trail is a walk without repeated edges, and a path, a trail without repeated vertices. A shortest path between two vertices $u, v \in V$ is a path between $u$ and $v$ whose length $\ell$ is a minimum. Such a path is also called a geodesic and its length, the geodesic distance between $u$ and $v$. In the case that the graph $G$ is weighted i.e., there is a collection of edge weights $\{w_e\}_{e \in E}$, where $w_e \geq 0$, shortest paths may be instead defined as paths for which the total sum of edge weights is a minimum. In the material that follows, we will restrict our exposition primarily to the case of unweighted graphs, but extensions to weighted graphs are straightforward. For additional background of this type, see, for example, the textbook [5].

Let $\sigma_{st}$ denote the total number of shortest paths that connect vertices $s$ and $t$ (with $\sigma_{ss} \equiv 1$), and let $\sigma_{st}(v)$ denote the number of shortest paths between $s$ and $t$ that also run over vertex $v$. Then we define the betweenness centrality of a vertex $v$ as a weighted sum of the number of paths through $v$,

$$B(v) = \sum_{s, t \in V \setminus \{v\}} \frac{\sigma_{st}(v)}{\sigma_{st}}. \quad (1)$$

Note that this definition excludes the shortest paths that start or end at $v$. However, in a connected graph we will have $\sigma_{st}(v) = \sigma_{st}$ whenever $s = v$ or $t = v$, so the exclusion amounts to removing a constant term that would otherwise be present in the betweenness centrality of every vertex.

As an illustration, which we will use throughout this section and the next, consider the network in Figure 1. This is the Abilene network, an Internet network that is part of the Internet2 project [7], a research project devoted to development of the ‘next generation’ Internet. It serves as a so-called ‘backbone’ network for universities and research labs across the United States, in a manner analogous to the federal highway system of roads. We use this network for illustration because, as a technological communication network, the notions of connectivity, information, flows, and paths are all explicit and physical, and hence facilitate our initial discussion of betweenness and co-betweeness. Later, in Section V we will illustrate further with two communication networks from the social network literature.

The information traversing this network takes the form of so-called ‘packets’, and the packets flow between origins and destinations on this network along paths strictly determined according to a set of underlying routing protocols (Technically, the Abilene network is more accurately described by a directed graph. But, given the fact that routing is typically symmetric in this network, we follow the Internet2 convention of displaying Abilene using an undirected graph.). A reasonable first approximation of the routing of information in this network is with respect to a set of unique shortest paths. In this case, the betweenness $B(v)$ of any given vertex $v \in V$ will be exactly equal to the number of shortest paths through $v$. The vertices in Figure 1 correspond to metropolitan regions, and have been laid out roughly with respect to their true geographical locations. Intuitively and according to earlier work on centrality in spatial networks [7], one might suspect that vertices near the central portion of the network, such as Denver or Indianapolis, have larger betweenness, being likely forced to support most of the flows of communication between east and west. We will see in Section III that such is indeed the case.

Until recently, standard algorithms for computing betweenness centralities $B(v)$ for all vertices in a network had $O(n_v^3)$ running times, which was a stumbling block to their application in large-scale network analyses. Faster algorithms now exist, such as those introduced in [1], which have running time of $O(n_v n_e)$ on unweighted networks and $O(n_v n_e + n_v^2 \log n_v)$ on weighted networks, with an $O(n_v + n_e)$ space requirement. These improvements derive from exploiting a clever recursive relation for the partial sums $\sum_{v \in V} \sigma_{st}(v) / \sigma_{st}$. As we will see, the need for efficient algorithms is even more important in the case of the co-betweeness, and we will make similar usage of recursions in developing an efficient algorithm for computing this quantity.

