A NEW DISCRETE CUCKER-SMALE FLOCKING MODEL UNDER HIERARCHICAL LEADERSHIP

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Abstract. This paper studies the flocking behavior in a new discrete-time Cucker-Smale model under hierarchical leadership. The features of this model are that each individual has its own intrinsic nonlinear dynamics and the interaction between individuals follows a hierarchical leadership structure. Based on a specific matrix norm, we prove that the conditional flocking indeed occurs. Numerical experiments are given to confirm the theoretical results.

1. Introduction. The emergence of flocking behavior in self-organized systems has attracted considerable attention in recent years, see, e.g., [1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. One of the most celebrated mathematical models for flocking behavior is proposed by Cucker and Smale in [3, 4]. This model, which we will refer to as the C-S model throughout this paper, describes the way that agents interact to align with their neighbors. All agents influence each other in a symmetric manner and they showed that the key role in the C-S model is played by a nonnegative parameter $\beta$ (see Section 2 below), which measures the strength of the interaction between agents. More specifically, it is shown that if $0 \leq \beta < 1/2$ then the convergence of flock occurs unconditionally, regardless the initial state. On the other hand, if $\beta \geq 1/2$, the convergence to flocking still occurs provided the initial state satisfies some certain conditions.

Motivated by the behavior of the primitive C-S model, many variants have been extensively studied in the literature. For agents influence each other in a symmetric manner, a general result of collision-avoiding flocking was proved in [1]. In [7], the C-S model with nonlinear velocity coupling was considered. The C-S model with inter-particle bonding forces and with bounded cohesive and repulsive forces were studied in [16] and [18], respectively. For C-S model with a non-symmetric structure, Shen [17] considered a model with hierarchical leadership and obtained similar results as that of [5]. Following Shen’s work, hierarchical leadership flocking
was extended in different aspects: to include random interactions [5], to consider a free-will leader [6], and to allow for individual preference [10]. Besides, C-S model under rooted leadership was investigated in [13, 14]. A model under joint rooted leadership was concerned in [12] and the C-S flocking with alternating leaders was studied in [11].

We should emphasize that most of the above mentioned works do not consider the intrinsic dynamics of agents within the system. Even if a flock includes the free-will leader or agents [6, 13, 17], the hypothesis on the free-will acceleration implies that the velocity of each agent eventually converges to a constant vector. Thus, the whole flock moves along a straight-line direction finally. However, in reality, a flock of birds may appear to move along some more complex trajectories. Therefore, it is interesting to explore the mechanism that could create complex motions. In this paper, we will study a discrete-time C-S model with a hierarchical leadership structure and each individual has its own intrinsic nonlinear dynamics. In other words, each agent’s behavior is not only influenced by its neighbors, but also determined by its own intrinsic dynamics. With a mild assumption on the intrinsic dynamics, we will employ a specific matrix norm introduced previously by Li [10] to find some sufficient conditions that ensure the occurrence of flocking behavior. We find that these conditions depend not only on the model parameters and the initial states, but also the coupling topology of the system. Numerical experiments will be given to verify the theoretical results. As we will see in Section 4, including the intrinsic dynamics of agents within the system. Even if a flock includes the free-will leader [6], and to allow for individual preference [10]. Besides, C-S model

2. Model formulation and preliminaries. We will consider a discrete-time C-S flocking model under hierarchical leadership. The dynamics of the system, composed of \( N + 1 \) agents labeled by \( \{0, 1, \ldots, N\} \), is described by

\[
\begin{align*}
\dot{x}_i(t + 1) &= x_i(t) + hv_i(t), \quad i = 0, 1, \ldots, N, \\
\dot{v}_i(t + 1) &= v_i(t) + hf(t, v_i(t)) + \sum_{j=0}^{N} \alpha_{ij}(x(t))(v_j(t) - v_i(t)),
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^3 \) and \( v_i(t) \in \mathbb{R}^3 \) are respectively the position and velocity of \( i \)th agent at time \( t \); \( h > 0 \) is the discrete time step and \( t \in \mathbb{N} \cup \{0\} := \{0, 1, 2, \cdots\} \) is the discrete time mark (in other words, we use \( x_i(t) \) and \( v_i(t) \) to represent \( x_i(th) \) and \( v_i(th) \) at time instant \( th \), respectively); \( f(t, v_i(t)) \) is a nonlinear function describing the intrinsic dynamics of \( i \)th agent at time \( t \) and we assume that \( f \) satisfies the following Lipschitz condition in the velocity variable:

**Assumption (H1).** There exists a constant \( \alpha > 0 \) such that for all \( t \in \mathbb{N} \cup \{0\} \) and \( u, w \in \mathbb{R}^3 \), we have

\[
|f(t, u) - f(t, w)|_2 \leq \alpha|u - w|_2,
\]

where \(| \cdot |_2 \) denotes the usual vector 2-norm in \( \mathbb{R}^3 \).

Indeed, \( f(t, v) \) represents the acceleration dynamics of each agent and the Lipschitz condition \( \alpha \) limits how fast the acceleration can change.