III. CO-BETWEENNESS

We extend the concept of vertex betweenness centrality to pairs of vertices $u$ and $v$ by letting $\sigma_{st}(u, v)$ denote the number of shortest paths between vertices $s$ and $t$ that pass through both $u$ and $v$, and defining the vertex co-
Let \( \Omega = \sum_{s,t} \sigma_{st} \mathbf{x} \mathbf{y}^T \) be the stochastic routing matrix. When shortest paths are not unique, the same results hold if the matrix \( \sigma_{st} \) is considered as a random variable, with uncorrelated elements, then its covariance matrix is simply equal to the \( n_p \times n_p \) identity matrix. The elements of \( \mathbf{y} \), however, will be correlated, and their covariance matrix takes the form \( \Omega = \mathbf{RR}^T \), by virtue of the linear relation between \( \mathbf{y} \) and \( \mathbf{x} \). Importantly, note that the diagonal elements of \( \Omega \) are the betweelessness \( B(v) \). Furthermore, the off-diagonal elements are the co-betweelessness \( C(u,v) \). When shortest paths are not unique, the same results hold if the matrix \( R \) is expanded so that each shortest path between a pair of vertices \( s \) and \( t \) is afforded a separate column, and the non-zero entries of each such column have the value \( \sigma^{-1}_{st} \), rather than \( 1 \). In this case, \( R \) may be interpreted as a stochastic routing matrix.

To illustrate, in Figure 2 we show a network graph representation of the matrix \( \Omega \) for the Abilene network. The vertices are again placed roughly with respect to their actual geographic location, but are now drawn in proportion to their betweelessness. Edges between pairs of vertices now represent non-zero co-betweelessness for the pair, and are also drawn with a thickness in proportion to their value. A number of interesting features are evident from this graph. First, we see that, as surmised earlier, the more centrally located vertices tend to have the largest betweelessness values. And it is these vertices that typically are involved with the larger co-betweelessness values. Since the paths going through both a vertex \( s \) and a vertex \( t \) are a subset of the paths going through either one or the other, this tendency for large co-betweelessness to associate with large betweelessness should not be a surprise. Also note that the co-betweelessness values tend to be smaller between vertices separated by a larger geographical distance, which again seems intuitive.

Somewhat more surprising perhaps, however, is the manner in which the network becomes disconnected. The Seattle vertex is now isolated, as there are no paths that route through that vertex – only to and from. Additionally, the vertices Houston, Atlanta, and Washington now form a separate component in this graph, indicating that information is routed on paths running through both the first two and the last two, but not through all three, and also not through any of these and some other vertex. Overall, one gets the impression of information being routed primarily over paths along the upper portion of the network in Figure 1. A similar observation has been made in [8], using different techniques.

While the raw co-betweelessness values appear to be quite informative, one can imagine contexts in which it would be useful to compare co-betweelessness’ across pairs of vertices in a manner that adjusts for the unequal betweelessness of the participating vertices. The value

\[
C_{\text{corr}}(u,v) = \frac{C(u,v)}{\sqrt{B(u)B(v)}},
\]

is a natural candidate for a standardized version of the co-betweelessness in (2), being simply the corresponding entry of the correlation matrix deriving from \( \Omega = \mathbf{RR}^T \).

Figure 3 shows a network graph representation of the quantities in \( C_{\text{corr}} \) for the Abilene network, with edges again drawn in proportion to the values and vertices now naturally all drawn to be the same size. Much of this network looks like that in Figure 2. The one notable exception is that the magnitude of the values between the three vertices in the lower subgraph component are now of a similar order to most of the other values in the other component. This fact may be interpreted as indicating that among themselves, adjusting for the lower levels of information flowing through this part of the network, these vertices are as strongly ‘correlated’ as many of the others.

The co-betweelessness may also be used to define a directed notion of the strength of pairwise relationships. Let

\[
C(u|v) = \frac{C(u,v)}{B(v)}
\]
denote the relative proportion of shortest paths through $v$ that also go through $u$. This quantity may be interpreted as a measure of the control that vertex $v$ has over the information that passes through vertex $u$. Alternatively, under uniqueness of shortest paths, if from among the set of shortest paths through $v$ one is chosen uniformly at random, the value $C(u|v)$ is the probability that the chosen path will also go through $u$. We call $C(u|v)$ the conditional betweenness of $u$, given $v$. Note that, in general, $C(u|v) \neq C(v|u)$.

Figure 4 shows a graph representation of the values $C(u|v)$ for the Abilene network. Due to the asymmetry of these values in $u$ and $v$, arcs are used, rather than edges, with an arc from $v$ to $u$ corresponding to $C(u|v)$. The thickness of the arcs is proportional to these values, and is therefore indicative of the control exercised on the vertex at the tail by the vertex at the head. For improved visualization, we have used a simple circular layout for the vertices. Examination of this figure shows symmetry in the relationships between some pairs of vertices, but a strong asymmetry between most others. For example, vertices like Indianapolis, which were seen previously to have a large betweenness, clearly exercise a strong degree of control over almost any other vertices with which they share paths. More interestingly, note that certain vertices that are neighbors in the original Abilene network have more symmetric relationships than others. The conditional betweenness’ for Atlanta and Washington, DC, are fairly similar in magnitude, while those for Los Angeles and Sunnyvale are quite dissimilar, with the latter evidently exercising a noticeably greater degree of control over the former.

IV. COMPUTATION OF CO-BETWEENNESS

We discuss here the calculation of the co-betweenness values $C(u,v)$ in (2), for all pairs $(u,v)$, from which the other quantities in (3) and (4) follow trivially. At a first glance, it would appear that an algorithm of $O(n^4)$ running time is necessary, given that the number of vertex pairs grows as the square of the number of vertices. Such an implementation would render the notion of co-betweenness infeasible to implement in any but network graphs of relatively modest size. However, exploiting ideas similar to those underlying the algorithms of [1] for calculating the betweenness’ $B(v)$, a decidedly more efficient implementation may be obtained, as we now describe briefly. Details may be found in the appendix.

Our algorithm for computing co-betweenness involves a three-stage procedure for each vertex $v \in V$. In the first stage, we perform a breadth-first traversal of the network graph $G$, to quickly compute intermediary quantities such as $\sigma_{sv}$, the number of shortest paths from a source $s$ to each other vertex $v$ in the network; in the process we form a directed acyclic graph that contains all shortest paths leading from vertex $s$. In the second stage, we iterate through each vertex in order of decreasing distance from $s$ and compute a score $\delta_s(v)$ for each vertex that is related to its contribution to the co-betweenness. These contributions are then aggregated in a depth-first traversal of the directed acyclic graph, which is carried out in the third and final stage.

In order to compute the number of shortest paths $\sigma_{sv}$ in the first stage, we note that the number of shortest paths from $s$ to a vertex $v$ is the sum of all shortest paths to each parent of $v$ in the directed acyclic graph rooted at $s$, namely,

$$\sigma_{sv} = \sum_{t \in \rho(v)} \sigma_{st}. \quad (5)$$

In the case of an undirected graph, this can be computed in the course of a breadth-first search with a running time of $O(n_e)$. 
In the second stage, we compute $\delta_s(v)$ using the recursive relation established in Theorem 6 of [1],

$$\delta_s(v) = \sum_{w \in C_s(w)} \frac{\sigma_{sv}}{\sigma_{sw}} \left(1 + \delta_s(w)\right),$$

where $c_s(v)$ denotes the set of child vertices of $v$ in the directed acyclic graph rooted at $s$.

Finally, in the third stage, we compute the co-betweennesses by interpreting the relation

$$C(u, v) = \sum_{s \in V \setminus \{u, v\}} \frac{\delta_s(v)}{\sigma_{sv}} \sigma_{su}(u)$$

as assigning a contribution of $\frac{\delta_s(v)}{\sigma_{sv}}$ to $C(u, v)$ for each of the $\sigma_{sv}(u)$ shortest paths to $v$ that run through $u$. We accumulate these contributions at each step of the depth-first traversal when we visit a vertex $v$ by adding $\frac{\delta_s(v)}{\sigma_{sv}}$ to $C(u, v)$ for every ancestor $u$ of the current vertex $v$.