The quantities \( \alpha_{ij}(x(t)) := \alpha_{ij}(x_0(t), x_1(t), \cdots, x_N(t)) \) describe the way that the agents influence each other. In this paper, we assume that the coupled system (1)
A new discrete Cucker-Smale flocking model is equipped with the following hierarchical leadership structure [17], which enforces some of $a_{ij}$ to disappear.

**Assumption (H2).** System (1) possesses the following hierarchical leadership structure: (i) $a_{ij} \neq 0$ implies that $j < i$; and (ii) if the leader set of each agent $i$ is defined by $L(i) := \{ j | a_{ij} > 0 \}$, then for any $i > 0$, $L(i)$ is non-empty. In other words, under such a hierarchical leadership, agent 0 is the overall leader, which is not influenced by any other agent, and the adjacency matrix $A_x := (a_{ij}(x))$ is a strictly lower triangular $(N + 1) \times (N + 1)$ matrix,

$$A_x = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.$$  \hspace{1cm} (2)

Moreover, we assume that each nonzero weight function is taken as that considered in the primitive C-S model [3, 4],

$$a_{ij}(x(t)) = \frac{K}{(1 + |x_i(t) - x_j(t)|^2)^{\beta}} > 0,$$

where $K > 0$ measures the strength of leader-follower interactions and $\beta \geq 0$ captures the rate of decay of the influence between agents in the flock as they separate in space.

In this paper, we will study the flocking behavior of system (1) under the above hierarchical leadership with an overall leader, namely, agent 0. We say that coupled system (1) converges to a flock if its solution satisfies the following conditions:

$$\sup_{t \in \mathbb{N}} |x_i(t) - x_j(t)|_2 < \infty \text{ and } \lim_{t \to \infty} |v_i(t) - v_j(t)|_2 = 0, \forall 0 \leq i, j \leq N.$$  \hspace{1cm} (3)

To this aim, we first introduce

$$a_{ij}(t) := a_{ij}(x(t)) \text{ and } d_i(t) := \sum_{j=0, j \neq i}^{N} a_{ij}(t)$$

to simplify the notation. One can verify that $d_i(t) > 0$ for all $i > 0$ because $L(i)$ is non-empty for $i = 1, 2, \cdots, N$. We define the following reduced position and velocity vectors [10],

$$X = (X_1, X_2, \cdots, X_N)^T = (x_1 - x_0, x_2 - x_0, \cdots, x_N - x_0)^T,$$
$$V = (V_1, V_2, \cdots, V_N)^T = (v_1 - v_0, v_2 - v_0, \cdots, v_N - v_0)^T,$$

and then define

$$F(t, V) = (f(t, v_1) - f(t, v_0), f(t, v_2) - f(t, v_0), \cdots, f(t, v_N) - f(t, v_0))^T.$$  \hspace{1cm} (4)

With these notations and system (1), we have the following error system which will be the base for flocking analysis made in Section 3,

$$X(t + 1) - X(t) = h \begin{pmatrix} v_1(t) - v_0(t) \\ v_2(t) - v_0(t) \\ \vdots \\ v_N(t) - v_0(t) \end{pmatrix} = hV(t),$$
\[
\begin{align*}
V(t + 1) - V(t) &= -h \begin{pmatrix}
d_1(t) & 0 & \cdots & 0 \\
-a_{21}(t) & d_2(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_{N1}(t) & -a_{N2}(t) & \cdots & d_N(t)
\end{pmatrix} V(t) \\
&\quad + h F(t, V(t)) \\
&:= -h L_t V(t) + h F(t, V(t)),
\end{align*}
\]

or rewritten in the compact form
\[
\begin{align*}
\begin{cases}
X(t + 1) = X(t) + h V(t), \\
V(t + 1) = P_t V(t) + h F(t, V(t)),
\end{cases}
\end{align*}
\]

where \( P_t := I_N - h L_t \) is given by
\[
P_t = \begin{pmatrix}
1 - h d_1(t) & 0 & \cdots & 0 \\
ha_{21}(t) & 1 - h d_2(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
ha_{N1}(t) & ha_{N2}(t) & \cdots & 1 - h d_N(t)
\end{pmatrix}_{N \times N}
\]

Let us remark that in error system (5), the notation \( P_t V(t) \) actually means the \( 3N \times 3N \) matrix \( P_t \otimes I_3 \) acting on the vector \( V(t) \in \mathbb{R}^3 \), i.e., \( P_t V(t) := (P_t \otimes I_3) V(t) \), where \( \otimes \) denotes the usual Kronecker product (cf. [8]).