Our proposed algorithms exploit recursions analogous to those of [1] to produce run-times that are in the worst case $O(n^3)$, but in empirical studies were found to vary like $O(n_e n_c + n_v^2 \log n_v)$ in general, or $O(n_v^2 \log n_v)$ in the case of sparse graphs. Here $p$ is related to the total number of shortest paths in the network and seems to lie comfortably between 0.1 and 0.5 in our experience. In the case of unique shortest paths, it may be shown rigorously that the running time reduces to $O(n_v n_e + n_v^2 \log n_v)$, and $O(n_v^2 \log n_v)$ if the network is sparse as well as ‘small-world’ (i.e., with diameter of size $O(\log n_v)$). See the appendix for details.

V. ADDITIONAL ILLUSTRATIONS

We provide in this section additional illustration of the use of co-betweenness, based on two other networks graphs. Both graphs originally derive from social network analyses in which one goal was to understand the flow of certain information among actors.

A. Michael’s Strike Network

Our first illustration involves the strike dataset of [3], which is also analyzed in detail in Chapter 7 of [10]. New management took over at a forest products manufacturing facility, and this management team proposed certain changes to the compensation package of the workers. The changes were not accepted by the workers, and a strike ensued, which was then followed by a halt in negotiations. At the request of management, who felt that the information about their proposed changes was not being communicated adequately, an outside consultant analyzed the communication structure among 24 relevant actors.

The social network graph in Figure 5 represents the communication structure among these actors, with an edge between two actors indicating that they communicated at some minimally sufficient level of frequency about the strike. Three subgroups are present in the network: younger, Spanish-speaking employees (black vertices), and older, English-speaking employees (white vertices). The two union negotiators, Sam and Wendle, are indicated by asterix’ next to their names. Edges indicate that the two incident actors communicated at some minimally sufficient level of frequency about the strike.

![FIG. 5: Original strike-group communication network of [3]. Three subgroups are represented in this network: younger, Spanish-speaking employees (black vertices), and older, English-speaking employees (white vertices). The two union negotiators, Sam and Wendle, are indicated by asterix’ next to their names. Edges indicate that the two incident actors communicated at some minimally sufficient level of frequency about the strike.](image-url)
FIG. 6: Co-betweenness for the strike-group communication network. Actors located apart from the network, in the corners, are isolated under this representation, as they have zero betweenness and hence no co-betweenness with any other actors. (Note: Isolated vertices are drawn to have unit diameter, and not in proportion to their (zero) betweenness.)

A plot of the standardized co-betweenness $C_{corr}$ shows similar patterns overall, and we have therefore not included it here. The conditional betweenness $C(u|v)$ for this network primarily shows most of the actors with large arcs pointing to Bob and Norm, and much smaller arcs pointing the opposite direction. This pattern further confirms the influence that these two actors can have on the other actors in the communication process. However, there are also some interesting asymmetrical relationships among the actors with smaller parts. For example, consider Figure 7, which shows the conditional betweenness among the older English-speaking employees. Ultrecht, for example, clearly has potential for a large amount of control on the communication of information passing through Russ, and similarly, Karl, on that through John.

FIG. 7: Conditional co-betweenness for the older English-speaking actors in the strike-group communication network.

FIG. 8: Karate club network of [11]. The gray vertices represent members of one of the two smaller clubs and the white vertices represent members who went to the other club. The edges are drawn with a width proportional to the number of situations in which the two members interacted.

B. Zachary’s Karate Club Network

Our second illustration uses the karate club dataset of [11]. Over the course of a couple of years in the 1970s, Zachary collected information from the members of a university karate club, including the number of situations (both inside and outside of the club) in which interactions occurred between members. During the course of this study, there was a dispute between the club’s administrator and the principal karate instructor. As a result, the club eventually split into two smaller clubs of approximately equal size—one centered around the administrator and the other centered around the instructor.