To deal with the discrete-time C-S flocking model with hierarchy and individual preference, Li developed a new matrix norm in [10], called \( \varepsilon \)-norm, to derive a time-successive decay estimate for the relative velocity. In this paper, we will also adopt such special matrix norm to perform our flocking analysis in next section. Now, we briefly recall some properties of this matrix norm. We denote the coupling topology of the coupled system (1) by a weighted directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \), where \( \mathcal{V} \) is the node set of the graph, \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the directed arc set of the graph, and matrix \( A = (a_{ij}) \) described above is the weighted adjacency matrix which assigns weight for each arc. In other words, \( a_{ij} > 0 \) if and only if \( (j, i) \in \mathcal{E} \), where \( (j, i) \) represents the directed arc from node \( j \) to node \( i \). The directed distance from node \( j \) to node \( i \), denoted by \( \text{dist}(j, i) \), is defined to be the number of edges in a shortest directed path from node \( j \) to node \( i \). The following definition gives a description of a parameter for system (1) under hierarchical leadership.

**Definition 2.1.** [14] The depth \( L \) of system (1) with the hierarchical leadership structure described in Assumption (H2) is the largest distance from the global leader to other agents, namely,
\[
L := \max\{\ell(i) \mid i = 0, 1, \ldots, N\},
\]

where \( \ell(i) := \text{dist}(0, i) \).

Obviously, we have \( L \leq N \). Let \( a_m(t) \) be the minimum value of positive \( a_{ij}(t) \) at time \( t \), that is,
\[
a_m(t) := \min\{a_{ij}(t) \mid (j, i) \in \mathcal{E}\} > 0.
\]

Since \( P_t \) given in (6) is a lower triangular matrix, the eigenvalues of \( P_t \) must lie on its main diagonal. Thus, the spectral radius of \( P_t \) satisfies \( \rho(P_t) \leq 1 - ha_m(t) \).
Now, let $\varepsilon \in (0, 1)$ and let $D$ denote the following $N \times N$ diagonal matrix,

$$
D := 
\begin{pmatrix}
\varepsilon^{(1)} & 0 & \cdots & 0 \\
0 & \varepsilon^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \varepsilon^{(N)}
\end{pmatrix}.
$$

Then for any real $N \times N$ matrix $Q$, we define the $\varepsilon$-norm of $Q$ as

$$
\|Q\|_\varepsilon := \|DQD^{-1}\|_\infty.
$$

It can be shown that $\| \cdot \|_\varepsilon$ is really a matrix norm (cf. [9, 10]), and we have the following estimate for $\varepsilon$

Lemma 2.2. [10] Assume that assumption (H2) holds and $h < 1/(NK)$. Then for all $t \in \mathbb{N} \cup \{0\}$, we have $\|P_t\|_\varepsilon \leq 1 - (1 - \varepsilon)h\alpha_m(t)$.

Moreover, we can show that the $\varepsilon$-norm is actually associated with some vector norm defined as follows. Given $u = (u_1, u_2, \cdots, u_N)\top \in \mathbb{R}^N$, we set $M : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ as

$$
M(u) := 
\begin{pmatrix}
u_1 & 0 & \cdots & 0 \\
u_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
u_N & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{N \times N} \quad \text{and} \quad |u|_\varepsilon := \|M(u)\|_\varepsilon.
$$

Lemma 2.3. [10]

(i) The norm $\| \cdot \|_\varepsilon$ is exactly the matrix norm induced by the vector norm $| \cdot |_\varepsilon$.

(ii) The vector norm $| \cdot |_\varepsilon$ is equivalent to the usual infinity vector norm: $\varepsilon^{L-1}|u|_\infty \leq |u|_\varepsilon \leq |u|_\infty$ for all $u \in \mathbb{R}^N$.

Remark 1. It has been shown in [10] that $P_t$ and $P_t \otimes I_3$ share the $\varepsilon$-norm, i.e., $\|P_t\|_\varepsilon = \|P_t \otimes I_3\|_\varepsilon$, where $\varepsilon$-norm of the $3N \times 3N$ matrix $P_t \otimes I_3$ is defined by $\|P_t \otimes I_3\|_\varepsilon := \|D(D \otimes I_3)(P_t \otimes I_3)(D \otimes I_3)^{-1}\|_\infty$. Therefore, in the following we can consider the $\varepsilon$-norm of $P_t$ instead of $P_t \otimes I_3$.

Remark 2. [10] For the $\varepsilon$-norm of vector $V(t) \in (\mathbb{R}^3)^N$, we can verify that

$$
|V(t)|_\varepsilon = \max \left\{|V_1(t)|_\infty, \varepsilon^{\ell(2) - \ell(1)}|V_2(t)|_\infty, \cdots, \varepsilon^{\ell(N) - \ell(1)}|V_N(t)|_\infty\right\},
$$

which implies

$$
\varepsilon^{L-1}|V(t)|_\infty \leq |V(t)|_\varepsilon \leq |V(t)|_\infty.
$$

In addition, by virtue of the equivalence of $\infty$-norm and 2-norm in $(\mathbb{R}^3)^N$, we have

$$
|V(t)|_\varepsilon \leq |V(t)|_2 \leq \varepsilon^{1-L}\sqrt{3N}|V(t)|_\varepsilon.
$$

We note that these results (11)-(13) still hold if we replace the vector $V(t)$ by $X(t)$ and $F(t, V)$.