Figure 8 displays the network of social interactions between club members. The gray vertices represent mem-
FIG. 9: Co-betweenness for the karate club network. Actors in the upper-left and lower-right corners, separated from the connected component, are isolated due to zero betweenness. The two actors in the lower right-hand corner (i.e., a5 and a11) have non-zero betweenness, but are bridges, in the sense that they only serve to connect to other vertices, and hence have zero co-betweenness. (Note: The vertices for actors with zero betweenness are drawn to have unit diameter, for purposes of visibility.)

members of one of the two smaller clubs and the white vertices represent members who went to the other club. The edges are drawn with a width proportional to the number of situations in which the two members interacted. The graph clearly shows that the original club was already polarized into two groups centered about actors 1 and 34, who were the key players in the dispute that split the club in two.

The co-betweenness for this network is shown in Figure 9. As in Figure 8, the layout is done using an energy minimization algorithm. Again, as in our other examples, the co-betweenness entries are dominated by a handful of larger values. As might be expected, actors 1 and 34, who were at the center of the dispute, have the largest betweenness centralities and are also involved in the largest co-betweenness. More interesting, however, is the fact that these two actors have a large co-betweenness with each other—despite not being directly connected in the original network graph. This indicates that they are nevertheless involved in connecting a large number of other pairs—probably through key intermediaries such as actors 3 and 32. These latter two actors, while certainly not cut-vertices, nevertheless seem to operate like conduits between the two groups, quite likely due to their direct ties to both actor 1 and either of actors 33 and 34, the latter of which are both central to the group of white vertices. The co-betweenness for actors 1 and 32 is in fact the largest in the entire network.

Also of potential interest are the 14 vertices that are isolated from the network in the co-betweenness representation. Some of these vertices, such as actor 8, have strong social interactions with certain other actors (i.e., with actors 1, 2, 3 and 4), but evidently play a peripheral role in the communication patterns of the network, as evidenced by their lack of betweenness. Alternatively, there are the vertices like those representing actors 5 and 11, who have some betweenness centrality but nonetheless find themselves cut off from the connected component in the co-betweenness graph. An examination of the definition of the co-betweenness tells us that such vertices must be bridge-vertices, in the sense that they only serve to connect pairs of other vertices, i.e., they only occur in the middle of paths of length two.

VI. DISCUSSION

We introduced in this paper the notion of co-betweenness as a natural and interpretable metric for quantifying the interplay between pairs of vertices in a network graph. As we discussed in different real world examples, this quantity has several interesting features. In particular, unlike the usual betweenness centrality which orders the vertices according to their importance in the information flow on the network, the co-betweenness gives additional information about the flow structure and the correlations between different actors. Using this quantity, we were able to identify vertices which are not the most central ones, but which however play a very important role in relaying the information and which therefore appear as crucial vertices in the control of the information flow.

In principle, of course, one could continue to define higher-order analogues, involving three or more vertices at a time. But the computational requirements associated with calculating such analogues would soon become burdensome. In the case of triplets of vertices, one can expect algorithms analogous to those presented here to scale no better than $O(n^3)$. Additionally, we remark that, in keeping with the statistics analogy made in Section III, it is likely that the pairwise ‘correlations’ picked up by co-betweenness captures to a large extent the more important elements of vertex interplay in the network, with respect to shortest paths.

Following the tendencies in the statistical physics literature on complex networks [12, 13], it can be of some interest to explore the statistical properties of co-betweenness in large-scale networks. Some work in this direction may be found in [14], where co-betweenness and functions thereof were examined in the context of standard network graph models. The most striking properties discovered were certain basic scaling relations with distance between vertices.

On a final note, we point out that, while our discussion here has been focused on co-betweenness for pairs of vertices in unweighted graphs, we have also developed the analogous quantities and algorithms for vertex co-betweenness on weighted graphs and for edge co-betweenness on unweighted and weighted graphs. Also
see [8], where a result is given relating edge betweenness to the eigen-values of the matrix edge-betweenness ‘covariance’ matrix, defined in analogy to the matrix \( \Omega \) in Section III.

This appendix contains details specific to the proposed algorithm for computing co-betweenness, including a derivation of key expressions, a rough analysis of algorithmic complexity. The pseudo-codes can be found at the address [15]. Actual software implementing our algorithm, written in the MATLAB software environment, is available at [16].

APPENDIX A: DERIVATION OF KEY EXPRESSIONS

Central to our algorithm are the expressions in [4] and [7], the derivations for which we present here. Before doing so, however, we need to introduce some definitions and relations. First note that a simple combinatorial argument will show that

\[
\sigma_{st}(v) = \begin{cases} 
\sigma_{sv} \sigma_{vt} & \text{if } d(s,t) = d(s,v) + d(v,t), \\
0 & \text{otherwise},
\end{cases} 
\]

(A1)

and

\[
\sigma_{st}(u,v) = \begin{cases} 
\sigma_{su} \sigma_{uv} \sigma_{vt} & \text{if } d(s,t) = d(s,u), \\
\sigma_{sv} \sigma_{su} \sigma_{ut} & \text{if } d(s,t) = d(s,v), \\
0 & \text{otherwise},
\end{cases} 
\]

(A2)

For the sake of notational simplicity, we will assume, without loss of generality, that

\[
d(s,u) \leq d(s,v). 
\]

(A3)

for the remainder of this discussion.

The remaining quantities we need to introduce are notions of the path-dependency of vertices. In the spirit of [1], we define the “dependency” of vertices \( s \) and \( t \) on the vertex pair \((u,v)\) as

\[
\delta_{st}(u,v) = \frac{\sigma_{st}(u,v)}{\sigma_{st}},
\]

(A4)

and we define the dependency of \( s \) alone on the pair of vertices \((u,v)\) as

\[
\delta_{s}(u,v) = \sum_{t \in V \setminus \{u,v\}} \delta_{st}(u,v) = \sum_{t \in V \setminus \{u,v\}} \frac{\sigma_{st}(u,v)}{\sigma_{st}}.
\]

(A5)

Similarly, we define the pair-wise dependency of \( s \) and \( t \) on a single vertex \( v \) as

\[
\delta_{st}(v) = \frac{\sigma_{st}(v)}{\sigma_{st}},
\]

(A6)

and the dependency of \( s \) alone on \( v \) as

\[
\delta_{s}(v) = \sum_{t \in V \setminus \{v\}} \delta_{st}(v) = \sum_{t \in V \setminus \{v\}} \frac{\sigma_{st}(v)}{\sigma_{st}}.
\]

(A7)

Note that unlike [1], we exclude \( t = v \) from the sum in (7). Two relations that follow immediately from these definitions, combined with (A1) and (A2), are

\[
\sigma_{st}(u,v) = \sigma_{su} \sigma_{uv} \sigma_{vt} = \sigma_{st}(u) \sigma_{vt} = \sigma_{sv}(u) \sigma_{st},
\]

(A8)

and

\[
\delta_{st}(u,v) = \frac{\sigma_{st}(u,v)}{\sigma_{st}} = \frac{\sigma_{sv}(u) \sigma_{st}(v)}{\sigma_{st}} = \delta_{sv}(u) \delta_{st}(v).
\]

(A9)

These two relations allow us to show that

\[
\delta_{s}(u,v) = \sum_{t \in V \setminus \{u,v\}} \delta_{st}(u,v)
\]

(A10)

\[
= \sum_{t \in V \setminus \{u,v\}} \delta_{sv}(u) \delta_{st}(v) \quad \text{by (A9)}
\]

(A11)

\[
= \delta_{sv}(u) \delta_{st}(v)
\]

(A12)

since \( \delta_{su}(v) = 0 \) by (A3) and using Eq. (A8), we obtain

\[
\delta_{s}(u,v) = \frac{\delta_{s}(v)}{\sigma_{sv}} \sigma_{sv}(u)
\]

(A13)

We use this result to re-express the co-betweenness defined in [4] as

\[
C(u,v) = \sum_{s,t \in V \setminus \{u,v\}} \delta_{st}(u,v)
\]

(A14)

\[
= \sum_{s \in V \setminus \{u,v\}} \left( \sum_{t \in V \setminus \{u,v\}} \delta_{st}(u,v) \right)
\]

(A15)

\[
= \sum_{s \in V \setminus \{u,v\}} \delta_{s}(u,v)
\]