3. Flocking analysis. In this section, we will analyze the flocking behavior of the discrete-time system (1) under hierarchical leadership. First, we estimate the evolution of relative velocity $V(t)$.

Lemma 3.1. Assume that assumptions (H1) and (H2) hold and $h < 1/(NK)$. Then for all $t \in \mathbb{N} \cup \{0\}$, we have

(i) $|F(t, V(t))|_\varepsilon \leq \varepsilon^{1-L}\alpha\sqrt{3}|V(t)|_\varepsilon$. 

conditions which are analogous to that in \[10\]:

On the other hand, to derive the flocking estimate, we also need the following two

This completes the proof.

Proof. (i) From Remark 2, we obtain that

\[
|F(t, V(t))|_\epsilon \leq |F(t, V(t))|_\infty = \max_{1 \leq i \leq N} |f(t, v_i) - f(t, v_0)|_\infty \\
\leq \max_{1 \leq i \leq N} |f(t, v_i) - f(t, v_0)|_2.
\]

According to the assumption (H1) and the equivalence of \(\infty\)-norm and 2-norm in \(\mathbb{R}^3\), we have

\[
\max_{1 \leq i \leq N} |f(t, v_i) - f(t, v_0)|_2 \leq \alpha \max_{1 \leq i \leq N} |v_i - v_0|_2 \leq \alpha \sqrt{3} \max_{1 \leq i \leq N} |v_i - v_0|_\infty = \alpha \sqrt{3} |V(t)|_\infty,
\]

which combining with (12) implies the assertion,

\[
|F(t, V(t))|_\epsilon \leq \alpha \sqrt{3} |V(t)|_\infty \leq \varepsilon^{1-L} \alpha \sqrt{3} |V(t)|_\epsilon.
\]

(ii) From the second equation of error system (5) and the compatibility of \(\| \cdot \|_\epsilon\) and \(\| \cdot \|_\epsilon\), we have

\[
|V(t + 1)|_\epsilon \leq \|\Pi_\epsilon\|_\epsilon |V(t)|_\epsilon + h|F(t, V(t))|_\epsilon.
\]

Combining Lemma 2.2 with part (i), we can conclude that

\[
|V(t + 1)|_\epsilon \leq (1 - (1 - \varepsilon)\alpha ha_m(t) + \varepsilon^{1-L} \alpha \sqrt{3}) |V(t)|_\epsilon.
\]

This completes the proof. \(\square\)

We also need the following lemma, whose proof can be found in [2]:

**Lemma 3.2.** Let \(c_1, c_2 > 0\) and \(s > q > 0\). Then the equation \(F(z) := z^s - c_1 z^q - c_2 = 0\) has a unique positive zero \(z_*\) satisfying \(0 < z_* \leq \max\{(2c_1)^{\frac{1}{s-q}}, (2c_2)^{\frac{1}{q}}\}\), and \(F(z) < 0\) for \(0 \leq z < z_*\).

Before presenting the main results, we specify some notation and assumptions related to the parameters and initial configuration. We define

\[
a := \frac{2\sqrt{6N}|V(0)|_\epsilon}{(1 - \varepsilon)\varepsilon^{L-1}K} > 0 \quad \text{and} \quad b := 1 + \sqrt{6N} \varepsilon^{1-L} |X(0)|_\epsilon > 1,
\]

and assume the discrete time step \(h\) is sufficiently small such that

\[
h < \min \left\{ \frac{1}{NK}, \frac{1}{\varepsilon^{1-L} \alpha \sqrt{3}} \right\}.
\]

To show that system (1) converges to a flock, we have to show that \(|V(t)|_\epsilon\) decreases to zero in time. In view of Lemma 3.1 (ii), a sufficient condition is given by

\[
1 - (1 - \varepsilon)\alpha ha_m(t) + \varepsilon^{1-L} \alpha \sqrt{3} h < 1,
\]

or equivalently, using the definition of \(a_m(t)\) given in (8),

\[
\left( 1 + \max_{i,j \in E} |x_i(t) - x_j(t)|_2^2 \right)^{\beta} < \frac{(1 - \varepsilon)\varepsilon^{L-1}K}{\alpha \sqrt{3}}.
\]

On the other hand, to derive the flocking estimate, we also need the following two conditions which are analogous to that in [10]:

\[
\gamma^{2\beta} < \frac{(1 - \varepsilon)\varepsilon^{L-1}K}{2^{1+\beta} \alpha \sqrt{3}}, \quad (2\gamma^2 + 12N \varepsilon^{2-2L} h^2 |V(0)|_2^2)^{\beta} < \frac{(1 - \varepsilon)\varepsilon^{L-1}K}{2\alpha \sqrt{3}},
\]

where the value of \(\gamma\), which depends on \(a\), \(b\) and \(\beta\), will be specified in Theorem 3.3. We are now in the position to state the main results of the present paper.
Theorem 3.3. Assume that assumptions (H1), (H2) and (15) hold and one of the following conditions is satisfied

(i) $0 \leq \beta < 1/2$ and (18) is satisfied with $\gamma := \max \left\{ (2a)^{\frac{1}{2\beta}}, 2b \right\}$.