(A16)

\[
= \sum_{s \in V \setminus \{u,v\}} \frac{\delta_{s}(v)}{\sigma_{sv}} \sigma_{sv}(u).
\]

(A17)

Lastly, to establish the recursive relation in [4], note that for a child vertex \( w \in c_{s}(v) \) every path to \( v \) gives rise to exactly one path to \( w \) by following the edge \((v,w)\). This means that

\[
\sigma_{sv}(w) = \sigma_{sv} \quad \text{for } w \in c_{s}(v),
\]

(A18)

and that

\[
\delta_{sw}(v) = \frac{\sigma_{sw}(v)}{\sigma_{sw}} = \frac{\sigma_{sv}}{\sigma_{sw}} \quad \text{for } w \in c_{s}(v).
\]

(A19)
Also note that for \( t = w \) we have
\[
\delta_{st}(w) = 1. \quad (A20)
\]
This allows us to decompose \( \delta_s(v) \) in essentially the same manner as [1], namely,
\[
\delta_s(v) = \sum_{t \in \mathcal{V} \setminus \{v\}} \delta_{st}(v) = \sum_{t \in \mathcal{V} \setminus \{v\}} \sum_{w \in \mathcal{C}_s(v)} \delta_{st}(v, w) = \sum_{w \in \mathcal{C}_s(v)} \sum_{t \in \mathcal{V} \setminus \{v\}} \delta_{sw}(v) \delta_{st}(w) = \sum_{w \in \mathcal{C}_s(v)} \frac{\sigma_{sw}}{\sigma_{sv}} \left(1 + \sum_{t \in \mathcal{V} \setminus \{v, w\}} \delta_{st}(w)\right) \quad (A21, A22, A23, A24, A25)
\]
\[
= \sum_{w \in \mathcal{C}_s(v)} \frac{\sigma_{sw}}{\sigma_{sv}} (1 + \delta_s(w)). \quad (A26)
\]
Where the last equality is due to the fact that since \( w \) is a child of \( v \) we have \( \sigma_{sv}(w) = 0 \) and thus \( \delta_{sv}(w) = 0 \).

**APPENDIX B: ALGORITHMIC COMPLEXITY**

Standard breadth-first search results put the running time for the first stage of our algorithm at \( O(n_v) \), and since we touch each edge at most twice when we compute the dependency scores \( \delta_s(v) \), the running time for the second stage is also \( O(n_v) \). Since we repeat each stage for each vertex in the network, the first two stages have a running time of \( O(n_v n_e) \). The running time for the depth-first traversal, that occurs during the third stage, depends on the number and length of all shortest paths in the network. Overall, we visit every shortest path once and compute a co-betweenness contribution for each edge of every shortest path. For ‘small-world’ networks i.e., networks with an \( O(\log n_v) \) diameter, we must compute \( O(\sigma \cdot \log n_v) \) contributions, where \( \sigma = \sum_{u,v \in \mathcal{V}} \sigma_{uv} \) is the total number of shortest paths in the network. So the overall running time for the algorithm is \( O(n_v n_e + \sigma \log n_v) \). Empirical evidence suggests that the upper bound for the average \( \frac{1}{|\mathcal{V}|} \sum_{u \in \mathcal{V}} \sigma_{uv} \) ranges from \( n_v^{0.19} \) to \( n_v^{0.32} \) for common random graph models, and at worst has been seen to reach \( n_v^{0.62} \) in the case of a network of airports. (In the latter case, there were extreme fluctuations in \( \frac{1}{|\mathcal{V}|} \sum_{u \in \mathcal{V}} \sigma_{uv} \) so the total number of shortest paths, \( \sigma \), might be much smaller than \( n_v(n_v - 1) \) times this upper bound.) This suggests a running time of \( O(n_v n_e + n_v^{2+p} \log n_v) \), though it is an open question to show this rigorously. In the case of sparse networks, where \( n_e \sim n_v \), this reduces to a running time of \( O(n_v^{2+p} \log n_v) \).

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