(ii) $\beta = 1/2$ and (18) is satisfied with $\gamma := \frac{b}{1-a} \frac{1}{2h_\beta}$ and $a < 1$.

(iii) $\beta > 1/2$ and (18) is satisfied with $\gamma := \frac{b}{2h_\beta - 1}$ and

$$\left( \frac{1}{a} \right)^{\frac{1}{2\beta}} \frac{1}{2\beta} - \left( \frac{1}{2\beta} \right)^{\frac{2\beta}{\beta - 1}} - b > \frac{6e^{-2L\varepsilon} |V(0)|_\varepsilon^2}{NK^2} \left( 2\beta a \right)^{\frac{1}{2\beta}} + \frac{2\varepsilon |V(0)|_\varepsilon}{\sqrt{NK^2}}.$$

Then the reduced velocity vector $V(t)$ converges to zero exponentially and the reduced position vector $X(t)$ is uniformly bounded, i.e., the coupled system (1) converges to a flock. Moreover, we have the estimates for all $t \in \mathbb{N}$,

$$|V(t)|_\varepsilon \leq \left( 1 - \varepsilon^{1-L} \alpha \sqrt{3h} \right) t |V(0)|_\varepsilon \quad \text{and} \quad |X(t)|_\varepsilon \leq \left( \frac{\gamma^2 - 1}{6N\varepsilon^{2-2L}} \right)^{1/2}.$$ \hspace{1cm} (19)

Proof. First, note that we always have $1 < b < \gamma$, since $a > 0$ and $b > 1$. Define the integer set,

$$\mathcal{T} = \left\{ t \in \mathbb{N} \cup \{ 0 \} : (1 + 6N\varepsilon^{2-2L} |X(t)|_\varepsilon^2)^{\beta} \leq \frac{(1 - \varepsilon^{L-1} K)}{2\alpha \sqrt{3}} \right\}.$$

From (14) and the first inequality in (18), we have

$$(1 + 6N\varepsilon^{2-2L} |X(t)|_\varepsilon^2)^{\beta} \leq b^{2\beta} < \gamma^{2\beta} < \frac{(1 - \varepsilon^{L-1} K)}{2\alpha \sqrt{3}},$$

which implies $0 \in \mathcal{T}$ and so $\mathcal{T}$ is nonempty. Assume that $\mathcal{T} \neq \mathbb{N} \cup \{ 0 \}$ and let $t_0 = \min \{ \mathbb{N} \setminus \mathcal{T} \} \geq 1$. Then we have

$$(1 + 6N\varepsilon^{2-2L} |X(t_0)|_\varepsilon^2)^{\beta} > \frac{(1 - \varepsilon^{L-1} K)}{2\alpha \sqrt{3}}. \hspace{1cm} (20)$$

Let $t < t_0$ and let $t_* \in \arg \max_{0 \leq \tau \leq t} |X(\tau)|_\varepsilon$. Thus, we have for all $\tau \in [0, t]$ that

$$a_m(\tau) = \min_{0 \leq i, j \leq N, \tau \in \mathcal{L}(i)} \frac{K}{(1 + |x_i(\tau) - x_j(\tau)|_2^2)^\beta}$$

$$= \frac{K}{\left( 1 + \max_{0 \leq i, j \leq N, \tau \in \mathcal{L}(i)} |x_i(\tau) - x_j(\tau)|_2^2 \right)^\beta}$$

$$= \frac{K}{\left( 1 + \max_{0 \leq i, j \leq N, \tau \in \mathcal{L}(i)} |X_i(\tau) - X_j(\tau)|_2^2 \right)^\beta} \geq \frac{K}{\left( 1 + 2 |X(\tau)|_2^2 \right)^\beta},$$

where we used the facts that $\mathcal{L}(0)$ is an empty set and

$$|X_i(\tau) - X_j(\tau)|_2 \leq \sqrt{2} |X(\tau)|_2, \quad \forall 1 \leq i, j \leq N.$$
Using the triangle inequality, (5), Lemma 3.1 (ii), and (23), we can derive that

\[ a_m(\tau) \geq \frac{K}{\left(1 + 2|X(\tau)|_2^2\right)^{\beta}} \geq \frac{K}{\left(1 + 6N\varepsilon^2 - 2L|X(\tau)|_2^2\right)^{\beta}} \geq \frac{K}{\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}}. \]  

(21)

Thanks to \(0 \leq t_* \leq t < t_0\), we have \(t_* \in \mathcal{T}\) and this implies that

\[ \frac{(1 - \varepsilon)K}{2\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}} \geq \varepsilon^{1-L}\alpha\sqrt{3}. \]  

(22)

Combining (21) with (22), we obtain for all \(\tau \in [0, t] \)

\[ (1 - \varepsilon)a_m(\tau) - \varepsilon^{1-L}\alpha\sqrt{3} \geq \frac{(1 - \varepsilon)K}{\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}} - \varepsilon^{1-L}\alpha\sqrt{3} \]

\[ \geq \frac{(1 - \varepsilon)K}{2\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}}. \]  

(23)

Using the triangle inequality, (5), Lemma 3.1 (ii), and (23), we can derive that

\[ |X(t)_e| \leq |X(0)_e| + \sum_{\tau = 0}^{t-1} |X(\tau + 1) - X(\tau)|_e \]

\[ \leq |X(0)_e| + h \sum_{\tau = 0}^{t-1} |V(\tau)|_e \]

\[ \leq |X(0)_e| + h|V(0)|_e \sum_{\tau = 0}^{t-1} \prod_{k = 0}^{\tau - 1} \left(1 - (1 - \varepsilon)h a_m(k) + \varepsilon^{1-L}\alpha\sqrt{3}h\right) \]

\[ \leq |X(0)_e| + h|V(0)|_e \sum_{\tau = 0}^{t-1} \left(1 - \frac{h(1 - \varepsilon)K}{2\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}}\right)^\tau \]

\[ \leq |X(0)_e| + |V(0)|_e \frac{2\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}}{(1 - \varepsilon)K}. \]  

(24)

Now if we replace the role \(t\) in (24) by \(t_*\), then there exists \(t_{**} \in [0, t_*] \) with \(t_{**} \in \arg \max_{0 \leq \tau \leq t_*} |X(\tau)|_e\) such that

\[ |X(t_*)_e| \leq |X(0)_e| + |V(0)|_e \frac{2\left(1 + 6N\varepsilon^2 - 2L|X(t_{**})|_2^2\right)^{\beta}}{(1 - \varepsilon)K} \]

\[ = |X(0)_e| + |V(0)|_e \frac{2\left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{\beta}}{(1 - \varepsilon)K}. \]  

(25)

Let \(Z = \left(1 + 6N\varepsilon^2 - 2L|X(t_*)|_2^2\right)^{1/2} > 0\), then it follows from (25) that

\[ Z \leq 1 + \varepsilon^{1-L}\sqrt{6N}|X(t_*)|_e \]

\[ \leq 1 + \varepsilon^{1-L}\sqrt{6N} \left(|X(0)_e| + \frac{2|V(0)|_e}{(1 - \varepsilon)K} Z^{2\beta}\right). \]
It follows from (28) and the first inequality in (18) that
\[
\sqrt{\frac{2\sqrt{6N\varepsilon^{1-L}|V(0)|\varepsilon}{(1-\varepsilon)K}}^{2} + 1 + \sqrt{6N\varepsilon^{1-L}|X(0)|\varepsilon}} = aZ^{2\beta} + b,
\]
where \(a\) and \(b\) are given in (14). Define \(F(z) = z - az^{2\beta} - b\). Next, the rest of the proof can be divided into three cases as that stated in (i), (ii) and (iii). However, we should emphasize that except for the first case \(0 \leq \beta < \frac{1}{2}\), the proof runs as that one of Theorem 4.1 in [10] step by step with slight modifications. Thus, in what follows, we consider only the case \(0 \leq \beta < \frac{1}{2}\) and we refer the reader to [10] for the other two cases.

Assume that \(0 \leq \beta < \frac{1}{2}\). By (26), we have \(F(Z) \leq 0\). According to Lemma 3.2, one can derive that
\[
0 < Z \leq \gamma = \max\{(2a)^{\frac{1}{1-2\beta}}, 2b\},
\]
which is equivalent to
\[
|X(t_{*})|^{2} \leq \frac{\gamma^{2} - 1}{6N\varepsilon^{2-2L}}.
\]
Particularly, taking \(t = t_{0} - 1\), we have
\[
|X(t_{0} - 1)|^{2} \leq \frac{\gamma^{2} - 1}{6N\varepsilon^{2-2L}},
\]
and consequently,
\[
|X(t_{0})|^{2}_{t_{0}} \leq \left( \frac{|X(t_{0} - 1)|^{2} + h|V(t_{0} - 1)|\varepsilon}{2} \right)^{2} \leq \frac{\gamma^{2} - 1}{3N\varepsilon^{2-2L}} + 2h^{2}|V(0)|^{2}_{\varepsilon}.
\]
Therefore, we can conclude from the second inequality in (18) that
\[
(1 + 6N\varepsilon^{2-2L}|X(t_{0})|^{2})^{\beta} \leq \left( 1 + 6N\varepsilon^{2-2L}\left( \frac{\gamma^{2} - 1}{3N\varepsilon^{2-2L}} + 2h^{2}|V(0)|^{2}_{\varepsilon} \right) \right)^{\beta} < \frac{(2\gamma^{2} + 12N\varepsilon^{2-2L}h^{2}|V(0)|^{2})^{\beta}}{2\alpha\sqrt{3}},
\]
which is, however, contrary to (20). Hence, no such \(t_{0}\) exists and so \(T = \mathbb{N} \cup \{0\}\). By virtue of (27), we obtain
\[
|X(t)|_{t} \leq \left( \frac{\gamma^{2} - 1}{6N\varepsilon^{2-2L}} \right)^{1/2}, \quad \forall t \in \mathbb{N}.
\]
From (21) with \(\tau = t\), we have
\[
(1 - \varepsilon)a_{m}(t) - \varepsilon^{1-L}a\sqrt{3} \geq \frac{(1 - \varepsilon)K}{\gamma^{2\beta}} - \varepsilon^{1-L}a\sqrt{3}.
\]
It follows from (28) and the first inequality in (18) that
\[
(1 - \varepsilon)a_{m}(t) - \varepsilon^{1-L}a\sqrt{3} \geq \frac{(1 - \varepsilon)K}{\gamma^{2\beta}} - \varepsilon^{1-L}a\sqrt{3} > \varepsilon^{1-L}2^{1+\beta}a\sqrt{3} = \varepsilon^{1-L}a\sqrt{3}.
\]
Finally, we conclude from Lemma 3.1 that
\[ |V(t)|_\varepsilon \leq (1 - \varepsilon^{1-L}\alpha \sqrt{3}h)^t |V(0)|_\varepsilon, \quad \forall \ t \in \mathbb{N}. \]
This completes the proof.

4. Numerical simulations. In this section, we will study three examples to demonstrate the three conditions in Theorem 3.3, respectively. We consider a system composed of 11 agents, namely, \( N = 10 \). The hierarchical leadership structure is given in Figure 1. Thus, the depth of the considered system is \( L = 2 \).

Example 4.1. We first consider the condition (i) in Theorem 3.3. Let \( \beta = 1/100, \ \varepsilon = 0.9, \ \text{and} \ K = 5 \). The initial state for this example is chosen so that \( |X(0)|_\varepsilon = 24.7858 \) and \( |V(0)|_\varepsilon = 40.5 \). This implies that \( a = 1129.4, \ b = 214.3225, \ \text{and} \ \gamma = 2304.8 \). The nonlinear function \( f \) describing the intrinsic dynamics is given by
\[ f(t,y) = \left( \sin(0.1y_1) + e^{0.1t} \cos(t), \cos(0.1y_2), \sin(0.05y_3) + \cos(\pi t) \right)^\top, \]
and the Lipschitz constant is given by \( \alpha = 0.1 \). From assumption (15), we need \( h < \min\{0.02, 5.1962\} \) and thus we take \( h = 0.01 \). Besides, one can claim that (18) is fulfilled. Consequently, by Theorem 3.3, system (1) will converge to a flock. The 3D plots of \( x_i(t) \) for \( 0 \leq i \leq 10 \) and \( 0 \leq t \leq 30 \) and the waveform of \( \text{Err}(t) := \max_{1 \leq i \leq 10} |v_i(t) - v_0(t)|_\infty \) are depicted in Figure 2. In the left panel of Figure 2, we use the symbol ‘◦’ to indicate the initial positions of all agents, and the blue curve stands for the trajectory of the leading agent 0. We can find that the flocking behavior occurs and the right panel of Figure 2 confirms that the velocity consensus can be reached exponentially. Again, we emphasize that all the three conditions in Theorem 3.3 are sufficient for ensuring the occurrence of flocking behavior. Thus, in order to examine the critical value of the Lipschitz constant \( \alpha \) for flocking in this example, we further consider the function \( f \) replaced by
\[ f_\eta(t,y) = \left( \sin(0.1\eta y_1) + e^{0.1t} \cos(t), \cos(0.1\eta y_2), \sin(0.05\eta y_3) + \cos(\pi t) \right)^\top, \]
where \( \eta \) is a positive parameter can be adjusted. Then we have the Lipschitz constant \( \alpha = 0.1\eta \) for function \( f_\eta \) in the velocity variable. Now, increasing the value of \( \eta \), we find that the critical value of \( \alpha \) for the coupled system to lose the flocking behavior is about \( \alpha \approx 22.1348 \), i.e., \( \eta \approx 221.348 \). Furthermore, closer inspection reveals that only the 5th agent asymptotically escapes from the coupled system as time evolves. The numerical results are reported in Figure 3.
Example 4.2. We now consider the condition (ii) in Theorem 3.3. Let $\beta = 1/2$, $\varepsilon = 0.8$, and $K = 225$. The initial state for this example is chosen so that $|X(0)|_\varepsilon = 21.5897$ and $|V(0)|_\varepsilon = 2.56$. This leads to $a = 0.7051 < 1$, $b = 210.041$, and $\gamma = 712.1364$. The nonlinear function $f$ is now given by

$$f(t, y) = (\sin(0.01y_1) + e^{0.1t}\cos(\sqrt{3}t), \cos(0.01y_2) + e^{0.2t}\sin(\sqrt{2}t),$$

$$\sin(0.01y_3) + e^{-t}\cos(\pi t))^T.$$ 

Then the Lipschitz constant is $\alpha = 0.01$. From (15) we have to require that $h < \min\{0.0004, 46.1880\}$. Thus, we take $h = 0.0001$ in this example. One can check that (18) is fulfilled and thus system (1) converges to a flock. The 3D plots of $x_i(t)$ for $0 \leq i \leq 10$ and $0 \leq t \leq 30$ and the waveform of $\text{Err}(t) := \max_{1 \leq i \leq 10} |v_i(t) - v_0(t)|_\infty$ are drawn in Figure 4, from which the flocking behavior can be clearly observed.

Example 4.3. In this example, we take $\beta = 0.6$, $\varepsilon = 0.9$, and $K = 290$. The initial state for this example is chosen so that $|X(0)|_\varepsilon = 3.2158$ and $|V(0)|_\varepsilon = 0.396$. Therefore, we have $a = 0.1904$, $b = 28.6774$ and $\gamma = 172.0645$. The nonlinear function $f$ is given by
and so the Lipschitz constant is \( \alpha = 0.01 \). From (15), \( h < \min\{0.0003, 51.9615\} \). Therefore, we take \( h = 0.0001 \). Moreover, one can verify that condition (iii) in Theorem 3.3 is fulfilled and so system (1) converges to a flock. The 3D plots of \( x_i(t) \) for \( 0 \leq i \leq 10 \) and \( 0 \leq t \leq 30 \) and the waveform of \( \text{Err}(t) := \max_{1 \leq i \leq 10} |v_i(t) - v_0(t)|_\infty \) are depicted in Figure 5, from which we can find that the flocking behavior occurs very quickly.

5. **Conclusions.** In this paper, we have proposed and analyzed a new discrete-time C-S flocking model, in which each individual has its own intrinsic nonlinear dynamics and the interaction between individuals within the system follows a hierarchical leadership structure. With the help of a specific matrix norm introduced previously by Li [10], we have derived three sufficient conditions to ensure the occurrence of flocking behavior. These conditions depend not only on the model parameters and the initial states, but also the coupling topology of the system. Numerical simulations have been performed to confirm the theoretical analysis. As one can see from the numerical results, the proposed discrete-time C-S flocking model indeed
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displays complex and interesting dynamical behaviors. Finally, we remark that the study of the corresponding continuous-time model will be our future work.

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REFERENCES

[1] F. Cucker and J. Dong, A general collision-avoiding flocking framework, IEEE Trans. Automat. Control, 56 (2011), 1124–1129.
[2] F. Cucker and S. Smale, Best choices for regularization parameters in learning theory: On the bias-variance problem, Found. Comput. Math., 2 (2002), 413–428.
[3] F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control, 52 (2007), 852–862.
[4] F. Cucker and S. Smale, On the mathematics of emergence, Jpn. J. Math., 2 (2007), 197–227.
[5] F. Dalmao and E. Moedecki, Cucker-Smale flocking under hierarchical leadership and random interactions, SIAM J. Appl. Math., 71 (2011), 1307–1316.
[6] J. Dong, Flocking under hierarchical behavior with a free-will leader, Internat. J. Robust Nonlinear Control, 23 (2013), 1891–1898.
[7] S. Ha, T. Ha and J. Kim, Emergent behavior of a Cucker-Smale type particle model with nonlinear velocity coupling, IEEE Trans. Automat. Control, 55 (2010), 1679–1683.
[8] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[9] R. A. Horn and C. R. Johnson, Matrix Analysis, second edition, Cambridge University Press, Cambridge, 2013.
[10] Z. Li, Effectual leadership in flocks with hierarchy and individual preference, Discrete Contin. Dyn. Syst., 34 (2014), 3683–3702.
[11] Z. Li and S. Ha, Cucker-Smale flocking with alternating leaders, Quart. Appl. Math., 73 (2015), 693–709.
[12] Z. Li, S. Ha and X. Xue, Emergent phenomena in an ensemble of Cucker-Smale particles under joint rooted leadership, Math. Models Methods Appl. Sci., 24 (2014), 1389–1419.
[13] Z. Li and X. Xue, Cucker-Smale flocking under rooted leadership with free-will agents, Phys. A, 410 (2014), 205–217.
[14] Z. Li and X. Xue, Cucker-Smale flocking under rooted leadership with fixed and switching topologies, SIAM J. Appl. Math., 70 (2010), 3156–3174.
[15] S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior, J. Stat. Phys., 144 (2011), 923–947.
[16] J. Park, H. Kim and S. Ha, Cucker-Smale flocking with inter-particle bonding forces, IEEE Trans. Automat. Control, 55 (2010), 2617–2623.
[17] J. Shen, Cucker-Smale flocking under hierarchical leadership, SIAM J. Appl. Math., 68 (2007), 694–719.
[18] Q. Song, F. Liu, J. Cao and J. Qiu, Cucker-Smale flocking with bounded cohesive and repulsive forces, Abstr. Appl. Anal., 2013, Art. ID 783279, 9 pp.
[19] J. Zhou, X. Wu, W. Yu, M. Small and J. Lu, Flocking of multi-agent dynamical systems based on pseudo-leader mechanism, Systems Control Lett., 61 (2012), 195–202.

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